Anul LI **2006**

S T U D I A
UNIVERSITATIS BABEŞ-BOLYAI

MATHEMATICA
2

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STUDIA UNIV. "BABEŞ–BOLYAI", MATHEMATICA, Volume **LI**, Number 2, June 2006

A STABILITY RESULT OF A PARAMETRIZED MINIMUM PROBLEM

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Abstract. This paper considers variational inequalities with pseudomonotone maps depending on a parameter and studies the behaviour of their solutions. The main result gives sufficient conditions for the stability of the initial minimum problem under small perturbation of the parameter.

1. **Introduction**

The parametrization is a welcome concept for almost every minimizing problem with solution and for the behaviour under perturbation.

The aim of this paper is to apply the result obtained in [5] for a particular type of parametric variational inequalities.

A lot of problems are reduced to looking for

$$(M) \qquad \inf\{\, I(u) : u \in C\,\},$$

where C is a nonempty subset of a real Banach space X and $I : C \to \mathbb{R}$ is given.

Some papers deal with the existence of the solution or with their regularity. Other papers study the "path" of the solution function provided by a family of parametrized problems, i.e. if it is single-valued, multivalued, continuous or not and so on.

For our purpose, let $\Omega \subset \mathbb{R}^n$ be a bounded domain and the minimizing problem in discussion

Received by the editors: 25.10.2005.

2000 *Mathematics Subject Classification.* 49J93D.

Key words and phrases. Parametric optimization, consistency, stability.

The author thanks Prof. Dr. Kolumbán Iosif for suggesting the topic and for the helpful comments.

$$(M)_0 \qquad \min\{\, I(u) = \int_\Omega f\big(t, \nabla u(t)\big)dt : u \in v_0 + X \,\},$$

where $X = H_0^{1,q}(\Omega), 1 < q < +\infty, v_0 \in X$ given with $I(v_0) < +\infty$, the integrand $f : \overline{\Omega} \times \mathbb{R}^n \to \mathbb{R}$ is continuous.

For I differentiable it is known that a (local) solution u_0 of $(M)_0$ has to satisfy the equilibrum equation $I'(u) = 0$.

For the real Banach space X, X^* denotes the dual space and $< x, u >$ the duality pairing between $x \in X$ and $u \in X^*$. If the admissible set C is a closed convex subset of X then u_0 has to satisfy the variational inequality

$$(VI) \qquad < I'(u), v - u > \geq 0, \quad \text{for each } v \in C.$$

The parametric form for the problem (VI) requires the following data. Let P be a topological space - the set of parameters, $K : P \to 2^X$ and $J : P \times X \to 2^{X^*}$ be given set-valued maps so that $K(p) \subseteq \mathrm{Dom}\, J(p, \cdot)$ for each $p \in P$, where $\mathrm{Dom} J(p, \cdot)$ denotes the domain of the map $J(p, \cdot) : X \to 2^{X^*}$, i.e. the set $\{\, u \in X \mid J(p, u) \neq \emptyset \,\}$.

For a given $p \in P$ we consider the following problem: find an element $u_p \in K(p)$ and $x \in J(p, u_p)$ so that

$$(VIP)_p \qquad < x, v - u_p > \geq 0, \qquad \text{for each } u \in K(p).$$

For a fixed $p_0 \in P$ suppose that $u_0 \in K(p_0)$ is the unique solution for $(VIP)_{p_0}$.

Then, the problem $(VIP)_{p_0}$ is called *stable under perturbations* if there exist a neighborhood U_0 of p_0 and a mapping $\bar{u} : U_0 \to X$ so that:

i) $\bar{u}(p)$ is a solution for $(VIP)_p$, for any $p \in U_0$;

ii) $\bar{u}(p_0) = u_0$;

iii) $\bar{u}$ is continuous at p_0.

Section 3 deals with sufficient conditions for the stability under perturbations of the initial problem $(M)_{p_0}$.

2. Definitions and auxiliary results

Consider $\alpha : (0, +\infty) \to (0, +\infty)$ a nondecreasing function.

The map $I : P \times X \to \mathbb{R}$ is called *uniformly α-pseudoconvex* on $U \subseteq P$, if for each $p \in U$ and $u, v \in X, u \neq v$ and $0 \leq s \leq 1$ one has

$$< I'(p,u), v-u > \geq 0 \Rightarrow I(p,v) \leq I(p, v + s(u-v)) + s(1-s)\alpha(\|v-u\|)||v-u||,$$

where $I'(p,u)$ denotes the gradient of $I(p,\cdot)$ at the point u.

The map $J : P \times X \to 2^{X^*}$ is called *uniformly α-pseudomonotone* on $U \subseteq P$, if for each $p \in U$ and $u, v \in X, u \neq v, x \in J(p,u), y \in J(p,v)$ one has

$$< x, v-u > \geq 0 \ \Rightarrow < y, v-u > \geq \alpha(\|v-u\|) \cdot \|v-u\|.$$

An important notion for some parametric problems is consistency. For the sequential case one can consult Grave's Theorem [2, pg. 95] while for the continuous case see [1], [5].

Definition 1. *Let $p_0 \in P$, $u_0 \in K(p_0)$ and $\gamma > 1$ be fixed. The map $J : P \times X \to 2^{X^*}$ is called* consistent *in p at (p_0, u_0) if for each $0 < r \leq 1$, there exist a neighborhood U_r of p_0 and a function $\beta : U_r \to \mathbb{R}$ continuous at p_0 with $\beta(p_0) = 0$ so that, for every $p \in U_r$, there exist $u_p \in K(p)$ and $x \in J(p, u_p)$ such that*

$$\|u_p - u_0\| \leq \beta(p)$$

and

$$< x, v - u_p > + \beta(p) \cdot \|v - u_p\| \geq 0,$$

for all $v \in K(p)$ with $r < \|v - u_p\| \leq \gamma$.

Note that for $p = p_0, u_{p_0}$ is u_0.

The mapping $A : X \to 2^{X^*}$ is said to be *upper semicontinuous* (*usc*) at $u_0 \in X$ if, for any open set V containing $A(u_0)$, there exist a neighborhood Δ of u_0 so that $A(\Delta) \subset V$.

Theorem 1. ([5]) *Let P be a topological space, X be a real Banach space, $K : P \to 2^X$ be with values closed convex sets in X and $J : P \times X \to 2^{X^*}$ be a set valued map.*

Let $p_0 \in P$ and $u_0 \in K(p_0)$ be fixed. Suppose that:

i) *u_0 is a solution of $(VIP)_{p_0}$;*

ii) *J is consistent in p at (p_0, u_0);*

iii) *there exists a neighborhood U of p_0 so that the mappings $J(p,\cdot)$ are uniformly α-pseudomonotone and $J(p,\cdot)$ is usc from the line segments in X to X^* for each $p \in U$;*

iv) *for each p, u the set $J(p,u)$ is compact.*

Then, the problem $(VIP)_{p_0}$ is stable under perturbations.

3. **Main Result**

In this section we are going to apply Theorem 1 to the solutions of $(M)_p$ in particular

$$(M)_p \qquad\qquad \min\{\, I(p,u) : u \in K(p)\,\},$$

where the functionals involving the parameter are given by

$$I(p,u) = \int_\Omega f_p\big(t, \nabla u(t)\big)\, dt.$$

Now, for $p_0 \in P$ fixed suppose that $u_0 \in K(p_0)$ is the unique solution of $(M)_{p_0}$.

In this case, the problem $(M)_{p_0}$ is called *stable under perturbations* if there exist a neighborhood U_0 of p_0 and a mapping $\bar{u} : U_0 \to X$ so that:

i) $\bar{u}(p)$ is a solution for $(M)_p$, for any $p \in U_0$;

ii) $\bar{u}(p_0) = u_0$;

iii) $\bar{u}$ is continuous at p_0.

Let P be a topological space, let X be a reflexive Banach space and Y a normed space. Let $C \subseteq X$ and $D \subseteq Y$ be nonempty closed convex sets and consider the mappings $a : P \to Y$, $L : P \to (X,Y)^*$ continuous, where $(X,Y)^*$ denotes the space of all linear, continuous mappings defined on X with values in Y.

The admissible set of the problem $(M)_p$ is considered the set

$$K(p) = \{u \in C \mid a(p) + L(p)(u) \in D\}.$$

For a $p \in P$ the admissible set $K(p)$ is called *regular* if

$$0 \in \operatorname{int}\{a(p) + L(p)(u) - y : u \in C, y \in D\}.$$

Lemma 1. ([7]) *Suppose that $K(p)$ is regular and $u_0 \in K(p_0)$. Then, for each $d > 0$, there exists a neighborhood U_d of p_0 such that $K(p) \cap B(u_0; d) \neq \emptyset$ for each $p \in U_d$. Moreover, there exists a constant $c_d > 0$ such that, for every $p_1, p_2 \in U_d$ one has*

$$\operatorname{dist}\big(u, K(p_2) \cap B(u_0; d)\big) \leq c_d[\|L(p_1) - L(p_2)\| + \|a(p_1) - a(p_2)\|],$$

for each $u \in K(p_1) \cap B(u_0; d)$.

Now, considering an initial problem and a small displacement of the data we state the stability under perturbation.

Theorem 2. *Suppose that $K(p_0)$ is regular and that:*

i) *u_0 is a solution of $(M)_{p_0}$;*
ii) *the map $(p, u) \longmapsto I'(p, u)$ is weakly continuous at (p_0, u_0);*
iii) *there exists a neighborhood U of p_0 such that for each $p \in U, t \in \Omega$, $\dfrac{\partial f_p}{\partial \nabla u}(t, \cdot)$ is continuous from $X = H^{1,q}(\Omega)$ to the weak* topology of X^* and $f_p(t, \cdot)$ are strictly convex on U;*
iv) *for each $p \in U, t \in \Omega$, $\dfrac{\partial f_p}{\partial \nabla u}(t, \cdot)$ is locally bounded around u_0.*

Then, the problem $(M)_{p_0}$ is stable under perturbations.

Proof. Since u_0 is a minimum point of the functional $I(p_0, \cdot)$ on the set $K(p_0)$ we have

$$< I'(p_0, u_0), u - u_0 > \geq 0, \quad \text{for each } \ u \in K(p_0).$$

Define $J : P \times X \to 2^{X^*}$ by $J(p, u) = \{I'(p, u)\}$, for each $p \in P$ and $u \in X$.

Let U_1 be the neighborhood of p_0, provided by Lemma 1. For each $p \in U_1$ let $u_p \in K(p) \cap B(u_0; 1)$ be the element such that

$$\|u_p - u_0\| \leq c_1[\|L(p) - L(p_0)\| + \|a(p) - a(p_0)\|].$$

Put $x = I'(p, u_p)$ (by Definition 1) and take the neighborhood U_γ and the constant c_γ given also by Lemma 1. Denote $c := \max\{c_1, c_\gamma\}$ and $U_0 := U_1 \cap U_\gamma$. For $v \in K(p)$ with $r < \|v - u_p\| \le \gamma$ define the control function

$$\beta(p) = \max\Big\{ -2\frac{1}{\|v-u_0\| + \|u_p - u_0\|} \cdot < I'(p,u_p) - I'(p_0,u_0), v - u_0 >,$$
$$\sqrt{c[\|L(p) - L(p_0)\| + \|a(p) - a(p_0)\|]}\ \Big\}.$$

From *iv)* $I'(p, u_p)$ is also locally bounded. Let $C_v > 0$ for which $\|I'(p, u_p)\| \le C_v$.

Choose $U_r \subset U_0$ a neighborhood of p_0 such that the restriction of the control function to U_r satisfies the following conditions:

$\beta(p) \le 1$, for each $p \in U_r$;
$\frac{1}{2}\|v - u_0\| - \beta(p)\left(C_v + 3\|I'(p_0, u_0)\| + \frac{3}{2}\beta(p)\right) \ge 0$, for each $p \in U_r$.

Observe that

$$\|u_p - u_0\| \le \beta^2(p) \le \beta(p).$$

By *ii)* β is continuous at p_0 and $\beta(p_0) = 0$.

Now, let $v \in K(p)$ for which $r < \|v - u_p\| \le \gamma$. We have

$$\|v - u_0\| \le \|v - u_p\| + \|u_p - u_0\| \le \gamma + 1.$$

Again by Lemma 1 there exists $v_0 \in K(p_0) \cap B(u_0; \gamma + 1)$ such that

$$\|v - v_0\| \le \beta^2(p).$$

The relationship we must verify is

$$< I'(p,u_p), v - u_p > + \beta(p) \cdot \|v - u_p\| \ge 0.$$

For simplicity denote by $I'_p := I'(p, u_p)$ and $I'_0 := I'(p_0, u_0)$. We will use

$$< I'_0, v_0 - u_0 > \ge 0,$$

due to the fact that $v_0 \in K(p_0)$.

So, we have

$$\begin{aligned}
& < I'_p, v - u_p > + \beta(p) \cdot \|v - u_p\| = \\
= \; & < I'_p - I'_0, v - u_p > + < I'_0, v - u_p > + \beta(p)\|v - u_p\| = \\
= \; & < I'_p - I'_0, v - u_0 > + < I'_p - I'_0, u_0 - u_p > + \\
& + < I'_0, v - v_0 > + < I'_0, v_0 - u_0 > + < I'_0, u_0 - u_p > + \beta(p)\|v - u_p\| \geq \\
\geq \; & -\frac{1}{2}(\|v - u_0\| + \|u_0 - u_p\|) \cdot \beta(p) - \|u_0 - u_p\| \cdot \|I'_p - I'_0\| + \\
& + < I'_0, v - v_0 > + < I'_0, u_0 - u_p > + \beta(p)(\|v - u_0\| - \|u_0 - u_p\|) \geq
\end{aligned}$$

$$\begin{aligned}
\geq \; & \frac{1}{2}\|v - u_0\| \cdot \beta(p) - \|u_0 - u_p\| \cdot \|I'_p - I'_0\| - \|v - v_0\| \cdot \|I'_0\| - \\
& - \|u_0 - u_p\| \cdot \|I'_0\| - \frac{3}{2}\beta(p)\|u_0 - u_p\| = \\
= \; & \frac{1}{2}\|v - u_0\| \cdot \beta(p) - \|u_0 - u_p\| \left(\|I'_p - I'_0\| + \|I'_0\| + \frac{3}{2}\beta(p) \right) - \\
& - \beta^2(p) \cdot \|I'_0\| \geq \\
\geq \; & \frac{1}{2}\|v - u_0\| \cdot \beta(p) - \beta^2(p) \left(C_v + 3\|I'_0\| + \frac{3}{2}\beta(p) \right) = \\
= \; & \beta(p) \left[\frac{1}{2}\|v - u_0\| - \beta(p) \left(C_v + 3\|I'_0\| + \frac{3}{2}\beta(p) \right) \right] \geq 0,
\end{aligned}$$

therefore $J \equiv I'$ is consistent in p at (p_0, u_0).

By *iii)* $I(p, \cdot)$ is strictly convex for each $p \in U$ so that $I(p, \cdot)$ is uniformly α-pseudoconvex, thus $J(p, \cdot) = I'(p, \cdot)$ are uniformly α-pseudomonotone (see [5], [3]). The conclusion follows by Theorem 1. □

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with Lipshitz frontier, and $f_p(t, \nabla u(t)) = g(t, p) \cdot h(t, \nabla u(t))$, for each $p \in P$ and each $t \in \Omega$.

Proposition 1. *If $h \in C^1$ and $g(t, \cdot)$ is continuous at p_0 for each $t \in \Omega$, then the mapping $(p, u) \longmapsto I'(p, u)$ is weakly continuous at (p_0, u_0).*

Proof. We estimate $\left|I_p'(u)(v) - I_0'(u_0)(v)\right| \leq$

$$\leq \int_\Omega \left|g(t,p)\cdot \frac{\partial h}{\partial \nabla u}\big(t,\nabla u(t)\big) - g(t,p_0)\cdot \frac{\partial h}{\partial \nabla u}\big(t,\nabla u_0(t)\big)\right| \cdot |\nabla v| dt \leq$$

$$\leq \int_\Omega \left|g(t,p) - g(t,p_0)\right| \cdot \left|\frac{\partial h}{\partial \nabla u}\big(t,\nabla u(t)\big)\right| \cdot |\nabla v| dt +$$

$$+ \int_\Omega |g(t,p_0)| \cdot \left|\frac{\partial h}{\partial \nabla u}\big(t,\nabla u(t)\big) - \frac{\partial h}{\partial \nabla u}\big(t,\nabla u_0(t)\big)\right| \cdot |\nabla v| dt \leq$$

$$\leq \|v\|_X \cdot \left(\int_\Omega \left|g(t,p) - g(t,p_0)\right|^{q'} \cdot \left|\frac{\partial h}{\partial \nabla u}\big(t,\nabla u_0(t)\big)\right|^{q'} dt\right)^{1/q'} +$$

$$+\|v\|_X \cdot \left(\int_\Omega |g(t,p_0)|^{q'} \cdot \left|\frac{\partial h}{\partial \nabla u}\big(t,\nabla u(t)\big) - \frac{\partial h}{\partial \nabla u}\big(t,\nabla u_0(t)\big)\right|^{q'} dt\right)^{1/q'} \to 0,$$

once that $p \to p_0$ and $u \to u_0$, for each $v \in X$. Here q' is the dual of q, i.e $\frac{1}{q}+\frac{1}{q'} = 1$. □

For the existence and unicity of the solution problem $(M)_p$ we refer to [4, pg. 87].

Proposition 2. *If g and h satisfy the following conditions:*

i) *$g(t,p) > 0$ and $h(t,\cdot)$ is strictly convex for each $t \in \Omega$;*

ii) *there exists $c > 0$ such that $h(t,\xi) \geq c(|\xi|^q - 1)$ for each $(t,\xi) \in \Omega \times \mathbb{R}^n$,*

then the problem $(M)_p$ has a unique solution in $H^{1,q}(\Omega)$ for each $p \in P$.

Example 1. *Let $X = H^1(\Omega)$ with $\Omega = (0,1), 0 < p_0 < 1$ fixed, $P = [p_0, 1)$ the set of parameters, and the initial problem*

$$(M)_{p_0} \qquad \min\{I(p_0,u) : u \in C\},$$

where $I(p_0,u) = \int_0^1 (t+p_0)\cdot u'^2(t)\,dt$ and $C = \{u \in X : u'(p_0) = 1, u(1) = 0\}$. The solution u_0 is given by $u_0(t) = 2p_0 \ln(t+p_0)/\ln(1+p_0)$.

The parametrized problem is

$$(M)_p \qquad \min\{I(p,u) : u \in K(p)\},$$

where $I(p,u) = \int_0^1 (t+p)\cdot u'^2(t)\,dt$ and $K(p) = \{u \in X : u'(p) = 1, u(1) = 0\}$. $(M)_{p_0}$ is stable under perturbations and the solution function $\bar{u}$ can be obtained explicitly $\bar{u}(p)(t) = u_p(t) = 2p\ln(t+p)/\ln(1+p)$.

Remark 1. *The following problem*

$$\min\{\int_0^1 t\cdot u'^2(t)\,dt : u'(0)=1, u(1)=0\},$$

has no solution in $X=H^{1,1}(\Omega)$ *because* $h(t,\xi)=t\cdot\xi^2$ *does not satisfy the hypotheses of Proposition* 2 *namely the existence of* $c>0$ (*the proof is similar to* [4, pg. 56].

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STUDIA UNIV. "BABEŞ–BOLYAI", MATHEMATICA, Volume **LI**, Number 2, June 2006

FRICTIONAL CONTACT PROBLEMS WITH NORMAL COMPLIANCE AND COULOMB'S LAW FOR NONLINEAR ELASTIC BODIES

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Abstract. The subject of this work is the study of a problem modeling the frictional contact between a non linear elastic body and a rigid foundation at the presence of rapel forces. First, we present variational formulation for this problem, after we indicate sufficient conditions in order to have the existence, the uniqueness and the Lipschitz continuous dependence of solution with respect to the data. Finally, we prove the dependence of the solution by the parameter θ. The proofs are based on results of topological degree theory as well as on convexity, monotonicity and fixed point arguments see [1].

1. **Introduction**

In this paper we consider perturbed quasivariational inequalities of the form

$$u \in V, \quad \langle Au, v-u\rangle_V + \langle Bu, v-u\rangle_{\mathbb{V}} + j(u,v) - j(u,u) \geq \langle f, v-u\rangle_V \quad \forall v \in V$$

where V denotes a real Hilbert space and $A : V \to V$ is a strongly monotone and Lipschitz continuous operator on V.

$$(h_1)\textbf{:} \begin{cases} a)\exists m > 0 \text{ such that } \langle Au - Av, u-v\rangle_V \geq m\,|u-v|_V^2 \quad \forall u, v \in V \\ b)\exists M > 0 \text{ such that } |Au - Av|_V \leq M\,|u-v|_V \quad \forall u, v \in V \end{cases}$$

Let $B : V \to V$, satisfies:

(h_2)**:** There exists $C \geq 0$ such that $\langle Bv, v\rangle_{\mathbb{V}} \geq -C\,|v|_V^2 \quad \forall v \in V$

Received by the editors: 15.08.2005.

2000 *Mathematics Subject Classification.* 73C50.

Key words and phrases. Elastic body, Coulomb's friction law , Strongly monotone operator, Lipschitz operator, Topological degree, Fixed point, Perturbed quasivariational inequality, Normal compliance.

(h_3): $\begin{cases} \text{For every sequence } \{\eta_n\} \subset V \text{ such that } \eta_n \to \eta \in V \text{ ,} \\ \quad \text{then there exist a subsequence } \{\eta_{n'}\} \subset V \\ \qquad B\eta_{n'} \to B\eta \text{ strongly in } V. \end{cases}$

(h_4): $\langle Bu - Bv, v - u \rangle_{\mathbb{V}} < (m - \alpha) |u - v|_V^2 \quad \forall u, v \in V, u \neq v.$

(h_5): $\exists \beta, 0 \leq \beta \leq (m - \alpha), \ \langle Bu - Bv, v - u \rangle_{\mathbb{V}} \leq \beta |u - v|_V^2 \quad \forall u, v \in V.$

The functional $j : V \times V \to \mathbb{R}$ satisfies

(h_6): $j(\eta, .) : V \to \mathbb{R}$ is a convex functional on V, for all $\eta \in V$,

It is well known that there exists the directional derivative $j_2^{'}$ given by

(h_7): $j_2^{'}(\eta, u; v) = \lim_{\lambda \to 0} [j(\eta, u + \lambda v) - j(\eta, v)] \quad \forall \eta, u, v \in V,$

We consider now the following assumptions:

(J_1): $\begin{cases} \text{For every sequence } \{u_n\} \subset V \text{ with } |u_n|_V \to \infty \\ \text{and every sequence } \{t_n\} \subset [0,1] \text{ one has} \\ \liminf_{n \to \infty} \left[\frac{1}{|u_n|_V^2} j_2^{'}(t_n u_n, u_n; -u_n) \right] < m - C \end{cases}$

(J_2): $\begin{cases} \text{For every sequence } \{u_n\} \subset V \text{ with } |u_n|_V \to \infty \\ \text{and every bounded sequence } \{\eta_n\} \subset V \text{ one has} \\ \liminf_{n \to \infty} \left[\frac{1}{|u_n|_V^2} j_2^{'}(\eta_n, u_n; -u_n) \right] < m. \end{cases}$

(J_3): $\begin{cases} \text{For every sequence } \{u_n\} \subset V \text{ and } \{\eta_n\} \subset V \text{ such that} \\ u_n \to u \in V, \ \eta_n \to \eta \in V \text{ and for every } v \in V \text{ then one has} \\ \limsup_{n \to \infty} [j(\eta_n, v_n) - j(\eta_n, u_n)] \leq j(\eta, v) - j(\eta, u). \end{cases}$

(J_4): $j(u, v) - j(u, u) + j(v, u) - j(v, v) < m |u - v|_V^2 \quad \forall u, v \in V, u \neq v$

(J_5): $\quad j(u, v) - j(u, u) + j(v, u) - j(v, v) \leq \alpha |u - v|_V^2$,

$\forall u, v \in V$,for some $\alpha \in \mathbb{R}$ with $\alpha < m$.

Theorem 1. *We consider the following problem :*

$$\langle Au, v - u \rangle_V + \langle Bu, v - u \rangle_{\mathbb{V}} + j(u, v) - j(u, u) \geq \langle f, v - u \rangle_V \quad \forall v \in V$$

Let $(h_1), (h_2)$ *and* (h_6) *hold.*

(1) Under the assumptions $(J_1), (J_2), (J_3), (J_5)$ and (h_3), the problem has at least a solution.

(2) Under the assumptions $(J_1), (J_2), (J_3), (J_5), (h_3)$ and (h_4),the problem has a unique solution. .

(3) Under the assumptions $(J_1), (J_2), (J_3), (J_5)$, (h_3) and (h_5), the problem has a unique solution $u = u(f)$ which depends Lipschitz continuously on $f \in V$ with the Lipschitz constant $(m - \alpha - \beta)^{-1}, i.e.$

$$|u(f_1) - u(f_2)|_V \leq \frac{1}{(m - \alpha - \beta)} |f_1 - f_2|_V \quad \forall f_1, f_2 \in V$$

Proof. It is based on results of topological degree theory as well as on convexity, monotonicity, compactness and fixed point arguments see [1].

Remark 1. *The coercivity conditions* $(J_1), (J_2)$ *and* (h_1) (a) *are needed in order to use the weakly sequential compactness property of the closed, bounded convex sets of* V*, see [1].*

2. The elastic contact problem

2.1. Formulation of the mechanical problem and assumptions.

Let us consider an elastic, homogeneous isotrop body whose material particles occupy a bounded domain $\Omega \subset \mathbb{R}^n (n = 1, 2, 3)$ and whose boundary Γ, assumed to be sufficiently smooth is partitioned into three disjoint measurable parts Γ_1, Γ_2 and Γ_3 such that meas $\Gamma_1 > 0$.

We denote by u the displacement vector, σ represents the stress field and $\varepsilon(u)$ is the small strain tensor such that that $\varepsilon = (\varepsilon_{ij}) : H_1 \to \mathcal{H}$

$$\varepsilon_{ij}(u) = \frac{1}{2} \left(\frac{\partial u_i}{\partial u_j} + \frac{\partial u_j}{\partial u_i} \right)$$

where the spaces H_1 and $\mathcal{H}$ are defined below. The elastic constitutive law of the material is assumed to be

$$\sigma = F(\varepsilon(u), \theta)$$

In which F is a given nonlinear function, and θ is a parameter.

We assume that the body is clamped on Γ_1 and thus the displacement field vanishes there, that the surface tractions h act on Γ_2 and that the body rests on a rigid foundation on the part Γ_3 of the boundary and that the normal stress σ_ν satisfies the normal compliance condition:

$$\sigma_\nu = -p_\nu(u_\nu)$$

where $\nu = (\nu_i)$ represents the outward unit normal vector on $\Gamma_j, (j = 1, 2, 3)$, u_ν represents the normal displacement $(u_\nu = u.\nu)$, p_ν is a prescribed nonnegative function and u_ν when it is positive, represents the penetration of the body in the foundation. The associated friction law on Γ_3 is chosen as

$$\begin{cases} |\sigma_\tau| \le p_\tau(u_\nu) \\ |\sigma_\tau| < p_\tau(u_\nu) \Rightarrow u_\tau = 0 \\ |\sigma_\tau| = p_\tau(u_\nu) \Rightarrow \sigma_\tau = -\lambda u_\tau, \lambda \ge 0 \end{cases}$$

here τ is the tangent unit vector in the positive sense on Γ_j $(j = 1, 2, 3)$, p_τ is a nonnegative function, the so-called friction bound, u_τ denotes the tangential displacement $(u_\tau = u - u_\nu\nu)$ and σ_τ represents the tangential force on the contact boundary.

For example, we can consider

(1): $\quad p_\nu(r) = c_\nu(r_+)^{m_\nu}, p_\tau(r) = c_\tau r_+$

where $m_\nu \in]0, 1], c_\nu$ and c_τ are positives constants and $r_+ = \max\{0, r\}$.

Also, the friction law can be used with

(2): $\quad p_\nu = \mu p_\nu$ or $p_\tau = \mu p_\nu(1 - \alpha p_\nu)_+$

where $\mu > 0$ is a coefficient of friction and α is a small positive coefficient related to the wear and hardness of the surface.

2.2. **Position of the problem.** The mechanical problem may be formulated as follows:

Problem (P): Find a displacement field $u : \Omega \to \mathbb{R}^n$ and a stress field $\sigma : \Omega \to S_n$ such that :

(3): $\quad Div\ \sigma + f_0 = 0 \quad in\ \Omega$

(4): $\quad \sigma = F(x, \varepsilon(u), \theta) \quad in\ \Omega$

(5): $\quad u = 0 \quad on \quad \Gamma_1$

(6): $\quad \gamma(\sigma\nu + \Phi(x, u)) = h\ on\ \Gamma_2$

and on Γ_3,

$$\textbf{(7):} \qquad \begin{cases} \sigma_\nu = -p_\nu(u_\nu) \\ |\sigma_\tau| \le p_\tau(u_\nu) \\ |\sigma_\tau| < p_\tau(u_\nu) \Rightarrow u_\tau = 0 \\ |\sigma_\tau| = p_\tau(u_\nu) \Rightarrow \sigma_\tau = -\lambda u_\tau, pour\ un\ certain\ \lambda \ge 0 \end{cases}$$

(6) is called rapel forces and it means that the surface tractions are proportional to the displacement. It's the case of building and matlats, ...).

To provide the variational analysis of the problem (P) we need additional notations. Let

$H = \left(\mathbb{L}^2(\Omega)\right)^n,\ \ H_1 = \left(H^1(\Omega)\right)^n.$

$\mathcal{H} = \left(\mathbb{L}^2(\Omega)\right)^{n\times n},\ \ \mathcal{H}_1 = \left(H^1(\Omega)\right)^{n\times n}.$

The spaces H, H_1 and $\mathcal{H}$ are real Hilbert spaces endowed with the canonical inner products denoted by $\langle .,.\rangle_H$, $\langle .,.\rangle_{H_1}$and $\langle .,.\rangle_{\mathcal{H}}$, respectively. The associate norms on H, H_1 and $\mathcal{H}$ are denoted by $|.|_H$, $|.|_{H_1}$and $|.|_{\mathcal{H}}$, respectively.

In the study of the mechanical problem (P) we assume that the elasticity operator $F : \Omega \times S_n \times \mathbb{R}^M \to S_n$ satisfies

$$\textbf{(H}_1\textbf{):} \begin{cases} (\textbf{a}) \quad \exists m_F > 0 \text{ such that } \forall \varepsilon_1, \varepsilon_2 \in S_n, \forall \theta \in \mathbb{R}^M \\ \left(F(x,\varepsilon_1,\theta) - F(x,\varepsilon_2,\theta)\right).(\varepsilon_1 - \varepsilon_2) \ge m_F \left|\varepsilon_1 - \varepsilon_2\right|^2 \ a.e.in\ \Omega, . \\ (\textbf{b}) \quad \exists L_1, L_2 > 0 \text{ such that } \forall \varepsilon_1, \varepsilon_2 \in S_2, \forall \theta_1, \theta_2 \in \mathbb{R}^M \\ \left|F(x,\varepsilon_1,\theta_1) - F(x,\varepsilon_2,\theta_2)\right| \le L_1 \left|\varepsilon_1 - \varepsilon_2\right| + L_2 \left|\theta_1 - \theta_2\right| \ a.e.in\ \Omega, \\ (\textbf{c}) x \to F(x,\varepsilon,\theta) \text{ is measurable function with respect to the} \\ \text{Lebesgue measure } a.e.in\ \Omega, \forall \varepsilon \in S_n, \forall \theta \in \mathbb{R}^M \\ (\textbf{d}) \quad F(x, 0_n, 0_M) = 0_n. \end{cases}$$

We assume that the forces and the tractions have the regularity

$$\textbf{(H}_2\textbf{):} \qquad f_0 \in H = \mathbb{L}^2(\Omega)^n,\ h \in \mathbb{L}^2(\Gamma_2)^n,$$

also,

$$\textbf{(H}_3\textbf{):} \qquad \theta \in \mathbb{L}^2(\Omega)^M$$

The function Φ is defined by:

$$\Phi : \Gamma_2 \times \mathbb{R}^n \to \mathbb{R}^n$$

such that

$$\textbf{(H}_4\textbf{):} \begin{cases} (\mathbf{a})\ \exists m_\Phi > 0 \text{ such that} \\ (\Phi(x,u_1) - \Phi(x,u_2)) \,.\, (u_1 - u_2) \geq m_\Phi \left|u_1 - u_2\right|^2 \\ \text{ a.e. in } \Gamma_2, \forall u_1, u_2 \in \mathbb{R}^n \\ (\mathbf{b})\ \exists L_\Phi > 0 \text{ such that} \\ \left|\Phi(x,u_1) - \Phi(x,u_2)\right| \leq L_\Phi \left|u_1 - u_2\right| \ a.e.in\ \Gamma_2, \forall u_1, u_2 \in \mathbb{R}^n \\ (\mathbf{c})\ x \mapsto \Phi(x,u) \text{ is measurable function with respect to the} \\ \quad \text{Lebesgue measure } a.e.in\ \Gamma_2, \forall u \in \mathbb{R}^n. \\ (\mathbf{d})\ \Phi(x, 0_n) = 0_n \end{cases}$$

We also assume that the normal compliance functions satisfy the following hypothesis for $r = \nu, \tau$:

$$\textbf{(H}_5\textbf{):} \begin{cases} (\mathbf{a})\ p_r : \Gamma_3 \times \mathbb{R} \to \mathbb{R}_+ \text{ such that} \\ \qquad p_r(.,r) \text{ is Lebesgue measurable on } \Gamma_3,\ \forall r \in \mathbb{R} \\ (\mathbf{b}) \text{ The mapping } p_\tau(.,r)\ = 0 \text{ for } r \leq 0; \\ (\mathbf{c}) \text{ There exists an } L_r > 0 \text{ such that} \\ \left|p_r(x,r_1) - p_r(x,r_2)\right| \leq L_r \left|r_1 - r_2\right|, \forall r_1, r_2 \in \mathbb{R},\ a.e.on\, \Gamma_3, \end{cases}$$

$(H_5^{'})$**:** $(p_\nu(x,r_1) - p_\nu(x,r_2)) \,.\, (r_1 - r_2) \geq 0, \forall r_1, r_2 \in \mathbb{R},\ a.e.on\, \Gamma_3,$

Remark 2. *Certainly the functions defined in* (1) *satisfy the conditions* (H_5) *and* $(H_5^{'})$. *Also, if* p_ν *defined in* (2) *is Lipschitz then the conditions* (H_5) *is satisfied.*

Using the hypothesis $(H_5)(b)$ and (c) it follows that:

(8): $\quad \left|p_r(x,t)\right| \leq L_\tau \left|t\right|, \forall t \in \mathbb{R}, a.e.on\, \Gamma_3.$

Remark 3. *Using* (H_1) *we find that for all* $\tau \in \mathcal{H}$ *the function* $x \to F(x, \tau(x), \theta(x))$ *belongs to* $\mathcal{H}$ *and hence we may consider* $F(.,\theta)$ *as an operator defined on* $\mathcal{H}$ *with range in* $\mathcal{H}$ *by:* $F(.,\theta) : \mathcal{H} \to \mathcal{H}$

$$F(\varepsilon, \theta)(x) = F(x, \varepsilon(x), \theta(x))\ a.e.in\ \Omega\ \forall \varepsilon \in \mathcal{H}$$

Moreover, $F(.,\theta)$ *is a strongly monotone Lipschitz continuous operator:*

(9): $\exists L_1 > 0 : |F(\varepsilon_1, \theta) - F(\varepsilon_2, \theta)|_{\mathcal{H}} \leq L_1 |\varepsilon_1 - \varepsilon_2|_{\mathcal{H}}$.

(10): $\langle F(\varepsilon_1, \theta) - F(\varepsilon_2, \theta), \varepsilon_1 - \varepsilon_2 \rangle_{\mathcal{H}} \geq m_F |\varepsilon_1 - \varepsilon_2|^2_{\mathcal{H}}$

The inequality (9) is a particular case of

(11): $\exists L_1, L_2 > 0 : |F(\varepsilon_1, \theta_1) - F(\varepsilon_2, \theta_2)|_{\mathcal{H}} \leq L_1 |\varepsilon_1 - \varepsilon_2|_{\mathcal{H}} + L_2 |\theta_1 - \theta_2|_{\mathbb{L}^2(\Omega)^M}$.

Therefore $F(.,\theta)$ is invertible and its inverse $F^{-1}(.,\theta) : \mathcal{H} \to \mathcal{H}$ is also a strongly Lipschitz continuous operator.

Remark 4. *The assumptions* (H_4) *allows us to consider the operator denoted by* $\Phi : H \to \mathbb{L}^2(\Gamma_2)^n$

$$\Phi(v)(x) = \Phi(x, v(x)) \text{ a.e. in } \Gamma_2 \ \forall v \in H$$

Moreover, Φ *is a strongly monotone Lipschitz continuous operator and therefore* Φ *is invertible and its inverse* $\Phi^{-1} : \mathbb{L}^2(\Gamma_2)^n \to H$ *is also a strongly Lipschitz continuous operator.*

We denote by V the closed subspace of H_1 given by

(12): $V = \{ v \in H_1 / \ \gamma v = 0 \ sur \ \Gamma_1 \}$

Since meas $\Gamma_1 > 0$, Korn's inequality holds:

$$|\varepsilon(v)|_{\mathcal{H}} \geq C |v|_{H_1} \ \ \forall v \in V$$

C denotes a strictly positive generic constant which may depend on Ω, Γ_1, Γ_2,Γ_3 and F.

We endow V with the inner product defined by

(13): $\langle u, v \rangle_V = \langle \varepsilon(u), \varepsilon(v) \rangle_{\mathcal{H}} \quad \forall u, v \in V$

and let $|.|_V$ the associated norm. It follows from the Korn's inequality that $|.|_V$ and $|.|_{H_1}$are equivalent norms on V. Therefore, $(V, |.|_V)$ is a real Hilbert space. Moreover, by the Sobolev trace theorem, Korn's inequality and (13) we have a constant C_0 depending on Ω, Γ_1 et Γ_3 such that:

(14): $|v|_{\mathbb{L}^2(\Gamma_3)^n} \leq C_0 |v|_V \ \ , \forall v \in V$.

The functional $v \to \langle f, v\rangle_H + \langle h, \gamma v\rangle_{\mathbb{L}^2(\Gamma_2)^n}$, $\forall v \in V$ is linear and continue on V; it results, by using the Riesz Fréchet theorem, the existence of an element $f \in V$ such that

(15): $\langle f, v\rangle_V = \langle f_0, v\rangle_H + \langle h, \gamma v\rangle_{\mathbb{L}^2(\Gamma_2)^n}\ \forall v \in V.$

For all fixed w in V and for all fixed θ in $\mathbb{L}^2(\Omega)^M$, the functional defined on V by: $v \to \langle F\varepsilon(w), \theta), \varepsilon(v)\rangle_{\mathcal{H}} + \langle \Phi(w), v\rangle_{\mathbb{L}^2(\Gamma_2)^n}$ is a continuous linear functional on V. Then using Riesz-Fréchet's theorem, there exists an element $A_\theta w \in V$ such that:

(16): $\langle A_\theta w, v\rangle_V = \langle F\varepsilon(w), \theta), \varepsilon(v)\rangle_{\mathcal{H}} + \langle \Phi(w), v\rangle_{\mathbb{L}^2(\Gamma_2)^n}\ \forall v \in V.$

Let $B : V \to V$ defined by

(17): $\langle Bu, v\rangle_V = \int\limits_{\Gamma_3} p_\nu(u_\nu - g)v_\nu ds, \forall u, v \in V.$

and let $j : V \times V \to \mathbb{R}$ be the functional

(18): $j(u, v) = \int\limits_{\Gamma_3} p_\tau(u_\nu - g)\,|v_\tau|\, ds, \forall u, v \in V.$

Using the conditions $(H_5)(b),(c)$ it follows that for all $v \in V$ the functions

(19): $x \longmapsto p_r(x, v(x)),\ (r = \nu, \tau),$

belong to $\mathbb{L}^2(\Gamma_3)$ and hence the integrals in (17) and (18) are well defined.

2.3. Variational Formulation.

Theorem 2. *If* $(u, \sigma) \in H_1 \times \mathcal{H}_1$ *are sufficiently smooth functions satisfying* $(3) - (7)$ *then*

(20): $$u \in V : \langle \sigma, \varepsilon(v) - \varepsilon(u)\rangle_{\mathcal{H}} + \langle \Phi(x, u), v - u\rangle_{\mathbb{L}^2(\Gamma_2)^n} + \langle Bu, v - u\rangle_{\mathbb{V}}$$
$$+ j(u, v) - j(u, u) \succeq \langle q, v - u\rangle_V, \forall v \in V.$$

Proof. Let $u, v \in U_{ad}$, by using the Green formula we obtain:

$$\begin{aligned}\langle f_0, v - u\rangle_H &= -\langle Div\sigma, \varepsilon(v) - \varepsilon(u)\rangle_{\mathcal{H}} \\ &= \langle \sigma, \varepsilon(v) - \varepsilon(u)\rangle_{\mathcal{H}} - \langle \sigma\nu, \gamma(v - u)\rangle_{H'_\Gamma \times H_\Gamma}\end{aligned}$$

but

$$\begin{aligned}\langle \sigma\nu, \gamma(v-u)\rangle_{H'_\Gamma\times H_\Gamma} &= \int\limits_{\Gamma} \sigma\nu(v-u)ds = \sum_{j=1}^{3}\int\limits_{\Gamma_j} \sigma\nu^j(v-u)ds \\ &= \int\limits_{\Gamma_2} h(v-u)ds - \int\limits_{\Gamma_2} \Phi(u)(v-u)ds + \int\limits_{\Gamma_3} \sigma\nu^1(v-u)ds\end{aligned}$$

Let $(u_\nu, u_\tau), (v_\nu, v_\tau)$ and $(\sigma_\nu, \sigma_\tau)$ the components of the vectors u, v and $\sigma\nu$ in the orthonorm system (ν, τ). From (5) and (7) it results that we obtain on Γ_3

$$\begin{aligned}\sigma\nu(v-u) &= \sigma_\nu(v_\nu - u_\nu) + \sigma_\tau(v_\tau - u_\tau) \\ &= -p_\nu(u_\nu)(v_\nu - u_\nu) + \sigma_\tau(v_\tau - u_\tau)\end{aligned}$$

Then

$$\begin{aligned}\langle f_0, v-u\rangle_H + \langle h, \gamma(v-u)\rangle_{\mathbb{L}^2(\Gamma_2)^n} &= \langle \sigma, \varepsilon(v-u)\rangle_{\mathcal{H}} + \langle \Phi(u), v-u\rangle_{\mathbb{L}^2(\Gamma_2)^n} + \\ &+ \int\limits_{\Gamma_3} p_\nu(u_\nu)(v_\nu - u_\nu)ds - \int\limits_{\Gamma_3} \sigma_\tau(v_\nu - u_\nu)ds\end{aligned}$$

So, by (15) we obtain:

$$\begin{aligned}\langle f, v-u\rangle_H &= \langle \sigma, \varepsilon(v) - \varepsilon(u)\rangle_{\mathcal{H}} + \langle \Phi(u), v-u\rangle_{\mathbb{L}^2(\Gamma_2)^n} + \\ &+ \langle Bu, v-u\rangle_{\mathbb{V}} - \int\limits_{\Gamma_3} \sigma_\tau(v_\nu - u_\nu)ds\end{aligned}$$

Using (7) it results

$$\begin{aligned}-\sigma_\tau(v_\nu - u_\nu) &= -\sigma_\tau v_\nu + \sigma_\tau u_\nu, \\ -\sigma_\tau v_\nu &\leq p_\nu(u_\nu)\,|v_\nu| \\ \sigma_\tau u_\nu &= -p_\nu(u_\nu)\,|u_\nu|\end{aligned}$$

It follows that

$$\begin{aligned}\langle f, v-u\rangle_H &\leq \langle \sigma, \varepsilon(v) - \varepsilon(u)\rangle_{\mathcal{H}} + \langle \Phi(u), v-u\rangle_{\mathbb{L}^2(\Gamma_2)^n} + \\ &+ \langle Bu, v-u\rangle_{\mathbb{V}} + j(u,v) - j(u,u)\end{aligned}$$

it results

$$\langle A_\theta u, v-u\rangle_V + \langle Bu, v-u\rangle_{\mathbb{V}} + j(u,v) - j(u,u) \geq \langle f, v-u\rangle_H$$

and using (4), yields to the following variational formulation of the problem (P):

Find a displacement field $u : \Omega \to H_1$, such that

(21): $u \in V,\ \langle F(\varepsilon(u),\theta), \varepsilon(v)-\varepsilon(u)\rangle_{\mathcal{H}} + \langle \Phi(u), v-u\rangle_{\mathbb{L}^2(\Gamma_2)^n} +$

$$\langle Bu, v-u\rangle_{\mathbb{V}} + j(u,v) - j(u,u) \geq \langle f, v-u\rangle_V, \forall v \in V.$$

Remark 5. *If u is a solution (21) then (u,σ) satisfy the mechanical problem (P), where σ is given by (4).*

2.4. **Existence and uniqueness results.** Let $L_0 > 0$ a constant such that:

$$L_0 = \frac{m_F + m_\Phi}{C_0^2}$$

Theorem 3. *Assume that $(H_1)-(H_5)$ and $L_\tau < L_0$ hold. Then*

1) the variational problem (P_V) has at least a solution $u \in V$

2) in addition to $(H_5^{'})$ the problem (P_V) has a unique solution which depends Lipschitz continuously on f.

Proof. The proof fellows from the abstract result provided by theorem1. It will be carried out in several steps. We are going to prove that if the hypothesis $(H_1)-(H_5)$ hold then the conditions $(h_1)-(h_6)$, $(j_2),(j_3)$ and (j_5) will be satisfied.

Lemma 4. *We suppose that $(H_1)-(H_5)$ hold, the we obtain that the conditions (h_1) and (h_6) are satisfied.*

Proof. **1)** We see that (16) et $(H_5)(b),(c)$ give $(h_1)(a)$ with $m = m_F + m_\Phi$ and $(h_1)(b)$ with $M = L_1 + L_\Phi$.

2) Moreover, from (18) we deduce that $j(u,.)$ is convex $\forall u \in V$.

Using (8) and (18) we obtain that $\forall u, v_1, v_2 \in V$

$$|j(u,v_1) - j(u,v_2)| \leq L_\tau\, |u_\nu|_{\mathbb{L}^2(\Gamma_2)}\, |v_1 - v_2|_{\mathbb{L}^2(\Gamma_2)^d} \leq C(u)\, |v_1 - v_2|_{\mathbb{V}}$$

Then $j(u,.)$ is continuous on V for all $u \in V$.

Lemma 5. *Under assumptions* $(H_4), (H_5)$*, and* $L_\tau < L_0$*, The functional* J *satisfies the conditions* $(J_2), (J_3)$ *and* (J_5)*.*

Proof. **1)** Using (18) it results that ,$\forall\ \eta, u \in V, \forall\lambda \in]0,1[$:

$$\frac{1}{\lambda}\left[j(\eta, u-\lambda u) - j(\eta, u)\right] = \int_{\Gamma_3} p_\tau(u_\nu)(-\left|u_\tau\right|)ds \leq 0$$

Then $j_2^{'}(\eta, u; -u) \leq 0 \quad \forall\ \eta, u \in V$. It follows that for every sequence $\{u_n\}$ and $\{\eta_n\}$ in V, we have

$$\liminf_{n\to\infty} \frac{1}{\left|u_n\right|_V^2}\left[j_2^{'}(\eta_n, u_n; -u_n)\right] \leq 0 < m$$

and we deduce that J satisfies (J_2).

2)Let now $\{u_n\} \subset V$, $\{\eta_n\} \subset V$ be two sequences such that $u_n \rightharpoonup u$ and $\eta_n \rightharpoonup \eta$ weakly in V.

Using the compactness property of the trace map of $H^1(\Omega)$ in $\mathbb{L}^2(\Gamma)$ it follows that

(30): $u_n \to u$ in $\mathbb{L}^2(\Gamma_3)$ strongly for a subsequence,

and

(31): $\eta_n \to \eta$ in $\mathbb{L}^2(\Gamma_3)$ strongly for a subsequence,

Using $(H_5)(c)$ and (31) we have

(32): $p_\tau(., \eta_{n\nu} - g) \to p_\tau(., \eta_\nu - g)$ in $\mathbb{L}^2(\Gamma_3)$ strongly for a subsequence,

Therefore we deduce that

(33): $j(\eta_n, v) \to j(\eta, v)\ \forall v \in V$.

Also, (30) gives

(34): $\left|u_{n\tau}\right| \to \left|u_\tau\right|$ in $\mathbb{L}^2(\Gamma_3)$ strongly for a subsequence,

So, by (18), (32) and (34) we obtain

(35): $j(\eta_n, u_n) \to j(\eta, v)$ for a subsequence.

Using (33) and (35) we have for all $v \in V$,

$$\limsup_{n\to\infty}\left[j(\eta_n, v) - j(\eta_n, u_n)\right] = j(\eta, v) - j(\eta, u)$$

The (J_3) is satisfied.

3) Let $u, v \in V$. Using $(H_5)(c)$ and (18) we obtain:

$$\begin{aligned} j(u,v) - j(u,u) + j(v,u) - j(v,v) &\leq \int_{\Gamma_3} |p_\tau(u_\nu) - p_\tau(v_\nu)| \, |v_\tau - u_\tau| \, ds \\ &\leq L_\tau \, |u - v|^2_{\mathbb{L}^2(\Gamma_3)} \end{aligned}$$

Using now (14) in the previous inequality we deduce

$$j(u,v) - j(u,u) + j(v,u) - j(v,v) \leq L_\tau C_0^2 \, |u - v|^2_{\mathbb{L}^2(\Gamma_3)}$$

Then (J_5) is satisfied with $\alpha = L_\tau C_0^2, \; L_\tau < L_0$.

Lemma 6. *Under assumptions* (H_4) *and* (H_5) *we deduce that* (J_1)*,* (h_2) *and* (h_3) *are satisfied, and under assumptions* $(H_4), (H_5)$ *and* $(H_5^{'})$ *we obtain the condition* (h_5)*.*

Proof. **1)** Using (17) we obtain $\langle Bu, v \rangle_V = \int_{\Gamma_3} p_\nu(u_\nu) v_\nu ds \quad \forall v \in V$.

Let $v_\nu \geq 0$, since $p_\nu \geq 0$, it results

$$\langle Bv, v \rangle_V \geq 0$$

then (h_2) is satisfied with $C = 0$.

2) By using (18) we have for all $\eta, u \in V$, $j_2^{'}(\eta, u; -u) \leq 0$. which results (J_1) with $C = 0$.

3)Let now $\eta_n \to \eta$ weakly in V. Using the compactness property of the trace map of $H^1(\Omega)$ in $\mathbb{L}^2(\Gamma)$ it follows that $\eta_n \to \eta$ in $\mathbb{L}^2(\Gamma_3)$ strongly for a subsequence,

It results from (17)

$$\langle Bu_1 - Bu_2, v \rangle_V = \langle p_\nu(u_{1\nu}) - p_\nu(u_{2\nu}), v_\nu \rangle_{\mathbb{L}^2(\Gamma_3)} \, \forall u_1, u_2, v \in V$$

Taking $v = Bu_1 - Bu_2$ in the previous equality, we have

$$\begin{aligned} |v|_V^2 &\leq |p_\nu(u_{1\nu}) - p_\nu(u_{2\nu})|_{\mathbb{L}^2(\Gamma_3)} \cdot |v|_{\mathbb{L}^2(\Gamma_3)} \\ &\leq C \, |p_\nu(u_{1\nu}) - p_\nu(u_{2\nu})|_{\mathbb{L}^2(\Gamma_3)} \cdot |v|_{\mathbb{V}} \, . \end{aligned}$$

Then

(36): $|Bu_1 - Bu_2| \leq C \, |p_\nu(u_{1\nu}) - p_\nu(u_{2\nu})|_{\mathbb{L}^2(\Gamma_3)}$.

So, by (H_5) we obtain

(37): $p_\nu(\eta_{n\nu} - g) \to p_\nu(\eta_\nu - g)$ in $\mathbb{L}^2(\Gamma_3)$ strongly for a subsequence.

Finally, (36) and (37) give $B\eta_n \to B\eta$ in V strongly for a subsequence.

4) Using (17) and $(H_5^{'})$ it follows that (h_5) is satisfied for $\beta = 0 : \forall u_1, u_2 \in V$:

(38): $\langle B, u_1 - Bu_2, u_2 - u_1 \rangle_V = \langle p_\nu(u_{1\nu}) - p_\nu(u_{2\nu}), u_{2\nu} - u_{1\nu} \rangle_{\mathbb{L}^2(\Gamma_3)} \leq 0.$

Proof of theorem 3.

The proof is based on the application of the theorem1. It follows by using the lemma 4, lemma 5 and lemma 6.

3. The dependence of the solution on the parameter

Theorem 7. *under the assumptions* $(H_1)-(H_5)$, *let* $(u_i, \sigma_i), (i = 1, 2)$ *the variational solution of the problem (P) associée to the parameter* θ_i *such that* $\theta_i \in \mathbb{L}^2(\Omega)^M$ *is satisfied. Then there exists a positive constant* $C > 0$ *which is depend to* Ω, Γ_1 *and* Γ *such that:*

$$|u_1 - u_2|_{H_1} + |\sigma_1 - \sigma_2|_{H_1} \leq C\,|\theta_1 - \theta_2|_{\mathbb{L}^2(\Omega)^M}$$

Proof. Let $(u_i, \sigma_i), (i = 1, 2)$, the variational solutions of the problem (P).

$$\langle \sigma_i, \varepsilon(v - u_i) \rangle_{\mathcal{H}} + \langle \Phi(u_1), v - u_i \rangle_{\mathbb{L}^2(\Gamma_2)^n} + \langle Bu_i, v - u_i \rangle_{\mathbb{V}} +$$

$$+ j(u_i, v) - j(u_i, u_i) \geq \langle f, v - u_i \rangle_H$$

Where $v = u_2$ for $i = 1$, and $v = u_1$ for $i = 2$.

$$\langle \sigma_1, \varepsilon(u_2 - u_1) \rangle_{\mathcal{H}} + \langle \Phi(u_1), u_2 - u_1 \rangle_{\mathbb{L}^2(\Gamma_2)^n} + \langle Bu_1, u_2 - u_1 \rangle_{\mathbb{V}} +$$

$$+ j(u_1, u_2) - j(u_1, u_1) + \succeq \langle f, u_2 - u_1 \rangle_V$$

and

$$\langle \sigma_2, \varepsilon(u_1 - u_2) \rangle_{\mathcal{H}} + \langle \Phi(u_2), u_1 - u_2 \rangle_{\mathbb{L}^2(\Gamma_2)^n} + \langle Bu_2, u_1 - u_2 \rangle_{\mathbb{V}} +$$

$$+ j(u_2, u_1) - j(u_2, u_2) + \succeq \langle f, u_1 - u_2 \rangle_V$$

it follows that

$$\langle \sigma_1 - \sigma_2, \varepsilon(u_2 - u_1) \rangle_{\mathcal{H}} + \langle \Phi(u_1) - \Phi(u_2), u_2 - u_1 \rangle_{\mathbb{L}^2(\Gamma_2)^n} +$$

$$+ \langle Bu_1 - Bu_2, u_2 - u_1 \rangle_{\mathbb{V}} + j(u_1, u_2) - j(u_1, u_1) + j(u_2, u_1) - j(u_2, u_2) \succeq 0$$

Then, using (j_5), we deduce that

$$j(u_1,u_2)-j(u_1,u_1)+j(u_2,u_1)-j(u_2,u_2)\leq\alpha\left|u_1-u_2\right|_V^2\ ,\ \alpha<m$$

by (33), we obtain that:

$$\langle\sigma_1-\sigma_2,\varepsilon(u_2-u_1)\rangle_{\mathcal{H}}+\langle\Phi(u_1)-\Phi(u_2),u_2-u_1\rangle_{\mathbb{L}^2(\Gamma_2)^n}+\alpha\left|u_1-u_2\right|_V^2\geq 0$$

it follows that

$$\langle\sigma_1-F(\varepsilon(u_2),\theta_1),\varepsilon(u_2-u_1)\rangle_{\mathcal{H}}+\langle F(\varepsilon(u_2),\theta_1)-\sigma_2,\varepsilon(u_2-u_1)\rangle_{\mathcal{H}}+$$

$$\langle\Phi(u_1)-\Phi(u_2),u_2-u_1\rangle_{\mathbb{L}^2(\Gamma_2)^n}+\alpha\left|u_1-u_2\right|_V^2\geq 0$$

Then

$$\left|\langle F(\varepsilon(u_2),\theta_1)-\sigma_1,\varepsilon(u_2-u_1)\rangle_{\mathcal{H}}+\langle\Phi(u_2)-\Phi(u_1),u_2-u_1\rangle_{\mathbb{L}^2(\Gamma_2)^n}\right|$$

$$\leq\left|\langle F(\varepsilon(u_2),\theta_1)-\sigma_2,\varepsilon(u_2-u_1)\rangle_{\mathcal{H}}\right|+\alpha\left|u_1-u_2\right|_V^2$$

Using the Cauchy-Schwartz inequality and $(H_1)(b)$on the right member of the previous inequality, and $(H_1)(a),(H_4)(a)$ and Korn's inequality on the left member, we obtain that

$$m\left|u_1-u_2\right|_V^2\leq cL_2\left|\theta_1-\theta_2\right|_{\mathbb{L}^2(\Omega)^M}\left|u_1-u_2\right|_V+\alpha\left|u_1-u_2\right|_V^2$$

Then

$$(m-\alpha)\left|u_1-u_2\right|_V\leq K\left|\theta_1-\theta_2\right|_{\mathbb{L}^2(\Omega)^M}\ ,\ \text{where } K \text{ is a constant} >0$$

Since $(m-\alpha)>0$, then there exists a constant $C>0$ such that

(39): $\left|u_1-u_2\right|_{H_1}\leq C\left|\theta_1-\theta_2\right|_{\mathbb{L}^2(\Omega)^M}$

Other way, we have:

$$\left|\sigma_1-\sigma_2\right|_{\mathcal{H}_1}=\left|\sigma_1-\sigma_2\right|_{\mathcal{H}}=\left|F(\varepsilon(u_1),\theta_1)-F(\varepsilon(u_2),\theta_2)\right|_{\mathcal{H}}$$

Using $(H_1)(b)$ and (34) we obtain that

(40): $\left|\sigma_1-\sigma_2\right|_{\mathcal{H}_1}\leq C\left|\theta_1-\theta_2\right|_{\mathbb{L}^2(\Omega)^M}$

The wanted inequality is now a consequence of (39) and (40)

This theorem prove well the dependence of the solution on the parameter θ and this result is very important from the mechanical point of view because it prove that small perturbations on the parameter θ gives small perturbations on the solution (u, σ) of the problem without frisher.

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STUDIA UNIV. "BABEŞ–BOLYAI", MATHEMATICA, Volume **LI**, Number 2, June 2006

PFAFFIAN TRANSFORMATIONS

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Abstract. Geometrical and structural properties are proved for manifolds possessing a particular locally conformal almost cosymplectic structure.

1. **Introduction**

Let $M(g, \Omega, \phi, \eta, \xi)$ be an $2m + 1$-dimensional Riemannian manifold with metric tensor g and associated Levi-Civita connection ∇. The quadruple $(\Omega, \phi, \xi, \eta)$ consists of a structure 2-form Ω of rank $2m$, an endomorphism ϕ of the tangent bundle, the Reeb vector field ξ, and its corresponding Reeb covector field η, respectively.

We assume that the 2-form Ω satisfies the relation

$$d\Omega = \lambda \; \eta \wedge \Omega \,, \tag{1}$$

where λ is constant, and that the 1-form η is given by

$$\eta = \lambda df \,, \tag{2}$$

for some scalar function f on M. We may therefore notice that a locally conformal almost cosymplectic structure [7] [10] is defined on the manifold M.

In addition, we assume that the field ϕ of endomorphisms of the tangent spaces defines a quasi-Sasakian structure, thus realizing in particular the identity

$$\phi^2 = -\mathrm{Id} + \eta \otimes \xi \,.$$

Received by the editors: 22.11.2005.

2000 *Mathematics Subject Classification.* 53B20.

Key words and phrases. Pfaffian transformation, almost cosymplectic structure.

Moreover, we will assume the presence on M of a structure vector field X satisfying the property

$$\nabla X = f dp + \lambda \nabla \xi \,. \tag{3}$$

In the present paper various properties involving the above mentioned objects are studied. In particular, for the Lie differential of Ω and η with respect to X, one has

$$\begin{aligned} \mathcal{L}_X \eta &= 0\,, \\ \mathcal{L}_X \Omega &= 0\,, \end{aligned}$$

which shows that η and Ω define Pfaffian transformations [3].

2. **Preliminaries**

Let (M, g) be an n-dimensional Riemannian manifold and let ∇ be the covariant differential operator defined by the metric tensor. We assume in the sequel that M is oriented and that the connection ∇ is symmetric.

Let $\Gamma TM = \Xi(M)$ be the set of sections of the tangent bundle TM, and

$$\flat : TM \xrightarrow{\flat} T^*M \qquad \text{and} \qquad \sharp : TM \xleftarrow{\sharp} T^*M$$

the classical isomorphisms defined by the metric tensor g (i.e. $^\flat$ is the index lowering operator, and $^\sharp$ is the index raising operator).

Following [12], we denote by

$$A^q(M, TM) = \Gamma \mathrm{Hom}(\Lambda^q TM, TM)\,,$$

the set of vector valued q-forms ($q < \dim M$) , and we write for the covariant derivative operator with respect to ∇

$$d^\nabla : A^q(M, TM) \to A^{q+1}(M, TM)\,. \tag{4}$$

It should be noticed that in general $d^{\nabla^2} = d^\nabla \circ d^\nabla \neq 0$, unlike $d^2 = d \circ d = 0$.

Furthermore, we denote by $dp \in A^1(M, TM)$ the canonical vector valued 1-form of M, which is also called the soldering form of M [3]; since ∇ is assumed to be symmetric, we recall that the identity $d^\nabla(dp) = 0$ is valid.

The operator

$$d^\omega = d + e(\omega)\,,$$

acting on ΛM is called the cohomology operator [5]. Here, $e(\omega)$ means the exterior product by the closed 1-form ω, i.e.

$$d^\omega u = du + \omega \wedge u\,,$$

with $u \in \Lambda M$. A form $u \in \Lambda M$ such that

$$d^\omega u = 0\,,$$

is said to be d^ω-closed, and ω is called the cohomology form.

A vector field $X \in \Xi(M)$ which satisfies

$$d^\nabla(\nabla X) = \nabla^2 X = \pi \wedge dp \in A^2(M, TM)\,, \qquad \pi \in \Lambda^1 M\,, \tag{5}$$

and where π is conformal to $X^\flat$, is defined to be an exterior concurrent vector field [14]. In this case, if $\mathcal{R}$ denotes the Ricci tensor field of ∇, one has

$$\mathcal{R}(X, Z) = -2m\lambda^3(\kappa + \eta) \wedge dp\,, \qquad Z \in \Xi(M)$$

3. **Geometrical properties**

In terms of a local field of adapted vectorial frames $\mathcal{O} = \mathrm{vect}\{e_A | A = 0, \cdots 2m\}$ and its associated coframe $\mathcal{O}^* = \mathrm{covect}\{\omega^A | A = 0, \cdots 2m\}$, the soldering form dp can be expressed as

$$dp = \sum_{A=0}^{2m} \omega^A \otimes e_A\,;$$

and we recall that E. Cartan's structure equations can be written as

$$\nabla e_A = \sum_{B=0}^{2m} \theta_A^B \otimes e_B \,, \tag{6}$$

$$d\omega^A = -\sum_{B=0}^{2m} \theta_B^A \wedge \omega^B \,, \tag{7}$$

$$d\theta_B^A = -\sum_{C=0}^{2m} \theta_B^C \wedge \theta_C^A + \Theta_B^A \,. \tag{8}$$

In the above equations θ (respectively Θ) are the local connection forms in the tangent bundle TM (respectively the curvature 2-forms on M).

In terms of the frame fields $\mathcal{O}$ and $\mathcal{O}^*$ with $e_0 = \xi$ and $\omega^0 = \eta$, the structure vector field X and the 2-form Ω can be expressed as

$$X = \sum_{a=1}^{2m} X^a e_a \,, \tag{9}$$

$$\Omega = \sum_{i=1}^{m} \omega^i \wedge \omega^{i^*} \,, \qquad i^* = i + m \,. \tag{10}$$

Taking the Lie differential of Ω and η with respect to X, one calculates

$$\mathcal{L}_X \eta = 0 \,, \tag{11}$$

$$\mathcal{L}_X \Omega = 0 \,. \tag{12}$$

According to [6] the above equations (11) and (12) prove that that η and Ω define a Pfaffian transformation [3].

Next, by (2) one gets that

$$\theta_0^a = \lambda \omega^a \,. \tag{13}$$

Since we also assume that

$$\nabla X = f dp + \lambda \nabla \xi \,, \tag{14}$$

we further also derive that

$$\nabla \xi = \lambda (dp - \eta \otimes \xi) \,. \tag{15}$$

Since the q-th covariant differential $\nabla^q Z$ of a vector field $Z \in \Xi(M)$ is defined inductively, i.e.

$$\nabla^q Z = d^{\nabla}(\nabla^{q-1} Z)\,,$$

this yields

$$\nabla^2 \xi = \lambda^2 \eta \otimes dp\,, \tag{16}$$

$$\nabla^3 \xi = 0\,. \tag{17}$$

Hence, one may say that the 3-covariant Reeb vector field ξ is vanishing.

Next, by (13), one derives that

$$\nabla^2 X = \lambda^3 (df + \eta) \wedge dp = \frac{1+\lambda}{\lambda} \eta \wedge dp\,, \tag{18}$$

and consecutively one gets that

$$\nabla^3 X = 0\,. \tag{19}$$

This shows that both vector fields ξ and X together define a 3-vanishing structure.

Moreover, by reference to [13], it follows from (18) that one may write that

$$\nabla^2 X = -\frac{1}{2m} \mathrm{Ric}(X) - X^{\flat} \wedge dp\,, \tag{20}$$

where Ric is the Ricci tensor.

Reminding that by the definition of the operator ϕ

$$\phi e_i = e_{i^*} \qquad i \in \{1\,, \cdots m\}\,,$$

$$\phi e_{i^*} = -e_i \qquad i^* = i + m\,,$$

one can check that indeed $\phi^2 = -\mathrm{Id}$. Acting with ϕ on the vector field X, one obtains in a first step that

$$\phi X = \sum_{i=1}^{m} X^i e_{i^*} - X^{i^*} e_i \qquad i^* = i + m\,. \tag{21}$$

Calculating the Lie derivative of ϕ w.r.t. ξ, one gets

$$(\mathcal{L}_\xi \phi)X = [\xi\,,\phi X] - \phi[\xi\,,X]\,. \tag{22}$$

Since clearly

$$[\xi\,,\phi X] = 0\,, \tag{23}$$

there follows that

$$(\mathcal{L}_\xi \phi)X = 0\,. \tag{24}$$

Hence, the Jacobi bracket corresponding to the Reeb vector field ξ vanishes.

By reference to the definition of the divergence

$$\text{div } Z = \sum_{A=0}^{2m} \omega^A\,(\nabla_{e_A} Z)$$

one obtains in the case under consideration that

$$\text{div } X = 2m(\lambda + f^2)\,, \tag{25}$$

and

$$\text{div } \phi X = 0\,. \tag{26}$$

Calculating the differential of the dual form $X^\flat$ of X, one gets

$$dX^\flat = \sum_{a=1}^{2m} \left(dX^a + \sum_{b=1}^{2m} X^b \theta_b^a \right) \wedge \omega^a\,. \tag{27}$$

Since

$$dX^a + \sum_{b=1}^{2m} X^b \theta_b^a = \lambda \omega^a\,, \tag{28}$$

one has that

$$dX^\flat = 0\,, \tag{29}$$

which means that the Pfaffian $X^\flat$ is closed. This implies that $X^\flat$ is an eigenfunction of the Laplacian Δ, and one can write that

$$\Delta X^\flat = f||X||^2 X^\flat\,.$$

If we set

$$2l = ||X||^2\,, \tag{30}$$

one also derives by (28) that

$$dl = \lambda X^\flat\,. \tag{31}$$

From (31) it follows that $dX^\flat = 0$ which is indeed in accordance with (29).

Returning to the operator ϕ, one calculates that

$$\nabla(\phi X) = \lambda\phi dp - \sum_{i=1}^{m}\left(\sum_{a=1}^{2m}(X^a\theta_a^i)\otimes e_{i^*} + \sum_{a=1}^{2m}(X^a\theta_a^{i^*})\otimes e_i\right)\,. \tag{32}$$

Hence there follows that

$$[\xi\,, X] \;=\; \rho\xi - \phi C\,, \tag{33}$$

$$[\xi\,, \phi X] \;=\; ((C^0)^2 + C^0(1-\lambda))\xi\,, \tag{34}$$

$$[X\,, \phi X] \;=\; \nabla_\xi\phi C = C^0\xi - C \tag{35}$$

which shows that the triple $\{X\,, \xi\,, \phi X\}$ defines a 3-distribution on M.

It is also interesting to draw the attention on the fact that X possesses the following property. From (14) and (15) one derives that

$$\nabla_X X = fX\,, \tag{36}$$

which means that X is an affine geodesic vector field.

Finally, if we denote by Σ the exterior differential system which defines X, it follows by Cartan's test [1] that the characteristic numbers are

$$r = 3\,, \qquad s_0 = 1\,, \qquad s_1 = 2\,.$$

Since $r = s_0 + s_1$, it follows that Σ is in involution and the existence of X depends on an arbitrary function of 1 argument.

Summarizing, we can organize our results into the following

Theorem 3.1. *Let M be a $2m+1$-dimensional Riemannian manifold and let ∇ be the Levi-Civita connection and ξ be the Reeb vector field and η the Reeb covector field on M. On has the following properties:*

- **(i):** *ξ and X define a 3-vanishing structure;*
- **(ii):** *the Jacobi bracket corresponding to ξ vanishes;*
- **(iii):** *the harmonic operator acting on $X^\flat$ gives*

$$\Delta X^\flat = f||X||^2 X^\flat \,,$$

  which proves that $X^\flat$ is an eigenfunction of Δ, having $f||X||^2$ as eigenvalue;
- **(iv):** *the 2-form Ω and the Reeb covector η define a Pfaffian transformation, i.e.*

$$\begin{aligned} \mathcal{L}_X \Omega &= 0\,, \\ \mathcal{L}_X \eta &= 0\,; \end{aligned}$$

- **(v):** *the Ricci tensor is determined by $\nabla^2 X$;*
- **(vi):** *one has*

$$\nabla_X X = fX\,, \qquad\qquad f = scalar\,,$$

  which shows that X is an affine geodesic;
- **(vii):** *the triple $\{X, \xi, \phi X\}$ is a 3-distribution on M and is in involution in the sense of Cartan.*

4. The structure 2-form Ω

In the present section, we derive some properties of the structure 2-form Ω. First, we recall that one has

$$d\Omega = \lambda \eta \wedge \Omega\,, \qquad\qquad \lambda = \text{constant}\,. \tag{37}$$

By Lie differentiation with respect to X, one gets

$$\mathcal{L}_X \Omega = 0\,. \tag{38}$$

Further, since $i_\xi \Omega = 0$, one calculates that

$$\begin{aligned} \mathcal{L}_\xi \Omega &= \lambda \Omega \,, \\ d\left(\mathcal{L}_\xi \Omega\right) &= \lambda^2 \eta \wedge \Omega \,. \end{aligned}$$

Moreover, by the Lie bracket [,] one also has that

$$i_{[X,\xi]} \Omega = 0 \,. \tag{39}$$

Next, we consider the vector field ϕX. By (32), one calculates that

$$\mathcal{L}_{\phi X} \Omega = -2\lambda \eta \wedge X^\flat \,, \qquad \lambda = \text{constant} \,. \tag{40}$$

Since $X^\flat$ is closed, this yields

$$d\left(\mathcal{L}_{\phi X} \Omega\right) = 0 \,. \tag{41}$$

This shows that ϕX defines a relative conformal transformation [15] [8] of Ω. In addition, one also derives that

$$\mathcal{L}_{[X,\xi]} \Omega = \mathcal{L}_X \mathcal{L}_\xi \Omega - \mathcal{L}_\xi \mathcal{L}_X \Omega = \mathcal{L}_X \mathcal{L}_\xi \Omega$$

and

$$\mathcal{L}_{fX} \Omega = f \mathcal{L}_X \Omega + df \wedge i_X \Omega = df \wedge i_X \Omega$$

Theorem 4.1. *The structure 2-form Ω satisfies the following relations*

(i):

$$d\Omega = \lambda \eta \wedge \Omega$$

(ii):

$$\begin{aligned} \mathcal{L}_\xi \Omega &= \lambda \Omega \\ d\left(\mathcal{L}_\xi \Omega\right) &= \lambda^2 \eta \wedge \Omega \end{aligned}$$

(iii):

$$i_{[X,\xi]} \Omega = 0$$

(iv):

$$d\left(\mathcal{L}_{\phi X} \Omega\right) = 0$$

(v):

$$\mathcal{L}_{[X,\xi]} \Omega = \mathcal{L}_X \mathcal{L}_\xi \Omega$$

(vi):

$$\mathcal{L}_{fX} \Omega = df \wedge i_X \Omega$$

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STUDIA UNIV. "BABEŞ–BOLYAI", MATHEMATICA, Volume **LI**, Number 2, June 2006

REMARKS ON SOME RECURRENCE RELATIONS IN LIFE ANNUITY

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Abstract. The problem of life annuity is one of the main items in the insurance theory. That's why, their computation is very important. The purpose of this paper is to give a possible parallel implementation for such a computation, by using some recurrence relations and the double recursive technique.

1. Introduction

A financial operation connected to the insurance of a person has a random character, both from the insurance institution part and from the insured person.

A fundamental principle for any kind of life insurance is the one of financial equilibrium: the mean value of the gain (of the insured person) has to be equal with the mean value of the gain of the institution which made that life assurance. This value is called **insurance premium**.

2. Life payments

According with [1], [3], [4], a life payment takes place only if the insured person is alive. It can be made both by the insurance institution and by the insured. In what follows, we want to determine the insurance premium that x aged old person has to pay, to get a certain amount of money after n years (if he'll be still alive). Keeping the ideas of [3], we denote by:

$p(x, x+n)$ the probability that a x aged old person to be alive after n years

Received by the editors: 14.11.2005.

2000 *Mathematics Subject Classification.* 91B30, 65Y05, 11C20.

Key words and phrases. Insurance, parallel computation, matrices.

$q(x, x+n) = 1 - p(x, x+n)$ the probability that a x aged person to be dead after n years

l_x the survival function, it means the mean value of the number of people considered who has the chance to live at the age x.

The formula is:

$$p(x, x+n) = \frac{l_{x+n}}{l_n}. \tag{1}$$

Then, denoting by v^n the discounted present value (where v is the discounter present factor from the compound interest), we get

$$M(x) = \frac{l_{x+n}}{l_x} v^n + \left(1 - \frac{l_{x+n}}{l_x}\right) \cdot 0. \tag{2}$$

For computation, in order to use some predefined tables, some notations are also made:

$$D_x = l_x \cdot v^x. \tag{3}$$

Note. D_x are called commutation numbers.

3. Payments in life anuities

In order to determine the amount of money that a x aged old person has to pay, once, to get one monetary unit per year, during all his life, we denote by

a_x the mean value of the posticipated life anuity (payed at the end of every year).

According with [3], and using the commutation numbers, we get the relation:

$$a_x = \frac{D_{x+1}}{D_x} + \frac{D_{x+2}}{D_x} + \cdots + \frac{D_{x+n}}{D_x} + \ldots \tag{4}$$

If we denote by

$$N_x = D_x + D_{x+1} + \cdots + D_\omega,$$

where ω is the age when the last person of the considered generation dies, it results that:

$$a_x = \frac{D_x + \cdots + D_\omega}{D_x} = \frac{N_{x+1}}{D_x}. \tag{5}$$

4. **Recurrence relations in life anuities**

Taking into account (5), and replacing x by $x+1$, we get

$$a_{x+1} = \frac{N_{x+2}}{D_{x+1}}. \tag{6}$$

But

$$N_x = D_x + D_{x+1} + \cdots + D_\omega$$

and

$$N_{x+1} = D_{x+1} + D_{x+2} + \cdots + D_\omega.$$

Subtracting (6) from (5), we get

$$D_x = N_x - N_{x+1}$$

$$D_{x+1} = N_{x+1} - N_{x+2}$$

and replacing in (5), the relation is, successively:

$$a_x = \frac{D_{x+1} + N_{x+2}}{D_x}$$

$$= \frac{D_{x+1}}{D_x}\left(1 + \frac{N_{x+2}}{D_{x+1}}\right) = \frac{D_{x+1}}{D_x}(1 + a_{x+1}). \tag{7}$$

Because

$$\frac{D_{x+1}}{D_x} = \frac{l_{x+1}v^{x+1}}{l_x v^x} = \frac{l_{x+1}}{l_x}v = p(x, x+1)v,$$

replacing in (7), we get the recurrence relation:

$$a_x = p(x, x+1) \cdot v \cdot (1 + a_{x+1}). \tag{8}$$

Or, making some calculation, we can write

$$a_{x+1} = \frac{a_x - p(x, x+1)v}{p(x, x+1)v} = \frac{1}{p(x, x+1)}a_x - 1. \tag{9}$$

5. Parallel computation

The formula (9) can be easily adapted to a parallel computation, using more than one processor. So, the following theorem holds:

Theorem. *The execution time needed to get the amount of money a person has to pay after* n *years is* $O(\log n)$ *using the double recursive technique on a binary tree communication among processors.*

Proof. Relation (9) can be written

$$\begin{bmatrix} a_{x+1} \\ -1 \end{bmatrix} = \begin{bmatrix} \dfrac{1}{p(x,x+1)v} & 1 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} a_x \\ -1 \end{bmatrix} \tag{10}$$

Analogous a_{x+2} depends on a_{x+1}, etc. Finally, in order to compute a_{x+n}, we have to compute only the matrices product which, on a binary tree connectivity, can be made in $O(\log n)$ (see [2]). So the theorem is proved.

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STUDIA UNIV. "BABEŞ–BOLYAI", MATHEMATICA, Volume **LI**, Number 2, June 2006

CRITERIA FOR UNIT GROUPS IN COMMUTATIVE GROUP RINGS

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Abstract. Suppose G is an arbitrary abelian group and F is a field of $charF = p \neq 0$. In the present paper criteria are found the group of all units $UF[G]$ in the group ring $F[G]$ and its subgroup $VF[G]$ of normed units to belong to some central classes of abelian groups under minimal restrictions on F and G. In many instances these necessary and sufficient conditions are in a final form and improve or supersede well-known and documented classical results in this aspect such as due to Karpilovsky (Arch. Math. Basel, 1983). The criteria obtained by us are a natural sequel to our recent results published in Glasgow Math. J. (September, 2001) and are generalizations to those stated and argued by us in Math. Balkanica (June, 2000) as well.

1. Introduction

Throughout the body of the text, let $F[G]$ be the group ring with prime characteristic p of the abelian group G over the field F of prime characteristic p. As usual, $n \in \mathbb{N}$ is a natural number and ζ_n is a primitive n-th root of unity, that is $\zeta_n^n = 1$ while $\zeta_n^k \neq 1 \ \forall \ k < n$. For an abelian group G, written via the multiplicative record as is customary when regarding group rings, G^* is the maximal p-divisible subgroup of G, $G[n] = \{g \in G | g^n = 1\}$ is the n-socle of G and $G_t = \cup_{n<\omega} G[n]$ (in the set-theoretic sense) jointly with G_p are the torsion part and its p-component in G, respectively. For a field F, F^- is the algebraic closure of F, $F(\zeta_n)$ is a cyclotomic extension of F by inserting ζ_n, $(F(\zeta_n) : F)$ is the binomial index of $F(\zeta_n)$ in F, F^* is

Received by the editors: 19.04.2005.

2000 *Mathematics Subject Classification.* Primary 16U60, 16S34; Secondary 20K10, 20K20, 20K21.

Key words and phrases. group rings, units, Σ-cyclic and Σ-countable groups, criteria.

the multiplicative group of F and $F_d = F^{p^\omega}$ is the maximal p-divisible subfield of F. In what follows, K denotes an algebraically closed field with characteristic p.

All other notions and notations from abelian group rings theory not explicitly defined herein will follow essentially our recent work [2]. For instance, $SF[G]$ is the normed Sylow p-subgroup in $F[G]$, $|M|$ is the cardinality of an arbitrary set M, $m_n = |\{g \in G | order(g) = n\}|/(F(\zeta_n) : F)$, etc. Apparently $m_n = 0 \Leftrightarrow G[n] \setminus G[k] = \emptyset \ \forall \ k < n \Leftrightarrow G[n] \setminus \cup_{k<n} G[k] = \emptyset$, and $|m_n| = |G[n] \setminus \cup_{k<n} G[k]| \geq \aleph_0$ for some, hence almost all, $n \in \mathbb{N}$ whenever $|G_t| \geq \aleph_0$ since $(F(\zeta_n) : F) < \aleph_0$ is ever fulfilled.

Concerning various technical terms and the terminology used in the abelian group theory, they are in agreement with the classical books [10-12]. Nevertheless, for the sake of completeness and for the convenience of the readers, we include some more specific details; for example, in all that follows, for any abelian group A, the cardinal number $r_0 A$ denotes the torsion-free rank of A, and $A^1 = \cap_n A^n = \cap_p \cap_m A^{p^m} = \cap_p A^{p^\omega}$ is the first Ulm subgroup of A. For simplicity of the exposition, we use the abbreviations Σ-cyclic and Σ-countable for direct sums of cyclic groups, respectively for direct sums of countable groups, with the exception of the definition of a Σ-group that is an abelian group whose high subgroups are direct sums of cyclics.

The main goal of this manuscript is to establish as applications to the structural theorems in [2] necessary and sufficient conditions for the groups $UF[G]$ and $VF[G]$ of all invertible elements (often called units) and normed invertible elements (often called normalised units), respectively, to possess some important properties and to compute explicitly their determinate numerical invariants. The given criteria and computations expand in some way classical facts in this direction proved in ([3,5]; see [6] too), [13] and [19-22; 23-28].

Conforming with the isomorphic descriptions of $UF[G]$ and $VF[G]$, given in [2], we have obtained in [4] certain additional algebraic properties for these groups, which properties are of some importance. Moreover, we indicate also that, a criterion for $VF[G]$ to be a direct sum of p-mixed countable abelian groups was established in [9,7], provided F is perfect.

2. Main results

Some of the main attainments presented here were previously announced in [1].

And so, we start with

Theorem 1. *$VF[G]$ is Σ-cyclic if and only if G is Σ-cyclic and at most one of the following conditions hold:*

1) $G_t = G_p$

or

2) $G_t \neq G_p$, $F \neq F^-$ and $F(\zeta_n)^*$ is Σ-cyclic for each $n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ which is an order of an element of G_t/G_p.

Proof. Certainly, $VF[G]$ being Σ-cyclic implies by the classical theorem of L. Kulikov ([10], p.110, Theorem 18.1) that $G \subseteq VF[G]$, being a subgroup, is also Σ-cyclic.

First of all, suppose F is algebraically closed and $VF[G]$ is Σ-cyclic. Hence $VF[M_t] \subseteq VF[G]$ is also Σ-cyclic, where M is a group so that $G = G_p \times M$. But besides $VF[M_t]$ is divisible (see [2], formula (8)), and therefore $VF[M_t] = 1$, i.e. $M_t = 1$. Thus $G_t = G_p$.

Conversely, take G to be Σ-cyclic and $G_t = G_p$. Hence G splits and so $G = G_p \times M$. Owing to Lemma 2.2 of [2], $VF[G] = VF[M] \times SG[G] \cong G/G_t \times SF[G]$ by using the well-known Higman's result on trivial units documented in [14]. Finally $VF[G]$ is Σ-cyclic because G/G_t is free and because Theorem 2.1 from [2] ensures that $SF[G]$ is Σ-cyclic.

Let now F be not algebraically closed, i.e. $F \neq F^-$. Suppose $VF[G]$ is Σ-cyclic. Hence $1 \neq VF[M_t]$ is as well, where M is such a group that $G = G_p \times M$ and $M_t \neq 1$, whence $G_t \neq G_p$. Consequently by formulas (3) and (4) of [2], $F(\zeta_n)^*$ is Σ-cyclic. The reverse inclusion follows applying formulas (17), (18) and Theorem 2.1(ii) in [2]. This ends the proof.

Theorem 2. *$UF[G]$ is Σ-cyclic if and only if G is Σ-cyclic and either*

1') $G_t = G_p$ and F^* is Σ-cyclic

or

2') $G_t \neq G_p$ and $F(\zeta_n)^*$ is Σ-cyclic for every n that is an order of an element of G_t/G_p.

Proof. It is analogous to the last theorem, since F is not algebraically closed provided $UF[G]$ is Σ-cyclic. Indeed, if $F = F^0$ then F^* is divisible Σ-cyclic, i.e. $F^* = 1$, and thereby $F = \{0, 1\}$, a contradiction with the infinite cardinality of F. Finally, we apply that $UF[G] = VF[G] \times F^*$ is Σ-cyclic only when so are $VF[G]$ and F^*. The proof is completed.

Example. The condition on n that $m_n \neq 0$ (i.e. that there is an element in G of order n) stated in the previous two theorems is necessary. In fact, inductively, let F_n be the finite field of order 2^{3^n}, and put $F = \cup_{n<\omega} F_n$; $F_n \subseteq F_{n+1}$ so F is a countable field of characteristic $p = 2$. Let G be the direct sum of $\aleph_0$ copies of a cyclic group of order 7. Thus $G_p = 1$ and $G = G_t \neq 1$ with $G^7 = 1$. In order to obtain that $VF[G]$ is Σ-cyclic, according to Theorems 1 and 2, $F(\zeta_n)^*$ should be Σ-cyclic only for $n = 7$ but not for every $n \in \mathbb{N}_0$. This is so since $F(\zeta_3)^*$ has 3-component isomorphic to $Z(3^\infty)$. As for $F(\zeta_3)$, we observe that (ζ_3) is the field of 4-elements, and in the formula $4^{3^{n+1}} - 1 = (4^{3^n} - 1)(4^{2\cdot 3^n} + 4^{3^n} + 1)$ the second factor is always divisible by 3.

To justify the example, since F contains a primitive 7-th root of unity, namely $\zeta_7 \in F_1$ since F_1^* is cyclic of power 7, whence $F(\zeta_7) = F$, we detect that $VF[G]$ will be Σ-cyclic if and only if F^* is Σ-cyclic. This follows from the formula $2^{3^{n+1}} - 1 = (2^{3^n} - 1)(2^{2\cdot 3^n} + 2^{3^n} + 1)$ because any prime dividing the first factor cannot divide the second one (note that the prime cannot be 3).

We continue in this way with

Theorem 3. *$UF[G]$ is bounded if and only if G is bounded and F^* is bounded.*

Proof. It is long-known that $SF[G]$ is bounded if and only if G_p is bounded (see for example [2]). We note that $m_k = 0$ precisely when G is bounded with exponent $\exp(G) < k$. That the statements G and F^* are both bounded, is equivalent to G is bounded and, either $F(\zeta_n)^*$ is bounded for each n dividing $|G_t/G_p| < \aleph_0$, or $\bigcup_{n=0}^{\infty} \times m_n F(\zeta_n)^*$ is bounded when $|G_t/G_p| \geq \aleph_0$, follows now easily, since F^* being bounded implies that F is a finite algebraic extension of a simple (hence a finite) field, whence it is finite as well. Appropriate arguments for this are that $F(\mathbb{Z}_n)^*$ is bounded $\Leftrightarrow$ F^* is bounded $\forall\ n < \omega$ and that $\bigcup_{n=0}^{\infty} \times m_n F(\zeta_n)^*$ reduces to $\bigcup_{n \leq \exp(G)} \times m_n F(\zeta_n)^*$. Therefore we wish only to apply Theorem 2.2 point (e) of [2]. The proof is complete.

Theorem 4. *$UF[G]$ is finitely generated if and only if $G_p \neq 1$, F and G are finite; or $G_p = 1$, G and F are finitely generated.*

Proof. First assume $G_p \neq 1$. Let $UF[G]$ be finitely generated. Then it is elementary that $1 \neq SF[G]$ is finite. But if $|F| \geq \aleph_0$ or $|G| \geq \aleph_0$, we derive as in [14] that $|SF[G]| = \max(|F|, |G|) \geq \aleph_0$, that is false. Thus obviously F and G are both finite. Conversely, if F and G are finite, then $UF[G]$ is finite, hence finitely generated.

Now let $G_p = 1$. In that case the proof goes by a standard application of Theorem 2.2 in [2] in view of the fact that a subgroup of a finitely generated group has the same property (cf. [10]). The equivalence of the second part half, namely that $G_p = 1$, G and F are finitely generated $\Leftrightarrow$ $G_p = 1$ and G along with $F(\zeta_n)^*$ are finitely generated for every n dividing $|G_t| < \aleph_0$, holds at once since F^* being finitely generated forces that so do both $F(\zeta_n)^*$ $\forall\ n$ and $F = F^* \cup \{0\}$.

The proof is finished in all generality.

Remark. A criterion for $UR[G]$ to be finitely generated was also founded by Karpilovsky (see [13, Theorem 3]) when R is a finitely generated commutative unitary ring of arbitrary characteristic and G is an arbitrary abelian group. However, in our situation, F need not be finitely generated a priory, as this fact follows easily from the same property for $UF[G]$.

Generally, does it follow that $UR[G]$ being finitely generated forces the same property for R? If yes, the problem of finding the criterion for $UR[G]$ to be finitely generated will be completely resolved. However, this question is quite difficult and its solution seems to be in the distant future.

In the next statement, we will use the simple but useful fact that G being Σ-countable yields that both G_t and G_p are Σ-countable groups as well.

Proposition 5. *Let G be splitting and F perfect. If G and $F(\zeta_n, \mu_q)^*$ are Σ-countable groups then the group $UF[G]$ is Σ-countable when $|G_t/G_p| \geq \aleph_0$ and if G and $F(\zeta_n)^*$ are Σ-countable groups then the group $UF[G]$ is Σ-countable when $|G_t/G_p| < \aleph_0$.*

Proof. This follows by a standard application of (19), (20) and of Claim 2.1, all from [2]. The proposition is verified.

Remark. By the same statements, as in the situation for Σ-cyclic groups, criteria can be established for $VF[G]$ to be bounded, finitely generated and Σ-countable. Nevertheless, we omit the reproduction of their explicit form.

The following two group-theoretic observations are well-known and have routine proofs - they shall be used below without further reference: an isotype subgroup of a direct product of a divisible and a bounded group inherits this group property; a pure subgroup of a divisible group is divisible. Moreover, it is not difficult to check that an outer direct sum of equal algebraically compact groups is also an algebraically compact group.

After this, we need one more technicality, which is crucial.

Lemma 6. *Suppose $G_t = G_p$. Then G is pure in $VF[G]$.*

Proof. We shall use the definition for the property "purity" by differing two basic cases:

Case 1. For each natural n so that $p \nmid n$ we write $n = p_1^{t_1} \dots p_s^{t_s}$ as the canonical form of n, where $p_1, \dots, p_s \neq p$ are distinct primes, $s \in \mathbb{N}$, $t_1, \dots, t_s \in \mathbb{N}_0$. Since $VF[G] = GSF[G]$ (see e.g. [21, 22] or [8]), by the usage of the modular law we conclude that $G \cap V^n F[G] = G \cap (GSF[G])^n = G \cap (G^n SF[G]) = G^n(G \cap SF[G]) = G^n G_p = G^n$.

Case 2. p/n, whence we write $n = p^{k_1} q_2^{k_2} \dots q_m^{k_m}$ to be the canonical form of n, where $q_2, \dots, q_m \neq p$ are different primes, $m \in \mathbb{N}$, $k_1, \dots, k_m \in \mathbb{N}_0$. As above we deduce $G \cap V^n F[G] = G \cap (GSF[G])^n = G \cap (G^n SF^{p^{k_1}}[G^{p^{k_1}}]) = G^n(G \cap SF^{p^{k_1}}[G^{p^{k_1}}]) = G^n G_p^{p^{k_1}} = G^n$. The proof is over.

Remark. When G is not p-mixed, that is $G_t \neq G_p$, G need not be a pure subgroup of $VF[G]$ in general (see the Remark after Corollary 9). Even more G_t is not pure in $V_tF[G] = SF[G]VF[G_t]$ assuming extra that $G_p \neq 1$. Another argumentation is when $G_p = 1$. Henceforth, in this situation, $V_tF[G] = VF[G_t]$ and thus $V_tK[G]$, by point (a') proved below, must be always divisible whereas G_t may not be so.

Now we are ready to attack the following.

Theorem 7. *Let $1 \neq G_t$ be p-torsion. Then*

(a) $VF[G]$ is divisible if and only if G is divisible and F is perfect.

(b) $VF[G]$ is a direct sum of a divisible group and of a bounded group if and only if G is a direct sum of a divisible group and of a bounded group and, either G_t is not reduced, G/G_t is p-divisible and F is perfect, or G_t is reduced.

(c) $VF[G]$ is algebraically compact if and only if G is algebraically compact and, either G_t is unbounded algebraically compact, G/G_t is p-divisible and FF is perfect, or G_t is bounded.

(d) $VF[G]$ is coperiodical if and only if G is coperiodical and, either G_t is unbounded coperiodical, G/G_t is p-divisible and F is perfect, or G_t is bounded.

Proof. (a) Choose $VF[G]$ to be divisible. Hence $G_t = G_p$ is divisible as it is a pure subgroup. Thus $G = G_t \times M$ and by formula (6) of [2], $VF[G] = VF[M] \times SF[G] \cong G/G_t \times SF[G]$ using again the classical Higman's result on the trivial units (cf. [14]). Further G/G_t is divisible, i.e. so is G, and moreover $SF[G]$ is also divisible. So, $S^pF[G] = SF^p[Gp] = SF[G]$, equivalently $F = F^p$, and F is perfect as asserted.

Conversely, assume G divisible and F perfect. Hence G_p is divisible as it is pure in G, and besides G/G_t is also divisible as it is a factor-group. Thus $G \cong G_t \times G/G_t$ and similarly to the above, $VF[G] \cong G/G_t \times SF[G]$. Finally, $SF[G]$ and $VF[G]$ are both divisible groups.

(b) Suppose $VF[G]$ is a direct sum of a divisible group and of a bounded group. Hence G_p as an isotype subgroup is one also. Therefore $G = G_p \times M$ (see [10]) and as above $VF[G] \cong G/G_t \times SF[G]$. Thus G/G_t is a direct sum of a divisible and a bounded group, i.e. the same is G. On the other hand $SF[G]$ belongs to this group class, i.e. it is algebraically compact (cf. [10]). But $SF[G] \cong SF[M][Gp]$ (see [0]) because $M_p = 1$, and thus $SF[M] = 1$. That is why, following [0], if G_p is not reduced, then $SF[G]$ algebraically compact yields FM is perfect since FM is without nilpotent elements (notice that F has a trivial nil-radical and M has no p-elements). Hence, F is perfect and G/G_p is p-divisible.

Oppositely, if the conditions from the text hold, then G_p is algebraically compact as an isotype subgroup in G. So, $G = G_p \times M$ (cf. [10]) and by equality (6) from [2], $VF[G] \cong G/G_t \times SF[G] \cong G/G_t \theta SF[M][G_p]$ (see [0]). We only need to apply [0] and the result follows immediately.

(c) If G is p-primary, the point follows directly from [0]. So, we may presume that $G \neq G_p$. Referring to Lemma 6 and ([10], p.190, Exercise 3), $VF[G]/G \cong SF[G]/G_p$ is algebraically compact provided that so is $VF[G]$. Therefore $SF[G]/G_p$ is a direct sum of a divisible and a bounded group (cf. [10]). But $(SF[G]/G_p)_d = (SF_d[G^*])G_p/G_p$ via [8], hence the quotient-group $SF[G]/SF_d[G^*]G_p$ is bounded, i.e. there is $k \in \mathbb{N}$ such that $SF^{p^k}[G^{p^k}] \subseteq SF_d[G^*]G_p$. The last reduces to $F^{p^k} = F_d$ and $G^{p^k} = G^*$ when G_p is not reduced. Indeed, consider the element $1 + rg(1 - g_p)$ where $r \in F^{p^k}$, $g \in G^{p^k} \setminus G_p^{p^k}$ and $g_p \in G_p^{p^k} \setminus \{1\}$. Thus $1 + rg(1 - g_p) = (f_1 a_1 + \cdots + f_t a_t)c_p$, where $f_i \in F_d$, $a_i \in G^*$, $c_p \in G_p$; $1 \leq i \leq t \in \mathbb{N}$. Henceforth, the canonical forms imply that $r \in F_d$ and $g \in G^*$, $g_p \in (G^*)_p$. Furthermore $(G^{p^k})_p = (G^*)_p$, i.e. $G_p^{p^k}$ is divisible, which is equivalent to G_p being algebraically compact by [10]. But $G^{p^k}/G_p^{p^k} \cong (G/G_p)^{p^k}$ is p-divisible, i.e. so is G/G_p. Finally, it is a plain exercise to verify that G^{p^k} is p-divisible, i.e. $G^{p^k} = G^*$. On the other hand, as we have already seen, $F^{p^k} = F^{p^{k+1}}$ whence F is perfect. Next, if G_p is reduced, we have $(G^*)_p = 1$ hence $SF_d[G^*] = 1$ and so the foregoing inclusion takes the form $SF^{p^k}[G^{p^k}] \subseteq G_p$ or equivalently $G_p^{p^{k+1}} = 1$. So, in both cases, G_p, being a pure subgroup, is a direct factor of $SF[G]$, hence G is a direct factor of $VF[G] = GSF[G]$. Then G is algebraically compact exploiting [10]. This verifies the first half.

For the converse implication, we observe that G_p is a direct factor of G, i.e. in other words G is p-splitting, whence G/G_t is algebraically compact. Thus by what we have shown above, $VF[G] \cong G/G_t \times SF[G] \cong G/G_t \times SF[G/G_p][G_p]$. By making use of [0] and [10], the point is exhausted.

(d) Since $VF[G]$ is coperiodical, we refer to [10] to infer that $VF[G]/G \cong SF[G]/G_p$ is coperiodical too. Therefore, again by using of [10], the proof goes on the same arguments and conclusions as in (c).

This proves the theorem.

After this, we proceed by proving the following.

Theorem 8.

(a') $VK[G]$ is divisible if and only if G/G_t and G_p are divisible.

(b') $VK[G]$ is a direct sum of a divisible and a bounded group if and only if G/G_t and G_p are a direct sum of a divisible and a bounded group, and G/G_p is p-divisible provided G_p is not reduced.

(c') $VK[G]$ is reduced algebraically compact if and only if G/G_t and G_p are reduced algebraically compact.

(d') $VK[G]$ is reduced coperiodical if and only if G/G_t and G_p are reduced coperiodical.

(e') Let G be p-splitting. $VK[G]$ is Σ-countable if and only if G/G_t and G_p are Σ-countable.

Proof. (a') $VK[G]$ divisible insures that G_p is divisible as its pure subgroup, whence G is p-splitting. Further the proof follows immediately from the description of $VK[G]$ in ([2], section 2, formulas (11)-(12)) and from the group-theoretic facts given in [10]. The reverse implication is similar.

(b') We firstly deal with the necessity. Certainly, the fact that G_p is isotype in $VK[G]$ yields that G_p is a direct sum of a divisible and a bounded group, so G is p-splitting. Further, the proof follows directly by virtue of formulae (11)-(12) in [2] and utilizing the criterion in [0] for $SK[G]$ to be algebraically compact combined with some group-theoretic facts obtained in [10]. The sufficiency is analogical.

(c') Foremost, assume that $VK[G]$ is reduced algebraically compact. Evidently G_p is reduced being a subgroup. Assume also that B is an unbounded basic subgroup of G_p. Therefore we write $B = \bigcup_{n=1}^{\infty} B_n$, where all subgroups

B_n are homogeneous of order p^n. We now construct the infinite sequence $g_n = \prod_{i=1}^{n}(1+b_i^{p^{i-1}}-b_{i+1}p^i)$, where $b_i \in B_i$; $n \in \mathbb{N}$. Clearly $g_n^p = 1$, and for each $k \in \mathbb{N}$ we have $g_{n+l}g_n^{-1} = \prod_{i=n+1}^{n+l}(1+b_i^{p^{i-1}}-b_{i+1}p^i) \in S^{p^k}K[G] \subseteq S^kK[G] \subseteq V^kK[G]$ for every $n \geq k$ and arbitrary positive integer L. We note that the first inclusion holds since if $p|/k$ we have $S^kK[G] = SK[G]$, while if p/k we have $k = p^sm$ for some $s, m \in \mathbb{N}$ with $(m,p) = 1$ and so $S^kK[G] = S^{p^s}K[G] \supseteq S^{p^k}K[G]$ by observing that $s < k$. That is why (g_n) is a Cauchy sequence in $VK[G]$ and consequently we can apply the well-known Kaplansky theorem ([10], p.191, Theorem 39.1) which guarantees that (g_n) must be convergent to an element of $VK[G]$ in its Z-adic topology. And so, let $g = \sum_{j=1}^{t}\alpha_jg_j \in VK[G]$ be the boundary of (g_n). Furthermore, for all $k \geq 1$ and $n \geq k$, we derive

$$\sum_{j=1}^{t}\alpha_jg_j = \left[\prod_{i=1}^{n}(1+b_i^{p^{i-1}}-b_{i+1}^{p^i})\right](r_{1n}(k)a_{1n}(k)p^k + \cdots + r_{s_nn}(k)a_{s_nn}(k)p^k),$$

where $r_{1n}(k), \ldots, r_{s_nn}(k) \in K$; $a_{1n}(k), \ldots, a_{s_nn}(k) \in G$; $s_n \in \mathbb{N}$. It is easily seen that the left hand-side of the last equality is constant about n, while the right hand-side depends on n and contains a number of elements in the canonical form that is $\geq n > t$. In fact, it is easy to see that there is $k \in \mathbb{N}$ so that all products of b_ip^{i-1}'s for different various indices i running $\mathbb{N}$ are not in G_p^k. If the reverse holds, these products belong to $B \cap G^{p^\omega} = 1$, which is demonstrably false because in that case $b_i^{p^{i-1}} = 1 \; \forall \; 1 \leq i \leq n$ whereas $order(b_i) = p^i$. Moreover, because of the direct decomposition of B, these products of $b_i^{p^{i-1}}$'s are independent and their number depend on n. By taking $n > t$, the claim really sustained. Finally, we deduce that (g_n) is not a convergent, i.e. it is a divergent, sequence in $VK[G]$ when B is unbounded. Thereby B is bounded, i.e. G_p must be so by referring to ([10, 12]). Henceforth, appealing to [10], G is p-splitting and the proof follows by means of formulas (11)-(12) from [2] and the simple observations stated before Lemma 6. The treatment of the converse question is similar.

(d') $VK[G]$ being coperiodical implies that $VK[G]/V^1K[G]$ is algebraically compact (see e.g. [10]), where $V^1K[G]$ is the first Ulm subgroup of $VK[G]$. Now we consider the sequence $(h_n) = (g_nV^1K[G])$ where (g_n) is constructed as in the previous point. Clearly $g_n \notin V^1K[G]$, otherwise $g_n \in V^{p^\omega}K[G] = VK[G^{p^\omega}]$ and so $b_i^{p^{i-1}} \in B \cap G^{p^\omega} = B^{p^\omega} = 1$, a contradiction. Besides, it is a routine technical work to check that (h_n) is a Cauchy sequence since (g_n) is. Further the proof goes by the same arguments as in the preceding statement. The sufficiency is analogous.

(e') Since a direct factor of a Σ-countable group is Σ-countable (see [10], a theorem of Kaplansky - C. Walker) and any divisible group is Σ-countable, then owing to the isomorphism (11) from [2], it is enough to show only that $SK[G]$ is Σ-countable if and only if G_p is Σ-countable. In fact, this is precisely Claim 2.1 of [2] and thus we are done. This deduces the theorem.

Corollary 9. *Let G be divisible. Then G is a direct factor of $VK[G]$ with divisible complementary factor. Thus $VK[G]$ is divisible.*

Remark. We can restate point (a') like this: $VK[G]$ is divisible if and only if G/G_t is divisible and G is p-divisible. From this, it follows that if $VK[G]$ is divisible, G need not be so. Consequently, a principal question is whether or not the divisibility of $VF[G]$ does imply that G is splitting. If yes, one can employ formulas (16) (and eventually (19) and (20)) from [2] to find a criterion for $VF[G]$ to be divisible.

If X is an arbitrary abelian group, as emphasized in the introduction, we shall say that $r_0(X)$ is the torsion-free rank of X. Mollov [24,25] has calculated the torsion-free rank of $UE[G]$ for semisimple EG whose G is torsion (see also [14]). Later on, Mollov and Nachev [26,27] have computed the torsion-free rank $r_0UE^t[G]$ of the group of units in a commutative semisimple twisted group algebra $E^t[G]$ in terms of E and of G, when G is torsion or torsion-free. Specifically, they calculated in a more general aspect this rank for semisimple abelian $E^t[G]$ when G is arbitrary, but in terms of E, G and $E^t[G_t]$. So, the result is incomplete, since a characterization of $E^t[G_t]$ that depends only of E, G_t and the system of factors of $E^t[G]$ was not given here.

Nevertheless, contrasting with their result, we compute $r_0UE[G]$ for a modular or a semisimple group ring $E[G]$ over a splitting or a torsion group G as well as over a p-splitting group G but over an algebraically closed field E, both in the two cases only in terms of E and G. That is, of course, more precise.

Before doing this, we require one more result.

Theorem 10. *The group $VF[G]$ is torsion if and only if G is p-torsion, or G and F^* are torsion provided $G \neq G_p$.*

Proof. If $G = G_p$, it is a simple matter to check that $VF[G]$ is a p-group. That is why we deal only with $G \neq G_p$. First assume $VF[G]$ is torsion. Hence G is torsion and so $G = G_p \times M$. Thus $VF[M]$ is torsion, and consequently by [25] or [14] we conclude that F is an algebraic extension of a finite field, i.e. F^* is torsion.

To treat the converse, write $G = G_p \times M$. Therefore, in accordance with Lemma 2.2 of [2], we obtain $VF[G] = VF[M] \times SF[G]$. But M and F^* are both torsion. By virtue of ([25], [14]), $VF[M]$ is torsion, i.e. so does $VF[G]$. This finishes the proof.

Our aims here are the following.

Theorem 11. *Let G be torsion. Then $r_0VF[G] = 0$ if F is an algebraic extension of a finite field or if $G = G_p$, and $r_0VF[G] = \max(|F|, |G/G_p|)$ otherwise.*

Proof. First take G to be p-primary or F to be an algebraic extension of a finite field. Consequently Theorem 10 assures that $VF[G]$ is torsion, and so $r_0VF[G] = 0$. In the remaining cases we write $G \cong G_p \times G/G_p$. Therefore formula (6) in [2] implies $VF[G] \cong VF[G/G_p] \times SF[G]$. Hence, $r_0VF[G] = r_0VF[G/G_p]$ (see [10]), whence we use [24,25] to conclude that $r_0VF[G] = \max(|F|, |G/G_p|)$, as stated. The theorem is proved.

Theorem 12. *Suppose G splits and E is a field. Then if $char(E) = 0$,*

$$r_0UE[G] = \max(|E|, |G_t|, r_0(G)); \tag{1}$$

and if $char(E) = p \neq 0$;

$$r_0UE[G] = \begin{cases} |G_t/G_p| r_0(G), & |G_t/G_p| \geq \aleph_0 \\ \sum_{d/|G_t/G_p|} m_d r_0(G), & |G_t/G_p| < \aleph_0 \end{cases} \tag{2}$$

provided E is an algebraic extension of a finite field, or

$$r_0UE[G] = \max(|E|, |G_t/G_p|, r_0(G)) \tag{3}$$

otherwise.

Proof. Given $char(E) = 0$. In virtue of the isomorphism (15) from [2] together with [10], we have $r_0UE[G] = r_0UE[G_t]+\Sigma_\alpha r_0(G/G_t)$, where α is computed as in [2]. But $r_0(G/G_t) = r_0(G)$ and thus [24,25] lead us to $r_0UE[G_t] = max(|E|, |G_t|)$, because E is infinite. Consequently by virtue of ([15], p.206, Theorem 7), $r_0UE[G] = max(|E|, G_t|)+(\alpha r_0(G) = max(|E|, |G_t|)+max(|Gt|, r_0(G)) = max(|E|, |G_t|, r_0(G))$.

For $char(E) = p > 0$ and E an algebraic extension of a simple (i.e. of a finite) field we derive via [14] that $UE[G_t/G_p]$ is torsion. In view of formula (16) in [2] and of ([15], p.206, Theorem 7) combined with [10], we deduce that $r_0UE[G] = \sum(|G_t/G_p|r_0(G) = |G_t/G_p|r_0(G)$ for the infinite situation or $r_0UE[G] = \sum_\beta r_0(G) = \beta r_0(G)$ where $\beta = \sum d/|G_t/G_p|m_d$ for the finite one.

In the remaining case, the same formula (16) plus [10], [15] and Theorem 11 are guarantors that $r_0UE[G] = \max(|E|, |G_t/G_p|) + \sum_{|G_t/G_p|} r_0(G) = \max(|E|, |G_t/G_p|) + |G_t/G_p|r_0(G) = max(|E|, |G_t/G_p|, r_0(G))$, as desired. So, the theorem is true.

Theorem 13. *Let E be a field. Then if $char(E) = 0$,*

$$r_0UE^-[G] = \max(|E^-|, |G_t|, r_0(G)); \tag{4}$$

and if $char(E) = p > 0$ and G is p-splitting,

$$r_0UE^-[G] = |G_t/G_p|r_0(G) \tag{5}$$

provided E is an algebraic extension of a finite field, or

$$r_0UE^-[G] = max(|E^-|, |G_t/G_p|, r_0(G)) \tag{6}$$

otherwise.

Proof. The result follows employing formulas (8)-(12) from [2] along with [10] and [15]. The conclusions are similar to these of the foregoing theorem. The proof is finished.

Now, we shall begin with other types of results by arguing the following (a part of the results presented here generalize those obtained by Mollov in [24] and [25]; see [14] and [19] as well).

Proposition 14. *Suppose G is a direct sum of finite cyclic groups. Then $VF[G]$ is nontrivial free modulo torsion if and only if $F(\zeta_n)^*$ is free modulo torsion for each n which is an order of an element of G.*

Proof. Clearly G is torsion and $G = G_p \times M$ for some group M. Referring to ratio (6) from [2], we may write $VF[G] = VF[M] \times SF[G]$. Thus $VF[G]/V_tF[G] \cong VF[M]/V_tF[M]$ and the result follows by application of [25] or [14]. The statement is shown.

We can extend the last affirmation to the next claim.

Proposition 15. *Let G be Σ-cyclic. Then $UF[G]$ is nontrivial free modulo torsion if and only if $F(\zeta_n)^*$ is free modulo torsion for every n which is an order of one element of G.*

Proof. It is not difficult to see by application of formulae (17-18) from [2] that $UF[G]/U_tF[G] \cong (\times_\delta G/G_t) \times (\Pi_n \times_{m_n} F(\zeta n)^*/F(\zeta n)^*_t)$, where δ is finite or infinite defined in the same manner as in [2]. This proves the result.

Proposition 16. *Suppose G p-splits. Then $UK[G]$ is nontrivial free modulo torsion if and only if G is free modulo torsion and K is an algebraic extension of a finite field.*

Proof. The isomorphism (11) of [2] obviously yields that $UK[G]/U_tK[G] \cong (\times_{|G_t/G_p|} G/G_t) \times (\times_{|G_t/G_p|} K^*/K^*_t)$. Thus $UK[G]$ is free modulo torsion precisely when G/G_t is free and $K^*/K^*_t = 1$ since the latter quotient is divisible. Finally, K^* is torsion, as desired. The affirmation is established.

Proposition 17. *Let G be torsion. If $F(\zeta_n)^*$ is divisible modulo torsion for each n which is an order of an element of G, then $VF[G]$ is divisible modulo torsion.*

Proof. Write $G = G_p \times M$. As we have seen, $VF[G] = VF[M] \times SF[G]$. Hence $VF[G]/V_tF[G] \cong VF[M]/V_tF[M]$. Finally either [25] or [14] gives the claim, thus completing the proof.

Proposition 18. *Let G be Σ-cyclic. Then $UF[G]$ is nontrivial divisible modulo torsion if and only if G is torsion and $F(\zeta_n)^*$ is divisible modulo torsion for every n which is an order of an element of G.*

Proof. By what we have already shown above, $UF[G]/U_tF[G]$ is divisible only when G/G_t is divisible free and $F(\zeta n)^*/F(\zeta_n)_t^*$ is divisible. Finally $G = G_t$ and $F(\zeta_n)^*$ is divisible modulo torsion, as promised, thus finishing the proof.

Proposition 19. *Let G be p-splitting. The group $UK[G]$ is divisible modulo torsion if and only if the group G is divisible modulo torsion.*

Proof. By what we have just given above, $UK[G]/U_tK[G]$ is divisible only if the same is valid for G/G_t, because K^* is divisible whence divisible modulo torsion. Thus G is really divisible modulo torsion, as expected. The proof is complete.

The following is our crucial tool for the further investigation (see, for instance, cf. [24] and [25]).

Definition 20. We recall that the field F belongs to the class $\mathcal{P}$ if $F(\zeta_n)^*$ splits for every primitive n-th root of unity ζ_n in F^-. Denote by $\mathcal{PI}$ and $\mathcal{PR}$ the subclasses of $\mathcal{P}$ which contain fields F with the following two corresponding properties: the torsion-free factor of $F(\zeta_n)^*$, that is, the quotient $F(\zeta_n)^*/F(\zeta_n)_t^*$, is free or divisible for each ζ_n.

An example for a field that belongs to $\mathcal{PI}$ is the following (e. g. see May [16] or [17,18]): If L is a field such that the multiplicative group E^* of every finite extension E of L is free modulo torsion, then all extensions F of L generated by the algebraic elements of a bounded degree over L belong to the class $\mathcal{PI}$. Besides, if K is algebraically closed but K is not an absolute algebraic field ($K^* \neq K_t^*$), then $K \in \mathcal{PR}$ ([11], p.298, Theorem 77.1 or [12]).

Proposition 21. *Suppose G is a torsion direct sum of cyclic groups such that $G \neq G_p$ and E is a neat transcedental extension of the field F. Then if $F \in \mathcal{P}$, the group $VE[G]$ splits; and if $F \in \mathcal{PI}$, the group $VE[G]$ is splitting of torsion-free rank* $\max(|E|, |G/G_p|)$.

Proof. Write $G = G_p \times M$, therefore $VE[G] = VE[M] \times SE[G]$. Hence, $VE[G]$ splits if and only if the same holds for $VE[M]$. So, we need only subsequently apply ([25], [14]) and Theorem 11. The proof is completed.

Proposition 22. *Suppose G is Σ-cyclic. Then if $F(\zeta_n)^*$ splits for each n which is an order of an element of G (in particular if $F \in \mathcal{P}$), the group $UF[G]$ is splitting.*

Proof. It follows obviously from dependencies (17) and (18) of [2] that $U_tF[G]$ is a direct factor of $UF[G]$, as claimed. This concludes the proof.

Proposition 23. *If G splits and, either F is an algebraic extension of a finite field (i.e. it is an absolute algebraic field), or G_t/G_p is Σ-cyclic and $F \in \mathcal{P}$, then $UF[G]$ splits.*

Proof. Consulting with formula (16) of [2], we argue $UF[G] \cong UF[G_t/G_p] \times (\times_\delta G/G_t) \times SF[G]$, where δ is finite when $|G_t/G_p|$ is finite or is infinite when $|G_t/G_p|$ is infinite. If now F is an absolute algebraic field, then $UF[G_t/G_p]$ is torsion. Thus, in this case, $UF[G]$ splits. In the remaining one, when G_t/G_p is Σ-cyclic and $F \in \mathcal{P}$, according to Proposition 22 we conclude that $UF[G_t/G_p]$ splits, therefore $UF[G]$ splits as well, as wanted. This is the end of the proof.

Corollary 24. *If G is Σ-cyclic and $F \in \mathcal{PI}$, then $UF[G]$ is splitting.*

We close the study with the following.

Proposition 25. *Assume G is p-splitting. Then $UK[G]$ splits.*

Proof. The group K^* is divisible, hence splitting. Therefore, the statement holds by application of formula (11) from [2]. The proof is deduced.

Remarks. The conditions $G \neq G_p$ in Theorems 10, 11 plus the restrictions $m_n \neq 0$ in Theorems 1 and 2 were omitted from [1] involuntarily. Their formulations in [1] are in an equivalent record.

Moreover, the condition $G_q^{p^\omega} \cong 1$ in Theorem 2.2 (f) on p.370 of [2] must be written and read as $G_q^{q^\omega} \neq 1$. The sentence on p.371-line 2(+) of [2], namely: "... E is an algebraic extension of finite field ..." must be assumed as "... E^- is an algebraic extension of a finite field ...", and on line 12(-) of the same page the reference "[36]" must be "[37]", although both the corrections are clear from the context.

Besides, the equality $A = \bigcup_{\alpha<\lambda} B_\alpha$ on p. 223 of [3] should be replaced by $A = \bigcup_{\alpha<\lambda} G_\alpha$. In that aspect, the letter $\bigcup_{\alpha<\lambda} \bigcup_{\mu<\alpha} G_\mu$ on p. 224 of [3] must be replaced by $\prod_{\alpha<\lambda} \bigcup_{\mu<\alpha} G_\mu$.

Also the identity $G = \bigcup_{\beta<\tau} C_\beta$ from [7, p. 258] would be interpreted as $G = \bigcup_{\beta<\tau} G_\beta$.

We terminate this article with problems of some interest and importance, which immediately arise, namely:

3. Open questions and conjectures

What are the general criteria for $UF[G]$ to be divisible or algebraically compact or coperiodical or Σ-countable or Warfield or simply presented or a Σ-group? The finding of such necessary and sufficient conditions for the classes of all quoted groups will definitely be of some significance. In the present research exploration we have partially settled some of these problems.

On the other hand, the calculation of the torsion-free rank of $UF[G]$ when F is not algebraic closed and G is absolute arbitrary is requisite for the description of the torsion-free part in $UF[G]$, and thus for the isomorphism structure of this group. In this work we have established only a partial answer.

A final question is does $UF[G]$ being splitting imply that the same holds for G, i.e., in other words, if $UF[G]$ is splitting is then G splitting? It seems to the author that this is not the case and even more that G is not p-splitting.

Acknowledgment. The author would like to express his deep thanks to the specialist referee for the helpful comments and suggestions included in the paper.

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STUDIA UNIV. "BABEŞ–BOLYAI", MATHEMATICA, Volume **LI**, Number 2, June 2006

NONEXISTENCE OF NONTRIVIAL PERIODIC SOLUTIONS FOR SEMILINEAR WAVE EQUATIONS

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Abstract. The semilinear wave equation

$$\begin{cases} \Box u + g(t,x,u) = \lambda u, & \text{in } \Omega \\ u(t,0) = u(t,\pi) = 0, & t \in [0,2\pi] \\ u \text{ is } 2\pi\text{-periodic in } t \end{cases}$$

is considered as an eigenvalue problem with parameter λ. The nonexistence of nontrivial periodic solutions in case $\lambda = 4k+2$ is treated and the other cases are studied under some uniform boundedness conditions on g.

1. **Introduction**

A large number of papers are devoted to the study of the nonexistence of nontrivial solutions (*i.e.* eigenfunctions) of semilinear eigenvalue problems. The great importance follows from the fact that these are closed related with the theory of bifurcation, in special with finding the bifurcation points and the bifurcation intervals. Such results for semilinear elliptic equations was established for example by Chiappinelli in [1] using critical point theory of Ljusternik Schnirelmann or by Berger in [2] in connection with bifurcation theory.

Received by the editors: 22.11.2005.

2000 *Mathematics Subject Classification.* 47F05, 47J10, 47J15.

Key words and phrases. Wave equations, bifurcation points, orthonormal bases, Caratheodory function, generalized solutions.

We consider here the semilinear wave equation in the form

$$\begin{cases} u_{tt} - u_{xx} + g(t,x,u) = \lambda u, & \text{in } \Omega \\ u(t,0) = u(t,\pi) = 0, & t \in [0,2\pi] \\ u \text{ is } 2\pi\text{-periodic in } t \end{cases} , \tag{1.1}$$

where $\Omega = (0,2\pi) \times (0,\pi)$ and λ is a real parameter.

Let $\widetilde{C}^2$ be the space of twice continuously differentiable functions $v : \overline{\Omega} \to \mathbf{R}$ such that $v(t,0) = v(t,\pi) = 0$ and $v(\cdot,x)$ is 2π-periodic. By $H = L^2(\Omega)$ denote the completion of the space $\widetilde{C}^2$ endowed with the inner product

$$(v,w) = \int_\Omega vw \quad , \quad v,w \in \widetilde{C}^2$$

and the corresponding norm

$$\|v\| = \sqrt{(v,v)} \quad , \quad v \in \widetilde{C}^2.$$

The set $(\psi_{nk})_{(n,k)\in \mathbf{N}\times\mathbf{Z}}$ forming an orthonormal basis in H and consists of eigenfunctions of the linear operator

$$\Box = \partial_t^2 - \partial_x^2$$

is defined by

$$\psi_{nk}(x,t) = \begin{cases} \dfrac{\sqrt{2}}{\pi} \sin nx \sin kt, & (n,k) \in \mathbf{N}\times\mathbf{N} \\ \dfrac{1}{\pi} \sin nx, & n \in \mathbf{N},\ k = 0 \\ \dfrac{\sqrt{2}}{\pi} \sin nx \cos kt, & n \in \mathbf{N},\ -k \in \mathbf{N} \end{cases} .$$

Obviously,

$$\Box\psi_{nk} = (n^2 - k^2)\psi_{nk}.$$

Let $L : D(L) \subset H \to H$ be given by

$$Lu = \sum_{n=1}^{\infty} \sum_{k=-\infty}^{\infty} (n^2 - k^2)(u,\psi_{nk})\psi_{nk} ,$$

with the domain

$$D(L) = \left\{ u \in H \mid \sum_{n=1}^{\infty} \sum_{k=-\infty}^{\infty} (n^2 - k^2) \left|(u, \psi_{nk})\right|^2 < \infty \right\}.$$

The operator L is densely defined, selfadjoint and with a closed range. Its spectrum

$$\sigma(L) = \left\{ \lambda_{nk} = n^2 - k^2 \mid (n,k) \in \mathbf{N} \times \mathbf{Z} \right\}$$

is unbounded from above and below and any non-zero eigenvalue has a finite algebraic multiplicity. More precisely,

$$\sigma(L) = \mathbf{Z} \setminus \{4k + 2 \mid k \in \mathbf{Z}\}$$

which follows easily.

Assume that $g : \Omega \times \mathbf{R} \to \mathbf{R}$ is a Carathéodory function and there exists $c > 0$ satisfying

$$|g(t,x,u)| \leq c\,|u| \quad , \quad \forall\,(t,x) \in \Omega,\ u \in \mathbf{R}.$$

In consequence,

$$g(t,x,0) = 0 \quad , \qquad \forall\,(t,x) \in \Omega$$

so the problem (1.1) has the trivial solution $u = 0$.

The Nemytskii operator

$$(Su)(t,x) = g(t,x,u(t,x))$$

generated by g is bounded and continuous from $L^2(\Omega)$ into itself. Consequently, the generalized solution of (1.1) is any function $u \in L^2(\Omega)$ such that

$$(u, v_{tt} - v_{xx}) + (Su, v) = \lambda(u,v)\ ,\ \ \forall\, v \in \widetilde{C}^2.$$

From now, we will write this equation in the operator form

$$Lu + S(u) = \lambda u, \tag{1.2}$$

where L inherets the properties of the generalized d'Alembertian with periodic boundary conditions.

2. **The results**

First we give a result in case $\lambda \in \mathbf{Z} \setminus \sigma(L)$, thus $\lambda = 4k+2$, $k \in \mathbf{Z}$.

THEOREM 2.1. *Assume that*

$$|g(t,x,u)| \leq c\,|u| \quad , \quad \forall\,(t,x) \in \Omega,\ u \in \mathbf{R}$$

for some $c \in (0,1)$. *Then the problem*

$$\begin{cases} u_{tt} - u_{xx} + g(t,x,u) = (4k+2)u, & \text{in } \Omega \\ u(t,0) = u(t,\pi) = 0, & t \in [0,2\pi] \\ u \text{ is } 2\pi\text{-periodic in } t \end{cases}$$

has no nontrivial solutions. Consequently, $4k+2$ *are not bifurcation points.*

Proof. Denote $\lambda_k = 4k+2$. The closest eigenvalues from $4k+2$ are $4k+1$ and $4k+3$, so

$$\mathrm{dist}(\lambda_k, \sigma(L)) = 1.$$

Let us suppose by contrary that there exists a nontrivial solution u of the equation

$$Lu + S(u) = \lambda_k u.$$

Since L is selfadjoint, then for $\lambda \notin \sigma(L)$, the resolvent $(L-\lambda I)^{-1}$ of L at λ is bounded, linear map with norm

$$\left|\left|(L-\lambda I)^{-1}\right|\right| = \frac{1}{\mathrm{dist}(\lambda, \sigma(L))},$$

(Kato [3], pp.272). Therefore,

$$(L - \lambda_k I)u = -S(u)$$

or

$$u = -(L-\lambda_k I)^{-1} S(u).$$

By taking the norm, we derive

$$||u|| = \left|\left|(L-\lambda_k I)^{-1} S(u)\right|\right| \leq$$

$$\leq \left|\left|(L-\lambda_k I)^{-1}\right|\right| \cdot ||S(u)|| \leq \frac{1}{\mathrm{dist}(\lambda_k, \sigma(L))} \cdot c\,||u||\,.$$

Now, if divide by $||u|| \neq 0$, we obtain

$$1 \leq \frac{1}{\operatorname{dist}(\lambda_k, \sigma(L))} \cdot c$$

or

$$c \geq \operatorname{dist}(\lambda_k, \sigma(L)) = 1,$$

a contradiction. This shows that $u = 0$. $\square$

Further we will give a more general result. For each real number $\lambda \notin \sigma(L)$, $\lambda \neq 4k+2$, we study cases when $\lambda \in I_k$, where the intervals I_k are given in the next table. If denote by $\mu_k \in \sigma(L)$ the closest eigenvalue from λ, we have the following situation:

I_k	μ_k	$\operatorname{dist}(\lambda, \sigma(L))$
$(4k, 4k+1)$	$4k$ or $4k+1$	$\min\{4k+1-\lambda, \lambda-4k\}$
$(4k+1, 4k+2)$	$4k+1$	$\lambda - 4k - 1$
$(4k+2, 4k+3)$	$4k+3$	$4k+3-\lambda$
$(4k+3, 4k+4)$	$4k+3$ or $4k+4$	$\min\{4k+4-\lambda, \lambda-4k-3\}$

Now we are in position to give the following

THEOREM 2.2. *Assume that* $\lambda \in (i, i+1)$, $i \in \mathbf{Z}$ *and*

$$|g(t,x,u)| \leq c\,|u| \quad , \quad \forall\,(t,x) \in \Omega,\ u \in \mathbf{R}$$

for some $0 < c < c(\lambda)$, *where*

$$c(\lambda) = \begin{cases} \lambda - i & , \ \textit{if} \ \ i = 4k+1 \\ i+1-\lambda & , \ \textit{if} \ \ i = 4k+2 \\ \min\{i+1-\lambda, \lambda - i\} & , \ \textit{if} \ \ i = 4k \ \ \textit{or} \ \ i = 4k+3 \end{cases}.$$

Then the problem

$$\begin{cases} u_{tt} - u_{xx} + g(t,x,u) = \lambda u, & \text{in } \Omega \\ u(t,0) = u(t,\pi) = 0, & t \in [0, 2\pi] \\ u \text{ is } 2\pi\text{-periodic in } t \end{cases}$$

has no nontrivial solutions. Consequently, these points λ are not bifurcation points.

Proof. We can easily see that

$$c(\lambda) = \text{dist}(\lambda, \sigma(L)),$$

so

$$0 < c < c(\lambda).$$

By assuming that there exists a nontrivial solution of the problem (1.2), we obtain as above that

$$u = -(L - \lambda I)^{-1} S(u)$$

and finally

$$1 \leq \frac{1}{\text{dist}(\lambda, \sigma(L))} \cdot c \Leftrightarrow c \geq \text{dist}(\lambda, \sigma(L)) = c(\lambda),$$

contradiction. Consequently, the problem admits only the solution $u = 0$. □

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STUDIA UNIV. "BABEŞ–BOLYAI", MATHEMATICA, Volume **LI**, Number 2, June 2006

A NONSMOOTH EXTENSION FOR THE BERNSTEIN-STANCU OPERATORS AND AN APPLICATION

CRISTINEL MORTICI AND INGRID OANCEA

Abstract. D.D. Stancu defined in [8] a class of approximation operators which are more general than the well-known Bernstein operators. We define here a new type of approximation operators which extend the Bernstein-Stancu operators. These new operators have the advantage that the points where the given function $f : [0,1] \to \mathbf{R}$ is calculated can be independently chosen in each interval of the equidistant division of the interval $[0,1]$. Moreover, all possible such choices of intermediary points cover the whole interval $[0,1]$. Finally, we consider a particular case as an application.

1. **Introduction**

The Bernstein approximations $B_m f$, $m \in \mathbb{N}$ associated to a given continuous function $f : [0,1] \to \mathbb{R}$ is the polynomial

$$\left(B_m f\right)(x) = \sum_{k=0}^{m} p_{m,k}(x) f\left(\frac{k}{m}\right),$$

where

$$p_{m,k}(x) = \binom{m}{k} x^k (1-x)^{m-k}$$

are called the Bernstein fundamental polynomials of $m-$th degree (see [2]). Bernstein used this approximation to give the first constructive proof of the Weierstrass theorem. One of the many remarkable properties of Bernstein approximation is that

Received by the editors: 22.11.2005.

2000 *Mathematics Subject Classification.* 41A20, 41A25, 41A35.

Key words and phrases. Integral operators, uniform convergence, Kantorovich-Stancu operators, delta operators.

each derivative of the polynomial function $B_m f$ of any order converges to the corresponding derivative of f ([6]). Other important properties are shape-preservation and variation-diminuation [4]. These many properties can be viewed as compensation for the slow convergence of $B_m f$ to f.

In 1968, D.D. Stancu defined in [8] a linear positive operator depending on two non-negative parameters α and β satisfying the condition $0 \leq \alpha \leq \beta$. Those operators defined for any non-negative integer m, associate to every function $f \in C([0,1])$ the polynomial $P_m^{(\alpha,\beta)} f$,

$$f \in C([0,1]) \longmapsto P_m^{(\alpha,\beta)} f,$$

in the following way:

$$\left(P_m^{(\alpha,\beta)} f\right)(x) = \sum_{k=0}^{m} p_{m,k}(x) f\left(\frac{k+\alpha}{m+\beta}\right).$$

Note that for $\alpha = \beta = 0$ the Bernstein-Stancu operators become the classical Bernstein operators B_m. It is known that the Bernstein-Stancu operators verify the following relations (e.g. [1]):

Lemma 1.1 *For Bernstein-Stancu operators* $P_m^{(\alpha,\beta)}$, $m \in \mathbb{N}$, *the following relations hold true:*

1) $\left(P_m^{(\alpha,\beta)} e_0\right)(x) = 1$
2) $\left(P_m^{(\alpha,\beta)} e_1\right)(x) = x + \dfrac{\alpha - \beta x}{m+\beta}$
3) $\left(P_m^{(\alpha,\beta)} e_2\right)(x) = x^2 + \dfrac{mx(1-x) + (\alpha - \beta x)(2mx + \beta x + \alpha)}{(m+\beta)^2}$,

where

$$e_j(x) = x^j \quad , \quad j = 0, 1, 2$$

are test functions.

For proofs and other comments see [1].

2. The Results

In order to define the new class of operators, for all non-negative integers m and $k = 0, 1, \ldots, m$ consider the non-negative reals α_{mk}, β_{mk} so that

$$\alpha_{mk} \leq \beta_{mk}.$$

Further, let us denote by A, respective B, the infinite dimensional lower triangular matrices

$$A = \begin{pmatrix} \alpha_{00} & 0 & 0 & 0 & 0 & ... \\ \alpha_{10} & \alpha_{11} & 0 & 0 & 0 & \dots \\ \alpha_{20} & \alpha_{21} & \alpha_{22} & 0 & 0 & \dots \\ \dots & \dots & \dots & \dots & ... & \dots \end{pmatrix}$$

and

$$B = \begin{pmatrix} \beta_{00} & 0 & 0 & 0 & 0 & ... \\ \beta_{10} & \beta_{11} & 0 & 0 & 0 & \dots \\ \beta_{20} & \beta_{21} & \beta_{22} & 0 & 0 & \dots \\ \dots & \dots & \dots & \dots & ... & \dots \end{pmatrix}.$$

Under these assumptions, we define an approximation operator denoted by

$$P_m^{(A,B)} : C([0,1]) \to C([0,1]),$$

with the formula

$$\left(P_m^{(A,B)} f\right)(x) = \sum_{k=0}^{m} p_{m,k}(x) f\left(\frac{k+\alpha_{mk}}{m+\beta_{mk}}\right) \quad , \qquad f \in C([0,1]).$$

Remark that the Bernstein-Stancu operators is a particular type of operators $P_m^{(A,B)}$, in case when the matrices A and B are of the form

$$A = \begin{pmatrix} \alpha & 0 & 0 & 0 & 0 & ... \\ \alpha & \alpha & 0 & 0 & 0 & \dots \\ \alpha & \alpha & \alpha & 0 & 0 & \dots \\ \dots & \dots & \dots & \dots & ... & \dots \end{pmatrix}$$

and

$$B = \begin{pmatrix} \beta & 0 & 0 & 0 & 0 & ... \\ \beta & \beta & 0 & 0 & 0 & \dots \\ \beta & \beta & \beta & 0 & 0 & \dots \\ \dots & \dots & \dots & \dots & ... & \dots \end{pmatrix},$$

or equivalent,

$$\alpha_{mk} = \alpha \quad , \qquad \beta_{mk} = \beta,$$

for all non-negative integers m and $k \leq m$.

The operators $P_m^{(A,B)}$ have some advantages in comparison with Bernstein-Stancu operators. The approximations of a given continuous function f are calculated using some known values of f. In case of Bernstein operators, the $m-$th approximation is given in function of the values of f at points

$$0 < \frac{1}{m} < \frac{2}{m} < ... < \frac{m-1}{m} < 1,$$

while in case of Stancu operators, the $m-$th approximation is given in terms of the values at points

$$\frac{\alpha}{m+\beta} < \frac{\alpha+1}{m+\beta} < \frac{\alpha+2}{m+\beta} < ... < \frac{\alpha+m}{m+\beta}.$$

This choice of the intermediary points are in some sense strictly, because they depend each other. In the first case, the intermediary points are in arithmetic progression and in the second case, the intermediary points are also under some restrictions. The success of the approximation method appears only if we know the values of the function f at that particular points. The $P_m^{(A,B)}$ operators defined here allow a great liberty for choice of the intermediary points. Indeed, we can independently choose intermediary points in each interval

$$\left[\frac{k}{m}, \frac{k+1}{m}\right] \quad , \quad 0 \leq k \leq m-1.$$

It is sufficient to have $\beta_{mk} \leq 1$, to imply

$$\frac{k+\alpha_{mk}}{m+\beta_{mk}} \leq \frac{k+\beta_{mk}}{m+\beta_{mk}} \leq \frac{k+1}{m+\beta_{mk}} \leq \frac{k+1}{m}.$$

Thus under similar weak assumptions, we can have

$$\frac{k+\alpha_{mk}}{m+\beta_{mk}} \in \left[\frac{k}{m}, \frac{k+1}{m}\right] \quad , \quad 0 \leq k \leq m-1,$$

so the possible values of the intermediary points cover the whole interval $[0,1]$. The operators $P_m^{(A,B)}$ are linear, in the sense that

$$P_m^{(A,B)}(\mu f + \lambda g) = \mu P_m^{(A,B)} f + \lambda P_m^{(A,B)} g,$$

for all real numbers λ, μ and $f, g \in C([0,1])$ and positive defined, *i.e.*

$$P_m^{(A,B)} f \geq 0 \quad , \quad if\ f \geq 0.$$

We will use the following result due to H. Bohman and P.P. Korovkin.

Theorem 2.1 *Let $L_m : C([a,b]) \to C([a,b])$, $m \in \mathbb{N}$ be a sequence of linear, positive operators such that*

$$(L_m e_0)(x) = 1 + u_m(x)$$

$$(L_m e_1)(x) = x + v_m(x)$$

$$(L_m e_2)(x) = x^2 + w_m(x)$$

with

$$\lim_{m\to\infty} u_m(x) = \lim_{m\to\infty} v_m(x) = \lim_{m\to\infty} w_m(x) = 0,$$

uniformly on $[0,1]$. *Then for every continuous function* $f \in C([0,1])$, *we have*

$$\lim_{m\to\infty} (L_m f)(x) = f(x),$$

uniformly on $[0,1]$.

For proofs and other results see [3], [5]. In order to prove that the operators $P_m^{(A,B)}$ are approximation operators, we give the following main result:

Theorem 2.2 *Given the infinite dimensional lower triangular matrices*

$$A = \begin{pmatrix} \alpha_{00} & 0 & 0 & \dots & ... & ... \\ \alpha_{10} & \alpha_{11} & 0 & 0 & ... & \dots \\ \alpha_{20} & \alpha_{21} & \alpha_{22} & 0 & 0 & \dots \\ \dots & \dots & \dots & \dots & ... & \dots \end{pmatrix}$$

and

$$B = \begin{pmatrix} \beta_{00} & 0 & 0 & \dots & ... & ... \\ \beta_{10} & \beta_{11} & 0 & 0 & ... & \dots \\ \beta_{20} & \beta_{21} & \beta_{22} & 0 & 0 & \dots \\ \dots & \dots & \dots & \dots & ... & \dots \end{pmatrix}$$

with the following properties:

a) $0 \leq \alpha_{mk} \leq \beta_{mk}$, *for every non-negative integers* m *and* $k \leq m$

b) $\alpha_{mk} \in [a,b]$, $\beta_{mk} \in [c,d]$ *for every non-negative integers* m *and* $k \leq m$ *and for some non-negative real numbers* $0 \leq a < b$ *and* $0 \leq c < d$.

Then for every continuous function $f \in C([0,1])$, *we have*

$$\lim_{m\to\infty} P_m^{(A,B)} f = f \quad , \quad uniformly\ on\ [0,1].$$

Proof. Let us compute the values of the operators $P_m^{(A,B)}$ on test functions e_j, $j = 0,1,2$. We have

$$\left(P_m^{(A,B)} e_0\right)(x) = \sum_{k=0}^{m} p_{m,k}(x) e_0\left(\frac{k+\alpha_{mk}}{m+\beta_{mk}}\right) = \sum_{k=0}^{m} p_{m,k}(x) = 1,$$

$$\left(P_m^{(A,B)} e_1\right)(x) = \sum_{k=0}^{m} p_{m,k}(x) e_1\left(\frac{k+\alpha_{mk}}{m+\beta_{mk}}\right) = \sum_{k=0}^{m} p_{m,k}(x) \frac{k+\alpha_{mk}}{m+\beta_{mk}},$$

respective

$$\left(P_m^{(A,B)} e_2\right)(x) = \sum_{k=0}^{m} p_{m,k}(x) e_2\left(\frac{k+\alpha_{mk}}{m+\beta_{mk}}\right) = \sum_{k=0}^{m} p_{m,k}(x) \left(\frac{k+\alpha_{mk}}{m+\beta_{mk}}\right)^2,$$

Now, from the inequalities

$$a \leq \alpha_{mk} \leq b \quad , \quad c \leq \beta_{mk} \leq d,$$

we obtain the estimations

$$\frac{k+a}{m+d} \leq \frac{k+\alpha_{mk}}{m+\beta_{mk}} \leq \frac{k+b}{m+c},$$

for all non-negative integers $k \leq m$. By multiplying each member of the inequality by $p_{m,k}(x)$ and taking the sum with respect to k it follows that

$$\sum_{k=0}^{m} p_{m,k}(x) \frac{k+a}{m+d} \leq \sum_{k=0}^{m} p_{m,k}(x) \frac{k+\alpha_{mk}}{m+\beta_{mk}} \leq \sum_{k=0}^{m} p_{m,k}(x) \frac{k+b}{m+c}$$

or

$$\left(P_m^{(a,d)} e_1\right)(x) \leq \left(P_m^{(A,B)} e_1\right)(x) \leq \left(P_m^{(b,c)} e_1\right)(x).$$

Now, by replacing $\left(P_m^{(a,d)} e_1\right)(x)$ and $\left(P_m^{(b,c)} e_1\right)(x)$ with their expressions from Lemma 1.1, we obtain the estimations

$$x + \frac{a-dx}{m+d} \leq \left(P_m^{(A,B)} e_1\right)(x) \leq x + \frac{b-cx}{m+c}.$$

Hence, for all $x \in [0,1]$, we have

$$\left| \left(P_m^{(A,B)} e_1 \right)(x) - x \right| \leq \max \left\{ \left| \frac{a - dx}{m + d} \right|, \left| \frac{b - cx}{m + c} \right| \right\}.$$

But, for all $x \in [0,1]$, we also have

$$\left| \frac{a - dx}{m + d} \right| \leq \frac{|a| + |d|}{m + d} \leq \frac{|a| + |d|}{m}$$

and

$$\left| \frac{b - cx}{m + c} \right| \leq \frac{|b| + |c|}{m + c} \leq \frac{|b| + |c|}{m}.$$

Now, with the notation

$$q = \max \left\{ |a| + |d|, |b| + |c| \right\},$$

we obtain

$$\left| \left(P_m^{(A,B)} e_1 \right)(x) - x \right| \leq \frac{q}{m} \to 0 \quad , \quad as\ m \to \infty,$$

for all $x \in [0,1]$, so

$$\lim_{m \to \infty} \left(P_m^{(A,B)} e_1 \right)(x) = x \quad , \quad uniformly\ on\ [0,1].$$

Moreover, from the inequality

$$\left(\frac{k + a}{m + d} \right)^2 \leq \left(\frac{k + \alpha_{mk}}{m + \beta_{mk}} \right)^2 \leq \left(\frac{k + b}{m + c} \right)^2$$

we obtain

$$\sum_{k=0}^{m} p_{m,k}(x) \left(\frac{k + a}{m + d} \right)^2 \leq \sum_{k=0}^{m} p_{m,k}(x) \left(\frac{k + \alpha_{mk}}{m + \beta_{mk}} \right)^2 \leq \sum_{k=0}^{m} p_{m,k}(x) \left(\frac{k + b}{m + c} \right)^2$$

or

$$\left(P_m^{(a,d)} e_2 \right)(x) \leq \left(P_m^{(A,B)} e_2 \right)(x) \leq \left(P_m^{(b,c)} e_2 \right)(x).$$

Now, by replacing $\left(P_m^{(a,d)} e_2 \right)(x)$ and $\left(P_m^{(b,c)} e_2 \right)(x)$ with their expressions from Lemma 1.1, we obtain the estimations

$$x^2 + \frac{mx(1-x) + (a - dx)(2mx + dx + a)}{(m+d)^2} \leq \left(P_m^{(A,B)} e_2 \right)(x) \leq$$

$$\leq x^2 + \frac{mx(1-x) + (b - cx)(2mx + cx + b)}{(m+c)^2}.$$

Hence

$$\left| \left(P_m^{(A,B)} e_2 \right)(x) - x^2 \right| \leq$$

$$\leq \max\left\{\left|\frac{mx(1-x)+(a-dx)(2mx+dx+a)}{(m+d)^2}\right|,\left|\frac{mx(1-x)+(b-cx)(2mx+cx+b)}{(m+c)^2}\right|\right\}.$$

Using that $x \in [0,1]$, we obtain

$$\left|\frac{mx(1-x)+(a-dx)(2mx+dx+a)}{(m+d)^2}\right| \leq \frac{m+(|a|+|d|)(2m+|d|+|a|)}{m^2}$$

and

$$\left|\frac{mx(1-x)+(b-cx)(2mx+cx+b)}{(m+c)^2}\right| \leq \frac{m+(|b|+|c|)(2m+|c|+|b|)}{m^2}.$$

Now, with the notation

$$w = \max\left\{(|a|+|d|)(2m+|d|+|a|),(|b|+|c|)(2m+|c|+|b|)\right\},$$

we obtain that

$$\left|\left(P_m^{(A,B)}e_2\right)(x) - x^2\right| \leq \frac{m+w}{m^2} \to 0\ , \quad as\ m \to \infty.$$

According with Bohman-Korovkin theorem,

$$\lim_{m\to\infty}\left(P_m^{(A,B)}e_2\right)(x) = x^2 \quad , \quad uniformly\ on\ [0,1].$$

Corollary 2.1 *Assume that* $0 \leq \alpha_{mk} \leq \beta_{mk}$, *for all non-negative integers* $k \leq m$. *For each integer* $k \geq 0$, *assume that the sequences* $(\alpha_{mk})_{m\in\mathbb{N}}$ *and* $(\beta_{mk})_{m\in\mathbb{N}}$ *are convergent to* α_k *and* β_k, *respectively, such that the sequences* $(\alpha_k)_{k\in\mathbb{N}}$ *and* $(\beta_k)_{k\in\mathbb{N}}$ *are bounded. Then for every continuous function* $f \in C([0,1])$,

$$\lim_{m\to\infty}\left(P_m^{(A,B)}f\right)(x) = f(x) \quad , \quad uniformly\ in\ [0,1].$$

Proof. We are in the hypotheses of the Theorem 2.1. From the fact that the sequences $(\alpha_{mk})_{m\in\mathbb{N}}$ and $(\beta_{mk})_{m\in\mathbb{N}}$ are convergent to α_k and β_k, respectively, we can find a positive integer m_1 for which

$$|\alpha_{mk} - \alpha_k| < 1,$$

for all integers $m \geq m_1$ and we can find m_2 for which

$$|\beta_{mk} - \beta_k| < 1,$$

for all integers $m \geq m_2$. Now, for every integer $m \geq m_1 + m_2$, we have

$$\alpha_{mk} < 1 + \alpha_k \leq 1 + M$$

and

$$\beta_{mk} < 1 + \beta_k \leq 1 + M,$$

where

$$M = \max\left\{\sup_{k\in\mathbb{N}} \alpha_k, \sup_{k\in\mathbb{N}} \beta_k\right\}.$$

3. **A particular case**

An interesting case is when the matrix B has all nonzero entries equal to a positive constant β,

$$B = \begin{pmatrix} \beta & 0 & 0 & \ldots & ... & ... \\ \beta & \beta & 0 & 0 & ... & \ldots \\ \beta & \beta & \beta & 0 & 0 & \ldots \\ \ldots & \ldots & \ldots & \ldots & ... & \ldots \end{pmatrix}.$$

We also impose the conditions

$$\alpha_{mk} \leq \beta,$$

for all non-negative integers $k \leq m$. Under these assumptions, we define an approximation operator denoted by

$$P_m^{(A,\beta)} : C[0,1] \to C([0,1],$$

with the formula

$$\left(P_m^{(A,\beta)} f\right)(x) = \sum_{k=0}^{m} p_{m,k}(x) f\left(\frac{k+\alpha_{mk}}{m+\beta}\right) \quad , \qquad f \in C\left[0,1\right].$$

Lemma 3.1 *For every continuous function $f \in C[0,1]$, the following relations hold true:*

a) $\left(P_m^{(A,\beta)} e_0\right)(x) = 1$

b) $\left(P_m^{(A,\beta)} e_1\right)(x) = x - \dfrac{\beta x}{m+\beta} + \dfrac{1}{m+\beta} \cdot \sum_{k=0}^{m} \alpha_{mk} p_{m,k}(x).$

Proof. a) We have

$$\left(P_m^{(A,\beta)}e_0\right)(x)=\sum_{k=0}^{m}p_{m,k}(x)e_0\left(\frac{k+\alpha_{mk}}{m+\beta}\right)=\sum_{k=0}^{m}p_{m,k}(x)=1.$$

b) We have

$$\left(P_m^{(A,\beta)}e_1\right)(x)=\sum_{k=0}^{m}p_{m,k}(x)e_1\left(\frac{k+\alpha_{mk}}{m+\beta}\right)=\sum_{k=0}^{m}p_{m,k}(x)\frac{k+\alpha_{mk}}{m+\beta}=$$

$$=\sum_{k=0}^{m}\frac{k}{m+\beta}\cdot p_{m,k}(x)+\frac{1}{m+\beta}\cdot\sum_{k=0}^{m}\alpha_{mk}p_{m,k}(x)=$$

$$=\left(P_m^{(0,\beta)}e_1\right)(x)+\frac{1}{m+\beta}\cdot\sum_{k=0}^{m}\alpha_{mk}p_{m,k}(x)=$$

$$=x-\frac{\beta x}{m+\beta}+\frac{1}{m+\beta}\cdot\sum_{k=0}^{m}\alpha_{mk}p_{m,k}(x).$$

Theorem 3.1 *Given the infinite dimensional lower triangular matrix*

$$A=\begin{pmatrix}\alpha_{00} & 0 & 0 & \dots & ... & ...\\ \alpha_{10} & \alpha_{11} & 0 & 0 & ... & \dots\\ \alpha_{20} & \alpha_{21} & \alpha_{22} & 0 & 0 & \dots\\ \dots & \dots & \dots & \dots & ... & \dots\end{pmatrix}$$

and a positive real number β with the following properties:

a) $\alpha_{mk}\le\beta$, for every non-negative integers m and $k\le m$

b) $\alpha_{mk}\in[a,b]$, for every non-negative integers m and $k\le m$ and for some non-negative real numbers $a<b$.

Then for every continuous function $f\in C([0,1])$, we have

$$\lim_{m\to\infty}P_m^{(A,\beta)}f=f\quad,\quad uniformly\ on\ [0,1].$$

Proof. Let us compute the values of the operators $P_m^{(A,\beta)}$ on test functions e_j, $j=0,1,2$. We have

$$\left(P_m^{(A,\beta)}e_0\right)(x)=\sum_{k=0}^{m}p_{m,k}(x)e_0\left(\frac{k+\alpha_{mk}}{m+\beta}\right)=\sum_{k=0}^{m}p_{m,k}(x)=1,$$

$$\left(P_m^{(A,\beta)}e_1\right)(x)=\sum_{k=0}^{m}p_{m,k}(x)e_1\left(\frac{k+\alpha_{mk}}{m+\beta}\right)=\sum_{k=0}^{m}p_{m,k}(x)\frac{k+\alpha_{mk}}{m+\beta},$$

respective

$$\left(P_m^{(A,\beta)}e_2\right)(x)=\sum_{k=0}^{m}p_{m,k}(x)e_2\left(\frac{k+\alpha_{mk}}{m+\beta}\right)=\sum_{k=0}^{m}p_{m,k}(x)\left(\frac{k+\alpha_{mk}}{m+\beta}\right)^2,$$

Now, from the inequalities

$$a\leq\alpha_{mk}\leq b,$$

we obtain the estimations

$$\frac{k+a}{m+\beta}\leq\frac{k+\alpha_{mk}}{m+\beta}\leq\frac{k+b}{m+\beta},$$

for all non-negative integers $k\leq m$. By multiplying each member of the inequality by $p_{m,k}(x)$ and taking the sum with respect to k it follows that

$$\sum_{k=0}^{m}p_{m,k}(x)\frac{k+a}{m+\beta}\leq\sum_{k=0}^{m}p_{m,k}(x)\frac{k+\alpha_{mk}}{m+\beta}\leq\sum_{k=0}^{m}p_{m,k}(x)\frac{k+b}{m+\beta}$$

or

$$\left(P_m^{(a,\beta)}e_1\right)(x)\leq\left(P_m^{(A,\beta)}e_1\right)(x)\leq\left(P_m^{(b,\beta)}e_1\right)(x)$$

and using the expressions of $\left(P_m^{(a,\beta)}e_1\right)(x)$ we obtain

$$x+\frac{a-\beta x}{m+\beta}\leq\left(P_m^{(A,B)}e_1\right)(x)\leq x+\frac{b-\beta x}{m+\beta}.$$

Hence, for all $x\in[0,1]$, we have

$$\left|\left(P_m^{(A,B)}e_1\right)(x)-x\right|\leq\max\left\{\left|\frac{a-\beta x}{m+\beta}\right|,\left|\frac{b-\beta x}{m+\beta}\right|\right\}.$$

Further, for all $x\in[0,1]$, we obtain

$$\left|\frac{a-\beta x}{m+\beta}\right|\leq\frac{|a|+|\beta|}{m+\beta}\leq\frac{|a|+|\beta|}{m}$$

and

$$\left|\frac{b-\beta x}{m+\beta}\right|\leq\frac{|b|+|\beta|}{m+\beta}\leq\frac{|b|+|\beta|}{m}.$$

Now, with the notation

$$q=\max\left\{|a|+|\beta|,|b|+|\beta|\right\},$$

we obtain

$$\left|\left(P_m^{(A,\beta)}e_1\right)(x)-x\right|\leq\frac{q}{m},$$

for all $x \in [0,1]$, so

$$\lim_{m\to\infty}\left(P_m^{(A,\beta)}e_1\right)(x)=x$$

uniformly on $[0,1]$.

Moreover, from the inequality

$$\left(\frac{k+a}{m+\beta}\right)^2 \le \left(\frac{k+\alpha_{mk}}{m+\beta}\right)^2 \le \left(\frac{k+b}{m+\beta}\right)^2$$

we obtain

$$\sum_{k=0}^{m} p_{m,k}(x)\left(\frac{k+a}{m+\beta}\right)^2 \le \sum_{k=0}^{m} p_{m,k}(x)\left(\frac{k+\alpha_{mk}}{m+\beta}\right)^2 \le \sum_{k=0}^{m} p_{m,k}(x)\left(\frac{k+b}{m+\beta}\right)^2$$

or

$$\left(P_m^{(a,\beta)}e_2\right)(x) \le \left(P_m^{(A,\beta)}e_2\right)(x) \le \left(P_m^{(b,\beta)}e_2\right)(x).$$

Now, by replacing $\left(P_m^{(a,\beta)}e_2\right)(x)$ with its expression we obtain the estimations

$$x^2+\frac{mx(1-x)+(a-\beta x)(2mx+\beta x+a)}{(m+\beta)^2} \le \left(P_m^{(A,\beta)}e_2\right)(x) \le$$

$$\le x^2+\frac{mx(1-x)+(b-\beta x)(2mx+\beta x+b)}{(m+\beta)^2}.$$

Hence

$$\left|\left(P_m^{(A,\beta)}e_2\right)(x)-x^2\right| \le$$

$$\le \max\left\{\left|\tfrac{mx(1-x)+(a-\beta x)(2mx+\beta x+a)}{(m+\beta)^2}\right|,\left|\tfrac{mx(1-x)+(b-\beta x)(2mx+\beta x+b)}{(m+\beta)^2}\right|\right\}.$$

Using that $x \in [0,1]$, we obtain

$$\left|\frac{mx(1-x)+(a-\beta x)(2mx+\beta x+a)}{(m+\beta)^2}\right| \le \frac{m+(|a|+|\beta|)(2m+|\beta|+|a|)}{m^2}$$

and

$$\left|\frac{mx(1-x)+(b-\beta x)(2mx+\beta x+b)}{(m+\beta)^2}\right| \le \frac{m+(|b|+|\beta|)(2m+|\beta|+|b|)}{m^2}.$$

Now, with the notation

$$w=\max\left\{(|a|+|\beta|)(2m+|\beta|+|a|),(|b|+|\beta|)(2m+|\beta|+|b|)\right\},$$

we obtain that

$$\left|\left(P_m^{(A,\beta)}e_2\right)(x)-x^2\right| \le \frac{m+w}{m^2} \to 0\ ,\quad as\ m\to\infty.$$

According with the Bohman-Korovkin theorem, we have

$$\lim_{m\to\infty}\left(P_m^{(A,\beta)}e_2\right)(x)=x^2,$$

uniformly on $[0,1]$.

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FUNCTIONAL-DIFFERENTIAL EQUATIONS OF MIXED TYPE, VIA WEAKLY PICARD OPERATORS

OLARU ION MARIAN

Abstract. In this paper we apply the weakly Picard operators technique to study the following second order functional differential equations of mixed type

$$-x''(t) = f(t, x(t), \int_{t-h}^{t} x(s)ds, \int_{t}^{t+h} x(s)ds), \quad t \in [a, b], h > 0.$$

1. Introduction

The purpose of this paper is to study,the following boundary value problem:

$$-x''(t) = f(t, x(t), \int_{t-h}^{t} x(s)ds, \int_{t}^{t+h} x(s)ds), \quad t \in [a, b], h > 0. \tag{1}$$

$$\begin{cases} x(t) = \varphi(t) \quad , \quad t \in [a-h, a] \\ x(t) = \psi(t) \quad , \quad t \in [b, b+h] \end{cases}. \tag{2}$$

Where:

(H_1) $f \in C([a, b] \times \mathbb{R}^3, \mathbb{R})$.

(H_2) There exists $L_f > 0$ such that:

$$|f(t, u_1, u_2, u_3) - f(t, v_1, v_2, v_3)| \le L_f \sum_{i=1}^{3} |u_i - v_i|,$$

Received by the editors: 10.01.2006.

2000 *Mathematics Subject Classification.* 34K10, 47H10.

Key words and phrases. Picard operators, weakly Picard operators, fixed points, equations of mixed type, data dependence.

for all $t \in [a,b], u_i, v_i \in \mathbb{R}, i = \overline{1,3}$.

(H_3) $\varphi \in C([a-h,a]), \psi \in C([b,b+h])$.

Let G be the Green function of the following problem:

$$\begin{cases} -x'' = \lambda \\ x(a) = 0 \\ x(b) = 0 \end{cases}.$$

From the definition of the Green function we have that,the problem (1)+(2), $x \in C([a-h,b+h]) \cap C^2([a,b])$, is equivalent with the fixed point equation:

$$x(t) = \begin{cases} \varphi(t), & t \in [a-h,a] \\ w(\varphi,\psi)(t) + \int\limits_a^b G(t,s) f(s, x(s), \int\limits_{s-h}^{s} x(u)du, \int\limits_{s}^{s+h} x(u)du)ds, & t \in [a,b] \\ \psi(t), & t \in [b,b+h] \end{cases} \tag{3}$$

$x \in C([a-h,b+h])$, where:

$$w(\varphi,\psi)(t) := \frac{t-a}{b-a} \cdot \psi(b) + \frac{b-t}{b-a} \cdot \varphi(a).$$

The equation (1) is equivalent with:

$$x(t) = \begin{cases} x(t), & t \in [a-h,a] \\ w(x|_{[a-h,a]}, x|_{[b,b+h]}) + \\ + \int\limits_a^b G(t,s) f(s, x(s), \int\limits_{s-h}^{s} x(u)du, \int\limits_{s}^{s+h} x(u)du)ds, & t \in [a,b] \\ x(t), & t \in [b,b+h] \end{cases}. \tag{4}$$

We consider the following operators:

$$B_f, E_f : C([a-h,b+h]) \to C([a-h,b+h])$$

where:

$B_f(x)(t)$:=second part of (3)

and

$E_f(x)(t)$:=second part of (4).

We denote by $X := C([a-h, b+h])$.

Let be

$$X_{\varphi,\psi} := \{x \in X \mid x \mid_{[a-h, a]} = \varphi, x \mid_{[b, b+h]} = \psi\}.$$

Then

$$X = \bigcup_{\substack{\varphi \in C([a-h,a]) \\ \psi \in C([b,b+h])}} X_{\varphi,\psi}$$

is a partition of X.

2. **Weakly Picard operators**

Led (X, d) be a metric space and $A : X \longrightarrow X$ an operator. We shall use the following notations:

$F_A := \{x \in X \mid A(x) = x\}$- the fixed point set of A.

$I(A) := \{Y \subset X \mid A(Y) \subset, Y \neq \emptyset\}$-the family of the nonempty invariant subsets of A.

$$A^{n+1} := A \circ A^n \; A^0 = 1_X, \;\; A^1 = A, \;\; n \in \mathbb{N}.$$

Definition 2.1. [1],[2] *An operator A is weakly Picard operator (WPO) if the sequence*

$$(A^n(x))_{n \in N}$$

converges , for all $x \in X$ and the limit (which depend on x) is a fixed point of A.

Definition 2.2. [1],[2] *If the operator A is WPO and $F_A = \{x^*\}$ then by definition A is Picard operator.*

Definition 2.3. [1],[2] *If A is WPO, then we consider the operator*

$$A^\infty : X \to X, A^\infty(x) = \lim_{n\to\infty} A^n(x).$$

We remark that $A^\infty(X) = F_A$.

Definition 2.4. [1],[2] *Let be A an WPO and* $c > 0$.*The operator A is c-WPO if*

$$d(x, A^{\infty}(x)) \leq c \cdot d(x, A(x)).$$

We have the following characterization of the WPOs:

Theorem 2.1. [1],[2]*Let* (X, d) *be a metric space and* $A : X \to X$ *an operator. The operator A is WPO (c-WPO) if and only if there exists a partition of X,*

$$X = \bigcup_{\lambda \in \Lambda} X_{\lambda}$$

such that

(a) $X_{\lambda} \in I(A)$

(b) $A \mid X_{\lambda} : X_{\lambda} \to X_{\lambda}$ *is a Picard (c-Picard) operator, for all* $\lambda \in \Lambda$.

For the class of c-WPOs we have the following data dependence result:

Theorem 2.2. [1],[2] *Let* (X, d) *be a metric space and* $A_i : X \to X, i = \overline{1, 2}$ *an operator.We suppose that :*

(i) *the operator* A_i *is* $c_i - WPO$, $i = \overline{1, 2}$.

(ii) *there exists* $\eta > o$ *such that*

$$d(A_1(x), A_2(x)) \leq \eta, (\forall) x \in X.$$

Then

$$H(F_{A_1}, F_{A_2}) \leq \eta \; max\{c_1, c_2\}.$$

Here stands for Hausdorff-Pompeiu functional.

We have:

Lemma 2.1. [1],[2] *Let*$(X, d, \leq)$ *be an ordered metric space and* $A : X \to X$ *an operator such that:*

a)A is monotone increasing.

b)A is WPO.

Then the operator A^{∞} *is monotone increasing.*

Lemma 2.2. [1],[2] *Let* $(X, d, \leq)$ *be an ordered metric space and* $A, B, C : X \to X$ *such that:*

(i)$A \leq B \leq C$.

(ii) the operators A,B,C are W.P.Os.

(iii) the operator B is monotone increasing.

Then

$$x \leq y \leq z \Longrightarrow A^{\infty}(x) \leq B^{\infty}(y) \leq C^{\infty}(z).$$

Lemma 2.3. [1],[2]*Let* $(X, d, \leq)$ *be an ordered metric space,* $A : X \longrightarrow X$ *an operator and* $x, y \in X$ *such that*

$$x < y, x \leq A(x), y \geq A(y).$$

We suppose that

(i) *A is W.P.O;*

(ii) *A is monotone increasing.*

Then

(a) $x \leq A^{\infty}(x) \leq A^{\infty}(y) \leq y$;

(b) $A^{\infty}(x)$ *is the minimal fixed point of A in* $[x, y]$ *and* $A^{\infty}(y)$ *is the maximal fixed point of A in* $[x, y]$

3. Boundary value problem

We consider the problem (1)+(2)

Theorem 3.1. *We suppose that*

(a) *The conditions* $(H_1) - (H_3)$ *are satisfied.*

(b) $\frac{1}{8}L_f(b-a)^2(1+2h) < 1$

Then the problem (1)+(2) has a unique solution in X.

Proof. The problem (1)+(2) is equivalent with the fixed point equation

$$B_f(x) = x, x \in X.$$

From the condition (H_2) we have

$$|B_f(x)(t) - B_f(y)(t)| \leq$$

$$\leq \int\limits_a^b G(t,s)|f(s,x(s),\int\limits_{s-h}^{s} x(u)du,\int\limits_{s}^{s+h} x(u)du)-f(s,y(s),\int\limits_{s-h}^{s} y(u)du,\int\limits_{s}^{s+h} y(u)du)|ds \leq$$

$$\leq L_f \int\limits_a^b G(t,s)[|x(s)-y(s)| + \int\limits_{s-h}^{s} |x(u)-y(u)|du + \int\limits_{s}^{s+h} |x(u)-y(u)|du]ds \leq$$

$$\leq \frac{L_f}{8}(b-a)^2\|x-y\|_C(1+2h),$$

for all $x, y \in X_{\varphi,\psi}$.

Then B_f is Picard operator on $X_{\varphi,\psi}$.

From this we have the conclusion.

Remark 3.1. *From the Theorem 3.1,using the Theorem 2.1, we have that the operator* E_f *is W.P.O and* $F_{E_f} \bigcap X_{\varphi,\psi} = \{x^\star_{\varphi,\psi}\}$ *where* $x^\star_{\varphi,\psi}$ *is the unique solution of (1)+(2).*

4. Inequalities of Čaplygin type

We have

Theorem 4.1. *We suppose that*

(a) *The conditions* $(H_1) - (H_3)$ *are satisfied;*
(b) $\frac{L_f}{8}(b-a)^2(1+2h) < 1$;
(c) *the operator* $f(t,\cdot,\cdot,\cdot) : \mathbb{R}^3 \longrightarrow \mathbb{R}$ *is monotone increasing for all* $t \in [a,b]$*;*

Let x *be a solution of the corresponding equation (1) and* y *a solution of the inequality*

$$-y''(t) \leq f(t,y(t),\int\limits_{t-h}^{t} y(s)ds,\int\limits_{t}^{t+h} y(s)ds).$$

Then

$$y(t) \leq x(t), (\forall)t \in [a-h,a] \cup [b,b+h] \Longrightarrow y \leq x$$

Proof. In the terms of the operator E_f we have that

$x = E_f(x)$,

$y \leq E_f(y)$

$w(y\mid_{[a-h,a]}, y\mid_{[b,b+h]}) \leq w(x\mid_{[a-h,h]}, x\mid_{[b,b+h]})$.

On the other hand, from the condition (c), using Lemma 2.1 we have that the operator E_f^∞ is monotone increasing.

From this using Lemma 2.3 we have that

$$y \leq E_f^\infty(y) = E_f^\infty(\widetilde{w}(y)) \leq E_f^\infty(\widetilde{w}(x)) = x,$$

where, for $z \in X$,

$$\widetilde{w}(z) = \begin{cases} z(t) & , \quad t \in [a-h,a] \\ w(z\mid_{[a-h,h]}, z\mid_{[b,b+h]}) & , \quad t \in [a,b] \\ z(t) & , \quad t \in [b,b+h] \end{cases}$$

5. Data dependence: Monotony

Now we shall study the monotony of the solutions of the equation (1) with respect to initial conditions.We have

Theorem 5.1. *Let $f_i \in C([a,b]\times\mathbb{R}^3,\mathbb{R})$,$i=\overline{1,3}$ be as in the Theorem 3.1.We suppose that*

(a) $f_2(t,\cdot,\cdot,\cdot) : \mathbb{R}^3 \longrightarrow \mathbb{R}$ *is monotone increasing;*

(b) $f_1 \leq f_2 \leq f_3$;

Let x_i, be a solution of the equation

$$-x''(t) = f_i(t, x(t), \int_{t-h}^{t} x(s)ds, \int_{t}^{t+h} x(s)ds), \quad i=\overline{1,3}.$$

If

$$x_1(t) \leq x_2(t) \leq x_3(t), (\forall)t \in [a-h,a] \cup [b,b+h]$$

then

$$x_1 \leq x_2 \leq x_3.$$

Proof. The operators E_{f_i} are $W.P.O.s$. From the condition (a) the operator E_{f_2} is monotone increasing. From (b) it follows that $E_{f_1} \leq E_{f_2} \leq E_{f_3}$.We remark that $x_i = E_{f_i}^{\infty}(\widetilde{w}(x_i))$, $i = \overline{1,3}$.

Now the proof follows from Lemma 2.2.

Theorem 5.2. *We consider the equation (1) under conditions of the Theorem 3.1. Let x, y be two solutions of the equations (1).We suppose that f is monotone increasing. If*

$$x(t) \leq y(t), \quad (\forall) t \in [a-h, a] \cup [b, b+h],$$

then

$$x \leq y,$$

on $[a-h, b+h]$.

Proof. The operator E_f is $W.P.O$. Because f is monotone increasing we obtain that E_f is monotone increasing. From Lemma 2.1 we have that E_f^{∞} is increasing. It follows that $E_f^{\infty}(\widetilde{w}(x)) \leq E_f^{\infty}(\widetilde{w}(y))$ and $x \leq y$.

6. **Data dependence: continuity**

Next, for $i = \overline{1,2}$, we consider the equations:

$$-x''(t) = f_i(t, x(t), \int_{t-h}^{t} x(s)ds, \int_{t}^{t+h} x(s)ds). \tag{5}$$

Theorem 6.1. *Let f_1 and f_2 be as in the Theorem 3.1.Let S_i be the solutions set of the equation (5) corresponding to $f_i, i = \overline{1,2}$.*

If $\eta > 0$ is such that

$$|f_1(t, u, v, w) - f_2(t, u, v, w)| \leq \eta,$$

for all $t \in [a, b], u, v, w \in \mathbb{R}$,

then

$$H(S_1, S_2) \leq \frac{\eta(b-a)^2}{8 - L(b-a)^2(1+2h)}$$

where $L := \max\{L_{f_1}, L_{f_2}\}$.

Proof. In the conditions of the Theorem 3.1 the operators $E_{f_i}, i = \overline{1,2}$ are c_i- $W.P.O.s$, with

$$c_i = (1-\alpha_i)^{-1}$$

where,

$$\alpha_i = \frac{1}{8} \cdot L_{f_i}(b-a)^2(1+2h).$$

From

$$|E_{f_1}(x)(t) - E_{f_2}(x)(t)| \leq$$

$$\leq \int_a^b G(t,s)|f_1(s,x(s),\int_{s-h}^{s} x(u)du, \int_s^{s+h} x(u)du) - f_2(s,x(s),\int_{s-h}^{s} x(u)du, \int_s^{s+h} x(u)du)|ds \leq$$

$$\leq \eta \int_a^b G(t,s)ds \leq \eta \frac{(b-a)^2}{8},$$

using the Theorem 2.2, we have the conclusions.

7. Smooth dependence on parameters

Consider the following boundary value problem with parameter

$$-x''(t) = f(t,x(t),\int_{t-h}^{t} x(s)ds, \int_t^{t+h} x(s)ds;\lambda), t \in [a,b], \lambda \in J \tag{6}$$

$$\begin{cases} x(t) = \varphi(t) & , \quad t \in [a-h,a] \\ x(t) = \psi(t) & , \quad t \in [b,b+h] \end{cases}. \tag{7}$$

We suppose that

(C_1) $J \subseteq \mathbb{R}$, a compact interval;

(C_2) $f \in C^1([a,b] \times \mathbb{R}^3 \times J, \mathbb{R})$;

(C_3) There exists $L_f > 0$ such that:

$$|\frac{\partial f}{\partial u_i}(t,u_1,u_2,u_3;\lambda)| \leq L_f,$$

for all $t \in [a,b], u_i \in \mathbb{R}, i = \overline{1,3}$.

(C_4) $\varphi \in C([a-h,a]), \psi \in C([b,b+h])$.

(C_5) $\frac{1}{8}L_f(b-a)^2 < 1$

In the above conditions from Theorem 3.1 we have that the problem (6)+(7) has a unique solution, $x^\star(\cdot;\lambda)$.

Now we prove that $x^\star(t,\cdot) \in C^1(J)$.For this we consider the equation

$$-x''(t,\lambda) = f(t, x(t,\lambda), \int_{t-h}^{t} x(s,\lambda)ds, \int_{t}^{t+h} x(s,\lambda)ds; \lambda), \tag{8}$$

for all $t \in [a,b], \lambda \in J, x \in C([a-h,b+h] \times J)$.

The problem (8)+(7) is equivalent with

$$x(t,\lambda) = \begin{cases} \varphi(t), & t \in [a-h,a], \lambda \in J \\ w(\varphi,\psi)(t) + \int_a^b G(t,s) f(s, x(s,\lambda), & \\ \int_{s-h}^{s} x(u,\lambda)du, \int_{s}^{s+h} x(u,\lambda)du)ds, & t \in [a,b], \lambda \in J \\ x(t), & t \in [b,b+h], \lambda \in J \end{cases} . \tag{9}$$

We consider the operator

$$B : C([a-h,b+h] \times J) \longrightarrow C([a-h,b+h] \times J),$$

where

$B(x)(t) =$second part of (9).

Let $X := C([a-h,b+h] \times J)$ and let, $\|\cdot\|$, be the Chebyshev norm on X. It is clear that in the condition $(C_1)-(C_5)$ the operator B is Picard operator.

Let $x^\star$ be the unique fixed point of B.We suppose that there exists $\dfrac{\partial x^\star}{\partial \lambda}$.Then for (9) we have that

$$\frac{\partial x^\star}{\partial \lambda}(t,\lambda) =$$

$$\begin{cases} 0, & t\in[a-h,a],\lambda\in J\\ \int\limits_a^b G(t,s)\frac{\partial f}{\partial u_1}(s,x^\star(s,\lambda),\int\limits_{s-h}^{s}x^\star(u,\lambda)du,\int\limits_{s}^{s+h}x^\star(u,\lambda)du)\cdot & \\ \cdot\frac{\partial x^\star}{\partial\lambda}(s,\lambda)ds+ & \\ +\int\limits_a^b G(t,s)\frac{\partial f}{\partial u_2}(s,x^\star(s,\lambda),\int\limits_{s-h}^{s}x^\star(u,\lambda)du,\int\limits_{s}^{s+h}x^\star(u,\lambda)du)\cdot & \\ \cdot\int\limits_{s-h}^{s}\frac{\partial x^\star}{\partial\lambda}(u,\lambda)duds+ & \\ +\int\limits_a^b G(t,s)\frac{\partial f}{\partial u_3}(s,x^\star(s,\lambda),\int\limits_{s-h}^{s}x^\star(u,\lambda)du,\int\limits_{s}^{s+h}x^\star(u,\lambda)du)\cdot & \\ \cdot\int\limits_{s}^{s+h}\frac{\partial x^\star}{\partial\lambda}(u,\lambda)duds+ & \\ +\int\limits_a^b G(t,s)\frac{\partial f}{\partial \lambda}(s,x^\star(s,\lambda),\int\limits_{s-h}^{s}x^\star(u,\lambda)du,\int\limits_{s}^{s+h}x^\star(u,\lambda)du)ds, & t\in[a,b],\lambda\in J\\ 0, & t\in[b,b+h] \end{cases}$$

This relation suggest us to consider the following operator

$$C: X\times X\longrightarrow X$$

$$(x,y)\longrightarrow C(x,y),$$

where

$$C(x,y)(t,\lambda)=$$

$$\begin{cases} 0, & t\in[a-h,a],\lambda\in J\\ \int\limits_a^b G(t,s)\frac{\partial f}{\partial u_1}(s,x(s,\lambda),\int\limits_{s-h}^{s}x(u,\lambda)du,\int\limits_{s}^{s+h}x(u,\lambda)du)\cdot & \\ \cdot y(s,\lambda)ds+ & \\ +\int\limits_a^b G(t,s)\frac{\partial f}{\partial u_2}(s,x(s,\lambda),\int\limits_{s-h}^{s}x(u,\lambda)du,\int\limits_{s}^{s+h}x(u,\lambda)du)\cdot & \\ \cdot\int\limits_{s-h}^{s}y(u,\lambda)duds+ & \\ +\int\limits_a^b G(t,s)\frac{\partial f}{\partial u_3}(s,x(s,\lambda),\int\limits_{s-h}^{s}x(u,\lambda)du,\int\limits_{s}^{s+h}x(u,\lambda)du)\cdot & \\ \cdot\int\limits_{s}^{s+h}y(u,\lambda)duds+ & \\ +\int\limits_a^b G(t,s)\frac{\partial f}{\partial \lambda}(s,x(s,\lambda),\int\limits_{s-h}^{s}x(u,\lambda)du,\int\limits_{s}^{s+h}x(u,\lambda)du)ds, & t\in[a,b],\lambda\in J\\ 0, & t\in[b,b+h],\lambda\in J \end{cases}$$

In this way we have that the operator

$$A : X \times X \longrightarrow X \times X$$

$$(x, y) \longrightarrow (B(x), C(x, y)),$$

where B is Picard operator and $C(x, \cdot) : X \longrightarrow X$ is a $\alpha-$ contraction, with

$$\alpha = L_f(1 + 2h)\frac{(b-a)^2}{8}.$$

From the theorem of fibre contraction(see [1],[5]) we have that the operator A is a Picard operator.So the sequences

$$x_{n+1} = B(x_n),$$

$$y_{n+1} = C(x_n, y_n),$$

converges uniformly (with respect to $t \in [a-h, b+h], \lambda \in J$) to $(x^\star, y^\star) \in F_A$, for all $x_0, y_0 \in C([a-h, b+h] \times J)$.

If we take, $x_0 = 0$, $y_0 = \dfrac{\partial x_0}{\partial \lambda} = 0$, then, $y_1 = \frac{\partial x_1}{\partial \lambda}$.

By induction,we prove that

$$y_n = \frac{\partial x_n}{\partial \lambda}, (\forall) n \in \mathbb{N}$$

Thus

$$x_n \longrightarrow x^\star, \; as \; n \longrightarrow \infty, \quad uniformly,$$

$$\frac{\partial x_n}{\partial \lambda} \longrightarrow y^\star \; as \; n \longrightarrow \infty, \quad uniformly.$$

These imply that there exists $\frac{\partial x^\star}{\partial \lambda}$ and, $\frac{\partial x^\star}{\partial \lambda} = y^\star$.

From the above consideration, we have that

Theorem 7.1. *Consider the problem (7)+(8) in the conditions* $(C_1) - (C_5)$.*Then*

(a) *The problem, (7)+(8), has in* $C([a-h, b+h])$ *a unique solution* $x^\star$.

(b) $x^\star(t, \cdot) \in C^1(J), (\forall) t \in [a-h, b+h]$.

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STUDIA UNIV. "BABEŞ–BOLYAI", MATHEMATICA, Volume **LI**, Number 2, June 2006

A NEW MONTE CARLO ESTIMATOR FOR SYSTEMS OF LINEAR EQUATIONS

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Abstract. We propose a new Monte Carlo estimator to solve systems of linear equations. We formulate and prove some results concerning the quality and the properties of this estimator. Using this estimator, we give error bounds and construct confidence intervals for the components of the solution. We also consider numerical examples. The numerical results indicate that the proposed estimator converges faster than another two estimators from the literature.

1. **Introduction**

Let us consider the system of linear algebraic equations:

$$x = Tx + c, \tag{1}$$

where $T = (t_{ij})_{i,j=1}^{n} \in \mathbb{R}^{n\times n}$, $c = (c_1, \ldots, c_n)^t \in \mathbb{R}^n$ and $I - T$ is an invertible matrix. The solution $x = (x_1, \ldots, x_n)^t \in \mathbb{R}^n$ of system (1) is unique and admits the Neumann series representation:

$$x = c + Tc + T^2c + T^3c + \ldots$$

or, detailed,

$$x_i = c_i + (Tc)_i + (T^2c)_i + \ldots, \quad i = 1, \ldots, n. \tag{2}$$

We assume that $\sum_{j=1}^{n} |t_{ij}| < 1$, $i = 1, \ldots, n$, which is a sufficient condition for the convergence of Neumann series to the solution.

Received by the editors: 10.01.2006.

2000 *Mathematics Subject Classification.* 65C05.

Key words and phrases. Monte Carlo method, linear systems, estimation.

Monte Carlo methods estimate the solution of system (1), by constructing unbiased estimators for the components of the solution (see [4], [5], [10]). Let $P = (p_{ij})_{i,j=1}^{n+1} \in \mathbb{R}^{(n+1)\times(n+1)}$ be a matrix, whose elements satisfy the conditions:

1. $p_{ij} \geq 0$ such that $t_{ji} \neq 0 \Longrightarrow p_{ij} \neq 0$,
2. $\sum_{j=1}^{n} p_{ij} \leq 1, \quad i = 1, \ldots, n$,
3. $p_{i,n+1} = 1 - \sum_{j=1}^{n} p_{ij}, \quad i = 1, \ldots, n$,
4. $p_{n+1,j} = 0,\ j < n+1$,
5. $p_{n+1,n+1} = 1$.

The notation p_i is also used to denote $p_{i,n+1}$. The matrix P describes a Markov chain with the set of states $\{1, \ldots, n+1\}$, where $n+1$ is an absorbing state and p_{ij}, $i, j = 1, \ldots, n+1$, is the one step transition probability from state i to state j.

Define the weights:

$$w_{ij} = \begin{cases} \frac{t_{ji}}{p_{ij}} & \text{if } p_{ij} \neq 0 \\ 0 & \text{if } p_{ij} = 0 \end{cases}, \quad i, j = 1, \ldots, n.$$

Denote by $\gamma = (i_0, i_1, \ldots, i_k, n+1)$ a trajectory that starts at the initial state $i_0 < n+1$ and passes successfully through the sequence of states $(i_1, \ldots, i_k)$, to finally get into the absorbing state $i_{k+1} = n+1$.

Consider a vector $\alpha = (\alpha_1, \ldots, \alpha_n)$, where α_i, $i = 1, \ldots, n$, is the probability that a trajectory starts in state i, i.e.,

$$P(i_0 = i) = \alpha_i, \qquad \alpha_i \geq 0,\ i = 1, \ldots, n, \qquad \sum_{i=1}^{n} \alpha_i = 1.$$

The probability to follow trajectory γ is $P(\gamma) = \alpha_{i_0} p_{i_0 i_1} \ldots p_{i_{k-1} i_k} p_{i_k}$.

Define the estimators θ_i, $i = 1, \ldots, n$, and λ_i, $i = 1, \ldots, n$, on the space of trajectories as follows. For a trajectory $\gamma = (i_0, i_1, \ldots, i_k, n+1)$, the values of these estimators are defined as:

$$\theta_i(\gamma) = W_k(\gamma) \frac{\delta_{i_k i}}{p_{i_k}}, \qquad \lambda_i(\gamma) = \sum_{m=0}^{k} W_m(\gamma) \delta_{i_m i}, \qquad i = 1, \ldots, n,$$

where $W_m,\ m = 0, \dots, k$, are random variables whose values are:

$$\begin{aligned} W_0(\gamma) &= \frac{c_{i_0}}{\alpha_{i_0}}, \\ W_m(\gamma) &= W_{m-1}(\gamma) w_{i_{m-1} i_m} \\ &= \frac{c_{i_0}}{\alpha_{i_0}} w_{i_0 i_1} w_{i_1 i_2} \dots w_{i_{m-1} i_m}, \qquad m = 1, \dots, k. \end{aligned}$$

These values are taken with probability $P(\gamma)$ (δ_{ij} is the Kronecker symbol, i.e., $\delta_{ij} = 1$ if $i = j$ and 0 otherwise).

It is proved in [8] that θ_i and λ_i are unbiased estimators of x_i, i.e., $E(\theta_i) = E(\lambda_i) = x_i,\ i = 1, \dots, n$.

For some particular systems, the variances of the estimators θ_i and λ_i are analytically compared in [6]. In [7], the complexity of the Monte Carlo method is calculated, when certain techniques to generate the trajectories of the Markov chain are used.

2. A new estimator

Definition 1. *We define the estimator U_i, $i = 1, \dots, n$, on the space of trajectories as follows. For an arbitrary trajectory $\gamma = (i_0, i_1, \dots, i_k, n+1)$, the value of U_i is defined as:*

$$U_i(\gamma) = c_i + W_k(\gamma) \frac{t_{i i_k}}{p_{i_k}}, \qquad i = 1, \dots, n,$$

and is taken with probability $P(\gamma) = \alpha_{i_0} p_{i_0 i_1} \dots p_{i_{k-1} i_k} p_{i_k}$.

Remark 2. *The distribution of the estimator U_i, $i = 1, \dots, n$, is:*

$$U_i : \begin{pmatrix} c_i + W_k(\gamma) \frac{t_{i i_k}}{p_{i_k}} \\ \alpha_{i_0} p_{i_0 i_1} \dots p_{i_{k-1} i_k} p_{i_k} \end{pmatrix}_{\substack{\gamma = (i_0, i_1, \dots, i_k, n+1) \\ i_0, i_1, \dots, i_k = 1, \dots, n}}.$$

Next, we formulate and prove some main results concerning the quality and the properties of the estimator U_i.

Theorem 3. *The expectation of U_i is equal to the component x_i of the solution of system (1), i.e.,*

$$E(U_i) = x_i, \quad i = 1, \dots, n. \tag{3}$$

In other words, U_i is an unbiased estimator of x_i, $i = 1, \ldots, n$.

Proof. We can write:

$$\begin{aligned}
E(U_i) &= \sum_{\gamma=(i_0,\ldots,i_k,n+1)} U_i(\gamma)P(\gamma) \\
&= \sum_{\gamma=(i_0,\ldots,i_k,n+1)} \left(c_i + W_k(\gamma)\frac{t_{ii_k}}{p_{i_k}}\right)P(\gamma) \\
&= \sum_{\gamma=(i_0,\ldots,i_k,n+1)} c_i P(\gamma) + \sum_{\gamma=(i_0,\ldots,i_k,n+1)} W_k(\gamma)\frac{t_{ii_k}}{p_{i_k}}P(\gamma) \\
&= c_i + \sum_{\gamma=(i_0,\ldots,i_k,n+1)} \frac{c_{i_0}}{\alpha_{i_0}} w_{i_0 i_1} \ldots w_{i_{k-1} i_k} \frac{t_{ii_k}}{p_{i_k}} \alpha_{i_0} p_{i_0 i_1} \ldots p_{i_{k-1} i_k} p_{i_k} \\
&= c_i + \sum_{\gamma=(i_0,\ldots,i_k,n+1)} c_{i_0} \frac{t_{i_1 i_0}}{p_{i_0 i_1}} \ldots \frac{t_{i_k i_{k-1}}}{p_{i_{k-1} i_k}} t_{ii_k} p_{i_0 i_1} \ldots p_{i_{k-1} i_k} \\
&= c_i + \sum_{k=0}^{\infty} \sum_{i_0=1}^{n} \ldots \sum_{i_k=1}^{n} t_{ii_k} t_{i_k i_{k-1}} \ldots t_{i_1 i_0} c_{i_0} \\
&= c_i + (Tc)_i + (T^2 c)_i + \ldots \\
&= x_i.
\end{aligned}$$

In the last equality, we used relation (2). ∎

Proposition 4. *The following relationship between the estimators U_i and θ_i holds:*

$$U_i = c_i + \sum_{j=1}^{n} \theta_j t_{ij}, \qquad i = 1, \ldots, n.$$

Proof. For any trajectory γ, we can write:

$$\begin{aligned}
U_i(\gamma) &= c_i + W_k(\gamma)\frac{t_{ii_k}}{p_{i_k}} = c_i + \sum_{j=1}^{n} W_k(\gamma)\frac{\delta_{i_k j}}{p_{i_k}} t_{ij} \\
&= c_i + \sum_{j=1}^{n} \theta_j(\gamma) t_{ij}, \qquad i = 1, \ldots, n.
\end{aligned}$$

∎

Theorem 5. *The following relationship between the variance of U_i and the variance of θ_i holds:*

$$Var(U_i) = \sum_{j=1}^{n} t_{ij}^2 Var(\theta_j) + \sum_{j<l} 2t_{ij}t_{il}Cov(\theta_j, \theta_l). \tag{4}$$

Proof. Using the result from Proposition 4 and some known properties of the variance, we can write:

$$\begin{aligned}
Var(U_i) &= Var\Big(c_i + \sum_{j=1}^{n} \theta_j t_{ij}\Big) \\
&= \sum_{j=1}^{n} Var\Big(t_{ij}\theta_j\Big) + \sum_{j<l} 2Cov(t_{ij}\theta_j, t_{il}\theta_l) \\
&= \sum_{j=1}^{n} t_{ij}^2 Var(\theta_j) + \sum_{j<l} 2t_{ij}t_{il}Cov(\theta_j, \theta_l).
\end{aligned}$$

□

Practically, to solve system (1), we generate N independent trajectories $\gamma_1, \ldots, \gamma_N$ and for each trajectory we compute the value of the estimator U_i. The values $U_i(\gamma_j)$, $j = 1, \ldots, N$, are values of the sample variables $U_{i1}, \ldots, U_{iN}$ that are independent identically distributed random variables and have the same distribution as U_i.

We use the notation $\overline{U}_{i,N}$ for the sample mean of the random variables U_{ij}, $j = 1, \ldots, N$, and $\overline{u}_{i,N}$ for its value, i.e.:

$$\overline{U}_{i,N} = \frac{\sum_{j=1}^{N} U_{ij}}{N}, \qquad \overline{u}_{i,N} = \frac{\sum_{j=1}^{N} U_i(\gamma_j)}{N}. \tag{5}$$

Proposition 6. *The estimator $\overline{U}_{i,N}$, $i = 1, \ldots, n$, has the following properties:*

$$E(\overline{U}_{i,N}) = x_i, \quad \text{(unbiased estimator of } x_i\text{)}, \tag{6}$$

$$\lim_{N\to\infty} Var(\overline{U}_{i,N}) = 0, \tag{7}$$

$$P(\lim_{N\to\infty} \overline{U}_{i,N} = x_i) = 1, \quad (\overline{U}_{i,N} \text{ converges almost surely to } x_i). \tag{8}$$

Proof. Properties (6) and (7) can be proved using known properties of the mean and variance. For property (8), we apply the Kolmogorov theorem ([1]) to the sequence of random variables $(U_{iN})_{N\geq 1}$ that are independent identically distributed and have finite means $E(U_{iN}) = x_i < \infty$. Under these conditions, the Kolmogorov theorem asserts that relation (8) is satisfied. $\square$

Taking into account these properties, the component x_i is approximated by:

$$x_i \approx \overline{u}_{i,N} = \frac{1}{N}\sum_{j=1}^{N} U_i(\gamma_j), \quad i = 1, \ldots, n. \tag{9}$$

The estimate of the solution is:

$$x_U = \left[\frac{1}{N}\sum_{j=1}^{N} U_1(\gamma_j), \ldots, \frac{1}{N}\sum_{j=1}^{N} U_n(\gamma_j)\right]^t. \tag{10}$$

Similar estimates x_θ and x_λ can be obtained by replacing the estimator U_i, $i = 1, \ldots, n$, by θ_i and λ_i respectively, i.e.,

$$x_\theta = \left[\frac{1}{N}\sum_{j=1}^{N} \theta_1(\gamma_j), \ldots, \frac{1}{N}\sum_{j=1}^{N} \theta_n(\gamma_j)\right]^t, \tag{11}$$

$$x_\lambda = \left[\frac{1}{N}\sum_{j=1}^{N} \lambda_1(\gamma_j), \ldots, \frac{1}{N}\sum_{j=1}^{N} \lambda_n(\gamma_j)\right]^t. \tag{12}$$

Remark 7. *The variance $Var(U_i)$ is in general unknown. It can be estimated using an unbiased estimation of it, given by the sample variance:*

$$\overline{\sigma}_{U,i}^2 = \frac{1}{N-1}\sum_{j=1}^{N}(U_{ij} - \overline{U}_{i,N})^2. \tag{13}$$

Remark 8. *Comparing the variances of estimators U_i and θ_i can be done either analytically (using, eventually, the result from Theorem 5) or experimentally. Experimentally, we can use the same N generated trajectories γ_j, $j = 1, \ldots, N$, and compute the values $\theta_i(\gamma_j)$, $j = 1, \ldots, N$. Let $\theta_{i1}, \ldots, \theta_{iN}$ be the corresponding sample variables. We use the same notation $\overline{\theta}_{i,N}$ for the sample mean of the random variables θ_{ij}, $j = 1, \ldots, N$, and respectively for its value, i.e.,*

$$\overline{\theta}_{i,N} = \frac{\sum_{j=1}^{N} \theta_{ij}}{N}, \qquad \overline{\theta}_{i,N} = \frac{\sum_{j=1}^{N} \theta_i(\gamma_j)}{N}.$$

We estimate $Var(\theta_i)$ by the following unbiased estimator:

$$\overline{\sigma}^2_{\theta,i} = \frac{1}{N-1}\sum_{j=1}^{N}(\theta_{ij} - \overline{\theta}_{i,N})^2.$$

Comparing the variances $Var(U_i)$ and $Var(\theta_i)$ reduces to comparing their estimations $\overline{\sigma}^2_{U,i}$ and $\overline{\sigma}^2_{\theta,i}$.

3. Error estimation

We evaluate (estimate) the error in formula (9). One way of doing this is by using the Chebyshev inequality ([1]). We have the following main result concerning the error:

Proposition 9. *The following estimation of the error of approximation of x_i holds:*

$$P\left(\left|\overline{U}_{i,N} - x_i\right| < \frac{\sigma(U_i)}{\sqrt{N\gamma}}\right) \geq 1-\gamma, \quad \gamma \in (0,1),$$

where $\sigma(U_i)$ is the standard deviation of U_i, i.e. $\sigma^2(U_i) = Var(U_i)$.

Proof. The proof is immediately, by applying the Chebyshev inequality for the estimator $\overline{U}_{i,N}$ and choosing $\varepsilon = \frac{\sigma(U_i)}{\sqrt{N\gamma}}$. □

Another modality of estimating the error is based on the Lindeberg's limit theorem ([1]). In this case, we have the following main result:

Proposition 10. *The following estimation of the error of approximation of x_i holds:*

$$P\left(\left|\overline{U}_{i,N} - x_i\right| < \lambda\frac{\sigma(U_i)}{\sqrt{N}}\right) \approx 2\phi(\lambda) - 1, \quad \lambda > 0,$$

where

$$\phi(\lambda) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\lambda} e^{-\frac{t^2}{2}}\,dt,$$

is the Laplace function.

Proof. The proof is immediately, by applying the Lindeberg's limit theorem to the sequence of random variables $(U_{iN})_{N\geq 1}$ that are independent and identically distributed and have the same distribution as U_i. □

4. **Confidence intervals**

We construct confidence intervals for x_i, $i = 1, \dots, n$. We consider the confidence level $\alpha \in (0, 1)$.

Proposition 11. *A* $(1-\alpha)\%$ *confidence interval for* x_i *is:*

$$\Bigg(\overline{U}_{i,N} - t_{N-1,1-\frac{\alpha}{2}} \frac{\overline{\sigma}_{U,i}}{\sqrt{N}}, \quad \overline{U}_{i,N} + t_{N-1,1-\frac{\alpha}{2}} \frac{\overline{\sigma}_{U,i}}{\sqrt{N}}\Bigg). \tag{14}$$

where $\overline{U}_{i,N}$ *is defined in (5),* $t_{N-1,1-\frac{\alpha}{2}}$ *is the* $(1-\frac{\alpha}{2})$*-th percentile of the Student distribution with* $N-1$ *degrees of freedom, and* $\overline{\sigma}_{U,i}$ *is the sample standard deviation (*$\overline{\sigma}^2_{U,i}$ *is defined in (13)).*

Proof. We consider the statistics:

$$T = \frac{\overline{U}_{i,N} - x_i}{\frac{\overline{\sigma}_{U,i}}{\sqrt{N}}},$$

that has the t (Student) distribution with $N-1$ degrees of freedom. We take $t_2 = t_{N-1,1-\frac{\alpha}{2}}$, $t_1 = -t_2$, i.e.,

$$F_{N-1}(t_2) = 1 - \frac{\alpha}{2}, \qquad F_{N-1}(t_1) = \frac{\alpha}{2},$$

where F_{N-1} is the distribution function of the t distribution with $N-1$ degrees of freedom. We have $P(t_1 < T < t_2) = 1 - \alpha$, which is equivalent to:

$$P\Bigg(\overline{U}_{i,N} - t_{N-1,1-\frac{\alpha}{2}} \frac{\overline{\sigma}_{U,i}}{\sqrt{N}} < x_i < \overline{U}_{i,N} + t_{N-1,1-\frac{\alpha}{2}} \frac{\overline{\sigma}_{U,i}}{\sqrt{N}}\Bigg) = 1 - \alpha.$$

Thus, a $(1-\alpha)\%$ confidence interval for x_i is given by (14). □

5. **Numerical example**

We consider the system:

$$\begin{cases} x_1 = 0.1x_1 + 0.5x_2 + 0.4 \\ x_2 = 0.3x_1 + 0.1x_2 + 0.6 \end{cases}$$

with the exact solution $x = (1, 1)$.

We choose the matrix P of the following form:

$$P = \begin{bmatrix} 0.1 & 0.3 & 0.6 \\ 0.5 & 0.1 & 0.4 \\ 0 & 0 & 1 \end{bmatrix}.$$

The matrix P describes a Markov chain with the set of states $\{1, 2, 3\}$, where state 3 is the absorbing one. As $p_{ij} = t_{ji}$, $i, j = 1, 2$, we have $w_{ij} = 1$, $i, j = 1, 2$. Since $c_1, c_2 \geq 0$ and $c_1 + c_2 = 1$, we take the vector $\alpha = c^t = (0.4, 0.6)$.

In order to get the initial state $i_0 \in \{1, 2\}$ of an arbitrary trajectory, we sample from the following discrete distribution:

$$Y_\alpha : \begin{pmatrix} 1 & 2 \\ \alpha_1 & \alpha_2 \end{pmatrix}.$$

Once the trajectory is in state $i_m = i \in \{1, 2\}$, we sample from the distribution:

$$Y_i : \begin{pmatrix} 1 & 2 & 3 \\ p_{i1} & p_{i2} & p_i \end{pmatrix},$$

described by the i-th line of matrix P, in order to determine the next state i_{m+1}. We repeat this procedure till absorbtion takes place. The sampling method is the inversion method ([2], [3]).

We generate N trajectories and we calculate the estimates x_θ, x_λ, x_U using formulas (11), (12) and (10), respectively. The following table contains: the number N of trajectories generated, the estimates x_θ, x_λ, x_U and the euclidian norm of the errors $\|x - x_\theta\|$, $\|x - x_\lambda\|$, $\|x - x_U\|$.

N	x_θ	x_λ	x_U	$\|x - x_\theta\|$	$\|x - x_\lambda\|$	$\|x - x_U\|$
5000	(0.9853 , 1.0220)	(0.9768, 0.9968)	(1.0095, 0.9978)	0.0264	0.0234	0.0098
10000	(0.9897, 1.0155)	(0.9859, 0.9948)	(1.0067, 0.9985)	0.0186	0.0150	0.0069
15000	(0.9939 , 1.0092)	(0.9875, 0.9930)	(1.0040, 0.9991)	0.0110	0.0144	0.0041
50000	(0.9945, 1.0083)	(0.9942, 0.9979)	(1.0036, 0.9992)	0.0100	0.0061	0.0037
100000	(0.9987 , 1.0020)	(0.9977, 0.9994)	(1.0009, 0.9998)	0.0024	0.0023	0.0009

The numerical results indicate that the proposed estimate x_U converges faster than the estimations x_θ and x_λ.

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STUDIA UNIV. "BABEŞ–BOLYAI", MATHEMATICA, Volume **LI**, Number 2, June 2006

BOUNDARY VALUE PROBLEMS FOR ITERATIVE FUNCTIONAL-DIFFERENTIAL EQUATIONS

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Abstract. We consider the following boundary value problem

$$\begin{aligned} -x''(t) &= f(t, x(t), x(x(t))), \quad t \in [a, b]; \\ x(t) &= \alpha(t), \quad a_1 \le t \le a, \\ x(t) &= \beta(t), \quad b \le t \le b_1. \end{aligned}$$

Using the weakly Picard operators technique we establish an existence and uniqueness theorem and some data dependence results.

1. **Introduction**

By an iterative functional-differential equation we understand an equation of the following type (see [1]–[5], [7], [9], [12]–[14])

$$x'(t) = f(t, x(t), \dots, x^m(t)),\ t \in J \subset \mathbb{R}$$

or (see [6], [8])

$$x''(t) = f(t, x(t), \dots, x^m(t)),\ t \in J \subset \mathbb{R}$$

where $x^k(t) := (x \circ x \circ \cdots \circ x)(t),\ k \in \mathbb{N}$.

The purpose of this paper is to study the following boundary value problem

$$-x''(t) = f(t, x(t), x(x(t))), \quad t \in [a, b]; \tag{1.1}$$

$$\begin{cases} x(t) = \alpha(t) & t \in [a_1, a], \\ x(t) = \beta(t) & t \in [b, b_1], \end{cases} \tag{1.2}$$

Received by the editors: 23.01.2006.

2000 *Mathematics Subject Classification.* 47H10, 34K10, 34K20.

Key words and phrases. iterative functional-differential equation, boundary value problem, weakly Picard operator, fibre Picard operator, data dependence.

where

(C_1) $a_1 \le a < b \le b_1$;

(C_2) $f \in C([a,b] \times [a_1,b_1]^2)$;

(C_3) $\alpha \in C([a_1,a],[a_1,b_1])$ and $\beta \in C([b,b_1],[a_1,b_1])$;

(C_4) there exists $L_f > 0$ such that:

$$|f(t,u_1,u_2) - f(t,v_1,v_2)| \le L_f \left(|u_1 - v_1| + |u_2 - v_2|\right),$$

for all $t \in [a,b]$, $u_i, v_i \in [a_1,b_1]$, $i = 1,2$.

By a solution of the problem (1.1)–(1.2) we understand a function $x \in C^2([a,b],[a_1,b_1]) \cap C([a_1,b_1],[a_1,b_1])$ which satisfies (1.1)–(1.2).

The problem (1.1)–(1.2) is equivalent with the following fixed point equation

$$x(t) = \begin{cases} \alpha(t), & t \in [a_1,a], \\ w(\alpha,\beta)(t) + \int_a^b G(t,s) f(s,x(s),x(x(s)))\,\mathrm{d}s, & t \in [a,b], \\ \beta(t), & t \in [b,b_1], \end{cases} \tag{1.3}$$

and $x \in C([a_1,b_1],[a_1,b_1])$, where

$$w(\alpha,\beta)(t) := \frac{t-a}{b-a}\beta(b) + \frac{b-t}{b-a}\alpha(a),$$

and G is the Green function of the problem

$$-x'' = \chi,\ x \in C[a,b]\ x(a) = 0,\ x(b) = 0.$$

On the other hand, the equation (1.1) is equivalent with

$$x(t) = \begin{cases} x(t), & t \in [a_1,a], \\ w(x|_{[a_1,a]}, x|_{[b,b_1]})(t) + \int_a^b G(t,s) f(s,x(s),x(x(s)))\,\mathrm{d}s, & t \in [a,b], \\ x(t), & t \in [b,b_1], \end{cases} \tag{1.4}$$

and $x \in C([a_1,b_1],[a_1,b_1])$.

In this paper we apply the weakly Picard operators technique to study the equations (1.3) and (1.4).

2. Weakly Picard operators

In this paper we need some notions and results from the weakly Picard operator theory (for more details see I. A. Rus [10] and [11]).

Let (X, d) be a metric space and $A : X \to X$ an operator. We shall use the following notations:

$F_A := \{x \in X \mid A(x) = x\}$- the fixed point set of A;

$I(A) := \{Y \subset X \mid A(Y) \subset Y, Y \neq \emptyset\}$- the family of the nonempty invariant subsets of A;

$A^{n+1} := A \circ A^n, \quad A^1 = A, \quad A^0 = 1_X, \quad n \in \mathbb{N}$;

$P(X) := \{Y \subset X | Y \neq \emptyset\}$;

$H(Y,Z) := \max\left\{\sup_{y\in Y}\inf_{z\in Z} d(y,z), \sup_{z\in Z}\inf_{y\in Y} d(y,z)\right\}$ -the Pompeiu–Hausdorff functional on $P(X) \times P(X)$.

Definition 2.1. *Let (X, d) be a metric space. An operator $A : X \to X$ is a Picard operator (PO) if there exists $x^* \in X$ such that:*

(i) $F_A = \{x^*\}$*;*

(ii) *the sequence $(A^n(x_0))_{n\in\mathbb{N}}$ converges to x^* for all $x_0 \in X$.*

Theorem 2.1 (Contraction principle). *Let (X, d) be a complete metric space and $A : X \to X$ a γ-contraction. Then*

(i) $F_A = \{x^*\}$,

(ii) $(A^n(x_0))_{n\in\mathbb{N}}$ *converges to x^* for all $x_0 \in X$,*

(iii) $d(x^*, A^n(x_0)) \leq \dfrac{\gamma^n}{1-\gamma}\, d(x_0, A(x_0))$*, for all $n \in \mathbb{N}$.*

Remark 2.1. *Accordingly to the definition, the contraction principle insures that, if $A : X \to X$ is a γ-contraction on the complet metric space X, then it is a Picard operator.*

Theorem 2.2. *Let (X, d) be a complete metric space and $A, B : X \to X$ two operators. We suppose that*

(i) *the operator A is a γ-contraction;*

(ii) $F_B \neq \emptyset$;

(iii) *there exists* $\eta > 0$ *such that*

$$d(A(x), B(x)) \leq \eta, \ \forall \ x \in X.$$

Then if $F_A = \{x_A^*\}$ *and* $x_B^* \in F_B$, *we have*

$$d(x_A^*, x_B^*) \leq \frac{\eta}{1-\gamma}.$$

Definition 2.2. *Let* (X, d) *be a metric space. An operator* A *is a weakly Picard operator (WPO) if the sequence* $(A^n(x))_{n\in\mathbb{N}}$ *converges for all* $x \in X$, *and its limit (which may depend on* x*) is a fixed point of* A.

Theorem 2.3. *Let* (X, d) *be a metric space and* $A : X \to X$ *an operator. The operator* A *is weakly Picard operator if and only if there exists a partition of* X,

$$X = \bigcup_{\lambda\in\Lambda} X_\lambda,$$

where Λ *is the indices' set of partition, such that*

(a) $X_\lambda \in I(A)$, *for all* $\lambda \in \Lambda$;

(b) $A|_{X_\lambda} : X_\lambda \to X_\lambda$ *is a Picard operator for all* $\lambda \in \Lambda$.

Definition 2.3. *If* A *is weakly Picard operator then we consider the operator* A^∞ *defined by*

$$A^\infty : X \to X, \quad A^\infty(x) := \lim_{n\to\infty} A^n(x).$$

It is clear that

$$A^\infty(X) = F_A \text{ and } \omega_A(x) = \{A^\infty(x)\},$$

where $\omega_A(x)$ is the ω-limit point set of A.

Definition 2.4. *Let* A *be a weakly Picard operator and* $c > 0$. *The operator* A *is* c*–weakly Picard operator if*

$$d(x, A^\infty(x)) \leq c\, d(x, A(x)), \ \forall \ x \in X.$$

Example 2.1. *Let* (X, d) *be a complete metric space and* $A : X \to X$ *a continuous operator. We suppose that there exists* $\gamma \in [0, 1)$ *such that*

$$d(A^2(x), A(x)) \leq \gamma\, d(x, A(x)), \ \forall \ x \in X.$$

Then A is c–weakly Picard operator with $c = \dfrac{1}{1-\gamma}$.

Theorem 2.4. *Let (X, d) be a metric space and $A_i : X \to X,\ i = 1, 2$. Suppose that*

(i) *the operator A_i is c_i–weakly Picard operator, $i = 1, 2$;*

(ii) *there exists $\eta > 0$ such that*

$$d(A_1(x), A_2(x)) \le \eta,\ \forall\ x \in X.$$

Then

$$H(F_{A_1}, F_{A_2}) \le \eta \max(c_1, c_2).$$

Theorem 2.5 (Fibre contraction principle). *Let (X, d) and (Y, ρ) be two metric spaces and $A : X \times X \to X \times Y,\ A = (B, C),\ (B : X \to X, C : X \times Y \to Y)$ a triangular operator. We suppose that*

(i) *(Y, ρ) is a complete metric space;*

(ii) *the operator B is PO;*

(iii) *there exists $l \in [0, 1)$ such that $C(x, \cdot) : Y \to Y$ is l-contraction, for all $x \in X$;*

(iv) *if $(x^*, y^*) \in F_A$, then $C(\cdot, y^*)$ is continuous in x^*.*

Then the operator A is PO.

3. Boundary value problem

In what follows we consider the fixed point equation (1.3). Let

$$B_f : C([a_1, b_1], [a_1, b_1]) \to C([a_1, b_1], \mathbb{R}),$$

where $B_f(x)(t) :=$ the right hand side of (1.3). Let $L > 0$ and

$$C_L([a_1, b_1], [a_1, b_1]) := \{x \in C([a_1, b_1], [a_1, b_1]) | |x(t_1) - x(t_2)| \le L|t_1 - t_2|,$$

$\forall\ t_1, t_2 \in [a_1, b_1]\}$.

It is clear that $C_L([a_1, b_1], [a_1, b_1])$ is a complete metric space with respect to the metric,

$$d(x_1, x_2) := \max_{a_1 \le t \le b_1} |x_1(t) - x_2(t)|.$$

We have

Theorem 3.1. *We suppose that*

(i) *the conditions* $(C_1)-(C_4)$ *are satisfied;*

(ii) $\alpha\in C_L([a_1,a],[a_1,b_1])$, $\beta\in C_L([b,b_1],[a_1,b_1])$;

(iii) m_f *and* $M_f\in\mathbb{R}$ *are such that* $m_f\le f(t,u_1,u_2)\le M_f$, $\forall\ t\in[a,b]$, $u_i\in[a_1,b_1]$, $i=1,2$, *and moreover,*

$$\begin{aligned}
a_1 &\le \min(\alpha(a),\beta(b))+m_f\frac{(b-a)^2}{8}, && \text{for } m_f<0,\\
a_1 &\le \min(\alpha(a),\beta(b)), && \text{for } m_f\ge 0,\\
b_1 &\ge \max(\alpha(a),\beta(b)), && \text{for } M_f\le 0,\\
b_1 &\ge \max(\alpha(a),\beta(b))+M_f\frac{(b-a)^2}{8}, && \text{for } M_f>0,
\end{aligned}$$

and

$$\frac{|\beta(b)-\alpha(a)|}{b-a}+|M_f|\frac{a^2+b^2-6ab}{2(b-a)}\le L;$$

(iv) $\dfrac{(b-a)^2}{8}L_f(L+2)<1$.

Then the boundary value problem (1.1)–(1.2) *has, in* $C_L([a_1,b_1],[a_1,b_1])$, *a unique solution. Moreover, the operator*

$$B_f: C_L([a_1,b_1],[a_1,b_1])\to C_L([a_1,b_1],C_L([a_1,b_1],[a_1,b_1]))$$

is a c–Picard operator with $c=\dfrac{8}{8-(b-a)^2L_f(L+2)}$.

Proof. First of all we remark that the condition (iii) implies that $C_L([a_1,b_1],[a_1,b_1])$ is an invariant subset for B_f. Indeed, we have $a_1\le B_f(x)(t)\le b_1$, $x(t)\in[a_1,b_1]$ for all $t\in[a,b]$. Actually, using the positivity of the Green function, for m_f and $M_f\in\mathbb{R}$ such that

$$m_f\le f(t,u_1,u_2)\le M_f,\ \forall\ t\in[a,b],\ u_i\in[a_1,b_1],\ i=1,2,$$

we have

$$G(t,s)m_f\le G(t,s)f(s,x(s),x(x(s)))\le G(t,s)M_f,\ \forall\ t\in[a,b].$$

This implies that

$$\int_a^b G(t,s)m_f \,\mathrm{d}s \le \int_a^b G(t,s)f(s,x(s),x(x(s)))\,\mathrm{d}s \le \int_a^b G(t,s)M_f\,\mathrm{d}s,\ \forall\ t\in[a,b],$$

that is,

$$w(\alpha,\beta)(t)+m_f\int_a^b G(t,s)\,\mathrm{d}s \le B_f(x)(t) \le w(\alpha,\beta)(t)+M_f\int_a^b G(t,s)\,\mathrm{d}s,\ \forall\ t\in[a,b].$$

It is easy to see that,

$$\min_{t\in[a,b]}\int_a^b G(t,s)\,\mathrm{d}s = \min_{t\in[a,b]}\frac{(t-a)(b-t)}{2} = 0$$

and

$$\max_{t\in[a,b]}\int_a^b G(t,s)\,\mathrm{d}s = \max_{t\in[a,b]}\frac{(t-a)(b-t)}{2} = \frac{(b-a)^2}{8}.$$

Therefore, if condition (iii) holds, we have satisfied the invariance property for the operator B_f in $C([a_1,b_1],[a_1,b_1])$.

Now, consider $t_1,t_2\in[a_1,a]$. Then,

$$|B_f(x)(t_1)-B_f(x)(t_2)| = |\alpha(t_1)-\alpha(t_2)| \le L|t_1-t_2|,$$

because of $\alpha\in C_L([a_1,a],[a_1,b_1])$.

Similarly, for $t_1,t_2\in[b,b_1]$

$$|B_f(x)(t_1)-B_f(x)(t_2)| = |\beta(t_1)-\beta(t_2)| \le L|t_1-t_2|,$$

that follows from (ii), too.

On the other hand, if $t_1,t_2\in[a,b]$, we have,

$$|B_f(x)(t_1)-B_f(x)(t_2)| =$$

$$= \left|w(\alpha,\beta)(t_1)-w(\alpha,\beta)(t_2)+\int_a^b [G(t_1,s)-G(t_2,s)]f(s,x(s),x(x(s)))\mathrm{d}s\right| =$$

$$= \left|\frac{t_1-t_2}{b-a}\,(\beta(b)-\alpha(a))+\int_a^b [G(t_1,s)-G(t_2,s)]f(s,x(s),x(x(s)))\mathrm{d}s\right| \le$$

$$\le \left|\frac{\beta(b)-\alpha(a)}{b-a}(t_1-t_2)\right| + \left|\int_a^b [G(t_1,s)-G(t_2,s)]f(s,x(s),x(x(s)))\mathrm{d}s\right| \le$$

$$\le \left|\frac{\beta(b)-\alpha(a)}{b-a}\right||t_1-t_2| + |M_f|\left|\int_a^b [G(t_1,s)-G(t_2,s)]\mathrm{d}s\right|.$$

But,

$$\int_a^b [G(t_1,s)-G(t_2,s)]\mathrm{d}s = \int_a^{t_1}\left[\frac{(s-a)(b-t_1)}{b-a}-\frac{(s-a)(b-t_2)}{b-a}\right]\mathrm{d}s+ \\ +\int_{t_1}^{t_2}\left[\frac{(t_1-a)(b-s)}{b-a}-\frac{(s-a)(b-t_2)}{b-a}\right]\mathrm{d}s+ \\ +\int_{t_2}^{b}\left[\frac{(t_1-a)(b-s)}{b-a}-\frac{(t_2-a)(b-s)}{b-a}\right]\mathrm{d}s.$$

After some calculation we obtain,

$$\int_a^b [G(t_1,s)-G(t_2,s)]\mathrm{d}s = \left[(a-b)(t_1+t_2)-a^2-4ab+b^2\right]\frac{t_1-t_2}{2(b-a)}.$$

Thus,

$$\left|\int_a^b [G(t_1,s)-G(t_2,s)]\mathrm{d}s\right| \le \frac{a^2+b^2-6ab}{2(b-a)}|t_1-t_2|.$$

So, we can affirm that

$$|B_f(x)(t_1)-B_f(x)(t_2)| \le \left[\frac{|\beta(b)-\alpha(a)|}{b-a}+|M_f|\frac{a^2+b^2-6ab}{2(b-a)}\right]|t_1-t_2|,$$

$\forall\ t_1,t_2\in[a,b],\ t_1\le t_2$, and due to (iii), $B_f(x)$ is L–Lipschitz.

Thus, according to the above, we have $C_L([a_1,b_1],[a_1,b_1])\in I(B_f)$.

From the condition (iv) it follows that, B_f is an L_{B_f}–contraction, with

$$L_{B_f} := \frac{(b-a)^2}{8}L_f(L+2).$$

Indeed, for all $t\in[a_1,a]\cup[b,b_1]$, we have $|B_f(x_1)(t)-B_f(x_2)(t)|=0$.

Otherwise, for $t \in [a,b]$

$$
\begin{aligned}
&|B_f(x_1)(t) - B_f(x_2)(t)| = \\
&= \left| \int_a^b G(t,s)\left[f(s,x_1(s),x_1(x_1(s))) - f(s,x_2(s),x_2(x_2(s)))\right] \mathrm{ds} \right| \leq \\
&\leq \max_{x\in[a,b]} \left| \int_a^b G(t,s)\mathrm{ds} \right| L_f \left(|x_1(s) - x_2(s)| + |x_1(x_1(s)) - x_2(x_2(s))| \right) \leq \\
&\leq \frac{(b-a)^2}{8} L_f \left(\|x_1 - x_2\|_C + |x_1(x_1(s)) - x_1(x_2(s))| + |x_1(x_2(s)) - x_2(x_2(s))| \right) \leq \\
&\leq \frac{(b-a)^2}{8} L_f \left(\|x_1 - x_2\|_C + L|x_1(s) - x_2(s)| + \|x_1 - x_2\|_C \right) \leq \\
&\leq \frac{(b-a)^2}{8} L_f (L+2) \|x_1 - x_2\|_C \,.
\end{aligned}
$$

So, B_f is a c–Picard operator, with $c = \dfrac{1}{1 - L_{B_f}}$. □

In what follows, consider the following operator

$$E_f : C_L([a_1,b_1],[a_1,b_1]) \to C_L([a_1,b_1],[a_1,b_1]),$$

where

$$E_f(x)(t) := \text{the right hand side of (1.4).}$$

Theorem 3.2. *In the conditions of the Theorem 3.1, the operator*

$$E_f : C_L([a_1,b_1],[a_1,b_1]) \to C_L([a_1,b_1],[a_1,b_1])$$

is WPO.

Proof. The operator E_f is a continuous operator but it is not a contraction operator.

Let take the following notation:

$$X_{\alpha,\beta} := \{x \in C([a_1,b_1],[a_1,b_1]) \,|\, x|_{[a_1,a]} = \alpha,\ x|_{[b,b_1]} = \beta\}.$$

Then we can write

$$C_L([a_1,b_1],[a_1,b_1]) = \bigcup_{\substack{\alpha\in C_L([a_1,a],[a_1,b_1]) \\ \beta\in C_L([b,b_1],[a_1,b_1])}} X_{\alpha,\beta}. \tag{3.5}$$

We have that $X_{\alpha,\beta} \in I(E_f)$ and $E_f|_{X_{\alpha,\beta}}$ is a Picard operator, because it is the operator which appears in the proof of the Theorem 3.1.

By applying the Theorem 2.3, we obtain that E_f is WPO. $\square$

4. **Increasing solutions of** (1.1)

4.1. **Inequalities of Čaplygin type.** We have

Theorem 4.1. *We suppose that*

(a) *the conditions of the Theorem 3.1 are satisfied;*

(b) $u_i, v_i \in [a_1, b_1],\ u_i \leq v_i,\ i = 1, 2,$ *imply that*

$$f(t, u_1, u_2) \leq f(t, v_1, v_2),$$

for all $t \in [a, b]$.

Let x *be a increasing solution of the equation* (1.1) *and* y *an increasing solution of the inequality*

$$-y''(t) \leq f(t, y(t), y(y(t))), \quad t \in [a, b].$$

Then

$$y(t) \leq x(t),\ \forall\, t \in [a_1, a] \cup [b, b_1] \Rightarrow y \leq x.$$

Proof. In the terms of the operator E_f, we have

$$x = E_f(x) \text{ and } y \leq E_f(y),$$

and

$$w(y|_{[a_1,a]}, y|_{[b,b_1]}) \leq w(x|_{[a_1,a]}, x|_{[b,b_1]}).$$

However, from the condition (b), we have that the operator E_f^∞ is increasing (see Lemma 7.1 in [11]), we have

$$y \leq E_f^\infty(y) = E_f^\infty(\widetilde{w}(y)) \leq E_f^\infty(\widetilde{w}(x)) = x,$$

thus $y \leq x$. Here, for $z \in C[a,b]$, we used the notation

$$\widetilde{w}(z)(t) := \begin{cases} z(z), t \in [a_1, a], \\ w(z|_{[a_1,a]}, z|_{[b,b_1]})(t)\, t \in [a,b], \\ z(b), t \in [b, b_1]. \end{cases}$$

□

4.2. **Comparison theorem.** In what follows we want to study the monotony of the solution of the problem (1.1)–(1.2), with respect to α, β and f. We will use the result below:

Lemma 4.1 (Abstract comparison lemma). *Let $(X, d, \leq)$ be an ordered metric space and $A, B, C : X \to X$ be such that:*

(i) $A \leq B \leq C$;
(ii) *the operators A, B, C are weakly Picard operators;*
(iii) *the operator B is increasing.*

Then

$$x \leq y \leq z \Rightarrow A^{\infty}(x) \leq B^{\infty}(y) \leq C^{\infty}(z).$$

In this case we can establish the theorem

Theorem 4.2. *Let $f_i \in C([a,b] \times [a_1, b_1]^2)$, $i = 1, 2, 3$. We suppose that*

(a) $f_2(t, \cdot, \cdot) : [a_1, b_1]^2 \to [a_1, b_1]^2$ *is increasing;*
(b) $f_1 \leq f_2 \leq f_3$.

Let x_i be a increasing solution of the equation

$$-x'' = f_i(t, x(t), x(x(t))), \quad t \in [a,b].$$

If

$$x_1(t) \leq x_2(t) \leq x_3(t), \ \forall\, t \in [a_1, a] \cap [b, b_1],$$

then

$$x_1 \leq x_2 \leq x_3.$$

Proof. The operators E_{f_i}, $i = 1, 2$ are weakly Picard operators. Taking into consideration the condition (a) the operator E_{f_2} is increasing. From (b) we have that

$$E_{f_1} \leq E_{f_2} \leq E_{f_3}.$$

We note that $x_i = E_{f_i}^{\infty}(\widetilde{w}(x_i))$, $i = 1, 2$. Now, using the Abstract comparison lemma, the proof is complete.

□

5. **Data dependence: continuity**

Consider the boundary value problem (1.1)–(1.2) and suppose the conditions of the Theorem 3.1 are satisfied. Denote by $x(\cdot; \alpha, \beta, f)$ the solution of this problem. We can state the following result:

Theorem 5.1. *Let α_i, β_i, f_i, $i = 1, 2$, be as in the Theorem 3.1. Furthermore, we suppose that*

(i) *there exists $\eta_1 > 0$, such that*

$$|\alpha_1(t) - \alpha_2(t)| \leq \eta_1, \ \forall \ t \in [a_1, a],$$

and

$$|\beta_1(t) - \beta_2(t)| \leq \eta_1, \ \forall t \in [b, b_1];$$

(ii) *there exists $\eta_2 > 0$ such that*

$$|f_1(t, u_1, u_2) - f_2(t, u_1, u_2)| \leq \eta_2, \ \forall \ t \in [a, b], \ \forall \ u_i \in [a_1, b_1], \ i = 1, 2.$$

Then

$$|x(t; \alpha_1, \beta_1, f_1) - x(t; \alpha_2, \beta_2, f_2)| \leq \frac{8\eta_1 + \eta_2(b-a)^2}{8 - L_f(L+2)(b-a)^2}$$

where $L_f = \max(L_{f_1}, L_{f_2})$.

Proof. Consider the operators B_{α_i,β_i,f_i}, $i = 1, 2$. From Theorem 3.1 these operators are contractions. Additionally,

$$||B_{\alpha_1,\beta_1,f_1}(x) - B_{\alpha_2,\beta_2,f_2}(x)||_C =$$

$$= \left| [w(\alpha_1, \beta_1)(t) - w(\alpha_2, \beta_2)(t)] + \int_a^b G(t,s) \left[f_1(s, x(s), x(x(s))) - f_2(s, x(s), x(x(s))) \right] \mathrm{d}s \right| \leq$$

$$\leq \left| \frac{t-a}{b-a} [\beta_1(b) - \beta_2(b)] + \frac{b-t}{b-a} [\alpha_1(a) - \alpha_2(a)] \right| + \max_{t \in [a,b]} \left| \int_a^b G(t,s) \mathrm{d}s \right| \eta_2$$

$$\leq \eta_1 + \eta_2 \frac{(b-a)^2}{8},$$

$\forall\ x \in C_L([a_1, b_1], [a_1, b_1])$.

Now, the proof follows from the Theorem 2.2, with

$$A := B_{\alpha_1, \beta_1, f_1}, \quad B := B_{\alpha_2, \beta_2, f_2}, \quad \eta := \eta_1 + \eta_2 \frac{(b-a)^2}{8}$$

and

$$\gamma := L_A = \frac{(b-a)^2}{8} L_f (L+2).$$

□

From the theorem above we have

Theorem 5.2. *Let α_i, β_i, f_i, $i \in \mathbb{N}$ and α, β, f be as in the Theorem 3.1. We suppose that*

$$\alpha_i \overset{univ.}{\longrightarrow} \alpha \text{ as } i \to \infty,$$

$$\beta_i \overset{univ.}{\longrightarrow} \beta \text{ as } i \to \infty,$$

$$f_i \overset{univ.}{\to} f \text{ as } i \to \infty.$$

Then

$$x(\cdot, \alpha_i, \beta_i, f_i) \overset{univ.}{\longrightarrow} x(\cdot, \alpha, \beta, f), \text{ as } i \to \infty.$$

Theorem 5.3. *Let f_1 and f_2 be as in the Theorem 3.1. Let $F_{E_{f_i}}$ be the solution set of equation* (1.1) *corresponding to f_i, $i = 1, 2$. Suppose that there exists $\eta > 0$ such that*

$$|f_1(t, u_1, u_2) - f_2(t, u_1, u_2)| \leq \eta, \tag{5.6}$$

for all $t \in [a, b]$, $u_i \in [a_1, b_1]$, $i = 1, 2$. Then

$$H_{||\cdot||_C}(F_{E_{f_1}}, F_{E_{f_2}}) \leq \frac{\eta (b-a)^2}{8 - L_f (L+2)(b-a)^2}$$

where $L_f := \max(L_{f_1}, L_{f_2})$ and $H_{||\cdot||_C}$ denotes the Pompeiu–Hausdorff functional with respect to $||\cdot||_C$ on $C_L([a_1, b_1], [a_1, b_1])$.

Proof. We will look for those c_i, for which in condition of the Theorem 3.1 the operators E_{f_i}, $i = 1, 2$, are c_i- weakly Picard operators.

Let $X_{\alpha,\beta} := \{x \in C_L([a_1, b_1], [a_1, b_1]) \mid x|_{[a_1,a]} = \alpha, x|_{[b,b_1]} = \beta\}$

It is clear that $E_{f_i}|_{X_{\alpha,\beta}} = B_{f_i}$. So, from Theorem 2.3 and Theorem 3.1 we have

$$||E_{f_i}^2(x) - E_{f_i}(x)||_C \leq L_{f_i}(L+2)\frac{(b-a)^2}{8}||E_{f_i}(x) - x||_C$$

for all $x \in C_L([a_1, b_1], [a_1, b_1])$, $i = 1, 2$.

Now, choosing $\lambda_i = \dfrac{(b-a)^2}{8}L_{f_i}(L+2)$, we get that E_{f_i} are c_i- weakly Picard operators, with $c_i = (1 - \lambda_i)^{-1}$.

From (5.6) we obtain that

$$||E_{f_1}(x) - E_{f_2}(x)||_C \leq \eta\frac{(b-a)^2}{8}, \text{ for all } x \in C_L([a_1, b_1], [a_1, b_1]).$$

Applying Theorem 2.4 we have that

$$H_{||\cdot||_C}(F_{E_{f_1}}, F_{E_{f_2}}) \leq \frac{\eta(b-a)^2}{8 - L_f(L+2)(b-a)^2}.$$

□

6. Data dependence: differentiability

Consider the following boundary value problem with parameter

$$-x''(t) = f(t, x(t), x(x(t)); \lambda), \quad t \in [a, b]; \tag{6.7}$$

$$\begin{cases} x(t) = \alpha(t) & t \in [a_1, a], \\ x(t) = \beta(t) & t \in [b, b_1]. \end{cases} \tag{6.8}$$

Suppose that we have satisfied the following conditions:

(P_1) $a_1 \leq a < b \leq b_1$; $J \subset \mathbb{R}$, a compact interval;

(P_2) $\alpha \in C_L^1([a_1, a], [a_1, b_1])$ and $\beta \in C_L^1([b, b_1], [a_1, b_1])$;

(P_3) $f \in C^1([a, b] \times [a_1, b_1]^2 \times J)$;

(P_4) there exists $L_f > 0$ such that

$$\left|\frac{\partial f(t, u_1, u_2; \lambda)}{\partial u_i}\right| \leq L_f,$$

for all $t \in [a, b]$, $u_i \in [a_1, b_1]$, $i = 1, 2$, $\lambda \in J$;

(P_5) m_f and $M_f \in \mathbb{R}$ are such that $m_f \le f(t,u_1,u_2) \le M_f,\ \forall\, t \in [a,b],\ u_i \in [a_1,b_1],\ i=1,2$, moreover we have

$$\begin{aligned} a_1 &\le \min(\alpha(a),\beta(b)) + m_f\frac{(b-a)^2}{8},\ \textit{for } m_f < 0,\\ a_1 &\le \min(\alpha(a),\beta(b)), \qquad\qquad\qquad \textit{for } m_f \ge 0,\\ b_1 &\ge \max(\alpha(a),\beta(b)), \qquad\qquad\qquad \textit{for } M_f \le 0,\\ b_1 &\ge \max(\alpha(a),\beta(b)) + M_f\frac{(b-a)^2}{8},\ \textit{for } M_f > 0, \end{aligned}$$

and

$$\frac{|\beta(b)-\alpha(a)|}{b-a} + |M_f|\frac{a^2+b^2-6ab}{2(b-a)} \le L;$$

(P_6) $\dfrac{(b-a)^2}{8}L_f(L+2) < 1.$

Then, from the Theorem 3.1, we have that the problem (6.7)–(6.8) has a unique solution, $x^*(\cdot,\lambda)$.

We will prove that $x^*(t,\cdot) \in C^1(J)$, for all $t \in [a_1,b_1]$.

For this, we consider the equation

$$\begin{aligned} -x''(t;\lambda) =& f(t,x(t;\lambda),x(x(t;\lambda);\lambda);\lambda),\ t\in[a,b],\ \lambda\in J,\\ & x \in C([a_1,b_1]\times J,[a_1,b_1]\times J)\cap C^2([a,b]\times J,[a_1,b_1]\times J). \end{aligned} \tag{6.9}$$

The problem (6.9)–(6.8) is equivalent with the following functional-integral equation

$$x(t;\lambda) = \begin{cases} \alpha(t), t\in[a_1,a], \quad \lambda\in J,\\ w(\alpha,\beta)(t) + \int_a^b G(t,s)f(s,x(s;\lambda),x(x(s;\lambda);\lambda);\lambda)\,\mathrm{d}s,\ t\in[a,b],\ \lambda\in J\\ \beta(t), t\in[b,b_1], \quad \lambda\in J. \end{cases} \tag{6.10}$$

Now, let take the operator

$$B : C_L([a_1,b_1]\times J,[a_1,b_1]\times J) \to C_L([a_1,b_1]\times J,[a_1,b_1]\times J),$$

where $B(x)(t;\lambda) :=$ the right hand side of (6.10).

Let $X := C_L([a_1,b_1]\times J,[a_1,b_1])$. It is clear from the proof of the Theorem 3.1 that in the conditions $(P_1)-(P_6)$, the operator $B : (X,\|\cdot\|_C) \to (X,\|\cdot\|_C)$

is a PO. Let x^* be the unique fixed point of B. We consider the subset $X_1 \subset X$, $X_1 := \left\{x \in X |\ \frac{\partial x}{\partial t} \in C[a_1, b_1]\right\}$. We remark that $x^* \in X_1$, $B(X_1) \subset X_1$ and $B : (X_1, \|\cdot\|_C) \to (X_1, \|\cdot\|_C)$ is PO. Let $Y := C([a_1, b_1] \times J)$.

Supposing that there exists $\frac{\partial x^*}{\partial \lambda}$, from (6.10) we have that

$$\frac{\partial x^*(t;\lambda)}{\partial \lambda} = \int_a^b G(t,s) \frac{\partial f(s, x^*(s;\lambda), x^*(x^*(s;\lambda);\lambda);\lambda)}{\partial u_1} \cdot \frac{\partial x^*(s;\lambda)}{\partial \lambda}\,\mathrm{d}s +$$

$$+ \int_a^b G(t,s) \frac{\partial f(s, x^*(s;\lambda), x^*(x^*(s;\lambda);\lambda);\lambda)}{\partial u_2} \cdot$$

$$\cdot \left[\frac{\partial x^*(x^*(s;\lambda);\lambda)}{\partial u_1} \cdot \frac{\partial x^*(s;\lambda)}{\partial \lambda} + \frac{\partial x^*(x^*(s;\lambda);\lambda)}{\partial \lambda}\right] \mathrm{d}s +$$

$$+ \int_a^b G(t,s) \frac{\partial f(s, x^*(s;\lambda), x^*(x^*(s;\lambda);\lambda);\lambda)}{\partial \lambda}\,\mathrm{d}s, \quad t \in [a,b],\ \lambda \in J.$$

This relation suggest us to consider the following operator

$$C : X_1 \times Y \to Y$$

$$(x, y) \mapsto C(x, y)$$

with

$$C(x,y)(t;\lambda) := \int_a^b G(t,s) \frac{\partial f(s, x(s;\lambda), x(x(s;\lambda);\lambda);\lambda)}{\partial u_1} \cdot y(s;\lambda)\,\mathrm{d}s +$$

$$+ \int_a^b G(t,s) \frac{\partial f(s, x(s;\lambda), x(x(s;\lambda);\lambda);\lambda)}{\partial u_2} \cdot$$

$$\cdot \left[\frac{\partial x(x(s;\lambda);\lambda)}{\partial u_1} \cdot y(s;\lambda) + \frac{\partial x(x(s;\lambda);\lambda)}{\partial \lambda}\right] \mathrm{d}s +$$

$$+ \int_a^b G(t,s) \frac{\partial f(s, x(s;\lambda), x(x(s;\lambda);\lambda);\lambda)}{\partial \lambda}\,\mathrm{d}s, \quad t \in [a,b],\ \lambda \in J$$

and

$$C(x,y)(t,\lambda) := 0, \text{ for } t \in [a_1, a] \cup [b, b_1],\ \lambda \in J.$$

In this way we have the triangular operator

$$A : X_1 \times Y \to X_1 \times Y$$

$$(x, y) \mapsto (B(x), C(x, y)),$$

where B is a Picard operator and $C(x,\cdot) : Y \to Y$ is an L_C- contraction, with $L_C = \frac{(b-a)^2}{8}\widetilde{L}_f(L+2)$, where $\widetilde{L}_f = max(L_f, LL_f)$.

From the fibre contraction theorem we have that the operator A is Picard operator, i.e. the sequences

$$x_{n+1} := B(x_n),$$

$$y_{n+1} := C(x_n, y_n), \quad n \in \mathbb{N}$$

converges uniformly, with respect to $t \in [a_1, b_1]$, $\lambda \in J$, to $(x^*, y^*) \in F_A$, for all $x_0 \in X_1$, $y_0 \in Y$.

If we take $x_0 = 0$, $y_0 = \frac{\partial x_0}{\partial \lambda} = 0$, then $y_1 = \frac{\partial x_1}{\partial \lambda}$.

By induction we prove that $y_n = \frac{\partial x_n}{\partial \lambda}$, $\forall\, n \in \mathbb{N}$.

So,

$$x_n \overset{unif.}{\longrightarrow} x^* \text{ as } n \to \infty,$$

$$\frac{\partial x_n}{\partial \lambda} \to y^* \text{ as } n \to \infty.$$

From these we have that there exists $\frac{\partial x^*}{\partial \lambda}$ and $\frac{\partial x^*}{\partial \lambda} = y^*$.

Taking into consideration the above, we can formulate the theorem

Theorem 6.1. *Consider the problem* (6.9)–(6.8)*, and suppose the conditions* $(P_1) - (P_6)$ *holds. Then,*

(i) (6.9)–(6.8) *has a unique solution,* x^*, *in* $C([a_1, b_1] \times J, [a_1, b_1])$,

(ii) $x^*(t,\cdot) \in C^1(J)$, $\forall\, t \in [a_1, b_1]$.

Remark 6.1. *By the same arguments we have that, if* $f(t,\cdot,\cdot) \in C^k$, *then* $x^*(t,\cdot) \in C^k(J)$, $\forall\, t \in [a_1, b_1]$.

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STUDIA UNIV. "BABEŞ–BOLYAI", MATHEMATICA, Volume **LI**, Number 2, June 2006

SOME INTEGRAL OPERATORS DEFINED ON p-VALENT FUNCTION BY USING HYPERGEOMETRIC FUNCTIONS

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Abstract. In the present paper we introduce some integral operators and verify the effect of these operators on p-valent functions and find radii of starlikeness and convexity for these operators, finally we introduce the concept of neighborhood.

1. **Introduction and Definitions**

Let $\mathcal{A}$ denote the family of functions analytic in unit disc $\Delta = \{z \in \mathbb{C} : |z| < 1\}$ with positive coefficient and let $\mathcal{A}_p$ be subclass of a consisting functions $f(z)$ of the form

$$f(z) = mz^p + \sum_{n=p-1}^{2p-1} t_{n-p+1} z^{n-p+1} - {}_2F_1(a, b; c; z), \; |z| < 1 \tag{1.1}$$

$$\text{where } {}_2F_1(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a, n)(b, n)}{(c, n)n!} z^n$$

$$(a, n) = \frac{\Gamma(a + n)}{\Gamma(a)} = a(a + 1, n - 1), \; c > b > 0, c > a + b, m > 0$$

$$\text{and } t_{n-p+1} = \frac{(a, n - p + 1)(b, n - p + 1)}{(c, n - p + 1)(n - p + 1)!}.$$

These functions are analytic in the punctured unit disk. For more details on hypergeometric functions ${}_2F_1(a, b; c; z)$ see [4] and [7].

Let $f \in \mathcal{A}$, then we denote by UCV^p the class of uniformly convex p-valent function in Δ and $\alpha - ST$ the class of α - starlike functions also denote by $\alpha - UCV^p$

Received by the editors: 11.01.2002.

2000 *Mathematics Subject Classification.* 30C45, 30C50.

Key words and phrases. Uniformly convex functions, fractional derivative, starlike and convex p-valent, hypergeometric, neighborhood.

the class of α-uniformly convex p-valent function in Δ which are introduced and investigated by Kanas, Wiśniwoska [6] and Silverman [10].

Definition 1. Let $f \in \mathcal{A}_P$ and $0 \leq \alpha < \infty$. Then $f \in \alpha - UCV^p$ if and only if

$$Re\left\{p + \frac{zf''}{f'}\right\} > \alpha\left|\frac{zf''}{f'}\right| \ z \in \Delta.$$

Definition 2. Let $f \in \mathcal{A}_p$. The class α - uniformly starlike functions $\alpha - UST^p$ is defined as

$$\alpha - UST^p = \left\{f \in \mathcal{A} : Re\left(\frac{zf'}{f}\right) > \alpha\left|\frac{zf'}{f} - p\right|, \ \alpha \geq 0, \ z \in \Delta\right\}$$

Definition 3. (cf. [7]; see also [11] and [12]). Let the function f be of the form $f(z) = z^p - \sum\limits_{n=2}^{\infty} a_n z^n$ and be analytic in Δ. The fractional derivative of f of order δ is defined by

$$D_z^\delta f(z) = \frac{1}{\Gamma(1-\delta)}\frac{d}{dz}\int_0^z \frac{f(\xi)}{(z-\xi)^\delta}d\xi \ (0 \leq \delta < 1) \tag{1.2}$$

where the multiplicity of $(z-\xi)^\delta$ is removed by requiring $\log(z-\xi)$ to be real when $z - \xi > 0$ and so we have

$$D_z^\delta f(z) = \frac{m}{\Gamma(2-\delta)}z^{p-\delta} - \sum_{n=2}^{\infty}\frac{\Gamma(n+p)}{\Gamma(n+p-\delta)} \ a_n z^{n-\delta}. \tag{1.3}$$

Making use of (1.2) and its known extensions involving fractional derivatives and fractional integrals, Owa and Srivastava [11] introduced the operator

$$\Omega_z^\delta f(z) := \Gamma(2-\delta)z^\delta D_z^\delta f(z), \ 0 \leq \delta < 1 \tag{1.4}$$

and for $\delta = 0$ we have $\Omega_z^0 f(z) = f(z)$.

Definition 4. Let $f(z) \in \mathcal{A}_p$ is said to be a member of the $\alpha - UCV_\delta^p(\eta, \phi)$ if $f(z)$ satisfies the inequality

$$\begin{aligned} &Re\left(\frac{z(\Omega_z^\delta f(z))' + \eta z^2(\Omega_z^\delta f(z))''}{(1-\eta)(\Omega_z^\delta f) + \eta z(\Omega_z^\delta f(z))'}\right) \\ &\geq \alpha\left|\frac{z(\Omega_z^\delta f(z))' + z^2(\Omega_z^\delta f)''}{(1-\eta)\Omega_z^\delta f(z) + \eta z(\Omega_z^\delta f(z))'} - 1\right| + \tan\phi \end{aligned} \tag{1.5}$$

where $0 \leq \eta \leq 1, 0 \leq \tan\phi < p, p \in \mathbb{N}, \alpha \geq 0$ and $0 \leq \delta < 1$.

We note that by specializing the parameters $\alpha, \phi, \eta, \delta$ we obtain the following subclasses studied by various authors (by putting $\tan\phi = \beta$).

(I) If $\alpha = 0, \delta = 0$ and $p = 1 \Rightarrow \alpha - UCV(\eta, 0) \equiv p_1(1, \lambda, \beta)$ was studied by Altintas [1].

(II) If $\eta = 1, \delta = 0, \alpha = 0, p = 1 \Rightarrow \alpha - UCV(1, \phi) \equiv C(\beta)$ was studied by Silverman [10].

(III) If $\eta = 0, \delta =, 0, p = 1 \Rightarrow \alpha - UCV(0, \phi) \equiv UCT(k, \beta)$ was studied by R. Bharati, R. Parvatham and A. Swaminathan [5].

(IV) If $p = 1, \eta = 0$ and $\beta = 0$ and $\delta = 0$, that is $k - ST$ introduced by Kanas and Wiśniowsak [6].

2. **Main Results**

In the first theorem we will obtain coefficient bounds, before it we need the following lemmas.

Lemma 1. *Let $w = u + iv$ then*

$$Re(w) \geq \alpha \Leftrightarrow |w - (1 + \alpha)| \leq |w + (1 - \alpha)|.$$

Lemma 2. *Let $w = u + iv$ and α, β be real numbers. Then*

$$Re(w) > \alpha|w - 1| + \beta \Leftrightarrow Re\{w(1 + \alpha e^{i\theta}) - \alpha e^{i\theta}\} > \beta.$$

Theorem 1. *The function $f(z)$ defined by (1.1) is in the class $\alpha - UCV^p_\delta(\eta, \phi)$ if and only if*

$$\sum_{n=p+1}^{\infty} \gamma^p(n, \delta)[(1 - \eta + n\eta)(n(1 + \alpha) - (\alpha + \tan\phi))]k_n \leq m(1 + \eta p - \eta)(p - \tan\phi + \alpha(p - 1)) \qquad (2.1)$$

where $\gamma^p(n, \delta) = \frac{\Gamma(2-\delta)\Gamma(n+p)}{\Gamma(n+p-\delta)}$ and $0 \leq \tan\phi < p, \alpha \geq 0, 0 \leq \eta \leq 1, p \in \mathbb{N}$ and $0 \leq \delta < 1$.

Proof. The function $f(z)$ in $\mathcal{A}_p$ can be expressed in the form

$$f(z) = mz^p - \sum_{n=p+1}^{\infty} k_n z^n, \ p \in \mathbb{N} \qquad (2.2)$$

such that $k_n = \frac{\Gamma(a+n)\Gamma(b+n)\Gamma(c)}{\Gamma(a)\Gamma(b)\Gamma(c+n)\Gamma(n+1)}$ $n \geq p+1$. Also

$$\begin{aligned} \Omega_z^\delta f(z) &= \Gamma(2-\delta) z^\delta D_z^\delta f(z) = mz^p - \sum_{n=p+1}^{\infty} \frac{\Gamma(n+p)}{\Gamma(n+p-\delta)} k_n z^n \\ &= mz^p - \sum_{n=p+1}^{\infty} \gamma^p(n,\delta) k_n z^n \end{aligned} \tag{2.3}$$

Now, let $f(z) \in \alpha - UCV_\delta^p(\eta,\phi)$ that is

$$Re\left\{\frac{z(\Omega_z^\delta f(z))' + \eta z^2 (\Omega_z^\delta f(z))''}{(1-\eta)(\Omega_z^\delta f(z)) + \eta z (\Omega_z^\delta f(z))'}\right\}$$
$$\geq \alpha \left|\frac{z(\Omega_z^\delta f(z))' + \eta z^2 (\Omega_z^\delta f(z))''}{(1-\eta)(\Omega_z^\delta f(z)) + \eta z (\Omega_z^\delta f(z))'} - 1\right| + \tan\phi$$

Using Lemma 2 we have

$$Re\left\{\frac{z(\Omega_z^\delta f(z))' + \eta z^2 (\Omega_z^\delta f(z))''}{(1-\eta)(\Omega_z^\delta f(z)) + \eta z (\Omega_z^\delta f(z))'}(1+\alpha e^{i\theta}) - \alpha e^{i\theta}\right\}$$
$$\geq \tan\phi, (0 \leq \tan\phi < p)$$

or equivalently

$$Re\{([z(\Omega_z^\delta f(z))' + \eta z^2 (\Omega_z^2 f(z))''](1+\alpha e^{i\theta}) - (\alpha e^{i\theta} + \tan\phi)$$
$$[(1-\eta)\Omega_z^\delta f(z) + \eta z (\Omega_z^\delta f(z))'])/((1-\eta)\Omega_z^\delta f(z) + \eta z(\Omega_z^\delta f(z))')\} \geq 0$$

Then, we can write

$$Re\{[m(1+\eta p-\eta)(p-\tan\phi) - \sum_{n=p+1}^{\infty} \gamma^p(n,\delta)((n+n\eta(n-1)$$
$$-\tan\phi(1-\eta+n\eta))k_n z^{n-p} - \alpha e^{i\theta}(m(1-\eta+p\eta)(p-1))$$
$$-\alpha e^{i\theta} \sum_{n=p+1}^{\infty} \gamma^p(n,\delta)(n+n\eta(n-1)-(1-\eta+n\eta))k_n z^{n-p}]$$
$$/[m(1-\eta+p\eta) - \sum_{n=p+1}^{\infty} \gamma^p(n,\delta)(1-\eta+n\eta)k_n z^{n-p}]\} > 0$$

The above inequality must hold for all z in Δ. Letting $z \to 1^-$ yields

$$Re\{[m(1+\eta p-\eta)(p-\tan\phi)-\sum_{p+1}^{\infty}\gamma^p(n,\delta)(1-\eta+\eta n)(n-\tan\phi)k_n$$
$$-\alpha e^{i\theta}(m(1-\eta+p\eta)(p-1))-\alpha e^{i\theta}\sum_{n=p+1}^{\infty}\gamma^p(n,\delta)(1-\eta+n\eta)(n-1)]$$
$$/[m(1-\eta+p\eta)-\sum_{n=p+1}^{\infty}\gamma^p(n,\delta)(1-\eta+n\eta)k_n]\}>0$$

and so by the mean value theorem we have

$$Re\{m(1+\eta p-\eta)(p-\tan\phi)-\sum_{p+1}^{\infty}\gamma^p(n,\delta)(1-\eta+\eta n)(n-\tan\phi)k_n$$
$$+\alpha e^{i\theta}[m(1-\eta+p\eta)(p-1)]-\alpha e^{i\theta}\sum_{p+1}^{\infty}\gamma^p(n,\delta)(1-\eta+n\eta)(n-1)k_n\}>0$$

Therefore we obtain

$$\sum_{n=p+1}^{\infty}\gamma^p(n,\delta)(1-\eta+n\eta)(n-\tan\phi+\alpha(n-1))k_n<m(1+\eta p-\eta)(p-\tan\phi+\alpha(p-1))$$

Conversely, let (2.1) hold true. We will show that (1.5) gets satisfied and then $f(z)\in\alpha-UCV_\delta^p(\eta,\phi)$. Using the Lemma 1 it is enough to show that

$$\begin{aligned}
E &= \left|\frac{z(\Omega_z^\delta f(z))'+\eta z^2(\delta_z^\delta f(z))''}{(1-\eta)(\Omega_z^\delta f(z))+\eta z(\Omega_z^\delta f(z)))'}\right.\\
&- \left.\left(1+\alpha\left|\frac{z(\Omega_z^\delta f(z))'+\eta z^2(\Omega_z^\delta f(z))''}{(1-\eta)(\Omega_z^\delta f(z))+\eta z(\Omega_z^\delta f(z))'}-1\right|+\tan\phi\right)\right|\\
&< \left|\frac{z(\Omega_z^\delta f(z))'+\eta z^2(\Omega_z^\delta f(z))''}{(1-\eta)(\Omega_z^\delta f(z))+\eta z(\Omega_z^\delta f(z))'}\right.\\
&+ \left(1-\alpha\left|\frac{z(\Omega_z^\delta f(z))'+\eta z^2(\Omega_z^\delta f(z))''}{(1-\eta)(\Omega_z^\delta f(z))+\eta z(\Omega_z^\delta f(z))'}-1\right|-\tan\phi\right)=F
\end{aligned}$$

We must show $E < F$ or $F-E > 0$. For letting $e^{i\theta} = \frac{B}{|B|}$ where $B = (1-\eta)(\Omega_z^\delta f(z)) + \eta z(\Omega_z^\delta f(z))'$, we may write

$$\begin{aligned} E &= \frac{1}{|B|}|z(\Omega_z^\delta f(z))' + \eta z^2(\Omega_z^\delta f(z))'' - (1+\tan\phi)[(1-\eta)(\Omega_z^\delta f(z)) \\ &\quad +\eta z(\Omega_z^\delta f(z))']| - \alpha e^{i\theta}|(1-\eta)z(\Omega_z^\delta f(z))' + \eta z^2(\Omega_z^\delta f(z))'' \\ &\quad -(1-\eta)(\Omega_z^\delta f(z))|| \\ &< \frac{|z|^p}{|B|}(m(1+\eta p-\eta)(p-1-\tan\phi+\alpha(p-1)) \\ &\quad + \sum_{n=p+1}^{\infty}\gamma^p(n,\delta)(1+\eta n-\eta)[(n-1-\tan\phi)+\alpha(n+1)]k_n \end{aligned}$$

Also, we have

$$\begin{aligned} F &= \frac{1}{|B|}|z(\Omega_z^\delta f(z))' + \eta z^2(\Omega_z^\delta f(z))'' + (1-\tan\phi)((1-\eta)(\Omega_z^\delta f(z)) \\ &\quad -\eta z(\Omega_z^\delta f(z))') - \alpha e^{i\theta}|(1-\eta)z(\Omega_z^\delta f(z))' + \eta z^2(\Omega_z^\delta f(z))'' - (1-\eta)(\Omega_z^\delta f(z))|| \\ &> \frac{|z|^p}{|B|}(m(1+\eta p-\eta)(p+1-\tan\phi+\alpha(p-1)) \\ &\quad - \sum_{n=p+1}^{\infty}\gamma^p(n,\delta)(1-\eta+\eta n)(n+1-\tan\phi+\alpha(n+1))k_n. \end{aligned}$$

It is easy to verify that $F-E>0$, if (2.1) holds and so the proof is complete. ☐

Corollary 1. *If $f(z) \in \alpha - UCV_\delta^p(\eta,\phi)$, then*

$$k_n \le \frac{m(\alpha(p-1)+p-\tan\phi)(1-\eta+\eta p)}{\gamma^p(n,\delta)[(1+n\eta-\eta)(n(1+\alpha)-(\alpha+\tan\phi)]},\ n \ge p+1$$

where $0 \le \tan\phi < p, \alpha \ge 0, 0 \le \eta \le 1, p \in \mathbb{N}$ and $\gamma^p(n,\delta) = \frac{\Gamma(2-\delta)\Gamma(n+p)}{\Gamma(n+p-\delta)}$.

Corollary 2. *$f(z) \in \alpha - UCV_0^1(\eta,\phi)$ if and only if*

$$\sum_{n=p+1}^{\infty}(1-\eta+n\eta)(n(1+\alpha)-(\alpha+\tan\phi))k_n \le m(1-\tan\phi),\ 0 \le \tan\phi < 1$$

that is a class introduced by E. Aqlan and S. R. Kulkarni [3].

Corollary 3. *$f(z) \in 0 - UCV_0^1(\eta,\phi)$ if and only if*

$$\sum_{n=p+1}^{\infty}(1-\eta+n\eta)(n-\tan\phi)k_n \le m(1-\tan\phi),\ 0 \le \tan\phi < 1$$

that is a class studied by Altintas [1].

Corollary 4. *$f(z) \in \alpha - UCV_0^1(0,\phi)$ if and only if*

$$\sum_{n=p+1}^{\infty} (n(1+\alpha) - (\alpha + \tan\phi))k_n \leq m(1 - \tan\phi),\ 0 \leq \tan\phi < 1.$$

That is class studied by R. Bharati, R. Parvatham and A. Swaminathan [5].

3. Special Functions and Integral Operators on $\alpha - UCV_\delta^p(\eta,\phi)$

Definition 5. Let c be a real number such that $c > -p$. For $f \in \alpha - UCV_\delta^p(\eta,\phi)$, we define F_c by

$$F_c(z) = \frac{c+p}{z^c} \int_0^z s^{c-1} f(s)ds \tag{3.1}$$

Theorem 2. *$F_c(z)$ defined by (3.1) belongs to $\alpha - UCV_\delta^p(\eta,\phi)$.*

Proof. Let $f(z) = mz^p - \sum_{n=p+1}^{\infty} k_n z^n \in \alpha - UCV_\delta^p(\eta,\phi)$ then

$$F_c(z) = \frac{c+p}{z^c} \int_0^z \left(ms^{c-1+p} - \sum_{n=p+1}^{\infty} k_n s^{n+c-1} \right) ds = mz^p - \sum_{n=p+1}^{\infty} \frac{c+p}{n+c} k_n z^n.$$

Since $f(z) \in \alpha - UCV_\delta^p(\eta,\phi)$ and $\frac{c+p}{c+n} < 1, n \geq p+1$ and by Theorem 1, $F_c(z) \in \alpha - UCV_\delta^p(\eta,\phi)$ if

$$\begin{aligned}
&\sum_{n=p+1}^{\infty} \gamma^p(n,\delta)[(1-\eta+n\eta)(n(1+\alpha) - (\alpha+\tan\phi))] \frac{c+p}{c+n} k_n \\
&\leq \sum_{n=p+1}^{\infty} \gamma^p(n,\delta)[(1-\eta+n\eta)(n(1+\alpha) - (\alpha+\tan\phi))] k_n \\
&\leq m(\alpha(p-1) + p - \tan\phi)(1-\eta+\eta p)
\end{aligned} \tag{3.2}$$

So $F_c(z) \in \alpha - UCV_\delta^p(\eta,\phi)$. □

Theorem 3. *The function $F_c(z)$ defined in 3.1 is starlike of order $\lambda (0 \leq \lambda < p)$ in $|z| < r_1(\eta,\phi,\alpha,\delta,n,p,c,\lambda)$ where*

$$r_1(\eta,\phi,\alpha,\delta,n,p,c,\lambda) = \inf_{n \geq p+1} \left\{ \frac{[(1-\eta+, n\eta)(n(1+\alpha) - (\alpha+\tan\phi)]}{m(\alpha(p-1) + p - \tan\phi)(1-\eta+\eta p)} \left(\frac{c+n}{c+p} \right) \left(\frac{m(p-\lambda)}{2p-n-\lambda} \right) \gamma^p(n,\delta) \right\}^{\frac{1}{n-p}}$$

The bound for $|z|$ is sharp for each n with extremal function being of the form

$$F_{c,n}(z) = mz^p - \frac{m(\alpha(p-1)+p-\tan\phi)(1-\eta+\eta p)}{\gamma^p(n,\delta)[(1-\eta+n\eta)(n(1+\alpha)-(\alpha+\tan\phi)]}\frac{c+n}{c+p}\ z^n, n \geq p+1.$$

Proof. We must show that

$$\left|\frac{zF_c'(z)}{F_c(z)} - p\right| < p - \lambda \tag{3.3}$$

But we have

$$\left|\frac{zF_c'(z)}{F_c(z)} - p\right| \leq \frac{\sum\limits_{n=p+1}^{\infty} \frac{c+p}{c+n} k_n (p-n)|z|^{n-p}}{m - \sum\limits_{n=p+1}^{\infty} \frac{c+p}{c+n} k_n |z|^{n-p}}.$$

Therefore (3.3) holds if

$$\sum_{n=p+1}^{\infty} \left(\frac{c+p}{c+n}\right)\left(\frac{2p-n-\lambda}{m(p-\lambda)}\right) k_n |z|^{n-p} < 1.$$

Now in view of (3.2) the last inequality holds if

$$|z|^{n-p} < \frac{(1-\eta+n\eta)(n(1+\alpha)-(\alpha+\tan\phi)]}{m(\alpha(p-1)+p-\tan\phi)(1-\eta+\eta p)}\left(\frac{m(p-\lambda)}{2p-n-\lambda}\right)\left(\frac{c+n}{c+p}\right)\gamma^p(n,\delta).$$

This gives the required result.

□

Corollary 5. *The function $F_c(z)$ defined in 3.1 is convex of order $\lambda (0 \leq \lambda < p)$ in $|z| < r_2 = r_2(\eta, \phi, \alpha, \delta, n, p, c, \lambda)$ where*

$$\begin{aligned} r_2(\eta, \phi, \alpha, \delta, n, p, c, \lambda) \quad &= \quad \inf_{n \geq p+1} \left\{ \frac{(1-\eta+n\eta)(n(1+\alpha)-(\alpha+\tan\phi))}{m(\alpha(p-1)+p-\tan\phi)(1-\eta+\eta p)} \right. \\ &\qquad \left. \left(\frac{m(p-\lambda)}{2p-n-\lambda}\right)\left(\frac{c+n}{c+p}\right)\gamma^p(n,\delta) \right\}^{\frac{1}{n-p}} \end{aligned}$$

Proof. We must show that $\left|\frac{zF_c''(z)}{F_C'(z)}\right| < p - \lambda$ for $|z| < r_2$ and $c > -p$.
But we have

$$\left|\frac{zF_c''(z)}{F_c'(z)}\right| \leq \frac{mp(p-1) + \sum\limits_{n=p+1}^{\infty} \frac{c+p}{c+n} k_n n(n-1)|z|^{n-p}}{mp - \sum\limits_{n=p+1}^{\infty} \frac{c+p}{c+n} k_n n |z|^{n-p}}.$$

Therefore $\left|\frac{zF_c''(z)}{F_C'(z)}\right| < p-\lambda$ holds if

$$\sum_{n=p+1}^{\infty} \frac{n(n-1+p-\lambda)}{mp(\lambda-1)} \left(\frac{c+p}{c+n}\right) k_n |z|^{n-p} < 1.$$

The last inequality holds if

$$|z|^{n-p} < \frac{(1-\eta+n\eta)(n(1+\alpha)-(\alpha+\tan\phi))}{m(\alpha(p-1)+p-\tan\phi)(1-\eta+\eta p)} \left(\frac{mp(\lambda-1)}{n(n-1+p-\lambda}\right)$$
$$\left(\frac{c+n}{c+p}\right) \gamma^p(n,\delta).$$

This gives the required result. □

Definition 6. Let c be a real number such that $c > -p$ and let $f \in \alpha - UCV_\delta^p(\eta,\phi)$, Komato operator in [8] is defined by

$$G(z) = \int_0^1 \frac{(c+1)^\xi}{\Gamma(\xi)} t^c (\log\frac{1}{t})^{\xi-1} \frac{f(tz)}{t^p} dt,\ c > -1, \xi \geq 0. \tag{3.4}$$

Theorem 4. *$G(z)$ defined in 3.4 belongs to $\alpha - UCV_\delta^p(\eta,\phi)$.*

Proof. Since $\int_0^1 t^c(-\log t)^{\xi-1}dt = \frac{\Gamma(\xi)}{(c+1)^\xi}$ and $\int_0^1 t^{n+c-p}(-\log t)^{\xi-1}dt$
$= \frac{\Gamma(\xi)}{(c+n-p+1)^\xi}$
$n \geq p+1$. Therefore we obtain

$$\begin{aligned} G(z) &= \frac{(c+1)^\xi}{\Gamma(\xi)} \left[\int_0^1 t^c z^p \log(\frac{1}{t})^{\xi-1} dt - \sum_{n=p+1}^{\infty} \int_0^1 \log(\frac{1}{t})^{\xi-1} t^{n-p+c} k_n z^n dt\right] \\ &= mz^p - \sum_{n=p+1}^{\infty} \left(\frac{c+1}{c+n-p+1}\right)^\xi k_n z^n. \end{aligned} \tag{3.5}$$

Therefore and with use of Theorem 1 and $\frac{c+1}{c+1+n-p} < 1$ for $n \geq p+1$ we can write

$$\sum_{n=p+1}^{\infty} \gamma^p(n,\delta)[(1-\eta+n\eta)(n(1+\alpha)-(\alpha+\tan\phi))] \left(\frac{c+1}{c+n-p+1}\right)^\xi k_n$$
$$\leq m(\alpha(p-1)+p-\sin\phi)(1-\eta+\eta p) \tag{3.6}$$

So $G(z) \in \alpha - UCV_\delta^p(\eta,\phi)$. □

Theorem 5. *The function $G(z)$ defined in 3.4 is starlike of order $\lambda (0 \leq \lambda < 1)$ in $|z| < r_1 = r_1(\eta,\phi,\alpha,\delta,n,p,c,\xi,\lambda)$ where*

$$r_1(\eta,\phi,\alpha,\delta,n,p,c,\xi,\lambda) \;=\; \inf_{n\geq p+1}\left\{\frac{(1-\eta+n\eta)(n(1+\alpha)-(\alpha+\tan\phi)}{m(\alpha(p-1)+p-\sin\phi)(1-\eta+\eta p)}\right.$$
$$\left.\left(\frac{m(p-\lambda)}{2p-n-\lambda}\right)\left(\frac{c+n-p+1}{c+1}\right)^{\xi}\gamma^{p}(n,\delta)\right\}^{\frac{1}{n-p}}$$

Proof. We must show that $\left|\frac{zG'(t)}{G(t)}-p\right|<p-\lambda$ or we must show

$$\left|\frac{zG'(t)}{G(t)}-p\right|\leq\frac{\sum\limits_{n=p+1}^{\infty}\left(\frac{c+1}{c+n-p+1}\right)^{\xi}(p-n)k_n|z|^{n-p}}{m-\sum\limits_{n=p+1}^{\infty}\left(\frac{c+1}{c+n-p+1}\right)^{\xi}k_n|z|^{n-p}}<p-\lambda.$$

The last inequality holds if

$$\sum_{n=p+1}^{\infty}\left(\frac{c+1}{c+n-p+1}\right)^{\xi}\frac{(2p-(n+\lambda))}{m(p-\lambda)}k_n|z|^{n-p}<1.$$

Now in view of (3.6), (3.5) the last inequality holds if

$$|z|^{n-p}\leq\frac{\gamma^{p}(n,\delta)(1-\eta+n\eta)(n(1+\alpha)-(\alpha+\tan\phi))}{m(\alpha(p-1)+p-\tan\phi)(1-\eta+\eta p)}$$
$$\left(\frac{m(p-\lambda)}{(2p-(n+\lambda))}\right)\left(\frac{c+n-p+1}{c+1}\right)^{\xi}$$

This gives the required result. ☐

Corollary 6. *The function $G(z)$ defined in (3.4) is convex of order $\lambda (0\leq\lambda<p)$ in $|z|<r_2=r_2(\eta,\phi,\alpha,\delta,n,p,c,\xi,\lambda)$ where*

$$r_2(\eta,\phi,\alpha,\delta,n,p,c,\xi,\lambda)=\inf_{n\geq p+1}\left\{\frac{(1-\eta+n\eta)(n(1+\alpha)-(\alpha+\tan\phi)}{m(\alpha(p-1)+p-\sin\phi)(1-\eta+\eta p)}\right.$$
$$\left.\left(\frac{c+n-p+1}{c+1}\right)^{\xi}\left(\frac{p(1-\lambda)}{n(p+n-\lambda-1)}\right)\gamma^{p}(n,\delta)\right\}^{\frac{1}{n-p}}$$

Proof. We must show that $\left|\frac{zG''(z)}{G'(z)}\right|<p-\lambda,\ |z|<r_2$ or

$$\left|\frac{zG''(z)}{G'(z)}\right|=\left|\frac{mp(p-1)z^{p-1}-\sum\limits_{n=p+1}^{\infty}\left(\frac{c+1}{c+n-p+1}\right)^{\xi}k_n n(n-1)z^{n-1}}{mpz^{p-1}-\sum\limits_{n=p+1}^{\infty}\left(\frac{c+1}{c+n-p+1}\right)^{\xi}_{k}nz^{n-1}}\right|<p-\lambda$$

Therefore

$$\sum_{n=p+1}^{\infty}\left(\frac{c+1}{c+n-p+1}\right)^{\xi}\left(\frac{n(p-\lambda+n-1)}{mp(1-\lambda)}\right)k_n|z|^{n-p}<1. \tag{3.7}$$

Therefore (3.7) holds if

$$|z|^{n-p}<\frac{\gamma^p(n,\delta)(1-\eta+n\eta)(n(1+\alpha)-(\alpha+\tan\phi))}{m(\alpha(p-1)+p-\tan\phi)(1-\eta+\eta p)}$$
$$\left(\frac{c+n-p+1}{c+1}\right)^{\xi}\left(\frac{mp(1-\lambda)}{n(p+n-\lambda-1)}\right)$$

□

Definition 7. Let $f\in\alpha-UCV_{\delta}^{p}(\eta,\phi)$. Function $H_\mu(z)$ defined by

$$H_\mu(z)=(1-\mu)mz^p+\mu p\int_0^z\frac{f(t)}{t}dt\ 0\leq\mu<1, z\in\Delta \tag{3.8}$$

Theorem 6. *The function $H_\mu(z)$ defined in (3.8) belongs to $\alpha-UCV_{\delta}^{p}(\eta,\phi)$ if $0\leq\mu\leq1$.*

Proof. Let $f(z)\in\alpha-UCV_{\delta}^{p}(\eta,\phi)$ and is of the form (1.1) so

$$H_\mu(z)=(1-\mu)mz^p+\mu p(\int_0^z(mt^{p-1}-\sum_{n=p+1}^{\infty}k_nt^{n-1})dt)=mz^p-\sum_{n=p+1}^{\infty}(\frac{\mu p}{n}k_n)z^n \tag{3.9}$$

By Theorem 1 we must show

$$\sum_{n=p+1}^{\infty}\gamma^p(n,\delta)[(1-\eta+n\eta)(n(1+\alpha)-(\alpha+\tan\phi))]\frac{\mu p}{n}k_n$$
$$\leq\sum_{n=p+1}^{\infty}\gamma^p(n,\delta)[(1-\eta+n\eta)(n(1+\alpha)-(\alpha+\tan\phi))]\frac{\mu p}{p+1}k_n$$
$$\leq\sum_{n=p+1}^{\infty}\gamma^p(n,\delta)[(1-\eta+n\eta)(n(1+\alpha)-(\alpha+\tan\phi))]k_n$$
$$\leq m(\alpha(p-1)+p-\tan\phi)(1-\eta+\eta p)$$

So $H_\mu(z)\in\alpha-UCV_{\delta}^{p}(\eta,\phi)$. □

Theorem 7. *By the similar method which we applied for Theorem 5 and Corollary 6, we obtain the radii of starlikeness and convexity of order* $\lambda(0 \leq \lambda \leq p)$ *for* $H_\mu(z)$ *respectively as following*

$$\begin{aligned} r_1(\eta,\phi,\alpha,\delta,n,p,\mu,\lambda) &= \inf_{n\geq p+1}\left\{\frac{(1-\eta+n\eta)(n(1+\alpha)-(\alpha+\tan\phi))\gamma^p(n,\delta)}{m(\alpha(p-1)+p-\tan\phi)(1-\eta+\eta p)}\right. \\ &\quad \left.\left(\frac{m(p-\lambda)}{2p-n-\lambda}\right)\left(\frac{n}{\mu p}\right)\right\}^{\frac{1}{n-p}} \\ r_2(\eta,\phi,\alpha,\delta,n,p,\mu,\lambda) &= \inf_{n\geq p+1}\left\{\frac{(1-\eta+n\eta)(n(1+\alpha)-(\alpha+\tan\phi))\gamma^p(n,\delta)}{m(\alpha(p-1)+p-\sin\phi)(1-\eta+\eta p)}\right. \\ &\quad \left.\left(\frac{mp(1-\lambda)}{\mu(p+n-\lambda-1)}\right)\right\}^{\frac{1}{n-p}} \end{aligned}$$

where $0 \leq \mu \leq 1$.

4. (n,λ) - Neighborhood

Definition 8. ([9], [2]) : Let $\lambda \geq 0$ and $f(z) \in \mathcal{A}_p$ and f defined by (1.1). We define the
(n,λ) - neighborhood of a function $f(z)$ by

$$N_{n,\lambda}(f) = \left\{g \in \mathcal{A}_p : g(z) = mz^p - \sum_{n=p+1}^{\infty} k'_n z^n \text{ and } \sum_{n=p+1}^{\infty} n|k_n - k'_n| \leq \lambda\right\} \tag{4.1}$$

For the identity function $e(z) = z$, we have

$$N_{n,\lambda}(e) = \left\{g \in \mathcal{A}_p : g(z) = mz^p - \sum_{n=p+1}^{\infty} k'_n z^n \text{ and } \sum_{n=p+1}^{\infty} n|k'_n| \leq \lambda\right\} \tag{4.2}$$

Theorem 8. *Let*

$$\lambda = \frac{(p+1)m(\alpha(p-1)+p-\tan\phi)(1-\eta+\eta p)}{\gamma^p(p+1,\delta)(1+p\eta)(p(1+\alpha)+1-\tan\phi)}.$$

where $\gamma^p(p+1,\delta) = \dfrac{\Gamma(2-\delta)\Gamma(2p+1)}{\Gamma(2p-\delta)}$. *Then*

$$\alpha - UCV^p_\delta(\eta,\phi) \subset N_{n,\lambda}(e).$$

Proof. For $f \in \alpha - UCV_\delta^p(\eta,\phi)$ we have from (2.1)

$$(1+p\eta)(p(1+\alpha)+1-\tan\phi)]\gamma^p(p+1,\delta)\sum_{n=p+1}^{\infty} k_n$$

$$\le \sum_{n=p+1}^{\infty} [(1-\eta+n\eta)(n(1+\alpha)-\alpha-\tan\phi)]\gamma^p(n,\delta)k_n$$

$$\le m(\alpha(p-1)+p-\tan\phi)(1-\eta+\eta p)$$

Therefore

$$\sum_{n=p+1}^{\infty} k_n \le \frac{m(\alpha(p-1)+p-\tan\phi)(1-\eta+\eta p)}{\gamma^p(p+1,\delta)(1+p\eta)(p(1+\alpha)+1-\tan\phi)}\ , \tag{4.3}$$

and on the other hand we have for $|z|<r$

$$|f'(z)| \le mp|z|^{p-1} + |z|^p \sum_{n=p+1}^{\infty} nk_n$$

$$\le mpr^{p-1} + r^p \sum_{n=p+1}^{\infty} nk_n$$

$$\text{(from (4.3))} \ \le pr^{p-1} + r^p \frac{(p+1)m(\alpha(p-1)+p-\tan\phi)(1-\eta+\eta p)}{\gamma^p(p+1,\delta)(1+p\eta)(p(1+\alpha)+1-\tan\phi)}.$$

From above inequalities we conclude

$$\sum_{n=p+1}^{\infty} nk_n \le \frac{(p+1)m(\alpha(p-1)+p-\tan\phi)(1-\eta+\eta p)}{\gamma^p(p+1,\delta)(1+p\eta)(p(1+\alpha)+1-\tan\phi)} = \lambda.$$

□

Definition 9. The function $f(z)$ defined by (1.1) is said to be a member of the class $\alpha - UCV_\delta^{p,\xi}(\eta,\phi)$ if there exists a function $g \in \alpha - UCV_\delta^p(\eta,\phi)$ such that

$$\left|\frac{f(z)}{g(z)} - 1\right| \le p-\xi,\ z\in\Delta,\ 0\le\xi<p.$$

Theorem 9. *If $g \in \alpha - UCV_\delta^p(\eta,\phi)$ and*

$$\xi = p - \frac{\lambda}{p+1}\mu(\eta,\phi,\alpha,\delta,p) \tag{4.4}$$

such that

$$\mu(\eta,\phi,\alpha,\delta,p) = [\gamma^p(p+1,\delta)(1+p\eta)(p(\alpha+1)+1-\tan\phi)]$$
$$/[m\gamma^p(p+1,\delta)(1+p\eta)(p(\alpha+1)+1-\tan\phi)$$
$$-m(\alpha(p-1)+p-\tan\phi)(1-\eta+\eta p)]$$

then $N_{n,\lambda}(g) \subset \alpha - UCV_\delta^{p,\xi}(\eta,\phi)$.

Proof. Let $f \in N_{n,\lambda}(g)$, then we have from (4.1) that $\sum\limits_{n=p+1}^{\infty} n|k_n - k_n'| \le \lambda$ which readily implies the coefficient inequality

$$\sum_{n=p+1}^{\infty} |k_n - k_n'| \le \frac{\lambda}{p+1}.$$

Also since $g \in \alpha - UCV_\delta^p(\eta,\phi)$ we have from (2.1)

$$\sum_{n=p+1}^{\infty} k_n' \le \frac{m(\alpha(p-1)+p-\tan\phi)(1-\eta+\eta p)}{\gamma^p(p+1,\delta)(1+p\eta)(p(\alpha+1)+1-\tan\phi)}$$

so that

$$\left|\frac{f(z)}{g(z)} - 1\right| < \frac{\sum\limits_{n=p+1}^{\infty} |k_n - k_n'|}{m - \sum\limits_{n=p+1}^{\infty} k_n'} \le \left(\frac{\lambda}{p+1}\right)$$
$$(\gamma^p(p+1,\delta)(1+p\eta)(p(\alpha+1)+1-\tan\phi)$$
$$/m\gamma^p(p+1,\delta)(1+p\eta)(p(\alpha+1)+1-\tan\phi)$$
$$-m(\alpha(p-1)+p-\tan\phi)(1-\eta+\eta p))$$
$$= \left(\frac{\lambda}{p+1}\right)\mu(\eta,\phi,\alpha,\delta,p) = p - \xi$$

Then $\left|\frac{f(z)}{g(z)} - 1\right| < p - \xi$. Thus, by definition 9, $f \in \alpha - UCV_\delta^{p,\xi}(\eta,\phi)$ for ξ given by (4.4). □

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BOOK REVIEWS

Jonathan M. Borwein and Qiji J. Zhu, ***Techniques of Variational Analysis,*** Canadian Mathematical Society (CMS) Books in Mathematics, Vol. 20, Springer 2005, vi+362 pp, ISBN 3-387-24298-8.

The term variational analysis concerns methods of proofs based on the fact that an appropriate auxiliary function attains a minimum, and has its roots in the physical principle of the least action. Probably that the first illustration of this method is Johann Bernoulli's solution to the Brachistocrone problem which led to the development of variational calculus.

A significant impact on variational analysis was done by the development of nonsmooth analysis, making possible the use of calculus of nonsmooth functions and enlarging substantially the area of applications. Other powerful tools are the decoupling method (a nonconvex substitute for Fenchel conjugacy and Hahn-Banach theorem from convex analysis), alongside with variational principles.

As it is well known, a lower semi-continuous (lsc) function attains its minimum on a compact set, a property that is not longer true in the absence of the compactness, even for bounded from below lsc functions. This drawback can be compensated by adding a small perturbation to the original function such that the perturbed function attains its minimum. The properties of the perturbation function depend on the geometric properties of the underlying space: the better these properties (smoothness) the nicer the perturbation function. This fact is well illustrated in the second chapter, *Variational Principles* - Ekeland variational principle holds in complete metric spaces, while the smooth Borwein-Preiss variational principle holds in Banach spaces with smooth norm. Another one, Stegall variational principle (proved in Chapter 6), holds in Banach spaces with the Radon-Nikodym property and ensures a continuous linear perturbation.

The aim of the book is to emphasize the strength of the variational techniques in various domains of analysis, optimization and approximation, dynamic systems, mathematical economics. These applications are arranged by chapters which are relatively independent and can be used for graduate topics courses.

The chapters are: 3. *Variational techniques in subdifferential theory* (Fréchet subdifferential and normal cone, sum rules, chain rules for Lyapunov functions, mean value theorems and inequalities, extremal principles); 4. *Variational techniques in convex analysis* (Fenchel conjugate, duality, entropy maximization); 5. *Variational techniques and multifunctions* (multifunctions, subdifferentials as multifunctions, distance functions, coderivatives of multifunctions, implicit multifunction theorems); 6. *Variational principles in nonlinear functional analysis* (subdifferential and Asplund spaces, nonconvex separation, Stegall variational principle, mountain pass theorem); 7. *Variational techniques in the presence of symmetry* (nonsmooth functions on smooth manifolds, manifolds of matrices and spectral functions, convex spectral functions).

The book contains a lot of exercises completing the main text, some of them, which are more difficult, being guided exercises with references.

Based mainly on developments and applications from the past several decades, the book is directed to graduate students in the field of variational analysis. The prerequisites for its reading are undergraduate analysis and basic functional analysis. Researchers who use variational techniques, or intend to do, will find the book very useful too.

S. Cobzaş

Dorin Bucur, Giuseppe Buttazzo, ***Variational Methods in Shape Optimization Problems***, Progress in Nonlinear Differential Equations and their Applications, Birkhäuser, 2005, ISBN 0-8176-4359-1.

Usually, problems of the calculus of variations concern optimization among an admissible class of functions. What is special about shape optimization problems is that the "competing objects" are shapes (domains of $\mathbf{R}^n$). Because of this, the existence of a solution is ensured only in certain cases, due to some geometrical restrictions on the admissible domains (shapes) or to a particular form of the cost function. In general, relaxed formulations of the problems have to be formulated.

The development of the field of shape optimization is due especially to the great number of applications in physics and engineering.

Several examples of shape optimization problems are presented in the first chapter of the book, in a detailed and clear manner: the isoperimetric problem, the Newton problem of minimal aerodynamical resistance, the optimal distribution of two different media in a fixed region, the optimal shape of a thin insulating layer.

The second chapter is about optimization problems over classes of convex domains and it deals with the case where an additional convexity constraint on the domains ensures the existence of an optimal shape (by providing some extra compactness). Some necessary conditions of optimality are given for the Newton problem.

Some shape optimization problems can be considered optimal control problems: the shape plays the role of the control and the state equation is usually a partial differential equation on the control domain. In Chapter 3, a topological framework for general optimization problems is given, together with the theory of relaxed controls and some examples of relaxed shape optimization problems.

Shape optimization problems with Dirichlet (Neumann) condition on the free boundary are treated in Chapters 4 (7, respectively). In both cases, is important to understand the stability of the solution to a PDE for nonsmooth perturbations of the geometric domain. This stability is related to the convergence in Mosco sense of the corresponding variational spaces. The relaxed form of a Dirichlet problem is given (in a case where the existence of an optimal solution does not occur), to understand the behavior of minimizing sequences. For Neumann boundary conditions, the problem of optimal cutting is treated completely.

Chapter 5 contains other particular cases where an unrelaxed optimal solution exists, in the family of classical admissible domains. The existence of solutions is ensured by some monotonicity properties of the cost functional or by some geometrical constraints on the domains.

Optimization problems for functions of eigenvalues are presented in Chapter 6. The case of the first two eigenvalues of the Laplace operator is studied, using the continuous Steiner symmetrization.

The book is addressed mainly to graduate students, applied mathematicians, engineers; it requires standard knowledge in the calculus of variations, differential equations and functional analysis.

The problems are treated from both the classical and modern perspectives, each chapter contains examples and illustrations and also several open problems for further research. A substantial bibliography is given, emphasizing the rapid development of the field.

Daniela Inoan

Stefaan Caenepeel and Freddy van Oystaeyen Editors, ***Hopf Algebras in Noncommutative Geometry and Physics***, Pure and Applied Mathematics; Vol. 239, Marcel Dekker, New York, 2005, 320 pp., ISBN 0-8247-5759-9.

The study of Hopf algebras and quantum groups has seen a great development during the last two decades. The present volume is devoted to these topics, and consists of high quality articles related to the lectures given at the meeting on "Hopf algebras and quantum groups" held at the Royal Academy in Brussels from May 28 to June 1, 2002. This volume contains refereed papers and surveys on different aspects of the subject, such as:

The list of contributors and their papers is as follows. *J. Abuhlail*, Morita contexts for corings and equivalences; *F. Aly and F. van Oystaeyen*, Hopf order module algebra orders; *G. Böhm*, An alternative notion of Hopf algebroid; *Ph. Bonneau and D. Sternheimer*, Topological Hopf algebras, quantum groups and deformation quantization; *T. Brzeziński, L. Kadison and R. Wisbauer*, On coseparable and biseparable corings; *D. Bulacu, S. Caenepeel and F. Panaite*, More properties of Yetter-Drinfeld modules over quasi-Hopf algebras; *S. Caenepeel, J. Vercruysse and S.H. Wang*, Rationality properties for Morita contexts associated to Corings; *L. El Kaoutit and J. Gómez-Torrecillas*, Morita duality for corings over quasi-Frobenius rings; *K.R. Goodearl and T.H. Lenagan*, Quantized coinvariants at transcendental q; *S. Majid*, Classification of differentials on quantum doubles and finite noncommutative geometry; *S. Majid*, Noncommutative differentials and Yang-Mills on permutation groups S_n; *C. Menini and G. Militaru*, The afineness criterion for Doi-Koppinen modules; *S. Montgomery*, Algebra properties invariant under twisting; *C. Ohn*, Quantum SL$(3, \mathbb{C})$'s: the missing case; *A Paolucci*, Cuntz algebras and dynamical quantum group $SU(2)$; *B. Pareigis*, On symbolic computations in braided monomial categories; *P. Schauenburg*, Quotients of finite quasi-Hopf algebras; *K. Szlachányi*; Adjointable monoidal functors and quantum groupoids; *R. Wisbauer*, On Galois corings.

The book is highly recommended to researchers in algebraic geometry, number theory and mathematical physics, who will find here an excellent overview of the most significant areas of research in this field. Some of the new results are presented here for the first time. It is a valuable addition to the literature, and I warmly recommend it to algebraists and theoretical physicists.

Andrei Marcus

Leszek Gasiński and Nikolaos S. Papageorgiou, ***Nonlinear Analysis***, Series in Mathematical Analysis and Applications, Vol. 9, Chapman & Hall/CRC, Taylor & Francis Group, Boca Raton, London, New York, Singapore, 2006, xi +971 pp., ISBN 1-58488-484-3.

The aim of the present volume is to provide the reader with a solid background in several areas related to some modern topics in nonlinear analysis as critical point theory, nonlinear differential operators and related regularity and comparison principles.

The first chapter, *Hausdorff measures and capacity,* is concerned with topics as Vitali and Besicovitch covering theorems, Hausdorff measure and dimension, differentiability of Hausdorff measures and of Lipschitz functions (Rademacher theorem), the area, coarea and change of variables formulae for Lipschitz transforms.

The second chapter, *Lebesgue-Bochner and Sobolev spaces,* contains a brief introduction to integration of vector-functions (weak and strong measurability, Pettis, Gelfand and Bochner integrals), a treatment of Banach spaces of continuous vector-functions, of Lebesgue-Bochner spaces (completeness, duality, compactness), and of Sobolev spaces of vector-functions.

Chapter 3, *Nonlinear operators and Young measures,* discusses some classes of nonlinear operators (monotone, accretive) and semigroups of operators, exemplified on the case of Nemytskii composition operator. Some results on compact and on Fredholm linear operators on Banach and Hilbert spaces are also included, in order to emphasize the similarities and the differences between the linear and nonlinear case. The chapter ends with an introduction to Young measures.

The fourth chapter, *Smooth and nonsmooth variational principles,* contains an introduction to differential calculus on Banach spaces (Gâteaux and Fréchet derivatives) with applications to the differentiability of convex functions - Mazur and Asplund generic differentiability theorems. Christensen theorem on almost everywhere differentiability of locally Lipschitz functions on Banach spaces (the extension of Rademacher theorem) with respect to Haar null sets is also proved. Subdifferential calculus for convex functions, as well as Clarke generalized subdifferential calculus for locally Lipschitz functions are considered too. The chapter ends with the proof of Ekeland and Borwein-Preiss variational principles with applications.

Chapter 5, *Critical point theory,* is concerned with applications of the critical point theory to minimax, saddle point and mountain pass theorems. Lusternik-Schnirelman theory with applications to eigenvalue problems is the topic of the last section of this chapter.

In Chapter 6, *Eigenvalue problems and maximum principles,* the techniques and methods developed so far are applied to the study of linear and nonlinear elliptic PDEs.

Fixed point theorems (FPT) constitute the basic tool in the proofs of the existence of solutions to various kinds of equations and inclusions. The last chapter of the book, Chapter 7, *Fixed point theorems,* is devoted to the proofs of the main FPT of metrical nature (Banach contraction principle with extensions and applications, normal structure in Banach spaces and FPT for nonexpansive mappings), and of topological nature as well – the fixed point theorems of Brouwer, Schauder, Borsuk, and Sadovskii. A special attention is paid to FPT in ordered structure (Tarski, Bourbaki-Kneser, Amann) and in ordered Banach spaces – Krasnoselskii FPT with applications to positive eigenvalues and to fixed point index.

An appendix collects the essential results from topology, measure theory, functional analysis, calculus and nonlinear analysis, used throughout the book.

Together with the books *Nonsmooth Critical Point Theory and Nonlinear Boundary Value Problems,* CRC 2005, by the same authors, and *An Introduction to Nonlinear Analysis,* Vol. I. *Theory*, Vol. II, *Applications,* by Z. Denkowski, S. Migorski & N. Papageorgiou, the present one provides a comprehensive and fairly self-contained presentation of some important results in nonlinear analysis and applications.

It (or parts of it) can be used for graduate or post-graduate course, but also as reference text by specialists.

S. Cobzaş