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ON BASIC FUZZY KOROVKIN THEORY

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Abstract. We prove the basic fuzzy Korovkin theorem via a fuzzy Shisha–Mond inequality given here. This determines the degree of convergence with rates of a sequence of fuzzy positive linear operators to the fuzzy unit operator. The surprising fact is that only the real case Korovkin assumptions are enough for the validity of the fuzzy Korovkin theorem, along with a natural realization condition fulfilled by the sequence of fuzzy positive linear operators. The last condition is fulfilled by almost all operators defined via fuzzy summation or fuzzy integration.

0. **Introduction**

Motivation for this work are the references [1], [2], [5], [6]. Our results Theorems 3 and 4 are simple, basic and very general, directly transferring the real case of the convergence with rates of positive linear operators to the unit, to the fuzzy one. The same real assumptions are kept here in the fuzzy setting, and they are the only assumptions we make along with the very natural and general realization condition (1). Condition (1) is fulfilled by almost all example — fuzzy positive operators, that is, by most fuzzy summation and fuzzy integration operators. At each step of our work we provide an example to justify our method. To the best of our knowledge our theorems are the first general fuzzy Korovkin type results.

1. **Background**

We start with

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Definition 1 (see [8]). Let $\mu\colon \mathbb{R} \to [0,1]$ with the following properties:

(i) is *normal*, i.e., $\exists x_0 \in \mathbb{R}$; $\mu(x_0) = 1$.

(ii) $\mu(\lambda x + (1-\lambda)y) \geq \min\{\mu(x), \mu(y)\}$, $\forall x, y \in \mathbb{R}$, $\forall \lambda \in [0,1]$ (μ is called a convex fuzzy subset).

(iii) μ is *upper semicontinuous* on $\mathbb{R}$, i.e., $\forall x_0 \in \mathbb{R}$ and $\forall \varepsilon > 0$, $\exists$ neighborhood $V(x_0)$: $\mu(x) \leq \mu(x_0) + \varepsilon$, $\forall x \in V(x_0)$.

(iv) The set $\overline{\operatorname{supp}(\mu)}$ is compact in $\mathbb{R}$ (where $\operatorname{supp}(\mu) := \{x \in \mathbb{R};\ \mu(x) > 0\}$).

We call μ a *fuzzy real number*. Denote the set of all μ with $\mathbb{R}_{\mathcal{F}}$.

E.g., $\mathcal{X}_{\{x_0\}} \in \mathbb{R}_{\mathcal{F}}$, for any $x_0 \in \mathbb{R}$, where $\mathcal{X}_{\{x_0\}}$ is the characteristic function at x_0.

For $0 < r \leq 1$ and $\mu \in \mathbb{R}_{\mathcal{F}}$ define $[\mu]^r := \{x \in \mathbb{R}\colon \mu(x) \geq r\}$ and

$$[\mu]^0 := \overline{\{x \in \mathbb{R} : \mu(x) > 0\}}.$$

Then it is well known [3] that for each $r \in [0,1]$, $[\mu]^r$ is a closed and bounded interval of $\mathbb{R}$. For $u, v \in \mathbb{R}_{\mathcal{F}}$ and $\lambda \in \mathbb{R}$, we define uniquely the sum $u \oplus v$ and the product $\lambda \odot u$ by

$$[u \oplus v]^r = [u]^r + [v]^r, \quad [\lambda \odot u]^r = \lambda [u]^r, \qquad \forall r \in [0,1],$$

where $[u]^r + [v]^r$ means the usual addition of two intervals (as subsets of $\mathbb{R}$) and $\lambda[u]^r$ means the usual product between a scalar and a subset of $\mathbb{R}$ (see, e.g., [4]). Notice $1 \odot u = u$ and it holds $u \oplus v = v \oplus u$, $\lambda \odot u = u \odot \lambda$. If $0 \leq r_1 \leq r_2 \leq 1$ then $[u]^{r_2} \subseteq [u]^{r_1}$. Actually $[u]^r = [u_-^{(r)}, u_+^{(r)}]$, where $u_-^{(r)} \leq u_+^{(r)}$, $u_-^{(r)}$, $u_+^{(r)} \in \mathbb{R}$, $\forall r \in [0,1]$. We denote $u \precsim v$ iff $u_-^{(r)} \leq v_-^{(r)}$ and $u_+^{(r)} \leq v_+^{(r)}$, all $r \in [0,1]$. Define

$$D\colon \mathbb{R}_{\mathcal{F}} \times \mathbb{R}_{\mathcal{F}} \to \mathbb{R}_+$$

by

$$D(u,v) := \sup_{r \in [0,1]} \max\{|u_-^{(r)} - v_-^{(r)}|, |u_+^{(r)} - v_+^{(r)}|\},$$

where $[v]^r = [v_-^{(r)}, v_+^{(r)}]$; $u, v \in \mathbb{R}_{\mathcal{F}}$. We have that D is a metric on $\mathbb{R}_{\mathcal{F}}$. Then $(\mathbb{R}_{\mathcal{F}}, D)$ is a complete metric space, see [7], with the properties

$$\begin{aligned} D(u \oplus w, v \oplus w) &= D(u, v), \quad \forall u, v, w \in \mathbb{R}_{\mathcal{F}}, \\ D(k \odot u, k \odot v) &= |k| D(u, v), \quad \forall u, v \in \mathbb{R}_{\mathcal{F}}, \ \forall k \in \mathbb{R}, \\ D(u \oplus v, w \oplus e) &\leq D(u, w) + D(v, e), \quad \forall u, v, w, e \in \mathbb{R}_{\mathcal{F}}. \end{aligned}$$

Let $f, g\colon [a, b] \to \mathbb{R}_{\mathcal{F}}$, $[a, b] \subseteq \mathbb{R}$, be *fuzzy real number valued functions*. The distance between f, g is defined by

$$D^*(f, g) := \sup_{x \in [a,b]} D(f(x), g(x)).$$

Here $\sum^*$ stands for the fuzzy summation.

We use the following

Definition 2. Let $f\colon [a, b] \to \mathbb{R}_{\mathcal{F}}$ be a fuzzy real number valued function. We define the (first) *fuzzy modulus of continuity* of f by

$$\omega_1^{(\mathcal{F})}(f, \delta) := \sup_{\substack{x, y \in [a,b] \\ |x-y| \leq \delta}} D(f(x), f(y)),$$

any $0 < \delta \leq b - a$.

Definition 3. Let $f\colon [a, b] \subseteq \mathbb{R} \to \mathbb{R}_{\mathcal{F}}$. We say that f is *fuzzy continuous at* $x_0 \in [a, b]$ iff whenever $x_n \to x_0$, then $D(f(x_n), f(x_0)) \to 0$, as $n \to \infty$, $n \in \mathbb{N}$. We call f *fuzzy continuous* iff it is fuzzy continuous $\forall x \in [a, b]$ and we denote the space of fuzzy continuous functions by $C_{\mathcal{F}}([a, b])$.

The space $C_{\mathcal{F}}([a, b])$ is only a cone and not a vector space, however any finite linear combination of its elements with scalars in $\mathbb{R}$ belongs there.

Denote $[f]^r = [f_-^{(r)}, f_+^{(r)}]$ and we mean

$$[f(x)]^r = \left[f_-^{(r)}(x), f_+^{(r)}(x)\right], \quad \forall x \in [a, b], \quad \text{all } r \in [0, 1].$$

Let $f, g \in C_{\mathcal{F}}([a, b])$ we say that f is *fuzzy larger* than g *pointwise* and we denote it by $f \succsim g$ iff $f(x) \succsim g(x)$ iff $f_-^{(r)}(x) \geq g_-^{(r)}(x)$ and $f_+^{(r)}(x) \geq g_+^{(r)}(x)$, $\forall x \in [a, b]$, $\forall r \in [0, 1]$, iff $f_-^{(r)} \geq g_-^{(r)}$, $f_+^{(r)} \geq g_+^{(r)}$, $\forall r \in [0, 1]$.

Let L be a map from $C_{\mathcal{F}}([a,b])$ into itself, we call it a *fuzzy linear operator* iff

$$L\big(c_1 \odot f_1 \oplus c_2 \odot f_2\big) = c_1 \odot L(f_1) \oplus c_2 \odot L(f_2),$$

for any $c_1, c_2 \in \mathbb{R}$, $f_1, f_2 \in C_{\mathcal{F}}([a,b])$. We say that L is a *fuzzy positive linear operator* iff for $f, g \in C_{\mathcal{F}}([a,b])$ with $f \succsim g$ we get $L(f) \succsim L(g)$ iff $(L(f))_-^{(r)} \geq (L(g))_-^{(r)}$ and $(L(f))_+^{(r)} \geq (L(g))_+^{(r)}$ on $[a,b]$ for all $r \in [0,1]$.

Example 1. Let $f \in C_{\mathcal{F}}([0,1])$, we define the *fuzzy Bernstein operator*

$$\big(B_n^{(\mathcal{F})}(f)\big)(x) = \sum_{k=0}^{n}{}^{*} \binom{n}{k} x^k (1-x)^{n-k} \odot f\left(\frac{k}{n}\right), \quad \forall x \in [0,1],\ n \in \mathbb{N}.$$

This is a fuzzy positive linear operator.

We mention the very interesting with rates approximation motivating this work.

Theorem 1 (see p. 642, [2], S. Gal). *If $f \in C_{\mathcal{F}}([0,1])$, then*

$$D^*\big(B_n^{(\mathcal{F})}(f), f\big) \leq \frac{3}{2}\omega_1^{(\mathcal{F})}\left(f, \frac{1}{\sqrt{n}}\right), \quad \forall n \in \mathbb{N}$$

i.e.

$$\lim_{n\to\infty} D^*\big(B_n^{(\mathcal{F})}(f), f\big) = 0,$$

that is $B_n^{(\mathcal{F})} f \to f$, $n \to \infty$ in fuzzy uniform convergence.

The last fact comes by the property that $\omega_1^{(\mathcal{F})}(f,\delta) \to 0$ as $\delta \to 0$, whenever $f \in C_{\mathcal{F}}([a,b])$.

We need to use

Theorem 2 (Shisha and Mond (1968), [6]). *Let $[a,b] \subseteq \mathbb{R}$. Let $(\tilde{L}_n)_{n\in\mathbb{N}}$ be a sequence of positive linear operators from $C([a,b])$ into itself. For $n = 1, 2, \ldots$, suppose $\tilde{L}_n(1)$ is bounded. Let $f \in C([a,b])$. Then for $n = 1, 2, \ldots$, we have*

$$\|\tilde{L}_n f - f\|_\infty \leq \|f\|_\infty \|\tilde{L}_n 1 - 1\|_\infty + \|\tilde{L}_n(1) + 1\|_\infty \omega_1(f, \mu_n),$$

where ω_1 is the standard real modulus of continuity and

$$\mu_n := \big\|(\tilde{L}_n((t-x)^2))(x)\big\|_\infty^{1/2},$$

and $\|\cdot\|_\infty$ *stands for the sup-norm over* $[a,b]$*. In particular, if* $L_n(1)=1$ *then*

$$\|\tilde{L}_n f - f\|_\infty \le 2\omega_1(f,\mu_n).$$

Note. One can easily see ([6]), for $n=1,2,\ldots$,

$$\mu_n^2 \le \left\|(\tilde{L}_n(t^2))(x)-x^2\right\|_\infty + 2c\left\|(\tilde{L}_n(t))(x)-x\right\|_\infty + c^2\left\|(\tilde{L}_n(1))(x)-1\right\|_\infty,$$

where $c := \max(|a|,|b|)$.

Assuming that $\tilde{L}_n(1) \xrightarrow{u} 1$, $\tilde{L}_n(id) \xrightarrow{u} id$, $\tilde{L}_n(id^2) \xrightarrow{u} id^2$ (id is the identity map), $n\to\infty$, uniformly, then from Theorem 2's main inequality we get $\tilde{L}_n(f) \xrightarrow{u} f$, $\forall f\in C([a,b])$, that is the famous Korovkin theorem (see [5]) in the real case.

We finally need

Lemma 1. *Let* $f\in C_{\mathcal{F}}([a,b])$*,* $[a,b]\subseteq\mathbb{R}$*. Then it holds*

$$\omega_1^{(\mathcal{F})}(f,\delta) = \sup_{r\in[0,1]} \max\left\{\omega_1(f_-^{(r)},\delta), \omega_1(f_+^{(r)},\delta)\right\},$$

for any $0<\delta\le b-a$*.*

Proof. Let $x,y\in[a,b]$: $|x-y|\le\delta$, $0<\delta\le b-a$. Then we have

$$\begin{aligned} D(f(x),f(y)) &= \sup_{r\in[0,1]} \max\left\{|(f(x))_-^{(r)}-(f(y))_-^{(r)}|, |(f(x))_+^{(r)}-(f(y))_+^{(r)}|\right\} \\ &\le \sup_{r\in[0,1]} \max\left\{\omega_1(f_-^{(r)},\delta), \omega_1(f_+^{(r)},\delta)\right\}. \end{aligned}$$

Thus

$$\omega_1^{(\mathcal{F})}(f,\delta) \le \sup_{r\in[0,1]} \max\left\{\omega_1(f_-^{(r)},\delta), \omega_1(f_+^{(r)},\delta)\right\}.$$

For any $r\in[0,1]$ and any $x,y\in[a,b]$: $|x-y|\le\delta$ we see that

$$\omega_1^{(\mathcal{F})}(f,\delta) \ge D(f(x),f(y)) \ge \left|(f(x))_-^{(r)}-(f(y))_-^{(r)}\right|, \left|(f(x))_+^{(r)}-(f(y))_+^{(r)}\right|.$$

Therefore

$$\omega_1(f_\pm^{(r)},\delta) \le \omega_1^{(\mathcal{F})}(f,\delta), \quad \forall r\in[0,1].$$

Hence

$$\sup_{r\in[0,1]} \max\left\{\omega_1(f_-^{(r)},\delta), \omega_1(f_+^{(r)},\delta)\right\} \le \omega_1^{(\mathcal{F})}(f,\delta),$$

proving the claim. ∎

Note. For $f \in C_{\mathcal{F}}([a,b])$ we get that f is fuzzy bounded and $\omega_1^{(\mathcal{F})}(f,\delta)$ is finite for all $0 < \delta \leq b-a$. Also $f_{\pm}^{(r)}$ are continuous on $[a,b]$ and $\omega_1(f_{\pm}^{(r)},\delta)$ are finite too, all $r \in [0,1]$.

2. **Main Results**

We present the fuzzy analog of Shisha–Mond inequality of Theorem 2.

Theorem 3. *Let $\{L_n\}_{n\in\mathbb{N}}$ be a sequence of fuzzy positive linear operators from $C_{\mathcal{F}}([a,b])$ into itself, $[a,b] \subseteq \mathbb{R}$. We assume that there exists a corresponding sequence $\{\tilde{L}_n\}_{n\in\mathbb{N}}$ of positive linear operators from $C([a,b])$ into itself with the property*

$$\big(L_n(f)\big)_{\pm}^{(r)} = \tilde{L}_n(f_{\pm}^{(r)}), \tag{1}$$

respectively, $\forall r \in [0,1]$, $\forall f \in C_{\mathcal{F}}([a,b])$. We assume that $\{\tilde{L}_n(1)\}_{n\in\mathbb{N}}$ is bounded. Then for $n \in \mathbb{N}$ we have

$$D^*(L_n f, f) \leq \|\tilde{L}_n 1 - 1\|_\infty D^*(f,\tilde{o}) + \|\tilde{L}_n(1) + 1\|_\infty \omega_1^{(\mathcal{F})}(f,\mu_n), \tag{2}$$

where

$$\mu_n := \Big(\big\|(\tilde{L}_n((t-x)^2))(x)\big\|_\infty\Big)^{1/2}, \tag{3}$$

$\forall f \in C_{\mathcal{F}}([a,b])$, $\tilde{o} := \mathcal{X}_{\{0\}}$ the neutral element for $\oplus$. If $\tilde{L}_n 1 = 1$, $n \in \mathbb{N}$, then

$$D^*(L_n f, f) \leq 2\omega_1^{(\mathcal{F})}(f,\mu_n). \tag{4}$$

Note. The fuzzy Bernstein operators $B_n^{(\mathcal{F})}$ and the real corresponding ones B_n acting on $C_{\mathcal{F}}([0,1])$ and $C([0,1])$, respectively, fulfill assumption (1).

We present now the Fuzzy Korovkin Theorem.

Theorem 4. *Let $\{L_n\}_{n\in\mathbb{N}}$ be a sequence of fuzzy positive linear operators from $C_{\mathcal{F}}([a,b])$ into itself, $[a,b] \subseteq \mathbb{R}$. We assume that there exists a corresponding sequence $\{\tilde{L}_n\}_{n\in\mathbb{N}}$ of positive linear operators from $C([a,b])$ into itself with the property*

$$\big(L_n(f)\big)_{\pm}^{(r)} = \tilde{L}_n(f_{\pm}^{(r)}), \tag{1}$$

respectively, $\forall r \in [0,1]$, $\forall f \in C_{\mathcal{F}}([a,b])$. *Furthermore assume that*

$$\tilde{L}_n(1) \xrightarrow{u} 1, \quad \tilde{L}_n(id) \xrightarrow{u} id, \quad \tilde{L}_n(id^2) \xrightarrow{u} id^2,$$

as $n \to \infty$, *uniformly. Then*

$$D^*(L_n f, f) \longrightarrow 0, \qquad \text{as } n \to \infty,$$

for any $f \in C_{\mathcal{F}}([a,b])$, *i.e.* $L_n f \xrightarrow{D^*} f$, *that is* $L_n \to I$ *unit operator in the fuzzy sense, as* $n \to \infty$.

Proof. Use of (2), property of (3), etc. □

Example for Theorem 4 the fuzzy Bernstein operators $B_n^{(\mathcal{F})}$.

Proof of Theorem 3. We would like to estimate

$$\begin{aligned}
D^*(L_n f, f) &= \sup_{x\in[a,b]} D\big((L_n f)(x), f(x)\big) \\
&= \sup_{x\in[a,b]} \sup_{r\in[0,1]} \max\big\{|(L_n f)_-^{(r)}(x) - (f)_-^{(r)}(x)|, |(L_n f))_+^{(r)}(x) - (f)_+^{(r)}(x)|\big\} \\
&= \sup_{x\in[a,b]} \sup_{r\in[0,1]} \max\big\{|\tilde{L}_n(f_-^{(r)}(x) - (f)_-^{(r)}(x)|, |\tilde{L}_n(f_+^{(r)})(x) - (f)_+^{(r)}(x)|\big\} \\
&= \sup_{r\in[0,1]} \max\big\{\|\tilde{L}_n f_-^{(r)} - f_-^{(r)}\|_\infty, \|\tilde{L}_n f_+^{(r)} - f_+^{(r)}\|_\infty\big\} \\
&\qquad\qquad \text{(by Theorem 2)} \\
&\le \sup_{r\in[0,1]} \max\big\{\big(\|f_-^{(r)}\|_\infty \|\tilde{L}_n 1 - 1\|_\infty + \|\tilde{L}_n(1) + 1\|_\infty \omega_1(f_-^{(r)}, \mu_n)\big), \\
&\qquad \big(\|f_+^{(r)}\|_\infty \|\tilde{L}_n 1 - 1\|_\infty + \|\tilde{L}_n(1) + 1\|_\infty \omega_1(f_+^{(r)}, \mu_n)\big)\big\} \\
&\le \|\tilde{L}_n 1 - 1\|_\infty \sup_{r\in[0,1]} \max\big(\|f_-^{(r)}\|_\infty, \|f_+^{(r)}\|_\infty\big) \\
&\quad + \|\tilde{L}_n(1) + 1\|_\infty \sup_{r\in[0,1]} \max\big\{\omega_1(f_-^{(r)}, \mu_n), \omega_1(f_+^{(r)}, \mu_n)\big\} \\
&\qquad\qquad \text{(by Lemma 1)} \\
&= \|\tilde{L}_n 1 - 1\|_\infty D^*(f, \tilde{o}) + \|\tilde{L}_n(1) + 1\|_\infty \omega_1^{(\mathcal{F})}(f, \mu_n),
\end{aligned}$$

proving (2). □

Application 1. Let $f \in C_{\mathcal{F}}([0,1])$ then by applying (2) we obtain

$$D^*(B_n^{(\mathcal{F})} f, f) \le 2\omega_1^{(\mathcal{F})}\left(f, \frac{1}{2\sqrt{n}}\right), \quad \forall n \in \mathbb{N}. \tag{5}$$

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SOME EXTENSION OF BIVARIATE TENSOR-PRODUCT FORMULA

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Abstract. In this article we construct bivariate approximation formulas with scattered data by extensions of bivariate tensor-product Lagrange formula, using spline univariate operators. The graphs of approximation functions are given.

Let $D \subseteq \mathbb{R}^2$ be an arbitrary domain, f a real-valued function defined on D, $Z = \{z_i |\ z_i = (x_i, y_i),\ i = \overline{1, N}\} \subset D$ and $I(f) = \{\lambda_k f |\ k = 1, \ldots, N\}$ a set of informations about f (evaluations of f and of certain of its derivatives at $z_1, \ldots, z_N$).

A general interpolation problem is: for a given function f find a function g that interpolates the data $I(f)$ i.e.

$$\lambda_k f = \lambda_k g, \quad k = \overline{1, N}.$$

Starting from bivariate Lagrange formula for the rectangular grid $\Pi = \{x_0, \ldots, x_m\} \times \{y_0, \ldots, y_n\}$:

$$f(x, y) = \sum_{i=0}^{m} \sum_{j=0}^{n} \frac{u(x)}{(x - x_i)u'(x_i)} \frac{v(y)}{(y - y_j)v'(y_j)} f(x_i, y_j) + (R_{mn}f)(x, y) \qquad (1)$$

where

$$(R_{mn}f)(x, y) = u(x)[x, x_0, \ldots, x_m; f(\cdot, y)] + \sum_{i=0}^{m} \frac{u(x)v(y)}{(x - x_i)u'(x_i)} [y, y_0, \ldots, y_n; f(x_i, \cdot)]$$

with $u(x) = (x - x_0) \ldots (x - x_m)$ and $v(y) = (y - y_0) \ldots (y - y_n)$, there are two generalizations.

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A first generalization of the formula (1) was given by J.F. Steffensen [7]:

$$f(x,y)=\sum_{i=0}^{m}\sum_{j=0}^{n_i}\frac{u(x)}{(x-x_i)u'(x_i)}\frac{v_i(y)}{(y-y_j)v_i'(y_j)}f(x_i,y_j)+(R_{m,n_i}f)(x,y) \qquad (2)$$

where

$$(R_{m,n_i}f)(x,y)=u(x)[x,x_0,\dots,x_m;f(\cdot,y)]+\sum_{i=0}^{m}\frac{u(x)v_i(y)}{(x-x_i)u_i'(x_i)}[y,y_0,\dots,y_{n_i};f(x_i,\cdot)]$$

with

$$v_i(y)=(y-y_0)\dots(y-y_{n_i}).$$

The interpolation grid here is $\Pi_1=\{(x_i,y_j)|\ i=\overline{0,m},\ j=\overline{0,n_i}\}$.

A second generalization of the Lagrange interpolation formula (1), that is also an extension of the Steffensen formula (2) was given by D.D. Stancu [5]:

$$f(x,y)=\sum_{i=0}^{m}\sum_{j=0}^{n_i}\frac{u(x)}{(x-x_i)u'(x_i)}\frac{v_i(y)}{(y-y_{ij})v_i'(y_{ij})}f(x_i,y_{ij})+(R_{m,n_i}f)(x,y) \qquad (3)$$

where

$$(R_{m,n_i}f)(x,y)=u(x)[x,x_0,\dots,x_m;f(\cdot,y)]+\sum_{i=0}^{m}\frac{u(x)v_i(y)}{(x-x_i)u'(x_i)}[y,y_{i0},\dots,y_{in_i};f(x,\cdot)]$$

with $v_i(y)=(y-y_{i0})\dots(y-y_{in_i})$ and the interpolation grid

$$\Pi_2=\{(x_i,y_{ij})|\ i=\overline{0,m},\ j=\overline{0,n_i}\}.$$

Remark. 1. *The Steffensen formula (2) does not solve the general interpolation problem, Π_1 is only a particular case of the interpolatory set $\{z_1,\dots,z_N\}$.. Formula (3) is really a solution of the considered general problem. Indeed, let $X_k\subset Z$ be the set of nodes (x_i,y_i), $i=\overline{1,N}$ with the same abscises x_k, i.e. $X_k=\{(x_k,y_{kj})|\ j=\overline{0,n_k}\}$ for all $k=0,1,\dots,m$. We have $X_i\neq X_j$ for $i\neq j$ and $Z=X_0\cup\dots\cup X_m$.Thus $\Pi_2=Z$. The set $\{x_0,\dots,x_m\}$ is obtained by projection of nodes set Z on Ox axis.*

If L_m^x is the Lagrange's operator for the interpolates nodes $x_0,\dots,x_m$ and $L_{n_i}^y$, $i=\overline{0,m}$ are the Lagrange's operators for the nodes $y_{i0},\dots,y_{in_i}$ respectively, then we have

$$f=L_m^x f+R_m^x f \qquad (4)$$

with

$$(L_m^x f)(x,y) = \sum_{i=0}^{m} \frac{u(x)}{(x-x_i)u'(x_i)} f(x_i,y)$$

and

$$f(x_i,\cdot) = (L_{n_i}^y f)(x_i,\cdot) + (R_{n_i}^y f)(x_i,\cdot), \quad i=\overline{0,m} \tag{5}$$

with

$$(L_{n_i}^y f)(x_i,y) = \sum_{j=0}^{n_i} \frac{v_i(y)}{(y-y_{ij})v_i'(y_{ij})} f(x_i,y_{ij}).$$

If the remainder terms are written with the divided differences, from (4) and (5) follows formula (3).

If we make the projection of nodes set Z on Oy axis(see [4],[8]), we consider the Lagrange's operator L_n^y which interpolates the nodes $y_0,\dots,y_n$ and Lagrange's operators $L_{m_i}^x$, $i=\overline{0,n}$ which interpolate the nodes $x_{i0},\dots,x_{im_i}$. In the first level of approximation we use the approximation formula

$$f = L_n^y f + R_n^y f \tag{6}$$

with

$$(L_n^y f)(x,y) = \sum_{i=0}^{n} \frac{v(y)}{(y-y_i)v'(y_i)} f(x,y_i)$$

where $v(y) = (y-y_0)\dots(y-y_n)$. For every $f(x,y_i)$, $i=\overline{0,n}$ we use in a second level of approximation the following formulas

$$f(\cdot,y_i) = (L_{m_i}^x f)(\cdot,y_i) + (R_{m_i}^x f)(\cdot,y_i), \quad i=\overline{0,n} \tag{7}$$

with

$$(L_{m_i}^x f)(x,y_i) = \sum_{j=0}^{m_i} \frac{u_i(x)}{(x-x_{ij})u_i'(x_{ij})} f(x_{ij},y_i).$$

where $u_i(x) = (x-x_{i0})\dots(x-x_{im_i})$. We obtain the following approximation formula

$$f(x,y) = \sum_{i=0}^{n}\sum_{j=0}^{m_i} \frac{u_i(x)}{(x-x_{ij})u_i'(x_{ij})}\frac{v(y)}{(y-y_i)v'(y_i)} f(x_{ij},y_i) + (R_{m_i,n}f)(x,y) \tag{8}$$

where

$$(R_{m_i,n}f)(x,y) = v(y)[y,y_0,\dots,y_n;f(x,\cdot)] + \sum_{i=0}^{n} \frac{u_i(x)v(y)}{(y-y_i)v'(y_i)}[x,x_{i0},\dots,x_{im_i};f(\cdot,y)]$$

Then interpolation grid of formula (8) is

$$\Pi_3 = \{(x_{ij}, y_i)|\ i = \overline{0,n},\ j = \overline{0,m_i}\}.$$

Remark. 2. *Formula (8) is also a solution of the considered general problem. Indeed, let $Y_k \subset Z$ be the set of nodes (x_i, y_i), $i = \overline{1,N}$ with the same ordinates y_k, i.e. $Y_k = \{(x_{kj}, y_k)|\ j = \overline{0,m_k}\}$ for all $k = 0,1,\dots,n$. We have $Y_i \neq Y_j$ for $i \neq j$ and $Z = Y_0 \cup \dots \cup Y_n$. Thus $\Pi_3 = Z$. The set $\{y_0,\dots,y_n\}$ is obtained by projection of nodes set Z on Oy axis.*

We denote

$$(P^x f)(x,y) = \sum_{i=0}^{m}\sum_{j=0}^{n_i} \frac{u(x)}{(x-x_i)u'(x_i)} \frac{v_i(y)}{(y-y_{ij})v'(y_{ij})} f(x_i, y_{ij}). \tag{9}$$

and

$$(P^y f(x,y) = \sum_{i=0}^{n}\sum_{j=0}^{m_i} \frac{u_i(x)}{(x-x_{ij})u_i'(x_{ij})} \frac{v(y)}{(y-y_i)v'(y_i)} f(x_{ij}, y_i) \tag{10}$$

The interpolation formula generated by the mean of operators P^x and P^y is

$$f(x,y) = (P^M f)(x,y) + (R^M f)(x,y) \tag{11}$$

where

$$(P^M f)(x,y) = \frac{(P^x f)(x,y) + (P^y f)(x,y)}{2} \tag{12}$$

and

$$(R^M f)(x,y) = \frac{(R_{m_i,n} f)(x,y) + (R_{m,n_i} f)(x,y)}{2}$$

The interpolation set of P^M is also Z.

Remark. 3. *Usually the degree m of the operator L_m^x is more greater than the largest degree of $L_{n_i}^y$ i.e. $m \gg \max\{n_0,\dots,n_m\}$ and degree n of the operator L_n^y is more greater than the largest degree of $L_{m_i}^x$ i.e. $n \gg \max\{m_0,\dots,m_n\}$, which imply a large computational complexity of the polynomials interpolation $(P^x f)(x,y)$ and $(P^y f)(x,y)$. From this reason and the another ones, in* [1],[2] *instead of Lagrange polynomial operator L_m^x is used a spline interpolation operator. In this article, in formula (6), instead of Lagrange polynomial operator L_n^y is used a spline interpolation operator.*

Let $S^y_{L,2r-1}$ be the spline interpolation operator of the degree $2r-1$, that interpolates the function f with regard to the variable y at the nodes (x, y_k), $k=\overline{0,n}$ i.e.

$$(S^y_{L,2r-1}f)(x,y)=\sum_{i=0}^{r-1}a_i^x y^i+\sum_{j=0}^{n}b_j^x(y-y_j)_+^{2r-1} \tag{13}$$

for which

$$\begin{cases}(S^y_{L,2r-1}f)(x,y_k)=f(x,y_k), & k=\overline{0,n}\\ (S^y_{L,2r-1}f)^{(0,p)}(x,\alpha)=0, & p=\overline{r,2r-1},\ \alpha>y_n\end{cases} \tag{14}$$

The spline function of Lagrange type can also be written in the form

$$(S^y_{L,2r-1}f)(x,y)=\sum_{k=0}^{n}s_k(y)f(x,y_k)$$

where s_k are the corresponding cardinal splines i.e., they are of the same form (13), but with the interpolatory conditions

$$s_k(y_j)=\delta_{kj},\quad k,j=\overline{0,n}.$$

This way, formula (8) becomes

$$f(x,y)=(P^y_S f)(x,y)+(R^y_S f)(x,y) \tag{15}$$

where

$$(P^y_S f)(x,y)=\sum_{i=0}^{n}\sum_{j=0}^{m_i}s_i(y)\frac{u_i(x)}{(x-x_{ij})u_i'(x_{ij})}f(x_{ij},y_i)$$

and $(R^y_S f)(x,y)$ is the remainder term.

Taking into account that for $f(x,\cdot)\in C^r[y_0,y_n]$

$$(R^y_{L,2r-1}f)(x,y)=\int_{y_0}^{y_n}\varphi_r(y,t)f^{(0,r)}(x,t)dt$$

with

$$\varphi_r(y,t)=\frac{(y-t)_+^{r-1}}{(r-1)!}-\sum_{i=0}^{n}s_i(y)\frac{(y_i-t)_+^{r-1}}{(r-1)!}$$

it follows

Theorem. 4. *If $f \in C^{0,r}(D)$ then*

$$(R_S^y f)(x,y) = \int_{y_0}^{y_n} \varphi_r(y,t) f^{(0,r)}(x,t)dt + \tag{16}$$

$$+ \sum_{i=0}^{n} s_i(y) u_i(x)[x, x_{i0}, \ldots, x_{im_i}; f(x_i, \cdot)]$$

and if $f \in C^{p+1,r}(D)$ with $p = \max\{m_0, \ldots, m_n\}$ we have

$$(R_S^y f))(x,y) = \int_{y_0}^{y_m} \varphi_r(y,t) f^{(0,r)}(x,t)dt + \tag{17}$$

$$+ \sum_{i=0}^{n} s_i(y) \int_{x_{i0}}^{x_{im_i}} \psi_{m_i}(x,s) f^{(m_i+1,0)}(s, y_i)ds$$

with

$$\psi_{m_i}(x,s) = \frac{(x-s)_+^{m_i}}{m_i!} - \sum_{j=0}^{m_i} \frac{u_i(x)}{(x-x_{ij})u_i'(x_{ij})} \frac{(x_{ij}-s)_+^{m_i}}{m_i!}.$$

If the projection of nodes set is on Ox axis, we have from [1], [2] the interpolation formula

$$f(x,y) = (P_S^x f)(x,y) + (R_S^x f)(x,y) \tag{18}$$

where

$$(P_S^x f)(x,y) = \sum_{i=0}^{m} \sum_{j=0}^{n_i} s_i(x) \frac{v_i(y)}{(y-y_{ij})v_i'(y_{ij})} f(x_i, y_{ij})$$

and $(R_S^x f)(x,y)$ is the remainder term.

Theorem. 5. *([1], [2]) If $f \in C^{r,p+1}(D)$ with $p = \max\{n_0, \ldots, n_m\}$ we have*

$$(R_S^x f)(x,y) = \int_{x_0}^{x_m} \varphi_r(x,s) f^{(r,0)}(s,y)ds + \tag{19}$$

$$+ \sum_{i=0}^{m} s_i(x) \int_{y_{i0}}^{y_{in_i}} \psi_{n_i}(y,t) f^{(0,n_i+1)}(x_i, t)$$

with

$$\varphi_r(x,s) = \frac{(x-s)_+^{r-1}}{(r-1)!} - \sum_{i=0}^{m} s_i(x) \frac{(x_i-s)_+^{r-1}}{(r-1)!}$$

$$\psi_{n_i}(y,t) = \frac{(y-t)_+^{n_i}}{n_i!} - \sum_{j=0}^{n_i} \frac{v_i(y)}{(y-y_{ij})v_i'(y_{ij})} \frac{(y_{ij}-t)_+^{n_i}}{n_i!}.$$

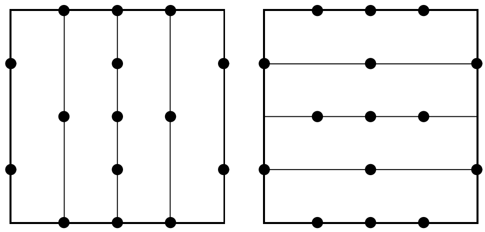


FIGURE 1. The grid I is projected on Ox and respective Oy axis

We obtain the following interpolation formula

$$f(x,y) = (P_S^M f)(x,y) + (R_S^M f)(x,y) \tag{20}$$

where

$$(P_S^M f)(x,y) = \frac{(P_S^x f)(x,y) + (P_S^y f)(x,y)}{2} \tag{21}$$

and

$$(R_S^M f)(x,y) = \frac{(R_S^x f)(x,y) + (R_S^y f)(x,y)}{2}$$

Example 6. . *Let*

$$f : [-1,1] \times [-1,1] \to \mathbb{R}, f(x,y) = \exp(-x^2 - y^2)$$

The sets of nodes are

-grid I

$P_1(-1,-0.5), P_2(-1,0.5), P_3(-0.5,-1), P_4(-0.5,0), P_5(-0.5,1), P_6(0,-1),$ $P_7(0,-0.5), P_8(0,0), P_9(0,0.5), P_{10}(0,1), P_{11}(0.5,-1), P_{12}(0.5,0), P_{13}(0.5,1),$ $P_{14}(1,-0.5), P_{15}(1,0.5)$

-grid II

$P_1(-1,-1), P_2(-1,0), P_3(-1,1), P_4(-0.5,-0.5), P_5(-0.5,0), P_6(-0.5,0.5),$ $P_7(0,-1), P_8(0,-0.5), P_9(0,0), P_{10}(0,0.5), P_{11}(0,1), P_{12}(0.5,-0.5), P_{13}(0.5,0),$ $P_{14}(0.5,0.5), P_{15}(1,-1), P_{16}(1,0), P_{17}(1,1)$

We plot the graph of functions f and the graphs of $P_S^x f$, $P_S^y f$, P_S^M. For a matrix Z we define

$$\|f - Pf\|_{2,Z} = \|\{(f - Pf)(x_i, y_j) | (x_i, y_j) \in Z\}\|_2$$

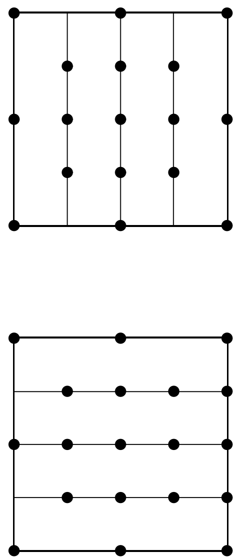
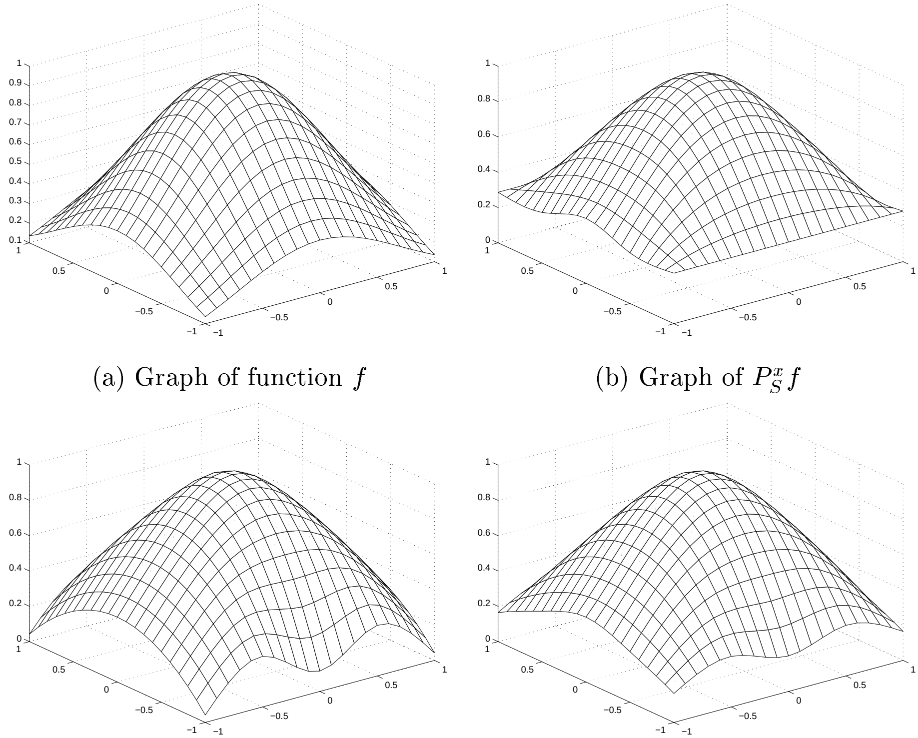


FIGURE 2. The grid II is projected on Ox and respective Oy axis

If we take $Z = [-1 : 0.1 : 1] \times [-1 : 0.1 : 1]$ *we obtain*

	P	$\|f - Pf\|_{2,Z}$
I	P_S^x	*0.9469*
	P_S^y	*0.8685*
	P_S^M	*0.7326*
II	P_S^x	*1.0244*
	P_S^y	*1.0244*
	P_S^M	*0.5004*



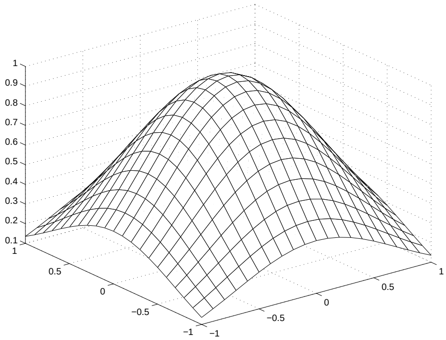
(a) Graph of function f

(b) Graph of $P_S^x f$

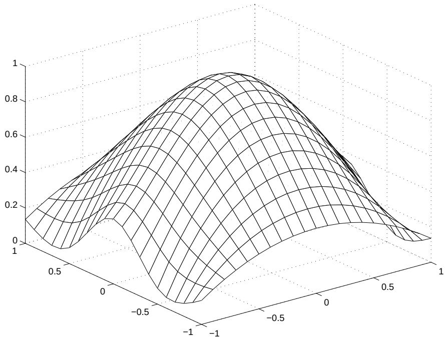
(c) Graph of $P_S^y f$

(d) Graph of $P_S^M f$

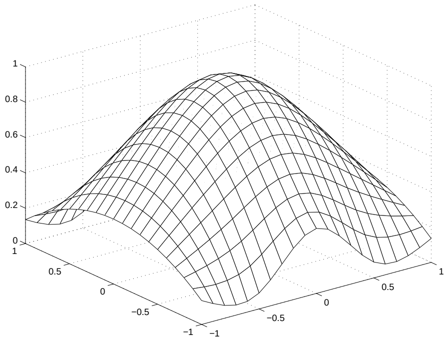
FIGURE 3. The graph of function f and the graphs of $P_S^x f$, $P_S^y f$, $P_S^M f$ for grid I



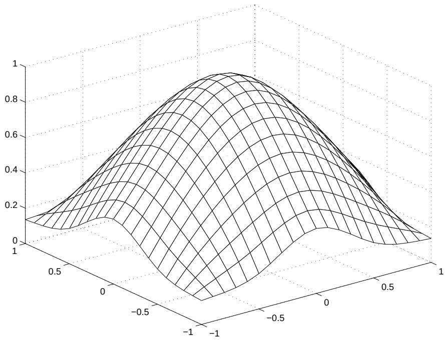
(a) Graph of function f



(b) Graph of $P_S^x f$



(c) Graph of $P_S^y f$



(d) Graph of $P_S^M f$

FIGURE 4. The graph of function f and the graphs of $P_S^x f$, $P_S^y f$, $P_S^M f$ for grid II

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AN APPLICATION OF MACKEY'S SELECTION LEMMA

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Abstract. Let G be a locally compact second countable groupoid. Let F be a subset of $G^{(0)}$ meeting each orbit exactly once. Let us denote by d_F the restriction of the domain map to G^F and by r' the restriction of the range map to the isotropy group bundle of G. We shall prove that if d_F is open, then r' is open and d_F has a regular Borel cross section. Conversely, we shall prove that if r' is open and d_F admits a regular cross section (a right inverse which carries each compact subset of $G^{(0)}$ into a relatively compact subset of G^F), then d_F is open. We shall also prove that, if d_F is open, then F is a closed subset of $G^{(0)}$, and the orbit space $G^{(0)}/G$ is a proper space. If F is closed and regular (the intersection of F with the saturated of any compact subset of $G^{(0)}$ is relatively compact) and $G^{(0)}/G$ is proper, then d_F is open.

1. **Introduction**

We shall consider a locally compact groupoid G and a set F containing exactly one element from each orbit of G. We shall study the connection between the openness of d_F, the restriction of the domain map to G^F, and the existence of a regular cross section of d_F (a right inverse which carries each compact subset of $G^{(0)}$ into a relatively compact subset of G^F). The motivation for studying the map d_F comes from the fact that if F is closed and d_F is open, then G and G_F^F are (Morita) equivalent locally compact groupoids (in the sense of Definition 2.1/p. 6 [4]). A result of Paul Muhly, Jean Renault and Dana Williams states that the C^*-algebras associated to (Morita)

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equivalent locally compact second countable groupoids are strongly Morita equivalent (Theorem 2.8/p. 10 [4]). Also the notion of topological amenability is invariant under the equivalence of groupoids (Theorem 2.2.7/p. 50 [1]). Consequently, if F is closed and d_F is open, then G and the bundle group G_F^F have strongly Morita equivalent C^*-algebras. Also, the equivalence of G and G_F^F implies that G is amenable if and only if each isotropy group G_u^u is amenable.

For establishing notation, we include some definitions that can be found in several places (e.g. [5]). A groupoid is a set G, together with a distinguished subset $G^{(2)} \subset G \times G$ (called the set of composable pairs), and two maps:

$$\begin{array}{rcl} (x,y) & \to & xy \left[: G^{(2)} \to G\right] \text{ (product map)} \\ x & \to & x^{-1} \ [: G \to G] \text{ (inverse map)} \end{array}$$

such that the following relations are satisfied:

(1) If $(x,y) \in G^{(2)}$ and $(y,z) \in G^{(2)}$, then $(xy,z) \in G^{(2)}$, $(x,yz) \in G^{(2)}$ and $(xy)\,z = x\,(yz)$.

(2) $\left(x^{-1}\right)^{-1} = x$ for all $x \in G$.

(3) For all $x \in G$, $\left(x, x^{-1}\right) \in G^{(2)}$, and if $(z,x) \in G^{(2)}$, then $(zx)\,x^{-1} = z$.

(4) For all $x \in G$, $\left(x^{-1}, x\right) \in G^{(2)}$, and if $(x,y) \in G^{(2)}$, then $x^{-1}\,(xy) = y$.

The maps r and d on G, defined by the formulae $r\,(x) = xx^{-1}$ and $d\,(x) = x^{-1}x$, are called the range and the source maps. It follows easily from the definition that they have a common image called the unit space of G, which is denoted $G^{(0)}$. Its elements are units in the sense that $xd\,(x) = r\,(x)\,x = x$. It is useful to note that a pair (x,y) lies in $G^{(2)}$ precisely when $d\,(x) = r\,(y)$, and that the cancellation laws hold (e.g. $xy = xz$ iff $y = z$). The fibers of the range and the source maps are denoted $G^u = r^{-1}(\{u\})$ and $G_v = d^{-1}(\{v\})$, respectively. More generally, given the subsets A, $B \subset G^{(0)}$, we define $G^A = r^{-1}(A)$, $G_B = d^{-1}(B)$ and $G_B^A = r^{-1}(A) \cap d^{-1}(B)$. G_A^A becomes a groupoid (called the reduction of G to A) with the unit space A, if we define $\left(G_A^A\right)^{(2)} = G^{(2)} \cap \left(G_A^A \times G_A^A\right)$. For each unit u, G_u^u is a group, called isotropy group at u. The group bundle

$$\{x \in G : r(x) = d(x)\}$$

is denoted G', and is called the isotropy group bundle of G. The relation $u\tilde{}v$ iff $G_v^u \neq \phi$ is an equivalence relation on $G^{(0)}$. Its equivalence classes are called orbits and the orbit of a unit u is denoted $[u]$. Let

$$R = (r, d)(G) = \{(r(x), d(x)),\ x \in G\}$$

be the graph of the equivalence relation induced on $G^{(0)}$. The quotient space for this equivalence relation is called the orbit space of G and denoted $G^{(0)}/G$.

A topological groupoid consists of a groupoid G and a topology compatible with the groupoid structure. We are exclusively concerned with topological groupoids which are locally compact Hausdorff. The Borel sets of a topological space are taken to be the σ-algebra generated by the open sets.

2. Necessary and sufficient conditions for the openness of d_F.

Definition 1. *Let X, Y be two topological spaces. A cross section of a map $f : X \to Y$ is a function $\sigma : Y \to X$ such that $f(\sigma(y)) = y$ for all $y \in Y$. We shall say that the cross section σ is regular if $\sigma(K)$ has compact closure in X for each compact set K in Y.*

We shall need the following lemma proved by Mackey (Lemma 1.1/p. 102 [3]):

Lemma 1. *If X and Y are second countable, locally compact spaces, and $f : X \to Y$ is a continuous open function onto Y, then f has a Borel regular cross section.*

Proposition 1. *Let G be a locally compact groupoid. Let F be a subset of $G^{(0)}$ meeting each orbit exactly once. Let us define the function $e : G^{(0)} \to G^{(0)}$ by*

$$e(u) = F \cap [u],\ u \in G^{(0)}$$

If the map $d_F : G^F \to G^{(0)}$, $d_F(x) = d(x)$, is open, then the function e is continuous and F is a closed subset of $G^{(0)}$.

Proof. Let $(u_i)_i$ be a net converging to u in $G^{(0)}$. Let $x \in G$ be such that $r(x) = e(u)$ and $d(x) = u$. Since $(u_i)_i$ converges to $d_F(x)$ and d_F is an open map, we may pass to a subnet and assume that there is a net $(x_i)_i$ converging to x in G^F such that $d_F(x_i) = u_i$. It is easy to see that $r(x_i) = e(u_i)$ ($r(x_i) \in F$ and $r(x_i) \in [d(x_i)] = [u_i]$). Thus $e(u_i) = r(x_i)$ converges to $r(x) = e(u)$. Since $G^{(0)}$ is Hausdorff, F is closed in $G^{(0)}$, being the image of the map e whose square is itself. □

Proposition 2. *Let G be a locally compact groupoid. Let F be a subset of $G^{(0)}$ meeting each orbit exactly once. If the map $d_F : G^F \to G^{(0)}$, $d_F(x) = d(x)$, is open, then graph*

$$R = \{(r(x), d(x)),\ x \in G\}$$

of the equivalence relation induced on $G^{(0)}$ is closed in $G^{(0)} \times G^{(0)}$, and the map $(r, d) : G \to R$, $(r, d)(x) = (r(x), d(x))$ is open, where R is endowed with the product topology induced from $G^{(0)} \times G^{(0)}$.

Proof. Let us define the function $e : G^{(0)} \to G^{(0)}$ by

$$e(u) = F \cap [u],\ u \in G^{(0)}.$$

By Proposition 1, the function e is continuous. Let $((u_i, v_i))_i$ be a net in R which converges to (u, v) in $G^{(0)} \times G^{(0)}$ (with respect to with the product topology). Then $(u_i)_i$ converges to u, $(v_i)_i$ converges to v, and $u_i\ \tilde{}\ v_i$ for all i. We have

$$\begin{aligned} \lim_i e(u_i) &= e(u) \\ \lim_i e(v_i) &= e(v) \end{aligned}$$

because e is continuous. On the other hand, the fact that $u_i\ \tilde{}\ v_i$ for all i implies that $e(u_i) = e(v_i)$ for all i. Hence $e(u) = e(v)$, or equivalently, $u \tilde{} v$. Therefore $(u, v) \in R$.

Let us prove that the map $(r, d) : G \to R$, $(r, d)(x) = (r(x), d(x))$ is open, where R is endowed with the product topology induced from $G^{(0)} \times G^{(0)}$. Let $x \in G$, and let $((u_i, v_i))_i$ be a net in R converging to $(r, d)(x)$. Then $(u_i)_i$ converges to $r(x)$, $(v_i)_i$ converges to $d(x)$, and $u_i\ \tilde{}\ v_i$ for all i. Let $s \in G$ be such that $r(s) = e(r(x))$

and $d(s) = r(x)$ and let $t = sx$. Obviously, $s, t \in G^F$ and

$$\lim_i u_i = r(x) = d(s)$$
$$\lim_i v_i = d(x) = d(sx) = d(t).$$

Since d_F is an open map, we may pass to subnets and assume that there is a net $(s_i)_i$ converging to s in G^F and there is a net $(t_i)_i$ converging to t in G^F such that $d_F(s_i) = u_i$ and $d_F(t_i) = v_i$. The fact that $e(u_i)$ is the only element of F, which is equivalent to $u_i \tilde{} \; v_i$, implies that $r(s_i) = e(u_i) = e(v_i) = r(t_i)$. We have

$$\lim_i s_i^{-1} t_i = s^{-1} t = x$$
$$r\left(s_i^{-1} t_i\right) = d(s_i) = u_i, \; d\left(s_i^{-1} t_i\right) = d(t_i) = v_i$$

Therefore the map (r, d) is open □

Corollary 1. *Let G be a locally compact groupoid having open range map. Let F be a subset of $G^{(0)}$ meeting each orbit exactly once. If the map $d_F : G^F \to G^{(0)}$, $d_F(x) = d(x)$, is open, then the orbit space $G^{(0)}/G$ is proper.*

Proof. The fact that $G^{(0)}/G$ is a proper space means that $G^{(0)}/G$ is Hausdorff and the map $(r, d) : G \to R$, $(r, d)(x) = (r(x), d(x))$ is open, where R is endowed with the product topology induced from $G^{(0)} \times G^{(0)}$. Let us note that the quotient map $\pi : G^{(0)} \to G^{(0)}/G$ is open (because the range map of G is open). Since the graph R of the equivalence relation is closed in $G^{(0)} \times G^{(0)}$, it follows that $G^{(0)}/G$ is Hausdorff. □

Lemma 2. *Let G be a locally compact groupoid having open range map. Let F be a subset of $G^{(0)}$ meeting each orbit exactly once. If the map $d_F : G^F \to G^{(0)}$, $d_F(x) = d(x)$, is open, then F and $G^{(0)}/G$ are homeomorphic spaces.*

Proof. Let $\pi : G^{(0)} \to G^{(0)}/G$ be the quotient map. We prove that the map $\pi_F : F \to G^{(0)}/G, \pi_F(x) = \pi(x)$ is a homeomorphism. It suffices to prove that π_F is an open map (because π_F is one-to-one from F onto $G^{(0)}/G$). Let $u \in F$ and $(\dot{u}_i)_i$ be a net converging to $\pi(u)$ in $G^{(0)}/G$. Since $\pi \circ d_F$ is open, we may pass to a subnet and assume that there is a net $(x_i)_i$ converging to u in G^F such that $\pi(d_F(x_i)) = \dot{u}_i$. Then $(r(x_i))_i$ is a net in F which converges to u. □

Remark 1. *Let G be a locally compact groupoid. If the map $(r,d): G \to R$ is open (where $R = \{(r(x), d(x)), x \in G\}$is endowed with the product topology induced from $G^{(0)} \times G^{(0)}$), then the map $r': G' \to G^{(0)}, r'(x) = r(x)$, is open, where*

$$G' = \{x \in G: \, r(x) = d(x)\},$$

is the isotropy group bundle of G.

Proposition 3. *Let G be a locally compact second countable groupoid. Let F be a subset of $G^{(0)}$ meeting each orbit exactly once. If the map $d_F: G^F \to G^{(0)}$, $d_F(x) = d(x)$, is open, then d_F has Borel regular cross section.*

Proof. If d_F is open, then according to Proposition 1, F is a closed subset of $G^{(0)}$. Therefore G^F is a locally compact space, and we may apply Lemma 1. □

We shall need a system of measures

$$\{\beta_v^u, \, (u,v) \in (r,d)(G)\}$$

satisfying the following conditions:

1. $supp(\beta_v^u) = G_v^u$ for all $u \,\tilde{}\, v$.
2. $\sup\limits_{u,v} \beta_v^u(K) < \infty$ for all compact $K \subset G$.
3. $\int f(y)\, d\beta_v^{r(x)}(y) = \int f(xy)\, d\beta_v^{d(x)}(y)$ for all $x \in G$ and $v \,\tilde{}\, r(x)$.

In Section 1 of [6] Jean Renault constructs a Borel Haar system for G'. One way to do this is to choose a function F_0 continuous with conditionally support, which is nonnegative and equal to 1 at each $u \in G^{(0)}$. Then for each $u \in G^{(0)}$ choose a left Haar measure β_u^u on G_u^u so the integral of F_0 with respect to β_u^u is 1. Renault defines $\beta_v^u = x\beta_v^v$ if $x \in G_v^u$ (where $x\beta_v^v(f) = \int f(xy)\, d\beta_v^v(y)$ as usual). If z is another element in G_v^u, then $x^{-1}z \in G_v^v$, and since β_v^v is a left Haar measure on G_v^v, it follows that β_v^u is independent of the choice of x. If K is a compact subset of G, then $\sup\limits_{u,v} \beta_v^u(K) < \infty$. We obtain another construction of a system a measures with the above properties if in the proof of Theorem 8/p. 331[2] we replace the regular cross section of $G^u \xrightarrow{d} G^{(0)}$ (in the transitive case) with a regular cross section of $G^F \xrightarrow{d} G^{(0)}$, where F is a subset of $G^{(0)}$ meeting each orbit exactly once.

Lemma 3. *Let G be a locally compact groupoid. Let F be a subset of $G^{(0)}$ meeting each orbit exactly once and let us denote $e(u)$ the unique element of F equivalent to u. If the map*

$$u \mapsto \int f(y)\, d\beta_u^{e(u)}(y) \ \left[: G^{(0)} \to \mathbf{C}\right]$$

is continuous for any continuous function with compact support, $f : G \to \mathbf{C}$, then the map

$$d_F : G^F \to G^{(0)},\ d_F(x) = d(x).$$

is open.

Proof. Let $x_0 \in G^F$ and let U be a nonempty compact neighborhood of x_0. Choose a nonnegative continuous function, f on G , with $f(x_0) > 0$ and $supp(f) \subset U$. Let W be the set of units u with the property that $\beta_u^{e(u)}(f) > 0$. Then W is an open neighborhood of $u_0 = d(x_0)$ contained in $d_F(U)$. □

Proposition 4. *Let G be a locally compact second countable groupoid. Let F be a subset of $G^{(0)}$ containing exactly one element from each orbit of G, and let us denote $e(u)$ the unique element of F equivalent to u. Let us assume that the map $r' : G' \to G^{(0)}, r'(x) = r(x)$ is open, where G' is the isotropy group bundle of G. If the map $d_F : G^F \to G^{(0)}$, $d_F(x) = d(x)$, has a regular cross section σ, then for each continuous with compact support function $f : G \to \mathbf{C}$, the map*

$$u \to \int f(y)\, d\beta_u^{e(u)}(y)$$

is continuous on G.

Proof. By Lemma 1.3/p. 6 [6], for each $f : G \to \mathbf{C}$ continuous with compact support, the function $u \to \int f(y)\, d\beta_u^u(y)\ \left[: G^{(0)} \to \mathbf{C}\right]$ is continuous. Let $(u_i)_i$ be a sequence in $G^{(0)}$ converging to u. Let $x_i = \sigma(u_i)^{-1}$. Since σ is regular, it follows that $(x_i)_i$ has a convergent subsequence in G^F. Let x be the limit of this subsequence. Let $f : G \to \mathbf{C}$ be a continuous function with compact support and let g be a continuous extension on G of $y \to f(xy) \left[: G^{d(x)} \to \mathbf{C}\right]$. Let K be the compact set $\left(\{x, x_i, i = 1, 2, ..\}^{-1} supp(f) \cup supp(g)\right) \cap r^{-1}(\{d(x), d(x_i), i = 1, 2, ...\})$. We have

$$\begin{aligned}
& \left| \int f(y)\, d\beta_u^{e(u)}(y) - \int f(y)\, d\beta_{u_i}^{e(u_i)}(y) \right| \\
= & \left| \int f(xy)\, d\beta_{d(x)}^{d(x)}(y) - \int f(x_i y)\, d\beta_{d(x_i)}^{d(x_i)}(y) \right| \\
= & \left| \int g(y)\, d\beta_{d(x)}^{d(x)}(y) - \int f(x_i y)\, d\beta_{d(x_i)}^{d(x_i)}(y) \right| \\
\leq & \left| \int g(y)\, d\beta_{d(x)}^{d(x)}(y) - \int g(y)\, d\beta_{d(x_i)}^{d(x_i)}(y) \right| + \\
& + \left| \int g(y)\, d\beta_{d(x_i)}^{d(x_i)}(y) - \int f(x_i y)\, d\beta_{d(x_i)}^{d(x_i)}(y) \right| \\
\leq & \left| \int g(y)\, d\beta_u^u(y) - \int g(y)\, d\beta_{u_i}^{u_i}(y) \right| + \\
& + \sup_{y \in G_{u_i}^{u_i}} |g(y) - f(x_i y)|\, \beta_{u_i}^{u_i}(K)
\end{aligned}$$

A compactness argument shows that $\sup_{y \in G_{u_i}^{u_i}} |g(y) - f(x_i y)|$ converges to 0. Also $\left| \int g(y)\, d\beta_{d(x_i)}^{d(x_i)}(y) - \int f(x_i y)\, d\beta_{d(x_i)}^{d(x_i)}(y) \right|$ converges to 0, because the function $u \to \int f(y)\, d\beta_u^u(y)$ is continuous on $G^{(0)}$. Hence

$$\left| \int f(y)\, d\beta_u^{e(u)}(y) - \int f(y)\, d\beta_{u_i}^{e(u_i)}(y) \right|$$

converges to 0. □

Corollary 2. *Let G be a locally compact second countable groupoid. Let F be a subset of $G^{(0)}$ meeting each orbit exactly once. If the restriction r' of the range map to the isotropy group bundle G' of G is open, and if the map $d_F : G^F \to G^{(0)}$, $d_F(x) = d(x)$, has a regular cross section, then d_F is an open map.*

Theorem 1. *Let G be a locally compact second countable groupoid. Let F be a subset of $G^{(0)}$ meeting each orbit exactly once, and let $d_F : G^F \to G^{(0)}$ be the map defined by $d_F(x) = d(x)$ for all $x \in G^F$. If d_F is open then d_F admits a Borel regular cross section. If the restriction r' of the range map to the isotropy group bundle G' of G is open and if d_F admits a regular cross section, then d_F is an open map.*

Proof. If d_F is an open map, then, according Proposition 3, d_F has a regular cross section. Conversely, if d_F admits a regular cross section, then applying Proposition 4 and Lemma 3, it follows that d_F is open. □

Remark 2. *Let us assume that $G^{(0)}/G$ is **proper**. There is a regular Borel cross section σ_0 of the quotient map $\pi : G^{(0)} \to G^{(0)}/G$. Let us assume that $F = \sigma_0\left(G^{(0)}/G\right)$is **closed** in $G^{(0)}$. Then the function $e : G^{(0)} \to G^{(0)}$ defined by $e(u) = F \cap [u]$ is continuous. If $\sigma_1 : R \to G$ is regular Borel cross section of (r, d), then $\sigma : G^{(0)} \to G^F$, $\sigma(u) = \sigma_1(e(u), u)$ is a Borel regular cross section of d_F. Therefore is that case d_F is open.*

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PARTIAL SUMS OF CERTAIN MEROMORPHIC P-VALENT FUNCTIONS

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Abstract. In this paper, we study the ratio of meromorphic p-valent functions in the punctured disk $\mathcal{D} = \{z : 0 < |z| < 1\}$ of the form $f(z) = \frac{1}{z^p} + \sum\limits_{k=1}^{\infty} a_{p+k-1} z^{p+k-1}$ to its sequence of partial sums of the form $f_n(z) = \frac{1}{z^p} + \sum\limits_{k=1}^{n} a_{p+k-1} z^{p+k-1}$. Also, we will determine sharp lower bounds for $\operatorname{Re}\{f(z)/f_n(z)\}$, $\operatorname{Re}\{f_n(z)/f(z)\}$, $\operatorname{Re}\{f'(z)/f'_n(z)\}$ and $\operatorname{Re}\left\{f'_n(z)/f'(z)\right\}$.

1. Introduction and definitions

Let Σ_p denotes the class of functions of the form:

$$f(z) = \frac{1}{z^p} + \sum_{k=1}^{\infty} a_{p+k-1} z^{p+k-1} \qquad (p \in \mathbb{N}), \tag{1}$$

which are analytic and p-valent in the punctured unit disk $\mathcal{D} = \{z : 0 < |z| < 1\}$. A function $f \in \Sigma_p$ is said to be in the class $\Sigma^*(p, \alpha)$ of meromorphic p-valently starlike functions of order α in $\mathcal{D}$ if and only if

$$\operatorname{Re}\left\{-\frac{zf'(z)}{f(z)}\right\} > \alpha \qquad (z \in \mathcal{D};\ 0 \le \alpha < p;\ p \in \mathbb{N}). \tag{2}$$

Furthermore, a function $f \in \Sigma_p$ is said to be in the class $\Sigma_{\mathcal{K}}(p, \alpha)$ of meromorphic p-valently convex functions of order α in $\mathcal{D}$ if and only if

$$\operatorname{Re}\left\{-1 - \frac{zf''(z)}{f'(z)}\right\} > \alpha \qquad (z \in \mathcal{D};\ 0 \le \alpha < p;\ p \in \mathbb{N}). \tag{3}$$

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The class $\Sigma^*(p,\alpha)$ and various other subclasses of Σ_p have been studied rather extensively by Aouf *et.al.* [1-3], Joshi and Srivastava [6], Kulkarni *et. al.* [7], Mogra [8], Owa *et. al.* [9], Srivastava and Owa [11], Uralegaddi and Somantha [12], and Yang [13].

Let $\Omega_p(\alpha)$ be the subclass of Σ_p consisting of functions $f(z)$ which satisfy the inequality

$$\operatorname{Re}\left\{-\frac{zf'(z)}{f(z)}\right\} < \alpha \qquad (z \in \mathcal{D};\ 0 \le \alpha < p;\ p \in \mathbb{N}). \tag{4}$$

And let $\Lambda_p(\alpha)$ be the subclass of Σ_p consisting of functions $f(z)$ which satisfy the inequality

$$\operatorname{Re}\left\{-1-\frac{zf''(z)}{f'(z)}\right\} < \alpha \qquad (z \in \mathcal{D};\ 0 \le \alpha < p;\ p \in \mathbb{N}). \tag{5}$$

The classes $\Omega_p(\alpha)$ and $\Lambda_p(\alpha)$ were introduced and studied by the authors [5].

In [5] the authors obtained the following sufficient conditions for a function of the form (1.1) to be in the classes $\Omega_p(\alpha)$ and $\Lambda_p(\alpha)$.

Lemma 1. *If* $f(z) \in \Sigma_p$ *satisfies*

$$\sum_{k=1}^{\infty}(p+k+\delta-1+|p+k+2\alpha-\delta-1|)a_{p+k-1} < 2(p-\alpha). \tag{6}$$

for some $\alpha(0 \le \alpha < p)$ *and some* $\delta(\alpha < \delta \le p)$, *then* $f(z) \in \Omega_p(\alpha)$.

Lemma 2. *If* $f(z) \in \Sigma_p$ *satisfies*

$$\sum_{k=1}^{\infty}(p+k-1)(p+k+\delta-1+|p+k+2\alpha-\delta-1|)a_{p+k-1} < 2(p-\alpha) \tag{7}$$

for some $\alpha(0 \le \alpha < p)$ *and some* $\delta(\alpha < \delta \le p)$, *then* $f(z) \in \Lambda_p(\alpha)$.

In view of Lemma 1 and Lemma 2, we now define the subclasses $\Omega_p^*(\alpha) \subset \Omega_p(\alpha)$ and $\Lambda_p^*(\alpha) \subset \Lambda_p(\alpha)$, which consist of functions $f(z) \in \Sigma_p$ satisfying the conditions (1.6) and (1.7), respectively.(see [5]).

In the present paper, and by following the earlier work of Silverman [10] (see also [4]), we will investigate the ratio of a function of the form (1.1) to its sequence of partial sums of the form

$$f_n(z) = \frac{1}{z^p} + \sum_{k=1}^{n} a_{p+k-1}z^{p+k-1} \qquad (p \in \mathbb{N}), \tag{8}$$

when the coefficients of $f(z)$ are satisfy the condition (1.6) or (1.7). More precisely, we will determine sharp lower bounds for $\operatorname{Re}\{f(z)/f_n(z)\}$, $\operatorname{Re}\{f_n(z)/f(z)\}$, $\operatorname{Re}\{f'(z)/f'_n(z)\}$ and $\operatorname{Re}\left\{f'_n(z)/f'(z)\right\}$.

For the notational convenience we shall henceforth denote

$$\sigma_k(p,\delta,\alpha) := p+k+\delta-1+|p+k+2\alpha-\delta-1| \tag{9}$$

2. Main results

Theorem 1. *If $f(z)$ of the form (1.1) satisfies the condition (1.6), then*

$$\operatorname{Re}\left\{\frac{f(z)}{f_n(z)}\right\} \geq \frac{\sigma_{n+1}(p,\delta,\alpha)-2(p-\alpha)}{\sigma_{n+1}(p,\delta,\alpha)} \qquad (z\in\mathcal{U}) \tag{1}$$

The results (2.1) is sharp for every n, with extermal function

$$f(z)=\frac{1}{z}+\frac{2(p-\alpha)}{\sigma_{n+1}(p,\delta,\alpha)}z^{p+n} \qquad (n\geq 0). \tag{2}$$

Proof. Define the function $w(z)$ by

$$\frac{1+w(z)}{1-w(z)} = \frac{\sigma_{n+1}(p,\delta,\alpha)}{2(p-\alpha)}\left[\frac{f(z)}{f_n(z)}-\left(\frac{\sigma_{n+1}(p,\delta,\alpha)-2(p-\alpha)}{\sigma_{n+1}(p,\delta,\alpha)}\right)\right]$$

$$= \frac{1+\sum\limits_{k=1}^{n} a_{p+k-1}z^{k+p}+\frac{\sigma_{n+1}(p,\delta,\alpha)}{2(p-\alpha)}\sum\limits_{k=n+1}^{\infty} a_{p+k-1}z^{k+p}}{1+\sum\limits_{k=1}^{n} a_{p+k-1}z^{k+p}} \tag{3}$$

It suffices to show that $|w(z)|\leq 1$. Now, from (2.3) we can write

$$w(z)=\frac{\frac{\sigma_{n+1}(p,\delta,\alpha)}{2(p-\alpha)}\sum\limits_{k=n+1}^{\infty} a_{p+k-1}z^{k+p}}{2+2\sum\limits_{k=1}^{n} a_{p+k-1}z^{k+p}+\frac{\sigma_{n+1}(p,\delta,\alpha)}{2(p-\alpha)}\sum\limits_{k=n+1}^{\infty} a_{p+k-1}z^{k+p}}$$

to find that

$$|w(z)|\leq\frac{\frac{\sigma_{k+1}(p,\delta,\alpha)}{2(p-\alpha)}\sum\limits_{k=n+1}^{\infty} |a_{p+k-1}|}{2-2\sum\limits_{k=1}^{n} a_{p+k-1}z^{k+1}-\frac{\sigma_{k+1}(p,\delta,\alpha)}{2(p-\alpha)}\sum\limits_{k=n+1}^{\infty} a_{p+k-1}z^{k+1}}.$$

Now $|w(z)|\leq 1$ if

$$2\left(\frac{\sigma_{n+1}(p,\delta,\alpha)}{2(p-\alpha)}\right)\sum_{k=n+1}^{\infty}|a_k|\leq 2-2\sum_{k=1}^{n}|a_{p+k-1}|,$$

which is equivalent to

$$\sum_{k=1}^{n} |a_{p+k-1}| + \sum_{k=n+1}^{\infty} \left(\frac{\sigma_{n+1}(p,\delta,\alpha)}{2(p-\alpha)} \right) |a_{p+k-1}| \leq 1.$$

From the condition (1.6), it is sufficient to show that

$$\sum_{k=1}^{n} |a_{p+k-1}| + \sum_{k=n+1}^{\infty} \left(\frac{\sigma_{n+1}(p,\delta,\alpha)}{2(p-\alpha)} \right) |a_{p+k-1}| \leq \sum_{k=1}^{\infty} \frac{\sigma_k(p,\delta,\alpha)}{2(p-\alpha)} |a_{p+k-1}| \tag{4}$$

which is equivalent to

$$\sum_{k=1}^{n} \frac{\sigma_k(p,\delta,\alpha) - 2(p-\alpha)}{2(p-\alpha)} |a_{p+k-1}| + \sum_{k=n+1}^{\infty} \frac{\sigma_k(p,\delta,\alpha) - \sigma_{n+1}(p,\delta,\alpha)}{2(p-\alpha)} |a_{p+k-1}| \geq 0.$$

To see that the function given by (2.2) gives the sharp result, we observe that for $z = re^{\pi i/(n+p+1)}$

$$\begin{aligned} \frac{f(z)}{f_n(z)} &= 1 + \frac{2(p-\alpha)}{\sigma_{n+1}(p,\delta,\alpha)} z^{n+p+1} \to 1 - \frac{2(p-\alpha)}{\sigma_{n+1}(p,\delta,\alpha)} \\ &= \frac{\sigma_{n+1}(p,\delta,\alpha) - 2(p-\alpha)}{\sigma_{n+1}(p,\delta,\alpha)} \quad \text{when } r \to 1^-. \end{aligned}$$

Therefore we complete the proof of Theorem 1.

Theorem 2. *If $f(z)$ of the form (1.1) satisfies the condition (1.7), then*

$$\operatorname{Re}\left\{ \frac{f(z)}{f_n(z)} \right\} \geq \frac{(p+n)\sigma_{n+1}(p,\delta,\alpha) - 2(p-\alpha)}{(p+n)\sigma_{n+1}(p,\delta,\alpha)} \qquad (z \in \mathcal{U}). \tag{5}$$

The results (2.5) is sharp for every n, with extremal function

$$f(z) = \frac{1}{z} + \frac{2(p-\alpha)}{(p+n)\sigma_{n+1}(p,\delta,\alpha)} z^{p+n} \qquad (n \geq 0). \tag{6}$$

Proof. We write

$$\frac{1+w(z)}{1-w(z)} = \frac{(p+n)\sigma_{n+1}(p,\delta,\alpha)}{2(p-\alpha)} \left[\frac{f(z)}{f_n(z)} - \left(\frac{(p+n)\sigma_{n+1}(p,\delta,\alpha) - 2(p-\alpha)}{(p+n)\sigma_{n+1}(p,\delta,\alpha)} \right) \right]$$

$$= \frac{1 + \sum_{k=1}^{n} a_{p+k-1} z^{k+p} + \frac{(p+n)\sigma_{n+1}(p,\delta,\alpha)}{2(p-\alpha)} \sum_{k=n+1}^{\infty} a_{p+k-1} z^{k+p}}{1 + \sum_{k=1}^{n} a_{p+k-1} z^{k+p}}$$

where

$$w(z) = \frac{\frac{(p+n)\sigma_{n+1}(p,\delta,\alpha)}{2(p-\alpha)} \sum_{k=n+1}^{\infty} |a_{p+k-1}|}{2 + 2\sum_{k=2}^{n} |a_{p+k-1}| + \frac{(p+n)\sigma_{n+1}(p,\delta,\alpha)}{2(p-\alpha)} \sum_{k=n+1}^{\infty} |a_{p+k-1}|}.$$

Now

$$|w(z)| \leq \frac{\frac{(p+n)\sigma_{n+1}(p,\delta,\alpha)}{2(p-\alpha)} \sum_{k=n+1}^{\infty} |a_{p+k-1}|}{2 - 2\sum_{k=1}^{n} |a_{p+k-1}| - \frac{(p+n)\sigma_{n+1}(p,\delta,\alpha)}{2(p-\alpha)} \sum_{k=n+1}^{\infty} |a_{p+k-1}|} \leq 1$$

if

$$\sum_{k=1}^{n} |a_{p+k-1}| + \sum_{k=n+1}^{\infty} \frac{(p+n)\sigma_{n+1}(p,\delta,\alpha)}{2(p-\alpha)} |a_{p+k-1}| \leq 1. \tag{7}$$

The left hand side of (2.7) is bounded above by
$\sum_{k=1}^{\infty} [(p+k-1)\sigma_k(p,\delta,\alpha)/(2(p-\alpha))] |a_{p+k-1}|$ if

$$\begin{aligned} & \frac{1}{2(p-\alpha)} \sum_{k=1}^{n} [(p+k-1)\sigma_k(p,\delta,\alpha) - 2(p-\alpha)] |a_{p+k-1}| \\ & + \sum_{k=n+1}^{\infty} [(p+k-1)\sigma_k(p,\delta,\alpha) - (p+n)\sigma_{n+1}(p,\delta,\alpha)] |a_{p+k-1}| \\ \geq \; & 0, \end{aligned}$$

and the proof is complete.

We next determine bounds for $f_n(z)/f(z)$.

Theorem 3. (a) *If $f(z)$ of the form (1.1) satisfies the condition (1.6), then*

$$\operatorname{Re}\left\{\frac{f_n(z)}{f(z)}\right\} \geq \frac{\sigma_{n+1}(p,\delta,\alpha)}{\sigma_{n+1}(p,\delta,\alpha) + 2(p-\alpha)} \qquad (z \in \mathcal{U}). \tag{8}$$

(b) *If $f(z)$ of the form (1.1) satisfies the condition (1.7), then*

$$\operatorname{Re}\left\{\frac{f_n(z)}{f(z)}\right\} \geq \frac{(p+n)\sigma_{n+1}(p,\delta,\alpha)}{(p+n)\sigma_{n+1}(p,\delta,\alpha) + 2(p-\alpha)} \qquad (z \in \mathcal{U}). \tag{9}$$

The results (2.8) and(2.9) are sharp for the functions given by (2.2) and (2.6), respectively.

Proof. We prove (a). The proof of (b) is similar to (a) and will be omitted. We write

$$\frac{1+w(z)}{1-w(z)} = \frac{\sigma_{n+1}(p,\delta,\alpha)+2(p-\alpha)}{2(p-\alpha)}\left[\frac{f_n(z)}{f(z)} - \left(\frac{\sigma_{n+1}(p,\delta,\alpha)}{\sigma_{n+1}(p,\delta,\alpha)+2(p-\alpha)}\right)\right]$$

$$= \frac{1+\sum\limits_{k=1}^{n} a_{p+k-1}z^{k+p} + \frac{\sigma_{n+1}(p,\delta,\alpha)}{2(p-\alpha)}\sum\limits_{k=n+1}^{\infty} a_{p+k-1}z^{k+p}}{1+\sum\limits_{k=1}^{\infty} a_{p+k-1}z^{k+p}}$$

where

$$|w(z)| \leq \frac{\frac{\sigma_{n+1}(p,\delta,\alpha)+2(p-\alpha)}{2(p-\alpha)}\sum\limits_{k=n+1}^{\infty}|a_{p+k-1}|}{2-2\sum\limits_{k=1}^{n}|a_{p+k-1}| - \left(\frac{2(p-\alpha)-\sigma_{n+1}(p,\delta,\alpha)}{2(p-\alpha)}\right)\sum\limits_{k=n+1}^{\infty}|a_{p+k-1}|} \leq 1.$$

This last inequality is equivalent to

$$\sum_{k=1}^{n}|a_{p+k-1}| + \sum_{k=n+1}^{\infty}\frac{\sigma_{n+1}(p,\delta,\alpha)}{2(p-\alpha)}|a_{p+k-1}| \leq 1. \tag{10}$$

The left hand side of (2.10) is bounded above by $\sum\limits_{k=1}^{\infty}[\sigma_k(p,\delta,\alpha)/(2(p-\alpha))]\,|a_{p+k-1}|$, the proof is completed.

We next turn to ratios involving derivatives

Theorem 4. *If $f(z)$ of the form (1.1) satisfies the condition (1.6), then for $z \in \mathcal{U}$,*

(a) $\mathrm{Re}\{f'(z)/f_n'(z)\} \geq [\sigma_{n+1}(p,\delta,\alpha)-2(n+1)(p-\alpha)]/\sigma_{n+1}(p,\delta,\alpha)$.

(b) $\mathrm{Re}\{f_n'(z)/f'(z)\} \geq \sigma_{n+1}(p,\delta,\alpha)/[\sigma_{n+1}(p,\delta,\alpha)+2(n+1)(p-\alpha)]$.

*The results in (*a*) and in (*b*) are sharp with the function given by (2.2)*

Proof. We prove only (a), which is similar to the proof of Theorem 1. The proof of (b) follows the pattern of that in Theorem 3(a). We write

$$\frac{1+w(z)}{1-w(z)} = \frac{\sigma_{n+1}(p,\delta,\alpha)}{2(n+1)(p-\alpha)}\left[\frac{f'(z)}{f_n'(z)} - \left(\frac{\sigma_{n+1}(p,\delta,\alpha)-2(n+1)(p-\alpha)}{\sigma_{n+1}(p,\delta,\alpha)}\right)\right]$$

where

$$w(z) = \frac{\left(\frac{\sigma_{n+1}(p,\delta,\alpha)}{2(n+1)(p-\alpha)}\right)\sum\limits_{k=n+1}^{\infty}ka_{p+k-1}z^{k+p}}{2+2\sum\limits_{k=2}^{n}ka_{p+k-1}z^{k+p} + \left(\frac{\sigma_{n+1}(p,\delta,\alpha)}{2(n+1)(p-\alpha)}\right)\sum\limits_{k=n+1}^{\infty}ka_{p+k-1}z^{k+p}}$$

Now $|w(z)| \leq 1$ if

$$\sum_{k=2}^{n} k\,|a_{p+k-1}| + \frac{\sigma_{n+1}(p,\delta,\alpha)}{2(n+1)(p-\alpha)} \sum_{k=n+1}^{\infty} k\,|a_{p+k-1}| \leq 1.$$

From the condition (1.6), it is sufficient to show that

$$\sum_{k=2}^{n} k\,|a_{p+k-1}| + \frac{\sigma_{n+1}(p,\delta,\alpha)}{2(n+1)(p-\alpha)} \sum_{k=n+1}^{\infty} k\,|a_{p+k-1}| \leq \sum_{k=2}^{\infty} \frac{\sigma_k(p,\delta,\alpha)}{2(p-\alpha)}\,|a_{p+k-1}|$$

which is equivalent to

$$\sum_{k=2}^{n} \left(\frac{\sigma_k(p,\delta,\alpha)}{2(p-\alpha)} - k\right)|a_{p+k-1}| + \sum_{k=n+1}^{\infty} \left(\frac{\sigma_k(p,\delta,\alpha)}{2(p-\alpha)} - \frac{\sigma_{n+1}(p,\delta,\alpha)}{2(n+1)(p-\alpha)}k\right)|a_{p+k-1}| \geq 0,$$

and the proof is complete.

Theorem 5. *If $f(z)$ of the form (1.1) satisfies the condition (1.7), then for $z \in \mathcal{U}$,*

(a) $\operatorname{Re}\{f'(z)/f_n'(z)\} \geq [(p+n)\sigma_{n+1}(p,\delta,\alpha) - 2(p-\alpha)(n+1)]/[(p+n)\sigma_{n+1}(p,\delta,\alpha)]$.

(b) $\operatorname{Re}\{f_n'(z)/f'(z)\} \geq [(p+n)\sigma_{n+1}(p,\delta,\alpha)]/[(p+n)\sigma_{n+1}(p,\delta,\alpha) + 2(p-\alpha)(n+1)]$.

*The results in (*a*) and in (*b*) are sharp with the function given by (2.6).*

Proof. It is well known that $f \in \Lambda_p(\alpha) \Leftrightarrow zf' \in \Omega_p(\alpha)$. In particular, f satisfies condition (1.7) if and only if zf' satisfies condition (1.6). Thus, (a) is an immediate consequence of Theorem 1 and (b) follows directly from Theorem 3(a).

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BIFURCATIONS OF THE LOGISTIC MAP PERTURBED BY ADDITIVE NOISE

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Abstract. In the paper we analyze, mainly numerically, the bifurcations of the logistic map perturbed by different type of additive noise: uniformly and normally distributed random variables. We prove the existence of the stationary density in both cases using some tools from [6], and study the bifurcations. In [4] there are numerical results for the uniform noise case. We extend the simulations for the logistic map perturbed by normally distributed random variables. In this case we get a different bifurcation scenario as in the case of perturbation by uniformly distributed random variables.

1. Basic Notions

Let (X, d) be a metric space and $S : X \to X$ be a discrete dynamical system. Let $x_0 \in X$. Then x_0, $x_1 = S(x_0)$, $x_2 = S(x_1)$,..., $x_n = S(x_{n-1})$,... is the orbit of x_0. In the deterministic case we usually study the orbit of different $x_0 \in X$ to find the dynamics of the system.

Now let ξ_0, ξ_1,...ξ_n,... be independent random variables and we use formula

$$x_{n+1} = S(x_n) + \xi_n, n \in \mathbb{N} \tag{1}$$

to find the orbit of a point from X. In this case the orbit of a point x_0 is different for different realization of the noise. Thus is more adequate to study the change of the initial density function.

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Let $X = \mathbb{R}$, g be the density function of the random variables ξ_0, ξ_1,...ξ_n,... and f_n the density function of x_n. Then we have to find a relation between f_n and f_{n+1}. For this we take an arbitrary bounded, measurable function $h : \mathbb{R} \to \mathbb{R}$ and calculate the expected value of $h(x_{n+1})$ in two different ways. Firstly,

$$\mathbb{E}(h(x_{n+1})) = \int_{\mathbb{R}} h(x) f_{n+1}(x) dx. \tag{2}$$

Secondly,

$$\begin{aligned} \mathbb{E}(h(x_{n+1})) &= \mathbb{E}(h(S(x_n) + \xi_n)) \\ &= \int_{\mathbb{R}} \int_{\mathbb{R}} h(S(y) + z) f_n(y) g(z) dy dz \\ &= \int_{\mathbb{R}} \int_{\mathbb{R}} h(x) f_n(y) g(x - S(y)) dx dy, \end{aligned} \tag{3}$$

using the change of variables $S(y) + z = x$.

From (2) and (3) we get

$$f_{n+1}(x) = \int_{\mathbb{R}} f_n(y) g(x - S(y)) dy. \tag{4}$$

We review some notions which we need to study the existence of a stationary density for a random dynamical system. In the followings let $(X, \mathcal{A}, \mu)$ be a measure space.

A linear operator $P : L^1 \to L^1$ is called **Markov operator** if

(a) $Pf \geq 0$, for all $f \geq 0$, $f \in L^1$;

(b) $\|Pf\| = \|f\|$, for all $f \geq 0$, $f \in L^1$.

A measurable function $K : X \times X \to \mathbb{R}$ is called **stochastic kernel** if

(a) $K(x, y) \geq 0$, for all $x, y \in X$;

(b) $\int_X K(x, y) \mu(dx) = 1$, for all $y \in X$.

Let $G \subset \mathbb{R}^d$ be a measurable, unbounded set, $K : G \times G \to \mathbb{R}$ a stochastic kernel. A measurable, nonnegative function $V : G \to \mathbb{R}$, for which

$$\lim_{|x| \to \infty} V(x) = \infty,$$

is called **Liapunov function**.

Returning to formula (4) we observe that

$$Pf(x) = \int_{\mathbb{R}} f(y)g(x - S(y))dy \tag{5}$$

is a Markov operator and

$$K(x, y) = g(x - S(y))$$

is a stochastic kernel. We can write formula (4) in the form $f_{n+1} = Pf_n$, which is equivalent with $f_{n+1} = P^{n+1}f_0$, thus we have to study the sequence $\{P^n\}$.

Let P be a Markov operator. A density function f is a **stationary density** if $Pf = f$.

$\{P^n\}$ is **asymptotically stable** if there exists a unique stationary density f_* such that

$$\lim_{n\to\infty} \|P^n - f_*\| = 0, \text{ for every density } f.$$

The proof of the following theorem can be found in [6]. The theorem gives a sufficient condition for the asymptotic stability of $\{P^n\}$.

Theorem 1.1. ([6], Theorem 5.7.1, pg 115) *Let $K : G \times G \to \mathbb{R}$ be a stochastic kernel, P the Markov operator given by (5). If K satisfies*

$$\int_G \inf_{|y|<r} K(x, y)dx > 0, \quad \textit{for all} \quad r > 0, \tag{6}$$

and there exists a Liapunov function $V : G \to \mathbb{R}$ such that

$$\int_G K(x, y)V(x)dx \leq \alpha V(y) + \beta, \ 0 \leq \alpha < 1, \ \beta \geq 0 \tag{7}$$

for every density f, then $\{P^n\}$ is asymptotically stable.

Consider a dynamical system which depends on a parameter r. A value r_0 of the parameter is a **bifurcation point**, if the system changes its dynamics for this value. There are two approaches in studying bifurcation: the phenomenological ((P)-bifurcation) and the dynamical ((D)-bifurcation) approach.

The (P)-bifurcation approach studies the qualitative changes of stationary densities. In the simulations we study the changes of the shape of the histogram for different values of the parameter. To draw the histogram we start with K initial points X_0^1, X_0^2, ...,X_0^K (K a big natural number) and we calculate the $(N+1)$th point of the orbit of every initial point getting X_N^1, X_N^2,...,X_N^K, $N \in \mathbb{N}$. Then we plot the histogram of the values X_N^1, X_N^2,...,X_N^K: we divide the interval $[0, 1]$ into 100 parts and count how many points are in each small interval. We are looking for parameter values r_0, for which the shape of the histogram changes.

The (D)-bifurcation approach focuses on the loss of stability of invariant measures. For this we study the Liapunov exponent which is calculated using the formula

$$\lambda_r(x) = \lim_{n\to\infty} \frac{1}{n} \log \left| \frac{\partial \varphi(n, x)}{\partial x} \right|,$$

where $\varphi(n, x) = x_n$. We are looking for parameter values for which the Liapunov exponent changes its sign.

It is also helpful to draw the bifurcation diagram. For this for every value r of the parameter we start with an arbitrary initial point x_0 and we calculate the points x_1, x_2,...,x_N of the orbit, for N big natural number. Then we calculate x_{N+1},..., x_M, $M > N + 1$, and plot the points (r, x_{N+1}), (r, x_{N+2}),...,(r, x_M).

2. The Deterministic Logistic Map

The deterministic case have been intensively studied. In this case the orbit of a point $x_0 \in \mathbb{R}$ can be calculated by the recursive formula

$$x_{n+1} = r x_n (1 - x_n), n \in \mathbb{N}.$$

In the followings we consider $x_0 \in [0, 1]$. The bifurcation scenario in this case is well known, see for example [2], [1] or [5]. We summarize this scenario for better comparison of the deterministic and stochastic case. In Figure 1 is plotted the bifurcation diagram and the Liapunov exponent. In simulations we approximate the Liapunov exponent by

$$\lambda_r(x) = \frac{1}{N} \sum_{k=0}^{N-1} \log |r(1 - 2\varphi(k, x)))|,$$

where N is a big natural number. If $0 < r < 1$, there are two fixed points: a stable fixed point 0 and an unstable fixed point $1 - \dfrac{1}{r}$, so the orbit of each point from [0,1] converges to 0. For $1 < r < 3$ the fixed point 0 becomes unstable and the orbit of

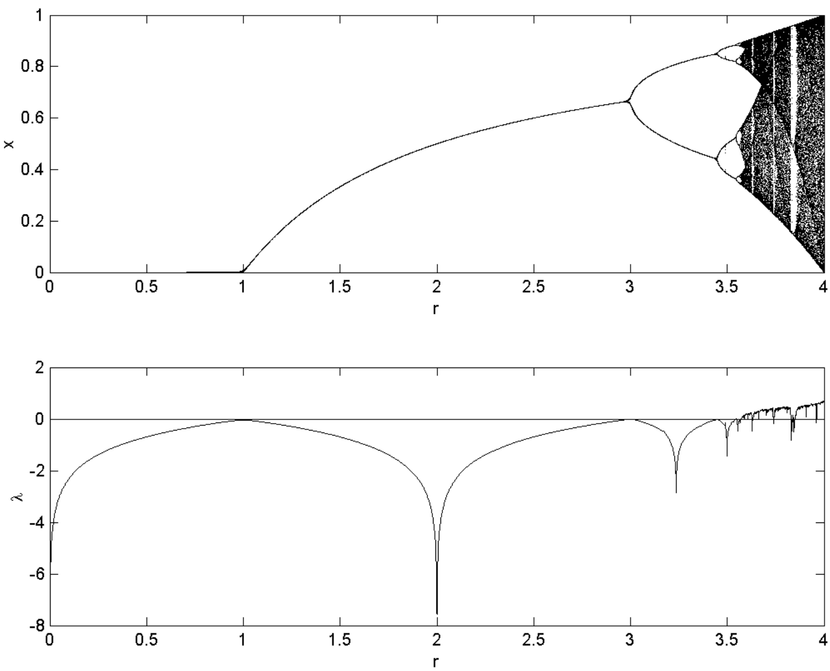

FIGURE 1. *Bifurcation diagram and Liapunov exponent in the deterministic case*

each point converges to $1 - \frac{1}{r}$. Thus $r = 1$ is a bifurcation point. Another bifurcation point is $r = 3$, where the orbit becomes an attractive period-2 orbit. In $r = 3.46$ the period-2 orbit becomes unstable and is replaced by a stable period-4 orbit. We can observe this behavior on the bifurcation diagram. As r increases this period doubling continues. This scenario is illustrated by Figure 2 too, where the shape of the histogram changes from a two-peaked to a four-peaked, then to an eight-peaked form. For $r = 3.57$ the dynamics becomes chaotic. For $r > 3.57$ the chaotic and period doubling behavior alternates. For $r = 3.83$ there is a stable period-3 orbit. If we study the Lyapunov exponent, this becomes zero for $r = 1$, $r = 3$, $r = 3,46$, etc., so these points are bifurcation points with this approach too.

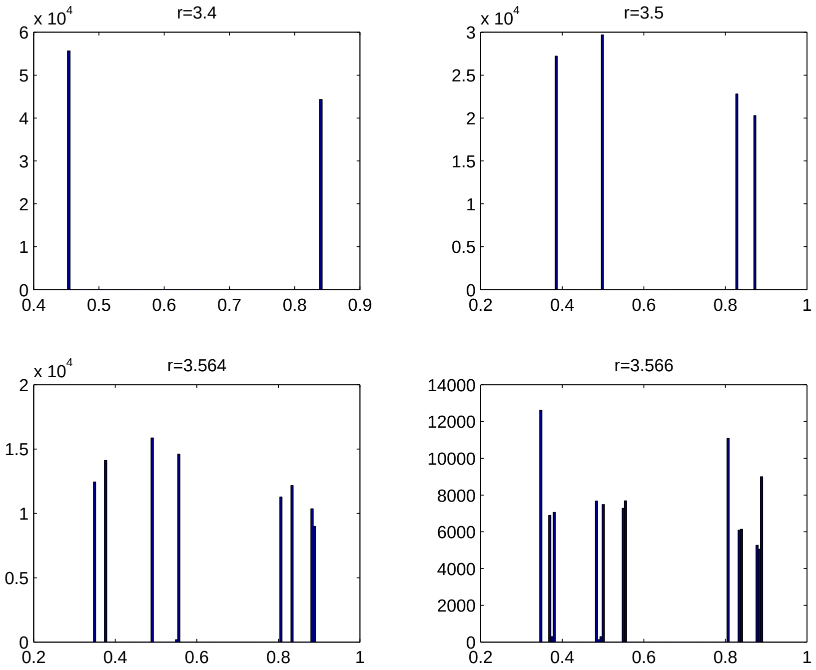

FIGURE 2. *Histogram in the neighborhood of* $r = 3.5$ *in deterministic case*

3. Perturbation with Uniformly Distributed Random Variables

Consider now the logistic map perturbed by uniformly distributed independent random variables ξ_0, ξ_1,...,ξ_n,... taking values in some interval $[a, b]$. The orbit of a point $x_0 \in [0, 1]$ can be calculated with the formula

$$x_{n+1} = rx_n(1 - x_n) + \xi_n, n \in \mathbb{N}.$$

We study the bifurcation points with two different approaches: the (P)-bifurcation and the (D)-bifurcation approach. In [4] there are some numerical results for this case. We extend them studying the changes of the histogram for different values of r.

Firstly using Theorem 1.1 we prove that for every $r \in (0, 4)$ there exists a stationary density function.

Theorem 3.1. *In case of the logistic map perturbed by uniformly distributed random variables, for every* $r \in (0, 4)$ *there exists a stationary density function.*

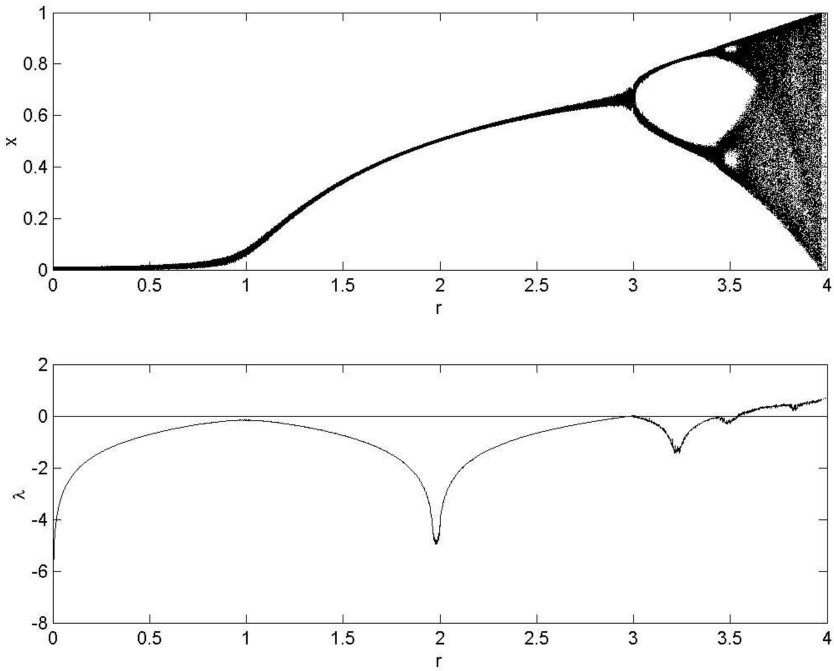

FIGURE 3. *Bifurcation diagram and Liapunov exponent for* $a = 0$ *and* $b = 0.01$

Proof. Let $V(x) = |x|$ be a Liapunov function. Then

$$\int_{\mathbb{R}} K(x,y)V(x)dx = \int_{\mathbb{R}} |x|g(x-ry)dx = \int_{\mathbb{R}} |s+ry|g(s)ds$$
$$\leq \int_{\mathbb{R}} |s|g(s)ds + |S(y)| \leq \frac{r}{4}V(y) + \frac{a+b}{2} + 1,$$

so $\alpha = \dfrac{r}{4}$ and $\beta = \dfrac{a+b}{2} + 1$, and $\alpha < 1$, if $r < 4$. So by Theorem 1.1 there exists a stationary density. □

In Figure 3 we plotted the bifurcation diagram for $a = 0$ and $b = 0.01$. The Liapunov exponent is not 0 in $r = 1$, so this point is not a (D)-bifurcation point. Studying the histogram in neighborhood of $r = 1$ leads to the conclusion, that this is not a (P)-bifurcation point, too (see Figure 4). So the dynamics of the system in $r = 1$ is different as in the deterministic case, where this point was a bifurcation point. In

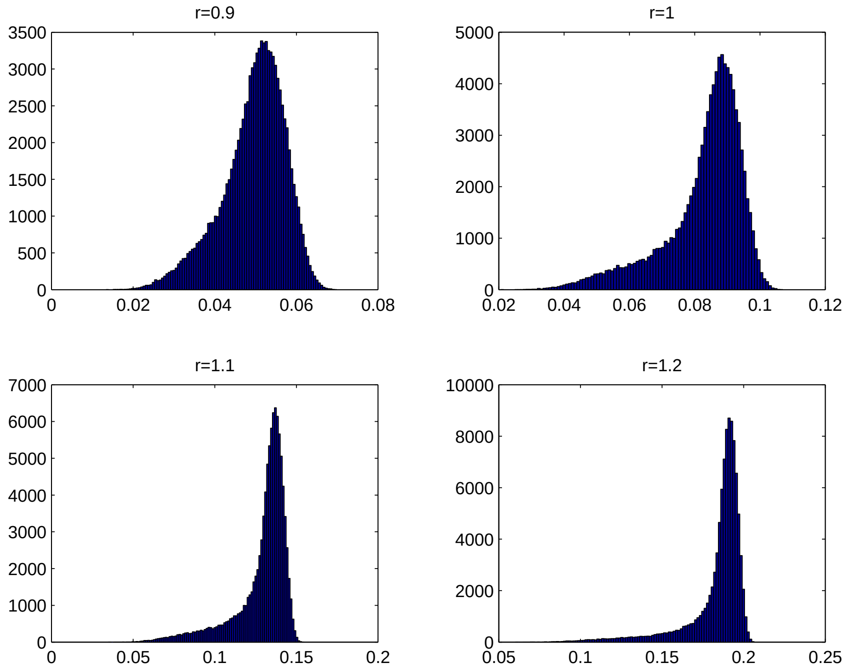

FIGURE 4. *Histogram in the neighborhood of* $r = 1$ *for* $a = 0$, $b = 0.01$

$r = 3$ the Liapunov exponent is also not 0, but this point is a (P)-bifurcation point, as the histogram changes its shape from a one-peaked to a two-peaked form (see Figure 5). Another (P)-bifurcation occurs between $r = 3.4$ and $r = 3.5$, where we observe a transition from a two-peaked histogram to a four-peaked histogram (see Figure 6). But the Liapunov exponent remains negative in this case, too.

Even if the Liapunov exponent for the deterministic case and for the small noise case is close to each other, the behavior of single trajectories can be very different, as the Liapunov exponent measures only the exponential of convergence (divergence) of two neighboring trajectories.

It is interesting that for $b \geq 0.05$ the period doubling behavior disappears (see Figure 7). Studying the histogram for values between 3.4 and 3.6 we observe that the shape doesn't become four-peaked as in the case of $b = 0.01$ (see Figure 8).

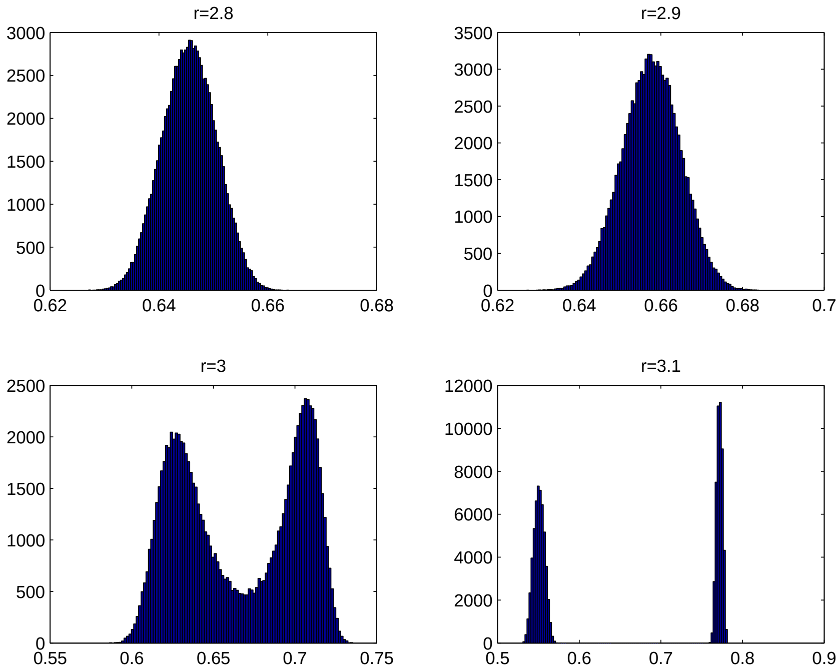

FIGURE 5. *Histogram in the neighborhood of* $r = 3$ *for* $a = 0$, $b = 0.01$

4. Perturbation with Normally Distributed Random Variables

Now consider ξ_0, ξ_1,...,ξ_n,... to be normally distributed independent random variables with mean m and variance σ^2.

Using Theorem 1.1 we prove that for every $r \in (0, 4)$ there exists a stationary density function.

Theorem 4.1. *In case of the logistic map perturbed by normally distributed random variables, for every* $r \in (0, 4)$ *there exists a stationary density function.*

Proof. Let $V(x) = |x|$ be a Liapunov function. Then

$$\int_{\mathbb{R}} K(x, y)V(x)dx = \int_{\mathbb{R}} |x|g(x - ry)dx = \int_{\mathbb{R}} |s + ry|g(s)ds$$
$$\leq \int_{\mathbb{R}} |s|g(s)ds + |S(y)| \leq \frac{r}{4}V(y) + m + 1,$$

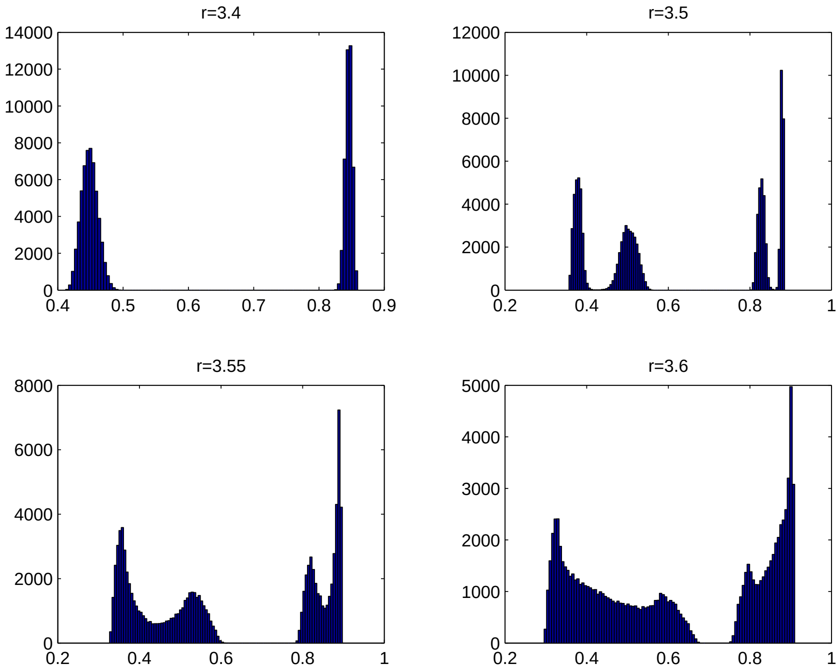

FIGURE 6. *Histogram in the neighborhood of* $r = 3.5$ *for* $a = 0$*,* $b = 0.01$

so $\alpha = \dfrac{r}{4}$ and $\beta = m + 1$ in Theorem 1.1. We have to have $\alpha < 1$, so $r < 4$. $\square$

In Figure 9 we see the bifurcation diagram and Liapunov exponent for $m = 0$ and $\sigma = 0.0001$. Comparing with Figure 1 we see that for small noise the bifurcation scenario is similar with the scenario in deterministic case. Here $r = 1$ is a bifurcation point as in the deterministic case (see Figure 10).

If the noise is bigger ($\sigma = 0.001$) the phenomena in $r = 1$ is interesting. Observe in Figure 11 that in neighborhood of $r = 1$ seems to be a chaotic region. The Lyapunov exponent becomes positive in $r = 1$.

In case of $\sigma = 0.01$ this region becomes larger (see Figure 12). The histogram in neighborhood of $r = 1$ (Figure 13) also tells this, see the histograms for $r = 0.9$, $r = 1$ and $r = 1.1$, where the values are spread to a large interval. Note that for $r = 1$ the scale of the $0x$ axis is multiplied by 10^{307}! $r = 3$ is a (P)-bifurcation

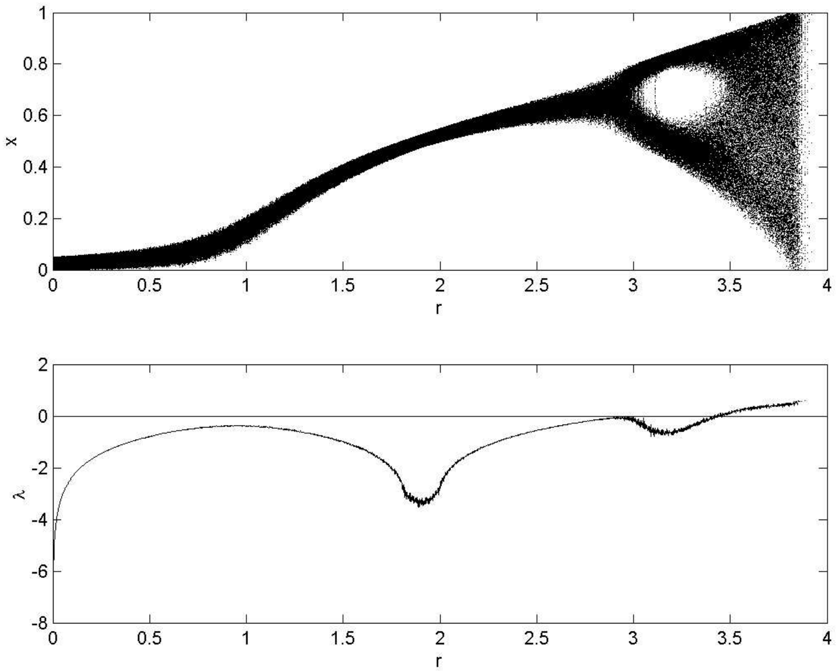

FIGURE 7. *Bifurcation diagram and Liapunov exponent for* $a = 0$ *and* $b = 0.05$

point, because the histogram changes its shape from one-peaked to two-peaked form, but this is not a (D)-bifurcation, because the Liapunov exponent stays negative. It is interesting that between 3.5 and 3.6 the Liapunov exponent changes its sign several times (see the zoomed in part of Figure 12 in Figure 15), so these point are (D)-bifurcation points, but the histogram doesn't changes its shape (Figure 16), so they are not (P)-bifurcation points.

We don't observe the period doubling behavior in this case (Figure 17) similarly with the case of the perturbation with uniformly distributed random variables on the interval $[a, b]$ for $b \geq 0.05$. So if the noise becomes bigger the period doubling behavior disappears. It is also interesting that for $r < 1.2$ the points of the orbit can have negative values too (as the random variables added can be negative), but for $r > 1.2$ the points are positive.

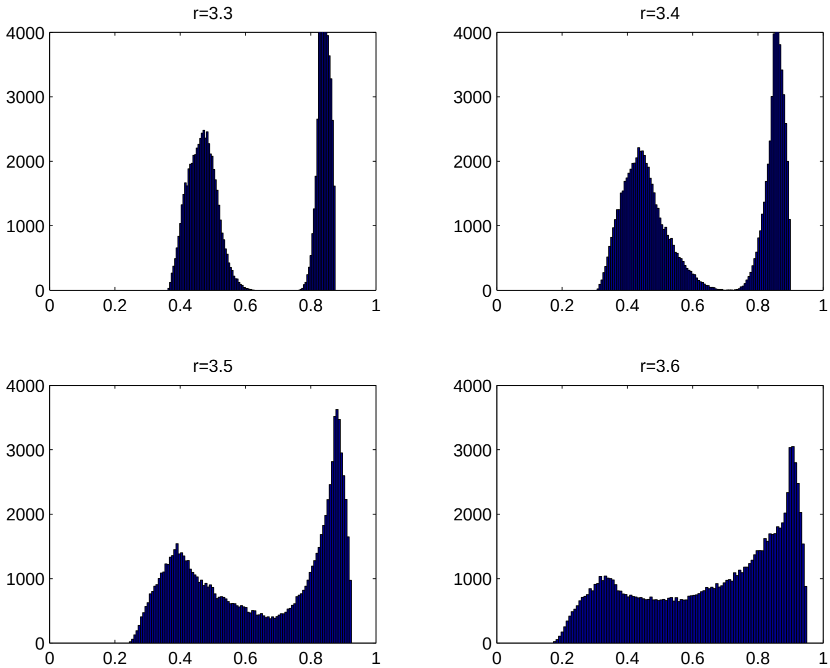

FIGURE 8. *Histogram in the neighborhood of* $r = 3.5$ *for* $a = 0$ *and* $b = 0.05$

Now change the mean of the normally distributed random variables. If the mean becomes positive the chaotic region around $r = 1$ disappears (see Figure 18 for $m = 0.01$ and $\sigma = 0.01$). If the mean becomes negative the length of the interval of the values of r for which we get a chaotic behavior increases as the mean decreases. In Figure 19 we observe, that for $m = -0.01$ the chaotic region is larger than in the case of $m = 0$.

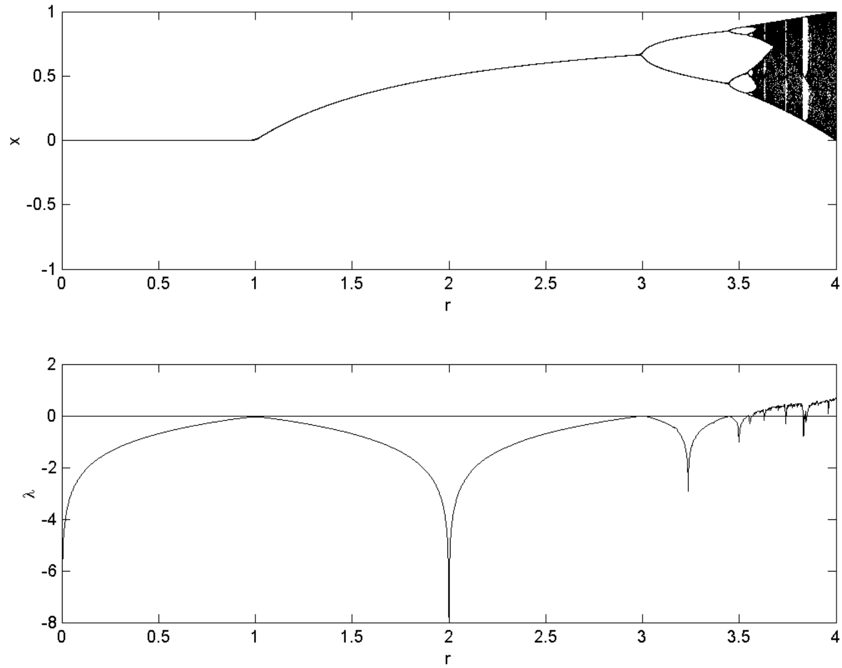

FIGURE 9. *Bifurcation diagram and Liapunov exponent for* $m = 0$ *and* $\sigma = 0.0001$

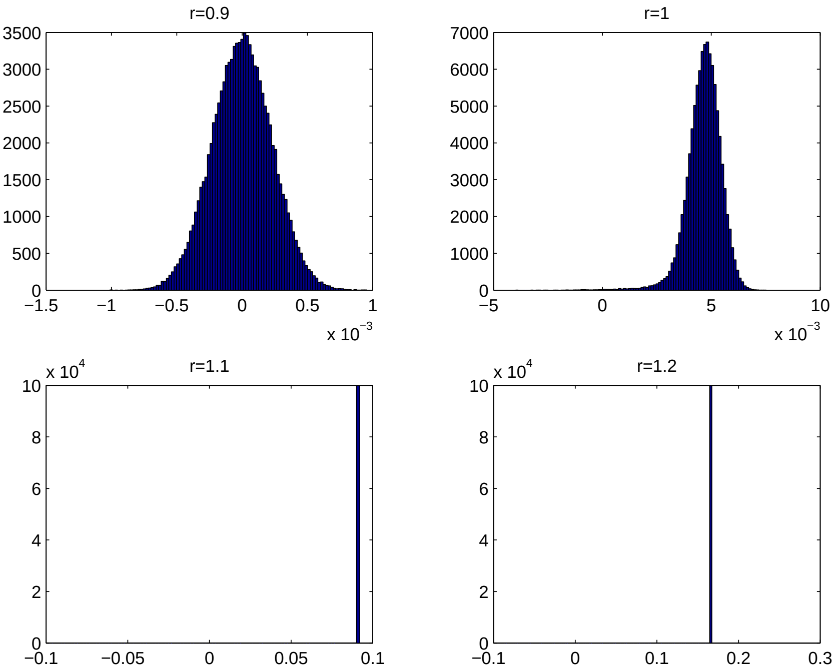

Figure 10. *Histogram in the neighborhood of* $r = 1$ *for* $m = 0$ *and* $\sigma = 0.0001$

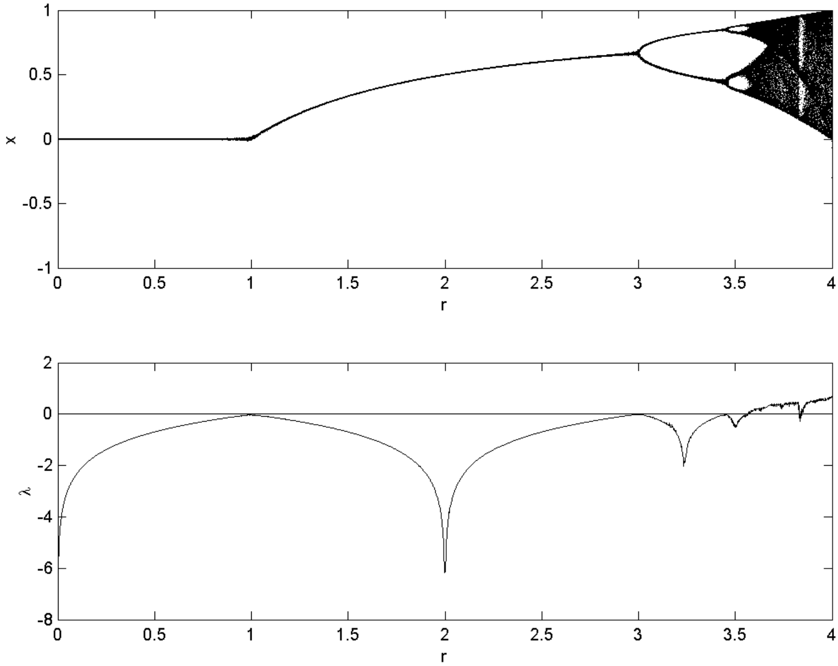

FIGURE 11. *Bifurcation diagram and Liapunov exponent for* $m = 0$ *and* $\sigma = 0.001$

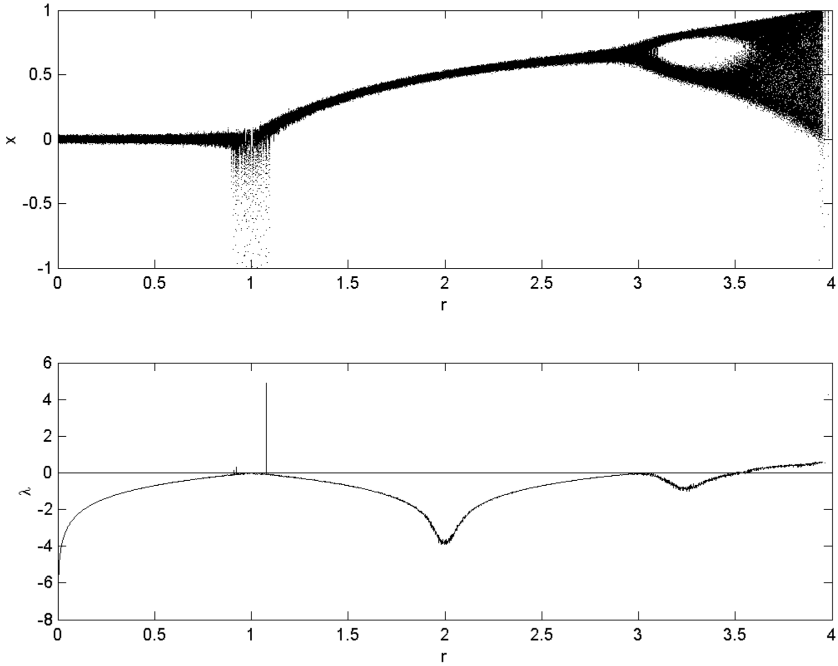

Figure 12. *Bifurcation diagram and Liapunov exponent for* $m = 0$ *and* $\sigma = 0.01$

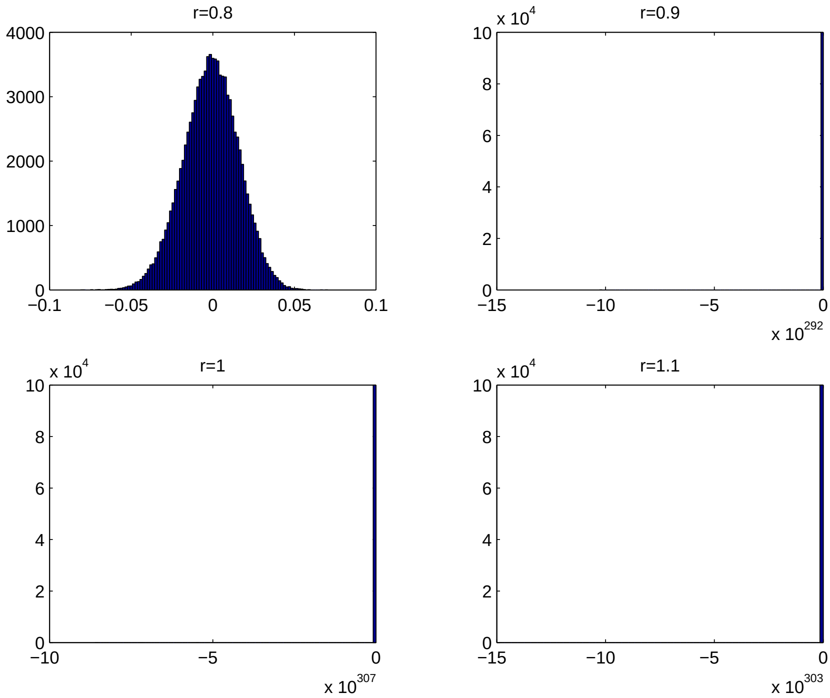

FIGURE 13. *Histogram in the neighborhood of* $r = 1$ *for* $m = 0$ *and* $\sigma = 0.01$

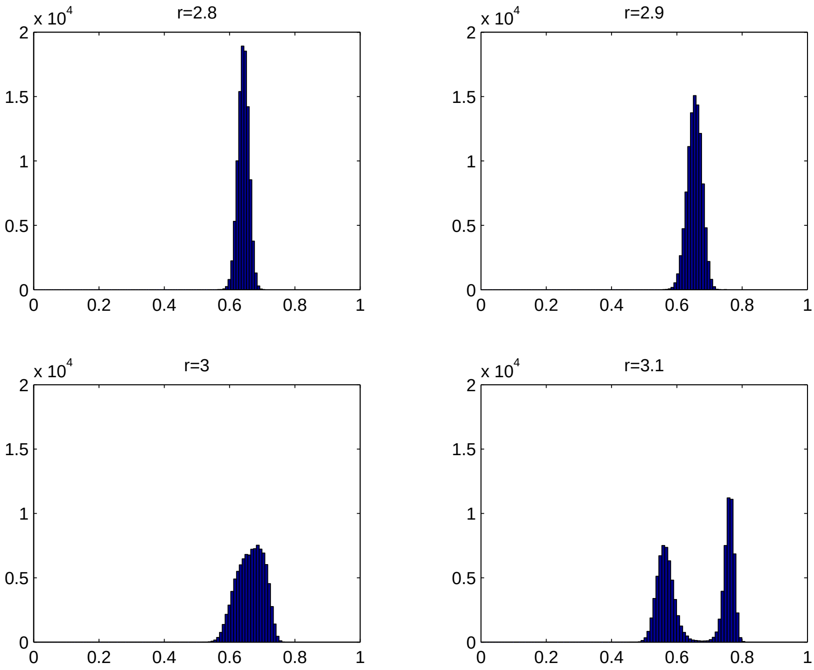

FIGURE 14. *Histogram in the neighborhood of* $r = 3$ *for* $m = 0$ *and* $\sigma = 0.01$

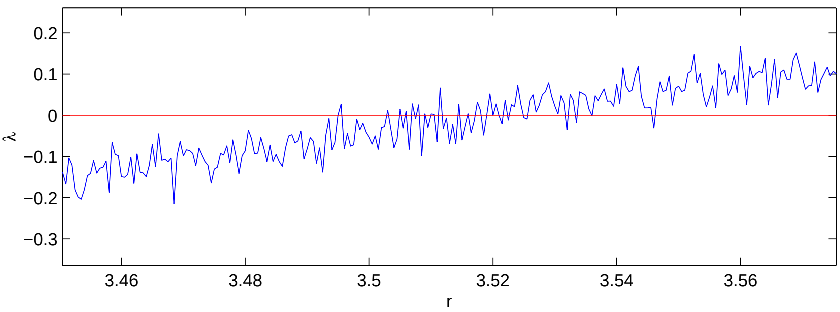

FIGURE 15. *Liapunov exponent for* $m = 0$ *and* $\sigma = 0.01$

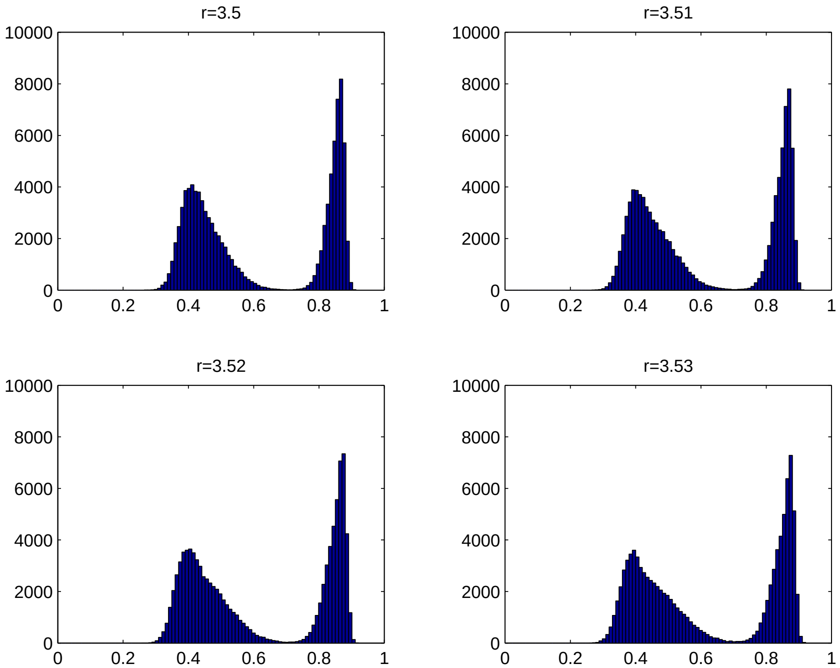

FIGURE 16. *Histogram in the neighborhood of* $r = 3.5$ *for* $m = 0$ *and* $\sigma = 0.01$

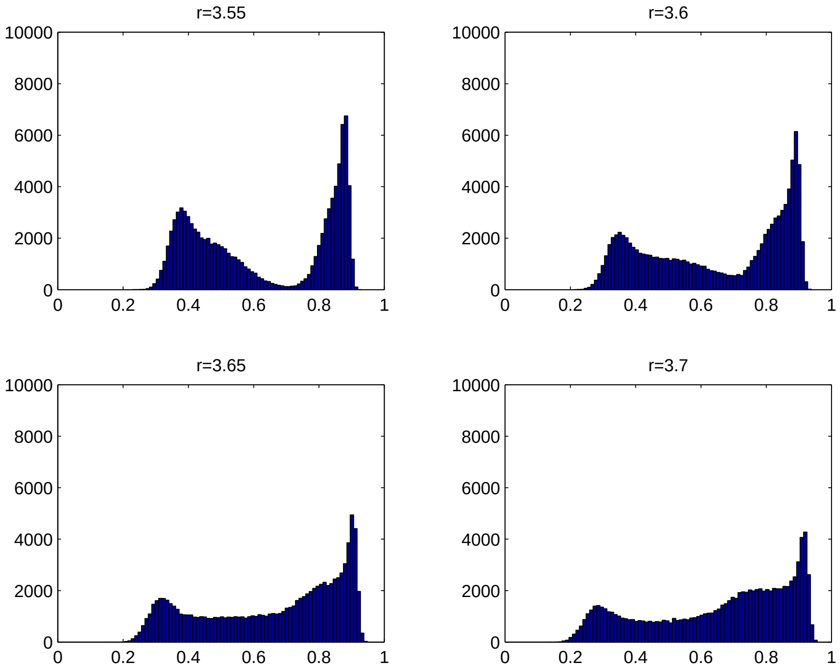

FIGURE 17. *Histogram in the neighborhood of* $r = 3.55$ *for* $m = 0$ *and* $\sigma = 0.01$

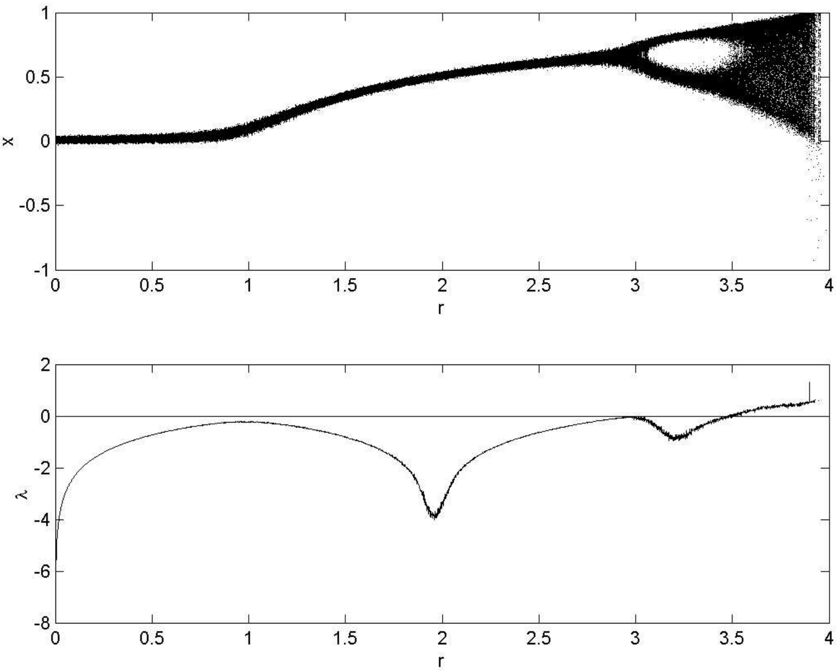

FIGURE 18. *Bifurcation diagram and Liapunov exponent for* $m = 0.01$ *and* $\sigma = 0.01$

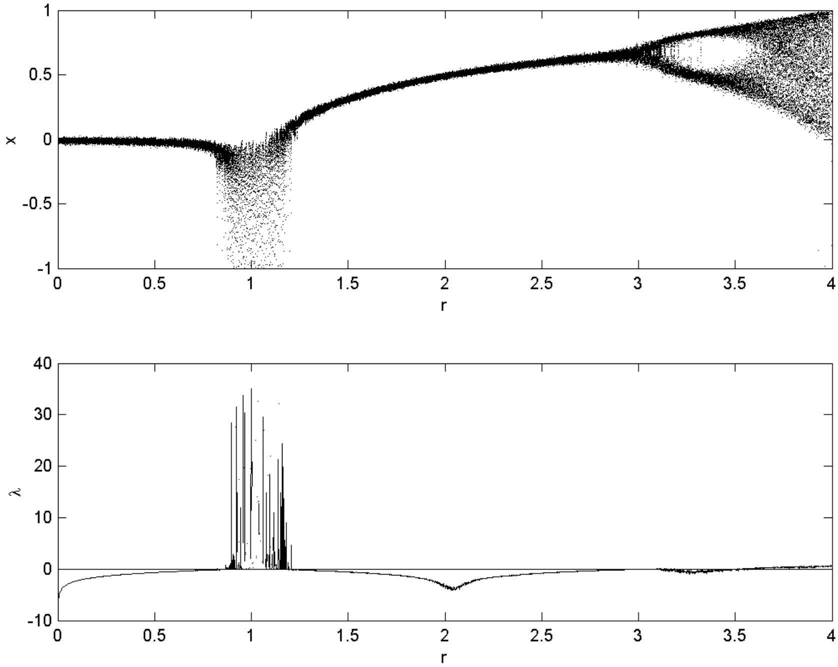

FIGURE 19. *Bifurcation diagram and Liapunov exponent for* $m = -0.01$ *and* $\sigma = 0.01$

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ON SOME INTEGRAL EQUATIONS WITH DEVIATING ARGUMENT

OLARU ION MARIAN

Abstract. The purpose of this paper is to study the following functional equation with modified argument:

$$x(t) = g(t, hx(t), x(t), x(0)) + \int_{-\theta t}^{\theta t} K(t, s, x(s))ds,$$

where $\theta \in (0, 1), t \in [-T, T], T > 0$.

1. **Introduction**

Let (X, d) be a metric space and $A : X \to X$ an operator.We shall use the following notations:

$F_A := \{x \in X \mid Ax = x\}$ the fixed points set of A.

$I(A) := \{Y \in P(X) \mid A(Y) \subset Y\}$ the family of the nonempty invariant subsets of A.

$A^{n+1} = A \circ A^n, A^0 = 1_X, A^1 = A, n \in N$.

Definition 1.1. [4] *An operator A is weakly Picard operator(WPO)if the sequence*

$$(A^n x)_{n\in N}$$

converges, for all $x \in X$ and the limit(which depend on x) is a fixed point of A.

Definition 1.2. [4],[1] *If the operator A is WPO and $F_A = \{x^*\}$then by definition A is Picard operator.*

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Definition 1.3. [4] *If A is WPO, then we consider the operator*

$$A^{\infty} : X \to X, A^{\infty}(x) = \lim_{n\to\infty} A^n x.$$

We remark that $A^{\infty}(X) = F_A$.

Definition 1.4. *[1] Let be A an WPO and* $c > 0$*.The operator A is c-WPO if* $d(x, A^{\infty}x) \leq d(x, Ax)$.

We have the following characterization of the WPOs

Theorem 1.1. [4]*Let* (X, d) *be a metric space and* $A : X \to X$ *an operator. The operator A is WPO (c-WPO) if and only if there exists a partition of X,*

$$X = \bigcup_{\lambda\in\Lambda} X_{\lambda}$$

such that

(a)$X_{\lambda} \in I(A)$

(b)$A \mid: X_{\lambda} \to X_{\lambda}$*is a Picard(c-Picard) operator,for all* $\lambda \in \Lambda$.

For the class of c-WPOs we have the following data dependence result.

Theorem 1.2. [4] *Let* (X, d) *be a metric space and* $A_i : X \to X, i = 1, 2$ *an operator.We suppose that :*

(i)the operator A_i *is* $c_i - WPOi$*=1,2.*

(ii)there exists $\eta > o$ *such that*

$$d(A_1x, A_2x) \leq \eta, (\forall)x \in X.$$

Then

$$H(F_{A_1}, F_{A_2}) \leq \eta max\{c_1, c_2\}.$$

Here stands for Hausdorff-Pompeiu functional

We have

Lemma 1.1. [4],[1] *Let$(X, d, \leq)$ be an ordered metric space and $A : X \to X$ an operator such that:*

a)A is monotone increasing.

b)A is WPO.

Then the operator A^∞ is monotone increasing.

2. Main results

Data dependence for functional-integral equations was study in [2],[3],[4],[1].

Let$(X, \|\cdot\|)$ a Banach space and the space $C([-T,T],X)$ endowed with the Bieleski norm $\|\cdot\|_\tau$ defined by

$$\|x\|_\tau = \max_{t\in[-T,T]} \|x(t)\| e^{-\tau(t+T)}.$$

In[1] Viorica Muresan was study the following functional integral equation:

$$x(t) = g(t, h(x)(t), x(t), x(0)) + \int_0^t K(t, s, x(\theta s))ds, t \in [0,b], \theta \in [0,1]$$

by the weakly Picard operators technique.

We consider the following functional-integral equations with modified argument:

$$x(t) = g(t, hx(t), x(t), x(0)) + \int_{-\theta t}^{\theta t} K(t, s, x(s))ds, \tag{1}$$

where:

i)$t \in [-T,T], , T > 0$.

ii)$h : C([-T,T],X) \longrightarrow C([-T,T],X), g \in C([-T,T] \times X^3, X), K \in C([-T,T] \times [-T,T] \times X^2, X)$.

We suppose that the following conditions are satisfied:

(c_1) there exists $l > 0$ such that

$$\|hx(t) - hy(t)\| \leq l\|x(t) - y(t)\|,$$

for all $x, y \in C([-T, T], X), t \in [-T, T]$.

(c_2) There exists $l_1 > 0, l_2 > 0$ such that

$$\|g(t, u_1, v_1, w) - g(t, u_2, v_2, w)\| \leq l_1 \|u_1 - u_2\| + l_2 \|v_1 - v_2\| .$$

for all $t \in [-T, T], u_i, v_i, w \in X, i = 1, 2$.

(c_3) There exists $l_3 > 0$ such that

$$\|K(t, s, u) - K(t, s, u_1)\| \leq l_3 \|u - u_1\|,$$

for all $t, s \in [-T, T], u, u_1 \in X$.

$(c_4) l_1 l + l_2 < 1$.

$(c_5) g(0, h(x)(0), x(0), x(0)) = x(0)$ for any $x \in C([-T, T], X)$.

Let $A : C([-T, T], X) \longrightarrow C([-T, T], X)$ be defined by

$$Ax(t) = g(t, hx(t), x(t), x(0)) + \int_{-\theta t}^{\theta t} K(t, s, x(s)) ds \tag{2}$$

Let $\lambda \in X$ and $X_\lambda = \{x \in C([-T, T], X) \mid x(0) = \lambda\}$. Then $C([-T, T], X) = \bigcup_{\lambda \in X} X_\lambda$ is a partition of $C([-T, T], X)$.From c_5 we have that $X_\lambda \in I(A)$.

For studding of data dependence we consider the following equations

$$x(t) = g_1(t, hx(t), x(t), x(0)) + \int_{-\theta t}^{\theta t} K_1(t, s, x(s)) ds \tag{3}$$

$$x(t) = g_2(t, hx(t), x(t), x(0)) + \int_{-\theta t}^{\theta t} K_2(t, s, x(s)) ds \tag{4}$$

Theorem 2.1. *We consider the equation (1) under following conditions:*

(i)The conditions $c_1 - c_5$ are satisfied.

(ii)The operators $h(\cdot), g(t, \cdot, \cdot, \cdot), K(t, s, \cdot, \cdot)$ are monotone increasing.

(iii)There exists $\eta_1, \eta_2 > 0$ such that

$$\|g_1(t, u, v, w) - g_2(t, u, v, w)\| < \eta_1,$$

$$\|K_1(t, s, u) - K_2(t, s,)\| \leq \eta_2$$

for all $t \in [-T, T], u, v, w \in X$*. Then:*

(a)For all x,y solutions of (1) with $x(0) \le y(0)$ *we have* $x(t) \le y(t)$*, for all* $t \in [-T, T]$.

(b) $H(S_1, S_2) \le \dfrac{\eta_1 + 2\eta_2 T}{(1 - l_1 l - l_2 - \frac{l_3}{\tau})}$*, where* S_1, S_2 *is the solutions set of(3),(4).*

Proof We denote with A_λ the restriction of the operator A at X_λ.First we show that $A_\lambda$is a contraction map onX_λ.From$c_1 - c_5$ we have that

$$\|A_\lambda x(t) - A_\lambda y(t)\| \le (l_1 l + l_2)\|x(t) - y(t)\| + \int_{-\theta t}^{\theta t} \|K(t, s, x(s)) - K(t, s, y(s))\| \, dsleq$$

$$\le (l_1 l + l_2)\|x - y\|_\tau e^{\tau(t+T)} + l_3 \|x - y\|_\tau \int_{-\theta t}^{\theta t} e^{\tau(t+T)} ds.$$

So A is c-WPO with

$$c = \frac{1}{1 - l_1 l - l_2 - \dfrac{l_3}{\tau}}.$$

Using the theorem 1.2 we obtain (b).

For proof of (a) let be x,y solutions for(1) with $x(0) \le y(0)$.Then $x \in X_{x(0)}, y \in X_{y(0)}$.We define

$$\widetilde{x}(t) = x(0), t \in [0, b]$$

$$\widetilde{y}(t) = y(0), t \in [0, b]$$

We have

$$\widetilde{x}(0) \in X_{x(0)}, \widetilde{y}(0) \in X_{y(0)}, \widetilde{x}(0) \le \widetilde{y}(0).$$

From lemma 1.1 we obtain that the operator A^∞ is increasing.It follows that

$$A^\infty(\widetilde{x}(0)) \le A^\infty(\widetilde{x}(0))$$

i.e $x \le y$

Next we define φ -contraction notion and use this for estimate distance between two weakly Picard operators.

Let $\varphi : R_+ \longrightarrow R_+$.

Definition 2.1. [5] *φ is a strict comparison function if φ satisfies the following:*

i) φ is continuous.

ii)φ is monotone increasing.

iii) $\varphi^n(t) \longrightarrow 0$, for all $t > 0$.

iv) t-$\varphi(t) \longrightarrow \infty$,for $t \longrightarrow \infty$.

Let (X, d) be a metric space and $f : X \longrightarrow X$ an operator.

Definition 2.2. [5] *The operator f is called a strict φ-contraction if:*

(i) φ is a strict comparison function.

(ii)$d(f(x), f(y)) \leq \varphi(d(x, y))$, for all $x, y \in X$.

Theorem 2.2. [5] *Let (X, d) be a complete metric space,$\varphi : R_+ \longrightarrow R_+$ a strict comparison and $f, g : X \longrightarrow X$ two orbitally continuous operators.We suppose that:*

(i) $d(f(x), f^2(x)) \leq \varphi(d(x, f(x)))$ for any $x \in X$ and $d(g(x), g^2(x)) \leq \varphi(d(x, g(x)))$ for any $x \in X$.

(ii)there exists $\eta > 0$ such that $d(f(x), g(x)) \leq \eta$, for any $x \in X$

Then:

(a) f,g are weakly Picard operators.

(b)$H(F_f, F_g) \leq \tau_\eta$where $\tau_\eta = sup\{t \mid t - \varphi(t) \leq \eta\}$.

Theorem 2.3. *We suppose that condition (c_5) is verified and the following conditions are satisfied:*

(H_1) there exists φ a strict comparison function such that

$$(i)\|hx(t) - hy(t)\| \leq \|x(t) - y(t)\|,$$

for all $x, y \in C([-T, T], X), t \in [-T, T]$.

$$(ii)g(t, u_1, v_1, w) - g(t, u_2, v_2, w)\| \leq a\varphi(\|u_1 - u_2\|) + b\varphi(\|v_1 - v_2\|).$$

for all $t \in [-T, T], u_i, v_i, w \in X, i = 1, 2$

$$(iii)\ \|K(t, s, u) - K(t, s, u_1)\| \leq l(t, s)\varphi(\|u - u_1\|),$$

for all $t, s \in [-T, T], u, u_1, \in X$,*where* $l(t, \cdot) \in L^1[-T, T]$.

(H_2) *There exists* $\eta_1, \eta_2 > 0$ *such that*

$$\|g_1(t, u, v, w) - g_2(t, u, v, w)\| \le \eta_1,$$

$$\|K_1(t, s, u) - K_2(t, s,)\| \le \eta_2$$

for all $t \in [-T, T], u, v, w \in X$.

(H_3)

$$a + b + \max_{t \in [-T,T]} \int_{-T}^{T} l(t, s) ds \le 1$$

Then:

(i)the equation (1) has at least solution.

(ii)$H(S_1, S_2) \le \tau_\eta$ *where* $\eta = \eta_1 + 2T\eta_2, S_1, S_2$ *is the solutions set of(3),(4).*

ProofLet be$A_1, A_2 : C([-T, T], X) \longrightarrow C([-T, T], X)$,

$$A_1 x(t) = g_1(t, hx(t), x(t), x(0)) + \int_{-\theta t}^{\theta t} K_1(t, s, x(s)) ds$$

$$A_2 x(t) = g_2(t, hx(t), x(t), x(0)) + \int_{-\theta t}^{\theta t} K_2(t, s, x(s)) ds.$$

From

$$\left\|A_i x(t) - A_i^2 x(t)\right\| \le \|g_i(t, hx(t), x(t), x(0)) - g_i(t, hA_i x(t), A_i x(t), A_i x(0))\| +$$

$$+ \int_{-\theta t}^{\theta t} \|K_i(t, s, x(s)) - K_i(t, s, A_i x(s))\| \, ds$$

$$\le a\varphi(\|hx(t) - hA_i x(t)\|) + b\varphi(\|x(t) - A_i x(t)\|) +$$

$$+ \int_{-\theta t}^{\theta t} l(t, s)\varphi(\|x(s) - A_i x(s))\| \, ds \le a\varphi(\|x(t) - A_i x(t)\|) + b\varphi(\|x(t) - A_i x(t)\|) +$$

$$\int_{-\theta t}^{\theta t} l(t, s)\varphi(\|x(s) - A_i x(s))\| \, ds \le (a + b + \max_{t \in [-T,T]} \int_{-T}^{T} l(t, s) ds)\varphi(\|x - A_i x\|_C) \le$$

$$\leq \varphi(\|x - A_i x\|_C)$$

we have that

$$\left\|A_i x - A_i^2 x\right\|_C \leq \varphi(\|x - A_i x\|_C), i = \overline{1, 2}.$$

Here $\|\cdot\|_C$is the Chebyshev norm on $C([-T, T], X)$.

We note that $\|A_1 x - A_2 x\|_C \leq \eta_1 + 2T\eta_2$. From this, using the theorem 2.2 we have the conclusions.

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COST OF TRACKING FOR DIFFERENTIAL STOCHASTIC EQUATIONS IN HILBERT SPACES

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Abstract. We consider the tracking problem for differential stochastic equations diffusions dependent on both state and control variables. The Riccati equation associated with this problem is in general different from the conventional Riccati equation. We establish that under stabilizability and uniform observability conditions this equation has a unique positive and bounded on $\mathbf{R}_+$ solution. Using this result we find the optimal control (and the optimal cost) for tracking problem (see also [11]).

Notations and statement of the problem

Let H, U, V be separable real Hilbert spaces. Let $J \subset \mathbf{R}_+ = [0, \infty)$ be an interval. If E is a Banach space we denote by $C(J, E)$ the space of all mappings $G(t) : J \rightarrow E$ that are continuous. We also denote by $C_s(J, L(H))$ the space of all strongly continuous mappings $G(t) : J \rightarrow L(H)$ and by $C_b(J, L(H))$ the subspace of $C_s(J, L(H))$, which consist of all mappings $G(t)$ such that $\sup_{t \in J} \|G(t)\| < \infty$. Given a signal $r \in C_b(\mathbf{R}_+, H)$ we want to minimize the cost

$$J(s,u) = \overline{\lim_{t\to\infty}} \frac{1}{t-s} E \int_s^t \|C(\sigma)\,(x(\sigma) - r(\sigma))\|^2 + \langle K(\sigma)u(\sigma), u(\sigma)\rangle \, d\sigma$$

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in a suitable class of control u subject to the equation (denoted $\{A : B; G_i : H_i\}$)

$$dx(t) = A(t)x(t)dt + B(t)u(t)dt + \sum_{i=1}^{m} (G_i(t)x(t) + H_i(t)u(t))\, dw_i(t) \qquad (1)$$

$$x(s) = x \in H.$$

We assume that the coefficients satisfy the following hypothesis:

$\mathbf{P}_1$: $A, G_i \in C_b(R_+, L(H)), i = 1, 2, ...m, m \in N^*$, $B, H_i \in C_b(R_+, L(U, H))$, $B^*, H_i^* \in C_b(R_+, L(H, U))$, $C \in C_b(R_+, L(H, V)), C^*C, G_i, G_i^* \in C_b(R_+, L(H))$, $K \in C_b(R_+, L(U))$ and there exist $\delta_0 > 0$ such that for all $t \in R_+, K(t) \geq \delta_0 I$. If $Z \in C_b(R_+, L(H, V))$, we will denote $\widetilde{Z} = \sup_{0 \leq r < \infty} \|Z(r)\| < \infty$.

1. **Stabilizability, detectability and uniform observability**

It is known (see Proposition 5 in [13] and Definition 5.3 in [4]) that if $A \in C_s(\mathbf{R}_+, L(H))$ then the family $A(t), t \geq 0$ generates an evolution operator $U(t, s)$ which has the following properties: 1. $(t, s) \to U(t, s)$ *is continuous in the uniform operator topology on* $\{(t, s)/0 \leq s \leq t \leq T\}$*; 2.* $\frac{\partial U(t,s)x}{\partial t} = L(t)U(t, s)x$ *and* $\frac{\partial U(t,s)x}{\partial s} = -U(t, s)L(s)x$ *for all* $x \in H$ *and* $0 \leq s \leq t \leq T$.

In the sequel we will assume that P_1 holds if we don't specify other conditions. Let $(\Omega, F, F_t, t \in [0, \infty), P)$ be a stochastic basis. We consider the equation

$$dy(t) = A(t)y(t)dt + \sum_{i=1}^{m} G_i(t)y(t)dw_i(t),\ y(s) = x \in H, \qquad (2)$$

denoted by $\{A; G_i\}$, where w_i's are independent real Wiener processes relative to F_t. It is known (see [5] and the notations therein) that (2) has a unique mild solution in $C([s, T]; L^2(\Omega; H))$ that is adapted to F_t; namely the solution of

$$y(t) = U(t, s)x + \sum_{i=1}^{m} \int_s^t U(t, r)G_i(r)y(r)dw_i(r). \qquad (3)$$

This solution is also a strong solution, that is $y(t)$ satisfies the integral equation

$$y(t) = x + \int_s^t A(r)y(r)dr + \sum_{i=1}^{m} \int_s^t G_i(r)y(r)dw_i(r).$$

Definition 1. *Let* $y(t,s;x)$ *be the mild solution of* $\{A;G_i\}$. *We say that (2) is uniformly exponentially stable if there exist constants* $M \geq 1, \omega > 0$ *such that* $E\|y(t,s;x)\|^2 \leq Me^{-\omega(t-s)}\|x\|^2$ *for all* $t \geq s \geq 0$ *and* $x \in H$.

If $C \in C_s([0,\infty), L(H,V))$ we consider the system formed by the equation (2) and the observation relation $z(t) = C(t)y(t,s,x)$ denoted by $\{A, G_i; C\}$.

Definition 2. *(see* [12]*) We say that the system* $\{A, C; G_i\}$ *is uniformly observable if there exist* $\tau > 0$ *and* $\gamma > 0$ *such that* $E\int_s^{s+\tau}\|C(t)y(t,s;x)\|^2 dt \geq \gamma\|x\|^2$ *for all* $s \in R_+$ *and* $x \in H$.

Definition 3. *(see* [5]*) We say that the system* $\{A, C; G_i\}$ *is detectable if there exists* $L \in C_b([0,\infty), L(V,H))$ *such that* $\{A + LC; G_i\}$ *is uniformly exponentially stable.*

Definition 4. *We say that* $\{A : B; G_i : H_i\}$ *is stabilizable if there exists* $F \in C_b([0,\infty), L(H,U))$ *such that* $\{A + BF; G_i + H_iF\}$ *is uniformly exponentially stable.*

In the deterministic case it is known (see [7] for the autonomous case) that uniform observability implies detectability. We proved in [12] that this assertion is not true in the stochastic case.

2. Bounded solutions of Riccati equation of stochastic control

Let us consider the linear and bounded operator
$\mathcal{B}$: $C_s(\mathbf{R}_+, L(H)) \to C_s(\mathbf{R}_+, L(H,U)), \mathcal{B}(P)(s) = B^*(s)P(s) + \sum_{i=1}^{m} H_i^*(s)P(s)G_i(s)$ and the function $\mathcal{K} : C_s(\mathbf{R}_+, L(H)) \to C_s(\mathbf{R}_+, L(U))$, $\mathcal{K}(P)(s) = K(s) + \sum_{i=1}^{m} H_i^*(s) P(s)H_i(s)$. Since K is uniformly positive, then it is easy to see that $\mathcal{K}(P)$ is uniformly positive. We consider the following Riccati equation in $C_s([0,\infty), L^+(H))$

$$P' + A^*P + PA + \sum_{i=1}^{m} G_i^*PG_i + C^*C - [\mathcal{B}(P)]^*[\mathcal{K}(P)]^{-1}\mathcal{B}(P) = 0, \qquad (4)$$

where the weak differentiability is considered. If $P \in C_s([0,\infty), L^+(H))$ we put

$$S(s) = -[\mathcal{K}(P)(s)]^{-1}\mathcal{B}(P)(s), s \geq 0. \qquad (5)$$

and we denote $\widehat{A} = A - BS$, $\widehat{G}_i = G_i - H_i S$. Then (4) can be written as it follows

$$P' + \widehat{A}^* P + P\widehat{A} + \sum_{i=1}^{m} \widehat{G}_i^* P \widehat{G}_i + C^*C + S^*KS = 0. \tag{6}$$

Arguing as in the proof of Proposition 4.64 [3] and using Dini's theorem we can prove the following lemma.

Lemma 1. *If $(P_n)_{n\in N^*}$ is an increasing sequence in $C_s([0,T], L^+(H))$ such as $P_n(t) \leq I$, for all $t \in [0,T]$ (I is the identity operator on H), then there exists $P \in C_s([0,T], L^+(H))$ such as $P_n(t)x \underset{n\to\infty}{\to} P(t)x, x \in H$, uniformly for $t \in [0,T]$.*

Theorem 1. *The Riccati equation (4) with the final condition $P(T) = R \in L^+(H), T \in \mathbf{R}_+^*$ has a unique solution in $C_s([0,T], L^+(H))$ denoted $P(T,s;R)$, which also belongs to $C([0,T], L^+(H))$ and has the following properties:*

a) It is the unique solution of the integral equation

$$P(s)x = U^*(T,s)P(T)U(T,s)x + \int_s^T U^*(r,s)[\sum_{i=1}^{m} G_i^*(r)P(r)G_i(r) \tag{7}$$

$$+C^*(r)C(r) - [\mathcal{B}(P)(r)]^* [\mathcal{K}(P)(r)]^{-1} \mathcal{B}(P)(r)]U(r,s)xdr.$$

b) It is monotone in the sense that $P(T,s;R_1) \leq P(T,s;R_2)$, if $R_1 \leq R_2$.

Proof. The existence of the solution. The proof is similar to that given in [1] for the finite dimensional case. We consider the following iterative scheme to construct the solution of (6). Let $P_0 = I$ (I is the identity operator on H), $S_0 = -[\mathcal{K}(P_0)]^{-1} \mathcal{B}(P_0)$, $\widehat{A}_0 = A - BS_0$, $\widehat{G}_{0,i} = G_i - H_i S_0$, $i = 1,..,m$. Using Lemma 1 in [7] we deduce that the following differential equation

$$P_{n+1}' + \widehat{A}_n^* P_{n+1} + P_{n+1}\widehat{A}_n + \sum_{i=1}^{m} \widehat{G}_{n,i}^* P_{n+1} \widehat{G}_{n,i} + C^*C + S_n^* K S_n = 0, \tag{8}$$

$$P_{n+1}(T) = R,$$

where $S_n = -[\mathcal{K}(P_n)]^{-1} \mathcal{B}(P_n)$, $\widehat{A}_n = A - BS_n$, $\widehat{G}_{n,i} = G_i - H_i S_n, i = 1,..,m$, $n = 0,1,2,...$ has a unique solution which belongs to $C_s([0,T], L^+(H))$. As in [1] we can establish that $\{P_n(.)\}$ is a decreasing sequence. Using the above lemma for the increasing sequence $\{I - P_n(.)\}$, it follows that there exists $P \in C_s([0,T], L^+(H))$

such that, for all $x \in H$, $P_n(t)x \underset{n\to\infty}{\to} P(t)x$, uniformly for $t \in [0,T]$. As $n \to \infty$ in (8) we deduce that P is weakly differentiable and satisfies (6). Thus (4) with the final condition $P(T) = R \in L^+(H)$, has a solution in $C_s(\mathbf{R}_+, L^+(H))$. Differentiating the function $f_x : [0,T] \to \mathbf{R}$ $f_x(\sigma) = \langle P(\sigma)U(\sigma,s)x, U(\sigma,s)x\rangle$ we get

$$\frac{\partial f_x(\sigma)}{\partial \sigma} = \langle P'(\sigma)U(\sigma,s)x, U(\sigma,s)x\rangle + 2\langle P(\sigma)A(\sigma)U(\sigma,s)x, U(\sigma,s)x\rangle .$$

Now, we integrate from s to T, $s \in [0,T]$ the above relation and we obtain (7). Using the Gronwall's lemma we deduce that (7) has a unique solution in $C_s(\mathbf{R}_+, L^+(H))$, and consequently (4) has a unique solution in $C_s(\mathbf{R}_+, L^+(H))$. It is not difficult to see that a solution of (7) belongs to $C(\mathbf{R}_+, L^+(H))$. Thus (4) has a unique solution in $C(\mathbf{R}_+, L^+(H))$ and a) holds.

Now we prove b).Let $R, R_1 \in L^+(H), R_1 \leq R$ and let $P(s) = P(T,s;R)$, $P_1(s) = P(T,s;R_1)$ be the corresponding solutions of (4). We use the notations $\Delta = P - P_1$, $S_1 = -[\mathcal{K}(P_1)]^{-1}\mathcal{B}(P_1)$, $\widehat{A}_1 = A - BS_1$, $\widehat{G}_{1,i} = G_i - H_iS_1$. Then, Δ is the solution of the following Lyapunov equation with the final condition $\Delta(T) = R - R_1$

$$\Delta' + \widehat{A}_1^*\Delta + \Delta\widehat{A}_1 + \sum_{i=1}^{m} \widehat{G}_{1,i}^*\Delta\widehat{G}_{1,i} + (S_1 - S)^*\mathcal{K}(P)(S_1 - S) = 0. \tag{9}$$

Thus it follows that $\Delta \geq 0$ and $P - P_1 \geq 0$ and we obtain the conclusion. □

Remark 1. *The function $F : [0,T] \to R, F(t,x) = \langle P(t)x, x\rangle$, where $P(t) = P(T,t;R)$ and the strong solution of (2) satisfy the conditions required by Ito's formula in infinite dimensions (see T. 3.8 in* [2]*).*

Moreover, if $P \in C_s(\mathbf{R}_+, L^+(H))$ is a solution of (4) and $\sup\limits_{s\in\mathbf{R}_+} \|P(s)\| < \infty$, then P is said to be a *bounded solution.* Assume that (4) has a bounded solution $P(s)$ and let $S(s)$ be given by (5). It is not difficult to see that $S, S^* \in C_b([0,\infty), L(H,U))$.

Definition 5. *A bounded solution of (4) is called stabilizing for $\{A; G_i\}$ if $\{A + BS; G_i + H_iS\}$ is uniformly exponentially stable, where $S(t)$ is given by (5).*

Proposition 1. *(see* [5]*)The Riccati equation (4) has at most a bounded solution, which is stabilizing for $\{A; G_i\}$.*

Proof. If P and P_1 are two bounded solutions of (4) and P_1 is stabilizing for $\{A; G_i\}$ then $\Delta = P - P_1$ is a solution of (9). As in the proof of the above theorem, we get

$$\Delta(s)x = U^*_{\widehat{A}_1}(T,s)\Delta(T)U_{\widehat{A}_1}(T,s)x + \int_s^T U^*_{\widehat{A}_1}(r,s)[\sum_{i=1}^m G_i^*(r)\Delta(r)G_i(r)+$$

$$[(S_1 - S)^* \mathcal{K}(P)(S_1 - S)](r)U_{\widehat{A}_1}(r,s)xdr,$$

where $U_{\widehat{A}_1}(t,s)$ is the evolution operator generated by $\widehat{A}_1$. From the uniform exponential stability of $\{A+BS; G_i+H_iS\}$ it follows that $U_{\widehat{A}_1}(t,s)$ is uniformly exponentially stable. Since it exists $m_1 \in \mathbf{R}_+$ such that $\|(S_1 - S)(r)\| < m_1 \|\Delta(r)\|$, we use Gronwall's inequality to deduce that there exists $M, a > 0$, such that $\|\Delta(s)\| \leq Me^{-a(T-s)}$. As $T \to \infty$ we obtain $\|\Delta(s)\| = 0$, for all $s \in [0, \infty)$. The conclusion follows. □

Reasoning as in [5], see Theorem 3.1 and stochasticize the proof we obtain the following result.

Proposition 2. *If $\{A; G_i\}$ is stabilizable then there exists a nonnegative boun- ded solution of the Riccati equation (4).*

Arguing as in [12] we can prove the following result:

Theorem 2. *Assume that $\{A, G_i; C\}$ is uniformly observable. If $P(t)$ is a nonnegative bounded solution of (4) then*

a) there exists $\delta > 0$ such that $P(t) \geq \delta I$ for all $t \in \mathbf{R}_+$ (P is uniformly positive on $\mathbf{R}_+$);

b) P is a stabilizing solution (for $\{A; G_i\}$).

The next theorem is a consequence of the above theorem and of Proposition 2.

Theorem 3. *Assume $\{A, G_i; B\}$ is stabilizable and $\{A, G_i; C\}$ is uniformly observable. Then the Riccati equation (4) has a unique nonnegative bounded on $\mathbf{R}_+$ solution $P(t)$, which is a stabilizing solution and there exists $\delta > 0$ such that $P(t) \geq \delta I$ for all $t \in [0, \infty)$.*

Proposition 3. *Assume that the hypotheses of the above theorem hod. If P is the unique and bounded on $\mathbf{R}_+$ solution of the Riccati equation (4) and S is the operator* *given by (5), then the equation*

$$g'(t) = -\left(A^* + S^*B^*\right)g(t) + C^*(t)C(t)r(t) \tag{10}$$

has a unique solution in $C_b([0,\infty), H)$*, where we consider the weak differentiability. Moreover, the function* $(t,x) \to \langle g'(t), x\rangle$ *is continuous on* $[0,\infty) \times H$.

Proof. Since $A + BS$ is the generator of an evolution operator $U_{A,B}(t,s)$, it is not difficult to see that the integral $g(s) = \int_s^\infty U^*_{A,B}(\sigma,s)C^*(\sigma)C(\sigma)r(\sigma)d\sigma$ is convergent in H and $g(s)$ is bounded on $\mathbf{R}_+$. Differentiating the function $t \to \langle g(t), y\rangle, y \in H$, we see that $\frac{\partial}{\partial t}\langle g(t), y\rangle = \langle -\left(A^* + S^*B^*\right)g(t) + C^*(t)C(t)r(t), y\rangle$ and $g(t)$ is a solution of (10). If h is an other bounded solution then $(h-g)'(t) = -\left(A^* + S^*B^*\right)(h-g)(t)$. The unique solution of the last equation with the final condition $(h-g)(t) = h(t) - g(t)$ is $(h-g)(s) = U^*_{A,B}(t,s)\left[h(t) - g(t)\right]$. As $t \to \infty$ and since $U_{A,B}(t,s)$ is exponentially stable and the functions g and h are bounded on $\mathbf{R}_+$, we deduce that $(h-g)(s) = 0$, for all $s \geq 0$. Thus $h \equiv g$, and (10) has a unique solution. The last statement follows from the hypothesis, if we see that $\langle g'(t), x\rangle = -\langle g(t), (A(t) + B(t)S(t))\,x\rangle + \langle C^*(t)C(t)r(t), x\rangle$. □

We take the set of admissible controls $U_{ad} = \{u$ is an U- valued random variable, F_s- measurable such as $\overline{\lim_{t\to\infty}} \frac{1}{t-s} E \int_s^t \|u(\sigma)\|^2 d\sigma < \infty$ and $\sup_{t\geq s} E\|x(t)\|^2 < \infty$, where x is the solution of (1)$\}$.

Theorem 4. *Assume that the hypotheses of the Theorem 3 hold. If P is the unique and bounded on $\mathbf{R}_+$ solution of the Riccati equation and $g(t)$ is the unique solution of (10) then the optimal control is*

$$u(t) = -\left[\mathcal{K}(P)(\sigma)\right]^{-1}\left[\mathcal{B}(P)(\sigma)x(\sigma) + B^*(\sigma)g(\sigma)\right]$$

and the optimal cost is

$$J(s) = \inf_{u\in U_{ad}} J(s,u) =$$

$$\overline{\lim_{t\to\infty}} \frac{1}{t-s}\left[\int_s^t \|C(\sigma)r(\sigma)\|^2 d\sigma - \int_s^t ||\left[\mathcal{K}(P)(\sigma)\right]^{-1/2} B^*(\sigma)g(\sigma)||^2 d\sigma\right].$$

Proof. We consider the function $F(t,x) = \langle P(t)x, x\rangle + 2\langle g(t), x\rangle$, which is continuous together its partial derivatives F_t, F_x, F_{xx} on $[0,\infty) \times H$, according the Remark 1 and the above proposition. Let $u \in U_{ad}$ and x be its response. Using Ito's formula for $F(t,x)$ and the strong solution of (1) we get

$$E\langle P(t)x(t), x(t)\rangle + 2\langle g(t), x(t)\rangle - E\langle P(s)x, x\rangle - 2\langle g(s), x\rangle =$$
$$-\int_s^t \|C(\sigma)[x(\sigma) - r(\sigma)]\|^2 + \langle K(\sigma)u(\sigma), u(\sigma)\rangle d\sigma +$$
$$\int_s^t \left\| \mathcal{K}(P)(\sigma)^{1/2}\left[u(\sigma) + [\mathcal{K}(P)(\sigma)]^{-1}[\mathcal{B}(P)(\sigma)x(\sigma) + B^*(\sigma)g(\sigma)]\right]\right\|^2$$
$$+\int_s^t \|C(\sigma)r(\sigma)\|^2 d\sigma - \int_s^t ||[\mathcal{K}(P)(\sigma)]^{-1/2} B^*(\sigma)g(\sigma)||^2 d\sigma.$$

Since $P(t)$ and $g(t)$ are bounded on $\mathbf{R}_+$ we multiply the last relation with $\frac{1}{t-s}$ and passing to the limit as $t \to \infty$ and, then, to the infimum we get the conclusion. □

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THE EXPONENTIAL MAP ON THE SECOND ORDER TANGENT BUNDLE

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Abstract. On the 2-tangent (or 2-jet) bundle T^2M of a Riemannian manifold endowed with geometrical objects as in [1] and [2], the first variation of the energy $E = \int_0^1 \langle \dot{c}, \dot{c} \rangle$ is computed and the conditions such that its extremal curves should be invariant to the group of homotheties are determined. In these conditions, by using homotheties, we define the exponential map on T^2M.

1. **Introduction**

The geometry of the second order tangent bundle T^2M (called as well "2-osculator bundle" and denoted by Osc^2M), constructed by R. Miron and Gh. Atanasiu, ([12]-[17]) represents the geometry of the jet-space J_0^2M, endowed with characteristic geometrical objects as: 2-tangent structure, nonlinear connections and N-linear connections. This construction allows the prolongation to T^2M of Riemannian and Finslerian structures of M. Within this geometrical framework, V. Balan and P.Stavrinos ([3], [4], [18]), defined geodesics of T^2M as stationary curves of the distance Lagrangian $L(c) = \sqrt{\langle \dot{c}, \dot{c} \rangle}$ and deduced their equations. In these papers, the authors use linear connections D with the property that the 2-tangent structure J is absolutely parallel with respect to D.

A notion which plays a major role in our considerations is that of homogeneity of a function given on T^2M (respectively, of a vector field on T^2M), defined and studied by M. de Leon and E. Vasquez, [5], R. Miron, [7], Gh. Atanasiu, [2].

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In this paper, we define geodesics as extremal curves of the energy Lagrangian $E = \langle \dot{c}, \dot{c} \rangle$ (not of the distance Lagrangian, as in [3], [4], [18]), deduce their equations (Theorem 5), study the conditions that an exponential map could be defined on T^2M (Theorem 7) and construct this application. It is worth mentioning the following facts:

1. for Lagrangians defined on T^2M, the integral of action $I(c)$ essentially depends on the parametrization of the curve c; this is why the classical technique of defining the exponential map (which relies on re-parametrizations) is here replaced by a technique which uses the group of homotheties;

2. throughout the paper, by N- linear connection we shall mean (as in Gh. Atanasiu, [1]) a linear connection which preserves by parallelism the distributions generated by a nonlinear connection N, but is not necessarily compatible with J.

2. **The 2-tangent bundle** T^2M

Let M be a real differentiable manifold of dimension n and class $\mathcal{C}^\infty$; the coordinates of a point $x \in M$ in a local chart (U, ϕ) will be denoted by $\phi(x) = \left(x^i\right)$, $i = 1, ..., n$. Let (Osc^2M, π^2, M) be its *2-osculator bundle* ([12]-[17]), which will be called in the following, *2-tangent bundle* and denoted by (T^2M, π^2, M), ([1], [2]. A point of T^2M will have in a local chart the coordinates $(x^i, y^{(1)i}, y^{(2)i})$.

Let N be a nonlinear connection on T^2M, given by its coefficients $(\underset{(1)}{N}{}^i_j, \underset{(2)}{N}{}^i_j)$, [1], [7], [8]. Then, the adapted basis to N is

$$\mathcal{B} = \{\delta_i := \frac{\delta}{\delta x^i} = \frac{\delta}{\delta y^{(0)i}}, \delta_{1i} := \frac{\delta}{\delta y^{(1)i}}, \delta_{2i} := \frac{\delta}{\delta y^{(2)i}}\},$$

where

$$\begin{cases} \dfrac{\delta}{\delta x^i} = \dfrac{\partial}{\partial x^i} - \underset{(1)}{N}{}^j_i \dfrac{\partial}{\partial y^{(1)j}} - \underset{(2)}{N}{}^j_i \dfrac{\partial}{\partial y^{(2)j}} \\ \dfrac{\delta}{\delta y^{(1)i}} = \dfrac{\partial}{\partial y^{(1)i}} - \underset{(1)}{N}{}^j_i \dfrac{\partial}{\partial y^{(2)j}} \\ \dfrac{\delta}{\delta y^{(2)i}} = \dfrac{\partial}{\partial y^{(2)i}}. \end{cases} \tag{1}$$

The dual basis of $\mathcal{B}$ is $\mathcal{B}^* = \left\{dx^i, \delta y^{(1)i}, \delta y^{(2)i}\right\}$, given by

$$\delta y^{(0)i} = dx^i,\ \delta y^{(1)i} = dy^{(1)i} + \underset{(1)}{M}{}^i_j dx^j,\ \delta y^{(2)i} = dy^{(2)i} + \underset{(1)}{M}{}^i_j dy^{(1)j} + \underset{(2)}{M}{}^i_j dx^j. \tag{2}$$

The bases above correspond to the direct-sum decomposition

$$\begin{aligned} T_u\left(T^2M\right) &= N_u \oplus N_{1u} \oplus V_{2u}, \\ T_u^*\left(T^2M\right) &= N_u^* \oplus N_{1u}^* \oplus V_{2u}^*, \forall u \in T^2M. \end{aligned} \tag{3}$$

Then, a vector field $X \in \mathcal{X}\left(T^2M\right)$ is represented in the local adapted basis as

$$X = X^{(0)i}\delta_i + X^{(1)i}\delta_{1i} + X^{(2)i}\delta_{2i}, \tag{4}$$

with the three right terms (called *d-vector fields*) belonging to the distributions N, N_1 and V_2 respectively.

A 1-form $\omega \in \mathcal{X}^*\left(T^2M\right)$ will be decomposed as

$$\omega = \omega_i^{(0)}dx^i + \omega_i^{(1)}\delta y^{(1)i} + \omega_i^{(2)}\delta y^{(2)i}. \tag{5}$$

Similarly, a tensor field $T \in \mathcal{T}_s^r\left(T^2M\right)$ can be split with respect to (3) into components , named *d-tensor fields.*

The $\mathcal{F}\left(T^2M\right)$-linear mapping $J : \mathcal{X}\left(T^2M\right) \to \mathcal{X}\left(T^2M\right)$ given by

$$J\left(\delta_i\right) = \delta_{1i}, J\left(\delta_{1i}\right) = \delta_{2i}, J\left(\delta_{2i}\right) = 0 \tag{6}$$

is called the *2-tangent structure on* T^2M,[7], [8].

Let

$$H = \left\{h_t \mid h_t : \mathbb{R} \to \mathbb{R},\ t \in \mathbb{R}_+^*\right\}$$

be the *group of homotheties*, ([1], [5], [7]), of the real numbers set. Then, H acts on T^2M as a one-parameter group of transformations, as follows:

$$\begin{aligned} \left(h_t, u\right) \mapsto h_t\left(u\right) : H \times T^2M &\to T^2M, \text{ where} \\ h_t\left(x, y^{(1)}, y^{(2)}\right) &= \left(x, ty^{(1)}, t^2y^{(2)}\right). \end{aligned} \tag{7}$$

A function $f : T^2M \to \mathbb{R}$, which is differentiable on $\widetilde{T^2M}$ and continuous on the null-section $0 : M \to T^2M$ is called *homogeneous of degree* r ($r \in \mathbb{Z}$) (or, shortly, r-homogeneous) on the fibres of T^2M, if

$$f \circ h_t = t^r f,\ \ \forall t \in \mathbb{R}_+^*, \tag{8}$$

A vector field $X \in \mathcal{X}\left(T^2M\right)$, $X = X^{(0)i}\dfrac{\partial}{\partial x^{\iota}} + X^{(1)i}\dfrac{\partial}{\partial y^{(1)\iota}} + X^{(2)i}\dfrac{\partial}{\partial y^{(2)\iota}}$, is *r-homogeneous* , [1], if and only if $X^{(0)i}$ are $(r-1)$-homogeneous, $X^{(1)i}$ are r-homogeneous and $X^{(2)i}$ are $(r+1)$-homogeneous functions.

3. *N*- linear connections

An *N-linear connection D*, [1], is a linear connection on T^2M, which preserves by parallelism the distributions N, N_1 and V_2. An N- linear connection, in the sense of the definition above, is not necessarily compatible to the 2-tangent structure J (an N-linear connection which is also compatible to J is called, [1], a *JN-linear connection*).

An N- linear connection is locally given by its coefficients

$$D\Gamma\left(N\right) = \left(\underset{(00)}{L}{}^{i}{}_{jk}, \underset{(10)}{L}{}^{i}{}_{jk}, \underset{(20)}{L}{}^{i}{}_{jk}, \underset{(01)}{C}{}^{i}{}_{jk}, \underset{(11)}{C}{}^{i}{}_{jk}, \underset{(21)}{C}{}^{i}{}_{jk}, \underset{(02)}{C}{}^{i}{}_{jk}, \underset{(12)}{C}{}^{i}{}_{jk}, \underset{(22)}{C}{}^{i}{}_{jk}\right), \quad (9)$$

where

$$\begin{cases} D_{\delta_k}\delta_j = \underset{(00)}{L}{}^{i}{}_{jk}\delta_i,\ D_{\delta_k}\delta_{1j} = \underset{(10)}{L}{}^{i}{}_{jk}\delta_{1i},\ D_{\delta_k}\delta_{2j} = \underset{(20)}{L}{}^{i}{}_{jk}\delta_{2i} \\ D_{\delta_{1k}}\delta_j = \underset{(01)}{C}{}^{i}{}_{jk}\delta_i,\ D_{\delta_{1k}}\delta_{1j} = \underset{(11)}{C}{}^{i}{}_{jk}\delta_{1i},\ D_{\delta_{1k}}\delta_{2j} = \underset{(21)}{C}{}^{i}{}_{jk}\delta_{2i} \\ D_{\delta_{2k}}\delta_j = \underset{(02)}{C}{}^{i}{}_{jk}\delta_i,\ D_{\delta_{2k}}\delta_{1j} = \underset{(12)}{C}{}^{i}{}_{jk}\delta_{1i},\ D_{\delta_{2k}}\delta_{2j} = \underset{(22)}{C}{}^{i}{}_{jk}\delta_{2i} \end{cases}. \quad (10)$$

In the particular case when D is J-compatible, we have

$$\begin{aligned} \underset{(00)}{L}{}^{i}{}_{jk} &= \underset{(10)}{L}{}^{i}{}_{jk} = \underset{(20)}{L}{}^{i}{}_{jk} =: L^{i}{}_{jk}, \\ \underset{(01)}{C}{}^{i}{}_{jk} &= \underset{(11)}{C}{}^{i}{}_{jk} = \underset{(21)}{C}{}^{i}{}_{jk} = \underset{(1)}{C}{}^{i}{}_{jk}, \\ \underset{(02)}{C}{}^{i}{}_{jk} &= \underset{(12)}{C}{}^{i}{}_{jk} = \underset{(22)}{C}{}^{i}{}_{jk} = \underset{(2)}{C}{}^{i}{}_{jk}. \end{aligned}$$

The torsion tensor of an N- linear connection D, $T\left(X,Y\right) = D_XY - D_YX - [X,Y]$, is well determined by the following components, which are d-tensors of $(1,2)$-type ([1], [7], [8]:

$$v_\gamma T(\delta_{\beta k}, \delta_{\alpha j}) =: \underset{(\alpha\beta)}{\overset{(\gamma)}{T}}{}^{i}{}_{jk}\delta_{\gamma i},\ \ \alpha,\beta,\gamma = 1,2;$$

the detailed expressions of $\underset{(\alpha\beta)}{\overset{(\gamma)}{T}}{}^{i}{}_{jk}$ can be found in [1].

The curvature of the N- linear connection D, namely, $R(X,Y)Z = D_X D_Y Z - D_Y D_X Z - D_{[X,Y]}Z$, is completely determined by its components (which are d-tensors):

$$R(\delta_{\gamma l}, \delta_{\beta k})\,\delta_{\alpha j} = \underset{(\alpha\beta\gamma)}{R}{}_{j\,kl}^{\ i}\,\delta_{\alpha i},\ \ \alpha, \beta, \gamma = 0, 1, 2.$$

4. **Metric structures and geodesics on T^2M**

A *Riemannian metric* on T^2M is a tensor field G of type $(0, 2)$, which is non-degenerate at each point $p \in T^2M$ and is positively defined on T^2M.

If G is a Riemannian metric on T^2M, we denote

$$\langle X, Y\rangle := G(X, Y),\ \forall X, Y \in \mathcal{X}(T^2M). \tag{11}$$

In this paper, we shall only consider metrics in the form

$$G = \underset{(0)}{g}{}_{ij}dx^i \otimes dx^j + \underset{(1)}{g}{}_{ij}\delta y^{(1)i} \otimes \delta y^{(1)j} + \underset{(2)}{g}{}_{ij}\delta y^{(2)i} \otimes \delta y^{(2)j}, \tag{12}$$

this is, so that the distributions N, N_1 and V_2 generated by the nonlinear connection N are orthogonal in pairs with respect to G.

An N- linear connection D is called *metrical* if $D_X G = 0$, $\forall X \in \mathcal{X}(T^2M)$. This means

$$X\langle Y, Z\rangle = \langle D_X Y, Z\rangle + \langle Y, D_X Z\rangle,\ \forall X, Y, Z \in \mathcal{X}\left(T^2M\right). \tag{13}$$

In the following, we shall consider throughout the paper T^2M endowed with:

- a nonlinear connection N;
- a Riemannian metric G;
- a metrical N- linear connection D with coefficients 9.

Let $c : [0, 1] \to T^2M$, $c(t) = \left(x^i(t), y^{(1)i}(t), y^{(2)i}(t)\right)$ be a piecewise smooth curve and $0 = t_0 < t_1 < ... < t_k = 1$ a division of $[0, 1]$ so that $c_{|[t_{i-1},t_i]}$ be of class C^∞ on each interval $[t_{i-1}, t_i]$. Let us denote $c(0) = p$, $c(1) = q$.

A *variation* of c (with fixed endpoints) is a mapping $\alpha : (-\varepsilon, \varepsilon) \times [0, 1] \to T^2M$, (where $\varepsilon > 0$), with the properties

1. $\alpha(0, t) = c(t)$, $\forall t \in [0, 1]$;
2. α is continuous on each $(-\varepsilon, \varepsilon) \times [t_{i-1}, t_i]$, $\forall i = 1, ..., k$.

Let $\overline{\alpha}$ the mapping defined on $(-\varepsilon, \varepsilon)$ by

$$\overline{\alpha}(u)(t) = \alpha(u,t).$$

If α is a variation with fixed endpoints of c, then the vector field $W \in \mathcal{X}\left(T^2M\right)$ along c, given by

$$W(t) = \frac{\partial \alpha}{\partial u}(0,t) \tag{14}$$

is called the *deviation vector field* , [3], [4], [18], attached to α. We obviously have

$$W(0) = W(1) = 0.$$

Let us denote, as in [3], [4], [18], $V = \dot{c}$. Then, V locally writes

$$\dot{c} = V = V^{(\alpha)i}\delta_{\alpha i},$$

with

$$V^{(0)i} = \frac{dx^i}{dt},\ V^{(1)i} = \frac{\delta y^{(1)i}}{dt},\ V^{(2)i} = \frac{\delta y^{(2)i}}{dt}.$$

Let also

$$A := \frac{DV}{dt} = A^{(0)i}\delta_i + A^{(1)i}\delta_{1i} + A^{(2)i}\delta_{2i}, \tag{15}$$

be the covariant acceleration, where, for $X \in \mathcal{X}\left(T^2M\right)$, we denoted

$$\frac{DX}{dt} := D_{\dot{c}}X,$$

and

$$\Delta_t X = X(t_+) - X(t_-),\ t \in [0,1],\ X \in \mathcal{X}\left(T^2M\right), \tag{16}$$

the *jump* of $X \in \mathcal{X}\left(T^2M\right)$ in t.

The *energy* of the curve c is

$$E(c) = \int_0^1 \underset{(0)}{g}_{ij}V^{(0)i}V^{(0)j} + \underset{(1)}{g}_{ij}V^{(1)i}V^{(1)j} + \underset{(2)}{g}_{ij}V^{(2)i}V^{(2)j}dt, \tag{17}$$

this is, $E(c) = \int_0^1 \langle V, V\rangle\, dt$.

Definition 1. *We call a* geodesic *of* T^2M, *a critical path* $c : [0,1] \to T^2M$ *of the energy* E, *which is* C^∞*-smooth on the whole* $[0,1]$.

By a direct computation, taking into account the metricity of the N- linear connection D, we obtain

Theorem 2. ***(The first variation of the energy):*** *If $c : [0,1] \to T^2M$ is a piecewise smooth path and $\alpha : (-\varepsilon, \varepsilon) \times [0,1] \to T^2M$ is a variation with fixed endpoints of c, then*

$$\frac{1}{2}\frac{dE\left(\overline{\alpha}\left(u\right)\right)}{du}\rfloor_{u=0} = -\sum_{i=0}^{k-1}\left\langle W, \Delta_{t_i}V\right\rangle + \int_0^1 \left\langle T\left(W,V\right),V\right\rangle - \left\langle W, A\right\rangle dt. \tag{18}$$

Remark 3. *If the curve c is C^∞-smooth on the whole $[0,1]$, then*

$$\frac{1}{2}\frac{dE\left(\overline{\alpha}\left(u\right)\right)}{du}\rfloor_{u=0} = \int_0^1 \left\langle T\left(W,V\right),V\right\rangle - \left\langle W, A\right\rangle dt.$$

In order to deduce the equations of the geodesics of T^2M, in (18), we write the term $\left\langle T\left(W,V\right),V\right\rangle$ in the form $\langle F, W\rangle$; in local coordinates, one obtains

Theorem 4. 1. *$F = \sum_{\alpha=0}^{2} F^{(\alpha)i}\delta_{\alpha i}$ given by*

$$F^{(\alpha)i} = \underset{(\alpha)}{g}{}^{il}\,\underset{(\gamma)}{g}{}_{kh}\,\overset{(\gamma)}{\underset{(\beta\alpha)}{T}}{}^{k}_{jl}V^{(\beta)j}V^{(\gamma)h}, \quad \alpha = 0,1,2 \tag{19}$$

is a vector field, globally defined along c.

2. *There holds the equality*

$$\left\langle T\left(W,V\right),V\right\rangle = \left\langle W,F\right\rangle. \tag{20}$$

3. *The vector field F does not depend on the variation α of c.*

Taking into account the previous theorem, we get

$$\frac{1}{2}\frac{dE\left(\overline{\alpha}\left(u\right)\right)}{du}\rfloor_{u=0} = -\sum_{i=0}^{k-1}\left\langle W, \Delta_{t_i}V\right\rangle - \int_0^1 \left\langle W, F - A\right\rangle dt.$$

We have proved this way

Theorem 5. *The C^∞-smooth curve $c : [0,1] \to T^2M$, $t \mapsto \left(x^i\left(t\right), y^{(1)i}\left(t\right), y^{(2)i}\left(t\right)\right)$ is a geodesic of T^2M if and only if*

$$\frac{D}{dt}\frac{dc}{dt} = F, \tag{21}$$

or, in local coordinates,

$$\begin{aligned}\frac{DV^{(0)i}}{dt} &= F^{(0)i},\\ \frac{DV^{(1)i}}{dt} &= F^{(1)i},\\ \frac{DV^{(2)i}}{dt} &= F^{(2)i}.\end{aligned} \tag{22}$$

It will be useful to write the last equalities in the following form

$$\begin{aligned}\frac{dV^{(0)i}}{dt} + \underset{(00)}{L}{}^{i}{}_{jk}V^{(0)k}V^{(0)j} + \underset{(01)}{C}{}^{i}{}_{jk}V^{(1)k}V^{(0)j} + \underset{(02)}{C}{}^{i}{}_{jk}V^{(2)k}V^{(0)j} &= F^{(0)i},\\ \frac{dV^{(1)i}}{dt} + \underset{(10)}{L}{}^{i}{}_{jk}V^{(0)k}V^{(1)j} + \underset{(11)}{C}{}^{i}{}_{jk}V^{(1)k}V^{(1)j} + \underset{(12)}{C}{}^{i}{}_{jk}V^{(2)k}V^{(1)j} &= F^{(1)i},\\ \frac{dV^{(2)i}}{dt} + \underset{(00)}{L}{}^{i}{}_{jk}V^{(0)k}V^{(2)j} + \underset{(01)}{C}{}^{i}{}_{jk}V^{(1)k}V^{(2)j} + \underset{(02)}{C}{}^{i}{}_{jk}V^{(2)k}V^{(2)j} &= F^{(2)i}.\end{aligned} \tag{23}$$

5. **Invariance to homotheties of the equations of geodesics**

We consider the homotheties h_λ in (7).

Definition 6. *Let* $c : [0,1] \to T^2M,\ t \mapsto c(t) = \left(x(t), y^{(1)}(t), y^{(2)}(t)\right)$ *be an arbitrary curve and* $\lambda > 0$ *a real number. We call the* homothetic *of* c *the curve*

$$\bar{c} : \left[0, \frac{1}{\lambda}\right] \to T^2M,\ \ \bar{c}\left(\frac{1}{\lambda}t\right) := h_\lambda\left(c(t)\right), \tag{24}$$

and we denote

$$\bar{c} = h_\lambda(c).$$

Let us remark that $h_\lambda(c) \neq h_\lambda \circ c$.

$\bar{c} = h_\lambda(c)$ locally writes

$$\begin{aligned}\bar{x}^i\left(\frac{1}{\lambda}t\right) &= x^i(t),\\ \bar{y}^{(1)i}\left(\frac{1}{\lambda}t\right) &= \lambda y^{(1)i}(t),\\ \bar{y}^{(2)i}\left(\frac{1}{\lambda}t\right) &= \lambda^2 y^{(2)i}(t).\end{aligned} \tag{25}$$

If we suppose that

$$\underset{(1)}{N}{}^{i}_{j} \text{ are 1-homogeneous, } \underset{(2)}{N}{}^{i}_{j} \text{ are 2-homogeneous,} \tag{26}$$

(which implies that $\underset{(1)}{M}{}^{i}_{j}$ are 1-homogeneous and, $\underset{(2)}{M}{}^{i}_{j}$ are 2-homogeneous), then, δ_i are 1-homogeneous, δ_{1i} are 0-homogeneous and δ_{2i} are -1-homogeneous; consequently, the tangent vectors of $\overline{c}$ are given by

$$\overline{V}^{(\alpha)i}\left(\frac{t}{\lambda}\right) = \lambda^{\alpha+1} V^{(\alpha)i}(t),\ \alpha = 0, 1, 2. \tag{27}$$

or $\overline{V}\left(\frac{t}{\lambda}\right) = \lambda h^{*}_{\lambda} V(t)$.

If we claim that, for any geodesic c of T^2M, the homothetic $\overline{c}$ should be a geodesic, too, we obtain:

Theorem 7. *Let $\underset{(1)}{N}{}^{i}_{j}$ be 1-homogeneous , $\underset{(2)}{N}{}^{i}_{j}$ be 2-homogeneous. If:*

1. $\underset{(00)}{L}{}^{i}_{jk}, \underset{(10)}{L}{}^{i}_{jk}, \underset{(20)}{L}{}^{i}_{jk}$ *are homogeneous of degree* 0;
2. $\underset{(01)}{C}{}^{i}_{jk}, \underset{(11)}{C}{}^{i}_{jk}, \underset{(21)}{C}{}^{i}_{jk}$ *- homogeneous of degree* -1;
3. $\underset{(02)}{C}{}^{i}_{jk}, \underset{(12)}{C}{}^{i}_{jk}, \underset{(22)}{C}{}^{i}_{jk}$ *- homogeneous of degree* -2;
4. $\underset{(\alpha)}{g}{}_{ij}$ *- homogeneous of degree* $-\alpha$, $\alpha = 0, 1, 2$,

then the equations of the geodesics of T^2M are invariant to the homotheties (24).

Proof. 1., 2. and 3. can be obtained by a direct computation.

In order to prove 4., we must take into account that:

- $V^{(\alpha)i}$ are $(\alpha + 1)$-homogeneous;
- in the expression of $F^{(\alpha)i}$, the term $\overset{(\gamma)}{\underset{(\beta\alpha)}{T}}{}^{k}_{jl} V^{(\beta)j} V^{(\gamma)h}$ is homogeneous of degree $\gamma - \beta - \alpha + \beta + 1 + \gamma + 1 = 2\gamma - \alpha + 2$;
- if $\underset{(\gamma)}{g}{}_{kh}$ are homogeneous of degree $-\gamma$, then $\underset{(\gamma)}{g}{}^{kh}$ are homogeneous of degree $+\gamma$.

□

6. **The exponential map of T^2M**

It is known, [7], that for regular Lagrangians defined on T^2M, the integral of action $I(c)$ essentially depends on the parametrization of the curve c (the Zermelo conditions); consequently, the equations of geodesics (22) are generally not invariant to re-parametrizations of the form $t \mapsto \frac{t}{\lambda}$, $\lambda > 0$. This is why, instead of the classical technique of defining the exponential map (which relies on such re-parametrizations), we shall use the homotheties $c \mapsto \bar{c}$ as defined above.

Let us remark, for the beginning, that the equations of geodesics (22) constitute a system of $6n$ ODE system with the unknown (real) functions x^i, $y^{(1)i}$, $y^{(2)i}$, $V^{(0)i}$, $V^{(1)i}$, $V^{(2)i}$. This allows us to state an existence and uniqueness result.

For $p \in T^2M$, let us denote in the following, $p := (x^i, y^{(1)i}, y^{(2)i})$ its coordinates in a local chart and, for $X \in \mathcal{X}(T^2M)$, $X := (X^{(0)i}, X^{(1)i}, X^{(2)i})$.

Let $p_1 := (x_1^i, y_1^{(1)i}, y_1^{(2)i}) \in T^2M$ and $V_1 := (V_1^{(0)i}, V_1^{(1)i}, V_1^{(2)i}) \in T_{p_1}\left(T^2M\right)$ be arbitrary. There holds

Theorem 8. *There exists a neighbourhood W of $(p_1, V_1) \in \mathbb{R}^{6n}$ and a real number $\varepsilon > 0$ so that, for any $(p_0, V_0) \in W$, the system (22) has a unique solution*

$$t \mapsto (p(t), V(t))$$

defined for $t \in (-\varepsilon, \varepsilon)$ and which satisfies the initial conditions

$$p(0) = p_0, \ \ V(0) = V_0. \tag{28}$$

Furthermore, the solution depends smoothly on the initial conditions (28).

In the conditions of Theorem 7, if c is a geodesic of T^2M, then $\bar{c} = h_\lambda(c)$ is also a geodesic. We are now able to state

Theorem 9. *In the conditions of Theorem 7, for any $p_0 \in T^2M$ there is an $\varepsilon > 0$ so that, for any tangent vector $V \in T_{p_0}\left(T^2M\right)$, with $\|V\| < \varepsilon$, there exists the geodesic*

$$c : (-2, 2) \to T^2M,\ t \mapsto \left(x^i(t), y^{(1)i}(t), y^{(2)i}(t)\right)$$

with the initial conditions

$$c(0) = p_0,\ \frac{dc}{dt}(0) = V.$$

Definition 10. *The point* $c(1) := \left(x^i(1), y^{(1)i}(1), y^{(2)i}(1)\right)$ *is called the* exponential of $V \in T_{p_0}\left(T^2M\right)$ *in* p_0 *and will be denoted by*

$$c(1) = \exp_{p_0}(V). \tag{29}$$

Let us prove Theorem 9:

Let $p_0 \in T^2M$, $\varepsilon > 0$ and $V \in T_{p_0}\left(T^2M\right)$ with $\|V\| < \varepsilon$ be arbitrary. Then, according to Theorem 8, for any $\overline{p}_0 \in T^2M$ and for any $\overline{V} \in T_{\overline{p}_0}\left(T^2M\right)$, there uniquely exists the geodesic $c_{\overline{V}} : (-2\varepsilon_2, 2\varepsilon_2) \to T^2M$ with

$$c_{\overline{V}}(0) = \overline{p}_0,\ \frac{dc_{\overline{V}}}{dt}(0) = \overline{V}. \tag{30}$$

We set

$$\begin{aligned} \overline{p}_0 \quad &: \quad = h_{\frac{1}{\varepsilon_2}}(p_0) \\ \overline{V} \quad &: \quad = \frac{1}{\varepsilon_2} h^*_{\frac{1}{\varepsilon_2},p_0}(V) \in T_{\overline{p}_0}\left(T^2M\right) \\ \varepsilon \quad &< \quad \varepsilon_1\varepsilon_2. \end{aligned} \tag{31}$$

($\overline{V}$ is the tangent vector field of $h_{\frac{1}{\varepsilon_2}}(c)$).

Because $\|V\| < \varepsilon$ and according to (31), we have

$$\left\|\overline{V}\right\| = \frac{1}{\varepsilon_2}\|V\| < \varepsilon_1;$$

consequently, there uniquely exists the geodesic $c_{\overline{V}}$ with the initial conditions (30). Furthermore, if $|t| < 2$, then $|\varepsilon_2 t| < 2\varepsilon_2$, which allows us to define

$$c(t) := h_{\varepsilon_2}\left(c_{\overline{V}}(t)\right) : (-2, 2) \to T^2M,$$

then c is obviously a geodesic and is uniquely defined by the above equality. Furthermore,

$$c(0) = h_{\varepsilon_2}\left(c_{\overline{V}}(0)\right) = h_{\varepsilon_2}\left(\overline{p}_0\right) = \left(h_{\varepsilon_2} \circ h_{\frac{1}{\varepsilon_2}}\right)(p_0) = p_0.$$

Let $\overline{Z}$ be the tangent vector field of $c_{\overline{V}}$; then, $\overline{Z}(0) = \overline{V} = \frac{1}{\varepsilon_2} h^*_{\frac{1}{\varepsilon_2},p_0}(V)$; taking into account that $h^*_{\varepsilon_2} = \left(h^*_{\frac{1}{\varepsilon_2}}\right)^{-1}$, we get

$$\frac{dc}{dt}(0) = \varepsilon_2 h^*_{\varepsilon_2,\overline{p}_0}\left(\overline{Z}\right) = \varepsilon_2 \frac{1}{\varepsilon_2} h^*_{\varepsilon_2,\overline{p}_0}\left(h^*_{\frac{1}{\varepsilon_2},p_0}(V)\right) = V,$$

which completes the proof.

It is worth mentioning that:

1. The exponential map in $p \in T^2M$ is generally defined only for small values of $\|V\|$. If it exists, the value $\exp_p(V)$ is unique.
2. If c is a geodesic of T^2M with $p_0 = c(0)$, $V = \dot{c}(0)$, then

$$c(t) = \exp_p(tV). \tag{32}$$

7. Example

Let (M, g) be a Riemannian manifold, (T^2M, π^2, M), its second order tangent bundle and $\widetilde{T^2M} = T^2M \backslash \{0\}$, i.e., T^2M without its null section. We consider the following geometric objects on $\widetilde{T^2M}$:

- the *canonical nonlinear connection* N, [8], given by its dual coefficients $\underset{(1)}{M}{}^i_j = \gamma^i_{jk} y^{(1)k}$, $\underset{(2)}{M}{}^i_j = \frac{1}{2}\left\{\mathbb{C}\left(\gamma^i_{jk} y^{(1)k}\right) + \underset{(1)}{M}{}^i_k \underset{(1)}{M}{}^k_j\right\}$,

where $\gamma^i_{jk} = \gamma^i_{jk}(x)$ are the Christoffel symbols of g and $\mathbb{C} = y^{(1)i}\frac{\partial}{\partial x^i} + 2y^{(2)i}\frac{\partial}{\partial y^{(1)i}}$;

- the *homogeneous N-lift* of the metric g, defined by prof. Gh. Atanasiu, [2],

$$\overset{o}{G} = g_{ij} dx^i \otimes dx^j + \frac{a^2}{\left\|y^{(1)}\right\|^2} g_{ij}\, \delta y^{(1)i} \otimes \delta y^{(1)j} + \frac{a^4}{\left\|y^{(1)}\right\|^4} g_{ij} \delta y^{(2)i} \otimes \delta y^{(2)j},$$

  where $\left\|y^{(1)}\right\| = \sqrt{g_{ij} y^{(1)i} y^{(1)j}}$;

- the *canonical N-linear connection*, $D\overset{o}{\Gamma}(N)$, [2], given by the coefficients

$$\underset{(00)}{L}{}^i{}_{jk} = \underset{(10)}{L}{}^i{}_{jk} = \underset{(20)}{L}{}^i{}_{jk} = \gamma^i_{jk}(x), \quad \underset{(01)}{C}{}^i{}_{jk} = 0,$$

$$\underset{(11)}{C}{}^i{}_{jk} = -\frac{1}{\left\|y^{(1)}\right\|^2}\left(\delta^i_j y^{(1)}_k + \delta^i_k y^{(1)}_l - g_{jk} y^{(1)i}\right), \quad \underset{(21)}{C}{}^i{}_{jk} = 2\underset{(11)}{C}{}^i{}_{jk},$$

$$\underset{(02)}{C}{}^i{}_{jk} = \underset{(12)}{C}{}^i{}_{jk} = \underset{(22)}{C}{}^i{}_{jk} = 0.$$

By a direct calculus, one proves that the conditions of Theorem 7 are accomplished; consequently, if $\widetilde{T^2M}$ is endowed with these structures, the exponential map can be defined on $\widetilde{T^2M}$.

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BOOK REVIEWS

***Nonlinear Evolution Equations and Related Topics –Dedicated to Philippe Bénilan*, W. Arendt, H. Brezis and M. Pierre Eds.**, Birkhäuser, Basel-Boston-Berlin, 2004, ISBN 3-7643-7107-2.

The volume is dedicated to Philippe Bénilan (October 6, 1940 - February 17, 2001), one of the most significant contributors to the theory of nonlinear evolution equations. It contains research papers related to Bénilan's work which cover a wide range of nonlinear and linear equations and appeared in regular issues of the Journal of Evolution Equations, Volumes 3 (2003) and 4 (2004). The main topics are Hamilton-Jacobi equations, the porous medium equation, reaction diffusion systems, integro-differential equations and viscoelasticity, maximal regularity for elliptic and parabolic equations and the Ornstein-Uhlenbeck operator. Thus new developments of nonlinear analysis are presented with applications to physics, mechanics, chemistry, biology and others.

The volume starts with an Introduction presenting Bénilan's main contributions to nonlinear analysis, a list of publications and a list of Ph.D.-Students of P. Bénilan. Furthermore the contains are as follows: F. Hirsch, intrinsic metrics and Lipschitz functions; S. Benachour and P. Laurençot, Decay estimates for "anisotropic" viscous Hamilton-Jacobi equations in R^N; F. Andreu, V. Caselles and J.M. Mazón, The Cauchy problem for linear growth functionals; J.L. Vázquez, Asymptotic behaviour for the porous medium equation posed in the whole space; W. Arendt and M. Warma, Dirichlet and Neumann boundary conditions: what is in between?; S.B. Angenent and D.G. Aronson, The focusing problem for the Eikonal equation; M. Pierre, Weak solutions and supersolutions in L^1 for reaction-diffusion systems; S.-O. Londen,

H. Petzeltová and J. Prüss, Global well-posedness and stability of a partial integro-differential equation with applications to viscoelasticity; P. Bénilan, L.C. Evans and R.F. Gariepy, On some singular limits of homogeneous semigroups; P. Bénilan and N. Igbida, Singular limit of changing sign solutions of the porous medium equation; L. Boccardo, On the regularizing effect of strongly increasing lower order terms; E. Bazhlekova and P. Clément, Global smooth solutions for a quasilinear fractional evolution equation; H. Gajewski and I.V. Skrypnik, On the uniqueness of solutions for nonlinear elliptic-parabolic equations; J. Carrillo, Conservation laws with discontinuous flux functions and boundary condition; V.G. Jakubowski and P. Wittbold, Regularity of solutions of nonlinear Volterra equations; J. Liang, R. Nagel and T.-J. Xiao, Nonautonomous heat equation with generalized Wentzell boundary conditions; H. Heck and M. Hieber, Maximal L^p-regularity for elliptic operators with VMO-coefficients; W.M. Ruess, Linearized stability for nonlinear evolution equations; D. Bothe, Nonlinear evolutions with Carathéodory forcing; H. Amann, Linear parabolic equations with singular potentials; L. Boccardo, L. Orsina and A. Porretta, Some noncoercive parabolic equations with lower order terms in divergence form; E. Feireisl, On the motion of rigid bodies in a viscous incompressible fluid; A. Henrot, Minimization problems for eigenvalues of the Laplacian; A. Haraux, M.A. Jendoubi and O. Kavian, Rate of decay to equilibrium in some semilinear parabolic equations; G. Da Prato, A new regularity result for Ornstein-Uhlenbeck generators and applications; J. Droniou, T. Gallouët and J. Vovelle, Global solution and smoothing effect for a non-local regularization of a hyperbolic equation; M. Gokieli and F. Simondon, Convergence to equilibrium for a parabolic problem with mixed boundary conditions in one space dimension; J. Escher and G. Simonett, Analyticity of solutions to fully nonlinear parabolic evolution equations on symmetric spaces; P. Bénilan and J.I. Díaz, Pointwise gradient estimates of solutions to onedimensional nonlinear parabolic equations; M. Maliki and H. Touré, Uniqueness of entropy solutions for nonlinear degenerate parabolic problems; C.G. Gal, G. Ruiz Goldstein and J.A. Goldstein, Oscillatory boundary conditions for acoustic wave equations; M. Marcus and L. Véron, Existence and uniqueness results for large solutions of general nonlinear elliptic equations; M.G. Crandall and

P.-Y. Wang, Another way to say caloric; P. Bénilan and H. Brezis, Nonlinear problems related to the Thomas-Fermi equation; P. Bénilan and H. Labani, Existence of attractors in $L^{\infty}(\Omega)$ for a class of reaction-diffusion systems; B.P. Andreianov and F. Bouhsiss, Uniqueness for an elliptic-parabolic problem with Neumann boundary condition.

The volume will interest all mathematicians working in nonlinear analysis and its applications. It is a nice tribute to one of the most original mathematicians with a deep and decisive impact on the theory of Evolution Equations.

Radu Precup

William Arveson, ***A Short Course on Spectral Theory***, Graduate Texts in Mathematics, Vol. 209, Springer, New York, Berlin, Heidelberg, 2002, x+135 pp., ISBN 0-387-95300-0.

The fundamental problem of operator theory is the calculation of spectra of operators on infinite dimensional spaces, especially on Hilbert spaces. The theory has deep applications to partial differential and integral operators, to mathematical foundation of quantum mechanics, noncommutative K-theory and the classification of simple C^{*}-algebras.

The aim of the present book, based on a fifteen-week course taught for several times by the author at the University of Berkeley, is to make the reader acquainted with the basic results in spectral theory, needed for the study of more advanced topics listed above. The prerequisites are elementary functional analysis and measure theory.

In the first chapter, *Spectral theory and Banach algebras,* the theory is developed in the natural framework of Banach algebras and includes spectral radius, regular representation, the spectral permanence theorem, and an introduction to analytic functional calculus. The abstract notions are illustrated on concrete examples of operators.

Ch. 2, *Operators on Hilbert space,* is concerned with spectral theory for operators on Hilbert space and their C^*-algebras, normal operators, compact operators, spectral measures. For the sake of clarity the treatment is restricted to separable Hilbert spaces. A good companion in reading this part could be another book by the same author: *An invitation to C^*-algebras,* Springer Verlag 1998.

Ch. 3, *Asymptotics: Compact perturbations and Fredholm theory,* contains the Calkin algebra, Riesz theory for compact operators, Fredholm operators and Fredholm index.

In the last chapter, Ch. 4, *Methods and applications,* a variety of operator theoretic methods are applied to determine the spectra of Toeplitz operators, the results being definitive only for Toeplitz operators with continuous symbol. An elementary theory of Hardy spaces H^2 is also developed. The book ends with the study of states on C^*-algebras and a proof of Gelfand-Naimark representation theorem.

The book is a clear, short and thorough introduction to spectral theory, accessible to first or second year graduate students. As the author points out in the Preface: "this material is the essential beginning for any serious student in modern analysis".

S. Cobzaş

François Bouchut, ***Nonlinear Stability of Finite Volume Methods for Hyperbolic Conservation Laws and Well-Balanced Schemes for Sources***, Birkhäuser Verlag, Basel-Boston-Berlin, 2004, 135 pp., ISBN 3-7643-6665-6.

This very good monograph is devoted to finite volume methods for hyperbolic systems of conservation laws. All examples included in the book are of gas dynamics type. The author presents systematically sufficient conditions for a scheme to preserve an invariant domain or to satisfy discrete entropy inequalities.

The book consists of two parts. The first part is concerned with the notion of approximate Riemann solver and the relaxation method. Certain practical formulas are obtained in a new variant of HLLC solver for the gas dynamics system, taking into account contact discontinuities, entropy conditions, and including vacuum.

The second part of the book is devoted to the numerical treatment of source terms that can appear additionally in hyperbolic conservation laws, with the extension of the notions of invariant domains, entropy inequalities, and approximate Riemann solvers. The author compares several methods that have been developed in the literature especially for the Saint Venant problem, concerning the positivity and the ability to treat resonant data. In particular, the hydrostatic reconstruction method is presented in details.

The book is clearly written, with rigorous proofs, in a pleasant and accessible style. It is warmly recommended as a useful guide for all engineers and researchers interested in the nonlinear stability of finite volume methods for hyperbolic systems of conservation laws.

Mirela Kohr

Calin, O., Chang, D.-C., ***Geometric Mechanics on Riemannian Manifolds. Applications to Partial Differential Equations***, Birkhäuser (Applied and Numerical Harmonic Analysis), 2004, Hardback, 278 pp., ISBN 0-8176-4354-0.

Unlike other parts of mathematics, in the theory of partial differential equations it is quite difficult to find results which apply to any equation or, at least, to a large class of equations. Instead, in this theory one studies *individual* equations, such as, for instance, the heat equation, the Laplace or the Poisson equations. It turns out that many equations are related, in a way or another, to mechanics and, as such, the geometrical approaches to mechanics can be used to investigate them, as alternative to the classical methods, such that that of the integral transformations. It is the aim of this monograph to give an introduction to this geometric approach to some important partial differential equations.

The book starts with an outline of the fundamentals of differential geometry, examines the Laplace operators on Riemannian manifolds and proceeds to discuss the main approaches to mechanics on Riemannian manifolds (Lagrangian, Hamiltonian and Hamilton-Jacobi). As important examples, there are discussed the harmonic maps and, in particular, the minimal hypersurfaces. It follows a analysis of radially symmetric spaces and a discussion of the fundamental solutions for heat operators with potentials and of elliptic operators on this kind of spaces.

The book ends with a chapter devoted to special classes of curves that the authors call "mechanical curves" and which appear as solution to different mechanical problems (for instance curves that minimize a potential, cycloids, astroids a.o.).

The differential operators which are treated in the book are among the most important, not only in the theory of partial differential equation, but they appear naturally in geometry, mechanics or theoretical physics (especially quantum mechanics). Thus, the book should be of interest for anyone working in these fields, from advanced undergraduate students to experts.

The book is written in a very pedagogical manner and does not assume many prerequisites, therefore it is quite appropriate to be used for special courses or for self-study. I have to mention that all chapter ends with a number of well-chosen exercises that will improve the understanding of the material and, also, that there are a lot of worked examples that will serve the same purpose.

Paul Blaga

Andrew J. Kurdila, Michael Zabarankin, ***Convex Functional Analysis***, Systems & Control: Foundations & Applications, Birkhäuser Verlag, Boston-Basel-Berlin 2005, xiv+228 pp, ISBN-10:3-7643-2198-9.

The aim of the present book is to make the students in applied mathematics and engineering acquainted with the basic principles and tools of functional and convex analysis, as required by modern treatments of some problems in variational calculus, mechanics and control. The emphasis is not on foundation and proofs, but rather on examples and applications. For this reason some results are given with full proofs, while for others one gives only references for detailed presentation. A good idea on the content of the book is done by the headings of its chapters: 1. *Classical abstract spaces in functional analysis*; 2. *Linear functionals and linear operators* (including a complete proof of the open mapping theorem and of Riesz's representation theorem for the dual of $C[a,b]$); 3. *Common function spaces in applications* (L^p and Sobolev spaces of scalar or Banach-valued functions); 4. *Differential calculus in normed vector spaces* (containing many examples of differential operators); 5. *Minimization of functionals* (including the Lagrange multipliers rule in constrained differential optimization, deduced via Ljusternik's theorem); 6. *Convex functionals* (including a section on ordered vector spaces and convex programming in such spaces); 7. *Lower semi-continuous functionals.*

The bibliography contains a list of basic textbooks and monographs covering the topics the book is dealing with.

The book is useful for students in engineering interested in a quick and accessible presentation of basic tools of functional analysis needed for applications, as well as for students in applied mathematics interested in possible applications of these disciplines.

S. Cobzaş

Jon P. Davis, ***Methods of Applied Mathematics with a MATLAB Overview***, Birkhäuser Verlag, Boston-Basel-Berlin 2004, ISBN 0-8176-4331-1.

This book is devoted to the application of Fourier Analysis. The author mixed in a remarkable way theoretical results and applications illustrating the results. Flexibility of presentation (increasing and decreasing level of rigor, accessibility) is a key feature.

The first chapter is an introductory one.

An introduction to Fourier series based mainly on inner product spaces is given in Chapter 2.

The third chapter treats elementary boundary value problems. Besides applications of the Fourier series, it presents standard boundary value problem models and their discrete analogous problems.

Higher-dimensional, non rectangular problems is the topic of the fourth chapter. These includes Sturm-Liouville Theory, series solutions, Bessel equations and nonhomogeneous boundary value problems.

Chapter 5 is an introduction to functions of complex variable. Here ones discuss basic results and their applications to problems of fluid flow and transform inversion.

The sixth chapter introduces Laplace transform and their applications to ordinary differential equations, circuit analysis and input-output analysis of linear systems.

Continuous Fourier transform is the topic of seventh chapter. Also applications of Fourier transform to ordinary differential equations, integral equations, partial differential equations are included here.

Chapter 8 is on discrete variable transforms. It treats discrete variable models, z-transform, discrete and fast Fourier transform and their properties. Computational aspects of fast Fourier transform are also pointed.

The last chapter "Additional Topics" introduces methods that are specialization of those treated previously such as two-sided and Walsh transform, wavelets analysis and integral transform.

The book contains extensive examples, presented in an intuitive way with high quality figure (some of them quite spectacular), useful MATLAB codes. MATLAB exercises and routines are well integrated within the text, and a concise introduction into MATLAB is given in an appendix. The emphasis is on program's numerical and graphical capabilities and its applications, not on its syntax. A large variety of problems graded from difficulty point of view. Applications are modern and up to date. Reach and comprehensive references are attached to each chapter.

Intended audience: especially students in pure and applied mathematics, physics and computer science, but also useful to applied mathematicians, engineers and computer scientists interested in applications of Fourier analysis.

Radu Trîmbiţaş

Cabral, Hildeberto, Diacu, Florin ***Classical and Celestial Mechanics (The Recife Lectures)***, Princeton University Press, 2002, Hardcover, 385 pages, ISBN 0-691-05022-8.

The University of Pernambuco (Brazil) invited, between 1991 and 1999, several international experts to lecture in Recife (Brazil) on different topics in classical or celestial mechanics. The editors managed to convince some of the lecturer to prepare an elaborate version of their lecture and gathered everything into a book: this one.

Due to the nature of the book, it doesn't have a unitary character (and it is not suppose to have!). The subjects treated include: "classical" celestial mechanics (the motion of the Moon (Dieter Schmidt) and the two-body problem (Alain Albouy)), the theory of equilibria and applications to celestial mechanics (central configurations and relative equilibria for the N-body problem (Dieter Schmidt), normal forms of Hamiltonian systems and stability of equilibria (Hildeberto Cabral)), geometrical methods in classical mechanics (Mark Levi, and Jair Koiller et al.), topological methods in celestial mechanics (Poincaré's compactification (Ernesto Pérez-Chavela)), singularities of the N-body problem (Florin Diacu) and bifurcations from families of periodic solutions (Jack Hale and Plácido Táboas).

Modern classical and celestial mechanics (and, especially, their mathematical tools) represent a vast field which is impossible to be described completely in a single monograph or textbook, not to mention the fact that there is no individual researcher who can call himself an expert in all the particular subjects. The summer schools are ideal opportunities for the discussion of the latest developments from a field or to describe a classical field from anew perspective. Unfortunately, the lectures from the summer schools are only rarely published and, besides, it is only very rarely possible to gather in the same place a large number of very good experts in a field. The editors of this book did a far better job. They managed to gather the lectures (most of them enlarged and polished) from several "schools" (or series of lectures, if you prefer) and, as such, they give the reader the possibility to interact with the science of some of the worldwide best experts in classical and celestial mechanics. This unique book (which, as argued above, is more a collection of graduate "minitextbooks" than a proceedings of a particular school) would be extremely useful especially for graduate students interested in the field, but, due to the wealth of the subjects treated, the researchers will also find, I am absolutely sure, many new results or new perspective on the classical material.

I would like to say, to finish, that the book benefits of a foreword written by Donald Saari.

Cristina Blaga

Bolsinov, A.V., ***Integrable Hamiltonian Systems: Geometry, Topology, Classification***, Chapman and Hall/CRC, 2004, 730 pp., Hardback, ISBN 0-415-29805-9.

The field of integrable systems is a very rich field that gave rise to several important developments in mathematics in the last decades. It has strong relations with domain as: symplectic and Poisson geometry, quantum groups, algebraic geometry and even quantum field theory.

This new book on the subject approaches a fundamental problem: that of the classification of integrable systems. Many dynamical systems (integrable or not) are described by means of a system of differential equations. Even if these systems are different from one dynamical system to another, their solutions do have, sometimes, similarities, they "behave" analogously, in a certain sense. It is the aim of the classification theory to spot such similarities and to exploit them.

In this book three kind of equivalence relations between dynamical systems are examined:

- *conjugacy*, which, loosely, means that the systems can be transformed on into the other through a change of variables;
- *orbital equivalence*, which means that between the manifolds on which the dynamical systems are defined there is a diffeomorphism that turns the trajectories of a systems into the the trajectories of the other (although the parameters along the trajectories are not necessarily preserved);
- *Liouville equivalence* (only for integrable systems): two integrable systems are said to be equivalent in the sense of Liouville if their phase spaces are foliated in the same manner into Liouville tori.

Certainly, it is hardly possible to solve the classifications problem for arbitrary Hamiltonian dynamical systems, even if they are integrable. Therefore, this book focuses on a particular, but very important class of systems: nondegenerate integrable Hamiltonian systems with two degrees of freedom. The first half of the book (the first 9 chapters) are devoted to foundational material on symplectic and Poisson geometry, followed by a discussion of the three equivalence relation and the solution of the Liouville and orbital classification problem for the aforementioned class of dynamical systems. The solution is based on a new approach to the qualitative theory of dynamical systems invented by A.T. Fomenko and developed by him and its collaborators in a series of papers.

The remaining of the book is devoted to applications of the classification theory to specific integrable Hamiltonian systems coming from mechanics and geometry. Two systems are considered more important and are treated in details: the integrable cases of the equation of motion for rigid bodies and the integrable geodesic flows of Riemannian metrics on two-dimensional surfaces.

The book is largely based on the works of the two authors (two well-known experts in the field) and of their collaborators and it is addressed, mainly, to researchers in dynamical systems, geometry and mechanics, managing, successfully, to fill a gap in the existing literature (in fact much of the material was never published into a book). However, the exposition is cursive and understandable, there is enough foundational material and there are enough worked examples, which makes it appropriate also for graduate students, both for self-study or as a textbook for an advanced course.

Paul Blaga

Giovanni P. Galdi, John G. Heywood, Rolf Rannacher (Editors), *Contributions to Current Challenges in Mathematical Fluid Mechanics*, Birkhäuser Verlag, Basel-Boston-Berlin, 2004, 151 pp., ISBN 3-7643-7104-8.

This volume consists of five very good research articles, each of them being dedicated to an important topic in the mathematical theory of the Navier-Stokes equations, for compressible and incompressible fluids. The results presented in this volume are all new and represent a key contribution to this topic, with particular interest to turbulence modelling, regularity of solutions to the initial-value problem, flow in region with an unbounded boundary and compressible flow.

The first article of this volume, due to Andrei Biryuk, deals with the Cauchy problem for a multidimensional Burgers type equation with periodic boundary conditions. The author obtains upper and lower bounds for derivatives of solutions for this equation, which are expressed in terms of powers of the viscosity. In addition, it is discussed how these bounds relate to the Kolmogorov-Obukhov spectral law. Moreover, these estimates are used to obtain bounds for derivatives of solutions to the Navier-Stokes system.

The second article, due to Dongho Chae and Jihoon Lee, is concerned with the problem of global well-posedness stability in the scale invariant Besov spaces for the modified $3D$ Navier-Stokes equations with the dissipation term, $-\triangle u$ replaced by $(-\triangle)^{\alpha}u$ for $0 \leq \alpha < 5/4$. The authors prove the unique existence of a global-in -time solution in $B^{5/2-2\alpha}_{2,1}$ for initial data having small $\dot{B}^{5/2-2\alpha}_{2,1}$ norm for $\alpha \in [1/2, 5/4)$.

In the next article the authors A. Dunca, V. John and W.J. Layton deal with the space averaged Navier-Stokes equations, which are the basic equations for the large eddy simulation of turbulent flows. In deriving these equations it is understood that differentiation and averaging operations can be interchanged. This procedure introduces a *commutation error* term that is typically ignored. However, in this paper the authors furnish a characterization of this term to be neglected. In fact, the authors show that the commutation error is asymptotically negligible in $L^p(\mathbb{R}^d)$ if and only if the fluid and the boundary exert exactly zero force on each other. In addition, the authors study the influence of the commutation error on the energy balance of the filtered equations.

The forth article of Toshiaki Hishida deals with the nonstationary Stokes and Navier-Stokes flows in aperture domains $\Omega \subset \mathbf{R}^n$, $n \geq 3$. The authors proves $L^q - L^r$ estimates of the Stokes semigroup. Then the author applies these estimates to the Navier-Stokes initial value problem, and obtains the global existence of a unique strong solution, which satisfies the vanishing flux condition through the aperture and some sharp decay properties as $t \to \infty$, when the initial velocity is sufficiently small in the space L^n.

The last article of T. Leonavičicene and K. Pileckas is concerned with steady compressible Navier-Stokes equations with zero velocity conditions at infinity in a three dimensional exterior domain. They consider the case of small perturbations of large potential forces. To solve this problem the authors apply a decomposition scheme and decompose the nonlinear problem into three linear problems of the following types: Neumann-type, modified Stokes problem and transport equation. Then they solve the resulted linear problems in weighted function spaces with detached asymptotics. Finally they prove certain results related to existence, uniqueness and asymptotics for the linearized problem and for the nonlinear problem.

Each paper from this volume is clearly written, with rigorous proofs, in a pleasant and accessible style. This voume is warmly recommended to all researchers interested in modern topics of the mathematical theory of fluid mechanics.

Mirela Kohr

Ram P. Kanwal, ***Generalized functions. Theory and applications***, 3rd revised ed. Boston, MA: Birkhäuser, 2004, xvii+476 pp., ISBN 0-8176-4343-5.

This book on generalized functions is suitable for physicist, engineers and applied mathematicians. The author presents the notion of generalized functions , their properties and their applications for solving ordinary differential equations and partial differential equations.

Chapters 1 to 8 contain a concise definition of distributions and their standard properties are proved. Chapters 9 to 15 deal with applications to ordinary differential equations, partial differential equations, boundary value problems, wave propagation, linear systems, probability and random processes, economics, microlocal theory.

The author demonstrate through various examples that familiarity with the generalizes functions is very helpful for students in mathematics, physical sciences and technology. The proposed exercises are very good for better understanding of notions and properties presented in the chapters. The book contains new topics and important features:

- Examination of the Poisson Summation Formula and the concepts of differential forms and the delta distribution on wave fronts.
- Enhaced presentation of the Schrödinger, Klein-Gordon, Helmholtz, heat and wave equations.
- Exposition driven by additional examples and exercises.

Marcel-Adrian Şerban

Stephen Lynch, ***Dynamical Systems with Applications Using MATLAB***, Birkhäuser Verlag, Boston-Basel-Berlin 2004, xviii+462 pp, ISBN 0-8176-4321-4.

The book is a good introduction to dynamical systems theory. In the first part real and complex discrete dynamical systems are considered, with examples taken from population dynamics, economics, biology, nonlinear optics, neural networks and electromagnetic waves. In the second part of the text, differential equations are used to model examples taken from mechanical systems, chemical kinetics, electric circuits, interacting species and economics. The theory and applications are presented with the aid of the MATLAB package. Throughout the book, MATLAB is viewed as a tool for solving systems or producing exciting graphics. The author suggests that the reader should save the relevant example programs. These programs can then be edited accordingly when attempting the exercises at the end of each chapter. The text is aimed at graduate students and working scientists in various branches of applied mathematics, natural sciences and engineering. The material is intelligible to readers with a general mathematical background. Fine details and theorems with proof are kept at a minimum. This book is informed by the research interests of the author which are nonlinear ordinary differential equations, nonlinear optics and fractals. Some chapters include recently published research articles and provide a useful resource for open problems in nonlinear dynamical systems. An efficient tutorial guide to MATLAB is included. The knowledge of a computer language would be beneficial but not essential. The MATLAB programs are kept as simple as possible and the author's experience has shown that this method of teaching using MATLAB works well with computer laboratory class of small sizes.

I recommend "Dynamical Systems with Applications using MATLAB" as a good handbook for a diverse readership, for graduates and professionals in mathematics, physics, science and engineering.

Damian Trif