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PROFESSOR ŞTEFAN COBZAŞ AT HIS 60^{TH} ANNIVERSARY

WOLFGANG W. BRECKNER

At the western border of Romania, where the Mureş river leaves the Romanian territory and runs into the Tisza, there is a small town called Nădlac. Ştefan Cobzaş was born there on the 11^{th} of December 1945 as son of the farmers Florea Cobzaş - father -, and Sofia Cobzaş - mother. At that time Nădlac was a village.

Ştefan Cobzaş attended elementary school (from 1952 to 1959) in his hometown. Then he studied at the most famous high school in Arad, the capital of the Mureş river plain. From 1963 to 1968, he was a student at the Faculty of Mathematics and Mechanics (nowadays the Faculty of Mathematics and Computer Science) of Babeş-Bolyai University, Cluj-Napoca, being awarded a diploma in Mathematics, with Mathematical Analysis as major. After graduation, he was a researcher at the Institute of Numerical Analysis of the Romanian Academy, the Cluj-Napoca branch, until 1977. In the same year, he was hired, following a contest, as instructor at the Department of Analysis (the current Department of Analysis and Optimization) of the faculty he had graduated nine years before. In this department Ştefan Cobzaş was successively promoted to the positions of assistant professor (1980), associate professor (1990) and finally full professor (1998).

In 1970, Ştefan Cobzaş married one of his fellow students, Lucia Maria Bordean. They have two children: Dana, born in 1975, and Alexandru, born in 1976. Both graduated from the same faculty as their parents (Faculty of Mathematics and Computer Science, Babeş-Bolyai University), but majored in different fields: Dana in Computer Science and Alexandru in Mathematics.

Professor Cobzaş is a specialist in analysis, in a broader sense of the term. Using a more precise language, we have to say that he is a specialist in mathematical analysis, functional analysis, real analysis, topology and measure theory. While a student, Ştefan Cobzaş was attracted to the wide field of analysis, which he has never abandoned since then. His graduation thesis, written under the guidance of professor Ioan Muntean (1931-1996), dealt with the non-Archimedean topological vector spaces. It is under the coordination of professor Tiberiu Popoviciu that he began to prepare his doctoral thesis in the field of best approximation with constraints. Unfortunately, the great mathematician died in 1976 and, consequently, Ştefan Cobzaş completed his doctoral thesis under the coordination of professor Dimitrie D. Stancu. Ştefan Cobzaş successfully defended his doctoral thesis at Babeş-Bolyai University in 1979. Our colleague had also the opportunity to participate in professional training sessions abroad, which helped him to further improve his knowledge: in 1972 in Sofia (Bulgaria); in 1973 in Moscow (USSR); and in 1998 in Perpignan (France). The direct beneficiaries of professor Cobzaş's broad culture in the field of analysis are the students of the Faculty of Mathematics and Computer Science of Babeş-Bolyai University. His lectures and seminars have a high scientific level and, as a distinctive note, our colleague has always made subtle observations enriching the course itself and making it more attractive. This is evidence that Ştefan Cobzaş is not only an excellent mathematician, but also a very witty professor, his textbook *Mathematical Analysis* (*Differential Calculus*), published in the Romanian language at Cluj University Press in 1997, offering many examples in this respect.

Professor Cobzaş's professional qualities are fully emphasized by his 49 scholarly articles (see the appendix), which can be grouped into five categories: best approximation and optimization ([A.1] - [A.25]), finitely additive measures and support functionals ([B.1] - [B.3]), condensation of singularities ([C.1] - [C.5]), Lipschitz functions ([D.1] - [D.9]), miscellaneous topics ([E.1] - [E.7]). Professor Cobzaş's works and papers have been welcomed by the international scholarly community. From among those who have cited him we mention the following: Amstrong Th. E., Balaganskii V. S., Borwein J. M., Breckner W. W., Edelstein M., Fonf V. P., Fitzpatrick S., Jebelean P., Jourani A., Konyagin S. V., Mitrea A. I., Phelps R. R., Precupanu A.-M., Precupanu T., Reich S., Smarzewski R., Trif T., Zaslavski A. J. and others.

On behalf of the members of the Department of Analysis and Optimization of the Faculty of Mathematics and Computer Science of Babeş-Bolyai University, as well as on behalf of other colleagues and students of our faculty, we warmly congratulate Professor Ştefan Cobzaş on his 60^{th} birthday wishing him good health and excellent achievements in his further research work.

List of Scientific Papers by Ştefan Cobzaş

A. Best approximation and optimization

A.1. *Strongly nonproximinal sets in* c_0. (Romanian). Rev. Anal. Numer. Teoria Aproximaţiei **2** (1973), 137-141

A.2. *Antiproximinal sets in some Banach spaces.* Math. Balkanica **4** (1974), 79-82

A.3. *Convex antiproximinal sets in the spaces* c_0 *and* c. (Russian). Mat. Zametki **17** (1975), 449-457

A.4. *Antiproximinal sets in Banach spaces of continuous functions.* Rev. Anal. Numér. Théor. Approx. **5** (1976), 127-143

A.5. *Antiproximinal sets in Banach spaces of* c_0*-type.* Rev. Anal. Numér. Théor. Approx. **7** (1978), 141-145

A.6. *Nonconvex optimization problems on weakly compact subsets of Banach spaces.* Rev. Anal. Numér. Théor. Approx. **9** (1980), 19-25

A.7. *Duality relations and characterizations of best approximation for p-convex sets.* Rev. Anal. Numér. Théor. Approx. **16** (1987), 95-108

A.8. *On a theorem of V. N. Nikolski on the characterization of best approximation for convex sets.* Rev. Anal. Numér. Théor. Approx. **19** (1990), 7-13

A.9. *Some remarks on the characterization of nearest points.* Studia Univ. Babeş-Bolyai, Mathematica **35** (1990), No. 2, 54-56

A.10. *Best approximation in spaces of bounded vector-valued sequences.* Rev. Anal. Numér. Théor. Approx. **23** (1994), 63-69

A.11. *Selections associated to the metric projection.* Rev. Anal. Numér. Théor. Approx. **24** (1995), 45-52 (with C. Mustăţa)

A.12. *Extension of bilinear operators and best approximation in 2-normed spaces.* Rev. Anal. Numér. Théor. Approx. **25** (1996), 63-75 (with C. Mustăţa)

A.13. *Extension of bilinear operators and best approximation in 2-normed spaces.* In: Göpfert A., Seeländer J., Tammer Chr. (eds.), *Methods of Multicriteria Decision Theory, Proceedings of the 6th Workshop of the DGOR-Working Group Multicriteria and Decision Theory,* Alexisbad 1996. Verlag Dr. Hänsel-Hohenhausen, Egelsbach, 1997, 19-29 (with C. Mustăţa)

A.14. *Antiproximinal sets in the Banach space c(X).* Comment. Math. Univ. Carolin. **38** (1997), 247-253

A.15. *Extension of bilinear functionals and best approximation in 2-normed spaces.* Studia Univ. Babeş-Bolyai, Mathematica **43** (1998), No. 2, 1-13 (with C. Mustăţa)

A.16. *Antiproximinal sets in the Banach space* $C(\omega^k, X)$. Rev. Anal. Numér. Théor. Approx. **27** (1998), 47-58

A.17. *Antiproximinal sets in Banach spaces.* Acta Univ. Carolin., Math. Phys. **40** (1999), 43-52

A.18. *Existence results for some optimization problems in Banach spaces.* In: Lupşa L., Ivan M. (eds.), *Analysis, Functional Equations, Approximation and Convexity.* Proceedings of the conference held in honour of Professor Elena Popoviciu on the occasion of her 75th birthday in Cluj-Napoca, October 15-16, 1999. Editura Carpatica, Cluj-Napoca, 1999, 39-44

A.19. *Generic existence of solutions for some perturbed optimization problems.* J. Math. Anal. Appl. **243** (2000), 344-356

A.20. *Antiproximinal sets in Banach spaces of continuous vector-valued functions.* J. Math. Anal. Appl. **261** (2001), 527-542

A.21. *Phelps type duality results in best approximation.* Rev. Anal. Numér. Théor. Approx. **31** (2002), 29-43

A.22. *Best approximation in random normed spaces.* In: Oyibo G. (ed.), *Advances in Mathematics Research.* Vol. 5. Nova Sci. Publ., Hauppage, New York, 2003, 1-18

A.23. *Extension of bounded linear functionals and best approximation in spaces with asymmetric norm.* Rev. Anal. Numér. Théor. Approx. **33** (2004), 39-50 (with C. Mustăţa)

A.24. *Separation of convex sets and best approximation in spaces with asymmetric norm.* Quaest. Math. **27** (2004), 275-296

A.25. *Geometric properties of Banach spaces and the existence of nearest and farthest points.* Abstr. Appl. Anal. **2005:3** (2005), 259-285

B. Finitely additive measures and support functionals

B.1. *Hahn decompositions of finitely additive measures.* Arch. Math. (Basel) **27** (1976), 620-621

B.2. *Support functionals of the unit ball in Banach spaces of bounded functions.* Babeş-Bolyai University Cluj-Napoca, Seminar on Mathematical Analysis, 1986, 85-90

B.3. *Support points and the convexity of sets in topological vector spaces.* An. Univ. Timişoara, Ser. Mat.-Inform. **36** (1998), No. 2, 237-242

C. Condensation of singularities

C.1. *Condensation of singularities and divergence results in approximation theory.* J. Approx. Theory **31** (1981), 138-153 (with I. Muntean)

C.2. *Double condensation of singularities for Walsh-Fourier series.* Rev. Anal. Numér. Théor. Approx. **21** (1992), 119-129

C.3. *A superdensity theorem.* Mathematica **39(62)** (1997), 37-44 (with I. Muntean)

C.4. *Triple condensation of singularities for some interpolation processes.* In: Stancu D. D., Coman G., Breckner W. W., Blaga P. (eds.), *Approximation and Optimization.* Proceedings of ICAOR: International Conference on Approximation and Optimization, Cluj-Napoca, July 29 - August 1, 1996. Vol. 1. Transilvania Press, Cluj-Napoca, 1997, 227-232 (with I. Muntean)

C.5. *Superdense a.e. unbounded divergence in some approximation processes of analysis.* Real Anal. Exchange **25** (1999/2000), 501-512 (with I. Muntean)

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D.2. *Norm-preserving extension of convex Lipschitz functions.* J. Approx. Theory **24** (1978), 236-244 (with C. Mustăţa)

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D.4. *Lipschitz properties of convex functions.* Babeş-Bolyai University Cluj-Napoca, Seminar on Mathematical Analysis, 1985, 77-84

D.5. *Extreme points in Banach spaces of Lipschitz functions.* Mathematica **31(54)** (1989), 25-33

D.6. *Extension of Lipschitz functions and best approximation.* In: Popoviciu E. (ed.), *Research on the Theory of Allure, Approximation, Convexity and Optimization.* SRIMA, Cluj-Napoca, 1999, 3-21 (with C. Mustăţa)

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D.9. *Adjoints of Lipschitz mappings.* Studia Univ. Babeş-Bolyai, Mathematica **48** (2003), No. 1, 49-54

E. Miscellaneous topics

E.1. *Separation theorems for convex sets in locally convex non-Archimedean spaces.* (Romanian). Rev. Anal. Numer. Teoria Aproximaţiei **3** (1974), 137-141

E.2. *On the starlikeness and convexity of holomorphic functions.* Babeş-Bolyai University Cluj-Napoca, Seminar on Geometric Function Theory, 1986, 80-90

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EQUICONTINUITY AND SINGULARITIES OF FAMILIES OF MONOMIAL MAPPINGS

WOLFGANG W. BRECKNER and TIBERIU TRIF

Dedicated to Professor Ştefan Cobzaş at his 60^{th} anniversary

Abstract. The starting-point for the present paper is the principle of condensation of the singularities of families consisting of continuous linear mappings that act between normed linear spaces. It is proved that this basic functional analytical principle can be generalized for families of continuous monomial mappings of degree n between topological linear spaces. The obtained principle yields a generalization of the principle of uniform boundedness published by I. W. Sandberg [IEEE Trans. Circuits and Systems CAS-32 (1985), 332–336] and recently rediscovered by R. Miculescu [Math. Reports (Bucharest) 5 (55) (2003), 57–59]. Furthermore, by applying the new nonlinear principle there are revealed Baire category properties of certain subsets of the normed linear space $C[a, b]$ involved with Riemann-Stieltjes integrability.

1. **Introduction**

One of the most important and most useful results in the theory of real or complex normed linear spaces is the following theorem, known as the principle of condensation of the singularities.

Theorem 1.1. *Let X and Y be normed linear spaces, and let $(F_j)_{j\in J}$ be a family of continuous linear mappings from X into Y such that*

$$\sup \{ \|F_j\| \mid j \in J \} = \infty.$$

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Then the set of all $x \in X$ *satisfying*

$$\sup\{\, \|F_j(x)\| \mid j \in J \,\} = \infty$$

is residual, i.e. its complement is a set of the first category.

This theorem immediately provides the next theorem called the principle of uniform boundedness and considered to represent also a major functional analytical result.

Theorem 1.2. *Let* X *and* Y *be normed linear spaces of which* X *is complete, and let* $(F_j)_{j \in J}$ *be a family of continuous linear mappings from* X *into* Y*. Then*

$$\sup\{\, \|F_j(x)\| \mid j \in J \,\} < \infty \qquad \text{for all } x \in X$$

if and only if

$$\sup\{\, \|F_j\| \mid j \in J \,\} < \infty.$$

Both these theorems have been extensively investigated and have been generalized in several directions. In some papers more general spaces have been considered instead of the normed linear spaces X and Y. For instance, Ş. Cobzaş and I. Muntean [5] dealt with the topological structure of the set of singularities associated with a nonequicontinuous family of continuous linear mappings from a topological linear space into another topological linear space and pointed out cases when this set of singularities is an uncountable infinite G_δ-set. In other papers the linear mappings F_j $(j \in J)$ have been replaced by nonlinear mappings of a certain type. Moreover, W. W. Breckner [1] has proved a very general principle of condensation of the singularities which does not require any algebraic structure of the considered spaces and neither assumptions as to the shape of the mappings that are concerned.

For a detailed information on diverse generalizations of the Theorems 1.1 and 1.2 the reader is referred to the surveys by W. W. Breckner [2, 3] as well as to T. Trif [14, 15].

In the present paper we deal with the equicontinuity of families of monomial mappings. Moreover, by following the general line of proving principles of condensation of the singularities we show that the Theorems 1.1 and 1.2 can be generalized for families of continuous monomial mappings of degree n acting between topological linear spaces. Consequently, these generalizations integrate well into the framework described in [1]. Besides, it should be mentioned that the new principle of uniform boundedness turns out to be a generalization of the principle of uniform boundedness proved by I. W. Sandberg [11] and recently rediscovered by R. Miculescu [9]. The paper ends with an application of the obtained nonlinear principle of condensation of singularities that directly reveals Baire category properties of certain subsets of the normed linear space $C[a, b]$ involved with Riemann-Stieltjes integrability.

2. **Monomial mappings**

All linear spaces as well as all topological linear spaces that will occur in this paper are over $\mathbf{K}$, where $\mathbf{K}$ is either the field $\mathbf{R}$ of real numbers or the field $\mathbf{C}$ of complex numbers. If X is a linear space, then its zero-element is denoted by o_X. The set of all positive integers is $\mathbf{N}$.

Throughout this section let X and Y be linear spaces. Furthermore, let n be a positive integer. A mapping $F\colon X^n \to Y$ is said to be:

(i) *symmetric* if

$$F(x_1, \ldots, x_n) = F(x_{\sigma(1)}, \ldots, x_{\sigma(n)})$$

for each $(x_1, \ldots, x_n) \in X^n$ and each bijection $\sigma\colon \{1, \ldots, n\} \to \{1, \ldots, n\}$;

(ii) n-*additive* if for each $i \in \{\, 1, \ldots, n \,\}$ the mapping

$$\forall\, x \in X \;\longmapsto\; F(x_1, \ldots, x_{i-1}, x, x_{i+1}, \ldots, x_n) \in Y$$

is additive whenever $x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n \in X$ are fixed.

If $F\colon X^n \to Y$ is an n-additive mapping, then it can be shown that

$$F(r_1 x_1, \ldots, r_n x_n) = r_1 \cdots r_n F(x_1, \ldots, x_n)$$

for all $x_1, \ldots, x_n \in X$ and all rational numbers $r_1, \ldots, r_n$. If in addition X and Y are topological linear spaces and F is continuous, then we even have

$$F(a_1 x_1, \ldots, a_n x_n) = a_1 \cdots a_n F(x_1, \ldots, x_n)$$

for all $x_1, \ldots, x_n \in X$ and all $a_1, \ldots, a_n \in \mathbf{R}$.

Given a mapping $F\colon X^n \to Y$, the mapping $F^*\colon X \to Y$, defined by

$$F^*(x) := F(\underbrace{x, \ldots, x}_{n \text{ times}}) \qquad \text{for all } x \in X,$$

is said to be the *diagonalization* of F. Any symmetric n-additive mapping can be expressed by means of its diagonalization as the following proposition points out (see A. M. McKiernan [8, Lemma 1] or D. Ž. Djoković [6, Lemma 2]).

Proposition 2.1. *If $F\colon X^n \to Y$ is a symmetric n-additive mapping, then*

$$F(u_1, \ldots, u_n) = \frac{1}{n!} \left(\Delta_{u_1} \cdots \Delta_{u_n} F^*\right)(x)$$

for all $u_1, \ldots, u_n, x \in X$, where $\Delta_u\colon Y^X \to Y^X$ is defined for each $u \in X$ by

$$\left(\Delta_u f\right)(x) := f(x+u) - f(x) \qquad \textit{for all } f \in Y^X \textit{ and all } x \in X.$$

A mapping $Q\colon X \to Y$ is said to be a *monomial mapping of degree n* if there exists a symmetric n-additive mapping $F\colon X^n \to Y$ such that $Q = F^*$. In virtue of Proposition 2.1 there exists for each monomial mapping $Q\colon X \to Y$ of degree n a single symmetric n-additive mapping $F\colon X^n \to Y$ such that $Q = F^*$.

A monomial mapping $Q\colon X \to Y$ of degree n has the homogeneity property $Q(rx) = r^n Q(x)$ for every $x \in X$ and every rational number r. If in addition X and Y are topological linear spaces and Q is continuous, this property implies

$$Q(ax) = a^n Q(x) \qquad \text{for every } x \in X \text{ and every } a \in \mathbf{R}.$$

Finally, we mention a useful characterization of the monomial mappings of degree n (see A. M. McKiernan [8, Corollary 3] or D. Ž. Djoković [6, Corollary 3]).

Proposition 2.2. *A mapping $Q\colon X \to Y$ is a monomial mapping of degree n if and only if*

$$\frac{1}{n!}\left(\Delta_u^n Q\right)(x) = Q(u) \qquad \text{for all } u, x \in X.$$

The monomial mappings of degree 1 coincide with the additive mappings, while the monomial mappings of degree 2 are called *quadratic*.

3. **Equicontinuity of families of monomial mappings**

Let X and Y be topological linear spaces, and let $\mathcal{F} := (F_j)_{j\in J}$ be a family of mappings from X into Y. If x is a point in X, then $\mathcal{F}$ is said to be *equicontinuous at x* if for every neighbourhood V of o_Y there exists a neighbourhood U of o_X such that

$$\{\, F_j(x+u) - F_j(x) \mid j \in J \,\} \subseteq V \qquad \text{for all } u \in U.$$

If $\mathcal{F}$ is equicontinuous at each point of X, then $\mathcal{F}$ is said to be *equicontinuous on X*.

For families of symmetric n-additive mappings the following characterization of the equicontinuity is valid.

Theorem 3.1. *Let n be a positive integer, let X and Y be topological linear spaces, let $\mathcal{F} := (F_j)_{j\in J}$ be a family of symmetric n-additive mappings from X^n into Y, and let $\mathcal{F}^* := (F_j^*)_{j\in J}$. Then the following assertions are equivalent:*

1° *$\mathcal{F}^*$ is equicontinuous on X.*

2° *$\mathcal{F}^*$ is equicontinuous at o_X.*

3° *$\mathcal{F}$ is equicontinuous at o_{X^n}.*

4° *$\mathcal{F}$ is equicontinuous on X^n.*

Proof. Since the implications 1° ⇒ 2° and 4° ⇒ 3° are obvious, it remains to prove that 2° ⇒ 3°, 3° ⇒ 1° and 1° ⇒ 4°.

We start by proving the implication 2° ⇒ 3°. Let V be any neighbourhood of o_Y. Choose a balanced neighbourhood V_0 of o_Y such that

$$\underbrace{V_0 + \cdots + V_0}_{2^n \text{ terms}} \subseteq V. \tag{1}$$

The equicontinuity of $\mathcal{F}^*$ at o_X ensures the existence of a neighbourhood U_0 of o_X such that

$$\{ F_j^*(u) \mid j \in J \} \subseteq V_0 \qquad \text{for all } u \in U_0. \tag{2}$$

Now select a neighbourhood U of o_X such that

$$\underbrace{U + \cdots + U}_{n \text{ terms}} \subseteq U_0. \tag{3}$$

We claim that

$$\{ F_j(u_1, \ldots, u_n) \mid j \in J \} \subseteq V \qquad \text{for all } (u_1, \ldots, u_n) \in U^n. \tag{4}$$

Indeed, let j be any index in J and let $(u_1, \ldots, u_n)$ be any point in U^n. According to Proposition 2.1 we have

$$\begin{aligned} F_j(u_1, \ldots, u_n) &= \frac{1}{n!} \left(\Delta_{u_1} \cdots \Delta_{u_n} F_j^* \right)(o_X) \\ &= \frac{1}{n!} \sum_{(a_1, \ldots, a_n) \in A} (-1)^{n-(a_1+\cdots+a_n)} F_j^*(a_1 u_1 + \cdots + a_n u_n), \end{aligned} \tag{5}$$

where $A := \{0,1\}^n$. Since

$$a_1 u_1 + \cdots + a_n u_n \in \underbrace{U + \cdots + U}_{n \text{ terms}} \subseteq U_0 \qquad \text{for all } (a_1, \ldots, a_n) \in A,$$

we conclude in virtue of (2) that

$$F_j^*(a_1 u_1 + \cdots + a_n u_n) \in V_0 \qquad \text{for all } (a_1, \ldots, a_n) \in A.$$

Taking into account that $\operatorname{card} A = 2^n$ and that V_0 is balanced, we get by (5)

$$F_j(u_1, \ldots, u_n) \in \underbrace{V_0 + \cdots + V_0}_{2^n \text{ terms}} \subseteq V.$$

Consequently, (4) is true. From (4) it follows that $\mathcal{F}$ is equicontinuous at o_{X^n}.

Next we prove that $3^\circ \Rightarrow 1^\circ$. Let x be any point in X, and let V be any neighbourhood of o_Y. Choose a balanced neighbourhood V_0 of o_Y such that

$$\underbrace{V_0 + \cdots + V_0}_{2^n - 1 \text{ terms}} \subseteq V.$$

The equicontinuity of $\mathcal{F}$ at o_{X^n} ensures the existence of a balanced neighbourhood U_0 of o_X such that

$$\{ F_j(u_1, \ldots, u_n) \mid j \in J \} \subseteq V_0 \qquad \text{for all } (u_1, \ldots, u_n) \in U_0^n. \tag{6}$$

Select a rational number $r \in \,]0,1]$ such that $rx \in U_0$. We assert that

$$\{ F_j^*(x + r^{n-1}u) - F_j^*(x) \mid j \in J \} \subseteq V \qquad \text{for all } u \in U_0. \tag{7}$$

Indeed, let $j \in J$ and $u \in U_0$ be arbitrarily chosen. Then we have

$$\begin{aligned} F_j^*(x{+}r^{n-1}u) - F_j^*(x) & \\ &= F_j(\underbrace{x + r^{n-1}u, \ldots, x + r^{n-1}u}_{n \text{ times}}) - F_j(\underbrace{x, \ldots, x}_{n \text{ times}}) \\ &= \sum_{k=1}^{n} \binom{n}{k} F_j(\underbrace{x, \ldots, x}_{n-k \text{ times}}, \underbrace{r^{n-1}u, \ldots, r^{n-1}u}_{k \text{ times}}) \\ &= \sum_{k=1}^{n} \binom{n}{k} F_j(\underbrace{rx, \ldots, rx}_{n-k \text{ times}}, r^{k-1}u, \underbrace{r^{n-1}u, \ldots, r^{n-1}u}_{k-1 \text{ times}}). \end{aligned} \tag{8}$$

Since U_0 is balanced and $r \in \,]0,1]$, we see that (6) implies

$$F_j(\underbrace{rx, \ldots, rx}_{n-k \text{ times}}, r^{k-1}u, \underbrace{r^{n-1}u, \ldots, r^{n-1}u}_{k-1 \text{ times}}) \in V_0$$

for each $k \in \{ 1, \ldots, n \}$. Therefore it follows from (8) that

$$F_j^*(x + r^{n-1}u) - F_j^*(x) \in \underbrace{V_0 + \cdots + V_0}_{2^n - 1 \text{ terms}} \subseteq V.$$

Hence (7) is true. If we set $U := r^{n-1}U_0$, then U is a neighbourhood of o_X satisfying

$$\{ F_j^*(x + u) - F_j^*(x) \mid j \in J \} \subseteq V \qquad \text{for all } u \in U.$$

Consequently, $\mathcal{F}^*$ is equicontinuous at x.

Finally, we prove that $1^\circ \Rightarrow 4^\circ$. Let $(x_1, \ldots, x_n)$ be any point in X^n, and let V be any neighbourhood of o_Y. Choose a balanced neighbourhood V_0 of o_Y such that (1) holds. Let $A := \{0,1\}^n$. Since $\mathcal{F}^*$ is equicontinuous on X, there exists for each $(a_1,\ldots,a_n) \in A$ a neighbourhood U_a of o_X such that

$$\{\, F_j^*(a_1x_1 + \cdots + a_nx_n + u) - F_j^*(a_1x_1 + \cdots + a_nx_n) \mid j \in J \,\} \subseteq V_0 \tag{9}$$

for all $u \in U_a$. Next we choose a neighbourhood U of o_X such that

$$\underbrace{U + \cdots + U}_{n \text{ terms}} \subseteq \bigcap_{a \in A} U_a.$$

Then it results from (9) that

$$\{\, F_j^*(a_1(x_1+u_1) + \cdots + a_n(x_n+u_n)) - F_j^*(a_1x_1 + \cdots + a_nx_n) \mid j \in J \,\} \subseteq V_0 \tag{10}$$

for all $(a_1,\ldots,a_n) \in A$ and all $(u_1,\ldots,u_n) \in U^n$. But, according to Proposition 2.1 we have

$$\begin{aligned} &F_j(x_1+u_1,\ldots,x_n+u_n) - F_j(x_1,\ldots,x_n) \\ &\quad = \frac{1}{n!}\left[\left(\Delta_{x_1+u_1}\cdots\Delta_{x_n+u_n}F_j^*\right)(o_X) - \left(\Delta_{x_1}\cdots\Delta_{x_n}F_j^*\right)(o_X)\right] \\ &\quad = \frac{1}{n!}\sum_{(a_1,\ldots,a_n)\in A}(-1)^{n-(a_1+\cdots+a_n)}\left[F_j^*(a_1(x_1+u_1)+\cdots+a_n(x_n+u_n))\right. \\ &\qquad\qquad\qquad\qquad\qquad\qquad \left. -F_j^*(a_1x_1+\cdots+a_nx_n)\right] \end{aligned}$$

for all $j \in J$ and all $(u_1,\ldots,u_n) \in U^n$. By (10) it follows that

$$\{\, F_j(x_1+u_1,\ldots,x_n+u_n) - F_j(x_1,\ldots,x_n) \mid j \in J \,\} \subseteq \underbrace{V_0 + \cdots + V_0}_{2^n \text{ terms}} \subseteq V$$

for all $(u_1,\ldots,u_n) \in U^n$. Consequently, $\mathcal{F}$ is equicontinuous at $(x_1,\ldots,x_n)$. □

Corollary 3.2. *Let n be a positive integer, let X and Y be topological linear spaces, and let $\mathcal{Q} := (Q_j)_{j\in J}$ be a family of monomial mappings of degree n from X into Y. Then $\mathcal{Q}$ is equicontinuous on X if and only if it is equicontinuous at some point of X.*

Proof. Necessity. Obvious.

Sufficiency. Suppose that $x \in X$ is a point at which $\mathcal{Q}$ is equicontinuous. Then $\mathcal{Q}$ is equicontinuous at o_X. Indeed, when $x = o_X$, then this assertion is trivial.

When $x \neq o_X$, then it can be proved as follows. Let V be any neighbourhood of o_Y. Choose a balanced neighbourhood V_0 of o_Y such that

$$\underbrace{V_0 + \cdots + V_0}_{n \text{ terms}} \subseteq V.$$

The equicontinuity of $\mathcal{Q}$ at x ensures the existence of a neighbourhood U_0 of o_X such that

$$\{\, Q_j(x+u) - Q_j(x) \mid j \in J \,\} \subseteq V_0 \qquad \text{for all } u \in U_0. \tag{11}$$

Select a neighbourhood U of o_X that satisfies (3). Taking into account that

$$\sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} = 0,$$

the Proposition 2.2 implies

$$\begin{aligned} Q_j(u) &= \frac{1}{n!} \left(\Delta_u^n Q_j\right)(x) \\ &= \frac{1}{n!} \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} Q_j(x+ku) \\ &= \frac{1}{n!} \sum_{k=1}^{n} (-1)^{n-k} \binom{n}{k} [Q_j(x+ku) - Q_j(x)] \end{aligned} \tag{12}$$

for all $j \in J$ and all $u \in U$. Since

$$ku \in \underbrace{U + \cdots + U}_{k \text{ terms}} \subseteq U_0$$

for all $k \in \{\, 1, \ldots, n \,\}$ and all $u \in U$, it follows from (11) that

$$\{\, Q_j(x+ku) - Q_j(x) \mid j \in J \,\} \subseteq V_0$$

for all $k \in \{\, 1, \ldots, n \,\}$ and all $u \in U$. Since V_0 is balanced, (12) implies that

$$\{\, Q_j(u) \mid j \in J \,\} \subseteq \underbrace{V_0 + \cdots + V_0}_{n \text{ terms}} \subseteq V \qquad \text{for all } u \in U.$$

Consequently, $\mathcal{Q}$ is equicontinuous at o_X.

By applying now the implication $2^\circ \Rightarrow 1^\circ$ stated in Theorem 3.1, it follows that $\mathcal{Q}$ is equicontinuous on X. $\square$

Corollary 3.3. *Let n be a positive integer, let X and Y be topological linear spaces, and let $Q\colon X \to Y$ be a monomial mapping of degree n. Then Q is continuous on X if and only if it is continuous at some point of X.*

In the special case when $n = 1$ this corollary is well-known. When $n = 2$ it generalizes a result stated by S. Kurepa [7, Theorem 2] under the assumption that X is a normed linear space and $Y = \mathbf{R}$. In addition we note that a similar continuity result involving quadratic set-valued mappings was obtained by W. Smajdor [12, Theorem 4.2].

Proposition 3.4. *Let n be a positive integer, let X and Y be normed linear spaces, and let $(F_j)_{j\in J}$ be a family of symmetric n-additive mappings from X^n into Y. Then the following assertions are equivalent:*

1° $(F_j^*)_{j\in J}$ *is equicontinuous at o_X.*

2° $\sup\{\, \|F_j(x_1,\dots,x_n)\| \mid j\in J,\ \|x_1\|\le 1,\ \dots,\ \|x_n\|\le 1 \,\} < \infty$.

3° $\sup\{\, \|F_j^*(x)\| \mid j\in J,\ \|x\|\le 1 \,\} < \infty$.

Proof. 1° ⇒ 2° According to the implication 2° ⇒ 3° in Theorem 3.1, the family $(F_j)_{j\in J}$ is equicontinuous at o_{X^n}. Therefore there exists a neighbourhood U of o_X such that

$$\|F_j(u_1,\dots,u_n)\| \le 1 \qquad \text{for all } j\in J \text{ and all } (u_1,\dots,u_n)\in U^n.$$

Let r be a positive rational number such that $\{\, x\in X \mid \|x\|\le r \,\}\subseteq U$. Then we have

$$\|F_j(x_1,\dots,x_n)\| = \frac{1}{r^n}\|F_j(rx_1,\dots,rx_n)\| \le \frac{1}{r^n}$$

for all $j\in J$ and all $(x_1,\dots,x_n)\in X^n$ satisfying $\|x_1\|\le 1,\ \dots,\ \|x_n\|\le 1$. Consequently, assertion 2° is true.

2° ⇒ 3° Obvious.

3° ⇒ 1° Let V be a neighbourhood of o_Y. Choose a positive real number a such that $\{\, y\in Y \mid \|y\|\le a \,\}\subseteq V$. In addition, choose a positive rational number r such that $br^n \le a$, where

$$b := \sup\{\, \|F_j^*(x)\| \mid j\in J,\ \|x\|\le 1 \,\}.$$

Then we have

$$\|F_j^*(u)\| = r^n \left\| F_j^* \left(\frac{1}{r} u \right) \right\| \le br^n \le a$$

for all $j \in J$ and all $u \in X$ with $\|u\| \le r$. Consequently, the neighbourhood $U := \{ x \in X \mid \|x\| \le r \}$ of o_X satisfies

$$\{ F_j^*(u) \mid j \in J \} \subseteq V \qquad \text{for all } u \in U.$$

Hence $(F_j^*)_{j \in J}$ is equicontinuous at o_X. □

4. **Singularities of families of monomial mappings**

Let X and Y be topological linear spaces, and let $\mathcal{F} := (F_j)_{j \in J}$ be a family of mappings from X into Y. If x is a point in X, then $\mathcal{F}$ is said to be *bounded at x* if the set $\{ F_j(x) \mid j \in J \}$ is bounded, i.e. for each neighbourhood V of o_Y there exists a positive real number a such that $\{ F_j(x) \mid j \in J \} \subseteq aV$. If M is a subset of X and $\mathcal{F}$ is bounded at each point of M, then $\mathcal{F}$ is said to be *pointwise bounded on M*.

Any point in X at which $\mathcal{F}$ is not bounded is said to be a *singularity* of $\mathcal{F}$. The set of all singularities of $\mathcal{F}$ is denoted by $S_{\mathcal{F}}$. Clearly, $\mathcal{F}$ is pointwise bounded on X if and only if $S_{\mathcal{F}} = \emptyset$.

Theorem 4.1. *Let n be a positive integer, let X and Y be topological linear spaces, and let $\mathcal{Q} := (Q_j)_{j \in J}$ be a family of monomial mappings of degree n from X into Y which is equicontinuous at o_X. Then $\mathcal{Q}$ is pointwise bounded on X.*

Proof. Let x be any point in X. We prove that $\mathcal{Q}$ is bounded at x. Let V be a neighbourhood of o_Y. Since $\mathcal{Q}$ is equicontinuous at o_X, there exists a neighbourhood U of o_X such that

$$\{ Q_j(u) \mid j \in J \} \subseteq V \qquad \text{for all } u \in U.$$

Choose a rational number $r \neq 0$ such that $rx \in U$. Then $\{ Q_j(rx) \mid j \in J \} \subseteq V$, whence

$$\{ Q_j(x) \mid j \in J \} \subseteq \frac{1}{r^n} V.$$

Consequently, $\mathcal{Q}$ is bounded at x. □

The converse of Theorem 4.1 is not true. The pointwise boundedness of $\mathcal{Q}$ on X does not imply the equicontinuity of $\mathcal{Q}$ at o_X, not even when $n = 1$. But, taking into consideration the next theorem, which is a principle of condensation of the singularities of families of continuous monomial mappings between topological linear spaces, we will be able to point out cases when the pointwise boundedness of $\mathcal{Q}$ implies the equicontinuity of $\mathcal{Q}$ at o_X (and therefore on the whole space X).

Theorem 4.2. *Let n be a positive integer, let X and Y be topological linear spaces, and let $\mathcal{Q} := (Q_j)_{j\in J}$ be a family of continuous monomial mappings of degree n from X into Y which is not equicontinuous at o_X. Then the following assertions are true:*

1° *$S_{\mathcal{Q}}$ is a residual set.*

2° *If, in addition, X is of the second category, then $S_{\mathcal{Q}}$ is of the second category, dense in X and with* $\operatorname{card} S_{\mathcal{Q}} \geq \aleph$.

Proof. 1° Since $\mathcal{Q}$ is not equicontinuous at o_X, there exists a neighbourhood V of o_Y such that for every neighbourhood U of o_X there is a $u \in U$ satisfying

$$\{\, Q_j(u) \mid j \in J \,\} \not\subseteq V.$$

Choose a closed balanced neighbourhood V_0 of o_Y such that

$$\underbrace{V_0 + \cdots + V_0}_{n+1 \text{ terms}} \subseteq V.$$

For each positive integer m put

$$S_m := \bigcap_{j\in J} \{\, x \in X \mid Q_j(x) \in mV_0 \,\}.$$

Since V_0 is closed and all the mappings Q_j $(j \in J)$ are continuous, it follows that all the sets S_m are closed. We claim that all these sets are nowhere dense. Indeed, otherwise there exists a positive integer m such that $\operatorname{int} S_m \neq \emptyset$. Choose any point $x_0 \in \operatorname{int} S_m$. Next select a neighbourhood U_0 of o_X such that $x_0 + U_0 \subseteq S_m$ and after that select a neighbourhood U of o_X satisfying (3). Fix any $j \in J$ and any $u \in U$. In virtue of Proposition 2.2 we have

$$Q_j(u) = \frac{1}{n!}\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}Q_j(x_0+ku). \tag{13}$$

Since

$$x_0 + ku \in x_0 + \underbrace{U + \cdots + U}_{k \text{ terms}} \subseteq x_0 + U_0 \subseteq S_m$$

for all $k \in \{0, 1, \ldots, n\}$, it follows that

$$Q_j(x_0+ku) \in mV_0 \qquad \text{for all } k \in \{0, 1, \ldots, n\}.$$

Taking into account that V_0 is balanced, we obtain from (13) that

$$Q_j(u) \in m(\underbrace{V_0 + \cdots + V_0}_{n+1 \text{ terms}}) \subseteq mV,$$

whence

$$Q_j\left(\frac{1}{m}\,u\right) = \frac{1}{m^n}\,Q_j(u) \in \frac{1}{m^{n-1}}\,V \subseteq V.$$

Since $j \in J$ and $u \in U$ were arbitrarily chosen, we have

$$\left\{\, Q_j\left(\frac{1}{m}\,u\right) \;\middle|\; j \in J \,\right\} \subseteq V \qquad \text{for all } u \in U,$$

which contradicts the choice of V. Consequently, all the sets S_m are nowhere dense, as claimed.

It is immediately seen that

$$X \setminus S_{\mathcal{Q}} \subseteq \bigcup_{m=1}^{\infty} S_m.$$

Therefore $X \setminus S_{\mathcal{Q}}$ is a set of the first category, i.e. $S_{\mathcal{Q}}$ is a residual set.

2° Since X is of the second category, it follows in virtue of a well-known result in the theory of topological linear spaces that X is a Baire space. Consequently, the residual set $S_{\mathcal{Q}}$ is of the second category and dense. Therefore $S_{\mathcal{Q}}$ is not empty. Let x be any point in $S_{\mathcal{Q}}$. Since $\mathcal{Q}$ is bounded at o_X, we have $x \neq o_X$. Besides we have

$$\{\, Q_j(ax) \mid j \in J \,\} = a^n \{\, Q_j(x) \mid j \in J \,\} \qquad \text{for all } a \in \mathbf{R}.$$

Since the set $\{\, Q_j(x) \mid j \in J \,\}$ is not bounded, it follows that

$$\{\, ax \mid a \in \mathbf{R} \setminus \{0\} \,\} \subseteq S_{\mathcal{Q}},$$

whence card $S_{\mathcal{Q}} \geq \aleph$. □

Together the Theorems 4.1 and 4.2 yield the following theorem revealing cases when the equicontinuity at o_X of a family $\mathcal{Q}$ of continuous monomial mappings of degree n from a topological linear space X into a topological linear space Y is equivalent to the pointwise boundedness of $\mathcal{Q}$ on X.

Theorem 4.3. *Let n be a positive integer, let X and Y be topological linear spaces, and let $\mathcal{Q}$ be a family of continuous monomial mappings of degree n from X into Y. Then the following assertions are equivalent:*

$1°$ *X is of the second category and $\mathcal{Q}$ is pointwise bounded on X.*

$2°$ *There exists a subset $M \subseteq X$ of the second category on which $\mathcal{Q}$ is pointwise bounded.*

$3°$ *X is of the second category and $\mathcal{Q}$ is equicontinuous at o_X.*

Proof. $1° \Rightarrow 2°$ Obvious.

$2° \Rightarrow 3°$ Since $M \subseteq X$, it follows that X is of the second category. Analogously, it follows from $M \subseteq X \setminus S_{\mathcal{Q}}$ that $X \setminus S_{\mathcal{Q}}$ is of the second category. In other words, $S_{\mathcal{Q}}$ is not a residual set. According to assertion $1°$ in Theorem 4.2 the family $\mathcal{Q}$ must be equicontinuous at o_X.

$3° \Rightarrow 1°$ Results by Theorem 4.1. □

Corollary 4.4. *Let n be a positive integer, let X and Y be normed linear spaces, and let $(F_j)_{j\in J}$ be a family of continuous symmetric n-additive mappings from X^n into Y. Then the following assertions are equivalent:*

$1°$ *X is of the second category and*

$$\sup\{\, \|F_j^*(x)\| \mid j \in J \,\} < \infty \qquad \text{for all } x \in X.$$

$2°$ *There exists a subset $M \subseteq X$ of the second category such that*

$$\sup\{\, \|F_j^*(x)\| \mid j \in J \,\} < \infty \qquad \text{for all } x \in M. \tag{14}$$

$3°$ *X is of the second category and*

$$\sup\{\, \|F_j(x_1,\dots,x_n)\| \mid j \in J,\ \|x_1\| \leq 1,\ \dots,\ \|x_n\| \leq 1 \,\} < \infty. \tag{15}$$

Proof. $1^\circ \Rightarrow 2^\circ$ Obvious.

$2^\circ \Rightarrow 3^\circ$ The inequality (14) expresses that the family $(F_j^*)_{j\in J}$ is pointwise bounded on M. By applying the implication $2^\circ \Rightarrow 3^\circ$ from Theorem 4.3 it follows that X is of the second category and that $(F_j^*)_{j\in J}$ is equicontinuous at o_X. Therefore, by the implication $1^\circ \Rightarrow 2^\circ$ in Proposition 3.4, the inequality (15) is true.

$3^\circ \Rightarrow 1^\circ$ Let $x \in X$ be arbitrarily chosen. When $x = o_X$, then

$$\sup\{\, \|F_j^*(x)\| \mid j \in J \,\} = 0.$$

When $x \neq o_X$, then the number $a := 1/\|x\|$ satisfies

$$\|F_j^*(x)\| = \frac{1}{a^n}\|F_j(\underbrace{ax, \ldots, ax}_{n \text{ times}})\| \leq \frac{b}{a^n}$$

for all $j \in J$, where

$$b := \sup\{\, \|F_j(x_1, \ldots, x_n)\| \mid j \in J,\ \|x_1\| \leq 1,\ \ldots,\ \|x_n\| \leq 1 \,\}.$$

Consequently, assertion 1° is true. $\square$

It should be remarked that Theorem 4.2 is a generalization of Theorem 1.1, while Theorem 4.3 and Corollary 4.4 are generalizations of Theorem 1.2. Besides, Corollary 4.4 is also a generalization of the principle of uniform boundedness proved by I. W. Sandberg [11, Theorem 2], which recently was rediscovered by R. Miculescu [9, Theorem 2].

5. An application to the theory of the Riemann-Stieltjes integral

Throughout this section a and b are real numbers satisfying the inequality $a < b$. Any finite sequence $(x_0, x_1, \ldots, x_n)$ of points of the interval $[a, b]$ such that $a = x_0 < x_1 < \cdots < x_n = b$ is called a *subdivision* of $[a, b]$.

If $\Delta := (x_0, x_1, \ldots, x_n)$ is a subdivision of $[a, b]$, then the number

$$\mu(\Delta) := \max\{\, x_1 - x_0, \ldots, x_n - x_{n-1} \,\}$$

is called the *mesh* of Δ and any finite sequence $(c_1, \dots, c_n)$ such that $c_j \in [x_{j-1}, x_j]$ for all $j \in \{1, \dots, n\}$ is called a *selection* assigned to Δ. The set of all selections assigned to Δ will be denoted by $\mathcal{S}_\Delta$.

A. Pelczynski and S. Rolewicz [10, Corollary] proved that a function $f : [a,b] \to \mathbf{R}$ is Riemann-Stieltjes integrable with respect to itself over $[a,b]$ if and only if for each $\varepsilon \in]0,\infty[$ there exists a $\delta \in]0,\infty[$ such that for any subdivision $\Delta := (x_0, x_1, \dots, x_n)$ of $[a,b]$ with $\mu(\Delta) < \delta$ the inequality

$$\sum_{j=1}^{n} [f(x_j) - f(x_{j-1})]^2 < \varepsilon$$

holds. According to this result each function $f : [a,b] \to \mathbf{R}$, which is Riemann-Stieltjes integrable with respect to itself over $[a,b]$, has to be continuous. On the other hand, the main result established by A. Pelczynski and S. Rolewicz [10, Theorem 3], concerning the Riemann-Stieltjes integrals of the form

$$\int_a^b \Phi(f(x))df(x),$$

reveals that not every continuous function $f : [a,b] \to \mathbf{R}$ is Riemann-Stieltjes integrable with respect to itself over $[a,b]$. Actually, the set consisting of all continuous functions $f : [a,b] \to \mathbf{R}$ having the property that f is not Riemann-Stieltjes integrable with respect to itself over $[a,b]$ is very large. More precisely, the following theorem holds.

Theorem 5.1. *Let $C[a,b]$ be the linear space of all real-valued continuous functions defined on $[a,b]$ endowed with the usual uniform norm*

$$\|f\| = \max\{\,|f(x)| \mid x \in [a,b]\,\} \qquad (f \in C[a,b]),$$

and let $\widetilde{RS}[a,b]$ be the set of all functions $f\colon [a,b] \to \mathbf{R}$ having the property that f is Riemann-Stieltjes integrable with respect to itself over $[a,b]$. Then the following assertions are true:

1° *The set $C[a,b] \setminus \widetilde{RS}[a,b]$ is residual, whence of the second category, dense in $C[a,b]$ and with* $\operatorname{card}(C[a,b] \setminus \widetilde{RS}[a,b]) \geq \aleph$.

2° *The set $\widetilde{RS}[a,b]$ is of the first category and dense in $C[a,b]$.*

Proof. Let $\varphi\colon [0,1] \to [a,b]$ be defined by $\varphi(t) := a + t(b-a)$. Taking into account that the mapping

$$\forall\ f \in C[a,b] \ \longmapsto\ f \circ \varphi \in C[0,1]$$

is an isometric isomorphism as well as that a function $f\colon [a,b] \to \mathbf{R}$ is Riemann-Stieltjes integrable with respect to itself over $[a,b]$ if and only if $f \circ \varphi$ is Riemann-Stieltjes integrable with respect to itself over $[0,1]$, it suffices to prove the theorem in the special case when $a = 0$ and $b = 1$.

Let $\mathcal{D}$ be the set consisting of all subdivisions of $[0,1]$. Given a subdivision $\Delta := (x_0, x_1, \ldots, x_n) \in \mathcal{D}$ and a selection $\xi := (c_1, \ldots, c_n) \in \mathcal{S}_\Delta$, let $Q_{\Delta,\xi}\colon C[0,1] \to \mathbf{R}$ be the mapping defined by

$$Q_{\Delta,\xi}(f) := \sum_{j=1}^{n} f(c_j)[f(x_j) - f(x_{j-1})] \qquad \text{for all } f \in C[0,1].$$

It is easily seen that $Q_{\Delta,\xi}$ is continuous. Besides, we notice that $Q_{\Delta,\xi}$ is a quadratic mapping, because it is the diagonalization of the symmetric bilinear mapping $F_{\Delta,\xi}\colon C[0,1] \times C[0,1] \to \mathbf{R}$, defined by

$$F_{\Delta,\xi}(f,g) := \frac{1}{2}\sum_{j=1}^{n} f(c_j)[g(x_j) - g(x_{j-1})] + \frac{1}{2}\sum_{j=1}^{n} g(c_j)[f(x_j) - f(x_{j-1})]$$

for all $f, g \in C[0,1]$.

1° Passing to the proof of the first assertion of the theorem, we consider for every positive integer n the family

$$\mathcal{Q}_n := \{\, Q_{\Delta,\xi} \mid \Delta \in \mathcal{D},\ \xi \in \mathcal{S}_\Delta,\ \mu(\Delta) \le 1/n \,\}\,.$$

We claim that for every positive integer n the family $\mathcal{Q}_n$ is not equicontinuous at the zero-element of $C[0,1]$.

Indeed, let n be any positive integer. Define the function $f\colon [0,1] \to \mathbf{R}$ by

$$f(x) := \begin{cases} 0 & \text{if} \quad x = 0 \\ \sqrt{x}\,\left|\cos \frac{\pi}{x}\right| & \text{if} \quad 0 < x \le \frac{1}{n} \\ \frac{1}{\sqrt{n}} & \text{if} \quad \frac{1}{n} < x \le 1. \end{cases}$$

For each positive integer p set

$$\Delta_p := \Big(0,\ \frac{1}{n+p},\ \frac{2}{2(n+p)-1},\ \frac{1}{n+p-1},\ \frac{2}{2(n+p)-3},\ \frac{1}{n+p-2},\ \dots,\ \frac{1}{n+1},\ \frac{2}{2n+1},\ \frac{1}{n},\ \frac{2}{n},\ \frac{3}{n},\ \dots,\ \frac{n-1}{n},\ 1\Big)$$

and

$$\xi_p := \Big(0,\ \frac{2}{2(n+p)-1},\ \frac{1}{n+p-1},\ \frac{2}{2(n+p)-3},\ \frac{1}{n+p-2},\ \dots,\ \frac{2}{2n+1},\ \frac{1}{n},\ \frac{2}{n},\ \dots,\ \frac{n-1}{n},\ 1\Big).$$

Obviously, Δ_p is a subdivision of $[0,1]$ with $\mu(\Delta_p) \le 1/n$ and ξ_p is a selection assigned to Δ_p. Since

$$\begin{aligned} Q_{\Delta_p,\xi_p}(f) &= \sum_{j=n}^{n+p-1} f\left(\frac{1}{j}\right)\left[f\left(\frac{1}{j}\right) - f\left(\frac{2}{2j+1}\right)\right] \\ &= \frac{1}{n} + \frac{1}{n+1} + \dots + \frac{1}{n+p-1} \end{aligned}$$

for every $p \in \mathbf{N}$, it follows that

$$\sup\{\, Q_{\Delta_p,\xi_p}(f) \mid p \in \mathbf{N}\,\} = \infty.$$

Consequently, f is a singularity of $\mathcal{Q}_n$. By applying Theorem 4.1 we conclude that $\mathcal{Q}_n$ is not equicontinuous at the zero-element of $C[0,1]$, as claimed.

By virtue of Theorem 4.2 we deduce that all the sets $S_{\mathcal{Q}_n}$ $(n \in \mathbf{N})$ are residual, hence the set

$$S := \bigcap_{n=1}^{\infty} S_{\mathcal{Q}_n}$$

is residual, too.

Since $S \subseteq C[0,1] \setminus \widetilde{RS}[0,1]$, it follows that the set $C[0,1] \setminus \widetilde{RS}[0,1]$ is residual, whence of the second category, dense in $C[0,1]$ and with

$$\operatorname{card}(C[0,1] \setminus \widetilde{RS}[0,1]) \ge \aleph.$$

2° The fact that $\widetilde{RS}[0,1]$ is of the first category follows from assertion 1°. On the other hand, since $\widetilde{RS}[0,1]$ contains the restrictions to $[0,1]$ of all polynomials, it follows that $\widetilde{RS}[0,1]$ is dense in $C[0,1]$. □

Remark. There is also another way to prove Theorem 5.1. The characterization of the functions that are Riemann-Stieltjes integrable with respect to themselves over $[a, b]$, recalled at the beginning of this section, yields that $\widetilde{RS}[a, b] \subseteq CBV_2[a, b]$, where $CBV_2[a, b]$ denotes the set of all real-valued functions defined on $[a, b]$ that are continuous and of bounded variation of order 2. Taking into consideration that $CBV_2[a, b]$ is a set of the first category in $C[a, b]$ (see, for instance, [4, Corollary 2.8]), it follows that $\widetilde{RS}[a, b]$ is also of the first category in $C[a, b]$. Apparently this proof avoids the condensation of singularities, but in reality the property of $CBV_2[a, b]$ to be of the first category in $C[a, b]$ is a consequence of a principle of condensation of the singularities of a family of nonnegative functions as shown in [4].

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PERIODIC AND ALMOST PERIODIC FUNCTIONS

EDWIN CASTRO and VERNOR ARGUEDAS

Dedicated to Professor Ştefan Cobzaş at his 60^{th} *anniversary*

Abstract. In this paper we present some synthesis results about almost periodic functions. Some of these results were discussed in ([3], [10], [11], [12]). A diagram which represents the function sets mentioned in the work is discussed.

1. Preliminaries

The periodic functions play a central role in mathematics. Unfortunately this class of functions is not linear since the sum of periodic functions which not have a non-zero period in common gives a non-periodic function.

A larger class is the class of almost periodic functions which is a linear space. This class was introduced by Harald Bohr ([7], [8]). Bohr's theory of almost periodic functions was studied in connection with differential equations an other theories. For example Riesz and Nagy present some applications for compact operators and Banach algebras ([17]).

Salomon Bochner presents some generalizations of Bohr's definition ([6]) for functions with values in abstract spaces which are useful in the study of differential equations, Fourier series and Fourier transforms ([14], [15]). Laurent Schwartz has presented a definition for almost periodic distributions ([18]).

Our work aims presenting three definitions and discussing some examples of periodic and almost periodic functions.

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We introduce the following sets of functions (The domain of the functions is $\mathbb{R}$ and the range is a subset of $\mathbb{C}$)

F: the set of functions $f : \mathbb{R} \to \mathbb{C}$

C: the set of continuous functions

B: the set of bounded functions

P: the set of periodic functions

UC: the set of uniform continuous functions

AP: the set of almost periodic functions

TP: the set of trigonometric polynomials

IAP: the set of almost periodic functions with an almost periodic primitive

$$PC = P \cap C, \quad BP = B \cap P, \quad BC = B \cap C.$$

2. **Some results about periodic functions**

The function

$$f(x) = \cos x + \cos \sqrt{2}x$$

([10], [16]) is clearly not periodic. On the other hand it is the sum of two periodic functions: $\cos\sqrt{2}x$ and $\cos x$, the function $f(x)$ does not attains its infimum -2 but attain its supremum 2.

The function

$$g(x) = \sin x + \sin \sqrt{2}x$$

([10], [13]) is also non-periodic and does not attain the infimum and the supremum.

In each case there exist a sequence $(x_n)_{n\in\mathbb{N}}$ such that $x_n \to +\infty$ and $f(x_n) \to -2$ and in the second example there is a sequence $(y_n)_{n\in\mathbb{N}}$ such that $y_n \to +\infty$ and $f(y_n) \to 2$.

For the class PC we have the following results.

Theorem 2.1. ([2], [10]) *If $f \in CP$ then f attains its infimum and supremum.*

Theorem 2.2. ([2], [10]) *Let $f \in CP$ with period T, $T \in \mathbb{R}\setminus\mathbb{Q}$ then the set:*

$$A = \{f(n) : \ n \in \mathbb{N}\}$$

is dense in $[m, M]$*, where* m *denotes the minimum and* M *the maximum of the function.*

If the function in the previous theorem has rational period, the theorem is not true for example for the function $f(x) = \sin \pi x$, in this case the set A is finite, $A = \{0\}$.

Theorem 2.3. ([2]) *Let* $f \in CP$ *with rational period, then the set:*

$$A_\theta = \{f(n\theta) : \ n \in \mathbb{N}\}$$

is dense in $[m, M]$*,* $\forall\ \theta \in \mathbb{R} \setminus \mathbb{Q}$*.*

Example 2.1. Consider the function

$$f(x) = p \cos ax + q \cos bx + r$$

with $a, b, p, q, r \in \mathbb{R}$.

(1) If $pq \neq 0$ and $a/b \in \mathbb{R} \setminus \mathbb{Q}$ then $f \in C \setminus P$.

(2) If $pq \neq 0$ and $a/b \in \mathbb{Q}$ with $a \neq l\pi$ and $b \neq s\pi$, $\forall\ l, s \in \mathbb{Q}$ then $f \in CP$ (see example 3.1) and the set

$$\{f(n) \mid n \in \mathbb{N}\}$$

is dense in $[m, M]$.

(3) If $pq \neq 0$ and $a = l\pi$, $b = s\pi$ for some $l, s \in \mathbb{Q} \setminus \{0\}$ then $f \in CP$ and the set

$$A_\theta = \{f(n\theta) : \ n \in \mathbb{N}\}$$

is dense in $[m, M]$.

For a discussion and examples about the maxima and minima for periodic and almost periodic functions see [19].

Example 2.2. The function $f(x) = e^{iax} + b$, $a \in \mathbb{R} \setminus \{0\}$, $b \in \mathbb{C}$ is periodic with period $2\pi/a$.

If a is a non rational multiple of π then the set:

$$\{f(n) : \ n \in \mathbb{N}\}$$

is dense in

$$\mathbb{T}_b = \{z \in \mathbb{C} : \ |z - b| = 1\}.$$

If a is a rational multiple of π then the set:

$$\{f(n\theta):\ n\in\mathbb{N}\}$$

is dense in $\mathbb{T}_b$, $\forall\,\theta\in\mathbb{R}\setminus\mathbb{Q}$.

3. Almost periodic functions ([1], [2], [4], [6], [9], [11])

Let $f\in C$, we call $f\in AP$ if it has one of the following mutually equivalent properties:

$(AP1)$ (Corduneanu, Besicovitch, Bohr, Bochner) $\forall\,\varepsilon>0$ there is a trigonometric polynomial

$$T_\varepsilon(x)=\sum_{k=1}^{n}C_k e^{i\lambda_k x}\quad\text{(depends of }\varepsilon)$$

$$C_k\in\mathbb{C},\ \lambda_k\in\mathbb{R},\ k=1,\ldots,n,\ \text{depending on }\varepsilon$$

such that $|f(x)-T_\varepsilon(x)|<\varepsilon,\ \forall\,x\in\mathbb{R}$.

$(AP2)$ (Bochner, Besicovitch) $\forall\,\varepsilon>0$ there is $l>0$ such that $\forall\,a\in\mathbb{R}$ there exists $\tau\in[a,a+l]$ such that:

$$|f(x+\tau)-f(x)|<\varepsilon,\ \forall\,x\in\mathbb{R}$$

$(AP3)$ (Bohr, Fink) For every sequence $(\alpha'_n)_{n\in\mathbb{N}}$ one can extract a subsequence $(\alpha_n)_{n\in\mathbb{N}}$ such that $\lim\limits_{n\to+\infty}f(-+\alpha_n)$ exists uniform on $\mathbb{R}$.

The three definitions are equivalent and useful in applications ([1], [5], [14], [15]).

For generalizations of these definitions see: ([1], [8], [6], [12], [18]).

Some properties of the almost periodic functions have been studied in ([1], [10], [14], [15], [16]).

Example 3.1. Any trigonometric polynomial $(AP1)$ is an almost periodic function. We have:

$$TP\subset AP$$

In particular the function of the example 2.1 is almost periodic.

We are interesting into study of the primitive of an almost periodic function.

Theorem 3.1. ([14], [15]) *Let $f \in AP$. Then a primitive F of f is almost periodic if and only F if is bounded on $\mathbb{R}$.*

Example 3.2. Let f be the function of the example 2.1. Then $f \in IAP$ if and only if $r = 0$.

Theorem 3.2. ([4]) *If $f \in CP$ is nonconstant and F is a primitive of f, then:*

$$F(x) = Ax + g(x),$$

where $T > 0$ is the period of f,

$$A = \frac{1}{T}\int_0^T f(t)dt$$

and g is a CP function.

In the paper ([4]) the preceding result has been proved for a function $f : \mathbb{R} \to \mathbb{R}$, $f \in CP$, for a function $f : \mathbb{R} \to \mathbb{C}$ were writing $f = f_1 + if_2$, f_1 the real part of f and f_2 the imaginary part of f and the theorem 3.2 follows.

The following theorems collects various results about the Fourier series theory for almost periodic functions.

Theorem 3.3. ([8], [5], [13], [14], [15]) *Let $f \in AP$ then:*

(1)
$$a(f,\lambda) := \lim_{T\to\infty} \frac{1}{T}\int_0^T f(t)e^{-i\lambda t}dt$$

exists and is equal to zero excepting a countable set Λ.

(2)
$$\lim_{T\to\infty} \frac{1}{T}\int_a^{a+T} f(t)e^{-i\lambda t}dt$$

exists uniformly for $a \in \mathbb{R}$.

For $\lambda = 0$ we denote its value by $M(f)$, and call it the mean of f.

(3) (Parseval's equality) The Parseval's equality holds:

$$M(|f|^2) = \sum_{\lambda\in\Lambda} |a(f,\lambda)|^2$$

where Λ is the set mentioned in (1).

(4) The series $\sum_{n=1}^{\infty} a(f,\lambda_n)e^{i\lambda_n x}$ is called the Fourier series of f and we write:

$$f \sim \sum_{n=1}^{\infty} a(f,\lambda_n)e^{i\lambda_n x}$$

If the precedent series converges uniform then:

$$f(x) = \sum_{n=1}^{\infty} a(f,\lambda_n)e^{i\lambda_n x}, \quad x \in \mathbb{R}$$

(5) If the derivate (primitive) of f is an almost periodic function then its Fourier series is obtained by formal derivation (integration) of the Fourier series of f.

Theorem 3.4. ([14], [15], [16]) *Let $f \in AP$.*

(1) *If a primitive of f is almost periodic then $M(f) = 0$.*

(2) *If the series $\sum_{n=1}^{\infty} \left|\frac{a(f,\lambda_n)}{\lambda_n}\right| < +\infty$ then $f \in IAP$ and:*

$$\int_0^x f(t)dt = a_0 + \sum_{n=1}^{\infty} \frac{a(f,\lambda_n)}{\lambda_n}e^{i\lambda_n x}$$

(3) *If the exponents in the Fourier series of f have the property:*

$$|\lambda_n| \geq \alpha > 0, \ \forall\ n \in \mathbb{N} \text{ then } f \in IAP.$$

(4) *If $\int_0^x f(t)dt = Ax^{\lambda} + g(x)$, $x \in \mathbb{R}_+$, $\lambda \geq 0$ and $g \in BC$ then $A = M(f)$ and $\lambda = 1$.*

Example 3.3. We consider the function

$$f(x) = \sum_{n=1}^{\infty} \frac{1}{n^2}e^{i2^n x}, \quad x \in \mathbb{R}$$

We see that $f(0) = \frac{\pi^2}{6}$ and $f(x) \neq \frac{\pi^2}{6}$, $\forall\ x \neq 0$ then $f \in C \setminus P$. On the other hand $f \in AP$ and $f \notin TP$. We have

$$\int_0^x f(t)dt = a_0 + \sum_{n=1}^{\infty} \frac{1}{i2^n n^2}e^{i2^n x}$$

and $f \in IAP$. We conclude

$$f \in IAP \setminus (CP \cup TP).$$

Example 3.4. ([14], [15]) We consider the function:

$$f(x) = \sum_{n=1}^{\infty} \frac{1}{n^2} e^{ix/n^2}, \quad x \in \mathbb{R}.$$

The function is almost periodic, non periodic. We observe that $\lambda_n = 1/n^2$, $n \in \mathbb{N}$ and $\lambda_n \to 0$ also $M(f) = 0$.

If a primitive of f would be almost periodic then:

$$\int_0^x f(t)dt \sim a_0 + \sum_{n=1}^{\infty} e^{ix/n^2}.$$

But this is not possible since the last series is violating the Parseval's equality.

The function f is an example of an almost periodic function which is not a trigonometric polynomial, has mean zero, $\lambda_n \to 0$ and the primitives are not almost periodic. We conclude

$$f \in AP \setminus (CP \cup IAP \cup TP).$$

4. Diagram of functions

The relations:

$$AP \subset BC, \quad AP \subset UC, \quad CP \subset AP$$

have been studied ([8], [5], [10], [13], [14], [16]).

The following examples shows the inclusions between the considered sets of functions are strict.

Examples

4.1. $f \in F \setminus (P \cup B \cup C)$

$$f(x) = \begin{cases} \frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

4.2. $f \in C \setminus (UC \cup P \cup B)$

$$f(x) = x^2$$

4.3. $f \in BC \setminus (UC \cup P)$

$$f(x) = e^{ix^2} \quad ([10])$$

4.4. $f \in (UC \cap B) \setminus (P \cup AP)$

$$f(x) = \text{artan} x \quad ([10])$$

4.5. $f \in B \setminus (C \cup P)$

$$f(x) = \begin{cases} \sin 1/x, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

4.6. $f \in AP \setminus (CP \cup IAP \cup TP)$

$$f(x) = \sum_{n=1}^{\infty} \frac{1}{n^2} e^{\frac{ix}{n^2}} \quad \text{(Example 3.4)}$$

4.7. $f \in UC \setminus (B \cup P)$

$$f(x) = x$$

4.8. $f \in (TP \cap IAP) \setminus P$ (Examples 2.1 and 3.2)

4.9. $f \in P \setminus (B \cup C)$

$$f(x) = \begin{cases} \tan x, & x \neq (2n+1)\frac{\pi}{2}, \quad n \in \mathbb{Z} \\ 0, & x = (2n+1)\frac{\pi}{2}, \quad n \in \mathbb{Z} \end{cases}$$

4.10. $f \in BP \setminus C$

$$f(x) = x - [x]$$

4.11. $f \in CP \setminus TP$

First we consider the function $\psi : [0, 2] \to [0, 1]$

$$\psi(x) = \begin{cases} x, & x \in [0, 1] \\ 2 - x, & x \in]1, 2] \end{cases}$$

The function f is the function:

$$f(x) = \psi(x - 2n), \quad x \in [2n, 2(n+1)], \quad n \in \mathbb{Z}.$$

4.12. $f \in TP \cap CP \cap IAP$

(Example 2.1 and 3.2)

4.13. $f \in IAP \setminus (CP \cup TP)$

(Example 3.3)

4.14. $g \in (CP \cap IAP) \setminus TP$

$$g(x) = f(x) - M(f)$$

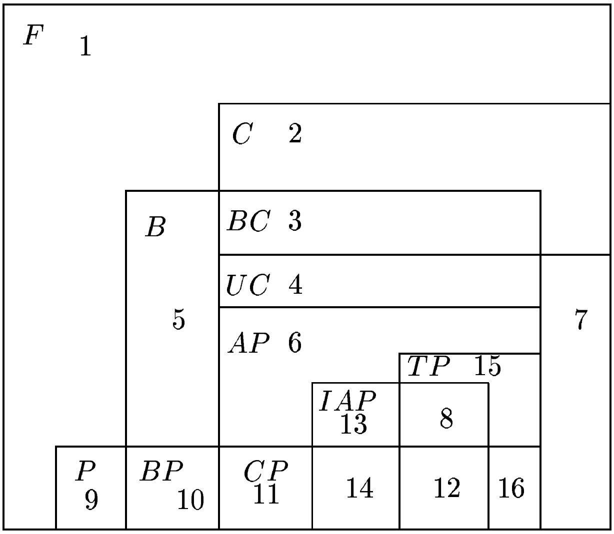
f is the function of the example 4.11

4.15. $f \in TP \setminus (IAP \cup P)$

(Example 2.1 and 3.2)

4.16. $f \in (TP \cap CP) \setminus IAP$

(Examples 2.1 and 3.2)



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OPTIMAL QUADRATURE FORMULAS WITH RESPECT TO THE ERROR

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Dedicated to Professor Ştefan Cobzaş at his 60^{th} anniversary

Abstract. Using the connection between the optimal approximation of linear operators and spline interpolation, estabilished by I.J. Schoenberg [12], is studied the optimality problem with respect to the error, for some quadrature formulas. Concrete examples are given.

Suppose that $f \in C^r[a,b]$ and $a = x_0 < x_1 < ... < x_n = b$. By $\varphi-$ function method [8], one obtains

$$\int_a^b f(x)dx = \sum_{k=0}^{n} \sum_{j=0}^{r-1} A_{kj} f^{(j)}(x_k) + R_n(f) \tag{1}$$

with

$$R_n(f) = (-1)^r \int_a^b \varphi(x) f^{(r)}(x)dx \tag{2}$$

where

$$\varphi(x) = \frac{(x-x_n)^r}{r!} + (-1)^{r+1} \sum_{k=0}^{n} \sum_{j=0}^{r-1} A_{kj} \frac{(x_k - x)_+^{r-j-1}}{(r-j-1)!} \tag{3}$$

and

$$A_{0\upsilon} = (-1)^{\upsilon+1} \varphi_1^{(r-\upsilon-1)}(x_0), \upsilon = 0, ..., r-1 \tag{4}$$

$$A_{i\upsilon} = (-1)^{\upsilon} (\varphi_i - \varphi_{i+1})^{(r-\upsilon-1)}(x_i), i = 1, ..., n-1, \upsilon = 0, ..., r-1$$

$$A_{n\upsilon} = (-1)^{\upsilon} \varphi_n^{(r-\upsilon-1)}(x_n), \upsilon = 0, ..., r-1,$$

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with

$$\varphi_i = \varphi \mid_{[x_{i-1}, x_i]}$$

i.e.

$$\varphi_i(x) = \frac{(x - x_n)^r}{r!} + (-1)^{r+1} \sum_{k=i}^{n} \sum_{j=0}^{r-1} A_{kj} \frac{(x_k - x)^{r-j-1}}{(r-j-1)!}. \tag{5}$$

Indeed, applying the integration by parts method, for $\varphi_k^{(r)} = 1, k = 1, ..., n$, one obtains

$$\int_a^b f(x)dx = \sum_{k=1}^{n} \int_{x_{k-1}}^{x_k} \varphi_k^{(r)}(x) f(x)\, dx =$$

$$= \sum_{k=1}^{n} \Big\{ \Big[\varphi_k^{(r-1)}(x) f(x) - \varphi_k^{(r-2)}(x) f'(x) + ... + (-1)^{r-1}$$

$$\varphi_k^{'}(x) f^{(r-1)}(x) \Big]_{x_{k-1}}^{x_k} + (-1)^r \int_{x_{k-1}}^{x_k} \varphi_k(x) f^{(r)}(x)\, dx \Big\}$$

$$= \sum_{v=0}^{r-1} (-1)^{v+1} \varphi_1^{(r-v-1)}(x_0) f^{(v)}(x_0)$$

$$+ \sum_{i=1}^{n-1} \sum_{v=0}^{r-1} (-1)^v [\varphi_i - \varphi_{i+1}]^{(r-v-1)}(x_i) f^{(v)}(x_i)$$

$$+ \sum_{v=0}^{r-1} (-1)^v \varphi_n^{(r-v-1)}(x_n) + (-1)^r \int_a^b \varphi(x) f^{(r)}(x) dx$$

and $(2) - (4)$ follows.

Remark 1. *φ_i is an algebraic polynomial of the degree r.*

Now, if $f \in H^{r,2}[a,b]$ then, from (2), one obtains

$$|R_n(f)| \le \left\| f^{(r)} \right\|_2 \left(\int_a^b |\varphi(x)|^2\, dx \right)^{1/2}$$

Definition 2. *The quadrature formula (1) for which*

$$F(A, X) := \int_a^b |\varphi(x)|^2\, dx$$

takes the minimum value, with respect to the coefficients $A := (A_{kj})_{k=\overline{o,n};j=\overline{0,r-1}}$ *and the nodes* $X := (x_k)_{k=1,n-1}$*, is called optimal with respect to the approximation error (or simple error).*

Remark 3. *From (2) it follows that the degree of exactness of the quadrature formula (1) is at least* $r-1$.

There are different ways to construct an optimal quadrature formula.

One of this way is based on the relationship between the optimal approximation of linear operators problem and the problem of spline interpolation, established by I.J.Schoemberg [12].

More precisely, if $S : H^{r,2}[a,b] \to \Im_{2r-1}(\Lambda_H)$ is the natural spline interpolation operator of the order $2r-1$, suitable to Λ_H:

$$\Lambda_H(f) = \left\{\lambda_{kj}(f) := f^{(j)}(x_k) \mid k = 0, ..., n; j = 0, ..., r-1\right\},$$

and

$$f = S_r f + R_r f$$

is the natural spline interpolation formula generated by S then

$$\int_a^b f(x)dx = \int_a^b (S_r f)(x)dx + \int_a^b (R_r f)(x)dx$$

is the corresponding optimal quadrature formula.

Now, if $s_{kj}, k = 0, ..., n; j = 0, ..., r-1$ are the corresponding cardinal splines, i.e.

$$S_r f = \sum_{k=0}^{n}\sum_{j=0}^{r-1} s_{kj} f^{(j)}(x_k) \tag{6}$$

then the optimal coefficients for fixed nodes $x_k, k = 1, ..., n$ are

$$\bar{A}_{kj} = \int_a^b s_{kj}(x)dx, k = 0, ..., n; j = 0, ..., r-1 \tag{7}$$

and

$$\bar{R}_n(f) = \int_a^b \bar{K}_r(t) f^{(r)}(t)dt$$

with

$$\bar{K}_r(t) := \int_a^b (Rf)(x)dx = \frac{(b-t)^r}{r!} - \sum_{k=0}^{n}\sum_{j=0}^{r-1} \bar{A}_{kj}\frac{(x_k-t)_+^{r-j-1}}{(r-j-1)!} \tag{8}$$

Remark 4. *If the quadrature nodes $x_k, k = 1, ..., n-1$ are given (fixed), then the quadrature formula*

$$\int_a^b f(x)\,dx = \sum_{k=0}^{n}\sum_{j=0}^{r-1} \bar{A}_{kj} f^{(j)}(x_k) + \bar{R}_n(f)$$

with

$$\left|\bar{R}_n(f)\right| \leq \left\|f^{(r)}\right\|_2 (\int_a^b \bar{K}_r^2(t)dt)^{1/2},$$

is optimal in sense of Sard and the optimality problem is solved.

Suppose, next, that the quadrature nodes $x_k, k = 1, ..., n$ are free. In this case, the optimality problem can be continued, in a secered step, by minimizing with respect to the free parameters $x_k, k = 1, ..., n$ the functional

$$F\left(\bar{A}, X\right) := \int_a^b \bar{K}_r^2(t)dt$$

Such, the optimal nodes, say $x_k^*, k = 1, ..., n-1$, are obtained as a solution of the system

$$\frac{\partial F\left(\bar{A}, X\right)}{\partial x_i} := 2\int_a^b \bar{K}_r(t)\left[-\sum_{j=0}^{r-1} \bar{A}_{ij} \frac{(x_i - t)_+^{r-j-2}}{(r-j-2)!}\right] dt = 0, i = 1, ..., n-1$$

Substituting in (7) the nodes x_k by x_k^* for all $k = 1, ..., n-1$, are obtained also the optimal coefficients $A_{kj}^*, k = 1, ..., n-1; j = 0, ..., r-1$.

Finally, for the optimal error, we have

$$|R_n^*(f)| \leq \left\|f^{(r)}\right\|_2 (\int_a^b [K_r^*(t)]^2 dt)^{1/2}$$

where

$$K_r^*(t) = \frac{(b-t)^r}{r!} - \sum_{k=0}^{n}\sum_{j=0}^{r-1} A_{kj}^* \frac{(x_n^* - t)_+^{r-1}}{(r-1)!}.$$

Next, two examples are given:

E1. Find the quadrature formula

$$\int_0^1 f(x)\,dx = \sum_{k=0}^{n} A_k f(x_k) + R_n(f)$$

that is optimal with respect to the error.

In the first step, one construct the linear spline function

$$S_1 f = \sum_{k=0}^{n} s_k f(x_k)$$

that interpolates the data $\Lambda(f) = \{f(0), f(x_1), ..., f(x_{n-1}), f(1)\}$, where $s_k, k = 0, ..., n$ are the corresponding cardinal splines, i.e.:

$$\begin{aligned}
s_0(x) &= 1 - \frac{1}{x_1}x + \frac{1}{x_1}(x - x_1)_+ \\
s_1(x) &= \frac{x}{x_1} - \frac{x_2}{x_1(x_2 - x_1)}(x - x_1)_+ + \frac{1}{x_2 - x_1}(x - x_2)_+ \\
s_k(x) &= \frac{1}{x_k - x_{k-1}}(x - x_{k-1})_+ - \frac{x_{k+1} - x_{k-1}}{(x_k - x_{k-1})(x_{k+1} - x_k)}(x - x_k)_+ + \\
&\quad + \frac{1}{x_{k+1} - x_k}(x - x_{k+1})_+, \; k = 2, ..., n-1 \\
s_n(x) &= \frac{1}{x_n - x_{n-1}}(x - x_{n-1})_+ - \frac{1}{x_n - x_{n-1}}(x - 1)_+
\end{aligned}$$

It follows that the optimal coefficients for fixed nodes $x_k, k = 1, ..., n-1$ and the corresponding kernel, are

$$\begin{aligned}
\bar{A}_0 &= \frac{x_1}{2} \\
\bar{A}_1 &= \frac{x_2}{2} \\
\bar{A}_k &= \frac{x_{k-1} - x_{k+1}}{2}, k = 2, ..., n-1 \\
\bar{A}_n &= \frac{1 - x_{n-1}}{2}
\end{aligned}$$

respectively

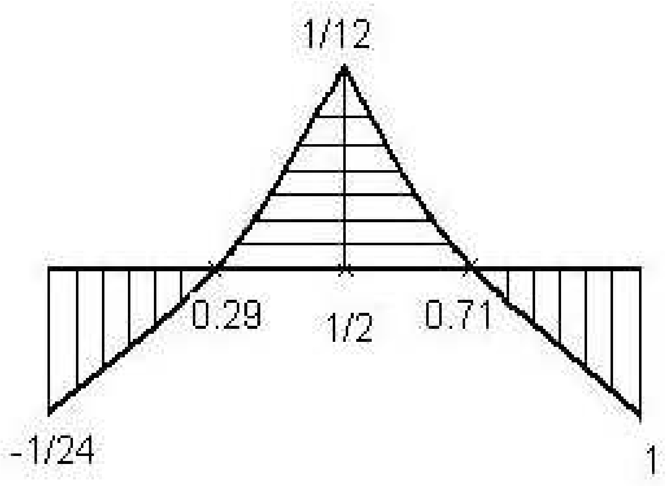
$$\bar{K}_1(t) = 1 - t - \sum_{k=0}^{n} \bar{A}_k (x_k - t)_+^0$$

So,

$$F\left(\bar{A}, X\right) := \int_0^1 \bar{K}_1^2(t)dt = -\frac{1}{12}\sum_{k=1}^{n}(x_k - x_{k-1})^3$$

Now, from the system

$$\frac{\partial F\left(\bar{A}, X\right)}{\partial x_j} := \frac{1}{4}[(x_j - x_{j-1})^2 - (x_{j+1} - x_j)^2] = 0, j = 1, ..., n-1$$



one obtains

$$x_j^* - x_{j-1}^* = x_{j+1}^* - x_j^*, j = 1, ..., n-1$$

i.e. the optimal nodes are

$$x_j^* = \frac{j}{n}, j = 0, 1, ..., n. \tag{9}$$

It follows that

$$A_0^* = \frac{1}{2n}, A_k^* = \frac{1}{n}, k = 1, ..., n-1; A_n^* = \frac{1}{2n} \tag{10}$$

are the optimal coefficients,

$$K_1^*(t) = 1 - t - \sum_{k=0}^{n} A_k^* (\frac{k}{n} - t)_+^0 \tag{11}$$

is the optimal kernel and

$$\int_0^1 (K^*(t))^2 dt = \frac{1}{12n^2}.$$

Finally, for the optimal error, we have

$$|R_n^*(f)| \leq \frac{1}{2n\sqrt{3}} \|f'\|_2$$

This way, is proved the following theorem:

Theorem 5. *If $f \in C^1[0,1]$ then the quadrature formula of the form*

$$\int_0^1 f(x)\,dx = \sum_{k=0}^{n} A_k f(x_k) + R_n(f)$$

which is optimal with respect to the error is given by the nodes and the coefficients of (9), respectively (10). Approximation error is estimated in (12)

Remark 6. *An interesting property of the kernel function K^* (figure 1), is that the domains placed upper respectively under the Ox axis have the equal areas.*

E2. The problem is to construct the quadrature formula of the form

$$\int_a^b f(x)\,dx = A_{01} f'(0) + A_{10} f(\alpha) + A_{11} f'(1) + R_2(f) \tag{12}$$

that is optimal with regard to the error.

Let

$$f = S_3 f + R_3 f$$

be the corresponding cubic spline interpolation formula. We have

$$(S_3 f)(x) = s_{01}(x) f'(0) + s_{10}(x) f(\alpha) + s_{21}(x) f'(1)$$

where

$$s_{01}(x) = \alpha\left(\frac{1}{2}\alpha - 1\right) + x - \frac{1}{2}x^2 + \frac{1}{2}(x-1)_+^2$$

$$s_{10}(x) = 1$$

$$s_{21}(x) = -\frac{1}{2}\alpha^2 + \frac{1}{2}x^2 - \frac{1}{2}(x-1)_+^2$$

Then

$$\bar{A}_{01} := \int_0^1 s_{01}(x)\,dx = \frac{1}{3} + \alpha\left(\frac{1}{2}\alpha - 1\right)$$

$$\bar{A}_{10} := \int_0^1 s_{10}(x)\,dx = 1$$

$$\bar{A}_{21} := \int_0^1 s_{21}(x)\,dx = \frac{1}{6} - \frac{1}{2}\alpha^2.$$

One obtains

$$\int_0^1 f(x)\,dx = \bar{A}_{01} f'(0) + \bar{A}_{10} f(\alpha) + \bar{A}_{21} f'(1) + \bar{R}_3(f)$$

with

$$\bar{R}_3(f) = \int_0^1 \bar{K}_3(t)\, f''(t)\,dt$$

where

$$\bar{K}_3(t) = \frac{(1-t)^2}{2} - (\alpha - t)_+ + \frac{1}{2}\left(\alpha^2 - \frac{1}{3}\right)$$

Taking into account that

$$|R_3(f)| \le \|f''\|_2 \,(\int_0^1 \bar{K}_3^2(t)\,dt)^{1/2}$$

next it must be minimized the function

$$F(\alpha) = \int_0^1 \bar{K}_3^2(t)\,dt$$

with respect to the parameter α.

We have

$$\min_{0<\alpha<1} F(\alpha) = F\left(\frac{1}{2}\right)$$

and

$$F\left(\frac{1}{2}\right) = \frac{1}{2^8 * 15}$$

It follows that the optimal parameters of the quadrature formula (12) are:

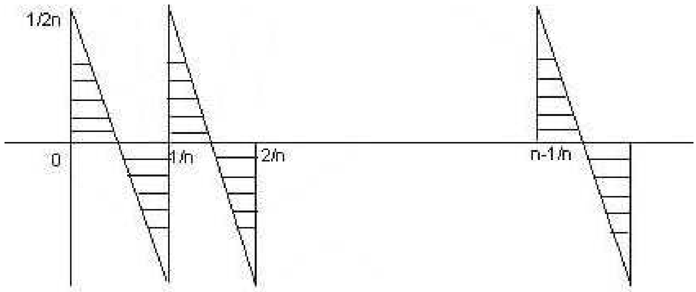
$$\alpha = \frac{1}{2};\, A_{01}^* = -\frac{1}{24};\, A_{10}^* = 1;\, A_{21}^* = \frac{1}{24} \qquad (13)$$

and the optimal kernel is:

$$K_3^*(t) = \frac{(1-t)^2}{2} - \left(\frac{1}{2} - t\right)_+ - \frac{1}{24}.$$

Theorem 7. *If $f \in C^2[0,1]$ then the quadrature formula of the form (12), that is optimal with respect to the error, is given by the parameter of (13), while for the optimal error we have*

$$|R_3^*(f)| \le \frac{1}{16\sqrt{15}}\,\|f''\|_2$$



Remark 8. *Also, the function K_3^* (figure 2) has the property that the area of the domains placed upper Ox axes is equal to the area of the domain placed under Ox axes.*

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HOMOMORPHS WITH RESPECT TO WHICH ANY HALL π-SUBGROUP OF A FINITE π-SOLVABLE GROUP IS A PROJECTOR

RODICA COVACI

Dedicated to Professor Ştefan Cobzaş at his 60^{th} *anniversary*

Abstract. Let π be a set of primes. The paper studies some special homomorphs of finite π-solvable groups, proving that some of them are Schunck classes. These homomorphs are used to give conditions on an arbitrary homomorph $\underline{X}$, such that any Hall π-subgroup of a finite π-solvable group G to be an $\underline{X}$-projector of G. Particularly, for π the set of all primes, one obtain the converse of a result given by W. Gaschütz in [8].

1. **Preliminaries**

In [4] we gave conditions with respect to which any $\underline{X}$-projector H of a finite π-solvable group G in a Hall π-subgroup of G, where $\underline{X}$ is a π-closed Schunck class with the P property. It is the aim of this paper to solve the converse problem: to give conditions on an arbitrary homomorph $\underline{X}$, such that any Hall π-subgroup H of a finite π-solvable group G to be an $\underline{X}$-projector of G. This problem leads us to the study of some special homomorphs, some of them being Schunck classes.

All groups considered in this paper are finite. Let π be a set of primes, π' the complement to π in the set of all primes and $O_{\pi'}(G)$ the largest normal π'-subgroup of a group G.

We first remind some useful definitions and theorems.

Definition 1.1. ([8], [11]) a) A class $\underline{X}$ of groups is a *homomorph* if $\underline{X}$ is epimorphically closed, i.e. if $G \in \underline{X}$ and N is a normal subgroup of G, then $G/N \in \underline{X}$.

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b) A group G is *primitive* if G has a *stabilizer*, i.e. a maximal subgroup H of G with $core_G H = \{1\}$, where $core_G H = \cap\{H^g / g \in G\}$.

c) A homomorph $\underline{X}$ is a *Schunck class* if $\underline{X}$ is *primitively closed*, i.e. if any group G, all of whose primitive factor groups are in $\underline{X}$, is itself in $\underline{X}$.

Definition 1.2. a) A positive integer n is said to be a *π-number* if for any prime divisor p of n we have $p \in \pi$.

b) A finite group G is a *p-group* if $|G|$ is a π-number.

Definition 1.3. ([6]) A group G is *π-solvable* if every chief factor of G is either a solvable π-group or a π'-group. For π the set of all primes, we obtain the notion of *solvable group.*

Definition 1.4. A class $\underline{X}$ of groups is said to be *π-closed* if

$$G/O_{\pi'}(G) \in \underline{X} \;\Rightarrow\; G \in \underline{X}.$$

A π-closed homomorph, respectively a π-closed Schunck class is called *π-homomorph*, respectively *π-Schunck class.*

Definition 1.5. ([7], [8]) Let $\underline{X}$ be a class of groups, G a group and H a subgroup of G.

a) H is an *$\underline{X}$-maximal subgroup* of G if:

i) $H \in \underline{X}$;

ii) $H \leq H^* \leq G$, $H^* \in \underline{X}$ imply $H = H^*$.

b) H is an *$\underline{X}$-projector* of G if, for any normal subgroup N of G, HN/N is $\underline{X}$-maximal in G/N.

c) H is an *$\underline{X}$-covering subgroup* of G if:

i) $H \in \underline{X}$;

ii) $H \leq K \leq G$, $K_0 \lhd K$, $K/K_0 \in \underline{X}$ imply $K = HK_0$.

Definition 1.6. ([3]) Let $\underline{X}$ be a class of groups. We say that $\underline{X}$ has the *P property* if, for any π-solvable group G and for any minimal normal subgroup M of G such that M is a π'-group, we have $G/M \in \underline{X}$.

Theorem 1.7. ([1]) *A solvable minimal normal subgroup of a finite group is abelian.*

Theorem 1.8. ([8]) *Let $\underline{X}$ be a class of groups, G a group and H a subgroup of G. H is an $\underline{X}$-projector of G if and only if:*

a) H is an $\underline{X}$-maximal subgroup of G;

b) HM/M is an $\underline{X}$-projector of G/M, for all minimal normal subgroups M of G.

Theorem 1.9. ([2]) *a) Let $\underline{X}$ be a class of groups, G a group and H a subgroup of G. If H is an $\underline{X}$-covering subgroup of G or H is an $\underline{X}$-projector of G, then H is $\underline{X}$-maximal in G.*

b) If $\underline{X}$ is a homomorph and G is a group, then a subgroup H of G is an $\underline{X}$-covering subgroup of G if and only if H is an $\underline{X}$-projector in any subgroup K of G with $H \subseteq K$.

Theorem 1.10. *Let $\underline{X}$ be a homomorph.*

a) ([7]) If H is an $\underline{X}$-covering subgroup of a group G and N is a normal subgroup of G, then HN/N is an $\underline{X}$-covering subgroup of G/N.

b) ([8]) If H is an $\underline{X}$-projector of a group G and N is a normal subgroup of G, then HN/N is an $\underline{X}$-projector of G/N.

c) ([7]) If H is an $\underline{X}$-covering subgroup of G and $H \leq K \leq G$, then H is an $\underline{X}$-covering subgroup of K.

Theorem 1.11. ([5]) *Let $\underline{X}$ be a π-homomorph. The following conditions are equivalent:*

(1) $\underline{X}$ is a Schunck class;

(2) any π-solvable group has $\underline{X}$-covering subgroups;

(3) any $\underline{X}$-solvable group has $\underline{X}$-projectors.

2. **Some properties of the Hall π-subgroups in finite π-solvable groups**

The Hall subgroups were introduced in [9], where Ph. Hall studied them in finite solvable groups. In [6], S. A. Čunihin extended this study to finite π-solvable groups.

Definition 2.1. Let G be a group and H a subgroup of G.

a) H is a *π-subgroup* of G if H is a π-group.

b) H is a *Hall π-subgroup* of G if:

i) H is a π-subgroup of G;

ii) $(|H|, |G : H|) = 1$.

We shall use some properties of the Hall π-subgroups, which we give below.

Theorem 2.2. ([10]) *Let G be a group and G a Hall π-subgroup of G.*

a) If $H \leq K \leq G$, then H is a Hall π-subgroup of K.

b) If N is a normal subgroup of G, then HN/N is a Hall π-subgroup of G/N.

Theorem 2.3. (Ph. Hall, S. A. Čunihin) ([10]) *If G is a π-solvable group, then:*

a) G has Hall π-subgroup and G has Hall π'-subgroups;

b) any two Hall π-subgroups of G are conjugate in G; any two Hall π'-subgroups of G are conjugate in G too.

We now prove a consequence of theorems 2.2 and 2.3.

Theorem 2.4. *Let G be a π-solvable group. If H is a Hall π-subgroup of G and H^* is a π-subgroup of G such that $H \subseteq H^*$, then $H = H^*$.*

Proof. By 2.2.a), H is a Hall π-subgroup of H^*. But H^* being a π-group and $|H^*|$ and $|H^* : H^*| = 1$ being coprime, it follows that H^* is a Hall π-subgroup of H^*. Applying now 2.3.b), we obtain that H and H^* are conjugate in H^*, i.e. there is an element $x \in H^*$ such that $H = (H^*)^x = H^*$. $\square$

Finally we give a result proved in [4]:

Theorem 2.5. ([4]) *Let G be a π-solvable group, H a subgroup of G and N a normal subgroup of G. If HN/N is a Hall π-subgroup of G/N and H is a Hall π-subgroup of HN, then H is a Hall π-subgroup of G.*

3. Some useful homomorphs

Let π be an arbitrary set of primes. Of special interest for our considerations will be the following classes of finite π-solvable groups:

Notations 3.1.

$\underline{W}_\pi = \{G/G \text{ finite } \pi - \text{solvable group}\}$;

$\underline{G}_\pi = \{G \in \underline{W}_\pi / G\pi - \text{group}\}$;

$\underline{G}_{\pi'} = \{G/G\pi' - \text{group}\}$;

$\underline{K}_\pi = \{G \in \underline{W}_\pi / O_{\pi'}(G) \neq 1\}$;

$\underline{M}_\pi = \underline{W}_\pi \setminus \underline{K}_\pi = \{G \in \underline{W}_\pi / O_{\pi'}(G) = 1\}$.

Remark 3.2. $\underline{G}_\pi \subseteq \underline{M}_\pi \subseteq \underline{W}_\pi$.

We now give some properties of the above classes.

Theorem 3.3. *$\underline{W}_\pi$ is a π-Schunck class.*

Proof. $\underline{W}_\pi$ is a homomorph. Indeed, if G is a π-solvable group and N is a normal subgroup of G, then G/N is a π-solvable group.

$\underline{W}_\pi$ is π-closed, since if $G/O_{\pi'}$ is a π-solvable group, then, observing that $O_{\pi'}(G)$ is π-solvable, we deduce that G is π-solvable.

In order to prove that the π-homomorph $\underline{W}_\pi$ is a Schunck class, it suffices to notice that any π-solvable group G is its own $\underline{W}_\pi$-covering subgroup. Applying 1.11, we obtain that $\underline{W}_\pi$ is a Schunck class. □

Theorem 3.4. *$\underline{G}_\pi$ is a homomorph.*

Proof. Let $G \in \underline{G}_\pi$ and let N be a normal subgroup of G. Then G/N is π-solvable and, $|G/N|$ being a divisor of $|G|$, G/N is a π-group. So $G/N \in \underline{G}_\pi$. □

Theorem 3.5. *a) $\underline{K}_\pi$ consists of all π-solvable groups G for which there is a minimal normal subgroup M of G, such that M is a π'-group.*

b) $\underline{K}_\pi$ is a homomorph.

Proof. a) Let $G \in \underline{K}_\pi$. It follows that G is π-solvable and $O_{\pi'}(G) \neq 1$. Hence there is a minimal normal subgroup M of G, such that $M \subseteq O_{\pi'}(G)$. So M is a π'-group.

Conversely, if G is a π-solvable group and there is a minimal normal subgroup M of G, such that M is a π'-group, then $M \subseteq O_{\pi'}(G)$ and so $O_{\pi'}(G) \neq 1$.

b) Let $G \in \underline{K}_\pi$ and let L be a normal subgroup of G. Then, G being π-solvable, G/L is also π-solvable. Let us prove that $O_{\pi'}(G/L) \neq 1$. Indeed, we notice that $O_{\pi'}(G)L$ is normal in G and so $O_{\pi'}(G)L/L$ is normal in G/L. But $O_{\pi'}(G)L/L \cong O_{\pi'}(G)/(O_{\pi'}(G) \cap L)$ is a π'-group. It follows that $O_{\pi'}(G)L/L \subseteq O_{\pi'}(G/L)$. From $O_{\pi'}(G) \neq 1$ we deduce that $O_{\pi'}(G)L/L \neq 1$ and so $O_{\pi'}(G/L) \neq 1$. □

Theorem 3.6. *$\underline{M}_\pi$ consists of all π-solvable groups G for which any minimal normal subgroup M of G is a solvable π-group.*

Proof. Let $G \in \underline{M}_\pi$. Then G is a π-solvable group and $O_{\pi'}(G) = 1$. Let M be a minimal normal subgroup of G. G being π-solvable, M is either a solvable

π-group or a π'-group. But π'-group implies $M \subseteq O_{\pi'}(G) = 1$, hence $M = 1$, which is a contradiction with the fact that M is a minimal normal subgroup of G. It follows that M is a solvable π-group.

Conversely, let G be a π-solvable group, such that any minimal normal subgroup M of G is a solvable π-group. This means that G has not minimal normal subgroups which are π'-groups. We must prove that $O_{\pi'}(G) = 1$. Suppose that $O_{\pi'}(G) \neq 1$. It follows that there is a minimal normal subgroup M of G, such that $M \subseteq O_{\pi'}(G)$. So M is a π'-group, in contradiction with the above. □

Theorem 3.7. *a)* $\underline{G}_{\pi'} \subseteq \underline{W}_{\pi}$*;*

b) $\underline{G}_{\pi'}$ *is a* π*-Schunck class. Furthermore, for any finite* π*-solvable group* G*,* H *is an* $\underline{G}_{\pi'}$*-covering subgroup of* G *if and only if* H *is a Hall* π'*-subgroup of* G*.*

Proof. Let G be a π'-group. Then any chief factor M/N of G is a π'-group. Hence G is π-solvable.

b) We prove that $\underline{G}_{\pi'}$ is a Schunck class using theorem 1.11. In [5], we proved that $\underline{G}_{\pi'}$ is a π-homomorph and that a subgroup H of a π-solvable group G is an $\underline{G}_{\pi'}$-covering subgroup of G if and only if H is a Hall π'-subgroup of G. So, by 1.11, $\underline{G}_{\pi'}$ is a π-Schunck class.

As a new fact, by using the properties given in 2.2 and 2.5, we give here a new proof of the following result: If G is a π-solvable group and H is an $\underline{G}_{\pi'}$-covering subgroup of G, then H is a Hall π'-subgroup of G.

Let G be a π-solvable group and H an $\underline{G}_{\pi'}$-covering subgroup of G. We prove, by induction on $|G|$, that H is a Hall π'-subgroup of G. Two cases are possible:

1) $H = G$. Then the result is obvious.

2) $H \neq G$. Let M be a minimal normal subgroup of G. By 1.10.a), HM/M is an $\underline{G}_{\pi'}$-covering subgroup of G/M, hence, by induction, HM/M is a Hall π'-subgroup of G/M. By 1.10.c), H is an $\underline{G}_{\pi'}$-covering subgroup of HM. We now consider two cases:

a) $HM \neq G$. By the induction, H is a Hall π'-subgroup of HM. Then, by 2.5, H is a Hall π'-subgroup of HM. Then, by 2.5, H is a Hall π'-subgroup of G.

b) $HM = G$. Then $HM/M = G/M$. But HM/M being a Hall π'-subgroup of G/M, we obtain that G/M is a π'-group. We prove that $|G : H|$ is a π-number. For the minimal normal subgroup M of the π-solvable group G we have two possibilities:

b_1) M is a solvable π-group. Then $|G : H| = |HM : H| = |M : M \cap H|$ divides $|M|$ and so $|G : H|$ is a π-number.

b_2) M is a π'-group. Then $|G| = |G/M||M|$ is a π'-number. So $G \in \underline{G}_{\pi'}$. But H being an $\underline{G}_{\pi'}$-covering subgroup of G, it follows that H is $\underline{G}_{\pi'}$-maximal in G. Then $H = G$, in contradiction with our assumption. □

The last results of this section refer to the connection of the classes $\underline{K}_\pi$ and $\underline{M}_\pi$ to the π-homomorphs with the P property studied in [3].

Theorem 3.8. *If $\underline{X}$ is a π-homomorph with the P property, then $\underline{K}_\pi \subseteq \underline{X}$.*

Proof. Let $G \in \underline{K}_\pi$. By 3.5.a), G is π-solvable and there is a minimal normal subgroup M of G, such that M is a π'-group. Then $M \subseteq O_{\pi'}(G)$, hence

$$G/O_{\pi'}(G) \cong (G/M)(O_{\pi'}(G)/M). \tag{1}$$

But $\underline{X}$ has the P property and so $G/M \in \underline{X}$ and $\underline{X}$ being a homomorph we deduce from (1) that $G/O_{\pi'}(G) \in \underline{X}$. By the π-closure of $\underline{X}$, $G \in \underline{X}$. So $\underline{K}_\pi \subseteq \underline{X}$. □

Theorem 3.9. *If $\underline{X}$ is a π-homomorph, such that $\underline{X} \subseteq \underline{M}_\pi$, then $\underline{X}$ has not the P property.*

Proof. Suppose that $\underline{X}$ has the P property. Then, by 3.8, we have $\underline{K}_\pi \subseteq \underline{X}$. But $\underline{X} \subseteq \underline{M}_\pi$. We obtain the contradiction $\underline{K}_\pi \subseteq \underline{M}_\pi$, where $\underline{M}_\pi = \underline{W}_\pi \setminus \underline{K}_\pi$. □

4. When are the Hall π-subgroups projectors in finite π-solvable groups

In [4], we gave conditions with respect to which an $\underline{X}$-projector H of a finite π-solvable G is a Hall π-subgroup of G, where $\underline{X}$ is a π-closed Schunck class with the P property.

Here we study the converse problem: to find conditions on the Schunck class $\underline{X}$, such that any Hall π-subgroup H of a finite π-solvable group G to be an $\underline{X}$-projector of G.

The main result is the following:

Theorem 4.1. *Let $\underline{X}$ be a homomorph, such that $\underline{G}_\pi \subseteq \underline{X} \subset \underline{M}_\pi$. If G is a finite π-solvable group and H is a Hall π-subgroup of G, then H is an $\underline{X}$-projector of G.*

Proof. By induction on $|G|$. Let G be a finite π-solvable group and H a Hall π-subgroup of G (H exists by 2.3.a)). We shall prove that H is an $\underline{X}$-projector of G, by verifying conditions (a) and (b) from theorem 1.8.

a) *H is $\underline{X}$-maximal in G.* Indeed, we shall prove below (i) and (ii) from 1.5.a).

i) $H \in \underline{X}$, since H being a Hall π-subgroup of G we have $H \in \underline{G}_\pi \subseteq \underline{X}$.

ii) $H \leq H^* \leq G$, $H^* \in \underline{X}$ imply $H = H^*$. In order to show this, we consider two cases:

α) $H^* \neq G$. In this case, $|H^*| < |G|$ and H being by 2.2.a) a Hall π-subgroup of H^*, we may apply the induction and obtain that H is an $\underline{X}$-projector of H^*, hence, by 1.9.a), H is $\underline{X}$-maximal in H^*. But $H^* \in \underline{X}$. So $H = H^*$.

β) $H^* = G$. Then $G \in \underline{X} \subset \underline{M}_\pi$. So we distinguish two cases:

β_1) There is a minimal normal subgroup M of G, such that $M \subseteq H$. By 2.2.b), H/M is a Hall π-subgroup of G/M. We notice that $|G/M| < |G|$. It follows by the induction that H/M is an $\underline{X}$-projector of G/M, hence, by 1.9.a), H/M is $\underline{X}$-maximal in G/M. But, $\underline{X}$ being a homomorph, $G \in \underline{X}$ implies $G/M \in \underline{X}$. So $H/M = G/M$, hence $H = G = H^*$.

β_2) For any minimal normal subgroup N of G, we suppose that N is not included in H. Since $G \in \underline{M}_\pi$, there is a minimal normal subgroup M of G, such that M is a solvable π-group. Then, by 1.7, M is abelian. We also have that M is not included in H.

By 2.2.b), HM/M is a Hall π-subgroup of G/M. By the induction, HM/M is an $\underline{X}$-projector of G/M, hence HM/M is $\underline{X}$-maximal in G/M. But, $\underline{X}$ being a homomorph, $G \in \underline{X}$ implies $G/M \in \underline{X}$. So $HM/M = G/M$, hence $HM = G$.

Let us prove that $H \cap M$ is normal in G. Let $g \in G$ and $x \in H \cap M$. Since $HM = G$, we have that $g = hm$, where $h \in H$, $m \in M$. Then

$$g^{-1}xg = (hm)^{-1}x(hm) = (m^{-1}h^{-1})x(hm) = m^{-1}(h^{-1}xh)m$$

$$= m^{-1}m(h^{-1}xh) = h^{-1}xh \in H \cap M,$$

where we applied that $H \cap M$ is normal in H and that M is abelian. So $H \cap M$ is normal in G. Furthermore, since M is not included in H, we have $H \cap M \neq M$ and M being a minimal normal subgroup of G, it follows that $H \cap M = 1$.

Finally we have

$$G/M = HM/M \cong H/M \cap M = H/1 \cong H,$$

which implies that $|G/M| = |H|$ and so G/M is a π-group. But M is a π-group too. So G is a π-group. But H is a Hall π-subgroup of G. Then, by 2.4, $H = G = H^*$. Condition a) is proved.

b) *HN/M is an $\underline{X}$-projector of G/M, for all minimal normal subgroups M of G.* Indeed, if M is a minimal normal subgroup of G, then by applying the induction for the π-solvable group G/M, with $|G/M| < |G|$, and for its Hall π-subgroup HM/M (see 2.2.b)), we obtain that HM/M is an $\underline{X}$-projector of G/M. □

Remark. Particularly, for π the set of all primes, theorem 4.1 represents the converse of a result given by W. Gaschütz in [8].

From the proof of theorem 4.1 we notice that this theorem can also be given in the following form:

Theorem 4.2. *Let $\underline{X}$ be a homomorph, such that $\underline{X} \subset \underline{M}_\pi$. If G is a finite π-solvable group and H is a Hall π-subgroup of G, such that we have $H \in \underline{X}$, then H is an $\underline{X}$-projector of G.*

Theorem 4.1 has the following important consequence:

Theorem 4.3. *Let $\underline{X}$ be a homomorph, such that $\underline{G}_\pi \subseteq \underline{X} \subset \underline{M}_\pi$. If G is a finite π-solvable group and H is a Hall π-subgroup of G, then H is an $\underline{X}$-covering subgroup of G.*

Proof. We use theorem 1.9.b). Let K be a subgroup of G, such that $H \subseteq K$. We prove that H is an $\underline{X}$-projector of K. By 2.2.a), H is a Hall π-subgroup of K. As a subgroup of the π-solvable group G, K is also a π-solvable group. Applying now theorem 4.1, H is an $\underline{X}$-projector of K. □

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ON CERTAIN PROPERTIES OF THE FRÉCHET DIFFERENTIAL OF HIGHER ORDER

ADRIAN DIACONU

Dedicated to Professor Ştefan Cobzaş at his 60^{th} *anniversary*

Abstract. In this paper we propose to give detailed proofs for different generalizations of the Leibnitz formula for the calculation of the derivative of the order n, with $n \in \mathbb{N}$, of the functions' product. We will consider the Fréchet derivative of certain composed functions with the help of certain multilinear mappings.

1. **Introduction**

The idea of this paper has its origin the well-known Leibniz's formula concerning the calculation of the derivative of the product of two real functions with real variables.

So, given the number $n \in \mathbb{N}$, the interval $\mathbb{I} \subseteq \mathbb{R}$ and the functions $f, g : \mathbb{I} \to \mathbb{R}$ that have the derivative of the order n, then the product function $fg : \mathbb{I} \to \mathbb{R}$ admits the derivative of the order n as well, and:

$$(fg)^{(n)} = \sum_{k=0}^{n} \binom{n}{k} f^{(n-k)} g^{(k)}, \quad \text{where} \quad \binom{n}{k} = \frac{n!}{k!\,(n-k)!},$$

for any function $h : \mathbb{I} \to \mathbb{R}$, $h^{(i)} : \mathbb{I} \to \mathbb{R}$ represents the derivative of the order i of the considered mapping.

A first generalization of this formula appears by considering the case of m functions with $m \in \mathbb{N}$, $f_1, \ldots, f_m : \mathbb{I} \to \mathbb{R}$. In this way, if these functions have

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derivatives of the order n, the same fact is true for the product function $f_1 \dots f_m : \mathbb{I} \to \mathbb{R}$ and:

$$(f_1 \dots f_m)^{(n)} = \sum_{\alpha_1 + \dots + \alpha_m = n} \frac{n!}{\alpha_1! \dots \alpha_m!} f_1^{(\alpha_1)} \dots f_m^{(\alpha_m)}.$$

We can raise the issue of extending these formulas to the case of using functions defined between linear normed spaces.

Of course in this case it is necessary to find a "substitute" for the notion of product, but it will be necessary to specify the definition used for the extension of the notion of derivative.

To begin with, we have:

Remark 1.1. *For the linear normed spaces* $(X, \|\cdot\|_X)$ *and* $(Y, \|\cdot\|_Y)$ *let us denote by* $(X, Y)^*$ *the set of the linear and continuous mappings* $T : X \to Y$*. The set* $(X, Y)^*$ *can be organized as a linear normed space with the usual operations that are the mappings' addition and multiplication with a real number, and the norm that for* $T \in (X, Y)^*$ *is defined through:*

$$\|T\| = \sup_{h \in X, \|h\|_X = 1} \|T(h)\|_Y .$$

It is easy to show that if $(Y, \|\cdot\|_Y)$ *is a Banach space, then the space* $\left((X, Y)^*, \|\cdot\|\right)$ *is a Banach space as well.*

Let us recall the following definition.

Definition 1.2. *Let be given the linear normed spaces* $(X, \|\cdot\|_X)$ *and* $(Y, \|\cdot\|_Y)$*, the set* $D \subseteq X$*, the function* $f : D \to Y$ *and the point* $x \in int(D)$*.*
The considered function is differentiable in the point x *in the Fréchet meaning that there exists a linear and continuous mapping* $T_x \in (X, Y)^*$ *and a mapping* $R_x : X \to Y$ *with:*

$$\lim_{h \to \theta_X} \|R_x(h)\|_Y = 0$$

so that for every $h \in X$ *the equality:*

$$f(x+h) - f(x) = T_x(h) + \|h\|_X R_x(h)$$

is true.

Now we have:

Remark 1.3. *For the linear normed spaces* $(X, \|\cdot\|_X)$, $(Y, \|\cdot\|_Y)$ *and a fixed element* $x \in X$ *let be the set:*

$$\mathcal{D}_x(X,Y) = \{ f / \exists_D \ D \subseteq X, x \in int(D), f : D \to Y, f \text{ differentiable at } x \}.$$

We can easily prove that if $f \in \mathcal{D}_x(X,Y)$ *the mapping* $T_x \in (X,Y)^*$ *exists with a unique determination. We will denote:*

$$f'(x) := T_x$$

and this mapping will be called a Fréchet differential of the mapping f *in the point* x.

Starting from the **definition 1.2** and using the successive differentiation and mathematical induction, we can introduce differentials of an order n, where $n \in \mathbb{N}$.

In order to clarify these questions, for $m \in \mathbb{N}$ we denote by $\left(X^{(m)}, Y\right)^*$ the set of the $m-$linear and continuous mappings which are defined from X^m to Y, where $X^m = \underbrace{X \times \cdots \times X}_{m \text{ times}}$.

We have:

Remark 1.4. *For any* $m \in \mathbb{N}$, *the set* $\left(X^{(m)}, Y\right)^*$ *can also be organized as a linear normed space using the mapping's addition and multiplication with a number. The norm in* $\left(X^{(m)}, Y\right)^*$ *for* $T \in \left(X^{(m)}, Y\right)^*$ *is defined through:*

$$\|T\| = \sup_{h_1,\ldots,h_n \in X, \|h_1\|_X = \cdots = \|h_n\|_X = 1} \|T(h_1, \ldots, h_n)\|_Y,$$

in addition, if $(Y, \|\cdot\|_Y)$ *is a Banach space,* $\left(X^{(m)}, Y\right)^*$ *is a Banach space as well.*

Therefore we have:

Definition 1.5. *In addition to the facts from the **definition 1.2** let us consider a number* $n \in \mathbb{N}$, $n \geq 2$. *If:*

a): *there exists a neighbourhood* V *of the points* x, *so that for every* $y \in V \cap D$ *it exists the differential of the order* $n-1$ *of the function* f *at the point* y *and* $f^{(n-1)}(y) \in \left(X^{(n-1)}, Y\right)^*$,

b): *the function* $f^{(n-1)}: V \cap D \to \left(X^{(n-1)}, Y\right)^*$ *is also differentiable at the point* x,

then $\left(f^{(n-1)}\right)'(x) \in \left(X^{(n)}, Y\right)^*$, *mapping which we will denote by* $f^{(n)}(x)$ *is called the differential of the order* n *of the function* f *at the point* x.

It is necessary to remind one more case. Let us consider the linear normed spaces:

$$\left(X_1, \|\cdot\|_{X_1}\right), \ \ldots \ , \left(X_m, \|\cdot\|_{X_m}\right), \ \left(Y, \|\cdot\|_Y\right)$$

and a mapping $T : X_1 \times \ \ldots \ \times X_m \to Y$. We can say that this mapping is an $m-$linear and continuous mapping, if this mapping is linear and continuous after every argument.

We denote by $(X_1, \ldots, X_m; Y)^*$ the set of all mappings that verify the aforementioned properties.

For $h = (h_1, \ldots, h_m) \in X_1 \times \ \ldots \ \times X_m$ we can define:

$$\|h\| = \max\left\{\|h_1\|_{X_1}, \ldots, \|h_m\|_{X_m}\right\}$$

and so $\left((X_1, \ldots X_m; Y)^*, \|\cdot\|\right)$ is a linear normed space. In the case if $\left(Y, \|\cdot\|_Y\right)$ is a Banach space, then $\left((X_1, \ldots X_m; Y)^*, \|\cdot\|\right)$ is a Banach space as well.

2. A generalization of Leibnitz's formula of derivation

Let us consider the linear normed spaces:

$$\left(X, \|\cdot\|_X\right), \ \left(Y_1, \|\cdot\|_{Y_1}\right), \ \ldots \ , \ \left(Y_m, \|\cdot\|_{Y_m}\right), \ \left(Z, \|\cdot\|_Z\right),$$

the set $D \subseteq X$, the nonlinear mappings $f_i : D \to Y_i; \ i = \overline{1, m}$ and the $m-$ linear mapping $L \in (Y_1, \ldots, Y_m; Z)^*$.

With the help of these elements we build the function:

$$F : D \to Z, \ \ F(x) = L\left(f_1(x), \ldots, f_m(x)\right). \qquad (1)$$

Our goal is to conclude, in the hypothesis of the differentiability of the functions $f_i : D \to Y_i; \ i = \overline{1, m}$, on the differentiability of the function (1) establishing connections between the differentials.

To start with, we have the following:

Lemma 2.1. *If the non-linear mappings* $f_i : D \to Y_i;\ i = \overline{1,m}$, *are differentiable at the point* $x \in int\,(D)$, *then the function* (1) *is also differentiable at the same point* x *and for any* $h \in X$ *we have the relation:*

$$F'(x)\,h = \\ = \sum_{k=1}^{m} L\left(f_1(x), \ldots, f_{k-1}(x), f_k'(x)\,h, f_{k+1}(x), \ldots, f_m(x)\right). \tag{2}$$

Proof. From the differentiability of the functions $f_i : D \to Y_i;\ i = \overline{1,m}$ at the point $x \in int\,(D)$ we deduce the existence, for any $i \in \{1, 2, \ldots, m\}$, of the linear mappings $f_i'(x) \in (X, Y_i)^*$ and of the non-linear mappings $R_x^{(i)} : X \to Y_i$, so that for any $h \in X$ we have:

$$f_i(x+h) = f_i(x) + f_i'(x)\,h + \|h\|_X\, R_x^{(i)}(h), \ \lim_{h \to \theta_X} \left\| R_x^{(i)}(h) \right\|_{Y_i} = 0.$$

So it is clear that:

$$F(x+h) = L\left(f_1(x+h), \ldots, f_m(x+h)\right)$$

is in fact the value of the mapping $L \in (Y_1, \ldots, Y_m; Z)^*$ on the arguments:

$$f_1(x) + f_1'(x)\,h + \|h\|_X\, R_x^{(1)}(h),\ \ldots\ ,\ f_m(x) + f_m'(x)\,h + \|h\|_X\, R_x^{(m)}(h).$$

In this way:

$$F(x+h) = L\left(f_1(x), \ldots, f_m(x)\right) + \\ + \sum_{k=1}^{m} L\left(f_1(x), \ldots, f_{k-1}(x), f_k'(x)\,h, f_{k+1}(x), \ldots, f_m(x)\right) +$$

$$+ \|h\|_X \sum_{k=1}^{m} L\left(f_1(x), \ldots, f_{k-1}(x), R_x^{(k)}(h), f_{k+1}(x), \ldots, f_m(x)\right) + \\ + \sum_{k=2}^{m} \sum_{1 \le i_1 < \cdots < i_k \le m} E_{i_1, \ldots, i_k}^{(k)}(f; x, h),$$

where $E^{(k)}_{i_1,\dots,i_k}(f;x,h) \in Z$ represents the value of the mapping $L \in (Y_1,\dots,Y_m;Z)^*$ on the arguments $f_1(x),\dots,f_m(x)$, with the exception of the positions $i_1,\dots,i_k \in \{1,2,\dots,m\}$ for which we have the arguments:

$$f'_{i_j}(x)\,h + \|h\|_X\, R^{(i_j)}_x(h)\,, \quad j=\overline{1,k};\; k=\overline{2,m}.$$

It is clear that if we define $F'(x) \in (X,Z)^*$, through the equality (2), and the mapping $R_x : X \to Z$ through:

$$R_x(h) = \begin{cases} \theta_Z & \text{for} \quad h=\theta_X, \\ P(x,h) + \dfrac{1}{\|h\|_X} Q(x,h) & \text{for} \quad h \neq \theta_X, \end{cases}$$

where we have denoted:

$$P(x,h) = \sum_{k=1}^{m} L\Big(f_1(x),\dots,f_{k-1}(x), R^{(k)}_x(h), f_{k+1}(x),\dots,f_m(x)\Big) \in Z$$

and:

$$Q(x,h) = \sum_{k=2}^{m} \sum_{1\le i_1<\cdots<i_k\le m} E^{(k)}_{i_1,\dots,i_k}(f;x,h) \in Z,$$

we will have:

$$F(x+h) - F(x) = F'(x)\,h + \|h\|_X\, R_x(h)\,. \tag{3}$$

It is clear that:

$$\|P(x,h)\|_Z \le \|L\| \sum_{k=1}^{m} \left(\left\| R^{(k)}_x(h) \right\|_{Y_k} \cdot \prod_{j=\overline{1,m};j\neq k} \|f_j(x)\|_{Y_j} \right)$$

and from $\lim\limits_{h\to\theta_X} \left\| R^{(k)}_x(h) \right\|_{Y_k} = 0$ we deduce:

$$\lim_{h\to\theta_X} \|P(x,h)\|_Z = 0. \tag{4}$$

Concerning the expression of $Q(x,h)$ we deduce:

$$\|Q(x,h)\|_Z \le \sum_{k=2}^{m} \sum_{1\le i_1<\cdots<i_k\le m} \left\| E^{(k)}_{i_1,\dots,i_k}(f;x,h) \right\|_Z$$

and for any $k \in \{2, 3, \ldots, m\}$ and $i_1, \ldots, i_k \in \{1, 2, \ldots, m\}$ with $1 \leq i_1 < \cdots < i_k \leq m$ we have:

$$\left\| E^{(k)}_{i_1,\ldots,i_k}(f; x, h) \right\|_Z \leq$$

$$\leq \|L\| \cdot \prod_{j \in \{1,\ldots,m\} \setminus \{i_1,\ldots,i_k\}} \|f_j(x)\|_{Y_j} \times \prod_{j=1}^{k} \left\| f'_{i_j}(x) h + \|h\|_X R^{(i_j)}_x(h) \right\|_{Y_{i_j}} \leq$$

$$\leq \|L\| \cdot \|h\|^k_X \cdot \mathbf{C}^{(k)}_{i_1,\ldots,i_k}(x, h),$$

where:

$$\mathbf{C}^{(k)}_{i_1,\ldots,i_k}(x, h) = \prod_{j \in \{1,\ldots,m\} \setminus \{i_1,\ldots,i_k\}} \|f_j(x)\|_{Y_j} \times \prod_{j=1}^{k} \left(\left\| f'_{i_j}(x) \right\| + \left\| R^{(i_j)}_x(h) \right\|_{Y_{i_j}} \right)$$

From the differentiability of the functions $f_1, \ldots, f_m$ we deduce clearly that:

$$\lim_{h \to \theta_X} \mathbf{C}^{(k)}_{i_1,\ldots,i_k}(x, h) = \prod_{j \in \{1,\ldots,m\} \setminus \{i_1,\ldots,i_k\}} \|f_j(x)\|_{Y_j} \times \prod_{j=1}^{k} \left\| f'_{i_j}(x) \right\|. \tag{5}$$

We have:

$$\|Q(x, h)\|_Z \leq \|L\| \cdot \|h\|_X \sum_{k=2}^{m} \|h\|^{k-1}_X \sum_{1 \leq i_1 < \cdots < i_k \leq m} \mathbf{C}^{(k)}_{i_1,\ldots,i_k}(x, h)$$

and from this relation we deduce for any $h \neq \theta_X$ the inequalities:

$$\begin{gathered} 0 \leq \|R_x(h)\|_Z \leq \\ \leq \|P(x, h)\|_Z + \|L\| \cdot \|h\|_X \sum_{k=2}^{m} \|h\|^{k-1}_X \sum_{1 \leq i_1 < \cdots < i_k \leq m} \mathbf{C}^{(k)}_{i_1,\ldots,i_k}(x, h) \end{gathered} \tag{6}$$

From the relations $(4) - (6)$ we deduce that:

$$\lim_{h \to \theta_X} \|R_x(h)\|_Z = 0. \tag{7}$$

The relations (3) and (7) indicate that the function (1) has a differential at the point $x \in int(D)$ and its value is given through the formula (2).

The lemma is proved. □

In order to pass to the expression of the differential of an order $n \in \mathbb{N}$ it is necessary to make certain specifications and to adopt certain notations.

To begin with, let be the set:

$$\mathbb{A}_{m,n} = \{\alpha / \alpha = (\alpha_1, \ldots, \alpha_m) \in (\mathbb{N} \cup \{0\})^m, \ \alpha_1 + \cdots + \alpha_m = n\}.$$

In certain cases we can use the notation $|\alpha|$ for $\alpha_1 + \cdots + \alpha_m$.

Considering a finite set $K \subseteq \mathbb{N}$, for a number $p \in \mathbb{N}$ we can consider the set:

$$\mathcal{C}_p(K) = \{i / i = (i_1, \ldots, i_p) \in \mathbb{K}^p, \ i_1 < \cdots < i_p\},$$

evidently $\mathcal{C}_p(K)$ represents the set of all subsets with p elements of the set K.

Evidently in the case in which the set K has q elements and $p \leq q$, then the set $\mathcal{C}_p(K)$ has $\binom{q}{p} = \frac{q!}{p!(q-p)!}$ elements, and if $p > q$ the set $\mathcal{C}_p(K)$ is a void set.

In the special case in which $K = \{1, 2, \ldots, n\}$, we will use the notation $\mathcal{C}_{n,k}$ for $\mathcal{C}_k(K)$, with $k \leq n$ and evidently this set has $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ elements.

Let us consider now the finite set $K \subseteq \mathbb{N}$ having n elements and we will build the sets $J_0, J_1, \ldots, J_m \subseteq K$ considering $J_0 = K$. Let us also consider for $m \in \mathbb{N}$ a system $\alpha = (\alpha_1, \ldots, \alpha_m) \in \mathbb{A}_{m,n}$.

Starting from these elements let us make the following construction.

To start with, we choose a system $\left(i_1^{(1)}, \ldots, i_{\alpha_1}^{(1)}\right) \in \mathcal{C}_{\alpha_1}(J_0)$.

Let be now the set $J_1 = J_0 \diagdown \left\{i_1^{(1)}, \ldots, i_{\alpha_1}^{(1)}\right\}$that has $n - \alpha_1$ elements. We choose a new system:

$$\left(i_1^{(2)}, \ldots, i_{\alpha_2}^{(2)}\right) \in \mathcal{C}_{\alpha_2}(J_1).$$

So there exist $\binom{n-\alpha_1}{\alpha_2} = \frac{(n-\alpha_1)!}{\alpha_2!(n-\alpha_1-\alpha_2)!}$ possibilities for the choice of this new system.

Further on, for the systems $\left(i_1^{(1)}, \ldots, i_{\alpha_1}^{(1)}\right)$ and $\left(i_1^{(2)}, \ldots, i_{\alpha_2}^{(2)}\right)$ that are chosen above and are fixed we consider the set $J_2 = J_1 \diagdown \left\{i_1^{(2)}, \ldots, i_{\alpha_2}^{(2)}\right\}$with $n - \alpha_1 - \alpha_2$ elements, then we choose a new system:

$$\left(i_1^{(3)}, \ldots, i_{\alpha_3}^{(3)}\right) \in \mathcal{C}_{\alpha_3}(J_2),$$

existing $\binom{n-\alpha_1-\alpha_2}{\alpha_3} = \frac{(n-\alpha-\alpha_2)!}{\alpha_3!(n-\alpha_1-\alpha_2-\alpha_3)!}$ possibilities for the choice of this new system.

We continue in this manner using mathematical induction.

Thus for the systems $\left(i_1^{(1)},\ldots,i_{\alpha_1}^{(1)}\right)$, ... , $\left(i_1^{(k-1)},\ldots,i_{\alpha_{k-1}}^{(k-1)}\right)$ already chosen and fixed, we consider the set:

$$J_{k-1}=J_{k-2}\diagdown\left\{i_1^{(k-1)},\ldots,i_{\alpha_{k-1}}^{(k-1)}\right\}=$$

$$=\{1,2,\ldots,n\}\diagdown\left\{i_1^{(1)},\ldots,i_{\alpha_1}^{(1)},\ldots,i_1^{(k-1)},\ldots,i_{\alpha_{k-1}}^{(k-1)}\right\},$$

that has $n-\alpha_1-\cdots-\alpha_{k-1}$ elements and we choose the new system $\left\{i_1^{(k)},\ldots,i_{\alpha_k}^{(k)}\right\}\in\mathcal{C}_{\alpha_k}\left(J_{k-1}\right)$ existing $\binom{n-\alpha_1-\cdots-\alpha_{k-1}}{\alpha_k}=\frac{(n-\alpha_1-\cdots-\alpha_{k-1})!}{\alpha_k!(n-\alpha_1-\cdots-\alpha_{k-1}-\alpha_k)!}$ possibilities for the choice of the new system.

At the end of this process we have already chosen and fixed the systems:

$$\left(i_1^{(1)},\ldots,i_{\alpha_1}^{(1)}\right),\ \ldots\ ,\ \left(i_1^{(m-1)},\ldots,i_{\alpha_{m-1}}^{(m-1)}\right)$$

we consider the set:

$$J_{m-1}=J_{m-2}\diagdown\left\{i_1^{(m-1)},\ldots,i_{\alpha_{m-1}}^{(m-1)}\right\}=$$

$$=\{1,2,\ldots,n\}\diagdown\left\{i_1^{(1)},\ldots,i_{\alpha_1}^{(1)},\ldots,i_1^{(m-1)},\ldots,i_{\alpha_{m-1}}^{(m-1)}\right\},$$

and we choose the new system $\left(i_1^{(m)},\ldots,i_{\alpha_m}^{(m)}\right)\in\mathcal{C}_{\alpha_m}\left(J_{m-1}\right)$ existing

$$\binom{n-\alpha_1-\cdots-\alpha_{m-1}}{\alpha_m}=\frac{(n-\alpha_1-\cdots-\alpha_{m-1})!}{\alpha_m!\,(n-\alpha_1-\cdots-\alpha_{m-1}-\alpha_m)!}$$

possibilities for the choice of the new system.

If we consider:

$$J_m=J_{m-1}\diagdown\left\{i_1^{(m)},\ldots,i_{\alpha_m}^{(m)}\right\}$$

this set has $n-\alpha_1-\cdots-\alpha_{m-1}-\alpha_m=0$ elements, therefore $J_m=\varnothing$ and so the process is finished.

We denote by

$$I=\left(\left(i_1^{(1)},\ldots,i_{\alpha_1}^{(1)}\right),\ \ldots\ ,\ \left(i_1^{(m)},\ldots,i_{\alpha_m}^{(m)}\right)\right)$$

a system composed of systems obtained through the process already presented.

For the numbers $m,n\in\mathbb{N}$ and $\alpha\in\mathbb{A}_{m,n}$ fixed, let us denote through $\mathcal{A}_{m,n}^{[\alpha]}(K)$ the set of all systems built in the manner already indicated.

It is clear that the number of elements of the set $\mathcal{A}_{m,n}^{[\alpha]}(K)$ is $\frac{n!}{\alpha_1!\ \ldots\ \alpha_m!}$.

In the case in which $K = \{1, 2, \ldots, n\}$, we will use the notation $\mathcal{A}_{m,n}^{[\alpha]}$ for $\mathcal{A}_{m,n}^{[\alpha]}(\{1, 2, \ldots, n\})$.

We can now enunciate the following:

Remark 2.2. *With the hypotheses of the* ***lemma 2.1*** *the relation concerning the value of $F'(x)h$ can be written under the form:*

$$F'(x)h = \sum_{\alpha \in \mathbb{A}_{m,1}} \sum_{I \in \mathcal{A}_{m,1}^{[\alpha]}} L\left(f_1^{(\alpha_1)}(x) h_{i_1^{(1)}} \ldots h_{i_{\alpha_1}^{(1)}}, \ \ldots \ , \ f_m^{(\alpha_m)}(x) h_{i_1^{(m)}} \ldots h_{i_{\alpha_m}^{(m)}}\right)$$

with $h_1 = h$.

Indeed, the fact that $\alpha \in \mathbb{A}_{m,1}$ means that $\alpha \in (\alpha_1, \ldots, \alpha_m) \in (\mathbb{N} \cup \{0\})^m$ (therefore $\alpha_i \in \mathbb{N} \cup \{0\}$ for any $i = \overline{1, m}$) with $|\alpha| = \alpha_1 + \cdots + \alpha_m = 1$, so we deduce that there exists a number $k \in \{1, 2, \ldots, m\}$, so that:

$$\alpha_i = \begin{cases} 0 & \text{for} \quad i \neq k, \\ 1 & \text{for} \quad i = k, \end{cases}$$

so the only possibility for the choice of

$$I = \left(\left(i_1^{(1)}, \ldots, i_{\alpha_1}^{(1)}\right), \ \ldots \ , \ \left(i_1^{(m)}, \ldots, i_{\alpha_m}^{(m)}\right)\right) = \left(i_1^{(k)}, \ldots, i_{\alpha_k}^{(k)}\right) = i_1^{(k)} \in \mathcal{A}_{m,1}^{[\alpha]}$$

is $i_1^{(k)})1$ and because $h_1 = h$, it is clear that:

$$L\left(f_1^{(\alpha_1)}(x) h_{i_1^{(1)}} \ldots h_{i_{\alpha_1}^{(1)}}, \ \ldots \ , \ f_m^{(\alpha_m)}(x) h_{i_1^{(m)}} \ldots h_{i_{\alpha_m}^{(m)}}\right) =$$
$$= L\left(f_1(x), \ldots, f_{k-1}(x), f_k'(x) h, f_{k+1}(x), \ldots, f_m(x)\right),$$

which justifies the proposition from this remark.

Taking into account the **remark 2.2** as well, we are now able to establish the theorem concerning the values of the differential of the order n of the non-linear mapping (1).

Thus we have:

Theorem 2.3. *If for $n \in \mathbb{N}$ the non-linear mappings $f_i : D \to Y_i, \ i = \overline{1, m}$ admit a differential of the order n at the point $x \in int(D)$, then the non-linear mapping (1)*

also admits a differential of the order n at the same point x and:

$$F^{(n)}(x)\,h_1\ldots h_n =$$
$$=\sum_{\alpha\in\mathbb{A}_{m,n}}\ \sum_{I\in\mathcal{A}^{[\alpha]}_{m,n}} L\left(f_1^{(\alpha_1)}(x)\,h_{i_1^{(1)}}\ldots h_{i_{\alpha_1}^{(1)}},\ \ldots\ ,\ f_m^{(\alpha_m)}(x)\,h_{i_1^{(m)}}\ldots h_{i_{\alpha_m}^{(m)}}\right).$$

Proof. We will proceed through mathematical induction after $n\in\mathbb{N}$.

For $n=1$ the proposition is true on account of the **lemma 2.1** and of the **remark 2.2**.

We suppose therefore that the property in discussion is true for a number $n\in\mathbb{N}$. We will prove that this property is true for n substituted by $n+1$.

Therefore we consider that the non-linear mappings $f_i : D\to Y_i,\ i=\overline{1,m}$ admit at the point $x\in int\,(D)$ differentials with the order $n+1$. On the basis of the definition there exists a neighbourhood V of the point x, so that the functions $f_i : D\to Y_i,\ i=\overline{1,m}$ admit differentials of the order n at every point $u\in V\cap D$.

On the basis of the hypothesis of the induction we deduce that the function $F : D\to Z$ defined through (1) also admits a differential of the order n at the point $u\in V\cap D$ and the equality in the conclusion of the theorem takes place with x replaced by u.

Choosing therefore $h_1\in X$ so that $x+h_1\in V\cap D$ and arbitrarily $h_2,\ldots,h_n,h_{n+1}\in X$ the equality in the conclusion of the theorem will be true for $h_1,\ldots,h_n$ replaced by $h_2,\ldots,h_{n+1}$ and $\mathcal{A}^{[\alpha]}_{m,n}$ by $\mathcal{A}^{[\alpha]}_{m,n}(\{2,\ldots,n+1\})$ and there will be another similar equality but with x replaced by $x+h_1$.

Subtracting these equalities member by member we obtain:

$$\left[F^{(n)}(x+h_1)-F^{(n)}(x)\right]h_2\ldots h_{n+1}=$$
$$=\sum_{\alpha\in\mathbb{A}_{m,n}}\ \sum_{I\in\mathcal{A}^{[\alpha]}_{m,n}(\{2,\ldots,n+1\})}\mathcal{L}^{(I)}_{\alpha}(x;h_1,h_2,\ldots,h_{n+1}),$$

where:

$$\mathcal{L}_{\alpha}^{(I)}(x;h_1,h_2,\ldots,h_{n+1}) =$$
$$= L\left(f_1^{(\alpha_1)}(x+h_1)\,h_{i_1^{(1)}}\ldots h_{i_{\alpha_1}^{(1)}},\ \ldots\ ,\ f_m^{(\alpha_m)}(x+h_1)\,h_{i_1^{(m)}}\ldots h_{i_{\alpha_m}^{(m)}}\right) -$$
$$-L\left(f_1^{(\alpha_1)}(x)\,h_{i_1^{(1)}}\ldots h_{i_{\alpha_1}^{(1)}},\ \ldots\ ,\ f_m^{(\alpha_m)}(x)\,h_{i_1^{(m)}}\ldots h_{i_{\alpha_m}^{(m)}}\right),$$

in the last expression $\alpha = (\alpha_1,\ldots,\alpha_m) \in \mathbb{A}_{m,n}$ and:

$$I = \left(\left(i_1^{(1)},\ldots,i_{\alpha_1}^{(1)}\right),\ \ldots\ ,\ \left(i_1^{(m)},\ldots,i_{\alpha_m}^{(m)}\right)\right) \in \mathcal{A}_{m,n}^{[\alpha]}\left(\{2,\ldots,n+1\}\right). \tag{8}$$

Let be the number $i \in \{1,2,\ldots,m\}$. From the existence of the Fréchet differential of the order $n+1$ of the function $f_i : D \to Y_i$ at the point $x \in int\,(D)$ we deduce the existence of these differentials for every $k \leq n+1$.

From this fact we deduce that for any $k \leq n$ and $h_1 \in X$ there exists $R_x^{(k,i)} : X \to \left(X^{(k)},Y_i\right)^*$ with $\lim_{h_1\to\theta_X}\left\|R_x^{(k,i)}(h_1)\right\| = 0$ so that:

$$f_i^{(k)}(x+h_1) = f_i^{(k)}(x) + f_i^{(k+1)}(x)\,h_1 + \|h_1\|_X\,R_x^{(k,i)}(h_1). \tag{9}$$

From $\alpha \in \mathbb{A}_{m,n}$ we deduce that $\alpha = (\alpha_1,\ldots,\alpha_m) \in (\mathbb{N}\cup\{0\})^m$ and $|\alpha| = \alpha_1+\cdots+\alpha_m = n$, therefore $\alpha_i \in \{0,1,\ldots,n\}$. So the relation (9) is true for $k = \alpha_i$.

Using a similar process with that from the **lemma 2.1** and taking into account the **remark 2.2**, we obtain for $\alpha \in \mathbb{A}_{m,n}$ and $I \in \mathcal{A}_{m,n}^{[\alpha]}$ the equality:

$$\mathcal{L}_{\alpha}^{(I)}(x;h_1,h_2,\ldots,h_{n+1}) =$$
$$= \sum_{\beta\in\mathbb{A}_{m,1}}\ \sum_{J\in\mathcal{A}_{m,1}^{[\beta]}} L\left(T_1^{(\alpha,\beta;I,J)},\ \ldots\ ,\ T_m^{(\alpha,\beta;I,J)}\right) + \tag{10}$$
$$+\|h_1\|_X\,\mathbf{R}_{\alpha}^{(I)}(x;h_1,h_2,\ldots,h_{n+1}),$$

with $\beta \in \mathbb{A}_{m,1}$ (therefore $\beta = (\beta_1,\ldots,\beta_m) \in (\mathbb{N}\cup\{0\})^m$ and $|\beta| = \beta_1+\cdots+\beta_m = 1$), while:

$$J = \left(\left(j_1^{(1)},\ldots,j_{\beta_1}^{(1)}\right),\ \ldots\ ,\ \left(j_1^{(m)},\ldots,j_{\beta_m}^{(m)}\right)\right) \in \mathcal{A}_{m,1}^{[\beta]} = \mathcal{A}_{m,1}^{[\beta]}\left(\{1\}\right), \tag{11}$$

where for $k = \overline{1, m}$ we have denoted:

$$T_k^{(\alpha,\beta;I,J)} = \left(f_k^{(\alpha_k)}\right)^{(\beta_k)}(x)\, h_{j_1^{(k)}} \dots h_{j_{\beta_k}^{(k)}} h_{i_1^{(k)}} \dots h_{i_{\alpha_k}^{(k)}} =$$
$$= f_k^{(\alpha_k+\beta_k)}(x)\, h_{j_1^{(k)}} \dots h_{j_{\beta_k}^{(k)}} h_{i_1^{(k)}} \dots h_{i_{\alpha_k}^{(k)}}.$$

The element $\mathbf{R}_\alpha^{(I)}(x; h_1, h_2, \dots, h_{n+1}) \in Z$ has the value θ_Z in the case in which $h_1 = \theta_X$ and the value that is deductible from (10) for $h_1 \neq \theta_X$.

So:

$$\begin{gathered}\left[F^{(n)}(x+h_1) - F^{(n)}(x)\right] h_2 \dots h_{n+1} = \\ = \mathcal{E}(x; h_1, h_2, \dots, h_n, h_{n+1}) + \|h_1\|_X\, \mathcal{R}(x; h_1, h_2, \dots, h_n, h_{n+1}),\end{gathered} \tag{12}$$

where:

$$\begin{gathered}\mathcal{E}(x; h_1, h_2, \dots, h_n, h_{n+1}) = \\ = \sum_{\alpha \in \mathbb{A}_{m,n}} \sum_{I \in \mathcal{A}_{m,n}^{[\alpha]}(\{2,\dots,n+1\})} \sum_{\beta \in \mathbb{A}_{m,1}} \sum_{J \in \mathcal{A}_{m,1}^{[\beta]}} L\left(T_1^{(\alpha,\beta;I,J)}, \ \dots\ , \ T_m^{(\alpha,\beta;I,J)}\right),\end{gathered} \tag{13}$$

while:

$$\mathcal{R}(x; h_1, h_2, \dots, h_n, h_{n+1}) =$$
$$= \begin{cases} \theta_Z, & \text{for } h_1 = \theta_X, \\ \dfrac{\left[F^{(n)}(x+h_1) - F^{(n)}(x)\right] h_2 \dots h_{n+1} - \mathcal{E}(x; h_1, \dots, h_{n+1})}{\|h_1\|_X} & \text{for } h_1 \neq \theta_X. \end{cases}$$

It is clear that for $h_1 \neq \theta_X$ we have:

$$\begin{gathered}\mathcal{R}(x; h_1, h_2, \dots, h_n, h_{n+1}) = \\ = \sum_{\alpha \in \mathbb{A}_{m,n}} \sum_{I \in \mathcal{A}_{m,n}^{[\alpha]}(\{2,\dots,n+1\})} \mathbf{R}_\alpha^{(I)}(x; h_1, h_2, \dots, h_n, h_{n+1}).\end{gathered} \tag{14}$$

Now let be:

$$\gamma = (\gamma_1, \dots, \gamma_m) = \alpha + \beta = (\alpha_1 + \beta_1, \dots, \alpha_m + \beta_m) \in (\mathbb{N} \cup \{0\})^m.$$

Because $|\alpha| = n$ and $|\beta| = 1$ we deduce that:

$$|\gamma| = \gamma_1 + \dots + \gamma_m = (\alpha_1 + \dots + \alpha_m) + (\beta_1 + \dots + \beta_m) = |\alpha| + |\beta| = n + 1,$$

therefore $\gamma \in \mathbb{A}_{m,n+1}$.

For the system I which verifies (8) and the system J which verifies (11), let us introduce:

$$\left(s_1^{(k)},\ldots,s_{\gamma_k}^{(k)}\right)=\left(j_1^{(k)},\ldots,j_{\beta_k}^{(k)},i_1^{(k)},\ldots,i_{\alpha_k}^{(k)}\right);\quad k=\overline{1,m}$$

and:

$$S=\left(\left(s_1^{(1)},\ldots,s_{\gamma_1}^{(1)}\right),\ldots,\left(s_1^{(m)},\ldots,s_{\gamma_m}^{(m)}\right)\right). \tag{15}$$

Because $\beta \in \mathbb{A}_{m,1}$ we deduce that there exists a number $r \in \{1,\ldots,m\}$ so that:

$$\beta_i=\begin{cases}0 & \text{for} \quad i\neq r,\\ 1 & \text{for} \quad i=r,\end{cases}$$

so the only possibility for the choosing of the index system:

$$J=\left(\left(j_1^{(1)},\ldots,j_{\beta_1}^{(1)}\right),\ \ldots\ ,\ \left(j_1^{(m)},\ldots,j_{\beta_m}^{(m)}\right)\right)=\left(j_1^{(r)},\ldots,j_{\beta_r}^{(r)}\right)=$$
$$=\left(j_1^{(r)}\right)\in\mathcal{A}_{m,1}^{[\beta]}\left(\{1\}\right),$$

is $j_1^{(k)}=1$.

Form here we deduce that the systems from S are identical with a system $I\in\mathcal{A}_{m,n}^{[\alpha]}\left(\{2,\ldots,n+1\}\right)$ (having the form (11)) except the subsystem situated on the position r. To this subsystem we add the element 1 on its first position. This indicates that $S\in\mathcal{A}_{m,n+1}^{[\gamma]}$.

Through the aforementioned process starting with the elements $\alpha\in\mathbb{A}_{m,n}$, $\beta\in\mathbb{A}_{m,1}$, $I\in\mathcal{A}_{m,n}^{[\alpha]}\left(\{2,\ldots,n+1\}\right)$ and $J\in\mathcal{A}_{m,1}^{[\beta]}\left(\{1\}\right)$ we obtain a $\gamma\in\mathbb{A}_{m,n+1}$ together with $S\in\mathcal{A}_{m,n+1}^{[\gamma]}$.

The inverted process starting from $\gamma\in\mathbb{A}_{m,n+1}$ together with $S\in\mathcal{A}_{m,n+1}^{[\gamma]}$ exists with a unique determination, a $\alpha\in\mathbb{A}_{m,n}$ together with $I\in\mathcal{A}_{m,n}^{[\alpha]}\left(\{2,\ldots,n+1\}\right)$ and $J\in\mathcal{A}_{m,1}^{[\beta]}\left(\{1\}\right)$, so that we obtain the systems from S through (15), the systems I and J having the forms (8) and (11) respectively.

So it is clear that for any $k=\overline{1,m}$ we have:

$$T_k^{(\alpha,\beta;I,J)}=f_k^{(\gamma_k)}\left(x\right)h_{s_1^{(k)}}\ldots h_{s_{\gamma_k}^{(k)}}$$

and from (13) we deduce that:

$$\mathcal{E}\left(x;h_1,h_2,\ldots,h_n,h_{n+1}\right)=\sum_{\gamma\in\mathbb{A}_{m,n+1}}\ \sum_{S\in\mathcal{A}^{[\gamma]}_{m,n+1}}\mathbf{L}^{(S)}_{\gamma}, \tag{16}$$

where:

$$\mathbf{L}^{(S)}_{\gamma}=L\left(f_1^{(\gamma_1)}\left(x\right)h_{s_1^{(1)}}\ldots h_{s_{\gamma_1}^{(1)}},\ldots,\ f_m^{(\gamma_m)}\left(x\right)h_{s_1^{(m)}}\ldots h_{s_{\gamma_m}^{(m)}}\right). \tag{17}$$

Let us denote:

$$\mathcal{H}_{n,X}=\{\,h/\,h=\left(h_1,\ldots,h_n\right)\in X^n,\ \left\|h_1\right\|_X=\cdots=\left\|h_n\right\|_X=1\}$$

and let us now evaluate $\left\|\mathcal{R}\left(x;h_1,h_2,\ldots,h_n,h_{n+1}\right)\right\|_Z$ supposing that $\left(h_2,\ldots,h_{n+1}\right)\in\mathcal{H}_{n,X}$ which means that $\left\|h_2\right\|_X=\cdots=\left\|h_{n+1}\right\|_X=1$.

First we notice that for any $h_1\neq\theta_X,\ \alpha\in\mathbb{A}_{m,n}$ and $I\in\mathcal{A}^{[\alpha]}_{m,n}$ we have:

$$\begin{gathered}\mathbf{R}^{(I)}_{\alpha}\left(x;h_1,h_2,\ldots,h_{n+1}\right)=\\=\sum_{j=1}^{m}\mathbf{G}^{(I)}_{j,\alpha}\left(x;h_1,h_2,\ldots,h_{n+1}\right)+\\+\frac{1}{\left\|h_1\right\|_X}\sum_{k=2}^{m}\ \sum_{1\leq r_1<\cdots<r_k\leq m}\mathbf{E}^{(k,\alpha,I)}_{r_1,\ldots,r_k}\left(x;h_1,h_2,\ldots,h_{n+1}\right).\end{gathered} \tag{18}$$

In (18) $\mathbf{G}^{(I)}_{j,\alpha}\left(x;h_1,h_2,\ldots,h_{n+1}\right)\in Z$ for $j\in\{1,2,\ldots,m\}$ represents the value of the mapping $L\in\left(Y_1,\ldots,Y_m;Z\right)^*$ with the arguments:

$$f_q^{(\alpha_q)}\left(x\right)h_{i_1^{(q)}}\ldots h_{i_{\alpha_q}^{(q)}}\in Y_q;\ q=\overline{1,m},$$

except the argument of the rank j, this argument being:

$$R_x^{(\alpha_j,j)}\left(h_1\right)h_{i_1^{(j)}}\ldots h_{i_{\alpha_j}^{(j)}}.$$

So:

$$\left\|\mathbf{G}^{(I)}_{j,\alpha}\left(x;h_1,h_2,\ldots,h_{n+1}\right)\right\|_Z\leq\left\|L\right\|\cdot\left\|R_x^{(\alpha_j,j)}\left(h_1\right)\right\|\prod_{q=\overline{1,m}\ q\neq k}\left\|f_q^{(\alpha_q)}\left(x\right)\right\|,$$

here we take into account that $I\in\mathcal{A}^{[\alpha]}_{m,n}\left(\{2,\ldots,n+1\}\right)$, therefore:

$$\prod_{q=1}^{m}\left(\left\|h_{i_1^{(q)}}\right\|_X\ \cdots\ \left\|h_{i_{\alpha_q}^{(q)}}\right\|_X\right)=\left\|h_2\right\|_X\ \cdots\ \left\|h_{n+1}\right\|_X=1.$$

In the same relation (18) for $k \in \{2,\ldots,n+1\}$ and $r_1,\ldots,r_k \in \mathbb{N}$ with $1 \le r_1 < \cdots < r_k \le m$ the expression $\mathbf{E}^{(k,\alpha,I)}_{r_1,\ldots,r_k}(x;h_1,h_2,\ldots,h_{n+1})$ is the value of the mapping $L \in (Y_1,\ldots,Y_m;Z)^*$ with the arguments $f_q^{(\alpha_q)}(x)\, h_{i_1^{(q)}}\ldots h_{i_{\alpha_q}^{(q)}} \in$ $\in Y_q;\ q=\overline{1,m}$ except the arguments situated in the position $r_1,\ldots,r_k$ where the arguments:

$$\left[f_p^{(\alpha_p+1)}(x)\, h_1 + \|h_1\|_X\, R_x^{(\alpha_p,p)}(h_1)\right] h_{i_1^{(p)}}\ldots h_{i_{\alpha_p}^{(p)}};\quad p\in\{r_1,\ldots,r_k\}$$

appear.

So:

$$\left\|\mathbf{E}^{(k,\alpha,I)}_{r_1,\ldots,r_k}(x;h_1,h_2,\ldots,h_{n+1})\right\| \le \|h_1\|_X^k \times \|L\| \times$$

$$\times \prod_{q=1}^{k}\left(\left\|f_{r_q}^{(\alpha_{r_q}+1)}(x)\right\| + \left\|R_x^{(\alpha_{r_q},r_q)}(h_1)\right\|\right) \times \prod_{q\in\{1,\ldots,m\}\setminus\{r_1,\ldots,r_k\}} \left\|f_q^{(\alpha_q)}(x)\right\|.$$

Therefore we can write that:

$$\left\|\mathbf{R}^{(I)}_{\alpha}(x;h_1,h_2,\ldots,h_{n+1})\right\|_Z \le \|L\|\, \mathbf{C}^{(I)}_{\alpha}(x,h_1) \tag{19}$$

where:

$$\mathbf{C}^{(I)}_{\alpha}(x,h_1) = \sum_{k=1}^{m}\left(\left\|R_x^{(\alpha_{r_k},r_k)}(h_1)\right\| \times \prod_{q=\overline{1,m}\ q\ne k}\left\|f_q^{(\alpha_q)}(x)\right\|\right) +$$

$$+\sum_{k=2}^{m}\|h_1\|_X^{k-1} \times \sum_{1\le r_1<\cdots<r_k\le m} \mathbf{D}^{(k,\alpha,I)}_{r_1,\ldots,r_k}(x,h_1),$$

while for $k\in\{2,\ldots,m\}$ and $r_1,\ldots,r_k\in\mathbb{N}$ with $1\le r_1<\cdots<r_k\le m$ we have:

$$\mathbf{D}^{(k,\alpha,I)}_{r_1,\ldots,r_k}(x,h_1) =$$

$$= \prod_{q=1}^{k}\left(\left\|f_{r_q}^{(\alpha_{r_q}+1)}(x)\right\| + \left\|R_x^{(\alpha_{r_q},r_q)}(h_1)\right\|\right) \times \prod_{q\in\{1,\ldots,m\}\setminus\{r_1,\ldots,r_k\}} \left\|f_q^{(\alpha_q)}(x)\right\|.$$

Thus, it is clear from the hypotheses on account of which for the specified values of k and of $r_1,\ldots,r_k$ we have:

$$\lim_{h_1\to\theta_X} \mathbf{D}^{(k,\alpha,I)}_{r_1,\ldots,r_k}(x,h_1) = \prod_{q=1}^{k}\left\|f_{r_q}^{(\alpha_{r_q}+1)}(x)\right\| \cdot \prod_{q\in\{1,\ldots,m\}\setminus\{r_1,\ldots,r_k\}} \left\|f_q^{(\alpha_q)}(x)\right\|,$$

that for any $\alpha \in \mathbb{A}_{m,n}$ and $I \in \mathcal{A}_{m,n}^{[\alpha]}$ we have:

$$\lim_{h \to \theta_X} \mathbf{C}_{\alpha}^{(I)} (x, h_1) = 0$$

and so, in the same situation as in (19) we deduce that:

$$\lim_{h_1 \to \theta_X} \sup_{(h_2,\ldots,h_{n+1}) \in \mathcal{H}_{n,X}} \left\| \mathcal{R} (x; h_1, h_2, \ldots, h_{n+1}) \right\|_Z = 0. \tag{20}$$

We define the mapping:

$$\begin{gathered} F^{(n+1)} (x) \in \left(X^{(n+1)}, Z \right)^*, \\ F^{(n+1)} (x) h_1 h_2 \ldots h_{n+1} = \mathcal{E} (x; h_1, h_2, \ldots, h_{n+1}), \end{gathered} \tag{21}$$

and it is clear that if we define $R_x (h_1) \in \left(X^{(n)}, Z \right)^*$ through:

$$R_x (h_1) = \begin{cases} \Theta_n & ;\ h_1 = \theta_X \\ \dfrac{F^{(n)} (x + h_1) - F^{(n)} (x) - F^{(n+1)} (x) h_1}{\|h_1\|_X} & ;\ h_1 \neq \theta_X \end{cases}$$

we have in $\left(X^{(n)}, Z \right)^*$ the equality:

$$F^{(n)} (x + h_1) - F^{(n)} (x) = F^{(n+1)} (x) h_1 + \|h_1\|_X R_x (h_1). \tag{22}$$

In the same time for $h_1 \neq \theta_X$ we have:

$$\begin{gathered} \|R_x (h_1) h_2 \ldots h_{n+1}\|_Z \leq \\ \leq \frac{\left\| \left[F^{(n)} (x + h_1) - F^{(n)} (x) \right] h_2 \ldots h_{n+1} - F^{(n+1)} (x) h_1 h_2 \ldots h_{n+1} \right\|_Z}{\|h_1\|_X} = \\ = \frac{\left\| \left[F^{(n)} (x + h_1) - F^{(n)} (x) \right] h_2 \ldots h_{n+1} - \mathcal{E} (x; h_1, h_2, \ldots, h_{n+1}) \right\|_Z}{\|h_1\|_X} = \\ = \|\mathcal{R} (x; h_1, h_2, \ldots, h_{n+1})\|_Z , \end{gathered}$$

therefore:

$$\begin{gathered} 0 \leq \|R_x (h_1)\| = \sup_{(h_2,\ldots,h_{n+1}) \in \mathcal{H}_{n,X}} \|R_x (h_1) h_2 \ldots h_{n+1}\|_Z \leq \\ \leq \sup_{(h_2,\ldots,h_{n+1}) \in \mathcal{H}_{n,X}} \|\mathcal{R} (x; h_1, h_2, \ldots, h_{n+1})\|_Z . \end{gathered}$$

From here, also using the relation (20), we deduce that:

$$\lim_{h\to\theta_X}\|R_x(h_1)\|=0. \tag{23}$$

From the relations (22) and (23) we deduce that the mapping $F:D\to Z$ has a Fréchet differential of the order $n+1$ at the point $x\in int(D)$, the expression of the mapping $F^{(n+1)}(x)\in\left(X^{(n+1)},Z\right)^*$ being specified through (21), therefore on account of the obtained expression (16) for $\mathcal{E}(x;h_1,h_2,\ldots,h_{n+1})$ we have:

$$F^{(n+1)}(x)\,h_1\ldots h_{n+1}=$$
$$=\sum_{\gamma\in\mathbb{A}_{m,n+1}}\ \sum_{S\in\mathcal{A}^{[\gamma]}_{m,n+1}} L\left(f_1^{(\gamma_1)}(x)\,h_{s_1^{(1)}}\ldots h_{s_{\gamma_1}^{(1)}},\ldots,\ f_m^{(\gamma_m)}(x)\,h_{s_1^{(m)}}\ldots h_{s_{\gamma_m}^{(m)}}\right).$$

The aforementioned assertion together with its corresponding equality indicates that the property expressed through this theorem is true for any $n\in\mathbb{N}$ replaced by $n+1$.

On account of the principle of mathematical induction this property is true for any $n\in\mathbb{N}$.

The theorem is proved. □

Remark 2.4. *In the case of $m=2$, case in which $L\in(L_1,L_2;Z)^*$, $f:D\to Y_1$, $g:D\to Y_2$ where $D\subseteq X$ and $x\in int(D)$, in the hypothesis of the existence of the differentials with the order n of the considered functions at the point x, it results the existence of the differential with the order n of the function $F:D\to Z$, $F(x)=L(f(x),g(x))$ together with the equality:*

$$F^{(n)}(x)\,h_1\ldots h_n=\sum_{k=0}^{n}\ \sum_{i\in\mathcal{C}_{m,k}} L\left(f^{(k)}(x)\,h_{i_1}\ldots h_{i_k},g^{(n-k)}(x)\,h_{j_1}\ldots h_{j_{n-k}}\right) \tag{24}$$

where we have denoted $i=(i_1,\ldots,i_k)\in\mathcal{C}_{m,k}$ and:

$$\{j_1,\ldots,j_{n-k}\}\in\{1,2,\ldots,n\}\diagdown\{i_1,\ldots,i_k\}$$

with $j_1<\cdots<j_{n-k}$.

Indeed, in this case:

$$\mathbb{A}_{2,1} = \left\{ \alpha / \, \alpha = (\alpha_1, \alpha_2) \in (\mathbb{N} \cup \{0\})^2 \, , \; \alpha_1 + \alpha_2 = n \right\} =$$

$$= \left\{ (k, n-k) / \, k = \overline{0, n} \right\}$$

and the set $\mathcal{A}_{2,n}^{[\alpha]} = \mathcal{A}_{2,n}^{(k,n-k)}$ is made of a pair of disjunct systems, the first system has k elements and the second $n-k$ elements. If we put together the elements from these systems we obtain the set $\{1, 2, \ldots, n\}$.

If we denote this pair of systems from $\mathcal{A}_{2,n}^{(k,n-k)}$ by:

$$(i, j) = ((i_1, \ldots, i_k), (j_1, \ldots, j_{n-k}))$$

because $1 \leq i_1 < \cdots < i_k \leq n$ and the system $(j_1, \ldots, j_{n-k})$ is obtained in the aforementioned manner, then $i \in \mathcal{C}_{n,k}$ and we also obtain the equality (24).

Remark 2.5. *In the case when $h_1 = \cdots = h_n = h \in X$ we have for the equality from the conclusion of the* ***theorem 2.3*** *the form:*

$$F^{(n)}(x) h^n = \sum_{\alpha \in \mathbb{A}_{m,n}} \frac{n!}{\alpha_1! \ldots \alpha_m!} L \left(f_1^{(\alpha_1)}(x) h^{\alpha_1}, \ldots, f_m^{(\alpha_m)}(x) h^{\alpha_m} \right) \tag{25}$$

and for the equality (24) *we have the form:*

$$F^{(n)}(x) h^n = \sum_{k=0}^{n} \binom{n}{k} L \left(f^{(k)}(x) h^k, g^{(n-k)}(x) h^{n-k} \right). \tag{26}$$

These relations are evident because the number of elements of the set $\mathcal{A}_{m,n}^{[\alpha]}$ is $\frac{n!}{\alpha_1! \ldots \alpha_m!}$, while the number of elements of the set $\mathcal{C}_{n,k}$ is $\frac{n!}{k!(n-k)!} = \binom{n}{k}$.

In the aforementioned writings it is clear that $f^{(k)}(x) h^k$ means:

$$f^{(k)}(x) (\underbrace{h, \ldots, h}_{k \text{ times}}).$$

3. An application to the differential of certain composed functions

Let us consider the number $m \in \mathbb{N}$, the linear normed spaces:

$$(X, \|\cdot\|_X), \; (Y_1, \|\cdot\|_{Y_1}), \; \ldots \;, \; (Y_m, \|\cdot\|_{Y_m}), \; (Z, \|\cdot\|_Z),$$

the set $D \subseteq X$ and the mappings:

$$U_i : D \to Y_i,\ i = \overline{1,m}; \qquad W : D \to (Y_1, \ldots, Y_m; Z)^* .$$

Using the aforemationed mappings we consider the composed mapping:

$$G : D \to Z,\ \ G(x) = [W(x)](U_1(x), \ldots, U_m(x)) . \tag{27}$$

Concerning the mapping (27), we have the following proposition:

Proposition 3.1. *If for an $n \in \mathbb{N}$ the mappings $W : D \to (Y_1, \ldots, Y_m; Z)^*$ and $U_i : D \to Y_i;\ i = \overline{1,m}$ admit Fréchet differentials with the order n at the point $x \in int(D)$, then the mapping $G : D \to Z$ defined through (27) also admits a Fréchet differential of the order n at the same point x, and for any $h_1, \ldots, h_n \in X$ we have the equality:*

$$G^{(n)}(x)\, h_1 \ldots h_n =$$
$$= \sum_{k=0}^{n} \sum_{\alpha \in \mathbb{A}_{m,n-k}} \sum_{S \in \mathcal{C}_{n,k}} \sum_{I \in \mathcal{A}^{[\alpha]}_{m,n-k}(M_{n,k}(S))} E^{(S,I)}_{k,\alpha}(W, U; x; h_1, \ldots, h_n) \tag{28}$$

where $U = (U_1, \ldots, U_m)$ and $E^{(S,I)}_{k,\alpha}(W, U; x; h_1, \ldots, h_n)$ is

$$\left[W^{(k)}(x)\, h_{s_1} \ldots h_{s_k}\right] \left(U_1^{(\alpha_1)}(x)\, h_{i_1^{(1)}} \ldots h_{i^{(1)}_{\alpha_1}}, \ldots, U_m^{(\alpha_m)}(x)\, h_{i_1^{(m)}} \ldots h_{i^{(m)}_{\alpha_m}}\right) \tag{29}$$

where for $S = (s_1, \ldots, s_k) \in \mathcal{C}_{n,k}$ we have denoted:

$$M_{n,k}(S) = \{1, \ldots, n\} \diagdown \{s_1, \ldots, s_k\} .$$

Proof. We will consider the mapping:

$$L : (Y_1, \ldots, Y_m; Z)^* \times Y_1 \times \cdots \times Y_m \to Z,\ L(T; y_1, \ldots, y_m) = T(y_1, \ldots, y_m)$$

where $y_i \in Y_i$ with $i = \overline{1,m}$ while $T \in (Y_1, \ldots, Y_m; Z)^*$.

From the definition of the operations in the set of mappings we deduce the linearity of the mapping L after the first argument, while from the linearity of the mapping $T : Y_1 \times \cdots \times Y_m \to Z$ we deduce the linearity of the mapping L after the last m arguments.

It is also clear that:

$$\|L(T; y_1, \ldots, y_m)\|_Z = \|T(y_1, \ldots, y_m)\|_Z \leq \|T\| \cdot \|y_1\|_{Y_1} \cdot \cdots \cdot \|y_m\|_{Y_m},$$

therefore:

$$L \in \left((Y_1, \ldots, Y_m; Z)^*, Y_1, \ldots, Y_m; Z\right)^*$$

and:

$$G(x) = L(W(x), U_1(x), \ldots, U_m(x))$$

as well.

In this way for the existence and the calculation of the differential with the order n of the non-linear mapping defined by (27) it is possible to use the theorem 2.3, therefore as the mappings $U_i : D \to Y_i;\ i = \overline{1, m}$ and $W : D \to (Y_1, \ldots, Y_m; Z)^*$ have the Fréchet differentials with the order n, at the point $x \in int(D)$, the same fact can be said about the mapping $G : X \to Z$, and for any $h_1, \ldots, h_n \in X$ we have:

$$G^{(n)}(x) h_1 \ldots h_n = \sum_{\gamma \in \mathbb{A}_{m+1,n}} \sum_{J \in \mathcal{A}^{[\gamma]}_{m+1,n}} \mathcal{G}_{\gamma,J}(x; h_1, \ldots, h_n),$$

where $\mathcal{G}_{\gamma,J}(x; h_1, \ldots, h_n)$ has the value:

$$L\left(W^{(\gamma_1)}(x) h_{j_1^{(1)}} \ldots h_{j_{\gamma_1}^{(1)}}, U_1^{(\gamma_2)}(x) h_{j_1^{(2)}} \ldots h_{j_{\gamma_2}^{(2)}}, \ldots, U_m^{(\gamma_{m+1})}(x) h_{j_1^{(m+1)}} \ldots h_{j_{\gamma_{m+1}}^{(m+1)}}\right)$$

for $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_{\bar{m}+1}) \in \mathbb{A}_{m+1,n}$ and

$$J = \left(\left(j_1^{(1)}, \ldots, j_{\gamma_1}^{(1)}\right), \left(j_1^{(2)}, \ldots, j_{\gamma_2}^{(2)}\right), \ldots, \left(j_1^{(m+1)}, \ldots, j_{\gamma_{m+1}}^{(m+1)}\right)\right) \in \mathcal{A}^{[\gamma]}_{m+1,n}.$$

The fact that $\gamma \in \mathbb{A}_{m+1,n}$ means that $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_{m+1}) \in (\mathbb{N} \cup \{0\})^{m+1}$ with $|\gamma| = \gamma_1 + \gamma_2 + \cdots + \gamma_{m+1} = n$.

We place:

$$k = \gamma_1,\ \alpha_1 = \gamma_2,\ \ldots\ ,\ \alpha_m = \gamma_{m+1}$$

and we deduce that in fact $k \in \{0, 1, \ldots, n\}$ and $\alpha = (\alpha_1, \ldots, \alpha_m) \in (\mathbb{N} \cup \{0\})^m$ with $|\alpha| = \alpha_1 + \cdots + \alpha_m = n - \gamma_1 = n - k$, therefore $\alpha \in \mathbb{A}_{m,n-k}$.

We then place:

$$S = (s_1, \ldots, s_k) = \left(j_1^{(1)}, \ldots, j_k^{(1)}\right) = \left(j_1^{(1)}, \ldots, j_{\gamma_1}^{(1)}\right)$$

and:

$$I=\left(\left(j_1^{(2)},\ldots,j_{\gamma_2}^{(2)}\right),\left(j_1^{(3)},\ldots,j_{\gamma_3}^{(3)}\right),\ldots,\left(j_1^{(m+1)},\ldots,j_{\gamma_{m+1}}^{(m+1)}\right)\right).$$

Thus it is evident that $J=(S,I)\in\mathcal{A}_{m+1,n}^{[k]}$ if and only if $S\in\mathcal{C}_{n,k}$ and:

$$I\in\mathcal{A}_{m,n-k}^{[\alpha]}\left(\{1,2,\ldots,n\}\setminus\{s_1,\ldots,s_k\}\right)=\mathcal{A}_{m,n-k}^{[\alpha]}\left(M_{n,k}(S)\right),$$

this fact results from the manner in which we have obtained the systems $J\in\mathcal{A}_{m+1,n}^{[\gamma]}$.

Thus the relations (28) and (29) are clear.

The proposition is proved. □

We have the following:

Remark 3.2. *In the case where $h_1=\cdots=h_n=h\in X$ in the hypotheses of the proposition 3.1 we have the equality:*

$$\begin{gathered}G^{(n)}(x)\,h^n=\\=\sum_{k=0}^{n}\frac{n!}{k!}\sum_{\alpha\in\mathbb{A}_{m,n-k}}\frac{\left[W^{(k)}(x)\,h^k\right]\left(U_1^{(\alpha_1)}(x)\,h^{\alpha_1},\ldots,U_m^{(\alpha_m)}(x)\,h_m\right)}{\alpha_1!\ldots\alpha_m!}.\end{gathered}\tag{30}$$

For $n=1$ we have:

Remark 3.3. *If the mappings $W:D\to(Y_1,\ldots,Y_m;Z)^*$ and $U_i:D\to Y_i;\ i=\overline{1,m}$ are Fréchet differentiable at the point $x\in int(D)$, then the mapping $G:D\to Z$ defined through (27) is also differentiable at the same point x, and for any $h_1,\ldots,h_n\in X$ we have the equality:*

$$\begin{gathered}G'(x)\,h=\left[W'(x)\,h\right]\left(U_1(x),\ldots,U_m(x)\right)+\\+\sum_{j=1}^{m}\left[W(x)\right]\left(U_1(x),\ldots,U_{j-1}(x),U_j'(x)\,h,U_{j+1}(x),\ldots,U_m(x)\right).\end{gathered}\tag{31}$$

For $n\in\mathbb{N}$ arbitrary and $m=1$ we have:

Remark 3.4. *If the linear normed spaces $(X,\|\cdot\|_X)$, $(Y,\|\cdot\|_Y)$, $(Z,\|\cdot\|_Z)$ and the functions $f:D\to(Y,Z)^*$, $g:D\to Y$, that admit Fréchet differentials with the order n at a point $x\in int(D)$ are given, then the function:*

$$G:D\to Z;\ G(x)=\left[f(x)\right]g(x)$$

also admits a Fréchet differential with the order n *at the same point* x*, and for any* $h_1, h_2, \ldots, h_n \in X$ *we have:*

$$\begin{aligned} &([f(x)]\, g(x))^{(n)}\, h_1 \ldots h_n = \\ &= \sum_{k=0}^{n} \sum_{i \in \mathcal{C}_{n,k}} \left[f^{(k)}(x)\, h_{i_1} \ldots h_{i_k}\right] g^{(n-k)}(x)\, h_{j_1} \ldots h_{j_{n-k}} \end{aligned} \tag{32}$$

where $i = (i_1, \ldots, i_k) \in \mathcal{C}_{n,k}$ *and* $\{j_1, \ldots, j_{n-k}\} = \{1, 2, \ldots, n\} \setminus \{i_1, \ldots, i_k\}$ *with* $j_1 < \cdots < j_{n-k}$.

The **remarks 3.2-3.4** are evident.

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TWO EXISTENCE RESULTS FOR VARIATIONAL INEQUALITIES

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Dedicated to Professor Ştefan Cobzaş at his 60^{th} *anniversary*

Abstract. In this paper we prove the existence of solutions for some variational inequalities, governed by two-variables set-valued mappings, in both stationary and evolution cases.

1. **Introduction**

Operators with two variables, having monotonicity properties with respect to one of the variables and continuity properties with respect to the other one, have been studied since more than 40 years (see [6], [13]). Such kind of operators appear in the theory of nonlinear elliptic operators in divergence form, which are monotone only in the highest order terms, and satisfy a compactness condition for the lower order terms (see [14]).

Existence results for variational inequalities governed by such operators were established in papers like [3], [5].

In this paper we continue some ideas from [5] for a more general class of variational problems, which includes, as a particular case, hemivariational inequalities.

The mathematical theory of hemivariational inequalities was introduced by P.D. Panagiatopoulos (see [11]) and studied by many authors (see for instance [9], [10]).

The main result of this paper is stated in Section 2 (Theorem 5). It gives sufficient conditions for the existence of solutions for a variational inequality governed

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by set-valued mappings of two variables, using the classical Ky Fan intersection theorem. Then, in Section 3, we use this result to study a class of evolution variational inequalities.

The same method that we applied can be used, when $\Phi = 0$, to prove a similar result, where (H1) is replaced by a condition of Karamardian pseudomontonicity (see [5]).

If A is a mapping of one variable, Brézis pseudomonotone, similar results on evolution hemivariational inequalities were established in [7] and [8].

2. An existence result for a stationary variational inequality

In what follows, V is a real Banach space, V^* is its dual and $\langle \cdot, \cdot \rangle$ is the usual duality pairing.

Let $C \subset V$ be a nonempty, closed, convex set and let $A : C \times C \to 2^{V^*}$ be a set-valued mapping. Let $\Phi : C \times V \to \mathbb{R} \cup \{+\infty\}$ be a weakly upper semi-continuous function, sublinear in the second variable. We suppose that for each $u \in C$ and for each $v \in T_C(u) + u$ we have $(u, v-u) \in D(\Phi)$, where by $T_C(u)$ we denote the tangent cone of C at u.

Consider the following variational problem:

$$(VI) \quad \text{Find } u \in C \text{ such that } \sup_{f \in A(u,u)} \langle v-u, f \rangle + \Phi(u, v-u) \geq 0, \quad \forall\, v \in C.$$

In what follows the set-valued mapping A will have the following properties:

(H1) $\sup_{f \in A(v,v)} \langle u-v, f \rangle + \Phi(v, u-v) \geq 0$ implies that $\sup_{f \in A(u,v)} \langle u-v, f \rangle + \Phi(v, u-v) \geq 0$, for each $u, v \in C$,

(H2) For each $v \in C$, $A(\cdot, v) : C \to 2^{V^*}$ is upper semi-continuous from the line segments of C to V^*,

(H3) For each $u \in C$, $A(u, \cdot) : C \to 2^{V^*}$ is weakly upper semi-continuous (from V with the weak topology, to V^* with the norm topology),

(H4) $A(u, v)$ is compact, for each $u, v \in C$.

Remark 1. *The hypothesis (H1) is true, for example, when it takes place*

(H1') $\sup\limits_{f\in A(u,v)} \langle u-v, f\rangle \geq \sup\limits_{f\in A(v,v)} \langle u-v, f\rangle$, *for each* $u, v \in C$,
in particular, when A is monotone with respect to the first variable, as it was considered in [3].

Remark 2. *Several particular cases of (VI) are:*

I) Suppose V is a reflexive Banach space, densely and compactly embedded into a separable Hilbert space H (then $V \subset H \subset V^*$ is an evolution triple). Let $J : H \to \mathbb{R}$ be a locally Lipschitz function, and by $J^0(u;v)$ denote the generalized Clarke derivative of J, at the point u, in the direction v:

$$J^0(u;v) = \limsup_{w\to u,\ \varepsilon\to 0_+} \frac{J(w+\varepsilon v) - J(w)}{\varepsilon}.$$

It is well known (see [4]*) that $J^0(\cdot;\cdot)$ is sublinear in the second variable and globally upper semi-continuous. This means we can take $\Phi = J|_V^0$.*

II) Consider the same evolution triple as above. Let Ω be a bounded and open subset of $\mathbb{R}^N$, let $T : H \to L^2(\Omega, \mathbb{R}^k)$ be a linear and continuous operator, and let $j : \Omega\times\mathbb{R}^k \to \mathbb{R}$ be a Caratheodory function, locally Lipschitz with respect to the second variable. Denote by $j^0(x,y)(h)$ the partial generalized Clarke derivative,

$$j^0(x,y)(h) = \limsup_{y'\to y, t\to 0^+} \frac{j(x,y'+th) - j(x,y')}{t}.$$

Suppose that there exist $h_1 \in L^2(\Omega)$ and $h_2 \in L^\infty(\Omega)$ such that

$$\|z\| \leq h_1(x) + h_2(x)\|y\| \text{ , a.e. } x \in \Omega, \text{ for all } y \in \mathbb{R}^k,\ z \in \partial j(x,y), \text{ where}$$

$$\partial j(x,y) = \{z \in \mathbb{R}^k, \langle z,h\rangle \leq j^0(x,y)(h),\ \forall h \in \mathbb{R}^k\}.$$

It is proved in [12], *that the mapping*
$(u,w) \in V\times V \mapsto \int_\Omega j^0(x,Tu(x))(Tw(x))dx$ *is weakly upper semi-continuous.*
Then we can take $\Phi(u,w) = \int_\Omega j^0(x,Tu(x))(Tw(x))dx$.

III) An example of a single-valued mapping that satisfies (H1'), (H2)-(H4) is the following (see [14]*): $A : H_0^1(\Omega) \to (H_0^1(\Omega))^*$, defined by*

$$\langle A(u,v), w\rangle = \int_\Omega G(x, v(x), \nabla u(x))\nabla w(x)dx + \int_\Omega g_0(x, v(x), \nabla v(x))w(x)dx,$$

where Ω *is a bounded domain in* $\mathbb{R}^N$ *and the functions* $g_j : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$, $G = (g_1, \ldots, g_N)$ *have the properties:*

(P1) $g_j(x, \eta, \xi)$ *is measurable in* $x \in \Omega$ *and continuous in* $(\eta, \xi) \in \mathbb{R} \times \mathbb{R}^N$*, for* $j = 0, \ldots, N$*,*

(P2) $|g_j(x, \eta, \xi)| \leq c(k(x) + |\eta| + \|\xi\|)$*, with* $k \in L^2(\Omega)$*, a.e.* $x \in \Omega$*, for every* $\eta \in \mathbb{R}$*,* $\xi \in \mathbb{R}^N$*,*

(P3) $\sum_{j=1}^{N}(g_j(x, \eta, \xi) - g_j(x, \eta, \tilde{\xi}))(\xi_j - \tilde{\xi}_j) > 0$*, a.e.* $x \in \Omega$*, for every* $\eta \in \mathbb{R}$*,* $\xi \neq \tilde{\xi} \in \mathbb{R}^N$*.*

We formulate also a coercivity condition:

(H5) There exists $K \subset C$, weakly compact, and $u_0 \in C$ such that

$$\sup_{f \in A(u,u)} \langle u_0 - u, f \rangle + \Phi(u, u_0 - u) < 0,$$

for each $u \in C \setminus K$.

In the study of existence of a solution for the problem (VI), the following lemmas will be needed.

Lemma 3. *(marginal function lemma)*[1] *Let* X *and* Y *be two topological spaces,* $G : X \to 2^Y$ *a set-valued mapping and* $g : X \times Y \to \mathbb{R}$*. Denote* $h : X \to \mathbb{R}$*,* $h(x) = \sup_{y \in G(x)} g(x, y)$ *the marginal function. If the following conditions*

(a) g *is u.s.c. on* $X \times Y$*,*

(b) $G(x_0)$ *is compact for some* $x_0 \in X$*,*

(c) G *is u.s.c. at* x_0*,*

are satisfied, then h *is u.s.c. at* x_0*.*

Lemma 4. *[KyFan](see* [2]*) Let* X *be a topological vector space,* H *a subset of* X *and* $F : H \to 2^X$ *a set-valued mapping with* $F(x)$ *closed for every* $x \in H$*, such that:*

(a) $F(x_0)$ *is compact for some* $x_0 \in H$*,*

(b) for every $x_1, x_2, \ldots, x_n \in H$*,* $co\{x_1, x_2, \ldots, x_n\} \subset \bigcup_{i=1}^{n} F(x_i)$*.*

Then $\bigcap_{x \in X} F(x) \neq \emptyset$*.*

Theorem 5. *In the hypotheses (H1)-(H5), the problem* (VI) *has at least one solution.*

Proof. Following the idea from [5], we divide the proof in several steps:

a) For each $w \in C$ we denote:

$$A_1(w) = \{u \in C \mid \sup_{f \in A(u,u)} \langle w - u, f \rangle + \Phi(u, w - u) \geq 0\}$$

It is obvious that $\bigcap_{w \in C} A_1(w)$ is the set of solutions of the problem (VI).

We verify the conditions of Ky Fan's Lemma for $F(w) = \text{w-cl}A_1(w)$ (the weak closure). Consider $u_0 \in C$, the element from the coercivity condition (H5). This condition implies that $A_1(u_0) \subset K$. But K is weakly compact and so $\text{w-cl}A_1(u_0)$ is also weakly compact.

b) Let $w_1, \ldots, w_n \in C$.

We want to prove that $\text{co}\{w_1, \ldots, w_n\} \subset \bigcup_{i=1}^{n} A_1(w_i) \subset \bigcup_{i=1}^{n} \text{w-cl}A_1(w_i)$.

Suppose that this is not true, that is there exist $\lambda_1, \ldots, \lambda_n \geq 0$, with $\sum_{j=1}^{n} \lambda_j = 1$ such that $\bar{w} = \sum_{j=1}^{n} \lambda_j w_j \notin A_1(w_i)$, for every $i = \overline{1, n}$, which implies

$$\langle w_i - \bar{w}, f \rangle + \Phi(\bar{w}, w_i - \bar{w}) < 0,$$

for each $f \in A(\bar{w}, \bar{w})$ and $i = \overline{1, n}$.

For a fixed $f \in A(\bar{w}, \bar{w})$, we have, using the previous inequality and the sublinearity of Φ,

$$\begin{aligned} 0 \leq & \langle \bar{w} - \bar{w}, f \rangle + \Phi(\bar{w}, \bar{w} - \bar{w}) \\ & = \langle \sum_{j=1}^{n} \lambda_j w_j - \sum_{j=1}^{n} \lambda_j \bar{w}, f \rangle + \Phi(\bar{w}, \sum_{j=1}^{n} \lambda_j w_j - \sum_{j=1}^{n} \lambda_j \bar{w}) \\ & \leq \sum_{j=1}^{n} (\lambda_j \langle w_j - \bar{w}, f \rangle + \lambda_j \Phi(\bar{w}, w_j - \bar{w}) < 0, \end{aligned}$$

which is a contradiction. This gives us that $\text{co}\{w_1, \ldots, w_n\} \subset \bigcup_{i=1}^{n} \text{w-cl}A_1(w_i)$. We obtain, by Lemma 4,

$$\bigcap_{w \in C} \text{w-cl}A_1(w) \neq \emptyset. \tag{1}$$

c) Denote, for $w \in C$

$$A_2(w) = \{u \in K \mid \sup_{f \in A(w,u)} \langle w - u, f \rangle + \Phi(u; w - u) \geq 0\}$$

We will prove that

$$\bigcap_{w\in C} A_2(w) \subset \bigcap_{w\in C} A_1(w). \tag{2}$$

Let $u \in C$, $u \in A_2(w)$, for every $w \in C$. Fix $v \in C$ and consider $v_t = tv+(1-t)u \in C$, for each $t \in [0,1]$. From $u \in A_2(v_t)$ we get:

$$\sup_{f\in A(v_t,u)} \langle v_t - u, f\rangle + \Phi(u, v_t - u) \geq 0, \qquad \forall t \in [0,1]$$

and further, using the fact that $\Phi(u,\cdot)$ is positively homogeneous and dividing by t

$$\sup_{f\in A(v_t,u)} \langle v - u, f\rangle + \Phi(u, v - u) \geq 0, \qquad \forall t \in (0,1]. \tag{3}$$

Define $G : [0,1] \to 2^{V^*}$, $G(t) = A(v_t,u)$. According to (H2), this is upper semi-continuous and according to (H4), $G(0) = A(u,u)$ is compact. The mapping $(f,v) \in V^* \times V \mapsto \langle v-u, f\rangle$ is continuous for V with the weak topology and V^* with the norm topology. Applying Lemma 3 we have that $h(t) = \sup\limits_{f\in G(t)} \langle v-u, f\rangle$ is upper semi-continuous at 0, that is

$$\limsup_{t\to 0} \sup_{f\in A(v_t,u)} \langle v-u, f\rangle \leq \sup_{f\in A(u,u)} \langle v-u, f\rangle.$$

From this and from (3) it follows that u is a solution for (VI), that is

$$u \in \bigcap_{w\in C} A_1(w).$$

d) At the final step we prove that

$$\bigcap_{w\in C} \text{w-cl}A_1(w) \subset \bigcap_{w\in C} A_2(w). \tag{4}$$

From step (a) we have that $\bigcap\limits_{w\in C} \text{w-cl}A_1(w) \subset K$.

Let $u \in C$, arbitrarily fixed and let $v \in \text{w-cl}A_1(u)$. We will prove that $v \in A_2(u)$. From $v \in \text{w-cl}A_1(u)$, there exists a net $\{v_j\}$ in $A_1(u)$ such that $v_j \rightharpoonup v$. The fact that $v_j \in A_1(u)$ means that

$$\sup_{f\in A(v_j,v_j)} \langle u - v_j, f\rangle + \Phi(v_j, u - v_j) \geq 0, \ \forall\, j \in I,$$

and using hypothesis (H1),

$$\sup_{f\in A(u,v_j)} \langle u-v_j, f\rangle + \Phi(v_j, u-v_j) \geq 0,\ \forall\, j\in I,$$

In order to use Lemma 3 we notice that the mapping $(f,z)\in V^*\times V \mapsto \langle u-z, f\rangle$ is continuous, $A(u,\cdot)$ is weakly upper semi-continuous and $A(u,v)$ is compact. It follows that $h(z) = \sup_{f\in A(u,z)} \langle u-z, f\rangle$ is weakly upper semi-continuous at v, that implies

$$\limsup_{v_j\rightharpoonup v} \sup_{f\in A(u,v_j)} \langle u-v_j, f\rangle \leq \sup_{f\in A(u,v)} \langle u-v, f\rangle.$$

On the other side, $\limsup_{v_j\rightharpoonup v} \Phi(v_j, u-v_j) \leq \Phi(v, u-v)$. Further on, we have

$$\begin{aligned} 0 &\leq \limsup_{v_j\rightharpoonup v}\{ \sup_{f\in A(u,v_j)} \langle u-v_j, f\rangle + \Phi(v_j, u-v_j)\} \\ &\leq \limsup_{v_j\rightharpoonup v} \sup_{f\in A(u,v_j)} \langle u-v_j, f\rangle + \limsup_{v_j\rightharpoonup v} \Phi(v_j, u-v_j) \\ &\leq \sup_{f\in A(u,v)} \langle u-v, f\rangle + \Phi(v, u-v), \end{aligned}$$

which means that $v\in A_2(u)$, for every $u\in C$. This proves (4). From (2), (1) and (4) we get $\bigcap_{w\in C} A_1(w) \neq \emptyset$, which concludes the proof.

Remark 6. *(see* [5]*) If in addition to the previous hypotheses, $A(u,u)$ is a convex set, then u is also a solution of the following problem:*

Find $u\in C$ and $f\in A(u,u)$ such that $\langle v-u, f\rangle + \Phi(u, v-u) \geq 0,\ \forall\, v\in C$.

Remark 7. *From step (d) of the proof, it is clear that hypothesis (H1) can be replaced with the supposition that the "diagonal" mapping $A(\cdot,\cdot)$ is weakly upper semi-continuous (from V with the weak topology to V^* with the norm topology).*

Remark 8. *A similar result can be obtained, for variational inequalities, by considering $C\subset V^*$, $A : C\times C \to V$, where V^* is equipped with the weak* topology and V with the norm topology (see* [5]*).*

3. An evolution variational inequality

Consider the following evolution variational inequality:

$$(EVI) \quad u \in C \quad \langle v-u, Lu\rangle + \sup_{f\in A(u,u)} \langle v-u, f\rangle + \Phi(u, v-u) \geq 0, \quad \forall\, v \in C,$$

where $C \subset V$ is nonempty, convex, closed,

$A : V \times V \to 2^{V^*}$ satisfies the hypotheses (H1'), (H2)-(H4) and $L : D(L) \subset V \to V^*$ is a closed densely linear maximal monotone operator.

It is known that, in these conditions, $W = D(L)$, endowed with the graph norm $\|u\|_W = \|u\|_V + \|Lu\|_{V^*}$, is a reflexive Banach space. Denote $\tilde{C} = C \cap D(L)$; it is a convex, closed, nonempty set.

Hypothesis (H5) will be replaced by

(H5') There exists $K \subset \tilde{C}$, weak compact, and $u_0 \in \tilde{C}$ such that

$$\langle u_0 - u, Lu\rangle + \sup_{f\in A(u,u)} \langle u_0 - u, f\rangle + \Phi(u, u_0 - u) < 0,$$

for each $u \in \tilde{C} \setminus K$.

Theorem 9. *In the hypotheses (H1'), (H2)-(H4) and (H5'), the problem (EVI) has at least one solution.*

Proof. We have that W is densely embedded in V. Denoting $i : W \to V$ the natural embedding of W in V and $i^* : V^* \to W^*$ its adjoint, we define the operator $B : \tilde{C} \times \tilde{C} \to 2^{W^*}$ by

$$B(u, v) = \tilde{L}(u) + \tilde{A}(u, v), \ \forall\, u, v \in \tilde{C},$$

where $\tilde{L} : W \to W^*$, $\tilde{L} = i^* \circ L \circ i$, that is

$$\langle v, \tilde{L}(u)\rangle_{W\times W^*} = \langle v, i^*(L(iu))\rangle_{W\times W^*} = \langle iv, L(iu)\rangle_{V\times V^*}, \ \forall\, u, v \in W.$$

The same, $\tilde{f} \in \tilde{A}(u, v)$ means that $\tilde{f} = i^* f$, with $f \in A(iu, iv) \subset V^*$, that is $\langle w, \tilde{f}\rangle_{W\times W^*} = \langle w, i^* f\rangle_{W\times W^*} = \langle iw, f\rangle_{V\times V^*}$.

With these notations, problem (EVI) can be written:

$$u \in \tilde{C} = C \cap D(L) \text{ such that } \sup_{g\in B(u,u)} \langle v-u, g\rangle + \Phi_{|W}(u, v-u) \geq 0, \ \forall\, v \in \tilde{C}$$

We will prove that the operator B defined above satisfies the hypotheses (H1')-(H4), in the space W with the weak topology and W^* with the norm topology.

(H1') Let $u, v \in \tilde{C}$, fixed. If $g \in B(u,v)$ then $g = \tilde{L}(u) + f$, with $f \in \tilde{A}(u,v)$ that is $f = i^*h$, $h \in A(iu, iv)$. We have, taking account of the monotonicity of L and of (H1):

$$\sup_{g\in B(u,v)} \langle u - v, g\rangle_{W\times W^*} = \langle iu - iv, L(iu)\rangle_{V\times V^*} + \sup_{h\in A(iu,iv)} \langle iu - iv, h\rangle_{V\times V^*}$$

$$\geq \langle iu - iv, L(iv)\rangle_{V\times V^*} + \sup_{h\in A(iv,iv)} \langle iu - iv, h\rangle_{V\times V^*} = \sup_{g\in B(v,v)} \langle u - v, g\rangle_{W\times W^*}.$$

(H2) For each $v \in W$ fixed, $B(\cdot, v) : \tilde{C} \to 2^{W^*}$ is upper semi-continuous from the line segments in $\tilde{C}$ to W^*.

Indeed, let $u_1, u_2 \in \tilde{C}$ arbitrarily fixed, $u_t = tu_1 + (1-t)u_2$, for $t \in [0,1]$ and define $\tilde{G} : [0,1] \to W^*$ by $\tilde{G}(t) = B(u_t, v) = \tilde{L}(u_t) + \tilde{A}(u_t, v)$.

The upper semi-continuity of $\tilde{G}$ at $t = 0$ follows from the upper semi-continuity of $A(\cdot, iv)$ and from the continuity of $\tilde{L}$ from W to W^*.

(H3) The weak upper semi-continuity of $B(u, \cdot) : \tilde{C} \to 2^{W^*}$, for $u \in \tilde{C}$ fixed, is a consequence of the fact that $\tilde{L}(u)$ does not depend on v and of the weak upper semi-continuity of $\tilde{A}(u, \cdot)$ at an arbitrary point of W.

(H4) We want to prove that, for each $u, v \in \tilde{C}$, $B(u,v)$ is compact. Consider a sequence $\{f_n\}$ in $B(u,v) \subset W^*$, $f_n = \tilde{L}(u) + g_n$, with $g_n = i^*h_n$, $h_n \in A(iu, iv)$. Since $A(iu, iv)$ is compact, there exists a subsequence (denoted in the same way), $h_n \to h \in A(iu, iv)$, in V^*. Then $f_n \to \tilde{L}(u) + i^*h \in B(u,v)$.

The fact that Φ is weakly upper semi-continuous in the topology of V implies directly that it is also upper semi-continuous in the topology of W, because $u_j \rightharpoonup u$ in W means $u_j \rightharpoonup u$ in V and $L(u_j) \rightharpoonup L(u)$ in V^*.

All the hypotheses (H1'), (H2)-(H4) being satisfied and having also (H5'), we can apply Theorem 5 and conclude the proof.

Remark 10. *The following particular case is frequently used: Let U be a real reflexive Banach space, densely and compactly imbedded into a separable Hilbert space H, $U \subset H \subset U^*$, i.e. an evolution triple. (For example $H_0^1(\Omega) \subset L^2(\Omega) \subset H^{-1}(\Omega)$, where Ω*

is a bounded domain in $\mathbb{R}^N$, $H_0^1(\Omega)$ *is the well known Sobolev space and* $H^{-1}(\Omega)$ *is its dual). Let* $V = L^2(0,\tau;U)$ *and* $V^* = L^2(0,\tau;U^*)$ *the dual of* V. *In this case,* L *can be the differentiation operator* $\frac{d^2\cdot}{dt^2}$.

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ON THE CONVERSES OF THE REDUCTION PRINCIPLE IN INNER PRODUCT SPACES

COSTICĂ MUSTĂŢA

Dedicated to Professor Ştefan Cobzaş at his 60^{th} *anniversary*

Abstract. Let H be an inner product space, X a complete subspace of H, and Y a closed subspace of X. The main result of this Note is the following converse of the Reduction Principle: if $x_0 \in X$, $h \in H \setminus X$ and $y_0 \in Y$ is the element of best approximation of both x_0 and h, $(x_0 - h, x_0 - y_0) = 0$ and $\operatorname{codim}_X Y = 1$, then x_0 is the element of best approximation of h in X.

1. Introduction

Let H be an inner product space, with real inner product $(\cdot, \cdot)$ and the norm $\|h\| = \sqrt{(h, h)}$, $h \in H$. For a subset M of H and $h \in H$, the distance of h to M is defined by

$$d(x, M) = \inf\{\|h - m\| : \ m \in M\}.$$

The set M is called **proximinal** if for every $h \in H$ there exists $m_0 \in M$ such that

$$\|h - m_0\| = d(h, M).$$

The set

$$P_M(h) := \{m \in M : \ \|h - m\| = d(h, M)\}, \quad h \in H$$

is called the set of **best approximation elements** of h by elements in M, and the application $P_M : H \to 2^M$ is called the metric projection of H on M.

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If card$P_M(h) = 1$ for every $h \in H$, then the set M is called a **Chebyshevian set** in H ([2], p.35).

The **existence** and the **uniqueness** of best approximation elements are treated in Chapter 3 of [2]: every complete convex set in an inner product space is a Chebyshev set ([2], Th.3.4).

Two elements $u, v \in H$ are called **orthogonal** if $(u, v) = 0$. The cosinus of the angle between the $u, v \in H \setminus \{0\}$ is defined by the formula

$$\cos \widehat{u, v} = \frac{(u, v)}{\|u\| \cdot \|v\|}.$$

Concerning the **characterization** of best approximation elements, the following result holds ([2], Th.4.9):

Let M be a subspace of H, $h \in H$ and $m_0 \in M$. Then $m_0 = P_M(h)$ iff

$$(h - m_0, m) = 0,$$

for all $m \in M$.

The geometric interpretation of this characterization result is that the element $h - P_M(h)$ is orthogonal to each element of M. This is the reason why $P_M(h)$ is often called the **orthogonal projection** of h on M.

The following result appears in [2], p.80 under the name "the Reduction Principle":

Let K be a convex subset of the inner product space H and let M be any Chebyshev subspace of H that contains K. Then

a) $P_K(P_M(h)) = P_K(h) = P_M(P_K(h)),\ h \in H$;

b) $d(h, K)^2 = d(h, M)^2 + d(P_M(h), K)^2$,

for every $h \in H$.

Obviously, if K is a closed and convex subset of a complete subspace M of the inner product space H, the properties a) and b) are also fulfilled (see Th.4.1 in [2], and Th. 2.2.6 in [3]).

2. Results

From now on, we consider the following particular case of the Reduction Principle:

Theorem 1. *Let H be an inner product space, X a complete subspace of H, and Y a closed subspace of X. Then*

a') $P_Y(h) = P_Y(P_X(h)) = P_X(P_Y(h))$, $h \in H$;

b') $d(h,Y)^2 = d(h,X)^2 + d(P_X(h),Y)^2$,

for every $h \in H$.

The proof of Theorem 1 is an immediate consequence of the characterization result ([2], Th.4.9) and the Pythagorean Law (see e.g. [1], Th.1, p.70).

A generalization of Theorem 1 is:

Theorem 2. *Let H be an inner product space and $M_1, M_2, \dots, M_n$ $(n \geq 2)$ be subspaces of H with the following properties:*

1) M_1 is complete;

2) M_i, $i = 2, 3, \dots, n$ are closed;

3) $M_1 \supset M_2 \supset \dots \supset M_n$.

a) For every $h \in H$ the following equalities hold

$$P_{M_n}(h) = P_{M_n} P_{M_{n-1}} \dots P_{M_1}(h) = P_{M_1} P_{M_2} \dots P_{M_n}(h).$$

b) Let $P_{M_1}(h) = m_1$, $P_{M_k} P_{M_{k-1}}(h) = m_k$, $k = 2, 3, \dots, n$.

The following equality holds:

$$d(h, M_n)^2 = \|h - m_1\|^2 + \sum_{k=2}^{h} \|m_k - m_{k-1}\|^2.$$

Proof. For every $y \in M_n$ we have

$$(h - P_{M_n} P_{M_{n-1}} \dots P_{M_1}(P_1), y)$$

$$= (h - P_{M_1}(h) + P_{M_1}(h) - P_{M_2} P_{M_1}(h) + \dots$$

$$+ P_{M_{n-1}} P_{M_{n-2}} \dots P_{M_1}(h) - P_{M_n} P_{M_{n-1}} \dots P_{M_1}(h), y)$$

$$= (h - P_{M_1}(h), y) + \sum_{k=2}^{n} (P_{M_{k-1}} \dots P_{M_1}(h) - P_{M_k} \dots P_{M_1}(h), y) = 0.$$

Using the characterization result ([2], Th.4.9) it follows that the element $P_{M_n} P_{M_{n-1}} \dots P_{M_1}(h)$ is the orthogonal projection of h on M_n.

On the other hand, $(h - P_{M_n}(h), y) = 0$ for every $y \in M_n$. Consequently

$$P_{M_n}(h) = P_{M_n} P_{M_{n-1}} \dots P_{M_1}(h).$$

The equality $P_{M_n}(h) = P_{M_1} P_{M_2} \dots P_{M_n}(h)$ is immediate.

For b) observe that

$$d(h, M_n)^2 = \|m_n - m_{n-1}\|^2 + \|h - m_{n-1}\|^2$$

$$= \|m_n - m_{n-1}\|^2 + \|m_{n-1} - m_{n-2}\|^2 + \|h - m_{n-2}\|^2 = \dots$$

$$= \|m_n - m_{n-1}\|^2 + \dots + \|m_2 - m_1\|^2 + \|h - m_1\|^2. \qquad \square$$

Remark. Obviously, Theorem 1 is also valid if H is a Hilbert space and X, Y are closed subspace of H, with $Y \subset X$. Also, Theorem 2 is valid if H is a Hilbert space and $M_1 \supset M_2 \supset \dots \supset M_n$ are closed subspaces of H.

A converse of the Reduction Principle is given in [3], Th.2.2.6:

Let H be an inner product space, X a complete subspace of H and K a closed and convex subset of X. If x is the orthogonal projection of $h \notin X$ on X, m is the metric projection of h on K, then m is the metric projection of x on K.

A **first** converse of Theorem 1 is:

Theorem 3. *Let H be an inner product space, X a complete subspace of H, and Y a closed subspace of X. Let $h \in H \setminus X$ and let $P_X(h)$ and $P_Y(h)$ be the orthogonal projections of h on X, respectively on Y. Then $P_Y(h)$ is the orthogonal projection of $P_X(h)$ on Y.*

Proof. Indeed, by hypothesis it follows:

$$(h - P_X(h), x) = 0, \ \forall\ x \in X,$$

$$(h - P_Y(h), y) = 0, \ \forall\ y \in Y,$$

so that for every $y \in Y$ one has:

$$(P_X(h) - P_Y(h), y) = (h - P_Y(h) - h + P_X(h), y)$$

$$= (h - P_Y(h), y) - (h - P_X(h), y) = 0.$$

It follows that $P_Y(h)$ is the orthogonal projection of $P_X(h)$ on Y. □

A second converse of Theorem 1 is:

Theorem 4. *Let H be an inner product space, X a complete subspace of H, and Y a closed subspace of X with* $\text{codim}_X Y = 1$. *Let $x_0 \in X \setminus Y$ and $P_Y(x_0)$ be the orthogonal projection of x_0 on Y. If $h \in H \setminus X$, $P_Y(h) = P_Y(x_0)$ and $(h - x_0, x_0 - P_Y(x_0)) = 0$, then $P_Y(h) = x_0$.*

Proof. If the equality $(h - x_0, x) = 0$ is fulfilled for every $x \in X$, then $P_X(h) = x_0$, i.e. the conclusion of the theorem.

For every $y \in Y$ we have

$$(h - x_0, y) = (h - P_Y(x_0) - (x_0 - P_Y(x_0)), y)$$

$$= (h - P_Y(x_0), y) - (x_0 - P_Y(x_0), y) = 0.$$

It follows that $h - x_0$ is orthogonal to Y.

Because, by hypothesis, $(h-x_0, x_0-P_Y(x_0)) = 0$ it follows that $(h-x_0, u) = 0$ for every $u \in \text{span}\{x_0 - P_Y(x_0)\}$. Because $x_0 - P_Y(x_0)$ is orthogonal to Y and Y is a closed subspace of the Hilbert space X, it follows that $X = \text{span}\{x_0 - P_Y(x_0)\} \oplus Y$, i.e. X is the direct sum of the subspaces $\text{span}\{x_0 - P_Y(x_0)\}$ and Y (see [2], Th.5.9 p.77 and [1], Th.4, p.65). Consequently $(h - x_0, x) = 0$ for every $x \in X$. □

Remark. The condition $\text{codim}_X Y = 1$ in Theorem 4 is essential. Indeed, let $\{e_1, e_2, e_3\}$ be the orthonormal basis of the Hilbert space $\mathbb{R}^3$, $X = \text{span}\{e_1, e_2\}$, $Y = \text{span}\{0\}$ and $h = 3e_1 + e_2 + 5e_3$. Let $x_0 = e_1 + 2e_2$. Then $P_Y(x_0) = 0$ and $P_Y(h) = 3e_1 + e_2$, $P_Y(h) = 0$. The conditions $P_Y(x_0) = P_Y(h)$ and $(h - x_0, x_0 - P_Y(x_0)) = (2e_1 - e_2, e_1 + 2e_2) = 0$ are fulfilled, but $P_X(h) = 3e_1 + e_2 \neq x_0 = e_1 + 2e_2$. Observe that $\text{codim}_X Y = 2$.

Examples. 1° Let $l_2 = l_2(\mathbb{N})$ be the space of all sequences $x = (x(i))$ of real numbers such that $\sum_{i=1}^{\infty} x^2(i) < \infty$. It is known that l_2 is a Hilbert space with respect to the inner product $(x, y) = \sum_{i=1}^{\infty} x(i)y(i)$ and the norm $\|x\| = \left(\sum_{i=1}^{\infty} x^2(i)\right)^{1/2}$. Let $\{e_1, e_2, \dots\}$ be the canonical basis of l_2. The closed subspace $X = \overline{\operatorname{span}\{e_{2n-1} \mid n = 1, 2, 3, \dots\}}$ is Chebyshevian in l_2 and the orthogonal projection of $h = (h(1), h(2), \dots) \in l_2$ is $P_X(h) = \sum_{i=1}^{\infty} h(2i-1)e_{2i-1}$, because $h - P_X(h) = \sum_{j=1}^{\infty} h(2j)e_{2j}$ is orthogonal on X.

Let $Y = \operatorname{span}\{e_1, e_3 + e_5\}$. Then Y is a Chebyshevian subspace of l_2 (and of X) and

$$P_Y(h) = h(1)e_1 + \frac{1}{2}[h(3) + h(5)](e_3 + e_5).$$

By Theorem 1 one obtains

$$P_Y(h) = P_Y P_X(h) = P_X P_Y(h).$$

By Theorem 3, the orthogonal projection of the element

$$x = \sum_{n=1}^{\infty} h(2n-1)e_{2n-1}$$

on Y is

$$y_0 = h(1)e_1 + \frac{1}{2}[h(3) + h(5)](e_3 + e_5).$$

Indeed,

$$x - y_0 = \frac{1}{2}[h(3) - h(5)]e_3 + \frac{1}{2}[h(5) - h(3)]e_5 + \sum_{n=4}^{\infty} h(2n-1)e_{2n-1}$$

is orthogonal to Y, so $y_0 = P_Y(x)$.

2° Let $l_2(4) = \operatorname{span}\{e_1, e_2, e_3, e_4\}$ where $e_i(j) = \delta_{ij}$, $i, j = 1, 2, 3, 4$ (see 1°), and $X = \operatorname{span}\{e_1, e_2, e_3\}$, $Y = \operatorname{span}\{e_1, e_2\}$ and $Z = \operatorname{span}\{e_1\}$.

If $x_0 = 2e_1 + e_2 + 2e_3$, then $P_Y(x_0) = 2e_1 + e_2$. For $\alpha, \beta \in \mathbb{R}$ let $h = 2e_1 + e_2 + \alpha e_3 + \beta e_4$. Then $P_Y(h) = 2e_1 + e_2$ and $(h - x_0, x_0 - P_Y(x_0)) = 2(\alpha - 2) = 0$ implies $\alpha = 2$.

Every element $h = 2e_1 + e_2 + 2e_3 + \beta e_4$, $\beta \in \mathbb{R}$ has as orthogonal projection on X

$$P_X(h) = 2e_1 + e_2 + 2e_3 = x_0.$$

Observe that $\text{codim}_X Y = 1$.

Consider now the orthogonal projections on Z ($\text{codim}_X Z = 2$). Then $P_Z(x_0) = 2e_1$, $P_Z(h) = 2e_1$ and $(h - x_0, x_0 - P_Z(x_0)) = \alpha + \beta - 3 = 0$ implies $\alpha + \beta = 3$.

Choosing the element $h = 2e_1 + 2e_2 + e_3 + 2e_4$ one obtains

$$P_X(h) = 2e_1 + 2e_2 + e_3 \neq 2e_1 + e_2 + 2e_3 = x_0.$$

3° Let $L_2[-1,1]$ be the Hilbert space of all (Lebesgue) measurable real-valued functions on $[-1,1]$ with the property that $\int_{-1}^{1} h^2(t)dt < \infty$. The inner product on $L_2[-1,1]$ is $(x,y) = \int_{-1}^{1} x(t)y(t)dt$ and the associated norm is $\|h\| = \left(\int_{-1}^{1} h^2(t)dt\right)^{1/2}$. Consider also the Legendre polynomials (see [2])

$$p_0(t) = \frac{1}{\sqrt{2}},\ p_1(t) = \frac{\sqrt{6}}{2}t,\ p_2(t) = \frac{\sqrt{10}}{4}(3t^2 - 1),\ p_3(t) = \frac{\sqrt{14}}{4}(5t^3 - 3t)$$

and in general

$$p_n(t) = \frac{(-1)^n\sqrt{2n+1}}{2^n \cdot \sqrt{2} \cdot n!} \cdot \frac{d^n}{dt^n}[(1-t^2)^n],$$

for $n \geq 0$.

The set $\{p_0, p_1, \ldots, p_n\}$, $n \geq 0$ is orthonormal in $L_2[-1,1]$. Consider the following subspaces of $L_2[-1,1]$:

$$X = \text{span}\{p_0, p_1, p_2, p_3\}, \quad Y = \text{span}\{p_0, p_1, p_2\} \quad \text{and}$$

$$Z = \text{span}\{p_0, p_1\}.$$

For every $h \in L_2[-1,1]$ one obtains ([2], Th.4.14)

$$P_X(h) = (h, p_0)p_0 + (h, p_1)p_1 + (h, p_2)p_2 + (h, p_3)p_3,$$

$$P_Y(h) = (h, p_0)p_0 + (h, p_1)p_1 + (h, p_2)p_2 \quad \text{and}$$

$$P_Z(h) = (h, p_0)p_0 + (h, p_1)p_1.$$

Obviously, $Z \subset Y \subset X \subset L_2[-1,1]$ and $P_Z(h) = P_Z P_Y P_X(h)$.

Let $x_0 = p_0 + 2p_1 + 2p_2 + p_3$. If $h \in L_2[-1,1] \setminus X$ then $P_Y(h) = P_Y(x_0)$ iff $(h, p_0) = 1$, $(h, p_1) = 2$ and $(h, p_2) = 2$. The condition $(x_0 - P_Y(x_0), h - x_0) = 0$ implies $(p_3, h - x_0) = 0$ and, consequently, $(p_3, h) = (p_3, x_0) = 1$. It follows $P_X(h) = x_0$. Observe that $\operatorname{codim}_X Y = 1$.

Now $P_Z(x_0) = p_0 + 2p_1$ and $P_Z(h) = P_Z(x_0)$ implies $(h, p_0) = 1$, $(h, p_1) = 2$. The condition $(x_0 - P_Z(x_0), h - x_0) = 0$ implies

$$(2p_2 + p_3, h - x_0) = 2(p_2, h) + (p_3, h) - 5 = 0.$$

Let $h_1 = p_0 + 2p_1 + p_2 + 3p_3 + p_4$ and $h_2 = p_0 + 2p_1 + \frac{1}{2}p_2 + 4p_3 + p_4$. Then $P_Z(h_i) = P_Z(x_0)$, $i = 1, 2$ and $(x_0 - P_Z(x_0), h_i - x_0) = 0$, $i = 1, 2$, but $P_X(h_1) \neq P_X(h_2) \neq x_0$. Observe that $\operatorname{codim}_X Z = 2$.

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BILATERAL APPROXIMATIONS OF SOLUTIONS OF EQUATIONS BY ORDER THREE STEFFENSEN-TYPE METHODS

ION PĂVĂLOIU

Dedicated to Professor Ştefan Cobzaş at his 60^{th} *anniversary*

Abstract. The convergence of method of Steffensen-type which is obtained from the Lagrange polynomial of inverse interpolation with controlled nodes - is studied in this paper. Conditions are given sequences which bilaterally approximates the solution of an equation.

1. **Introduction**

In order to approximate the solutions of scalar equations it is suitable to use iteration methods which lead to monotone sequences. Suppose that such a method generates two such sequences, i.e., an increasing sequence $(u_n)_{n\geq 0}$ and a decreasing sequence $(v_n)_{n\geq 0}$. If both converge to the solution $\bar{x}$ of a given equation, then at each step one obtains the following error control:

$$\max\{\bar{x} - u_n, v_n - \bar{x}\} \leq v_n - u_n$$

Such methods can be generated, for example, by combining simultaneously both Newton and chord methods [1], [2], [3], [10].

Conditions for Steffensen and Aitken-Steffensen methods which lead to monotone sequences which bilaterally approximate the root of a given equation, were studied in [6], [10].

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It is known that both the Steffensen and Aitken-Steffensen methods are obtained from the chord method in which the interpolation nodes are controlled.

In this paper we shall consider a Steffensen-type method, obtained from the inverse interpolation polynomial of second degree, using three controlled interpolation nodes.

More exactly, consider the following equation

$$f(x) = 0, \tag{1}$$

where $f : [a, b] \longrightarrow \mathbb{R}$, $a, b \in \mathbb{R}$, $a < b$.

Denote by $F = f([a, b])$ the range of f for $x \in [a, b]$.

Suppose that $f : [a, b] \longrightarrow F$ is a bijection, that is, there exists $f^{-1} : F \longrightarrow [a, b]$

Let a_1, a_2, $a_3 \in [a, b]$, $a_i \neq a_j$, for $i \neq j$, $i; j = \overline{1, 3}$, three distinct interpolation nodes and let $b_1 = f(a_1)$, $b_2 = f(a_2)$, $b_3 = f(a_3)$. The inverse interpolation Lagrange polynomial for f^{-1} on the nodes b_1, b_2, $b_3 \in F$ is given by the following relation:

$$\begin{aligned} L(b_1, b_2, b_3; f^{-1} \mid y) =& a_1 + [b_1, b_2; f^{-1}](y - b_1) \\ &+ [b_1, b_2, b_3; f^{-1}](y - b_1)(y - b_2) \end{aligned} \tag{2}$$

with the remainder given by:

$$f^{-1}(y) - L(b_1, b_2, b_3; f^{-1} \mid y) = [y, b_1, b_2, b_3; f^{-1}](y - b_1)(y - b_2)(y - b_3). \tag{3}$$

It is known that (2) is symmetric with respect to nodes order. Thus, if (i_1, i_2, i_3) is a permutation of $(1, 2, 3)$, then the following relations are satisfied:

$$L(b_1, b_2, b_3; f^{-1} \mid y) = a_{i_1} + [b_{i_1}, b_{i_2}; f^{-1}](y - b_{i_1}) + [b_{i_1}, b_{i_2}, b_{i_3}; f^{-1}](y - b_{i_1})(y - b_{i_2}) \tag{4}$$

Apart from (2), these relations lead to five more representations for Lagrange's polynomial.

In order to obtain a Steffensen-type method, and to approximate the solutions of equations (1), we shall consider one additional equation:

$$x - g(x) = 0, \quad g : [a, b] \longrightarrow [a, b],$$

which we shall assume is equivalent with (1).

If equation (1) has one root $\bar{x} \in [a, b]$, then obviously $\bar{x} = f^{-1}(0)$, and from (3) one obtains

$$\bar{x} = L(b_1, b_2, b_3; f^{-1} \mid 0) - [0, b_1, b_2, b_3; f^{-1}]b_1 b_2 b_3,$$

and if we neglect the remainder, we obtain:

$$\bar{x} \simeq L(b_1, b_2, b_3; f^{-1} \mid 0). \tag{5}$$

For divided differences of first and second order of f^{-1}, one knows that [5], [7], [8], [10]:

$$[b_1, b_2; f^{-1}] = \frac{1}{[a_1, a_2; f]} \tag{6}$$

and

$$[b_1, b_2; b_3; f^{-1}] = -\frac{[a_1, a_2, a_3; f]}{[a_1, a_2; f][a_1, a_3; f][a_2, a_3; f]}. \tag{7}$$

Relations (2), (5), (6) and (7) lead to the following approximation of $\bar{x}$:

$$\bar{x} \simeq a_1 - \frac{f(a_1)}{[a_1, a_2; f]} - \frac{[a_1, a_2, a_3; f] f(a_1) f(a_2)}{[a_1, a_2; f][a_2, a_3; f][a_1, a_3; f]} \tag{8}$$

or the equivalent formal representations from (4). Supposing that f has third degree derivatives at each point from $[a, b]$, then function f^{-1} has third degree derivatives at each point of F.

The following relation is satisfied for the third order derivative of f^{-1} [4], [10], [11], [12]:

$$[f^{-1}(y)]''' = \frac{3[f''(x)]^2 - f'(x) f'''(x)}{[f'(x)]^5} \tag{9}$$

where $y = f(x)$.

Denote $x_n \in [a, b]$ an approximation to the root $\bar{x}$ of (1).

We consider the following nodes in (8):

$$a_1 = x_n, a_2 = g(x_n), a_3 = g(g(x_n)).$$

Taking into account all six approximative representations of $\bar{x}$, obtained by permutations of set $(1,2,3)$ one obtains the following representations for the Steffensen-type method.

If

$$D(x_n) = \frac{[x_n, g(x_n), g(g(x_n)); f]}{[x_n, g(x_n); f][x_n, g(g(x_n)); f][g(x_n), g(g(x_n)); f]}$$

then the above considerations lead us to the following:

$$x_{n+1} = x_n - \frac{f(x_n)}{[x_n, g(x_n); f]} - D(x_n) f(x_n) \cdot f(g(x_n)); \tag{10}$$

$$x_{n+1} = x_n - \frac{f(x_n)}{[x_n, g(g(x_n)); f]} - D(x_n) f(x_n) \cdot f(g(g(x_n))); \tag{11}$$

$$x_{n+1} = g(x_n) - \frac{f(g(x_n))}{[x_n, g(x_n); f]} - D(x_n) f(x_n) f(g(x_n)); \tag{12}$$

$$x_{n+1} = g(x_n) - \frac{f(g(x_n))}{[g(x_n), g(g(x_n)); f]} - D(x_n) f(g(x_n)) f(g(g(x_n))); \tag{13}$$

$$x_{n+1} = g(g(x_n)) - \frac{f(g(g(x_n)))}{[x_n, g(g(x_n)); f]} - D(x_n) f(x_n) f(g(g(x_n))); \tag{14}$$

$$x_{n+1} = g(g(x_n)) - \frac{f(g(g(x_n)))}{[g(x_n), g(g(x_n)); f]} - D(x_n) f(g(x_n)) f(g(g(x_n))). \tag{15}$$

From Newton's identity (3) one obtains the error representation:

$$\bar{x} - x_{n+1} = -[0, f(x_n), f(g(x_n)), f(g(g(x_n))); f^{-1}] f(x_n) f(g(x_n)) f(g(g(x_n))). \tag{16}$$

From the mean value formulas for divided differences one obtains for a fixed $x \in [a,b]$ the existence of $\eta \in F$ such that:

$$[0, f(x), f(g(x)), f(g(g(x))); f^{-1}] = \frac{[f^{-1}(\eta)]}{3!}.$$

Since $\eta \in F$ and f is a bijection, using (9) there results the existence of $\xi \in [a,b]$ such that

$$[0, f(x), f(g(x)), f(g(g(x))); f^{-1}] = \frac{3[f''(\xi)]^2 - f'(\xi) f'''(\xi)}{6[f'(\xi)]^5}.$$

Denote

$$E(x) = 3[f''(x)]^2 - f'(x)f'''(x) \tag{17}$$

so that we obtain from (16)

$$\bar{x} - x_{n+1} = -\frac{E(\xi_n)}{6[f'(\xi_n)]^5} f(x_n)f(g(x_n))f(g(g(x_n))). \tag{18}$$

where $\xi_n \in [a,b]$ is assigned to $x = x_n$.The x_{n+1} term is given by each of the relations (10)-(15).

2. The convergence of Steffensen-type method

In this section we shall see that conditions for the Steffensen-type method of third order given by any relations (10)-(15), lead to sequences which bilaterally approximate the root of (1).

We suppose that g satisfies the following conditions:

a) *there exists $l \in \mathbb{R}$, $0 < l < 1$ such that for all $x \in [a,b]$:*

$$|g(x) - g(\bar{x})| \le l\,|x - \bar{x}|\,, \tag{19}$$

where $\bar{x}$ is the common root of (1) *and $x = g(x)$;*

b) *the function g is decreasing on $[a,b]$;*

c) *the equations* (1) *and $x = g(x)$ ar equivalent.*

The following result holds:

Theorem 1. *If functions f, g and element $x_0 \in [a,b]$ satisfy the conditions:*

i$_1$. *if $x_0 \in [a,b]$, then $g(x_0) \in [a,b]$;*

ii$_2$. *f has third order derivatives on $[a,b]$;*

iii$_3$. *$f'(x) > 0$, $f''(x) \ge 0$, for all $x \in [a,b]$;*

iv$_1$. *$E(x) \le 0$ for all $x \in [a,b]$, where E is given by (17);*

v$_1$. *function g satisfies a)-c);*

vi$_1$. *equation (1) has a root $\bar{x} \in [a,b]$.*

Then the following properties are true:

j$_1$. *The elements of sequence* $(x_n)_{n\geq 0}$ *generated by (10), where* x_0 *satisfies* i_1*, remain in* $[a,b]$ *and for each* $n=0,1,\ldots$*, the following relations are satisfied:*

$$x_n \leq x_{n+1} \leq \bar{x} \leq g(x_{n+1}) \leq g(x_n) \tag{20}$$

if $f(x_0)<0$*, or*

$$x_n \geq x_{n+1} \geq \bar{x} \geq g(x_{n+1}) \geq g(x_n) \tag{21}$$

if $f(x_0)>0$.

jj$_1$. $\lim x_n = \lim g(x_n) = \bar{x}$;

jjj$_1$. $\max\{|x_{n+1}-\bar{x}|, |g(x_n)-\bar{x}|\} \leq |x_{n+1}-g(x_n)|$, *for each* $n=0,1,\ldots$.

Proof. Let $x_n \in [a,b]$, $n\geq 0$ for which $g(x_n)\in[a,b]$.

We consider first: $f(x_n)<0$, that is, $x_n<\bar{x}$. Since g is decreasing and using $g(\bar{x})=\bar{x}$ one obtains:

$$g(x_n) > \bar{x}$$

and $g(g(x_n)) < \bar{x}$.

Relation (19) implies:

$$|g(g(x_n)) - \bar{x}| \leq l\,|g(x_n)-\bar{x}| \leq l^2\,|x_n - \bar{x}|$$

from which one obtains:

$$a \leq x_n < g(g(x_n)) < \bar{x} < g(x_n) \leq b. \tag{22}$$

By use of iii$_1$, (22) and (10) one gets

$$x_{n+1} \geq x_n. \tag{23}$$

The assumptions ii$_1$-iv$_1$and from (22) and (18) we get

$$\bar{x} - x_{n+1} > 0,$$

i.e., $\bar{x} > x_{n+1}$, that is $f(x_{n+1})<0$.

Using (23) and assumption $c)$ on g one obtains $g(x_{n+1}) \leq g(x_n)$ and $g(x_{n+1}) > g(\bar{x}) = \bar{x}$. Hence we have shown (20).

We consider now the case $f(x_n) > 0$, that is $x_n > \bar{x}$.

Taking in consideration (11) instead of (10) and by use of

$$g(x_n) < \bar{x}$$

and $g(g(x_n)) > \bar{x}$, one gets:

$$f(g(x_n)) < 0, \quad f(g(g(x_n))) > 0.$$

It is obvious to note relations (21).

Eventually, (20) and (21) show that sequences $(x_n)_{n\geq 0}$ and $(g(x_n))_{n\geq 0}$ converge. Denote $\lim x_n = a$, then,we obtain $\lim g(x_n) = g(a)$. Using (10) or (11) as $n \to \infty$, it results that $f(a) = 0$ and therefore $a = \bar{x}$, the unique solution of (1) on $[a, b]$. □

Remark 2. *Suppose the assumptions from Theorem 1 are satisfied and excepting* iii_1, *the following assumption holds.*

$f'(x) < 0$ *and* $f''\,(x) < 0$ *for each* $x \in [a, b]$ *and consider instead of (1) the following equation:*

$$h(x) = 0, \tag{24}$$

where h *is given by* $h(x) = -f(x)$.

Note that Theorem 1 holds for (24).

The proof is obvious, since $h'(x) > 0$ *and* $h''(x) > 0$, *for all* $x \in [a, b]$ *and* $E(x) = 3[h''(x)]^2 - h'(x)h''(x) < 0$, *that is* E *remains invariant.*

A result similar to Theorem 1 holds, for the case in which f is decreasing and convex.

Theorem 3. *If functions* f, g *and element* $x_0 \in [a, b]$ *satisfy the following conditions:*

i$_2$. *if* $x_0 \in [a, b]$, *then* $g(x_0) \in [a, b]$;

ii$_2$. f *has third order derivative on* $[a, b]$;

iii$_2$. $f'(x) < 0$ *and* $f''(x) > 0$, *for all* $x \in [a, b]$;

iv$_2$. $E(x) \leq 0$, *for all* $x \in [a, b]$;

v$_2$. *function* g *satisfies* a)-c);

vi$_2$. *equation* (1) *has one root* $\bar{x} \in [a, b]$.

Then $(x_n)_{n \geq 0}$ *generated by (10) or (11), remains in* $[a, b]$, *and relation* j_1- jjj_1 *from Theorem 1 are satisfied, when* x_0 *satisfies* i_2.

Proof. The assumption iii$_1$ leads to $D(x) < 0$ for all $x \in [a, b]$. Let $x_n \in [a, b]$, $n \geq 0$, an element for which $g(x_n) \in [a, b]$.

If $x_n > \bar{x}$, then $f(x_n) < 0$ and $g(x_n) < \bar{x}$, $g(g(x_n)) > \bar{x}$.

From (19) one gets

$$|g(g(x_n)) - \bar{x}| \leq l^2 \left| x_n - \bar{x} \right|,$$

that is the following relations hold:

$$a \leq g(x_n) < \bar{x} < g(g(x_n)) < x_n \leq b.$$

From iii$_2$, $f(x_2) < 0$ and using $D(x_n) < 0$, (10) one obtains

$$x_{n+1} < x_n.$$

The assumptions ii$_2$-iv$_2$ and relations (22), and (18) imply

$$\bar{x} - x_{n+1} < 0,$$

that is, $x_{n+1} > \bar{x}$, $f(x_{n+1}) < 0$. Obviously relations (21) hold.

Relations (20) and consequences jj$_1$ and jjj$_1$ are proven analogously to Theorem 1. □

Remark 4. *If* f *is increasing and concave, that is,* $f'(x) > 0$ *and* $f'(x) < 0$, *then obviously* $h = -f$ *is decreasing and convex.*

If we replace in Theorem 3: function f *by function* h, *and if we take into account that function* E *remains invariant by this replacement, then we note that the statements of Theorem 3 remain true.*

3. Determination of the auxiliary function

In the following, by use of function f, we give a method to determine auxiliary function g, which could assure the control of interpolatory nodes.

If f is a convex function, i.e. $f''(x) > 0$, then for function g we consider

$$g(x) = x - \frac{f(x)}{f'(x)}. \tag{25}$$

If f is a concave function, then we can set

$$g(x) = x - \frac{f(x)}{f'(b)}. \tag{26}$$

Obviously in both cases we have $g'(x) < 0$ and thus function g satisfies assumption b).

It is clear that function g given by either (25) or (26), assures the equivalence of (1) and $x = g(x)$, i.e., g satisfies assumption c).

In order that g also satisfies assumption a), it is enough that the following relations hold:

$$\left|1 - \frac{f'(x)}{f(a)}\right| < 1,$$

for all $x \in [a, b]$, if f is convex function, or

$$\left|1 - \frac{f'(x)}{f(b)}\right| < 1,$$

for all $x \in [a, b]$, if f is a concave function.

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METRIC SPACE WITH FIXED POINT PROPERTY WITH RESPECT TO CONTRACTIONS

IOAN A. RUS

Dedicated to Professor Ştefan Cobzaş at his 60^{th} *anniversary*

Abstract. In this paper we present some equivalent statements with the fixed point property of a metric space with respect to contractions. These statements are in terms of completeness, Picard operators, fractal operators, minimal displacement, well posedness of fixed point problem and the limit shadowing property.

1. Introduction

Let X be a nonempty set and $(X, S(X), M)$ a fixed point structure on X ([18] and [19]). Let $S_1(X) \subset P(X)$ such that $S(X) \subset S_1(X)$. By definition $(X, S(X), M)$ is maximal in $S_1(X)$ if we have

$$S(X) = \{A \in S_1(X) \mid f \in M(A) \;\Rightarrow\; F_f \neq \emptyset\}.$$

Is an open problem to establish if a given fixed point structure is maximal or not. For example in some concrete structured sets this problem take the following forms:

- Characterize the ordered sets with fixed point property with respect to increasing operators ([3], [5], [10], [11], [18], [19], [22]-[24]).
- Characterize the metric space with fixed point property with respect to continuous operators ([1], [2], [6], [10]).

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Key words and phrases. fixed point property, contraction, completeness, fractal operator, minimal displacement, well posedness, shadowing property.

- Characterize the metric space with fixed point property with respect to contractions ([4], [21], [6]-[10]).
- Characterize the Banach space X with the following property ([10], [19], [20]):

$$Y \in P_{b,cl,cv}(X),\ f : Y \to Y \text{ nonexpansive } \Rightarrow F_f \neq \emptyset.$$

- Characterize the Banach space with the following property ([10], [19], [20]):

$$Y \in P_{wcp,cv}(X),\ f : Y \to Y \text{ nonexpansive } \Rightarrow F_f \neq \emptyset.$$

The aim of this paper is to present some equivalent statements with the fixed point property of a metric space with respect to contractions.

2. Notations and notions

Let (X, d) be a metric space and $(P_{cp}(X), H_d, \subset)$ the corresponding ordered metric space of fractals. In what follow we shall use the following notations. We denote

$$CT(X, X) := \{f : X \to X \mid f \text{ is a contraction}\}.$$

If $f \in CT(X, X)$ then we denote by $\widehat{f} : P_{cp}(X) \to P_{cp}(X)$, $A \mapsto f(A) := \bigcup_{a \in A} f(a)$, the corresponding fractal operator.

$$(UF)_{\widehat{f}} := \{A \in P_{cp}(X) \mid \widehat{f}(A) \subset A\},$$

$$(LF)_{\widehat{f}} := \{A \in P_{cp}(X) \mid \widehat{f}(A) \supset A\}.$$

For an operator $f : X \to X$ we denote by $d(f) := \inf\{d(x, f(x)) \mid x \in X\}$ the minimal displacement of f (K. Goebel (1973) ([10], 586)).

To present our results we need the following notions:

Definition 2.1. (F.S. De Blasi and J. Myjak (1989) ([17])). Let (X, d) be a metric space and $f : X \to X$ an operator. The fixed point problem for f is well posed iff

(a) $F_f = \{x^*\}$;

(b) if $x_n \in X$, $n \in \mathbb{N}$ and $d(x_n, f(x_n)) \to 0$ as $n \to \infty$, then $d(x_n, x^*) \to 0$ as $n \to \infty$.

Definition 2.2. ([13]) An operator $f : X \to X$ has the limit shadowing property iff $x_n \in X$, $n \in \mathbb{N}$, $d(x_{n+1}, f(x_n)) \to 0$ as $n \to \infty$ imply that there exists $x \in X$ such that $d(x_n, f^n(x)) \to 0$ as $n \to \infty$.

Definition 2.3. A metric space is complete with respect to $CT(X, X)$ iff $f \in CT(X, X)$ implies that $(f^n(x))_{n\in\mathbb{N}}$ converges for all $x \in X$.

Remark 2.1. If $f \in CT(X, X)$ then $(f^n(x))_{n\in\mathbb{N}}$ is a Cauchy sequence for all $x \in X$.

3. Equivalent statements

The main result of this paper is the following

Theorem 3.1. *Let (X, d) be a metric space. The following statements are equivalent:*

(i) (X, d) has the fixed point property with respect to $CT(X, X)$.

(ii) $f \in CT(X, X)$ implies that f is Picard operator.

(iii) (X, d) is complete with respect to $CT(X, X)$.

(iv) $f \in CT(X, X)$ implies that there exists $x^*_f \in X$ such that $d(f) = d(x^*_f, f(x^*_f))$.

(v) $f \in CT(X, X)$ implies that the fixed point problem for f is well posed.

(vi) $f \in CT(X, X)$ implies that $F_{\widehat{f}} \neq \emptyset$.

(vii) $f \in CT(X, X)$ implies that $(UF)_{\widehat{f}} \neq \emptyset$.

(viii) $f \in CT(X, X)$ implies that $(LF)_{\widehat{f}} \neq \emptyset$.

(ix) $f \in CT(X, X)$ implies that there exists $x \in X$ such that $(f^n(x))_{n\in\mathbb{N}}$ converges.

Proof. (i) $\Rightarrow$ (ii). Let f be an α-contraction with $F_f = \{x^*\}$. Then $d(f^n(x), x^*) = d(f^n(x), f^n(x^*)) \leq \alpha^n d(x, x^*) \to 0$ as $n \to \infty$, for all $x \in X$. So, f is Picard operator.

(ii) $\Rightarrow$ (iii). Follows from the definition of Picard operators.

(iii) $\Rightarrow$ (iv). Let $f \in CT(X,X)$. It is clear that (iii) implies (ii). So, x_f^* is the fixed point of f.

(iv) $\Rightarrow$ (v). Let f be an α-contraction and $d(f) = d(x_f^*, f(x_f^*))$. If $x_f^* \neq f(x_f^*)$, then we have

$$d(f(x_f^*), f^2(x_f^*)) \leq \alpha d(x_f^*, f(x_f^*)) < d(x_f^*, f(x_f^*)).$$

This implies that $F_f = \{x_f^*\}$. Let $x_n \in X$, $n \in \mathbb{N}$, be such that $d(x_n, f(x_n)) \to 0$ as $n \to \infty$. We have

$$\begin{aligned} d(x_n, x_f^*) &\leq d(x_n, f(x_n)) + d(f(x_n), x_f^*) \\ &\leq d(x_n, f(x_n)) + \alpha d(x_n, x_f^*). \end{aligned}$$

Hence,

$$d(x_n, x^*) \leq \frac{1}{1-\alpha} d(x_n, f(x_n)) \to 0 \text{ as } n \to \infty.$$

(v) $\Rightarrow$ (vi). By a theorem of Nadler f contraction implies that $\widehat{f}$ is contraction. Let $f \in CT(X,X)$ with $F_f = \{x^*\}$. Then by definition of $\widehat{f}$, $\{x^*\} \in F_{\widehat{f}}$. But, $\widehat{f}$ contraction implies $F_{\widehat{f}} = \{\{x^*\}\}$.

(vi) $\Rightarrow$ (vii). Let $f \in CT(X,X)$. The condition (vi) implies $F_{\widehat{f}} = \{A^*\}$. But $\delta(\widehat{f}(A^*)) = \delta(A^*) \leq \alpha\delta(A^*)$. This implies $A^* = \{a^*\}$. We remark that $A^* \in (UF)_{\widehat{f}} \neq \emptyset$.

(vii) $\Rightarrow$ (viii). Let $f \in CT(X,X)$ and $A^* \in (UF)_{\widehat{f}}$. These imply $A^* = \{a^*\}$. So, $A^* \in (LF)_{\widehat{f}}$.

(viii) $\Rightarrow$ (ix). $f \in CT(X,X)$ and $A^* \in (LF)_{\widehat{f}}$ imply $A^* = \{a^*\}$. So, $F_f = \{x^*\}$ and $f^n(x) \to x^*$ as $n \to \infty$, for all $x \in X$.

(ix) $\Rightarrow$ (i). Let $f \in CT(X,X)$, and $x \in X$ such that $f^n(x) \to y^*$. From the continuity of f we have that $y^* \in F_f$. So, $F_f = \{y^*\}$.

Remark 3.1. It is well known that there exist incomplete metric spaces with fixed point property with respect to contraction (see [4], [21]). On the other hand there exists some equivalent statements with completeness ([1], [5], [7]-[9], [12],...).

Remark 3.2. Condition (ii) implies that each $f \in CT(X,X)$ has the limit shadowing property.

Indeed, let f be an α-contraction with $F_f = \{x^*\}$ and $x_n \in X$, $n \in \mathbb{N}$, such that

$$d(x_{n+1}, f(x_n)) \to 0 \text{ as } n \to \infty.$$

We have

$$\begin{aligned} d(x_n, x^*) &\le d(x_n, f(x_{n-1})) + d(f(x_{n-1}), x^*) \\ &\le d(x_n, f(x_{n-1})) + \alpha d(x_{n-1}, x^*) \le \dots \\ &\le d(x_n, f(x_{n-1})) + \alpha d(x_{n-1}, f(x_{n-2})) + \dots \\ &+ \alpha^{n-1} d(x_1, f(x_0)) + \alpha^n d(x_0, x^*). \end{aligned}$$

From the Cauchy's lemma we have that

$$d(x_n, x^*) \to 0 \text{ as } n \to \infty.$$

So, $d(x_n, f^n(x_0)) \le d(x_n, x^*) + d(x^*, f^n(x_0)) \to 0$ as $n \to \infty$.

Remark 3.3. Let X be a nonempty and $f : X \to X$ an operator. We suppose that there exists $A \in P(X)$ such that

$$A \subset f(A).$$

For the fixed point theory for such operators see J. Andres [2] and the references therein.

The above considerations give rise to

Open problem 3.1. Extend the results of this paper to generalized metric spaces.

Open problem 3.2. Extend the results of this paper to generalized contractions.

Open problem 3.3. Extend the results of this paper to the case of multivalued generalized contractions.

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BOOK REVIEWS

Antonio Ambrosetti and Andrea Malchiodi, ***Perturbation Methods and Semilinear Elliptic Problems on*** $\mathbf{R}^n$, Progress in Mathematics (series editors: H. Bass, J. Oesterlé and A. Weinstein), vol. 240, Birkhäuser Verlag, Basel-Boston-Berlin, 2006, xii+183 pp; ISBN-10:3-7643-7321-0, ISBN-13:978-3-7643-7321-4, e-ISBN 3-7643-7396-2.

The monograph is based on the authors' own papers carried out in the last years, some of them in collaboration with other people like D. Arcoya, M. Badiale, M. Berti, S. Cingolani, V. Coti Zelati, J.L. Gamez, J. Garcia Azorero, V. Felli, Y.Y. Li, W.M. Ni, I. Peral and S. Secchi.

The book is concerning with perturbation methods in critical point theory together with their applications to semilinear elliptic equations on $\mathbf{R}^n$ having a variational structure.

The contents are as follows: Foreword; Notation; 1 Examples and motivations (giving an account of the main nonlinear variational problems studied by the monograph); 2 Perturbation in critical point theory (where some abstract results on the existence of critical points of perturbed functionals are presented); 3 Bifurcation from the essential spectrum; 4 Elliptic problems on $\mathbf{R}^n$; 5 Elliptic problems with critical exponent; 6 The Yamabe problem; 7 Other problems in conformal geometry; 8 Nonlinear Schrödinger equations; 9 Singularly perturbed Neumann problems; 10 Concentration at spheres for radial problems; Bibliography (147 titles) and Index.

The topics are presented in a systematic and unified way and the large range of applications talks about the power of the critical point methods in nonlinear analysis.

I recommend the book to researchers in topological methods for partial differential equations, especially to those interested in critical point theory and its applications.

Radu Precup

Jonathan M. Borwein and Qiji J. Zhu, ***Techniques of Variational Analysis,*** Canadian Mathematical Society (CMS) Books in Mathematics, Vol. 20, Springer 2005, vi+362 pp, ISBN 3-387-24298-8.

The term variational analysis concerns methods of proofs based on the fact that an appropriate auxiliary function attains a minimum, and has its roots in the physical principle of the least action. Probably that the first illustration of this method is Johann Bernoulli's solution to the Brachistocrone problem which led to the development of variational calculus.

A significant impact on variational analysis was done by the development of nonsmooth analysis, making possible the use of calculus of nonsmooth functions and enlarging substantially the area of applications. Other powerful tools are the decoupling method (a nonconvex substitute for Fenchel conjugacy and Hahn-Banach theorem from convex analysis), alongside with variational principles.

As it is well known, a lower semi-continuous (lsc) function attains its minimum on a compact set, a property that is not longer true in the absence of the compactness, even for bounded from below lsc functions. This drawback can be compensated by adding a small perturbation to the original function such that the perturbed function attains its minimum. The properties of the perturbation function depend on the geometric properties of the underlying space: the better these properties (smoothness) the nicer the perturbation function. This fact is well illustrated in the second chapter, *Variational Principles* - Ekeland variational principle holds in complete metric spaces, while the smooth Borwein-Preiss variational principle holds in Banach spaces with smooth norm. Another one, Stegall variational principle (proved in Chapter 6), holds in Banach spaces with the Radon-Nikodym property and ensures a continuous linear perturbation.

The aim of the book is to emphasize the strength of the variational techniques in various domains of analysis, optimization and approximation, dynamic systems, mathematical economics. These applications are arranged by chapters which are relatively independent and can be used for graduate topics courses.

The chapters are: 3. *Variational techniques in subdifferential theory* (Fréchet subdifferential and normal cone, sum rules, chain rules for Lyapunov functions, mean value theorems and inequalities, extremal principles); 4. *Variational techniques in convex analysis* (Fenchel conjugate, duality, entropy maximization); 5. *Variational techniques and multifunctions* (multifunctions, subdifferentials as multifunctions, distance functions, coderivatives of multifunctions, implicit multifunction theorems); 6. *Variational principles in nonlinear functional analysis* (subdifferential and Asplund spaces, nonconvex separation, Stegall variational principle, mountain pass theorem); 7. *Variational techniques in the presence of symmetry* (nonsmooth functions on smooth manifolds, manifolds of matrices and spectral functions, convex spectral functions).

The book contains a lot of exercises completing the main text, some of them, which are more difficult, being guided exercises with references.

Based mainly on developments and applications from the past several decades, the book is directed to graduate students in the field of variational analysis. The prerequisites for its reading are undergraduate analysis and basic functional analysis. Researchers who use variational techniques, or intend to do, will find the book very useful too.

S. Cobzaş

Dorin Bucur, Giuseppe Buttazzo, ***Variational Methods in Shape Optimization Problems***, Progress in Nonlinear Differential Equations and their Applications, Birkhäuser, 2005, ISBN 0-8176-4359-1.

Usually, problems of the calculus of variations concern optimization among an admissible class of functions. What is special about shape optimization problems is that the "competing objects" are shapes (domains of $\mathbf{R}^n$). Because of this, the existence of a solution is ensured only in certain cases, due to some geometrical restrictions on the admissible domains (shapes) or to a particular form of the cost function. In general, relaxed formulations of the problems have to be formulated.

The development of the field of shape optimization is due especially to the great number of applications in physics and engineering.

Several examples of shape optimization problems are presented in the first chapter of the book, in a detailed and clear manner: the isoperimetric problem, the Newton problem of minimal aerodynamical resistance, the optimal distribution of two different media in a fixed region, the optimal shape of a thin insulating layer.

The second chapter is about optimization problems over classes of convex domains and it deals with the case where an additional convexity constraint on the domains ensures the existence of an optimal shape (by providing some extra compactness). Some necessary conditions of optimality are given for the Newton problem.

Some shape optimization problems can be considered optimal control problems: the shape plays the role of the control and the state equation is usually a partial differential equation on the control domain. In Chapter 3, a topological framework for general optimization problems is given, together with the theory of relaxed controls and some examples of relaxed shape optimization problems.

Shape optimization problems with Dirichlet (Neumann) condition on the free boundary are treated in Chapters 4 (7, respectively). In both cases, is important to understand the stability of the solution to a PDE for nonsmooth perturbations of the geometric domain. This stability is related to the convergence in Mosco sense of the corresponding variational spaces. The relaxed form of a Dirichlet problem is given (in a case where the existence of an optimal solution does not occur), to understand the behavior of minimizing sequences. For Neumann boundary conditions, the problem of optimal cutting is treated completely.

Chapter 5 contains other particular cases where an unrelaxed optimal solution exists, in the family of classical admissible domains. The existence of solutions is ensured by some monotonicity properties of the cost functional or by some geometrical constraints on the domains.

Optimization problems for functions of eigenvalues are presented in Chapter 6. The case of the first two eigenvalues of the Laplace operator is studied, using the continuous Steiner symmetrization.

The book is addressed mainly to graduate students, applied mathematicians, engineers; it requires standard knowledge in the calculus of variations, differential equations and functional analysis.

The problems are treated from both the classical and modern perspectives, each chapter contains examples and illustrations and also several open problems for further research. A substantial bibliography is given, emphasizing the rapid development of the field.

Daniela Inoan

Stefaan Caenepeel and Freddy van Oystaeyen Editors, ***Hopf Algebras in Noncommutative Geometry and Physics***, Pure and Applied Mathematics, Vol. 239, Marcel Dekker, New York, 2005, 320 pp., ISBN 0-8247-5759-9.

The study of Hopf algebras and quantum groups has seen a great development during the last two decades. The present volume is devoted to these topics, and consists of high quality articles related to the lectures given at the meeting on "Hopf algebras and quantum groups" held at the Royal Academy in Brussels from May 28 to June 1, 2002. This volume contains refereed papers and surveys on different aspects of the subject, such as:

The list of contributors and their papers is as follows. *J. Abuhlail*, Morita contexts for corings and equivalences; *F. Aly and F. van Oystaeyen*, Hopf order module algebra orders; *G. Böhm*, An alternative notion of Hopf algebroid; *Ph. Bonneau and D. Sternheimer*, Topological Hopf algebras, quantum groups and deformation quantization; *T. Brzeziński, L. Kadison and R. Wisbauer*, On coseparable and biseparable corings; *D. Bulacu, S. Caenepeel and F. Panaite*, More properties of Yetter-Drinfeld modules over quasi-Hopf algebras; *S. Caenepeel, J. Vercruysse and S.H. Wang*, Rationality properties for Morita contexts associated to Corings; *L. El Kaoutit and J. Gómez-Torrecillas*, Morita duality for corings over quasi-Frobenius rings; *K.R. Goodearl and T.H. Lenagan*, Quantized coinvariants at transcendental q; *S. Majid*, Classification of differentials on quantum doubles and finite noncommutative geometry; *S. Majid*, Noncommutative differentials and Yang-Mills on permutation groups S_n; *C. Menini and G. Militaru*, The afineness criterion for Doi-Koppinen modules; *S. Montgomery*, Algebra properties invariant under twisting; *C. Ohn*, Quantum $\mathrm{SL}(3,\mathbb{C})$'s: the missing case; *A Paolucci*, Cuntz algebras and dynamical quantum group $SU(2)$; *B. Pareigis*, On symbolic computations in braided monomial categories; *P. Schauenburg*, Quotients of finite quasi-Hopf algebras; *K. Szlachányi*; Adjointable monoidal functors and quantum groupoids; *R. Wisbauer*, On Galois corings.

The book is highly recommended to researchers in algebraic geometry, number theory and mathematical physics, who will find here an excellent overview of the most significant areas of research in this field. Some of the new results are presented here for the first time. It is a valuable addition to the literature, and I warmly recommend it to algebraists and theoretical physicists.

Andrei Marcus

Farid M.L. Amirouche, ***Fundamentals of Multibody Dynamics: Theory and Applications***, Birkhäuser, Boston-Basel-Berlin, 2006, XVIII+684 pp, ISBN 0-8176-4236-6.

Multibody dynamics has grown in the past two decades to be an important tool for designing, prototyping, and simulating complex articulated mechanical systems. This is mainly due to its versatility in analyzing a broad range of applications. This textbook – a result of the author's many years of research and teaching – brings together diverse concepts of dynamics, combining the efforts of many researchers in the field of mechanics. Bridging the gap between dynamics and engineering applications such as microrobotics, virtual reality simulation of interactive mechanical systems, nanomechanics, flexible biosystems, crash simulation, and biomechanics, the book puts into perspective the importance of modelling in the dynamic simulation and solution of problems in these fields.

To help engineering students and practicing engineers understand the rigid-body dynamics concepts needed for the book, the author presents a compiled overview of particle dynamics and Newton's second law of motion in the first chapter. A particular strength of the work is its use of matrices to generate kinematic coefficients associated with the formulation of the governing equations of motion, facilitating the computational investigation of the presented problems. Additional features of the book include:

- numerous worked examples at the end of each section;
- introduction of boundary-element methods (BEM) in the description of flexible systems;
- up-to-date solution techniques for rigid and flexible multibody dynamics using finite-element methods (FEM);
- inclusion of MATLAB-based simulations and graphical solutions;
- in-depth presentation of constrained systems;
- presentation of the general form of equations of motion ready for computer implementation;

• two unique chapters on stability and linearization of the equations of motion;

• numerous illustrations facilitating the understanding of the used models and methods;

• specific references at the end of each chapter and a comprehensive list of reference at the end of the book;

• supplementary material and solutions manual available upon request.

Junior/senior undergraduates and first-year graduate engineering students taking a course in dynamics, physics, control, robotics, or biomechanics will find this a useful book with a strong computer orientation towards the subject. The work may also be used as a self-study resource or research reference for practitioners in the above-mentioned fields.

Ferenc Szenkovits

Leszek Gasiński and Nikolaos S. Papageorgiou, ***Nonlinear Analysis***, Series in mathematical Analysis and Applications, Vol. 9, Chapman & Hall/CRC, Taylor & Francis Group, Boca Raton, London, New York, Singapore, 2006, xi +971 pp., ISBN 1-58488-484-3.

The aim of the present volume is to provide the reader with a solid background in several areas related to some modern topics in nonlinear analysis as critical point theory, nonlinear differential operators and related regularity and comparison principles.

The first chapter, *Hausdorff measures and capacity,* is concerned with topics as Vitali and Besicovitch covering theorems, Hausdorff measure and dimension, differentiability of Hausdorff measures and of Lipschitz functions (Rademacher theorem), the area, coarea and change of variables formulae for Lipschitz transforms.

The second chapter, *Lebesgue-Bochner and Sobolev spaces,* contains a brief introduction to integration of vector-functions (weak and strong measurability, Pettis, Gelfand and Bochner integrals), a treatment of Banach spaces of continuous vector-functions, of Lebesgue-Bochner spaces (completeness, duality, compactness), and of Sobolev spaces of vector-functions.

Chapter 3, *Nonlinear operators and Young measures,* discusses some classes of nonlinear operators (monotone, accretive) and semigroups of operators, exemplified on the case of Nemytskii composition operator. Some results on compact and on Fredholm linear operators on Banach and Hilbert spaces are also included, in order to emphasize the similarities and the differences between the linear and nonlinear case. The chapter ends with an introduction to Young measures.

The fourth chapter, *Smooth and nonsmooth variational principles,* contains an introduction to differential calculus on Banach spaces (Gâteaux and Fréchet derivatives) with applications to the differentiability of convex functions - Mazur and Asplund generic differentiability theorems. Christensen theorem on almost everywhere differentiability of locally Lipschitz functions on Banach spaces (the extension of Rademacher theorem) with respect to Haar null sets is also proved. Subdifferential calculus for convex functions, as well as Clarke generalized subdifferential calculus for locally Lipschitz functions are considered too. The chapter ends with the proof of Ekeland and Borwein-Preiss variational principles with applications.

Chapter 5, *Critical point theory,* is concerned with applications of the critical point theory to minimax, saddle point and mountain pass theorems. Lusternik-Schnirelman theory with applications to eigenvalue problems is the topic of the last section of this chapter.

In Chapter 6, *Eigenvalue problems and maximum principles,* the techniques and methods developed so far are applied to the study of linear and nonlinear elliptic PDEs.

Fixed point theorems (FPT) constitute the basic tool in the proofs of the existence of solutions to various kinds of equations and inclusions. The last chapter of the book, Chapter 7, *Fixed point theorems,* is devoted to the proofs of the main FPT of metrical nature (Banach contraction principle with extensions and applications, normal structure in Banach spaces and FPT for nonexpansive mappings), and of topological nature as well – the fixed point theorems of Brouwer, Schauder, Borsuk, and Sadovskii. A special attention is paid to FPT in ordered structure (Tarski, Bourbaki-Kneser, Amann) and in ordered Banach spaces – Krasnoselskii FPT with applications to positive eigenvalues and to fixed point index.

An appendix collects the essential results from topology, measure theory, functional analysis, calculus and nonlinear analysis, used throughout the book.

Together with the books *Nonsmooth Critical Point Theory and Nonlinear Boundary Value Problems,* CRC 2005, by the same authors, and *An Introduction to Nonlinear Analysis,* Vol. I. *Theory*, Vol. II, *Applications,* by Z. Denkowski, S. Migorski & N. Papageorgiou, the present one provides a comprehensive and fairly self-contained presentation of some important results in nonlinear analysis and applications.

It (or parts of it) can be used for graduate or post-graduate course, but also as reference text by specialists.

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