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PROFESSOR GRIGORE CĂLUGĂREANU AT HIS 60$^{\text{TH}}$ ANNIVERSARY

SIMION BREAZ AND COSMIN PELEA

Grigore Călugăreanu was born on the 20$^{\text{th}}$ of January 1947, in Cluj-Napoca, in a family with strong academic traditions and important influences on the scientific life, as both his father and grandfather were members of the Romanian Academy of Sciences. He graduated from high-school in 1964 and the same year he became a student at the Faculty of Mathematics and Mechanics of the "Babeş-Bolyai" University in Cluj-Napoca. In 1969 he graduated with a dissertation on "Embedding Theorems in Categories" and he began to work at the Department of Algebra of the same university, first as Probation Assistant (September 1969–September 1970), then as Assistant (September 1970–March 1978). In 1977, he defended his Ph.D. thesis "Contributions to the theory of enriched modules and to the problem of endomorphisms" under the supervision of Professor Ionel Bucur and Professor Alexandru Solianu from the Faculty of Mathematics of the University of Bucharest. From March 1978 until March 1990 he was Lecturer and in 1990 he became Associate Professor in the Department of Algebra of the "Babeş-Bolyai" University.

It is in that period that we first met Professor Grigore Călugăreanu as students in an introductory course in Algebra. This meeting was undoubtedly one of the most important events in our students life and it convinced us to attend an Algebra course whenever this was possible. Later, we had the opportunity to meet Professor Călugăreanu again at an optional undergraduate course on the connections between Lattice Theory and Abelian Group Theory and at a graduate level course on Abelian Group Theory. As a matter of fact, these two topics are the core of Grigore Călugăreanu's research. From February 1998, Grigore Călugăreanu is Professor at

"Babeş-Bolyai" University, and from 2001, he is also Associate Professor at the Department of Mathematics and Computer Science of the Faculty of Science of Kuwait University.

Grigore Călugăreanu had an important influence on the mathematical activity in Cluj-Napoca (and not only). He obtained many valuable and interesting results concerning Abelian groups and lattices, results published in more that 40 research papers concerning the structure of an abelian group in connection with some objects attached to it, such as the endomorphism ring or the subgroup lattice. We mention the results concerning the structure of some generalizations of extending abelian groups, obtained in collaboration with L. Fuchs, who can be considered the "father" of Abelian Group Theory. We also mention a characterization for abelian groups with semilocal endomorphism ring, a characterization of n-root property using subgroup lattices, and the results on the structure of abelian groups with continuous subgroup lattice (in collaboration with K. Benabdallah), and the structure of abelian groups with breaking point subgroup lattice (in collaboration with M. Deaconescu and S. Breaz). Along the time, Professor Călugăreanu's communication abilities lead to numerous cooperations which are not confined to the field of mathematics. Thus, we can complete the list of collaborators by adding G. Birkenmeier, B. Charles, P. Goeters, P. Hamburg, R. Khazal, V. Leoreanu, C. Modoi, A. Orsatti, C. Pelea, D. Vălcan, H. Wiesler.

Last but not least, we should stress that all his mathematical activity was influenced by his teaching abilities. He is author and coauthor of 10 books for students or for experts in algebra. All these books are the fruit of his rich and successful teaching activity. Three of them were published by Kluwer Academic Publisher (now a part of Springer Verlag): *Exercices in Basic Ring Theory* (with P. Hamburg), *Lattice Concepts in Module Theory*, *Exercises in Abelian Group Theory* (with S. Breaz, C. Modoi, C. Pelea, D. Vălcan).

For a complete image on the exceptional work of Grigore Călugăreanu, both a teacher and a researcher, we present here the most important issues of his mathematical activity.

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Visits Abroad

• 1993 COST - Research with Prof. Bernard Charles, Universit Technique du Languedoc, Montpellier 2

• 1995 TEMPUS - Teaching Algebra in University - Universita degli Studi din Padova

• 1997 NATO-CNR - Research with Prof. Adalberto Orsatti, Universita degli Studi di Padova

• 1999 FULBRIGHT - Research with Prof. Laszlo Fuchs, Tulane University, New Orleans, LA

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ON ANALOGS OF THE DUAL BRUNN-MINKOWSKI INEQUALITY FOR WIDTH-INTEGRALS OF CONVEX BODIES

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Abstract. In this paper we prove two new inequalities about width-integrals of centroid and projection bodies. Two analogs of the dual Brunn-Minkowski inequality for width-integral of convex bodies are established.

0. Definitions and preliminary results

The setting for this paper is n-dimensional Euclidean space $\mathbb{R}^n (n > 2)$. Let $\mathcal{K}^n$ denote the set of convex bodies (compact, convex subsets with non-empty interiors) in $\mathbb{R}^n$. Let φ^n denote the set of star bodies in $\mathbb{R}^n$. The subset of φ^n consisting of the centred star bodies will be denoted by φ_c^n. We reserve the letter u for unit vectors, and the letter B is reserved for the unit ball centered at the origin. The surface of B is S^{n-1}. For $u \in S^{n-1}$, let E_u denote the hyperplane, through the origin, that is orthogonal to u. We will use K^u to denote the image of K under an orthogonal projection onto the hyperplane E_u.

We use $V(K)$ for the n-dimensional volume of convex body K. Let $h(K, \cdot) : S^{n-1} \to R$, denote the support function of $K \in \mathcal{K}^n$; i.e.

$$h(K, u) = Max\{u \cdot x : x \in K\}, u \in S^{n-1}, \tag{1}$$

where $u \cdot x$ denotes the usual inner product u and x in $\mathbb{R}^n$.

Let δ denote the Hausdorff metric on $\mathcal{K}^n$; i.e., for $K, L \in \mathcal{K}^n$,

$$\delta(K, L) = |h_K - h_L|_\infty,$$

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where $|\cdot|_\infty$ denotes the sup-norm on the space of continuous functions, $C(S^{n-1})$.

For a convex body K and a nonnegative scalar $\lambda, \lambda K$, is used to denote $\{\lambda x : x \in K\}$. For $K_i \in \mathcal{K}^n, \lambda_i \geq 0, (i = 1, 2, \ldots, r)$,the Minkowski linear combination $\lambda_1 K_1 + \cdots + \lambda_r K_r \in \mathcal{K}^n$ is defined by

$$\lambda_1 K_1 + \cdots + \lambda_r K_r = \{\lambda_1 x_1 + \cdots + \lambda_r x_r \in \mathcal{K}^n : x_i \in K_i\}. \tag{2}$$

It is trivial to verify that

$$h(\lambda_1 K_1 + \cdots + \lambda_r K_r, \cdot) = \lambda_1 h(K_1, \cdot) + \cdots + \lambda_r h(K_r, \cdot). \tag{3}$$

1.1 Mixed volumes

If $K_i \in \mathcal{K}^n (i = 1, 2, \ldots, r)$ and $\lambda_i (i = 1, 2, \ldots, r)$are nonnegative real numbers, then of fundamental impotence is the fact that the volume of $\lambda_1 K_1 + \cdots + \lambda_r K_r$ is a homogeneous polynomial in λ_i given by [4,p.275]

$$V(\lambda_1 K_1 + \cdots + \lambda_r K_r) = \sum_{i_1, \ldots, i_n} \lambda_{i_1} \ldots \lambda_{i_n} V_{i_1 \ldots i_n}, \tag{4}$$

where the sum is taken over all n-tuples $(i_1, \ldots, i_n)$ of positive integers not exceeding r. The coefficient $V_{i_1 \ldots i_n}$ depends only on the bodies $K_{i_1}, \ldots, K_{i_n}$, and is uniquely determined by (8), it is called the mixed volume of $K_i, \ldots, K_{i_n}$, and is written as $V(K_{i_1}, \ldots, K_{i_n})$. Let $K_1 = \ldots = K_{n-i} = K$ and $K_{n-i+1} = \ldots = K_n = L$, then the mixed volume $V(K_1 \ldots K_n)$ is usually written $V_i(K, L)$. If $L = B$, then $V_i(K, B)$ is the ith projection measure(Quermassintegral) of K and is written as $W_i(K)$.
If $K_i (i = 1, 2, \ldots, n-1) \in \mathcal{K}^n$, then the mixed volume of the convex figures $K_i^u (i = 1, 2, \ldots, n-1)$ in the $(n-1)$-dimensional space E_u will be denoted by $v(K_1^u, \ldots, K_{n-1}^u)$. It is well known, and easily shown [5,p.45], that for $K_i \in \mathcal{K}^n (i = 1, 2, \ldots, n-1)$, and $u \in S^{n-1}$

$$v(K_1^u, \ldots, K_{n-1}^u) = nV(K_1, \ldots, K_{n-1}, [u]) \tag{5}$$

where $[u]$ denotes the line segment joining $u/2$ and $-u/2$.

1.2 Width-integrals of convex bodies

For $u \in S^{n-1}$, $b_K = \frac{1}{2}(h(K, u) + h(K, -u))$ is called as half the width of K in the direction u. Two convex bodies K and L are said to have similar width if there exists a constant $\lambda > 0$ such that $b_K = \lambda b_L$ for all $u \in S^{n-1}$. The width-integral of index i is defined by: For $K \in \mathcal{K}^n, i \in \mathbb{R}$

$$B_i(K) = \frac{1}{n}\int_{S^{n-1}} b_K^{n-i} dS(u), \tag{6}$$

where dS is the $(n-1)$-dimensional volume element on S^{n-1}.
The width-integral of index i is a map

$$B_i : \mathcal{K}^n \to \mathbb{R}.$$

It is positive, continuous, homogeneous of degree $n - i$ and invariant under motion. In addition, for $i \leq n$ it is also bounded and monotone under set inclusion.

The following result easy is proved, for $K_j \in \mathcal{K}^n (j = 1, \ldots, m)$

$$b_{K_1+\cdots+K_m} = b_{K_1} + \cdots + b_{K_m}, \tag{7}$$

1.3 The Blaschke linear combination and the harmonic Blaschke linear combination

A convex body K is said to have a positive continuous curvature function [8],

$$f(K,\cdot) : S^{n-1} \to [0,\infty),$$

if for each $L \in \varphi^n$, the mixed volume $V_1(K, L)$ has the integral representation

$$V_1(K,L) = \frac{1}{n}\int_{S^{n-1}} f(K,u)h(L,u)dS(u).$$

The subset of $\mathcal{K}^n$ consisting of bodies which have a positive continuous curvature function will be denoted by κ^n. Let κ_c^n denote the set of centrally symmetric member of κ^n.

The following result is true [9], for $K \in \kappa^n$

$$\int_{S^{n-1}} uf(K,u)dS(u) = 0.$$

Suppose $K, L \in \kappa^n$ and $\lambda, \mu \geq 0$(not both zero). From above it follows that the function $\lambda f(K,\cdot) + \mu f(L,\cdot)$ satisfies the hypothesis of Minkowski's existence theorem (see [5]). The solution of the Minkowski problem for this function is denoted by $\lambda \cdot K \ddot{+} \mu \cdot L$ that is

$$f(\lambda \cdot K \ddot{+} \mu \cdot L,\cdot) = \lambda f(K,\cdot) + \mu f(L,\cdot), \tag{8}$$

where the linear combination $\lambda \cdot K \ddot{+} \mu \cdot L$ is called a Blaschke linea combination.

The relationship between Blaschke and Minkowski scalar multiplication is given by

$$\lambda \cdot K = \lambda^{1/(n-1)} K. \tag{9}$$

A new addition, harmonic Blaschke addition, be defined by Lutwak [8]. Suppose $K, L \in \varphi^n$, and $\lambda, \mu \geq 0$(not both zero). To define the harmonic Blaschke linear combination, $\lambda K \hat{+} \mu L$, first define $\xi > 0$ by

$$\xi^{1/(n+1)} = \frac{1}{n} \int_{S^{n-1}} [\lambda V(K)^{-1} \rho(K, u)^{n+1} + \mu V(L)^{-1} \rho(L, u)^{n+1}]^{n/(n+1)} dS(u). \tag{10}$$

The body $\lambda K \hat{+} \mu L \in \varphi^n$ is defined as the body whose radial function is given by

$$\xi^{-1} \rho(\lambda K \hat{+} \mu L, \cdot)^{n+1} = \lambda V(K)^{-1} \rho(K, \cdot)^{n+1} + \mu V(L)^{-1} \rho(L, \cdot)^{n+1}. \tag{11}$$

It follows immediately that $\xi = V(\lambda K \hat{+} \mu L)$, and hence

$$V(\lambda K \hat{+} \mu L)^{-1} \rho(\lambda K \hat{+} \mu L, \cdot)^{n+1} = \lambda V(K)^{-1} \rho(K, \cdot)^{n+1} + \mu V(L)^{-1} \rho(L, \cdot)^{n+1}.$$

Lutwak [10] define a mapping:

$$\Lambda : \varphi_c^n \to \kappa_c^n$$

and point out that Λ tansforms harmonic Blaschke linear combination into Blaschke linear combinations, i.e.

If $K, L \in \varphi_c^n$ and $\lambda, \mu \geq 0$, then

$$\Lambda(\lambda K \hat{+} \mu L) = \lambda \cdot \Lambda K \ddot{+} \mu \cdot \Lambda L.$$

Further, We obtain that

If $K_j \in \varphi_c^n (j = 1, \ldots, m)$, and $\lambda_j \geq 0 (j = 1, \ldots, m)$, then

$$\Lambda(\lambda_1 K_1 \hat{+} \cdots \hat{+} \lambda_m K_m) = \lambda_1 \cdot \Lambda K_1 \ddot{+} \cdots \ddot{+} \lambda_m \cdot \Lambda K_m. \tag{12}$$

and

$$\Lambda(\lambda K) = \lambda \Lambda K \tag{13}$$

1.4 Projection bodies and Centroid bodies

The projection bodies, ΠK, of the body $K \in \mathcal{K}^n$ is defined as the convex figure whose support function is given, for $u \in S^{n-1}$, by [6]

$$h(\Pi K, u) = v(K^u) \tag{14}$$

It is easy to see, that a projection body is always centered(symmetric about the origin), and if K has interior points then ΠK will have interior point as well.

Here, we introduce the following property.

If $K, L \in \mathcal{K}^n$ and $\lambda, \mu \geq 0$, then [7]

$$\Pi(\lambda \cdot K \ddot{+} \mu \cdot L) = \lambda \Pi K + \mu \Pi L.$$

Further, we may prove that

If $K_j \in \mathcal{K}^n (j = 1, \ldots, m)$ and $\lambda_j \geq 0 (j = 1, \ldots, m)$, then

$$\Pi(\lambda_1 \cdot K_1 \ddot{+} \cdots \ddot{+} \lambda_m \cdot K_m) = \lambda_1 \Pi K_1 + \cdots + \lambda_m \Pi K_m. \tag{15}$$

The centroid body, ΓK, of $K \in \varphi^n$, is the convex body whose support function, at $x \in \mathbb{R}^n$, is given by [8]:

$$h(\Gamma K, x) = \frac{1}{V(K)} \int_K \mid x \cdot y \mid dy. \tag{16}$$

Here, we give the following property:

If $K_j \in \varphi^n (j = 1, \ldots, m)$, and $\lambda_j \geq 0 (j = 1, \ldots, m)$, then

$$\Gamma(\lambda_1 K_1 \hat{+} \cdots \hat{+} \lambda_m K_m) = \lambda_1 \Gamma K_1 + \cdots + \lambda_m \Gamma K_m. \tag{17}$$

If $K \in \varphi^n$, then from (20) it follows that ΓK is centered.

Please see the next section for above interrelated notations, definitions and their background material.

1. **Main results**

Width-integrals were first considered by Blaschke [1,p.85] and later by Hadwiger [2,p.266]. In [3], Lutwak also introduced the width-integral of index i and proved some important results, one of them is the following Theorem:

Theorem A. *If $K, L \in \mathcal{K}^n$ and $i < n-1$, then*

$$B_i(K+L)^{1/(n-i)} \leq B_i(K)^{1/(n-i)} + B_i(L)^{1/(n-i)} \tag{18}$$

with equality if and only if K and L have similar width.

Since inequality (1) is a new result similar to the following Brunn-Minkowski inequality for the cross-sectional measures [2, p.249].

Theorem B. *If $K, L \in \mathcal{K}^n$ and $i < n-1$, then*

$$W_i(K+L)^{1/(n-i)} \leq W_i(K)^{1/(n-i)} + W_i(L)^{1/(n-i)} \tag{19}$$

with equality if and only if K and L are homothetic.

Hence, inequality (1) is called as the dual Brunn-Minkowski inequality for width-integral of convex bodies.

The main purpose of this paper is to establish two analogs of inequality (1), them can be stated as:

Theorem C. If $K_1, \ldots, K_m \in \varphi^n$, $\lambda_1, \ldots, \lambda_m > 0$ *and* $i < n-1$, *then*

$$B_i(\Gamma(\lambda_1 K_1 \hat{+} \cdots \hat{+} \lambda_m K_m))^{1/(n-i)} \leq \lambda_1 B_i(\Gamma K_1))^{1/(n-i)} + \cdots + \lambda_m B_i(\Gamma K_m)^{1/(n-i)}, \tag{20}$$

with equality if and only if $\Gamma K_j (j = 1, 2, \ldots, m)$ have similar width.

Theorem D. *If $K_1, \ldots, K_m \in \varphi_c^n$, $\lambda_1, \ldots, \lambda_m > 0$ and $i < n-1$, then*

$$\begin{aligned} &B_i(\Pi(\Lambda(\lambda_1 K_1 \hat{+} \ldots \hat{+} \lambda_m K_m)))^{1/(n-i)} \\ &\leq \lambda_1 B_i(\Pi(\Lambda K_1))^{1/(n-i)} + \ldots + \lambda_m B_i(\Pi(\Lambda K_m))^{1/(n-i)}, \end{aligned} \tag{21}$$

with equality if and only if $\Pi(\Lambda K_j)(j = 1, 2, \ldots, m)$ have similar width.

2. A dual Brunn-Minkowski inequality about the width-integrals of centroid bodies for the harmonic Blaschke linear combination

The following dual Brunn-Minkowski inequality about the width-integrals of centroid bodies will be proved.

Theorem C. If $K_1, \ldots, K_m \in \varphi^n$, $\lambda_1, \ldots, \lambda_m > 0$ *and* $i < n-1$, *then*

$$B_i(\Gamma(\lambda_1 K_1 \hat{+} \cdots \hat{+} \lambda_m K_m))^{1/(n-i)} \leq \lambda_1 B_i(\Gamma K_1))^{1/(n-i)} + \cdots + \lambda_m B_i(\Gamma K_m)^{1/(n-i)}, \tag{22}$$

with equality if and only if $\Gamma K_j (j = 1, 2, \ldots, m)$ *have similar width.*

Proof. From (7),(10),(11),(21) and in view of Minkowski inequality for integral [11, p.147], we obtain that

$$B_i(\Gamma(\lambda_1 K_1 \hat{+} \cdots \hat{+} \lambda_m K_m))^{\frac{1}{n-i}} = \left(\frac{1}{n}\int_{S^{n-1}} b^{n-i}_{\Gamma(\lambda_1 K_1 \hat{+} \cdots \hat{+} \lambda_m K_m)} dS(u)\right)^{\frac{1}{n-i}}$$

$$= \left(\frac{1}{n}\int_{S^{n-1}} (b_{\lambda_1 \Gamma K_1 + \cdots + \lambda_m \Gamma K_m})^{n-i}\, dS(u)\right)^{\frac{1}{n-i}}$$

$$= \left(\frac{1}{n}\int_{S^{n-1}} (\lambda_1 b_{\Gamma K_1} + \cdots + \lambda_m b_{\Gamma K_m})^{n-i}\, dS(u)\right)^{\frac{1}{n-i}}$$

$$\leq \lambda_1 \left(\frac{1}{n}\int_{S^{n-1}} b^{n-i}_{\Gamma K_1} dS(u)\right)^{\frac{1}{n-i}} + \cdots + \lambda_m \left(\frac{1}{n}\int_{S^{n-1}} b^{n-i}_{\Gamma K_m} dS(u)\right)^{\frac{1}{n-i}}$$

$$= \lambda_1 B_i(\Gamma K_1)^{\frac{1}{n-i}} + \cdots + \lambda_m B_i(\Gamma K_m)^{\frac{1}{n-i}},$$

with equality if and only if $\Gamma K_j (j = 1, \ldots, m)$ have similar width.

The proof is complete. □

Taking $m = 2$ to (23), we have

Corollary 1. *If* $K, L \in \varphi^n$, $\lambda, \mu > 0$ *and* $i < n-1$, *then*

$$B_i(\Gamma(\lambda K \hat{+} \mu L))^{1/(n-i)} \leq \lambda B_i(\Gamma K)^{1/(n-i)} + \mu B_i(\Gamma L)^{1/(n-i)}, \tag{23}$$

with equality if and only if ΓK and ΓL *have similar width.*

Another important consequence is obtained when $\lambda = \mu = 1$.

Corollary 2. *If*$K, L \in \varphi^n$ and $i < n-1$, then

$$B_i(\Gamma(K \hat{+} L))^{1/(n-i)} \leq B_i(\Gamma K)^{1/(n-i)} + B_i(\Gamma L)^{1/(n-i)}, \tag{24}$$

with equality if and only if ΓK and ΓL have similar width.

3. A dual Brunn-Minkowski inequality about the width-integrals of projection bodies for the harmonic Blaschke linear combination

The following dual Brunn-Minkowski inequality about the width-integrals of projection bodies will be proved.

Theorem D. *If* $K_1, \ldots, K_m \in \varphi^n_c$, $\lambda_1, \ldots, \lambda_m > 0$ *and* $i < n-1$, *then*

$$B_i(\Pi(\Lambda(\lambda_1 K_1 \hat{+} \ldots \hat{+} \lambda_m K_m)))^{1/(n-i)}$$

$$\leq \lambda_1 B_i(\Pi(\Lambda K_1))^{1/(n-i)} + \ldots + \lambda_m B_i(\Pi(\Lambda K_m))^{1/(n-i)}, \tag{25}$$

with equality if and only if $\Pi(\Lambda K_j)(j = 1, 2, \ldots, m)$ *have similar width.*

Proof. From (7),(10),(11),(16),(19) and in view of Minkowski inequality for integral [11, p.147], we obtain that

$$B_i(\Pi(\Lambda(\lambda_1 K_1 \hat{+} \cdots \hat{+} \lambda_m K_m)))^{\frac{1}{n-i}} = \left(\frac{1}{n}\int_{S^{n-1}} b^{n-i}_{\Pi(\Lambda(K_1 \hat{+} \cdots \hat{+} K_m))} dS(u)\right)^{\frac{1}{n-i}}$$

$$= \left(\frac{1}{n}\int_{S^{n-1}} b^{n-i}_{\Pi(\lambda_1 \Lambda K_1 \ddot{+} \cdots \ddot{+} \lambda_m \Lambda K_m}) dS(u)\right)^{\frac{1}{n-i}}$$

$$= \left(\frac{1}{n}\int_{S^{n-1}} b^{n-i}_{\sum_{j=1}^m \lambda_j \Pi(\Lambda K_j)} dS(u)\right)^{\frac{1}{n-i}}$$

$$= \left(\frac{1}{n}\int_{S^{n-1}} \left(\sum_{j=1}^m \lambda_j b_{\Pi(\Lambda K_j)}\right)^{n-i} dS(u)\right)^{\frac{1}{n-i}}$$

$$\leq \sum_{j=1}^m \lambda_j \left(\frac{1}{n}\int_{S^{n-1}} b^{n-i}_{\Pi(\Lambda K_j)} dS(u)\right)^{\frac{1}{n-i}} = \sum_{j=1}^m \lambda_j B_i(\Pi(\Lambda K_j))^{\frac{1}{n-i}},$$

with equality if and only if $\Pi(\Lambda K_j)(j = 1, \ldots, m)$ have similar width.

The proof is complete. □

Taking $m = 2$ to (26), we have

Corollary 3. *If* $K, L \in \varphi^n_c$, $\lambda, \mu > 0$ *and* $i < n - 1$, *then*

$$B_i(\Pi(\Lambda(\lambda K \hat{+} \mu L)))^{1/(n-i)} \leq \lambda B_i(\Pi(\Lambda K))^{1/(n-i)} + \mu B_i(\Pi(\Lambda L))^{1/(n-i)}, \tag{26}$$

with equality if and only if $\Pi(\Lambda K)$ and $\Pi(\Lambda L)$ *have similar width.*

Another remarkable case is obtained for $\lambda = \mu = 1$.

Corollary 4. *If* $K, L \in \varphi^n_c$ and $i < n - 1$, then

$$B_i(\Pi(\Lambda(K \hat{+} L)))^{1/(n-i)} \leq B_i(\Pi(\Lambda K))^{1/(n-i)} + B_i(\Pi(\Lambda L))^{1/(n-i)}, \tag{27}$$

with equality if and only if $\Pi(\Lambda K)$ and $\Pi(\Lambda L)$ have similar width.

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PROJECTORS IN FINITE π-SOLVABLE GROUPS AND ASSOCIATED PRIMITIVE CLASSES

RODICA COVACI

Dedicated to Professor Grigore Călugăreanu on his 60th birthday

Abstract. In [7], W. Gaschütz introduced the notion of *primitive class* and studied its connection with projectors in finite solvable groups. Let π be a set of primes. Introduced by S.A. Čunihin in [6], the π-solvable groups are more general than the soluble groups. It is the aim of this paper to give conditions under which we can establish a connection between projectors in finite π-solvable groups and their associated primitive classes. The main results establishing this connection are based on some new properties of projectors in finite π-solvable groups which are also proved in this paper.

1. Preliminaries

All groups considered in this paper are finite. Let π be a set of primes and π' the complement to π in the set of all primes.

We first remind some useful definitions and theorems.

Definition 1.1. a) A positive integer n is said to be a π*-number* if for any prime divisor p of n we have $p \in \pi$.

b) A finite group G is a π*-group* if the order of G is a π-number.

c) A subgroup H of a group G is said to be a π*-subgroup* if H is a π-group.

Notation 1.2. Let G be a group. We denote by $O_{\pi'}(G)$ the largest normal π'-subgroup of G.

Definition 1.3. Let G be a group.

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a) M is a *minimal normal subgroup* of G if:

(i) $M \lhd G$ and $M \neq 1$;

(ii) if $M^* \lhd G$, $M^* \neq 1$ and $M^* \leq M$ then $M^* = M$.

b) Let M and N be normal subgroups of G such that $N \subseteq M$. The factor M/N is called a *chief factor* of G if M/N is a minimal normal subgroup of G/N.

c) Let

$$C : G = G_0 > G_1 > G_2 > \cdots > G_r = 1$$

be a chain of normal subgroups of G. C is called a *chief chain* of G if G_i/G_{i+1} is a chief factor of G, for any i.

The following theorem given by R. Baer in [1] is of special interest for our considerations.

Theorem 1.4. ([1]) *A solvable minimal normal subgroup of a finite group is abelian.*

Introduced by S.A. Čunihin in [6], the π-solvable groups are more general than the solvable groups.

Definition 1.5. ([6]) A group G is *π-solvable* if every chief factor of G is either a solvable π-group or a π'-group. For π the set of all primes, we obtain the notion of solvable group.

Remark 1.6. a) A finite group G is π-solvable if and only if G is a chief chain

$$C : G = G_0 > G_1 > G_2 > \cdots > G_r = 1$$

such that any factor G_i/G_{i+1} is either a solvable π-group or a π'-group. Moreover, if G is a finite π-solvable group then any chief chain of G has the above property.

b) We use in the paper the following basic properties of π-solvable groups:

(i) If G is a π-solvable group and H is a subgroup of G, then H is also a π-solvable group.

(ii) If G is a π-solvable group and N is a normal subgroup of G, then G/N is a π-solvable group.

We define below the special classes of groups and subgroups which will appear in the paper.

Definition 1.7. a) A class X of groups is a *homomorph* if X is epimorphically closed, i.e. if $G \in X$ and N is a normal subgroup of G, then $G/N \in X$.

b) A group G is *primitive* if G has a *stabilizer*, i.e. a maximal subgroup H of G with $core_G H = 1$, where $core_G H = \cap\{H^g/g \in G\}$. We remind here that $core_G H$ is a normal subgroup of G.

c) ([7]) A homomorph X is called a *Schunck class* if X is *primitively closed*, i.e. if any group G, all of whose primitive factor groups are in X, is itself in X.

d) A class X of groups is said to be π*-closed* if:

$$G/O_{\pi'}(G) \in X \;\Rightarrow\; G \in X.$$

A π-closed homomorph, respectively a π-closed Schunck class will be called π*-homomorph*, respectively π*-Schunck class*.

Definition 1.8. ([7]) Let X be a class of groups, G a group and H a subgroup of G.

a) H is an X*-maximal subgroup* of G if:

(i) $H \in X$;

(ii) $H \leq H^* \leq G$, $H^* \in X \;\Rightarrow\; H = H^*$.

b) H is an X*-projector* of G if, for any normal subgroup N of G, HN/N is X-maximal in G/N.

Remark 1.9. Let X be a class of groups.

a) Any X-projector of a group G is X-maximal in G.

b) If H is an X-maximal subgroup of G and $H \leq K \leq G$, then H is also an X-maximal subgroup of K.

Theorem 1.10. ([7]) *Let X be a class of groups, G a group and H a subgroup of G. H is an X-projector of G if and only if:*

(a) H is an X-maximal subgroup of G;

(b) HM/M is an X-projector of G/M, for all minimal normal subgroups M of G.

Theorem 1.11. ([7]) *Let X be a homomorph, G a group and H a subgroup of G. If H is an X-projector of G and N is a normal subgroup of G, then HN/N is an X-projector of G/N.*

In [3], we introduced the P property, which is important for the present paper.

Definition 1.12. Let X be a class of groups and G a π-solvable group. We say that G has the *property* (α) with respect to X if the following implication is true:

$$M \text{ minimal normal subgroup of } G,\ M\ \pi' - \text{subgroup} \ \Rightarrow\ G/M \in X. \qquad (\alpha)$$

Definition 1.13. ([3]) Let X be a class of groups and π a set of primes. We say that X has the P *property* with respect to π if any π-solvable group G has the property (α) with respect to X.

The following two results on properties (α) and P will be used in our considerations.

Theorem 1.14. *Let X be a π-homomorph and G a π-solvable group having property (α) with respect to X and suppose that there is a minimal normal subgroup M of G such that M is a π'-group. Then $G \in X$.*

Proof. From $M \triangleleft G$ and M π'-group follows that $M \leq O_{\pi'}(G)$ and so

$$G/O_{\pi'}(G) \cong (G/M)/(O_{\pi'}(G)/M). \qquad (1)$$

By property (α), $G/M \in X$. Hence, by (1) and X being a homomorph, $G/O_{\pi'}(G) \in X$. This implies by the π-closure of X that $G \in X$. $\square$

Theorem 1.15. *Let X be a π-homomorph with the P property with respect to π and G a π-solvable group such that there is a minimal normal subgroup M of G, M π'-group. Then $G \in X$.*

Proof. Since X has the P property with respect to π and G is a π-solvable group, it follows that G has property (α) with respect to X. Applying now 1.14, we obtain that $G \in X$. $\square$

2. Some new properties of projectors in finite π-solvable groups

We first give some sufficient conditions on a subgroup of a π-solvable group to be an X-projector, where X is a π-Schunck class.

Theorem 2.1. ([5]) *Let X be a π-Schunck class, G a finite π-solvable group such that for any minimal normal subgroup M of G which is a π'-group we have $G/M \in X$ and let B be a normal abelian subgroup of G and S a subgroup of G such that:*

(i) S is X-maximal in BS;

(ii) BS/B is an X-projector of G/B.

Then S is an X-projector of G.

Theorem 2.2. *Let X be a π-Schunck class with the P property with respect to π, G a finite π-solvable group and let B be a normal abelian subgroup of G and S a subgroup of G such that:*

(i) S is X-maximal in BS;

(ii) BS/B is an X-projector of G/B.

Then S is an X-projector of G.

Proof. X having the P property with respect to π and G being a π-solvable group, we deduce that G has the property (α) with respect to X. Applying now 2.1, it follows that S is an X-projector of G. $\square$

Theorem 2.3. *Let X be a π-Schunck class and G a finite π-solvable group with $O_{\pi'}(G) = 1$. If B is a normal abelian subgroup of G and S is a subgroup of G such that*

(i) S is X-maximal in BS;

(ii) BS/B is an X-projector of G/B,

then S is an X-projector of G.

Proof. The proof is similar with that of 2.1 given in [5]. Two cases are considered:

1) $B = 1$. From (ii) follows that S is an X-projector of G.

2) $B \neq 1$. We prove that S is an X-projector of G by using 1.10.

(a) S is X-maximal in G. This follows like in the proof of 2.1 given in [5].

(b) Let M be a minimal normal subgroup of G. Since G is π-solvable, it follows that M is either a solvable π-group or a π'-group. If M is a solvable π-group, we prove like in the proof of 2.1 given in [5] that SM/M is an X-projector of G/M. If M is a π'-group, we deduce that $M \leq O_{\pi'}(G) = 1$, which leads to the contradiction $M = 1$. So this case cannot happen. $\square$

We continue by giving two new characterizations of projectors in finite π-solvable groups.

Theorem 2.4. *Let X be a π-Schunck class with the P property with respect to π, G a finite π-solvable group and S a subgroup of G. Let*

$$C : G = G_0 > G_1 > \cdots > G_{r-1} > G_r = 1$$

be a chief chain of G. Then S is an X-projector of G if and only if G_iS/G_i is X-maximal in G/G_i, for any i.

Proof. We consider two cases: 1) $S = G$; 2) $S \neq G$.

1) $S = G$. If S is an X-projector of G, then G is an X-projector of G, and so $G \in X$. But X being a homomorph, we have also $G/G_i \in X$, for any i. It follows that $G/G_i = G_iS/G_i$ is X-maximal in G/G_i, for any i. Conversely, let G_iS/G_i be X-maximal in G/G_i, for any i. Then G/G_i is X-maximal in G/G_i, for any i. In particular, for $i = r$ we obtain that G is X-maximal in G, hence $S = G \in X$ and so $S = G$ is an X-projector of G.

2) $S \neq G$. Let S be an X-projector of G. For any i, G_i is normal in G. Hence, by 1.11, G_iS/G_i is an X-projector of G/G_i, for any i. Then, by 1.9.a), G_iS/G_i is X-maximal in G/G_i, for any i. The converse is proved by induction on $|G|$. Suppose that G_iS/G_i is X-maximal in G/G_i, for any i. In particular, for $i = r$, we obtain that S is X-maximal in G. By 1.6.a), G_i/G_{i+1} is either a solvable π-group or a π'-group, for any i. In particular, for $i = r - 1$, $G_{r-1} \cong G_{r-1}/G_r$ is either a solvable π-group or a π'-group. We consider the two cases and prove in each of them that S is an X-projector of G.

a) G_{r-1} *is a solvable* π*-group.* Then, by 1.4, G_{r-1} is abelian. In this case, we are in the hypotheses of theorem 2.2, taking G_{r-1} as a normal abelian subgroup of G and S as a subgroup of G. Indeed, we will prove below that (i) and (ii) from 2.2 are true.

(i) S is X-maximal in $G_{r-1}S$, since S is X-maximal in G and we apply 1.9.b).

(ii) $G_{r-1}S/G_{r-1}$ is an X-projector of G/G_{r-1}. Indeed, this can be obtained if we observe that the group G/G_{r-1} is also π-solvable (by (1.6.b)) and apply for G/G_{r-1} the induction. It follows, by Theorem 2.2, that S is an X-projector of G.

b) G_{r-1} *is a* π'*-group.* In this case, we are in the hypotheses of theorem 1.15, where $G_{r-1} \cong G_{r-1}/G_r$ is the minimal normal subgroup of $G \cong G/G_r$ which is a π'-group. By 1.15, $G \in X$. From this and S being X-maximal in G, it follows that $S = G$. But $G \in X$ implies that G is its own X-projector. So $S = G$ is an X-projector of G. $\square$

In order to prove the second characterization of projectors in finite π-solvable groups, we first give a lemma.

Lemma 2.5. ([2]) *If X is a π-homomorph, G a π-solvable group, H a subgroup of G, $H \neq G$, H X-maximal in G and N is a minimal normal subgroup of G with $HN = G$, then N is abelian.*

Theorem 2.6. *Let X be a π-Schunck class, G a finite π-solvable group with the property (α) with respect to X and S a subgroup of G. Let*

$$C : G = G_0 > G_1 > \cdots > G_{r-1} > G_r = 1$$

be a chief chain of G such that $G/G_{r-1} \in X$. Then S is an X-projector of G if and only if G_iS/G_i is X-maximal in G/G_i, for any i.

Proof. We consider two cases: 1) $S = G$; 2) $S \neq G$.

1) $S = G$. This case has the same proof as the proof of case 1) for theorem 2.4.

2) $S \neq G$. If S is an X-projector of G, then we prove like in the proof of case 2) for theorem 2.4 that G_iS/G_i is X-maximal in G/G_i, for any i. Conversely, suppose that G_iS/G_i is X-maximal in G/G_i, for any i. In particular, for $i = r$, we obtain that

S is X-maximal in G. Obviously, $G_{r-1} \cong G_{r-1}/G_r$ is a minimal normal subgroup of $G/G_r \cong G$. Putting $i = r-1$ in our hypothesis, we have that $G_{r-1}S/G_{r-1}$ is X-maximal in G/G_{r-1}. But $G/G_{r-1} \in X$. It follows that $G_{r-1}S/G_{r-1} = G/G_{r-1}$, and so $G_{r-1}S = G$. We are now in the hypotheses of lemma 2.5, for $H = S$ and $N = G_{r-1}$. Applying 2.5, we obtain that G_{r-1} is abelian. In order to prove that S is an X-projector of G we use theorem 2.1, in our case the abelian normal subgroup of G being G_{r-1} and S verifying the conditions (i) and (ii) from 2.1. Indeed, we have:

(i) S is X-maximal in $G_{r-1}S$, since S is X-maximal in G and we apply 1.9.b).

(ii) $G_{r-1}S/G_{r-1}$ is an X-projector of G/G_{r-1}, since $G_{r-1}S = G$ implies that $G_{r-1}S/G_{r-1} = G/G_{r-1} \in X$.

By applying 2.1, it follows that S is an X-projector of G. $\square$

We finally give conditions under which the property of being projector in a finite π-solvable group is hereditary to subgroups. In preparation for the main result, we first give a lemma.

Lemma 2.7. *Let G be a finite group and let*

$$C : G = G_0 > G_1 > \cdots > G_{r-1} > G_r = 1$$

be a chief chain of G. Let H be a characteristic subgroup of G. We put $H_i = H \cap G_i$, for any i. Then

$$C^* : H = H_0 > H_1 > \cdots > H_{r-1} > H_r = 1$$

is a chief chain of H.

Proof. We have to prove that H_i/H_{i+1} is a minimal normal subgroup of H/H_{i+1}, for any i. Obviously H_i/H_{i+1} is a normal $\neq 1$ subgroup of H/H_{i+1}. Let N/H_{i+1} be a normal $\neq 1$ subgroup of G/H_{i+1} such thar $N/H_{i+1} \subseteq H_i/H_{i+1}$. We will prove that $N/H_{i+1} = H_i/H_{i+1}$. From $H_i \lhd H$ and H characteristic in G follows that $H_i \lhd G$, for any i. Furthermore,

$$H_i/H_{i+1} = (H \cap G_i)/(H \cap G_{i+1})$$

$$= (H \cap G_i)/(H \cap G_i) \cap G_{i+1} \cong (H \cap G_i)G_{i+1}/G_{i+1} \leq G_i/G_{i+1}.$$

This means that H_i/H_{i+1} is isomorphic with a normal $\neq 1$ subgroup of G/G_{i+1} included in G_i/G_{i+1}. Since G_i/G_{i+1} is a minimal normal subgroup of G/G_{i+1}, this subgroup coincides with G_i/G_{i+1}. So, for any i, we have:

$$H_i/H_{i+1} \cong G_i/G_{i+1}. \tag{2}$$

Similarly, since

$$H/H_{i+1} = (H \cap G)/(H \cap G_{i+1})$$

$$= (H \cap G)/(H \cap G) \cap G_{i+1} \cong (H \cap G)G_{i+1}/G_{i+1} \leq G/G_{i+1}$$

we deduce that H/H_{i+1} is isomorphic with a subgroup G^*/G_{i+1} of G/G_{i+1}. But from $N \lhd H$ and H characteristic in G follows that $N \lhd G$. From this and from $N/H_{i+1} \subseteq H_i/H_{i+1}$, it follows by (2) that N/H_{i+1} is isomorphic with a normal $\neq 1$ subgroup of G/G_{i+1} included in G_i/G_{i+1}. Since G_i/G_{i+1} is a minimal normal subgroup of G/G_{i+1}, this subgroup coincides with G_i/G_{i+1}, which means that

$$N/H_{i+1} \cong G_i/G_{i+1}. \tag{3}$$

From (2) and (3) follows that $N/H_{i+1} \cong H_i/H_{i+1}$. But $N/H_{i+1} \subseteq H_i/H_{i+1}$. So $N/H_{i+1} = H_i/H_{i+1}$. $\square$

Theorem 2.8. *Let X be a π-Schunck class with the P property with respect to π, G a finite π-solvable group and $S \leq H \leq G$ such that H is characteristic in G. If S is an X-projector of G, then S is an X-projector of H.*

Proof. By 1.6.a), G has a chief chain

$$C : G = G_0 > G_1 > \cdots > G_{r-1} > G_r = 1$$

such that G_i/G_{i+1} is either a solvable π-group or a π'-group, for any i. Since S is an X-projector of G, by applying theorem 2.4, we deduce that G_iS/G_i is X-maximal in G/G_i, for any i. We put $H_i = H \cap G_i$, for any i. Applying lemma 2.7 for the group G, its chief chain C and its characteristic subgroup H, it follows that

$$C^* : H = H_0 > H_1 > \cdots > H_{r-1} > H_r = 1$$

is a chief chain of H. Since G is π-solvable, by 1.6.b) follows that H is also π-solvable. In order to prove that S is an X-projector of H we use theorem 2.4 for the π-solvable group H and its chief chain C^*. It remains only to show that H_iS/H_i is X-maximal in H/H_i, for any i. Indeed, we have that

$$H_iS/H_i \cong S/S \cap H_i = S/S \cap (H \cap G_i) = S/(S \cap H) \cap G_i = S/S \cap G_i \cong G_iS/G_i$$

and so

$$H_iS/H_i \cong G_iS/G_i. \tag{4}$$

We also have that

$$H/H_i = H/H \cap G_i \cong HG_i/G_i \leq G/G_i.$$

From G_iS/G_i X-maximal in G/G_i and $G_iS/G_i \leq G_iH/G_i \leq G/G_i$ follows by 1.9.b) that

$$G_iS/G_i \text{ is } X-\text{maximal in } HG_i/G_i. \tag{5}$$

We also have

$$HG_i/G_i \cong H/H \cap G_i = H/H_i \tag{6}$$

and

$$H_iS/H_i \leq H/H_i. \tag{7}$$

(5) becomes by using (4), (6) and (7):

$$H_iS/H_i \text{ is } X-\text{maximal in } H/H_i, \text{ for any } i.$$

Finally, applying theorem 2.4, it follows that S is an X-projector of H. $\square$

3. Primitive classes. Their connection with projectors in finite π-solvable groups

In 1.7.b), we defined the notion of *primitive group*. If G is a group and N is a normal subgroup of G, we will call a primitive quotient group G/N simply by a *primitive factor* of G.

Proposition 3.1. ([7]) Let G be a finite group.

a) If N is a normal subgroup of G, then: G/N is primitive if and only if there is a maximal subgroup W of G such that $N = core_G W$.

b) If W is a maximal subgroup of G, then $G/core_G W$ is primitive.

Notation 3.2. We put Pr for the class of all finite primitive groups.

Remark 3.3. If G is a primitive group and N is a normal subgroup of G, it doesn't generally follow that G/N is primitive. So the class Pr is neither a homomorph nor a Schunck class.

In [7], W. Gaschütz introduced the notion of *primitive class* and gave some theorems which reveal a connection between primitive classes and projectors in the universe of finite solvable groups. It is the main aim of this paper to give conditions under which such a connection also holds in the more general case of π-solvable groups.

Definition 3.4. ([7]) A class P of finite groups is said to be a *primitive class* if P satisfies the following two conditions:

(i) if $G \in P$, then G is primitive;

(ii) if $G \in P$ and G/N is a primitive factor of G, then $G/N \in P$.

In other words, a primitive class P is a class of finite groups such that $P \subset Pr$ and which together with a group G contains all primitive factors of G.

Notation 3.5. ([7]) Let G be a finite group. We put

$$P(G) = \{G/N \ / \ G/N \text{ is primitive}\}.$$

Proposition 3.6. If G is a finite group, then $P(G)$ is a primitive class.

Proof. Obviously $P(G)$ satisfies (i). In order to verify (ii), let $G/N \in P(G)$ and let $(G/N)/(M/N)$ be a primitive factor of G/N. Since

$$(G/N)/(M/N) \cong G/M, \tag{8}$$

we deduce that G/M is a primitive factor of G, and so $G/M \in P(G)$. It follows by (8) that $(G/N)/(M/N) \in P(G)$. So (ii) is also satisfied. $\square$

Definition 3.7. Let G be a finite group. We call $P(G)$ given by

$$P(G) = \{G/N \ / \ G/N \text{ is primitive}\}$$

the primitive class associated to the group G.

Notation 3.8. ([7]) Let X be an arbitrary class of groups. We denote by

$$P(X) = \cup\{P(G)/G \in X\}$$

the union of the primitive classes associated to all groups belonging to the class X.

b) Let P be a primitive class. We denote by

$$X(P) = \{G \ / \ G/N \in P, \text{ for all primitive factors } G/N \text{ of } G\},$$

i.e. $X(P)$ is the class of those finite groups G whose primitive factors are in P.

Proposition 3.9. ([7]) If X is a Schunck class and P is a primitive class, then:

a) $X = X(P(X))$;

b) $P = P(X(P))$.

We give below the main results of this paper.

Theorem 3.10. *Let X be a π-Schunck class with the P property with respect to π, G a finite π-solvable group and S a subgroup of G. If S is an X-projector of G, then:*

(i) $P(S) \subseteq X$;

(ii) $S \leq W < H \leq G$, with W maximal in H and H characteristic in G imply that $H/core_H W \notin X$.

Proof. (i) S being an X-projector of G, by 1.9.a) S is X-maximal in G, hence $S \in X$. Let $S/N \in P(S)$. Since $S \in X$ and X being a homomorph, we obtain that $S/N \in X$. So $P(S) \subseteq X$.

(ii) Let $S \leq W < H \leq G$, with W maximal in H and H characteristic in G. Suppose that $H/core_H W \in X$. By theorem 2.8, it follows that S is an X-projector of H. Hence by theorem 1.11 we obtain that $S\ core_H W/core_H W$ is an X-projector in $H/core_H W$, and so by 1.9.a) $S\ core_H W/core_H W$ is X-maximal in $H/core_H W$. But we supposed that $H/core_H W \in X$. Hence $S\ core_H W/core_H W = H/core_H W$ and so $S\ core_H W = H$. It follows that

$$H = S\ core_H W \leq W,$$

in contradiction with the hypothesis that W is a maximal subgroup of H. So we conclude that $H/core_H W \notin X$. $\square$

Theorem 3.11. *Let X be a π-Schunck class with the P property with respect to π, G a finite π-solvable group and S a subgroup of G. If the following two conditions are verified:*

(i) $P(S) \subseteq X$;

(ii) $S \leq W < H \leq G$, with W maximal in $H \;\Rightarrow\; H/core_H W \notin X$,

then S is an X-projector of G.

Proof. We prove by induction on $|G|$.

We first remark that $S \in X$. Indeed, from (i) follows that any primitive factor S/N of S is in X, which, by the primitive closure of the Schunck class X, leads to $S \in X$.

Let M be a minimal normal subgroup of G. We put $S^* = SM$. Applying the induction for the π-solvable group G/M (see 1.6.b)) and its subgroup S^*/M, we deduce that S^*/M is an X-projector of G/M.

We now prove that S is X-maximal in S^*. We saw that $S \in X$. Let $S \leq T \leq S^*$ and $T \in X$. We will prove that $S = T$. Suppose that $S \neq T$. Let then W be a maximal subgroup of T such that $S \leq V < T \leq S^* \leq G$. Applying (ii), we obtain that $T/core_T W \notin X$, in contradiction with $T/core_T W \in X$, which comes from $T \in X$ by applying the hypothesis that X is a homomorph. So $S = T$.

G being a π-solvable group and M being a minimal normal subgroup of G, it follows that M is either a solvable π-group or a π'-group. We continue the proof by considering two cases:

1) M is a solvable π-group. Then, by theorem 1.4, M is abelian. So we are in the hypotheses of theorem 2.2, where we take M as normal abelian subgroup of G. Applying theorem 2.2, we obtain that S is an X-projector of G.

2) M is a π'-group. Since X has the P property with respect to π, it follows that $G/M \in X$. From the fact that S^*/M is an X-projector of G/M we deduce, by 1.9.a), that S^*/M is X-maximal in G/M, which gives, using $G/M \in X$, that $S^*/M = G/M$. So $S^* = G$. We saw that S is X-maximal in S^*. It follows that

S is X-maximal in G. On the other side, by applying theorem 1.15, we obtain that $G \in X$. This leads to $S = G \in X$. So $S = G$ is its own X-projector. □

In our last considerations, we generalize the definition of the notion of *projector* given by W. Gaschütz in [7] and give some connection of projectors with primitive classes in the universe of finite π-solvable groups.

Definition 3.12. Let G be a finite π-solvable group and S a subgroup of G. We say that S is a *projector* of G if there is some π-Schunck class X with the P property with respect to π, such that S is an X-projector of G.

Theorem 3.13. *Let G be a finite π-solvable group and S a subgroup of G. If S is a projector of G, then the following implication is true: $S \leq W < H \leq G$, with W maximal in H and H characteristic in $G \Rightarrow H/core_H W \notin P(S)$.*

Proof. Let S be a projector of G and let $S \leq W < H \leq G$, with W maximal in H and H characteristic in G. It follows that there is a π-Schunck class X with the P property with respect to π, such that S is an X-projector of G. Applying now theorem 3.10, it follows that $P(S) \subseteq X$ and also that $S \leq W < H \leq G$, with W maximal in H and H characteristic in G imply $H/core_H W \notin X$. Then obviously $H/core_H W \notin P(S)$. □

Theorem 3.14. *Let G be a finite π-solvable group and S a subgroup of G. If*

(i) $X = X(P(S))$ is a π-Schunck class with the P property with respect to π;

(ii) $S \leq V < H \leq G$, with W maximal in $H \Rightarrow H/core_H W \notin P(S)$,

then S is a projector of G.

Proof. It is easy to see that $P(S) \subseteq X(P(S)) = X$. Let $S \leq W < H \leq G$, with W maximal in H. By 3.1.b), $H/core_H W$ is primitive. By (ii), we have that $H/core_H W \notin P(S)$. From this and from $P(S) \subseteq X(P(S)) = X$ and $H/core_H W$ primitive, we deduce that $H/core_H W \notin X = X(P(S))$. So we proved that conditions (i) and (ii) from theorem 3.11 are verified. Applying now theorem 3.11, we obtain that S is an X-projector of G, where $X = X(P(S))$ is a π-Schunck class with the P property with respect to π. Hence S is a projector of G. □

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LEBESGUE TYPE CONVERGENCE THEOREMS

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Abstract. In [9], the author introduced a set-valued integral for multifunctions with respect to a multimeasure φ. If $\mathcal{P}_k(X)$ is the family of nonempty compact subsets of a Hausdorff locally convex algebra X, then both the multifunctions and the multimeasure take values in a subset $\widetilde{X}$ of $\mathcal{P}_k(X)$ which satisfies certain conditions. In this paper, we present Lebesgue type convergence theorems for the integral defined in [9].

1. **Terminology and notations**

The theory of multifunctions has multiple applications in the theory of mathematical economics or in the theory of games. In an earlier article [9], we constructed an integration theory for multifunctions $F : S \to \widetilde{X}$ with respect to a multimeasure $\varphi : \mathcal{A} \to \widetilde{X}$. If $\mathcal{P}_k(X)$ is the family of nonempty compact subsets of a Hausdorff locally convex algebra X, then both the multifunctions and the multimeasure take values in a subset $\widetilde{X}$ of $\mathcal{P}_k(X)$, where $\widetilde{X}$ satisfies certain conditions. For different choices of the space X, of the multifunctions F and of the multimeasure φ, this set-valued integral contains, like particular cases, the classical integrals of Dunford [11], Brooks [5] and Martellotti-Sambucini [14]. In this paper, we obtain Lebesgue and Vitali type theorems of passing to the limit into the set-valued integral defined in [9].

Let S be a nonempty set and $\mathcal{A}$ an algebra of subsets of S. Let X be a Hausdorff locally convex vector space and Q a filtering family of seminorms which defines the topology of X.

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We consider $(x, y) \mapsto xy$ having the following properties for every $x, y, z \in X, \alpha, \beta \in \mathbb{R}, p \in Q$:

(i) $x(yz) = (xy)z$,

(ii) $xy = yx$,

(iii) $x(y + z) = xy + xz$,

(iv) $(\alpha x)(\beta y) = (\alpha\beta)(xy)$,

(v) $p(xy) \leq p(x)p(y)$.

1.1. Examples. (a) $X=\{f \mid f : T\to\mathbb{R}\}$ where T is a nonempty set and

$$Q = \{p_t | t \in T\}, p_t(f) = |f(t)|, \ \forall f \in X.$$

(b) $X = \{f \mid f : T \to \mathbb{R}$ is bounded $\}$ where T is a topological space. Let $\mathcal{K} = \{K \subset T | K$ is compact $\}$ and $Q = \{p_k | K \in \mathcal{K}\}$ where $p_K(f) = \sup\limits_{t\in K} |f(t)|, \forall f \in X$.

Let $\mathcal{P}_k(X) = \mathcal{P}_k$ be the family of all nonempty compact subsets of X. If $A, B \in \mathcal{P}_k$ and $\alpha \in \mathbb{R}$, then:

$$A + B = \{x + y | x \in A, y \in B\}$$
$$\alpha A = \{\alpha x | x \in A\}$$
$$A \cdot B = \{xy | x \in A, y \in B\}.$$

For every $p \in Q, A, B \in \mathcal{P}_k$, let $e_p(A, B) = \sup\limits_{x\in A} \inf\limits_{y\in B} p(x - y)$ be the p-excess of A over B and let $h_p(A, B) = \max\{e_p(A, B), e_p(B, A)\}$ be the Hausdorff-Pompeiu semimetric defined by p on $\mathcal{P}_k$. We define $\|A\|_p = h_p(A, O), \ \forall A \in \mathcal{P}_k$, where $O = \{0\}$. Then $\{h_p\}_{p\in Q}$ is a filtering family of semimetrics on $\mathcal{P}_k$ which defines a Hausdorff topology on $\mathcal{P}_k$.

Let $\widetilde{X} \subset \mathcal{P}_k$ be a subset of $\mathcal{P}_k$, satisfying the conditions:

$(\widetilde{1})$ $\widetilde{X}$ is complete with respect to $\{h_p\}_{p\in Q}$;

$(\widetilde{2})$ $O \in \widetilde{X}$;

$(\widetilde{3})$ $A + B, A \cdot B \in \widetilde{X}$, for all $A, B \in \widetilde{X}$;

$(\widetilde{4})$ $A \cdot (B + C) = A \cdot B + A \cdot C$, for all $A, B, C \in \widetilde{X}$.

1.2. Examples.

(a) $\widetilde{X} = \{\{x\} | x \in X\}$ for X like in the examples 1.1-(a), (b).

(b) $\widetilde{X} = \{A \in \mathcal{P}_k | A \subset [0, +\infty)\}$ for $X = \mathbb{R}$.

(c) For X like in the example 1.1-(a), let $\widetilde{X} = \{[f,g] \mid f,g \in X, 0 \le f \le g\}$, where $[f,g] = \{u \in X \mid f \le u \le g\}$ and $[f,f] = \{f\}$, for every $f,g \in X$.

1.3. Definition. A multifunction $\varphi : \mathcal{A} \to \mathcal{P}_k$ is said to be a *multimeasure* if:

(i) $\varphi(\emptyset) = O$;

(ii) $\varphi(A \cup B) = \varphi(A) + \varphi(B),\ \forall A,B \in \mathcal{A}, A \cap B = \emptyset$.

1.4. Definition. Let $\varphi : \mathcal{A} \to \mathcal{P}_k$. For every $p \in Q$, the p - *variation* of φ is the non - negative (possibly infinite) set function $v_p(\varphi, \cdot)$ defined on $\mathcal{A}$ as follows: for every $A \in \mathcal{A}$,

$$v_p(\varphi, A) = \sup \left\{ \sum_{i=1}^{n} \|\varphi(E_i)\|_p \middle| (E_i)_{i=1}^{n} \subset \mathcal{A} \text{ is a partition of } A \right\}.$$

We denote $v_p(\varphi, \cdot)$ by ν_p if there is no ambiguity.

If φ is a multimeasure, then ν_p is finitely additive, $\forall p \in Q$.

Throughout this paper, $\varphi : \mathcal{A} \to \widetilde{X}$ will be a multimeasure. We shall work under a weaker condition than that presumed in Croitoru [9], that is: we shall assume $\nu_p(S) < \infty$, for every $p \in Q$.

2. Set-valued integral

We recall in this paragraph some basic definitions introduced in [9].

2.1. Definition. A multifunction $F : S \to \widetilde{X}$ is said to be a *simple multifunction* if $F = \sum\limits_{i=1}^{n} C_i \cdot \mathcal{X}_{A_i}$, where $C_i \in \widetilde{X},\ A_i \in \mathcal{A},\ i \in \{1,\ldots,n\},\ A_i \cap A_j = \emptyset$ $(i \ne j)$, $\bigcup\limits_{i=1}^{n} A_i = S$ and $\mathcal{X}_{A_i}$ is the characteristic function of A_i.

The integral of F over $E \in \mathcal{A}$ with respect to φ is defined to be:

$$\int_E F d\varphi = \sum_{i=1}^{n} C_i \cdot \varphi(A_i \cap E) \in \widetilde{X}.$$

2.2. Definition. A multifunction $F : S \to \widetilde{X}$ is said to be φ - *totally measurable* if there is a sequence $(F_n)_n$ of simple multifunctions $F_n : S \to \widetilde{X}$ satisfying the following condition for every $p \in Q$:

$h_p(F_n, F)$ converges to 0 in ν_p- measure (in the sense of Dunford-Schwartz [11]) (denoted $h_p(F_n, F) \xrightarrow{\nu_p} 0$).

2.3. Remarks. (a) Every simple multifunction is φ-totally measurable.

(b) Let $F : S \to \widetilde{X}$ be a φ-totally measurable multifunction and, by the previous definition, let $(F_n)_{n\in\mathbb{N}}$ be the sequence of simple multifunctions $F_n : S \to \widetilde{X}$ such that, for every $p \in Q$:

$$h_p(F_n, F) \xrightarrow{\nu_p} 0.$$

Then, for every $n \in \mathbb{N}$, $h_p(F_n, F)$ and $\|F\|_p$ are ν_p-measurable in the sense of Dunford-Schwartz [11].

2.4. Definition. Let $F : S \to \widetilde{X}$ be a φ - totally measurable multifunction. F is said to be φ - *integrable* (*over* S) if there is a sequence $(F_n)_{n\in\mathbb{N}}$ of simple multifunctions $F_n : S \to \widetilde{X}$ satisfying the following conditions for every $p \in Q$:

(i) $h_p(F_n, F) \xrightarrow{\nu_p} 0$,

(ii) $\lim\limits_{n,m\to\infty} \int_S h_p(F_n, F_m) d\nu_p = 0$.

The sequence $(F_n)_n$ is said to be *a defining sequence* for F. For every $E \in \mathcal{A}$, we define *the integral of* F *over* E *with respect to* φ by:

$$\int_E F d\varphi = \lim_{n\to\infty} \int_E F_n d\varphi \in \widetilde{X}.$$

2.5. Remarks. (a) Every simple multifunction is φ-integrable.

(b) If $X = \mathbb{R}, \widetilde{X} = \{\{x\}|x \in \mathbb{R}\}, F = \{f\}(f$ is a function), $\varphi = \{\mu\}(\mu$ is finitely additive) and F is φ - integrable, then $\int_E F d\varphi = \{\int_E f d\mu\}, E \in \mathcal{A}$, where $\int_E f d\mu$ is the Dunford integral [11].

(c) If $X = \mathbb{R}, \widetilde{X} = \{\{x\}|x \in \mathbb{R}\}, F = \{f\}$ (f is a function) and F is φ - integrable, then f is Brooks - integrable with respect to φ and $[B]\int_E f d\varphi = \int_E F d\varphi, E \in \mathcal{A}$, where $[B]\int_E f d\varphi$ is the Brooks integral [5].

(d) If $X = \mathbb{R}$ and $\varphi = \{\mu\}$ (μ is finitely additive), then we get the integral defined by Martellotti - Sambucini [14] for F with respect to μ.

(e) If X is a real Banach algebra, then we obtain the integral defined in [8].

2.6. Definition. A multifunction $F : S \to \widetilde{X}$ is said to be *strong* φ *-integrable* if there is a sequence $(F_n)_n$ of simple multifunctions such that:

$$\left.\begin{array}{l}(i)\ h_p(F_n, F) \xrightarrow{\nu_p} 0,\\ (ii)\ \lim\limits_{n,m\to\infty} \displaystyle\int_S h_p(F_n, F_m) d\nu_p = 0,\end{array}\right\} \text{uniformly in } p \in Q.$$

The sequence $(F_n)_n$ is said to be *a strong defining sequence* for F.

2.7. Remarks. (i) Every simple multifunction is strong φ-integrable.

(ii) If $F : S \to \widetilde{X}$ is strong φ-integrable, then F is φ-integrable.

3. **Main results**

3.1. Theorem. *Let $F, G : S \to \widetilde{X}$ be φ-totally measurable multifunctions and $\alpha \in \mathbb{R}$. Then:*

(i) $h_p(F, G)$ is ν_p-measurable for every $p \in Q$,

(ii) $F + G$ and αF are φ-totally measurable.

Proof. Since F, G are φ-totally measurable, there exist $(F_n)_n, (G_n)_n$ sequences of simple multifunctions such that for every $p \in Q$:

$$h_p(F_n, F) \xrightarrow{\nu_p} 0, \ \ h_p(G_n, G) \xrightarrow{\nu_p} 0. \tag{1}$$

(i) Since $|h_p(F_n, G_n) - h_p(F, G)| \le h_p(F_n, F) + h_p(G_n, G), \ \ \forall n \in \mathbb{N}$, from (1) it follows:

$$h_p(F_n, G_n) \xrightarrow{\nu_p} h_p(F, G).$$

But $h_p(F_n, G_n)$ are simple functions, so $h_p(F, G)$ is ν_p-measurable.

(ii) Since the relations:

$$h_p(F_n + G_n, F + G) \le h_p(F_n, F) + h_p(G_n, G) \ \text{ and}$$

$$h_p(\alpha F_n, \alpha F) = |\alpha| h_p(F_n, F),$$

from (1) it follows:

$$h_p(F_n + G_n, F + G) \xrightarrow{\nu_p} 0 \text{ and } h_p(\alpha F_n, \alpha F) \xrightarrow{\nu_p} 0$$

which show that $F + G$ and αF are φ-totally measurable. $\square$

3.2. Theorem. *Let $F, G : S \to \widetilde{X}$ be φ-integrable (strong φ-integrable respectively) and let $\alpha \in \mathbb{R}$. Then:*

(i) $h_p(F,G)$ *is* ν_p*-integrable for every* $p \in Q$.

(ii) $F+G, \alpha F$ *are* φ*-integrable (strong* φ*-integrable respectively) and we have for every* $E \in \mathcal{A}$:

$$\int_E (F+G)d\varphi = \int_E Fd\varphi + \int_E Gd\varphi,$$

$$\int_E (\alpha F)d\varphi = \alpha \int_E Fd\varphi.$$

Proof. Suppose F, G are φ-integrable. Then there exist $(F_n)_n, (G_n)_n$ defining sequences for F, G respectively. From definition 2.4, it follows that $(h_p(F_n, G_n))_n, (F_n + G_n)_n$ and $(\alpha F_n)_n$ are defining sequences for $h_p(F,G)$, $F+G$ and αF respectively. Thus the function $h_p(F,G)$ is ν_p-integrable for every $p \in Q$ and the multifunctions $F+G$ and αF are φ-integrable.

Moreover, since

$$h_p\left(\int_E (F_n+G_n)d\varphi, \int_E Fd\varphi + \int_E Gd\varphi\right) \leq$$

$$\leq h_p\left(\int_E F_n d\varphi, \int_E Fd\varphi\right) + h_p\left(\int_E G_n d\varphi, \int_E Gd\varphi\right) \text{ and}$$

$$h_p\left(\int_E (\alpha F_n)d\varphi, \alpha\int_E Fd\varphi\right) = |\alpha| h_p\left(\int_E F_n d\varphi, \int_E Fd\varphi\right),$$

it results:

$$\int_E (F+G)d\varphi = \int_E Fd\varphi + \int_E Gd\varphi \text{ and } \int_E (\alpha F)d\varphi = \alpha\int_E Fd\varphi, \ \forall E \in \mathcal{A}.$$

The case of strong φ-integrability of F and G is proving analogously. $\square$

3.3. Theorem. *Let* $F : S \to \widetilde{X}$ *be a strong* φ*-integrable multifunction and let* $(F_n)_n$ *be a strong defining sequence for* F*. Then we have the following limits uniformly in* $p \in Q$ *and* $E \in \mathcal{A}$:

(i) $\lim\limits_{n\to\infty} \int_E h_p(F_n, F)d\nu_p = 0$,

(ii) $\lim\limits_{n\to\infty} \int_E \|F_n\|_p d\nu_p = \int_E \|F\|_p d\nu_p$.

Proof. (i) First, from definition of Dunford-Schwartz, p.112-[11], it follows that $h_p(F_n, F)$ is ν_p-integrable, $\forall n \in \mathbb{N}, p \in \mathbb{Q}$ and:

$$\lim_{n\to\infty} \int_S h_p(F_m, F_n)d\nu_p = \int_S h_p(F_m, F)d\nu_p, \ \forall m \in \mathbb{N}. \tag{2}$$

From definition 2.6-(ii) we obtain:

$$\forall\varepsilon > 0, \exists n_0 \in \mathbb{N} \text{ such that } \int_S h_p(F_m, F_n)d\nu_p < \frac{\varepsilon}{2},\ \forall m, n \geq n_0,\ \text{uniformly in } p \in Q. \tag{3}$$

Since (2), we have:

$$\exists n_1 \in \mathbb{N} \text{ such that } \left|\int_S h_p(F_m, F_n)d\nu_p - \int_S h_p(F, F_n)d\nu_p\right| < \frac{\varepsilon}{2},\ \ \forall m \geq n_1, \tag{4}$$

uniformly in $p \in Q$. Finally, from (3) and (4), it results:

$$\lim_{n\to\infty}\int_E h_p(F_n, F)d\nu_p = 0,\ \text{uniformly in } p \in Q \text{ and } E \in \mathcal{A}.$$

(ii) Since $\big|\|F_n\|_p - \|F\|_p\big| \leq h_p(F_n, F)$ and $\big|\|F_n\|_p - \|F_m\|_p\big| \leq h_p(F_n, F_m)$, from definition 2.6 it results the following conditions uniformly in $p \in Q$:

$$\|F_n\|_p \xrightarrow{\nu_p} \|F\|_p \ \text{ and } \lim_{n,m\to\infty}\int_S \big|\|F_n\|_p - \|F_m\|_p\big|d\nu_p = 0,$$

which show that $\|F\|_p$ is ν_p-integrable and $\lim\limits_{n\to\infty}\int_E \|F_n\|_p d\nu_p = \int_E \|F\|_p d\nu_p$, uniformly in $p \in Q$ and $E \in \mathcal{A}$. $\square$

3.4. Theorem (Vitali). *Suppose there exists $\alpha > 0$ such that $\nu_p(S) < \alpha$ for every $p \in Q$. Let $F : S \to \widetilde{X}$ be a multifunction and let $F_n : S \to \widetilde{X}$ be a sequence of strong φ-integrable multifunctions satisfying the following conditions uniformly in $p \in Q$:*

(i) $h_p(F_n, F) \xrightarrow{\nu_p} 0$,

(ii) $\forall\varepsilon > 0, \exists\delta(\varepsilon) = \delta > 0$ s.t. $\int_E \|F_n\|_p d\nu_p < \varepsilon$ for every $E \in \mathcal{A}$ with $\nu_p(E) < \delta, \forall n \in \mathbb{N}$. Then F is strong φ-integrable and $\int_E F d\varphi = \lim\limits_{n\to\infty}\int_E F_n d\varphi, \forall E \in \mathcal{A}$.

Proof. Since (i),

$$\lim_{n,m\to\infty}\nu_p(\{s \in S | h_p(F_n(s), F_m(s)) > \varepsilon\}) = 0,\ \ \forall\varepsilon > 0.$$

If we denote $A_{nm}(\varepsilon) = A_{nm} = \{s \in S | h_p(F_n(s), F_m(s)) > \varepsilon\}, \forall n, m \in \mathbb{N}$, then from theorem 3.10 - [9], it follows:

$$\int_E h_p(F_n, F_m)d\nu_p \leq \int_{A_{nm}} \|F_n\|_p d\nu_p + \int_{A_{nm}} \|F_m\| d\nu_p + \varepsilon\nu_p(S). \tag{5}$$

Since (i) and (ii), there exists $n_0 \in \mathbb{N}$ such that $\nu_p(A_{nm}) < \delta, \int_{A_{nm}} \|F_n\|_p d\nu_p < \varepsilon$ and $\int_{A_{nm}} \|F_m\|_p d\nu_p < \varepsilon$ for every $n, m \geq n_0$. Thus from (5) and theorem 3.9 - [9], it results:

$$h_p\left(\int_E F_n d\varphi, \int_E F_m d\varphi\right) < (2+\nu_p(S))\varepsilon, \quad \forall n, m \geq n_0, E \in \mathcal{A}, p \in Q,$$

which shows that the sequence $(\int_E F_n d\varphi)_n$ is Cauchy in $\widetilde{X}$ and consequently, converges in $\widetilde{X}$. Now, we follow the proof of Vitali theorem 3.17 - [9]. So, let $(G_k^n)_{k\in\mathbb{N}}$ be a strong defining sequence for F_n and let $H_{n,k}^p = \{s \in S | h_p(G_k^n(s), F_n(s)) > \frac{1}{2^n}\}$. Then,for every $n \in \mathbb{N}$, there is $\overline{k}(n) \in \mathbb{N}$ such that $\nu_p(H_{n,k}^p) < \frac{1}{2^n}$ and

$$\int_S h_p(G_k^n, F_n) d\nu_p < \frac{1}{2^n}, \quad \forall k \geq \overline{k}(n), \text{ uniformly in } p \in Q. \tag{6}$$

If we denote $G_n = G_{\overline{k}(n)}^n$ for every $n \in \mathbb{N}$, then we obtain:

$$h_p(G_n, F_n) \xrightarrow{\nu_p} 0. \tag{7}$$

From (7) and (i), it results that $h_p(G_n, F)$ is ν_p-measurable and $h_p(G_n, F) \xrightarrow{\nu_p} 0$, uniformly in $p \in Q$. Like in the beginning of the proof, it follows:

$$\int_E h_p(F_n, F_m) d\nu_p < (2+\alpha)\varepsilon, \quad \forall n, m \geq n_0, E \in \mathcal{A}, p \in Q. \tag{8}$$

Since (6), it results:

$$\int_S h_p(G_n, G_m) d\nu_p < \frac{1}{2^n} + (2+\alpha)\varepsilon + \frac{1}{2^m}, \tag{9}$$

for all sufficiently large n, m and for every $p \in Q$. Consequently, we have $\lim\limits_{n,m\to\infty} \int_S h_p(G_n, G_m) d\nu_p = 0$. So, F is φ-integrable and

$$\int_E F d\varphi = \lim_{n\to\infty} \int_E G_n d\varphi, \quad \forall E \in \mathcal{A}. \tag{10}$$

According to theorem 3.9 - [9] and (6), we obtain:

$$\lim_{n\to\infty} h_p\left(\int_E G_n d\varphi, \int_E F_n d\varphi\right) = 0, \quad \forall p \in Q, E \in \mathcal{A}. \tag{11}$$

From (10) and (11) it results now that $\int_E F d\varphi = \lim\limits_{n\to\infty} \int_E F_n d\varphi, \forall E \in \mathcal{A}$.

Since (i) and (9), it follows that $(G_n)_n$ is a strong defining sequence for F and from theorem 3.3-(i), it results that the limit of (10) is uniform. □

3.5. Theorem (Lebesgue). *Let $F : S \to \widetilde{X}$ be a multifunction such that $\|F\|_p$ is ν_p-integrable for every $p \in Q$. Suppose there exists a sequence $(F_n)_{n\in\mathbb{N}}$ of φ-integrable multifunctions having the following properties:*

$$h_p(F_n, F) \xrightarrow{\nu_p} 0, \quad \forall p \in Q, \tag{i}$$

(ii) *there exists $\alpha > 0$, such that $\|F_n(s)\|_p \leq \alpha$, $\forall s \in S, p \in Q, n \in \mathbb{N}$.*
Then F is φ-integrable and $\displaystyle\int_E F d\varphi = \lim_{n\to\infty} \int_E F_n d\varphi$, $\forall E \in \mathcal{A}$.

Proof. Since F_n is φ-integrable for every $n \in \mathbb{N}$, there exists $(G_k^n)_{k\in\mathbb{N}}$ a defining sequence for F_n. Let $H_k^n(p) = H_k^n = \{s \in S \mid h_p(G_k^n(s), F_n(s)) > \frac{1}{2^n}\}$. Then for every $n \in \mathbb{N}$, since proposition 3.7-b) of [9], there exists $\overline{k}(n) \in \mathbb{N}$ such that:

$$\nu_p(H_k^n) < \frac{1}{2^n} \text{ and } \int_S h_p(G_k^n, F_n) d\nu_p \leq \frac{1}{2^n}, \quad \forall k \geq \overline{k}(n). \tag{12}$$

Let $G_n = G_{\overline{k}(n)}^n$, $\forall n \in \mathbb{N}$. Then we have

$$h_p(G_n, F_n) \xrightarrow{\nu_p} 0. \tag{13}$$

Since $h_p(G_n, F) \leq h_p(G_n, F_n) + h_p(F_n, F)$, $\forall n \in \mathbb{N}$, from (13) and (i) we obtain:

$$h_p(G_n, F) \xrightarrow{\nu_p} 0. \tag{14}$$

By the definition 2.2, F is φ-totally measurable. Since theorem 3.1,it results $h_p(G_n, F)$ is ν_p-measurable, $\forall n \in \mathbb{N}$. Since F_n is φ-integrable and F is φ-totally measurable, from theorem 3.1-(i) it follows that $h_p(F_n, F)$ is ν_p-measurable. Now we have:

$$h_p(F_n, F) \leq \|F_n\|_p + \|F\|_p, \quad \forall n \in \mathbb{N}. \tag{15}$$

From proposition 3.7-a) of [9], it results $\|F_n\|_p$ is ν_p-integrable and from hypothesis we have $\|F\|_p$ is ν_p-integrable. But $h_p(F_n, F)$ is ν_p-measurable so, from (15) it follows that $h_p(F_n, F)$ is ν_p-integrable, $\forall n \in \mathbb{N}$. For every $k \in \mathbb{N}^*$, let

$$A_k(p, n) = \left\{ s \in S \mid h_p(F_n(s), F(s)) > \frac{1}{k \cdot \nu_p(S)} \right\}.$$

Since (i), for $\varepsilon = \frac{1}{k}$, there exists $n_k > k$ such that $\nu_p(A_k(p, n)) > \frac{1}{k}$, $\forall n \geq n_k$. Particularly,

$$\nu_p(A_k(p, n_k)) < \frac{1}{k}, \quad \forall k \in \mathbb{N}^*. \tag{16}$$

Let us denote $A_k(p, n_k) = B_k$ and let $\varepsilon > 0$. Denoting $\Gamma(E) = \int_E \|F\|_p d\nu_p$, $\forall E \in \mathcal{A}$, since $\Gamma \ll \nu_p$, there is $\delta(p, \varepsilon) = \delta > 0$, such that

$$\int_E \|F\|_p d\nu_p < \varepsilon, \ \ \forall E \in \mathcal{A} \text{ with } \nu_p(E) < \delta. \tag{17}$$

Then, for every $k \in \mathbb{N}^*$ with $\frac{1}{k} < \min\{\delta, \varepsilon\}$, from (ii), (16) and (17) we have:

$$\int_S h_p(F_k, F) d\nu_p = \int_{B_k} h_p(F_k, F) d\nu_p + \int_{cB_k} h_p(F_k, F) d\nu_p <$$

$$< \int_{B_k} \|F_k\|_p d\nu_p + \int_{B_k} \|F\|_p d\nu_p + \frac{1}{k} <$$

$$< \alpha \nu_p(B_k) + 2\varepsilon < \frac{\alpha}{k} + 2\varepsilon < (\alpha + 2)\varepsilon,$$

that is

$$\lim_{k \to \infty} \int_S h_p(F_k, F) d\nu_p = 0. \tag{18}$$

Since the inequality:

$$\int_S h_p(F_n, F_m) d\nu_p \leq \int_S h_p(F_n, F) d\nu_p + \int_S h_p(F, F_m) d\nu_p$$

and from (18) it follows

$$\lim_{n,m \to \infty} \int_S h_p(F_n, F_m) d\nu_p = 0. \tag{19}$$

For all sufficiently large n and m, since (12) and (19) we have:

$$\int_S h_p(G_n, G_m) d\nu_p \leq \int_S h_p(G_n, F_n) d\nu_p + \int_S h_p(F_n, F_m) d\nu_p +$$

$$+ \int_S h_p(F_m, G_m) d\nu_p < \frac{1}{2^n} + \varepsilon + \frac{1}{2^m}, \ \text{ that is}$$

$$\lim_{n,m \to \infty} \int_S h_p(G_n, G_m) d\nu_p = 0. \tag{20}$$

Finally, from (14) and (20), it results that F is φ-integrable.

Now, since theorem 3.9 - [9], we have:

$$h_p\left(\int_E F_n d\varphi, \int_E F d\varphi\right) \leq \int_E h_p(F_n, F) d\nu_p \leq$$

$$\leq \int_S h_p(F_n, F) d\nu_p, \ \forall E \in \mathcal{A}, n \in \mathbb{N}$$

and from (18) it follows that $\int_E F d\varphi = \lim\limits_{n \to \infty} \int_E F_n d\varphi, \ \ \forall E \in \mathcal{A}$.$\square$

3.6. Theorem (Lebesgue). *Let $F : S \to \widetilde{X}$ be a φ-totally measurable multifunction such that $\|F\|_p$ is ν_p-integrable for every $p \in Q$. Suppose there exists a sequence $(G_n)_n$ of simple multifunctions satisfying the conditions:*

(i) $h_p(G_n, F) \xrightarrow{\nu_p} 0, \forall p \in Q$,

(ii) there is $\alpha > 0$ such that $\|G_n(s)\|_p \leq \alpha, \forall s \in S, p \in Q, n \in \mathbb{N}$.

Then F is φ-integrable.

Proof. Since $\|F\|_p$ is ν_p-integrable, from the inequality $h_p(G_n, F) \leq \|G_n\|_p + \|F\|_p$, it follows that $h_p(G_n, F)$ is ν_p-integrable for every $p \in Q, n \in \mathbb{N}$. For every $k \in \mathbb{N}^*$, let

$$A_k(p,n) = \left\{ s \in S | h_p(G_n(s), F(s)) > \frac{1}{k\nu_p(S)} \right\}.$$

For now on, acting like in the proof of the previous theorem 3.5, we obtain:

$$\lim_{k\to\infty} \int_S h_p(G_k, F) d\nu_p = 0. \tag{21}$$

Since the inequality:

$$\int_S h_p(G_n, G_m) d\nu_p \leq \int_S h_p(G_n, F) d\nu_p + \int_S h_p(G_m, F) d\nu_p$$

and from (21), it follows:

$$\lim_{n,m\to\infty} \int_S h_p(G_n, G_m) d\nu_p = 0. \tag{22}$$

Finally, from (i) and (22), it results that F is φ-integrable.

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RESTRICTION TO SUBGROUPS AND SEPARABILITY

ANDREI MARCUS

Dedicated to Professor Grigore Călugăreanu on his 60th birthday

Abstract. Let G be a group, $R = \bigoplus_{g\in G} R_g$ a strongly G-graded ring, and let H be a subgroup of G. In this note we prove that the functors $(\mathrm{Ind}_H^G, \mathrm{Res}_G^H)$ form a Frobenius pair if and only if $[G;H] \leq \infty$, and that $\mathrm{Res}_G^H \colon R_H\text{-Mod} \to R\text{-Mod}$ is a separable functor if and only if $[G;H] \leq \infty$ and the trace map $\mathrm{Tr}_H^G \colon Z(R_1)^H \to Z(R_1)^G$ is surjective.

1. **Introduction and preliminaries**

1.1. Let $\mathcal{F}\colon \mathcal{A} \to \mathcal{B}$ be a covariant functor, and $\mathcal{G}\colon \mathcal{B} \to \mathcal{A}$ a right adjoint of F. Recall that $\mathcal{F}$ is *separable* if the unit $\eta\colon \mathbf{1}_{\mathcal{C}} \to \mathcal{G} \circ \mathcal{F}$ of the adjunction splits. Similarly, G is separable if the counit $\epsilon\colon \mathbf{1}_{\mathcal{D}} \to \mathcal{F} \circ \mathcal{G}$ of the adjunction cosplits.

The functors $\mathcal{F}$ and $\mathcal{G}$ are said to form a *Frobenius pair* if G is also a left adjoint of $\mathcal{F}$. Frobenius and separability properties have attracted the interest of many authors, and an excellent account of the subject is given in [1].

1.2. In this note we consider a group G, a strongly G-graded ring $R = \bigoplus_{g\in G} R_g$, and a subgroup H of G. Denote by R-Mod the category of left R-modules. We are concerned with the ring extension $R_H \to R$. It is easy to see that the induction functor

$$\mathrm{Ind}_H^G = R \otimes_{R_H} - \colon R\text{-Mod} \to R_H\text{-Mod}$$

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is separable, therefore we investigate the separability of its right adjoint

$$\mathrm{Res}_H^G\colon R_H\text{-Mod} \to R\text{-Mod}.$$

This is related to [1, Section 3.2] and [1, Section 3.2], but we give here a direct proof of a separability criterion for Res_H^G in terms of the action of G on the center $Z(R_1)$ of R_1, which generalize [4, Proposition 2.1] and [2, Proposition 1.5].

More precisely, we prove in Section 2 that $(\mathrm{Ind}_H^G, \mathrm{Res}_H^G)$ is a Frobenius pair if and only if the index of H in G is finite, and the functor Res_H^G is separable if and only if $[G:H]<\infty$ and the trace map

$$\mathrm{Tr}_H^G\colon Z(R_1)^H \to Z(R_1)^G$$

is surjective.

We recall some well-known facts about the ungraded case.

1.3. Let $\iota\colon S\to R$ be a ring homomorphism, and let $\mu\colon R\otimes_S R\to R$ be the multiplication. Then the restriction of scalars induced by ι is a separable functor if and only if the ring extension R/S is separable, that is, there is an (R,R)-bimodule map

$$\zeta\colon R\to R\otimes_S R$$

such that $\zeta\circ\mu = \mathrm{id}_R$. Clearly, the existence of an (R,R)-bimodule map $\zeta\colon R\to R\otimes_S R$ is equivalent to the existence of an element

$$x=\sum x^{(1)}\otimes x^{(2)}\in R\otimes_S R$$

such that $rx = xr$ for all $r\in R$, and then ζ splits μ if and only if $\sum x^{(1)}x^{(2)}=1$.

1.4. The restriction of scalars induced by ι and the extension of scalars

$$R\otimes_S -\colon S\text{-Mod}\to R\text{-Mod}$$

form a Frobenius pair if and only if there is an (S,S)-bimodule map

$$\nu\colon R\to S$$

and an (R,R)-bimodule map

$$\zeta\colon R\to R\otimes_S R$$

such that the diagram

$$\begin{tikzcd} & R \otimes_S R \arrow[dl, "\nu\otimes\mathrm{id}"'] \arrow[dr, "\mathrm{id}\otimes\nu"] & \\ S \otimes_S R \arrow[dr, "\simeq"'] & & R \otimes_S S \arrow[dl, "\simeq"] \\ & R & \end{tikzcd}$$

is commutative, that is,

$$\sum \nu(x^{(1)})x^{(2)} = \sum x^{(1)}\nu(x^{(2)}) = 1.$$

2. **Strongly graded rings and restriction to subgroups**

Let G be a group and let $R = \bigoplus_{g\in G} R_g$ be a strongly G-graded ring.

2.1. There is an action of G on $Z(R_1)$ defined as follows.

For each $g \in G$, we have that $R_g R_{g^{-1}} = R_1$, so there are elements $r_{g,i} \in R_g$ and $r'_{g,i} \in R_{g^{-1}}$ such that $\sum_i r_{g,i} r'_{g,i} = 1$. By definition, for each $c \in Z(R_1)$, we have

$$^{g}c = \sum_i r_{g,i} c r'_{g,i} = 1.$$

This definition does not depend on the choices we have made, and $(g, c) \mapsto {}^{g}c$ is indeed a (left) action of G on $Z(R_1)$. Note also that the element ${}^{g}c \in Z(R_1)$ is uniquely defined by the property

$$r_g c = {}^{g}c r_g$$

for each $r_g \in R_g$.

If $[G : H]$ is finite, we may define the trace map

$$\mathrm{Tr}_H^G \colon Z(R_1)^H \to Z(R_1)^G, \qquad a \mapsto \sum_{g\in[G/H]} {}^{g}a,$$

where $[G/H]$ denotes a full set of representatives for the left cosets of H in G. The image of Tr_H^G is an ideal of $Z(R_1)^G$, so Tr_H^G is surjective if and only if there is $a \in Z(R_1)^H$ such that $\mathrm{Tr}_H^G(a) = 1$.

We may state the main result of this note. We use the notation introduced in 1.2.

Theorem 2.2. 1) *The functors* Ind_H^G *and* Res_H^G *form a Frobenius pair if and only if* $[G:H]<\infty$.

2) *The functor* Res_H^G *is separable if and only if* $[G:H]<\infty$ *and trace map* $\mathrm{Tr}_H^G\colon Z(R_1)^H\to Z(R_1)^G$ *is surjective.*

Proof. Assume that $[G:H]$ is finite. Then it is well-known (see for instance [3, 3.1] for details) that Ind_H^G is both a left and a right adjoint of Res_H^G. In fact, let us define an (R_H,R_H)-bimodule map $\nu\colon R\to R_H$ and an (R,R)-bimodule map $\zeta\colon R\to R\otimes_{R_H}R$ such that 1.4 holds. First, let

$$\nu\colon R\to R_H,\quad \sum_{g\in G}r_g\mapsto\sum_{g\in H}r_g$$

be the projection onto R_H, which is obviously (R_H,R_H)-linear. Next, let $c\in Z(R_1)^H$, and let

$$\zeta_c\colon R\to R\otimes_{R_H}R,\quad r\mapsto r\sum_{\sigma\in G/H}\sum_i r_{\sigma,i}c\otimes r'_{\sigma,i}=r\zeta_c(1),$$

where we choose $g\in\sigma$ and then $r_{\sigma,i}:=r_{g,i}$ and $r'_{\sigma,i}:=r'_{g,i}$. Observe that $\zeta_c(1)$ does not depend on the choices made in 2.1. Indeed, let $s_{g,j}\in R_g$ and $s'_{g,j}\in R_{g^{-1}}$ such that

$$\sum_j s_{g,j}s'_{g,j}=1.$$

Choose another full set of representatives for the left cosets of H in G, and define $s_{\sigma,j}$ and $s'_{\sigma,j}$ by the above convention, so in particular, $r'_{\sigma,i}s_{\sigma,j}\in R_H$. Then, since $c\in Z(R_H)$, we have

$$\begin{aligned}\sum_{\sigma\in G/H}\sum_j s_{\sigma,j}c\otimes s'_{\sigma,j}&=\sum_{\sigma\in G/H}\sum_j\sum_i r_{\sigma,i}r'_{\sigma,i}s_{\sigma,j}c\otimes s'_{\sigma,j}\\&=\sum_{\sigma\in G/H}\sum_j\sum_i r_{\sigma,i}c\otimes r'_{\sigma,i}s_{\sigma,j}s'_{\sigma,j}\\&=\sum_{\sigma\in G/H}\sum_i r_{\sigma,i}c\otimes r'_{\sigma,i}.\end{aligned}$$

Now let $h \in G$ and $r_h \in R_h$. Then

$$\begin{aligned} r_h\zeta_c(1) &= \sum_{g\in[G/H]}\sum_i r_h r_{g,i} c \otimes r'_{g,i} \\ &= \sum_{g\in[G/H]}\sum_i\sum_j s_{hg,j}s'_{hg,j} r_h r_{g,i} c \otimes r'_{g,i} \\ &= \sum_{g\in[G/H]}\sum_i\sum_j s_{hg,j} \otimes s'_{hg,j} r_h r_{g,i} c r'_{g,i} \\ &= \sum_{g\in[G/H]}\sum_j s_{hg,j} c \otimes s'_{hg,j} r_h = \zeta_c(1) r_h. \end{aligned}$$

It follows that ζ_c is an (R,R)-bimodule map. It is easy to see that

$$((\nu\otimes \mathrm{id})\circ\zeta_c)(1) = ((\mathrm{id}\otimes\nu)\circ\zeta_c)(1) = c,$$

hence 1.4 holdy by taking $c = 1$. Moreover, $(\mu\circ\zeta_c)(r) = r\,\mathrm{Tr}_H^G(c)$, hence by 1.3, Res_H^G is a separable functor if there is $c \in Z(R_H)^H$ such that $\mathrm{Tr}_H^G(c) = 1$.

To prove the converses, assume that $\zeta\colon R \to R\otimes_{R_H} R$ is an (R,R)-bimodule map, and let $\zeta(1) = \sum x^{(1)}\otimes x^{(2)} \in R\otimes_{R_H} R$. If $(\mathrm{Ind}_H^G, \mathrm{Res}_H^G)$ is a Frobenius pair, there is an (R_H,R_H)-bimodule map $\nu\colon R\to R_H$ such that for all $r\in R_H$

$$r = \nu(rx^{(1)})x^{(2)} = x^{(1)}\nu(x^{(2)}r).$$

It follows that the family $\{x^{(2)}, \nu((-)x^{(1)})\}$ is a dual basis for R as a left R_H-module, hence in particular, R is a finitely generated left R_H-module. Since R is strongly graded, it follows that $[G:H]$ is finite.

We have that $R = \bigoplus_{g\in[G/H]} R_{gH}$ as (R_1,R_H)-bimodules. We denote by α the isomorphism

$$\begin{aligned} R \to R\otimes_{R_H} R &\simeq \bigoplus_{g\in[G/H]} R_{gH}\otimes_{R_H} R \\ &\simeq \bigoplus_{g\in[G/H]} R_g\otimes_{R_1} R \\ &\simeq \bigoplus_{g\in[G/H]}\bigoplus_{h\in G} R_g\otimes R_h \end{aligned}$$

of (R_1, R_1)-bimodules. Denote also $\zeta' := \alpha \circ \zeta$ and $\mu' := \mu \circ \alpha^{-1}$. Then $\zeta'(1)$ is a finite sum of monomials of the form $a_g \otimes b_h$, where $g \in [G/H]$, $h \in G$, $a_g \in R_g$ and $b_h \in R_h$.

Assume that Res^G_H is separable, so $\mu'(\zeta'(1) = 1 \in R_1$. By looking at the homogeneous components, we deduce that $\zeta'(1)$ is of the form

$$\zeta'(1) = \sum_{g \in [G/H]} \sum{}' a_g \otimes b_g^{-1}.$$

Denote $c_{g,g^{-1}} = \mu'(\sum' a_g \otimes b_g^{-1}) \in R_1$, so we have that $\sum_{g\in[G/H]} c_{g,g^{-1}} = 1$. We claim that $c_{g,g^{-1}} = {}^g c_{1,1}$ for all $g \in [G/H]$. Indeed, we know that $r_g\zeta'(1) = \zeta'(1)r_g$ for all $g \in G$ and $r_g \in R_g$. By taking homogeneous components, we deduce

$$\sum{}' r_g a_1 \otimes b_1 = \sum{}' a_g \otimes b_{g^{-1}} r_g.$$

By applying μ, we get $r_g c_{1,1} = c_{g,g^{-1}} r_g$, so by 2.1, the claim is proved. It follows that

$$\sum_{g\in[G/H]} {}^g c_{1,1} = \sum_{g\in[G/H]} c_{g,g^{-1}} = 1.$$

This also implies that $[G : H]$ is finite, since otherwise there would exist $g \in [G/H]$ such that $c_{g,g^{-1}} = 0$; then $c_{1,1} = {}^{g^{-1}} c_{g,g^{-1}} = 0$, which is a contradiction.

Finally, it remains to show that we may take $c_{1,1} \in Z(R_1)^H$, that is $r_h c_{1,1} = c_{1,1} r_h$ for all $h \in H$ and $r_h \in R_h$. Indeed, the representation of $\zeta'(1)$ can be chosen such that $1 \in [G/H]$ and $\sum' a_1 \otimes b_1 = 1 \otimes b_1$, and then $c_{1,1} = b_1$. The claim follows from the fact that $r_h\zeta'(1) = \zeta'(1)r_h$ for all $h \in H$ and $r_h \in R_h$. $\square$

Remark 2.3. Assume that $[G : H]$ is finite and let $M, M' \in R$-Mod. Then we have a trace map

$$\mathrm{Tr}^G_H\colon \mathrm{Hom}_{R_H}(M, M') \to \mathrm{Hom}_R(M, M'), \quad \mathrm{Tr}^G_H(f)(m) = \sum_{g\in[G/H]} \sum_i r_{g,i} f(r'_{g,i} m).$$

If moreover $c \in Z(R_1)^H$ satisfies $\mathrm{Tr}^G_H(c) = 1$, then for any $f \in \mathrm{Hom}_R(M, M')$, we have that $\mathrm{Tr}^G_H(cf) = f$.

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ON THE COMPLETENESS OF THE SEMIHYPERGROUPS ASSOCIATED TO BINARY RELATIONS

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Dedicated to Professor Grigore Călugăreanu on his 60th birthday

Abstract. We will determine the complete hypergroupoids, semihypergroups or hypergroups determined by binary relations (more exacly, by monounary multialgebras) an we will study some closure properties of their classes with respect to the products of some categories where they are contained.

1. **Introduction**

On the basis of [1, 7], in [4, 5] C. Pelea and I. Purdea started an investigation on some constructions of hypergroupoids associated to binary relations. This paper continues the investigation of C. Pelea and I. Purdea from the point of view of the completeness of the multialgebras involved in this discussion, which is another problem studied by Pelea and Purdea (for general multialgebras) in [3]. So, we will give a characterization for the complete hypergroupoids, complete semihypergroups and complete hypergroups associated to monounary multialgebras (hence with binary relations). Even if the subcategory of the hypergroupoids determined by monounary multialgebras is not closed under direct product (i.e. under the product from the category of hypergroupoids) the completeness condition seems to fix this problem. This is not surprising since we will see that the completeness condition on a hypergroupoid determined by a monounary multialgebra is very restrictive. We will see that the

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complete hypergroupoids determined by monounary multialgebras coincide with the complete semihypergroups determined by monounary multialgebras, so, we deal most of the time with semihypergroups determined by monounary multialgebras. We will also be able to adapt the results we obtain to hypergroups determined by monounary multialgebras. We mention that the categorical notions are not complicated and they can be found in [6].

2. **Preliminaries**

Let H be a set and let R be a binary relation on H. Denote the inverse of the relation R by $\overset{-1}{R}$. For $x_1, \dots, x_n \in H$, $X \subseteq H$ we denote

$$R(X) = \{y \in H \mid \exists x \in X : xRy\} \text{ and } R(x_1, \dots, x_n) = R(\{x_1, \dots, x_n\}).$$

As in [7], one can associate to R the partial hypergroupoid $H_R = (H, \circ)$ defined by

$$x \circ y = R(x, y).$$

It is obvious that $x^2 = x \circ x = R(x) = \{y \in H \mid xRy\}$ and

$$x \circ y = x^2 \cup y^2,\ \forall x, y \in H. \tag{1}$$

Lemma 1. [7, Lemma 1] *Let H be a set and let R be a binary relation on H. The partial hypergroupoid $H_R = (H, \circ)$ is a hypergroupoid if and only if $\overset{-1}{R}(H) = H$.*

An element $x \in H$ is an *outer element* of (the relation) R if there exists $h \in H$ such that $(h, x) \notin R^2$.

Proposition 1. [7, Proposition 2] *If H be a set and R is a binary relation on H with $\overset{-1}{R}(H) = H$ then H_R is a semihypergroup if and only if $R \subseteq R^2$ and*

$$(a, x) \in R^2 \Rightarrow (a, x) \in R$$

whenever x is an outer element of R.

Proposition 2. [7] *Let $H \neq \emptyset$ and let R be a binary relation on H. The hypergroupoid H_R is a hypergroup if and only if the following conditions hold:*

1) $\overset{-1}{R}(H) = H$;

2) *$R(H) = H$;*

3) *$R \subseteq R^2$;*

4) *whenever x is an outer element of R we have*

$$(a, x) \in R^2 \Rightarrow (a, x) \in R.$$

Let (H, R), (H', R') be relational systems with binary relations and $h : H \to H'$. One says that h is a *homomorphism* of relational systems if

$$xRy \Rightarrow h(x)R'h(y).$$

Let $(H, \circ)$, $(H', \circ')$ be hypergroupoids. A mapping $h : H \to H'$ is called *homomorphism* (of hypergroupoids) if

$$h(x \circ y) \subseteq h(x) \circ' h(y),\ \forall x, y \in H.$$

Remark 1. If R is a binary relation on H with $\overset{-1}{R}(H) = H$, we can see (H, R) as the multialgebra (H, f) with one unary multioperation $f : H \to P^*(H)$ defined by

$$xRy \Leftrightarrow y \in f(x). \tag{2}$$

An element $x \in H$ is *outer element* (of (H, f)) if there exists $h \in H$ such that $x \notin f(f(h))$. An element $x \in H$ is an *inner element* if it is not an outer element.

Remark 2. If (H', R') is also a relational system for which $\overset{-1}{R'}(H') = H'$ and (H', f') is the corresponding monounary multialgebra then h is a relational homomorphism between (H, R) and (H', R') if and only if h is a homomorphism between the multialgebras (H, f) and (H', f'). If $\mathcal{R}_2$ denotes the category of the relational systems with one binary relation (having as morphisms the homomorphisms of relational systems and as product the usual composition of homomorphisms) and $\mathcal{R}_2'$ its (full) subcategory consisting in the relational systems (H, R) for which $\overset{-1}{R}(H) = H$. The identification we made in the previous remark gives a categorical isomorphism between $\mathcal{R}_2'$ and the category **Malg**(1) of the monounary multialgebras (i.e. the multialgebras of type (1)), where the morphisms are the multialgebra homomorphisms and the product of two morphisms is the usual composition of homomorphisms.

The hypergroupoids (or semihypergroups, or hypergroups) associated to binary relations can be seen as hypergroupoids (or semihypergroups, or hypergroups) associated to monounary multialgebras (H, f) using the translation of (1) in the terms of the unary multioperation f. Thus we have

$$f(X) = \bigcup_{x \in X} f(x) \text{ and } x \circ y = f(\{x, y\}) = f(x) \cup f(y) (= x^2 \cup y^2)$$

for any $X \subseteq H,\ X \neq \emptyset$, $x, y \in H$ and Lema 1 can be rewritten as below:

Lemma 2. *For any multialgebra (H, f) with one unary multioperation, the equality*

$$x \circ y = f(\{x, y\})$$

defines a hypergroupoid $H_f = (H, \circ)$.

Propositions 1 and 2 can be restated as follows:

Proposition 3. *Let (H, f) be a multialgebra with one unary multioperation. The hypergroupoid H_f is a semihypergroup if and only if*

$$f(x) \subseteq f(f(x)),\ \forall x \in H$$

and for any outer element $x \in H$,

$$x \in f(f(a)) \Rightarrow x \in f(a).$$

Proposition 4. *Let $H \neq \emptyset$ and let (H, f) be a multialgebra with one unary multioperation. The hypergroupoid H_f is a hypergroup if and only if the following conditions hold:*

i) $f(H) = H$;

ii) $f(x) \subseteq f(f(x)),\ \forall x \in H$;

iii) *whenever x is an outer element we have*

$$x \in f(f(a)) \Rightarrow x \in f(a).$$

In [7, Proposition 3], Rosenberg determines the semihypergroups which can be obtained from a binary relations using (1). We restated the result of Rosenberg as follows:

Proposition 5. *Let $(H,*)$ be a hypergroupoid. There exists a binary relation R on H such that $(H,*) = H_R$ if and only if* (1) *holds. A hypergroupoid $(H,*)$ which satisfies the condition* (1) *is a semihypergroup if and only if it verifies the conditions:*

$$x^2 \subseteq (x^2)^2 \text{ and } (x^2)^2 \cap (H \setminus (y^2)^2) \subseteq x^2,\ \forall x, y \in H. \tag{3}$$

A hypergroupoid $(H,*)$ which satisfies the conditions (1) *and* (3) *is a hypergroup if and only if $\bigcup_{x\in H} x^2 = H$.*

The binary relation $R \subseteq H \times H$ from Rosenberg's proof is defined by

$$xRy \Leftrightarrow y \in x * x,$$

$\overset{-1}{R}(H) = H$ and the corresponding unary multioperation is

$$f_* : H \to P^*(H),\ f_*(x) = x * x.$$

Remark 3. The conditions (3) are the (equivalent) translation of the conditions 3) and 4) from Proposition 2 in the terms of the hyperoperation $*$. So, a hypergroupoid (or semihypergroup, or hypergroup) $(H,*)$ is determined by a unary multioperation f on H if and only if $(H,*)$ satisfies the condition (1).

In the category **Malg**(2) of hypergroupoids – the morphisms are the hypergroupoid homomorphisms and the product of two morphisms is the usual composition of homomorphisms – we consider the following subcatgories: the subcategory $\mathbf{Malg}'(2)$ of the hypergroupoids satisfying (1), the subcategory $\mathbf{SHG}'$ whose objects are the semihypergroups which satisfy (1) and the subcategory $\mathbf{HG}'$ whose objects are the hypergroups which satisfy (1). We also denote by $\mathbf{Malg}'(1)$ the full subcategory of **Malg**(1) whose objects are the monounary multialgebras (H,f) which satisfy the conditions ii),iii) from Proposition 4 and by $\mathbf{Malg}''(1)$ the full subcategory of **Malg**(1) whose objects are the monounary multialgebras (H,f), with $H \neq \emptyset$, which satisfy the conditions i), ii), iii) from Proposition 4.

Remark 4. [4, Corollaries 3, 4] The correspondences

$$(H,f) \mapsto H_f,\ h \mapsto h$$

define three covariant functors

$$F : \mathbf{Malg}(1) \to \mathbf{Malg}'(2),\ F' : \mathbf{Malg}'(1) \to \mathbf{SHG}',\ \text{and}\ F'' : \mathbf{Malg}''(1) \to \mathbf{HG}'.$$

These functors are isomorphisms of categories and their inverses are the functors

$$G : \mathbf{Malg}'(2) \to \mathbf{Malg}(1),\ G' : \mathbf{SHG}' \to \mathbf{Malg}'(1),\ \text{and}\ G'' : \mathbf{HG}' \to \mathbf{Malg}''(1),$$

respectively, given by

$$(H, *) \mapsto (H, f_*),\ h \mapsto h.$$

3. **Complete semihypergroups associated to monounary multialgebras**

In this section we will determine those monounary multialgebras which determine complete semihypergroups and complete hypergroups.

First, remember that a *multialgebra* $\mathfrak{A} = (A, (f_\gamma)_{\gamma < o(\tau)})$ of type τ is *complete* if for any $m, n \in \mathbb{N}$, any

$$\mathbf{q} \in P^{(m)}(\tau) \setminus \{\mathbf{x}_i \mid i \in \{0, \dots, m-1\}\},\ \mathbf{r} \in P^{(n)}(\tau) \setminus \{\mathbf{x}_i \mid i \in \{0, \dots, n-1\}\},$$

and any $a_0, \dots, a_{m-1}, b_0, \dots, b_{n-1} \in A$,

$$q(a_0, \dots, a_{m-1}) \cap r(b_0, \dots, b_{n-1}) \neq \emptyset\ \Rightarrow\ q(a_0, \dots, a_{m-1}) = r(b_0, \dots, b_{n-1}). \tag{4}$$

Remark 5. For a monounary multialgebra (H, f), the images of the term functions involved in (4) are the nonempty subsets

$$f^n(x) = \underbrace{f(f(\dots(f}_{n}(x))\dots)),$$

with $n \in \mathbb{N}^*$ and $x \in H$, hence the monounary multialgebra (H, f) is complete if and only if for any $m, n \in \mathbb{N}$, and any $x, y \in H$,

$$f^m(x) \cap f^n(y) \neq \emptyset \Rightarrow f^m(x) = f^n(y).$$

Lemma 3. *Let $(H, \circ)$ be a semihypergroup which satisfy* (1). *The semihypergroup $(H, \circ)$ is complete only if the multialgebra $(H, f_\circ)$ satisfies the identity*

$$\mathbf{f}(\mathbf{f}(\mathbf{x})) = \mathbf{f}(\mathbf{x}) \tag{5}$$

(**f** *denotes the symbol of the multioperation corresponding to the monounary multialgebra type*).

Proof. According to Proposition 5, in the semihypergroup $(H, \circ) = H_{f_\circ}$ we have

$$x^2 \subseteq (x^2)^2, \ \forall x \in H,$$

and the completeness of $(H, \circ)$ leads us to the equalities

$$x^2 = (x^2)^2, \ \forall x \in H.$$

Let us remember that $x^2 = f_\circ(x)$ and

$$\begin{aligned}(x^2)^2 &= (x \circ x) \circ (x \circ x) = \bigcup\{y \circ z \mid y, z \in x \circ x\} = \bigcup\{y^2 \cup z^2 \mid y, z \in f_\circ(x)\} \\ &= \bigcup\{y^2 \mid y \in f_\circ(x)\} = \bigcup\{f_\circ(y) \mid y \in f_\circ(x)\} = f_\circ(f_\circ(x)).\end{aligned}$$

Thus we have

$$f_\circ(f_\circ(x)) = f_\circ(x), \ \forall x \in H,$$

hence the identity (5) is satisfied on the multialgebra $(H, f_\circ)$. $\square$

Remark 6. From Remark 5 one deduce easily that a monounary multialgebra (H, f) satisfying (5) is complete if and only if $\{f(x) \mid x \in H\}$ is a partition of $f(H)$. In the terms of binary relations, if R is the binary relation from (2), this happens when the restriction of R to $R(H)$ is an equivalence relation on $R(H)$.

The condition that the monounary multialgebra (H, f) satisfies the identity (5) is stronger than the conditions from Proposition 3, but it is not sufficient for obtaining a complete semihypergroup H_f.

Example 1. Let $H = \{1, 2, 3\}$ and $f : H \to P^*(H), \ f(x) = x$. Clearly, (H, f) satisfies (5). Yet, the corresponding hypergroupoid $H_f = (H, \circ)$ is a hypergroup which is not complete since

$$1 \circ 2 = \{1, 2\} \neq \{2, 3\} = 2 \circ 3, \text{ even if } (1 \circ 2) \cap (2 \circ 3) = \{2\} \neq \emptyset.$$

Lemma 4. *Let $(H, \circ)$ be a hypergroupoid determined by a monounary multialgebra (H, f). If the hypergroupoid $(H, \circ)$ is complete then:*

a) $x \circ x = x \circ y,\ \forall x, y \in H$;

b) $f(x) = f(H),\ \forall x \in H$;

c) (H, f) *is a complete multialgebra;*

d) $(H, \circ)$ *is a commutative* (*complete*) *semihypergroup.*

Proof. a) Since $(H, \circ)$ is a complete hypergroupoid and

$$(x \circ x) \cap (x \circ y) = f(x) \cap (f(x) \cup f(y)) = f(x) \neq \emptyset,$$

for all $x, y \in H$, we have

$$x \circ x = x \circ y,\ \forall x, y \in H.$$

b) From a) follows that $f(x) = f(\{x, y\})$ for all $x, y \in H$, thus for each $x \in H$,

$$f(x) = \bigcup_{y \in H} f(\{x, y\}) = f\left(\bigcup_{y \in H} \{x, y\}\right) = f(H).$$

c) We apply Remark 5 and from b) follows that

$$f^n(x) = f(H),\ \forall n \in \mathbb{N}^*,\ \forall x \in H.$$

d) It is clear that any hypergroupoid determined by a monounary multialgebra is commutative. For any $x, y, z \in H$, using b) we have

$$(x \circ y) \circ z = f(\{x, y\}) \circ z = f(H) \circ z = f(H).$$

Analogously, $x \circ (y \circ z) = f(H)$. $\square$

Lemma 5. *Any commutative hypergroupoid* $(H, \circ)$ *which satisfies the identity*

$$\mathbf{x} \circ \mathbf{x} = \mathbf{x} \circ \mathbf{y} \tag{6}$$

is a complete semihypergroup determined by a monounary multialgebra.

Proof. If a commutative hypergroupoid $(H, \circ)$ satisfies the identity (6) then

$$x \circ y = x \circ x = y \circ y = x^2 \cup y^2,\ \forall x, y \in H,$$

so $(H, \circ)$ satisfies (1). This means that $(H, \circ)$ is determined by (H, f), where

$$f : H \to P^*(H),\ f(x) = x \circ x.$$

Under these circumstances, (6) leads us, as in the proof of the previous lemma, to

$$x \circ y = f(\{x, y\}) = f(x) = f(H),\ \forall x, y \in H$$

and to the fact that the hypergroupoid $(H, \circ)$ is a semihypergroup. The completeness of this semihypergroup follows from the form of its hyperproducts: it is easy to prove by induction on $n \in \mathbb{N}^*$, $n \geq 2$ that

$$x_1 \circ \cdots \circ x_n = f(H),$$

for any $x_1, \ldots, x_n \in H$. □

Theorem 1. *Let $(H, \circ)$ be a hypergroupoid determined by a monounary multialgebra (H, f). The following conditions are equivalent:*

1) *$(H, \circ)$ is a complete hypergroupoid;*
2) *$(H, \circ)$ satisfies the identity (6);*
3) *$f(x) = f(H),\ \forall x \in H$;*
4) *$(H, \circ)$ is a complete semihypergroup.*

Proof. 1) ⇒ 2) ⇒ 3) follows as in the proof of Lemma 4.
3) ⇒ 4) follows as in the final part of the proof of Lemma 5.
4) ⇒ 1) is obvious. □

Corollary 1. *A hypergroupoid $(H, \circ)$ determined by a nonempty monounary multialgebra (H, f) is a complete hypergroup if and only if*

$$f(x) = H,\ \forall x \in H$$

and this happens exactly when the $(H, \circ)$ is the total hypergroup on H.

Indeed, if $(H, \circ)$ is a hypergroup then $f(H) = H$, hence its completeness implies $f(x) = f(H) = H$ for each $x \in H$. Conversely, if $f(x) = H$ for each $x \in H$ then $f(H) = H$, thus $(H, \circ)$ is a complete semihypergroup which is a hypergroup. The hyperproduct of this hypergroup is defined by

$$x \circ y = f(\{x, y\}) = f(H) = H,\ \forall x, y \in H$$

(i.e. $(H, \circ)$ is *the total hypergroup* on H).

Remark 7. The condition on (H, f) to be complete is not equivalent with the conditions from the Theorem 1. For instance, the monounary multialgebra (H, f) from Example 1 is complete and determines a (semi)hypergroup which is not complete since (H, f) does not satisfy the condition 3) from Theorem 1.

The following corollary is the translation of the above results in the terms of binary relations.

Corollary 2. *Let $H_R = (H, \circ)$ be the hypergroupoid determined by the binary relation R. Then H_R is a complete semihypergroup if and only if*

$$R(x) = R(H),\ \forall x \in H.$$

If $H \neq \emptyset$ then H_R is a complete hypergroup if and only if

$$R(x) = H,\ \forall x \in H.$$

4. Products of complete semihypergroups associated to monounary multialgebras

Let $((H_i, f_i) \mid i \in I)$ be a family of monounary multialgebras. The direct product of the multialgebras (H_i, f_i) is the multialgebra $\left(\prod_{i\in I} H_i, f\right)$ with

$$f((x_i)_{i\in I}) = \prod_{i\in I} f_i(x_i).$$

This multialgebra, with the projections $e_i^I : \prod_{j\in I} H_j \to H_i$, $e_i^I((x_j)_{j\in I}) = x_i$ $(i \in I)$ is the product of the multialgebras (H_i, f_i) in the category $\mathbf{Malg}(1)$.

Proposition 6. [2, Proposition 4] *If $n \in \mathbb{N}$, $\mathbf{q}, \mathbf{r} \in \mathbf{P}^{(n)}(\tau)$, and $(\mathfrak{A}_i \mid i \in I)$ is a family of multialgebras of type τ such that $\mathbf{q} = \mathbf{r}$ is satisfied on each multialgebra $\mathfrak{A}_i$ then $\mathbf{q} = \mathbf{r}$ is also satisfied on the multialgebra $\prod_{i\in I} \mathfrak{A}_i$.*

Corollary 3. *If $((H_i, f_i) \mid i \in I)$ is a family of monounary multialgebras satisfying the identity* (5) *then the direct product $\left(\prod_{i\in I} H_i, f\right)$ also satisfies* (5).

Remark 8. [5, Remark 9] If K_2' is the subclass of $\mathbf{Malg}(1)$ which consists in multialgebras which satisfies (5) then K_2' is a subclass of $\mathbf{Malg}'(1)$ closed under the formation of the direct products. The subclass K_2'' of $\mathbf{Malg}(1)$ which consists in nonempty multialgebras (H, f) which satisfy (5) and $f(H) = H$ is a subclass of $\mathbf{Malg}''(1)$ closed under the formation of the direct products.

Theorem 2. *The monounary multialgebras which determine complete hypergroupoids form a subclass of K_2' closed under the formation of direct products. Also, the monounary multialgebras which determine complete hypergroups form a subclass of K_2'' closed under the formation of direct products.*

Proof. From Lemma 3 it follows that a monounary multialgebra (H, f) which determines a complete hypergroupoid H_f satisfies (5), hence (H, f) is in K_2'. According to Theorem 1, the complete hypergroupoid H_f is a semihypergroup. It is immediate that if H_f is a hypergroup then (H, f) is in K_2''. Let I be a set and for each $i \in I$, let (H_i, f_i) be a monounary multialgebra for which

$$f_i(x_i) = f_i(H_i),\ \forall x_i \in H_i.$$

If $\left(\prod_{i\in I} H_i, f\right)$ is the direct product of the multialgebras (H_i, f_i) then

$$f((x_i)_{i\in I}) = \prod_{i\in I} f_i(x_i) = \prod_{i\in I} f_i(H_i) = f\left(\prod_{i\in I} H_i\right),$$

for any $(x_i)_{i\in I} \in \prod_{i\in I} H_i$, hence $\left(\prod_{i\in I} H_i, f\right)$ determines a complete semihypergroup. If, in addition, for any $i \in I$ we have $H_i \neq \emptyset$ and $f_i(H_i) = H_i$ then

$$f((x_i)_{i\in I}) = \prod_{i\in I} f_i(H_i) = \prod_{i\in I} H_i,$$

for any $(x_i)_{i\in I} \in \prod_{i\in I} H_i$, so the multialgebra $\left(\prod_{i\in I} H_i, f\right)$ determines a complete hypergroup. $\square$

Let us denote by $\mathbf{SHG_c}'$ the subcategory of $\mathbf{SHG}'$ whose objects are the complete semihypergroups determined by monounary multialgebras and by $\mathbf{HG_c}'$ the subcategory of $\mathbf{HG}'$ whose objects are the complete hypergroups determined by monounary multialgebras. Since the direct product of monounary multialgebras is their product in $\mathbf{Malg}(1)$ from the above theorem, using Remark 4 and Remark 8 we obtain:

Corollary 4. *The subcategory* $\mathbf{SHG_c}'$ *of* $\mathbf{SHG}'$ *is closed under products. Moreover, if* I *is a set and for each* $i \in I$, (H_i, f_i) *is a monounary multialgebra which determines a complete semihypergroup* $(H_i)_{f_i} = (H_i, \circ_i)$ *then the product of* $((H_i, \circ_i) \mid i \in I)$ *in* $\mathbf{SHG_c}'$ *is the (complete) semihypergroup determined by the direct product* $\left(\prod_{i\in I} H_i, f\right)$.

Corollary 5. *The subcategory* $\mathbf{HG_c}'$ *of* $\mathbf{HG}'$ *is closed under products. Moreover, if* I *is a set and for each* $i \in I$, (H_i, f_i) *is a monounary multialgebra which determines a complete hypergroup* $(H_i)_{f_i} = (H_i, \circ_i)$ *then the product of* $((H_i, \circ_i) \mid i \in I)$ *in* $\mathbf{HG_c}'$ *is the (complete semi)hypergroup determined by the direct product* $\left(\prod_{i\in I} H_i, f\right)$.

If $((H_i, f_i) \mid i \in I)$ is a family of monounary multialgebras which determine the complete semihypergroups (hypergroups) $((H_i)_{f_i} = (H_i, \circ_i) \mid i \in I)$ then

$$x_i \circ y_i = f_i(x_i) = f_i(y_i) = f_i(H_i),\ \forall x_i, y_i \in H_i, \forall i \in I.$$

The product of $((H_i)_{f_i} \mid i \in I)$ in $\mathbf{SHG_c}'$ ($\mathbf{HG_c}'$) is the (complete) semihypergroup (hypergroup) $\left(\prod_{i\in I} H_i, \circ\right)$ determined by the direct product $\left(\prod_{i\in I} H_i, f\right)$. The hyperproduct $\circ$ is defined as follows: if $(x_i)_{i\in I}, (y_i)_{i\in I} \in \prod_{i\in I} H_i$ then

$$\begin{aligned}(x_i)_{i\in I} \circ (y_i)_{i\in I} &= f((x_i)_{i\in I}) \cup f((y_i)_{i\in I}) = \prod_{i\in I} f_i(x_i) \cup \prod_{i\in I} f_i(y_i)\\ &= \prod_{i\in I} f_i(H_i) = \prod_{i\in I}(x_i \circ y_i),\end{aligned}$$

hence $\left(\prod_{i\in I} H_i, \circ\right)$ is the direct product of $((H_i, \circ_i) \mid i \in I)$, i.e. the product of $((H_i, \circ_i) \mid i \in I)$ in $\mathbf{Malg}(2)$. Thus we have proved the following result:

Corollary 6. *The categories* $\mathbf{SHG_c}'$ *and* $\mathbf{HG_c}'$ *are subcategories of* $\mathbf{Malg}(2)$ *which are closed under products.*

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ON THE INTEGRABILITY OF A SYMPLECTIC STRUCTURE ON TANGENT BUNDLE

MONICA PURCARU

Abstract. In this paper we determine the set of all semi-symmetric metrical d-linear connections for a fixed nonlinear connection. We consider the group: $\overset{ms}{\mathcal{T}}_N$ of transformations of semi-symmetric metrical d-linear connections on TM, having the same nonlinear connection N and we give some important invariants. We study the 2-forms on TM and we define the integrability of a 2-form. We study the integrability of an almost symplectic d-structure on TM.

1. **Preliminaries**

The geometry of the tangent bundle (TM, π, M) has been studied by M.Matsumoto in [4], by R.Miron and M.Anastasiei in [5], [6], by R.Miron and M.Hashiguchi in [7], by V.Oproiu in [8], by Gh.Atanasiu and I.Ghinea in [1], by R.Bowman in [2], by K.Yano and S.Ishihara in [10],etc. Concerning the terminology and notations, we use those from [6].

Let M be a real n-dimensional C$^\infty$-differentiable manifold and (TM, π, M) its tangent bundle.

If (x^i) is a local coordinates system on a domain U of a chart on M, the induced system of coordinates on $\pi^{-1}(U)$ is (x^i, y^i), $(i = 1, ..., n)$.

Let N be a nonlinear connection on TM, with the coefficients $N^i{}_j(x, y)$, $(i, j = 1, ..., n)$.

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We consider on TM a metrical structure G defined by:

(1.1) $$G(x,y) = \frac{1}{2}g_{ij}(x,y)dx^i \otimes dx^j + \frac{1}{2}\tilde{g}_{ij}(x,y)\delta y^i \otimes \delta y^j,$$

where $(dx^i, \delta y^i), (i = 1, ..., n)$ is the dual basis of $\left(\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^i}\right)$, and $(g_{ij}(x,y),\ \tilde{g}_{ij}(x,y))$ is a pair of given d-tensor fields on TM, of the type (0,2), each of them symmetric and nondegenerate.

We asociate to the lift G the Obata's operators:

(1.2) $$\begin{cases} \Omega^{ir}_{sj} = \frac{1}{2}(\delta^i_s\delta^r_j - g_{sj}g^{ir}),\ \Omega^{*ir}_{sj} = \frac{1}{2}(\delta^i_s\delta^r_j + g_{sj}g^{ir}), \\ \tilde{\Omega}^{ir}_{sj} = \frac{1}{2}(\delta^i_s\delta^r_j - \tilde{g}_{sj}\tilde{g}^{ir}),\ \tilde{\Omega}^{*ir}_{sj} = \frac{1}{2}(\delta^i_s\delta^r_j + \tilde{g}_{sj}\tilde{g}^{ir}). \end{cases}$$

Obata's operators have the same properties as the ones associated with a Finsler space [7].

2. Semi-symmetric metrical d-linear connections on TM

Let N and $\overset{0}{N}$ be two nonlinear connections on TM with the coefficients $N^i_j(x,y)$ and $\overset{0}{N^i_j}(x,y)$ respectively, (i,j=1,...,n).

Definition 2.1. *A d-linear connection, D, on TM, with local coefficients $D\Gamma(N) = ({L^i}_{jk}, {\tilde{L}^i}_{jk}, {\tilde{C}^i}_{jk}, {C^i}_{jk})$, is called metrical d-linear connection on TM if:*

(2.1) $$g_{ij|k} = 0,\ g_{ij}|_k = 0,\ \tilde{g}_{ij|k} = 0,\ \tilde{g}_{ij}|_k = 0,$$

where $|$, $\big|$ *denote the h-and v-covariant derivatives respectively, with respect to D.*

Using a well known method given by R.Miron in [7] for the case of Finsler connections we obtain:

Theorem 2.1. *Let $\overset{0}{D}$ be a given d-linear connection on TM, with local coefficients $\overset{0}{D}\Gamma(\overset{0}{N}) = (\overset{0}{L^i}_{jk}, \overset{0}{\tilde{L}^i}_{jk}, \overset{0}{\tilde{C}^i}_{jk}, \overset{0}{C^i}_{jk})$. The set of all metrical d-linear connections on TM, with local coefficients $D\Gamma(N) = ({L^i}_{jk}, {\tilde{L}^i}_{jk}, {\tilde{C}^i}_{jk}, {C^i}_{jk})$ is given by:*

$$(2.2)\quad \begin{cases} N^i_j = \overset{0}{N}{}^i_j - X^i{}_j, \\ L^i{}_{jk} = \overset{0}{L}{}^i{}_{jk} + \overset{0}{\tilde{C}}{}^i{}_{jm} X^m{}_k + \frac{1}{2} g^{is}(g_{sj\overset{0}{|}k} + g_{sj} \overset{0}{|}_m X^m{}_k) + \Omega^{ir}_{hj} X^h{}_{rk}, \\ \tilde{L}^i{}_{jk} = \overset{0}{\tilde{L}}{}^i{}_{jk} + \overset{0}{C}{}^i{}_{jm} X^m{}_k + \frac{1}{2} \tilde{g}^{is}(\tilde{g}_{sj\overset{0}{|}k} + \tilde{g}_{sj} \overset{0}{|}_m X^m{}_k) + \tilde{\Omega}^{ir}_{hj} \tilde{X}^h{}_{rk}, \\ \tilde{C}^i{}_{jk} = \overset{0}{\tilde{C}}{}^i{}_{jk} + \frac{1}{2} g^{is} g_{sj} \overset{0}{|}_k + \Omega^{ir}_{hj} \tilde{Y}^h{}_{rk}, \\ C^i{}_{jk} = \overset{0}{C}{}^i{}_{jk} + \frac{1}{2} \tilde{g}^{is} \tilde{g}_{sj} \overset{0}{|}_k + \tilde{\Omega}^{ir}_{hj} Y^h{}_{rk}, \quad X^i{}_{j\overset{0}{|}k} = 0, \end{cases}$$

where $X^i_j,\ X^i_{jk},\ \tilde{X}^i_{jk},\ \tilde{Y}^i_{jk},\ Y^i_{jk}$ are arbitrary tensor fields on TM, $\overset{0}{|}$, $\overset{0}{\Big|}$ denote the h-and v-covariant derivatives respectively, with respect to $\overset{0}{D}$.

If we take $X^i_j = 0$ in theorem 2.1, we obtain a theorem given by R.Miron and M. Anastasiei in their papers [5], [6]:

Theorem 2.2. [5], [6] *Let $\overset{0}{D}$ be a given metrical d-linear connection on TM, with local coefficients: $\overset{0}{D}\Gamma(\overset{0}{N}) = (\overset{0}{L}{}^i{}_{jk}, \overset{0}{\tilde{L}}{}^i{}_{jk}, \overset{0}{\tilde{C}}{}^i{}_{jk}, \overset{0}{C}{}^i{}_{jk})$. The set of all metrical d-linear connections on TM, corresponding to the same nonlinear connection $\overset{0}{N}$, with local coefficients $D\Gamma(\overset{0}{N}) = (L^i{}_{jk}, \tilde{L}^i{}_{jk}, \tilde{C}^i{}_{jk}, C^i{}_{jk})$ is given by:*

$$(2.3)\quad \begin{cases} L^i{}_{jk} = \overset{0}{L}{}^i{}_{jk} + \Omega^{ir}_{hj} X^h{}_{rk}, \\ \tilde{L}^i{}_{jk} = \overset{0}{\tilde{L}}{}^i{}_{jk} + \tilde{\Omega}^{ir}_{hj} \tilde{X}^h{}_{rk}, \\ \tilde{C}^i{}_{jk} = \overset{0}{\tilde{C}}{}^i{}_{jk} + \Omega^{ir}_{hj} \tilde{Y}^h{}_{rk}, \\ C^i{}_{jk} = \overset{0}{C}{}^i{}_{jk} + \tilde{\Omega}^{ir}_{hj} Y^h{}_{rk}, \end{cases}$$

where $X^i_j,\ X^i_{jk},\ \tilde{X}^i_{jk},\ \tilde{Y}^i_{jk},\ Y^i_{jk}$ are arbitrary tensor fields on TM.

If we shall try to replace the arbitrary tensor fields $X^i{}_{jk}, Y^i_{jk}$ in theorem 2.2 by the torsion tensor fields $T_{(0)}{}^i{}_{jk}, S^i{}_{jk}$, and if we put:

$$(2.4)\quad \begin{cases} T^*{}^i{}_{jk} = \frac{1}{2} g^{ir}(g_{rh} T_{(0)}{}^h{}_{jk} - g_{jh} T_{(0)}{}^h{}_{rk} + g_{kh} T_{(0)}{}^h{}_{jr}), \\ \\ S^{*i}_{jk} = \frac{1}{2} \tilde{g}^{ir}(\tilde{g}_{rh} S^h_{jk} - \tilde{g}_{jh} S^h_{rk} + \tilde{g}_{kh} S^h_{jr}), \end{cases}$$

we find a result obtained by R.Miron and M.Anastasiei in [5], [6]:

Theorem 2.3. [5], [6] *Let* ${T_{(0)}}^{i}{}_{jk}$ *and* S^{i}_{jk} *be two given alternate tensor fields. Then there exists a unique metrical d-linear connection with local coefficients:* $D\Gamma(N) = (L^{i}{}_{jk}, \tilde{L}^{i}{}_{jk}, \tilde{C}^{i}{}_{jk}, C^{i}{}_{jk})$, *having* ${T_{(0)}}^{i}{}_{jk}$ *and* $S^{i}{}_{jk}$ *as the torsion tensor fields. It is given by:*

$$
(2.5)\qquad
\begin{cases}
L^{i}{}_{jk} = \overset{c}{L}{}^{i}{}_{jk} + T^{*i}{}_{jk}, \\
\tilde{L}^{i}{}_{jk} = \overset{c}{\tilde{L}}{}^{i}{}_{jk}, \\
\tilde{C}^{i}{}_{jk} = \overset{c}{\tilde{C}}{}^{i}{}_{jk}, \\
C^{i}{}_{jk} = \overset{c}{C}{}^{i}{}_{jk} + S^{*i}{}_{jk},
\end{cases}
$$

where $C\Gamma(N) = (\overset{c}{L}{}^{i}{}_{jk}, \overset{c}{\tilde{L}}{}^{i}{}_{jk}, \overset{c}{\tilde{C}}{}^{i}{}_{jk}, \overset{c}{C}{}^{i}{}_{jk})$ *are the local coefficients of the canonical d-linear connection of G.*

Definition 2.2. *A d-linear connection, D, on TM, with local coefficients* $D\Gamma(N) = (L^{i}{}_{jk}, \tilde{L}^{i}{}_{jk}, \tilde{C}^{i}{}_{jk}, C^{i}{}_{jk})$, *is called semi-symmetric d-linear connection if the torsion tensor fields* ${T_{(0)}}^{i}{}_{jk}$ *and* $S^{i}{}_{jk}$ *have the form:*

$$
(2.6)\qquad
\begin{cases}
{T_{(0)}}^{i}{}_{jk} = \frac{1}{n-1}(T_j\delta^i_k - T_k\delta^i_j) = \sigma_j\delta^i_k - \sigma_k\delta^i_j, \\
\\
S^{i}{}_{jk} = \frac{1}{n-1}(S_j\delta^i_k - S_k\delta^i_j) = \tau_j\delta^i_k - \tau_k\delta^i_j,
\end{cases}
$$

where $T_j = {T_{(0)}}^{i}{}_{ji}$, $S_j = S^{i}{}_{ji}$ and $\sigma_j = \frac{T_j}{n-1}, \tau_j = \frac{S_j}{n-1}$.

Then (2.4) become:

$$
(2.7)\qquad T^{*\,i}{}_{jk} = 2\Omega^{mi}_{jk}\sigma_m, \quad S^{*\,i}{}_{jk} = 2\tilde{\Omega}^{mi}_{jk}\tau_m,
$$

Using the theorem 2.3 and the relations (2.7) we have:

Theorem 2.4. *The set of all semi-symmetric metrical d-linear connections corresponding to the same nonlinear connection N, with local coefficients:* $D\Gamma(N) = (L^{i}{}_{jk}, \tilde{L}^{i}{}_{jk}, \tilde{C}^{i}{}_{jk}, C^{i}{}_{jk})$, *is given by:*

$$
(2.8)\quad \begin{cases} L^i{}_{jk} = \overset{c}{L^i}{}_{jk} + \sigma_j \delta^i_k - g_{jk} g^{im} \sigma_m, \\ \tilde{L}^i{}_{jk} = \overset{c}{\tilde{L}^i}{}_{jk}, \\ \tilde{C}^i{}_{jk} = \overset{c}{\tilde{C}^i}{}_{jk}, \\ C^i{}_{jk} = \overset{c}{C^i}{}_{jk} + \tau_j \delta^i_k - \tilde{g}_{jk} \tilde{g}^{im} \tau_m, \end{cases}
$$

where $C\Gamma(N) = (\overset{c}{L^i}{}_{jk}, \overset{c}{\tilde{L}^i}{}_{jk}, \overset{c}{\tilde{C}^i}{}_{jk}, \overset{c}{C^i}{}_{jk})$ *are the local coefficients of the canonical d-linear connection of G.*

3. The group of transformations of semi-symmetric metrical d-linear connections

Let N be a given nonlinear connection. Then any semi-symmetric metrical d-linear connection with local coefficients $\bar{D}\Gamma(N) = (\bar{L}^i{}_{jk}, \bar{\tilde{L}}^i{}_{jk}, \bar{\tilde{C}}^i{}_{jk}, \bar{C}^i{}_{jk})$ is given by (2.5) with (2.7).

From theorem 2.4 we have:

Theorem 3.1. *Two semi-symmetric metrical d-linear connections: D and $\bar{D}$, with local coefficients $D\Gamma(N) = (L^i{}_{jk}, \tilde{L}^i{}_{jk}, \tilde{C}^i{}_{jk}, C^i{}_{jk})$ and $\bar{D}\Gamma(N) = (\bar{L}^i{}_{jk}, \bar{\tilde{L}}^i{}_{jk}, \bar{\tilde{C}}^i{}_{jk}, \bar{C}^i{}_{jk})$ are related as follows:*

$$
(3.1)\quad \begin{cases} \bar{L}^i{}_{jk} = L^i{}_{jk} + \sigma_j \delta^i_k - g_{jk} g^{im} \sigma_m, \\ \bar{\tilde{L}}^i{}_{jk} = \tilde{L}^i{}_{jk}, \\ \bar{\tilde{C}}^i{}_{jk} = \tilde{C}^i{}_{jk}, \\ \bar{C}^i{}_{jk} = C^i{}_{jk} + \tau_j \delta^i_k - \tilde{g}_{jk} \tilde{g}^{im} \tau_m, \end{cases} \quad or: \begin{cases} \bar{L}^i{}_{jk} = L^i{}_{jk} + 2\Omega^{im}_{kj} \sigma_m, \\ \bar{\tilde{L}}^i{}_{jk} = \tilde{L}^i{}_{jk}, \\ \bar{\tilde{C}}^i{}_{jk} = \tilde{C}^i{}_{jk}, \\ \bar{C}^i{}_{jk} = C^i{}_{jk} + 2\tilde{\Omega}^{im}_{kj} \tau_m. \end{cases}
$$

Conversely, given $\sigma_j \in \mathcal{X}^*(M), \tau_j \in \mathcal{X}^*(M)$ the above (3.1) is thought to be a transformation of a semi-symmetric metrical d-linear connection D, with local coefficients $D\Gamma(N) = (L^i{}_{jk}, \tilde{L}^i{}_{jk}, \tilde{C}^i{}_{jk}, C^i{}_{jk})$ to a semi-symmetric metrical d-linear connection $\bar{D}$, with local coefficients $\bar{D}\Gamma(N) = (\bar{L}^i{}_{jk}, \bar{\tilde{L}}^i{}_{jk}, \bar{\tilde{C}}^i{}_{jk}, \bar{C}^i{}_{jk})$.

We shall denote this transformation by: $t(\sigma_j, \tau_j)$.

Thus we have:

Theorem 3.2. *The set:* $\overset{ms}{\mathcal{T}}_N$ *of all transformations* $t(\sigma_j, \tau_j) : D\Gamma(N) \to \bar{D}\Gamma(N)$ *of semi-symmetric metrical d-linear connections given by (3.1) is an abelian group, together with the mapping product:*

$$t(\bar{\sigma}_j, \bar{\tau}_j) \circ t(\sigma_j, \tau_j) = t(\sigma_j + \bar{\sigma}_j, \tau_j + \bar{\tau}_j).$$

This group acts on the set of all semi-symmetric metrical d-linear connections, corresponding to the same nonlinear connection N, transitively.

In order to find invariants of the group $\overset{ms}{\mathcal{T}}_N$, let us consider the transformation formulas of the torsion tensor fields by a transformation $t(\sigma_j, \tau_j) : D\Gamma(N) \to \bar{D}\Gamma(N)$ of semi-symmetric metrical d-linear connections on TM, with respect to G, given by (3.2)-(3.3):

$$(3.2)\quad \begin{cases} \bar{L}^i{}_{jk} = L^i{}_{jk} - B^i{}_{jk}, \\ \bar{\tilde{L}}^i{}_{jk} = \tilde{L}^i{}_{jk} - \tilde{B}^i{}_{jk}, \\ \bar{\tilde{C}}^i{}_{jk} = \tilde{C}^i{}_{jk} - \tilde{D}^i{}_{jk}, \\ \bar{C}^i{}_{jk} = C^i{}_{jk} - D^i{}_{jk}, \end{cases}$$

where:

$$(3.3)\quad \begin{cases} B^i{}_{jk} = g_{jk} g^{im} \sigma_m - \sigma_j \delta^i_k, \\ \tilde{B}^i{}_{jk} = 0, \quad \tilde{D}^i{}_{jk} = 0, \\ D^i{}_{jk} = \tilde{g}_{jk} \tilde{g}^{im} \tau_m - \tau_j \delta^i_k. \end{cases}$$

Proposition 3.1. *By the transformations (3.2)-(3.3) of semi-symmetric metrical d-linear connections, corresponding to the same nonlinear connection N:* $D\Gamma(N) \to \bar{D}\Gamma(N)$*, the torsion tensor fields,* $\underset{(0)}{T}{}^i{}_{jk}, \underset{(1)}{T}{}^i{}_{jk}, \underset{(1)}{P}{}^i{}_{jk}, \underset{(2)}{P}{}^i{}_{jk}, S^i{}_{jk}$ *are transformed as follows:*

$$(3.4)\quad \begin{cases} \underset{(0)}{\bar{T}}{}^i{}_{jk} = \underset{(0)}{T}{}^i{}_{jk} + (\sigma_j \delta^i_k - \sigma_k \delta^i_j), \\ \underset{(1)}{\bar{T}}{}^i{}_{jk} = \underset{(1)}{T}{}^i{}_{jk}, \\ \underset{(1)}{\bar{P}}{}^i{}_{jk} = \underset{(1)}{P}{}^i{}_{jk}, \\ \underset{(2)}{\bar{P}}{}^i{}_{jk} = \underset{(2)}{P}{}^i{}_{jk}, \\ \bar{S}^i{}_{jk} = S^i{}_{jk} + (\tau_j \delta^i_k - \tau_k \delta^i_j). \end{cases}$$

We denote with:

(3.5) $\quad t^i{}_{jk} = \mathcal{A}_{jk}\{\frac{\delta N^i_j}{\delta y^k}\},$

where $\mathcal{A}_{jk}\{...\}$ denotes the alternate summation: $\mathcal{A}_{jk}\{B^i{}_{jk}\} = B^i{}_{jk} - B^i{}_{kj}$

and with:

$$(3.6)\quad \begin{cases} t^*{}_{ijk} = \sum_{ijk}\{\tilde{g}_{im}t^m{}_{jk}\}, \\ T_{(0)}{}^*{}_{ijk} = \sum_{ijk}\{g_{im}T_{(0)}{}^m{}_{jk}\}, \\ R^*{}_{ijk} = \sum_{ijk}\{\tilde{g}_{im}R^m{}_{jk}\}, \\ S^*{}_{ijk} = \sum_{ijk}\{\tilde{g}_{im}S^m{}_{jk}\}, \\ \tilde{C}^*{}_{ijk} = \sum_{ijk}\{g_{im}\tilde{C}^m{}_{jk}\}, \\ \tilde{L}^*{}_{ijk} = \sum_{ijk}\{\tilde{g}_{im}\tilde{L}^m{}_{jk}\}, \\ P_{(1)}{}^*{}_{ijk} = \sum_{ijk}\{g_{im}P_{(1)}{}^m{}_{jk}\}, \\ P_{(2)}{}^*{}_{ijk} = \sum_{ijk}\{\tilde{g}_{im}P_{(2)}{}^m{}_{jk}\}, \end{cases}$$

where $\sum_{ijk}\{...\}$ denotes the cyclic summation: $\sum_{ijk}\{A_{ijk}\} = A_{ijk}+A_{jki}+A_{kij}$ and with:

$$(3.7)\quad \begin{cases} \overset{1}{\mathcal{K}}_{ijk} = -g_{km}T_{(0)}{}^m{}_{ij} + \mathcal{A}_{ij}\{\tilde{g}_{im}P_{(2)}{}^m{}_{jk}\}, \\ \overset{2}{\mathcal{K}}_{ijk} = \tilde{g}_{im}S^m{}_{jk} - \mathcal{A}_{jk}\{\tilde{g}_{km}C^m{}_{ij}\}, \\ \overset{3}{\mathcal{K}}_{ijk} = \mathcal{A}_{jk}\{\tilde{g}_{km}P_{(2)}{}^m{}_{ij}\}, \\ \overset{4}{\mathcal{K}}_{ijk} = \tilde{g}_{mj}C^m{}_{ik} + \tilde{g}_{im}C^m{}_{jk}, \\ \overset{5}{\mathcal{K}}_{ijk} = g_{mj}\tilde{C}^m{}_{ik} + g_{im}\tilde{C}^m{}_{jk}. \end{cases}$$

Remark 3.1 It is noted that: $t^*{}_{ijk}, T_{(0)}{}^*{}_{ijk}, R^*{}_{ijk}, S^*{}_{ijk}$ are alternate, $\overset{1}{\mathcal{K}}_{ijk}$ is alternate with respect to: i,j and $\overset{2}{\mathcal{K}}_{ijk}$, $\overset{3}{\mathcal{K}}_{ijk}$ are alternate with respect to: j,k.

By direct calculation from (3.1) and (3.4) and using the notations (3.5), (3.6) and (3.7) we have:

Theorem 3.3. *The tensor fields:* $t^i{}_{jk}, R^i{}_{jk}, P_{(1)}{}^i{}_{jk}, P_{(2)}{}^i{}_{jk}$, $t^*{}_{ijk}, T_{(0)}{}^*{}_{ijk}, R^*{}_{ijk}$, $S^*{}_{ijk}, P_{(1)}{}^*{}_{ijk}$, $P_{(2)}{}^*{}_{ijk}$, $\overset{1}{\mathcal{K}}_{ijk}$, $\overset{2}{\mathcal{K}}_{ijk}$, $\overset{3}{\mathcal{K}}_{ijk}$, $\overset{4}{\mathcal{K}}_{ijk}$, $\overset{5}{\mathcal{K}}_{ijk}$, *are invariants of the group* $\overset{ms}{\mathcal{T}}_N$.

Theorem 3.4. *Between the invariants in theorem 3.3 there exists the following relations:*

$$(3.8)\quad \begin{cases} \sum_{ijk}\{\overset{1}{\mathcal{K}}_{ijk}\} = -T_{(0)}{}^{*}{}_{ijk} + t^{*}{}_{ijk} + \sum_{ijk}\{\tilde{g}_{im}\mathcal{A}_{jk}\{\tilde{L}^{m}{}_{jk}\}\}, \\ \sum_{ijk}\{\overset{2}{\mathcal{K}}_{ijk}\} = 0, \\ \sum_{ijk}\{\overset{3}{\mathcal{K}}_{ijk}\} = t^{*}{}_{ijk} + \sum_{ijk}\{\tilde{g}_{im}\mathcal{A}_{jk}\{\tilde{L}^{m}{}_{jk}\}\}. \end{cases}$$

4. About the integrability of a symplectic structure on tangent bundle

Let $\Lambda^k(TM)$ be the $\mathcal{F}$-module of all k-forms on the tangent bundle (TM, π, M) where $\mathcal{F}$ is the ring of all differentiable functions on TM. If N is a given nonlinear connection, then $\{dx^i, \delta y^i\}$ is a local basis of $\Lambda^1(TM)$, which is dual to $\{\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^i}\}$.

If $f \in \mathcal{F}$, then the 1-form df is written as:

$$(4.1)\quad df = \frac{\delta f}{\delta x^i} dx^i + \frac{\partial f}{\partial y^i}\delta y^i,$$

and the exterior differential of δy^i is given by:

$$(4.2)\quad d(\delta y^i) = \frac{1}{2} R^{j}{}_{ik} dx^k \wedge dx^i + \frac{\partial N^{j}{}_{i}}{\partial y^k}\delta y^k \wedge dx^i.$$

In general, $\omega \in \Lambda^2(TM)$ is written in the form:

$$(4.3)\quad \omega = \frac{1}{2} a_{ij} dx^i \wedge dx^j + b_{ij} dx^i \wedge \delta y^j + \frac{1}{2}\ c_{ij}\delta y^i \wedge \delta y^j,$$

where $a_{ij} = -a_{ji},\ c_{ij} = -c_{ji}$.

The exterior differential $d\omega$ is given by:

$$(4.4)\quad \begin{aligned} d\omega = {} & \frac{1}{6}\ \overset{1}{\omega}_{ijk}\ dx^i \wedge dx^j \wedge dx^k + \frac{1}{2}\ \overset{2}{\omega}_{ijk}\ dx^i \wedge dx^j \wedge \delta y^k + \\ & + \frac{1}{2}\ \overset{3}{\omega}_{ijk}\ dx^i \wedge \delta y^j \wedge \delta y^k + \frac{1}{6}\ \overset{4}{\omega}_{ijk}\ \delta y^i \wedge \delta y^j \wedge \delta y^k, \end{aligned}$$

where:

$$(4.5)\quad \begin{cases} \overset{1}{\omega}_{ijk} = \sum_{ijk}\{\frac{\delta a_{ij}}{\delta x^k} + b_{im} R^{m}_{jk}\}, \\ \overset{2}{\omega}_{ijk} = \frac{\partial a_{ij}}{\partial y^k} + c_{km} R^{m}_{ij} + \mathcal{A}_{ij}\{\frac{\delta b_{jk}}{\delta x^i} + b_{im}\frac{\partial N^{m}{}_{j}}{\partial y^k}\}, \\ \overset{3}{\omega}_{ijk} = \frac{\delta c_{jk}}{\delta x^i} + \mathcal{A}_{jk}\{\frac{\partial b_{ij}}{\delta y^k} + c_{km}\frac{\partial N^{m}{}_{i}}{\partial y^j}\}, \\ \overset{4}{\omega}_{ijk} = \sum_{ijk}\{\frac{\partial c_{ij}}{\partial y^k}\}. \end{cases}$$

Proposition 4.1. *If a d-linear connection D is given on TM, with local coefficients: $D\Gamma(N) = (L^{i}{}_{jk}, \tilde{L}^{i}{}_{jk}, \tilde{C}^{i}{}_{jk}, C^{i}{}_{jk})$, then the coefficients $\overset{1}{\omega}_{ijk}, \overset{2}{\omega}_{ijk}, \overset{3}{\omega}_{ijk}$, $\overset{4}{\omega}_{ijk}$, of (4.5) have the following expressions:*

$$(4.6)\qquad \begin{cases} \overset{1}{\omega}_{ijk}=\sum_{ijk}\{a_{ij|k}+a_{im}T^m_{(0)jk}+b_{im}R^m_{\ jk}\},\\ \overset{2}{\omega}_{ijk}=a_{ij}|_k+b_{mk}T^m_{(0)ji}+c_{km}R^m_{\ ij}+\mathcal{A}_{ij}\{b_{jk|i}+a_{im}\tilde{C}^m_{\ jk}+b_{im}P^m_{(2)jk}\},\\ \overset{3}{\omega}_{ijk}=c_{jk|i}+b_{im}S^m_{\ jk}+\mathcal{A}_{jk}\{b_{ij}|_k+b_{mj}\tilde{C}^m_{\ ik}+c_{mj}P^m_{(2)ik}\},\\ \overset{4}{\omega}_{ijk}=\sum_{ijk}\{c_{ij}|_k+c_{im}S^m_{\ jk}\}. \end{cases}$$

Definition 4.1. *A 2-form* $\omega\in\Lambda^2(TM)$ *written in the form (4.3), for which the matrix* $B=\begin{pmatrix} a_{ij} & b_{ij}\\ -b_{ji} & c_{ij}\end{pmatrix}$ *is nondegenerate is called integrable if:* $d\omega=0$.

Theorem 4.1. *A 2-form* $\omega\in\Lambda^2(TM)$*, for which the matrix B is nondegenerate, is integrable, if and only if the tensor fields* $\overset{1}{\omega}_{ijk}=0,\overset{2}{\omega}_{ijk}=0,\overset{3}{\omega}_{ijk}=0$ *and* $\overset{4}{\omega}_{ijk}=0$*, where* $\overset{1}{\omega}_{ijk},\overset{2}{\omega}_{ijk},\overset{3}{\omega}_{ijk}$ *and* $\overset{4}{\omega}_{ijk}$ *are given in (4.6).*

Let $(g_{ij}(x,y))$ be a GL-metric and $(N^i_j(x,y))$ the local coefficients of a nonlinear connection ([6]).

The 2-form ω above considerated doesn't define a metrical structure on TM, because the coefficients a_{ij} and c_{ij} are alternate and b_{ij} cannot be the coefficients of the metrical structure G from (1.1). Therefore it isn't possible to consider the problem of the integrability of a metrical structure on TM.

On the other hand, ω is in fact an almost symplectic structure on TM. It defines a symplectic structure if it is closed (i.e. $d\omega=0$).

In the remainder of this section we shall present the integrability problem of the symplectic structure defined by the 2-form ω on TM, ([9]).

Assume that a nonlinear connection N on TM is given, then an almost symplectic structure on the base manifold M ($A(x,y)=\frac{1}{2}a_{ij}(x,y)dx^i\wedge dx^j+\frac{1}{2}\tilde{a}_{ij}(x,y)\delta y^i\wedge\delta y^j$, where $(dx^i,\delta y^i),(i=1,...,n)$ is the dual basis of $\left(\frac{\delta}{\delta x^i},\frac{\partial}{\partial y^i}\right)$, and $(a_{ij}(x,y),\ \tilde{a}_{ij}(x,y))$ is a pair of given d-tensor fields on TM, of the type (0,2), each of them alternate and nondegenerate) is lifted to a 2-form ω on TM in the following way: we consider the 2-forms ω of two single types I, II and one combined type εI+II, $\varepsilon\in R:$

I: $\omega=\frac{1}{2}a_{ij}dx^i\wedge dx^j$; II: $\omega=\frac{1}{2}\tilde{a}_{ij}\delta y^i\wedge\delta y^j$,

εI+II: $\omega=\frac{1}{2}\varepsilon a_{ij}dx^i\wedge dx^j+\frac{1}{2}\tilde{a}_{ij}\delta y^i\wedge\delta y^j$.

Proposition 4.2. *Each 2-form ω of the type $\varepsilon I + II$ is nondegenerate and defines an almost symplectic structure on TM.*

Proposition 4.3. *The coefficients $\overset{1}{\omega}_{ijk}, \overset{2}{\omega}_{ijk}, \overset{3}{\omega}_{ijk}, \overset{4}{\omega}_{ijk}$, of the exterior differential of the 2-form εI+II given in Proposition 4.2 are invariants of the transformation group of the set of all almost symplectic d-linear connections with the same nonlinear connection N: $\mathcal{G}_{as}$, and are given in the following form:*

$$\varepsilon\text{I+II: } \overset{1}{\omega}_{ijk} = \varepsilon \tilde{T}^*_{(0)ijk},\ \overset{2}{\omega}_{ijk} = \alpha \overset{\tilde{4}}{K}_{ijk} + \tilde{a}_{km} R^m{}_{ij},\ \overset{3}{\omega}_{ijk} = \overset{\tilde{3}}{K}_{ijk},\ \overset{4}{\omega}_{ijk} = \tilde{S}^*_{ijk}.$$

where:

$$(4.7)\quad \begin{cases} \tilde{T}^*_{(0)ijk} = \mathcal{S}_{ijk}\{a_{im} T^m_{(0)jk}\}, \quad \tilde{S}^*_{ijk} = \mathcal{S}_{ijk}\{\tilde{a}_{im} S^m{}_{jk}\}, \\ \tilde{t}^*_{ijk} = \mathcal{S}_{ijk}\{a_{im} t^m{}_{jk}\}, \quad \tilde{R}^*_{ijk} = \mathcal{S}_{ijk}\{\tilde{a}_{im} R^m{}_{jk}\}, \end{cases}$$

$$(4.8)\quad \begin{cases} \overset{\tilde{1}}{K}_{ijk} = a_{km} T^m_{(0)ij} + \mathcal{A}_{ij}\{\tilde{a}_{im} P^m_{(2)jk}\}, \\ \overset{\tilde{2}}{K}_{ijk} = \tilde{a}_{im} S^m{}_{jk} + \mathcal{A}_{jk}\{\tilde{a}_{km} C^m{}_{ij}\}, \\ \overset{\tilde{3}}{K}_{ijk} = \mathcal{A}_{jk}\{\tilde{a}_{km} P^m_{(2)ij}\}, \\ \overset{\tilde{4}}{K}_{ijk} = \mathcal{A}_{ij}\{a_{im} P^m_{(1)jk}\} = \mathcal{A}_{ij}\{a_{im} \tilde{C}^m{}_{jk}\}. \end{cases}$$

Definition 4.2. *An almost symplectic structure A on a differentiable manifold M is called integrable of the types I, II or $\varepsilon I + II$, if there exists an almost symplectic d-linear connection D on TM such that the corresponding lifted 2-forms on TM are integrable.*

Theorem 4.2. *An almost symplectic structure A on a differentiable manifold M is integrable of the type $\varepsilon I + II$ if and only if there exists an almost symplectic d-linear connection D on TM with local coefficients $D\Gamma(N) = (L^i{}_{jk}, \tilde{L}^i{}_{jk}, \tilde{C}^i{}_{jk}, C^i{}_{jk})$, satisfying the following conditions:*

$$\varepsilon\text{I+II: } T^i{}_{(0)jk} = S^i{}_{jk} = 0,\ \varepsilon \overset{\tilde{4}}{K}_{ijk} + \tilde{a}_{km} R^m{}_{ij} = 0, \overset{\tilde{3}}{K}_{ijk} = 0.$$

Theorem 4.3. *An almost symplectic structure A on a differentiable manifold M, integrable of the type $\varepsilon I + II, \varepsilon \in R^*$, does not depend on y if and only if $R^k{}_{ij} = 0$.*

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ALMOST OPTIMAL CUBATURE FORMULAS WITH REGARD TO THE EFFICIENCY

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Abstract. We introduce the definition of the almost optimal efficiency of the cubature formulas obtained with the tensor product and boolean-sum of the numerical quadrature operators, and we also give some applications and examples for such formulas.

The purpose of this note is to study the cubature formulas from efficiency point of view. We will consider the case when we have a rectangular domain and the cubature formula is constructed with the boolean-sum and tensor product of the one dimensional approximation operators. We will give the definition of the almost optimal formulas with regard to the efficiency and also some examples.
Let be f a function defined and integrable on the rectangular domain $D_n = [a_1, b_1] \times [a_2, b_2] \times ... \times [a_n, b_n]$ and $\prod_n$ a rectangular partition of the domain $D_n : \prod_n = \Delta x_1 \times \Delta x_2 \times ... \times \Delta x_n$, where $\Delta x_k = \{x_{k,1}, ..., x_{k,m_k}\}$ with $a_k \leq x_{k,1} < ... < x_{k,m_k} \leq b_k$.
First of all we will consider the tensor product cubature formula

$$\begin{aligned}\int_D f(x_1, ..., x_n)dx_1...dx_n &= \int_{a_1}^{b_1} ... \int_{a_n}^{b_n} f(x_1, ..., x_n)dx_1...dx_n = \\ &= Q_1^1...Q_1^n f + R_1^1 I^{2,...,n} + ... + R_1^n I^{1,..,n-1} - \\ &- R_1^1 R_1^2 I^{3,...,n} - ...(-1)^{n-1} R_1^1...R_1^n \qquad (1)\end{aligned}$$

where we use the following partial quadrature formulas

$$I^k f = Q_1^k f + R_1^k f, \qquad (2)$$

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for k=1,2,...,n

$$I^k f = \int_{a_k}^{b_k} f(x_1, ..., x_k, ..., x_n) dx_k$$

$$Q_1^k f = \sum_{i_k=1}^{m_k} A_{i_k}^k f(x_1, ..., x_{k-1}, x_{k,i_k}, x_{k+1}, ..., x_n)$$

and $R_1^k f$ is the corresponding remainder term. The formula (1) is a numerical approximation formula because

$$Q_1^1 ... Q_1^n f = \sum_{i_1=1}^{m_1} ... \sum_{i_n=1}^{m_n} A_{i_1}^1 ... A_{i_n}^n f(x_{1,i_1}, ..., x_{n,i_n}).$$

Definition 1. *The cubature formula (1) is almost optimal with regard to the efficiency, if all the quadrature formulas (2) are optimal with respect to the efficiency.*

In bidimensional case we have the following almost optimal efficient cubature formula :

Theorem 1. *Let $D = [a, b] \times [c, d] \subset \mathbb{R}^2$ and $f : D \to \mathbb{R}$ an integrable function on D. If $f^{(2,0)}, f^{(0,2)}, f^{(2,2)} \in C(D)$ then the cubature formula*

$$\begin{aligned}\int_a^b \int_c^d f(x, y) dxdy = & \frac{(b-a)(d-c)}{4}[f(a, c) + f(a, d) + f(b, c) + f(b, d)] - \\ & - \frac{(b-a)^3}{12} \int_c^d f^{(2,0)}(\xi_1, y) dy - \frac{(d-c)^3}{12} \int_a^b f^{(0,2)}(x, \eta_1) dx - \\ & - \frac{(b-a)^3 (d-c)^3}{144} f^{(2,2)}(\xi, \eta) \end{aligned} \tag{3}$$

is almost optimal with regard to the efficiency.

Proof. From (1), in the bidimensional case we have:

$$\int_D f(x, y) dxdy = Q_1^x Q_1^y f + (R_1^x I^y + R_1^y I^x - R_1^x R_1^y) f \tag{4}$$

Let Q_1^x, Q_1^y be the trapezes quadrature rules

$$(Q_1^x f)(\cdot, y) = \frac{b-a}{2}[f(a, y) + f(b, y)] \tag{5}$$

$$(Q_1^y f)(x, \cdot) = \frac{d-c}{2}[f(x, c) + f(x, d)] \tag{6}$$

then, if we use this rules in (4) with being R_1^x, R_1^y the corresponding remainder terms, we obtain the formula (3). But from [1] we know that in the class of Newton-Cotes quadratures $\mathbb{A}_{N-C}(Q;f)$, we have

$$E(Q_1;f) = \max_{Q\in\mathbb{A}_{N-C}(Q;f)} E(Q;f)$$

i.e. the trapezium quadrature is the optimal formula regard to the efficiency. $\square$

Theorem 2. *Let $f : D \to \mathbb{R}$, if $f \in C^2(D)$ then*

$$\int_a^b\int_c^d f(x,y)dxdy = (b-a)(d-c)[f(\frac{a+b}{2},c) + f(\frac{a+b}{2},d)+$$
$$+ f(a,\frac{c+d}{2}) + f(b,\frac{c+d}{2})] + \frac{(b-a)^3}{24}\int_c^d f^{(2,0)}(\xi_1,y)dy+$$
$$+ \frac{(d-c)^3}{24}\int_a^b f^{(0,2)}(x,\eta_1)dx - \frac{(b-a)^3(d-c)^3}{576}f^{(2,2)}(\xi,\eta) \quad (7)$$

is a cubature formula which is almost optimal from efficiency point of view.

Proof. In the formula (4) we will choose Q_1^x, Q_1^y to be the rectangular quadrature, and we will use the relation proved in [1]

$$E(Q_D;f) = \max_{Q\in\mathbb{A}_G(Q;f)} E(Q;f),$$

in the class of Gauss quadrature rules the optimal quadrature with respect to the efficiency is the rectangular quadrature. $\square$

We will consider now the boolean-sum cubature formula:

$$I = Q_s + R_p \quad (8)$$

where

$$Q_s = Q_1^1 I^{2,\dots,n} + \dots + Q_1^n I^{1,\dots,n-1} - Q_1^1Q_1^2 I^{3,\dots,n} - \dots$$
$$+ (-1)^{n-1}Q_1^1\dots Q_1^n, \quad (9)$$

respectively

$$R_p = R_1^1\dots R_1^n$$

with

$$I^{\nu_1,\ldots,\nu_p} f = \int\limits_{a_{\nu_1}}^{b_{\nu_1}} \ldots \int\limits_{a_{\nu_p}}^{b_{\nu_p}} f(x_1,...,x_n) dx_{\nu_1}...dx_{\nu_p}.$$

It follows that $Q_s f$ in formula (9) contains (n-1),...,2 multiple integrals, thus (8) is not a numerical integration formula. In order to obtain a numerical integration formula we have to use n-1 levels of approximation.

If we want to reduce the approximation levels we can combine the two methods. In this case

-if $n = 2k$, $k \geq 1$, we have the following decomposition of the integral operator:

$$I = \bigoplus_{j=0}^{\frac{n-2}{2}} Q_{2j+1} Q_{2j+2} + \prod_{j=0}^{\frac{n-2}{2}} (R_{2j+1} \oplus R_{2j+2}); \qquad (10)$$

-if $n = 2k + 1$,

$$I = \left(\bigoplus_{j=0}^{\frac{n-3}{2}} Q_{2j+1} Q_{2j+2} \right) Q_n + \left(\prod_{j=0}^{\frac{n-3}{2}} R_{2j+1} R_{2j+2} \right) \oplus R_n. \qquad (11)$$

The identities (10) and (11) lead us to the following product-boolean sum cubature formulas :

$$If = Q_1 Q_2 \oplus Q_3 Q_4 \oplus ... \oplus Q_{n-1} Q_n f + (R_1 \oplus R_2)...(R_{n-1} \oplus R_n) f, \text{ if n=2k} \qquad (12)$$

and

$$If = (Q_1 Q_2 \oplus ... \oplus Q_{n-2} Q_{n-1}) Q_n f + (R_1 \oplus R_2)...(R_{n-2} \oplus R_{n-1}) \oplus R_n f \qquad (13)$$

if $n = 2k + 1$.

Definition 2. *The boolean-sum respectively the product-boolean sum formula are almost optimal with regard to the efficiency, if the quadrature formulas used in each level of approximation are optimal with regard to the efficiency.*

Now, we will give some examples for such a formulas.

Theorem 3. *Let $D_h \subset \mathbb{R}^2$ be the standard domain $D_h = [0,h] \times [0,h]$. If $f^{(4,0)}, f^{(0,4)}, f^{(2,2)} \in C(D_h)$ then the homogeneous cubature formula*

$$\int\int_{D_h} f(x,y)dxdy = -\frac{h^2}{12}\left[f(0,0)+f(0,h)+f(h,0)+f(h,h)\right]+$$
$$+\frac{4h^2}{12}\left[f(0,\frac{h}{2})+f(h,\frac{h}{2})+f(\frac{h}{2},0)+f(\frac{h}{2},h)\right]+R(f), \qquad (14)$$

where

$$R(f) = -\frac{h^6}{144}\left[\frac{1}{20}f^{(4,0)}(\xi_1,\eta_1)+\frac{1}{20}f^{(0,4)}(\xi_2,\eta_2)+f^{(2,2)}(\xi_3,\eta_3)\right].$$

is almost optimal with regard to the efficiency.

Proof. From (8) in bidimensional case we have the following boolean-sum cubature formula:

$$I^{xy}f = (Q_1^x I^y + Q_1^y I^x - Q_1^x Q_1^y)f + R_1^x R_1^y f. \qquad (15)$$

Let Q_1^x, Q_1^y be the trapezoidal rules (5), (6), then the formula (15) become:

$$\int\int_{D_h} f(x,y)dxdy = \frac{h}{2}\int_0^h [f(0,y)+f(h,y)]dy + \frac{h}{2}\int_0^h [f(x,0)+f(x,h)]dx$$
$$-\frac{h^2}{4}[f(0,0)+f(h,0)+f(0,h)+f(h,h)]+R_S(f) \qquad (16)$$

where

$$R_S(f) = -\frac{h^6}{144}f^{(2,2)}(\xi,\eta).$$

The formula (16) is not a numerical integration formula, so we have to use a next level of approximation. If in the second level of approximation we use the Simpson's quadrature, we obtain the formula (14).

But, from [1], the two quadrature formulas which we use in booth levels are optimal with regard to the efficiency in the corresponding class of quadrature formulas. □

Theorem 4. *Let $D \subset \mathbb{R}^3$, $D = [a_1, b_1] \times [a_2, b_2] \times [a_3, b_3]$, $f : D \to \mathbb{R}$ an integrable function on D. On some differentiability condition of the function f the product-boolean sum cubature formula:*

$$\int_{a_1}^{b_1}\int_{a_2}^{b_2}\int_{a_3}^{b_3} f(x, y, z)dxdydz = Q_{xyz}f + R_{xyz}f, \tag{17}$$

where

$$\begin{aligned} Q_{xyz}f &= \frac{(b_1 - a_1)(b_2 - a_2)(b_3 - a_3)}{8} [f(a_1, a_2, a_3) + f(b_1, a_2, a_3)+ \\ &+f(b_1, b_2, a_3) + f(a_1, a_2, b_3) + f(b_1, a_2, b_3) + f(a_1, b_2, b_3)+ \\ &+f(a_1, b_2, a_3) + f(b_1, b_2, b_3)] \end{aligned}$$

and

$$\begin{aligned} R_{xyz}f &= \frac{(b_1 - a_1)^3(b_2 - a_2)^3}{144} \int_{a_3}^{b_3} f^{(2,2,0)}(\xi_1, \eta_1, z)dz - \\ &-\frac{(b_3 - a_3)^3}{12} \int_{a_1}^{b_1}\int_{a_2}^{b_2} f^{(0,0,2)}(x, y, \rho_1)dxdy + \\ &+\frac{(b_1 - a_1)^3(b_2 - a_2)^3(b_3 - a_3)^3}{1728} f^{(2,2,2)}(\xi_2,\eta_2, \rho_2) - \\ &-\frac{(b_1 - a_1)(b_2 - a_2)^3(b_3 - a_3)}{48} \Big[f^{(0,2,0)}(a_1, \gamma_1, a_3)+ \\ &+f^{(0,2,0)}(a_1, \gamma_2, b_3) + f^{(0,2,0)}(b_1, \gamma_3, a_3) + f^{(0,2,0)}(b_1, \gamma_4, b_3)\Big] - \\ &-\frac{(b_1 - a_1)^3(b_2 - a_2)(b_3 - a_3)}{48} \Big[f^{(2,0,0)}(\delta_1, a_2, a_3)+ \\ &+f^{(2,0,0)}(\delta_2, a_2, b_3) + f^{(2,0,0)}(\delta_3, b_2, a_3) + f^{(2,0,0)}(\delta_4, b_2, b_3)\Big] \end{aligned}$$

is an almost optimal formula with regard to the efficiency.

Proof. In 3-dimensional case we have the following product-boolean sum cubature formula:

$$If = (Q^1_x \oplus Q^1_y)Q^1_z f + (R^1_x R^1_y \oplus R^1_z)f. \tag{18}$$

In order to obtain a numerical approximation formula we have to use two level of approximation, and the formula (18) become:

$$If = Q_{xyz}f + R_{xyz}f \tag{19}$$

where

$$Q_{xyz} = Q_x^1 Q_y^2 Q_z^1 + Q_x^2 Q_y^1 Q_z^1 - Q_x^1 Q_y^1 Q_z^1 \tag{20}$$

and

$$R_{xyz} = R_x^1 R_y^1 I^z + I^x I^y R_z^1 + R_x^1 R_y^1 R_z^1 + Q_x^1 R_y^2 Q_z^1 + R_x^2 Q_y^1 Q_z^1. \tag{21}$$

Let Q_x^1, Q_y^1 and Q_z^1 be the trapezes approximation operators:

$$(Q_x^1 f)(\cdot, y, z) = \frac{b_1 - a_1}{2}\left[f(a_1, y, z) + f(b_1, y, z)\right]$$

$$(Q_y^1 f)(x, \cdot, z) = \frac{b_2 - a_2}{2}\left[f(x, a_2, z) + f(x, b_2, z)\right]$$

$$(Q_z^1 f)(x, y, \cdot) = \frac{b_3 - a_3}{2}\left[f(x, y, a_3) + f(x, y, b_3)\right]$$

then we have

$$\begin{aligned} Q_{xyz}f \quad = \quad & \frac{(b_1 - a_1)(b_3 - a_3)}{4} \int\limits_{a_2}^{b_2} [f(a_1, y, a_3) + f(a_1, y, b_3) + f(b_1, y, a_3) + \\ & + f(b_1, y, b_3)]\, dy + \frac{(b_2 - a_2)(b_3 - a_3)}{4} \int\limits_{a_1}^{b_1} [f(x, a_2, a_3) + \\ & + f(x, a_2, b_3) + f(x, b_2, a_3) + f(x, b_2, b_3)]\, dx - \\ & \frac{(b_1 - a_1)(b_2 - a_2)(b_3 - a_3)}{8} [f(a_1, a_2, a_3) + \\ & + f(a_1, a_2, b_3) + f(a_1, b_2, a_3) + f(a_1, b_2, b_3) + \\ & + f(b_1, a_2, a_3) + f(b_1, a_2, b_3) + f(b_1, b_2, a_3) + f(b_1, b_2, b_3)], \end{aligned}$$

where for the representation of the error we assume that $f^{(2,0,0)}, f^{(0,2,0)}, f^{(0,0,2)} \in C(D)$.

If in the second level we also use the trapezoidal rules where $f^{(2,2,0)}, f^{(2,2,2,)} \in C(D)$ we get the formula (17). The quadrature rules used in each level are optimal with regard to the efficiency, so the proof follows. $\square$

Theorem 5. *Let $D_h \subset \mathbb{R}^2$ be the standard domain $D_h = [0,h]\times[0,h]$ and $f : D_h \to \mathbb{R}$ a derivable function on D_h. If $CP(f') < CP(f) - 3$ and $f^{(2,0)}, f^{(0,2)}, f^{(2,2)} \in C(D_h)$ then the cubature formula*

$$\begin{aligned}\int\int_{D_h} f(x,y)dxdy = Q_2 f + R_{Q_2} f = \\ &= \frac{h^2}{4}\left[3f(0,0) + f(0,h) + f(h,0) - f(h,h)\right] + \\ &+ \frac{h^2}{4}\Big[f^{(0,1)}(0,h) + f^{(0,1)}(h,h) + f^{(1,0)}(h,0) + \\ &+ f^{(1,0)}(h,h)\Big] + R_{Q_2}(f)\end{aligned} \tag{22}$$

where

$$\begin{aligned}R_{Q_2}(f) &= \frac{h^6}{144} f^{(2,2)}(\xi,\eta) - \frac{h^4}{6}\Big[f^{(0,2)}(0,\eta_1) + f^{(0,2)}(h,\eta_2) + \\ &+ f^{(2,0)}(\xi_1,0) + f^{(2,0)}(\xi_2,h)\Big].\end{aligned}$$

is almost optimal with regard to the efficiency.

Proof. First of all we will consider the boolean-sum cubature formula in bidimensional case, when Q_1^x, Q_1^y are the trapezes approximation operators. In this case we obtain the formula (16). In the second level of approximation we will consider the following interpolation quadrature formula:

$$\int_0^h f(x)dx = h[f(0) + \frac{h}{2} f'(h)] - \frac{h^3}{3} f"(\theta), \theta \in (0,h). \tag{23}$$

We got this formula from an Abel-Gonciarov interpolation formula corresponding to the linear functionals $L_i f = f^{(i)}(x_i), i = 0,1, x_0 = 0, x_1 = h$. The efficiency of this quadrature formula is

$$E(Q_2, f) = \frac{\log_2 3}{CP(f) + CP(f') + 9}.$$

If $CP(f) - CP(f') > 3$ then the efficiency of the formula (23) is better than the trapezoidal quadrature formula. $\square$

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FUNCTIONAL-DIFFERENTIAL EQUATION WITH RETARDED ARGUMENT

ALEXANDRINA ALINA TARŢA

Dedicated to Professor Grigore Călugăreanu on his 60th birthday

Abstract. Sufficient conditions are obtained for all positive solutions of

$$\frac{dx(t)}{dt} = [f(x(t), x(t-\tau))]x(t)$$

to converges as $t \to \infty$ to a positive equilibrium solution.

1. **Introduction**

Trade cycles, business cycles, and fluctuations in the price and supply of various commodities have attracted the attention of economists for well over 100 years and possible more than thousands of years. Early authors often attribute these fluctuations to random factors, e.g. the weather for agricultural commodities, see for instance Slutzky [17] and Kalecki [11].

Other workers speculated that economic cycling of fluctuations might be an inherent endogenous dynamical behavior characteristic of instable economic systems, (Ezekiel [3] and the references therein). A number of business cycle models postulating the existence of nonlinearities to account for limit cycle behavior have played a fundamental role in sharpening the debate between the proponent of the exogenous versus endogenous (or stochastic versus deterministic) school (cf. Zarnowitz [19]).

The development of modern dynamical system theory (Guckenheimer and Holmes [10], Lasota and Mackey [12], Glass and Mackey [6]) have shed new light on this debate. The possibility that economic fluctuations may reflect underlying

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periodic or chaotic dynamics in nonlinear economics systems have been explored in various context, se for instance Goodwin at all [7] , Grandomnd and Malgrange [9], Gabisch and Lorenz [5] and references therein.

Mackey [13] developed a price adjustment model for a single commodity market with state dependent production and storage delays. Conditions for the equilibrium price to be stable are derived in terms of a variety of economic parameters. Also Blaire and Mackey [2] developed a model for the dynamics of price adjustment model in a single commodity market where nonlinearities in both supply and demand functions are considered explicitly. Farahani and Grove [4] have studied a special case of a general model studied of Blaire and Mackey [2] which it calls the case of naive consumer.

Our purpose here is to study the following model.

$$x'(t) = [f(x(t), x(t-\tau))]x(t), \quad t \in \mathbb{R}_+ \tag{1}$$

$$x(t) = \varphi(t), \quad t \in [-\tau, 0] \tag{2}$$

where $\tau > 0$, $f, g \in C(\mathbb{R}_+, \mathbb{R}_+)$ and $\varphi \in C([-\tau, 0], \mathbb{R}_+^*)$. Sufficient conditions are obtained for all positive solutions of (1) to converges as $t \to \infty$ to a positive equilibrium solution. We say that the function $x^*(t)$ oscillate about r^* if $x^*(t) - r^*$ has arbitrarily large zeros. If is not the case that $x^*(t)$ oscillate about r^*, then we say that $x^*(t)$ is nonoscillatory about r^*.

2. **The main result**

Consider the problem (1)+(2). The following theorem establish sufficient conditions that $x^*(t)$ oscillate about r^* where x^* is the unique positive solution of problem (1)+(2) and r^* is the unique positive equilibrium solution of (1).

Theorem 1. *Suppose that*

(i) $f \in C^1(\mathbb{R}_+ \times \mathbb{R}_+, \mathbb{R}_+)$, $\quad \varphi \in C([-\tau, 0], \mathbb{R}_+^*)$,

(ii) $f(\,\cdot\,, y)$ *is locally Lipschitz,*

(iii) *There exists* $M_f > 0$ *such that* $|f(u, v)| \leq M_f$ *for all* $u, v \in \mathbb{R}_+$,

(iv)

$$\frac{\partial f(u,v)}{\partial x}M + M_f \leq 0, \frac{\partial f(u,v)}{\partial y} \leq 0 \quad \textit{for all} \quad (u,v) \in \mathbb{R}_+ \times \mathbb{R}_+$$

and $M > 0$ *such that* $x^*(t) \leq M$ *for all* $t \in \mathbb{R}_+$,

(v) $1 + \lambda\dfrac{\partial f(u,u)}{\partial x} + \lambda\dfrac{\partial f(u,u)}{\partial y} \leq M_1 < 1$ *for all* $u \in \mathbb{R}_+$.

Then

(a) *Problem* (1)+(2) *has an unique positive solution* $x^*(t)$.

(b) *There exists* $m, M \in \mathbb{R}_+, 0 < m < M$ *such that* $m \leq x^*(t) \leq M$ *for all* $t \in \mathbb{R}_+$.

(c) *Equation* (1) *has a unique positive equilibrium solution* r^*.

(d) *if* x^* *is* r^* *-nonoscillatory then*

$$\lim_{t\to\infty} x^*(t) = r^*.$$

Proof:

(a) Let $x^* \in C([-\tau, t_+), \mathbb{R}_+) \cap C^1([0, t_+), \mathbb{R}_+)$ be a maximal solution of (1)+(2). We can rewrite the equation (1) in the form

$$\frac{x'(t)}{x(t)} = f(x(t), x(t-\tau)). \tag{3}$$

Integrating the equation (3) from 0 to t we obtain

$$\ln x(t) - \ln x(0) = \int_0^t f(x(s), x(s-\tau))ds.$$

From (2) we have

$$\ln\frac{x(t)}{\varphi(0)} = \int_0^t f(x(s), x(s-\tau))ds,$$

and

$$x(t) = \varphi(0)\exp\int_0^t f(x(s), x(s-\tau))ds.$$

From (iii) we have that

$$x(t) \leq \varphi(0)e^{M_f t}, \quad \text{for all} \qquad t \in [0, t_+).$$

From steps method and the Theorem of the maximal solution (see [1] and [16]) we have that there exists a unique x^* and $t_+ = +\infty$

(b) Follows from (a).

(c) Let $F : \mathbb{R}^2_+ \to \mathbb{R}^2_+,\ F(x,y) = (F_1(x), F_1(y))$

where

$$F_1(x) = x + \lambda f(x,x)$$

$$||F(x,y) - F(u,v)||_{\mathbb{R}^2}$$
$$\leq \left(\begin{array}{c} 1+\lambda\frac{\partial f}{\partial x}(u,u)+\lambda\frac{\partial f}{\partial y}(u,u) \quad 001+\lambda\frac{\partial f}{\partial x}(v,v)+\lambda\frac{\partial f}{\partial y}(v,v) \end{array} \right) ||(x,y)-(u,v)||\,,$$

where $(\mathbb{R}^2_+, ||\circ||)$, and $||x,y|| = \left(\begin{array}{c} |x| \\ |y| \end{array} \right).$

From (v) we have that

$$||F(x,y) - F(u,v)||_{\mathbb{R}^2} \leq \left(\begin{array}{cc} M_1 & 0 \\ 0 & M_1 \end{array} \right) ||(x,y)-(u,v)||\,,$$

and

$$S^n \to O_2, \text{ as } n \to \infty,$$

where

$$S = \left(\begin{array}{cc} M_1 & 0 \\ 0 & M_1 \end{array} \right).$$

It follows that F is a contraction and from Contraction Principle of Perov we have that F has a unique fixed point, i.e.

$$x = x + \lambda f(x,x).$$

This implies that the equation

$$f(x,x) = 0$$

has a unique solution and consequently that equation (1) has a positive equilibrium solution r^*.

(d) We rewrite equation (1) in the form

$$\frac{dy}{dt} = G(y(t), y(t-\tau)) - G(0,0), \tag{4}$$

where

$$G(y(t), y(t-\tau)) = [f(y(t+r^*, y(t-\tau+r^*)](y(t)+r^*),$$

and

$$y(t) = x(t) - r^*.$$

It is now sufficient to show that $y(t) \to 0$ as $t \to \infty$. An application of the mean-value Theorem to (4) leads to

$$\frac{dy}{dt} = -a(t)y(t) - b(t)y(t-\tau), \tag{5}$$

where

$$-a(t) = \frac{\partial G}{\partial y}(u(t), v(t)),$$
$$-b(t) = \frac{\partial G}{\partial y}(u(t), v(t)),$$

and $(u(t), v(t))$ lies on the line segment joining $(0,0)$ and $(y(t), y(t-\tau))$. It is found that

$$-a(t) = \frac{\partial f(u,v)}{\partial y} \cdot (y(t) + r^*) + f(y(t) + r^*, y(t-\tau) + r^*),$$

and

$$-b(t) = \frac{\partial f(u,v)}{\partial y} \cdot (y(t) + r^*).$$

Note that $a(t)$, and $b(t)$ are positive and bounded away from zero. The existence of solution of (5) for all $t \geq 0$ is a consequence of boundedness of $x(t)$ for all $t \geq 0$. If y is nonoscillatory then $|y(t)| > 0$, for all $t > 0$.

If $y(t) > 0$ for all $t > T$ then we have from (5) that $y'(t) < 0$ and so $\lim_{t\to\infty} \lim y(t)$ exists.

Since $y(t) > 0$ eventually, $\lim_{t\to\infty} \lim y(t) = l \geq 0$. We claim that $l = 0$; suppose that $l > 0$. Then there exists $t_0 > 0$ such that

$$y(t) \geq \frac{l}{2}, \text{ for } t \geq t_0.$$

We have directly from (5) that

$$\frac{dy(t)}{dt} \leq -a(t)\frac{l}{2}$$

leading to

$$y(t) - y(t_0) \leq -\frac{l}{2}\int_{t_0}^{t} a(s)ds,$$

which implies that

$$y(t) \to -\infty \text{ as } t \to \infty.$$

But this contradicts the eventual positivity of y.

Thus

$$\lim_{t\to\infty} y(t) = l = 0.$$

If $y(t) < 0$ for $t > T$, the arguments are similar.

Thus the result follows from

$$\lim_{t\to\infty} y(t) = 0.$$

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ON THE HOMOGENIZATION OF A CLIMATIZATION PROBLEM

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Abstract. This paper deals with the homogenization of a nonlinear model for heat conduction through the exterior of a domain containing periodically distributed conductive grains. We assume that on the walls of the grains we have climatizators governing the heat flux through the boundary. The effective behavior of this nonlinear flow is described by a new elliptic boundary-value problem containing an extra zero-order term which captures the effect of the boundary climatization.

1. **Introduction**

The aim of this paper is to study the homogenization of some nonlinear thermal flows through periodically perforated domains. We will focus our attention on a nonlinear problem which describes the heat conduction through the exterior of a domain containing periodically distributed conductive grains (or conductive obstacles). We suppose that on the walls of the grains we have climatizators governing the heat flux through the boundary.

Let Ω be an open bounded set in $\mathbb{R}^n$ and let us perforate it by holes. As a result, we obtain an open set Ω^ε which will be referred to as being the *perforated domain*; ε represents a small parameter related to the characteristic size of the perforations.

The nonlinear problem studied in this paper concerns the stationary flow of a fluid confined in Ω^ε, of temperature u^ε, with a given heat flux on the boundary of

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the grains:

$$\begin{cases} -D_f \Delta u^\varepsilon = f \quad \text{in } \Omega^\varepsilon, \\ -D_f \dfrac{\partial u^\varepsilon}{\partial \nu} = a\varepsilon g(u^\varepsilon) \quad \text{on } S^\varepsilon, \\ u^\varepsilon = 0 \quad \text{on } \partial\Omega. \end{cases} \tag{1}$$

Here, ν is the exterior unit normal to Ω^ε, $a > 0$, $f \in L^2(\Omega)$ and S^ε is the boundary of our porous medium $\Omega \setminus \overline{\Omega^\varepsilon}$. Moreover, the fluid is assumed to be homogeneous and isotropic, with a constant diffusion coefficient $D_f > 0$.

In the semilinear boundary condition on S^ε in problem (1) the function g is assumed to be given. We shall address here the case of a single-valued maximal monotone graph with $g(0) = 0$, i.e. the case in which g is the derivative of a convex lower semicontinuous function G. This situation is well illustrated by the following example, which is of practical importance in climatization problems:

$$g(r) = \begin{cases} 1 & r \geq \dfrac{1}{k}, \\ kr & \mid r \mid < \dfrac{1}{k}, \\ -1 & r \leq -\dfrac{1}{k}, \end{cases}$$

for a given $k > 0$.

The existence and uniqueness of a weak solution of (1) can be settled by using the classical theory of semilinear monotone problems (see [1]). As a result, we know that there exists a unique weak solution $u^\varepsilon \in V^\varepsilon \bigcap H^2(\Omega^\varepsilon)$, where

$$V^\varepsilon = \{v \in H^1(\Omega^\varepsilon) \mid v = 0 \text{ on } \partial\Omega\}.$$

If with Ω^ε we associate the following nonempty convex subset of V^ε:

$$K^\varepsilon = \{v \in V^\varepsilon \mid G(v)|_{S^\varepsilon} \in L^1(S^\varepsilon)\}, \tag{2}$$

then u^ε is also known to be the unique solution of the following variational problem:

$$\begin{cases} \text{Find } \; u^\varepsilon \in K^\varepsilon \; \text{ such that} \\ D_f \displaystyle\int_{\Omega^\varepsilon} Du^\varepsilon D(v^\varepsilon - u^\varepsilon)dx - \int_{\Omega^\varepsilon} f(v^\varepsilon - u^\varepsilon)dx + a\,\langle \mu^\varepsilon, G(v^\varepsilon) - G(u^\varepsilon)\rangle \geq 0, \\ \forall v^\varepsilon \in K^\varepsilon, \end{cases} \tag{3}$$

where μ^ε is the linear form on $W_0^{1,1}(\Omega)$ defined by

$$\langle \mu^\varepsilon, \varphi \rangle = \varepsilon \int\limits_{S^\varepsilon} \varphi d\sigma \quad \forall \varphi \in W_0^{1,1}(\Omega).$$

From a geometrical point of view, we shall just consider periodic structures obtained by removing periodically from Ω, with period εY (where Y is a given hyper-rectangle in $\mathbb{R}^n$), an elementary hole T which has been appropriated rescaled and which is strictly included in Y, i.e. $\overline{T} \subset Y$.

We shall prove that the solution u^ε, properly extended to the whole of Ω, converges to the unique solution of the following variational inequality:

$$\begin{cases} u \in H_0^1(\Omega) \\ \int\limits_\Omega QDuD(v-u)dx \geq \int\limits_\Omega f(v-u)dx - a\dfrac{|\partial T|}{|Y \setminus \overline{T}|} \int\limits_\Omega (G(v) - G(u))dx, \\ \forall v \in H_0^1(\Omega). \end{cases} \tag{4}$$

Here, $Q = ((q_{ij}))$ is the homogenized matrix (symmetric and positive-definite):

$$q_{ij} = D_f \left(\delta_{ij} + \frac{1}{|Y \setminus \overline{T}|} \int\limits_{Y \setminus \overline{T}} \frac{\partial \chi_j}{\partial y_i} dy \right), \tag{5}$$

defined in terms of the functions χ_i, $i = 1, ..., n$, solutions of the cell problems

$$\begin{cases} -\Delta \chi_i = 0 \text{ in } Y \setminus \overline{T}, \\ \dfrac{\partial(\chi_i + y_i)}{\partial \nu} = 0 \text{ on } \partial T, \\ \chi_i \;\; Y- \text{ periodic.} \end{cases} \tag{6}$$

We can treat in a similar manner the case of a multi-valued maximal monotone graph, which includes various semilinear boundary-value problems, such as Dirichlet or Neumann problems, Robin boundary conditions, Signorini's unilateral conditions, problems arising in chemistry (see [2], [4] and [5]).

The structure of the paper is as follows: first, let us mention that we shall just focus on the case $n \geq 3$. The case $n = 2$ is much more simpler and we shall omit to treat it. Section 2 is devoted to the setting of our problem and to the formulation of the main result of this paper. Section 3 contains some necessary preliminary results. In the last section we give the proof of our main result.

2. **Setting of the problem and the main result**

Let Ω be a smooth bounded connected open subset of $\mathbb{R}^n$ $(n \geq 3)$ and let $Y = [0, l_1[\times...[0, l_n[$ be the representative cell in $\mathbb{R}^n$. Denote by T an open subset of Y with smooth boundary ∂T such that $\overline{T} \subset Y$.

Let ε be a real parameter taking values in a sequence of positive numbers converging to zero. For each ε and for any integer vector $k \in \mathbb{Z}^n$, set $T_k^\varepsilon = \varepsilon(kl + T)$ the translated image of εT by the vector $kl = (k_1 l_1, ..., k_n l_n)$ and denote by T^ε the set of all the holes contained in Ω, i.e. $T^\varepsilon = \bigcup \left\{ T_k^\varepsilon \mid \overline{T_k^\varepsilon} \subset \Omega,\ k \in \mathbb{Z}^n \right\}$. Set $\Omega^\varepsilon = \Omega \setminus \overline{T^\varepsilon}$ and $S^\varepsilon = \cup\{\partial T_k^\varepsilon \mid \overline{T_k^\varepsilon} \subset \Omega,\ k \in \mathbb{Z}^n\}$. Also, let $Y^* = Y \setminus \overline{T}$ and $\rho = \dfrac{|Y^*|}{|Y|}$. Moreover, for an arbitrary function $\psi \in L^2(\Omega^\varepsilon)$, we shall denote by $\widetilde{\psi}$ its extension by zero inside the holes.

As already mentioned, we are interested in studying the asymptotic behavior of the solution of problem (1). We shall treat the case in which the function g appearing in (1) has a single-valued maximal monotone graph in $\mathbb{R} \times \mathbb{R}$, with $g(0) = 0$. Also, if we denote by $D(g)$ the domain of g, i.e. $D(g) = \{\xi \in \mathbb{R} \mid g(\xi) \neq \emptyset\}$, then we suppose that $D(g) = \mathbb{R}$. Moreover, we assume that g is continuous and there exist $C \geq 0$ and an exponent q, with $0 \leq q < n/(n-2)$, such that

$$|g(v)| \leq C(1 + |v|^q). \tag{7}$$

We know that in this case there exists a lower semicontinuous convex function G from $\mathbb{R}$ to $]-\infty, +\infty]$, G proper, i.e. $G \not\equiv +\infty$ such that g is the subdifferential of G, $g = \partial G$ (G is an indefinite "integral" of g). Let $G(v) = \int\limits_0^v g(s)ds$.

If the convex set K^ε is defined by (2), then, for a given function $f \in L^2(\Omega)$, the weak solution of the problem (1) is also the unique solution of the variational inequality

(3). Also, notice that u^ε is the unique solution of the minimization problem:

$$\begin{cases} u^\varepsilon \in K^\varepsilon, \\ J^\varepsilon(u^\varepsilon) = \inf\limits_{v \in K^\varepsilon} J^\varepsilon(v), \end{cases}$$

where

$$J^\varepsilon(v) = \frac{1}{2} D_f \int\limits_{\Omega^\varepsilon} |Dv|^2 \, dx + a \, \langle \mu^\varepsilon, G(v) \rangle - \int\limits_{\Omega^\varepsilon} fv dx.$$

Let us introduce the following functional defined on $H_0^1(\Omega)$:

$$J^0(v) = \frac{1}{2} \int\limits_\Omega QDvDvdx + a \frac{|\partial T|}{|Y^*|} \int\limits_\Omega G(v)dx - \int\limits_\Omega fvdx.$$

The main result of this paper is the following one:

Theorem 2.1. *One can construct an extension $P^\varepsilon u^\varepsilon$ of the solution u^ε of the variational inequality (3) such that*

$$P^\varepsilon u^\varepsilon \rightharpoonup u \quad \textit{weakly in } \ H_0^1(\Omega),$$

where u is the unique solution of the minimization problem

$$\begin{cases} \textit{Find } u \in H_0^1(\Omega) \textit{ such that} \\ J^0(u) = \inf\limits_{v \in H_0^1(\Omega)} J^0(v). \end{cases} \tag{8}$$

Moreover, $G(u) \in L^1(\Omega)$. Here, $Q = ((q_{ij}))$ is the classical homogenized matrix, whose entries were defined by (5)-(6).

3. **Preliminary results**

In order to extend the solution u^ε of problem (1) to the whole of Ω, let us recall the following well-known result (see [3]):

Lemma 3.1. *There exists a linear continuous extension operator $P^\varepsilon \in \mathcal{L}(L^2(\Omega^\varepsilon); L^2(\Omega)) \cap \mathcal{L}(V^\varepsilon; H_0^1(\Omega))$ and a positive constant C, independent of ε, such that, for any $v \in V^\varepsilon$,*

$$\|P^\varepsilon v\|_{L^2(\Omega)} \leq C \, \|v\|_{L^2(\Omega^\varepsilon)}$$

and

$$\|\nabla P^\varepsilon v\|_{L^2(\Omega)} \leq C \, \|\nabla v\|_{L^2(\Omega^\varepsilon)} \, .$$

For getting the effective behavior of our solution u^ε, we have to pass to the limit in (3). In order to do this, let us introduce, for any $h \in L^{s'}(\partial T)$, $1 \leq s' \leq \infty$, the linear form μ_h^ε on $W_0^{1,s}(\Omega)$ defined by

$$\langle \mu_h^\varepsilon, \varphi \rangle = \varepsilon \int\limits_{S^\varepsilon} h(\frac{x}{\varepsilon})\varphi d\sigma \quad \forall \varphi \in W_0^{1,s}(\Omega),$$

with $1/s + 1/s' = 1$. It is proved in [2] that

$$\mu_h^\varepsilon \to \mu_h \quad \text{strongly in } (W_0^{1,s}(\Omega))', \tag{9}$$

where $\langle \mu_h, \varphi \rangle = \mu_h \int\limits_\Omega \varphi dx$, with $\mu_h = \dfrac{1}{|Y|} \int\limits_{\partial T} h(y) d\sigma$.
In the particular case in which $h \in L^\infty(\partial T)$ or even h is constant, we have

$$\mu_h^\varepsilon \to \mu_h \quad \text{strongly in } W^{-1,\infty}(\Omega). \tag{10}$$

We shall denote by μ^ε the above introduced measure in the case in which $h = 1$.

Let F be a continuously differentiable function, monotonously non-decreasing and such that $F(v) = 0$ iff $v = 0$. We shall suppose that there exist a positive constant C and an exponent q, with $0 \leq q < n/(n-2)$, such that $\left|\dfrac{\partial F}{\partial v}\right| \leq C(1 + |v|^q)$. It is not difficult to prove (see [4]) that for any $\varphi \in \mathcal{D}(\Omega) = C_0^\infty(\Omega)$ and for any $z^\varepsilon \rightharpoonup z$ weakly in $H_0^1(\Omega)$, we get

$$\varphi F(z^\varepsilon) \rightharpoonup \varphi F(z) \quad \text{weakly in } W_0^{1,\overline{q}}(\Omega), \tag{11}$$

where $\overline{q} = \dfrac{2n}{q(n-2)+n}$.

4. **Proof of the main result**

Proof of Theorem 2.1. Let u^ε be the solution of the variational inequality (3) and let $P^\varepsilon u^\varepsilon$ be the extension of u^ε given by Lemma 3.1. It is not difficult to see that $P^\varepsilon u^\varepsilon$ is bounded in $H_0^1(\Omega)$. So by extracting a subsequence, one has

$$P^\varepsilon u^\varepsilon \rightharpoonup u \quad \text{weakly in } H_0^1(\Omega).$$

Let $\varphi \in \mathcal{D}(\Omega)$. By classical regularity results $\chi_i \in L^\infty$. Using the boundedness of χ_i and φ, there exists $M \geq 0$ such that $\left\| \dfrac{\partial \varphi}{\partial x_i} \right\|_{L^\infty} \left\| \chi_i \right\|_{L^\infty} < M$. Let

$$v^\varepsilon = \varphi + \sum_i \varepsilon \frac{\partial \varphi}{\partial x_i}(x) \chi_i (\frac{x}{\varepsilon}).$$

Then, $v^\varepsilon \in K^\varepsilon$, which will allow us to take it as a test function in (3). Moreover, $v^\varepsilon \to \varphi$ strongly in $L^2(\Omega)$. If we compute Dv^ε, we get:

$$Dv^\varepsilon = \sum_i \frac{\partial \varphi}{\partial x_i}(x)(\mathbf{e}_i + D\chi_i (\frac{x}{\varepsilon})) + \varepsilon \sum_i D\frac{\partial \varphi}{\partial x_i}(x)\chi_i (\frac{x}{\varepsilon}),$$

where $\mathbf{e}_i$, $1 \leq i \leq n$, are the elements of the canonical basis in $\mathbb{R}^n$.
Using v^ε as a test function in (3), we can write

$$D_f \int_\Omega DP^\varepsilon u^\varepsilon \widetilde{(Dv^\varepsilon)} dx \geq \int_{\Omega^\varepsilon} f(v^\varepsilon - u^\varepsilon)dx +$$

$$+ D_f \int_{\Omega^\varepsilon} Du^\varepsilon Du^\varepsilon dx - a \left\langle \mu^\varepsilon, G(v^\varepsilon) - G(u^\varepsilon) \right\rangle . \tag{12}$$

Denote

$$\rho Q \mathbf{e}_j = \frac{1}{|Y^*|} D_f \int_{Y^*} (D\chi_j + \mathbf{e}_j)dy, \tag{13}$$

where $\rho = |Y^*| \, / \, |Y|$. Neglecting the term $\varepsilon \sum\limits_i D\frac{\partial\varphi}{\partial x_i}(x)\chi_i (\frac{x}{\varepsilon})$ which actually tends strongly to zero, we can pass easily to the limit in the left-hand side of (12). Hence

$$D_f \int_\Omega DP^\varepsilon u^\varepsilon \widetilde{Dv^\varepsilon} dx \to \int_\Omega \rho Q Du D\varphi dx. \tag{14}$$

For the first term of the right-hand side of (12) we get

$$\int_{\Omega^\varepsilon} f(v^\varepsilon - u^\varepsilon)dx = \int_\Omega f\chi_{\Omega^\varepsilon}(v^\varepsilon - P^\varepsilon u^\varepsilon)dx \to \int_\Omega f\rho(\varphi - u)dx. \tag{15}$$

For the third term of the right-hand side of (12), assuming (7) for the maximal monotone graph g and using (11) written for G and for $z^\varepsilon = P^\varepsilon u^\varepsilon$, we get

$$G(P^\varepsilon u^\varepsilon) \rightharpoonup G(u) \quad \text{weakly in } W_0^{1,\overline{q}}(\Omega).$$

Combining this with the convergence (10) written for $h = 1$, we have

$$\langle \mu^\varepsilon, G(P^\varepsilon u^\varepsilon)\rangle \to \frac{|\partial T|}{|Y|} \int_\Omega G(u)dx.$$

Using a similar technique for the convergence of $\langle \mu^\varepsilon, G(v^\varepsilon)\rangle$, we obtain

$$a\,\langle \mu^\varepsilon, G(v^\varepsilon) - G(P^\varepsilon u^\varepsilon)\rangle \to a\frac{|\partial T|}{|Y|} \int_\Omega (G(\varphi) - G(u))dx. \tag{16}$$

For passing to the limit in the second term of the right-hand side of (12) let us write down the subdifferential inequality

$$D_f \int_{\Omega^\varepsilon} Du^\varepsilon Du^\varepsilon dx \geq D_f \int_{\Omega^\varepsilon} Dw^\varepsilon Dw^\varepsilon dx + 2D_f \int_{\Omega^\varepsilon} Dw^\varepsilon (Du^\varepsilon - Dw^\varepsilon)dx, \tag{17}$$

for any $w^\varepsilon \in H^1_0(\Omega)$. Reasoning as before and choosing $w^\varepsilon = \overline{\varphi} + \sum_i \varepsilon \dfrac{\partial\overline{\varphi}}{\partial x_i}(x)\chi_i(\dfrac{x}{\varepsilon})$, where $\overline{\varphi}$ enjoys similar properties as the corresponding φ, the right-hand side of the inequality (17) passes to the limit and one has

$$\liminf_{\varepsilon\to 0} D_f \int_{\Omega^\varepsilon} Du^\varepsilon Du^\varepsilon dx \geq \int_\Omega \rho Q D\overline{\varphi} D\overline{\varphi} dx + 2\int_\Omega \rho Q D\overline{\varphi}(Du - D\overline{\varphi})dx,$$

for any $\overline{\varphi} \in \mathcal{D}(\Omega)$ and, by density, for any $\overline{\varphi} \in H^1_0(\Omega)$. So, for $u \in H^1_0(\Omega)$, we have

$$\liminf_{\varepsilon\to 0} D_f \int_{\Omega^\varepsilon} Du^\varepsilon Du^\varepsilon dx \geq \int_\Omega \rho Q DuDu dx. \tag{18}$$

Putting together (14)-(16) and (18), we get

$$\int_\Omega \rho Q DuD\varphi dx \geq \int_\Omega f\rho(\varphi - u)dx + \int_\Omega \rho Q DuDu dx - a\frac{|\partial T|}{|Y|}\int_\Omega (G(\varphi) - G(u))dx,$$

for any $\varphi \in \mathcal{D}(\Omega)$ and hence, by density, for any $v \in H^1_0(\Omega)$. So, finally, we obtain

$$\int_\Omega QDuD(v-u)dx \geq \int_\Omega f(v-u)dx - a\frac{|\partial T|}{|Y^*|}\int_\Omega (G(v) - G(u))dx,$$

which gives exactly the limit problem (4). This ends the proof of Theorem 2.1. $\square$

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NEW UNIVALENCE CRITERIA

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Abstract. In this paper we obtain sufficient conditions for univalence, which generalize some well known univalence criteria for analytic functions in the unit disk of the complex plane.

1. **Introduction**

Let A be the class of analytic functions f in the unit disk

$$U = \{z \in \mathbf{C} : |z| < 1\}$$

of the form

$$f(z) = z + a_2 z^2 + \dots \quad , z \in U. \tag{1}$$

In order to prove our results a brief summary of Loewner parametric method is needed.

A family of functions $L(z,t),\ z \in U,\ t \in [0,\infty)$ is a Loewner chain if $L(z,t)$ is analytic and univalent in U and $L(z,t)$ is subordinate to $L(z,s)$ for all $0 \leq s < t$.

Theorem 1. *[4] Let r be a real number such that $r \in (0,1]$ and let $L(z,t) = a_1(t)z + \dots$ be an analytic function in $U_r = \{z \in \mathbf{C} : |z| < r\}$, for all $t \geq 0$. If*

i) *$L(z,t)$ is locally absolutely continuous in $[0,\infty)$, locally uniformly with respect to U_r;*
ii) *$a_1 \neq 0$, $\lim_{t\to\infty} |a_1(t)| = \infty$ and $\{L(z,t)/a_1(t)\}$ forms a normal family in U_r;*

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iii) $z\frac{\partial L(z,t)}{\partial z} = p(z,t)\frac{\partial L(z,t)}{\partial t}$, $z \in U_r$, *a.e.* $t \in [0,\infty)$ *where* $p(z,t)$ *is analytic in* U *and satisfies* $Re\, p(z,t) > 0$, *for all* $z \in U$, $t \in [0,\infty)$,

then $L(z,t)$ *has an analytic and univalent extension to the whole unit disk* U.

2. **Main results**

Theorem 2. *Let* α *be a complex number such that* $Re\ \alpha > \frac{1}{2}$ *and let* $f \in A$. *Let* g *and* h *be two analytic functions in* U, $g(z) = 1 + b_1 z + \ldots$, $h(z) = c_0 + c_1 z + \ldots$. *If the following inequalities are satisfied*

$$\left|\frac{1}{\alpha}\cdot\frac{f'(z)}{g(z)} - 1\right| < 1,\ z \in U \tag{2}$$

$$\left|\left(\frac{1}{\alpha}\cdot\frac{f'(z)}{g(z)} - 1\right)|z|^4 + \right. \tag{3}$$

$$(1-|z|^2)|z|^2 z\left[\left(\frac{1}{\alpha}-1\right)\frac{f'(z)}{f(z)} + \frac{2}{\alpha}\cdot\frac{f'(z)h(z)}{g(z)} + \frac{g'(z)}{g(z)}\right] + (1-|z|^2)^2 z^2\cdot$$

$$\left.\left[\left(\frac{1}{\alpha}-1\right)\frac{f'(z)h(z)}{f(z)} + \frac{1}{\alpha}\cdot\frac{f'(z)h^2(z)}{g(z)} + \frac{g'(z)h(z)}{g(z)} - h'(z)\right]\right| \le |z|^2$$

for all $z \in U$, *then the function* f *is univalent in* U.

Proof. We define

$$L(z,t) = f^{1-\alpha}(e^{-t}z)\left[f(e^{-t}z) + \frac{(e^t - e^{-t})zg(e^{-t}z)}{1+(e^t-e^{-t})zh(e^{-t}z)}\right]^{\alpha}$$

and we will prove that $L(z,t)$ satisfies theorem 1.

From the analyticity of the functions f, g and h it follows that $L(z,t)$ is analytic in a neighborhood U_r, $r \in (0,1]$ of $z = 0$.

Elementary calculation shows that

$$L(z,t) = a_1(t)z + \ldots \quad \text{where} \quad a_1(t) = e^{(2\alpha-1)t}.$$

We have $a_1(t) \neq 0$ and $\lim_{t\to\infty}|a_1(t)| = \infty$.

Since $L(z,t)$ is an analytic function in U_r, for all $t \in (0,\infty)$ we obtain that there exist a number $0 < r_1 < r$ and a constant $k = k(r_1)$ such that

$$\left|\frac{L(z,t)}{a_1(t)}\right| < k,\ z \in U_{r_1},\ t \in (0,\infty)$$

and hence $\{L(z,t)/a_1(t)\}$ forms a normal family in U_{r_1}.

It can be easy see that $\frac{\partial L(z,t)}{\partial t}$ is an analytic function in U_{r_1} and hence we obtain the absolute continuity requirements of theorem 1.

We define

$$p(z,t) = z\frac{\partial L(z,t)}{\partial z} \bigg/ \frac{\partial L(z,t)}{\partial t}$$

and we will prove that the function $p(z,t)$ has an analytic extension with positive real part in U, for all $t \geq 0$.

Let $W(z,t)$ be the function defined by

$$W(z,t) = \frac{p(z,t)-1}{p(z,t)+1}.$$

Elementary calculation shows that

$$W(z,t) = \left[\frac{1}{\alpha}\frac{f'(e^{-t}z)}{g(e^{-t}z)} - 1\right] e^{-2t} + \tag{4}$$

$$(1-e^{-2t})e^{-t}z \cdot \left[\left(\frac{1}{\alpha}-1\right)\frac{f'(e^{-t}z)}{f(e^{-t}z)} + \frac{2}{\alpha}\frac{f'(e^{-t}z)h(e^{-t}z)}{g(e^{-t}z)} + \frac{g'(e^{-t}z)}{g(e^{-t}z)}\right] + (1-e^{-2t})^2 z^2 \cdot$$

$$\left[\left(\frac{1}{\alpha}-1\right)\frac{f'(e^{-t}z)h(e^{-t}z)}{f(e^{-t}z)} + \frac{1}{\alpha}\frac{f'(e^{-t}z)h^2(e^{-t}z)}{g(e^{-t}z)} + \frac{g'(e^{-t}z)h(e^{-t}z)}{g(e^{-t}z)} - h'(e^{-t}z)\right]$$

We have

$$|W(z,0)| = \left|\frac{1}{\alpha} \cdot \frac{f'(z)}{g(z)} - 1\right|$$

and

$$|W(0,t)| = \left|\left(\frac{1}{\alpha} \cdot \frac{f'(0)}{g(0)} - 1\right) e^{-2t} + \left(\frac{1}{\alpha} - 1\right)(1-e^{-2t})\right| = \left|\frac{1}{\alpha} - 1\right|$$

From (2) and since $\operatorname{Re} \alpha > \frac{1}{2}$ we obtain that

$$|W(z,0)| < 1 \text{ and also } |W(0,t)| < 1. \tag{5}$$

Let t be a fixed positive number. Since $|e^{-tz}| \leq e^{-t} < 1$ for all $z \in \bar{U} = \{z \in \mathbf{C} : |z| \leq 1\}$ it results that the function $W(z,t)$ is analytic in $\bar{U}$. By using the maximum principle we obtain

$$|W(z,t)| < \max_{|\xi|=1} |W(\xi,t)| = |W(e^{i\theta,t})|, \tag{6}$$

where $\theta = \theta(t) \in \mathbf{R}$.

We denote $u = e^{-t} \cdot e^{i\theta}$. Then $|u| = e^{-t} < 1$ and from (4) it results

$$|W(e^{i\theta}, t)| = \left| \left(\frac{1}{\alpha} \cdot \frac{f'(u)}{g(u)} - 1 \right) |u|^2 + \right.$$

$$(1 - |u|^2) u \left[\left(\frac{1}{\alpha} - 1 \right) \frac{f'(u)}{f(u)} + \frac{2}{\alpha} \cdot \frac{f'(u)h(u)}{g(u)} + \frac{g'(u)}{g(u)} \right] +$$

$$\left. \frac{(1 - |u|^2)^2}{|u|^2} u^2 \left[\left(\frac{1}{\alpha} - 1 \right) \frac{f'(u)h(u)}{f(u)} + \frac{1}{\alpha} \cdot \frac{f'(u)h^2(u)}{g(u)} + \frac{g'(u)h(u)}{g(u)} - h'(u) \right] \right|$$

The inequality (3) implies $|W(e^{i\theta}, t)| \leq 1$ and by using (6) we obtain $|W(z, t)| < 1$ for all $z \in U$ and $t > 0$. From (5) it follows that $|W(z, t)| < 1$ for all $z \in U$ and $t \geq 0$. Hence the requirements for the function $p(z, t)$ are satisfied.

Finally, from Theorem 1 we obtain that the function $L(z, t)$ has an analytic and univalent extension to the whole unit disk U. For $t = 0$ we have $L(z, t) = f(z)$, $z \in U$ and thus the function f is univalent in U. □

Remark 1. The univalence criterion which results from Theorem 2 when $\alpha = 1$ is due to H. Ovesea-Tudor [2].

Specific choices for the functions g and h in Theorem 2 gives us various univalence criteria, between them being the very well known Nehari's criterion [1] and also Ozaki's criterion [3].

Corollary 1. *Let α be a complex number, with Re $\alpha > \frac{1}{2}$ and let $f \in A$. Suppose there exists an analytic function h in U, $h(z) = c_0 + c_1 z + \ldots$ such as*

$$\left| \left(\frac{1}{\alpha} - 1 \right) |z|^4 + (1 - |z|^2)|z|^2 z \left[\left(\frac{1}{\alpha} - 1 \right) \frac{f'(z)}{f(z)} + \frac{2}{\alpha} \cdot h(z) + \frac{f''(z)}{f'(z)} \right] + \right.$$

$$\left. + (1 - |z|^2)^2 z^2 \left[\left(\frac{1}{\alpha} - 1 \right) \frac{f'(z)h(z)}{f(z)} + \frac{1}{\alpha} \cdot h^2(z) + \frac{f''(z)h(z)}{f'(z)} - h'(z) \right] \right|$$

$$\leq |z|^2, \ z \in U \tag{7}$$

then the function f is univalent in U.

Proof. It results from Theorem 2 with $g = f'$. □

If we choose $g = f'$ and $h = -\frac{1}{2} \cdot \frac{f''}{f'}$ in Theorem 2 we obtain the following univalence criterion.

Corollary 2. **[5]***Let α be a complex number with Re $\alpha > \frac{1}{2}$ and let $f \in A$. Suppose*

$$\left|\left(\frac{1}{\alpha}-1\right)|z|^4+(1-|z|^2)|z|^2\left(\frac{1}{\alpha}-1\right)\left(\frac{zf'(z)}{f(z)}-\frac{zf''(z)}{f'(z)}\right)+\right.$$
$$\left.+(1-|z|^2)^2\left\{\frac{1}{2}z^2\{f;z\}+\frac{1}{2}\left(\frac{1}{\alpha}-1\right)\left[\frac{1}{2}\left(\frac{zf''(z)}{f'(z)}\right)^2-\frac{z^2f''(z)}{f(z)}\right]\right\}\right|$$
$$\leq |z|^2,\ z \in U \tag{8}$$

then f is univalent in the unit disk U.

Remark 2. If we consider $\alpha = 1$ in Corollary 2 we obtain the univalence criterion due to Nehari [1].

Corollary 3. *Let α be a complex number with Re $\alpha > \frac{1}{2}$ and let $f \in A$. If there exists a function $h : U \to \mathbf{C}$ $h(z) = b_0 + b_1 z + \ldots$ such as*

$$\left|\frac{1}{\alpha}\cdot\frac{z^2f'(z)}{f^2(z)}-1\right|<1,\ z\in U \tag{9}$$

$$\left|\left(\frac{1}{\alpha}\cdot\frac{z^2f'(z)}{f^2(z)}-1\right)|z|^4+\right. \tag{10}$$
$$(1-|z|^2)|z|^2z\left[\frac{\alpha+1}{\alpha}\cdot\frac{f'(z)}{f(z)}+\frac{2}{\alpha}\cdot\frac{z^2f'(z)h(z)}{f^2(z)}-\frac{2}{z}\right]+$$
$$\left.+(1-|z|^2)z^2\left[\frac{\alpha+1}{\alpha}\cdot\frac{f'(z)h(z)}{f(z)}+\frac{1}{\alpha}\cdot\frac{z^2f'(z)h^2(z)}{f^2(z)}-\frac{2h(z)}{z}-h'(z)\right]\right|\leq|z|^2,$$

for all $z \in U$, then f is univalent in U.

Proof. It results from Theorem 2 with $g(z) = \left(\frac{f(z)}{z}\right)^2$. □

Finally, if we choose $g(z) = \left(\frac{f(z)}{z}\right)^2$ and $h(z) = \frac{1}{z} - \frac{f(z)}{z^2}$ in Theorem 2 we obtain the following corollary.

Corollary 4. **[6]***Let α be a complex number with Re $\alpha > \frac{1}{2}$ and let $f \in A$. Suppose*

$$\left|\frac{1}{\alpha}\cdot\frac{z^2f'(z)}{f^2(z)}-1\right|<1,\ z\in U \tag{11}$$

$$\left|\frac{1}{\alpha}\cdot\frac{z^2f'(z)}{f^2(z)}-1+\frac{\alpha-1}{\alpha}(1-|z|^2)\frac{zf'(z)}{f(z)}\right|<|z|^2,\ z\in U \tag{12}$$

then the function f is univalent in the unit disk.

Remark 3. If we consider $\alpha = 1$ in Corollary 4 we obtain the univalence criterion due to Ozaki [3].

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INDECOMPOSABLE SUBGROUPS OF TORSION-FREE ABELIAN GROUPS

DUMITRU VĂLCAN

Dedicated to Professor Grigore Călugăreanu on his 60th birthday

Abstract. The present work gives descriptions of some classes of torsion-free abelian groups which have indecomposable subgroups, and for such a group we will present the structure of these subgroups.

All through this paper by group we mean abelian group in additive notation, and we will mark with P the set of all prime numbers, with $r(A)$ - the rank of the group A and with $t(A)$ - the type of A. According to [2,27.4], if a group A is indecomposable, then it is either torsion-free or cocyclic. Since the case of cocyclic groups (so of torsion groups) is well-known, in this paper we will present certain classes of torsion-free abelian groups which have indecomposable subgroups, properly constructing these subgroups.

Let G be a group of rank r and let $L = \{x_i\}_{i \in I}$ be a maximal independent system in G; I is a index set with $|I| = r$. Then, according to [2,16.1], $\langle L\rangle = \bigoplus_{i\in I}\langle x_i\rangle$. For every $i \in I$, we mark with X_i the pure subgroup of G, generated by $\{x_i\}$. Then, for every $i \in I$, the subgroup X_i is (homogeneous) of rank one and $t(X_i) = t(x_i) = t_i$. Therefore any group of rank r has a completely decomposable subgroup of the same rank. This motivates the study of our problem for completely decomposable groups.

For the beginning we have:

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Theorem 1. *Let* $G = \bigoplus_{i \in I} G_i$ *be a torsion-free group with the following properties:*

1) I is at most countable index set;

2) For every $i \in I$, G_i is a reduced group and $r(G_i) = 1$;

3) There is at most countable set $P_0 = \{p_1, p_2, \ldots, p_n, \ldots\}$ of distinct prime numbers, with the following properties:

i) for every $p_k \in P_0$, G_k is p_k-divisible,

ii) there is $p_{l_k} \in P_0 \setminus \{p_k\}$ such that G_k is not p_{l_k}-divisible.

Then G has an indecomposable subgroup A with $r(A) = |P_0|$.

Proof. Let G be a group as in the statement. For every $p_k \in P_0$, we consider the groups $E_{p_k} = \langle p_k^{-\infty} g_k \rangle$ and $H_{p_k} = \langle p_k^{-\infty} g_k, p_{l_k}^{-1} \overline{g}_k \rangle$, where $p_{l_k} \in P_0 \setminus \{p_k\}$ and $g_k \in G_k \setminus \{0\}$. Then, for every $p_k \in P_0$, E_{p_k} is a subgroup in H_{p_k} and $r(E_{p_k}) = 1$. Let be $A = \langle \bigoplus_{p_k \in P_0} E_{p_k}, p_{l_1}^{-1} \overline{g}_1 + p_{l_2}^{-1} \overline{g_2}, p_{l_1}^{-1} \overline{g}_1 + p_{l_3}^{-1} \overline{g}_3, \ldots, p_{l_1}^{-1} \overline{g}_1 + p_{l_n}^{-1} \overline{g}_n, \ldots \rangle$. Then $A \leq \bigoplus_{p_k \in P_0} H_{p_k} \leq \bigoplus_{i \in I} G_i$ and, for every $p_k \in P_0$, neither g_k nor $\overline{g}_k$ is divisible by $p_{l_k} \in P_0 \setminus \{p_k\}$ in A. (1)

Since, for every $p_k \in P_0$, all elements of E_{p_k} are divisible by every power of p_k, and, for every $p_s \in P_0 \setminus \{p_k\}$, E_{p_s} does not contain any such except for 0, it follows that, for every $p_k \in P_0$, E_{p_k} is fully invariant in A. On the other hand, since, for every $p_k \in P_0$, H_{p_k}/E_{p_k} is torsion, it follows that $(\bigoplus_{p_k \in P_0} H_{p_k})/(\bigoplus_{p_k \in P_0} E_{p_k})$ is torsion too. According to [1,1.6.12], it follows that $\bigoplus_{p_k \in P_0} E_{p_k}$ is an essential subgroup of A. (2)

We are going to show that A is indecomposable. In this way we suppose that $A = B \oplus C$. Then, according to [2,9.3], for every $p_k \in P_0$, $E_{p_k} = (E_{p_k} \cap B) \oplus (E_{p_k} \cap C)$. Since each E_{p_k} is indecomposable, it follows that, for every $p_k \in P_0$, either $E_{p_k} \cap B = 0$ or $E_{p_k} \cap C = 0$. So, for every $p_k \in P_0$, either $E_{p_k} \subseteq B$ or $E_{p_k} \subseteq C$. We suppose that there is $k \neq 1$ such that $E_{p_1} \subseteq B$ and $E_{p_k} \subseteq C$. In this case we consider the element $p_{l_1}^{-1} \overline{g}_1 + p_{l_k}^{-1} \overline{g}_k = b + c$, with $b \in B$ and $c \in C$. It follows that $p_{l_k} \overline{g}_1 + p_{l_1} \overline{g}_k = p_{l_1} p_{l_k} (b + c)$, that is $p_{l_k}(\overline{g}_1 - p_{l_1} b) = p_{l_1}(\overline{g}_k - p_{l_k} c) = 0$, which is impossible, according to the statement (1). Therefore, for every $p_k \in P_0$, either $E_{p_k} \subseteq B$ in which case $C = 0$, or $E_{p_k} \subseteq C$ in which case $B = 0$, according to the statement (2). It follows that A is indecomposable and since $r(A) = |P_0|$, the theorem is completely proved.

Corollary 2. *If $G = \bigoplus_{i \in I} G_i$ is a group which satisfies the conditions from Theorem 1 and the sets I and P_0 are equipotent, then G has indecomposable subgroups of every rank $m \leq r(G)$.*

Now we obtain the example 2 from [3,p.123]:

Corollary 3. *Let $G = \bigoplus_{i \in I} G_i$ be a group which satisfies the conditions from Theorem 1. If there is $q \in P \setminus P_0$ such that in the condition 3) of Theorem 1, for every $p_k \in P_0$, p_{l_k} may be replaced by q, then G has indecomposable subgroups of every rank $m \leq |P_0|$.*

Proof. Keeping the notations from Theorem 1, for every cardinal $m \leq |P_0|$, we consider the indecomposable subgroup $A_m = \langle \bigoplus_{p_k \in P_0^{(m)}} E_{p_k}, q^{-1}(\overline{g}_1 + \overline{g}_2), q^{-1}(\overline{g}_1 + \overline{g}_3), \ldots, q^{-1}(\overline{g}_1 + \overline{g}_m)\rangle$, where $P_0^{(m)}$ is a subset of cardinal m of P_0.

Other consequences of Theorem 1:

Corollary 4. *Let $G = \bigoplus_{p \in P} G_p$ be a torsion-free group with the following properties:*

1) For every $p \in P$, G_p is p-divisible and $r(G_p) = 1$;

2) For every $p \in P$, there is a $q_p \in P \setminus \{p\}$ for which G_p is not q_p-divisible.

Then G has indecomposable subgroups of every rank $m \leq r(G)$.

Proof. Let G be a group as in the statement. According to the condition 1), for every $p \in P$, there is a $\overline{g}_p \in G_p$ such that $h_p^{G_p}(\overline{g}_p) = \infty$. From the condition 2) it follows that, for every $p \in P$, there is a $q_p \in P \setminus \{p\}$ for which there is a $g_p \in G_p$ such that $h_{q_p}^{G_p}(g_p) = 1$. Since $r(G_p) = 1$, it follows that $\overline{g}_p$ and g_p are linear dependent; so $h_p^{G_p}(g_p) = \infty$. Now, for every $p_k \in P$, we consider the groups $E_{p_k} = \langle p_k^{-\infty}, g_{p_k}\rangle$ and $H_{p_k} = \langle p_k^{-\infty} g_{p_k}, q_{p_k}^{-1} g_{p_k}\rangle$. Then, for every $p_k \in P$, E_{p_k} is a subgroup of index q_{p_k} in H_{p_k} and $r(E_{p_k}) = 1$. For any cardinal $m \leq r(G)$, we consider the subgroup $A_m = \langle \bigoplus_{p_k \in P^{(m)}} E_{p_k}, q_{p_1}^{-1} g_{p_1} + q_{p_2}^{-1} g_{p_2}, q_{p_1}^{-1} g_{p_1} + q_{p_3}^{-1} g_{p_3}, \ldots, q_{p_1}^{-1} g_{p_1} + q_{p_m}^{-1} q_{p_m}\rangle$, where

$P^{(m)}$ is a subset of cardinal m of P. Following the same reasoning as in Theorem 1, we obtain that A_m is indecomposable.

Corollary 5. *For every $p \in P$, we consider the group $Q^{(p)}$ of all rational numbers whose denominators are powers of p. Then the group $G = \bigoplus_{p \in P} Q^{(p)}$ has indecomposable subgroups of every rank $m \leq r(G)$.*

Proof. For every $p \in P$, $t(Q^{(p)}) = (0, \dots, 0, \infty, 0, \dots)$, where ∞ stands at the proper place of the p-height h_p. So the group G satisfies the conditions from Corollary 4.

Corollary 6. *If I is a index set with $|I| \leq |P|$, then the group $Q^* = \oplus_I Q$ has indecomposable subgroups of every rank $m \leq |I|$.*

From Corollary 3 it follows:

Corollary 7. *We consider G a reduced, torsion-free of rank one group, I at most countable index set and let be $G^* = \oplus_I G$. If there is a set $P_0 = \{p_1, p_2, \dots, p_n, \dots\}$ of distinct prime numbers with the property that, for every $p_k \in P_0$ there is $g_k \in G$ (not necessarily distinct) such that $h^G_{p_k}(g_k) = \infty$, and there is $q \in P \setminus P_0$, for which there is $\overline{g}_k \in G_k$ such that $h^G_q(\overline{g}_k) = 1$, then G has indecomposable subgroups of every rank $m \leq |P_0|$.*

One can notice that there is a basic condition in all the cases we have mentioned above: the direct summands of group G have elements of infinite p-height, for certain prime numbers p. Afterwards this condition is replaced by another: the existence of a rigid system in group G. For the beginning we generalize [3,88.3].

Theorem 8. *Let be $\{H_i | i \in I\}$ a family of torsion-free groups such that, for every $i \in I$, there is $G_i \leq H_i$, where $\{G_i | i \in I\}$ is a rigid system of groups, with the property that there is a set $P_0 = \{p_i | i \in I\}$ of prime numbers (not necessarily distinct) such that, for every $p_i \in P_0$, there is a $g_i \in G_i$ with $h^{H_i}_{p_i}(g_i) = 1$ and which is not divisible by p_i in G_i. Then the group $H = \bigoplus_{i \in I} H_i$ has indecomposable subgroups of every rank $m \leq |I|$.*

Proof. Let m be any cardinal, $m \leq |I|$ and let $I^{(m)}$ be a subset of cardinal m of I. According to the hypothesis, for every $i \in I$, there is a $p_i \in P_0$ for which there is a $g_i \in G_i$ which is not divisible by p_i in G_i. Now we consider the subgroup $A_m = \langle \bigoplus_{i_j \in I^{(m)}} G_{i_j}, p_{i_1}^{-1} g_{i_1} + p_{i_2}^{-1} g_{i_2}, p_{i_1}^{-1} g_{i_1} + p_{i_3}^{-1} g_{i_3}, \ldots, p_{i_1}^{-1} g_{i_1} + p_{i_m}^{-1} g_{i_m} \rangle$ of H. Then, for every $p_i \in P_0$, g_i is not divisible by p_i in A. Since $\{G_i | i \in I\}$ is a rigid system of groups, for every $i \in I$, G_i is fully invariant in H; so, for every $i \in I$, G_i is fully invariant in A_m. We suppose that $A_m = B_m \oplus C_m$. Then, for every $i_j \in I^{(m)}$, $G_{i_j} = (G_{i_j} \cap B_m) \oplus G_{i_j} \cap C_m)$. Since each G_{i_j} is indecomposable, it follows that, for every $i_j \in I^{(m)}$, either $G_{i_j} \cap B_m = 0$ or $G_{i_j} \cap C_m = 0$. So, for every $i_j \in I^{(m)}$, either $G_{i_j} \subseteq B_m$ or $G_{i_j} \subseteq C_m$. We suppose that there is $j \neq 1$ such that $G_{i_1} \subseteq B_m$ and $G_{i_j} \subseteq C_m$. In this case we consider the element $p_{i_1}^{-1} g_{i_1} + p_{i_j}^{-1} g_{i_j} = b_m + c_m$, with $b_m \in B_m$ and $c_m \in C_m$. It follows that $p_{i_1} | g_{i_1}$ and $p_{i_j} | g_{i_j}$ in A_m, which is impossible, according to the hypothesis. Therefore, for every $i_j \in I^{(m)}$, either $G_{i_j} \subseteq B_m$ in which case $C_m = 0$ or $G_{i_j} \subseteq C_m$ in which case $B_m = 0$, because $\bigoplus_{i_j \in I^{(m)}} G_{i_j}$ is essential in A_m (in this way it is straightforward to verify that $A_m / (\bigoplus_{i_j \in I^{(m)}} G_{i_j})$ is torsion). It follows that A_m is indecomposable and since $r(A_m) = |I^{(m)}|$, the theorem is completely proved.

An immediate consequence of Theorem 8 is:

Corollary 9. *Let $\{H_i | i \in I\}$ be a family of torsion-free groups such that, for every $i \in I$, there is $G_i \leq H_i$, where $\{G_i | i \in I\}$ is a family of reduced of rank one groups, with the property that, for every $i_1, i_2 \in I$, $i_1 \neq i_2$, $t(G_{i_1})$ and $t(G_{i_2})$ are incomparable. Then the group $H = \bigoplus_{i \in I} H_i$ has indecomposable subgroups of every rank $m \leq |I|$.*

Proof. According to the hypothesis, $\{G_i | i \in I\}$ is a rigid system of groups and there is a set $P_0 = \{p_i | i \in I\}$ of prime numbers (not necessarily distinct) such that, for every $p_i \in P_0$, there is a $g_i \in G_i$ with $h_{p_i}^{H_i}(g_i) = 1$ and which is not divisible by p_i in G_i. Since $|I| = r(G)$, the statement follows from Theorem 8.

Let $G = B \oplus C$ be any group and A a subgroup of G. According to [2,p.44], there are subgroups B_2, B_1 of B and there are subgroups C_2, C_1 of C such that $B_2 \leq B_1$, $C_2 \leq C_1$, $B_1 \oplus C_1$ is the minimal direct sum containing A, and $B_2 \oplus C_2$ is the maximal direct sum contained in A, with components in B, respectively C. So $B_2 \oplus C_2 \leq A \leq B_1 \oplus C_1$ and it is straightforward to verify that A is a subdirect sum of B_1 and C_1 with kernels B_2, respectively C_2. Then, according to [2,p.43,44] the following relationships hold:

$$A/(B_2 \oplus C_2) \cong B_1/B_2 \cong C_1/C_2 \tag{3}$$

$$(B_1 \oplus C_1)/A \cong A/(B_2 \oplus C_2) \tag{4}$$

$$A/B_2 \cong C_1 \tag{5}$$

$$A/C_3 \cong B_1. \tag{6}$$

Remark 10. *Let $G = B \oplus C$ be any group and A a subgroup of G. If C is free and A is indecomposable, then either $r(A) = I$ or $A \subseteq B$.*

Proof. According to the hypothesis, keeping the above notation, C_1 is free. In this case, the relationship (4) and [2,14.4] show that B_2 is a direct summand in A - which is in contradiction to the hypothesis.

Now, we suppose that $G = B \oplus C$ is a torsion-free group, with $r(B) = r(C) = 1$. Then, according to the relationships (3), $B_2 = 0$ if and only if $C_2 = 0$; in this case, according to the condition (4), $(B_1 \oplus C_1)/A \cong A$ - which is impossible, because $(B_1 \oplus C_1)/A$ is torsion and A is torsion-free. Therefore $B_2 \neq 0$ and $C_2 \neq 0$. On the other hand, if $B_1 = B_2$ then $C_1 = C_2$ (see (3)) and in this case $A = B_2 \oplus C_2$.

Of course $B_2 \oplus C_2$ is essential in A ($A/(B_2 \oplus C_2)$ is torsion), $B_2 = B \cap A$, and $C_2 = C \cap A$. It follows that if B_2 is a proper subgroup of B_1, then also C_2 is a proper subgroup of C_1 and

$$A = \langle B_2 \oplus C_2, a_1, a_2, \ldots \rangle \tag{7}$$

where, for every $i = 1, 2, \ldots$, there is a $b_1^i \in B_1 \setminus \{0\}$ and there is a $c_1^i \in C_1 \setminus \{0\}$ such that $a_i = b_1^i + c_1^i$.

Now we can present the structure of indecomposable subgroups of completely decomposable groups of rank 2.

Theorem 11. *Let $G = B \oplus C$ be a torsion-free group with $r(B) = r(C) = 1$ and let A be any subgroup, of the form (7), of G. Then the following statements are equivalent:*

a) A is indecomposable;

b) i) for every $a_i \in A \setminus (B_2 \oplus C_2)$, there are $b_2^i \in B_2 \setminus \{0\}$ and $c_2^i \in C_2 \setminus \{0\}$ for which there are the prime numbers p_2^i and q_2^i (not necessarily distinct) such that b_2^i is not divisible by p_2^i in B_2, c_2^i is not divisible by q_2^i in C_2 and $a_i = (p_2^i)^{-1} b_2^i + (q_2^i)^{-1} c_2^i$;

ii) the subgroups B_2 and C_2 are fully invariant in A.

Proof. In view of Theorem 8, suffice it to show that a) implies b). Let $A = \langle B_2 \oplus C_2, b_1 + c_1, b_2 + c_2, \ldots\rangle$ be a subgroup, of the form (7), of G, where $b_1, b_2, \dots \in B \setminus B_2$ and $c_1, c_2, \dots \in C \setminus C_2$. According to the hypothesis, B_2 and C_2 are reduced and not pure in B and C respectively. Let p be a prime number and let $b + c + (B_2 \oplus C_2)$ be an element of order p from $A/(B_2 \oplus C_2)$. Then $pb = x \in B_2$ and $pc = y \in C_2$. If x is divisible by p in B_2, then $b \in B_2$, what is in contradiction to the hypothesis. It follows that x is not divisible by p in B_2 and $b = p^{-1}x$. Analogously it follows that $c = p^{-1}y$ and the statement i) from point b) is completely proved.

If $b + c + (B_2 \oplus C_2)$ is an element of order p^r, with $r \geq 2$, then we follow the same reasoning.

For the proof of the second statement from point b) we distinguish two cases:

Case 1. $t(B_2)$ and $t(C_2)$ are incomparable. Then this gives the required result.

Case 2. $t(B_2) \leq t(C_2)$. In this case there is a monomorphism $f : B_2 \to C_2$; so $B_2 \cong f(B_2) = B_2^* \leq C_2$. We consider the group $A^* = \langle B_2^* \oplus C_2, a_1^*, a_2^*, \ldots\rangle$, where, for every $i = 1, 2, \ldots$ $a_i^* = (p_2^i)^{-1}(b_2^i)^* + (q_2^i)^{-1}c_2^i$, and $(b_2^i)^* = f(b_2^i) \in B_2^*$; also we consider the subgroup $C_3 = \langle C_2, a_1^*, a_2^*, \ldots\rangle$ of A^*. Then, for every $i = 1, 2, \ldots$ there is $n_i \in N^*$ such that $n_i a_i^* \in C_2$. We are going to show that $A^* = B_2^* \oplus C_3$. Of course $A^* = B_2^* + C_3$. Let a^* be any element from A^*. We suppose that there are $x^*, y^* \in B_2$ and there are $u, v \in C_2$, such that $a^* = x^* + u + a_i^* = y^* + v + a_j^*$. Let be $n \in N^*$ such that $n(a_j^* - a_i^*) \in C_2$. Then $n(x^* - y^*) = n(v - u) + n(a_j^* - a_i^*) = 0$. Since G is torsion-free, it follows that $x^* = y^*$ and $u + a_i^* = v + a_j^*$, that is a^* may be written in a unique way of the form $b^* + c$, with $b^* \in B_2^*$ and $c \in C_3$. Since $A \cong A^*$, it follows that A is completely decomposable, what is in contradiction to the hypothesis.

From Remark 10 or Theorem 11 we obtain:

Corollary 12. *If B is a torsion-free of rank one group, then the group $G = B \oplus Z$ has no indecomposable subgroups of rank 2.*

Proof. If G is a group as in the statement, then there is no direct sum in G which is not made up of fully invariant direct summands. Now the statement follows from Theorem 11.

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INDECOMPOSABLE SUBGROUPS OF TORSION-FREE ABELIAN GROUPS

DUMITRU VĂLCAN

Dedicated to Professor Grigore Călugăreanu on his 60th birthday

Abstract. The present work gives descriptions of some classes of torsion-free abelian groups which have indecomposable subgroups, and for such a group we will present the structure of these subgroups.

All through this paper by group we mean abelian group in additive notation, and we will mark with P the set of all prime numbers, with $r(A)$ - the rank of the group A and with $t(A)$ - the type of A. According to [2,27.4], if a group A is indecomposable, then it is either torsion-free or cocyclic. Since the case of cocyclic groups (so of torsion groups) is well-known, in this paper we will present certain classes of torsion-free abelian groups which have indecomposable subgroups, properly constructing these subgroups.

Let G be a group of rank r and let $L = \{x_i\}_{i\in I}$ be a maximal independent system in G; I is a index set with $|I| = r$. Then, according to [2,16.1], $\langle L\rangle = \bigoplus_{i\in I}\langle x_i\rangle$. For every $i \in I$, we mark with X_i the pure subgroup of G, generated by $\{x_i\}$. Then, for every $i \in I$, the subgroup X_i is (homogeneous) of rank one and $t(X_i) = t(x_i) = t_i$. Therefore any group of rank r has a completely decomposable subgroup of the same rank. This motivates the study of our problem for completely decomposable groups.

For the beginning we have:

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Theorem 1. *Let* $G = \bigoplus_{i \in I} G_i$ *be a torsion-free group with the following properties:*

1) I is at most countable index set;

2) For every $i \in I$, G_i is a reduced group and $r(G_i) = 1$;

3) There is at most countable set $P_0 = \{p_1, p_2, \ldots, p_n, \ldots\}$ of distinct prime numbers, with the following properties:

i) for every $p_k \in P_0$, G_k is p_k-divisible,

ii) there is $p_{l_k} \in P_0 \setminus \{p_k\}$ such that G_k is not p_{l_k}-divisible.

Then G has an indecomposable subgroup A with $r(A) = |P_0|$.

Proof. Let G be a group as in the statement. For every $p_k \in P_0$, we consider the groups $E_{p_k} = \langle p_k^{-\infty} g_k \rangle$ and $H_{p_k} = \langle p_k^{-\infty} g_k, p_{l_k}^{-1} \overline{g}_k \rangle$, where $p_{l_k} \in P_0 \setminus \{p_k\}$ and $g_k \in G_k \setminus \{0\}$. Then, for every $p_k \in P_0$, E_{p_k} is a subgroup in H_{p_k} and $r(E_{p_k}) = 1$. Let be $A = \langle \bigoplus_{p_k \in P_0} E_{p_k}, p_{l_1}^{-1} \overline{g}_1 + p_{l_2}^{-1} \overline{g_2}, p_{l_1}^{-1} \overline{g}_1 + p_{l_3}^{-1} \overline{g}_3, \ldots, p_{l_1}^{-1} \overline{g}_1 + p_{l_n}^{-1} \overline{g}_n, \ldots \rangle$. Then $A \leq \bigoplus_{p_k \in P_0} H_{p_k} \leq \bigoplus_{i \in I} G_i$ and, for every $p_k \in P_0$, neither g_k nor $\overline{g}_k$ is divisible by $p_{l_k} \in P_0 \setminus \{p_k\}$ in A. (1)

Since, for every $p_k \in P_0$, all elements of E_{p_k} are divisible by every power of p_k, and, for every $p_s \in P_0 \setminus \{p_k\}$, E_{p_s} does not contain any such except for 0, it follows that, for every $p_k \in P_0$, E_{p_k} is fully invariant in A. On the other hand, since, for every $p_k \in P_0$, H_{p_k}/E_{p_k} is torsion, it follows that $(\bigoplus_{p_k \in P_0} H_{p_k})/(\bigoplus_{p_k \in P_0} E_{p_k})$ is torsion too. According to [1,1.6.12], it follows that $\bigoplus_{p_k \in P_0} E_{p_k}$ is an essential subgroup of A. (2)

We are going to show that A is indecomposable. In this way we suppose that $A = B \oplus C$. Then, according to [2,9.3], for every $p_k \in P_0$, $E_{p_k} = (E_{p_k} \cap B) \oplus (E_{p_k} \cap C)$. Since each E_{p_k} is indecomposable, it follows that, for every $p_k \in P_0$, either $E_{p_k} \cap B = 0$ or $E_{p_k} \cap C = 0$. So, for every $p_k \in P_0$, either $E_{p_k} \subseteq B$ or $E_{p_k} \subseteq C$. We suppose that there is $k \neq 1$ such that $E_{p_1} \subseteq B$ and $E_{p_k} \subseteq C$. In this case we consider the element $p_{l_1}^{-1} \overline{g}_1 + p_{l_k}^{-1} \overline{g}_k = b + c$, with $b \in B$ and $c \in C$. It follows that $p_{l_k} \overline{g}_1 + p_{l_1} \overline{g}_k = p_{l_1} p_{l_k} (b + c)$, that is $p_{l_k}(\overline{g}_1 - p_{l_1} b) = p_{l_1}(\overline{g}_k - p_{l_k} c) = 0$, which is impossible, according to the statement (1). Therefore, for every $p_k \in P_0$, either $E_{p_k} \subseteq B$ in which case $C = 0$, or $E_{p_k} \subseteq C$ in which case $B = 0$, according to the statement (2). It follows that A is indecomposable and since $r(A) = |P_0|$, the theorem is completely proved.

Corollary 2. *If $G = \bigoplus_{i \in I} G_i$ is a group which satisfies the conditions from Theorem 1 and the sets I and P_0 are equipotent, then G has indecomposable subgroups of every rank $m \leq r(G)$.*

Now we obtain the example 2 from [3,p.123]:

Corollary 3. *Let $G = \bigoplus_{i \in I} G_i$ be a group which satisfies the conditions from Theorem 1. If there is $q \in P \setminus P_0$ such that in the condition 3) of Theorem 1, for every $p_k \in P_0$, p_{l_k} may be replaced by q, then G has indecomposable subgroups of every rank $m \leq |P_0|$.*

Proof. Keeping the notations from Theorem 1, for every cardinal $m \leq |P_0|$, we consider the indecomposable subgroup $A_m = \langle \bigoplus_{p_k \in P_0^{(m)}} E_{p_k}, q^{-1}(\overline{g}_1 + \overline{g}_2), q^{-1}(\overline{g}_1 + \overline{g}_3), \ldots, q^{-1}(\overline{g}_1 + \overline{g}_m) \rangle$, where $P_0^{(m)}$ is a subset of cardinal m of P_0.

Other consequences of Theorem 1:

Corollary 4. *Let $G = \bigoplus_{p \in P} G_p$ be a torsion-free group with the following properties:*

1) For every $p \in P$, G_p is p-divisible and $r(G_p) = 1$;

2) For every $p \in P$, there is a $q_p \in P \setminus \{p\}$ for which G_p is not q_p-divisible.

Then G has indecomposable subgroups of every rank $m \leq r(G)$.

Proof. Let G be a group as in the statement. According to the condition 1), for every $p \in P$, there is a $\overline{g}_p \in G_p$ such that $h_p^{G_p}(\overline{g}_p) = \infty$. From the condition 2) it follows that, for every $p \in P$, there is a $q_p \in P \setminus \{p\}$ for which there is a $g_p \in G_p$ such that $h_{q_p}^{G_p}(g_p) = 1$. Since $r(G_p) = 1$, it follows that $\overline{g}_p$ and g_p are linear dependent; so $h_p^{G_p}(g_p) = \infty$. Now, for every $p_k \in P$, we consider the groups $E_{p_k} = \langle p_k^{-\infty}, g_{p_k} \rangle$ and $H_{p_k} = \langle p_k^{-\infty} g_{p_k}, q_{p_k}^{-1} g_{p_k} \rangle$. Then, for every $p_k \in P$, E_{p_k} is a subgroup of index q_{p_k} in H_{p_k} and $r(E_{p_k}) = 1$. For any cardinal $m \leq r(G)$, we consider the subgroup $A_m = \langle \bigoplus_{p_k \in P^{(m)}} E_{p_k}, q_{p_1}^{-1} g_{p_1} + q_{p_2}^{-1} g_{p_2}, q_{p_1}^{-1} g_{p_1} + q_{p_3}^{-1} g_{p_3}, \ldots, q_{p_1}^{-1} g_{p_1} + q_{p_m}^{-1} q_{p_m} \rangle$, where

$P^{(m)}$ is a subset of cardinal m of P. Following the same reasoning as in Theorem 1, we obtain that A_m is indecomposable.

Corollary 5. *For every $p \in P$, we consider the group $Q^{(p)}$ of all rational numbers whose denominators are powers of p. Then the group $G = \bigoplus_{p \in P} Q^{(p)}$ has indecomposable subgroups of every rank $m \leq r(G)$.*

Proof. For every $p \in P$, $t(Q^{(p)}) = (0, \dots, 0, \infty, 0, \dots)$, where ∞ stands at the proper place of the p-height h_p. So the group G satisfies the conditions from Corollary 4.

Corollary 6. *If I is a index set with $|I| \leq |P|$, then the group $Q^* = \oplus_I Q$ has indecomposable subgroups of every rank $m \leq |I|$.*

From Corollary 3 it follows:

Corollary 7. *We consider G a reduced, torsion-free of rank one group, I at most countable index set and let be $G^* = \oplus_I G$. If there is a set $P_0 = \{p_1, p_2, \dots, p_n, \dots\}$ of distinct prime numbers with the property that, for every $p_k \in P_0$ there is $g_k \in G$ (not necessarily distinct) such that $h^G_{p_k}(g_k) = \infty$, and there is $q \in P \setminus P_0$, for which there is $\overline{g}_k \in G_k$ such that $h^G_q(\overline{g}_k) = 1$, then G has indecomposable subgroups of every rank $m \leq |P_0|$.*

One can notice that there is a basic condition in all the cases we have mentioned above: the direct summands of group G have elements of infinite p-height, for certain prime numbers p. Afterwards this condition is replaced by another: the existence of a rigid system in group G. For the beginning we generalize [3,88.3].

Theorem 8. *Let be $\{H_i | i \in I\}$ a family of torsion-free groups such that, for every $i \in I$, there is $G_i \leq H_i$, where $\{G_i | i \in I\}$ is a rigid system of groups, with the property that there is a set $P_0 = \{p_i | i \in I\}$ of prime numbers (not necessarily distinct) such that, for every $p_i \in P_0$, there is a $g_i \in G_i$ with $h^{H_i}_{p_i}(g_i) = 1$ and which is not divisible by p_i in G_i. Then the group $H = \bigoplus_{i \in I} H_i$ has indecomposable subgroups of every rank $m \leq |I|$.*

Proof. Let m be any cardinal, $m \leq |I|$ and let $I^{(m)}$ be a subset of cardinal m of I. According to the hypothesis, for every $i \in I$, there is a $p_i \in P_0$ for which there is a $g_i \in G_i$ which is not divisible by p_i in G_i. Now we consider the subgroup $A_m = \langle \bigoplus_{i_j \in I^{(m)}} G_{i_j}, p_{i_1}^{-1} g_{i_1} + p_{i_2}^{-1} g_{i_2}, p_{i_1}^{-1} g_{i_1} + p_{i_3}^{-1} g_{i_3}, \ldots, p_{i_1}^{-1} g_{i_1} + p_{i_m}^{-1} g_{i_m} \rangle$ of H. Then, for every $p_i \in P_0$, g_i is not divisible by p_i in A. Since $\{G_i | i \in I\}$ is a rigid system of groups, for every $i \in I$, G_i is fully invariant in H; so, for every $i \in I$, G_i is fully invariant in A_m. We suppose that $A_m = B_m \oplus C_m$. Then, for every $i_j \in I^{(m)}$, $G_{i_j} = (G_{i_j} \cap B_m) \oplus G_{i_j} \cap C_m)$. Since each G_{i_j} is indecomposable, it follows that, for every $i_j \in I^{(m)}$, either $G_{i_j} \cap B_m = 0$ or $G_{i_j} \cap C_m = 0$. So, for every $i_j \in I^{(m)}$, either $G_{i_j} \subseteq B_m$ or $G_{i_j} \subseteq C_m$. We suppose that there is $j \neq 1$ such that $G_{i_1} \subseteq B_m$ and $G_{i_j} \subseteq C_m$. In this case we consider the element $p_{i_1}^{-1} g_{i_1} + p_{i_j}^{-1} g_{i_j} = b_m + c_m$, with $b_m \in B_m$ and $c_m \in C_m$. It follows that $p_{i_1} | g_{i_1}$ and $p_{i_j} | g_{i_j}$ in A_m, which is impossible, according to the hypothesis. Therefore, for every $i_j \in I^{(m)}$, either $G_{i_j} \subseteq B_m$ in which case $C_m = 0$ or $G_{i_j} \subseteq C_m$ in which case $B_m = 0$, because $\bigoplus_{i_j \in I^{(m)}} G_{i_j}$ is essential in A_m (in this way it is straightforward to verify that $A_m / (\bigoplus_{i_j \in I^{(m)}} G_{i_j})$ is torsion). It follows that A_m is indecomposable and since $r(A_m) = |I^{(m)}|$, the theorem is completely proved.

An immediate consequence of Theorem 8 is:

Corollary 9. *Let $\{H_i | i \in I\}$ be a family of torsion-free groups such that, for every $i \in I$, there is $G_i \leq H_i$, where $\{G_i | i \in I\}$ is a family of reduced of rank one groups, with the property that, for every $i_1, i_2 \in I$, $i_1 \neq i_2$, $t(G_{i_1})$ and $t(G_{i_2})$ are incomparable. Then the group $H = \bigoplus_{i \in I} H_i$ has indecomposable subgroups of every rank $m \leq |I|$.*

Proof. According to the hypothesis, $\{G_i | i \in I\}$ is a rigid system of groups and there is a set $P_0 = \{p_i | i \in I\}$ of prime numbers (not necessarily distinct) such that, for every $p_i \in P_0$, there is a $g_i \in G_i$ with $h_{p_i}^{H_i}(g_i) = 1$ and which is not divisible by p_i in G_i. Since $|I| = r(G)$, the statement follows from Theorem 8.

Let $G = B \oplus C$ be any group and A a subgroup of G. According to [2,p.44], there are subgroups B_2, B_1 of B and there are subgroups C_2, C_1 of C such that $B_2 \leq B_1$, $C_2 \leq C_1$, $B_1 \oplus C_1$ is the minimal direct sum containing A, and $B_2 \oplus C_2$ is the maximal direct sum contained in A, with components in B, respectively C. So $B_2 \oplus C_2 \leq A \leq B_1 \oplus C_1$ and it is straightforward to verify that A is a subdirect sum of B_1 and C_1 with kernels B_2, respectively C_2. Then, according to [2,p.43,44] the following relationships hold:

$$A/(B_2 \oplus C_2) \cong B_1/B_2 \cong C_1/C_2 \tag{3}$$

$$(B_1 \oplus C_1)/A \cong A/(B_2 \oplus C_2) \tag{4}$$

$$A/B_2 \cong C_1 \tag{5}$$

$$A/C_3 \cong B_1. \tag{6}$$

Remark 10. *Let $G = B \oplus C$ be any group and A a subgroup of G. If C is free and A is indecomposable, then either $r(A) = I$ or $A \subseteq B$.*

Proof. According to the hypothesis, keeping the above notation, C_1 is free. In this case, the relationship (4) and [2,14.4] show that B_2 is a direct summand in A - which is in contradiction to the hypothesis.

Now, we suppose that $G = B \oplus C$ is a torsion-free group, with $r(B) = r(C) = 1$. Then, according to the relationships (3), $B_2 = 0$ if and only if $C_2 = 0$; in this case, according to the condition (4), $(B_1 \oplus C_1)/A \cong A$ - which is impossible, because $(B_1 \oplus C_1)/A$ is torsion and A is torsion-free. Therefore $B_2 \neq 0$ and $C_2 \neq 0$. On the other hand, if $B_1 = B_2$ then $C_1 = C_2$ (see (3)) and in this case $A = B_2 \oplus C_2$.

Of course $B_2 \oplus C_2$ is essential in A ($A/(B_2 \oplus C_2)$ is torsion), $B_2 = B \cap A$, and $C_2 = C \cap A$. It follows that if B_2 is a proper subgroup of B_1, then also C_2 is a proper subgroup of C_1 and

$$A = \langle B_2 \oplus C_2, a_1, a_2, \ldots \rangle \tag{7}$$

where, for every $i = 1, 2, \ldots$, there is a $b_1^i \in B_1 \setminus \{0\}$ and there is a $c_1^i \in C_1 \setminus \{0\}$ such that $a_i = b_1^i + c_1^i$.

Now we can present the structure of indecomposable subgroups of completely decomposable groups of rank 2.

Theorem 11. *Let $G = B \oplus C$ be a torsion-free group with $r(B) = r(C) = 1$ and let A be any subgroup, of the form (7), of G. Then the following statements are equivalent:*

a) A is indecomposable;

b) i) for every $a_i \in A \setminus (B_2 \oplus C_2)$, there are $b_2^i \in B_2 \setminus \{0\}$ and $c_2^i \in C_2 \setminus \{0\}$ for which there are the prime numbers p_2^i and q_2^i (not necessarily distinct) such that b_2^i is not divisible by p_2^i in B_2, c_2^i is not divisible by q_2^i in C_2 and $a_i = (p_2^i)^{-1} b_2^i + (q_2^i)^{-1} c_2^i$;

ii) the subgroups B_2 and C_2 are fully invariant in A.

Proof. In view of Theorem 8, suffice it to show that a) implies b). Let $A = \langle B_2 \oplus C_2, b_1 + c_1, b_2 + c_2, \ldots \rangle$ be a subgroup, of the form (7), of G, where $b_1, b_2, \cdots \in B \setminus B_2$ and $c_1, c_2, \cdots \in C \setminus C_2$. According to the hypothesis, B_2 and C_2 are reduced and not pure in B and C respectively. Let p be a prime number and let $b + c + (B_2 \oplus C_2)$ be an element of order p from $A/(B_2 \oplus C_2)$. Then $pb = x \in B_2$ and $pc = y \in C_2$. If x is divisible by p in B_2, then $b \in B_2$, what is in contradiction to the hypothesis. It follows that x is not divisible by p in B_2 and $b = p^{-1}x$. Analogously it follows that $c = p^{-1}y$ and the statement i) from point b) is completely proved.

If $b + c + (B_2 \oplus C_2)$ is an element of order p^r, with $r \geq 2$, then we follow the same reasoning.

For the proof of the second statement from point b) we distinguish two cases:

Case 1. $t(B_2)$ and $t(C_2)$ are incomparable. Then this gives the required result.

Case 2. $t(B_2) \leq t(C_2)$. In this case there is a monomorphism $f : B_2 \to C_2$; so $B_2 \cong f(B_2) = B_2^* \leq C_2$. We consider the group $A^* = \langle B_2^* \oplus C_2, a_1^*, a_2^*, \ldots \rangle$, where, for every $i = 1, 2, \ldots$ $a_i^* = (p_2^i)^{-1}(b_2^i)^* + (q_2^i)^{-1} c_2^i$, and $(b_2^i)^* = f(b_2^i) \in B_2^*$; also we consider the subgroup $C_3 = \langle C_2, a_1^*, a_2^*, \ldots \rangle$ of A^*. Then, for every $i = 1, 2, \ldots$ there is $n_i \in N^*$ such that $n_i a_i^* \in C_2$. We are going to show that $A^* = B_2^* \oplus C_3$. Of course $A^* = B_2^* + C_3$. Let a^* be any element from A^*. We suppose that there are $x^*, y^* \in B_2$ and there are $u, v \in C_2$, such that $a^* = x^* + u + a_i^* = y^* + v + a_j^*$. Let be $n \in N^*$ such that $n(a_j^* - a_i^*) \in C_2$. Then $n(x^* - y^*) = n(v - u) + n(a_j^* - a_i^*) = 0$. Since G is torsion-free, it follows that $x^* = y^*$ and $u + a_i^* = v + a_j^*$, that is a^* may be written in a unique way of the form $b^* + c$, with $b^* \in B_2^*$ and $c \in C_3$. Since $A \cong A^*$, it follows that A is completely decomposable, what is in contradiction to the hypothesis.

From Remark 10 or Theorem 11 we obtain:

Corollary 12. *If B is a torsion-free of rank one group, then the group $G = B \oplus Z$ has no indecomposable subgroups of rank 2.*

Proof. If G is a group as in the statement, then there is no direct sum in G which is not made up of fully invariant direct summands. Now the statement follows from Theorem 11.

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BOOK REVIEWS

K. Burns and M. Gidea, ***Differential Geometry and Topology With a View to Dynamical Systems***, Chapman & Hall/CRC (Studies in Advanced Mathematics), 2005, ISBN 1-58488-253-0, 978-1-58488-253-4, Hardcover, IX+389 pp.

This is a graduate course on the topology and differential geometry of smooth manifolds, introducing, in parallel, the basic notions of smooth dynamical systems.

The first two chapters of the book introduce the basics of differential topology (manifolds and maps, the tangent bundle, immersions, submersions, embeddings, submanifolds, critical points, the Sard's theorem) and vector fields and the associated dynamical systems. The following three chapters make up a concise introduction to Riemannian geometry, covering most of the standard material (Riemannian metrics, connections, geodesics, the exponential map, minimal geodesics, the Riemannian distance, Riemannian curvature, Riemannian submanifolds, sectional and Ricci curvature, Jacobi fields and conjugate points,manifolds of constant curvature). Chapter 6, *Tensors and Differential Forms*, is devoted, essentially, to integration theory of manifolds, as well to the de Rham cohomology. It is also, introduced the singular homology and it is given a proof of the de Rham theorem. Chapter 7 is concerned with some global results in the theory of smooth manifolds and Riemannian geometry (the Brouwer degree, the intersection number, the fixed point index, the Lefschetz number, the Euler characteristic and the Gauss-Bonnet theorem), while the chapter 8 covers the basic notions and results of Morse theory. Finally, the chapter 9 provides a short introduction to the theory of hyperbolic dynamical systems.

There are plenty of worked examples in the book and each chapter ends with a comprehensive list of exercises. Another feature that has to be remarked is the presence of a great number of very suggestive and well realized graphical illustration.

The book is very well written, in a very pedagogical manner and it covers a lot of material in a very clear way. I think this is an ideal introduction to differential geometry and topology for beginning graduate students or advanced undergraduate students in mathematics, but it will be, also, useful to physicists or other scientists with an interests in differential geometry and dynamical systems.

Paul Blaga

Donal O'Regan, Yeol Je Cho and Yu-Qing Chen, *Topological Degree Theory and Applications*, Series in Mathematical Analysis and Applications (R.P. Agarwal and D. O'Regan eds.), Vol. 10, Chapman & Hall/CRC, Taylor & Francis Group, Boca Raton, 2006, 221 pp, ISBN 1-58488-648-X.

The degree theory for continuous maps on finite dimensional spaces was created by Brouwer in 1910-1912, and later, for compact maps on infinite dimensional spaces, by Leray and Schauder in 1934, and it has become one of the most useful tool in nonlinear analysis. Since the 1960s, several extensions have been done for various classes of non-compact type maps. The present book focuses on topological degree theory in normed spaces and its applications to integral, ordinary differential and partial differential equations.

The Contents are as follows: Chapter 1: *Brouwer degree theory*, presents the construction of Brouwer degree, the degree for vanishing mean oscillation functions, and in particular for Sobolev maps, of Brezis and Nirenberg (1995), and applications to periodic and anti-periodic problems for ordinary differential equations in $\mathbf{R}^n$.

Chapter 2: *Leray-Schauder degree theory*, starts with the basic result on the approximation of a compact map by finite dimensional maps and presents the Leray-Schauder extension of Brouwer degree to compact maps in Banach spaces. Then a degree theory is described for upper semicontinuous compact maps with closed convex values. Applications are given to bifurcations and to the existence of solutions of the Cauchy problem, the Dirichlet problem for a second order partial differential equation, and anti-periodic problems in Hilbert spaces.

Chapter 3: *Degree theory for set contractive maps*, presents the degree theory for k-set contraction maps and condensing maps and some applications to the initial value problem and anti-periodic problems for ordinary differential equations in Banach spaces.

Chapter 4: *Generalized degree theory for A-proper maps*, is devoted to Petryshyn's generalized degree theory and some typical applications to periodic problem for second order differential equations and semilinear wave equation.

Chapter 5: *Coincidence degree theory*, introduces Mawhin's degree for L-compact maps and gives application to periodic ordinary differential equations.

Chapter 6: *Degree theory for monotone-type maps*, presents basic contributions of Skrypnik, Browder, Berkovitz, Mustonen, Kartsatos and others, to the construction of the degree for monotone-type maps. Applications to evolution equations are included.

Chapter 7: *Fixed point index theory*, is in connection with the problem of the existence of non-negative solutions (in a cone) to operator equations. After defining the fixed point index, the authors present a variety of fixed point theorems of compression-expansion type in cones of Banach spaces and give applications to integral and differential equations.

The book is very well written, presents essential ideas and results with typical applications, being extremely useful to the beginners in nonlinear analysis. Each chapter of the book is concluded by a section of exercises and the bibliography contains 314 titles.

This is really a valuable text for self-study and special courses in nonlinear analysis and also a good reference for anyone applying topological methods to integral, ordinary and partial differential equations.

Radu Precup

Rajendra Bhatia, ***Positive Definite matrices,*** Princeton Series in Applied Mathematics, Princeton University Press, Princeton and Oxford 2007, ix + 254 pp., ISBN 0-691-12918-5.

Denote by $\mathcal{H}$ be the n-dimensional Hilbert space $\mathbb{C}^n$ with inner product $\langle x, y\rangle$. Let $\mathcal{L}(\mathcal{H})$ be the space of all linear operators on $\mathcal{H}$ and $\mathbb{M}_n = \mathbb{M}_n(\mathbb{C})$ the space of $n\times n$-matrices over $\mathbb{C}$. An operator $A \in \mathcal{L}(\mathcal{H})$ is identified with the associated matrix, denoted also by A, with respect to the standard basis $\{e_j\}$ of $\mathbb{C}^n$. A matrix A in $\mathbb{M}_n$ is called positive if $\langle x, Ax\rangle \geq 0$ for all $x \in \mathbb{C}^n$, and positive definite (or strictly positive) if $\langle x, Ax\rangle > 0$ for all $x \neq 0$.

The present book is devoted to the study of positive matrices, positive linear maps, and positive definite functions. This is a rich field with numerous interesting results and with deep and far reaching applications. One of the domains of application, of great interest in the last time, is quantum information theory, where the quantum communication channels are thought as completely positive trace preserving linear maps. The author presents in the fourth chapter two fundamental results in quantum entropy - the inequalities of Lieb-Ruskai and Furuta's inequality. Recall that the quantum entropy of a positive definite matrix A was defined in 1927 by J. von Neumann by the formula $S(A) = -\mathrm{tr}(A\log A)$.

The first chapter of the book, 1. *Positive matrices,* presents the basic notions and results: characterizations of positivity, the Schur product, block matrices. Note that, as it was shown by T. Ando and M.-D. Choi in 1986, the $2\times$ 2-block matrices play a crucial role in the proofs of many results on positive matrices, a point made very clear by the author of the book too.

Although many results in Chapters 2. *Positive linear maps,* and 3. *Completely positive maps,* hold in the more general framework of C^*-algebras, the presentation is restricted to their finite dimensional versions (called "toy versions" by the author), which are sufficient for matrix theory and for the applications as well.

As we did mention, in Chapter 4. *Matrix means,* some spectacular applications of various means for matrices to convex matrices and to quantum information theory are presented.

Chpater 5. *Positive definite functions,* is devoted to the study of positive definite functions from $\mathbb{R}$ to $\mathbb{C}$. There are proved the fundamental theorems of Herglotz and Bochner and applications to various matrix inequalities and to the study of Loewner matrices are given.

In the last chapter of the book, 6. *Geometry of positive matrices,* the set of positive matrices is studied as a Riemannian manifold of nonpositive curvature, a domain of very active current research, mainly due to the results of M. Gromov. This is a promising area of investigation, of great interest to analysts and geometers as well.

Written by an expert in the area, the book presents in an accessible manner a lot of important results from the realm of positive matrices and of their applications. Although, in some places, references to the author's book, *Matrix Analysis* (MA), Springer 1997, are made, the present one is practically self-contained and can be read independently of MA.

The book can be used for graduate courses in linear algebra, or as supplementary material for courses in operator theory, and as a reference book by engineers and researchers working in the applied field of quantum information.

S. Cobzaş

M. de Gosson, *Symplectic Geometry and Quantum Mechanics*, Birkhäuser (Operator Theory and Applications, vol. 166), 2006, Hardback, 367 pp., ISBN-10: 3-7643-7574-4, ISBN-13: 978-3-7643-7574-4.

Hamiltonian formalism lies at the very heart of quantum mechanics. In the recent decades, Hamiltonian mechanics "happily married" differential geometry, giving birth to one of the most beautiful parts of geometry, symplectic geometry. This kind of geometry was quite successfully applied to quantum mechanics in the so-called geometric quantization approach. Several monographs on geometric quantization are available by now, but the focus mainly on the geometrical formalism without doing justice to quantum mechanics. It is the aim of this book to correct this deficiency.

The book has three parts. The first one is a detailed exposition of the basic notions of symplectic geometry, as well as of those of an extension of it, the so-called multiply-oriented symplectic geometry, or q-symplectic geometry. In particular, there are studied a series of indices, essential in this field (e.g. the Arnold-Leray-Maslov and the Conley-Zehnder indices).

The second part is dedicated to the Heisenberg group, the Weyl calculus and metaplectic group, while the final part is more physically-oriented. It begins with a geometrical approach to the uncertainty principle from quantum mechanics and its connections to the symplectic capacity. It follows a rigorous treatment of the density matrix, by using the Hilbert-Schmidt and trace-class operators and, finally, the Weyl pseudo-differential calculus is extended to the phase space, via the Stone-von Neumann theorem on the irreducible representation of the Heisenberg group. Several appendices review some standard mathematical material (classical Lie groups, covering spaces, pseudo-differential operators, elementary probability theory).

The book is very clearly written, by one of the most active researchers in the field, and, in my opinion, it successfully manages to fill a gap in the mathematical physics literature. It will be very useful for graduate students and researchers both in theoretical physics and geometry.

Paul A. Blaga

A. Mallios, *Modern Differential Geometry in Gauge Theories, Maxwell Fields*, volume I, Birkhäuser, 2006, 293 pp., Paperback, ISBN-10: 0-8176-4378-8, ISBN-13: 978-0-8176-4378-2.

Much of the differential geometry of a smooth manifold X can be built staring from a small number of objects: the sheaf of smooth functions $\mathcal{C}_X^\infty$, the sheaf of differential forms and the differential associating to functions 1-forms. For instance, vector bundles are just projective, locally free $\mathcal{C}_X^\infty$-modules, the connections can also be constructed easily out of the three mentioned objects.

This book starts with a more general framework: a space X (which is not necessarily a manifold), two sheaves $\mathcal{A}$ and $\mathcal{E}$ on X and a sheaf morphism $\partial\mathcal{A} \to \mathcal{E}$, having similar properties to those of the ordinary differential of functions (linearity and a kind of Leibniz property). Such a triple is called a *triad* on the space X. There are introduced, then, some generalizations of vector bundles, through the so-called *vector sheaves*, which are just locally-free sheaves of $\mathcal{A}$-modules.

All the constructions from differential geometry (connections, metrics, curvature and torsion tensors) can be carried out in this generalized context. The theory obtained is called *abstract differential geometry* (ADG).

The book under review is the first volume of a two-volume work dedicated to the applications of ADG to gauge theories. This first volume focuses only on electromagnetic fields (Maxwell theory). It first gives a review of ADG, then it recasts the classification of elementary particles by the spin structure in terms of sheaves. The next two chapters are devoted to electromagnetic fields and their classification in terms of sheaf cohomology. The final chapter is dedicated to the reformulation of geometric quantization in the language of abstract differential geometry.

Many of the results from this monographs belong to the author or to his collaborators. He, is, in fact, one of the founders of ADG. The book is very well written and it brings a fresh approach to gauge theories, that will probably be of a great help both to theoretical physicists and geometers.

Paul Blaga