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ON THE APPROXIMATION NUMBERS OF CERTAIN VOLTERRA INTEGRAL OPERATORS BETWEEN LEBESGUE SPACES

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Abstract. The aim of the paper is to study certain Volterra integral operators of the form $Tf(x) = v(x) \int_0^x u(t)f(t)dt$ which defines linear maps between the Lebesgue spaces $L^p(\mathbb{R}^+)$ and $L^q(\mathbb{R}^+)$ where $1 \leq q < p \leq +\infty$. Under some conditions of the integrability on its kernel, we show some important properties for T such as boundedness, compactness, the measure of non compactness and estimating, upper and lower bounds for its approximation numbers. These estimates have an application to the spectral properties of Sturm-Liouville differential operators.

1. **Introduction**

In this paper we study certain linear integral operators of the form

$$Tf(x) = v(x) \int_0^x u(t)f(t)dt \quad (x \in \mathbb{R}^+ := [0, +\infty[). \tag{1.1}$$

These operators appear naturally in the theory of differential equations and it is important to establish when operators of this kind have properties (under some conditions of the integrability on the kernel) such as boundedness, compactness and to estimate their eigenvalues, or their singular numbers (approximation numbers) if these exist. Our concern in this paper lies with the problem which arises when integrability conditions on the kernel are weakened to local integrability requirements. Here, we consider in (1) that u and v are functions satisfying the local integrability and integrability conditions respectively and our objective is to give precise estimates for the approximation numbers of T. This operator has been studied extensively during

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the last several decades [2, 7, 11] in $L^2(\mathbb{R}^+)$ and $L^p(\mathbb{R}^+)$ spaces, respectively. Lastly Edmunds et al [4] generalized their results to the sitting in which T maps $L^p(\mathbb{R}^+)$ to $L^q(\mathbb{R}^+)$ with $1 \leq p \leq q \leq +\infty$ by considering the lower and upper bounds for the approximation numbers and hence for the measure of non compactness of T. This paper extends their results to the case $1 \leq q < p \leq +\infty$. The paper is organized as follows. In the next section we state our results concerning some properties such that the boundedness, compactness and of course the measure of non compactness of the operator T. In section 3 we give precise estimates for the approximation numbers of T. In section 4 we give an example to illustrate our ideas.

2. **Properties of T**

Throughout the paper, we use the same notation as in [4], and we shall assume that $p, q \in [1, +\infty[$, that $p' = \dfrac{p}{p-1}$, and u and v are prescribed real-valued functions such that

$$u \in L^{p'}_{loc}\left(\mathbb{R}^+\right) \tag{2.1}$$

and

$$v \in L^q\left(\mathbb{R}^+\right). \tag{2.2}$$

Given any measure μ on $\mathbb{R}^+$, any $\mu-$ measurable subset S of $\mathbb{R}^+$ and any function f in $L^p(S,\mu)$, we shall write:

$$\|f\|_{p,S,\mu} = \left(\int_S |f|^p \, d\mu\right)^{\frac{1}{p}} \; (1 \leq p < +\infty),$$

$$\|f\|_{\infty,S,\mu} = \mu - ess\sup|f(t)|$$

if μ is Lebesgue measure we shall omit the suffix μ and simply write $\|f\|_{p,S}$ if no ambiguity will result. For any $a \in \mathbb{R}^+$ we put

$$J_a = \left(\int_0^{+\infty}\left(\left(\int_a^x |u(t)|^{p'} dt\right)^{q-1}\left(\int_x^{+\infty}|v(z)|^q\right)dz\right)^{\frac{p}{p-q}} |u(x)|^{p'} dx\right)^{\frac{p-q}{pq}} \tag{2.3}$$

which define a continuous nonnegative function on $\mathbb{R}^+$.

2.1. **Boundedness.**

Theorem 1. *Let* $1 \leqslant q < p \leqslant +\infty$. *Then the operator* T *defined in* (1) *is a bounded linear map of* $L^p(\mathbb{R}^+)$ *to* $L^q(\mathbb{R}^+)$ *if, and only if,* $J_0 < \infty$, *and in this case, when* $1 < q < p \leq +\infty$,

$$(\frac{p-q}{p-1})^{\frac{1}{q'}} q^{\frac{1}{q}} J_0 \leqslant \|T\| \leqslant (p')^{\frac{1}{q'}} q^{\frac{1}{q}} J_0 \tag{2.4}$$

while

$$\|T\| = J_0 \quad \text{if} \quad q = 1 \text{ and } 1 < p \leq +\infty \tag{2.5}$$

$$\|T\| = q^{\frac{1}{q}} J_0 \quad \text{for } 1 < q < p = +\infty \tag{2.6}$$

where J_0 *is defined in (4) with* $a = 0$.

Proof. For all the cases $1 \leqslant q < p \leqslant +\infty$, we have

$$\|Tf(x)\|_q^q = \int_0^{+\infty} \left| v(x) \int_0^x u(t)f(t)dt \right|^q dx$$

and so the result follows from Maz'ja [**9**, theorem 1.3.2/1]. ∎

2.2. **Compactness and measure of non-compactness.** Let $\mathcal{K}_{p,q}$ denotes the set of all compact linear mappings from $L^p(\mathbb{R}^+)$ to $L^q(\mathbb{R}^+)$, and stand by $\mathcal{F}_{p,q}$ for all those elements of $\mathcal{K}_{p,q}$ which are of finite rank, put $\mathcal{K}_{p,p}= \mathcal{K}_p$, $\mathcal{F}_{p,p}= \mathcal{F}_p$ and write

$$\alpha(T) = \inf \{\|T - P\| \, ; \; P \in \mathcal{F}_{p,q}\} . \tag{2.7}$$

Since $L^q(\mathbb{R}^+)$ has the approximation property, it follows that $\alpha(T)$ is the distance of T from $\mathcal{K}_{p,q}$ ([4]).

Theorem 2. *Let* $1 \leq q < p \leq +\infty$. *If* $J_0 < \infty$, *then*

$$(\frac{p-q}{p-1})^{\frac{1}{q'}} q^{\frac{1}{q}} \lim_{a \mapsto +\infty} J_a \leq \alpha(T) \leq (p')^{\frac{1}{q'}} q^{\frac{1}{q}} \lim_{a \mapsto +\infty} J_a, \;\; \text{if } q > 1 \tag{2.8}$$

and

$$\alpha(T) = \lim_{a \mapsto +\infty} J_a \text{ for } q = 1. \tag{2.9}$$

To prove theorem 2.2 we use the following lemma (decomposition lemma).

Lemma 3. ([4, *theorem* 2]). *Suppose* T *is a bounded linear operator from* $L^p(\mathbb{R}^+)$ *to* $L^q(\mathbb{R}^+)$. *Then for every* $X \in (0,\infty)$ *there exist integral operators* P_X *and* T_X *both of the same type of* T *such that:*1)P_X *is a compact linear operator from* $L^p(\mathbb{R}^+)$ *to* $L^q(\mathbb{R}^+)$,

2) T_X *is a bounded linear operator from* $L^p(\mathbb{R}^+)$ *to* $L^q(\mathbb{R}^+)$ *with*

$$\|T_X\| \leqslant q^{\frac{1}{q}} p^{\frac{1}{\acute{q}}} J_X \quad \textit{for } q > 1$$

and

$$\|T_X\| = J_X \quad \textit{for } q = 1$$

3) $Tf = T_X f + P_X f \quad$ *for all* $f \in L^p(\mathbb{R}^+)$.

Proof. of the theorem 2.2 Choose $X \in [0,\infty)$ and let $T = T_X + P_X$ be the decomposition in the lemma above. By recalling some properties of the measure of the non-compactness (cf. [3, 12]) for more details) and theorem 2.1, we see that T_X is bounded linear operator from $L^p(\mathbb{R}^+)$ to $L^q(\mathbb{R}^+)$ with

$$\alpha(T) = \alpha(T_X) \leqslant \|T_X\| \leqslant q^{\frac{1}{q}} p^{\frac{1}{\acute{q}}} J_X \quad \text{for } q > 1.$$

Hence we have established that since $J_X < \lim_{X\to\infty} J_X$ then

$$\alpha(T) \leqslant \|T_X\| \leqslant q^{\frac{1}{q}} p^{\frac{1}{\acute{q}}} \lim_{X\mapsto+\infty} J_X \quad \text{for } q > 1$$

and

$$\alpha(T) \leqslant \lim_{X\to\infty} J_X \quad \text{for} \quad q = 1.$$

To establish the lower bound for $\alpha(T)$, we use the method employed by Evans and Harris [5,**§2**] to prove similar results for embedding maps. Let $1 \leqslant q < p \leqslant \infty$ and $\lambda > \alpha(T)$. Then there exists $P \in \mathcal{F}_{p,q}(L^p(\mathbb{R}^+), L^q(\mathbb{R}^+))$ with rank$P \leqslant N$, such that for all $f \in L^p(\mathbb{R}^+)$, $\|Tf - Pf\| \leqslant \lambda \|f\|_p$. By the argument of the [5, lemma 2.2], we may and shall suppose that there exists $Y \in \mathbb{R}^+$ such that for all $f \in L^p(\mathbb{R}^+)$, $\mathrm{Supp}Pf \subset [0, Y]$. Hence

$$\int_Y^{+\infty} |Tf(x)|^q \, dx = \int_Y^{+\infty} \left| v(x) \int_0^X u(t)f(t)dt \right|^q dx. \tag{2.10}$$

Now observe that

$$\begin{aligned} \int_Y^{+\infty} |Tf(x)|^q \, dx &= \int_Y^{+\infty} \left| v(x) \int_0^X u(t)f(t)dt \right|^q dx \\ &\geqslant \int_Y^{+\infty} \left| v(x) \int_Y^X u(t)f(t)dt \right|^q dx. \end{aligned}$$

We may assume that $uf \geqslant 0$ and by theorem 2.1 ensure that there exists f in $L^p(\mathbb{R}^+)$ such that

$$\int_Y^{+\infty} \left| v(x) \int_Y^X u(t)f(t)dt \right|^q dx \geqslant \left(\frac{p-q}{p-1}\right)^{\frac{1}{q'}} q^{\frac{1}{q}} J_Y \, \|f\|_p \, .$$

From (11) we have

$$\lambda \, \|f\|_p \geqslant \left(\frac{p-q}{p-1}\right)^{\frac{1}{q'}} q^{\frac{1}{q}} J_Y \, \|f\|_p \ \text{ for } 1 \leqslant q < p \leqslant \infty$$

and as λ may be chosen arbitrary close to $\alpha(T)$, we finally obtain:

$$\left(\frac{p-q}{p-1}\right)^{\frac{1}{q'}} q^{\frac{1}{q}} \lim_{Y \to \infty} J_Y \leqslant \alpha(T).$$

This completes the proof. ∎

Corollary 4. *The linear integral operator T from $L^p(\mathbb{R}^+)$ to $L^q(\mathbb{R}^+)$ is compact if, and only if,* $\lim\limits_{a \longmapsto +\infty} J_a = 0$.

The proof of the corollary follows immediately from theorem 2.2.

3. **Approximation numbers**

Our objective in this section is to estimate the approximation numbers $a_n(T)$ of T. This section follows closely the argument developed in [4] for the case $1 \leq p \leq q \leq \infty$ and several new features emerge because of the interchange of the order of p and q. To achieve this goal the following lemmas are crucial. Some notation will be useful. Given any interval I with end points (a,b) with $(0 \leqslant a \leqslant b)$ and any f in $L^p(\mathbb{R}^+)$, set

$$l(I,f) = l(a,b,f) = \int_a^b \int_a^b \left| v(x)u(x) \int_x^y u(t)f(t)dt \right|^q dxdy \tag{3.1}$$

and define

$$L(I) = \begin{cases} \left[\dfrac{1}{\mu(I)} \sup \left\{ l(a,b) : \|f\|_{p,I} \leq 1 \right\} \right]^{\frac{1}{q}} & \text{if} \quad \mu(I) \neq 0 \\ 0 & \text{otherwise} \end{cases} \tag{3.2}$$

where I is any interval in $\mathbb{R}^+$ with end points a and b, and μ is the finite measure defined by $d\mu(x) = |v(x)|^q \, dx$, so that $\mu(I) = \int_I |v(x)|^q \, dx$. In fact the two quantities satisfy the following lemmas.

Lemma 5. *For all bounded intervals $I \subset \mathbb{R}^+$ with end points a and b, the quantity* $\sup \left\{ l(a,b,f) : \|f\|_{p,I,\mu} \leq 1 \right\}$ *depends continuously upon a and b.*

Lemma 6. *The second quantity $L(a,b)$ is monotonic decreasing as a increases and monotonic increasing as b increases.*

Remark 1. *To deal with infinite interval we set $L(a,\infty) = \lim_{b \mapsto \infty} L(a,b)$ and it is possible that $L(a,\infty)$ may be infinite.*

Another piece of notation will be useful. We shall write

$$F(x) = \int_0^x u(t)f(t)dt \;\; (\; f \text{ in } L^p(\mathbb{R}^+), \;\; x \geq 0) \tag{3.3}$$

$$F_I(x) = \frac{1}{\mu(I)} \int_I F(x) d\mu(x) \quad \text{if} \quad \mu(I) \neq 0. \tag{3.4}$$

Given any $a, b \geq 0$ with $b > a$ and any c in $I =]a,b[$, we put

$$A_c = \left(\int_a^c \left(\left(\int_x^c |u(t)|^{p'} \, dt \right)^{q-1} \left(\int_a^x |v(z)|^q dz \right) \right)^{\frac{p}{p-q}} |u(x)|^{p'} \, dx \right)^{\frac{p-q}{pq}} \tag{3.5}$$

and

$$B_c = \left(\int_c^b \left(\left(\int_c^x |u(t)|^{p'} dt \right)^{q-1} \left(\int_x^b |v(z)|^q dz \right) \right)^{\frac{p}{p-q}} |u(x)|^{p'} dx \right)^{\frac{p-q}{pq}} \tag{3.6}$$

and we write $W(I) = \max(A_c, B_c)$ where c is the minimum point in I such that :

$$\int_a^c |v(y)|^q \, dy = \frac{1}{2} \int_a^b |v(y)|^q \, dy.$$

Remark 2. *$W(I)$ is a right continuous function of b since A_c and B_c are both right continuous as functions of b and if for almost all x in $I, v(x) \neq 0,$ then c is unique and $W(I)$ depends continuously on b.*

Finally we write

$$K(I) = \begin{cases} \sup \left\{ \|F - F_I\|_{q,I,\mu} / \|f\|_{p,I} \text{ in } L^p(I), \; f \neq 0 \right\} & \text{if} \quad \mu(I) \neq 0 \\ 0 & \text{if} \quad \mu(I) = 0. \end{cases} \tag{3.7}$$

We refer this section to the [4, section 4] for full discussion of the significance of these definitions and for the proofs of the previous lemmas.

Lemma 7. *Let $1 \leq q < p \leq +\infty$. Then given any a , b with $a < b$,*

$$K(I) \leq \begin{cases} 2^{\frac{1}{p'} + \frac{1}{q}} q^{\frac{1}{q}} (p')^{\frac{1}{q}} W(I) & \textit{if } q \neq 1 \textit{ and } p < \infty \\ 2^{\frac{3}{2} - \frac{1}{p}} (p')^{\frac{1}{2}} W(I) & \textit{if } q = 2 \textit{ and } p < \infty \end{cases} \qquad \textit{and}$$

$$K(I) > (1 - 2^{-\frac{1}{q}}) (\frac{p-q}{p-1})^{\frac{1}{q'}} q^{\frac{1}{q}} W(I) \qquad \textit{if } 1 < q < p < \infty.$$

Proof. First consider the case $q \neq 2$ and $p < +\infty$. To establish the upper bound for $K(I)$, let c be any point in $]a, b[$. We have

$$\begin{aligned} \|F - F(c)\|^q_{q,[a,c],\mu} &= \int_a^c |F(x) - F(c)|^q \, d\mu(x) \text{ in } [a, c] \\ &= \int_a^c \left| \int_a^x u(t) f(t) dt \right|^q d\mu(x) \\ &= \|Tf\|^q_{q,[a,c],\mu} \, . \end{aligned}$$

By theorem 2.1 and a simple transformation that for every f in $L^p([a,c])$, we have

$$\|F-F(c)\|^{q}_{q,[a,c],\mu} \leq q^{\frac{1}{q}} (p^{'})^{\frac{1}{q^{'}}} A_c \|f\|_{p,[a,c],\mu} . \tag{3.8}$$

Similarly in $[c,b]$ we have

$$\|F-F(c)\|^{q}_{q,[c,b],\mu} = \|Tf\|^{q}_{q,[c,b],\mu}$$

and also by theorem 2.1 shows that for every f in $L^p([c,b])$

$$\|F-F(c)\|_{q,[c,b],\mu} \leq q^{\frac{1}{q}} (p^{'})^{\frac{1}{q^{'}}} B_c \|f\|_{p,[c,b],\mu} . \tag{3.9}$$

Hence from (18) and (19) we have

$$\begin{aligned}
\|F-F(c)\|^{p}_{q,[a,b],\mu} &= \left\{\|F-F(c)\|^{q}_{q,[a,c],\mu} + \|F-F(c)\|^{q}_{q,[c,b],\mu}\right\}^{p/q} \\
&\leqslant q^{\frac{p}{q}} (p^{'})^{\frac{p}{q^{'}}} \left\{A_c^{q} \|f\|^{q}_{p,[a,c],\mu} + B_c^{q} \|f\|^{q}_{p,[c,b],\mu}\right\}^{p/q} \\
&\leqslant q^{\frac{p}{q}} (p^{'})^{\frac{p}{q^{'}}} \left\{ \begin{array}{c} \left(\|f\|^{p}_{p,[a,c],\mu} + \|f\|^{p}_{p,[c,b],\mu}\right) \\ \times \left(A_c^{qp/p-q} + B_c^{qp/p-q}\right)^{(p-q)/p} \end{array} \right\}^{p/q} \\
&\leqslant q^{\frac{p}{q}} (p^{'})^{\frac{p}{q^{'}}} 2^{(p-q)/q} (\max(A_c,B_c))^{p} \|f\|_{p,[a,b],\mu} ,
\end{aligned}$$

therefore

$$\|F-F(c)\|_{q,[a,b],\mu} \leqslant q^{\frac{1}{q}} (p^{'})^{\frac{1}{q^{'}}} 2^{\frac{p-q}{pq}} W(I) \|f\|_{p,[a,b],\mu} ,$$

since

$$\begin{aligned}
\|F-F_I\|_{q,I,\mu} &\leqslant \|F-F(c)\|_{q,I,\mu} + \|(F-F(c))_I\|_{q,I,\mu} \\
&= 2\|F-F(c)\|_{q,I,\mu} \\
&\leqslant q^{\frac{1}{q}} (p^{'})^{\frac{1}{q^{'}}} 2^{\frac{p-q}{pq}+1} W(I) \|f\|_{p,I,\mu} ,
\end{aligned}$$

which shows the first inequality (upper bound for $K(I)$ in the lemma).

However, if $q = 2$, then

$$\begin{aligned} \|F - F_I\|_{2,I,\mu} &= \inf\left\{\|F - F(c)\|_{2,I,\mu} : c \text{ constant}\right\} \\ &\leqslant \|F - F(c)\|_{2,I,\mu}, \\ &\leqslant 2^{\frac{3p-2}{2p}} (p')^{\frac{1}{2}} W(I) \|f\|_{p,I,\mu}. \end{aligned}$$

To establish the lower bounds for $K(I)$. First consider the case $q \neq 1$ and let c be any point in $]a, b[$. Assume first that $x \in [a, c]$ then $\|F_1\|_{q,[a,c],\mu} = \|Tf\|_{q,[a,c],\mu}$ since $F_1(x) = \int_c^x u(t) f_1(t) dt$, it follows by theorem 2.1, that there exists f_1 in $L^p[a, c]$ such that

$$\|F_1\|_{q,[a,c],\mu} \geqslant \left(\frac{p-q}{p-1}\right)^{\frac{1}{q'}} q^{\frac{1}{q}} A_c \|f_1\|_{p,[a,c],\mu}. \tag{3.10}$$

Now define

$$g(x) = \begin{cases} f_1 & \text{in } [a, c] \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad G(x) = \int_0^x u(t) g(t) dt.$$

We want to estimate $\|G - G_I\|_{q,I,\mu} \geqslant \|G\|_{q,I,\mu} - \|G_I\|_{q,I,\mu}$ with $I = [a, b]$. Since $G_{[a,b]} = \dfrac{\mu[a, c]}{\mu[a, b]} F_1[a, c]$, by applying Holder's inequality to the above

$$\left|G_{[a,b]}\right| \leqslant \frac{1}{\mu[a, b]} \|F_1\|_{q,[a,c],\mu} (\mu[a, c])^{\frac{1}{q}},$$

consequently

$$\|G_I\| \leqslant \left(\frac{\mu[a, c]}{\mu[a, b]}\right)^{\frac{1}{q'}} \|F_1\|_{q,[a,c],\mu}.$$

Therefore

$$\begin{aligned} \|G-G_I\|_{q,I,\mu} &\geqslant \|G\|_{q,I,\mu}-\|G_I\|_{q,I,\mu} \\ &\geqslant \left\{1-\left(\frac{\mu\,[a,c]}{\mu\,[a,b]}\right)^{\frac{1}{q'}}\right\}\|F_1\|_{q,[a,c],\mu} \\ &\geqslant \left\{1-\left(\frac{\mu\,[a,c]}{\mu\,[a,b]}\right)^{\frac{1}{q'}}\right\}\left(\frac{p-q}{p-1}\right)^{\frac{1}{q'}} q^{\frac{1}{q}} A_c\,\|f_1\|_{p,[a,c],\mu} \\ &\geqslant \left\{1-\left(\frac{\mu\,[a,c]}{\mu\,[a,b]}\right)^{\frac{1}{q'}}\right\}\left(\frac{p-q}{p-1}\right)^{\frac{1}{q'}} q^{\frac{1}{q}} A_c\,\|g\|_{p,I}\,. \end{aligned} \tag{3.11}$$

Similarly for $x\in[c,b]$

$$\|F_2\|_{q,[c,b],\mu}=\|Tf\|_{q,[c,b],\mu}\geqslant\left(\frac{p-q}{p-1}\right)^{\frac{1}{q'}} q^{\frac{1}{q}} B_c\,\|f_2\|_{p,[c,b],\mu} \tag{3.12}$$

and also by theorem 1 there exists an $f_2\in L^P[c,b]$ corresponding to c such that (23) holds. We define also

$$h(x)=\begin{cases} 0 & \text{otherwise} \\ f_2 & \text{in } [c,b] \end{cases}$$

and

$$H(x)=\int_0^x u(t)h(t)dt,$$

then

$$H(x)=\begin{cases} 0 & \text{otherwise} \\ F_2 & \text{in } [c,b] \end{cases},$$

therefore

$$\begin{aligned}
\|H - H_I\|_{q,[c,b],\mu} &\geqslant \|H\|_{q,[c,b],\mu} - \|H_I\|_{q,[c,b],\mu} \\
&\geqslant \left\{1 - \left(\frac{\mu\,[c,b]}{\mu\,[a,b]}\right)^{\frac{1}{q'}}\right\} \|F_2\|_{q,[c,b],\mu} \qquad (3.13)\\
&\geqslant \left(1 - \left(\frac{\mu\,[c,b]}{\mu\,[a,b]}\right)^{\frac{1}{q'}}\right) \left(\frac{p-q}{p-1}\right)^{\frac{1}{q'}} q^{\frac{1}{q}} B_c \|f_1\|_{p,[c,b],\mu} \\
&\geqslant \left(1 - \left(\frac{\mu\,[c,b]}{\mu\,[a,b]}\right)^{\frac{1}{q'}}\right) \left(\frac{p-q}{p-1}\right)^{\frac{1}{q'}} q^{\frac{1}{q}} B_c \|h\|_{p,I,\mu}\,.
\end{aligned}$$

If we take

$$\Phi(x) = \begin{cases} G(x) & \text{if } A_c < B_c \\ H(x) & \text{if } A_c \geqslant B_c \end{cases} \quad \text{and} \quad \theta(x) = \begin{cases} g(x) & \text{if } A_c < B_c \\ h(x) & \text{if } A_c \geqslant B_c \end{cases}$$

We get

$$\|\Phi - \Phi_I\|_{q,I,\mu} \geqslant \left\{1 - \left(\frac{\mu\,[a,c]}{\mu\,[a,b]}\right)^{\frac{1}{q'}}\right\} \left(\frac{p-q}{p-1}\right)^{\frac{1}{q'}} q^{\frac{1}{q}} W(I)\, \|\theta\|_{p,I,\mu}\,.$$

If c is the minimum point in $]a,b[$ for which $\mu\,[a,c] = \mu\,[c,b] = \frac{1}{2}\mu\,[a,b]$ then

$$\|\Phi - \Phi_I\|_{q,I,\mu} \geqslant \left(1 - 2^{-\frac{1}{q'}}\right) \left(\frac{p-q}{p-1}\right)^{\frac{1}{q'}} q^{\frac{1}{q}} W(I)\, \|\theta\|_{p,I,\mu}$$

which the last inequality for $K(I)$. The proof of the lemma for $1 < q < p \leq +\infty$ is complete . ∎

Remark 3. *Inspection of the proof shows that the lemma holds with $W(I)$ replaced by* $\inf\{\max(A_\alpha, B_\alpha) : \alpha \in\,]a,b[\}$.

Lemma 8. [**4**, *lemma* 6] *Let $1 \leq q < p \leq +\infty$ and given any a, b with $a < b$ then*

$$K(I) = \frac{1}{\sqrt{2}} L(I) \qquad \textit{if } q = 2$$

and

$$\frac{1}{2}L(I) \leqslant K(I) \leqslant L(I) \qquad \text{if } q \neq 2$$

where $I = [a, b]$.

Now we are ready to state the result concerning the approximation numbers of T.

Lemma 9. *Let* $1 \leq q < p \leq \infty$. *Let* $\varepsilon > 0$ *and suppose that there exists* $N \in \mathbb{N}$($\mathbb{N}$ *denotes the set of all natural numbers) and numbers* $c_k(k = 0, 1, ..., N + 1)$ *with* $0 = c_0 < c_1 < \cdots < c_{N+1} = \infty$ *such that* $L(I_k) \leq \varepsilon$ *for* $k = 0, 1, ..., N$ *where* $I_k =]c_k, c_{k+1}[$. *Then with* $\sigma_q = 1$ *if* $q \neq 2$ *and* $\sigma_q = \frac{1}{\sqrt{2}}$ *if* $q = 2$, *we have*

$$a_{N+2} \leqslant \sigma_q \varepsilon (N + 1) \frac{p - q}{pq}.$$

Proof. Let $f \in L^p(\mathbb{R}^+)$ be such that $\|f\|_{p,R^+} = 1$ and we write $Pf = \sum_{k=1}^{N} P_{I_k} f$ where $P_I = \chi_I \; v(x) F_I$. Then P is a bounded linear map from $L^P(\mathbb{R}^+)$ to $L^q(\mathbb{R}^+)$ with rank $P \leqslant N + 1$. Also we have

$$\begin{aligned}
\|Tf - Pf\|^q_{q,\mathbb{R}^+} &= \sum_{k=1}^{N} \|Tf - P_{I_k} f\|^q_{q,I_K} \\
&= \sum_{k=1}^{N} \|F - F_{I_k}\|^q_{q,I_k,\mu}, \\
&\leqslant \sum_{k=1}^{N} \sup \left\{ \frac{\|F - F_{I_k}\|^q_{q,I_k,\mu}}{\|f\|_{p,I_k}} \right\}^q \|f\|^q_{p,I_k} \\
&\leqslant \sum_{k=1}^{N} K^q(I_k) \|f\|^q_{p,I_k}.
\end{aligned}$$

By the hypothesis of the lemma and the lemma 3.3 we have

$$K(I_k) \leqslant L(I_k)\sigma_q \leqslant \varepsilon \sigma_q$$

then

$$\begin{aligned}\|Tf-Pf\|_{q,R^+}^q &< \sum_{k=1}^{N}\varepsilon^q\sigma_q^q\,\|f\|_{p,I_k}^q\\ &= \varepsilon^q\sigma^q\sum_{k=1}^{N}\|f\|_{q,I_k}^q\,.\end{aligned}$$

Applying Holder's inequality to the $\sum_{k=1}^{N}\|f\|_{q,I_k}^q$, we have

$$\begin{aligned}\sum_{k=1}^{N}\|f\|_{q,I_k}^q &\leqslant \left(\sum_{k=1}^{N}1\right)^{1-\frac{q}{p}}\left(\sum_{k=1}^{N}\|f\|_{q,I_k}^q\right)^{\frac{q}{p}}\\ &< (N+1)^{1-\frac{q}{p}}\left(\sum_{k=1}^{N}\|f\|_{q,I_k}^q\right)^{\frac{q}{p}}\end{aligned}$$

therefore

$$\|Tf-Pf\|_{q,R^+}^q < \varepsilon^q\sigma^q(N+1)^{1-\frac{q}{p}}\left(\sum_{k=1}^{N}\|f\|_{p,I_k}^p\right)^{\frac{1}{p}}.$$

Hence $\|Tf-Pf\|_{q,R^+} \leqslant \varepsilon\sigma_q(N+1)^{1-\frac{q}{p}}\,\|f\|_{p,I_k}$ and since

$$\|f\|_{p,I_k} \leqslant \|f\|_{p,R^+} = 1,$$

then

$$\|Tf-Pf\|_{q,R^+} \leqslant \varepsilon\sigma_q(N+1)^{1-\frac{q}{p}}$$

which shows that

$$a_{N+2} \leq \sigma_q\varepsilon(N+1)\frac{p-q}{pq}.$$

∎

Lemma 10. *Let* $1 \leq q < p \leq \infty$*. Let* $\varepsilon > 0$ *and suppose that there exists* $N \in \mathbb{N}$ ($\mathbb{N}$ *denotes the set of all natural numbers) and numbers* $c_k (k = 0, 1, ..., N+1)$ *with* $0 = c_0 < c_1 < \cdots < c_N < \infty$*, such that* $L(I_k) \geq \varepsilon$ *for* $k = 0, 1, ..., N-1$ *where* $I_k =]c_k, c_{k+1}[$. *Then*

$a_N(T) \geq \nu_q\varepsilon$ *where* $\nu_2 = \frac{1}{\sqrt{2}}$ *and* $\nu_q = \frac{1}{4}$ *if* $q \neq 2$.

Proof. Let $\lambda \in \left]0,1\right[$. By lemma 3.3 and the hypothesis that $L(I_k) \geqslant \varepsilon$ for $k = 0, 1, ..., N-1$, there exists $\theta_k \in L^p(I_k)$ such that

$$\|\Phi_k - (\Phi_k)_{I_k}\|_{q,I,\mu} \,/\, \|\theta_k\|_{p,I_k} > \lambda \eta_q \varepsilon,$$

where $\Phi_k(x) = \int\limits_{I_k} u(t)\theta_k(t)dt$, and $\eta_q = 1/2$ if $q \neq 2$ and $\eta_2 = 1/\sqrt{2}$; set $\theta_k(x) = 0$ for all $x \in R^+ - I_k$. Let $P \in \mathcal{F}(L^p(\mathbb{R}^+), L^q(\mathbb{R}^+))$ with rank $N-1$, then there are constants $\lambda_1, \lambda_2, ..., \lambda_{N-1}$ not all zero, such that $P(\sum\limits_{k=0}^{N-1} \lambda_k\theta_k) = 0$ with $\left\|\sum\limits_{k=0}^{N-1} \lambda_k\theta_k\right\|_{p,\mathbb{R}^+} = 1$. Put $\theta = \sum\limits_{k=1}^{n} \lambda_k\theta_k$ and $\Phi(x) = \int\limits_0^x u(t)\theta(t)dt$ $(x \in \mathbb{R}^+)$. For all $x \in I_k, \Phi(x) = \lambda_k\theta_k + \mu_k$ for some constant μ_k, for all constant C, we have

$$\begin{aligned} \|F - F_I\|_{q,I,\mu} &\leqslant \|F - C\|_{q,I,\mu} + \|(C-F)_I\|_{q,I,\mu} \\ &\leqslant 2\,\|F - C\|_{q,I,\mu} \qquad \text{when } q \neq 2, \end{aligned}$$

while for $q = 2$ we have

$$\|F - F_I\|_{2,I,\mu} = \inf\left\{\|F - C\|_{2,\ I,\ \mu} : C \text{ constant }\right\},$$

where the infinimum is taken over all constant C. Hence

$$\|F - F_I\|_{q,I,\mu} \leqslant \delta_q \inf\left\{\|F - C\|_{q,\ I,\ \mu}\right\}$$

where $\delta_q = 2$ if $q \neq 2$ and $\delta_q = 1$ if $q = 2$. Thus

$$\|T\theta - P\theta\|_{q,R^+}^q = \|T\theta\|_{q,\mathbb{R}^+}^q \geqslant \sum_{k=0}^{N-1} \|\lambda_k\Phi_k + \mu_k\|_{q,I_k,\mu}^q$$

$$= \sum_{k=0}^{N-1} \|\Phi\|_{q,I_k,\mu}^q \geqslant \delta_q^{-q} \sum_{k=0}^{N-1} \|\lambda_k\Phi_k - (\lambda_k\Phi_k)_{I_k}\|_{q,I_k,\mu}^q$$

$$= \delta_q^{-q} \sum_{k=0}^{N-1} \|\lambda_k\Phi_k - (\lambda_k\Phi_k)_{I_k}\|_{q,I_k,\mu}^q$$

$$= \delta_q^{-q} \sum_{k=0}^{N-1} |\lambda_k|^q \,\|\Phi_k - (\Phi_k)_{I_k}\|_{q,I_k,\mu}^q$$

$$> \delta_q^{-q} (\lambda\eta_q\varepsilon)^q \sum_{k=0}^{N-1} |\lambda_k|^q \,\|\theta_k\|_{q,I_k,\mu}^q$$

$$= \delta_q^{-q}(\lambda\eta_q\varepsilon)^q \sum_{k=0}^{N-1} \|\lambda_k\theta_k\|_{p,I_k,\mu}^p .$$

Since $q < p$ and $\sum\limits_{k=0}^{N-1} \|\lambda_k\theta_k\|_{p,I_k,\mu}^p = 1$, then

$$\|\lambda_k\theta_k\|_{p,I_k,\mu}^q > \|\lambda_k\theta_k\|_{p,I_k,\mu}^p .$$

It follows that

$$\|T\theta - P\theta\|_{q,\mathbb{R}^+}^q \geqslant \delta_q^{-q}(\lambda\eta_q\varepsilon)^q \|\theta\|_{p,\mathbb{R}^+}^p \text{, and } \|\theta\|_{p,\mathbb{R}^+}^p = 1.$$

Therefore

$$\|T\theta - P\theta\|_{q,R^+} \geqslant \delta_q^{-1}(\lambda\eta_q\varepsilon)$$

which shows that

$$a_N(T) \geqslant (\lambda\eta_q\varepsilon)\delta_q^{-1}.$$

Since λ may be chosen arbitrarily close to 1 it follows that $a_N(T) \geqslant (\eta_q\varepsilon)\delta_q^{-1}$. ∎

Provided with these lemmas (lemma 3.5 and lemma 3.6) we may produce our main result concerning the approximation numbers of T. Given any $\varepsilon > 0$, define numbers c_k by the rule that

$$c_0 = 0,\ c_{k+1} = \inf\{t; L(c_k, t) > \varepsilon\} \tag{3.14}$$

with understanding the $\inf\emptyset = \infty$. We shall refer to these numbers as forming the (ε, L)-sequence for a given ε there are two possibilities :(i) the (ε, L)-sequence is finite. Then there is an integer N such that $c_0 < c_1 < ... < c_N < c_{N+1} = \infty$, and by the continuity of L we have

$$\{L(c_k, c_{k+1}) = \varepsilon \quad \text{for } k = 0, 1, ..., N-1 \text{ and } L(c_N, c_{N+1}) \leqslant \varepsilon.$$

By the length of the (ε, L)-sequence we shall mean the integer $N + 1$.

(ii) The (ε, L)-sequence is infinite. Then $L(c_k, c_{k+1}) = \varepsilon$ for all $k \in \mathbb{N}$.

If (i) holds, then by lemma 3.5 and lemma 3.6 we see that

$$a_{N+2} \leqslant \sigma_q\varepsilon(N+1)\frac{p-q}{pq} \text{ and } a_{N+2} \geqslant \varepsilon\nu_q.$$

If (ii) holds,then by lemma 3.6 shows that for all $n \in \mathbb{N}$;

$$a_N(T) \geqslant \varepsilon \nu_q;$$

note also that in this case, $c_k \mapsto \infty$ as $k \mapsto \infty$; for if not

$$c_k \to c < \infty, |c_k - c_{k+1}| \to 0$$

and thus

$$L(c_k, c_{k+1}) = \varepsilon$$

for all k.

Theorem 11. *Let $1 \leq q < p \leq \infty, \nu_q = \frac{1}{4}$, $\sigma_q = 1 (q \neq 2)$ and $\nu_2 = \nu_q = \frac{1}{\sqrt{2}}$. Then* 1) *$T$ is bounded if, and only, if $L(0, \infty) < \infty$.*

2) *Let $\varepsilon \in (\sigma, L(0, \infty))$ (where $\sigma = \lim_{x \mapsto \infty} L(x, \infty)$) and let $N+1$ be the length of the (ε, L)-sequence. Then we have*

$$a_{N+2} \leq \sigma_q \varepsilon (N+1) \frac{p-q}{pq}$$

and

$$a_N(T) \geq \nu_q \varepsilon.$$

Proof. (1) First suppose $L(0, \infty) < \infty$ and take $\varepsilon = L(0, \infty)$. In view of the monotonicity of L we see that the length of the $(\varepsilon, L)-$sequence is 1 that is $N = 0$.This is by lemma 3.5, $a_2(T) \leqslant \sigma_q \varepsilon$, that is

$$\inf \{\|T - P\|\} \leqslant \varepsilon \sigma_q$$

where the infinimum is taken over all bounded maps P of rank 1 from $L^p(\mathbb{R}^+)$ to $L^q(\mathbb{R}^+)$. Since each such map is bounded, it follows that is bounded. Conversely, suppose that is bounded. Then for any interval I,

$$K(I) \leqslant 2L(I).$$

Hence by the lemma 3.5 is bounded, uniformly in $|I|$. Thus $L(0, \infty)$ is bounded.

(2) Let $\varepsilon \in (\sigma, L(\mathbb{R}^+))$. First, suppose $\varepsilon > \sigma$ and suppose if possible that the (ε, L)-sequence is infinite, then $c_k \to \infty$ as $k \to \infty$.

We have

$$\varepsilon = L(c_k, c_{k+1}) \leqslant L(c_k, \infty)$$

and hence $\varepsilon \leqslant \sigma$ which contradicts $\varepsilon > \sigma$. Therefore the sequence is finite and by lemma 3.5,

$$a_{N+2} \leq (N+1)\frac{p-q}{pq}\varepsilon\sigma_q.$$

Second, suppose if possible that the sequence is finite of length $N+1$.

Then

$$\sigma = \lim_{x\to\infty} L(x, \infty) \leqslant L(c_N, \infty) = L(c_N, c_{N+1}) \leqslant \varepsilon$$

and we have a contradiction. Therefore the sequence is infinite. Then by lemma 3.6, it follows that

$$a_N(T) \geq \nu_q \varepsilon.$$

Therefore

$$\alpha(T) = \lim_{N\to\infty} a_N(T) \geqslant \varepsilon\nu_q.$$

Since this holds for arbitrary $\varepsilon < \sigma$, then

$$\alpha(T) \geqslant L\nu_q.$$

This completes the proof. ∎

4. **An example**

To illustrate the scope of the theorem 3.7 we deal with the situation in which

$$u(x) = e^{Ax}, v(x) = e^{-Bx} \tag{4.1}$$

for all $x \in \mathbb{R}^+$ where $0 < A < B$. From theorem 2.1 and theorem 2.2

(Boundedness and compactness of T), we see that:

$$J_a = (Ap')^{\frac{-1}{p'}} (Bq)^{\frac{-1}{q}} e^{-(B-A)a} \boldsymbol{\beta}\left[\frac{(p-1)q}{p-q}, \frac{(p-1)q}{p-q}(\frac{B-A}{A})\right]^{\frac{p-q}{pq}} \tag{4.2}$$

(where $\boldsymbol{\beta}$ denotes the beta function) which shows that

$$J_0 < \infty \text{ and } \lim_{a\mapsto\infty} J_a = 0.$$

Thus T is a compact linear map from $L^p(\mathbb{R}^+)$ to $L^q(\mathbb{R}^+)$ and

$$(Ap^{'})^{\frac{-1}{p^{'}}}(B(\frac{p-1}{p-q}))^{\frac{-1}{q}}\boldsymbol{\beta}\left[\frac{(p-1)q}{p-q},\frac{(p-1)q}{p-q}(\frac{B-A}{A})\right]^{\frac{p-q}{pq}} \leqslant \|T\| \leqslant$$

$$(Ap^{'})^{\frac{-1}{p^{'}}}(Bq)^{\frac{-1}{q}}\boldsymbol{\beta}\left[\frac{(p-1)q}{p-q},\frac{(p-1)q}{p-q}(\frac{B-A}{A})\right]^{\frac{p-q}{pq}} \tag{4.3}$$

Now, we establish lower and upper bounds for $a_n(T)$. For the lower bound, we obtain it by considering the expression for $L(I)$ directly rather than the expressions of $K(I)$ or $W(I)$ which are difficult to compute in practice . First, we state the following lemma which is useful in developing the previous task.

Lemma 12. *If* $0 \leqslant x < y$ *and* $\alpha \in \,]0,1[$ *. Then*

$$\alpha \leqslant \frac{x^\alpha - y^\alpha}{(x-y)x^{\alpha-1}} \leqslant 1.$$

Proof. We have

$$\frac{d}{dt}(\frac{1-t^\alpha}{1-t}) = \frac{-\alpha t^{\alpha-1}+(\alpha-1)t^\alpha+1}{(1-t)^2} = \frac{\phi(t)}{(1-t)^2},$$

so

$$1 \geqslant \frac{1-t^\alpha}{1-t} \geqslant \lim_{t\to 1-0}\frac{1-t^\alpha}{1-t} = \alpha.$$

The results follow on putting $t = \frac{y}{x}$. ∎

Now we have

$$\begin{aligned} l(I,f) &\geqslant \left(\int_{c_2}^{b} e^{-qBy}dy\right)\left(\int_{c_1}^{c_2} e^{-qBx}dx\right)\left(\left|\int_x^y e^{At}f(t)dt\right|^q\right) \\ &\geqslant \left(\int_{c_1}^{b} e^{-qBy}dy\right)\left(\int_{a}^{c_1} e^{-qBx}dx\right)\left(\int_{c_1}^{c_2} e^{Ap't}dt\right)^{\frac{q}{p'}} \end{aligned}$$

by a suitable choice of f with $\|f\|_{L^P(I)} = 1$. Take c_1, c_2 to be such

$$\left(\int_a^{c_1} e^{-qBy}dy\right) = \left(\int_{c_2}^{b} e^{-qBy}dy\right) = \frac{1}{3}\left(\int_a^{b} e^{-qBy}dy\right).$$

By holder's inequality if $\frac{1}{r} = \frac{1}{p'} + \frac{1}{q}$, we have

$$\left(\int_{c_1}^{c_2} e^{-r(B-A)t} dt\right)^{\frac{1}{r}} \leqslant \left(\int_{c_1}^{c_2} e^{-qBy} dy\right)^{\frac{1}{q}} \left(\int_{a}^{c_1} e^{-Ap't} dt\right)^{\frac{q}{p'}}.$$

Hence

$$L(I) = \left(\frac{\sup\limits_{\|f\|_{p,I}=1} (l(I,f)}{\int_I e^{-qBy} dy}\right)^{\frac{1}{q}} \geqslant \left(\int_{c_1}^{c_2} e^{-r(B-A)t} dt\right)^{\frac{1}{r}}.$$

Let now

$$v_1 = e^{-qBa} \text{ and } v_2 = e^{-qBb},$$

then

$$\begin{aligned} e^{-qBc_1} &= \frac{2}{3} v_1 + \frac{1}{3} v_2, \\ e^{-qBc_2} &= \frac{2}{3} v_2 + \frac{1}{3} v_1. \end{aligned}$$

We have

$$\begin{aligned} X &= \left(\int_{c_1}^{c_2} e^{-r(B-A)t} dt\right)^{\frac{1}{r}} = (r(B-A))^{-\frac{1}{r}} (e^{-r(B-A)c_1} - e^{-r(B-A)c_2}), \\ &= \left(r(B-A))^{-\frac{1}{r}} (\frac{2}{3} v_1 + \frac{1}{3} v_2)^{\frac{1}{q}\left(\frac{B-A}{B}\right)} - (\frac{2}{3} v_2 + \frac{1}{3} v_1)^{\frac{1}{q}\left(\frac{B-A}{B}\right)} \right). \end{aligned}$$

Then by the above lemma we have

$$X \geqslant \frac{\frac{1}{q}\left(\frac{B-A}{B}\right)}{(r(B-A))^{\frac{1}{r}}} 3^{-r} (v_1 - v_2)^{\frac{1}{r}} \left(\frac{2}{3} v_1 + \frac{1}{3} v_2\right)^{\frac{1}{q}\left(\frac{B-A}{B}\right) - \frac{1}{r}}. \tag{4.4}$$

Again

$$Y = \left(\int_{a}^{b} e^{-r(B-A)t} dt\right)^{\frac{1}{r}} = (r(B-A))^{-\frac{1}{r}} \left(e^{-r(B-A)a} - e^{-r(B-A)b}\right)^{\frac{1}{r}}.$$

Hence

$$Y \leqslant (r(B-A))^{-\frac{1}{r}} (v_1 - v_2)^{\frac{1}{r}} (v_1)^{\frac{1}{q}\left(\frac{B-A}{B}\right)-\frac{1}{r}}. \tag{4.5}$$

From (29) and (30) we see that

$$X \geqslant \left(\frac{2}{3}\right)^{\frac{1}{q}\left(\frac{B-A}{B}\right)-\frac{1}{r}} (r(B-A))^{-\frac{1}{r}} Y. \tag{4.6}$$

Then from (31) we have

$$L(I) \geqslant c\left(\int_a^b e^{-r(B-A)t}dt\right)^{\frac{1}{r}} \tag{4.7}$$

where c is a constant. Let $c_0, c_1, ..., c_N, c_{N+1} = \infty$ be the (ε, L)-sequence defined in (25). Since $L(I_k) = \varepsilon$ for $k = 0, 1, ..., N-1,$ it follows from (31) that

$$N\varepsilon^r = \sum_{k=0}^{N-1} L^r(I_k) \geqslant c \sum_{k=0}^{N-1} \int_{c_k}^{c_{k+1}} e^{-r(B-A)x}dx.$$

We summarize our results as follows

Theorem 13. *Let u and v be defined in* (26) *and let* $1 \leq q < p \leq \infty$. *Then as* $n \longrightarrow \infty$, $a_n(T) \geq cn^{-\frac{1}{r}}$ *(c is constant) where* $\frac{1}{r} = \frac{1}{p'} + \frac{1}{q}$.

To derive an upper bound for $a_n(T)$ we use the result in [4] that for the case $p = q$, $a_n(T) = 0(\frac{1}{n})$. Let $T_1 f(x) = e^{-B_1 x} \int e^{At} f(t)dt$ where $B > B_1 > A$ so that

$$Tf(x) = e^{-(B-B_1)x} T_1 f(x).$$

Now the map $T_1 : L^p(\mathbb{R}^+) \to L^q(\mathbb{R}^+)$ has approximation numbers

$$a_n(T_1) = O(\frac{1}{n}).$$

The map $U : L^p(\mathbb{R}^+) \to L^q(\mathbb{R}^+)$ defined by

$$Uf(x) = e^{-(B-B_1)x} f(x)$$

is bounded (by holder's inequality). It follows from the fact that

$$a_n(T) \leqslant a_n(T_1)\, \|U\| = O(\frac{1}{n}).$$

We summarize our results as follows.

Theorem 14. *Let u and v be defined in* (26) *and let* $1 \leq q < p \leq \infty$. *Then as* $n \longrightarrow \infty$, $a_n(T) = O\left(\frac{1}{n}\right)$.

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OPTIMAL MULTIVARIATE OSTROWSKI EULER TYPE INEQUALITIES

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Abstract. In [8] we derived general tight multivariate high order Ostrowski type inequalities for the estimate of the error of a multivariate function f evaluated at a point from its average. The estimates involve only the single partial derivatives of f and are with respect to $\|\cdot\|_p$, $1 \le p \le \infty$. We give here specific applications of these results to the multivariate trapezoid and midpoint rules for functions f differentiable up to order 6. We prove sharpness of these inequalities for differentiation orders $m = 1, 2, 4$ and with respect to $\|\cdot\|_\infty$.

1. **Introduction**

We mention as inspiration to our work the great Ostrowski inequality, see [12], [3], [5].

Theorem 1. *It holds*

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)dt \right| \le \left[\frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2} \right](b-a)\|f'\|_\infty, \tag{1}$$

where $f \in C^1([a,b])$, $x \in [a,b]$, which is a sharp inequality.

Here $B_k(x)$, $k \ge 0$, are the Bernoulli polynomials, $B_k = B_k(0)$, $k \ge 0$, the Bernoulli numbers, and $B_k^*(x)$, $k \ge 0$, are periodic functions of period one, related to the Bernoulli polynomials as

$$B_k^*(x) = B_k(x), \quad 0 \le x < 1, \tag{2}$$

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and

$$B_k^*(x+1) = B_k^*(x), \quad x \in \mathbb{R}. \tag{3}$$

Some basic properties of Bernoulli polynomials follow (see [1, 23.1]). We have

$$B_0(x) = 1, \quad B_1(x) = x - \frac{1}{2}, \quad B_2(x) = x^2 - x + \frac{1}{6},$$

and

$$B_k'(x) = kB_{k-1}(x), \quad k \in \mathbb{N}, \tag{4}$$

$$B_k(x+1) - B_k(x) = kx^{k-1}, \quad k \geq 0. \tag{5}$$

Clearly $B_0^* = 1$, B_1^* *is a discontinuous function with a jump of* -1 *at each integer, and* B_k^*, $k \geq 2$, *is a continuous function. Notice that* $B_k(0) = B_k(1) = B_k$, $k \geq 2$.

We make

Assumption 1. *Let* f *and the existing* $\frac{\partial^\ell f}{\partial x_j^\ell}$, *all* $\ell = 1, \ldots, m$; $j = 1, \ldots, n$, *be continuous real valued functions on* $\prod_{i=1}^n [a_i, b_i]$; $m, n \in \mathbb{N}$, $a_i, b_i \in \mathbb{R}$.

A general set of suppositions follow

Assumption 2. *Here* $m \in \mathbb{N}$, $j = 1, \ldots, n$. *We suppose*

1) $f\colon \prod_{i=1}^n [a_i, b_i] \to \mathbb{R}$ *is continuous.*
2) $\frac{\partial^\ell f}{\partial x_j^\ell}$ *are existing real valued functions for all* $j = 1, \ldots, n$; $\ell = 1, \ldots, m-2$.
3) *For each* $j = 1, \ldots, n$ *we assume that*

$$\frac{\partial^{m-1} f}{\partial x_j^{m-1}}(x_1, \ldots, x_{j-1}, \cdot, x_{j+1}, \ldots, x_n)$$

is a continuous real valued function.

4) *For each* $j = 1, \ldots, n$ *we assume that*

$$g_j(\cdot) := \frac{\partial^m f}{\partial x_j^m}(x_1, \ldots, x_{j-1}, \cdot, x_{j+1}, \ldots, x_n)$$

exists and is real valued with the possibility of being infinite only over an at most countable subset of (a_j, b_j).

5) *Parts #3, #4 are true for all*

$$(x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n) \in \prod_{\substack{i=1 \\ i \neq j}}^n [a_i, b_i].$$

6) *The functions for* $j = 2, \dots, n$; $\ell = 1, \dots, m-2$,

$$q_j \left(\overbrace{\cdot, \cdot, \cdots, \cdot}^{j-1} \right) := \frac{\partial^\ell f}{\partial x_j^\ell} \left(\overbrace{\cdot, \cdot, \cdot, \cdots, \cdot}^{j-1}, x_j, x_{j+1}, \dots, x_n \right)$$

are continuous on $\prod_{i=1}^{j-1} [a_i, b_i]$, *for each*

$$(x_j, x_{j+1}, \dots, x_n) \in \prod_{i=j}^{n} [a_i, b_i].$$

7) *The functions for each* $j = 1, \dots, n$,

$$\varphi_j \left(\overbrace{\cdot, \cdot, \cdot, \cdots, \cdot, \cdot, \cdot}^{j} \right) := \frac{\partial^m f}{\partial x_j^m} \left(\overbrace{\cdot, \cdot, \cdot, \cdots, \cdot}^{j}, x_{j+1}, \dots, x_n \right) \in L_1 \left(\prod_{i=1}^{j} [a_i, b_i] \right),$$

for any $(x_{j+1}, \dots, x_n) \in \prod_{i=j+1}^{n} [a_i, b_i]$.

Some weaker general suppositions follow.

Assumption 3. *Here* $m \in \mathbb{N}$, $j = 1, \dots, n$, *and only the Parts #1, #2, #6, #7 of Assumption 2 remain the same. We further assume that for each* $j = 1, \dots, n$ *and over* $[a_j, b_j]$, *the function*

$$\frac{\partial^{m-1}}{\partial x_j^{m-1}} f(x_1, \dots, x_{j-1}, \cdot, x_{j+1}, \dots, x_n)$$

is absolutely continuous, and this is true for all

$$(x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_n) \in \prod_{\substack{i=1 \\ i \neq j}}^{n} [a_i, b_i].$$

In [8] we proved the following multivariate Euler type identity.

Theorem 2. *All as in Assumption 1 or 2 or 3 for* $m, n \in \mathbb{N}$, $x_i \in [a_i, b_i]$, $i = 1, 2, \dots, n$. *Then*

$$E_m^f(x_1, x_2, \dots, x_n) := f(x_1, x_2, \dots, x_n) \tag{6}$$

$$- \frac{1}{\prod_{i=1}^{n} (b_i - a_i)} \int_{\prod_{i=1}^{n} [a_i, b_i]} f(s_1, \dots, s_n) ds_1 \cdots ds_n - \sum_{j=1}^{n} A_j = \sum_{j=1}^{n} B_j,$$

where for $j = 1, \dots, n$ we have

$$A_j := A_j(x_j, x_{j+1}, \dots, x_n) \tag{7}$$

$$= \frac{1}{\left(\prod_{i=1}^{j-1}(b_i - a_i)\right)} \left\{ \sum_{k=1}^{m-1} \frac{(b_j - a_j)^{k-1}}{k!} B_k\left(\frac{x_j - a_j}{b_j - a_j}\right) \right.$$

$$\times \left(\int_{\prod_{i=1}^{j-1}[a_i,b_i]} \left(\frac{\partial^{k-1} f}{\partial x_j^{k-1}}(s_1, s_2, \dots, s_{j-1}, b_j, x_{j+1}, \dots, x_n) \right.\right.$$

$$\left.\left.\left. - \frac{\partial^{k-1} f}{\partial x_j^{k-1}}(s_1, s_2, \dots, s_{j-1}, a_j, x_{j+1}, \dots, x_n) \right) ds_1 \cdots ds_{j-1} \right) \right\},$$

and

$$B_j := B_j(x_j, x_{j+1}, \dots, x_n) \tag{8}$$

$$:= \frac{(b_j - a_j)^{m-1}}{m!\left(\prod_{i=1}^{j-1}(b_i - a_i)\right)} \left\{ \int_{\prod_{i=1}^{j}[a_i,b_i]} \left(\left(B_m\left(\frac{x_j - a_j}{b_j - a_j}\right) \right.\right.\right.$$

$$\left.\left.\left. - B_m^*\left(\frac{x_j - s_j}{b_j - a_j}\right)\right) \frac{\partial^m f}{\partial x_j^m}(s_1, s_2, \dots, s_j, x_{j+1}, \dots, x_n) \right) ds_1 ds_2 \cdots ds_j \right\}.$$

When $m = 1$ then $A_j = 0$, $j = 1, \dots, n$.

Also in [8] we proved the following tight multivariate Ostrowski type inequalities.

Theorem 3. *Suppose Assumptions 1 or 2 or 3. Let $E_m^f(x_1, x_2, \dots, x_n)$ as in (6) and A_j for $j = 1, \dots, n$ as in (7), $m \in \mathbb{N}$. In particular we assume that*

$$\frac{\partial^m f}{\partial x_j^m}(\overbrace{\cdots}^{j}, x_{j+1}, \dots, x_n) \in L_\infty\left(\prod_{i=1}^{j}[a_i, b_i]\right),$$

for any $(x_{j+1}, \dots, x_n) \in \prod_{i=j+1}^{n}[a_i, b_i]$, all $j = 1, \dots, n$. Then

$$|E_m^f(x_1, \dots, x_n)| \tag{9}$$

$$= \left| f(x_1, \dots, x_n) - \frac{1}{\prod_{i=1}^{n}(b_i - a_i)} \int_{\prod_{i=1}^{n}[a_i,b_i]} f(s_1, \dots, s_n) ds_1 \cdots ds_n - \sum_{j=1}^{n} A_j \right|$$

$$\leq \frac{1}{m!} \sum_{j=1}^{n} \left[(b_j - a_j)^m \left(\sqrt{\frac{(m!)^2}{(2m)!}|B_{2m}| + B_m^2\left(\frac{x_j - a_j}{b_j - a_j}\right)} \right) \right.$$

$$\times \left\| \frac{\partial^m f}{\partial x_j^m} (\overbrace{\cdots}^{j}, x_{j+1}, \ldots, x_n) \right\|_{\infty, \prod_{i=1}^{j} [a_i, b_i]} \Bigg].$$

Theorem 4. *Suppose Assumptions 1 or 2 or 3. Let $E_m^f(x_1, \ldots, x_n)$ as in (6), $m \in \mathbb{N}$. Let $p_j, q_j > 1$: $\frac{1}{p_j} + \frac{1}{q_j} = 1$; $j = 1, \ldots, n$. In particular we assume that*

$$\frac{\partial^m f}{\partial x_j^m} (\ldots, x_{j+1}, \ldots, x_n) \in L_{q_j} \left(\prod_{i=1}^{j} [a_i, b_i] \right),$$

for any $(x_{j+1}, \ldots, x_n) \in \prod_{i=j+1}^{n} [a_i, b_i]$, for all $j = 1, \ldots, n$. Then

$$|E_m^f(x_1, \ldots, x_n)| \tag{10}$$

$$\leq \frac{1}{m!} \sum_{j=1}^{n} \Bigg[(b_j - a_j)^{m - \frac{1}{q_j}} \left(\prod_{i=1}^{j-1} (b_i - a_i) \right)^{-\frac{1}{q_j}} \left(\int_0^1 \left| B_m \left(\frac{x_j - a_j}{b_j - a_j} \right) \right.\right.$$

$$\left.\left. -B_m(t_j) \right|^{p_j} dt_j \right)^{1/p_j} \left\| \frac{\partial^m f}{\partial x_j^m} (\ldots, x_{j+1}, \ldots, x_n) \right\|_{q_j, \prod_{i=1}^{j} [a_i, b_i]} \Bigg].$$

When $p_j = q_j = 2$, all $j = 1, \ldots, n$, then we get

$$|E_m^f(x_1, \ldots, x_n)| \leq \frac{1}{m!} \sum_{j=1}^{n} \Bigg[(b_j - a_j)^{m - \frac{1}{2}} \left(\prod_{i=1}^{j-1} (b_i - a_i) \right)^{-1/2} \tag{11}$$

$$\times \left(\sqrt{ \frac{(m!)^2}{(2m)!} |B_{2m}| + B_m^2 \left(\frac{x_j - a_j}{b_j - a_j} \right) } \right)$$

$$\times \left\| \frac{\partial^m f}{\partial x_j^m} (\ldots, x_{j+1}, \ldots, x_n) \right\|_{2, \prod_{i=1}^{j} [a_i, b_i]} \Bigg].$$

Theorem 5. *Suppose Assumptions 1 or 2 or 3. Let $E_m^f(x_1, \ldots, x_n)$ as in (6), $m \in \mathbb{N}$. In particular we assume for $j = 1, \ldots, n$ that*

$$\frac{\partial^m f}{\partial x_j^m} (\ldots, x_{j+1}, \ldots, x_n) \in L_1 \left(\prod_{i=1}^{j} [a_i, b_i] \right),$$

for any $(x_{j+1}, \ldots, x_n) \in \prod_{i=j+1}^{n} [a_i, b_i]$. Then

$$|E_m^f(x_1, \ldots, x_n)| \leq \frac{1}{m!} \sum_{j=1}^{n} \Bigg[\frac{(b_j - a_j)^{m-1}}{\left(\prod_{i=1}^{j-1} (b_i - a_i) \right)} \tag{12}$$

$$\left(\left\|\frac{\partial^m f}{\partial x_j^m}(\ldots,x_{j+1},\ldots,x_n)\right\|_{1,\prod_{i=1}^{j}[a_i,b_i]}\right)\left\|B_m(t)-B_m\left(\frac{x_j-a_j}{b_j-a_j}\right)\right\|_{\infty,[0,1]}\Bigg].$$

The special cases are calculated and estimated further as follows:

1) *When* $m = 2r$, $r \in \mathbb{N}$, *then*

$$|E_{2r}^f(x_1,\ldots,x_n)| \tag{13}$$
$$\leq \frac{1}{(2r)!}\sum_{j=1}^{n}\left\{\frac{(b_j-a_j)^{2r-1}}{\left(\prod_{i=1}^{j-1}(b_i-a_i)\right)}\left(\left\|\frac{\partial^{2r} f}{\partial x_j^{2r}}(\ldots,x_{j+1},\ldots,x_n)\right\|_{1,\prod_{i=1}^{j}[a_i,b_i]}\right)\right.$$
$$\left.\times\left[(1-2^{-2r})|B_{2r}|+\left|2^{-2r}B_{2r}-B_{2r}\left(\frac{x_j-a_j}{b_j-a_j}\right)\right|\right]\right\}.$$

2) *When* $m = 2r+1$, $r \in \mathbb{N}$, *then*

$$|E_{2r+1}^f(x_1,\ldots,x_n)| \tag{14}$$
$$\leq \frac{1}{(2r+1)!}\sum_{j=1}^{n}\left\{\frac{(b_j-a_j)^{2r}}{\left(\prod_{i=1}^{j-1}(b_i-a_i)\right)}\left(\left\|\frac{\partial^{2r+1} f}{\partial x_j^{2r+1}}(\ldots,x_{j+1},\ldots,x_n)\right\|_{1,\prod_{i=1}^{j}[a_i,b_i]}\right)\right.$$
$$\left.\times\left[\frac{2(2r+1)!}{(2\pi)^{2r+1}(1-2^{-2r})}+\left|B_{2r+1}\left(\frac{x_j-a_j}{b_j-a_j}\right)\right|\right]\right\}.$$

And at last

3) *When* $m = 1$, *then*

$$|E_1^f(x_1,\ldots,x_n)| \tag{15}$$
$$\leq \sum_{j=1}^{n}\left\{\frac{1}{\left(\prod_{i=1}^{j-1}(b_i-a_i)\right)}\left[\left\|\frac{\partial f}{\partial x_j}(\ldots,x_{j+1},\ldots,x_n)\right\|_{1,\prod_{i=1}^{j}[a_i,b_i]}\right]\right.$$
$$\left.\times\left[\frac{1}{2}+\left|x_j-\left(\frac{a_j+b_j}{2}\right)\right|\right]\right\}.$$

In this article we give lots of specific and important applications of Theorems 3, 4, 5. see Theorems 6–28. There are produced many multivariate Ostrowski type inequalities for differentiation orders $m = 1,\ldots,6$, mostly related to multivariate trapezoid and midpoint rules. When we impose some basic and natural boundary conditions, then inequalities become very simple and elegant, see Theorems 25–28. The surprising fact there is, that only a very small number of sets of boundary conditions is needed comparely to the higher order of differentiation of the involved functions. At the end we establish sharpness of our inequalities with respect to $\|\cdot\|_\infty$ and for differentiation orders $m = 1, 2, 4$, see Theorems 30-34.

2. Main Results

Here we apply Theorems 3, 4, 5. We get

Theorem 6. *Suppose Assumptions 1 or 2 or 3, case $m = 1$.*

i) *Assume*

$$\frac{\partial f}{\partial x_j}(\ldots, x_{j+1}, \ldots, x_n) \in L_\infty\left(\prod_{i=1}^{j}([a_i, b_i])\right),$$

for any $(x_{j+1}, \ldots, x_n) \in \prod_{i=j+1}^{n}[a_i, b_i]$, all $j = 1, \ldots, n$. Then

$$\left| f(x_1, x_2, \ldots, x_n) - \frac{1}{\prod_{i=1}^{n}(b_i - a_i)} \int_{\prod_{i=1}^{n}[a_i,b_i]} f(s_1, \ldots, s_n) ds_1 \cdots ds_n \right| \tag{16}$$

$$\leq \sum_{j=1}^{n}\left[(b_j - a_j)\left(\sqrt{\frac{1}{12} + \left(\frac{x_j - a_j}{b_j - a_j} - \frac{1}{2}\right)^2}\right) \times \right.$$

$$\left. \times \left\| \frac{\partial f}{\partial x_j}(\ldots, x_{j+1}, \ldots, x_n) \right\|_{\infty, \prod_{i=1}^{j}[a_i,b_i]}\right].$$

ii) *Let $p_j, q_j > 1$: $\frac{1}{p_j} + \frac{1}{q_j} = 1$; $j = 1, \ldots, n$. In particular we assume that*

$$\frac{\partial f}{\partial x_j}(\ldots, x_{j+1}, \ldots, x_n) \in L_{q_j}\left(\prod_{i=1}^{j}[a_i, b_i]\right),$$

for any $(x_{j+1}, \ldots, x_n) \in \prod_{i=j+1}^{n}[a_i, b_i]$, for all $j = 1, \ldots, n$. Then

$$\left| f(x_1, x_2, \ldots, x_n) - \frac{1}{\prod_{i=1}^{n}(b_i - a_i)} \int_{\prod_{i=1}^{n}[a_i,b_i]} f(s_1, \ldots, s_n) ds_1 \cdots ds_n \right| \tag{17}$$

$$\leq \sum_{j=1}^{n}\left[(b_j - a_j)^{1-\frac{1}{q_j}}\left(\prod_{i=1}^{j-1}(b_i - a_i)\right)^{-\frac{1}{q_j}}\left(\int_0^1\left|\left(\frac{x_j - a_j}{b_j - a_j}\right) - t_j\right|^{p_j} dt_j\right)^{1/p_j}\right.$$

$$\times \left\| \frac{\partial f}{\partial x_j}(\ldots, x_{j+1}, \ldots, x_n) \right\|_{q_j, \prod_{i=1}^{j}[a_i,b_i]} \Bigg].$$

When $p_j = q_j = 2$*, all* $j = 1, \ldots, n$*, then*

$$\left| f(x_1, \ldots, x_n) - \frac{1}{\prod_{i=1}^{n}(b_i - a_i)} \int_{\prod_{i=1}^{n}[a_i,b_i]} f(s_1, \ldots, s_n) ds_1 \cdots ds_n \right| \tag{18}$$

$$\leq \sum_{j=1}^{n} \left[\left(\sqrt{(b_j - a_j)} \Big/ \sqrt{\prod_{i=1}^{j-1}(b_i - a_i)} \right) \left(\sqrt{\frac{1}{12} + \left(\frac{x_j - a_j}{b_j - a_j} - \frac{1}{2} \right)^2} \right) \right.$$

$$\left. \times \left\| \frac{\partial f}{\partial x_j}(\ldots, x_{j+1}, \ldots, x_n) \right\|_{2, \prod_{i=1}^{j}[a_i,b_i]} \right].$$

iii) *Here assume for* $j = 1, \ldots, n$ *that*

$$\frac{\partial f}{\partial x_j}(\ldots, x_{j+1}, \ldots, x_n) \in L_1\left(\prod_{i=1}^{j}[a_i, b_i] \right),$$

for any $(x_{j+1}, \ldots, x_n) \in \prod_{i=j+1}^{n}[a_i, b_i]$*. Then*

$$\left| f(x_1, \ldots, x_n) - \frac{1}{\prod_{i=1}^{n}(b_i - a_i)} \int_{\prod_{i=1}^{n}[a_i,b_i]} f(s_1, \ldots, s_n) ds_1 \cdots ds_n \right| \tag{19}$$

$$\leq \sum_{j=1}^{n} \left\{ \frac{1}{\left(\prod_{i=1}^{j-1}(b_i - a_i)\right)} \left[\left\| \frac{\partial f}{\partial x_j}(\ldots, x_{j+1}, \ldots, x_n) \right\|_{1, \prod_{i=1}^{j}[a_i,b_i]} \right] \right.$$

$$\left. \times \left[\frac{1}{2} + \left| x_j - \left(\frac{a_j + b_j}{2} \right) \right| \right] \right\}.$$

Notice 1. We have for $j = 1, \ldots, n$:

$$\begin{aligned} \lambda_j &:= \frac{x_j - a_j}{b_j - a_j} = 0 && \textit{iff } x_j = a_j, \\ \lambda_j &= 1 && \textit{iff } x_j = b_j, \\ \lambda_j &= \frac{1}{2} && \textit{iff } x_j = \frac{a_j + b_j}{2}. \end{aligned} \tag{20}$$

We continue with

Theorem 7. *Suppose Assumptions 1 or 2 or 3, Case* $m=1$, *all* $x_j = a_j$, $j=1,\dots,n$.

i) *Assume*

$$\frac{\partial f}{\partial x_j}(\dots,a_{j+1},\dots,a_n) \in L_\infty\left(\prod_{i=1}^{j}[a_i,b_i]\right), \quad \text{all } j=1,\dots,n.$$

Then

$$\left| f(a_1,a_2,\dots,a_n) - \frac{1}{\prod_{i=1}^{n}(b_i-a_i)}\int_{\prod_{i=1}^{n}[a_i,b_i]} f(s_1,\dots,s_n)ds_1\cdots ds_n \right| \tag{21}$$

$$\leq \frac{\sqrt{3}}{3}\left\{\sum_{j=1}^{n}\left[(b_j-a_j)\left\|\frac{\partial f}{\partial x_j}(\dots,a_{j+1},\dots,a_n)\right\|_{\infty,\prod_{i=1}^{j}[a_i,b_i]}\right]\right\}.$$

ii) *Let* $p_j,q_j>1$: $\frac{1}{p_j}+\frac{1}{q_j}=1$; $j=1,\dots,n$. *In particular we assume that*

$$\frac{\partial f}{\partial x_j}(\dots,a_{j+1},\dots,a_n) \in L_{q_j}\left(\prod_{i=1}^{j}[a_i,b_i]\right),$$

for all $j=1,\dots,n$. *Then*

$$\left| f(a_1,\dots,a_n) - \frac{1}{\prod_{i=1}^{n}(b_i-a_i)}\int_{\prod_{i=1}^{n}[a_i,b_i]} f(s_1,\dots,s_n)ds_1\cdots ds_n \right| \tag{22}$$

$$\leq \sum_{j=1}^{n}\left[\frac{(b_j-a_j)^{1-\frac{1}{q_j}}\left(\prod_{i=1}^{j-1}(b_i-a_i)\right)^{-1/q_j}}{(p_j+1)^{1/p_j}} \times \left\|\frac{\partial f}{\partial x_j}(\dots,a_{j+1},\dots,a_n)\right\|_{q_j,\prod_{i=1}^{j}[a_i,b_i]}\right].$$

When $p_j=q_j=2$, *all* $j=1,\dots,n$, *then*

$$\left| f(a_1,\dots,a_n) - \frac{1}{\prod_{i=1}^{n}(b_i-a_i)}\int_{\prod_{i=1}^{n}[a_i,b_i]} f(s_1,\dots,s_n)ds_1\cdots ds_n \right| \tag{23}$$

$$\leq \frac{\sqrt{3}}{3}\left(\sum_{j=1}^{n}\left[\left(\sqrt{(b_j-a_j)}\Big/\sqrt{\prod_{i=1}^{j-1}(b_i-a_i)}\right)\right.\right.$$

$$\times \left\| \frac{\partial f}{\partial x_j}(\ldots, a_{j+1}, \ldots, a_n) \right\|_{2, \prod_{i=1}^{j} [a_i, b_i]} \Bigg] \Bigg).$$

iii) *Here assume for $j = 1, \ldots, n$, that*

$$\frac{\partial f}{\partial x_j}(\ldots, a_{j+1}, \ldots, a_n) \in L_1\left(\prod_{i=1}^{j}[a_i, b_i]\right).$$

Then

$$\left| f(a_1, \ldots, a_n) - \frac{1}{\prod_{i=1}^{n}(b_i - a_i)} \int_{\prod_{i=1}^{n}[a_i, b_i]} f(s_1, \ldots, s_n) ds_1 \cdots ds_n \right| \tag{24}$$

$$\leq \frac{1}{2} \sum_{j=1}^{n} \left\{ \frac{(b_j - a_j + 1)}{\left(\prod_{i=1}^{j-1}(b_i - a_i)\right)} \left\| \frac{\partial f}{\partial x_j}(\ldots, a_{j+1}, \ldots, a_n) \right\|_{1, \prod_{i=1}^{j}[a_i, b_i]} \right\}.$$

Proof. By Theorem 6. □

Theorem 8. *Suppose Assumptions 1 or 2 or 3, case $m = 1$, all $x_j = b_j$, $j = 1, \ldots, n$.*

i) *Assume*

$$\frac{\partial f}{\partial x_j}(\ldots, b_{j+1}, \ldots, b_n) \in L_\infty\left(\prod_{i=1}^{j}[a_i, b_i]\right),$$

for all $j = 1, \ldots, n$. Then

$$\left| f(b_1, \ldots, b_n) - \frac{1}{\prod_{i=1}^{n}(b_i - a_i)} \int_{\prod_{i=1}^{n}[a_i, b_i]} f(s_1, \ldots, s_n) ds_1 \cdots ds_n \right| \tag{25}$$

$$\leq \frac{\sqrt{3}}{3} \sum_{j=1}^{n} \left[(b_j - a_j) \left\| \frac{\partial f}{\partial x_j}(\ldots, b_{j+1}, \ldots, b_n) \right\|_{\infty, \prod_{i=1}^{j}[a_i, b_i]} \right].$$

ii) *Let $p_j, q_j > 1$: $\frac{1}{p_j} + \frac{1}{q_j} = 1$; $j = 1, \ldots, n$. In particular we assume that*

$$\frac{\partial f}{\partial x_j}(\ldots, b_{j+1}, \ldots, b_n) \in L_{q_j}\left(\prod_{i=1}^{j}[a_i, b_i]\right),$$

for all $j = 1, \ldots, n$. *Then*

$$\left| f(b_1, \ldots, b_n) - \frac{1}{\prod_{i=1}^{n}(b_i - a_i)} \int_{\prod_{i=1}^{n}[a_i,b_i]} f(s_1, \ldots, s_n) ds_1 \cdots ds_n \right| \tag{26}$$

$$\leq \sum_{j=1}^{n} \left[(p_j + 1)^{-1/p_j} (b_j - a_j)^{1-\frac{1}{q_j}} \left(\prod_{i=1}^{j-1}(b_i - a_i) \right)^{-\frac{1}{q_j}} \times \right.$$

$$\left. \times \left\| \frac{\partial f}{\partial x_j}(\ldots, b_{j+1}, \ldots, b_n) \right\|_{q_j, \prod_{i=1}^{j}[a_i,b_i]} \right].$$

When $p_j = q_j = 2$, *all* $j = 1, \ldots, n$, *then*

$$\left| f(b_1, \ldots, b_n) - \frac{1}{\prod_{i=1}^{n}(b_i - a_i)} \int_{\prod_{i=1}^{n}[a_i,b_i]} f(s_1, \ldots, s_n) ds_1 \cdots ds_n \right| \tag{27}$$

$$\leq \frac{\sqrt{3}}{3} \sum_{j=1}^{n} \left[\left(\sqrt{b_j - a_j} \Big/ \sqrt{\prod_{i=1}^{j-1}(b_i - a_i)} \right) \left\| \frac{\partial f}{\partial x_j}(\ldots, b_{j+1}, \ldots, b_n) \right\|_{2, \prod_{i=1}^{j}[a_i,b_i]} \right].$$

iii) *Here assume for* $j = 1, \ldots, n$ *that*

$$\frac{\partial f}{\partial x_j}(\ldots, b_{j+1}, \ldots, b_n) \in L_1 \left(\prod_{i=1}^{j}[a_i, b_i] \right).$$

Then

$$\left| f(b_1, \ldots, b_n) - \frac{1}{\prod_{i=1}^{n}(b_i - a_i)} \int_{\prod_{i=1}^{n}[a_i,b_i]} f(s_1, \ldots, s_n) ds_1 \cdots ds_n \right| \tag{28}$$

$$\leq \frac{1}{2} \sum_{j=1}^{n} \left\{ \frac{(1 + b_j - a_j)}{\left(\prod_{i=1}^{j-1}(b_i - a_i) \right)} \left\| \frac{\partial f}{\partial x_j}(\ldots, b_{j+1}, \ldots, b_n) \right\|_{1, \prod_{i=1}^{j}[a_i,b_i]} \right\}.$$

Proof. By Theorem 6. □

Next come the multivariate midpoint rule inequalities.

Theorem 9. *Suppose Assumptions 1 or 2 or 3, case* $m = 1$, *all* $x_j = \frac{a_j + b_j}{2}$, $j = 1, \ldots, n$.

i) *Assume*

$$\frac{\partial f}{\partial x_j}\left(\ldots,\frac{a_{j+1}+b_{j+1}}{2},\ldots,\frac{a_n+b_n}{2}\right)\in L_\infty\left(\prod_{i=1}^{j}[a_i,b_i]\right),\quad \text{all } j=1,\ldots,n.$$

Then

$$\left|f\left(\frac{a_1+b_1}{2},\frac{a_2+b_2}{2},\ldots,\frac{a_n+b_n}{2}\right)\right. \tag{29}$$

$$\left.-\frac{1}{\prod_{i=1}^{n}(b_i-a_i)}\int_{\prod_{i=1}^{n}[a_i,b_i]}f(s_1,\ldots,s_n)ds_1\cdots ds_n\right|$$

$$\leq \frac{1}{2\sqrt{3}}\sum_{j=1}^{n}\left[(b_j-a_j)\left\|\frac{\partial f}{\partial x_j}\left(\ldots,\frac{a_{j+1}+b_{j+1}}{2},\ldots,\frac{a_n+b_n}{2}\right)\right\|_{\infty,\prod_{i=1}^{j}[a_i,b_i]}\right].$$

ii) *Let* $p_j,q_j>1$: $\frac{1}{p_j}+\frac{1}{q_j}=1$; $j=1,\ldots,n$. *In particular we assume that*

$$\frac{\partial f}{\partial x_j}\left(\ldots,\frac{a_{j+1}+b_{j+1}}{2},\ldots,\frac{a_n+b_n}{2}\right)\in L_{q_j}\left(\prod_{i=1}^{j}[a_i,b_i]\right),$$

for all $j=1,\ldots,n$. *Then*

$$\left|f\left(\frac{a_1+b_1}{2},\ldots,\frac{a_n+b_n}{2}\right)-\frac{1}{\prod_{i=1}^{n}(b_i-a_i)}\int_{\prod_{i=1}^{n}[a_i,b_i]}f(s_1,\ldots,s_n)ds_1\cdots ds_n\right| \tag{30}$$

$$\leq \frac{1}{2}\sum_{j=1}^{n}\left[(p_{j+1})^{-\frac{1}{p_j}}(b_j-a_j)^{1-\frac{1}{q_j}}\left(\prod_{i=1}^{j-1}(b_i-a_i)\right)^{-\frac{1}{q_j}}\right.$$

$$\left.\times\left\|\frac{\partial f}{\partial x_j}\left(\ldots,\frac{a_{j+1}+b_{j+1}}{2},\ldots,\frac{a_n+b_n}{2}\right)\right\|_{q_j,\prod_{i=1}^{j}[a_i,b_i]}\right].$$

When $p_j=q_j=2$, *all* $j=1,\ldots,n$, *then*

$$\left|f\left(\frac{a_1+b_1}{2},\ldots,\frac{a_n+b_n}{2}\right)-\frac{1}{\prod_{i=1}^{n}(b_i-a_i)}\int_{\prod_{i=1}^{n}[a_i,b_i]}f(s_1,\ldots,s_n)ds_1\cdots ds_n\right| \tag{31}$$

$$\leq \frac{1}{2\sqrt{3}}\sum_{j=1}^{n}\left[\left(\sqrt{b_j-a_j}\Big/\sqrt{\prod_{i=1}^{j-1}(b_i-a_i)}\right)\right.$$

$$\times \left\| \frac{\partial f}{\partial x_j}\left(\ldots, \frac{a_{j+1}+b_{j+1}}{2}, \ldots, \frac{a_n+b_n}{2}\right) \right\|_{2, \prod_{i=1}^{j}[a_i,b_i]} \Bigg].$$

iii) *Here assume for $j = 1, \ldots, n$ that*

$$\frac{\partial f}{\partial x_j}\left(\ldots, \frac{a_{j+1}+b_{j+1}}{2}, \ldots, \frac{a_n+b_n}{2}\right) \in L_1\left(\prod_{i=1}^{j}[a_i, b_i]\right).$$

Then

$$\left| f\left(\frac{a_1+b_1}{2}, \ldots, \frac{a_n+b_n}{2}\right) - \frac{1}{\prod_{i=1}^{n}(b_i-a_i)} \int_{\prod_{i=1}^{n}[a_i,b_i]} f(s_1, \ldots, s_n) ds_1 \cdots ds_n \right| \tag{32}$$

$$\leq \frac{1}{2}\sum_{j=1}^{n}\left\{ \frac{1}{\left(\prod_{i=1}^{j-1}(b_i-a_i)\right)} \left\| \frac{\partial f}{\partial x_j}\left(\ldots, \frac{a_{j+1}+b_{j+1}}{2}, \ldots, \frac{a_n+b_n}{2}\right) \right\|_{1, \prod_{i=1}^{j}[a_i,b_i]} \right\}.$$

Proof. By Theorem 6. □

Next we treat the case of $m = 2$ and only for the norms $\|\cdot\|_\infty$, $\|\cdot\|_2$, and specifically for $\lambda_j = 0, 1, \frac{1}{2}$, $j = 1, \ldots, n$. The multivariate trapezoid rule estimates follow immediately.

Theorem 10. *Suppose Assumptions 1 or 2 or 3, case $m = 2$, all $x_j = a_j$, $j = 1, \ldots, n$.*

i) *Assume*

$$\frac{\partial^2 f}{\partial x_j^2}(\ldots, a_{j+1}, \ldots, a_n) \in L_\infty\left(\prod_{i=1}^{j}[a_i, b_i]\right),$$

all $j = 1, \ldots, n$. Then

$$K_2 := \left| \left(\frac{f(a_1, a_2, \ldots, a_n) + f(b_1, a_2, \ldots, a_n)}{2} \right) \right. \tag{33}$$

$$- \frac{1}{\prod_{i=1}^{n}(b_i-a_i)} \int_{\prod_{i=1}^{n}[a_i,b_i]} f(s_1, \ldots, s_n) ds_1 \cdots ds_n$$

$$+ \frac{1}{2}\Bigg\{ \sum_{j=2}^{n} \Bigg[\frac{1}{\left(\prod_{i=1}^{j-1}(b_i-a_i)\right)} \Bigg\{ \int_{\prod_{i=1}^{j-1}[a_i,b_i]} \big(f(s_1, s_2, \ldots, s_{j-1}, b_j, a_{j+1}, \ldots, a_n)$$

$$- f(s_1,\ldots,s_{j-1},a_j,a_{j+1},\ldots,a_n)\big)ds_1\cdots ds_{j-1}\Big\}\Big]\Big\}\Bigg|$$

$$\leq \frac{1}{2\sqrt{30}}\left\{\sum_{j=1}^{n}\left[(b_i-a_j)^2\left\|\frac{\partial^2 f}{\partial x_j^2}(\ldots,a_{j+1},\ldots,a_n)\right\|_{\infty,\prod_{i=1}^{j}[a_i,b_i]}\right]\right\}.$$

ii) *Assume*

$$\frac{\partial^2 f}{\partial x_j^2}(\ldots,a_{j+1},\ldots,a_n)\in L_2\left(\prod_{i=1}^{j}[a_i,b_i]\right),$$

all $j=1,\ldots,n$. *Then*

$$K_2 \leq \frac{1}{2\sqrt{30}}\left\{\sum_{j=1}^{n}\left[(b_j-a_j)^{3/2}\left(\prod_{i=1}^{j-1}(b_i-a_i)\right)^{-1/2}\times\right.\right. \tag{34}$$

$$\left.\left.\times\left\|\frac{\partial^2 f}{\partial x_j^2}(\ldots,a_{j+1},\ldots,a_n)\right\|_{2,\prod_{i=1}^{j}[a_i,b_i]}\right]\right\}.$$

Proof. By Theorem 6. □

We continue with trapezoid rule estimates.

Theorem 11. *Suppose Assumptions 1 or 2 or 3, case* $m=2$, *all* $x_j=b_j$, $j=1,\ldots,n$.

i) *Assume*

$$\frac{\partial^2 f}{\partial x_j^2}(\ldots,b_{j+1},\ldots,b_n)\in L_\infty\left(\prod_{i=1}^{j}[a_i,b_i]\right),$$

all $j=1,\ldots,n$. *Then*

$$\Lambda_2 := \Bigg|\frac{f(b_1,\ldots,b_n)+f(a_1,b_2,\ldots,b_n)}{2} \tag{35}$$

$$-\frac{1}{\prod_{i=1}^{n}(b_i-a_i)}\int_{\prod_{i=1}^{n}[a_i,b_i]} f(s_1,\ldots,s_n)ds_1\cdots ds_n$$

$$-\frac{1}{2}\left\{\sum_{j=2}^{n}\left[\frac{1}{\left(\prod_{i=1}^{j-1}(b_i-a_i)\right)}\left\{\int_{\prod_{i=1}^{j-1}[a_i,b_i]}\big(f(s_1,\ldots,s_{j-1},b_j,b_{j+1},\ldots,b_n)\right.\right.\right.$$

$$- f(s_1,\ldots,s_{j-1},a_j,b_{j+1},\ldots,b_n)\big)ds_1\cdots ds_{j-1}\Big\}\Big]\Big\}\Bigg|$$

$$\leq \frac{1}{2\sqrt{30}} \left\{ \sum_{j=1}^{n} (b_j - a_j)^2 \left\| \frac{\partial^2 f}{\partial x_j^2}(\ldots, b_{j+1}, \ldots, b_n) \right\|_{\infty, \prod_{i=1}^{j} [a_i, b_i]} \right\}.$$

ii) *Assume*

$$\frac{\partial^2 f}{\partial x_j^2}(\ldots, b_{j+1}, \ldots, b_n) \in L_2 \left(\prod_{i=1}^{j} [a_i, b_i] \right),$$

all $j = 1, \ldots, n$. *Then*

$$\Lambda_2 \leq \frac{1}{2\sqrt{30}} \left\{ \sum_{j=1}^{n} \left[(b_j - a_j)^{3/2} \left(\prod_{i=1}^{j-1} (b_i - a_i) \right)^{-1/2} \right. \right. \tag{36}$$

$$\left. \left. \times \left\| \frac{\partial^2 f}{\partial x_j^2}(\ldots, b_{j+1}, \ldots, b_n \right\|_{2, \prod_{i=1}^{j} [a_i, b_i]} \right] \right\}.$$

Proof. By Theorem 6. $\square$

The multivariate midpoint rule estimates follow.

Theorem 12. *Suppose Assumptions 1 or 2 or 3, case* $m = 2$, *all* $x_j = \frac{a_j + b_j}{2}$, $j = 1, \ldots, n$.

i) *Assume*

$$\frac{\partial^2 f}{\partial x_j^2}\left(\ldots, \frac{a_{j+1} + b_{j+1}}{2}, \ldots, \frac{a_n + b_n}{2}\right) \in L_\infty \left(\prod_{i=1}^{j} [a_i, b_i] \right),$$

all $j = 1, \ldots, n$. *Then*

$$M_2 := \left| f\left(\frac{a_1 + b_1}{2}, \ldots, \frac{a_n + b_n}{2}\right) \right. \tag{37}$$

$$\left. - \frac{1}{\prod_{i=1}^{n} (b_i - a_i)} \int_{\prod_{i=1}^{n} [a_i, b_i]} f(s_1, \ldots, s_n) ds_1 \cdots ds_n \right|$$

$$\leq \frac{1}{8\sqrt{5}} \left\{ \sum_{j=1}^{n} \left[(b_j - a_j)^2 \left\| \frac{\partial^2 f}{\partial x_j^2}\left(\ldots, \frac{a_{j+1} + b_{j+1}}{2}, \ldots, \frac{a_n + b_n}{2}\right) \right\|_{\infty, \prod_{i=1}^{j} [a_i, b_i]} \right] \right\}.$$

ii) *Assume*

$$\frac{\partial^2 f}{\partial x_j^2}\left(\ldots, \frac{a_{j+1} + b_{j+1}}{2}, \ldots, \frac{a_n + b_n}{2}\right) \in L_2 \left(\prod_{i=1}^{j} [a_i, b_i] \right),$$

all $j = 1, \ldots, n$. *Then*

$$M_2 \leq \frac{1}{8\sqrt{5}} \left\{ \sum_{j=1}^{n} \left[(b_j - a_j)^{3/2} \left(\prod_{i=1}^{j-1} (b_i - a_i) \right)^{-1/2} \times \right. \right. \tag{38}$$

$$\left. \left. \times \left\| \frac{\partial^2 f}{\partial x_j^2} \left(\ldots, \frac{a_{j+1} + b_{j+1}}{2}, \ldots, \frac{a_n + b_n}{2} \right) \right\|_{2, \prod_{i=1}^{j} [a_i, b_i]} \right] \right\}.$$

Proof. By Theorem 6. □

We continue with trapezoid and midpoint rules inequalities for $m = 3$.

Theorem 13. *Suppose Assumptions 1 or 2 or 3, case* $m = 3$, *all* $x_j = a_j$, $j = 1, \ldots, n$.

i) *Assume*

$$\frac{\partial^3 f}{\partial x_j^3}(\ldots, a_{j+1}, \ldots, a_n) \in L_\infty \left(\prod_{i=1}^{j} [a_i, b_i] \right),$$

all $j = 1, \ldots, n$. *Then*

$$K_3 := \left| \left(\frac{f(a_1, \ldots, a_n) + f(b_1, a_2, \ldots, a_n)}{2} \right) \right. \tag{39}$$

$$- \frac{1}{\prod_{i=1}^{n} (b_i - a_i)} \int_{\prod_{i=1}^{n} [a_i, b_i]} f(s_1, \ldots, s_n) ds_1 \cdots ds_n$$

$$- \sum_{\substack{j=1 \\ \text{with } j=k\neq 1}}^{n} \left\{ \frac{1}{\prod_{i=1}^{j-1} (b_i - a_i)} \left\{ \sum_{k=1}^{2} \frac{(b_j - a_j)^{k-1}}{k!} B_k(0) \right. \right.$$

$$\times \left(\int_{\prod_{i=1}^{j-1} [a_i, b_i]} \left(\frac{\partial^{k-1} f}{\partial x_j^{k-1}} (s_1, \ldots, s_{j-1}, b_j, a_{j+1}, \ldots, a_n) \right. \right.$$

$$\left. \left. \left. \left. - \frac{\partial^{k-1} f}{\partial x_j^{k-1}} (s_1, \ldots, s_{j-1}, a_j, a_{j+1}, \ldots, a_n) \right) ds_1 \cdots ds_{j-1} \right) \right\} \right\}$$

$$\leq \frac{1}{12\sqrt{210}} \left\{ \sum_{j=1}^{n} \left[(b_j - a_j)^3 \left\| \frac{\partial^3 f}{\partial x_j^3} (\ldots, a_{n+1}, \ldots, a_n) \right\|_{\infty, \prod_{i=1}^{j} [a_i, b_i]} \right] \right\}.$$

ii) *Assume*

$$\frac{\partial^3 f}{\partial x_j^3}(\ldots, a_{j+1}, \ldots, a_n) \in L_2 \left(\prod_{i=1}^{j} [a_i, b_i] \right),$$

all $j = 1, \ldots, n$. *Then*

$$K_3 \leq \frac{1}{12\sqrt{210}} \sum_{j=1}^{n} \left[(b_j - a_j)^{5/2} \left(\prod_{i=1}^{j-1} (b_i - a_i) \right)^{-1/2} \times \right. \tag{40}$$

$$\left. \times \left\| \frac{\partial^3 f}{\partial x_j^3} (\ldots, a_{j+1}, \ldots, a_n) \right\|_{2, \prod_{i=1}^{j} [a_i, b_i]} \right].$$

Proof. By Theorem 6. □

Theorem 14. *Suppose Assumptions 1 or 2 or 3, case* $m = 3$, *all* $x_j = b_j$, $j = 1, \ldots, n$.

i) *Assume*

$$\frac{\partial^3 f}{\partial x_j^3} (\ldots, b_{j+1}, \ldots, b_n) \in L_\infty \left(\prod_{i=1}^{j} [a_i, b_i] \right),$$

all $j = 1, \ldots, n$. *Then*

$$\Lambda_3 := \left| \left(\frac{f(b_1, \ldots, b_n) + f(a_1, b_2, \ldots, b_n)}{2} \right) \right. \tag{41}$$

$$- \frac{1}{\prod_{i=1}^{n} (b_i - a_i)} \int_{\prod_{i=1}^{n} [a_i, b_i]} f(s_1, \ldots, s_n) ds_1 \cdots ds_n$$

$$- \sum_{\substack{j=1 \\ \text{with } j=k\neq 1}}^{n} \left\{ \frac{1}{\prod_{i=1}^{j-1} (b_i - a_i)} \left\{ \sum_{k=1}^{2} \frac{(b_j - a_j)^{k-1}}{k!} B_k(1) \right. \right.$$

$$\times \left(\int_{\prod_{i=1}^{j-1} [a_i, b_i]} \left(\frac{\partial^{k-1} f}{\partial x_j^{k-1}} (s_1, s_2, \ldots, s_{j-1}, b_j, b_{j+1}, \ldots, b_n) \right. \right.$$

$$\left. \left. \left. \left. - \frac{\partial^{k-1} f}{\partial x_j^{k-1}} (s_1, s_2, \ldots, s_{j-1}, a_j, b_{j+1}, \ldots, b_n) \right) ds_1 \cdots ds_{j-1} \right) \right\} \right\} \right|$$

$$\leq \frac{1}{12\sqrt{210}} \sum_{j=1}^{n} \left[(b_j - a_j)^3 \left\| \frac{\partial^3 f}{\partial x_j^3} (\ldots, b_{j+1}, \ldots, b_n) \right\|_{\infty, \prod_{i=1}^{j} [a_i, b_i]} \right].$$

ii) *Assume*

$$\frac{\partial^3 f}{\partial x_j^3} (\ldots, b_{j+1}, \ldots, b_n) \in L_2 \left(\prod_{i=1}^{j} [a_i, b_i] \right),$$

all $j = 1, \ldots, n$. *Then*

$$\Lambda_3 \leq \frac{1}{12\sqrt{210}} \sum_{j=1}^{n} \left[(b_j - a_j)^{5/2} \left(\prod_{i=1}^{j-1} (b_i - a_i) \right)^{-1/2} \times \right. \tag{42}$$

$$\left. \times \left\| \frac{\partial^3 f}{\partial x_j^3}(\ldots, b_{j+1}, \ldots, b_n) \right\|_{2, \prod_{i=1}^{j} [a_i, b_i]} \right].$$

Proof. By Theorem 6. □

Theorem 15. *Suppose Assumptions 1 or 2 or 3, case* $m = 3$, *all* $x_j = \frac{a_j + b_j}{2}$, $j = 1, \ldots, n$.

i) *Assume*

$$\frac{\partial^3 f}{\partial x_j^3} \left(\ldots, \frac{a_{j+1} + b_{j+1}}{2}, \ldots, \frac{a_n + b_n}{2} \right) \in L_\infty \left(\prod_{i=1}^{j} [a_i, b_i] \right),$$

all $j = 1, \ldots, n$. *Then*

$$M_3 := \left| f \left(\frac{a_1 + b_1}{2}, \ldots, \frac{a_n + b_n}{2} \right) \right. \tag{43}$$

$$- \frac{1}{\prod_{i=1}^{n} (b_i - a_i)} \int_{\prod_{i=1}^{n} [a_i, b_i]} f(s_1, \ldots, s_n) ds_1 \cdots ds_n$$

$$+ \frac{1}{24} \sum_{j=1}^{n} \left\{ \frac{(b_j - a_j)}{\left(\prod_{i=1}^{j-1} (b_i - a_i) \right)} \times \right.$$

$$\times \left(\int_{\prod_{i=1}^{j-1} [a_i, b_i]} \left(\frac{\partial f}{\partial x_j} (s_1, s_2, \ldots, s_{j-1}, b_j, \frac{a_{j+1} + b_{j+1}}{2}, \ldots, \frac{a_n + b_n}{2} \right) \right.$$

$$\left. \left. \left. - \frac{\partial f}{\partial x_j} \left(s_1, \ldots, s_{j-1}, a_j, \frac{a_{j+1} + b_{j+1}}{2}, \ldots, \frac{a_n + b_n}{2} \right) \right) ds_1 \cdots ds_{j-1} \right\} \right|$$

$$\leq \frac{1}{12\sqrt{210}} \sum_{j=1}^{n} \left[(b_j - a_j)^3 \left\| \frac{\partial^3 f}{\partial x_j^3} \left(\ldots, \frac{a_{j+1} + b_{j+1}}{2}, \ldots, \frac{a_n + b_n}{2} \right) \right\|_{\infty, \prod_{i=1}^{j} [a_i, b_i]} \right].$$

ii) *Assume*

$$\frac{\partial^3 f}{\partial x_j^3} \left(\ldots, \frac{a_{j+1} + b_{j+1}}{2}, \ldots, \frac{a_n + b_n}{2} \right) \in L_2 \left(\prod_{i=1}^{j} [a_i, b_i] \right),$$

all $j = 1, \ldots, n$. *Then*

$$M_3 \leq \frac{1}{12\sqrt{210}} \sum_{j=1}^{n} \Bigg[(b_j - a_j)^{5/2} \left(\prod_{i=1}^{j-1} (b_i - a_i) \right)^{-1/2} \tag{44}$$

$$\times \left\| \frac{\partial^3 f}{\partial x_j^3} \left(\ldots, \frac{a_{j+1} + b_{j+1}}{2}, \ldots, \frac{a_n + b_n}{2} \right) \right\|_{2, \prod_{i=1}^{j} [a_i, b_i]} \Bigg].$$

Proof. By Theorem 6. □

Next we present trapezoid and midpoint rules inequalities for $m = 4$.

Theorem 16. *Suppose Assumptions 1 or 2 or 3, case* $m = 4$, *all* $x_j = a_j$, $j = 1, \ldots, n$.

i) *Assume*

$$\frac{\partial^4 f}{\partial x_j^4} (\ldots, a_{j+1}, \ldots, a_n) \in L_\infty \left(\prod_{i=1}^{j} [a_i, b_i] \right),$$

all $j = 1, \ldots, n$. *Then*

$$K_3 = \Bigg| \left(\frac{f(a_1, \ldots, a_n) + f(b_1, a_2, \ldots, a_n)}{2} \right) \tag{45}$$

$$- \frac{1}{\prod_{i=1}^{n} (b_i - a_i)} \int_{\prod_{i=1}^{n} [a_i, b_i]} f(s_1, \ldots, s_n) ds_1 \cdots ds_n$$

$$- \sum_{\substack{j=1 \\ (\text{with } j = k \neq 1)}}^{n} \Bigg\{ \frac{1}{\prod_{i=1}^{j-1} (b_i - a_i)} \Bigg\{ \sum_{k=1}^{2} \frac{(b_j - a_j)^{k-1}}{k!}$$

$$\times B_k(0) \Bigg(\int_{\prod_{i=1}^{j-1} [a_i, b_i]} \Bigg(\frac{\partial^{k-1} f}{\partial x_j^{k-1}} (s_1, \ldots, s_{j-1}, b_j, a_{j+1}, \ldots, a_n)$$

$$- \frac{\partial^{k-1} f}{\partial x_j^{k-1}} (s_1, s_2, \ldots, s_{j-1}, a_j, a_{j+1}, \ldots, a_n) \Bigg) ds_1 \cdots ds_{j-1} \Bigg) \Bigg\} \Bigg\} \Bigg|$$

$$\leq \frac{1}{24\sqrt{630}} \sum_{j=1}^{n} \left[(b_j - a_j)^4 \left\| \frac{\partial^4 f}{\partial x_j^4} (\ldots, a_{j+1}, \ldots, a_n) \right\|_{\infty, \prod_{i=1}^{j} [a_i, b_i]} \right].$$

ii) *Assume*

$$\frac{\partial^4 f}{\partial x_j^4} (\ldots, a_{j+1}, \ldots, a_n) \in L_2 \left(\prod_{i=1}^{j} [a_i, b_i] \right),$$

all $j = 1, \ldots, n$. *Then*

$$K_3 \le \frac{1}{24\sqrt{630}} \sum_{j=1}^{n} \left[(b_j - a_j)^{7/2} \left(\prod_{i=1}^{j-1} (b_i - a_i) \right)^{-1/2} \times \right. \tag{46}$$

$$\left. \times \left\| \frac{\partial^4 f}{\partial x_j^4} (\ldots, a_{j+1}, \ldots, a_n) \right\|_{2, \prod_{i=1}^{j} [a_i, b_i]} \right].$$

Proof. By Theorem 6. □

Theorem 17. *Suppose Assumptions 1 or 2 or 3, case* $m = 4$, *all* $x_j = b_j$, $j = 1, \ldots, n$.

i) *Assume*

$$\frac{\partial^4 f}{\partial x_j^4} (\ldots, b_{j+1}, \ldots, b_n) \in L_\infty \left(\prod_{i=1}^{j} [a_i, b_i] \right),$$

all $j = 1, \ldots, n$. *Then*

$$\Lambda_3 = \left| \left(\frac{f(b_1, \ldots, b_n) + f(a_1, b_2, \ldots, b_n)}{2} \right) \right. \tag{47}$$

$$- \frac{1}{\prod_{i=1}^{n} (b_i - a_i)} \int_{\prod_{i=1}^{n} [a_i, b_i]} f(s_1, \ldots, s_n) ds_1 \cdots ds_n$$

$$- \sum_{\substack{j=1 \\ (\text{with } j=k\neq 1)}}^{n} \left\{ \frac{1}{\left(\prod_{i=1}^{j-1} (b_i - a_i) \right)} \left\{ \sum_{k=1}^{2} \frac{(b_j - a_j)^{k-1}}{k!} \right. \right.$$

$$\times B_k(1) \left(\int_{\prod_{i=1}^{j-1} [a_i, b_i]} \left(\frac{\partial^{k-1} f}{\partial x_j^{k-1}} (s_1, s_2, \ldots, s_{j-1}, b_j, b_{j+1}, \ldots, b_n) \right. \right.$$

$$\left. \left. \left. \left. \left. - \frac{\partial^{k-1} f}{\partial x_j^{k-1}} (s_1, s_2, \ldots, s_{j-1}, a_j, b_{j+1}, \ldots, b_n) \right) ds_1 \cdots ds_{j-1} \right) \right\} \right\} \right|$$

$$\le \frac{1}{24\sqrt{630}} \left\{ \sum_{j=1}^{n} \left[(b_j - a_j)^4 \left\| \frac{\partial^4 f}{\partial x_j^4} (\ldots, b_{j+1}, \ldots, b_n) \right\|_{\infty, \prod_{i=1}^{j} [a_i, b_i]} \right] \right\}.$$

ii) *Assume*

$$\frac{\partial^4 f}{\partial x_j^4} (\ldots, b_{j+1}, \ldots, b_n) \in L_2 \left(\prod_{i=1}^{j} [a_i, b_i] \right),$$

all $j = 1, \dots, n$. *Then*

$$\Lambda_3 \le \frac{1}{24\sqrt{630}} \sum_{j=1}^{n} \left[(b_j - a_j)^{7/2} \left(\prod_{i=1}^{j-1} (b_i - a_i) \right)^{-1/2} \times \right. \tag{48}$$

$$\left. \times \left\| \frac{\partial^4 f}{\partial x_j^4} (\dots, b_{j+1}, \dots, b_n) \right\|_{2, \prod_{i=1}^{j} [a_i, b_i]} \right].$$

Proof. By Theorem 6. □

Theorem 18. *Suppose Assumptions 1 or 2 or 3, case* $m = 4$, *all* $x_j = \frac{a_j + b_j}{2}$, $j = 1, \dots, n$.

i) *Assume*

$$\frac{\partial^4 f}{\partial x_j^4} \left(\dots, \frac{a_{j+1} + b_{j+1}}{2}, \dots, \frac{a_n + b_n}{2} \right) \in L_\infty \left(\prod_{i=1}^{j} [a_i, b_i] \right),$$

all $j = 1, \dots, n$. *Then*

$$M_3 = \left| f\left(\frac{a_1 + b_1}{2}, \dots, \frac{a_n + b_n}{2} \right) \right. \tag{49}$$

$$- \frac{1}{\prod_{i=1}^{n} (b_i - a_i)} \int_{\prod_{i=1}^{n} [a_i, b_i]} f(s_1, \dots, s_n) ds_1 \cdots ds_n$$

$$+ \frac{1}{24} \left\{ \sum_{j=1}^{n} \left\{ \frac{(b_j - a_j)}{\left(\prod_{i=1}^{j-1} (b_i - a_i) \right)} \times \right. \right.$$

$$\times \left(\int_{\prod_{i=1}^{j-1} [a_i, b_i]} \left(\frac{\partial f}{\partial x_j} \left(s_1, s_2, \dots, s_{j-1}, b_j, \frac{a_{j+1} + b_{j+1}}{2}, \dots, \frac{a_n + b_n}{2} \right) \right. \right.$$

$$\left. \left. \left. \left. \left. - \frac{\partial f}{\partial x_j} \left(s_1, s_2, \dots, s_{j-1}, a_j, \frac{a_{j+1} + b_{j+1}}{2}, \dots, \frac{a_n + b_n}{2} \right) \right) ds_1 \cdots ds_{j-1} \right) \right\} \right\} \right|$$

$$\le \frac{1}{152} \sqrt{\frac{107}{35}} \left\{ \sum_{j=1}^{n} \left[(b_j - a_j)^4 \left\| \frac{\partial^4 f}{\partial x_j^4} \left(\dots, \frac{a_{j+1} + b_{j+1}}{2}, \dots, \frac{a_n + b_n}{2} \right) \right\|_{\infty, \prod_{i=1}^{j} [a_i, b_i]} \right] \right\}.$$

ii) *Assume*

$$\frac{\partial^4 f}{\partial x_j^4} \left(\dots, \frac{a_{j+1} + b_{j+1}}{2}, \dots, \frac{a_n + b_n}{2} \right) \in L_2 \left(\prod_{i=1}^{j} [a_i, b_i] \right),$$

all $j = 1, \ldots, n$. *Then*

$$M_3 \leq \frac{1}{1152}\sqrt{\frac{107}{35}}\left\{\sum_{j=1}^{n}\left[(b_j - a_j)^{7/2}\left(\prod_{i=1}^{j-1}(b_i - a_i)\right)^{-1/2}\right.\right. \tag{50}$$

$$\left.\left.\times\left\|\frac{\partial^4 f}{\partial x_j^4}\left(\ldots, \frac{a_{j+1}+b_{j+1}}{2}, \ldots, \frac{a_n+b_n}{2}\right)\right\|_{2,\prod_{i=1}^{j}[a_i,b_i]}\right]\right\}.$$

Proof. By Theorem 6. □

Also we present trapezoid and midpoint rules inequalities for $m = 5$.

Theorem 19. *Suppose Assumptions 1 or 2 or 3, case* $m = 5$, *all* $x_j = a_j$, $j = 1, \ldots, n$.

i) *Assume*

$$\frac{\partial^5 f}{\partial x_j^5}(\ldots, a_{j+1}, \ldots, a_n) \in L_\infty\left(\prod_{i=1}^{j}[a_i, b_i]\right),$$

all $j = 1, \ldots, n$. *Then*

$$K_5 := \left|\left(\frac{f(a_1, \ldots, a_n) + f(b_1, a_2, \ldots, a_n)}{2}\right)\right. \tag{51}$$

$$-\frac{1}{\prod_{i=1}^{n}(b_i - a_i)}\int_{\prod_{i=1}^{n}[a_i,b_i]} f(s_1, \ldots, s_n)ds_1 \cdots ds_n$$

$$-\sum_{\substack{j=1\\(\text{with } j=k\neq 1)}}^{n}\left\{\frac{1}{\left(\prod_{i=1}^{j-1}(b_i - a_i)\right)}\left\{\sum_{k\in\{1,2,4\}}\frac{(b_j - a_j)^{k-1}}{k!}\right.\right.$$

$$\times B_k(0)\left(\int_{\prod_{i=1}^{j-1}[a_i,b_i]}\left(\frac{\partial^{k-1} f}{\partial x_j^{k-1}}(s_1, \ldots, s_{j-1}, b_j, a_{j+1}, \ldots, a_n)\right.\right.$$

$$\left.\left.\left.\left.\left.-\frac{\partial^{k-1} f}{\partial x_j^{k-1}}(s_1, \ldots, s_{j-1}, a_j, a_{j+1}, \ldots, a_n)\right)ds_1 \cdots ds_{j-1}\right)\right\}\right\}\right|$$

$$\leq \frac{1}{720}\sqrt{\frac{5}{462}}\left\{\sum_{j=1}^{n}\left[(b_j - a_j)^5\left\|\frac{\partial^5 f}{\partial x_j^5}(\ldots, a_{j+1}, \ldots, a_n)\right\|_{\infty,\prod_{i=1}^{j}[a_i,b_i]}\right]\right\}.$$

ii) *Assume*

$$\frac{\partial^5 f}{\partial x_j^5}(\ldots, a_{j+1}, \ldots, a_n) \in L_2\left(\prod_{i=1}^{j}[a_i, b_i]\right),$$

all $j = 1, \dots, n$. *Then*

$$K_5 \leq \frac{1}{720}\sqrt{\frac{5}{462}}\left\{\sum_{j=1}^{n}\left[(b_j - a_j)^{9/2}\left(\prod_{i=1}^{j-1}(b_i - a_i)\right)^{-1/2} \times \left\|\frac{\partial^5 f}{\partial x_j^5}(\dots, a_{j+1}, \dots, a_n)\right\|_{2,\prod_{i=1}^{j}[a_i,b_i]}\right]\right\}. \tag{52}$$

Proof. By Theorem 6. □

Theorem 20. *Suppose Assumptions 1 or 2 or 3, case* $m = 5$, *all* $x_j = b_j$, $j = 1, \dots, n$.

i) *Assume*

$$\frac{\partial^5 f}{\partial x_j^5}(\dots, b_{j+1}, \dots, b_n) \in L_\infty\left(\prod_{i=1}^{j}[a_i, b_i]\right),$$

all $j = 1, \dots, n$. *Then*

$$\begin{aligned}\Lambda_5 := \Bigg| &\left(\frac{f(b_1, \dots, b_n) + f(a_1, b_2, \dots, b_n)}{2}\right) \\ &- \frac{1}{\prod_{i=1}^{n}(b_i - a_i)}\int_{\prod_{i=1}^{n}[a_i,b_i]} f(s_1, \dots, s_n)ds_1 \cdots ds_n \\ &- \sum_{\substack{j=1 \\ (\text{with } j=k\neq 1)}}^{n}\left\{\frac{1}{\left(\prod_{i=1}^{j-1}(b_i - a_i)\right)}\left\{\sum_{k\in\{1,2,4\}}\frac{(b_j - a_j)^{k-1}}{k!}\right.\right. \\ &\times B_k(1)\left(\int_{\prod_{i=1}^{j-1}[a_i,b_i]}\left(\frac{\partial^{k-1} f}{\partial x_j^{k-1}}(s_1, \dots, s_{j-1}, b_j, b_{j+1}, \dots, b_n)\right.\right. \\ &\left.\left.\left.\left.- \frac{\partial^{k-1} f}{\partial x_j^{k-1}}(s_1, \dots, s_{j-1}, a_j, b_{j+1}, \dots, b_n)\right)ds_1 \cdots ds_{j-1}\right)\right\}\right\}\Bigg| \\ \leq \frac{1}{720}\sqrt{\frac{5}{462}}&\left\{\sum_{j=1}^{n}\left[(b_j - a_j)^5\left\|\frac{\partial^5 f}{\partial x_j^5}(\dots, b_{j+1}, \dots, b_n)\right\|_{\infty,\prod_{i=1}^{j}[a_i,b_i]}\right]\right\}.\end{aligned} \tag{53}$$

ii) *Assume*

$$\frac{\partial^5 f}{\partial x_j^5}(\dots, b_{j+1}, \dots, b_n) \in L_2\left(\prod_{i=1}^{j}[a_i, b_i]\right),$$

all $j = 1, \ldots, n$. *Then*

$$\Lambda_5 \leq \frac{1}{720}\sqrt{\frac{5}{462}}\left\{\sum_{j=1}^{n}\left[(b_j - a_j)^{9/2}\left(\prod_{i=1}^{j-1}(b_i - a_i)\right)^{-1/2}\right.\right. \tag{54}$$

$$\left.\left.\times \left\|\frac{\partial^5 f}{\partial x_j^5}(\ldots, b_{j+1}, \ldots, b_n)\right\|_{2, \prod_{i=1}^{j}[a_i, b_i]}\right]\right\}.$$

Proof. By Theorem 6. □

Theorem 21. *Suppose Assumptions 1 or 2 or 3, case* $m = 5$, *all* $x_j = \frac{a_j + b_j}{2}$, $j = 1, \ldots, n$.

i) *Assume*

$$\frac{\partial^5 f}{\partial x_j^5}\left(\ldots, \frac{a_{j+1} + b_{j+1}}{2}, \ldots, \frac{a_n + b_n}{2}\right) \in L_\infty\left(\prod_{i=1}^{j}[a_i, b_i]\right),$$

all $j = 1, \ldots, n$. *Then*

$$M_5 := \left| f\left(\frac{a_1 + b_1}{2}, \ldots, \frac{a_n + b_n}{2}\right)\right. \tag{55}$$

$$- \frac{1}{\prod_{i=1}^{n}(b_i - a_i)} \int_{\prod_{i=1}^{n}[a_i, b_i]} f(s_1, \ldots, s_n) ds_1 \cdots ds_n$$

$$- \sum_{j=1}^{n}\left\{\frac{1}{\left(\prod_{i=1}^{j-1}(b_i - a_i)\right)}\left\{\sum_{k \in \{2,4\}} \frac{(b_j - a_j)^{k-1}}{k!}\right.\right.$$

$$\times B_k\left(\frac{1}{2}\right)\left(\int_{\prod_{i=1}^{j-1}[a_i, b_i]}\left(\frac{\partial^{k-1} f}{\partial x_j^{k-1}}\left(s_1, \ldots, s_{j-1}, b_j, \frac{a_{j+1} + b_{j+1}}{2}, \ldots, \frac{a_n + b_n}{2}\right)\right.\right.$$

$$\left.\left.\left.\left.\left. - \frac{\partial^{k-1} f}{\partial x_j^{k-1}}\left(s_1, \ldots, s_{j-1}, a_j, \frac{a_{j+1} + b_{j+1}}{2}, \ldots, \frac{a_n + b_n}{2}\right)\right) ds_1 \cdots ds_{j-1}\right)\right\}\right\}\right|$$

$$\leq \frac{1}{720}\sqrt{\frac{5}{462}}\left\{\sum_{j=1}^{n}\left[(b_j - a_j)^5\left\|\frac{\partial^5 f}{\partial x_j^5}\left(\ldots, \frac{a_{j+1} + b_{j+1}}{2}, \ldots, \frac{a_n + b_n}{2}\right)\right\|_{\infty, \prod_{i=1}^{j}[a_i, b_i]}\right]\right\}.$$

ii) *Assume*

$$\frac{\partial^5 f}{\partial x_j^5}\left(\ldots, \frac{a_{j+1} + b_{j+1}}{2}, \ldots, \frac{a_n + b_n}{2}\right) \in L_2\left(\prod_{i=1}^{j}[a_i, b_i]\right),$$

all $j = 1, \dots, n$. *Then*

$$M_5 \le \frac{1}{720}\sqrt{\frac{5}{462}}\left\{\sum_{j=1}^{n}\left[(b_j - a_j)^{9/2}\left(\prod_{i=1}^{j-1}(b_i - a_i)\right)^{-1/2} \times \left\|\frac{\partial^5 f}{\partial x_j^5}\left(\dots, \frac{a_{j+1}+b_{j+1}}{2}, \dots, \frac{a_n+b_n}{2}\right)\right\|_{2, \prod_{i=1}^{j}[a_i,b_i]}\right]\right\}. \tag{56}$$

Proof. By Theorem 6. □

Finally we present trapezoid and midpoint rules inequalities for $m = 6$.

Theorem 22. *Suppose Assumptions 1 or 2 or 3, case* $m = 6$, *all* $x_j = a_j$, $j = 1, \dots, n$.

i) *Assume*

$$\frac{\partial^6 f}{\partial x_j^6}(\dots, a_{j+1}, \dots, a_n) \in L_\infty\left(\prod_{i=1}^{j}[a_i, b_i]\right),$$

all $j = 1, \dots, n$. *Then*

$$K_5 = \Bigg|\left(\frac{f(a_1,\dots,a_n) + f(b_1, a_2, \dots, a_n)}{2}\right) - \frac{1}{\prod_{i=1}^{n}(b_i - a_i)}\int_{\prod_{i=1}^{n}[a_i,b_i]} f(s_1,\dots,s_n)ds_1\cdots ds_n - \sum_{\substack{j=1 \\ (\text{with } j=k\neq 1)}}^{n}\left\{\frac{1}{\left(\prod_{i=1}^{j-1}(b_i - a_i)\right)}\left\{\sum_{k\in\{1,2,4\}}\frac{(b_j-a_j)^{k-1}}{k!} \times B_k(0)\left(\int_{\prod_{i=1}^{j-1}[a_i,b_i]}\left(\frac{\partial^{k-1} f}{\partial x_j^{k-1}}(s_1,\dots,s_{j-1}, b_j, a_{j+1},\dots,a_n) - \frac{\partial^{k-1} f}{\partial x_j^{k-1}}(s_1,\dots,s_{j-1}, a_j, a_{j+1},\dots,a_n)\right)ds_1\cdots ds_{j-1}\right)\right\}\right\}\Bigg| \tag{57}$$

$$\le \frac{1}{1440}\sqrt{\frac{101}{30030}}\left\{\sum_{j=1}^{n}\left[(b_j - a_j)^6\left\|\frac{\partial^6 f}{\partial x_j^6}(\dots, a_{j+1}, \dots, a_n)\right\|_{\infty, \prod_{i=1}^{j}[a_i,b_i]}\right]\right\}.$$

ii) *Assume*

$$\frac{\partial^6 f}{\partial x_j^6}(\dots, a_{j+1}, \dots, a_n) \in L_2\left(\prod_{i=1}^{j}[a_i, b_i]\right),$$

all $j = 1, \dots, n$. *Then*

$$K_5 \leq \frac{1}{1440}\sqrt{\frac{101}{30030}}\left\{\sum_{j=1}^{n}\left[(b_j - a_j)^{11/2}\left(\prod_{i=1}^{j-1}(b_i - a_i)\right)^{-1/2}\right.\right. \tag{58}$$

$$\left.\left.\times\left\|\frac{\partial^6 f}{\partial x_j^6}(\dots, a_{j+1}, \dots, a_n)\right\|_{2, \prod_{i=1}^{j}[a_i,b_i]}\right]\right\}.$$

Proof. By Theorem 6. □

Theorem 23. *Suppose Assumptions 1 or 2 or 3, case* $m = 6$, *all* $x_j = b_j$, $j = 1, \dots, n$.

i) *Assume*

$$\frac{\partial^6 f}{\partial x_j^6}(\dots, b_{j+1}, \dots, b_n) \in L_\infty\left(\prod_{i=1}^{j}[a_i, b_i]\right),$$

all $j = 1, \dots, n$. *Then*

$$\Lambda_5 = \left|\left(\frac{f(b_1, \dots, b_n) + f(a_1, b_2, \dots, b_n)}{2}\right)\right. \tag{59}$$

$$-\frac{1}{\prod_{i=1}^{n}(b_i - a_i)}\int_{\prod_{i=1}^{n}[a_i,b_i]} f(s_1, \dots, s_n)ds_1 \cdots ds_n$$

$$-\sum_{\substack{j=1 \\ (\text{with } j=k\neq 1)}}\left\{\frac{1}{\left(\prod_{i=1}^{j-1}(b_i - a_i)\right)}\left\{\sum_{k\in\{1,2,4\}}\frac{(b_j - a_j)^{k-1}}{k!}\right.\right.$$

$$\times B_k(1)\left(\int_{\prod_{i=1}^{j-1}[a_i,b_i]}\left(\frac{\partial^{k-1} f}{\partial x_j^{k-1}}(s_1, \dots, s_{j-1}, b_j, b_{j+1}, \dots, b_n)\right.\right.$$

$$\left.\left.\left.\left.\left.-\frac{\partial^{k-1} f}{\partial x_j^{k-1}}(s_1, \dots, s_{j-1}, a_j, b_{j+1}, \dots, b_n)\right)ds_1 \cdots ds_{j-1}\right)\right\}\right\}\right|$$

$$\leq \frac{1}{1440}\sqrt{\frac{101}{30030}}\left\{\sum_{j=1}^{n}\left[(b_j - a_j)^6\left\|\frac{\partial^6 f}{\partial x_j^6}(\dots, b_{j+1}, \dots, b_n)\right\|_{\infty, \prod_{i=1}^{j}[a_i,b_i]}\right]\right\}.$$

ii) *Assume*

$$\frac{\partial^6 f}{\partial x_j^6}(\dots, b_{j+1}, \dots, b_n) \in L_2\left(\prod_{i=1}^{j}[a_i, b_i]\right),$$

all $j = 1, \ldots, n$. *Then*

$$\Lambda_5 \leq \frac{1}{1440}\sqrt{\frac{101}{30030}}\left\{\sum_{j=1}^{n}\left[(b_j - a_j)^{11/2}\left(\prod_{i=1}^{j-1}(b_i - a_i)\right)^{-1/2}\right.\right. \tag{60}$$

$$\left.\left.\times\left\|\frac{\partial^6 f}{\partial x_j^6}(\ldots, b_{j+1}, \ldots, b_n)\right\|_{2, \prod_{i=1}^{j}[a_i,b_i]}\right]\right\}.$$

Proof. By Theorem 6. □

Theorem 24. *Suppose Assumptions 1 or 2 or 3, case* $m = 6$*, all* $x_j = \frac{a_j+b_j}{2}$*,* $j = 1, \ldots, n$.

i) *Assume*

$$\frac{\partial^6 f}{\partial x_j^6}\left(\ldots, \frac{a_{j+1}+b_{j+1}}{2}, \ldots, \frac{a_n+b_n}{2}\right) \in L_\infty\left(\prod_{i=1}^{j}[a_i, b_i]\right),$$

all $j = 1, \ldots, n$. *Then*

$$M_5 = \left| f\left(\frac{a_1+b_1}{2}, \ldots, \frac{a_n+b_n}{2}\right)\right. \tag{61}$$

$$-\frac{1}{\prod_{i=1}^{n}(b_i - a_i)}\int_{\prod_{i=1}^{n}[a_i,b_i]} f(s_1, \ldots, s_n)ds_1 \cdots ds_n$$

$$-\sum_{j=1}^{n}\left\{\frac{1}{\left(\prod_{i=1}^{j-1}(b_i - a_i)\right)}\left\{\sum_{k\in\{2,4\}}\frac{(b_j - a_j)^{k-1}}{k!}\right.\right.$$

$$\times B_k\left(\frac{1}{2}\right)\left(\int_{\prod_{i=1}^{j-1}[a_i,b_i]}\left(\frac{\partial^{k-1}f}{\partial x_j^{k-1}}\left(s_1, \ldots, s_{j-1}, b_j, \frac{a_{j+1}+b_{j+1}}{2}, \ldots, \frac{a_n+b_n}{2}\right)\right.\right.$$

$$\left.\left.\left.\left.\left.-\frac{\partial^{k-1}f}{\partial x_j^{k-1}}\left(s_1, \ldots, s_{j-1}, a_j, \frac{a_{j+1}+b_{j+1}}{2}, \ldots, \frac{a_n+b_n}{2}\right)\right)ds_1 \cdots ds_{j-1}\right)\right\}\right\}\right|$$

$$\leq \frac{1}{46080}\sqrt{\frac{7081}{2145}}\left\{\sum_{j=1}^{n}\left[(b_j - a_j)^6\right.\right.$$

$$\left.\left.\times\left\|\frac{\partial^6 f}{\partial x_j^6}\left(\ldots, \frac{a_{j+1}+b_{j+1}}{2}, \ldots, \frac{a_n+b_n}{2}\right)\right\|_{\infty, \prod_{i=1}^{j}[a_i,b_i]}\right]\right\}.$$

ii) *Assume*

$$\frac{\partial^6 f}{\partial x_j^6}\left(\ldots,\frac{a_{j+1}+b_{j+1}}{2},\ldots,\frac{a_n+b_n}{2}\right)\in L_2\left(\prod_{i=1}^{j}[a_i,b_i]\right),$$

all $j=1,\ldots,n$. *Then*

$$M_5\leq\frac{1}{46080}\sqrt{\frac{7081}{2145}}\left\{\sum_{j=1}^{n}\left[(b_j-a_j)^{11/2}\left(\prod_{i=1}^{j-1}(b_i-a_i)\right)^{-1/2}\right.\right. \tag{62}$$

$$\left.\left.\times\left\|\frac{\partial^6 f}{\partial x_j^6}\left(\ldots,\frac{a_{j+1}+b_{j+1}}{2},\ldots,\frac{a_n+b_n}{2}\right)\right\|_{2,\prod_{i=1}^{j}[a_i,b_i]}\right]\right\}.$$

Proof. By Theorem 6. □

At the end we give a simplified special case of Theorems 3 and 4.

Theorem 25. *Suppose Assumptions 1 or 2 or 3. We further assume that*

$$\frac{\partial^\ell f}{\partial x_j^\ell}(\ldots,b_j,\ldots)=\frac{\partial^\ell f}{\partial x_j^\ell}(\ldots,a_j,\ldots), \tag{63}$$

for all $j=1,\ldots,n$ *and all* $\ell=0,1,\ldots,m-2$. *Here* $m,n\in\mathbb{N}$, $x_i\in[a_i,b_i]$, $i=1,2,\ldots,n$.

i) *In particular we assume that*

$$\frac{\partial^m f}{\partial x_j^m}(\ldots,x_{j+1},\ldots,x_n)\in L_\infty\left(\prod_{i=1}^{j}[a_i,b_i]\right),$$

for any $(x_{j+1},\ldots,x_n)\in\prod_{i=j+1}^{n}[a_i,b_i]$, *all* $j=1,\ldots,n$. *Then*

$$\left|f(x_1,\ldots,x_n)-\frac{1}{\prod_{i=1}^{n}(b_i-a_i)}\int_{\prod_{i=1}^{n}[a_i,b_i]}f(s_1,\ldots,s_n)ds_1\cdots ds_n\right| \tag{64}$$

$$\leq\frac{1}{m!}\left\{\sum_{j=1}^{n}\left[(b_j-a_j)^m\left(\sqrt{\frac{(m!)^2}{(2m)!}|B_{2m}|+B_m^2\left(\frac{x_j-a_j}{b_j-a_j}\right)}\right)\right.\right.$$

$$\left.\left.\times\left\|\frac{\partial^m f}{\partial x_j^m}(\ldots,x_{j+1},\ldots,x_n)\right\|_{\infty,\prod_{i=1}^{j}[a_i,b_i]}\right]\right\},$$

true $\forall(x_1,\ldots,x_n)\in\prod_{i=1}^{n}[a_i,b_i]$.

ii) *Assume*

$$\frac{\partial^m f}{\partial x_j^m}(\ldots, x_{j+1}, \ldots, x_n) \in L_2\left(\prod_{i=1}^{j}[a_i, b_i]\right),$$

for any $(x_{j+1}, \ldots, x_n) \in \prod_{i=j+1}^{n}[a_i, b_i]$, *for all* $j = 1, \ldots, n$. *Then*

$$\left| f(x_1, \ldots, x_n) - \frac{1}{\prod_{i=1}^{n}(b_i - a_i)} \int_{\prod_{i=1}^{n}[a_i,b_i]} f(s_1, \ldots, s_n) ds_1 \cdots ds_n \right| \tag{65}$$

$$\leq \frac{1}{m!}\left\{\sum_{j=1}^{n}\left[(b_j - a_j)^{m-\frac{1}{2}}\left(\prod_{i=1}^{j-1}(b_i - a_i)\right)^{-1/2}\left(\sqrt{\frac{(m!)^2}{(2m)!}|B_{2m}| + B_m^2\left(\frac{x_j - a_j}{b_j - a_j}\right)}\right)\right.\right.$$
$$\left.\left.\times \left\| \frac{\partial^m f}{\partial x_j^m}(\ldots, x_{j+1}, \ldots, x_n) \right\|_{2, \prod_{i=1}^{j}[a_i,b_i]}\right]\right\},$$

true $\forall (x_1, \ldots, x_n) \in \prod_{i=1}^{n}[a_i, b_i]$.

Proof. Clearly here $A_j = 0$, $j = 1, \ldots, n$. Then proof is obvious. □

Similarly as in Theorem 25 we obtain

Theorem 26. *Suppose Assumptions 1 or 2 or 3, case* $m = 6$, *all* $x_j = a_j$, $j = 1, \ldots, n$. *Also assume*

$$\frac{\partial^\ell f}{\partial x_j^\ell}(\ldots, b_j, a_{j+1}, \ldots, a_n) = \frac{\partial^\ell f}{\partial x_j^\ell}(\ldots, a_j, a_{j+1}, \ldots, a_n), \tag{66}$$

for all $j = 1, \ldots, n$, *and all* $\ell = 0, 1, 3$.

i) *Assume*

$$\frac{\partial^6 f}{\partial x_j^6}(\ldots, a_{j+1}, \ldots, a_n) \in L_\infty\left(\prod_{i=1}^{j}[a_i, b_i]\right), \quad \text{all } j = 1, \ldots, n.$$

Then

$$\left| f(a_1, \ldots, a_n) - \frac{1}{\prod_{i=1}^{n}(b_i - a_i)} \int_{\prod_{i=1}^{n}[a_i,b_i]} f(s_1, \ldots, s_n) ds_1 \cdots ds_n \right| \tag{67}$$

$$\leq \frac{1}{1440}\sqrt{\frac{101}{30030}}\left\{\sum_{j=1}^{n}\left[(b_j - a_j)^6 \left\| \frac{\partial^6 f}{\partial x_j^6}(\ldots, a_{j+1}, \ldots, a_n) \right\|_{\infty, \prod_{i=1}^{j}[a_i,b_i]}\right]\right\}.$$

ii) *Assume*

$$\frac{\partial^6 f}{\partial x_j^6}(\dots, a_{j+1}, \dots, a_n) \in L_2\left(\prod_{i=1}^{j}[a_i, b_i]\right), \quad \textit{all } j = 1, \dots, n.$$

Then

$$\left| f(a_1, \dots, a_n) - \frac{1}{\prod_{i=1}^{n}(b_i - a_i)} \int_{\prod_{i=1}^{n}[a_i,b_i]} f(s_1, \dots, s_n) ds_1 \cdots ds_n \right| \tag{68}$$

$$\leq \frac{1}{1440}\sqrt{\frac{101}{30030}}\left\{\sum_{j=1}^{n}\left[(b_j - a_j)^{11/2}\left(\prod_{i=1}^{j-1}(b_i - a_i)\right)^{-1/2} \times \left\| \frac{\partial^6 f}{\partial x_j^6}(\dots, a_{j+1}, \dots, a_n)\right\|_{2, \prod_{i=1}^{j}[a_i,b_i]}\right]\right\}.$$

Proof. By Theorem 22. □

Theorem 27. *Suppose Assumptions 1 or 2 or 3, case $m = 6$, all $x_j = b_j$, $j = 1, \dots, n$. Also assume*

$$\frac{\partial^\ell f}{\partial x_j^\ell}(\dots, b_j, b_{j+1}, \dots, b_n) = \frac{\partial^\ell f}{\partial x_j^\ell}(\dots, a_j, b_{j+1}, \dots, b_n), \tag{69}$$

for all $j = 1, \dots, n$ and all $\ell = 0, 1, 3$.

i) *Assume*

$$\frac{\partial^6 f}{\partial x_j^6}(\dots, b_{j+1}, \dots, b_n) \in L_\infty\left(\prod_{i=1}^{j}[a_i, b_i]\right), \quad \text{all } j = 1, \dots, n.$$

Then

$$\left| f(b_1, \dots, b_n) - \frac{1}{\prod_{i=1}^{n}(b_i - a_i)} \int_{\prod_{i=1}^{n}[a_i,b_i]} f(s_1, \dots, s_n) ds_1 \cdots ds_n \right| \tag{70}$$

$$\leq \frac{1}{1440}\sqrt{\frac{101}{30030}}\left\{\sum_{j=1}^{n}\left[(b_j - a_j)^6 \left\| \frac{\partial^6 f}{\partial x_j^6}(\dots, b_{j+1}, \dots, b_n)\right\|_{\infty, \prod_{i=1}^{j}[a_i,b_i]}\right]\right\}.$$

ii) *Assume*

$$\frac{\partial^6 f}{\partial x_j^6}(\dots, b_{j+1}, \dots, b_n) \in L_2\left(\prod_{i=1}^{j}[a_i, b_i]\right), \quad \textit{all } j = 1, \dots, n.$$

Then

$$\left| f(b_1,\ldots,b_n) - \frac{1}{\prod_{i=1}^{n}(b_i-a_i)} \int_{\prod_{i=1}^{n}[a_i,b_i]} f(s_1,\ldots,s_n)ds_1\cdots ds_n \right| \tag{71}$$

$$\leq \frac{1}{1440}\sqrt{\frac{101}{30030}}\left\{\sum_{j=1}^{n}\left[(b_j-a_j)^{11/2}\left(\prod_{i=1}^{j-1}(b_i-a_i)\right)^{-1/2}\right.\right.$$

$$\left.\left.\times\left\|\frac{\partial^6 f}{\partial x_j^6}(\ldots,b_{j+1},\ldots,b_n)\right\|_{2,\prod_{i=1}^{j}[a_i,b_i]}\right]\right\}.$$

Proof. By Theorem 23. □

Theorem 28. *Suppose Assumptions 1 or 2 or 3, case* $m=6$, *all* $x_j=\frac{a_j+b_j}{2}$, $j=1,\ldots,n$. *Also assume*

$$\frac{\partial^\ell f}{\partial x_j^\ell}\left(\ldots,b_j,\frac{a_{j+1}+b_{j+1}}{2},\ldots,\frac{a_n+b_n}{2}\right) \tag{72}$$

$$=\frac{\partial^\ell f}{\partial x_j^\ell}\left(\ldots,a_j,\frac{a_{j+1}+b_{j+1}}{2},\ldots,\frac{a_n+b_n}{2}\right),$$

for all $j=1,\ldots,n$ *and* $\ell=1,3$.

i) *Assume*

$$\frac{\partial^6 f}{\partial x_j^6}\left(\ldots,\frac{a_{j+1}+b_{j+1}}{2},\ldots,\frac{a_n+b_n}{2}\right)\in L_\infty\left(\prod_{i=1}^{j}[a_i,b_i]\right),$$

all $j=1,\ldots,n$. *Then*

$$\left| f\left(\frac{a_1+b_1}{2},\ldots,\frac{a_n+b_n}{2}\right) - \frac{1}{\prod_{i=1}^{n}(b_i-a_i)} \int_{\prod_{i=1}^{n}[a_i,b_i]} f(s_1,\ldots,s_n)ds_1\cdots ds_n \right| \tag{73}$$

$$\leq \frac{1}{46080}\sqrt{\frac{7081}{2145}}\left\{\sum_{j=1}^{n}\left[(b_j-a_j)^6\right.\right.$$

$$\left.\left.\times\left\|\frac{\partial^6 f}{\partial x_j^6}\left(\ldots,\frac{a_{j+1}+b_{j+1}}{2},\ldots,\frac{a_n+b_n}{2}\right)\right\|_{\infty,\prod_{i=1}^{j}[a_i,b_i]}\right]\right\}.$$

ii) *Assume*

$$\frac{\partial^6 f}{\partial x_j^6}\left(\ldots,\frac{a_{j+1}+b_{j+1}}{2},\ldots,\frac{a_n+b_n}{2}\right)\in L_2\left(\prod_{i=1}^{j}[a_i,b_i]\right),$$

all $j = 1, \ldots, n$. *Then*

$$\left| f\left(\frac{a_1+b_1}{2}, \ldots, \frac{a_n+b_n}{2}\right) - \frac{1}{\prod_{i=1}^{n}(b_i - a_i)} \int_{\prod_{i=1}^{n}[a_i,b_i]} f(s_1, \ldots, s_n) ds_1 \cdots ds_n \right| \tag{74}$$

$$\leq \frac{1}{46080}\sqrt{\frac{7081}{2145}} \left\{ \sum_{j=1}^{n} \left[(b_j - a_j)^{11/2} \left(\prod_{i=1}^{j-1} (b_i - a_i) \right)^{-1/2} \right.\right.$$

$$\left.\left. \times \left\| \frac{\partial^6 f}{\partial x_j^6}\left(\ldots, \frac{a_{j+1}+b_{j+1}}{2}, \ldots, \frac{a_j+b_n}{2}\right) \right\|_{2, \prod_{i=1}^{j}[a_i,b_i]} \right] \right\}.$$

Proof. By Theorem 24. □

Comment 1. One can apply similar conditions to (63) for the cases of $m = 2, 3, 4, 5$ and simplify a lot the results of Theorems 10, 11 and of Theorems 13-21, exactly as we did in Theorems 25-28 for general $m \in \mathbb{N}$ and $m = 6$, etc.

3. Sharpness

We need to include

Theorem 29. *Suppose Assumptions 1 or 2 or 3. Let* $E_m^f(x_1, \ldots, x_n)$ *as in (6),* $m \in \mathbb{N}$. *In particular we assume that*

$$\frac{\partial^m f}{\partial x_j^m}\left(\overbrace{\cdots}^{j}, x_{j+1}, \ldots, x_n\right) \in L_\infty\left(\prod_{i=1}^{j}[a_i, b_i]\right),$$

for any $(x_{j+1}, \ldots, x_n) \in \prod_{i=j+1}^{n}[a_i, b_i]$, *all* $j = 1, \ldots, n$. *Then*

$$|E_m^f(x_1, \ldots, x_n)| \tag{75}$$

$$\leq \frac{1}{m!}\sum_{j=1}^{n}\left[(b_j - a_j)^m \left(\int_0^1 \left| B_m\left(\frac{x_j - a_j}{b_j - a_j}\right) - B_m(t_j) \right| dt_j \right) \right.$$

$$\left. \times \left\| \frac{\partial^m f}{\partial x_j^m}(\ldots, x_{j+1}, \ldots, x_n) \right\|_{\infty, \prod_{i=1}^{j}[a_i,b_i]} \right].$$

Proof. By Remark 4, see there (55) in [8]. □

We give the important

Theorem 30. *Suppose Assumptions 1 or 2 or 3. Let* $E_m^f(x_1,\dots,x_n)$ *as in (6),* $m \in \mathbb{N}$. *In particular we assume that*

$$\frac{\partial^m f}{\partial x_j^m}\left(\overbrace{\cdots}^{j}, x_{j+1},\dots,x_n\right) \in L_\infty\left(\prod_{i=1}^{j}[a_i,b_i]\right),$$

for any $(x_{j+1},\dots,x_n) \in \prod_{i=j+1}^{n}[a_i,b_i]$, *all* $j = 1,\dots,n$. *And also assume* $\frac{\partial^m f}{\partial x_j^m} \in L_\infty\left(\prod_{i=1}^{n}[a_i,b_i]\right)$, $j = 1,\dots,n-1$. *Call*

$$D_m(f) := \max_{1\le j\le n}\left\{\left\|\frac{\partial^m f}{\partial x_j^m}\right\|_\infty\right\}. \tag{76}$$

Then

$$|E_m^f(x_1,\dots,x_n)| \tag{77}$$

$$\le \frac{D_m(f)}{m!}\sum_{j=1}^{n}\left[(b_j-a_j)^m\left(\int_0^1\left|B_m\left(\frac{x_j-a_j}{b_j-a_j}\right)-B_m(t_j)\right|dt_j\right)\right].$$

Comment 2. We see that (see also [9])

$$I_1(\lambda_j) := \int_0^1 |B_1(\lambda_j)-B_1(t_j)|dt_j = \frac{1}{4}+\frac{\left(x_j-\left(\frac{a_j+b_j}{2}\right)\right)^2}{(b_j-a_j)^2}, \tag{78}$$

where $\lambda_j = \frac{x_j-a_j}{b_j-a_j}$, $j = 1,\dots,n$. Notice that

$$\max_{\lambda_j\in[0,1]} I_1(\lambda_j) = I_1(0) = I_1(1) = \frac{1}{2}, \tag{79}$$

i.e. when $x_j = a_j$ or b_j.

Thus we have

Theorem 31. *All here assumed as in Theorem 30 when* $m = 1$. *Then*

$$\left|f(x_1,\dots,x_n) - \frac{1}{\prod_{i=1}^{n}(b_i-a_i)}\int_{\prod_{i=1}^{n}[a_i,b_i]} f(s_1,\dots,s_n)ds_1\cdots ds_n\right| \tag{80}$$

$$\le \frac{D_1(f)}{2}\left(\sum_{j=1}^{n}(b_j-a_j)\right).$$

Inequality (80) is sharp, that is attained by $f_1(s_1, \ldots, s_n) := \sum_{j=1}^{n}(s_j - a_j)$ *when* $x_j = a_j$, $j = 1, \ldots, n$, *and by* $f_2(s_1, \ldots, s_n) := \sum_{j=1}^{n}(b_j - s_j)$ *when* $x_j = b_j$, $j = 1, \ldots, n$.

Proof. i) Case of $x_j = a_j$, $j = 1, \ldots, n$. Then $\frac{\partial f_1}{\partial x_j} = 1$, $j = 1, \ldots, n$, i.e. $\left\| \frac{\partial f_1}{\partial x_j} \right\|_\infty = 1$ and $D_1(f_1) = 1$. Clearly then we have

$$\text{L.H.S.}(80) = \text{R.H.S.}(80) = \frac{1}{2}\sum_{j=1}^{n}(b_j - a_j),$$

proving sharpness.

ii) Case of $x_j = b_j$, $j = 1, \ldots, n$. Then $\frac{\partial f_2}{\partial x_j} = -1$, $j = 1, \ldots, n$, i.e. $\left\| \frac{\partial f_2}{\partial x_j} \right\|_\infty = 1$ and $D_1(f_2) = 1$. Clearly we have

$$\text{L.H.S.}(80) = \text{R.H.S.}(80) = \frac{1}{2}\sum_{j=1}^{n}(b_j - a_j),$$

proving again sharpness. □

Comment 3. We see that ([9])

$$I_2(\lambda_j) := \int_0^1 |B_2(\lambda_j) - B_2(t_j)| dt_j \tag{81}$$
$$= \frac{8}{3}\delta_j^3(x) - \delta_j^2(x) + \frac{1}{12}, \quad j = 1, \ldots, n,$$

where

$$\delta_j(x_j) := \frac{\left| x_j - \frac{a_j + b_j}{2} \right|}{b_j - a_j}, \quad x_j \in [a_j, b_j].$$

Also from [9] we have that

$$\max_{0 \le \lambda_j \le 1} I_2(\lambda_j) = I_2(0) = I_2(1) = \frac{1}{6}, \tag{82}$$

i.e. when $x_j = a_j$ or b_j.

We continue with

Theorem 32. *All here assumed as in Theorem 30 when* $m = 2$. *Then*

$$|E_2^f(x_1, \ldots, x_n)| \le \frac{D_2(f)}{12}\sum_{j=1}^{n}(b_j - a_j)^2. \tag{83}$$

Inequality (83) is sharp, that is attained by $f_1(s_1, \ldots, s_n) := \sum_{j=1}^{n}(s_j - a_j)^2$ *when* $x_j = a_j$, $j = 1, \ldots, n$ *and by* $f_2(s_1, \ldots, s_n) := \sum_{j=1}^{n}(s_j - b_j)^2$ *when* $x_j = b_j$, $j = 1, \ldots, n$.

Proof. i) Case of $x_j = a_j$, $j = 1, \ldots, n$. Then $\frac{\partial f_1}{\partial x_j} = 2(s_j - a_j)$, $\frac{\partial^2 f_1}{\partial x_j^2} = 2$, and $\left\| \frac{\partial^2 f_1}{\partial x_j^2} \right\|_\infty = 2$, with $D_2(f) = 2$. Clearly then we have

$$\text{L.H.S.}(83) = \text{R.H.S.}(83) = \frac{1}{6} \sum_{j=1}^{n} (b_i - a_j)^2,$$

proving sharpness.

ii) Case of $x_j = b_j$, $j = 1, \ldots, n$. Then $\frac{\partial f_2}{\partial x_j} = 2(s_j - b_j)$, $\frac{\partial^2 f_2}{\partial x_j^2} = 2$, and $\left\| \frac{\partial^2 f_2}{\partial x_j^2} \right\|_\infty = 2$, with $D_2(f) = 2$. Clearly again we have

$$\text{L.H.S.}(83) = \text{R.H.S.}(83) = \frac{1}{6} \sum_{j=1}^{n} (b_i - a_j)^2,$$

proving again sharpness. □

Comment 4. By [2] we have that

$$\max_{0 \le \lambda_j \le 1} I_3(\lambda_j) = I_3\left(\frac{3 - \sqrt{3}}{6}\right) = I_3\left(\frac{3 + \sqrt{3}}{6}\right) = \frac{\sqrt{3}}{36}, \tag{84}$$

where

$$I_3(\lambda_j) := \int_0^1 |B_3(\lambda_j) - B_3(t_j)| dt_j, \quad j = 1, \ldots, n. \tag{85}$$

Consequently we have

Theorem 33. *All here assumed as in Theorem 30 when $m = 3$. Then*

$$|E_3^f(x_1, \ldots, x_n)| \le \frac{\sqrt{3} D_3(f)}{216} \sum_{j=1}^{n} (b_j - a_j)^3. \tag{86}$$

Comment 5. We call

$$I_m(\lambda_j) := \int_0^1 |B_m(\lambda_j) - B_m(t_j)| dt_j, \tag{87}$$

where $\lambda_j := \frac{x_j - a_j}{b_j - a_j}$, $j = 1, \ldots, n, m \in \mathbb{N}$. In [6] we found that

$$\max_{\lambda_j \in [0,1]} I_4(\lambda_j) = I_4(0) = I_4(1) = \frac{1}{30}. \tag{88}$$

So we give

Theorem 34. *All here assumed as in Theorem 30 when $m = 4$. Then*

$$|E_4^f(x_1, \ldots, x_n)| \le \frac{D_4(f)}{720} \left(\sum_{j=1}^{n} (b_j - a_j)^4 \right). \tag{89}$$

Inequality (89) is sharp, that is attained by $f_1(s_1,\dots,s_n) := \sum_{j=1}^{n}(s_j - a_j)^4$ *when* $x_j = a_j$, $j = 1,\dots,n$ *and by* $f_2(s_1,\dots,s_n) := \sum_{j=1}^{n}(s_j - b_j)^4$ *when* $x_j = b_j$, $j = 1,\dots,n$.

Proof. Case of $x_j = a_j$, $j = 1,\dots,n$. Then

$$\frac{\partial f_1}{\partial x_j} = 4(s_j - a_j)^3,\quad \frac{\partial^2 f_1}{\partial x_j^2} = 12(s_j - a_j)^2,\quad \frac{\partial^3 f_1}{\partial x_j^3} = 24(s_j - a_j),\quad \frac{\partial^4 f_1}{\partial x_j^4} = 24,$$

with $\left\|\frac{\partial^4 f_1}{\partial x_j^4}\right\|_\infty = 24$ and $D_4(f) = 24$. Clearly then we have

$$\text{L.H.S.}(89) = \text{R.H.S.}(89) = \frac{1}{30}\left(\sum_{j=1}^{n}(b_j - a_j)^4\right),$$

proving sharpness.

ii) Case $x_j = b_j$, $j = 1,\dots,n$. Then

$$\frac{\partial f_2}{\partial x_j} = 4(s_j - b_j)^3,\quad \frac{\partial^2 f_2}{\partial x_j^2} = 12(s_j - b_j)^2,\quad \frac{\partial^3 f_2}{\partial x_j^3} = 24(s_j - b_j),\quad \frac{\partial^4 f_2}{\partial x_j^4} = 24,$$

with $D_4(f_2) = 24$. Clearly again we have

$$\text{L.H.S.}(89) = \text{R.H.S.}(89) = \frac{1}{30}\left(\sum_{j=1}^{n}(b_j - a_j)^4\right),$$

proving again sharpness. □

Comment 6. Inequality (75) is sharper than (9), however the integral $Im(\lambda_j)$ (see (87)) in its right hand side, is difficult to compute and find its maximum value for $m \geq 5$. That is why (9) is more practical, also less restrictive, and we used it extensively here in the applications.

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BOUNDARY VALUE PROBLEMS FOR SYSTEMS OF SECOND ORDER DIFFERENTIAL EQUATIONS OF MIXED TYPE

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Abstract. Applying the Perov's fixed point theorem is approached boundary value problems for systems of second order differential equations of mixed type.

1. **Introduction**

The aim of this paper is to present some results about the existence and uniqueness, subsolutions and suprasolutions, continuity, monotony and data dependence of the solution of (1)+(2). We apply the W.P.O's technique as in [12].
Consider the problem

$$-x''(t) = f(t, x(t), x(g(t)), x(h(t))),\ t \in [a, b] \tag{1}$$

$$\begin{cases} l_1(x(t)) = \alpha(t),\ \text{for}\ \ t \in [a_1, a] \\ l_2(x(t)) = \beta(t),\ \text{for}\ \ t \in [b, b_1] \end{cases} \tag{2}$$

where $\alpha \in C([a_1, a], \mathbb{R}^m)$, $\beta \in C([b, b_1], \mathbb{R}^m)$, $g, h \in C([a, b], [a_1, b_1])$ and $a_1 \leq a < b \leq b_1$.
Here,

$$l_1 : C^1([a_1, a], \mathbb{R}^m) \longrightarrow C([a_1, a], \mathbb{R}^m)$$

and

$$l_2 : C^1([b, b_1], \mathbb{R}^m) \longrightarrow C([b, b_1], \mathbb{R}^m)$$

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are linear functions.

We suppose that the boundary value problem

$$\begin{cases} -u'' = \chi, \quad t \in [a,b] \\ l_1(u(t)) = \alpha(t), \text{ for } t \in [a_1,a] \\ l_2(u(t)) = \beta(t), \text{ for } t \in [b,b_1] \end{cases} \tag{3}$$

has a unique solution $u \in C([a_1,b_1],\mathbb{R}^m) \cap C^2([a,b],\mathbb{R}^m)$,

$$u(t) = \begin{cases} \varphi(t), \quad t \in [a_1,a] \\ w(\alpha,\beta)(t) + \int_a^b G(t,s)\chi(s)\,ds, \quad t \in [a,b] \\ \psi(t), \quad t \in [b,b_1] \end{cases}$$

and there exist

$$G \in C([a,b] \times [a,b], M_{m,m}(\mathbb{R})),$$

where G is the corresponding Green function and

$$w(\alpha,\beta)(t) = \frac{t-a}{b-a}\psi(b) + \frac{b-t}{b-a}\varphi(a), \; t \in [a,b].$$

Consider the following conditions:

$(C1)$ $g,h \in C([a,b],[a_1,b_1])$; $\alpha \in C([a_1,a],\mathbb{R}^m)$, $\beta \in C([b,b_1],\mathbb{R}^m)$;

$(C2)$ $f = (f_1,f_2,...,f_m) \in C([a,b] \times \mathbb{R}^{3m},\mathbb{R}^m)$;

$(C3)$ There exist a matrix $L_f \in M_{m,m}(\mathbb{R}_+)$ such that

$$\left\|f(t,u^1,u^2,u^3) - f(t,v^1,v^2,v^3)\right\|_{\mathbb{R}^m} \le L_f\left(\left\|u^1 - v^1\right\|_{\mathbb{R}^m} + \left\|u^2 - v^2\right\|_{\mathbb{R}^m} + \left\|u^3 - v^3\right\|_{\mathbb{R}^m}\right)$$

$$\forall t \in [a,b], \quad u^i, v^i \in \mathbb{R}^m, \quad i = \overline{1,3},$$

where

$$\|u\|_{\mathbb{R}^m} := \begin{pmatrix} |u_1| \\ \vdots \\ |u_m| \end{pmatrix}$$

is the vectorial norm on $\mathbb{R}^m$.

In this study we will use the Weakly Picard Operator's technique and the Perov's fixed point theorem(see [1] and [8]). A variant of this theorem on vector valued normed spaces and the above vectorial norm is used in [1].

We need the following notions and notations:

Let (X, d) be a generalized metric space, $d(x, y) \in \mathbb{R}^m$ and $A\colon X \to X$ an operator.

We shall use:

$F_A := \{x \in X \mid A(x) = x\}$-the fixed point set of the operator A;

$I(A) := \{Y \subset X \mid A(Y) \subset Y, Y \neq \varnothing\}$-the family of the nonempty invariant subsets of A;

$A^{n+1} := A \circ A^n$; $A^0 = 1_X$; $A^1 = A$; $n \in \mathbb{N}$.

Definition 1. *([10]) An operator A is Weakly Picard Operator(W.P.O.) if the sequence $(A^n(x))_{n\in\mathbb{N}}$ converges , for all $x \in X$ and the limit (which may depend on x) is a fixed point of A.*

Definition 2. *([10]) If the operator A is W.P.O. and $F_A = \{x^*\}$, then by definition, the operator A is Picard Operator(P.O.).*

Definition 3. *([10]) If A is W.P.O., then we consider the operator A^∞ defined by*

$$A^\infty : X \to X,\ A^\infty(x) := \lim_{n\to\infty} A^n(x).$$

2. Existence and uniqueness

The problem (1)+(2) is equivalent in $C([a_1, b_1], \mathbb{R}^m) \cap C^2([a, b], \mathbb{R}^m)$ with the fixed point equation

$$x(t) = \begin{cases} \varphi(t), \quad t \in [a_1, a] \\ w(\alpha, \beta)(t) + \int_a^b G(t, s) f(s, x(s), x(g(s)), x(h(s)))ds, \quad t \in [a, b] \\ \psi(t), \quad t \in [b, b_1] \end{cases} \tag{4}$$

where

$$w(\alpha, \beta)(t) = \frac{t-a}{b-a}\psi(b) + \frac{b-t}{b-a}\varphi(a),\ t \in [a, b].$$

Then, in $C([a_1,b_1],\mathbb{R}^m)\cap C^2([a,b],\mathbb{R}^m)$, the equation (1) is equivalent with

$$x(t)=\begin{cases} x(t), \quad t\in[a_1,a] \\ w(x\mid_{[a_1,a]},x\mid_{[b,b_1]})(t)+\int\limits_a^b G(t,s)f(s,x(s),x(g(s)),x(h(s)))ds, \ \ t\in[a,b] \\ x(t), \quad t\in[b,b_1] \end{cases} \tag{5}$$

Consider the following operators:

$$B_f,E_f:C([a_1,b_1],\mathbb{R}^m)\longrightarrow C([a_1,b_1],\mathbb{R}^m)$$

where

$$B_f(x)(t):=\mathit{second\ part\ of}\ (4)$$

$$E_f(x)(t):=\mathit{second\ part\ of}\ (5).$$

Consider the functional spaces

$$X:=C([a_1,b_1],\mathbb{R}^m),$$

$$X_{\varphi,\psi}:=\{x\in X:x\mid_{[a_1,a]}=\varphi,\ x\mid_{[b,b_1]}=\psi\}.$$

Then

$$X=\bigcup_{\substack{\varphi\in C([a_1,a],\mathbb{R}^m)\\ \psi\in C([b,b_1],\mathbb{R}^m)}} X_{\varphi,\psi}\ \ \mathit{is\ a\ partition\ of}\ \ X.$$

Lemma 4. *(see* [12]*) We suppose that the conditions* $(C1)$*,* $(C2)$*,* $(C3)$ *are satisfied. Then:*

(a) $B_f(X)\subset X_{\varphi,\psi}$ *and* $B_f(X_{\varphi,\psi})\subset X_{\varphi,\psi}$*;*

(b) $B_f\mid_{X_{\varphi,\psi}}=E_f\mid_{X_{\varphi,\psi}}$.

Proof. Is similar as in [12] taking $X=C([a_1,b_1],\mathbb{R}^m)$. ∎

Let

$$M_G=\left(\|G_{ij}\|\right)_{i,j=\overline{1,m}}\in M_{m,m}\left(\mathbb{R}_+\right),$$

where $\|G_{ij}\|=\max\left\{|G_{ij}\left(x,s\right)|:(x,s)\in[a,b]\times[a,b]\right\},\ \ \forall i,j=\overline{1,m}.$

and

$$Q=3\left(b-a\right)M_G\cdot L_f\in M_{m,m}\left(\mathbb{R}_+\right).$$

We have the following existence and uniqueness theorem:

Theorem 5. *We suppose that:*

(i) the conditions $(C1)-(C3)$ *are satisfied;*

(ii) $Q^n \longrightarrow 0$ *as* $n \longrightarrow \infty$.

Then the problem (1)+(2) has a unique solution

$$x_f^* = (x_f^{*^1}, ..., x_f^{*^m}) \in C([a_1, b_1], \mathbb{R}^m) \cap C^2([a, b], \mathbb{R}^m).$$

Proof. Consider the Banach space $X = C([a_1, b_1], \mathbb{R}^m)$ with generalized Chebyshev's norm

$$\|u\|_C := \begin{pmatrix} \|u_1\|_C \\ \vdots \\ \|u_m\|_C \end{pmatrix}, \quad \textit{where} \quad \|u_i\|_C := \max_{a_1 \le t \le b_1} |u_i(t)|, \ \forall i = \overline{1, m}.$$

The problem $(1) + (2)$ is equivalent on X with the fixed point equation:

$$B_f(x) = x.$$

We prove now that the operator $B_f = (B_{f_1}, ..., B_{f_m})$ is Picard Operator. For $y, z \in X$ we have:

$$\|B_f(y)(t) - B_f(z)(t)\|_{\mathbb{R}^m} \le$$

$$\le \int_a^b \|G(t,s)[f(s, y(s), y(g(s)), y(h(s))) - f(s, z(s), z(g(s)), z(h(s)))]\|_{\mathbb{R}^m} ds \le$$

$$\le \int_a^b M_G \cdot L_f \cdot [\|y(s) - z(s)\|_{\mathbb{R}^m} + \|y(g(s)) - z(g(s))\|_{\mathbb{R}^m} +$$

$$+ \|y(h(s)) - z(h(s))\|_{\mathbb{R}^m}] ds \le$$

$$\le 3(b-a) M_G L_f \cdot \|y - z\|_C = Q \cdot \|y - z\|_C, \quad \forall t \in [a, b].$$

Then,

$$\|B_f(y) - B_f(z)\|_C \le Q \|y - z\|_C$$

and by (ii), the operator B_f is $Q-contraction$. From the Perov's fixed point theorem we infer that the operator B_f is P.O. and has a unique fixed point

$$x_f^* = (x_f^{*^1}, ..., x_f^{*^m}) \in X.$$

Since f is continuous, deriving (4) two times by t, we infer that

$$x_f^* \in C^2([a,b], \mathbb{R}^m)$$

is the unique solution of (1)+(2). ∎

Remark 6. *Since from Theorem 5 we have that the operator B_f is P.O. and because*

$$B_f \mid_{X_{\varphi,\psi}} = E_f \mid_{X_{\varphi,\psi}},$$

$$X := C([a_1, b_1], \mathbb{R}^m) = \bigcup_{\varphi,\psi} X_{\varphi,\psi}, \quad X_{\varphi,\psi} \in I(E_f)$$

we infer that the operator E_f is W.P.O. and

$$F_{E_f} \cap X_{\varphi,\psi} = \{x^*_{\varphi,\psi}\}, \quad \forall \varphi \in C([a_1, a], \mathbb{R}^m), \ \forall \psi \in C([b, b_1], \mathbb{R}^m)$$

where $x^*_{\varphi,\psi}$ is the unique solution of the problem $(1) + (2)$.

3. **Data dependence**

In this paragraph we shall study the subsolutions and suprasolutions of equation (1).

For the problem $(1) + (2)$ we have:

Theorem 7. *We suppose that:*
(a) the conditions $(C1) - (C3)$ are satisfied;
(b) $Q^n \longrightarrow 0$ as $n \longrightarrow \infty$;
(c) the operator $f(t, \bullet, \bullet, \bullet)$:$\mathbb{R}^{3m} \longrightarrow \mathbb{R}^m$ is increasing,where on $\mathbb{R}^m$ we have the order relation:

$$x \leq y \Longleftrightarrow x_i \leq y_i, \forall i = \overline{1, m}.$$

Let $x^* = (x^{*^1}, ..., x^{*^m})$ *be a solution of* (1) *and*
$y^* = (y_1^*, ..., y_m^*)$ *a solution of the inequality:*

$$-y''(t) \leq f(t, y(t), y(g(t)), y(h(t))),\ \forall t \in [a, b].$$

Then

$$y^*(t) \leq x^*(t),\ \forall t \in [a_1, a] \cup [b, b_1] \implies y^* \leq x^*. \quad (**)$$

Proof. In terms of the operator E_f we have

$$x = E_f(x),\ y \leq E_f(y)$$

and

$$w(y\,|_{[a_1,a]}, y\,|_{[b,b_1]}) \leq w(x\,|_{[a_1,a]}, x\,|_{[b,b_1]}).$$

From (c) we have that the operator E_f^∞ is increasing and

$$y \leq E_f^\infty(y) = E_f^\infty(\widetilde{w}(y)) \leq E_f^\infty(\widetilde{w}(x)) = x \implies y \leq x,$$

where

$$\widetilde{w}(z)(t) := \begin{cases} z(t), & t \in [a_1, a] \\ w(z\,|_{[a_1,a]}, z\,|_{[b,b_1]})(t), & t \in [a, b] \\ z(t), & t \in [b, b_1]. \end{cases},$$

for $z \in C([a_1, b_1], \mathbb{R}^m)$. According to Theorem 5 this lead to the inequality $(**)$. ∎

Now we shall study the monotony of the solution of the problem (1) + (2) with respect to φ, ψ and f. In this aim we need the following abstract result:

Lemma 8. *(see* [10]*) Let* $(X, d, \leq)$ *be an ordered generalized metric space with* $d(x, y) \in \mathbb{R}^m$ *and* $A, B, C : X \longrightarrow X$ *be such that:*
(i) $A \leq B \leq C$;
(ii) the operators A, B, C *are W.P.O.' s;*
(iii) the operator B *is increasing.*
Then $x \leq y \leq z \implies A^\infty(x) \leq B^\infty(y) \leq C^\infty(z)$.

Theorem 9. *Let* $f^i \in C([a,b]\times\mathbb{R}^{3m},\mathbb{R}^m)$, $i=\overline{1,3}$, g *and* h *as in the Theorem 5. We suppose that:*

(a) $f^2(t,\bullet,\bullet,\bullet):\mathbb{R}^{3m}\longrightarrow\mathbb{R}^m$ *is increasing;*

(b) $f^1\le f^2\le f^3$.

Let x^i *be a solution of the system:*

$$\left(-x^i\right)''(t)=f^i(t,x(t),x(g(t)),x(h(t))),\ t\in[a,b],\ i=\overline{1,3}.$$

If $x^1(t)\le x^2(t)\le x^3(t)$, $\forall t\in[a_1,a]\cup[b,b_1]$ *then* $x^1\le x^2\le x^3$.

Proof. From Remark 6 we have that the operators E_{f^1}, E_{f^2} , E_{f^3} are *W.P.O.*' s.
From condition (a) we infer that the operator E_{f^2} is increasing.
From (b) it follows that $E_{f^1}\le\ E_{f^2}\le\ E_{f^3}$.
But $x^1=E^{\infty}_{f^1}(\widetilde{w}(x^1))$, $x^2=E^{\infty}_{f^2}(\widetilde{w}(x^2))$, $x^3=E^{\infty}_{f^3}(\widetilde{w}(x^3))$.
Using Lemma 9 we have that

$$x^1\le x^2\le x^3.$$

Now, let $f^1,f^2\in C([a,b]\times\mathbb{R}^{3m},\mathbb{R}^m)$ and $L_{f^1},L_{f^2}\in M_{m,m}(\mathbb{R}_+)$ as in the condition $(C3)$. Consider $L_f\in M_{m,m}(\mathbb{R}_+)$ with

$$L_f(i,j)=\max\left(L_{f^1}(i,j),L_{f^2}(i,j)\right),\quad \forall i,j=\overline{1,m}.$$

According to the result of Theorem 5, let $x(\bullet;\varphi,\psi,f)$ the notation for the unique solution of (4). We investigate now, the dependence of $x(\bullet;\varphi,\psi,f)$ by φ, ψ, f.
Let $Q_1=3(b-a)M_G\cdot L_{f^1}$, $Q_2=3(b-a)M_G\cdot L_{f^2}$ and $Q=3(b-a)M_G\cdot L_f$ being in $M_{m,m}(\mathbb{R}_+)$. We will denote $Q=\max\{Q_1,Q_2\}$ (only formally). ∎

Theorem 10. *Let* $\alpha_1,\alpha_2\in C([a_1,a],\mathbb{R}^m)$, $\beta_1,\beta_2\in C([b,b_1],\mathbb{R}^m),\varphi_i,\ \psi_i,\ i=\overline{1,2}$, *and* f^1,f^2 *as in the Theorem 5. We suppose that:*

(i) there exists $\eta_1\in\mathbb{R}^m_+$ *such that*

$$\left\|\varphi^1(t)-\varphi^2(t)\right\|_{\mathbb{R}^m}\le\eta_1,\ \forall t\in[a_1,a]$$

and

$$\left\|\psi^1(t)-\psi^2(t)\right\|_{\mathbb{R}^m}\le\eta_1,\ \forall t\in[b,b_1],$$

(ii) there exists $\eta_2 \in \mathbb{R}^m_+$ *such that*

$$\left\| f^1(t,u_1,u_2,u_3) - f^2(t,u_1,u_2,u_3) \right\|_{\mathbb{R}^m} \le \eta_2,\ \forall t \in [a,b],\ \forall u_1,u_2,u_3 \in \mathbb{R}^m.$$

Then

$$\left\| x(\bullet;\varphi^1,\psi^1,f^1) - x(\bullet;\varphi^2,\psi^2,f^2) \right\|_C \le (I_m - Q)^{-1} \cdot (2\eta_1 + M_G\,(b-a)\cdot \eta_2)\,.$$

Proof. Consider the operators B_{φ^1,ψ^1,f^1} and B_{φ^2,ψ^2,f^2} as in the Theorem 5. It follows that

$$\left\| B_{\varphi^i,\psi^i,f^i}(x) - B_{\varphi^i,\psi^i,f^i}(y) \right\|_C \le Q\,\|x-y\|_{\mathbb{C}}\,,\ \forall x,y,\quad i=\overline{1,2}.$$

Moreover, we have

$$\left\| B_{\varphi^1,\psi^1,f^1}(x)(t) - B_{\varphi^2,\psi^2,f^2}(x)(t) \right\|_{\mathbb{R}^m} \le \left\| \varphi^1(a) - \varphi^2(a) \right\|_{\mathbb{R}^m} + \left\| \psi^1(b) - \psi^2(b) \right\|_{\mathbb{R}^m} +$$

$$+ \int_a^b \left\| G(t,s)\cdot \left[f^1(s,x(s),x(g(s)),x(h(s))) - f^2(s,x(s),x(g(s)),x(h(s))) \right] ds \right\|_{\mathbb{R}^m} \le$$

$$\le 2\eta_1 + M_G\,(b-a)\cdot \eta_2,\quad \forall t \in [a,b].$$

Since

$$\left\| x^*(\bullet;\varphi^1,\psi^1,f^1) - x^*(\bullet;\varphi^2,\psi^2,f^2) \right\|_C =$$

$$= \left\| B_{\varphi^1,\psi^1,f^1}(x^*(\bullet;\varphi^1,\psi^1,f^1)) - B_{\varphi^2,\psi^2,f^2}(x^*(\bullet;\varphi^2,\psi^2,f^2)) \right\|_C \le$$

$$\le \left\| B_{\varphi^1,\psi^1,f^1}(x^*(\bullet;\varphi^1,\psi^1,f^1)) - B_{\varphi^1,\psi^1,f^1}(x^*(\bullet;\varphi^2,\psi^2,f^2)) \right\|_C +$$

$$+ \left\| B_{\varphi^1,\psi^1,f^1}(x^*(\bullet;\varphi^2,\psi^2,f^2)) - B_{\varphi^2,\psi^2,f^2}(x^*(\bullet;\varphi^2,\psi^2,f^2)) \right\|_C \le$$

$$\le Q \cdot \left\| x^*(\bullet;\varphi^1,\psi^1,f^1) - x^*(\bullet;\varphi^2,\psi^2,f^2) \right\|_C + 2\eta_1 + M_G\,(b-a)\cdot \eta_2,$$

and because $Q^n \longrightarrow 0$ as $n \longrightarrow \infty$ imply that

$$(I_m - Q)^{-1} \in M_{m,m}(\mathbb{R}_+)\,,$$

we obtain,

$$\left\| x(\bullet;\varphi^1,\psi^1,f^1) - x(\bullet;\varphi^2,\psi^2,f^2) \right\|_C \le (I_m - Q)^{-1} \cdot (2\eta_1 + M_G\,(b-a)\cdot \eta_2)\,.$$

∎

Corollary 11. *Let* φ^i, ψ^i, f^i, $i \in \mathbb{N}$ *and* φ, ψ, f *be such in the Theorem 10. Let* $Q, Q_i \in M_{m,m}(\mathbb{R}_+)$, $i \in \mathbb{N}$ *as above such that exist* $\overline{Q}_i = \max\{Q, Q_i\}$, $\forall i \in \mathbb{N}$. *We suppose that:*

$\overline{Q}_i^n \longrightarrow 0$ *as* $n \longrightarrow \infty$, $\forall i \in \mathbb{N}$ *and*

$\varphi^i \overset{unif}{\longrightarrow} \varphi$ *as* $i \longrightarrow \infty$;

$\psi^i \overset{unif}{\longrightarrow} \psi$ *as* $i \longrightarrow \infty$;

$f^i \overset{unif}{\longrightarrow} f$ *as* $i \longrightarrow \infty$.

Then $x(\bullet; \varphi^i, \psi^i, f^i) \overset{unif}{\longrightarrow} x(\bullet; \varphi, \psi, f)$ *as* $i \longrightarrow \infty$.

4. **Smooth dependence by parameter**

In this section we present the dependence by parameter λ of the solution of problem $(6) + (7)$.

Consider the following boundary value problem with parameter:

$$-x''(t;\lambda) = f(t, x(t;\lambda), x(g(t);\lambda), x(h(t);\lambda);\lambda), t \in [a,b], \lambda \in [c,d] \subset \mathbb{R} \qquad (6)$$

$$\begin{cases} l_1(x(t)) = \alpha(t), & \text{for } t \in [a_1, a] \\ l_2(x(t)) = \beta(t), & \text{for } t \in [b, b_1] \end{cases} \qquad (7)$$

We suppose that:

$(D1)$ $g, h \in C([a,b],[a_1,b_1])$;

$(D2)$ $f = (f_1, f_2, ..., f_m) \in C^1([a,b] \times \mathbb{R}^{3m} \times [c,d], \mathbb{R}^m)$;

$(D3)$ There exist $L_f \in M_{m,m}(\mathbb{R}_+)$such that:

$$\left[\left(\left|\frac{\partial f_i(t,u,v,w;\lambda)}{\partial u_j}\right|\right)_{i,j=\overline{1,m}.}\right]_{M_{mm}(\mathbb{R})} \leq L_f,$$

$$\left[\left(\left|\frac{\partial f_i(t,u,v,w;\lambda)}{\partial v_j}\right|\right)_{i,j=\overline{1,m}.}\right]_{M_{mm}(\mathbb{R})} \leq L_f,$$

$$\left[\left(\left|\frac{\partial f_i(t,u,v,w;\lambda)}{\partial w_j}\right|\right)_{i,j=\overline{1,m}.}\right]_{M_{mm}(\mathbb{R})} \leq L_f,$$

$\forall t \in [a,b]$, $\forall u, v, w \in \mathbb{R}^m$, $i = \overline{1,m}$, $j = \overline{1,m}$, and $\lambda \in [c,d]$, in respect by the componentwise order on $M_{m,m}(\mathbb{R}_+)$.

$(D4)$ $\alpha \in C([a_1,a],\mathbb{R}^m)$, $\beta \in C([b,b_1],\mathbb{R}^m)$.

$(D5)$ For $Q = 3\,(b-a)\,M_G \cdot L_f \in M_{m,m}\,(\mathbb{R}_+)$ we have $Q^n \longrightarrow 0$ as $n \longrightarrow \infty$.

In the above conditions, from Theorem 5 we have that the problem (6) + (7) has a unique solution $x^*(\bullet;\lambda)$, for any $\lambda \in [c,d]$.

Now we prove that $x^*(t;\bullet) \in C^1([c,d]\,;\mathbb{R}^m)$, for all $t \in [a,b]$.

For this we consider the equation

$$-x''(t;\lambda) = f(t, x(t;\lambda), x(g(t);\lambda), x(h(t);\lambda);\lambda),\ t \in [a,b], \lambda \in [c,d] \tag{8}$$

The equation (8) is equivalent with the following system:

$$x(t;\lambda) = \begin{cases} x(t), \quad t \in [a_1, a] \\ w(\varphi,\psi)(t) + \int\limits_a^b G(t,s) f(s, x(s;\lambda), x(g(s);\lambda), x(h(s);\lambda);\lambda) ds, \\ \qquad t \in [a,b], \lambda \in [c,d] \\ x(t), \quad t \in [b, b_1] \end{cases} \tag{9}$$

where

$$w(\alpha,\beta)(t) = \frac{t-a}{b-a}\psi\,(b) + \frac{b-t}{b-a}\varphi\,(a)\,,\ t \in [a,b].$$

Let $X := C([a_1,b_1] \times [c,d]\,, \mathbb{R}^m)$ with the Chebyshev norm

$$\|x\|_C := \begin{pmatrix} \|x_1\|_C \\ \vdots \\ \|x_m\|_C \end{pmatrix} \in \mathbb{R}^m_+.$$

Now we consider the operator

$$B : C([a_1,b_1] \times [c,d]\,, \mathbb{R}^m) \longrightarrow C([a_1,b_1] \times [c,d]\,, \mathbb{R}^m)$$

where

$$B(x)(t;\lambda) := second\ part\ of\ (9)$$

Analogous as in Theorem 5 it probes that in the conditions $(D1)-(D5)$ the operator B is P.O., since

$$\|B(y) - B(z)\|_C \le Q \cdot \|y - z\|_C\,.$$

This implies that B has a unique fixed point x^*.

We suppose that there exists $\frac{\partial x^*}{\partial \lambda}$.

From relation (9) and condition $(D3)$ we have:

$$\frac{\partial x^*(t;\lambda)}{\partial \lambda} = \begin{cases} 0, \quad \text{for } t \in [a_1, a] \\ \int\limits_a^b G(t,s)\cdot \left(\frac{\partial f_i(s, x^*(s;\lambda), x^*(g(s);\lambda), x^*(h(s);\lambda);\lambda)}{\partial u_j} \right)_{i,j} \cdot \frac{\partial x^*(s;\lambda)}{\partial \lambda} ds \\ + \int\limits_a^b G(t,s)\cdot \left(\frac{\partial f_i(s, x^*(s;\lambda), x^*(g(s);\lambda), x^*(h(s);\lambda);\lambda)}{\partial v_j} \right)_{i,j} \cdot \frac{\partial x^*(g(s);\lambda)}{\partial \lambda} ds \\ + \int\limits_a^b G(t,s)\cdot \left(\frac{\partial f_i(s, x^*(s;\lambda), x^*(g(s);\lambda), x^*(h(s);\lambda);\lambda)}{\partial w_j} \right)_{i,j} \cdot \frac{\partial x^*(h(s);\lambda)}{\partial \lambda} ds \\ + \int\limits_a^b G(t,s)\cdot \left(\frac{\partial f_i(s, x^*(s;\lambda), x^*(g(s);\lambda), x^*(h(s);\lambda);\lambda)}{\partial \lambda} \right)_{i,j} ds, \quad \text{for } t \in [a,b], \\ \qquad \lambda \in [c,d] \\ 0, \quad \text{for } t \in [b, b_1] \end{cases}$$

This relation suggest us to consider the following operator

$$C : X \times X \longrightarrow X$$

$$C(x,y)(t;\lambda) := \int\limits_a^b G(t,s)\cdot \left(\frac{\partial f_i(s, x(s;\lambda), x(g(s);\lambda), x(h(s);\lambda);\lambda)}{\partial u_j} \right)_{i,j} \cdot y(s;\lambda) ds +$$

$$+ \int\limits_a^b G(t,s)\cdot \left(\frac{\partial f_i(s, x(s;\lambda), x(g(s);\lambda), x(h(s);\lambda);\lambda)}{\partial v_j} \right)_{i,j} \cdot y(g(s);\lambda) ds +$$

$$+ \int\limits_a^b G(t,s)\cdot \left(\frac{\partial f_i(s, x(s;\lambda), x(g(s);\lambda), x(h(s);\lambda);\lambda)}{\partial w_j} \right)_{i,j} \cdot y(h(s);\lambda) ds +$$

$$+ \int\limits_a^b G(t,s)\cdot \left(\frac{\partial f_i(s, x(s;\lambda), x(g(s);\lambda), x(h(s);\lambda);\lambda)}{\partial \lambda} \right)_{i,j} ds$$

$$\forall t \in [a,b], \ \lambda \in [c,d] \,.$$

$$C(x,y)(t;\lambda) = 0 \quad for \quad t \in [a_1,a] \cup [b,b_1], \ \lambda \in [c,d]\,.$$

In this way we have the triangular operator

$$A\colon X \times X \longrightarrow X \times X,$$

$$A(x,y) = (B(x),C(x,y))$$

where B is a Picard operator and

$$C(x^*,\bullet) : X \longrightarrow X$$

is $Q - contraction.$

Indeed, we have

$$\left\|C(x^*,u)(t;\lambda) - C(x^*,v)(t;\lambda)\right\|_{\mathbb{R}^m} \le Q \cdot \|u-v\|_C\,, \quad \forall t \in [a,b], \forall \lambda \in [c,d]\,,$$

which implies that

$$\left\|C(x^*,u) - C(x^*,v)\right\|_C \le Q \cdot \|u-v\|_C\,, \quad \forall u,v \in X.$$

Since $Q^n \longrightarrow 0$ as $n \longrightarrow \infty$, applying the Fiber Generalized Contraction Theorem (see [16]), follows that A is P.O. and has a unique fixed point $(x^*,y^*) \in X \times X$.

So the sequences

$$\left(x^{n+1},y^{n+1}\right) = (B(x^n),C(x^n,y^n))\,, \ \ n \in \mathbb{N}$$

converges uniformly (with respect to $t \in [a_1,b_1], \lambda \in [c,d]$) to (x^*,y^*) for any $x^0,y^0 \in C([a_1,b_1] \times [c,d]\,,\mathbb{R}^m)$.

If we take $x^0 = 0$ and $y^0 = \frac{\partial x^0}{\partial \lambda} = 0$ then $y^1 = \frac{\partial x^1}{\partial \lambda}$.

By induction we prove that

$y^n = \frac{\partial x^n}{\partial \lambda}, \forall n \in \mathbb{N}$.

Thus $x^n \overset{unif}{\longrightarrow} x^*$, as $n \longrightarrow \infty$

and $\frac{\partial x^n}{\partial \lambda} \overset{unif}{\longrightarrow} y^*$, as $n \longrightarrow \infty$.

These imply that there exists $\frac{\partial x^*}{\partial \lambda}$ and $\frac{\partial x^*}{\partial \lambda} = y^*$.

Theorem 12. *We consider the problem* $(6) + (7)$ *in the conditions* $(D1) - (D5)$*. Then:*

(i) the problem $(6) + (7)$ *has a unique solution*

$$x^* = (x_1^*, ..., x_m^*) \in C([a_1, b_1] \times [c, d], \mathbb{R}^m);$$

(ii) $x^*(t, \bullet) \in C^1([c, d], \mathbb{R}^m), \ \forall t \in [a_1, b_1]$.

Remark 13. *If we consider* $l_1(x) = x$, $l_2(x) = x$, $\alpha = \varphi$ *and* $\beta = \psi$*, we obtain the vectorial variant of the boundary value problem from* [12]*. In this case,*

$$G = \begin{pmatrix} g & 0 & 0 & \cdots & 0 & 0 \\ 0 & g & 0 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & g \end{pmatrix} \in M_{m,m}(\mathbb{R}_+),$$

where g *is the Green function of the problem*

$$\begin{cases} -x'' = \chi \\ x(a) = 0, \ x(b) = 0. \end{cases}$$

If $l_1(x) = \alpha_{11}x + \alpha_{12}x'$, $l_2(x) = \alpha_{21}x + \alpha_{22}x'$, $\alpha(t) = \alpha \in \mathbb{R}^m$, $\forall t \in [a_1, a]$, $\beta(t) = \beta \in \mathbb{R}^m$, $\forall t \in [b, b_1]$, *we obtain the vectorial variant of the boundary value problem from* [3] *and* [4]*. Here,*

$$G = \begin{pmatrix} g & \cdots & 0 \\ \cdots & \cdots & \cdots \\ 0 & \cdots & g \end{pmatrix},$$

where

$$g(t,s) = \begin{cases} \frac{\alpha_{11}\alpha_{21}}{\alpha_{11}\alpha_{22} - \alpha_{12}\alpha_{21} + (b-a)\alpha_{11}\alpha_{21}} \left(t - a - \frac{\alpha_{12}}{\alpha_{11}}\right)\left(b - s + \frac{\alpha_{22}}{\alpha_{21}}\right), & a \le t < s \le b \\ \frac{\alpha_{11}\alpha_{21}}{\alpha_{11}\alpha_{22} - \alpha_{12}\alpha_{21} + (b-a)\alpha_{11}\alpha_{21}} \left(s - a - \frac{\alpha_{12}}{\alpha_{11}}\right)\left(b - t + \frac{\alpha_{22}}{\alpha_{21}}\right), & a \le s < t \le b. \end{cases}$$

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ON SOME NUMERIC METHODS TO DETERMINATE THE GUARANTEED OPTIMAL VALUES

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Abstract. Let's take two sets $A, B \neq \emptyset$ and $f : A \times B \to R$. We consider the quantities:

$$a_1 = \sup_{x \in A} \inf_{y \in B} f(x, y) \quad (1)$$

$$a_2 = \inf_{y \in B} \sup_{x \in A} f(x, y) \quad (2)$$

between which there always exist the inequality $a_1 \leq a_2$ [1]. We assume the problem of determination of the conditions in which bounds from (1) and (2) can be effectively reached as well as methods of determining of these bounds. The first part of the paper deals with results regarding the existence and the properties of guaranteed optimal strategies (the case of simple strategies with no informational exchange, the case of simple strategies with informational exchange and the case of mixed strategies). The second part of the paper presents some methods to determine optimal values. Practically, there are penalty methods which solve the above mentioned problem using a more general approach than other methods known in specialized literature.

1. The existence and the properties of the optimal and guaranteed strategies

1.1. Simple strategies in the case the exchange of information is not permitted

We start from the structure of a non-cooperative game $J_1 = (F, D_1, D_2)$ with two deciders in which:

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1) D_1, D_2 represent the sets of the decision of the two deciders;

2) F stands for the function of gain.

Definition 1.1. Quantities:

$$V_1 = \sup_{d_1 \in D_1} \inf_{d_2 \in D_2} F(d_1, d_2) \tag{1.1}$$

$$V_2 = \inf_{d_2 \in D_2} \sup_{d_1 \in D_1} F(d_1, d_2) \tag{1.2}$$

represent the optimal guaranteed values of the two deciders.

In the case the bounds from (1.1), (1.2) are reached, then the strategies for which these bounds are reached are named optimal guaranteed strategies.

Remark 1.1. The significance of the optimal and guaranteed strategies and values in the context of the theory of non-cooperative games is the following:

1) if (d_1^2, d_2^1) is the optimal guaranteed strategy of the first decider, then the quantity:

$$V_1 = \max_{d_1 \in D_1} \min_{d_2 \in D_2} F(d_1, d_2) = F(d_1^1, d_2^1)$$

represents the maximum sure gain which the first decider can obtain;

2) if (d_1^2, d_2^2) is the optimal guaranteed strategy of the second decider, then the quantity:

$$V_2 = \min_{d_2 \in D_2} \max_{d_1 \in D_1} F(d_1, d_2) = F(d_1^2, d_2^2)$$

represents the maximum loss the second decider could wait for.

If we consider the game $J_{2n} = (F_1, D_1, D_2)$ at which take part $2n$ coalized deciders in the coalitions $C_1 = \{1, 3, \ldots, 2n-1\}$, $C_2 = \{2, 4, \ldots, 2n\}$ with opposite interest in which:

D_i represents the set of strategies of the decider $i \in C_1$

$$D_1 = \prod_{i=1}^{n} D_i$$

$\overline{D}_j$ represents the set of strategies of decider $j \in C_2$

$$D_2 = \prod_{j=1}^{n} \overline{D}_j$$

F_1 is the functional of gain, then we can define in the same way with the game J_1 the optimal guaranteed values for C_1 and C_2:

$$V_1 = \sup_{d_1\in D_1} \inf_{\overline{d}_1\in\overline{D}_1} \dots \sup_{d_n\in D_n} \inf_{\overline{d}_n\in\overline{D}_n} F(d_1,\overline{d}_1,\dots,d_n,\overline{d}_n)$$

$$\stackrel{not}{=} \left[\sup_{d_i\in D_i} \inf_{\overline{d}_i\in\overline{D}_i}\right]^{(n)} F(d_1,\overline{d}_1,\dots,d_n,\overline{d}_n)$$

$$V_2 = \inf_{\overline{d}_1\in\overline{D}_1} \sup_{d_1\in D_1} \dots \inf_{\overline{d}_n\in\overline{D}_n} \sup_{d_n\in D_n} F(d_1,\overline{d}_1,\dots,d_n,\overline{d}_n)$$

$$\stackrel{not}{=} \left[\inf_{\overline{d}_i\in\overline{D}_i} \sup_{d_i\in D_i}\right]^{(n)} F(d_1,\overline{d}_1,\dots,d_n,\overline{d}_n)$$

as well as the optimal guaranteed strategies.

We assume the problem of determining conditions in which there exists optimal guaranteed strategies for J_1 and J_{2n} $(n > 1)$.

Theorem 1.1. [4] *If the conditions hold true:*

1) $(D_1,\rho_1),(D_2,\rho_2)$ *there are compact metrical spaces;*

2) F *is continue in both arguments*

then there exist optimal guaranteed strategies for J_1*.*

Remark 1.2. The theorem 1.1 which applies to J_1 can be generalized to J_n, $n > 1$.

In case that F_1 is continue and the sets $D_i,\overline{D}_i$, $i=\overline{1,n}$, are compact metric spaces, then for J_{2n} we can prove the existence of the optimal guaranteed strategies.

An extremely important problem which appears in real decisional situations is that when from any reason the function of efficiency F is replaced by another function of efficiency F_1 (easily expressed analytically or with more properties).

It arises the problem of deviation from the optimal guaranteed value, knowing that two functions of efficiency verify the requisite $|F(d_1,d_2)-F_1(d_1,d_2)|<\varepsilon$, $\varepsilon>0$ small enough.

Theorem 1.2. [3] *If* $|F(d_1,d_2)-F_1(d_1,d_2)|\le\varepsilon$ *then the inequality holds true:*

$$\left|\sup_{d_1\in D_1}\inf_{d_2\in D_2} F(d_1,d_2) - \sup_{d_1\in D_1}\inf_{d_2\in D_2} F_1(d_1,d_2)\right| \le \varepsilon.$$

1.2. Simple strategies in the case when the informational exchange is permitted

The study of the decisional processes in which the informational transfer is allowed represents relatively a recent problem that dealt with the non-cooperative games with the exchange of information (Fedorov [3], Mitran [4]), differential games with memory, decisional processes and systems of classification.

In case the informational transfer is admitted the non-cooperative games present properties connected with the optimal guaranteed solution different from the properties of the games in which the informational exchange is not permitted.

For the beginning we assume a game with two deciders $J_1(D_1, D_2, F)$ where (D_1, d_{D_1}), (D_2, d_{D_2}) are metric spaces (D_1, D_2 represents the sets of strategies of two deciders).

We construct Haussdorff's metric:

$$d : \mathcal{P}(D_2) \times \mathcal{P}(D_2) \to R, \quad d(A, B) = \max\{\rho(A, B); \rho(B, A)\},\ \forall\ A, B \subset D_2$$

where $\rho(A, B) = \sup\limits_{x \in A} \inf\limits_{y \in B} d_{D_2}(x, y)$.

Let $T : D_1 \to \mathcal{P}(D_2)$ be the informational (multivocal) application of the first decider [3], [4].

Definition 1.2. The set $Td_1 \subseteq D_2$ is called the informational set of decider 1 corresponding to the strategy $d_1 \in D_1$ and it represents the set of the strategies that decider 2 can take if decider 1 adopted the strategy $d_1 \in D_1$.

Definition 1.3. The multivocal application T is said:

1) s.c.s. in $D_1^0 \in D_1$, if $\lim\limits_{d_1 \to d_1^0} \rho(Td_1, Td_1^0) = 0$;

2) s.c.i. in $d_1^0 \in D_1$, if from conditions:

$$\lim_n d_1^n = d_1^0, \quad d_2^0 \in Td_1^0$$

results that there exists $(d_2^n :\ d_2^n \in Td_1^n)_n$ so that $d_2^0 = \lim\limits_n d_2^0$;

3) closed in $d_1^0 \in D_1$, if from conditions:

$$\lim_n d_1^n = d_1^0, \quad \lim_n d_2^n = d_2^0, \quad d_2^n \in Td_1^n$$

results $d_2^0 \in Td_1^0$;

4) continue in $d_1^0 \in D_1$, if $\lim\limits_{d_1 \to d_1^0} d(Td_1, Td_1^0) = 0$.

Remark 1.3. The conditions of s.c.i. and s.c.s. according to definition 1.2 in $d_1^0 \in D_1$ for the multivocal application T is generally not for assuring the continuity in d_1^0; but if D_2 is sufficient compact, it is assured. As it follows through the content of §1.1 and §1.2 we suppose that D_1 and D_2 are compact.

Consider the functional:

$$R : D_1 \to R, \quad R(d_1) = \inf_{d_2 \in Td_1} F(d_1, d_2).$$

Theorem 1.3. *If F is s.c.s. on $D_1 \times D_2$, T is continuous in $d_1^0 \in D_1$, then functional R is s.c.s. in $d_1^0 \in D_1$.*

Proof. Take any $\varepsilon > 0$ and small enough.

As $R(d_1^0) = \inf\limits_{d_2 \in Td_1^0} F(d_1^0, d_2)$ there exists $d_2^0 \in Td_1^0$ so that the inequality:

$$F(d_1^0, d_2^0) \leq R(d_1^0) + \frac{\varepsilon}{2} \tag{1.3}$$

holds true.

From condition of s.c.s. of F in (d_1^0, d_2^0) for the chosen $\varepsilon > 0$ it will exist $\delta > 0$ so that:

$$F(d_1^0, d_2^0) \geq F(d_1, d_2) - \frac{\varepsilon}{2}, \ \forall \ (d_1, d_2) \in D_1 \times D_2, \quad d_{D_1 \times D_2}((d_1, d_2), (d_1^0, d_2^0)) \leq \delta$$

From continuity condition of T occurs that there is a $\gamma > 0$ so that $d_{D_2}(Td_1, Td_1^0) \leq \delta$ for any $d_1 \in D_1$ with the property $d_{D_1}(d_1, d_1^0) \leq \gamma$.

Take $V_{d_1^0} = \{d_1 \in D_1 \mid d_{D_1}(d_1, d_1^0) \leq \min\{\delta, \gamma\}\}$. Regardless $d_1 \in V_{d_1^0}$, there is a $d_2 \in Td_1$ so that $d_{D_2}(d_2, d_2^0) \leq \delta$.

So for any $d_1 \in V_{d_1^0}$ there is a $d_2 \in Td_1$ so that:

$$R(d_1^0) + \frac{\varepsilon}{2} \geq F(d_1^0, d_2^0) \geq F(d_1, d_2) - \frac{\varepsilon}{2} \tag{1.4}$$

as

$$R(d_1) \leq F(d_1, \widetilde{d}_2), \ \forall \ \widetilde{d}_2 \in Td_1 \tag{1.5}$$

From (1.4) and (1.5) the inequality occurs:

$$R(d_1^0) \geq R(d_1) - \varepsilon, \ \forall \ d_1 \in V_{d_1^0}$$

which means that R is s.c.s. in d_1^0.

Consequence 1.3.1. *If F is s.c.s. on $D_1 \times D_2$, T is continuous on D_1 then there is a $d_1^* \in D_1$ so that:*

$$R(d_1^*) = \max_{d_1 \in D_1} \inf_{d_2 \in T d_1} F(d_1, d_2).$$

The proof is evident considering theorem 1.1 and the fact that any functional s.c.s. on a compact touches its superior bound on that compact.

Remark 1.4. We assume F continue. As the functional of gain of the second decider is $G = -F$, if it is defined the functional $\widetilde{f} : D_1 \times D_2 \to R$:

$$\widetilde{f}(d_1, d_2) = G(d_1, d_2) - \max_{d_2 \in D_2} G(d_1, d_2)$$

then the best guaranteed result of the first decider will be [3], [4], [5]:

$$\sup_{d_1 \in D_1} \min_{d_2 \in T d_1} F(d_1, d_2)$$

if

$$T d_1 = \{d_2 \in D_2 \mid \widetilde{f}(d_1, d_2) = 0\} \neq \emptyset \tag{1.6}$$

The form in which the informational set of the first decider occurs in (1.6) leads to consider the informational application T to be defined as follows:

$$T d_1 = \{d_2 \in D_2 \mid f(d_1, d_2) \geq 0\} \tag{1.7}$$

where $f : D_1 \times D_2 \to R$ is a known functional.

We note $T^0 d_1 = \{d_2 \in D_2 \mid f(d_1, d_2) > 0\}$.

Theorem 1.4. [4] *The following results occur:*

1) if F is continuous and $\overline{T^0 d_1} = T d_1$, $\forall \ d_1 \in D_1$, then T is continuous on D_1;

2) if in addition to 1) we assume that F is continuous on $D_1 \times D_2$ and Td_1 is compact $\forall\ d_1 \in D_1$, then R is continuous, so there is a $d_1^* \in D_1$ thus:

$$R(d_1^*) = \max_{d_1 \in D_1} \min_{d_2 \in Td_1} F(d_1, d_2).$$

Remark 1.5. The conditions of continuity of T proved at point 1) acted essentially only to assure the condition of closeness and the equalities:

$$F(d_1^0, \overline{d}_2^0) = \min_{d_2 \in Td_1^0} F(d_1^0, d_2) = R(d_1^0) \tag{1.8}$$

we have:

$$\lim_n R(d_1^n) = \lim_n \min_{d_2 \in Td_1^n} F(d_1^n, d_2) = \lim_n F(d_1^n, d_2^n) = F(d_1^0, d_2^0) = R(d_1^0)$$

and so the theorem is proved.

Remark 1.6. The condition of compactity of D_1, D_2 and the continuity of F does not assure in the case of the games with an informational exchange, the contingence of the optimal guaranteed value:

$$V = \max_{d_1 \in D_1} \min_{d_2 \in Td_1} F(d_1, d_2)$$

(likewise in the case of the games without an informational exchange). But if F, T, f are continuous, $Td_1 \neq \emptyset$, $\forall\ d_1 \in D_1$, then the contingence of quantity V is realizable [3].

Let's notice that in comparison with conditions of theorem 1.4, these conditions are modified (in the sense that have been weaker and others are harder).

The performances obtained in the case of game J_1 can be extended to a game

$$J_n = \left(\prod_{i=1}^{n} D_i \times \overline{D}_i, F_1 \right)$$

in which the deciders are coalized into two coalitions $C_1 = \{1, 3, \ldots, 2n-1\}$, $C_2 = \{2, 4, \ldots, 2n\}$ with opposite interests (we marked with F_1 the functional of gain and D_i, $\overline{D}_i$ the set of the strategies of decider i, $i = \overline{1, n}$).

In this case the best guaranteed result of the first decider is given by the quantity:

$$\sup_{d_1\in D_1}\inf_{\overline{d}_1\in\overline{D}_1}\dots\sup_{d_n\in D_n}\inf_{\overline{d}_n\in\overline{D}_n}F(d_1,\overline{d}_1,\dots,d_n,\overline{d}_n)$$
$$=\left[\sup_{d_i\in D_i}\inf_{\overline{d}_i\in\overline{D}_i}\right]^{(n)}F(d_1,\overline{d}_1,\dots,d_n,\overline{d}_n)$$

Let's assume that in game J_n the informational exchange is permitted; the informational sets of those $2n$ deciders are A_i, $i=\overline{1,n}$ for the deciders from C_1 and B_i, $i=\overline{1,n}$ for the deciders from C_2, being constructed by means of the informational applications T_i (for the deciders from C_1), $\overline{T}_i$ (from the deciders from C_2), $i=\overline{1,n}$:

$$A_i=T_i(d_1,\overline{d}_1,\dots,d_{i-1},\overline{d}_{i-1})$$

$$=\{d_i\in D_i\mid f_i(d_1,\overline{d}_1,\dots,d_{i-1},\overline{d}_{i-1},d_i)\geq 0\},\quad i=\overline{2,n},\ A_1=D_1$$

$$B_i=\overline{T}_i(d_1,\overline{d}_1,\dots,d_{i-1},\overline{d}_{i-1},d_i)=\{\overline{d}_i\in\overline{D}_i\mid \overline{f}_i(d_1,\overline{d}_1,\dots,d_i,\overline{d}_i)\geq 0\},\quad i=\overline{1,n}$$

$$f_i:\prod_{j=1}^{i=1}(D_j\times\overline{D}_j)\times D_i\to R,\quad i=\overline{2,n},\quad \overline{f}_i:\prod_{j=1}^{i}D_j\times\overline{D}_j\to R,\quad i=\overline{1,n}$$

being known.

The best guaranteed result of the decider 1 in this case is given by the quantity:

$$V_n=\sup_{d_i\in A_i}\inf_{\overline{d}_i\in B_i}F(d_1,\overline{d}_1,\dots,d_n,\overline{d}_n)\tag{1.9}$$

Let's notice that in the case which D_i, $\overline{D}_i$, $i=\overline{1,n}$, are compact metric spaces, $F,T_i,\overline{T}_i,f_i,\overline{f}_i$, $i=\overline{1,n}$ are continuous, $A_i\neq\emptyset$, $i=\overline{2,n}$, $B_i\neq\emptyset$, $i=\overline{1,n}$, then the superior and inferior bounds from (1.9) can be reached (remark 1.6). That's why there naturally arises the problem of establishing of some methods for determining of the optimal guaranteed values of the first decider in games J_n and J_1.

1.3. Mixed Strategies

The aim of this paragraph is that of presenting the generalization of the notion of simple strategy through introduction of the notion of mixed strategy.

Besides the definitions, the interpretations and the immediate properties there will occur properties connected to the existence of the non-null components.

For the beginning we assume the game $J_1 = (F, D_1, D_2)$ introduced in 1.1 in which the sets of strategies D_1 and D_2 are finite:

$$D_1 = \{d_1^1, d_1^2, \ldots, d_1^m\}, \quad D_2 = \{d_2^1, d_2^2, \ldots, d_2^n\}$$

Definition 1.4. The distribution of probabilities $p = (p_1, p_2, \ldots, p_m)$, $p_1 \geq 0$, $i = \overline{1, m}$. $\sum_{i=1}^{m} p_i = 1$, is called a mixed strategy for the decider 1, if he uses the strategy d_1^i with the probability p_i, $i = \overline{1, m}$.

The distribution of the probabilities $q = (q_1, q_2, \ldots, q_n)$, $q_j \geq 0$, $j = \overline{1, n}$, $\sum_{j=1}^{n} q_j = 1$, is named mixed strategy for the decider 2, if he uses the strategy d_2^j with the probability q_j, $j = \overline{1, n}$.

We mark $\overline{D}_1, \overline{D}_2$ the sets of mixed strategies of the two deciders.

If $a_{ij} = F(d_1^j, d_2^j)$, $i = \overline{1, m}$, $j = \overline{1, n}$, the following quantities are defined:

$$\overline{V}_1 = \max_{p \in \overline{D}_1} \min_{q \in \overline{D}_2} \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} p_i q_j \tag{1.10}$$

$$\overline{V}_2 = \min_{q \in \overline{D}_2} \max_{p \in \overline{D}_1} \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} p_i q_j \tag{1.11}$$

between which we can easily show that there always exists the relationship $\overline{V}_1 \leq \overline{V}_2$.

Definition 1.5. Quantities $\overline{V}_1, \overline{V}_2$ are called the inferior value, respectively, the superior value of the game J_1.

If $\overline{V}_1 = \overline{V}_2$, then the common value V of these quantities is value of the game.

The resolution of a matrixed game with the matrix of paying $(a_{ij})_{\substack{i=\overline{1,n} \\ j=\overline{1,m}}}$ is equivalent to the resolution of the couple of the dual problems of linear optimization:

$$1) \begin{cases} \min \sum_i x_i \\ x_i \geq 0, \ \sum_i a_{ij} x_i \geq 1, \ j = \overline{1, m} \end{cases}$$

$$2) \begin{cases} \max \sum_j y_j \\ y_j \geq 0, \ \sum_j a_{ij} y_j \leq 1, \ i = \overline{1, n} \end{cases}$$

The value V of the game is

$$V = \frac{1}{\sum_i x_i^0} = \frac{1}{\sum_j y_j^0}$$

and the optimal strategies p_i^0, q_j^0 are given by $p_i^0 = Vx_i^0$, $q_j^0 = VY_j^0$.

Remark 1.6. The use of the mixed strategies is justified in the case if the decisional process is repeatable. In the case of the repetition of the decisional process, by $n$times, then, according to a version of the law of the large numbers with a probability next to 1, the gain which will be obtained is approximate nV.

Remark 1.8. If decider 1 has got the possibility of getting information about decider 2, and this information doesn't lead to the restriction of the set of strategies D_1, then it isn't favorable (for decider 1) the use of the mixed strategies [4].

The notion of mixed strategy for the finite games can be extended to the infinite games, as well.

Let D_1, D_2 be compact sets in the Euclidean spaces E^n, respectively E^m.

We call mixed strategies for the two deciders any measures of probability ν and μ defined on D_1, respectively D_2.

The optimal guaranteed values of the two deciders are:

$$V_1 = \sup_\nu \inf_\mu \int_{D_1} \int_{D_2} f(d_1, d_2) d\mu d\nu$$

$$V_2 = \inf_\mu \sup_\nu \int_{D_1} \int_{D_2} f(d_1, d_2) d\mu d\nu$$

Between these two quantities there always exists the relationship $V_1 \leq V_2$. In the case of achieving the equality we shall say that the game has got a saddle point and that the value $V = V_1 = V_2$ is called the value of the game [2], [3].

2. **Determination of the guaranteed optimum solution**

2.1. Simple strategies in the case when the exchange of information is not permitted

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ON THE APPROXIMATION BY ASSOCIATED GBS OPERATORS OF EXPONENTIAL TYPE

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Abstract. The sequence of GBS operators of exponential type is constructed and some approximation properties of this sequence are established. By particularization, we obtain statements verified by the GBS operators of Bernstein, Mirakjan-Favard-Szász, Baskakov, Ismail-May, Post-Widder and Gauss-Weierstrass.

1. **Introduction**

The aim of the present note is to demonstrate a general formula for the approximation of a bivariate function by associated GBS operators of exponential type.

In Section 1 we recall some notions and results which we will use in this paper. In Section 2 we recall the definition of exponential operators. The last section is devoted to estimating the approximation of bivariate functions by associated GBS operators of exponential type.

The term of "GBS operator" (Generalized Boolean Sum Operator) was introduced by C. Badea and C. Cottin (see [5]).

In the end, by particularization in Theorem 3.2 and Theorem 3.3 for some known operators, we give some applications.

In the following, let X and Y be real intervals.

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A function $f : X \times Y \to \mathbb{R}$ is called B-continuous in $(x_0, y_0) \in X \times Y$ iff $\lim\limits_{(x,y)\to(x_0,y_0)} \Delta f[(x,y),(x_0,y_0)] = 0$, where $\Delta f[(x,y)(x_0,y_0)] = f(x,y) - f(x_0,y) - f(x,y_0) + f(x_0,y_0)$ denotes a so-called mixed difference of f.

A function $f : X \times Y \to \mathbb{R}$ is called B-continuous on $X \times Y$ iff is B-continuous in any point of $X \times Y$.

A function $f : X \times Y \to \mathbb{R}$ is called B-differentiable in $(x_0, y_0) \in X \times Y$ iff it exists and if the limit is finite $\lim\limits_{(x,y)\to(x_0,y_0)} \dfrac{\Delta f[(x,y),(x_0,y_0)]}{(x-x_0)(y-y_0)}$.
This limit is named the B-differential of f in the point (x_0, y_0) and is denoted by $D_B f(x_0, y_0)$.

A function $f : X \times Y \to \mathbb{R}$ is called B-differentiable on $X \times Y$ iff is B-differentiable in any point of $X \times Y$.

The definition of B-continuity and B-differentiability was introduced by K. Bögel in the papers [7] and [8].

The function $f : X \times Y \to \mathbb{R}$ is B-bounded on $X \times Y$ iff there exists $k > 0$ such that $|\Delta f[(x,y),(s,t)]| \le k$ for any $(x,y),(s,t) \in X \times Y$.

We shall use the following function sets: $B(X \times Y) = \{f | f : X \times Y \to \mathbb{R},\ f$ bounded on $X \times Y\}$ with the usual sup-norm $\|\cdot\|_\infty$, $B_b(X \times Y) = \{f | f : X \times Y \to \mathbb{R}$, f B-bounded on $X \times Y\}$, $C_b(X \times Y) = \{f | f : X \times Y \to \mathbb{R},\ f$ B-continuous on $X \times Y\}$ and $D_b(X \times Y) = \{f | f : X \times Y \to \mathbb{R},\ f$ B-differentiable on $X \times Y\}$.

Let $f \in B_b(X \times Y)$. The function $\omega_{\text{mixed}}(f; \cdot, \cdot) : [0,\infty) \times [0,\infty) \to \mathbb{R}$ defined by $\omega_{\text{mixed}}(f; \delta_1, \delta_2) = \sup\{|\Delta f[(x,y),(s,t)]| : |x-s| \le \delta_1, |y-t| \le \delta_2\}$ for any $(\delta_1, \delta_2) \in [0,\infty) \times [0,\infty)$ is called the mixed modulus of smoothness.

Theorem 1.1. *Let X and Y be compact real intervals and $f \in B_b(X \times Y)$. Then $\lim\limits_{\delta_1,\delta_2\to 0} \omega_{mixed}(f; \delta_1, \delta_2) = 0$ iff $f \in C_b(X \times Y)$.*

For other information, see the following papers: [2] and [4].

Theorem 1.2. *Let $L : C_b(X \times Y) \to B(X \times Y)$ be a linear positive operator and $UL : C_b(X \times Y) \to B(X \times Y)$ the associated GBS operator defined for any function*

$f \in C_b(X \times Y)$ and any $(x,y) \in X \times Y$ by

$$(ULf)(x,y) = \left(L(f(\cdot,y) + f(x,*) - f(\cdot,*))\right)(x,y), \tag{1.1}$$

where "$\cdot$" and "$*$" stand for the first and second variable. Assuming that the operator L has the property

$$\left(L(\cdot - x)^{2i}(* - y)^{2j}\right)(x,y) = \left(L(\cdot - x)^{2i}\right)(x,y)\left(L(* - y)^{2j}\right)(x,y) \tag{1.2}$$

for any $(x,y) \in X \times Y$ and any $i,j \in \{1,2\}$. The following statements are true.

(i) For any function $f \in C_b(X \times Y)$, any $(x,y) \in X \times Y$ and any $\delta_1, \delta_2 > 0$, we have

$$\begin{aligned} |f(x,y) - (ULf)(x,y)| \le |f(x,y)||1 - (Le_{00})(x,y)| + \Big[(Le_{00})(x,y)+ \\ + \delta_1^{-1}\sqrt{(L(\cdot - x)^2)(x,y)} + \delta_2^{-1}\sqrt{(L(* - y)^2)(x,y)}+ \\ + \delta_1^{-1}\delta_2^{-1}\sqrt{(L(\cdot - x)^2)(x,y)(L(* - y)^2)(x,y)}\Big]\omega_{mixed}(f;\delta_1,\delta_2). \end{aligned} \tag{1.3}$$

(ii) For any $f \in D_b(X \times Y)$ with $D_B f \in B(X \times Y)$, any $(x,y) \in X \times Y$ and any $\delta_1, \delta_2 > 0$, we have

$$\begin{aligned} |f(x,y) - (ULf)(x,y)| \le |f(x,y)||1 - (Le_{00})(x,y)|+ \\ + 3\|D_B f\|_\infty \sqrt{(L(\cdot - x)^2)(x,y)(L(* - y)^2)(x,y)}+ \\ + \Big[\sqrt{(L(\cdot - x)^2)(x,y)(L(* - y)^2)(x,y)}+ \\ + \delta_1^{-1}\sqrt{(L(\cdot - x)^4)(x,y)(L(* - y)^2)(x,y)}+ \\ + \delta_2^{-1}\sqrt{(L(\cdot - x)^2)(x,y)(L(* - y)^4)(x,y)}+ \\ + \delta_1^{-1}\delta_2^{-1}(L(\cdot - x)^2)(x,y)(L(* - y)^2)(x,y)\Big]\omega_{mixed}(D_B f;\delta_1,\delta_2). \end{aligned} \tag{1.4}$$

The inequality of Corollary 5 from [5], in the condition of (1.2), becomes the (1.3) inequality. The (1.4) inequality is demonstrated in [17].

2. **Preliminaries**

We set $\mathbb{N} = \{1, 2, \dots\}$. Let a and b such that $-\infty \le a < b \le \infty$. In this paper we consider the notations

$$I(a,b) = \begin{cases} [a,b], & \text{if} \quad a,b \in \mathbb{R} \\ (-\infty, b], & \text{if} \quad a = -\infty, b \in \mathbb{R} \\ [a,\infty)\,, & \text{if} \quad a \in \mathbb{R}, b = \infty \\ (-\infty,\infty) = \mathbb{R}, & \text{if} \quad a = -\infty, b = \infty\,. \end{cases} \tag{2.1}$$

Let $m \in \mathbb{N}$. The kernel $W_m : I(a,b) \times I(a,b) \to \mathbb{R}$ satisfies

$$W_m(x,t) \ge 0 \tag{2.2}$$

for any $(x,t) \in I(a,b) \times (a,b)$,

$$\int_a^b W_m(x,t)dt = 1 \tag{2.3}$$

for any $x \in I(a,b)$,

$$\frac{\partial}{\partial x} W_m(x,t) = \frac{m(t-x)}{p(x)} W_m(x,t) \tag{2.4}$$

for any $(x,t) \in I(a,b) \times I(a,b)$, where $p(x)$ is polynomial in x and $p(x)$ is strictly positive for any $x \in I(a,b)$.

We define the operators $S_m : \mathcal{F}(I(a,b)) \to C(I(a,b))$, for any function $f \in \mathcal{F}(I(a,b))$ by

$$(S_m f)(x) = \int_a^b W_m(x,t)f(t)dt \tag{2.5}$$

for any $x \in I(a,b)$, $m \in \mathbb{N}$, where $\mathcal{F}(I(a,b)) = \{f | f : I(a,b) \to \mathbb{R}, \int_a^b W_m(x,t)f(t) < \infty$ for any $x \in I(a,b)$, any $m \in \mathbb{N}\}$.

The operators S_m, $m \ge 1$ are introduced and are studied by C. P. May in the paper [12]. These operators are referred to us like exponential operators.

The following results contained in the Lemma 2.1 are known (see [18]).

Lemma 2.1. *The operators S_m, $m \geq 1$ verify*

$$(S_m e_0)(x) = 1, \tag{2.6}$$

$$(S_m e_1)(x) = x, \tag{2.7}$$

$$(S_m e_2)(x) = x^2 + \frac{p(x)}{m} \tag{2.8}$$

for any $x \in I(a,b)$ and $m \in \mathbb{N}$.

3. Main results

Lemma 3.1. *The operators S_m, $m \geq 1$ verify*

$$\left(S_m \varphi_x^2\right)(x) = \frac{p(x)}{m}\,, \tag{3.1}$$

$$\left(S_m \varphi_x^4\right)(x) = \frac{3p^2(x)}{m^2} + \frac{p(x)\left[p''(x)p(x) + (p'(x))^2\right]}{m^3} \tag{3.2}$$

and

$$\left(S_m \varphi_x^4\right)(x) \leq \frac{p(x)\left[3p(x) + |p''(x)p(x) + (p'(x))^2|\right]}{m^2} \tag{3.3}$$

for any $x \in I(a,b)$, any $m \in \mathbb{N}$.

Proof. Differentiating the relation (2.8) with respect to x, we have

$$\int_a^b t^2 \frac{\partial}{\partial x} W_m(x,t)dt = 2x + \frac{p'(x)}{m}$$

and taking (2.4) into account we obtain

$$\frac{m}{p(x)} \int_a^b t^2(t-x) W_m(x,t)dt = 2x + \frac{p'(x)}{m}\,.$$

Thus

$$(S_m e_3)(x) = x^3 + \frac{3xp(x)}{m} + \frac{p'(x)p(x)}{m^2}\,. \tag{3.4}$$

Similarly, differentiating the relation (3.4), we have

$$(S_m e_4)(x) = x^4 + \frac{6x^2 p(x)}{m} + \frac{4xp'(x)p(x) + 3p^2(x)}{m^2} + \\ + \frac{p''(x)p^2(x) + (p'(x))^2 p(x)}{m^3}\,. \tag{3.5}$$

Because

$$\left(S_m\varphi_x^2\right)(x) = (S_m e_2)(x) - 2x(S_m e_1)(x) + x^2(S_m e_0)(x),$$

$$\left(S_m\varphi_x^4\right)(x) = (S_m e_4)(x) - 4x(S_m e_3)(x) + 6x^2(S_m e_2)(x) -$$

$$- 4x^3(S_m e_1)(x) + x^4(S_m e_0)(x),$$

taking (2.6) - (2.8) and (3.4), (3.5) into account, we obtain (3.1) and (3.2). From (3.2), the inequality (3.3) results immediately. □

Definition 3.1. Let $(m,n) \in \mathbb{N} \times \mathbb{N}$. The operator $S_{m,n} : \mathcal{F}(I(a,b) \times I(a,b)) \to C(I(a,b) \times I(a,b))$ defined for any function $f \in \mathcal{F}(I(a,b) \times I(a,b)) = \Big\{ f | f : I(a,b) \times I(a,b) \to \mathbb{R}$, $\int\limits_a^b W_m(x,s)W_n(y,t)f(s,t)dsdt < \infty$ for any $(x,y) \in I(a,b) \times I(a,b)$ and any $(m,n) \in \mathbb{N} \times \mathbb{N}\}$, any $(x,y) \in I(a,b) \times I(a,b)$ by

$$(S_{m,n}f)(x,y) = \int\limits_a^b \int\limits_a^b W_m(x,s)W_n(y,t)f(s,t)dsdt \tag{3.6}$$

is called a bivariate exponential operator.

Lemma 3.2. *The operators $S_{m,n}$, $(m,n) \in \mathbb{N} \times \mathbb{N}$ are linear and positive on $\mathcal{F}(I(a,b) \times I(a,b))$.*

Proof. The assertion follows from the definition of the operators $S_{m,n}$, $(m,n) \in \mathbb{N} \times \mathbb{N}$. □

Lemma 3.3. *We have*

$$\left(S_{m,n}(\cdot - x)^{2i}(* - y)^{2j}\right)(x,y) = \left(S_m(\cdot - x)^{2i}\right)(x)\left(S_n(* - y)^{2j}\right)(y) \tag{3.7}$$

for any $(x,y) \in I(a,b) \times I(a,b)$, any $i,j \in \{1,2\}$ and any $(m,n) \in \mathbb{N} \times \mathbb{N}$.

The proof is immediate, so we omit it.

Lemma 3.4. *The operators $S_{m,n}$, $(m,n) \in \mathbb{N} \times \mathbb{N}$, verify*

$$(S_{m,n}e_{00})(x,y) = 1 \tag{3.8}$$

for any $(x,y) \in I(a,b) \times I(a,b)$ and any $(m,n) \in \mathbb{N} \times \mathbb{N}$.

The proof is immediate, taking into account the definition of $S_{m,n}$, $(m,n) \in \mathbb{N}\times\mathbb{N}$ operators.

Definition 3.2. Let $(m,n) \in \mathbb{N}\times\mathbb{N}$. The operator $US_{m,n} : \mathcal{F}(I(a,b)\times I(a,b)) \to C(I(a,b)\times I(a,b))$ defined for any function $f \in \mathcal{F}(I(a,b)\times I(a,b))$, any $(x,y) \in I(a,b)\times I(a,b)$ by

$$(US_{m,n}f)(x,y) = (S_{m,n}(f(\cdot,y)+f(x,*)-f(\cdot,*))\,(x,y) = \tag{3.9}$$
$$= \int_a^b \int_a^b W_m(x,s)W_n(y,t)\,(f(s,y)+f(x,t)-f(s,t))\,dsdt$$

is called the GBS operator of exponential type.

In the following, we note $p^*(x) = \sqrt{3p(x)+|p''(x)p(x)+(p'(x))^2|}$, $x \in I(a,b)$.

Theorem 3.1. *(i) For any function $f \in C_b(I(a,b)\times I(a,b)) \cap \mathcal{F}(I(a,b)\times I(a,b))$, any $(x,y) \in I(a,b)\times I(a,b)$, any $(m,n) \in \mathbb{N}\times\mathbb{N}$, we have*

$$|f(x,y)-(US_{m,n}f)(x,y)| \le \left(1+\delta_1^{-1}\sqrt{\frac{p(x)}{m}}+\right. \tag{3.10}$$
$$\left.+\,\delta_2^{-1}\sqrt{\frac{p(y)}{n}}+\delta_1^{-1}\delta_2^{-1}\sqrt{\frac{p(x)p(y)}{mn}}\right)\omega_{mixed}(f;\delta_1,\delta_2)$$

for any $\delta_1,\delta_2 > 0$ and

$$|f(x,y)-(US_{m,n}f)(x,y)| \le \tag{3.11}$$
$$\le \left(1+\sqrt{p(x)}\right)\left(1+\sqrt{p(y)}\right)\omega_{mixed}\left(f;\frac{1}{\sqrt{m}},\frac{1}{\sqrt{n}}\right).$$

(ii) For any function $f \in D_b(I(a,b)\times I(a,b)) \cap \mathcal{F}(I(a,b)\times I(a,b))$ with $D_Bf \in B(I(a,b)\times I(a,b))$, any $(x,y) \in I(a,b)\times I(a,b)$, any $(m,n) \in \mathbb{N}\times\mathbb{N}$, we have

$$|f(x,y)-(US_{m,n}f)(x,y)| \le \left\{3\|D_Bf\|_\infty+\right. \tag{3.12}$$
$$+\left[1+\delta_1^{-1}\frac{p^*(x)}{\sqrt{m}}+\delta_2^{-1}\frac{p^*(y)}{\sqrt{n}}+\delta_1^{-1}\delta_2^{-1}\frac{\sqrt{p(x)p(y)}}{\sqrt{mn}}\right]\cdot$$
$$\left.\cdot\,\omega_{mixed}(D_Bf;\delta_1,\delta_2)\right\}\frac{\sqrt{p(x)p(y)}}{\sqrt{mn}}$$

for any $\delta_1, \delta_2 > 0$ *and*

$$|f(x,y) - (US_{m,n}f)(x,y)| \leq \Bigg[3\|D_B f\|_\infty + \Big(1 + p^*(x) + \tag{3.13}$$

$$+ p^*(y) + \sqrt{p(x)p(y)}\Big)\omega_{mixed}\left(D_B f; \frac{1}{\sqrt{m}}, \frac{1}{\sqrt{n}}\right)\Bigg] \frac{\sqrt{p(x)p(y)}}{\sqrt{mn}}.$$

Proof. Taking Lemma 3.1 - Lemma 3.4 and Theorem 1.2 into account, we obtain the relations (3.10) and (3.12). If we choose $\delta_1 = \dfrac{1}{\sqrt{m}}$, $\delta_2 = \dfrac{1}{\sqrt{n}}$ in (3.10) and (3.12), we obtain (3.11) and (3.13). □

In the following, let the real numbers $\alpha, \beta, \gamma, \delta$ with $\alpha < \beta$ and $\gamma < \delta$, such that $[\alpha, \beta]$, $[\gamma, \delta] \subset I(a,b)$. We note by $A^{\gamma,\delta}_{\alpha,\beta}$ a constant dependent on $\alpha, \beta, \gamma, \delta$ such that

$$\left(1 + \sqrt{p(x)}\right)\left(1 + \sqrt{p(y)}\right) \leq A^{\gamma,\delta}_{\alpha,\beta} \tag{3.14}$$

for any $(x,y) \in [\alpha,\beta] \times [\gamma,\delta]$, and if $f : X \times Y \to \mathbb{R}$ is a function, $f \in D_b\left(I(a,b) \times I(a,b)\right) \cap \mathcal{F}\left(I(a,b) \times I(a,b)\right)$ with $D_B f \in B\left(I(a,b) \times I(a,b)\right)$, we note by $B^{\gamma,\delta}_{\alpha,\beta}(f,m,n)$ a constant dependent on $\alpha, \beta, \gamma, \delta$ on the function f and $m, n \in \mathbb{N}$, such that

$$\Bigg[3\|D_B f\|_\infty + \Big(1 + p^*(x) + p^*(y) + \sqrt{p(x)p(y)}\Big)\omega_{mixed}\left(D_B; \frac{1}{\sqrt{m}}, \frac{1}{\sqrt{n}}\right)\Bigg] \cdot \tag{3.15}$$

$$\cdot \sqrt{p(x)p(y)} \leq B^{\gamma,\delta}_{\alpha,\beta}(f,m,n)$$

for any $(x,y) \in [\alpha,\beta] \times [\gamma,\delta]$.

Theorem 3.1 implies the next theorem.

Theorem 3.2. *The following statements are true.*

(i) For any function $f \in C_b(I(a,b) \times I(a,b)) \cap \mathcal{F}(I(a,b) \times I(a,b))$, *any* $(x,y) \in [\alpha,\beta] \times [\gamma,\delta]$ *and any* $(m,n) \in \mathbb{N} \times \mathbb{N}$, *we have*

$$|f(x,y) - (US_{m,n}f)(x,y)| \leq A^{\gamma,\delta}_{\alpha,\beta}\omega_{mixed}\left(f; \frac{1}{\sqrt{m}}, \frac{1}{\sqrt{n}}\right). \tag{3.16}$$

(ii) For any function $f \in D_b(I(a,b) \times I(a,b)) \cap \mathcal{F}(I(a,b) \times I(a,b))$ *with* $D_B f \in B(I(a,b) \times I(a,b))$, *any* $(x,y) \in [\alpha,\beta] \times [\gamma,\delta]$ *and any* $(m,n) \in \mathbb{N} \times \mathbb{N}$, *we*

have

$$|f(x,y)-(US_{m,n}f)(x,y)| \leq B^{\gamma,\delta}_{\alpha,\beta}(f,m,n)\frac{1}{\sqrt{mn}}\,. \tag{3.17}$$

From Theorem 1.1 and Theorem 3.2 the following theorem results.

Theorem 3.3. *If* $f \in C_b(I(a,b)\times I(a,b)) \cap \mathcal{F}(I(a,b)\times I(a,b))$*, then*

$$\lim_{m,n\to\infty}(US_{m,n}f)(x,y) = f(x,y) \tag{3.18}$$

and the convergence is uniform on any compact $[\alpha,\beta]\times[\gamma,\delta]\subset I(a,b)\times I(a,b)$*.*

In the following, by particularization and applying both Theorem 3.2 and Theorem 3.3, we give some approximations and convergence theorems for some known operators.

Application 3.1. If $a=0$, $b=1$ and $p(x)=x(1-x)$, $x\in[0,1]$, we obtain the Bernstein operators. Because $C([0,1])\subset\mathcal{F}([0,1])$, Theorem 3.2 and Theorem 3.3 hold for any function $f\in C([0,1]\times[0,1])$ and taking into account that $x(1-x)\leq \frac{1}{4}$ for any $x\in[0,1]$, we have that $A^{0,1}_{0,1}=\frac{9}{4}$. If $f\in D_b([0,1]\times[0,1])$ with $D_Bf\in B([0,1]\times[0,1])$, then we can take $B^{0,1}_{0,1}(f,m,n)=\frac{1}{4}\left[3\|D_Bf\|_\infty+\frac{5+4\sqrt{7}}{4}\omega_{\text{mixed}}\left(D_Bf;\frac{1}{\sqrt{m}},\frac{1}{\sqrt{n}}\right)\right]$, where $(m,n)\in\mathbb{N}\times\mathbb{N}$.

Application 3.2. If $a=0$, $b=\infty$ and $p(x)=x$, $x\in[0,\infty)$, we obtain the Mirakjan-Favard-Szász operators. If $\alpha,\beta,\gamma,\delta\in[0,\infty)$, then Theorem 3.2 and Theorem 3.3 hold for any function $f\in C_2([0,\infty)\times[0,\infty))\cap\mathcal{F}([0,\infty)\times[0,\infty))$, $A^{\gamma,\delta}_{\alpha,\beta}=\left(1+\sqrt{\beta}\right)\left(1+\sqrt{\delta}\right)$ and if $f\in D_b([\alpha,\beta]\times[\gamma,\delta])$ with $D_Bf\in B([\alpha,\beta]\times[\gamma,\delta])$, then

$$B^{\gamma,\delta}_{\alpha,\beta}(f,m,n)=\left[3\|D_Bf\|_\infty+\left(1+\sqrt{3\beta+1}+\sqrt{3\delta+1}+\right.\right.$$
$$\left.\left.+\sqrt{\beta\delta}\right)\omega_{\text{mixed}}\left(D_Bf;\frac{1}{\sqrt{m}},\frac{1}{\sqrt{n}}\right)\right]\sqrt{\beta\delta},$$

for $(m,n)\in\mathbb{N}\times\mathbb{N}$.

Application 3.3. If $a = 0$, $b = \infty$ and $p(x) = x(1+x)$, $x \in [0,\infty)$, we obtain the Baskakov operators. If $\alpha, \beta, \gamma, \delta \in [0,\infty)$, then the Theorem 3.2 and Theorem 3.3 hold for any function $f \in C([0,\infty) \times [0,\infty)) \cap \mathcal{F}([0,\infty) \times [0,\infty))$, $A_{\alpha,\beta}^{\gamma,\delta} = \left(1 + \sqrt{\beta(1+\beta)}\right)\left(1 + \sqrt{\delta(1+\delta)}\right)$ and if $f \in D_b([\alpha,\beta] \times [\gamma,\delta])$ with $D_b f \in B([\alpha,\beta] \times [\gamma,\delta])$, then

$$B_{\alpha,\beta}^{\gamma,\delta}(f,m,n) = \Big[3\|D_B f\|_\infty + \Big(1 + \sqrt{9\beta^2 + 9\beta + 1} + \sqrt{9\delta^2 + 9\delta + 1} + \\ + \sqrt{\beta\delta(1+\beta)(1+\delta)}\Big)\omega_{\text{mixed}}\left(D_B f; \frac{1}{\sqrt{m}}, \frac{1}{\sqrt{n}}\right)\Big]\sqrt{\beta\delta(1+\beta)(1+\delta)},$$

for $(m,n) \in \mathbb{N} \times \mathbb{N}$.

Application 3.4. If $a = 0$, $b = \infty$ and $p(x) = x(1+x)^2$, $x \in [0,\infty)$ we obtain the Ismail-May operators (see [11]). For $\alpha, \beta, \gamma, \delta \in [0,\infty)$, the Theorem 3.2 and Theorem 3.3 hold for any function $f \in C([0,\infty) \times [0,\infty)) \cap \mathcal{F}([0,\infty) \times [0,\infty))$, $A_{\alpha,\beta}^{\gamma,\delta} = \left(1 + (1+\beta)\sqrt{\beta}\right)\left(1 + (1+\delta)\sqrt{\delta}\right)$ and if $f \in D_b([\alpha,\beta] \times [\gamma,\delta])$ with $D_B f \in B([\alpha,\beta] \times [\gamma,\delta])$, then

$$B_{\alpha,\beta}^{\gamma,\delta}(f,m,n) = \Big[3\|D_B f\|_\infty + \\ + \Big(1 + (1+\beta)\sqrt{1 + 13\beta + 15\beta^2} + (1+\delta)\sqrt{1 + 13\delta + 15\delta^2} + \\ + (1+\beta)(1+\delta)\sqrt{\beta\delta}\Big)\omega_{\text{mixed}}\left(D_B f; \frac{1}{\sqrt{m}}, \frac{1}{\sqrt{n}}\right)\Big](1+\beta)(1+\delta)\sqrt{\beta\delta}\,,$$

$(m,n) \in \mathbb{N} \times \mathbb{N}$.

Application 3.5. If $a = 0$, $b = \infty$ and $p(x) = x^2$, $x \in [0,\infty)$, we obtain the Post-Widder operators. If $\alpha, \beta, \gamma, \delta \in [0,\infty)$, then the Theorem 3.2 and Theorem 3.3 hold for any function $f \in C([0,\infty) \times [0,\infty)) \cap \mathcal{F}([0,\infty) \times [0,\infty))$, $A_{\alpha,\beta}^{\gamma,\delta} = (1+\beta)(1+\delta)$ and if $f \in D_b([\alpha,\beta] \times [\gamma,\delta])$ with $D_B f \in B([\alpha,\beta] \times [\gamma,\delta])$, then

$$B_{\alpha,\beta}^{\gamma,\delta}(f,m,n) = \left[3\|D_B f\|_\infty + (1 + 3\beta + 3\delta + \beta\delta)\omega_{\text{mixed}}\left(D_B f; \frac{1}{\sqrt{m}}, \frac{1}{\sqrt{n}}\right)\right]\beta\delta,$$

$(m,n) \in \mathbb{N} \times \mathbb{N}$.

Application 3.6. If $a = -\infty$, $b = \infty$ and $p(x) = 1$, $x \in \mathbb{R}$, we obtain the Gauss-Weierstrass operators. If $\alpha, \beta, \gamma, \delta \in \mathbb{R}$, then the Theorem 3.2 and Theorem 3.3 hold for any function $f \in C(\mathbb{R} \times \mathbb{R}) \cap \mathcal{F}(\mathbb{R} \times \mathbb{R})$, $A_{\alpha,\beta}^{\gamma,\delta} = 4$ and if $f \in D_b([\alpha,\beta] \times [\gamma,\delta])$ with $D_B f \in B([\alpha,\beta] \times [\gamma,\delta])$, then

$$B_{\alpha,\beta}^{\gamma,\delta}(f,m,n) = 3\|D_B f\|_\infty + \left(2+2\sqrt{3}\right)\omega_{\text{mixed}}\left(D_B f; \frac{1}{\sqrt{m}}, \frac{1}{\sqrt{n}}\right),$$

where $(m,n) \in \mathbb{N} \times \mathbb{N}$.

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ON THE TRANSFORMATIONS OF N-LINEAR CONNECTIONS IN THE k-OSCULATOR BUNDLE

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Abstract. In the present paper we determine the transformations for the coefficients of an N-linear connection on Osc^2M, $Osc^3M, Osc^4M, ...,$ Osc^kM, $(k \geq 2, k \in N)$ by a transformation of nonlinear connections. We prove that the set $\mathcal{T}$ of the transformations of N-linear connections on $Osc^kM, (k \geq 2, k \in N)$, together with the composition of mappings isn't a group, but we give some groups which keep invariant a part of components of the local coefficients of an N-linear connection.

1. **Preliminaries**

The geometry of J_0^kM, $k \in N^*$, the k-jet bundle, discovered by Ch. Ehresmann [4], was largely investigated by many scholars: P. Liebermann [9], M. Crampin [3], A. Kawaguchi [6], I. Kolar [7], D. Krupka [8], M. de Léon [10], W. Sarlet [3], F. Cantrjin [3], W.M. Tulczyew [19], D. Grigore [5], R. Miron [4] et al. [12, 13, 14, 15, 16].

Generally, the geometries of higher order defined as the study of the category of bundles of jet $(J_0^kM\ ,\ \pi^k,\ M)$ are based on a direct approach of the properties of objects and morphisms in this category, without local coordinates.

But, many mathematical models from Lagrangian Mechanics, Theoretical Physics and Variational Calculus used multivariate Lagrangians of higher order acceleration, L$(x,\ \dfrac{dx}{dt}\,(t),\ ...\ ,\ \frac{1}{k!}\dfrac{d^kx}{dtk}\,(t)\ )$, (see E. Cartan, [2], for k=2, etc.).

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From here one can see the reason of construction of the geometry of the total space of the bundle of higher order accelerations (or the oscuator bundle of higher order) in local coordinates.

Recently, this construction was achived by R. Miron and Gh. Atanasiu in their joint papers [13, 14].

Let M be a real C^∞ manifold with n-dimensions, and $\left(Osc^k M, \pi^k, M\right)$ $(k \geq 2, k \in \mathbb{N})$ its k-osculator bundle. The local coordinates on the $(k+1)\,n$-dimensional manifold $Osc^k M, (k \geq 2, k \in \mathbb{N})$ are denoted by $\left(x^i, y^{(1)i}, y^{(2)i}, ..., y^{(k)i}\right)$.

Let N be a nonlinear connection on $Osc^k M, (k \geq 2, k \in \mathbb{N})$ with the coefficients

$$\left(\underset{(1)}{N}{}^i_j,\ \underset{(2)}{N}{}^i_j, ...,\ \underset{(k)}{N}{}^i_j\right), (k \geq 2, k \in \mathbb{N}), \left(i, j = \overline{1, n}\right).$$

Hence, the tangent space of $Osc^k M, (k \geq 2, k \in \mathbb{N})$ in the point

$u = \left(x, y^{(1)}, y^{(2)}, ..., y^{(k)}\right) \in Osc^k M, (k \geq 2, k \in \mathbb{N})$ is given by the direct sum of the vector spaces:

$$T_u Osc^k M, (k \geq 2, k \in \mathbb{N}) = N_0(u) \oplus N_1(u) \oplus ... \oplus N_{k-1}(u) \oplus V_k(u), \quad \forall u \in Osc^k M, (k \geq 2, k \in \mathbb{N}). \quad (1.1)$$

An adapted basis to (1.1) is given by:

$$\left\{\frac{\delta}{\delta x^i}, \frac{\delta}{\delta y^{(1)i}}, \frac{\delta}{\delta y^{(2)i}}, ..., \frac{\delta}{\delta y^{(k)i}}\right\} (k \geq 2, k \in \mathbb{N}), \left(i = \overline{1, n}\right), \quad (1.2)$$

where

$$\begin{cases} \frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - \underset{(1)}{N}{}^j_i \frac{\partial}{\partial y^{(1)j}} - \underset{(2)}{N}{}^j_i \frac{\partial}{\partial y^{(2)j}} - ... - \underset{(k)}{N}{}^j_i \frac{\partial}{\partial y^{(k)j}}, \\ \frac{\delta}{\delta y^{(1)i}} = \frac{\partial}{\partial y^{(1)i}} - \underset{(1)}{N}{}^j_i \frac{\partial}{\partial y^{(2)j}} - \underset{(2)}{N}{}^j_i \frac{\partial}{\partial y^{(3)j}} - ... - \underset{(k-1)}{N}{}^j_i \frac{\partial}{\partial y^{(k)j}}, \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ \frac{\delta}{\delta y^{(k-1)i}} = \frac{\partial}{\partial y^{(k-1)i}} - \underset{(1)}{N}{}^j_i \frac{\partial}{\partial y^{(k)j}}, \\ \frac{\delta}{\delta y^{(k)i}} = \frac{\partial}{\partial y^{(k)i}}, (k \geq 2, k \in \mathbb{N}), \left(i, j = \overline{1, n}\right). \end{cases} \quad (1.3)$$

Let us consider the dual basis of (1.2) :

$$\left\{dx^i, \delta y^{(1)i}, \delta y^{(2)i}, ..., \delta y^{(k)i}\right\}, (k \geq 2, k \in \mathbb{N}), \left(i = \overline{1,n}\right), \tag{1.4}$$

where:

$$\begin{cases} \delta x^i = dx^i, \\ \delta y^{(1)i} = dy^{(1)i} + \underset{(1)}{M}{}^i_j dx^j, \\ \delta y^{(2)i} = dy^{(2)i} + \underset{(1)}{M}{}^i_j dy^{(1)j} + \underset{(2)}{M}{}^i_j dx^j, \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ \delta y^{(k)i} = dy^{(k)i} + \underset{(1)}{M}{}^i_j dy^{(k-1)j} + ... + \underset{(k-1)}{M}{}^i_j dy^{(1)j} + \underset{(k)}{M}{}^i_j dx^j, \end{cases} \tag{1.5}$$

where

$$\begin{cases} \underset{(1)}{M}{}^i_j = \underset{(1)}{N}{}^i_j, \underset{(2)}{M}{}^i_j = \underset{(2)}{N}{}^i_j + \underset{(1)}{N}{}^i_m \underset{(1)}{M}{}^m_j, \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ \underset{(k)}{M}{}^i_j = \underset{(k)}{N}{}^i_j + \underset{(k-1)}{N}{}^i_m \underset{(1)}{M}{}^m_j + ... + \underset{(2)}{N}{}^i_m \underset{(k-2)}{M}{}^m_j + \underset{(1)}{N}{}^i_m \underset{(k-1)}{M}{}^m_j. \end{cases} \tag{1.6}$$

Let D be an N-linear connection on $Osc^k M, (k \geq 2, k \in \mathbb{N})$ with the local coefficients in the adapted basis: $D\Gamma(N) = \left(L^i_{jk}, \underset{(\alpha)}{C}{}^i_{jk}\right), (k \geq 2, k \in \mathbb{N})$.

The terminology and notations are usually retained, which are essentially based on the R. Miron's book: [12].

2. The set of the transformations of N-linear connections

Let $\overline{N}$ be another nonlinear connection on $Osc^k M, (k \geq 2, k \in \mathbb{N})$ with the coefficients $\left(\underset{(1)}{\overline{N}}{}^i_j, \underset{(2)}{\overline{N}}{}^i_j, ..., \underset{(k)}{\overline{N}}{}^i_j\right)$ .Then there exists the uniquely determined tensor fields $\underset{(\alpha)}{A}{}^i_j \in \tau^1_1\left(Osc^k M, (k \geq 2, k \in \mathbb{N})\right), \left(\alpha = \overline{1,k}\right)$ on $Osc^k M, (k \geq 2, k \in \mathbb{N})$ such that:

$$\underset{(\alpha)}{\overline{N}}{}^i_j = \underset{(\alpha)}{N}{}^i_j - \underset{(\alpha)}{A}{}^i_j, \left(\alpha = \overline{1,k}\right)(k \geq 2, k \in \mathbb{N}). \tag{2.1}$$

Conversely, if $\underset{(\alpha)}{N}{}^i_j$ and $\underset{(\alpha)}{A}{}^i_j$ $\left(\alpha = \overline{1,k}\right)(k \geq 2, k \in \mathbb{N})$ are given, then $\underset{(\alpha)}{\overline{N}}{}^i_j$, $\left(\alpha = \overline{1,k}\right)(k \geq 2, k \in \mathbb{N})$, given by (2.1) is a nonlinear connection.

Let us suppose that the mapping $N \to \overline{N}$ is given by (2.1)

Let $\overline{D}$ be an $\overline{N}$-linear connection on $Osc^k M, (k \geq 2, k \in \mathbb{N})$ with the local coefficients in the adapted basis: $D\overline{\Gamma}\left(\overline{N}\right) = \left(\overline{L}^i_{jk}, \underset{(\alpha)}{\overline{C}}{}^i_{jk}\right), (k \geq 2, k \in \mathbb{N})$.

According to [12] we have:

$$\begin{cases} D_{\frac{\delta}{\delta x^j}} \frac{\delta}{\delta x^i} = L^m_{ij} \frac{\delta}{\delta x^m}, D_{\frac{\delta}{\delta x^j}} \frac{\delta}{\delta y^{(\alpha)i}} = L^m_{ij} \frac{\delta}{\delta y^{(\alpha)m}}, \left(\alpha = \overline{1,k}\right) (k \geq 2, k \in \mathbb{N}), \\ D_{\frac{\delta}{\delta y^{(\beta)j}}} \frac{\delta}{\delta x^i} = \underset{(\beta)}{C}{}^m_{ij} \frac{\delta}{\delta x^m}, D_{\frac{\delta}{\delta y^{(\beta)j}}} \frac{\delta}{\delta y^{(\alpha)i}} = \underset{(\beta)}{C}{}^m_{ij} \frac{\delta}{\delta y^{(\alpha)m}}, \left(\alpha, \beta = \overline{1,k}\right) (k \geq 2, k \in \mathbb{N}), \end{cases} \tag{2.2}$$

The adapted basis corresponding to the nonlinear connection $\overline{N}$ is:

$$\begin{cases} \frac{\overline{\delta}}{\delta x^i} = \frac{\partial}{\partial x^i} - \underset{(1)}{\overline{N}}{}^j_i \frac{\partial}{\partial y^{(1)j}} - \underset{(2)}{\overline{N}}{}^j_i \frac{\partial}{\partial y^{(2)j}} - ... - \underset{(k)}{\overline{N}}{}^j_i \frac{\partial}{\partial y^{(k)j}}, \\ \frac{\delta}{\delta y^{(1)i}} = \frac{\partial}{\partial y^{(1)i}} - \underset{(1)}{\overline{N}}{}^j_i \frac{\partial}{\partial y^{(2)j}} - \underset{(2)}{\overline{N}}{}^j_i \frac{\partial}{\partial y^{(3)j}} - ... - \underset{(k-1)}{\overline{N}}{}^j_i \frac{\partial}{\partial y^{(k)j}}, \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ \frac{\overline{\delta}}{\delta y^{(k-1)i}} = \frac{\partial}{\partial y^{(k-1)i}} - \underset{(1)}{\overline{N}}{}^j_i \frac{\partial}{\partial y^{(k)j}}, \\ \frac{\delta}{\delta y^{(k)i}} = \frac{\partial}{\partial y^{(k)i}}. \end{cases} \tag{2.3}$$

It follows first of all that the transformations (2.1) preserve the coefficients $\underset{(k)}{C}{}^i_{jk}$.
From $(1.3), (2.3)$ and (2.1) we obtain:

$$\begin{cases} \frac{\overline{\delta}}{\delta x^i} = \frac{\delta}{\delta x^i} + \underset{(1)}{A}{}^j_i \frac{\partial}{\partial y^{(1)j}} + \underset{(2)}{A}{}^j_i \frac{\partial}{\partial y^{(2)j}} + ... + \underset{(k)}{A}{}^j_i \frac{\partial}{\partial y^{(k)j}}, \\ \frac{\overline{\delta}}{\delta y^{(1)i}} = \frac{\delta}{\delta y^{(1)i}} + \underset{(1)}{A}{}^j_i \frac{\partial}{\partial y^{(2)j}} + \underset{(2)}{A}{}^j_i \frac{\partial}{\partial y^{(3)j}} + ... + \underset{(k-1)}{A}{}^j_i \frac{\partial}{\partial y^{(k)j}}, \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ \frac{\overline{\delta}}{\delta y^{(k-1)i}} = \frac{\delta}{\delta y^{(k-1)i}} + \underset{(1)}{A}{}^j_i \frac{\partial}{\partial y^{(k)j}}, \\ \frac{\overline{\delta}}{\delta y^{(k)i}} = \frac{\delta}{\delta y^{(k)i}}. \end{cases} \tag{2.4}$$

Using $(2.2), (2.4)$and (1.3) we have:

$$D_{\frac{\overline{\delta}}{\delta x^j}} \frac{\overline{\delta}}{\delta y^{(k)i}} = \overline{L}^m_{ij} \frac{\overline{\delta}}{\delta y^{(k)m}} = \overline{L}^m_{ij} \frac{\delta}{\delta y^{(k)m}},$$

$$D_{\frac{\overline{\delta}}{\delta x^j}} \frac{\overline{\delta}}{\delta y^{(k)i}} = D_{\left(\frac{\delta}{\delta x^j} + \underset{(1)}{A}{}^l_j \frac{\partial}{\partial y^{(1)l}} + \underset{(2)}{A}{}^l_j \frac{\partial}{\partial y^{(2)l}} + ... + \underset{(k)}{A}{}^l_j \frac{\partial}{\partial y^{(k)l}}\right)} \frac{\delta}{\delta y^{(k)i}} =$$

$$= D_{\frac{\delta}{\delta x^j}} \frac{\delta}{\delta y^{(k)i}} + \underset{(1)}{A}{}^l_j D_{\frac{\partial}{\partial y^{(1)l}}} \frac{\delta}{\delta y^{(k)i}} + \underset{(2)}{A}{}^l_j D_{\frac{\partial}{\partial y^{(2)l}}} \frac{\delta}{\delta y^{(k)i}} + ... + \underset{(k)}{A}{}^l_j D_{\frac{\partial}{\partial y^{(k)l}}} \frac{\delta}{\delta y^{(k)i}} =$$

$$= L^m_{ij} \frac{\delta}{\delta y^{(k)m}} + \underset{(1)}{A}{}^l_j D_{\frac{\delta}{\delta y^{(1)l}} + \underset{(1)}{N}{}^r_l \frac{\partial}{\partial y^{(2)r}} + \underset{(2)}{N}{}^r_l \frac{\partial}{\partial y^{(3)r}} + ... + \underset{(k-1)}{N}{}^r_l \frac{\partial}{\partial y^{(k)r}}} \frac{\delta}{\delta y^{(k)i}} +$$

$$+ \underset{(2)}{A}{}^l_j D_{\frac{\delta}{\delta y^{(2)l}} + \underset{(1)}{N}{}^r_l \frac{\partial}{\partial y^{(3)r}} + \underset{(2)}{N}{}^r_l \frac{\partial}{\partial y^{(4)r}} + ... + \underset{(k-2)}{N}{}^r_l \frac{\partial}{\partial y^{(k)r}}} \frac{\delta}{\delta y^{(k)i}} + ... + \underset{(k)}{A}{}^l_j D_{\frac{\delta}{\delta y^{(k)l}}} \frac{\delta}{\delta y^{(k)i}} =$$

$$= L^m_{ij} \frac{\delta}{\delta y^{(k)m}} + \left(\underset{(1)}{A}{}^l_j D_{\frac{\delta}{\delta y^{(1)l}}} \frac{\delta}{\delta y^{(k)i}} + \underset{(1)}{A}{}^l_j \underset{(1)}{N}{}^r_l D_{\frac{\partial}{\partial y^{(2)r}}} \frac{\delta}{\delta y^{(k)i}} + \underset{(1)}{A}{}^l_j \underset{(2)}{N}{}^r_l D_{\frac{\partial}{\partial y^{(3)r}}} \frac{\delta}{\delta y^{(k)i}} + ... \right.$$

$$\left. + \underset{(1)}{A}{}^l_j \underset{(k-1)}{N}{}^r_l D_{\frac{\delta}{\delta y^{(k)r}}} \frac{\delta}{\delta y^{(k)i}} \right) + \left(\underset{(2)}{A}{}^l_j D_{\frac{\delta}{\delta y^{(2)l}}} \frac{\delta}{\delta y^{(k)i}} + \underset{(2)}{A}{}^l_j \underset{(1)}{N}{}^r_l D_{\frac{\partial}{\partial y^{(3)r}}} \frac{\delta}{\delta y^{(k)i}} + \right.$$

$$\left. + \underset{(2)}{A}{}^l_j \underset{(2)}{N}{}^r_l D_{\frac{\partial}{\partial y^{(4)r}}} \frac{\delta}{\delta y^{(k)i}} + ... + \underset{(2)}{A}{}^l_j \underset{(k-2)}{N}{}^r_l D_{\frac{\delta}{\delta y^{(k)r}}} \frac{\delta}{\delta y^{(k)i}} \right) + ... + \underset{(k)}{A}{}^l_j \underset{(k)}{C}{}^m_{il} \frac{\delta}{\delta y^{(k)m}} =$$

$$= L^m_{ij} \frac{\delta}{\delta y^{(k)m}} + \underset{(1)}{A}{}^l_j \underset{(1)}{C}{}^m_{il} \frac{\delta}{\delta y^{(k)m}} + \underset{(2)}{A}{}^l_j \underset{(2)}{C}{}^m_{il} \frac{\delta}{\delta y^{(k)m}} + ... + \underset{(k)}{A}{}^l_j \underset{(k)}{C}{}^m_{il} \frac{\delta}{\delta y^{(k)m}} +$$

$$+ \underset{(1)}{A}{}^l_j \underset{(1)}{N}{}^r_l D_{\left(\frac{\delta}{\delta y^{(2)r}} + \underset{(1)}{N}{}^s_r \frac{\partial}{\partial y^{(3)s}} + \underset{(2)}{N}{}^s_r \frac{\partial}{\partial y^{(4)s}} + ... + \underset{(k-2)}{N}{}^s_r \frac{\partial}{\partial y^{(k)s}} \right)} \frac{\delta}{\delta y^{(k)i}} +$$

$$+ \underset{(1)}{A}{}^l_j \underset{(2)}{N}{}^r_l D_{\left(\frac{\delta}{\delta y^{(3)r}} + \underset{(1)}{N}{}^s_r \frac{\partial}{\partial y^{(4)s}} + \underset{(2)}{N}{}^s_r \frac{\partial}{\partial y^{(5)s}} + ... + \underset{(k-3)}{N}{}^s_r \frac{\partial}{\partial y^{(k)s}} \right)} \frac{\delta}{\delta y^{(k)i}} + ... +$$

$$+ \underset{(1)}{A}{}^l_j \underset{(k-1)}{N}{}^r_l \underset{(k)}{C}{}^m_{ir} \frac{\delta}{\delta y^{(k)m}} + ...$$

$$D_{\frac{\bar{\delta}}{\delta y^{(1)j}}} \frac{\bar{\delta}}{\delta y^{(k)i}} = \underset{(1)}{\overline{C}}{}^m_{ij} \frac{\delta}{\delta y^{(k)m}}, D_{\frac{\bar{\delta}}{\delta y^{(1)j}}} \frac{\bar{\delta}}{\delta y^{(k)i}} = D_{\left(\frac{\delta}{\delta y^{(1)j}} + \underset{(1)}{A}{}^l_j \frac{\partial}{\partial y^{(2)l}} + ... + \underset{(k-1)}{A}{}^l_j \frac{\partial}{\partial y^{(k)l}} \right)} \frac{\delta}{\delta y^{(k)i}}$$

$$= \underset{(1)}{C}{}^m_{ij} \frac{\delta}{\delta y^{(k)m}} + \underset{(1)}{A}{}^l_j D_{\frac{\partial}{\partial y^{(2)l}}} \frac{\delta}{\delta y^{(k)i}} + ... + \underset{(k-2)}{A}{}^l_j D_{\frac{\partial}{\partial y^{(k-1)l}}} \frac{\delta}{\delta y^{(k)i}} + \underset{(k-1)}{A}{}^l_j \underset{(k)}{C}{}^m_{il} \frac{\delta}{\delta y^{(k)m}}.$$

....

$$D_{\frac{\bar{\delta}}{\delta y^{(k-1)j}}} \frac{\bar{\delta}}{\delta y^{(k)i}} = \underset{(k-1)}{\overline{C}}{}^m_{ij} \frac{\delta}{\delta y^{(k)m}}, D_{\frac{\bar{\delta}}{\delta y^{(k-1)j}}} \frac{\bar{\delta}}{\delta y^{(k)i}} = D_{\left(\frac{\delta}{\delta y^{(k-1)j}} + \underset{(1)}{A}{}^l_j \frac{\partial}{\partial y^{(k)l}} \right)} \frac{\delta}{\delta y^{(k)i}} =$$

$$\underset{(k-1)}{C}{}^m_{ij} \frac{\delta}{\delta y^{(k)m}} + \underset{(1)}{A}{}^l_j \underset{(k)}{C}{}^m_{il} \frac{\delta}{\delta y^{(k)m}}.$$

Therefore the change we are looking for is:

$$\begin{cases} \overline{L}_{ij}^{m} = L_{ij}^{m} + \underset{(1)_j}{A^{\,l}}\left[\underset{(1)_{il}}{C^{\,m}} + \underset{(1)_l}{N^{\,r}}\underset{(2)_{ir}}{C^{\,m}} + ... + \underset{(k-1)_l}{N^{\,r}}\underset{(k)_{ir}}{C^{\,m}} + \underset{(1)_l}{N^{\,r}}\underset{(1)_r}{N^{\,s}}\underset{(3)_{is}}{C^{\,m}} + \right. \\ ... + \left(\underset{(1)_l}{N^{\,r}}\underset{(k-2)_r}{N^{\,s}} + \underset{(2)_l}{N^{\,r}}\underset{(k-3)_r}{N^{\,s}} + ... + \underset{(k-2)_l}{N^{\,r}}\underset{(2)_r}{N^{\,s}} + \underset{(k-1)_l}{N^{\,r}}\underset{(1)_r}{N^{\,s}}\right)\underset{(k)_{is}}{C^{\,m}} + \\ \left. ... + \underbrace{\underset{(1)_l}{N^{\,r}}\underset{(1)_r}{N^{\,s}}...\underset{(1)}{N}}_{(k-1)}\underset{k}{C}\right] + \underset{(2)_j}{A^{\,l}}\left[\underset{(2)_{il}}{C^{\,m}} + \underset{(1)_l}{N^{\,r}}\underset{(3)_{ir}}{C^{\,m}} + ... + \right. \\ \left. + \underset{(k-2)_l}{N^{\,r}}\underset{(k)_{ir}}{C^{\,m}} + ... + \underbrace{\underset{(1)_l}{N^{\,r}}\cdot ... \cdot\underset{(1)}{N}}_{(k-2)}\underset{(k)}{C}\right] + ... + \\ + \underset{(k-1)_j}{A^{\,l}}\left[\underset{(k-1)_{il}}{C^{\,m}} + \underset{(1)_l}{N^{\,r}}\underset{(k)_{ir}}{C^{\,m}}\right] + \underset{(k)_j}{A^{\,l}}\underset{(k)_{il}}{C^{\,m}}, (k \geq 2, k \in \mathbb{N}), \\ \underset{(1)_{ij}}{\overline{C}^{\,m}} = \underset{(1)_{ij}}{C^{\,m}} + \underset{(1)_j}{A^{\,l}}\left[\underset{(2)_{il}}{C^{\,m}} + \underset{(1)_l}{N^{\,r}}\underset{(3)_{ir}}{C^{\,m}} + ... + \underset{(k-2)_l}{N^{\,r}}\underset{(k)_{ir}}{C^{\,m}} + ... + \right. \\ \left. + \underbrace{\underset{(1)_l}{N^{\,r}}\cdot ... \cdot\underset{(1)}{N}}_{(k-2)}\underset{(k)}{C}\right] + ... + \underset{(k-2)_j}{A^{\,l}}\left[\underset{(k-1)_{il}}{C^{\,m}} + \underset{(1)_l}{N^{\,r}}\underset{(k)_{ir}}{C^{\,m}}\right] + \underset{(k-1)_j}{A^{\,l}}\underset{(k)_{il}}{C^{\,m}}, (k \geq 2, k \in \mathbb{N}), \\ .... \\ \underset{(k-1)_{ij}}{\overline{C}^{\,m}} = \underset{(k-1)_{ij}}{C^{\,m}} + \underset{(1)_j}{A^{\,l}}\underset{(k)_{il}}{C^{\,m}}, (k \geq 2, k \in \mathbb{N}), \\ \underset{(k)_{ij}}{\overline{C}^{\,m}} = \underset{(k)_{ij}}{C^{\,m}}, (k \geq 2, k \in \mathbb{N}). \end{cases} \tag{2.5}$$

So, we have proved:

Proposition 1. *The transformation* (2.1) *of nonlinear connections imply the transformations* (2.5)*for the coefficients*

$$D\Gamma(N) = \left(L_{jk}^{i}, \underset{(\alpha)_{jk}}{C^{\,i}}\right), \left(\alpha = \overline{1,k}\right), (k \geq 2, k \in \mathbb{N}).$$

Particular cases:

1. If we take $k = 2$ in (2.5) then we obtain a result given in [17]:

$$\begin{cases} \overline{L}^m_{ij} = L^m_{ij} + \underset{(1)}{A}{}^l_j \left(\underset{(1)}{C}{}^m_{il} + \underset{(1)}{N}{}^r_l \underset{(2)}{C}{}^m_{ir} \right) + \underset{(2)}{A}{}^l_j \underset{(2)}{C}{}^m_{il}, \\ \underset{(1)}{\overline{C}}{}^m_{ij} = \underset{(1)}{C}{}^m_{ij} + \underset{(1)}{A}{}^l_j \underset{(2)}{C}{}^m_{il}, \\ \underset{(2)}{\overline{C}}{}^m_{ij} = \underset{(2)}{C}{}^m_{ij}. \end{cases} \tag{2.6}$$

2.If we take $k = 3$ in (2.5), then we obtain the transformations for the coefficients of an N-linear connection on Osc^3M by a transformation of nonlinear connections, result given in [18]:

$$\begin{cases} \overline{L}^m_{ij} = L^m_{ij} + \underset{(1)}{A}{}^l_j \left(\underset{(1)}{C}{}^m_{il} + \underset{(1)}{N}{}^r_l \underset{(2)}{C}{}^m_{ir} + \underset{(1)}{N}{}^r_l \underset{(1)}{N}{}^s_r \underset{(3)}{C}{}^m_{is} + \underset{(2)}{N}{}^r_l \right) \underset{(3)}{C}{}^m_{ir} + \\ + \underset{(2)}{A}{}^l_j \left(\underset{(2)}{C}{}^m_{il} + \underset{(1)}{N}{}^r_l \underset{(3)}{C}{}^m_{ir} \right) + \underset{(3)}{A}{}^l_j \underset{(3)}{C}{}^m_{il}, \\ \underset{(1)}{\overline{C}}{}^m_{ij} = \underset{(1)}{C}{}^m_{ij} + \underset{(1)}{A}{}^l_j \left(\underset{(2)}{C}{}^m_{il} + \underset{(1)}{N}{}^r_l \underset{(3)}{C}{}^m_{ir} \right) + \underset{(2)}{A}{}^l_j \underset{(3)}{C}{}^m_{il}, \\ \underset{(2)}{\overline{C}}{}^m_{ij} = \underset{(2)}{C}{}^m_{ij} + \underset{(1)}{A}{}^l_j \underset{(3)}{C}{}^m_{il}, \\ \underset{(3)}{\overline{C}}{}^m_{ij} = \underset{(3)}{C}{}^m_{ij}. \end{cases} \tag{2.7}$$

3. If we consider $k = 4$ in (2.5), then we obtain the transformations for the coefficients of an N-linear connection on Osc^4M by a transformation of nonlinear connections.

$$
\left\{
\begin{aligned}
&\overline{L}^m_{ij} = L^m_{ij} + \underset{(1)}{A}{}^{l}_{j}\Big(\underset{(1)}{C}{}^{m}_{il} + \underset{(1)}{N}{}^{r}_{l}\underset{(2)}{C}{}^{m}_{ir} + \underset{(2)}{N}{}^{r}_{l}\underset{(3)}{C}{}^{m}_{ir} + \underset{(3)}{N}{}^{r}_{l}\underset{(4)}{C}{}^{m}_{ir} + \\
&+\underset{(1)}{N}{}^{r}_{l}\underset{(1)}{N}{}^{s}_{r}\underset{(3)}{C}{}^{m}_{is} + \underset{(1)}{N}{}^{r}_{l}\underset{(2)}{N}{}^{s}_{r}\underset{(4)}{C}{}^{m}_{is} + \underset{(2)}{N}{}^{r}_{l}\underset{(1)}{N}{}^{s}_{r}\underset{(4)}{C}{}^{m}_{is} + \\
&+\underset{(1)}{N}{}^{r}_{l}\underset{(1)}{N}{}^{s}_{r}\underset{(1)}{N}{}^{t}_{s}\underset{(4)}{C}{}^{m}_{it}\Big) + \underset{(2)}{A}{}^{l}_{j}\Big(\underset{(2)}{C}{}^{m}_{il} + \underset{(3)}{C}{}^{m}_{ir} + \underset{(2)}{N}{}^{r}_{l}\underset{(4)}{C}{}^{m}_{ir} + \\
&+\underset{(1)}{N}{}^{r}_{l}\underset{(1)}{N}{}^{s}_{r}\underset{(4)}{C}{}^{m}_{is}\Big) + \underset{(3)}{A}{}^{l}_{j}\Big(\underset{(3)}{C}{}^{m}_{il} + \underset{(1)}{N}{}^{r}_{l}\underset{(4)}{C}{}^{m}_{ir}\Big) + \underset{(4)}{A}{}^{l}_{j}\underset{(4)}{C}{}^{m}_{il}, \\
&\underset{(1)}{\overline{C}}{}^{m}_{ij} = \underset{(1)}{C}{}^{m}_{ij} + \underset{(1)}{A}{}^{l}_{j}\Big(\underset{(2)}{C}{}^{m}_{il} + \underset{(1)}{N}{}^{r}_{l}\underset{(3)}{C}{}^{m}_{ir} + \underset{(2)}{N}{}^{r}_{l}\underset{(4)}{C}{}^{m}_{ir} + \underset{(1)}{N}{}^{r}_{l}\underset{(1)}{N}{}^{s}_{r}\underset{(4)}{C}{}^{m}_{is}\Big) + \\
&+\underset{(2)}{A}{}^{l}_{j}\Big(\underset{(3)}{C}{}^{m}_{il} + \underset{(1)}{N}{}^{r}_{l}\underset{(4)}{C}{}^{m}_{ir}\Big) + \underset{(3)}{A}{}^{l}_{j}\underset{(4)}{C}{}^{m}_{il}, \\
&\underset{(2)}{\overline{C}}{}^{m}_{ij} = \underset{(2)}{C}{}^{m}_{ij} + \underset{(1)}{A}{}^{l}_{j}\Big(\underset{(3)}{C}{}^{m}_{il} + \underset{(1)}{N}{}^{r}_{l}\underset{(4)}{C}{}^{m}_{ir}\Big) + \underset{(2)}{A}{}^{l}_{j}\underset{(4)}{C}{}^{m}_{il}, \\
&\underset{(3)}{\overline{C}}{}^{m}_{ij} = \underset{(3)}{C}{}^{m}_{ij} + \underset{(1)}{A}{}^{l}_{j}\underset{(4)}{C}{}^{m}_{il}, \\
&\underset{(4)}{\overline{C}}{}^{m}_{ij} = \underset{(4)}{C}{}^{m}_{ij}.
\end{aligned}
\right. \tag{2.8}
$$

etc.

Now, we can prove:

Theorem 1. *Let N and $\overline{N}$ be two nonlinear connections on $Osc^k M$, $(k \geq 2$, $k \in \mathbb{N})$ with coefficients*

$$\left(\underset{(1)}{N}{}^{i}_{j}, \underset{(2)}{N}{}^{i}_{j}, ..., \underset{(k)}{N}{}^{i}_{j}\right), \ \left(\underset{(1)}{\overline{N}}{}^{i}_{j}, \underset{(2)}{\overline{N}}{}^{i}_{j}, ..., \underset{(k)}{\overline{N}}{}^{i}_{j}\right), \ \left(\alpha = \overline{1,k}\right), \ (k \geq 2, k \in \mathbb{N})$$

respectively. If

$$D\Gamma\left(N\right) = \left(L^m_{ij}, \underset{(\alpha)}{C}{}^{m}_{ij}\right)$$

and

$$D\overline{\Gamma}\left(\overline{N}\right) = \left(\overline{L}^m_{ij}, \underset{(\alpha)}{\overline{C}}{}^{m}_{ij}\right), \left(\alpha = \overline{1,k}\right), (k \geq 2, k \in \mathbb{N})$$

are the local coefficients of two N-, respectively $\overline{N}-$linear connections, D, respectively $\overline{D}$ on the differentiable manifold $Osc^k M, (k \geq 2, k \in \mathbb{N})$, then there exists only one system of tensor fields $\left(\underset{(1)}{A}{}^{i}_{j}, \underset{(2)}{A}{}^{i}_{j}, ..., \underset{(k)}{A}{}^{i}_{j}, B^m_{ij}, \underset{(1)}{D}{}^{m}_{ij}, \underset{(2)}{D}{}^{m}_{ij}, ..., \underset{(k)}{D}{}^{m}_{ij}\right)$ such that:

$$\begin{cases} \underset{(\alpha)j}{\overline{N}^{\,i}} = \underset{(\alpha)j}{N^{\,i}} - \underset{(\alpha)j}{A^{\,i}}, \left(\alpha = \overline{1,k}\right), (k \geq 2, k \in \mathbb{N}), \\ \overline{L}^{m}_{ij} = L^{m}_{ij} + \underset{(1)j}{A^{\,l}} \Big[\underset{(1)il}{C^{\,m}} + \underset{(1)l}{N^{\,r}} \underset{(2)ir}{C^{\,m}} + ... + \underset{(k-1)l}{N^{\,r}} \underset{(k)ir}{C^{\,m}} + \underset{(1)l}{N^{\,r}} \underset{(1)r}{N^{\,s}} \underset{(3)is}{C^{\,m}} + \\ ... + \left(\underset{(1)l}{N^{\,r}} \underset{(k-2)r}{N^{\,s}} + \underset{(2)l}{N^{\,r}} \underset{(k-3)r}{N^{\,s}} + ... + \underset{(k-2)l}{N^{\,r}} \underset{(2)r}{N^{\,s}} + \underset{(k-1)l}{N^{\,r}} \underset{(1)r}{N^{\,s}} \right) \underset{(k)is}{C^{\,m}} + \\ ... + \underbrace{\underset{(1)l}{N^{\,r}} \underset{(1)r}{N^{\,s}} ... \underset{(1)}{N}}_{(k-1)} \underset{(k)}{C} \Big] + \underset{(2)j}{A^{\,l}} \Big[\underset{(2)il}{C^{\,m}} + \underset{(1)l}{N^{\,r}} \underset{(3)ir}{C^{\,m}} + ... + \\ + \underset{(k-2)l}{N^{\,r}} \underset{(k)ir}{C^{\,m}} + ... + \underbrace{\underset{(1)l}{N^{\,r}} \cdot ... \cdot \underset{(1)}{N}}_{(k-2)} \underset{(k)}{C} \Big] + ... + \\ + \underset{(k-1)j}{A^{\,l}} \Big[\underset{(k-1)il}{C^{\,m}} + \underset{(1)l}{N^{\,r}} \underset{(k)ir}{C^{\,m}} \Big] + \underset{(k)j}{A^{\,l}} \underset{(k)il}{C^{\,m}} - B^{m}_{ij}, (k \geq 2, k \in N), \\ \underset{(1)ij}{\overline{C}^{\,m}} = \underset{(1)ij}{C^{\,m}} + \underset{(1)j}{A^{\,l}} \Big[\underset{(2)il}{C^{\,m}} + \underset{(1)l}{N^{\,r}} \underset{(3)ir}{C^{\,m}} + ... + \underset{(k-2)l}{N^{\,r}} \underset{(k)ir}{C^{\,m}} + ... + + \underbrace{\underset{(1)l}{N^{\,r}} \cdot ... \cdot \underset{(1)}{N}}_{(k-2)} \underset{(k)}{C} \Big] \\ + \cdots + \underset{(k-2)j}{A^{\,l}} \Big[\underset{(k-1)il}{C^{\,m}} + \underset{(1)l}{N^{\,r}} \underset{(k)ir}{C^{\,m}} \Big] + \underset{(k-1)j}{A^{\,l}} \underset{(k)il}{C^{\,m}} - \underset{(1)ij}{D^{\,m}}, (k \geq 2, k \in \mathbb{N}), \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \underset{(k-1)ij}{\overline{C}^{\,m}} = \underset{(k-1)ij}{C^{\,m}} + \underset{(1)j}{A^{\,l}} \underset{(k)il}{C^{\,m}} - \underset{(k-1)ij}{D^{\,m}}, (k \geq 2, k \in \mathbb{N}), \\ \underset{(k)ij}{\overline{C}^{\,m}} = \underset{(k)ij}{C^{\,m}} - \underset{(k)ij}{D^{\,m}}, (k \geq 2, k \in \mathbb{N}). \end{cases} \tag{2.9}$$

Proof. The first equality (2.9) determines uniquely the tensor fields $\underset{(\alpha)j}{A^{\,i}}, \left(\alpha = \overline{1,k}\right), (k \geq 2, k \in \mathbb{N})$. Since $\underset{(\alpha)ij}{C^{\,m}}, \left(\alpha = \overline{1,k}\right), (k \geq 2, k \in \mathbb{N})$ are tensor fields, the second equation (2.9) determines uniquely the tensor field B^{m}_{ij}. Similarly the third, the fourth,...and the last equation (2.9)determine the tensor fields $\underset{(1)ij}{D^{\,m}}, \underset{(2)ij}{D^{\,m}}, ...$ and $\underset{(k)ij}{D^{\,m}}, (k \geq 2, k \in \mathbb{N})$ respectively. □

We have immediately:

Theorem 2. *If* $D\Gamma(N) = \left(L^m_{ij}, \underset{(\alpha)}{C}{}^m_{ij}\right), \left(\alpha = \overline{1,k}\right), (k \geq 2, k \in \mathbb{N})$ *are the local coefficients on an N-linear connection D on* $Osc^k M$ $(k \geq 2, k \in \mathbb{N})$ *and*

$$\left(\underset{(1)}{A}{}^i_j, \underset{(2)}{A}{}^i_j, ..., \underset{(k)}{A}{}^i_j, B^m_{ij}, \underset{(1)}{D}{}^m_{ij}, \underset{(2)}{D}{}^m_{ij}, ..., \underset{(k)}{D}{}^m_{ij}\right), (k \geq 2, k \in \mathbb{N})$$

is a system of tensor fields on $Osc^k M$ $(k \geq 2, k \in \mathbb{N})$ *then*

$$D\overline{\Gamma}\left(\overline{N}\right) = \left(\overline{L}^m_{ij}, \underset{(\alpha)}{\overline{C}}{}^m_{ij}\right), \left(\alpha = \overline{1,k}\right), (k \geq 2, k \in \mathbb{N})$$

given by (2.9) *are the local coefficients of an* $\overline{N}-$*linear connection*, $\overline{D}$, *on* $Osc^k M$; $(k \geq 2, k \in \mathbb{N})$.

The system of tensor fields

$$\left(\underset{(1)}{A}{}^i_j, \underset{(2)}{A}{}^i_j, ..., \underset{(k)}{A}{}^i_j, B^m_{ij}, \underset{(1)}{D}{}^m_{ij}, \underset{(2)}{D}{}^m_{ij}, ..., \underset{(k)}{D}{}^m_{ij}\right), \ (k \geq 2, k \in \mathbb{N})$$

is called the difference tensor fields of $D\Gamma(N)$ to $D\overline{\Gamma}\left(\overline{N}\right)$ and the mapping $D\Gamma(N) \to D\overline{\Gamma}\left(\overline{N}\right)$ given by (2.9) is called a transformation of N-linear connection to $\overline{N}-$linear connection on $Osc^k M$ $(k \geq 2, k \in \mathbb{N})$ and is noted by:

$$t\left(\underset{(1)}{A}{}^i_j, \underset{(2)}{A}{}^i_j, ..., \underset{(k)}{A}{}^i_j, B^m_{ij}, \underset{(1)}{D}{}^m_{ij}, \underset{(2)}{D}{}^m_{ij}, ..., \underset{(k)}{D}{}^m_{ij}\right), (k \geq 2, k \in \mathbb{N}).$$

Theorem 3.*The set* $\mathcal{T}$ *of the transformations of N-linear connections to* $\overline{N}-$*linear connection on* $Osc^k M$ $(k \geq 2, k \in \mathbb{N})$ *together with the composition of mappings ist't a group.*

Proof. Let

$$t\left(\underset{(1)}{A}{}^i_j, \underset{(2)}{A}{}^i_j, ..., \underset{(k)}{A}{}^i_j, B^m_{ij}, \underset{(1)}{D}{}^m_{ij}, \underset{(2)}{D}{}^m_{ij}, ..., \underset{(k)}{D}{}^m_{ij}\right) : D\Gamma(N) \to D\overline{\Gamma}\left(\overline{N}\right)$$

and

$$t\left(\underset{(1)}{\overline{A}}{}^i_j, \underset{(2)}{\overline{A}}{}^i_j, ..., \underset{(k)}{\overline{A}}{}^i_j, B^m_{ij}, \underset{(1)}{D}{}^m_{ij}, \underset{(2)}{D}{}^m_{ij}, ..., \underset{(k)}{D}{}^m_{ij}\right) : D\overline{\Gamma}\left(\overline{N}\right) \to D\overline{\overline{\Gamma}}\left(\overline{\overline{N}}\right), \ (k \geq 2, k \in \mathbb{N})$$

be two transformations from $\mathcal{T}$, given by (2.9).

From (2.9) we have:

$$\underset{(\alpha)}{\overline{\overline{N}}}{}^i_j = \underset{(\alpha)}{N}{}^i_j - \left(\underset{(\alpha)}{A}{}^i_j + \underset{(\alpha)}{\overline{A}}{}^i_j\right), \left(\alpha = \overline{1,k}\right), (k \geq 2, k \in \mathbb{N}).$$

We obtain for example:

$$\overline{\overline{C}}_{(k-1)ij}^{\ \ m} = \underset{(k-1)ij}{C}^{\ \ m} + \underset{(k)il}{C}^{m}\left(\underset{(1)j}{A}^{l} + \underset{(1)j}{\overline{A}}^{l}\right)\left(\underset{(k)il}{D}^{m}\underset{(1)j}{\overline{A}}^{l} + \underset{(k-1)ij}{D}^{\ \ m} + \underset{(k-1)ij}{\overline{D}}^{\ \ m}\right),$$

So, $\overline{\overline{C}}_{(k-1)ij}^{\ \ m}$ hasn't the form (2.9) . Result that the composition of two transformations from $\mathcal{T}$, isn't a transformation from $\mathcal{T}$, so $\mathcal{T}$ together with the composition of mappings isn't a group. □

Remark 1. If we consider $\underset{(\alpha)j}{A}^{i}, \left(\alpha = \overline{1,k}\right), (k \geq 2, k \in \mathbb{N})$ in (2.9) we obtain the set $\mathcal{T}_N$ of transformations of N-linear connections corresponding to the same nonlinear connection N:

$$\mathcal{T}_N = \left\{ t\left(\underbrace{0,0,...,0}_{(k)}, B_{ij}^{m}, \underset{(1)ij}{D}^{m}, \underset{(2)ij}{D}^{m}, ..., \underset{(k)ij}{D}^{m}\right) \in \mathcal{T}\,(k \geq 2, k \in \mathbb{N}) \right\}.$$

We have:

Theorem 4. The set $\mathcal{T}_N$ of the transformations of N-linear connections to N-linear connections on $Osc^k M$ $(k \geq 2, k \in \mathbb{N})$ together with the composition of mappings ist't a group. This group $\mathcal{T}_N$ acts effectively and transitivelly on the set of N-linear connections.

Proof. Let $t\left(\underbrace{0,0,...,0}_{(k)}, B_{ij}^{m}, \underset{(1)ij}{D}^{m}, \underset{(2)ij}{D}^{m}, ..., \underset{(k)ij}{D}^{m}\right) : D\Gamma(N) \rightarrow D\overline{\Gamma}(N), (k \geq 2, k \in \mathbb{N})$ be a transformation from $\mathcal{T}_N$ given by (2.10) : □

Proof.

$$\begin{cases} \underset{(\alpha)j}{\overline{N}}^{i} = \underset{(\alpha)j}{N}^{i}, \left(\alpha = \overline{1,k}\right), (k \geq 2, k \in \mathbb{N}), \\ \underset{(\alpha)ij}{\overline{L}}^{m} = L_{ij}^{m} - B_{ij}^{m}, \\ \underset{(\alpha)ij}{\overline{C}}^{m} = \underset{(\alpha)ij}{C}^{m} - \underset{(\alpha)ij}{N}^{m}, \left(\alpha = \overline{1,k}\right), (k \geq 2, k \in \mathbb{N}). \end{cases} \tag{2.10}$$

The composition of two transformations from $\mathcal{T}_N$ is a transformation from $\mathcal{T}_N$, given by:

$$t\left(\underbrace{0,0,...,0}_{(k)}, \overline{B}_{ij}^{m}, \underset{(1)ij}{\overline{D}}^{m}, \underset{(2)ij}{\overline{D}}^{m}, ..., \underset{(k)ij}{\overline{D}}^{m}\right) \circ t\left(\underbrace{0,0,...,0}_{(k)}, B_{ij}^{m}, \underset{(1)ij}{D}^{m}, \underset{(2)ij}{D}^{m}, ..., \underset{(k)ij}{D}^{m}\right) =$$

$$t\left(\underbrace{0,0,...,0}_{(k)},B_{ij}^{m}+\overline{B}_{ij}^{m},\underset{(1)_{ij}}{D}{}^{m}+\underset{(1)_{ij}}{\overline{D}}{}^{m},\underset{(2)_{ij}}{D}{}^{m}+\underset{(2)_{ij}}{\overline{D}}{}^{m},...,\underset{(k)_{ij}}{D}{}^{m}+\underset{(k)_{ij}}{\overline{D}}{}^{m}\right).$$

The inverse of a transformation from $\mathcal{T}_N$ is the transformation:

$$t\left(\underbrace{0,0,...,0}_{(k)},-B_{ij}^{m},-\underset{(1)_{ij}}{D}{}^{m},-\underset{(2)_{ij}}{D}{}^{m},...,-\underset{(k)_{ij}}{D}{}^{m}\right):D\Gamma\left(N\right)\to D\overline{\Gamma}\left(N\right).$$

The transformation (2.10) preserves all the N-linear connections D if $B_{ij}^{m}=\underset{(\alpha)_{ij}}{D}{}^{m}=0,\left(\alpha=\overline{1,k}\right),\left(k\geq 2,k\in\mathbb{N}\right).$ Therefore $\mathcal{T}_N$ acts effectively on the set of N-linear connections. From the theorem 1. results that $\mathcal{T}_N$ acts transitively on this set. □

Let be:

$$\mathcal{T}_{NL}=\left\{t\left(\underbrace{0,0,...,0}_{(k)},\underset{(1)_{ij}}{D}{}^{m},\underset{(2)_{ij}}{D}{}^{m},...,\underset{(k)_{ij}}{D}{}^{m}\right)\in\mathcal{T}_N,\left(k\geq 2,k\in\mathbb{N}\right)\right\},$$

$$\mathcal{T}_{N\underset{(1)}{C}}=\left\{t\left(\underbrace{0,0,...,0}_{(k)},B_{ij}^{m},0,\underset{(2)_{ij}}{D}{}^{m},...,\underset{(k)_{ij}}{D}{}^{m}\right)\in\mathcal{T}_N,\left(k\geq 2,k\in\mathbb{N}\right)\right\},$$

$$\mathcal{T}_{N\underset{(2)}{C}}=\left\{t\left(\underbrace{0,0,...,0}_{(k)},B_{ij}^{m},\underset{(1)_{ij}}{D}{}^{m},0,\underset{(3)_{ij}}{D}{}^{m},...,\underset{(k)_{ij}}{D}{}^{m}\right)\in\mathcal{T}_N,\left(k\geq 2,k\in\mathbb{N}\right)\right\},$$

....

$$\mathcal{T}_{N\underset{(k)}{C}}=\left\{t\left(\underbrace{0,0,...,0}_{(k)},B_{ij}^{m},\underset{(1)_{ij}}{D}{}^{m},\underset{(2)_{ij}}{D}{}^{m},...,\underset{(k-1)_{ij}}{D}{}^{m},0\right)\in\mathcal{T}_N,\left(k\geq 2,k\in\mathbb{N}\right)\right\},$$

$$\mathcal{T}_{N\underset{(1)}{C}\underset{(2)}{C}}=\left\{t\left(\underbrace{0,0,...,0}_{(k)},B_{ij}^{m},\underbrace{0,0,...,0}_{(k)}\right)\in\mathcal{T}_N,\left(k\geq 2,k\in\mathbb{N}\right)\right\}.$$

Proposition 2. $\mathcal{T}_{NL},\mathcal{T}_{N\underset{(1)}{C}},\mathcal{T}_{N\underset{(2)}{C}},...,\mathcal{T}_{N\underset{(k)}{C}}$ and $\mathcal{T}_{N\underset{(1)}{C}\underset{(2)}{C}...\underset{(k)}{C}}$ are Abelian subgroups of $\mathcal{T}_N$.

Proposition 3. The group $\mathcal{T}_N$ preserves the nonlinear connection N,$\mathcal{T}_{NL}$ preserves the nonlinear connection N and the component L of the local coefficients $\mathrm{D}\Gamma\left(N\right),\mathcal{T}_{N\underset{(1)}{C}}$ preserves the nonlinear connection N and the component $\underset{(1)}{C}$ of local coefficients $D\Gamma(N), ..., \mathcal{T}_{N \underset{(k)}{C}}$ preserves the nonlinear connection N and the component $\underset{(k)}{C}$ of the local coefficients $D\Gamma(N)$ and $\mathcal{T}_{N \underset{(1)}{C} \underset{(2)}{C} \cdots \underset{(k)}{C}}$ preserves the nonlinear connection N and the components $\underset{(1)}{C} \underset{(2)}{C}, ..., \underset{(k)}{C}$ of the local coefficients $D\Gamma(N)$.

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A COMBINED MONTE CARLO AND QUASI-MONTE CARLO METHOD FOR ESTIMATING MULTIDIMENSIONAL INTEGRALS

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Abstract. In this paper, we propose a method to estimate a multidimensional integral I. The method combines the ideas of the Monte Carlo (MC) and Quasi-Monte Carlo (QMC) methods. We call our method *random sampling from non-uniform low-discrepancy sequences.* The method is based on a new estimator for the integral I, for which some theoretical properties are given. A statistical as well as a deterministic analysis of the error are performed. In the statistical analysis, the accuracy is measured by constructing confidence intervals for I. In the deterministic analysis, deterministic upper bounds for the error of approximation are given. The method is applied to a numerical example. The numerical results indicate that our method performs better than the MC and QMC methods.

1. **Introduction**

We consider the problem of approximating the integral of a real valued function f defined over the unit hypercube $[0,1]^s$, given by

$$I = \int_{[0,1]^s} f(x)dx.$$

Two frequently used approaches are the Monte Carlo (MC) and the Quasi-Monte Carlo (QMC) methods.

In the MC method, we generate N independent sample variables $X_1, \ldots, X_N$, from the uniform distribution on $[0,1]^s$. The integral I is estimated by the sample

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mean

$$\hat{I}_N = \frac{1}{N}\sum_{k=1}^{N} f(X_k).$$

The estimator $\hat{I}_N$ is an unbiased estimator of the integral I. The strong law of large numbers tells us that

$$P\Big(\lim_{N\to\infty} \hat{I}_N = I\Big) = 1.$$

In other words, the MC estimator converges almost surely to I, as $N \to \infty$.

The practical advantage of the MC method is that we can easily measure the accuracy of the MC estimate, by using the sample variance $\frac{1}{N-1}\sum_{k=1}^{N}\big(f(X_k)-\hat{I}_N\big)^2$. By constructing confidence intervals for I, we get probabilistic error bounds of order $\mathcal{O}\big(1/\sqrt{N}\big)$.

The QMC method can be defined by analogy with the MC method, by replacing the random samples by a sequence of "well distributed" deterministic points. In this approach, the integral I is approximated by sums of the form $\frac{1}{N}\sum_{k=1}^{N} f(x_k)$, where $(x_1,\ldots,x_N)$ is a sequence of deterministic points, with $x_k \in [0,1]^s$, $k = 1,\ldots,N$.

An important advantage of the QMC method is that we get deterministic upper bounds for the error of approximation, given by the Koksma-Hlawka inequality ([7])

$$\left|\int_{[0,1]^s} f(x)dx - \frac{1}{N}\sum_{k=1}^{N} f(x_k)\right| \le V_{HK}(f) D_N^*(x_1,\ldots,x_N),$$

where $V_{HK}(f) < \infty$ is the variation of f in the sense of Hardy and Krause and $D_N^*(x_1,\ldots,x_N)$ is the discrepancy of sequence $(x_1,\ldots,x_N)$.

When uniformly distributed low-discrepancy sequences are used, the error of approximation in QMC method is of order $\mathcal{O}((\log N)^s/N)$, which is better than the order of MC error. This is due to the fact that, for each dimension s, the inequality $(\log N)^s/N < 1/\sqrt{N}$ holds for a sufficiently large N.

Nevertheless, the error estimation, given by the Koksma-Hlawka inequality, while possible in theory, is intractable in practice. This is due to the difficulty of computing the factors $V_{HK}(f)$ and $D_N^*(x_1,\ldots,x_N)$.

The goal of our paper is to design a method of estimating the integral I that combines the theoretical advantages of the QMC method with the practical advantages of the MC method. We call our method *random sampling from non-uniform low-discrepancy sequences*. We perform a statistical as well as a deterministic analysis of the error. In the statistical analysis, the accuracy can be measured in a practical way, as in MC methods, by constructing confidence intervals for I. In the deterministic analysis, we provide deterministic upper bounds for the error of approximation. The method is applied to a numerical example and compared to the MC and QMC methods.

2. Basic notions and results

We first recall some useful notions and results.

Definition 1 (discrepancy). *Let $P = (x_1, \dots, x_N)$ be a sequence of points in $[0,1]^s$. The discrepancy of sequence P is defined as*

$$D_N^*(P) = \sup_{J \subseteq [0,1]^s} \left| \frac{1}{N} A_N(J,P) - Vol(J) \right|,$$

where the supremum is calculated over all subintervals J of $[0,1]^s$ of the form $\prod_{i=1}^s [0, a_i]$; $Vol(J)$ denotes the volume of J; A_N counts the number of elements of sequence P, falling into the interval J, i.e.,

$$A_N(J,P) = \sum_{k=1}^N 1_J(x_k),$$

1_J is the characteristic function of J.

The sequence P is called uniformly distributed if $D_N^*(P) \to 0$ as $N \to \infty$.

The uniformly distributed sequence P is said to be a low-discrepancy sequence if $D_N^*(P) = \mathcal{O}\big((\log N)^s/N\big)$.

Uniformly distributed low-discrepancy sequences are constructed in [4], [5], [6] and [7]. The definition of discrepancy can be generalized in a straightforward way.

Definition 2 (G-discrepancy). *Consider an s-dimensional continuous distribution on $[0,1]^s$, with distribution function G. Let λ_G be the probability measure induced* *by G. Let* $P = (x_1, \ldots, x_N)$ *be a sequence of points in* $[0,1]^s$*. The G-discrepancy of sequence P is defined as*

$$D^*_{N,G}(P) = \sup_{J \subseteq [0,1]^s} \left| \frac{1}{N} A_N(J,P) - \lambda_G(J) \right|,$$

where the supremum is calculated over all subintervals J of $[0,1]^s$ *of the form* $\prod_{i=1}^s [0, a_i]$.

The sequence P is called G-distributed if $D^*_{N,G}(P) \to 0$ *as* $N \to \infty$.

The G-distributed sequence P is said to be a low-discrepancy sequence if $D^*_{N,G}(P) = \mathcal{O}\big((\log N)^s / N\big)$.

The non-uniform Koksma-Hlawka inequality ([2]) gives an upper bound for the error of approximation in QMC integration, when G-distributed low-discrepancy sequences are used.

Theorem 3 (non-uniform Koksma-Hlawka inequality). *Let* $f : [0,1]^s \to \mathbb{R}$ *be a function of bounded variation in the sense of Hardy and Krause and* $(x_1, \ldots, x_N)$ *be a sequence of points in* $[0,1]^s$*. Consider an s-dimensional continuous distribution on* $[0,1]^s$*, with distribution function G. Then, for any* $N > 0$

$$\left| \int_{[0,1]^s} f(x) dG(x) - \frac{1}{N} \sum_{k=1}^N f(x_k) \right| \le V_{HK}(f) D^*_{N,G}(x_1, \ldots, x_N),$$

where $V_{HK}(f)$ *is the variation of f in the sense of Hardy and Krause.*

In order to generate G-distributed low-discrepancy sequences in $[0,1]^s$, we use the one-dimensional marginal distributions defined below.

Definition 4. *Consider an s-dimensional continuous distribution on* $[0,1]^s$*, with density function g. For a point* $u = \big(u^{(1)}, \ldots, u^{(s)}\big) \in [0,1]^s$*, the marginal density functions* g_l*,* $l = 1, \ldots, s$*, are defined by*

$$g_l\big(u^{(l)}\big) = \underbrace{\int \ldots \int}_{[0,1]^{s-1}} g\big(t^{(1)}, \ldots, t^{(l-1)}, u^{(l)}, t^{(l+1)}, \ldots t^{(s)}\big) dt^{(1)} \ldots dt^{(l-1)} dt^{(l+1)} \ldots dt^{(s)},$$

and the marginal distribution functions G_l*,* $l = 1, \ldots, s$*, are defined by*

$$G_l\big(u^{(l)}\big) = \int_0^{u^{(l)}} g_l(t) dt.$$

In this paper, we consider s-dimensional continuous distributions on $[0,1]^s$, with independent marginals. Moreover, it is assumed that the functions G_l, $l = 1,\dots,s$, are invertible on $[0,1]$ and their inverses are given explicitly in analytical form.

In this case, G-distributed low-discrepancy sequences in $[0,1]^s$ can be constructed as follows. First, we consider a uniformly distributed low-discrepancy sequence in $[0,1]^s$, $\alpha = (\alpha_1,\dots,\alpha_N)$, with $\alpha_k = \big(\alpha_k^{(1)},\dots,\alpha_k^{(s)}\big)$, $k = 1,\dots,N$. Then, we construct the sequence $\beta = (\beta_1,\dots,\beta_N)$ in $[0,1]^s$, with $\beta_k = \big(\beta_k^{(1)},\dots,\beta_k^{(s)}\big)$, $k = 1,\dots,N$, defined by

$$\beta_k^{(1)} = G_1^{-1}\big(\alpha_k^{(1)}\big), \quad \beta_k^{(2)} = G_2^{-1}\big(\alpha_k^{(2)}\big),\ \dots\ ,\beta_k^{(s)} = G_s^{-1}\big(\alpha_k^{(s)}\big), \quad \text{sequentially.}$$

Such a transformation preserves the discrepancy, as shown in the following theorem.

Theorem 5. *(see [9]) Let $\alpha = (\alpha_1,\dots,\alpha_N)$ be a sequence in $[0,1]^s$. Consider an s-dimensional continuous distribution on $[0,1]^s$, with distribution function G. Assume that $G(u) = \prod_{l=1}^{s} G_l(u^{(l)})$, $\forall u = \big(u^{(1)},\dots,u^{(s)}\big) \in [0,1]^s$, and that the functions G_l, $l = 1,\dots,s$, are invertible on $[0,1]$. Let $\beta = (\beta_1,\dots,\beta_N)$ be the sequence constructed as above. Then*

$$D^*_{N,G}(\beta_1,\dots,\beta_N) = D^*_N(\alpha_1,\dots,\alpha_N).$$

From Theorem 5, it follows that, if α is a uniformly distributed low-discrepancy sequence in $[0,1]^s$, then β is a G-distributed low-discrepancy sequence in $[0,1]^s$.

For the case when the inverse functions G_l^{-1}, $l = 1,\dots,s$, are not explicitly available, modalities to generate G-distributed low-discrepancy sequences in $[0,1]^s$ are proposed in [11] and [12].

3. The method of random sampling from non-uniform low-discrepancy sequences

In the following, we propose a method to estimate the integral I. The method combines the ideas of the MC and QMC methods. Our method uses an s-dimensional continuous distribution on $[0,1]^s$, with distribution function G and density function

g (g is nonnegative and $\int_{[0,1]^s} g(u)du = 1$). It is assumed that $G(u) = \prod_{l=1}^{s} G_l(u^{(l)})$, $\forall u = \left(u^{(1)}, \ldots, u^{(s)}\right) \in [0,1]^s$. Moreover, it is assumed that the functions G_l, $l = 1, \ldots, s$, are invertible on $[0,1]$, and G_l^{-1}, $l = 1, \ldots, s$, are given explicitly in analytical form.

In our method, we consider Ω to be a set of G-distributed low-discrepancy sequences in $[0,1]^s$, $\Omega = \{\beta_1, \ldots, \beta_r\}$, where sequence β_i, $i = 1, \ldots, r$, has the following form:

$$\beta_i = (\beta_{1,i}, \ldots, \beta_{N,i}),$$

with $\beta_{k,i} = (\beta_{k,i}^{(1)}, \ldots, \beta_{k,i}^{(s)}) \in [0,1]^s$, $k = 1, \ldots, N$.

An arbitrary sequence β_i is obtained from a uniformly distributed low-discrepancy sequence α_i in $[0,1]^s$, $\alpha_i = (\alpha_{1,i}, \ldots, \alpha_{N,i})$, with $\alpha_{k,i} = \left(\alpha_{k,i}^{(1)}, \ldots, \alpha_{k,i}^{(s)}\right)$, $k = 1, \ldots, N$, as follows:

$$\beta_{k,i}^{(1)} = G_1^{-1}\left(\alpha_{k,i}^{(1)}\right), \quad \beta_{k,i}^{(2)} = G_2^{-1}\left(\alpha_{k,i}^{(2)}\right) \quad , \ldots, \quad \beta_{k,i}^{(s)} = G_s^{-1}\left(\alpha_{k,i}^{(s)}\right), \quad \text{sequentially.} \tag{1}$$

For instance, the sequences α_i, $i = 1, \ldots, r$, may be Halton sequences in prime bases $p_{1,i}, \ldots, p_{s,i}$ ([6]) or SQRT sequences ([10]).

According to Theorem 5, the G-discrepancies of the constructed sequences $\beta_i = (\beta_{1,i}, \ldots, \beta_{N,i})$ are given by

$$D_{N,G}^*(\beta_{1,i}, \ldots, \beta_{N,i}) = D_N^*(\alpha_{1,i}, \ldots, \alpha_{N,i}), \qquad i = 1, \ldots, r.$$

We rewrite the integral in the following form:

$$I = \int_{[0,1]^s} f(x)dx = \int_{[0,1]^s} \frac{f(x)}{g(x)} g(x)dx = \int_{[0,1]^s} \frac{f(x)}{g(x)} dG(x).$$

The integral $I = \int_{[0,1]^s} f(x)/g(x)dG(x)$ can be approximated by sums of the form $\frac{1}{N}\sum_{k=1}^{N} \frac{f(\beta_{k,i})}{g(\beta_{k,i})}$, which represent QMC approximations. An upper bound for the error of the approximation, when the G-distributed low-discrepancy sequence β_i is used, is given by the following theorem.

Theorem 6. *If f/g is a function of bounded variation in the sense of Hardy and Krause, then, for any $N > 0$ and for all $i = 1, \ldots, r$, we have*

$$\left| I - \frac{1}{N} \sum_{k=1}^{N} \frac{f(\beta_{k,i})}{g(\beta_{k,i})} \right| \leq V_{HK}\left(\frac{f}{g}\right) D^*_{N,G}(\beta_{1,i}, \ldots, \beta_{N,i}).$$

The above theorem is derived from Theorem 3.

Corollary 7. *For any G-distributed low-discrepancy sequence $\beta_i \in \Omega$, $\beta_i = (\beta_{1,i}, \ldots, \beta_{N,i})$, $i = 1, \ldots, r$, we have*

$$\frac{1}{N} \sum_{k=1}^{N} \frac{f(\beta_{k,i})}{g(\beta_{k,i})} \to I, \qquad N \to \infty.$$

We define the random variable X_N on the space Ω as follows.

Definition 8. *For an arbitrary sequence $\beta_i = (\beta_{1,i}, \ldots, \beta_{N,i}) \in \Omega$, the value of the random variable X_N is defined as*

$$X_N(\beta_i) = \frac{1}{N} \sum_{k=1}^{N} \frac{f(\beta_{k,i})}{g(\beta_{k,i})},$$

and is taken with probability $1/r$.

Remark 9. *The distribution of the random variable X_N is*

$$X_N : \begin{pmatrix} \frac{1}{N} \sum_{k=1}^{N} \frac{f(\beta_{k,i})}{g(\beta_{k,i})} \\ 1/r \end{pmatrix}_{\substack{\beta_i=(\beta_{1,i},\ldots,\beta_{N,i}) \\ i=1,\ldots,r}}.$$

Next, we give some properties of the random variable X_N. These properties will be used later, in the statistical analysis.

Theorem 10. *The random variable X_N has the following properties:*

$$\lim_{N\to\infty} E(X_N) = I, \tag{2}$$

$$\lim_{N\to\infty} Var(X_N) = 0. \tag{3}$$

Proof. 1) We have

$$\begin{aligned}
\lim_{N\to\infty} E(X_N) &= \lim_{N\to\infty} \sum_{i=1}^{r} \frac{1}{r} \frac{1}{N} \sum_{k=1}^{N} \frac{f(\beta_{k,i})}{g(\beta_{k,i})} \\
&= \frac{1}{r} \sum_{i=1}^{r} \lim_{N\to\infty} \left(\frac{1}{N} \sum_{k=1}^{N} \frac{f(\beta_{k,i})}{g(\beta_{k,i})} \right) \\
&= \frac{1}{r} \sum_{i=1}^{r} I = I.
\end{aligned}$$

For one of the previous identities, we used Corollary 7.

2) We know that $Var(X_N) = E(X_N^2) - (E(X_N))^2$. We first calculate $E(X_N^2)$.

$$E(X_N^2) = \sum_{i=1}^{r} \frac{1}{r} \frac{1}{N^2} \left(\sum_{k=1}^{N} \frac{f(\beta_{k,i})}{g(\beta_{k,i})} \right)^2.$$

It follows that

$$Var(X_N) = \sum_{i=1}^{r} \frac{1}{r} \left(\frac{\sum_{k=1}^{N} \frac{f(\beta_{k,i})}{g(\beta_{k,i})}}{N} \right)^2 - \left(\sum_{i=1}^{r} \frac{1}{r} \frac{1}{N} \sum_{k=1}^{N} \frac{f(\beta_{k,i})}{g(\beta_{k,i})} \right)^2.$$

By letting $N \to \infty$, and using Corollary 7, we obtain

$$\lim_{N\to\infty} Var(X_N) = \left(\frac{1}{r} \sum_{i=1}^{r} I^2 \right) - I^2 = 0.$$

□

Once we have defined the random variable X_N, we select the integers $i_1, \ldots, i_M$ at random from the uniform distribution on $\{1, \ldots, r\}$, and consider the corresponding sequences $\beta_{i_1}, \ldots, \beta_{i_M}$. For each sequence, we compute the value of the random variable X_N. The values $X_N(\beta_{i_l})$, $l = 1, \ldots, M$, are values of the sample variables $X_{N,i_1}, \ldots, X_{N,i_M}$ that are independent identically distributed random variables and have the same distribution as X_N.

We use the notation $\overline{X}_{N,M}$ for the sample mean of the random variables $X_{N,i_1}, \ldots, X_{N,i_M}$, and $\overline{x}_{N,M}$ for its value, i.e.,

$$\begin{aligned}\overline{X}_{N,M} &= \frac{X_{N,i_1} + \ldots + X_{N,i_M}}{M}, \\ \overline{x}_{N,M} &= \frac{\sum_{l=1}^{M} X_{N,i_l}(\beta_{i_l})}{M} = \frac{\sum_{l=1}^{M}\left(\frac{1}{N}\sum_{k=1}^{N}\frac{f(\beta_{k,i_l})}{g(\beta_{k,i_l})}\right)}{M}.\end{aligned}$$

We formulate some properties of the estimator $\overline{X}_{N,M}$.

Proposition 11. *For a fixed N, the estimator $\overline{X}_{N,M}$ has the following properties:*

$$E(\overline{X}_{N,M}) = E(X_N), \quad (\textit{unbiased estimator of } E(X_N)), \tag{4}$$

$$\lim_{M\to\infty} Var(\overline{X}_{N,M}) = 0, \tag{5}$$

$$P\Big(\lim_{M\to\infty} \overline{X}_{N,M} = E(X_N)\Big) = 1, \quad (\overline{X}_{N,M} \textit{ converges almost surely to } E(X_N)) \tag{6}$$

Proof. Properties (4) and (5) can be proved using known properties of the mean and variance. For property (6), we apply the Kolmogorov theorem ([1]) to the sequence of random variables $(X_{N,i})_{i\geq 1}$, that are independent identically distributed and have finite means $E(X_{N,i}) = E(X_N) < \infty$. Under these conditions, the Kolmogorov theorem asserts that relation (6) is satisfied. □

Proposition 12. *For a fixed M, we have the following properties of the estimator $\overline{X}_{N,M}$:*

$$\begin{aligned}\lim_{N\to\infty} E(\overline{X}_{N,M}) &= I, \\ \lim_{N\to\infty} Var(\overline{X}_{N,M}) &= 0.\end{aligned}$$

Proof. The proof is immediately, by using property (4) and Theorem 10. □

Taking into account these properties, in our method the integral I is approximated by

$$I \cong \overline{x}_{N,M} = \frac{\sum_{l=1}^{M} X_{N,i_l}(\beta_{i_l})}{M} = \frac{\sum_{l=1}^{M}\left(\frac{1}{N}\sum_{k=1}^{N}\frac{f(\beta_{k,i_l})}{g(\beta_{k,i_l})}\right)}{M}. \tag{7}$$

We call our method *random sampling from non-uniform low-discrepancy sequences* (RSNU method). The method is based on the estimator $\overline{X}_{N,M}$, which shall be referred as an RSNU estimator. We call the value $\overline{x}_{N,M}$ an RSNU estimate.

Our method is described in the Algorithm 13.

Algorithm 13. *The method of random sampling from non-uniform low-discrepancy sequences*

Input data:

- the function f, the density function g;
- the marginal distribution functions G_l, $l = 1, \ldots, s$, and their inverses;
- the integers N, M, r;
- the uniformly distributed low-discrepancy sequences $\alpha_i = (\alpha_{1,i}, \ldots, \alpha_{N,i})$, $i = 1, \ldots, r$;

Step 1. Construct the G-distributed low-discrepancy sequences $\beta_i = (\beta_{1,i}, \ldots, \beta_{N,i})$, $i = 1, \ldots, r$, using formula (1).

Step 2. Select the integers $i_1, \ldots, i_M$ at random from the uniform distribution on $\{1, \ldots, r\}$ and consider the corresponding sequences $\beta_{i_1}, \ldots, \beta_{i_M}$.

Step 3. For each sequence β_{i_l}, $l = 1, \ldots, M$, compute the value of the sample variable X_{N,i_l}:

$$x_{N,i_l} = \frac{1}{N} \sum_{k=1}^{N} \frac{f(\beta_{k,i_l})}{g(\beta_{k,i_l})}.$$

Step 4. Compute the sample mean

$$\overline{x}_{N,M} = \frac{\sum_{l=1}^{M} x_{N,i_l}}{M}.$$

Output data: The value $\overline{x}_{N,M}$, which approximates the integral I.

4. **Confidence intervals**

We first construct confidence intervals for $E(X_N)$ and then give an important remark concerning the confidence intervals for the integral I.

We consider the confidence level $\alpha \in (0,1)$. We use the sample standard deviation

$$\overline{\sigma}_{X_N} = \sqrt{\frac{1}{M-1}\sum_{l=1}^{M}\left(X_{N,i_l} - \overline{X}_{N,M}\right)^2}.$$

Proposition 14. *A $(1-\alpha)\%$ confidence interval for $E(X_N)$ is*

$$\left(\overline{X}_{N,M} - t_{M-1,1-\frac{\alpha}{2}}\frac{\overline{\sigma}_{X_N}}{\sqrt{M}},\quad \overline{X}_{N,M} + t_{M-1,1-\frac{\alpha}{2}}\frac{\overline{\sigma}_{X_N}}{\sqrt{M}}\right). \tag{8}$$

where $t_{M-1,1-\frac{\alpha}{2}}$ is the $(1-\frac{\alpha}{2})$-th percentile of the Student distribution with $M-1$ degrees of freedom.

Proof. We consider the statistics

$$T = \frac{\overline{X}_{N,M} - E(X_N)}{\frac{\overline{\sigma}_{X_N}}{\sqrt{M}}},$$

that has the t (Student) distribution with $M-1$ degrees of freedom. Thus, a $(1-\alpha)\%$ confidence interval for $E(X_N)$ is given by (8). □

Remark 15. *We proved that $E(X_N) \to I$, as $N \to \infty$ (property (2)). Therefore, for N sufficiently large, we consider $E(X_N) \cong I$. Consequently, for large enough values of N, the confidence interval for I is well approximated by the confidence interval for $E(X_N)$, given by (8).*

5. **Deterministic error bounds**

In what follows, we give deterministic upper bounds for the error of approximation in formula (7). We have the following main result.

Theorem 16. *The error of approximation in RSNU method is bounded by*

$$\left|I - \overline{x}_{N,M}\right| \le \frac{1}{M} V_{HK}\left(\frac{f}{g}\right)\sum_{l=1}^{M} D^*_{N,G}(\beta_{i_l}).$$

Proof. We can write

$$\begin{aligned} \left|I - \overline{x}_{N,M}\right| &= \left|I - \frac{\sum_{l=1}^{M} X_{N,i_l}(\beta_{i_l})}{M}\right| \\ &\leq \frac{1}{M}\sum_{l=1}^{M}\left|I - X_{N,i_l}(\beta_{i_l})\right| \\ &\leq \frac{1}{M} V_{HK}\left(\frac{f}{g}\right)\sum_{l=1}^{M} D^*_{N,G}(\beta_{i_l}). \end{aligned}$$

For the last inequality, we used Theorem 6. □

Corollary 17. *For a fixed M, the RSNU estimate satisfies the following property:*

$$\lim_{N\to\infty} \overline{x}_{N,M} = I.$$

Next, we compare, from the error point of view, the RSNU method with the QMC method. As $\alpha_{i_l} = (\alpha_{1,i_l}, \ldots, \alpha_{N,i_l})$, $l = 1, \ldots M$, are uniformly distributed low-discrepancy sequences in $[0,1]^s$, there exists the explicitly computable constants C_{i_l} such that

$$D^*_N(\alpha_{i_l}) \leq C_{i_l}\frac{(\log N)^s}{N}.$$

We define $C' = \min_{l=\overline{1,M}} C_{i_l}$.

Theorem 18. *A necessary condition for the error bound in RSNU method to be smaller than each error bound in QMC method, obtained when the sequence α_{i_l}, $l = 1, \ldots M$, is used, is*

$$V_{HK}\left(\frac{f}{g}\right) < V_{HK}(f)\frac{MC'}{\sum_{l=1}^{M} C_{i_l}}.$$

Proof. The error bound in QMC method, when the sequence α_{i_l} is used, is given by

$$\begin{aligned} \left|I - \frac{1}{N}\sum_{k=1}^{N} f(\alpha_{k,i_l})\right| &\leq V_{HK}(f) D^*_N(\alpha_{1,i_l}, \ldots, \alpha_{N,i_l}) \\ &\leq V_{HK}(f) C_{i_l}\frac{(\log N)^s}{N}, \qquad l = 1, \ldots, M. \end{aligned}$$

The error bound for the RSNU method is given by

$$\begin{aligned} \left|I - \overline{x}_{N,M}\right| &\leq \frac{1}{M} V_{HK}\left(\frac{f}{g}\right) \sum_{l=1}^{M} D^*_{N,G}(\beta_{i_l}) = \frac{1}{M} V_{HK}\left(\frac{f}{g}\right) \sum_{l=1}^{M} D^*_{N}(\alpha_{i_l}) \\ &\leq \frac{1}{M} V_{HK}\left(\frac{f}{g}\right) \sum_{l=1}^{M} C_{i_l} \frac{(\log N)^s}{N}. \end{aligned}$$

We impose the following conditions:

$$\frac{1}{M} V_{HK}\left(\frac{f}{g}\right) \sum_{l=1}^{M} C_{i_l} \frac{(\log N)^s}{N} < V_{HK}(f) C_{i_j} \frac{(\log N)^s}{N}, \qquad \forall j = 1, \ldots, M.$$

We obtain

$$V_{HK}\left(\frac{f}{g}\right) \frac{\sum_{l=1}^{M} C_{i_l}}{M} < V_{HK}(f) C_{i_j}, \qquad \forall j = 1, \ldots, M,$$

which is equivalent to

$$V_{HK}\left(\frac{f}{g}\right) < V_{HK}(f) \frac{MC'}{\sum_{l=1}^{M} C_{i_l}}.$$

□

6. **Numerical example**

We consider a numerical example to illustrate our method and to compare it with the MC and QMC methods. We want to estimate the integral

$$I = \int_0^1 \int_0^1 \int_0^1 16 x y^3 z^2 e^{xz} dx dy dz,$$

whose exact value is $4(3-e) \approx 1.1268726$. We choose the density function $g(x, y, z) = 12 x y^2 z$. We determine the marginal distribution functions

$$G_1(x) = x^2, \qquad G_2(y) = y^3, \qquad G_3(z) = z^2,$$

and the inverses of these functions

$$G_1^{-1}(x) = \sqrt{x}, \qquad G_2^{-1}(y) = \sqrt[3]{y}, \qquad G_3^{-1}(z) = \sqrt{z}.$$

To apply the RSNU method, we need to populate the space Ω. For this, we first generate a set A that contains the first 30 prime numbers

$$A = \{2, 3, 5, 7, \ldots, 113\}.$$

Next, we construct all the subsets with 3 elements of the set A. There are $r = C_{30}^3 = 4060$ such subsets of A. For each subset $A_i = \{p_{i,1}, p_{i,2}, p_{i,3}\}$, we define the SQRT sequence $\alpha_i = (\alpha_{1,i}, \dots, \alpha_{N,i})$, by

$$\alpha_{k,i} = (\{k\sqrt{p_{i,1}}\}, \{k\sqrt{p_{i,2}}\}, \{k\sqrt{p_{i,3}}\}), \qquad k = 1, \dots, N.$$

The defined SQRT sequences α_i, $i = 1, \dots, r$, are uniformly distributed low-discrepancy sequences in $[0,1]^3$.

Then, we construct the space Ω of G-distributed low-discrepancy sequences, $\Omega = \{\beta_1, \dots, \beta_r\}$, where β_i, $i = 1, \dots, r$, has the following form:

$$\beta_i = (\beta_{1,i}, \dots, \beta_{N,i}),$$

with $\beta_{k,i} = \left(\beta_{k,i}^{(1)}, \beta_{k,i}^{(2)}, \beta_{k,i}^{(3)}\right) \in [0,1]^3$. An arbitrary sequence β_i is obtained from the sequence α_i, as follows:

$$\beta_{k,i}^{(1)} = \sqrt{\alpha_{k,i}^{(1)}}, \qquad \beta_{k,i}^{(2)} = \sqrt[3]{\alpha_{k,i}^{(2)}}, \qquad \beta_{k,i}^{(3)} = \sqrt{\alpha_{k,i}^{(3)}}, \qquad \text{sequentially.}$$

Next, we select the integers $i_1, \dots, i_M$ at random from the uniform distribution on $\{1, \dots, r\}$ and consider the corresponding G-distributed low-discrepancy sequences $\beta_{i_1}, \dots, \beta_{i_M}$.

We calculate the following estimates:

$$\text{RSNU estimate} \qquad \hat{I}_{RSNU} = \frac{\sum_{l=1}^{M} \left(\frac{1}{N} \sum_{k=1}^{N} \frac{f(\beta_{k,i_l})}{g(\beta_{k,i_l})} \right)}{M},$$

$$\text{MC estimate} \qquad \hat{I}_{MC} = \frac{1}{NM} \sum_{k=1}^{NM} f(x_k), \tag{9}$$

$$\text{QMC estimate} \qquad \hat{I}_{QMC} = \frac{1}{NM} \sum_{k=1}^{NM} f(x_k). \tag{10}$$

In (9), x_k are random numbers uniformly distributed in $[0,1]^3$. In (10), $(x_1, \dots, x_{NM})$ is the SQRT sequence in $[0,1]^3$

$$x_k = (\{k\sqrt{2}\}, \{k\sqrt{3}\}, \{k\sqrt{5}\}), \qquad k = 1, \dots, NM.$$

After performing several experiments, we give the results for the case $M = 20$. The following table contains: the value of N, the estimates $\hat{I}_{MC}$, $\hat{I}_{QMC}$, $\hat{I}_{RSNU}$ and the absolute values of the errors $|I - \hat{I}_{MC}|$, $|I - \hat{I}_{QMC}|$, $|I - \hat{I}_{RSNU}|$.

N	$\hat{I}_{MC}$	$\hat{I}_{QMC}$	$\hat{I}_{RSNU}$	$\|I - \hat{I}_{MC}\|$	$\|I - \hat{I}_{QMC}\|$	$\|I - \hat{I}_{RSNU}\|$
1000	1.1606	1.1301	1.1267	0.0338	0.0032	0.00021
1200	1.1562	1.1299	1.1261	0.0294	0.0031	0.00072
1400	1.1440	1.1313	1.1266	0.0171	0.0045	0.00024
1600	1.1535	1.1315	1.1265	0.0266	0.0047	0.00036
1800	1.1553	1.1300	1.1272	0.0285	0.0031	0.00036
2000	1.1554	1.1291	1.1266	0.0286	0.0022	0.00026

Table 1: Case M=20.

The numerical results indicate that the proposed RSNU estimate converges much faster than the MC and QMC estimates. The error in RSNU method is smaller than the error in QMC method by approximately a factor of 10. The error in RSNU method gives approximately a factor of 100 improvement over the error in MC method.

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BOOK REVIEWS

Joseph A. Cima, Alec L. Matheson and William T. Ross, *The Cauchy Transform,* Mathematical Surveys and Monographs , Vol. 125, American Mathematical Society 2006, ix + 272 pp., ISBN 0-8218-3871-7.

Denote by M the space of all finite, complex Borel measures on the unit circle $\mathbb{T} = \partial\mathbb{D}$, where $\mathbb{D}$ is the open unit disk in $\mathbb{C}$. The Cauchy transform of a measure $\mu \in M$ is defined by $(K\mu)(z) := \int_{\mathbb{T}}(1 - \bar{\zeta}z)^{-1}d\mu(\zeta)$, $z \in \mathbb{D}$. It turns out that $K\mu$ is analytic in $\mathbb{D}$, its power extension being $(K\mu)(z) = \sum_{n=0}^{\infty} \hat{\mu}(n)z^n$, where $\hat{\mu}(n) = \int \bar{\zeta} d\mu(\zeta)$, $n \in \mathbb{Z}$ are the Fourier coefficients of the measure μ. One denotes by $\mathcal{K}$ the space of all analytic function representable by the Cauchy transforms.

The monograph is dedicated to a thorough study of many aspects of the Cauchy transform: function theoretic properties, properties of the operator $\mu \mapsto K\mu$, functional analytic properties of the space $\mathcal{K}$, characterizations of analytic functions representable by the Cauchy transform, multipliers (functions ϕ such that $\phi\mathcal{K} \subset \mathcal{K}$), some classical operators on $\mathcal{K}$ (shifts, composition operators), and the properties of the distribution function $y \mapsto m(|K\mu| > y)$, where m is the normalized Lebesgue measure on $\mathbb{T}$.

Some background material from functional analysis, complex analysis, Hardy spaces, interpolation, is outlined in the first chapter of the book.

The matter starts in the second chapter, *The Cauchy transform as function,* dealing with growth estimates, boundary behavior for the Cauchy transform, Plemelj's formula, and others. In Chapter 3, *The Cauchy transform as an operator,* one studies operator theoretic properties of the Cauchy transform and contains results of Privalov, Riesz, Kolmogorov, the spaces BMO and BMOA, and an introduction to Hilbert transform.

The functional analysis of the Banach space $\mathcal{K}$ is developed in Chapter 4, *Topologies on the space of Cauchy transforms.* Since the dual of the space $L^1/\bar{H}_0^1$ is isometrically isomorphic to H^∞, the dual of $\mathcal{K}$ can also be identified with H^∞, so that $\mathcal{K}$ is not reflexive. This chapter also comprises a study of the weak* topology of the space $\mathcal{K}$.

Chapter 5, *Which functions are Cauchy integrals ?* addresses the problem of characterization of analytic functions representable by the Cauchy transform and contains important results of Havin, Tumarkin and Hruščev. Multipliers are studied in Chapter 6, *Multipliers and divisors,* establishing some interesting connections with Toeplitz operators and inner functions. Some recent results of Goluzina, Hruščev and Vinogradov are included. The distribution function is studied in Chapter 7, *The distribution function for the Cauchy transform.* As a curiosity, a result of G. Boole from 1857 (rediscovered several times later) on the distribution function is used to prove a theorem of Hruščev and Vinogradov (1981).

The rest of the book, Chapters 8. *The backward shift on H^2,* 9. *Clark measures ,* 10. *The normalized Cauchy transform,* and 11. *Other operators on the Cauchy transforms,* is devoted to recent advances of Aleksandrov and Poltoratski based on a seminal paper by D. Clark (1972) relating the Cauchy transform and perturbation theory.

Combining both classical and recent result, the book presents a great interest for students, teachers and researchers interested mainly in functional analysis methods in complex analysis. The topics are presented in an elegant manner, with many comments, detours and historical references. The result is a fine book that deserves to be on the bookshelf of each analyst.

S. Cobzaş

Jorge Ize, Alfonso Vignoli, *Equivariant Degree Theory*, Walter de Gruyter, Berlin - New York, 2003, 361 pages, ISBN 3110175509.

In the last two decades many paper were dedicated to the study of the symmetry breaking for differential equations, Hopf bifurcation problems, periodic solutions of Hamiltonians systems. A very useful tool in the study of these problems is the equivariant degree theory. There are many equivalent methods to construct the degree theory, depending on the possible applications or on the particular taste of the user.

In this book the authors present in a very elegant way a new degree theory for maps which commute with a group of symmetries. The book contains four chapters. The first chapter is devoted to the presentation of the basic tools - representation theory, equivariant homotopy theory and differential equations - needed in the text.

The second chapter is devoted to the definition and the study of the basic properties of the equivariant degree. The construction is done first in the finite dimensional case, and then the notion of degree is extended to infinite dimensions using approximations by finite dimensional maps, as in the case of Leray-Schauder degree. The orthogonal degree is also defined and studied. At the end of this chapter one defines the usual operators of the degree: symmetry breaking, product and composition.

Chapter 3, *Equivariant Homotopy Groups of Spheres,* is divided into seven sections. The first section is concerned with the extension problem, which will be used in the next two sections to calculate the homotopy groups of Γ-maps and of Γ-classes. The following three sections are dealing with Borsuk-Ulam type results and orthogonal maps. In the last section of this chapter it is shown how the Γ-homotopy groups of spheres behave under different operation: suspension, reduction of the group, products and composition.

The last chapter, Chapter 4. *Equivariant Degree and Applications*, is devoted to various applications of the equivariant degree defined in the second chapter. Here we mention: differential equations with fixed period and with first integral, symmetry breaking for differential equations, periodic solutions of Hamiltonian systems, spring-pendulum equations and Hopf bifurcations.

Due to the included results and examples and to the self-contained and unifying approach, this book can be helpful to researchers and postgraduate students working in nonlinear analysis, differential equations, topology, and in quantitative aspects of applied mathematics.

Csaba Varga

Jan Brinkhuis & Vladimir Tikhomirov, ***Optimization: Insights and Applications,*** Princeton Series in Applied Mathematics, Princeton University Press, Princeton and Oxford 2005, xxiv + 658 pp., ISBN 0-691-10287-2.

This is a self contained informal book on optimization presented by means of numerous examples and applications at various level of sophistication, depending on the mathematical background of the reader. The authors call metaphorically these levels lunch, dinner and dessert, nicely illustrated by the painting of Floris van Dijk, "Still life with cheeses", reproduced on the front cover of the book.

One supposes that the reader has already had the *breakfast,* meaning a first course on vectors, matrices, continuity, differentiation. For his/her convenience, some *snacks* are supplied in the appendices A (on vectors and matrices), B (on differentiation), C (on continuity) - three refreshment courses - and in the introductory chapter *Necessary Conditions: What is the point*?

The *lunch* is a light, simple and enjoyable meal, devoted to those interested mainly in applications. This part is formed by the chapters 1. *Fermat: One variable without constraints*; 2. *Fermat: Two or more variables without constraints*; 3. *Lagrange: Equality constraints*; 4. *Inequality constraints and convexity*; 6. *Basic algorithms.* In this part proofs are optional as well as the related Chapter 5. *Second order conditions,* and Appendix D. *Crash course on problem solving.*

The base meal is the *dinner,* a substantial, refined and tasty meal requiring more effort for preparation and for its appreciation as well. This refers to chapters 5. *Second order conditions*; 7. *Advanced algorithms*; 10. *Mixed smooth-convex problems*; 12. *Dynamic optimization in continuous time*, and the appendices E. *Crash course on optimization: Geometrical style*; F. *Crash course on optimization: Analytical style*, and G. *Conditions of extremum: From Fermat to Pontryagin.* This part contains also full proofs of the results from the "lunch sections", where they are only sketched.

The dessert is delicious and without special motivation, at the choice of the reader, just for fun and pleasure, and concerns applications of optimization methods. Some of these are gathered in the chapters 8. *Economic applications*; 9. *Mathematical applications*, and in the chapters on numerical methods: 6. *Basic algorithms,* and 7. *Advanced algorithms.* Other applications are contained in the numerous problems and exercises scattered throughout the book.

In many places in the book there are indications for a shortcut to applications (the dessert) under the heading *royal road*, showing that, in spite to the famous Euclid answer to the pharaoh of Egypt: "There is no royal road to geometry", there are such roads. "Insights" in the title reflects one of the overarching points of the book, namely that most problems can be solved by the direct application of the theorems of Fermat, Lagrange and Weierstrass. All the proofs are preceded by simple explanatory geometric figures, which make the writing of the rigorous analytic proofs a routine task, a principle nicely motivated by a quotation from Plato: "Geometry draws the soul to the truth".

Beside those mentioned above, the book contains a lot of quotations from scholars - mathematicians, physicists, economists, philosophers, historical comments and some anecdotes as, for instance, that with the trace of tzar's finger on the Moscow-Sankt Petersburg rail road line. The very interesting and witty examples and puzzles from economics, physics, mechanics, economics and everyday life, rise the quality of the book and make its reading a pleasant and instructive enterprise.

Written in a live and informal style, containing a lot of examples treated first at an elementary, heuristical level, and solved rigorously later, the book appeals to a large audience including economists, engineers, physicists, mathematicians or people interested to learn something about some famous problems and puzzles from the humanity spiritual thesaurus. The book is also of interest for the experts who can find some simpler and ingenious proofs of some results, culminating with that of the Pontryagin extremum principle, presented in appendix G.

S. Cobzaş

Andrzej Ruszczyński, ***Nonlinear Optimization,*** Princeton University Press, Princeton and Oxford 2006, xii + 448 pp., ISBN 13: 978-0-691-11915-1 and 10: 0-691-11915-5.

The book is based on a course on optimization theory taught by the author for a period of 25 years at Warsaw University, Princeton University, University of Wisconsin-Madison, and Rutgers University, for students of engineering, applied mathematics, and management sciences. It is organized in two parts, 1. THEORY and 2. METHODS, allowing the treatment of applications of optimization theory on a rigorous mathematical foundation. The applications, contained in the numerous examples and exercises spread throughout the book, concern approximation theory, probability theory, structure design, chemical process control, routing in telecommunication networks, image reconstruction, experiment design, radiation therapy, asset valuation, portfolio management, supply chain management, facility location. In Chapter 1. *Introduction,* the author briefly explains on some examples how the optimization theory can help in solving some practical problems.

The theory, which is the matter of the first part of the book, is covered in the chapters 2. *Elements of convex analysis,* 3. *Optimality conditions,* and 4. *Lagrangian* *duality.* One works within the framework of the space $\mathbb{R}^n$ with emphasis on differentiability, subdifferentiability and conjugation properties of convex functions, with applications to necessary and sufficient conditions and duality for minimization and maximization problems. These problems are considered with respect to various orderings on $\mathbb{R}^n$ generated by cones. Some important cones in optimization, such as those of feasible directions, normal, polar and recession cones, and their relevance to differentiability and subdifferentiability properties of convex functions and to optimization problems are studied in detail.

The second part of the book, dedicated to applications, contains the presentation of the main algorithms and iterative methods for solving optimization problems, along with a careful study of the convergence and error evaluations. This is done in Chapters 5. *Unconstrained optimization of differentiable functions,* 6. *Unconstrained optimization of nondifferentiable functions,* and 7. *Nondifferentiable optimization,* where algorithms and methods such as the steepest descent method, Newton-type methods, the conjugate gradient method, feasible point methods, proximal point methods, subgradient methods, are presented, analyzed and exemplified within the corresponding context.

The book is clearly written, with numerous practical examples and figures, illustrating and clarifying the theoretical notions and results, and providing the reader with a solid background in the area of optimization theory and its applications. The prerequisites are linear algebra and multivariate calculus. It (or parts of it) can be used for one year (or one-semester) graduate courses for students in engineering, applied mathematics, or management science, with no prior knowledge of optimization theory. The part on nondifferentiable optimization can be used as supplementary material for students who have already had a first course in optimization.

Nicolae Popovici

B. S. Mordukhovich, ***Variational Analysis and Generalized Differentiation,*** Vol. I: ***Basic Theory,*** Vol. II: ***Applications,*** Springer, Berlin-Heidelberg-New York, 2006, Grundlehren der mathematischen Wissenschaften, Volumes 300, 301, ISBN 3-540-25437-4 and ISBN 3-540-25438-2.

Prof. Mordukhovich starts his book with the well-known sentence of Euler that "... nothing in all of the world will occur in which no maximum or minimum rule is somehow shining forth." Paraphrasing Euler's sentence we can doubtless state that the book under review is a maximum in the topic of variational analysis and applications. Moreover, two times the word "perfection" appears into the preface of the book. Certainly this was the desire of the author, namely to achieve the perfection by this book. He indeed attained the perfection!

Two fundamental books on variational analysis are corner stones on this topic. The finite dimensional case has been addressed in the book "Variational Analysis" by R. T. Rockafellar and R. J.-B. Wets (Springer, Berlin, 1998), while some fundamental techniques of modern variational analysis for the infinite dimensional case are discussed in "Techniques of Variational Analysis" by J. M. Borwein and Q. J. Zhu, Springer, New York, 2005.

Modern variational analysis is an outgrowth of the calculus of variations and mathematical programming. The focus is on optimization of functions relative to various constraints and on sensitivity and stability of optimization-related problems with respect to perturbations. Many problems of optimal control and mathematical programming have nonsmooth intrinsic nature (the value function to simple problems is discontinuous). Therefore since the nonsmoothness is a usual ingredient into this topic, many fundamental objects frequently appearing have to be redesign. One of them is the (generalized) differential of a function not differentiable in the usual sense. Generalized differentiation lies at the heart of variational analysis and its applications. It is systematically developed a geometric dual-space approach to generalized differential theory around the extremal principle. The extremal principle is a local variational counterpart of the classical convex separation in nonconvex settings. It allows to deal with nonconvex derivative-like constructions for sets (normal cones), set-valued mappings (coderivatives), and extended-real-valued functions (subdifferentials).

The first volume (Basic Theory) is structured on four chapters, while the second volume (Applications) also contains four chapters, lists of references, statements, a glossary of notation, and a subject index.

In the first chapter, *Generalized differentiation in Banach spaces,* there are introduced the fundamental notions of basic normals, subgradients, and coderivatives, one studies their properties (Lipschitz stability, metric regularity) and one elaborates first-order and second-order calculus rules.

The second chapter, *Extremal principle in variational analysis,* is dedicated to the study of this important notion (a term coined by Mordukhovich, J. Math. Anal. Appl. **183** (1994), 250-288, a preliminary version being published in Dokl. Akad. Nauk BSSR **24** (1980), 684-687, jointly with A. Y. Kruger), which is the main tool of the book. The extremal principle is proved first in finite-dimensional spaces based on a smoothing penalization principle, while in infinite-dimensional case the setting is that of Asplund spaces, based on the method of metric approximation.

The third chapter, *Full calculus in Banach spaces,* contains the basic theory of the generalized differential theory, namely the calculus rules for basic normals, subgradients, and coderivatives in the framework of Asplund spaces. For the infinite-dimensional case it is necessary to add sufficient amount of compactness expressed by the so-called sequential normal compactness, introduced in the first chapter of the book.

The fourth chapter, *Characterizations of well-posedness and sensitivity analysis,* is devoted to the study of Lipschitzian, metric regularity, and linear openness properties of set-valued mappings, and to their applications to sensitivity analysis of parametric constraint and variational systems.

Volume II, *Applications,* is mostly devoted to applications of basic principles in variational analysis and generalized differential calculus to topics in constrained optimization and equilibria, optimal control of ordinary and distributed-parameter models, and models of welfare economics.

In the fifth chapter, *Constrained optimization and equilibria,* the use of variational methods based on extremal principles and generalized differentiation allows the treatment of a large variety of problems, including even problems with smooth data.

In the sixth chapter, *Optimal control of evolution systems in Banach spaces,* by using methods of discrete approximations one obtains necessary optimality conditions in the extended Euler-Lagrange form for nonconvex differential inclusions in infinite dimension. Constraint optimal control systems governed by ordinary evolution equations of smooth dynamics in arbitrary Banach spaces are also studied.

The seventh chapter, *Optimal control of distributed systems,* contains a further development of the study of optimal control problems by applications of modern methods of variational analysis. One establishes a strong variational convergence of discrete approximations and derived extended optimality conditions for continuous-time systems in both Euler-Lagrange and Hamiltonian forms.

The eighth chapter, *Applications to economics,* is devoted to the applications of variational analysis to economic modelling. The focus is on the welfare economics in the nonconvex setting with infinite-dimensional commodity spaces. The extremal principle is a proper tool to study Pareto optimal allocations and associated price equilibria for such models.

Each chapter ends with a section of commentaries,where the author exhibits connections of the results just introduced with other results. The commentaries are deep and pertinent. We just mention that one can find such a section having more than 30 pages.

The book ends with references, a list of statements, a glossary of notation, and a subject index. The list of references containing 1379 titles, most of them very recent. This references reflect, on one side, the author's contribution to this topic and, on the other side, the contributions of many other researchers all over the world.

At the end of this short review, we can state doubtless that in front of us there is a masterpiece on the topic of variational analysis and generalized differentiation. Certainly this wonderful work will be included in many libraries all over the world.

Marian Mureşan

Proceedings of the International Workshop on Small Sets in Analysis, (Held at the Technion - Israel Institute of Technology, June 25-30, 2003). Edited by **Eva Matoušková, Simeon Reich and Alexander Zaslavski**, Hindawi Publishing Corporation, New York and Cairo, 2005, ISBN: 977-5945-23-2.

The smallness of a set can be understood in a topological (sets of first Baire category) or measure-theoretical (sets of Lebesgue measure zero) sense, or even by its cardinality (finite, at most countable). The complement of a small set is called a big set. A well known classical result asserts that the real line can be written as the union of a set of first Baire category and of a set of Lebesgue measure zero, showing that these two notions are strongly unrelated. In spite of this, by the duality principle of Siepiński and Erdös, there exists a bijection $f : \mathbb{R} \to \mathbb{R}$ such that $f = f^{-1}$ and $E \subset \mathbb{R}$ is of first Baire category iff E is of Lebesgue measure zero. A stronger notion is that of porosity - a porous subset of a finite dimensional normed space being of first Baire category and, at the same time, of Lebesgue measure zero, but not vice versa. In analysis there are a lot of classical results asserting that some sets of functions are big (topologically) or that some properties hold excepting a small set. From the first category we mention the Banach-Steinhaus principle of the condensation of singularities, and Banach's result that the set of nowhere differentiable continuous functions is topologically big in the space of continuous functions. Two famous results from the second category are Rademacher's theorem on the generic (i.e., excepting a set of first Baire category) differentiability of Lipschitz functions, and Alexandroff's theorem on the twice almost everywhere differentiability of convex functions. In the attempt to extend these results to infinite dimensions, new notions of null sets were introduced - Gauss (Aronszajn) null sets, Haar null sets, Christensen null sets, Γ-null sets - that led to a revitalization of research in this area, and to new concepts in the geometry of Banach spaces as well.

Taking into account the growing interest of the mathematical community in these topics, the idea of a conference emerged and it took place at The Technion - Israel Institute of Technology, Haifa, from 25 to 30 of June, 2003, under the name "The International Workshop on Small Sets in Analysis". The workshop was very successful, being attended by researchers from thirteen countries, prominent specialists in various areas of analysis.

The present volume contains the refereed proceedings of this workshop, many of the papers being revised and extended versions of the lectures delivered at the workshop. The papers have been previously published in three issues of the journal Abstract and Applied Analysis (Hindawi), and this volume brings them together.

The included papers cover a wide spectrum pertaining to small sets of various kinds and their relations to other notions such as, for instance, the descriptive theory of sets. On this line we mention the contributions of L. Zajiček and M. Zelený (σ-porous and Suslin sets), the survey paper by Zajiček on σ-porous sets, the paper by S. Solecki on analytic P-ideals and that by J. Myjak on dimension and measure. Very well are represented the applications of small sets to various domains of analysis. Among these topics we mention those on generic results in optimization and the geometry of Banach spaces (papers by S. Cobzaş, P. G. Howlett, A. Ioffe, R. Lucchetti, A. M. Rubinov, T. Zamfirescu, A. J. Zaslavski), infinite dimensional holomorphy (M. Budzyńska and S. Reich), Markov operators (T. Szarek), differentiability (M. Csörnyei, D. Preiss, J. Tišer, R. Deville), convex geometry (M. Kojman, F. S. de Blasi and N. V. Zhivkov), weak Asplund spaces (W. Moors), generic existence in optimal control (A. J. Zaslavski) Lipschitz functions (O. Maleva), and dynamics of random Ramanujan fractions (J. M. Borwein and D. R. Luke).

By surveying and discussing various topics connected by the unifying idea of a small set, posing open questions and assessing possible future directions of investigation, the present volume appeals to a large audience, researchers and scholars interested in the areas mentioned above or, generally speaking, in analysis understood in a broad (i.e., complementary to a small) sense.

Ioan V. Şerb

Simeon Reich and David Shoikhet, ***Nonlinear Semigroups, Fixed Points, and Geometry of Domanins in Banach Spaces,*** Imperial College Press, World Scientific, London and Singapore, 2005, xv+354 pp, ISBN: 1-86094-575-9.

The main concern of the book is the theory of semigroups of holomorphic mappings defined on a domain D in a complex Banach space X and with values in X. Beside their intrinsic mathematical interest, the study of these semigroups is also motivated by the applications to Markov stochastic processes and branching processes, to the geometry of complex Banach spaces, to control and optimization and to complex analysis.

As it is well known an important question in the theory of nonlinear semigroups of operators is whether they are generated by one operator. M. Abate proved in 1992 that, in the finite dimensional case, any continuous semigroup of holomorphic mappings is everywhere differentiable with respect to the parameter or, equivalently it is generated by one operator, a result that is no longer true in infinite dimensions. E. Vesentini considered in 1987 semigroups of fractional-linear transformations of the unit ball $\mathbb{B}$ of a Hilbert space H which are isometries with respect to the hyperbolic metric on $\mathbb{B}$. His approach, based on a correspondence between the holomorphic semigroups and some semigroups of nonlinear operators on a Pontryagin space, revealed that these semigroups are not everywhere differentiable. It seems that Vesentini was the first who considered semigroups of holomorphic mappings in infinite dimensional setting.

In order to make the book self-contained, the first two chapters, 1. *Mappings in metric and normed spaces,* and 2. *Differentiable and holomorphic mappings in Banach spaces,* collect some results from topology, functional analysis, and differentiability and holomorphy in infinite dimensional setting.

Of crucial importance for the book is Chapter 3. *Hyperbolic metrics on domains in complex Banach spaces,* where one introduces the Poincaré metric, the Carathéodory and Kobayashi pseudometrics and Finsler infinitesimal pseudometrics.

Chapter 4. *Some fixed point principles,* is concerned with the classical fixed point theorems of Banach, Brouwer and Schauder (the last two without proofs), along with some fixed point theorems for holomorphic mappings, from which the Earle-Hamilton theorem is basic for the book.

A classical result of Denjoy and Wolff asserts that if $F \in \mathrm{Hol}(\Delta)$ ($\Delta =$ the unit disk in $\mathbb{C}$) is not the identity nor an automorphism with exactly one fixed point in Δ, then there is a point $a \in \bar{\Delta}$ such that the sequence $\{F^n\}$ of iterates of F converges to $h(z) \equiv a$, uniformly on compact subsets of Δ. This result and some of its extensions to infinite dimensional setting (Hilbert and Banach spaces) are presented in Chapter 5. *The Denjoy-Wolff fixed point theory.*

The study of nonlinear semigroups is carried out in the chapters: 6. *Generation theory for one-parameter semigroups,* 7. *Flow-invariance conditions,* 8. *Stationary points of continuous semigroups,* and 9. *Asymptotic behavior of continuous flows.* The framework is that of nonlinear semigroups of mappings which are nonexpansive with respect to some special metrics on domains in Banach spaces, with emphasis on nonlinear semigroups of holomorphic mappings, in which case the description is more complete.

The last chapter of the book, 10. *Geometry of domains in Banach spaces,* is devoted to the geometric theory of functions in infinite dimensions - starlike, convex and spirallike mappings - a topic less developed in the literature. The approach is based on the unifying idea of a dynamical system, and uses in an essential way the results on asymptotic behavior of semigroups of holomorphic mappings, developed in Chapter 9.

Written by two experts in the area and incorporating their original contributions, the book contains a lot of interesting results, most of them appearing for the first time in book form. The excellent typographical layout of the book must also be mentioned.

The book appeals to a large audience, including specialists in functional analysis, complex analysis, dynamical systems, abstract differential equations, and can be used for advanced graduate or post-graduate courses, or as a reference by experts.

S. Cobzaş

Štefan Schwabik and Ye Guoju, ***Topics in Banach Space Integration,*** Series in Real Analysis - Volume 10, World Scientific, London and Singapore, 2005, xiii+298 pp, ISBN: 981-256-428-4.

An extension of Riemann method of integration was discovered around 1960 by Jaroslav Kurzweil and, independently, by Ralph Henstock. This theory, which covers Lebesgue integration and, at a same time, nonabsolutely convergent improper integrals, is based only on Riemann type sums which are fine with respect to some gauge functions. The avoidance of any measure theoretical considerations makes it appropriate for teaching advanced topics in integration theory at an elementary level. There are several books on integration based on Riemann type sums for real-valued functions of one or several variables.

The first who considered the case of vector-valued functions was Russel A. Gordon around 1990. The aim of the present book is to present these integration theories for functions defined on compact intervals $I \subset \mathbf{R}^m$ and with values in a Banach space X. Possible extensions to noncompact intervals are briefly discussed in Section 3.7. Since the authors treat the relations of these integrals with other integrals for vector-valued functions, the first two chapters, 1. *Bochner integral,* and 2. *Dunford* *and Pettis integrals,* present shortly the main properties of these integrals. The study of the integrals of vector-valued functions based on Riemann type sums starts in Chapters 3. *McShane and Henstock-Kurzweil integrals,* and 4. *More on McShane integral.* These chapters contain the basic results, including convergence theorems for these integrals, a hard topic in the Lebesgue integration. The relations with other types of integrals for vector functions are studied in Chapters 5. *Comparison of the Bochner and McShane integrals,* and 6. *Comparison of the Pettis and McShane integrals.* It turns out that the class of Bochner integrable functions is strictly contained in the class of McShane integrable functions, and these two classes agree if the space X is finite dimensional. In its turn, the class of McShane integrable functions is contained in the class of Pettis integrable functions, and the two classes agree if the Banach space X is separable.

As it is well known, one of the most delicate question in the case of Lebsesgue integration is the relation between differentiability, absolute continuity and the properties of the primitive function. These problems, within the framework of generalized Riemann type integrals are examined in Chapter 7. *Primitive of the McShane and Henstock-Kurzweil integrals.* The last chapter of the book 8. *Generalizations of some integrals,* is concerned with possible extensions of Bochner and Pettis integrals, following Denjoy and Henstock-Kurzweil approaches. An appendix contains a summary of some results from functional analysis used in the book.

The authors have done substantial contributions to the field, which are incorporated in the book. The book is clearly written and can be recommended for graduate courses on integration or as a companion book for a course in functional analysis.

Valer Anisiu

Antonio Ambrosetti and Andrea Malchiodi, *Perturbation Methods and Semilinear Elliptic Problems on* $\mathbf{R}^n$, Progress in Mathematics (series editors: H. Bass, J. Oesterlé and A. Weinstein), vol. 240, Birkhäuser Verlag, Basel-Boston-Berlin, 2006, xii+183 pp; ISBN-10:3-7643-7321-0, ISBN-13:978-3-7643-7321-4, e-ISBN: 3-7643-7396-2.

The monograph is based on the authors' own papers carried out in the last years, some of them in collaboration with other people like D. Arcoya, M. Badiale, M. Berti, S. Cingolani, V. Coti Zelati, J.L. Gamez, J. Garcia Azorero, V. Felli, Y.Y. Li, W.M. Ni, I. Peral and S. Secchi.

The book is concerning with perturbation methods in critical point theory together with their applications to semilinear elliptic equations on $\mathbf{R}^n$ having a variational structure.

The contents are as follows: Foreword; Notation; 1 Examples and motivations (giving an account of the main nonlinear variational problems studied by the monograph); 2 Perturbation in critical point theory (where some abstract results on the existence of critical points of perturbed functionals are presented); 3 Bifurcation from the essential spectrum; 4 Elliptic problems on $\mathbf{R}^n$; 5 Elliptic problems with critical exponent; 6 The Yamabe problem; 7 Other problems in conformal geometry; 8 Nonlinear Schrödinger equations; 9 Singularly perturbed Neumann problems; 10 Concentration at spheres for radial problems; Bibliography (147 titles) and Index.

The topics are presented in a systematic and unified way and the large range of applications talks about the power of the critical point methods in nonlinear analysis.

I recommend the book to researchers in topological methods for partial differential equations, especially to those interested in critical point theory and its applications.

Radu Precup

Spiros A. Argyros, Stevo Todorcevic, *Ramsey Methods in Analysis,* Advanced Courses in Mathematics CRM Barcelona, Birkhäuser Verlag, Basel, Boston, Berlin, 2005.

This excellent book contains two sets of notes presented by the authors for the Advanced Course on Ramsey Methods in Analysis given at the Centre de Recerca Mathematica, Barcelona in 2004. The modern area of the research lying on the borderline between functional analysis and combinatorics is at this moment a very active area of research. An important part of this area is presented in this book.

The first example of W.T. Gowers and B. Maurey of a reflexive Banach space with no unconditional basis is also an example of hereditarily indecomposable (HI) space. This means that no infinite dimensional closed subspace is the topological direct sum of two infinite dimensional closed subspaces of it. As a consequence, no HI space is isomorphic to any proper subspace answering in negative the long standing hyperplane problem. On the other hand the Gowers' famous dichotomy: every Banach space either is unconditionally saturated or contains an HI space provides a positive solution of the homogeneous problem. Finally, W.T. Gowers and B. Maurey have shown that every bounded linear operator on a complex HI space is of the form $\lambda I + S$ with S strictly singular.This means that HI spaces are spaces with few operators.

The goal of the first set of notes written by S.A. Argyros is to describe a general method of building norms with desired properties in order to obtain examples and general geometric properties of infinite dimensional Banach spaces. Here are constructed Tsirelson and Mixed Tsirelson spaces, HI extensions with a Schauder basis and are presented examples of HI extensions. For instance, quasi-reflexive and non separable HI spaces are described. General properties of HI spaces and the space of operators acting on a HI space are also presented.

The goal of the second set of notes written by S. Todorcevic is to present combinatorial theoretic methods, especially Ramsey methods, relevant for the description of the rough structure of infinite dimensional Banach spaces. For instance, finite dimensional Ramsey Theorem, spreading models of Banach spaces, finite representability of Banach spaces, Ramsey theory of finite and infinite sequences or block sequences, approximate and strategic Ramsey theory of Banach spaces, Gowers dichotomy, an application to Rough classification of Banach spaces are samples of subjects in the second part.

The book is a valuable, concise, and systematic text for mathematicians who wish to understand and to work in a fascinating area of mathematics.

Ioan Şerb

Klaus Gürlebeck, Klaus Habetha, Wolfgang Sprössig, *Funktionentheorie in der Ebene und in Raum*, Grundstudium Mathematik Birkhäuser Verlag, Basel-Boston-Berlin, 2006, ISBN 10: 3-7643-7369-6, xiii+406 pp., (CD included).

The theory of complex holomorphic functions of one complex variable is a 200 years old and well established field of mathematics. In the 1930s the Romanian mathematicians G. C. Moisil and N. Teodorescu and the Swiss mathematician R. Fueter started to develop the function theory in quaternion fields and Clifford algebras. This study was systematically continued and developed in the 1960s in the works of a group of Belgian mathematicians headed by R. Delanghe, followed by a lot of other ones all around the world, so that the authors succeeded to count over than 9000 entries in the area.

The aim of the present monograph is to give a systematic account on basic facts in this relatively recent and rapidly growing domain of research. The first chapter of the book I. *Zahlen,* is concerned with the basic properties of the fields $\mathbb{R}$ of real numbers, $\mathbb{C}$ of complex numbers and the quaternions $\mathbb{H}$. All these can be treated within the more general notion of Clifford algebra $C\ell(n)$ in $\mathbb{R}^{n+1}$.

The treatment of function theory starts in Chapter II. *Funktionen,* with some continuity questions and then with the differentiability, holomorphy, power functions and Möbius transforms in $\mathbb{C}$ and in higher dimensions. The holomorphy for functions $f : G \to \mathbb{H}$, where $G \subset \mathbb{H}$ is nonempty open, is defined by the generalized Cauchy-Riemann (CR) conditions, and similarly in $C\ell(n)$.

In Chapter III, *Integration und Integral Sätze,* besides the extension of integral theorems of Morera and Cauchy to the $C\ell(n)$ setting, a special attention is paid to the integral formula of Borel-Pompeiu and its applications, as, e.g., to the Teodorescu transform.

Chapter IV, *Reihenentwicklungen und lokales Verhalten,* is concerned with series in $C\ell(n)$, Taylor series, Laurent series, and their applications to the study of holomorphic functions in $C\ell(n)$. Elementary and special functions are introduced and the theory of residues is applied to the calculation of integrals.

An *Appendix* contains some results on integration of differential forms on differentiable manifolds, on spherical functions and on function spaces.

The book is clearly written, in a pleasant and informal style. A lot of historical notes along with pictures and short biographies of the main contributors to the domain are included.

The prerequisites are minimal and concern only the basic notions in algebra, calculus and analytic functions.

The book can be recommended as material for complementary courses on algebra, geometry and function theory.

P. T. Mocanu

George Grätzer, *The Congruences of a Finite Lattice. A Proof-by-Picture Approach*, Birkhäuser, Boston-Basel-Berlin, 2006, ISBN 0-8176-3224-7, xxii+282 p., 110 illus.

The study of the lattices formed by the congruence relations of a lattice (called congruence lattices) has been an important field in the algebra of the past half-century. The problems concerning the congruence lattices drew the attention of many valuable algebraists. Consequently, there exist a lot of interesting results in this area of lattice theory, and some of them are presented in this book.

The monograph under review is an exceptional work in lattice theory, like all the others contributions by this author. This work points out, once again, the rich experience of George Grätzer in the study of lattices. The way this book is written makes it extremely interesting for the specialists in the field but also for the students in lattice theory. Moreover, the author provides a series of companion lectures which help the reader to approach the Proof-by-Picture sections. These can be found on his homepage

http://www.math.umanitoba.ca/homepages/gratzer.html

(in the directory */MathBooks/lectures.html*). Each chapter from 4 to 18 (from 19 chapters) has at least one Proof-by-Picture section. As mentioned by George Grätzer, his proof-by-picture "is not a proof" but "an attempt to convey the idea of proof".

The book contains a Glossary of Notation, a Picture Gallery, an abundant Bibliography, and an Index (of names and subjects). The chapters of the book are: Part I. *A Brief Introduction to Lattices:* 1. Basic Concepts; 2. Special Concepts; 3. Congruences; Part II. *Basic Techniques:* 4. Chopped Lattices; 5. Boolean Triples; 6. Cubic Extensions; Part III. *Representation Theorems:* 7. The Dilworth Theorem; 8. Minimal Representations; 9. Semimodular Lattices; 10. Modular Lattices; 11. Uniform Lattices; Part IV. *Extensions:* 12. Sectionally Complemented Lattices; 13. Semimodular Lattices; 14. Isoform Lattices; 15. Independence Theorems; 16. Magic Wands;Part V. *Two Lattices:* 17. Sublattices; 18. Ideals; 19. Tensor Extensions.

Cosmin Pelea

Vladimir A. Marchenko and Evgueni Ya. Khruslov, *Homogenization of Partial Differential Equations,* Progress in Mathematical Physics, Vol. 46, Birkhäuser, Boston-Basel-Berlin, 2006, xii+398 pp., ISBN - 10 0-8176-4351-6.

The aim of homogenization theory is to establish the macroscopic behaviour of a microinhomogeneous system, in order to describe some characteristics of the given heterogeneus medium. From mathematical point of view, this signifies mainly that the solutions of a boundary value problem, depending on a small parameter, converge to the solution of a (homogenized) limit boundary value problem which is explicitly described. In this case, the main problem is to determine the effective parameters of the homogenized equation.

The aim of this book is to present some basic results on microinhomogeneous media leading to nonstandard mathematical models. For such media, homogenized models of physical processes may have various forms differing substantially from the microscopic model, and the macroscopic description cannot be reduced to the determination of the effective characteristics only. The homogenized models can have nonlocal character (integro-differential equations) or can appear as models with memory.

The book is divided into eight chapters. The first chapter contains some typical examples of nonstationary heat conduction processes in microinhomogeneous media of various types. In the next chapters necessary and sufficient conditions for the convergence of solutions of the original (microscopic) problems to solutions of the corresponding homogenized equations are given. This part of the book is devoted to the following topics: the Dirichlet boundary value problem in strongly perforated domains with fine-grained boundary, the Dirichlet value problem in strongly perforated domains with complex boundary, strongly connected domains, the Neumann boundary value problems in strongly perforated domains, nonstationary problems and spectral problems, differential equations with rapidly oscillating coefficients, and homogenized conjugation conditions.

The book is an excellent, practice oriented, and well written introduction to homogenization theory bringing the reader to the frontier of current research in the area. It is highly recommended to graduate students in applied mathematics as well as to researchers interested in mathematical modelling and asymptotical analysis.

J. Kolumbán

Steven G. Krantz, ***Geometric Function Theory. Explorations in Complex Analysis***, Birkhäuser-Basel-Berlin, 2006; 314 pp. ISBN -10 0-8176-4339-7; ISBN -13 978-0-8176-4339-3; ISBN 0-8176-4440-7.

This book provides a very good and deep point of view of modern and advanced topics in complex analysis.

The book is divided into three parts. The first part consists of six chapters devoted to classical function theory. The first chapter begins with special topics of invariant geometry, like conformality and invariance, the Bergman metric and the Bergman kernel function and its properties. Also there are presented some applications of invariant metrics on planar domains. The second chapter explores the Schwarz lemma and its variants. To this end, it is presented a geometric view of the Schwarz lemma, which leads to the study of the Poincaré metric on the unit disk. This chapter also contains the Ahlfors version of the Schwarz lemma as well as the geometric approaches of the Liouville and Picard theorems. This chapter concludes with the presentation of the Schwarz lemma at the boundary. In the third chapter the author is concerned with the concept of "normal family" and its applications to questions in complex analysis. There are various applications of normal families of holomorphic functions. They are used in the modern proof of the famous Riemann mapping theorem as well as in the proof of Picard's theorems. This chapter also contains advanced results on normal families such as Robinson's principle concerning the relationship between normal families and entire functions. Chapter five is devoted to boundary regularity of conformal maps. This chapter also offers certain practical applications. The Riemnann mapping theorem asserts that if Ω is a simply connected domain in the complex plane, not all of $\mathbb{C}$, then there exists a conformal mapping f of Ω onto the unit disk D. Note that this result has no analogue in the case of several complex variables. A deeper understanding of the Riemann mapping theorem naturally raises the question of whether the mapping f extends in a nice way to the boundary. But this is not always possible, and it is necessary to require certain conditions in order to obtain a positive result. One of the deep results in this direction is the Carathéodory theorem presented in Section 5.1. Chapter six deals with the boundary behavior of holomorphic functions. Basic tools in this subject are reproducing kernels (the Poisson and Cauchy kernels), harmonic measure and conformal mapping.

The second part of the book contains many ideas and results in real and complex analysis, based on he Cauchy-Riemann equations and the Laplacian, harmonic analysis, singular integral operators and Banach algebras. Chapter seven deals with solution of the inhomogeneous Cauchy-Riemann equations, and development and application of the $\overline{\partial}$ equation. Chapter eight is concerned with several problems related to the Laplacian and its fundamental solution, the Green function, Poisson kernel.

Chapter nine is devoted to the idea of harmonic measure which is a device for estimating harmonic functions on a domain. It is also a key tool in potential theory and in the study of the corona theorem. Chapter ten deals with special topics related to conjugate functions and the Hilbert transform. The next chapter is devoted to the Wolff proof of the very deep "corona theorem".

The last part of the book is concerned with certain algebraic topics, which illustrate the symbiosis with other parts of mathematics that complex analysis has enjoyed. Algebra is encountered in various guises throughout the book. It plays a role in the group-theoretic aspects of automorphisms and in the treatment of Banach algebra techniques. It plays a main role in the study of sheaves. Chapter 12 contains various results related to automorphism groups of domains in the plane, while the last chapter is devoted to Cousin problems, cohomology and sheaves.

Each chapter contains a rich collection of exercises of different level, examples and illustrations.

The book ends with an extensive list of monographs and research papers.

The book is very clearly written, with rigorous proofs, in a pleasant and accessible style. It is warmly recommended to advanced undergraduate and graduate students with a basic background in complex analysis, as well as to all researchers that are interested in modern and advanced topics in complex analysis.

Gabriela Kohr

Paul F. X. Müller, *Isomorphisms between H^1 Spaces,* Monografie Matematyczne (New Series), Vol. 66, Birkhäuser Verlag, Boston-Basel-Berlin, 2005, xiv+453 pp, ISBN-10:3-7643-2431-7 and 13:978-3-7643-2431-5.

H^1 spaces form one of the most important classes of Banach spaces in functional analysis, complex analysis, harmonic analysis and probability theory. H^1 spaces appear in several variants. The first one is the classical Hardy space $H^1(\mathbb{T})$ of integrable functions on the unit circle $\mathbb{T}$ for which the harmonic extension to the unit disk is analytic. Its foundation has been laid by several deep theorems of G. H. Hardy (1915), F. and M. Riesz (1916), Hardy and Littlewood (1930), R. E. A. C. Paley (1933). Beside this space, by 1977 there were known also two other classes of H^1 spaces: the atomic H^1_{at} spaces linked to analytic functions via Fefferman's duality theorem, and the martingale H^1 spaces consisting of martingales for which Doob's maximal function is integrable. Many results from atomic H^1 spaces have direct analogues results in martingale class, and their study has put in evidence a remarkable object - the dyadic H^1 spaces.

The book is concerned with dyadic H^1 spaces, their invariants and their position within the two classes of atomic and martingale H^1 spaces. A key tool in this study is formed by the Haar function orthogonal system, allowing to reach directly the straight point of some difficult results as, for instance, Johnson's factorization theorem, the uniform approximation property of H^1 – namely the combinatorial difficulty which is inherent to the problem. The Haar system is studied in the first chapter which contains the proofs of classical inequalities of Khintchin, Burkholder, Fefferman, and Hardy-Littlewood. Walsh expansions and Figiel's representation of singular integral operators are also presented. The basic combinatorial tools are elaborated in Chapter 3, *Combinatorics of colored dyadic intervals.*

The second chapter, *Projections, isomorphisms, interpolation,* contains a review of basic concepts of functional analysis, with emphasis on complemented subspaces of H^1 and analytic families of operators on H^p spaces.

A remarkable conjecture of A. Pelczynski asked whether dyadic H^1 and $H^1(\mathbb{T})$ spaces are isomorphic as Banach spaces. In Chapter 4 of the book one gives a complete detailed proof of Maurey's isomorphism theorem between the martingale space $H^1[(\mathcal{F}_n)]$ and a special space $X[\mathcal{E}]$. This isomorphism opened the way to the proof given by L. Carleson that $H^1(\mathbb{T})$ has an unconditional basis.

In Chapter 5, *Isomorphic invariants for H^1*, one establishes dichotomies for complemented subspaces of H^1, one proves that H^1 and $H^1(\ell^2)$ are not isomorphic, and that H^1 has the uniform approximation property. Chapter 6, *Atomic H^1 spaces,* contains a careful presentation of Carleson's biorthogonal system, with the proof that it is an unconditional basis for H^1_{at}, yielding another isomorphism result of Maurey, namely that H^1 and H^1_{at} are isomorphic.

The book contains deep results combining methods from functional analysis, real analysis, complex analysis and probability theory, exposed in an accessible way - the prerequisites are standard courses in functional analysis, complex function and probability. The proofs, often long, technical and difficult, are presented in detail.

The book is of great interest to researchers in functional analysis and its applications, complex analysis and probability. It can be also used for post-graduate and doctoral courses.

I. V. Şerb

Tomáš Roubiček, ***Nonlinear Partial Differential Equations with Applications***, International Series in Numerical Mathematics - ISNM, Vol. 153, Birkhäuser Verlag, Basel-Boston-Berlin 2005, ISBN 10: 3-7643-7293-1 and 13: 978-3-7643-7293-4.

The present book focuses on partial differential equations (PDE) involving various nonlinearities, related to concrete applications in engineering, physics, (thermo)mechanics, biology, medicine, chemistry, etc. The exposition combines the rigorous abstract presentation with applications to concrete real-world problems, when a part of rigor is sacrificed to reach the scope in a reasonable fashion. As the author points out in the Preface, although the abstract approach has its own interest and beauty, usually it does not fit with a concrete problem involving PDEs and whose solution requires many specific technicalities not supplied by the abstract theory.

The book is concerned mainly with boundary-value problems for semilinear and quasilinear PDEs, and with variational inequalities. The book is divided into two parts: I. Steady-State Problems, containing the chapters 2. *Pseudomonotone or weakly continuous mappings,* 3. *Accretive mappings,* 4. *Potential problems: smooth case,* 5. *Nonsmooth problems: variational inequalities,* 6. *Systems of equations: particular examples,* and II. Evolution Problems, containing the chapters 7. *Special auxiliary tools,* 8. *Evolution by pseudomonotone or weakly continuous mappings,* 9. *Evolution governed by accretive mappings,* 10. *Evolution governed by certain set-valued mappings,* 11. *Doubly-nonlinear problems,* 12. *Systems of equations: particular examples.*

The first chapter has an introductory character, presenting some results from functional analysis and function spaces needed in the rest of the book.

The numerous exercises (with solutions sketched in footnotes) and concrete real-world examples illustrate and complete the main text.

Combining the abstract approach with numerous worked examples, the book reflects the research interests of the author as well as his teaching experience at Charles University in Prague, where he taught between the years 1996 and 2005 a course on mathematical modelling. The choice of some examples is motivated also by the electrical-engineering background of the author.

The book, or parts of it (a scheme in the preface suggests possible selections of the chapters), can be used for one year graduate courses for students in mathematics, physics or chemistry, interested in applications of partial differential equations and in mathematical modelling.

Damian Trif

Volker Scheidemann, ***Introduction to Complex Analysis in Several Variables***, Birkhäuser Verlag, Basel-Boston-Berlin, 2005; 171 pp. ISBN 3-7643-7490-X.

The present book gives a very good and comprehensive introduction to complex analysis in several variables. It consists of eight chapters as follows. In the first chapter there are presented certain elementary results in the theory of several complex variables, such as the geometry of $\mathbb{C}^n$, the definition of a holomorphic function, the compact-open topology on the space $\mathcal{O}(U)$ of holomorphic functions on an open set U in $\mathbb{C}^n$ etc.

The second chapter deals with extension phenomena for holomorphic functions based on the geometry of their domain of definition. The next chapter is devoted to the study of biholomorphic maps of domains in $\mathbb{C}^n$ and it is proved the biholomorphic inequivalence of the unit ball and the unit polydisc in $\mathbb{C}^n$, $n \geq 2$. In the chapter four it is given an introduction to analytic sets. To this end, there are presented elementary properties of analytic sets and the Riemann removable singularity theorems. The aim of the fifth chapter is to state and prove the well known "Kugelsatz" result due to Hartogs. To this end, this chapter begins with a brief introduction to holomorphic differential forms in $\mathbb{C}^n$, followed by the study of the inhomogeneous Cauchy-Riemann differential equations and Dolbeaut's lemma. Chapter six is devoted to the proof of a continuation theorem due to Bochner, which states that any holomorphic function on a tubular domain D can be holomorphically extended to the convex hull of D. Chapter seven deals with the Cartan-Thullen theory. There are presented the notions of holomorphically convex sets, domains of holomorphy, holomorphically convex Reinhardt domains. The last chapter is concerned with local properties of holomorphic functions. To this end, it is not taken into account the domain of definition of a holomorphic function, but only its local representation. This leads to the concept of germ of a holomorphic function.

Each chapter contains a useful collection of examples and exercises of different level, that help the reader to become acquainted with the theory of several complex variables.

The book is clearly written, with rigorous proofs, in an accessible style. It is warmly recommended to students that start to work in the field of complex analysis in several variables, as well as to all researchers that are interested in modern and advanced topics in the theory of several complex variables.

Gabriela Kohr

Hans Triebel, ***Theory of Function Spaces,*** Birkhäuser Verlag, Boston – Basel – Berlin.
Volume II - Monographs in Mathematics, Vol. 84, 1992, viii+370 pp, ISBN: 3-7643-2639-5 and 0-8176-2639-5;
Volume III - Monographs in Mathematics, Vol. 100, 2006, xii+426 pp, ISBN-10: 3-7643-7581-7 and 13: 978-3-7643-7581-2.

The first volume of this treatise was published by Birkhäuser Verlag in 1983, the second one in 1992 and the third one in 2006, all dealing with function spaces of type B^s_{pq} and F^s_{pq} and reflecting the situation approximatively up to the year of their publication. These two scales of function spaces cover many well-known spaces of functions and distributions such as Hölder-Zygmund spaces, Sobolev spaces, fractional Sobolev spaces, Besov spaces, inhomogeneous Hardy spaces, spaces of BMO-type and local approximation spaces which are closely related to Morrey-Campanato spaces. Although these three volumes can be considered as parts of a unitary treatise on function spaces, the author made the second and the third volume essentially self-contained. Each new volume reflects the developments made since the publication of the previous one - simpler proofs to old results, new results and new applications. Each of these two volumes starts with a consistent chapter entitled *How to measure smoothness* - 86 pages in the second volume and 125 pages in the third one. As devices to measure smoothness one can mention: derivatives, differences of functions, boundary values of harmonic and thermic functions, local approximations, sharp maximal functions, interpolation methods, Fourier-analytical representations, atomic decompositions, etc. The main point is that all these devices when put together yield the same classes of function spaces, giving a high degree of flexibility, unknown and even unexpected at the time when the first volume was written. This is one of the aims of this introductory chapters - to show that all these apparently unrelated devices are, in fact, only different ways to characterize the same function spaces. The second one is to provide the non-specialists which are not interested in the technical details, with a readable survey on recent trends in function spaces from a historical perspective. Some of the topics surveyed in these parts are treated in detail in the subsequent chapters.

The main feature in the second volume is the use of local means and local methods with applications to pseudo-differential operators. The headings of the chapter give a general idea about its content: 2. *The spaces B^s_{pq} and F^s_{pq}*; 3. *Atoms, oscillations, and distinguished representations*; 4. *Key theorems* (containing new simple proofs for some crucial theorems for the spaces B^s_{pq} and F^s_{pq} - invariance under diffeomorphic maps of $\mathbb{R}^n$, pointwise multipliers, traces, extensions from $\mathbb{R}^n_+$ to $\mathbb{R}^n$); 5. *Spaces on domains* (dealing mainly with intrinsic characterizations); 6. *Mapping properties of pseudo-differential operators*; 7. *Spaces on Riemannian manifolds and Lie groups.*

The third volume exposes the theory of B^s_{pq} and F^s_{pq} spaces as it stands at the beginning of this century and focusses on applications of function spaces to some neighboring areas such as numerics, signal processing, and fractal analysis. The fractal quantities of measures and spectral properties of fractal elliptic operators are treated by the author in other two books published with Birkhäuser Verlag too: *Fractals and Spectra* (1992) and *The Structure of Functions* (2001). The topics covered in the third volume are quite well illustrated by the headings of the chapters: 2. *Atoms and pointwise multipliers* ; 3. *Wavelets*; 4. *Spaces on domains, wavelets, sampling numbers*; 5. *Anisotropic function spaces*; 6. *Weighted function spaces*; 7. *Fractal analysis*; 8. *Function spaces on quasi-metric spaces*; 9. *Function spaces on sets.*

The author is a leading expert in the area with outstanding contribution to function spaces and their applications, contained in the 9 books written by him (one in cooperation), and in the numerous research or survey papers he published. The present books will be an indispensable tool for all working in function spaces, partial differential equations, fractal analysis and wavelets, or in their applications as well.

S. Cobzaş