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Redacţia: 400084 Cluj-Napoca, str. M. Kogălniceanu nr. 1 • Telefon: 405300

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DIFFERENTIAL SUBORDINATION AND STARLIKENESS OF ANALYTIC FUNCTIONS

R. AGHALARY AND S. B. JOSHI

Abstract. In the present paper by using the method of differential subordination we aim to prove some classical results in univalent function theory. In particular we give some new sufficient condition for an analytic function to be starlike and convex in the unit disc U. Also by applying Ruscheweyh derivative we investigate some argument properties of some subclasses of univalent functions.

1. Introduction

Let $\mathcal{A}$ denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

which are analytic in the unit disc $U = \{z : |z| < 1\}$. For f and g which are analytic in U, we say that f is subordinate to g,written $f(z) \prec g(z)$, if there exists an analytic function ω in U such that $\omega(0) = 0$, $|\omega(z)| < 1$ and $f(z) = g(\omega(z))$.
For $0 < b \leq a$, the function $p \in \mathcal{A}$ is said to be in $P(a, b)$ if and only if

$$|p(z) - a| < b, \qquad z \in U.$$

Without loss of generality we omit the trivial case $p(z) = 1$ and assume that $|1-a| < b$.
For $-1 \leq B < A \leq 1$, the function $p \in \mathcal{A}$ is said to be in $P[A, B]$ if and only if

$$p(z) \prec \frac{1 + Az}{1 + Bz}, \qquad z \in U.$$

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Here the symbol $'\prec'$ stands for subordination.For $0 < b \leq a$, there is a correspondence between $P(a,b)$ and $P[A,B]$, namely

$$P(a,b) \equiv P\left[\frac{b^2 - a^2 + a}{b}, \frac{1-a}{b}\right].$$

Two subclasses of $P(a,b)$ and $P[A,B]$ that have been studied extensively by other authors (e.g.see [2])are $P(1,b)$ and $P[A,-1]$.
The object of the present paper is to investigate some argument properties of analytic functions. We also obtain new sufficient condition for starlikeness and convexity.

First we introduce a subordination criterion for $p(z)$ which is subordinate to $\left(\frac{1+z}{1-z}\right)^\eta$.
To establishing our main results,we shall need the following results, which are due to Miller and Mocanu [4], Nunokawa [5] and Miller and Mocanu [4], respectively.

Lemma 1.1. *Let h be a convex function in U and let λ be analytic in U with $\Re\lambda(z) \geq 0$. If q is analytic in U and $q(0) = h(0)$, then*

$$q(z) + \lambda(z)zq'(z) \prec h(z),$$

implies

$$q(z) \prec h(z) \quad (z \in U).$$

Lemma 1.2. *Let q be analytic in U with $q(0) = 0$ and $q(z) \neq 0$ in U. Suppose that there exists a point $z_0 \in U$ such that*

$$|\arg q(z)| < \frac{\pi\eta}{2} \text{ for } |z| < |z_0| \tag{1}$$

and

$$|\arg q(z_0)| = \frac{\pi\eta}{2}, \tag{2}$$

where $0 < \eta \leq 1$. Then we have

$$\frac{z_0 q'(z_0)}{q(z_0)} = ik\eta, \tag{3}$$

where

$$k \geq \frac{1}{2}(a + \frac{1}{a}) \quad \text{when} \quad \arg q(z_0) = \frac{\pi\eta}{2}, \tag{4}$$

$$k \leq \frac{-1}{2}(a + \frac{1}{a}) \quad \text{when} \quad \arg q(z_0) = \frac{-\pi\eta}{2}, \tag{5}$$

and

$$q(z_0)^{\frac{1}{\eta}} = \pm ia \quad (a > 0). \tag{6}$$

Lemma 1.3. *Let F be analytic in U and let G be analytic and univalent on $\bar{U}$, with $F(0) = G(0)$. If F is not subordinate to G, then there exist points $z_0 \in U$ and $\xi_0 \in \partial U$, and $m \geq 1$ for which $F(|z| < |z_0|) \subset G(|z| < |z_0|), F(z_0) = G(\xi_0)$ and $z_0 F'(z_0) = m\xi_0 G'(\xi_0)$.*

2. **Main Results**

We now state and prove our main results.

Lemma 2.1. *Let p be analytic in U with $p(0) = 1$. If*

$$\left|\arg\left[p(z) + \frac{\lambda}{S(z)} zp'(z)\right]\right| < \frac{\pi}{2}\delta \quad (0 < \delta \leq 1, \lambda \geq 0),$$

for some $S(z)$ where $S(z) \in P(a, b)$, then

$$|\arg p(z)| < \frac{\pi}{2}\eta,$$

where $\eta \quad (0 < \eta \leq 1)$ is the solution of the equation

$$\delta = \eta + \frac{2}{\pi}\tan^{-1}\left(\frac{\lambda\eta\sqrt{a^2 - b^2}}{a^2 + ab + \lambda\eta b}\right). \tag{7}$$

Proof. Let $h(z) = \left(\frac{1+z}{1-z}\right)^{\delta}$, we observe that h is convex and $h(0) = 1$. Applying Lemma 1.1 for this h with $\lambda(z) = \frac{\lambda}{S(z)}$, we see that $\Re p(z) > 0$ in U and hence $p(z) \neq 0$ in U. If there exists a point $z_0 \in U$ such that the conditions (1) and (2) are satisfied, then (by Lemma 1.2) we obtain (3) under the restrictions (4),(5) and (6). Since $S(z) \in P(a, b)$ we have

$$S(z) = re^{i\frac{\pi}{2}\phi},$$

where $a - b < r < a + b$ *and* $\frac{-2}{\pi}\sin^{-1}(\frac{b}{a}) < \phi < \frac{2}{\pi}\sin^{-1}(\frac{b}{a})$

At first, suppose that $p(z_0)^{\frac{1}{\eta}} = ia \quad (a > 0)$, we obtain

$$\begin{aligned}
\arg\left[p(z_0) + \frac{\lambda}{S(z_0)} z_0 p'(z_0)\right] &= \arg p(z_0) + \arg\left(1 + \frac{\lambda}{S(z_0)} \frac{z_0 p'(z_0)}{p(z_0)}\right) \\
&= \frac{\pi}{2}\eta + \arg\left(1 + \frac{\lambda}{re^{i\frac{\pi}{2}\phi}} i\eta k\right) \\
&= \frac{\pi}{2}\eta + \tan^{-1}\left(\frac{\lambda\eta k \cos(\frac{\pi}{2}\phi)}{r + \lambda\eta k \sin(\frac{\pi}{2}\phi)}\right) \\
&\geq \frac{\pi}{2}\eta + \tan^{-1}\left(\frac{\lambda\eta \cos(\sin^{-1}(\frac{b}{a}))}{a + b + \lambda\eta\frac{b}{a}}\right) \\
&= \frac{\pi}{2}\eta + \tan^{-1}\left(\frac{\lambda\eta\sqrt{a^2 - b^2}}{a^2 + ab + \lambda\eta b}\right) \\
&= \frac{\pi}{2}\delta
\end{aligned}$$

This is a contradiction to the assumption of our lemma.

Next, suppose that $p(z_0) = -ia \quad (a > 0)$. Applying the same method as the above, we have

$$\begin{aligned}
\arg\left[p(z_0) + \frac{\lambda}{S(z_0)} z_0 p'(z_0)\right] &= \frac{-\pi}{2}\eta - \tan^{-1}\left(\frac{\lambda\eta\sqrt{a^2 - b^2}}{a^2 + ab + \lambda\eta b}\right) \\
&= \frac{-\pi}{2}\delta,
\end{aligned}$$

where δ is given by (7) which contradict the assumption. This completes the proof of our lemma.

Theorem 2.2. *Let η be as defined by (7). Let $M(z) = z^n + ...$ and $N(z) = z^n + ...$ be analytic in U and such that, N satisfies*

$$\frac{zN'(z)}{N(z)} \in P(a, b).$$

Then

$$\left|\arg\left[(1 - \lambda)\frac{M(z)}{N(z)} + \lambda\frac{M'(z)}{N'(z)}\right]\right| < \frac{\pi}{2}\delta.$$

implies

$$\left|\arg\frac{M(z)}{N(z)}\right| < \frac{\pi}{2}\eta.$$

Proof. Consider the function $p(z) = \frac{M(z)}{N(z)}$ and let $S(z) = \frac{zN'(z)}{N(z)}$. Then by hypothesis, p is analytic and $p(0) = 1$. Hence all the conditions of Lemma 2.1 are satisfied. Now it is elementary to show that

$$(1-\lambda)\frac{M(z)}{N(z)} + \lambda\frac{M'(z)}{N'(z)} = p(z) + \frac{\lambda}{S(z)}zp'(z).$$

And hence Theorem 2.2 follows from Lemma 2.1.

The $\nu - th$ order Ruscheweyh Derivative [6] D^ν of a function $f \in \mathcal{A}$ is defined by

$$D^\nu f(z) = \frac{z}{(1-z)^{1+\nu}} * f(z) = z + \sum_{k=2}^{\infty} B_k(\nu)a_k z^k,$$

where

$$B_k(\nu) = \frac{(1+\nu)(2+\nu)...(\nu+k-1)}{(k-1)!}.$$

The operator $'*'$ stands for the convolution or Hadamard product of two power series $f(z) = \sum_{i=1}^{\infty} a_i z^i$ and $g(z) = \sum_{i=1}^{\infty} b_i z^i$ defined by

$$(f * g)(z) = f(z) * g(z) = \sum_{i=1}^{\infty} a_i b_i z^i.$$

From the definition of D^ν and the properties of convolution $'*'$ follows the identity

$$z(D^\nu f(z))' = (1+\nu)D^{1+\nu} f(z) - \nu D^\nu f(z). \tag{8}$$

Corollary 2.3. *Let $f \in \mathcal{A}$. If*

$$\left|\arg\left[(1-\lambda)\frac{D^\nu f(z)}{D^\mu g(z)} + \lambda\frac{(D^\nu f(z))'}{(D^\mu g(z))'}\right]\right| < \frac{\pi}{2}\delta, \quad (\nu \geq 0, \mu \geq 0, \lambda \geq 0, 0 < \delta \leq 1)$$

for some g where $D^\mu g(z) \in P[A,B], (-1 < B < A \leq 1)$, then

$$\left|\arg\frac{D^\nu f(z)}{D^\mu g(z)}\right| < \frac{\pi}{2}\eta,$$

where $\eta, (0 < \eta \leq 1)$ is the solution of the equation

$$\delta = \eta + \frac{2}{\pi}\tan^{-1}\left(\frac{\lambda\eta\cos\left(\sin^{-1}\frac{A-B}{1-AB}\right)}{\frac{1+A}{1+B} + \lambda\eta\frac{A-B}{1-AB}}\right). \tag{9}$$

Proof. If we let $a = \frac{1-AB}{1-B^2}.b = \frac{A-B}{1-B^2}, M(z) = D^\nu f(z)$ and $N(z) = D^\mu g(z)$ then in this case η is given by (9) and the corollary now follows from Theorem 2.2.

Taking $B \mapsto A$ and $g(z) = z$ in Corollary 2.3, we have

Corollary 2.4. *Let $f \in \mathcal{A}$. If*

$$\left|\arg\left[(1-\lambda)\frac{D^{\nu}f(z)}{z}+\lambda(D^{\nu}f(z))'\right]\right|<\frac{\pi}{2}\delta, \quad (\nu\geq 0, \lambda\geq 0, 0<\delta\leq 1),$$

then

$$\left|\arg\frac{D^{\nu}f(z)}{z}\right|<\frac{\pi}{2}\eta,$$

where $\eta, (0<\eta\leq 1)$ is the solution of the equation

$$\delta=\eta+\frac{2}{\pi}tan^{-1}\lambda\eta.$$

By using the same technique as in the proof of Lemma 2.1, we obtain

Theorem 2.5. *Let $f \in \mathcal{A}$.If*

$$\left|\arg\left[(1-\lambda)\frac{D^{\nu+1}f(z)}{z}+\lambda(D^{\nu+1}f(z))'\right]\right|<\frac{\pi}{2}\delta, \quad (\nu\geq 0, \lambda\geq 0, 0<\delta\leq 1),$$

then

$$\left|\arg\left[(1-\lambda)\frac{D^{\nu}f(z)}{z}+\lambda(D^{\nu}f(z))'\right]\right|<\frac{\pi}{2}\eta,$$

where $\eta, (0<\eta\leq 1)$ is the solution of the equation

$$\delta=\eta+\frac{2}{\pi}tan^{-1}\left(\frac{\eta}{1+\nu}\right).$$

Corollary 2.6. *Let $f \in \mathcal{A}$. If*

$$|\arg(f'(z)+\lambda zf''(z))|<\frac{\pi}{2}\delta,$$

then

$$\left|\arg\left[(1-\lambda)\frac{f(z)}{z}+\lambda f'(z)\right]\right|<\frac{\pi}{2}\eta$$

where $\eta, (0<\eta\leq 1)$ is the solution of the equation

$$\delta=\eta+\frac{2}{\pi}\tan^{-1}\eta.$$

We note that, by making use of Theorem 2.3, one can construct several new results for Bazilevic functions.

Theorem 2.7. *Let $f \in \mathcal{A}$. If*

$$\left|\arg\left[(1-\lambda)\frac{D^\nu f(z)}{z} + \lambda\frac{(D^\nu f(z))'}{z}\right]\right| < \frac{\pi}{2}\delta, \quad (\nu > 0, \lambda \geq 0, 0 < \delta \geq 1)$$

then

$$\left|\arg\frac{D^\nu f(z)}{D^{\nu-1}f(z)}\right| < \frac{\pi}{2}(\eta_1 + \eta_2),$$

where η_1, η_2 are the solutions of the equations

$$\delta = \eta_1 + \frac{2}{\pi}\tan^{-1}\lambda\eta_1, \tag{10}$$

and

$$\delta = \eta_2 + \frac{2}{\pi}tan^{-1}\lambda\eta_2 + \frac{2}{\pi}\tan^{-1}\left(\frac{\eta_2 + \tan^{-1}\lambda\eta_2}{\nu}\right). \tag{11}$$

Proof. Using Corollary 2.4 and Theorem 2.5, we obtain

$$\left|\arg\frac{D^\nu f(z)}{z}\right| < \frac{\pi}{2}\eta_1, \tag{12}$$

and

$$\left|\arg\frac{D^{\nu-1}f(z)}{z}\right| < \frac{\pi}{2}\eta_2, \tag{13}$$

where η_1 and η_2 are defined by (10) and (11). Hence by using (12) and (13) we get our result.

Letting $\nu = 1$ and $\lambda = 1$ in Theorem 2.7 we have

Corollary 2.8. *Let $f \in \mathcal{A}$. If*

$$|\arg(f'(z) + zf''(z))| < \frac{\pi}{2}\delta, \qquad (0 < \delta \leq 1)$$

then

$$\left|\arg\frac{zf'(z)}{f(z)}\right| < \frac{\pi}{2}(\eta_1 + \eta_2)$$

where η_1 and η_2 are the solutions of the equations

$$\delta = \eta_1 + \frac{2}{\pi}\tan^{-1}\eta_1,$$

and

$$\delta = \eta_2 + \frac{2}{\pi}\tan^{-1}\eta_2 + \frac{2}{\pi}\tan^{-1}(\eta_2 + \tan^{-1}\eta_2).$$

Lemma 2.9. *Let λ be a function defined on U satisfies*

$$\eta = \inf_{z\in U}\left(\Re\lambda(z) - \cot\frac{\pi}{2}\delta|\Im\lambda(z)|\right) > 0, \tag{14}$$

and let

$$\beta(\eta,\delta) = (t_0)^{\delta}\left[\cos\frac{\pi}{2}\delta - \frac{\delta\eta}{2}\sin\frac{\pi}{2}\delta(t_0+\frac{1}{t_0})\right], \tag{15}$$

be such that $2\beta(\eta,\delta)+\eta \ge 0$ with $t_0 = \frac{\cot\frac{\pi}{2}\delta+\sqrt{\cot^2\frac{\pi}{2}\delta+\eta^2(1-\delta^2)}}{\eta(1+\delta)}$.
If p be analytic in U with $p(0)=1$,satisfies

$$\Re[p(z)+\lambda(z)zp'(z)] > \beta(\eta,\delta)$$

then

$$|\arg p(z)| < \frac{\pi}{2}\delta.$$

Proof. Let $h(z) = (\frac{1+z}{1-z})^{\delta}$, we observe that h is convex and $h(0)=1$. Applying Lemma 1.1 for this h with $\lambda(z)$, we see that $\Re p(z) > 0$ and hence $p(z)\neq 0$ in U. For completing the proof of lemma we need only to show that $p(z)\prec h(z)$. If $p(z)$ is not subordinate to h, then by Lemma 1.3 there exist points $z_0\in U$ and $\xi_0\in\partial U$, and $m\ge 1$ such that

$$p(|z|\subset|z_0|)\subset q(U), p(z_0)=q(\xi_0) \quad \text{and} \quad z_0p'(z_0) = m\xi_0 q'(\xi_0).$$

Since $p(z_0)\neq 0, \xi_0\neq \pm 1$, by letting X and Y be the real and imaginary part of $\lambda(z_0)$,from (14), we find that

$$X+\cot\frac{\pi}{2}\delta Y \ge X - \cot\frac{\pi}{2}\delta|Y| \ge \eta > 0,$$

and

$$X-\cot\frac{\pi}{2}\delta Y \ge X - \cot\frac{\pi}{2}\delta|Y| \ge \eta > 0. \tag{16}$$

Further if we put $ix = \frac{1+\xi_0}{1-\xi_0}$ and use the above observations, we obtain

$$p(z_0)+\lambda(z_0)z_0p'(z_0) = (ix)^{\delta}\left[1+i\frac{m\delta}{2}(X+iY)\frac{1+x^2}{x}\right].$$

For $x\neq 0$,

$$\Re(p(z_0) + \lambda(z_0)z_0p'(z_0))$$

$$= \Re \quad \begin{cases} |x|^\delta \left(\cos\frac{\pi}{2}\delta + i\sin\frac{\pi}{2}\delta\right)\left(1 - \frac{m\delta}{2}(Y - iX)\frac{1+x^2}{|x|}\right) & if x > 0 \\ |x|^\delta \left(\cos\frac{\pi}{2}\delta - i\sin\frac{\pi}{2}\delta\right)\left(1 + \frac{m\delta}{2}(Y - iX)\frac{1+x^2}{|x|}\right) & if x < 0 \end{cases}$$

$$= \quad \begin{cases} |x|^\delta \left[\cos\frac{\pi}{2}\delta - \frac{m\delta}{2}\sin\frac{\pi}{2}\delta(X + \cot\frac{\pi}{2}\delta Y)\frac{1+x^2}{|x|}\right] & if x > 0 \\ |x|^\delta \left[\cos\frac{\pi}{2}\delta - \frac{m\delta}{2}\sin\frac{\pi}{2}\delta(X - \cot\frac{\pi}{2}\delta Y)\frac{1+x^2}{|x|}\right] & if x < 0. \end{cases}$$

Therefore, for $x \neq 0$, since $\lambda(z_0)$ satisfies (14) and $m \geq 1$, we obtain

$$\Re(p(z_0) + \lambda(z_0)z_0p'(z_0)) \leq |x|^\delta \left[\cos\frac{\pi}{2}\delta - \frac{\delta\eta}{2}\sin\frac{\pi}{2}\delta\left(|x| + \frac{1}{|x|}\right)\right] = f(|x|)$$

Since $f(t)$ with $t = |x|$ attains its maximum value at point

$$t_0 = \frac{\cot\frac{\pi}{2}\delta + \sqrt{\cot^2\frac{\pi}{2}\delta + \eta^2(1-\delta^2)}}{\eta(1+\delta)}.$$

We have

$$\Re(p(z_0) + \lambda(z_0)z_0p'(z_0)) \leq f(|x|) \leq f(t_0) = \beta(\eta, \delta).$$

This is contradiction with our assumption. Hence we must have $p(z) \prec h(z)$. This completes the proof.

Theorem 2.10. *Let $\beta(\eta, \delta)$ be as defined by (15) so that $2\beta(\eta, \delta) + \eta \geq 0$. Let $M(z) = z^n + ...$ and $N(z) = z^n + ...$ be analytic in U such that for some $\alpha \in C$, N satisfies*

$$\left|\Im\frac{\alpha N(z)}{zN'(z)}\right| \leq \tan\frac{\pi}{2}\delta\left(\Re\frac{\alpha N(z)}{zN'(z)} - \eta\right),$$

Then

$$\Re\left[(1-\alpha)\frac{M(z)}{N(z)} + \alpha\frac{M'(z)}{N'(z)}\right] > \beta(\eta, \delta),$$

implies

$$\left|\arg\frac{M(z)}{N(z)}\right| < \frac{\pi}{2}\delta.$$

Proof. Consider the function $p(z) = \frac{M(z)}{N(z)}$ and let $\lambda(z) = \frac{\alpha N(z)}{zN'(z)}$. Then by hypothesis,$p$ is analytic and $p(0) = 1$ and all conditions of Lemma 2.9 are satisfied.Now it is elementary to show that,

$$(1-\alpha)\frac{M(z)}{N(z)} + \alpha\frac{M'(z)}{N'(z)} = p(z) + \lambda(z)zp'(z),$$

and hence Theorem 2.10 follows from Lemma 2.9.

Taking $M(z) = D^{\nu} f(z)$ and $N(z) = z$ in Theorem 2.10 we have.

Corollary 2.11. *Let $f \in \mathcal{A}$, and let α be complex number satisfies*

$$|\Im\alpha| \leq (\tan \frac{\pi}{2}\delta)\Re\alpha.$$

Then

$$\Re\left[(1-\alpha)\frac{D^{\nu} f(z)}{z} + \alpha(D^{\nu} f(z))'\right] > \beta(\eta, \delta) \quad (0 < \delta \leq 1, \nu \geq 0)$$

implies

$$\left|\arg \frac{D^{\nu} f(z)}{z}\right| < \frac{\pi}{2}\delta,$$

where $\eta = [\Re\alpha - \cot \frac{\pi}{2}\delta|\Im\alpha|]$.

Theorem 2.12. *Let $f \in \mathcal{A}$. If*

$$\Re\left[(1-\lambda)\frac{D^{1+\nu} f(z)}{z} + \lambda(D^{1+\nu} f(z))'\right] > \beta(\eta, \delta) \quad (0 < \delta \leq 1, \nu \geq 0, \lambda \in \mathbb{C})$$

then

$$\arg\left[(1-\lambda)\frac{D^{\nu} f(z)}{z} + \lambda(D^{\nu} f(z))'\right] < \frac{\pi}{2}\delta,$$

where $\beta(\eta, \delta)$ is defined by (15) with $\eta = \frac{1}{1+\nu}$.

Proof. Suppose

$$p(z) = (1-\lambda)\frac{D^{\nu} f(z)}{z} + \lambda(D^{\nu} f(z))'. \tag{17}$$

It is clear that p is analytic and $p(0) = 1$. Differentiating of (17) with respect to z,multiplying by z and using the identity (8) we obtain

$$(1-\lambda)\frac{D^{1+\nu} f(z)}{z} + \lambda(D^{1+\nu} f(z))' = p(z) + \frac{1}{1+\nu} zp'(z). \tag{18}$$

Hence the result follows from(18) and Lemma 2.9.

To prove our next theorem, we shall need the following result, which is due to Miller and Mocanu [3].

Lemma 2.13. *Let Ω be a set in the complex plane $\mathbb{C}$ and suppose that the function $\psi : \mathbb{C}^2 \times U \mapsto \mathbb{C}$ satisfies $\psi(ix, y, z) \notin \Omega$, for all real x, y with $y \leq -\frac{1+x^2}{2}$ and $z \in U$.If the function $zp \in \mathcal{A}$ satisfies $\psi(p(z), zp'(z), z) \in \Omega, z \in U$, then $\Re p(z) > 0$ holds for all $z \in U$.*

Theorem 2.14. *Let $f \in \mathcal{A}$ and $\nu \geq 1$. Also let $\delta \approx 0.638324$ and $\gamma > 0$ be the roots of the equations (respectively)*

$$\delta \tan \frac{\pi}{2}\delta = 1, \tag{19}$$

and

$$\gamma = \nu \tan \frac{\pi}{2}(\delta - \gamma). \tag{20}$$

If $\alpha \geq 1$ and

$$\Re\left((1-\alpha)\frac{D^\nu f(z)}{z} + \alpha(D^\nu f(z))'\right) > \frac{-(1-\frac{1}{\alpha})(2\xi(\alpha)-1)}{1-(1-\frac{1}{\alpha})(2\xi(\alpha)-1)}, \tag{21}$$

then

$$\Re\left(\frac{D^\nu f(z)}{D^{\nu-1}f(z)}\right) > \beta, \tag{22}$$

where $\xi(\alpha) = \int_0^1 \frac{dt}{1+t^\alpha}$ and β is the smallest positive root of the equation

$$\frac{2\sqrt{[\nu(1-\beta)+\frac{1}{2}][\beta\nu+\frac{1}{2}-\frac{\beta}{1-\beta}]}}{|1-\nu+2\beta\nu|} = \tan\frac{\pi}{2}\gamma. \tag{23}$$

Proof. From (21) and using the well-known result of Hallenbeck and Ruscheweyh [1] with identity

$$\frac{D^\nu f(z)}{z} + z\left(\frac{D^\nu f(z)}{z}\right)' = \left(1-\frac{1}{\alpha}\right)\frac{D^\nu f(z)}{z} + \frac{1}{\alpha}\left[\frac{D^\nu f(z)}{z} + \alpha z\left(\frac{D^\nu f(z)}{z}\right)'\right],$$

we observe

$$\Re\left[\frac{D^\nu f(z)}{z} + z\left(\frac{D^\nu f(z)}{z}\right)'\right] > 0. \tag{24}$$

Applying Corollary 2.11 to (20) we get

$$\left|\arg\frac{D^\nu f(z)}{z}\right| < \frac{\pi}{2}\delta,$$

where δ is defined by (19).

Now by using the identity $\frac{D^\nu f(z)}{z} = \frac{\nu-1}{\nu}\frac{D^{\nu-1}}{z} + \frac{1}{\nu}(D^{\nu-1}f(z))'$ and Corollary 2.4 we obtain

$$\left|\arg\frac{D^{\nu-1}f(z)}{z}\right| < \frac{\pi}{2}\gamma,$$

where γ is defined by (20).

Setting $p(z) = \left(\frac{D^{\nu}f(z)}{D^{\nu-1}f(z)} - \beta\right)\frac{1}{1-\beta}, F(z) = \frac{D^{\nu-1}f(z)}{z}$, and by performing differentiation and some algebraic simplifications, (24) deduces to

$$\Re\psi(p(z), zp'(z), z) > 0$$

where

$$\begin{aligned}\psi(r,s,z) &= F(z)\left(\beta^2\nu + \beta - \beta\nu + r[(1-\beta)(1-\nu) + 2\beta(1-\beta)\nu]\right)\\ &+ F(z)\left(r^2\nu(1-\beta)^2 + (1-\beta)s\right).\end{aligned}$$

Let us now put $F(z) = X + iY$ and apply Lemma 2.13.Then for all x, y real and $z \in U$ we have

$$\begin{aligned}\Re\psi(ix,y,z) &= X\left((\beta^2\nu + \beta - \beta\nu) - \nu(1-\beta)^2x^2 + (1-\beta)y\right)\\ &- Yx[(1-\beta)(1-\nu) + 2\beta(1-\beta)].\end{aligned}$$

From this we observe that

$$\Re\psi(ix,y,z) \leq -(ax^2 + bx + c),$$

for all x real,$y \leq \frac{-(1+x^2)}{2}$ and $z \in U$, where

$$a = X\left[\nu(1-\beta)^2 + \frac{(1-\beta)}{2}\right], b = Y\left[(1-\beta)(1-\nu) + 2\beta(1-\beta)\nu\right] \quad \textit{and}$$

$$c = X\left[\beta\nu(1-\beta) - \beta + \frac{1-\beta}{2}\right].$$

Therefore $\Re\psi(ix,y,z) \leq 0$ if and only if $b^2 \leq 4ac$ this indeed equivalent to

$$|\arg F(z)| < \frac{2\sqrt{[\nu(1-\beta)+\frac{1}{2}][\beta\nu + \frac{1}{2} - \frac{\beta}{1-\beta}]}}{|1-\nu+2\beta\nu|} = \tan\frac{\pi}{2}\gamma.$$

Hence if β be the smallest root of the equation (23) then $\Re\psi(ix,y,z) \leq 0$ and so by Lemma 2.13 we obtain $\Re p(z) > 0$ which is desired conclution.Therefore the proof is complete.

Corollary 2.15. *Let $f \in \mathcal{A}$ and $\delta \approx 0.638324$ and $\gamma \approx 0.39747$ be the roots of the equations (respectively),*

$$\delta \tan\frac{\pi}{2}\delta = 1 \qquad \textit{and} \qquad \gamma = \tan\frac{\pi}{2}(\delta - \gamma).$$

If

$$\Re(f'(z) + zf''(z)) > 0,$$

then

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > \beta,$$

where $\beta \approx 0.46085$ *is the smallest positive root of the equation*

$$\frac{\sqrt{(\frac{3}{2}-\beta)\left(\beta+\frac{1}{2}-\frac{\beta}{1-\beta}\right)}}{\beta} = \tan\frac{\pi}{2}\gamma.$$

Corollary 2.16. *Let* $f \in \mathcal{A}$ *and* $\delta \approx 0.638324$ *and* $\gamma \approx 0.4864$ *be the smallest roots of the equations (respectively),*

$$\delta \tan\frac{\pi}{2}\delta = 1 \qquad \textit{and} \qquad \gamma = 2\tan\frac{\pi}{2}(\delta-\gamma).$$

If

$$\Re(f'(z) + 2zf''(z) + \frac{1}{2}z^2 f'''(z)) > 0,$$

then

$$\Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > 2\beta - 1,$$

where $\beta \approx 0.57669$ *is the the smallest positive root of the equation*

$$\frac{2\sqrt{(\frac{5}{2}-2\beta)\left(2\beta+\frac{1}{2}-\frac{\beta}{1-\beta}\right)}}{4\beta-1} = \tan\frac{\pi}{2}\gamma.$$

*Proof.*Put $\nu = 2$ in Theorem 2.14.

We also note that by using Corollary 2.15 one can get the other new sufficient condition for convexity such as

Corollary 2.17. *Let* $f \in \mathcal{A}$ *and* $\delta \approx 0.638324$ *and* $\gamma \approx 0.39747$ *be the roots of the equations,(respectively)*

$$\delta \tan\frac{\pi}{2}\delta = 1 \qquad \textit{and} \qquad \gamma = \tan\frac{\pi}{2}(\delta-\gamma).$$

If

$$\Re(f'(z) + 3zf''(z) + z^2 f'''(z)) > 0.$$

Then

$$\Re\left(1+\frac{zf''(z)}{f'(z)}\right) > \beta,$$

where $\beta \approx 0.46085$ *is the smallest positive root of the equation*

$$\frac{\sqrt{\left(\frac{3}{2}-\beta\right)\left(\beta+\frac{1}{2}-\frac{\beta}{1-\beta}\right)}}{\beta} = \tan\frac{\pi}{2}\gamma.$$

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Department of Maths, University of Urmia, Urmia, Iran

E-mail address: raghalary@yahoo.com

Department of Maths, Walchand College of Engineering, Sangli, India

E-mail address: joshisb@hotmail.com

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ON THE CONVERGENCE OF COLLOCATION SPLINE METHODS FOR INTEGRAL DELAY PROBLEMS

F. CALIÒ, E. MARCHETTI AND R. PAVANI

Abstract. In some recent works we proposed a collocation method by deficient splines to approximate the solution of Neutral Delay Differential Equations and Volterra Delay Integral Equations. In this work we extend that method to integro-differential equations. The existence and uniqueness of the numerical solution is proved. Consistency and convergence of this method are studied.

1. The problem

In this work we present some remarks about the convergence of a collocation spline method for a problem which is the synthesis of models recently studied in collaboration with Professor Georghe Micula.

Precisely we consider the following non linear first-order Fredholm integro-differential problem with delay:

$$\begin{gathered} y'(x) = f(x, y(x), y(g(x)), \int_0^T K(x,t,y(t),y(g(t)))dt), \quad x \in [0,T] \\ y(0) = y_0, \quad y(x) = \psi(x), \quad x \in [\alpha, 0], \quad \alpha \leq 0, \quad \alpha = \underset{x\in[0,T]}{Inf}(g(x)) \\ \alpha \leq g(x) \leq x \ , \qquad x \in [\alpha, T] \end{gathered} \tag{1}$$

where $f : [0,T] \times \mathcal{R}^3 \to \mathcal{R}$, $K : [0,T] \times [0,T] \times \mathcal{R}^2 \to \mathcal{R}$, $g \in \mathcal{C}[\alpha, T]$, $\psi \in \mathcal{C}^{m-1}[\alpha, 0]$, $m > 1, \ m \in N$.

(1) can be considered Volterra delay integro-differential problem by replacing the upper limit of integration T by x.

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As usual we write problem (1) in the following form

$$\begin{array}{c} y'(x) = f(x, y(x), y(g(x)), z(x)), \quad x \in [0, T] \\ z(x) = \int_0^T K(x, t, y(t), y(g(t)))dt \\ y(0) = y_0, \quad y(x) = \psi(x), \quad x \in [\alpha, 0], \quad \alpha \leq 0, \quad \alpha = \underset{x \in [0,T]}{Inf}(g(x)) \\ \alpha \leq g(x) \leq x \ , \quad x \in [\alpha, T] \end{array} \tag{2}$$

In the following we assume $g(x) = x - \tau$, where $\tau \in \mathcal{R}$, $\tau > 0$ is the constant delay. Let $y_\tau = y(x - \tau)$ and $T = M\tau$ for some $M \in \mathcal{N}$.

1. Suppose that $f(x, y, y_\tau, z)$ is a smooth function satisfying the following Lipschitz condition

$||f(x, y_1, y_{\tau 1}, z_1) - f(x, y_2, y_{\tau 2}, z_2)|| \leq$
$L_1(||y_1 - y_2|| + ||y_{\tau 1} - y_{\tau 2}|| + ||z_1 - z_2||)$
$\forall \ (x, y_1, y_{\tau 1}, z_1), \ (x, y_2, y_{\tau 2}, z_2) \in [0, T] \times \mathcal{R}^3$.

2. Suppose also that the kernel $K(x, t, y, y_\tau)$ is a smooth bounded function satisfying the following Lipschitz condition

$||K(x, t, y_1, y_{\tau 1}) - K(x, t, y_2, y_{\tau 2})|| \leq$
$L_2(||y_1 - y_2|| + ||y_{\tau 1} - y_{\tau 2}||)$
$\forall \ (x, t, y_1, y_{\tau 1}), \ (x, t, y_2, y_{\tau 2}) \in [0, T] \times [0, T] \times \mathcal{R}^2$.

In these conditions, the problem (2) has a unique solution (see for example [2]).

To face this mathematical model we propose a numerical model based on direct collocation spline method using the well known advantages of a collocation method and of a spline approximation. In particular we construct splines pertaining to low regularity class and with weak regularity conditions in the junction points.

The collocation allows to recursively define a piecewise approximating polynomial and is characterized (differently from what is suggested by the literature) by the fact that knowledge gathered in previous steps is completely utilized, thus refining the approximating solution, even at price of a heavier computational load.

Let $r \in \mathcal{N}$, $\quad N = rM \quad$ and Δ be the following uniform partition of the interval $[0, T]$:

$$\Delta : 0 = x_0 < x_1 < ... < x_k < x_{k+1} < ... < x_N = T \ , \ x_k = kh, \ \ h = \frac{T}{N}.$$

Here we approximate the solution of (2) by means of functions pertaining to the class of spline of degree $m \geq 2$ and deficiency 2, denoted by $s : [0, T] \to \mathcal{R}$, $(s \in \mathcal{S}_m,\ s \in \mathcal{C}^{m-2})$.

Precisely, the spline function s is defined in $I_k = [x_k, x_{k+1}]$ as:

$$s_k(x) := \sum_{j=0}^{m-2} s_{k-1}^{(j)}(x_k)(x - x_k)^j / j! + \frac{a_k}{(m-1)!}(x - x_k)^{m-1} + \frac{b_k}{m!}(x - x_k)^m$$

We choose to determine coefficients $a_k,\ b_k$ by the following system of collocation conditions

$$\begin{cases} s'_k(x_k + \frac{h}{2}) = f(x_k + \frac{h}{2}, s_k(x_k + \frac{h}{2}), s_{k-r}(x_k + \frac{h}{2} - \tau), z_k(x_k + \frac{h}{2})) \\ s'_k(x_{k+1}) = f(x_{k+1}, s_k(x_{k+1}), s_{k-r}(x_{k+1} - \tau), z_k(x_{k+1})) \end{cases} \quad (3)$$

where

$$\begin{aligned} z_k(x_k + \frac{h}{2}) &= \sum_{j=0}^{k-1} \int_{jh}^{(j+1)h} K(x_k + \frac{h}{2}, t, s_j(t), s_{j-r}(t - \tau))dt + \\ &+ \int_{kh}^{kh + \frac{h}{2}} K(x_k + \frac{h}{2}, t, s_k(t), s_{k-r}(t - \tau))dt \end{aligned}$$

and

$$z_k(x_{k+1}) = \sum_{j=0}^{k} \int_{jh}^{(j+1)h} K(x_k + h, t, s_j(t), s_{j-r}(t - \tau))dt$$

provided that

$$s_k^{(i)}(x_k) = \lim_{x \to x_k} s_{k-1}^{(i)}(x),\ x \in [x_{k-1}, x_k] \text{ for } i = 0, ..., m-2$$

$$s_i(x) = \psi(x),\ \text{ for } i = -r, ..., -1.$$

Our model is thus reduced to compute the solution of the system (3), through which the spline is determined on the interval I_k.The system can be either non-linear or linear according to $f(x, y(x), y(g(x)), z(x))$.

2. Existence and uniqueness of numerical solution

If we set in $[x_k, x_{k+1}],\ k = 0, 1, ..., N-1$:
$A_k(x) = \sum_{j=0}^{m-2} s_{k-1}^{(j)}(x_k)(x-x_k)^j/j!$,

$$\mathbf{B}_k = \begin{bmatrix} a_k\,(\frac{h}{2})^{m-2} \\ b_k\,(\frac{h}{2})^{m-1} \end{bmatrix}, \qquad \mathbf{Y}_k = \begin{bmatrix} A'_k(x_k+\frac{h}{2}) \\ A'_k(x_{k+1}) \end{bmatrix},$$

$$\mathbf{P} = \frac{1}{(m-2)!}\mathbf{P}_0 \quad \text{with} \quad \mathbf{P}_0 = \begin{bmatrix} 1 & \frac{1}{m-1} \\ 2^{m-2} & \frac{2^{m-1}}{m-1} \end{bmatrix}$$

$$\mathbf{\Phi}_k(\mathbf{B}_k) = \begin{bmatrix} f(x_k+\frac{h}{2}, A_k(x_k+\frac{h}{2}) + \frac{a_k}{(m-1)!}(\frac{h}{2})^{m-1} + \frac{b_k}{m!}(\frac{h}{2})^m, \\ s_{k-r}(x_k+\frac{h}{2}-\tau), z_k(x_k+\frac{h}{2})) \\ f(x_{k+1}, A_k(x_{k+1}) + \frac{a_k}{(m-1)!}h^{m-1} + \frac{b_k}{m!}h^m, \\ s_{k-r}(x_{k+1}-\tau), z_k(x_{k+1})) \end{bmatrix}$$

then (3) becomes:

$$\mathbf{P}\mathbf{B}_k = \mathbf{\Phi}_k(\mathbf{B}_k) - \mathbf{Y}_k$$

Taking into account that $\mathbf{P}_0$ is non singular $\forall m > 1$, system (3) is equivalent to

$$\mathbf{B}_k = (m-2)!\mathbf{P}_0^{-1}(\mathbf{\Phi}_k(\mathbf{B}_k) - \mathbf{Y}_k) \tag{4}$$

Theorem 1. *Let us consider the nonlinear first-order Fredholm integro-differential equation with delay in (2). If functions f and K satisfy the Lipschitz conditions 1. and 2. and if h is small enough, then there exists a unique spline approximation solution $s(x)$ of the problem (2) given by the above construction.*

Proof. The proof of Theorem 1 consists of showing that (4) defines for all sufficiently small h, a contraction mapping. This comes straightforward from the hypotheses and we omit the details of the proof. □

In general, $z_k(x_k+\frac{h}{2}), z_k(x_{k+1})$ in (3) have to be approximated by numerical quadrature: $z_k(x_k+\frac{h}{2}) \simeq Z_k(x_k+\frac{h}{2}),\ z_k(x_{k+1}) \simeq Z_k(x_{k+1})$, where

$$\begin{aligned} Z_k(x_k+\frac{h}{2}) &= \sum_{l=0}^{k-1}\left[\sum_{j=0}^{n(l)} w_j^{(l)}\, K(x_k+\frac{h}{2}, t_j^{(l)}, s_l(t_j^{(l)}), s_{l-r}(t_j^{(l)}-\tau))\right] + \\ &\quad + \sum_{j=0}^{n(k)} w_j^{(k)}\, K(x_k+\frac{h}{2}, t_j^{(k)}, s_k(t_j^{(k)}), s_{k-r}(t_j^{(k)}-\tau)) \end{aligned}$$

with $x_l \le t_j^{(l)} \le x_{l+1}\,(l=0,1,...,k-1) \qquad x_k \le t_j^{(k)} \le x_k+\frac{h}{2}$

$$Z_k(x_{k+1}) = \sum_{l=0}^{k}\left[\sum_{j=0}^{n(l)} w_j^{(l)}\, K(x_{k+1}, t_j^{(l)}, s_l(t_j^{(l)}), s_{l-r}(t_j^{(l)}-\tau))\right]$$

with $x_l \le t_j^{(l)} \le x_{l+1}(l=0,1,...,k)$ and we assume that $\max_{j,l}\left|w_j^{(l)}\right| \le W < \infty$.

System (3) is then reduced to

$$\mathbf{B}_k = (m-2)!\mathbf{P}_0^{-1}(\mathbf{\Psi}_k(\mathbf{B}_k) - \mathbf{Y}_k) \tag{5}$$

with

$$\mathbf{\Psi}_k(\mathbf{B}_k) = \begin{bmatrix} f(x_k+\frac{h}{2}, s_k(x_k+\frac{h}{2}), s_{k-r}(x_k+\frac{h}{2}-\tau), Z_k(x_k+\frac{h}{2})) \\ f(x_{k+1}, s_k(x_{k+1}), s_{k-r}(x_{k+1}-\tau), Z_k(x_{k+1})) \end{bmatrix}$$

Theorem 2. *Under the assumptions stated above and if h is small enough, there exists a unique solution of system (5).*

Proof. As in Theorem 1, the proof consists of showing that (5) defines for all sufficiently small h, a contraction mapping. □

3. Consistency and convergence of the collocation method

Let $y(x) \in \mathcal{C}^{m+1}[0,T]$, $s_k(x)$ be the deficient spline approximating $y(x)$ in $[x_k, x_{k+1}]$, $(k = 0,1,...,N-1)$ and denote with $e_k(x) = s_k(x) - y(x)$ the error function for $x \in [x_k, x_{k+1}]$.

Considering $y(x) = y(x_k) + y'(x_k)(x - x_k) + ... + y^{(m+1)}(\eta_k)\frac{(x-x_k)^{m+1}}{(m+1)!}$ $(x_k < \eta_k < x)$ then

$$\begin{aligned} e_k(x) &= [\sum_{j=0}^{m-2} \frac{s_{k-1}^{(j)}(x_k)}{j!}(x - x_k)^j + \frac{a_k}{(m-1)!}(x - x_k)^{m-1} + \frac{b_k}{m!}(x - x_k)^m] + \\ &\quad - \left[y(x_k) + y'(x_k)(x - x_k) + ... + y^{(m+1)}(\eta_k)\frac{(x - x_k)^{m+1}}{(m+1)!}\right] \\ (x_k &< \eta_k < x) \end{aligned}$$

consequently

$$\begin{aligned} e_k(x) &= e_k(x_k) + \sum_{j=1}^{m-2} \frac{s_{k-1}^{(j)}(x_k) - y^{(j)}(x_k)}{j!}(x - x_k)^j + \\ &\quad + \frac{a_k - y^{(m-1)}(x_k)}{(m-1)!}(x - x_k)^{m-1} + \frac{b_k - y^{(m)}(x_k)}{m!}(x - x_k)^m + \\ &\quad - \frac{y^{(m+1)}(\eta_k)}{(m+1)!}(x - x_k)^{m+1} \\ (x_k &< \eta_k < x) \end{aligned}$$

If we set for $k = 0, 1, ..., N - 1$

$$\begin{aligned} \beta_{k,m-1} &= \frac{a_k - y^{(m-1)}(x_k)}{h^2} \\ \beta_{k,m} &= \frac{b_k - y^{(m)}(x_k)}{h} \\ \gamma_{k,j} &= \frac{s_{k-1}^{(j)}(x_k) - y^{(j)}(x_k)}{h^{m-j+1}}, \ j = 1, ..., m - 2 \\ \varphi_{k,i}(x) &= \frac{(x - x_k)^i}{h^i} \qquad (i = 1, 2, ...) \end{aligned}$$

and

$$T_k(y(x)) = \frac{y^{(m+1)}(\eta_k)}{(m+1)!}, \ x_k < \eta_k < x$$

then the error becomes

$$\begin{gathered} e_k(x) = e_k(x_k) + h^{m+1} \textstyle\sum_{j=1}^{m-2} \frac{\gamma_{k,j}}{j!}\varphi_{k,j}(x) + \\ + h^{m+1}\left[\frac{\beta_{k,m-1}}{(m-1)!}\varphi_{k,m-1}(x) + \frac{\beta_{k,m}}{m!}\varphi_{k,m}(x) - T_k(y(x))\varphi_{k,m+1}(x)\right] \end{gathered} \tag{6}$$

with

$$\begin{array}{c} e'_k(x) = e'_k(x_k) + h^m \sum_{j=2}^{m-2} \frac{\gamma_{k,j}}{(j-1)!}\varphi_{k,j-1}(x)+ \\ +h^m \left[\frac{\beta_{k,m-1}}{(m-2)!}\varphi_{k,m-2}(x) + \frac{\beta_{k,m}}{(m-1)!}\varphi_{k,m-1}(x) - T'_k(y(x))\varphi_{k,m}(x)\right] \end{array} \tag{7}$$

where

$$T'_k(y(x)) = \frac{y^{(m+1)}(\mu_k)}{m!},\ x_k < \mu_k < x$$

Lemma 3. *Let the hypotheses 1. and 2. hold for f and K, then there exists a constant c independent of h such that*

$$\sum_{j=1}^{m-2} \gamma_{k,j} = O(h) \ \ \textit{for all} \ \ k = 0, 1, ..., N-1$$

Proof. The proof comes straightforward from Lemma 4.3 [1]. □

Lemma 4. *(i) Let $f(x, y, y_\tau, z)$ have continuous derivatives of order one with respect to y, y_τ, z in $[0, T]$*

(ii) Let $K(x, t, y, y_\tau)$ have continuous derivatives of order one with respect to y, y_τ in T

then $|\beta_{k,m-1}| + |\beta_{k,m}| \le B$ for all $k = 0, 1, ..., N-1$, where B is a real constant.

Proof. Let $k = 0$, then $e_0(0) = e'_0(0) = 0$ as $x_0 = 0$. We observe that $\varphi_{0,\nu}(\frac{h}{2}) = \frac{1}{2^\nu}$ and $\varphi_{0,\nu}(h) = 1$, $\nu = 1, 2, ..., m+1$, taking account of Lemma 3, from (7) we obtain:

$$\left\{ \begin{array}{c} \frac{\beta_{0,m-1}}{(m-2)!}\frac{1}{2^{m-2}} + \frac{\beta_{0,m}}{(m-1)!}\frac{1}{2^{m-1}} = T'_0(y(\frac{h}{2}))\frac{1}{2^m} + e'_0(\frac{h}{2}) + O(h) \\ \frac{\beta_{0,m-1}}{(m-2)!} + \frac{\beta_{0,m}}{(m-1)!} = T'_0(y(h)) + e'_0(h) + O(h) \end{array} \right. \tag{8}$$

In order to prove that (8) has a unique limited solution, we follow Theorem 1 in [2], taking account of the delay terms.

We observe that a simple calculation yields for $k = 0, 1, ..., N-1$, using the hypotheses on f and K

$$\begin{aligned} e'_k(x_k + \frac{h}{2}) \ &= \ \frac{\partial}{\partial y} f(x_k + \frac{h}{2}, y^*_k, y^*_{k\tau}, z^*_k)\, e_k(x_k + \frac{h}{2}) + \\ &+ \frac{\partial}{\partial y_\tau} f(x_k + \frac{h}{2}, y^*_k, y^*_{k\tau}, z^*_k)\, e_k(x_k + \frac{h}{2} - \tau) + \\ &+ \frac{\partial}{\partial z} f(x_k + \frac{h}{2}, y^*_k, y^*_{k\tau}, z^*_k)\, \delta_k(x_k + \frac{h}{2}) \end{aligned}$$

where:

y_k^* is between $y(x_k + \frac{h}{2})$ and $s_k(x_k + \frac{h}{2})$,

$y_{k\tau}^*$ between $y(x_k + \frac{h}{2} - \tau)$ and $s_k(x_k + \frac{h}{2} - \tau)$,

z_k^* between $z(x_k + \frac{h}{2})$ and $z_k(x_k + \frac{h}{2})$,

$e_k(x_k + \frac{h}{2} - \tau) = s_k(x_k + \frac{h}{2} - \tau) - y(x_k + \frac{h}{2} - \tau)$ and

$$\begin{aligned}\delta_k(x_k + \frac{h}{2}) &= \int_{x_0}^{x_k+\frac{h}{2}} [\frac{\partial}{\partial y} K(x_k + \frac{h}{2}, t, y^*(t), y_\tau^*(t-\tau))e(t) + \\ &\frac{\partial}{\partial y_\tau} K(x_k + \frac{h}{2}, t, y^*(t), y_\tau^*(t-\tau))e(t-\tau)]dt\end{aligned}$$

$y^*(t)$ being between $y(t)$ and $s(t)$, $y_\tau^*(t-\tau)$ between $y(t-\tau)$ and $s(t-\tau)$.

In the same way we obtain

$$\begin{aligned}e_k'(x_{k+1}) &= \frac{\partial}{\partial y} f(x_{k+1}, \overline{y}^*, \overline{y}_\tau^*, \overline{z}^*)\, e_k(x_{k+1}) + \\ &+ \frac{\partial}{\partial y_\tau} f(x_{k+1}, \overline{y}^*, \overline{y}_\tau^*, \overline{z}^*)\, e_k(x_{k+1} - \tau) + \\ &+ \frac{\partial}{\partial z} f(x_{k+1}, \overline{y}^*, \overline{y}_\tau^*, \overline{z}^*)\, \delta_k(x_{k+1})\end{aligned}$$

with suitable $\overline{y}_k^*$, $\overline{y}_{k\tau}^*$, $\overline{z}_k^*$, $\overline{y}^*(t)$, $\overline{y}_\tau^*(t-\tau)$ and an obvious definition of $e_k(x_{k+1} - \tau)$ and $\delta_k(x_{k+1})$.

Consequently we obtain

$$\begin{aligned}e_0'(\frac{h}{2}) &= \frac{\partial}{\partial y} f(\frac{h}{2}, y_0^*, y_{0\tau}^*, z_0^*)\, e_0(\frac{h}{2}) + \\ &+ \frac{\partial}{\partial y_\tau} f(\frac{h}{2}, y_0^*, y_{0\tau}^*, z_0^*)\, e_0(\frac{h}{2} - \tau) + \\ &+ \frac{\partial}{\partial z} f(\frac{h}{2}, y_0^*, y_{0\tau}^*, z_0^*)\, \delta_0(\frac{h}{2})\end{aligned}$$

and

$$\begin{aligned}e_0'(h) &= \frac{\partial}{\partial y} f(h, \overline{y}_0^*, \overline{y}_{0\tau}^*, \overline{z}_0^*)\, e_0(h) + \\ &+ \frac{\partial}{\partial y_\tau} f(h, \overline{y}_0^*, \overline{y}_{0\tau}^*, \overline{z}_0^*)\, e_0(h - \tau) + \\ &+ \frac{\partial}{\partial z} f(h, \overline{y}_0^*, \overline{y}_{0\tau}^*, \overline{z}_0^*)\, \delta_0(h)\end{aligned}$$

so that, according to Lemma 1 in [2], the unique solution $\overline{\beta}_{0,m-1}, \overline{\beta}_{0,m}$ of the system

$$\begin{cases} \frac{\beta_{0,m-1}}{(m-2)!}\frac{1}{2^{m-2}} + \frac{\beta_{0,m}}{(m-1)!}\frac{1}{2^{m-1}} = \frac{1}{2^m}T_0'(y(\frac{h}{2})) \\ \frac{\beta_{0,m-1}}{(m-2)!} + \frac{\beta_{0,m}}{(m-1)!} = T_0'(y(h)) \end{cases}$$

can be regarded as the solution $\beta_{0,m-1}, \beta_{0,m}$ of the system (8) for $h \to 0$ and

$$\beta_{0,\nu} = \overline{\beta}_{0,\nu} + O(h),\ \nu = m-1, m.$$

Let $k = 1$, we observe from (6) and (7) that $e_1(x_1) = e_0(x_1) = O(h^{m+1})$ and $e_1'(x_1) = e_0'(x_1) = O(h^{m+1})$, we proceed by induction on k in the same way as for $k = 0$. The proof of the Lemma follows immediately. □

Theorem 5. *Under the assumptions stated in Lemma 4, then there exists a constant C independent of h such that the error function $e(x)$ satisfies for all $x \in [0,T]$ the following inequalities*

$$|e(x)| \leq C\,h^{m+1}$$
$$|e'(x)| \leq C\,h^{m}$$

Proof. We initially prove the Theorem for $x = x_k$. If we set $M_{m+1} = \max_{k,x\in[0,T]} |T_k'(y(x))|$, then $|T_k(y(x))| \leq M_{m+1}$ for all $x \in [0,T]$; from (6) and Lemma 3 the following relation holds:

$$|e_k(x_k)| \leq \left|e_{k-1}(x_{k-1})\right| + h^{m+1}(c + B + M_{m+1})$$

where c is real constant.

Taking into account that $|e_1(x_1)| \leq h^{m+1}(B + M_{m+1})$ then

$|e_2(x_2)| \leq |e_1(x_1)| + h^{m+1}(c + B + M_{m+1}) \leq h^{m+1}(c + 2(B + M_{m+1}))$,

and

$$|e_k(x_k)| \leq N h^{m+1}(c + B + M_{m+1}) \tag{9}$$

It follows that $|e_k(x_k)| \leq C_1 h^{m+1}$.

Taking into account of (6) we obtain for $x \in [x_k, x_{k+1}]$

$$|e_k(x)| \leq |e_k(x_k)| + h^{m+1}\sum_{j=1}^{m-2}\frac{|\gamma_{k,j}|}{j!}\varphi_{k,j}(x) +$$
$$+ h^{m+1}\left|\frac{\beta_{k,m-1}}{(m-1)!}\varphi_{k,m-1}(x) + \frac{\beta_{k,m}}{(m)!}\varphi_{k,m}(x) - T_k(y(x))\varphi_{k,m+1}(x)\right|$$

from (9), Lemma 3 and Lemma 4 and $|\varphi_{k,j}(x)| \leq 1,\ j = 1, ..., m+1$ we obtain

$$\begin{aligned} |e_k(x)| &\leq Nh^{m+1}(c + B + M_{m+1}) + c\, h^{m+1} \sum_{j=1}^{m-2} \frac{1}{j!} + \\ &+ h^{m+1} \left| B(\frac{1}{(m-1)!} + \frac{1}{(m)!}) + M_{m+1} \right| \end{aligned}$$

it follows $|e_k(x)| \leq C\, h^{m+1}$.

Analogously we obtain

$$\begin{aligned} |e_k'(x)| &\leq Nh^{m}(c + B + M_{m+1}) + c\, h^{m} \sum_{j=2}^{m-2} \frac{1}{(j-1)!} + \\ &+ h^{m} \left| B(\frac{1}{(m-2)!}1 + \frac{1}{(m-1)!}) + M_{m+1} \right| \end{aligned}$$

It follows $|e_k'(x)| \leq C_1\, h^m$.

Because any upper bound for $|e_k(x)|$ and for $|e_k'(x)|$ is independent of k, the thesis follows. □

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Politecnico di Milano – Dipartimento di Matematica,
Piazza Leonardo da Vinci 32, 20133 Milano, Italy

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AN INTEGRAL EQUATION WITH MODIFIED ARGUMENT

MARIA DOBRIŢOIU

Abstract. By the fixed point theorem given in the first part of Rus [3] and an idea of Sotomayor [9], a theorem of differentiability of the solution of the equation

$$x(t) = \int_a^b K(t, s, x(s), x(\varphi(s)))ds + g(t), \quad t \in [\alpha, \beta]$$

is given.

1. **Notations and preliminaries**

Let X be a nonempty set, $A : X \to X$ an operator and we shall use the following notation:

$$F_A := \{x \in X |\ A(x) = x\} \text{ - the fixed point set of } A.$$

Definition 1.1. (Rus [6] or [7]) Let (X, d) be a metric space. An operator $A : X \to X$ is *Picard operator* if there exists $x^* \in X$ such that:

(a) $F_A = \{x^*\}$

(b) the sequence $(A^n(x_0))_{n\in N}$ converges to x^*, for all $x_0 \in X$.

Definition 1.2. (Rus [6] or [7]) Let (X, d) be a metric space. An operator $A : X \to X$ is *weakly Picard operator* if the sequence $(A^n(x_0))_{n\in N}$ converges for all $x_0 \in X$ and the limit (which may depend on x_0) is a fixed point of A.

If A is a weakly Picard operator, then we consider the following operator

$$A^\infty : X \to X, \quad A^\infty(x) = \lim_{n\to\infty} A^n(x)$$

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It is clear that $A^{\infty}(X) = F_A$.

In the section 2 we need the following results (see [4] and [3]).

Perov's theorem. *Let* (X, d)*, with* $d(x, y) \in R^m$*, be a complete generalized metric space and* $A : X \to X$ *an operator. We suppose that there exists a matrix* $Q \in M_{mm}(R_+)$*, such that*

(i) $d(A(x), A(y)) \leq Qd(x, y)$*, for all* $x, y \in X$*;*

(ii) $Q \to 0$ *as* $n \to \infty$.

Then

(a) $F_A = \{x^*\}$,

(b) $A^n(x) \to x^*$ *as* $n \to \infty$ *and*

$$d(A^n(x), x^*) \leq (I - Q)^{-1} Q^n d(x_0, A(x_0)).$$

Rus theorem. (Rus [3]) *Let* (X, d) *be a metric space (generalized or not) and* (Y, ρ) *be a complete generalized metric space* ($\rho(x, y) \in R^m$).

Let $A : X \times Y \to X \times Y$ *be a continuous operator. We suppose that:*

(i) $A(x, y) = (B(x), C(x, y))$*, for all* $x \in X$*,* $y \in Y$*;*

(ii) $B : X \to X$ *is a weakly Picard operator;*

(iii) There exists a matrix $Q \in M_{mm}(R_+)$*,* $Q^n \to 0$ *as* $n \to \infty$*, such that*

$$\rho(C(x, y_1), C(x, y_2)) \leq Q\rho(y_1, y_2),$$

for all $x \in X$*,* y_1 *and* $y_2 \in Y$.

Then the operator A *is weakly Picard operator. Moreover, if* B *is Picard operator, then* A *is Picard operator.*

In the section 3 we need the following definition and result (see [8]).

Definition 1.3. (Rus [8]) A matrix $Q \in M_{nn}(\mathbb{R})$ converges to zero if Q^k converges to the zero matrix as $k \to \infty$.

Theorem 1.1. (Rus [8]) *Let* $Q \in M_{nn}(\mathbb{R}_+)$*. The following statements are equivalent:*

(i) $Q^k \to 0$ *as* $k \to \infty$*;*

(ii) The eigenvalues λ_k, $k = \overline{1,n}$ *of the matrix* Q, *verify the condition* $|\lambda_k| < 1$, $k = \overline{1,n}$;

(iii) The matrix $I - Q$ *is non-singular and* $(I-Q)^{-1} = I + Q + \cdots + Q^n + \ldots$.

2. **The main result**

We consider the following Fredholm integral equation with modified argument

$$x(t) = \int_a^b K(t,s,x(s),x(\varphi(s)))ds + g(t), \quad t \in [\alpha,\beta], \tag{1}$$

where $\alpha, \beta \in R$, $\alpha \leq \beta$, $a, b \in [\alpha, \beta]$, $g \in C([\alpha,\beta], R^m)$, $K \in C([\alpha,\beta] \times [\alpha,\beta] \times R^m \times R^m, R^m)$, $x \in C([\alpha,\beta], R^m)$ and $\varphi \in C([\alpha,\beta],[\alpha,\beta])$.

We have

Theorem 2.1. *We suppose that there exists* $Q \in M_{mm}(R_+)$ *such that:*
(i) $[(\beta-\alpha)Q]^n \to 0$ *as* $n \to \infty$;

(ii) $$\begin{pmatrix} |K_1(t,s,u,v) - K_1(t,s,w,z)| \\ \cdots \\ |K_m(t,s,u,v) - K_m(t,s,w,z)| \end{pmatrix} \leq Q \begin{pmatrix} |u_1 - w_1| + |v_1 - z_1| \\ \cdots \\ |u_m - w_m| + |v_m - z_m| \end{pmatrix}$$
for all $u, v, w, z \in R^m$, $t, s \in [\alpha,\beta]$.

Then

(a) the equation (1) has in $C([\alpha,\beta], R^m)$ *a unique solution,* $x^*(\cdot, a, b)$;

(b) for all $x^0 \in C([\alpha,\beta], R^m)$ *the sequence* $(x^n)_{n\in N}$, *defined by*

$$x^{n+1}(t;a,b) := \int_a^b K(t,s,x^n(s;a,b),x^n(\varphi(s);a,b))ds + g(t)$$

converges uniformly to x^*, *for all* $t, a, b \in [\alpha,\beta]$, *and*

$$\begin{pmatrix} |x_1^n(t;a,b) - x_1^*(t;a,b)| \\ \cdots \\ |x_m^n(t;a,b) - x_m^*(t;a,b)| \end{pmatrix} \leq$$

$$\leq [I - (\beta-\alpha)Q]^{-1}[(\beta-\alpha)Q]^n \begin{pmatrix} |x_1^0(t;a,b) - x_1^1(t;a,b)| \\ \cdots \\ |x_m^0(t;a,b) - x_m^1(t;a,b)| \end{pmatrix}$$

(c) the function

$$x^* : [\alpha,\beta] \times [\alpha,\beta] \times [\alpha,\beta] \to R^m, \quad (t,a,b) \to x^*(t;a,b)$$

is continuous;

(d) if $K(t,s,\cdot,\cdot) \in C^1(R^m \times R^m, R^m)$*, for all* $t,s \in [\alpha,\beta]$*, then* $x^*(t;\cdot,\cdot) \in C^1([\alpha,\beta] \times [\alpha,\beta], R^m)$*, for all* $t \in [\alpha,\beta]$*.*

Proof. Let $\|\cdot\|$ be a generalized Chebyshev norm on $X := C([\alpha,\beta]^3, R^m)$ i.e.

$$\|x\| := \begin{pmatrix} \|x_1\|_\infty \\ \dots \\ \|x_m\|_\infty \end{pmatrix}.$$

Let we consider the operator $B : X \to X$ defined by

$$B(x)(t;a,b) := \int_a^b K(t,s,x(s;a,b),x(\varphi(s);a,b))ds$$

for all $t,a,b \in [\alpha,\beta]$.

From (i) and (ii) and the Perov's theorem we have (a)+(b)+(c).

(d) Let we prove that there exists $\frac{\partial x^*}{\partial a}$ and $\frac{\partial x^*}{\partial a} \in X$.

If we suppose that there exists $\frac{\partial x^*}{\partial a}$, then from (1) we have

$$\begin{aligned} \frac{\partial x^*(t;a,b)}{\partial a} &= -K(t,a,x^*(a;a,b),x^*(\varphi(a);a,b)) + \\ &+ \int_a^b \left[\left(\frac{\partial K_j(t,s,x^*(s;a,b),x^*(\varphi(s);a,b))}{\partial x_i} \right) \frac{\partial x^*(s;a,b)}{\partial a} + \right. \\ &+ \left. \left(\frac{\partial K_j(t,s,x^*(s;a,b),x^*(\varphi(s);a,b))}{\partial x_i} \right) \frac{\partial x^*(\varphi(s);a,b)}{\partial a} \right] ds. \end{aligned}$$

This relation suggest to consider the following operator

$$C : X \times X \to X,$$

$$C(x,y)(t;a,b) := -K(t,a,x(a;a,b),x(\varphi(a);a,b)) + \qquad (2)$$
$$+\int_a^b \left[\left(\frac{\partial K_j(t,s,x(s;a,b),x(\varphi(s);a,b))}{\partial x_i}\right) y(s;a,b) + \right.$$
$$\left. + \left(\frac{\partial K_j(t,s,x(s;a,b),x(\varphi(s);a,b))}{\partial x_i}\right) y(\varphi(s);a,b)\right] ds.$$

From (ii), we remark that

$$\left(\left|\frac{\partial K_j(t,s,u,v)}{\partial x_i}\right|\right) \leq Q \qquad (3)$$

for all $t,s \in [\alpha,\beta]$ and $u,v \in R^m$.

From (2) and (3) it follows that

$$\|C(x,y_1) - C(x,y_2)\| \leq (\beta-\alpha)Q,$$

for all $x,y_1,y_2 \in X$.

If we take the operator

$$A : X \times X \to X \times X, \quad A = (B,C),$$

then we are in the conditions of the Rus theorem. From this theorem, the operator A is a Picard operator and the sequences

$$x^{n+1}(t;a,b) = \int_a^b K(t,s,x^n(s;a,b),x^n(\varphi(s);a,b))ds + g(t)$$

$$y^{n+1}(t;a,b) := -K(t,a,x^n(a;a,b),x^n(\varphi(a);a,b)) +$$
$$+\int_a^b \left[\left(\frac{\partial K_j(t,s,x^n(s;a,b),x^n(\varphi(s);a,b))}{\partial x_i}\right) y^n(s;a,b) + \right.$$
$$\left. + \left(\frac{\partial K_j(t,s,x^n(s;a,b),x^n(\varphi(s);a,b))}{\partial x_i}\right) y^n(\varphi(s);a,b)\right] ds$$

converges uniformly (with respect to $t,a,b \in [\alpha,\beta]$) to $(x^*,y^*) \in F_A$, for all $x^0,y^0 \in X$.

If we take $x^0 = y^0 = 0$, then $y^1 = \dfrac{\partial x^1}{\partial a}$. By induction we prove that $y^n = \dfrac{\partial x^n}{\partial a}$. Thus

$$x^n \overset{unif.}{\longrightarrow} x^* \text{ as } n \to \infty,$$

$$\frac{\partial x^n}{\partial a} \overset{unif.}{\longrightarrow} y^* \text{ as } n \to \infty.$$

These imply that there exists $\dfrac{\partial x^*}{\partial a}$ and $\dfrac{\partial x^*}{\partial a} = y^*$.

By a similar way we prove that there exists $\dfrac{\partial x^*}{\partial b}$. □

3. **Example**

In what follows we consider the following system of Fredholm integral equations

$$\begin{cases} x_1(t) = \displaystyle\int_a^b \left[\frac{1}{8}(t+s)x_1(s) + \frac{1}{4}x_1(s/2)\right] ds + 1 - \cos t \\ x_2(t) = \displaystyle\int_a^b \left[\frac{1}{2}x_1(x) + \frac{2t+s}{4}x_2(s) + \frac{3}{4}x_2(s/2)\right] ds + \sin t \end{cases}, \tag{4}$$

$t, a, b \in [0,1]$, where $a, b \in [0,1]$, $g \in C([0,1], \mathbb{R}^2)$, $g(t) = (g_1(t), g_2(t))$, $g_1(t) = 1 - \cos t$, $g_2(t) = \sin t$, $K \in C([0,1] \times [0,1] \times \mathbb{R}^2 \times \mathbb{R}^2, \mathbb{R}^2)$,

$$K(t,s,x(s),x(\varphi(s))) = (K_1(t,s,x(s),x(\varphi(s))), K_2(t,s,x(s),x(\varphi(s)))),$$

$$K_1 = \frac{1}{8}(t+s)x_1(s) + \frac{1}{4}x_1(s/2), \quad K_2 = \frac{1}{2}x_1(x) + \frac{2t+s}{4}x_2(s) + \frac{3}{4}x_2(s/2),$$

$\varphi \in C([0,1],[0,1])$, $\varphi(s) = s/2$ and $x \in C([0,1], \mathbb{R}^2)$.

From the condition (ii) of the theorem 2.1 we have

$$\begin{pmatrix} |K_1(t,s,x(s),x(s/2)) - K_1(t,s,x(s),z(s/2))| \\ |K_2(t,s,x(s),x(s/2)) - K_2(t,s,x(s),z(s/2))| \end{pmatrix} \leq$$

$$\leq \begin{pmatrix} 1/4 & 0 \\ 1/2 & 3/4 \end{pmatrix} \begin{pmatrix} |x_1(s) - z_1(s)| + |x_1(s/2) - z_1(s/2)| \\ |x_2(s) - z_2(s)| + |x_2(s/2) - z_2(s/2)| \end{pmatrix}, \quad t, s \in [0,1],$$

which lead to matrix

$$Q = \begin{pmatrix} 1/4 & 0 \\ 1/2 & 3/4 \end{pmatrix}, \quad Q \in M_{22}(\mathbb{R}_+),$$

that according to the theorem 1.1 and definition 1.3, converges to zero,

Therefore the conditions of the theorem 2.1 are satisfies and we have

- the system of equations (4) has in $C([0,1], \mathbb{R}^2)$ a unique solution $x^*(\cdot, a, b)$;

- for all $x^0 \in C([0,1],\mathbb{R}^2)$ the sequence $(x^n)_{n\in\mathbb{N}}$, defined by

$$x^{n+1}(t;a,b) := \int_a^b K(t,s,x^n(s;a,b),x^n(\varphi(s);a,b))ds + g(t)$$

converges uniformly to x^*, for all $t,a,b \in [0,1]$, and

$$\begin{pmatrix} |x_1^n(t;a,b) - x_1^*(t;a,b)| \\ \dots \\ |x_m^n(t;a,b) - x_m^*(t;a,b)| \end{pmatrix} \leq [I-Q]^{-1}Q^n \begin{pmatrix} |x_1^0(t;a,b) - x_1^1(t;a,b) \\ \dots \\ |x_m^0(t;a,b) - x_m^1(t;a,b) \end{pmatrix}$$

- the function

$$x^* : [0,1]\times[0,1]\times[0,1] \to \mathbb{R}^2, \quad (t;a,b) \to x^*(t;a,b)$$

is continuous;

- if $K(t,s,\cdot,\cdot) \in C^1(\mathbb{R}^2\times\mathbb{R}^2,\mathbb{R}^2)$, for all $t,s\in[0,1]$, then $x^*(t;\cdot,\cdot) \in C^1([0,1]\times[0,1],\mathbb{R}^2)$, for all $t \in [0,1]$.

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Faculty of Science, University of Petroşani, Petroşani, Romania

E-mail address: mariadobritoiu@yahoo.com

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NOTE ON THE SOLVABILITY OF A SYSTEM OF EQUATIONS

ZOLTÁN FINTA

Abstract. In this note we formulate sufficient conditions for the solvability of a system of equations in R^d $(d \geq 1)$ using attached polynomial system of equations. The solution of the last system tends to the solution of the original system and the approximation error will be estimated by means of the modulus of smoothness and $K-$functional, respectively.

1. **Introduction**

Let $(X, \|\cdot\|_X)$ be a real or complex normed space and denote by $L(X)$ the space of all continuous linear operators from X to X. For an operator $A \in L(X)$ and an element $y \in X$ let us consider the equation $(I - A)(x) = y$. Approximating the operator A by another operator $\tilde{A} \in L(X)$ and the element y by $\tilde{y} \in X$, we arrive at a new equation $(I - \tilde{A})(\tilde{x}) = \tilde{y}$. This equation usually is easier to solve and it is called the near equation of $(I - A)(x) = y$. The problem to give estimations of the error $\|x - \tilde{x}\|_X$ with the aid of A, $\tilde{A}$, y and $\tilde{y}$ has been studying extensively (see e.g. [6]).

The algorithm described in [4] provides the solutions of the system of equations $f_i(x_1, x_2, \ldots, x_d) = 0$, $i \in \{1, 2, \ldots, d\}$, located in $\Pi_{i=1}^d\ [0, 1]$, and a polynomial system of equations is used in place of the near equation.

The purpose of this paper is to give sufficient conditions regarding the functions f_i which imply the solvability of the system of equations $f_i(x_1, x_2, \ldots, x_d) = 0$, $i \in \{1, 2, \ldots, d\}$, $d \geq 2$, using different attached polynomial system in comparison

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with [4]. This last system will be given by means of the multivariate Bernstein - Durrmeyer polynomials defined on a simplex. The approximation error will be estimated using a $K-$functional. The case $d = 1$ is treated separately, where the attached equation contains the well - known Bernstein polynomial, and the approximation error is estimated by the Ditzian - Totik modulus of smoothness.

2. **Main results**

For a function $f : [0, 1] \to R$ the Bernstein polynomials are defined by

$$(B_n f)(x) \;=\; \sum_{k=0}^{n} p_{n,k}(x)\; f\left(\frac{k}{n}\right),$$

where

$$p_{n,k}(x) \;=\; \left(\begin{array}{c} n \\ k \end{array}\right) x^k (1-x)^{n-k}, \quad x \in [0,1], \quad n \geq 1.$$

Let us consider the equation

$$f(x) \;=\; 0 \tag{1}$$

and let

$$(B_n f)(x) \;=\; 0 \tag{2}$$

be the attached equation to (1). Our first result is:

Theorem 1. *Let $f : [0, 1] \to R$ be a continuously differentiable function such that $f(0) \cdot f(1) < 0$ and there exists $q > 0$ with the property $|f'(x)| \geq q$ for all $x \in [0, 1]$. If $y \in [0, 1]$ is a solution of the equation (1) then there exists a sequence $(x_n)_{n\geq 1}$ such that x_n is a solution of (2) for all $n \geq 1$ and $\lim_{n\to\infty} x_n = y$. Moreover, we have the estimations*

$$|x_n - y| \;\leq\; \frac{1}{q}\,(10 + 4\sqrt{3})\; \omega_\varphi^2\left(f, n^{-1/2}\right), \qquad n \geq 1,$$

where

$$\omega_\varphi^2\left(f, n^{-1/2}\right) \;=\; \sup_{0<t\leq n^{-1/2}} \; \sup_{x\in[0,1]} \; |f(x + t\varphi(x)) - 2f(x) + f(x - t\varphi(x))|,$$

$\varphi(x) = \sqrt{x(1-x)}$, $x \in [0,1]$ *is the Ditzian - Totik modulus of smoothness. Furthermore, if* $f \in C^2[0,1]$ *then*

$$\lim_{n\to\infty} n(B_n f)(y) \ = \ \frac{1}{2}\ y(1-y)f''(y)$$

and

$$|x_n - y| \ \le \ \frac{1}{qn}\ \left(\frac{5}{2} + \sqrt{3}\right)\ \|f''\|_\infty,$$

where $n \ge 1$ *and* $\|\cdot\|_\infty$ *is the sup - norm on* $[0,1]$.

Proof. The hypotheses $f(0)\cdot f(1) < 0$ and $|f'(x)| \ge q$, $x \in [0,1]$ imply that y is the unique solution of (1). Furthermore, $(B_n f)(0) = f(0)$ and $(B_n f)(1) = f(1)$. So, by $f(0)\cdot f(1) < 0$ we obtain $(B_n f)(0)\cdot(B_n f)(1) < 0$, which implies the existence of a solution x_n of the equation (2) for all $n \ge 1$. On the other hand, in view of Lagrange's mean - value theorem there exists z_n between y and x_n such that

$$f(x_n) = f(x_n) - f(y) = f'(z_n)\cdot(x_n - y).$$

Hence, by $|f'(x)| \ge q$, $x \in [0,1]$ we get

$$\begin{aligned} |x_n - y| \ &\le\ \frac{1}{q}\cdot|f(x_n)| = \frac{1}{q}\cdot|f(x_n) - (B_n f)(x_n)| \\ &\le\ \frac{1}{q}\cdot\max\{|f(x) - (B_n f)(x)|\ :\ x \in [0,1]\} \\ &=\ \frac{1}{q}\cdot\|f - B_n f\|_\infty \to 0 \quad \text{as} \quad n \to \infty \end{aligned} \tag{3}$$

This means that $\lim_{n\to\infty} x_n = y$. Using (3) and [5, p. 452, Corollary 11] we have

$$|x_n - y| \ \le\ \frac{1}{q}\ (10 + 4\sqrt{3})\ \omega_\varphi^2\left(f, n^{-1/2}\right) \tag{4}$$

If $f \in C^2[0,1]$ then, in view of Voronovskaja theorem [3, p. 307, Theorem 3.1] and $f(y) = 0$,

$$\lim_{n\to\infty} n(B_n f)(y) \ = \ \frac{1}{2}\ y(1-y)f''(y).$$

Using the definition of $\omega_\varphi^2\left(f, n^{-1/2}\right)$ and [7, p. 47, (2)], we obtain

$$f(x + t\varphi(x)) - 2f(x) + f(x - t\varphi(x)) \ = \ \int_{-\frac{t}{2}\varphi(x)}^{\frac{t}{2}\varphi(x)} \int_{-\frac{t}{2}\varphi(x)}^{\frac{t}{2}\varphi(x)} f''(x + u_1 + u_2)\ du_1 du_2.$$

Hence

$$\omega_\varphi^2\left(f, n^{-1/2}\right) \;\leq\; \|f''\|_\infty \cdot \sup_{0<t\leq n^{-1/2}} \; \sup_{x\in[0,1]} \; t^2\varphi^2(x) \;\leq\; \frac{1}{4n}.$$

By (4) we arrive at the estimation

$$|x_n - y| \;\leq\; \frac{1}{qn} \cdot \left(\frac{5}{2} + \sqrt{3}\right) \cdot \|f''\|_\infty,$$

which completes the proof.

Using same ideas it can be proved the following:

Corollary 1. *Let $f : [0,1] \to R$ be a continuous function such that $f(0) \cdot f(1) < 0$ and there exists $q > 0$ with the property $q|x - x'| \leq |f(x) - f(x')|$ for all $x, x' \in [0,1]$. If $y \in [0,1]$ is a solution of the equation (1) then there exists a sequence $(x_n)_{n\geq 1}$ such that x_n is a solution of (2) for all $n \geq 1$ and*

$$|x_n - y| \;\leq\; \frac{1}{q} \left(10 + 4\sqrt{3}\right) \omega_\varphi^2\left(f, n^{-1/2}\right).$$

Remark 1. *A solution x_n of the equation $(B_n f)(x) = 0$, $n \geq 1$, can be obtained by Bairstow's method* [9, pp. 301 - 303].

In what follows we consider the multivariate Bernstein - Durrmeyer polynomials introduced by Derriennic [2] as

$$(M_n f)(x) \;=\; \frac{(n+d)!}{n!} \sum_{(\beta/n)\in T} P_{n,\beta}(x) \int_T P_{n,\beta}(u) f(u)\, du,$$

where $x, u \in R^d$, $x = (x_1, \ldots, x_d)$, $u = (u_1, \ldots, u_d)$, $\beta = (k_1, \ldots, k_d)$ with k_i integers, and $T = \{u : 0 \leq u_i, \sum_{i=1}^d \; u_i \leq 1\}$. Furthermore, $P_{n,\beta}(u)$ is given by

$$P_{n,\beta}(u) \;=\; \frac{n!}{\beta!(n-|\beta|)!} \; u^\beta (1 - |u|)^{n-|\beta|},$$

where $\beta! = k_1! \ldots k_d!$, $u^\beta = u_1^{k_1} \ldots u_d^{k_d}$ ($u_i^{k_i} = 1$ if $k_i = u_i = 0$), $|u| = \sum_{i=1}^d \; u_i$ and $|\beta| = \sum_{i=1}^d \; k_i$. We define, by virtue of [1, p. 112, (2.9)],

$$P(D) \;=\; \sum_{i=1}^d \frac{\partial}{\partial x_i} \, x_i(1-|x|) \, \frac{\partial}{\partial x_i} \;+\; \sum_{i<j} \left(\frac{\partial}{\partial x_i} - \frac{\partial}{\partial x_j}\right) x_i x_j \left(\frac{\partial}{\partial x_i} - \frac{\partial}{\partial x_j}\right).$$

Now we consider the following system of equations:

$$f_i(x) \;=\; 0, \qquad i \in \{1, 2, \ldots, d\}, \tag{5}$$

where $f_i : T \to R$, and let

$$(M_n f_i)(x) \;=\; 0, \qquad i \in \{1, 2, \ldots, d\} \tag{6}$$

be the attached system of equations to (5). We denote by $\|\cdot\|$ a norm on R^d.

Theorem 2. *Let $(f_1, \ldots, f_d) : T \to R^d$ be a continuous function. If $y \in T$ is a solution of (5), $(x^n)_{n\geq 1}$, $x^n = (x_1^n, \ldots, x_d^n)$ is a solution of (6) for all $n \geq 1$ and there exist $q > 0$ and $i_0 \in \{1, 2, \ldots, d\}$ with the property $q\|x - x'\| \leq |f_{i_0}(x) - f_{i_0}(x')|$ for all $x, x' \in T$ then $\lim_{n\to\infty} \|x^n - y\| = 0$. Moreover, we have the estimation*

$$\|x^n - y\| \;\leq\; \frac{2}{q}\, K\left(f_{i_0}, n^{-1}\right), \qquad n \geq 1,$$

where

$$K\left(f_{i_0}, n^{-1}\right) \;=\; \inf\,\{\; \|f_{i_0} - g\|_\infty + n^{-1}\; \|P(D)g\|_\infty \;:\; g \in C^2(T)\;\}$$

and $\|\cdot\|_\infty$ is the sup - norm on T. If $f \in C^2(T)$ then

$$\lim_{n\to\infty} n(M_n f_{i_0})(y) = P(D) f_{i_0}(y).$$

Proof. We have

$$\begin{aligned}
\|x^n - y\| \;&\leq\; \frac{1}{q} \cdot |f_{i_0}(x^n) - f_{i_0}(y)| = \frac{1}{q} \cdot |f_{i_0}(x^n)| \\
&=\; \frac{1}{q} \cdot |f_{i_0}(x^n) - (M_n f_{i_0})(x^n)| \\
&\leq\; \frac{1}{q} \cdot \|M_n f_{i_0} - f_{i_0}\|_\infty \leq \frac{2}{q} \cdot K\left(f_{i_0}, n^{-1}\right), \quad n \geq 1,
\end{aligned}$$

in view of [1, p. 115, (3.2)]. Hence $\lim_{n\to\infty} \|x^n - y\| = 0$.

If $f \in C^2(T)$, then, by [1, p. 112, Lemma 2.1] and $f_{i_0}(y) = 0$ we obtain

$$\lim_{n\to\infty} n(M_n f_{i_0})(y) \;=\; P(D) f_{i_0}(y),$$

which was to be proved.

Remark 2. *Let $q > 0$ and $f = (f_1, \ldots, f_d) : R^d \to R^d$ be a differentiable function with $\|f'(x)\|_* = \sup\{\|f'(x)(z)\| \;:\; \|z\| \leq 1\} \geq q$ for all $x \in R^d$. Then the condition $\|f(x) - f(x')\| \geq q\|x - x'\|$ for all $x, x' \in R^d$ is not necessarily true for $d \geq 2$ (see [8, p. 81, 3.23]).*

Remark 3. Following [4] we have: *the system of equations (6) can be transformed into an equivalent "triangular" polynomial system using the Gröbner basis algorithm. So the solvability of the last system of equations can be traced back to the solvability of a polynomial equation with one unknown. To solve this equation we apply again Bairstow's method on* $[0,1]$. *After that we generate all solutions of the "triangular" polynomial system located in* T. *Thus we arrive at the solutions* x^n *of (6),* $n \geq 1$. *It may happen that the polynomial equation with one unknown has not solutions in* $[0,1]$. *In this case the polynomial system of equation has not solution either and the polynomial system must be rephrased.*

Remark 4. *In* [4] *another attached system of equation is given, namely*

$$(\tilde{B}_n f_i)(x) = 0, \qquad i \in \{1, 2, \dots, d\},$$

where $x = (x_1, \dots, x_d) \in D$, $D = \prod_{i=1}^{d} [0,1]$, $f_i : D \to R$ *and*

$$(\tilde{B}_n f_i)(x) = \sum_{i_1=1}^{n} \sum_{i_2=1}^{n} \cdots \sum_{i_d=1}^{n} f_i\left(\frac{i_1}{n}, \dots, \frac{i_d}{n}\right) \prod_{j=1}^{d} \begin{pmatrix} n \\ i_j \end{pmatrix} (x_j)^{i_j} \cdot (1 - x_j)^{n - i_j}.$$

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Babeş-Bolyai University, Department of Mathematics and Computer Science, 1, M. Kogălniceanu st., 3400 Cluj-Napoca, Romania
E-mail address: fzoltan@math.ubbcluj.ro

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A STUDY ON METRICS AND STATISTICAL ANALYSIS

DĂNUŢ MARCU

Abstract. The purpose of this article is to introduce some classes of metrics, to describe their importance to mathematics and the sciences, to state the basic theorems concerning these classes, to state some new theorems which we have obtained by using topological methods, and even provide a proof here and there. But, the main purpose, is to state many of the open problems around these concepts and to show how much of this subject might be understood by topological means.

1. **Introduction**

Everyone is familiar with the *triangle inequality.* This inequality played a major role in the definition of a topological space.

$$\rho(a, b) \leq \rho(a, c) + \rho(b, c)$$

Still familiar to topologists is the *ultrametric inequality.*

$$\rho(a, b) \leq \max\{\rho(a, c), \rho(b, c)\}$$

But there are more inequalities of importance to mathematics which topologists are not familiar with. For example, there is the *four-point inequality*,

$$\rho(a, b) + \rho(c, d) \leq \max\{\rho(b, c) + \rho(a, d), \rho(a, c) + \rho(b, d)\}$$

and there is the *pentagon inequality*

$$\rho(a, b) + \rho(c, d) + \rho(c, e) + \rho(d, e) \leq \rho(a, c) + \rho(a, d) + \rho(a, e) + \rho(b, c) + \rho(b, d) + \rho(b, e)$$

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and there is the *negative-type inequality*

$$\rho(a,b)+\rho(b,c)+\rho(a,c)+\rho(d,e)+\rho(d,f)+\rho(e,f)$$

$$\leq \rho(a,d)+\rho(a,e)+\rho(a,f)+\rho(b,d)+\rho(b,e)+\rho(b,f)+\rho(c,d)+\rho(c,e)+\rho(c,f)$$

All of these inequalities turn out to be important in various parts of mathematics and, especially, in the applications of mathematics to the sciences.

2. **Statistics**

The standard definition states that multivariate statistical analysis and, especially, that more applied part of multivariate statistical analysis which is called multivariate data analysis, is concerned with data collected on several dimensions of the same individual. A cursory examination of the literature of that subject reveals that a major concern, worthy of a few chapters in a typical textbook, is the following situation and resulting problem: For each of n objects, each of k tests is performed with a result which might be a real number. This gives us an $n \times k$ matrix. We wish to combine this test data and produce an $n \times n$ matrix of non-negative reals which measures the "similarity" or "dissimilarity" of the objects so far as their test results indicate. If the tests have been designed to give a reasonable notion of similarity, then this *similarity matrix* usually satisfies the axioms of a metric space. We wish to determine what kind of distance concept has been isolated, that is, what kind of metric space has been constructed.[1] Of course, with real data, things are not as simple as we have described. In most cases, the data has to be *transformed*, some data is missing and has to be *reconstructed* and the data has error or even spurious entries and has to be *approximated*. Only then can the data be *represented* in some fashion which makes it possible to use our human facilites to understand this data. So, before analysing distance data, we need some means of *classifying* metric spaces and some compendium of reasonable representations or *embeddings*.

[1] This topic is a huge one. There are many textbooks devoted to the various aspects of this problem. A bibliography listing only articles which appeared up to 1975 has 7530 entries.

3. **Kinds of Metrics**

Here is a list of the basic kinds of metrics:

- 1. ultrametric
- 2a. L_2-embeddable
- 2b. four-point property
- 3. L_1-embeddable
- 4. hypermetric
- 5. spherical
- 6. negative-type
- 7. one positive eigenvalue
- 8. L_∞-embeddable

Each property implies those properties listed below it, except that 2a. does not imply 2b.

The purpose of this article is to introduce these classes of metrics, to describe their importance to mathematics and the sciences, to state the basic theorems concerning these classes, to state some new theorems which we have obtained by using topological methods, and even provide a proof here and there. But, the main purpose, is to state many of the open problems around these concepts and to show how much of this subject might be understood by topological means.

4. **Ultrametrics**

Ultrametric spaces are well-known to topologists and perhaps even better known to number theorists and analysts. K. Hensel invented the p-adic numbers in 1897. These numbers carry a natural ultrametric structure and there are now textbooks on “Ultrametric Calculus” and “Non-Archimedean Functional Analysis”. A closely related topic which has attracted attention of many topologists is spherical completeness. The ultrametric inequality was formulated at least as early as 1934 by Hausdorff, but the term *ultrametric* was coined only in 1944 by M. Krasner. In 1956, deGroot characterized the ultrametric spaces, up to homeomorphism, as the strongly zero-dimensional metric spaces.[2]

But, questions at a topological level of generality, remain open. It does not seem to be known which non-metric spaces are "essentially" ultrametric:

Problem 1. *Characterize those topological spaces X such that, for every continuous pseudometric ρ on X, there is a continuous ultra-pseudometric σ on X which generates a larger topology.*

Ultrametric spaces have emerged in the last fifteen years as a major concern in statistical mechanics, in neural networks and in optimization theory. The history of this emergence is quite interesting.

In 1984, Mezard, Parisi, Sourlas, Toulouse and Virasoro published an article on the mean-field theory of spin glasses in which they established that the distribution of "pure states" in the configuration space is an ultrametric subspace. Within a few years, it was shown that the "graph partitioning problem" in finite combinatorics could be "mapped onto" the spin glass problem and thus that the solution space for this problem also has an ultrametric structure. S. Kirkpatrick then found numerically that the solutions for certain *travelling salesman problems*[3] seem to scatter in an ultrametric fashion. J. P. Bouchaud and P. Le Doussal have conjectured that, in optimization problems in which "the imposed constraints cannot all be satisfied simultaneously, the optimal configurations (i.e., those which minimize the number of unsatisfied constraints) spread in an ultrametric way in the configuration space". These kinds of problems are known as *frustrated optimization problems.*

No results of this kind have actually been proven, except in special classes of spin glasses. All other indications are numerical or by reduction. It would be of major significance to many fields to show that this phenomenon occurs under some general circumstances.

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[2] Nyikos and Purisch have extensively investigated the relationship between ultrametrics and generalized metrics and orderability

[3] Is there an infinitary version of the travelling salesman problem? Examples might be "When do metric spaces admit space-filling curves of finite length?" or "When do they admit ϵ-dense curves of finite length for each $\epsilon > 0$?"

Problem 2. *Give some reasonable conditions on non-negative continuous real-valued functions* $\{f_i : i < n\}$ *on a metric space* X *so that, if* K *is minimal for* $Y = \{x \in X : \sum\{f_i(x) : i < n\} = K\}$ *non-empty, then* Y *is ultrametric. Formulate this question more accurately.*

A recent and effective strategy in handling optimization problems is to use simulated annealing and *random walks* to find global solutions. In problems where the local solutions have an ultrametric structure, it is therefore essential to understand random walks on ultrametric spaces. There has been much work already on different ways in which to define such random walks.

There are, undoubtedly, quite general theorems which show that the natural metric on sufficiently few independent stochastic processes which are nontrivial on sufficiently few of sufficiently many coordinates is arbitrarily close to being ultrametric. It seems likely that, to obtain a statement and proof of such a theorem, we should state and prove an infinite version first.

Problem 3. *Let* $\{R^x : x \in X\}$ *be a finite set of independent stochastic processes acting on* $\mathbb{R}^\omega$*, independently of the coordinates, so that*

$$(\forall x \in X)(\forall t \in \mathbb{R}) Prob_t(|\{n \in \omega : R^x(n) \neq 0\}| < \omega) = 1$$

Let d *be the metric defined on* X *by* $d(x, x') = E(L_1(|R^x - R^{x'}|))$ *for a suitable measure on* ω*. Prove that* d *is an ultrametric.*

Problem 4. *Can an asymptotic finitary version of problem 3 be stated and proved? Can the assumptions be made sufficiently reasonable so as to show that the numerical evidence for ultrametricity of phylogenetic trees in evolution is inevitable?*

In examining the numerical evidence for ultrametricity, and in proving theoretical results about the tendency of finite data to approach ultrametricity, there is a need for answering a fundamental question: How can we measure how far a given metric is from being an ultrametric?

The main method used in spin glasses for answering this question is based on the following:

Proposition 1 (Jardine, 1967). *If ρ is a metric on a finite set, then there is an ultrametric τ which minimizes* $\sup\{|\rho(x,y)-\tau(x,y)| : x,y \in X\}$ *among those τ such that* $(\forall x,y \in X)\tau(x,y) \leq \rho(x,y)$.

This analog of the subharmonic in potential theory which is called the subdominant ultrametric can be quite pathological. R. Rammal, G. Toulouse and M. Virasoro in their article *Ultrametricity for Physicists* ask whether there are optimal l_p ultrametric approximations for a given metric where $1 \leq p \leq \infty$ (and specifically ask it for 1 and ∞). Noting that the proposition can be viewed as an optimal l_∞ ultrametric approximation among those ultrametrics below a given metric, we have obtained the next result:

Theorem 1. *If ρ is a metric on a finite space, then there is an ultrametric τ which minimizes* $\sup\{|\rho(x,y)-\tau(x,y)| : x,y \in X\}$ *among all ultrametrics τ.*

There may be several choices for the ultrametric in theorem 1 but perhaps this duplication only occurs in a trivial way.

Problem 5. *Is there, up to some kind of manipulation, always an unique ultrametric τ which minimizes* $\sup\{|\rho(x,y)-\tau(x,y)| : x,y \in X\}$ *among all ultrametrics τ?*

But, our construction in theorem 1, seems to take exponential time, while Jardine's only takes polynomial time.

Problem 6. *Is there a polynomial algorithm for computing an ultrametric τ which minimizes* $\sup\{|\rho(x,y)-\tau(x,y)| : x,y \in X\}$ *among all ultrametrics τ?*

Krivanek showed that computing the closest ultrametric above a given metric is NP-complete.

Problem 7. *Show that the subdominant ultrametric can be quite pathological. That is, show that the subdominant ultrametric of a given metric ρ can be arbitrarily close to zero, even when there is an ultrametric quite close to ρ in the supremum norm.*

Jardine's theorem was extended by Bayod and Martinez-Maurica, in 1990, to totally disconnected locally compact spaces. But, they failed to obtain a characterization.

Problem 8. *Characterize those metric spaces which have a subdominant ultrametric.*

Problem 9. *Can theorem 1 be extended to a reasonable class of infinite metric spaces?*

Returing to the problem of Rammal, Toulouse and Virasoro:

Problem 10. *If* ρ *is a metric on a finite (or arbitrary) set, then is there an ultrametric* τ *which minimizes* $\sum\{|\rho(x,y)-\tau(x,y)| : x,y \in X\}$ *among all ultrametrics* τ*? How does one construct* τ*?*

Problem 11. *Which metric spaces have an (uniformly) equivalent metric* ρ *for which there is an ultrametric* τ *such that* $\sum\{|\rho(x,y)-\tau(x,y)| : x,y \in X\}$ *is finite?*

It would be quite useful to associate, to each metric, an ultrametric which is somehow derived from it in a natural way. But, this seems unlikely.

Problem 12. *Let the family of all metrics on a (finite, countable or arbitrary) set* X *be equipped with an* l_p *metric. Is there a continuous retraction of metrics onto ultrametrics?*

Note that when $p = \infty$, this problem is entirely topological.

Ultrametric spaces can be embedded in linearly ordered spaces, but this is not an isometric embedding. To provide an isometric representation, we must use another device, well-known to natural scientists as a *dendrogram* (see p. 769 of Rammal). This method is equally valid for infinite spaces.

5. **Additive Trees**

The representation of ultrametrics by dendrograms leads one to consider a more general kind of diagram called an additive tree in the social sciences literature or a phylogenetic tree (this term has many inexact definitions) in the biological literature. Suppose (V, E) is a tree (a graph without cycles or loops) in which each edge has an "weight" which is a non-negative real number. The distance between any two vertices $x, y \in V$ is defined to be the sum of the weights of the edges which make up the unique minimal path from x to y. It is an exercise in graph theory to show that this distance is a metric which satisfies the four-point property.

Theorem 2. *Any ultrametric space satisfies the four-point property.*

In 1971, Bunemann showed that, in fact, any metric on a finite set satisfying the four-point property could be represented as the vertices of a graph equipped with this "path distance".

Definition 1. *An* R-tree *is an (uniquely) arcwise connected metric space in which each arc is isometric to a subarc of the reals.*

In 1985, Mayer and Oversteegen constructed an universal R-tree of a given weight. This construction allows us to prove that the path metric or *intrinsic* metric on an R-tree satisfies the four-point condition and that, conversely, any metric space satisfying the four-point condition can be represented as a set of points in an R-tree.[4]

Indeed, the concept of an additive tree may be valuable for arbitrary completely regular spaces:

Problem 13. *Characterize those topological spaces X such that, for every continuous pseudometric ρ on X, there is a continuous pseudometric σ on X with the four-point property which generates a larger topology.*

Any linearly ordered connected compactum satisfies problem 13.

This representation by additive trees is not, by any means, only a theoretical concern. It is an useful way of representing data which satisfies the four-point condition (see p. 395 of Shepard). Note that this is the right diagram for representing evolution in which rates of evolution may be different for different species. Dendrograms assume that the rates are uniform for all species.

Additive trees are obviously easy to interpret. A topologist might ask whether one can use the intrinsic metric of more general spaces to represent metric spaces of a broader kind. The answer is yes.

Proposition 2. *Any separable metric space can be represented as a subset of a subspace of $\mathbb{R}^3$ equipped with the intrinsic metric.*

[4]Rudnik and Borsuk have asked whether there is an one-dimensional subset X of $\mathbb{R}^2$ in which every two points is joined by an arc of finite length and in which every intrinsic isometry in $\mathbb{R}^2$ is an isometry.

But, this proposition shows by its strength, its uselessness. We must keep in mind that, to be useful, a representation must take advantage of human facilities.[5]

Problem 14. *Characterize those metric spaces which can be represented as a subset of a (simply connected) continuum in* $\mathbb{R}^2$ *with the intrinsic metric.*

For example, any ultrametric space, such as K_5 with the graph metric,[6] can be so represented but $K(3,3)$ cannot be so represented.

Problem 15. *Is there a version of Kuratowski's test for planarity of graphs which answers problem 14 for graph metrics? That is, is there a finite list of "forbidden" graphs?*

While testing a metric for ultrametricity requires testing each set of three points (and thus can be done in $O(n^3)$ computing time), testing a metric for the four-point condition seems to require testing each set of four points and that would require $O(n^4)$ time. But, there is a beautiful way of converting additive trees into ultrametrics.

Definition 2. *If* ρ *is a metric on a set* X *and* $v \in X$ *and* c *is an appropriate constant, then, for each* $x, y \in X$, *define* $\delta(x,y) = c + \rho(x,y) - \rho(x,v) - \rho(y,v)$. δ *is the* Farris transform *of* ρ.

Proposition 3 (Farris, 1970). δ *is an ultrametric if and only if* ρ *satisfies the four-point condition.*

This theorem is not hard to prove: it just requires some manipulation. Of course δ and ρ do not generate the same topology even if we choose c carefully.

But, Farris' lemma is quite useful. We see immediately that we can test the four-point condition in just $O(n^3)$ time. Actually, testing ultrametricity and thus the four-point condition can even be done in $O(n^2 \log n)$ time.

Problem 16. *Which metric spaces can be represented up to uniform equivalence by a subset of a space (or an R-tree) with the intrinsic metric and finite total length?*

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[5] But, despite this, many articles in the optimization literature ask for minimizing the total length of a graph which represents a given finite metric space. This should also be explored for infinite metric spaces.

[6] Any connected graph has a *graph metric* which is the largest metric in which the distance between any two vertices which are joined by an edge is 1

6. L_1-Embeddable Metrics and their Decompositions

A metric space (X, ρ) where X is finite is said to be l_1-*embeddable* if we can embed X isometrically into l_1.

Do such metric spaces occur in nature? Is this class useful for statistical analysis? It is often true that real-life estimates of similarity are obtained by forming a linear combination of various criteria. Such estimates, such metrics are precisely the L_1-embeddable metrics ! Let us make this exact.

Definition 3. *Let $(X, \mathcal{M}, \sigma)$ be a measure space. For $A, B \in \mathcal{M}$, define $\rho(A, B) = \int_{A \triangle B} d\sigma$. We call ρ L_1-*embeddable*.*

Since we use integration, we are restricted to estimating similarity by linear combinations of various criteria. But, this still allows us to represent a broad range of metrics.

Proposition 4. *Let ρ be a metric on a finite set. Then, ρ is l_1-embeddable if and only if ρ is L_1-embeddable.*

Theorem 3. *If a metric ρ on X satisfies the four-point-condition, then ρ is L_1-embeddable.*

Proof. Represent (X, ρ) by a subset of an R-tree Y with the intrinsic metric. Choose $v \in X$. For each $x \in X$, let A_x be the unique shortest path in Y from x to v. Let $\mathcal{M}$ be the set of all Borel sets of Y. Let μ be the measure which assigns to each Borel set B the sum of the lengths of all disjoint families of paths in B. Let f be the constant one function. Now, the intrinsic metric between x and y coincides with the L_1 metric on $(Y, \mathcal{M}, \sigma)$.

In the analysis of statistical data, it is not only important to recognize L_1-embeddable distances but also to be able to decompose distance data into an L_1-combination of more primitive distances. That is, we want to be able to carry out "linear decompositions" whenever this is possible and to identify when this is not possible.

Definition 4. *Suppose (X, ρ) is a metric space. If there are metric spaces $\{(X_i, \rho_i) : i \in I\}$ and a one-to-one map $\pi : X \to \prod\{X_i : i \in I\}$ such that $(\forall x, y \in X)\rho(x, y) = \sum_{i \in I} |\pi(x)(i) - \pi(y)(i)|$ and if $\{\pi(x)(i) : x \in X\} = X_i$, then we say π is a decomposition (X, ρ) as a subdirect L_1-product of metric spaces.*

This is motivated by the important existence of subdirect representations in algebra.

Theorem 4. *Every metric space can be decomposed in a "maximal" manner as a subdirect L_1-product of subsets of the reals and one more irreducible metric space. Every L_1-embeddable metric space is decomposed completely into a subdirect L_1-product of subsets of the reals.*

Proof. Construct π, inductively, on an well-ordered set I.[7] If this has been done on an initial segment $J \subset I$ and i is the least element of $I - J$, then define $\rho^*(x, y) = \rho(x, y) - \sum\{|\pi(x)(i) - \pi(y)(i)| : i \in J\}$ and let $\Sigma = \{\sigma \in \mathbb{R}^X : \rho^* - \sigma$ satisfies the triangle inequality $\}$ be partially ordered by defining $\sigma \leq \sigma'$ if, for all $x, x' \in X$, $\sigma(x, x') \leq \sigma'(x, x')$. Choose a maximal $\sigma \in \Sigma$ and define, for each $x \in X$, $\pi(x)(i) = \sigma(i)$.

Problem 17. *Is there a "maximal" decomposition of metric spaces as a subdirect L_1-product of additive trees (or Hilbert spaces) and one more irreducible metric space so that every additive tree (or Hilbert space) remains its unique factor?*

The notion of L_1-decomposition is well-motivated by the central importance of "dimension reduction" in multivariate data analysis. In his influential textbook, Kshirsagar said "The aim of the statistician undertaking multivariate analysis is to reduce the number of variables by employing suitable linear transformations ... thus reduces the dimensionality of the problem." Reasonable decompositions accomplish this by removing the interaction between coordinates.

Problem 18. *Are there reasonable L_p decompositions for $1 < p \leq \infty$?*

[7] The reals themselves can be decomposed into two copies of the reals, namely as the line $y = x$, and this is why we require an well-ordering of the factors. With a restriction to integer-valued metrics, this is no longer an issue.

A more useful L_1-decomposition would do more and break down the remaining irreducible factor in theorem 4 into an L_1-product of other irreducible factors whenever possible. We are able neither to prove such a theorem or even to formulate this accurately. The criterion by which such a decomposition should be judged is that it should have as a corollary the following result of R. L. Graham and P. M. Winkler and reported in Proc. Nat. Acad. Sci. 81 (1984) 7259.

Theorem 5 (Graham, Winkler). *Any finite graph can be canonically embedded isometrically into a maximum cartesian product of irreducible factors.*

The existence of the decomposition by subdirect products for varieties is a true theorem of universal algebra but, this is not a variety and so this seems to be of no help.

The general problem of identifying L_1-embeddability turns out to be significant in operations research. The problem of *multicommodity flows* is set in a graph in which each edge has a capacity and a demand. We seek a flow on the edges of the graph so that flow on each edge meets demand and does not exceed capacity. The so-called Japanese theorem of 1971 states that a capacity and demand are *feasible* i.e., can be met if there is a metric ρ on the vertices of the graph so that $(c-r)\rho \geq 0$. The celebrated Ford-Fulkerson theorem in operations research is just this theorem in the special and tractable case of single commodity flows in which the demand occurs on a single edge. Usually, the Ford-Fulkerson condition is not sufficient when the demand is more complicated. However, Lomonosov showed in 1978 that this condition is still sufficient when the demand lies on an L_1-embeddable subgraph.

7. Graphs and Hamming Distance

Indeed, theorem 5 illustrates the intimate connection between L_1-embeddability and Hamming distance. If we use factors in which all non-zero distances are 1 and a counting measure, then the L_1-distance is precisely the Hamming distance. This Hamming distance is useful in estimating distances between binary strings, since error-correcting codes can be designed which do nothing more than replace a string with the "closest" string of a certain kind. Although Avis showed that any finite L_1-embeddable metric space embeds in a "weighted" hypercube, it is not true that an integer-valued L_1-embeddable metric can be embedded in the *hypercube* 2^κ with the Hamming distance.

Problem 19. *Give necessary and sufficient conditions for an integer-valued (L_1-embeddable) metric to be embeddable in 2^κ with the Hamming distance.*

For example, a necessary condition is that triangles must have even perimeter.

There is a huge literature on graphs which can be embedded in hypercubes and metrics which can be embedded in graphs[8], but this beautiful theory carries us too far away from our topic.

8. Compactness and L_∞-Embeddable Metrics

A classical result of Banach and Mazur, published in 1932, states that any separable metric space can be isometrically embedded in $L_\infty(\kappa)$ when κ is the continuum. But, more is true. Suppose (X, d) is a metric space. Fix $a \in X$ and define an isometric embedding π of X into $C^*(X) \subset L_\infty(|X|)$ by defining $\pi(x)$ by setting $\pi(x)(x') = d(x, x') - d(a, x')$.

Theorem 6 (Banach, Mazur; 1932). *Any metric space can be isometrically embedded in $L_\infty(\kappa)$ for sufficiently large κ.*

This theorem, surprisingly, is essentially finitary.

Theorem 7. *If every finite subset of a metric space X is L_∞-embeddable, then X is L_∞-embeddable.*

Proof. Define, for each finite $F \subset X$, $E(F)$ to be the set of all mappings ϕ from X into $\mathbb{R}^\kappa$ which are isometric when restricted to F and achieve the supremum, for any pair, on a coordinate specifically assigned to that pair. These form a centred family of closed sets. If we restrict ourselves to maps which, for some $x \in X$, satisfy

[8] Djoković characterized graphs that can be embedded into hypercubes in 1973.

$\phi(x) \equiv 0$, then each $E(F)$ is a subset of a fixed compact set and so we have a nonempty intersection.[9]

This leads us to the three basic compactness problems for L_∞-embeddable (or L_1-embeddable, or L_p-embeddable) metrics.

- If every finite subset of a metric space X can be embedded in l_∞ (or l_1, or l_p), then must X be embeddable in some L_∞ (or L_1, or L_p)?
- If $n \in \omega$, then what is the minimal $k_n \in \omega$ (if it exists) such that any (l_1-embeddable, l_p-embeddable) finite metric space of size n can be embedded in $l_\infty^{k_n}$ ($l_1^{k_n}$, $l_p^{k_n}$)?
- If $n \in \omega$, then what is the minimal $k_n \leq \omega$ (if it exists) such that any metric space which cannot be embedded in l_∞^n (l_1^n, l_p^n) has a subspace of size k_n which also cannot be embedded in l_∞^n (l_1^n, l_p^n)?

For the first of these problems, Witsenhausen showed that, if every finite subset of a metric space X is embeddable in l_1, then X is embeddable in some L_1. Results of Yang and Zhang show that, if every finite subset of a metric space X is embeddable in l_2, then X is embeddable in some L_2. The situation for L_p seems to be unclear:

Problem 20. *If every finite subset of a metric space X can be embedded in l_p, then must X be embeddable in some L_p?*

Problem 21. *Find a general compactness theorem which implies that the solution to the first compactness problem is positive for all p.*

For the second problem, Schoenberg noted in 1938 that, although the construction in the proof of theorem 6 above seems to require n coordinates, we can omit one coordinate without difficulty. This shows that $k_n \leq n-1$ for l_∞. Wolfe showed that, in fact, $k_n \leq n-2$ for l_∞. Witsenhausen has obtained the lower and upper bounds $n-2 \leq k_n \leq n(n-1)/2$ for l_1 and, later, Ball showed that $k_n \leq n(n-1)/2$ for any l_p. But, none of these results solve the problem completely:

[9] Of course, L_p might not be locally compact but this is irrelevant. We work in the Tychonoff product topology.

Problem 22. *If $n \in \omega$, then what is the minimal $k_n \in \omega$ (if it exists) such that any l_1-embeddable finite metric space of size n can be embedded in $l_1^{k_n}$? What about for l_p when $1 < p \leq \omega$?*

This second problem has an interesting variation. Suppose $D = \{1, 2, 3\}$ has the "distance" in which 1 and 3 are distance one apart and all other pairs are at distance zero. What is the least k_n such that any connected graph on n vertices can be embedded in a product of k_n many copies of D with the L_1 distance? It is not obvious that k_n exists and is finite.

This may seem a strange problem, but this is exactly the "addressing problem for loop switching" posed by R. L. Graham and H. O. Pollak in 1971 in the Bell System Technical Journal and solved by P. M. Winkler in 1983. The answer is $k_n = n - 1$.[10]

The third problem is quite interesting. It may involve finite approximations to topological orientability.

Proposition 5 (S. Malitz and J. Malitz, 1992). *If a metric space X cannot be embedded in $\mathbb{R}^2$ with the l_∞-norm (or, equivalently, the l_1-norm), then X has a subspace of size 11 which cannot be embedded in $\mathbb{R}^2$ with the l_∞-norm (or, equivalently, the l_1-norm). Thus, determining whether a finite metric space can be embedded in $\mathbb{R}^2$ with the l_∞-norm can be done in polynomial time.*

They state the existence of such a number (like 11), for $\mathbb{R}^n$ when $n \geq 3$ is an open question, and that their methods get "wildly complicated".

But, we have obtained the following results.

Theorem 8. *There is no N such that a finite metric space X cannot be embedded in $\mathbb{R}^3$ with the l_∞-norm if and only if X has a subspace of size N which cannot be embedded in $\mathbb{R}^3$ with the l_∞-norm.*

Proof. Use a Mobius strip in which the width of the strip is much smaller than N times the radius of the circle. Apply compactness to get a finite subset which is still sufficiently "Mobius".

[10] These are "squashed cubes", but the problem for graphs in ordinary cubes remains open.

Problem 23. *Is it true that there is no N such that a finite metric space X cannot be embedded in $\mathbb{R}^3$ with the l_1-norm if and only if X has a subspace of size N which cannot be embedded in $\mathbb{R}^3$ with the l_1-norm? Is this true for some $\mathbb{R}^n$? Can the construction in theorem 8 be carried out in some $\mathbb{R}^n$ with the l_1-norm?*

Theorem 9. *Determining whether a finite metric space X can be embedded in $\mathbb{R}^6$ with the l_∞-norm is NP-complete.*

Proof. The axes of a cube can be each be assigned one of three dimensions in exactly six ways. This assignment must be constant on the product of a cube and a line. If we join together two such products in such a way that all coordinates change, then knowing the assignment on one side of the join gives us exactly two possibilities on the other side of the join. Thus, using three more dimensions we can code the 3-colorability of graphs which is NP-complete.

Problem 24. *Let $3 \leq n \leq 5$. Is determining whether a finite metric space X can be embedded in $\mathbb{R}^n$ with the l_∞-norm NP-complete?*

Problem 25. *Is determining whether a finite metric space X can be embedded in $\mathbb{R}^n$ with the l_1-norm NP-complete?*

9. L_2-Embeddable Metrics

The problem of characterizing metric spaces which embed in Euclidean space of some dimension is a classical one and was solved by Menger in the 1930's. There is a book by Blumenthal entitled *Distance Geometry* and even a Mathematical Reviews section 51K devoted to this topic. But, in fact, this is an easy problem in $\mathbb{R}^2$ with the Euclidean (l_2) metric. For if a space embeds in $\mathbb{R}^2$ and a, b, c are points in that space which do not satisfy the equality $\rho(a,b) + \rho(b,c) = \rho(a,c)$ under any permutation, then a is, without loss of generality, embedded arbitrarily. Now, b is embedded on some circle centred at a, but otherwise its position is arbitrary. We deduce that c must be placed in one of two positions, but this choice is again arbitrary. But, now any further point must occupy an uniquely determined position. Thus, the position of any point is determined uniquely once we have three points "in general position". In the general setting of the Euclidean metric on $\mathbb{R}^n$, the situation is analogous.

Much of the work in distance geometry is devoted to characterizing Euclidean spaces, Banach spaces, hyperbolic spaces, inner product spaces and so forth entirely from the combinatorial properties of their metrics. But, we will not discuss here this fascinating topic and its intense activity since 1932 nor will we discuss the interesting work on the "distance-one-preserving" maps of A. D. Aleksandrov.

What is surprising and important to us is that ultrametrics are L_2-embeddable.

Theorem 10 (Lemin, 1985; Vestfrid and Timan, 1979 for l_∞). *Any ultrametric space of cardinality κ can be embedded isometrically in generalized Hilbert space $\{f \in \mathbb{R}^\kappa : \sum\{f(\alpha) : \alpha \in \kappa\} < \infty\}$.*

This requires some work.

Another surprising fact is that L_2-embeddable metrics are L_1-embeddable.

Theorem 11. *Any L_2-embeddable space is L_1-embeddable.*

Problem 26. *Give a direct proof that any L_2 embeds isometrically into some $L_1(\mu)$. Can this be done by integration over projections onto hyperplanes of codimension 1? What happens for $p \neq \infty$?*

But, the most important fact about L_2-embeddable metrics is that they are the basic notion of MDS: *multi-dimensional scaling.* This is a huge topic about which entire books have been written and for which there are many software packages being sold.

The basic purpose of MDS, the thing that these packages accomplish, is to take a set of data, either an $n \times k$ matrix showing the results of tests or an $n \times n$ matrix which already exhibits similarity data, and to do the best job possible in representing this data as points in the plane or in a higher-dimension Euclidean space.

There is a lot involved here. Scaling the similarity data with real numbers, reconstruction of missing and spurious data, approximation to a metric which is embeddable in some Euclidean space. The problem of reconstructing missing data is an important one. Sippl and Scheraga, Proc. Nat. Acad. Sci. USA 83 (1986) 2283 and Schlitter 1987 in pursuit of reconstructing distance data in problems on nuclear magnetic resonance, showed that we need only a $4 \times n$ submatrix of the distance matrix to reconstruct effectively in $\mathbb{R}^3$ so long as the 4 points chosen are in general position.

Problem 27. *What happens in the reconstruction problem for the L_1 or L_∞ metric?*

Problem 28. *If (X, ρ) is a metric space, then what are necessary and sufficient conditions on $A \subset X^2$ so that, whenever ρ' is another metric on X such that $\rho \restriction A = \rho' \restriction A$, we must have $\rho = \rho'$. What if we only want ρ and ρ' to be equivalent or uniformly equivalent?*

Problem 29. *Find $k(n)$ so that, if A is a metric space which can be embedded in l^n_∞, then is there a finite set $B \subset A$ of size $k(n)$ such that knowing all the distances between points of A and points of B allows one to reconstruct the distance matrix.*

Problem 30. *Where does L_p-embeddable fit into the scheme we have given? Does ultrametric imply L_p-embeddable which implies L_1-embeddable, when $p \neq \infty$? Are the classes of L_p-embeddable metrics comparable?*

10. **Hypermetric Spaces and Spaces of Negative Type**

The notion of L_1-embeddable differs greatly from additive trees and ultrametrics in that it does not seem to have a definition by a simple inequality. It is suspected that there are no simple characterizations of L_1-embeddable metrics, but this has never been established.

Problem 31. *Is there a first-order characterization of L_1-embeddability?*

A. Neyman showed in 1984 that there is no characterization which is a finite conjunction of inequalities. Of course, by compactness, there is an infinite conjunction of first-order formulas which characterizes L_1-embeddable.

The attempts to characterize L_1-embeddable by means of inequalities has led to some interesting inequalities which must be satisfied by any L_1-embeddable metric. These include the *hypermetric inequalities.*

Definition 5. *A* hypermetric inequality *is defined for each* $b : X \to \mathbb{Z}$ *such that* $\sum\{b(x) : x \in X\} = 1$ *and states that* $\sum\{b(x)b(y)d(x,y) : x, y \in X\} \leq 0$. *A metric space which satisfies each hypermetric inequality is said to be a* hypermetric *space.*

While this scheme is a little hard to understand at first, there are relatively few instances which are not satisfied automatically. In fact, the least complicated instance is accomplished by the b's which are $1, 1, 1, -1, -1$. This yields the *pentagon inequality*, cited in the introduction. The easiest way to understand the hypermetric inequalities is to note that they forbid the bipartite graphs $K(n, n+1)$ when $n \geq 2$.

Theorem 12. *L_1-embeddable metrics are hypermetric.*

Proof. A *cut pseudometric* on a set X is a binary-valued pseudometric induced by any $A \subset X$ which is defined by letting $\rho(x, x') = 1$ iff $|\{x, x'\} \cap A| = 1$. Any L_1-embeddable metric is a linear combination of cut pseudometrics. Hypermetricity is clearly preserved by linear combinations. So, it suffices to show that cut pseudometrics are hypermetric. This means that we must show that, whenever $a, b, c, d \geq 0$, we have $a + c - b - d = 1 \Rightarrow (a - b)(c - d) \leq 0$ which is easy.

Nevertheless, these inequalities do not characterize L_1-embeddable metrics. In 1977, Assouad and, independently, Avis in 1981, showed that the graph obtained by deleting two adjacent edges from K_7 is hypermetric, but not L_1-embeddable. More sophisticated inequality schemes valid for L_1-embeddable metrics were devised by Deza and Laurent in 1992.

Despite their humble birth as approximations to L_1-embeddability, hypermetrics are significant to geometry. Consider the problem of identifying the metrics on $\mathbb{R}^n$ which are scalar multiples of the usual metric on each straight line (these are called projective metrics). This is Hilbert's fourth problem. In 1974, Pogorolev characterized projective metrics in $\mathbb{R}^2$. In 1986, Szabo defined a complicated example of a projective metric on $\mathbb{R}^3$ which does not satisfy Pogorolev's characterization. To see how hypermetrics are closely related to the fourth problem, we need a concept from convex geometry. A *zonoid* is a convex set which is arbitrarily close in the Hausdorff metric to convex polytopes in $\mathbb{R}^n$. Alexander showed in 1988 that whenever the dual unit ball of a finite-dimensional normed linear space M (with a projective metric) is not zonoid, Pogorolev's characterization does not work. In 1975, Kelly proved that this problem is equivalent to determining whether the dual space of M is hypermetric. To get a projective metric on $\mathbb{R}^3$ which does not satisfy Pogorolev's characterization, we need only a projective metric which is not hypermetric. $L_\infty(\mathbb{R}^3)$ works!

Problem 32. *Does $L_\infty(\mathbb{R}^3)$ satisfy the pentagonal inequality? Characterize the projective metrics on $\mathbb{R}^3$ which disobey the pentagonal inequality or hypermetric inequalities (or weaker properties).*

It was proved in 1993 however by Deza, Grishukhin and Laurent, making use of Voronoi theory, that hypermetric spaces can be described by a finite list of inequalities. This is amazing, since the hypermetric scheme is infinite and does not seem to contain any redundancies. We don't know if this follows from logical considerations alone.

Another surprising aspect of the hypermetric inequalities is that, despite their failure to characterize the L_1-embeddable metrics, they do carry some power. Indeed, any hypermetric space still has some "Euclidean" structure.

Consider the example of a "distance" space consisting of the the points on the n-sphere with the metric defined by the square of the Euclidean metric. Of course, if we examine any three nearby and nearly collinear points, we see that this is not a metric space, but it certainly has many metric subspaces.

Definition 6. *If a metric space X can be isometrically embedded in some n-sphere with the square of the Euclidean metric, then we say that X is* spherical.

Theorem 13 (Deza, Grishukhin, Laurent). *Every finite hypermetric space is spherical.*

Problem 33. *Is any (countable, separable, arbitrary) hypermetric space isometrically embeddable in some appropriately defined κ-sphere? What is the correct infinitary notion of spherical?*

Problem 34. *There is at least an example of a spherical space which is not hypermetric?*

Note that it does not suffice to take an appropriate sphere, since this will not satisfy the triangle inequality.

Moving even further into weak properties, we can identify the *negative-type inequalities*. These are defined exactly like the hypermetric inequalities, except that we require only $\sum\{b(x) : x \in X\} = 0$.

Definition 7. *A* negative-type inequality *is defined for each* $b : X \to \mathbb{Z}$ *such that* $\sum\{b(x) : x \in X\} = 0$ *and states that* $\sum\{b(x)b(y)d(x,y) : x, y \in X\} \leq 0$. *A metric space which satisfies each negative-type inequality is said to be a* space of negative-type.

Again, it is easiest to understand the negative-type inequalities as forbidding the graph $K(n,n)$ when $n \geq 3$.[11]

So, hypermetric spaces and spaces of negative-type are defined by analogous schemes of inequalities, and spherical spaces are characterized by embeddability in a specific Euclidean-style space. Nevertheless, spherical spaces interpolate hypermetric spaces and spaces of negative-type !

Theorem 14 (Deza, Grishukhin). *Every spherical space has negative-type and thus every hypermetric space has negative-type.*

Of course, metric spaces of negative-type need not be hypermetric. The graph $K(2,3)$ demonstrates this. This graph also answers one of the two parts of the next question, but which one?

Problem 35. *What is an example of a negative-type metric space which is not spherical? What is an example of a spherical space which is not hypermetric?*

In the application to Hilbert's fourth problem, we used the fact that $L_\infty(\mathbb{R}^3)$ is not hypermetric.

Problem 36. *Is* $L_\infty(\mathbb{R}^3)$ *of negative type? For which* n *is* $L_\infty(\mathbb{R}^n)$ *of negative type?*

[11] One easily embeds $K(2,2)$ in $\mathbb{R}^3$ and, of course, K_n is ultrametric.

The next classical result is beautiful and surprising and demonstrates immediately why spherical metrics are of negative-type.

Theorem 15 (I. J. Schoenberg, 1938). *A metric space is of negative-type if and only if it can be embedded in some $\mathbb{R}^n$ with the metric which is the square of the Euclidean metric.*

Actually, in the language of linear algebra, this was first proved by Cayley !

Ponder theorem 15. It says that any metric of negative type can be squared and suddenly it is embedded in Euclidean space. But, this squaring is such a "nice" transformation ! The reason that we have not discussed the topological level of generality, since leaving additive trees becomes clear. All of these properties: L_2-embeddable, L_1-embeddable, hypermetric, spherical, negative type all coincide up to homeomorphism, up to uniform homeomorphism, even up to composition of the metric with a monotone function.

Let us call this composition a "scaling" and then be more exact.

Definition 8. *If $f : [0, \infty) \to [0, \infty)$ is a function whose limit at zero is zero, then the scaling of a metric ρ by f is the function ρ_f defined by $\rho_f(x, y) = f(\rho(x, y))$.*

Proposition 6. *Any scale which is concave up preserves the triangle inequality.*

Delistathis has noted the well-known transformation $x \to \frac{x}{1+x}$ which is used to bound metrics provides the most common example of an application of proposition 6.

The notion of scaling can be used to approach the problem of deciding how "geometric" these weaker metric concepts are.[12] Certainly, all separable metric spaces can be embedded by an uniform homeomorphism into Hilbert space (this was proved first by Mysior, it seems). But, not all separable metric spaces can be embedded by a re-scaling into Hilbert space.

Theorem 16. *There is a separable metric space which cannot be scaled to embed in a pentagonal (and thus, Euclidean or negative-type) space.*

[12]Note that scaling preserves ultrametricity, but maybe not additive tree distances.

Proof. Take the bipartite graph $K(n,n)$ for all possible choices of n and multiplied by all possible choices of positive rational numbers.

Every finite metric space has a scale which embeds it into l_2 but whether one can get these scales in an uniform manner is unknown.

Problem 37 (Maehara, 1986). *Is there a scale which embeds all metric spaces of fixed size n (even size 5) into l_2 simultaneously?*

11. Lipschitz Constants and Eigenvalues

Another property of a transformation weaker than uniform homeomorphism but incomparable to scaling is that of an α-Lipschitz map. We say that two metrics ρ and π are α-Lipschitz where $\alpha \geq 1$ if every quotient $\frac{\rho(x,y)}{\pi(x,y)}$ and its inverse is at most α. Of course, two metrics are 1-Lipschitz if and only if they coincide. This notion enables us to ask whether an arbitrary metric is α-Lipschitz to an Euclidean metric and so forth.

Note that the square root scaling is not α-Lipschitz for any constant α. So, there is no reason to expect L_2-embeddable, L_1-embeddable, hypermetric, and negative-type to be α-Lipschitz for any constant α.

Proposition 7 (J. Bourgain, T. Figiel, V. Milman). *There is a finite metric space which is not 2-Lipschitz isometrically embeddable in l_2.*

Theorem 17. *There is, for each $\alpha > 2$, a finite metric space which is not α-Lipschitz to a space of negative type (or a subset of l_2).*

Note that $K(n,n)$ is easily shown not to be $(\sqrt{2}-\epsilon)$-Lipschitz isometrically embeddable in l_2.

Problem 38. *Is there a metric space of negative type which is not α-Lipschitz isometrically embeddable to a subset of l_2?*

In their pursuit of pathological examples in the geometry of Banach spaces, Bourgain, Milman and Wolfson did establish a Ramsey-theoretic theorem showing that in the disorder of arbitrary finite metric spaces can be found a certain amount of "Euclidean behavior". That is, arbitrary finite metric spaces do have fair-sized subsets which do embed into l_2.

Theorem 18 (J. Bourgain, T. Figiel, V. Milman). *For every $\alpha > 1$, there is $C > 0$ such that every finite metric space contains a subset which is α-Lipschitz embeddable in l_2 and has size at least $C \log |X|$.*

Indeed, Bourgain, Milman and Wolfson defines their own metric inequality which says that a metric space has type 2 if there is $\epsilon > 0$ so that, for any labelling of points by the vertices of an n-cube, the l_2-sum of the diagonals is less than ϵ times the l_2-sum of the edges. They show that a metric space of type 2 contains copies of l_1^n up to a Lipschitz constant.

Problem 39. *Does type 2 fit naturally into the scheme of hypermetric and negative-type inequalities?*

Problem 40. *What Lipschitz constants, if any, exhibit the distinction between L_2-embeddable, L_1-embeddable, hypermetric, negative-type and one positive eigenvalue?*

Another transformation of metrics derives from the notion of a Robinsonian metric. This is a metric ρ whose underlying set admits a linear order $\leq$ such that $a \leq b \leq c \leq d \Rightarrow \rho(a, d) \leq \rho(b, c)$. Thus, Robinsonian metrics are metrics which are "compatible" with a linear order. Ultrametrics are Robinsonian, but we know little more than this.

Problem 41. *Are additive metrics Robinsonian? Are Robinsonian metrics of negative type (or hypermetric)? What if we allow $\leq$ to be a partial order of some kind?*

Let us now turn to eigenvalues. Suppose we are given any n points in some Euclidean space and compute the distance matrix. This matrix is symmetric and thus has all real eigenvalues. It has zero entries along the diagonal and has exactly one positive and $n - 1$ negative eigenvalues. It turns out that if a metric has negative type, then it is still true that the distance matrix has exactly one positive eigenvalue.

Theorem 19. *Any metric space which is of negative type has a single positive eigenvalue.*

The existence of a single positive eigenvalue represents the weakest metric property which has so far been isolated.

Definition 9. *If* (X, ρ) *is a metric space and, for each finite* $\{a_i : i \in n\} \subset X$*, the* $n \times n$ *distance matrix whose* (i, j)*-th entry is* $\rho(a_i, a_j)$ *has exactly one positive eigenvalue, then we say that* (X, ρ) has one positive eigenvalue*.*

To see that this definition is reasonable, one should note that if a matrix has a particular eigenvalue, then any square submatrix also has that eigenvalue. $K(3, 3)$ is not negative-type and, indeed, it has two positive eigenvalues.

Problem 42. *What are the metric spaces (of smallest cardinality) which do not have one positive eigenvalue?*

An example due to Winkler of a metric space with one positive eigenvalue which is not of negative type is the bipartite graph $K(5, 2)$ with a single edge added between the two points on the "side" with only two points.[13]

Problem 43. *Can any metric space be scaled to have one positive eigenvalue?*

The scaling method we described (taking the square root) shows that any metric of negative type can be scaled to be Euclidean, but it is unknown what happens for metrics with one positive eigenvalue.

Problem 44. *Is there a metric space which has one positive eigenvalue which cannot be scaled to have negative type (equivalently, to be Euclidean)?*

Problem 45. *Which Tychonoff spaces have, for each continuous pseudometric, an equivalent (or generating a larger topology) continuous pseudometric with one positive eigenvalue?*

Further work has been done on investigating the characteristic polynomial of distance matrices of graphs by R. L. Graham and L. Lovász. This work is beyond the scope of this article, but, no doubt, investigating the characteristic polynomial of an arbitrary metric space would be rewarding.

[13] An elegant proof of this was given by Deza and Maehara in 1990 and Marcu in 1991.

Problem 46. *Is there an useful class of metric spaces strictly weaker than those with exactly one positive eigenvalue?*

12. **Quasi-Metrics**

The notion of asymmetric distances occur frequently in the literature. In optimization theory, for example, the "windy postman" problem is a version of the travelling salesman problem in which the quasi-metric represents times needed to cover a distance and so, "depending on the wind", there is asymmetry.

Another significant application of asymmetric distances is in psychological measurement. The influential 1978 article by Cunningham explains why this is so. "There are some situations in which the direction of the dissimilarity measurement may make a difference." He continues: "As an example, consider the case of people judging the similarity of two stimuli which differ markedly in their prominence or number of known traits". In 1977, Tversky found that people gave a consistently higher rating when asked questions like "How similar is North Korea to Red China" than when asked questions like "How similar is Red China to North Korea".

The notion of an additive tree and the notion of the four-point property both generalize to the asymmetric case naturally, but these generalizations do not seem to be equivalent. Bandelt in 1990 found equations which characterize the asymmetric generalization of additive trees.

Besides, these generalizations from the symmetric case, there is no available means of classifying asymmetric distances.

The distance matrices for finite subsets of a quasi-metric spaces are not symmetric and thus these matrices may have some eigenvalues which are not real.

Problem 47. *Do all quasi-metric spaces have an equivalent quasi-metric with all real eigenvalues?*

Problem 48. *Let X be a completely regular (topological) space. Is there, for every continuous quasi-metric on X, another continuous quasi-metric which generates a* *larger topology and all of whose eigenvalues are real? What if we require these quasi-metrics to generate completely regular topologies?*

Problem 49. *Formulate problems whose solution would make progress towards the understanding of asymmetric distance data.*

13. **Conclusion**

The understanding of distance data is a fundamental goal of the natural and social sciences. To create this understanding, there are problems of reconstruction and approximation which are perhaps mainly problems in optimization theory and thus in linear algebra or non-linear analysis. But, the problems of transformation, representation and classification are topological problems. Although the data is finite, solving the corresponding infinitary problems gives asymptotic and efficient methods for solving the finite problems.[14] Moreover, finite combinatorists find all but the most graph-theoretic of these problems far too geometric or topological.[15] Although the use of distances suggests that this is a geometric problem, the importance of transforming the data in a non-linear manner, and the key role of approximation and reconstruction eliminates geometers from all but the most artificial and rigid of these problems. The importance of L_p in the classification may suggest that these problems lie in the territory of Banach space experts, but the absence of linearity immediately disqualifies these problems from consideration by all but the most heretical of functional analysts.

This is a problem which is directly adjacent to graph theory, optimization theory, operations research, geometry, and the theory of stochastic processes. This is a problem of immediate and great importance to communications theory, to statistical mechanics, to mathematical psychology, to mathematical taxonomy and to

[14]The importance of algorithms and complexity of computation is key to making the infinite important. If the uncountable fails, we must need enumeration and there will often be no algorithm. If the countably infinite fails, we must need to quantify over subsets and this often gives a lower bound on complexity.

[15]But, it seems that a large part of the theory of distances in graphs may be extended usefully, with some work, to a theory of L_1-embeddable metrics.

multivariate statistical analysis whose significance will only increase when a more sophisticated theory is developed. This is a problem whose solution can be developed by topologists.

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Str. Pasului 3, Sector 2, 020795-Bucharest, Romania

E-mail address: drmarcu@yahoo.com

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AN INTERPOLATION BASED COLLOCATION METHOD FOR SOLVING THE DIRICHLET PROBLEM

SANDA MICULA

Abstract. In this paper we study the numerical solution of a boundary integral equation reformulation of the Dirichlet problem. We give a brief outline of both this problem and its solvability and of a collocation method based on interpolation. We conclude the paper by giving an error analysis of this collocation method.

1. **The Exterior Dirichlet Problem**

We will study only the *exterior* Dirichlet problem, but would like to mention that all the results hold for the *interior* Dirichlet problem, as well, since their integral equation reformulations are very similar.

Let D denote a bounded open simply-connected region in $\mathbb{R}^3$, and let S denote its boundary. Let $\overline{D} = D \cup S$ and denote by $D_e = \mathbb{R}^3 - \overline{D}$ the region complementary to D. Let $\overline{D}_e = D_e \cup S$. At a point $P \in S$, let $\mathbf{n}_P$ denote the unit normal directed into D, provided that such a normal exists. Also assume that S is a piecewise smooth surface that can be decomposed into a finite union of smooth surfaces intersecting each other along common edges at most. In addition, assume that S has a triangulation $\mathcal{T}_n = \{\Delta_{n,k} \mid 1 \le k \le n\}$ with mesh size h (such a triangulation can be obtained as the image of a composition of bijections m_k from the unit simplex σ onto a planar triangle Δ_k and bijections F_j from a right triangle onto each smooth piece S_j of S; for details, see Micula [6, Chapter 2]).

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The Exterior Dirichlet Problem. Find $u \in C^1(\overline{D}_e) \cap C^2(D_e)$ that satisfies

$$\begin{aligned} \Delta u(P) &= 0,\ P \in D_e \\ u(P) &= f(P),\ P \in S \\ u(P) = O(P^{-1}),\ \frac{\partial u(P)}{\partial r} &= O(|P|^{-2}) \quad \text{, as} \quad r = |P| \to \infty \text{ uniformly in } \frac{P}{|P|} \end{aligned} \tag{1}$$

with $f \in C(S)$ a given boundary function.

The boundary value problem (1) has been studied extensively (see Mikhlin [8], Günter [5], Colton [4]). Here we only give a very brief outlook at results on the solvability of the problem (1).

The Divergence Theorem (see Atkinson [2, Theorem 7.1.2]) can be used to obtain a representation formula for harmonic functions.

We seek a solution of (1) in the form of a *double layer potential*

$$u(A) = \int_S \rho(Q) \cdot \frac{\partial}{\partial \mathbf{n}_Q} \left[\frac{1}{|A-Q|} \right] dS_Q, \quad A \in D_e \tag{2}$$

Using a limiting argument, we obtain the second kind integral equation

$$2\pi\rho(P) - \int_S \rho(Q) \cdot \frac{\partial}{\partial \mathbf{n}_Q} \left[\frac{1}{|P-Q|} \right] dS_Q = f(P), \quad P \in S \tag{3}$$

The kernel function in (3) is given by

$$\frac{\partial}{\partial \mathbf{n}_Q} \left[\frac{1}{|P-Q|} \right] = \frac{\mathbf{n}_Q \cdot (P-Q)}{|P-Q|^3} = \frac{\cos\theta_Q}{|P-Q|^2} \tag{4}$$

where θ_Q denotes the angle between $\mathbf{n}_Q$ and $(P-Q)$. Equation (3) can now be written as

$$\rho(P) - \frac{1}{2\pi} \int_S \rho(Q) \cdot \frac{\cos\theta_Q}{|P-Q|^2} dS_Q = \hat{f}(P), \quad P \in S \tag{5}$$

where $\hat{f}(P) = \frac{1}{2\pi} f(P)$. For simplicity, we will write $f(P)$ instead of $\hat{f}(P)$.

Write the equation (5) in operator form:

$$(\mathcal{I} - K)\rho = f \tag{6}$$

We have (see Mikhlin [8, Chapters 12 and 16]):

Theorem 1.1. Let S be a C^2 surface. Then the equation (6) has a unique solution $\rho \in X$ for each given function $f \in X$, with $X = C(S)$ or $X = L^2(S)$.

Theorem 1.2. Let S be a smooth surface with $\overline{D_e}$ a region to which the Divergence Theorem can be applied. Assume the function $f \in C(S)$. Then, the Dirichlet problem (1) has a unique solution $u \in C^\infty(D_e)$.

2. A Collocation Method

We will use a collocation method where the collocation nodes are the interpolation (of order r) nodes, chosen the following way:

$$q_{i,j} = \left(\frac{i + (r - 3i)\alpha}{r}, \frac{j + (r - 3j)\alpha}{r} \right), \quad i, j \geq 0, \quad i + j \leq r \tag{7}$$

for some $0 < \alpha < 1/3$ (these are points interior to the unit simplex, but they get mapped into points interior to each triangle in $\mathcal{T}_n$). For corresponding *Lagrange* functions (see Micula [6, pg. 7-11]), for $g \in C(S)$ define an operator $\mathcal{P}_n$ by

$$\mathcal{P}_n g(P) = \sum_{j=1}^{f_r} g\left(m_k(q_j)\right) l_j(s,t), \quad (s,t) \in \sigma, \quad P = m_k(s,t) \in \Delta_k \tag{8}$$

This interpolates $g(P)$ over each triangular element $\Delta_k \in S$, with the interpolating function polynomial in the parametrization variables s and t.

Define a collocation method with (7). Denote $v_{k,j} = m_k(q_i)$. Substitute

$$\begin{aligned} \rho_n(P) &= \sum_{j=1}^{f_r} \rho_n\left(v_{k,j}\right) l_j(s,t) \\ P &= m_k(s,t) \in \Delta_k, \quad k = 1, ..., n \end{aligned} \tag{9}$$

into (5). To determine the values $\{\rho_n(v_{k,j})\}$, force the equation resulting from the substitution to be true at the collocation nodes $\{v_1, ..., v_{nf_r}\}$. This leads to the linear system

$$\begin{aligned} \rho_n(v_i) \quad - \quad & \frac{1}{2\pi} \sum_{k=1}^{n} \sum_{j=1}^{f_r} \rho_n(v_{k,j}) \int_\sigma \frac{\cos\theta_{v_{k,j}}}{|v_i - m_k(s,t)|^2} \\ \cdot \quad & |(D_s m_k \times D_t m_k)(s,t)| \, d\sigma = f(v_i), \quad i = 1, ..., nf_r \end{aligned} \tag{10}$$

which we write abstractly as

$$(\mathcal{I} - P_n\mathcal{K})\rho_n = \mathcal{P}_n f \tag{11}$$

which will be compared to (6). We have the following result.

Theorem 2.1. Let S be a C^2 surface as described earlier, with $F_j \in C^{r+2}$. Then for all sufficiently large n, say $n \geq n_0$, the operators $\mathcal{I} - P_n\mathcal{K}$ are invertible on $L^\infty(S)$ and have uniformly bounded inverses. For the solution ρ of (6) and the solution ρ_n of (10)

$$\|\rho - \rho_n\|_\infty \leq \left\|(\mathcal{I} - P_n\mathcal{K})^{-1}\right\| \cdot \|\rho - \mathcal{P}_n\rho\|_\infty, \quad n \geq n_0 \tag{12}$$

Furthermore, if $f \in C^{r+1}(S)$, then

$$\|\rho - \rho_n\|_\infty = O(h^{r+1}), \quad n \geq n_0 \tag{13}$$

For the proof, see, for example, Atkinson [1].

So interpolation of order r, leads to an error of order $O(h^{r+1})$. But superconvergent methods can be developed. Next, we want to explore in more detail the collocation method based on piecewise constant interpolation (the centroid method) and show that it is superconvergent at the collocation points. Define the operator $\mathcal{P}_n$ by

$$\mathcal{P}_n g(P) = g(P_k),\ P \in \Delta_k,\ k = 1, ..., n \tag{14}$$

for $g \in C(S)$. Then, $\mathcal{P}_n$ is a bounded operator on $C(S)$ with $\|\mathcal{P}_n\| = 1$. Define a collocation method with (14). Substitute

$$\rho_n(P) = \rho_n(P_k),\ P = m_k(s,t) \in \Delta_k,\ k = 1, ..., n \tag{15}$$

into (5). To determine the values $\{\rho_n(P_k)\}$, force the equation resulting from the substitution to be true at the collocation nodes $\{P_k \mid k = 1, ..., n\}$. This leads to the linear system

$$\begin{aligned} \rho_n(P_i) \quad &+ \quad \frac{1}{2\pi}\sum_{k=1}^{n} \rho_n(P_k) \cdot \int_\sigma \frac{\cos\theta_{Q_k}}{|P_k - m_k(s,t)|^2} \\ &\cdot \quad |(D_s m_k \times D_t m_k)(s,t)| \ d\sigma = f(P_k),\ i = 1, ..., n \end{aligned} \tag{16}$$

which can be rewritten abstractly as

$$(\mathcal{I} + P_n\mathcal{K})\,\rho_n = \mathcal{P}_n f \tag{17}$$

which will be compared to (6).

By Theorem 2.1., for the true solution ρ of (6) and the solution ρ_n of the collocation equation (17), we have

$$\|\rho - \rho_n\|_\infty = O(h), \quad n \geq n_0 \tag{18}$$

For $g \in C(\sigma)$, consider the interpolation formula (14), which has degree of precision 0. Integrating it over σ, we obtain

$$\int\limits_\sigma g(s,t)\; d\sigma \approx \int\limits_\sigma \mathcal{L}_\tau g(s,t)\; d\sigma = \frac{1}{2} g\left(\frac{1}{3}, \frac{1}{3}\right) \tag{19}$$

which has degree of precision 1.

For $\tau \subset \mathbb{R}^2$, a planar triangle and for a function $g \in C(\tau)$, the function

$$\mathcal{L}_\tau g(x,y) = g\left(m_\tau\left(\frac{1}{3}, \frac{1}{3}\right)\right) = g(P_\tau) \tag{20}$$

is the constant polynomial interpolating g at the node $m_\tau\left(\frac{1}{3}, \frac{1}{3}\right) = P_\tau$ (the centroid of τ). We have the following.

Lemma 2.2. Let τ be a planar right triangle and assume the two sides which form the right angle have length h. Let $g \in C^2(\tau)$. Let $\Phi \in L^1(\tau)$ be differentiable with the first derivatives $D_x\Phi$, $D_y\Phi \in L^1(\tau)$. Then

$$\left|\int\limits_\tau \Phi(x,y)\,(\mathcal{I} - L_\tau)\,g(x,y)\; d\tau\right| \leq ch^2 \left[\int\limits_\tau (|\Phi| + |D\Phi|)\;\, d\tau\right] \cdot \max_\tau \left\{|Dg|, |D^2|g\right\} \tag{21}$$

For the proof, see Micula [6, pg 74-75].

This result can be extended to general triangles, provided

$$\sup_n \left[\max_{\Delta_{n,k}\in\mathcal{T}_n} r(\Delta_{n,k})\right] < \infty \tag{22}$$

where

$$r(\tau) = \frac{h(\tau)}{h^*(\tau)} \tag{23}$$

with $h(\tau)$ and $h^*(\tau)$ denoting the diameter of τ and the radius of the circle inscribed in τ, respectively.

Corollary 2.3. Let τ be a planar triangle of diameter h, let $g \in C^2(\tau)$, and let $\Phi \in L^1(\tau)$ with both first derivatives in $L^1(\tau)$. Then

$$\begin{aligned} \left| \int_\tau \Phi(x,y)(\mathcal{I} - L_\tau)g(x,y) \right| &\leq c\,(r(\tau))\, h^2 \left[\int_\tau (|\Phi| + |D\Phi|)\ d\tau \right] \\ &\cdot \max_\tau \left\{ \|Dg\|_\infty, \|D^2 g\|_\infty \right\} \end{aligned} \tag{24}$$

where $c\,(r(\tau))$ is some multiple of $r(\tau)$ of (23).

Since formula (22) has degree of precision 1 (odd) over σ, extending it to a square would not improve the degree of precision, which means the same error bound as in Lemma 2.2 is true for a parallelogram formed by two symmetric triangles.

We want to apply the above results to the individual subintegrals in

$$\begin{aligned} \mathcal{K}g(P_i) &= \frac{1}{2\pi} \sum_{k=1}^{n} \int_\sigma \frac{\cos\theta_{Q_k}}{|P_k - m_k(s,t)|^2} \rho\,(m_k(s,t)) \\ &\quad \cdot |(D_s m_k \times D_t m_k)\,(s,t)|\ d\sigma \end{aligned} \tag{25}$$

with the role of g played by $\rho\,(m_k(s,t))\,|(D_s m_k \times D_t m_k)\,(s,t)|$, and the role of Φ played by $\dfrac{\cos\theta_{Q_k}}{|P_k - m_k(s,t)|^2}$. For the derivatives of this last function, we have

Theorem 2.4. Let i be an integer and S be a smooth C^{i+1} surface. Then

$$\left| D_Q^i \left(\frac{\cos\theta_Q}{|P-Q|^2} \right) \right| \leq \frac{c}{|P-Q|^{i+1}}, \quad P \neq Q \tag{26}$$

with c a generic constant independent of P and Q.

For details of the proof, see Micula [6, pg.76].

For the error at the collocation node points, we have the following.

Theorem 2.5. Assume the hypotheses of Theorem 2.1, with each $F_j \in C^2$. Assume $\rho \in C^2$. Assume the triangulation $\mathcal{T}_n$ of S satisfies (22) and is symmetric. For those integrals in (25) for which $P_i \in \Delta_k$, assume that all such integrals are evaluated with an error of $O(h^2)$. Then

$$\max_{1\leq i\leq n} |\rho(P_i) - \widehat{\rho}_n(P_i)| \leq ch^2 \log h \tag{27}$$

Proof. We will bound

$$\max_{1\leq i\leq n} |\mathcal{K}(I-P)_n u(v_i)|$$

For a given node point v_i, denote Δ^* the triangle containing it and denote:

$$\mathcal{T}_n^* = \mathcal{T}_n - \{\Delta^*\}$$

By our assumption, the error in evaluating the integral of (25) over Δ^* will be $O(h^2)$.

Partition $\mathcal{T}_n^*$ into parallelograms to the maximum extent possible. Denote by $\mathcal{T}_n^{(1)}$ the set of all triangles making up such parallelograms and let $\mathcal{T}_n^{(2)}$ contain the remaining triangles. Then

$$\mathcal{T}_n^* = \mathcal{T}_n^{(1)} \cup \mathcal{T}_n^{(2)}.$$

It is easy to show that the number of triangles in $\mathcal{T}_n^{(1)}$ is $O(n) = O(h^{-2})$, and the number of triangles in $\mathcal{T}_n^{(2)}$ is $O(\sqrt{n}) = O(h^{-1})$.

It can be shown that all but a finite number of the triangles in $\mathcal{T}_n^{(2)}$, bounded independent of n, will be at a minimum distance from v_i. That means that the triangles in $\mathcal{T}_n^{(2)}$ are "far enough" from v_i, so that the function $G(v_i, Q)$ is uniformly bounded for Q being in a triangle in $\mathcal{T}_n^{(2)}$ (where we denote by $G(P,Q) = \dfrac{\cos\theta_Q}{|P-Q|^2}$).

First, consider the contribution to the error coming from the triangles in $\mathcal{T}_n^{(2)}$. By Lemma 2.2. the error over each such triangle is $O\left(h^2\|D^2 g\|_\infty\right)$, since the area of each triangle is $O(h^2)$ and using our earlier observation. Having $O(h^{-1})$ such triangles in $\mathcal{T}_n^{(2)}$, the total error coming from triangles in $\mathcal{T}_n^{(2)}$ is $O\left(h^3\|D^2 g\|_\infty\right)$.

Next, consider the contribution to the error coming from triangles in $\mathcal{T}_n^{(1)}$. By Lemma 2.2., the error will be of size $O(h^2)$ multiplied times the integral over each such parallelogram of the maximum of the first derivatives of $G(v_i, Q)$ with respect to Q. Combining these we will have a bound

$$ch^2 \int\limits_{S-\Delta^*} (|G| + |DG|)\, dS_Q \tag{28}$$

By Theorem 2.4., the quantity in (28) is bounded by

$$ch^2 \int\limits_{S-\Delta^*} \left(\frac{1}{|P-Q|} + \frac{1}{|P-Q|^2} \right) dS_Q \tag{29}$$

Using a local representation of the surface and then using polar coordinates, the expression in (29) is of order

$$ch^2 \left(h + \log h \right)$$

Thus, the error arising from the triangles in $\mathcal{T}_n^{(1)}$ is $O(h^2 \log h)$. Combining the error arising from the integrals over Δ^*, $\mathcal{T}_n^{(1)}$, and $\mathcal{T}_n^{(2)}$, we have (27).

□

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Babeş-Bolyai University, Faculty of Mathematics and Computer Science, Str. Kogălniceanu 1, Cluj-Napoca, Romania

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FEEDBACK DIFFERENTIAL SYSTEMS: APPROXIMATE AND LIMITING TRAJECTORIES

ŞTEFAN MIRICĂ

Abstract. A "feedback differential system" is defined as a (generally discontinuous) parameterized differential inclusion (in particular, differential equation), that usually appears in the description of the complete solution of an optimal control problem or a differential game. In this article one obtains certain invariant characterizations of the uniform limits of two types of approximate trajectories: the well-known "Euler polygonal lines" and the less known "Isaacs approximate trajectories" suggested by the natural assumption of the discrete (step-by-step) "action" of a player in optimal control and differential games. The main results state that under very general hypotheses on the data, the limiting Euler and, respectively, Isaacs-Krassovskii-Subbotin trajectories are Carathéodory solutions of two distinct associated differential inclusions defined by corresponding "u.s.c.-convexified" limits of the original orientor fields. In particular, one provides a counter-example of a "conjecture" in Krassovskii and Subbotin(1974) and one gives a complete proof of the correct variant of this conjecture.

1. **Introduction**

The aim of this paper is to obtain certain "invariant" characterizations of the uniform limits of the well-known "Euler polygonal lines" in the general theory of Ordinary Differential Equations (ODE), on one hand and, on the other hand, of the less known "Isaacs approximate trajectories" of "proper" feedback differential

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systems, naturally appearing in the description of the complete solutions of optimal control problems and differential games.

We recall that the "limiting Euler trajectories" are frequently used not only in the general theory of ODE (e.g. Kurzweil(1986)[9]) and differential inclusions (e.g. Aubin and Cellina(1984)[1]) to prove the existence of classical (Newton or Carathéodory) solutions but also in the description of corresponding numerical algorithms; in a more general setting of certain types of differential inclusions, the study of the limiting Euler trajectories has been recently taken-up in Clarke et al(1998)[3].

On the other hand, the limiting "Isaacs-Krassovskii-Subbotin" trajectories have been considered first in Krassovskii and Subbotin(1974)[8] in an attempt to put on a more rigorous basis the rather heuristical approach in Isaacs(1965)[7] which referred only to the corresponding "approximate" trajectories; however, Krassovskii and Subbotin(1974), without mentioning Isaacs' name, identified "Isaacs approximate trajectories" as "Euler polygonal lines" (which is not true, as the definitions below show) and, moreover, formulated a "conjecture" (contradicted by the counterexample in Remark 4.3 below) according to which these limiting trajectories are Carathéodory solutions of a certain associated differential inclusion.

The main results of this paper are contained in Sections 3 and 4 below and show that under some mild hypotheses on the data, these types of limiting trajectories are *Carathéodory solutions* of certain associated *u.s.c.-convexified* differential inclusions which are closely related to concepts introduced by Cesari(1983)[2], Filippov(1988)[5] and Mirică(1992)[10] in different contexts.

As a general idea, we point out that, as in [1], [2], [3], [11], etc., in the proofs of the main results in Sections 3 and 4 below, we shall use, in a more explicit manner, a string of arguments based, first, on the "compactness" [Theorem 0.3.4 in Aubin and Cellina(1984)[1]] (which may be considered a refinement of the well-known Arzelà-Ascoli theorem), then the so called Banach-Saks-Mazur Lemma in Functional Analysis (e.g. Yosida(1974)[15]) and, finally, some arguments from basic Measure Theory (e.g. Dunford and Schwartz(1958)[4]).

One may note also that the results in Sections 3 and 4 below may explain some of the apparent "anomalies" pointed out in [Clarke et al(1998)[3], Section 4.1.2] and the role of the usual hypotheses in the general theory of ODE.

The paper is organized as follows: in Section 2 we present the necessary notations, definitions and preliminary results needed in the sequel and in Section 3 we prove the main result and some comments concerning the characterization of the limiting Euler(**E-**) trajectories; in Section 4 we prove in the same way the more complicated analogous results regarding the limiting Isaacs-Krassovskii-Subbotin(**IKS**) trajectories.

2. Notations, definitions and preliminary results

In this paper we shall be concerned mainly with a **feedback differential system** (actually a "parameterized differential inclusion") of the form

$$x' \in F(t,x) := f(t,x,U(t,x)),\ x(t_0) = x_0,\ t \in I = [t_0,t_1], \tag{2.1}$$

defined by a non-empty set (of "control parameters") U, a *parameterized vector field* $f(.,.,.) : D \times U \to R^n$ and a *multifunction* ("feedback strategy") $U(.,.) : D \subseteq R \times R^n \to \mathcal{P}(U)$ where $\mathcal{P}(U)$ denotes the family of all subsets of U; we note that in the particular case in which $U \subseteq R^n$, $f(t,x,u) \equiv u$ one has a "general" (non-parameterized) differential inclusion

$$x' \in F(t,x) := U(t,x),\ x(t_0) = x_0,\ t \in I = [t_0,t_1],\ (t_0,x_0) \in D \tag{2.2}$$

while in the case the multifunction $U(.,.)$ is either absent or "reduces" to a point, $U(t,x) \equiv \{u_0\}$, for some fixed point $u_0 \in U$, the inclusion in (2.1) becomes an "ordinary differential equation"

$$x' = g(t,x) := f(t,x,u_0),\ x(t_0) = x_0,\ (t_0,x_0) \in D. \tag{2.3}$$

In what follows, a $\Delta -$ *approximate solution* is related to a *partition ("division") of the interval* $I = [t_0,t_1]$ denoted by $\Delta = \{\tau^j; j \in \{0,1,...k+1\}\}$ where $t_0 = \tau^0 < \tau^1 < ... \tau^k < \tau^{k+1} = t_1$ and whose "norm"(or "mesh size") is defined by $|\Delta| := max\{\tau^{j+1} - \tau^j;\ 0 \leq j \leq k\}$.

Definition 2.1. If $\Delta = \{\tau^j; j = \{0, 1, ...k+1\}\}$ is a partition of the interval $I = [t_0, t_1]$ then $x_\Delta(.) \in AC(I; R^n)$ is said to be:

(i) *an Euler* $\Delta-$ *approximate solution* of (2.1) if there exists a finite subset, $\{v^j;\ j \in \{0, 1, ...k\}\} \subset R^n$ such that:

$$x_\Delta(t_0) = x_\Delta(\tau^0) = x_0,\ v^j \in F(\tau^j, x_\Delta(\tau^j)),\ j \in \{0, 1, ...k\} \tag{2.7}$$

$$x_\Delta(t) = x_\Delta(\tau^j) + (t - \tau^j)v^j\ \forall\ t \in I^j = [\tau^j, \tau^{j+1}]; \tag{2.8}$$

(ii) *an Isaacs* $\Delta-$*approximate solution* if there exists a finite subset, $\{u^j;\ j \in \{0, 1, ...k\}\} \subset U$ such that the mappings $f(., x_\Delta(.), u^j)$ are (Lebesgue) integrable and satisfy the following relations:

$$x_\Delta(t_0) = x_\Delta(\tau^0) = x_0,\ u^j \in U(\tau^j, x_\Delta(\tau^j)),\ j \in \{0, 1, ...k\} \tag{2.9}$$

$$x_\Delta(t) = x_\Delta(\tau^j) + \int_{\tau^j}^{t} f(s, x_\Delta(s), u^j)ds\ \forall\ t \in I^j = [\tau^j, \tau^{j+1}]. \tag{2.10}$$

Remark 2.2. We note first that the mappings $x_\Delta(.)$ in (2.8), (2.10) are defined "recurrently" on the sub-intervals $I^j = [\tau^j, \tau^{j+1}] \subset I,\ j \in \{0, 1, ...k\}$ starting from the initial value $x_\Delta(\tau^0) = x_0$ and choosing, at each step, a point $v^j \in F(\tau^j, x_\Delta(\tau^j))$ (respectively, $u^j \in U(\tau^j, x_\Delta(\tau^j))$); moreover, on each sub-interval I^j the mapping $x_\Delta(.)$ in (2.10) is a Carathéodory solution of the O.D.E.

$$x'(t) = f(t, x(t), u^j)\ a.e.(I^j),\ j \in \{0, 1, ...k\},\ I^j = [\tau^j, \tau^{j+1}] \tag{2.11}$$

while the corresponding mapping in (2.8) is a *piecewise affine mapping* with the constant derivative $v^j \in F(\tau^j, x_\Delta(\tau^j))$ on the sub-interval $Int(I^j) = (\tau^j, \tau^{j+1})$; one may note that while an Euler $\Delta-$*solution* may be defined for general (non-parameterized) differential inclusions, the Isaacs $\Delta-$ *solutions* in (2.10) are specific to the "properly parameterized" differential inclusions in (2.1) since in the case of the general ones in (2.2), they become Euler $\Delta-$ *solutions*.

Further on, as it is well known, the "integrability" condition in Def.1(ii) (which is rather difficult to verify in the general case) is implied by the fact that the mappings $f(., ., u),\ u \in U$ are *Carathéodory vector fields* (e.g. [5], [9], [11], etc.); for the proof of the main result in Section 3 below we need the following more restrictive property:

Hypothesis 2.3. *The data of the problem (2.1) have the following properties:*

(i): $U \neq \emptyset,\ D = Int(D) \subseteq R \times R^n$ *(i.e. is open) and* $U(.,.) : D \to \mathcal{P}(U)$ *has non-empty values at each point;*

(ii): the mapping $f(.,.,.) : D \times U \to R^n$ *is such that there exists a null subset* $I_f \subset pr_1 D$ *such that:*

(ii_1)*: the mappings* $f(.,x,u),\ x \in pr_2 D,\ u \in U$ *are measurable;*

(ii_2)*: the mappings* $f(t,.,u),\ t \in pr_1 D \setminus I_f,\ u \in U$ *are continuous;*

(iii): the multifunctions $F(.,.) := f(.,.,U(.,.))$ *and* $U(.,.)$ *are "jointly" locally integrably-bounded in the sense that for any compact subset* $D_0 \subset D$ *there exists an integrable mapping* $c(.) \in L^1(pr_1 D_0; R_+)$ *and a null subset* $I_0 \subset pr_1 D_0$ *such that:*

$$||f(t,x,u)|| \leq c(t)\ \forall\ (t,x) \in D_0,\ t \in pr_1 D_0 \setminus I_0,\ u \in U(D_0) \qquad (2.12)$$

where $U(D_0) := \bigcup\{U(s,y);\ (s,y) \in D_0\}$.

One may note that property (iii) is implied by the usual hypothesis according to which U is a compact topological space and $f(.,.,)$ is continuous (with respect to all variables); moreover, property (ii) implies the fact that $f(.,.,u),\ u \in U$ are Carathéodory vector fields hence the definition of the Isaacs $\Delta-$*solutions* in Def.2.4(ii) makes sense without the "artificial" requirement of the integrability condition.

On the other hand, for the study of the limiting Euler-trajectories in Section 3 we need only a simpler "local boundedness" property of the orientor field $F(.,.)$ (see Th.3.1 below).

In what follows we shall study the corresponding types of "limiting trajectories" defined as "uniform limits" of the approximate trajectories in Def.2.1; we recall that the "limiting Euler trajectories" are frequently used not only in the general theory of ODE (e.g. Kurzweil(1986)[9]) and differential inclusions (e.g. Aubin and Cellina(1984)[1]) to prove the existence of classical (Newton or Carathéodory) solutions but also in the description of certain numerical algorithms; on the other hand, the limiting "Isaacs-Krassovskii-Subbotin" (**IKS-** trajectories have been considered first in Krassovskii and Subbotin(1974)[8] in an attempt to put on a more rigorous basis the rather heuristical approach in Isaacs(1965)[7] which referred only to the corresponding "approximate" trajectories.

Definition 2.4. The continuous mapping $x(.) \in C(I;R^n)$ is said to be:

(i): *an Euler*(**E**)*-trajectory* of the problem in (2.1) if there exist a sequence of partitions $\Delta_m = \{\tau_m^j;\ j \in \{0,1,...k_m+1\}\},\ m \in N$ of the interval $I = [t_0, t_1]$, the subsets $\{v_m^j;\ j \in \{0,1,...k_m\}\} \subset R^n$ and the corresponding Euler $\Delta_m - solutions,\ x_m(.) := x_{\Delta_m}(.),\ m \in N$ in (2.7),(2.8) such that:

$$|\Delta_m| \to 0,\ x_m(t) \to x(t)\ uniformly\ for\ t \in I\ as\ m \to \infty; \tag{2.13}$$

(ii): *an Isaacs-Krassovskii-Subbotin*(**IKS**)*-trajectory* of the problem (2.1) if there exist a sequence of partitions $\Delta_m = \{\tau_m^j;\ j \in \{0,1,...k_m+1\}\},\ m \in N$ of the interval $I = [t_0, t_1]$, the subsets $\{u_m^j;\ j \in \{0,1,...k_m\}\} \subset U$ and the corresponding $Isaacs\ \Delta_m - solutions,\ x_m(.) := x_{\Delta_m}(.),\ m \in N$ in (2.9), (2.10) such that the properties in (2.13) are satisfied.

Note that, in general the uniform limit of $x_m(.)$ (in the topology generated by the norm $\|x(.)\|_C := \max\{\|x(t)\|;\ t \in I\}$ of the space $C(I;R^n)$ of continuous mappings) need not be absolutely continuous (**AC**), not even a.e. differentiable; a sufficient condition for this property is given in the following [*compactness theorem* 0.3.4 in Aubin and Cellina[1]] which seems to be more suitable than the classical Arzelà-Ascoli theorem, in the study of Carathéodory-type differential inclusions and differential equations.

Theorem 2.5 (**compactness**). *Let X be a Banach space, let $I \subset R$ be an interval and let $\{x_m(.)\} \subset AC(I;X)$ be a sequence of* **AC** *mappings with the following properties:*

(i): for each $t \in I$ the subset $\{x_m(t);\ m \in N\} \subset X$ is relatively compact;

(ii): there exists an integrable function $c(.) \in L^1(I;R_+)$ such that

$$\|x'_m(t)\| \le c(t)\ a.e.(I)\ \forall\ m \in N.$$

Then there exists a subsequence $\{x_{m_j}(.)\}$ *and a mapping* $x(.) \in AC(I;X)$ *such that*

(1): $x_{m_j}(.) \to x(.)$ *uniformly on each compact subset of* I*;*

(2): $x'_{m_j}(.) \to x'(.)$ *weakly in the space* $L^1(I;X)$ *of integrable mappings.*

One may note that in the case $X = R^n$, if $x_m(.)$ are *equally bounded* and property (ii) is satisfied then $x_m(.)$ are also uniformly equi-continuous hence one may apply the Arzelà-Ascoli Theorem but the conclusion in Theorem 2.5 is stronger, stating not only the fact that the limit is **AC** but also the weak convergence (in L^1) of the derivatives.

As in the study of many other problems (e.g. [11]), at a certain stage of the proofs of the main results, we shall use the following important theorem in Functional Analysis which seems to belong, jointly, to Banach, Saks and Mazur though in some books and monographs only one, two or no names are mentioned.

Theorem 2.6 (**Banach-Saks-Mazur**). *Let* X *be a normed space, let* X^* *be its dual and let* $x_m,\ x \in X,\ m \in N$ *be such that* $x_m \to x$ *weakly i.e. such that* $x^*(x_m) \to x^*(x)\ \forall\ x^* \in X^*$.

Then for each $m \in N$ *there exist the integer* $i_m \geq m$ *and the real numbers,* $c_m^i \in R$ *such that*

$$c_m^i \geq 0,\ \sum_m^{i_m} c_m^i = 1 \text{ and } \|y_m - x\| \to 0 \text{ if } y_m := \sum_m^{i_m} c_m^i x_i.$$

For the proof and equivalent statements of this important result we refer to Yosida [15], to Dunford and Schwartz [4] and to the references therein.

Finally, we shall use also the following result in Measure Theory which is very often used as a piece of "Mathematical folklore".

Theorem 2.7. (**Measure Theory**). *Let* X *be a Banach space, let* $I = [a,b] \subset R$ *be a compact interval and let* $x_m(.),\ x(.) \in L^1(I;X)$ *be such that* $x_m(.) \to x(.)$ *strongly in* L^1.

Then there exist a subsequence $x_{m_j}(.)$ *such that* $x_{m_j}(t) \to x(t)$ *a.e.*(I).

For a proof of this theorem we refer to Theorem 3.3.6 and Corollary 3.6.13 in Dunford and Schwartz [4].

In what follows $\|.\|$ denotes the *Euclidean norm* on R^n, if $r > 0$ and $x \in R^n$ then $B_r(x) := \{y \in R^n;\ \|y-x\| < r\}$ and if $A \subset R^n$ then $Int(A),\ Cl(A),\ co[A],\ \overline{co}[A]$ denote the interior, the closure, the convex hull and, respectively, the closed convex hull of A.

3. **Limiting Euler trajectories**

In this section we use Theorems 2.5, 2.6, 2.7 to obtain certain "invariant" characterizations of the *limiting Euler trajectories* in Def.2.4(i) and, in particular, of existence theorems for solutions of upper semicontinuous convex-valued differential inclusions and differential equations.

Theorem 3.1. *If the "orientor field" $F(.,.) : D = Int(D) \subseteq R \times R^n \to \mathcal{P}(R^n)$* **is locally bounded** *in the sense that for any compact subset $D_0 \subset D$ there exists $c > 0$ such that*

$$\|v\| \leq c\ \forall\ v \in F(D_0) := \bigcup\{F(t,x);\ (t,x) \in D_0\} \tag{3.1}$$

*and $x(.) \in C(I;R^n)$ is an Euler trajectory in the sense of Def.2.4(i) of the differential inclusion in (2.2) then $x(.)$ is a Carathéodory (**AC**) solution of the u.s.c.-convexified differential inclusion*

$$x' \in F^{co}(t,x) := \bigcap_{\delta>0} \overline{co}[F((t-\delta,t+\delta) \times B_\delta(x))]. \tag{3.2}$$

Proof. From Def.2.4(i) it follows that there exist a sequence of partitions $\Delta_m = \{\tau_m^j;\ j \in \{0,1,...k_m+1\}\},\ m \in N$ of the interval $I = [t_0,t_1]$, the subsets $\{v_m^j;\ j \in \{0,1,...k_m\}\} \subset R^n$ and the corresponding Euler Δ_m-*solutions*, $x_m(.) := x_{\Delta_m}(.), m \in N$ in (2.7),(2.8), hence such that the following relations are satisfied on the intervals $I_m^j = [\tau_m^j, \tau_m^{j+1}]$:

$$x_m(t) = x_m(\tau_m^j) + (t-\tau_m^j)v_m^j, t \in I_m^j, v_m^j \in F(\tau_m^j, x_m(\tau_m^j)) \tag{3.3}$$

and such that the properties in (2.13) hold true; obviously, the property in (3.3) is equivalent with the fact that

$$x_m'(t) = v_m^j\ \forall\ t \in (\tau_m^j, \tau_m^{j+1}),\ j \in \{0,1,...k_m\},\ x(t_0) = x_0. \tag{3.4}$$

On the other hand, since $D \subseteq R \times R^n$ is open and $x(.)$ is continuous, there exists $r > 0$ and a rank $m_r \in N$ such that

$$D_0 := \{(t, y);\ t \in I,\ y \in \overline{B}_r(x(t)) := Cl(B_r(x(t))\} \subset D \tag{3.5}$$

$$(t, x_m(t)) \in D_0\ \forall\ t \in I,\ m \geq m_r. \tag{3.6}$$

As already stated, the general idea is to show that Theorems 2.5, 2.6, 2.7 are successively applicable to the above sequence $\{x_m(.);\ m \geq m_r\} \subset AC(I; R^n)$; to this end we note that from the fact that $(\tau_m^j, x_m(\tau_m^j)) \in D_0\ \forall\ m \geq m_r,\ j \in \{0, 1, ...k_m\}$ and from the properties in (3.1) and (3.4) it follows that

$$\|x'_m(t)\| \leq c\ \forall\ t \in I \setminus \{\tau_m^j;\ j \in \{0, 1, ...k_m\},\ m \geq m_r. \tag{3.7}$$

Therefore Th.2.5 is applicable to the sequence $\{x_m(.);\ m \geq m_r\}$ hence taking possibly a subsequence, without loss of generality, we may assume that $x(.) \in AC(I; R^n)$ and that $x'_m(.) \rightarrow x'(.)$ *weakly in* $L^1(I; R^n)$; next, we apply first Th.2.6 to obtain the existence of the non-negative numbers $c_m^i \geq 0$ and of the natural numbers $i_m \geq m$ such that

$$\sum_m^{i_m} c_m^i = 1,\ \|\sum_m^{i_m} c_m^i x'_i(.) - x'(.)\|_{L^1} \rightarrow 0\ as\ m \rightarrow \infty \tag{3.8}$$

while from Th.2.7 it follows that, taking possibly a subsequence, one may assume that there exists a null subset $I_2 \subset I$ such that:

$$c_m^i \geq 0,\ \sum_m^{i_m} c_m^i = 1,\ y_m(t) := \sum_m^{i_m} c_m^i x'_i(t) \rightarrow x'(t)\ \forall\ t \in I \setminus I_2. \tag{3.9}$$

From (2.13) it follows now that for each $\delta > 0$ there exists a rank $m_\delta \geq m_r$ such that $\forall\ t \in I,\ m \geq m_\delta\ \exists j = j(t, m) \in \{0, 1, ...k_m\}$ such that $(\tau_m^j, x_m(\tau_m^j)) \in (t - \delta, t + \delta) \times B_\delta(x(t))$ which, in view of (3.4) and of the fact that $v_m^j \in F(\tau_m^j, x_m(\tau_m^j))$ implies:

$$x'_m(t) \in F((t - \delta, t + \delta) \times B_\delta(x(t))\ \forall\ t \in I \setminus \{\tau_m^j; j \in \{0, 1, ...k_m + 1\}\}, m \geq m_\delta$$

and which, in turn, in view of (3.9), implies the fact that $x(.)$ is a Carathéodory solution of the differential inclusion in (3.2).

In the particular case of locally bounded but otherwise arbitrary vector fields in (2.3) we obtain immediately the following result.

Corollary 3.2. *If $g(.,.) : D = Int(D) \subseteq R \times R^n \to R^n$ is a vector field that is* **locally bounded** *in the sense of (3.1) and $x(.) \in C(I; R^n)$ is an Euler-trajectory of the ODE in (2.3) in the sense of Def.2.4(i) then $x(.)$ is a Lipschitzian (Carathéodory) solution of the differential inclusion*

$$x' \in g^{co}(t,x) := \bigcap_{\delta>0} \overline{co}[g((t-\delta, t+\delta) \times B_\delta(x))]. \qquad (3.10)$$

In particular, if $g(.,.)$ is continuous (with respect to both variables) then $x(.)$ is a continuously differentiable ("Newton's") solution of the same equation.

Remark 3.3. One may note that the statement in Cor.3.2 is much weaker than the corresponding one in Cor.4.2 below for **IKS**-trajectories of ODE and simple examples (e.g. [Clarke et al.[3], Section 4.1.2]) show that it cannot be significantly improved; according to these examples, even if $g(.,.)$ is continuous, an **E**-trajectory may not be a **C**-solution of (2.3) and, on the other hand, a Newton (i.e., of class C^1) solution may not be an **E**-trajectory. The only case in which an equivalence analogous to the one in Cor.4.2 may hold seems to be that of the "Peano-Lipschitz vector fields", $g(.,.)$, which are continuous with respect to both variables and locally-Lipschitz with respect to the second one or, slightly more general, that have the uniqueness property in the theory of ODE.

4. Limiting Isaacs-Krassovskii-Subbotin trajectories

The main result of this section is the following theorem giving the correct variant of the "conjecture" in [Krassovskii and Subbotin (1974)[8], Section 2.7].

Theorem 4.1. *If Hypothesis 2.3 is satisfied and $x(.) \in C(I; R^n)$ is a* **IKS**-*trajectory in the sense of Def.2.4(ii) then $x(.)$ is a Carathéodory solution of the u.s.c.-convexified differential inclusion*

$$\begin{aligned} x' \in F_u^{co}(t,x) := \bigcap_{\delta>0} \overline{co}[\bigcup \{f(t,y,U(s,z));\ y \in B_\delta(x),\\ (s,z) \in (t-\delta, t+\delta) \times B_\delta(y)\}]. \end{aligned} \qquad (4.1)$$

Proof. From Def.2.4(ii) it follows that there exist a sequence of partitions $\Delta_m = \{\tau_m^j;\ j \in \{0, 1, ...k_m + 1\}\}$, $m \in N$, of the interval $I = [t_0, t_1]$, the subsets $\{u_m^j;\ j \in$

$\{0,1,...k_m\}\} \subset U$ and the corresponding Isaacs $\Delta_m-solutions,\ x_m(.) := x_{\Delta_m}(.), m \in N$ in (2.9),(2.10), hence such that:

$$x_m(t_0) = x_m(\tau^0) = x_0,\ u_m^j \in U(\tau_m^j, x_m(\tau_m^j)),\ j \in \{0,1,...k_m\} \tag{4.2}$$

$$x_m(t) = x_m(\tau_m^j) + \int_{\tau_m^j}^t f(s, x_m(s), u_m^j)ds\ \forall\ t \in I_m^j = [\tau_m^j, \tau_m^{j+1}] \tag{4.3}$$

and such that the properties in (2.13) hold true; as already noted, the property in (4.3) is equivalent with the fact that there exists a null subset, $I_1 \subset I,\ I_1 \supset \{\tau_m^j;\ m \in N,\ j \in \{0,1,...k_m+1\}$ such that:

$$x_m'(t) = f(t, x_m(t), u_m^j)\ \forall\ t \in (\tau_m^j, \tau_m^{j+1}) \setminus I_1,\ x(t_0) = x_0. \tag{4.4}$$

On the other hand, since $D \subseteq R \times R^n$ is open and $x(.)$ is continuous (hence $x(I) \subset R^n$ is compact), there exist $r, m_r > 0$ such that (3.5) and (3.6) hold.

As in the proof of Th.3.1, the general idea is to show that Theorems 2.5, 2.6, 2.7 are successively applicable to the above sequence $\{x_m(.);\ m \geq m_r\} \subset AC(I; R^n)$; to this end we note that from Hypothesis 2.3(iii) it follows that for the compact subset $D_0 \subset D$ in (3.5) there exists an integrable function $c(.) \in L^1(I; R_+)$ and a null subset $I_0 \subset I = pr_1 D_0$ such that (2.12) holds; further on, since from (3.6) it follows that, in particular, $(\tau_m^j, x_m(\tau_m^j)) \in D_0$, hence $u_m^j \in U(\tau_m^j, x_m(\tau_m^j)) \subset U(D_0)$, from (2.12) and (4.4) it follows that:

$$\|x_m'(t)\| \leq c(t)\ \forall\ t \in I \setminus (I_0 \cup I_1),\ m \geq m_r. \tag{4.5}$$

Therefore Th.2.5 is applicable to the sequence $\{x_m(.);\ m \geq m_r\}$ hence taking possibly a subsequence (without loss of generality), we may assume that $x(.) \in AC(I; R^n)$ and also that $x_m'(.) \to x'(.)$ *weakly in* $L^1(I; R^n)$; next, we apply Th.2.6 to obtain the existence of the non-negative numbers $c_m^i \geq 0$ and of the natural numbers $i_m \geq m$ such that (3.8) holds while from Th.2.7 it follows that, taking possibly a subsequence, one may assume that there exists a null subset $I_2 \subset I,\ I_2 \supset I_1$, such that (3.9) holds.

We shall prove now that for each $\delta > 0$ there exists a rank $m_\delta \geq m_r$ such that $\forall\ t \in I \setminus I_1,\ m \geq m_\delta$ one has:

$$x_m'(t) \in \bigcup \{f(t, x_m(t), U(s,z));\ (s,z) \in (t-\delta, t+\delta) \times B_\delta(x_m(t))\} \tag{4.6}$$

which, in view of (2.13) and (3.9), implies in the following way the fact that $x(.)$ is a Carathéodory solution of the differential inclusion in (4.1): from (2.13) it follows that for $\delta > 0$ there exists a rank $m_\delta \geq m_r$ such that $x_m(t) \in B_\delta(x(t)) \ \forall\ t \in I,\ m \geq m_\delta$ hence from (4.6) it follows that for each $\delta > 0,\ t \in I \setminus I_1$ one has:

$$x'_m(t) \in \bigcup \{f(t, y, U(s, z)); y \in B_\delta(x(t)), (s, z) \in (t - \delta, t + \delta) \times B_\delta(y)\}$$

which, in view of (3.9), implies the fact that $x(.)$ is a **C**-solution of (4.1).

To prove (4.6) we use first the well-known *absolute continuity of the Lebesgue integral*, $J \mapsto \int_J c(s)ds$ to obtain that for $\delta > 0$ there exists $\eta_\delta > 0$ such that

$$\int_J c(s)ds < \delta \ \forall\ J \subset I,\ \mu(J) < \eta_\delta;$$

next, we use the property in (2.13) to obtain the existence of a rank $m_\delta \geq m_r$ such that

$$|\Delta_m| < \min\{\delta, \eta_\delta\},\ \|x_m(t) - x(t)\| < \delta \ \forall\ m \geq m_\delta,\ t \in I$$

hence, in particular, such that:

$$\int_{\tau_m^j}^t c(s)ds < \delta,\ t - \tau_m^j \leq |\Delta_m| < \delta \ \forall\ t \in I_m^j = [\tau_m^j, \tau_m^{j+1}].$$

Therefore from (3.3), (3.4) and (4.5) it follows that:

$$\|x_m(t) - x_m(\tau_m^j)\| \leq \int_{\tau_m^j}^t c(s)ds < \delta,\ \forall\ t \in I_m^j = [\tau_m^j, \tau_m^{j+1}]$$

hence $(\tau_m^j, x_m(\tau_m^j)) \in (t - \delta, t + \delta) \times B_\delta(x_m(t))$ and the relation in (4.6) follows from (3.4) and from the fact that $u_m^j \in U(\tau_m^j, x_m(\tau_m^j))$.

In the particular case of the Carathéodory ODE in (2.3) we obtain the following result.

Corollary 4.2. *If $g(.,.) : D = Int(D) \subseteq R \times R^n \to R^n$ is a Carathéodory vector field in the sense of Hypothesis 2.3(ii),(iii) then $x(.) \in AC(I; R^n)$ is an* **IKS**-*trajectory of the ODE in (2.3)* **iff** *it is a Carathéodory solution of the same equation.*

Proof. If $x(.) \in AC(I; R^n)$ is a Carathéodory(**C**) solution of (2.3) then, obviously, for any partition Δ of the interval I it is an Isaacs $\Delta-$*solution* hence $x(.)$ is an **IKS**-trajectory in the sense of Def.2.4(ii).

Conversely, if $x(.) \in C(I; R^n)$ is a **IKS**-trajectory then according to Th.3.1 it is a **C**-solution of the differential inclusion in (4.1) which, in this case, is defined by the orientor field

$$F_u^{co}(t,x) = (g(t,.))^{co}(x) := \bigcap_{\delta>0} \overline{co}[g(t, B_\delta(x))];$$

finally, since $g(t,.)$ is assumed to be continuous for $t \in pr_1 D \setminus I_g$ for some null subset $I_g \subset pr_1 D$, it is easy to see that

$$F_u^{co}(t,x) = (g(t,.))^{co}(x) = \{g(t,x)\} \ \forall\ t \in pr_1 D \setminus I_g, x \in pr_2 D$$

hence $x(.)$ is a **C**-solution of (2.3).

Remark 4.3. We recall that the "conjecture" in [Krassovskii and Subbotin (1974)[8], Section 2.7] (in the case of the single-valued feedback strategies $U(.,.) = \{u(.,.)\}$), states that "using standard tools in the theory of ODE one may prove that any "perfect" (i.e. **IKS**) trajectory is a "generalized" trajectory in the sense that it is a Carathéodory solution of the associated differential inclusion" in (3.2).

Besides the fact that Theorems 2.5, 2.6, 2.7 above (that have been used essentially in the proof of Th.3.1, 4.1) may hardly be taken as "standard tools in the theory of ODE", the conjecture may be considered justified only in the case the multifunctions in (4.1) and (3.2) are related as follows: $F_u^{co}(t,x) \subseteq F^{co}(t,x) \ \forall\ (t,x) \in D$ as it is the case of the *vector fields* $g(.,.)$ in Cor.4.2, since one may write successively: $F_u^{co}(t,x) \equiv g(t,.)^{co}(x) \equiv \{g(t,x)\} \subseteq F^{co}(t,x) \equiv g^{co}(t,x)$; in the general case, the orientor fields in (4.1) and (3.2) may not be related in this way and very simple examples show that Krassovskii-Subbotin conjecture is false. For instance, if $d(.)$ is the well-known "Dirichlet function"

$$d(t) := \begin{cases} 1 \ if \ t \in Q \\ 0 \ if \ t \in R \setminus Q \end{cases}$$

and $f(t,x,u) \equiv d(t) + u,\ U(t,x) \equiv \{1 - d(t)\} \subset U := [0,1]$ then obviously $F(t,x) \equiv F^{co}(t,x) \equiv \{1\}$ while the convexified u.s.c.-limit in (4.1) is given by: $F_u^{co}(t,x) \equiv d(t) + [0,1]$; therefore, the only **C**-solution of (3.2) that satisfies $x(0) = 0 \in R$ is the function $x(t) = t,\ t \in I = [0,1]$ while taking a sequence $\{\Delta_m\}$ of partitions of

$I = [0, 1]$ such that $\tau_m^j \in Q$ it follows that $x_0(t) \equiv 0$ is an **IKS**-trajectory of (2.1) that it is not a **C**-solution of (3.2).

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University of Bucharest, Faculty of Mathematics,
Academiei 14, 010014 Bucharest, Romania
E-mail address: mirica@math.math.unibuc.ro

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ON STRONGLY NONLINEAR PARABOLIC FUNCTIONAL DIFFERENTIAL EQUATIONS OF DIVERGENCE FORM

LÁSZLÓ SIMON

Abstract. We consider initial boundary value problems for second order strongly nonlinear parabolic equations where also the main part contains functional dependence on the unknown function.

Introduction

This investigation was motivated by works [4], [5] of M. Chipot on "nonlocal evolution problems" for the equation

$$D_t u - \sum_{i,j=1}^{n} D_i[a_{ij}(l(u(\cdot,t))D_i u] + a_0(l(u(\cdot,t))u = f \text{ in } \Omega \times R^+ \qquad (0.1)$$

where $\Omega \subset R^n$ is a bounded domain with sufficiently smooth boundary,

$$\sum_{i,j=1}^{n} a_{ij}(\zeta)\xi_i\xi_j \geq \lambda|\xi|^2 \text{ for all } \xi \in R^n, \quad \zeta \in R$$

with some constant $\lambda > 0$,

$$l(u(\cdot,t)) = \int_\Omega g(x)u(x,t)dx$$

with a given function $g \in L^2(\Omega)$. Existence and asymptotic properties (as $t \to \infty$) of solutions of initial-boundary value problems for (0.1) were proved. That problem was motivated by diffusion process (for heat or population), where the diffusion coefficient depends on a nonlocal quantity.

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Our aim is to consider similar problems for quasilinear parabolic functional differential equations of the form

$$D_t u - \sum_{i=1}^{n} D_i[a_i(t,x,u(t,x),Du(t,x);u)] + a_0(t,x,u(t,x),Du(t,x);u)+ \qquad (0.2)$$

$$b(t,x,u(t,x);u) = f \text{ in } Q_{T_0} = (0,T_0)\times\Omega$$

with homogeneous Dirichlet boundary and initial conditions, where the functions

$$a_i : Q_{T_0} \times R^{n+1} \times L^p(0,T_0;V) \to R$$

(with $V = W_0^{1,p}(\Omega)$, $2 \le p < \infty$) satisfy conditions which are generalizations of conditions for strongly nonlinear parabolic differential equations, considered in [3], [7], [8] by using the theory of monotone type operators; a_i have polynomial ($p-1$ power) growth with respect to $u(t,x), Du(t,x)$ and b may be quickly increasing in $u(t,x)$.

1. **Existence in** $[0,T_0]$

Let $\Omega \subset R^n$ be a bounded domain having the uniform C^1 regularity property (see [1]) and $V = W_0^{1,p}(\Omega)$ the usual Sobolev space of real valued functions which is the completion of $C_0^\infty(\Omega)$ with respect to the norm

$$\| u \| = \left[\int_\Omega (|Du|^p + |u|^p)\right]^{1/p}.$$

Denote by $L^p(0,T_0;V)$ the Banach space of the set of measurable functions $u : (0,T_0) \to V$ such that $\| u \|^p$ is integrable and define the norm by

$$\| u \|^p_{L^p(0,T_0;V)} = \int_0^{T_0} \| u(t) \|_V^p \, dt.$$

The dual space of $L^p(0,T_0;V)$ is $L^q(0,T_0;V^\star)$ where $1/p+1/q=1$ and $V^\star$ is the dual space of V (see, e.g., [6], [11]).

Assume that

I. The functions $a_i : Q_T \times R^{n+1} \times L^p(0,T_0;V) \to R$ satisfy the Carathéodory conditions for arbitrary fixed $v \in L^p(0,T_0;V)$ $(i = 0,1,...,n)$.

II. There exist bounded (nonlinear) operators $g_1 : L^p(0,T_0;V) \to R^+ =$ and $k_1 : L^p(0,T_0;V) \to L^q(Q_{T_0})$ such that

$$|a_i(t,x,\zeta_0,\zeta;v)| \leq g_1(v)[|\zeta_0|^{p-1} + |\zeta|^{p-1}] + [k_1(v)](t,x)$$

for a.e. $(t,x) \in Q_{T_0}$, each $(\zeta_0,\zeta) \in R^{n+1}$ and $v \in L^p(0,T_0;V)$.

III. $\sum_{i=1}^n [a_i(t,x,\zeta_0,\zeta;v) - a_i(t,x,\zeta_0,\zeta^\star;v)](\zeta_i - \zeta_i^\star) > 0$ if $\zeta \neq \zeta^\star$.

IV. There exist bounded operators $g_2 : L^p(0,T_0;V) \to R^+$, $k_2 : L^p(0,T_0;V) \to L^1(Q_{T_0})$ such that

$$\sum_{i=0}^n a_i(t,x,\zeta_0,\zeta;v)\zeta_i \geq g_2(v)[|\zeta_0|^p + |\zeta|^p] - [k_2(v)](t,x)$$

for a.e. $(t,x) \in Q_{T_0}$, all $(\zeta_0,\zeta) \in R^{n+1}$, $v \in L^p(0,T_0;V)$ and $g_2(v) \geq c_2$ with some constant $c_2 > 0$,

$$\lim_{\|v\|_X \to \infty} \frac{\| k_2(v) \|_{L^1(Q_{T_0})}}{\| v \|_X^p} = 0 \tag{1.3}$$

where we used the notation $X = L^p(0,T_0;V)$. Further, if the sequence $(v_k)$is bounded in $L^p(0,T_0;V)$ and convergent in $L^p(Q_{T_0})$ then the sequence $[k_2(v_k)](t,x)$is equiintegrable in Q_{T_0}.

V. If $(u_k) \to u$ weakly in $L^p(0,T_0;V)$ and strongly in $L^p(Q_{T_0})$ then

$$\lim_{k\to\infty} \| a_i(t,x,u_k(t,x),Du_k(t,x);u_k) - a_i(t,x,u_k(t,x),Du_k(t,x);u) \|_{L^q(Q_{T_0})} = 0.$$

VI. $b : Q_{T_0} \times R \times L^p(0,T_0;V)$ satisfies the Carathéodory condition for each fixed $v \in L^P(0,T_0;V)$,

$$0 \leq b(t,x,\zeta_0;v)\zeta_0 \leq \psi(\zeta_0)\zeta_0 \leq \text{const}[b(t,x,\zeta_0;v)\zeta_0 + 1]$$

with some continuous nondecreasing function ψ with $\psi(0) = 0$.

VII. If $(u_k) \to u$ in the norm of $L^p(Q_{T_0})$ then for a suitable subsequence

$$b(t,x,u_k(t,x);u_k) \to b(t,x,u(t,x);u) \text{ for a.e. } (t,x) \in Q_{T_0}.$$

Theorem 1.1. *Assume I - VII. Then for any $f \in L^q(0,T_0;V^\star)$ there exists*

$$u \in L^p(0,T_0;V) \cap C([0,T_0];L^2(\Omega)) \textit{ such that } u(0) = 0,$$

$$b(t,x,u(t,x);u), \quad u(t,x)b(t,x,u(t,x);u) \in L^1(Q_{T_0}),$$

u is a distributional solution of (0.2). Further, for arbitrary $T \in [0, T_0]$,

$$v \in L^p(0,T_0;V) \cap C^1([0,T_0];L^2(\Omega)) \text{ with } v(0)=0, \quad v \in L^\infty(Q_{T_0})$$

we have

$$\int_0^T \langle D_t v(t), u(t)-v(t)\rangle dt + \tag{1.4}$$

$$\int_{Q_T} \left[\sum_{i=1}^n a_i(t,x,u,Du;u)(D_i u - D_i v) + a_0(t,x,u,Du;u)(u-v) \right] dtdx +$$

$$\frac{1}{2} \parallel u(T)-v(T) \parallel^2_{L^2(\Omega)} + \int_{Q_T} b(t,x,u(t,x);u)(u-v)dtdx =$$

$$\int_0^T \langle f(t), u(t)-v(t)\rangle dt.$$

Proof. Define

$$b_k(t,x,\zeta_0;v) = b(t,x,\zeta_0;v) \text{ if } b(t,x,\zeta_0;v) < k,$$

$$b_k(t,x,\zeta_0;v) = k \text{ if } b(t,x,\zeta_0;v) \geq k,$$

$$b_k(t,x,\zeta_0;v) = -k \text{ if } b(t,x,\zeta_0;v) \leq -k,$$

$$[A(u),v]_T =$$

$$\int_{Q_T} \left[\sum_{i=1}^n a_i(t,x,u(t,x),Du(t,x);u)D_i v + a_0(t,x,u(t,x),Du(t,x);u)v \right] dtdx,$$

$$[B_k(u),v]_T = \int_{Q_T} b_k(t,x,u(t,x);u)v dtdx, \quad u,v \in X = L^p(0,T_0;V);$$

with a fixed $u_0 \in X$

$$[\tilde{A}_{u_0}(u),v]_T =$$

$$\int_{Q_T} \left[\sum_{i=1}^n a_i(t,x,u(t,x),Du(t,x);u_0)D_i v + a_0(t,x,u(t,x),Du(t,x);u_0)v \right] dtdx.$$

It is not difficult to show that by I, II, IV (for fixed k)

$$(A+B_k) : L^p(0,T_0;V) \to L^q(0,T_0;V^\star)$$

is bounded (i.e. it maps bounded sets into bounded sets) and coercive, i.e.

$$\lim_{\|v\|_X \to \infty} \frac{[(A+B_k)(v),v]_{T_0}}{\parallel v \parallel_X} = +\infty.$$

Further, it is well known (see, e.g., [2]) that $\tilde{A}_{u_0} : X \to X^\star$ is demicontinuous (i.e. if $(u_j) \to u$ strongly in X then $(\tilde{A}_{u_0}(u_j)) \to \tilde{A}_{u_0}(u)$ weakly in $X^\star$) and pseudomonotone with respect to

$$D(L) = \{v \in X : D_t v \in X^\star, \quad v(0) = 0\},$$

i.e. if

$$(u_j) \to u \text{ weakly in } X, \quad (D_t u_j) \to D_t u \text{ weakly in } X^\star \text{ and}$$

$$\limsup_{j\to\infty} [\tilde{A}_{u_0}(u_j), u_j - u]_{T_0} \le 0$$

then

$$\lim_{j\to\infty} [\tilde{A}_{u_0}(u_j), u_j - u]_{T_0} = 0 \text{ and } (\tilde{A}_{u_0}(u_j)) \to \tilde{A}_{u_0}(u) \text{ weakly in } X^\star.$$

By using assumption V, it is easy to show that also $A + B_k : X \to X^\star$ is demicontinuous and pseudomonotone with respect to $D(L)$ (see [10]).

Consequently, for each k there exists $u_k \in D(L)$ such that

$$D_t u_k + (A + B_k)(u_k) = f \text{ in } [0, T_0]. \tag{1.5}$$

(See, e.g., [2].) Applying (1.5) to $v = u_k$, we obtain by IV and Hölder's inequality for any $T \in [0, T_0]$

$$\frac{1}{2} \parallel u_k(T) \parallel^2_{L^2(\Omega)} + c_2 \parallel u_k \parallel^p_{L^p(0,T;V)} - \int_{Q_T} k_2(u_k) dt dx + \tag{1.6}$$

$$[B_k(u_k), u_k]_T \le \parallel f \parallel_{L^q(0,T;V^\star)} \parallel u_k \parallel_{L^p(0,T;V)} .$$

According to VI $[B_k(u_k), u_k]_T \ge 0$, thus (1.3), (1.6), II imply that

$$\parallel u_k \parallel_{L^p(0,T_0;V)}, \quad \parallel A(u_k) \parallel^p_{L^p(0,T_0;V^\star)}, \quad [B_k(u_k), u_k]_{T_0} \text{ are bounded.} \tag{1.7}$$

Consequently, (1.6) and boundedness of k_2 imply that

$$\parallel u_k \parallel_{L^\infty(0,T_0;L^2(\Omega))} \text{ is bounded.} \tag{1.8}$$

By using VI, $|b_k| \le |b| \le |\psi|$, we find

$$|b_k(t, x, u_k(t, x); u_k)| \le [\psi(1) + \psi(-1)| + b_k(t, x, u_k(t, x); u_k) u_k$$

which implies by (1.7) that

$$\int_{Q_{T_0}} |b_k(t,x,u_k(t,x);u_k)|dtdx \text{ is bounded.} \tag{1.9}$$

According to (1.5)

$$D_t u_k = [f - A(u_k)] - B_k(u_k)) \tag{1.10}$$

where the first term is bounded in $L^q(0,T;V^\star)$ and the second term is bounded in $L^1(Q_{T_0})$. Thus Proposition 1 of [3] implies that there is a subsequence of (u_k) (for simplicity denoted again by (u_k)) such that

$$(u_k) \to u \text{ weakly in } L^p(0,T_0;V), \text{ strongly in } L^p(Q_{T_0}) \text{ and a.e. in } Q_{T_0}. \tag{1.11}$$

Further, by (1.7) there exists $w \in L^q(0,T_0;V^\star)$ such that

$$(A(u_k)) \to w \text{ weakly in } L^q(0,T_0;V^\star). \tag{1.12}$$

Since by IV $k_2(u_k)(t,x)$ is equiintegrable in Q_{T_0}, we obtain from (1.6), (1.8), (1.11)

$$u \in L^\infty(0,T_0;L^2(\Omega)), \quad \lim_{T\to 0} \| u \|_{L^\infty(0,T_0;L^2(\Omega))} = 0. \tag{1.13}$$

We obtain from (1.11), assumption VII and the definition of b_k that

$$b_k(t,x,u_k(t,x);u_k) \to b(t,x,u(t,x);u) \text{ a.e. in } Q_{T_0}, \text{ so} \tag{1.14}$$

$$u_k b_k(t,x,u_k(t,x);u_k) \geq 0, \tag{1.15}$$

(1.7), Fatou's lemma imply

$$ub(t,x,u(t,x);u) \in L^1(Q_{T_0}) \quad \text{and so by VI} \quad u\psi(u) \in L^1(Q_{T_0}). \tag{1.16}$$

From (1.14), (1.16), VI and Vitali's theorem we obtain

$$b_k(t,x,u_k(t,x);u_k) \to b(t,x,u(t,x);u) \text{ in } L^1(Q_{T_0}), \quad \psi(u) \in L^1(Q_{T_0}) \tag{1.17}$$

because for arbitrary $\varepsilon > 0$

$$|b_k(t,x,\zeta_0;u_k)| \leq |b(t,x,\zeta_0;u_k)| \leq |\psi(\zeta_0)| \leq \varepsilon\psi(\zeta_0)\zeta_0 + \psi(1/\varepsilon) + |\psi(-1/\varepsilon)|$$

if $|\zeta_0| > 1/\varepsilon$, so by (1.7) $(b_k(t,x,u_k(t,x);u_k))$ is equiintegrable in Q_{T_0}.

From (1.5), (1.11), (1.12), (1.17) we obtain as $k \to \infty$

$$D_t u + w + b(t, x, u(t, x); u) = f \tag{1.18}$$

in distributional sense.

In order to show $w = A(u)$, we prove

$$\limsup_{k\to\infty} [A(u_k), u_k - u]_{T_0} \leq 0. \tag{1.19}$$

Since by (1.11), V

$$\lim_{k\to\infty} [A(u_k) - \tilde{A}_u(u_k), u_k - u]_{T_0} = 0,$$

(1.19) will imply

$$\limsup_{k\to\infty} [\tilde{A}_u(u_k), u_k - u]_{T_0} \leq 0,$$

thus we obtain from (1.11), (1.12) $w = \tilde{A}_u(u) = A(u)$ (see, e.g., Remark 4 in [8]).

Applying (1.5) to $u_k - v$ with some

$$v \in L^p(0, T_0; V) \cap C^1([0, T_0]; L^2(\Omega)) \cap L^\infty(Q_{T_0}) \text{ with } v(0) = 0,$$

we have for any $T \in [0, T_0]$

$$\int_0^T \langle D_t v, u_k - v\rangle dt + \frac{1}{2} \parallel u_k(T) - v(T) \parallel^2_{L^2(\Omega)} + \int_0^T \langle A(u_k), u_k - v\rangle dt + \tag{1.20}$$

$$\int_{Q_T} b_k(t, x, u_k(t, x); u_k)(u_k - v) dt dx = \int_0^T \langle f(t), u_k - v\rangle dt.$$

Since

$$[A(u_k), u_k - v]_T = [A(u_k), u_k - u]_T + [A(u_k), u - v]_T$$

and by Fatou's lemma, (1.7), (1.14), (1.15)

$$\liminf_{k\to\infty} \int_{Q_T} b_k(t, x, u_k(t, x); u_k) u_k dt dx \geq \int_{Q_T} b(t, x, u(t, x); u) u dt dx, \tag{1.21}$$

we obtain from (1.20) (by using (1.11), (1.12), (1.17))

$$\limsup_{k\to\infty} [A(u_k), u_k - u]_T \leq \int_0^T \langle D_t v, v - u\rangle dt + \tag{1.22}$$

$$\int_{Q_T} b(t, x, u(t, x); u)(v - u) dt dx + \int_0^T \langle f(t) - w(t), u - v\rangle dt.$$

Consider the sequence (v_ν) of Theorem 3 in [3], approximating the function u which satisfies all the conditions of that theorem by (1.13), (1.17), and apply (1.22) to $v = v_\nu$. Then Proposition 3 of [3] implies (as $\nu \to \infty$) (1.19). Thus we have also

$$\lim_{k\to\infty} [A(u_k), u_k - u]_T = 0, \quad (A(u_k)) \to A(u) \text{ weakly in } L^q(0, T_0; V^\star) \tag{1.23}$$

(see, e.g., [8]). So, (1.18), $w = A(u)$ imply that u satisfies (0.2) in distributional sense.

Finally, we show $u \in C([0, T_0]; L^2(\Omega))$, $u(0) = 0$ and (1.4). From (1.11), (1.17), (1.20), (1.23) one obtains as $k \to \infty$

$$\limsup_{k\to\infty} \int_{Q_T} b_k(t, x, u_k(t, x); u_k) u_k dtdx \leq \int_0^T \langle D_t v, v - u\rangle dt + \tag{1.24}$$

$$\int_{Q_T} b(t, x, u(t, x); u) v dtdx + [f - A(u), u - v]_T.$$

Applying (1.24) again to $v = v_\nu$ (approximating u), we find

$$\limsup_{k\to\infty} \int_{Q_T} b_k(t, x, u_k(t, x); u_k) u_k dtdx \leq \int_{Q_T} b(t, x, u(t, x); u) u dtdx. \tag{1.25}$$

Further, by (1.11) for a.e. $T \in [0, T_0]$

$$(u_k(T)) \to u(T) \text{ a.e. in } \Omega,$$

so by (1.8) for a.e. $T \in [0, T_0]$

$$(u_k(T)) \to u(T) \text{ in } L^2(\Omega).$$

Consequently, from (1.20), (1.21), (1.25) one derives (1.4) for a.e. $T \in [0, T_0]$. Since all the terms in (1.4) are continuous in T, except possibly the term

$$\| u(T) - v(T) \|_{L^2(\Omega)}, \tag{1.26}$$

the latter can be extended to a continuous function in T and (1.4) holds for all $T \in [0, T_0]$.

For any smooth testing function w (defined in Ω) $(u(T), w)_{L^2(\Omega)}$ is continuous in T because (0.2) holds in distributional sense and the term in (1.26) is continuous in T, thus $u \in C([0, T_0]; L^2(\Omega))$ and so by (1.13) the initial condition $u(0) = 0$ is satisfied which completes the proof of Theorem 1.1.

2. **Boundedness and stabilization**

Denote by $L^p_{loc}(0,\infty;V)$ the set of functions $v:(0,\infty)\to V$ such that for each fixed finite $T_0>0$, $v|_{(0,T_0)}\in L^p(0,T_0;V)$ and let $Q_\infty=(0,\infty)\times\Omega$, $L^\alpha_{loc}(Q_\infty)$ the set of functions $v:Q_\infty\to R$ such that $v|_{Q_{T_0}}\in L^\alpha(Q_{T_0})$ for any finite T_0. By using a "diagonal process", it is not difficult to prove (see, e.g., [9])

Theorem 2.1. *Assume that we have functions* $a_i: Q_\infty\times R^{n+1}\times L^p_{loc}(0,\infty;V)\to R$, $b:Q_\infty\times R\times L^p_{loc}(0,\infty;V)\to R$ *such that they satisfy I - VII for any finite* $T_0>0$ *and* $a_i(t,x,\zeta_0,\zeta;v)|_{Q_{T_0}}$, $b(t,x,\zeta_0;v)|_{Q_{T_0}}$ *depend only on* $v|_{(0,T_0)}$ *(Volterra property). Then for any* $f\in L^q_{loc}(0,\infty;V^\star)$ *there exists* $u\in L^p_{loc}(0,\infty;V)$ *which is a solution for any finite* T_0 *(in the sense of Theorem 1.1).*

Theorem 2.2. *Let the assumptions of Theorem 2.1 be satisfied such that in IV we have* $g_2: L^p_{loc}(0,\infty;V)\to R^+$ *and* $k_2: L^p_{loc}(0,\infty;V)\to L^1_{loc}(Q_\infty)$, *satisfying for any* $v\in L^p_{loc}(0,\infty;V)$, $g_2(v)\ge c_2>0$ *and*

$$\int_\Omega |k_2(v)|dx\le c_4\left[\sup_{[0,t]}|y|^{p_1/2}+\varphi(t)\sup_{[0,t]}|y|^{p/2}+1\right]$$

with some constants c_4, $p_1<p$, $p>2$ *and* $\lim_\infty\varphi=0$ *where*

$$y(t)=\int_\Omega v(t,x)^2dx;$$

finally, $\| f(t)\|_{V^\star}$ *is bounded.*

Then for the solutions u, *formulated in Theorem 2.1,* $\int_\Omega u(t,x)^2dx$ *is bounded for* $t\in[0,\infty)$.

The idea of the proof. If u is a solution in $(0,\infty)$ then the assumptions of the theorem imply that $y(t)=\int_\Omega u(t,x)^2dx$ satisfies the inequality

$$y(T_2)-y(T_1)+c_5\int_{T_1}^{T_2}[y(t)]^{p/2}dt\le$$

$$c_6\int_{T_1}^{T_2}\left[\sup_{[0,t]}y^{p_1/2}+\varphi(t)\sup_{[0,t]}y^{p/2}+1\right]dt,\quad 0<T_1<T_2<\infty$$

with some constants $c_5>0, c_6$. It is not difficult to show that this inequality and $p>2$, $p_1<p$ imply the boundedness of y.

3. **Examples**

1. The conditions of Theorem 1.1 are satisfied if

$$a_i(t,x,\zeta_0,\zeta;v) = [H(v)](t,x)a_i^1(t,x,\zeta_0,\zeta) + [G(v)](t,x)a_i^2(t,x,\zeta_0,\zeta), \quad i=1,...,n,$$

$$a_0(t,x,\zeta_0,\zeta;v) = [H(v)](t,x)a_0^1(t,x,\zeta_0,\zeta) + [G_0(v)](t,x)a_0^2(t,x,\zeta_0,\zeta)$$

where $H : L^p(Q_{T_0}) \to L^\infty(Q_{T_0})$ is bounded and continuous operator with the property: There exists a constant $c_2 > 0$ such that $H(v) \geq c_2$ for all v;

$$G, G_0 : L^p(Q_{T_0}) \to L^{\frac{p}{p-1-\rho}}(Q_{T_0}), \quad (0 \leq \rho < p-1)$$

are bounded and continuous operators, $G(v) \geq 0$ for all v and

$$\lim_{\|v\|_X \to \infty} \frac{\int_{Q_{T_0}} |G_0(v)|^{\frac{p}{p-1-\rho}}}{\| v \|_X^p} = 0.$$

Further, a_i^1, a_i^2 satisfy the usual conditions: They are Carathéodory functions,

$$|a_i^1(t,x,\zeta_0,\zeta)| \leq c_1(|\zeta_0|^{p-1} + |\zeta|^{p-1}) + k_1(x)$$

with some constant c_1, $k_1 \in L^q(\Omega)$, $i = 0,1,...,n$;

$$\sum_{i=1}^n [a_i^1(t,x,\zeta_0,\zeta) - a_i^1(t,x,\zeta_0,\zeta^\star)](\zeta_i - \zeta_i^\star) > 0 \text{ if } \zeta \neq \zeta^\star;$$

$$\sum_{i=0}^n a_i^1(t,x,\zeta_0,\zeta)\zeta_i \geq c_3(|\zeta_0|^p + |\zeta|^p) - k_2(x)$$

with some constant $c_3 > 0$, $k_2 \in L^1(\Omega)$;

$$|a_i^2(t,x,\zeta_0,\zeta)| \leq c_1(|\zeta_0|^\rho + |\zeta|^\rho), \quad 0 \leq \rho < p-1, \quad i = 0,1,...,n;$$

$$\sum_{i=1}^n [a_i^2(t,x,\zeta_0,\zeta) - a_i^2(t,x,\zeta_0,\zeta^\star)](\zeta_i - \zeta_i^\star) \geq 0;$$

$$\sum_{i=1}^n a_i^2(t,x,\zeta_0,\zeta)\zeta_i \geq 0.$$

By using Young's and Hölder's inequalities it is not difficult to show that the conditions I - V are fulfilled.

A simple special case for a_i^1, a_i^2 are:

$$a_i^1(t,x,\zeta_0,\zeta) = \zeta_i|\zeta|^{p-2}, \quad i=1,...,n, \quad a_0^1(t,x,\zeta_0,\zeta) = \zeta_0|\zeta_0|^{p-2}, a_i^2 = 0.$$

The operator H may have e.g. one of the forms:

$\varphi\left(\int_{Q_t} bv\right)$ where $\varphi : R \to R$ is a continuous function, $\varphi \geq c_2 > 0$ (constant), $b \in L^q(Q_T)$;

$\varphi\left(\left[\int_{Q_t} |v|^\beta\right]^{1/\beta}\right)$ with some $1 \leq \beta \leq p$;

The operators G, G_0 may have e.g. one of the forms:

$$\psi_0\left(\int_0^t a(\tau,x)v(\tau,x)d\tau\right), \quad \psi_0\left(\int_\Omega a(t,x)v(t,x)dx\right),$$

$$\psi_0\left(\left[\int_0^t |v(\tau,x)|^\beta d\tau\right]^{\frac{1}{\beta}}\right),$$

where $\psi_0 : R \to R$ is continuous, $|\psi_0(\theta)| \leq \text{const}|\theta|^{p-1-\rho_0}$ with some $\rho_0 > \rho$, $\psi_0(\theta) \geq 0$ for G, $a \in L^\infty$.

The operators G, G_0 may have also the forms

$$\int_0^t h(t,\tau,x,v(\tau,x))d\tau \text{ or } h(t,x,v(\chi(t),x))$$

where

$$|h(t,\tau,x,\theta)|, \quad |h(t,x,\theta)| \leq \text{const}|\theta|^{p-1-\rho_0},$$

$0 \leq \chi(t) \leq t$, $\chi \in C^1$ and $h \geq 0$ for G.

2. The conditions on a_i of Theorem 1.1 are satisfied if

$$a_i(t,x,\zeta_0,\zeta;v) = [H_i(v)](t,x)\tilde{a}_i^1(t,x,\zeta_0,\zeta_i) + [G_i(v)](t,x)\tilde{a}_i^2(t,x,\zeta_0,\zeta_i)$$

where $\zeta_i \mapsto \tilde{a}_i^1(t,x,\zeta_0,\zeta_i)$ is strictly increasing for $i = 1,...,n$;

$$|\tilde{a}_i^1(t,x,\zeta_0,\zeta_i)| \leq c_1(|\zeta_0|^{p-1} + |\zeta_i|^{p-1}) + k_1(x)$$

with some constant c_1, $k_1 \in L^q(\Omega)$, $i = 0,1,...,n$;

$$\tilde{a}_i^1(t,x,\zeta_0,\zeta_i)\zeta_i \geq c_2|\zeta_i|^p - k_2(x), \quad i = 1,...,n$$

with some constant $c_2 > 0$, $k_2 \in L^1(\Omega)$; $\zeta_i \mapsto \tilde{a}_i^2(t,x,\zeta_0,\zeta_i)$ is monotone nondecreasing such that $\tilde{a}_i^2(t,x,\zeta_0,\zeta_i) = 0$ if $\zeta_i = 0$ $(i = 1,...,n)$;

$$|\tilde{a}_i^2(t,x,\zeta_0,\zeta_i)| \leq c_1(|\zeta_0|^\rho + |\zeta_i|^\rho) \text{ with } 0 \leq \rho < p-1, \quad i = 0,1,...,n.$$

Operators H_i satisfy the same conditions as H in Example 1 and operators G_i satisfy the same conditions as G, G_0, respectively, in Example 1.

Example on b. $b(t,x,\zeta_0;v) = \psi(\zeta_0)\tilde{G}(v)$ where $\tilde{G} : L^p(Q_{T_0}) \to L^\infty(Q_{T_0})$ is a continuous operator with the property

$$0 < c_1 \le \tilde{G}(v) \le c_2 < \infty \text{ for any } v$$

with some constants c_1, c_2.

The conditions of Theorem 2.1 are fulfilled for the Examples 1,2 if

$$H, H_i : L^p_{loc}(Q_\infty) \to L^\infty(Q_\infty), \quad G, G_i : L^p_{loc}(Q_\infty) \to L^{\frac{p}{p-1-\rho}}(Q_\infty)$$

satisfy the above conditions for any finite T_0 and they have the Volterra property; further, $a_i^1, a_i^2, \tilde{a}_i^1, \tilde{a}_i^2$ satisfy the above conditions for any t.

The conditions of Theorem 2.2 are satisfied if the following additional condition is fulfilled:

$$\int_\Omega |G_0(v)|^{\frac{p}{p-1-\rho}} dx \le c_4 \left[\sup_{[0,t]} |y|^{p_1/2} + \varphi(t) \sup_{[0,t]} |y|^{p/2} + 1\right]$$

for any $v \in L^p_{loc}(0,\infty;V)$ with $y(t) = \int_\Omega v(t,x)^2 dx$ and $\| f(t) \|_{V^\star}$ is bounded.

The operator G_0 may have e.g. one of the forms

$$\psi_0\left(\int_\Omega a(t,x)v(t,x)dx\right), \quad \psi_0\left(\left[\int_\Omega |a(t,x)||v(t,x)|^\beta dx\right]^{1/\beta}\right),$$

$$\varphi_0(t)\chi_0\left(\left[\int_\Omega |a(t,x)||v(t,x)|^2 dx\right]^{1/2}\right)$$

where $1 \le \beta \le 2$, $a \in L^\infty$, $\psi_0, \varphi_0, \chi_0 : R \to R$ are continuous,

$$|\psi_0(\theta)| \le \text{const}|\theta|^{p-1-\rho_0} \text{ with some } \rho_0 > \rho,$$

$$|\chi_0(\theta)| \le \text{const}|\theta|^{p-1-\rho}, \quad \lim_\infty \varphi_0 = 0.$$

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Department of Applied Analysis, L. Eötvös University of Budapest,
Pázmány Péter sétány 1/C, H-1117 Budapest, Hungary
E-mail address: `simonl@ludens.elte.hu`

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CONSTRUCTION OF GAUSS-KRONROD-HERMITE QUADRATURE AND CUBATURE FORMULAS

DANIEL VLADISLAV

Abstract. We study Gauss-Kronrod quadrature formula for Hermite weight function for the particular cases $n = 1, 2, 3$, we introduce a new Gauss-Kronrod-Hermite cubature formula and we describe the form of the weights and nodes.

1. Introduction. Quadrature and cubature rules of Gauss-Hermite type

Let us consider the weight function $\rho(x) = e^{-x^2}$, defined and positive on $(-\infty, \infty)$. The quadrature rule of Gauss-Hermite type corresponding to this weight function is:

$$\int_{\mathbb{R}} e^{-x^2} f(x)dx = \sum_{k=0}^{m} A_{m,k} f(a_k) + R_m[f]. \tag{1}$$

The nodes a_k, $k = \overline{0,m}$, the coefficients $A_{m,k}$, $k = \overline{0,m}$ and the remainder term can be determined using the properties of Hermite orthogonal polynomials, defined as follows:

$$H_m(x) = (-1)^m e^{x^2} \frac{d^m}{dx^m}[e^{-x^2}]. \tag{2}$$

It has been proved (see [1]) that:

(i) the nodes a_k, $k = \overline{0,m}$, are the zeros of the Hermite orthogonal polynomial of degree $m + 1$;

(ii) the coefficients $A_{m,k}$, $k = \overline{0,m}$ would be computed with the formula:

$$A_{m,k} = \frac{2^{m+1} m! \sqrt{\pi}}{H_m(a_k) H'_{m+1}(a_k)} \tag{3}$$

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(iii) the remainder term $R_m[f]$, for $f \in C^{2m+2}(\mathbb{R})$, has the representation:

$$R_m[f] = \frac{(m+1)!\sqrt{\pi} f^{(2m+2)}(\xi)}{2^{m+1}(2m+2)!}, \quad -\infty < \xi < \infty. \tag{4}$$

We also consider the Gauss-Hermite cubature rule of the form:

$$\int\int_{\mathbb{R}^2} P(x,y) f(x,y) dxdy = \sum_{i=0}^{m} \sum_{j=0}^{n} A_{i,j} f(x_i, y_j) + R_{m,n}[f]. \tag{5}$$

It has been proved (see [1]) that in this formula the coefficients are computed with:

$$A_{i,j} = A^{[1]}_{m,i} A^{[2]}_{n,j} = \frac{2^{m+1} m! \sqrt{\pi}}{H_m(x_i) H'_{m+1}(x_i)} \cdot \frac{2^{n+1} n! \sqrt{\pi}}{H_n(y_j) H'_{m+1}(y_j)} \tag{6}$$

where the nodes x_i, $i = \overline{0,m}$ and y_j, $j = \overline{0,n}$ are respectively the zeros of Hermite orthogonal polynomials H_{m+1}, H_{n+1}.

If $f \in C^{m+1,n+1}(\mathbb{R}^2)$ then the remainder term has the expression:

$$R_{m,n}[f] = \frac{\pi(m+1)!}{2^{m+1}(2m+2)!} f^{(2m+2,0)}(\xi_1, \eta_1) + \frac{\pi(n+1)!}{2^{n+1}(2n+2)!} f^{(0,2n+2)}(\xi_2, \eta_2) \tag{7}$$
$$- \frac{\sqrt{\pi}(m+1)!}{2^{m+1}(2m+2)!} \cdot \frac{\sqrt{\pi}(n+1)!}{2^{n+1}(2n+2)!} f^{(2m+2,2n+2)}(\xi_3, \eta_3).$$

2. Study upon the quadrature rule of Gauss-Kronrod type with Hermite weight function

In this section we consider the Gauss-Kronrod quadrature formula with Hermite weight function $\rho(x) = e^{-x^2}$, nonnegative and defined on $\mathbb{R}$

$$\int_{\mathbb{R}} \rho(x) f(x) dx = \sum_{i=1}^{m} \sigma_i f(x_i) + \sum_{k=1}^{m+1} \sigma^*_k f(x^*_k) + R_m[f] \tag{8}$$

where $x_i = x_i^{(m)}$ are the Gaussian nodes (i.e. the zeros of $H_m(\cdot, \rho)$, the mth degree orthogonal polynomial relative to the measure $d\sigma(t) = \rho(t)dt$ on $\mathbb{R}$) and the nodes x^*_k (the Kronrod nodes) and weights $\sigma_i = \sigma_i^{(m)}$, $\sigma^*_k = \sigma_k^{(m)*}$ are determined such that (8) has maximum degree of exactness $3m+1$, i.e.

$$R_m[f] = 0, \ \forall \ f \in \mathbf{P}_{3m+1}. \tag{9}$$

It is well known that x^*_k must be the zeros of the (monic) orthogonal polynomial H^*_{m+1} of degree $m+1$ relative to the measure $\rho^*(x) = H_m(x,\rho)\rho(x)$ on $\mathbb{R}$.

Even through H_m, and hence ρ^*, changes sign on $\mathbb{R}$, it is known that H^*_{m+1} exists uniquely (see e.g. [3]). There is no guarantee, however, that the zeros x^*_k of H^*_{m+1} are real, neither that the interlacing property of Gauss nodes with nodes of Kronrod type holds.

We study in the following the cases $m = 1$, $m = 2$ and $m = 3$, i.e. we check the existence of quadrature rules with 3, 5 and 7 nodes.

For case $m = 1$ we found the Gauss-Kronrod quadrature formula with 3 nodes:

$$\int_{\mathbb{R}} e^{-x^2} f(x)dx = \frac{\sqrt{\pi}}{6} f\left(-\frac{\sqrt{3}}{2}\right) + \frac{2\sqrt{\pi}}{3} f(0) + \frac{\sqrt{\pi}}{6} f\left(\frac{\sqrt{3}}{2}\right) + R_2[f]$$

where $H_1(x) = x$ represent the Hermite polynomial with zeros $x_1 = 0$ and the polynomial $H^*_2(x) = x^2 - \frac{3}{2}$ has been determined from the orthogonality condition:

$$\int_{\mathbb{R}} e^{-x^2} H^*_2(x) H_1(x) x^k dx = 0, \quad k = 0, 1.$$

For the computation of the coefficients we used the formula (see [5])

$$\sigma_i = \gamma_i + \frac{\|H_m\|^2_{d\rho}}{H^*_{m+1}(x_i) H'_m(x_i)}, \quad i = 1, 2, \ldots, m$$

and

$$\sigma^*_k = \frac{\|H_m\|^2_{d\sigma}}{H_m(x^*_k) H^{*1}_{m+1}(x^*_k)}, \quad k = 1, 2, \ldots, m+1$$

where $\gamma_i = \gamma_i^{(m)}$ are the Christoffel numbers (i.e. the weights in the Gaussian quadrature rule and $\|\cdot\|_{d\rho}$ the L_2-norm for the weight function).

One can observe that all the zeros of polynomial H^*_2 are real and they interlace with the zero of polynomial H_1.

For the case $m = 2$, one gets the following quadrature formula

$$\int_{\mathbb{R}} e^{-x^2} f(x)dx =$$

$$= \frac{\sqrt{\pi}}{30} \left[f(-\sqrt{3}) + 9f\left(-\frac{\sqrt{2}}{2}\right) + 10f(0) + 9f\left(\frac{\sqrt{2}}{2}\right) + f(\sqrt{3}) \right] + R_2[f].$$

Here one can observe that all the nodes are real and the interlacing property is satisfied. All the coefficients of formula are positive.

If $m = 3$ Stieltjes polynomial, respective H_4^* has two real zeros and two complex zeros, fact that doesn't assure us the existence of Gauss-Kronrod quadrature formula in this case.

3. Construction of Gauss-Kronrod-Hermite type cubature formula

Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a integrable Riemann function. Applying the Gauss-Kronrod quadrature formula with Hermite measure

$$\int_{\mathbb{R}} e^{-x^2} f(x,y)dx = \sum_{i=1}^{m} A_{m,i} f(x_i, y) + \sum_{k=1}^{m+1} A_{m,k}^* f(x_k^*, y) + R_m[f]$$

which will multiply with measure $\rho(y) = e^{-y^2}$, obtaining the measure $p(x,y) = e^{-(x^2+y^2)}$ of the double integrals, after that we integrate, term by term and we obtain:

$$\int\int_{\mathbb{R}^2} e^{-(x^2+y^2)} f(x,y)dxdy = \sum_{i=1}^{m} A_{m,i} \int_{\mathbb{R}} e^{-y^2} f(x_i, y)dy + \sum_{k=1}^{m+1} A_{m,k}^* e^{-y^2} f(x_k^*, y) +$$

$$+ R_m[f] \int_{\mathbb{R}} e^{-y^2} dy.$$

For the integrals above we apply again one of quadrature rule of Gauss-Kronrod type:

$$\int e^{-y^2} f(x_i, y)dy = \sum_{j=1}^{n} A_{n,j} f(x_i, y_j) + \sum_{l=1}^{n+1} A_{n,l}^* f(x_i, y_l^*) + R_n[f]$$

and

$$\int e^{-y^2} f(x_k^*, y)dy = \sum_{j=1}^{n} A_{n,j} f(x_k^*, y_j) + \sum_{l=1}^{n+1} A_{n,l}^* f(x_k^*, y_l^*) + R_n[f]$$

respectively.

From here, it results the cubature rule:

$$\int\int_{\mathbb{R}^2} e^{-(x^2+y^2)} f(x,y)dxdy \approx \sum_{i=1}^{m} \sum_{j=1}^{n} A_{m,i} A_{n,j} f(x_i, y_j)$$

$$+ \sum_{i=1}^{m} \sum_{l=1}^{n+1} A_{m,i} A_{n,l}^* f(x_i, y_l^*) + \sum_{k=1}^{m+1} \sum_{j=1}^{n} A_{m,k}^* A_{n,j} f(x_k^*, y_j)$$

$$+ \sum_{k=1}^{m+1} \sum_{l-1}^{n+1} A_{m,k}^* A_{n,l}^* f(x_k^*, y_l^*)$$

with Gauss nodes (x_i, y_j), $i = \overline{1,m}$, $j = \overline{1,n}$ and Kronrod nodes (x_k^*, y_l^*), $k = \overline{1,m+1}$, $l = \overline{1,n+1}$, respectively mixed nodes of form (x_k^*, y_j) and (x_i, y_l^*).

The coefficients of Gauss-Kronrod-Hermite type cubature rules could be determined from:

$$A_{i,j} = A_{m,i}A_{n,j}, \quad A_{i,l}^* = A_{m,i}A_{n,l}^*,$$

$$A_{k,j}^* = A_{m,k}^*A_{n,j}, \quad A_{k,l}^* = A_{m,k}^*A_{n,l}^*$$

where

$$A_{m,i} = \gamma_i + \frac{\|H_m\|^2}{H_{m+1}^*(x_i)H_n'(x_i)}, \quad i = \overline{1,m}$$

$$A_{n,j} = \gamma_j + \frac{\|H_n\|^2}{H_{n+1}^*(y_j)H_n'(y_j)}, \quad j = \overline{1,n}$$

$$A_{n,l}^* = \frac{\|H_n\|^2}{H_n(y_l^*)H_{n+1}^{*\prime}(y_l^*)}, \quad l = \overline{1,m+1}$$

$$A_{m,k}^* = \frac{\|H_m\|^2}{H_m(x_k^*)H_{m+1}'(x_k^*)}, \quad k = \overline{1,m+1}.$$

4. **Example**

1. For the case $m = n = 1$ we have:

$$\iint_{\mathbb{R}^2} e^{-(x^2+y^2)}dxdy \simeq (A_{1,1})^2 f(x_1,y_1) + A_{1,1}A_{1,1}^* f(x_1,y_1^*) + A_{1,1}A_{1,2}^* f(x_1,y_2^*) +$$

$$+A_{1,1}^*A_{1,1}f(x_1^*,y_1) + A_{1,2}^*A_{1,1}f(x_2^*,y_1) + (A_{1,1}^*)^2 f(x_1^*,y_1^*) + A_{1,1}^*A_{1,2}^* f(x_1^*,y_2^*) +$$

$$+A_{1,2}^*A_{1,1}^* f(x_2^*,y_1^*) + (A_{1,2}^*)^2 f(x_2^*,y_2^*),$$

where $x_1 = 0 = y_1$ and $x_1^* = -\sqrt{\frac{3}{2}} = y_1^*$, $x_2^* = \sqrt{\frac{3}{2}} = y_2^*$.

The values of the weights of this formula are:

$$A_{1,1} = \frac{2\sqrt{\pi}}{3} \quad \text{respectively} \quad A_{1,1}^* = \frac{\sqrt{\pi}}{6} = A_{1,2}^*.$$

From here result the following cubature formula:

$$\iint_{\mathbb{R}^2} e^{-(x^2+y^2)} f(x,y)dxdy \simeq \frac{4\pi}{9} f(0,0) +$$

$$+\frac{\pi}{9}\left[f\left(0,-\sqrt{\frac{3}{2}}\right) + f\left(0,\sqrt{\frac{3}{2}}\right) + f\left(-\sqrt{\frac{3}{2}},0\right) + f\left(\sqrt{\frac{3}{2}},0\right)\right] +$$

$$+\frac{\pi}{36}\left[f\left(-\sqrt{\frac{3}{2}},-\sqrt{\frac{3}{2}}\right)+f\left(-\sqrt{\frac{3}{2}},\sqrt{\frac{3}{2}}\right)+f\left(\sqrt{\frac{3}{2}},-\sqrt{\frac{3}{2}}\right)+f\left(\sqrt{\frac{3}{2}},\sqrt{\frac{3}{2}}\right)\right].$$

For $f(x,y)=x^2y^2$ we have

$$\iint_{\mathbb{R}} e^{-(x^2+y^2)}x^2y^2dxdy=\int_{\mathbb{R}} x^2e^{-x^2}dx\int_{\mathbb{R}} y^2e^{-y^2}dy=\frac{\sqrt{\pi}}{2}\cdot\frac{\sqrt{\pi}}{2}=\frac{\pi}{4}$$

representing the exact value of this integral.

Applying cubature formula we obtain:

$$\iint_{\mathbb{R}^2} e^{-(x^2+y^2)}x^2y^2dxdy=\frac{\pi}{35}\left[\frac{3}{2}\cdot\frac{3}{2}+\frac{3}{2}\cdot\frac{3}{2}+\frac{3}{2}\cdot\frac{3}{2}+\frac{3}{2}\cdot\frac{3}{2}\right]=\frac{\pi}{36}\cdot 4\cdot\frac{9}{4}=\frac{\pi}{4}$$

2. For the case $m=n=2$ we have:

$$\iint_{\mathbb{R}^2} e^{-(x^2+y^2)}f(x,y)dxdy\simeq(A_{2,1})^2f(x_1,y_1)+A_{2,1}A_{2,2}f(x_1,y_2)+A_{2,2}A_{2,1}f(x_2,y_1)+$$

$$+(A_{2,2})^2f(x_2,y_2)+A_{2,1}A^*_{2,1}f(x_1,y^*_1)+A_{2,1}A^*_{2,2}f(x_1,y^*_2)+A_{2,1}A^*_{2,3}f(x_1,y^*_3)+$$

$$+A_{2,2}A^*_{2,1}f(x_2,y^*_1)+A_{2,2}A^*_{2,2}f(x_2,y^*_2)+A_{2,2}A^*_{2,3}f(x_2,y^*_3)+$$

$$+A^*_{2,1}A_{2,1}f(x^*_1,y_1)+A^*_{2,1}A_{2,2}f(x^*_1,y_2)+A^*_{2,2}A_{2,1}f(x^*_2,y_1)+A^*_{2,2}A_{2,2}f(x^*_2,y_2)+$$

$$+A^*_{2,3}A_{2,1}f(x^*_3,y_1)+A^*_{2,3}A_{2,2}f(x^*_3,y_2)+(A^*_{2,1})^2f(x^*_1,y^*_1)+A^*_{2,1}A^*_{2,2}f(x^*_1,y^*_2)+$$

$$+A^*_{2,1}A^*_{2,3}f(x^*_1,y^*_3)+A^*_{2,2}A^*_{2,1}f(x^*_2,y^*_1)+(A^*_{2,2})^2f(x^*_2,y^*_2)+A^*_{2,2}A^*_{2,3}f(x^*_2,y^*_3)+$$

$$+A^*_{2,3}A^*_{2,1}f(x^*_3,y^*_1)+A^*_{2,3}A^*_{2,2}f(x^*_3,y^*_2)+(A^*_{2,3})^2f(x^*_3,y^*_3),$$

where the gaussian nodes are: $x_1=-\frac{\sqrt{2}}{2}=y_1$ and $x_2=\frac{\sqrt{2}}{2}=y_2$ the roots of the orthogonal polynomial of Hermite type: $H_2(x)=x^2-\frac{1}{2}$. The Stieltjes polynomial is: $H^*_3(x)=x(x^2-3)$ with the roots: $x^*_1=-\sqrt{3}=y^*_1$, $x^*_2=0=y^*_2$, $x^*_3=\sqrt{3}=y^*_3$, representing the Kronrod nodes.

The values of the weights of this formula are:

$$A_{2,1}=\frac{3\sqrt{\pi}}{10}=A_{2,2}$$

and

$$A^*_{2,1}=\frac{\sqrt{\pi}}{30}=A^*_{2,3},\quad A^*_{2,2}=\frac{\sqrt{\pi}}{3}.$$

$$\iint_{\mathbb{R}^2} e^{-(x^2+y^2)}f(x,y)dxdy=$$

$$= \frac{9\pi}{100}\left[f\left(-\frac{\sqrt{2}}{2},-\frac{\sqrt{2}}{2}\right)+f\left(-\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2}\right)+f\left(\frac{\sqrt{2}}{2},-\frac{\sqrt{2}}{2}\right)+f\left(\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2}\right)\right]+$$
$$+\frac{\pi}{100}\left[f\left(-\frac{\sqrt{2}}{2},-\sqrt{3}\right)+f\left(-\frac{\sqrt{2}}{2},\sqrt{3}\right)+f\left(\frac{\sqrt{2}}{2},-\sqrt{3}\right)+f\left(-\frac{\sqrt{2}}{2},\sqrt{3}\right)+\right.$$
$$\left.+f\left(-\sqrt{3},-\frac{\sqrt{2}}{2},\right)+f\left(-\sqrt{3},\frac{\sqrt{2}}{2},\right)+f\left(\sqrt{3},-\frac{\sqrt{2}}{2},\right)+f\left(\sqrt{3},\frac{\sqrt{2}}{2},\right)\right]+$$
$$+\frac{\pi}{10}\left[f\left(-\frac{\sqrt{2}}{2},0\right)+f\left(\frac{\sqrt{2}}{2},0\right)+f\left(0,-\frac{\sqrt{2}}{2}\right)+f\left(0,\frac{\sqrt{2}}{2}\right)\right]+$$
$$+\frac{\pi}{900}\left[f\left(-\sqrt{3},-\sqrt{3}\right)+f\left(-\sqrt{3},\sqrt{3}\right)+f\left(\sqrt{3},-\sqrt{3}\right)+f\left(\sqrt{3},\sqrt{3}\right)\right]+$$
$$+\frac{\pi}{90}\left[f\left(-\sqrt{3},0\right)+f\left(0,-\sqrt{3}\right)+f\left(0,\sqrt{3}\right)+f\left(\sqrt{3},0\right)\right]+\frac{\pi}{9}f(0,0).$$

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Univ. Petroşani, Str. Institutului nr. 20, Romania

E-mail address: vladislav_dan@upet.ro

BOOK REVIEWS

Pietro Aiena, ***Fredholm and Local Spectral Theory, with Applications to Multipliers***, Kluwer Academic Publishers, Dordrecht-Boston-London 2004, xiv + 444 pp, ISBN: 1-4020-1830-4.

Let T be an operator acting on a complex Banach space X. The local resolvent of T at a point $x \in X$ is the set $\rho_T(x)$ of all $\lambda \in \mathbb{C}$ for which there exist an open neighborhood $U_\lambda \subset \mathbb{C}$ of λ and an analytic function $f : U_\lambda \to X$ such that (1) $(\mu I - T)f(\mu) = x$, for all $\mu \in U_\lambda$. Obviously that the analytic function $f_x(\mu) = (\mu I - T)^{-1}x$ satisfies this relation on the resolvent set $\rho(T)$ of the operator T, but it could exist other analytic functions satisfying (1), even on neighborhoods of some points in the spectrum $\sigma(T)$ of T. The set $\rho_T(x)$ is open and contains $\rho(T)$. The local spectrum of T at x is $\sigma_T(x) = \mathbb{C} \setminus \rho_T(x)$, and $\sigma_T(x) \subset \sigma(T)$.

The local spectral theory is systematically treated in a recent book by K. B. Laursen and M. M. Neumann, *An Introduction to Local Spectral Theory,* Oxford University Press 2000, containing also some elements of Fredholm theory, mainly those which can be approached by the methods of local spectral theory.

The emphasis of the present book is first on Fredholm theory, developed in connection with Kato decomposition and a property called the single-valued extension property (SVEP). This property, which means that the only analytic function f satisfying the relation $(\mu I - T)f(\mu) = 0$ for any μ in a neighborhood of an arbitrary point $\lambda \in \mathbb{C}$ is the null function $f \equiv 0$, has deep connections with Fredholm theory. It was considered first by Dunford in 1952 and is systematically studied in the treatise on operator theory by Dunford and Schwartz, and in other books on spectral theory of operators. The author presents also the abstract Fredholm theory in semi-prime

Banach algebras. The main applications of Fredholm theory considered in the book are to multipliers on Banach algebras.

Another direction of investigation studied in the book is that of perturbation theory for classes of operators which occur in Fredholm theory, completing the existing monographs on this topic (as, e.g., Kato's classical treatise) with more recent results.

The author tried to make the book as self-contained as possible, by proving some elementary facts about the notions considered. Of course that, as a research monograph, it requires from the reader an acquaintance with basic complex function theory and functional analysis, including classical Fredholm theory.

The book can be used for graduate courses in operator theory or by professional mathematicians working in the subject or interested in its applications to other areas of investigation.

S. Cobzaş

Fundamental Directions in Mathematical Fluid Mechanics, **G. P. Galdi, J. G. Heywood, R. Rannacher** (Editors), Birkhäuser Verlag, 2000.

The present volume consists of six articles, written by very good experts in the field, each article treating an important topic in the theory of the Navier-Stokes equations, at the research level. As it is well known, the most famous problem in this area is to go beyond the presently known global existence of weak solutions, to the global existence of smooth solutions, for which uniqueness and continuous dependence on the data can be proved. In fact, Galdi's article, *An introduction to the Navier-Stokes initial-boundary value problem*, gives an overview of the state of research regarding this subject.

Then the book moves on to a discussion of new developments of the finite element Galerkin method. The article by Rannacher, *Finite element methods for the incompressible Navier-Stokes equations*, treats both the theory and implementation of the finite element methods, with an emphasis on a priori and a posteriori error estimation and adaptive mesh refinement.

The article by Gervasio, Quarteroni and Saleri *Spectral approximation of Navier-Stokes equations* is devoted on spectral Galerkin methods and their extension to domains with complicated geometries, by employing the techniques of domain decomposition.

The article by Heywood and Nagata *Simple proofs of bifurcation theorems* introduces in a rigorous way bifurcation theory in a general setting that is convenient for application to the Navier-Stokes equations.

The two articles by Heywood and Padula, *On the steady transport equation* and *On the existence and uniqueness theory for the steady compressible viscous flow*, yield a simplified approach to the theory of steady compressible viscous flow. The extension of Navier-Stokes theory to compressible viscous flows, studied in these papers, opens up a beautiful point of view of theoretical and numerical problems.

The book is very well written and enjoyable. It is addressed to researchers, advanced students, and all mathematicians interested to the research level on some of the most important topics in the field of fluid mechanics.

Mirela Kohr

Leszek Gasiński and Nikolaos S. Papageorgiou, *Nonsmooth critical point theory and nonlinear boundary value problems*, Chapman & Hall/CRC, 2004.

One often encounters practical situations where the associated energy functional to a nonlinear elliptic problem is not smooth. Several methods have been elaborated in the last decades in order to handle such kind of problems, see the theories of Chang (1981), Szulkin (1986), Degiovanni and Marzocchi (1994), Frigon (1998), Motreanu and Panagiotopoulos (1999).

The aim of the monograph of L. Gasiński and N. S. Papageorgiou is to present a comprehensive exposition of the aforementioned (non-smooth) critical point theories, as well as to provide us with various applications and concrete examples.

The book is as self-contained as possible and it is made more interesting by the perspectives in various sections, where the authors mention the historical background and development of the material and provide the reader with detailed explanations and updated references.

The first chapter is dedicated to the background material used throughout the book, as basic elements from Sobolev spaces, Set-Valued analysis, Non-smooth analysis (Clarke's calculus of locally Lipschitz functions, weak slope), Nonlinear Operators.

In the second chapter the authors present the existing nonsmooth critical point theories. This part is very well written; the reader obtains a complete picture about these theories. In the first two sections the locally Lipschitz functionals (Chang's theory) as well as constrained locally Lipschitz functionals (the non-smooth version of Struwe's theory) are treated. In the third section the critical point theory of locally Lipschitz functions is developed which are perturbed by a convex, proper and lower semicontinuous functional. This part unifies the theories of Chang and Szulkin. We point out that Motreanu and Panagioutopoulos (1999) were the first authors, and not Kourogenis, Papadrianos and Papageorgiou (2002) as it is mentioned in the book (page 204, paragraph 2.3), who considered this class of functionals. In the fourth section the classical local linking theorem is extended to locally Lipschitz functions, while the last two sections are devoted to the theory of weak slopes, in the sense of Degiovanni-Marzocchi (for continuous functionals), and Frigon (for multivalued functionals). In all the cases, deformation and minimax results are obtained (with Palais-Smale, or Cerami compactness conditions).

The rest of the book deals with applications. Chapter 3 is devoted to the study of nonlinear boundary value problems for ordinary differential equations. Several kind of problems are treated: Dirichlet problems, periodic problems, Hamiltonian inclusions, problems with nonlinear boundary conditions. A great variety of methods and techniques are used, as upper-lower solutions, fixed-point and degree theory arguments, nonsmooth analysis, set-valued analysis.

The biggest part of this book is Chapter 4 (more than 250 pages), which is devoted to the study of nonlinear elliptic equations. The theoretical material, presented in the second chapter, is consistently applied in order to establish existence and multiplicity results for several type of resonance problems (like semilinear, nonlinear, variational-hemivariational inequalities and strongly resonant problems); Neumann problems (homogeneous and non-homogeneous type); problems with an area-type term; problems which involve discontinuous nonlinearities.

In my opinion, the book is very readable, and it can serve as a start point for researchers and students in order to carry out further investigations in the nonsmooth critical point theory as well as in its applications in mechanics, mathematical physics and engineering.

A. Kristály

E. I. Gordon, A. G. Kusraev and S. S. Kutateladze, *Infinitesimal Analysis*, Mathematics and Its Applications, Vol. 544, Kluwer A. P. , Dordrecht-Boston-London, 2002, xiii + 422 pp, ISBN: 1-4020-0738-8.

Infinitesimals or infinitely small quantities, and infinitely large quantities were used for two millennia by scientists and philosophers, starting with Archimedes. The infinitesimals were basic tools in the foundation of mathematical analysis by Leibniz and Newton, and were used by their followers as well, e.g. Euler, until the 19th century when Bolzano, Cauchy and Weierstrass founded the analysis on the notion of limit and $\epsilon - \delta$ technique. After that the use of infinitesimals was considered as lacking of rigor, until the sixties of the 20th century when A. Robinson created nonstandard analysis and put firm basis for the use of infinitely small and infinitely large quantities in mathematics. A brief survey on the historical evolution of ideas in mathematical analysis is presented in the first chapter of the book *Excursus in the history of calculus.*

The term infinitesim al analysis is used to designate a technique of studying general mathematical objects by discriminating between standard and nonstandard ones. The present book is the third in the series "Nonstandard Methods of Analysis" published at Novosibirsk by Sobolev Institute Press under the guidance of Professor Kutateladze. The previous two books were *Boolean Valued Analysis*, Kluwer 1999, by the same authors, and a collection of papers Nonstandard Analysis and Vector Lattices, Kluwer 2000. All these books were written in Russian and then translated (in a revised form) into English and published by Kluwer, as well as another book of the authors *Nonstandard Methods of Analsyis*, Kluwer 1994, which gave the name to the series.

The purpose of the present book is to make the methods of nonstandard analysis more accessible to a larger audience. To this end the second chapter, *Naive foundation of infinitesimal analysis*, contains an intuitive and illustrative introduction to the subject, but sufficient for effective applications, without appealing to any logical formalism.

The cantorian set theory is presented in Ch. 3, *Set-theoretic formalisms of infinitesimal analysis.* Beside the Zermelo-Frenkel system, Nelson internal set theory and the external set theories of Hrbaçek and Kawai are included.

The rest of the book is dedicated to applications of nonstandard analysis to various branches of mathematics – topology in Ch. 4, *Monads in general topology,* and subdifferential calculus and non-smooth analysis in Ch. 5, *Infinitesimals and subdifferentials.* Remark that another book of the authors on the same topic *Subdifferentials: Theory and Applications*, Kluwer 1995, makes extensive use of nonstandard methods. Ch. 6, *Technique of hyperaproximation* deals with nonstandard hulls of normed spaces defined by Luxemburg, and Loeb measures. The technique of hyperapproximation for the Fourier transform on a locally compact abelian group is considered in Ch. 7, *Infinitesimals in harmonic analysis.*

The last chapter of the book, Ch. 8, *Exercises and problems*, contains some exercises along with some open problems of varying difficulty.

The authors have included in the book a lot of philosophical and historical comments. The bibliography at the end of the book contains 542 items.

The book is aimed first to researchers in various branches of mathematics desiring to be acquainted with the powerful tools of nonstandard analysis. Teachers will find in the book a lot of interesting things: – methodological, historical and philosophical.

S. Cobzaş

Martin Väth, *Integration Theory*, World Scientific, New Jersey - Singapore - London, 2002, viii + 27 pp, ISBN: 981-238-115-5.

This book on measure and integration proposes a very general approach to the subject, allowing the simultaneous treatment of both scalar and vector cases. The framework is that of a measure space (S, Σ, μ) and of functions on S taking values in a space $Y = [0, \infty,\ [-\infty, \infty]$, or a Banach space with an ideal element ∞. This approach, presented in the first chapter of the book, Ch. 1, *Abstract Integration*, is based on some results such as the exhaustion theorem, the covering theorem and a theorem on approximation of measurable functions, appearing for the first time in this general form. The Carathéodori method of constructing a measure from an outer measure along with some extension theorems are also included, with applications to Tonelli and Fubini theorems.

Radon measures are treated in the second chapter which contains also some basic results from topology, including Urysohn and Tychonov theorems. Luzin measurability theorem is proved. The highlight of the chapter is Riesz representation theorem for positive linear functionals on the space of continuous functions with compact support defined on a locally compact Hausdorff space.

The existence, uniqueness and basic properties of Haar invariant measure on a locally compact group are considered in the third chapter.

These first three chapters form Part 1, *Basic Integration Theory*, of the book. The first chapter of Part 2, *Advanced Topics*, is concerned with Lebesgue-Bochner function spaces $L_p(S, \Sigma, \mu)$ and their duals, treated as particular cases of ideal spaces. Orlicz spaces are discussed in exercises.

The fundamental properties of convolutions, a basic tool in harmonic analysis and in approximation theory, are discussed in the fifth chapter. As application one proves an extension of a famous result of H. Steinhaus: if M is a subset of positive measure of a Hausdorff locally compact group S with a left invariant Haar measure, then $M^{-1}M$ is a neighborhood of e. H. Steinhaus (1920) proved the result for $S = \mathbb{R}^n$.

Chapter 6 contains a fine discussion on the connections of some results in measure theory with mathematical logic and set theory. Some famous paradoxes, such as Hausdorff's and Banach-Tarski, are presented along with their consequences for the problem of the existence of finitely additive measures on $\mathbb{R}^n$.

The fundamental results on Lebesgue integration on $\mathbb{R}^n$ – absolutely continuous functions, a.e. differentiability, change of variable formula – are presented in Chapter 7. The last chapter of the book, Chapter 8, is concerned with some useful formulas in Lebesgue integration theory, as ,e.g., the differentiation under integral sign, the change of the order of integration, the Cavalieri principle.

All the notions and results presented in the book are accompanied by comments and examples warning the reader about some delicate points of the subject, or on errors that could be done (or were done). The exercises at the end of each chapter complete the main text with related results and examples.

The result is a fine book on measure theory and integration, based on a general approach to the subject and discussing many difficult topics in the area. It can be recommended for advanced courses in measure theory, but it is suitable also for self-study by graduate students.

V. Anisiu

Vladimir A. Zorich, ***Mathematical Analysis***, Springer Verlag, Berlin-Heidelberg 2004, Vol. I: xviii + 574 pages, ISBN: 3-540-40386-8; Vol. II: xv + 681 pages, ISBN: 3-540-40633-6.

This is the translation of the fourth edition of a well known course on mathematical analysis, taught for several years by the author at the Moscow State University (MSU) and at other universities. Together with V.I. Arnold and S.P. Novikov, the author is one of the organizers of advanced experimental courses at MSU, this experience being reflected in the book too. Written in the good tradition of Russian mathematical textbooks, the present one combines intuition and accessibility with modern mathematical rigor.

The book is divided into two volumes. The main part of the first volume is concerned with the calculus of functions of one variable, developed in the first 6 chapters: 1. *Some logical and mathematical concepts and notation*; 2. *The real number system* (introduced axiomatically); 3. *Limits* (including a treatment of limits with respect to a filter base that are used in several places throughout the book, as e.g. in integration theory); 4. *Continuous functions*; 5. *Differential calculus* (including the calculus of primitives, complex numbers and power series of complex numbers which are used to define e^z); 6. *Integration* (meaning Riemann integration and improper Riemann integrals). Beside the basic theoretical material, these chapters contain many worked examples of applications of the methods of mathematical analysis to other branches of mathematics (as, for instance, a proof of the fundamental theorem of algebra), or from natural and physical sciences (the barometric formula, the motion of a body with variable mass, the falling of a body in atmosphere, radioactive decay, etc).

The last two chapters of the first volume deal with functions of several variables – 7. *Functions of several variables* (continuity questions), and 8. *Differential calculus in several variables.* This last chapter contains some deep results, as the implicit function and inverse function theorems, the tangent space to a k-dimensional surface in $\mathbb{R}^n$ and constrained extrema.

The volume ends with some midterm examination problems, as well as some final examination problems for the first semester (one variable theory) and the second semester (integration and multivariate calculus).

The second volume contains more advanced topics and basically correspond to the second year curriculum in the mathematics departments at MSU. It can be read independently of the first volume, because the first two chapters, 9. *Continuous mappings - General theory* and 10. *Differential calculus from a general viewpoint*, contain in a compressed and generalized form the results on continuity and differentiability from the first volume: basic properties and constructions for metric and topological spaces, continuous mappings, differential calculus for mappings between normed spaces, higher-order differentials, Taylor's formula, and a general implicit function theorem. Multiple Riemann integration and improper multiple Riemann integrals are treated in Chapter 11, *Multiple integrals.* Chapter 12, *Surfaces and differential forms in* $\mathbb{R}^n$, is concerned with surfaces, orientation, area surface and elementary properties of differential forms, preparing the ground for the next two chapters, 13, *Line and surface integrals,* which contains the proofs of the fundamental integral formulas of Green Ostrogradski-Gauss and Stokes, and 14, *Elements of vector analysis.* Chapter 15, *Integration and differential forms on manifolds*, can be viewed as a synthesis at a higher level of abstractization of the topics treated in chapters 11-14. Uniform and pointwise convergence of sequences of functions are treated in Chapter 16, which contains also proofs of the Arzela-Ascoli compactness theorem and of Stone approximation theorem. The integrals depending on a parameter (including improper and multiple integrals) are treated in Chapter 16, with applications to Euler's functions Beta and Gamma. Convolutions and generalized functions are also briefly discussed in this chapter.

The last two chapters of the book are 18, *Fourier series and Fourier transform* and 19, *Asymptotic expansions.*

As the first one, this volume ends also with some midterm and final examination questions.

The bibliography is grouped in four categories: 1. Classical books; 2. Textbooks; 3. Classroom material; 4. Further reading. For the convenience of the readers, some English titles were added for this edition.

There are a lot of exercises and problems, of varying difficulty, spread through the book, needed for a better understanding of the subject, as well as historical notes about the great names who contributed along the centuries to the building of the edifice of mathematical analysis.

This comprehensive course on mathematical analysis provides the readers, first of all students specializing in mathematics, with rigorous proofs of the fundamental theorems, but also with its applications in mathematics itself and outside it. It is correlated with subsequent disciplines relying on its methods and results, as differential equations, differential geometry, functions of a complex variable and functional analysis.

T. Trif