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SUMAR – CONTENTS – SOMMAIRE

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ON EDGE-CONNECTIVITY OF INSERTED GRAPHS

M. R. ADHIKARI, AND L. K. PRAMANIK

Abstract. The aim of this paper is to estimate the edge-connectivity of the inserted graph with the help of the degree of vertices of the inserted graph and the edge-connectivity of the original graph.

1. **Introduction**

Throughout the paper we consider ordinary graphs (finite, undirected, with no loops or multiple edges) and G denotes a graph with vertex set V_G and edge set E_G. Each member of $V_G \cup E_G$ will be called an element of G. A graph G is called trivial graph if it has a vertex set with single vertex and a null edge set. If e be an edge of a graph G with end vertices x and y, then we denote the edge e, by $e = xy$.

We introduce the notions of box graph $B(G)$ and inserted graph $I(G)$ of a non-trivial graph G in [3]. It is an elementary basic fact that the inserted graph $I(G)$ of a non-trivial connected graph G in connected. The edge-connectivity $\lambda(G)$ of a graph G is the least number of edges whose removal disconnects G; and a set of $\lambda(G)$ edges satisfying this condition is called a minimal separating edge set of G. Clearly, G is m-edge-connected if and only if $\lambda(G) \geq m$.

In §2, we recall some definitions and results which will be used in §3 and also give an example of edge-connectivity of a graph G and its inserted graph $I(G)$.

In [1], we investigate the relations between the connectivity and edge-connectivity of a graph and its inserted graph. In §3 of this paper we obtain more

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results about edge-connectivity and give an alternative proof of some corollaries stated in [1].

2. Preliminaries

Definition 2.1. [3] *A graph can be constructed by inserting a new vertex on each edge of G, the resulting graph is called Box graph of G, denoted by $B(G)$.*

Definition 2.2. [3] *Let I_G be the set of all inserted vertices in $B(G)$. A graph $I(G)$ with vertex set I_G is called the inserted graph in which any two vertices are adjacent if they are joined by a path of length two in $B(G)$.*

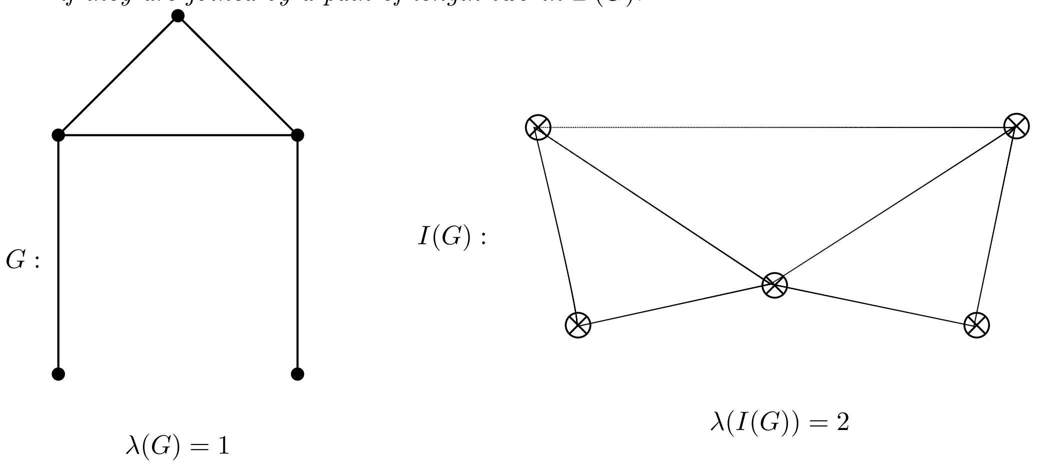

Figure 1 : The edge - connectivity of a graph and its inserted graph

These concepts are illustrated for a graph G and its inserted graph $I(G)$ in the Fig.1. Here $\otimes$ marked vertices are the newly inserted vertices.

Now we recall the following theorems:

Theorem 2.3. [4] *A graph G is m-edge-connected if and only if for every non-empty proper subset A of the vertex set V_G of the graph G, the number of edges joining A and $V_G - A$ is at least m.*

The next observation is due to Whitney [5].

Theorem 2.4. *For any graph G, $\lambda(G) \leq min\ deg\ G$.*

The order of a graph is the cardinality of its vertex set. If $G^{'}$ is a subgraph of G and $V_{G'}$, V_G are the vertex sets of $G^{'}$ and G respectively, then the degree of

$G^{'}$ in G is the number of all edges of G joining vertices in $V_{G'}$ with the vertices in $V_G - V_{G'}$.

3. **Edge-connectivity of $I(G)$**

To begin with let us prove the following lemma.

Lemma 3.1. *If*

$$\lambda(I(G)) < \lambda(G)[\frac{\lambda(G)+1}{2}],$$

then there exists a connected subgraph of G of order 2 and degree $\lambda(I(G)$ in G.

Proof: Let Y denote any nonempty proper subset of the edge set E_G of G. Thus Y induces a nonempty proper subset $\overline{Y}$ of the vertex set $V_{I(G)}$. For each vertex u in G, denote the number of edges of Y incident with u by $\delta(u)$ and the number of edges of $E_G - Y$ incident with u by $\delta^{'}(u)$; and set $W = \{u; \delta(u) > 0, \delta^{'}(u) > 0\}$. Suppose that each connected subgraph of G with two vertices has degree at least $\lambda(I(G)) + 1$ in G. We shall show that

$$\sum_{u \in W} \delta(u)\delta^{'}(u) \geq \lambda(I(G)) + 1.$$

First, suppose that no two vertices of W are adjacent. Now from the Theorem 2.4, $deg \quad u \geq \lambda(G)$ for every vertex $u \in W$. Thus one of the numbers $\delta(u)$ and $\delta^{'}(u)$ must be $[\frac{\lambda(G)+1}{2}]$. Consequently,

$$\sum_{u \in W} \delta(u)\delta^{'}(u) \geq [\frac{\lambda(G)+1}{2}] \sum_{u \in W} \delta_u(u),$$

where δ_u means δ or $\delta^{'}$. From the $\lambda(G)$-edge-connectivity of G it follows that

$$\sum_{u \in W} \delta_u(u) \geq \lambda(G),$$

and hence

$$\sum_{u \in W} \delta(u)\delta^{'}(u) \geq \lambda(G)[\frac{\lambda(G)+1}{2}] > \lambda(I(G)).$$

Suppose now that two adjacent vertices, say v and w, belonging to W. We assume that the degree of the subgraph generated by v and w is at least $\lambda(I(G)) + 1$

in G, i.e.

$$\delta(v) + \delta^{'}(v) + \delta(w) + \delta^{'}(w) \geq \lambda(I(G)) + 3.$$

Since for any natural numbers N_1 and N_2, $N_1 N_2 \geq N_1 + N_2 - 1$, we may write

$$\sum_{u \in W} \delta(u)\delta^{'}(u) \geq \delta(v)\delta^{'}(v) + \delta(w)\delta^{'}(w) \geq \delta(v) + \delta^{'}(v) - 1 + \delta(w) + \delta^{'}(w) - 1 \geq \lambda(I(G)) + 1.$$

By application of Theorem 2.3, the inequality

$$\sum_{u \in W} \delta(u)\delta^{'}(u) \geq \lambda(I(G)) + 1$$

proved above for a set W derived from an arbitrary proper subset $\overline{Y}$ of $V_{I(G)}$ shows that $I(G)$ is $(\lambda(I(G)) + 1)$-edge-connected, which is by definition impossible.

Therefore there exist a connected subgraph $G^{'}$ of G of order 2 and of degree at most $\lambda(I(G))$; if this degree becomes smaller than $\lambda(I(G))$, then the corresponding vertex of $I(G)$ have degree smaller than $\lambda(I(G))$, contradicting the Theorem 2.4. Hence $G^{'}$ has precisely the degree $\lambda(I(G))$ in G.

We now show that Corollaries 3.5 and 3.6 of [1] follows from the above Lemma.

Corollary 3.2. [1] $\lambda(I(G)) \geq 2\lambda(G) - 2$.

Proof: We prove the corollary by the method of contradiction. Suppose that $\lambda(I(G)) < 2\lambda(G) - 2$. Since

$$2\lambda(G) - 2 \leq \lambda(G)[\frac{\lambda(G)) + 1}{2}],$$

Lemma 3.1 implies the existence of a connected subgraph $G^{'}$ of G with two vertices of degree $\lambda(I(G))$ in G; since this degree is smaller than $2\lambda(G) - 2$, the degree of one of the vertices of $G^{'}$ is at most $\lambda(G) - 1$, contradicting Theorem 2.4.

Corollary 3.3. [1] *If $\lambda(G) \neq 2$, then $\lambda(I(G)) = 2\lambda(G) - 2$ if and only if there exist two adjacent vertices in G with degree $\lambda(G)$.*

Proof: For $\lambda(G) \neq 2$,

$$2\lambda(G) - 2 < \lambda(G)[\frac{\lambda(G)) + 1}{2}].$$

Hence by using Lemma 3.1, it follows that if

$$\lambda(I(G)) = 2\lambda(G) - 2,$$

then there exist two adjacent vertices v, w in G so that

$$deg \quad v + deg \quad w = \lambda(I(G)) + 2.$$

Since both v and w have degree $\geq \lambda(G)$ and

$$deg \quad v + deg \quad w = 2\lambda(G),$$

it follows immediately by Theorem 2.4 that

$$deg \quad v = deg \quad w = \lambda(G).$$

Conversely, if v and w are adjacent vertices of G and

$$deg \quad v = deg \quad w = \lambda(G),$$

then the vertex in $I(G)$ corresponding to the edge joining v and w has degree $2\lambda(G)-2$. Hence by Theorem 2.4,

$$\lambda(I(G)) \leq 2\lambda(G) - 2.$$

Now by Corollary 3.2, it follows that

$$\lambda(I(G)) = 2\lambda(G) - 2.$$

Corollary 3.4. *If $\lambda(G) \geq 3$, then $\lambda(I(G)) = 2\lambda(G)-1$ only if there exist two adjacent vertices in G, one of degree $\lambda(G)$ and the other of degree $\lambda(G)+1$.*

Proof: Proof is similar to that of Corollary 3.3

This procedure can be continued finitely as the graph is finite. Now we prove the following significant theorem.

Theorem 3.5. *If*

$$min \quad deg \quad I(G) \leq \lambda(G)[\frac{\lambda(G)+1}{2}],$$

then $\lambda(I(G)) = min \quad deg \quad I(G)$. Also if

$$min \quad deg \quad I(G) \geq \lambda(G)[\frac{\lambda(G)+1}{2}],$$

then

$$\lambda(G)[\frac{\lambda(G)+1}{2}] \leq \lambda(I(G)) \leq min \quad deg \quad I(G).$$

Proof: Theorem 2.4 implies $\lambda(I(G)) \leq min \quad deg \quad I(G)$. Now for the case

$$min \quad deg \quad I(G) \leq \lambda(G)[\frac{\lambda(G)+1}{2}]$$

suppose that $\lambda(I(G)) < min \quad deg \quad I(G)$. Then Lemma 3.1 asserts that there exists a connected subgraph of order 2 and degree $\lambda(I(G))$ in G; this means that there is a vertex in $I(G)$ of degree $\lambda(I(G))$, violating the assumed inequality. Consequently,

$$\lambda(I(G)) = min \quad deg \quad I(G).$$

For the case

$$min \quad deg \quad I(G) \geq \lambda(G)[\frac{\lambda(G)+1}{2}],$$

it remains to be shown that

$$\lambda(G)[\frac{\lambda(G)+1}{2}] \leq \lambda(I(G)).$$

Suppose on the contrary that

$$\lambda(G)[\frac{\lambda(G)+1}{2}] > \lambda(I(G)).$$

Then by Lemma 3.1 some vertex in $I(G)$ has degree $\lambda(I(G))$. Hence

$$min \quad deg \quad I(G) \leq \lambda(I(G)).$$

Thus it follows that

$$\lambda(G)[\frac{\lambda(G)+1}{2}] \leq \lambda(I(G)),$$

contradicting the inequality assumed above.

References

[1] Adhikari, M.R. and Pramanik, L.K., *The Connectivity of Inserted Graphs,* J. Chung. Math. Soc, 18 (1), 2005, 61-68.

[2] Adhikari, M.R., Pramanik, L.K. and Parui, S., *On Planar Graphs,* Rev. Bull. Cal. Math. Soc 12, 2004, 119-122.

[3] Adhikari, M.R., Pramanik, L.K. and Parui, S., *On Box Graph and its Square,* Communicated.

[4] Ore, O., *Theory of graphs,* Amer. Math. Soc. Providence, R.I., 1962.

[5] Whitney, H., *Congruent graphs and the connectivity of graphs,* Amer. J. Math. 54(1932), 150-168.

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APPROXIMATE FIXED POINT THEOREMS

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Abstract. Two general lemmas are given regarding $\varepsilon-$fixed points of operators on metric spaces. Using these results we prove qualitative and quantitative theorems for various types of well known generalized contractions on metric spaces.

1. **Introduction**

There are plenty of problems in applied mathematics which can be solved by means of fixed point theory. Still, practice proves that in many real situations an approximate solution is more than sufficient, so the existence of fixed points is not strictly required, but that of "nearly" fixed points. Another type of practical situations that lead to this approximation is when the conditions that have to be imposed in order to guarantee the existence of fixed points are far too strong for the real problem one has to solve.

It is then natural to introduce the concepts of $\varepsilon-$*fixed point (*or *approximate fixed point)*, which is a "nearly" fixed point, and that of function with the *approximate fixed point property* and to formulate a proper theory regarding them.

In this paper, starting from the article of Tijs, Torre and Branzei [10], we study some well known types of operators on metric spaces, and we give some qualitative and quantitative results regarding $\varepsilon-$fixed points of such operators.

We have to mention that we consider operators on metric spaces, not on complete metric spaces, the usual framework for fixed point problems. Weakening

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the conditions by giving up the completeness of the space we can still guarantee the existence of $\varepsilon-$fixed points for various types of operators.

We begin with two lemmas. The first one is the qualitative result that indicates under which conditions the operator f has the approximate fixed point property. This will be used in order to prove all the results given in the second section. The second lemma is the quantitative result which will be used in order to prove all the results given in the third section.

Let (X, d) be a metric space.

Definition 1.1. *Let $f : X \to X$, $\varepsilon > 0$, $x_0 \in X$. Then x_0 is an $\varepsilon-$**fixed point (approximate fixed point) of** f if*

$$d(f(x_0), x_0) < \varepsilon.$$

Remark 1.1. *As many authors we prefer the terminology with ε, as being more suggestive throughout the paper.*

Remark 1.2. *In this paper we will denote the set of all $\varepsilon-$fixed points of f, for a given ε, by:*

$$F_\varepsilon(f) = \{x \in X \mid x \text{ is an } \varepsilon - \text{fixed point of } f\}.$$

Definition 1.2. *Let $f : X \to X$. Then f has **the approximate fixed point property (a.f.p.p.)** if*

$$\forall \varepsilon > 0,\ F_\varepsilon(f) \neq \varnothing.$$

The following result guarantees the existence of $\varepsilon-$fixed points for an operator on a metric space.

Lemma 1.1. *Let (X, d) be a metric space, $f : X \to X$ such that f is asymptotic regular, i.e.,*

$$d(f^n(x), f^{n+1}(x)) \to 0 \text{ as } n \to \infty,\ \forall x \in X.$$

Then f has the approximate fixed point property.

Proof. Let $x_0 \in X$. Then:

$$d(f^n(x_0), f^{n+1}(x_0)) \to 0 \text{ as } n \to \infty \ \Leftrightarrow$$

$$\forall \varepsilon > 0,\ \exists n_0(\varepsilon) \in \mathbb{N}^* \text{ such that } \forall n \geq n_0(\varepsilon),\ d(f^n(x_0), f^{n+1}(x_0)) < \varepsilon \ \Leftrightarrow$$

$$\forall \varepsilon > 0,\ \exists n_0(\varepsilon) \in \mathbb{N}^* \text{ such that } \forall n \geq n_0(\varepsilon),\ d(f^n(x_0), f(f^n(x_0))) < \varepsilon\ .$$

Denoting

$$y_0 = f^n(x_0),$$

it follows that:

$$\forall \varepsilon > 0,\ \exists y_0 \in X \text{ such that } d(y_0, f(y_0)) < \varepsilon\ ,$$

so for each $\varepsilon > 0$ there exists an $\varepsilon-$fixed point of f in X, namely y_0.

This means exactly that f has the approximate fixed point property. □

Remark 1.3. *The following result (see [5]) gives conditions under which the existence of fixed points for a given mapping is equivalent to that of approximate fixed points.*

Proposition.: *Let A be a closed subset of a metric space (X, d) and $f : A \to X$ a compact map. Then f has a fixed point if and only if it has the approximate fixed point property.*

In the following, by $\delta(A)$ for a set $A \neq \varnothing$ we will understand the diameter of the set A, i.e.,

$$\delta(A) = \sup\{d(x, y) \mid x, y \in A\}.$$

Lemma 1.2. *Let (X, d) be a metric space, $f : X \to X$ an operator and $\varepsilon > 0$. We assume that:*

$\imath$)**:** $F_\varepsilon(f) \neq \varnothing$;

$\imath\imath$)**:** $\forall \eta > 0,\ \exists \varphi(\eta) > 0$ *such that*

$$d(x, y) - d(f(x), f(y)) \leq \eta \Rightarrow d(x, y) \leq \varphi(\eta),\ \forall x, y \in F_\varepsilon(f).$$

Then:

$$\delta(F_\varepsilon(f)) \leq \varphi(2\varepsilon).$$

Proof. Let $\varepsilon > 0$ and $x, y \in F_\varepsilon(f)$. Then:

$$d(x, f(x)) < \varepsilon,\ d(y, f(y)) < \varepsilon.$$

We can write:

$$d(x, y) \leq d(x, f(x)) + d(f(x), f(y)) + d(y, f(y)) \leq d(f(x), f(y)) + 2\varepsilon$$

$$\Rightarrow d(x,y) - d(f(x), f(y)) \leq 2\varepsilon.$$

Now by (ii) it follows that

$$d(x,y) \leq \varphi(2\varepsilon),$$

so

$$\delta(F_\varepsilon(f)) \leq \varphi(2\varepsilon).$$

□

Remark 1.4. *Condition (i) in Lemma 1.2 can be replaced by the asymptotic regularity condition, as, by Lemma 1.1, the latter ensures (i). So Lemma 1.2 can be given in the form:*

Lemma 1.3. *Let (X,d) be a metric space and $f : X \to X$ such that for $\varepsilon > 0$ the following hold:*

$\imath$)**:** $d(f^n(x), f^{n+1}(x)) \to 0$ *as* $n \to \infty$, $\forall x \in X$;
$\imath\imath$)**:** $\forall \eta > 0$, $\exists \varphi(\eta) > 0$ *such that*

$$d(x,y) - d(f(x), f(y)) \leq \eta \Rightarrow d(x,y) \leq \varphi(\eta), \ \forall x, y \in F_\varepsilon(f).$$

Then:

$$\delta(F_\varepsilon(f)) \leq \varphi(2\varepsilon).$$

2. Qualitative results for operators on metric spaces

In this section we will formulate and prove, using Lemma 1.1, qualitative results for various types of operators on a metric space, results that establish the conditions under which the mappings considered have the approximate fixed point property.

Let (X,d) be a metric space. Note that the completeness of the space is not required, as in fixed point theorems.

Definition 2.1. *([8]) A mapping $f : X \to X$ is an **a-contraction** if*

$$\exists a \in]0,1[\text{ such that } d(f(x), f(y) \leq a d(x,y), \ \forall x, y \in X.$$

Theorem 2.1. *Let (X, d) be a metric space and $f : X \to X$ an a-contraction.*

Then:

$$\forall \varepsilon > 0,\ \ F_\varepsilon(f) \neq \varnothing.$$

Proof. Let $\varepsilon > 0$, $x \in X$.

$$d(f^n(x), f^{n+1}(x)) = d(f(f^{n-1}(x)), f(f^n(x))) \leq$$

$$\leq ad(f^{n-1}(x), f^n(x)) \leq ... \leq a^n d(x, f(x))$$

But $a \in]0, 1[\Rightarrow$

$$d(f^n(x), f^{n+1}(x)) \to 0, \text{ as } n \to \infty,\ \forall x \in X.$$

Now by Lemma 1.1 it follows that $F_\varepsilon(f) \neq \varnothing,\ \forall \varepsilon > 0.$ □

Remark 2.1. *Theorem 2.1 is a result presented and proved, by means of a different method, in [10].*

Any operator satisfying the condition in Definition 2.1 is Lipschitz and implicitly continuous, which means a relatively small class of mappings. Still, the theory of fixed points and consequently $\varepsilon-$fixed points deals also with non-continuous mappings. In 1968, Kannan (see [6],[2]) proved a fixed point theorem for operators which need not be continuous, by considering the following contraction condition.

Definition 2.2. *([6],[8]) A mapping $f : X \to X$ is a* ***Kannan operator*** *if*

$$\exists a \in]0, \frac{1}{2}[\text{ such that } d(f(x), f(y)) \leq a[d(x, f(x)) + d(y, f(y))], \forall x, y \in X.$$

Theorem 2.2. *Let (X, d) be a metric space and $f : X \to X$ a Kannan operator.*

Then:

$$\forall \varepsilon > 0,\ \ F_\varepsilon(f) \neq \varnothing.$$

Proof. Let $\varepsilon > 0$ and $x \in X$.

$$d(f^n(x), f^{n+1}(x)) = d(f(f^{n-1}(x)), f(f^n(x))) \leq$$

$$\leq a[d(f^{n-1}(x), f(f^{n-1}(x))) + d(f^n(x), f(f^n(x)))] =$$

$$= ad(f^{n-1}(x), f^n(x)) + ad(f^n(x), f^{n+1}(x))$$

$$\Rightarrow (1-a)d(f^n(x), f^{n+1}(x)) \le ad(f^{n-1}(x), f^n(x)) \Rightarrow$$

$$d(f^n(x), f^{n+1}(x)) \le \frac{a}{1-a} d(f^{n-1}(x), f^n(x)) \le ... \le a1 - a^n d(x, f(x))$$

But $a \in]0, \frac{1}{2}[\Rightarrow \dfrac{a}{1-a} \in]0, 1[\Rightarrow$

$$d(f^n(x), f^{n+1}(x)) \to 0, \text{ as } n \to \infty, \ \forall x \in X.$$

Now by Lemma1.1 it follows that $F_\varepsilon(f) \neq \varnothing, \forall \varepsilon > 0$. □

In 1972, Chatterjea considered another contraction condition, similar to that of Kannan but independent of this one, and which again does not impose the continuity of the operator.

Definition 2.3. *([4],[8]) A mapping $f : X \to X$ is a **Chatterjea operator** if*

$$\exists a \in]0, \frac{1}{2}[\text{ such that } d(f(x), f(y)) \le a[d(x, f(y)) + d(y, f(x))], \forall x, y \in X.$$

Theorem 2.3. *Let (X, d) be a metric space and $f : X \to X$ a Chatterjea operator. Then:*

$$\forall \varepsilon > 0, \ F_\varepsilon(f) \neq \varnothing.$$

Proof. Let $\varepsilon > 0$ and $x \in X$.

$$d(f^n(x), f^{n+1}(x)) = d(f(f^{n-1}(x)), f(f^n(x))) \le$$

$$\le a[d(f^{n-1}(x), f(f^n(x))) + d(f^n(x), f(f^{n-1}(x)))] =$$

$$= a[d(f^{n-1}(x), f^{n+1}(x)) + d(f^n(x), f^n(x))] = ad(f^{n-1}(x), f^{n+1}(x))$$

On the other hand

$$d(f^{n-1}(x), f^{n+1}(x)) \le d(f^{n-1}(x), f^n(x)) + d(f^n(x), f^{n+1}(x)) \Rightarrow$$

$$(1-a)d(f^n(x), f^{n+1}(x)) \le ad(f^{n-1}(x), f^n(x)) \Rightarrow$$

$$d(f^n(x), f^{n+1}(x)) \le \frac{a}{1-a} d(f^{n-1}(x), f^n(x)) \le ... \le a1 - a^n d(x, f(x)).$$

But $a \in]0, \frac{1}{2}[\Rightarrow \dfrac{a}{1-a} \in]0, 1[\Rightarrow$

$$d(f^n(x), f^{n+1}(x)) \to 0, \text{ as } n \to \infty, \ \forall x \in X.$$

Now by Lemma 1.1 it follows that $F_\varepsilon \neq \varnothing, \forall \varepsilon > 0$. □

In 1972, by combining the three independent (see [7]) contraction conditions above, Zamfirescu (see [11]) obtained another fixed point result for operators which satisfy the following.

Definition 2.4. *([8],[11]) A mapping $f : X \to X$ is a* ***Zamfirescu operator*** *if*

$$\exists a, k, c \in \mathbb{R},\ a \in [0, 1[,\ k \in [0, \frac{1}{2}[,\ c \in [0, \frac{1}{2}[\text{ such that}$$

$$\forall x, y \in X, \text{ at least one of the following is true:}$$

$\imath$)**:** $d(f(x), f(y)) \leq ad(x, y)$;

$\imath\imath$)**:** $d(f(x), f(y)) \leq k[d(x, f(x)) + d(y, f(y))]$;

$\imath\imath\imath$)**:** $d(f(x), f(y)) \leq c[d(x, f(y)) + d(y, f(x))]$.

Theorem 2.4. *Let (X, d) be a metric space and $f : X \to X$ a Zamfirescu operator. Then:*

$$\forall \varepsilon > 0,\ F_\varepsilon(f) \neq \varnothing.$$

Proof. First we will try to concentrate the three independent conditions into a single one they all imply, see the proof of Zamfirescu's fixed point theorem given in [3].

Let $x, y \in X$.

Supposing $\imath\imath$) holds, we have that:

$$d(f(x), f(y)) \leq k[d(x, f(x)) + d(y, f(y))] \leq$$

$$\leq kd(x, f(x)) + k[d(y, x) + d(x, f(x)) + d(f(x), f(y))] =$$

$$= 2kd(x, f(x)) + kd(x, y) + kd(f(x), f(y)) \Rightarrow$$

$$d(f(x), f(y)) \leq \frac{2k}{1-k} d(x, f(x)) + \frac{k}{1-k} d(x, y). \tag{1}$$

Supposing $\imath\imath\imath$) holds, we have that:

$$d(f(x), f(y)) \leq c[d(x, f(y)) + d(y, f(x))] \leq$$

$$\leq c[d(x, y) + d(y, f(y))] + c[d(y, f(y)) + d(f(y), f(x))] =$$

$$= cd(f(x), f(y)) + 2cd(y, f(y)) + cd(x, y) \Rightarrow$$

$$d(f(x), f(y)) \leq \frac{2c}{1-c} d(y, f(y)) + \frac{c}{1-c} d(x, y). \tag{2a}$$

Similarly:

$$d(f(x),f(y)) \leq c[d(x,f(y)) + d(y,f(x))] \leq$$

$$\leq c[d(x,f(x)) + d(f(x),f(y))] + c[d(y,x) + d(x,f(x))] =$$

$$= cd(f(x),f(y)) + 2cd(x,f(x)) + cd(x,y) \Rightarrow$$

$$d(f(x),f(y)) \leq \frac{2c}{1-c}d(x,f(x)) + \frac{c}{1-c}d(x,y). \tag{2b}$$

Now looking at $\imath), (1), (2a), (2b)$ we can denote:

$$\delta = \max\{a, \frac{k}{1-k}, \frac{c}{1-c}\},$$

and it is easy to see that $\delta \in [0,1[$.

For f satisfying at least one of the conditions $\imath), \imath\imath), \imath\imath\imath)$ we have that

$$d(f(x),f(y)) \leq 2\delta d(x,f(x)) + \delta d(x,y) \tag{3a}$$

$$\text{and } d(f(x),f(y)) \leq 2\delta d(y,f(y)) + \delta d(x,y) \tag{3b}$$

hold.

Using these conditions implied by $\imath) - \imath\imath\imath)$ and taking $x \in X$, we have:

$$d(f^n(x), f^{n+1}(x)) = d(f(f^{n-1}(x)), f(f^n(x))) \overset{(3a)}{\leq}$$

$$\leq 2\delta d(f^{n-1}(x), f(f^{n-1}(x))) + \delta d(f^{n-1}(x), f^n(x)) = 3\delta d(f^{n-1}(x), f^n(x)) \Rightarrow$$

$$d(f^n(x), f^{n+1}(x)) \leq ... \leq (3\delta)^n d(x, f(x)) \Rightarrow$$

$$d(f^n(x), f^{n+1}(x)) \to 0, \text{ as } n \to \infty, \ \forall x \in X.$$

Now by Lemma 1.1 it follows that $F_\varepsilon \neq \varnothing, \forall \varepsilon > 0$. □

Remark 2.2. *Theorems 2.1, 2.2, 2.3 are actually contained in Theorem 2.4, as any a-contraction, Kannan operator or Chatterjea operator is also a Zamfirescu operator (see Definitions 2.1, 2.2, 2.3, 2.4.).*

If we go further generalizing, we may consider the contraction condition given in 2004 by V. Berinde, who also formulated a corresponding fixed point theorem, see [2], for example.

Definition 2.5. *A mapping $f : X \to X$ is a* ***weak contraction*** *if*

$$\exists a \in]0,1[\text{ and } L \geq 0 \text{ such that } d(f(x), f(y)) \leq ad(x,y) + Ld(y, f(x)), \forall x, y \in X.$$

Theorem 2.5. *Let (X, d) be a metric space and $f : X \to X$ a weak contraction. Then:*

$$\forall \varepsilon > 0, \ F_\varepsilon(f) \neq \varnothing.$$

Proof. Let $x \in X$.

$$d(f^n(x), f^{n+1}(x)) = d(f(f^{n-1}(x)), f(f^n(x))) \leq$$

$$\leq ad(f^{n-1}(x), f^n(x)) + Ld(f^n(x), f^n(x)) =$$

$$= ad(f^{n-1}(x), f^n(x)) \leq ... \leq a^n d(x, f(x))$$

But $a \in]0,1[\Rightarrow$

$$d(f^n(x), f^{n+1}(x)) \to 0, \text{ as } n \to \infty, \ \forall x \in X.$$

Now by Lemma 1.1 it follows that $F_\varepsilon \neq \varnothing, \forall \varepsilon > 0$.

Remark 2.3. *Theorem 2.5 is even more general than the others above, as any of the above mentioned mappings is also a weak contraction, see Proposition 1 in [2].*

Remark 2.4. *An analogous result could be given for quasi-contractions with $0 < h < \frac{1}{2}$, see again [2].*

□

Similar results concerning the existence of $\varepsilon-$fixed points for other classes of operators on metric spaces will be the subject of future papers.

3. **Quantitative results for operators on metric spaces**

For the same operators we have studied in the previous section, from the qualitative point of view, we will now use Lemma 1.2 in order to obtain quantitative results.

Theorem 3.1. *Let (X, d) be a metric space and $f : X \to X$ an a-contraction.*

Then:

$$\delta(F_\varepsilon(f)) \leq \frac{2\varepsilon}{1-a}, \forall \varepsilon > 0.$$

Proof. Let $\varepsilon > 0$. Condition $\imath)$ in Lemma 1.2 is satisfied, as one can see in the proof of Theorem 2.1.

We will show now that $\imath\imath)$ also holds for a-contractions.

Let $\eta > 0$ and $x, y \in F_\varepsilon(f)$. We also assume that

$$d(x, y) - d(f(x), f(y)) \leq \eta$$

and aim to show that there exists an $\varphi(\eta) > 0$ such that $d(x, y) \leq \varphi(\eta)$.

We have that:

$$d(x, y) \leq d(f(x), f(y)) + \eta \leq ad(x, y) + \eta$$

$$\Rightarrow (1-a)d(x, y) \leq \eta,$$

which implies $d(x, y) \leq \dfrac{\eta}{1-a}$.

So $\forall \eta > 0, \exists \varphi(\eta) = \dfrac{\eta}{1-a} > 0$ such that

$$d(x, y) - d(f(x), f(y)) \leq \eta \Rightarrow d(x, y) \leq \varphi(\eta).$$

Now by Lemma 1.2 it follows that

$$\delta(F_\varepsilon(f)) \leq \varphi(2\varepsilon), \forall \varepsilon > 0,$$

which means exactly that

$$\delta(F_\varepsilon(f)) \leq \frac{2\varepsilon}{1-a}, \forall \varepsilon > 0.$$

□

Theorem 3.2. *Let (X,d) be a metric space and $f : X \to X$ a Kannan operator.*

Then:

$$\delta(F_\varepsilon(f)) \leq 2\varepsilon(1+a), \forall \varepsilon > 0.$$

Proof. Let $\varepsilon > 0$. As in the proof of Theorem 3.1 we only verify that condition $ii)$ in Lemma 1.2 holds.

Let $\eta > 0$ and $x, y \in F_\varepsilon(f)$ and assume that $d(x,y) - d(f(x), f(y)) \leq \eta$. Then

$$d(x,y) \leq a[d(x, f(x)) + d(y, f(y))] + \eta.$$

As $x, y \in F_\varepsilon(f)$, we know that $d(x, f(x)) < \varepsilon$ and $d(y, f(y)) < \varepsilon$.

$$\Rightarrow d(x,y) \leq 2a\varepsilon + \eta$$

So $\forall \eta > 0, \exists \varphi(\eta) = \eta + 2a\varepsilon > 0$ such that

$$d(x,y) - d(f(x), f(y)) \leq \eta \Rightarrow d(x,y) \leq \eta.$$

Now by Lemma 1.2 it follows that

$$\delta(F_\varepsilon(f)) \leq \varphi(2\varepsilon), \forall \varepsilon > 0,$$

which means exactly that

$$\delta(F_\varepsilon(f)) \leq 2\varepsilon(1+a), \forall \varepsilon > 0.$$

□

Theorem 3.3. *Let (X,d) be a metric space and $f : X \to X$ a Chatterjea operator.*

Then:

$$\delta(F_\varepsilon(f)) \leq \frac{2\varepsilon(1+a)}{1-2a}, \forall \varepsilon > 0.$$

Proof. Let $\varepsilon > 0$. Again we will only show that condition $ii)$ in Lemma 1.2 holds.

Let $\eta > 0$ and $x, y \in F_\varepsilon(f)$ and assume that

$$d(x,y) - d(f(x), f(y)) \leq \eta.$$

Then

$$d(x,y) \leq a[d(x, f(y)) + d(y, f(x))] + \eta \leq$$

$$\leq ad(x, f(y)) + ad(y, f(x)) + \eta \leq$$

$$\leq a[d(x, y) + d(y, f(y))] + a[d(y, x) + d(x, f(x))] + \eta.$$

As $x, y \in F_\varepsilon(f)$, it follows that

$$d(x, y) \leq 2ad(x, y) + 2\varepsilon a + \eta.$$

$$\Rightarrow (1 - 2a)d(x, y) \leq 2\varepsilon a + \eta \Rightarrow$$

$$d(x, y) \leq \frac{\eta + 2\varepsilon a}{1 - 2a}$$

So $\forall \eta > 0, \exists \varphi(\eta) = \dfrac{\eta + 2\varepsilon a}{1 - 2a} > 0$ such that

$$d(x, y) - d(f(x), f(y)) \leq \eta \Rightarrow d(x, y) \leq \varphi(\eta).$$

Now by Lemma 1.2 it follows that

$$\delta(F_\varepsilon(f)) \leq \varphi(2\varepsilon), \forall \varepsilon > 0,$$

which means exactly that

$$\delta(F_\varepsilon(f)) \leq \frac{2\varepsilon(1 + a)}{1 - 2a}, \forall \varepsilon > 0.$$

□

Theorem 3.4. *Let (X, d) be a metric space and $f : X \to X$ a Zamfirescu operator.*

Then

$$\delta(F_\varepsilon(f)) \leq 2\varepsilon \frac{1 + \rho}{1 - \rho}, \forall \varepsilon > 0,$$

where $\rho = \max\{a, \dfrac{k}{1 - k}, \dfrac{c}{1 - c}\}$ *and a, k, c as in Definition 2.4.*

Proof. In the proof of Theorem 2.4 we have already shown that if f satisfies at least one of the conditions $\imath), \imath\imath)$ or $\imath\imath\imath)$ from Definition 2.4, then

$$d(f(x), f(y)) \leq 2\rho d(x, f(x)) + \rho d(x, y)$$

and

$$d(f(x), f(y)) \leq 2\rho d(y, f(y)) + \rho d(x, y)$$

hold.

Let $\varepsilon > 0$. Again we will only show that condition $\imath\imath)$ in Lemma 1.2 is satisfied, as $\imath)$ holds, see the Proof of Theorem 2.4.

Let $\eta > 0$ and $x, y \in F_\varepsilon(f)$, and assume that

$$d(x,y) - d(f(x), f(y)) \leq \eta.$$

Then

$$d(x,y) \leq d(f(x), f(y)) + \eta \leq 2\rho d(x, f(x)) + \rho d(x,y) + \eta \Rightarrow$$

$$(1-\rho)d(x,y) \leq 2\rho\varepsilon + \eta \Rightarrow$$

$$d(x,y) \leq \frac{\eta + 2\rho\varepsilon}{1-\rho}.$$

So $\forall \eta > 0, \exists \varphi(\eta) = \dfrac{\eta + 2\rho\varepsilon}{1-\rho} > 0$ such that

$$d(x,y) - d(f(x), f(y)) \leq \eta \Rightarrow d(x,y) \leq \eta.$$

Now by Lemma 1.2 it follows that

$$\delta(F_\varepsilon(f)) \leq \varphi(2\varepsilon), \forall \varepsilon > 0,$$

which means exactly that

$$\delta(F_\varepsilon(f)) \leq 2\varepsilon \frac{1+\rho}{1-\rho}, \forall \varepsilon > 0.$$

□

Remark 3.1. *In the case of weak contractions we have to add a condition, namely* $a + L < 1$, *with the same notations as above, in order to get the result.*

Theorem 3.5. *Let* (X, d) *be a metric space and* $f : X \to X$ *a weak contraction with* $a + L < 1$.

Then

$$\delta(F_\varepsilon(f)) \leq \frac{2+L}{1-a-L}\varepsilon, \forall \varepsilon > 0.$$

Proof. Let $\varepsilon > 0$. We show again only that condition (ii) in Lemma 1.2 holds.

Let $\eta > 0$ and $x, y \in F_\varepsilon(f)$, and assume that

$$d(x,y) - d(f(x), f(y)) \leq \eta.$$

Then

$$d(x,y) \le d(f(x),f(y)) + \eta \le ad(x,y) + Ld(y,f(x)) + \eta \le$$

$$\le ad(x,y) + Ld(x,y) + Ld(x,f(x)) + \eta \le (a+L)d(x,y) + L\varepsilon + \eta.$$

$$\Rightarrow (1-a-L)d(x,y) \le L\varepsilon + \eta \Rightarrow d(x,y) \le \frac{L\varepsilon+\eta}{1-a-L}$$

So $\forall \eta > 0, \exists \varphi(\eta) = \dfrac{L\varepsilon+\eta}{1-a-L} > 0$ such that

$$d(x,y) - d(f(x),f(y)) \le \eta \Rightarrow d(x,y) \le \eta.$$

Now by Lemma 1.2 it follows that

$$\delta(F_\varepsilon(f)) \le \varphi(2\varepsilon), \forall \varepsilon > 0,$$

which means exactly that

$$\delta(F_\varepsilon(f)) \le \frac{2+L}{1-a-L}\varepsilon, \forall \varepsilon > 0.$$

□

4. **Conclusions**

The theory of $\varepsilon-$fixed points is not less interesting than that of fixed points and many results formulated in the latter can be adapted to a less restrictive framework in order to guarantee the existence of the $\varepsilon-$fixed points and the fact that the diameter of the set containing these points goes to zero when ε goes to zero.

We proved results referring to some types of contractive operators on metric spaces, starting from a result presented in [10] for $a-$contractions, but the study may go further to other classes of operators, which will be the subject of future papers.

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References

[1] Berinde, V., *Iterative Approximation of Fixed Points,* Editura Efemeride, Baia Mare, 2002.

[2] Berinde, V., *On the Approximation of Fixed Points of Weak Contractive Mappings,* Carpathian J. Math., **19**(2003), No.1, 7-22.

[3] Berinde, V., *On Zamfirescu's Fixed Point Theorem,* Revue Roum. Math. Pures et Appl. (to appear).

[4] Chatterjea, S.K., *Fixed-point Theorems,* C.R. Acad. Bulgare Sci., **25**(1972), 727-730.

[5] Granas, A., Dugundji, J., *Fixed Point Theory,* Springer-Verlag, New York, 2003.

[6] Kannan, R., *Some Results on Fixed Points,* Bull. Calcutta Math. Soc., **10**(1968), 71-76.

[7] Rhoades, B. E., *A Comparison of Various Definitions of Contractive Mappings,* Trans. Amer. Math. Soc., **226**(1977), 257-290.

[8] Rus, I. A., *Generalized Contractions and Applications,* Cluj University Press, Cluj-Napoca, 2001.

[9] Rus, I. A., *Principles and Applications of Fixed Point Theory* (in Romanian), Editura Dacia, Cluj-Napoca, 1979.

[10] Tijs, S., Torre, A., Branzei, R., *Approximate Fixed Point Theorems,* Libertas Mathematica, **23**(2003), 35-39.

[11] Zamfirescu, T., *Fixed Point Theorems in Metric Spaces,* Arch. Math. (Basel), **23**(1972), 292-298.

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BOOLEAN SHEPARD INTERPOLATION

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Abstract. Using Shepard univariate interpolation projectors which form the chains and boolean methods we construct Biermann-Shepard projector. We study the approximation order of Biermann-Shepard operator for two particular cases. The convergence of this operator is mark out by graphs and numerical examples.

1. Preliminaries

Let X, Y be the linear spaces on $\mathbb{R}$ or $\mathbb{C}$.

The linear operator P defined on space X is called projector if $P^2 = P$.

The operator $P^C = I - P$, where I is identity operator, is called the remainder projector of P.

The set of interpolation points of projector P is denoted by $\mathcal{P}(P)$. If P, Q are commutative projectors then we have

$$\mathcal{P}(P \oplus Q) = \mathcal{P}(P) \cup \mathcal{P}(Q) \tag{1}$$

If P_1, P_2 are projectors on space X, we define relation "$\leq$":

$$P_1 \leq P_2 \Leftrightarrow P_1 P_2 = P_1 \tag{2}$$

Let be $f \in \mathcal{C}(X \times Y)$ and $x \in X$. We define $f^x \in C(Y)$ by

$$f^x(t) = f(x,t) \ , \ t \in Y$$

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For $y \in Y$ we define ${}^y f \in C(X)$ by

$${}^y f(s) = f(s, y)\ ,\ s \in X$$

Let P be a linear and bounded operator on $C(X)$. The parametric extension P' of P is defined by

$$(P'f)(x, y) = (P\, {}^y f)(x) \tag{3}$$

If Q is a linear and bounded operator on $C(Y)$, then the parametric extension Q'' of Q is defined by

$$(Q''f)(x, y) = (Q f^x)(y) \tag{4}$$

Proposition 1. *Let $r \in \mathbb{N}$, $P_1, \ldots, P_r$ univariate interpolation projectors on $C(X)$ and $Q_1, \ldots, Q_r$ univariate interpolation projectors on $C(Y)$. Let $P_1', \ldots, P_r'$, $Q_1'', \ldots, Q_r''$ be the corresponding parametric extension. We assume that*

$$P_1 \leq P_2 \leq \cdots \leq P_r,\ Q_1 \leq Q_2 \leq \cdots \leq Q_r \tag{5}$$

Then

$$B_r = P_1' Q_r'' \oplus P_2' Q_{r-1}'' \oplus \cdots \oplus P_r' Q_1'' \tag{6}$$

is projector and it has representation

$$B_r = \sum_{m=1}^{r} P_m' Q_{r+1-m}'' - \sum_{m=1}^{r-1} P_m' Q_{r-m}'' \tag{7}$$

Moreover, we have

$$B_r^C = P_r'^C + P_{r-1}'^C Q_1''^C + \cdots + P_1'^C Q_{r-1}''^C + Q_r''^C - (P_r'^C Q_1''^C + \cdots + P_1'^C Q_r''^C) \tag{8}$$

where $P^C = I - P$, I is identity operator.

For the proof of this proposition see [3].

Remark 2. *If $P_1, \ldots, P_r$ and $Q_1, \ldots, Q_r$ are Lagrange univariate operators which form the chains (i.e. satisfy the relation (5)) the operator B_r given by (6) is called Biermann interpolation projectors. In this article, we instead the Lagrange univariate operators by Shepard univariate operators.*

2. Main result

Let be the univariate interpolation projectors of Shepard type $P_1,\ldots,$ P_r, $Q_1,\ldots,$ Q_r which are given by relations

$$(P_m f)(x) = \sum_{i=1}^{k_m} A_{i,m}(x) f(x_i),\ 1 \le m \le r \tag{9}$$

$$(Q_n g)(y) = \sum_{j=1}^{l_n} \widetilde{A}_{j,n}(y) g(y_j),\ 1 \le n \le r$$

The interpolation points satisfy

$$\{x_1, \ldots, x_{k_m}\} \subseteq [a,b] \text{ and } \{y_1, \ldots, y_{l_n}\} \subseteq [c,d]$$

with

$$1 \le k_1 < k_2 < \cdots < k_r \text{ and } 1 \le l_1 < l_2 < \cdots < l_r \tag{10}$$

The cardinal functions are given by

$$A_{i,m}(x) = \frac{|x-x_i|^{-\mu}}{\sum_{k=1,\ k\neq i}^{k_m} |x-x_k|^{-\mu}},\ 1 \le i \le k_m \tag{11}$$

$$\widetilde{A}_{j,n}(y) = \frac{|y-y_j|^{-\mu}}{\sum_{l=1,\ l\neq j}^{l_n} |y-y_l|^{-\mu}},\ 1 \le j \le l_n$$

with $\mu \in \mathbb{R}$ and satisfy the relations

$$A_{i,m}(x_\nu) = \delta_{i\nu},\ i,\nu = \overline{1,k_m}$$

$$\widetilde{A}_{j,n}(y_\sigma) = \delta_{j\sigma},\ j,\sigma = \overline{1,l_n}$$

and

$$\sum_{i=1}^{k_m} A_{i,m}(x) = 1$$

$$\sum_{j=1}^{l_n} \widetilde{A}_{j,n}(y) = 1$$

Theorem 3. *The parametric extensions*

$$P_1', \ldots, P_r',\ Q_1'', \ldots, Q_r''$$

are bivariate interpolation projectors which form the chains

$$P'_1 \le P'_2 \le \cdots \le P'_r,\ Q''_1 \le Q''_2 \le \cdots \le Q''_r \tag{12}$$

Proof. Let be $1 \le m_1 \le m_2 \le r$. From (10) we have

$$k_{m_1} \le k_{m_2} \tag{13}$$

We have that

$$(P'_{m_1}P'_{m_2}f)(x,y) = \sum_{i_1=1}^{k_{m_1}} A_{i_1,m_1}(x) \sum_{i_2=1}^{k_{m_2}} A_{i_2,m_2}(x_{i_1}) f(x_{i_2},y) \tag{14}$$

But

$$A_{i_2,m_2}(x_{i_1}) = \delta_{i_2,i_1} \tag{15}$$

From (13), (14) and (15) we have that

$$(P'_{m_1}P'_{m_2}f)(x,y) = \sum_{i_1=1}^{k_{m_1}} A_{i_1,m_1}(x) f(x_{i_1},y) = (P'_{m_1}f)(x,y)$$

i.e. $P'_{m_1} \le P'_{m_2}$. Thus the projectors $P'_1, \ldots, P'_r$ form the chain. Analogous $Q''_1, Q''_2, \ldots, Q''_r$ are projectors which form a chain. □

We have that

$$P'_m Q''_n = Q''_n P'_m,\ 1 \le m, n \le r$$

and the tensor product projector has the representation:

$$(P'_m Q''_n f)(x,y) = \sum_{i=1}^{k_m} \sum_{j=1}^{l_n} A_{i,m}(x) \widetilde{A}_{j,n}(y) f(x_i, y_j)$$

with interpolation properties

$$(P'_m Q''_n f)(x_i, y_j) = f(x_i, y_j),\ 1 \le i \le k_m,\ 1 \le j \le l_n$$

The projectors $P'_1, \ldots, P'_r,\ Q''_1, \ldots, Q''_r$ generate a distributive lattice of projectors on $C([a,b] \times [c,d])$. A special element in this lattice is

$$B^S_r = P'_1 Q''_r \oplus P'_2 Q''_{r-1} \oplus \cdots \oplus P'_r Q''_1,\ r \in \mathbb{N} \tag{16}$$

called Biermann-Shepard projector and which has the interpolation properties

$$(B_r^S f)(x_i, y_j) = f(x_i, y_j),\ 1 \leq i \leq k_m,\ 1 \leq j \leq l_{r+1-m},\ 1 \leq m \leq r$$

The set of interpolation points of the Biermann-Shepard projector given by (16) has the disjoint representation

$$\mathcal{P}(B_r^S) = \bigcup_{m=1}^{r} \bigcup_{n=0}^{r-m} \{(x_i, y_j) : k_{m-1} < i < k_m,\ l_{r-m-n} < j \leq l_{r-m-n+1}\} \qquad (17)$$

with $k_0 = 0$, $l_0 = 0$. The number of interpolation points of Biermann-Shepard operator B_r^S given by (16) is

$$\left|\mathcal{P}(B_r^S)\right| = \sum_{m=1}^{r} k_m (l_{r+1-m} - l_{r-m})$$

with $l_0 = 0$.

Using the disjoint representation (17) of interpolation set we obtain the Lagrange representation of Biermann-Shepard interpolant

$$B_r(f) = \sum_{m=1}^{r} \sum_{n=0}^{r-m} \sum_{i=1+k_{m-1}}^{k_m} \sum_{j=1+l_{r-m-n}}^{l_{r+1-m-n}} f(x_i, y_j) S_{ij} \qquad (18)$$

The cardinal functions of Biermann-Shepard interpolation projector are given by

$$S_{ij}(x, y) = \sum_{s=m}^{m+n} A_{i,s}(x) \widetilde{A}_{j,r+1-s}(y) - \sum_{s=m}^{m+n-1} A_{i,S}(x) \widetilde{A}_{j,r-s}(y), \qquad (19)$$

with $k_{m-1} \leq i \leq k_m,\ l_{r-m-n} \leq j \leq l_{r+1-m-n},\ 0 \leq n \leq r-m,\ 1 \leq m \leq r.$

For the remainder term we can use formula (8) and integral representation of remainder [1]

$$(P_m^C f)(x) = \int_a^b \varphi_m(x, s) f'(s) \mathrm{d}s \qquad (20)$$

$$(Q_n^C g)(y) = \int_c^d \psi_n(t, y) g'(t)\, \mathrm{d}t$$

where

$$\varphi_m(x,s) = (x-s)_+^0 - \sum_{i=1}^{k_m} A_{i,m}(x)(x_i - s)_+^0$$

$$\psi_n(y,t) = (y-t)_+^0 - \sum_{j=1}^{l_n} \widetilde{A}_{j,n}(y)(y_j - t)_+^0$$

Also

$$\left|(P_m^C f)(x)\right| \leq H_m(x) M_1 f$$

$$\left|(Q_n^C g)(y)\right| \leq K_n(y) M_1 g$$

where

$$H_m(x) = x - \sum_{i=1}^{k_m} x_i A_i(x) + 2\sum_{i=1}^{k_m} A_i(x)(x_i - x)_+$$

$$K_n(y) = y - \sum_{j=1}^{l_n} y_j \widetilde{A_j}(y) + 2\sum_{j=1}^{l_n} \widetilde{A}_j(y)(y_j - y)_+$$

$$M_1 f = \sup_{a \leq x \leq b} |f'(x)|$$

$$M_1 g = \sup_{c \leq y \leq d} |g'(y)|$$

If $f \in C^{1,1}([a,b] \times [c,d])$ we have the following estimation for remainder term of Biermann-Shepard interpolant

$$\left|f(x,y) - B_r^S f(x,y)\right| \leq H_r(x)\left\|f^{(1,0)}\right\| + K_r(y)\left\|f^{(0,1)}\right\| + \sum_{i=1}^{r-1} H_{r-i}(x) K_i(y)\left\|f^{(1,1)}\right\|$$

$$+ \sum_{i=1}^{r} H_{r+1-i}(x) K_i(y)\left\|f^{(1,1)}\right\|$$

where $\left\|f^{(i,j)}\right\| = \max_{(x,y)\in[a,b]\times[c,d]} \left|f^{(i,j)}(x,y)\right|$.

3. **Examples**

Using relations (8) we determine the approximation order of Biermann-Shepard projector (16) for two particular case.

Example 1

Let be

$$k_m = 2^m + 1,\ 1 \le m \le r$$
$$l_n = 2^n + 1,\ 1 \le n \le r$$

and the univariate Shepard interpolation projectors on $[0, 1]$ with equidistant nodes

$$(P_m f)(x) = (S_{2^m,\mu} f)(x) = \frac{\sum_{k=0}^{2^m} f(\frac{k}{2^m}) \left| x - \frac{k}{2^m} \right|^{-\mu}}{\sum_{k=0}^{2^m} \left| x - \frac{k}{2^m} \right|^{-\mu}},\ 1 \le \mathrm{m} \le \mathrm{r} \tag{21}$$
$$(Q_n g)(y) = (S_{2^n,\mu} g)(y) = \frac{\sum_{j=0}^{2^n} g(\frac{j}{2^n}) \left| y - \frac{j}{2^n} \right|^{-\mu}}{\sum_{j=0}^{2^n} \left| y - \frac{j}{2^n} \right|^{-\mu}}, 1 \le \mathrm{n} \le \mathrm{r}$$

We have that the extension projectors form the chains

$$P_1' \le P_2' \le \cdots \le P_r',\ Q_1'' \le Q_2'' \le \cdots \le Q_r''$$

and we can define the Biermann-Shepard operator

$$B_r^S = P_1' Q_r'' \oplus P_2' Q_{r-1}'' \oplus \cdots \oplus P_r' Q_1'' \tag{22}$$

From [5], if ${}^y f \in Lip_{[0,1]} 1$, we have that

$$\|{}^y f - (S_{2^m,\mu} {}^y f)\| = \begin{cases} O(\frac{1}{2^m}) & \mu > 2 \\ O(\frac{m}{2^m}) & \mu = 2 \\ O(\frac{m}{2^{m(\mu-1)}}) & \mu \in (1,2) \\ O(\frac{1}{m}) & \mu = 1 \end{cases} \tag{23}$$

If $f^x \in Lip_{[0,1]} 1$ we obtain a analogous estimation for $\|f^x - S_{2^n,\mu} f^x\|$.

Theorem 4. *If $f \in Lip_{[0,1]} 1 \times Lip_{[0,1]} 1$, the approximation orders of the B_r^S interpolant given by (22) are*

$$\left\| f - B_r^S f \right\| = \begin{cases} O(\frac{r}{2^r}) & , \quad \mu > 2 \\ O(\frac{r^3}{2^r}) & , \quad \mu = 2 \\ O(\frac{r^3}{2^{r(\mu-1)}}) & , \quad \mu \in (1,2) \\ O(\frac{1}{r}) & , \quad \mu = 1 \end{cases}$$

Proof. From (8) we have

$$(B_r^S)^C = (S'_{2^r,\mu})^C + (S''_{2^r,\mu})^C +$$

$$+\sum_{m=1}^{r-1}(S'_{2^{r-m},\mu})^C(S''_{2^m,\mu})^C - \sum_{m=1}^{r}(S'_{2^{r+1-m},\mu})^C(S''_{2^m,\mu})^C$$

Taking into account (23) on obtain

- in the case $\mu > 2$

$$\begin{aligned}\left\|(B_r^S f)^C\right\| &\leq \frac{c}{2^r}+\frac{c}{2^r}+\sum_{m=1}^{r-1}\frac{c}{2^{r-m}}\cdot\frac{c}{2^m}+\sum_{m=1}^{r}\frac{c}{2^{r+1-m}}\cdot\frac{c}{2^m} = \\ &= O(\frac{r}{2^r}).\end{aligned}$$

- in the case $\mu = 2$

$$\begin{aligned}\left\|(B_r^S f)^C\right\| &\leq c\frac{r}{2^r}+c\frac{r}{2^r}+\sum_{m=1}^{r-1}\frac{c(r-m)}{2^{r-m}}\cdot\frac{cm}{2^m}+\sum_{m=1}^{r}\frac{c(r+1-m)}{2^{r+1-m}}\cdot\frac{cm}{2^m} = \\ &= O(\frac{r^3}{2^r}).\end{aligned}$$

- in the case $\mu \in (1,2)$

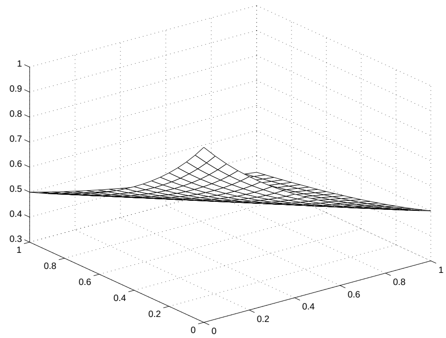
$$\begin{aligned}\left\|(B_r^S f)^C\right\| &\leq c\frac{r}{2^{r(\mu-1)}}+c\frac{r}{2^{r(\mu-1)}}+\sum_{m=1}^{r-1}\frac{c(r-m)}{2^{(r-m)(\mu-1)}}\cdot\frac{cm}{2^{m(\mu-1)}}+ \\ &+\sum_{m=1}^{r}\frac{c(r+1-m)}{2^{(r+1-m)(\mu-1)}}\cdot\frac{cm}{2^{m(\mu-1)}} \\ &= O(\frac{r^3}{2^{r(\mu-1)}}).\end{aligned}$$

- in the case $\mu = 1$

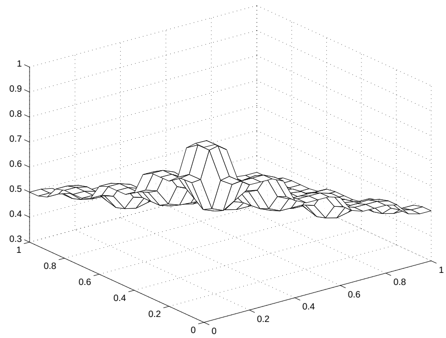
$$\begin{aligned}\left\|(B_r^S f)^C\right\| &\leq \frac{c}{r}+\frac{c}{r}+\sum_{m=1}^{r-1}\frac{c}{r-m}\cdot\frac{c}{m}+\sum_{m=1}^{r}\frac{c}{r+1-m}\cdot\frac{c}{m} = \\ &= O(\frac{1}{r}).\end{aligned}$$

□

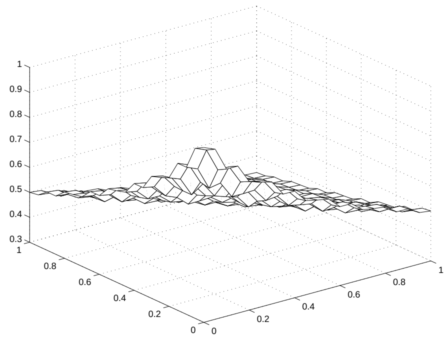
In Figure 1 we approximate the functions $f : [0,1]\times[0,1] \to R$ $f(x,y) = \frac{1}{1+x+y}$ by B_r^S for $\mu = 4$ and $r = 2,3,4$.



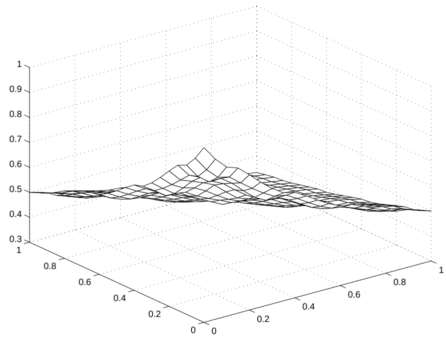
(a) The function f



(b) The approximant B_2^S



(c) The approximant B_3^S



(d) The approximant B_4^S

FIGURE 1. The graph of function $f(x,y) = 1/(1+x+y)$ and the graphs of B_r^S for $\mu = 4$ and $r = 2,3,4$

We have the estimations

r	$\lVert f - B_r^S f \rVert$	$\lvert \mathcal{P}(B_r^S f) \rvert$
2	0.0998	21
3	0.0531	49
4	0.0295	113
5	0.0167	257

Remark 5. *Under stronger restrictions on f, see* [5], [6]

$$f^{(1,0)}(0,y)=f^{(1,0)}(1,y),\ y\in[0,1] \tag{24}$$
$$f^{(0,1)}(x,0)=f^{(0,1)}(x,1),\ x\in[0,1]$$
$$f\in C^{1,1}([0,1]\times[0,1])$$

we have

$$\|^{y}f-(S_{2^m,2}{}^{y}f)\|=O(\frac{1}{2^m})$$
$$\|f^{x}-S_{2^n,2}f^{x})\|=O(\frac{1}{2^n})$$

which implies

$$\left\|f-B_r^Sf\right\|=O(\frac{r}{2^r}),\ \textit{for}\ \mu=2 \tag{25}$$

Remark 6. *The approximation orders of product operator $S'_{2^r,\mu}S''_{2^r,\mu}$ are*

$$\left\|f-(S'_{2^r,\mu}S''_{2^r,\mu}f)\right\|=\begin{cases} O(\frac{1}{2^r}) & \mu>2\\ O(\frac{r}{2^r}) & \mu=2\\ O(\frac{r}{2^{r(\mu-1)}}) & \mu\in(1,2)\\ O(\frac{1}{r}) & \mu=1\end{cases}$$

But,

$$\left|\mathcal{P}(B_r^S)\right|=2^r(r+3)+1$$
$$\left|\mathcal{P}(S'_{2^r,\mu}S''_{2^r,\mu})\right|=(2^r+1)^2$$

It follows the Biermann-Shepard operators B_r^S is more efficient that operator $S'_{2^r,\mu}S''_{2^r,\mu}$.

Example 2

Let be $r=2$ and

$$k_1=N+1,\ k_2=N^2+1$$
$$l_1=N+1,\ l_2=N^2+1$$

and the univariate Shepard interpolation projectors on $[0,1]$ with equidistant nodes

$$(P_m f)(x) = (S_{N^m,\mu} f)(x) = \frac{\sum_{k=0}^{N^m} f(\frac{k}{N^m}) \left|x - \frac{k}{N^m}\right|^{-\mu}}{\sum_{k=0}^{N^m} \left|x - \frac{k}{N^m}\right|^{-\mu}},\ m = 1,2 \tag{26}$$

$$(Q_n g)(y) = (S_{N^n,\mu} g)(y) = \frac{\sum_{j=0}^{N^n} g(\frac{j}{N^n}) \left|y - \frac{j}{N^n}\right|^{-\mu}}{\sum_{j=0}^{N^n} \left|y - \frac{j}{N^n}\right|^{-\mu}},\ n = 1,2$$

The Biermann-Shepard interpolation projector is given by

$$B_2^S = P_1' Q_2'' \oplus P_2' Q_1'' \tag{27}$$

If ${}^y f \in Lip_{[0,1]}1$, we have that (from [5])

$$\|{}^y f - (S_{N^m,\mu}{}^y f)\| = \begin{cases} O(\frac{1}{N^m}) & \mu > 2 \\ O(\frac{\log N}{N^m}) & \mu = 2 \\ O(\frac{1}{N^{m(\mu-1)}}) & \mu \in (1,2) \\ O(\frac{1}{\log N}) & \mu = 1 \end{cases} \quad m = 1.2 \tag{28}$$

If $f^x \in Lip_{[0,1]}1$ we obtain a analogous estimation for $\|f^x - S_{N^m,\mu} f^x\|$.

Theorem 7. *If $f \in Lip_{[0,1]}1 \times Lip_{[0,1]}1$, the approximation orders of the B_2^S interpolant given by (27) are*

$$\|f - B_2^S f\| = \begin{cases} O(\frac{1}{N^2}) & \mu > 2 \\ O(\frac{\log^2 N}{N^2}) & \mu = 2 \\ O(\frac{1}{N^{2(\mu-1)}}) & \mu \in (1,2) \\ O(\frac{1}{\log N}) & \mu = 1 \end{cases}$$

Proof. From (8) we have

$$(B_2^S)^C = (S'_{N^2,\mu})^C + (S''_{N^2,\mu})^C + (S'_{N,\mu})^C (S''_{N,\mu})^C$$

$$-(S'_{N^2,\mu})^C (S''_{N,\mu})^C - (S'_{N,\mu})^C (S''_{N^2,\mu})^C$$

Taking into account (28) on obtain

- in the case $\mu > 2$

$$\left\|(B_2^S f)^C f\right\| \leq \frac{c}{N^2} + \frac{c}{N^2} + \frac{c}{N} \cdot \frac{c}{N} + \frac{c}{N^2} \cdot \frac{c}{N} + \frac{c}{N} \cdot \frac{c}{N^2} = O(\frac{1}{N^2})$$

- in the case $\mu = 2$

$$\begin{aligned}\left\|(B_2^S f)^C f\right\| &\leq c\frac{\log N}{N^2} + c\frac{\log N}{N^2} + c\frac{\log N}{N} \cdot c\frac{\log N}{N} + \\ &+ c\frac{\log N}{N^2} \cdot c\frac{\log N}{N} + c\frac{\log N}{N} \cdot c\frac{\log N}{N^2} \\ &= O(\frac{\log^2 N}{N^2})\end{aligned}$$

- in the case $\mu \in (1, 2)$

$$\begin{aligned}\left\|(B_2^S f)^C f\right\| &\leq \frac{c}{N^{2(\mu-1)}} + \frac{c}{N^{2(\mu-1)}} + \frac{c}{N^{(\mu-1)}} \cdot \frac{c}{N^{(\mu-1)}} + \\ &+ \frac{c}{N^{2(\mu-1)}} \cdot \frac{c}{N^{(\mu-1)}} + \frac{c}{N^{(\mu-1)}} \cdot \frac{c}{N^{2(\mu-1)}} \\ &= O(\frac{1}{N^{2(\mu-1)}})\end{aligned}$$

- in the case $\mu = 1$

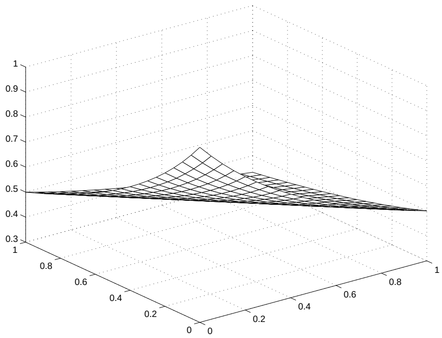
$$\begin{aligned}\left\|(B_2^S f)^C f\right\| &\leq \frac{c}{\log N} + \frac{c}{\log N} + \frac{c}{\log N} \cdot \frac{c}{\log N} + \\ &+ \frac{c}{\log N} \cdot \frac{c}{\log N} + \frac{c}{\log N} \cdot \frac{c}{\log N} \\ &= O(\frac{c}{\log N})\end{aligned}$$

□

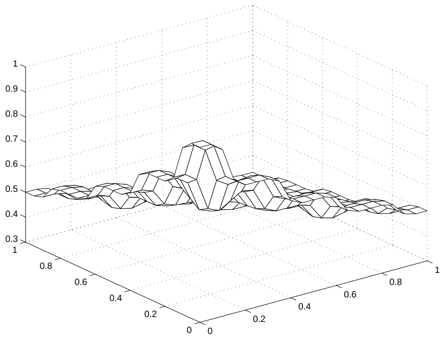
In Figure 2 we approximate the functions $f : [0, 1] \times [0, 1] \to R$ $f(x, y) = \frac{1}{1+x+y}$ by B_2^S for $\mu = 4$ and $N = 2, 3, 4$.

We have the following estimations

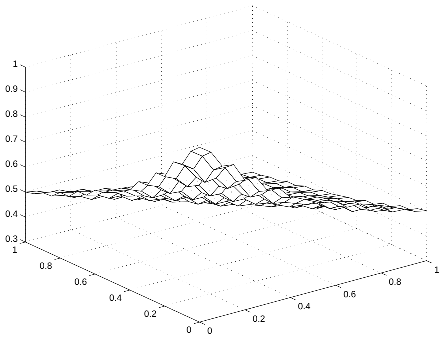
N	$\left\|f - B_2^S f\right\|$	$\left|\mathcal{P}(B_2^S f)\right|$
2	0.0998	21
3	0.0365	64
4	0.0283	145
5	0.0135	276



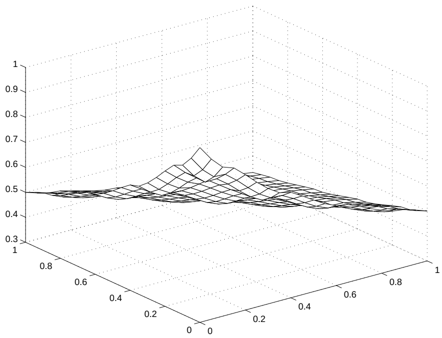
(a) The function f



(b) The approximant B_2^S, $N = 2$



(c) The approximant B_2^S, $N = 3$



(d) The approximant B_2^S, $N = 4$

FIGURE 2. The graph of function $f(x, y) = 1/(1 + x + y)$ and the graphs of B_2^S for $\mu = 4$ and $N = 2, 3, 4$

Remark 8. *Under the same stronger restrictions on f given by (24) we have that*

$$\|{}^y f - (S_{N^m,2}{}^y f)\| = O(\frac{1}{N^m})$$
$$\|f^x - (S_{N^n,2} f^x)\| = O(\frac{1}{N^n})$$

which implies

$$\left\|f - B_2^S f\right\| = O(\frac{1}{N^2}), \textit{ for } \mu = 2. \tag{29}$$

Remark 9. *The approximation orders of operator $S'_{N^2}S''_{N^2}$ are*

$$\|f-(S'_{N^2}S''_{N^2}f)\| = \begin{cases} O(\frac{1}{N^2}) & \mu>2 \\ O(\frac{\log N}{N^2}) & \mu=2 \\ O(\frac{1}{N^{2(\mu-1)}}) & \mu\in(1,2) \\ O(\frac{1}{\log N}) & \mu=1 \end{cases} \tag{30}$$

But,

$$\left|\mathcal{P}(S'_{N,\mu}S''_{N^2,\mu}\oplus S'_{N^2,\mu}S''_{N,\mu})\right| = 2N^3+N^2+1$$

$$\left|\mathcal{P}(S'_{N^2,\mu}S''_{N^2,\mu})\right| = (N^2+1)^2$$

It follows that the Biermann-Shepard operator B_r^S given by (27) is more efficient that operator $S'_{N^2}S''_{N^2}$.

References

[1] Coman, Gh., *The remainder of certain Shepard type interpolation formulas*, Studia Univ. "Babeş-Bolyai" Mathematica, XXXII, **4**(1987), 24-32.

[2] Delvos, F.-S., Posdorf, H., *Generalized Biermann interpolation,* Resultate Math, **5**(1982), no.1, 6-18.

[3] Delvos, F.-S., Schempp, W., *Boolean methods in interpolation and approximation*, Pitman Research Notes in Mathematics, serie 230, New York, 1989.

[4] Gordon, W.J., Hall C.A., *Transfinite element methods blending functions interpolation over arbitrary curved element domains*, Numer. Math., **21**(1973/1974), 109-129.

[5] Szabados, J., *Direct and converse approximation theorems for the Shepard operators*, J. Approx. Theory and Appl. **7**(1991), 63-76.

[6] Vecchia Della, B., Mastroianni, G., Totik, V., *Saturation of Shepard operators*, J. Approx. Theory and Appl., **6**(1990), 76-84.

[7] Vecchia Della, B., Mastroianni, G., *On function approximation by Shepard type operators - a survey,* Approximation theory, wavelets and applications, Maratea, 1994, (Dordrecht), Ser. C, Math. Phys. Sci., vol. 454, NATO Adv. Sci. Inst., Kluwer Acad. Publ., 1995,. 335-346.

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ON THE MORITA INVARIANCE OF THE HOCHSCHILD HOMOLOGY OF SUPERALGEBRAS

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Abstract. We provide a direct proof that the Hochschild homology of a $\mathbb{Z}_2$-graded algebra is Morita invariant.

1. Introduction

The goal of this paper is to show that if R is an arbitrary superalgebra (i.e. $\mathbb{Z}_2$-graded algebra) while $M_{p,q}(R)$ is the (super)algebra of (p,q)-supermatrices over R, then the two algebras have the same Hochschild homology (in the $\mathbb{Z}_2$-graded sense, see (Kassel, 1986)). This, naturally, suggest the idea of introducing the notion of Morita equivalence between two superrings and of proving that, in general, the Hochschild homology for superalgebras should be Morita invariant. We should discuss this issues at the end of the paper.

2. The Hochschild homology of superalgebras

The Hochschild complex for superalgebras (Kassel, 1986), is very similar to the analogous complex for ungraded case. Namely, the chain groups are, as in the classical case, $C_m(R) = R^{\otimes m+1}$, where, of course, the tensor product should be understood in the graded sense, while the face maps and degeneracies are given by

$$\delta_i^m(a_0 \otimes \cdots \otimes a_m) = a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \dots a_n, \quad \text{if} \quad 0 \leq i < m, \tag{1}$$

$$\delta_m^m(a_0 \otimes \cdots \otimes a_m) = (-1)^{|a_m|(|a_0|+\cdots+|a_{m-1})} a_m a + 0 \otimes a_1 \otimes \cdots \otimes a_{m-1}, \tag{2}$$

$$s_i^m(a_0 \otimes \cdots \otimes a_m) = a_0 \otimes \cdots \otimes a_i \otimes 1 \otimes a_{i+1} \otimes \cdots \otimes a_m, \quad 0 \leq i \leq m. \tag{3}$$

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Now the differential is defined in the usual way, meaning $d^m : C_m(R) \to C_{m-1}(R)$,

$$d^m = \sum_{i=0}^{m} (-1)^i \delta_i^m. \tag{4}$$

and the Hochschild homology of the superalgebra is just the homology of the complex $(C(R), d)$. In particular, it is easy to see that for any superalgebra R we have

$$H_0(R) = R/\{R, R\}, \tag{5}$$

where $\{R, R\}$ is the subspace generated by the supercommutators. of that element.

3. **The Morita invariance**

We shall simply give the definition of the Morita equivalence here. For a detailed approach, see for, instance, the book of Bass ([1]). The definition is completely analogous to that from the ungraded case.

Definition 1. If A and B are two unital, associative superalgebras over a graded commutative superring R, then A and B are said to be *Morita equivalent* if there exists an $A-B$-bimodule P and a $B-A$-bimodule Q such that $P \otimes_B Q \simeq A$ (as $A-A$-bimodules), while $Q \otimes_A P \simeq B$ (as $B-B$-bimodulea). The tensor products should be taken in the graded sense.

Theorem 1. *Let R be a commutative superring and A and B – two unital R-superalgebras (not necessarily commutative). Let, also, P be an $A-B$-bimodule which is projective over both rings and Q – an arbitrary $B-A$-bimodule. Then there is an isomorphism*

$$F_* : H_*(A, P \otimes_B Q) \to H_*(B, Q \otimes_A P),$$

which is functorial in the 4-tuple $(A, B; P, Q)$.

Before actually proving the theorem, let us, first, prove a technical lemma.

Lemma 1. *Let A be a unital, associative superalgebra over a commutative superring. If M is an arbitrary left A-module, while Q is a* projective *right A-module, then*

$$H_n(A, M \otimes Q) = \begin{cases} Q \otimes_A M & \text{if}\, n = 0 \\ 0 & \text{if}\, n \geq 1 \end{cases}.$$

Dually, if N is a right A-module, while P is a projective *left A-module, then*

$$H_n(A, P \otimes N) = \begin{cases} N \otimes_A P & \text{if } n = 0 \\ 0 & \text{if } n \geq 1 \end{cases}.$$

Proof. We shall assume, first, that $Q = A$, which is, clearly, projective, when regarded as right A-module. Moreover, in this case we have $A \otimes_A M \cong M$, so what we have to prove is that

$$H_n(A, M \otimes A) = \begin{cases} M & \text{if } n = 0 \\ 0 & \text{if } n \geq 1 \end{cases}.$$

It is easily seen, however, that the standard complex for computing the Hochschild homology of A with coefficients in the module $M \otimes A$ is, essentially, the (unnormalized) bar resolution β of the M, which has non-vanishing homology only in degree zero and the zero degree homology is M.

To prove now the general case, take Q an arbitrary projective right A-module. Then the functor $Q \otimes_A -$ is exact and the result follows from the isomorphism $(M \otimes Q) \otimes A^n \cong Q \otimes_A (M \otimes A \otimes A^n)$ established by the maps

$$f : (M \otimes Q) \otimes A^n \to Q \otimes_A (M \otimes A \otimes A^n),$$

$$f((m \otimes q) \otimes (a_1 \otimes \cdots \otimes a_n)) = (-1)^{|m||q|} q \otimes (m \otimes 1 \otimes a_1 \otimes \cdots \otimes a_n)$$

and

$$g : Q \otimes_A (M \otimes A^{n+1}) \to (M \otimes Q) \otimes A^n,$$

$$g(q \otimes (m \otimes a_0 \otimes \cdots \otimes a_n)) = (-1)^{|m||q|} (m \otimes q) \otimes a_0 a_1 \otimes a_2 \otimes \cdots \otimes a_n.$$

The proof of the second part of the lemma is completely similar. □

Proof of the theorem 1. We consider the following family of modules and maps: $(C_{p,q}, d', d'')$, where

$$C_{m,n} = P \otimes B^n \otimes Q \otimes A^m,$$

where

$$B^n = \underbrace{B \otimes B \otimes \cdots \otimes B}_{n \text{ factors}}$$

and

$$A^m = \underbrace{A \otimes A \otimes \cdots \otimes A}_{m \text{ factors}},$$

and all the tensor products are considered over the ground superring R. Before defining the maps d' and d'', several remarks are in order.

First of all, it is very clear that

$$C_{m,n} = C_m(A, P \otimes B^n \otimes Q),$$

i.e. $C_{m,n}$ is the group of the Hochschild m-chains of the superalgebra A, with the coefficients in the A-bimodule $P \otimes B^n \otimes Q$. On the other hand, up to a cyclic permutation of the factors in the tensor product, $C_{m,n}$ is, also, the group of the Hochschild n-chains of the superalgebra B with coefficients in a $B - B$-bimodule. More specifically, we have

$$C_{m,n} = \omega_{m+1,n+1}\left(C_n(B, Q \otimes A^m \otimes P)\right),$$

where $\omega_{m+1,n+1} : Q \otimes A^m \otimes P \otimes B^n \to P \otimes B^n \otimes Q \otimes A^m$ is the cyclic permutation of factors given by

$$\omega_{m+1,n+1}(p \otimes b_1 \otimes \cdots \otimes b_n \otimes q \otimes a_1 \otimes \cdots \otimes a_m) =$$
$$= (-1)^{|p|+|q|+\sum\limits_{i=1}^{m}|a_i|+\sum\limits_{j=1}^{n}|b_j|} q \otimes a_1 \otimes \cdots \otimes a_m \otimes p \otimes b_1 \otimes \cdots \otimes b_n.$$

Now we can use the Hochschild differentials to build the maps d' and d''. Let $m, n \in \mathbb{N}$ two given natural numbers. We define now, for any pair of natural numbers, $m, n \in \mathbb{N}$, $d'_{m,n} : C_{m,n} \to C_{m-1,n}$ to be the Hochschild differential for A, with coefficients in $P \otimes B^n \otimes Q$. Thus, on the columns we have Hochschild complexes. On the other hand, also for any pair of natural numbers m, n we define the horizontal differentials $d''_{m,n} : C_{m,n} \to C_{m,n-1}$,

$$d''_{m,n} = (-1)^m b_{m,n} \circ \omega_{m+1,n+1},$$

where $b_{m,n} : C_n(B, Q \otimes A^m \otimes P) \to C_{n-1}(B, Q \otimes A^m \otimes P)$ is the Hochschild differential. From the construction, it is obvious that both d' and d'' are differentials. We will prove now that they anticommute. We have

$$d''d'(p\otimes b_1\otimes\cdots\otimes b_n\otimes q\otimes a_1\otimes\cdots\otimes a_n)=d''\Big(p\otimes b_1\otimes\cdots\otimes b_n\otimes qa_1\otimes a_2\otimes\cdots\otimes a_m+$$

$$+\sum_{i=1}^{m-1}(-1)^i p\otimes b_1\otimes\cdots\otimes b_n\otimes q\otimes a_1\otimes\cdots\otimes a_ia_{i+1}\otimes\cdots\otimes a_m+$$

$$+(-1)^{m+|a_m|\left(|p|+|q|+\sum_{j=1}^{m-1}|a_j|+\sum_{j=1}^{n}|b_j|\right)}a_mp\otimes b_1\otimes\cdots\otimes b_n\otimes q\otimes a+1\otimes\cdots\otimes a_{m-1}\Big)=$$

$$=(-1)^m\Big[pb_1\otimes\cdots\otimes b_n\otimes qa_1\otimes a_2\otimes\cdots\otimes a_m+$$

$$+\sum_{j=1}^{n-1}p\otimes b_1\otimes\cdots\otimes b_jb_{j+1}\otimes\cdots\otimes b_n\otimes qa_1\otimes a_2\otimes\cdots\otimes a_m+$$

$$+(-1)^{n+|b_n|\left(|p|+|q|+\sum_{j=1}^{m}|a_j|+\sum_{j=1}^{n-1}|b_j|\right)}p\otimes b_1\otimes\cdots\otimes b_{n-1}\otimes b_nqa_1\otimes a_2\otimes\cdots\otimes a_m+$$

$$+\sum_{i=1}^{m-1}(-1)^i\Big(pb_1\otimes b_2\otimes\cdots\otimes b_n\otimes q\otimes a_1\otimes\ldots a_ia_{i+1}\otimes\cdots\otimes a_m+$$

$$+(-1)^{n+|b_n|\left(|p|+|q|+\sum_{j=1}^{m}|a_j|+\sum_{j=1}^{n-1}|b_j|\right)}p\otimes b_1\otimes\cdots\otimes b_{n-1}\otimes b_nq\otimes a_1\otimes\cdots\otimes a_ia_{i+1}\otimes\cdots\otimes a_m\Big)$$

$$+(-1)^{m+|a_m|\left(|p|+|q|+\sum_{j=1}^{m-1}|a_j|+\sum_{j=1}^{n}|b_j|\right)}\Big(a_mpb_1\otimes b_1\otimes\cdots\otimes b_n\otimes q\otimes a_1\otimes\cdots\otimes a_{m-1}+$$

$$+\sum_{j=1}^{n-1}(-1)^j a_mp\otimes b_1\otimes\ldots b_jb_{j+1}\otimes\ldots b_n\otimes q\otimes a_1\otimes\cdots\otimes a_{m-1}+$$

$$+(-1)^{n+|b_n|\left(|p|+|q|+\sum_{j=1}^{m}|a_j|+\sum_{j=1}^{n-1}|b_j|\right)}a_mp\otimes b_1\otimes b_{n-1}\otimes b_nq\otimes a_1\otimes\cdots\otimes a_{m-1}\Big)\Big].$$

On the other hand,

$$d'd''(p\otimes b_1\otimes\ldots b_n\otimes q\otimes a_1\otimes\cdots\otimes a_m)=(-1)^{m-1}d'\Big(pb_1\otimes b_2\otimes\cdots\otimes b_n\otimes q\otimes a_1\otimes\cdots\otimes a_m+$$

$$+\sum_{i=1}^{n-1}(-1)^i p\otimes b_1\otimes\cdots\otimes b_ib_{i+1}\otimes\cdots\otimes b_n\otimes q\otimes a_1\otimes\cdots\otimes a_m+$$

$$+(-1)^{n+|b_n|\left(|p|+|q|+\sum_{j=1}^{m}|a_j|+\sum_{j=1}^{n-1}|b_j|\right)}p\otimes b_1\otimes\cdots\otimes b_{n-1}\otimes b_nq\otimes a_1\otimes\cdots\otimes a_m\Big)=$$

$$= (-1)^{m-1}\Bigg[pb_1 \otimes b_2 \otimes \dots \otimes b_n \otimes qa_1 \otimes a_2 \otimes \dots \otimes a_m +$$

$$+ \sum_{j=1}^{m-1} (-1)^j pb_1 \otimes b_2 \otimes \dots \otimes b_n \otimes a \otimes a_1 \otimes \dots \otimes a_j a_{j+1} \otimes \dots \otimes a_m +$$

$$+(-1)^{m+|a_m|\left(|p|+|q|+\sum\limits_{k=1}^{m-1}|a_k|+\sum\limits_{k=1}^{n}|b_k|\right)} a_m pb_1 \otimes b_2 \otimes \dots \otimes b_n \otimes q \otimes a_1 \otimes \dots \otimes a_m +$$

$$+ \sum_{i=1}^{n-1} (-1)^i \Bigg(p \otimes b_1 \otimes \dots \otimes b_i b_{i+1} \otimes \dots \otimes b_n \otimes q \otimes a_1 \otimes \dots \otimes a_m +$$

$$+ \sum_{j=1} (-1)^j p \otimes b_1 \otimes \dots \otimes b_i b_{i+1} \otimes \dots \otimes b_n \otimes q \otimes a_1 \otimes \dots \otimes a_j a_{j+1} \otimes \dots \otimes a_m +$$

$$+(-1)^{m+|a_m|\left(|p|+|q|+\sum\limits_{j=1}^{m-1}|a_j|+\sum\limits_{j=1}^{n}|b_j|\right)} a_m p \otimes b_1 \otimes \dots \otimes bib_{i+1} \otimes \dots \otimes b_n \otimes q \otimes a_1 \otimes \dots \otimes a_{m-1} \Bigg)$$

$$+(-1)^{n+|b_n|\left(|p|+|q|+\sum\limits_{j=1}^{m}|a_j|+\sum\limits_{j=1}^{n-1}|b_j|\right)} \Bigg(p \otimes b_1 \otimes \dots b_{n-1} \otimes b_n q a_1 \otimes a_2 \otimes \dots \otimes a_m +$$

$$+ \sum_{j=1}^{m-1} (-1)^j p \otimes b_1 \otimes \dots \otimes b_{n-1} \otimes b_n q \otimes a_1 \otimes \dots \otimes a_j a_{j+1} \otimes \dots \otimes a_m +$$

$$+(-1)^{m+|a_m|\left(|p|+|q|+\sum\limits_{j=1}^{m-1}|a_j|+\sum\limits_{j=1}^{n}|b_j|\right)} a_m p \otimes b_1 \otimes \dots \otimes b_{n-1} \otimes b_n q \otimes a_1 \otimes \dots \otimes a_{m-1} \Bigg)\Bigg].$$

An inspection shows immediately that the quantities between the square brackets in the expressions of $d'd''$ and $d''d'$ coincide, while the signs in front of these brackets are opposite, which means that we have

$$d'd'' + d''d' = 0.$$

Thus, as we saw previously that ${d'}^2 = {d''}^2 = 0$, it follows that the family of modules and morphisms $(C_{m,n}, d', d'')_{m,n\in\mathbb{N}}$ is a *double complex* of modules. We consider now its total complex, given, for any $n \geq 0$, by

$$Tot_n = \bigoplus_{p+q=n} C_{p,q}$$

and

$$d_n : Tot_n \to Tot_{n-1}, \quad , d_n = \sum_{p+q=n} (d'_{p,q} + d''_{p,q}).$$

As it is well-known (see [3], from where the notations, classical, in fact, are taken), the total complex has two canonical filtrations (a horizontal and a vertical one) and to each of this filtration we can associate a spectral sequence. The two spectral sequences both converge to the homology of the total sequence. We shall show that in our case both spectral sequences collapse at the second step. In fact, the second order terms of the two sequences are

$$^{I}E^2_{p,q} = H'_p H''_{p,q}(C)$$

and

$$^{II}E^2_{p,q} = H''_p H'_{q,p}(C).$$

In our particular case, due to the particular form of the vertical and horizontal complexes, we get

$$H''_{p,q}(C) = H_q(B, Q \otimes A^p \otimes P) \tag{6}$$

and

$$H'_{q,p}(C) = H_q(A, P \otimes B^p \otimes Q). \tag{7}$$

As P is a bimodule which is projective at both sides, applying the previous lemma, we can write

$$H''_{p,q}(C) = \begin{cases} P \otimes_B Q \otimes A^p & \text{for } \ q = 0 \\ 0 & \text{for } \ q \geq 1 \end{cases}$$

$$H'_{q,p}(C) = \begin{cases} B^p \otimes_A Q \otimes P & \text{for } \ q = 0 \\ 0 & \text{for } \ q \geq 1 \end{cases}$$

As a consequence, we obtain for the second terms of the two spectral sequences:

$$^{I}E^2_{p,q} = \begin{cases} H_p(A, P \otimes_B Q) & \text{for } \ q = 0 \\ 0 & \text{for } \ q \geq 1 \end{cases}$$

$$^{II}E^2_{p,q} = \begin{cases} H_p(B, Q \otimes_A P) & \text{for } \ q = 0 \\ 0 & \text{for } \ q \geq 1 \end{cases}$$

Since, as we see, the two spectral sequences collapse, their limits coincide, in fact, with the second terms. Therefore, as they should converge to the same limit (the homology of the total complex), we have, in particular, that, for any $n \geq 0$, we should have

$$^{I}E^{2}_{n,0} = ^{II}E^{2}_{n,0},$$

i.e.

$$H_n(A, P \otimes_B Q) = H_n(B, Q \otimes_A P)$$

which concludes the proof (the functoriality follows from the way we constructed the double complex). □

Corollary. *If A and B are Morita equivalent superalgebras, then they have isomorphic Hochschild homologies.*

References

[1] Bass, H., *Algebraic K-Theory*, Benjamin-Cummings, 1968.

[2] Blaga, P.A., *Relative derived functors and Hochschild (co)homology for superalgebras*, in D. Andrica, P. Blaga, Z. Kása şi F. Szenkovits (eds.), *Proceedings of "Bolyai 200" International Conference on Geometry and Topology*, Cluj University Press, 2003, 47-60.

[3] Rotman, J.J., *An Introduction to Homological Algebra*, Academic Press, 1979.

[4] Seibt, P., *Cyclic Homology of Algebras*, World Scientific, 1987.

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OPTIMAL QUADRATURE FORMULAS BASED ON THE φ-FUNCTION METHOD

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Abstract. In this survey paper it is studied the optimality in sense of Nikolski for some classes of quadrature formulas, using the method of φ-function. It is presented the one-to-one correspondence between φ-functions and the quadrature formulas. Also, there are given some examples of quadrature formulas which are optimal in sense of Nikolski with regard to the error.

1. **Introduction**

Let H be a linear space of real-valued functions, defined and integrable on a finite interval $[a,b]\subset\mathbb{R}$, and $S : H \to \mathbb{R}$ be the integration operator defined by

$$S(f) = \int_a^b f(x)dx.$$

Let

$$\Lambda = \{\lambda_i \,|\lambda_i : H \to \mathbb{R},\, i = 1, ..., n\}$$

be a set of linear functionals. For $f \in H$, one considers the quadrature formula

$$S(f) = Q_n(f) + R_n(f), \tag{1}$$

where

$$Q_n(f) = \sum_{i=1}^{n} A_i\lambda_i(f)$$

and $R_n(f)$ denotes the remainder term.

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Remark 1. *Usually,* $\lambda_i(f)$, $i = 1, .., n$ *are the values of the function* f *or of certain of its derivatives on the quadrature nodes from* $[a, b]$.

An important problem regarding the quadrature formulas is the optimality problem with respect to the error. In this paper it is studied the optimality in sense of Nikolski for some classes of quadrature formulas, using the one-to-one corespondence between φ-functions and quadrature formulas.

Definition 2. *The quadrature formula (1) is called optimal in the sense of Nikolski, in the space* H, *if*

$$F_n(H, A, X) = \sup_{f \in H} |R_n(f)|,$$

attains the minimum value with regard to A *and* X, *where* $A = (A_1, ..., A_n)$ *are the coefficients and* $X = (x_1, ..., x_n)$ *are the quadrature nodes.*

2. The method of $\varphi-$ function

Suppose that $f \in C^r[a, b]$ and for some given $n \in \mathbb{N}$ consider the nodes $a = x_0 < \ldots < x_n = b$. On each interval $[x_{k-1}, x_k]$, $k = 1, ..., n$, it is considered a function φ_k, $k = 1, ..., n$, with the property that

$$\varphi_k^{(r)} = 1, \quad k = 1, ..., n. \tag{2}$$

One defines the function φ as follows:

$$\varphi|_{[x_{k-1}, x_k]} = \varphi_k, \quad k = 1, ..., n, \tag{3}$$

i.e., the restriction of the function φ to the interval $[x_{k-1}, x_k]$ is φ_k. Based on the additivity property of the defined integral and on the relations (2), we have

$$S(f) := \int_a^b f(x)dx = \sum_{k=1}^{n} \int_{x_{k-1}}^{x_k} \varphi_k^{(r)}(x) f(x) dx.$$

Using the integration by parts, one obtains

$$S(f) = \sum_{k=1}^{n} \Bigg\{ \left[\varphi_k^{(r-1)}(x)f(x) - \varphi_k^{(r-2)}(x)f'(x) + ... + (-1)^{r-1}\varphi_k(x)f^{(r-1)}(x)\right]\Big|_{x_{k-1}}^{x_k} \tag{4}$$

$$+ (-1)^r \int_{x_{k-1}}^{x_k} \varphi_k(x)f^{(r)}(x)dx \Bigg\}$$

$$= -\varphi_1^{(r-1)}(x_0)f(x_0) + \left[\varphi_1^{(r-1)}(x_1) - \varphi_2^{(r-1)}(x_1)\right] f(x_1) + ...+$$

$$+ \left[\varphi_{n-1}^{(r-1)}(x_{n-1}) - \varphi_n^{(r-1)}(x_{n-1})\right] f(x_{n-1}) + \varphi_n^{(r-1)}(x_n)f(x_n)-$$

$$- \Big\{-\varphi_1^{(r-2)}(x_0)f'(x_0) + \left[\varphi_1^{(r-2)}(x_1) - \varphi_2^{(r-2)}(x_1)\right] f'(x_1) + ...+$$

$$+ \left[\varphi_{n-1}^{(r-2)}(x_{n-1}) - \varphi_n^{(r-2)}(x_{n-1})\right] f'(x_{n-1}) + \varphi_n^{(r-2)}(x_n)f'(x_n)\Big\} +$$

$$+ ...+$$

$$+ (-1)^{r-1} \Big\{-\varphi_1(x_0)f^{(r-1)}(x_0) + \left[\varphi_1(x_1) - \varphi_2(x_1)\right] f^{(r-1)}(x_1) + ...+$$

$$+ \left[\varphi_{n-1}(x_{n-1}) - \varphi_n(x_{n-1})\right] f^{(r-1)}(x_{n-1}) + \varphi_n(x_n)f^{(r-1)}(x_n)\Big\}$$

$$+ (-1)^r \int_a^b \varphi(x)f^{(r)}(x)dx.$$

For

$$A_{0j} = (-1)^{j+1}\varphi_1^{(r-j-1)}(x_0), \tag{5}$$

$$A_{kj} = (-1)^j(\varphi_k - \varphi_{k+1})^{(r-j-1)}(x_k), \quad k = 1, ..., n-1,$$

$$A_{nj} = (-1)^j\varphi_n^{(r-j-1)}(x_n), \quad j = 0, 1, ..., r-1,$$

relation (4) becomes

$$\int_a^b f(x)dx = \sum_{k=0}^{n}\sum_{j=0}^{r-1} A_{kj}f^{(j)}(x_k) + R_n(f), \tag{6}$$

with

$$R_n(f) = (-1)^r \int_a^b \varphi(x)f^{(r)}(x)dx. \tag{7}$$

Remark 3. *Knowing the function φ, one can find the coefficients A_{kj}, $k = 0, ..., n$, $j = 0, ..., r-1$, and the nodes x_k, $k = 1, ..., n-1$, based on the relations (5). This method of constructing the quadrature formulas is called the φ-function method* [10].

Remark 4. *From (7) it follows that the degree of exactness of the quadrature formula (6) is at least* $r-1$.

3. The one-to-one corespondence between $\varphi-$ functions and quadrature formulas

First of all, one remarks that to a function φ, which satisfies (3) and (2), corresponds the quadrature formula (6).

Conversely, let us consider the quadrature formula (6), which has the degree of exactness $r-1$. By Peano's theorem it follows that

$$R_n(f) = \int_a^b R_n^t\left[\tfrac{(t-x)_+^{r-1}}{(r-1)!}\right] f^{(r)}(x)dx,$$

where

$$R_n^t\left[\tfrac{(t-x)_+^{r-1}}{(r-1)!}\right] = \tfrac{(x_n-x)_+^r}{r!} - \sum_{k=0}^{n}\sum_{j=0}^{r-1} A_{kj}\tfrac{(x_k-x)_+^{r-j-1}}{(r-j-1)!}.$$

So,

$$(-1)^r R_n^t\left[\tfrac{(t-x)_+^{r-1}}{(r-1)!}\right] = \tfrac{(x-x_n)_+^r}{r!} + (-1)^{r+1}\sum_{k=0}^{n}\sum_{j=0}^{r-1} A_{kj}\tfrac{(x_k-x)_+^{r-j-1}}{(r-j-1)!},$$

i.e,

$$(-1)^r R_n^t\left[\tfrac{(t-x)_+^{r-1}}{(r-1)!}\right] = \varphi(x).$$

If

$$\varphi_i = \varphi|_{[x_{i-1},x_i]}, \quad i = 1, ..., n,$$

then

$$\varphi_i(x) = \tfrac{(x-x_n)_+^r}{r!} + (-1)^{r+1}\sum_{k=i}^{n}\sum_{j=0}^{r-1} A_{kj}\tfrac{(x_k-x)_+^{r-j-1}}{(r-j-1)!},$$

$$\varphi_{i+1}(x) = \tfrac{(x-x_n)_+^r}{r!} + (-1)^{r+1}\sum_{k=i+1}^{n}\sum_{j=0}^{r-1} A_{kj}\tfrac{(x_k-x)_+^{r-j-1}}{(r-j-1)!},$$

and we get that

$$(\varphi_i - \varphi_{i+1})(x) = (-1)^{r+1}\sum_{j=0}^{r-1} A_{ij}\tfrac{(x_i-x)_+^{r-j-1}}{(r-j-1)!}.$$

Further,

$$(\varphi_i - \varphi_{i+1})^{(r-\nu-1)}(x) = (-1)^\nu \sum_{j=0}^{r-1} A_{ij} \frac{(x_i - x)_+^{(\nu-j)}}{(\nu-j)!},$$

$$(\varphi_i - \varphi_{i+1})^{(r-\nu-1)}(x_i) = (-1)^\nu A_{i\nu}.$$

It follows that

$$A_{0\nu} = (-1)^{\nu+1} \varphi_1^{(r-\nu-1)}(x_0),$$

$$A_{i\nu} = (-1)^\nu (\varphi_i - \varphi_{i+1})^{(r-\nu-1)}(x_i), \quad i = 1, ..., n-1,$$

$$A_{n\nu} = \varphi_n^{(r-\nu-1)}(x_n), \quad \nu = 0, 1, ..., r-1.$$

So, the correspondence is proved.

4. **The optimality problem**

We consider $H^{m,2}[a,b]$, $m \in \mathbb{N}$, the space of functions f in C^{m-1}, with the $m-1$th derivative absolute continuous on $[a,b]$ and with f^m in $L^2[a,b]$. Suppose that $f \in H^{m,2}[a,b]$, $m \in \mathbb{N}$. From (7) one obtains

$$|R_n(f)| \leq \left\| f^{(m)} \right\|_2 \left(\int_a^b \varphi^2(x) dx \right)^{1/2}.$$

So, the optimal quadrature formula of the form (6) is determined by the parameters A and X for which

$$F(A, X) = \int_a^b \varphi^2(x) dx = \sum_{k=1}^n \int_{x_{k-1}}^{x_k} \varphi_k^2(x) dx$$

attains the minimum value.

Remark 5. *Taking into account the property of minimal $L_w^2[a,b]-$norm, (w is a weight function), of the orthogonal polynomials, the function $F(A,X)$ takes the minimal value when φ_k is the orthogonal polynomial on $[x_{k-1}, x_k]$, $k = 1, \ldots, n$, with regard to the weight w.*

For example, if $w = 1$ the corresponding orthogonal polynomial on $[a, b]$ is the Legendre polynomial

$$l_r(x) = \frac{d^r}{dx^r}\left[(x-a)^r(y-b)^r\right].$$

It means that the parameters of the optimal quadrature formula can be obtained by identifying the functions $\varphi_k = \varphi|_{[x_{k-1},x_k]}$ with the corresponding orthogonal polynomials on $[x_{k-1}, x_k]$, $k = 1, ..., n$.

Example 6. *One considers the quadrature formula*

$$\int_0^1 f(x)dx = \sum_{k=0}^{n} A_k f(x_k) + R_n(f), \tag{8}$$

obtained from (6) for $r = 1$, with

$$R_n(f) = \int_0^1 \varphi(x) f'(x)dx.$$

Theorem 7. *For $f \in H^{1,2}[0,1]$, the quadrature formula of the form (8), optimal with regard to the error, is*

$$\int_0^1 f(x)dx = \frac{1}{2n}\left[f(0) + 2\sum_{i=1}^{n-1} f(\frac{i}{n}) + f(1)\right] + R_n^*(f),$$

with

$$|R_n^*(f)| \le \frac{1}{2n\sqrt{3}}\left\|f'\right\|_2.$$

Proof. Relations (5) become

$$\begin{aligned} A_0 &= -\varphi_1(0), \\ A_k &= \varphi_k(x_k) - \varphi_{k+1}(x_k), \quad k = 1, ..., n-1, \\ A_n &= \varphi_n(1), \end{aligned} \tag{9}$$

and from (2) we get

$$\varphi_k' = 1, \ \ k = 1, ..., n. \tag{10}$$

From (9) and (10) it follows

$$\varphi_1(x) = x - A_0,$$

$$\varphi_2(x) = x - A_0 - A_1,$$

$$\ldots$$

$$\varphi_k(x) = x - A_0 - A_1 - \ldots - A_{k-1},$$

$$\ldots$$

$$\varphi_n(x) = x - A_0 - A_1 - \ldots - A_{n-2} - A_{n-1}.$$

As the quadrature formula (8) has the degree of exactness zero, i.e., $R_n(e_0) = 0$ ($e_0(x) = 1$) we have

$$A_0 + \ldots + A_n = 1.$$

It follows that for φ_n we have

$$\varphi_n(x) = x - 1 + A_n.$$

Now, the optimal coefficients A_k, $k = 0, \ldots, n$ and the optimal nodes x_k, $k = 1, \ldots, n-1$ are obtained by minimizing the functions

$$F_1(A, X) = \int_0^1 \varphi^2(x)dx = \sum_{k=1}^{n} \int_{x_{k-1}}^{x_k} \varphi_k^2(x)dx.$$

But, $\int_{x_{k-1}}^{x_k} \varphi_k^2(x)dx$ takes its minimum value for $\varphi_k \equiv l_1$, the Legendre polynomial of degree one, on the interval $[x_{k-1}, x_k]$, i.e.,

$$\varphi_k(x) = x - \frac{x_{k-1} + x_k}{2}$$

and

$$\int_{x_{k-1}}^{x_k} \varphi_k^2(x)dx = \frac{(x_k - x_{k-1})^3}{12}.$$

It follows that

$$\sum_{i=0}^{k-1} A_i = \frac{x_{k-1} + x_k}{2} \tag{11}$$

and

$$\int_0^1 \varphi^2(x)dx = \frac{1}{12} \sum_{k=1}^{n} (x_k - x_{k-1})^3.$$

Hence,

$$\bar{F}_1(X) := \min_A F_1(A, X) = \frac{1}{12}\sum_{k=1}^{n}(x_k - x_{k-1})^3.$$

As

$$\frac{\partial \bar{F}_1(X)}{\partial x_k} = \frac{1}{4}\left[(x_k - x_{k-1})^2 - (x_{k+1} - x_k)^2\right],$$

the optimal nodes constitute the solution of the system

$$x_k - x_{k-1} = x_{k+1} - x_k, \quad k = 1, ..., n-1,$$

with

$$x_0 = 0, \quad x_n = 1,$$

i.e.,

$$x_k^* = \frac{k}{n}, \quad k = 0, ..., n \tag{12}$$

and

$$\bar{F}_1(X^*) = \frac{1}{12n^2}.$$

From (11) and (12) one obtains the optimal coefficients

$$A_0^* = \frac{1}{2n}$$
$$A_1^* = ... = A_{n-1}^* = \frac{1}{n}$$
$$A_n^* = \frac{1}{2n}.$$

Finally, we have

$$F_1(A^*, X^*) := \min_{A,X} F_1(A, X) = \frac{1}{12n^2},$$

and the proof follows. □

Example 8. *For $f \in H^{2,2}[0,1]$ one considers the quadrature formula of the form*

$$\int_0^1 f(x)dx = \sum_{k=0}^{n} A_k f(x_k) + R_n(f), \tag{13}$$

with $0 = x_0 < x_1 < ... < x_n = 1$.

Theorem 9. *For $f \in H^{2,2}[0,1]$, the quadrature formula of the form (13), optimal with regard to the error, is*

$$\int_0^1 f(x)dx = \sum_{k=0}^{n} A_k^* f(x_k^*) + R_n^*(f),$$

with

$$A_0^* = A_n^* = \frac{3}{4}\mu,$$

$$A_1^* = A_{n-1}^* = \frac{5+2\sqrt{6}}{4}\mu,$$

$$A_k^* = \sqrt{6}\mu, \quad k = 2, ..., n-2,$$

$$x_k^* = [2+(k-1)\sqrt{6}]\mu, \quad k = 1, ..., n-1,$$

and

$$|R_n^*(f)| \le \frac{\mu^2}{2\sqrt{5}} \left\| f'' \right\|_2,$$

where

$$\mu = \frac{1}{4+(n-2)\sqrt{6}}.$$

Proof. For $r = 2$ relation (4) becomes

$$\int_0^1 f(x)dx = -\varphi_1'(0)f(0) + \sum_{k=1}^{n-1}(\varphi_k' - \varphi_{k+1}')(x_k)f(x_k) + \varphi_n'(1)f(1) \qquad (14)$$

$$+ \varphi_1(0)f'(0) - \sum_{k=1}^{n-1}(\varphi_k - \varphi_{k+1})(x_k)f'(x_k) - \varphi_n(1)f'(1)$$

$$+ \int_0^1 \varphi(x)f''(x)dx.$$

Taking into account (13), we have

$$A_0 = -\varphi_1'(0),$$

$$A_k = (\varphi_k' - \varphi_{k+1}')(x_k), \quad k = 1, ...n-1,$$

$$A_n = \varphi_n'(1),$$

and

$$\varphi_1(0) = 0, \tag{15}$$

$$(\varphi_k - \varphi_{k+1})(x_k) = 0, \quad k = 1, ...n-1,$$

$$\varphi_n(1) = 0,$$

respectively,

$$R_n(f) = \int_0^1 \varphi(x) f''(x) dx. \tag{16}$$

Relation (2) becomes

$$\varphi_k'' = 1, \qquad k = 1, ..., n. \tag{17}$$

From (15) and (17) it follows that

$$\varphi_1(x) = \frac{x^2}{2} - A_0 x, \tag{18}$$

$$\varphi_k(x) = \frac{x^2}{2} - \sum_{j=0}^{k-1} A_k (x - x_k), \quad k = 2, ...n-1,$$

$$\varphi_n(x) = \frac{(1-x)^2}{2} - A_n(1-x).$$

By (16) one obtains

$$|R_n(f)| \leq \left(\int_0^1 \varphi^2(x) dx \right)^{1/2} \|f''\|_2 .$$

Next, the problem is to minimize the function

$$F_2(A, X) = \int_0^1 \varphi^2(x) dx$$
$$= \int_0^{x_1} \varphi_1^2(x) dx + \sum_{k=2}^{n-1} \int_{x_{k-1}}^{x_k} \varphi_k^2(x) dx + \int_{x_{n-1}}^1 \varphi_n^2(x) dx$$

with regard to the parameters $A = (A_0, ..., A_n)$ and $X = (x_1, ..., x_{n-1})$.

By (18) it follows that the integrals

$$\int_{x_{k-1}}^{x_k} \varphi_k^2(x) dx, \quad k = 2, ..., n-1,$$

attain the minimum values for

$$\varphi_k \equiv \frac{1}{2} \tilde{l}_{2,k}, \quad k = 2, ..., n-1, \tag{19}$$

where $\tilde{l}_{2,k}$ is the two degree Legendre polynomial on the interval $[x_{k-1}, x_k]$,

$$\tilde{l}_{2,k}(x) = x^2 - (x_{k-1} + x_k)x + \frac{1}{6}(x_{k-1}^2 + 4x_{k-1}x_k + x_k^2).$$

We have

$$\int_{x_{k-1}}^{x_k} \tilde{l}_{2,k}^2(x)dx = \frac{4}{45}\left(\frac{x_k - x_{k-1}}{2}\right)^5.$$

From (18) and (19), one obtains

$$\sum_{i=0}^{k-1} A_i = \frac{x_k + x_{k-1}}{2}, \quad k = 2, ..., n-1, \tag{20}$$

and, also, from

$$\frac{d}{dA_0}\left[\int_0^{x_1} (\frac{x^2}{2} - A_0 x)^2 dx\right] = 0,$$
$$\frac{d}{dA_n}\left\{\int_{x_{n-1}}^1 \left[\frac{(1-x)^2}{2} - A_n(1-x)\right]^2 dx\right\} = 0$$

it follows

$$A_0 = \frac{3}{8}x_1, \quad A_n = \frac{3}{8}(1 - x_{n-1}), \tag{21}$$

respectively,

$$\int_0^{x_1} (\frac{x^2}{2} - \frac{3}{8}x_1 x)^2 dx = \frac{1}{32}x_1^5$$
$$\int_{x_{n-1}}^1 \left[\frac{(1-x)^2}{2} - \frac{3}{8}(1 - x_n)(1-x)\right]^2 dx = \frac{1}{320}(1 - x_{n-1})^5.$$

So,

$$\bar{F}_2(X) := \min_A F_2(A, X) = \frac{1}{32}x_1^5 + \frac{1}{720}\sum_{k=2}^{n-1}(x_k - x_{k-1})^5 + \frac{1}{320}(1 - x_{n-1})^5. \tag{22}$$

Now, from

$$\frac{\partial}{\partial x_k}\sum_{i=2}^{n-1}(x_i - x_{i-1})^5 = 5[(x_k - x_{k-1})^4 - (x_{k+1} - x_k)^4] = 0, \quad k = 2, ..., n-1$$

one obtains

$$x_k - x_{k-1} = \frac{x_n - x_1}{n-2}, \quad k = 2, ..., n-1. \tag{23}$$

For

$$\tilde{F}_2(x_1, x_{n-1}) = \min_{x_2,\dots,x_{n-2}} \bar{F}_2(X)$$

we have

$$\tilde{F}_2(x_1, x_{n-1}) = \frac{1}{32}x_1^5 + \frac{(x_{n-1} - x_1)^5}{720(n-2)^4} + \frac{1}{320}(1 - x_{n-1})^5.$$

From the following system

$$\begin{cases} \frac{\partial \tilde{F}_2(x_1, x_{n-1})}{\partial x_1} = 0 \\ \frac{\partial \tilde{F}_2(x_1, x_{n-1})}{\partial x_{n-1}} = 0 \end{cases}$$

one obtains

$$x_1^* = 1 - x_{n-1}^* = 2\mu \tag{24}$$

and

$$\tilde{F}_2(x_1^*, x_{n-1}^*) = \frac{1}{20}\mu^4. \tag{25}$$

Finally, the proof follows from (20)–(25). □

Theorem 10. *For a function $f \in H^{2,2}[0,1]$, the quadrature formula of the form*

$$\int_0^1 f(x)dx = \sum_{k=0}^{n} A_k f(x_k) + B_0 f'(0) + B_1 f'(1) + R_n(f), \tag{26}$$

is optimal with regard to the error for

$$\begin{aligned} A_0 &= A_n = \frac{1}{2n}, \\ A_k &= \frac{1}{n}, \quad k = 1, ..., n-1, \\ B_0 &= \frac{1}{12n^2}, \\ B_1 &= -B_0, \end{aligned}$$

$$x_0 = 0, \quad x_k = \frac{k}{n}, \;\; k = 1, ..., n-1, \quad x_n = 1$$

and

$$|R_n(f)| \leq \frac{1}{12n^2\sqrt{5}} \left\| f'' \right\|_2.$$

Proof. From (14), we get

$$A_0 = -\varphi_1'(0),$$
$$A_k = (\varphi_k' - \varphi_{k+1}')(x_k), \quad k = 1, ...n-1,$$
$$A_n = \varphi_n'(1),$$
$$B_0 = \varphi_1(0),$$
$$B_1 = -\varphi_n(1),$$

and

$$(\varphi_k - \varphi_{k+1})(x_k) = 0, \quad k = 1, ...n-1.$$

It follows that

$$\varphi_k(x) = \frac{x^2}{2} - \sum_{i=0}^{k-1} A_i(x - x_i) + B_0, \quad k = 1, ..., n.$$

As the integral

$$\int_0^1 \varphi^2(x)dx = \sum_{k=1}^{n} \int_{x_{k-1}}^{x_k} \varphi_k^2(x)dx$$

attains the minimum value for

$$\varphi_k \equiv \frac{1}{2}\tilde{l}_{2,k}, \quad k = 1, ..., n,$$

from these last identities, using the fact that the degree of exactness of the quadrature formula is one, the proof follows. □

Remark 11. *In an analogous way, for $f \in H^{2,1}[0,1]$ one can prove that the quadrature formula of the form (26), optimal with regard to the error, has the coefficients:*

$$A_0^* = A_n^* = \frac{1}{2n},$$
$$A_k^* = \frac{1}{n}, \quad k = 1, ..., n-1,$$
$$B_0^* = \frac{3}{32n^2},$$
$$B_1^* = -B_0^*,$$

the nodes

$$x_0^* = 0,$$
$$x_k^* = \frac{k}{n}, \quad k = 1, ..., n-1,$$
$$x_n^* = 1,$$

and

$$|R_n^*(f)| \le \frac{1}{32n^2} \|f''\|_1 .$$

It is important in the proof that the functions $\frac{1}{2}\varphi_k$*,* $k = 1, ..., n$*, are identified with the Cebyshev polynomials of the second kind.*

Now, let us consider the general case, i.e., the quadrature formula (6), with the remainder term given by (7), for $r \ge 1$ and for $f \in H^{r,p}[0,1]$. The problem is to find the values of the parameters A_{kj} and x_k, $k = 0, ..., n$, $j = 0, ..., r-1$ for which

$$F(A, X) := \int_0^1 |\varphi(x)|^p \, dx$$

attains the minimum value. We have

$$F(A, X) = \sum_{i=1}^{n} \int_{x_{i-1}}^{x_i} |\varphi_i(x)|^p \, dx,$$

where

$$\varphi_i(x) = \frac{x^r}{r!} + \sum_{k=0}^{i-1} \sum_{j=0}^{r-1} A_{kj} \frac{(x - x_k)^j}{j!}, \quad x \in [x_{i-1}, x_i].$$

As the polynomials φ_i are independent, the function $F(A, X)$ can be minimized, first with regard to the coefficients A_{kj}, $k = 0, ..., n$, $j = 0, ..., r-1$, considering the nodes fixed, and then, with regard to the nodes $x_1, ..., x_{n-1}$.

Using the notation $\frac{A_{kj}}{j!} = \frac{B_{kj}}{r!}$ one obtains

$$\varphi_i = \frac{1}{r!}\psi_i,$$

with

$$\psi_i(x) = x^r + \sum_{k=0}^{i-1} \sum_{j=0}^{r-1} B_{kj} (x - x_k)^j$$

and

$$F(A, X) = \frac{1}{(r!)^p} \sum_{i=1}^{n} \int_{x_{i-1}}^{x_i} |\psi_i(x)|^p \, dx.$$

Using the minimum norm property of the orthogonal polynomials, the integrals

$$I_i = \int_{x_{i-1}}^{x_i} |\psi_i(x)|^p \, dx, \quad i = 1, ..., n,$$

can be minimized by identifying the polynomials ψ_i with the corresponding orthogonal polynomials, say θ_i, for different values of p. One obtains

$$\tilde{F}(x_1, ..., x_{n-1}) = \tfrac{1}{(r!)^p} \sum_{i=1}^{n} \int_{x_{i-1}}^{x_i} |\theta_i(x)|^p \, dx,$$

that is further minimized with regard to x_i, $i = 1, ..., n-1$.

References

[1] Coman, Gh., *On some optimal quadrature formulae,* Studia Universitatis "Babeş-Bolyai", Series Mathematica-Mechanica, **2**, 1970, pp. 39–54.

[2] Coman, Gh., *Monosplines and optimal quadrature formulae in L_p*, Rendiconti di Matematica, **5** (1972) no.3, pp. 567–577.

[3] Coman, Gh., *Optimal quadratures with regard to the efficiency,* Calcolo, **24** (1987), no. 1, pp. 85–100.

[4] Coman, Gh., Cătinaş, T., Birou, M., Oprişan, A., Oşan, C., Pop, I., Somogyi, I., Todea, I., *Interpolation operators*, Ed. Casa Cărţii de Ştiinţă, Cluj-Napoca, 2004.

[5] Ghizzetti, A., Ossicini, A., *Quadrature Formulae,* Akademie-Verlag, Berlin, 1970.

[6] Ionescu, D.V., *Numerical Quadratures,* Bucharest, 1957.

[7] Ionescu, D.V., *Sur une classe de formules de cubature*, C.R. Acad. Sc. Paris, **266**, 1968, pp. 1155–1158.

[8] Ionescu, D.V., *Extension de la formule de quadrature de Gauss à une classe de formules de cubature*, C.R. Acad. Sc. Paris, **269**, 1969, pp. 655–657.

[9] Ionescu, D.V., *L'extension d'une formule de cubature*, Acad. Royale de Belgique, Bull. de la Classe des Science, **56**, 1970, pp. 661–690.

[10] Ionescu, D.V., *La méthode de la foction φ en analyse numérique,* Mathematical contributions of D.V. Ionescu, (Ed. I.A. Rus), Babeş-Bolyai University, Department of Applied Mathematics, Cluj-Napoca, 2001.

[11] Meyers, L.S., Sard, A., *Best approximate integration formulas, J. Math. Physics,* **29** (1950), no. 2, pp. 118–123.

[12] Nikolski, S.M., *Quadrature Formulae,* Ed. Tehnică, Bucharest, 1964 (translation from Russian).

[13] Sard, A., *Best approximate integration formulas, best approximate formulas,* Amer. J. of Math., **71** (1949), no. 1, pp. 80–91.

[14] Sard, A., *Linear Approximation,* AMS, 1963.

[15] Schoenberg, I.J., *On monosplines of least deviation and best quadrature formulae,* J. SIAM Numer. Anal., Ser. B, **2** (1965) no. 1, pp. 145–170.

[16] Schoenberg, I.J., *On monosplines of least deviation and best quadrature formulae II,* J. SIAM Numer. Anal., Ser. B, **3** (1966), no. 2, pp. 321–328.

[17] Stancu, D.D., *Sur quelques formules générales de quadrature du type Gauss-Christoffel,* Mathematica (Cluj), **24** (1959), no. 1, pp. 167–182.

[18] Stancu, D.D., Coman, Gh., Blaga, P., *Numerical Analysis and Approximation Theory,* Vol. II, Presa Universitară Clujeană, Cluj-Napoca, 2002 (in Romanian).

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UNBOUNDED SOLUTIONS OF EQUATION $\dot{y}(t) = \beta(t)[y(t-\delta) - a(t)y(t-\tau)]$

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Abstract. This contribution is devoted to asymptotic behavior (for $t \to \infty$) of solutions of first-order differential equation with two delays

$$\dot{y}(t) = \beta(t)[y(t-\delta) - a(t)y(t-\tau)].$$

Representation of solutions in an exponential form is discussed and inequalities for such solutions are given. As a consequence, existence of unbounded solutions is proved. An overview of known results and illustrative examples are considered, too.

1. Introduction

1.1. **The aim of the contribution.** In this contribution we deal with asymptotic behavior of solutions to a linear homogeneous differential equation with two delayed terms containing two discrete delays

$$\dot{y}(t) = \beta(t)[y(t-\delta) - a(t)y(t-\tau)] \tag{1}$$

for $t \to \infty$. In (1) $\delta, \tau \in \mathbb{R}^+$, $\mathbb{R}^+ := (0, +\infty)$, $\tau > \delta$, $\beta : I_{-1} \to \mathbb{R}^+$ is a continuous function, $I_{-1} := [t_0 - \tau, \infty)$, $t_0 \in \mathbb{R}$ and $a : I \to [0, 1]$, where $I := [t_0, \infty)$, is a continuous function. The symbol " $\dot{}$ " denotes (at least) the *right-hand* derivative. Similarly, if necessary, the value of a function at a point of I_{-1} is understood (at least) as value of the corresponding limit *from the right*. We show that increasing solutions of (1) have representation

$$\exp\left[\int_{t_0-\tau}^{t} \tilde{\varepsilon}(s)\beta(s)\,\mathrm{d}s\right] \tag{2}$$

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with a function $\tilde{\varepsilon} : I_{-1} \to (0,1)$. Such representation we call *exponential*. Representation (2) is then specified and a criterion connecting it with an integral inequality is formulated. Since the equation considered is linear, the corresponding statements formulated for increasing solutions are (under obvious modification) valid for decreasing solutions etc. Let us note that close investigation of asymptotic behaviour of a solution of delayed functional differential equations is performed e.g. in papers [1]–[24]. The studied Eq. (1) (with $a \equiv 1$) occurs e.g. in the number theory [23].

The contribution is organized as follows: In Section 2 a basic auxiliary inequality is studied and the relationship of its solutions with solutions of Eq. (1) is established. Exponential representation of monotone solutions is discussed in Section 3. Section 4 contains main results of the paper concerning inequalities for solutions of Eq. (1) and existence of unbounded solutions. An overview of known results and illustrative examples are contained in Section 5. The paper ends with an open problem formulated in Section 6.

1.2. **Some definitions.** Let us shortly recall basic definitions. Let $\mathcal{C} := C([-\tau, 0], \mathbb{R})$ be Banach space of continuous functions mapping the interval $[-\tau, 0]$ into $\mathbb{R}$ equipped with the supremum norm.

A function $y(t)$ is said to be a *solution of Eq.* (1) *on* $[\nu - \tau, \nu + A)$ with $\nu \in I$ and $A > 0$, if $y \in C([\nu - \tau, \nu + A), \mathbb{R}) \cap C^1([\nu, \nu + A), \mathbb{R})$, and $y(t)$ satisfies the Eq. (1) for $t \in [\nu, \nu + A)$.

For given $\nu \in I$, $\varphi \in \mathcal{C}$, we say that $y(\nu, \varphi)$ is a *solution* of Eq. (1) *through* (ν, φ) (or that $y(\nu, \varphi)$ *corresponds to the initial point* ν), if there is an $A > 0$ such that $y(\nu, \varphi)$ is a solution of Eq. (1) on $[\nu - \tau, \nu + A)$ and $y(\nu, \varphi)(\nu + \theta) = \varphi(\theta)$ for $\theta \in [-\tau, 0]$.

Due to linearity of equation (1), the solution $y(\nu, \varphi)$ is unique and is defined on $[\nu - \tau, \infty)$, i.e. in previous definitions we can put $A := \infty$.

2. An auxiliary inequality

Auxiliary inequality

$$\dot{\omega}(t) \le \beta(t)[\omega(t - \delta) - a(t)\omega(t - \tau)] \tag{3}$$

plays a main role in analysis of equation (1). A function $\omega(t)$ is said to be a *solution of* (3) *on* $[\nu-\tau, \nu+A)$ with $\nu \in I$ and $A > 0$, if $\omega \in C([\nu-\tau,\nu+A),\mathbb{R}) \cap C^1([\nu,\nu+A),\mathbb{R})$, and $\omega(t)$ satisfies the inequality (3) for $t \in [\nu, \nu+A)$.

2.1. **Inequalities between solutions of inequality (3) and equation (1).** Below we discuss some properties of solutions of inequalities of the type (3) and inequalities between solutions of (1) and inequality (3).

Theorem 1. *Suppose that $\omega(t)$ is a solution of inequality* (3) *on I_{-1}. Then there exists a solution $y(t)$ of* (1) *on I_{-1} such that an inequality*

$$y(t) \geq \omega(t) \tag{4}$$

holds on I_{-1}. In particular, a solution $y(t_0,\phi)$ of Eq. (1) *with $\phi \in \mathcal{C}$ defined by relation*

$$\phi(\theta) := \omega(t_0+\theta), \quad \theta \in [-\tau, 0], \tag{5}$$

is a such solution.

Proof. Let $\omega(t)$ be a solution of inequality (3) on I_{-1}. Let us show that the solution $y(t) := y(t_0,\phi)(t)$ of (1) satisfies inequality (4) i.e.

$$y(t_0,\phi)(t) \geq \omega(t) \tag{6}$$

on I_{-1}. Due to definition of $y(t)$ we have $y(t) \equiv \omega(t)$, $t \in [t_0-\tau, t_0]$ and (4) holds on initial interval $[t_0-\tau, t_0]$. Define on I_{-1} a continuous function

$$W(t) := y(t) - \omega(t).$$

Function W is continuously differentiable on I. Then (taking into account inequality (3)) the estimation

$$\dot{W}(t) = \dot{y}(t) - \dot{\omega}(t) \geq Z(t)$$

with

$$Z(t) := \beta(t)[y(t-\delta) - a(t)y(t-\tau)] - \beta(t)[\omega(t-\delta) - a(t)\omega(t-\tau)] = \beta(t)[W(t-\delta) - a(t)W(t-\tau)]$$

is valid on I . Let $t \in (t_0, t_0 + \delta]$. In view of (5) $W(t-\delta) \equiv W(t-\tau) \equiv 0$, $Z(t) \equiv 0$ and $\dot{W}(t) \geq 0$, i.e. (4) holds on $(t_0, t_0 + \delta]$. Let $t \in (t_0 + \delta, t_0 + \tau]$. In this case $W(t-\tau) \equiv 0$ and

$$Z(t) \equiv \beta(t)[y(t-\delta) - \omega(t-\delta)] = \beta(t)W(t-\delta) \geq 0.$$

Consequently, $\dot{W}(t) \geq 0$, i.e. (4) holds on $(t_0 + \delta, t_0 + \tau]$, too. Let us show that inequality $\dot{W}(t) \geq 0$ holds on the whole interval I. For it *suppose the contrary*, i.e. suppose existence of a point $t_1 > t_0 + \tau$ such that

$$\begin{aligned} \dot{W}(t) &\geq 0, \quad t \in [t_0, t_1), \\ \dot{W}(t_1) &= 0, \\ \dot{W}(t) &< 0, \quad t \in (t_1, t_1 + \varepsilon), \end{aligned} \tag{7}$$

where $\varepsilon < \delta$ is a small positive number. Due to continuity of $W(t)$ on I_{-1}, our construction and suppositions, such point t_1 exists. Let $t_2 \in (t_1, t_1 + \varepsilon)$. Taking into account that $W(t)$ is nondecreasing on $[t_0, t_1]$ we conclude $W(t_2 - \delta) \geq W(t_2 - \tau) \geq 0$. Then

$$\dot{W}(t_2) = \dot{y}(t_2) - \dot{\omega}(t_2) \geq Z(t_2) = \beta(t_2)[W(t_2 - \delta) - a(t_2)W(t_2 - \tau)] \geq$$
$$\beta(t_2)(1 - a(t_2))W(t_2 - \tau) \geq 0.$$

The resulting inequality $\dot{W}(t_2) \geq 0$ contradicts (7). □

Remark 1. *Let us note that an affirmation, opposite in a sense with the statement of Theorem 1 is obvious. Namely, if a solution $y(t)$ of (1) on I_{-1} is given, then there exists a solution $\omega(t)$ of inequality (3) on I_{-1} such that inequality*

$$\omega(t) \geq y(t) \tag{8}$$

holds on I_{-1}, since it can be put $\omega(t) \equiv y(t)$.

2.2. **A comparison lemma.** Let us consider an inequality of the type (3)

$$\dot{\omega}^*(t) \le \beta_1(t)[\omega^*(t-\delta) - a_1(t)\omega^*(t-\tau)] \tag{9}$$

where $\beta_1 : I_{-1} \to \mathbb{R}^+$ and $a_1 : I \to [0,1]$ are continuous functions satisfying inequalities $\beta_1(t) \le \beta(t)$, $a_1(t) \ge a(t)$ on I_{-1}. The following comparison lemma will be used below.

Lemma 1. *Let the inequality* (9) *have a nondecreasing positive solution on* I_{-1}. *Then this solution is a solution of the inequality* (3) *on* I_{-1}, *too.*

Proof. Let ω^* be a nondecreasing solution of inequality (9) on I_{-1}. Then

$$\begin{aligned}\dot{\omega}^* \le \beta_1(t)[\omega^*(t-\delta) - a_1(t)\omega^*(t-\tau)] &\le \beta(t)[\omega^*(t-\delta) - a_1(t)\omega^*(t-\tau)] \\ &\le \beta(t)[\omega^*(t-\delta) - a(t)\omega^*(t-\tau)].\end{aligned}$$

Consequently, the function $\omega := \omega^*$ solves the inequality (3), too. □

2.3. **A solution of the inequality (3).** It is easy to get a solution of inequality (3) in an exponential form.

Lemma 2. *Suppose that there exists a function* $\varepsilon : I_{-1} \to \mathbb{R}$, *continuous on* $I_{-1} \setminus \{t_0\}$ *with at most first order discontinuity at the point* $t = t_0$ *and satisfying on* I *the inequality*

$$\exp\left[-\int_{t-\delta}^{t} \varepsilon(s)\beta(s)\,\mathrm{d}s\right] \ge \varepsilon(t) + a(t)\exp\left[-\int_{t-\tau}^{t} \varepsilon(s)\beta(s)\,\mathrm{d}s\right]. \tag{10}$$

Then on I_{-1}, *there exists a solution* $\omega(t) = \omega_{\mathrm{e}}(t)$ *of inequality* (3) *having the form*

$$\omega_{\mathrm{e}}(t) := \exp\left[\int_{t_0-\tau}^{t} \varepsilon(s)\beta(s)\,\mathrm{d}s\right]. \tag{11}$$

Proof. Inequality (10) follows immediately from inequality (3) if a possible solution $\omega(t)$ is taken in the form (11). □

3. **Properties of solutions of equation (1)**

In this part we prove auxiliary results concerning solutions of equation (1).

Lemma 3. *Let* $\varphi \in \mathcal{C}$ *is increasing and positive on* $[-\tau, 0]$. *Then the corresponding solution* $y(t^*, \varphi)(t)$ *of* (1) *with* $t^* \in I$ *is increasing in* $[t^* - \tau, \infty)$, *too.*

Proof. Immediately, from the form of (1), we get $\operatorname{sign} \dot{y}(t^*, \varphi)(t^*) = +1$ in the case when the function φ increases on $[-\tau, 0]$. The case $\dot{y}(t^*, \varphi)(t^{**}) = 0$ for a $t^{**} \in (t^*, \infty)$ and simultaneously $\operatorname{sign} \dot{y}(t^*, \varphi)(t) \neq 0$ on interval $t \in (t^*, t^{**})$ is impossible because, as it follows from (1) and from the properties of function φ, the inequality $y(t^{**} - \delta) > y(t^{**} - \tau)$ holds and, consequently,

$$y(t^{**} - \delta) - a(t^{**}) y(t^{**} - \tau) \neq 0.$$

I.e. $\dot{y}(t^*, \varphi)(t^{**}) \neq 0$. □

3.1. **Exponential representation of solutions of equation (1).**

Theorem 2. *Every continuously increasing on I_{-1} and continuously differentiable on $I_{-1} \setminus \{t_0\}$ solution $y(t)$ of (1) with $y(t_0 - \tau) = 1$ is on I_{-1} representable in exponential form:*

$$y(t) = \exp\left[\int_{t_0-\tau}^{t} \tilde{\varepsilon}(s)\beta(s)\, \mathrm{d}s\right] \tag{12}$$

where $\tilde{\varepsilon} : I_{-1} \to \mathbb{R}_+ := [0, \infty)$ is a continuous function on $I_{-1} \setminus \{t_0\}$ with at most first order discontinuity at t_0 and $0 < \tilde{\varepsilon}(t) < 1$ on I.

Proof Let $\varphi \in \mathcal{C}$, $\varphi(t_0 - \tau) = 1$ be increasing and continuously differentiable initial function generating solution $y(t) = y(t_0, \varphi)(t)$. By Lemma 3 is $y(t)$ increasing in I_{-1}. Define

$$\tilde{\varepsilon}(t) := \begin{cases} \dfrac{\varphi'(t)}{\beta(t)\varphi(t)} & \text{on} \quad [t_0 - \tau, t_0), \\[2ex] \dfrac{\dot{y}(t)}{\beta(t) y(t)} & \text{on} \quad I. \end{cases}$$

Then on I_{-1} representation (12) holds. Really, for $t \in [t_0 - \tau, t_0)$ we have

$$\exp\left[\int_{t_0-\tau}^{t} \tilde{\varepsilon}(s)\beta(s)\, \mathrm{d}s\right] = \exp\left[\ln \frac{\varphi(t)}{\varphi(t_0 - \tau)}\right] = \varphi(t)$$

and for $t \in I$

$$\exp\left[\int_{t_0-\tau}^{t} \tilde{\varepsilon}(s)\beta(s)\, \mathrm{d}s\right] = \exp\left[\int_{t_0-\tau}^{t_0} \tilde{\varepsilon}(s)\beta(s)\, \mathrm{d}s + \int_{t_0}^{t} \tilde{\varepsilon}(s)\beta(s)\, \mathrm{d}s\right] =$$

$$\exp\left[\ln \frac{\varphi(t_0)}{\varphi(t_0 - \tau)} + \ln \frac{y(t)}{y(t_0)}\right] = y(t).$$

Function $\tilde{\varepsilon}$ is on $[t_0-\tau, t_0)$ nonnegative, since obviously $\varphi > 0$, $\varphi' \geq 0$ and $\beta > 0$. Positivity of $\tilde{\varepsilon}$ on I is obvious, too since

$$\tilde{\varepsilon}(t) = \frac{\dot{y}(t)}{\beta(t)y(t)} = \frac{y(t-\delta) - a(t)y(t-\tau)}{y(t)} > \frac{(1-a(t))y(t-\tau)}{y(t)} \geq 0,$$

i.e. $\tilde{\varepsilon}(t) > 0$. Moreover, on I,

$$\tilde{\varepsilon}(t) = \frac{\dot{y}(t)}{\beta(t)y(t)} = \frac{y(t-\delta) - a(t)y(t-\tau)}{y(t)} \leq \frac{y(t-\delta)}{y(t)} < \frac{y(t)}{y(t)} = 1,$$

i.e. $\tilde{\varepsilon}(t) < 1$. □

Below is given a modification of previous result.

Corollary 1. *There exists continuously increasing on I_{-1} and continuously differentiable on $I_{-1} \setminus \{t_0\}$ solution $y(t)$ of* (1) *with $y(t_0-\tau) = 1$, representable in exponential form* (12)*, where*

$$\tilde{\varepsilon} : I_{-1} \to (0,1)$$

is a continuous function on $I_{-1} \setminus \{t_0\}$ with at most first order discontinuity at t_0.

The proof remains exactly the same if the initial function $\varphi \in \mathcal{C}$ is defined as

$$\varphi(\theta) := \exp\left[\int_{t_0-\tau}^{t_0+\theta} \varepsilon^*(s)\beta(s)\,\mathrm{d}s\right], \quad \theta \in [-\tau, 0],$$

where $\varepsilon^* : [t_0-\tau, t_0] \to (0,1)$ is a continuous function. Then we can define corresponding function $\tilde{\varepsilon}$ e.g. in the following way:

$$\tilde{\varepsilon}(t) := \begin{cases} \varepsilon^*(t) & \text{on} \quad [t_0-\tau, t_0), \\[2ex] \dfrac{y(t-\delta) - a(t)y(t-\tau)}{y(t)} & \text{on} \quad [t_0, \infty). \end{cases}$$

Remark 2. *From the statement of Theorem 2 it follows that every continuously increasing on I_{-1} and continuously differentiable on $I_{-1} \setminus \{t_0\}$ solution $y(t)$ of* (1) *with $y(t_0-\tau) = 1$ satisfies on I the inequality*

$$y(t) < \exp\left[\int_{t_0-\tau}^{t} \beta(s)\,\mathrm{d}s\right]. \tag{13}$$

Moreover (as it follows from Corollary 1) there exists continuously increasing on I_{-1} and continuously differentiable on $I_{-1} \setminus \{t_0\}$ solution $y(t)$ of (1) *with $y(t_0-\tau) = 1$, such that inequality* (13) *holds on $I_{-1} \setminus \{t_0-\tau\}$.*

4. Main results

The purpose of this part is to give an equivalence between existence of a certain type of exponential behavior of solutions of (1) and existence of a solution of inequality (3). The following result can be useful in the case when we need a concrete inequality for indicated solution $y = y(t)$ of (1).

4.1. Two equivalent statements.

Theorem 3. *Let $q : I_{-1} \to (0,1)$ be a given function such that the integral $\int_{t_0-\tau}^{t} q(s)\beta(s)\,\mathrm{d}s$ exists for any $t \in I_{-1}$. Then the following two statements are equivalent:*

a) *There exists a continuously increasing on I_{-1} and continuously differentiable on $I_{-1} \setminus \{t_0\}$ solution $y = y(t)$ of (1) representable in the form*

$$y(t) = \exp\left[\int_{t_0-\tau}^{t} \tilde{\varepsilon}(s)\beta(s)\,\mathrm{d}s\right] \tag{14}$$

on I_{-1}, where $\tilde{\varepsilon} : I_{-1} \to (0,1)$ is a continuous function on $I_{-1} \setminus \{t_0\}$ with at most first order discontinuity at the point $t = t_0$, such that

$$y(t) \geq \exp\left[\int_{t_0-\tau}^{t} q(s)\beta(s)\,\mathrm{d}s\right] \tag{15}$$

on I_{-1}.

b) *There exists a function $\varepsilon : I_{-1} \to (0,1)$ continuous on $I_{-1} \setminus \{t_0\}$ with at most first order discontinuity at the point $t = t_0$ such that*

$$\int_{t_0-\tau}^{t} \varepsilon(s)\beta(s)\mathrm{d}s \geq \int_{t_0-\tau}^{t} q(s)\beta(s)\mathrm{d}s \tag{16}$$

on I_{-1}, and satisfying the integral inequality (10) on I.

Proof

Part b) $\Longrightarrow$ a). In this case there exists (by Lemma 2) a solution $\omega(t) \equiv \omega_{\mathrm{e}}(t)$ of inequality (10) given by formula (14). Define

$$\varphi(\theta) := \omega_{\mathrm{e}}(t_0 + \theta),\ \ \theta \in [-\tau, 0].$$

Since $\varphi \in \mathcal{C}$ is increasing and positive on $[-\tau, 0]$, then (by Lemma 3) solution $y(t) = y(t_0, \varphi)(t)$ is increasing in I_{-1} and, by Theorem 1, satisfies on I_{-1} inequality (4), i.e.

$$y(t) \geq \exp\left[\int_{t_0-\tau}^{t} \varepsilon(s)\beta(s)\,\mathrm{d}s\right], \quad t \in I_{-1}.$$

Now is the inequality (15) a straightforward consequence of inequality (16). The part *b*) $\implies$ *a*) is proved.

Part a) $\implies$ b). Let $y(t)$ be a solution of (1) on I_{-1}, having form (14), with properties indicated in the part a). Then on $I_{-1} \setminus \{t_0\}$:

$$\dot{y}(t) = \tilde{\varepsilon}(t)\beta(t) \cdot \exp\left[\int_{t_0-\tau}^{t} \tilde{\varepsilon}(s)\beta(s)\,\mathrm{d}s\right].$$

Let us put $y(t)$ into (1). Then on I:

$$\tilde{\varepsilon}(t) = \exp\left[-\int_{t-\delta}^{t} \tilde{\varepsilon}(s)\beta(s)\,\mathrm{d}s\right] - a(t)\exp\left[-\int_{t-\tau}^{t} \tilde{\varepsilon}(s)\beta(s)\,\mathrm{d}s\right].$$

Define function $\varepsilon : I_{-1} \setminus \{t_0\} \to (0,1)$ as $\varepsilon := \tilde{\varepsilon}$, and rewrite the last equality. For $t \in I$ we get

$$\exp\left[-\int_{t-\delta}^{t} \varepsilon(s)\beta(s)\,\mathrm{d}s\right] = \varepsilon(t) + a(t)\exp\left[-\int_{t-\tau}^{t} \varepsilon(s)\beta(s)\,\mathrm{d}s\right],$$

i.e. the integral inequality (10) holds on I. Moreover, due to (15) we have

$$y(t) = \exp\left[\int_{t_0-\tau}^{t} \varepsilon(s)\beta(s)\,\mathrm{d}s\right] \geq \exp\left[\int_{t_0-\tau}^{t} q(s)\beta(s)\,\mathrm{d}s\right],$$

i.e. the inequality (16) holds, too. This ends the proof. □

Remark 3. *Note that Theorem 3 remains valid if, instead of the supposition $q : I_{-1} \to (0,1)$, a more general supposition $q : I_{-1} \to \mathbb{R}$ is used. But for some specifications of the function q the equivalence between statements a) and b) can lose sense since the existence of solution $y = y(t)$ satisfying inequality* (15) *can follows directly from the statements of Theorem 2 or Corollary 1. E.g. the choice $q(t) := 0$ gives no new information as well as the choice $q(t) := \tilde{\varepsilon}(t)$. Theorem 3 generalizes and improves Theorem 2 from* [14], *where the equation* (1) *with $a(t) \equiv 1$ was investigated. The authors are grateful to R. Hakl for corresponding remark during discussions on*

International Conference on Nonlinear Operators, Differential Equations and Applications in Cluj-Napoca, Romania, August 2004, indicating a gap in formulation of this result.

Remark 4. *Let us underline that Theorem 3 together with Remark 2 give for solution $y(t)$ on I_{-1} estimation*

$$\exp\left[\int_{t_0-\tau}^{t} q(s)\beta(s)\,\mathrm{d}s\right] \le y(t) \le \exp\left[\int_{t_0-\tau}^{t} \beta(s)\,\mathrm{d}s\right].$$

4.2. **Sufficient conditions for divergence.** Conditions guarantee existence of unbounded solution can be derived easily from previous results. Let us formulate some of them. From Theorem 1 we get

Theorem 4. *Suppose that $\omega(t)$ is a solution of inequality (3) on I_{-1} such that*

$$\limsup_{t\to\infty} \omega(t) = +\infty.$$

Then there exists unbounded solution $y(t)$ of (1) on I_{-1}.

From Lemma 2, Theorem 1 and Theorem 3 (putting $q(t) := \varepsilon(t)$) we get

Theorem 5. *Suppose there exists a function $\varepsilon : I_{-1} \to \mathbb{R}$, continuous on $I_{-1} \setminus \{t_0\}$ with at most first order discontinuity at the point $t = t_0$ satisfying $\int^{\infty} \varepsilon(s)\beta(s)\mathrm{d}s = \infty$, and on I the inequality (10). Then there exists unbounded solution $y(t)$ of (1) on I_{-1} satisfying inequality*

$$y(t) \ge \exp\left[\int_{t_0-\tau}^{t} \varepsilon(s)\beta(s)\,\mathrm{d}s\right] \tag{17}$$

on I_{-1}. If, moreover ε is on $[t_0-\tau, t_0]$ positive then there exists increasing unbounded solution $y(t)$ of (1) on I_{-1}, satisfying inequality (17).

5. Summary of known results and examples

5.1. **Known results relative to equation (1).** Let us recall some known partial results concerning equation (1). In paper [12] conditions for convergence of all solutions of equation (1) with $a(t) \equiv 1$ and $\delta = 1$, i.e. the equation

$$\dot{y}(t) = \beta(t)[y(t) - y(t-\tau)]. \tag{18}$$

are given. We reproduce one result as the first statement of following theorem. The second part concerns of equation (1) with $a(t) \equiv 1$, i.e. the equation

$$\dot{y}(t) = \beta(t)[y(t-\delta) - y(t-\tau)]. \tag{19}$$

and follows from results given in [3, 6].

Theorem 6. *Let for all $t \in I_{-1}$ and a constant $p > 1$:*

$$\beta(t) \leq \frac{1}{\tau} - \frac{p}{2t} \ . \tag{20}$$

Then each solution of (18) corresponding to the initial point t_0 converges.
Let for all $t \in I_{-1}$ exists a constant ρ such that

$$\beta(t) \leq \rho < \frac{1}{\tau - \delta} \ . \tag{21}$$

Then each solution of (19) corresponding to the initial point t_0 converges.

In the paper [14] is proved following result concerning existence of unbounded increasing solutions of (19).

Theorem 7. *Let for all $t \in I_{-1}$ with sufficiently large t_0 and for a constant $p \in (0,1)$:*

$$\beta(t) \geq \frac{1}{\tau - \delta} - \frac{p}{2t} \ . \tag{22}$$

Then there exists an increasing and unbounded solution of (19) as $t \to \infty$.

5.2. **Examples.** In this part we give two examples to demonstrate the influence of the coefficient a to appearance of unbounded solutions.

Example 1. *The first remark is obvious - the presence of coefficient a in Eq. (1) enlarges, in the case $a(t) \not\equiv 1$, the range for coefficient β. Consider the following result to illustrate this phenomenon.*

Theorem 8. *Let for all $t \in I_{-1}$ inequalities*

$$\beta(t) \geq \frac{1}{\tau - \delta} + \frac{p}{t} \ , \quad 0 \leq a(t) \leq 1 - \frac{b}{t^2} \tag{23}$$

with constants $p \in \mathbb{R}$, $b \in \mathbb{R}^+$ hold. Then there exists increasing and unbounded solution $y(t)$ of (1) as $t \to \infty$.

Proof. *Let us verify that the integral inequality* (10) *have (for sufficiently large values* t*) a solution of the form* $\varepsilon(t) := \alpha/t$ *with* $\alpha \in \mathbb{R}^+$*. Put in* (10)

$$\beta(t) := \frac{1}{\tau-\delta} + \frac{p}{t}\,, \quad a(t) := 1 - \frac{b}{t^2}\,, \quad \varepsilon(t) := \frac{\alpha}{t}\,.$$

Then the left-hand side $\mathcal{L}(t)$ *of* (10) *equals*

$$\mathcal{L}(t) \equiv \exp\left[-\int_{t-\delta}^{t} \varepsilon(s)\beta(s)\,\mathrm{d}s\right] = \exp\left[-\int_{t-\delta}^{t} \frac{\alpha}{s}\left[\frac{1}{\tau-\delta} + \frac{p}{s}\right]\mathrm{d}s\right] =$$

$$\left(\frac{t-\delta}{t}\right)^{\dfrac{\alpha}{\tau-\delta}} \cdot \exp\left[\frac{-\alpha\delta p}{t(t-\delta)}\right].$$

Now we asymptotically decompose $\mathcal{L}(t)$ *for* $t \to \infty$ *with sufficient accuracy for further application. We get:*

$$\mathcal{L}(t) = \left[1 - \frac{\alpha\delta}{(\tau-\delta)t} + \frac{\alpha\delta^2}{2(\tau-\delta)} \cdot \left(\frac{\alpha}{\tau-\delta} - 1\right)\frac{1}{t^2} + O\left(\frac{1}{t^3}\right)\right] \times \left[1 - \frac{\alpha\delta p}{t^2} + O\left(\frac{1}{t^3}\right)\right]$$

$$= 1 - \frac{\alpha\delta}{(\tau-\delta)t} + \left[\frac{\alpha\delta^2}{2(\tau-\delta)} \cdot \left(\frac{\alpha}{\tau-\delta} - 1\right) - \alpha\delta p\right]\frac{1}{t^2} + O\left(\frac{1}{t^3}\right)$$

where O *is the Landau order symbol. Decomposition of the right-hand side* $\mathcal{R}(t)$ *of* (10) *leads to*

$$\mathcal{R}(t) \equiv \varepsilon(t) + a(t)\exp\left[-\int_{t-\tau}^{t} \varepsilon(s)\beta(s)\,\mathrm{d}s\right]$$

$$= \frac{\alpha}{t} + \left(1 - \frac{b}{t^2}\right) \cdot \exp\left[-\int_{t-\tau}^{t} \frac{\alpha}{s}\left[\frac{1}{\tau-\delta} + \frac{p}{s}\right]\mathrm{d}s\right]$$

$$= \frac{\alpha}{t} + \left(1 - \frac{b}{t^2}\right) \cdot \left(\frac{t-\tau}{t}\right)^{\dfrac{\alpha}{\tau-\delta}} \cdot \exp\left[\frac{-\alpha\tau p}{t(t-\tau)}\right]$$

$$= \frac{\alpha}{t} + \left(1 - \frac{b}{t^2}\right) \cdot \left[1 - \frac{\alpha\tau}{(\tau-\delta)t} + \frac{\alpha\tau^2}{2(\tau-\delta)} \cdot \left(\frac{\alpha}{\tau-\delta} - 1\right)\frac{1}{t^2} + O\left(\frac{1}{t^3}\right)\right] \times$$

$$\left[1 - \frac{\alpha\tau p}{t^2} + O\left(\frac{1}{t^3}\right)\right]$$

$$= 1 + \frac{\alpha}{t} - \frac{\alpha\tau}{\tau-\delta} \cdot \frac{1}{t} + \left[\frac{\alpha\tau^2}{2(\tau-\delta)} \cdot \left(\frac{\alpha}{\tau-\delta} - 1\right) - \alpha\tau p - b\right]\frac{1}{t^2} + O\left(\frac{1}{t^3}\right).$$

Comparing $\mathcal{L}(t)$ and $\mathcal{R}(t)$, we see that for $\mathcal{L}(t) \geq \mathcal{R}(t)$ it is necessary to compare coefficients of the terms t^{-2} because coefficients of the terms t^0 and t^{-1} are equal. It means we need the inequality

$$\frac{\alpha\delta^2}{2(\tau-\delta)} \cdot \left(\frac{\alpha}{\tau-\delta} - 1\right) - \alpha\delta p > \frac{\alpha\tau^2}{2(\tau-\delta)} \cdot \left(\frac{\alpha}{\tau-\delta} - 1\right) - \alpha\tau p - b.$$

We see that for sufficiently small positive α this inequality holds since taking limit for $\alpha \to 0^+$, the limiting inequality $0 > -b$ is valid due to positivity of b. Consequently, a function

$$\omega_e(t) := \exp\left[\int_{t_0-\tau}^{t} \varepsilon(s)\beta(s)\mathrm{d}s\right] = \exp\left[\int_{t_0-\tau}^{t} \frac{\alpha}{s}\left(\frac{1}{\tau-\delta} + \frac{p}{s}\right)\mathrm{d}s\right]$$

is (under supposition that t_0 is sufficiently large) a positive solution of the integral inequality (3) and, moreover, it is easy to verify that $\omega_e(\infty) = +\infty$. Let us show that this solution solves every inequality of the type (10) (perhaps starting with a different value t_0) if the above fixed functions *β and a (defined at beginning of the proof) are changed by any functions β and a specifying in formulation of theorem by inequalities (23). This statement is a straightforward consequence of Lemma 1 if in its formulation*

$$\beta_1(t) := \frac{1}{\tau-\delta} + \frac{p}{t}\,, \quad a_1(t) := 1 - \frac{b}{t^2}\,.$$

Finally, by Theorem 4 with $\omega := \omega_e$, there exists increasing and unbounded solution $y(t)$ of (1) as $t \to \infty$. □

Remark 5. *The discussed above influence of the coefficient a can be now treated as follows. Slight perturbation of the coefficient $a(t) := 1$, in situation when Theorem 8 holds, leads to substantial enlargement or the range of the coefficient β (compare inequalities (22) and (23)) such that the property of existence of increasing unbounded solutions remains preserved.*

Example 2. *Let us show that unbounded increasing solution of (1) as $t \to \infty$ can exists even in the case when the inequality (20) holds. This can be caused due to smallness of a.*

Theorem 9. *Put* $\beta(t) := 1/\sqrt{t}$ *on* I_{-1} *with* $t_0 > \tau$. *Let there exists a constant* $q \in (0,1)$ *such that* $a : I \to [0,q]$. *Then there exists increasing and unbounded solution* $y(t)$ *of* (1) *as* $t \to \infty$.

Proof. *Let us verify that the integral inequality* (10) *have a solution given by formula* $\varepsilon(t) := 1/\sqrt{t}$. *We proceed similarly as in the proof of Theorem 8. The left-hand side* $\mathcal{L}(t)$ *of* (10) *equals*

$$\mathcal{L}(t) \equiv \exp\left[-\int_{t-\delta}^{t} \varepsilon(s)\beta(s)\,\mathrm{d}s\right] = \exp\left[-\int_{t-\delta}^{t} \frac{1}{s}\,\mathrm{d}s\right] = \exp\left[-\ln\frac{t}{t-\delta}\right] = 1 - \frac{\delta}{t}$$

Computation of the right-hand side $\mathcal{R}(t)$ *of* (3) *leads to*

$$\mathcal{R}(t) \equiv \varepsilon(t) + a(t)\exp\left[-\int_{t-\tau}^{t} \varepsilon(s)\beta(s)\,\mathrm{d}s\right] = \frac{1}{\sqrt{t}} + a(t)\exp\left[-\int_{t-\tau}^{t} \frac{1}{s}\,\mathrm{d}s\right] =$$
$$\frac{1}{\sqrt{t}} + a(t)\exp\left[-\ln\frac{t}{t-\tau}\right] = \frac{1}{\sqrt{t}} + a(t)\left(1-\frac{\tau}{t}\right) < \frac{1}{\sqrt{t}} + q\left(1-\frac{\tau}{t}\right).$$

Inequality $\mathcal{L}(t) \geq \mathcal{R}(t)$ *will be valid if*

$$1 - \frac{\delta}{t} > \frac{1}{\sqrt{t}} + q\left(1-\frac{\tau}{t}\right).$$

This inequality obviously holds for sufficiently large t *since, by supposition,* $q < 1$. *So, function*

$$\omega_e(t) := \exp\left[\int_{t_0-\tau}^{t} \varepsilon(s)\beta(s)\,\mathrm{d}s\right] = \exp\left[\int_{t_0-\tau}^{t} \frac{1}{s}\,\mathrm{d}s\right] = \frac{t}{t_0-\tau}$$

is (under supposition that t_0 *is sufficiently large) a solution of the integral inequality* (3) *and* $\omega_e(\infty) = +\infty$. *By Theorem 5, there exists increasing and unbounded solution* $y(t)$ *of* (1) *as* $t \to \infty$ *satisfying inequality* $y(t) \geq t/(t_0-\tau)$. $\square$

6. **Open problem**

Problem 1. *Comparing inequalities* (22), (23) *the following open question arises. Can be the affirmation of Theorem 8 improved in the following sense? Exists a function* b^* *satisfying on* I_{-1} *inequalities*

$$1 - \frac{b}{t^2} < b^*(t) < 1$$

such that formulated statement remains valid if for the function a inequalities

$$0 \le a(t) \le b^*(t)$$

on I_{-1} hold?

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References

[1] Arino, O., Györi, I., Pituk, M., *Asymptotically diagonal delay differential systems,* J. Math. Anal. Appl. **204** (1996), 701-728.

[2] Arino, O., Pituk, M., *Convergence in asymptotically autonomous functional differential equations,* J. Math. Anal. Appl. **237** (1999), 376-392.

[3] Arino, O., Pituk, M., *More on linear differential systems with small delays,* J. Diff. Equat. **170** (2001), 381-407.

[4] Atkinson, F.V., Haddock, J.R., *Criteria for asymptotic constancy of solutions of functional differential equations,* J. Math. Anal. Appl. **91** (1983), 410-423.

[5] Bellman, R., Cooke, K.L., *Differential-difference Equations,* Mathematics in science and engineering, A series of Monographs and Textbooks, Academic Press, 1963.

[6] Bereketoglu, H., Pituk, M., *Asymptotic constancy or nonhomogeneous linear differential equations with unbounded delays,* Discrete Contin. Dyn. Syst. 2003, Suppl. Vol., (2003), 100-107.

[7] Castillo, S., Pinto, M., *L^p perturbations in delay differential equations,* Electronic. J. Diff. Equat. **2001** (2001), 1-11.

[8] Čermák, J., *A change of variables in the asymptotic theory of differential equations with unbounded delays,* J. Comput. Appl. Math., **143** (2002), 81-93.

[9] Čermák, J., *On the asymptotic behaviour of solutions of certain functional differential equations,* Math. Slovaca **48** (1998), 187-212.

[10] Čermák, J., *The asymptotic bounds of solutions of linear delay systems,* J. Math. Anal. Appl. **225** (1998), 373-388.

[11] Cooke, K., Yorke, J., *Some equations modelling growth processes and gonorrhea epidemics,* Math. Biosci., **16** (1973), 75-101.

[12] Diblík, J., *Asymptotic convergence criteria of solutions of delayed functional differential equations,* J. Math. Anal. Appl., **274** (2002) 349-373.

[13] Diblík, J., *Asymptotic representation of solutions of equation* $\dot{y}(t) = \beta(t)[y(t) - y(t - \tau(t))]$, J. Math. Anal. Appl., **217** (1998) 200-215.

[14] Diblík, J., Růžičková, M., *Exponential solutions of equation* $\dot{y}(t) = \beta(t)[y(t-\delta)-y(t-\tau)]$, J. Math. Anal. Appl., **294** (2004) 273-287.

[15] Domshlak, Y., Stavroulakis, I.P., *Oscillations of differential equations with deviating arguments in a critical state,* Dyn. Sys. Appl. **7** (1998), 405-414.

[16] Džurina, J., *Comparison Theorems for Functional Differential Equations,* EDIS, Žilina, 2002.

[17] Erbe, L.H., Qingkai Kong, Zhang, B.G., *Oscillation Theory for Functional Differential Equations,* Marcel Dekker, Inc., 1995.

[18] Györi, I., Pituk, M., *Comparison theorems and asymptotic equilibrium for delay differential and difference equations,* Dynam. Systems Appl. **5** (1996), 277-302.

[19] Györi, I., Ladas, G., *Oscillation Theory of Delay Differential Equations,* Clarendon Press (1991).

[20] Györi, I., Pituk, M., *Special solutions for neutral functional differential equations,* J. of Inequal. & Appl. **6** (2001), 99-117.

[21] Györi, I., Pituk, M., L^2-*perturbation of a linear delay differential equation,* J. Math. Anal. Appl. **195** (1995), 415-427.

[22] Krisztin, T., *Asymptotic estimation for functional differential equations via Lyapunov functions,* Coll. Math. Soc. János Bolyay **53** (1988), Szeged, Hungary, 365-376.

[23] Mahler, K., *On a special functional equation,* J. London Math. Soc. **15** (1940), 115-123.

[24] Murakami, K., *Asymptotic constancy for systems of delay differential equations,* Nonl. Anal. TMA **30** (1997), 4595-4606.

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ANALYSIS OF AN INTEGRAL EQUATION WITH MODIFIED ARGUMENT

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Abstract. This paper contains a study of the Fredholm integral equation with modified argument

$$x(t) = \int_a^b K(t,s,x(s),x(g(s)),x(a),x(b))ds + f(t), \;\; t \in [a,b], \tag{1}$$

concerning:

- the existence and uniqueness of the solution using Schauder's theorem and Contractions Principle;

- continuous dependence on data of the solution using data dependence general theorem;

- approximation of the solution using successive approximations method with two quadrature formula: the trapezoidal rule and the rectangle quadrature formula.

1. Notations and preliminaries

Let X be a nonempty set, $A : X \to X$ an operator and we shall use the following notation:

$$F_A := \{x \in X|\ A(x) = x\} \text{ - the fixed point set of } A.$$

We consider the Banach space $X = C[a,b]$ endowed with the Chebyshev norm $\|\cdot\|$.

In the section 2 we need the following results (see [2], [8], [9], [10] and [12]).

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Theorem 1.1. (*Schauder*). Let X be a Banach space and $Y \subset X$ a nonempty, bounded, convex and closed set. If $A : Y \to Y$ is a completely continuous operator, then A has at least one fixed point.

Theorem 1.2. (*Contractions Principle*). Let (X, d) be a complete metric space and $A : X \to X$ an α-contraction $(\alpha < 1)$. In these conditions we have:

(i) $F_A = \{x^*\}$;

(ii) $A^n(x_0) \to x^*$, as $n \to \infty$;

(iii) $d(x^*, A^n(x_0)) \leq \dfrac{\alpha^n}{1-\alpha} d(x_0, A(x_0))$.

In the section 3 we need the following result (see [2], [8], [9], [10] and [12]).

Theorem 1.3. (*Dependence on data).* Let (X, d) be a complete metric space and $A, B : X \to X$ two operators. We suppose that:

(i) A is an α-contraction $(\alpha < 1)$ and $F_A = \{x^*\}$;

(ii) $x_B^* \in F_B$;

(iii) there exist $\eta > 0$ such that $d(A(x), B(x)) < \eta$ for all $x \in X$.

In these conditions we have

$$d(x_A^*, x_B^*) \leq \frac{\eta}{1-\alpha} .$$

In the section 4 we need the following results (see [2], [7], [8], [9] and [12]).

We will use for the calculus of the integrals of the succesive approximations sequence, two quadrature formulae:

1) The trapezoidal rule

$$\int_a^b f(x)dx = \frac{b-a}{2n}\left[f(a) + 2\sum_{i=1}^{n-1} f(x_i) + f(b)\right] + R^T(f)\,, \tag{1}$$

with a very sharp division of the interval $[a, b]$ through the points $a = x_0 < x_1 < \ldots < x_n = b$ and $f \in C^2[a, b]$. We use for the rest of the formula $R^T(f) = \sum\limits_{i=1}^{n} R_i^T(f)$ the following estimation:

$$\left|R^T(f)\right| \leq M^T \frac{(b-a)^3}{12n^2} . \tag{2}$$

2) The rectangle quadrature formula

(a) If we consider the intermediary points of the division of the interval $[a,b]$ at the left terminal point of the partial intervals $[x_i,\, x_{i+1}]$, $\xi_i = x_i$, we will have the following formula:

$$\int_a^b f(x)dx = \frac{b-a}{n}\left[f(a) + \sum_{i=1}^{n-1} f(x_i)\right] + R^D(f)\,, \tag{3}$$

or

(b) If we consider the intermediary points of the division of the interval $[a,b]$ at the right terminal point of the partial intervals $[x_i,\, x_{i+1}]$, $\xi_i = x_{i+1}$, we will have the following formula:

$$\int_a^b f(x)dx = \frac{b-a}{n}\left[\sum_{i=1}^{n-1} f(x_i) + f(b)\right] + R^D(f)\ , \tag{4}$$

with a very sharp division of the interval $[a,b]$ through the points $a = x_0 < x_1 < \ldots < x_n = b$ and $f \in C^1[a,b]$. We use for the rest of the formula $R^D(f) = \sum\limits_{i=1}^{n} R_i^D(f)$ the following estimation:

$$\left|R^D(f)\right| \le M^D \frac{(b-a)^2}{n}\,. \tag{5}$$

2. **Existence of the solution**

Theorems of existence of the solution for several type of integral equations with modified argument have been presented in the papers [1], [2], [5], [9], [10], [11], [12] **.**

In what follows we will establish theorems of existence of the solution of the integral equation (1) in $C[a,b]$ and in the $\overline{B}(f;R)$ sphere.

A. Existence of the solution in $C[a,b]$

Let us consider the Fredholm integral equation with modified argument (1) and assume that the following conditions are satisfied:

(a_1) $K \in C([a,b]\times[a,b]\times\mathbb{R}^4)$;

(a_2) $f \in C[a,b]$;

(a_3) $g \in C\left([a,b],\, [a,b]\right)$.

Theorem 2.1. Suppose (a_1)-(a_3) are satisfied. In addition suppose

(a_4) there exist $M_K > 0$ such that

$$|K(t,s,u_1,u_2,u_3,u_4)| \leq M_K\,, \quad for\ \ all\ \ t \in [a,b],\ u_1,u_2,u_3,u_4 \in \mathbb{R}\,.$$

Then the integral equation (1) has at least one solution $x^* \in C[a,b]$.

Proof. We attach to the integral equation (1), the operator $A : C[a,b] \to C[a,b]$, defined by

$$A(x)(t) := \int_a^b K(t,s,x(s),x(g(s)),x(a),x(b))ds + f(t), \tag{6}$$

for all $t \in [a,b]$.

We have

$$\|A(x)\|_{C[a,b]} \leq \|f\|_{C[a,b]} + M_K(b-a)\,, \quad for\ \ all\ \ x \in C[a,b]\ .$$

Let $Y \subset C[a,b]$ be a nonempty, bounded subset. Then $A(Y)$ is also a bounded subset. From the uniform continuity of K with respect to t, it follows that the operator A is continuous and that the subset $A(Y)$ is equicontinuous. Therefore $\overline{A(Y)}$ is a compact subset.

Let be $Y = \overline{conv}A(C[a,b])$ and now Y is a nonempty, bounded, convex and closed subset. We consider the operator $A : Y \to Y$ also noted with A and defined by same relation (7). Y is an invariant subset by A.

On the other hand, by *Arzela-Ascoli theorem*, A is completely continuous.

The conditions of the *Schauder's theorem* are satisfied. □

We have the following theorem of existence and uniqueness of the solution of the integral equation (1) in $C[a,b]$:

Theorem 2.2. Suppose (a_1)-(a_3) are satisfied. In addition suppose

(a_5) there exist $L > 0$ such that

$$|K(t,s,u_1,u_2,u_3,u_4) - K(t,s,v_1,v_2,v_3,v_4)| \leq$$

$$\leq L\left(|u_1-v_1| + |u_2-v_2| + |u_3-v_3| + |u_4-v_4|\right),$$

for all $t,s \in [a,b],\ u_i,\ v_i \in \mathbb{R}$, $i = \overline{1,4}$;

(a_6) $4L(b-a) < 1$.

Then the integral equation (1) has a unique solution $x^* \in C[a,b]$.

Proof. We attach to the integral equation (1), the operator $A : C[a,b] \to C[a,b]$, defined by the relation (7). The set of the solutions of the integral equation (1) coincide with the set of fixed points of the operator A. By (a_5) and using the Chebyshev norm, we have

$$\|A(x_1) - A(x_2)\|_{C[a,b]} \leq 4L\,(b-a)\,\|x_1 - x_2\|_{C[a,b]}$$

and therefore, by (a_6) it result that the operator A is an α-contraction with the coefficient $\alpha = 4L\,(b-a)$. The conclusion result from the *Contractions Principle.* □

B. Existence of the solution in the $\overline{B}(f;R)$ sphere

We suppose the following conditions are satisfied:

(a_1') $K \in C([a,b] \times [a,b] \times J^4)$, $J \subset \mathbb{R}$ closed interval;

and (a_2), (a_3).

In addition, we denote M_K a positive constant such that, for the restriction $K|_{[a,b]\times[a,b]\times J^4}$, $J \subset \mathbb{R}$ compact, we have

$$|K(t,s,u_1,u_2,u_3,u_4)| \leq M_K\,, \quad for\ \ all\ \ t \in [a,b],\ u_1,u_2,u_3,u_4 \in J\,. \tag{7}$$

We have the following theorem of existence of the solution of the integral equation (1) in $\overline{B}(f;R) \subset C[a,b]$:

Theorem 2.3. Suppose (a_1'), (a_2), (a_3) are satisfied. In addition suppose

(b_1) $M_K(b-a) \leq R$ (the invariability condition of the $\overline{B}(f;R)$ sphere).

Then the integral equation (1) has at least one solution $x^* \in \overline{B}(f;R) \subset C[a,b]$.

Proof. We attach to the integral equation (1), the operator $A : \overline{B}(f;R) \to C[a,b]$, defined by the relation (7), where R is a real positive number which satisfies the condition below:

$$\left[x \in \overline{B}(f;R)\right] \Longrightarrow [x(t) \in J \subset \mathbb{R}]$$

and we suppose that there exist at least one number R with this property.

We establish under what conditions, the $\overline{B}(f;R)$ sphere is an invariant set for the operator A . We have

$$|A(x)(t)-f(t)| = \left| \int_a^b K(t,s,x(s),x(g(s)),x(a),x(b))ds \right| \leq$$

$$\leq \int_a^b |K(t,s,x(s),x(g(s)),x(a),x(b))|\, ds$$

and by (8) we have

$$|A(x)(t)-f(t)| \leq M_K(b-a), \;\; for \;\; all \;\; t \in [a,b]\,,$$

and then by (b_1) it result that the $\overline{B}(f;R)$ sphere is an invariant set for the operator A. Now we have the operator $A : \overline{B}(f;R) \to \overline{B}(f;R)$, also noted with A, defined by same relation, where $\overline{B}(f;R)$ is a closed subset of the Banach space $C[a,b]$.

Next we assure the conditions of the *Schauder's theorem.*

We have

$$\|A(x)\|_{C[a,b]} \leq \|f\|_{C[a,b]} + R\,, \;\; for \;\; all \;\; x \in \overline{B}(f;R)$$

and it follows that the subset $A(\overline{B}(f;R))$ is bounded. From the uniform continuity of K with respect to t, it follows that the subset $A(\overline{B}(f;R))$ is equicontinuous. Now it result that $A(\overline{B}(f;R))$ is a compact subset.

Also, from the uniform continuity of K with respect to t, it follows that the operator A is continuous. On the other hand, by *Arzela-Ascoli theorem*, A is completely continuous. The proof follows the *Schauder's theorem.* □

Theorem 2.4. Suppose the conditions (a_1'), (a_2), (a_3), (b_1) and (a_6) are satisfied. In addition suppose

(b_2) there exist $L > 0$ such that

$$|K(t,s,u_1,u_2,u_3,u_4) - K(t,s,v_1,v_2,v_3,v_4)| \leq$$

$$\leq L\left(|u_1-v_1| + |u_2-v_2| + |u_3-v_3| + |u_4-v_4|\right),$$

for all $t,s \in [a,b]$, u_i, $v_i \in J$, $i = \overline{1,4}$;

Then the integral equation (1) has a unique solution $x^* \in \overline{B}(f;R) \subset C[a,b]$.

Proof. We attach to the integral equation (1), the operator $A : \overline{B}(f;R) \to C[a,b]$, defined by the relation (7), where R is a real positive number which satisfies the condition below:

$$\left[x \in \overline{B}(f;R)\right] \Longrightarrow [x(t) \in J \subset \mathbb{R}]$$

and we suppose that there exist at least one number R with this property.

If we use a reasoning as the one used in the proof of theorem 2.3, we will obtain that the $\overline{B}(f;R)$ sphere is an invariant set for the operator A, and the invariability condition (b_1), of the $\overline{B}(f;R)$ sphere is hold.

Now we have the operator $A : \overline{B}(f;R) \to \overline{B}(f;R)$, also noted with A, defined by same relation, where $\overline{B}(f;R)$ is a closed subset of the Banach space $C[a,b]$. The set of the solutions of the integral equation (1) coincide with the set of fixed points of the operator A.

By a similar reasoning as in the proof of theorem 2.2 and using the condition (b_2) it result that the operator A is an α-contraction with the coefficient $\alpha = 4L\,(b-a)$
.

Now the proof result from the *Contractions Principle.* □

3. Dependence on data

Theorems of dependence on data for several type of integral equations with modified argument have been presented in the papers [5], [6], [9], [12].

In what follows we consider the integral equation (1) and we will study the dependence of the solution of the integral equation (1) with respect to K and f .

Now we consider the perturbed integral equation

$$y(t) = \int_a^b H(t,s,y(s),y(g(s)),y(a),y(b))ds + h(t)\,, \quad t \in [a,b] \tag{8}$$

and we have the following theorem of dependence on data of the solution of the integral equation (1):

Theorem 3.1. Suppose

(i) the conditions of the theorem 2.2 are satisfied and denote x^* the unique solution of the integral equation (1).

(ii) $H \in C([a,b] \times [a,b] \times \mathbb{R}^4)$ and $h \in C[a,b]$;

(iii) there exist $\eta_1,\ \eta_2 > 0$ such that

$$|K(t,s,u_1,u_2,u_3,u_4) - H(t,s,u_1,u_2,u_3,u_4)| \leq \eta_1 \, ,$$

for all $t,s \in [a,b]$, $u_1,u_2,u_3,u_4 \in \mathbb{R}$ and

$$|f(t) - h(t)| \leq \eta_2 \quad for \ \ all \quad t \in [a,b] \, .$$

In these conditions, if y^* is a solution of the integral equation (9), then we have:

$$\|x^* - y^*\| \leq \frac{\eta_1(b-a) + \eta_2}{1 - 4L(b-a)} \, .$$

Proof. We consider the operator A which appear in the proof of the theorem 2.2.

Let $B : C[a,b] \to C[a,b]$ be an operator defined by

$$B(y)(t) = \int_a^b H(t,s,y(s),y(g(s)),y(a),y(b))ds + h(t) \, , \quad t \in [a,b] \ .$$

By the condition (iii) we have

$$\|A(x) - B(x)\| \leq \eta_1(b-a) + \eta_2 \ .$$

The proof result from *data dependence general theorem.* □

4. **Approximation of the solution**

Approximative methods for various type of integral equations with modified argument have been presented in the papers [1], [2], [3], [4], [7], [8], [9] .

We will determine as follows, a method for the approximation of the solution of the integral equation (1).

We suppose that the conditions of one of the two existence and uniqueness theorems from section 2 are satisfied. In order to lay down the ideas we consider the case of the integral equation (1) with a unique solution in the sphere $\overline{B}(f;R) \subset C[a,b]$

(theorem 2.4), called x^* and established using the successive approximation method. We have the sequence of the successive approximations:

$$\begin{aligned} x_0(t) &= f(t) \\ x_1(t) &= \int_a^b K(t,s,x_0(s),x_0(g(s)),x_0(a),x_0(b))ds + f(t) \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ x_m(t) &= \int_a^b K(t,s,x_{m-1}(s),x_{m-1}(g(s)),x_{m-1}(a),x_{m-1}(b))ds + f(t) \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{aligned}$$

and we consider a division of the interval $[a,b]$ through the points $a = x_0 < x_1 < \ldots < x_n = b$.

A. Approximation of the solution using the trapezoidal rule

We suppose that:

(h_{11}) $K \in C^2([a,b] \times [a,b] \times J^4)$, $J \subset \mathbb{R}$ closed interval ;

(h_{12}) $f \in C^2[a,b]$;

(h_{13}) $g \in C^2([a,b],[a,b])$

and we will approximate the terms of the successive approximations sequence using the trapezoidal rule (2) with the rest from (3). Generally, for the term $x_m(t_k)$ we have

$$\begin{aligned} x_m(t_k) = \frac{b-a}{2n} & [K(t_k,a,x_{m-1}(a),x_{m-1}(g(a)),x_{m-1}(a),x_{m-1}(b)) + \\ & +2\sum_{i=1}^{n-1} K(t_k,t_i,x_{m-1}(t_i),x_{m-1}(g(t_i)),x_{m-1}(a),x_{m-1}(b)) + \\ & +K(t_k,b,x_{m-1}(b),x_{m-1}(g(b)),x_{m-1}(a),x_{m-1}(b))] + f(t_k) + R^T_{m,k} , \end{aligned} \tag{9}$$

$k = \overline{0,n}$, $m \in \mathbb{N}$, with the estimation of the rest

$$\left|R^T_{m,k}\right| \le \frac{(b-a)^3}{12n^2} \cdot \max_{s\in[a,b]} \left|[K(t_k,s,x_{m-1}(s),x_{m-1}(g(s)),x_{m-1}(a),x_{m-1}(b))]''_s\right| .$$

According to (h_{11}) it result that the derivative of the function K from the expression of the rest $R^T_{m,k}$ exist and has the following form:

$$[K(t_k, s, x_{m-1}(s), x_{m-1}(g(s)), x_{m-1}(a), x_{m-1}(b))]''_s = \frac{\partial^2 K}{\partial s^2} +$$

$$+2\frac{\partial^2 K}{\partial s \partial x_{m-1}} \cdot x'_{m-1}(s) + 2\frac{\partial^2 K}{\partial s \partial x_{m-1}} \cdot \frac{\partial x_{m-1}}{\partial g} \cdot g'(s) +$$

$$+\frac{\partial^2 K}{\partial x^2_{m-1}} \left(x'_{m-1}(s)\right)^2 + 2\frac{\partial^2 K}{\partial x^2_{m-1}} \cdot \frac{\partial x_{m-1}}{\partial g} \cdot x'_{m-1}(s) \cdot g'(s) +$$

$$+\frac{\partial K}{\partial x_{m-1}} \cdot x''_{m-1}(s) + \frac{\partial^2 K}{\partial x^2_{m-1}} \cdot \left(\frac{\partial x_{m-1}}{\partial g}\right)^2 \cdot \left(g'(s)\right)^2 +$$

$$+\frac{\partial K}{\partial x_{m-1}} \cdot \frac{\partial^2 x_{m-1}}{\partial g \partial s} \cdot x'_{m-1}(s) \cdot g'(s) + \frac{\partial K}{\partial x_{m-1}} \cdot \frac{\partial x_{m-1}}{\partial g} \cdot g''(s),$$

where

$$x^{(\alpha)}_{m-1}(t) = \int_a^b \frac{\partial^{(\alpha)} K(t, s, x_{m-2}(s), x_{m-2}(g(s)), x_{m-2}(a), x_{m-2}(b))}{\partial t^{(\alpha)}} ds + f^{(\alpha)}(t),$$

$\alpha = 1, 2$.

If we denote

$$M^T_1 = \max_{|\alpha| \le 2, t, s \in [a,b]} \left| \frac{\partial^{|\alpha|} K}{\partial t^{\alpha_1} \partial s^{\alpha_2} \partial u_1^{\alpha_3} \partial u_2^{\alpha_4} \partial u_3^{\alpha_5} \partial u_4^{\alpha_6}} \right| ,$$

$$M^T_2 = \max_{\alpha \le 2, t \in [a,b]} \left| f^{(\alpha)}(t) \right| , \quad M^T_3 = \max_{\alpha \le 2, t \in [a,b]} \left| g^{(\alpha)}(t) \right| ,$$

then we obtain for $x_{m-1}(t)$ and its derivative, the following estimations:

$$\left| x^{(\alpha)}_{m-1}(t) \right| \le M^T_1(b-a) + M^T_2 , \quad \alpha = \overline{0,2}$$

while for the derivative of function K, we have

$$[K(t_k, s, x_{m-1}(s), x_{m-1}(g(s)), x_{m-1}(a), x_{m-1}(b))]''_s \le M^T_1 \left\{1 + 3\left[M^T_1(b-a) + \right.\right.$$

$$\left.\left. +M^T_2\right]\left(1 + M^T_3\right) + \left[M^T_1(b-a) + M^T_2\right]^2 \left[1 + 3M^T_3 + \left(M^T_3\right)^2\right]\right\} = M^T_0 .$$

It is obvious that M^T_0 doesn't depend on m and k, so the estimation of the rest is

$$\left|R^T_{m,k}\right| \le M^T_0 \cdot \frac{(b-a)^3}{12n^2} , \tag{10}$$

where $M_0^T = M_0^T\left(K, D^{(\alpha)}K, f, D^{(\alpha)}f, g, D^{(\alpha)}g\right)$, $|\alpha| \leq 2$, and we obtain a formula for the approximative calculus of the integrals of the successive approximations sequence. Using the method of successive approximations and the formula (10) with the estimation of the rest resulted from (11), we suggest further on an algorithm in order to solve the integral equation (1) approximately. To this end, we will calculate approximately the terms of the successive approximations sequence and through induction we obtain

$$\begin{aligned} x_m(t_k) &= \frac{b-a}{2n}\left[K(t_k, a, \widetilde{x}_{m-1}(a), \widetilde{x}_{m-1}(g(a)), \widetilde{x}_{m-1}(a), \widetilde{x}_{m-1}(b))+\right.\\ &\quad +2\sum_{i=1}^{n-1} K(t_k, t_i, \widetilde{x}_{m-1}(t_i), \widetilde{x}_{m-1}(g(t_i)), \widetilde{x}_{m-1}(a), \widetilde{x}_{m-1}(b))+\\ &\quad \left.+K(t_k, b, \widetilde{x}_{m-1}(b), \widetilde{x}_{m-1}(g(b)), \widetilde{x}_{m-1}(a), \widetilde{x}_{m-1}(b))\right]+\\ &\quad +f(t_k) + \widetilde{R}^T_{m,k} = \widetilde{x}_m(t_k) + \widetilde{R}^T_{m,k}\ , \quad k = \overline{0,n} \end{aligned}$$

where

$$\left|\widetilde{R}^T_{m,k}\right| \leq \frac{(b-a)^3}{12n^2} M_0^T\left[4^{m-1}L^{m-1}(b-a)^{m-1} + \ \ldots\ + 1\right], \quad k = \overline{0,n}\ .$$

Since the conditions of theorem 2.4 are satisfied we have $4L(b-a) < 1$, and it result the estimation:

$$\left|\widetilde{R}^T_{m,k}\right| \leq \frac{(b-a)^3}{12n^2\left[1-4L(b-a)\right]} M_0^T\ .$$

We have thus obtained the sequence $\left(\widetilde{x}_m(t_k)\right)_{m\in\mathbb{N}}$, $k = \overline{0,n}$, that estimates the successive approximations sequence $(x_m)_{m\in\mathbb{N}}$ using a division of the interval $[a,b]$, $a = x_0 < x_1 < \ldots < x_n = b$, with the following error in calculus:

$$|x_m(t_k) - \widetilde{x}_m(t_k)| \leq \frac{(b-a)^3}{12n^2\left[1-4L(b-a)\right]} M_0^T\ .$$

B. Approximation of the solution using the rectangle quadrature formula

We suppose that:

(h_{21}) $K \in C^1([a,b]\times[a,b]\times J^4)$, $J \subset \mathbb{R}$ closed interval ;

(h_{22}) $f \in C^1[a,b]$;

(h_{23}) $g \in C^1([a,b],[a,b])$

and we will approximate the terms of the successive approximations sequence using the rectangle quadrature formula (4) with the rest from (5). Generally, for the term $x_m(t_k)$ we have

$$x_m(t_k) = \frac{b-a}{2n}\left[K(t_k,a,x_{m-1}(a),x_{m-1}(g(a)),x_{m-1}(a),x_{m-1}(b)) + \right. \tag{11}$$
$$\left. + \sum_{i=1}^{n-1} K(t_k,t_i,x_{m-1}(t_i),x_{m-1}(g(t_i)),x_{m-1}(a),x_{m-1}(b))\right] +$$
$$+ f(t_k) + R^D_{m,k}\,, \quad k=\overline{0,n},\ m\in\mathbb{N}$$

with the estimation of the rest

$$\left|R^D_{m,k}\right| \le \frac{(b-a)^2}{n}\cdot \max_{s\in[a,b]}\left|\left[K(t_k,s,x_{m-1}(s),x_{m-1}(g(s)),x_{m-1}(a),x_{m-1}(b))\right]'_s\right| .$$

According to (h_{21}) it result that the derivative of the function K from the expression of the rest $R^D_{m,k}$ exist and has the following form:

$$\left[K(t_k,s,x_{m-1}(s),x_{m-1}(g(s)),x_{m-1}(a),x_{m-1}(b))\right]'_s = \frac{\partial K}{\partial s} +$$
$$+ \frac{\partial K}{\partial x_{m-1}}\cdot x'_{m-1}(s) + \frac{\partial K}{\partial x_{m-1}}\cdot\frac{\partial x_{m-1}}{\partial g}\cdot g'(s)\ ,$$

where

$$x'_{m-1}(t) = \int_a^b \frac{\partial K(t,s,x_{m-2}(s),x_{m-2}(g(s)),x_{m-2}(a),x_{m-2}(b))}{\partial t}ds + f'(t)\ .$$

If we denote

$$M^D_1 = \max_{|\alpha|\le 1,t,s\in[a,b]}\left|\frac{\partial^{|\alpha|}K}{\partial t^{\alpha_1}\partial s^{\alpha_2}\partial u_1^{\alpha_3}\partial u_2^{\alpha_4}\partial u_3^{\alpha_5}\partial u_4^{\alpha_6}}\right| ,$$
$$M^D_2 = \max_{\alpha\le 1,t\in[a,b]}\left|f^{(\alpha)}(t)\right| ,\quad M^D_3 = \max_{\alpha\le 1,t\in[a,b]}\left|g^{(\alpha)}(t)\right|$$

then we obtain for $x'_{m-1}(t)$ the following estimation:

$$\left|x'_{m-1}(t)\right| \le M^D_1(b-a) + M^D_2\ ,$$

while for the derivative of function K, we have

$$\left[K(t_k,s,x_{m-1}(s),x_{m-1}(g(s)),x_{m-1}(a),x_{m-1}(b))\right]'_s \le$$

$$\leq M_1^D \left\{1 + \left[M_1^D(b-a) + M_2^D\right]\left(1 + M_3^D\right)\right\} = M_0^D \ .$$

It is obvious that M_0^D doesn't depend on m and k, so the estimation of the rest is

$$\left|R_{m,k}^D\right| \leq M_0^D \cdot \frac{(b-a)^2}{n}\,, \tag{12}$$

where $M_0^D = M_0^D\left(K, D^{(\alpha)}K, f, D^{(\alpha)}f, g, D^{(\alpha)}g\right)$, $\alpha = 1$, and we obtain a formula for the approximative calculus of the integrals of the successive approximations sequence. Using the method of successive approximations and the formula (12) with the estimation of the rest resulted from (13), we suggest further on an algorithm in order to solve the integral equation (1) approximately. To this end, we will calculate approximately the terms of the successive approximations sequence and through induction we obtain

$$\begin{aligned} x_m(t_k) \quad = \quad & \frac{b-a}{n}\Big[K(t_k, a, \widetilde{x}_{m-1}(a), \widetilde{x}_{m-1}(g(a)), \widetilde{x}_{m-1}(a), \widetilde{x}_{m-1}(b)) + \\ & + \sum_{i=1}^{n-1} K(t_k, t_i, \widetilde{x}_{m-1}(t_i), \widetilde{x}_{m-1}(g(t_i)), \widetilde{x}_{m-1}(a), \widetilde{x}_{m-1}(b))\Big] + \\ & + f(t_k) + \widetilde{R}_{m,k}^D = \widetilde{x}_m(t_k) + \widetilde{R}_{m,k}^D \ , \quad k = \overline{0,n} \quad , \end{aligned}$$

where

$$\left|\widetilde{R}_{m,k}^D\right| \leq \frac{(b-a)^2}{n} M_0^D\left[4^{m-1}L^{m-1}(b-a)^{m-1} + \ \dots \ + 1\right]\,, \quad k = \overline{0,n} \ .$$

Since the conditions of theorem 2.4 are satisfied we have $4L(b-a) < 1$, and it result the estimation:

$$\left|\widetilde{R}_{m,k}^D\right| \leq \frac{(b-a)^2}{n\left[1 - 4L(b-a)\right]} M_0^D \ ,$$

and we have thus obtained the sequence $\left(\widetilde{x}_m(t_k)\right)_{m\in\mathbb{N}}$, $k = \overline{0,n}$, that estimates the successive approximations sequence $(x_m)_{m\in\mathbb{N}}$ using a division of the interval $[a,b]$, $a = x_0 < x_1 < \ldots < x_n = b$, with the following error in calculus:

$$|x_m(t_k) - \widetilde{x}_m(t_k)| \leq \frac{(b-a)^2}{n\left[1 - 4L(b-a)\right]} M_0^D \ .$$

References

[1] Ambro, M., *Aproximarea soluţiilor unei ecuaţii integrale cu argument modificat*, Studia Univ. Babeş-Bolyai, Math., **2**(1978), 26-32.

[2] Coman, Gh., Pavel, G., Rus, I., Rus, I.A., *Introducere în teoria ecuaţiilor operatoriale*, Ed. Dacia, Cluj-Napoca, 1976.

[3] Dobriţoiu, M., *The rectangle method for approximating the solution to a Fredholm integral equation with a modified argument*, Lucrările ştiinţifice a celei de a XXX-a Sesiuni de comunicări ştiinţifice cu participare internaţională "TEHNOLOGII MODERNE ÎN SECOLUL XXI", Academia Tehnică Militară Bucureşti, secţiunea Matematică, 2003, 36-39.

[4] Dobriţoiu, M., *A Fredholm integral equation - numerical methods*, Bulletins for Applied&Computer Mathematics, Budapest, BAM - CVI / 2004, Nr. 2188, 285-292.

[5] Dobriţoiu, M., *An integral equation with modified argument*, Studia Univ. Babeş-Bolyai, Cluj-Napoca, Mathematica, vol. XLIX, nr. 3/2004, 27-33.

[6] Dobriţoiu, M., *Existence and continuous dependence on data of the solution of an integral equation*, Bulletins for Applied&Computer Mathematics, Budapest, BAM - CVI / 2005 , to appear.

[7] Ionescu, D.V., *Cuadraturi numerice*, Ed. Tehnică, Bucureşti, 1957.

[8] Precup, R., *Methods in nonlinear integral equations*, Dordrecht ; Boston : Kluwer Academic Publishers, 2002.

[9] Rus, I.A., *Principii şi aplicaţii ale teoriei punctului fix*, Ed. Dacia, Cluj-Napoca, 1979.

[10] Rus, I.A., *Generalized contractions*, Univ. of Cluj-Napoca, Preprint Nr. 3, 1983, 1-130.

[11] Rus, I.A., *Picard operators and applications*, Babeş-Bolyai Univ. of Cluj-Napoca, Preprint Nr. 3, 1996.

[12] Rus, I.A., *Ecuaţii diferenţiale, ecuaţii integrale şi sisteme dinamice*, Casa de editură Transilvania Press, Cluj-Napoca, 1996.

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NEW INVERSE INTERPOLATION METHODS

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Abstract. The goal of this paper is to give some numerical methods for the solution of nonlinear equations, generated by inverse interpolation of Abel Goncharov type and a particular case of Lidstone inverse interpolation.

1. Preliminars

Let $\Omega \subset \mathbf{R}$ and $f : \Omega \to \mathbf{R}$. Consider the equation

$$f(x) = 0, \quad x \in \Omega, \tag{1}$$

and attach to it a mapping

$$F : D \to D, \quad D \subset \Omega^n.$$

Let $x_0, ..., x_{n-1} \in D$. Using the mapping F and the numbers $x_0, ..., x_{n-1}$ we construct iteratively the sequence

$$x_0, x_1, ..., x_{n-1}, x_n, ... \tag{2}$$

where

$$x_i = F(x_{i-n}, ..., x_{i-1}), \quad i = n, ... \tag{3}$$

The problem is to choose F and the numbers $x_0, ..., x_{n-1} \in D$ such that sequence (2) converges to a solution of equation (1).

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Definition 1. *The method of approximating a solution of equation (1) by the elements of sequence (2), computed as in (3) is called F - method attached to equation (1) and to the values $x_0, ..., x_{n-1}$. Numbers $x_0, ..., x_{n-1}$ are called starting values, and the pth element of sequence (2) is called pth order approximation of the solution. If the set of the starting values consists of a single element, the corresponding F - method is called one step method, otherwise it is called multi-step method.*

Definition 2. *If sequence (2) converges to a solution of equation (1), F - method is said to be convergent, otherwise is divergent.*

Definition 3. *Let $x^* \in \Omega$ be a solution of equation (1) and let $x_0, ..., x_n, ...$ be a sequence generated by a given F - method. Number $p = p(F)$ having the property*

$$\lim_{x_i \to x^*} \frac{x^* - F(x_{i-n+1}, ..., x_i)}{(x^* - x_i)^p} = C \neq 0, \tag{4}$$

is called order of the F - method, and constant C is the asymptotical error.

Let $x^* \in \Omega$ be a solution of the equation (1) and $V(x^*)$ a neighborhood of x^*. Assume that f has inverse on $V(x^*)$ and denote $q = f^{-1}$. Since $f(x^*) = 0$, it follows that $x^* = g(0)$. This way, the approximation of the solution x^* is reduced to the approximation of the $g(0)$. The approximation of the inverse g by means of a certain interpolating method, and x^* by the value of the interpolating element at point zero is called inverse interpolation procedure. This approach generates a large number of approximation methods for the solution of an equation (thus for the zeros of a function), according to the employed interpolation method.

Such examples of methods, based on Taylor, Lagrange and Hermite inverse interpolation are:

Let x^* be a solution of $f(x) = 0$, $V(x^*)$ a neighbourhood of x^*, $f \in C^m[V(x^*)], f'(x) \neq 0$ for $x \in V(x^*)$ and $x_i \in V(x^*)$. Using Taylor polynomial of the degree $m-1$, that interpolates the function $g = f^{-1}$, one obtains the one step method [2]:

$$F_m^T(x_i) = x_i + \sum_{k=1}^{m-1} \frac{(-1)^k}{k!}[f(x_i)]^k g^{(k)}(f(x_i)). \tag{5}$$

Also, if $g^{(m)}(0) \neq 0$, we have $ord(F_m^T) = m$.

Based on Lagrange interpolation, it follows the multistep method [2]

$$F_m^L(x_0, ..., x_m) = \sum_{k=0}^{m} \frac{f_0 ... f_{k-1} f_{k+1} ... f_m}{(f_0 - f_k)... / ...(f_m - f_k)} x_k \tag{6}$$

where $f_k = f(x_k)$, is a multistep method based on inverse Lagrange interpolation.

The order of this method is the solution of equation:

$$t^{m+1} - t^m - ... - t - 1 = 0.$$

More general methods are generated by Hermite and Birkhoff interpolation [2], [5]. Such, let x^* be a solution of the equation (1), $V(x^*)$ a neighbourhood of x^* and $x_0, x_1 ..., x_m \in V(x^*)$.For $n = r_0 + ... + r_m + m$, where r_k represents the multiplicity order of the point x_k, $k = 0, ..., m$, if $f \in C^{n+1}(V(x^*))$ and $f'(x) \neq 0$ for $x \in V(x^*)$, we have the following Hermite approximation method:

$$F_n^H(x_0, ..., x_m) = \sum_{k=0}^{m} \sum_{j=0}^{r_k} \sum_{\upsilon=0}^{r_k - j} \frac{(-1)^{j+\upsilon}}{j! \upsilon!} f_k^{j+\upsilon} v_k(0) \left(\frac{1}{v_k(y)}\right)_{y=f_k}^{(\upsilon)} g^{(j)}(f_k) \tag{7}$$

where $f_k = f(x_k), k = 0, ..., m, g = f^{-1}$, and

$$v_k(y) = (y - f_0)^{r_0+1} ... (y - f_{k-1})^{r_{k-1}+1} (y - f_{k+1})^{r_{k+1}+1} ... (y - f_m)^{r_m+1}$$

The order of F_n^H, is [5] the unique real positive root of the equation:

$$t^{m+1} - r_m t^m - r_{m-1} t^{m-1} - .... - r_1 t - r_0 = 0. \tag{8}$$

where $r_0, ..., r_m$ are permutation of the multiplicity orders of the nodes x_k, $k = 0, ..., m$ satisfying the conditions:

$$(1) \quad r_0 + r_1 + ... + r_m > 1$$

$$(2) \quad r_m \geq r_{m-1} \geq ... \geq r_1 \geq r_0,$$

respectively of the equation:

$$t^{m+1} - (r+1)\sum_{j=0}^{m} t^j = 0. \tag{9}$$

if $r_0 = ... = r_m$.

2. Abel-Goncharov inverse interpolation method

On the base of Abel-Goncharov interpolation, we have the following method for the solution of equation $f(x) = 0$:

Theorem 4. *Let $n \in N$; $a, b \in R$; $a < b$; $f : [a,b] \to R$ be a function having n derivatives $f^{(i)}, i = 1, 2, ..., n$. The values $x_i \in [a,b]$, $i = 0, ..., n$ and $f^{(i)}(x_i)$, $i = 0, ..., n$, with $x_i \neq x_j$ for $i \neq j$ are given. Let x^* be the solution of the equation $f(x) = 0$ and $V(x^*)$ a neighborhood of x^*. If $f \in C^{n+1}(V(x^*))$ and $f^{(i)}(x_i) \neq 0$, $i = 0, ..., n$ then we have the following method of Abel-Gonciarov type:*

$$F_n^{AG}(x_0, ..., x_n) = q(y_0) - y_0.q'(y_1) - \sum_{k=2}^{n} \frac{q^{(k)}(y_k)}{k!} \left(\sum_{j=0}^{k-1} g_j(0) \tbinom{k}{j} y_j^{k-1} \right) \tag{10}$$

Proof. Suppose that $\exists\ q = f^{-1}$. Then

$$q = P_n q + R_n q$$

with

$$(P_n q)(y) = \sum_{k=0}^{n} g_k(y) q^{(k)}(y_k)$$

and

$$g_0(y) = 1$$

$$g_1(y) = y - y_0$$

$$g_k(y) = \frac{1}{k!} \left[y^k - \sum_{j=0}^{k-1} g_j(y) \tbinom{k}{j} y_j^{k-1} \right]$$

Because $x^* = q(0)$, $q \simeq P_n q \Longrightarrow x^* \simeq (P_n q)(0)$

$$(P_n q)(0) = \sum_{k=0}^{n} g_k(0) q^{(k)}(y_k)$$

$$(P_n q)(0) = q(y_0) - y_0 . q'(y_1) - \sum_{k=2}^{n} \frac{q^{(k)}(y_k)}{k!} \left(\sum_{j=0}^{k-1} g_j(0) \binom{k}{j} y_j^{k-1} \right)$$

$$\Longrightarrow x^* \simeq q(y_0) - y_0 . q'(y_1) - \sum_{k=2}^{n} \frac{q^{(k)}(y_k)}{k!} \left(\sum_{j=0}^{k-1} g_j(0) \binom{k}{j} y_j^{k-1} \right) :=$$

$$:= F_n^{AG}(x_0, ..., x_n).$$

□

Particular cases.

1). $n = 1$ (nodes x_0, x_1 and $f(x_0), f'(x_1)$ given)

$$F_1^{AG}(x_0, x_1) = q(y_0) - y_0 . q'(y_1)$$

$$F_1^{AG}(x_0, x_1) = q(y_0) - y_0 \frac{1}{f'(x_1)}$$

$$\Longrightarrow F_1^{AG}(x_0, x_1) = x_0 - \frac{f(x_0)}{f'(x_1)} \tag{11}$$

$\Longrightarrow F_1^{AG}(x_0, x_1) = F_1^B(x_0, x_1)$ and the method F_1^{AG} coincide with the method F_1^B generated by the Birkhoff inverse interpolation.

Remark 5. *If $x_0 = x_1 := x_i$ (the nodes coincide), then:*
$F_1^{AG}(x_i) = x_i - \frac{f(xi)}{f'(x_i)} \Longrightarrow$
$F_1^{AG}(x_i) = F_2^T(x_i)$ *and the method coincide with the method F_2^T generated by inverse interpolation Taylor for two nodes.*

The order of this method is the solution of the equation:

$$t^2 - t - 1 = 0$$

so

$$ord(F_1^{AG}) = \frac{1 + \sqrt{5}}{2}$$

2). $n = 2$. ($x_0, f(x_0), x_1, f'(x_1), x_2, f''(x_2)$ given)

$$g_0(0) = 1$$

$$g_1(0) = -y_0$$

$$g_2(0) = \frac{1}{2}[2y_0y_1 - y_0^2]$$

$$\Longrightarrow (P_2q)(0) = q(y_0) - y_0.q'(y_1) - \tfrac{1}{2}[2y_0y_1 - y_0^2].q''(y_2) =$$
$$= x_0 - \frac{f(x_0)}{f'(x_1)} - \frac{1}{2}\frac{f''(x_2)}{[f'(x_2)]^3}[2f(x_0)f(x_1) - f(x_0)^2] \Longrightarrow$$

$$F_2^{AG}(x_0, x_1, x_2) = x_0 - \frac{f(x_0)}{f'(x_1)} - \frac{1}{2}\frac{f''(x_2)}{[f'(x_2)]^3}[2f(x_0)f(x_1) - f(x_0)^2]. \tag{12}$$

Remark 6. *For $x_0 = x_1 = x_2 := x_i$, the method coincide with the method generated by Taylor inverse interpolation, for $n = 3$.*

$$F_3^T(x_i) = x_i - \frac{f(x_i)}{f'(x_i)} - \frac{1}{2}\left[\frac{f(x_i)}{f'(x_i)}\right]^2 \frac{f''(x_i)}{f'(x_i)}.$$

The order of this method is the solution of the equation:

$$t^3 - t^2 - t - 1 = 0$$

so

$$ord(F_2^{AG}) = 1.839$$

3. **Lidstone inverse interpolation method**

For the particular case of Lidstone interpolation, on $[x_0, x_1], x_0 \neq x_1, i = \overline{0,1}, m = 2$, and

$$\begin{cases} L_{2i+1}f = f^{(2i)}(x_0) \\ L_{2i+2}f = f^{(2i)}(x_1) \end{cases}$$

it follows that

$$\left(L_2^{\Delta} f\right)\big|_{[x_0,x_1]}(x) = \sum_{k=0}^{1}\left[\Lambda_k\left(\frac{x_1 - x}{h}\right) f^{(2k)}(x_0) + \Lambda_k\left(\frac{x - x_0}{h}\right) f^{(2k)}(x_1)\, h^{2k}\right]$$

where

$$\begin{cases} \Lambda_0(x) = x \\ \Lambda_1''(x) = \Lambda_0(x) = x \\ \Lambda_1(0) = \Lambda_1(1) = 0 \end{cases}$$

The interpolation polynomial is:

$$\left(L_2^{\Lambda} f\right)(x) = \sum_{i=0}^{1}\sum_{j=0}^{1} r_{m,i,j}(x)\, f^{(2j)}(x_i)$$

$\Longrightarrow \left(L_2^\Lambda f\right)(x) = r_{2,0,0}(x) f(x_0) + r_{2,0,1}(x) f''(x_0) + r_{2,1,0}(x) f(x_1) + r_{2,1,1}(x) f''(x_1)$ where

$$r_{2,0,j}(x) = \Lambda_j\left(\frac{x_1 - x}{h}\right) h^{2j}, 0 \le x \le x_1; i = 0$$

$$r_{2,1,j}(x) = \Lambda_j\left(\frac{x - x_0}{h}\right) h^{2j}, x_0 \le x \le x_1; i = 1$$

$r_{2,0,0}(x) = \Lambda_0\left(\frac{x_1 - x}{h}\right) h = x_1 - x$

$r_{2,0,1}(x) = \Lambda_1\left(\frac{x_1 - x}{h}\right) h^2$

$r_{2,1,0}(x) = \Lambda_0\left(\frac{x - x_0}{h}\right) h = x - x_0$

$r_{2,1,1}(x) = \Lambda_1\left(\frac{x - x_0}{h}\right) h^2$ but

$$\Lambda_1(x) = \int_0^1 g_1(x,s)\, s ds = \int_0^x (x-1)\, s^2 s ds + \int_x^1 (s-1)\, xss ds = \frac{x^3 - x}{6} + c$$

$\Lambda_1(0) = \Lambda_1(1) = 0 \Longrightarrow c = 0$

and

$$r_{2,0,1}(x) = \Lambda_1\left(\frac{x_1 - x}{h}\right) h^2 = \frac{1}{6h}(x_1 - x)(x_1 - x - h)(x_1 - x + h)$$

$$r_{2,1,1}(x) = \Lambda_1\left(\frac{x - x_0}{h}\right) h^2 = \frac{1}{6h}(x - x_0)(x - x_0 - h)(x - x_0 + h)$$

We know that for $g = f^{-1}$,

$$g = L_2^\Lambda g + R_2^\Lambda g$$

and $x^* = g(0), g \simeq L_2^\Lambda g \Longrightarrow x^* \simeq L_2^\Lambda g(0)$.

$$L_2^\Lambda g(0) = x_1 g(x_0) + \frac{x_1}{6h}\left(x_1^2 - h^2\right) g''(x_0) - x_0 g(x_1) + \frac{x_0}{6h}\left(h^2 - x_0^2\right) g''(x_1)$$

$$\Longrightarrow x^* = x_1 g(x_0) + \frac{x_1}{6h}\left(x_1^2 - h^2\right) g''(x_0) - x_0 g(x_1) + \frac{x_0}{6h}\left(h^2 - x_0^2\right) g''(x_1)$$

and so we have the following method:

$$F_2^\Lambda(x_0, x_1) = x_1 g(x_0) + \frac{x_1}{6h}\left(x_1^2 - h^2\right) g''(x_0) - x_0 g(x_1) + \frac{x_0}{6h}\left(h^2 - x_0^2\right) g''(x_1)$$

References

[1] Agarwal, R., Wong, P., *Explicit error bounds for the derivatives of piecewise Lidstone interpolation,* Journal of Computational and Applied Mathematics, 58 (1995), 67-81.

[2] Agratini, O., Chiorean, I., Coman, Gh., Trîmbiţaş, R., *Analiza Numerica si Teoria Aproximarii,* vol. III, Presa Universitara Clujeana, Cluj Napoca, 2002.

[3] Cătinaş, T., *The combined Shepard-Abel-Goncharov univariate operator*, Rev. Anal. Numer. Theor. Approx., 32 (2003), no. 1, pp.11-20

[4] Coman, Gh., Cătinaş, T., Birou, M., Oprişan, A., Oşan, C., Pop, I., Somogyi, I., Todea, I., *Interpolation Operators*, Ed. Casa Cartii de Stiinta, Cluj Napoca, 2004.

[5] Oprişan, A., *About convergence order of the iterative methods generated by inverse interpolation*, Seminar on Numerical and Statistical Calculus, 2004, pp. 97-109.

[6] Sendov, B., Andreev, A., *Approximation and Interpolation Theory*, Handbook of Numerical Analysis, vol. III, ed. P.G. Ciarlet and J.L. Lions, North Holland, Amsterdam, 1994.

[7] Traub, J.F., *Iterativ methods for the solutions of equations,* Prentia Hall, Inc. Englewood Cliffs, 1964

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MONTE CARLO METHODS FOR SYSTEMS OF LINEAR EQUATIONS

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Abstract. We study Monte Carlo methods for solving systems of linear equations. We propose three methods to generate the trajectories of the Markov chain associated to the system. We calculate the average complexity of generating the trajectories using these methods. From the complexity point of view, the proposed methods are better than other methods reported in the literature.

1. **Introduction**

We consider the system of linear algebraic equations:

$$Ax = b, \tag{1}$$

where $A = (a_{ij})_{i,j=1}^{n} \in \mathbb{R}^{n\times n}$ is a given invertible matrix and $b \in \mathbb{R}^n$ is a given vector, $b = (b_1, \ldots, b_n)^t$. We are interested in estimating the solution $x = (x_1, \ldots, x_n)^t \in \mathbb{R}^n$ of system (1), using Monte Carlo methods. For this, we write the system in the following form:

$$x = Tx + c, \tag{2}$$

where $T = (t_{ij})_{i,j=1}^{n} \in \mathbb{R}^{n\times n}$, $c = (c_1, \ldots, c_n)^t \in \mathbb{R}^n$ and $I - T$ is an invertible matrix. The solution x admits the Neumann series representation:

$$x = c + Tc + T^2c + T^3c + \ldots \tag{3}$$

It is assumed that $\sum_{j=1}^{n} |t_{ij}| < 1,\ i = 1, \ldots, n$, which is a sufficient condition for the convergence of Neumann series to the solution.

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The first Monte Carlo method for solving systems of linear equations was proposed by von Neumann and Ulam, and extended by Forsythe and Leibler [5]. For further details, see [6] and [9]. The method is efficient when we are interested in estimating one component of the solution.

2. Monte Carlo methods to estimate the solution of the system

There is also a Monte Carlo method for solving systems of linear equations, which allows to estimate the entire solution, by constructing unbiased estimators for the components of the solution.

To solve system (2), let $P = (p_{ij})_{i,j=1}^{n+1} \in \mathbb{R}^{(n+1)\times(n+1)}$ be a matrix, whose elements satisfy the conditions:

1. $p_{ij} \geq 0$ such that $t_{ji} \neq 0 \Longrightarrow p_{ij} \neq 0$,
2. $\sum_{j=1}^{n} p_{ij} \leq 1$, $i = 1, \dots, n$,
3. $p_{i,n+1} = 1 - \sum_{j=1}^{n} p_{ij}, \quad i = 1, \dots, n$,
4. $p_{n+1,j} = 0$, $j < n+1$,
5. $p_{n+1,n+1} = 1$.

We also use the notation p_i for $p_{i,n+1}$. Furthermore, define the weights:

$$w_{ij} = \begin{cases} \frac{t_{ji}}{p_{ij}} & \text{if } p_{ij} \neq 0 \\ 0 & \text{if } p_{ij} = 0 \end{cases}, \quad i,j = 1,\dots,n. \tag{4}$$

The matrix P describes a Markov chain with states $\{1, \dots, n+1\}$, where $n+1$ is an absorbing state and p_{ij}, $i,j = 1, \dots, n+1$ is the one step transition probability from state i to state j. Such a Markov chain is also called a *random walk*, as it is homogeneous and finite.

Denote by $\gamma = (i_0, i_1, \dots, i_k, n+1)$ a trajectory that starts at the initial state $i_0 < n+1$ and passes successfully through the sequence of states $(i_1, \dots, i_k)$, to finally get into the absorbing state $i_{k+1} = n+1$. Consider a vector $\alpha = (\alpha_1, \dots, \alpha_n)$, where α_i, $i = 1, \dots, n$ is the probability that a trajectory starts in state i, in other words,

$$P(i_0 = i) = \alpha_i, \qquad \alpha_i \geq 0,\ i = 1, \dots, n, \qquad \sum_{i=1}^{n} \alpha_i = 1.$$

The probability to follow trajectory γ is $P(\gamma) = \alpha_{i_0} p_{i_0 i_1} \dots p_{i_{k-1} i_k} p_{i_k}$.

Define the estimators $\theta_i,\ i = 1,\dots,n$ and $\lambda_i,\ i = 1,\dots,n$ on the space of trajectories as follows. For a trajectory $\gamma = (i_0, i_1, \dots, i_k, n+1)$, the values of these estimators are defined as:

$$\theta_i(\gamma) = W_k(\gamma)\frac{\delta_{i_k i}}{p_{i_k}}, \qquad \lambda_i(\gamma) = \sum_{m=0}^{k} W_m(\gamma)\delta_{i_m i}, \qquad i = 1,\dots,n,$$

where $W_m,\ m = 0,\dots,k$ are random variables whose values are:

$$\begin{aligned} W_0(\gamma) &= \frac{c_{i_0}}{\alpha_{i_0}}, \\ W_m(\gamma) &= W_{m-1}(\gamma)w_{i_{m-1}i_m} \\ &= \frac{c_{i_0}}{\alpha_{i_0}} w_{i_0 i_1} w_{i_1 i_2} \dots w_{i_{m-1} i_m}, \qquad m = 1,\dots,k. \end{aligned}$$

The above values are taken with probability $P(\gamma)$ (δ_{ij} is the Kronecker symbol, i.e., $\delta_{ij} = 1$ if $i = j$ and 0 otherwise).

It can be proved that θ_i and λ_i are unbiased estimators of x_i, i.e.: $E(\theta_i) = E(\lambda_i) = x_i,\ i = 1,\dots,n.$

The Monte Carlo Algorithm to estimate the solution of system (2) is the following:

Algorithm 1. *Monte Carlo Algorithm to estimate the solution x*

1. Input data: the matrix $T$and P, the vectors c and α, the integer n.
2. Generate N trajectories $\gamma_1,\dots,\gamma_N$.
3. Compute the Monte Carlo estimate of the solution:

$$\hat{x} = \left[\frac{\theta_1(\gamma_1) + \dots + \theta_1(\gamma_N)}{N}, \dots, \frac{\theta_n(\gamma_1) + \dots + \theta_n(\gamma_N)}{N}\right]^t. \tag{5}$$

or, the estimate:

$$\tilde{x} = \left[\frac{\lambda_1(\gamma_1) + \dots + \lambda_1(\gamma_N)}{N}, \dots, \frac{\lambda_n(\gamma_1) + \dots + \lambda_n(\gamma_N)}{N}\right]^t. \tag{6}$$

3. **Complexity of the Monte Carlo Algorithm**

To compute the complexity of Algorithm 1, we assume that:

1. The costs of all arithmetical operations are equal, i.e., $CP(+) = CP(-) = CP(*) = CP(:) = 1$.

2. The cost of testing any of the inequalities $x < y$, $x > y$, $x \leq y$ or $x \geq y$ or the equality $x = y$ is d arithmetical operations.

3. The cost of generating one random number uniformly distributed on $[0, 1)$ is 3 arithmetical operations, as we use the linear congruential generator to generate random numbers.

Next, we analyse the complexity of each step of Algorithm 1.

3.1. **Complexity of generating the trajectories.** To start a trajectory, we sample from the following discrete distribution:

$$Y_\alpha : \begin{pmatrix} 1 & 2 & \dots & n \\ \alpha_1 & \alpha_2 & \dots & \alpha_n \end{pmatrix}$$

described by the probability vector α, in order to get the initial state $i_0 \in \{1, 2, \dots, n\}$. Once the trajectory is in state $i_m = i$, $i \in \{1, 2, \dots, n\}$, we sample from the discrete distribution:

$$Y_i : \begin{pmatrix} 1 & 2 & \dots & n & n+1 \\ p_{i1} & p_{i2} & \dots & p_{in} & p_i \end{pmatrix}$$

described by the i-th line of matrix P, in order to determine the next state i_{m+1}. We repeat this procedure till absorbtion takes place.

The total number of steps before absorption is $\sum_{i=1}^{n} C_i$, where C_i denotes the number of times a trajectory visits the non-absorbing state i. Let $z = (z_1, \dots, z_n)$ be the solution of system $z = \bar{P}z + \alpha$, where $\bar{P}$ is the transpose of matrix $(p_{ij})_{i,j=1}^{n}$. The expectation of the random variable C_i is $E(C_i) = z_i$, $i = 1, \dots, n$ ([7]).

Denote by CP the (computational) complexity of generating a trajectory, defined as the number of arithmetical operations needed to generate it. For a trajectory, we sample from Y_α once, at the beginning of the generation process. The number of times we sample from Y_i is C_i. Let CP_{Y_α} and CP_{Y_i} denote the number of operations needed to generate a sample from Y_α and Y_i, respectively. It follows that the average complexity of generating a trajectory is given by:

$$E(CP) = E(CP_{Y_\alpha}) + \sum_{i=1}^{n} z_i E(CP_{Y_i}). \tag{7}$$

There are several methods for sampling from discrete distributions (see [3] or [4]). In [7] three such methods are used: the inversion, the acceptance-rejection and the alias method. The following results for the average complexities of generating a trajectory were obtained:

$$E(CP_{inv}) \leq (d+1)(\|z\|_1+1)n, \tag{8}$$

$$E(CP_{rej}) \leq (d+9)(\|z\|_1+1)n, \tag{9}$$

$$E(CP_{alias}) = (d+9)(\|z\|_1+1). \tag{10}$$

In the following, we use three methods: the decomposition, the economical and the table look-up method to sample from Y_α and Y_i, $i=1,\dots,n$. We calculate the average complexity of generating a trajectory and compare our results with the results (8)-(10).

3.2. **Generating trajectories using decomposition method.** We describe how we can generate a trajectory using the decomposition method to sample from Y_α and Y_i, $i=1,\dots,n$. Decomposition method is based on the following result (see [3]):

Theorem 1. *Any discrete distribution Y with m possible values can be written as the weighted sum of m distributions $\xi_1,\xi_2,\dots,\xi_m$, each taking two possible values and having weight $1/m$.*

Next, we consider $Y=Y_\alpha$ and we describe how we can construct the distributions $\xi_1,\dots,\xi_n$ (in this case $m=n$). For the sake of simplicity, we denote the values $1,2,\dots,n$ of distribution Y_α by $y_1,\dots,y_n$, respectively. Thus, Y_α has the following form:

$$Y_\alpha = \begin{pmatrix} y_1 & y_2 & \dots & y_n \\ \alpha_1 & \alpha_2 & \dots & \alpha_n \end{pmatrix}.$$

We assume that $\alpha_1 \leq 1/n$ and $\alpha_2 \geq 1/n$, otherwise we look for two such probabilities in distribution Y_α and re-index them to 1 and 2, respectively. First, we decompose the distribution Y_α into the two-point distribution ξ_1 and the $n-1$ point distribution η_1 with weights $1/n$ and $(n-1)/n$ respectively, i.e.,

$$Y_\alpha = \frac{1}{n}\xi_1 + \frac{n-1}{n}\eta_1. \tag{11}$$

It can be shown that these distributions have the following form:

$$\xi_1 = \begin{pmatrix} y_1 & y_2 \\ q_1 & q_2 \end{pmatrix} \qquad \eta_1 = \begin{pmatrix} y_2 & y_3 & \dots & y_n \\ \alpha_2' & \alpha_3' & \dots & \alpha_n' \end{pmatrix},$$

where $q_1 = n\alpha_1$ and $q_2 = 1 - n\alpha_1$ and

$$\alpha_2' = \frac{n(\alpha_1 + \alpha_2) - 1}{n - 1}, \qquad \alpha_j' = \frac{n}{n-1}\alpha_j, \qquad j = 3, \dots, n.$$

Distribution η_1 is further decomposed into the two-point distribution ξ_2 with weight $1/(n-1)$ and the $(n-2)$-point distribution η_2 with weight $(n-2)/(n-1)$, i.e.,

$$\eta_1 = \frac{1}{n-1}\xi_2 + \frac{n-2}{n-1}\eta_2. \tag{12}$$

These distributions can be constructed as described above. Substituting (12) into (11), one obtains that the weight of ξ_2 in the decomposition of Y_α is $1/n$ as well. In a similar way distributions $\xi_3, \dots, \xi_n$ are constructed. Their weights are $1/n$. Thus, Y_α can be written as:

$$Y_\alpha = \frac{1}{n}\xi_1 + \frac{1}{n}\xi_2 + \dots + \frac{1}{n}\xi_n, \tag{13}$$

where the distributions ξ_i, $i = 1, \dots, n$ have the following form :

$$\xi_i = \begin{pmatrix} y_{i1} & y_{i2} \\ q_{i1} & q_{i2} \end{pmatrix}$$

with $y_{i1}, y_{i2} \in \{y_1, \dots, y_n\}$ (i.e. $y_{i1}, y_{i2} \in \{1, \dots, n\}$) , $i = 1, \dots, n$.

Now, we give the procedure that generates a sample from Y_α.

Algorithm 2. *Decomposition Algorithm*

1. [*Set-up step*] Construct distributions $\xi_1, \dots, \xi_n$.
2. [*Selecting the distribution* ξ_i] Generate u uniformly distributed on $[0, 1)$ and set $i = [nu] + 1$ (i is uniformly distributed over $\{1, 2, \dots, n\}$).
3. [*Generating a sample from the distribution* ξ_i] Generate v uniformly distributed on $[0, 1)$, if $v < q_{i1}$ then return y_{i1}, otherwise return y_{i2}.

A similar algorithm can be written for sampling from $Y_i, i = 1, \dots, n$.

Concerning the complexity, we obtain the following main result.

Theorem 2. *The average complexity of generating a trajectory using the decomposition method is:*

$$E(CP_{dec}) = (d+8)(\|z\|_1 + 1). \tag{14}$$

Proof. Algorithm 2 requires the generation of 2 random numbers, 1 comparison, 1 multiplication and 1 addition. We omitted the integer part operation and the complexity of the set-up step. From formula (7), we obtain that the average complexity of generating a trajectory with decomposition method is given by:

$$\begin{aligned} E(CP_{dec}) &= E(CP_{Y_\alpha}) + \sum_{i=1}^{n} z_i E(CP_{Y_i}) = (d+8) + \sum_{i=1}^{n} z_i (d+8) \\ &= (d+8)(\|z\|_1 + 1). \end{aligned}$$

□

Corollary 3. *The average complexity of generating N trajectories using the decomposition method is equal to* $(d+8)(\|z\|_1+1)N$.

Remark. From (14) and (10), we obtain $E(CP_{dec}) < E(CP_{alias})$, which is an improvement from the complexity point of view.

3.3. **Generating trajectories using economical method.** We describe how to generate a trajectory using the economical method to sample from Y_α and Y_i, $i = 1, \ldots, n$. The economical method is a variant of the acceptance-rejection method, where no generated value is rejected. This will lead to a decrease in the complexity of generating a trajectory.

As previously illustrated, distribution Y_α can be written as:

$$Y_\alpha = \frac{1}{n}\xi_1 + \frac{1}{n}\xi_2 + \ldots + \frac{1}{n}\xi_n,$$

where the distributions ξ_i have the following form:

$$\xi_i = \begin{pmatrix} y_{i1} & y_{i2} \\ q_{i1} & q_{i2} \end{pmatrix}, \quad i = 1, \ldots, n.$$

Recall that y_{i1}, y_{i2}, $i = 1, \ldots, n$ are among the values of distribution Y_α.

We assume that $q_{i1} \le q_{i2}, i = 1, \ldots, n$, otherwise q_{i1} and q_{i2} are inverted. In the economical method, degenerated distributions with $P(\xi_i = y_{i2}) = 1$ have to be transformed into $P(\xi_i = y_{i1}) = 1/2$, $P(\xi_i = y_{i2}) = 1/2$, where $y_{i1} = y_{i2}$.

The probabilities $q_{i1}, q_{i2}, i = 1, \ldots, n$ will be arranged into a vector $r = (r_1, \ldots, r_{2n})$, and correspondingly the values $y_{i1}, y_{i2}, i = 1, \ldots, n$ will be placed into a vector $v = (v_1, \ldots, v_{2n})$ as described below.

Algorithm 3. *Set-up Step for Economical Algorithm*

Initialize $j = 1$, $m = 1$, $i = 1$.

WHILE $i \le n$ DO

IF $q_{i1} < q_{i2}$ THEN [case of a non-degenerated distribution ξ_i]

Set $r_j \leftarrow q_{i1}$, $r_{2n-j+1} \leftarrow q_{i2}$, $v_j \leftarrow y_{i1}$, $v_{2n-j+1} \leftarrow y_{i2}$,

Increase $j \leftarrow j + 1$.

ELSE [case of a degenerated distribution ξ_i]

Set $r_{n-m+1} \leftarrow q_{i1}$, $r_{n+m} \leftarrow q_{i2}$, $v_{n-m+1} \leftarrow y_{i1}$, $v_{n+m} \leftarrow y_{i2}$,

Increase $m \leftarrow m + 1$.

END IF

Increase $i \leftarrow i + 1$.

END WHILE

Save $n_1 \leftarrow j$, $n_2 \leftarrow n + m$.

Note that the probabilities $q_{i1} = q_{i2} = 1/2$ occupy the positions $r_s, s = n_1, n_1 + 1, \ldots, n_2 - 1$, which are central positions of vector r. The probabilities $q_{i1} < q_{i2}$ occupy symmetrical positions in vector r.

The procedure that generates a sample from Y_α is the following:

Algorithm 4. *Economical Algorithm*

Generate u_1 uniformly distributed on $[0, 1)$.

Compute $j \leftarrow [2nu_1] + 1$ (j is uniformly distributed over $\{1, \ldots, 2n\}$).

IF $j \ge n_1$ THEN RETURN v_j

ELSE Generate u_2 uniformly distributed on $[0, 1)$.

IF $\frac{u_2}{2} < r_j$ THEN RETURN v_j

ELSE RETURN v_{2n-j+1}

END IF

END IF.

A similar algorithm can be written for sampling from $Y_i, i = 1, \dots, n$.

Concerning the complexity, we obtain the following main result.

Theorem 4. *The average complexity of generating a trajectory using the economical method is bounded by:*

$$E(C_{econ}) \le (2d + 12)(\|z\|_1 + 1). \tag{15}$$

Proof. In Algorithm 4, the worst case scenario is the situation when both ELSE instructions are executed. In this case, we have 2 random numbers generated, 2 multiplications (we count the multiplication $2n$ only once), 2 additions, 1 substraction, 1 division and 2 comparisons. We omitted the integer part operation and the complexity of the set-up step. From formula (7), we get that the average complexity of generating a trajectory with the economical method is:

$$\begin{aligned} E(CP_{econ}) &= E(CP_{Y_\alpha}) + \sum_{i=1}^{n} z_i E(CP_{Y_i}) \\ &\le (2d + 12) + \sum_{i=1}^{n} z_i (2d + 12) = (2d + 12)(\|z\|_1 + 1). \end{aligned}$$

□

Corollary 5. *The average complexity of generating N trajectories using the economical method is bounded by $(2d + 12)(\|z\|_1 + 1)N$.*

Remark. In the economical method the size n of matrix T is not included in the upper bound, whereas in the acceptance-rejection method, the complexity is proportional to n. As a consequence, the computing time is substantially reduced in the economical method, comparing to the acceptance-rejection method.

3.4. **Generating trajectories using table look-up method.** The table look-up method is a fast method to sample from Y_α, in the particular case when the probabilities α_i are rational numbers with common denominator M, i.e., $\alpha_i = m_i/M$, with $\alpha_i > 0, \ i = 1, \dots, n$ and $\sum_{i=1}^{n} m_i = M$.

First, we construct a vector D of size M with m_1 entries 1, m_2 entries 2, ..., m_n entries n. Then, one element of this vector is picked up randomly (uniformly). Obviously, this element is a sample from distribution Y_α. The algorithm that generates a sample from Y_α is:

Algorithm 5. *Table Look-up Algorithm*

1. [*Set-up step*] Construct a vector $D = (D(1), \ldots, D(M))$, where m_i entries are i, $i = 1, \ldots, n$.
2. Generate u uniformly distributed on $[0, 1)$ and set $j = [Mu] + 1$ (j is uniformly distributed over $\{1, \ldots, M\}$).
3. Return $D(j)$.

If the transition probabilities are rational numbers with common denominator, a similar algorithm can be written to sample from $Y_i, i = 1, \ldots, n$.

Concerning the complexity, we get the following theorem.

Theorem 6. *The average complexity of generating a trajectory using the table look-up method is:*

$$E(CP_{tab}) = 5 + \sum_{i=1}^{n} 5z_i = 5(\|z\|_1 + 1). \tag{16}$$

Proof. Algorithm 5 requires the generation of 1 random number, 1 multiplication and 1 addition. We omitted the integer part operation and the complexity of the set-up step. From formula (7), we obtain the average complexity of generating a trajectory with the table look-up method:

$$\begin{aligned} E(CP_{tab}) &= E(CP_{Y_\alpha}) + \sum_{i=1}^{n} z_i E(CP_{Y_i}) = 5 + \sum_{i=1}^{n} 5z_i \\ &= 5(\|z\|_1 + 1). \end{aligned}$$

□

Corollary 7. *The average complexity of generating N trajectories using the table look-up method is equal to* $5(\|z\|_1 + 1)N$.

3.5. **Complexity of evaluating the estimators.** The average complexity of computing (5) is equal to $(2\|z\|_1 + 1)N + n$ ([7]). The average complexity of computing (6) is bounded by $\big(d(\|z\|_1 - 1) + 3\big)\|z\|_1 N + n$.

3.6. **Total Complexity.** The average complexity of the Monte Carlo Algorithm 1 is the sum of the average complexity of generating N trajectories and the average complexity of evaluating the estimator.

The following table contains bounds for the average complexity of the Monte Carlo Algorithm 1, when the decomposition (DEC), the economical (ECON) and the table-look up (TAB) methods are used to generate the trajectories.

Method	Est.	Upper bound for the average complexity
DEC	θ_i	$(d+8)(\|z\|_1 + 1)N + (2\|z\|_1 + 1)N + n$
ECON	θ_i	$(2d+12)(\|z\|_1 + 1)N + (2\|z\|_1 + 1)N + n$
TAB	θ_i	$5(\|z\|_1 + 1)N + (2\|z\|_1 + 1)N + n$
DEC	λ_i	$(d+8)(\|z\|_1 + 1)N + \big(d(\|z\|_1 - 1) + 3\big)\|z\|_1 N + n$
ECON	λ_i	$(2d+12)(\|z\|_1 + 1)N + \big(d(\|z\|_1 - 1) + 3\big)\|z\|_1 N + n$
TAB	λ_i	$5(\|z\|_1 + 1)N + \big(d(\|z\|_1 - 1) + 3\big)\|z\|_1 N + n$

Thus, the total average complexity of Algorithm 1 is $O(N) + n$.

4. Concluding remarks

1. We described how to generate the trajectories using decomposition method and calculated the average complexity of this procedure. We found this is less then the average complexity for the alias method, which is an improvement from the complexity point of view.

2. We used the economical method to generate the trajectories. This leads to a substantial decrease in the average complexity of generating the trajectories, comparing to the acceptance-rejection method.

3. We used the table look-up method to generate the trajectory, in the case when the initial and transition probabilities are rational numbers with a common denominator. This leads to the smallest complexity.

References

[1] Blaga, P., *Probability Theory and Mathematical Statistics*, II, Babes-Bolyai University Press, Cluj-Napoca, 1994 (In Romanian).

[2] Coman, Gh., *The Complexity of Algorithms*, Proceedings of the Conference in Differential Equations, Cluj-Napoca, November 21-23, 1985, 25-42.

[3] Deak, I., *Random Number Generators and Simulation*, Akademiai Kiado, Budapest, 1990.

[4] Devroye, L., *Non-Uniform Random Variate Generation*, Springer-Verlag, New York, 1986.

[5] Forsythe, G.E., and Leibler, R.A., *Matrix Inversion by a Monte Carlo Method*, Mathematical Tables and Other Aids to Computation 4(1950), 127-129.

[6] Hammersley, J.M., and Handscomb, D.C., *Monte Carlo Methods*, Methuen, London, 1964.

[7] Okten, G., *Solving Linear Equations by Monte Carlo Simulation*, SIAM Journal on Scientific Computing, in process of editing, 2005.

[8] Spanier, J., and Gelbard, E.M., *Monte Carlo Principles and Neutron Transport Problems*, Addison-Wesley, 1969.

[9] Wasow, W., *A Note on the Inversion of Matrices by Random Walks*, Mathematical Tables and Other Aids to Computation 6(1952), 78-81.

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ON THE ASYMPTOTIC BEHAVIOUR OF THE SOLUTIONS OF A CERTAIN FOURTH-ORDER DIFFERENTIAL EQUATION

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Abstract. The main purpose of this paper is to establish sufficient conditions under which any solution of (1.1) is uniformly bounded and tend to zero as $t \to \infty$.

1. Introduction and Statement of the Result

As we know from the relevant literature, up to now, many results have been obtained on the asymptotic behaviour of solutions of certain non-linear differential equations of the fourth- order (see, e.g., Hara [2-4], Abou-el-Ela, A.M.A and Sadek, A.I. [1] , Sadek and Elaiw [7] and Tunç, C. and Tunç, E. [5], Tunç [9-10].

In this paper we investigate the asymptotic behaviour of solutions of the real non-linear ordinary differential equation of fourth order:

$$x^{(4)} + a(t)f_1(x, \dot{x}, \ddot{x}, \dddot{x}) + b(t)f_2(x, \dot{x}, \ddot{x}) + c(t)f_3(x, \dot{x}) + d(t)f_4(x) = p(t, x, \dot{x}, \ddot{x}, \dddot{x}), \tag{1.1}$$

in which the functions $a, b, c, d, f_1, f_2, f_3, f_4,$ and p are continuous for all values of their respective arguments. We assume that the functions a, b, c, d are positive definite and differentiable in $R^+ = [0, \infty)$, and that the derivatives $\frac{\partial}{\partial y} f_2(x, y, z)$, $\frac{\partial}{\partial x} f_3(x, y)$, $\frac{\partial}{\partial y} f_3(x, y)$, $\frac{\partial}{\partial x} f_2(x, y, z)$ and $f_4'(x)$ exist and are continuous for all x, y, z and w. The dots indicate differentiation with respect to t.

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The main purpose of this work is to prove the following

Theorem. *In addition to the basic assumptions on the functions* $a, b, c, d,$ $f_1, f_2, f_3, f_4,$ *and* p, *suppose that*

(i) $A \geq a(t) \geq a_0 > 0, B \geq b(t) \geq b_0 > 0, C \geq c(t) \geq c_0 > 0, D \geq d(t) \geq d_0 > 0$ *for* $t \in R^+$;

(ii) $0 < \left[\frac{f_1(x,y,z,w)}{w} - \alpha_1\right] \leq \min\left\{\frac{c_0\alpha_3}{2\sqrt{3}\alpha_4 DA}\sqrt{(\varepsilon-\varepsilon_0)c_0\alpha_3\varepsilon a_0\alpha_1}, \frac{\sqrt{6}}{3A}\sqrt{\frac{\delta_0\varepsilon}{c_0\alpha_3}}\right\}$ *for all* x, y, z, w; $\alpha_1 > 0$, $\alpha_2 > 0$, $\alpha_3 > 0$, $\alpha_4 > 0$;

(iii) $f_3(x, 0) = 0$ *and* $\frac{\partial}{\partial y} f_3(x, y) \geq \alpha_3 > 0$ *for all* x *and* y;

(iv) *There is a finite constant* $\delta_0 > 0$ *such that*

$$a_0b_0c_0\alpha_1\alpha_2\alpha_3 - C^2\alpha_3\frac{\partial}{\partial y}f_3(x,y) - A^2D\alpha_1^2\alpha_4 \geq \delta_0$$

for all x, y *and* z;

(v) $0 \leq \frac{\partial}{\partial y} f_3(x,y) - \frac{f_3(x,y)}{y} \leq \delta_1 < \frac{2D\delta_0\alpha_4}{Ca_0\alpha_1 c_0^2\alpha_3^2}$ *for all* x *and* $y \neq 0$,

(vi) $yz\frac{\partial}{\partial x} f_2(x, y, z) \leq 0$ *for all* x, y *and* z

(vii) $f_2(x, y, 0) = 0$, $\frac{\partial}{\partial y} f_2(x, y, z) \leq 0$ *and* $0 \leq \frac{f_2(x,y,z)}{z} - \alpha_2 \leq \frac{\varepsilon_0 c_0^3\alpha_3^3}{BD^2\alpha_4^2}$ $(z \neq 0)$, *where* ε_0 *is a positive constant such that*

$$\varepsilon_0 < \epsilon = \min\left\{\frac{1}{a_0\alpha_1}, \frac{D\alpha_4}{c_0\alpha_3}, \frac{\delta_0}{4a_0c_0\alpha_1\alpha_3\Delta_0}, \frac{Cc_0\alpha_3}{4D\alpha_4\Delta_0}\left(\frac{2D\delta_0\alpha_4}{Ca_0\alpha_1c_0^2\alpha_3^2} - \delta_1\right)\right\} \qquad (1.2)$$

with $\Delta_0 = \frac{a_0b_0c_0\alpha_1\alpha_2}{C} + \frac{a_0b_0c_0\alpha_2\alpha_3}{AD\alpha_4}$;

(viii) $\frac{1}{y}\int_0^y \frac{\partial}{\partial x} f_3(x,\zeta)d\zeta \leq \frac{c_0\alpha_3(\varepsilon-\varepsilon_0)}{4C}$ *for all* x *and* $y \neq 0$, *and* $\left\{\frac{\partial}{\partial x} f_3(x,y)\right\}^2 \leq \frac{a_0\delta_0\alpha_1(\varepsilon-\varepsilon_0)}{16C^2}$ *for all* x *and* y;

(ix) $f_4(0) = 0, f_4(x)sgnx > 0$ $(x \neq 0), F_4(x) \equiv \int_0^x f_4(\zeta)d\zeta \to \infty$ *as* $|x| \to \infty$ *and*

$0 \leq \alpha_4 - f_4'(x) \leq \frac{\varepsilon\Delta_0 a_0^2\alpha_1^2}{D}$ *for all* x;

(x) $\int_0^\infty \gamma_0(t)dt < \infty, d'(t) \to 0$ *as* $t \to \infty$, *where* $\gamma_0(t) := |a'(t)| + b'_+(t) + |c'(t)| + |d'(t)|$,

$b'_+(t) = \max\{b'(t), 0\}$;

(xi) $|p(t,x,y,z,w)| \leq p_1(t) + p_2(t)[F_4(x)+y^2+z^2+w^2]^{\delta/2} + \Delta(y^2+z^2+w^2)^{1/2}$, *where δ and Δ are constants such that $0 \leq \delta \leq 1, \Delta \geq 0$ and $p_1(t), p_2(t)$ are nonnegative continuous functions satisfying*

$$\int_0^\infty p_i(t)dt < \infty \qquad (i = 1, 2). \tag{1.3}$$

If Δ is sufficiently small, then every solution $x(t)$ of (1.1) is uniformly bounded and satisfies

$$x(t) \to 0, \dot{x}(t) \to 0, \ddot{x}(t) \to 0, \dddot{x}(t) \to 0, \quad \text{as } t \to \infty. \tag{1.4}$$

Remark. *Our result includes those of Abou-el-Ela and Sadek [1], Sadek and AL-Elaiw [7].*

2. The function $V_0(t, x, y, z, w)$

In what follows it will be convenient to use the equivalent differential system

$$\dot{x} = y, \dot{y} = z, \dot{z} = w,$$

$$\dot{w} = -a(t)f_1(x, y, z, w) - b(t)f_2(x, y, z) - c(t)f_3(x, y) - d(t)f_4(x) + p(t, x, y, z, w), \tag{2.1}$$

which is obtained from (1.1) by setting $\dot{x} = y, \ddot{x} = z$ and $\dddot{x} = w$.

For the proof of the theorem our main tool is the function $V_0 = V_0(t, x, y, z, w)$ defined as follows:

$$\begin{aligned} 2V_0 = \quad & 2\Delta_2 d(t) \int_0^x f_4(\zeta)d\zeta + 2c(t) \int_0^y f_3(x, \zeta)d\zeta \\ & +[\Delta_2\alpha_2 b(t) - \Delta_1\alpha_4 d(t)]y^2 + a(t)\alpha_1 z^2 + 2\Delta_1 b(t) \int_0^z f_2(x, y, \zeta)d\zeta \\ & -\Delta_2 z^2 + \Delta_1 w^2 + 2d(t)yf_4(x) + 2\Delta_1 d(t)zf_4(x) \\ & +2\Delta_2 a(t)\alpha_1 yz + 2\Delta_1 c(t)zf_3(x, y) + 2\Delta_2 yw + 2zw + k, \end{aligned} \tag{2.2}$$

where

$$\Delta_1 = \frac{1}{a_0\alpha_1} + \varepsilon, \quad \Delta_2 = \frac{\alpha_4 D}{c_0\alpha_3} + \varepsilon \tag{2.3}$$

and k is a positive constant to be determined later in the proof.

Now we will obtain some basic inequalities which will be used in the proof of the result.

By noting (2.3), (i) and (iii) we obtain

$$\Delta_1 - \frac{1}{a(t)\alpha_1} \geq \varepsilon, \quad \text{for all } x, y, z \text{ and all } t \in R^+, \tag{2.4}$$

$$\Delta_2 - \frac{D\alpha_4 y}{c(t)f_3(x,y)} \geq \varepsilon, \quad \text{for all } x, y \neq 0 \text{ and all } t \in R^+. \tag{2.5}$$

In view of (2.3), (i) and (iv) it follows that

$$\alpha_2 b(t) - \Delta_1 c(t)\tfrac{\partial}{\partial y} f_3(x,y) - \Delta_2 a(t)\alpha_1$$

$$\geq \tfrac{1}{a_0c_0\alpha_1\alpha_3}\left[a_0b_0c_0\alpha_1\alpha_2\alpha_3 - C^2\alpha_3\tfrac{\partial}{\partial y} f_3(x,y) - A^2D\alpha_1^2\alpha_4\right]$$

$$-\left[c(t)\tfrac{\partial}{\partial y} f_3(x,y) + a(t)\alpha_1\right]\varepsilon$$

$$\geq \tfrac{\delta_0}{a_0c_0\alpha_1\alpha_3} - \left[c(t)\tfrac{\partial}{\partial y} f_3(x,y) + a(t)\alpha_1\right]\varepsilon.$$

Also (iv) implies that

$$\frac{\partial}{\partial y} f_3(x,y) < \frac{a_0b_0c_0\alpha_1\alpha_2}{C^2}, \quad \alpha_1 < \frac{a_0b_0c_0\alpha_2\alpha_3}{A^2D\alpha_4}. \tag{2.6}$$

Hence

$$\alpha_2 b(t) - \Delta_1 c(t)\frac{\partial}{\partial y} f_3(x,y) - \Delta_2 a(t)\alpha_1 \geq \frac{\delta_0}{a_0c_0\alpha_1\alpha_3} - \varepsilon\Delta_0, \tag{2.7}$$

for all x, y, z and all $t \in R^+$.

Let Φ_3 be the function defined by

$$\Phi_3(x,y) = \begin{cases} \frac{f_3(x,y)}{y}, & y \neq 0 \\ \\ \frac{\partial}{\partial y} f_3(x,0), & y = 0. \end{cases} \tag{2.8}$$

Then from (iii) and (v) we have

$$\Phi_3(x,y) \geq \alpha_3 \quad \text{for all } x \text{ and } y, \tag{2.9}$$

$$0 \leq \frac{\partial}{\partial y} f_3(x,y) - \Phi_3(x,y) \leq \delta_1 \quad \text{for all } x \text{ and } y. \tag{2.10}$$

From (2.9), (i) and (2.3) we get

$$\Delta_2 - \frac{D\alpha_4}{c(t)\Phi_3(x,y)} \geq \varepsilon, \quad \text{for all } x, y \text{ and all } t \in R^+. \tag{2.11}$$

To prove the present theorem we need the following two lemmas:

Lemma 1. *Subject to the conditions (i)-(ix) of the theorem, there are positive constants D_1 and D_2 such that*

$$D_1[F_4(x) + y^2 + z^2 + w^2 + k] \leq V_0 \leq D_2[F_4(x) + y^2 + z^2 + w^2 + k] \tag{2.12}$$

for all x, y, z and w.

Proof. Since $f_2(x,y,0) = 0$ and $\frac{f_2(x,y,z)}{z} \geq \alpha_2$ $(z \neq 0)$, it is clear that

$$2\Delta_1 b(t) \int_0^z f_2(x,y,\zeta)d\zeta \geq \Delta_1 b(t)\alpha_2 z^2.$$

Therefore it follows from (2.2) that

$$\begin{aligned} 2V_0 \geq \quad & 2\Delta_2 d(t) \int_0^x f_4(\zeta)d\zeta + 2c(t)\int_0^y f_3(x,\zeta)d\zeta + [\Delta_2\alpha_2 b(t) - \Delta_1\alpha_4 d(t)]y^2 \\ & + a(t)\alpha_1 z^2 + \Delta_1 b(t)\alpha_2 z^2 - \Delta_2 z^2 + \Delta_1 w^2 + 2d(t)yf_4(x) + 2\Delta_1 d(t)zf_4(x) \\ & + 2\Delta_2 a(t)\alpha_1 yz + +2\Delta_1 c(t)zf_3(x,y) + 2\Delta_2 yw + 2zw + k. \end{aligned}$$

Rewrite above inequality as follows:

$$2V_0 \geq \frac{c(t)}{\Phi_3(x,y)}\left[\frac{d(t)}{c(t)}f_4(x) + y\Phi_3(x,y) + \Delta_1 z\Phi_3(x,y)\right]^2$$

$$+\frac{a(t)}{\alpha_1}\left[\frac{w}{a(t)} + \alpha_1 z + \Delta_2\alpha_1 y\right]^2 + \left[2\Delta_2 d(t)\int_0^x f_4(\zeta)d\zeta - \frac{d^2(t)f_4^2(x)}{c(t)\Phi_3(x,y)}\right]$$

$$+[\Delta_2 b(t)\alpha_2 - \Delta_1 d(t)\alpha_4 - \Delta_2^2 a(t)\alpha_1]y^2 + 2c(t)\int_0^y f_3(x,\zeta)d\zeta - c(t)\Phi_3(x,y)y^2$$

$$+[\Delta_1\alpha_2 b(t) - \Delta_2 - \Delta_1^2 c(t)\Phi_3(x,y)]z^2 + \left[\Delta_1 - \frac{1}{a(t)\alpha_1}\right]w^2 + k.$$

From (2.4) we get

$$\left[\Delta_1 - \frac{1}{a(t)\alpha_1}\right]w^2 \geq \varepsilon w^2.$$

Then

$$2V_0 \geq V_1 + V_2 + V_3 + \varepsilon w^2 + k, \tag{2.13}$$

where

$$V_1 := 2\Delta_2 d(t)\int_0^x f_4(\zeta)d\zeta - \frac{d^2(t)f_4^2(x)}{c(t)\Phi_3(x,y)},$$

$$V_2 := [\Delta_2\alpha_2 b(t) - \Delta_1\alpha_4 d(t) - \Delta_2^2 a(t)\alpha_1]y^2 + 2c(t)\int_0^y f_3(x,\zeta)d\zeta - c(t)\Phi_3(x,y)y^2,$$

$$V_3 := [\Delta_1\alpha_2 b(t) - \Delta_2 - \Delta_1^2 c(t)\Phi_3(x,y)]z^2.$$

From (2.3), (2.9) and (i) we find

$$\begin{aligned} V_1 &\geq 2\varepsilon d(t)\int_0^x f_4(\zeta)d\zeta + \tfrac{Dd(t)}{c_0\alpha_3}\left[2\alpha_4\int_0^x f_4(\zeta)d\zeta - f_4^2(x)\right] \\ &\geq 2\varepsilon d(t)\int_0^x f_4(\zeta)d\zeta + \tfrac{2Dd(t)}{c_0\alpha_3}\int_0^x [\alpha_4 - f_4'(\zeta)]f_4(\zeta)d\zeta. \end{aligned}$$

Since the second integral on the right hand side is non-negative by (ix), it clear that

$$2\alpha_4\int_0^x f_4(\zeta)d\zeta - f_4^2(x) \geq 0. \tag{2.14}$$

So $V_1 \geq 2\varepsilon d_0 \int_0^x f_4(\zeta)d\zeta$. Also from (2.3), (iii), (i) and (2.7) we obtain

$$\begin{aligned}
& \Delta_2\alpha_2 b(t) - \Delta_1\alpha_4 d(t) - \Delta_2^2 a(t)\alpha_1 \\
= \; & \Delta_2\left[\alpha_2 b(t) - \Delta_1 c(t)\tfrac{\partial}{\partial y} f_3(x,y) - \Delta_2 a(t)\alpha_1\right] \\
& +\Delta_1\left[\Delta_2 c(t)\tfrac{\partial}{\partial y} f_3(x,y) - \alpha_4 d(t)\right] \\
> \; & \Delta_2\left[\alpha_2 b(t) - \Delta_1 c(t)\tfrac{\partial}{\partial y} f_3(x,y) - \Delta_2 a(t)\alpha_1\right] \\
> \; & \tfrac{D\alpha_4}{c_0\alpha_3}\left(\tfrac{\delta_0}{a_0 c_0\alpha_1\alpha_3} - \varepsilon\Delta_0\right).
\end{aligned}$$

Since

$$\begin{aligned}
\int_0^y \zeta \tfrac{\partial}{\partial \zeta} f_3(x,\zeta)d\zeta \quad & \equiv y f_3(x,y) - \int_0^y f_3(x,\zeta)d\zeta \\
& = y^2\Phi_3(x,y) - \int_0^y f_3(x,\zeta)d\zeta,
\end{aligned}$$

then

$$\begin{aligned}
2c(t)\int_0^y f_3(x,\zeta)d\zeta - c(t)\Phi_3(x,y)y^2 \quad & = c(t)\left[\int_0^y f_3(x,\zeta)d\zeta - \int_0^y \zeta\tfrac{\partial}{\partial\zeta} f_3(x,\zeta)d\zeta\right] \\
& = c(t)\int_0^y \left[\Phi_3(x,y) - \tfrac{\partial}{\partial\zeta} f_3(x,\zeta)\right]\zeta d\zeta \\
& \geq -\tfrac{C\delta_1}{2} y^2, \quad \text{by (2.10)}.
\end{aligned}$$

Therefore we have

$$V_2 \geq \left[\frac{D\alpha_4}{c_0\alpha_3}\left(\frac{\delta_0}{a_0 c_0\alpha_1\alpha_3} - \varepsilon\Delta_0\right) - \frac{C\delta_1}{2}\right] y^2 \geq \frac{C}{4}\left(\frac{2\alpha_4 D\delta_0}{C a_0\alpha_1 c_0^2\alpha_3^2} - \delta_1\right) y^2, \text{ by (1.2).}$$

Similarly, from (2.3), (i), (2.10) and (2.7) we obtain

$$
\begin{aligned}
& \Delta_1\alpha_2 b(t) - \Delta_2 - \Delta_1^2 c(t)\Phi_3(x,y) \\
= \;& \Delta_1[\alpha_2 b(t) - \Delta_1 c(t)\Phi_3(x,y) - \Delta_2 a(t)\alpha_1] + \Delta_2[\Delta_1 a(t)\alpha_1 - 1] \\
> \;& \Delta_1[\alpha_2 b(t) - \Delta_1 c(t)\tfrac{\partial}{\partial y} f_3(x,y) - \Delta_2 a(t)\alpha_1] \\
> \;& \tfrac{1}{a_0\alpha_1}\left(\tfrac{\delta_0}{a_0 c_0 \alpha_1 \alpha_3} - \varepsilon\Delta_0\right).
\end{aligned}
$$

Therefore we obtain

$$V_3 \geq \frac{1}{a_0\alpha_1}\left(\frac{\delta_0}{a_0 c_0 \alpha_1 \alpha_3} - \varepsilon\Delta_0\right) z^2, \text{ by } (1.2).$$

Combining the estimates for V_1, V_2 and V_3 with (2.13) we find

$$2V_0 \geq 2\varepsilon d_0 F_4(x) + \frac{C}{4}\left(\frac{2\alpha_4 D\delta_0}{C a_0 \alpha_1 c_0^2 \alpha_3^2} - \delta_1\right) y^2 + \left(\frac{3\delta_0}{4a_0^2 c_0 \alpha_1^2 \alpha_3}\right) z^2 + \varepsilon w^2 + k.$$

Then there exists a positive constant D_1 such that

$$V_0 \geq D_1[F_4(x) + y^2 + z^2 + w^2 + k].$$

Easily,by noting the hypothesis of the theorem, it can be followed that there exists a positive constant D_2 such that

$$V_0 \leq D_2[F_4(x) + y^2 + z^2 + w^2 + k].$$

Therefore (2.12) is verified.

Lemma 2. *Under the conditions of the theorem there exist positive constants* D_4, D_5 *and* D_6 *such that*

$$
\begin{aligned}
\dot{V_0} \leq \;& -D_5(y^2 + z^2 + w^2) + \sqrt{3}D_6(y^2 + z^2 + w^2)^{1/2}[p_1(t) + p_2(t)] \\
& + \sqrt{3}D_6 p_2(t)[F_4(x) + y^2 + z^2 + w^2] + D_4\gamma_0 V_0.
\end{aligned}
\tag{2.15}
$$

Proof. An easy calculation from (2.2) and (2.1) yields that

$$\frac{d}{dt}V_0 = \frac{\partial V_0}{\partial w}\ \dot{w} + \frac{\partial V_0}{\partial z}w + \frac{\partial V_0}{\partial y}z + \frac{\partial V_0}{\partial x}y + \frac{\partial V_0}{\partial t}$$

$$= -\Delta_1 a(t) w f_1(x,y,z,w) - \Delta_2 b(t) y f_2(x,y,z) - \Delta_2 c(t) y f_3(x,y) - b(t) z f_2(x,y,z)$$

$$+w^2 + c(t)y\int_0^y \frac{\partial}{\partial x} f_3(x,\zeta)d\zeta + \Delta_1 b(t) z\int_0^z \frac{\partial}{\partial y} f_2(x,y,\zeta)d\zeta + \Delta_1 b(t) y\int_0^z \frac{\partial}{\partial x} f_2(x,y,\zeta)d\zeta$$

$$+\Delta_2 a(t)\alpha_1 z^2 + [\Delta_2\alpha_2 b(t) - \Delta_1\alpha_4 d(t)]yz + \Delta_1 c(t) z^2 \frac{\partial}{\partial y} f_3(x,y)$$

$$+\Delta_1 c(t) yz \frac{\partial}{\partial x} f_3(x,y) + d(t) y^2 f_4'(x) + \Delta_1 d(t) yz f_4'(x)$$

$$-\Delta_2 a(t) y f_1(x,y,z,w) + \Delta_2 a(t)\alpha_1 yw$$

$$-a(t) z f_1(x,y,z,w) + a(t)\alpha_1 zw + (\Delta_2 y + z + \Delta_1 w) p(t,x,y,z,w) + \frac{\partial V_0}{\partial t}.$$

Since

$$z\int_0^z \frac{\partial}{\partial y} f_2(x,y,\zeta)d\zeta \le 0, \text{ by (vii) and } \ y\int_0^z \frac{\partial}{\partial x} f_2(x,y,\zeta)d\zeta, \quad \text{by (vi).}$$

Then we find that

$$\begin{aligned} \tfrac{d}{dt}V_0 \quad &= -(V_4+V_5+V_6+V_7+V_8) - \Delta_2 a(t) y f_1(x,y,z,w) + \Delta_2 a(t)\alpha_1 yw \\ &\quad -a(t) z f_1(x,y,z,w) + a(t)\alpha_1 zw + (\Delta_2 y + z + \Delta_1 w) p(t,x,y,z,w) + \tfrac{\partial V_0}{\partial t}, \end{aligned} \tag{2.16}$$

where

$$V_4 := \Delta_2 c(t) y f_3(x,y) - \alpha_4 d(t) y^2 - c(t) y\int_0^y \frac{\partial}{\partial x} f_3(x,\zeta)d\zeta - \Delta_1 c(t) yz\frac{\partial}{\partial x} f_3(x,y),$$

$$V_5 := \left[\alpha_2 b(t) - \Delta_1 c(t)\frac{\partial}{\partial y} f_3(x,y) - \Delta_2 a(t)\alpha_1\right] z^2,$$

$$V_6 := [\Delta_1 a(t)\frac{f_1(x,y,z,w)}{w} - 1]w^2,$$

$$V_7 := z b(t) f_2(x,y,z) - \alpha_2 b(t) z^2 + \Delta_2 b(t) y f_2(x,y,z) - \Delta_2\alpha_2 b(t) yz,$$

$$V_8 := \alpha_4 d(t) y^2 - d(t) f_4'(x) y^2 + \Delta_1\alpha_4 d(t) yz - \Delta_1 d(t) f_4'(x) yz.$$

But

$$V_4 = c(t)\Phi_3(x,y)\left[\Delta_2 - \frac{D\alpha_4}{c(t)\Phi_3(x,y)}\right]y^2 - c(t)y\int_0^y \frac{\partial}{\partial x}f_3(x,\zeta)d\zeta - \Delta_1 c(t)yz\frac{\partial}{\partial x}f_3(x,y)$$

$$\geq \varepsilon c_0\alpha_3 y^2 - Cy\int_0^y \frac{\partial}{\partial x}f_3(x,\zeta)d\zeta - \Delta_1 Cyz\frac{\partial}{\partial x}f_3(x,y), \qquad (2.17)$$

by (i), (2.9) and (2.11).

$$\begin{aligned} V_5 &= \left[\alpha_2 b(t) - \Delta_1 c(t)\tfrac{\partial}{\partial y}f_3(x,y) - \Delta_2 a(t)\alpha_1\right]z^2 \\ &\geq \left(\tfrac{\delta_0}{a_0 c_0\alpha_1\alpha_3} - \varepsilon\Delta_0\right)z^2, \quad \text{by (2.7)}, \end{aligned} \qquad (2.18)$$

$$V_6 = [\Delta_1 a(t)\frac{f_1(x,y,z,w)}{w} - 1]w^2 \geq \varepsilon\alpha_0\alpha_1 w^2, \qquad (2.19)$$

by (i), (ii) and (2.3).

$$\begin{aligned} V_7 &= b(t)\left[\tfrac{f_2(x,y,z)}{z} - \alpha_2\right](z^2 + \Delta_2 yz), \quad \text{for } z \neq 0 \\ &\geq -\tfrac{\Delta_2^2}{4}b(t)\left[\tfrac{f_2(x,y,z)}{z} - \alpha_2\right]y^2, \quad \text{by (vii)}. \end{aligned}$$

By using (vii) and (2.3) we get for $z \neq 0$

$$\begin{aligned} \frac{\Delta_2^2}{4}b(t)\left[\frac{f_2(x,y,z)}{z} - \alpha_2\right] &\leq \frac{1}{4}b(t)\left(\frac{D\alpha_4}{c_0\alpha_3} + \varepsilon\right)^2 \frac{\varepsilon_0 c_0^3\alpha_3^3}{BD^2\alpha_4^2} \\ &= \frac{1}{4}b(t)\left(1 + \frac{c_0\alpha_3}{D\alpha_4}\varepsilon\right)^2 \frac{\varepsilon_0 c_0\alpha_3}{B} \leq \varepsilon_0 c_0\alpha_3, \end{aligned}$$

since $\varepsilon < \frac{D\alpha_4}{c_0\alpha_3}$ by (1.2). Then

$$V_7 \geq -\varepsilon_0 c_0\alpha_3 y^2 \quad \text{for all } x, y \text{ and } z \neq 0,$$

but $V_7 = 0$ when $z = 0$, so

$$V_7 \geq -\varepsilon_0 c_0\alpha_3 y^2 \quad \text{for all } x, y \text{ and } z. \qquad (2.20)$$

By (ix)

$$V_8 = d(t)[\alpha_4 - f_4'(x)](y^2 + \Delta_1 yz) \geq -\frac{\Delta_1^2}{4}d(t)[\alpha_4 - f_4'(x)]z^2.$$

From (ix) and (2.3) we find

$$\frac{\Delta_1^2}{4}d(t)[\alpha_4 - f_4'(x)] \le \frac{1}{4}d(t)\left(\frac{1}{a_0\alpha_1} + \varepsilon\right)^2 \frac{\varepsilon\Delta_0 a_0^2\alpha_1^2}{D}$$
$$= \frac{1}{4}d(t)\left(1 + a_0\alpha_1\varepsilon\right)^2 \frac{\varepsilon_0\Delta_0}{D} \le \varepsilon\Delta_0,$$

since $\varepsilon < \frac{1}{a_0\alpha_1}$ by (1.2). Thus it follows that

$$V_8 \ge -\varepsilon\Delta_0 z^2. \tag{2.21}$$

From (2.17) and (2.20) we have, for $y \ne 0$,

$$\begin{aligned}
V_4 + V_7 &\ge \left[(\varepsilon - \varepsilon_0)c_0\alpha_3 - \tfrac{C}{y}\int_0^y \tfrac{\partial}{\partial x} f_3(x,\zeta)d\zeta\right] y^2 - \Delta_1 C y z \tfrac{\partial}{\partial x} f_3(x,y) \\
&\ge \tfrac{3}{4}(\varepsilon - \varepsilon_0)c_0\alpha_3 y^2 - \Delta_1 C y z \tfrac{\partial}{\partial x} f_3(x,y), \quad \text{by(viii)} \\
&= \tfrac{1}{2}(\varepsilon - \varepsilon_0)c_0\alpha_3 y^2 + \tfrac{1}{4}(\varepsilon - \varepsilon_0)c_0\alpha_3\left[y^2 - \tfrac{4\Delta_1 C}{(\varepsilon-\varepsilon_0)c_0\alpha_3} y z \tfrac{\partial}{\partial x} f_3(x,y)\right] \\
&\ge \tfrac{1}{2}(\varepsilon - \varepsilon_0)c_0\alpha_3 y^2 - \tfrac{\Delta_1^2 C^2}{(\varepsilon-\varepsilon_0)c_0\alpha_3}\left[\tfrac{\partial}{\partial x} f_3(x,y)\right]^2 z^2 \\
&\ge \tfrac{1}{2}(\varepsilon - \varepsilon_0)c_0\alpha_3 y^2 - \tfrac{\delta_0}{4a_0\alpha_1 c_0\alpha_3} z^2,
\end{aligned}$$

by using (vii), (2.3) and (1.2). But $V_4 + V_7 = 0$, when $y = 0$,by (2.17) and (2.20); therefore we have

$$V_4 + V_7 \ge \frac{1}{2}(\varepsilon - \varepsilon_0)c_0\alpha_3 y^2 - \frac{\delta_0}{4a_0\alpha_1 c_0\alpha_3} z^2, \quad \text{for all } y \text{ and } z. \tag{2.22}$$

From the estimates given by (2.18), (2.19), (2.21) and (2.22) we get

$$\begin{aligned}
\dot{V_0} &\le -\frac{1}{2}(\varepsilon - \varepsilon_0)c_0\alpha_3 y^2 - \left(\frac{3\delta_0}{4a_0c_0\alpha_1\alpha_3} - 2\varepsilon\Delta_0\right) z^2 \\
&\quad -\varepsilon a_0\alpha_1 w^2 - a(t) z f_1(x,y,z,w) + a(t)\alpha_1 z w \\
&\quad -\Delta_2 a(t) y f_1(x,y,z,w) + \Delta_2 a(t)\alpha_1 y w + (\Delta_2 y + z + \Delta_1 w) p(t,x,y,z,w) + \frac{\partial V_0}{\partial t} \\
&\le -\frac{1}{2}(\varepsilon - \varepsilon_0)c_0\alpha_3 y^2 - \frac{1}{4}\frac{\delta_0}{a_0c_0\alpha_1\alpha_3} z^2 - \varepsilon a_0\alpha_1 w^2 - a(t) z f_1(x,y,z,w) + a(t)\alpha_1 z w \\
&\quad -\Delta_2 a(t) y f_1(x,y,z,w) + \Delta_2 a(t)\alpha_1 y w + (\Delta_2 y + z + \Delta_1 w) p(t,x,y,z,w) + \frac{\partial V_0}{\partial t},
\end{aligned} \tag{2.23}$$

since $\varepsilon < \frac{\delta_0}{4a_0c_0\alpha_1\alpha_3\Delta_0}$ by (1.2).Consider the expressions

$$W_1 = -\frac{1}{4}(\varepsilon-\varepsilon_0)c_0\alpha_3y^2 - \frac{1}{4}(\varepsilon-\varepsilon_0)c_0\alpha_3y^2 - \frac{1}{3}\varepsilon a_0\alpha_1w^2$$
$$-\Delta_2a(t)\left[\frac{f_1(x,y,z,w)}{w} - \alpha_1\right]yw$$

and

$$W_2 = -\frac{1}{2}\frac{\delta_0}{a_0c_0\alpha_1\alpha_3}z^2 - \frac{1}{2}\frac{\delta_0}{a_0c_0\alpha_1\alpha_3}z^2 - \frac{1}{3}\varepsilon a_0\alpha_1w^2 - a(t)\left[\frac{f_1(x,y,z,w)}{w} - \alpha_1\right]zw$$

which is contained in (2.23). Because of the inequalities

$$-W_1 = \tfrac{1}{4}(\varepsilon-\varepsilon_0)c_0\alpha_3y^2 + \tfrac{1}{4}(\varepsilon-\varepsilon_0)c_0\alpha_3y^2 + \tfrac{1}{3}\varepsilon a_0\alpha_1w^2$$

$$+\Delta_2a(t)\left[\tfrac{f_1(x,y,z,w)}{w} - \alpha_1\right]yw$$

$$\geq \tfrac{1}{4}(\varepsilon-\varepsilon_0)c_0\alpha_3y^2 + \left[\tfrac{1}{2}\sqrt{(\varepsilon-\varepsilon_0)c_0\alpha_3}\,|y| \pm \sqrt{\tfrac{1}{3}\varepsilon a_0\alpha_1}\,|w|\right]^2$$

$$\geq 0, \quad \text{by (ii)},$$

and

$$-W_2 = \tfrac{1}{2}\tfrac{\delta_0}{a_0c_0\alpha_1\alpha_3}z^2 + \tfrac{1}{2}\tfrac{\delta_0}{a_0c_0\alpha_1\alpha_3}z^2 + \tfrac{1}{3}\varepsilon a_0\alpha_1w^2 + a(t)\left[\tfrac{f_1(x,y,z,w)}{w} - \alpha_1\right]zw$$

$$\geq \tfrac{1}{2}\tfrac{\delta_0}{a_0c_0\alpha_1\alpha_3}z^2 + \left[\sqrt{\tfrac{1}{2}\tfrac{\delta_0}{a_0c_0\alpha_1\alpha_3}}\,|z| \pm \sqrt{\tfrac{1}{3}\varepsilon a_0\alpha_1}\,|w|\right]^2$$

$$\geq 0, \quad \text{by(ii)},$$

it follows that

$$W_1 \leq -\frac{1}{4}(\varepsilon-\varepsilon_0)c_0\alpha_3y^2,$$
$$W_2 \leq -\frac{1}{2}\frac{\delta_0}{a_0c_0\alpha_1\alpha_3}z^2.$$

Hence, a combination of the estimates W_1 and W_2 with (2.23) yields that

$$\dot{V}_0 \quad \leq -\tfrac{1}{4}(\varepsilon-\varepsilon_0)c_0\alpha_3y^2 - \tfrac{1}{2}\tfrac{\delta_0}{a_0c_0\alpha_1\alpha_3}z^2 - \tfrac{1}{3}\varepsilon a_0\alpha_1w^2$$

$$+(\Delta_2y + z + \Delta_1w)p(t,x,y,z,w) + \tfrac{\partial V_0}{\partial t}$$

From (2.2) we obtain

$$\begin{aligned}\frac{\partial V_0}{\partial t} = {} & a'(t)\left[\tfrac{1}{2}\alpha_1 z^2 + \tfrac{1}{2}\Delta_2\alpha_1 yz\right] \\ & + b'(t)\left[\Delta_1\int_0^z f_2(x,y,\zeta)d\zeta + \tfrac{1}{4}\Delta_2\alpha_2 y^2\right] + c'(t)\left[\int_0^y f_3(x,\zeta)d\zeta + \Delta_1 z f_3(x,y)\right] \\ & + d'(t)\left[\Delta_2\int_0^x f_4(\zeta)d\zeta - \tfrac{1}{2}\Delta_1\alpha_4 y^2 + y f_4(x) + \Delta_1 z f_4(x)\right].\end{aligned}$$

From the assumptions in the theorem, (2.6) and (2.14) we have a positive constant D_3 satisfying

$$\frac{\partial V_0}{\partial t} \le D_3[|a'(t)| + b'_+(t) + |c'(t)| + |d'(t)|][F_4(x) + y^2 + z^2 + w^2] \le D_4\gamma_0 V_0,$$

by using the inequality (2.12), where $D_4 = \frac{D_3}{D_1}$.Therefore one can find a positive constant D_5 such that

$$\dot{V_0} \le -2D_5(y^2+z^2+w^2) + (\Delta_2 y + z + \Delta_1 w)p(t,x,y,z,w) + D_4\gamma_0 V_0.$$

Let $D_6 = \max(\Delta_2, 1, \Delta_1)$, then

$$\begin{aligned}\dot{V_0} & \le -2D_5(y^2+z^2+w^2) + \sqrt{3}D_6(y^2+z^2+w^2)^{1/2}\,|p(t,x,y,z,w)| + D_4\gamma_0 V_0 \\ & \le -2D_5(y^2+z^2+w^2) + \sqrt{3}D_6(y^2+z^2+w^2)^{1/2}\,\{p_1(t) \\ & + p_2(t)[F_4(x)+y^2+z^2+w^2]^{\delta/2} + \Delta(y^2+z^2+w^2)^{1/2}\big\} + D_4\gamma_0 V_0.\end{aligned}$$

Let Δ be fixed, in what follows, to satisfy $\Delta = \frac{D_5}{\sqrt{3}D_6}$ with this limitation on Δ we have

$$\begin{aligned}\dot{V_0} \le {} & -D_5(y^2+z^2+w^2) + \sqrt{3}D_6(y^2+z^2+w^2)^{1/2}\,\{p_1(t) \\ & + p_2(t)[F_4(x)+y^2+z^2+w^2]^{\delta/2}\big\} + D_4\gamma_0 V_0.\end{aligned} \tag{2.24}$$

Note that

$$[F_4(x)+y^2+z^2+w^2]^{\delta/2} \le 1 + [F_4(x)+y^2+z^2+w^2]^{1/2}. \tag{2.25}$$

From (2.24) and (2.25) we find

$$\dot{V_0} \leq \quad -D_5(y^2+z^2+w^2)+\sqrt{3}D_6(y^2+z^2+w^2)^{1/2}[p_1(t)+p_2(t)]$$

$$+\sqrt{3}D_6p_2(t)[F_4(x)+y^2+z^2+w^2]+D_4\gamma_0V_0$$

3. **Completion of the Proof**

We define

$$V(t,x,y,z,w)=\exp\left(-\int_0^t \gamma(\tau)d\tau\right)V_0(t,x,y,z,w), \tag{3.1}$$

where

$$\gamma(t)=D_4\gamma_0+\frac{2\sqrt{3}D_6}{D_1}[p_1(t)+p_2(t)]. \tag{3.2}$$

Then it is easy to see that there exist two functions $U_1(r), U_2(r)$ satisfying

$$U_1(\|\overline{x}\|)\leq V(t,x,y,z,w)\leq U_2(\|\overline{x}\|), \tag{3.3}$$

for all $\overline{x}\in R^4$ and $t\in R^+$ where $U_1(r)$ is a continuous increasing positive definite function, $U_1(r)\to\infty$ as $r\to\infty$ and $U_2(r)$ is a continuous increasing function.

From (3.1), (2.15), (3.2) and (2.12) we have

$$\dot{V}=\exp\left(-\int_0^t \gamma(\tau)d\tau\right)\left[\dot{V_0}-\gamma(t)V_0\right]$$

$$\leq \exp\left(-\int_0^t \gamma(\tau)d\tau\right)\Big\{-D_5(y^2+z^2+w^2)+\sqrt{3}D_6(y^2+z^2+w^2)^{1/2}[p_1(t)+p_2(t)]$$

$$-\sqrt{3}D_6[p_1(t)+p_2(t)][F_4(x)+y^2+z^2+w^2+2k]\Big\}$$

$$\leq \exp\left(-\int_0^t \gamma(\tau)d\tau\right)\big\{-D_5(y^2+z^2+w^2)$$

$$-\sqrt{3}D_6[p_1(t)+p_2(t)]\left[\left(\sqrt{y^2+z^2+w^2}-\frac{1}{2}\right)^2-\frac{1}{4}+2k]\right]\Bigg\}.$$

Setting $k\geq\frac{1}{8}$, we can find a positive constant D_7 such that

$$\dot{V}\leq -D_7(y^2+z^2+w^2)=-U(\|\overline{x}\|). \tag{3.4}$$

From inequalities (3.3) and (3.4) it follows that all the solutions $(x(t), y(t), z(t), w(t))$ of (2.1) are uniformly bounded [12; Theorem 10.2].

Auxiliary Lemma

We consider a system of differential equations

$$\dot{\overline{x}} = F(t, \overline{x}) + G(t, \overline{x}), \tag{3.5}$$

where $F(t, \overline{x})$ and $G(t, \overline{x})$ are continuous vector functions on $R^+ \times Q$ (Q is an open set in R^n). We assume

$$\|G(t, \overline{x})\| \leq G_1(t, \overline{x}) + G_2(\overline{x}),$$

where $G_1(t, \overline{x})$ is non-negative continuous scalar function on $R^+ \times Q$ and $\int\limits_0^t G_1(\tau, \overline{x}) d\tau$ is bounded for all t whenever $\overline{x}$ belongs to any compact subset of Q and $G_2(\overline{x})$ is a non-negative continuous scalar function on Q.

The following lemma is a simple extension of the well-known result obtained by Yoshizawa [12; Theorem 14.2].

Lemma 3. *Suppose that there exists a non-negative continuously differentiable scalar function $V(t, \overline{x})$ on $R^+ \times Q$ such that $\dot{V}_{(3.5)}\ (t, \overline{x}) \leq -U(\|\overline{x}\|)$, where $U(\|\overline{x}\|)$ is positive definite with respect to a closed set Ω of Q. Moreover, suppose that $F(t, \overline{x})$ of system (3.5) is bounded for all t when $\overline{x}$ belongs to an arbitrary compact set in Q and that $F(t, \overline{x})$ satisfies the following two conditions with respect to Ω*

(1) $F(t, \overline{x})$ tends to a function $H(\overline{x})$ for $\overline{x} \in \Omega$ as $t \to \infty$, and on any compact set in Ω this convergence is uniform;

(2) Corresponding to each $\varepsilon > 0$ and each $\overline{y} \in \Omega$, there exist a δ, $\delta = \delta(\varepsilon, \overline{y})$ and a $T = T(\varepsilon, \overline{y})$ such that if $t \geq T$ and $\|\overline{x} - \overline{y}\|\, \delta$, we have $\|F(t, \overline{x}) - F(\varepsilon, \overline{y})\| < \varepsilon$. And suppose that

(3) $G_2(\overline{x})$ is positive definite with respect to a closed set Ω of Q.

Then every bounded solution of (3.5) approaches the largest semi-imvariant set of the system $\dot{\overline{x}} = H(\overline{x})$ contained in Ω as $t \to \infty$.

Proof. (See [7]) From (2.1) we set F and G in (3.5) as follows

$$F(t,\overline{x}) = \begin{bmatrix} y \\ z \\ w \\ -a(t)f_1(x,y,z,w)w - b(t)f_2(x,y,z) - c(t)f_3(x,y) - d(t)f_4(x) \end{bmatrix},$$

$$G(t,\overline{x}) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ p(t,x,y,z,w) \end{bmatrix}.$$

Thus from (xi) we find

$$\|G(t,\overline{x})\| \le p_1(t) + p_2(t)[F_4(x) + y^2 + z^2 + w^2]^{\delta/2} + \Delta(y^2 + z^2 + w^2)^{1/2}.$$

Let

$$G_1(t,\overline{x}) = p_1(t) + p_2(t)[F_4(x) + y^2 + z^2 + w^2]^{\delta/2} \text{ and } G_2(\overline{x}) = \Delta(y^2 + z^2 + w^2)^{1/2}.$$

Then $F(t,\overline{x})$ and $G(t,\overline{x})$ clearly satisfy the conditions of Lemma 3.

Now $U(\|\overline{x}\|)$ in (3.4) is positive definite with respect to the closed set $\Omega = \{(x,y,z,w) \mid x \in R^+, y = 0, z = 0, w = 0\}$, it follows that, in Ω,

$$F(t,\overline{x}) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -d(t)f_4(x) \end{bmatrix}.$$

From (i) and (x), we have $d(t) \to d_\infty$ as $t \to \infty$ where $0 \le d_0 < d_\infty \le D$. If we set

$$H(\overline{x}) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -d_\infty f_4(x) \end{bmatrix}, \tag{3.6}$$

then the conditions on $H(\overline{x})$ of Lemma 3 are satisfied. Moreover $G_2(\overline{x})$ is positive definite with respect to a closed set Ω.

Since all of the solutions of (2.1) are bounded, it follows from Lemma 3 that every solution of (2.1) approaches the largest semi-imvariant set of the system $\dot{\overline{x}}= H(\overline{x})$ contained in Ω as $t \to \infty$. From (3.6), $\dot{\overline{x}}= H(\overline{x})$ is the system

$$\dot{x}= 0, \dot{y}= 0, \dot{z}= 0, \dot{w}= -d_\infty f_4(x),$$

which has the solutions $x = k_1, y = k_2, z = k_3, w = k_4 - d_\infty f_4(k_1)(t-t_0)$. To remain in Ω; $k_2 = k_3 = 0$ and $k_4 - d_\infty f_4(k_1)(t-t_0) = 0$ for all $t \geq t_0$ which implies $k_1 = k_4 = 0$.

Therefore the only solution of $\dot{\overline{x}}= H(\overline{x})$ remaining in Ω is $\overline{x} = \overline{0}$, that is, the largest semi-invariant set of $\dot{\overline{x}}= H(\overline{x})$ contained in Ω is the point $(0, 0, 0, 0)$. Then it follows that

$$x(t) \to 0, y(t) \to 0, z(t) \to 0, w(t) \to 0 \quad \text{as} \quad t \to \infty,$$

which are equivalent to (1.4).

This completes the proof of the theorem.

References

[1] Abou-el-Ela, A.M.A and Sadek, A.I., *On the asymptotic behaviour of solutions of some differential equations of the fourth order,* Ann. of Diff. Eqs., 8, 1(1992), 1-12.

[2] Hara, T., *On the asymptotic behaviour of solutions of certain third-order ordinary differential equations*, Proc. Japan Acad. 47, II(1971), 903-908.

[3] Hara, T., *On the asymptotic behaviour of the solutions of certain non-autonomous differential equations*, Osaka, J. Math., 12, 2(1975), 267-282.

[4] Hara, T., *On the asymptotic behaviour of solutions of some third and fourth order non-autonomous differential equations*, Publ. RIMS. Kyota Univ., 9(1974), 649-673.

[5] Liu, Jun., *Asymptotic behavior of solutions for a class of fourth-order nonlinear differential equations*, (Chinese) J. Math. (Wuhan) 22 (2002), no. 4, 468-474.

[6] Sadek, A. I., *On the asymptotic behaviour of solutionsof certain fourth-order ordinary differential equations*, Math. Japon. 45(1997), no. 3, 527-540.

[7] Sadek, A. I., AL-Elaiw, A. S., *Asymptotic behaviour of the solutions of a certain fourth-order differential equation*, Ann. Differential Equations, 20, 3(2004), 221-224.

[8] Tunç, C. and Tunç, E., *On the asymptotic behaviour of solutions of certain non-autonomous differential equations*, Applied Mathematics and Computation, 151 (2004), 363-378.

[9] Tunç, Cemil., *On the asymptotic behaviour of solutions of certain fourth order non-autonomous differential equations*, Studia Univ. Babes-Bolyai Math. 41 (1996), no. 3, 95-105.

[10] Tunç, Cemil., *On the asymptotic behaviour of solutions of some differential equations of the fourth order*, Studia Univ. Babes-Bolyai Math. 39 (1994), no. 2, 87-96.

[11] Yoshizawa, T., *Stability theory by Liapunov's second method*, The Mathematical Society of Japan, 1966.

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BOOK REVIEWS

Handbook of Computational and Numerical Methods in Finance, **Svetlozar T. Rachev and George A. Anastassiou** (Eds.), Birkhäuser Verlag 2004, VI+435 p., 15 illus., Hardcover ISBN 0-8176-3219-0.

Svetlozar Todorov Rachev, one of the editors of this book, belongs both to the Department of Statistics and Applied Probability from University of California USA and to the Department of Economics and Business Engineering from Universität Karlsruhe Germany.

The merit of the editors was that they collected twelves articles, each of them treating a computational or a numerical method applied in finance.

Many years the Nobel prize for economics was awarded to mathematicians who applied their knowledge of probability theory, numerical analysis, partial derivatives equations, statistics, in one word the mathematics, to find models in economics. After the establishing of the economical models, some of them become difficult to solve and, as a consequence, impossible to performed their results.

As in other fields of science, i.e. mechanics, where computational and numerical methods were successfully applied, it is the turn of finance to be the ,,reason" of research for mathematicians and a field in which those who work with applied mathematics find a fertile ground to develop them ideas.

In this book are fruitfully putted together themes from financial analysis like: computation of complex derivatives; market credit and operational risk assessment, asset liability management, optimal portfolio theory, financial econometrics, as well as recent studied themes from computational and numerical methods in finance, and risk management like: Genetic Algorithms, Neural Networks, Monte-Carlo methods, Finite Difference Methods, Stochastic Portfolio Optimization and others.

As the editor Svetlozar Rachev mentioned, the subject of computational and numerical methods in finance has recently emerged as a new discipline at the intersection of probability theory, finance and numerical analysis. The methods employed bridge the gap between financial theory and computational practice and provide solutions for complex problems that are difficult to solve by traditional analytical methods.

The book presents some of the current research collected by the editor and survey articles focusing on various numerical methods in finance. The articles and the authors, in order of appearance are:

1. O.J. Blaskowitz, W.K. H Kärdle and P. Schmidt, *Skewness and Kurtosis Trades*; 2. D.D' Souza, K. Amir-Atefi and B. Racheva-Jotova, *Valuation of a Credit Spread Put Option: The Stable Paretian model with Copulas*; 3. I. Khindanova, Z. Atakhanova, and S. Rachev, *GARCH - Tupe Processes in Modeling Energy Prices*; 4. Kohatsu-Higa and M. Monteri, *Malliavin Calculus in Finance* ; 5. P. Kokoszka and A. Parfionovas, *Bootsfrap Unit Root Tests for Heavy-Tailed Time Series*; 6. S. Ortobelli, S. Rachev, I. Huber and A. Briglova, *Optimal Portfolio Selection and Risk Management: A Comparison between the Stable Paretian Approach and the Gaussian One*; 7. G. Pages, H. Pham, and J. Printems, *Optimal Quantization Methods and Applications to Numerical Problems in Finance*; 8. S. Stoianov and B. Racheva - Jotova, *Numerical Methods for Stable Modeling in Financial Risk Management;* 9. F. Schlotttman and D. Seese, *Modern Heuristics for Finance Problems: A Survey of Selected Methods and Applications*; 10. C. E. Testuri and S. Uryasev, *On Relation Between Expected Regret and Conditional Value-at-Risk*; 11. S. Trück and E. Özturkmen, *Estimation Adjustment and Application of Transition Matrices in Credit Risk Models*; 12. Z. Zheng, *Numerical Analysis of Stochastic Differential Systems and its Applications in Finance.*

Diana Andrada Filip

Songmu Zheng, ***Nonlinear Evolution Equations***, Monographs and Surveys in Pure and Applied Mathematics, Vol. 133, Chapman & Hall/CRC, Boca Raton, London, New York, Washington DC, 2004, xiv +287 pp., ISBN 1-58488-452-5.

Nonlinear evolution equations are partial differential equations with time t as one of the independent variables. Beside their presence in various fields of mathematics, nonlinear evolution equations are also important for their applications in physics, mechanics, material science. For instance, the Navier-Stokes and Euler equations in fluid mechanics, the nonlinear Klein-Gordon equations in quantum mechanics, the Cahn-Hilliard equations in material science are particular cases of nonlinear evolution equations.

The first question in the study of nonlinear evolution equations is that of existence and uniqueness of the solution (at least locally), which is that of the existence and uniqueness of the solution (at least locally) which is usually solved by fixed point methods (the contraction principle and Leray-Schauder fixed point theorem). Another fundamental question, which is vital in applications, is that of global existence and uniqueness and the long time behavior of a solution as the time goes to infinity.

The aim of the present book is to develop in a detailed and accessible manner the basic methods and tools for the treatment of nonlinear evolution equations – the semigroup method, compactness and monotone operators method, monotone iterative methods. These are developed in the six chapters of the boo: 1. *Preliminaries;* 2. *Semigroup Method;* 3. *Compactness method and Monotone Operator Method;* 4. *Monotone Iterative Methods and Invariant Regions;* 5. *Global Solutions and Small Initial Data;* 6. *Asymptotic Behavior of Solutions and Global Attractors.*

Most of the included material appears for the first time in book form, some of it being based on the research work of the author.

The prerequisites for the reading of the book are familiarity with Sobolev spaces and embedding theorems, distribution theory, elements of functional analysis. The required results are enounced at the beginning of each chapter with exact references.

The bibliography counts 171 items.

The book is well written and provide the reader with a good introduction to this area of investigation.

Treating a topic of great importance in nonlinear science, with applications to mechanics, material science and biological sciences, the book is of interest for mathematician (graduate students and researchers) working in this area, as well for people working in the applied domains mentioned above.

Radu Precup

Daniel Li, Hervé Queffelec, ***Introduction à l'étude des espaces de Banach – Analyse et probabilités,*** Cours Spécialisés 12, Société Mathématique de France, 2004, xxiv + 627 pp., ISBN 1-58488-452-5.

The aim of the present book is to show how probabilistic methods can be used to solve difficult problems in Banach space theory and, at a same time, to emphasize the interplay between Banach space theory and classical analysis. It is an advanced course, so the reader is supposed familiar with basic results in functional analysis, real analysis, measure theory and complex analysis. But modulo these standard results (taught in the 2nd cycle of French universities) the book is fairly selfcontained, one of the main targets of the authors being to avoid the use of "by a well known result" or references to other places. To this end some special results in classical analysis, which are not usually taught in general courses as, for instance, Rademacher's theorem on a.e. differentiability of Lipschitz functions, Riesz-Thorin and Marcinkiewicz interpolation theorems, M. Riesz theorem, F. and M. Riesz theorem, and others of this kind, are included with full proofs. Also, Chapter 0, *Notions fondamentales de probabilités,* contains an introduction to classical probability theory, and the *Annexe* to the book is concerned with harmonic analysis on compact abelian groups. Some probabilistic results in Banach spaces, needed in the rest of the book, are developed in Chapters 3, *Variables aléatoires* and 10, *Processus gaussiens.* As the authors point out in the introduction, the book is not on probability in Banach spaces but rather on their applications to the study of Banach spaces.

In the following we shall present some highlights of the book. Chapters 1 and 2 are concerned with bases and unconditional bases in Banach spaces, including Maurey's proof of Gowers' dichotomy theorem.

Type and cotype are discussed in the fourth chapter, culminating with the proof of Kwapien's result that a Banach space which is both of type and cotype 2 is isomorphic to a Hilbert space. Stegall's proof of Lindenstrauss-Rosenthal local reflexivity principle is also included.

Chapter 5 deals with p-summing operators, emphasizing the key role Grothendieck's theorem (every linear operator from ℓ_1 to ℓ_2 is 1-summing) played in the development of the subject initiated by Pietsch. This chapter contains also a proof of Dvoretzky-Rogers theorem on unconditionally convergent series and an introduction to Sidon sets. In fact, Sidon sets and, more generally, thin sets in harmonic analysis and their interplay with Banach space theory form one of the central themes of the book.

Chapter 6 is concerned with the spaces L^p $-$ Vitali-Hahn-Saks and Dunford-Pettis theorems in L^1, the Haar basis in L^p, and a new proof of the Grothendieck's theorem based on a result of Paley. The space ℓ_1 is studied in Chapter 7, having as central result Rosenthal's theorem on Banach spaces containing ℓ_1 and some of its consequences. The main result of chapter 8, *Sections euclidiennes,* is the famous theorem of Dvoretzky on almost spherical sections of convex bodies in finite dimensional spaces. There are included two proofs of this result – one by Gordon (1985) covering only the real case, and the other one by Pisier (1986), based on the phenomenon of concentration of measures developed by Maurey and Pisier, and valid in both real and complex cases.

Chapter 9 is devoted to Davie's construction of a separable Banach space without approximation theory and related results.

Chapter 11 is concerned with reflexive subspaces of L^1, characterizations in terms of the convergence in measure, relations to sets closed in measure and other results. In Chapter 12, *Quelques exemples d'utilisations de la méthode des sélaecteurs,* selectors obtained from independent Bernoulli random variables (ϵ_n) by the condition $I_\omega = \{n \geq 1 : \epsilon(\omega) = 1\}$ are used to study Sidon sets, the vector Hilbert transform, and K-convexity. The results of this chapter belong, in essence, to Bourgain.

The last chapter of the book, Chapter 13, *Espaces de Pisier des fonctions presque sûrement continues. Applications,* is concerned with Pisier spaces $\mathcal{C}^{ps}$.

Each chapter of the book ends with a section of Exercises with results completing those from the main text. They are accompanied by hints, meaning that the proof is decomposed in several steps and the reader has to fill in the details.

By collecting a lot of fundamental results in modern Banach space theory and exposing them in an accessible way, with full details and auxiliary results, the authors have done a great service to mathematical community. The book can be used for advanced graduate or postgraduate course or as a reference text as well.

S. Cobzaş

Elias M. Stein & Rami Shakarchi, ***Real Analysis – Measure Theory, Integration and Hilbert Spaces***, Princeton Lectures in Analysis III, Princeton University Press, Princeton and Oxford 2005, xix + 402 pp., ISBN 0-691-11386-6.

This is a third one of a four-volume treatise on analysis, based on four one-semester courses taught at the Princeton University, and having as purpose to emphasize the organic unity between various parts of the subject and, at a same time, to illustrate its wide applicability to other fields of mathematics and science. The first two volumes were concerned with Fourier Analysis (Part I) and Complex Analysis (Part II). A fourth volume on functional analysis, distribution theory and probability theory is planned. The emphasis in the presentation is on the historical order in which the main ideas and results emerged and shaped the field. For this reason some results are reconsidered and reexamined at various stages, with interconnections and applications to other areas. A typical example is that of Fourier series considered in the first volume within the framework of Riemann integration with applications to the infinitude of prime numbers in arithmetic progression and to X-ray and Radon transform, and which reappear in the third volume (within Lebesgue integration this time) with applications to Besicovich sets and Fatou theorem on the boundary values of bounded holomorphic functions.

Let us pass to a detailed description of the content. Chapters 1, *Measure theory,* and 2, *Integration theory,* are concerned with the Lebesgue measure and integral in $\mathbb{R}^N$. As applications, one proves the Brunn-Minkowski inequality and the inversion formula for the Fourier transform. The third chapter, *Differentiation and integration,* is devoted to a presentation of the deep results on the differentiation of functions with bounded variation and of absolutely continuous functions and the relations with integrability. After presenting in the fourth chapter, *Hilbert spaces: An introduction,* the basic results on Hilbert spaces and Hilbert space operators, the authors study in Chapter 5, *Hilbert spaces: Several examples,* the Fourier transform on L^2, the Hardy space on the upper half-plane and some applications to PDEs. The results on Lebesgue measure and integration from chapters 1 and 2 are reconsidered in Chapter 6, *Abstract measure and integration theory,* from an abstract point of view, with applications to ergodic theory and spectral theory of Hilbert space operators. The last chapter of the book, Chapter 7, *Hausdorf measure and fractals,* contains a short presentation of these topics with applications to space-filling curves and Besicovich like sets.

Each chapter ends with a set of exercises and problems. The exercises are accompanied by hints, smoothing the way to their solutions. The "Problems" sections contain more challenging problems, some of them, marked by an asterisk, of higher difficulty.

The result is a fine book, which together with the previous one and the forthcoming fourth volume, will give a comprehensive and well motivated approach to a lot of core results of analysis. It can be used for graduate or advanced graduate courses on analysis and its applications.

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