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ON THE BASIC PROPERTIES OF DISCONTINUOUS FLOWS

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Abstract. In this paper, we define discontinuous dynamical systems which can be used as models of various processes in mechanics, electronics, biology and medicine. We find sufficient conditions to guarantee the existence of such systems. These conditions are easy to verify.

1. Introduction and preliminaries

A book [1] edited by D.V. Anosov and V.I. Arnold considers two fundamentally different Dynamical Systems (DSs) : flows and cascades. Roughly speaking, flows are DSs with continuous time and cascades are DSs with discrete time. One of the most important theoretical problem is to consider *Discontinuous Dynamical Systems* ($DDSs$). That is systems whose trajectories are piecewise continuous curves. It is well-recognized (for example, see [2]) that the general notion of such systems was introduced by Th. Pavlidis [3], although particular examples (the mathematical model of clock [4]-[6] and so on) had been discussed before. Some basic elements of the theory are given in [7]-[11]. Analysing the behavior of the trajectories we can conclude that $DDSs$ combine features of vector fields and maps, they can not be reduced to flows or cascades, but are close to flows since time is continuous. That is why we propose to call them also *Discontinuous Flows* (DFs). Applications of $DDSs$ in mechanics, electronics, biology and medicine were considered in [3], [12] - [15]. Chaotic behavior of discontinuous processes was investigated in [13, 16]. One must emphasize that DFs are not *differential equations with discontinuous right side* which often have

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been accepted as *DDSs* [17]. However, theoretical problems of nonsmooth dynamics and discontinuous maps [18]-[25] are also very close to the subject of our paper. One should also agree that *nonautonomous impulsive differential equations*, which were thoroughly described in [8] and [11], are not *DFs*.

The paper embodies results that provide sufficient conditions for the existence of a *differentiable DF*. Since *DFs* have specific smoothness of solutions we call these systems *B-differentiable DFs.* Apparently, it is the first time when notions of $B-$ continuous and $B-$ differentiable dependence of solutions on initial values [27] are applied to described *DDSs* and sufficient conditions for the continuation of solutions and the group property are obtained. A central auxiliary result of the paper is the construction of a new form of the general autonomous impulsive equation (system (1)). Effective methods of investigation of systems with variable time of impulsive actions were considered in [8, 11], [27]- [31].

Let $\mathbb{Z}, \mathbb{N}$ and $\mathbb{R}$ be the sets of all integers, natural and real numbers, respectively. Denote by $||\cdot||$ the Euclidean norm in $\mathbb{R}^n$, $n \in \mathbb{N}$. Consider a set of strictly ordered real numbers $\{\theta_i\}$, where the set $\mathcal{A}$ of indices is an interval of $\mathbb{Z}/\{0\}$.

Definition 1.1. *The set* $\{\theta_i\}$ *is said to be a sequence of* $\beta-$ *type if the product* $i\theta_i, \quad i \geq 0$ *for all* i *and one of the following alternative cases holds:*

(a) $\{\theta_i\} = \emptyset$*;*
(b) $\{\theta_i\}$ *is a finite and nonempty set;*
(c) $\{\theta_i\}$ *is an infinite set such that* $|\theta_i| \to \infty$ *as* $|i| \to \infty$.

From the definition, it follows immediately that a sequence of $\beta-$ *type* does not have a finite accumulation point in $\mathbb{R}$.

Definition 1.2. *A function* $\varphi : \mathbb{R} \longrightarrow \mathbb{R}^n$ *is said to be from a space* $\mathcal{PC}(\mathbb{R})$ *if*

1. $\varphi(t)$ *is left continuous on* $\mathbb{R}$;
2. *there exists a sequence* $\{\theta_i\}$ *of* $\beta-$ *type such that* φ *is continuous if* $t \neq \theta_i$ *and* φ *has discontinuities of the first kind at the points* θ_i.

Particularly, $C(\mathbb{R}) \subset \mathcal{PC}(\mathbb{R})$.

Definition 1.3. *A function* $\varphi(t)$ *is said to be from a space* $\mathcal{PC}^1(\mathbb{R})$ *if* $\varphi' \in \mathcal{PC}(\mathbb{R})$.

Let T be an interval in $\mathbb{R}$.

Definition 1.4. *We denote by $\mathcal{PC}(T)$ and $\mathcal{PC}^1(T)$ the sets of restrictions of all functions from $\mathcal{PC}(\mathbb{R})$ and $\mathcal{PC}^1(\mathbb{R})$ on T respectively.*

Let G be an open subset of $\mathbb{R}^n$, G_r be an $r-$ neighbourhood of G in $\mathbb{R}^n$ for a fixed $r > 0$ and $\hat{G} \subset G_r$ be an open subset of $\mathbb{R}^n$. Denote as $\Phi : \hat{G} \longrightarrow \mathbb{R}$ be a function from $C^1(\hat{G})$ and assume that a surface $\Gamma = \Phi^{-1}(0)$ is a subset of $\bar{G}$, where $\bar{G}$ denotes the closure of the set G in $\mathbb{R}^n$. Moreover, define a function $J : \Gamma_r \to \bar{G}$, where Γ_r is an $r-$ neighbourhood of Γ. We shall need the following assumptions.

C1) $\nabla\Phi(x) \neq 0$, $\forall x \in \Gamma$;
C2) $J \in C^1(\Gamma_r), \det[\frac{\partial J(x)}{\partial x}] \neq 0$, for all $x \in \Gamma$.

One can see that the restriction $J|_\Gamma$ is a one-to-one function. Let also $\tilde{\Gamma} = J(\Gamma)$, $\tilde{\Gamma} \subset \bar{G}$. If $\tilde{\Phi}(x) = \Phi(J^{-1}(x))$, $x \in \tilde{\Gamma}$ then $\tilde{\Gamma} = \left\{x \in G|\, \tilde{\Phi}\,(x) = 0\right\}$. It is easy to verify that $\nabla\tilde{\Phi}(x) \neq 0$, $\forall x \in \tilde{\Gamma}$.

Consider the following impulsive differential equation in the domain $D = \left[G \cup \Gamma \cup \tilde{\Gamma}\right] \setminus \left[(\bar{\Gamma}\setminus\Gamma) \cup \left(\bar{\tilde{\Gamma}}\setminus\tilde{\Gamma}\right)\right]$

$$
\begin{aligned}
&x'(t) = f(x(t)), \{x(t) \notin \Gamma \wedge t \geq 0\} \vee \{x(t) \notin \tilde{\Gamma} \wedge t \leq 0\},\\
&x(t+)|_{x(t-)\in\Gamma\wedge t\geq 0} = J(x(t-)),\\
&x(t-)|_{x(t+)\in\tilde{\Gamma}\wedge t\leq 0} = J^{-1}(x(t+)).
\end{aligned}
\tag{1}
$$

We make the following assumptions which will be needed throughout the paper.

C3) $f \in C^1(G_r)$.
C4) $\Gamma \cap \tilde{\Gamma} = \emptyset$, $\Gamma \cap \left(\bar{\tilde{\Gamma}}\setminus\tilde{\Gamma}\right) = \emptyset$, $\left(\bar{\Gamma}\setminus\Gamma\right) \cap \tilde{\Gamma} = \emptyset$.
C5) $\langle\nabla\Phi(x), f(x)\rangle \neq 0$ if $x \in \Gamma$.
C6) $\left\langle\nabla\tilde{\Phi}(x), f(x)\right\rangle \neq 0$ if $x \in \tilde{\Gamma}$.

2. Main results

Definition 2.1. *A function* $x(t) \in \mathcal{PC}^1(T)$ *with a set of discontinuity points* $\{\theta_i\} \subset T$ *is said to be a solution of (1) on the interval* $T \subset \mathbb{R}$ *if it satisfies the following conditions:*

(i) *equation (1) is satisfied at each point* $t \in T \backslash \{\theta_i\}$ *and* $x'(\theta_i-) = f(x((\theta_i))), i \in \mathcal{A}$, *where* $x'(\theta_i-)$ *is the left-sided derivative;*

(ii) $x(\theta_i+) = J(x((\theta_i))$ *for all* θ_i.

Theorem 2.1. *Assume that conditions* $C1) - C6)$ *hold. Then for every* $x_0 \in D$ *there exists an interval* $(a, b) \subset \mathbb{R}, a < 0 < b$, *such that the solution* $x(t) = x(t, 0, x_0)$ *of (1) exists on the interval.*

Definition 2.2. *A solution* $x(t) : [a, \infty) \to \mathbb{R}^n, a \in \mathbb{R}$, *of (1) is said to be continuable to* ∞.

Definition 2.3. *A solution* $x(t) : (-\infty, b] \to \mathbb{R}^n, b \in \mathbb{R}$, *of (1) is said to be continuable to* $-\infty$.

Definition 2.4. *A solution* $x(t)$ *of (1) is said to be continuable on* $\mathbb{R}$ *if it is continuable to* ∞ *and to* $-\infty$.

Definition 2.5. *A solution* $x(t) = x(t, 0, x_0)$ *of (1) is said to be continuable to a set* $S \subset \mathbb{R}^n$ *as time decreases (increases) if there exists a moment* $\xi \in \mathbb{R}$, *such that* $\xi \leq 0\, (\xi \geq 0)$ *and* $x(\xi) \in S$.

Denote by $B(x_0, \xi) = \{x \in \mathbb{R}^n | ||x - x_0|| < \xi\}$ a ball with centre $x_0 \in \mathbb{R}^n$ and radius $\xi \in \mathbb{R}$.

The following Theorem provides sufficient conditions for the continuation of solutions of (1).

Theorem 2.2. *Assume that*

(a) *every solution* $y(t, 0, x_0), x_0 \in D$, *of*

$$y' = f(y). \tag{2}$$

satisfies the following conditions:

(a1) *it is continuable either to* ∞ *or to* Γ *as time increases,*

(a2) *it is continuable either to* $-\infty$ *or to* $\tilde{\Gamma}$ *as time decreases;*

(b) *for every* $x \in \tilde{\Gamma}$ *there exists a number* $\epsilon_x > 0$ *such that* $\bar{B}(x, \epsilon_x) \cap \Gamma = \emptyset$;

b′) *for every* $x \in \Gamma$ *there exists a number* $\tilde{\epsilon}_x > 0$ *such that* $\bar{B}(x, \tilde{\epsilon}_x) \cap \tilde{\Gamma} = \emptyset$;

(c) $\inf_{(x,\epsilon_x)\in\tilde{\Gamma}\times\mathbb{R}} \frac{\epsilon_x}{\sup_{\bar{B}(x,\epsilon_x)}\|f(x)\|} > 0$;

c′) $\inf_{(x,\tilde{\epsilon}_x)\in\Gamma\times R} \frac{\tilde{\epsilon}_x}{\sup_{B(x,\tilde{\epsilon}_x)}\|f(x)\|} > 0.$

Then every solution $x(t) = x(t, 0, x_0)$, $x_0 \in D$, *of (1) is continuable on* $\mathbb{R}$.

Consider a solution $x(t) : \mathbb{R} \to \mathbb{R}^n$ of (1). Let $\{\theta_i\}$ be the sequence of discontinuity points of $x(t)$. Fix $\theta \in \mathbb{R}$ and introduce a function $\psi(t) = x(t + \theta)$.

Lemma 2.1. *The set* $\{\theta_i - \theta\}$ *is a set of all solutions of the equation*

$$\Phi(\psi(t)) = 0. \tag{3}$$

The following condition is one of the main assumptions for DFs.

$C7$) $\Gamma, \tilde{\Gamma} \subset \partial G$;

$\exists\epsilon > 0$ such that $\forall x \in \Gamma_\epsilon \cap G$ function $\Phi(x)$ is either positive or negative;

$\exists\epsilon > 0$ such that $\forall x \in \tilde{\Gamma}_\epsilon \cap G$ function $\tilde{\Phi}(x)$ is either positive or negative.

Lemma 2.2. *Assume that* $C1) - C7)$ *hold. Then* $x(-t, 0, x(t, 0, x_0)) = x_0$ *for all* $x_0 \in D, t \in \mathbb{R}$.

Lemma 2.3. *If* $x(t) : T \to \mathbb{R}^n$ *is a solution of (1) then* $x(t + \theta), \theta \in \mathbb{R}$, *is also a solution of (1).*

Lemmas 2.1-2.3 imply that the following theorem is valid.

Theorem 2.3. *Assume that conditions* $C1) - C7)$ *are fulfilled. Then*

$$x(t_2, x(t_1, x_0)) = x(t_2 + t_1, x_0), \tag{4}$$

for all $t_1, t_2 \in \mathbb{R}$.

Let $x^0(t) : [a, b] \to \mathbb{R}^n, a \le 0 \le b$, be a solution of (1), $x^0(t) = x(t, 0, x_0), \theta_i, i = -k, \ldots, -1, 1, \ldots, m$, are the points of discontinuity of $x^0(t)$, such that $a \leq \theta_{-k} < \cdots < \theta_{-1} \leq 0 \leq \theta_1 < \cdots < \theta_m \leq b$. Denote by $x(t) = x(t, 0, \bar{x})$ another solution of (1).

Definition 2.6. *The solution $x(t) : [a, b] \to \mathbb{R}^n$ is said to be in an ϵ-neighbourhood of $x^0(t)$ if:*

1. *every point of discontinuity of $x(t)$ lies in an ϵ-neighbourhood of a point of discontinuity of $x^0(t)$;*
2. *For each $t \in [a, b]$ which is outside of ϵ-neighbourhood of points of discontinuity of $x^0(t)$ the inequality $\left\|x^0(t) - x(t)\right\| < \epsilon$ holds.*

Definition 2.7. *Hausdorff's topology, which is built on the basis of all ϵ-neighbourhoods, $0 < \epsilon < \infty$, of piecewise solutions will be called $B_{[a,b]}$-topology.*

Theorem 2.4. *Assume that conditions $C1) - C7)$ are satisfied. Then the solution $x(t)$ continuously depends on initial value in $B_{[a,b]}$ topology .*
Moreover, if all $\theta_i, i = -k, \ldots, -1, 1, \ldots, m$, are interior points of $[a, b]$, then, for sufficiently small $||x_0 - \bar{x}||$, the solution $x(t) = x(t, 0, \bar{x}), x(t) : [a, b] \to \mathbb{R}^n$, meets the surface Γ exactly $m + k - 1$ times.

Without loss of generality, assume that all points of discontinuity of $x^0(t)$ are interior. Denote by $x_j(t), j = \overline{1, n}$, a solution of (1) such that $x_j(t_0) = x_0 + \xi e_j = (x_1^0, x_2^0, \ldots, x_{j-1}^0, x_j^0 + \xi, x_{j+1}^0, \ldots, x_n^0), \xi \in \mathbb{R}$, $(t_0, x_0 + \xi e_j, \mu_0) \in C_0(\delta)$ and let θ_i^j be the moments of discontinuity of $x_j(t)$. By Theorem 2.4, for sufficiently small $|\xi|$ the solution $x_j(t)$ is defined on $[a, b]$.

Definition 2.8. *The solution $x^0(t)$ is said to be differentiable in $x_j^0, j = \overline{1, n}$, if*
A) there exist such constants $\nu_{ij}, i = -k, \ldots, -1, 1, \ldots, m$, that

$$\theta_i^j - \theta_i = \nu_{ij}\xi + o(|\xi|); \tag{5}$$

B) for all $t \in [a, b] \setminus \cup_{i=-k}^{m} \widehat{(\theta_i, \theta_i^j]}$, the following equality is satisfied

$$x_j(t) - x^0(t) = u_j(t)\xi + o(|\xi|), \tag{6}$$

where $u_j(t)$ is a piecewise continuous function, with discontinuities of the first kind at the points $t = \theta_i, i = -k, \ldots, -1, 1, \ldots, m$.

The pair $\{u_j, \{\nu_{ij}\}_i\}$ *is said to be a* $B-$ *derivative of* $x^0(t)$ *in initial value* x_0^j *on* $[a,b]$.

The following theorem is valid.

Theorem 2.5. *Assume that conditions* $C1) - C7)$ *are satisfied. Then the solution* $x^0(t)$ *of (1) has* $B-$ *derivatives in the initial value on* $[a,b]$.

3. The B-smooth discontinuous flow

Let $G \subset \mathbb{R}^n$ be an open set and $\Gamma, \tilde{\Gamma}$ be disjoint subsets of $\bar{G}$. Denote $D = G \cup \Gamma \cup \tilde{\Gamma}$.

Definition 3.1. *We say that a* $B-$ *smooth DF is a map* $\phi : \mathbb{R} \times D \to D$, *which satisfies the following properties:*

$I)$ *The group property:*

(i) $\phi(0,x) : D \to D$ *is the identity;*

(ii) $\phi(t, \phi(s,x)) = \phi(t+s, x)$, *is valid for all* $t, s \in \mathbb{R}$ *and* $x \in D$.

$II)$ *If* $x \in D$ *is fixed then* $\phi(t,x) \in \mathcal{PC}^1(\mathbb{R})$, *and* $\phi(\theta_i, x) \in \Gamma, \phi(\theta_i+, x) \in \tilde{\Gamma}$ *for every discontinuity point* θ_i *of* $\phi(t,x)$.

$III)$ *The function* $\phi(t,x)$ *is* $B-$ *differentiable in* $x \in D$ *on* $[a,b] \subset \mathbb{R}$ *for every* $\{a,b\} \subset \mathbb{R}$, *assuming that all discontinuity points of* $\phi(t,x)$ *are interior points of* $[a,b]$.

One can see that the system (1) defines a $B-$ smooth DF provided conditions $C1) - C7)$ and the conditions of the continuation theorem are fulfilled.

Definition 3.2. *We say that a DF is a map* $\phi : \mathbb{R} \times D \to D$, *which satisfies the property* $I)$ *of Definition 3.1 and the following conditions are valid:*

$IV)$ *If* $x \in D$ *is fixed then* $\phi(t,x) \in \mathcal{PC}(\mathbb{R})$, *and* $\phi(\theta_i, x) \in \Gamma, \phi(\theta_i+, x) \in \tilde{\Gamma}$ *for every discontinuity point* θ_i *of* $\phi(t,x)$.

$V)$ *The function* $\phi(t,x)$ *is* $B-$ *continuous in* $x \in D$ *on* $[a,b] \subset \mathbb{R}$ *for every* $\{a,b\} \subset \mathbb{R}$.

Comparing definitions of the $B-$ differentiability and the $B-$ continuity one can conclude that every $B-$ smooth DF is a DF.

Example 3.1. *Consider the following model for simple neural nets from* [3]. *We have modified its form according to the proposed equation (1).*

$$x_1' = x_2,\, x_2' = -\beta^2 x_1,\, p' = -\gamma p + x_1 + B_0, if\ (x(t) \notin \Gamma \wedge t \geq 0) \vee (x(t) \notin \tilde{\Gamma} \wedge t \leq 0),$$

$$x_1(t+) = x_2(t-),\, x_2(t+) = x_2(t-),\, p(t+) = 0, if\ x(t) \in \Gamma \wedge t \geq 0,$$

$$x_1(t-) = x_1(t+),\, x_2(t-) = x_2(t+),\, p(t-) = r, if\ x(t) \in \tilde{\Gamma} \wedge t \leq 0,$$

where $\beta, B_0 \in \mathbb{R}$ *are constants,* $\Gamma = \{(x_1, x_2, p)|\, p = r\}, \tilde{\Gamma} = \{(x_1, x_2, p)|\, p = 0\}, \Phi(x) = p - r,\ f(x) = (x_2, \beta^2 x_1, -\gamma p + x_1 + B_0), J(x) = (x_1, x_2, r), \beta, \gamma, r > 0,$ *are constants. We assume that* $G = \{(x_1, x_2, p)|0 < p < r, x_1^2 + \frac{x_2^2}{\beta^4} < 1\}$. *In the system the variable* $p(t)$ *is a scalar input of a neural trigger and* x_1, x_2, *are other variables. The value of* r *is the threshold. One can verify that the functions and the sets satisfy* $C1) - C7)$ *and the conditions of Theorem 2.2. That is, the system defines a DF.*

Remark 3.1. *The extended version of the paper has been submitted to Mathematical and Computer Modelling.*

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TWO- AND THREE-DIMENSIONAL INVERSE PROBLEM OF DYNAMICS

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Abstract. For a given a monoparametric family of curves $f(x,y)=c$, we present the partial differential equations satisfied by the potentials $V = V(x,y)$ under whose action a particle of unit mass can describe the curves of the family. Szebehely's equation depends on the total energy of the particle, while Bozis' one relates merely the potential and the given family. Therefore the last one is also adequate for the direct problem of dynamics. A similar program is accomplished for a two-parametric spatial family of curves $\varphi(x,y,z)=c_1$, $\psi(x,y,z)=c_2$ and potentials $\mathcal{V}=\mathcal{V}(x,y,z)$.

1. **Introduction**

The first result concerning the inverse problem of dynamics is due to Newton [24], who presented the form of the gravitational potential on the basis of Kepler's laws. Kepler has had at his disposal the very accurate tables of observations made by Tycho Brache (whose assistant he was in Prague); these observations allowed him to discover that the orbit of Mars is an ellipse and to formulate the three laws of planetary motion.

Later on, Bertrand [7] showed that Kepler's first law suffices to derive the Newtonian universal force; Dainelli [18] obtained the expressions of general force fields producing given planar or spatial families of curves.

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The two-dimensional problem, this time for conservative systems, has renewed the interest in the inverse problem of dynamics by means of Szebehely's [29] partial differential equation. This equation relates the potential to the given monoparametric family of curves and to the total energy. Puel [26] derived a Szebehely-type equation which is independent of the coordinate system. Another basic result for the two-dimensional inverse problem is the energy-free partial differential equation obtained by Bozis [9] from Szebehely's equation, and later derived directly by Anisiu [3].

The conservative three-dimensional problem was considered by Érdi [19] for a monoparametric family of orbits, and then for two-parametric families by Váradi and Érdi [30]. Puel [25] used the least action principle of Maupertuis to obtain the equations satisfied by the potential in the two- and three-dimensional inverse problem of dynamics. The existence of such a potential and its relation with the energy in the three-dimensional case was subject to further papers, as those of Gonzales-Gascon et al [21], Bozis and Nakhla [15] and Shorokhov [28]. Puel [27] obtained the intrinsic equations of the three-dimensional inverse problem, using the Frenet reference frame. A review of the basic results in the inverse problem of dynamics, including the three-dimensional ones, can be found in [10].

2. The planar inverse problem of dynamics

We consider the following version of the inverse problem for one material point of unit mass, moving in the xy inertial Cartesian plane. Given a family of curves

$$f\left(x,y\right)=c \tag{1}$$

with f of C^3-class (continuous and with continuous derivatives up to third order on a domain of the plane), find the potentials $V(x,y)$ under whose action, for appropriate initial conditions, the particle will describe the curves of that family. The equations of motion are

$$\ddot{x}=-V_x \qquad \ddot{y}=-V_y, \tag{2}$$

where the dots denote derivatives with respect to the time t and the subscripts partial derivatives. By making use of the energy integral, Szebehely [29] proved that the potential V is a solution of the first order partial differential equation

$$f_x V_x + f_y V_y + \frac{2\,(V - E(f))}{f_x^2 + f_y^2}\left(f_{xx} f_y^2 - 2 f_{xy} f_x f_y + f_{yy} f_x^2\right) = 0, \tag{3}$$

where $E(f)$ denotes the total energy, which is constant on each curve of the family (1). Bozis [8] wrote Szebehely's equation in the simpler form

$$V_x + \gamma V_y + \frac{2\Gamma\,(E(f) - V)}{1 + \gamma^2} = 0, \tag{4}$$

making use of the functions

$$\gamma = \frac{f_y}{f_x} \quad \text{and} \quad \Gamma = \gamma\gamma_x - \gamma_y \tag{5}$$

related to the geometry of the family (γ representing the slope and Γ being proportional to the curvature). By eliminating the energy from (4) (using the fact that $E_y/E_x = f_y/f_x$) Bozis [9] obtained the energy-free equation of second order

$$-V_{xx} + \kappa V_{xy} + V_{yy} = \lambda V_x + \mu V_y, \tag{6}$$

where

$$\kappa = \frac{1}{\gamma} - \gamma,\ \lambda = \frac{\Gamma_y - \gamma\Gamma_x}{\gamma\Gamma},\ \mu = \lambda\gamma + \frac{3\Gamma}{\gamma}. \tag{7}$$

The basic equations (4) and (6) of the planar inverse problem of dynamics present the connection between geometry and dynamics. Their derivation and other related results are exposed in [10], [2], [1], [3].

Szebehely obtained the first order equation intending to determine the potential of the earth by means of satellite observations, while Bozis used equation (6) to check if a given family of orbits may be generated in the plane of symmetry outside a material concentration.

2.1. **Basic tools.** Let us consider a particle whose motion is described by equations (2), where V is of C^2-class on a domain of the xy plane. We shall use a procedure exposed by Anisiu [3], related to that followed by Kasner [22] while he has obtained the differential equation of the trajectories corresponding to a general (not necessarily conservative) force field. By differentiating (1) with respect to t we get $f_x\dot{x}+f_y\dot{y}=0$, or, using notation (5),

$$\gamma = -\frac{\dot{x}}{\dot{y}}. \tag{8}$$

By differentiating (8) we get

$$-\Gamma = \frac{\dot{x}\ddot{y} - \dot{y}\ddot{x}}{\dot{y}^3}. \tag{9}$$

Inserting in (9) $\ddot{x}$ and $\ddot{y}$ from (2), and $\dot{x}$ from (8) we obtain

$$\Gamma\dot{y}^2 = -\left(V_x + \gamma V_y\right).$$

The function Γ is related to the curvature K of the family (1) by $K = |\Gamma| / \left(\gamma^2+1\right)^{3/2}$. It follows that $\Gamma = 0$ if and only if the family (1) contains only straight lines. In this case, which was studied in [11], we have by necessity

$$V_x + \gamma V_y = 0, \tag{10}$$

which represents Szebehely's equation for this special case. The straight lines are traced with arbitrary energy.

Let us consider now a general family (1) with $\Gamma \neq 0$. In this case we have

$$\dot{y}^2 = -\frac{V_x + \gamma V_y}{\Gamma}. \tag{11}$$

We differentiate (9), divide both members by $\dot{y}$ and get

$$\gamma\Gamma_x - \Gamma_y = \frac{\dot{y}\left(\dot{x}\dddot{y} - \dot{y}\dddot{x}\right) - 3\ddot{y}\left(\dot{x}\ddot{y} - \dot{y}\ddot{x}\right)}{\dot{y}^5}. \tag{12}$$

We remark that (8), (9) and (12) express the relations between the geometry of the family of curves (1) and the kinematics derivatives.

Two additional equations are obtained by differentiating equations (2) with respect to t, namely

$$\begin{aligned} \dddot{x} &= -\left(V_{xx}\dot{x} + V_{xy}\dot{y}\right) \\ \dddot{y} &= -\left(V_{xy}\dot{x} + V_{yy}\dot{y}\right). \end{aligned} \tag{13}$$

Now we eliminate the derivatives $\dot{x}, \dot{y}, \ddot{x}, \ddot{y}, \dddot{x}, \dddot{y}$ between the seven relations in (2), (8), (11), (12) and (13), and get the partial differential equation

$$\Gamma\left(-\gamma V_{xx} + V_{xy} - \gamma^2 V_{xy} + \gamma V_{yy}\right) = -\left(V_x + \gamma V_y\right)\left(\gamma\Gamma_x - \Gamma_y\right) + 3V_y\Gamma^2. \tag{14}$$

We divide both members of (14) by $\gamma\Gamma$ and obtain Bozis' equation (6), with λ and μ given in (7).

A straightforward calculation shows that equation (6) can be written as

$$\gamma W_x - W_y = 0, \tag{15}$$

where

$$W = V - \frac{1+\gamma^2}{2\Gamma}\left(V_x + \gamma V_y\right). \tag{16}$$

Equation (15) has the general solution $W = E(f)$, where E denotes an arbitrary function. It follows that

$$V - \frac{1+\gamma^2}{2\Gamma}\left(V_x + \gamma V_y\right) = E(f). \tag{17}$$

In view of relations (2), (8) and (9) we obtain

$$V + \frac{\dot{x}^2 + \dot{y}^2}{2} = E(f), \tag{18}$$

which means that $E(f)$ represents the total energy, constant on each curve of the family (1). Therefore equation (17), obtained this time from Bozis' equation, is in fact Szebehely's equation. From (18) we obtain $E(f) - V \geq 0$, and from (17) it follows that only the curves of the family (1) or parts of them which are situated in the plane region

$$\frac{V_x + \gamma V_y}{\Gamma} \leq 0 \tag{19}$$

can be described by the unit mass particle. Inequality (19) was obtained by Bozis and Ichtiaroglou [12].

Remark 1. *Bozis* [10] *arranged equation (6) in a form adequate for the direct problem of dynamics, namely*

$$\gamma^2\gamma_{xx} - 2\gamma\gamma_{xy} + \gamma_{yy} = h, \tag{20}$$

where

$$h = \frac{\gamma\gamma_x - \gamma_y}{V_y\gamma + V_x}\left(-\gamma_x V_x + (2\gamma\gamma_x - 3\gamma_y)\, V_y + \gamma\,(V_{xx} - V_{yy}) + (\gamma^2 - 1)V_{xy}\right). \tag{21}$$

Relations (20)-(21) have been used to find families of curves satisfying auxiliary conditions, supposing that a potential is given, in [16], [17], [6].

2.2. **Examples.**

Example 2. *From the class of Hénon-Heiles potentials*

$$V(x, y) = ax^2 + by^2 + cx^2y + dy^3 \tag{22}$$

with $a, b, c, d \in \mathbb{R},\ a, b > 0$, *Anisiu and Pal* [5] *looked for those compatible with the family of polytropic curves* $f(x, y) = x^{-p}y$, *where* $p \in \mathbb{Z} \setminus \{0, 1\}$. *The potential*

$$V_1(x, y) = a\left(x^2 + 16y^2\right) + c\left(x^2 + (16/3)y^2\right)y$$

was found to generate the family $f_1(x, y) = x^{-4}y$ *in the region described by* $y(cx^2 + 8cy^2 + 24ay) \le 0$, *with the energy* $E_1(f_1) = -c/(24f_1)$. *Another potential is*

$$V_2(x, y) = a\left(x^2 + 4y^2\right) + dy^3,$$

which produces the family $f_2(x, y) = x^2y$ *in the region* $dy + 4a \le 0$, *with the energy* $E_2(f_2) = -df_2/4$.

It was shown in [11] *that no potential of the form (22) allows for families of straight lines.*

Example 3. *For the family* $f = y - 1/x^2$, *the potential*

$$V(x, y) = 8y^2 + 4x^2y - x^8 - 6x^2$$

was found in [17]. *The particle describes the curves of the given family in the region* $y \le x^4 + 1/(2x^2)$ *with the energy* $E(f) = 8f^2$.

3. The three-dimensional inverse problem

We consider the three-dimensional family of curves

$$\varphi(x,y,z) = c_1, \quad \psi(x,y,z) = c_2. \tag{23}$$

with φ, ψ of C^3-class and with

$$\begin{vmatrix} \varphi_y & \varphi_z \\ \psi_y & \psi_z \end{vmatrix} \neq 0. \tag{24}$$

We can suppose that any other determinant (containing derivatives with respect to x and y, or to x and z) is different from zero, and proceed accordingly.

We deal with the following version of the inverse problem: find the potentials $\mathcal{V}(x,y,z)$ under whose action, for appropriate initial conditions, a material point of unit mass, whose motion is described by

$$\ddot{x} = -\mathcal{V}_x \quad \ddot{y} = -\mathcal{V}_y \quad \ddot{z} = -\mathcal{V}_z, \tag{25}$$

will trace the curves of the family (23). The partial differential equations satisfied by $\mathcal{V}$ will be derived as in [4], where the geometrical methods used by Kasner [23] were adapted to this problem.

3.1. **Basic tools.** In order to obtain the equations satisfied by $\mathcal{V}$, we differentiate both sides of equations (23) with respect to t, and get

$$\frac{\dot{y}}{\dot{x}} = \alpha, \quad \frac{\dot{z}}{\dot{x}} = \beta, \tag{26}$$

where

$$\alpha = \frac{\varphi_z\psi_x - \varphi_x\psi_z}{\varphi_y\psi_z - \varphi_z\psi_y}, \quad \beta = \frac{\varphi_x\psi_y - \varphi_y\psi_x}{\varphi_y\psi_z - \varphi_z\psi_y}. \tag{27}$$

We remark that at least one of the functions α and β, say α, is not identically null (otherwise condition (24) fails to be fulfilled).

The notation (27) was introduced by Bozis and Kotoulas [13], where it was emphasized that the family (23) leads to a unique pair α, β and, conversely, the pair α, β determines uniquely the family (23).

We differentiate both relations in (26) and get

$$\frac{\dot{x}\ddot{y}-\ddot{x}\dot{y}}{\dot{x}^3}=A, \quad \frac{\dot{x}\ddot{z}-\ddot{x}\dot{z}}{\dot{x}^3}=B, \tag{28}$$

where

$$A=\alpha_x+\alpha\alpha_y+\beta\alpha_z, \quad B=\beta_x+\alpha\beta_y+\beta\beta_z. \tag{29}$$

Using (26) and equations (25), we obtain from (28)

$$\frac{\alpha\mathcal{V}_x-\mathcal{V}_y}{\dot{x}^2}=A, \quad \frac{\beta\mathcal{V}_x-\mathcal{V}_z}{\dot{x}^2}=B. \tag{30}$$

We have to analyze the special case when $A=B=0$. It is obvious that, in view of relation (28), it follows that also $\dot{y}\ddot{z}-\ddot{y}\dot{z}=0$, hence the curvature $K=|\dot{\overline{r}}\times\ddot{\overline{r}}|/|\dot{\overline{r}}|^3$ of each member of the family (23) vanishes. We have denoted by $\overline{r}=x\,(t)\,\overline{i}+y\,(t)\,\overline{j}+z\,(t)\,\overline{k}$, where $\overline{i}$, $\overline{j}$, $\overline{k}$ are unit vectors along the axes Ox, Oy, Oz.

It follows that we have $A=B=0$ if and only if the family (23) consists of straight lines. This case was analyzed in detail in [13]. Relations (30) give rise to two linear partial differential equations to be necessarily satisfied by $\mathcal{V}$, namely

$$\alpha\mathcal{V}_x-\mathcal{V}_y=0, \quad \beta\mathcal{V}_x-\mathcal{V}_z=0. \tag{31}$$

These equations will admit of a solution only if α and β satisfy, besides the two equations obtained from (29) for $A=B=0$, a supplementary equation (see [20])

$$\alpha\beta_x-\beta\alpha_x=\beta_y-\alpha_z. \tag{32}$$

So, generally, the inverse problem is not expected to have a solution for arbitrary families of straight lines.

Let us consider now $A\neq 0$ and $B\neq 0$. By eliminating $\dot{x}^2$ between the two relations in (30) we obtain a first necessary condition to be satisfied by $\mathcal{V}$,

$$\frac{\alpha\mathcal{V}_x-\mathcal{V}_y}{A}=\frac{\beta\mathcal{V}_x-\mathcal{V}_z}{B}, \tag{33}$$

where α,β from (27) and A,B from (29) depend on the derivatives of φ and ψ up to the second order. Because of $\dot{x}^2\geq 0$, it follows that the motion is possible only in the region determined by

$$\frac{\alpha\mathcal{V}_x-\mathcal{V}_y}{A}\geq 0. \tag{34}$$

Differentiating both members of the equality $\dot{x}^2 = (\alpha\mathcal{V}_x - \mathcal{V}_y)/A$ with respect to t and replacing $\ddot{x}$ from the first equation in (25), respectively $\dot{y}/\dot{x}$ and $\dot{z}/\dot{x}$ from (26), we obtain a second differential relation to be satisfied by $\mathcal{V}$

$$-\mathcal{V}_{xx} + k\mathcal{V}_{xy} + \mathcal{V}_{yy} + p\mathcal{V}_{yz} + q\mathcal{V}_{xz} = l\mathcal{V}_x + m\mathcal{V}_y, \tag{35}$$

where

$$\begin{gathered} k = \frac{1}{\alpha} - \alpha,\ p = \frac{\beta}{\alpha},\ q = -\beta \\ l = \frac{3A}{\alpha} - \alpha m,\ m = \frac{A_x + \alpha A_y + \beta A_z}{\alpha A}. \end{gathered} \tag{36}$$

Summarizing the above reasoning, we assert that a potential which produces as orbits the curves of the family (23) satisfies by necessity the two differential relations (33) and (35), the motion of the particle being possible in the region determined by inequality (34). We remark that equation (35) is of second order in $\mathcal{V}$ and does not involve the energy (constant on each curve of the family), hence it is the corresponding for the three-dimensional case of Bozis' equation (6) satisfied by planar potentials.

In the following we shall derive the equation from which the total energy can be expressed. Denoting by

$$\mathcal{W} = \left(1 + \alpha^2 + \beta^2\right)\frac{\alpha\mathcal{V}_x - \mathcal{V}_y}{2A} + \mathcal{V}, \tag{37}$$

one can check by direct calculation that (35) is equivalent to

$$\mathcal{W}_x + \alpha\mathcal{W}_y + \beta\mathcal{W}_z = 0. \tag{38}$$

The characteristic system for (38) is

$$\frac{dx}{\varphi_y\psi_z - \varphi_z\psi_y} = \frac{dy}{\psi_x\varphi_z - \varphi_x\psi_z} = \frac{dz}{\varphi_x\psi_y - \varphi_y\psi_x}$$

and one obtains easily that $\varphi_x dx + \varphi_y dy + \varphi_z dz = 0$ and $\psi_x dx + \psi_y dy + \psi_z dz = 0$. It follows that $\varphi(x, y, z) = c_1$ and $\psi(x, y, z) = c_2$ are integrals, hence the general solution of (38) is $\mathcal{W} = \mathcal{E}(\varphi, \psi)$ with $\mathcal{E}$ an arbitrary function.

In view of relations (26) and (30), we get from (37) that

$$\mathcal{E}(\varphi, \psi) = \left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right)/2 + \mathcal{V}, \tag{39}$$

i.e. $\mathcal{W} = \mathcal{E}(\varphi, \psi)$ is the total energy, constant on each curve of the family (23). It follows that the equation

$$\mathcal{E}(\varphi, \psi) = \left(1 + \alpha^2 + \beta^2\right) \frac{\alpha \mathcal{V}_x - \mathcal{V}_y}{2A} + \mathcal{V}, \tag{40}$$

which was derived by Váradi and Érdi [30] using the energy integral (and which corresponds to Szebehely's planar equation), can be obtained as a consequence of the second order partial differential equation (35).

The two equations (33) and (35) for a single unknown function $\mathcal{V}$ will not have always a solution; compatibility conditions are to be checked. The advantage of this formulation consists in the fact that it is free of energy.

Remark 4. *Equations (33) and (35) are suitable for the direct problem of dynamics: given a three-dimensional potential, find families of curves of the form (23) generated by it. We can rearrange the mentioned equations and obtain a linear partial differential equation of first order in α and β*

$$\left(\mathcal{V}_x \beta - \mathcal{V}_z\right)\left(\alpha_x + \alpha \alpha_y + \beta \alpha_z\right) - \left(\mathcal{V}_x \alpha - \mathcal{V}_y\right)\left(\beta_x + \alpha \beta_y + \beta \beta_z\right) = 0, \tag{41}$$

and a nonlinear one of second order

$$\begin{aligned} &\alpha_{xx} + \alpha^2 \alpha_{yy} + \beta^2 \alpha_{zz} + 2\alpha \alpha_{xy} + 2\beta \alpha_{xz} + 2\alpha\beta\alpha_{yz} = \\ &\frac{A}{\mathcal{V}_x \alpha - \mathcal{V}_y} \cdot \left(3\mathcal{V}_x \alpha_x + \left(2\mathcal{V}_x \alpha + \mathcal{V}_y\right)\alpha_y + \left(2\mathcal{V}_x \beta + \mathcal{V}_z\right)\alpha_z \right. \\ &\left. + \mathcal{V}_{xx}\alpha - \mathcal{V}_{xy}\left(1 - \alpha^2\right) - \mathcal{V}_{yy}\alpha - \mathcal{V}_{yz}\beta + \mathcal{V}_{xz}\alpha\beta\right). \end{aligned} \tag{42}$$

If $B = 0$ and $A \neq 0$, we still have inequality (34); instead of (33), the relation $\beta \mathcal{V}_x - \mathcal{V}_z = 0$ holds, beside the second order partial differential equation (35).

If $A = 0$ and $B \neq 0$, the inequality to be satisfied is $\left(\beta \mathcal{V}_x - \mathcal{V}_z\right)/B \geq 0$, and (33) is replaced by $\alpha \mathcal{V}_x - \mathcal{V}_y = 0$. Starting with $\dot{x}^2 = \left(\beta \mathcal{V}_x - \mathcal{V}_z\right)/B$, we follow the steps from the case when both A and B were different from zero and obtain instead of (35)

$$-\mathcal{V}_{xx} + \tilde{k}\mathcal{V}_{xz} + \mathcal{V}_{zz} + \tilde{p}\mathcal{V}_{yz} + \tilde{q}\mathcal{V}_{xy} = \tilde{l}\mathcal{V}_x + \tilde{m}\mathcal{V}_z, \tag{43}$$

where

$$\tilde{k} = \frac{1}{\beta} - \beta,\ \tilde{p} = \frac{\alpha}{\beta},\ \tilde{q} = -\alpha$$
$$\tilde{l} = \frac{3B}{\beta} - \beta\tilde{m},\ \tilde{m} = \frac{B_x + \alpha B_y + \beta B_z}{\beta B}. \tag{44}$$

3.2. **Examples.**

Example 5. *The two-parametric family of straight lines*

$$\frac{y}{x} = c_1, \quad \frac{z}{x} = c_2$$

was found in [13] *to be compatible with the (central) potential*

$$\mathcal{V}(x, y, z) = F(x^2 + y^2 + z^2),$$

where F is an arbitrary function of its argument.

Shorokhov [28] *presented a family of straight lines*

$$\frac{x}{y} = c_1, \quad y + z = c_2$$

which cannot be described by a particle under the action of any potential. This family has $\alpha = y/x$ and $\beta = -y/x$, hence condition (31) does not hold.

Example 6. *The family of curves*

$$\frac{z}{x} = c_1, \quad x^2 + y^2 = c_2$$

was considered in [30] *and* [15]*. It can be traced all over the space under the action of the potential*

$$\mathcal{V}(x, y, z) = (x^2 + y^2 + z^2)/2,$$

with the energy $\mathcal{E}(\varphi, \psi) = \psi(\varphi^2 + 2)/2$. This example illustrates the case $A \neq 0$, $B = 0$.

Example 7. *For the family of curves*

$$x^2 + y^2 = c_1, \quad \frac{x^2 - y^2}{z} = c_2$$

one has $A \neq 0$ and $B \neq 0$. The potential

$$\mathcal{V}(x, y, z) = x^2 + y^2 + 4z^2$$

given in [14] *produces the given family with the energy* $\mathcal{E}(\varphi, \psi) = 2\varphi\left(2\varphi + \psi^2\right)/\psi^2$.

4. Conclusions

The energy-free equations have a basic role in the inverse problem of dynamics. When we have no a priori information on the energy of the given family, it is natural to work with equations (6), respectively (33) and (35) in order to obtain potentials compatible with the given family. These equations can be used also when the search of the potentials is restricted to a class of theoretical or practical interest.

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SMOOTH DEPENDENCE OF SOLUTION ON PARAMETERS FOR THE VOLTERRA-FREDHOLM INTEGRAL EQUATION

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Abstract. In this paper we will give conditions that ensures the differentiability with respect to parameters of the solution of Volterra-Fredholm nonlinear integral equation.

1. **Introduction**

In the present paper consider the nonlinear integral equation of Volterra-Fredholm type:

$$u(x,t) = f(x,t) + \int_0^t \int_a^b K(x,t,y,s,u(y,s))dyds \qquad (1)$$

$\forall t \in [0,c]$, $\forall x \in [\alpha,\beta]$, where $[a,b] \subset [\alpha,\beta]$.

Applying fiber Picard operators theory, we will prove the differentiability of the solution of (1) with respect to a and b.

2. **Fiber Picard operators**

Let (X,d) be a metric space and $A : X \to X$ an operator. In this paper we will use the following notations:

$$F_A := \{x \in X : A(x) = x\};$$

$$A^0 := 1_X \,,\; A^{n+1} := A \circ A^n \; \forall n \in \mathbb{N}.$$

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Definition 2.1. *(I. A. Rus [1]) The operator A is said to be:*
(i) ***weakly Picard operator (wPo)*** *if $\forall x_0 \in X$ $A^n(x_0) \to x_0^*$, and the limit x_0^* is a fixed point of A, which may depend on x_0.*
(ii) ***Picard operator (Po)*** *if $F_A = \{x^*\}$ and $\forall x_0 \in X$ $A^n(x_0) \to x^*$.*

In the next section we need the following result:

Theorem 2.1. *(Fiber Contraction Principle, I. A. Rus [3]) Let (X, d), (Y, ρ) be two metric spaces and $B : X \to X$, $C : X \times Y \to Y$ two operators such that:*
(i) (Y, ρ) is complete;
(ii) B is a Picard operator, $F_B = \{x^*\}$;
(iii) $C(\cdot, y) : X \to Y$ is continuous $\forall y \in Y$;
(iv) $\exists a \in]0, 1[$ such that the operator $C(x, \cdot) : Y \to Y$ is an a-contraction for all $x \in X$; let y^* be the unique fixed point of $C(x^*, \cdot)$.
Then

$$A : X \times Y \to X \times Y, \quad A(x, y) := (B(x), C(x, y))$$

is a Picard operator and $F_A = \{(x^*, y^*)\}$.

This theorem is very useful for proving solutions of operatorial equations to be differentiable with respect to parameters. For such results see [6], [3], [2], [4], [5], etc.

3. Main result

Theorem 3.1. *Consider the equation (1) in the next conditions:*
(i) $f \in C([a, b] \times [0, c])$ and $K \in C([a, b] \times [0, c] \times [a, b] \times [0, c] \times \mathbb{R})$;
(ii) there exists $L_K > 0$ such that:

$$|K(x, t, y, s, u) - K(x, t, y, s, v)| \leq L_K |u - v| \tag{2}$$

$\forall (x, t, y, s) \in [\alpha, \beta] \times [0, c] \times [\alpha, \beta] \times [0, c]$, $\forall u, v \in \mathbb{R}$.
Then:
a) for all $a < b \in [\alpha, \beta]$, the equation (1) has in $C([\alpha, \beta] \times [0, c])$ a unique solution

$u^*(\cdot,\cdot,a,b)$.

b) for all $u_0 \in C([\alpha,\beta]\times[0,c])$, *the sequence* $(u_n)_{n\geq 0}$ *defined by:*

$$u_n(x,t,a,b) = f(x,t) + \int_0^t \int_a^b K(x,t,y,s,u_{n-1}(y,s,a,b))dyds$$

converges uniformly to u^*, $\forall(x,t,a,b) \in [\alpha,\beta]\times[0,c]\times[\alpha,\beta]\times[\alpha,\beta]$.

c) The function u^*, $(x,t,a,b) \mapsto u^*(x,t,a,b)$ *is continuous:* $u^* \in C([\alpha,\beta]\times[0,c]\times[\alpha,\beta]\times[\alpha,\beta])$*;*

d) If $K(x,t,y,s,\cdot) \in C^1(\mathbb{R})$, $\forall(x,t,y,s) \in [\alpha,\beta]\times[0,c]\times[\alpha,\beta]\times[0,c]$, *then* $u^*(x,t,\cdot,\cdot) \in C^1([\alpha,\beta]\times[\alpha,\beta])$, $\forall(x,t) \in [\alpha,\beta]\times[0,c]$.

Proof. Let the space $C([a,b]\times[0,c],\mathbb{R})$ be endowed with a suitable norm,

$$\|u\|_{BC} := \sup\{\|u(x,t)\| e^{-\tau t} : x \in [a,b], t \in [0,c]\}, \quad \tau > 0 \tag{3}$$

Let $X := C([\alpha,\beta]\times[0,c]\times[\alpha,\beta]\times[\alpha,\beta])$. We consider the operator $B : X \to X$ defined by:

$$B(u)(x,t,a,b) := f(x,t) + \int_0^t \int_a^b K(x,t,y,s,u(y,s,a,b))dyds$$

From (ii), applying the Contraction Principle, it follows that B is a contraction, so we have a), b) and c).

For all $a < b \in [\alpha,\beta]$, there is a unique solution $u^*(\cdot,\cdot,a,b) \in C([\alpha,\beta]\times[0,c])$, so we have:

$$u^*(x,t,a,b) = f(x,t) + \int_0^t \int_a^b K(x,t,y,s,u^*(y,s,a,b))dyds \tag{4}$$

Let us prove that $\dfrac{\partial u^*(x,t,a,b)}{\partial a}$ and $\dfrac{\partial u^*(x,t,a,b)}{\partial b}$ exists and they are continuous.

1. Supposing that $\dfrac{\partial u^*(x,t,a,b)}{\partial a}$ exists, from (4) we obtain:

$$\frac{\partial u^*(x,t,a,b)}{\partial a} = -\int_0^t K(x,t,a,s,u^*(a,s,a,b))ds +$$

$$+\int_0^t \int_a^b \frac{\partial K(x,t,y,s,u^*(y,s,a,b))}{\partial u} \cdot \frac{\partial u^*(y,s,a,b)}{\partial a} dyds$$

This relationship suggest us to consider the next operator:

$C : X \times X \to X$, defined by:

$$C(u,v)(x,t,a,b) := -\int_0^t K(x,t,a,s,u(a,s,a,b))ds+$$

$$+\int_0^t\int_a^b \frac{\partial K(x,t,y,s,u(y,s,a,b))}{\partial a}\cdot v(y,s,a,b)dyds$$

Let u^* be the unique fixed point of B. The operator $C(u,\cdot)$ is a contraction $\forall u \in X$ and let v^* be the unique fixed point of $C(u^*,\cdot))$.

If we define the operator $A : X \times X \to X \times X$,

$$A(u,v)(x,t,a,b) := \left(B(u)(x,t,a,b), C(u,v)(x,t,a,b)\right),$$

then the conditions of the Theorem 2.1 are fulfilled. It follows that A is a Picard operator and $F_A = \{(u^*,v^*)\}$.

Consider the sequences $(u_n)_{n\geq 0}$ and $(v_n)_{n\geq 0}$ defined by:

$$u_n(x,t,a,b) := B(u_{n-1}(x,t,a,b)) =$$

$$= f(x,t) + \int_0^t\int_a^b K(x,t,y,s,u_{n-1}(y,s,a,b))dyds \quad \forall n \geq 1$$

$$v_n(x,t,a,b) := C(u_{n-1}(x,t,a,b), v_{n-1}(x,t,a,b)) =$$

$$= -\int_0^t K(x,t,a,s,u_{n-1}(a,s,a,b))ds+$$

$$+\int_0^t\int_a^b \frac{\partial K(x,t,y,s,u_{n-1}(y,s,a,b))}{\partial u}\cdot v_{n-1}(y,s,a,b)dyds \quad \forall n \geq 1$$

We have:

$$u_n \rightrightarrows u^* \ (n\to\infty), \quad v_n \rightrightarrows v^* \ (n\to\infty) \tag{5}$$

uniformly for $(x,t,a,b) \in [\alpha,\beta]\times[0,c]\times[\alpha,\beta]\times[\alpha,\beta]$.

We take $u_0 = v_0 := 0$, so $v_1 = \dfrac{\partial u_1}{\partial a}$.

By induction we can prove that $v_n = \dfrac{\partial u_n}{\partial a}$ $\forall n$ and from (5) results:

$$\frac{\partial u_n}{\partial\ a} \rightrightarrows v^* \ (n\to\infty)$$

Using a Weierstrass theorem, it follows that $\frac{\partial u^*}{\partial\ a}$ exists and

$$\frac{\partial u^*(x,t,a,b)}{\partial a} = v^*(x,t,a,b).$$

2. By a similar way, we can prove the existence and the continuity of $\frac{\partial u^*}{\partial\ b}$.$\square$

Remark 3.1. *We can also consider the following integral equation of Volterra-Fredholm type:*

$$u(x,t) = f(x,t) + \int_0^t \int_a^b K(x,t,y,s,u(y,s),\lambda)dyds \tag{6}$$

$\forall t \in [0,c]$, $\forall x \in [a,b]$, *where* $\lambda \in \mathbb{R}$ *and we can prove the differentiability of the solution with respect to the parameter* λ.

This case will be presented elsewhere.

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PERIODIC AND QUASIPERIODIC MOTION IN THE PERIODICALLY FORCED RAYLEIGH SYSTEM

PETRE BĂZĂVAN

Abstract. In this paper we present a numerical study of the periodic and quasiperiodic motion in the dynamical system associated with the generalized Rayleigh equation. Numerical results describe the system dynamics changes (in particular bifurcations), when the forcing amplitude is varied.

1. Introduction

The autonomous second order nonlinear ordinary differential equation (ODE),

$$\ddot{x} + \frac{\dot{x}^3}{3} - \dot{x} + x = 0, \tag{1}$$

introduced in 1883 by Lord Rayleigh, is the nonlinear equation which appears to be the closest to the ODE of the harmonic oscillator with dumping [1]. Some aspects concerning *canard* bifurcations are analyzed in [1] and [2] for the periodically forced generalization of Rayleigh equation,

$$\varepsilon\ddot{x} + \frac{\dot{x}^3}{3} - \dot{x} + ax = g \sin \omega t. \tag{2}$$

From mathematical perspective the nonautonomous system of nonlinear ODEs associated with (2) is one of a class of periodically forced nonlinear oscillators, as the Van der Pol and Bonhoeffer Van der Pol systems are.

The behavior of these systems was much numerically investigated in [3], [4] and [14], due to their applications in electronics and physiology.

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With system (2) there are associated the two-dimensional nonlinear non-autonomous system of ODEs

$$\begin{cases} \dot{x}_1 = x_2, \\ \dot{x}_2 = -\frac{a}{\varepsilon}x_1 + \frac{1}{\varepsilon}x_2 - \frac{x_2^3}{3} + g\sin\omega t, \end{cases} \tag{3}$$

and the three-dimensional nonlinear autonomous system

$$\begin{cases} \dot{x}_1 = x_2, \\ \dot{x}_2 = -\frac{a}{\varepsilon}x_1 + \frac{1}{\varepsilon}\left(x_2 - \frac{x_2^3}{3}\right) + \frac{g}{\varepsilon}\sin x_3, \\ \dot{x}_3 = \omega mod 2\pi. \end{cases} \tag{4}$$

A three-dimensional dynamical system with the phase space $\mathbb{R}^2 \times S^1$ can be associated with (4).

Periodic solutions and the dynamics of the systems associated with (3) and (4) are studied in [6] and [7]. The succession of the periodic and chaotic attractors for the system (4) and then, the transition between the periodic and chaotic motion are numerically studied in [8].

The dynamical system associated with (4) involves the interaction between two periodic motions, each with a different frequency. When the ratio of the frequencies is irrational the dynamical system behaves in a manner which is neither periodic or chaotic. This motion is called *quasiperiodic*. More precisely, the natural periodic motion, studied in [6] for the unforced case, i.e. Eq. (1), is modulated by a second periodic motion given by the sinusoidal term when $g > 0$. The system behaves in a manner with the motion never quite repeating any previous motion. This behavior is generically followed by the system locking into a periodic motion, as the control parameter for the system is varied [14].

The aim of our numerical analysis is to establish the parameter region where the system (4) presents a quasiperiodic motion and structural changes which may lead to any subsequent mode locked region of periodic motion.

The mathematical model used in our numerical study is presented in Sec. 2. Numerical results in Sec. 3 are concerned with the proof of the existence of the

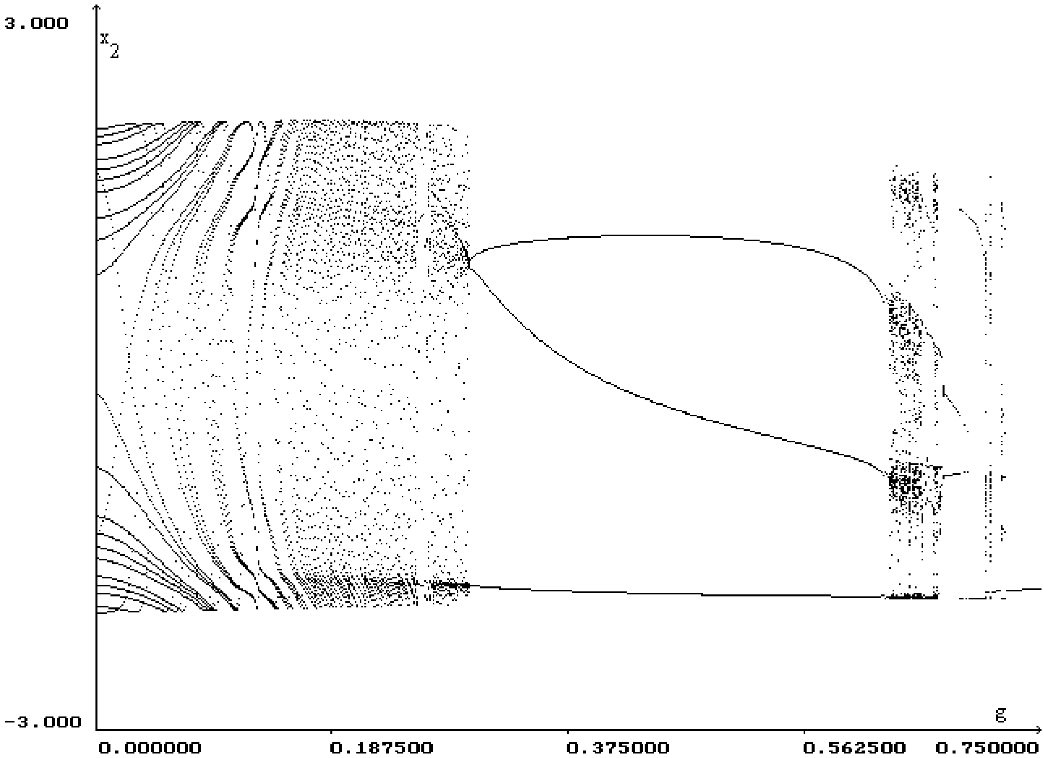

FIGURE 1. Bifurcation diagram for the dynamical system (4).

quasiperiodic motion and the study of the transition from quasiperiodic to periodic motion in the system (4).

2. **The mathematical model**

In order to present the mathematical model used in the numerical study from Sec. 3, we shortly write (4) in the form

$$\dot{x} = f(x), \qquad (5)$$

where f is defined on the $\mathbb{R}^2 \times S^1$ cylinder. We define a Poincaré map as follows. Let

$$\sum = \left\{ (x_1, x_2, x_3) \in \mathbb{R}^2 \times S^1, x_3 = 0 mod \frac{2\pi}{\omega} \right\}$$

be a surface of section, which is transversally crossed by the orbits of (5). The Poincaré map $P:\Sigma\to\Sigma$ is defined by

$$P\left(x_0\right)=x\left(2\pi/\omega,x_0\right)=\int_0^{2\pi/\omega}f\left(x\left(t,x_0\right)\right)dt \tag{6}$$

where $x_0\in\Sigma$ and $x\left(t,x_0\right)$ is the solution of the Cauchy problem $x\left(0\right)=x_0$ for (5). We denote by P^n the n-times iterated map.

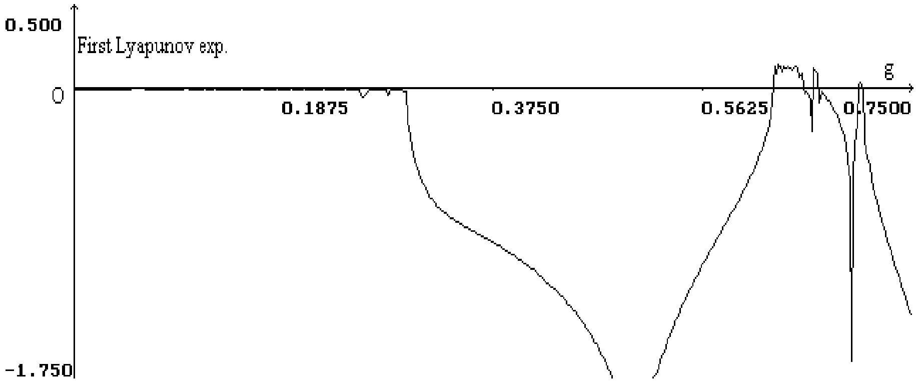

FIGURE 2. The first Lyapunov exponent for the dynamical system (4).

Let be $\xi\left(t,x_0\right)$ a periodic solution of (5) with period $T=n\cdot2\pi/\omega$, $n\geq1$, lying on a closed orbit and consider the map P of the initial point x_0. Then, to this closed orbit an n-periodic orbit of P corresponds. Numerically, the period T (i.e. n from the expression of T) can be determined by integrating Eq. (5) with the initial condition x_0 and sampling the orbit points $x_k=P\left(x_{k-1}\right)$, $k\geq1$ at discrete times $t_k=k\cdot2\pi/\omega$, until $P^k\left(x_0\right)=x_0$. Then, $n=k$.

The stability discussion of the periodic orbit $\xi\left(t,x_0\right)$ is reduced to the stability discussion of the fixed point x_0 of P^n since the stability of the periodic solution ξ is determined by the eigenvalues of the matrix DP^n, [9], [10], [11]. The linear stability of the n-periodic orbit of P is determined from the linearized-map matrix DP^n of P^n. The matrix DP^n can be obtained by integrating the linearized system (5) for a small perturbation $y\in\mathbb{R}^2\times S^1$, [9], [10]. We note [9] that one of the eigenvalues of this matrix always equals 1, and that the remained two eigenvalues, also called the

Poincaré map multipliers, influence the stability. We denote these eigenvalues by λ_1 and λ_2.

The diagnostics used to establish structural changes of the system (4) involve two-dimensional $x_1 - x_2$ phase plane diagrams, Poincaré sections at intervals of forcing period $2\pi/\omega$, bifurcation diagrams with $g - x_2$ coordinates, evaluations of the eigenvalues of the linearized Poincaré map-matrix, evaluations of the Lyapunov exponents.

All numerical calculations were carried out through the application of a variable step-size algorithm for Runge-Kutta methods [12], [13]. This algorithm is a variant of an algorithm [14], [15] which controls the time step with a Richardson extrapolation method [16]. For the calculation of Lyapunov exponents we used the method described in [17]. The 3D-representation uses a center projection [18].

3. **Periodic and quasiperiodic motion**

In our numerical study we investigated a region in the four-dimensional parameter space $(\varepsilon, a, g, \omega)$ given by $0 < \varepsilon \leq 1$, $0 < a \leq 1$, $1 < \omega \leq 3$ and $0 < g \leq 2$. By logistic reasons we restrict the presentation to the region space

$$\varepsilon = 0.125, \qquad a = 0.5, \qquad \omega = 2.84, \qquad 0 < g \leq 0.75. \tag{7}$$

An overview of the numerical results which typify the system is given by the bifurcation diagram in Fig. 1.

In the first part of the subinterval $0 < g < 0.3$ we observe an apparent regularity of the return points. This region which can indicate a quasiperiodic or chaotic behavior is followed by a region with clear periodic motion. This last region is interrupted by short chaotic regions. We prove the existence of the quasiperiodic behavior in two ways.

The first argument is the first Lyapunov exponent value. Recall that a leading Lyapunov exponent of zero verifies quasiperiodic behavior [14]. Fig. 2 is a graph of the control parameter (the forcing amplitude g) against the first Lyapunov exponent for the same parameter range as the bifurcation diagram of Fig. 1. In the interval

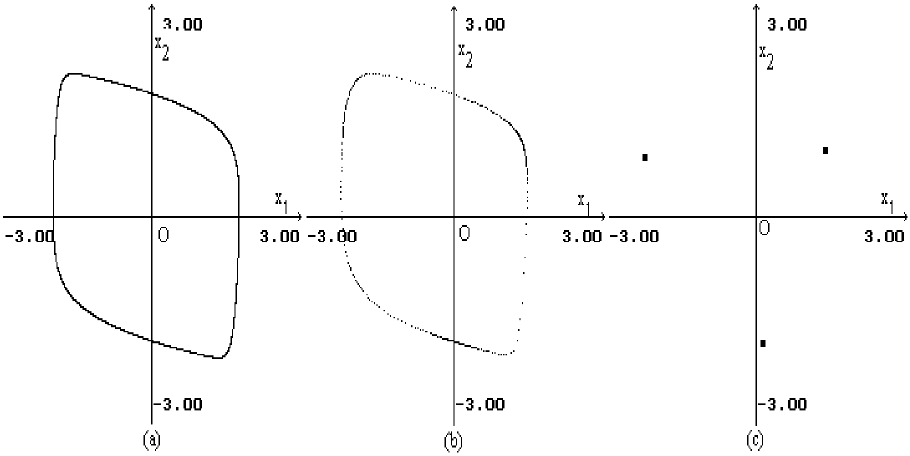

FIGURE 3. Poincaré sections for the dynamical system (4).

$0 < g < 0.3$ the exponent was consistently within -0.01 of 0. This is the first numerical confirmation of the quasiperiodic behavior.

The intersection points of the trajectories of the system (4) with the associated Poincaré section represent the second argument. At $g_1 = 0.07$ the section is represented in the Fig. 3a. The drift ring is associated with quasiperiodic motion. Integrating with a large period, the curve does not modify the shape. The fact that the points are situated on a closed curve and the constant shape related to the integration time confirm the quasiperiodic behavior [14].

In proportion as g increases in the interval $0 < g < 0.3$ the return points remain on the same curve but the density increases markedly in some locations (Fig. 3b for $g_2 = 0.25$). At $g_3 = 0.3$ there are only three intersection points in the Poincaré section (Fig. 3c) and on the bifurcation diagram the quasiperiodic region is replaced by a periodic window. The motion changes from quasiperiodic to periodic, with the emergence of a period-3 attractor. This is due to the saddle-node bifurcation of the Poincaré map P^3,

$$x_{n+3} = P^3(x_n), \quad x_0 \in \mathbb{R}^2 \times S^1, \quad n \geq 0.$$

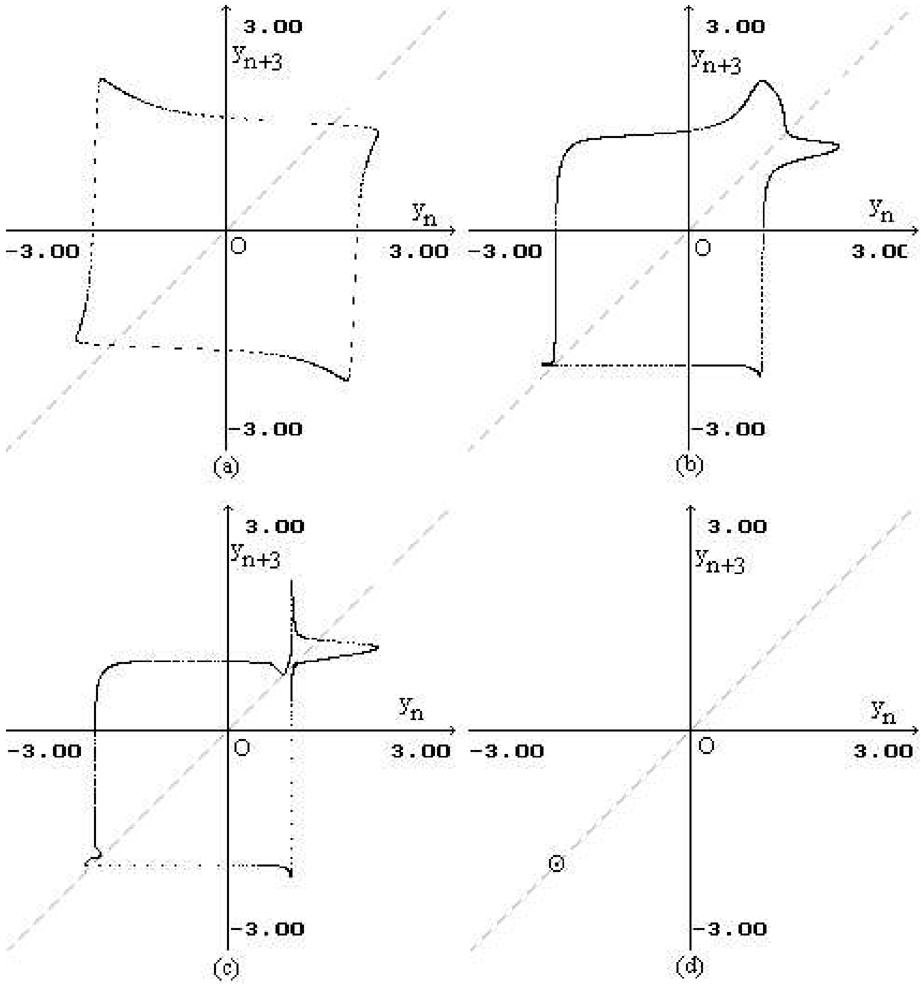

FIGURE 4. The Poincaré map P^3 associated with the dynamical system (4).

We numerically prove this fact. We use the projection of the graph of P^3 on the plane (y_n, y_{n+3}), $n \geq 0$, where we denote by y the x_2 coordinate of the point $x \in \mathbb{R}^2 \times S^1$.

In Fig. 4a for $g_4 = 0.07$, when the motion is quasiperiodic, there are two intersection points of P^3 with the diagonal $y_n = y_{n+3}$. At the intersection the magnitude of the slope not equals 1. As g increases the curve approaches the diagonal in other locations (Fig. 4b for $g_5 = 0.28$). These locations suggest the imminent tangential intersections. At $g_6 = 0.2961$ there are three tangential intersections (Fig. 4c) and we have a saddle-node bifurcation of the map P^3. When $g_7 = 0.3$ (Fig. 4d) the graph of the map P^3 is a single point which is situated on the diagonal. This fact confirms the existence of the period-3 attractor.

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INTEGRAL $\lambda - \tau$ BIVARIATE SPLINE OPERATORS IN COMPUTER GRAPHICS PROBLEMS

F. CALIÒ, E. MIGLIO, G. MORONI, AND M. RASELLA

Abstract. In the present work we propose and analyze a particular class of bivariate tensor VDS splines defined by an integral operator and depending on two shape parameters (λ and τ). These functions are used to generate surface models. Precisely we generate and algebrically formalize a $\lambda - \tau$ parametric integral spline family and advocate its use in the field of computer graphics. We apply such models to the problem of reconstructing, starting from a set of measured points, "smooth" surfaces (where the optimal value of the shape parameters is obtained minimizing suitable functionals).

Introduction

It is well known that variation diminishing splines (VDS), introduced in the approximation theory during the eighties of the last century, have found many important applications in the field of integral-differential problems (see for example a survey in [1]).

In [2] Milovanovic and Kocic present an interesting application of the spline functional class in the field of computer graphics. Precisely, they propose an integral operator depending on a real parameter and based on variation dimininishing spline: the underlying properties of this new class of splines are particularly interesting in the field of free form curve modelling. We recall that a curve or surface is said to have a free form if it is possible to alter its shape by changing one or a few parameters with a priori knowledge of how this changing will affect the shape of the curve or surface.

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Following the idea suggested in [2], in the present paper we propose a particular class of bivariate tensor splines defined through an integral operator and depending on two parameters. It is named $\lambda - \tau$ integral VDS spline operator and is applied in the field of computer graphics, in order to obtain regularly behaving and pleasantly shaped surfaces, called B-spline integral models, with $\lambda - \tau$ shape parameters.

This paper is organized as follows: first we introduce the problem, going through the most significant results on univariate splines linked to an integral operator. In the second section we propose and analyse, as an extension of the univariate case, the bivariate case of the integral tensor splines operators. The third section is dedicated to the operator matrix expression which is used for the theoretical analysis and for the algorithm construction. The parameter optimization procedure is discussed in section four. An example illustrating the effectiveness of the proposed algorithm is presented in section five.

1. **Generation and properties of univariate integral parametric spline**

In this section we recall the basic concepts about VDS splines and the genesis of univariate integral parametric splines proposed in [2], to acquire the terminology and the motivations to build and study a new bivariate operator.

Given a set of vector points (*control points*) $\underline{P}_0$, $\underline{P}_1$,..., $\underline{P}_m$,(e.g. in a three-dimensional space) and a knots vector t:

$$0 = t_{-k} = ... = t_0 < t_1 < ... < t_{n-1} < t_n = ... = t_m = 1 \quad n = m - k,$$

the expression

$$(S_m P)(t) = \sum_{i=0}^{m} \underline{P}_i B_i^k(t) \quad 0 \le t \le 1 \tag{1}$$

is called a k-order variation diminishing spline operator (VDS operator) and generates a curve model called B-spline curve.

The basis function $B_i^k(t)$ $(i = 0, 1, \dots m)$ are recursively defined as:

$$B_i^k(t) = \frac{t - t_{i-k}}{t_{i-1} - t_{i-k}} B_i^{k-1}(t) + \frac{t_i - t}{t_i - t_{i-k+1}} B_{i+1}^{k-1}(t)$$

$$B_i^0(t) = 1 \quad t_i \le t \le t_{i+1}$$
$$B_i^0(t) = 0 \quad otherwise$$

In matrix form the VDS operator is:

$$(S_m P)(t) = \underline{b}_m(t)P \quad 0 \le t \le 1 \tag{2}$$

where:

$$\underline{b}_m = (B_0^k(t), B_1^k(t), ..., B_m^k(t)), \qquad P = (\underline{P}_0, \underline{P}_1, ..., \underline{P}_m)^T.$$

The authors in [2] proposed modifications to this class of splines by introducing a family of integral spline operators depending on a real parameter. We designate this new class as Univariate Integral $\lambda-$Variation Diminishing Splines.

Assuming that t_i is the value of the parameter corresponding to the given control point $\underline{P}_i$ we define:

$$\xi_i^k = \frac{t_{i-k+1} + ...t_i}{k}. \tag{3}$$

These points in the field of approximation are called Schonberg points [3]. We will call "correspondence points" such ξ_i^kvalues.

Let $x_j^i, (j = 1, 2, 3)$ be the generic component of vector $\underline{P}_i$ and $\varphi_j, (j = 1, 2, 3)$ the piecewise linear function interpolating points (ξ_i^k, x_j^i) and whose graphic is the control polygon.

The S_m operator on j-th component of P can then be expressed as:

$$(S_m P)_j = (S_m \varphi_j) = \sum_{i=0}^{m} \varphi_j(\xi_i^k) B_i^k(t), \quad j = 1, 2, 3$$

If we substitute $\varphi_j(\xi_i^k)$ by the integral mean:

$$\mu_i \varphi_j(t) = \frac{\int_{\xi_i^{k+1}}^{\xi_{i+1}^{k+1}} \varphi_j(u) du}{\xi_{i+1}^{k+1} - \xi_i^{k+1}} \tag{4}$$

we obtain the following operator T_m (integral VDS operator):

$$(S_m \mu_i \varphi_j) = (T_m \varphi_j) = (T_m P)_j \qquad (j = 1, 2, 3).$$

The T_m operator can be used to generate a new curve model and in matrix form it can be written as:

$$(T_m P)(t) = \underline{b}_m(t)(MP) \qquad 0 \leq t \leq 1 \tag{5}$$

Equation (5) shows that the integral spline can be regarded as the VDS operator produced by a new control points set Q, transforming in the global way the given set P. That is: $Q = MP$. Where matrix M has the following form:

$$M = \begin{bmatrix} \beta_0 & \gamma_0 & 0 & \ldots & 0 \\ \alpha_1 & \beta_1 & \gamma_1 & \ldots & 0 \\ 0 & \alpha_2 & \beta_2 & \ldots & 0 \\ 0 & \ldots & \ldots & & \gamma_{m-1} \\ 0 & \ldots & \ldots & \beta_m & \gamma_m \end{bmatrix}$$

$$\begin{aligned} \alpha_0 &= 0, \alpha_i = \frac{(\delta_i^l)^2}{2\Delta_{i-1}^k \Delta_i^{k+1}}, i = 1, ...m; \\ \beta_i &= 1 - \alpha_i - \gamma_i, i = 1, ..., m \\ \gamma_i &= \frac{(\delta_i^r)^2}{2\Delta_i^k \Delta_i^{k+1}}, i = 1, ..., m, \gamma_m = 0 \end{aligned} \qquad \begin{aligned} \Delta_i^k &= \xi_{i+1}^k - \xi_i^k, \\ \delta_i^r &= \xi_{i+1}^{k+1} - \xi_i^k, \\ \delta_i^l &= \xi_i^k - \xi_i^{k+1}, \\ \xi_i^{k+1} &< \xi_i^k < \xi_{i+1}^{k+1} \end{aligned}$$

It follows that

$$(T_m P)(t) = (S_m Q)(t) \qquad 0 \leq t \leq 1$$

The obtained curve model is characterized by the following properties:

- it is invariant under affine transformations of the coordinate system;

- the whole curve lies inside the convex hull of the control polygon (the piecewise line whose vertices are the control points);

- it is uniquely determined by its control polygon and no two polygons produce the same curve;

- it crosses an arbitrary plane no more then does the control polygon ;

- it reproduces points and lines.

In [2] the authors introduce a shape parameter λ in the VDS integral operator. The integral mean expression in (4) is replaced by:

$$\mu_i^\lambda \varphi_j(t) = \frac{\int_{\zeta_i}^{\eta_i} \varphi_j(u)du}{\eta_i - \zeta_i} \tag{6}$$

where:

$$\begin{aligned} \zeta_i &= (1-\lambda)\xi_i^k + \lambda \xi_i^{k+1} \\ \eta_i &= (1-\lambda)\xi_i^k + \lambda \xi_{i+1}^{k+1} \end{aligned}$$

with $0 \le t \le 1$ and $0 \le \lambda \le 1$.

In matrix form this new operator can be written as:

$$(T_m^\lambda P)(t) = \underline{b}_m(t)(M^\lambda(\lambda)P) \qquad 0 \le \lambda \le 1 \tag{7}$$

where

$$M^\lambda(\lambda) = \begin{bmatrix} \beta_0^\lambda & \gamma_0^\lambda & 0 & ... & 0 \\ \alpha_1^\lambda & \beta_1^\lambda & \gamma_1^\lambda & ... & 0 \\ 0 & \alpha_2^\lambda & \beta_2^\lambda & ... & 0 \\ 0 & ... & ... & & \gamma_{m-1}^\lambda \\ 0 & ... & ... & \beta_m^\lambda & \gamma_m^\lambda \end{bmatrix} \tag{8}$$

$$\begin{aligned} \alpha_i^\lambda &= \lambda\alpha_i \qquad i = 0, ..., m \\ \beta_i^\lambda &= 1 - \lambda(\alpha_i + \gamma_i), \qquad i = 0, ..., m \\ \gamma_i^\lambda &= \lambda\gamma_i \qquad i = 0, ..., m-1. \end{aligned}$$

$(T_m^\lambda P)$ is called integral spline VDS operator, with shape parameter. It can be shown that the λ parameter allows to control the global shape of the curve (whereas with the conventional spline only a local control can be achieved).

2. **The bivariate spline operator**

Now we extend the previously seen concepts of integral λ-VDS operator to the field of splines depending on two parameters (t and s).

This gives rise to a technique for describing surfaces in a three-dimensional space.

Let us organize our control points into $p+1$ sets of $m+1$ elements each, i.e.: $\underline{P}_{ij}$ $i = 0, 1, \ldots, p$ and $j = 0, 1, \ldots, m$ where $\underline{P}_{ij}$ is a three–dimensional vector. We call P the global set of control points.

We express the bivariate tensor VDS as:

$$(S_{mp}P)(t,s) = \sum_{i=0}^{p} \sum_{j=0}^{m} \underline{P}_{ij} C_{ij}^{kh}(t,s) \tag{9}$$

where the basis functions are obtained as a product of two univariate basis splines (of order k for the t parameter and h for the s parameter, respectively):

$$C_{ij}^{kh}(t,s) = B_i^k(t) B_j^h(s).$$

It can be easily seen from (9) that the bivariate tensor VDS is built on two classes of univariate VDS: a first one (*control curves*) controlled by the vector points $\underline{P}_{ij}$ and a second one (*swept curves*) controlled by points evaluated on first function class.

3. **The matrix expression of the bivariate spline operator**

We suggest to express the l-th component of bivariate tensor VDS, in matrix form, as follows:

$$(S_{mp}P)(t,s)_l = \underline{b}^{km}(t) \begin{bmatrix} p_l^{00} & \ldots & p_l^{0p} \\ \ldots & \ldots & \ldots \\ p_l^{m0} & \ldots & p_l^{mp} \end{bmatrix} (\underline{b}^{hp}(s))^T$$

i.e.:

$$(S_{mp}(t,s))_l = \underline{b}^{km} P_l (\underline{b}^{hp})^T \tag{10}$$

where:

$$\underline{b}^{wr} = (B_0^w(t), B_1^w(t), \ldots, B_r^w(t))$$

and p_l^{ij} is the l-th component of vector $\underline{P}_{ij}$.

By exploiting the separability of the tensor product basis functions, it is possible to extend the formalism introduced in the univariate case to the bivariate case.

Therefore: the control points for the *control spline* function are modified by the same matrix used in the univariate case, then the points on these splines (control points for *swept splines*) are modified by another similar matrix.

We get:

$$(T_{mp}^{\lambda\tau}P)_l(t,s) = \underline{b}^{km}(t)M^{\tau}(\tau)P_l M^{\lambda}(\lambda)(\underline{b}^{hp}(s))^T \tag{11}$$

where $M^{\tau}(\tau)$ is a $(m+1)$-order square matrix, depending on the knots of an h-order B-spline, having the same expression as for univariate splines.

Similarly, $M^{\lambda}(\lambda)$ is a $(p+1)$-order square matrix, depending on the knots of an h-order B-spline.

Theorem 1. *Let us consider the nonlinear operator(11). An algebraic and sintetic expression of it is the following:* $(1-\tau)(1-\lambda)(S_{mp}^{\lambda\tau}P)_l + \tau(1-\lambda)(S_{mp}^{\lambda\tau}Q^{\tau})_l + \lambda(1-\tau)(S_{mp}^{\lambda\tau}Q^{\lambda})_l + \lambda\tau(S_{mp}^{\lambda\tau}Q^{\lambda\tau})_l$, *where:* $Q^{\tau} = M^{\tau}(1)P$, $Q^{\lambda} = PM^{\lambda}(1)$, $Q^{\lambda\tau} = M^{\tau}(1)PM^{\lambda}(1)$

Proof. The proof is based on the following relationship: $M^{\alpha}(\alpha) = (1-\alpha)I + \alpha M(1)$. Substituing it into (11) we get: $\underline{b}^{km}((1-\tau)I^{(m+1)} + \tau M^{\tau}(1))P_l((1-\lambda)I^{(p+1)} + \lambda M^{\lambda}(1))(\underline{b}^{hp}(s))^T$ through some algebraic steps the thesis follows. ☐

4. Parameters optimization

In this section we will deal with the problem of finding optimal values for λ and τ.

The aim is to obtain the “best” reconstruction of a surface starting from a cloud of measured points.

The first possibility to find optimal λ and τ parameters is to minimize a quadratic functional expressing the global (Euclidean) distance of the given data points from the correspondence points on the reconstructed surface. The functional has the following expression:

$$F(\lambda,\tau) = \sum_{l=1}^{3}\sum_{j=0}^{m}\sum_{i=0}^{p}\delta^2(\underline{P}_l^{ij},((T_{mp}^{\lambda\tau}P)(\xi_i^k,\xi_j^h))_l)$$

This solution gives the most precise representation of a given set of points, but is very sensitive to digitizing errors.

A second approach consists in minimizing the energy-functional:

$$F(\lambda,\tau) = \sum_{l=1}^{3}\int_D(\frac{\partial^2}{\partial t^2}(T_{mp}^{\lambda\tau}P)(\xi_i^k,\xi_j^h))_l + \frac{\partial^2}{\partial s^2}(T_{mp}^{\lambda\tau}P)(\xi_i^k,\xi_j^h))_l d\lambda d\tau$$

The D domain corresponds to the whole variation of the t and s parameters, relevant to the considered surface.

This algorithm gives satisfactory results as far as the surface smoothness is involved.

5. **Test example**

The following example higlights the noise sensitivity of the computed surface. The saddle surface whose equation is $z = x^2 - y^2$ (hyperbolic paraboloid) has been used for testing.

The left part Figure 1 shows the "measured" points on the surface; on the right of the same figure the surface reconstructed using usual splines function is shown.

The left part of Figure 2 represents the surface reconstructed using the minimization of the distance-functional while on the right part the reconstructed surface by means of minimizing the energy functional is shown. The first surface, which is satisfactory as algorithm test, furthemore still presents some irregularity, on the contrary the second one looks very smooth.

6. **Conclusion**

We have proposed a non linear bivariate operator based on an integral parametric spline family. By this operator it is possible to obtain a smooth surface, without modifying each single control point; such surfaces exhibit interesting properties as far as engineering applications are involved. The next activities we intend to carry out

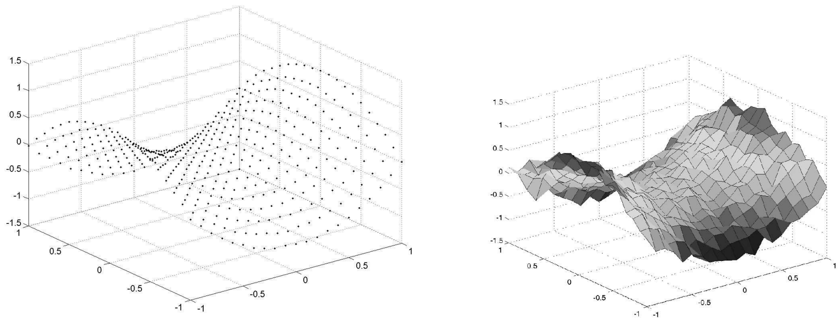

FIGURE 1. Left: "Measured" points. Right: Reconstructed surface using conventional splines functions.

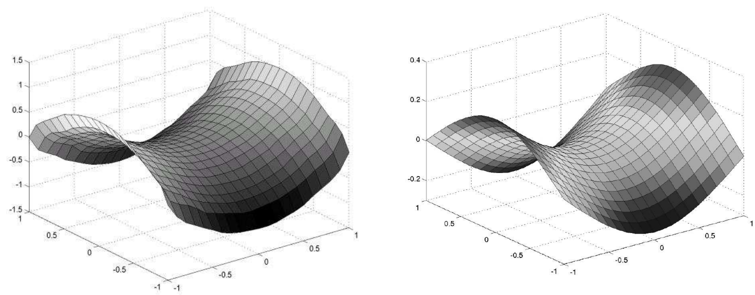

FIGURE 2. Left: Reconstructed surface obtained minimizing the distance-functional. Right: Reconstructed surface obtained minimizing the energy-functional.

are: the theoretical investigation of geometrical properties, to acquire a wider record of application cases and finally to study other functionals to obtain the optimal values of thye shape parameters.

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ON REVERSE HILBERT TYPE INEQUALITIES

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Abstract. In this paper we establish a new inverse inequality of Hilbert type for a finite number of positive sequences of real numbers. The integral analogue of the inequality are also proved. The results of this paper reduce to that of B. G. Pachpatte.

1. **Introduction**

In recent years several authors(see [1], [2], [3], [4], [5], [6], [7], [8]) have given considerable attention to Hilbert's inequalities and Hilbert type inequalities and their various generalizations. In particular, in 1988, B. G. Pachpatte[1] proved two new inequalities similar to Hilbert's inequality[9,P.226]. These two new results can be stated as follows, respectively:

Theorem A. *Let $\{a_m\}$ and $\{b_n\}$ be two nonnegative sequences of real numbers defined for $m = 1, 2, \ldots, k$ and $n = 1, 2, \ldots, r$ with $a_0 = b_0 = 0$ and let $\{p_m\}$ and $\{q_n\}$ be two positive sequences of real numbers defined for $m = 1, 2, \ldots, k$ and $n = 1, 2, \ldots, r$, where k, r are natural numbers and define $P_m = \sum_{s=1}^{m} p_s$ and $Q_n = \sum_{t=1}^{n} q_t$. Let ϕ and ψ be two real-valued nonnegative, convex and submultiplicative functions defined on $R_+ = [0, \infty)$. Then*

$$\sum_{m=1}^{k}\sum_{n=1}^{r}\frac{\phi(a_m)\psi(b_n)}{m+n} \leq M(k,r)\Bigg(\sum_{m=1}^{k}\Big(k-m+1\Big)\Big(p_m\phi(\frac{\nabla(a_m)}{p_m})\Big)^2\Bigg)^{1/2}$$

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$$\times\Big(\sum_{n=1}^{r}\Big(r-n+1\Big)\Big(q_n\psi\big(\frac{\nabla(b_n)}{q_n}\big)\Big)^2\Big)^{1/2}, \tag{1}$$

where

$$M(k,r)=\frac{1}{2}\left(\sum_{m=1}^{k}\Big(\frac{\phi(P_m)}{P_m}\Big)^2\right)^{1/2}\left(\sum_{n=1}^{r}\Big(\frac{\psi(Q_n)}{Q_n}\Big)^2\right)^{1/2},$$

and $\nabla(a_m)=a_m-a_{m-1}, \nabla(b_n)=b_n-b_{n-1}$.

Theorem B. *Let* $f\in C^1[[0,x),R_+], g\in C^1[[0,y),R_+]$ *with* $f(0)=g(0)=0$ *and let* $p(\sigma)$ *and* $q(\tau)$ *be two positive functions defined for* $\sigma\in[0,x)$ *and* $\tau\in[0,y)$, *and* $P(s)=\int_0^s p(\sigma)d\sigma$ *and* $Q(t)=\int_0^t q(\tau)d\tau$ *for* $s\in[0,x)$ *and* $t\in[0,y)$, *where* x,y *are positive real numbers. Let* ϕ *and* ψ *be as in Theorem A. Then*

$$\int_0^x\int_0^y\frac{\phi\Big(f(s)\Big)\psi\Big(g(t)\Big)}{s+t}dsdt\le L(x,y)\left(\int_0^x(x-s)\Big(p(s)\phi\Big(\frac{f'(s)}{p(s)}\Big)\Big)^2ds\right)^{1/2}$$

$$\times\left(\int_0^y(y-t)\Big(q(t)\psi\Big(\frac{g'(t)}{q(t)}\Big)\Big)^2dt\right)^{1/2}, \tag{2}$$

where

$$L(x,y)=\frac{1}{2}\left(\int_0^x\Big(\frac{\phi\big(P(s)\big)}{P(s)}\Big)^2ds\right)^{1/2}\left(\int_0^y\Big(\frac{\psi\big(Q(t)\big)}{Q(t)}\Big)^2dt\right)^{1/2},$$

and ′ *denotes the derivative of a function.*

The main purpose of this paper is to establish reverse forms of the above two inequalities.

2. **Main results**

Theorem 1. *Let* $\{a_{i,m_i}\}(i=1,2,\ldots,n)$ *be* n *positive sequences of real numbers defined for* $m_i=1,2,\ldots,k_i$ *with* $a_{i,0}=0(i=1,2,\ldots,n)$, *where* $k_i(i=1,\ldots,n)$ *are the natural numbers. Let* $\{p_{i,m_i}\}$ *be* n *positive sequences of real numbers defined for* $m_i=1,2,\ldots,k_i(i=1,2,\ldots,n)$. *Set* $P_{i,m_i}=\sum_{s_i=1}^{m_i}p_{i,s_i}(i=1,2,\ldots,n)$. *Let* $\phi_i(i=1,2,\ldots,n)$ *be* n *real-valued nonnegative concave, supermultiplicative and non-decreasing functions defined on* $R_+=[0,+\infty)$. *Let* $\frac{1}{\alpha_i}+\frac{1}{\beta_i}=1, 0<\beta_i<1$

and $\sum_{i=1}^{n}\frac{1}{\alpha_i}=\frac{1}{\alpha}$. *Sut* $A_{i,m_i}^{(p_i)}=\nabla(a_{i,m_i})\cdot a_{i,m_i}^{p_i-1}$, *where the operator* ∇ *is defined by* $\nabla(a_{i,m_i})=a_{i,m_i}-a_{i,m_i-1}(i=1,2,\ldots,n)$ *and* $0\le p_i\le 1$ *are real numbers. Then*

$$\sum_{m_1=1}^{k_1}\cdots\sum_{m_n=1}^{k_n}\frac{\prod_{i=1}^{n}\phi_i(\frac{a_{i,m_i}^{p_i}}{p_i})}{\left(\alpha\sum_{i=1}^{n}\frac{1}{\alpha_i}m_i\right)^{1/\alpha}}$$

$$\ge M(k_1,k_2,\ldots,k_n)\prod_{i=1}^{n}\left(\sum_{m_i=1}^{k_i}(k_i-m_i+1)\left(p_{i,m_i}\phi_i\left(\frac{A_{i,m_i}^{(p_i)}}{p_{i,m_i}}\right)\right)^{\beta_i}\right)^{1/\beta_i}, \quad (3)$$

where

$$M(k_1,k_2,\ldots,k_n)=\prod_{i=1}^{n}\left(\sum_{m_i=1}^{k_i}\left(\frac{\phi_i(P_{i,m_i})}{P_{i,m_i}}\right)^{\alpha_i}\right)^{1/\alpha_i}.$$

Proof. By using the following inequality(see Hardy *et al.* [9,P.39])

$$h_i x_{i,m_i}^{h_i-1}(x_{i,m_i}-y_{i,m_i})\le x_{i,m_i}^{h_i}-y_{i,m_i}^{h_i}\le h_i y_{i,m_i}^{h_i-1}(x_{i,m_i}-y_{i,m_i}),$$

where $x_{i,m_i}>0$ and $y_{i,m_i}>0$ and $0\le h_i\le 1(i=1,2,\ldots,n)$, we obtain that

$$a_{i,m_i+1}^{p_i}-a_{i,m_i}^{p_i}\ge p_i(a_{i,m_i+1})^{p_i-1}(a_{i,m_i+1}-a_{i.m_i})=p_i(a_{i,m_i+1})^{p_i-1}\cdot\nabla(a_{i,m_i+1}).$$

Consequently

$$\sum_{m_i=0}^{k_i-1}a_{i,m_i+1}^{p_i}-a_{i,m_i}^{p_i}=a_{i,k_i}^{p_i}\ge p_i\sum_{m_i=0}^{k_i-1}\nabla(a_{i,m_i+1})\cdot a_{i,m_i+1}^{p_i-1}=p_i\sum_{m_i=1}^{k_i}A_{i,m_i}^{(p_i)}.$$

Hence

$$\frac{a_{i,m_i}^{p_i}}{p_i}\ge\sum_{s_i=1}^{m_i}A_{i,s_i}^{(p_i)}. \quad (4)$$

On the other hand, from the following theorem of the Arithmetic and Geometric means[9,p.17]

$$\prod_{i=1}^{n}b_i^{q_i}\le\left(\frac{\sum_{i=1}^{n}q_ib_i}{\sum_{i=1}^{n}q_i}\right)^{\sum_{i=1}^{n}q_i},$$

where $q_i>0, b_i>0$, we easy get the following result

$$\prod_{i=1}^{n}m_i^{1/\alpha_i}\ge\left(\alpha\sum_{i=1}^{n}\frac{1}{\alpha_i}m_i\right)^{1/\alpha}, \quad (5)$$

where $\alpha_i<0$.

From (4), (5) and in view of Jensen's inequality and inverse Hölder's inequality[10], we obtain that

$$\prod_{i=1}^{n}\phi_i(\frac{a_{i,m_i}^{p_i}}{p_i}) \geq \prod_{i=1}^{n}\phi_i\left(\frac{P_{i,m_i}\sum_{s_i=1}^{m_i}p_{i,s_i}\left(\frac{A_{i,s_i}^{(p_i)}}{p_{i,s_i}}\right)}{\sum_{s_i=1}^{m_i}p_{i,s_i}}\right)$$

$$\geq \prod_{i=1}^{n}\phi_i(P_{i,m_i})\cdot\phi_i\left(\frac{\sum_{s_i=1}^{m_i}p_{i,s_i}\left(\frac{A_{i,s_i}^{(p_i)}}{p_{i,s_i}}\right)}{\sum_{s_i=1}^{m_i}p_{i,s_i}}\right)$$

$$\geq \prod_{i=1}^{n}\frac{\phi_i(P_{i,m_i})}{P_{i,m_i}}\sum_{s_i=1}^{m_i}p_{i,s_i}\phi_i\left(\frac{A_{i,s_i}^{(p_i)}}{p_{i,s_i}}\right)$$

$$\geq \prod_{i=1}^{n}\frac{\phi_i(P_{i,m_i})}{P_{i,m_i}}m_i^{1/\alpha_i}\left(\sum_{s_i=1}^{m_i}\left(p_{i,s_i}\phi_i\left(\frac{A_{i,s_i}^{(p_i)}}{p_{i,s_i}}\right)\right)^{\beta_i}\right)^{1/\beta_i}$$

$$\geq \left(\alpha\sum_{i=1}^{n}\frac{1}{\alpha_i}m_i\right)^{1/\alpha}\prod_{i=1}^{n}\frac{\phi_i(P_{i,m_i})}{P_{i,m_i}}\left(\sum_{s_i=1}^{m_i}\left(p_{i,s_i}\phi_i\left(\frac{A_{i,s_i}^{(p_i)}}{p_{i,s_i}}\right)\right)^{\beta_i}\right)^{1/\beta_i}. \quad (6)$$

Dividing both sides of (6) by $\left(\alpha\sum_{i=1}^{n}\frac{1}{\alpha_i}m_i\right)^{1/\alpha}$ and then taking the sum over m_i from 1 to $k_i (i = 1, 2, \ldots, n)$ and in view of inverse Hölder's inequality, we have

$$\sum_{m_1=1}^{k_1}\cdots\sum_{m_n=1}^{k_n}\frac{\prod_{i=1}^{n}\phi_i(\frac{a_{i,m_i}^{p_i}}{p_i})}{\left(\alpha\sum_{i=1}^{n}\frac{1}{\alpha_i}m_i\right)^{1/\alpha}}$$

$$\geq \prod_{i=1}^{n}\left(\sum_{m_i=1}^{k_i}\frac{\phi_i(P_{i,m_i})}{P_{i,m_i}}\left(\sum_{s_i=1}^{m_i}\left(p_{i,s_i}\phi_i\left(\frac{A_{i,s_i}^{(p_i)}}{p_{i,s_i}}\right)\right)^{\beta_i}\right)^{1/\beta_i}\right)$$

$$\geq \prod_{i=1}^{n}\left(\sum_{m_i=1}^{k_i}\left(\frac{\phi_i(P_{i,m_i})}{P_{i,m_i}}\right)^{\alpha_i}\right)^{1/\alpha_i}\left(\sum_{m_i=1}^{k_i}\sum_{s_i=1}^{m_i}\left(p_{i,s_i}\phi_i\left(\frac{A_{i,s_i}^{(p_i)}}{p_{i,s_i}}\right)\right)^{\beta_i}\right)^{1/\beta_i}$$

$$= M(k_1,k_2,\ldots,k_n)\prod_{i=1}^{n}\left(\sum_{m_i=1}^{k_i}\sum_{s_i=1}^{m_i}\left(p_{i,s_i}\phi_i\left(\frac{A_{i,s_i}^{(p_i)}}{p_{i,s_i}}\right)\right)^{\beta_i}\right)^{1/\beta_i}$$

$$= M(k_1,k_2,\ldots,k_n)\prod_{i=1}^{n}\left(\sum_{m_i=1}^{k_i}(k_i-m_i+1)\left(p_{i,m_i}\phi_i\left(\frac{A_{i,m_i}^{(p_i)}}{p_{i,m_i}}\right)\right)^{\beta_i}\right)^{1/\beta_i}.$$

The proof is complete.

Remark 1. Taking for $\beta_i = \frac{n-1}{n}(i = 1, \ldots, n)$ in (3), (3) changes to

$$\sum_{m_1=1}^{k_1} \cdots \sum_{m_n=1}^{k_n} \frac{\prod_{i=1}^{n} \phi_i(\frac{a_{i,m_i}^{p_i}}{p_i})}{(m_1 + \cdots + m_n)^{-n/(n-1)}}$$

$$\geq \bar{M}(k_1, k_2, \ldots, k_n) \prod_{i=1}^{n} \left(\sum_{m_i=1}^{k_i} (k_i - m_i + 1) \left(p_{i,m_i} \phi_i \left(\frac{A_{i,m_i}^{(p_i)}}{p_{i,m_i}} \right) \right)^{(n-1)/n} \right)^{n/(n-1)}, \tag{7}$$

where

$$\bar{M}(k_1, k_2, \ldots, k_n) = n^{n/(n-1)} \prod_{i=1}^{n} \left(\sum_{m_i=1}^{k_i} \left(\frac{\phi_i(P_{i,m_i})}{P_{i,m_i}} \right)^{-(n-1)} \right)^{-1/(n-1)}.$$

Taking for $n = 2$ and $p_i = 1(i = 1, 2)$ in (7), (7) becomes

$$\sum_{m_1=1}^{k_1} \sum_{m_2=1}^{k_2} \frac{\phi_1(a_{1,m_1})\phi_2(a_{2,m_2})}{(m_1 + m_2)^{-2}} \geq$$

$$\geq M(k_1, k_2) \Big(\sum_{m_1=1}^{k_1} \Big(k_1 - m_1 + 1\Big) \Big(p_{1,m_1} \phi_1 (\frac{\nabla(a_{1,m_1})}{p_{1,m_1}})\Big)^{1/2} \Big)^2$$

$$\times \Big(\sum_{m_2=1}^{k_2} \Big(k_2 - m_2 + 1\Big) \Big(p_{2,m_2} \phi_2 (\frac{\nabla(a_{2,m_2})}{p_{2,m_2}})\Big)^{1/2} \Big)^2, \tag{8}$$

where

$$M(k_1, k_2) = 4 \left(\sum_{m_1=1}^{k_1} \Big(\frac{\phi_1(P_{1,m_1})}{P_{1,m_1}} \Big)^{-1} \right)^{-1} \left(\sum_{m_2=1}^{k_2} \Big(\frac{\phi_2(P_{2,m_2})}{P_{2,m_2}} \Big)^{-1} \right)^{-1},$$

and

$$\nabla(a_{1,m_1}) = a_{1,m_1} - a_{1,m_1-1}, \nabla(a_{2,m_2}) = a_{2,m_2} - a_{2,m_2-1}.$$

Inequality (8) is just an reverse form of inequality (1) which was stated in the introduction.

Theorem 2. *Let* $f_i(\sigma_i) \in C^1[[0, x_i), [0, \infty)], i = 1, \ldots, n,$ *with* $f_i(0) = 0$, *Let* $p_i(\sigma_i)$ *be n positive functions defined for* $\sigma_i \in [0, x_i)(i = 1, 2, \cdots, n)$ *and define* $P_i(s_i) = \int_0^{s_i} p_i(\sigma_i) d\sigma_i$, for $s_i \in [0, x_i)$, *where* x_i *are positive real numbers and set* $F_{i,\sigma_i}^{(p_i)} = f_i'(\sigma_i) f_i^{p_i - 1}(\sigma_i)$, *where* p_i *are real numbers. Let* $\phi_i(i = 1, 2, \ldots, n)$ *be n* *real-valued nonnegative concave and supermultiplicative functions defined on* $R_+ = [0, +\infty)$. *Let* α_i, β_i *and* α *be as in Theorem 1. Then*

$$\int_0^{x_1} \cdots \int_0^{x_n} \frac{\prod_{i=1}^n \phi_i\left(\frac{f_i^{p_i}(s_i)}{p_i}\right)}{\left(\alpha \sum_{i=1}^n \frac{1}{\alpha_i} s_i\right)^{1/\alpha}} ds_1 \cdots ds_n$$

$$\geq L(x_1, \cdots, x_n) \prod_{i=1}^n \left(\int_0^{x_i} (x_i - s_i)\left(p_i(s_i)\phi_i(\frac{F_{i,s_i}^{(p_i)}}{p_i(s_i)})\right)^{\beta_i} ds_i\right)^{1/\beta_i}, \qquad (9)$$

where

$$L(x_1, \cdots, x_n) = \prod_{i=1}^n \left(\int_0^{x_i} \left(\frac{\phi_i\left(P_i(s_i)\right)}{P_i(s_i)}\right)^{\alpha_i} ds_i\right)^{1/\alpha_i}.$$

Proof. From the hypotheses, we have

$$f_i^{p_i}(s_i) = p_i \int_0^{s_i} F_{i,\sigma_i}^{(p_i)} d\sigma_i, s_i \in [0, x_i).$$

By using Jensen integral inequality and inverse Hölder integral inequality and notice that $\phi_i (i = 1, 2, \cdots, n)$ are n real-valued supermultiplicative functions, it is easy to observe that

$$\prod_{i=1}^n \phi_i\left(\frac{f_i^{p_i}(s_i)}{p_i}\right) = \prod_{i=1}^n \phi_i\left(\frac{P_i(s_i)\int_0^{s_i} p_i(\sigma_i)\frac{F_{i,\sigma_i}^{(p_i)}}{p_i(\sigma_i)}d\sigma_i}{\int_0^{s_i} p_i(\sigma_i)d\sigma_i}\right)$$

$$\geq \prod_{i=1}^n \phi_i(P_i(s_i))\phi_i\left(\frac{\int_0^{s_i} p_i(\sigma_i)\frac{F_{i,\sigma_i}^{(p_i)}}{p_i(\sigma_i)}d\sigma_i}{\int_0^{s_i} p_i(\sigma_i)d\sigma_i}\right) \geq \prod_{i=1}^n \frac{\phi_i\left(P_i(s_i)\right)}{P_i(s_i)} \int_0^{s_i} p_i(\sigma_i)\phi_i\left(\frac{F_{i,\sigma_i}^{(p_i)}}{p_i(\sigma_i)}\right) d\sigma_i$$

$$\geq \prod_{i=1}^n \left(\frac{\phi_i\left(P_i(s_i)\right)}{P_i(s_i)}\right) s_i^{1/\alpha_i} \left(\int_0^{s_i} \left(p_i(\sigma_i)\phi_i\left(\frac{F_{i,\sigma_i}^{(p_i)}}{p_i(\sigma_i)}\right)\right)^{\beta_i} d\sigma_i\right)^{1/\beta_i}. \qquad (10)$$

In view of inequality (5) and integrating two sides of (10) over s_i from 0 to $x_i (i = 1, 2, \ldots, n)$ and noticing reverse Hölder integral inequality, we observe that

$$\int_0^{x_1} \cdots \int_0^{x_n} \frac{\prod_{i=1}^n \phi_i(\frac{f_i^{p_i}(s_i)}{p_i})}{\left(\alpha \sum_{i=1}^n \frac{1}{\alpha_i} s_i\right)^{1/\alpha}} ds_1 \cdots ds_n$$

$$\geq \prod_{i=1}^{n} \int_0^{x_i} \left(\frac{\phi_i\left(P_i(s_i)\right)}{P_i(s_i)} \right) \left(\int_0^{s_i} \left(p_i(\sigma_i)\phi_i \left(\frac{F_{i,\sigma_i}^{(p_i)}}{p_i(\sigma_i)} \right) \right)^{\beta_i} d\sigma_i \right)^{1/\beta_i} ds_i$$

$$\geq \prod_{i=1}^{n} \left(\int_0^{x_i} \left(\frac{\phi_i\left(P_i(s_i)\right)}{P_i(s_i)} \right)^{\alpha_i} ds_i \right)^{1/\alpha_i} \left(\int_0^{x_i} \int_0^{s_i} \left(p_i(\sigma_i)\phi_i \left(\frac{F_{i,\sigma_i}^{(p_i)}}{p_i(\sigma_i)} \right) \right)^{\beta_i} d\sigma_i ds_i \right)^{1/\beta_i}$$

$$= L(x_1, \ldots, x_n) \prod_{i=1}^{n} \left(\int_0^{x_i} (x_i - s_i) \left(p_i(s_i)\phi_i(\frac{F_{i,s_i}^{(p_i)}}{p_i(s_i)}) \right)^{\beta_i} ds_i \right)^{1/\beta_i}.$$

This completes the proof of Theorem 2.

Remark 2. Taking for $\beta_i = \frac{n-1}{n} (i = 1, \ldots, n)$ in (9), (9) changes to

$$\int_0^{x_1} \cdots \int_0^{x_n} \frac{\prod_{i=1}^{n} \phi_i(\frac{f_i^{p_i}(s_i)}{p_i})}{(s_1 + \cdots + s_n)^{-n/(n-1)}} ds_1 \cdots ds_n$$

$$\geq \bar{L}(x_1, \cdots, x_n) \prod_{i=1}^{n} \left(\int_0^{x_i} (x_i - s_i) \left(p_i(s_i)\phi_i(\frac{F_{i,s_i}^{(p_i)}}{p_i(s_i)}) \right)^{(n-1)/n} ds_i \right)^{n/(n-1)}, \quad (11)$$

where

$$\bar{L}(x_1, \cdots, x_n) = n^{n/(n-1)} \prod_{i=1}^{n} \left(\int_0^{x_i} \left(\frac{\phi_i\left(P_i(s_i)\right)}{P_i(s_i)} \right)^{-(n-1)} ds_i \right)^{-1/(n-1)}.$$

Taking $n = 2$ and $p_i = 1$ to (11), (11) changes to

$$\int_0^{x_1} \int_0^{x_2} \frac{\phi_1\Big(f_1(s_1)\Big)\phi_2\Big(f_2(s_2)\Big)}{(s_1 + s_2)^{-2}} ds_1 ds_2 \geq$$

$$\geq L(x_1, x_2)\left(\int_0^{x_1} (x_1 - s_1)\left(p_1(s_1)\phi_1\Big(\frac{f_1'(s_1)}{p_1(s_1)}\Big) \right)^{1/2} ds_1 \right)^2$$

$$\times \left(\int_0^{x_2} (x_2 - s_2)\left(p_2(s_2)\phi_2\Big(\frac{f_2'(s_2)}{p_2(s_2)}\Big) \right)^{1/2} ds_2 \right)^2, \quad (12)$$

where

$$\bar{L}(x_1, x_2) = 4\left(\int_0^{x_1} \left(\frac{\phi_1\Big(P_1(s_1)\Big)}{P_1(s_1)} \right)^{-1} ds_1 \right)^{-1} \left(\int_0^{x_2} \left(\frac{\phi_2\Big(P_2(s_2)\Big)}{P_2(s_2)} \right)^{-1} ds_2 \right)^{-1}.$$

Inequality (12) is just an reverse form of inequality (2) which was stated in the introduction.

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UNIQUENESS ALGEBRAIC CONDITIONS IN THE STUDY OF SECOND ORDER DIFFERENTIAL SYSTEMS

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Abstract. The purpose of this paper is to give some algebraic conditions for the coefficients of an second order differential system in order to obtain some uniqueness and comparison results.

1. **Introduction**

Let us consider the following second order differential system:

$$Lu := \delta^2 I_n \frac{d^2u}{dx^2} + B(x)\frac{du}{dx} + C(x)u = 0, \delta > 0, \tag{1}$$

where $B, C \in C([a,b], M_n(\mathbb{R}))$, and the following statement:

$$\left.\begin{array}{c} u \in C^2([a,b], \mathbb{R}^n) \\ Lu = 0,\ in\ ]a,b[\\ u(a) = u(b) = 0 \end{array}\right\} \Longrightarrow u \equiv 0\, in\ [a,b] \tag{2}$$

It is well known the fact that if u satisfies a maximum principle, then the statement (2) automatically take place. The aim of this paper is to determine effective algebraic conditions for B, C such that the statement (2) to take place, without using a maximum principle. Let $A \in M_n(\mathbb{R}), J$ the Jordan normal form of A. We know that there exist a nonsingular matrix T such that $A = TJT^{-1}$.

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We will denote:

$$\begin{aligned} \widetilde{\alpha} &= \begin{cases} \frac{1}{n}\sum\limits_{k=1}^{s} n_k\lambda_k, \lambda_k \in \mathbb{R} \\ \frac{1}{n}\sum\limits_{k=1}^{s} n_k \ Re\lambda_k, \lambda_k \in \mathbb{C}\backslash\mathbb{R} \end{cases} \\ \gamma_F &= \|T\|_F \cdot \left\|T^{-1}\right\|_F \\ m_F &= \|J - \widetilde{\alpha} I\|_F \end{aligned}$$

where λ_k are the eigenvalues of A, n_k is the number of λ_k which appears in Jordan blocks (generated by λ_k) and $\|\cdot\|_F$ is the euclidean norm of a matrix (see [2]).

We shall use the following result given in [2]:

Theorem 1. *Let* $\varphi_{\|\cdot\|} : \mathbb{R} \to \mathbb{R}, \varphi_{\|\cdot\|}(\alpha) = \|A - \alpha I_n\|$, $\|\cdot\|$ *being one of the following norms:* $\|\cdot\|_F$, $\|\cdot\|_1$, $\|\cdot\|_2$, $\|\cdot\|_\infty$. *In these conditions:*

$$\varphi_{\|\cdot\|}(\widetilde{\alpha}) \leq \sqrt{n}\gamma_F m_F.$$

Remark 1. *In case of euclidean norm* $\|\cdot\|_F$ *and spectral norm* $\|\cdot\|_2$ *we have that* $\varphi_{\|\cdot\|}(\widetilde{\alpha}) \leq \gamma_F m_F$ *(see [2]). Because* $n \geq 2$, *if* $m_F \neq 0$, *then:*

$$\varphi_{\|\cdot\|}(\widetilde{\alpha}) < \sqrt{n}\gamma_F m_F.$$

Conditions determined here will be very useful to obtain comparison results (see Section 3 of the paper).

2. **Establishing the conditions in which the statement** (2) **take place**

Let $u \in C^2([a,b], \mathbb{R}^n)$, $u \neq 0$, a solution of the system (1) with the property that $u(a) = u(b) = 0$. We have:

$$u^* Lu = \delta^2 u^* \frac{d^2u}{dx^2} + u^* B(x) \frac{du}{dx} + u^* C(x) u$$

$$\delta^2 \frac{d}{dx}\left(u^* \frac{du}{dx}\right) = u^* Lu - u^* B(x) \frac{du}{dx} - u^* C(x) u + \delta^2 \frac{du^*}{dx}\frac{du}{dx}.$$

If we integrate on $[a,b]$, we obtain:

$$\int_a^b \left(\delta^2 \frac{du^*}{dx}\frac{du}{dx} - u^* B(x)\frac{du}{dx} - u^* C(x)\, u\right) dx = 0.$$

Let us denote

$$E := \delta^2 \frac{du^*}{dx}\frac{du}{dx} - u^* B(x)\frac{du}{dx} - u^* C(x)\, u.$$

We shall show that under some assumptions for the coefficients B and C this expression is positive which will imply the fact that the integral can not be zero on $[a,b]$, only if $E \equiv 0$. Let $u = R \cdot e$, where $R = \|u\| = \left(\sum\limits_{i=1}^{n} u_i^2\right)^{\frac{1}{2}}$,

$$e \in C^2([a,b], \mathbb{R}^n),\, e = \begin{pmatrix} e_1 \\ . \\ . \\ . \\ e_n \end{pmatrix}, e^* = (e_1, ..., e_n), \|e\| = \left(\sum_{i=1}^{n} e_i^2\right)^{\frac{1}{2}} = 1.$$

A simple computation shows us that

$$E = \delta^2 (R')^2 + e^* B(x)\, e R R' - (e^* L e)\, R^2 \tag{3}$$

where

$$e^* L e = -\delta^2 \|e'\|^2 + e^* B(x)\, e' + e^* C(x)\, e.$$

The quadric form (3) is positive if and only if

$$[e^* B(x)\, e]^2 + 4\delta^2 e^* L e \le 0. \tag{4}$$

It is simple to see that:

$$e^* L e \le \frac{1}{4\delta^2}\left\| B(x) - \widetilde{\beta}(x)\, I_n \right\|^2 + e^* C(x)\, e$$

From Theorem 1 we know that for every $x \in]a,b[$ there exist $\widetilde{\beta}(x) \in \mathbb{R}$ such that

$$\left\| B(x) - \widetilde{\beta}(x)\, I_n \right\| \le \gamma_F m_F.$$

If we suppose that $m_F \neq 0$, than we have:

$$\left\| B(x) - \widetilde{\beta}(x)\, I_n \right\| < \gamma_F m_F.$$

Under assumption that

$$e^{*}C\left(x\right)e\leq-\frac{1}{4\delta^{2}}n\left(\gamma_{F}m_{F}\right)^{2},\forall x\in\left]a,b\right[\tag{5}$$

we obtain:

$$e^{*}Le\leq\frac{1}{4\delta^{2}}\left[\left\|B\left(x\right)-\widetilde{\beta}\left(x\right)I_{n}\right\|^{2}-n\left(\gamma_{F}m_{F}\right)^{2}\right]:=-p^{2}\left(x\right)<0. \tag{6}$$

Supposing that

$$e^{*}B\left(x\right)e\leq2\delta p\left(x\right), \tag{7}$$

we observe that the relation (4) take place. In conclusion if (5) and (7) take place, then the quadric form (3) is positive and that means that the integral can not be identically null, only if $E\equiv0$. But, if $E\equiv0$, then

$$\frac{d}{dx}\left(u^{*}\frac{du}{dx}\right)=0,$$

meaning that $\left\|u\right\|^{2}$ is constant. Because $u\left(a\right)=u\left(b\right)=0$, we obtain that $u\equiv0$. In this way, if $m_{F}\neq0$, we obtain the following result:

Theorem 2. *Suppose that:*

1. $e^{*}C\left(x\right)e\leq-\frac{1}{4\delta^{2}}n\left(\gamma_{F}m_{F}\right)^{2},\forall x\in\left]a,b\right[;$
2. $e^{*}B\left(x\right)e\leq2\delta p\left(x\right);$

$\forall e\in C^{2}\left(\left[a,b\right],\mathbb{R}^{n}\right),\left\|e\right\|=\left(\sum\limits_{i=1}^{n}e_{i}^{2}\right)^{\frac{1}{2}}=1,$ *with* p *as in* (6). *In these conditions the statement* (2) *take place.*

Example 1. *If we consider system* (1), *in the case* $n=2$, *with* $B=\begin{pmatrix}a_{1}&a_{2}\\a_{3}&a_{1}\end{pmatrix}$, $a_{2},a_{3}>0$, $a_{1}^{2}\leq a_{2}^{2}+4a_{2}a_{3}+a_{3}^{2}$, *we have an example of matrix* B *which verifies* (7), *with* $p^{2}=\frac{1}{4\delta^{2}}\left(3a_{2}^{2}+8a_{2}a_{3}+3a_{3}^{2}\right).$

Remark 2. *If* $m_{F}=0$, *then the statement* (2) *still take place if*

$$e^{*}C\left(x\right)e\leq-\frac{1}{4\delta^{2}}\left(\widetilde{\beta}\left(x\right)\right)^{2},\forall x\in\left]a,b\right[.$$

3. **A comparison result**

Let us consider the following second order differential systems:

$$Lu := \lambda^2 \frac{d^2u}{dx^2} + B(x) \frac{du}{dx} + C(x) u = 0 \tag{8}$$

$$Mv := \mu^2 \frac{d^2u}{dx^2} + Q(x) v = 0, \tag{9}$$

with $B \in C^1([a,b], M_n(\mathbb{R})), C, Q \in C([a,b], M_n(\mathbb{R})), \lambda > \mu > 0$. Using the same method as in section 2 of this paper we obtain the following result (in case $m_F \neq 0$):

Theorem 3. *Suppose:*

1. *Q is symmetric;*
2. *There exist a solution matrix S of system* (9) *such that* $\det S(x) \neq 0$ *in* $[a,b]$ *and the matrix* $\frac{dS}{dx} S^{-1}$ *is symmetric..*

If:

(i): $e^* (C(x) - Q(x)) e \leq -\frac{1}{4\delta^2} n (\gamma_F m_F)^2, \forall x \in]a,b[;$

(ii): $e^* B(x) e \leq 2\delta p(x), \forall x \in]a,b[.$

$\forall e \in C^2([a,b], \mathbb{R}^n), \|e\| = \left(\sum\limits_{i=1}^{n} e_i^2 \right)^{\frac{1}{2}} = 1,$*with* $\delta^2 = \lambda^2 - \mu^2,$

$p^2(x) = \frac{1}{4\delta^2} \left[n (\gamma_F m_F)^2 - \left\| B(x) - \widetilde{\beta}(x) I_n \right\|^2 \right] > 0,$ *then the system* (8) *is non-oscillatory.*

Remark 3. *If $m_F = 0$, conditions* (*i*) *and* (*ii*) *from Theorem 3 are reduced to:*

$$e^* (C(x) - Q(x)) e \leq -\frac{1}{4\delta^2} \left(\widetilde{\beta}(x) \right)^2, \forall x \in]a,b[.$$

Remark 4. *Theorem 3 improve a result given by I.A. Rus in* [5].

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ON THE INVARIANCE PROPERTY OF THE FISHER INFORMATION (I)

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Abstract. The objective of this paper is to give some properties for the Fisher's information measure when $X_{a\leftrightarrow b}$ represents a bilateral truncated random variable that corresponds to a normal random variable X with the probability density function $f(x;\theta)$, where $\theta = (m, \sigma^2)$, $\theta \in D_\theta$, $D_\theta \subseteq \mathbb{R}^2$, $m \in \mathbb{R}$, $\sigma^2 \in \mathbb{R}^+$.

The Fisher's invariance property will be studied in the case of a truncated normal distribution.

Let X be a normal distribution with probability density function

$$f(x;m,\sigma^2) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left\{-\frac{1}{2}\left(\frac{x-m}{\sigma}\right)^2\right\}, x \in \mathbb{R}, \tag{1}$$

where the parameters m and σ have their usual significance, namely: $m = E(X)$, $\sigma^2 = Var(X)$, $m \in \mathbb{R}$, $\sigma > 0$.

Definition 1. [1] *We say that the random variable X has a normal distribution truncated to the left at $X = a, a \in \mathbb{R}$ and to the right at $X = b, b \in \mathbb{R}$, denoted by $X_{a\leftrightarrow b}$, if its probability density function, denoted by $f_{a\leftrightarrow b}(x;m,\sigma^2)$, has the form*

$$f_{a\leftrightarrow b}(x;m,\sigma^2) = \begin{cases} \dfrac{k(a,b)}{\sqrt{2\pi}\sigma} \exp\left\{-\dfrac{1}{2}\left(\dfrac{x-m}{\sigma}\right)^2\right\} & if\ a \le x \le b, \\ 0 & if\ x < a\ or\ x > b, \end{cases} \tag{2}$$

where

$$k(a,b) = \frac{1}{A} = \frac{1}{\Phi\left(\dfrac{b-m}{\sigma}\right) - \Phi\left(\dfrac{a-m}{\sigma}\right)}, \tag{3}$$

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$$\Phi(z)=\frac{1}{\sqrt{2\pi}}\int\limits_{-\infty}^{z}\exp\left(-\frac{t^2}{2}\right)dt, \tag{4}$$

$$\Phi(-\infty)=0,\ \Phi(0)=\frac{1}{2},\ \Phi(+\infty)=1,\ \Phi(-z)=1-\Phi(z), \tag{5}$$

$\Phi(z)$ *is the standard normal distribution function corresponding to the standard normal random variable*

$$Z=\frac{X-m}{\sigma},\ E(Z)=0, Var(Z)=1. \tag{6}$$

The probability density function of the random variable Z has the form

$$f(z;0,1)=f(z)=\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{t^2}{2}\right), z\in(-\infty,+\infty). \tag{7}$$

Remark 1. *A truncated probability distribution can be regarded as a conditional probability distribution in the sense that if X has an unrestricted distribution with probability density function $f(x)$ then $f_{a\leftrightarrow b}(x)$, as defined above, is the probability density function which governs the behavior of X subject to the condition that X is known to lie in $[a,b]$.*

Theorem 1. [2] *Let $X_{a\leftrightarrow b}$ be a random variable with a normal distribution truncated to the left at $X=a$ and to the right at $X=b$. Then*

$$E(X_{a\leftrightarrow b})=m-\frac{\sigma^2}{A}\left[f(b;m,\sigma^2)-f(a;m,\sigma^2)\right], \tag{8}$$

where

$$f(a;m,\sigma^2)=f(x;m,\sigma^2)\mid_{x=a}=\frac{1}{\sqrt{2\pi}\sigma}\exp\left(-\frac{1}{2}\left(\frac{a-m}{\sigma}\right)^2\right), \tag{9}$$

$$f(b;m,\sigma^2)=f(x;m,\sigma^2)\mid_{x=b}=\frac{1}{\sqrt{2\pi}\sigma}\exp\left(-\frac{1}{2}\left(\frac{b-m}{\sigma}\right)^2\right). \tag{10}$$

Theorem 2. [2] *Let $X_{a\leftrightarrow b}$ be a random variable with a normal distribution truncated to the left at $X=a$ and to the right at $X=b$. Then*

$$E(X^2_{a\leftrightarrow b})=m^2+\sigma^2-\frac{\sigma^2}{A}\left((m+b)f(b;m,\sigma^2)-(m+a)f(b;m,\sigma^2)\right). \tag{11}$$

Corollary 1. [2] *If $X_{a\leftrightarrow b}$ is a random variable with a normal distribution truncated to the left at $X=a$ and to the right at $X=b$, then*

$$Var(X_{a\leftrightarrow b}) = \sigma^2 + \frac{\left(\sigma^2\right)^2}{A^2}\left(f(b;m,\sigma^2) - f(a;m,\sigma^2)\right)^2 + \tag{12}$$

$$+ \frac{\sigma^2}{A}\left((m-b)f(b;m,\sigma^2) - (m-a)f(a;m,\sigma^2)\right). \tag{13}$$

Corollary 2. [1] *For the random variables $X_{a\leftarrow}$, $X_{\rightarrow b}$ and X we have*

$$\lim_{a\to-\infty} f_{a\leftrightarrow b}(x;m,\sigma^2) = f_{\rightarrow b}(x;m,\sigma^2) = \tag{14}$$

$$= \begin{cases} \dfrac{1}{\Phi\left(\dfrac{b-m}{\sigma}\right)}\cdot f(x;m,\sigma^2) & if\ x \le b \\ 0 & if\ x > b, \end{cases} \tag{15}$$

$$\lim_{b\to+\infty} f_{a\leftrightarrow b}(x;m,\sigma^2) = f_{a\leftarrow}(x;m,\sigma^2) = \tag{16}$$

$$= \begin{cases} \dfrac{1}{1-\Phi\left(\dfrac{a-m}{\sigma}\right)}\cdot f(x;m,\sigma^2) & if\ x \ge a \\ 0 & if\ x < a, \end{cases} \tag{17}$$

and

$$\lim_{\substack{a\to-\infty,\\ b\to+\infty}} f_{a\leftrightarrow b}(x;m,\sigma^2) = f(x;m,\sigma^2) = \tag{18}$$

$$= \frac{1}{\sqrt{2\pi}\sigma}\exp\left(-\frac{1}{2}\left(\frac{x-m}{\sigma}\right)^2\right) \quad if\ x\in\mathbb{R}, \tag{19}$$

where $f_{\rightarrow b}(x;m,\sigma^2)$ is the probability density function when $X_{\rightarrow b}$ has a normal distribution truncated to the right at $X=b$; $f_{a\leftarrow}(x;m,\sigma^2)$ is the probability density function when $X_{a\leftarrow}$ has a normal distribution truncated to the left at $X=a$ and $f(x;m,\sigma^2)$ is the probability density function when X has an ordinary normal distribution.

Corollary 3. [1] *For the random variables* $X_{a\leftarrow}$, $X_{\rightarrow b}$ *and* X *we have*

$$E(X_{a\leftarrow}) = \lim_{b\to+\infty} E(X_{a\leftrightarrow b}) = \tag{20}$$

$$= m + \frac{\sigma^2}{1-\Phi\left(\frac{a-m}{\sigma}\right)} f(a;m,\sigma^2), \tag{21}$$

$$E(X_{\rightarrow b}) = \lim_{a\to-\infty} E(X_{a\leftrightarrow b}) = \tag{22}$$

$$= m - \frac{\sigma^2}{\Phi\left(\frac{b-m}{\sigma}\right)} f(b;m,\sigma^2), \tag{23}$$

and

$$E(X) = \lim_{\substack{a\to-\infty,\\ b\to+\infty}} E(X_{a\leftrightarrow b}) = \tag{24}$$

$$= m. \tag{25}$$

Corollary 4. [1] *For the random variables* $X_{a\leftarrow}$, $X_{\rightarrow b}$ *and* X *we have*

$$Var(X_{a\leftarrow}) = \lim_{b\to+\infty} Var(X_{a\leftrightarrow b}) = \tag{26}$$

$$= \sigma^2 + \frac{\left(\sigma^2\right)^2 f^2(a;m,\sigma^2)}{\left(1-\Phi(\frac{a-m}{\sigma})\right)^2} - \frac{\sigma^2(m-a)f(a;m,\sigma^2)}{1-\Phi(\frac{a-m}{\sigma})}, \tag{27}$$

$$Var(X_{\rightarrow b}) = \lim_{a\to-\infty} Var(X_{a\leftrightarrow b}) = \tag{28}$$

$$= \sigma^2 + \frac{\left(\sigma^2\right)^2 f^2(b;m,\sigma^2)}{\Phi^2(\frac{b-m}{\sigma})} + \frac{\sigma^2(m-b)f(b;m,\sigma^2)}{\Phi(\frac{b-m}{\sigma})}, \tag{29}$$

and

$$Var(X) = \lim_{\substack{a\to-\infty,\\ b\to+\infty}} Var(X_{a\leftrightarrow b}) = \sigma^2. \tag{30}$$

Let consider the case: $m-$*an unknown parameter*, σ^2-*a known parameter.*

Theorem 3. [2] *If the random variable $X_{a\leftrightarrow b}$ has a bilateral truncated normal distribution, that is, its probability distribution has the form (2), then the Fisher's information measure, about the unknown parameter m, has the following form*

$$I_{X_{a\leftrightarrow b}}(m) = \int_a^b \left(\frac{\partial \ln f_{a\leftrightarrow b}(x;m,\sigma^2)}{\partial m}\right)^2 f_{a\leftrightarrow b}(x;m,\sigma^2)dx = \tag{31}$$

$$= \frac{1}{\sigma^2} - \frac{[f(b;m,\sigma^2) - f(a;m,\sigma^2)]^2}{\sqrt{2\pi}\sigma A^2} + + \frac{(m-b)f(b;m,\sigma^2) - (m-a)f(a;m,\sigma^2)}{\sigma^2 A}, \tag{32}$$

where $f(a;m,\sigma^2)$ and $f(b;m,\sigma^2)$ are given in (9) and (10).

Corollary 5. *If $a = m-\sigma$, $b = m+\sigma$, then the Fisher's information measure, relative to the unknown parameter m, has the following value*

$$I_{X_{m-\sigma\leftrightarrow m+\sigma}}(m) = \frac{1}{\sigma^2}\left(1 - \frac{1}{0,341\sqrt{2\pi e}}\right), \tag{33}$$

moreover, we obtain the inequality

$$I_{X_{m-\sigma\leftrightarrow m+\sigma}}(m) < I_X(m). \tag{34}$$

Corollary 6. *(Invariance of the Fisher information - the first form) If we consider values $a = m$, $b = m+\sigma$ or $a = m-\sigma$, $b = m$, then the Fisher's information measures, relative to the unknown parameter m, has the same value, namely*

$$I_{X_{m\leftrightarrow m+\sigma}}(m) = I_{X_{m-\sigma\leftrightarrow m}}(m) = \frac{1}{\sigma^2}\left\{1 - \left(\frac{\left(1-\sqrt{e}\right)^2}{\left(\sqrt{2\pi e}.0,341\right)^2} + \frac{1}{\sqrt{2\pi e}.0,341}\right)\right\}, \tag{35}$$

moreover, we have the following inequality

$$I_{X_{m\leftrightarrow m+\sigma}}(m) = I_{X_{m-\sigma\leftrightarrow m}}(m) < I_X(m). \tag{36}$$

Corollary 7. *If $a = m - k\sigma$, $b = m + k\sigma$, $k \in \mathbb{N}^*$, then the Fisher's information measure, relative to the unknown parameter m, can be written like*

$$I_{X_{m-k\sigma\leftrightarrow m+k\sigma}}(m) = \frac{1}{\sigma^2}\left\{1 - \frac{2k}{\sqrt{2\pi e^{k^2}}(2\Phi(k)-1)}\right\},\ k \in \mathbb{N}^*, \tag{37}$$

moreover, we obtain the inequality

$$I_{X_{m-k\sigma\leftrightarrow m+k\sigma}}(m) < \frac{1}{\sigma^2} = I_X(m),\ k \in \mathbb{N}^*. \tag{38}$$

Remark 2. *In the particular case $k = 3$ we obtain a bilateral truncated random variable $X_{m-k\sigma\leftrightarrow m+k\sigma}$ and the Fisher's information measure, relative to the unknown parameter m, can be written like*

$$I_{X_{m-3\sigma\leftrightarrow m+3\sigma}}(m) = \frac{1}{\sigma^2}\left[1 - \frac{1}{\sqrt{2\pi e e^4}.0,166}\right], \tag{39}$$

moreover, we obtain the inequality

$$I_{X_{m-3\sigma\leftrightarrow m+3\sigma}}(m) < \frac{1}{\sigma^2} = I_X(m). \tag{40}$$

Corollary 8. *For the random variables $X_{a\leftarrow}$, $X_{\rightarrow b}$ and X the Fisher's information measures have the following forms*

$$I_{X_{a\leftarrow}}(m) = \lim_{b\to+\infty} I_{X_{a\leftrightarrow b}}(m) = \tag{41}$$

$$= \frac{1}{\sigma^2} - \frac{(m-a)f(a;m,\sigma^2)}{\sigma^2\left(1-\Phi\left(\frac{a-m}{\sigma}\right)\right)} - \frac{.f^2(a;m,\sigma^2)}{\left(1-\Phi\left(\frac{a-m}{\sigma}\right)\right)^2}, \tag{42}$$

$$I_{X_{\rightarrow b}}(m) = \lim_{a\to-\infty} I_{X_{a\leftrightarrow b}}(m) = \tag{43}$$

$$=\frac{1}{\sigma^2} + \frac{(m-b)f(b;m,\sigma^2)}{\sigma^2\Phi\left(\frac{b-m}{\sigma}\right)} - \frac{f^2(b;m,\sigma^2)}{\Phi^2\left(\frac{a-m}{\sigma}\right)}, \tag{44}$$

and

$$I_X(m) = \lim_{\substack{a\to-\infty,\\ b\to+\infty}} I_{X_{a\leftrightarrow b}}(m) = \frac{1}{\sigma^2}. \tag{45}$$

Corollary 9. *If $b = m$, then from (5) we obtain $\Phi(0) = \frac{1}{2}$ and from (44) it results the inequality*

$$I_{X_{\to m}}(m) = \frac{1}{\sigma^2}\left(1 - \frac{2}{\pi}\right) < \frac{1}{\sigma^2} = I_X(m). \quad (46)$$

Corollary 10. *If $b = m - \sigma$, then from (5) we obtain*

$$\Phi(-1) = 1 - \Phi(1) = 0,159, \quad (47)$$

and from (44), the following relations

$$I_{X_{\to m-\sigma}}(m) = \frac{1}{\sigma^2}\left(1 + \frac{1}{\sqrt{2\pi e}\Phi(-1)} - \frac{1}{\left(\sqrt{2\pi e}\Phi(-1)\right)^2}\right), \quad (48)$$

moreover, the inequalities

$$I_{X_{\to m}}(m) < I_X(m) < I_{X_{\to m-\sigma}}(m). \quad (49)$$

Corollary 11. *If $b = m + \sigma$, we have the following relations*

$$I_{X_{\to m+\sigma}}(m) = \frac{1}{\sigma^2}\left\{1 - \left(\frac{1}{\sqrt{2\pi e}\Phi(1)} + \frac{1}{\left(\sqrt{2\pi e}\Phi(1)\right)^2}\right)\right\}, \quad (50)$$

moreover, the inequalities

$$I_{X_{\to m+\sigma}}(m) < I_{X_{\to m}}(m) < I_X(m) < I_{X_{\to m-\sigma}}(m). \quad (51)$$

Proof. From (44), it results the equality (50) which imply the inequality

$$I_{X_{\to m+\sigma}}(m) = \frac{1}{\sigma^2}\left\{1 - \left(\frac{1}{\sqrt{2\pi e}\Phi(1)} + \frac{1}{\left(\sqrt{2\pi e}\Phi(1)\right)^2}\right)\right\} < \frac{1}{\sigma^2} = I_X(m). \quad (52)$$

Then, from (49) and (52) it results the inequalities

$$I_{X_{\to m+\sigma}}(m) < I_X(m) < I_{X_{\to m-\sigma}}(m). \quad (53)$$

Now, from (46), the inequality (51) is reduced to the following inequality

$$I_{X_{\to m+\sigma}}(m) < I_{\to m}(m). \quad (54)$$

Using the relations (46) and (50), we observe that this last inequality is equivalent to the inequalities

$$\frac{1}{\sqrt{2\pi e}\Phi(1)} + \frac{1}{\left(\sqrt{2\pi e}\Phi(1)\right)^2} < \frac{2}{\sqrt{2\pi e}\Phi(1)} < \frac{2}{\pi},$$

or to the inequality

$$\pi < \sqrt{2\pi e}\Phi(1).$$

This last inequality results using the approximations: $\pi \approx 3,14$, $e \approx 2,72$, $\Phi(1) = 0,841$. □

Corollary 12. *If $a = m$, then from (5) we obtain $\Phi(0) = \frac{1}{2}$ and from (42) it results the inequality*

$$I_{X_{m\leftarrow}}(m) = \frac{1}{\sigma^2}\left(1 - \frac{2}{\pi}\right) < \frac{1}{\sigma^2} = I_X(m). \tag{55}$$

Corollary 13. *If $a = m - \sigma$, then from (5) we obtain*

$$1 - \Phi(-1) = \Phi(1) = 0,841, \tag{56}$$

and from (42) it results the equality

$$I_{X_{m-\sigma\leftarrow}}(m) = \frac{1}{\sigma^2}\left\{1 - \left(\frac{1}{\sqrt{2\pi e}\Phi(1)} + \frac{1}{\left(\sqrt{2\pi e}\Phi(1)\right)^2}\right)\right\}, \tag{57}$$

moreover, the inequality

$$I_{X_{m-\sigma\leftarrow}}(m) < I_X(m). \tag{58}$$

Corollary 14. *If $a = m + \sigma$, then from (5) we obtain $\Phi(-1) = 0,159$, and from (42) it results the equality*

$$I_{X_{m+\sigma\leftarrow}}(m) = \frac{1}{\sigma^2}\left\{1 + \left(\frac{1}{\sqrt{2\pi e}\Phi(-1)} - \frac{1}{\left(\sqrt{2\pi e}\Phi(-1)\right)^2}\right)\right\}, \tag{59}$$

moreover, the inequalities

$$I_{X_{m-\sigma\leftarrow}}(m) < I_{m\leftarrow}(m) < I_X(m) < I_{X_{m+\sigma\leftarrow}}(m) \tag{60}$$

Proof. From the relation (42), we obtain the equality (59) which imply the inequality

$$I_{X_{m+\sigma\leftarrow}}(m) = \frac{1}{\sigma^2}\left\{1 + \left(\frac{1}{\sqrt{2\pi e}\Phi\left(-1\right)} - \frac{1}{\left(\sqrt{2\pi e}\Phi\left(-1\right)\right)^2}\right)\right\} > \frac{1}{\sigma^2} = I_X(m). \quad (61)$$

From (58) and (61) it results the inequalities

$$I_{X_{m-\sigma\leftarrow}}(m) < I_X(m) < I_{X_{m+\sigma\leftarrow}}(m). \quad (62)$$

Now, regarding the inequalities (55) and (62), we observe that the inequality (60) is reduced to the inequality

$$I_{X_{m-\sigma\leftarrow}}(m) < I_{X_{m\leftarrow}}(m). \quad (63)$$

By the relations (55) and (57), we observe that this last inequality is equivalent to the inequality

$$\frac{1}{\sigma^2}\left\{1 - \left(\frac{1}{\sqrt{2\pi e}\Phi\left(1\right)} + \frac{1}{\left(\sqrt{2\pi e}\Phi\left(1\right)\right)^2}\right)\right\} < \frac{1}{\sigma^2}\left(1 - \frac{2}{\pi}\right),$$

or to the inequalities

$$\frac{1}{\sqrt{2\pi e}\Phi\left(1\right)} + \frac{1}{\left(\sqrt{2\pi e}\Phi\left(1\right)\right)^2} < \frac{2}{\sqrt{2\pi e}\Phi\left(1\right)} < \frac{2}{\pi}.$$

The last inequality is equivalent to the inequality $\sqrt{2\pi e}\Phi\left(1\right) < \left(\sqrt{2\pi e}\Phi\left(1\right)\right)^2$ which imply the inequality

$$\pi < \sqrt{2\pi e}\Phi\left(1\right). \quad (64)$$

Using the approximations: $\pi \approx 3,14$, $e \approx 2,72$ and $\Phi(1) = 0,841$, the last inequality is true, because

$$\sqrt{2\pi e}\Phi\left(1\right) \approx \sqrt{2 \times 3,14 \times 2,72}.0,841 \approx 4,13.0,841 \approx 3,475.$$

□

The invariance of Fisher's information is ilustrated in the following corollaries.

Corollary 15. *(the second form)*

$$I_{X_{\to m+\sigma}}(m) = I_{X_{-\infty\leftrightarrow m+\sigma}}(m) = \tag{65}$$

$$= \frac{1}{\sigma^2}\left\{1 - \left(\frac{1}{\sqrt{2\pi e}\Phi(1)} + \frac{1}{\left(\sqrt{2\pi e}\Phi(1)\right)^2}\right)\right\} = \tag{66}$$

$$= I_{X_{m-\sigma\leftarrow}}(m) = I_{X_{m-\sigma\leftrightarrow+\infty}}(m). \tag{67}$$

Proof. Using the relations (50) and (57), the proof is obviously. □

Corollary 16. *(the third form)*

$$I_{X_{\to m-\sigma}}(m) = I_{X_{-\infty\leftrightarrow m-\sigma}}(m) = \tag{68}$$

$$= \frac{1}{\sigma^2}\left(1 + \frac{1}{\sqrt{2\pi e}\Phi(-1)} - \frac{1}{\left(\sqrt{2\pi e}\Phi(-1)\right)^2}\right) = I_{X_{m+\sigma\leftrightarrow+\infty}}(m). \tag{69}$$

Proof. Using the relations (48) and (59), the proof is obviously. □

Corollary 17. *(the fourth form)*

$$I_{X_{\to m}}(m) = I_{X_{-\infty\leftrightarrow m}}(m) = \frac{1}{\sigma^2}\left(1 - \frac{2}{\pi}\right) = I_{X_{m\leftarrow}}(m) = I_{X_{m\leftrightarrow+\infty}}. \tag{70}$$

Proof. Using the relations (46) and (55), the proof is obviously. □

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ON SOME INTERPOLATION PROBLEM ON TRIANGLE

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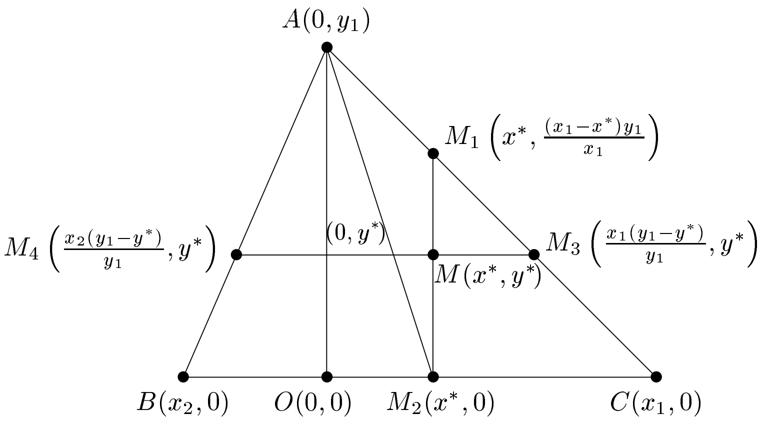
Abstract. Many of the theoretical and practical problems of numerical analysis consist in approximating of some types of functions on some kinds of domains like the triangular or rectangular domains. On the triangular domains the most of the approximations are made on some interior points of the triangle or on some derivatives values of the mentioned points. But also there it exists some types of functions which approximate the values of an entire side of the triangle or an entire interior line of this. The purpose of this paper is to present a this type of above mentioned function.

We shall suppose that we have given an interior point $M(x^*, y^*)$ on the triangle. Using an appropriate coordinates transform we shall suppose that the origin of the Oxy coordinates system and the triangle are situated as shows the next picture:



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Building from M the parallels to AO and BC and denoting by M_1, M_2 and M_3, M_4, respectively, the obtained intersections. In other words, the coordinates of the M_1 will be given by the solution of the system:

$$\begin{cases} AC: \ xy_1 + x_1y - x_1y_1 = 0 \\ x = x^* \end{cases}$$

So we shall find $M_1\left(\frac{(x_1 - x^*)y_1}{x_1}, x^*\right)$. Similar, we shall find the coordinates of the others above mentioned points: $M_2(x^*, 0)$, $M_3\left(\frac{x_1(y_1 - y^*)}{y_1}, y^*\right)$, $M_4\left(\frac{x_2(y_1 - y^*)}{y_1}, y^*\right)$ as being the intersections between $x = x^*$ and OC, $y = y^*$ and AC, and AB.

Let us consider the Lagrange operator L_1^y who interpolate the f function with respect to y in the points $(x^*, 0)$ and $\left(x^*, \frac{(x_1 - x^*)y_1}{x_1}\right)$. Also let us consider the operators L_1^x, L_2^x who interpolate the function with respect to x in the points $(0, y^*)$, (x^*, y^*) and $\left(\frac{x_1(y_1 - y^*)}{y_1}, y^*\right)$ respectively. The expressions of these operators will be given by:

$$(L_1^x f)(x, y) = \frac{(x - x^*)\left(x - \frac{x_1(y - y^*)}{y_1}\right)}{(-x^*)\left(0 - \frac{x_1(y - y^*)}{y_1}\right)} f(0, y^*) +$$

$$+ \frac{(x - 0)\left(x - \frac{x_1(y - y^*)}{y_1}\right)}{(x^* - 0)\left(x^* - \frac{x_1(y - y^*)}{y_1}\right)} f(x^*, y^*) +$$

$$+ \frac{(x - 0)(x - x^*)}{\left(\frac{x_1(y - y^*)}{y_1} - 0\right)\left(\frac{x_1(y - y^*)}{y_1} - x^*\right)} f\left(\frac{x_1(y - y^*)}{y_1}, y^*\right) \Leftrightarrow$$

$$(L_1^x f)(x, y) = \frac{(x - x^*)(y_1x - x_1y + x_1y^*)}{x^*x_1(y - y^*)} f(0, y^*) + \frac{x(y_1x - x_1y + x_1y^*)}{x^*(x^*y_1 - x_1y + x_1y^*)} f(x^*, y^*) +$$
$$\frac{y_1^2 x(x - x^*)}{x_1(y - y^*)(x_1y - x_1y^* - x^*y_1)} f\left(\frac{x_1(y - y^*)}{y_1}, y^*\right)$$

respective

$$(L_2^x)(x, y) = \frac{y_1^2 x(x - x^*)}{x_2(y_1 - y^*)(x_2y_1 - x_2y^* - x^*y_1)} f\left(\frac{x_2(y_1 - y^*)}{y_1}, y^*\right) +$$

$$+\frac{(y_1x - x_2y_1 + x_2y^*)(x - x^*)}{x^*x_2(y_1 - y^*)}f(0, y^*) + \frac{(y_1x - x_2y_1 + x_2y_1^*)x}{(x^*y_1 - x_2y_1 + x_2y^*)}f(x^*, y^*),$$

$$(L_1^y f)(x, y) = \frac{yx_1 - x_1y_1 + x^*y_1}{(x^* - x_1)y_1}f(x^*, 0) + \frac{x_1y}{(x_1 - x^*)y_1}f\left(x^*, \frac{(x_1 - x^*)y_1}{x_1}\right)$$

We denote by T the ABC triangle, by T_1 the AOC triangle and by T_2 the ABO triangle. We define the operators:

$$G_1 : T_1 \to R, \quad G_1 f = L_1^x \oplus L_1^y f,$$

respective

$$G_2 : T_2 \to R, \quad G_2 f = L_2^x \oplus L_1^y f.$$

Remark 1. We can easily verify that G_1 interpolate the f function on the frontier of T_1 and on the interior of AM_2 and G_2 interpolate the function on the frontier of T_2 and on the interior of AM_2, that means:

1) $G_1 f = f$ on $\partial T_1 \cup AM_2$;

2) $G_2 f = f$ on $\partial T_2 \cup AM_2$.

Remark 2. We can also verify that $dex(G_1) = 3$ and $dex(G_2) = 3$.

We shall build the F function who will interpolate the f function on the frontier of the T triangle, on the height AO and on the interior line AM_2 as follows:

$$F : T \to R, \quad F(x, y) = \begin{cases} G_1(x, y), & (x, y) \in T_1 \\ G_2(x, y), & (x, y) \in T_2 \setminus [AO] \end{cases}$$

We shall consider that, starting from the expression of G_1, we can give the next approximation formula:

$$f = G_1 f + R_{G_1} f.$$

Regarding the remainder of the above mentioned approximation formula, the next theorem show us how can be expressed this using the well known Peano's theorem.

Theorem 1. If $G_1 \in B_{2,2}(0, 0)$ then:

$$(R_{G_1} f)(x, y) = \int_0^{x_1} \varphi_{04}(x, y, t) f^{(0,4)}(0, t)dt + \int_0^{y_1} \varphi_{13}(x, y, t) f^{(1,3)}(0, t)dt +$$

$$+\iint\limits_{T_1} \varphi_{22}(x,y,s,t)f^{(2,2)}(s,t)dsdt$$

where

$$\varphi_{04}(x,y,t) = R^{xy}_{G_1}\left[\frac{(y-t)^3_+}{6}\right], \quad \varphi_{13}(x,y,t) = R^{xy}_{G_1}\left[x\frac{(y-t)^2_+}{2}\right],$$

$$\varphi_{22}(x,y,s,t) = R^{xy}_{G_1}[(x-s)_+(y-t)_+].$$

We can also give, starting from G_2, the next approximation formula:

$$f = G_2 f + R_{G_2} f$$

and if we take care of Theorem 1 we have that for $G_2 \in B_{2,2}(0,0)$ the remainder has the next approximation formula:

$$(R_{G_2}f)(x,y) = \int_0^{x_2} \varphi_{04}(x,y,t)f^{(0,4)}(0,t)dt + \int_0^{y_1} \varphi_{13}(x,y,t)f^{(1,3)}(0,t)dt +$$

$$+\iint\limits_{T_2} \varphi_{22}(x,y,s,t)f^{(2,2)}(s,t)dsdt$$

where $\varphi_{04}(x,y,t), \varphi_{13}(x,y,t), \varphi_{22}(x,y,s,t)$ have the same mentioned expressions.

Let us consider the approximation formula on T:

$$f = Tf + R_T f$$

Regarding the remainder $R_T f$ we can define it as follows:

$$R_T(f) = \begin{cases} R_{T_1} f, & (x,y) \in T_1 \\ R_{T_2} f, & (x,y) \in T_2 \setminus [AO] \end{cases}$$

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A NEW DIFFERENTIAL INEQUALITY II

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Abstract. We find conditions on the complex-valued functions A and B, in the unit disc U such that the differential inequality

$$Re\,[A(z)p^2(z)+B(z)p(z)-\alpha(zp'(z))^{2k}+\beta(zp'(z))^{2k-1}-$$
$$-\gamma(zp'(z))^{2k-2}+\delta(zp'(z))^{2k-3}+\eta]>0$$

implies $Re\,p(z)>0$, where $\alpha,\beta,\gamma,\delta\geq 0$, $\eta<\dfrac{\alpha n^{2k}}{2^{2k}}+\dfrac{\beta n^{2k-1}}{2^{2k-1}}+\dfrac{\gamma n^{2k-2}}{2^{2k-2}}+\dfrac{\delta n^{2k-3}}{2^{2k-3}}$, $p\in\mathcal{H}[1,n]$ and $k\in\mathbb{N}, k\geq 2$

1. **Introduction and preliminaries**

We let $\mathcal{H}[U]$ denote the class of holomorphic functions in the unit disc

$$U=\{z\in\mathbb{C}:\quad |\, z\,|<1\}.$$

For $a\in\mathbb{C}$ and $n\in\mathbb{N}^*$ we let

$$\mathcal{H}[a,n]=\{f\in\mathcal{H}[U],\quad f(z)=a+a_nz^n+a_{n+1}z^{n+1}+...,\ z\in U\}$$

and

$$\mathcal{A}_n=\{f\in\mathcal{H}[U],\quad f(z)=z+a_{n+1}z^{n+1}+a_{n+2}z^{n+2}+...,\ z\in U\}$$

with $\mathcal{A}_1=\mathcal{A}$.

In order to prove the new results we shall use the following lemma, which is a particular form of Theorem 2.3.i[1,p.35].

Lemma A. [1,p.35] *Let $\psi:\mathbb{C}^2\times U\to\mathbb{C}$ a function which satisfies*

$$Re\ \psi(\rho i,\sigma;z)\leq 0,$$

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where $\rho, \sigma \in \mathbb{R}$, $\sigma \leq -\frac{n}{2}(1+\rho^2)$ $z \in U$ *and* $n \geq 1$.

If $p \in \mathcal{H}[1, n]$ and

$$Re\, \psi\left(p(z), zp'(z); z\right) > 0$$

then

$$Re\, p(z) > 0.$$

2. Main results

Theorem 1. *Let* $\alpha, \beta, \gamma, \delta \geq 0$, $\eta < \frac{\alpha n^{2k}}{2^{2k}} + \frac{\beta n^{2k-1}}{2^{2k-1}} + \frac{\gamma n^{2k-2}}{2^{2k-2}} + \frac{\delta n^{2k-3}}{2^{2k-3}}$ *and let* n, k *be positive integers,* $k \geq 2$. *Suppose that the functions* $A, B : U \to \mathbb{C}$ *satisfy*

$$(i) Re\, A(z) > -\frac{\alpha n^{2k}}{2^{2k}} 2k - \frac{\beta n^{2k-1}}{2^{2k-1}}(2k-1) -$$

$$-\frac{\gamma n^{2k-2}}{2^{2k-2}}(2k-2) - \frac{\delta n^{2k-3}}{2^{2k-3}}(2k-3)$$

$$(ii) (Im B(z))^2 \leq 4\left(\frac{\alpha n^{2k}}{2^{2k}} 2k + \frac{\beta n^{2k-1}}{2^{2k-1}}(2k-1) + \right. \quad (1)$$

$$\left. + \frac{\gamma n^{2k-2}}{2^{2k-2}}(2k-2) + \frac{\delta n^{2k-3}}{2^{2k-3}}(2k-3) + Re\, A(z)\right) \cdot$$

$$\cdot \left(\frac{\alpha n^{2k}}{2^{2k}} + \frac{\beta n^{2k-1}}{2^{2k-1}} + \frac{\gamma n^{2k-2}}{2^{2k-2}} + \frac{\delta n^{2k-3}}{2^{2k-3}} - \eta\right)$$

If $p \in \mathcal{H}[1, n]$ *and*

$$Re\, [A(z)p^2(z) + B(z)p(z) - \alpha (zp'(z))^{2k} + \beta (zp'(z))^{2k-1} -$$

$$-\gamma (zp'(z))^{2k-2} + \delta (zp'(z))^{2k-3} + \eta] > 0 \quad (2)$$

then

$$Re\, p(z) > 0.$$

Proof. We let $\psi : \mathbb{C}^2 \times U \to \mathbb{C}$ be defined by

$$\psi(p(z), zp'(z); z) = A(z)p^2(z) + B(z)p(z) - \alpha (zp'(z))^{2k} +$$

$$+ \beta (zp'(z))^{2k-1} - \gamma (zp'(z))^{2k-2} + \delta (zp'(z))^{2k-3} + \eta \quad (3)$$

From (2) we have

$$Re\,\psi(p(z), zp'(z); z) > 0 \text{ for } z \in U. \tag{4}$$

For $\rho, \sigma \in \mathbb{R}$ satisfying $\sigma \leq -\frac{n}{2}(1+\rho^2)$, hence

$$-\sigma^{2k} \leq -\frac{n^{2k}}{2^{2k}}(1+\rho^2)^{2k};\ \sigma^{2k-1} \leq -\frac{n^{2k-1}}{2^{2k-1}}(1+\rho^2)^{2k-1}$$

$$-\sigma^{2k-2} \leq -\frac{n^{2k-2}}{2^{2k-2}}(1+\rho^2)^{2k-2};\ \sigma^{2k-3} \leq -\frac{n^{2k-3}}{2^{2k-3}}(1+\rho^2)^{2k-3}$$

and $z \in U$, by using (1) we obtain

$$Re\,\psi(\rho i, \sigma; z) = Re\,[A(z)(\rho i)^2 + B(z)\rho i - \alpha\sigma^{2k} +$$

$$+\beta\sigma^{2k-1} - \gamma\sigma^{2k-2} + \delta\sigma^{2k-3} + \eta] =$$

$$= -\rho^2 Re\,A(z) - \rho\, Im\,B(z) - \alpha\sigma^{2k} + \beta\sigma^{2k-1} - \gamma\sigma^{2k-2} + \delta\sigma^{2k-3} + \eta \leq$$

$$\leq -\rho^2 Re\,A(z) - \rho\, Im\,B(z) - \frac{\alpha n^{2k}}{2^{2k}}(1+\rho^2)^{2k} - \frac{\beta n^{2k-1}}{2^{2k-1}}(1+\rho^2)^{2k-1} -$$

$$-\frac{\gamma n^{2k-2}}{2^{2k-2}}(1+\rho^2)^{2k-2} - \frac{\delta n^{2k-3}}{2^{2k-3}}(1+\rho^2)^{2k-3} + \eta =$$

$$= -\frac{\alpha n^{2k}}{2^{2k}}(\rho^2)^{2k} - \left(\frac{\alpha n^{2k}}{2^{2k}}\mathrm{C}_{2\mathrm{k}}^{2\mathrm{k}-1} + \frac{\beta n^{2k-1}}{2^{2k-1}}\mathrm{C}_{2\mathrm{k}-1}^{2\mathrm{k}-1}\right)(\rho^2)^{2k-1} -$$

$$-\left(\frac{\alpha n^{2k}}{2^{2k}}\mathrm{C}_{2\mathrm{k}}^{2\mathrm{k}-2} + \frac{\beta n^{2k-1}}{2^{2k-1}}\mathrm{C}_{2\mathrm{k}-1}^{2\mathrm{k}-2} + \frac{\gamma n^{2k-2}}{2^{2k-2}}\mathrm{C}_{2\mathrm{k}-2}^{2\mathrm{k}-2}\right)(\rho^2)^{2k-2} - \ldots -$$

$$-\left[\left(\frac{\alpha n^{2k}}{2^{2k}}2k + \frac{\beta n^{2k-1}}{2^{2k-1}}(2k-1) + \frac{\gamma n^{2k-2}}{2^{2k-2}}(2k-2) +\right.\right.$$

$$\left. + \frac{\delta n^{2k-3}}{2^{2k-3}}(2k-3) + Re\,A(z)\right)\rho^2 + \rho\, Im\,B(z) +$$

$$\left. + \frac{\alpha n^{2k}}{2^{2k}} + \frac{\beta n^{2k-1}}{2^{2k-1}} + \frac{\gamma n^{2k-2}}{2^{2k-2}} + \frac{\delta n^{2k-3}}{2^{2k-3}} - \eta\right] \leq 0.$$

By using Lemma A we have that $Re\,p(z) > 0$. □

If $\alpha = 0$ and $\beta = 0$, then Theorem 1 can be rewritten as follows:

Corollary 1. *Let* $\gamma, \delta \geq 0$, $\eta < \dfrac{\gamma n^{2k-2}}{2^{2k-2}} + \dfrac{\delta n^{2k-3}}{2^{2k-3}}$ *and let* n, k *be positive integers,* $k \geq 2$*. Suppose that the functions* $A, B : U \to \mathbb{C}$ *satisfy*

$$(i) Re\, A(z) > -\frac{\gamma n^{2k-2}}{2^{2k-2}}(2k-2) - \frac{\delta n^{2k-3}}{2^{2k-3}}(2k-3)$$

$$(ii)(Im B(z))^2 \leq 4\left(\frac{\gamma n^{2k-2}}{2^{2k-2}}(2k-2) + \frac{\delta n^{2k-3}}{2^{2k-3}}(2k-3) + Re A(z)\right) \tag{5}$$

$$\cdot \left(\frac{\gamma n^{2k-2}}{2^{2k-2}} + \frac{\delta n^{2k-3}}{2^{2k-3}} - \eta\right)$$

If $p \in \mathcal{H}[1, n]$ *and*

$$Re\, [A(z)p^2(z) + B(z)p(z) - \gamma(zp'(z))^{2k-2} + \delta(zp'(z))^{2k-3} + \eta] > 0 \tag{6}$$

then

$$Re\, p(z) > 0.$$

Remarks. 1. This result from Corollary 1 was obtained in Theorem 1 from [2].

2. For $\alpha = 0$, $k = 2$ we obtain Theorem 1 from [3].

If $\alpha = 0$, then Theorem 1 can be rewritten as follows:

Corollary 2. *Let* $\beta, \gamma, \delta \geq 0$, $\eta < \dfrac{\beta n^{2k-1}}{2^{2k-1}} + \dfrac{\gamma n^{2k-2}}{2^{2k-2}} + \dfrac{\delta n^{2k-3}}{2^{2k-3}}$ *and let* n, k *be positive integers,* $k \geq 2$*. Suppose that the functions* $A, B : U \to \mathbb{C}$ *satisfy*

$$(i) Re\, A(z) > -\frac{\beta n^{2k-1}}{2^{2k-1}}(2k-1) - \frac{\gamma n^{2k-2}}{2^{2k-2}}(2k-2) -$$

$$-\frac{\delta n^{2k-3}}{2^{2k-3}}(2k-3)$$

$$(ii)(Im B(z))^2 \leq 4\left(\frac{\beta n^{2k-1}}{2^{2k-1}}(2k-1) + + \frac{\gamma n^{2k-2}}{2^{2k-2}}(2k-2)\right. \tag{7}$$

$$\left. + \frac{\delta n^{2k-3}}{2^{2k-3}}(2k-3) + Re A(z)\right)\left(\frac{\beta n^{2k-1}}{2^{2k-1}} + \frac{\gamma n^{2k-2}}{2^{2k-2}} + \frac{\delta n^{2k-3}}{2^{2k-3}} - \eta\right)$$

If $p \in \mathcal{H}[1, n]$ *and*

$$Re\, [A(z)p^2(z) + B(z)p(z) + \beta(zp'(z))^{2k-1} -$$

$$-\gamma(zp'(z))^{2k-2} + \delta(zp'(z))^{2k-3} + \eta] > 0 \quad (8)$$

then

$$Re\, p(z) > 0.$$

If $\beta = 0$, then Theorem 1 can be rewritten as follows.

Corollary 3. *Let* $\alpha, \gamma, \delta \geq 0$, $\eta < \frac{\alpha n^{2k}}{2^{2k}} + \frac{\gamma n^{2k-2}}{2^{2k-2}} + \frac{\delta n^{2k-3}}{2^{2k-3}}$ *and let* n, k *be positive integers,* $k \geq 2$. *Suppose that the functions* $A, B : U \to \mathbb{C}$ *satisfy*

$$(i) Re\, A(z) > -\frac{\alpha n^{2k}}{2^{2k}} 2k - \frac{\gamma n^{2k-2}}{2^{2k-2}}(2k-2) - \frac{\delta n^{2k-3}}{2^{2k-3}}(2k-3)$$

$$(ii) (Im B(z))^2 \leq 4\left(\frac{\alpha n^{2k}}{2^{2k}} 2k + \frac{\gamma n^{2k-2}}{2^{2k-2}}(2k-2) + \right. \quad (9)$$

$$\left. + \frac{\delta n^{2k-3}}{2^{2k-3}}(2k-3) + Re\, A(z)\right)\left(\frac{\alpha n^{2k}}{2^{2k}} + \frac{\gamma n^{2k-2}}{2^{2k-2}} + \frac{\delta n^{2k-3}}{2^{2k-3}} - \eta\right)$$

If $p \in \mathcal{H}[1, n]$ *and*

$$Re\, [A(z)p^2(z) + B(z)p(z) - \alpha(zp'(z))^{2k} -$$

$$-\gamma(zp'(z))^{2k-2} + \delta(zp'(z))^{2k-3} + \eta] > 0 \quad (10)$$

then

$$Re\, p(z) > 0.$$

If $\gamma = 0$, then Theorem 1 can be rewritten as follows.

Corollary 4. *Let* $\alpha, \beta, \delta \geq 0$, $\eta < \frac{\alpha n^{2k}}{2^{2k}} + \frac{\beta n^{2k-1}}{2^{2k-1}} + \frac{\delta n^{2k-3}}{2^{2k-3}}$ *and let* n, k *be positive integers,* $k \geq 2$. *Suppose that the functions* $A, B : U \to \mathbb{C}$ *satisfy*

$$(i) Re\, A(z) > -\frac{\alpha n^{2k}}{2^{2k}} 2k - \frac{\beta n^{2k-1}}{2^{2k-1}}(2k-1) - \frac{\delta n^{2k-3}}{2^{2k-3}}(2k-3)$$

$$(ii) (Im B(z))^2 \leq 4\left(\frac{\alpha n^{2k}}{2^{2k}} 2k + \frac{\beta n^{2k-1}}{2^{2k-1}}(2k-1) + \right. \quad (11)$$

$$\left. + \frac{\delta n^{2k-3}}{2^{2k-3}}(2k-3) + Re A(z)\right)\left(\frac{\alpha n^{2k}}{2^{2k}} + \frac{\beta n^{2k-1}}{2^{2k-1}} + \frac{\delta n^{2k-3}}{2^{2k-3}} - \eta\right)$$

If $p \in \mathcal{H}[1, n]$ *and*

$$Re\, [A(z)p^2(z) + B(z)p(z) - \alpha(zp'(z))^{2k} + \beta(zp'(z))^{2k-1} -$$

$$+\delta(zp'(z))^{2k-3}+\eta]>0 \tag{12}$$

then

$$Re\,p(z)>0.$$

If $\delta=0$, then Theorem 1 can be rewritten as follows:

Corollary 5. *Let* $\alpha,\beta,\gamma\geq 0$, $\eta<\frac{\alpha n^{2k}}{2^{2k}}+\frac{\beta n^{2k-1}}{2^{2k-1}}+\frac{\gamma n^{2k-2}}{2^{2k-2}}$ *and let* n, k *be positive integers,* $k\geq 2$. *Suppose that the functions* $A,B:U\to\mathbb{C}$ *satisfy*

$$(i) Re\,A(z)>-\frac{\alpha n^{2k}}{2^{2k}}2k-\frac{\beta n^{2k-1}}{2^{2k-1}}(2k-1)-\frac{\gamma n^{2k-2}}{2^{2k-2}}(2k-2)$$

$$(ii)(ImB(z))^2\leq 4\left(\frac{\alpha n^{2k}}{2^{2k}}2k+\frac{\beta n^{2k-1}}{2^{2k-1}}(2k-1)+\right. \tag{13}$$

$$\left.+\frac{\gamma n^{2k-2}}{2^{2k-2}}(2k-2)+ReA(z)\right)\left(\frac{\alpha n^{2k}}{2^{2k}}+\frac{\beta n^{2k-1}}{2^{2k-1}}+\frac{\gamma n^{2k-2}}{2^{2k-2}}-\eta\right)$$

If $p\in\mathcal{H}[1,n]$ *and*

$$Re\,[A(z)p^2(z)+B(z)p(z)-\alpha(zp'(z))^{2k}+\beta(zp'(z))^{2k-1}-$$

$$-\gamma(zp'(z))^{2k-2}+\eta]>0 \tag{14}$$

then

$$Re\,p(z)>0.$$

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HÖLDER ESTIMATES OF HIGHER ORDER DERIVATIVES FOR EVOLUTIONARY MONGE-AMPÉRE EQUATION ON A RIEMANNIAN MANIFOLD

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Abstract. Let (V, g) be a compact Riemannian manifold. For $u \in C^2(V)$ we consider the form $g_{ij} + \nabla_{ij}u$. If the form is positive definite, it gives a new metric on V. The Monge-Ampére operator on V is the quotient of determinants: $M(u) = |g_{ij} + \nabla_{ij}u|/|g_{ij}|$. The paper deals with the Cauchy problem for the evolutionary Monge-Ampére type equation:

$$-\frac{\partial u}{\partial t} + \ln M(u) = f(t, x, u), \quad (t, x) \in [0, T] \times V,$$
$$u(0, x) = u_0(x).$$

Hölder estimates for higher order derivatives u_t and $\nabla_{ij}u$ of a solution u are proved.

1. Introduction

The paper deals with the apriory estimates of solutions of the Cauchy problem for the evolutionary Monge-Ampére type equation on Riemannian manifolds and continues [1],[2].

Let (V, g) be a smooth compact Riemannian manifold, $\dim V = m$. We consider the Levi-Civita connection on V, it defines the covariant differentiation on V. The Levi-Civita connection is the unique symmetric connection with vanishing torsion tensor, for which the covariant derivative of the metric tensor is zero. Let $x^1, \ldots, x^m$ be a local coordinate system on V, and $\partial_1, \ldots, \partial_m$, where $\partial_k = \frac{\partial}{\partial x^k}$, be

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the corresponding basic vector fields. Suppose $u(x)$ is a function on V at least twice continuously differentiable. By $\nabla_i u = \nabla_{\partial_i} u$ we denote the covariant derivative, and $\nabla_{ij} u = \nabla_i(\nabla_j u)$ second order covariant derivative. Let $(g_{ij}(x))$ be the matrix of the metric g in a local coordinates. We consider the form g_u with matrix $g_{ij}(x) + \nabla_{ij} u(x)$ in the local coordinate system. A function $u(x)$ is said to be **admissible** provided the form g_u is positive definite. An admissible function gives a new metric on V.

By $|g_{ij}|$ and $|g^u_{ij}|$ we denote determinants of metrics. The quotient $M(u)(x) = |g^u_{ij}(x)|/|g_{ij}(x)|$ is a positive function on V. We call $u \to M(u)$ **the Monge-Ampere type operator** by analogy with the classical Monge-Ampére operator. The distinction between $M(u)$ and the classical operator is the following. The classical operator is the Hesse matrix of a function u, but $M(u)$ contains sum of the matrix (g_{ij}) and Hesse matrix. The classical operator is defined on the convex set of symmetric positive definite matrix, for $M(u)$ we shall consider a bundle with convex fibres.

We consider the product $[0, T] \times V$ with the same metric g and connection ∇ for each $t \in [0, t]$. Let $u(t, x)$ be at least twice continuously differentiable function on $[0, T] \times V$ with respect to spatial variables. The function $u(t, x)$ is said to be *admissible* provided the form $g^u_{ij}(t) = g_{ij} + \nabla_{ij} u(t, \cdot)$ is positive definite for all $t \in [0, 1]$. An admissible function $u(t, x)$ defines the family of metrics $g^u(t)$, $t \in [0, 1]$ on V. Applying M to $u(t, x)$, we obtain the function $M(u)(t, x)$ depending on two variables.

We consider the evolutionary equation

$$-\frac{\partial u}{\partial t} + \ln M(u) = f(t, x, u), \qquad (t, x) \in [0, T] \times V, \tag{1}$$

with initial condition:

$$u(0, x) = u_0(x). \tag{2}$$

The stationary equation with $M(u)$ arises in some geometrical problem. For example, the condition that describes Einstein-Kähler manifolds is proportionality of the Ricci tensor and the metric tensor, it was first proposed by Einstein as the equation of the gravity field in vacuum. The question of existence of Einstein-Káhler metric leads to the stationary Monge-Ampére type equation. The proof of the famous Calabi conjecture, which asserts that every form representing the first Chern class is the Ricci form of some Kähler metric, proved in 1976 by S.T.Yau and T.Aubin independently, is based on the existence theorem for stationary Monge-Ampére equation (see [3], [4], [5],[6] for more details).

Evolutionary equations with classical Monge-Ampére operator on a bounded domain in the n-dimensional space arise in the problem of deformation of a hypersurface with rapid controlled by the mean curvature. Papers of many authors are devoted to the last problem, e.g. papers of N.Uraltseva, V.Oliker, N.Ivochkina, K.Tso, G.Huisken and others.

The aim of the paper is the Hölder constant estimates for the higher order derivatives for solutions of (1-2). In the proof we use the following estimates obtained in [1],[2].

Theorem 1. *([1], th.1) Let $u(t,x)$ be an admissible function and belong to $C\left([0,T],C^2(V)\right)$. By D denote the diameter of V. Then we have*

$$\max_{[0,t]\times V} |\nabla_x u| \le 2D.$$

Theorem 2. *([1], th.2) Suppose $u(t,x)$ is an admissible solution of (1)-(2) and belongs $C\left([0,T],C^3(V)\right)\bigcap C^1\left([0,T],C^2(V)\right)\bigcap C^2\left([0,T],C(V)\right)$. Let the right hand side $f(t,x,u)$of equation (1) be bounded and have bounded first order partial derivatives, $f_u(t,x,u)\ge\delta>0$ on $[0,T]\times V\times R^1$. Then*

$$|u_t(x,t)| \le M_1,$$

where M_1 depends on the diameter D, metric g, $\|u_0\|_{C^1(V)}$, $\|f\|_{C^1(V)}$, and δ.

As usual we denote by (g_{ij}) elements of matrix g in a local coordinates, (g^{ij}) elements of inverse matrix, also we denote by (g^u_{ij}) elements of matrix g_u and (g^{ij}_u) elements of corresponding inverse matrix.

Theorem 3. *Let $u(t,x)$ be an admissible solution of (1)-(2) and belong $C\left([0,T],C^4(V)\right)\bigcap C^1\left([0,T],C^2(V)\right)\bigcap C^2\left([0,T],C(V)\right)$. Suppose the right hand side $f(t,x,u)$ is bounded and has bounded partial derivatives up to the second order, $f_u(t,x,u)\ge\delta>0$ on $[0,T]\times V\times R^1$. Then all metrics generated by the solution $u(t,x)$ of (1)-(2) are uniformly equivalent, i.e. there are positive constants*

c_1, c_2, depending on the diameter D, metric tensor g, curvature tensor of V, $\|f\|_{C^2(V)}$, δ, $\|u_0\|_{C^2(V)}$, and independent on (t,x) such that

$$c_1 g_{ij}\xi^i\xi^j \le g^u_{ij}\xi^i\xi^j \le c_2 g_{ij}\xi^i\xi^j; \tag{3}$$

$$1/c_2 g^{ij}\xi_i\xi_j \le g^{ij}_u\xi_i\xi_j \le 1/c_1 g^{ij}\xi_i\xi_j. \tag{4}$$

for all $\xi = (\xi_1, \dots, \xi_m) \in R^m$.

Theorem 4. *Let $u(t,x)$ be a solution of (1-2) and belong to $C\left([0,T], C^4(V)\right) \bigcap C^1\left([0,T], C^2(V)\right) \bigcap C^2\left([0,T], C(V)\right)$. Under the assumptions of theorem 3 we have*

$$0 < m - \Delta u \le K,$$

where K depends on the same values as c_1, c_2 in theorem 3.

2. Some properties of the operator $M(u)$

Let us consider the set of all square matrix of order m, we identify it with R^{m^2}. Denote by $S \in R^{m^2}$ the subset of symmetric positive definite matrix. S is open and convex. Write $a = (a_{ij})$ for elements of S.

Let us cover V by finite number of local charts $(\Omega_k, \varphi_k)_{k=1}^q$ and choose open sets Ω'_k, $\bar\Omega'_k \subset \Omega_k$, such that $\varphi_k(\Omega'_k)$ convex in R^m and $\bigcup_{k=1}^q \Omega'_k = V$. Fix an index k, we shall proceed throughout $\bar\Omega'_k$ in the local coordinates of chart (Ω_k, φ_k).

Fix $x \in \bar\Omega'_k$, then $g(x) \in S$. Denote by S_x the following subset in R^{m^2}:

$$S_x = \{a \in R^{m^2} \mid g(x) + a = (g_{ij}(x) + a_{ij}) \in S\}.$$

We consider the fibre bundle $\pi\colon \mathbf{S} \to \bar\Omega'_k$ with fibre $\pi^{-1}(x) = S_x$ and total space $\mathbf{S} = \bigcup_{x\in\bar\Omega'_k} S_x$.

Fibres of the bundle π are open convex subset in R^{m^2} and every fibre is homeomorphic to S. The bundle π is trivializable, i.e. there is a homeomorphism $\varphi\colon \mathbf{S} \to \bar\Omega'_k \times S$ such that the following diagram is commutative:

$$\begin{array}{ccc} \mathbf{S} & \xrightarrow{\varphi} & \bar\Omega'_k \times S \\ \pi \searrow & & \swarrow p_1 \\ & \bar\Omega'_k & \end{array},$$

where $p_1\colon \bar{\Omega}'_k \times S \to \bar{\Omega}'_k$ is the projection on the first factor, $p_1(x,a) = x$.

Indeed, we put $\varphi(x,a) = (x, a + g(x))$. The map φ is one-to-one:

1. if $(x_1, a_1 + g(x_1)) = (x_2, a_2 + g(x_2))$, then $x_1 = x_2$ and $a_1 + g(x_1) = a_2 + g(x_1) \quad \Rightarrow \quad a_1 = a_2$.
2. if $(x,b) \in \bar{\Omega}'_k \times S$, then $\varphi(x, b - g(x)) = (x, b - g(x) + g(x)) = (x,b)$.

The map φ is continuous due to continuity of g, the inverse map $\varphi^{-1}(x,b) = (x, b - g(x))$ is also continuous .

Together with $\pi\colon \mathbf{S} \to \bar{\Omega}'_k$ we consider the bundle $\bar{\pi}\colon [0,T] \times \mathbf{S} \to [0,T] \times \bar{\Omega}'_k$ whose fibre over (t,x) coincides with the fibre S_x of π over x: $\bar{\pi}^{-1}(t,x) = \pi^{-1}(x) = S_x$. By $\bar{\mathbf{S}} = [0,T] \times \mathbf{S}$ we denote the total space of the bundle $\bar{\pi}$.

On the bundle $\mathbf{S}$ we consider the following function $F\colon \mathbf{S} \to R^1$:

$$F(x,a) = \ln \frac{|g(x) + a|}{|g(x)|}.$$

We extend F identically to $\bar{\mathbf{S}}$: $F(t,x,a) = F(x,a)$. It is easily seen that the restriction F to a fibre S_x of the bundles $\mathbf{S}$ and $\bar{\mathbf{S}}$ is a convex function of m^2 variables. Indeed, $\frac{\partial F}{\partial (a_{ij})} = g_a^{ij}$, where g_a^{ij} is an element of the inverse matrix $(g(x)+a)^{-1}$. Then $\frac{\partial^2 F}{\partial (a_{ij}) \partial (a_{kl})} = -g_a^{ik} g_a^{lj}$ ([1], lemma 1). Thus $\frac{\partial^2 F}{\partial a_{ij} \partial a_{kl}} \xi_{ij} \xi_{kl} = -g_a^{ik} g_a^{lj} \xi_{ij} \xi_{kl}$ is a negative definite form, i.e. the function $F|_{S_x}$ is a convex function of m^2 variables.

Let $\Lambda\colon [0,T] \times \bar{\Omega}'_k \to \bar{\mathbf{S}}$, $\Lambda(t,x) = (t, x, \lambda(t,x))$, $\lambda(t,x) \in S_x$, be a continuous section of the bundle $\bar{\pi}$. Assume that there are exist positive constants c_1, c_2 such that

$$c_1 |\xi|^2 \le (g_{ij}(x) + \lambda_{ij}(t,x)) \xi^i \xi^j \le c_2 |\xi|^2 \tag{5}$$

for all $(t,x) \in [0,T] \times \bar{\Omega}'_k$.

We consider the superposition

$$F(x, \lambda(t,x)) = \ln \frac{|g(x) + \lambda(t,x)|}{|g(x)|}.$$

Denote by

$$\rho(z_1, z_2) = |t - \tau|^{1/2} + |x - y| \tag{6}$$

the parabolic distance between points $z_1 = (t,x), z_2 = (\tau, y) \in [0,T] \times \bar{\Omega}'_k$.

Lemma 1. *Suppose the metric g and section Λ_{ij} are $C^{2+\alpha}$. Let M^α_Λ be the maximal Hölder constant for λ_{ij}. Then*

$$|F(x, \lambda(z_1)) - F(y, \lambda(z_2))| \le N_1 \rho^\alpha(z_1, z_2) + N_2 |x - y|^\alpha,$$

for all $z_1 = (t, x), z_2 = (\tau, y) \in [0, T] \times \bar{\Omega}'_k$ with constant N_1 depending on $c_1, m, \|g\|_{C^{2+\alpha}}, M^\alpha_\Lambda$ and constant N_2 depending on m, $\|g\|_{C^{2+\alpha}}$.

Proof.

$$\begin{gathered} F(x, \lambda(t, x)) - F(y, \lambda(\tau, y)) = \ln \tfrac{|g(x)+\lambda(t,x)|}{|g(x)|} - \ln \tfrac{|g(y)+\lambda(\tau,y)|}{|g(y)|} = \\ = [\ln |g(x) + \lambda(t, x)| - \ln |g(y) + \lambda(\tau, y)|] + [\ln |g(y)| - \ln |g(x)|]. \end{gathered} \tag{7}$$

We start with the first term. For all $\theta \in [0, 1]$ the form $g_\theta = \theta[g(x) + \lambda(t, x)] + (1 - \theta)[g(y) + \lambda(\tau, y)]$ is positive definite. Let us consider the function $\varphi(\theta) = \ln |g_\theta|$. We have

$$\varphi'(\theta) = \frac{1}{|g_\theta|} g^{ij}_\theta |g_\theta| \frac{d}{d\theta} (g_\theta)_{ij} = g^{ij}_\theta [g_{ij}(x) - g_{ij}(y) + \lambda_{ij}(t, x) - \lambda_{ij}(\tau, y)],$$

where $(g_\theta)_{ij}$ are elements of matrix g_θ, g^{ij}_θ are elements of the inverse matrix. Then

$$\begin{gathered} \ln |g(x) + \lambda(t, x)| - \ln |g(y) + \lambda(\tau, y)| = \varphi(1) - \varphi(0) = \int_0^1 \varphi'(\theta)\, d\theta = \\ = \sum_{ij} d^{ij} [g_{ij}(x) - g_{ij}(y) + \lambda_{ij}(t, x) - \lambda_{ij}(\tau, y)], \end{gathered} \tag{8}$$

where

$$d^{ij} = \int_0^1 g^{ij}_\theta \, d\theta. \tag{9}$$

Since matrices $g(x) + \lambda(t, x)$ satisfy condition (5), then the matrix g_θ satisfies (5) as well, and for elements of the inverse matrix we have

$$\frac{1}{c_2} |\xi|^2 \le g^{ij}_\theta \xi_i \xi_j \le \frac{1}{c_1} |\xi|^2.$$

Integrating the above inequality with respect to θ from 0 to 1 we get

$$\frac{1}{c_2} |\xi|^2 \le d^{ij} \xi_i \xi_j \le \frac{1}{c_1} |\xi|^2. \tag{10}$$

Then

$$\begin{aligned} &\ln|g(x)+\lambda(t,x)|-\ln|g(y)+\lambda(\tau,y)| \le \\ &\le \sum_{ij}|d^{ij}|(|g_{ij}(x)-g_{ij}(y)|+|\lambda_{ij}(t,x)-\lambda_{ij}(\tau,y)|) \le \\ &\le \tfrac{1}{c_1}\sum_{ij}(M_g^\alpha|x-y|^\alpha+M_\Lambda^\alpha\rho(z_1,z_2)^\alpha) \le \tfrac{m^2}{c_1}(M_g^\alpha+M_\Lambda^\alpha)\rho^\alpha(z_1,z_2), \end{aligned} \tag{11}$$

where M_g^α is the maximal Hölder constant for g_{ij}.

To obtain estimate for the second term we denote for a while $g(x)=a$, $g(y)=b$ and consider $\ln|s|$ as a function of m^2 variables $s=(s_{ij})\in S$.

$$|\ln|a|-\ln|b|\,| \le \sum_{ij}\sup_{t\in[0,1]}\frac{\partial \ln|s|}{\partial s_{ij}}(ta+(1-t)b)|b_{ij}-a_{ij}|.$$

Since for any matrix $s=(s_{ij})$ we have $\frac{\partial \ln|s|}{\partial s_{ij}}=s^{ij}$, then

$$|\ln|a|-\ln|b|\,| \le \sum_{ij}\sup_{t\in[0,1]}(ta+(1-t)b)^{ij}|b_{ij}-a_{ij}|.$$

Put $G=\{g(x), x\in\bar\Omega_k'\}$ and let $\overline{\mathrm{co}}\,G$ be its convex hull. Let M_g be the bound for elements of matrices that are inverse to matrices from $\overline{\mathrm{co}}\,G$. Then

$$|\ln|g(x)|-\ln|g(y)|| \le M_g M_g^\alpha m^2|x-y|^\alpha. \tag{12}$$

Combining (11) - (12) we obtain the estimate that we need. □

We shall use equality (8) ones more to obtain the following:

Lemma 2. *Under the assumptions of lemma 1 we have*

$$\sum_{ij} d^{ij}[\lambda_{ij}(t,x)-\lambda_{ij}(\tau,y)] = F(x,\lambda(t,x))-F(y,\lambda(\tau,y))+F_1(x,y),$$

where d^{ij} are given by (9) and $F_1(x,y)$ satisfies Hölder condition

$$|F_1(x,y)| \le \hat M_g|x-y|^\alpha$$

with $\hat M_g = M_g^\alpha(1/c_1+M_g)m^2$.

Proof. From (8) and(7) we have:

$$\begin{aligned}
&\sum_{ij} d^{ij}[\lambda_{ij}(t,x) - \lambda_{ij}(\tau,y)] = \\
&\ln|g(x) + \lambda(t,x)| - \ln|g(y) + \lambda(\tau,y)| + \sum_{ij} d^{ij}[g_{ij}(y) - g_{ij}(x)] = \\
&F(x,\lambda(t,x)) - F(y,\lambda(\tau,y)) + [\ln|g(x)| - \ln|g(y)|] + \\
&\sum_{ij} d^{ij}[g_{ij}(y) - g_{ij}(x)].
\end{aligned} \tag{13}$$

Consider $F_1(x,y) = [\ln|g(x)| - \ln|g(y)|] + \sum_{ij} d^{ij}[g_{ij}(y) - g_{ij}(x)]$. Inequalities (10) and (12) give

$$|F_1(x,y)| \le M_g^\alpha(1/c_1 + M_g)m^2|x-y|^\alpha \tag{14}$$

□

3. Hölder Estimate for u_t

Theorem 5. *Let $u(t,x)$ be a solution of (1-2) from the space $C\left([0,T],C^4(V)\right)\bigcap C^1\left([0,T],C^2(V)\right)\bigcap C^2\left([0,T],C(V)\right)$. Assume that the right hand side $f(t,x,u)$ bounded and has bounded derivatives up to the second order, $f_u(t,x,u) \ge \delta > 0$ on $[0,T]\times V\times R^1$. Let u_0 be an admissible function from $C^{2+\alpha}(V)$. Then*

$$|u_t(z_1) - u_t(z_2)| \le N\rho^\beta(z_1,z_2) \tag{15}$$

with some power $\beta \in (0,\alpha]$ depending on dimension m and constants c_1, c_2 from theorem 3. The constant N depends on β, m, c_1 c_2, D, g, $\|u_0\|_{C^{2+\alpha}}$, $\|f\|_{C^2}$, and on δ.

Proof. Fix a number ρ_0, $0 < \rho_0 < 1/2$, we begin with estimate for u_t on the cylinder $[\rho_0,T]\times V$.

Suppose that the manifold V is covered by charts (Ω_k,φ_k) whose images coincide with $B_1(0)$, where $B_r(0)$ is the ball in the Euclidean space R^m of radius r centered at the origin, and preimages Ω'_k of balls $B_{1/2}(0)$ cover V as well. Differentiating (1) in t within local coordinates of chart Ω_k, we have:

$$-\frac{\partial u_t}{\partial t} + g_u^{\alpha\beta}\nabla_{\alpha\beta}u_t = f_t + f_u u_t$$

Write $v = u_t$. We have got a linear equation with respect to v:

$$Lv = f_t, \tag{16}$$

where

$$L = -\partial/\partial t + g_u^{\alpha\beta}\nabla_{\alpha\beta} - f_u$$

is a uniformly parabolic operator due to theorem 3.

By Q and Q_ρ we denote cylinders $Q = (0,T) \times B_1(0)$, $Q_\rho = (\rho, T) \times B_{1/2}(0)$ in R^{m+1}. By $\partial' Q$ denote the parabolic boundary of the cylinder Q: $\partial' Q = \big(\{0\} \times \bar{B}_1(0)\big) \bigcup \big((0,T) \times \partial B_1(0)\big)$. Let $\rho(z,z')$ be the parabolic distance (6) between points $z = (t,x)$, $z' = (t', z')$, for a point $z \in Q$ we write

$$\rho(z) = \inf\{\rho(z,z'), z' = (t',x') \in \partial Q, t' < t\}, \tag{17}$$

$\rho(z)$ is said to be the parabolic distance from z to the boundary of Q.

Note that $\inf\{\rho(z), z \in Q_{\rho_0}\} = \rho_0$.

For a solution $v = u_t$ of uniformly parabolic equation (16) there is the following Hölder estimate ([7],theorem IV.2.7, p.120): for $z_1 = (x_1,t_1)$, $z_2 = (x_2,t_2)$, $z_1, z_2 \in Q_{\rho_0}$,

$$|u_t(z_1) - u_t(z_2)| \le N(\sup_Q |u_t| + \|Lu_t\|_{L_{m+1}(Q)})\rho^\gamma(z_1,z_2)$$

with some power $\gamma \in (0,1)$, depending on m and constants c_1, c_2 from theorem 3. The constant N depends on m, c_1 c_2 as well, and extra on $\sup|f_u|$ and distance ρ_0 from the parabolic boundary.

Using the estimate of $|u_t|$ (theorem 2) and the equality $Lu_t = f_t$, we get

$$|u_t(z_1) - u_t(z_2)| \le N_1\rho^\gamma(z_1,z_2) \tag{18}$$

with N_1 depending on $m, c_1, c_2, \delta, \rho_0, D$, metric tensor g, initial function u_0, right hand side f, and their derivatives up to the second order.

Before getting an estimate of $v = u_t$ for small $t \in (0,\rho_0)$, let us consider the case $t = 0$.

If $t = 0$, then from equation (1) and initial condition (2) we have

$$u_t(0, x) = \ln M(u)(0, x) - f(0, x, u_0). \tag{19}$$

The initial function $u_0 \in C^{2+\alpha}(V)$ defines the continuous section $\Lambda_0 \colon \bar{\Omega}'_k \to \mathbf{S}$ of the bundle $\pi \colon \mathbf{S} \to \bar{\Omega}'_k$ as follows: $\Lambda_0(x) = (x, \nabla_{ij} u_0(x))$. For the section Λ_0 we have constants c_1 and c_2 in (5) are equal to the minimal and maximal eigenvalues of matrices $(g_{ij}(x) + \nabla_{ij} u_0(x))$ and depend on the initial metric g and second order derivatives of the initial function.

Application of lemma 1 gives

$$|\ln M(u_0)(x) - \ln M(u_0)(y)| \le (N_1 + N_2)|x - y|^\alpha,$$

where $N = N_1 + N_2$ depends on m, $\|g\|_{C^{2+\alpha}}$, $\|u_0\|_{C^{2+\alpha}}$.

On the other hand,

$$\begin{aligned}
&|f(0, x, u_0(x)) - f(0, y, u_0(y))| = \\
&\left|\sum_{i=1}^m \frac{\partial f}{\partial x^i}(0, \theta x + (1-\theta)y, \theta u_0(x) + (1-\theta)u_0(y))(x^i - y^i) + \right. \\
&f_u(0, \theta x_1 + (1-\theta)x_2, \theta u_0(x_1) + (1-\theta)u_0(x_2))(u_0(x_1) - u_0(x_2))| \le \\
&\sup\left|\frac{\partial f}{\partial x^i}\right| |x - y| + \sup|f_u| \sup|\frac{\partial u_0}{\partial x^i}||x - y|.
\end{aligned}$$

Thus from (19) we have

$$|u_t(0, x) - u_t(0, y)| \le N_0|x - y|^\alpha,$$

where α is the Hölder power of u_0 and N_0 depends on m, $\|g\|_{C^{2+\alpha}}$, $\|u_0\|_{C^{2+\alpha}}$, and first order derivatives of f.

To estimate u_t on the cylinder $(0, \rho_0) \times V$ we use another theorem ([7],th. IV.4.5, p.142). Choose a covering (Ω_k, φ_k) of V such that images of Ω_k in the space R^m coincide with balls of radius $r = 3\sqrt{2}$ centered at $(3, 0, \ldots, 0) \in R^m$ and preimages Ω'_k of sets $\{(x_1, \ldots, x_m) : 1/2 < x_1 < 2, |x_i| < 1, i = 2, \cdots, m\}$ cover V as well. Then we apply the theorem mentioned above to uniformly parabolic equation (16). It claims existence of a constant $\tilde{\gamma}_0 \in (0, 1)$, $\tilde{\gamma}_0 \le \alpha$, depending on m, c_1 c_2 such that for every $\tilde{\gamma} \in (0, \tilde{\gamma}_0]$ we have the following estimate

$$|u(z_1) - u(z_2)| \le \rho^{\tilde{\gamma}}(z_1, z_2)(M_{\tilde{\gamma}} + M_2 + M_1 q^{-\tilde{\gamma}})N \tag{20}$$

with constant N depending on α, m, c_1, c_2, $\sup|f_u|$, where $M_{\tilde{\gamma}}$ is the Hölder constant for $u_t(0,x)$ on the lower base $t=0$, which corresponds to the power $\tilde{\gamma}$, M_2 is the bound for the right hand side f_t of equation (16), M_1 is a constant from theorem 2, and $q=1/2$ due to the choice of charts.

Then inequalities (18) and (20) give the estimate that we need on the hole cylinder $[0,T]\times V$ with power $\beta=\min\{\gamma,\tilde{\gamma}\}$. □

4. **Hölder Estimates for $\nabla_{ij}u$**

Theorem 6. *Let $u(t,x)$ be a solution of (1-2) from the space $C\left([0,T],C^4(V)\right)$ $\bigcap C^1\left([0,T],C^2(V)\right)\bigcap C^2\left([0,T],C(V)\right)$. Assume that the right hand side $f(t,x,u)$ is bounded and has bounded derivatives up to the second order, $f_u(t,x,u)\geq \delta>0$ on $[0,T]\times V\times R^1$. Suppose that u_0 is an admissible function and belongs to $C^{2+\alpha}(V)$. Then*

$$|\nabla_{ij}u(z_1)-\nabla_{ij}u(z_2)|\leq N\rho^{\beta}(z_1,z_2) \tag{21}$$

with some power $\beta\in(0,\alpha]$ depending on m and constants c_1, c_2 from theorem 3. The constant N depends on β, m, c_1 c_2, diameter D, metric g, $\|u_0\|_{C^{2+\alpha}}$, $\|f\|_{C^2}$, and δ.

Proof. Suppose that V is covered by local charts in the same way as in the proof of theorem 5. Let $z=(t,x)$ be a fixed point in $(0,T]\times\varphi_k(\Omega_k)$. Let γ be an arbitrary direction in the model space. Differentiating (1) with respect to γ, we have:

$$-\frac{\partial}{\partial t}\nabla_\gamma u+g_u^{\alpha\beta}\nabla_{\gamma\alpha\beta}u=f_\gamma+f_u\nabla_\gamma u.$$

Differentiating once more, we get

$$\begin{aligned}-\tfrac{\partial}{\partial t}\nabla_{\gamma\gamma}u\quad &+\nabla_\gamma(g_u^{\alpha\beta})\nabla_{\gamma\alpha\beta}u+g_u^{\alpha\beta}\nabla_{\gamma\gamma\alpha\beta}u=\\ &=\nabla_\gamma(f_\gamma)+\nabla_\gamma(f_u)\nabla_\gamma u+f_u\nabla_{\gamma\gamma}u.\end{aligned}$$

Write $w=\nabla_{\gamma\gamma}u$, then

$$\begin{aligned}-w_t\,-\,&g_u^{\alpha k}g_u^{l\beta}\nabla_{\gamma kl}u\nabla_{\gamma\alpha\beta}u+g_u^{\alpha\beta}\nabla_{\alpha\beta}w+E\\ &=f_{\gamma\gamma}+2f_{u\gamma}\nabla_\gamma u+f_{uu}(\nabla_\gamma u)^2+f_uw,\end{aligned} \tag{22}$$

where $E = g_u^{\alpha\beta}(\nabla_{\gamma\gamma\alpha\beta}u - \nabla_{\alpha\beta\gamma\gamma}u)$. Commutation formulas for forth order covariant derivatives, which contain coefficients of the curvature tensor and second order covariant derivatives, imply the following estimate ([5], lemma 2):

$$|E| \leq [a(m - \Delta u) + b]g_u^{\lambda\mu}g_{\lambda\mu} + c, \tag{23}$$

where a, b, c are positive constants, depending on diameter and curvature tensor of V. Using the estimate of $(m - \Delta u)$ (theorem 2) and uniformly equivalence of metrics g_u (theorem 3), we obtain

$$|E| \leq \frac{1}{c_1}(aK + b)g^{\lambda\mu}g_{\lambda\mu} + c = \frac{1}{c_1}(aK + b)m + c \overset{def}{=} M.$$

Let $u(t, x)$ be a solution of (1-2). Denote by L the linear differential operator $Lw = -w_t + g_u^{\alpha\beta}\nabla_{\alpha\beta}w - f_u w$. Coefficients $g_u^{\alpha\beta}$ at higher order derivatives continuous if the solution $u(t, x)$ has continuous derivatives with respect to spatial variables up to the second order. The second term in (22) nonnegative since F is convex. Therefore we get the following linear differential inequality :

$$Lw \geq -E + f_{\gamma\gamma} + 2f_{u\gamma}\nabla_\gamma u + f_{uu}(\nabla_\gamma u)^2.$$

Second order derivatives of the right hand side f are bounded, and we have the estimate $|\nabla_\gamma u| \leq 2D$, thus we obtain the inequality

$$Lw \geq -K_1, \tag{24}$$

with a constant $K_1 > 0$ depends on diameter and curvature tensor of V, and on $\|f\|_{C^2}$.

We are going to use Hölder estimates for solutions of a system of uniformly parabolic inequalities ([7]), but we need one more inequality. It will be obtained separately for interior points and for points near the base $\{0\} \times V$ of cylinder. Fix a number ρ_0, $0 < \rho_0 < 1/2$, and choose cylinders Q and Q_ρ as in theorem 5.

Each solution $u(t, x)$ of (1) is an admissible function and determine the continuous solution Λ_u of the bundle $\bar{\mathbf{S}}$:

$$\Lambda_u(t, x) = (t, x, \lambda_u(t, x)) = (t, x, \nabla_{ij}u(t, x)).$$

The above section satisfies condition (5) (theorem 3) with c_1, c_2 depending on diameter of V, metric g, curvature tensor, $\|f\|_{C^2}$, δ, and $\|u_0\|_{C^2}$. Lemma 2 implies

$$\sum_{ij} d^{ij}[\nabla_{ij}u(t,x) - \nabla_{ij}u(\tau,y)] = F(x,\lambda_u(t,x)) - F(y,\lambda_u(\tau,y)) + F_1(x,y), \quad (25)$$

where $F_1(x,y)$ satisfies Hölder's condition with power α and Hölder's constant $\hat{M}_g = M_g^{\alpha}(1/c_1 + M_g)m^2$. In (25) we have

$$F(x,\lambda_u(t,x)) = F(x,\nabla_{ij}u(t,x)) = \ln M(u)(t,x), \quad (26)$$

Let us write equation (1) at points $z = (t,x), z' = (\tau,y) \in Q_{\rho_0}$:

$$-u_t(t,x) + \ln M(u)(t,x) = f(t,x,u(t,x)),$$

$$-u_t(\tau,y) + \ln M(u)(\tau,y) = f(\tau,y,u(\tau,y)).$$

Subtracting yields:

$$\begin{aligned}&\ln M(u)(t,x) - \ln M(u)(\tau,y) = \\ &[u_t(t,x) - u_t(\tau,y)] + [f(t,x,u(t,x)) - f(\tau,y,u(\tau,y)].\end{aligned}$$

Then using the Hölder estimate for u_t (theorem 5), mean value theorem for $f(t,x,u)$ regarded as a function of three variables, and estimates from theorems 1, 2, we get

$$\begin{aligned}&|\ln M(u)(t,x) - \ln M(u)(\tau,y)| \le |u_t(t,x) - u_t(\tau,y)| + \\ &|f(t,x,u(t,x)) - f(\tau,y,u(\tau,y)| \le N\rho^{\beta}(z_1,z_2) + \\ &\sup|f_t||t-\tau| + \sup|\nabla_x f||x-y| + \sup|f_u||u(t,x) - u(\tau,y)| \le \\ &N_1\rho^{\beta}(z_1,z_2),\end{aligned} \quad (27)$$

with β is Hölder's power for u_t; N_1 depends on β, m, c_1 c_2, D, g, $\|u_0\|_{C^{2+\alpha}}$, $\|f\|_{C^2}$, and δ.

Therefore from (25), (26), and (27) we have

$$\begin{aligned}&\sum_{ij} d^{ij}(\nabla_{ij}u(t,x) - \nabla_{ij}u(\tau,y)) \le \\ &N_1\rho^{\beta}(z_1,z_2) + \hat{M}_g|x-y|^{\alpha} \le N_2\rho^{\beta}(z_1,z_2).\end{aligned} \quad (28)$$

where N_2 depends on the same values as N_1 and Hölder's constant of coefficients of g.

Now we use lemma from [7](p.212, lemma V.5.4)(see also [8], lemma 5.2, for another wording). It claims that for all positive definite matrices (a_{ij}) satisfying the condition

$$d_1|\xi|^2 \leq a_{ij}\xi_i\xi_j \leq d_2|\xi|^2, \tag{29}$$

there exists a natural number n, unit vectors $\gamma_1, \cdots, \gamma_n$, and $d \in (0,1)$, depending on m, d_1, d_2, such that the following inequality holds

$$a_{ij}u_{ij} \geq d\sum_{i=1}^{n}(u_{\gamma_i\gamma_i})_+ - 3d_2\sum_{i=1}^{n}(u_{\gamma_i\gamma_i})_-, \tag{30}$$

where u_{γ_i} denotes the derivative in the direction of vector γ_i and $c_+ = \max\{0, c\}$, $c_- = \max\{0, -c\}$.

The above claim contains partial derivatives, but it is true for covariant derivatives due to linearity of the covariant derivative with respect to subscript vector field. Indeed, inequality (30) is based on presentation of a matrix $A = (a_{ij})$ in the form $A = \sum_{i=1}^{n}\beta_i(A)\gamma_i \otimes \gamma_i$, which implies presentation of a linear operator: $Lu = \mathrm{tr}(A \cdot D^2u) = \sum_{i=1}^{n}\beta_i(A)\gamma_i^k\gamma_i^l\nabla_{kl}u$. Here $\gamma_i^k\gamma_i^l\nabla_{kl}u = \nabla_{\gamma_i\gamma_j}u$. Indeed, let γ be a direction, $\gamma = \gamma_k\partial_k$, where γ_k are constant coefficients. Then $\nabla_\gamma u = \nabla_{\gamma_1\partial_k+\cdots+\gamma_m\partial_m}u = \sum_k \gamma_k\nabla_k u$ and $\nabla_{\gamma\gamma}u = \nabla_\gamma(\sum_k \gamma_k u_k) = \sum_k \gamma_k\nabla_\gamma(\nabla_k u) = \sum_{k,l}\gamma_k\gamma_l\nabla_l(\nabla_k u) = \sum_{k,l}\gamma_k\gamma_l\nabla_{lk}u = \sum_{k,l}\gamma_k\gamma_l\nabla_{kl}u$.

Applying (28) to $d^{ij}\nabla_{ij}u(t,x)$ and $d^{ij}\nabla_{ij}(-u(\tau,y))$, and note that $(-c)_+ = c_-$, $(-c)_- = c_+$ we get:

$$\sum_{ij} d^{ij}(\nabla_{ij}u(t,x) - \nabla_{ij}u(\tau,y)) \geq$$
$$d\sum_{i=1}^{n}((\nabla_{\gamma_i\gamma_i}u(t,x) - \nabla_{\gamma_i\gamma_i}u(\tau,y)))_+ - \tfrac{3}{c_1}\sum_{i=1}^{n}((\nabla_{\gamma_i\gamma_i}u(t,x) - \nabla_{\gamma_i\gamma_i}u(\tau,y)))_-.$$

Write $w_i = \nabla_{\gamma_i\gamma_i}u$. The above inequality together with (28) imply:

$$\begin{aligned} &\tfrac{N_2c_1}{3}\rho^\beta(z_1,z_2) \geq \\ &\tfrac{dc_1}{3}\sum_{i=1}^{n}(w_i(t,x) - w_i(\tau,y))_+ - \sum_{i=1}^{n}(w_i(t,x) - w_i(\tau,y))_-. \end{aligned} \tag{31}$$

Therefore, for every point $z = (t,x)$ in the fixed local chart we have uniformly parabolic inequality (24) and for all $z = (t,x), z' = (\tau,y) \in Q_{\rho_0}$ inequality (31). Put

$K_2 = \max\{K_1, N_2c_1/3\}$, then we shall consider the same constant K_2 in the right hand sides of both inequalities (24), (31).

Now we are ready to use theorem [7](. IV.3.1, .122), which gives Hölder estimates for solutions of system of linear parabolic inequalities. In this theorem we take $f_i(r) \equiv r$ and $\nu = \alpha = \beta$, where β is the Hólder power for u_t (theorem 5). The theorem mentioned above claims existence of power $\beta_0 \in (0,1)$, depending on n, d, m, c_1, c_2, such that for all $\beta' \le \min\{\beta_0, \beta\}$ and all $z_1, z_2 \in Q_{\rho_0}$ the following inequality holds

$$\sum_{i=1}^{m} |w_i(z_1) - w_i(z_2)| \le \tilde{\rho}^{-\beta'} \rho^{\beta'}(z_1, z_2) N (K_2 \tilde{\rho}^{\beta} + \sum_{i=1}^{m} \sup_Q |w_i|), \tag{32}$$

where $\tilde{\rho} = \min\{\rho(z_1), \rho(z_2), 1\}$ and $\rho(z)$ is the parabolic distance from z to the boundary Q_{ρ_0} (evidently $\tilde{\rho} \ge \rho_0$). The constant N depends on the same values as β_0 and extra on $\sup |f_u|$, β.

Thus substituting ρ_0 for $\tilde{\rho}$ in denominator and 1 for $\tilde{\rho}$ in numerator we get the estimate on $[\rho_0, T] \times V$:

$$\sum_{i=1}^{m} |w_i(z_1) - w_i(z_2)| \le {\rho_0}^{-\beta'} \rho^{\beta'}(z_1, z_2) N_1, \tag{33}$$

with N_1 depending on diameter, $\|g\|_{C^{0+\alpha}}$, curvature tensor, $\|f\|_{C^2}$, δ, $\|u_0\|_{C^{2+\alpha}}$ and β, where β is the Hölder power for u_t.

To obtain the estimate on $[0, \rho_0) \times V$ we use ([7], theorem IV.5.1, p.147). Proceeding in the same way as in theorem 5 we cover V with charts (Ω_k, φ_k) such that images of Ω_k in the space R^m coincide with balls of radius $r = 3\sqrt{2}$ centered at $(3, 0, \ldots, 0) \in R^m$ and preimages Ω'_k of sets $\{(x_1, \ldots, x_m) : 1/2 < x_1 < 2, |x_i| < 1, i = 2, \cdots, m\}$ cover V as well. Then the theorem mentioned above claims that inequalities (24) and (31) imply existence of a constant $\tilde{\gamma}_0 \in (0,1)$, depending on n, m, c_1, c_2, d, such that for every $\tilde{\gamma} \in (0, \min\{\tilde{\gamma}_0, \beta\}]$ the following inequality holds

$$\sum_{i=1}^{n} |w_i(z_1) - w_i(z_2)| \le \rho^{\tilde{\gamma}}(z_1, z_2)(M_{\tilde{\gamma}} + K_2 + M q^{-\tilde{\gamma}}) N \tag{34}$$

with N depending on n, m, c_1, c_2, $\sup|f_u|$, γ, β, where $M_{\tilde{\gamma}}$ is the largest Hólder constant of $\nabla_{\gamma_i\gamma_i}u_0$ with power $\tilde{\gamma}$, $M = \sup\nabla_{\gamma_i\gamma_i}u$, and $q = 1/2$ due to the choice of charts. □

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FRACTIONAL BROWNIAN MOTION USING CONTRACTION METHOD IN PROBABILISTIC METRIC SPACES

A. SOÓS

Abstract. In this paper we show how the random scaling law can be generalized such that the fractional Brownian motion satisfies it. Using the contraction method in probabilistic metric spaces, we give existence and uniqueness conditions for fractional Brownian motion.

The fractional Brownian motion (fBm) has been introduced in 1968 by Mandelbrot and Van Ness. For any H in $[0,1]$ we denote by $\{B_t^H : t \in [0,1]\}$ the fractional Brownian motion of index H (Hurst parameter), and it is the centered Gaussian process whose covariance kernel is given by

$$R_H(s,t) = E(B_s^H B_t^H) := \frac{V_H}{2}\left(s^{2H} + t^{2H} - |t-s|^{2H}\right),$$

where

$$V_H := \frac{\Gamma(2-2H)cos(\pi H)}{\pi H(1-2H)}.$$

The theoretical study of the fractional Brownian motion was originally motivated by new problems in mathematical finance and telecommunication networks. In engineering applications of stochastic processes it is often used to model the input of system. These real inputs exhibit long-range dependence: the behavior of a real process after a given time t does not only depend on the situation at time t but also on the whole history of the process up to time t.

Another property of the fBm encountered in applications is the self similarity: the behavior of fBm is stochastically the same, up to a space-scaling, i.e. the process

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$\{X_{\alpha t}, t \in [0,1]\}$ has the same law as the process $\{\alpha^H X_t, t \in [0,1]\}$, where $H \in]0,1[$ and $\alpha > 0$.

Since R_H is a positive definite operator, the Bochner-Milos theorem ensures that, for any value of $H \in [0,1]$, there exists a unique probability measure on $C_0([0,1];\mathbb{R})$ such that the canonical process is a fBm.

Using fractal theory methods Hutchinson and Rüschendorf [2] have obtained the classical Brownian motion ($H = \frac{1}{2}$) as the invariant set for an iterated function system.

A first theory of selfsimilar fractal sets and measures was developed in Hutchinson [1]. Falconer, Graf, Mouldin and Williams, and Arbeiter randomized each step in the approximation process to obtain self-similar random fractal sets and measures. Recently Hutchinson and Rüschendorf [3] gave a simple proof for the existence and uniqueness of random fractal sets, measures and fractal functions using probability metrics defined by expectation.

In this paper we use probabilistic metric spaces techniques in order to prove that the fBm can be characterized as the fixed point of a scaling law.

1. Invariant sets in E-spaces

Let X be a nonempty set. We denote by Δ^+ denote the set of all distribution functions F with $F(0) = 0$. A *Menger space* is a triplet $(X, \mathcal{F}, T)$, where $\mathcal{F} : X \times X \to \Delta^+$ is a mapping with the following properties:

1^0. $F_{x,y}(t) = F_{y,x}(t)$ for all $x, y \in X$ and $t \in \mathbb{R}$;

2^0. $F_{x,y}(t) = 1$, for every $t > 0$, if and only if $x = y$;

3^0. $F_{x,y}(s + t) \geq T(F_{x,z}(s), F_{z,y}(t))$ for all $x, y, z \in X$ and $s, t \in \mathbb{R}_+$,

and T is a t-norm.

A mapping $T : [0,1] \times [0,1] \to [0,1]$ is called a *t-norm* if the following conditions are satisfied:

4^0. $T(a, 1) = a$ for every $a \in [0,1]$;

5^0. $T(a, b) = T(b, a)$ for every $a, b \in [0,1]$

6^0. if $a \geq c$ and $b \geq d$ then $T(a, b) \geq T(c, d)$;

7^0. $T(a, T(b, c)) = T(T(a, b), c)$ for every $a, b, c \in [0, 1]$.

The mapping $f : X \to X$ is said to be a *contraction* if there exists $r \in]0, 1[$ such that

$$F_{f(x),f(y)}(rt) \geq F_{x,y}(t)$$

for every $x, y \in X$ and $t \in \mathbb{R}_+$.

A sequence $(x_n)_{n\in\mathbb{N}}$ from X is said to be *Cauchy* if

$$\lim_{n,m\to\infty} F_{x_m,x_n}(t) = 1 \text{ for all } t > 0.$$

The element $x \in X$ is called *limit* of the sequence $(x_n)_{n\in\mathbf{N}}$ if $\lim_{n\to\infty} F_{x,x_n}(t) = 1$ for all $t > 0$. A probabilistic metric (Menger) space is said to be *complete* if every Cauchy sequence in this space is convergent.

The notion of *E-space* was introduced by Sherwood [7] in 1969. Let $(\Omega, \mathcal{K}, P)$ be a probability space and let (Y, ρ) be a metric space. The ordered pair $(\mathcal{E}, \mathcal{F})$ is an *E-space over the metric space* (Y, ρ) if the elements of $\mathcal{E}$ are random variables from Ω into Y and $\mathcal{F} : \mathcal{E} \times \mathcal{E} \to \Delta^+$ defined by $\mathcal{F}(x, y) = F_{x,y}$, where

$$F_{x,y}(t) = P(\{\omega \in \Omega | \; d(x(\omega), y(\omega)) < t\})$$

for every $t \in \mathbb{R}$. The E-space $(\mathcal{E}, \mathcal{F})$ is said to be complete if the Menger space $(\mathcal{E}, \mathcal{F}, T_m)$ is complete, where $T_m(x, y) = \max\{x + y - 1, 0\}$.

In the sequel we will use the following result proved in [4]:

Theorem 1.1. *Let $(\mathcal{E}, \mathcal{F})$ be a complete E- space, $N \in \mathbb{N}^*$, and let $f_1, ..., f_N : \mathcal{E} \to \mathcal{E}$ be contractions with ratio $r_1, ...r_N$, respectively. Suppose that there exists an element $z \in \mathcal{E}$ and a real number γ such that*

$$P(\{\omega \in \Omega | \rho(z(\omega), f_i(z(\omega)) \geq t\}) \leq \frac{\gamma}{t}, \tag{1}$$

for all $i \in \{1, .., N\}$ and for all $t > 0$. Then there exists a unique nonempty closed bounded and compact subset K of $\mathcal{E}$ such that

$$f_1(K) \cup ... \cup f_N(K) = K.$$

Corollary 1.1. *Let $(\mathcal{E},\mathcal{F})$ be a complete E- space, and let $f:\mathcal{E}\to\mathcal{E}$ be a contraction with ratio r. Suppose that there exists $z\in\mathcal{E}$ and a real number γ such that*

$$P(\{\omega\in\Omega|\ \rho(z(\omega),f(z)(\omega))\geq t\})\leq\frac{\gamma}{t}\ \textit{for all}\, t>0.$$

Then there exists a unique $x_0\in\mathcal{E}$ such that $f(x_0)=x_0$.

2. **Scaling law and Brownian motion**

Denote by (X,d) a complete separable metric space. Let $g:I\to X$, where $I\subset\mathbb{R}$ is a closed bounded interval, $N\in\mathbb{N}$ and let $I=I_1\cup I_2\cup\cdots\cup I_N$ be a partition of I into disjoint subintervals. Let $\Phi_i:I\to I_i$ be increasing Lipschitz maps with $p_i=Lip\Phi_i$. If $g_i:I_i\to X$, for $i\in\{1,...,N\}$ define $\sqcup_i g_i:I\to X$ by

$$\left(\sqcup_i g_i\right)(x)=g_j(x)\quad\text{for}\quad x\in I_j.$$

A *scaling law* $\mathbb{S}$ is an N-tuple $(S_1,....,S_N)$, $N\geq 2$, of Lipschitz maps $S_i:X\to X$. Denote $r_i=LipS_i$.

A random scaling law $\mathbb{S}=(S_1,S_2,...,S_N)$ is a random variable whose values are scaling laws. We write $\mathcal{S}=dist\mathbb{S}$ for the probability distribution determined by $\mathbb{S}$ and $\stackrel{d}{=}$ for the equality in distribution.

Let $\mathbb{S}=(S_1,...,S_N)$ be a random scaling law and let $G=(G_t)_{t\in I}$ be a stochastic process or a random function with state space X. The trajectory of the process G is the function $g:I\to X$. The trajectory of the random function $\mathbb{S}g$ is defined up to probability distribution by

$$\mathbb{S}g\stackrel{d}{=}\sqcup_i S_i\circ g^{(i)}\circ\Phi_i^{-1},$$

where $\mathbb{S},g^{(1)},...,g^{(N)}$ are pairwise independent and $g^{(i)}\stackrel{d}{=}g$, for $i\in\{1,...,N\}$. We say that g or $\mathcal{G}$ *satisfies the scaling law* $\mathbb{S}$, or is a *random fractal function*, if

$$\mathbb{S}g\stackrel{d}{=}g,$$

The fBm can be characterized as the fixed point of a scaling law. Next we will contruct this scaling law.

Let $(\Omega, \mathcal{K}, P)$ be a probability space. The fBm with Hurst exponent H is a stochastic process $B^\alpha = (B^\alpha_t)_{t\in\mathbb{R}}$ characterised by $B^H_0(\omega) = 0$ a.s. and

$$B^H(t+h) - B^H(t) \stackrel{d}{=} N(0, h^H), \quad \text{for} \quad t > 0 \,\text{and}\, h > 0,$$

where $N(0, h^H)$ denotes the normal distribution with mean 0 and variance h^{2H}.

For each $H > 0$, let $B^H : [0,1] \to \mathbb{R}$ denote the *constrained fBm* given by

$$B^H(0) = 0 \; a.s. \quad \text{and} \quad B^H(1) = 1 \; a.s.$$

For a fixed $p \in \mathbb{R}$ consider the fBm $B^H\Big|_{B^H(\frac{1}{2})=p}$ constrained by $B^H(\frac{1}{2}) = p$.

Let $S_1, S_2 : \mathbb{R} \to \mathbb{R}$ be the unique affine transformations characterized by $S_1(0) = 0$, $S_1(1) = S_2(0) = p$, $S_2(1) = 1$. If $r_1 = LipS_1 = |p|, \quad r_2 = LipS_2 = |1-p|$, then

$$B^H|_{B^H(\frac{1}{2})=p}(t) \stackrel{d}{=} S_1 \circ B^{\frac{H}{2r_1^2}}(2t), \quad t \in [0, \frac{1}{2}].$$

Similarly

$$B^H|_{B^H(\frac{1}{2})=p}(t) \stackrel{d}{=} S_2 \circ B^{\frac{H}{2r_1^2}}(2t-1), \quad t \in [\frac{1}{2}, 1].$$

Let $I = [0,1]$, and define

$$\Phi_1 : I \to [0, \frac{1}{2}], \quad \Phi_1(s) = \frac{s}{2}, \,\text{and}\Phi_2 : I \to [\frac{1}{2}, 1], \quad \Phi_1(s) = \frac{s+1}{2}.$$

It follows that

$$B^H|_{B^H(\frac{1}{2})}(t) \stackrel{d}{=} \sqcup_i S_i \circ B^{\frac{H}{2r_i^2}} \circ \Phi_i^{-1}(t), \quad t \in [0,1].$$

Now let p^H be a random variable with distribution $N(0, \frac{H}{2})$. For each $H > 0$ let us define the random scaling law $\mathbb{S}^H = (S^H_1, S^H_2)$ in the same manner that (S_1, S_2) was previously defined from the point p.

Let $r^H_i = Lip^H_i$ for $i = 1, 2$ and let $r^\alpha = \max\{r^H_1, r^H_2\}$. It follows for each $H > 0$ that

$$B^H \stackrel{d}{=} \sqcup_i S^H_i \circ B^{\frac{H}{2r_i^2}(i)} \circ \Phi_i^{-1},$$

where $\mathbb{S}$ is first chosen as above, and then after conditioning on $\mathbb{S}$, $B^{\frac{H}{2r_1^2}(1)} \stackrel{d}{=} B^{\frac{H}{2r_1^2}}$ and $B^{\frac{H}{2r_2^2}(2)} \stackrel{d}{=} B^{\frac{H}{2r_2^2}}$ are chosen independently of one another.

Thus the family of constrained Brownian motion $\{B^H|H > 0\}$ satisfies the family of scaling laws $\mathbb{S} = \{\mathbb{S}^H|H > 0\}$.

3. Generalized scaling law

In this section we generelize the notion of random scaling law. Let p^H be a random variable in $\mathbb{R}$ with distribution $N(0, \frac{H}{2})$ and denote $I = [a, b]$. Let $S_1^H, S_2^H : \mathbb{R} \to \mathbb{R}$ be the unique affine transformations characterized by $S_1^H(a) = a$, $S_1^H(b) = S_2^H(a) = p^H$, $S_2H(b) = b$. Let $\Phi_i : I \to I_i, \quad i = 1, 2$ be increasing Lipschitz maps, such that $I_1 \cup I_2 = I$ and $\overset{\circ}{I}_1 \cap \overset{\circ}{I}_2 = \emptyset$.

The *generalized random scaling law* is a family of scaling laws

$$\mathbb{S} = \{\mathbb{S}^H|H > 0\}.$$

If $f^{\omega,H}(t) = f^{\omega}(H,t) :]0, \infty[\times I \to \mathbb{R}$ is a stochastic process, then the stochastic process $(\mathbb{S}f)^H$ is defined up to probability distribution by

$$(\mathbb{S}f)^H \overset{d}{=} \sqcup_i S_i^H \circ f^{\frac{H}{2r_i^2}(i)} \circ \Phi_i^{-1},$$

where $\mathbb{S}$ is first chosen as before, and then after conditioning on $\mathbb{S}$, $f^{\frac{H}{2r_1^2}(1)} \overset{d}{=} f^{\frac{H}{2r_1^2}}$ and $f^{\frac{H}{2r_2^2}(2)} \overset{d}{=} f^{\frac{H}{2r_2^2}}$ are chosen independently of one another.

The family of stochastic processes or random functions f^H *satisfies the generalized scaling law* $\mathbb{S}$ or is a *fractal stochastic process* if

$$(\mathbb{S}f)^H \overset{d}{=} f^H.$$

Theorem 3.1. *Denote by $\mathcal{E}^H$ the set of random functions $g^H : \Omega \times I \to \mathbb{R}$ with the following property: there exist $h^H \in \mathcal{E}^H$ and a positive number γ such that*

$$P(\{\omega \in \Omega| \sup_H H^{-\frac{1}{2}} \int_I |h^H(x)|dx \geq t\}) \leq \frac{\gamma}{t}$$

for all $t > 0$.

Then there exists a family of stochastic processes $g* \in \mathcal{E}^H$ satisfying $\mathbb{S}$.

Proof. Let $f : \mathcal{E}^\alpha \to \mathcal{E}^\alpha$ defined by

$$f(g^H) = (\mathbb{S}g)^H = \sqcup_i S_i^H \circ g^{\frac{H}{2r_i^2}(i)} \circ \Phi_i^{-1},$$

where $\mathbb{S}$ is first chosen as in the previous section, and then after conditioning on $\mathbb{S}$, $g^{\frac{H}{2r_i^2}(i)} \stackrel{d}{=} g^{\frac{H}{2r_i^2}}$, $i = 1, 2$ are chosen independently of one another.

We first claim that, if $g^H \in \mathcal{E}^H$ then $f(g^H) \in \mathcal{E}^H$ as well. For this, choose $g^{\frac{H}{2r_i^2}(i)} \stackrel{d}{=} g^{\frac{H}{2r_i^2}}$, $i = 1, 2$, independently of one another and $\mathbb{S}^H = (S_1^H, S_2^H)$. Then, for $t > 0$,

$$P(\{\omega \in \Omega | \sup_H H^{-\frac{1}{2}} \int_I |(\mathbb{S}h)^H(x)| dx \geq t\}) \leq$$

$$\leq P(\{\omega \in \Omega | \frac{1}{2} \sup_H H^{-\frac{1}{2}} \sum_{i=1}^{2} r_i^H \int_{I_i} |h^{\frac{\alpha}{2(r_i^\alpha)^2}(i)}(x)| dx \geq t\}) \leq \frac{\gamma\sqrt{2}}{t}.$$

To establish the contraction property let us consider $g_1^H, g_2^H \in \mathcal{E}^H$. Since

$$F_{f(g_1^H), f(g_2^H)}(t) \geq F_{g_1^H, g_2^H}(\frac{t}{\sqrt{2}})$$

for all $t > 0$, f is a contraction. Then we can apply Corollary 1.1 and existence and uniqueness follows. □

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SPECTRAL RADIUS OF QUOTIENT BOUNDED OPERATOR

SORIN MIREL STOIAN

Abstract. We introduce the spectral radius $r_{\mathcal{P}}(T)$ for a quotient bounded operator on a locally convex space X. Similarly to the case of bounded operator on a Banach space we prove that the Neumann series $\sum_{n=0}^{\infty} \frac{T^n}{\lambda^{n+1}}$ converges to $R(\lambda, T)$, whenever $|\lambda| > r_{\mathcal{P}}(T)$, and $|\sigma(Q_{\mathcal{P}}, T)| = r_{\mathcal{P}}(T)$.

1. **Introduction**

The spectral theory for a linear operator on Banach space X is well developed and we have useful tools for use this theory. For example, the spectral radius of such operator T is defined by the Gelfand formula $r(T) = \lim_{n\to\infty} \sqrt[n]{\|T^n\|}$ and $|\sigma(Q, T)| = r(T)$.

Further it is known that the rezolvent $R(\lambda, T)$ is given by the Neumann series $\sum_{n=0}^{\infty} \frac{T^n}{\lambda^{n+1}}$, whenever $|\lambda| > r(T)$.

If we want to generalize this theory on locally convex space X one major difficulty is that is not clear which class of operators we can use, because there are several non-equivalent ways of defining bounded operators on X. The concept of bounded element of a locally convex algebra was introduced by Allan [1]. An element is said to be bounded if some scalar multiple of it generates a bounded semigroup.

Definition 1.1. Let X be a locally convex algebra. The radius of boundness of an element $x \in X$ is the number

$$\beta(x) = \inf\{\alpha > 0 |\ \text{the set } \{(\alpha x)^n\}_{n\geq 1} \text{ is bounded}\}.$$

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In this paper we consider the class of quotient bounded operators, which was introduced in Appendix A by A. Michael [8], and later was studied by T. Moore [9] and A. Chilana [2].

Throught this paper all locally convex spaces will be assumed Hausdorff, over complex field $\mathbb{C}$, and all operators will be linear. If X and Y are topological vector spaces we denote by $L(X,Y)$ ($\mathcal{L}(X,Y)$) the algebra of linear operators (continuous operators) from X to Y.

Any family $\mathcal{P}$ of seminorms who generate the topology of locally convex space X (in the sense that the topology of X is the coarsest with respect to which all seminorms of $\mathcal{P}$ are continuous) will be called a calibration on X. A calibration is characterized by the property, that for every seminorms $p \in \mathcal{P}$ and every constant $\varepsilon > 0$ the sets

$$S(p,\varepsilon) = \{x \in X|\ p(x) < \varepsilon\},$$

constitute a neighbourhoods sub-base at 0. A calibration on X will be principal if it is directed. The set of calibration for X is denoted by $\mathcal{C}(X)$.

Any family of seminorms on a linear space is partially ordered by relation "$\leq$", where

$$p \leq q \ \Leftrightarrow\ p(x) \leq q(x),\ \forall\ x \in X.$$

A family of seminorms is preordered by relation "$\prec$", where

$$p \prec q \ \Leftrightarrow \ \text{ there exists some } r > 0 \text{ such that } p(x) \leq rq(x),\ \forall\ x \in X.$$

If $p \prec q$ and $q \prec p$, we write $p \approx q$.

Definition 1.2. Two families $\mathcal{P}_1$ and $\mathcal{P}_2$ of seminorms on a linear space are called Q-equivalent (denoted $\mathcal{P}_1 \approx \mathcal{P}_2$) provided:

a) for each $p_1 \in \mathcal{P}_1$ there exists $p_2 \in \mathcal{P}_2$ such that $p_1 \approx p_2$;

b) for each $p_2 \in \mathcal{P}_2$ there exists $p_1 \in \mathcal{P}_1$ such that $p_2 \approx p_1$.

It is obvious that two Q-equivalent and separating families of seminorms on a linear space generate the same locally convex topology.

Similar to the norm of an operator on a normed space we define the mixed operator seminorm of an operator between locally convex spaces. If $(X, \mathcal{P})$, $(Y, \mathcal{Q})$ are locally convex spaces, then for each $p, q \in \mathcal{P}$ the application $m_{pq} : L(X,Y) \to \mathbb{R} \cup \{\infty\}$, defined by

$$m_{pq}(T) = \sup_{p(x) \neq 0} \frac{q(Tx)}{p(x)},$$

is called the mixed operator seminorm of T associated with p and q. When $X = Y$ and $p = q$ we use notation $\widehat{p} = m_{pp}$.

Lemma 1.3. (V. Troistky [10]) *If $(X, \mathcal{P}), (Y, \mathcal{Q})$ are locally convex spaces and $T \in L(X,Y)$, then*

1) $m_{pq}(T) = \sup\limits_{p(x)=1} q(Tx) = \sup\limits_{p(x)\leq 1} q(Tx),\ \forall\ p \in \mathcal{P},\ \forall\ q \in \mathcal{Q}$;

2) $q(Tx) \leq m_{pq}(T)p(x),\ \forall\ x \in X$, whenever $m_{pq}(T) < \infty$.

Corollary 1.4. *If $(X, \mathcal{P}), (Y, \mathcal{Q})$ are locally convex spaces and $T \in L(X,Y)$, then*

$$m_{pq}(T) = \inf\{M > 0 |\ q(Tx) \leq Mp(x),\ \forall\ x \in X\},$$

whenever $m_{pq}(T) < \infty$.

Proof. If $p, q \in \mathcal{P}$ then from previous lemma we have

$$q(Tx) \leq m_{pq}(T)p(x),\ \forall\ x \in X.$$

If $M > 0$ such that

$$q(Tx) \leq Mp(x),\ \forall\ x \in X,$$

then using lemma 1.3.(1) we obtain

$$m_{pq}(T) = \sup_{p(x)=1} q(Tx) \leq M.$$

Definition 1.5. An operator T on a locally convex space X is quotient bounded with respect to a calibration $\mathcal{P} \in \mathcal{C}(X)$ if for every seminorm $p \in \mathcal{P}$ there exists some $c_p > 0$ such that

$$p(Tx) \leq c_p p(x),\ \forall\ x \in X.$$

The class of all quotient bounded operators with respect to a calibration $\mathcal{P} \in \mathcal{C}(X)$ is denoted by $Q_{\mathcal{P}}(X)$.

Lemma 1.6. *If X is a locally convex space and $\mathcal{P} \in \mathcal{C}(X)$, then for every $p \in \mathcal{P}$ the application $\widehat{p} : Q_{\mathcal{P}}(X) \to \mathbb{R}$ defined by*

$$\widehat{p}(T) = \{r > 0|\ p(Tx) \leq rp(x),\ \forall\ x \in X\},$$

is a submultiplicative seminorm on $Q_{\mathcal{P}}(X)$, satisfying $\widehat{p}(I) = 1$.

We denote by $\widehat{\mathcal{P}}$ the family $\{\widehat{p}|p \in \mathcal{P}\}$.

Proposition 1.7. (G. Joseph [7]) *Let X be a locally convex space and $\mathcal{P} \in \mathcal{C}(X)$.*

1) $Q_{\mathcal{P}}(X)$ is a unital subalgebra of the algebra of continuous linear operators on X;

2) $Q_{\mathcal{P}}(X)$ is a unital locally multiplicative convex algebra (l.m.c.-algebra) with respect to the topology determined by $\widehat{\mathcal{P}}$;

3) If $\mathcal{P}' \in \mathcal{C}(X)$ such that $\mathcal{P} \approx \mathcal{P}'$, then $Q_{\mathcal{P}'}(X) = Q_{\mathcal{P}}(X)$ and $\widehat{\mathcal{P}} = \widehat{\mathcal{P}}'$;

4) The topology generated by $\widehat{\mathcal{P}}$ on $Q_{\mathcal{P}}(X)$ is finer than the topology of uniform convergence on bounded subsets of X.

Lemma 1.8. *If X is a sequentially complete convex space, then $Q_{\mathcal{P}}(X)$ is a sequentially complete m-convex algebra for all $\mathcal{P} \in \mathcal{C}(X)$.*

Proof. Let $\mathcal{P} \in \mathcal{C}(X)$ and $(T_n)_n \subset Q_{\mathcal{P}}(X)$ be a Cauchy sequence. Then, for each $\varepsilon > 0$ and each $\widehat{p} \in \widehat{\mathcal{P}}$ there exists some index $n_{p,\varepsilon} \in \mathbb{N}$ such that

$$|\widehat{p}(T_n) - \widehat{p}(T_m)| \leq \widehat{p}(T_n - T_m) < \varepsilon,\ \forall\ n, m \geq n_{p,\varepsilon}. \tag{1}$$

From the previous relation it follows that $(\widehat{p}(T_n))_n$ is convergent sequence of real numbers, for each $\widehat{p} \in \widehat{\mathcal{P}}$. If $x \in X$, then

$$p(T_n x - T_m x) \leq \widehat{p}(T_n - T_m)p(x),\ \forall\ p \in \mathcal{P}, \tag{2}$$

so $(T_n(x))_n \subset X$ is a Cauchy sequence. But, since X is sequentially complete and Hausdorff, there exists an unique element $y \in X$ such that

$$\lim_{n\to\infty} T_n x = y.$$

Therefore, the operator $T : X \to X$ defined by

$$T(x) = \lim_{n\to\infty} T_n x, \ \forall \ x \in X,$$

is well defined. It is obvious that T is linear operator. Using the continuity of seminorms $\widehat{p} \in \widehat{\mathcal{P}}$ we have

$$p(Tx) = p\left(\lim_{n\to\infty} T_n x\right) = \lim_{n\to\infty} p(T_n x) \leq \lim_{n\to\infty} \widehat{p}(T_n)p(x) = c_p p(x),$$

for all $x \in X$ and for each $p \in \mathcal{P}$ (where $c_p = \lim_{n\to\infty} \widehat{p}(T_n)$).

This implies that $T \in Q_{\mathcal{P}}(X)$. Now we prove that $T_n \to T$ in $Q_{\mathcal{P}}(X)$.

From relations (1) and (2) it follows that for each $\varepsilon > 0$ and each $\widehat{p} \in \widehat{\mathcal{P}}$ there exists $n_{p,\varepsilon} \in \mathbb{N}$ such that

$$p(T_n x - T_m x) < \varepsilon p(x), \ \forall \ n, m \geq n_{p,\varepsilon}$$

so

$$p(T_n x - Tx_) \leq \varepsilon p(x), \ \forall \ n \geq n_{p,\varepsilon}.$$

This implies that

$$\widehat{p}(T_n - T) \leq \varepsilon, \ \forall \ n \geq n_{p,\varepsilon},$$

which prove that $T_n \to T$ in $Q_{\mathcal{P}}(X)$ and $Q_{\mathcal{P}}(X)$ is a sequentially complete m-convex algebra. □

Given $(X, \mathcal{P})$, for each $p \in \mathcal{P}$ let N^p denote the null space $\{x| \ p(x) = 0\}$ and X_p the quotient space X/N^p. For each $p \in \mathcal{P}$ consider the natural mapping

$$x \to x_p \equiv x + N^p \text{ (from } X \text{ to } X_p).$$

It is obvious that X_p is normed space, for each $p \in \mathcal{P}$, with norm defined by $\|x_p\|_p = p(x)$. Consider the algebra homomorphism $T \to T^p$ of $Q_{\mathcal{P}}(X)$ into $\mathcal{L}(X_p)$ defined by

$$T^p(x_p) = (Tx)_p, \ \forall \ x \in X.$$

This operator are well defined because $T(N^p) \subset N^p$. Moreover, for each $p \in \mathcal{P}$, $\mathcal{L}(X_p)$ is a unital normed algebra and we have

$$\|T_p\|_p = \sup\{\|T_p x_p\|_p | \; \|x_p\|_p \leq 1 \text{ for } x_p \in X_p\} =$$

$$= \sup\{p(Tx) | \; p(x) \leq 1 \text{ for } x \in X\}.$$

For $p \in \mathcal{P}$ consider the normed space $(\widetilde{X}_p, \|\cdot\|_p)$ the completion of $(X_p, \|\cdot\|_p)$. If $T \in Q_{\mathcal{P}}(X)$, then the operator T^p has a unique continuous linear extension $\widetilde{T}^p$ on $(\widetilde{X}, \|\cdot\|_p)$.

Definition 1.9. Let $(X, \mathcal{P})$ be a locally convex space and $T \in Q_{\mathcal{P}}(X)$. We say that $\lambda \in \rho(Q_{\mathcal{P}}, T)$ if the inverse of $\lambda I - T$ exists and $(\lambda I - T)^{-1} \in Q_{\mathcal{P}}(X)$.

Spectral sets $\sigma(Q_{\mathcal{P}}T)$ are defined to be complements of rezolvent sets $\rho(Q_{\mathcal{P}}, T)$.

For each $p \in \mathcal{P}$ we denote by $\sigma(X_p, T^p)$ $(\sigma(\widetilde{X}_p, \widetilde{T}^p))$ the spectral set of the operator T^p in $\mathcal{L}(X_p)$ (respectively the rezolvent set of $\widetilde{T}^p$ in $\mathcal{L}(\widetilde{X}_p)$). The rezolvent set of the operator T^p in $\mathcal{L}(X_p)$ (respectively the spectral set of $\widetilde{T}^p$ in $\mathcal{L}(\widetilde{X}_p)$) is denoted by $\rho(X_p, T^p)$ $(\rho(\widetilde{X}_p, \widetilde{T}^p))$.

Lemma 1.10. (J. R. Gilles, G. Joseph, B. Sims [6]) *Let $(X, \mathcal{P})$ be a sequentially complete convex space and $T \in Q_{\mathcal{P}}(X)$. Then T is invertible in $Q_{\mathcal{P}}(X)$ if and only if $\widetilde{T}^p$ is invertible in $\mathcal{L}(\widetilde{X}_p)$ for all $p \in \mathcal{P}$.*

Corollary 1.11. (J. R. Gilles, G. Joseph, B. Sims [6]) *If $(X, \mathcal{P})$ is a sequentially complete convex space and $T \in Q_{\mathcal{P}}(X)$, then*

$$\sigma(Q_{\mathcal{P}}, T) = \cup\{\sigma(X_p, T^p) | \; p \in \mathcal{P}\} = \cup\{\sigma(\widetilde{X}_p, \widetilde{T}^p) | \; p \in \mathcal{P}\}.$$

2. Spectral radius of quotient bounded operators

Let $(X, \mathcal{P})$ be a locally convex space and $T \in Q_{\mathcal{P}}(X)$. We said that T is bounded element of the algebra $Q_{\mathcal{P}}(X)$ if it is a bounded element of $Q_{\mathcal{P}}(X)$ in the sense of G. R. Allan [1]. The class of bounded elements of $Q_{\mathcal{P}}(X)$ is denoted by $(Q_{\mathcal{P}}(X))_0$.

Definition 2.1. If $(X, \mathcal{P})$ is a locally convex space and $T \in Q_{\mathcal{P}}(X)$ we denote by $r_{\mathcal{P}}(T)$ the radius of boundness of operator T in $Q_{\mathcal{P}}(X)$, i.e.

$$r_{\mathcal{P}}(T) = \inf\{\alpha > 0 | \ \alpha^{-1}T \text{ generates a bounded semigroup in } Q_{\mathcal{P}}(X)\}.$$

We said that $r_{\mathcal{P}}(T)$ is the $\mathcal{P}$-spectral radius of the operator T.

Proposition 1.7(3) implies that for each $\mathcal{P}' \in \mathcal{C}(X)$, $\mathcal{P} \approx \mathcal{P}'$, we have $Q_{\mathcal{P}'}(X) = Q_{\mathcal{P}}(X)$, so if $\mathcal{H}$ is a Q-equivalence class in $\mathcal{C}(X)$, then

$$r_{\mathcal{P}}(T) = r_{\mathcal{P}'}(T), \ \forall \ \mathcal{P}, \mathcal{P}' \in \mathcal{H}.$$

Since $Q_{\mathcal{P}}(X)$ is a m-convex algebra, for each $\mathcal{P} \in \mathcal{C}(X)$, the propositions 2.2-2.5 follows from the results proved by G. A. Allan [1] and I. Colojoara [3].

Proposition 2.2. *If X is a locally convex algebra and $\mathcal{P} \in \mathcal{C}(X)$, then for each $T \in Q_{\mathcal{P}}(X)$ we have:*

1) $r_{\mathcal{P}}(T) \geq 0$ and

$$r_{\mathcal{P}}(\lambda T) = |\lambda| r_{\mathcal{P}}(T), \ \forall \ \lambda \in \mathbb{C},$$

where by convention $0\infty = \infty$;

2) $r_{\mathcal{P}}(T) < +\infty$ if and only if $T \in (Q_{\mathcal{P}}(X))_0$;

3) $r_{\mathcal{P}}(T) = \inf\left\{\lambda > 0 | \ \lim\limits_{n\to\infty} \dfrac{T^n}{\lambda^n} = 0\right\}$.

Proposition 2.3. *If X is a locally convex algebra and $\mathcal{P} \in \mathcal{C}(X)$, then for each $T \in Q_{\mathcal{P}}(X)$ we have:*

$$r_{\mathcal{P}}(T) = \sup\left\{\limsup_{n\to\infty} (\widehat{p}(T^n))^{1/n} | \ p \in \mathcal{P}\right\} =$$

$$= \sup\left\{\lim_{n\to\infty} (\widehat{p}(T^n))^{1/n} | \ p \in \mathcal{P}\right\} = \sup\left\{\inf_{n\geq 1} (\widehat{p}(T^n))^{1/n} | \ p \in \mathcal{P}\right\}.$$

Proposition 2.4. *Let X be a locally convex algebra and $\mathcal{P} \in \mathcal{C}(X)$.*

1) If $T \in (Q_{\mathcal{P}}(X))$, then

$$\lim_{n\to\infty} \frac{T^n}{\lambda^n} = 0, \ \forall \ |\lambda| > r_{\mathcal{P}}(T);$$

2) If $T \in (Q_{\mathcal{P}}(X))_0$ and $0 < |\lambda| < r_{\mathcal{P}}(T)$, then the set $\left\{\dfrac{T^n}{\lambda^n}\right\}_{n\geq 1}$ *is unbounded;*

3) For each $T \in Q_{\mathcal{P}}(X)$ and every $n > 0$ we have

$$r_{\mathcal{P}}(T^n) = r_{\mathcal{P}}(T)^n.$$

Proposition 2.5. *Let X be a locally convex algebra and $\mathcal{P} \in \mathcal{C}(X)$. Then:*

1) $r_{\mathcal{P}}(T+S) \leq r_{\mathcal{P}}(T) + r_{\mathcal{P}}(S)$, $\forall\, T, S \in Q_{\mathcal{P}}(X)$ which have property $TS = ST$;

2) $r_{\mathcal{P}}(TS) \leq r_{\mathcal{P}}(T) r_{\mathcal{P}}(S)$, $\forall\, T, S \in Q_{\mathcal{P}}(X)$ which have property $TS = ST$.

From real analysis we have the following lemma.

Lemma 2.6. (V. Troistky [10]) *If $(t_n)_n$ is a sequence in $\mathbb{R}^* \cup \{\infty\}$ then*

$$\limsup_{n\to\infty} \sqrt[n]{t_n} = \inf\left\{ v > 0 | \lim_{n\to\infty} \frac{t_n}{v^n} = 0 \right\}.$$

This lemma implies that for a bounded operator on Banach space we have

$$r(T) = \lim_{n\to\infty} \sqrt[n]{\|T^n\|} = \inf\Big\{ v > 0 |\ \text{sequence } \left(\frac{T^n}{v^n}\right)_n \text{ converge to zero}$$

$$\text{in operator norm topology}\Big\}.$$

If we consider this relation as an alternative definition of the spectral radius, then proposition 2.2(3) implies that $\mathcal{P}$-spectral radius of an quotient bounded operator can be considered to be natural generalization of the spectral radius of bounded operator on Banach space.

Proposition 2.7. *Let X be a sequentially complete locally convex algebra and $\mathcal{P} \in \mathcal{C}(X)$. If $T \in (Q_{\mathcal{P}}(X))_0$ and $|\lambda| > r_{\mathcal{P}}(T)$, then the Neumann series* $\displaystyle\sum_{n=0}^{\infty} \frac{T^n}{\lambda^{n+1}}$ *converges to $R(\lambda, T)$ (in $Q_{\mathcal{P}}(X)$) and $R(\lambda, T) \in Q_{\mathcal{P}}(X)$.*

Proof. If $|\lambda| > r_{\mathcal{P}}(T)$, then there exists $\beta \in \mathbb{C}$ such that $0 < |\beta| < 1$ and $r_{\mathcal{P}}(T) < \beta\lambda$. From proposition 2.4(1) we obtain that for each $\varepsilon > 0$ and every $p \in \mathcal{P}$, there exists some index $n_{p,\varepsilon} \in \mathbb{N}$, with property

$$\widehat{p}\left(\frac{T^n}{(\beta\lambda)^n}\right) < \varepsilon,\ \forall\ n \geq n_{p,\varepsilon}.$$

Therefore, using corollary 1.4 we obtain

$$p\left(\frac{T^n}{(\beta\lambda)^n}x\right) \leq \widehat{p}\left(\frac{T^n}{(\beta\lambda)^n}\right)p(x) < \varepsilon p(x),\ \forall\ n \geq n_{p,\varepsilon},\ \forall\ x \in X.$$

Since $0 < |\beta| < 1$, there exists $n_0 \in \mathbb{N}$, such that

$$\sum_{k=n}^{m} |\beta|^k < 1,\ \forall\ m > n \geq n_0.$$

From a previous relation result that for each $\varepsilon > 0$ and every $p \in \mathcal{P}$ there exists an index $m_{p,\varepsilon} = \max\{n_{p,\varepsilon}, n_0\} \in \mathbb{N}$, for which we have

$$p\left(\sum_{k=n}^{m} \frac{T^k}{\lambda^k}x\right) \leq \varepsilon\left(\sum_{k=n}^{m} |\beta|^k\right)p(x) < \varepsilon p(x), \tag{3}$$

for every $m > n \geq m_{p,\varepsilon}$ and every $x \in X$.

Therefore, for each $x \in X$, $\left(\sum_{k=0}^{m} \frac{T^k}{\lambda^{k+1}}x\right)_{m\geq 0}$ is a Cauchy sequence.

But X is sequentially complete, so for every $x \in X$ there exists an unique element $y \in X$ such that

$$y = \lim_{m\to\infty} \sum_{k=0}^{m} \frac{T^k}{\lambda^{k+1}}x.$$

We consider the operator $S : X \to X$ given by

$$S(x) = \lim_{m\to\infty} \sum_{k=0}^{m} \frac{T^k}{\lambda^{k+1}}x,\ \forall\ x \in X.$$

It is obvious that S is linear operator. Moreover, from equality

$$\sum_{k=0}^{m} \frac{T^k}{\lambda^{k+1}}(\lambda x - Tx) = x - \frac{T^{m+1}}{\lambda^{m+1}}x,\ \forall\ x \in X,$$

result that if $m \to \infty$ then

$$S(\lambda x - Tx) = x,\ \forall\ x \in X.$$

Hence $S(\lambda I - T) = I$, we prove $(\lambda I - T)S = I$. From continuity of the operator T result that

$$STx = \lim_{m\to\infty} \sum_{k=0}^{m} \frac{T^k}{\lambda^{k+1}} Tx = \lim_{m\to\infty} T\left(\sum_{k=0}^{m} \frac{T^k}{\lambda^{k+1}} x\right) =$$

$$= T\left(\lim_{m\to\infty} \sum_{k=0}^{m} \frac{T^k}{\lambda^{k+1}} x\right) = TSx,$$

for all $x \in X$, therefore

$$S(\lambda I - T) = (\lambda I - T)S = I.$$

The definition of $\mathcal{P}$-spectral radius implies that family $\left(\frac{T^n}{(\beta\lambda)^n}\right)_n$ is bounded in $Q_{\mathcal{P}}(X)$, therefore for every $p \in \mathcal{P}$ there exists a constant $\varepsilon_p > 0$ with property

$$\widehat{p}\left(\frac{T^n}{(\beta\lambda)^n}\right) < \varepsilon_p, \ \forall\, n \geq 1.$$

Using again corollary 1.4 we have

$$p\left(\frac{T^n}{\lambda^n} x\right) < \varepsilon_p |\beta|^n p(x), \ \forall\, n \geq 1, \ \forall\, x \in X.$$

Therefore, for every $p \in \mathcal{P}$ there exists some $\varepsilon_p > 0$ with property

$$p\left(\sum_{k=0}^{m} \frac{T^k}{\lambda^{k+1}} x\right) < \frac{\varepsilon_p}{|\lambda|}\left(\sum_{k=0}^{m} |\beta|^k\right) p(x) < \frac{\varepsilon_p}{|\lambda|} \frac{1}{1-|\beta|} p(x),$$

for every $m \geq 1$ and every $x \in X$, which implies that $S = R(\lambda, T) \in Q_{\mathcal{P}}(X)$.

If we write relation (3) under the form

$$p\left(\sum_{k=0}^{m} \frac{T^k}{\lambda^{k+1}} x - \sum_{k=0}^{n} \frac{T^k}{\lambda^{k+1}} x\right) < \frac{\varepsilon}{|\lambda|} p(x),$$

then for $m \to \infty$ result that for every $\varepsilon > 0$ and every $p \in \mathcal{P}$ there exists some index $n_{p,\varepsilon} \in \mathbb{N}$, such that

$$p\left(Sx - \sum_{k=0}^{n} \frac{T^k}{\lambda^{k+1}} x\right) \leq \frac{\varepsilon}{|\lambda|} p(x), \ \forall\, n \geq n_{p,\varepsilon}, \ \forall\, x \in X.$$

This implies that the Neumann series $\sum_{n=0}^{\infty} \frac{T^n}{\lambda^{n+1}}$ converges to $R(\lambda, T)$ in $Q_{\mathcal{P}}(X)$. □

Proposition 2.8. *Let X be a sequentially complete locally convex algebra and $\mathcal{P} \in \mathcal{C}(X)$. If $T \in Q_{\mathcal{P}}(X)$, then*

$$|\sigma(Q_{\mathcal{P}}, T)| = r_{\mathcal{P}}(T).$$

Proof. Inequality $|\sigma(Q_{\mathcal{P}}, T)| \leq r_{\mathcal{P}}(T)$ is implied by previous proposition.

We prove now the reverse inequality. From corollary 1.11 we have

$$\sigma(Q_{\mathcal{P}}, T) = \cup\{\sigma(X_p, T^p) | \ p \in \mathcal{P}\} = \cup\{\sigma(\widetilde{X}_p, \widetilde{T}^p) | \ p \in \mathcal{P}\}.$$

So, if $|\lambda| > |\sigma(Q_{\mathcal{P}}, T)|$, then

$$|\lambda| > |\sigma(\widetilde{X}_p, \widetilde{T}^p)|, \ \forall \ p \in \mathcal{P}.$$

But, $\widetilde{X}_p$ is Banach space for each $p \in \mathcal{P}$, therefore

$$|\sigma(\widetilde{S}_p, \widetilde{T}^p)| = r(\widetilde{X}_p, \widetilde{T}^p)$$

where $r(\widetilde{X}_p, \widetilde{T}^p)$ is spectral radius of bounded operator $\widetilde{T}^p$ in $\widetilde{X}_p$.

This observation implies that for each $p \in \mathcal{P}$ we have $\dfrac{T^{p^n}}{\lambda^n} \to 0$ in $\mathcal{L}(\widetilde{X}_p)$. This means that for any $\varepsilon > 0$ we must have $\|T^n\|_p \leq (\varepsilon + |\sigma(Q_{\mathcal{P}}, T)|)^n$ for large n. Hence, by proposition 2.3 we have $r_{\mathcal{P}}(T) \leq |\sigma(Q_{\mathcal{P}}, T)|$.

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BOOK REVIEWS

Free Boundary Value Problems – Theory and Applications, **Pierluigi Colli, Claudio Verdi, Augusto Visintin,** Editors, Birkhäuser Verlag, ISNM Vol. 147, 2004, ISBN 3-7643-2193-8.

The volume collects the proceedings of the conference on free boundary problems, Trento (Italy), June 2002. The contributions concern problems which are either directly related to free boundaries, or may be so in perspective. Special emphasis was put on interdisciplinarity and on issues of applicative relevance. Talks included twenty plenary addresses, and seven sessions were devoted to selected topics: free boundary problems in polymer science, image processing, grain boundary motion, numerical aspects of free boundary problems, free boundary problems in biomathematics, modelling in crystal growth and transition in anisotropic materials. I remark some contributions about numerical aspects:

Structural Optimization by the Level-Set Method by G. Allaire et al. describes a new numerical method based on a combination of the classical shape derivative and of the level-set method for front propagation.

Finite Element Methods for Surface Diffusion by E. Bnsch et al. presents a novel variational formulation for the parametric case of this 4th order highly nonlinear geometric driven motion of a surface. The authors also develop a finite element method and propose a Schur complement approach to solve the resulting linear system.

Upscaling of Well Singularities in the Flow Transport through Heterogeneous Porous Media by Z. Chen and X. Yue presents a method based on the recently introduced over-sampling multiscale finite element method and on the introduction of new base functions that locally resolve the well singularities.

On Plasma Expansion in Vacuum by P. Degond et al. formally and numerically justifies why electron emission produces a reaction pressure which slows down the plasma expansion .

A Posteriori Error Control of Free Boundary Problems by R. H. Nochetto assesses the derivation of a posteriori error estimators (including interface error estimators): computable quantities depending on the discrete solutions and the data, which provide upper and lower bounds for the error.

Shape Deformations and Analytic Continuation in Free Boundary Problems by F. Reitich presents an analysis of stability and bifurcation of steady states and travelling waves for a class of free boundary problems. These results lead to an understanding of the mechanisms behind the observed performance of a class of numerical algorithms based on shape-perturbation theory.

A Multi-mesh Finite Element Method for 3D Phase Field Simulations by A. Schmidt presents a general framework for the adaptive solution of coupled systems and its application to phase field simulations, making 3D simulations possible even on desktop computers.

Of course, all the 26 contributions included, which reflect and study the free boundary problems with applications in industry, make this volume interesting for a large spectrum of readers and offer opportunities of collaboration among mathematicians, physicists, engineers, material scientists, biologists and other researchers.

Damian Trif

George Grätzer, ***General Lattice Theory***, Second Edition, Birkhäuser Verlag, 2003, ISBN 3-7643-6996-5.

From its first edition George Grätzer's *General Lattice Theory* was a fundamental work in the lattice theory. It can be used as a course of lattice theory for students as well as a source of research problems for specialists. The last edition is enriched by 8 appendices and a new (and updated) bibliography. In this form, the book covers so well the study of lattices that it is almost impossible for someone who works in this field not to find here something useful for his research. Besides the rich valuable information concerning the developments of the last two decades, the first appendix (**Retrospective**) is the history of the last 20years of lattice theory. The other 7 appendices are surveys on various topics of lattice theory. They are recommended by the value of their authors.The original chapters of the book are: **I. First Concepts;II. Distributive Lattices; III. Congruences and Ideals; IV.Modular and Semimodular Lattices; V. Varieties of Lattices; VI.Free Products; Concluding Remarks**, and the appendices are **A. Retrospective; B. Distributive Lattices and Duality** by B.Davey and H. Priestley; **C. Congruence Lattice** by G.Grätzer and E. T. Schmidt; **D. Continuous Geometry** by F.Wehrung; **E. Projective Lattice Geometries** by M. Greferathand S. Schmidt; **F. Varieties of Lattices** by P. Jipsen and H. Rose; **G. Free Latices** by R. Freese; **H. Formal Concept Analysis** by B. Ganter and R. Wille. The book ends with the new bibliography which contains 530 title.

C. Pelea

Alfred Göpfert, Hassan Riahi, Christiane Tammer, and Constantin Zălinescu, ***Variational Methods in Partially Ordered Spaces***, Springer-Verlag, New York, 2003 (CMS Books in Mathematics/Ouvrages de Mathématiques de la SMC, 17), xiv+350 pp, ISBN 0-387-00452-1.

This book is intended to provide a systematic and self-contained presentation of recent significant developments in vector optimization and adjacent fields, in connection with the own research of the four authors. The targeted audience includes researchers and graduate students in pure and applied mathematics, economics, engineering, geography, and town planning, who want to study modern variational methods in general partially ordered linear spaces and their concrete applications.

Since vector optimization is nowadays an attractive and quickly growing field, it is understood that authors have not intended to present an exhaustive treatment, and coercive choices were imposed in order to pack all of the topics surveyed into the following format:

Chapter 1. Examples: Section 1.1 (Göpfert-Tammer-Zălinescu): *Cones in vector spaces*; Sections 1.2–1.6 (Göpfert-Tammer): *Equilibrium problems, Location problems in town planning, Multicriteria control problems, Multicriteria fractional programming problems, Stochastic efficiency in a set*;

Chapter 2. Functional analysis over cones: Sections 2.1–2.3 (Göpfert-Tammer-Zălinescu): *Order structures, Functional analysis and convexity, Separation theorems for not necessarily convex sets*; Sections 2.4–2.7 (Zălinescu): *Convexity notions for sets and multifunctions, Continuity notions for multifunctions, Continuity properties of multifunctions under convexity assumptions, Tangent cones and differentiability of multifunctions.*

Chapter 3. Optimization in partially ordered spaces: Sections 3.1 (Göpfert-Tammer-Zălinescu): *Solution concepts*; Sections 3.2–3.6 (Zălinescu): *Existence results for efficient points, Continuity properties with respect to a scalarization parameter, Well-posedness of vector optimization problems, Continuity properties, Sensitivity of vector optimization problems*; Section 3.7 (Göpfert-Tammer): *Duality*; Sections 3.8–3.9 (Riahi): *Vector equilibrium problems and related topics, Applications to vector variational inequalities*; Section 3.10 (Göpfert-Tammer-Zălinescu): *Minimal-point theorems in product spaces and corresponding variational principles*; Section 3.11 (Göpfert-Tammer): *Optimality conditions*

Chapter 4. Applications: Section 4.1 (Göpfert-Tammer): *Approximation problems*; Section 4.2: *Solution Procedures*; Subsections 4.2.1–4.2.3 (Göpfert-Tammer): *A proximal-point algorithm for real-valued control approximation problems, Computer programs for the application of the proximal-point algorithm, An interactive* *algorithm for the vector control approximation problem*; Subsections 4.2.4–4.2.5 (Riahi): *Proximal algorithms for vector equilibrium problems, Relaxation and penalization*; Sections 4.3–4.6 (Göpfert-Tammer): *Location problems, Multicriteria fractional programming, Multicriteria control problems, Stochastic efficiency in a set.*

The bibliography counts almost four hundreds items and allows the reader to easily find up-to-date literature on the field. The book also contains two lists providing an overview of illustrative figures, abbreviations and notations, and an index of selective terminology used throughout. In contrast to the wide spectrum of surveyed topics, the book reads easily, an unifying approach being visible throughout the whole text.

On one hand, specialists will certainly enjoy this monograph, especially because of the following features: many important results from functional analysis and partially ordered space theory are stated in a very general setting without undue abstraction; a large variety of relevant notions currently used in the literature are presented in a systematic way, the relationship between them being illustrated by diagrams; the authors use advanced techniques from different modern fields.

On the other hand, since the book is written in an rigorous, understandable, and teachable way, it may certainly serve to support courses on vector optimization, applied functional analysis, set-valued analysis, etc., targeted at the graduate level. For designing accompanying exercises, instructors will find in the text a good number of qualitative examples, which can be used to illustrate the results or to justify the stated assumptions. In particular, the emphasis on location theory applications will be especially appealing to graduate students of geography and researchers dealing with town planning. The final part of the book could also serve as a know-how support for practitioners who need to design multiple criteria decision software.

As a whole, this book can be strongly recommended as an excellent reference of general interest in vector optimization.

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