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PROFESSOR PETRU BLAGA AT HIS 60^{TH} ANNIVERSARY

OCTAVIAN AGRATINI AND GHEORGHE COMAN

Petru Blaga was born on 7^{th} September 1947 in Oradea, a rich cultural city at the western border of Romania. He attended elementary school (1954-1958) in Roit village, secondary school (1958-1961) in Sînnicolau Român village and high-school (1962-1968) in Salonta city.

Between 1966 and 1971 he was student at the Faculty of Mathematics and Mechanics (nowadays the Faculty of Mathematics and Computer Science), Babeş-Bolyai University. During his student years he had the privilege to be under the influence of three refined masters: Tiberiu Popoviciu (1906-1975), D.V. Ionescu (1901-1985) and D.D. Stancu (born on 1927). After graduation he was hired at Faculty of Mathematics and Mechanics as probation assistant (1971-1975) at the Chair of Numerical and Statistical Calculus. Holding a continuous academic career at this department, he successively advanced assistant (1975), lecturer (1990), associate professor (1993) and, finally, full professor (1995).

Petru Blaga has obtained PhD in 1983, the scientific advisor being professor D.D. Stancu.

The major coordinates of his private life: he got married in 1971, his wife Livia is pedagogue and they have two boys, Alin born in 1973 and Daniel born in 1974. The first of them works in computer science field in Toulouse, France, and the second is economist in Cluj.

His devotion to academic life is certified by the following features. As teacher, Blaga Petru had given courses on Computer Science, Numerical Analysis, Probability Theory, Statistics. As member of Babeş-Bolyai community, he was the manager of Applied Mathematics Department (2002-2004), the head of the Chair of Numerical and Statistical Calculus (2002-2004) and starting from 2004 he is in the position of Dean of our faculty. This way, we found out that the patience is another specific feature of our colleague. As PhD scientific advisor, so far, five students completed their doctoral studies: Ban Ioan, Barnabas Bede, Breaz Nicoleta, Crainic Nicolae Ioan, Otrocol Diana.

Professor Petru Blaga is member of the Editorial Board of the journals: *Studia Universitatis Babeş-Bolyai, Mathematica* and *Studia Universitatis Babeş-Bolyai, Informatica*. Since 1993 he is reviewer at *Mathematical Reviews* and member of the American Mathematical Society. Based on his abilities, he was an active member of scientific and organizing committees of many international meetings held in Romania.

On behalf of all colleagues of our faculty, we warmly congratulate Professor Petru Blaga on his 60^{th} birthday wishing him health and achievements in his further work.

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SOME CUBATURES WITH CHEBYSHEV NODES

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Dedicated to Professor Petru Blaga at his 60^{th} anniversary

Abstract. In this article we construct boolean cubature formulas using univariate Lagrange interpolation projectors with Chebyshev nodes of second type. We compute the coefficients of these cubature formulas using coefficients of corresponding Fejer-Clenshaw-Curtis quadratures. The remainder terms have minim properties in a class of cubature formulas with the same number of nodes. Some numerical examples are given.

1. **Preliminaries**

First, we present the construction of Biermann projector and some properties from [2].

Let be the univariate Lagrange interpolation projectors

$$P_1, \ldots, P_r, Q_1, \ldots, Q_r$$

given by

$$\begin{aligned} (P_m f_1)(x) &= \sum_{i=1}^{k_m} l_{im}(x) f_1(x_i), \quad 1 \le m \le r \\ (Q_n f_2)(y) &= \sum_{j=1}^{l_n} \widetilde{l}_{jn}(y) f_2(y_j), \quad 1 \le n \le r \end{aligned} \tag{1}$$

where $f_1 : [a, b] \to \mathbb{R}$ and $f_2 : [c, d] \to \mathbb{R}$.

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The sets of interpolations nodes

$$\{x_1, \dots, x_{k_m}\} \subseteq [a,b], \quad \{y_1, \dots, y_{l_n}\} \subseteq [c,d]$$

satisfy the conditions

$$1 \le k_1 < k_2 < \cdots < k_r, \quad 1 \le l_1 < l_2 < \cdots < l_r. \tag{2}$$

The cardinal functions of Lagrange interpolation are given by

$$l_{im}(x) = \prod_{\substack{k=1\\k\neq i}}^{k_m} \frac{x - x_k}{x_i - x_k}, \quad 1 \le i \le k_m$$

$$\widetilde{l}_{jn}(y) = \prod_{\substack{l=1\\l\neq j}}^{l_n} \frac{y - y_l}{y_j - y_l}, \quad 1 \le j \le l_n.$$

If $f : [a,b] \times [c,d] \to \mathbb{R}$, $f \in C([a,b] \times [c,d])$ then we have the parametric extensions, which are bivariate projectors

$$(P'_m f)(x,y) = \sum_{i=1}^{k_m} l_{im}(x) f(x_i, y),\ 1 \le m \le r$$

$$(Q'_n f)(x,y) = \sum_{j=1}^{l_n} \widetilde{l}_{jn}(y) f(x, y_j),\ 1 \le n \le r$$

From (2) it follows that the parametric extensions

$$P'_1, \dots, P'_r, Q''_1, \dots, Q''_r$$

are bivariate interpolation projectors which form the chains

$$P'_1 \le \cdots \le P'_r, \quad Q''_1 \le \cdots \le Q''_r. \tag{3}$$

where relation order "$\le$" is defined by: $P \le Q$ if and only if $PQ = P$.

The interpolation projector B_r defined by relation

$$B_r = P'_1 Q''_r \oplus \cdots \oplus P'_r Q''_1, \quad r \in \mathbb{N} \tag{4}$$

is called Biermann interpolation projector.

The remainder operator in Bierman interpolation is

$$B_r^c = P_r^{'c} + P_{r-1}^{'c} Q_1^{''c} + \cdots + P_1^{'c} Q_{r-1}^{''c} + Q_r^{''c} - P_1^{'c} Q_r^{''c} - \cdots - P_r^{'c} Q_1^{''c} \tag{5}$$

where $P^c = I - P$ and I is identity operator.

The Biermann interpolation projector has the representation

$$B_r(f) = \sum_{m=1}^{r} \sum_{n=0}^{r-m} \sum_{i=1+k_{m-1}}^{k_m} \sum_{j=1+l_{r-m-n}}^{l_{r+1-m-n}} \Phi_{ij} f(x_i, y_j). \tag{6}$$

where the cardinal functions of Biermann interpolations are given by

$$\Phi_{ij}(x,y) = \sum_{s=m}^{m+n} l_{is}(x)\widetilde{l}_{j,r+1-s}(y) - \sum_{s=m}^{m+n-1} l_{is}(x)\widetilde{l}_{j,r-s}(y) \tag{7}$$

$$k_{m-1} < i \le k_m, \quad k_{r-m-n} < j \le l_{r+1-m-n}, \quad 0 \le n \le r-m, \quad 1 \le m \le r.$$

If $f \in C^{k_r,l_r}([a,b] \times [c,d])$ then for remainder term, we have Cauchy representation

$$f(x,y) - (B_r f)(x,y) \tag{8}$$

$$= (x-x_1)\dots(x-x_{k_r})\frac{f^{(k_r,0)}(\xi_r,y)}{k_r!} + (y-y_1)\dots(y-y_{l_r})\frac{f^{(0,l_r)}(x,\eta_r)}{l_r!}$$

$$+\sum_{m=1}^{r-1}\prod_{i=1}^{k_{r-m}}(x-x_i)\prod_{j=1}^{l_m}(y-y_j)\frac{f^{(k_{r-m},l_m)}(\xi_{r-m},\eta_m)}{k_{r-m}!l_m!}$$

$$-\sum_{m=1}^{r}\prod_{i=1}^{k_{r+1-m}}(x-x_i)\prod_{j=1}^{l_m}(y-y_j)\frac{f^{(k_{r+1-m},l_m)}(\sigma_{r+1-m},\tau_m)}{k_{r+1-m}!l_m!}.$$

where $\xi_i, \sigma_i \in [a,b]$, $\eta_i, \tau_i \in [c,d]$, $1 \le i \le r$

Next we obtain cubature formulas by integrating Biermann interpolation formula with P_i and Q_j univariate Lagrange interpolation projectors with Chebyshev nodes of second type.

2. **First type cubature**

We consider the following univariate Lagrange interpolation projectors

$$(P_m f_1)(x) = \sum_{i=1}^{2^m-1} l_{im}(x) f_1(x_{im}), \quad f_1 \in C[-1,1],\ 1 \le m \le r$$

$$(Q_n f_2)(y) = \sum_{j=1}^{2^n-1} \widetilde{l}_{jn}(y) f_2(y_{jn}), \quad f_2 \in C[-1,1],\ 1 \le n \le r$$

with Chebyshev nodes of second type

$$x_{im} = \cos\frac{i\pi}{2^m}, \quad i = \overline{1, 2^m - 1},\ m = \overline{1, r}$$

$$y_{jn} = \cos\frac{j\pi}{2^n}, \quad j = \overline{1, 2^n - 1},\ n = \overline{1, r}.$$

The cardinal functions are given by

$$l_{im}(x) = \prod_{\substack{k=1\\k\neq i}}^{2^m-1} \frac{x - x_{km}}{x_{im} - x_{km}}$$

$$\widetilde{l}_{jn}(y) = \prod_{\substack{l=1\\l\neq j}}^{2^n-1} \frac{y - y_{ln}}{y_{jn} - y_{ln}}.$$

We construct the sets of nodes

$$(u_k)_{k=\overline{1,2^r-1}}, \quad u_k = \cos\frac{2i+1}{2^j}\pi, \quad j = \overline{1, r},\ i = \overline{0, 2^{j-1} - 1}, \quad k = 2^{j-1} + i$$

$$(v_l)_{l=\overline{1,2^r-1}}, \quad v_l = u_l, \quad l = \overline{1, 2^r - 1}.$$

If $f \in C([-1,1]\times[-1,1])$ we have the Biermann interpolation formula

$$f = B_r f + Rf \tag{9}$$

where

$$B_r = P_1' Q_r'' \oplus P_2' Q_{r-1}'' \oplus \cdots \oplus P_r' Q_1''$$

and remainder operator

$$R = P_r^{'c} + P_{r-1}^{'c} Q_1^{''c} + \cdots + P_1^{'c} Q_{r-1}^{''c} + Q_r^{''c} - P_1^{'c} Q_r^{''c} - \cdots - P_r^{'c} Q_1^{''c}.$$

The representation of Biermann interpolation projector is

$$B_r f = \sum_{m=1}^{r} \sum_{n=0}^{r-m} \sum_{i=2^{m-1}}^{2^m-1} \sum_{j=2^{r-m-n}}^{2^{r-m-n+1}-1} \Phi_{ij} f(u_i, v_j)$$

where

$$\Phi_{ij}(x,y) = \sum_{s=m}^{m+n} l_{is}(x)\widetilde{l}_{j,r+1-s}(y) - \sum_{s=m}^{m+n-1} l_{is}(x)\widetilde{l}_{j,r-s}(y)$$

$$2^{m-1} < i \le 2^m - 1,\ 2^{r-m-n} - 1 < j \le 2^{r-m-n+1},\ 0 \le n \le r - m,\ 1 \le m \le r.$$

By integrating Biermann interpolation formula (9) on domain $D = [-1,1] \times [-1,1]$ we obtain boolean cubature formula

$$\int_{-1}^{1}\int_{-1}^{1} f(x,y)dxdy = \sum_{m=1}^{r}\sum_{n=0}^{r-m}\sum_{i=2^{m-1}}^{2^m-1}\sum_{j=2^{r-m-n}}^{2^{r-m-n+1}-1} C_{ij} f(u_i, v_j) + R(f) \tag{10}$$

with

$$C_{ij} = \sum_{s=m}^{m+n} A_{is}B_{j,r+1-s} - \sum_{s=m}^{m+n-1} A_{is}B_{j,r-s}$$

$$2^{m-1}-1 < i \le 2^m - 1,\ 2^{r-m-n}-1 < j \le 2^{r-m-n+1}-1,\ 0 \le n \le r-m,\ 1 \le m \le r$$

where the numbers A_{is} and B_{js} are coefficients of some Fejer-Clenshaw-Curtis quadratures

$$A_{\sigma_s(i),s} = \frac{4\sin\dfrac{i\pi}{2^s}}{2^s}\sum_{j=0}^{2^{s-1}-1}\frac{\sin\left(\dfrac{i(2j+1)}{2^s}\pi\right)}{2j+1}, \quad i=\overline{1,2^s-1},\ s=\overline{1,r}$$

$$B_{js} = A_{js}, \quad j = \overline{1,2^s-1},\ s=\overline{1,r}.$$

σ_s being the permutations of numbers $1,...,2^s-1$ so that $x_{\sigma_s(i),s} = u_i$.

For remainder term of cubature (10), we have the following estimations

$$|R(f)| \le \frac{2M_{2^r-1,0}f}{(2^r-1)!}\int_{-1}^{1}|u_r(x)|dx + \frac{2M_{0,2^r-1}f}{(2^r-1)!}\int_{-1}^{1}|v_r(y)|dy$$

$$+\sum_{m=1}^{r-1}\frac{M_{2^{r-m}-1,2^m-1}f}{(2^{r-m}-1)!(2^m-1)!}\int_{-1}^{1}|u_{r-m}(x)|dx\int_{-1}^{1}|v_m(y)|dy$$

$$+\sum_{m=1}^{r}\frac{M_{2^{r+1-m}-1,2^m-1}f}{(2^{r+1-m}-1)!(2^m-1)!}\int_{-1}^{1}|u_{r+1-m}(x)|dx\int_{-1}^{1}|v_m(y)|dy$$

where

$$u_m(x) = (x-x_{1m})\dots(x-x_{2^m-1,m})$$

$$v_n(y) = (y-y_{1n})\dots(y-y_{2^n-1,n})$$

$$M_{ij}f = \sup_{(x,y)\in[-1,1]\times[-1,1]} f^{(i,j)}(x,y)$$

We notice that

$$\int_{-1}^{1}|u_m(x)|dx = \min_{c_1,\dots,c_{2^m-1}}\int_{-1}^{1}|(x-c_1)\dots(x-c_{2^m-1})|dx$$

and

$$\int_{-1}^{1} |v_n(y)|dy = \min_{d_1,\dots,d_{2^n-1}} \int_{-1}^{1} |(y-d_1)\dots(y-d_{2^n-1})|dy.$$

Next, we propose to approximate the double integral

$$\int_{-1}^{1}\int_{-1}^{1} e^{(-x^2-y^2)}dxdy$$

The approximative value given by Maple is

$$va := 2.230985141$$

In table are given the results obtained using cubature formula (10)

r	The approximation	The absolut error
2	1.901496850	0.329488291
3	2.249623509	0.018638368
4	2.230704125	0.000281016
5	2.230983295	0.000001846
6	2.230985150	0.000000009

3. **Second type cubature**

Let be the Chebyshev nodes of second type

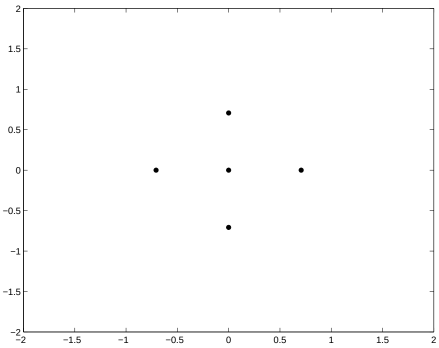
$$x_{im} = \cos\frac{i\pi}{N^m}, \quad i = \overline{1, N^m - 1},\ m = 1, 2$$

$$y_{jn} = \cos\frac{j\pi}{N^n}, \quad j = \overline{1, N^n - 1},\ n = 1, 2$$

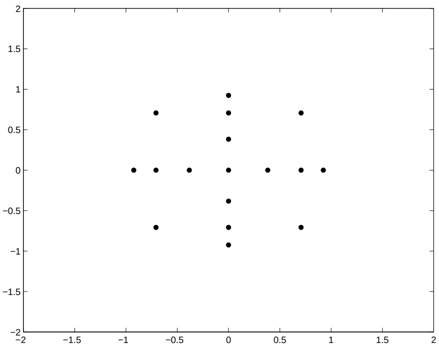
and univariate Lagrange interpolation projectors

$$(P_m f_1)(x) = \sum_{i=1}^{N^m-1} l_{im}(x) f_1(x_{im}), \quad m = 1, 2,\ f_1 \in C[-1, 1]$$

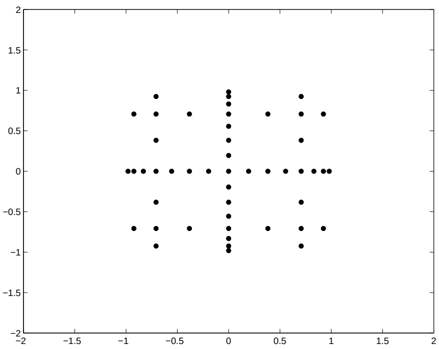
$$(Q_n f_2)(y) = \sum_{j=1}^{N^n-1} l_{jn}(y) f_2(y_{jn}), \quad n = 1, 2,\ f_2 \in C[-1, 1].$$



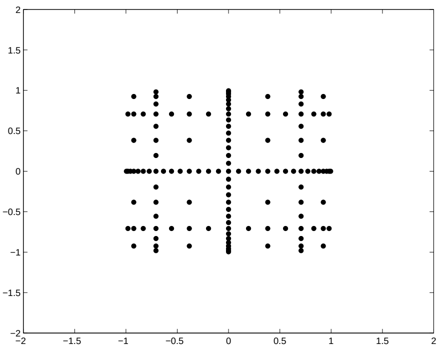
(a) r=2



(b) r=3



(c) r=4



(d) r=5

FIGURE 1. The distribution of nodes in cubature formula (10)

The cardinal functions are given by

$$l_{im}(x) = \prod_{\substack{i=1 \\ i\neq k}}^{N^m-1} \frac{x - x_{km}}{x_{im} - x_{km}}$$

$$\widetilde{l}_{jn}(y) = \prod_{\substack{l=1 \\ l\neq j}}^{N^n-1} \frac{y - y_{ln}}{y_{jn} - y_{ln}}.$$

We construct the sets of nodes $(u_k)_{k=\overline{1,N^2-1}}$, $(v_l)_{l=\overline{1,N^2-1}}$

$$u_1 = \cos\frac{\pi}{N},\ u_2 = \cos\frac{2\pi}{N}, \ldots, u_{N-1} = \cos\frac{N-1}{N}\pi,$$

$$u_N = \cos\frac{\pi}{N^2},\ u_{N+1} = \cos\frac{2\pi}{N^2}, \ldots, u_{2N-2} = \cos\frac{N-1}{N^2}\pi,$$

$$u_{2N-1}=\cos\frac{N+1}{N^2}\pi,\ u_{2N}=\cos\frac{N+2}{N^2}\pi,\dots,u_{3N-3}=\cos\frac{2N-1}{N^2}\pi$$

$$\dots$$

$$u_{N^2-N+1}=\cos\frac{N^2-N+1}{N^2}\pi,\dots,u_{N^2-1}=\cos\frac{N^2-1}{N^2}\pi$$

and

$$v_l=u_l,\quad l=\overline{1,N^2-1}$$

If $f\in C([-1,1]\times[-1,1])$ then we have discrete blending interpolation formula

$$f=B_2f+Rf \tag{11}$$

where

$$B_2=P_1'Q_2''\oplus P_2'Q_1''$$

and remainder operator

$$R=P_2^{'c}+P_1^{'c}Q_1^{''c}+Q_2^{''c}-P_1^{'c}Q_2^{''c}-P_2^{'c}Q_1^{''c}.$$

The blending discrete interpolant has the representation

$$B_2f=\sum_{m=1}^{2}\sum_{n=0}^{2-m}\sum_{i=N^{m-1}}^{N^m-1}\sum_{j=N^{2-m-n}}^{N^{3-m-n}-1}\Phi_{ij}f(u_i,v_j)$$

where

$$\Phi_{ij}(x,y)=\sum_{s=m}^{m+n}l_{is}(x)\widetilde{l}_{j,r+1-s}(y)-\sum_{s=m}^{m+n-1}l_{is}(x)\widetilde{l}_{j,r-s}(y)$$

$N^{m-1}-1<i\le N^m-1,\ N^{2-m-n}-1<j\le N^{3-m-n}-1,\ 0\le n\le 2-m,\ m=1,2.$

By integrating the interpolation formula (11) on domain $D=[-1,1]\times[-1,1]$ we obtain boolean cubature formula

$$\int_{-1}^{1}\int_{-1}^{1}f(x,y)dxdy=\sum_{m=1}^{2}\sum_{n=0}^{2-m}\sum_{i=N^{m-1}}^{N^m-1}\sum_{j=N^{2-m-n}}^{N^{3-m-n}-1}C_{ij}f(u_i,x_j)+R(f) \tag{12}$$

with

$$C_{ij}=\sum_{s=m}^{m+n}A_{is}B_{j,r+1-s}-\sum_{s=m}^{m+n-1}A_{is}B_{j,r-s}$$

$N^{m-1}-1<i\le N^m-1,\ N^{2-m-n}-1<j\le N^{3-m-n}-1,\ 0\le n<2-m,\ m=1,2$

where the numbers A_{is} and B_{js} are coefficients of some quadratures of Fejer-Clenshaw-Curtis type

$$A_{\sigma_s(i),s} = \frac{4\sin\dfrac{i\pi}{N^s}}{N^s}\sum_{j=0}^{[\frac{N^s}{2}]-1}\frac{\sin\dfrac{i(2j+1)\pi}{N^s}}{2j+1}, \quad i=\overline{1,N^s-1},\ s=\overline{1,2}$$

$$B_{js} = A_{js}, \quad j=\overline{1,N^s-1},\ s=\overline{1,2}.$$

σ_s being the permutations of the numbers $1,...,N^s-1$ so that $x_{\sigma_s(i),s} = u_i$.

For the remainder term of cubature (12), we have the following estimation

$$|R(f)| \leq \frac{2M_{N^2-1,0}f}{(N^2-1)!}\int_{-1}^{1}|u_2(x)|dx + \frac{2M_{0,N^2-1}f}{(N^2-1)!}\int_{-1}^{1}|v_2(y)|dy$$

$$+\frac{M_{N^2-1,N-1}f}{(N^2-1)!(N-1)!}\int_{-1}^{1}|u_2(x)|dx\int_{-1}^{1}|v_1(y)|dy$$

$$+\frac{M_{N-1,N^2-1}f}{(N-1)!(N^2-1)!}\int_{-1}^{1}|u_1(x)|dx\int_{-1}^{1}|v_2(y)|dy$$

$$+\frac{M_{N-1,N-1}f}{(N-1)!(N-1)!}\int_{-1}^{1}|u_1(x)|dx\int_{-1}^{1}|v_1(y)|dy$$

where

$$u_m(x) = (x-x_{1m})\dots(x-x_{N^m-1,m})$$

$$v_n(y) = (y-y_{1n})\dots(y-y_{N^n-1,n})$$

$$M_{ij}f = \sup_{(x,y)\in[-1,1]\times[-1,1]} f^{(i,j)}(x,y).$$

We notice that

$$\int_{-1}^{1}|u_m(x)|dx = \min_{c_1,\dots,c_{N^m-1}}\int_{-1}^{1}|(x-c_1)\dots(x-c_{N^m-1})|dx$$

and

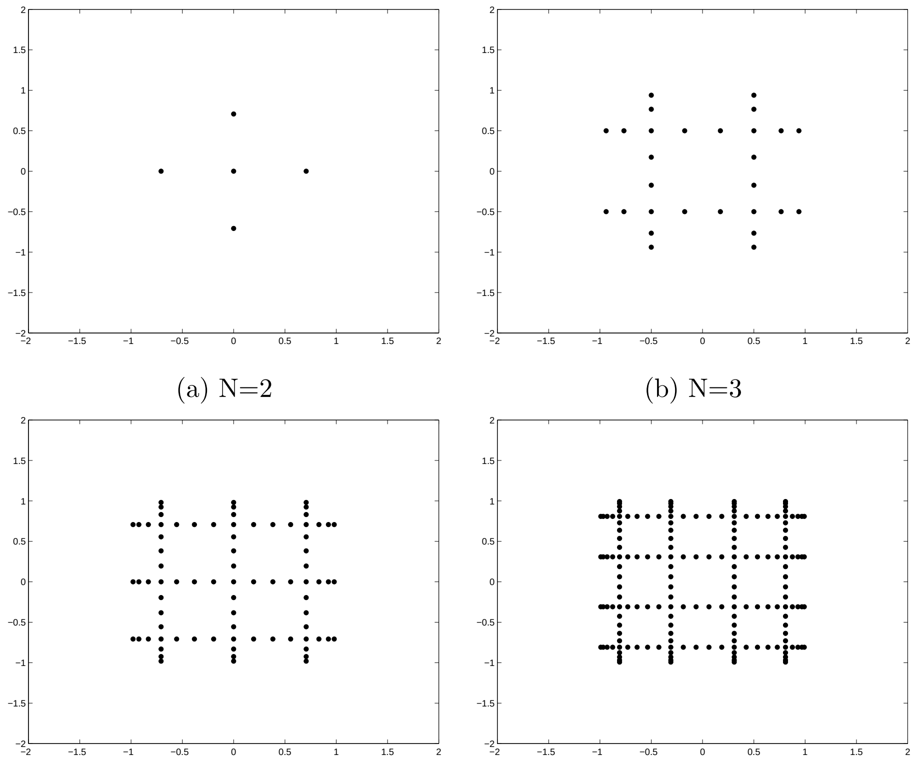
$$\int_{-1}^{1}|v_n(y)|dy = \min_{d_1,\dots,d_{N^n-1}}\int_{-1}^{1}|(y-d_1)\dots(y-d_{N^n-1})|dy.$$

We approximate the same double integral from previous section

$$\int_{-1}^{1}\int_{-1}^{1} e^{(-x^2-y^2)}dxdy$$

In table are given the results which are obtained using cubature formula (12)

r	The approximation	The absolut error
2	1.901496850	0.329488291
3	2.226829679	0.004155462
4	2.230651197	0.000333944
5	2.230936110	0.000049031
6	2.230983949	0.000001192



(a) N=2 (b) N=3

(c) N=4 (d) N=5

FIGURE 2. Distribution of nodes in cubature formula (12).

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SOME INTERPOLATION OPERATORS ON A SIMPLEX DOMAIN

TEODORA CĂTINAŞ AND GHEORGHE COMAN

Dedicated to Professor Petru Blaga at his 60^{th} anniversary

Abstract. This paper studies with certain operators, their product and boolean sum, which interpolate a given function on the triangle and tetrahedron. The interpolation properties and the degree of exactness for these operators are studied. Also, the remainders of the corresponding interpolation formulas are studied.

Our goal is to study some interpolation formulas for bivariate and trivariate functions.

Bivariate case. We consider the standard triangle

$$T_h = \{(x, y) \in \mathbb{R} \mid x \geq 0, y \geq 0, x + y \leq h, h \in \mathbb{R}_+\}$$

and the function $f : T_h \to \mathbb{R}$.

Let P_1, P_2 and P_3 be the operators that interpolate the function f at two edges of the triangle T_h, as seen in Figure 1.

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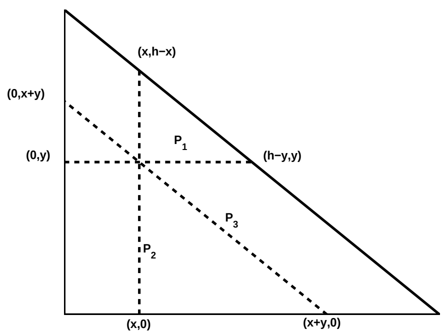


Triangle T_h.

Figure 1.

We have

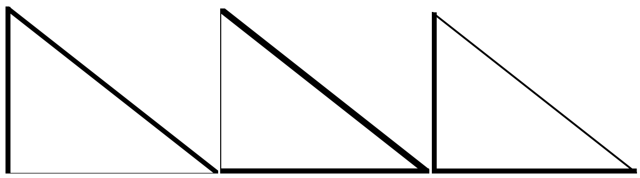
$$
\begin{aligned}
(P_1 f)(x,y) &= \frac{h-x-y}{h-y} f(0,y) + \frac{x}{h-y} f(h-y,y),\\
(P_2 f)(x,y) &= \frac{h-x-y}{h-x} f(x,0) + \frac{y}{h-x} f(x,h-x),\\
(P_3 f)(x,y) &= \frac{x}{x+y} f(x+y,0) + \frac{y}{x+y} f(0,x+y).
\end{aligned}
$$

As seen, these operators have the following interpolation properties:

$$
\begin{array}{ll}
(P_1 f)(0,y) = f(0,y), & y \in [0,h]\\
(P_1 f)(h-y,y) = f(h-y,y), & y \in [0,h]\\
(P_2 f)(x,0) = f(x,0) & x \in [0,h]\\
(P_2 f)(x,h-x) = f(x,h-x), & x \in [0,h]\\
(P_3 f)(x,0) = f(x,0) & x \in [0,h]\\
(P_3 f)(0,y) = f(0,y), & y \in [0,h]
\end{array}
$$

These properties are illustrated in Figure 2.

Remark 1. *In the following figures, we use bold lines and points to indicate the interpolation domains of the corresponding operators.*



P_1 P_2 P_3

Figure 2.

Remark 2. *The degree of exactness of each of the operators P_1, P_2 and P_3 is 1, i.e.,*

$$\operatorname{dex}(P_1) = 1, \quad \operatorname{dex}(P_2) = 1, \quad \operatorname{dex}(P_3) = 1.$$

Our goal is to study some interpolation formulas generated by the operators P_1, P_2 and P_3.

1. Let us consider the interpolation formula generated by P_1 :

$$f = P_1 f + R_1 f,$$

where $R_1 f$ denotes the remainder. Regarding this remainder, we have the following result.

Theorem 3. *If $f \in B_{11}(0,0)$ ($B_{11}(0,0)$ denotes the Sard space, see e.g., [9]) then*

$$(R_1 f)(x,y) = \frac{x(x+y-h)}{2} f^{(2,0)}(\xi,0) + \frac{xy(h-x-y)}{h-y}[f^{(1,1)}(\xi_1,\eta_1) - f^{(1,1)}(\xi_2,\eta_2)], \tag{1}$$

with $\xi \in [0,h]$, $(\xi_1,\eta_1) \in [0,x] \times [0,y]$ and $(\xi_2,\eta_2) \in [x,h-y] \times [0,y]$, and

$$|(R_1 f)(x,y)| \leq \frac{h}{8}\left[\left\|f^{(2,0)}(\cdot,0)\right\|_{L_\infty[0,h]} + \left\|f^{(1,1)}\right\|_{L_\infty(T_h)}\right], \text{ respectively} \tag{2}$$

for all $(x,y) \in T_h$.

Proof. We have

$$\operatorname{dex}(P_1) = 1,$$

which implies that

$$\ker(R_1) = \mathbb{P}_1^2,$$

where $\mathbb{P}_1^2$ denotes the set of bivariate polynomials of degree at most 1. Therefore, by Peano's Theorem it follows that

$$(R_1 f)(x,y) = \int_0^h K_{20}(x,y,s) f^{(2,0)}(s,0)ds + \int_0^h K_{02}(x,y,t) f^{(0,2)}(0,t)dt$$
$$+ \iint_{T_h} K_{11}(x,y,s,t) f^{(1,1)}(s,t)dsdt,$$

with

$$K_{20}(x,y,s) = (x-s)_+ - \frac{x}{h-y}(h-y-s)_+$$
$$K_{02}(x,y,t) = 0$$
$$K_{11}(x,y,s,t) = (y-t)_+^0[(x-s)_+^0 - \frac{x}{h-y}(h-y-s)_+^0].$$

As

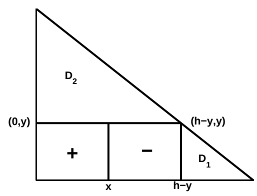
$$\begin{aligned} &K_{20}(x,y,s) \le 0, && s \in [0,h], \\ &K_{11}(x,y,s,t) \ge 0, && (s,t) \in [0,x] \times [0,y], \\ &K_{11}(x,y,s,t) \le 0, && (s,t) \in [x,h-y] \times [0,y], \\ &K_{11}(x,y,s,t) = 0, && (s,t) \in D_1 \cup D_2, \end{aligned}$$

by the mean value theorem one obtains the formula (1). (The domains D_1 and D_2 are represented in Figure 3.)

Taking into account that

$$\max_{(x,y)\in T_h} \frac{x(h-x-y)}{2} = \frac{h^2}{8},$$
$$\max_{(x,y)\in T_h} \frac{xy(h-x-y)}{h-y} = \frac{h^2}{16},$$

the inequality (2) follows. □



The sign of the kernel K_{11}.

Figure 3.

Remark 4. *Analogous formulas are generated by P_2 and P_3.*

2. Let P_iP_j, $i,j=\overline{1,3}$, $i\neq j$ be the product of two of the operators P_1, P_2, P_3, previously given. We have

$$\begin{aligned}
(P_{12}f)(x,y) &= \frac{h-x-y}{h}f(0,0)+\frac{y(h-x-y)}{h(h-y)}f(0,h)+\frac{x}{h-y}f(h-y,y),\\
(P_{13}f)(x,y) &= \frac{h-x-y}{h-y}f(0,y)+\frac{x}{h}f(h,0)+\frac{xy}{h(h-y)}f(0,h),\\
(P_{23}f)(x,y) &= \frac{h-x-y}{h-x}f(x,0)+\frac{y}{h}f(0,h)+\frac{xy}{h(h-x)}f(0,h).
\end{aligned}$$

It is easy to verify the following properties.

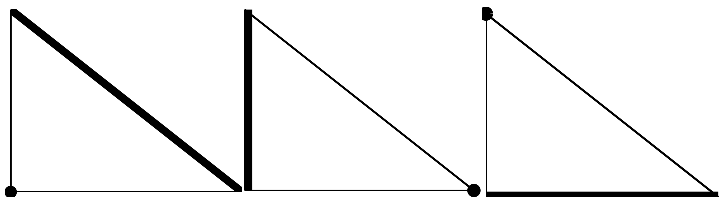
- The interpolation properties:

$$\begin{aligned}
&(P_{12}f)(0,0)=f(0,0), && (P_{12}f)(h-y,y)=f(h-y,y), && y\in[0,h]\\
&(P_{13}f)(h,0)=f(h,0), && (P_{13}f)(0,y)=f(0,y), && y\in[0,h]\\
&(P_{23}f)(0,h)=f(0,h) && (P_{23}f)(x,0)=f(x,0) && x\in[0,h].
\end{aligned}$$

- The degree of exactness is

$$\mathrm{dex}(P_{ij})=1,\qquad i,j=\overline{1,3},\ i\neq j.$$

Remark 5. *The operator P_{ij} has the same interpolation properties as the operator P_{ji}, $i,j=\overline{1,3}$, $i\neq j$. These properties are illustrated in Figure 4.*



P_{12} P_{13} P_{23}

Figure 4.

We consider the interpolation formula generated by P_{12}, namely

$$f=P_{12}f+R_{12}f.$$

Similarly with Theorem 3, for the remainder $R_{12}f$, it is proved the following result.

Theorem 6. *If* $f \in B_{11}(0,0)$ *then*

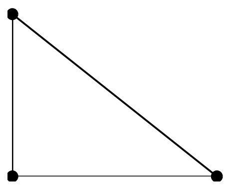
$$|(R_1 f)(x,y)| \leq \frac{h^2}{8}\left[\left\|f^{(2,0)}(\cdot,0)\right\|_\infty + \left\|f^{(0,2)}(0,\cdot)\right\|_\infty + \left\|f^{(1,1)}\right\|_\infty\right],$$

for all $(x,y) \in T_h$.

3. Consider the product operator $P = P_i P_j P_k$, $i,j,k = \overline{1,3}$, $i \neq j \neq k \neq i$, namely

$$(Pf)(x,y) = \frac{h-x-y}{h} f(0,0) + \frac{x}{h} f(h,0) + \frac{y}{h} f(0,h),$$

which interpolates the function f at the vertices of the triangle T_h (as we can see in Figure 5), and $\mathrm{dex}(P) = 1$.



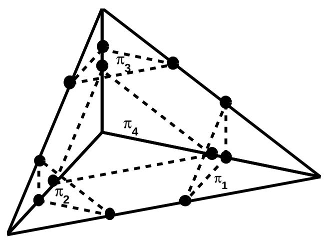
P_{123}

Figure 5.

Trivariate case. As an extension of the previous results we consider the standard tetrahedron

$$\mathcal{T}_h = \left\{(x,y,z) \in \mathbb{R}^3 \mid x,y,z \geq 0,\ x+y+z \leq h,\ h > 0\right\},$$

and $f : \mathcal{T}_h \to \mathbb{R}$. Let π_i be the parallel planes to the tetrahedron faces, as we can see in Figure 6.



Tetrahedron $\mathcal{T}_h$.

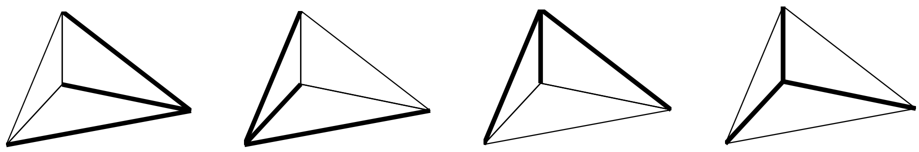
Figure 6.

Denote by Q_i, $i = \overline{1,4}$ the operator that interpolates the function f at the intersection points of the plane π_i with the tetrahedron edges. We have

$$\begin{aligned}
(Q_1 f)(x,y,z) &= \frac{h-x-y-z}{h-y} f(0,y,0) + \frac{x}{h-y} f(h-y,y,0) + \frac{z}{h-y} f(0,y,h-y),\\
(Q_2 f)(x,y,z) &= \frac{h-x-y-z}{h-x} f(x,0,0) + \frac{y}{h-x} f(x,h-x,0) + \frac{z}{h-x} f(x,0,h-x),\\
(Q_3 f)(x,y,z) &= \frac{h-x-y-z}{h-z} f(0,0,z) + \frac{x}{h-z} f(h-z,0,z) + \frac{y}{h-z} f(0,h-z,z),\\
(Q_4 f)(x,y,z) &= \frac{x}{x+y+z} f(x+y+z,0,0) + \frac{y}{x+y+z} f(0,x+y+z,0)\\
&\quad + \frac{z}{x+y+z} f(0,0,x+y+z).
\end{aligned}$$

Theorem 7. *Each operator Q_i, $i = \overline{1,4}$ interpolates the function f at three edges of the tetrahedron (see Figure 7) and it has the degree of exactness equal to* 1.

Proof. The proof is a straightforward computation. □



Q_1 Q_2 Q_3 Q_4

Figure 7.

Next, we shall study the product of two, three and four operators Q_i, $i = \overline{1,4}$.

4. Let us consider the product $Q_{ij} = Q_i Q_j$, $i,j = \overline{1,4}$, $i \neq j$. We have

$$\begin{aligned}
(Q_{12} f)(x,y,z) &= \tfrac{h-x-y-z}{h} f(0,0,0) + \tfrac{x}{h-y} f(h-y,y,0) + \tfrac{y(h-x-y)}{h(h-y)} f(0,h,0) + \tfrac{z}{h} f(0,0,h),\\
(Q_{13} f)(x,y,z) &= \tfrac{h-x-y-z}{h} f(0,0,0) + \tfrac{x}{h} f(h,0,0) + \tfrac{y(h-y-z)}{h(h-y)} f(0,h,0) + \tfrac{z}{h-y} f(0,y,h-y)\\
(Q_{14} f)(x,y,z) &= \tfrac{h-x-y-z}{h-y} f(0,y,0) + \tfrac{x}{h} f(h,0,0) + \tfrac{y(x+z)}{h(h-y)} f(0,h,0) + \tfrac{z}{h} f(0,0,h),\\
(Q_{23} f)(x,y,z) &= \tfrac{h-x-y-z}{h} f(0,0,0) + \tfrac{x(h-x-z)}{h(h-x)} f(h,0,0) + \tfrac{y}{h} f(0,h,0) + \tfrac{z}{h-x} f(x,0,h-x),\\
(Q_{24} f)(x,y,z) &= \tfrac{h-x-y-z}{h-x} f(x,0,0) + \tfrac{x(y+z)}{h(h-x)} f(h,0,0) + \tfrac{y}{h} f(0,h,0) + \tfrac{z}{h} f(0,0,h),\\
(Q_{34} f)(x,y,z) &= \tfrac{h-x-y-z}{h-z} f(0,0,z) + \tfrac{x}{h} f(h,0,0) + \tfrac{y}{h} f(0,h,0) + \tfrac{z(x+y)}{h(h-z)} f(0,0,h).
\end{aligned}$$

The main properties of the operators Q_{ij}, $i, j = \overline{1, 4}$ are:

- $Q_{ij}f$, $i, j = \overline{1, 4}$ interpolates the function f at one edge and two vertices of the tetrahedron $\mathcal{T}_h$, as one can see in Figure 8.
- $Q_{ij}f$ has the same interpolation properties as $Q_{ji}f$, $i, j = \overline{1, 4}$.
- $\mathrm{dex}(Q_{ij}) = 1$, for $i, j = \overline{1, 4}$, $i \neq j$.

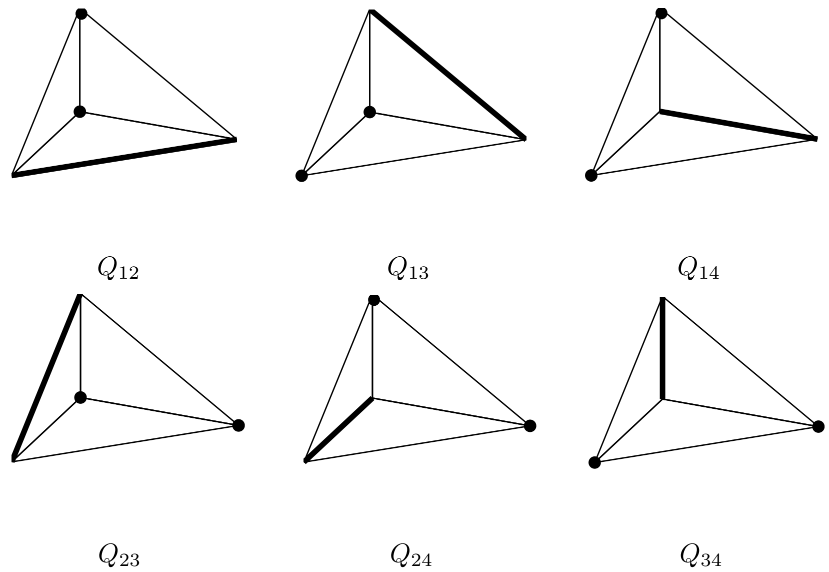


Figure 8.

5. Let Q_{ijk} be the product of Q_i, Q_j and Q_k, $i, j, k = \overline{1, 4}$, $i \neq j \neq k \neq i$. We have

$$(Q_{ijk}f)(x, y, z) = \tfrac{h-x-y-z}{h} f(0, 0, 0) + \tfrac{x}{h} f(h, 0, 0) + \tfrac{y}{h} f(0, y, 0) + \tfrac{z}{h} f(0, 0, h),$$

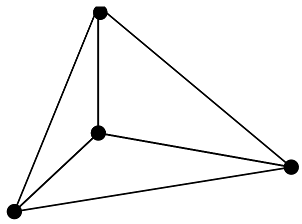
for all $i, j, k = \overline{1, 4}$, $i \neq j \neq k \neq i$.

We notice that $Q_iQ_jQ_kQ_lf = Q_{ijk}f$, $i, j, k, l = \overline{1, 4}$, $i \neq j \neq k \neq l$.

It is easy to verify the following properties.

- Each operator Q_{ijk}, $i, j, k = \overline{1, 4}$, $i \neq j \neq k \neq i$ interpolates the function f at the vertices of the tetrahedron $\mathcal{T}_h$. (See Figure 9.)
- The degree of exactness is

$$\mathrm{dex}(Q_{ijk}) = 1, \quad i, j, k = \overline{1, 4},\ i \neq j \neq k \neq i.$$



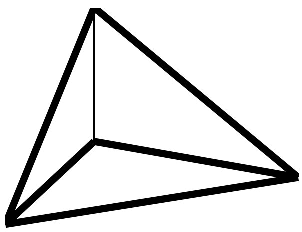
$$Q_{ijk},\ i,j,k=\overline{1,4},$$

Figure 9.

6. Some useful operators are obtained using the boolean sum of the operators Q_i, $i=\overline{1,4}$.

For example, each of the operators $S_{ij}=Q_i\oplus Q_j$, $i,j=\overline{1,4}$, $i\neq j$ has the property that $S_{ij}f$ inetrpolates the function f at five of the tetrahedron edges (see Figure 10) and

$$\operatorname{dex}(S_{ij})=2,\quad i,j=\overline{1,4}.$$



S_{12}

Figure 10.

7. We also have $S_{ijk}f=f$, $i,j,k=\overline{1,4}$, $i\neq j\neq k\neq i$ on all the edges of the tetrahedron $\mathcal{T}_h$ and

$$\operatorname{dex}(S_{ijk})=2,\quad i,j=\overline{1,4},$$

where $S_{ijk}=Q_i\oplus Q_j\oplus Q_k$, $i\neq j\neq k\neq i$.

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ON SOME IMPLICITE SCHEME IN MATHEMATICAL FINANCE

IOANA CHIOREAN

Dedicated to Professor Petru Blaga at his 60^{th} *anniversary*

Abstract. The aim of this paper is to give a parallel approach for the Crank-Nicsloson method applied to the discretized form of the Black-Scholes equation.

1. Introduction in the value of an option

One of the key problems in Mathematical Finance is the determining the value of an option.

According to [8] the simplest financial option, a European call option, is a contract with the following properties:

- at a prescribed time in the future, known as the *expiry date or expiration date* (denoted by T), the holder of the option may

- purchase a prescribed asset, known as the *underlying asset* (denoted by S), for a
- prescribed amount of money, known as the *exercise price* (denoted by E).

Note 1.: The word "may" in this description implies that for the holder of an option, this contract is a "right" and not an "obligation".

The other part, who is known as the *writer,* has a potential obligation: he "must" sell the asset if the holder chooses to buy it. Since the option confers on its holder a right with no obligation, it has some value. Moreover, it must be paid for

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at the time of opening the contract. Conversely, the writer of the option must be compensated for the obligation he has assumed. So the following questions arise:

- How much would one pay for this right, i.e. what is the value of an option?
- How can the writer minimize the risk associated with his obligation?

Note 2.: There are Call Options (which means the options *to buy* assets) and Put Options (which means *to sell* assets). Whereas the holder of a call option wants the asset prince to rise - the higher the asset price at expiry, the greater the profit - the holder of a put option wants the asset price to fall as low as possible.

2. **The mathematical model**

The problem of determining the value of an option is mathematically modeled by the well known *Black-Scholes equation:*

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 \cdot s^2 \cdot \frac{\partial^2 V}{\partial s^2} + r \cdot s \cdot \frac{\partial V}{\partial s} - r \cdot V = 0 \tag{1}$$

where the following notations are used:

- V- the value of an option, where $V_s = V(S,t)$, with t-the time and S - the underlying asset. If we have a call option, V will be replaced by C, and if we have a put option, it will be replaced by P.
- σ - the volatility of the underlying asset
- r - the interest rate

Note 3.: For a Call option, e.g., the boundary conditions are:

$$V(S,T) = \max(S-E, 0) \tag{2}$$
$$V(0,t) = 0$$

where we denoted by

- E - the exercise price
- T - the expiry.

The exact solution of equation (1) with boundary condition (2) can be determined, but in practice it is difficult to handle. This is the reason for which a numerical approach is preferred.

3. **Solving numericaly the Black-Scholes equation**

In order to obtain the numerical solution of equation (1), one has to discretize it.

The most common way to do this, is by using finite - difference methods.

In the literature, there exist many results in this direction. So, in [6], [7] and [8] one may find the basic tools for numerical option pricing. In [5], a backward differentiations formula is used and in [2], some results are obtained by using an explicit technique.

In what follows, we recall another technique, known as the Crank-Nicolson method.

3.1. **The Crank-Nicolson method.**

As is presented in [8], the Black-Scholes equation can be reduced to a diffusion equation, where the numerical solutions are easier to determine. Then, by a change of variable, these are converted into numerical solutions of the Black-Scholes equation.

So, let us consider the general form of the transformed Black-Scholes model for the value of a European option,

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2} \tag{3}$$

with the boundary conditions

$$\begin{aligned} & u(x,\tau) \backsim u_{-\infty}(x,\tau), u(x,\tau) = u_{\infty}(x,\tau) \text{ as } x \to \pm\infty \\ & u(x,0) = u_0(x) \end{aligned} \tag{4}$$

Using grid points with the x-axis divided into equally spaced nodes at distance δx apart, and the τ-axis into equally spaced nodes at a distance $\delta \tau$ apart, the grid points have the form $(n\delta x,\ m\delta\tau)$. We denote by

$$u_n^m = u(n\delta x, m\delta\tau) \tag{5}$$

the value of $u(x,\tau)$ at the grid point $(n\delta x,\ m\delta\tau)$. Considering that, on the grid,

$$N^-\delta x \le x \le N^+\delta x,\ 0 \le t \le M\delta\tau$$

where N^-, N^+ and M are large positive integers, we may write equation (3) with the boundary conditions (4), in the following manner:

$$\frac{u_n^{m+1} - u_n^m}{\delta\tau} + 0(\delta\tau) = \frac{u_{n+1}^m - 2u_n^m + u_{n-1}^m}{(\delta x)^2} + 0\left((\delta x)^2\right) \tag{6}$$

by using an explicit formula and

$$\frac{u_n^{m+1} - u_n^m}{\delta\tau} + 0(\delta\tau) = \frac{u_{n+1}^{m+1} - 2u_n^{m+1} + u_{n-1}^{m+1}}{(\delta x)^2} + 0\left((\delta x)^2\right) \tag{7}$$

by using an implicit formula, where the discretized boundary conditions are:

$$u_{N^-}^m = u_{-\infty}(N^-\delta x, m\delta\tau),\ 0 < m \le M$$
$$u_{N^+}^m = u_{\infty}(N^+\delta x, m\delta\tau),\ 0 < m \le M.$$

Making the average of (6) and (7), we obtain the Crank-Nicolson formula which, ignoring the error terms, is the following:

$$u_n^{m+1} - \frac{1}{2}\alpha\left(u_{n-1}^{m+1} - 2u_n^{m+1} + u_{n+1}^{m-1}\right) = \tag{8}$$
$$u_n^m + \frac{1}{2}\alpha\left(u_{n-1}^m - 2u_n^m + u_{n+1}^m\right)$$

where

$$\alpha = \frac{\delta\tau}{(\delta x)^2}.$$

In a matriceal form, (8) can be written as follows:

$$A \cdot u^{m+1} = B \cdot u^m \tag{9}$$

where

$$A = \begin{bmatrix} 1+\alpha & -\frac{1}{2}\alpha & 0 & \cdots & 0 \\ -\frac{1}{2}\alpha & 1+\alpha & -\frac{1}{2}\alpha & \cdots & 0 \\ 0 & -\frac{1}{2}\alpha & 1+\alpha & \cdots & \\ \vdots & \vdots & \vdots & & -\frac{1}{2}\alpha \\ 0 & \cdots & 0 & -\frac{1}{2}\alpha & 1+\alpha \end{bmatrix}$$

$$B = \begin{bmatrix} 1-\alpha & \frac{1}{2}\alpha & 0 & \cdots & 0 & \\ \frac{1}{2}\alpha & 1-\alpha & \frac{1}{2}\alpha & & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & & \frac{1}{2}\alpha \\ 0 & 0 & \cdots & & \frac{1}{2}\alpha & 1-\alpha \end{bmatrix}$$

By successive replacing (9) becomes:

$$u^{m+1} = A^{-1} \cdot B \cdot u^m = (A^{-1} \cdot B)^2 \cdot u^{m+1} = \cdots = (A^{-1} \cdot B)^m \cdot u^0 \tag{10}$$

where u^0 contains the option values at the initial moment.

In (10) we have to compute the $m^{th}-$power of a matrix product. The complexity of this computation, performed in a usual manner (it means with a serial computer) is $O(n^{3m})$, where n is the dimension of the matrix. In order to improve this complexity, it means to reduce the effort of computation, one way is to use parallel calculus.

4. **Parallel approaches**

Parallel calculus implies the execution of the corresponding algorithm by means of several processors. For more details about parallel computation, in general, see [4]. Many authors use more than one processor to reduce the execution time, for different types of algorithms. Connected with the numerical methods for the Black-Scholes formula, in [1] a parallel approach is proposed which generates an effort of computation of order $O(\log\, n)$, where n is the dimension of the problem. Also, in [3], by using another parallel technique, a similar result is given.

4.1. **Using the recursive doubling technique.** One possibility to gain speed is to apply the recursive doubling technique (see [4]) to evaluate the matriceal product in (10).

As presented in [4], having enough processors (let's say p, with $p \geq m$) the matriceal product can be performed on a binary tree network: every leaf processor memorizes a pair $A^{-1}B$, and exactly in $[\log_2 m]$ steps, the final product will be obtained in the root processor. The computation effort at every level is of order $O(n^3)$. So the total computational effort will be of order $O\left(n^3 \cdot [\log_2 m]\right)$.

4.2. **Using a parallel matriceal product.** Another possibility to gain speed is to use the p processors (with $p \geq n^3$, this time) to compute in parallel one matriceal product $A^{-1} \cdot B$. According with some technique presented in [4], this can be done exactly in the time needed to perform one single scalar multiplication. So, the total time involved (the computational effort), will be of order $O(m*$ complexity of a scalar multiplication).

5. Conclusions

The previous parallel approaches presented above reduce the computational effort and can be used if there are enough processors in the system. Otherwise, the matrices can be divided into blocks, and then some block parallel techniques may be applied.

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COMPLETENESS WITH RESPECT TO THE PROBABILISTIC POMPEIU-HAUSDORFF METRIC

ŞTEFAN COBZAŞ

Dedicated to Professor Petru Blaga at his 60^{th} *anniversary*

Abstract. The aim of the present paper is to prove that the family of all closed nonempty subsets of a complete probabilistic metric space L is complete with respect to the probabilistic Pompeiu-Hausdorff metric H. The same is true for the families of all closed bounded, respectively compact, nonempty subsets of L. If L is a complete random normed space in the sense of Šerstnev, then the family of all nonempty closed convex subsets of L is also complete with respect to H. The probabilistic Pompeiu-Hausdorff metric was defined and studied by R.J. Egbert, Pacific J. Math. **24** (1968), 437-455, in the case of Menger probabilistic metric spaces, and by R.M. Tardiff, Pacific J. Math. **65** (1976), 233-251, in general probabilistic metric spaces. The completeness with respect to probabilistic Pompeiu-Hausdorff metric of the space of all closed bounded nonempty subsets of some Menger probabilistic metric spaces was proved by J. Kolumbán and A. Soós, Studia Univ. Babes-Bolyai, Mathematica, **43** (1998), no. 2, 39-48, and **46** (2001), no. 3, 49-66.

1. Introduction

The study of probabilistic metric spaces (PM spaces for short) was initiated by K. Menger [17] and A. Wald [28], in connection with some measurements problems in physics. The positive number expressing the distance between two points p, q of a metric space is replaced by a distribution function (in the sense of probability theory)

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$F_{p,q} : \overline{\mathbb{R}} \to [0,1]$, whose value $F_{p,q}(x)$ at the point $x \in \mathbb{R}$ can be interpreted as the probability that the distance between p and q be less than x. Since then the subject developed in various directions, an important one being that of fixed points in PM spaces. Important contributions to the subject have been done by A.N. Šerstnev and the Kazan school of probability theory, see [21, 22, 23, 24] and the bibliography in [19].

A clear and thorough presentation of the results up to 1983 is given in the book by B. Schweizer and A. Sklar [19]. Beside this book, at the present there are several others dealing with various aspects of analysis in probabilistic metric spaces and in probabilistic normed spaces - V. Istrăţescu [11], I. Istrăţescu and Gh. Constantin [4, 5], V. Radu [18], S.-S. Chang and Y. J. Cho [3], O. Hadžić [8], O. Hadžić and E. Pap [9]. In the present paper we shall follow the treatise [19].

The probabilistic Pompeiu-Hausdorff metric on the family of nonempty closed subsets of a PM space was defined by Egbert [6] in the case of Menger PM spaces, and by Tardiff [27] in general PM spaces (see also [19, §12.9]), by analogy with the classical case. Sempi [20] used the probabilistic Pompeiu-Hausdorff metric to prove the existence of a completion of a PM space. Some results have been obtained also by Beg and Ali [2].

As it is well known, the family of nonempty closed bounded subsets of a complete metric space is complete with respect to the Pompeiu-Hausdorff distance (see, e.g., [10, Chapter 1]). The aim of the present paper is to prove the probabilistic analog of this result for the family of all nonempty closed subsets of a probabilistic metric space. We shall prove that the families of all nonempty closed bounded, respectively compact, subsets of a complete probabilistic metric space L are also complete with respect to the probabilistic Pompeiu-Hausdorff metric. If L is a complete random normed space in the sense of Šerstnev, then the family of all nonempty closed convex subsets of L is complete with respect to the Pompeiu-Hausdorff metric too. In the case of Menger PM spaces (L, ρ, Min), and (L, ρ, W), with t-norms $\mathrm{Min}(s,t) = \min\{s,t\}$, $s,t \in [0,1]$, respectively $W(s,t) = \max\{s+t-1, 0\}$, the completeness of the space of all closed bounded nonempty subsets of L with respect to the probabilistic Pompeiu-Hausdorff metric was proved by Kolumbán and Soós in [13] and [14]. In the case of a Menger PM space (L, ρ, Min), they proved also in [13] the completeness of the family of all compact nonempty subsets of L. These completeness results were applied in [13, 14, 15] to prove the existence of invariant sets for finite families of contractions in PM spaces of random variables (E-spaces in the sense of Sherwood [25], or [19, Ch. 9, Sect. 1]).

As in Aubin's book [1], I have adopted the term Pompeiu-Hausdorff metric. For a short comment on this fact, as well as on the similar case of the Painlevé-Kuratowski convergence for sequences of sets, see [1, page xiv].

2. **Preliminary notions**

Denote by Δ the set of *distribution functions*, meaning nondecreasing, left continuous functions $F : \overline{\mathbb{R}} \to [0,1]$ with $F(-\infty) = 0$ and $F(\infty) = 1$. Let D be the subclass of Δ formed by all functions $F \in \Delta$ such that

$$\lim_{x \to -\infty} F(x) = 0 \quad \text{and} \quad \lim_{x \to \infty} F(x) = 1.$$

The *weak convergence* of a sequence (F_n) in Δ to $F \in \Delta$, denoted by $F_n \xrightarrow{w} F$, means that the equality

$$\lim_{n \to \infty} F_n(x) = F(x) \tag{2.1}$$

holds for every continuity point x of F. Since F is non-decreasing the set of its discontinuity points is at most countable, so that the set of continuity points of F is dense in $\mathbb{R}$. In order that $F_n \xrightarrow{w} F$ it is sufficient that the relation (2.1) holds for every x in an arbitrary dense subset of $\mathbb{R}$. An important result concerning weak convergence of distribution functions is Helly's First Theorem: every sequence in Δ contains a weakly convergent subsequence (see Loève [16, Sect. 11.2]).

The topology of weak convergence in Δ is metrizable. The first who realized this was P. Lévy (see the Appendix to Fréchet's book [7]), and for this reason the metrics generating the weak convergence in Δ are called Lévy metrics. Since the original Lévy metric characterizes the weak convergence only in D, Sibley [26] proposed a modification of Lévy metric that generates the weak convergence in Δ. We shall work with a further modification proposed by Schweizer and Sklar [19] and denoted by d_L. The distance $d_L(F,G)$ between two functions $F, G \in \Delta$ is defined as the infimum of all numbers $h > 0$ such that the inequalities

$$F(x-h) - h \leq G(x) \leq F(x+h) + h$$

and

$$G(x-h) - h \leq F(x) \leq G(x+h) + h$$

hold for every $x \in (-h^{-1}; h^{-1})$. One shows that d_L is a metric on Δ and, for any sequence (F_n) in Δ and $F \in \Delta$, we have

$$F_n \xrightarrow{w} F \iff d_L(F_n, F) \to 0.$$

By Helly's First Theorem the space (Δ, d_L) is compact, hence complete (see [19, §4.2]).

The sets of *distance functions* are:

$$\Delta^+ = \{F \in \Delta : F(0) = 0\} \quad \text{and} \quad D^+ = D \cap \Delta^+.$$

It follows that for $F \in \Delta^+$ we have $F(x) = 0$, $\forall x \leq 0$. The set Δ^+ is closed in the metric space Δ, hence compact and complete too.

Two important distance functions are

$$\begin{aligned} \epsilon_0(x) &= 0 \quad \text{for} \quad x \leq 0 \\ &= 1 \quad \text{for} \quad x > 0 \end{aligned} \qquad \text{and} \qquad \begin{aligned} \epsilon_\infty(x) &= 0 \quad \text{for} \quad x < \infty \\ &= 1 \quad \text{for} \quad x = \infty \end{aligned}$$

The order in Δ^+ is defined as the punctual order: for $F, G \in \Delta^+$ we put

$$F \leq G \iff \forall x > 0 \;\; F(x) \leq G(x).$$

It follows that ϵ_0 is the maximal element of Δ^+ and of D^+ as well, and ϵ_∞ is the minimal element of Δ^+.

In the following we shall define some functions, say F, on $\mathbb{R}$ and consider them automatically extended to $\overline{\mathbb{R}}$ by $F(-\infty) = 0$ and $F(\infty) = 1$.

If $\{F_i : i \in I\}$ is a family of functions in Δ^+ then the function $F : \overline{\mathbb{R}} \to [0, 1]$ defined by

$$F(x) = \sup\{F_i(x) : i \in I\},\ x \in \mathbb{R},$$

is the supremum of the family $\{F_i\}$ in the ordered set $(\Delta^+, \leq) :\ F = \sup_{i \in I} F_i$.

To define the infimum of the family $\{F_i\}$ put

$$\Gamma(x) = \inf\{F_i(x) : i \in I\},\ x \in \mathbb{R}. \tag{2.2}$$

Since the function Γ is nondecreasing, but not necessarily left continuous on $\mathbb{R}$, we have to regularize it by taking the left limit

$$G(x) = \ell^-\Gamma(x) := \lim_{x' \nearrow x} \Gamma(x') = \sup_{x' < x} \Gamma(x'),\ x \in \mathbb{R}. \tag{2.3}$$

Then $G(x) \leq \Gamma(x),\ \forall x \in R$, the function G belongs to Δ^+ and $G = \inf_{i \in I} F_i$ - the infimum of the family $\{F_i\}$ in the ordered set $(\Delta^+, \leq)$.

A *triangle function* is a binary operation τ on $\Delta^+,\ \tau : \Delta^+ \times \Delta^+ \to \Delta^+,$ that is commutative, associative, non-decreasing in each place $(\tau(F_1, G_1) \leq \tau(F_2, G_2),$ if $F_1 \leq F_2$ and $G_1 \leq G_2)$, and has ϵ_0 as identity: $\tau(F, \epsilon_0) = F,\ F \in \Delta^+$. The triangle function τ is called continuous if it is continuous with respect to the d_L-topology of Δ^+. It follows that τ is, in fact, uniformly continuous, since the metric space (Δ^+, d_L) is compact.

3. **Probabilistic metric spaces**

A *probabilistic metric space* (PM space) is a triple (L, ρ, τ), where L is a set, ρ is a mapping from $L \times L$ to Δ^+, and τ is a continuous triangle function. The value of ρ at $(p, q) \in L \times L$ is denoted by F_{pq}, i.e., $\rho(p, q) = F_{pq}$.

One supposes that the following conditions are satisfied for all $p, q, r \in L$:

(PM1) $F_{pp} = \epsilon_0$,

(PM2) $F_{pq} = \epsilon_0 \ \Rightarrow\ p = q$,

(PM3) $F_{pq} = F_{qp}$,

(PM4) $F_{pr} \geq \tau(F_{pq}, F_{qr})$.

The mapping ρ is called the *probabilistic metric* on L and the condition (PM4) is the probabilistic analogue of the triangle inequality.

The *strong topology* on a PM space is defined by the neighborhood system:

$$U_t(p) = \{q \in L : F_{pq}(t) > 1 - t\}, \quad t > 0. \tag{3.1}$$

Putting

$$\bar{U}_t(p) = \{q \in L : F_p(t) \geq 1 - t\} \tag{3.2}$$

we have $U_t(p) \subset \bar{U}_t(p)$ and $\bar{U}_{t'}(p) \subset U_t(p)$ for $t' < t$, showing that the family (3.2) of subsets of L forms also a neighborhood base for the strong topology of L.

Observe that $U_t(p) = L$, for $t > 1$, and $\bar{U}_t(p) = L$, for $t \geq 1$, so that we can restrict to $t \in (0, 1)$ when working with strong neighborhoods. In fact, we can suppose that t is as small as we need.

The strong topology on a PM space (L, ρ, τ) is derived from the uniformity $\mathcal{U}$ generated by the vicinities:

$$U_t = \{(p, q) \in L \times L : F_{pq}(t) > 1 - t\}, \quad t > 0. \tag{3.3}$$

The strong topology is metrizable since $\{U_{1/n} : n \in \mathbb{N}\}$ is a countable base for the uniformity $\mathcal{U}$. The probabilistic metric ρ is uniformly continuous mapping from $L \times L$ with the product topology to (Δ^+, d_L), meaning that

$$p_n \to p \text{ and } q_n \to q \text{ in } L \quad \Rightarrow \quad F_{p_n q_n} \xrightarrow{w} F_{pq}. \tag{3.4}$$

The convergence of a sequence (p_n) in L to $p \in L$ is characterized by

$$\begin{aligned} p_n \to p \quad &\iff \forall t > 0 \, \exists n_0 \, \forall n \geq n_0 \quad p_n \in U_t(p) \\ &\iff F_{p_n p} \xrightarrow{w} \epsilon_0 \\ &\iff d_L(F_{p_n p}, \epsilon_0) \to 0. \end{aligned}$$

A sequence (p_n) in L is called a *Cauchy sequence*, or *fundamental*, if

$$F_{p_n p_m} \xrightarrow{w} \epsilon_0 \quad \text{for} \quad n, m \to \infty,$$

or, equivalently,

$$\forall t > 0 \; \exists n_0 \text{ such that } \forall n, m \geq n_0 \;\; (p_n, p_m) \in U_t \; (\iff F_{p_n p_m}(t) > 1 - t).$$

A convergent sequence in L is a Cauchy sequence, and the PM space L is called *complete* (with respect to the strong topology) if every Cauchy sequence is convergent.

For these and other questions concerning the strong topology of a PM space, see [19, Chapter 12].

Throughout this paper all the topological notions concerning a PM space will be considered with respect to the strong topology.

4. **The probabilistic Pompeiu-Hausdorff metric**

For a metric space (X, d), two nonempty bounded subsets A, B of X and a point $p \in X$, one introduces the following notations and notions :

$$d(p, B) = \inf\{d(p, q) : q \in B\} \; - \; \text{ the distance from } p \text{ to } B,$$

$$h^*(A, B) = \sup\{d(p, B) : p \in A\} \; - \; \text{the excess of } A \text{ over } B,$$

and let

$$h(A, B) = \max\{h^*(A, B), h^*(B, A)\}$$

be the Pompeiu-Hausdorff distance between the sets A, B.

Denoting by $P_{fb}(X)$ the family of all nonempty closed bounded subset of X it follows that h is a metric on $P_{fb}(X)$, and the metric space $(P_{fb}(X), h)$ is complete if (X, d) is complete (see, e.g., [10, Chapter 1]).

In the case of a PM space (L, ρ, τ) the definitions are similar but, taking into account the fact that the probabilistic triangle inequality (PM4) is written in reversed form with respect to the usual triangle inequality, sup and inf will change their places.

For two nonempty subsets A, B of L and $p \in L$ denote by

$$F_{pB} = \sup\{F_{pq} : q \in B\} \iff F_{pB}(x) = \sup\{F_{pq}(x) : q \in B\}, \; x \in \mathbb{R}, \tag{4.1}$$

the *probabilistic distance* from p to B, and let

$$F^*_{AB} = \inf\{F_{pB} : p \in A\}. \tag{4.2}$$

Taking into account the formulae (2.2) and (2.3), it follows

$$F^*_{AB} = \ell^- \Gamma^*_{AB},$$

where

$$\Gamma^*_{AB}(x) = \inf\{F_{pB}(x) : p \in A\},\ x \in \mathbb{R}.$$

The *probabilistic Pompeiu-Hausdorff distance* between the sets A, B is defined by
$H(A,B) = F_{AB}$, where

$$F_{AB}(x) = \min\{F^*_{AB}(x), F^*_{BA}(x)\}, \quad x \in \mathbb{R}. \tag{4.3}$$

The probabilistic Pompeiu-Hausdorff metric was defined and studied by Egbert [6] in the case of Menger PM spaces and by Tardiff [27] in general PM spaces (see also [19, §12.9]). The mapping $H(A,B) = F_{AB}$ satisfies the following properties, where cl denotes the closure with respect to the strong topology:

Proposition 4.1. ([19, Th. 12.9.2])

1. $F_{\{p\}\{q\}} = F_{pq}$ *for* $p, q \in L$;

2. For nonempty $A, B \subset L$, $F_{AB} = F_{BA}$, $F_{AB} = F_{\mathrm{cl}(A)\,\mathrm{cl}(B)}$, *and* $F_{AB} = \epsilon_0$ *if and only if* $\mathrm{cl}(A) = \mathrm{cl}(B)$.

In order that H satisfy the probabilistic triangle inequality (PM4), we have to impose a supplementary condition on the triangle function τ. The triangle function τ is called *sup-continuous* if

$$\tau(\sup_{i\in I} F_i, G) = \sup_{i\in I} \tau(F_i, G) \tag{4.4}$$

for any family $\{F_i : i \in I\} \subset \Delta^+$ of distance functions and any $G \in \Delta^+$.

Denote by $P_f(L)$ the family of all nonempty closed subsets of a PM space (L, ρ, τ).

Theorem 4.2. ([19, Th. 12.9.5]) *If the triangle function τ is sup-continuous then the mapping $H(A,B) = F_{AB}$, where F_{AB} is defined by* (4.3)*, is a probabilistic metric on $P_f(L)$.*

In the following proposition we collect some properties which will be used in the proof of the completeness of $P_f(L)$ with respect to the probabilistic Pompeiu-Hausdorff metric.

Proposition 4.3. *Let (L, ρ, τ) be a PM space with sup-continuous triangle function τ, and let $A, B \in P_f(L)$ and $p \in L$. Then*

1. $F_{pB} \geq \tau\left(F_{pA}, F^*_{AB}\right)$;

and

2. $F_{pB} \geq \Gamma^*_{AB} \geq F^*_{AB} \geq F_{AB}$.
3. *If $F_{AB}(s) > 1 - s$ for some s, $0 < s < 1$, then*

$$\forall p \in A \; \exists q \in B \text{ such that } F_{pq}(s) > 1 - s, \tag{4.5}$$

and

$$\forall q \in B \; \exists p \in A \text{ such that } F_{pq}(s) > 1 - s. \tag{4.6}$$

Proof. For $x \in \mathbb{R}$ we have

$$\forall a \in A \; \forall b \in B \quad F_{pB}(x) \geq F_{pb}(x) \geq \tau(F_{pa}F_{ab})(x).$$

Taking the supremum with respect to $b \in B$ and taking in account that τ is sup continuous and monotonic in each place, we get

$$\forall a \in A \;\; F_{pB}(x) \geq \tau(F_{pa}, F_{aB})(x) \geq \tau(F_{pA}, F^*_{AB})(x).$$

Taking now the supremum with respect to $a \in A$ one obtains the inequality 1.

The inequalities 2 are immediate from definitions.

To prove 3, observe that

$$F_{AB}(s) > 1 - s \iff F^*_{AB}(s) > 1 - s \text{ and } F^*_{BA}(s) > 1 - s.$$

It follows

$$\inf\{F_{p'B}(s) : p' \in A\} = \Gamma^*(s) \geq F^*_{AB}(s) > 1 - s,$$

so that

$$\sup\{F_{pq}(s) : q \in B\} > 1 - s,$$

implying (4.5).

The inequality (4.6) can be proved similarly. $\square$

The completeness result will be obtained under a further restriction imposed to τ. We say that the triangle function τ satisfies the condition (W) if

$$\text{(W)} \qquad F(x) > \alpha \text{ and } G(x) > \beta \;\Rightarrow\; \tau(F,G)(x) > \max\{\alpha + \beta - 1, 0\},$$

for all $x > 0$, where $F, G \in \Delta^+$, and $\alpha, \beta \in \mathbb{R}$.

Remark. Considering the t-norm

$$W(x,y) = \max\{x + y - 1, 0\}, \quad (x,y) \in [0,1]^2,$$

(see [19, p. 5]) and the associated triangle function $\mathbf{W}$, defined for $F, G \in \Delta^+$ by

$$\mathbf{W}(F,G)(x) = W(F(x), G(x)),\ x \in \mathbb{R},$$

(see [19, p. 97]), the condition (W) essentially means that $\tau \geq \mathbf{W}$.

Now we are ready to state and prove the completeness result.

Theorem 4.4. *Let (L, ρ, τ) be a PM space with sup-continuous triangle function τ satisfying the condition (W).*

If the PM space L is complete then the space $P_f(L)$ is complete with respect to the probabilistic Pompeiu-Hausdorff metric.

Proof. Let (A_n) be a sequence in $P_f(L)$ that is fundamental with respect to the probabilistic Pompeiu-Hausdorff metric H.

Put

$$A = \bigcap_{n \geq 1} \operatorname{cl}\left(\bigcup_{m \geq n} A_m\right),$$

and show that $A \in P_f(L)$ (meaning that $A \subset L$ is nonempty closed) and that the sequence (A_n) converges to A with respect to the probabilistic Pompeiu-Hausdorff metric H.

Observe that

$$p \in A \quad \iff \quad \exists \ n_1 < n_2 < ... \ \exists p_k \in A_{n_k} : \quad p_k \to p. \tag{4.7}$$

For $0 < t < 1/2$ fixed, choose $n_0 \in \mathbb{N}$ such that

$$\forall n, m \geq n_0 \quad F_{A_n A_m}(t) > 1 - t.$$

For $m \geq n_0$ fixed, put $n_1 := m$ and pick an element $p_1 \in A_{n_1}$.

Let now $n_2 > n_1$ be such that

$$\forall n, n' \geq n_2 \quad F_{A_n A_{n'}}(\frac{t}{2}) > 1 - \frac{t}{2}.$$

The inequalities

$$F^*_{A_{n_1} A_{n_2}}(t) \geq F_{A_{n_1} A_{n_2}}(t) > 1 - t$$

and the fact that p_1 belongs to A_{n_1} imply $F_{p_1 A_{n_2}}(t) > 1 - t$, so that there exists $p_2 \in A_{n_2}$ such that

$$F_{p_1 p_2}(t) > 1 - t.$$

Take now $n_3 > n_2$ such that

$$\forall n, n' \geq n_3 \quad F_{A_n A_{n'}}(\frac{t}{2^2}) > 1 - \frac{t}{2^2}.$$

Reasoning like above, we can find an element $p_3 \in A_{n_3}$ such that

$$F_{p_2 p_3}(\frac{t}{2}) > 1 - \frac{t}{2}.$$

Continuing in this way, we obtain a strictly increasing sequence of indices $n_1 < n_2 < ...$ and the elements $p_k \in A_{n_k}, \ k \in \mathbb{N},$ such that

$$F_{p_k p_{k+1}}(\frac{t}{2^{k-1}}) > 1 - \frac{t}{2^{k-1}}, \tag{4.8}$$

for all $k \in \mathbb{N}$.

Claim I. $\quad \forall i \in \mathbb{N} \, \forall k \in \mathbb{N} \quad F_{p_k p_{k+i}}(\frac{t}{2^{k-1}}) > 1 - (\frac{1}{2^{k-1}} + \frac{1}{2^k} + ... + \frac{1}{2^{k+i-2}})t.$

We proceed by induction on i. For $i = 1$ the assertion is true by the choice of the elements p_k (see (4.8)).

Suppose that the assertion is true for i and prove it for $i+1$. Appealing to condition (W) we have

$$F_{p_kp_{k+i+1}}(\frac{t}{2^{k-1}}) \geq \tau\left(F_{p_kp_{k+1}}, F_{p_{k+1}p_{k+i+1}}\right)(\frac{t}{2^{k-1}}) > 1-(\frac{1}{2^{k-1}}+\frac{1}{2^k}+...+\frac{1}{2^{k+i-1}})t,$$

since $F_{p_kp_{k+1}}(\frac{t}{2^{k-1}}) > 1-\frac{1}{2^{k-1}}$ and, by the induction hypothesis,

$$F_{p_{k+1}p_{k+i+1}}(\frac{t}{2^{k-1}}) \geq F_{p_{k+1}p_{k+i+1}}(\frac{t}{2^k}) > 1-(\frac{1}{2^k}+...+\frac{1}{2^{k+i-1}})t.$$

Claim II. The sequence (p_k) is fundamental in the PM space L.

For $0<s<1$ choose $k_0 \in \mathbb{N}$ such that $2^{-k_0+1} < s$. Then for any $k \geq k_0$ and arbitrary $i \in \mathbb{N}$ we have

$$F_{p_kp_{k+i}}(s) \geq F_{p_kp_{k+i}}(\frac{t}{2^{k-1}}) > 1-(\frac{t}{2^{k-1}}+...+\frac{t}{2^{k+i-1}}) > 1-\frac{t}{2^k} > 1-s.$$

Since the PM space L is complete, there exists $p \in L$ such that $p_k \to p$ in the strong topology of L. The choice of the elements p_k and (4.7) yield $p \in A$. Since the set A is obviously closed it follows $A \in P_f(L)$.

By Claim I we have

$$F_{p_1p_k}(t) > 1-(1+\frac{1}{2}+...+\frac{1}{2^{k-2}})t > 1-2t.$$

Let now t', $t<t'<2t$, be a continuity point of the distribution function F_{p_1p}. The continuity of the distance function (see (3.4)) and the inequalities

$$F_{p_1p_k}(t') \geq F_{p_1p_k}(t) > 1-2t$$

yield, for $k\to\infty$, $F_{p_1p}(t') \geq 1-2t$, so that

$$F_{p_1A}(t') = \sup_{q\in A} F_{p_1q}(t') \geq 1-2t.$$

As p_1 was arbitrarily chosen in A_m, it follows

$$\Gamma^*_{A_mA}(t') = \inf\{F_{p'A}(t') : p' \in A_m\} \geq 1-2t.$$

But then

$$F^*_{A_mA}(2t) = \sup_{t'} \Gamma^*_{A_mA}(t') \geq 1 - 2t,$$

where the supremum is taken over all continuity points t' of the function F_{p_1p} lying in the interval $(t, 2t)$. The fact that the set of these points is dense in the interval $(t, 2t)$ justifies the equality sign in the first of the above relations.

Taking into account that $m \geq n_0$ was arbitrarily chosen too, we finally obtain

$$\forall m \geq n_0 \quad F^*_{A_mA}(2t) \geq 1 - 2t, \tag{4.9}$$

Let now $p \in A$ and let $n_1 < n_2 < ...$ and $p_k \in A_{n_k}$ be such that $p_k \to p$ in the strong topology of the PM space L.

Choose $k_0 \in \mathbb{N}$ such that

$$\forall k \geq k_0 \quad F_{p_kp}(t) > 1 - t.$$

Proposition 4.3, the inequality $F_{pA_{n_{k_0}}} \geq F_{pp_{k_0}}$, and condition (W) give, for any $t',\ t < t' < 2t$,

$$F_{pA_m}(t') \geq F_{pA_m}(t) \geq \tau\left(F_{pA_{n_{k_0}}}, F^*_{A_{n_{k_0}}A_m}\right)(t) \geq \tau\left(F_{pp_{k_0}}, F^*_{A_{n_{k_0}}A_m}\right)(t) > 1 - 2t.$$

Since $p \in A$ was arbitrarily chosen, it follows

$$\forall t',\ t < t' < 2t, \quad \Gamma^*_{AA_m}(t') \geq 1 - 2t,$$

so that

$$\forall m \geq n_0 \quad F^*_{AA_m}(2t) \geq 1 - 2t. \tag{4.10}$$

The inequalities (4.9) and (4.10) yield

$$\forall m \geq n_0 \quad H(A_m, A)(2t) = F_{A_mA}(2t) \geq 1 - 2t,$$

i.e., the sequence (A_m) converges to A with respect to the probabilistic Pompeiu-Hausdorff metric H.

The proof of the completeness is complete. $\square$

The *diameter* of a subset A of a PM space (L, ρ, τ) is defined by

$$D_A(t) = \ell^- \Phi_A(t)$$

where

$$\Phi_A(t) = \inf\{F_{pp'}(t) : p, p' \in A\}.$$

The set A is called *bounded* if $D_A \in D^+$, i.e. $\sup\{D_A(t) : t > 0\} = 1$ (see [19, pages 200-201]). This is equivalent to

$$\sup\{\Phi_A(t) : t > 0\} = 1. \tag{4.11}$$

Now we shall show that the families $P_{fb}(L)$ and $P_k(L)$ of all closed bounded nonempty subsets of a PM space L, respectively of all nonempty compact subsets of L, are complete in $P_f(L)$ with respect to the Pompeiu-Hausdorff metric, provided the PM space L is complete. To prove the assertion concerning the compact sets, we shall use the characterization of compactness in uniform spaces in terms of total boundedness (see [12, Ch. 6]). Let $(X, \mathcal{U})$ be a uniform space. For $U \in \mathcal{U}$ and a subset A of X put $U(A) = \{x \in X : \exists y \in A \text{ such that } (x, y) \in U\}$. It follows that $U(x) = U(\{x\})$ is a neighborhood of x and $\{U(x) : U \in \mathcal{U}\}$ forms a neighborhood base at x. A subset Y of X is called *totally bounded* if for every $U \in \mathcal{U}$ there exists a finite subset Z of X such that $Y \subset U(Z)$. Then a subset of a uniform space $(X, \mathcal{U})$ is compact if and only if it is complete and totally bounded ([12, Ch. 6, Th. 32]). If L is a PM space then, considering L as a uniform space with respect to the uniformity generated by the vicinities (3.3), denote by $P_{ftb}(L)$ the family of all nonempty, closed and totally bounded subsets of L.

Theorem 4.5. *If (L, ρ, τ) is a PM space with sup-continuous triangle function τ satisfying the condition (W), then the subspaces $P_{fb}(L)$ and $P_{ftb}(L)$ are closed in $P_f(L)$.*

Consequently, if the PM space L is complete then the subspaces $P_{fb}(L)$ and $P_k(L)$ are complete with respect to the probabilistic Pompeiu-Hausdorff metric.

Proof. Let (A_n) be a sequence of closed bounded nonempty sets converging to $A \in P_f(L)$ with respect to probabilistic Pompeiu-Hausdorff metric H. We have to show that A is bounded too, i.e. that

$$\sup\{\Phi_A(t) : t > 0\} = 1. \tag{4.12}$$

Let $0 < \epsilon < 1/3$ and let $m \in \mathbb{N}$ be such that

$$\forall n \geq m \quad F_{AA_n}(\epsilon) > 1 - \epsilon. \tag{4.13}$$

Since $\sup\{\Phi_{A_m}(t) : t > 0\} = 1$ there exists $t > 0$ such that $\Phi_{A_m}(t) > 1 - \epsilon$, so that

$$\forall q, q' \in A_m \quad F_{qq'}(t) > 1 - \epsilon. \tag{4.14}$$

We can suppose also that $t \geq \epsilon$. By (4.13) and (4.5), for any $p, p' \in A$ there exist $q, q' \in A_m$ such that

$$F_{pq}(\epsilon) > 1 - \epsilon \quad \text{and} \quad F_{p'q'}(\epsilon) > 1 - \epsilon. \tag{4.15}$$

Since $t \geq \epsilon$ we have $F_{pq}(t) \geq F_{pq}(\epsilon) > 1 - \epsilon$ and $F_{p'q'}(t) \geq F_{p'q'}(\epsilon) > 1 - \epsilon$, so that, by (4.14) and condition (W), we have

$$F_{qp'}(t) \geq \tau(F_{qq'}, F_{q'p'})(t) > 1 - 2e,$$

and

$$F_{pp'}(t) \geq \tau(F_{pq}, F_{qp'})(t) > 1 - 3\epsilon.$$

We have proved that for any ϵ, $0 < \epsilon < 1/3$, there exists $t > 0$ such that $F_{pp'}(t) > 1 - \epsilon$ for all $p, p' \in A$. It follows $\Phi_A(t) \geq 1 - 3\epsilon$, so that (4.12) holds.

Suppose now that (A_n) is a sequence of nonempty compact subsets of L converging with respect to the probabilistic Pompeiu-Hausdorff metric H to a set $A \in P_f(L)$. We shall show that A is totally bounded with respect to the uniformity having as vicinities the sets U_t given by (3.3).

Let $0 < \epsilon < 1/2$ and let $n \in \mathbb{N}$ be such that $F_{AA_n}(\epsilon) > 1 - \epsilon$. By (4.5) it follows

$$\forall p \in A \ \exists q \in A_n \text{ such that } F_{pq}(\epsilon) > 1 - \epsilon. \tag{4.16}$$

Now, since the set A_n is totally bounded, there exits a finite set $Z \subset L$ such that

$$\forall q \in A_n \ \exists z \in Z \text{ such that } F_{qz}(\epsilon) > 1 - \epsilon. \tag{4.17}$$

For an arbitrary $p \in A$ choose first an element $q \in A_n$ according to (4.16) and then, for this q select $z \in Z$ according to (4.17). Taking into account the condition (W) we get

$$F_{pq}(\epsilon) \geq \tau(F_{pq}, F_{qz})(\epsilon) > \max\{1 - 2\epsilon, 0\} = 1 - 2\epsilon.$$

It follows $A \subset U_{2\epsilon}(Z)$, i.e. the set A is totally bounded.

Now, if the PM space L is complete and A is closed in L, it follows that A is complete too, hence compact, as complete and totally bounded. □

Remark. As we have yet mentioned, in the case of Menger PM spaces (L, ρ, Min), and (L, ρ, W), the completeness of the space of all closed bounded subsets of L was proved by Kolumbán and Soós in [13] and [14], respectively. Since $\mathrm{Min} \geq W$, both of these results are contained in the above completeness result. The completeness of $P_k(L)$ in the case of a Menger PM space (L, ρ, Min) was proved in [13].

For a subset A of a PM space (L, ρ, τ) and $0 < \epsilon \leq 1$ let

$$A_\epsilon = \{q \in L : \exists p \in A \ F_{pq}(\epsilon) > 1 - \epsilon\} = \bigcup\{U_\epsilon(p) : p \in A\}.$$

As in the case of ordinary metric spaces we have:

Proposition 4.6. *(i)* $\operatorname{cl} A = \bigcap_{\epsilon > 0} A_\epsilon$

If τ satisfies (W) then

(ii) $A \subset B_\epsilon \ \Rightarrow \ \operatorname{cl} A \subset B_{2\epsilon}$.

Proof. Let $q \in \operatorname{cl} A$ and $\epsilon > 0$. Choosing $p \in U_\epsilon(q) \cap A$ it follows

$$q \in U_\epsilon(p) \subset A_\epsilon.$$

i.e. $\operatorname{cl} A \subset \cap_\epsilon A_\epsilon$. To prove the reverse inclusion we shall show that

$$\cap_{n \geq 1} A_{1/n} \subset \operatorname{cl} A.$$

If $q \in \cap_{n\geq 1} A_{1/n}$ then

$$\forall n\ \exists p_n \in A \text{ such that } F_{pp_n} > 1 - \frac{1}{n},$$

which implies that (p_n) converges to p in the strong topology of the PM space L, i.e. $p \in \operatorname{cl} A$.

To prove (ii), let $p \in \operatorname{cl} A$. It follows $U_\epsilon(p) \cap A \neq \emptyset$, so that $F_{pq}(\epsilon) > 1 - \epsilon$, for some $q \in A$.

Since $A \subset B_\epsilon$ it follows $F_{qr}(\epsilon) > 1 - \epsilon$, for some $r \in B$. But then, taking into account the condition (W), we have for $0 < \epsilon \leq 1/2$

$$F_{pr}(\epsilon) \geq \tau(F_{pq}, F_{qr})(\epsilon) > \max\{1 - 2\epsilon, 0\} = 1 - 2\epsilon,$$

showing that $p \in B_{2\epsilon}$. If $\epsilon > 1/2$ then $B_{2\epsilon} = L$. □

In the following proposition we give two expressions for the probabilistic Pompeiu-Hausdorff limit of a sequence of sets in $P_f(L)$, inspired by a well known result for the usual Pompeiu-Hausdorff metric (see [10, Proposition 1.3]).

Proposition 4.7. *Let (L, ρ, τ) be a PM space with sup-continuous triangle function τ satisfying the condition (W). If (A_n) is sequence in $P_f(L)$ converging to $A \in P_f(L)$ with respect to the probabilistic Pompeiu-Hausdorff metric H then*

$$A = \bigcap_{n\geq 1} \operatorname{cl}\left(\bigcup_{m\geq n} A_m\right) = \bigcap_{\epsilon>0} \bigcup_{n\geq 1} \bigcap_{m\geq n} (A_m)_\epsilon. \tag{4.18}$$

Proof. Show first that

$$A \subset \bigcap_{n\geq 1} \operatorname{cl}\left(\bigcup_{m\geq n} A_m\right). \tag{4.19}$$

Let $p \in A$ and let $n_1 \in \mathbb{N}$ be such that

$$\forall m \geq n_1\ F_{AA_m}(\frac{1}{2}) > 1 - \frac{1}{2}.$$

By (4.5),

$$\exists p_1 \in A_{n_1}\ F_{pp_1}(\frac{1}{2}) > 1 - \frac{1}{2}.$$

Continuing in this way we obtain a sequence $n_1 < n_2 < ...$ of indices and the elements $p_k \in A_{n_k}$ such that

$$F_{pp_k}(\frac{1}{2^k}) > 1 - \frac{1}{2^k}.$$

It follows $p_k \to p$, so that

$$p \in \bigcap_{n\geq 1} \mathrm{cl}\left(\bigcup_{m\geq n} A_m\right).$$

Let now $0 < \epsilon < 1/2$ and let $n_0 \in \mathbb{N}$ be such that

$$\forall m \geq n_0\ F_{AA_m}(\epsilon) > 1 - \epsilon.$$

By (4.6) it follows

$$\forall m \geq n_0\ \forall q \in A_m\ \exists p \in A \text{ such that } F_{pq}(\epsilon) > 1 - \epsilon,$$

so that

$$\forall m \geq n_0\ A_m \subset A_\epsilon,$$

or, equivalently,

$$\bigcup_{m\geq n_0} A_m \subset A_\epsilon.$$

But then

$$\bigcap_{n\geq 1} \mathrm{cl}\left(\bigcup_{m\geq n} A_m\right) \subset \mathrm{cl}\left(\bigcup_{m\geq n_0} A_m\right) \subset A_{2\epsilon}.$$

Since $0 < \epsilon < 1/2$ is arbitrary we have

$$\bigcap_{n\geq 1} \mathrm{cl}\left(\bigcup_{m\geq n} A_m\right) \subset \bigcap_{0<\epsilon<1/2} A_{2\epsilon} = \mathrm{cl}\, A = A.$$

It follows

$$A = \bigcap_{n\geq 1} \mathrm{cl}\left(\bigcup_{m\geq n} A_m\right). \tag{4.20}$$

Let's prove now that

$$A \subset \bigcap_{\epsilon>0} \bigcup_{n\geq 1} \bigcap_{m\geq n} (A_m)_\epsilon. \tag{4.21}$$

For $0 < \epsilon < 1/2$ choose $n_0 \in \mathbb{N}$ such that

$$\forall m \geq n_0\ F_{AA_m}(\epsilon) > 1 - \epsilon.$$

By (4.5) we have

$$\forall m \geq n_0 \; \forall p \in A \; \exists q \in A_m \; F_{pq}(\epsilon) > 1 - \epsilon,$$

implying

$$A \subset \bigcap_{m \geq n_0} (A_m)_\epsilon \subset \bigcup_{n \geq 1} \bigcap_{m \geq n} (A_m)_\epsilon$$

Again, since $0 < \epsilon < 1/2$ was arbitrarily chosen, we get (4.21).

Finally, prove that

$$B := \bigcap_{\epsilon > 0} \bigcup_{n \geq 1} \bigcap_{m \geq n} (A_m)_\epsilon \subset \bigcap_{n \geq 1} \operatorname{cl} \left(\bigcup_{m \geq n} A_m \right) =: C. \tag{4.22}$$

If $p \in B$ then

$$\forall \epsilon, \; 0 < \epsilon < 1, \; \exists n_0(\epsilon) \; \forall m \geq n_0(\epsilon) \; p \in (A_m)_\epsilon.$$

For $n \geq 1$ letting $m = \max\{n, n_0(\epsilon)\}$ we have

$$p \in (A_m)_\epsilon \subset \left(\bigcup_{m' \geq n} A_{m'} \right)_\epsilon .$$

We have obtained

$$\forall n \geq 1 \; \forall \epsilon > 0 \; p \in \left(\bigcup_{m' \geq n} A_{m'} \right)_\epsilon ,$$

implying

$$\forall n \geq 1 \quad p \in \operatorname{cl} \left(\bigcup_{m' \geq n} A_{m'} \right),$$

so that

$$p \in \bigcap_{n \geq 1} \operatorname{cl} \left(\bigcup_{m' \geq n} A_{m'} \right).$$

Combining now (4.20), (4.21) and (4.22) we obtain (4.18). □

Now we shall prove that the family $P_{fc}(L)$ of all nonempty closed convex subsets of a complete Šerstnev random normed space L is complete with respect to the probabilistic Pompeiu-Hausdorff metric H.

A *Šerstnev random normed space* (RN space) is a triple (L, ν, τ) where L is a real linear space, τ is a continuous triangle function such that $\tau(D^+ \times D^+) \subset D^+$, and ν is a mapping $\nu : L \to D^+$ satisfying the following conditions:

(RN1) $\nu(p) = \epsilon_0 \iff p = \theta$;

(RN2) $\nu(ap)(x) = \nu(p)(\frac{x}{|a|})$ for $x \geq 0$ and $a \neq 0$;

(RN3) $\nu(p + q) \geq \tau(\nu(p), \nu(q)),\ p, q \in L$.

If (L, ν, τ) is a Šerstnev RN space then

$$\rho(p, q) = \nu(p - q),\ p, q \in L, \tag{4.23}$$

is a random metric on L. The topology of L is the strong topology corresponding to the random metric (4.23), and L is a metrizable topological vector space with respect to this topology. Random normed spaces were defined and studied by A. N. Šerstnev [21, 22, 24] (see also [19, Ch. 15, Sect. 1]).

The following result holds:

Theorem 4.8. *Let (L, ν, τ) be a Šestnev random normed space with sup-continuous triangle function satisfying the condition*

$$\tau(F, G)(x) \geq \sup_{t \in [0,1]} \min\{F(tx), G((1 - t)x)\}, \tag{4.24}$$

for $x \geq 0$ and $F, G \in D^+$.

Then the family $P_{fc}(L)$ of all nonempty closed convex subsets of L is closed in $P_f(L)$ with respect to the probabilistic Pompeiu-Hausdorff metric H, hence complete if the random normed space L is complete.

If L is complete then the family $P_{kc}(L)$ of all nonempty compact convex subsets of L is complete with respect to the probabilistic Pompeiu-Hausdorff metric.

Proof. Observe first that if the set $A \subset L$ is convex then the set A_ϵ is convex too.

Indeed, let $q_1, q_2 \in A_\epsilon$ and $t_1, t_2 > 0, ; t_1 + t_2 = 1$. If $p_1, p_2 \in A$ are such that $\nu(p_i - q_i)(\epsilon) > 1 - \epsilon,\ i = 1, 2$, then $t_1 p_1 + t_2 p_2 \in A$ and, by (4.24) and (RN2),

$$\nu(t_1p_1 + t_2p_2 - (t_1q_1 + t_2q_2))(\epsilon) \geq$$
$$\geq \min\{\nu(t_1(p_1 - q_1))(t_1\epsilon), \nu(t_2(p_2 - q_2))(t_2\epsilon)\}$$
$$= \min\{\nu(p_1 - q_1)(\epsilon), \nu(p_2 - q_2)(\epsilon)\} > 1 - \epsilon,$$

showing that $t_1q_1 + t_2q_2 \in A_\epsilon$.

Let now (A_n) be a sequence of nonempty closed convex subsets of L converging to $A \in P_f(L)$ with respect to H. By Proposition 4.7

$$A = \bigcap_{\epsilon>0} \bigcup_{n\geq 1} \bigcap_{m\geq n} (A_m)_\epsilon.$$

Since each A_m is convex, the same is true for $(A_m)_\epsilon$, as well as for

$$B_{n,\epsilon} = \bigcap_{m\geq n} (A_m)_\epsilon,\ n = 1, 2, ...$$

The union of the increasing sequence $B_{1,\epsilon} \subset B_{2,\epsilon} \subset ...$ of convex sets will be convex too, so that their intersection for all $\epsilon > 0$ is a convex set.

The assertion concerning the family $P_{kc}(L)$ of all nonempty compact convex subsets of L follows from Theorem 4.5 and the first assertion of the theorem. $\square$

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HOMOTOPIC EMBEDDINGS IN n-GROUPS

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Dedicated to Professor Petru Blaga at his 60^{th} *anniversary*

Abstract. In this paper we prove necessary and sufficient conditions for homotopic embeddings of an n-groupoid in an n-group. The results obtained are generalization for $n > 2$ of the Malcev [2], [3] and Rado [4] results. We prove that an $\mathcal{A}$ semirectangular partial groupoid can be homotopic embedded in a group if and only if it is with cancellation and in $\mathcal{A}$ the Malcev conditions are satisfied. Also we proved that a n-groupoid can be homotopic embedded in a n-group if and only if it is with cancellation and in it all the Malcev conditions are satisfied.

In this paper we prove necessary and sufficient conditions for homotopic embeddings of an n-groupoid in an n-group.

Let $A_n = \{L_i^k, \overline{L}_i^k, R_i^k, \overline{R}_i^k \mid k = \overline{1, n-1}, i \in \mathbb{N}\}$ be the alphabet of the n-ary Malcev symbols. An n-ary Malcev sequence is a word in the A_n alphabet which satisfies the following conditions:

i) each symbols from A_n appears in a word at the most once;

ii) the L_i^k, R_i^k symbols appear in the natural order of inferior index;

iii) if in a word appears the $L_i^k(R_i^k)$ symbol, then also the $\overline{L}_i^k, (\overline{R}_i^k)$ appears and $L_i^k(R_i^k)$ symbol precedes the $\overline{L}_i^k(\overline{R}_i^k)$;

iv) if the $L_j^p(R_j^p)$ symbol is between the L_i^k and $\overline{L}_i^k$ (respectively R_i^k and $\overline{R}_i^k$), then also $\overline{L}_j^p(\overline{R}_j^p)$ is between L_i^k and $\overline{L}_i^k$ (respectively R_i^k and $\overline{R}_i^k$).

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Let be $\mathcal{A}$ an n-groupoid. To each n-ary Malcev sequence I an equalities system is attached by the following table as in the binary case $\sigma(I)$

L^k	$\overline{L}^k$	R^{n-k}	$\overline{R}^{n-k}$
$\alpha(x_1^k, u_{k+1}^n)$	$\alpha(\overline{x}_1^k, \overline{u}_{k+1}^n)$	$\alpha(v_1^k, y_{k+1}^n)$	$\alpha(\overline{v}_1^k, \overline{y}_{k+1}^n)$
$\alpha(\overline{x}_1^k, u_{k+1}^n)$	$\alpha(x_1^k, \overline{u}_{k+1}^n)$	$\alpha(v_1^k, \overline{y}_{k+1}^n)$	$\alpha(\overline{v}_1^k, y_{k+1}^n)$

Tabel nr.1

The closing equality to the $\sigma(I)$ system is formed using the table as in the binary case.

We will say that in $\mathcal{A}$ the Malcev conditions are satisfied if for any I Malcev sequence from the realisation in $\mathcal{A}$ of the $\sigma(I)$ equalities system it results the realisation in $\mathcal{A}$ of the closing equalities of this system.

Let be $\mathcal{A}$ an n-groupoid which does not have isotopes with unit.

Lemma 1. *Let be $\mathcal{A}$ a cancellative n-groupoid. If in $\mathcal{A}$ are satisfied the Malcev conditions, then $\mathcal{A}$ can be embedded in a partial n-groupoid having an unit which belongs to the center.*

Proof. Let $(a_1^n) \in A^n$, $e = \alpha(a_1^n)$ and we consider the translations $T_i, i = \overline{1,n}$, determined by (a_1^n). On the A set we define the β n-ary partial operation by $D(\beta) = \{(T_1(x_1), \ldots, T_n(x_n)) \,|\, (x_1^n) \in A^n\} \cup \{(\overset{i-1}{e}, x, \overset{n-i}{e}) \,|\, i = \overline{1,n}, x \in A\}$, $\beta(T_1(x_1), \ldots, T_n(x_n)) = \alpha(x_1^n), \beta(\overset{i-1}{e}, x, \overset{n-i}{e}) = x$, $i = \overline{1,n}, x \in A$.

Because $\mathcal{A}$ is cancellative $[\{T_1^n\}, 1_A] : \mathcal{A} \to (A, \beta)$ is a monotopy.

Let $(y_1^{n-1}) \in A^{n-1}$ and $(y_1^{i-1}, e, y_i^{n-1}), (y_1^{j-1}, e, y_j^{n-1}) \in \mathcal{D}(\beta)$ and we suppuse that $i > j$.. From the definition of the β operation it results that there are $x_1, \ldots, x_{n-1}, \overline{x}_j, \ldots, \overline{x}_{i-1} \in A$ so that

$$y_1 = T_1(x_1), \ldots, y_{j-1} = T_{j-1}(x_{j-1}), y_j = T_j(x_j) = T_{j+1}(\overline{x}_j), \ldots,$$
$$y_{i-1} = T_{i-1}(x_{i-1}) = T_i(\overline{x}_{i-1}), y_i = T_{i+1}(x_i), \ldots, y_{n-1} = T_n(x_{n-1}) \qquad (1)$$

We obtain the equalities system

$$\alpha(x_1^{i-1}, a_i, x_i^{n-1}) = \alpha(x_1^{i-2}, a_{i-1}, \overline{x}_{i-1}, x_i^{n-1})$$

from $T_{i-1}(x_{i-1}) = T_i(\overline{x}_{i-1})$ with $D_{i-1,i}$

$$\alpha(x_1^{i-2}, a_{i-1}, \overline{x}_{i-1}, x_i^{n-1}) = \alpha(x_1^{i-3}, a_{i-2}, \overline{x}_{i-2}^{i-1}, x_i^{n-1})$$

from $T_{i-2}(x_{i-2}) = T_{i-1}(\overline{x}_{i-2})$ with $D_{i-2,i-1}$;

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\alpha(x_1^{j}, a_{j+1}, \overline{x}_{j+1}^{i-1}, x_i^{n-1}) = \alpha(x_1^{j-1}, a_j, \overline{x}_j^{i-1}, x_i^{n-1})$$

from $T_j(x_j) = T_{j+1}(\overline{x}_j)$ with $D_{j,j+1}$. Therefore

$$\alpha(x_1^{i-2}, a_i, x_i^{n-1}) = \alpha(x_1^{j-1}, a_j, \overline{x}_j^{i-1}, x_i^{n-1}),$$

that is $\beta(y_1^{i-1}, e, y_i^{n-1}) = \beta(y_i^{j-1}, e, y_j^{n-1})$. $\square$

We consider the (A,γ) extension of (A,β) defined by

$$D(\gamma) = D(\beta) \cup \bigcup_{i=1}^{n} \{(y_1^{i-1}, e, y_i^{n-1}) \,|\, \exists j \in \{1, 2, \ldots, n\}$$

such that $(y_1^{j-1}, e, y_j^{n-1}) \in \mathcal{D}(\beta)\}, \gamma\big|_{D(\beta)} = \beta$ and $\gamma(y_1^{i-1}, e, y_i^{n-1}) = \beta(y_1^{j-1}, e, y_j^{n-1}), i = \overline{1,n}$. From the lemma 1 it results that the γ operation is well defined.

Lemma 2. *In the partial n-groupoid (A,γ) the Malcev conditions are satisfied.*

Proof. Let be the table

L^k	$\overline{L}^k$
$\gamma(u_1^k, u_{k+1}^n)$	$\gamma(v_1^k, v_{k+1}^n)$
$\gamma(v_1^k, u_{k+1}^n)$	$\gamma(u_1^k, v_{k+1}^n)$

$\square$

We study only the nontrivial cases. For example:

$$\begin{aligned}
\gamma(u_1^k, u_{k+1}^n) &= \beta(u_1^{p-1}, e, u_p^k, u_{k+1}^{j-1}, u_{j+1}^n)\\
\gamma(v_1^k, u_{k+1}^n) &= \beta(v_1^{q-1}, v_{q+1}^k, u_{k+1}^{r-1}, e, u_r^n)\\
\gamma(v_1^k, v_{k+1}^n) &= \beta(v_1^{s-1}, e, v_s^k, v_{k+1}^{t-1}, v_{t+1}^n)\\
\gamma(u_1^k, v_{k+1}^n) &= \beta(u_1^{i-1}, u_{i+1}^k, v_{k+1}^{g-1}, e, v_{g+1}^n)
\end{aligned} \tag{2}$$

We prove from (2) it follow:

$$\begin{aligned}
\beta(u_1^{p-1}, e, u_p^k, \overset{n-k-1}{e}) &= \beta(u_1^{i-1}, u_{i+1}^k, \overset{n-k+1}{e})\\
\beta(\overset{k+1}{e}, u_{k+1}^{j-1}, u_{j+1}^n) &= \beta(\overset{k-1}{e}, u_{k+1}^{r-1}, e, u_r^n)\\
\beta(v_1^{s-1}, e, v_s^k, \overset{n-k-1}{e}) &= \beta(v_1^{q-1}, v_{q+1}^k, \overset{n-k+1}{e})\\
\beta(\overset{k+1}{e}, v_{k+1}^{t-1}, v_{t+1}^n) &= \beta(\overset{k-1}{e}, v_{k+1}^{g-1}, e, v_g^n)
\end{aligned} \tag{3}$$

Let $i < p$. From the definition of β operation it results that:

$$\begin{aligned}
&u_1 = T_1(x_1), \ldots, u_{i-1} = T_{i-1}(x_{i-1}), u_{i+1} = T_{i+1}(x_{i+1}) = T_i(\overline{x}_{i+1}), \ldots,\\
&u_{p-1} = T_{p-1}(x_{p-1}) = T_{p-2}(\overline{x}_{p-1}),\\
&u_p = T_{p+1}(x_p) = T_{p-1}(\overline{x}_p), \ldots, u_k = T_{k+1}(x_k) = T_{k-1}(\overline{x}_k).
\end{aligned} \tag{4}$$

From (4) we get the equalities system:

$$\alpha(x_1^{i-1}, a_i, x_{i+1}^{p-1}, a_p, x_p^k, a_{k+2}^n) = \alpha(x_1^{i-1}, \overline{x}_{i+1}, a_{i+1}, x_{i+2}^{p-1}, a_p, x_p^k, a_{k+2}^n)$$

from $T_{i+1}(x_{i+1}) = T_i(\overline{x}_{i+1})$ with $D_{i,i+1}$;

$$\alpha(x_1^{i-1}, \overline{x}_{i+1}, a_{i+1}, x_{i+2}^{p-1}, a_p, x_p^k, a_{k+2}^n) = \alpha(x_1^{i-1}, \overline{x}_{i+1}^{i+2}, a_{i+2}, x_{i+3}^{p-1}, a_p, x_p^k, a_{k+2}^n)$$

from $T_{i+2}(x_{i+2}) = T_{i+1}(\overline{x}_{i+2})$ with $D_{i+1,i+2}$;

. . . .

$$\alpha(x_1^{i-1}, \overline{x}_{i+1}^{p-2}, a_{p-2}, x_{p-1}, a_p, x_p^k, a_{k+2}^n) = \alpha(x_1^{i-1}, \overline{x}_{i+1}^{p-1}, a_{p-1}, a_p, x_p^k, a_{k+2}^n)$$

from $T_{p-1}(x_{p-1}) = T_{p-2}(\overline{x}_{p-1})$ with $D_{p-2,p-1}$;

$$\alpha(x_1^{i-1}, \overline{x}_{i+1}^{p-1}, a_{p-1}, a_p, x_p^k, a_{k+2}^n) = \alpha(x_1^{i-1}, \overline{x}_{i+1}^{p}, a_p, a_{p+1}, x_{p+1}^k, a_{k+2}^n)$$

from $T_{p+1}(x_p) = T_{p-1}(\overline{x}_p)$ with $D_{p-1,p+1}$;

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\alpha(x_1^{i-1}, \overline{x}_{i+1}^{k-1}, a_{k-1}, a_k, x_k, a_{k+2}^n) = \alpha(x_1^{i-1}, \overline{x}_{i+1}^k, a_k^n)$$

from $T_{k+1}(x_k) = T_{k-1}(\overline{x}_k)$ with $D_{k-1,k+1}$. Therefore

$$\alpha(x_1^{i-1}, a_i, x_{i+1}^{p-1}, a_p, x_p^k, a_{k+2}^n) = \alpha(x_1^{i-1}, \overline{x}_{i+1}^k, a_k^n)$$

that is$\beta(u_1^{p-1}, e, u_p^k, \overset{n-k-1}{e}) = \beta(u_1^{i-1}, u_{i+1}^k, \overset{n-k+1}{e})$.

Similarly we study the other cases. In (A, β) are satisfied the $D_{i,j}$ conditions: $\beta(y_1^{i-1}, u_i^j, y_{j+1}^n) = \beta(y_1^{i-1}, v_i^j, y_{j+1}^n)$ and

$$(z_1^{i-1}, u_i^j, z_{j+1}^n), (z_1^{i-1}, v_i^j, z_{j+1}^n) \in \mathcal{D}(\beta) \Rightarrow$$
$$\beta(z_1^{i-1}, u_i^j, z_{j+1}^n) = \beta(z_1^{i-1}, v_i^j, z_{j+1}^n).$$

Thus from (3) we obtain the equalities system:

$$\beta(u_1^{p-1}, e, u_p^k, u_{k+1}^{j-1}, u_{j+1}^n) = \beta(u_1^{i-1}, u_{i+1}^k, \overset{2}{e}, u_{k+1}^{j+1})$$

from (3) with $D_{1,k+1}$;

$$\beta(v_1^{s-1}, e, v_s^k, v_{k+1}^{t-1}, v_{t+1}^n) = \beta(v_1^{q-1}, v_{q+1}^k, \overset{2}{e}, v_{k+1}^{t-1}, v_{t+1}^n)$$

from (3) with $D_{1,k+1}$;

$$\beta(v_1^{q-1}, v_{q+1}^k, u_{k+1}^{r-1}, e, u_r^n) = \beta(v_1^{q-1}, v_{q+1}^k, \overset{2}{e}, u_{k+1}^{j-1}, u_{j+1}^n)$$

from (3) with $D_{k-1,n}$;

$$\beta(u_1^{i-1}, u_{i+1}^k, v_{k+1}^{g-1}, e, v_g^n) = \beta(u_1^{i-1}, u_{i+1}^k, \overset{2}{e}, v_{k+1}^{t-1}, v_{t+1}^n)$$

from (3) with $D_{k-1,n}$.

Therefore the initial table becomes

L^k	$\overline{L}^k$
$\beta(u_1^{i-1}, u_{i+1}^k, \overset{2}{e}, u_{k+1}^{j-1}, u_{j+1}^n)$	$\beta(v_1^{q-1}, v_{q+1}^k, \overset{2}{e}, v_{k+1}^{t-1}, v_{t+1}^n)$
$\beta(v_1^{q-1}, v_{q+1}^k, \overset{2}{e}, u_{k+1}^{j-1}, u_{j+1}^n)$	$\beta(u_1^{i-1}, u_{i+1}^k, \overset{2}{e}, v_{k+1}^{t-1}, v_{t+1}^n)$

Similarly it is demonstrated that a $\begin{array}{c|c} R^k & \overline{R}^k \\ \hline & \end{array}$ table in (A,γ) becomes a table of a similar type $\begin{array}{c|c} R^k & \overline{R}^k \\ \hline & \end{array}$. Thus, to every equalities system $\sigma(I)$ in (A,γ) it corresponds an equalities system $\tilde{\sigma}(I)$ in (A,β) for the same Malcev sequence I. The closing equality of $\sigma(I)$ becomes the closing equality of the $\tilde{\sigma}(I)$ system. Because in $\mathcal{A}$ are satisfied the Malcev conditions it results that also in (A,β) are satisfied the Malcev conditions, so they are satisfied also in (A,γ).

Lemma 3. *i)* $(u_1^n) \in \mathcal{D}(\gamma) \Rightarrow (u_1^{n-1}, e), (e, u_2^n) \in \mathcal{D}(\gamma)$

ii) $(u_1^n), (v_1^n) \in \mathcal{D}(\gamma)$ *and* $\gamma(u_1^{n-1}, e) = v_1, \gamma(e, v_2^n) = u_n \Rightarrow (v_1, \overset{n-2}{e}, u_n) \in \mathcal{D}(\gamma)$ *and* $\gamma(u_1^n) = \gamma(v_1^n) = \gamma(v_1, \overset{n-2}{e}, u_n)$.

Proof. If $v_1 = e$ or $u_n = e$, then is obvious that $(v_1, \overset{n-2}{e}, u_n) \in \mathcal{D}(\gamma)$. We suppose that $v_1 \neq e$ and $u_n \neq e$. Then $u_n \notin T_n(A) \Rightarrow u_n \in T_{n-1}(A)$ and $v_1 \notin T_1(A) \Rightarrow v_1 \in T_2(A)$. We suppose that $u_n \notin T_n(A)$. From $(u_1^n) \in \mathcal{D}(\gamma) \Rightarrow \gamma(u_1^n) = \beta(u_1^{i-1}, u_{i+1}^n, e)$, that is $u_n \in T_{n-1}(A)$. If $v_1 \notin T_1(A)$, then from $(v_1^n) \in \mathcal{D}(\gamma) \Rightarrow \gamma(v_1^n) = \beta(e, v_1^{j-1}, v_{j+1}^n)$, that is $v_1 \in T_2(A)$. We suppose that $(v_1, \overset{n-2}{e}, u_n) \notin \mathcal{D}(\gamma)$. Then $u_n \in T_{n-1}(A), v_1 \in T_2(A)$ and $n-1 = 2$. But then $\gamma(u_1^3) = \beta(u_1, u_3, e)$ (or $\beta(u_2, u_3, e)$), so $v_1 = \gamma(u_1^2, e) = u_1$ (or u_2), that is $(v_1, e, u_3) \in \mathcal{D}(\gamma)$ which is false.

From $\gamma(u_1^{n-1}, e) = v_1 = \gamma(v_1, \overset{n-1}{e})$ with $D_{1,n-1} \Rightarrow \gamma(u_1^n) = \gamma(v_1, \overset{n-2}{e}, u_n)$, and from $\gamma(e, v_2^n) = u_n = \gamma(\overset{n-1}{e}, u_n)$ with $D_{2,n} \Rightarrow \gamma(v_1^n) = \gamma(v_1, \overset{n-2}{e}, u_n)$. $\square$

We consider the (A,δ) extension of (A,γ) defined by $D(\delta) = D(\gamma) \cup D$, where $D \subseteq A^n$ is defined by $(y_1^n) \in D \Leftrightarrow$

i) there are $i, j, k \in \mathbb{N}$ so that $(y_1^n) = (\overset{i}{e}, \gamma(u_1^{n-1}, e), \overset{j}{e}, u_n, \overset{k}{e})$ and $(u_1^n) \in D(\gamma)$

or

ii) there are $r, s, t \in \mathbb{N}$ so that $(y_1^n) = (\overset{r}{e}, v_1, \overset{s}{e}, \gamma(e, v_2^n), \overset{t}{e})$ and $(v_1^n) \in D(\gamma)$;

$\delta/_{D(\gamma)} = \gamma$ and $\delta(\overset{i}{e}, \gamma(u_1^{n-1}, e), \overset{j}{e}, u_n, \overset{k}{e}) = \gamma(u_1^n)$, $\delta(\overset{r}{e}, v_1, \overset{s}{e}, \gamma(e, v_2^n), \overset{t}{e}) = \gamma(v_1^n)$.

From the 2 and 3 lemmas it results that the δ operation is well defined. (A,δ) is a partial n-grupoid with unit e which is in the centre.

On the set A we define the binary operation " $\circ$ " by $(y_1, y_2) \in D(\circ) \Leftrightarrow (y_1, \overset{n-2}{e}, y_2) \in D(\delta)$ and $y_1 \circ y_2 = \delta(y_1, \overset{n-2}{e}, y_2)$.

Lemma 4. *If* $(y_1^n) \in D(\gamma)$, *then* $\gamma(y_1^n) = ((y_1 \circ y_2) \circ \cdots \circ y_{n-1}) \circ y_n$.

Proof.

$$(y_1^n) \in D(\gamma) \Rightarrow (y_1^{n-1}, e) \in D(\gamma),$$

$$\gamma(y_1^n) = \delta(\gamma(y_1^{n-1}, e), \overset{n-2}{e}, y_n) = \gamma(y_1^{n-1}, e) \circ y_n$$

$$(y_1^{n-1}, e) \in D(\gamma) \Rightarrow (y_1^{n-2}, e, y_{n-1}) \in D(\gamma) \Rightarrow$$

$$\gamma(y_1^{n-1}, e) = \gamma(y_1^{n-2}, e, y_{n-1}) = \delta(\gamma(y_1^{n-2}, \overset{2}{e}), \overset{n-2}{e}, y_{n-1}) =$$

$$= \gamma(y_1^{n-2}, e^2) \circ y_{n-1}, \text{etc.} \quad \square$$

Lemma 5. *In the partial groupoid* $(A, \circ)$ *the Malcev conditions are satisfied.*

Proof. We consider the table

L	L^*
$y_1 \circ y_2$	$\overline{y}_1 \circ \overline{y}_2$
$\overline{y}_1 \circ y_2$	$y_1 \circ \overline{y}_2$

Table nr.2

$\square$

We have several possibilities. We will consider only two of them

1. $y_1 = \gamma(u_1^{n-1}, e), y_2 = u_n, \overline{y}_1 = \gamma(v_1^{n-1}, e)$, $\overline{y}_2 = v_n$, $y_1 \circ y_2 = \gamma(u_1^n)$, $y_1 \circ \overline{y}_2 = \gamma(u_1^{n-1}, v_n), \overline{y}_1 \circ \overline{y}_2 = \gamma(v_1^n)$ and $\overline{y}_1 \circ y_2 = \gamma(v_1^{n-1}, u_n)$.

The above table becomes

L^{n-1}	$\overline{L}^{n-1}$
$\gamma(u_1^{n-1}, u_n)$	$\gamma(v_1^{n-1}, v_n)$
$\gamma(v_1^{n-1}, u_n)$	$\gamma(u_1^{n-1}, v_n)$

Table nr.3

2. $y_1 = \gamma(u_1^{n-1}, e) = t_1, y_2 = u_n = \gamma(e, v_2^n)$, $\overline{y}_1 = \gamma(s_1^{n-1}, e) = v_1$, $\overline{y}_2 = s_n = \gamma(e, t_2^n)$, $y_1 \circ y_2 = \gamma(u_1^n), \overline{y}_1 \circ y_2 = \gamma(v_1^n)$, $\overline{y}_1 \circ \overline{y}_2 = \gamma(s_1^n), y_1 \circ \overline{y}_2 = \gamma(t_1^n)$.

Similarly to the prove of lemma 3 we obtain:

$$(t_1, \overset{n-2}{e}, u_n), (v_1, \overset{n-2}{e}, u_n), (v_1, \overset{n-2}{e}, s_n), (t_1, \overset{n-2}{e}, s_n) \in D(\gamma).$$

So

$$
\begin{array}{lll}
\gamma(u_1^{n-1}, e) = \gamma(t_1, \overset{n-1}{e}) & \text{cu} & D_{1,n-1} \Rightarrow \gamma(u_1^n) = \gamma(t_1, \overset{n-2}{e}, u_n), \\
\gamma(s_1^{n-1}, e) = \gamma(v_1, \overset{n-1}{e}) & \text{cu} & D_{1,n-1} \Rightarrow \gamma(s_1^n) = \gamma(v_1, \overset{n-2}{e}, s_n), \\
\gamma(e, v_2^n) = \gamma(\overset{n-1}{e}, u_n) & \text{cu} & D_{2,n} \Rightarrow \gamma(v_1^n) = \gamma(v_1, \overset{n-2}{e}, u_n), \\
\gamma(e, t_2^n) = \gamma(\overset{n-1}{e}, s_n) & \text{cu} & D_{2,n} \Rightarrow \gamma(t_1^n) = \gamma(t_1, \overset{n-2}{e}, s_n),
\end{array}
$$

Therefore the table 3 becomes:

L^{n-1}	$\overline{L}^{n-1}$
$\gamma(t_1, \overset{n-2}{e}, u_n)$	$\gamma(v_1, \overset{n-2}{e}, s_n)$
$\gamma(v_1, \overset{n-2}{e}, u_n)$	$\gamma(t_1, \overset{n-2}{e}, s_n)$

Table nr.4

The other cases are studied in a similar way.

By a similar reasoning it is demonstrated that a $\begin{array}{c|c} R & R^* \\ \hline & \end{array}$ table in $(A, \circ)$ becomes a $\begin{array}{c|c} R^{n-1} & \overline{R}^{n-1} \\ \hline & \end{array}$ table in (A, γ).

In accordance with lemma 2 we obtain that in the partial groupoid $(A, \circ)$ the Malcev conditions are satisfied.

Definition 1. A binary groupoid $(A, \cdot)$ is called semirectangular if the following conditions are satisfied.

i) $(x_1, y_1), (x_1, y_2), (x_2, y_2), (x_3, y_1) \in D(\cdot) \Rightarrow (x_2, y_1) \in D(\cdot)$ or $(x_3, y_2) \in D(\cdot)$;

ii) $(x_1, y_1), (x_1, y_2), (x_2, y_2), (x_2, y_3) \in D(\cdot) \Rightarrow (x_1, y_3) \in D(\cdot)$ or $(x_2, y_1) \in D(\cdot)$.

Lemma 6. *The partial groupoid $(A, \circ)$ is semirectangular.*

Proof. i) Let $x_1 = \gamma(u_1^{n-1}, e) = v_1, y_1 = u_n = \gamma(e, v_2^n)$, $x_2 = \gamma(t_1^{n-1}, e) = s_1$, $y_2 = t_n = \gamma(e, s_2^n)$, $x_3 = w_1 = \gamma(r_1^{n-1}, e)$ and $x_1 \circ y_1 = \gamma(u_1^n), x_1 \circ y_2 = \gamma(v_1, s_2^n), x_2 \circ y_2 = \gamma(t_1^n), x_3 \circ y_1 = \gamma(w_1, v_2^n)$.

For $n > 3$ using the prove of lemma 3 ii) we obtain that $(w_1, \overset{n-2}{e}, t_n) \in D(\circ)$, that is$(x_3, y_2) \in D(\circ)$. We suppose that $n = 3$ and $(w_1, \overset{n-2}{e}, t_n) \notin D(\circ)$. Then $t_3 \in T_2(A) \Rightarrow \gamma(t_1^3) = \beta(t_2, t_3, e)$ (that is $t_1 = e$) or $\gamma(t_1^3) = \beta(t_1, t_3, e)$ (that is $t_2 = e$).

From $s_1 = \gamma(t_1^2, e) \Rightarrow s_1 = t_2$ or $s_1 = t_1$, so $s_1 \in T_1(A)$; $w_1 \in T_2(A) \Rightarrow \gamma(w_1, v_2^3) = \beta(e, w_1, v_3)$, that is $v_2 = e$, so $u_3 = \gamma(e, v_2^3) = v_3 \in T_3(A)$ or $\gamma(w_1, v_2^3) = \beta(e, w_1, v_2)$, that is $v_3 = e$, so $u_3 = \gamma(e, v_2^3) = v_2 \in T_3(A)$. In conclusion, $s_1 \in T_1(A)$ and $u_3 \in T_3(A) \Rightarrow (s_1, e, u_3) \in D(\circ)$, that is $(x_2, y_1) \in D(\circ)$.

The other cases are studied similarly. $\square$

Lemma 7. *The groupoid $(A, \circ)$ can be homomorphic embedded in a group.*

Proof. The groupoid $(A, \circ)$ can be homomorphic embedded in a group if and only if $\mathcal{R}((A, \circ))$ can be embedded in a g-regular classic net, that is a classic net in which the Reidemeister closure condition is satisfied [1]. We use Radó's results [4] for defined nets associated to n-groupoids: a 2-seminet ρ can be embedded in a 2 g-regular classic net if and only if it is quasiregular and all M-configurations of ρ are closed.

In accordance with [4] the Malcev conditions are exactly $M_3^{\square}$ closing conditions. From the lemmas 5 and 6 it results that in $\mathcal{R}((A, \circ))$ all M_i closing conditions are satisfied $(i = 1, 2, 3)$ (cf. [4]).

We prove that in $\mathcal{R}((A, \circ))$ all M_3 closing conditions are satisfied. Let be a M_3-condition of which M-model is given in fig. 1.

Applying the lemma 6 it results the existence in $\mathcal{R}((A, \circ))$ of at least one of q or r points. We suppose that point q exists. We consider the M model from fig. 2, for the pairs of the nods (p_2', p_2), respectively $(p_1', p_1), (p_3', p_3)$ and (q', q) corresponding respectively the same points in $\mathcal{R}((A, \circ))$.

Repeating this procedure after a finite number of steps (in our case 3) we obtain a M-model to which correspond an equalities system in $(A, \circ)$ associated to a Malcev sequence. By applying this procedure the closing equality is kept. As a consequence of lemma 5 in $\mathcal{R}((A, \circ))$ all M_3-closing conditions are satisfied.

Let be now a M_1 - condition of which M-model is given in fig. 3.

We consider the polygonal line from fig. 4

In accordance with the lemma 6 in $\mathcal{R}((A, \circ))$ there is at least one of the q or r points.

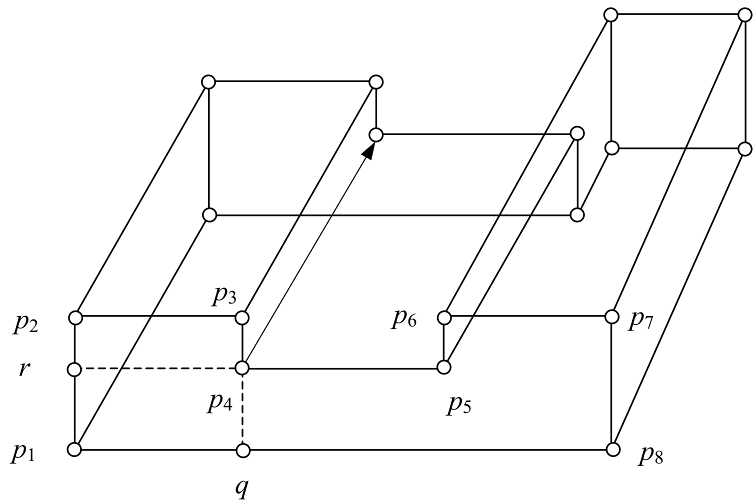

FIGURE 1

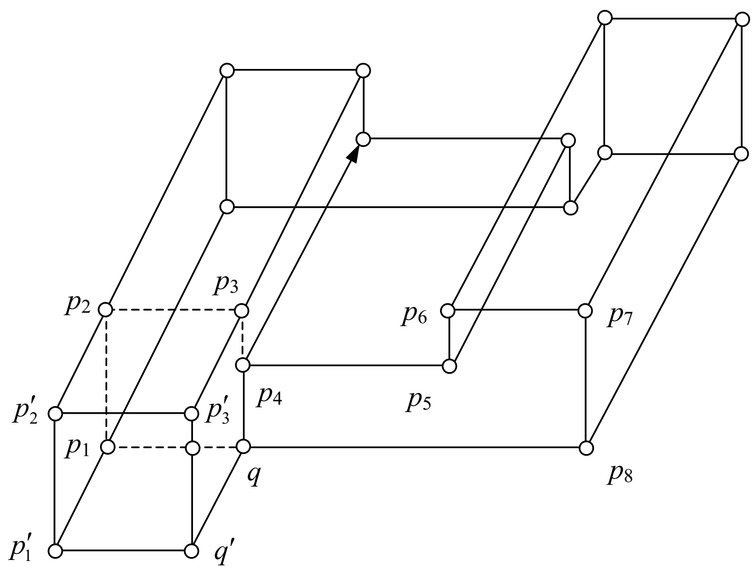

FIGURE 2

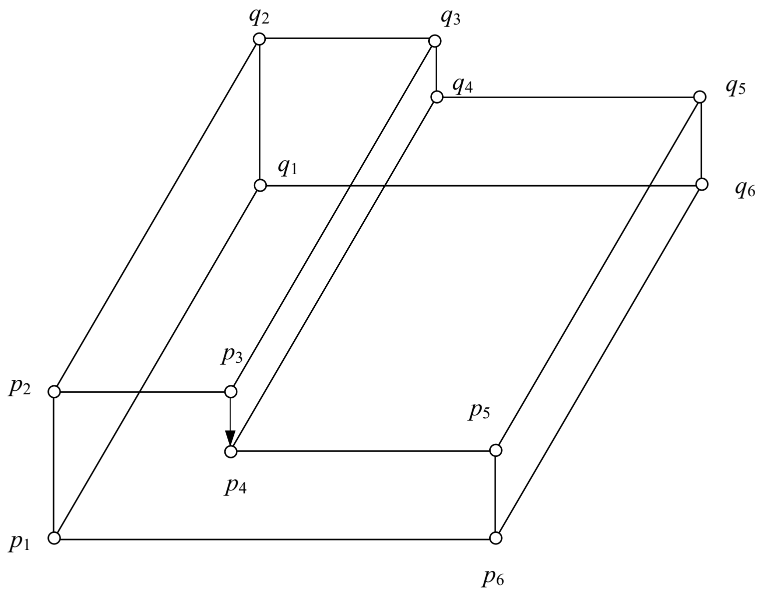

Figure 3

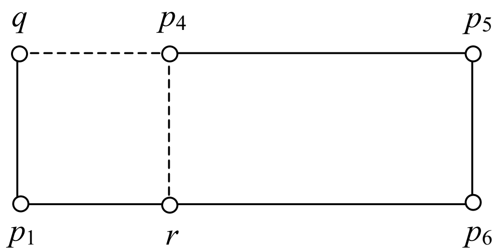

Figure 4

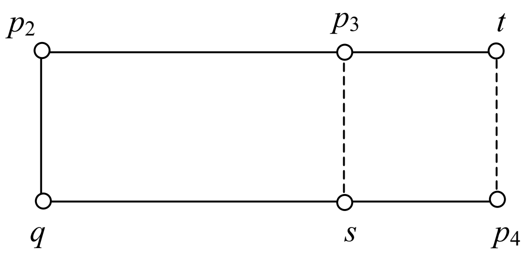

Figure 5

If q exists applying the lemma 6 to the polygonal line it results the existence in $\mathcal{R}((A,\circ))$ of at least one of the t or s points. If r exists, applying the lemma 6 to the polygonal line it results the existence in

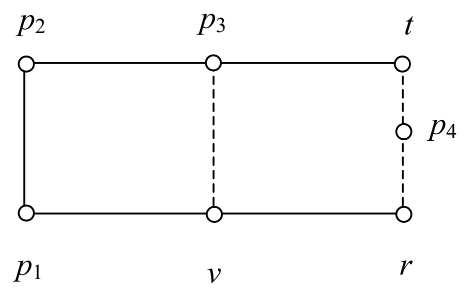

FIGURE 6

$\mathcal{R}((A,\circ))$ of at least one of the t or v points.

If v exists, applying the lemma 6 to the polygonal line from fig. 7 results the existence in $\mathcal{R}((A,\circ))$ of at least one of the t or u points.

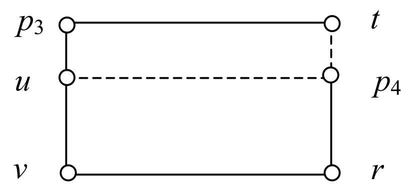

FIGURE 7

Thus we can always consider one of the M-models from fig. 8 and fig. 9. $\square$

To each model corresponds a M_3-condition. From the above results $\mathcal{R}((A,\circ))$ all M_3-conditions are satisfied. If $p_3(x_1', y_1), q_3(x_2, y_2), p_4(x_1, y_3)$, then in the first case $t(x_1, y_1)$, so $x_1 \circ y_1 = x_2 \circ y_2$ and $x_1' \circ y_1 = x_2 \circ y_2 \Rightarrow x_1' = x_1$ (because $(A,\circ)$ has a unit from the lemma 5 it result that $(A,\circ)$ is cancellative).

Therefore the M_1 considerated condition is satisfied.

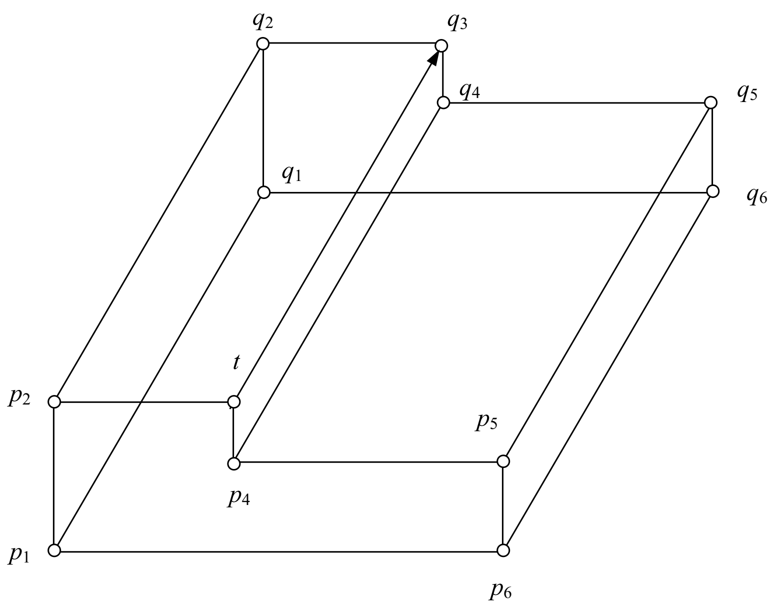

FIGURE 8

In a similar way the second case is proved. Similarly in $\mathcal{R}((A, \circ))$ the M_2 closing condition are satisfied.

From the above proof we obtain, in particular, the following generalization of a result from [4].

Corollary 1. *An $\mathcal{A}$ semirectangular partial groupoid can be homotopic embedded in a group if and only if it is cancellative and in $\mathcal{A}$ the Malcev conditions are satisfied.*

Theorem 1. *An n-grupoid can be homotopic embedded in an n-group if and only if it is cancellative and in it the Malcev conditions are satisfied.*

Proof. Let be a $\mathcal{A}$ a cancellative n-groupoid in which the Malcev conditions are satisfied. From the lemma 1 it results that $[\{T_1^n\}, 1_A] : \mathcal{A} \to (A, \gamma)$ is a homotopic embedding. In accordance with the lemma 7, $(A, \circ)$ can be homomorphic embedded in $(G, \cdot)$; $f : (A, \circ) \to (G, \cdot)$. From the 4 results that $f(\gamma(y_1^n)) = f(y_1) \cdot f(y_2) \cdot \ldots \cdot f(y_n)$,

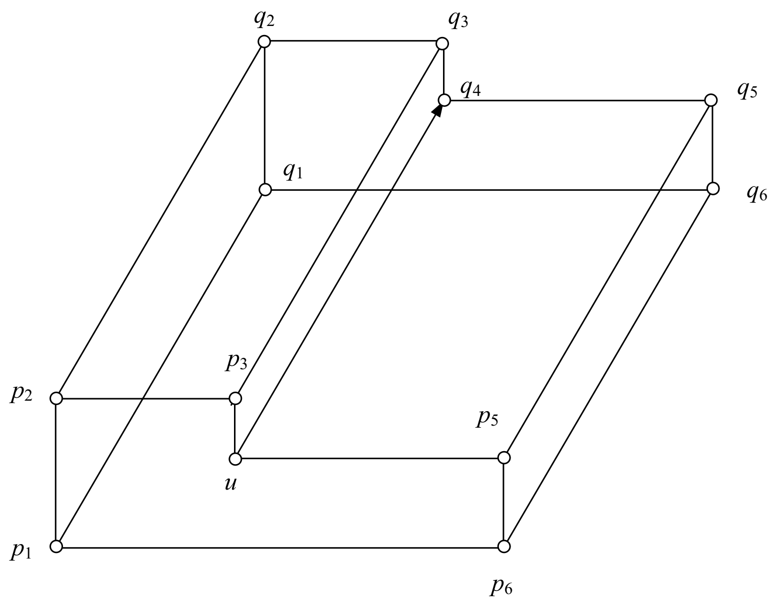

FIGURE 9

that is (A, γ) is homomorphic embedded in n-derivate $\mathcal{G} = (G, \overline{\gamma})$ of group: $(G, \cdot)$ $\overline{\gamma}(g_1^n) = g_1 \cdot g_2 \cdot \ \ldots \ \cdot g_n$.

In conclusion $f[\{T_1^n\}, 1_A] : \mathcal{A} \to \mathcal{G}$ is a homomorphic embedding of $\mathcal{A}$ in an n-group. $\square$

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AN INTEGRAL EQUATION ARISING FROM INFECTIOUS DISEASES, VIA PICARD OPERATOR

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Dedicated to Professor Petru Blaga at his 60^{th} *anniversary*

Abstract. In this paper we use the Picard operators technique to study the following integral equation arising from infectious diseases

$$x(t) = \int_{t-\tau}^{t} f(s, x(s))ds, \ \ t \in \mathbb{R},$$

regarding the existence and uniqueness of the solutions and periodic solutions, lower and upper subsolutions of the integral equation, the data dependence of the solution, the differentiability of the solutions with respect to a parameter.

1. Introduction

We consider the nonlinear integral equation

$$x(t) = \int_{t-\tau}^{t} f(s, x(s))ds, \ \ t \in \mathbb{R}, \tag{1.1}$$

when $\tau > 0$ is a parameter.

This equation can be interpreted as a model for the spread of certain infectious diseases with periodic contact rate that varies seasonally and has been studied in [2], [3], [9], [10], [13] [16], [17], [25].

We will study the solutions of this nonlinear integral equation.

Let X be a nonempty set, d a metric on X and $A : X \to X$ an operator. In this paper we shall use the following notations:

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$$\begin{aligned}
P(X) &: \ = \{Y \subset X |\ Y \neq \varnothing\} \\
F_A &: \ = \{x \in X |\ A(x) = x\} \text{ - the fixed point set of } A \\
(UF)_A &: \ = \{x \in X |\ A(x) \leq x\} \\
(LF)_A &: \ = \{x \in X |\ A(x) \geq x\} \\
I(A) &: \ = \{Y \in P(X) |\ A(Y) \subset Y\} \text{ - the family of invariant and nonempty sets of A} \\
\delta(Y) &: \ = \sup\{d(x,y) |\ x \in Y,\ y \in Y\} \\
I_b(A) &: \ = \{Y \in I(A) |\ \delta(Y) < +\infty\} \\
I_{cl}(A) &: \ = \{Y \in I(A) |\ Y = \overline{Y}\} \\
I_{b,cl}(A) &: \ = \{Y \in I_b(A) |\ Y = \overline{Y}\}.
\end{aligned}$$

Definition 1.1. *(Rus* [17] *or* [18]*) Let (X,d) be a metric space. An operator $A : X \to X$ is a Picard operator (PO) if there exists $x^* \in X$ such that:*

(a) $F_A = \{x^*\}$;

(b) *the sequence $(A^n(x_0))_{n\in N}$ converges to x^*, for all $x_0 \in X$.*

Definition 1.2. *(Rus* [17] *or* [18]*) Let (X,d) be a metric space. An operator $A : X \to X$ is a weakly Picard operator (WPO) if the sequence $(A^n(x_0))_{n\in N}$ converges for all $x_0 \in X$ and the limit (which may depend on x_0) is a fixed point of A.*

If A is a WPO, then we consider the following operator

$$A^\infty : X \to X, \quad A^\infty(x) = \lim_{n\to\infty} A^n(x)$$

It is clear that $A^\infty(X) = F_A$.

In section 3 we need the following results (see [14], [15], [19]).

Lemma 1.1. *(Rus* [19]*) Let $(X,d,\leq)$ be an ordered metric space and $A : X \to X$ an operator, such that:*

(i) *the operator A is increasing ;*

(ii) *A is WPO.*

Then, the operator A^∞ is increasing.

Lemma 1.2. *(Comparison abstract lemma) Let $(X,d,\le)$ be an ordered metric space and $A,B,C : X \to X$ three operators, such that:*

(i) *$A \le B \le C$;*

(ii) *A,B,C are WPOs;*

(iii) *the operator B is increasing.*

Then

$$x \le y \le z \;\;\Rightarrow\;\; A^\infty(x) \le B^\infty(y) \le C^\infty(z).$$

Lemma 1.3. *(Abstract Gronwall lemma) (Rus [15]) Let $(X,d,\le)$ be an ordered metric space and $A : X \to X$ an operator. We suppose that:*

(i) *A is Picard operator;*

(ii) *the operator A is increasing.*

If we denote with x_A^* the unique fixed point of A, then

(a) $x \le A(x) \;\;\Rightarrow\;\; x \le x_A^*$;

(b) $x \ge A(x) \;\;\Rightarrow\;\; x \ge x_A^*$.

Lemma 1.4. *(Rus [19]) Let $(X,d,\le)$ be an ordered metric space and $A : X \to X$ an increasing operator. If $A|_{(UF)_A \cup (LF)_A}$ is Picard operator, then*

$$x \le x_A^* \le y\,,$$

for every $x \in (LF)_A$ and $y \in (UF)_A$.

In section 4 we need the general data dependence theorem (see [12], [16] and [20]).

Theorem 1.1. *(General data dependence theorem). Let (X,d) be a complete metric space and $A,B : X \to X$ two operators. We suppose that:*

(i) *A is an α-contraction ($\alpha < 1$) and $F_A = \{x^*\}$;*

(ii) *$x_B^* \in F_B$;*

(iii) *there exists $\eta > 0$ such that*

$$d(A(x),B(x)) < \eta$$

for all $x \in X$.

In these conditions we have

$$d(x_A^*, x_B^*) \leq \frac{\eta}{1-\alpha}\,.$$

In section 5 we need the fiber Picard operators theorem (see [13], [15]).

Theorem 1.2. *(Fiber Picard operators theorem). Let* (X, d) *and* (Y, ρ) *be two metric spaces,* $B : X \to X$, $C : X \times Y \to Y$ *and* $A = (B, C) : X \times Y \to X \times Y$ *a triangular operator. We suppose that:*

(i) (Y, ρ) *is a complete metric space;*

(ii) *the operator* $B : X \to X$ *is WPO;*

(iii) *there exists* $\alpha \in [0, 1)$, *such that* $C(x, \cdot)$ *is an* α*-contraction, for all* $x \in X$*;*

(iv) *if* $(x^*, y^*) \in F_A$, *then* $C(\cdot, y^*)$ *is continuous in* x^* .

Then the operator A *is WPO.*

If B *is a PO then* A *is a PO.*

The existence and uniqueness and the data dependence on data of the solutions for some nonlinear integral equations have been studied in [1], [4], [5], [6], [7], [8], [9], [10], [11], [18], [21], [22], [23], [24]. In what follows we shall study the equation (1.1)

2. Existence and uniqueness in a subset of $C(\mathbb{R}, I)$

We consider the equation (1.1) in the following conditions:

(c_1) $I,\, J \subset \mathbb{R}$ two compact intervals and $f \in C(\mathbb{R} \times I,\, J)$;

(c_2) $f\,(t, \cdot) : I \to J$ is L_f - Lipschitz for all $t \in \mathbb{R}$;

(c_3) $L_f \cdot \tau < 1$;

(c_4) there exists $U \subset C(\mathbb{R}, I)$ such that $U \in I_{cl}\,(A)$.

We have:

Theorem 2.1. *Under the conditions* $(c_1) - (c_4)$ *the equation (1.1) has a unique solution in* U .

Proof. We consider the Banach space $(C(\mathbb{R}, I), \|\cdot\|_C)$ with supremum norm

$$\|x\|_C = \sup_{t\in\mathbb{R}} |x(t)|$$

and the operator defined by

$$A(x)(t) := \int_{t-\tau}^{t} f(s, x(s))ds, \ \ t \in \mathbb{R}\,. \tag{2.1}$$

From condition (c_4) it results that $A(U) \subset U$, and we take the operator $A : U \to U$, defined by (2.1). The set of the solutions of the integral equation (1.1) coincides with the set of fixed points of the operator A.

From (c_2) we have:

$$|A(x_1)(t) - A(x_2)(t)| = \left| \int_{t-\tau}^{t} [f(s, x_1(s)) - f(s, x_2(s))]\, ds \right| \leq$$

$$\leq \left| \int_{t-\tau}^{t} |f(s, x_1(s)) - f(s, x_2(s))|\, ds \right| \leq \left| \int_{t-\tau}^{t} L_f\, |x_1(s) - x_2(s)|\, ds \right|$$

and using the supremum norm, we obtain

$$\|A(x_1) - A(x_2)\|_C \leq L_f \cdot \tau\ \|x_1 - x_2\|_C\ .$$

Therefore, by (c_3) it results that the operator A is an α-contraction with the coefficient $\alpha = L_f \cdot \tau$. Now, the conclusion of the theorem results from the *Contraction Principle.*

□

Remark 2.1. *Under the conditions* $(c_1) - (c_4)$ *the operator* $A : (U, d_{\|\cdot\|_C}) \to (U, d_{\|\cdot\|_C})$ *is a PO.*

Let $0 < m < M\,,\ 0 < \alpha < \beta,\ \ I = [\alpha, \beta],\ \ J = [m, M]\,.$

Corollary 2.1. *We suppose that:*

(i) *the conditions* $(c_1) - (c_3)$ *are satisfied;*

(ii) $\alpha \leq m\tau,\ \beta \geq M\tau$.

Then the equation (1.1) has a unique solution in $C(\mathbb{R}, I)$.

Proof. We take $U := C(\mathbb{R}, I)$, where $I = [\alpha, \beta]$ and we consider the operator A defined by (2.1).

By the definition of the function f it results that

$$f(t, x(t)) \in [m, M], \quad for\ all\ \ t \in \mathbb{R}\,,\ x \in U$$

and we have

$$\int_{t-\tau}^{t} f(s, x(s))ds \in [m\tau, M\tau], \quad for\ all\ \ t \in \mathbb{R}\,,\ x \in U,$$

i.e.

$$A(x)(t) \in [m\tau, M\tau], \quad for\ all\ \ t \in \mathbb{R}\,,\ x \in U.$$

Condition (ii) implies that

$$A(x)(t) \in [\alpha, \beta], \quad for\ all\ \ t \in \mathbb{R}\,,\ x \in U.$$

Therefore, U is an invariant set for the operator A. Now the proof is obtained applying Theorem 2.1 □

Corollary 2.2. *We suppose that the conditions of the Corollary 2.1 hold. Moreover, we suppose that there exists* $\omega > 0$ *such that:*

$$f(t + \omega, u) = f(t, u), \ \ for\ all\ \ t \in \mathbb{R},\, u \in I.$$

Then, the equation (1.1) has a unique ω*-periodic solution.*

Proof. We take

$$U := X_\omega := \{x \in C(\mathbb{R}, I)\, :\, x(t + \omega) = x(t),\ for\ all\ t \in \mathbb{R}\}$$

and we consider the operator A defined by (2.1).

Since the function f is ω-periodic with respect to t and from (c_1) and the condition (ii) of the Corollary 2.1, we have $A(U) \subset U$, i.e. $U \in I(A)$. Thus all the conditions of Theorem 2.1 are satisfied and therefore we obtain the conclusion. □

3. Lower and upper subsolutions

We consider the integral equation (1.1) under the conditions $(c_1) - (c_4)$ and we denote by $x_A^* \in U$ the unique fixed point of the operator A. In addition, we suppose that:

(c_5) $f(t, \cdot) : I \to J$ is increasing, for every $t \in \mathbb{R}$.

We have:

Theorem 3.1. *We suppose that the conditions* $(c_1) - (c_5)$ *are satisfied.*

If

$$x \in U, \ \ x(t) \leq \int_{t-\tau}^{t} f(s, x(s))ds\,,$$

then

$$x \leq x_A^*\,.$$

Proof. We consider $A : U \to U$, defined by (2.1). Under the conditions $(c_1) - (c_4)$ the operator A is a PO and by (c_5) we have that the operator A is increasing. Since all the conditions of the *Abstract Gronwall lemma* (Lemma1.3) are satisfied, we obtain

$$x \leq x_A^*$$

and the proof is complete. □

Let $0 < m < M$, $0 < \alpha < \beta$, $I = [\alpha, \beta]$, $J = [m, M]$. We have the following theorem:

Theorem 3.2. *Let* f_i , $i = 1, 2, 3$ *be three functions and suppose that:*

(i) $f_i \in C(\mathbb{R} \times I,\, J)$, $i = 1, 2, 3$, *where* $I,\, J \subset \mathbb{R}$ *two compact intervals;*
(ii) $f_2(t, \cdot)$ *is increasing for every* $t \in \mathbb{R}$*;*
(iii) $f_1 \leq f_2 \leq f_3$*;*
(iv) $f_i\,(t, \cdot) : I \to J$ *is* L_{f_i} *- Lipschitz for all* $t \in R$, $i = 1, 2, 3$*;*
(v) $L_{f_i} \cdot \tau < 1$, $i = 1, 2, 3$*;*
(vi) $\alpha \leq m\tau$, $\beta \geq M\tau$.

Let x_i^*, $i = 1, 2, 3$ *be the unique solution of the integral equation (1.1) correspondent to each of* f_i , $i = 1, 2, 3$. *Then*

$$x_1^* \leq x_2^* \leq x_3^* .$$

Proof. We consider the operators $A_i : C(\mathbb{R}, I) \to C(\mathbb{R}, I)$, defined by

$$A_i(x)(t) := \int_{t-\tau}^{t} f_i(s, x(s))ds, \ \ t \in R\, , \ i = 1, 2, 3\, . \tag{1}$$

From condition (ii) we get that the operator A_2 is increasing and from the condition (iii) we have

$$A_1 \leq A_2 \leq A_3 .$$

Using conditions (i), (iv), (v) we obtain that the operators A_i are α_i-contractions with the constants $\alpha_i = L_{f_i} \cdot \tau$, $i = 1, 2, 3$, therefore A_i, $i = 1, 2, 3$ are POs.

By *Comparison abstract lemma* (Lemma1.2), it results that

$$x_1 \leq x_2 \leq x_3 \quad \Rightarrow \quad A_1^\infty(x_1) \leq A_2^\infty(x_2) \leq A_3^\infty(x_3)\, ,$$

but A_i, $i = 1, 2, 3$ are POs and therefore, we obtain

$$x_1^* \leq x_2^* \leq x_3^* .$$

This completes the proof. □

Theorem 3.3. *We suppose that the conditions* $(c_1)-(c_3)$ *and* (c_5) *are satisfied. Then*

$$x \leq x_A^* \leq y\, ,$$

for all $x \in (LF)_A$ *and* $y \in (UF)_A$.

Proof. We take $U := (LF)_A \cup (UF)_A$. Now we consider the operator A defined by the relation (2.1)

$$A(x)(t) := \int_{t-\tau}^{t} f(s, x(s))ds, \ \ t \in \mathbb{R}\, .$$

By the condition (c_5) the operator A is increasing and therefore, we have $(LF)_A \in I(A)$ and $(UF)_A \in I(A)$, so $(LF)_A \cup (UF)_A$ is an invariant set for A.

From conditions $(c_1) - (c_3)$ and the above condition we have that $A : U \to U$ is a PO. Theorem 2.1 implies that the operator A has a unique fixed point in U, which we denote by x_A^*. Since the conditions of *Lemma* 1.4 are satisfied, we obtain the conclusion of the theorem. □

4. **Data dependence**

In what follows we will study the dependence of the solution of the integral equation (1.1), with respect to f.

Now we consider the perturbed integral equation

$$y(t) = \int_{t-\tau}^{t} g(s, y(s))ds, \quad t \in \mathbb{R}, \tag{4.1}$$

where $g \in C(\mathbb{R} \times I,\, J)$, with $I,\, J \subset \mathbb{R}$ two compact intervals.

We have:

Theorem 4.1. *We suppose:*

(i) *the conditions of the Theorem 2.1 are satisfied and we denote by x^* the unique solution of the integral equation (1.1);*

(ii) *there exists $\eta > 0$ such that*

$$|f(t,u) - g(t,u)| \le \eta\,, \quad for \;\; all \; t \in \mathbb{R}\,, \;\; u \in I\,.$$

In these conditions, if y^* is a solution of the integral equation (4.1), then we have

$$\|x^* - y^*\|_C \le \frac{\eta\tau}{1 - L_f \cdot \tau}\,.$$

Proof. We consider the operator $A : U \to U$ defined by the relation (2.1).

Let $B : U \to U$ be the operator defined by

$$B(y)(t) := \int_{t-\tau}^{t} g(s, y(s))ds\,, \quad t \in \mathbb{R}\;. \tag{4.2}$$

From the condition (ii) we have

$$|A(x)(t) - B(x)(t)| = \left| \int_{t-\tau}^{t} \left[f(s, x(s)) - g(s, x(s)) \right] ds \right| \le$$

$$\le \left| \int_{t-\tau}^{t} |f(s, x(s)) - g(s, x(s))|\, ds \right| \le \left| \int_{t-\tau}^{t} \eta ds \right| \le \eta\,\tau$$

and using the supremum norm, we obtain

$$\|A(x) - B(x)\|_C \leq \eta\tau.$$

The proof of the theorem follows from the *General data dependence theorem, Theorem 1.2.* □

Also, we have the following data dependence theorem of the periodic solution of the integral equation (1.1):

Theorem 4.2. *We suppose that:*

(i) *the conditions of the Corollary 2.2 are satisfied and we denote by x^* the unique ω-periodic solution of the integral equation (1.1);*

(ii) $g(t+\omega, u) = g(t, u)$, *for all* $t \in \mathbb{R}$, $u \in I$;

(iii) *there exists* $\eta > 0$ *such that*

$$|f(t,u) - g(t,u)| \leq \eta\,, \quad for\ \ all\ t \in \mathbb{R}\,,\ u \in I\,.$$

In these conditions, if y^* *is a ω-periodic solution of the integral equation (4.1), then we have*

$$\|x^* - y^*\|_C \leq \frac{\eta\tau}{1 - L_f\ \cdot \tau}\,.$$

Proof. The proof of this theorem is similar to the proof of the Theorem 4.1. □

5. **Differentiability with respect to a parameter**

In this section we will study the differentiability of the solution of the integral equation (1.1) (see [9], [10], [19]) with respect to parameter λ

$$x(t,\lambda) = \int_{t-\tau}^{t} f(s, x(s); \lambda) ds\,, \ \ t \in \mathbb{R},\ \ \lambda \in K\,, \tag{5.1}$$

where $f \in C(\mathbb{R}\times I\times K,\ J)$, with $I = [\alpha, \beta]$, $0 < \alpha < \beta$, $J = [m, M]$, $0 < m < M$ and $K \subset \mathbb{R}$ a compact interval.

Let

$$X_\omega := \{x \in C(\mathbb{R} \times K, I)\ :\ x(t+\omega, \lambda) = x(t, \lambda),\ for\ all\ \ t \in \mathbb{R}, \lambda \in K\}\,,$$

where $\omega > 0$.

Theorem 5.1. *We suppose that:*

(i) $\alpha \leq m\tau, \ \beta \geq M\tau$;

(ii) $f(t,u;\lambda) \in [m,M]$, *for all* $t \in \mathbb{R}$, $u \in I$, $\lambda \in K$;

(iii) $f(t+\omega,u;\lambda) = f(t,u;\lambda)$, *for all* $t \in \mathbb{R}$, $u \in I$, $\lambda \in K$;

(iv) $f(t,\cdot;\lambda) : I \to J$ *is* L_f *- Lipschitz for all* $t \in \mathbb{R}$, $\lambda \in K$;

(v) $L_f \cdot \tau < 1$.

Then:

(a) *the equation (1.1) has a unique solution* x^* *in* X_ω ;

(b) *for all* $x_0 \in X_\omega$, *the sequence* $(x_n)_{n\in\mathbb{N}}$, *defined by*

$$x_{n+1}(t,\lambda) = \int_{t-\tau}^{t} f(s, x_n(s,\lambda))\, ds$$

converges uniformly to x^*;

(c) *if* $f(t,\cdot,\cdot) \in C^1(I \times K)$ *then* $x^*(t,\cdot) \in C^1(K)$.

Proof. $(a) + (b)$. We consider the operator $B : X_\omega \to C(\mathbb{R} \times K)$ defined by

$$B(x)(t,\lambda) := \int_{t-\tau}^{t} f(s, x(s;\lambda)) ds\, .$$

From (i) and (iii) we have that $X_\omega \in I(B)$. From (iv) and (v) it follows that B is an α-contraction with the constant $\alpha = L_f \cdot \tau$. By the *Contraction Principle* we have that B is a PO.

(c). We prove that there exists $\dfrac{\partial x^*}{\partial \lambda}$ and $\dfrac{\partial x^*}{\partial \lambda} \in C(\mathbb{R} \times K)$.

If we suppose that there exists $\dfrac{\partial x^*}{\partial \lambda}$, then from

$$x(t;\lambda) = \int_{t-\tau}^{t} f(s, x(s;\lambda);\lambda) ds$$

we have

$$\frac{\partial x(t;\lambda)}{\partial t} = \int_{t-\tau}^{t} \frac{\partial f(s,x(s;\lambda);\lambda)}{\partial x} \cdot \frac{\partial x(s;\lambda)}{\partial \lambda} ds + \int_{t-\tau}^{t} \frac{\partial f(s,x(s;\lambda);\lambda)}{\partial \lambda} ds.$$

This relation suggests to us to consider the following operator

$$T : X_\omega \times X_\omega \to X_\omega \times X_\omega\, ,$$

defined by

$$T = (B, C), \quad T(x, y) = (B(x), C(x, y)),$$

where

$$C(x,y)(t,\lambda) := \int_{t-\tau}^{t} \frac{\partial f(s, x(s;\lambda);\lambda)}{\partial x} \cdot y(s,\lambda)ds + \int_{t-\tau}^{t} \frac{\partial f(s, x(s;\lambda);\lambda)}{\partial \lambda} ds.$$

We have:

$$|C(x,y)(t,\lambda) - C(x,z)(t,\lambda)| \leq \int_{t-\tau}^{t} \left| \frac{\partial f(s, x(s;\lambda);\lambda)}{\partial x} \right| \cdot |y(s,\lambda) - z(s,\lambda)|\, ds \leq$$

$$\leq L_f \cdot \|y - z\|_C \int_{t-\tau}^{t} ds = L_f \cdot \tau \cdot \|y - z\|_C,$$

for all $x, y, z \in X_\omega$. Now, the conditions of the *fiber Picard operators theorem* are satisfied. From this theorem we obtain that the operator T is a PO and the sequences

$$x_{n+1} = B(x_n)$$

and

$$y_{n+1} = C(x_n, y_n)$$

converge uniformly to $(x^*, y^*) \in F_T$, for all $x_0, y_0 \in X_\omega$.

If we take $x_0, y_0 \in X_\omega$ such that

$$y_0 = \frac{\partial x_0}{\partial \lambda},$$

then we have that

$$y_n = \frac{\partial x_n}{\partial \lambda}, \quad for \ all \ n \in \mathbb{N}.$$

So

$$x_n \overset{unif.}{\longrightarrow} x^* \text{ as } n \to \infty,$$

$$\frac{\partial x_n}{\partial \lambda} \overset{unif.}{\longrightarrow} y^* \text{ as } n \to \infty.$$

Using a Weierstrass argument we conclude that x^* is differentiable, i.e. there exists $\frac{\partial x^*}{\partial \lambda}$, and $y^* = \frac{\partial x^*}{\partial \lambda}$. □

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ON A CLASS OF ORTHOGONAL POLYNOMIALS

I. GAVREA

Dedicated to Professor Petru Blaga at his 60^{th} *anniversary*

Abstract. In this note we study a system of polynomials $\{\widehat{P}_k\}$ orthogonal with respect to the modified measure

$$d\widehat{\lambda}(t) = \frac{t-d}{t-c}w(t)dt, \quad t \in [0,1]$$

where $d, c < 0$ and w is a weight function, using orthogonal polynomials $\{P_k\}$ with respect to the measure $dw(t)$.

1. Introduction

In [3] G.V. Milovanovic, A.S. Cvetković and M.M. Matejic investigated polynomials orthogonal with respect to the moment functional

$$\mathcal{L}(P) = \int_{-1}^{1} P(t)\frac{t+\frac{1}{2}c+\frac{1}{c}}{t+\frac{1}{2}c+\frac{1}{2c}}\sqrt{1-t^2}dt, \quad P \in \Pi$$

where $c \in \mathbb{R} \setminus \{0\}$.

Similar measures, e.g. with the weight function $(1-t^2)(1-k^2t^2)^{-1/2}$, $k^2 < 1$ were studied in [4]. For the Chebyshev measure of the first kind the same modification has been studied in [2].

In this note we investigate polynomials orthogonal with respect to the moment functional

$$A(P) = \int_0^1 P(t)\frac{t-d}{t-c}w(t)dt, \quad P \in \Pi \tag{1.1}$$

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where $dw(t)$ is a positive measure on $[0,1]$ having finite moments of all orders, d, c being fixed negative numbers. We will denote by $\{P_k\}$ the set of orthogonal polynomials with respect to the measure $dw(t)$ and by $\{\widehat{P}_k\}$ the system of orthogonal polynomials with respect to the measure

$$dw_1(t) = \frac{t-d}{t-c} w(t)dt.$$

The existence of $\{\widehat{P}_k\}$ is guaranteed, since $dw_1(t)$ is a positive measure on $[0,1]$ having finite moments of all orders

$$A(t^k) = \int_0^1 t^k \frac{t-d}{t-c} w(t)dt.$$

We will solve the problem in two steps as well as in [3]. First, we consider modification of $dw(t)$ measure by linear factor $t-d$ computing the coefficients of the three-term recurrence relation and then we consider the modification of $dw(t)$ measure by the linear divisor.

In the following we will suppose that

$$\int_0^1 w(t)dt = 1. \tag{1.2}$$

2. **Linear factors**

We denote by $\widetilde{w}$ the weight function defined by

$$\widetilde{w}(t) = (t-d)w(t). \tag{2.1}$$

It is well known (see [5]) that the orthogonal polynomial of degree n relative to the weight function $\widetilde{w}(t)$ is given by

$$\widetilde{P}_n(t) = \sum_{k=0}^{n} P_k(t)P_k(d). \tag{2.2}$$

It is known ([5]) that there exists a relation of the form

$$\widetilde{P}_n(t) = (A_n t + B_n)\widetilde{P}_{n-1}(t) - c_n \widetilde{P}_{n-2}(t), \quad n = 2, 3, \ldots \tag{2.3}$$

Theorem 2.1. *The coefficients A_n, B_n, C_n are given by:*

$$A_n = \frac{P_{n,n}(d)}{P_{n-1,n-1}(d)} \tag{2.4}$$

$$B_n = 1 + \frac{a_{n-1,n-1}}{a_{n,n}} \cdot \frac{P_{n,n}^2(d)}{P_{n-1,n-1}^2(d)} - \frac{dP_{n,n}(d)}{P_{n-1,n-1}(d)} \tag{2.5}$$

$$C_n = \frac{a_{n-1,n-1}}{a_{n,n}} \cdot \frac{P_{n,n}^2(d)}{P_{n-1,n-1}^2(d)} \tag{2.6}$$

where $P_{n,n}$ is the polynomial of degree n orthogonal with respect to $w(t)$ normalized by

$$\int_0^1 x^n P_{n,n}(x)w(x)dx = 1$$

and $a_{n,n}$, $n = 0, 1, \ldots$ is the coefficient of x^n of the polynomial $P_{n,n}$.

Proof. For $k \in \{0, 1, \ldots, n\}$ we denote by $P_{n,k}$ the polynomial of degree n defined by the equalities:

$$\int_0^1 w(t)t^i P_{n,k}(t) = \delta_{k,i}, \quad i = \overline{0,n}. \tag{2.7}$$

$P_{n,k}$ is well defined by the relations (2.7).

If P is a polynomial of degree n then P can be written in the following form:

$$P(t) = \sum_{k=0}^{n} M_k P_{n,k}(t) \tag{2.8}$$

where

$$M_k = \int_0^1 t^k P(t)w(t)dt.$$

The polynomial $\widetilde{P}_n$ can be written as

$$\widetilde{P}_n(t) = \sum_{k=0}^{n} d^k P_{n,k}(t) = \sum_{k=0}^{n} t^k P_{n,k}(d). \tag{2.9}$$

We note that

$$\int_0^1 P_n(t)w(t)dt = 1. \tag{2.10}$$

From (2.9) and (2.10) the equality (2.3) is equivalent with

$$\sum_{k=0}^{n} d^k P_{n,k}(t) = (A_n t + B_n) \sum_{k=0}^{n-1} d^k P_{n-1,k}(t) - C_n \sum_{k=0}^{n-1} d^k P_{n-2,k}(t), \quad n \geq 2. \tag{2.11}$$

From (2.10) and (2.11) we get:

$$1 = B_n - C_n + dA_n. \tag{2.12}$$

On the other hand (2.11) can be written as:

$$\sum_{k=0}^{n} t^k P_{n,k}(d) = (A_n t + B_n) \sum_{k=0}^{n-1} t^k P_{n-1,k}(d) - C_n \sum_{k=0}^{n-2} t^k P_{n-2,k}(d). \tag{2.13}$$

From (2.13) we get

$$A_n = \frac{P_{n,n}(d)}{P_{n-1,n-1}(d)} \tag{2.14}$$

and

$$P_{n,n-1}(d) = A_n P_{n-1,n-2}(d) + B_n P_{n-1,n-1}(d). \tag{2.15}$$

$P_{n,n-1}$ can be written as:

$$P_{n,n-1}(x) = P_{n-1,n-1}(x) + aP_{n,n}(x) \tag{2.16}$$

where

$$a = -\int_0^1 t^n P_{n-1,n-1}(t) dt.$$

By (2.15), (2.14) and (2.16) we obtain

$$B_n = \frac{P_{n-1,n-1}^2(d) - P_{n,n}(d) P_{n-2,n-2}(d)}{P_{n-1,n-1}^2(d)}$$

$$+ \frac{\int_0^1 t^{n-1}(P_{n-2,n-2}(t) - tP_{n-1,n-1}(d)) w(t) dt \cdot P_{n-1,n-1}(d) P_{n,n}(d)}{P_{n-1,n-1}^2(d)}. \tag{2.17}$$

There is the constants $\alpha_n, \beta_n, \gamma_n$ such that

$$P_{n,n}(t) = (\alpha_n t + \beta_n) P_{n-1,n-1}(t) - \gamma_n P_{n-2,n-2}(t) \tag{2.18}$$

and we have

$$\alpha_n = \gamma_n = \frac{a_{n,n}}{a_{n-1,n-1}} \tag{2.19}$$

and

$$\beta_n = \frac{a_{n,n}}{a_{n-1,n-1}} \int_0^1 t^{n-1}(P_{n-2,n-2}(t) - tP_{n-1,n-1}(t)) w(t) dt. \tag{2.20}$$

From (2.17), (2.18), (2.19) and (2.20) we get

$$B_n = 1 + \frac{a_{n-1,n-1}}{a_{n,n}} \cdot \frac{P_{n,n}^2(d)}{P_{n-1,n-1}^2(d)} - d \frac{P_{n,n}(d)}{P_{n-1,n-1}(d)}. \tag{2.21}$$

By (2.12) and (2.21) we obtain

$$C_n = \frac{a_{n-1,n-1}}{a_{n,n}} \cdot \frac{P^2_{n,n}(d)}{P^2_{n-1,n-1}(d)}$$

and the theorem is proved.

3. **Linear divisors**

Let us consider the weight function

$$w_2(t) = \frac{\widetilde{w}(t)}{t-c}.$$

The orthogonal polynomials $\{\widehat{P}_n\}$, $n = 0, 1, \ldots$ are orthogonal polynomials relative to the weight function $w_2(t)$.

Theorem 3.1. *The polynomial $\widehat{P}_n$ is given by*

$$\widehat{P}_n(t) = \widetilde{P}_{n-1,n-1}(t) - \widetilde{P}_{n,n}(t) \frac{\displaystyle\int_0^1 \frac{\widetilde{P}_{n-1,n-1}(t)}{t-c} w(t)dt}{\displaystyle\int_0^1 \frac{\widetilde{P}_{n,n}(t)}{t-c} w(t)dt} \tag{3.1}$$

where $\widetilde{P}_{n,n}$ is orthogonal polynomial of degree n relative to the weight function $\widetilde{w}$ normalized by

$$\int_0^1 x^n \widetilde{P}_{n,n}(x)\widetilde{w}(x)dx = 1.$$

$\widehat{P}_n$ is normalized by

$$\int_0^1 x^n \widehat{P}_n(x) w(x)dx = 1.$$

Proof. By the conditions

$$\int_0^1 x^k \widehat{P}_n(x) w_2(x) = 0 \text{ for } k = 0, 1, \ldots, n-1$$

we get

$$0 = \int_0^1 \frac{x^k}{x-c} \widehat{P}_n(x)\widetilde{w}(x)dx = \int_0^1 \frac{x^k - c^k}{x-c} \widetilde{P}_n(x)\widetilde{w}(x)dx$$

$$+c^k \int_0^1 \frac{\widetilde{P}_n(x)\widetilde{w}(x)}{x-c} dx = \int_0^1 \frac{x^k - c^k}{x-c} \widetilde{P}_n(x)\widetilde{w}(x)dx$$

or

$$0=\sum_{i=0}^{k-1}c^i\int_0^1 t^{k-i-1}\widehat{P}_n(t)\widetilde{w}(t)dt,\quad k=1,2,\dots,n-1. \tag{3.2}$$

If we denote by

$$M_i=\int_0^1 t^i\widetilde{w}(t)\widehat{P}_n(t)dt$$

from (3.2) we get

$$M_0=M_1=\cdots=M_{n-2}=0$$

and so, the polynomial $\widehat{P}_n$ can be written

$$\widehat{P}_n(t)=M_{n-1}\widetilde{P}_{n,n-1}(t)+M_n\widetilde{P}_{n,n}(t). \tag{3.3}$$

By the condition

$$\int_0^1\frac{t^n\widehat{P}_n(t)\widetilde{w}(t)}{t-c}dt=1$$

we get $M_{n-1}=1$.

From the condition

$$\int_0^1\frac{\widehat{P}_n(t)}{t-c}\widetilde{w}(t)dt=0$$

we obtain

$$M_n=-\frac{\displaystyle\int_0^1\frac{\widetilde{P}_{n,n-1}(t)\widetilde{w}(t)}{t-c}dt}{\displaystyle\int_0^1\frac{P_{n,n}(t)\widetilde{w}(t)}{t-c}dt}. \tag{3.4}$$

The polynomial $\widetilde{P}_{n,n-1}$ can be written as

$$\widetilde{P}_{n,n-1}(t)=\widetilde{P}_{n-1,n-1}(t)+u_n\widetilde{P}_{n,n}(t) \tag{3.5}$$

where

$$u_n=-\int_0^1 t^n\widetilde{P}_{n-1,n-1}(t)\widetilde{w}(t)dt. \tag{3.6}$$

From (3.6), (3.5), (3.4) and (3.3) we get

$$\widehat{P}_n(t)=\widetilde{P}_{n-1,n-1}(t)-\widetilde{P}_{n,n}(t)\frac{\displaystyle\int_0^1\frac{\widetilde{P}_{n-1,n-1}(t)}{t-c}\widetilde{w}(t)dt}{\displaystyle\int_0^1\frac{\widetilde{P}_{n,n}(t)}{t-c}\widetilde{w}(t)dt}.$$

The last relation proves the theorem.

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ON THE DEGREE OF APPROXIMATION IN VORONOVSKAJA'S THEOREM

HEINER GONSKA

Dedicated to Professor Petru Blaga at his 60^{th} anniversary

Abstract. The present article continues earlier research by P. Piţul, I. Raşa and the author on quantitative versions of E.V. Voronovskaja's 1932 result concerning the asymptotic behavior of Bernstein polynomials.

1. Introduction and historical remarks

In two recent notes P. Piţul, I. Raşa and the author discussed E.V. Voronovskaja's [20] classical theorem on the asymptotic behavior of Bernstein polynomials $B_n(f;\cdot)$ for twice continuously differentiable functions given on $[0,1]$. We recall that for $f \in \mathbb{R}^{[0,1]}, n \geq 1$ and $x \in [0,1]$ one puts

$$\begin{aligned} B_n(f;x) &:= \sum_{k=0}^{n} f\left(\tfrac{k}{n}\right) \cdot p_{n,k}(x) \\ &:= \sum_{k=0}^{n} f\left(\tfrac{k}{n}\right) \binom{n}{k} x^k (1-x)^{n-k}. \end{aligned}$$

Normally Voronovskaja's result is given today in its local version as follows.

Theorem 1.1. *(see R.A. DeVore & G.G. Lorentz* [3]*)*

If f is bounded on $[0,1]$, differentiable in some neighborhood of x and has second derivative $f''(x)$ for some $x \in [0,1]$, then

$$n \cdot [B_n(f;x) - f(x)] - \frac{x(1-x)}{2} \cdot f''(x) \to 0, n \to \infty.$$

If $f \in C^2[0,1]$, the convergence is uniform.

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In an article following directly that of Voronovskaja S.N. Bernstein [1] generalized the uniform version as given below.

Theorem 1.2. *If $q \in \mathbb{N}$ is even, $f \in C^q[0,1]$, then, uniformly in $x \in [0,1]$,*

$$n^{\frac{q}{2}} \cdot \left\{ B_n(f;x) - f(x) - \sum_{r=1}^{q} B_n((e_1 - x)^r; x) \cdot \frac{f^{(r)}(x)}{r!} \right\} \to 0, n \to \infty.$$

Before we continue here is a word of warning:

"Voronovskaja" ist just one possible way to spell the Russian ВОРОНОВСКАЯ in Latin characters. Other possibilities to be observed in the literature are Voronovskaya, Woronowskaja, Woronowskaya and even Voronovsky (as given on the original 1932 article).

It is the aim of this note to present new quantitative results which also cover Bernstein's above case, among others. Before going into details we give three examples of quantitative Voronovskaja theorems. Some authors call them "strong Voronovskaja-type theorems" because, in addition to the convergence of $n \cdot [B_n g - g]$ towards Ag, they also express a degree of approximation depending on smoothness properties of the function g.

Example 1.3. *Write $A(f;x) := \frac{x(1-x)}{2} \cdot f''(x), \varphi(x) := \sqrt{x(1-x)}$. Then for*

- $g \in C^4[0,1] : \|n \cdot [B_n g - g] - Ag\|_\infty \leq \frac{24}{n}(\|g'''\| + \|g^{(4)}\|)$ *(see [8]);*
- $g \in C^3[0,1] : \|n \cdot [B_n g - g] - Ag\|_\infty \leq \frac{C}{\sqrt{n}} \cdot \|\varphi^3 \cdot g'''\|$ *(see [4]).*

A full quantitative pointwise version of Voronovskaja's uniform result reads

- $f \in C^2[0,1] : |n \cdot [B_n(f;x) - f(x)] - A(f;x)| \leq \frac{x(1-x)}{2} \cdot \tilde{\omega}(f''; \frac{1}{3 \cdot \sqrt{n}})$ (see [6]).

All the proofs are based on Taylor's formula; the last one uses the "Peano remainder without Landau" as recalled in the next section. During the writing of this note it was brought to the author's attention that already in 1985 V.S. Videnskij published in [19] (see Theorem 15.2 on p. 49) the following:

- $f \in C^2[0.1] : |n \cdot [B_n(f;x) - f(x)] - A(f;x)| \leq x(1-x) \cdot \omega(f''; \sqrt{\frac{2}{n}})$.

Videnskij's inequality follows from ours given in [6]; later in this note an even more precise pointwise estimate will be given.

2. An auxiliary result

Here we recall an estimate of the remainder in Taylor's formula which (strange enough!) we were unable to locate in the literature.

Theorem 2.1. *(see [6]) Let $\omega(f;\varepsilon)$ denote the classical first order modulus of continuity of $f \in C[a,b], \varepsilon > 0$. The least concave majorant $\tilde{\omega}(f;\varepsilon)$ is given by*

$$\tilde{\omega}(f;\varepsilon) = \begin{cases} \sup\limits_{\substack{0 \le x \le \varepsilon \le y \le b-a \\ x \ne y}} \frac{(\varepsilon - x)\omega(f;y) + (y-\varepsilon)\omega(f;x)}{y-x}, & 0 \le \varepsilon \le b-a; \\ \omega(f;b-a), & \varepsilon > b-a. \end{cases}$$

Suppose that $f \in C^q[a,b], q \ge 0$. Then for the remainder in Taylor's formula we have

$$|R_q(f;x,t)| \le \frac{|t-x|^q}{q!}\tilde{\omega}\left(f^{(q)};\frac{|t-x|}{q+1}\right).$$

Here $x \in [a,b]$ is fixed, and $t \in [a,b]$.

Remark 2.2. *Since $\tilde{\omega}(f^{(q)};\frac{|t-x|}{q+1}) = o(1), t \to x$, this is a more explicit form of Peano's remainder in Taylor's formula.*

3. A general quantitative Voronovskaja-type theorem

As mentioned before, Bernstein's generalization can be turned into a quantitative statement. However, we intend to be more general. For historical reasons we recall the following

Theorem 3.1. *(R.G. Mamedov [13])*
Let $q \in \mathbb{N}$ be even, $f \in C^q[0,1]$, and $L_n : C[0,1] \to C[0,1]$ be a sequence of positive linear operators such that

$$L_n(e_0;x) = 1, x \in [0,1];$$

$$\lim_{n\to\infty} \frac{L_n((e_1-x)^{q+2j};x)}{L_n((e_1-x)^q;x)} = 0 \text{ for at least one } j \in \{1,2,\ldots\}.$$

Then

$$\frac{1}{L_n((e_1-x)^q;x)}\{L_n(f;x) - f(x) - \sum_{r=1}^{q} L_n((e_1-x)^r;x)\cdot\frac{f^{(r)}(x)}{r!}\} \to 0, n \to \infty.$$

The quantitative version of the above result will be based upon

Theorem 3.2. *Let $q \in \mathbb{N}_0, f \in C^q[0,1]$ and $L : C[0,1] \to C[0,1]$ be a positive linear operator. Then*

$$|L(f;x) - \sum_{r=0}^{q} L((e_1-x)^r;x) \cdot \frac{f^{(r)}(x)}{r!}| \leq \frac{L(|e_1-x|^q;x)}{q!} \cdot \tilde{\omega}(f^{(q)}; \frac{1}{q+1} \cdot \frac{L(|e_1-x|^{q+1};x)}{L(|e_1-x|^q;x)}.$$

Sketch of proof. For x fixed and $t \in [0,1]$ write Taylor's formula as

$$f(t) = \sum_{r=0}^{q} \frac{f^{(r)}(x)}{r!}(t-x)^r + R_q(f;x,t), \text{ i.e.,}$$

$$f(t) - \sum_{r=0}^{q} \frac{f^{(r)}(x)}{r!}(t-x)^r = R_q(f;x,t).$$

Applying L to both sides (as functions of t) yields

$$|L(f;x) - \textstyle\sum_{r=0}^{q} \frac{f^{(r)}(x)}{r!} \cdot L((e_1-x)^r;x)|$$

$$\leq L(|R_q(f;x,\cdot)|;x)$$

$$\leq L(\tfrac{|e_1-x|^q}{q!} \cdot \tilde{\omega}(f^{(q)}; \tfrac{|e_1-x|}{q+1};x)$$

$$\leq \tfrac{L(|e_1-x|^q;x}{q!} \cdot \tilde{\omega}(f^{(q)}; \tfrac{1}{q+1} \cdot \tfrac{L(|e_1-x|^{q+1};x)}{L(|e_1-x|^q;x)}).$$

For further details (such as the intermediate use of a K-functional) see [6].□

Corollary 3.3. *(Mamedov's situation) Suppose that we consider a sequence (L_n) of positive linear operators, q is even, $L_n(e_0;x) = 1$ and that for at least one $j \in \mathbb{N}$ one has*

$$\lim_{n\to\infty} \frac{L_n((e_1-x)^{q+2j};x)}{L_n((e_1-x)^q;x)} = 0.$$

In this case $L_n(|e_1-x|^q;x) = L_n((e_1-x)^q;x)$.

Using the Cauchy-Schwarz inequality for positive linear functionals (possibly repeatedly) we obtain

$$\frac{L_n(|e_1-x|^{q+1};x)}{L_n(|e_1-x|^q;x)} = \frac{L_n(|e_1-x|^{\frac{q}{2}} \cdot |e_1-x|^{\frac{q}{2}+1};x)}{L_n((e_1-x)^q;x)}$$

$$\leq \sqrt[2]{\frac{L_n((e_1-x)^{q+2};x)}{L_n((e_1-x)^q;x)}} \cdots \leq \sqrt[2j]{\frac{L_n((e_1-x)^{q+2j};x)}{L_n((e_1-x)^q;x)}}.$$

And from here Mamedov's statement follows, since according to his assumption the latter quantity tends to 0 as n goes to ∞. □

4. **Some special cases**

Here we briefly discuss the cases $q = 0, q = 1$ and $q = 2$.

Example 4.1. *In case $q = 0$ we may assume that $L(e_0; x) > 0$. Since otherwise, for $f \in C[0, 1]$ arbitrary, we would have*

$$|L(f; x)| \leq L(|f|; x) \leq \|f\| \cdot L(e_0; x) = 0,$$

leading to a trivial inequality. Making the above assumption we get

$$|L(f; x) - L(e_0; x) \cdot f(x)| \leq L(e_0; x) \cdot \tilde{\omega}\left(f; \frac{L(|e_1 - x|; x)}{L(e_0; x)}\right);$$

thus

$$|L(f; x) - f(x) + f(x) - L(e_0; x) \cdot f(x)| \leq L(e_0; x) \cdot \tilde{\omega}\left(f; \frac{L(|e_1 - x|; x)}{L(e_0; x)}\right),$$

or

$$|L(f; x) - f(x)| \leq |L(e_0; x) - 1| \cdot |f(x)| + L(e_0; x) \cdot \tilde{\omega}\left(f; \frac{L(|e_1 - x|; x)}{L(e_0; x)}\right),$$

or, for $L(e_0; x) = 1$,

$$|L(f; x) - f(x)| \leq \tilde{\omega}(f; L(|e_1 - x|; x).$$

This is an inequality which can already be found in [5], Theorem 3.1.

Example 4.2. *For $q = 1$, i.e., $f \in C^1[0, 1]$ we arrive at*

$$|L(f; x) - L(e_0; x) \cdot f(x) - L(e_1 - x; x) \cdot f'(x)| \leq L(|e_1 - x|; x) \cdot \tilde{\omega}\left(f'; \frac{1}{2} \cdot \frac{L((e_1 - x)^2; x)}{L(|e_1 - x|; x)}\right).$$

Proceeding as in the previous case we find

$$\begin{aligned}
|L(f; x) - f(x)| &\leq |(L(e_0; x) - 1) \cdot f(x) + L(e_1 - x; x) \cdot f'(x)| \\
&\quad + L(|e_1 - x|; x) \cdot \tilde{\omega}(f'; \tfrac{1}{2} \tfrac{L((e_1 - x)^2; x)}{L(|e_1 - x|; x)}) \\
&\leq |L(e_0; x) - 1| \cdot |f(x)| + |L(e_1 - x; x)| \cdot |f'(x)| \\
&\quad + L(|e_1 - x|; x) \cdot \tilde{\omega}(f'; \tfrac{1}{2} \tfrac{L((e_1 - x)^2; x)}{L(|e_1 - x|; x)}).
\end{aligned}$$

If L reproduces linear functions this simplifies to

$$|L(f;x)-f(x)| \le L(|e_1-x|;x)\cdot\tilde{\omega}(f';\frac{1}{2}\cdot\frac{L((e_1-x)^2;x)}{L(|e_1-x|;x)}).$$

A similar inequality was given in [5], *Section 4.*

Example 4.3. *For $q=2$ we get*

$$|L(f;x)-L(e_0;x)\cdot f(x)-L((e_1-x);x)\cdot f'(x)-\tfrac{1}{2}\cdot L((e_1-x)^2;x)\cdot f''(x)|$$

$$\le \tfrac{1}{2}\cdot L((e_1-x)^2;x)\cdot\tilde{\omega}(f'';\tfrac{1}{3}\cdot\tfrac{L(|e_1-x|^3;x)}{L((e_1-x)^2;x)}).$$

If $L(e_0;x)=1$ and $L((e_1-x);x)=0$, then this turns into

$$|L(f;x)-f(x)-\tfrac{1}{2}\cdot L((e_1-x)^2;x)\cdot f''(x)|$$

$$\le \tfrac{1}{2}\cdot L((e_1-x)^2;x)\cdot\tilde{\omega}(f'';\tfrac{1}{3}\cdot\tfrac{L(|e_1-x|^3;x)}{L((e_1-x)^2;x)})$$

(see [6], *proof of Theorem 6.2).*

For Bernstein operators B_n we arrive at

$$|B_n(f;x)-f(x)-\frac{x(1-x)}{2n}\cdot f''(x)| \le \frac{x(1-x)}{2n}\cdot\tilde{\omega}(f'';\frac{n}{3}\frac{B_n(|e_1-x|^3;x)}{x(1-x)})$$

$$\le \frac{x(1-x)}{2n}\cdot\tilde{\omega}(f'';\frac{1}{3\sqrt{n}})$$

(see [6], Proposition 7.2). This is the example mentioned earlier. But we can do better as is shown in the next section.

5. Application to Bernstein-type operators

Theorem 5.1. *For $f\in C^2[0,1], x\in[0,1]$ and $n\in\mathbb{N}$ one has*

$$|n\cdot[B_n(f;x)-f(x)]-\frac{x(1-x)}{2}\cdot f''(x)| \le \frac{x(1-x)}{2}\cdot\tilde{\omega}(f'';\sqrt{\frac{1}{n^2}+\frac{x(1-x)}{n}}).$$

Proof. We discriminate two cases.

(i) $x\in[\frac{1}{n},1-\frac{1}{n}]$. In this situation we showed in Remark 9.4 of [6] that, using the Cauchy-Schwarz inequality,

$$\frac{B_n(|e_1-x|^3;x)}{B_n((e_1-x)^2;x)} \le \sqrt{\frac{B_n((e_1-x)^4;x}{B_n((e_1-x)^2;x)}} \le 2\cdot\sqrt{\frac{x(1-x)}{n}}.$$

(ii) $x \in [0, \frac{1}{n}] \cup [1 - \frac{1}{n}, 1]$. In Remark 7.3 of [6] it was proved that in this case

$$\frac{B_n(|e_1 - x|^3; x)}{B_n((e_1 - x)^2; x)} \le \frac{3}{n}.$$

Thus for all $x \in [0, 1]$ we get

$$\frac{B_n(|e_1 - x|^3; x)}{B_n((e_1 - x)^2; x)} \le 3 \cdot \sqrt{\frac{1}{n^2} + \frac{x(1-x)}{n}}$$

which, using also

$$B_n((e_1 - x)^2; x) = \frac{x(1-x)}{n},$$

gives the inequality in the theorem. □

Remark 5.2. *The above inequality can also be written as*

$$B_n(|e_1 - x|^3; x) \le 3 \cdot \sqrt{\frac{1}{n^2} + \frac{x(1-x)}{n}} \cdot B_n((e_1 - x)^2; x).$$

This shows that - in this particular case - the absolute moment of "high" order 3 may be estimated by the product of a function vanishing uniformly of order $o(\frac{1}{\sqrt{n}})$ *and the moment of "low" order 2. It would be interesting to know if this can be proved for more general positive linear operators.* □

A similar improvement close to the endpoints 0 and 1 is also possible for the so-called "genuine Bernstein-Durrmeyer operators" defined by

$$U_n(f; x) = f(0) \cdot p_{n,0}(x) + f(1) \cdot p_{n,n}(x) + \sum_{k=1}^{n-1} p_{n,k}(x) \cdot \int_0^1 p_{n-2,k-1}(t) \cdot f(t) dt.$$

A survey on these and related operators was recently prepared by D. Kacsó [9]. For our purposes the information given about them in [6] and [7] will suffice. We obtain

Theorem 5.3. *For* $f \in C^2[0, 1], x \in [0, 1]$ *and* $n \in \mathbb{N}, n \ge 2$, *the following inequality holds*

$$|(n+1)[U_n(f; x) - f(x)] - x(1-x)f''(x)| \le \frac{x(1-x)}{n+1} \cdot \tilde{\omega}(f''; 4 \cdot \sqrt{\frac{1}{(n+1)^2} + \frac{x(1-x)}{n+1}}).$$

Proof. Again we consider two cases.

(i) $x \in [\frac{1}{n}, 1 - \frac{1}{n}]$. In Section 7 of [6] we noted that

$$U_n((e_1 - x)^2; x) = \frac{2x(1-x)}{n+1},$$

$$U_n((e_1 - x)^4; x) = \frac{12x^2(1-x)^2(n-7) + 24x(1-x)}{(n+1)(n+2)(n+3)}.$$

From this we get

$$\frac{U_n(|e_1 - x|^3; x)}{U_n((e_1 - x)^2; x)} \le \sqrt{\frac{U_n((e_1 - x)^4; x)}{U_n((e_1 - x)^2; x)}} = \sqrt{\frac{6x(1-x)(n-7) + 12}{(n+2)(n+3)}}.$$

For $n \ge 2$ and $x \in [\frac{1}{n}, 1 - \frac{1}{n}]$

$$\frac{6x(1-x)(n-7) + 12}{(n+2)(n+3)} \le \frac{18x(1-x)}{n+1}, \text{ so that}$$

$$\frac{U_n(|e_1 - x|^3; x)}{U_n((e_1 - x)^2; x)} \le \sqrt{\frac{18x(1-x)}{n+1}}.$$

(ii) $x \in [0, \frac{1}{n}] \cup [1 - \frac{1}{n}, 1]$. We only consider the left interval, the second one can be treated analogously. In this case we write

$$U_n(|e_1 - x|^3; x) = U_n(|e_1 - x|^3; x) - U_n((e_1 - x)^3; x) + U_n((e_1 - x)^3; x)$$

$$= U_n(|e_1 - x|^3 - (e_1 - x)^3; x) + U_n((e_1 - x)^3; x)$$

$$= U_n(2 \cdot (e_1 - x)^2 (x - e_1)_+; x) + U_n((e_1 - x)^3; x).$$

Here $(x - e_1)_+ := \max\{0, x - e_1\}$.

From the definition of U_n it follows that

$$\begin{aligned} &U_n(|e_1 - x|^3; x) \\ &= 2x^3 \cdot p_{n,0}(x) + 2 \sum_{k=1}^{n-1} p_{n,k}(x) \cdot \int_0^x p_{n-2,k-1}(t) \cdot (x-t)^3 dt + U_n((e_1 - x)^3; x) \\ &\le 2x^3 \cdot (1-x)^n + \frac{2}{n^2} x(1-x) \cdot \sum_{k=1}^{n-1} p_{n,k}(x) + \frac{6x(1-x)(1-2x)}{(n+1)(n+2)}. \end{aligned}$$

The representation of $U_n((e_1 - x)^3; x)$ can be found in [7].

The latter expression is bounded from above by

$$\begin{aligned}&\frac{2}{n^2}x(1-x)\cdot[(1-x)^{n-1}+1+3(1-2x)]\\&\le\frac{10}{n^2}x(1-x)\le\frac{45}{2}\cdot\frac{x(1-x)}{(n+1)^2}\text{ for } n\ge 2.\end{aligned}$$

Thus in this case we get

$$\frac{U_n(|e_1-x|^3;x)}{U_n((e_1-x)^2;x)}\le\frac{45}{2}\cdot\frac{x(1-x)}{(n+1)^2}\cdot\frac{n+1}{2x(1-x)}\le\frac{12}{n+1}.$$

Combining the two cases entails, for $x\in[0,1]$ and $n\ge 2$,

$$\frac{U_n(|e_1-x|^3;x)}{U_n(e_1-x)^2;x)}\le 12\cdot\sqrt{\frac{1}{(n+1)^2}+\frac{x(1-x)}{n+1}},$$

and from this the theorem follows. □

6. **Concluding remarks**

Remark 6.1. *Here we make some further remarks concerning the case when $q\ge 3$ is odd. In this case the right hand side in the inequality of Theorem 3.1 is*

$$\frac{L(|e_1-x|^q;x)}{q!}\cdot\tilde{\omega}(f^{(q)};\frac{1}{q+1}\cdot\frac{L((e_1-x)^{q+1};x)}{L(|e_1-x|^q;x)}.$$

Furthermore we assume that $L(e_0;x)=1$. A Hölder-type inequality for positive linear operators (see [6], Theorem 5.1) then implies for $1\le s<r$:

$$L(|e_1-x|^s;x)^{\frac{1}{s}}\le L(|e_1-x|^r;x)^{\frac{1}{r}}.$$

Taking $s=q-1\ge 2$ and $r=q$ gives

$$L((e_1-x)^{q-1};x)^{\frac{1}{q-1}}\le L(|e_1-x|^q;x)^{\frac{1}{q}}$$

or

$$L((e_1-x)^{q-1};x)^{\frac{q}{q-1}}\le L(|e_1-x|^q;x).$$

Thus the left side in Theorem 3.1 is bounded from above by

$$\frac{L(|e_1-x|^q;x)}{q!}\cdot\tilde{\omega}\left(f^{(q)};\frac{1}{q+1}\cdot\frac{L((e_1-x)^{q+1};x)}{L((e_1-x)^{q-1};x)^{\frac{q}{q-1}}}\right).$$

Now the moments inside $\tilde{\omega}(f^{(q)};\cdot)$ are both of even order and can more easily be evaluated. The absolute moment in front of $\tilde{\omega}(f^{(q)};\cdot)$ can also be estimated using Hölder's inequality.

Our quantitative Voronovskaja-type theorem is not only applicable to polynomial operators as can be seen from the following

Example 6.2. *(see* [7]*) For variation-diminishing spline operators* S_{Δ_n} *giving piecewise linear interpolators at equidistant knots in* $[0,1]$ *the quantitative Voronovskaja theorem in case* $q=2$ *reads*

$$|\frac{n^2}{z_n(x)(1-z_n(x))}[S_{\Delta_n}(f;x)-f(x)]-\frac{1}{2}\cdot f''(x)|\leq\frac{1}{2}\cdot\tilde{\omega}(f'';\frac{1}{3n}).$$

This is (again) obtained via representations of the second and fourth central moments as given, for example, in Lupaş' Romanian Ph. D. thesis [11] *on p. 46:*

$$S_{\Delta_n}((e_1-x)^2;x)=\frac{1}{n^2}z_n(x)\cdot(1-z_n(x)),$$

$$S_{\Delta_n}((e_1-x)^4;x)=\frac{1}{n^2}z_n(x)(1-z_n(x))\cdot[1-3z_n(x)(1-z_n(x))].$$

Here $z_n(x)=\{nx\}:=nx-[nx]$ *is the fractional part of* nx*.*

Voronovskaja-type results are also known for other cases of Schoenberg's variation-diminishing spline operators. See the cited articles by Marsden, Riemenschneider and Schoenberg for non-quantitative versions. It would be of interest to find quantitative statements also for cases other than S_{Δ_n}.

Remark 6.3. *We noted before (see Theorem 1.1) that Voronovskaja's theorem is pointwise in nature, i.e., it does not only hold for functions* $f\in C^2[0,1]$*. One example is the negative "entropy function"*

$$f(x)=x\log x+(1-x)\log(1-x),x\in(0,1),f(0):=0,f(1):=0.$$

Here $f''(x)=[x(1-x)]^{-1},x\in(0,1)$*, so that the local version of Voronovskaja's theorem gives*

$$\lim_{n\to\infty}n\cdot[B_n(f;x)-f(x)]=\frac{1}{2},x\in(0,1),$$

while $B_n(f;0)-f(0)=0=B_n(f;1)-f(1),n\in\mathbb{N}$.

In [2] *several interesting phenomena concerning the approximation of the entropy function f by $B_n f$ are discussed.*

Moreover, Lupaş showed in [12] *that for this function one has*

$$\frac{x(1-x)}{2} \leq n[B_n(f;x) - f(x)] \leq \sqrt{2nx(1-x)};$$

see also the related question in [18].

Remark 6.4. *It was not easy to find out details about the life of Elizaveta Vladimirovna Voronovskaja who was born in 1898 or 1899 in Sankt Peterburg (Russia) and died in 1972, most likely in Leningrad (Soviet Union). Voronovskaja held university degrees in mathematics and history and was influenced in her mathematical work by S.N. Bernstein and V.I. Smirnov . Her main scientific achievement is - besides the famous 1932 paper on the asymptotic behavior of Bernstein polynomials - the monography "The functional method and its applications" which was published in Russian in 1963 and in English in 1970. Since 1946 she was the chairperson of the department of higher mathematics in the Leningrad Institute of Aerospace Instrumentation (now St. Petersburg State University of Aerospace Instrumentation). The last years of her life she also worked as a chairperson, now in the department of higher mathematics in the Leningrad Institute of Telecommunications (now St. Petersburg State University of Telecommunications).*

This and more information on E.V. Voronovskaja including a photograph can be found at the following Russian internet pages (operative on April 20, 2007):

http://www.spbstu.ru/public/m_v/N_002/Yarv/Voronovskaya.html

http://www.spbstu.ru/phmech/math/persons/HISTORY/Voronovskaia_E_V.html

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ON GRAPH-INVARIANT MEASURES

IULIANA MARCHIŞ

Dedicated to Professor Petru Blaga at his 60^{th} *anniversary*

Abstract. The goal of this paper is to prove some basic properties of invariant set list and invariant list of measures defined by Mauldin-Williams graphs. The main result of this article is the so called oscillation lemma for graph self-similar sets, similar to that used in the homogenization theory of partial differential equations.

1. **Introduction**

A theory of self-similar fractal sets and measures was developed by Hutchinson. Construction mapping methods for showing the existence and uniqueness of self-similar fractal sets and measures were first used in [3]. There is a generalization of self-similarity that provides a way to study a larger class of sets. A definitive formulation is due to Mauldin and Williams [6]. In this case there is a list of nonempty compact sets to be constructed simultaneously. Each of them is decomposed into parts obtained from this list using certain similarities. For each such construction corresponds a directed multigraph: there is one node for each set in the list and the edges from a node correspond to the subsets into which the corresponding set is decomposed. A well know example of graph self-similar fractal is the golden rectangle fractal [1]. Another important examples are sets constructed for number systems with complex base. For example for the number system with base $-i+1$ and digit set

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$\{0,1\}$ we get the so called twindragon, which has a graph-similar fractal boundary [1].

The goal of this paper is to prove some basic properties of invariant set list and invariant list of measures defined by Mauldin-Williams graphs. The graph-similar measures are the generalizations of the self-similar measures introduced by Hutchinson [3]. We prove also a so called oscillation lemma, similar to that used in the homogenization theory of partial differential equations. For self-similar sets an oscillation lemma was given in [4]. This result was used in homogenization with multiple scale expansion on self-similar structures in [5].

2. **Preliminaries**

Let (V, E, i, t) be a strongly connected directed multigraph, where V is the set of the vertices, E the set of the edges of the graph, and $i : E \to V$, $t : E \to V$ two functions such that, for each edge $e \in E$, $i(e)$ is the initial vertex of e and $t(e)$ is the terminal vertex of e. Denote by E_{uv} the set of all edges from u to v, where $u, v \in V$. Each edge from E belongs exactly to one of these subsets E_{uv}. Denote by E_u the set of all edges leaving the vertex u.

A **finite path** in the graph is a finite string $\alpha = e_1e_2...e_q$ of edges, where the terminal vertex of each edge e_i is the initial vertex of the next edge e_{i+1}. The length of a path is the number of the edges contained by it, so the length of α is $|\alpha| = q$. Denote by $E_u^{(q)}$ the set of all paths of length q with the initial vertex u, and by $E_{uv}^{(q)}$ the set of all paths of length q from vertex u to v. For each $u \in V$ the set $E_{uu}^{(0)}$ has one element, the empty path from u to itself, denoted by Λ_u. Denote by $E_u^{(*)}$ the set of all finite paths with initial vertex u, $E^{(*)}$ the set of all finite paths.

$\sigma = e_1e_2...e_q...$ is an **infinite path**. Denote by $E_u^{(\omega)}$ the set of all infinite paths with initial vertex u, $E^{(\omega)}$ the set of all infinite paths.

Because (V, E, i, t) is a strongly connected multigraph, there is a path from any vertex to any other, along the edges of the graph, taken in the proper direction.

Let $\alpha \in E^{(*)}$. Then

$$[\alpha] = \{\sigma \in E^{(\omega)} : \alpha \leq \sigma\}$$

is the set of all infinite paths that begin with the finite path α.

Let $(S_v, d)_{v \in V}$ be a family of nonempty complete metric spaces, $(f_e)_{e \in E}$ a family of Lipschitz functions, $f_e : S_v \to S_u$ such that

$$d(f_e(x), f_e(y)) \leq r_e d(x, y),$$

for $x, y \in S_v$, $e \in E_{uv}$.

A directed multigraph (V, E, i, t) together with a positive number r_e for each edge $e \in E$, is a **Mualdin-Williams graph**. A definitive formulation is due to Mauldin and Williams [6].

3. **Invariant set list**

An **invariant set list** for the iterated function system $(f_e)_{e \in E}$ is a list of nonempty sets $(K_v)_{v \in V}$, $K_v \subseteq S_v$, such that

$$K_u = \bigcup_{v \in V} \bigcup_{e \in E_{uv}} f_e(K_v),$$

for all $u \in V$.

Theorem 3.1 (Theorem 4.3.5. in [1])**.** *There exists a unique invariant set list $(K_v)_{v \in V}$ of nonempty compact sets.*

Let $(f_e)_{e \in E}$ be an iterated function system. Denote by $f_{e_1 \ldots e_q} = f_{e_1} \circ \ldots \circ f_{e_q}$, and $K_u^{e_1 \ldots e_q} = f_{e_1 \ldots e_q}(K_{v_q})$, for $u, v_q \in V$, $v_q = t(e_q)$ and $e_1, \ldots, e_q \in E$.

The following results are the adaptation of the results for self-similar sets from [3] to graph self-similar sets.

Theorem 3.2. *Let $\sigma \in E_u^{(\omega)}$, $\sigma = e_1 e_2 \ldots e_q \ldots$. Then*

$$K_u \supset K_u^{e_1} \supset K_u^{e_1 e_2} \supset \ldots \supset K_u^{e_1 \ldots e_q} \supset \ldots$$

and

$$\bigcap_{q=1}^{\infty} K_u^{e_1 \ldots e_q \ldots}$$

is a singleton, whose unique member we denote by $k_u^{e_1\dots e_q\dots}$.

Proof. We have $K_{i(e)} \supset f_e(K_{t(e)})$, for every $e \in E$.

So, $K_u \supset f_{e_1}(K_{t(e_1)}) = K_u^{e_1}$.

Also $K_{i(e_2)} \supset f_{e_2}(K_{t(e_2)})$, from where $f_{e_1}(K_{i(e_2)}) \supset f_{e_1}(f_{e_2}(K_{t(e_2)}))$. But $f_{e_1}(K_{i(e_2)}) = K_u^{e_1}$ and $f_{e_1}(f_{e_2}(K_{t(e_2)})) = f_{e_1}(K_{i(e_2)}^{e_2}) = K_u^{e_1e_2}$, so $K_u^{e_1} \supset K_u^{e_1e_2}$.

Therefore

$$K_u \supset K_u^{e_1} \supset K_u^{e_1e_2} \supset \dots \supset K_u^{e_1\dots e_q} \supset \dots$$

But $\operatorname{diam}(K_u^{e_1\dots e_q}) \to 0$, as $q \to \infty$, so $\bigcap_{q=1}^{\infty} K_u^{e_1\dots e_q\dots}$ is a singleton. We denote the element of this singleton by $k_u^{e_1\dots e_q\dots}$. $\square$

Theorem 3.3. *Let* $\alpha \in E_u^{(*)}$, $\beta \in E_{t(\alpha)}^{(*)}$, $\sigma \in E_{t(\alpha)}^{(\omega)}$. *The following relations hold:*
i) $f_\alpha(K_{t(\alpha)}^{\beta}) = K_u^{\alpha\beta}$,
ii) $f_\alpha(k_{t(\alpha)}^{\sigma}) = k_u^{\alpha\sigma}$.

Proof. i) $f_\alpha(K_{t(\alpha)}^{\beta}) = f_\alpha(f_\beta(K_{t(\beta)})) = f_{\alpha\beta}(K_{t(\beta)}) = K_u^{\alpha\beta}$

ii) $f_\alpha(k_{t(\alpha)}^{\sigma}) \in f_\alpha\left(\bigcap_{q=1}^{\infty} K_{t(\alpha)}^{\sigma}\right) = \bigcap_{q=1}^{\infty} K_u^{\alpha\sigma} = \{k_u^{\alpha\sigma}\}$.

$\square$

Let $u \in V$ and $A_u \subset S_u$. We denote by $A_u^{e_1\dots e_q} = f_{e_1\dots e_q}(A_{t(e_q)})$.

Proposition 3.4. *If* A_u *is a non-empty bounded set, then*

$$d(A_u^{e_1\dots e_q}, k_u^{e_1\dots e_q\dots}) \to 0 \text{ uniformly as } q \to \infty.$$

Proof.

$$\begin{aligned} d(A_u^{e_1\dots e_q}, k_u^{e_1\dots e_q\dots}) &= d(f_{e_1\dots e_q}(A_{t(e_q)}), f_{e_1\dots e_q}(k_{t(e_q)}^{e_{q+1}\dots})) \\ &\le r_{e_1\dots e_q} d(A_{t(e_q)}, k_u^{e_{q+1}\dots}) \\ &\le r_{e_1\dots e_q} \sup\{d(a,b) : a \in A_{t(e_q)}, b \in K_{t(e_q)}\} \\ &\le C\left(\max_{1\le i\le |E|} r_{e_i}\right)^q \to 0, \text{ as } q \to \infty, \end{aligned}$$

where C is a constant, $r_{e_1e_2\dots e_q} = r_{e_1}r_{e_2}\dots r_{e_q}$, and we denote by $|E|$ the number of the edges of the graph. $\square$

Define the following map:

$$\pi_u : E_u^{(\omega)} \to K_u, \pi_u(\sigma) = k_u^\sigma, \sigma \in E_u^{(\omega)}.$$

Proposition 3.5. *π_u is a continuous map onto K_u.*

Proof. Let $\sigma \in E_u^{(\omega)}$ and $\epsilon > 0$. Then $\pi_u(\sigma) = k_u^\sigma$ and there exists q such that $K_u^{\sigma_1 \ldots \sigma_q} \subset \{x \in K_u : d(x, \pi_u(\sigma)) < \frac{\epsilon}{2}\}$. Let $\sigma' \in E_u^{(\omega)}$ such that $\sigma_1 \ldots \sigma_q$ is the longest common prefix of σ and σ' . Then

$$d(k_u^\sigma, k_u^{\sigma'}) < d(x, k_u^\sigma) + d(x, k_u^{\sigma'}) < \epsilon,$$

so π_u is continuous. □

4. **Invariant list of measures**

We assign weights $p_e > 0$ to each edge $e \in E$ such that

$$\sum_{v \in V} \sum_{e \in E_{uv}} p_e = 1, \text{ for all } u \in V,$$

so the weight on edges leaving a node must sum to 1.

For an S metric space we denote by $\mathcal{B}(S)$ the set of all tight Borel probability measures.

Let $\nu = (\nu_u)_{u \in V}$, $\mu = (\mu_u)_{u \in V}$, where $\nu_u, \mu_u \in \mathcal{B}(S_u)$, for all $u \in V$.

Define

$$\rho(\mu, \nu) = \max_{u \in V} \sup\{\mu_u(\Phi) - \nu_u(\Phi) : \Phi : X \to \mathbb{R}, \text{ Lip } \Phi \leq 1\},$$

which is a metric.

Let $\nu = (\nu_u)_{u \in V}$ and $f \in C(\mathbb{R}^n)$. We define the push-forward measure by

$$f_\# \nu_u(\phi) = \nu_u(\phi \circ f), \text{ for } \phi \in C(\mathbb{R}^n), \text{ for every } u \in V.$$

For every $u \in V$ define

$$G_u(\nu) = \sum_{v \in V} \sum_{e \in E_{uv}} p_e f_{e\#} \nu_v, \tag{1}$$

and $G(\nu) = (G_u(\nu))_{u \in V}$.

Theorem 4.1. *G is a contraction map in metric ρ.*

Proof. Let $\Phi : X \to \mathbb{R}$, Lip $\Phi \leq 1$ and $\mu, \nu \in \mathcal{B}$. Then we have

$$\begin{aligned} G_u(\mu)(\Phi) - G_u(\nu)(\Phi) &= \sum_{v\in V} \sum_{e\in E_{uv}} p_e(f_{e\#}\mu_v(\Phi) - f_{e\#}\nu_v(\Phi)) \\ &= \sum_{v\in V} \sum_{e\in E_{uv}} p_e(\mu_v(\Phi \circ f_e) - \nu_v(\Phi \circ f_e)) \\ &= \sum_{v\in V} \sum_{e\in E_{uv}} p_e r(\mu_v(r^{-1}\Phi \circ f_e) - \nu_v(r^{-1}\Phi \circ f_e)), \end{aligned}$$

where $r = \max_{e\in E} r_e < 1$. Then $\rho(G(\mu), G(\nu)) \leq r\rho(\mu, \nu)$, so G is a contraction. □

The result of the following theorem follows from Theorem 4.1 (it is given in [2] without proof).

Theorem 4.2. *There exists a unique invariant list of measures* $(\mu_v)_{v\in V}$, $\mu_v \in \mathcal{B}(S_v)$, *for all* $v \in V$, *such that*

$$\mu_u = \sum_{v\in V} \sum_{e\in E_{uv}} p_e f_{e\#}\mu_v,$$

for all $u \in V$.

We call this invariant list of measures **graph-invariant measure**. The graph-similar measures are the generalization of the self-similar measures introduced by Hutchinson [3].

Define $G^k(\nu)$ recursively. Denote $G^1(\nu) = G(\nu)$. Define $G^k(\nu) := G \circ G^{k-1}(\nu)$. Then G^k can be written in the following form:

$$G^k_u(\nu) = \sum_{v\in V} \sum_{\sigma\in E^{(k)}_{uv}} p_\sigma f_{\sigma\#}\mu_v,$$

where $p_\sigma = p_{\sigma_1} \cdot p_{\sigma_2} \cdot ... \cdot p_{\sigma_k}$. and

$$\lim_{k\to\infty} \rho(G^k(\nu), \mu) = 0.$$

Let $e \in E_{uv}$. Define

$$\theta_e : E^{(\omega)}_v \to E^{(\omega)}_u, \theta_e(\sigma) = e\sigma, \text{ for all } \sigma \in E^{(\omega)}_v,$$

which is called **right-shift map**.

For $\sigma, \tau \in E_v^{(\omega)}$ define the metric ρ_1 by

$$\rho_1(\sigma, \gamma) = r_\alpha,$$

where α is the longest common prefix of σ and γ, and $r_\alpha = r_{e_1} r_{e_2} ... r_{e_q}$, where $\alpha = e_1 e_2 ... e_q$.

The following two results are stated in [1] without proof. As we did not find the proofs in the literature, for completeness, we give it here.

Lemma 4.3. *With the metric ρ_1 the right-shift θ_e is a similarity with ratio r_e.*

Proof. $\rho_1(e\sigma, e\gamma) = r_{e\alpha} = r_e r_\alpha = r_e \rho_1(\sigma, \gamma)$, where α is the longest common prefix of σ and γ. □

Fix a vertex $v \in V$. Suppose nonnegative numbers w_α are given, one for each $\alpha \in E_v^{(*)}$ such that

$$w_\alpha = \sum_{\substack{e \in E \\ i(e) = t(\alpha)}} w_{\alpha e}.$$

Theorem 4.4. *The method I outer measure defined by the set function $C([\alpha]) = w_\alpha$ is a metric outer measure τ_v on $E_v^{(\omega)}$ with $\tau_v([\alpha]) = w_\alpha$.*

Proof. Let $\mathcal{A}_v = \{[\alpha] : \alpha \in E_v^{(*)}\}$ and $\mathcal{A}_v^\epsilon = \{D \in \mathcal{A} : \text{ diam } D \le \epsilon\}$.

Let ν_v^ϵ be the method I measure defined by the set function C restricted to $\mathcal{A}_v^\epsilon$. If $D \in \mathcal{A}_v^\epsilon$, then $\tau_v(D) \le C(D)$, so by the method I theorem (Theorem 5.2.2. in [1])

$$\nu_v^\epsilon(A) \ge \tau_v(A), \text{ for all } A.$$

Therefore the method II measure ν_v defined by

$$\nu_v(A) = \lim_{\epsilon \to 0} \nu_v^\epsilon(A)$$

satisfies $\nu_v \ge \tau_v$.

For any $\alpha \in E_v^{(*)}$ we have

$$C([\alpha]) = w_\alpha = \sum_{\substack{e\in E\\ i(e)=t(\alpha)}} w_{\alpha e} = \sum_{\substack{e\in E\\ i(e)=t(\alpha)}} C([\alpha e]).$$

Applying this repeatedly, knowing that

$$\lim_{k\to\infty} (\ \sup\ \{\text{diam}\ [\alpha] : \alpha \in E_v^{(k)}\}) = 0,$$

we have that for any $\epsilon > 0$, any set $D \in \mathcal{A}$ is finite union of sets from $\mathcal{A}_v^\epsilon$, i.e.

$$D = \bigcup_{i=1}^n D_i, D_i \in \mathcal{A}_v^\epsilon \text{ with } C(D) = \sum_{i=1}^n C(D_i).$$

From this follows that

$$\nu_v^\epsilon(D) = \nu_v^\epsilon(\bigcup_{i=1}^n D_i) \leq \sum_{i=1}^n \nu_v^\epsilon(D_i) \leq \sum_{i=1}^n C(D_i) = C(D),$$

so by method I theorem

$$\nu_v^\epsilon(A) \leq \tau_v(A), \text{ for all } A,$$

from where

$$\nu_v(A) \geq \tau_v(A), \text{ for all } A.$$

Therefore $\tau_v = \nu_v$ is a method II outer measure, so it is a metric outer measure. □

Let $\tau = (\tau_v)_{v\in V}$.

Lemma 4.5. *The measure $(\tau_u)_{u\in V}$ is $(\theta_e, p_e)_{e\in E}$ invariant. i.e. for all $u \in V$*

$$\sum_{v\in V}\sum_{e\in E_{uv}} p_e\theta_{e\#}\tau_v = \tau_u.$$

Proof. Let $\alpha \in E_v^{(*)}$. Then

$$\begin{aligned}\sum_{v\in V}\sum_{e\in E_{uv}} p_e\theta_{e\#}\tau_v([\alpha]) &= \sum_{v\in V}\sum_{e\in E_{uv}} p_e\tau_v\theta_e^{-1}([\alpha])\\ &= \sum_{v\in V}\sum_{e\in E_{uv}} p_e\tau_v([e^{-1}\alpha])\\ &= \sum_{v\in V}\sum_{e\in E_{uv}} p_e w_{e^{-1}\alpha} = w_\alpha,\end{aligned}$$

where we denote by $e^{-1}\alpha$ the following path: we remove the edge e from the beginning of the path α.

On the other side of the equation $\tau_u([\alpha]) = w_\alpha$. Therefore the equality holds. $\square$

Theorem 4.6. *The following two relations hold:*

i) $\mu_u = \pi_{u\#}\tau_u$

ii) $\mathrm{spt}\mu_u = K_u$.

Proof. i) First we prove that $f_e \circ \pi_v = \pi_u \circ \theta_e$. For $\sigma \in E_u^{(\omega)}$

$$f_e(\pi_v(\sigma)) = f_e(k_v^\sigma) = k_u^{e\sigma},$$

using Theorem 3.3 and

$$\pi_u(\theta_e(\sigma)) = \pi_u(e\sigma) = k_u^{e\sigma},$$

so the above relation is true.

Now

$$\begin{aligned}\sum_{v\in V}\sum_{e\in E_{uv}} p_e f_{e\#}(\pi_{v\#}\tau_v) &= \sum_{v\in V}\sum_{e\in E_{uv}} p_e \pi_{u\#}(\theta_{e\#}\tau_v)\\ &= \pi_{u\#}\left(\sum_{v\in V}\sum_{e\in E_{uv}} p_e\theta_{e\#}\tau_v\right)\\ &= \pi_{u\#}\tau_u\end{aligned}$$

ii) Let A be an open set such that $\overline{A} \cap K_u = \emptyset$. Then $\pi_{u\#}(\tau_u(A)) = \tau_u(\pi_u^{-1}(A)) = \emptyset$, thus $\mathrm{spt}\mu_u = K_u$. $\square$

5. **Oscillation lemma**

Theorem 5.1 (Oscillation lemma). *Let $K = (K_u)_{u\in V}$ be the list of invariant sets, $\mu = (\mu_u)_{u\in V}$ the list of invariant measures, $\nu = (\nu_u)_{u\in V}$ a list of probabilistic Borel measures with compact support,* $\mathrm{spt}\nu_u = \Omega_u$. *If $\phi_u \in L^2(\Omega_u, C_b(\mathbb{R}^n))$, $u \in V$ then*

$$\lim_{k\to\infty}\sum_{\sigma\in E_u^{(k)}} p_\sigma \int_{\Omega_u}\phi_u(f_\sigma(z), z)d\nu_u(z) = \int_{K_u}\left[\int_{\Omega_u}\phi_u(x,y)d\nu_u(y)\right]d\mu_u(x), \qquad (2)$$

for every $u \in V$, where $p_\sigma = p_{\sigma_1}\cdot p_{\sigma_2}\cdot \ldots \cdot p_{\sigma_k}$.

Proof. Take first $\phi_u \in C(\Omega_u, C_b(\mathbb{R}^n))$, $u \in V$. Let h be the Hausdorff metric in the space of all nonempty compact sets in $\mathbb{R}^n$. Since

$$\lim_{k\to\infty} \rho(G^k(\nu), \mu) = 0,$$

we have

$$\lim_{k\to\infty} h(\text{spt } ((G^k(\nu))_u), K_u) = 0.$$

Then

$$\lim_{k\to\infty} h(\Omega_u^\sigma, K_u^\sigma) = 0$$

and

$$\lim_{k\to\infty} \text{diam } (\Omega_u^\sigma) = 0.$$

Let $\epsilon > 0$. Then there exists $k_0 \in \mathbb{N}$ such that

$$\max\{|\phi(x,y) - \phi(t,y)| : y \in \Omega_u\} < \epsilon,$$

for all $x \in K_u^\sigma$, $t \in \Omega_u^\sigma$, $k \geq k_0$.

Choose $z \in \Omega_u$ such that $t = f_\sigma(z)$.

Then

$$\left| \int_{\Omega_u} \phi_u(x,z) d\nu_u(z) - \int_{\Omega_u} \phi_u(f_\sigma(z), z) d\nu_u(z) \right| < \epsilon.$$

Multiplying this equality by p_σ and summing, using that

$$\sum_{\sigma\in E_u^{(k)}} p_\sigma = 1,$$

we get

$$-\epsilon < \int_{\Omega_u} \phi_u(x,y) d\nu_u(y) - \sum_{\sigma\in E_u^{(k)}} p_\sigma \int_{\Omega_u} \phi_u(f_\sigma(z), z) d\nu_u(z) < \epsilon.$$

Integrating on $f_\sigma(K_u)$ and summing

$$-\epsilon < \int_{K_u} \int_{\Omega_u} \phi_u(x,y) d\nu_u(y) d\mu_u(x) - \sum_{\sigma\in E_u^{(k)}} p_\sigma \int_{\Omega_u} \phi_u(f_\sigma(z), z) d\nu_u(z) < \epsilon.$$

As $C(\Omega, C_b(\mathbb{R}^n))$ is dense in $L^2(\Omega_u, C_b(\mathbb{R}^n))$, we get the result. □

For self-similar sets formula like (2) was given in [4]. This result was used in homogenization with multiple scale expansion on self-similar structures in [5].

Remark 5.2. *If ϕ depends only on its first variable, the convergence (2) has the following form:*

$$\lim_{k\to\infty} \sum_{\sigma\in E_u^{(k)}} p_\sigma \int_{\Omega_u} \phi_u(f_\sigma(z))d\nu_u(z) = \int_{K_u} \phi_u(x)d\mu_u(x),$$

for every $u \in V$, which means the weak convergence of the sequence $G^k(\nu)$ to the invariant vector measure μ.

Remark 5.3. *A well know example of graph self-similar fractal is the golden rectangle fractal. For the construction of this fractal see for example* [1].

Another important examples are sets constructed for number systems with complex base. For example for the number system with base $-i+1$ and digit set $\{0,1\}$ we get the so called twindragon. The twindragon has a graph-similar fractal boundary (see [1]*).*

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STUDIA UNIV. "BABEŞ–BOLYAI", MATHEMATICA, Volume **LII**, Number 3, September 2007

ON SOME GEOMETRIC PROPERTIES OF THE RADIOSITY EQUATION

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Dedicated to Professor Petru Blaga at his 60^{th} *anniversary*

Abstract. In this paper we study some properties of the kernel of the radiosity equation. The geometric term $G(P,Q)$ from the kernel is of great importance in the study of solvability and regularity of the radiosity equation. We state some of its properties and give a complete proof of the main one, namely the fact that

$$\left| D_Q^i G(P,Q) \right| \leq \frac{c}{|P-Q|^i}, \quad P \neq Q.$$

1. The radiosity equation

Radiosity, an important quantity in image synthesis, is defined as being the energy per unit solid angle that leaves a surface. The photometric equivalent is *luminosity*. Radiosity is a method of describing illumination based on a detailed analysis of light reflections off diffuse surfaces. It is typically used to render images of the interior of buildings, and it can achieve extremely photo-realistic results for scenes that are comprised of diffuse reflecting surfaces. In computer graphics, the computation of lighting can be done via radiosity. The *radiosity equation* is a mathematical model for the brightness of a collection of one or more surfaces. The equation is

$$u(P) - \frac{\rho(P)}{\pi} \int\limits_S u(Q)G(P,Q)V(P,Q)dS_Q = E(P), \quad P \in S, \tag{1}$$

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where $u(P)$ is the *radiosity*, or the brightness, at $P \in S$. $E(P)$ is the *emissivity* at $P \in S$, the energy per unit area emitted by the surface.

The function $\rho(P)$ gives the *reflectivity* at $P \in S$, i. e. the bidirectional reflection distribution function. We have that $0 \leq \rho(P) < 1$, with $\rho(P)$ being 0 where there is no reflection at all at P. The radiosity equation is derived from the rendering equation under the *radiosity assumption*: all surfaces in the environment are *Lambertian diffuse reflectors* (see [1], [3], [4]). What this means is that the reflectivity $\rho(P)$ is independent of the incoming and outgoing directions and, hence, of the angle at which the reflection takes place. Thus, $\rho(P)$ can be taken out from under the integral of a more general formulation (the rendering equation, see [4]), leading to (1).

The function $V(P, Q)$ is a *visibility* function. It is 1 if the points P and Q are "mutually visible" (meaning they can "see each other" along a straight line segment which does not intersect S at any other point), and 0 otherwise. Surfaces S for which $V \equiv 1$ on S are called *unoccluded*. More about the radiosity equation can be found in [4].

The function G, a geometric term, is given by

$$\begin{aligned} G(P,Q) &= \frac{[(Q-P)\cdot \mathbf{n}_P]\,[(P-Q)\cdot \mathbf{n}_Q]}{|P-Q|^4} \\ &= \frac{\cos\theta_P \cdot \cos\theta_Q}{|P-Q|^2}, \end{aligned} \tag{2}$$

where $\mathbf{n}_P$ is the inner unit normal to S at P, θ_P is the angle between $\mathbf{n}_P$ and $Q-P$, and $\mathbf{n}_Q$ and θ_Q are defined analogously.

We can write (1) in the form

$$u(P) - \int_S K(P,Q)u(Q)dS_Q = E(P), \quad P \in S, \tag{3}$$

with

$$K(P,Q) = \frac{\rho(P)}{\pi} G(P,Q)V(P,Q), \quad P, Q \in S, \tag{4}$$

or, in operator form

$$(I - K)u = E. \tag{5}$$

The function $G(P,Q)$ given in (2) has a singularity at $P=Q$ and is smooth otherwise. The study of the solvability and regularity of equation (1) (see [2], [3], [7], [8], [9]) relies heavily on special properties of this function, which is why we want to focus on some of these properties.

2. **Properties of the function** $G(P,Q)$

Lemma 1. *Let S be a smooth surface to which the Divergence Theorem can be applied. Let $P \in S$. Then*

a) $|G(P,Q)| \leq c, \quad P,Q \in S, \quad P \neq Q$ *(c independent of P and Q);*

b) $G(P,Q) \geq 0,$ for $Q \in S$*;*

c) $\int\limits_S G(P,Q)\, dS_Q = \pi$*;*

d) if S is the unit sphere, then $G(P,Q) \equiv \dfrac{1}{4}$.

The proofs of these properties are straight forward computations and can be found in [5], [6], [10]. □

Next, let us make some simplifying notations. First, by $\dfrac{\partial F(P)}{\partial P}$ we denote generically the derivatives $\dfrac{\partial F(P)}{\partial x}$, $\dfrac{\partial F(P)}{\partial y}$, where $P=P(x,y)$. Denote by

$$\begin{aligned} F^P(P,Q) &= \frac{\cos\theta_P}{|P-Q|}, \\ F^Q(P,Q) &= \frac{\cos\theta_Q}{|P-Q|}. \end{aligned} \tag{6}$$

Then we can write

$$G(P,Q) = F^P(P,Q)\cdot F^Q(P,Q). \tag{7}$$

Now we can state the main property of the function G.

Theorem 2. *Let $i \geq 0$ be an integer and let S be a smooth C^{i+1} surface to which the Divergence Theorem can be applied. Then*

$$\left|D_Q^i G(P,Q)\right| \leq \frac{c}{|P-Q|^i}, \quad P \neq Q, \tag{8}$$

for the function $G(P,Q)$ of (2), with c a generic constant independent of P and Q.

Proof. Since

$$|\cos\theta_P| \le c\,|P-Q|\,,\ |\cos\theta_Q| \le c\,|P-Q|\,, \tag{9}$$

we have

$$|F^P|, |F^Q| \le c. \tag{10}$$

For $n \ge 0$ we can write

$$D_Q^n G = \sum_{k=0}^{n} \binom{n}{k} D_Q^{n-k}(F^P)(P,Q) \cdot D_Q^k(F^Q)(P,Q). \tag{11}$$

Claim.

$$\left|D_Q^i F^P\right|, \left|D_Q^i F^Q\right| \le \frac{c}{|P-Q|^i}. \tag{12}$$

Proof of claim. Fix $P \in S$. The proof of (12) is very delicate. We will use both a local parametrization of the surface as well as formal reasoning. For a better understanding, one could see [9], [10], [11]. Assume the surface S can be represented locally by

$$z = f(x,y), \tag{13}$$

with $f \in C^{i+2}$. We consider P to be the origin of a coordinate system and Q an arbitrary point in S. Then we have

$$\begin{aligned} P &= (0,0,0), \\ Q &= (x,y,f(x,y)), \\ \mathbf{n}_P &= (0,0,1), \\ \mathbf{n}_Q &= (-f_x(x,y), -f_y(x,y), 1). \end{aligned} \tag{14}$$

(Implicitly, we then also have that $f(0,0) = f_x(0,0) = f_y(0,0) = 0$).

We can write

$$\begin{aligned} \cos\theta_P = \ & \frac{(Q-P)\cdot\mathbf{n}_P}{|P-Q|\cdot|\mathbf{n}_P|} = \frac{Q}{|Q|}\cdot N^P, \\ \cos\theta_Q = \ & \frac{(Q-P)\cdot\mathbf{n}_Q}{|P-Q|\cdot|\mathbf{n}_Q|} = \frac{Q}{|Q|}\cdot N^Q, \end{aligned} \tag{15}$$

where we denoted by $N^P = \mathbf{n}_P$ and by $N^Q = \dfrac{\mathbf{n}_Q}{|\mathbf{n}_Q|}$. Note that by (14), N^P is independent of Q (and, hence, of x and y), while N^Q is a function of Q, i.e. of x and y. The inequalities (10) can now be written

$$\begin{aligned} \left| \frac{Q}{|Q|^2} \cdot N^P \right| &\leq c, \\ \left| \frac{Q}{|Q|^2} \cdot N^Q \right| &\leq c. \end{aligned} \tag{16}$$

Let us proceed first with the derivative of F^P. In what follows we will use the notation g_x, rather than $\dfrac{\partial g}{\partial x}$, for the derivative of a function g with respect to x. We have

$$\begin{aligned} |P-Q| \frac{\partial F^P}{\partial x} &= |Q| \left(\frac{Q}{|Q|^2} \cdot N^P \right)_x \\ &= |Q| \left(\frac{Q_x \cdot N^P}{|Q|^2} - 2 \frac{\left(Q \cdot N^P\right)(Q \cdot Q_x)}{|Q|^4} \right) \\ &= \frac{Q_x \cdot N^P}{|Q|} - 2 \frac{\left(Q \cdot N^P\right)(Q \cdot Q_x)}{|Q|^3}. \end{aligned} \tag{17}$$

For the first term on the right of (17), we have

$$\begin{aligned} \frac{Q_x \cdot N^P}{|Q|} &= \frac{(1,0,f_x) \cdot (0,0,1)}{\sqrt{x^2+y^2+(f(x,y))^2}} \\ &= \frac{f_x}{\sqrt{x^2+y^2+(f(x,y))^2}} \\ &= O\left(\frac{|x|+|y|}{\sqrt{x^2+y^2}} \right), \end{aligned} \tag{18}$$

which is bounded. The second term on the right of (17) can be rewritten as

$$2 \left(\frac{Q}{|Q|^2} \cdot N^P \right) \left(\frac{Q}{|Q|} \cdot Q_x \right). \tag{19}$$

The first term in (19) is clearly bounded because of (16). For the second term in (19), note that by (14), $Q_x = (1, 0, f_x)$. Then by our assumption on the smoothness of f , $|f_x|$ is bounded, and hence, so is $\left| \dfrac{Q}{|Q|} \cdot Q_x \right|$.

We have just proved that $\left|\dfrac{\partial F^P}{\partial x}\right| \le \dfrac{c}{|P-Q|}$. An identical argument will lead to the result $\left|\dfrac{\partial F^P}{\partial y}\right| \le \dfrac{c}{|P-Q|}$. So we have that

$$\left|D_Q F^P\right| \le \frac{c}{|P-Q|}. \tag{20}$$

In a similar way we prove that the claim is also true for F^Q. We have

$$\begin{aligned} |P-Q|\frac{\partial F^Q}{\partial x} &= |Q|\left(\frac{Q}{|Q|^2}\cdot N^Q\right)_x \\ &= |Q|\left(\frac{Q_x\cdot N^Q}{|Q|^2}+\frac{Q\cdot N^Q_x}{|Q|^2}\right. \\ &- \left. 2\,\frac{\left(Q\cdot N^Q\right)\left(Q\cdot Q_x\right)}{|Q|^4}\right) \\ &= \frac{Q_x\cdot N^Q}{|Q|}+\frac{Q\cdot N^Q_x}{|Q|}-2\frac{\left(Q\cdot N^Q\right)\left(Q\cdot Q_x\right)}{|Q|^3}. \end{aligned} \tag{21}$$

The first term on the right of (21) is obviously 0. The second term on the right of (21) is bounded because $\dfrac{Q}{|Q|}$ is a unit vector (so bounded) and

$$N^Q_x = \frac{1}{|n_Q|}\left(-f_{xx},-f_{yx},0\right)+\frac{f_x f_{xx}+f_y f_{yx}}{|n_Q|^3}\left(f_x,f_y,-1\right),$$

and we assumed $f \in C^{i+2}$. The third term on the right of (21) can be rewritten (similarly with (19)) as

$$2\left(\frac{Q}{|Q|}\cdot Q_x\right)\left(\frac{Q}{|Q|^2}\cdot N^Q\right), \tag{22}$$

which is bounded by (16) and by our earlier discussion following (19).

The same argument (with x replacing y) proves that $\left|\dfrac{\partial F^Q}{\partial y}\right| \le \dfrac{c}{|P-Q|}$ and so

$$\left|D_Q F^Q\right| \le \frac{c}{|P-Q|}. \tag{23}$$

The computations for higher order derivatives get more complicated, but the idea of the proof is the same. Use the inequalities (16) and the fact that the norm of a vector of the form $\dfrac{Q}{|Q|}\cdot A$ is bounded if the components of A involve f and/or its derivatives (e.g. $\dfrac{Q}{|Q|}\cdot Q_x$, $\dfrac{Q}{|Q|}\cdot N^Q$, $\dfrac{Q}{|Q|}\cdot N^Q_x$, etc.)

This concludes the proof of the claim.

For the derivatives of G, we have

$$\left|D_Q^i G\right| |P-Q|^i = \sum_{k=0}^{i} \binom{i}{k} \left|D_Q^{i-k} F^P (P-Q)^{i-k}\right| \left|D_Q^k F^Q (P-Q)^k\right| \leq c,$$

which proves (8). □

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A NOTE ON THREE-STEP ITERATIVE METHODS FOR NONLINEAR EQUATIONS

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Dedicated to Professor Petru Blaga at his 60^{th} *anniversary*

Abstract. In this short note we give certain comments and improvements of some three-step iterative methods recently considered by N.A. Mir and T. Zaman (Appl. Math. Comput. (2007) doi: 10.1016/j.amc.2007.03.071).

1. **Introduction**

Very recently, N.A. Mir and T. Zaman [1] have considered three-step quadrature based iterative methods for finding a single zero $x = \alpha$ of a nonlinear equation

$$f(x) = 0. \tag{1.1}$$

All variants of their methods include the formula

$$x_{n+1} = x_n - \frac{f(x_n)f'(x_n)}{f'(x_n)^2 - \lambda f(x_n)f''(x_n)}, \tag{1.2}$$

obtained from the rectangular quadrature formula. It is clear that (1.2) reduces to Newton and Halley method for $\lambda = 0$ and $\lambda = 1/2$, respectively.

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As a variant with maximal order of convergence they have proposed the following three-step method

$$\left.\begin{aligned} y_n &= x_n - \frac{f(x_n)}{f'(x_n)}, \\ z_n &= y_n - \frac{f(y_n)f'(y_n)}{f'(y_n)^2 - \lambda f(y_n)f''(y_n)}, \\ x_{n+1} &= z_n - \frac{(y_n - z_n)f(z_n)}{f(y_n) - 2f(z_n)}, \end{aligned}\right\} \tag{1.3}$$

proving that for a sufficiently smooth function f and a starting point x_0 sufficiently close to the single zero $x = \alpha$, this method has eighth order convergence for $\lambda = 1/2$, i.e.,

$$e_{n+1} = (-c_3c_2^5 + c_2^7)e_n^8 + O(e_n^9),$$

where $e_n = x_n - \alpha$ and

$$c_k = \frac{1}{k!}\frac{f^{(k)}(\alpha)}{f'(\alpha)}, \quad k = 2, 3, \dots\ . \tag{1.4}$$

As we can see this three-step method need six function evaluations per iteration: $f(x_n)$, $f(y_n)$, $f(z_n)$, $f'(x_n)$, $f'(y_n)$, and $f''(y_n)$. Without new function evaluations, in this note we show that the formula

$$x_{n+1} = S(y_n, z_n) = z_n - \frac{f(z_n)}{f'(y_n) + (z_n - y_n)f''(y_n)} \tag{1.5}$$

is a much better choice than the third formula in (1.3). In that case the corresponding three-step method has tenth order convergence. Moreover, the formula (1.5) is numerically stable in comparing with the previous one.

The paper is organized as follows. In Section 2 we give certain auxiliary formulae, which can be used also in other investigations in convergence analysis. The main results and a numerical example are given in Section 3.

2. **Some auxiliary formulae**

We suppose that the equation (1.1) has a single zero $x = \alpha$ in certain neighborhood $U_\varepsilon(\alpha) := (\alpha - \varepsilon, \alpha + \varepsilon)$, $\varepsilon > 0$, and that the function f is sufficiently differentiable in $U_\varepsilon(\alpha)$. Evidently, $f'(\alpha) \neq 0$.

Let $x_n \in U_\varepsilon(\alpha)$ and

$$e_n := x_n - \alpha, \quad \widetilde{e}_n := y_n - \alpha, \quad \hat{e}_n := z_n - \alpha.$$

Using (1.4) it is easy to get the following formula

$$\begin{aligned}\frac{f(x_n)}{f'(x_n)} &= e_n - c_2 e_n^2 + 2(c_2^2 - c_3)e_n^3 - (4c_2^3 - 7c_3c_2 + 3c_4)e_n^4 \\ &\quad + (8c_2^4 - 20c_3c_2^2 + 10c_4c_2 + 6c_3^2 - 4c_5)e_n^5 \\ &\quad - [16c_2^5 - 52c_3c_2^3 + 28c_4c_2^2 + (33c_3^2 - 13c_5)c_2 - 17c_3c_4 + 5c_6]e_n^6 \\ &\quad + O(e_n^7). \end{aligned} \tag{2.1}$$

This formula is an inverse of the well-known Schröder formula (cf. [2, pp. 352–354]).

Therefore, in the case of the Newton method

$$\Phi_N(x_n) = x_n - \frac{f(x_n)}{f'(x_n)} \tag{2.2}$$

we have

$$\begin{aligned}\widetilde{e}_n &= \Phi_N(x_n) - \alpha \\ &= c_2 e_n^2 - 2(c_2^2 - c_3)e_n^3 + (4c_2^3 - 7c_3c_2 + 3c_4)e_n^4 \\ &\quad - (8c_2^4 - 20c_3c_2^2 + 10c_4c_2 + 6c_3^2 - 4c_5)e_n^5 \\ &\quad + [16c_2^5 - 52c_3c_2^3 + 28c_4c_2^2 + (33c_3^2 - 13c_5)c_2 - 17c_3c_4 + 5c_6]e_n^6 \\ &\quad + O(e_n^7). \end{aligned} \tag{2.3}$$

Also, we need the corresponding expression for

$$\widetilde{C}_2(y_n) := \frac{f''(y_n)}{2f'(y_n)} = \frac{1}{2}\,\frac{f''(\alpha) + \dfrac{f'''(\alpha)}{1!}\,\widetilde{e}_n + \dfrac{f^{iv}(\alpha)}{2!}\,\widetilde{e}_n^2 + \cdots}{f'(\alpha) + \dfrac{f''(\alpha)}{1!}\,\widetilde{e}_n + \dfrac{f'''(\alpha)}{2!}\,\widetilde{e}_n^2 + \cdots},$$

i.e.,

$$\widetilde{C}_2(y_n) = \frac{1}{2}\cdot\frac{1\cdot 2\, c_2 + 2\cdot 3\, c_3\, \widetilde{e}_n + 3\cdot 4\, c_4\, \widetilde{e}_n^2 + \cdots}{1 + 2c_2\, \widetilde{e}_n + 3c_3\, \widetilde{e}_n^2 + \cdots},$$

where c_k are defined by (1.4). It gives

$$\widetilde{C}_2(y_n) = A_0 + A_1\widetilde{e}_n + A_2\widetilde{e}_n^2 + A_3\widetilde{e}_n^3 + \cdots, \tag{2.4}$$

where

$$\begin{aligned}
A_0 &= c_2, \quad A_1 = 3c_3 - 2c_2^2, \quad A_2 = 4c_2^3 - 9c_3c_2 + 6c_4,\\
A_3 &= -8c_2^4 + 24c_3c_2^2 - 16c_4c_2 - 9c_3^2 + 10c_5,\\
A_4 &= 16c_2^5 - 60c_3c_2^3 + 40c_4c_2^2 + 5(9c_3^2 - 5c_5)c_2 + 15(c_6 - 2c_3c_4),\\
A_5 &= -32c_2^6 + 144c_3c_2^4 - 96c_4c_2^3 + (60c_5 - 162c_3^2)c_2^2 + 36(4c_3c_4 - c_6)c_2\\
&\quad +3(9c_3^3 - 15c_5c_3 - 8c_4^2 + 7c_7),\\
A_6 &= 64c_2^7 - 336c_3c_2^5 + 224c_4c_2^4 + 28(18c_3^2 - 5c_5)c_2^3 - 84(6c_3c_4 - c_6)c_2^2\\
&\quad -7(27c_3^3 - 30c_5c_3 - 16c_4^2 + 7c_7)c_2 + 7(18c_4c_3^2 - 9c_6c_3 - 10c_4c_5 + 4c_8),
\end{aligned}$$

etc.

Now, for the Halley method

$$\Phi_H(y_n) = y_n - \frac{f(y_n)/f'(y_n)}{1 - \widetilde{C}_2(y_n)(f(y_n)/f'(y_n))} \tag{2.5}$$

we have

$$\hat{e}_n = \Phi_H(y_n) - \alpha = \widetilde{e}_n - g_n\Big(1 + \widetilde{C}_2(y_n)g_n + [\widetilde{C}_2(y_n)g_n]^2 + \cdots\Big), \quad g_n = \frac{f(y_n)}{f'(y_n)}.$$

Using (2.1), in this case, we get

$$\begin{aligned}
\hat{e}_n &= (c_2^2 - c_3)\widetilde{e}_n^3 - 3(c_2^3 - 2c_3c_2 + c_4)\widetilde{e}_n^4 + 6(c_2^4 - 3c_3c_2^2 + 2c_4c_2 + c_3^2 - c_5)\widetilde{e}_n^5\\
&\quad -[9c_2^5 - 37c_3c_2^3 + 29c_4c_2^2 + 4(7c_3^2 - 5c_5)c_2 - 19c_3c_4 + 10c_6]\widetilde{e}_n^6 + O(\widetilde{e}_n^7).
\end{aligned} \tag{2.6}$$

In our analysis we also need an expansion of $f(z_n)/f'(y_n)$ in terms of $\widetilde{e}_n$ $(= y_n - \alpha)$, where $z_n - \alpha = \hat{e}_n$ is given by (2.6). Thus, we have

$$v_n = \frac{f(z_n)}{f'(y_n)} = \frac{\hat{e}_n + c_2\,\hat{e}_n^2 + c_3\,\hat{e}_n^3 + \cdots}{1 + 2c_2\,\widetilde{e}_n + 3c_3\,\widetilde{e}_n^2 + \cdots},$$

i.e.,

$$\begin{aligned} v_n &= (c_2^2 - c_3)\widetilde{e}_n^3 - (5c_2^3 - 8c_3c_2 + 3c_4)\widetilde{e}_n^4 \\ &\quad +(16c_2^4 - 37c_3c_2^2 + 18c_4c_2 + 9c_3^2 - 6c_5)\widetilde{e}_n^5 \\ &\quad -(40c_2^5 - 124c_3c_2^3 + 69c_4c_2^2 - (32c_5 - 69c_3^2)c_2 - 32c_3c_4 + 10c_6)\widetilde{e}_n^6 \\ &\quad +O(\widetilde{e}_n^7). \end{aligned} \tag{2.7}$$

In the case when $y_n = \Phi_N(x_n)$ and

$$z_n = y_n - \frac{(x_n - y_n)f(y_n)}{f(x_n) - 2f(y_n)}\,, \tag{2.8}$$

we are interested in

$$u_n = \frac{f(x_n)}{f(y_n)}\,, \quad t_n = \frac{f(x_n)}{f'(z_n)}\,, \quad s_n = \frac{f'(x_n)}{f'(z_n)}\,, \tag{2.9}$$

i.e.,

$$u_n = \frac{e_n + c_2\, e_n^2 + c_3\, e_n^3 + \cdots}{\widetilde{e}_n + c_2\, \widetilde{e}_n^2 + c_3\, \widetilde{e}_n^3 + \cdots}\,, \tag{2.10}$$

$$t_n = \frac{e_n + c_2\, e_n^2 + c_3\, e_n^3 + \cdots}{1 + 2c_2\, \bar{e}_n + 3c_3\, \bar{e}_n^2 + \cdots}\,, \quad \text{and} \quad s_n = \frac{1 + 2c_2\, e_n + 3c_3\, e_n^2 + \cdots}{1 + 2c_2\, \bar{e}_n + 3c_3\, \bar{e}_n^2 + \cdots}\,,$$

where $e_n = x_n - \alpha$, $\widetilde{e}_n = y_n - \alpha$, and $\bar{e}_n = z_n - \alpha$.

Here, $\bar{e}_n = \widetilde{e}_n - (e_n - \widetilde{e}_n)/(u_n - 2)$. According to (2.3) and (2.10) we get

$$\begin{aligned} \bar{e}_n &= c_2(c_2^2 - c_3)e_n^4 - 2(2c_2^4 - 4c_3c_2^2 + c_4c_2 + c_3^2)e_n^5 \\ &\quad +[10c_2^5 - 30c_3c_2^3 + 12c_4c_2^2 + 3c_2(6c_3^2 - c_5) - 7c_3c_4]e_n^6 \\ &\quad +O(e_n^7). \end{aligned} \tag{2.11}$$

For t_n and s_n we obtain

$$\begin{aligned} t_n &= e_n + c_2e_n^2 + c_3e_n^3 + c_4e_n^4 - (2c_2^4 - 2c_3c_2^2 - c_5)e_n^5 \\ &\quad +(6c_2^5 - 14c_3c_2^3 + 4c_4c_2^2 + 4c_3^2c_2 + c_6)e_n^6 + O(e_n^7) \end{aligned} \tag{2.12}$$

and

$$
\begin{aligned}
s_n \;=\;& 1+2c_2e_n+3c_3e_n^2+4c_4e_n^3-(2c_2^4-2c_3c_2^2-5c_5)e_n^4 \\
&+(4c_2^5-12c_3c_2^3+4c_4c_2^2+4c_3^2c_2+6c_6)e_n^5 \\
&-[4c_2^6-22c_3c_2^4+16c_4c_2^3-(6c_5-22c_3^2)c_2^2-14c_3c_4c_2-7c_7]e_n^6 \\
&+O(e_n^7), \qquad (2.13)
\end{aligned}
$$

respectively.

3. **Main results**

We consider now the third-step iterative formula given by (2.2), (2.5), and (1.5), i.e.,

$$
y_n=\Phi_N(x_n), \quad z_n=\Phi_H(y_n), \quad e_{n+1}=S(y_n,z_n), \quad n=0,1,\dots\,, \qquad (3.1)
$$

for finding a simple zero $x=\alpha$ of the equation (1.1).

Theorem 3.1. *For a sufficiently differentiable function f in $U_\varepsilon(\alpha)$ and x_0 sufficiently close to α, the third-step method* (3.1) *has tenth order of convergence, i.e.,*

$$
e_{n+1}=3c_2^5c_3(c_3-c_2^2)e_n^{10}+30c_2^4(c_2^2-c_3)^2c_3e_n^{11}+O(e_n^{12}) \qquad (3.2)
$$

where $e_n=x_n-\alpha$ and c_k are given in (1.4)*.*

Proof. According to (3.1) (and (1.5)) we have

$$
e_{n+1}=x_{n+1}-\alpha=S(y_n,z_n)-\alpha=\hat{e}_n-\frac{f(z_n)/f'(y_n)}{1+2(\hat{e}_n-\widetilde{e}_n)\widetilde{C}_2(y_n)}\,,
$$

where $\widetilde{e}_n=y_n-\alpha$, $\hat{e}_n=z_n-\alpha$, and $\widetilde{C}_2(y_n)=f''(y_n)/(2f'(y_n))$. Replacing $\widetilde{C}_2(y_n)$ and $v_n=f(z_n)/f'(y_n)$ by the corresponding expressions (2.4) and (2.7), we obtain

$$
e_{n+1}=3c_3(c_3-c_2^2)\widetilde{e}_n^5+(c_2^5+7c_3c_2^3-8c_4c_2^2-17c_3^2c_2+17c_3c_4)\widetilde{e}_n^6+O(\widetilde{e}_n^7).
$$

Finally, using (2.3) we get (3.2). □

In [1] the authors also considered the following three-step method

$$\left.\begin{aligned} y_n &= x_n - \frac{f(x_n)}{f'(x_n)}, \\ z_n &= y_n - \frac{(x_n - y_n)f(y_n)}{f(x_n) - 2f(y_n)}, \\ x_{n+1} &= z_n - \frac{f(z_n)f'(z_n)}{f'(z_n)^2 - \lambda\, f(z_n)\left\{\dfrac{2(f(z_n) - f(x_n))}{(z_n - x_n)^2} - \dfrac{2f'(x_n)}{z_n - x_n}\right\}}, \end{aligned}\right\} \tag{3.3}$$

with five function evaluations per iteration: $f(x_n)$, $f(y_n)$, $f(z_n)$, $f'(x_n)$, and $f'(z_n)$. Their Theorem 3 states that this method has seventh order of convergence for any value of λ. However, the order of convergence is bigger than seven. Namely, we have the following result:

Theorem 3.2. *For a sufficiently differentiable function f in $U_\varepsilon(\alpha)$ and x_0 sufficiently close to α, the third-step method* (3.3) *has eighth order of convergence for any $\lambda \neq 1/2$, except for $\lambda = 1/2$ when the convergence is of the order nine. Then,*

$$\begin{aligned} e_{n+1} &= -2c_3c_2^2(c_2^2 - c_3)^2e_n^9 \\ &\quad + c_2(c_2^2 - c_3)(16c_3c_2^4 - 3c_4c_2^3 - 32c_3^2c_2^2 + 11c_2c_3c_4 + 8c_3^3)e_n^{10} \\ &\quad + O(e_n^{11}), \end{aligned} \tag{3.4}$$

where $e_n = x_n - \alpha$ and c_k are given in (1.4).

Proof. Using the expansion (2.1) for the Newton correction $f(x_n)/f'(x_n) =: h(e_n)$, we have $f(z_n)/f'(z_n) = h(\bar{e}_n)$, where $\bar{e}_n$ is given by (2.11). According to (2.9), for the third formula in (3.3) we get

$$e_{n+1} = \bar{e}_n - \frac{h(\bar{e}_n)}{1 - 2\lambda\left\{\dfrac{h(\bar{e}_n) - t_n}{(\bar{e}_n - e_n)^2} - \dfrac{s_n}{\bar{e}_n - e_n}\right\}}, \tag{3.5}$$

where the expansions for t_n and s_n are given by (2.12) and (2.13), respectively. This gives

$$\begin{aligned} e_{n+1} &= (1 - 2\lambda)c_2^3(c_2^2 - c_3)^2e_n^8 + 4c_2^2(c_2^2 - c_3)\big[2(2\lambda - 1)c_2^4 + (4 - 9\lambda)c_3c_2^2 \\ &\qquad + (2\lambda - 1)c_4c_2 + (3\lambda - 1)c_3^2\big]e_n^9 + \cdots \end{aligned}$$

For $\lambda = 1/2$ it reduces to (3.4). $\square$

Thus, the computational efficiency of the method (3.3), for $\lambda = 1/2$, is EFF$=$ $9^{1/5} \approx 1.55185$. With the same function evaluations we can get a slightly simpler method of order nine with the same efficiency.

Theorem 3.3. *For a sufficiently differentiable function f in $U_\varepsilon(\alpha)$ and x_0 sufficiently close to α, the third-step method*

$$\left.\begin{aligned} y_n &= x_n - \frac{f(x_n)}{f'(x_n)}, \\ z_n &= y_n - \frac{(x_n - y_n)f(y_n)}{f(x_n) - 2f(y_n)}, \\ x_{n+1} &= z_n - \frac{f(z_n)f'(z_n)}{f'(z_n)^2 - \dfrac{1}{2}\, f(z_n)\dfrac{f'(z_n) - f'(x_n)}{z_n - x_n}}\,, \end{aligned}\right\} \tag{3.6}$$

has ninth order of convergence, i.e.,

$$\begin{aligned} e_{n+1} &= -\frac{3}{2}c_3c_2^2(c_2^2 - c_3)^2e_n^9 \\ &\quad +2c_2(c_2^2 - c_3)(6c_2^4c_3 - 12c_2^2c_3^2 + 3c_3^3 - c_2^3c_4 + 4c_2c_3c_4)e_n^{10} \\ &\quad +O(e_n^{11}), \end{aligned} \tag{3.7}$$

where $e_n = x_n - \alpha$ and c_k are given in (1.4).

Proof. Similarly as in the proof of the previous theorem, we have now

$$e_{n+1} = \bar{e}_n - \frac{h(\bar{e}_n)}{1 - \dfrac{1}{2}\, h(\bar{e}_n)\dfrac{1 - s_n}{\bar{e}_n - e_n}}$$

instead of (3.5). This gives (3.7). $\square$

The number of function evaluations in (3.6) can be reduced to four if we take an approximation of $f'(z_n)$ in the form

$$f'(z_n) \approx \tilde{f}'(z_n) = p_nf(x_n) + q_nf(y_n) + r_nf(z_n) + w_nf'(x_n),$$

obtained by the Hermite interpolation (cf. [3, pp. 51–58]), where

$$p_n = \frac{(y_n - z_n)(z_n + 2y_n - 3x_n)}{(x_n - y_n)^2(x_n - z_n)}, \quad q_n = \frac{(x_n - z_n)^2}{(x_n - y_n)^2(y_n - z_n)},$$

$$r_n = \frac{3z_n - 2y_n - x_n}{(x_n - z_n)(y_n - z_n)}, \qquad w_n = \frac{y_n - z_n}{x_n - y_n}.$$

For a such modified three-step method, in notation (3.6^M), the following result holds:

Theorem 3.4. *For a sufficiently differentiable function f in $U_\varepsilon(\alpha)$ and x_0 sufficiently close to α, the third-step method (3.6^M) has eight order of convergence, i.e.,*

$$\begin{aligned} e_{n+1} = {} & (c_2^2 - c_3)c_2^2 c_4 e_n^8 - \frac{1}{2}\Big[3c_2^5(c_2c_3 + 4c_4) - 2c_2^4(3c_3^2 + 2c_5) \\ & + c_2^2(3c_3^3 + 4c_3c_5 + 4c_4^2) + 8c_2c_3c_4(c_3 - 3c_2^2)\Big] e_n^9 + O(e_n^{10}), \end{aligned}$$

where $e_n = x_n - \alpha$ and c_k are given in (1.4).

The corresponding computational efficiency is now much better, EFF$=8^{1/4} \approx 1.68179$.

Example 3.1. Consider the equation

$$f(x) = xe^{x^2} - \sin^2 x + 3\cos x + 5 = 0,$$

with a simple zero

$$\alpha = -1.207647827130918927009416758356084097760235818949538815205 92\ldots$$

In order to show the behavior of three-step methods (1.3), (3.1), (3.3), (3.6) and (3.6^M) we need a multi-precision arithmetics. Starting with $x_0 = -1$, we use Mathematica with 10000 significant digits. The errors $e_n = x_n - \alpha$ are given in Table 3.1. Numbers in parentheses indicate decimal exponents. Besides the convergence order (r) we give also the corresponding computational efficiency (EFF).

TABLE 3.1. The errors $e_n = x_n - \alpha$, $n = 0, 1, 2, 3, 4$, in three-step methods

method	(1.3)	(3.1)	(3.3)	(3.6)	(3.6^M)
order	$r = 8$	$r = 10$	$r = 9$	$r = 9$	$r = 8$
EFF	1.41421	1.46780	1.55185	1.55185	1.68179
$n = 0$	$2.08(-1)$	$2.08(-1)$	$2.08(-1)$	$2.08(-1)$	$2.08(-1)$
$n = 1$	$-1.05(-5)$	$3.70(-6)$	$-1.19(-7)$	$-9.24(-8)$	$-2.25(-6)$
$n = 2$	$-2.87(-40)$	$5.66(-54)$	$2.74(-63)$	$2.15(-64)$	$-8.57(-46)$
$n = 3$	$-8.87(-317)$	$3.93(-532)$	$-5.05(-564)$	$-4.26(-574)$	$-3.77(-361)$
$n = 4$	$-7.48(-2529)$	$1.02(-5313)$	$1.26(-5070)$	$2.204(-5161)$	$-5.32(-2884)$

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PICARD OPERATORS AND WELL-POSEDNESS OF FIXED POINT PROBLEMS

IOAN A. RUS

Dedicated to Professor Petru Blaga at his 60^{th} *anniversary*

Abstract. In this paper we study the following problems:
• For which Picard operators the fixed point problem is well-posed?
• For which operators from those which appear in continuation principles, with a unique fixed point, the fixed point problem is well-posed?

1. **Introduction**

The notion of well-posedness of fixed point problem for an operator was introduced in [3] and studied by S. Reich and A.J. Zaslavski [8] and E. Matouškova, S. Reich and A.J. Zaslavski [6]. In this paper we shall study the following problems:

Question 1.1. For which Picard operators the fixed point problem is well-posed?

Question 1.2. For which operators from those which appear in continuation principles, with a unique fixed point, the fixed point problem is well-posed?

Throughout this paper we follow the terminologies and the notations in [11] and [12]. For the convenience of the reader we shall recall some of them.

2. **Picard operators**

Let (X, d) a metric space and $A : X \to X$ an operator. We shall use the following notations and definitions:

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$P(X) := \{Y \subset X |\ Y \neq \emptyset\}$;

$F_A := \{x \in X |\ A(x) = x\}$;

$I(A) := \{Y \in P(X) |\ A(Y) \subset Y\}$;

$A^0 := 1_X,\ A^1 := A,\ A^{n+1} := A \circ A^n,\ n \in \mathbb{N}$.

Definition 2.1. ([11], [12]). An operator $A : X \to X$ is weakly Picard operator (WPO) if the sequence $(A^n(x))_{n\in\mathbb{N}}$ converges, for all $x \in X$, and the limit (which may depend on x) is a fixed point of A.

Definition 2.2. ([11], [12]). If an operator A is WPO and $F_A = \{x^*\}$, then by definition the operator A is Picard operator (PO).

Definition 2.3. ([11], [12]). If $A : X \to X$ is WPO, then we define the operator

$$A^\infty : X \to X, \quad A^\infty(x) := \lim_{n\to\infty} A^n(x).$$

Definition 2.4. ([11], [12]). Let $c > 0$. An WPO $A : X \to X$ is by definition a c-WPO iff

$$d(x, A^\infty(x)) \leq cd(x, A(x)),\ \forall\ x \in X.$$

For some examples and properties of POs and WPOs see [11] and [12]. See also [10] and [13].

3. **Well-posedness of fixed point problems**

Let (X, d) be a metric space and $A : X \to X$ an operator. Let (X, d) be a metric space and $A : X \to X$ an operator.

Definition 3.1. ([7]) The fixed point problem for an operator A is well posed iff

(a) $F_A = \{x^*\}$;

(b) if $x_n \in X$, $n \in \mathbb{N}$ and $d(x_n, A(x_n)) \to 0$ as $n \to \infty$, then $d(x_n, x^*) \to 0$ as $n \to \infty$.

This paper is motivated by a recent result of S. Reich and A.J. Zaslavski [8] who proved that if the iterates of a uniformly continuous self-operator A of X

converge to its unique fixed point uniformly on X (a bounded and complete metric space), then the fixed point problem for A is well posed.

We begin our considerations with some remarks and examples.

Lemma 3.1. *Let X be a nonempty set and d, ρ two metrics on X, metrically equivalent. Let $A : x \to X$ be an operator. If the fixed point problem for A is well posed w.r.t. the metric d then it is well posed also w.r.t. the metric ρ.*

Proof. Let $c_1 > 0$, $c_2 > 0$ such that

$$d(x, y) \leq c_1 \rho(x, y) \text{ and } \rho(x, y) \leq c_2 d(x, y),\ \forall\ x, y \in X.$$

We denote by x^* the unique fixed point of A. Let $x_n \in X$, $n \in \mathbb{N}$, be such that $\rho(x_n, A(x_n)) \to 0$ as $n \to \infty$. We have

$$d(x_n, A(x_n)) \leq c_1 \rho(x_n, A(x_n)) \to 0 \text{ as } n \to \infty.$$

From this it follows that $x_n \xrightarrow{d} x^*$ as $n \to \infty$. But $\rho(x_n, x^*) \leq c_2 d(x_n, x^*) \to 0$ as $n \to \infty$.

In a similar way we prove

Lemma 3.2. *Let X be a nonempty set, d, ρ two metric on X and $A : X \to X$ an operator. We suppose that:*

(i) d and ρ are topologically equivalent;

(ii) there exists $c > 0$: $d \leq c\rho$;

(iii) the fixed point problem is well posed for S w.r.t. the metric d.

Then the fixed point problem is well posed for A w.r.t. the metric ρ.

Lemma 3.3. *Let (X, d) be a metric space and $A : X \to X$ a uniformly continuous operator. If there exists $k \in \mathbb{N}^*$ such that the fixed point problem is well posed for A^k, then the fixed point problem is well posed for the operator A.*

Proof. Let $x_n \in X$, $n \in \mathbb{N}$, such that $d(x_n, A(x_n)) \to 0$ as $n \to \infty$. Then from the uniform continuity of A it follows that $d(A^s(x_n), A^{s+1}(x_n)) \to 0$ as $n \to \infty$, for $s = 1, 2, \dots, k-1$. We have

$$d(x_n, A^k(x_n)) \leq d(x_n, A(x_n)) + d(A(x_n), A^2(x_n)) + \cdots +$$

$$+d(A^{k-1}(x_n), A^k(x_n)).$$

Hence

$$d(x_n, A^k(x_n)) \to 0 \text{ as } n \to \infty.$$

So, $x_n \to x^*$ as $n \to \infty$.

Example 3.1. Let (X, d) be a complete metric space and $A : X \to X$ an α-contraction. Then the fixed point problem is well posed for the operator A.

Indeed, $F_A = \{x^*\}$ and let $x_n \in X$, $n \in \mathbb{N}$, be such that $d(x_n, A(x_n)) \to 0$ as $n \to \infty$. We have

$$\begin{aligned} d(x_n, x^*) &\leq d(x_n, A(x_n)) + d(A(x_n), x^*) \\ &\leq d(x_n, A(x_n)) + \alpha d(x_n, x^*). \end{aligned}$$

So,

$$d(x_n, x^*) \leq \frac{1}{1-\alpha} d(x_n, A(x_n)) \to 0 \text{ as } n \to \infty.$$

Example 3.2. Let (X, d) be a compact metric space and $A : X \to X$ a continuous operator such that $F_A = \{x^*\}$. Then the fixed point problem for A is well posed.

Indeed, let $x_n \in X$, $n \in \mathbb{N}$, be such that $d(x_n, A(x_n)) \to 0$ as $n \to \infty$. Let $(x_{n_i})_{i \in \mathbb{N}}$ be a convergent subsequence of $(x_n)_{n \in \mathbb{N}}$. If $x_{n_i} \to y^*$ as $i \to \infty$, then $A(x_{n_i}) \to A(y^*)$ as $i \to \infty$. From $d(x_{n_i}, A(x_{n_i})) \to 0$ as $i \to \infty$, it follows that $y^* = x^*$.

From this property and from the compactness of x it follows that $x_n \to x^*$ as $n \to \infty$.

Example 3.3. Let (X, d) a compact metric space and $A : X \to X$ a contractive operator, i.e., $d(A(x), A(y)) < d(x, y)$, $\forall\ x, y \in X$, $x \neq y$. Then the fixed point problem is well posed for the operator A.

This follows from the Example 3.2.

Example 3.4. Let X be a nonempty set and $A : X \to X$ a Bessaga operator ([9]), i.e.,

$$F_A = F_{A^n} = \{x^*\},\ \forall\ n \in \mathbb{N}.$$

Then there exists a metric d on X such that the fixed point problem is well posed for the operator A w.r.t. the metric d.

Indeed, from the Bessaga theorem (see [9], p.31) there exists a complete metric d on X such that $A : (X, d) \to (X, d)$ is a contraction.

Now we are in the conditions of Example 3.1.

Remark 3.1. There exists an operator which isn't a Bessaga operator but for which the fixed point problem is well posed. For example,

$$A : [-1, 1] \to [-1, 1], \quad A(x) = -x.$$

For this function the fixed point problem is well posed but $F_{A^2} = [-1, 1]$.

The above considerations give rise to the following problem.

Question 3.1. For which generalized contractions the fixed point problem is well posed?

For some results for this problem see [6] and [8]. Other results shall be given in the following sections.

For the generalized contractions see B.E. Rhoades [9], I.A. Rus [10] E. De Pascale and P.P. Zabreiko [4].

4. ψ-Picard operators

Let (X, d) be a metric space, $A : X \to X$ a WPO and $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ an increasing function which is continuous in 0 and $\psi(0) = 0$.

Definition 4.1. The operator A is ψ-WPO iff $d(x, A^\infty(x)) \leq \psi(d(x, A(x))), \forall\, x \in X$.

If $\psi(t) = ct$, with $c > 0$, then we say that A is $c - WPO$. For $c - Pos$ and $c - WPOs$ see [12].

Example 4.1. ([12]) Let (X, d) be a complete metric space, $A : X \to X$ an operator. We suppose that

(i) A is continuous;

(ii) there exists $\alpha \in (0, 1)$ such that

$$d(A^2(x), A(x)) \leq \alpha d(x, A(x)),\ \forall\, x \in X.$$

Then S is $\dfrac{1}{1-\alpha} - WPO$.

Example 4.2. Let (X,d) be a complete metric space and $A : X \to X$ a Ciric-Reich-Rus operator, i.e., there exist $a,b,c \in \mathbb{R}_+$, $a+b+c,1$, such that

$$d(A(x),A(y)) \leq ad(x,y) + bd(x,A(x)) + cd(y,A(y)),\ \forall\ x,y \in X.$$

Then A is $\dfrac{1-c}{1-a-b-c} - PO$.

Indeed, from a theorem by Ciric-Reich-Rus (see [1], [2]) the operator A is PO and

$$d(A^2(x),A(x)) \leq \frac{a+b}{1-c} d(x,A(x)),\ \forall\ x \in X.$$

Example 4.3. Let $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ a strict comparison function (see [10], pp.41-42, 69), i.e.,

(a) φ is increasing;

(b) $\varphi^n(t) \to 0$ as $n \to \infty$, for all $t \in \mathbb{R}_+$;

(c) $t - \varphi(t) \to +\infty$ as $t \to +\infty$.

Let $\psi_\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ be defined by

$$\psi_\varphi(\eta) := \sup\{t \in \mathbb{R}_+ |\ t - \varphi(t) \leq \eta\}.$$

Let (X,d) be a complete metric space and $A : X \to X$ a strict φ-contraction (see [10], p. 50), i.e., φ is a strict comparison function and

$$d(f(x),f(y)) \leq \varphi(d(x,y)),\ \forall\ x,y \in X.$$

Then A is a $\psi_\varphi - PO$.

Indeed, by Theorem 4.3.1 in [10], the operator A is PO. Let $F_A = \{x^*\}$. Then

$$d(x,x^*) \leq d(x,A(x)) + d(A(x),x^*) \leq d(x,A(x)) + \varphi(d(x,x^*)).$$

Hence

$$d(x,x^*) - \varphi(d(x,x^*)) \leq d(x,A(x)),\ \forall\ x \in X.$$

So,

$$d(x,x^*) \leq \psi_\varphi(d(x,A(x))),\ \forall\ x \in X,$$

i.e., the operator A is $\psi_\varphi - PO$.

We have

Theorem 4.1. *Let (X, d) be a metric space and $A : X \to X$ a ψ-Picard operator. Then the fixed point problem is well posed for A.*

Proof. Let x^* be the unique fixed point of A. Let $x_n \in X$, $n \in \mathbb{N}$, be such that $d(x_n, A(x_n)) \to 0$ as $n \to \infty$. We have

$$d(x_n, x^*) \leq \psi(d(x_n, A(x_n))) \to 0 \text{ as } n \to \infty.$$

Theorem 4.2. *Let X be a nonempty set and $A : X \to X$ an operator such that*

$$F_A = F_{A^n} \neq \emptyset, \ \forall\, n \in \mathbb{N}^*.$$

Then there exists a partition of X, $X = \bigcup_{i \in I} X_i$ such that

(i) $A(X_i) \subset X_i, \ \forall\, i \in I$;

(ii) for each $i \in I$ there exists a complete metric d_i on X_i such that the fixed point problem for $A|_{X_i} : X_i \to X_i$ is well posed for all $i \in I$.

Proof. By the Theorem 4.1 in [11], there exists a partition of X, $X = \bigcup_{i \in I} X_i$, such that $A(X_i) \subset X_i, \ \forall\, i \in I$ and for each $i \in I$ there exists a complete metric d_i on X_i such that the operator $A|_{X_i} : X_i \to X_i$ is a contraction. So, the fixed point problem is well posed for $A|_{X_i}$, for all $i \in I$.

5. **Asymptotically regular operators**

Let (X, d) be a metric space. By definition an operator $A : X \to X$ is asymptotically regular iff

$$d(A^n(x), A^{n+1}(x)) \to 0 \text{ as } n \to \infty, \ \forall\, x \in X.$$

We have

Theorem 5.1. *Let (X, d) be a metric space and $A : X \to X$ an operator. We suppose that:*

(i) A is asymptotically regular;

(ii) the fixed point problem for the operator A is well posed.

Then the operator A is a PO.

Proof. From (ii) we have that $F_A = \{x^*\}$.

Let $x \in X$ and $x_n := A^n(x)$. From (i), $d(x_n, A(x_n)) \to 0$ as $n \to \infty$. From (ii), it follows that $A^n(x) \to x^*$ as $n \to \infty$, for all $x \in X$. So, A is PO.

Theorem 5.2. *Let X be a Banach space, $Y \subset X$ a bounded closed convex subset and $A : Y \to Y$ an affine nonexpansive operator. Let $\lambda \in (0,1)$ and $A_\lambda := \lambda 1_Y + (1-\lambda)A$. If the fixed point problem for A is well posed, then the operator A_λ is PO, for all $\lambda \in (0,1)$.*

Proof. We have that

$$\|x - A_\lambda(x)\| = (1-\lambda)\|x - A(x)\|, \ \forall \ x \in Y.$$

This implies that the fixed point problem is well posed for A if and only if is well posed for A_λ.

On the other hand by Ishikawa's theorem (see [9], p.105) the operator A_λ is asymptotically regular.

Now the proof follows from Theorem 5.1.

6. **Non self-operators**

In this section we shall make some remarks on Question 1.2.

Let (X,d) be a metric space and $Y \subset X$ a nonempty subset.

Definition 6.1. The fixed point problem is well posed for an operator $A : Y \to X$ iff:

(i) $F_A = \{x^*\}$;

and

(ii) if $x_n \in Y$, $n \in \mathbb{N}$, and $d(x_n, A(x_n)) \to 0$ as $n \to \infty$, then $x_n \to x^*$ as $n \to \infty$.

As in the case of self-operators (see section 3) we have

Lemma 6.1. *Let X be a nonempty set and d, ρ two metrics on X, metrically equivalent. Let $Y \subset X$ and $A : Y \to X$ be an operator. If the fixed point problem for A is well posed w.r.t. the metric d, then it is well posed, also, w.r.t. the metric ρ.*

Lemma 6.2. *Let d, ρ be two metrics on $X, Y \subset X$ and $A : Y \to X$ an operator. We suppose that:*

(i) d and ρ are topologically equivalent;

(ii) there exists $c > 0$: $\rho \leq cd$;

(iii) the fixed point problem is well posed for A w.r.t. the metric ρ.

Then the fixed point problem is well posed for A w.r.t. the metric d.

Let $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ an increasing function which is continuous in 0 and $\psi(0) = 0$. The corresponding result of the Theorem 4.1 is the following

Theorem 6.1. *Let (X, d) be a metric space, $Y \subset X$ and $A : Y \to X$ an operator. We suppose that:*

(i) $F_A = \{x^*\}$;

(ii) $d(x, x^*) \leq \psi(d(x, A(x)))$, $\forall\ x \in X$.

Then the fixed point problem is well posed for A.

Proof. Let $x_n \in X$, $n \in \mathbb{N}$, such that

$$d(x_n, A(x_n)) \to 0 \text{ as } n \to \infty.$$

From (ii), we have

$$d(x_n, x^*) \leq \psi(d(x_n, A(x_n))) \to 0 \text{ as } n \to \infty.$$

From the Theorem 6.1 we have

Theorem 6.2. *Let (X, d) a metric space, $Y \in P(X)$ and $A : Y \to X$. We suppose that*

(i) $F_A = \{x^*\}$;

(ii) the operator A is a strict φ-contraction.

Then the fixed point problem is well posed for A.

Theorem 6.3. *The fixed point problem for the strict φ-contractions which appear in continuation principles ([5], [7],...) is well posed.*

Proof. From the continuation principle we have that $F_A = \{x^*\}$. So, we are in the conditions of the Theorem 6.2.

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STATISTICAL APPROXIMATION BY AN INTEGRAL TYPE OF POSITIVE LINEAR OPERATORS

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Dedicated to Professor Petru Blaga at his 60^{th} anniversary

Abstract. In this paper we will construct an integral type generalization of operators defined and investigated by M.A. Ozarslan, O. Duman, O. Dogru in [7]. We also present a statistical approximation result for these operators.

1. Introduction

In [7] the following positive linear operators defined on $C[0,b], 0<b<1$,

$$T_n(f;x) = \frac{u_n}{F_n(x,t)} \sum_{v=0}^{\infty} f\left(\frac{v}{a_n(v)}\right) C_v^{(n)}(t)x^v, \quad f \in [0,b], \tag{1.1}$$

have been introduced, where $u_n \geq 0$, $x \in [0,b]$, $t \in (-\infty, 0]$. In the above $\{F_n(x,t)\}$ is the set of generating functions for the sequence of functions $\{C_v^{(n)}(t)\}_{v\in\mathbb{N}_0}$ in the form

$$F_n(x,t) = \sum_{v=0}^{\infty} C_v^{(n)}(t)x^v \tag{1.2}$$

and $C_v^{(n)}(t) \geq 0$ for $t \in (-\infty, 0]$. This general sequence includes many well-known operators in approximation theory.

In the present paper we construct an integral type generalization of operators defined by (1.1) and we present a Korovkin type approximation theorem via A-statistical convergence.

At first we recall some notation on A-statistical convergence.

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Let $A := (a_{jn})$, $j, n = 1, 2, \ldots$, be an infinite summability matrix. For a given sequence $x = (x_n)$, the A-transform of x, denoted by $Ax := ((Ax)_j)$, is given by

$$(Ax)_j = \sum_{n=1}^{\infty} a_{jn} x_n,$$

provided that the series converges for each $j \in \mathbb{N}$.

We say that A is regular if $\lim_j (Ax)_j = L$ whenever $\lim_j x_j = L$. Assume that A is a non-negative regular summability matrix. A sequence $x = (x_n)$ is called A-statistically convergent to L if for every $\varepsilon > 0$,

$$\lim_j \sum_{n:|x_n - L| \geq \varepsilon} a_{jn} = 0.$$

We denote this limit by $st_A - \lim x = L$ (see [2]).

Observe that, if A is the identity matrix, then I-statistical convergence reduces to ordinary convergence.

It is not hard to see that every convergent sequence is A-statistically convergent. E. Kolk [2] proved that A-statistical convergence is stronger than convergence when $A = (a_{jn})$ is a regular summability matrix such that

$$\lim_j \lim_n |a_{jn}| = 0.$$

2. **Auxiliary results**

In this section we define an integral type generalization of operators defined by (1.1) and present a statistical approximation result for these operators.

We introduce the sequence of operators $\{T_n^*\}$ as follows

$$(T_n^* f)(x) = \frac{u_n}{F_n(x,t)} \sum_{v=0}^{\infty} C_v^{(n)}(t) x^v \int_v^{v+c_{n,v}} f\left(\frac{\xi}{a_n(v)}\right) d\xi, \quad n \in \mathbb{N}, \tag{2.1}$$

$x \in [0, b]$, where f is an integrable function on the interval $(0, 1)$ and $(c_{n,v})$ is a sequence such that

$$0 < c_{n,v} \leq 1, \quad (n, v) \in \mathbb{N} \times \mathbb{N}, \tag{2.2}$$

$u_n \geq 0$ for any $n \in \mathbb{N}$ and

$$st_A - \lim_n u_n = 1. \tag{2.3}$$

The set $\{F_n(x,t)\}$, $t \in (-\infty, 0]$, is described as in (1.2).

Assume that the next conditions hold

(i) $F_{n+1}(x,t) = p(x)F_n(x,t),\ p(x) < M < \infty,\ x \in (0,1)$,

(ii) $BtC_{v-1}^{(n+1)}(t) = a_n(v)C_{v-1}^{(n)}(t) - vC_v^{(n)}(t),\ B \in [0,a],\ C_v^{(n)}(t) = 0$ for $v \in \mathbb{Z}^- := \{\ldots, -3, -2, -1\}$,

(iii) $\max\{v, n\} \leq a_n(v) \leq a_n(v+1)$.

In what follows we prove inequalities for the operators T_n^* given by (2.1).

We set e_j, $e_j(x) = x^j$, $j \geq 0$.

Lemma 2.1. *Let T_n^* be the positive linear operator given by (2.1). Then, for each $x \in [0,b]$, $t \in (-\infty, 0]$ and $n \in \mathbb{N}$ we have*

$$\|T_n e_1 - e_1\|_{C[0,b]} \leq \frac{u_n}{2n} + abM|t|\frac{u_n}{n} + b|u_n - 1|,$$

where $e_1(x) = x$ and M, a are given as in (i) and (ii) respectively.

Proof. Using (2.1), (2.2), (2.3), (2.4), (i), (ii) and (iii) respectively we get

$$(T_n^* e_1)(x) = \frac{u_n}{F_n(x,t)} \sum_{v=0}^{\infty} C_v^{(n)}(t)x^v \int_v^{v+c_{n,v}} \frac{\xi}{a_n(v)} d\xi$$

$$= \frac{u_n}{F_n(x,t)} \sum_{v=0}^{\infty} C_v^{(n)}(t)x^v \frac{1}{a_n(v)} \cdot \frac{\xi^2}{2}\Big|_v^{v+c_{n,v}}$$

$$= \frac{u_n}{F_n(x,t)} \sum_{v=0}^{\infty} \frac{C_v^{(n)}(t)}{a_n(v)} \cdot x^v \cdot \frac{1}{2}(c_{n,v}^2 + 2vc_{n,v})$$

$$\leq \frac{u_n}{2F_n(x,t)} \sum_{v=0}^{\infty} \frac{C_v^{(n)}(t)}{a_n(v)} x^v + \frac{u_n}{F_n(x,t)} \sum_{v=0}^{\infty} C_v^{(n)}(t) \frac{v}{a_n(v)} x^v$$

$$\leq \frac{u_n}{2F_n(x,t)n} \sum_{v=0}^{\infty} C_v^{(n)}(t)x^v + \frac{u_n}{F_n(x,t)} \sum_{v=1}^{\infty} \left[C_{v-1}^{(n)}(t) - \frac{Bt}{a_n(v)} C_{v-1}^{(n+1)}(t)\right] x^v$$

It follows that

$$(T_n^* e_1)(x) - x \leq \frac{u_n}{2n} + u_n x - x + \left| \frac{u_n}{F_n(x,t)} \sum_{v=1}^{\infty} \frac{Bt}{a_n(v)} C_{v-1}^{(n+1)}(t) x^v \right|$$

$$\leq \frac{u_n}{2n} + u_n x - x + \left| \frac{Btxu_n}{F_n(x,t)n} \sum_{v=1}^{\infty} C_{v-1}^{(n+1)}(t) x^{v-1} \right|$$

$$= \frac{u_n}{2n} + x(u_n - 1) + \left| \frac{Btxu_n}{nF_n(x,t)} \sum_{v=0}^{\infty} C_v^{(n+1)}(t) x^v \right|$$

$$= \frac{u_n}{2n} + x(u_n - 1) + \left| \frac{Btxu_n}{n} \cdot \frac{F_{n+1}(x,t)}{F_n(x,t)} \right|$$

$$= \frac{u_n}{2n} + x(u_n - 1) + \left| \frac{Btxu_n}{n} p(x) \right|.$$

Consequently, we have

$$(T_n^* e_1)(x) - x \leq \frac{u_n}{2n} + x(u_n - 1) + \left| \frac{Btxu_n}{n} p(x) \right|.$$

Hence, by taking the supremum over $x \in [0,b]$ on both sides of the above inequality, the proof is completed.

$$\|T_n^* e_1 - e_1\|_{C[0,b]} \leq \frac{u_n}{2n} + abM|t|\frac{u_n}{n} + b|u_n - 1|.$$

Lemma 2.2. *For each $x \in [0,b]$, $t \in (-\infty, 0]$ and $n \in \mathbb{N}$ we have*

$$\|T_n^* e_2 - e_2\|_{C[0,b]} \leq \frac{u_n}{3n^2} + abM|t|\frac{u_n}{n^2} + \frac{u_n}{n} b(abM|t| + aM|t| + 2) + b^2|u_n - 1|,$$

where $e_2(x) = x^2$ and M are as in Lemma 2.1.

Proof. We have from (2.1) that

$$(T_n e_2)(x) = \frac{u_n}{F_n(x,t)} \sum_{v=0}^{\infty} C_v^{(n)}(t) x^v \int_v^{v+c_{n,v}} \frac{\xi^2}{(a_n(v))^2} d\xi$$

$$= \frac{u_n}{F_n(x,t)} \sum_{v=0}^{\infty} C_v^{(n)}(t) x^v \frac{1}{(a_n(v))^2} \cdot \frac{\xi^3}{3} \Big|_v^{v+c_{n,v}}$$

$$= \frac{u_n}{F_n(x,t)} \sum_{v=0}^{\infty} C_v^{(n)}(t) x^v \frac{1}{[a_n(v)]^2} \cdot \frac{1}{3}(v^3 + 3v^2 c_{n,v} + 3v c_{n,v}^2 + c_{n,v}^3 - v^3)$$

$$= \frac{u_n}{F_n(x,t)} \sum_{v=0}^{\infty} C_v^{(n)}(t) x^v \frac{v^2}{[a_n(v)]^2} c_{n,v} + \frac{u_n}{F_n(x,t)} \sum_{v=0}^{\infty} C_v^{(n)}(t) x^v \frac{v}{[a_n(v)]^2} c_{n,v}^2$$

$$+ \frac{u_n}{F_n(x,t)} \sum_{v=0}^{\infty} C_v^{(n)}(t) x^v \frac{1}{3[a_n(v)]^2} c_{n,v}^3$$

$$\leq \frac{u_n}{F_n(x,t)} \sum_{v=0}^{\infty} C_v^{(n)}(t) x^v \frac{v^2}{[a_n(v)]^2} + \frac{u_n}{F_n(x,t)} \sum_{v=0}^{\infty} C_v^{(n)}(t) x^v \frac{v}{[a_n(v)]^2}$$

$$+\frac{u_n}{3F_n(x,t)}\sum_{v=0}^{\infty}C_v^{(n)}(t)\frac{x^v}{[a_n(v)]^2}$$

Using the recurrence formula (ii) twice, we may write

$$\frac{v^2}{[a_n(v)]^2}C_v^{(n)}(t)=\frac{a_n(v-1)}{a_n(v)}C_{v-2}^{(n)}(t)-\frac{Bt}{a_n(v)}C_{v-2}^{(n+1)}(t) \tag{2.5}$$

$$+\frac{1}{a_n(v)}C_{v-1}^{(n)}(t)-\frac{Btv}{[a_n(v)]^2}C_{v-1}^{(n+1)}(t).$$

Taking into account (2.5) we get respectively

$$(T_n^*e_2)(x)-e_2(x)\le\left(\frac{u_n}{F_n(x,t)}\sum_{v=2}^{\infty}\frac{a_n(v-1)}{a_n(v)}C_{v-2}^{(n)}(t)x^v-x^2\right) \tag{2.6}$$

$$+\left|\frac{Btu_n}{F_n(x,t)}\sum_{v=2}^{\infty}\frac{1}{a_n(v)}C_{v-2}^{(n+1)}(t)x^v\right|+\left|\frac{u_n}{F_n(x,t)}\sum_{v=1}^{\infty}\frac{1}{a_n(v)}C_{v-1}^{(n)}(t)x^v\right|$$

$$+\left|\frac{Btu_n}{F_n(x,t)}\sum_{v=1}^{\infty}\frac{v}{[a_n(v)]^2}C_{v-1}^{(n+1)}(t)\right|+\left|\frac{u_n}{nF_n(x,t)}\sum_{v=0}^{\infty}C_v^{(n)}(t)x^v\frac{v}{a_n(v)}\right|$$

$$+\left|\frac{u_n}{3F_n(x,t)}\sum_{v=0}^{\infty}C_n^{(v)}(t)x^v\frac{1}{(a_n(v))^2}\right|$$

By using the requirement (iii) we have

$$\frac{v+1}{[a_n(v+1)]^2}\le\frac{1}{n},\ \frac{1}{a_n(v+2)}\le\frac{1}{n},\ \frac{1}{a_n(v+1)}\le\frac{1}{n},\ a_n(v-1)\le a_n(v).$$

Considering (2.2), (i), (ii), (iii) and the above relations results

$$\left|\frac{Btu_n}{F_n(x,t)}\sum_{v=1}^{\infty}\frac{v}{[a_n(v)]^2}C_{v-1}^{(n+1)}(t)x^v\right|\le\frac{B|t|xu_n}{F_n(x,t)}\sum_{v=0}^{\infty}\frac{v+1}{[a_n(v+1)]^2}C_v^{(n+1)}(t)x^v$$

$$\le a|t|xp(x)\frac{u_n}{n}$$

$$\left|\frac{Btu_n}{F_n(x,t)}\sum_{v=2}^{\infty}\frac{1}{a_n(v)}C_{v-2}^{(n+1)}(t)x^v\right|\le\frac{a|t|x^2u_n}{F_n(x,t)}\sum_{v=0}^{\infty}\frac{1}{a_n(v+2)}C_v^{(n+1)}(t)x^v$$

$$\le\frac{a|t|x^2u_n}{nF_n(x,t)}\sum_{v=0}^{\infty}C_v^{(n+1)}(t)x^v=a|t|p(x)x^2\frac{u_n}{n}$$

$$\left|\frac{u_n}{F_n(x,t)}\sum_{v=1}^{\infty}\frac{1}{a_n(v)}C_{v-1}^{(n)}(t)x^v\right|\le\frac{xu_n}{n},$$

$$\frac{u_n}{F_n(x,t)}\sum_{v=2}^{\infty}\frac{a_n(v-1)}{a_n(v)}C_{v-2}^{(n)}(t)x^v-x^2\le x^2(u_n-1).$$

The above inequalities and (2.6) imply

$$(T_n^* e_2)(x) - x^2 \le x^2(u_n - 1) + ax^2|t|p(x)\frac{u_n}{n} + \frac{xu_n}{n} + a|t|xp(x)\frac{u_n}{n}$$

$$+x\frac{u_n}{n} + a|t|p(x)x\frac{u_n}{n^2} + \frac{u_n}{3n^2}$$

Consequently, we have

$$\|T_n^* e_2 - e_2\|_{C[0,b]} \le \frac{u_n}{3n^2} + abM|t|\frac{u_n}{n^2} + \frac{bu_n}{n}(abM|t| + aM|t| + 2] + b^2|u_n - 1|.$$

3. **Statistical approximation**

In this section, we provide a Korovkin type theorem via A-statistical convergence for the sequence of positive linear operators defined by (2.1).

Lemma 3.1. *Let $A = (a_{jn})$ be a non-negative regular summability matrix. Then we have*

$$st_A - \lim_n \|T_n^* e_1 - e_1\|_{C[0,b]} = 0,$$

where T_n^* is defined by (2.1).

Proof. We conclude from Lemma 2.1 that

$$\|T_n^* e_1 - e_1\|_{C[0,b]} \le \frac{u_n}{2n} + BbM|t|\frac{u_n}{n} + b|u_n - 1| \tag{2.7}$$

$$= \frac{u_n}{2n}(1 + 2bBM|t|) + b|u_n - 1| \le B_1\left(\frac{u_n}{2n} + |u_n - 1|\right),$$

where $B_1 = \max\{(1 + 2bBM|t|), b\}$.

We can conclude according to (2.3) that

$$st_A - \lim_n \frac{u_n}{2n} = 0.$$

Now, for a given $\varepsilon > 0$, define

$$U := \left\{n : \ \frac{u_n}{2n} + |u_n - 1| \ge \frac{\varepsilon}{B_1}\right\},$$

$$U_1 := \left\{n : \ \frac{u_n}{2n} \ge \frac{\varepsilon}{2B_1}\right\}, \quad U_2 := \left\{n : \ |u_n - 1| \ge \frac{\varepsilon}{2B_1}\right\}.$$

We see that $U \subseteq U_1 \cup U_2$.

The inequality (2.7) yields

$$\sum_{n:\|T_n^* e_1 - e_1\| \geq \varepsilon} a_{jn} \leq \sum_{n\in U} a_{jn}, \leq \sum_{n\in U_1} a_{jn} + \sum_{n\in U_2} a_{jn},$$

and taking $j \to \infty$ the result follows.

Lemma 3.2. *Let $A = (a_{jn})$ be a non-negative regular summability matrix. Then we have*

$$st_A - \lim_n \|T_n^* e_2 - e_2\| = 0,$$

where T_n^* is defined by (2.1).

Proof. It follows from Lemma 2.2 that

$$\|T_n^* e_2 - e_2\|_{C[0,b]} \leq \frac{u_n}{3n^2}(1 + 3bBM|t|)$$
$$+\frac{u_n}{n}(b^2BM|t| + bBM|t| + 2) + b^2|u_n - 1|.$$

Hence, we get

$$\|T_n^* e_2 - e_2\|_{C[0,b]} \leq B_2 \left(\frac{u_n}{3n^2} + \frac{u_n}{n} + |u_n - 1|\right), \tag{2.8}$$

where

$$B_2 = \max\{(1 + 3bBM|t|), (b^2BM|t| + bBM|t| + 2), b^2\}.$$

By (2.3) we have

$$st_A - \lim_n u_n = 1, \quad st_A - \lim_n \frac{u_n}{n} = 0 \quad \text{and} \quad st_A - \lim_n \frac{u_n}{3n^2} = 0.$$

For a given $\varepsilon > 0$ we define

$$U := \left\{n : \ \frac{u_n}{3n^2} + \frac{u_n}{n} + |u_n - 1| \geq \frac{\varepsilon}{B_2}\right\},$$
$$U_1 := \left\{n : \ \frac{u_n}{3n^2} \geq \frac{\varepsilon}{3B_2}\right\}, \quad U_2 := \left\{n : \ \frac{u_n}{n} \geq \frac{\varepsilon}{3B_2}\right\},$$
$$U_3 := \left\{n : \ |u_n - 1| \geq \frac{\varepsilon}{3B_2}\right\}.$$

Then we have $U \subseteq U_1 \cup U_2 \cup U_3$.

By using (2.8) we can write successively

$$\sum_{n:\|T_n e_2 - e_2\|_{C[0,b]} \geq \varepsilon} a_{jn} \leq \sum_{n\in U} a_{jn} \leq \sum_{n\in U_1} a_{jn} + \sum_{n\in U_2} a_{jn} + \sum_{n\in U_3} a_{jn}.$$

Taking limit as $j \to \infty$ the proof is complete.

We recall the following important result established by A.P. Gadjiev and C. Orhan.

Theorem 3.1. ([4], Theorem 1) *If the sequence of positive linear operators* $L_n : C_M[a,b] \to B[a,b]$ *satisfies the conditions*

(1) $st - \lim \|L_n e_0 - e_0\|_B = 0$

(2) $st - \lim \|L_n e_1 - e_1\|_B = 0$

(3) $st - \lim \|L_n e_2 - e_2\|_B = 0$

then for any function $f \in C_M[a,b]$ *we have*

(4) $st - \lim \|L_n f - f\|_B = 0,$

where $C_M[a,b] = \{f : \mathbb{R} \to \mathbb{R},\ f$ *continuous on* $[a,b]$ *and bounded on the whole real axis*$\}$ *and* $\|f\|_B := \sup\limits_{a \le x \le b} |f(x)|$.

We mention that the above Theorem is given for statistical convergence, but the proof also works for A-statistical convergence.

Now we provide a Korovkin type approximation theorem for the operators T_n^* via A-statistical convergence.

Theorem 3.2. *Let* $A = (a_{jn})$ *be a non-negative regular summability matrix. Then, for all* $f \in C[0,b]$, *we have*

$$st_A - \lim_n \|T_n^* f - f\|_{C[0,b]} = 0.$$

Proof. By $(T_n^* e_0)(x) \le u_n$, Lemmas 3.1 and 3.2 we get

$$st_A - \lim_n \|T_n^* e_i - e_i\|_{C[0,b]} = 0, \quad i = 0, 1, 2.$$

The result follows from Theorem 1 in [4].

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A COUNTER-EXAMPLE CONCERNING STARLIKE FUNCTIONS

RÓBERT SZÁSZ

Dedicated to Professor Petru Blaga at his 60^{th} *anniversary*

Abstract. Let A denote the Alexander integral operator, let C and S^* denote the class of close-to-convex functions and the class of starlike functions, respectively. In the paper it is proved that the inclusion $A(C) \subset S^*$ does not hold.

1. Introduction

Let $\mathcal{A}$ be the class of analytic functions defined in the unit disk

$$U = \{z \in \mathbb{C} : |z| < 1\}$$

and having the form $f(z) = z + a_2 z^2 + a_3 z^3 + \dots$

The analytic description of the class S^* and C are given by

$$S^* = \left\{ f \in \mathcal{A} \mid \operatorname{Re} \frac{zf'(z)}{f(z)} > 0,\ z \in U \right\}$$

and

$$C = \left\{ f \in \mathcal{A} \mid \exists\ g \in S^* : \operatorname{Re} \frac{zf'(z)}{g(z)} > 0,\ z \in U \right\}.$$

In this article we discuss a relation between these two classes which involve the integral operator of Alexander defined by

$$A(f)(z) = \int_0^z \frac{f(t)}{t} dt.$$

In [2] pp. 310 and [3] pp. 361 the authors have proved the following theorem:

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Theorem 1. *Let $g \in \mathcal{A}$ be a function which has the property:*

$$\text{Re}\,\frac{zg'(z)}{g(z)} \geq \left| \text{Im}\,\left(\frac{z(zg'(z))'}{g(z)}\right)\right|,\ z \in U.$$

If $f \in \mathcal{A}$ and

$$\text{Re}\,\frac{zf'(z)}{g(z)} > 0,\ z \in U$$

or

$$\text{Re}\,\frac{zf'(z)}{g'(z)} > 0,\ z \in U$$

then $F = A(f) \in S^*$.

This Theorem rises the question, if the following better results can be valid:

$$A(C) \subset S^*$$

2. **Preliminaries**

To prove the main result, we need the Lemma

Lemma 1. [1] *pp. 18 (2.8)*

The functions

$$f(z) = \frac{z - \frac{1}{2}(x+y)z^2}{(1-yz)^2},\ |x| = |y| = 1$$

belong to class C.

These functions are the extreme points of class C if $x \neq y$.

3. **The Main Result**

Theorem 2.

$$A(C) \not\subset S^*.$$

Proof. We shall prove that there are two complex numbers $x, y \in \partial U$, $x \neq y$ so that

$$A(f) \notin S^*$$

where

$$f(z) = \frac{z - \frac{1}{2}(x+y)z^2}{(1-yz)^2}$$

and

$$\begin{aligned}
A(f)(z) &= \int_0^z \frac{f(t)}{t}dt. \\
A(f)(z) &= \int_0^z \frac{1-\frac{1}{2}(x+y)t}{(1-yt)^2}dt = \\
&= \frac{1}{2}\left(1-\frac{x}{y}\right)\frac{z}{1-yz} - \frac{1}{2}\left(1+\frac{x}{y}\right)\frac{\log(1-yz)}{y}.
\end{aligned}$$

The branch of $\log(1-yz)$ is chosen so that $\mathrm{Im}\,(\log(1-y(z)) \in [-\pi,\pi]$.

Let F be the function defined by the equality:

$$F(z) = \frac{z(A(f))'(z)}{A(f)(z)} = \frac{2-\left(1+\frac{x}{y}\right)yz}{\left(1-\frac{x}{y}\right)(1-yz)-\left(1+\frac{x}{y}\right)(1-yz)^2\frac{\log(1-yz)}{yz}}$$

If $A(f) \in S^*$ then $\mathrm{Re}\ F(z) > 0$, $z \in U$ and from the continuity it follows that

$$\mathrm{Re}\ F(z) \geq 0 \text{ for every } z \in \partial U \text{ for which } F(z) \text{ is defined.} \tag{1}$$

We will prove that the assertion (1) is not valid.

If we introduce the notations

$$\frac{x}{y} = \cos u + i\sin u$$
$$yz = \cos\alpha + i\sin\alpha$$

we get that

$$\begin{aligned}
F(z) &= \frac{1-i\,\mathrm{tg}\,\frac{u}{2}-\cos\alpha - i\sin\alpha}{-i\,\mathrm{tg}\,\frac{u}{2}(1-\cos\alpha-i\sin\alpha)+4\sin^2\frac{\alpha}{2}\log(1-\cos\alpha-i\sin\alpha)} = \\
&= \frac{1-\cos\alpha - i\left(\mathrm{tg}\,\frac{u}{2}+\sin\alpha\right)}{4\sin^2\frac{\alpha}{2}\ln\left(2\sin\frac{\alpha}{2}\right) - \,\mathrm{tg}\,\frac{u}{2}\sin\alpha - 2i\sin^2\frac{\alpha}{2}\left(\pi-\alpha+\,\mathrm{tg}\,\frac{u}{2}\right)}
\end{aligned}$$

$$Re\ F(z) =$$

$$= \frac{(1-\cos\alpha)\left(-\,\mathrm{tg}\,\frac{u}{2}\sin\alpha+4\sin^2\frac{\alpha}{2}\ln\left(2\sin\frac{\alpha}{2}\right)\right)+\left(\,\mathrm{tg}\,\frac{u}{2}+\sin\alpha\right)2\sin^2\frac{\alpha}{2}\left(\,\mathrm{tg}\,\frac{u}{2}+\pi-\alpha\right)}{\left(-\,\mathrm{tg}\,\frac{u}{2}\sin\alpha+4\sin^2\frac{\alpha}{2}\ln\left(2\sin\frac{\alpha}{2}\right)\right)^2+4\sin^4\frac{\alpha}{2}\left(\,\mathrm{tg}\,\frac{u}{2}+\pi-\alpha\right)^2}$$

The numerator of $\mathrm{Re}\ F(z)$ is a polinomial of degree two with respect to $\mathrm{tg}\,\frac{u}{2}$.

The discriminant of the polinomial is

$$\Delta(\alpha) = 4\sin^4\frac{\alpha}{2}\left[(\pi-\alpha)^2-4(\pi-\alpha)\sin\alpha-16\sin^2\frac{\alpha}{2}\ln\left(2\sin\frac{\alpha}{2}\right)\right].$$

Because

$$\lim_{\alpha\to 0}\frac{\Delta(\alpha)}{\sin^4\frac{\alpha}{2}} = 4\pi^2$$

there are $\alpha \in (-\pi,\pi)$, for which $\Delta(\alpha) > 0$.

The inequality $\Delta(\alpha) > 0$ for some $\alpha \in (-\pi,\pi)$, means that there are two points $x, y \in \partial U$, $x \neq y$ and $z \in \partial U$ so that $\operatorname{Re} F(z) < 0$, which contradicts (1). $\square$

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BOOK REVIEWS

Barry Simon, *Orthogonal Polynomials on the Unit Circle*,
Part 1: *Classical Theory,* xxv+pages 1-466 (ISBN:0-8218-3446-0),
Part 2: *Spectral Theory*, xxi+pages 467-1044 (ISBN:0-8218-3675-7),
American Mathematical Society, Colloquium Publications, Volume 54, Providence, Rhode Island 2005, ISBN:0-8218-3757-5 (set).

This monumental two volume treatise contains a comprehensive study of orthogonal polynomials on the unit circle (OPUC) corresponding to nontrivial (i.e., with infinite support) probability measures on the unit circle $\partial\mathbb{D}$, in the complex plane, where $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ is the unit disk. This is viewed as a counterpart of the well established theory of orthogonal polynomials on the real line (OPRL) as presented, for instance, in the classical treatises of G. Szegö (first edition, AMS 1939) and G. Freud (Pergamon Press 1971). At the same time, OPUC theory supplies the study of OPRLs with new tools and methods. If μ is a nontrivial probability measure on $\partial\mathbb{D}$, then $1, z, z^2, \ldots$ are linearly independent in the Hilbert space $L^2(\partial\mathbb{D}, \mu)$, so that the Gram-Schmidt orthogonal procedure produces an orthogonal system $\{\Phi_n\}$ given by $\Phi_n = P_n[z^n]$, where P_n denotes the projection onto $\{1, z, \ldots z^{n-1}\}^{\perp}$. By normalization one obtains the orthonormal system $\varphi_n = \Phi_n/\|\Phi_n\|$. The orthogonal polynomials Φ_n satisfy Szegő's recurrence $\Phi_{n+1}(z) = z\Phi_n(z) - \bar{\alpha}_n\Phi_n^*(z)$, where $\Phi_n^*(z) = z^n\overline{\Phi_n(1/\bar{z})}$ is the reversed polynomials of Φ_n. The parameters $\alpha_0, \alpha_1, \ldots,$ called Verblunsky coefficients, all belong to $\mathbb{D}$ (i.e., $|\alpha_j| < 1$). These were considered by S. Verblunsky in two remarkable papers published in 1934 and 1935 in Proceedings of the London Mathematical Society, where he proved, among other things, that there is a bijective correspondence between the nontrivial probability measures on $\partial\mathbb{D}$ and the sequences $\{\alpha_j\}_{j=0}^{\infty}$ in $\mathbb{D}$. The author gives four proofs for this result. The irony is that the results of Verblunsky were largely overlooked by the mathematical community, some of them being rediscovered later.

One of the central topic of the book, and of the whole theory of OPUC, is how the properties of the measure μ correspond to properties of the Verblunsky coefficients, and viceversa. The major result in this respect is Szegő Theorem asserting that $\prod_{j=0}^{\infty}(1-|\alpha_j|^2) = \exp\left(\int_0^{2\pi}\log(w(\theta))\frac{d\theta}{2\pi}\right)$, where $w(\theta)d\theta$ is the absolutely continuous part of μ. In particular, $\sum_j |\alpha_j|^2 < \infty$ if and only if $\int \log w(\theta)d\theta > -\infty$. Four proofs of this remarkable result are given. An extension of this result is the so called Strong Szegő Theorem, which can be rephrased as an assertion about the asymptotics of Toeplitz determinants. Verblunsky coefficients are related also with other important quantities in the theory of OPUC as, for instance, with Fourier coefficients $\hat{\mu}_n$ of the measure μ: $\sum_j |\alpha_j| < \infty$ if and only if $\sum_j |\hat{\mu}_j| < \infty$ (Baxter's Theorem (1961)).

Another important tools, relating OPUC with the theory of analytic functions are the Carathéodori functions defined by $F(z) = \int [e^{i\theta} + z][e^{i\theta} + z]^{-1} d\mu(\theta)$ and the associated Schur functions f given by the relation $F(z) = [1 + zf(z)][1 + zf(z)]^{-1}$. Section 1.3 contains an overview of their properties. Schur proved that Schur functions admit continued fraction expansion with coefficients $|\gamma_j| < 1$, called Schur parameters. A remarkable result proved by Ya. L. Geronimus in the fifties of the last century asserts that Schur parameters agree with Verblunsky coefficients: $\gamma_j = \alpha_j$, $j \in \mathbb{N}$. Again five proofs of this result are included in the book. The method of Schur functions and some real variable methods were used by S. Kruschev in a series of papers published between 1993 and 2005 to obtain some important results on OPUC.

The book is very well organized and covers a lot of important results. In fact, the aim of the inclusion of several proofs for some important results is to emphasize that different approaches shed new light on the subject and, at the same time, allow to systematize and organize the study of OPUCs. For the convenience of the reader, beside the section on Carathéodori and Schur functions we did yet mention, a section containing a survey of principal results and methods from OPRL theory, as well as one on spectral theory of operators on Hilbert space are included. At the end of the second volume there are four very useful appendices: A. *Reader's Guide: Topics and Formulae*; B. *Perspective* (OPRL vs OPUC); C. *Twelve Great Theorems*; D. *Conjectures and Open Questions*. Concerning the "twelve great theorems", the author quote a nice remark of his father who said once that "to pick ten people from twenty for some positive reasons, you don't make ten friends but ten enemies". Apparently, the only way to make ten friends from twenty people is to pick ten for some negative reasons.

The bibliography counts 1119 items with references to the pages where each of them is quoted. The remarks and the historical notes following each section present the evolution of the subject, putting in evidence some corner points and seminal papers.

Undoubtedly that, as Szegő's book, published first in 1939 as the volume 23 in the same prestigious series, this book will become a standard reference in the field tracing the way for future investigations on orthogonal polynomials and their applications. Combining methods from various areas of analysis (calculus, real analysis, functional analysis, complex analysis) as well as by the importance of the orthogonal polynomials in applications, the book will have a large audience including researchers in mathematics, physics, engineering.

S. Cobzaş

Athanase Papadopoulos, ***Metric spaces, convexity and nonpositive curvature,*** IRMA Lectures in Mathematics and Theoretical Physics, Vol. 6, European Mathematical Society, Zürich 2005, xi+287 pp, ISBN: 3-03719-010-8.

Geodesic metric spaces form a class of metric spaces in which convexity of subsets can be defined as well as other related analytic concepts. Buseman spaces are geodesic metric spaces whose length function satisfies a convexity condition. Beside their intrinsic geometric interest, geodesic metric spaces are important for their applications to complex analysis and nonlinear analysis - fixed point theory for nonexpansive mappings, generalized differentiability and optimization. Classical examples of geodesic metrics are the Riemannian metric, the Poincaré metric on the hyperbolic ball $\mathbb{H}^n$, the Carathéodori and the Kobayashi distances for complex manifolds, Thusrston's metric on complex projective surfaces, the Teichmüller metric and Teichmüller spaces. The book starts with a short historical overview emphasizing some corner points in its development - the pioneering work of J. Hadamard, the contributions of K. Menger, A. Wald, H. Busemann and A. D. Alexandrov.

In order to make the book self-contained the author systematically develop in the first two chapters 1. *Lengths of paths in metric spaces,* 2. *Length spaces and geodesic spaces,* the basic construction and the properties of length spaces and geodesic spaces, including convexity - geodesic convexity and Menger convexity, this last being defined through the betweeness relation. Chapter 3. *Maps between metric spaces,* is concerned with Lipschitz maps and fixed points for contractive and for nonexpansive mappings on geodesic spaces. The analog of Hausdorff distance for subsets of a geodesic metric space, called the Busemann-Hausdorff distance, with applications to limits of subsets is considered in Chapter 4. *Distances.*

Chapters 5. *Convexity in vector spaces,* 6. *Convex functions,* and 7. *Strictly convex normed spaces,* are concerned with convexity in vector and normed spaces, emphasizing connections with geodesic metric spaces and the geometry of Minkowski space (finite dimensional normed spaces).

The rest of the book, chapters 8. *Busemann spaces,* 9. *Locally convex spaces* (meaning geodesic metric spaces such that every point has a neighborhood which is a Busemann space), 10. *Asymptotic rays and the visual boundary,* 11. *Isometries,* 12. *Busemann functions, co-rays and horospheres,* is devoted to the theory of Busemann spaces. Again convexity is the main topic and the unifying idea of the development of Busemann spaces.

Written in a clear and pleasant style, with numerous examples from geometry and analysis, the book is accessible to graduate students interested in this topic of intense current research due to its intrinsic importance and to its numerous applications as well.

S. Cobzaş

The essential of John Nash, Edited by Harold W. Kuhn and Sylvia Nasar, Princeton University Press, 2007, ISBN-13: 978-0-691-09610-0, ISBN-10: 0-691-09610-4.

The brilliant mathematician John Forbes Nash was born in 1928 in Bluefield, West Wirginia. After finishing high school in Bluefield he went to Carnegie Tech in Pittsburgh with major chemical engineering, but he eventually switched to mathematics. After graduation he was offered fellowship at either Harvard or Princeton, and he choose the more generous Princeton fellowship, which was also nearer to his hometown. At Princeton he wrote in 1950 a 27 pages Ph.D. thesis on *Non-cooperative games* which initiated a new era in game theory with applications to economics, social behavior, war questions, etc. The basic idea was that which is called now the Nash equilibrium, which finally led to a Nobel Prize in Economics attributed to him in 1994, shared with John Harsanyi and Reinhard Selten. As it is mentioned in the motivation of the Nobel committee the Nash equilibrium has become "the analytical structure for studying all situations of conflict and cooperation".

Other important contributions of John Nash concern the imbedding of Riemannian manifolds in the Euclidean space, the Nash implicit function theorem, real algebraic varieties, and parabolic partial differential equations.

But after these astonishing and deep contributions J. Nash suffered at the age of thirty one of mental illness, being diagnosed as paranoid schizophrenia, so he had to retire from MIT where he was affiliated. After a long period of absence (30 years) he recovered himself, due in good part to the recognition of his achievements by the Nobel prize committee, started to travel and to work again. The illness prevented him to obtain a Fields medal which he fully deserved for his outstanding results on elliptic and parabolic partial differential equations. Because in 1958 these results were still unpublished, the Fields Committee postponed Nash as a virtual winner of the 1962 Medal, but the mental illness destroyed his career for a long period of time.

His situation is known to the general public due to the book *A beautiful mind* by Sylvia Nasar (one of the editors of the present volume) and by the movie with the same name with Russell Crowe starring as John Nash.

While the biography by Sylvia Nasar was concerned mainly with the life of J. Nash, the present volume deals with his scientific work. It contains contributions by Harold W. Kuhn (the other editor of the book), a Princeton fellow of J. Nash and a life friend, an introduction by Sylvia Nasar, the press release of the Swedish Academy on Nobel prize and an autobiography written by J. Nash with this occasion, a nice collection of photos, a short note by John Milnor on the Hex game (known also as the Nash game), the facsimile of the Ph.D. thesis of J. Nash and several of his seminal papers.

The book is written in a pleasant and informal style, being addressed to a large audience. It's nice that Princeton University Press released this cheeper paperback version making the book accessible to a large public.

P. T. Mocanu

J. Baik, T. Kriecherbauer, K.T.-R. McLaughlin, P.D. Miller, *Discrete Orthogonal Polynomials. Asymptotics and Applications,* Annals of Mathematics Studies, Number 164, Princeton University Press, 2007, 170 pp. ISBN-13: 978-0-691-12734-7, ISBN-10: 0-691-12734-4.

As the title of the book suggests, the main aim of this monograph is to present asymptotic properties of polynomials that are orthogonal with respect to pure point measures supported on finite sets of nodes. Further on, the authors use these results to establish various statistical properties of discrete orthogonal polynomial ensembles. In particular, the authors apply their results to the problem of computing asymptotic of statistics for random rhombus tilings of a large hexagon. They also obtain new results for the continuum limit of the Toda lattice. Working with a general class of weights that contains Krawtchouk and Hahn weights as special cases, the authors compute the asymptotic of the associated discrete orthogonal polynomials for all values of the variable in the complex plane.

The starting point is the following basic interpolation problem: given a natural number N, a set X_N of nodes and a set of corresponding weights $\{w_{N,n}\}$, consider the possibility of finding a 2×2 matrix $P(z; N, k)$, $k \in \mathbb{Z}$, with certain properties. After a comprehensive introduction providing the reader with a thorough mathematical background, the main results for the discrete orthogonal polynomials and for corresponding applications are presented in Chapter 2 and Chapter 3, respectively. Chapters 4 and 5 contain the complete asymptotic analysis of the matrix $P(z; N, k)$ in the limit N tends to infinity. In the last two chapters the authors prove the theorems stated in Chapters 2 and 3.

The contents of the book is enriched with 3 Appendices. I mention the first of them summarizes construction of the solution of a limiting Riemann-Hilbert problem by means of hyperelliptic function theory and the second of them gives a proof of the determination of the equilibrium measure of the Hahn weight. At the same time good references are inserted.

The authors' style is pleasant offering detailed and clear explanations of every concept and method.

The book includes the authors' own research results developed over the last years. Their approach and proofs are straightforward constructive making this monograph accessible and valuable to undergraduate and graduate students, PhD students and researchers involved in the asymptotic analysis of systems of discrete orthogonal polynomials.

Octavian Agratini

David L. Applegate, Robert E. Bixby, Vasek Chvtal and William J. Cook *The Traveling Salesman Problem:A Computational Study,* Princeton University Press, 2007, 606 pp., ISBN13: 978-0-691-12993-8.

The present book contains an exhaustive and interesting presentation of various questions related to the famous Travelling Salesman Problem (TSP). Its goal is to set down the techniques that have led to the solution of a number of large instances of the TSP, including the full set of examples from TSPLIB challenge collection.

The first and the second chapters of the book cover history and applications of the TSP. This part is very interesting and accessible to a wide audience. A short definition of the TSP is the following: given a set of cities along with the cost of travel between each part of them, the travelling salesman problem is to find the way of visiting all the cities and returning to the start point with minimal cost. The origins of the study of the TSP as a mathematical problem are somewhat obscure. M. Flood said that the TSP was posed, in 1934, by Hassler Whitney in a seminar talk at Princeton University. Therefore as father of the TSP can be considered Hassler Whitney. The third chapter is dedicated to present the work of Danzig, Fulkerson and Johnson for solving the 49-city problem and, indirectly, to make a short presentation of the cutting plane method. The fourth chapter contents a history of TSP computation. It includes, among other things, the bases of the branch-and-bound method, the Gomory's cuts, the TSP's cuts, the Lin-Kernighan's heuristic, the Crowder-Padberg's code, the dynamic programming etc. It is mentioned that the dynamic programming algorithm can solve any n-city TSP instance in time that is at most proportional to $n^2 \cdot 2^n$.

Chapters 5 - 10 describe (some) methods for finding cutting planes for the TSP. In the chapter 11 it is presented a separation method that disdains all understanding of the TSP polytope and bashes on regardless of all prescribed templates.

The twelfth chapter presents machinery to handle the flow of cuts and edges into and out of the linear programming relaxations, as well as methods used for interacting with a linear programming solver. The actual solution of the linear programming problems is described in the chapter 13.

The branch-and-cut algorithm embeds the cutting-plane method within a branch-and-bound search. Its specialization to TSP is described in the chapter 14. There is a growing literature devoted to the study of heuristic algorithms for the TSP and to their various aspects. The fifteenth chapter includes some of them.

The algorithmic components described in this book are brought together in the *Concorde* computer code for the TSP. *Concorde* is described in the chapter 16. Also, in this chapter a report on computational studies learning to the solution of the full set of TSPLIB instances is given.

Chapter 17 contents a short survey of recent work dedicated to the TSP by various research group.

The book includes a bibliography of 561 titles.

Liana Lupşa