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ON SOME APPLICATIONS OF INTERPOLATION OPERATORS

GH. COMAN AND I. TODEA

Abstract. The goal of this paper is to give applications of interpolation operators, with a special emphasis on optimal approximation of some linear functionals and the construction of methods for the solution of equations on $\mathbb{R}$.

Let $\mathcal{B}$ be a linear space of real-valued functions defined on a domain $\Omega \subset \mathbb{R}^n$, $\mathcal{A} \subset \mathcal{B}$ and $\Lambda = \{\lambda_i |\ \lambda_i : \mathcal{B} \to \mathbb{R},\ i = 1, \ldots, N\}$, a set of linear functionals. For $f \in \mathcal{B}$, is denoted by $\Lambda(f) = \{\lambda_i(f)|\ \lambda_i \in \Lambda,\ i = 1, \ldots, N\}$, the informations on f suitable to Λ.

An operator $P : \mathcal{B} \to \mathcal{A}$, for which

$$\lambda_i(Pf) = \lambda_i(f), \quad i = 1, \ldots, N,$$

$f \in \mathcal{B}$, is an interpolation operator that interpolates the set Λ, while

$$f = Pf + Rf$$

is the interpolation formula generated by P, with R the remainder operator.

The number $r \in \mathbb{N}$ for which $Pf = f$, for all $f \in \mathbf{P}_r^n$ and there exists $g \in \mathbf{P}_{r+1}^n$, such that $Pg \neq g$, where $\mathbf{P}_r^n$ is the set of all polynomial functions of the total degree at most r, is the degree of exactness of the operator P.

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The purpose of this paper is to discuss applications of interpolation operators to the approximation of some linear functionals, the construction of some homogeneous cubature formulas, the construction of numerical methods for solving of some operatorial equations.

1. **Optimal approximation in sense of Sard**

One suppose here that $\Lambda := \Lambda_B = \{\lambda_{kj} :\ H^{m,2}[a,b] \to \mathbb{R},\ k = 1,\dots,n,\ j \in I_k\}$, with $I_k \subset \{0,1,\dots,r_k\}$, $r_k \in \mathbb{N}$, $r_k < m$, is a set of Birkhoff-type functionals, i.e. $\lambda_{kj}(f) = f^{(j)}(x_k)$, $x_k \in [a,b]$, $x_k \neq x_j$ for $k \neq j$. Let $\lambda : H^{m,2}[a,b] \to \mathbb{R}$, be a given linear functional such that the elements of the set $\Lambda_B \cup \{\lambda\}$ to be linear independent. One considers the approximation formula

$$\lambda(f) = \sum_{k=1}^{n} \sum_{j\in I_k} A_{kj} f^{(j)}(x_k) + R_N(f), \tag{1}$$

where $N = |I_1| + \dots |I_n| - 1$.

Definition 1. Formula (1), with prescribed points $x_k \in [a,b]$, $k = 1,\dots,n$, for which:

i) $R_N(e_\nu) = 0,\ \nu = 0,1,\dots,m-1$

ii) $\int_a^b K_m^2(t)dt \to minim,$

where K is the corresponding Peano kernel:

$$\begin{aligned} K(t) &:= R_N\left(\frac{(\cdot - t)_+^{m-1}}{(m-1)!}\right) \\ &= \lambda\left[\frac{(\cdot - t)_+^{m-1}}{(m-1)!}\right] - \sum_{k=1}^{n}\sum_{j\in I_k} A_{kj}\frac{(x_k - t)_+^{m-j-1}}{(m-j-1)!}, \end{aligned}$$

is called optimal in sense of Sard.

In 1964, I. J. Schoenberg [10] has established a relationship between the optimal approximation of linear operators, in particular, optimality in sense of Sard, and spline interpolation problems.

So, let $S : H^{m,2}[a,b] \to \mathcal{S}_{2m-1}(\Lambda_B)$ be a natural spline interpolation operator of the order $2m-1$, suitable to Λ_B.

Remark 2. [3] If Λ_B contains at least m functionals of Hermite-type then S exists and is unique.

For $f \in H^{m,2}[a,b]$, let

$$f = Sf + Rf$$

be the natural spline interpolation formula generated by S.

It follows [10], that

$$\lambda(f) = \lambda(Sf) + \lambda(Rf) \tag{2}$$

is the formula of the form (1) that is optimal in sense of Sard.

For example, if

$$\lambda(f) = \int_a^b f(x)dx$$

then (2) becomes an optimal, in sense of Sard, quadrature formula.

As an application, let us find the quadrature formula of the form

$$\int_0^1 f(x)dx = A_{00}f(0) + A_{10}F\left(\frac{1}{2}\right) + A_{21}f'\left(\frac{1}{2}\right) + A_{30}d(1) + R(f)$$

that is optimal in sense of Sard. Using, for example, the cubic spline interpolation formula

$$f(x) = s_{00}(x)f(0) + s_{10}(x)f\left(\frac{1}{2}\right) + s_{11}(x)f'\left(\frac{1}{2}\right) + s_{20}(x)f(1) + (Rf)(x)$$

where

$$s_{00}(x) = 1 - 3x + 4x^3 - 4\left(x - \frac{1}{2}\right)_+^3 - 6\left(x - \frac{1}{2}\right)_+^2$$

$$s_{10}(x) = 3x - 4x^3 + 8\left(x - \frac{1}{2}\right)_+^3 - 4(x-1)_+^3$$

$$s_{11}(x) = -\frac{1}{2}x + 2x^3 - 6\left(x - \frac{1}{2}\right)_+^2 - 2(x-1)_+^3$$

$$s_{20}(x) = -4\left(x - \frac{1}{2}\right)_+^3 + 6\left(x - \frac{1}{2}\right)_+^2 + 4(x-1)_+^3$$

and

$$(Rf)(x) = \int_0^1 \varphi_1(x,t)f''(t)dt$$

with

$$\varphi_1(x,t) = (x-t)_+ - s_{10}(x)\left(\frac{1}{2}-t\right)_+ - s_{11}(x)\left(\frac{1}{2}-t\right)_+^0 - s_{20}(x)(1-t)_+$$

it follows that the optimal coefficients A_{ij}^* respectively the optimal kernel K_1^* are

$$A_{ij}^* = \int_0^1 s_{ij}(x)dx$$

and

$$K_1^*(t) = \int_0^1 \varphi_1(x,t)dx.$$

One obtains:

$$A_{00}^* = \frac{3}{16}, \quad A_{10}^* = \frac{5}{8}, \quad A_{11}^* = 0, \quad A_{20}^* = \frac{3}{16}$$

and

$$K_1^*(t) = \frac{(1-t)^2}{2} - \frac{5}{8}\left(\frac{1}{2}-t\right)_+ - \frac{3}{16}(1-t).$$

We also have

$$\int_0^1 (K_1^*(t))^2 dt = \frac{1}{2^{10}\cdot 5}$$

Hence

$$|R(f)| \le \frac{1}{32\sqrt{5}}\|f''\|_2.$$

Remark 2. The gaussian quadratures are optimal in sense of Sard - all the coefficients and nodes are determined from the condition i). But, the nodes in the gaussian quadrature formula of the form

$$\int_{-1}^1 f(x)dx = \sum_{i=1}^n A_i f(x_i) + R_n(f) \tag{3}$$

are the zeros of the Cebyshev polynomial T_n, i.e.

$$x_k^* = \cos\frac{2k-1}{2n}\pi, \quad k = 1,\dots,n.$$

It means that the optimal coefficients of the quadrature form (3) are:

$$A_i^* = \int_{-1}^1 l_i^*(x)dx, \quad i = 1,\dots,n$$

where l_i^* are the fundamental Lagrange interpolation polynomials corresponding to the interpolation nodes x_i^*, $i = 1,\dots,n$.

2. **Homogeneous cubature formulas**

Let D be a domain in $\mathbb{R}^2$, $f : D \to \mathbb{R}$ an integrable function on D and $\Lambda(f) = \{\lambda_k(f)|\ k = 1,\dots,N\}$ some informations on f. Next, one suppose that $\lambda_k(f)$, $k = 1,\dots,N$, are punctual values of f or of certain of its derivatives, i.e. the values of them at the cubature nodes.

One considers the cubature formula

$$I^{xy} := \int\int_D f(x,y)dxdy = \sum_{i=1}^{N} A_i\lambda_i(f) + R_N(f) \tag{4}$$

where A_i, $i = 1,\dots,N$ are of the cubature coefficients and $R_N(f)$ is the remainder term.

The problem is to find the parameters of such a cubature formula (coefficients and nodes) and to study the corresponding remainder term. Most solutions for this problem has been obtained when D is a regular domain in $\mathbb{R}^2$ (rectangle, triangle, etc.) and the informations are regularly spaced. For example, the product and the boolean sum rules belong to this class of cubature procedures.

So, let $D \subset \mathbb{R}^2$ be a rectangle, $D = [a,b] \times [c,d]$ and $\lambda^x(f) = \{\lambda_i^x(f)|\ i = 0,1,\dots,m\}$, $\Lambda^y(f) = \{\lambda_j^y(f)|\ j = 0,1,\dots,n\}$, $m,n \in \mathbb{N}$ some given partial information on f, with regard to x respectively y.

One considers the quadrature formulas:

$$I^x f := \int_a^b f(x,y)dx = (Q_1^x f)(\cdot,y) + (R_1^x f)(\cdot,y) \tag{5}$$

and

$$I^y f := \int_c^d f(x,y)dy = (Q_1^y f)(x,\cdot) + (R_1^y f)(x,\cdot) \tag{6}$$

where

$$(Q_1^x f)(\cdot,y) = \sum_{i=0}^{m} A_i\lambda_i^x(f)$$

$$(Q_1^y f)(x,\cdot) = \sum_{j=0}^{n} B_j\lambda_j^y(f)$$

and R_1^x, R_1^y the corresponding remainder operators:

$$R_1^x = I^x - Q_1^x$$

$$R_1^y = I^y - Q_1^y.$$

We have the following decompositions of the double integral operator I^{xy}:

$$I^{xy} = Q_1^x Q_1^y + (I^y R_1^x + I^x R_1^y - R_1^x R_1^y) \tag{7}$$

and

$$I^{xy} = (Q_1^x I^y + I^x Q_1^y - Q_1^x Q_1^y) + R_1^x R_1^y \tag{8}$$

The identities (7) and (8) generate so called product cubature formula

$$I^{xy} f = (Q_1^x Q_1^y) f + (R_1^x I^y + I^x R_1^y - R_1^x R_1^y) f$$

respectively, the boolean-sum cubature formula

$$I^{xy} f = (Q_1^x I^y + I^x Q_1^y - Q_1^x Q_1^y) f + R_1^x R_1^y f.$$

Now, if p_1 and q_1 are the approximation orders of Q_1^x respectively Q_1^y ($ord(Q_1^x) = p_1$, $ord(Q_1^y) = q_1$), it follows that the order of the product cubature formula is $\min\{p_1, q_1\}$, while the order of boolean-sum cubature formula is $p_1 + q_1$ [5].

Hence, the boolean-sum cubature rule has the remarkable property that it has a high approximation order. Otherwise, the boolean-sum cubature formula contains the simple integrals $I^x f$ and $I^y f$. But, this simple integrals can be approximated, in a second level of approximation, using new quadrature procedures, i.e.

$$I^x f = Q_2^x f + R_2^x f$$

respectively

$$I^y f = Q_2^y f + R_2^y f.$$

This way, from (8), is obtained

$$I^{xy} = Q^{xy} + R^{xy} \tag{9}$$

where

$$Q^{xy} = Q_1^x Q_2^y + Q_2^x Q_1^y - Q_1^x Q_1^y \tag{10}$$

and

$$R^{xy} = Q_1^x R_2^y + Q_1^y R_2^x + R_1^x R_1^y \tag{11}$$

As can be seen, from (11) follows

$$ord(Q^{xy}) = \min\{ord(Q_1^x) + ord(Q_1^y), ord(Q_2^x) + 1, ord(Q_2^y) + 1\}$$

If

$$ord(Q_2^x) = ord(Q_1^x) + ord(Q_1^x) - 1$$
$$ord(Q_2^y) = ord(Q_1^y) + ord(Q_1^y) - 1$$

then the cubature formula given by (9) is called a homogeneous cubature formula [5].

One of the main procedure to construct homogeneous cubature formulas is based on the interpolation formulas. It is well known that each interpolation formula give rise to a quadrature or cubature formula.

Remark 3. [6] If the multivariate interpolation formula is a homogeneous one, then the suitable cubature formula is also an homogeneous cubature formula.

To illustrate it, we give some simple examples:

Example 2. Let $f : D_h \to \mathbb{R}$, with $D_h = [0, b] \times [0, b]$ be given and $\Lambda(f) = \{f(0,0), f(h,0), f(0,h), f(h,h)\}$. For the partial informations on f: $\Lambda_1^x(f) = \{f(0,y), f(h,y)\}$ respectively $\Lambda_1^y(f) = \{f(x,0), f(x,h)\}$, one considers the Lagrange's operators L_1^x and L_1^y:

$$(L_1^x f)(x,y) = \frac{h-x}{h} f(0,y) + \frac{x}{h} f(h,y),$$
$$(L_1^y f)(x,y) = \frac{h-y}{h} f(x,0) + \frac{y}{h} f(x,h).$$

If $R_1^x = I - L_1^x$ and $R_1^y = I - L_1^y$, with I the identical operator, then we have

$$I = L_1^x \oplus L_1^y + R_1^x R_1^y, \tag{12}$$

the boolean sum decomposition of the identity operator. Also, we have

$$f = L_1^x \oplus L_1^y f + R_1^x R_1^y f$$

or

$$f = (L_1^x + L_1^y - L_1^x L_1^y) f + R_1^x R_1^y f$$

Now, if $L_1^x f$ and $L_1^y f$ are interpolated, in a second level, by the Hermite's operators H_3^y respectively H_3^x suitable to the information sets

$$\Lambda_2^y(f) = \{f(x,0), f^{(0,1)}(x,0), f(x,h), f^{(0,1)}(x,h)\}$$

and

$$\Lambda_2^x(f) = \{f(0,y), f^{(1,0)}(0,y), f(h,y), f^{(1,0)}(h,y)\},$$

one obtains

$$f = (L_1^x H_3^y + H_3^x L_1^y - L_1^x L_1^y)f + (L_1^x R_3^y + L_1^y R_3^x + R_1^x R_1^y)f, \qquad (13)$$

where $R_3^x = I - H_3^x$ and $R_3^y = I - H_3^y$. Taking into account that

$$ord(L_1^x L_1^y) = ord(H_3^x) = ord(H_3^y) = 4$$

(13) is a homogeneous interpolation formula of order 4 [4].

Theorem 4. *The cubature formula*

$$\iint_{D_h} f(x,y)dxdy = Q(f) + R(f)$$

where

$$Q(f) = \iint_{D_h} ((L_1^x H_3^y + H_3^x L_1^y - L_1^x L_1^y)f)(x,y)dxdy$$

or

$$Q = Q_1^x Q_3^y + Q_3^x Q_1^y + Q_1^x Q_1^y$$

and

$$R(f) = \iint_{D_h} ((L_1^x R_3^y + L_1^y R_3^x + R_1^x R_1^y)f)(x,y)dxdy$$

is a homogeneous cubature formula of order 6.

Proof. Suppose that $f \in C^{4,4}(D_h)$. Then

$$(R_1^x f)(x,y) = \frac{x(x-h)}{2} f^{(2,0)}(\xi, y)$$

$$(R_1^y f)(x,y) = \frac{y(y-h)}{2} f^{(0,2)}(x, \eta_1)$$

$$(R_3^x f)(x,y) = \frac{x^2(x-h)^2}{24} f^{(4,0)}(\xi_2, y)$$

$$(R_3^y f)(x,y) = \frac{y^2(y-h)^2}{24} f^{(0,4)}(x,\eta_2)$$

with $\xi_1, \xi_2, \eta_1, \eta_2 \in [0,h]$.

As

$$\int_0^h (R_1^x f)(x,y)dx = \frac{h^3}{12} f^{(2,0)}(\mu_1, y)$$

$$\int_0^h (R_1^y f)(x,y)dy = \frac{h^3}{12} f^{(0,2)}(x, \nu_1)$$

$$\int_0^h (R_3^x f)(x,y)dx = \frac{h^5}{720} f^{(4,0)}(\mu_2, y)$$

$$\int_0^h (R_3^y f)(x,y)dy = \frac{h^5}{720} f^{(0,4)}(x, \nu_2)$$

$\mu_1, \mu_2, \nu_1, \nu_2 \in [0,h]$, it follows that $ord(Q_1^x) = ord(Q_1^y) = 3$ and $ord(Q_3^x) = ord(Q_3^y) = 5$. Hence, $ord(Q_3^x) = ord(Q_3^y) = ord(Q_1^x) + ord(Q_1^y) - 1$.

Example 3. Let T_h be the standard triangle, $T_h = \{(x,y) \in \mathbb{R}^2 |\ x \geq 0,\ y \geq 0,\ x+y \leq h\}$, $f : T_h \to \mathbb{R}$ and $\Lambda(f) = \left\{ \left(0, \frac{h}{2}\right), f\left(\frac{h}{2}, 0\right), f\left(\frac{h}{2}, \frac{h}{2}\right) \right\}$. Let P be the operator that interpolate the information $\Lambda(f)$, i.e.

$$(Pf)(x,y) = \frac{h-2x}{h} f\left(0, \frac{h}{2}\right) + \frac{h-2y}{h} f\left(\frac{h}{2}, 0\right) + \frac{2x+2y-h}{h} f\left(\frac{h}{2}, \frac{h}{2}\right)$$

and

$$f = Pf + Rf \tag{14}$$

the interpolation formula suitable to P.

Let

$$\int\int_{T_h} f(x,y)dxdy = Q(f) + R(f) \tag{15}$$

be the cubature formula generated by (14), i.e.

$$Q(f) = \frac{h^2}{6}\left[f\left(0, \frac{h}{2}\right) + f\left(\frac{h}{2}, 0\right) + f\left(\frac{h}{2}, \frac{h}{2}\right) \right]$$

and

$$R(f) = \int\int_{T_h} (Rf)(x,y)dxdy.$$

Remark 7. $dex(Q) = 2$, although $dex(P) = 1$.

Theorem 8. *Formula (15) is a homogeneous cubature formula of interpolations type.*

Proof. Suppose that $f \in B_{12}(0,0)$ on T_h. By Peano's theorem ($dex(Q) = 2$), we have

$$R(f) = \int_0^h K_{30}(x,y,s) f^{(3,0)}(s,0)ds + \int_0^h K_{21}(x,y,s) f^{(2,1)}(s,0)ds \tag{16}$$

$$+ \int_0^h K_{03}(x,y,t) f^{(0,3)}(0,t)dt + \iint_{T_h} K_{12}(x,y,s,t) f^{(1,2)}(s,t)dsdt$$

where

$$K_{30}(s) = \frac{(h-s)^4}{24} - \frac{h^2}{6}\left(\frac{h}{2} - s\right)_+^2$$

$$K_{21}(s) = \frac{(h-s)^4}{24} - \frac{h^3}{12}\left(\frac{h}{2} - s\right)_+$$

$$K_{03}(t) = K_{30}(t)$$

$$K_{12}(s,t) = \frac{(h-s-t)^3}{6} - \frac{h^2}{6}\left(\frac{h}{2} - s\right)_+^0 \left(\frac{h}{2} - t\right)_+$$

As $K_{30} \geq 0$, $K_{03} \geq 0$ on $[0,h]$ and

$$\int_0^h K_{30}(s)ds = \frac{1}{720}h^5, \quad \int_0^h K_{03}(t)dt = \frac{1}{720}h^5$$

respectively

$$\max_{0 \leq s \leq h} |K_{21}(s)| = \frac{1}{384}h^4, \quad \max_{T_h} |K(s,t)| = \frac{1}{12}h^3,$$

the proof follows from (16).

Example 4. An interesting homogeneous formula, for $f : T_h \to \mathbb{R}$, is obtained from the interpolation formula

$$f = Pf + Rf$$

when

$$(Pf)(x,y) = f\left(\frac{h}{3}, \frac{h}{3}\right)$$

namely

$$\iint_{T_h} f(x,y)dxdy = Q(f) + R(f) \tag{17}$$

with

$$Q(f) = \frac{h^2}{2} f\left(\frac{h}{3}, \frac{h}{3}\right).$$

It is easy to verify that $dex(Q) = 1$, while $dex(P) = 0$.

Theorem 9. *Formula (17) is a homogeneous cubature formula.*

Proof. If $f \in B_{11}(0,0)$ on T_h, by Peano's theorem one obtain

$$|R(f)| \leq \frac{h^4}{72}\left[\|f^{(2,0)}(\cdot,0)\| + \|f^{(0,2)}(0,\cdot)\| + \frac{89}{27}\|f^{(1,1)}\|\right].$$

3. Methods for nonlinear equations on $\mathbb{R}$

For $f : \Omega \to \mathbb{R}$, $\Omega \subset \mathbb{R}$, one considers the equation

$$f(x) = 0, \quad x \in \Omega. \tag{18}$$

Let $F : D^n \to D$, $D \subset \Omega$, be an iterative method for solutions of the equation (18), i.e. for given $(x_0, \ldots, x_{n-1}) \in D^n$, F generates the sequence

$$x_0, x_1, \ldots, x_{n-1}, x_n, \ldots \tag{19}$$

where

$$x_i = F(x_{i-n}, \ldots, x_{i-1}), \quad i = n, \ldots$$

If the sequence (19) converges to a solution, say x^*, of the equation (18), F is a convergent method.

The number p with the property that

$$\lim_{x_i \to x^*} \frac{x^* - F(x_{i-n+1}, \ldots, x_i)}{(x^* - x_i)^p} = c, \quad c \in \mathbb{R} \setminus \{0\},$$

is the order of F $(ord(F) = p)$.

An efficient procedure to construct numerical methods for the solution of the equation (18), is based on inverse interpolation. Namely, if g is the inverse of f, $g = f^{-1}$, and $x^* \in D$ is a solution of the equation (18), $f(x^*) = 0$, then $x^* = g(0)$. Inverse interpolation procedure means that the inverse function g is approximated by an interpolation operator P and $x^* \approx (Pg)(0)$.

For example [7], using Taylor interpolation is obtained the one step method

$$F_n^T(x_i) = x_i - \sum_{k=1}^{n-1} \frac{(-1)^k}{k!}(f(x_i))^k g^{(k)}(f(x_i))$$

with $ord(F_n^T) = n$.

Also, using Lagrange interpolation is obtained the multistep method

$$F_n^L(x_{i-n},\dots,x_i)=\sum_{k=0}^{n} \frac{f_{i-n}\dots f_{i-n+k-1}f_{i-n+k+1}\dots f_i}{(f_{i-n}-f_k)\dots(f_{i-n+k-1}-f_k)(f_{i-n+k+1}-f_k)\dots(f_i-f_k)} x_k$$

with $ord(F_n^L) = \rho$, the positive solution of the equation

$$t^{n+1} - t^n - \dots - t - 1 = 0$$

i.e., $1 < \rho < 2$.

An interesting class of methods is given by Abel-Goncharov interpolation operator P, defined by

$$(Pf)(x) = \sum_{k=0}^{n} p_k(x) f^{(k)}(x_k)$$

where

$$p_0(x) = 1$$
$$p_1(x) = x - x_0$$
$$p_k(x) = \frac{1}{k!}\left[x^k - \sum_{j=0}^{k-1} p_j(x) x_j^{k-j}\right],\ k = 2,\dots,n.$$

Applying the operator P to the function $g = f^{-1}$, for the interpolation nodes $x_{i-n},\dots,x_i$, one obtains

$$F_n^{AG}(x_{i-n},\dots,x_i) = \sum_{k=i-n}^{i} p_{n-i-k}(0) g^{(n-i-k)}(f(x_k))$$

For example,

$$F_1^{AG}(x_{i-1},x_i) = x_{i-1} - \frac{f(x_{i-1})}{f'(x_i)},$$

is a new modified of Newton-Raphson method.

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PROJECTORS AND HALL π-SUBGROUPS IN FINITE π-SOLVABLE GROUPS

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Abstract. Let π be a set of primes and $\underline{X}$ be a π-closed Schunck class with the P property. The paper gives conditions with respect to which an $\underline{X}$-projector H of a finite π-solvable group G is an Hall π-subgroup of G, and consequently we have that $N_G(N_G(H)) = N_G(H)$.

1. Preliminaries

All groups considered in the paper are finite. Let π be a set of primes, π' the complement to π in the set of all primes and $O_{\pi'}(G)$ the largest normal π'-subgroup of a group G.

We first give some useful definitions.

Definition 1.1. ([8], [11]) a) A class $\underline{X}$ of groups is a *homomorph* if $\underline{X}$ is epimorphically closed, i.e. if $G \in \underline{X}$ and N is a normal subgroup of G, then $G/N \in \underline{X}$.

b) A group G is *primitive* if G has a *stabilizer*, i.e. a maximal subgroup H with $core_G H = \{1\}$, where $core_G H = \cap\{H^g / g \in G\}$.

c) A homomorph $\underline{X}$ is a *Schunck class* if $\underline{X}$ is *primitively closed*, i.e. if any group G, all of whose primitive factor groups are in $\underline{X}$, is itself in $\underline{X}$.

Definition 1.2. a) A positive integer n is said to be a π-*number* if for any prime divisor p of n we have $p \in \pi$.

b) A finite group G is a π-*group* if $|G|$ is a π-number.

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Definition 1.3. ([6]) A group G is *π-solvable* if every chief factor of G is either a solvable π-group or a π'-group. For π the set of all primes, we obtain the notion of *solvable group.*

Definition 1.4. A class $\underline{X}$ of groups is said to be *π-closed* if

$$G/O_{\pi'}(G) \in \underline{X} \;\Rightarrow\; G \in \underline{X}.$$

A π-closed homomorph, respectively a π-closed Schunck class is called *π-homomorph*, respectively *π-Schunck class.*

Definition 1.5. ([7], [8]) Let $\underline{X}$ be a class of groups, G a group and H a subgroup of G.

a) H is an *$\underline{X}$-maximal subgroup* of G if: (i) $H \in \underline{X}$; (ii) $H \leq H^* \leq G$, $H^* \in \underline{X}$ imply $H = H^*$.

b) H is an *$\underline{X}$-projector* of G if, for any normal subgroup N of G, HN/N is $\underline{X}$-maximal in G/N.

c) H is an *$\underline{X}$-covering subgroup* of G if: (i) $H \in \underline{X}$; (ii) $H \leq K \leq G$, $K_0 \trianglelefteq K$, $K/K_0 \in \underline{X}$ imply $K = HK_0$.

Definition 1.6. ([3], [4]) Let $\underline{X}$ be a class of groups. We say that $\underline{X}$ has the *P property* if, for any π-solvable group G and for any minimal normal subgroup M of G such that M is a π'-group, we have $G/M \in \underline{X}$.

The following results are used in this paper.

Theorem 1.7. ([1]) *A solvable minimal normal subgroup of a group is abelian.*

Theorem 1.8. ([1]) *Suppose that G has a $\neq \{1\}$ normal solvable subgroup and let S be a maximal subgroup of G with $core_G S = \{1\}$. Then, the existence of a $\neq \{1\}$ normal solvable subgroup of S implies the existence of a normal subgroup $N \neq \{1\}$ of S with $(|N|, |G:S|) = 1$.*

Theorem 1.9. ([2]) *a) Let $\underline{X}$ be a class of groups, G a group and H a subgroup of G. If H is an $\underline{X}$-covering subgroup of G or H is an $\underline{X}$-projector of G, then H is $\underline{X}$-maximal in G.*

b) If $\underline{X}$ is a homomorph and G is a group, then a subgroup H of G is an $\underline{X}$-covering subgroup of G if and only if H is an $\underline{X}$-projector in any subgroup K of G with $H \subseteq K$.

Theorem 1.10. *Let $\underline{X}$ be a homomorph.*

a) ([7]) If H is an $\underline{X}$-covering subgroup of a group G and N is a normal subgroup of G, then HN/N is an $\underline{X}$-covering subgroup of G/N.

b) ([8]) If H is an $\underline{X}$-projector of a group G and N is a normal subgroup of G, then HN/N is an $\underline{X}$-projector of G/N.

c) ([7]) If H is an $\underline{X}$-covering subgroup of G and $H \leq K \leq G$, then H is an $\underline{X}$-covering subgroup of K.

Theorem 1.11. ([5]) *Let $\underline{X}$ be a π-homomorph. The following conditions are equivalent:*

(1) $\underline{X}$ is a Schunck class;

(2) any π-solvable group has $\underline{X}$-covering subgroups;

(3) any π-solvable group has $\underline{X}$-projectors.

2. **Hall π-subgroups in finite π-solvable groups**

Of special interest in this paper will be the Hall π-subgroups and some of their properties. The Hall subgroups were given in [9]. Ph. Hall studied them in finite solvable groups. In [6], S. A. Čunihin extended this study to finite π-solvable groups.

Definition 2.1. Let G be a group and H a subgroup of G.

a) H is a *π-subgroup* of G if H is a π-group.

b) H is an *Hall π-subgroup* of G if: (i) H is a π-subgroup of G; (ii) $(|H|, |G : H|) = 1$, i.e. $|G : H|$ is a π'-number.

We shall use some properties of the Hall π-subgroups given in [10]:

Theorem 2.2. ([10]) (Ph. Hall, S. A. Čunihin) *If G is a π-solvable group, then:*

a) G has Hall π-subgroups and G has Hall π'-subgroups;

b) any two Hall π-subgroups of G are conjugate in G; any two Hall π'-subgroups of G are conjugate in G too.

Theorem 2.3. ([10]) *Let G be a group and H an Hall π-subgroup of G.*

a) If $H \leq K \leq G$, then H is an Hall π-subgroup of K.

b) If N is a normal subgroup of G, then HN/N is an Hall π-subgroup of G/N.

We complete these properties with two new ones, which will be used in the formation theory considerations in the main section of this paper.

Theorem 2.4. *Let G be a π-solvable group, H a subgroup of G and N a normal subgroup of G. If HN/N is an Hall π-subgroup of G/N and H is an Hall π-subgroup of HN, then H is an Hall π-subgroup of G.*

Proof. (i) H is a π-subgroup of G, since H is a π-subgroup of HN.

(ii) We shall prove that $|G : H|$ is a π'-number. Indeed, we know that $|G : HN| = |G/N : HN/N|$ is a π'-number. Further, $|HN : H|$ is a π'-number too. Then $|G : H| = |G : HN||HN : H|$ is a π'-number. $\square$

Theorem 2.5. *If G is a π-solvable group and H is a Hall π-subgroup of G, then $N_G(N_G(H)) = N_G(H)$.*

Proof. We know that

$$N_G(H) = \{g \in G/H^g = H\} \supseteq H$$

and so we have $N_G(H) \subseteq N_G(N_G(H))$. We now prove that $N_G(N_G(H)) \subseteq N_G(H)$. Let $x \in N_G(N_G(H))$. It is known that $N_G(H) \trianglelefteq N_G(N_G(H))$. It follows that $N_G(H)^x = N_G(H)$, hence $H^x \subseteq N_G(H)^x = N_G(H)$, which implies by 2.3.a) that H and H^x are Hall π-subgroups of $N_G(H)$. Applying Hall-Čunihin Theorem 2.2.b), we obtain that H and H^x are conjugate in $N_G(H)$. So there is an element $y \in N_G(H)$ such that $(H^x)^y = H$. It follows that $H^{xy} = H$, hence $xy \in N_G(H)$. But $y \in N_G(H)$ implies $y^{-1} \in N_G(H)$ and so $x = (xy)y^{-1} \in N_G(H)$. $\square$

3. Projectors which are Hall π-subgroups in finite π-solvable groups

In [8], W. Gaschütz gives for finite solvable groups the following result: If $\underline{X}$ is a Schunck class, G a solvable group and S an $\underline{X}$-projector of G such that S is a p-group, then S is a Sylow p-subgroup of G.

It is the aim of this paper to study similar properties in the more general case of finite π-solvable groups.

All groups considered in this section are finite π-solvable.

Theorem 3.1. *Let $\underline{X}$ be a π-Schunck class with the P property. If G is a π-solvable group, such that there is a minimal normal subgroup M of G which is a π'-group, and if H is an $\underline{X}$-projector of G which is a π-group, then H is an Hall π-subgroup of G.*

Proof. We will show that $|G : H|$ is a π'-number. Let M be a minimal normal subgroup of G, such that M is a π'-group. We know that $\underline{X}$ has the P property, and so, by 1.6., we have $G/M \in \underline{X}$.

On the other side, H being an $\underline{X}$-projector of G, we have, by 1.10., that HM/M is an $\underline{X}$-projector of G/M. Now 1.9.a) implies that HM/M is $\underline{X}$-maximal in G/M. But $G/M \in \underline{X}$. It follows that $HM/M = G/M$, hence $HM = G$. From this and from $HM/M \cong H/H \cap M$, we obtain that

$$|G : H| = |HM : H| = |M : H \cap M|.$$

Since $|M : H \cap M|$ divides $|M|$ which is a π'-number, we obtain that $|M : H \cap M|$ is also a π'-number. Hence $|G : H|$ is a π'-number. □

In order to renounce to the condition on the group G of having a minimal normal subgroup M which is a π'-group, the next theorem contains the assumption that H is an $\underline{X}$-covering subgroup of G. This means, by 1.9.b), that H is a particular $\underline{X}$-projector.

Theorem 3.2. *Let $\underline{X}$ be a π-Schunck class with the P property. If G is a π-solvable group and H is an $\underline{X}$-covering subgroup of G which is a π-group, then H is an Hall π-subgroup of G.*

Proof. By induction on $|G|$. We consider two cases:

1) There is a minimal normal subgroup M of G, such that M is a π'-group. By 1.9.b), H is an $\underline{X}$-projector of G. Applying theorem 3.1., it follows that H is an Hall π-subgroup of G.

2) Any minimal normal subgroup M of G is a solvable π-group. Hence, by 1.7., M is abelian. If $H = G$, it follows from H π-group that H is an Hall π-subgroup of $G = H$. Let now $H \neq G$. We distinguish two possibilities:

a) For any minimal normal subgroup M of G we have $HM = G$.

Let us first prove that H is a maximal subgroup of G. Indeed, we have $H < G$. Further, if $H \leq H^* < G$, we prove that $H = H^*$. Suppose that $H < H^*$, and let $h^* \in H^* \setminus H$. Let M be a minimal normal subgroup of G. By the above, we have that M is abelian and $G = HM$. So $h^* = hm$, where $h \in H$, $m \in M$. It follows that $m = h^{-1}h^* \in M \cap H^*$. Let us prove that $M \cap H^* = \{1\}$. Suppose that $M \cap H^* \neq \{1\}$. We have $M \cap H^* \trianglelefteq H^*$. Further, $M \cap H^* \trianglelefteq G$, since if $x \in G = HM = H^*M = MH^*$ and $m \in M \cap H^*$, then $x = m_1h^*$, where $m_1 \in M$, $h^* \in H^*$, and M being abelian, we have:

$$x^{-1}mx = (m_1h^*)^{-1}m(m_1h^*) = (h^*)^{-1}m_1^{-1}mm_1h^* = (h^*)^{-1}mm_1^{-1}m_1h^* =$$

$$= (h^*)^{-1}mh^* \in M \cap H^*.$$

So $M \cap H^* \trianglelefteq G$, $M \cap H^* \subseteq M$, $M \cap H^* \neq \{1\}$. But M is a minimal normal subgroup. Hence $M \cap H^* = M$, which implies that $M \subseteq H^*$ and so $G = H^*M = H^*$, a contradiction with $H^* < G$. It follows that $M \cap H^* = \{1\}$. Hence $m = 1$ and so $h^* = h \in H$, in contradiction with the choice of h^*. We proved that $H = H^*$. So H is a maximal subgroup of G.

Let us notice that $core_G H = \{1\}$. Indeed, if we suppose that $core_G H \neq \{1\}$, it follows since $core_G H \trianglelefteq G$ that there exists a minimal normal subgroup M of G such that $M \subseteq core_G H$. We obtain $G = HM \subseteq H core_G H = H$, in contradiction with $H \neq G$. So $core_G H = \{1\}$.

We are now in the hypotheses of theorem 1.8.. By 1.8., it follows the existence of a normal subgroup $N \neq \{1\}$ of H, such that $(|N|, |G : H|) = 1$. But H being a

π-group, N is also a π-group. Then $|G:H|$ is a π'-number. It follows that H is an Hall π-subgroup of G.

b) There is a minimal normal subgroup M of G such that $HM \neq G$.

We apply the induction to the π-solvable group HM, with $|HM| < |G|$. By 1.10.c), H is an $\underline{X}$-covering subgroup of HM. Further, H is a π-group. By the induction, H is an Hall π-subgroup of HM.

We now apply the induction to the π-solvable group G/M, with $|G/M| < |G|$. By 1.10.a), HM/M is an $\underline{X}$-covering subgroup of G/M. Further, we have that $|HM/M| = |H/H \cap M|$ divides $|H|$, and so HM/M is a π-group. By the induction, HM/M is an Hall π-subgroup of G/M.

Finally, theorem 2.4. leads us to the conclusion that H is an Hall π-subgroup of G. □

Corollary 3.3. *Let $\underline{X}$ be a π-Schunck class with the P property. If G is a π-solvable group and H is an X-covering subgroup of G which is a π-group, then* $N_G(N_G(H)) = N_G(H)$.

Proof. Follows from 3.2. and 2.5.. □

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THE P-LAPLACIAN OPERATOR ON THE SOBOLEV SPACE $W^{1,p}(\Omega)$

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Abstract. In this paper p-Laplacian operator is defined on $W^{1,p}(\Omega)$ in connection with the duality mapping of $W^{1,p}(\Omega)$.

1. **Introduction and preliminary results**

Let Ω be an open bounded subset in $\mathbf{R}^N$, $\mathbf{N} \geq 2$, with smooth boundary and $1 < p < \infty$.

We shall use the standard notations:

$$W^{1,p}(\Omega) = \left\{ u \in L^p(\Omega) : \frac{\partial u}{\partial x_i} \in L^p(\Omega), i = \overline{1, N} \right\},$$

equipped with the norm

$$\|u\|_{1,p}^p = \|u\|_{0,p}^p + \sum_{i=1}^{N} \left\| \frac{\partial u}{\partial x_i} \right\|_{0,p}^p,$$

where $\|\cdot\|_{0,p}$ is the usual norm on $L^p(\Omega)$.

It is well known that $\left(W^{1,p}(\Omega), \|\cdot\|_{1,p}\right)$ is separable, reflexive and uniformly convex Banach space (see e.g. [1], theorem 3.5).

If $u \in W^{1,p}(\Omega)$ we can speak about $u|_{\partial\Omega}$ in the sense of the trace: there is a unique linear and continuous operator $\gamma : W^{1,p}(\Omega) \to W^{1-\frac{1}{p},p}(\partial\Omega)$ such that γ is surjective and for $u \in W^{1,p}(\Omega) \cap C(\overline{\Omega})$ we have $\gamma u = u|_{\partial\Omega}$.

Then the closure of $C_0^\infty(\Omega)$ in the space $W^{1,p}(\Omega)$ is

$$W_0^{1,p}(\Omega) = \left\{ u \in W^{1,p}(\Omega) : u|_{\partial\Omega} = 0 \right\} = Ker\gamma.$$

The dual space $(W_0^{1,p}(\Omega))^*$ will be denoted by $W^{-1,p'}(\Omega)$, where $\frac{1}{p} + \frac{1}{p'} = 1$.

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For each $u \in W^{1,p}(\Omega)$ we put

$$\nabla u = \left(\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, ..., \frac{\partial u}{\partial x_N} \right) \quad , \quad |\nabla u| = \left(\sum_{i=1}^{N} \left(\frac{\partial u}{\partial x_i} \right)^2 \right)^{\frac{1}{2}}$$

and let us remark that

$$|\nabla u| \in L^p(\Omega) \quad , \quad |\nabla u|^{p-2} \frac{\partial u}{\partial x_i} \in L^{p'}(\Omega) \ , \quad i = \overline{1, N}.$$

By the Poincaré inequality

$$\|u\|_{0,p} \le const\,(\Omega, N)\,||\nabla u||_{0,p} \text{ , for all } u \in W_0^{1,p}(\Omega) \ ,$$

the functional

$$W_0^{1,p}(\Omega) \ni u \to \|u\|_{1,p} := ||\nabla u||_{0,p}$$

is a norm on $W_0^{1,p}(\Omega)$, equivalent with $\|\cdot\|_{W^{1,p}(\Omega)}$.

The p-Laplacian operator $\Delta_p u = div \left(|\nabla u|^{p-2} \nabla u \right)$ may be action (see [2] or [6]) from $W_0^{1,p}(\Omega)$ into $W^{-1,p'}(\Omega)$ by

$$< -\Delta_p u, v > \ = \int_\Omega |\nabla u|^{p-2} \nabla u \nabla v \text{ , for } u, v \in W_0^{1,p}(\Omega).$$

Now we define the p-Laplacian operator on the space $W^{1,p}(\Omega)$.

We define a new equivalent norm on the space $W^{1,p}(\Omega)$:

$$|||u|||_{1,p}^p = ||u||_{0,p}^p + \|\nabla u\|_{0,p}^p = \int_\Omega |u|^p + \int_\Omega \left(\sum_{i=1}^{N} \left(\frac{\partial u}{\partial x_i} \right)^2 \right)^{\frac{p}{2}} .$$

The space $\left(W^{1,p}(\Omega), ||| \cdot |||_{1,p} \right)$ is separable, reflexive and uniformly convex Banach space (see [5]).

The dual norm on $(W^{1,p}(\Omega), ||| \cdot |||_{1,p})^*$ is denoted by $||| \cdot |||_*$.

If $u \in W^{1,p}(\Omega)$ and $div \left(|\nabla u|^{p-2} \nabla u \right) \in L^{p'}(\Omega)$ we can speak about $|\nabla u|^{p-2} \frac{\partial u}{\partial n}\big|_{\partial\Omega}$ and $|\nabla u|^{p-2} \frac{\partial u}{\partial n}\big|_{\partial\Omega} \in W^{-\frac{1}{p'},\, p'}(\partial\Omega)$ is defined (see [5] and [8]) by

$$< |\nabla u|^{p-2} \frac{\partial u}{\partial n}\bigg|_{\partial\Omega}, v|_{\partial\Omega} > \ = \int_\Omega |\nabla u|^{p-2} \nabla u \nabla v + \int_\Omega div \left(|\nabla u|^{p-2} \nabla u \right) v,$$

$$(\forall) \ \ v \in W^{1,p}(\Omega).$$

If $|\nabla u|^{p-2}\ \frac{\partial u}{\partial n}\big|_{\partial\Omega} = 0$ it follows that

$$\int_\Omega -div\left(|\nabla u|^{p-2}\nabla u\right)v = \int_\Omega |\nabla u|^{p-2}\nabla u\nabla v,\ (\forall)\ v \in W^{1,p}(\Omega).$$

Because the integral $\int_\Omega |\nabla u|^{p-2}\nabla u\nabla v$ exists for each $u, v \in W^{1,p}(\Omega)$ we define the operator

$$-\Delta_p : (W^{1,p}(\Omega), |||\cdot|||_{1,p}) \to (W^{1,p}(\Omega), |||\cdot|||_{1,p})^* \text{ by}$$

$$< -\Delta_p u, v >= \int_\Omega |\nabla u|^{p-2}\nabla u\nabla v \text{ , for all } u, v \in W^{1,p}(\Omega).$$

Let us remark that if $u \in W^{1,p}(\Omega)$ then $-\Delta_p u \in (W^{1,p}(\Omega), |||\cdot|||_{1,p})^*$.

Indeed, if $u \in W^{1,p}(\Omega)$ the application $W^{1,p}(\Omega) \ni v \to < -\Delta_p u, v >$ is linear and, since for all $v \in W^{1,p}(\Omega)$:

$$\begin{aligned} |< -\Delta_p u, v >| &= \left|\int_\Omega |\nabla u|^{p-2}\nabla u\nabla v\right| \le \int_\Omega |\nabla u|^{p-1}|\nabla v| \le \\ &\le \left(\int_\Omega |\nabla u|^p\right)^{\frac{1}{p'}}\left(\int_\Omega |\nabla v|^p\right)^{\frac{1}{p}} \le |||u|||_{1,p}^{p-1}|||v|||_{1,p}, \end{aligned}$$

it follows that $-\Delta_p u \in (W^{1,p}(\Omega), |||\cdot|||_{1,p})^*$.

2. **Basic results concerning the duality mapping**

Let $(X, ||\cdot||)$ be a real Banach space and X^* its dual.

For a multivalued operator $A : X \to \mathcal{P}(X^*)$, the range of A is defined by

$$R(A) = \bigcup_{x\in D(A)} Ax,$$

where $D(A) = \{x \in X : Ax \neq \phi\}$ is the domain of A.

The operator A is said to be monotone if

$$< x_1^* - x_2^*, x_1 - x_2 >\geq 0 \text{ , for all } x_1, x_2 \in D(A) \text{ and}$$

$$x_1^* \in Ax_1,\ x_2^* \in Ax_2.$$

A continuous function $\varphi : \mathbf{R}_+ \to \mathbf{R}_+$ is called a normalization function if it is strictly increasing, $\varphi(0) = 0$ and $\varphi(r) \to \infty$ with $r \to \infty$.

By duality mapping corresponding to the normalization function φ, we mean the set valued operator $J_\varphi : X \to \mathcal{P}(X^*)$ defined by

$$J_\varphi x = \{x^* \in X^* :< x^*, x >= \varphi(\|x\|)\|x\| \ , \ \|x^*\| = \varphi(\|x\|)\} \ ,$$

for $x \in X$.

By the Hahn-Banach theorem one has that $D(J_\varphi) = X$.

We need of the following result:

Theorem 2.1. *If φ is a normalization function, then:*

(i) for each $x \in X$, $J_\varphi x$ is a bounded, closed and convex subset of X^*;

(ii) J_φ is monotone:

$$< x_1^* - x_2^*, x_1 - x_2 >\geq (\varphi(\|x_1\|) - \varphi(\|x_2\|))(\|x_1\| - \|x_2\|) \geq 0 \ ,$$

for each $x_1, x_2 \in X$ and $x_1^* \in J_\varphi x_1$, $x_2^* \in J_\varphi x_2$;

(iii) for each $x \in X$, $J_\varphi x = \partial\Phi(x)$, where $\Phi(x) = \int_0^{\|x\|} \varphi(t)\, dt$ and $\partial\Phi : X \to \mathcal{P}(X^*)$ is the subdifferential of Φ in the sense of convex analysis, i.e.

$$\partial\Phi(x) = \{x^* \in X^* : \Phi(y) - \Phi(x) \geq < x^*, y - x > \ , (\forall)\ y \in X\} \, .$$

For proof we refer to Browder [3], Lions [7], Ciorãnescu [4].

Remark 2.1. We recall that a functional $f : X \to \mathbf{R}$ is said to be Gâteaux differentiable at $x \in X$, if there exists $f'(x) \in X^*$ such that

$$\lim_{t \to 0} \frac{f(x + th) - f(x)}{t} =< f'(x), h > \ , \text{for all } h \in X.$$

If the convex function $f : X \to \mathbf{R}$ is Gâteaux differentiable at $x \in X$ then $\partial f(x) = \{f'(x)\}$.

For example, if $X = (W_0^{1,p}(\Omega), ||\cdot||_{1,p})$, $1 < p < \infty$ and $\varphi(t) = t^{p-1}$, then (see e.g. [6] or [7]) the duality mapping J_φ on the space $W_0^{1,p}(\Omega)$ is exactly the p-Laplacian operator $-\Delta_p$,

$$J_\varphi : W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega),$$

$$J_\varphi u = -\Delta_p u \ , (\forall)\ u \in W_0^{1,p}(\Omega).$$

The surjectivity of the duality mapping (see [6]) achieves the existence of the $W_0^{1,p}(\Omega)$-solution for the equation $-\Delta_p u = f$, with $f \in W^{-1,p'}(\Omega)$.

3. **The main result**

In the sequel, $W^{1,p}(\Omega)$ will be endowed with the norm $|||\cdot|||_{1,p}$.

Theorem 3.1. *The duality mapping on the space* $(W^{1,p}(\Omega), |||\cdot|||_{1,p})$, *corresponding to the normalization function* $\varphi(t) = t^{p-1}$, $1 < p < \infty$, *is the single-valued map*

$$J_\varphi : (W^{1,p}(\Omega), |||\cdot|||_{1,p}) \to (W^{1,p}(\Omega), |||\cdot|||_{1,p})^*$$

$$J_\varphi u = -\Delta_p u + |u|^{p-2}\, u \ , \textit{ for each } u \in W^{1,p}(\Omega) \ ,$$

where $-\Delta_p$ *is the p-Laplacian operator on the space* $(W^{1,p}(\Omega), |||\cdot|||_{1,p})$.

Proof. By the theorem 2.1. $J_\varphi u = \partial\Phi\,(u)$, $(\forall)\ u \in W^{1,p}(\Omega)$, where $\Phi : (W^{1,p}(\Omega), |||\cdot|||_{1,p}) \to \mathbf{R}$, $\Phi\,(u) = \int_0^{|||u|||_{1,p}} \varphi\,(t)\,dt = \frac{1}{p}|||u|||_{1,p}^p = \frac{1}{p}\,\|u\|_{0,p}^p +$ $+\frac{1}{p}\,\|\nabla u\|_{0,p}^p$ and $\partial\Phi : (W^{1,p}(\Omega), |||\cdot|||_{1,p}) \to \mathcal{P}\left((W^{1,p}(\Omega), |||\cdot|||_{1,p})^*\right)$ is the subdifferential in the sense of convex analysis.

We define the functionals

$$\begin{aligned} \tilde{\Phi}_1 \quad &: \quad L^p(\Omega) \to \mathbf{R},\ \ \tilde{\Phi}_1(u) = \frac{1}{p}\,\|u\|_{0,p}^p \ = \frac{1}{p}\int_\Omega |u|^p \\ \Phi_2 \quad &: \quad W^{1,p}(\Omega) \to \mathbf{R},\ \Phi_2(u) = \frac{1}{p}\,\|\nabla u\|_{0,p}^p = \frac{1}{p}\int_\Omega |\nabla u|^p \end{aligned}$$

and $\Phi_1 : W^{1,p}(\Omega) \to \mathbf{R}$, $\Phi_1 = \tilde{\Phi}_1 / W^{1,p}(\Omega)$

The functional $\tilde{\Phi}_1$ is Gâteaux differentiable (see [9]) and

$$< \tilde{\Phi}_1'(u), v >=< |u|^{p-1}\, sgn\, u, v >, \text{ for all } u, v \in L^p\,(\Omega)\,.$$

By the imbedding $(W^{1,p}(\Omega), |||\cdot|||_{1,p}) \to \left(L^p\,(\Omega)\,, \|\cdot\|_{0,p}\right)$ we have that Φ_1 is Gâteaux differentiable on $(W^{1,p}(\Omega), |||\cdot|||_{1,p})$.

Let the operator $P : W^{1,p}(\Omega) \to L^p\,(\Omega)$ be defined by $P(u) = |\nabla u|\,.$

If $u \in W^{1,p}(\Omega), u = 0$, then $< \Phi_2'(0), v >= 0$, $(\forall)\ v \in W^{1,p}(\Omega)$.

If $u \neq 0$, a simple computation shows that

$$< P'(u), v >= \frac{\nabla u \cdot \nabla v}{|\nabla u|}\ ,\ (\forall)\ \ v \in W^{1,p}(\Omega).$$

Since the functional $W^{1,p}(\Omega) \ni v \to < P'(u), v >$ is linear and

$$|< P'(u), v >| = \left| \int_\Omega \frac{\nabla u \cdot \nabla v}{|\nabla u|} \right| \leq \int_\Omega |\nabla v| \leq (meas\ \Omega)^{\frac{1}{p'}} \left(\int_\Omega |\nabla v|^p \right)^{\frac{1}{p}} \leq$$

$\leq c|||v|||_{1,p}$, where $c = (meas\ \Omega)^{\frac{1}{p'}}$, it follows that the operator P is Gâteaux differentiable at u.

Since $\Phi_2 = \tilde{\Phi}_1 oP$ one has that the functional Φ_2 is Gâteaux differentiable at u and

$$\begin{aligned} < & \Phi_2^{'}(u), v >=< \tilde{\Phi}_1^{'}(Pu), < P'(u), v >>= \\ = & < |\nabla u|^{p-1}, \frac{\nabla u \cdot \nabla v}{|\nabla u|} >= \int_\Omega |\nabla u|^{p-2} \nabla u \nabla v =< -\Delta_p u, v >, (\forall)\ v \in W^{1,p}(\Omega). \end{aligned}$$

Consequently, the functional $\Phi = \Phi_1 + \Phi_2$ is Gâteaux differentiable on the space $W^{1,p}(\Omega)$ and

$$< \Phi_1^{'}(u), v >=< -\Delta_p u + |u|^{p-2} u, v >, (\forall)\ u, v \in W^{1,p}(\Omega).$$

Using the convexity of the functional Φ, by remark 2.1 it follows that $J_\varphi u = \Phi'(u) = -\Delta_p u + |u|^{p-2} u$, for all $u \in W^{1,p}(\Omega)$. $\Box$

Remark 3.1. By the theorem 2.1 we have

$$|||J_\varphi u|||_* = \varphi\left(|||u|||_{1,p}\right) = |||u|||_{1,p}^{p-1},$$

where $|||\cdot|||_*$ is dual norm of $|||\cdot|||_{1,p}$.

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QUASIPOSITIVE STURM-LIOUVILLE PROBLEM

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Abstract. We explain a new approach for investigation of quasilinear boundary problem by means of Sturm-Liouville problem.

1. **The main result**

In this paper, we consider the following nonlinear problem: find a classical solution $u \in C^2[0,1]$ of equation

$$-u''(x) + p(u(x), u'(x), x) \cdot u(x) = f(x), \tag{1}$$

under Dirichlet condition

$$u(0) = u(1) = 0. \tag{2}$$

We assume that

$$p \in C^0(\mathbf{R} \times \mathbf{R} \times [0,1]), \quad f \in C^0[0,1] \tag{3}$$

We assume that the function p is non-negative:

$$p(u,t,x) \geq 0 \text{ for any } (u,t,x) \in \mathbf{R} \times \mathbf{R} \times [0,1] \tag{4}$$

and there are such constant $C > 0$ and continuous function $c : \mathbf{R} \to \mathbf{R}$ that

$$p(u,t,x) \leq C(c(u) + t^2). \tag{5}$$

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Let u^* be a solution of the *nonlinear* problem (1), (2). Then u^* is the solution of the *linear* problem

$$[-\frac{d^2}{dx^2} + q(x)]u(x) = f(x), \quad u(0) = u(1) = 0 \tag{6}$$

with the positive operator $[-d^2/dx^2 + q(x)]$ and the non-negative potential

$$q(x) = p(u^*(x), {u^*}'(x), x) \geq 0.$$

Therefore the problem (1), (2) will be named *quasipositive*. Looking at the solution of nonlinear problem (1), (2) as a solution of linear problem (6), we shall introduce a new approach of passage from the boundary problem to a fixed point equation. There are several methods of passage (see [1-3]). Our approach is analogous to D.Gilbarg and N.Trudinger one ([2], chapter 11.3). The eigenfunction theory for quasipositive operators is developed in our papers [4, 5].

We formulate the principal result. Let

$$K(x,\tau) = \begin{cases} (1-x)\tau, \ 0 \leq \tau \leq x, \\ x(1-\tau), \ x < \tau \leq 1. \end{cases}$$

be the Green function of boundary problem

$$-u''(x) = f(x), \quad u(0) = u(1) = 0.$$

Theorem 1. *The problem (1)-(5) has at least one solution* $u \in C^2[0,1]$. *For any solution the estimate*

$$\int_0^1 (u'(x))^2 dx \leq \int_0^1 \left(\int_0^1 K'_x(x,\tau) f(\tau) d\tau \right)^2 dx. \tag{7}$$

is true.

We note that the estimate (7) does not depend on "potential" $p = p(u,t,x)$. It is a direct consequence of the non-negative condition (4).

2. The operator equation

Introduce following notations. As usual, we denote $L_k(0,1)$ $(k=1,2)$ the space of functions on $(0,1)$ which are k integrable. The Sobolev space of functions $u \in L_2(0,1)$ with distributional derivative which are integrable square we denote by $W_2^1(0,1)$; $\overset{\circ}{W_2^1}(0,1) \subset W_2^1(0,1)$ is the closure in $W_2^1(0,1)$ of subspace of C^∞-functions, which are equal to zero outside some segment $[\alpha,\beta] \subset (0,1)$. The norm of $u \in W_2^1(0,1)$ is $||u||_1 = \sqrt{\int_0^1((u')^2(x)+u^2(x))dx}$; the norm of $u \in \overset{\circ}{W_2^1}(0,1)$ is $||u||_1^\circ = \sqrt{\int_0^1 (u')^2(x)dx}$. The norms $||u||_1$ and $||u||_1^\circ$ are equivalent on the space $\overset{\circ}{W_2^2}(0,1)$ due to the boundary condition (2) (see [1], chapter 13.7). Moreover, space $\overset{\circ}{W_2^1}(0,1)$ is Hilbert one with the inner product $(u,v)^\circ = \int_0^1 u'v'dx$.

First we are interested in solutions (of problem (1), (2)) from the space $\overset{\circ}{W_2^1}(0,1)$. As usual, multiplying both sides of the equation (1) by $v \in \overset{\circ}{W_2^1}(0,1)$ and integrating by parts, we get

$$\int_0^1 u'v'dx + \int_0^1 \left(\int_0^1 K(x,\tau)p(u(\tau),u'(\tau),\tau)u(\tau)d\tau\right)' v'dx$$
$$= \int_0^1 \left(\int_0^1 K(x,\tau)f(\tau)\right)' v'dx. \tag{8}$$

A function $u \in \overset{\circ}{W_2^1}(0,1)$ is called a weak solution of the problem (1), (2) if for every $v \in \overset{\circ}{W_2^1}(0,1)$ the equation (8) is valid. By the inner product, the identity (8) is of the following form

$$(u,v)^\circ + (P(u),v)^\circ = (\mathbf{f},v)^\circ, \tag{9}$$

where

$$P : \overset{\circ}{W_2^1}(0,1) \to \overset{\circ}{W_2^1}(0,1), \quad P(u) = \int_0^1 K(x,\tau)p(u(\tau),u'(\tau),\tau)u(\tau)d\tau, \tag{10}$$

$$\mathbf{f} \in \overset{\circ}{W_2^1}(0,1), \quad \mathbf{f} = \int_0^1 K(x,\tau)f(\tau)d\tau. \tag{11}$$

Since (9) is valid for every function $v \in \overset{\circ}{W_2^1}(0,1)$, the identity (9) is equivalent to the operator equation

$$u + P(u) = \mathbf{f}. \tag{12}$$

3. **The quasilinear representation of** P

Now we investigate the operator P (see (10)) in more detail. By $L(\overset{\circ}{W}{}_2^1(0,1))$ denote the Banach space of continuous linear maps $\mathbf{A}$ which operate in $\overset{\circ}{W}{}_2^1(0,1)$ and by $Lis(\overset{\circ}{W}{}_2^1(0,1)) \subset L(\overset{\circ}{W}{}_2^1(0,1))$ the open subset of linear isomorphisms. As usually, the norm $||\mathbf{A}|| = \sup ||\mathbf{A}v||_1^\circ$ where $||v||_1^\circ = 1$. Consider the map

$$A : \overset{\circ}{W}{}_2^1(0,1) \to L(\overset{\circ}{W}{}_2^1(0,1)), \quad A(u) = \mathbf{A} \ \text{ that } \ \forall v \in \overset{\circ}{W}{}_2^1(0,1)$$

$$\mathbf{A}v = \int_0^1 K(x,\tau)p(u(\tau),u'(\tau),\tau)v(\tau)d\tau,$$

Clearly $P(u) = A(u)u$. We shall call A the quasilinear representation of the map P [5]. Now the equation (12) is of the following form

$$(\mathbf{E} + A(u))u = \mathbf{f}, \tag{13}$$

where $\mathbf{E}$ is identity mapping. Properties of the map A is in next lemma.

Lemma 1. *1) For every* $u \in \overset{\circ}{W}{}_2^1(0,1)$ *the linear operator* $A(u) \in L(\overset{\circ}{W}{}_2^1(0,1))$ *is completely continuous.*

2) The map A *is completely continuous.*

3) For every $u \in \overset{\circ}{W}{}_2^1(0,1)$ *the map* $\mathbf{E} + A(u) \in Lis(\overset{\circ}{W}{}_2^1(0,1))$ *and*

$$||(\mathbf{E} + A(u))^{-1}|| < 1. \tag{14}$$

Proof. Since for any $u, v \in \overset{\circ}{W}{}_2^1(0,1)$

$$((A(u)v)(x))' = \int_0^1 K'_x(x,\tau)p(u(\tau),u'(\tau),\tau)v(\tau)dt$$

and the function $r(\xi) = p(u(\xi),u'(\xi),\xi)v(\xi) \in L_1(0,1)$ is integrable one (see (5)), the function $(A(u)v)' \in C^0[0,1]$. Thus the map $\mathbf{A} = A(u) : \overset{\circ}{W}{}_2^1(0,1) \to \{C^1[0,1] \cap (2)\}$ is continuous. Embedding $im : \{C^1[0,1] \cap (2)\} \subset \overset{\circ}{W}{}_2^1(0,1)$ is completely continuous ([1], chapter 26.24). Hence the linear operator $A(u) = im \cdot A(u)$ is completely continuous as the composition of continuous and completely continuous maps [6].

To prove the second statement, we represent the map A in the form

$$A(u)v = -x \int_0^1 \left(\int_0^\tau \left(\int_0^\xi p(u(\nu),u'(\nu),\nu)d\nu \right) v'(\xi)d\xi \right) d\tau +$$

$$\int_0^x \left(\int_0^\tau \left(\int_0^\xi p(u(\nu), u'(\nu), \nu) d\nu \right) v'(\xi) d\xi \right) d\tau.$$

Write map A as the composition of four maps: $A = \delta \cdot \gamma \cdot \beta \cdot \alpha$, where

$$\alpha : \overset{\circ}{W}{}_2^1 (0,1) \to L_1(0,1), \quad \alpha(u) := p(u(\xi), u'(\xi), \xi) = q$$

$$\beta : L_1(0,1) \to C^0[0,1], \quad \beta(q) := \int_0^\xi q(\nu) d\nu = s;$$

$$\gamma : C^0[0,1] \subset L_2(0,1), \quad \gamma(s) := s \text{ is the natural embedding;}$$

$$\delta : L_2(0,1) \to L(\overset{\circ}{W}{}_2^1 (0,1)), \quad \delta(s) = \mathbf{A} \text{ that } \forall v \in \overset{\circ}{W}{}_2^1 (0,1)$$

$$\mathbf{A}(v) = -x \int_0^1 \left(\int_0^\tau s(\xi) v'(\xi) d\xi \right) d\tau + \int_0^x \left(\int_0^\tau s(\xi) v'(\xi) d\xi \right) d\tau.$$

These maps are continuous and the map γ is completely continuous. This completes the proof of the second statement.

For any $u, v \in \overset{\circ}{W}{}_2^1 (0,1)$ we have

$$(||(\mathbf{E} + A(u))v||_1^\circ)^2 = (v,v)^\circ + 2(A(u)v,v)^\circ + (A(u)v, A(u)v)^\circ \geq$$

$$(||v||_1^\circ)^2 + 2(A(u)v,v)^\circ.$$

Since (see (4))

$$(A(u)v,v)^\circ = \int_0^1 p(u(x), u'(x), x) v^2(x) dx \geq 0,$$

then

$$(||(\mathbf{E} + A(u))v||_1^\circ)^2 \geq (||v||_1^\circ)^2.$$

Whence we obtain the third statement. □

4. **Proof of Theorem**

Next step is the passage from the non-homogeneous equation (13) to a fixed point equation.

Lemma 2. *1) The equation (13) is equivalent to the operator equation*

$$u = (\mathbf{E} + A(u))^{-1}\mathbf{f}. \tag{15}$$

2) For any weak solution u the following a priori estimate is valid:

$$||u||_1^\circ \leq ||\mathbf{f}||_1^\circ. \tag{16}$$

3) The map

$$B : \overset{\circ}{W}{}_2^1 (0,1) \to \overset{\circ}{W}{}_2^1 (0,1), \quad B(u) := (\mathbf{E} + A(u))^{-1}\mathbf{f}$$

is completely continuous.

Proof. The first statement follows from the third statement of Lemma 1. The second statement follows from the first one and third statement of Lemma 1 (see (14)).

The map B is the composition:

$$u \to A(u) \to \mathbf{E} + A(u) \to (\mathbf{E} + A(u))^{-1} \to (\mathbf{E} + A(u))^{-1}\mathbf{f}.$$

The first map is completely continuous (see the second statement of Lemma 1) and the others maps are continuous. This completes the proof [6]. □

Note that the map $B = (\mathbf{E} + A(u))^{-1}\mathbf{f}$ depends on the u in the operator part only. Thus properties of equation (15) follow from properties of map A.

To proof Theorem, we apply Leray-Schauder degree. Let the ball $T_R = \{u \in \overset{\circ}{W}{}_2^1 (0,1) : ||u||_1^\circ \leq R\}$, where the constant $R > ||\mathbf{f}||_1^\circ$. Let the sphere $S_R = \{u \in \overset{\circ}{W}{}_2^1 (0,1) : ||u||_1^\circ = R\}$. By (14) and (15), for any $u \in S_R$ we obtain $||u||_1^\circ > ||B(u)||_1^\circ$. Therefore on S_R the completely continuous vector field $u - B(u) \neq 0$ and degree of B is equal to one [6]. Consequently there is a solution $u \in T_R$ of equation (15). The existence of a weak solution is proved.

By (16) and (11) we obtain

$$\int_0^1 (u'(x))^2 dx \le \int_0^1 \left\{ \left(\int_0^1 K(x,\tau) f(\tau) d\tau \right)'_x \right\}^2 dx = \int_0^1 \left(\int_0^1 K'_x(x,\tau) f(\tau) d\tau \right)^2 dx.$$

The estimate (7) is proved.

Actually, the weak solution $u \in \overset{\circ}{W}{}_2^1 (0,1)$ is the classical solution, i.e. $u \in C^2[0,1]$. This follows from well known theorem about regularity of weak solution (see [1], §17). Theorem is proved. □

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ON THE INVARIANCE PROPERTY OF THE FISHER INFORMATION (II)

CRISTINA-IOANA FĂTU

Abstract. The objective of this paper is to give some properties for the Fisher's information measure when $X_{a\leftrightarrow b}$ represents a bilateral truncated random variable that corresponds to a normal random variable X with the probability density function $f(x;\theta)$, where $\theta=(m,\sigma^2)$, $\theta\in D_\theta$, $D_\theta\subseteq\mathbb{R}^2$, $m\in\mathbb{R}, m-$known parameter, $\sigma^2\in\mathbb{R}^+,\sigma^2-$unknown parameter.

1. Bilateral truncation effect of a normal distribution on Fisher's information

The Fisher's invariance property will be studied in the case of a truncated normal distribution.

Let X be a normal distribution with probability density function

$$f(x;m,\sigma^2)=\frac{1}{\sqrt{2\pi}\sigma}\exp\left\{-\frac{1}{2}\left(\frac{x-m}{\sigma}\right)^2\right\},x\in\mathbb{R}, \tag{1}$$

where the parameters m and σ have their usual significance, namely: $m=E(X)$, $\sigma^2=Var(X)$, $m\in\mathbb{R}$, $\sigma>0$.

Definition 1. [2] *We say that the random variable X has a normal distribution truncated to the left at $X=a, a\in\mathbb{R}$ and to the right at $X=b, b\in\mathbb{R}$, denoted by $X_{a\leftrightarrow b}$, if its probability density function, denoted by $f_{a\leftrightarrow b}(x;m,\sigma^2)$, has the form*

$$f_{a\leftrightarrow b}(x;m,\sigma^2)=\begin{cases}\dfrac{k(a,b)}{\sqrt{2\pi}\sigma}\exp\left\{-\dfrac{1}{2}\left(\dfrac{x-m}{\sigma}\right)^2\right\} & if\ a\le x\le b,\\ 0 & if\ x<a\ or\ x>b,\end{cases} \tag{2}$$

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where

$$k(a,b)=\frac{1}{A}=\frac{1}{\Phi\left(\dfrac{b-m}{\sigma}\right)-\Phi\left(\dfrac{a-m}{\sigma}\right)},\tag{3}$$

and

$$\Phi(z)=\frac{1}{\sqrt{2\pi}}\int\limits_{-\infty}^{z}\exp\left(-\frac{t^2}{2}\right)dt,\tag{4}$$

is the standard normal distribution function.

Theorem 1. *If the random variable $X_{a\leftrightarrow b}$ has a bilateral truncated normal distribution, that is its probability distribution has the form (2), then the Fisher's information measure about the unknown parameter σ^2,then the parameter m is known, has the following form*

$$\begin{aligned}I_{X_{a\leftrightarrow b}}(\sigma^2)=&-\frac{(a-m)^3f(a;m,\sigma^2)-(b-m)^3f(b;m,\sigma^2)}{4\sigma^6A}-\\&-\frac{3}{4\sigma^4}\left\{\frac{[(b-m)f(b;m,\sigma^2)-(a-m)f(a;m,\sigma^2)]-A}{A}\right\}-\\&-\frac{1}{4\sigma^4}\left\{\frac{[(b-m)f(b;m,\sigma^2)-(a-m)f(a;m,\sigma^2)]-A}{A}\right\}^2,\end{aligned}\tag{5}$$

where

$$f(a;m,\sigma^2),\ f(b;m,\sigma^2)\in\mathbb{R}^+,\tag{6}$$

Proof. We have

$$\begin{aligned}I_{X_{a\leftrightarrow b}}(\sigma^2)&=I_{X_{a\leftrightarrow b}}(\theta)=\\&=\int\limits_a^b\left(\frac{\partial\ln f_{a\leftrightarrow b}(x;m,\sigma^2)}{\partial\sigma^2}\right)^2f_{a\leftrightarrow b}(x;m,\sigma^2)dx=\\&=\int\limits_a^b\left(\frac{\partial\ln f_{a\leftrightarrow b}(x;m,\theta)}{\partial\theta}\right)^2f_{a\leftrightarrow b}(x;m,\theta)dx.\end{aligned}$$

Using (2) and (3), we obtain

$$\begin{aligned}
\ln f_{a\leftrightarrow b}(x;m,\sigma^2) &= -\ln\sqrt{2\pi}-\tfrac{1}{2}\ln\sigma^2-\ln A-\tfrac{1}{2}\left(\tfrac{x-m}{\sigma}\right)^2 = \\
&= -\ln\sqrt{2\pi}-\tfrac{1}{2}\ln\theta-\ln\left[\Phi\left(\frac{b-m}{\sqrt{\theta}}\right)-\Phi\left(\frac{a-m}{\sqrt{\theta}}\right)\right]- \\
&- \tfrac{1}{2}\tfrac{(x-m)}{\theta}^2 = \\
&= \ln f_{a\leftrightarrow b}(x;m,\theta)
\end{aligned}$$

and, it follows

$$\frac{\partial\ln f_{a\leftrightarrow b}(x;m,\theta)}{\partial\theta}=-\frac{1}{2\theta}-\frac{\frac{\partial}{\partial\theta}\left[\Phi\left(\frac{b-m}{\sqrt{\theta}}\right)\right]-\frac{\partial}{\partial\theta}\left[\Phi\left(\frac{b-m}{\sqrt{\theta}}\right)\right]}{A}+\frac{(x-m)}{2\theta^2}^2.$$

Using the relations

$$\begin{aligned}
\frac{\partial}{\partial\theta}\left[\Phi\left(\frac{b-m}{\sqrt{\theta}}\right)\right] &= \frac{\partial}{\partial\theta}\left[\frac{1}{\sqrt{2\pi}}\int\limits_{-\infty}^{\frac{b-m}{\sqrt{\theta}}}\exp\{-\frac{1}{2}z^2\}dz\right] = \\
&= -\frac{b-m}{2}\frac{1}{\theta^3}\frac{1}{\sqrt{2\pi}}\exp\left\{-\frac{1}{2}\left(\frac{b-m}{\sqrt{\theta}}\right)^2\right\} = \\
&= -\frac{b-m}{2}\frac{1}{\theta^2}f(b;m,\theta),
\end{aligned}$$

$$\begin{aligned}
\frac{\partial}{\partial\theta}\left[\Phi\left(\frac{a-m}{\sqrt{\theta}}\right)\right] &= \frac{\partial}{\partial\theta}\left[\frac{1}{\sqrt{2\pi}}\int\limits_{-\infty}^{\frac{a-m}{\sqrt{\theta}}}\exp\{-\frac{1}{2}z^2\}dz\right] = \\
&= -\frac{a-m}{2}\frac{1}{\theta^2}f(a;m,\theta),
\end{aligned}$$

it results

$$\begin{aligned}
\frac{\partial\ln f_{a\leftrightarrow b}(x;m,\theta)}{\partial\theta} &= \frac{\partial\ln f_{a\leftrightarrow b}(x;m,\sigma^2)}{\partial\sigma^2}=\frac{1}{2\sigma^2}\left\{\frac{(x-m)^2}{\sigma^2}+\right. \\
&\left.+\frac{[(b-m)f(b;m,\sigma^2)-(a-m)f(a;m,\sigma^2)]}{A}-1\right\}
\end{aligned}$$

and Fisher's information will be written

$$I_{X_{a\leftrightarrow b}}(\sigma^2) = \int_a^b \left(\frac{\partial \ln f_{a\leftrightarrow b}(x;m,\sigma^2)}{\partial \sigma^2}\right)^2 f_{a\leftrightarrow b}(x;m,\sigma^2)dx = \frac{1}{4\sigma^8 A}\{I_1 +$$

$$+\sigma^4 \left[\frac{(b-m)f(b;m,\sigma^2)-(a-m)f(a;m,\sigma^2)}{A} - 1\right]^2 I_2 +$$

$$+2\sigma^2 \left[\frac{(b-m)f(b;m,\sigma^2)-(a-m)f(a;m,\sigma^2)}{A} - 1\right] I_3 \Bigg\},$$

where

$$I_2 = \frac{1}{\sqrt{2\pi}\sigma}\int_a^b \exp\left\{-\frac{1}{2}\left(\frac{x-m}{\sigma}\right)^2\right\}dx,$$

$$I_3 = \frac{1}{\sqrt{2\pi}\sigma}\int_a^b (x-m)^2 \exp\left\{-\frac{1}{2}\left(\frac{x-m}{\sigma}\right)^2\right\}dx,$$

$$I_1 = \frac{1}{\sqrt{2\pi}\sigma}\int_a^b (x-m)^4 \exp\left\{-\frac{1}{2}\left(\frac{x-m}{\sigma}\right)^2\right\}dx.$$

By making the change of variables

$$z = \frac{x-m}{\sigma},$$

and, if we consider the formula for integration by parts

$$\int_\alpha^\beta u dv = uv \mid_\alpha^\beta - \int_\alpha^\beta v du,$$

it results

$$I_2 = A = \Phi\left(\frac{b-m}{\sigma}\right) - \Phi\left(\frac{a-m}{\sigma}\right),$$

$$I_3 = -\sigma^2 \left\{\left[(b-m)f(b;m,\sigma^2)-(a-m)f(a;m,\sigma^2)\right] - A\right\},$$

$$I_1 = -\sigma^2\left[(b-m)^3 f(b;m,\sigma^2) - (a-m)^3 f(a;m,\sigma^2)\right] -$$

$$-3\sigma^4\left[(b-m)f(b;m,\sigma^2)-(a-m)f(a;m,\sigma^2)\right] + 3\sigma^4 A.$$

Using the final values of the integrals I_2, I_3 and I_1, we obtain (8). □

2. Invariance of the Fisher information

Corollary 1. *If* $a = m - \sigma$, $b = m + \sigma$, *then*

$$f_{m-\sigma\leftrightarrow m+\sigma}(x; m, \sigma^2) = \begin{cases} \dfrac{C(m-\sigma, m+\sigma)}{\sqrt{2\pi}\sigma} \exp\left\{-\frac{1}{2}\left(\dfrac{x-m}{\sigma}\right)^2\right\} \\ \quad if\ m-\sigma \leq x \leq m+\sigma, \\ 0\ if\ x < m-\sigma\ or\ x > m+\sigma, \end{cases} \tag{7}$$

where

$$C(m-\sigma, m+\sigma) = \frac{1}{2\Phi(1)-1} \approx 2,93 \tag{8}$$

and the Fisher's information measure, relative to the unknown parameter σ^2, *has the value*

$$I_{X_{m-\sigma\leftrightarrow m+\sigma}}(\sigma^2) \approx 0,03 I_X(\sigma^2). \tag{9}$$

Proof. Using (8), we obtain

$$\begin{aligned} I_{X_{m-\sigma\leftrightarrow m+\sigma}}(\sigma^2) &= \frac{-2}{4\sigma^4\sqrt{2\pi e}[2\Phi(1)-1]} - \\ &- \frac{3}{4\sigma^4}\left\{\frac{2}{\sqrt{2\pi e}[2\Phi(1)-1]} - 1\right\} - \\ &- \frac{1}{4\sigma^4}\left\{\frac{2}{\sqrt{2\pi e}[2\Phi(1)-1]} - 1\right\}^2 = \\ &= -\frac{1}{4\sigma^4}\left\{-3 + \frac{4}{\sqrt{2\pi e}[\Phi(1)-0,5]} + \right. \\ &\left. + \left(\frac{1}{\sqrt{2\pi e}[\Phi(1)-0,5]} - 1\right)^2\right\} = \\ &= -\frac{1}{4\sigma^4}\{-3 + 2,86 + 0,08\} = \frac{0,03}{2\sigma^4}. \end{aligned}$$

$\square$

Corollary 2. *(Invariance of the Fisher information - the first form) If $a = m$, $b = m + \sigma$ or if $a = m - \sigma$, $b = m$, then*

$$f_{m\leftrightarrow m+\sigma}(x;m,\sigma^2) = \begin{cases} \dfrac{C(m,m+\sigma)}{\sqrt{2\pi}\sigma}\exp\left\{-\dfrac{1}{2}\left(\dfrac{x-m}{\sigma}\right)^2\right\} \\ \quad if\ m \le x \le m+\sigma, \\ 0\ if\ x < m\ or\ x > m+\sigma, \end{cases} \tag{10}$$

and

$$f_{m-\sigma\leftrightarrow m}(x;m,\sigma^2) = \begin{cases} \dfrac{C(m-\sigma,m)}{\sqrt{2\pi}\sigma}\exp\left\{-\dfrac{1}{2}\left(\dfrac{x-m}{\sigma}\right)^2\right\} \\ \quad if\ m-\sigma \le x \le m, \\ 0\ if\ x < m-\sigma\ or\ x > m, \end{cases} \tag{11}$$

where

$$C(m,m+\sigma) = \frac{1}{\Phi(1)-\Phi(0)} = C(m-\sigma,m) = \frac{1}{\Phi(0)-\Phi(-1)} \approx 2,93 \tag{12}$$

and the Fisher's information measures relative to the unknown parameter σ^2 has the same value, namely

$$I_{X_{m\leftrightarrow m+\sigma}}(\sigma^2) = I_{X_{m-\sigma\leftrightarrow m}}(\sigma^2) = \frac{0,03}{2\sigma^4} = I_{X_{m-\sigma\leftrightarrow m+\sigma}}(\sigma^2). \tag{13}$$

Remark 1. *If we consider the normal variable*

$$Y = X - m, \tag{14}$$

then $E(Y) = 0$, $Var(Y) = Var(X) = \sigma^2$ and the probability density function has the form

$$f_Y(y;\sigma^2) = \frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{y^2}{2\sigma^2}}, y \in \mathbb{R}. \tag{15}$$

In this case, the random variable $Y_{a\leftrightarrow b}$ has a bilateral truncated normal distribution:

$$f_{a\leftrightarrow b}(y;\sigma^2) = \begin{cases} \frac{C_0(a,b)}{\sqrt{2\pi}\sigma}e^{-\frac{y^2}{2\sigma^2}} & if\ a \le y \le b \\ 0 & if\ y < a\ or\ y > b, \end{cases} \tag{16}$$

where

$$C_0 = C_0(a,b) = \frac{1}{[\Phi\left(\frac{b}{\sigma}\right) - \Phi\left(\frac{a}{\sigma}\right)]}. \tag{17}$$

*Using (8)***,** *the Fisher's information measure,* $I_{Y_{a\leftrightarrow b}}(\sigma^2)$ *relative to the unknown parameter* σ^2, *can be written like*

$$I_{Y_{a\leftrightarrow b}}(\sigma^2) = \frac{a^3 f_Y(a;\sigma^2) - b^3 f_Y(b;\sigma^2)}{4\sigma^6 C_0} -$$
$$-\frac{3}{4\sigma^4}\left[\frac{bf_Y(b;\sigma^2) - af_Y(a;\sigma^2}{C_0} - 1\right] -$$
$$-\frac{1}{4\sigma^4}\left[\frac{bf_Y(b;\sigma^2) - af_Y(a;\sigma^2}{C_0} - 1\right]^2. \tag{18}$$

Corollary 3. *(Invariance of the Fisher information - the second form)*

$$I_{X_{-\sigma\leftrightarrow\sigma}}(\sigma^2) = I_{X_{m-\sigma\leftrightarrow m+\sigma}}(\sigma^2) = I_{X_{m\leftrightarrow m+\sigma}}(\sigma^2) = I_{X_{m-\sigma\leftrightarrow m}}(\sigma^2) \approx$$
$$\approx \frac{0,03}{2\sigma^4} = 0,03 I_X(\sigma^2). \tag{19}$$

Theorem 2. *(Invariance of the Fisher information - the third form) If* $a = m - k\sigma$, $b = m + k\sigma$, *or* $a = -k\sigma$, $b = k\sigma$, *then the probability density function, denoted by* $f_{a\leftrightarrow b}(x;m,\sigma^2)$, *in(2), has the form*

$$f_{m-k\sigma\leftrightarrow m+k\sigma}(x;m,\sigma^2) = \begin{cases} \dfrac{C(m-k\sigma, m+k\sigma)}{\sqrt{2\pi}\sigma}\exp\left\{-\dfrac{1}{2}\left(\dfrac{x-m}{\sigma}\right)^2\right\} \\ \quad if\ m - k\sigma \le x \le m + k\sigma, \\ 0\ if\ x < m - k\sigma\ or\ x > m + k\sigma, \end{cases} \tag{20}$$

or the form

$$f_{-k\sigma\leftrightarrow k\sigma}(x;m,\sigma^2) = \begin{cases} \dfrac{C(-k\sigma, k\sigma)}{\sqrt{2\pi}\sigma}\exp\left\{-\dfrac{1}{2}\left(\dfrac{x}{\sigma}\right)^2\right\}\ if\ -k\sigma \le x \le k\sigma, \\ 0\ if\ x < -k\sigma\ or\ x > k\sigma, \end{cases} \tag{21}$$

where

$$C(m-k\sigma, m+k\sigma) = C(-k\sigma, k\sigma) = \frac{1}{2\Phi(k) - 1}, k \in \mathbb{N}^*. \tag{22}$$

The Fisher's information measures, relative to the unknown parameter σ^2, *have the same values*

$$I_{X_{m-k\sigma\leftrightarrow m+k\sigma}}(\sigma^2) = I_{X_{-k\sigma\leftrightarrow k\sigma}}(\sigma^2) = -\frac{1}{4\sigma^4}\left\{-3 + \frac{(k^3+3k)}{\sqrt{2\pi}e^{\frac{k^2}{2}}[\Phi(k) - 0,5]} + \right.$$
$$\left. + \left(\frac{k}{\sqrt{2\pi}e^{\frac{k^2}{2}}[\Phi(k) - 0,5]} - 1\right)^2\right\}, k \in \mathbb{N}^*. \tag{23}$$

Proof. Indeed, using the relations (2), (3) as well as the Theorem 1 and the Corollary 3, we obtain just the above value. □

Corollary 4. *(Invariance of the Fisher information - extended form)*

$$I_{X_{m\leftrightarrow m+k\sigma}}(\sigma^2)=I_{X_{m-k\sigma\leftrightarrow m}}(\sigma^2)=I_{X_{m-k\sigma\leftrightarrow m+k\sigma}}(\sigma^2)=I_{X_{-k\sigma\leftrightarrow k\sigma}}(\sigma^2)=$$

$$=-\frac{1}{4\sigma^4}\left\{-3+\frac{(k^3+3k)}{\sqrt{2\pi}e^{\frac{k^2}{2}}[\Phi(k)-0,5]}+\left(\frac{k}{\sqrt{2\pi}e^{\frac{k^2}{2}}[\Phi(k)-0,5]}-1\right)^2\right\},k\in\mathbb{N}^*. \tag{24}$$

Remark 2. *Using Theorem 2 we obtain :*

a) for $k=1$*, from (23) it results (19).*

b) for $k=2$*, from (23) it results*

$$I_{X_{m-2\sigma\leftrightarrow m+2\sigma}}(\sigma^2)=I_{X_{-2\sigma\leftrightarrow 2\sigma}}\ (\sigma^2)=$$

$$=-\frac{1}{4\sigma^4}\left\{-3+\frac{14}{\sqrt{2\pi}e^2[\Phi(2)-0,5]}+\right.$$

$$\left.+\left(\frac{3}{\sqrt{2\pi}e^2[\Phi(2)-0,5]}-1\right)^2\right\}\approx \tag{25}$$

$$\approx-\frac{1}{4\sigma^4}(-3+1,6+0,60)=$$

$$=\frac{0,40}{2\sigma^4}=0,40I_X(\sigma^2). \tag{26}$$

c) for $k=3$ *we obtain*

$$I_{X_{m-3\sigma\leftrightarrow m+3\sigma}}(\sigma^2)=I_{X_{-3\sigma\leftrightarrow 3\sigma}}\ (\sigma^2)=$$

$$=-\frac{1}{4\sigma^4}\left\{-3+\frac{36}{\sqrt{2\pi e}e^4[\Phi(3)-0,5]}+\right.$$

$$\left.+\left(\frac{2}{\sqrt{2\pi e}e^4[\Phi(3)-0,5]}-1\right)^2\right\}\approx \tag{27}$$

$$\approx-\frac{1}{4\sigma^4}(-3+0,33+0,95)=\frac{0,86}{2\sigma^4}=0,86I_X(\sigma^2). \tag{28}$$

Conclusion 1. *The invariance properties of the Fisher information, relative to the unknown parameter* σ^2, *take place then when the normal variable* X *is truncated on intervals of the forms:*

$$[m-k\sigma, m+k\sigma], [m, m+k\sigma], [m-k\sigma, m], [-k\sigma, k\sigma], k \in \mathbb{N}^*. \tag{29}$$

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ON THE NUMERICAL SIMULATION OF A LOW-MACH NUMBER FLOW

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Abstract. In the present work we investigate numerically a flow model used to simulate convection problems such as tunnel fires. This model is based on an asymptotic approach for Navier-Stokes equations first derived in [2]. We will show that this model is capable to combine the low-Mach number limit with large temperature gradients. Two sets of calculations are included in this work to show the capabilities of the proposed model and also the usefulness of the standard Boussinesq approximation.

1. Introduction

Because of many fire accidents in tunnels, the interest in the description, modeling and the simulation of such events has been increased in the last years. In practice, to simulate a complete fire accident is not possible due to many parameters involved: the tunnel geometry, the number of cars inside, the intensity and position of the fire, ventilation rules etc. In time two main features of fire events have been observed, namely characteristic velocities in the tunnel of the order of $1\,m/s$ and characteristic temperature differences which are quite large [1].

In [2], [3] and [4] a mathematical model which combines these two features has been developed and numerically tested. The modeling starts with the description of the air flow using the compressible Navier-Stokes equations. Then, using appropriate scales

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(see [2]), the two-dimensional compressible system is written as:

$$
\begin{aligned}
(\rho)_t + div(\rho \boldsymbol{u}) &= 0 \\
\boldsymbol{u}_t + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} + (\gamma M^2)^{-1}\frac{1}{\rho}\nabla p &= \frac{1}{\rho}\left(Re^{-1}\Delta\boldsymbol{u} + \frac{Re^{-1}}{3}\nabla(div(\boldsymbol{u}))\right) + \boldsymbol{f} \\
(\rho T)_t + div(\boldsymbol{u}\rho T) + (\gamma-1)p\,div(\boldsymbol{u}) &= \gamma Pr^{-1}Re^{-1}\Delta T + \boldsymbol{q}
\end{aligned} \tag{1}
$$

where $\rho, \boldsymbol{u}, p$, and T represent the density, the velocity field, the pressure and the temperature, respectively. The functions $\boldsymbol{f}$, $\boldsymbol{q}$ are the external force (e.g gravity) and the heat source due to the fire which acts as a volume indicator function over the fire. The dimensionless constants γ, M, Re, Pr and Fr are the adiabatic exponent, the Mach number, the Reynolds number and the Prandtl number, respectively. All these quantities and reference values are detalied in [2].

Since $M \ll 1$, a compressible flow solver will suffer severe deficiencies, both in efficiency and accuracy. Two distinct techniques have been proposed to capture solution convergence for low-Mach number flows: preconditioning and asymptotic expansion methods. In fact these techniques rescale the condition number of the system. The first one is to multiply time derivatives by suitable preconditioning matrix, in the sense that they scale the eigenvalues of the system to similar orders of magnitude and remove the disparity in wave speeds, leading to a well-conditioned system [5].

In this work, we will follow the second technique, the asymptotic or perturbation method. This approach consists in a Taylor series expansion of variables (in our case the pressure) in power terms of the Mach number. The basic philosophy behind this technique is to decrease the numerical representation of the speed of sound artificially, by substracting a constant pressure p_0 across the entire domain:

$$p = p_0 + (\gamma M^2)p_1 + O((\gamma M^2)^2),$$

where p_0 is the ground pressure and p_1 is the fluctuation pressure part. It turns out that the ground pressure can be only a function of time, i.e $p_0 = p_0(t)$, but since the tunnel is a open domain this ground pressure will also not change in time. Therefore, considering $p_0 = constant$ and that in leading order we have $T = p_0/\rho$, the system

(1) can be rewritten as [2]

$$
\begin{aligned}
(\rho)_t + div(\rho \boldsymbol{u}) &= 0 & (2)\\
\boldsymbol{u}_t + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} + \frac{1}{\rho}\nabla p_1 &= \frac{1}{\rho}\left(Re^{-1}\Delta\boldsymbol{u} + \frac{Re^{-1}}{3}\nabla(div(\boldsymbol{u}))\right) + \boldsymbol{f} & (3)\\
div(\boldsymbol{u}) &= \gamma Pr^{-1}Re^{-1}\Delta(\frac{1}{\rho}) + \frac{q}{\gamma p_0}. & (4)
\end{aligned}
$$

This system represents a density-dependent flow with a non-vanishing divergence of the velocity field.

The system (2)-(4) is solved numerically by a modified first order projection method described in [3]. For the numerical scheme we prescribe the following boundary conditions

$$
\begin{aligned}
\frac{\partial \boldsymbol{u}}{\partial \boldsymbol{x}}(\boldsymbol{x},t) &= 0, \quad \boldsymbol{x}\in\Gamma_1\cup\Gamma_3\\
\boldsymbol{u}(\boldsymbol{x},t) &= 0, \quad \boldsymbol{x}\in\Gamma_2\cup\Gamma_4\\
\rho(\boldsymbol{x},t) &= \rho_0, \quad if\, \boldsymbol{u}(\boldsymbol{x},t) > 0, \quad \boldsymbol{x}\in\Gamma_1\\
\rho(\boldsymbol{x},t) &= \rho_1, \quad if\, \boldsymbol{u}(\boldsymbol{x},t) < 0, \quad \boldsymbol{x}\in\Gamma_3\\
p(\boldsymbol{x},t) &= p_0, \quad \boldsymbol{x}\in\Gamma_1\\
p(\boldsymbol{x},t) &= p_1, \quad \boldsymbol{x}\in\Gamma_3\\
\nabla p\cdot\vec{n} &= \frac{1}{\rho}\left(Re^{-1}\Delta\boldsymbol{u} + \frac{Re^{-1}}{3}\nabla(div(\boldsymbol{u}))\right)\cdot\vec{n} + \boldsymbol{f}\cdot\vec{n}, \quad \boldsymbol{x}\in\Gamma_2\cup\Gamma_4
\end{aligned}
$$

where Γ_1, Γ_3 denote the entrance and the exit of the tunnel and Γ_2, Γ_4 the lower and upper fixed walls, respectively.

2. The validity of the Boussinesq approximation in the case of large temperature differences

The Boussinesq approximation starts by considering the compressible Navier-Stokes equations for fluid flow. At this stage all fluid properties are assumed to be functions of temperature T and presssure P, i.e.

$$\rho = \rho(T,P), \qquad c_p = c_p(T,P)$$
$$\mu = \mu(T,P), \qquad \alpha = \alpha(T,P)$$
$$k = k(T,P)$$

Because these functions are not known completely, one assumes that each function may be approximated by a first order Taylor expansion, i.e.

$$\begin{aligned}
\rho &= \rho_r(1-\alpha_r(T-T_r)+\beta_r(P-P_r)) \\
c_p &= c_{p_r}(1+a_r(T-T_r)+b_r(P-P_r)) \\
\mu &= \mu_r(1+c_r(T-T_r)+d_r(P-P_r)) \\
\alpha &= \alpha_r(1+e_r(T-T_r)+f_r(P-P_r)) \\
k &= k_r(1+m_r(T-T_r)+n_r(P-P_r)) \qquad (5)
\end{aligned}$$

with $\boldsymbol{x}_r = (\rho_r, c_{p_r}, \mu_r, \alpha_r, k_r)$ where $w_r = (\alpha_r, a_r, c_r, e_r, m_r)$ represents the reference states of $(1/\boldsymbol{x}_r)\partial\boldsymbol{x}_r/\partial T$ and $\boldsymbol{y}_r = (\beta_r, b_r, d_r, f_r, n_r)$ represents the reference states of $(1/\boldsymbol{x}_r)\partial\boldsymbol{x}_r/\partial P$, respectively.

According to [6] the following criteria must be checked in order to ensure the validity of the Boussinesq approximation:

$$c_1 = |\alpha_r\theta| \leq 0.1, \qquad c_2 = |\beta_r\rho_r\boldsymbol{g}L| \leq 0.1 \qquad (6)$$

$$c_3 = |c_r\theta| \leq 0.1, \qquad c_4 = |d_r\rho_r\boldsymbol{g}L| \leq 0.1 \qquad (7)$$

$$c_5 = |a_r\theta| \leq 0.1, \qquad c_6 = |b_r\rho_r\boldsymbol{g}L| \leq 0.1 \qquad (8)$$

$$c_7 = |m_r\theta| \leq 0.1, \qquad c_8 = |n_r\rho_r\boldsymbol{g}L| \leq 0.1 \qquad (9)$$

$$c_9 = |e_r\theta| \leq 0.1, \qquad c_{10} = |f_r\rho_r\boldsymbol{g}L| \leq 0.1 \qquad (10)$$

$$c_{11} = |\frac{\alpha_r\boldsymbol{g}L}{c_{p_0}}| \leq 0.1 \qquad c_{12} = |\frac{\alpha_r\boldsymbol{g}LT_r}{c_{p_0}\theta}| \leq 0.1 \qquad (11)$$

$$c_{13} = |\frac{\alpha_r \boldsymbol{g} L}{c_{p_0}}|(Pr\,Ra^{-1})^{1/2} \quad \leq \quad 0.1(Pr\,Ra)^{-1/2} \tag{12}$$

where θ, $\boldsymbol{g}$, L are the maximal temperature variations around T_r, the gravitational force and the reference length, respectively. In the case of air at $T_r = 15\,°C$ and $P_r = 10^5 Pa$ the following values for the criteria $c_1 - c_{11}$ are given in [6]:

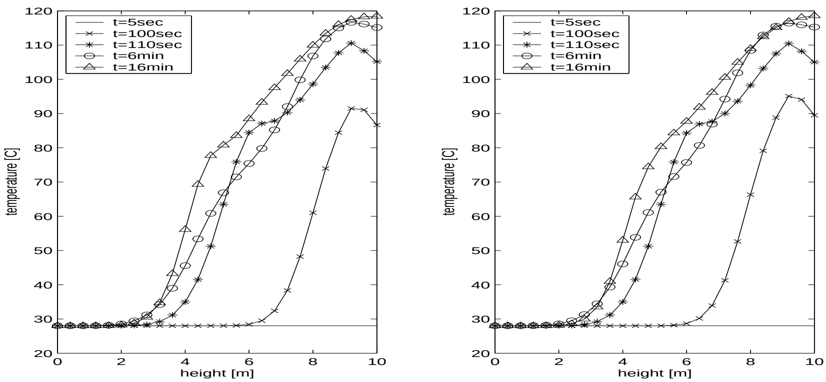
c_1	c_2	c_3	c_4	c_5	c_6
$3.5 \cdot 10^{-3}\theta$	$1.2 \cdot 10^{-6}L$	$2.8 \cdot 10^{-3}\theta$	0	$4.5 \cdot 10^{-5}\theta$	$2.3 \cdot 10^{-9}L$

c_7	c_8	c_9	c_{10}	c_{11}
$2.4 \cdot 10^{-3}\theta$	0	$-3.6 \cdot 10^{-3}\theta$	0	$3.6 \cdot 10^{-7}L$

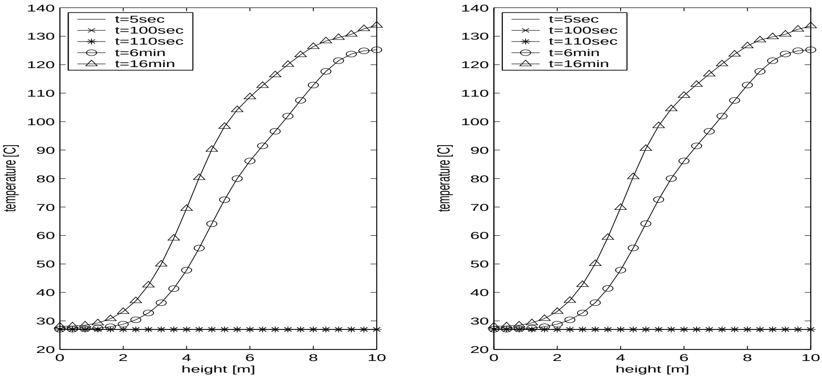
If the maximal temperature difference θ is very large (e.g. $1000\,°C$) then it is quite easy to check that the criteria c_1, c_3, c_7, c_9 are not fullfiled, hence the Boussinesq approximation does not apply.

3. Numerical results

In the following we will compare the numerical results in the case of two realistic tunnel fire events described in [3] with the Boussinesq approximation [6]. In both cases the heat source is placed exactly in the middle of the tunnel and it is distributed over a rectangular area of size 10 m x 4 m.



(a)



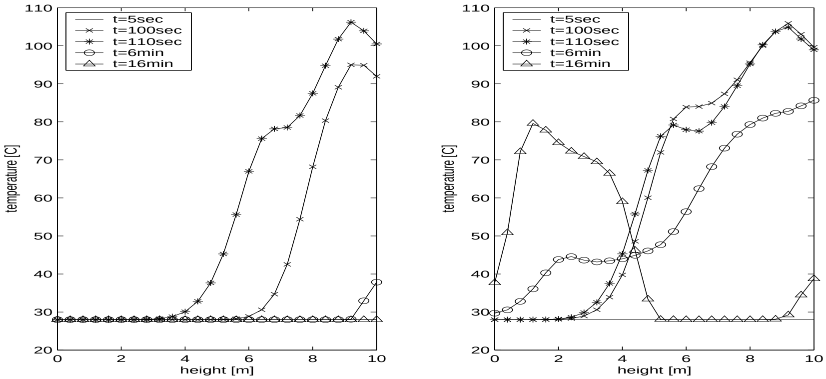
(b)

Figure 1. Vertical temperature profiles (in °C) for a tunnel without slope 100 m left and right from the heat source at various times: (a) the low-Mach number model (1), (b) the standard incompressible Navier-Stokes model with Boussinesq approximation.

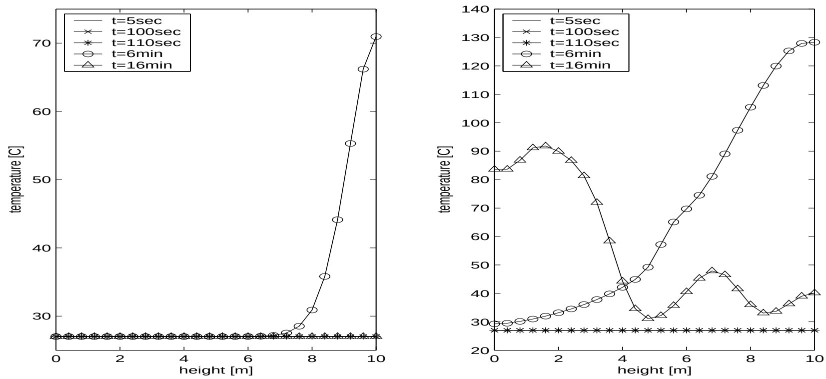
3.1. **Tunnel without slope.** The tunnel configuration data are listed in Table I. More information about the numerical method and other relevand data are given in [3]. Figure 1 show the temperature profiles along a vertical axis, which is placed 100 m to the left and right of the middle of the tunnel in the case of the low-Mach number model (1) (1a), and the standard Boussinesq approximation model (1b). First of all the results show that the flow field is symmetric with respect to the location of the heat source.

Table I. Test configuration

Length	$1000m$
Height	$10m$
Heat source	$1MW$
Initial velocity	0.0
Pressure difference(bottom-top)	120Pa
Re number	2500
Simulation time	30 min



(a)



(b)

Figure 2. Vertical temperature profiles (in °C) for a tunnel without slope 100 m left and right from the heat source at various times: (a) the low-Mach number model (1), (b) the standard incompressible Navier-Stokes model with Boussinesq approximation.

If we compare the Figure (1a) with the Figure (1b) we see that the temperature fronts are moving with different velocities, i.e. the velocity coming from the Boussinesq approximation model is lower than in the low-Mach number model (1). This fact is clear in the literature where it is claimed that the bouyancies forces are not so strong when simulated with the Boussinesq approximation model. Indeed because the heat transfer towards the tunnel ends is not so fast as in the low-Mach number model, the temparatures in the Boussinesq approximation are higher.

3.2. **Tunnel with slope.** As in the previous example the tunnel configuration data are listed in Table I. The only modification here is that the tunnel has a slope of 3% upwards from the left to the right end. The same features of both simulations are seen also in this case (see Figure 2a,b).

4. Conclusions

Mathematical models which describe fire accidents in tunnels should model low-Mach number flows together with large temperature gradients. In the present paper we compared the low-Mach number model proposed in [3] with the standard Boussinesq approach for fluid flow in the case of two fire examples. As written in Table I we do not use the real Reynolds number, indeed the numerical examples presented here have not to be seen as a comparison with the real experiment data. This is a preliminary step in this direction. The effect of turbulence will be the subject of further investigations.

Acknowledgements

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STUDIA UNIV. "BABEŞ–BOLYAI", MATHEMATICA, Volume **L**, Number 1, March 2005

ON SPECTRAL PROPERTIES OF SOME CHEBYSHEV-TYPE METHODS DIMENSION VS. STRUCTURE

CĂLIN-IOAN GHEORGHIU

Abstract. The aim of the present paper is to analyze the non-normality of the matrices (finite dimensional operators) which result when some Chebyshev-type methods are used in order to solve second order differential two-point boundary value problems. We consider in turn the classical Chebyshev-tau method as well as two variants of the Chebyshev-Galerkin method. As measure of non-normality we use the non-normality ratio introduced in a previous paper. The competition between the dimension of matrices (the order of approximation) and their structure (the numerical method itself) with respect to normality is the core of our study. It is observed that for quasi normal matrices, i.e., non-normality ratio close to 0, exhibiting pure real spectrum, this measure remains the unique indicator of non-normality. In such cases the pseudospectrum tells nothing about non-normality.

1. **Introduction**

With the scalar measure of non-normality introduced in our paper [1] we try to quantify the non-normality of three Chebyshev-type methods corresponding to differential operators involved in a second order two-point boundary value problem. To be more specific we work with Chebyshev-tau method (CT for short), a Chebyshev-Galerkin method suggested by J. Shen (CGS for short) in his paper [7],

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and a Chebyshev-Galerkin method (CG for short) with different trial and test basis functions analyzed in our previous papers [5], [4], [2] and [6].

In the most significant cases we also display the corresponding pseudospectra. We want to point out the fact that, besides the pseudospectra, our scalar measure of non-normality can be thought of as a fairly reasonable characteristic of non-normality of square matrices. More, it has an important advantage. By use of this scalar measure the matrices, and consequently the numerical methods, can be compared.

2. **The non-normality of C T, C G S and C G methods**

In order to quantify the non-normality of the first three differential operators when they are discretized using the above mentioned methods we consider the following two-point boundary value problem:

$$u'' + \mu \cdot u' - \lambda \cdot u = f(x), \quad u(-1) = u(1) = 0. \tag{1}$$

It is well known that a matrix is non-normal if it does not commute with its conjugate transpose, i.e., $A^*A - AA^* \neq 0$ - the null matrix. We recall that for a square (non null) matrix A of dimension N with complex entries, its non-normality ratio, introduced in [1], reads as follows

$$H(A) := \frac{\sqrt{\varepsilon(A^*A - AA^*)}}{\varepsilon(A)},$$

where * stands for the conjugate transpose of A and $\varepsilon(A)$ means the Frobenius norm of A. This is indeed a scalar measure (see [1]) and it satisfies the sharp inequality

$$0 \leq H(A) \leq \sqrt[4]{2}.$$

For the classical CT method we refer to the well known monograph Gottlieb and Orszag [3], pp. 119-120.

For the CGS method we found out the matrices from the paper [7] P. 4.

Eventually, all the technicalities implied by CG method are available in the report of I. S. Pop [4]. For various values of parameters μ and λ the non-normality ratios are displayed in the following three tables.

	N=8	N=64	N=128	N=512	The variation
CT	1.0254	0.9852	0.9847	0.9845	→
CG	0.3958	0.2220	0.1616	0.0825	↘
CGS	0.2926	0.1238	0.0891	0.0452	↘

Table 1: The non-normality ratios for mu=0. and lambda=0.1 in (1).

	N=8	N=64	N=128	N=512	The variation
CT	0.0510	0.9713	0.9845	0.9845	→
CG	0.3584	0.1359	0.0986	0.0728	↘
CGS	0.0076	0.0121	0.0200	0.0538	↗

Table 2: The non-normality ratios for mu=0. and lambda=256ˆ2 in (1).

	N=8	N=64	N=128	N=512	The variation
CG	0.4759	0.1972	0.1435	0.0796	↘
CGS	0.1979	0.0628	0.0538	0.0414	↘

Table 3: The non-normality ratios for mu=256. and lambda=0.1 in (1).

It is well known that the non-normality of matrices is also investigated using the notion of pseudospectrum i.e., the spectrum of the randomly perturbed matrix with an arbitrary small quantity. Up to our knowledge a direct connection between scalar measures of non-normality and pseudospectra does not exist.

For example, the pseudospectra reported in Figures 1 and 2 look very different even if they correspond to matrices with close values of non-normality ratios.

While for the CG method the spectrum contains complex eigenvalues, and the pseudospectrum underlines the spectral instability, in case of CGS method all eigenvalues are pure real and the spectral instability is almost absent. Thus, in spite of the fact that a matrix is non-normal, its pseudospectrum does not perceive this anomaly. In this situation we must resort to a scalar measure in order to observe the non-normality. We also observe that the complex part of the spectrum is much more instable than the real counterpart.

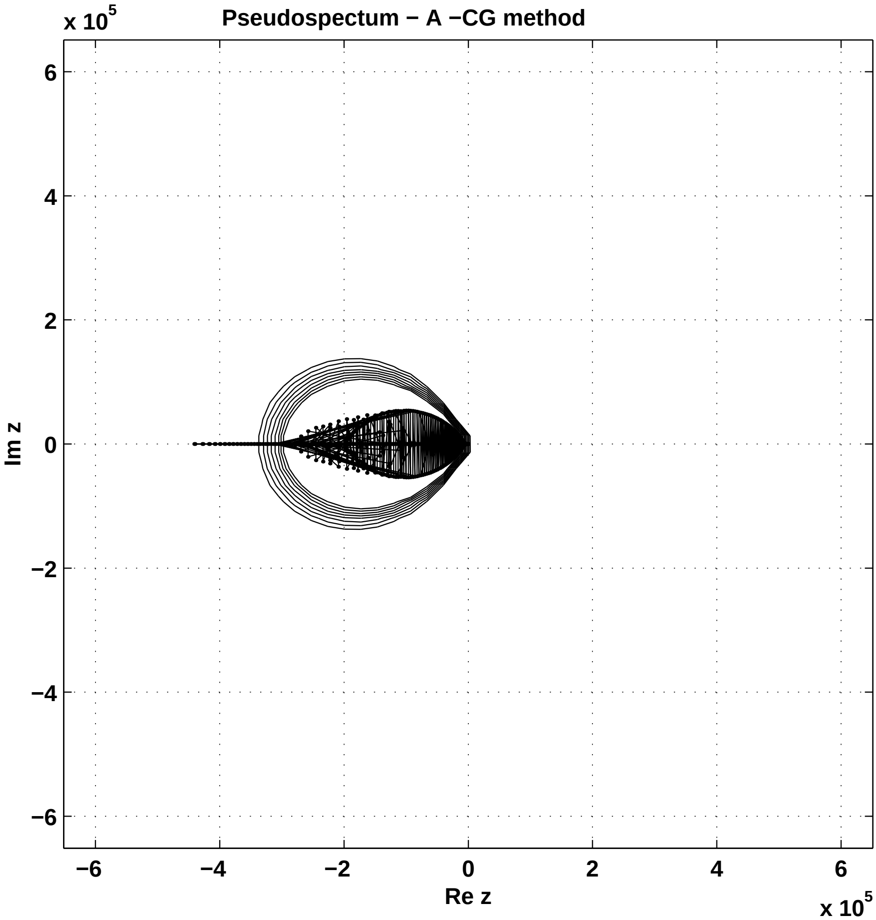

FIGURE 1. The pseudospectrum corresponding to position 2, 4 in Table 2

Remark 1. *A Matlab code was used in order to work out the entries of Tables 1,2 and 3. The pseudospectra were depicted using a slightly modified code from the paper of L. N. Trefethen* [8].

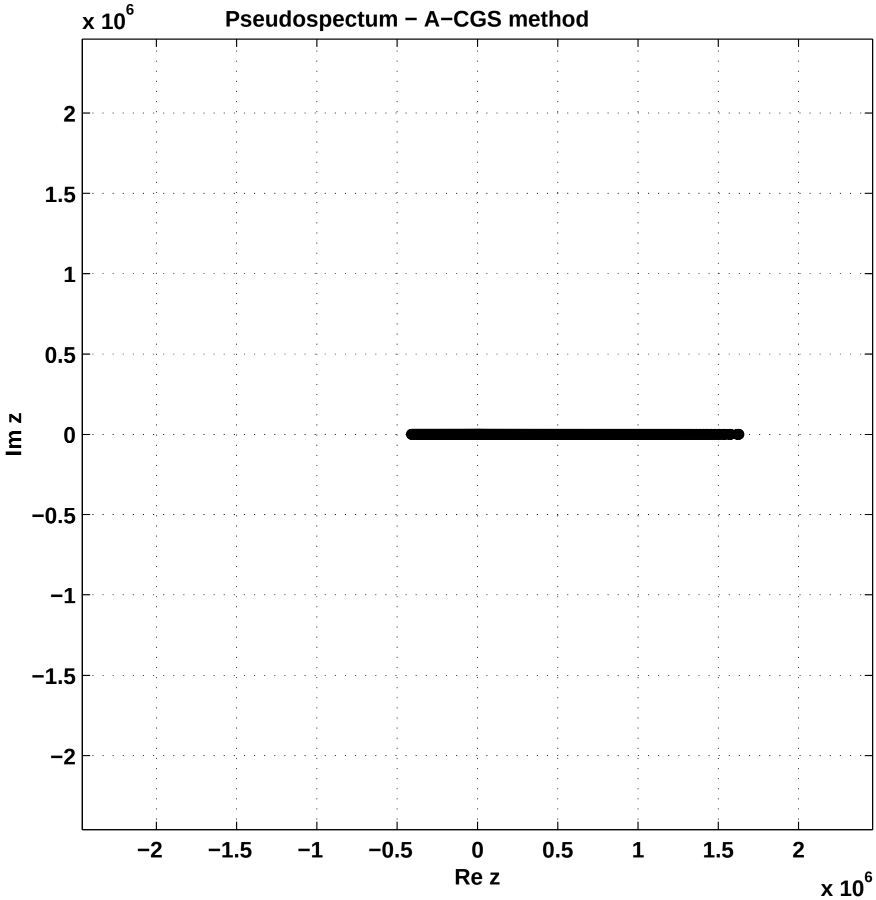

FIGURE 2. The pseudospectrum corresponding to position 3, 4 in Table 2

3. Concluding remarks

First of all, it is very clear from Tables 1 and 2 that the CT method is worse with respect to normality and its non-normality does not vary with the dimension N of the approximation.

The most normal method seems to be CGS. Anyway, for large N the CG and CGS methods become closer and closer. At the same time, it is quite surprising that in the absence of the first order term (see Table 3) these methods seem to converge to the same value of non-normality, CG decreasing and CGS increasing for large values of cutoff parameter N.

Finally, we have to remark that in spite of the fact that in cases considered in Figures 1 and 2 the non-normality ratios are quite closed, the pseudospectra are incomparable. It seems that in such cases the information furnished by pseudospectrum could be misleading.

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STUDIA UNIV. "BABEŞ–BOLYAI", MATHEMATICA, Volume **L**, Number 1, March 2005

RADIATION EFFECTS ON FREE CONVECTION FROM A VERTICAL CONE EMBEDDED IN A FLUID SATURATED POROUS MEDIUM

TEODOR GROŞAN AND IOAN POP

Abstract. The radiation effects on the steady free convection boundary layer over a vertical cone embedded in a fluid saturated porous medium are studied. We adopt for the radiative model the well-known Rosseland model. It has been found that similarity solutions exist and the ordinary differential equations were solved using a combined Runge-Kutta method and shooting technique.

1. Introduction

Heat transfer in porous media is involved in many practical applications in geophysics, energy related problems, environment problems, etc. The monographs: Ingham and Pop (1998, 2002), Vafai (2002), Pop and Ingham (2001) and Ingham et al. (2004) give an excellent summary of the work on this subject.

If the heat transfer process take place at high temperature radiative effects can't be neglected (Modest, 2003; Siegel and Howell, 1992). The radiative models used for fluids are not always appropriate for porous media. A synthesis of radiative models in porous media is given by Kaviany and Singh (1993). Using Rosseland approximation (see Rosseland, 1936), Hossain and Takhar (1996), Raptis (1998), Hossain and Pop (2001) and Bakier (2001) studied the free mixed convection from vertical surfaces placed in porous media. Chamka (1997), Chamka et al. (2001, 2002) considered the solar radiation case or, in addition, the mass transfer.

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In this paper we will study the radiation effect in the free convection from a vertical cone embedded in a fluid saturated porous medium using the Rosseland radiative model.

2. Basic equations

We consider a vertical cone having a constant surface temperature, T_w, while the cone is placed in an opaque fluid saturated porous medium having the temperature T_∞ (see Fig. 1). Under boundary layer Boussinesq approximations and using the Rosseland radiative model the governing equations are given by:

$$\frac{\partial(ru)}{\partial x}+\frac{\partial(rv)}{\partial y}=0 \tag{1}$$

$$u=\frac{g\cos\gamma K\beta}{v}(T-T_\infty) \tag{2}$$

$$u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}=\alpha_m\frac{\partial^2 T}{\partial y^2}-\frac{1}{(\rho_\infty c)_f}\frac{\partial q^r}{\partial y} \tag{3}$$

where $r=x\sin\gamma$ is the cone's radius, v is the kinematic viscosity, K is the permeability, α_m is the thermal diffusivity, ρ is the density and c is the specific heat. The subscripts w and ∞ are related to the surface and to the ambient medium, respectively. The radiative heat flux, q^r, has the form:

$$q^r=-\left(\frac{4\sigma}{3\chi}\right)\frac{\partial T^4}{\partial y} \tag{4}$$

where σ is the Stefan-Boltzman's constant and χ is the mean absorption coefficient in the Rosseland approximation.

The boundary conditions for eqs. (1)-(3) are:

$$\begin{aligned} &v=0,\quad T=T_w \quad \text{for}\quad y=0\\ &u\to 0,\quad T\to T_\infty \quad \text{for}\quad y\to\infty \end{aligned} \tag{5}$$

In order to obtain similar solutions the following transformations are introduced:

$$\psi=\alpha_m r Ra_x^{1/2}f(\eta),\quad \theta(\eta)=\frac{T-T_\infty}{T_w-T_\infty},\quad \eta=Ra_x^{1/2}(y/x) \tag{6}$$

where ψ is the stream function, $ru = \dfrac{\partial\psi}{\partial x}$, $rv = -\dfrac{\partial\psi}{\partial y}$, η is the similar variable and Ra_x is the local Rayleigh number defined as:

$$Ra_x = \frac{g\beta K \cos\gamma (T_w - T_\infty) x}{v\alpha_m} \tag{7}$$

Using (6) in eqs. (1)-(3) and in boundary conditions (5) the governing equations became:

$$f' = \theta \tag{8}$$

$$\left\{\left[1 + \frac{4}{3}N[1 + (\theta_w - 1)\theta]^3\right]\theta'\right\}' + \frac{3}{2}f\theta' = 0 \tag{9}$$

$$f(0) = 0, \quad \theta(0) = 1, \quad f'(\infty) = 0 \tag{10}$$

where the radiative and temperature parameters N and θ_w, respectively, have the form:

$$N = \frac{4\sigma T_\infty^3}{k\chi}, \quad \theta_w = \frac{T_w}{T_\infty} \tag{11}$$

From the energetic balance on the cone's surface it is possible to deduce the convective heat transfer coefficient, h:

$$-k\frac{\partial T}{\partial y}\bigg|_{y=0} + q^r = h(T_w - T_\infty) \tag{12}$$

and thus the local Nusselt number is given by:

$$Nu_x = -\theta'(0) Ra_x^{1/2}\left[1 + \frac{4}{3}N\theta_w^3\right] \tag{13}$$

We must mention that in the absence of the radiation effect ($N = 0$), eqs. (8)-(10) reduce to those obtained by Cheng et al. (1985).

3. **Results and discussions**

Eqs. (8)-(10) have been solved numerically using a combined Runge-Kutta and shooting method for the following values of the radiative parameter $N = 0,\ 1,\ 5$ and 10 for the temperature parameter $\theta_w = 1.1,\ 1.5$ and 2. In the case $N = 0$ (i.e. radiation effects are negligible), the calculated value for the local Nusselt number, $-\theta_w$, is in very good agreement with that obtained by Cheng et al. (1985). The values for the Nusselt number are given in Table 1 for different values of parameters N and θ_w. Figs. 2-4 present the dimensionless temperature's profiles variation with

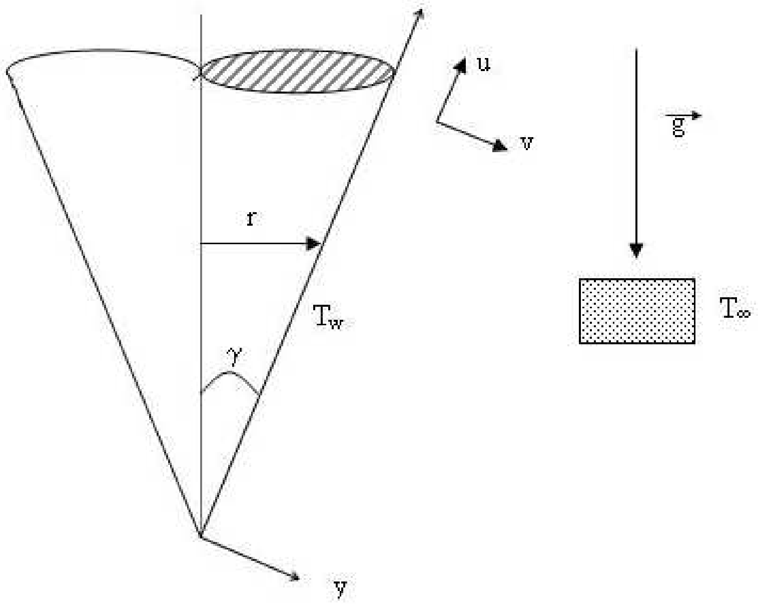

FIGURE 1. Geometry of the problem

the variation of the radiation parameter N. It is observed that the thickness of the boundary layer increase with the increasing of the parameter N. It is also observed in Figs. 5-7 that the dimensionless temperature increase with the increasing of the parameter θ_w.

N	$-\theta'(0)$		
	$\theta_w = 1.1$	$\theta_w = 1.5$	$\theta_w = 2.0$
0	0.768596(*0.769)	0.768596	0.768596
1	0.250263	0.212828	0.187691
5	0.107103	0.094502	0.085511
10	0.074887	0.066666	0.060616

Table 1. Values of the local Nusselt number, $\theta'(0)$

*Result obtained by Cheng et al. (1985)

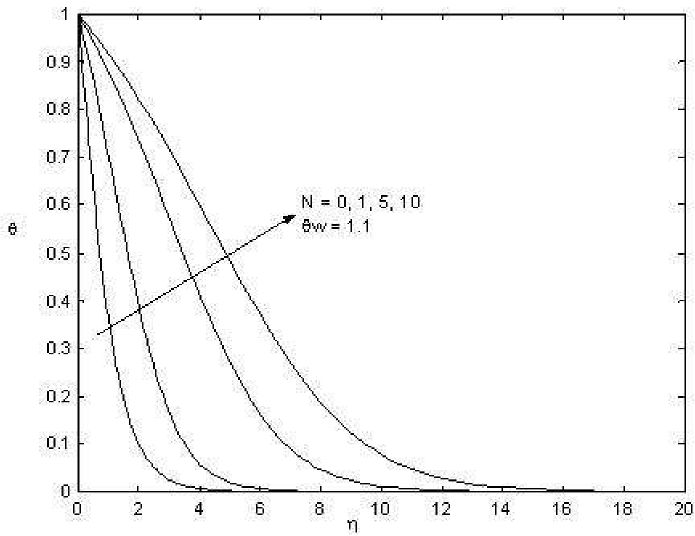

FIGURE 2. Dimensionless temperature profiles for $N = 0, 1, 5, 10$ and $\theta_w = 1.1$

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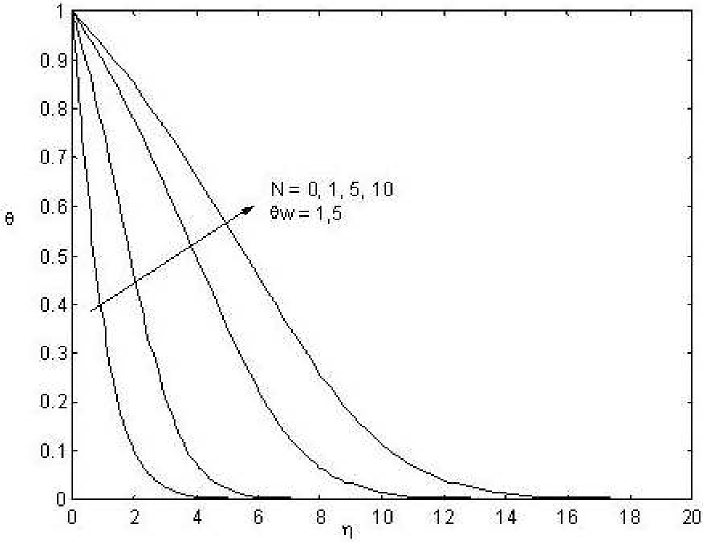

FIGURE 3. Dimensionless temperature profiles for $N = 0, 1, 5, 10$ and $\theta_w = 1.5$

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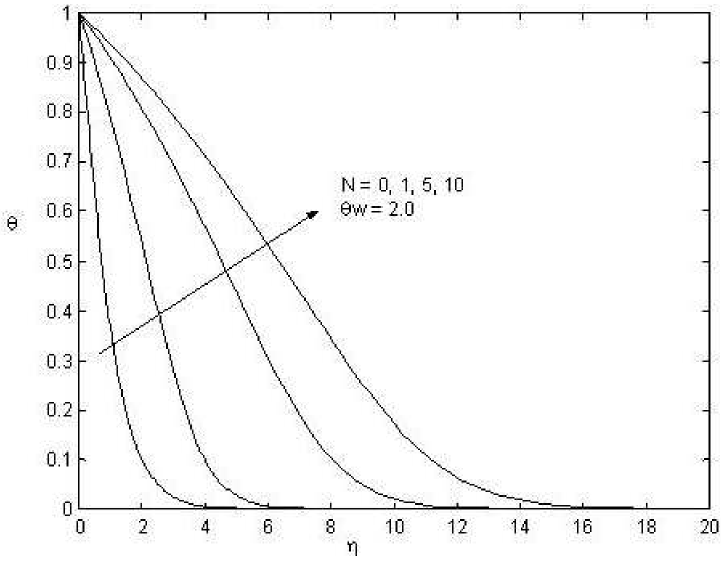

FIGURE 4. Dimensionless temperature profiles for $N = 0, 1, 5, 10$ and $\theta_w = 2$

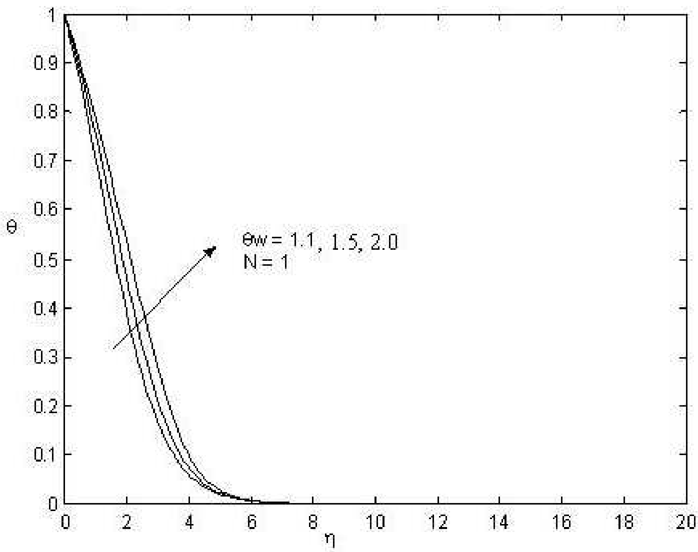

FIGURE 5. Dimensionless temperature profiles for $N = 1$ and $\theta_w = 1.1, 1.5, 2$

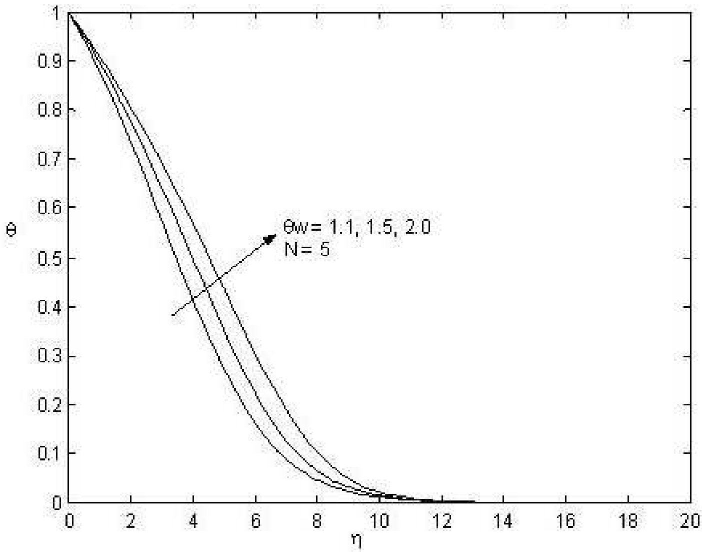

FIGURE 6. Dimensionless temperature profiles for $N = 5$ and $\theta_w = 1.1, 1.5, 2$

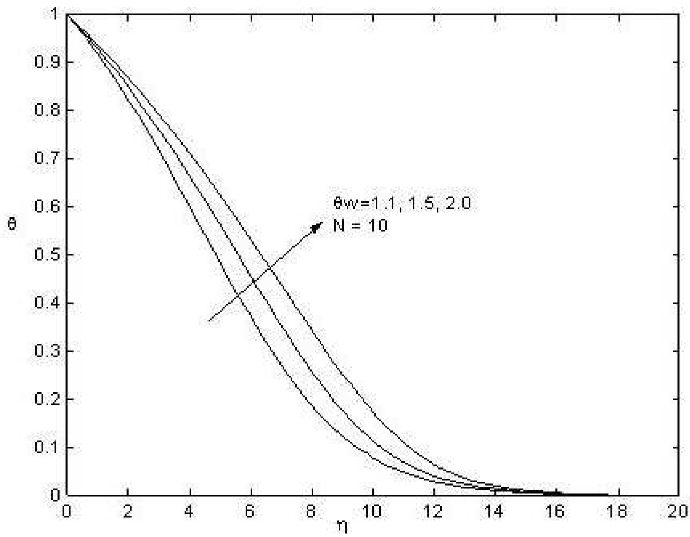

FIGURE 7. Dimensionless temperature profiles for $N = 10$ and $\theta_w = 1.1, 1.5, 2$

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STUDIA UNIV. "BABEŞ–BOLYAI", MATHEMATICA, Volume **L**, Number 1, March 2005

ASYMPTOTIC PROPERTIES OF THE DISCRETIZED PANTOGRAPH EQUATION

PETR KUNDRÁT

Abstract. We are going to deal with the asymptotic properties of all solutions of the delay difference equation

$$\Delta x_n = -ax_n + bx_{\left\lfloor \frac{\tau(t_n)-t_0}{h} \right\rfloor}, \qquad n = 0, 1, 2, \dots ,$$

where $a > 0$, $b \neq 0$ are reals. This equation represents the discretization of the corresponding delay differential equation. Our aim is to show the resemblance in the asymptotic bounds of solutions of the discrete and continuous equation and discuss some numerical problems connected with this investigation.

1. **Introduction**

We discuss the numerical discretization of the delay differential equation

$$\dot{x}(t) = -ax(t) + bx(\tau(t)), \qquad t \in I := [t_0, \infty) \tag{1}$$

in the form

$$x_{n+1} - x_n = -ahx_n + bhx_{\tau_n}, \tag{2}$$

where $a > 0$, $b \neq 0$ are reals, $\tau_n := \left\lfloor \frac{\tau(t_n)-t_0}{h} \right\rfloor$, $t_n := t_0 + nh$, $n = 0, 1, 2, \dots$, $h > 0$ is the stepsize and the symbol $\lfloor\ \rfloor$ is an integer part. Then x_n means the approximation of $x(t_n)$.

Equation (2) is a difference equation obtained from (1) via the modified Euler method. It has been shown in [2] that numerical schema (2) is convergent. Our aim

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Key words and phrases. Differential equation, difference equation.

is to describe some asymptotic properties of equation (2) (more precisely, to find conditions under which asymptotic behaviour of (1) and (2) is similar).

We especially investigate equations with unbounded lag, i.e. such that $t - \tau(t) \to \infty$ as $t \to \infty$. As the prototype of this equation may serve the so called pantograph equation (i.e. equation (1) with $\tau(t) = qt$, $0 < q < 1$). The name of the equation has its origin in the application on British railways [9], where the motion of pantograph of electric locomotive along trolley wire has been described.

In the connection with the investigation of asymptotic properties of solutions of these equations we recall papers dealing with relative problems, e.g., Čermák [1], Heard [4], Iserles [5], Liu [7], Kato and McLeod [6] and many others in the continuous case, and Györi and Pituk [3], Makay and Terjéki [8], Péics[10] and others in the discrete case.

The paper is organized as follows. In the next section we recall the asymptotic estimate of all solutions of (1) (valid under certain assumptions). In Section 4 we derive the analogous asymptotic estimate valid for all solutions of difference equation (2).

2. **Continuous case**

In this section we mention the result describing the asymptotics of the investigated delay differential equation.

Theorem 2.1 (Heard [4]**).** *Let $a > 0$, $b \neq 0$ be scalars, $\tau \in C^2(I)$ be such that $\dot{\tau}$ is positive and decreasing on I and $q = \dot{\tau}(t_0) < 1$. Then for any solution x of (1) there exists a continuous periodic function g of period $\log q^{-1}$ such that*

$$x(t) = (\varphi(t))^{\alpha}\, g(\log \varphi(t)) + O\left((\varphi(t))^{\alpha_r - 1}\right) \qquad \text{as} \quad t \to \infty,$$

where φ is a solution of

$$\varphi(\tau(t)) = q\varphi(t), \qquad t \in I, \tag{3}$$

$\alpha = \log\left(b/a\right) \big/ \log q^{-1}$ *and* $\alpha_r = \Re(\alpha)$.

Remark 2.2. *Particularly, it follows from Theorem 2.1 that for any solution x of (1) holds*

$$x(t) = O\left(\psi(t)\right) \qquad \text{as} \quad t \to \infty,$$

where $\psi(t) = \left(\varphi(t)\right)^{\alpha_r}$ is a solution of the functional equation

$$a\psi(t) = |b|\psi(\tau(t)), \qquad t \in I. \tag{4}$$

3. **Preliminaries**

In this section we summarize the assumptions necessary to formulate the result for discrete case. First, let us denote (H) the assumptions on function τ:

(H): Let τ be an increasing continuous function on I such that $\tau(t) < t$ for all $t \in I$ (the case $\tau(t_0) = t_0$ is also possible), $\tau(t + \tilde{h}) - \tau(t)$ is nonincreasing for arbitrary $\tilde{h}$ fulfilling $0 < \tilde{h} \leq h$ on I and let $\lim_{t\to\infty} \tau(t) = \infty$.

Further, throughout this paper we denote $T_{-1} = \tau(t_0)$ and $T_k = \tau^{-k}(t_0)$, $k = 0, 1, 2, \ldots$, where τ^{-k} means the k-th iteration of the inverse τ^{-1}. If we set $I_m := [T_{m-1}, T_m]$ for all $m = 0, 1, 2, \ldots$, then τ is mapping I_{m+1} onto I_m.

Instead of the above functional equation (4) we consider the functional inequality

$$a\rho(t) \geq |b|\rho(t_0 + \left\lfloor \frac{\tau(t) - t_0}{h} \right\rfloor h), \qquad t \in I. \tag{5}$$

Now we can formulate the proposition ensuring some required properties of solutions of the inequality (5).

Proposition 3.1. *Consider the inequality (5), where $a > 0$, $b \neq 0$ are reals and let (H) be fulfilled.*

- **(i):** *If $|b|/a \geq 1$, then there exists a positive continuous nondecreasing solution ρ of inequality (5).*
- **(ii):** *If $|b|/a < 1$, then there exists a positive continuous decreasing solution ρ of inequality (5) such that $\rho(t+\tilde{h}) - \rho(t)$ is nondecreasing on I for arbitrary real $0 < \tilde{h} \leq h$.*

Proof. Using the step method we can easily verify that there exists a positive continuous solution $\rho(t)$ of (5) which is nondecreasing or decreasing according to $|b|/a \geq 1$ or $|b|/a < 1$, respectively.

Further, we assume that $|b|/a < 1$ and show that the function $\rho(t+\tilde{h}) - \rho(t)$ is nondecreasing on I for all $0 < \tilde{h} \leq h$. It is easy to check that any solution of functional equation

$$a\rho(t) = |b|\rho(\tau(t) - h) \tag{6}$$

is fulfilling the inequality (5). We choose the decreasing function ρ defined on the initial interval I_0 such that $\rho(t_0) = (|b|/a)\rho(T_{-1} - h)$ and let $\rho(t+\tilde{h}) - \rho(t)$ be nondecreasing on I_0. Further let $t^*, t^{**} \in I_1$, $t^* < t^{**}$. If we denote $h^* := \tau(t^* + \tilde{h}) - \tau(t^*)$, $h^{**} := \tau(t^{**} + \tilde{h}) - \tau(t^{**})$, then $h^* \geq h^{**}$ and we can write

$$\begin{aligned}
&\rho(t^* + \tilde{h}) - \rho(t^*) = \tfrac{|b|}{a}\Big(\rho(\tau(t^* + \tilde{h}) - h) - \rho(\tau(t^*) - h)\Big)\\
&= \tfrac{|b|}{a}\Big(\rho(\tau(t^*) + h^* - h) - \rho(\tau(t^*) - h)\Big)\\
&\leq \tfrac{|b|}{a}\Big(\rho(\tau(t^{**}) + h^* - h) - \rho(\tau(t^{**}) - h)\Big)\\
&\leq \tfrac{|b|}{a}\Big(\rho(\tau(t^{**}) + h^{**} - h) - \rho(\tau(t^{**}) - h)\Big)\\
&= \tfrac{|b|}{a}\Big(\rho(\tau(t^{**} + \tilde{h}) - h) - \rho(\tau(t^{**}) - h)\Big) = \rho(t^{**} + \tilde{h}) - \rho(t^{**})
\end{aligned}$$

by use of the assumptions of proposition. Thus $\rho(t+\tilde{h}) - \rho(t)$ is nondecreasing on $I_0 \cup I_1$ and repeating this procedure for intervals $I_2, I_3, \dots$ we obtain that the function $\rho(t+\tilde{h}) - \rho(t)$ is nondecreasing on I. □

4. **Main result**

Theorem 4.1. *Let x_n, $n = 0, 1, 2, \dots$ be a solution of (2), where $0 < ah < 1$, $b \neq 0$ are reals. Let (H) be fulfilled, let ρ be a positive solution of (5) with the properties guaranteed by Proposition 3.1 and let $\rho_n := \rho(t_n)$.*

(i): *If $|b|/a \geq 1$, then $x_n = O(\rho_n)$ as $n \to \infty$.*

(ii): *If $|b|/a < 1$ and moreover*

$$\sum_{k=1}^{\infty} \frac{\rho(T_{k-1}) - \rho(T_{k-1} + h)}{\rho(T_{k+1})} < \infty,$$

then $x_n = O(\rho_n)$ as $n \to \infty$.

Proof. First we rewrite the difference equation (2) as

$$x_{n+1} = \tilde{a}^h x_n + hbx_{\tau_n}, \quad n = 1, 2, 3, \dots, \tag{7}$$

where $\tilde{a}$ is a (unique) positive real such that $\tilde{a}^h = 1 - ah$.

We introduce the substitution $y_n = x_n/\rho_n$ in (7) to obtain

$$\rho_{n+1} y_{n+1} = \tilde{a}^h \rho_n y_n + bh\rho_{\tau_n} y_{\tau_n}, \quad n = 1, 2, 3, \dots \tag{8}$$

and show that every solution y_n of (8) is bounded as $n \to \infty$. Multiplying the previous equality by $1/\tilde{a}^{t_n+h}$ we get

$$\frac{\rho_{n+1} y_{n+1}}{\tilde{a}^{t_n+h}} = \frac{\rho_n y_n}{\tilde{a}^{t_n}} + \frac{bh}{\tilde{a}^{t_n+h}} \rho_{\tau_n} y_{\tau_n},$$

i.e.,

$$\Delta \left(\frac{\rho_n y_n}{\tilde{a}^{t_n}} \right) = \frac{bh}{\tilde{a}^{t_n+h}} \rho_{\tau_n} y_{\tau_n}. \tag{9}$$

Now we take any $\bar{t} \in I_{m+1}$, $m = 1, 2, \dots$. We define nonnegative integers $k_m(\bar{t}) := \lfloor (\bar{t} - T_m)/h \rfloor$. Denote $\bar{t}_m := \bar{t} - k_m(\bar{t})h - h$. Summing the equation (9) from $\bar{t}_m$ to $\bar{t} - h$, we get

$$y(\bar{t}) = \frac{\rho(\bar{t}_m)\tilde{a}^{\bar{t}}}{\rho(\bar{t})\tilde{a}^{\bar{t}_m}} y(\bar{t}_m) + \frac{\tilde{a}^{\bar{t}}}{\rho(\bar{t})} \sum_{s=\bar{t}_m}^{\bar{t}-h} \frac{bh}{\tilde{a}^{s+h}} \rho_{\tau_s} y_{\tau_s}.$$

Let us denote $M_m := \sup \left\{ |y(t)|, \quad t \in \bigcup_{j=0}^{m} I_j \right\}$. In accordance with (5) we obtain

$$|y(\bar{t})| \le \frac{\rho(\bar{t}_m)\tilde{a}^{\bar{t}}}{\rho(\bar{t})\tilde{a}^{\bar{t}_m}} M_m + \frac{\tilde{a}^{\bar{t}}}{\rho(\bar{t})} \sum_{s=\bar{t}_m}^{\bar{t}-h} \frac{(1-\tilde{a}^h)\rho_s}{\tilde{a}^{s+h}} M_m.$$

Using the relation $\frac{(1-\tilde{a}^h)}{\tilde{a}^{s+h}} = \Delta \left(\frac{1}{\tilde{a}}\right)^s$ we get

$$|y(\bar{t})| \le \frac{\rho(\bar{t}_m)\tilde{a}^{\bar{t}}}{\rho(\bar{t})\tilde{a}^{\bar{t}_m}} M_m + \frac{\tilde{a}^{\bar{t}}}{\rho(\bar{t})} \sum_{s=\bar{t}_m}^{\bar{t}-h} \left(\Delta \left(\frac{1}{\tilde{a}} \right)^s \right) \rho_s M_m$$

and summing by parts we finally have

$$\begin{aligned} |y(\bar{t})| &\leq M_m \left\{ \frac{\rho(\bar{t}_m)\tilde{a}^{\bar{t}}}{\rho(\bar{t})\tilde{a}^{\bar{t}_m}} + \frac{\tilde{a}^{\bar{t}}}{\rho(\bar{t})} \left(\frac{\rho(\bar{t})}{\tilde{a}^{\bar{t}}} - \frac{\rho(\bar{t}_m)}{\tilde{a}^{\bar{t}_m}} - \sum_{s=\bar{t}_m}^{\bar{t}-h} \left(\frac{1}{\tilde{a}}\right)^{s+h} \Delta\rho_s \right) \right\} \\ &= M_m \left\{ 1 - \frac{\tilde{a}^{\bar{t}}}{\rho(\bar{t})} \sum_{s=\bar{t}_m}^{\bar{t}-h} \left(\frac{1}{\tilde{a}}\right)^{s+h} \Delta\rho_s \right\}. \end{aligned} \tag{10}$$

The common part of the proof ends here and we continue for the cases (i) and (ii) separately.

ad (i): If $|b|/a \geq 1$, then in accordance with Proposition 3.1 we choose a non-decreasing function $\rho(t)$ on I. Then $\Delta\rho(t)$ is nonnegative on I, hence $|y(\bar{t})| \leq M_m$. Since $\bar{t} \in I_{m+1}$ was arbitrary, we have $M_{m+1} \leq M_m$, i.e., M_m is bounded as $m \to \infty$. Hence the function $y(t)$ is bounded and the statement (i) is proved.

ad (ii): If $|b|/a < 1$, then in accordance with Proposition 3.1 we choose a decreasing function $\rho(t)$ on I such that $\Delta\rho(t)$ is nondecreasing on I. Then from (10) we have

$$\begin{aligned} |y(\bar{t})| &\leq M_m \left\{ 1 + \frac{\rho(\bar{t}_m) - \rho(\bar{t}_m + h)}{\rho(\bar{t})} \sum_{s=\bar{t}_m}^{\bar{t}-h} \left(\frac{\tilde{a}^{\bar{t}}}{\tilde{a}^{s+h}} \right) \right\} \\ &\leq M_m \left\{ 1 + \frac{\rho(\bar{t}_m) - \rho(\bar{t}_m + h)}{\rho(\bar{t})} \xi \right\} \leq M_m \left\{ 1 + \xi \frac{-\Delta\rho(T_m - h)}{\rho(T_{m+1})} \right\}, \end{aligned}$$

where $\xi := 1/(1 - \tilde{a}^h)$. The repeated application of this procedure yields

$$|y(\bar{t})| \leq M_1 \prod_{j=1}^{m} \left(1 + \xi \frac{-\Delta\rho(T_j - h)}{\rho(T_{j+1})} \right),$$

i.e.,

$$M_{m+1} \leq M_1 \prod_{j=1}^{m} \left(1 + \xi \frac{-\Delta\rho(T_j - h)}{\rho(T_{j+1})} \right).$$

By our assumption, the product converges as $m \to \infty$, hence M_m is bounded as $m \to \infty$. The theorem is proved. □

Remark 4.2. The assumption on the stepsize h ($h < 1/a$) enables us to preserve the correlation of asymptotic estimates of discrete and continuous case. In other words, the estimates of solutions in the discrete case and the continuous case are expressed via the same function, resp. sequence (provided the stepsize h is sufficiently small).

Remark 4.3. In the estimate concerning the case $|b|/a < 1$ it is also possible to take a solution ψ of functional equation (4) instead of a function ρ (which is a solution of (5)). Using the fact that the term $\rho(\tau(t)) - \rho(\tau(t) - h)$ is a positive nonincreasing function it could be shown that there exists a solution ψ of (4) such that $\psi(t) > \rho(t)$ for all $t > t_0$. In some cases the utilizing of ψ instead of ρ can be more applicable.

5. Examples

Corollary 5.1. Consider the scalar pantograph equation

$$\dot{x}(t) = -ax(t) + bx(qt), \tag{11}$$

where $a > 0$, $b \neq 0$, $0 < q < 1$ are reals. The qualitative theory yields the estimate

$$x(t) = O(t^r), \qquad r = \frac{\log \frac{|b|}{a}}{\log q^{-1}}, \qquad \text{as} \quad t \to \infty \tag{12}$$

for every solution x of the equation (11). The corresponding difference equation is

$$x_{n+1} = (1 - ah)x_n + bhx_{\lfloor \frac{qt_n - t_0}{h} \rfloor}, \qquad t \geq t_0 > 0, \tag{13}$$

where the above assumptions on a, b, q are fulfilled and $0 < ah < 1$. Then the following estimate

$$x_n = O(t_n^r), \quad r = \frac{\log \frac{|b|}{a}}{\log q^{-1}} \qquad \text{as} \quad n \to \infty \tag{14}$$

is valid for all solutions $\{x_n\}_{n=0}^{\infty}$ of difference equation (13).

Example 5.2. Consider the initial problem:

$$\dot{x}(t) = -2x(t) + x(t/2), \quad x(0) = 1, \quad t \in [0, \infty). \tag{15}$$

In accordance with (12) we get the asymptotic estimate

$$x(t) = O(1/t) \qquad \text{as} \quad t \to \infty.$$

In the corresponding discrete case we consider formula (13) in the form

$$x_{n+1} = (1 - 2h)x_n + hx_{\lfloor t_n/2h \rfloor}, \qquad t_0 = 0, \qquad x_0 = 1. \tag{16}$$

Then $x_n = O(1/t_n)$ as $t \to \infty$ provided $h < 1/a$. If we violate the condition on stepsize, this asymptotic formula is not valid. Indeed, if $h = 1 > 1/a$, then the corresponding discrete equation $x_{n+1} = -x_n + x_{\lfloor t_n/2 \rfloor}$ admits solutions not tending to zero as $n \to \infty$.

It is obvious, that the assumption $0 < ah < 1$ has its relevance in the choice of suitable stepsize h to preserve the same behaviour of difference case as in the continuous case.

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CLASSIFICATION OF NEAR EARTH ASTEROIDS WITH ARTIFICIAL NEURAL NETWORK

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Abstract. Asteroids that can pass inside the orbit of Mars are said to be Near-Earth Asteroids (NEAs) or Earth-Approaching asteroids. The NEAs are subdivided into several groups based on their orbital characteristics. There are three important groups: Amor, Apollo and Aten. In this paper we show that fundamental characteristics of this classification, for which these groups are linear separable, are the focal distance and the semimajor axis. Starting from this property we construct a perceptron-type artificial neural network to classify automatically these objects into groups Amor, Apollo or Aten.

1. Introduction

Asteroids are rocky and metallic objects that orbit the Sun but are too small to be considered planets. They are also known as minor planets. Asteroids are divided into groups and families based on their orbital characteristics. Usually a group of asteroids is named after the first discovered member of the group. These groups are relatively loose dynamical associations.

Asteroids that can pass inside the orbit of Mars are known as Near-Earth Asteroids (NEAs) or Earth-Approaching asteroids. Rigorously NEAs are the asteroids with the perihelion distance $q < 1.3$ AU and the aphelion distance $Q > 0.983$ AU (see [2]). These asteroids probably came from the main asteroid belt, but were jolted from the belt by collisions or by interactions with the gravitational fields of other objects (primarily Jupiter). According to astronomers there are at least 1,000 NEAs

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whose diameter is greater than 1 kilometer and which could do catastrophic damage to the Earth (see for example [6], [1]). Even smaller NEAs could cause substantial destruction if they were to collide with the Earth.

From the point of view of the geometry of the orbit, there are four types of NEAs ([5], [4]):

1. The group of Aten asteroids (Atens) was named after 2062 Aten (discovered by E. F. Helin in 1976, with semimajor axis $a = 0.967$ AU, eccentricity $e = 0.183$ and inclination $i = 18^\circ.9$) . They have semimajor axes less than 1 AU and aphelion distance greater than or equal to 0.983 AU (the present perihelion distance of the Earth), namely

$$a(1 + e) \geq 0.983 \text{ AU and } a < 1 \text{ AU}, \tag{1}$$

placing them inside the Earth's orbit.

2. The Apollos, named after 1862 Apollo (K. Reinmuth, 1932, $a = 1.471$ AU, $e = 0.560$, $i = 6^\circ.4$), have semimajor axes greater than or equal to 1 AU and perihelion distances less than or equal to 1.017 AU (the present aphelion distance of the Earth), namely

$$a \geq 1 \text{ AU and } a(1 - e) \leq 1.017 \text{ AU}. \tag{2}$$

Some Apollos have eccentric orbits that cross the orbit of the Earth, making them a potential threat to our planet.

3. The Amors, named after the asteroid 1221 Amor (E.J. Delporte, 1932, $a = 1.920$ AU, $e = 0.435$ and $i = 11^\circ.9$), have perihelion distances between 1.017 AU and 1.3 AU (the present perihelion distance of the Mars), namely

$$1.017 \text{ AU} < a(1 - e) < 1.3 \text{ AU}. \tag{3}$$

Amors often cross the orbit of Mars (if the orbit is eccentric enough), but they do not cross the orbit of Earth.

4. Inside of the orbit of the Earth, with perihelion distances less then 0.983 AU, orbit the Apoheles, for which

$$a(1 + e) < 0.983 \text{ AU}. \tag{4}$$

"Apohele" is Hawaiian for "orbit". Other proposed names for this group are Inner-Earth Objects (IEOs) and Anons (as in "Anonymous"). Until May 2004 there are only two known Apoheles: 2003 CP20 and 2004 JG6.

In the presented classification (1–4) the semimajor axis (a) and the eccentricity (e) are used as fundamental parameters. In the plane of these parameters the separatrices between the different groups are mainly hyperbolas (see Figure 1). Generally is more convenient, if the separatrices of the groups are linear. Linear separatrices make possible for example the use of linear statistical methods in the different studies, and simple classification with a parallel computing artificial neural network.

In this paper we point out that if the semimajor axis and the focal distance, $c = ea$ are used as fundamental parameters, then the separatrices between the presented classical groups are all linear. Using this property we constructed a parallel computing artificial neural network to classify the NEAs.

2. Linear separation of the NEA groups

The different groups of asteroids (Atens, Apollos, Amos, Apoheles and other asteroids), defined in the above presented classification (1–4), are separated – in the plane of parameters (a, e) –, by curves of equation (see Figure 1)

$$f_i(a, e) = 0,\ i = 1, 2, 3, 4, \tag{5}$$

where functions $f_i : (0, \infty) \times [0, 1) \to \mathbb{R}$ are

$$\begin{aligned} f_1(a, e) &= a + ae - 0.983, \qquad f_2(a, e) = a - 1, \\ f_3(a, e) &= -a + ae + 1.017, \quad f_4(a, e) = a - ae - 1.3. \end{aligned}$$

The presence of the focal distance $c = ea$ in the hyperbolas $f_1 = 0, f_2 = 0$ and $f_4 = 0$ suggests us to transform the plane of parameters (a, e) in the plane (a, c). The corresponding function of transformation is $T : (0, \infty) \times [0, 1) \to (0, \infty) \times [0, \infty)$, given by

$$T(a, e) = (a, ea). \tag{6}$$

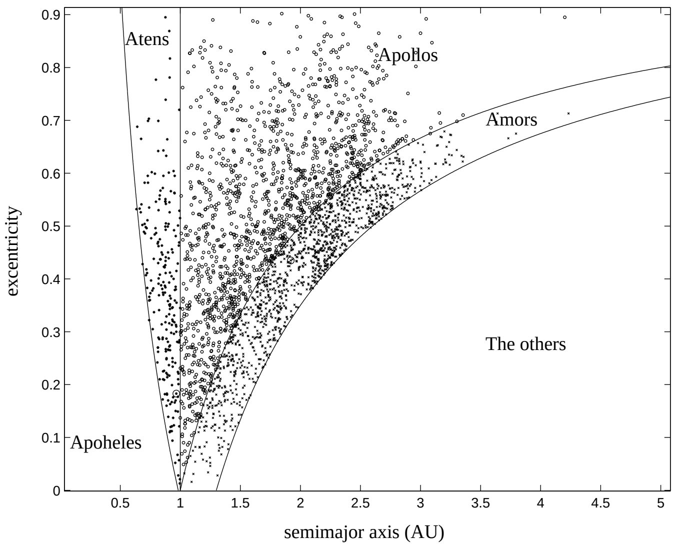

FIGURE 1. Distribution of NEAs in the (a, e) plane.

The four separatrices $f_i = 0$ are transformed by T in linear separatrices $g_i = 0$, $i = 1, 2, 3, 4$, where functions $g_i : (0, \infty) \times [0, \infty) \to \mathbb{R}$ are

$$\begin{aligned} g_1(a,c) &= a + c - 0.983, \qquad g_2(a,c) = a - 1, \\ g_3(a,c) &= -a + c + 1.017, \quad g_4(a,c) = a - c - 1.3. \end{aligned}$$

The distribution of the above defined groups of NEAs, in the plane of parameters (a, c), is illustrated in Figure 2.

In this plane the separatrices are all linear. The characterization of different groups by using the sign of the g_i separator functions is presented in Table 1 (here "+" means positive value or zero and "−" means negative value).

3. Classification of NEAs with artificial neural network

Artificial neural networks are composed of simple elements (*neurons*) operating in parallel. These elements are inspired by biological nervous systems. As in

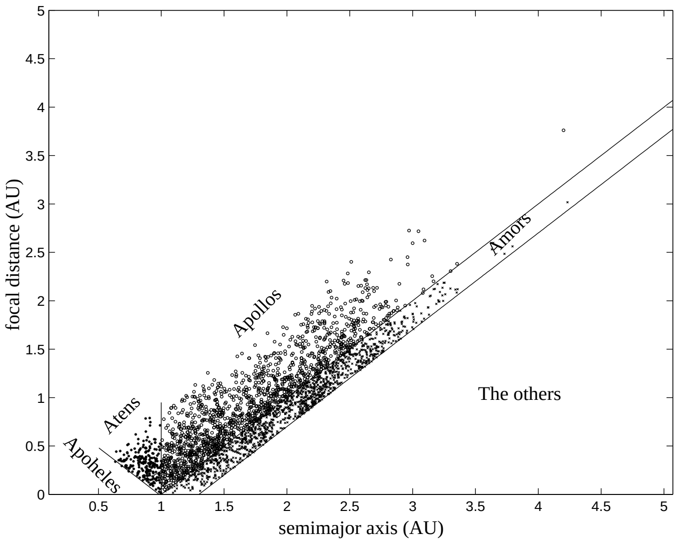

FIGURE 2. Distribution of NEAs in the (a, c) plane.

TABLE 1. The signs of the NEA groups in the plane (a, c).

Group	g_1	g_2	g_3	g_4
Apoheles	−	−	+	−
Atens	+	−	+	−
Apollos	+	+	+	−
Amors	+	+	−	−
others	+	+	−	+

nature, the network function is determined largely by the connections between elements. We can train a neural network to perform a classification by adjusting the values of the connections (*weights*) between elements ([3]).

A neuron with an input vector (a_1, a_2) and with bias scalar b appears on the Figure 3.

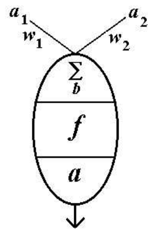

FIGURE 3. The neuron.

This neuron, denoted by a, transmits the input vector (a_1, a_2) to output scalar o, by using:

$$o = f\left(w_1 a_1 + w_2 a_2 + b\right), \tag{7}$$

where $f : \mathbb{R} \to \mathbb{R}$ is the *transfer function*, (w_1, w_2) is the *weights vector* and b is the *bias scalar*. Here f is a hardlim or a linear transfer function. Note that $\mathbf{w}$ and b are both adjustable parameters of the neuron.

The perceptron-type neural network, developed to classify exactly the NEAs in the above defined five groups is presented in Figure 4.

The transfer functions of our network is the hardlim function

$$H : \mathbb{R} \to \{-1, 1\}, \qquad H(x) = \begin{cases} 1, & \text{if } x \geq 0, \\ -1, & \text{if } x < 0, \end{cases}$$

in the neurons a_1, a_2, a_3 and a_4, and the identity function

$$I : \mathbb{R} \to \mathbb{R}, \qquad I(x) = x$$

in the neuron a_5 (Figure 4).

The input values of our network are the semimajor axis $a \in (0, \infty)$, and the focal distance $c = ea \in [0, \infty)$. The output value of this network is computed by formula

$$o = \frac{1}{2}\left[H\left(g_1(a,c)\right) + H\left(g_2(a,c)\right) - H\left(g_3(a,c)\right) + H\left(g_4(a,c)\right)\right] + 3. \tag{8}$$

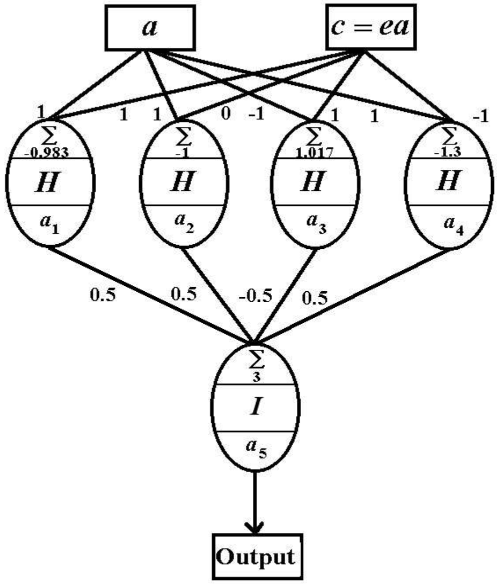

FIGURE 4. The neural network.

This value is: 1 for Apoheles, 2 for Atens, 3 for Apollos, 4 for Amors, and 5 for other asteroids.

4. **Conclusions**

In this study we proved that in the classification of the NEAs is more convenient to use as parameters the semimajor axis and the focal distance instead of the semimajor axis and eccentricity, because in this plane of parameters (a,c) the separatrices between different NEA groups are linear. This parameters make also possible the development of an artificial neural network to perform a parallel computing classification of NEAs. Another advantage of our linear classification is that it can be easily compared with other linear classifications.

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AN APPLICATION OF BRIOT-BOUQUET DIFFERENTIAL SUPERORDINATIONS AND SANDWICH THEOREM

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Abstract. Let $f \in A$. We consider the following integral operator

$$F(z) = \frac{2}{z}\int_0^z f(t)dt. \qquad (1)$$

By using this integral operator we obtain a Briot-Bouquet differential superordination and sandwich theorem.

1. Introduction

Let $\mathcal{H}(U)$ denote the class of functions analytic in the unit disc

$$U = \{z \in \mathbb{C}, |z| < 1\}.$$

For n a positive integer and $a \in \mathbb{C}$, let

$$\mathcal{H}[a,n] = \{f \in H(U) : \ f(z) = a + a_n z^n + a_{n+1}z^{n+1} + \dots, \ z \in U\},$$

and $A_n = \{f \in \mathcal{H}(U), \ f(z) = z + a_{n+1}z^{n+1} + \dots, \ z \in U\}$ with $A_1 = A$.

A function $f \in \mathcal{H}[a,n]$ is convex in U if it is univalent and $f(U)$ is convex. It is well known that f is convex if and only if $f'(0) \neq 0$ and

$$\text{Re}\,\frac{zf''(z)}{f'(z)} + 1 > 0, \quad z \in U.$$

Let Q denote the set of functions f that are analytic and injective on the set $\overline{U} \setminus E(f)$, where

$$E(f) = \left\{\zeta \in \partial U, \ \lim_{z\to\zeta} f(z) = \infty\right\}$$

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and are such that $f'(\zeta) \neq 0$ for $\zeta \in \partial U \setminus E(f)$. The subclass of Q for which $f(0) = a$ is denoted by $Q(a)$.

Many of the inclusion results that follow can be written very easily in terms of subordination and superordination. We recall these definitions. Let $f, F \in \mathcal{H}(U)$ and let F be univalent in U. The function F is said to be superordinate to f, or f is subordinate to F, written $f \prec F$, if $f(0) = F(0)$ and $f(U) \subset F(U)$.

Let β and γ be complex numbers. Let Ω_2 and Δ_2 be sets in the complex plane, and let p be analytic in the unit disc U. In a series of articles the authors and many others [1, p. 80-119] have determined conditions so that

$$\left\{p(z) + \frac{zp'(z)}{\beta p(z) + \gamma} \mid z \in U\right\} \subset \Omega_2 \ \Rightarrow \ p(U) \subset \Delta_2. \tag{2}$$

The differential operator on the left is known as the Briot-Bouquet differential operator. The main concern in this subject is to find the smallest set Δ_2 in $\mathbb{C}$ for which (2) holds.

In [2] the authors consider the dual problem of determining conditions so that

$$\Omega_1 \subset \left\{p(z) + \frac{zp'(z)}{\beta p(z) + \gamma} \mid z \in U\right\} \ \Rightarrow \ \Delta_1 \subset p(U). \tag{3}$$

In particular we are interested in determining the largest set Δ_1 in $\mathbb{C}$ for which (3) holds.

If the sets Ω and Δ in (2) and (3) are simply connected domains not equal to $\mathbb{C}$, then it is possible to rephrase these expressions very neatly in terms of subordination and superordination in the forms:

$$p(z) + \frac{zp'(z)}{\beta p(z) + \gamma} \prec h_2(z) \ \Rightarrow \ p(z) \prec q_2(z) \tag{2'}$$

$$h(z) \prec p(z) + \frac{zp'(z)}{\beta p(z) + \gamma} \ \Rightarrow \ q_1(z) \prec p(z). \tag{3'}$$

The left side of (2′) is called a Briot-Bouquet differential subordination, and the function q_2 is called a dominant of the differential subordination. The best dominant which provides a sharp result, is the dominant that is subordinate to all other dominants.

In a recent paper [3] the authors introduced the dual concept of a differential superordination. In light of those results we call the left side of (3′) a Briot-Bouquet differential superordination, and the function q, is called a subordinant of the differential superordination. The best subordinant, which provides a sharp result is the subordinant which is superordinate to all other subordinants.

In [3] the authors combine (2′) and (3′) and obtain a condition so that the Briot-Bouquet sandwich

$$h_1(z) \prec p(z) + \frac{zp'(z)}{\beta p(z) + \gamma} \prec h_2(z) \tag{4}$$

implies that $q_1(z) \prec p(z) \prec q_2(z)$.

In order to prove the new results we shall use the following lemma:

Lemma A. [3, Corollary 1.1] *Let $\beta, \gamma \in \mathbb{C}$ and let h be convex in U, with $h(0) = a$. Suppose that the differential equation*

$$q(z) + \frac{zq'(z)}{\beta q(z) + \gamma} = h(z)$$

has a univalent solution q that satisfies $q(0) = a$ and $q(z) \prec h(z)$. If $p \in \mathcal{H}[a, 1] \cap Q$ and $p(z) + \dfrac{zp'(z)}{\beta p(z) + \gamma}$ is univalent in U, then

$$h(z) \prec p(z) + \frac{zp'(z)}{\beta p(z) + \gamma}$$

implies

$$q(z) \prec p(z).$$

The function q is the best subordinant.

Lemma B. [1, Th. 3.2.b, p. 83] *Let h be a convex function in U, with $h(0) = a$ and let n be a positive integer. Suppose that the Briot-Bouquet differential equation*

$$q(z) + \frac{nzq'(z)}{q(z) + 1} = h(z)$$

has a univalent solution q that satisfies $q(z) \prec h(z)$.

If $p \in \mathcal{H}[a, n]$ satisfies

$$p(z) + \frac{zp'(z)}{p(z) + 1} \prec h(z)$$

then $p(z) \prec q(z)$ *and* q *is the best* (a,n) *dominant.*

2. **Main results**

Theorem 1. *Let* $R \in (0,1]$ *and let* h *be convex in* U*, with* $h(0)=1$*, defined by*

$$h(z) = 1 + Rz + \frac{zR}{2+Rz}, \quad z \in U.$$

If $f \in A$ *and* $\frac{zf'(z)}{f(z)}$ *is univalent,* $\frac{zF'(z)}{F(z)} \in \mathcal{H}[1,1] \cap Q$ *and*

$$h(z) \prec \frac{zf'(z)}{f(z)}, \quad z \in U \tag{5}$$

then

$$q(z) = 1 + Rz \prec \frac{zF'(z)}{F(z)}, \quad z \in U,$$

where F *is given by (1).*

The function q *is the best subordinant.*

Proof. In [4] the authors have showed that

$$h(z) = 1 + Rz + \frac{zR}{2+Rz}, \quad R \in (0,1] \tag{6}$$

is convex, and $q(z) = 1 + Rz$ is a univalent solution of (3) which satisfies $q(0) = 1$ and $q(z) \prec h(z)$, $z \in U$.

From (1) we have

$$zF(z) = 2\int_0^z f(t)dt, \quad z \in U.$$

By using the derivative of this equality, with respect to z, after a short calculation, we obtain

$$zF'(z) + F(z) = 2f(z).$$

This equality is equivalent to

$$F(z)\left[1 + \frac{zF'(z)}{F(z)}\right] = 2f(z), \quad z \in U. \tag{7}$$

If we let

$$p(z) = \frac{zF'(z)}{F(z)}, \tag{8}$$

then (7) becomes

$$F(z)[1+p(z)] = 2f(z), \quad z \in U. \tag{9}$$

By using the derivative of (9) with respect to z, after a short calculation, we obtain

$$\frac{zF'(z)}{F(z)} + \frac{zp'(z)}{1+p(z)} = \frac{zf'(z)}{f(z)}$$

which, using (8), is equivalent to

$$p(z) + \frac{zp'(z)}{1+p(z)} = \frac{zf'(z)}{f(z)}.$$

Using (5) we have

$$1 + Rz + \frac{Rz}{2+Rz} \prec p(z) + \frac{zp'(z)}{1+p(z)}, \quad z \in U.$$

By using Lemma A we deduce that

$$q(z) \prec p(z) = \frac{zF'(z)}{F(z)}, \quad 1 + Rz \prec \frac{zF'(z)}{F(z)}.$$

Theorem 2. *Let h be convex in U, with $h(0) = 1$, defined by*

$$h(z) = 1 + z + \frac{z}{z+2}, \quad z \in U.$$

If $f \in A$ and

$$\frac{zf'(z)}{f(z)} \prec h(z), \quad z \in U \tag{10}$$

then

$$\frac{zF'(z)}{F(z)} \prec 1 + z,$$

where F is given by (1). The function $q(z) = 1 + z$ is best dominant.

Proof. In [4] the authors have showed that

$$h(z) = 1 + z + \frac{z}{z+2}$$

is convex.

From (1) we have

$$zF(z) = 2\int_0^z f(t)dt, \quad z \in U.$$

Following the steps from the proof of Theorem 1 we obtain:

$$p(z)+\frac{zp'(z)}{1+p(z)}=\frac{zf'(z)}{f(z)}.$$

Using (10) we have

$$p(z)+\frac{zp'(z)}{1+p(z)}\prec h(z).$$

By applying Lemma B we obtain

$$p(z)=\frac{zF'(z)}{F(z)}\prec q(z)=1+z,\quad z\in U.$$

The function $q(z)=1+z$ is the best dominant.

Using the conditions from Theorem 1 and Theorem 2 we can write the following

Corollary. *If* $f\in A$ *and*

$$1+Rz+\frac{zR}{2+Rz}\prec\frac{zf'(z)}{f(z)}\prec 1+z+\frac{z}{2+z},\quad z\in U$$

then

$$1+Rz\prec\frac{zF'(z)}{F(z)}\prec 1+z,\quad z\in U.$$

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A NUMERICAL METHOD FOR APPROXIMATING THE SOLUTION OF A LOTKA-VOLTERRA SYSTEM WITH TWO DELAYS

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Abstract. In this paper, using the step method, we established the existence and uniqueness of solution for the system (1.2) with initial condition (1.3). The aim of this paper is to present a numerical method for this system.

1. The statement of the problem

Consider the following Lotka-Volterra type delay differential system:

$$\begin{cases} x_i'(t) = x_i(t)r_i(t)\left\{c_i - a_i x_i(t) - \sum\limits_{j=1}^{n}\sum\limits_{k=0}^{m} a_{ij}^k x_j(\tau_{ij}^k(t))\right\},\ t \geq t_0,\ 1 \leq i \leq n \\ x_i(t) = \phi_i(t) \geq 0,\ t \leq t_0 \text{ and } \phi_i(t_0) > 0,\ 1 \leq i \leq n \end{cases} \tag{1}$$

There have been many studies on this subject (see [2], [5], [7]). In particular, for $n = 2$, $r_i(t) \equiv 1$, $a_i = 0$ and $\tau_{ij}^k(t) = t - \tau_{ij}^k$, $1 \leq i,j \leq 2$, $0 \leq k \leq m$, the fact that time delays are harmless for the uniform persistence of solutions, is established by Wang and Ma for a predator-prey system, by Lu and Takeuchi and Takeuchi for competitive systems.

Recently, Saito, Hara and Ma [7] have derived necessary and sufficient conditions for the permanence (uniform persistence) and global stability of a symmetrical Lotka-Volterra-type predator-prey system with $a_i > 0$, $i = 1, 2$ and two delays.

For a nonautonomous competitive Lotka-Volterra system with no delays, recently Ahmad and Lazer have established the average conditions for the persistence,

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which are weaker than those of Gopalsamy and Tineo and Alvarez for periodic or almost-periodic cases.

In this paper, using the step method [6], we established the existence and uniqueness of solution for the following system

$$\begin{cases} x'(t) = f_1(t, x(t), y(t), x(t-\tau_1), y(t-\tau_2)) \\ y'(t) = f_2(t, x(t), y(t), x(t-\tau_1), y(t-\tau_2)) \end{cases}, \ t \in [t_0, b], \ t_0 < b \tag{2}$$

with initial condition

$$\begin{cases} x(t) = \varphi(t), \ t \in [t_0 - \tau_1, t_0] \\ y(t) = \psi(t), \ t \in [t_0 - \tau_2, t_0] \end{cases} \tag{3}$$

Here τ_1 and τ_2 are constants with $\tau_1 \geq 0, \ \tau_2 \geq 0, \ \tau_1 \leq \tau_2$ and φ, ψ are continuous functions.

On the basis of these results, the aim of this paper is to present a numerical method for obtaining the solutions of system (2) with initial condition (3).

2. The existence and uniqueness of solution

We consider the system (2) with initial condition (3) and we established the existence and uniqueness of the solution for the problem (2) + (3).

We have

$$\begin{aligned} x &\in C[t_0 - \tau_1, b] \cap C^1[t_0, b] \\ y &\in C[t_0 - \tau_2, b] \cap C^1[t_0, b] \end{aligned}$$

If we suppose that

(i) $f_i \in C([t_0, b] \times \mathbb{R}^4), \ i = 1, 2$

(ii)$|f_i(t, u_1, v_1, u, v) - f_i(t, u_2, v_2, u, v)| \leq L_1 |u_1 - u_2| + L_2 |v_1 - v_2|,$

$$\forall u_1, u_2, v_1, v_2, u, v \in \mathbb{R}, \forall t \in [t_0, b]$$

(ii')$|f_i(t, u_1, u_2, u_3, u_4) - f_i(t, v_1, v_2, v_3, v_4)| \leq L(|u_1 - v_1| + |u_2 - v_2| + |u_3 - v_3| + |u_4 - v_4|)$

$$\forall u_i, v_i \in \mathbb{R}, i = \overline{1, 4}, \forall t \in [t_0, b]$$

then the following result is given.

Theorem 1. *We consider the system (2) with initial condition (3). If the conditions (i) and (ii) are satisfied, then the problem (2)+(3) has a unique solution.*

Proof. We use the step method.

$t \in [t_0, t_0 + \tau_1]$

$$\begin{cases} x'(t) = f_1(t, x(t), y(t), \varphi(t-\tau_1), \psi(t-\tau_2)) \\ y'(t) = f_2(t, x(t), y(t), \varphi(t-\tau_1), \psi(t-\tau_2)) \\ x(t_0) = \varphi(t_0) \\ y(t_0) = \psi(t_0) \end{cases}$$

So we have the Cauchy problem with f_i continuous functions, $i = 1, 2$. But $f_i(t, \cdot, \cdot, u, v) : \mathbb{R} \to \mathbb{R}$ are Lipschitz. Then it results from the Cauchy theorem that:

$$\begin{aligned} \exists!\, x_1 &\in C^1[t_0, t_0 + \tau_1] \\ \exists!\, y_1 &\in C^1[t_0, t_0 + \tau_1] \end{aligned}$$

solution of the problem (2) + (3).

$t \in [t_0 + \tau_1, t_0 + 2\tau_1]$

$$\begin{cases} x'(t) = f_1(t, x(t), y(t), x_1(t-\tau_1), y_1(t-\tau_2)) \\ y'(t) = f_2(t, x(t), y(t), x_1(t-\tau_1), y_1(t-\tau_2)) \\ x(t_0 + \tau_1) = x_1(t_0 + \tau_1) \\ y(t_0 + \tau_1) = y_1(t_0 + \tau_1) \end{cases}$$

$$\begin{aligned} &\Rightarrow \quad \exists! x_2 \in C[t_0 + \tau_1, t_0 + 2\tau_1] \\ &\Rightarrow \quad \exists! y_2 \in C[t_0 + \tau_1, t_0 + 2\tau_1] \end{aligned}$$

solution of the problem (2) + (3).

$t \in [t_0 + n\tau_1, t_0 + \tau_2]$

$$\begin{cases} x'(t) = f_1(t, x(t), y(t), x_n(t-\tau_1), y_n(t-\tau_2)) \\ y'(t) = f_2(t, x(t), y(t), x_n(t-\tau_1), y_n(t-\tau_2)) \\ x(t_0 + n\tau_1) = x_n(t_0 + n\tau_1) \\ y(t_0 + n\tau_1) = y_n(t_0 + n\tau_1) \end{cases}$$

$$\Rightarrow \quad \exists! x_{n+1} \in C[t_0 + n\tau_1, t_0 + \tau_2]$$

$$\Rightarrow \quad \exists! y_{n+1} \in C[t_0 + n\tau_1, t_0 + \tau_2]$$

So we obtained:

$$(x(t), y(t)) = \begin{cases} (x_1(t), y_1(t)), & t \in [t_0, t_0 + \tau_1] \\ (x_2(t), y_2(t)), & t \in [t_0 + \tau_1, t_0 + 2\tau_1] \\ \dots \\ (x_{n+1}(t), y_{n+1}(t)), & t \in [t_0 + n\tau_1, t_0 + \tau_2] \end{cases}$$

solution of the problem (2) + (3). □

Remark 1. *We consider the system (2) with initial condition (3). If the conditions (i) si (ii') are satisfied, then the problem (2)+(3) has a unique solution which can be obtained by the method of successive approximations.*

3. The approximation of the solution

We consider the system (2) with initial condition (3)

This problem is equivalent with the delayed integral Volterra equations:

$$x(t) = \begin{cases} \varphi(t), & t \in [t_0 - \tau_1, t_0] \\ \varphi(t_0) + \int_{t_0}^{t} f_1(s, x(s), y(s), x(s-\tau_1), y(s-\tau_2))\mathrm{d}s, & t \in [t_0, b] \end{cases}$$

$$y(t) = \begin{cases} \psi(t), & t \in [t_0 - \tau_2, t_0] \\ \psi(t_0) + \int_{t_0}^{t} f_2(s, x(s), y(s), x(s-\tau_1), y(s-\tau_2))\mathrm{d}s, & t \in [t_0, b] \end{cases}$$

where $f_i \in C([t_0, b] \times \mathbb{R}^4),\ i = 1, 2.$

We suppose that the hypotheses of Remark 1 are satisfied. Then the problem (2) + (3) has a unique solution

$$\begin{aligned} x &\in C[t_0 - \tau_1, t_0] \cap C^1[t_0, b] \\ y &\in C[t_0 - \tau_2, t_0] \cap C^1[t_0, b]. \end{aligned}$$

Let (α, β) be the solution, which, by virtue of Remark 1, can be obtained by successive approximation method. So, we have

$$\begin{aligned}\alpha(t) &= \varphi(t),\ t \in [t_0 - \tau_1, t_0]\\ \beta(t) &= \psi(t),\ t \in [t_0 - \tau_2, t_0]\end{aligned}$$

For $t \in [t_0, b]$ we have:

$$\begin{cases}\alpha_0(t) = \varphi(t)\\ \beta_0(t) = \psi(t)\end{cases} \tag{4}$$

$$\begin{cases}\alpha_1(t) = \varphi(t_0) + \int_{t_0}^{t} f_1(s, \alpha_0(s), \beta_0(s), \alpha_0(s-\tau_1), \beta_0(s-\tau_2))ds\\ \beta_1(t) = \psi(t_0) + \int_{t_0}^{t} f_2(s, \alpha_0(s), \beta_0(s), \alpha_0(s-\tau_1), \beta_0(s-\tau_2))ds\end{cases}$$

$$\dots$$

$$\begin{cases}\alpha_m(t) = \varphi(t_0) + \int_{t_0}^{t} f_1(s, \alpha_{m-1}(s), \beta_{m-1}(s), \alpha_{m-1}(s-\tau_1), \beta_{m-1}(s-\tau_2))ds\\ \beta_m(t) = \psi(t_0) + \int_{t_0}^{t} f_2(s, \alpha_{m-1}(s), \beta_{m-1}(s), \alpha_{m-1}(s-\tau_1), \beta_{m-1}(s-\tau_2))ds\end{cases}$$

To obtain the sequence of successive approximations (4), it is necessary to calculate the integrals which appear in the right-hand side. In general, this problem is difficult. We shall use the trapezoidal rule.

Let an interval $[a, b] \subseteq \mathbb{R}$ be given, and the function $f \in C^2[a, b]$.

Divide the interval $[a, b]$ by points

$$a = x_0 < x_1 < x_2 < \ldots < x_n = b$$

into n equal parts of length $\Delta x = \dfrac{b-a}{n}$.

Then we have the trapezoidal formula:

$$\int_a^b f(x)dx = \frac{b-a}{2n}\left[f(a) + f(b) + 2\sum_{i=1}^{n-1} f(x_i)\right] + r_n(f) \tag{5}$$

where $r_n(f)$ is the remainder of the formula.

To evaluate the approximation error of the trapezoidal formula there exists the following result.

Theorem 2. *For every function $f \in C^2[a, b]$, the remainder $r_n(f)$ from the trapezoidal formula (5), satisfies the ineguality:*

$$|r_n(f)| \leq \frac{(b-a)^3}{12n^2} = \max_{x\in[a,b]} |f''(x)| \tag{6}$$

3.1. **The calculation of the integrals which appear in the successive approximations methods.** Now we suppose that $f_i \in C([t_0,b]\times\mathbb{R}^4),\ i=1,2$, and in order to calculate the integral α_m and β_m from (4), we apply the formula (5). Then we divide the interval $[t_0,b]$ by the points:

$$0=t_0\leq t_1<\ldots<t_n=b \tag{7}$$

where: $t_i=t_{i-1}+h,\ h=\dfrac{t-t_0}{2^v},\ v=0,1,2,...,\ i=\overline{1,n},\ n=\left[\dfrac{b}{h}\right]$ ($[\cdot]$ is integer part). Thus we have

$$\begin{aligned}\alpha_m(t_k)=&\int_{t_0}^{t_k} f_1(s,\alpha_{m-1}(s),\beta_{m-1}(s),\alpha_{m-1}(s-\tau_1),\beta_{m-1}(s-\tau_2))ds=\\ &\frac{t-t_0}{2n}[f_1(t_0,\alpha_{m-1}(t_0),\beta_{m-1}(t_0),\alpha_{m-1}(t_0-\tau_1),\beta_{m-1}(t_0-\tau_2))+\\ &f_1(t_k,\alpha_{m-1}(t_k),\beta_{m-1}(t_k),\alpha_{m-1}(t_k-\tau_1),\beta_{m-1}(t_k-\tau_2))+\\ &2\sum_{i=1}^{n-1}f_1(t_i,\alpha_{m-1}(t_i),\beta_{m-1}(t_i),\alpha_{m-1}(t_i-\tau_1),\beta_{m-1}(t_i-\tau_2))]+\\ &\varphi(t_0)+r_{m,k}(f_1)\end{aligned} \tag{8}$$

where, for the remainder $r_{m,k}(f_1)$, we have the estimation:

$$|r_{m,k}(f_1)|\ \leq \tfrac{(t-t_0)^3}{12n^2}\max_{s\in[t_0,b]}\left|[f_1(,\alpha_{m-1}(s),\beta_{m-1}(s),\alpha_{m-1}(s-\tau_1),\beta_{m-1}(s-\tau_2))]''_s\right|,$$

$$k=\overline{0,n}, m\in\mathbb{N}$$

We denote by $\alpha_{m-1}(s)=u,\ \beta_{m-1}(s)=v,\ \alpha_{m-1}(s-\tau_1)=w,\ \beta_{m-1}(s-\tau_2)=z$. Taking into account the fact that:

$$\begin{aligned}&[f_1(s,\alpha_{m-1}(s),\beta_{m-1}(s),\alpha_{m-1}(s-\tau_1),\beta_{m-1}(s-\tau_2))]''_s=\\ &\frac{\partial^2 f_1}{\partial s^2}+\frac{\partial^2 f_1}{\partial s\partial u}u'+\frac{\partial^2 f_1}{\partial s\partial v}v'+\frac{\partial^2 f_1}{\partial s\partial w}w'+\frac{\partial^2 f_1}{\partial s\partial z}z'+\\ &\frac{\partial^2 f_1}{\partial s\partial u}u'+\frac{\partial^2 f_1}{\partial u^2}(u')^2+\frac{\partial^2 f_1}{\partial u\partial v}u'v'+\frac{\partial^2 f_1}{\partial u\partial w}u'w'+\frac{\partial^2 f_1}{\partial u\partial z}u'z'+\\ &\frac{\partial^2 f_1}{\partial s\partial v}v'+\frac{\partial^2 f_1}{\partial u\partial v}u'v'+\frac{\partial^2 f_1}{\partial v^2}(v')^2+\frac{\partial^2 f_1}{\partial v\partial w}v'w'+\frac{\partial^2 f_1}{\partial v\partial z}v'z'+\\ &\frac{\partial^2 f_1}{\partial s\partial w}w'+\frac{\partial^2 f_1}{\partial u\partial w}u'w'+\frac{\partial^2 f_1}{\partial v\partial w}v'w'+\frac{\partial^2 f_1}{\partial w^2}(w')^2+\frac{\partial^2 f_1}{\partial w\partial z}w'z'+\\ &\frac{\partial^2 f_1}{\partial s\partial z}z'+\frac{\partial^2 f_1}{\partial u\partial z}u'z'+\frac{\partial^2 f_1}{\partial v\partial z}v'z'+\frac{\partial^2 f_1}{\partial w\partial z}w'z'+\frac{\partial^2 f_1}{\partial z^2}(z')^2+\end{aligned}$$

$$\frac{\partial f_1}{\partial u}u''+\frac{\partial f_1}{\partial v}v''+\frac{\partial f_1}{\partial w}w''+\frac{\partial f_1}{\partial z}z''=$$
$$\frac{\partial^2 f_1}{\partial s^2}+2\frac{\partial^2 f_1}{\partial s\partial u}u'+2\frac{\partial^2 f_1}{\partial s\partial v}v'+2\frac{\partial^2 f_1}{\partial s\partial w}w'+2\frac{\partial^2 f_1}{\partial s\partial z}z'+$$
$$\frac{\partial^2 f_1}{\partial u^2}(u')^2+\frac{\partial^2 f_1}{\partial v^2}(v')^2+\frac{\partial^2 f_1}{\partial w^2}(w')^2+\frac{\partial^2 f_1}{\partial z^2}(z')^2+2\frac{\partial^2 f_1}{\partial u\partial v}u'v'+$$
$$2\frac{\partial^2 f_1}{\partial u\partial w}u'w'+2\frac{\partial^2 f_1}{\partial u\partial z}u'z'+2\frac{\partial^2 f_1}{\partial v\partial w}v'w'+\frac{\partial^2 f_1}{\partial v\partial z}v'z'+$$
$$2\frac{\partial^2 f_1}{\partial w\partial z}w'z'+\frac{\partial f_1}{\partial u}u''+\frac{\partial f_1}{\partial v}v''+\frac{\partial f_1}{\partial w}w''+\frac{\partial f_1}{\partial z}z''$$

and

$$\begin{aligned}
\alpha_{m-1}(t) &= \varphi(t_0)+\int_{t_0}^{t} f_1(s,\alpha_{m-2}(s),\beta_{m-2}(s),\alpha_{m-2}(s-\tau_1),\beta_{m-2}(s-\tau_2))ds\\
\alpha'_{m-1}(t) &= \int_{t_0}^{t}\frac{\partial f_1(s,\alpha_{m-2}(s),\beta_{m-2}(s),\alpha_{m-2}(s-\tau_1),\beta_{m-2}(s-\tau_2))}{\partial s}ds\\
\alpha''_{m-1}(t) &= \int_{t_0}^{t}\frac{\partial^2 f_1(s,\alpha_{m-2}(s),\beta_{m-2}(s),\alpha_{m-2}(s-\tau_1),\beta_{m-2}(s-\tau_2))}{\partial s^2}ds
\end{aligned}$$

and denoting by

$$M_0=\max_{\substack{|\alpha|\le 2\\ s\in[t_0,b]\\ |u|,|v|,|w|,|z|\le R}}\left|\frac{\partial^{|\alpha|}f_1(s,u,v,w,z)}{\partial s^{\alpha_1}\partial u^{\alpha_2}\partial v^{\alpha_3}\partial w^{\alpha_4}\partial z^{\alpha_5}}\right|,$$

we obtain

$$|\alpha_{m-1}(t)|\le(t-t_0)M_0;\ \left|\alpha'_{m-1}(t)\right|\le(t-t_0)M_0;\ \left|\alpha''_{m-1}(t)\right|\le(t-t_0)M_0.$$

Again from here we have:

$$[f_1(s,\alpha_{m-1}(s),\beta_{m-1}(s),\alpha_{m-1}(s-\tau_1),\beta_{m-1}(s-\tau_2))]''_s\le M_1$$

where $M_1=M_0+12(t-t_0)M_0^2+16(t-t_0)M_0^3$ and M_1does not depend on m and k.

For the remainder $r_{m,k}(f_1)$, from the formula (8) we have:

$$|r_{m,k}(f_1)|\le\frac{(t-t_0)^3}{12n^2}M_1,\ m=0,1,2,...,\ k=\overline{0,n}. \tag{9}$$

In this way we have obtained a formula for the approximative calculation of the integrals from (4).

3.2. **The approximate calculation of the terms of the successive approximations sequence.** Using the approximation (4) and the formula (8) with the remainder estimation (9), we shall present further down an algorithm for the approximate solution of system (2) with initial condition (3).

So, we have:

$$\alpha_1(t_k) = \int_{t_0}^{t_k} f_1(s, \alpha_0(s), \beta_0(s), \alpha_0(s-\tau_1), \beta_0(s-\tau_2))ds =$$

$$\frac{t-t_0}{2n}[f_1(t_0, \alpha_0(t_0), \beta_0(t_0), \alpha_0(t_0-\tau_1), \beta_0(t_0-\tau_2))+$$

$$2\sum_{i=1}^{n-1} f_1(t_i, \alpha_0(t_i), \beta_0(t_i), \alpha_0(t_i-\tau_1), \beta_0(t_i-\tau_2))+$$

$$f_1(t_k, \alpha_0(t_k), \beta_0(t_k), \alpha_0(t_k-\tau_1), \beta_0(t_k-\tau_2))]+$$

$$\varphi(t_0) + r_{1,k}(f_1) =$$

$$\widetilde{\alpha}_1(t_k) + r_{1,k}(f_1), \ k = \overline{0,n}$$

$$\alpha_2(t_k) = \int_{t_0}^{t_k} f_1(s, \alpha_1(s), \beta_1(s), \alpha_1(s-\tau_1), \beta_1(s-\tau_2))ds =$$

$$\frac{t-t_0}{2n}[f_1(t_0, \widetilde{\alpha}_1(t_0) + r_{1,0}(f_1), \widetilde{\beta}_1(t_0) + r_{1,0}(f_1),$$

$$\widetilde{\alpha}_1(t_0-\tau_1) + r_{1,0}(f_1), \widetilde{\beta}_1(t_0-\tau_2) + r_{1,0}(f_1))+$$

$$2\sum_{i=1}^{n-1} f_1(t_i, \widetilde{\alpha}_1(t_i) + r_{1,i}(f_1), \widetilde{\beta}_1(t_i) + r_{1,i}(f_1),$$

$$\widetilde{\alpha}_1(t_i-\tau_1) + r_{1,i}(f_1), \widetilde{\beta}_1(t_i-\tau_2) + r_{1,i}(f_1))+$$

$$f_1(t_k, \widetilde{\alpha}_1(t_k) + r_{1,n}(f_1), \widetilde{\beta}_1(t_k) + r_{1,n}(f_1),$$

$$\widetilde{\alpha}_1(t_k-\tau_1) + r_{1,n}(f_1), \widetilde{\beta}_1(t_k-\tau_2) + r_{1,n}(f_1))]+$$

$$\varphi(t_0) + r_{2,k}(f_1) =$$

$$\frac{t-t_0}{2n}[f_1(t_0, \widetilde{\alpha}_1(t_0), \widetilde{\beta}_1(t_0), \widetilde{\alpha}_1(t_0-\tau_1), \widetilde{\beta}_1(t_0-\tau_2))+$$

$$2\sum_{i=1}^{n-1} f_1(t_i, \widetilde{\alpha}_1(t_i), \widetilde{\beta}_1(t_i), \widetilde{\alpha}_1(t_i-\tau_1), \widetilde{\beta}_1(t_i-\tau_2))+$$

$$f_1(t_k, \widetilde{\alpha}_1(t_k), \widetilde{\beta}_1(t_k), \widetilde{\alpha}_1(t_k-\tau_1), \widetilde{\beta}_1(t_k-\tau_2))]+$$

$$\varphi(t_0) + \widetilde{r}_{2,k}(f_1) =$$

$$\widetilde{\alpha}_2(t_k) + \widetilde{r}_{2,k}(f_1)$$

Observe that

$$r_{1,k}(f_1) = \alpha_1(t_k) - \widetilde{\alpha}_1(t_k) \text{ and } \widetilde{r}_{2,k}(f_1) = \alpha_2(t_k) - \widetilde{\alpha}_2(t_k).$$

and we pass from

$$f_1(t_i, \widetilde{\alpha}_1(t_i) + r_{1,i}(f_1), \widetilde{\beta}_1(t_i) + r_{1,i}(f_1), \widetilde{\alpha}_1(t_i - \tau_1) + r_{1,i}(f_1), \widetilde{\beta}_1(t_i - \tau_2) + r_{1,i}(f_1))$$

to

$$f_1(t_i, \widetilde{\alpha}_1(t_i), \widetilde{\beta}_1(t_i), \widetilde{\alpha}_1(t_i - \tau_1), \widetilde{\beta}_1(t_i - \tau_2)) \ + \ \text{same remainder}$$

so that the remainders cumulated after i it gives us $\widetilde{r}_{2,k}(f_1)$.

We use the Taylor formula with respect to the last four variables from f_1 around $\widetilde{\alpha}_1(t_i)$.

$$|\widetilde{r}_{2,k}(f_1)| \leq \frac{t-t_0}{2n} L \left[|r_{1,0}(f_1)| + \sum_{i=1}^{n-1} |r_{1,i}(f_1)| + |r_{1,n}(f_1)| \right] + |r_{2,k}(f_1)| \leq$$

$$\frac{t-t_0}{2n} L \left(\frac{(t-t_0)^3}{12n^2} M_1 + (n-1)\frac{(t-t_0)^3}{12n^2} M_1 + \frac{(t-t_0)^3}{12n^2} M_1 \right) +$$

$$\frac{(t-t_0)^3}{12n^2} M_1 \leq \frac{t-t_0}{2n} L \frac{(t-t_0)^3}{12n^2} M_1 \left(1 + n - 1 + 1\right) + \frac{(t-t_0)^3}{12n^2} M_1 \leq$$

$$\frac{(t-t_0)^3}{12n^2} M_1 \left[\frac{(n+1)(t-t_0)}{2n} L + 1 \right] \leq \frac{(t-t_0)^3}{12n^2} M_1 \left[(t-t_0)L + 1 \right]$$

We continue in this manner, for $m = 3, 4, ...,$ by induction, and obtain:

$$\alpha_m(t_k) = \frac{t-t_0}{2n} [f_1(t_0, \widetilde{\alpha}_{m-1}(t_0) + \widetilde{r}_{m-1,0}(f_1), \widetilde{\beta}_{m-1}(t_0) + \widetilde{r}_{m-1,0}(f_1),$$

$$\widetilde{\alpha}_{m-1}(t_0 - \tau_1) + \widetilde{r}_{m-1,0}(f_1), \widetilde{\beta}_{m-1}(t_0 - \tau_2) + \widetilde{r}_{m-1,0}(f_1)) +$$

$$2 \sum_{i=1}^{n-1} f_1(t_i, \widetilde{\alpha}_{m-1}(t_i) + \widetilde{r}_{m-1,i}(f_1), \widetilde{\beta}_{m-1}(t_i) + \widetilde{r}_{m-1,i}(f_1),$$

$$\widetilde{\alpha}_{m-1}(t_i - \tau_1) + \widetilde{r}_{m-1,i}(f_1), \widetilde{\beta}_{m-1}(t_i - \tau_2) + \widetilde{r}_{m-1,i}(f_1)) +$$

$$f_1(t_k, \widetilde{\alpha}_{m-1}(t_k) + \widetilde{r}_{m-1,n}(f_1), \widetilde{\beta}_{m-1}(t_k) + \widetilde{r}_{m-1,n}(f_1),$$

$$\widetilde{\alpha}_{m-1}(t_k - \tau_1) + \widetilde{r}_{m-1,n}(f_1), \widetilde{\beta}_{m-1}(t_k - \tau_2) + \widetilde{r}_{m-1,n}(f_1))] +$$

$$\varphi(t_0) + r_{m,k}(f_1) =$$

$$\frac{t-t_0}{2n} [f_1(t_0, \widetilde{\alpha}_{m-1}(t_0), \widetilde{\beta}_{m-1}(t_0), \widetilde{\alpha}_{m-1}(t_0 - \tau_1), \widetilde{\beta}_{m-1}(t_0 - \tau_2)) +$$

$$2\sum_{i=1}^{n-1} f_1(t_i, \widetilde{\alpha}_{m-1}(t_i), \widetilde{\beta}_{m-1}(t_i), \widetilde{\alpha}_{m-1}(t_i-\tau_1), \widetilde{\beta}_{m-1}(t_i-\tau_2))+$$

$$f_1(t_k, \widetilde{\alpha}_{m-1}(t_k), \widetilde{\beta}_{m-1}(t_k), \widetilde{\alpha}_{m-1}(t_k-\tau_1), \widetilde{\beta}_{m-1}(t_k-\tau_2))]+$$

$$\varphi(t_0)+\widetilde{r}_{m,k}(f_1)=\widetilde{\alpha}_m(t_k)+r_{m,k}(f_1),\ k=\overline{0,n}.$$

where

$$|\widetilde{r}_{m,k}(f_1)|=|\alpha_m(t_k)-\widetilde{\alpha}_m(t_k)|\leq \frac{(t-t_0)^3}{12n^2}M_1\left[(t-t_0)^{m-1}L^{m-1}+...+1\right],\ k=\overline{0,n}.$$

or

$$|\widetilde{r}_{m,k}(f_1)|\leq \frac{(t-t_0)^3}{12n^2}M_1\frac{1-(t-t_0)^m L^m}{1-(t-t_0)L}\leq \frac{(t-t_0)^3 M_1}{12n^2[1-(t-t_0)L]}, k=\overline{0,n},\ m\in\mathbb{N}^*.$$

In this way we got the sequence

$$(\widetilde{\alpha}_m(t_k))_{m\in\mathbb{N}},\ k=\overline{0,n}$$

which approximates the sequence of successive approximation (4) on the knots (7), with the error

$$|\alpha_m(t_k)-\widetilde{\alpha}_m(t_k)|\leq \frac{(t-t_0)^3 M_1}{12n^2[1-(t-t_0)L]} \tag{10}$$

By Picard's theorem [1], we have the following estimation

$$|\alpha(t_k)-\alpha_m(t_k)|\leq \frac{(t-t_0)^m L^m}{1-(t-t_0)L}\left\|\alpha_0-\alpha_1\right\|_{C[t_0,b]}. \tag{11}$$

Analogously we calculate for β_m.

In this way there was obtained the main result of our paper:

Theorem 3. *Consider the system (2) with initial condition (3) under the conditions of Remark 1. If the exact solution* (α,β) *is approximated by the sequence* $\left((\widetilde{\alpha}_m(t_k)),(\widetilde{\beta}_m(t_k))\right)_{m\in\mathbb{N}}$, $k=\overline{0,n}$, $m<n$ *on the knots (7), by the successive approximations method (4) combined with the trapezoidal rule (5), then the following error estimation holds:*

$$|\alpha(t_k)-\widetilde{\alpha}_m(t_k)|\leq \frac{(t-t_0)^3}{1-(t-t_0)L}\left[(t-t_0)^{m-3}L^m\left\|\alpha_0-\alpha_1\right\|_{C[t_0,b]}+\frac{M_1}{12n^2}\right], \tag{12}$$
$$m=1,2,...,\ k=\overline{0,n}$$

$$\left|\beta(t_k) - \widetilde{\beta}_m(t_k)\right| \leq \frac{(t-t_0)^3}{1-(t-t_0)L}\left[(t-t_0)^{m-3}L^m \left\|\alpha_0 - \alpha_1\right\|_{C[t_0,b]} + \frac{M_2}{12n^2}\right], \qquad (13)$$
$$m = 1, 2, ..., \ k = \overline{0, n}$$

Proof. We have

$$\begin{aligned} |\alpha(t_k) - \widetilde{\alpha}_m(t_k)| &= |\alpha(t_k) - \alpha_m(t_k) + \alpha_m(t_k) - \widetilde{\alpha}_m(t_k)| \leq \\ &\leq |\alpha(t_k) - \alpha_m(t_k)| + |\alpha_m(t_k) - \widetilde{\alpha}_m(t_k)| \end{aligned}$$

which, by virtue of formula (10) and (11), can also be written

$$|\alpha(t_k) - \widetilde{\alpha}_m(t_k)| \leq \frac{(t-t_0)^3 M_1}{12n^2\left[1-(t-t_0)L\right]} + \frac{(t-t_0)^m L^m}{1-(t-t_0)L}\left\|\alpha_0 - \alpha_1\right\|_{C[t_0,b]}$$

and, from here, it results immediately (12) and analogue (13). The theorem is proved. □

Remark 2. *For* $L < \dfrac{1}{t-t_0}$ *the errors from Theorem 3 converges.*

4. **Example**

Consider the following Lotka-Volterra-type predator-prey system with two delays τ_1 and τ_2:

$$\begin{cases} x'(t) = x(t) + x(t-2) + y(t-5) \\ y'(t) = y(t) + x(t-2) - y(t-5) \end{cases}, \ t \geq 0$$

with initial condition

$$\begin{cases} x(t) = 1, \ t \in [-2, 0] \\ y(t) = 0, \ t \in [-5, 0] \end{cases}$$

We apply the step method for this system:

$t \in [0, 2]$

$$\begin{cases} x'(t) = x(t) + 1 \\ x(0) = 1 \end{cases} \Rightarrow x_1(t)$$

$$\begin{cases} y'(t) = y(t) + 1 \\ y(0) = 0 \end{cases} \Rightarrow y_1(t)$$

$t \in [2, 4]$

$$\begin{cases} x'(t) = x(t) + x_1(t-2) \\ x(2) = x_1(2) \end{cases} \Rightarrow x_2(t)$$

$$\begin{cases} y'(t) = y(t) + x_1(t-2) \\ y(2) = y_1(2) \end{cases} \Rightarrow y_2(t)$$

$t \in [4, 5]$

$$\begin{cases} x'(t) = x(t) + x_2(t-2) \\ x(4) = x_2(4) \end{cases} \Rightarrow x_3(t)$$

$$\begin{cases} y'(t) = y(t) + x_2(t-2) \\ y(4) = y_2(4) \end{cases} \Rightarrow y_3(t)$$

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EXTREMAL PROBLEMS OF TURÁN TYPE

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Abstract. We give estimations of certain weighted L^2-norms of the k-th derivative of polynomials which have a curved majorant. They are obtained as applications of special quadrature formulae.

1. **Introduction**

The following problem was raised by P. Turán.

Let $\varphi(x) \geq 0$ *for* $-1 \leq x \leq 1$ *and consider the class* $P_{n,\varphi}$ *of all polynomials of degree* n *such that* $|p_n(x)| \leq \varphi(x)$ *for* $-1 \leq x \leq 1$.

How large can $max_{[-1,1]} \left|p_n^{(k)}(x)\right|$ *be if* p_n *is arbitrary in* $P_{n,\varphi}$ *?*

The aim of this paper is to consider the solution in the weighted L^2-norm for the majorant

$$\varphi(x) = \frac{\alpha - \beta x}{\sqrt{1-x^2}}, 0 \leq \beta \leq \alpha.$$

Let us denote by

$$x_i = \cos\frac{(2i-1)\pi}{2n}, i = 1, 2, ..., n, \text{ the zeros of } T_n(x) = \cos n\theta, x = \cos\theta, \tag{1.1}$$

the Chebyshev polynomial of the first kind,

$$y_i^{(k)} \text{ the zeros of } U_{n-1}^{(k)}(x), U_{n-1}(x) = \sin n\theta / \sin\theta, x = \cos\theta, \tag{1.2}$$

the Chebyshev polynomial of the second kind and

$$G_{n-1}(x) = \alpha U_{n-1}(x) - \beta U_{n-2}(x), 0 \leq \beta \leq \alpha. \tag{1.3}$$

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Let $\Pi_{\alpha,\beta}$ be the class of all polynomials p_{n-1}, of degree $\leq n-1$ such that

$$|p_{n-1}(x_i)| \leq \frac{\alpha-\beta x_i}{\sqrt{1-x_i^2}}, i=1,2,...,n, \tag{1.4}$$

where the x_i's are given by (1.1) and $0 \leq \beta \leq \alpha$.

2. **Results**

Theorem 2.1. *If* $p_{n-1} \in \Pi_{\alpha,\beta}$ *then we have*

$$\int_{-1}^{1} \frac{1}{\sqrt{1-x^2}} \left[p'_{n-1}(x)\right]^2 dx \tag{2.1}$$
$$\leq \frac{2\pi n (n-1) \left[\left(\alpha^2+\beta^2\right)(n-2)\left(n^2-2n+2\right)+5\beta^2\left(n^2-n+1\right)\right]}{15}$$

with equality for $p_{n-1} = G_{n-1}$.

Two cases are of special interest:

I. Case $\alpha=\beta=1$, $\varphi(x)=\sqrt{\frac{1-x}{1+x}}$,

$G_{n-1}(x) = V_{n-1}(x) = \frac{\cos\left[\left(n-\frac{1}{2}\right)\arccos x\right]}{\cos\left[\frac{1}{2}\arccos x\right]}$.

Note that $P_{n-1,\varphi} \subset \Pi_{1,1}$, $V_{n-1} \notin P_{n-1,\varphi}$, $V_{n-1} \in \Pi_{1,1}$.

Corollary 2.2. *If* $p_{n-1} \in \Pi_{1,1}$ *then we have*

$$\int_{-1}^{1} \frac{1}{\sqrt{1-x^2}} \left[p'_{n-1}(x)\right]^2 dx \leq \frac{2\pi n (n-1)(2n-1)\left(n^2-n+3\right)}{15} \tag{2.2}$$

with equality for $p_{n-1} = V_{n-1}$.

II. Case $\alpha=1$, $\beta=0$, $\varphi(x)=\frac{1}{\sqrt{1-x^2}}$, $G_{n-1}=U_{n-1}$.

Note that $P_{n-1,\varphi} \subset \Pi_{1,0}$, $U_{n-1} \in P_{n-1,\varphi}$, $U_{n-1} \in \Pi_{1,0}$.

Corollary 2.3. *If* $p_{n-1} \in \Pi_{1,0}$ *then we have*

$$\int_{-1}^{1} \frac{1}{\sqrt{1-x^2}} \left[p'_{n-1}(x)\right]^2 dx \leq \frac{2\pi n \left(n^4-1\right)}{15} \tag{2.3}$$

with equality for $p_{n-1} = U_{n-1}$.

In this second case we have a more general result:

Theorem 2.4. *If* $p_{n-1} \in \Pi_{1,0}$ *and* $0 \leq b \leq a$ *then we have*

$$\int_{-1}^{1} (a-bx)^3 \left(1-x^2\right)^{k-1/2} \left[p_{n-1}^{(k+1)}(x)\right]^2 dx \tag{2.4}$$

$$\leq \frac{\pi a (n+k+1)!}{(n-k-2)!} \left[\frac{2\left(n^2-(k+2)^2\right)\left(a^2+3b^2\right)}{(2k+1)(2k+3)(2k+5)} + \frac{2(k+1)a^2+3b^2}{(2k+1)(2k+3)} \right]$$

$k=0,...,n-2$, *with equality for* $p_{n-1}=U_{n-1}$.

Setting $a=1, b=1$ one obtains the following

Corollary 2.5. *If* $p_{n-1} \in \Pi_{1,1}$ *then we have*

$$\int_{-1}^{1} (1-x)^{k+5/2} (1+x)^{k-1/2} \left[p_{n-1}^{(k+1)}(x)\right]^2 dx \tag{2.5}$$

$$\leq \frac{\pi (n+k+1)!}{(n-k-2)!} \times \frac{8\left(n^2-(k+2)^2\right)+(2k+5)^2}{(2k+1)(2k+3)(2k+5)}$$

$k=0,...,n-2$, *with equality for* $p_{n-1}=U_{n-1}$.

Setting $a=1, b=0$ one obtains the following

Corollary 2.6. *If* $p_{n-1} \in \Pi_{1,0}$ *then we have*

$$\int_{-1}^{1} \left(1-x^2\right)^{k-1/2} \left[p_{n-1}^{(k+1)}(x)\right]^2 dx \tag{2.6}$$

$$\leq \frac{2\pi (n+k+1)!}{(n-k-2)!} \times \frac{n^2+k^2+3k+1}{(2k+1)(2k+3)(2k+5)}$$

$k=0,...,n-2$, *with equality for* $p_{n-1}=U_{n-1}$.

3. Lemmas

Here we state some lemmas which help us in proving our theorems.

Lemma 3.1. *Let* p_{n-1} *be such that* $|p_{n-1}(x_i)| \leq \frac{\alpha-\beta x_i}{\sqrt{1-x_i^2}}, i=1,2,...,n,$ *where the* x_i*'s are given by (1.1). Then we have*

$$\left|p'_{n-1}(y_j)\right| \leq \left|G'_{n-1}(y_j)\right|,\ k=0,1,...,n-1, \text{ and} \tag{3.1}$$

$$\left|p'_{n-1}(1)\right| \leq \left|G'_{n-1}(1)\right|,\ \left|p'_{n-1}(-1)\right| \leq \left|G'_{n-1}(-1)\right| . \tag{3.2}$$

Proof. By the Lagrange interpolation formula based on the zeros of T_n and using $T_n'(x_i) = \frac{(-1)^{i+1}n}{(1-x_i^2)^{1/2}}$, we can represent any polynomial p_{n-1} by $p_{n-1}(x) = \frac{1}{n}\sum_{i=1}^{n} \frac{T_n(x)}{x-x_i}(-1)^{i+1}\left(1-x_i^2\right)^{1/2} p_{n-1}(x_i)$.

From $G_{n-1}(x_i) = (-1)^{i+1}\frac{\alpha-\beta x_i}{\sqrt{1-x_i^2}}$ we have $G_{n-1}(x) = \frac{1}{n}\sum_{i=1}^{n}\frac{T_n(x)}{x-x_i}(\alpha-\beta x_i)$.

Differentiating with respect to x we obtain

$p_{n-1}'(x) = \frac{1}{n}\sum_{i=1}^{n}\frac{T_n'(x)(x-x_i)-T_n(x)}{(x-x_i)^2}(-1)^{i+1}\left(1-x_i^2\right)^{1/2}p_{n-1}(x_i)$.

On the roots of $T_n'(x) = nU_{n-1}(x)$ and using (1.4) we find

$\left|p_{n-1}'(y_j)\right| \le \frac{1}{n}\sum_{i=1}^{n}\frac{|T_n(y_j)|}{(y_j-x_i)^2}(\alpha-\beta x_i) = \frac{|T_n(y_j)|}{n}\sum_{i=1}^{n}\frac{\alpha-\beta x_i}{(y_j-x_i)^2} = \left|G_{n-1}'(y_j)\right|$.

For $l_i(x) = \frac{T_n(x)}{x-x_i}$ taking into account that $l_i'(1) > 0$ (see [5]) it follows

$\left|p_{n-1}'(1)\right| \le \frac{1}{n}\sum_{i=1}^{n} l_i'(1)(\alpha-\beta x_i) = \left|G_{n-1}'(1)\right|$.

Similarly $\left|p_{n-1}'(-1)\right| \le \left|G_{n-1}'(-1)\right|$. □

Lemma 3.2. (*Duffin − Schaeffer*)[2] *If* $q(x) = c\prod_{i=1}^{n}(x-x_i)$ *is a polynomial of degree* n *with* n *distinct real zeros and if* $p \in P_n$ *is such that*

$$|p'(x_i)| \le |q'(x_i)| \quad (i = 1, 2, ..., n),$$

then for $k = 1, 2, ..., n-1$,

$$\left|p^{(k+1)}(x)\right| \le \left|q^{(k+1)}(x)\right| \text{ whenever } q^{(k)}(x) = 0.$$

Lemma 3.3. *Let* p_{n-1} *be such that* $|p_{n-1}(x_i)| \le \frac{1}{\sqrt{1-x_i^2}}, i = 1, 2, ..., n,$ *where the* x_i*'s are given by (1.1). Then we have*

$$\left|p_{n-1}^{(k+1)}(y_j^{(k)})\right| \le \left|U_{n-1}^{(k+1)}(y_j^{(k)})\right|, \text{ whenever } U_{n-1}^{(k)}(y_j^{(k)}) = 0, \tag{3.3}$$

$$k = 0, 1, ..., n-1, and$$

$$\left|p_{n-1}^{(k+1)}(1)\right| \le \left|U_{n-1}^{(k+1)}(1)\right|, \left|p_{n-1}^{(k+1)}(-1)\right| \le \left|U_{n-1}^{(k+1)}(-1)\right| . \tag{3.4}$$

Proof. For $\alpha = 1$, $\beta = 0$, $G_{n-1} = U_{n-1}$ and (3.1) give $\left|p_{n-1}'(y_j)\right| \le \left|U_{n-1}'(y_j)\right|$ and (3.2) $\left|p_{n-1}'(1)\right| \le \left|U_{n-1}'(1)\right|$, $\left|p_{n-1}'(-1)\right| \le \left|U_{n-1}'(-1)\right|$.

Now the proof ends by applying Duffin-Schaeffer Lemma. □

We need the following quadrature formulae:

Lemma 3.4. *For any given* n *and* k, $0 \le k \le n-1$, *let* $y_i^{(k)}$, $i = 1, ..., n-k-1$, *be the zeros of* $U_{n-1}^{(k)}$.

Then the quadrature formulae

$$\int_{-1}^{1} \left(1-x^2\right)^{k-1/2} f(x)\, dx = A_0 \left[f(-1)+f(1)\right] + \sum_{i=1}^{n-k-1} s_i f\left(y_i^{(k)}\right), \tag{3.5}$$

$$A_0 = \frac{2^{2k-1}(2k+1)\,\Gamma(k+1/2)^2\,(n-k-1)!}{(n+k)!}, s_i > 0$$

and

$$\int_{-1}^{1} \left(1-x^2\right)^{k-1/2} f(x)\, dx = B_0 \left[f(-1)+f(1)\right] \tag{3.6}$$

$$+C_0 \left[f'(-1)-f'(1)\right] + \sum_{i=1}^{n-k-2} v_i f\left(y_i^{(k+1)}\right)$$

$$C_0 = \frac{2^{2k}(2k+3)\,\Gamma(k+3/2)^2\,(n-k-2)!}{(n+k+1)!},$$

$$B_0 = C_0 \frac{2\left(n^2-(k+2)^2\right)(2k+3)+4(k+1)(2k+5)}{(2k+1)(2k+5)}$$

have algebraic degree of precision $2n-2k-1$.

For $r(x) = (a-bx)^3$, $0 \le b \le a$ *the formulae*

$$\int_{-1}^{1} r(x) \left(1-x^2\right)^{k-1/2} f(x)\, dx = A_1 f(-1) + B_1 f(1) \tag{3.7}$$

$$+ \sum_{i=1}^{n-k-1} s_i r\left(y_i^{(k)}\right) f\left(y_i^{(k)}\right)$$

$$A_1 = \frac{2^{2k-1}(2k+1)\,\Gamma(k+1/2)^2\,(n-k-1)!\,(a+b)^3}{(n+k)!},$$

$$B_1 = \frac{2^{2k-1}(2k+1)\,\Gamma(k+1/2)^2\,(n-k-1)!\,(a-b)^3}{(n+k)!}$$

and

$$\int_{-1}^{1} r(x)\left(1-x^2\right)^{k-1/2} f(x)\,dx = C_1 f(-1) + D_1 f(1) \tag{3.8}$$

$$+C_2 f'(-1) - D_2 f'(1) + \sum_{i=1}^{n-k-2} v_i r\left(y_i^{(k+1)}\right) f\left(y_i^{(k+1)}\right),$$

$$C_1 = B_0(a+b)^3 - 3C_0 b(a+b)^2, D_1 = B_0(a-b)^3 + 3C_0 b(a-b)^2,$$

$$C_2 = C_0(a+b)^3, D_2 = C_0(a-b)^3,$$

have algebraic degree of precision $2n-2k-4$.

Proof. The first quadrature formula (3.5) is the Bouzitat quadrature formula of the second kind [3, *formula* (4.8.1)], for the zeros of $U_{n-1}^{(k)} = cP_{n-k-1}^{\left(k+\frac{1}{2},k+\frac{1}{2}\right)}$.
Setting $\alpha = \beta = k-1/2, m = n-k-1$ in [3, *formula* (4.8.5)] we find A_0 and $s_i > 0$ (cf. [3, *formula* (4.8.4)]).

If in the above quadrature formula (3.6), we put

$$f(x) = (1-x)(1+x)^2 P_{n-k-2}^{\left(k+\frac{3}{2},k+\frac{3}{2}\right)}(x),$$

$$U_{n-1}^{(k+1)}(x) = cP_{n-k-2}^{\left(k+\frac{3}{2},k+\frac{3}{2}\right)}(x),$$

we obtain C_0, and for

$$f(x) = (1+x)^2 P_{n-k-2}^{\left(k+\frac{3}{2},k+\frac{3}{2}\right)}(x)$$

we find B_0.
If in formula (3.5) we replace $f(x)$ with $r(x)f(x)$ we get (3.7) and if in formula (3.6) we replace $f(x)$ with $r(x)f(x)$ we get (3.8). □

4. **Proof of the Theorems**

Proof of Theorem 2.1

Setting $k=0$ in (3.5) we find the formula

$$\int_{-1}^{1} \frac{f(x)}{\sqrt{1-x^2}}dx = \frac{\pi}{2n}\left[f(-1)+f(1)\right] + \frac{\pi}{n}\sum_{i=1}^{n-1} f(y_i) \tag{4.1}$$

According to this quadrature formula and using (3.1) and (3.2) we have

$$\int_{-1}^{1} \frac{1}{\sqrt{1-x^2}} \left[p'_{n-1}(x)\right]^2 dx = \frac{\pi}{2n}\left(p'_{n-1}(-1)\right)^2 + \frac{\pi}{2n}\left(p'_{n-1}(1)\right)^2 + \frac{\pi}{n}\sum_{i=1}^{n-1}\left(p'_{n-1}(y_i)\right)^2$$

$$\leq \frac{\pi}{2n}\left(G'_{n-1}(-1)\right)^2 + \frac{\pi}{2n}\left(G'_{n-1}(1)\right)^2 + \frac{\pi}{n}\sum_{i=1}^{n-1}\left(G'_{n-1}(y_i)\right)^2 = \int_{-1}^{1}\frac{1}{\sqrt{1-x^2}}\left[G'_{n-1}(x)\right]^2 dx.$$

Using the following formula ($k=0$ in (3.6))

$$\int_{-1}^{1}\frac{f(x)}{\sqrt{1-x^2}}dx = \frac{3\pi\left(3n^2-2\right)}{10n(n^2-1)}\left[f(-1)+f(1)\right] + \frac{3\pi}{4n(n^2-1)}\left[f'(-1)-f'(1)\right] + \sum_{i=1}^{n-2} v_i f\left(y'_i\right)$$

we find $\int_{-1}^{1}\frac{\left[U'_{n-1}(x)\right]^2}{\sqrt{1-x^2}} = \frac{2\pi n\left(n^4-1\right)}{15}$, $\int_{-1}^{1}\frac{\left[U'_{n-2}(x)\right]^2}{\sqrt{1-x^2}} = \frac{2\pi n(n-1)(n-2)\left(n^2-2n+2\right)}{15}$

and $\int_{-1}^{1}\frac{\left[G'_{n-1}(x)\right]^2}{\sqrt{1-x^2}}dx = \frac{2\pi n(n-1)\left[\left(\alpha^2+\beta^2\right)(n+1)\left(n^2+1\right)-5\beta^2\left(n^2-n+1\right)\right]}{15}$.

Proof of Theorem 2.4

According to the quadrature formula (3.7), positivitiness of s_i's, and using (3.3) and (3.4) we have

$$\int_{-1}^{1}(a-bx)^3\left(1-x^2\right)^{k-1/2}\left[p_{n-1}^{(k+1)}(x)\right]^2 dx$$

$$= A_1\left[p_{n-1}^{(k+1)}(-1)\right]^2 + B_1\left[p_{n-1}^{(k+1)}(1)\right]^2 + \sum_{i=1}^{n-k-1} s_i r\left(y_i^{(k)}\right)\left[p_{n-1}^{(k+1)}\left(y_i^{(k)}\right)\right]^2$$

$$\leq A_1\left[U_{n-1}^{(k+1)}(-1)\right]^2 + B_1\left[U_{n-1}^{(k+1)}(1)\right]^2 + \sum_{i=1}^{n-k-1} s_i r\left(y_i^{(k)}\right)\left[U_{n-1}^{(k+1)}\left(y_i^{(k)}\right)\right]^2$$

$$= \int_{-1}^{1}(a-bx)^3\left(1-x^2\right)^{k-1/2}\left[U_{n-1}^{(k+1)}(x)\right]^2 dx$$

In order to complete the proof we apply formula (3.8) to $f=\left[U_{n-1}^{(k+1)}(x)\right]^2$. Having in mind $U_{n-1}^{(k+1)}\left(y_i^{(k+1)}\right)=0$ and the following relations deduced from [1]

$$U_{n-1}^{(k+1)}(1) = \frac{n\left(n^2-1^2\right)...\left(n^2-(k+1)^2\right)}{1.3...(2k+3)},\ U_{n-1}^{(k+2)}(1) = \frac{n^2-(k+2)^2}{2k+5}U_{n-1}^{(k+1)}(1),$$

$$U_{n-1}^{(k+1)}(-1)\,U_{n-1}^{(k+2)}(-1) = -U_{n-1}^{(k+1)}(1)\,U_{n-1}^{(k+2)}(1),$$

we find

$$\int_{-1}^{1}(a-bx)^3\left(1-x^2\right)^{k-1/2}\left[p_{n-1}^{(k+1)}(x)\right]^2 dx = C_1\left[U_{n-1}^{(k+1)}(-1)\right]^2 + D_1\left[U_{n-1}^{(k+1)}(1)\right]^2$$

$$+2C_2U_{n-1}^{(k+1)}(-1)\,U_{n-1}^{(k+2)}(-1) - 2D_2U_{n-1}^{(k+1)}(1)\,U_{n-1}^{(k+2)}(1)$$

$$= \frac{\pi a(n+k+1)!}{(n-k-2)!}\left[\frac{2\left[n^2-(k+2)^2\right]\left(a^2+3b^2\right)}{(2k+1)(2k+3)(2k+5)} + \frac{2(k+1)a^2+3b^2}{(2k+1)(2k+3)}\right].$$

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ON GENERALIZED DIFFERENCE LACUNARY STATISTICAL CONVERGENCE

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Abstract. A lacunary sequence is an increasing integer sequence $\theta = (k_r)$ such that $k_0 = 0$, $k_r - k_{r-1} \to \infty$ as $r \to \infty$. A sequence x is called $S_\theta(\Delta^m)-$convergent to L provided that for each $\varepsilon > 0$, $\lim_r (k_r - k_{r-1})^{-1}$ {the number of $k_{r-1} < k \leq k_r : |\Delta^m x_k - L| \geq \varepsilon\} = 0$, where $\Delta^m x_k = \Delta^{m-1} x_k - \Delta^{m-1} x_{k+1}$. The purpose of this paper is to introduce the concept of Δ^m-lacunary statistical convergence and Δ^m-lacunary strongly convergence and examine some properties of these sequence spaces. We establish some connections between Δ^m-lacunary strongly convergence and Δ^m-lacunary statistical convergence. It is shown that if a sequence is Δ^m-lacunary strongly convergent then it is Δ^m-lacunary statistically convergent. We also show that the space $S_\theta(\Delta^m)$ may be represented as a $[f, p, \theta](\Delta^m)$ space.

1. **Introduction**

Throughout the article w, ℓ_∞, c, c_0, $\bar{c}$, and $\bar{c}_0$ denote the spaces of all, bounded, convergent, null, statistically convergent and statistically null complex sequences. The notion of statistical convergence was introduced by Fast [6] and Schoenberg [19] independently. Subsequently statistical convergence have been discussed in ([5], [7], [8], [12], [16], [18]).

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The notion depends on the density of subsets of the set $\mathbb{N}$ of natural numbers. A subset E of $\mathbb{N}$ is said to have density $\delta(E)$, if

$$\delta(E) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \chi_E(k) \text{ exists,}$$

where χ_E is the characteristic function of E.

A sequence (x_n) is said to be statistically convergent to L if for every $\varepsilon > 0$, $\delta(\{k \in \mathbb{N} : |x_k - L| \geq \varepsilon\}) = 0$. In this case we write $S - \lim x_k = L$ or $x_k \to L(S)$.

The notion of difference sequence spaces was introduced by Kizmaz [10]. Later on the notion was generalized by Et and Çolak [3] and was studied by Et and Basarir [4], Malkowsky and Parashar [14], Et and Nuray [5], Çolak [2] and many others.

Let m be a non-negative integer, then

$$X(\Delta^m) = \{x = (x_k) : (\Delta^m x_k) \in X\}$$

for $X = \ell_\infty$, c and c_0, where $m \in \mathbb{N}$, $\Delta^0 x = (x_k)$ and $\Delta^m x = (\Delta^m x_k) = \left(\Delta^{m-1} x_k - \Delta^{m-1} x_{k+1}\right)$.

The generalized difference has the following binomial representation:

$$\Delta^m x_k = \sum_{v=0}^{m} (-1)^v \binom{m}{v} x_{k+v}.$$

The sequence spaces $\ell_\infty(\Delta^m)$, $c(\Delta^m)$ and $c_0(\Delta^m)$ are BK-spaces, normed by

$$\|x\|_\Delta = \sum_{i=0}^{m} |x_i| + \|\Delta^m x\|_\infty.$$

We call these sequence spaces Δ^m-bounded, Δ^m-convergent and Δ^m-null sequences, respectively. The classes $\bar{c}(\Delta^m)$ and $\bar{c}_0(\Delta^m)$ was studied by Et and Nuray [5].

Let $\theta = (k_r)$ be the sequence of positive integers such that $k_0 = 0$, $0 < k_r < k_{r+1}$ and $h_r = k_r - k_{r-1} \to \infty$ as $r \to \infty$. Then θ is called a lacunary sequence. The intervals determined by θ will be denoted by $I_r = (k_{r-1}, k_r]$ and the ratio k_r/k_{r-1} will be denoted by q_r.

Let $E, F \subset w$. Then we shall write

$$M(E,F) = \underset{x \in E}{\cap} x^{-1} * F = \{a \in w : ax \in F \text{ for all } x \in E\} \ [20].$$

The set $E^{\alpha} = M(E, l_1)$ is called Köthe-Toeplitz dual space or $\alpha-$dual of E.

A sequence space E is said to be solid (or normal) if $(\alpha_k x_k) \in E$ whenever $(x_k) \in E$ for all sequences (α_k) of scalars with $|\alpha_k| \leq 1$ for all $k \in \mathbb{N}$,

A sequence space E is said to be symmetric if $(x_k) \in E$ implies $\left(x_{\pi(k)}\right) \in E$, where $\pi(k)$ is a permutation of $\mathbb{N}$,

A sequence space E is said to be convergence free when, if x is in E and if $y_k = 0$ whenever $x_k = 0$, then y is in E,

A sequence space E is said to be monotone if it contains the canonical preimages of its step spaces,

A sequence space E is said to be sequence algebra if $x.y \notin E$ whenever $x, y \in E$,

A sequence space E is said to be perfect if $E = E^{\alpha\alpha}$ [9].

It is well known that if E is perfect $\Longrightarrow E$ is normal.

The following inequality will be used throughout this paper.

$$|a_k + b_k|^{p_k} \leq C\left\{|a_k|^{p_k} + |b_k|^{p_k}\right\}, \tag{1}$$

where $a_k, b_k \in \mathbb{C}$, $0 < p_k \leq \sup_k p_k = H$, $C = \max\left(1, 2^{H-1}\right)$.

The notion of modulus function was introduced by Nakano [15]. We recall that a modulus f is a function from $[0,\infty)$ to $[0,\infty)$ such taht

i) $f(x) = 0$ if and only if $x = 0$, ii) $f(x+y) \leq f(x) + f(y)$ for $x, y \geq 0$, iii) f is increasing, iv) f is continuous from the right at 0.

It follows that f must be continuous everwhwre on $[0,\infty)$. A modulus may be unbounded or bounded. Ruckle [17] and Maddox [12] used a modulus f to construct sequence spaces.

2. Definitions and Preliminaries

The notion of almost convergence of sequences was introduced by Lorentz [11]. The notion was generalized by Et and Başarır [4].

Definition 2.1 [4] The sequence (x_n) is said to be Δ^m-almost convergent to L if

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=k+1}^{k+n}(\Delta^m x_i - L) = 0, \text{ uniformly in } k.$$

We denote the class of all Δ^m-almost convergent sequences by $AC(\Delta^m)$.

Definition 2.2 [4] The sequence (x_n) is said to be Δ^m- strongly almost convergent to L if

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=k+1}^{k+n}|\Delta^m x_i - L| = 0, \text{ uniformly in } k.$$

We denote the class of all Δ^m-strongly almost convergent sequences by $|AC|(\Delta^m)$.

Definition 2.3 [8] The sequence (x_k) is said to be lacunary statistically convergent to L if for each $\varepsilon > 0$,

$$\lim_{r\to\infty}\frac{1}{h_r}\operatorname{card}\{k \in I_r : |x_k - L| \geq \varepsilon\} = 0.$$

The class of all lacunary statistically convergent sequences is denoted by S_θ.

Definition 2.4 A sequence (x_n) is said to be Δ^m-Cesàro summable to L if

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}(\Delta^m x_k - L) = 0.$$

The class of all Δ^m-Cesàro summable sequences is denoted by $\sigma_1(\Delta^m)$.

Definition 2.5 A sequence (x_n) is said to be Δ^m-strongly Cesàro summable to L if

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}|\Delta^m x_k - L| = 0.$$

The class of all Δ^m-strongly Cesàro summable sequences is denoted by $|\sigma_1|(\Delta^m)$.

Now we introduce the definitions of Δ^m-lacunary statistically convergence, Δ^m- lacunary strongly convergence and Δ^m-lacunary strongly convergence with respect to a modulus f.

Definition 2.6 Let θ be a lacunary sequence, the number sequence x is Δ^m-lacunary statistically convergent to the number L provided that for every $\varepsilon > 0$,

$$\lim_{r\to\infty} \frac{1}{h_r} \operatorname{card}\left\{k \in I_r : |\Delta^m x_k - L| \geq \varepsilon\right\} = 0.$$

In this case we write $S_\theta\left(\Delta^m\right) - \lim x_k = L$ or $x_k \to L\left(S_\theta\left(\Delta^m\right)\right)$. We denote Δ^m-lacunary statistically convergent sequence by $S_\theta\left(\Delta^m\right)$.

Definition 2.7 Let θ be a lacunary sequence. Then a sequence (x_k) is said to be $C_\theta\left(\Delta^m\right)-$summable to L if

$$\lim_{r\to\infty} \frac{1}{h_r} \sum_{k\in I_r} \left(\Delta^m x_k - L\right) = 0.$$

We denote the class of all $C_\theta\left(\Delta^m\right)-$summable sequences by $C_\theta\left(\Delta^m\right)$.

A sequence (x_k) is said to be Δ^m- lacunary strongly summable to L if

$$\lim_{r\to\infty} \frac{1}{h_r} \sum_{k\in I_r} |\Delta^m x_k - L| = 0.$$

We denote the class of all Δ^m- lacunary strongly summable sequences by $N_\theta\left(\Delta^m\right)$. In the case $L = 0$ we shall write $N_\theta^0\left(\Delta^m\right)$ instead of $N_\theta\left(\Delta^m\right)$. It can be shown that the sequence space $N_\theta\left(\Delta^m\right)$ is a Banach space with norm by

$$\|x\|_{\Delta\theta} = \sum_{i=1}^{m} |x_i| + \sup_r \frac{1}{h_r} \sum_{k\in I_r} |\Delta^m x_k|.$$

If we take $m = 0$ then we obtain the sequence space N_θ which were introduced by Freedman et al.[1].

Definition 2.8 Let f be a modulus function and $p = (p_k)$ be any sequence of strictly positive real numbers. We define the following sequence set

$$[f, p, \theta]\left(\Delta^m\right) = \left\{x = (x_k) : \lim_r \frac{1}{h_r} \sum_{k\in I_r} \left[f\left(|\Delta^m x_k - L|\right)\right]^{p_k} = 0, \quad \text{for some } L\right\},$$

If $x \in [f, p, \theta]\left(\Delta^m\right)$, then we will write $x_k \to L\,[f, p, \theta]\left(\Delta^m\right)$ and will be called Δ^m-lacunary strongly summable with respect to a modulus f. In the case $p_k = 1$ for all $k \in \mathbb{N}$, we shall write $[f, \theta]\left(\Delta^m\right)$ instead of $[f, p, \theta]\left(\Delta^m\right)$. It may be noted here that the space $[f, \theta]\left(\Delta^m\right)$ was discussed by Colak [2].

3. **Main Results**

In this section we prove the results of this article. The proof of the following results is a routine work.

Proposition 3.1 Let θ be a lacunary sequence, then $S_\theta\left(\Delta^{m-1}\right) \subset S_\theta\left(\Delta^m\right)$. In general $S_\theta\left(\Delta^i\right) \subset S_\theta\left(\Delta^m\right)$, for all $i = 1, 2, \ldots, m-1$. Hence $S_\theta \subset S_\theta\left(\Delta^m\right)$ and the inclusions are strict.

Theorem 3.2 If a Δ^m-bounded sequence is Δ^m-statistically convergent to L then it is Δ^m-Cesàro summable to L.

Proof. Without loss of generality we may assume that $L = 0$. Then,

$$\begin{aligned} \left|\frac{1}{n}\sum_{k=1}^{n}\Delta^m x_k\right| &\leq \frac{1}{n}\sum_{k=1}^{n}\left|\Delta^m x_k\right| = \frac{1}{n}\sum_{\substack{1\leq k\leq n \\ |\Delta^m x_k|\geq\varepsilon}}\left|\Delta^m x_k\right| + \frac{1}{n}\sum_{\substack{1\leq k\leq n \\ |\Delta^m x_k|<\varepsilon}}\left|\Delta^m x_k\right| \\ &< \frac{1}{n}K\,\mathrm{card}\left\{k\leq n : \left|\Delta^m x_k\right|\geq\varepsilon\right\} + \frac{n}{n}\varepsilon. \end{aligned}$$

Thus $x \in \sigma_1\left(\Delta^m\right)$. Converse of Theorem 3.2 does not holds, for example, the sequence $x = (0, -1, -1, -2, -2, -3, -3, -4, -4, ...)$ belongs to $\sigma_1(\Delta)$ and does not belong to $S(\Delta)$.

Theorem 3.3 Let θ be a lacunary sequence, then

i) If a sequence is Δ^m-lacunary strongly convergent to L, then it is Δ^m-lacunary statistically convergent to L and the inclusion is strict.

ii) If a Δ^m-bounded sequence is Δ^m-lacunary statistically convergent to L then it is Δ^m-lacunary strongly convergent to L.

iii) $\ell_\infty\left(\Delta^m\right) \cap S_\theta\left(\Delta^m\right) = \ell_\infty\left(\Delta^m\right) \cap N_\theta\left(\Delta^m\right)$.

Proof. We give the proof of (i) only. If $\varepsilon > 0$ and $x_k \to L\left(N_\theta\left(\Delta^m\right)\right)$ we can write

$$\sum_{k\in I_r}\left|\Delta^m x_k - L\right| \geq \sum_{\substack{k\in I_r \\ |\Delta^m x_k - L|\geq\varepsilon}}\left|\Delta^m x_k - L\right| \geq \varepsilon.\left|\left\{k\in I_r : \left|\Delta^m x_k - L\right|\geq\varepsilon\right\}\right|.$$

Hence $x_k \to L\left(S_\theta\left(\Delta^m\right)\right)$. The inclusion is strict. In order to establish this, let θ be given and define $\Delta^m x_k$ to be $1, 2, ..., \left[\sqrt{h_r}\right]$ at the first $\left[\sqrt{h_r}\right]$ integers in I_r, and $\Delta^m x_k = 0$ otherwise. Then x is not Δ^m-bounded, $x_k \to 0\left(S_\theta\left(\Delta^m\right)\right)$ and $x_k \nrightarrow 0\left(N_\theta\left(\Delta^m\right)\right)$.

Note that any Δ^m-bounded $S_\theta\left(\Delta^m\right)-$summable sequence is $C_\theta\left(\Delta^m\right)-$summable.

Theorem 3.4 Let θ be a lacunary sequence, then $S\left(\Delta^m\right) = S_\theta\left(\Delta^m\right)$ if and only if $1 < \lim_r \inf q_r \leq \lim_r \sup q_r < \infty$.

The proof of Theorem 3.4, we need the following lemmas.

Lemma 3.5 For any lacunary sequence θ, $S\left(\Delta^m\right) \subset S_\theta\left(\Delta^m\right)$ if and only if $\lim_r \inf q_r > 1$.

Proof. If $\liminf_r q_r > 1$ there exists a $\delta > 0$ such that $1+\delta \leq q_r$ for sufficiently large r. Since $h_r = k_r - k_{r-1}$, we have $\frac{k_r}{h_r} \leq \frac{1+\delta}{\delta}$. Let $x_k \to L\left(S_\theta\left(\Delta^m\right)\right).$ Then for every $\varepsilon > 0$,

$$\begin{aligned}
\frac{1}{k_r}\left|\left\{k \leq k_r : \left|\Delta^m x_k - L\right| \geq \varepsilon\right\}\right| &\geq \frac{1}{k_r}\left|\left\{k \in I_r : \left|\Delta^m x_k - L\right| \geq \varepsilon\right\}\right| \\
&\geq \frac{\delta}{1+\delta}\frac{1}{h_r}\left|\left\{k \in I_r : \left|\Delta^m x_k - L\right| \geq \varepsilon\right\}\right|.
\end{aligned}$$

Hence $S\left(\Delta^m\right) \subset S_\theta\left(\Delta^m\right).$

Conversely suppose that $\liminf_r q_r = 1$. If we consider the sequence defined by,

$$\Delta^m x_i = \begin{cases} 1, & \text{if } i \in I_{r_j} \quad \text{for some } j = 1,2,3,... \\ 0, & \text{otherwise} \end{cases}$$

then $x \in \ell_\infty\left(\Delta^m\right)$ but $x \notin N_\theta\left(\Delta^m\right).$ However, $x \in \left|\sigma_1\right|\left(\Delta^m\right).$ Theorem 3.3 (ii) implies that $x \notin S_\theta\left(\Delta^m\right).$ On the other hand if a sequence is strongly Δ^m-strongly Cesàro summable to L then it is Δ^m-statistically convergent to L (Theorem 4.2, Et and Nuray [5]). Hence $S\left(\Delta^m\right) \nsubseteq S_\theta\left(\Delta^m\right)$ and the proof is complete.

Lemma 3.6 For any lacunary sequence θ, $S_\theta\left(\Delta^m\right) \subset S\left(\Delta^m\right)$ if and only if $\limsup_r q_r < \infty$.

Proof. Sufficiency can be proved using the same technique of Lemma 3 of [8]. Now suppose that $\limsup_r q_r = \infty$. Consider the sequence defined by

$$\Delta^m x_i = \begin{cases} 1, & \text{if } k_{r_j-1} < i \leq 2k_{r_j-1} \quad \text{for some } j = 1, 2, 3, ... \\ 0, & \text{otherwise} \end{cases}.$$

Then $x \in N_\theta(\Delta^m)$ but $x \notin |\sigma_1|(\Delta^m)$. Clearly we have $x \in S_\theta(\Delta^m)$, but Theorem 4.2 of Et and Nuray [5] $x \notin S(\Delta^m)$. Hence $S_\theta(\Delta^m) \nsubseteq S(\Delta^m)$. This completes the proof.

Lemma 3.7 If £ denotes the set of all lacunary sequences, then

$$|AC|(\Delta^m) = \ell_\infty(\Delta^m) \cap (\cap_{\theta \in £} S_\theta(\Delta^m)).$$

Proof. Omitted.

Lemma 3.8 Let E be any of the spaces σ_1, $|\sigma_1|$, C_θ, N_θ, N_θ^0, AC, $|AC|$ and S_θ. Then the sequence spaces $E(\Delta^m)$ are neither solid nor symmetric nor sequence algebra nor convergence free nor perfect.

Proof. Proof follows from the following examples.

Example 1. Let $\theta = (2^r)$. Then $x = (k) \in N_\theta^0(\Delta^2)$, but $\alpha x = (\alpha_k x_k) \notin N_\theta^0(\Delta^2)$, for $\alpha_k = (-1)^k$ for all $k \in \mathbb{N}$. Hence $N_\theta^0(\Delta^m)$ is not solid.

Example 2. Let $\theta = (2^r)$. Then $x = (k) \in (N_\theta)(\Delta)$. Let (y_k) be a rearrangement of (x_k), which is defined as follows:

$$(y_k) = \{x_1, x_2, x_4, x_3, x_9, x_5, x_{16}, x_6, x_{25}, x_7, x_{36}, x_8, x_{49}, x_{10}, ...\}.$$

Then $(y_k) \notin (N_\theta)(\Delta)$.

Example 3. Let $\theta = (2^r)$. Then $x = (k) \in N_\theta^0(\Delta^2)$. Let (y_k) be a rearrangement of (x_k), which is defined as above, then $(y_k) \notin N_\theta^0(\Delta^2)$.

Example 4. Let $\theta = (2^r)$. Consider the sequences $x = (k)$, $y = (k^{m-1})$, then $x, y \in N_\theta^0(\Delta^m)$ but $x.y \notin N_\theta^0(\Delta^m)$. For the others spaces consider the sequences $x = (k)$, $y = (k^m)$.

Example 5. Let $\theta = (2^r)$. Then $(x_k) = (1)$ is in $N_\theta^0(\Delta)$. The sequence (y_k) defined as $y_k = k$ for all $k \in \mathbb{N}$ does not belong to $N_\theta^0(\Delta)$. Hence $N_\theta^0(\Delta)$ is not convergence free.

Note. Similarly different examples can be constructed for the other spaces.

Now we will give some relations between Δ^m-lacunary statistically convergent sequences and Δ^m- lacunary strongly summable sequences with respect to a modulus function.

Theorem 3.9 The inclusion $[f,p,\theta](\Delta^{m-1}) \subset [f,p,\theta](\Delta^m)$ is strict. In general $[f,p,\theta](\Delta^i) \subset [f,p,\theta](\Delta^m)$ for all $i = 1,2,\ldots,m-1$ and the inclusion is strict.

Proof. Straight forward and hence omitted.

Theorem 3.10 Let f, f_1, f_2 be modulus functions. Then we have

i) $[f,\theta](\Delta^m) \subset [f \circ f_1,\theta](\Delta^m)$,

ii) $[f_1,p,\theta](\Delta^m) \cap [f_2,p,\theta](\Delta^m) \subset [f_1+f_2,p,\theta](\Delta^m)$.

Proof. i) Let $\varepsilon > 0$ and choose δ with $0 < \delta < 1$ such that $f(t) < \varepsilon$ for $0 \le t \le \delta$. Write $y_k = f_1(|\Delta^m x_k - L|)$ and consider

$$\sum_{k \in I_r} f(y_k) = \sum_1 f(y_k) + \sum_2 f(y_k)$$

where the first summation is over $y_k \le \delta$ and second summation is over $y_k > \delta$. Since f is continuous, we have

$$\sum_1 f(y_k) < h_r \varepsilon \tag{2}$$

and for $y_k > \delta$, we use the fact that

$$y_k < \frac{y_k}{\delta} \le 1 + \frac{y_k}{\delta}.$$

By the definition of f we have for $y_k > \delta$,

$$f(y_k) < 2f(1)\frac{y_k}{\delta}.$$

Hence

$$\sum_2 f(y_k) \le 2f(1)\delta^{-1} \sum_{k=1}^{n} y_k. \tag{3}$$

From(2) and (3), we obtain $[f,\theta](\Delta^m) \subset [f \circ f_1,\theta](\Delta^m)$.

ii) The proof of (ii) follows from the following inequality

$$[(f_1+f_2)(|\Delta^m x_k - L|)]^{p_k} \le C[f_1(|\Delta^m x_k - L|)]^{p_k} + C[f_2(|\Delta^m x_k - L|)]^{p_k}.$$

The following result is a consequence of Theorem 3.10 (i).

Proposition 3.11 ([2]) Let f be a modulus function. Then $N_\theta\left(\Delta^m\right) \subset [f,\theta]\left(\Delta^m\right)$.

Theorem 3.12 Let $0 < p_k \leq q_k$ and (q_k/p_k) be bounded. Then $[f,q,\theta]\left(\Delta^m\right) \subset [f,p,\theta]\left(\Delta^m\right)$.

Proof: If we take $w_k = \left[f\left(\left|\Delta^m x_k - L\right|\right)\right]^{q_k}$ for all k. Following the technique applied for establishing Theorem 5 of Maddox [13], we can easily prove the theorem.

Theorem 3.13 The sequence space $[f,p,\theta]\left(\Delta^m\right)$ is neither solid nor symmetric nor sequence algebra nor convergence free nor perfect for $m \geq 1$.

To show these, consider the examples cited in Lemma 3.8.

Theorem 3.14 Let f be modulus function and $\sup_k p_k = H$. Then $[f,p,\theta]\left(\Delta^m\right) \subset S_\theta\left(\Delta^m\right)$.

Proof. Let $x \in [f,p,\theta]\left(\Delta^m\right)$ and $\varepsilon > 0$ be given. Then

$$\frac{1}{h_r}\sum_{k\in I_r}\left[f\left(\left|\Delta^m x_k - L\right|\right)\right]^{p_k} = \frac{1}{h_r}\sum_{\substack{k\in I_r\\ \left|\Delta^m x_k - L\right|\geq\varepsilon}}\left[f\left(\left|\Delta^m x_k - L\right|\right)\right]^{p_k}$$

$$+\frac{1}{h_r}\sum_{k\in I_r\left|\Delta^m x_k - L\right| < \varepsilon}\left[f\left(\left|\Delta^m x_k - L\right|\right)\right]^{p_k}$$

$$\geq \frac{1}{h_r}\sum_{\substack{k\in I_r\\ \left|\Delta^m x_k - L\right|\geq\varepsilon}}\left[f\left(\left|\Delta^m x_k - L\right|\right)\right]^{p_k} \geq \frac{1}{h_r}\sum_{k\in I_r}\left[f\left(\varepsilon\right)\right]^{p_k}$$

$$\geq \frac{1}{h_r}\sum_{k\in Ir}\min\left(\left[f\left(\varepsilon\right)\right]^{\inf p_k}, \left[f\left(\varepsilon\right)\right]^{H}\right)$$

$$\geq \frac{1}{h_r}\left|\left\{k\in I_r : \left|\Delta^m x_k - L\right|\geq\varepsilon\right\}\right| \min\left(\left[f\left(\varepsilon\right)\right]^{\inf p_k}, \left[f\left(\varepsilon\right)\right]^{H}\right).$$

Hence $x \in S_\theta\left(\Delta^m\right)$.

Theorem 3.15 Let f be bounded and $0 < h = \inf_k p_k \leq p_k \leq \sup_k p_k = H < \infty$. Then $S_\theta\left(\Delta^m\right) \subset [f,p,\theta]\left(\Delta^m\right)$.

Proof. Suppose that f is bounded and let $\varepsilon > 0$ be given. Then

$$\frac{1}{h_r}\sum_{k\in I_r}\left[f\left(\left|\Delta^m x_k - L\right|\right)\right]^{p_k} = \frac{1}{h_r}\sum_{k\in I_r\left|\Delta^m x_k - L\right| \geq \varepsilon}\left[f\left(\left|\Delta^m x_k - L\right|\right)\right]^{p_k}$$

$$+\frac{1}{h_r}\sum_{k\in I_r |\Delta^m x_k - L| < \varepsilon} [f(|\Delta^m x_k - L|)]^{p_k}$$

$$\leq \frac{1}{h_r}\sum_{k\in I_r} \max\left(K^h, K^H\right) + \frac{1}{h_r}\sum_{k\in I_r} [f(\varepsilon)]^{p_k}$$

$$\leq \max\left(K^h, K^H\right)\frac{1}{h_r}\left|\{k\in I_r : |\Delta^m x_k - L| \geq \varepsilon\}\right|$$

$$+\max\left(f(\varepsilon)^h,\ f(\varepsilon)^H\right).$$

Hence $x \in [f,p,\theta](\Delta^m)$.

Theorem 3.16 Let f be bounded and $0 < h = \inf_k p_k \leq p_k \leq \sup_k p_k = H < \infty$. Then $S_\theta(\Delta^m) = [f,p,\theta](\Delta^m)$ if and only if f is bounded.

Proof. Let f be bounded. By Theorem 3.14 and Theorem 3.15 we have $S_\theta(\Delta^m) = [f,p,\theta](\Delta^m)$.

Conversely suppose that f is unbounded. Then there exists a sequence (t_k) of positive numbers with $f(t_k) = k^2$, for $k = 1, 2, ...$. If we choose

$$\Delta^m x_i = \begin{cases} t_k, & i = k^2,\ i = 1, 2, \ldots \\ 0, & \text{otherwise} \end{cases}$$

then we have

$$\frac{1}{n}\left|\{k \leq n : |\Delta^m x_k| \geq \varepsilon\}\right| \leq \frac{\sqrt{n}}{n}$$

for all n and so $x \in S_\theta(\Delta^m)$, but $x \notin [f,p,\theta](\Delta^m)$ for $\theta = (2^r)$ and $p_k = 1$ for all $k \in \mathbb{N}$. This contradicts to $S_\theta(\Delta^m) = [f,p,\theta](\Delta^m)$.

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BOOK REVIEWS

Jürgen Appel, Espedito De Pascale and Alfonso Vignoli, ***Nonlinear Spectral Theory,*** De Gruyter Series in Nonlinear Analysis and Applications, Vol. 10, Walter de Gruyter, Berlin - New York 2004, xi + 408 pages, ISBN: 3-11-018143-6.

The spectral theory of bounded linear operators on Banach spaces is one of the most important branches of functional analysis and operator theory, with deep and far reaching applications to spectral theory of differential operators and to classical quantum mechanics.

It is expected that a reasonable definition of the spectrum of a continuous nonlinear operator F acting on a Banach space X should agree with the usual one when F is linear and, at a same time, to retain some of its essential properties, as nonemptiness, compactness, to contain the eigenvalues, etc. By a sequence of 8 simple examples given in the introduction the authors show some of the drawbacks of various natural definitions of the spectrum of a nonlinear operator, leading them to the conclusion that the main matter is not the intrinsic structure of the spectrum, but rather its usefulness in the study of nonlinear operator equations.

The book contains a systematic presentation of various spectra for nonlinear operators, along with some applications. Numerous examples and tables illustrate the relations between these spectra, as well as some shortcomings arising for each of them.

The first chapter contains an overview of the spectral theory of bounded linear operators, the second one is concerned with various metric and topological properties of nonlinear operators (the Lipschitz property, α-contractibility, etc), while the third one presents some results on the invertibility of nonlinear operators, a question closely related to the solvability of nonlinear equations - the main target of the nonlinear spectral theory.

Various kinds of spectra for nonlinear operators are presented in Chapters 4 through 9: 4. *The Rhodius and Neuberger spectra*, 5. *The Kachurovskij and Dörfner spectra*, 6. *The Furi-Martelli-Vignoli spectrum*, 7. *The Feng spectrum*, 8. *The Väth phantom*, 9. *Other spectra*.

Chapter 10 is concerned with the quite subtle notion of eigenvalue of a nonlinear operator. Again, a direct transpose of the definition to the nonlinear case does not fit best the needs of the theory, this being done by other equivalent definitions, apparently different from the familiar one. Chapter 11 emphasizes through an appropriate definition of the numerical range of a nonlinear operator, the influence of the geometry of the underlying Banach space in the study of nonlinear spectra. The last chapter of the book, Chapter 12, is devoted to applications to general solvability of nonlinear equations and to bifurcation theory. A nonlinear Fredholm theory is applied existence and perturbation results for p-Laplacian.

Each chapter ends with a section of bibliographical and historical notes and remarks. The book is fairly self-contained, the prerequisites being a modes background in nonlinear functional analysis and spectral theory.

As the authors point out in the introduction, the theory is far from being complete - in fact there is no a satisfactory definition of the spectrum in the nonlinear case. The book can be considered as a systematic introduction to this area, emphasizing the diversity of directions in which current research in nonlinear spectral is developing.

S. Cobzaş

Jon P. Davis, ***Methods of Applied Mathematics with a MATLAB overview***, Birkhäuser Verlag, 2004, XII, 721 p., ISBN: 0-8176-4331-1.

This book is devoted to the application of Fourier Analysis. The author mixed in a remarkable way theoretical results and application illustrating the results. Flexibility of presentation (increasing and decreasing level of rigor, accessibility) is a key feature.

The first chapter is an introductory one.

An introduction to Fourier series based mainly on inner product spaces is given in chapter 2.

The third chapter treats elementary boundary value problems. Besides applications of the Fourier series, it presents standard boundary value problem models and their discrete analogous problems.

Higher-dimensional, non rectangular problems is the topic of the fourth chapter. These includes Sturm-Liouville Theory, series solutions, Bessel equations and nonhomogeneous boundary value problems.

Chapter 5 is an introduction to functions of complex variable. Here ones discuss basic results and their applications to problems of fluid flow and transform inversion.

The sixth chapter introduces Laplace transform and their applications to ordinary differential equations, circuit analysis and input-output analysis of linear systems.

Continuous Fourier transform is the topic of seventh chapter. Also applications of Fourier transform to ordinary differential equations, integral equations, partial differential equations are included here.

Chapter eight is on discrete variable transforms. It treats discrete variable models, z-transform, discrete and fast Fourier transform and their properties. Computational aspects of fast Fourier transform are also pointed.

The last chapter "Additional Topics" introduces methods that are specialization of those treated previously such as two-sided and Walsh transform, wavelets analysis and integral transform.

The book contains extensive examples, presented in an intuitive way with high quality figure (some of them quite spectacular), useful MATLAB codes. MATLAB exercises and routines are well integrated within the text, and a concise introduction into MATLAB is given in an appendix. The emphasis is on program's numerical and graphical capabilities and its applications, not on its syntax. A large variety of problems graded from difficulty point of view. Applications are modern and up to date. Reach and comprehensive references are attached to each chapter.

Intended audience: especially students in pure and applied mathematics, physics and computer science, but also useful to applied mathematicians, engineers and computer scientists interested in applications of Fourier analysis.

Radu Trîmbiţaş

Donaldson, S.K., Eliashberg, Y., Gromov, M. (Eds.), ***Different Faces of Geometry***, Kluwer Academic / Plenum Press (International Mathematical Series), 2004, Hardback, 404 pp., ISBN 0-306-48657-1.

Everybody knows how difficult can be to give a proper definition. This is, particularly, true when it comes to geometry. I think it's quite impossible to give a definition of contemporary geometry. Definitely, the old ethymological definition doesn't do the job anymore. In fact, the editors (three of the most influential mathematicians of our times, who don't need any formal introduction) claim that "there is, perhaps, no branch of mathematics which cannot be considered a part of geometry, when approached in the right spirit". Their idea, therefore is that it is probably better to think of geometry as being rather a collection of subjects than a single field. To put it another way, the geometry has many "faces".

The aim of the editors of this book is to provide a readable description of some of these faces, by asking leading specialists to discuss the current state and prospects of their fields of expertise. These fields include (but are not restricted to): amoebas and tropical geometry, convex geometry, differential geometry of 4-manifolds, 3-dimensional contact geometry, Lagrangian and Special Lagrangian submanifolds, Floer homology. It is probably no accident that many of these topics are closely related to the research interests of the editors themselves.

While I didn't mentioned all the subjects touched in this book, I would like, nevertheless, to mention at least the authors of the contributions: G. Mikhalkin, V.D. Milman, A.A. Giannopoulos, C.LeBrun, Ko Honda, P. Ozsváth, Z. Szabó, C. Simpson, D. Joyce, P. Seidel and S. Bauer.

The books of this kind, providing a rapid access to reliable information on different fields of mathematics are of a great help for many people, from graduate students and researchers. Nowadays is quite difficult to find your way through a field which is not exactly your own and a hand lent by an expert is always very helpful. The book under review is no exception. The subjects chosen belong to the most active fields of research in the last period and the authors manage to describe them in an accessible way. I would gladly recommend it to anyone with an interest in geometry, even if he/she has no intention whatsoever to specialize in one of the fields described in the book. It's always nice to know what your neighbors are doing.

I'd like to finish this review by mentioning that this book (like any other in this series, edited by the Russian mathematician Tamara Rozhkovskaya) was simultaneously published in Russian.

Paul A. Blaga

Pei-Kee Lin, ***Köthe-Bochner Function Spaces,*** Birkhäuser Verlag, Boston-Basel-Berlin 2004, xii+370 pp, ISBN: 0-8176-3521-1.

Let (Ω, Σ, μ) be a complete measure space. A real Banach space E consisting of equivalence classes (modulo equality a.e.) of locally integrable real-valued functions is called a Köthe function space provided:

(i) if $h \in E$ and $g : \Omega \to \mathbb{R}$ is measurable and $g(\omega)| \leq |h(\omega)|$ a.e. on Ω, then $g \in E$ and $\|g\| \leq \|h\|$;

(ii) for every $A \in \Sigma$ with $\mu(A) < \infty$ the characteristic function 1_A of A belongs to E.

Every Köthe function space is a Banach lattice with respect to the pointwise order: $f \leq g \iff f(\omega) \leq g(\omega)$ a.e. on Ω. Köthe function spaces form an important class of Banach function spaces and Banach lattices as can be seen, for instance, from the second volume of the treatise J. Lindenstrauss and L. Tzafriri, *Classical Banach spaces,* Springer Verlag, Berlin 1979. If X is a Banach space and E is a Köthe function space over the complete measure space (Ω, μ), then the Köthe-Bochner function space $E(X)$ is formed by all strongly measurable functions $f : \Omega \to X$ such that the function $\omega \mapsto \|f(\omega)\|_X$ belongs to E. Equipped with the norm $\|\|f(\cdot)\|_X\|_E$, $E(X)$ is a Banach space.

The main questions the author of the present book addresses are: if both of the spaces E and X have a geometric property P, then does the space $E(X)$ have the same property, and conversely, if $E(X)$ has the property P, then must E and X have the property P ? For P one takes various rotundity and smoothness conditions (strict convexity, local uniform convexity, uniform convexity, smoothness, etc) or other properties of geometric or topological nature as Dunford-Pettis, Radon-Nikodým, Kadec-Klee properties. Chapters 5.I and 5.II, both headed *Stability properties,* are concerned with the following problem: if f is an extreme (smooth, exposed, etc) point of the unit ball of $E(X)$ then is $f(\omega)/\|f(\cdot)\|$ and extreme (smooth, exposed, etc) point of the unit ball of X for a.e. $\omega \in \text{supp} f$, and, conversely, is this property sufficient for f to be an extreme (smooth, exposed, etc) point of the unit ball of $E(X)$? These chapters contain also a discussion of the containment of c_0 and ℓ_1 in $E(X)$.

The basic properties of Köthe and Köthe-Bochner function spaces are treated in the third chapter *Köthe-Bochner function spaces.* Chapters 1, *Classical theorems* and 2, *Convexity and smoothness,* contain some basic results (most of them with complete proofs) on Banach spaces as strict convexity, uniform convexity, smoothness, Dunford-Pettis property, conditional expectations and martingales, tensor products. The last chapter of the book, Chapter 6, *Continuous function spaces,* is concerned with the Banach space $C(K, X)$.

Each chapter contains a set of exercises completing the main text, open questions for further study, remarks and historical notes, and bibliography.

The book is clearly written and succeeds to present in an accessible manner some deep and difficult results in the domain. It can be recommended for advanced graduate students and for researchers in functional analysis, probability theory, operator theory and related fields.

S. Cobzaş

Ole Christensen and Khadija L. Christensen, ***Approximation theory - From Taylor Polynomials to Wavelets***, Applied and Numerical Harmonic Analysis Series, Birkhäuser Verlag, Boston-Basel-Berlin 2004, xi+156 pp, ISBN:0-8176-3600-5.

This book contains an elementary introduction to approximation theory, in a way which naturally leads to the modern field of wavelets. One of the main goals of this presentation is to make it clear to the reader that the mathematics is a subject in a state of continuous evolution. The exposition demonstrates the dynamic nature of mathematics and how the classical disciplines influence many areas of modern mathematics and their applications.

The focus here is on ideas rather than on technical details. The book may be used in courses on infinite series and Fourier series, where ideas and motivation are more important than proofs. Some of the material from the two chapters on wavelets can be used as a guide towards more recent research. The wavelets are presented as a natural continuation of the material from the previous chapters.

The information is accessible to readers at several levels. Some basic material is placed at the beginning of each chapter preparing the reader for the more advanced concepts and topics in the latter part of that chapter. Only selected results are proved, while more technical proofs are included in an appendix.

The first chapter, dedicated to approximation by polynomials, contains elementary results. It also gives an idea about the content of the entire book. The next chapter presents the infinite series. It contains several classical entertaining examples and constructions. The Fourier analysis is treated in Chapter 3.

Wavelet analysis can be considered a modern supplement to classical Fourier analysis. Therefore, the chapters 4 and 5 are dedicated to this subject. Chapter 4 describes wavelets more in words rather than in symbols, but it gives the reader an understanding of the fundamental questions and concepts involved. It also tells the story of how the wavelets era began and discusses the applications in the signal processing.

In Chapter 5, which is slightly more technical, the multiscale representation associated to wavelets in the special case of the Haar wavelets is explained. It also presents the Gabor system. In this chapter the role of wavelets in digital signal processing and data compression is discussed, along with the FBI's manner of using wavelets to store fingerprints.

Each of the chapters contains more examples and ends with a few exercises. The book can be used as a good textbook or for self-study reference for students. Readers find the motivation and the background material pointing towards advanced literature and research topics in pure and applied harmonic analysis and related areas.

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