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ACADEMICIAN PROFESSOR DIMITRIE D. STANCU AT HIS 80TH BIRTHDAY ANNIVERSARY

PETRU BLAGA AND OCTAVIAN AGRATINI

We are pleased to have the opportunity to celebrate this anniversary of academician professor Dr. Dimitrie D. Stancu, a distinguish Romanian mathematician.

He is an Emeritus member of American Mathematical Society and a Honorary member of the Romanian Academy.

He has obtained the scientific title of Doctor Honoris Causa from the University "Lucian Blaga" from Sibiu and also a similar title from the "North University" of Baia Mare.

Professor D.D. Stancu was born on February 11, 1927, in a farmer family, from the township Călacea, situated not far from Timişoara, the capital of Banat, a south-west province of Romania. In his shoolage he had many difficulties being orphan and very poor, but with the help of his mathematics teachers he succeeded to make progress in studies at the prestigious Liceum "Moise Nicoară" from the large city Arad.

In the period 1947-1951 he studied at the Faculty of Mathematics of the University "Victor Babeş", from Cluj, Romania. When he was a student he was under the influence of professor Tiberiu Popoviciu (1906-1975), a great master of Numerical Analysis and Approximation Theory. He stimulated him to do research work.

After his graduation, in 1951, he was named assistant at the Department of Mathematical Analysis, University of Cluj. He has obtained the Ph.D. in Mathematics in 1956. His scientific advisor for the doctoral dissertation was professor Tiberiu Popoviciu.

In a normal succession he advanced up to the rank of full professor, in 1969.

He was happy to benefit from a fellowship at the University of Wisconsin, at Madison, Numerical Analysis Department, conducted by the late professor Preston C. Hammer. He spent at the University of Wisconsin, in Madison, the academic year 1961-1962.

Professor D.D. Stancu has participated in different events in USA. Among them we mention that he has presented contributed papers at several regional meetings organized by the American Mathematical Society in the cities Milwakee, Chicago and New York.

After his returning from America, he was named deputy dean at the Faculty of Mathematics of University of Cluj and head of the Chair of Numerical and Statistical Calculus.

He hold a continuous academic career at the Cluj University.

He has a nice family. His wife dr. Felicia Stancu was a lecturer of mathematics at the same University "Babeş-Bolyai" from Cluj.

They have two wonderful daughters: Angela (1957) and Mirela (1958), both teaching mathematics at secondary schools in Cluj-Napoca. From them they have three grandchildren: Alexandru-Mircea Scridon (1983) and George Scridon (1992) (the sons of Mirela) and Ştefana-Ioana Muntean (1991), the daughter of Angela. Their oldest grandson: Alexandru-Mircea Scridon, graduated in 2006 the Romanian-German section of Informatics at the University "Babeş-Bolyai" from Cluj-Napoca.

Professor D.D. Stancu loved all deeply and warmly.

At the University of Cluj-Napoca professor D.D. Stancu has taught several courses: Mathematical Analysis, Numerical Analysis, Approximation Theory, Informatics, Probability Theory and Constructive Theory of Functions.

He has used probabilistic, umbral and spline methods in Approximation Theory.

He had a large number of doctoral students from Romania, Germany and Vietnam.

Besides the United States of America, professor D.D. Stancu has participated in many scientific events in Germany (Stuttgart, Hannover, Hamburg, Gottingen, Dortmund, Münster, Siegen, Würzburg, Berlin, Oberwolfach), Italy (Roma, Napoli, Potenza, L'Aquila), England (Lancaster, Durham), Hungary (Budapest), France (Paris), Bulgaria (Sofia, Varna), Czech Republic (Brno).

His publication lists about 160 items (papers and books).

In different research journals there are more than 60 papers containing his name in their titles.

Since 1961 he is a member of American Mathematical Society and a reviewer of the international journal "Mathematical Reviews".

He is also a member of the German Society: "Gesellschaft für Angewandte Mathematik und Mechanik" and a reviewer of the journal "Zentralblatt für Mathematik".

At present he is Editor in Chief of the journal published by the Romanian Academy: "Revue d'Analyse Numérique et de Théorie de l'Approximation". For many years he is a member of the Editorial Board of the Italian mathematical journal "Calcolo", published now by Springer-Verlag, in Berlin.

In 1968 he has obtained one of the Research Awards of the Department of Education in Bucharest for his research work in Numerical Analysis and Approximation Theory.

In 1995 the University "Lucian Blaga", from Sibiu, has accorded him the scientific title of "Doctor Honoris Causa". The "North University" of Baia Mare, from which he has several doctoral students, has distinguished him with a similar honorary scientific title.

In 1999 professor D.D. Stancu was elected a Honorary Member of the Romanian Academy.

In the same year he has participated with a lecture at the "Alexits Memorial Conference", held in Budapest.

In May 2000 he was invited to participate at the International Symposium: "Trends in Approximation Theory", dedicated to the 60^{th} birthday anniversary of

Professor Larry L. Schumacker, held in Nashville, TN, where he presented a contributed paper, in collaboration with professor Wanzer Drane, from University of South Carolina, Columbia, S.C. With this occasion of visiting America, Professor D.D. Stancu was invited to present colloquium talks at several American Universities: Ohio State University from Columbus, OH, University of South Carolina, Columbia, S.C., Vanderbilt University, Nashville, TN, PACE University, Pleasantville, N.Y.

The main contributions of research work of Professor D.D. Stancu fall into the following list of topics: Interpolation theory, Numerical differentiation, Orthogonal polynomials, Numerical quadratures and cubatures, Taylor-type expansions, Approximation of functions by means of linear positive operators, Representation of remainders in linear approximation procedures, Probabilistic methods for construction and investigation of linear positive operators of approximation, spline approximation, use of Interpolation and Calculus of finite differences in Probability theory and Mathematical statistics.

In 1996 Professor D.D. Stancu has organized in Cluj-Napoca an "International Conference on Approximation and Optimization" (ICAOR), in conjunction with the Second European Congress, held in Budapest. There were participated around 150 mathematicians from 20 countries around the world. The Proceedings of ICAOR were published in two volumes having the title: "Approximation and Optimization", by Transilvania Press, Cluj-Napoca, Romania, 1997.

In May 9-11, 2002 was organized by the "Babeş-Bolyai" University, Cluj-Napoca, an "International Symposium on Numerical Analysis and Approximation Theory", dedicated to the 75^{th} anniversary of Professor D.D. Stancu. The Proceedings of this symposium were edited by Radu T. Trîmbiţaş and published in 2002 in Cluj-Napoca by Cluj University Press.

In the period July 5-8, 2006 an "International Conference on Numerical Analysis and Approximation Theory" was held at Babeş-Bolyai University. Professor D.D. Stancu was the honorary chair of this Conference.

This Conference was attended by over 60 mathematicians coming from 12 countries. The programme included 8 invited lectures and over 50 research talks.

The invited speakers were: Francesco Altomare (Italy), George Anastassiou (USA), Heiner Gonska (Germany), Bohdan Maslowski (Czech Republic), Giuseppe Mastroianni (Italy), Gradimir Milovanović (Serbia), Jozsef Szabados and Peter Vertesi (Hungary). The proceedings of this International Conference were published under the title: "Numerical Analysis and Approximation Theory", Cluj-Napoca, Romania, 2006, edited by "Casa Cărţii de Ştiinţă" (418 pages).

The intensive research work and the important results obtained in Numerical Analysis and Approximation Theory by Professor D.D. Stancu has brought to him recognition and appreciation in his country and abroad.

We conclude by wishing him health and happiness on his 80^{th} birthday and for many years to come.

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ON CERTAIN NUMERICAL CUBATURE FORMULAS FOR A TRIANGULAR DOMAIN

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Dedicated to Professor D. D. Stancu on his 80^{th} *birthday*

Abstract. The purpose of this article is to discuss certain cubature formulas for the approximation of the value of a definite double integral extended over a triangular domain. We start by using the Biermann's interpolation formula [5], [19]. Then we consider also the results obtained by D.V. Ionescu in the paper [12], devoted to the construction of some cubature formulas for evaluating definite double integrals over an arbitrary triangular domain. In the recent papers [6] and [7] there were investigated some homogeneous cubature formulas for a standard triangle. In the case of the triangle having the vertices $(0,0)$, $(1,0)$, $(0,1)$ there were constructed by H. Hillion [11] several cubature formulas by starting from products of Gauss-Jacobi formulas and using an affine transformation, which can be seen in the book of A.H. Stroud [22].

1. Use of Biermann's interpolation formula

1.1. Let us consider a triangular grid of nodes $M_{i,k}(x_i, y_k)$, determined by the intersection of the distinct straight lines $x = x_i$, $y = y_k$ $(0 \leq i + k \leq m)$. We assume that $a = x_0 < x_1 < \cdots < x_{m-1} < x_m = b$ and $c = y_0 < y_1 < \cdots < y_{m-1} < y_m = d$. We denote by $T = T_{a,b,c}$ the triangle having the vertices $A(a, c)$, $B(b, c)$, $C(a, d)$.

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It is known that the Biermann's interpolation formula [5], [19], which uses the triangular array of base points $M_{i,k}$ can be written in the following form:

$$f(x,y)=\sum_{i=0}^{m}\sum_{j=0}^{m-i}u_{i-1}(x)v_{j-1}(y)\left[\begin{array}{c}x_0,x_1,\ldots,x_i\\ y_0,y_1,\ldots,y_j\end{array};\ f\right]+(r_mf)(x,y), \tag{1.1}$$

where

$$u_{i-1}(x)=(x-x_0)(x-x_1)\ldots(x-x_{i-1}),\ u_{-1}(x)=1,\ u_m(x)=u(x)$$

and

$$v_{j-1}(y)=(y-y_0)(y-y_1)\ldots(y-y_{j-1}),\ v_{-1}(y)=1,\ v_m(y)=v(y),$$

with the notations

$$u(x)=(x-x_0)(x-x_1)\ldots(x-x_m),$$

$$v(y)=(y-y_0)(y-y_1)\ldots(y-y_m).$$

The brackets used above represent the symbol for bidimensional divided differences. By using a proof similar to the standard methods for obtaining the expression of the remainder of Taylor's formula for two variables, Biermann has shown in [5] that this remainder $(r_mf)(x,y)$ may be expressed under the remarkable form:

$$(r_mf)(x,y)=\frac{1}{(m+1)!}\sum_{k=0}^{m+1}\binom{m+1}{k}u_{k-1}(x)v_{m-k}(y)f^{(k,m+1-k)}(\xi,\eta)$$

where $(\xi,\eta)\in T$ and we have used the notation for the partial derivatives:

$$g^{(p,q)}(x,y)=\frac{(\partial^{p+q}g)(x,y)}{\partial^p x^p\partial^q y^q}.$$

1.2. By using the interpolation formula (1.1) we can construct cubature formulas for the approximation of the value of the double integral

$$I(f)=\int\int_T f(x,y)dxdy.$$

We obtain a cubature formula of the following form:

$$J(f)=\sum_{i=0}^{m}\sum_{j=0}^{m-i}A_{i,j}\left[\begin{array}{c}x_0,x_1,\ldots,x_i\\ y_0,y_1,\ldots,y_j\end{array};\ f\right]+R_m(f), \tag{1.2}$$

where the coefficients have the expressions:

$$A_{i,j} = \int\int_{\Gamma} u_{i-1}(x) v_{j-1}(y) dxdy. \tag{1.3}$$

The remainder of the cubature formula (1.2) can be expressed as follows

$$R_m(f) = \sum_{k=0}^{m+1} \frac{1}{k!(m+1-k)!} \cdot A_{k,m+1-k} f^{(k,m+1-k)}(\xi, \eta).$$

Developing the computation in (1.2), (1.3) we can obtain the following form for our cubature formula:

$$J(f) = \sum_{i=1}^{m+1} \sum_{j=1}^{m+1-i} C_{i,j} f(x_i, y_j) + R_m(f). \tag{1.4}$$

This formula has, in general, the degree of exactness $D = m$, but by special selections of the nodes one can increase this degree of exactness.

A necessary and sufficient condition that the degree of exactness to be $m+p$ is that

$$K_{n,s} = \int\int_{T} x^r y^s u_{i-1}(x) v_{m-i}(y) dxdy, \quad i,j = \overline{0,m}$$

vanishes if $r+s = 0, 1, \ldots, p-1$ and to be different from zero if $r+s = p$. This result can be established if we take into account the expression of the remainder $R_m(f)$.

1.3. For illustration we shall give some examples. First we introduce a notation:

$$I_{p,q} = \int\int_{T} x^p y^q dxdy,$$

where p and q are nonnegative integers.

If we take $m = 0$ then we obtain the interpolation formula

$$f(x,y) = f(x_0,y_0) + (x-x_0)\begin{bmatrix} x_0, x & ; & f \\ y_0 & & \end{bmatrix} + (y-y_0)\begin{bmatrix} x_0 & ; f \\ y_0, y & \end{bmatrix}.$$

Imposing the conditions

$$\int\int_{T} (x-x_0) dxdy = 0,$$

$$\int\int_{T} (y-y_0) dxdy = 0$$

we deduce that $x_0 = I_{1,0}/I_{0,0}$, $y_0 = I_{0,1}/I_{0,0}$ and thus we get a cubature formula of the form

$$\int\int_T f(x,y)dxdy = Af(G) + R_0(f),$$

where $G = (x_G, y_G)$ is the barycentre of the triangle T.

The remainder has an expression of the following form

$$R_0(f) = Af^{(2,0)}(\xi,\eta) + 2Bf^{(1,1)} + Cf^{(0,2)}(\xi,\eta)$$

with

$$A = \frac{1}{2}(I_{0,0}I_{2,0} - I_{1,0}^2)/I_{0,0}$$
$$B = \frac{1}{2}(I_{0,0}I_{1,1} - I_{1,0}I_{0,1})/I_{0,0}$$
$$C = \frac{1}{2}(I_{0,0}I_{0,2} - I_{0,1}^2)/I_{0,0}$$

where (ξ,η) are points from the interior of the triangle T.

1.4. If we take $m = 1$ and we determine x_1 and y_1 by imposing the conditions

$$\int\int_T (x-a)(x-x_1)dxdy = 0$$
$$\int\int_T (y-c)(y-y_1)dxdy = 0$$

we can deduce that we must have: $x_1 = \dfrac{a+b}{2}$, $y_1 = \dfrac{c+d}{2}$.

So we obtain a cubature formula of the form

$$\int\int_T f(x,y)dxdy$$
$$= \frac{(b-a)(d-c)}{6}\left[f\left(\frac{a+b}{2},c\right) + f\left(a,\frac{c+d}{2}\right) + f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right] + R_1(f),$$

where the remainder has the expression:

$$R_1(f) = \frac{(b-a)^4(d-c)}{720}f^{(3,0)}(\xi,\eta) - \frac{(b-a)^3(d-c)^2}{480}f^{(2,1)}(\xi,\eta)$$
$$-\frac{(b-a)^2(d-c)^3}{480}f^{(1,2)}(\xi,\eta) + \frac{(b-a)(d-c)^4}{720}f^{(0,3)}(\xi,\eta).$$

By starting from this cubature formula we can construct a numerical integration formula, with five nodes, for a rectangle domain $D = [a,b]\times[c,d]$, namely

$$\int\int_D f(x,y)dxdy = \frac{(b-a)(d-c)}{6}\left[f\left(a,\frac{c+d}{2}\right) + f\left(\frac{a+b}{2},c\right)\right.$$

$$+2f\left(\frac{a+b}{2},\frac{c+d}{2}\right)+f\left(\frac{a+b}{2},d\right)+f\left(b,\frac{c+d}{2}\right)\Bigg]+R(f).$$

For the remainder of this formula we are able to obtain the following estimation

$$|R(f)|\le\frac{h^2k^3}{80}\left[\frac{h}{3}M_1+\frac{k}{2}M_2\right],$$

where $h=b-a$, $k=d-c$ and $M_1=\sup_D|f^{(2,2)}(x,y)|$, $M_2=\sup_D|f^{(1,3)}(x,y)|$.

As was indicated by S.E. Mikeladze [13] this formula was obtained earlier by N.K. Artmeladze in the paper [1].

1.5. Now we establish a cubature formula for the triangle $T=T_{a,b,c}$ having the total degree of exactness equal with three. Imposing the conditions that the remainder vanishes for the monomials $e_{i,j}(x,y)=x^iy^j$ $(0\le i+j\le 3)$, one finds the equations

$$\int\int_T(x-a)(x-x_2)(x-b)dxdy=0$$

$$\int\int_T(x-a)(x-x_2)(y-c)dxdy=0$$

$$\int\int_T(x-a)(y-c)(y-y_2)dxdy=0$$

$$\int\int_T(y-c)(y-y_2)(y-d)dxdy=0$$

This system of equations has the solution $x_2=\dfrac{3x+2b}{5}$, $y_2=\dfrac{3c+2d}{5}$.

Consequently we obtain a cubature formula with six nodes, having the degree of exactness three, namely:

$$\int\int_T f(x,y)dxdy=\frac{(b-a)(d-c)}{288}\left[3f(a,c)+8f(a,d)+8f(b,c)+25f\left(a,\frac{3c+2d}{5}\right)\right.$$

$$\left.+25f\left(\frac{3a+2b}{5},c\right)+75f\left(\frac{3a+2b}{5},\frac{3c+2d}{5}\right)\right]+R_3(f),$$

where the remainder has the expression:

$$R_3(f)=\frac{1}{7200}[-h^5kf^{(4,0)}(\xi,\eta)+2h^4f^{(3,1)}(\xi,\eta)-2h^3k^3f^{(2,2)}(\xi,\eta)$$

$$+2h^2k^4f^{(1,3)}(\xi,\eta)-hk^5f^{(0,4)}(\xi,\eta)].$$

If we choose $m = 3$ and we want to obtain a cubature formula of degree of exactness four, it is necessary and sufficient to be satisfied the conditions:

$$\int\int_T (x-a)(x-x_2)(x-x_3)(x-b)dxdy = 0$$

$$\int\int_T (x-a)(x-x_2)(x-x_3)(y-c)dxdy = 0$$

$$\int\int_T (x-a)(x-x_2)(y-c)(y-y_2)dxdy = 0$$

$$\int\int_T (x-a)(y-c)(y-y_2)(y-y_3)dxdy = 0$$

$$\int\int_T (y-c)(y-y_2)(y-y_3)(y-d)dxdy = 0$$

This system of equations has the following solution:

$$x_2 = \frac{7a+2b}{9}, \quad x_3 = \frac{3a+5b}{8}, \quad y_2 = \frac{3c+d}{4}, \quad y_3 = \frac{c+2d}{3}$$

By using it we arrive at the following cubature formula with ten nodes and degree of exactness four:

$$\int\int_T f(x,y)dxdy \cong \frac{(b-a)(d-c)}{4384800}\Bigg[34713f(a,c) + 83835f\left(\frac{7a+2b}{9}, c\right)$$

$$+77952f\left(a, \frac{3c+d}{4}\right) + 172032f\left(\frac{3a+5b}{8}, c\right) + 653184f\left(\frac{7a+2b}{9}, \frac{3c+d}{4}\right)$$

$$+147987f\left(a, \frac{c+2d}{3}\right) + 38280f(b,c) + 516096f\left(\frac{3a+5b}{8}, \frac{c+2d}{4}\right)$$

$$+443961f\left(\frac{7a+2b}{9}, \frac{c+2d}{3}\right) + 24360f(a,d)\Bigg]$$

Now we present a cubature formula more simple:

$$\int\int_T f(x,y)dxdy \cong \Big\{\frac{hk}{45} - \{8[f(a-2h,b) + f(a,b-2k) + f(a,b)]$$

$$+32[f(a-h,b-2k) + f(a-2h,b-k) + f(a+h,b-2k) + f(a-2h,b+k)$$

$$+f(a+h,b-k) + f(a-h,b+k)] + 64[f(a-h,b) - f(a,b-k) + f(a-h,b-k)]\}\Big\},$$

1.6. By using a similar procedure we can construct formulas of global degree of exactness equal with five. For this purpose we have to resolve the equations:

$$\int\int_T (x-a)(x-x_2)(x-x_3)(x-x_4)(x-b)dxdy = 0$$

$$\int\int_T (x-a)(x-x_2)(x-x_3)(x-x_4)(y-c)dxdy = 0$$
$$\int\int_T (x-a)(x-x_2)(x-x_3)(y-c)(y-y_2)dxdy = 0$$
$$\int\int_T (x-a)(x-x_2)(y-c)(y-y_2)(y-y_3)dxdy = 0$$
$$\int\int_T (x-a)(y-c)(y-y_2)(y-y_3)(y-y_4)dxdy = 0$$
$$\int\int_T (y-c)(y-y_2)(y-y_3)(y-y_4)(y-d)dxdy = 0$$

with six unknown quantities. We mention that only four of these equations are distinct.

Now we can take advantage of having two degree of liberty and to construct a cubature formula using 13 nodes and having the degree of exactness equal with five, but we prefer to establish a cubature formula with 15 nodes represented by rational numbers.

By using the following solution of the preceding system of equations:

$$x_2 = \frac{3a+b}{2}, \quad x_3 = \frac{5a+2b}{7}, \quad x_4 = \frac{3a+5b}{8}$$
$$I_2 = \frac{3c+d}{4}, \quad I_3 = \frac{5c+2d}{7}, \quad I_4 = \frac{3c+5d}{8},$$

we obtain the following cubature formula with 15 nodes:

$$\int\int_T f(x,y)dxdy \cong \frac{(b-a)(d-c)}{2872800}\left[-11571f(a,c) + 493696f\left(\frac{3a+b}{4}, c\right)\right.$$
$$+493696f\left(a, \frac{3c+d}{4}\right) - 424977f\left(\frac{5a+2b}{7}, c\right) - 424977f\left(a, \frac{5c+2b}{7}\right)$$
$$-4085760f\left(\frac{3a+b}{4}, \frac{3c+d}{4}\right) + 211456f\left(\frac{3a+5b}{8}, c\right) + 211456f\left(a, \frac{3c+5d}{8}\right)$$
$$+211456f\left(\frac{3a+5b}{8}, c\right) + 211456f\left(a, \frac{3c+5d}{8}\right) + 4609920f\left(\frac{5a+2b}{7}, \frac{3c+d}{4}\right)$$
$$+4609920f\left(\frac{3a+b}{4}, \frac{5c+2d}{7}\right) + 24472f(b,c) + 24472f(a,d)$$
$$+247296f\left(\frac{3a+5b}{8}, \frac{3c+d}{4}\right) + 247296f\left(\frac{3a+b}{4}, \frac{3c+5d}{8}\right)$$
$$\left.-4789995f\left(\frac{5a+2b}{7}, \frac{5c+2d}{7}\right)\right].$$

2. **Use of a method of D.V. Ionescu for numerical evaluation double integrals over an arbitrary triangular domain**

2.1. In the paper [12] D.V. Ionescu has given a method for construction certain cubature formulas for an arbitrary triangular domain D, with vertices $A(x_1, y_1)$, $B(x_2, y_2)$, $C(x_3, y_3)$.

One denotes by L, M and N the barycentres of the masses $(\alpha, 1, 1)$, $(1, \alpha, 1)$, $(1, 1, \alpha)$ situated in the vertices A, B, C of triangle D. The new triangle LMN is homotethic with the triangle ABC. Giving to α the real values $\alpha_1, \alpha_2, \ldots, \alpha_n$ one obtains the nodes L_i, M_i, N_i $(i = \overline{1, n})$. In the paper [12] were considered cubature formulas with the fixed nodes L_i, M_i, N_i of the following form:

$$\int\int_D f(x, y)dxdy = \sum_{i=1}^{n} A_i[f(L_i) + f(M_i) + f(N_i)]. \tag{2.1}$$

The coefficients A_i will be determined so that this cubature formula to have the degree of exactness equal with n.

It was proved that this problem is possible if and only if $n \leq 5$.

One observes that in the special case $\alpha = 1$ and $n = 1$, one obtains the cubature formula

$$\int\int_D f(x, y)dxdt = Sf(G),$$

where G is the barycentre of the triangle T, while S is the area of this triangle.

For $n = 2$ and $\alpha_2 = 1$ we get the cubature formula

$$\int\int_D f(x, y)dxdy = \frac{S}{12}\left\{\frac{(\alpha_1 + 2)^2}{(\alpha_1 - 1)^2}[f(L_1) + f(M_1) + f(N_1)] + 9\frac{\alpha_1^2 - 4a_1}{(\alpha_1 - 1)^2}f(G)\right\}.$$

If $\alpha_1 = 0$ we obtain the cubature formula:

$$\int\int_D f(x, y)dxdy = \frac{S}{3}[f(A') + f(B') + f(C')],$$

where A', B', C' are the middles of the sides of the triangle T.

In the case $n = 3$, $\alpha_2 = 3$ and $\alpha_3 = 1$ we find the cubature formula:

$$\int\int_T f(x, y)dxdy = \frac{S}{48}\{25[f(L) + f(M) + f(N)] - 27f(G)\},$$

where G is the barycenter of the masses (3,1,1) placed in the vertices of the triangle D.

It should be mentioned that the above cubature formulas can be used for the numerical calculation of an integral extended to a polygonal domain, since it can be decomposed in triangles and then we can apply these particular cubature formulas.

The above cubature formulas can be extended to three or more dimensions.

We mention that Hortensia Roşcău [15] has proved that the problem is possible only for $n \leq 3$ in the case when the masses are distinct.

3. Some recent methods for numerical calculation of a double integral over a triangular domain

3.1. One can make numerical evaluation of a double integral extended over a triangular domain T, having the vertices (0,0), (0,1) and (1,0) using as basic domain the square $D = [0,1] \times [0,1]$.

Let T and D be related to each other by means of the transformations $x = g(u,v)$, $y = h(u,v)$.

It will be assumed that g and h have continuous partial derivatives and that the Jacobian $J(u,v)$ does not vanish in D. We have:

$$I(f) = \int\int_T f(x,y)dxdy = \int\int_D f(g(u,v),h(u,v))J(u,v)dudv$$

For $g = xu$, $h = x(1-u)$ we have $J(u,v) = x$ and the integral $I(f)$ becomes:

$$I(f) = \int_0^1 \int_0^1 xf(xu, x(1-u))dxdy \tag{3.1}$$

3.2. Several classes of numerical integration formulas can be obtained by products of Gauss-Jacobi quadrature formulas based on the transformation (3.1). In the paper [11] by Hillion Pierre [11] has been described some applications of this transformation, including one to the solution of a Dirichlet problem using the finite element method.

3.3. Since for any triangle from the plane xOy there exists an affine transformation which leads to the standard triangle:

$$T_h = \{(x,y) \in \mathbb{R}^2;\ x \geq 0,\ y \geq 0,\ x+y \leq h,\ h \in \mathbb{R}_+\},$$

Gh. Coman has considered and investigated, in the paper [6] (see also the book [7]), the so called homogeneous cubature formulas of interpolation type for the triangle T_h, characterized by the fact that each term of the remainder has the same order of approximation.

A simple example of such a cubature formula is represented by

$$\int\int_{T_h} f(x,y)dxdy = \frac{h^2}{120}\Bigg[3f(0,0)+3f(h,0)+3f(0,h)$$

$$+8f\left(\frac{h}{2},0\right)+8f\left(\frac{h}{2},\frac{h}{2}\right)+8f\left(0,\frac{h}{2}\right)+27f\left(\frac{h}{3},\frac{h}{3}\right)\Bigg]+R_3(f),$$

having the degree of exactness equal with three.

The remainder can be evaluated by using the partial derivatives of the function f of order (4,0), (3,1), (1,3), (0,4) and (2,2).

Some new homogeneous cubature formulas for a triangular domains were investigated in the recent paper [14] by I. Pop-Purdea.

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SOME PROBLEMS ON OPTIMAL QUADRATURE

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Dedicated to Professor D. D. Stancu on his 80^{th} *birthday*

Abstract. Using the connection between optimal approximation of linear operators and spline interpolation established by I. J. Schoenberg [35], the φ-function method of D. V. Ionescu [17], and a more general method given by A. Ghizzetti and A. Ossicini [14], the one-to-one correspondence between the monosplines and quadrature formulas given by I. J. Schoenberg [36, 37], and the minimal norm property of orthogonal polynomials, the authors study optimal quadrature formulas in the sense of Sard [33] and in the sense of Nikolski [27], respectively, with respect to the error criterion. Many examples are given.

1. **Introduction**

Optimal quadrature rules with respect to some given criterion represent an important class of quadrature formulas.

The basic optimality criterion is the error criterion. More recently the efficiency criterion has also been used, which is based on the approximation order of the quadrature rule and its computational complexity.

Next, the error criterion will be used.

Let

$$\Lambda = \left\{\lambda_i \mid \lambda_i \colon H^{m,2}\,[a,b] \to \mathbb{R},\ i=\overline{1,N}\right\} \tag{1.1}$$

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be a set of linear functionals and for $f \in H^{m,2}[a,b]$, let

$$\Lambda(f) = \left\{ \lambda_i(f) \mid i = \overline{1,N} \right\} \tag{1.2}$$

be the set of information on f given by the functionals of Λ.

Remark 1.1. *Usually, the information $\lambda_i(f)$, $i = \overline{1,N}$, are the pointwise evaluations of f or some of its derivatives at distinct points $x_i \in [a,b]$, $i = \overline{0,n}$, i.e. the pointwise information.*

For $f \in H^{m,2}[a,b]$, one considers the quadrature formula

$$\int_a^b w(x) f(x)\, dx = Q_N(f) + R_N(f), \tag{1.3}$$

where

$$Q_N(f) = \sum_{i=1}^{N} A_i \lambda_i(f),$$

$R_N(f)$ is the remainder, w is a weight function and $\boldsymbol{A} = (A_1, \ldots, A_N)$ are the coefficients. If $\lambda_i(f)$, $i = \overline{1,N}$, represent pointwise information, then $\boldsymbol{X} = (x_0, \ldots, x_n)$ are the quadrature nodes.

Definition 1.1. *The number $r \in \mathbb{N}$ with the property that $Q_N(f) = f$ (or $R_N(f) = 0$) for all $f \in \mathcal{P}_r$ and that there exists $g \in \mathcal{P}_{r+1}$ such that $Q_N(g) \neq g$, (or $R_N(g) \neq 0$) where $\mathcal{P}_s$ is the set of polynomial functions of degree at most s, is called* the degree of exactness *of the quadrature rule Q_N (quadrature formula* (1.3)*) and is denoted by* $\operatorname{dex}(Q_N)$ *($\operatorname{dex}(Q_N) = r$).*

The problem with a quadrature formula is to find the quadrature parameters (coefficients and nodes) and to evaluate the corresponding remainder (error).

Let

$$E_N(f, \boldsymbol{A}, \boldsymbol{X}) = |R_N(f)|$$

be the quadrature error.

Definition 1.2. *If for a given $f \in H^{m,2}[a,b]$, the parameters $\boldsymbol{A}$ and $\boldsymbol{X}$ are found from the conditions that $E_N(f, \boldsymbol{A}, \boldsymbol{X})$ takes its minimum value, then the quadrature formula is called* locally optimal *with respect to the error.*

If $\boldsymbol{A}$ *and* $\boldsymbol{X}$ *are obtained such that*

$$E_N\left(H^{m,2}[a,b], \boldsymbol{A}, \boldsymbol{X}\right) = \sup_{f \in H^{m,2}[a,b]} E_N(f, \boldsymbol{A}, \boldsymbol{X})$$

takes its minimum value, the quadrature formula is called globally optimal *on the set* $H^{m,2}[a,b]$, *with respect to the error.*

Remark 1.2. *Some of the quadrature parameters can be fixed from the beginning. Such is the case, for example, with quadrature formulas with uniformly spaced nodes or with equal coefficients. Also, the quadrature formulas with a prescribed degree of exactness are frequently considered.*

Subsequently we will study the optimality problem for some classes of quadrature formulas with pointwise information $\lambda_i(f)$, $i = \overline{1, N}$.

2. **Optimality in the sense of Sard**

Suppose that Λ is a set of Birkhoff-type functionals

$$\Lambda := \Lambda_B = \left\{ \lambda_{kj} \mid \lambda_{kj}(f) = f^{(j)}(x_k),\ k = \overline{0, n},\ j \in I_k \right\},$$

where $x_k \in [a,b]$, $k = \overline{0,n}$, and $I_k \subset \{0, 1, \ldots, r_k\}$, with $r_k \in \mathbb{N}$, $r_k < m$, $k = \overline{0,n}$.

For $f \in H^{m,2}[a,b]$ and for fixed nodes $x_k \in [a,b]$, $k = \overline{0,n}$, (for example, uniformly spaced nodes), consider the quadrature formula

$$\int_a^b f(x)\,dx = \sum_{k=0}^{n} \sum_{j \in I_k} A_{kj} f^{(j)}(x_k) + R_N(f). \tag{2.1}$$

Definition 2.1. *The quadrature formula* (2.1) *is said to be* optimal in the sense of Sard *if*

(i) $R_N(e_i) = 0, \quad i = \overline{0, m-1}, \quad$ with $e_i(x) = x^i$,

(ii) $\displaystyle\int_a^b K_m^2(t)\,dt \quad$ is minimum,

where K_m *is Peano's kernel, i.e.*

$$K_m(t) := R_N\left[\frac{(\cdot - t)_+^{m-1}}{(m-1)!}\right] = \frac{(b-t)^m}{m!} - \sum_{k=0}^{n} \sum_{j \in I_k} A_{kj} \frac{(x_k - t)_+^{m-j-1}}{(m-j-1)!}.$$

Such formulas for uniformly spaced nodes ($x_k = a + kh$, $h = (b-a)/n$) and for Lagrange-type functionals, $\lambda_k(f) = f(x_k)$, $k = \overline{0,n}$, were first studied by A. Sard [32] and L. S. Meyers and A. Sard [24], respectively.

In 1964, I. J. Schoenberg [34, 35] has established a connection between optimal approximation of linear operators (including definite integral operators) and spline interpolation operators. For example, if S is the natural spline interpolation operator with respect to the set Λ and

$$f = Sf + Rf$$

is the corresponding spline interpolation formula, then the quadrature formula

$$\int_a^b f(x)\,dx = \int_a^b (Sf)(x)\,dx + \int_a^b (Rf)(x)\,dx \tag{2.2}$$

is optimal in the sense of Sard.

More specifically, let us suppose that the uniqueness condition of the spline operator is satisfied and that

$$(Sf)(x) = \sum_{k=0}^{n} \sum_{j \in I_k} s_{kj}(x)\, f^{(j)}(x_k),$$

where s_{kj}, $k = \overline{0,n}$, $j \in I_k$, are the cardinal splines and S is the corresponding spline operator. Then the optimal quadrature formula (2.2) becomes

$$\int_a^b f(x)\,dx = \sum_{k=0}^{n} \sum_{j \in I_k} A_{kj}^{\star} f^{(j)}(x_k) + R^{\star}(f),$$

with

$$A_{kj}^{\star} = \int_a^b s_{kj}(x)\,dx, \quad k = \overline{0,n}, \quad j \in I_k,$$

and

$$R^{\star}(f) = \int_a^b (Rf)(x)\,dx.$$

Example 2.1. *Let $f \in H^{2,2}[0,1]$ and let the set of Birkhoff-type functionals $\Lambda_B(f) = \left\{f'(0), f\left(\frac{1}{4}\right), f\left(\frac{3}{4}\right), f'(1)\right\}$ be given. Also, let*

$$\begin{aligned}(S_4 f)(x) = s_{01}(x) f'(0) + s_{10}(x) f\left(\frac{1}{4}\right) \\ + s_{20}(x) f\left(\frac{3}{4}\right) + s_{31}(x) f'(1),\end{aligned}$$

be the corresponding cubic spline interpolation function, where s_{01}, s_{10}, s_{20} and s_{31} are the cardinal splines. For the cardinal splines, we have

$$\begin{aligned}s_{01}(x) = & -\frac{11}{64} + x - \frac{5}{4}(x-0)_+^2 + \left(x - \frac{1}{4}\right)_+^3 - \left(x - \frac{3}{4}\right)_+^3 \\ & - \frac{1}{4}(x-1)_+^2, \\ s_{10}(x) = & \frac{19}{16} - 3(x-0)_+^2 + 4\left(x - \frac{1}{4}\right)_+^3 - 4\left(x - \frac{3}{4}\right)_+^3 \\ & - 3(x-1)_+^2, \\ s_{20}(x) = & -\frac{3}{16} + 3(x-0)_+^2 - 4\left(x - \frac{1}{4}\right)_+^3 + 4\left(x - \frac{3}{4}\right)_+^3 \\ & + 3(x-1)_+^2, \\ s_{31}(x) = & \frac{1}{64} - \frac{1}{4}(x-0)_+^2 + \left(x - \frac{1}{4}\right)_+^3 - \left(x - \frac{3}{4}\right)_+^3 - \frac{5}{4}(x-1)_+^2,\end{aligned}$$

while the remainder is

$$(R_4 f)(x) = \int_0^1 \varphi_2(x,t) f''(t)\, dt,$$

with

$$\varphi_2(x,t) = (x-t)_+ - \left(\frac{1}{4} - t\right)_+ s_{10}(x) - \left(\frac{3}{4} - t\right)_+ s_{20}(x) - s_{31}(x).$$

It follows that the optimal quadrature formula is given by

$$\int_0^1 f(x)\, dx = A_{01}^\star f'(0) + A_{10}^\star f\left(\frac{1}{4}\right) + A_{20}^\star f\left(\frac{3}{4}\right) + A_{31}^\star f'(1) + R_4^\star(f),$$

where

$$A_{01}^\star = -\frac{1}{96}, \quad A_{10}^\star = \frac{1}{2}, \quad A_{20}^\star = \frac{1}{2}, \quad A_{31}^\star = \frac{1}{96},$$

and

$$R_4^{\star}(f) = \int_0^1 K_2^{\star}(t)\, f''(t)\, dt,$$

with

$$\begin{aligned} K_2^{\star}(t) &:= \int_0^1 \varphi_2(x,t)\, dx \\ &= \frac{1}{2}(1-t)^2 - \frac{1}{2}\left(\frac{1}{4}-t\right)_+ - \frac{1}{2}\left(\frac{3}{4}-t\right)_+ - \frac{1}{96}\,. \end{aligned}$$

Finally, we have

$$\left|R_4^{\star}(f)\right| \leq \|f''\|_2 \left(\int_0^1 \left[K_2^{\star}(t)\right]^2 dt\right)^{\frac{1}{2}}$$

i.e.

$$\left|R_4^{\star}(f)\right| \leqslant \frac{1}{48\sqrt{5}}\, \|f''\|_2\,.$$

3. **Optimality in the sense of Nikolski**

Suppose now, that all the parameters of the quadrature formula (2.1) (the coefficients $\boldsymbol{A}$ and the nodes $\boldsymbol{X}$) are unknown.

The problem is to find the coefficients $\boldsymbol{A}^{\star}$ and the nodes $\boldsymbol{X}^{\star}$ such that

$$E_n(f, \boldsymbol{A}^{\star}, \boldsymbol{X}^{\star}) = \min_{\boldsymbol{A},\boldsymbol{X}} E_n(f, \boldsymbol{A}, \boldsymbol{X})$$

for local optimality, or

$$E_n\left(H^{m,2}[a,b], \boldsymbol{A}^{\star}, \boldsymbol{X}^{\star}\right) = \min_{\boldsymbol{A},\boldsymbol{X}} \sup_{f\in H^{m,2}[a,b]} E_n(f, \boldsymbol{A}, \boldsymbol{X})$$

in the global optimality case.

Definition 3.1. *The quadrature formula with the parameters $\boldsymbol{A}^{\star}$ and $\boldsymbol{X}^{\star}$ is called* optimal in the sense of Nikolski *and $\boldsymbol{A}^{\star}$, $\boldsymbol{X}^{\star}$ are called* optimal coefficients *and* optimal nodes*, respectively.*

Remark 3.1. *If $f \in H^{m,2}[a,b]$ and the degree of exactness of the quadrature formula* (2.1) *is $r-1$ ($r < m$) then by Peano's theorem, one obtains*

$$R_N(f) = \int_a^b K_r(t)\, f^{(r)}(t)\, dt, \tag{3.1}$$

where

$$K_r(t) = \frac{(b-t)^r}{r!} - \sum_{k=0}^{n} \sum_{j \in I_k} A_{kj} \frac{(x_k - t)_+^{r-j-1}}{(r-j-1)!} \cdot$$

From (3.1), *one obtains*

$$|R_N(f)| \leqslant \left\| f^{(r)} \right\|_2 \left(\int_a^b K_r^2(t)\, dt \right)^{\frac{1}{2}} . \tag{3.2}$$

It follows that the optimal parameters $\boldsymbol{A}^\star$ and $\boldsymbol{X}^\star$ are those which minimize the functional

$$F(\boldsymbol{A}, \boldsymbol{X}) = \int_a^b K_r^2(t)\, dt.$$

There are many ways to find the functional F.

1. One of them is described above and is based on Peano's theorem.

Remark 3.2. *In this case, the quadrature formula is assumed to have degree of exactness* $r-1$.

2. Another approach is based on the φ-function method [17].

Suppose that $f \in H^{r,2}[a,b]$ and that $a = x_0 < \ldots < x_n = b$. On each interval $[x_{k-1}, x_k]$, $k = \overline{1,n}$, consider a function φ_k, $k = \overline{1,n}$,, with the property that

$$D^r \varphi_k := \varphi_k^{(r)} = 1, \quad k = \overline{1,n}. \tag{3.3}$$

We have

$$\int_a^b f(x)\, dx = \sum_{k=1}^{n} \int_{x_{k-1}}^{x_k} \varphi_k^{(r)}(x) f(x)\, dx.$$

Using the integration by parts formula, one obtains

$$\int_a^b f(x)\, dx = \sum_{k=1}^{n} \left\{ \left[\varphi_k^{(r-1)} f - \varphi_k^{(r-2)} f' + \cdots + (-1)^{r-1} \varphi_k f^{(r-1)} \right] \Bigg|_{x_{k-1}}^{x_k} + (-1)^r \int_{x_{k-1}}^{x_k} \varphi_k(x) f^{(r)}(x)\, dx \right\}$$

and subsequently,

$$
\begin{aligned}
\int_a^b f(x)dx &= \sum_{j=1}^{r} (-1)^j \varphi_1^{(r-j)}(x_0) f^{(j-1)}(x_0) \\
&+ \sum_{k=1}^{n-1}\sum_{j=1}^{r} (-1)^{j-1} (\varphi_k - \varphi_{k+1})^{(r-j)}(x_k) f^{(j-1)}(x_k) \\
&+ \sum_{j=1}^{r} (-1)^{j-1} \varphi_n^{(r-j)}(x_n) f^{(j-1)}(x_n) \\
&+ (-1)^r \int_a^b \varphi(x) f^{(r)}(x)\, dx,
\end{aligned}
\tag{3.4}
$$

where

$$
\varphi\Big|_{[x_{k-1},x_k]} = \varphi_k, \quad k = \overline{1,n}. \tag{3.5}
$$

For

$$
\begin{aligned}
(-1)^j \varphi_1^{(r-j)}(x_0) &= \begin{cases} A_{0j}, & j \in I_0, \\ 0, & j \in J_0, \end{cases} \\
(-1)^{j-1} (\varphi_k - \varphi_{k+1})^{(r-j)}(x_k) &= \begin{cases} A_{kj}, & j \in I_k, \\ 0, & j \in J_k, \end{cases} \\
(-1)^{j-1} \varphi_n^{(r-j)}(x_n) &= \begin{cases} A_{nj}, & j \in I_n, \\ 0, & j \in J_n, \end{cases}
\end{aligned}
\tag{3.6}
$$

with $J_k = \{0, 1, \ldots, r_k\} \setminus I_k$, formula (3.4) becomes the quadrature formula (2.1), with the remainder

$$
R_N(f) = (-1)^r \int_a^b \varphi(x) f^{(r)}(x)\, dx.
$$

It follows that

$$
K_r = (-1)^r \varphi
$$

and

$$
F(\boldsymbol{A}, \boldsymbol{X}) = \int_a^b \varphi^2(x)\, dx.
$$

Remark 3.3. *From* (3.3), *it follows that* φ_k *is a polynomial of degree* r : $\varphi_k(x) = \frac{x^r}{r!} + P_{r-1,k}(x)$, *with* $P_{r-1,k} \in \mathcal{P}_{r-1}$, $k = \overline{1,n}$, *satisfying the conditions of* (3.6).

Example 3.1. *Let* $f \in H^{2,2}[0,1]$, $\Lambda(f) = \left\{ f(x_k) \mid k = \overline{0,n} \right\}$, *with* $0 = x_0 < x_1 < \cdots < x_{n-1} < x_n = 1$, *and let*

$$\int_0^1 f(x)\,dx = \sum_{k=0}^{n} A_k f(x_k) + R_n(f) \tag{3.7}$$

be the corresponding quadrature formula. Find the functional $F(\boldsymbol{A}, \boldsymbol{X})$, *where* $\boldsymbol{A} = (A_0, \ldots, A_n)$ *and* $\boldsymbol{X} = (x_0, \ldots, x_n)$.

Using the φ*-function method, on each interval* $[x_{k-1}, x_k]$ *one considers a function* φ_k, *with* $\varphi_k'' = 1$.

Formula (3.4) *becomes*

$$\begin{aligned}\int_0^1 f(x)\,dx = &- \varphi_1'(0) f(0) + \sum_{k=1}^{n-1} \left(\varphi_k' - \varphi_{k+1}'\right)(x_k) f(x_k) \\ &+ \varphi_n'(1) f(1) + \varphi_1(0) f'(0) \\ &- \sum_{k=1}^{n-1} (\varphi_k - \varphi_{k+1})(x_k) f'(x_k) - \varphi_n(1) f'(1) \\ &+ \int_0^1 \varphi(x) f''(x)\,dx.\end{aligned} \tag{3.8}$$

Now, for

$$\begin{aligned}&\varphi_1'(0) = -A_0,\ \left(\varphi_k' - \varphi_{k+1}'\right)(x_k) = A_k,\ k = \overline{1,n-1},\ \varphi_n'(1) = A_n, \\ &\varphi_1(0) = 0, \qquad \varphi_k(x_k) = \varphi_{k+1}(x_k),\ k = \overline{1,n-1},\ \varphi_n(1) = 0,\end{aligned} \tag{3.9}$$

formula (3.8) *becomes the quadrature formula of* (3.7).

From the conditions $\varphi_k'' = 1$, $k = \overline{1,n}$, *and using* (3.9), *it follows that*

$$\begin{aligned}
\varphi_1(x) &= \frac{x^2}{2} - A_0 x, \\
\varphi_2(x) &= \frac{x^2}{2} - A_0 x - A_1(x - x_1), \\
&\vdots \\
\varphi_n(x) &= \frac{x^2}{2} - A_0 x - A_1(x - x_1) - \cdots - A_{n-1}(x - x_{n-1}).
\end{aligned}$$

Finally, we have

$$F(\boldsymbol{A}, \boldsymbol{X}) = \int_0^1 \varphi^2(x)\,dx = \sum_{k=1}^{n} \int_{x_{k-1}}^{x_k} \varphi_k^2(x)\,dx$$

or

$$F(\boldsymbol{A}, \boldsymbol{X}) = \sum_{k=1}^{n} \int_{x_{k-1}}^{x_k} \left[\frac{x^2}{2} - x\sum_{i=0}^{k-1} A_i + \sum_{i=1}^{k-1} A_i x_i\right]^2 dx.$$

Remark 3.4. *A generalization of the* φ*-function method was given in the book of A. Ghizzetti and A. Ossicini* [14], *where a more general linear differential operator of order* r *is used instead of the differential operator* D^r.

3. A third method was given by I. J. Schoenberg [36, 37, 38] and it uses the one-to-one correspondence between the set of so called monosplines

$$M_r(x) = \frac{x^r}{r!} + \sum_{k=0}^{n} \sum_{j \in I_k} A_{kj} (x - x_k)_+^j$$

and the set of quadrature formulas of the form (2.1), with degree of exactness $r - 1$.

The one-to-one correspondence is described by the relations

$$\begin{aligned}
A_{0j} &= (-1)^{j+1} M_r^{(r-j-1)}(x_0), \quad j \in I_0, \\
A_{kj} &= (-1)^j \left[M_r^{(r-j-1)}(x_k - 0) - M_r^{(r-j-1)}(x_k + 0)\right], k = \overline{1, n-1}, j \in I_k, \\
A_{nj} &= (-1)^{j+1} M_r^{(r-j-1)}(x_n), \quad j \in I_n,
\end{aligned}$$

and the remainder is given by

$$R_N(f) = (-1)^r \int_a^b M_r(x) f^{(r)}(x)\,dx.$$

So

$$F(\boldsymbol{A}, \boldsymbol{X}) = \int_a^b M_r^2(x)\,dx.$$

In fact, there is a close relationship between monosplines and φ-functions.

Remark 3.5. *One of the advantages of the φ-function method is that the* degree of exactness *condition is not necessary, it follows from the remainder representation*

$$R_N(f) = (-1)^r \int_a^b \varphi(x) f^{(r)}(x)\,dx.$$

3.1. **Solutions for the optimality problem.** In order to obtain an optimal quadrature formula, in the sense of Nikolski, we have to minimize the functional $F(\boldsymbol{A}, \boldsymbol{X})$.

1. A two-step procedure

First step. The functional $F(\boldsymbol{A}, \boldsymbol{X})$ is minimized with respect to the coefficients, the nodes being considered fixed. For this, we use the relationship with spline interpolation.

So let

$$f = Sf + Rf$$

be the spline interpolation formula with $\boldsymbol{X} = (x_0, \ldots, x_n)$ the interpolation nodes. If

$$(Sf)(x) = \sum_{k=0}^{n} \sum_{j \in I_k} s_{kj}(x) f^{(j)}(x_k)$$

is the interpolation spline function, then

$$\overline{A}_{kj} := \overline{A}_{kj}(x_0, \ldots, x_n) = \int_a^b s_{kj}(x)\,dx, \quad k = \overline{0, n}, \;\; j \in I_k,$$

are the corresponding optimal (in the sense of Sard) coefficients for the fixed nodes $\boldsymbol{X}$ and

$$\overline{R}_N(f) = \int_a^b (Rf)(x)\,dx$$

is the remainder. So

$$\overline{R}_N(f) = \int_a^b \overline{K}_r(t)\, f^{(r)}(t)\, dt,$$

with

$$\overline{K}_r(t) = \frac{(b-t)^r}{r!} - \sum_{k=0}^{n} \sum_{j \in I_k} \overline{A}_{kj} \frac{(x_k - t)_+^{r-j-1}}{(r-j-1)!}.$$

Second step. The functional

$$F\left(\overline{\boldsymbol{A}}, \boldsymbol{X}\right) := \int_a^b \overline{K}_r^2(t)\, dt$$

is minimized with respect to the nodes $\boldsymbol{X}$.

Let $\boldsymbol{X}^\star = (x_0^\star, \ldots, x_n^\star)$ be the minimum point of $F\left(\overline{\boldsymbol{A}}, \boldsymbol{X}\right)$, i.e. the optimal nodes of the quadrature formula. It follows that $A_{kj}^\star := \overline{A}_{kj}(x_0^\star, \ldots, x_n^\star)$, $k = \overline{0,n}$, $j \in I_k$, are the optimal coefficients and that

$$R_N^\star(f) = \int_a^b K_r^\star(t)\, f^{(r)}(t)\, dt,$$

with

$$K_r^\star(t) = \frac{(b-t)^r}{r!} - \sum_{k=0}^{n} \sum_{j \in I_k} A_{kj}^\star \frac{(x_k^\star - t)_+^{r-j-1}}{(r-j-1)!},$$

is the optimal error. We also have

$$\left|R_N^\star(f)\right| \leqslant \left\|f^{(r)}\right\|_2 \left(\int_a^b \left[K_r^\star(t)\right]^2 dt\right)^{\frac{1}{2}}.$$

Example 3.2. *For* $f \in H^{2,2}[0,1]$ *and* $\Lambda_B = \{f'(0), f(x_1), f'(x_1), f'(1)\}$, *with* $x_1 \in (0,1)$, *find the quadrature formula of the type*

$$\int_0^1 f(x)\, dx = A_{01} f'(0) + A_{10} f(x_1) + A_{11} f'(x_1) + A_{21} f'(1) + R_3(f)$$

that is optimal in the sense of Nikolski, i.e. find the optimal coefficients

$$\boldsymbol{A}^\star = (A_{01}^\star, A_{10}^\star, A_{11}^\star, A_{21}^\star)$$

and the optimal nodes $\boldsymbol{X}^\star = (0, x_1^\star, 1)$.

First step. *The spline interpolation formula is given by*

$$f(x) = s_{01}(x) f'(0) + s_{10}(x) f(x_1) + s_{11}(x) f'(x_1) + s_{21}(x) f'(1) + (R_4 f)(x),$$

where

$$s_{01}(x) = -\frac{x_1}{2} + x - \frac{(x-0)_+^2}{2x_1} + \frac{(x-1)_+^2}{2x_1},$$
$$s_{10}(x) = 1,$$
$$s_{11}(x) = -\frac{x_1}{2} + \frac{(x-0)_+^2}{2x_1} - \frac{(x-x_1)_+^2}{2x_1(1-x_1)} - \frac{(x-1)_+^2}{1-x_1},$$
$$s_{21}(x) = \frac{(x-x_1)_+^2}{2(1-x_1)} - \frac{(x-1)_+^2}{2(1-x_1)}.$$

It follows that

$$\overline{A}_{01} = -\frac{x_1^2}{6},\ \overline{A}_{10} = 1,\quad \overline{A}_{11} = \frac{1-2x_1}{3},\ \overline{A}_{21} = \frac{(1-x_1)^2}{6}, \tag{3.10}$$

$$\overline{K}_2(t) = \frac{(1-t)^2}{2} - (x_1-t)_+ - \frac{1-2x_1}{3}(x_1-t)_+^0 - \frac{(1-x_1)^2}{6}$$

and

$$F\left(\overline{\boldsymbol{A}}, \boldsymbol{X}\right) = \int_0^1 \overline{K}_2^2(t)\,dt = \frac{1}{45} - \frac{1}{9}x_1(1-x_1)\left(1-x_1+x_1^2\right).$$

Second step. *We have to minimize* $F\left(\overline{\boldsymbol{A}}, \boldsymbol{X}\right)$ *with respect to* x_1. *From the equation*

$$\frac{\partial F\left(\overline{\boldsymbol{A}}, \boldsymbol{X}\right)}{\partial x_1} = -\frac{1}{9}(1-2x_1)\left[x_1^2 + (1-x_1)^2\right] = 0,$$

we obtain $x_1 = \frac{1}{2}$. *Also,* (3.10) *implies that*

$$A_{01}^\star = -\frac{1}{24},\quad A_{10}^\star = 1,\quad A_{11}^\star = 0,\quad A_{21}^\star = \frac{1}{24}.$$

Finally, we have

$$\left|R_4^\star(f)\right| \leqslant \left\|f''\right\|_2 \left(\int_0^1 \left[K_2^\star(t)\right]^2 dt\right)^{\frac{1}{2}} = \frac{1}{12\sqrt{5}}\left\|f''\right\|_2.$$

3.2. **Minimal norm of orthogonal polynomials.** Let $\widetilde{\mathcal{P}}_n \subset \mathcal{P}_n$ be the set of polynomials of degree n with leading coefficient equal to one. If $P_n \in \widetilde{\mathcal{P}}_n$ and $P_n \perp \mathcal{P}_{n-1}$ on $[a,b]$ with respect to the weight function w, then

$$\|P_n\|_{w,2} = \min_{P \in \widetilde{\mathcal{P}}_n} \|P\|_{w,2}\,,$$

where

$$\|P\|_{w,2} = \left(\int_a^b w(x)\, P^2(x)\, dx \right)^{\frac{1}{2}}.$$

It follows that the parameters of the functional $F(\boldsymbol{A}, \boldsymbol{X})$ can be determined such that the restriction of the kernel K_r to the interval $[x_{k-1}, x_k]$ is identical to the orthogonal polynomial on the same interval with respect to the corresponding weight function.

Example 3.3. *Consider the functional of Example 3.1*

$$F(\boldsymbol{A}, \boldsymbol{X}) = \sum_{k=1}^{n} \int_{x_{k-1}}^{x_k} \varphi_k^2(x)\, dx,$$

with

$$\begin{aligned}
\varphi_1(x) &= \frac{x^2}{2} - A_0 x, \\
\varphi_k(x) &= \frac{x^2}{2} - x \sum_{i=0}^{k-1} A_i + \sum_{i=1}^{k-1} A_i x_i, \quad k = \overline{2, n-1}, \\
\varphi_n(x) &= \frac{(1-x)^2}{2} - A_n (1-x)\,.
\end{aligned}$$

Since for $w = 1$ the corresponding orthogonal polynomial on $[x_{k-1}, x_k]$ is the Legendre polynomial of degree two

$$\ell_{2,k}(x) = \frac{x^2}{2} - \frac{x_{k-1} + x_k}{2} x + \frac{(x_{k-1} + x_k)^2}{8} - \frac{(x_k - x_{k-1})^2}{24}$$

and

$$\int_{x_{k-1}}^{x_k} \ell_{2,k}^2(x)\, dx = \frac{1}{720} (x_k - x_{k-1})^5\,,$$

from $\varphi_k \equiv \ell_{2,k}$, $k = \overline{2, n-1}$, *one obtains*

$$\begin{aligned} \sum_{i=0}^{k-1} \overline{A}_i &= \frac{x_{k-1} + x_k}{2}, \\ \sum_{i=0}^{k-1} \overline{A}_i x_i &= \frac{(x_{k-1} + x_k)^2}{8} - \frac{(x_k - x_{k-1})^2}{24}, \quad k = \overline{2, n-1} \end{aligned} \tag{3.11}$$

and thus

$$\sum_{k=2}^{n-1} \int_{x_{k-1}}^{x_k} \varphi_k^2 (x)\, dx = \frac{1}{720} \sum_{k=2}^{n-1} (x_k - x_{k-1})^5 . \tag{3.12}$$

Taking into account that φ_1 *and* φ_n *are particular polynomials of second degree to which the above identities do not apply, from the equations*

$$\frac{\partial}{\partial A_0} \left[\int_0^{x_1} \varphi_1^2 (x)\, dx \right] = 0, \quad \frac{\partial}{\partial A_n} \left[\int_{x_{n-1}}^{1} \varphi_n^2 (x)\, dx \right] = 0,$$

one obtains

$$\overline{A}_0 = \frac{3}{8} x_1, \quad \overline{A}_n = \frac{3}{8} (1 - x_{n-1}) \tag{3.13}$$

and hence

$$\begin{aligned} \int_0^{x_1} \left(\frac{x^2}{2} - \overline{A}_0 x \right)^2 dx &= \frac{1}{320} x_1^5, \\ \int_{x_{n-1}}^{1} \left[\frac{(1-x)^2}{2} - \overline{A}_n (1-x) \right]^2 dx &= \frac{1}{320} (1 - x_{n-1})^5 . \end{aligned} \tag{3.14}$$

From (3.12) *and* (3.14), *it follows that*

$$F\left(\overline{\boldsymbol{A}}, \boldsymbol{X}\right) = \frac{1}{320} x_1^5 + \frac{1}{720} \sum_{k=2}^{n-1} (x_k - x_{k-1})^5 + \frac{1}{320} (1 - x_{n-1})^5 ,$$

which can be minimized with respect to the nodes $\boldsymbol{X}$.

First, we have that

$$\frac{\partial}{\partial x_k} \left[\sum_{i=2}^{n-1} (x_i - x_{i-1})^5 \right] = 5 \left[(x_k - x_{k-1})^4 - (x_{k+1} - x_k)^4 \right] = 0,$$

which implies that

$$x_k - x_{k-1} = \frac{x_{n-1} - x_1}{n-2}, \quad k = \overline{2, n-1}, \tag{3.15}$$

and thus

$$F\left(\overline{\boldsymbol{A}},\boldsymbol{X}\right) = \frac{1}{320}x_1^5 + \frac{1}{720\,(n-2)^4}\left(x_{n-1}-x_1\right)^5 + \frac{1}{320}\left(1-x_{n-1}\right)^5. \tag{3.16}$$

Next, the minimum value of $F\left(\overline{\boldsymbol{A}},\boldsymbol{X}\right)$ *with respect to* x_1 *and* x_{n-1} *is attained for*

$$x_1^\star = 1 - x_{n-1}^\star = 2\mu,$$

where

$$\mu = \frac{1}{4+(n-2)\sqrt{6}}.$$

Finally, from (3.15), (3.11), (3.13) *and* (3.16), *one obtains*

$$x_0^\star = 0;\quad x_k^\star = \left[2+(k-1)\sqrt{6}\right]\mu,\ k=\overline{1,n-1};\quad x_n^\star = 1;$$

$$A_0^\star = A_n^\star = \frac{3}{4}\mu;\ A_1^\star = A_{n-1}^\star = \frac{5+2\sqrt{6}}{4}\mu;\ A_k^\star = \mu\sqrt{6},\ k=\overline{2,n-2};$$

and

$$F\left(\boldsymbol{A}^\star,\boldsymbol{X}^\star\right) = \frac{1}{20}\mu^4,$$

which is the minimum value of $F\left(\boldsymbol{A},\boldsymbol{X}\right)$.

4. Optimal quadrature formulas generated by Lagrange interpolation formula

Let $\Lambda(f) = \left\{f(x_i) \mid i=\overline{0,n}\right\}$, with $x_i \in [a,b]$, be a set of Lagrange-type information.

Consider the Lagrange interpolation formula

$$f = L_n f + R_n f, \tag{4.1}$$

where

$$(L_n f)(x) = \sum_{k=0}^{n} \frac{u(x)}{(x-x_k)\,u'(x_k)} f(x_k),$$

with $u(x) = (x-x_0)\ldots(x-x_n)$ and for $f \in C^{n+1}[a,b]$,

$$(R_n f)(x) = \frac{u(x)}{(n+1)!} f^{(n+1)}(\xi),\quad a<\xi<b.$$

If $w\colon [a,b] \to \mathbb{R}$ is a weight function, from (4.1) one obtains

$$\int_a^b w(x) f(x)\, dx = \sum_{k=0}^{n} A_k f(x_k) + R_n(f), \tag{4.2}$$

where

$$A_k = \int_a^b w(x) \frac{u(x)}{(x - x_k)\, u'(x_k)} dx \tag{4.3}$$

and

$$R_n(f) = \frac{1}{(n+1)!} \int_a^b w(x)\, u(x)\, f^{(n+1)}(\xi)\, dx.$$

We also have

$$\left| R_n(f) \right| \leqslant \frac{1}{(n+1)!} \left\| f^{(n+1)} \right\|_\infty \int_a^b w(x) \left| u(x) \right| dx. \tag{4.4}$$

Theorem 4.1. *Let $w\colon [a,b] \to \mathbb{R}$ be a weight function and $f \in C^{n+1}[a,b]$. If $u \perp \mathcal{P}_n$, then the quadrature formula (4.2), with the coefficients (4.3) and the nodes $\boldsymbol{X} = (x_0, \ldots, x_n)$ - the roots of the polynomial u, is optimal with respect to the error.*

Proof. From (4.4), we have

$$\left| R_n(f) \right| \leqslant \frac{1}{(n+1)!} \left\| f^{(n+1)} \right\|_\infty \int_a^b \sqrt{w(x)} \sqrt{w(x)} \left| u(x) \right| dx \tag{4.5}$$

or

$$\left| R_n(f) \right| \leqslant \frac{1}{(n+1)!} \left\| f^{(n+1)} \right\|_\infty \left[\int_a^b w(x)\, dx \right]^{\frac{1}{2}} \left[\int_a^b w(x) \left| u(x) \right|^2 dx \right]^{\frac{1}{2}}.$$

So

$$\left| R_n(f) \right| \leqslant C_{w,2}^f \left\| u \right\|_{w,2}, \tag{4.6}$$

where

$$C_{w,2}^f = \frac{1}{(n+1)!} \left\| f^{(n+1)} \right\|_\infty \left\| \sqrt{w} \right\|_2 .$$

If $u \perp \mathcal{P}_n$ on $[a,b]$ with respect to the weight function w, then $\left\| u \right\|_{w,2}$ is minimum, i.e. the error $\left| R_n(f) \right|$ is minimum. □

Remark 4.1. *Theorem 4.1 implies that the optimal nodes* $x_k^\star$, $k=\overline{0,n}$, *are the roots of the orthogonal polynomial on* $[a,b]$ *with respect to the weight function* w, *say* P_{n+1}, *and the optimal coefficients* $A_k^\star$, $k=\overline{0,n}$, *are given by*

$$A_k^\star=\int_a^b w(x)\,\frac{\tilde P_{n+1}(x)}{(x-x_k^\star)\,\tilde P'_{n+1}(x_k^\star)}dx,\quad k=\overline{0,n}.$$

For the optimal error, we have

$$|R_n^\star(f)|\leqslant C_{w,2}^f\left\|\tilde P_{n+1}\right\|_{w,2}.$$

4.1. **Particular cases.** *Case 1.* $[a,b]=[-1,1]$ and $w=1$.

The orthogonal polynomial is the Legendre polynomial

$$\tilde\ell_{n+1}(x)=\frac{(n+1)!}{(2n+2)!}\frac{d^{n+1}}{dx^{n+1}}\left[\left(x^2-1\right)^{n+1}\right].$$

The corresponding optimal quadrature formula has the nodes $x_k^\star$, $k=\overline{0,n}$, and the coefficients $A_k^\star$, $k=\overline{0,n}$, of the Gauss quadrature rule. For the error, we have

$$\left|R_n^\star(f)\right|\leqslant\frac{(n+1)!2^{n+2}}{(2n+2)!\sqrt{2n+3}}\left\|f^{(n+1)}\right\|_\infty.$$

Case 2. $[a,b]=[-1,1]$ and $w(x)=\frac{1}{\sqrt{1-x^2}}$.

The orthogonal polynomial is the Chebyshev polynomial of the first kind

$$T_{n+1}(x)=\cos\left[(n+1)\arccos(x)\right].$$

The optimal parameters are

$$x_k^\star=\cos\frac{2k+1}{2(n+1)}\pi,\quad k=\overline{0,n},$$

$$A_k^\star=\int_{-1}^1\frac{1}{\sqrt{1-x^2}}\frac{\tilde T_{n+1}(x)}{(x-x_k^\star)\,\tilde T'_{n+1}(x_k^\star)}dx=\frac{\pi}{n+1},\quad k=\overline{0,n},$$

and we have

$$\begin{aligned}\left|R_n^\star(f)\right|&\leqslant\frac{1}{(n+1)!}\left\|f^{(n+1)}\right\|_\infty\left(\int_{-1}^1\frac{1}{\sqrt{1-x^2}}dx\right)^{\frac12}\left\|\tilde T_{n+1}\right\|_{w,2}\\&=\frac{\pi}{\sqrt2\,(n+1)!2^n}\left\|f^{(n+1)}\right\|_\infty.\end{aligned}$$

Case 3. $[a,b]=[-1,1]$ and $w(x)=\sqrt{1-x^2}$.

The orthogonal polynomial is the Chebyshev polynomial of the second kind

$$Q_{n+1}(x) = \frac{1}{\sqrt{1-x^2}} \sin\left[(n+2)\arccos(x)\right].$$

We have

$$x_k^\star = \cos\frac{k+1}{n+2}\pi, \quad k = \overline{0,n},$$

$$\begin{aligned} A_k^\star &= \int_{-1}^{1} \sqrt{1-x^2} \frac{\tilde{Q}_{n+1}(x)}{(x-x_k^\star)\,\tilde{Q}'_{n+1}(x_k^\star)} dx \\ &= \frac{\pi}{n+2} \sin^2\left(\frac{k+1}{n+2}\pi\right), \quad k = \overline{0,n}, \end{aligned}$$

and

$$\begin{aligned} \left|R_n^\star(f)\right| &\leqslant \frac{1}{(n+1)!} \left\|f^{(n+1)}\right\|_\infty \left(\int_{-1}^{1} \sqrt{1-x^2}\, dx\right)^{\frac{1}{2}} \left\|\tilde{Q}_{n+1}\right\|_{w,2} \\ &= \frac{\pi}{(n+1)!2^{n+2}} \left\|f^{(n+1)}\right\|_\infty. \end{aligned}$$

4.2. **Special cases.** $[a,b] = [-1,1]$ and $w = 1$.

Case 4. From (4.4), we obtain

$$|R_n(f)| \leqslant \frac{2}{(n+1)!} \left\|f^{(n+1)}\right\|_\infty \|u\|_\infty.$$

Since

$$\left\|\tilde{T}_{n+1}\right\|_\infty \leqslant \|P\|_\infty, \quad P \in \tilde{\mathcal{P}}_{n+1},$$

it follows that for $u = \tilde{T}_{n+1}$, the error $|R_n(f)|$ is minimum. So

$$x_k^\star = \cos\frac{2k+1}{2(n+1)}\pi, \quad k = \overline{0,n},$$

$$\begin{aligned} A_k^\star &= \int_{-1}^{1} \frac{\tilde{T}_{n+1}(x)}{(x-x_k^\star)\,\tilde{T}'_{n+1}(x_k^\star)} dx \\ &= \frac{2}{n+1}\left[1 - 2\sum_{i=1}^{\left[\frac{n}{2}\right]} \frac{1}{4i^2-1} \cos\frac{(2k+1)\,i}{n+1}\pi\right], \quad k = \overline{0,n}, \end{aligned} \tag{4.7}$$

and

$$\left|R_n^\star(f)\right| \leqslant \frac{1}{(n+1)!2^{n-1}} \left\|f^{(n+1)}\right\|_\infty.$$

Case 5. From (4.4), we also have

$$|R_n(f)| \leqslant \frac{1}{(n+1)!} \left\| f^{(n+1)} \right\|_\infty \|u\|_1 .$$

In this case the minimum $L_1[-1,1]$-norm is given by the Chebyshev polynomial of the second kind Q_{n+1}. So

$$x_k^\star = \cos \frac{k+1}{n+2}\pi, \quad k = \overline{0,n},$$

$$\begin{aligned} A_k^\star &= \int_{-1}^{1} \frac{\tilde{Q}_{n+1}(x)}{(x - x_k^\star)\, \tilde{Q}'_{n+1}(x_k^\star)} dx \\ &= \frac{4 \sin\left(\frac{k+1}{n+2}\pi\right)}{n+2} \sum_{i=0}^{\left[\frac{n}{2}\right]} \frac{\sin\left[\frac{(2i+1)(k+1)}{n+2}\pi\right]}{2i+1}, \quad k = \overline{0,n}, \end{aligned} \tag{4.8}$$

and

$$\left| R_n^\star(f) \right| \leqslant \frac{1}{(n+1)!} \left\| f^{(n+1)} \right\|_\infty \left\| \tilde{Q}_{n+1} \right\|_1 = \frac{1}{(n+1)!2^n} \left\| f^{(n+1)} \right\|_\infty .$$

4.3. **Other cases.** Let

$$\int_a^b f(x)\, dx = \sum_{k=0}^{n} A_k f(x_k) + R_n(f) \tag{4.9}$$

be the quadrature formula generated by the Lagrange interpolation formula

$$f(x) = \sum_{k=0}^{n} \frac{u(x)}{(x - x_k)\, u'(x_k)} f(x_k) + (R_n f)(x),$$

with $u(x) = (x - x_0) \dots (x - x_n)$ and

$$(R_n f)(x) = \frac{u(x)}{(n+1)!} f^{(n+1)}(\xi), \quad a < \xi < b.$$

We have

$$\left| R_n(f) \right| \leqslant \frac{1}{(n+1)!} \left\| f^{(n+1)} \right\|_\infty \int_a^b |u(x)|\, dx.$$

If w is a weight function, then

$$\int_a^b |u(x)|\, dx = \int_a^b \frac{1}{\sqrt{w(x)}} \sqrt{w(x)}\, |u(x)|\, dx \leqslant \left[\int_{-1}^{1} \frac{dx}{w(x)} \right]^{\frac{1}{2}} \|u\|_{w,2} .$$

Finally, we have

$$|R_n(f)| \leqslant C_{f,w} \|u\|_{w,2} ,$$

with

$$C_{f,w} = \frac{1}{(n+1)!} \left\| f^{(n+1)} \right\|_{\infty} \left[\int_{-1}^{1} \frac{1}{w(x)} dx \right]^{\frac{1}{2}}.$$

It follows that the quadrature formula (4.9) is optimal when $\|u\|_{w,2}$ is minimum, i.e. u is orthogonal on $[a,b]$ with respect to the weight function w.

Case 6. $[a,b] = [-1,1]$ and $w(x) = \frac{1}{\sqrt{1-x^2}}\cdot$

We get

$$x_k^{\star} = \cos \frac{2k+1}{2(n+1)} \pi, \quad k = \overline{0,n},$$

$$A_k^{\star} = \int_{-1}^{1} \frac{\tilde{T}_{n+1}(x)}{(x - x_k^{\star}) \tilde{T}'_{n+1}(x_k^{\star})} dx, \quad k = \overline{0,n}, \quad \text{(see (4.7))},$$

and hence

$$\begin{aligned} \left| R_n^{\star}(f) \right| &\leqslant \frac{1}{(n+1)!} \left\| f^{(n+1)} \right\|_{\infty} \left(\int_{-1}^{1} \sqrt{1-x^2}\, dx \right)^{\frac{1}{2}} \left\| \tilde{T}_{n+1} \right\|_{w,2} \\ &= \frac{\pi}{(n+1)! 2^{n+1}} \left\| f^{(n+1)} \right\|_{\infty}. \end{aligned}$$

Case 7. $[a,b] = [-1,1]$ and $w(x) = \sqrt{1-x^2}$.

It follows that

$$x_k^{\star} = \cos \frac{k+1}{n+2} \pi, \quad k = \overline{0,n},$$

$$A_k^{\star} = \int_{-1}^{1} \frac{\tilde{Q}_{n+1}(x)}{(x - x_k^{\star}) \tilde{Q}'_{n+1}(x_k^{\star})} dx, \quad k = \overline{0,n}, \quad \text{(see (4.8))},$$

and thus

$$\begin{aligned} \left| R_n^{\star}(f) \right| &\leqslant \frac{1}{(n+1)!} \left\| f^{(n+1)} \right\|_{\infty} \left(\int_{-1}^{1} \frac{1}{\sqrt{1-x^2}}\, dx \right)^{\frac{1}{2}} \left\| \tilde{Q}_{n+1} \right\|_{w,2} \\ &= \frac{\pi}{\sqrt{2}(n+1)! 2^{n+1}} \left\| f^{(n+1)} \right\|_{\infty}. \end{aligned}$$

Remark 4.2. *From* (4.5), *we also have*

$$|R_n(f)| \leqslant C_{w,\infty}^{f} \|u\|_{w,2},$$

with

$$C_{w,\infty}^{f} = \frac{\sqrt{b-a}}{(n+1)!} \left\| f^{(n+1)} \right\|_{\infty} \left\| \sqrt{w} \right\|_{\infty}.$$

For particular orthogonal polynomials, we can obtain new upper bounds for the quadrature error.

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REMARKS ON COMPUTING THE VALUE OF AN OPTION WITH BINOMIAL METHODS

IOANA CHIOREAN

Dedicated to Professor D. D. Stancu on his 80^{th} *birthday*

Abstract. The purpose of this paper is to give a formula for computing the value of a financial option, using the binomial method.

1. Introduction

Binomial methods for valuing options and other derivative securities arise from discrete random walk models of the underlying security. This happens because the movement of asset prices is a random walk. It can be modeled, but any such model must incorporate a degree of randomness.

In valuating an option, the Black-Scholes formula is mostly used, the solution being obtained numerically, using the finite difference method, with serial and/or parallel algorithms (see [1], [2], [4]).

As is stated in [3] and [5], the binomial method is a particular case of the explicit finite difference method. Using this method, several serial and parallel algorithms are given. In what follows, we give a general formula for computing the value of an option, starting with discrete values at expiry date and using binomial methods.

2. Asset Price Random Walk

The theory of option pricing is based on the assumption that we do not know tomorrow's values of asset prices. We may use, anyway, the past history of the asset

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value, which tells us what are the likely jumps in asset price, what are their mean and variance and, more generally, what is the likely distribution of future asset prices.

It is known that asset prices move randomly. In order to model this movement, for each change in asset price, a **return** is associated, defined to be the change in the price divided by the original value (for more details, see [5]).

In order to get the equation which modeled this random walk, we consider that at time t, the asset price is S. In a small subsequence time interval, dt, the value S changes to $S+dS$. The corresponding return, $\frac{dS}{S}$, will be decomposed in two parts. One is predictable, deterministic, denoted by μdt, where μ is a measure of the average rate of growth of the asset price.

Note. In simple models, μ is taken to be a constant.

The second contribution to $\frac{dS}{S}$ models the random change in the asset price in response to external effects, such as unexpected news. It is represented by a random sample drawn from a normal distribution with mean zero and adds a term, σdX. Here, σ is a number called the **volatility**, which measures the standard deviation of the returns. The quantity dX is the sample from a normal distribution, with the mean zero and variance, dt.

We all this in mind, we obtain the stochastic differential equation

$$\frac{dS}{S} = \sigma dX + \mu dt \tag{2.1}$$

which is the mathematical representation of our simple recipe for generating asset prices.

3. Binomial Methods

3.1. Discrete random walks

In order to obtain binomial methods, we started from the idea that the continuous random walk given by (2.1) may be modeled by a discrete random walk, with the following properties:

• the asset price S changes only at the discrete times $\delta t, 2\delta t, 3\delta t, \dots$ up to $M\delta t = T$, the expiry date of derivative security. We use δt instead of dt to denote the small but non-infinitesimal time-steps between movements in asset price.

• if the asset price is S^m at time step $m\delta t$ then at time $(m+1)\delta t$ it will take one of only two possible values; $uS^m > S^m$ or $vS^m > S^m$. It means that the asset price may move from S up to uS or down to vS. This is equivalent to the fact that there are only two returns $\frac{\delta S}{S}$ possible at each time step: $u - 1 > 0$ and $v - 1 < 0$, and these two returns are the same for all time steps.

• the probability, p, of S moving up to uS is known (as the probability $(1-p)$ of S moving down to vS).

Starting with a given value of the asset price (for example, to day's asset price) the remaining life-time of the derivative security is divided up into M time-steps of size $\delta t = (T-t)/M$. The asset price S is assumed to move only at times $m\delta t$ for $m = 1, 2, \dots, M$. Then, a **tree** of all possible asset prices is created. This tree is constructed by starting with the given value S, generating the two possible asset prices (uS and vS) at the first time-step, then the three possible asset prices (u^2S, uvS and v^2S) at the second time-step, and so on, until the expiry time is reached.

Remark. We observe that after m time-steps, there are only $m+1$ possible asset prices.

3.2. Risk-neutral world

Another assumption in getting the binomial methods is a risk-neutral world. Under this circumstances, we may assume that the investitors are risk-neutral, and that the return from the underlying is the risk-free interest rate. Then, μ from (2.1), which does not appear into the Black-Scholes equation, is replaced by r, which appears in it and defined the interest rate.

So, in a risk-neutral world, equation (2.1) is replaced by

$$\frac{dS}{S} = \sigma dX + r dt. \tag{3.1}$$

The value of an option is then determined by calculating the present value of its expected return at expiry with the previous modification to the random walk. Having this in mind and, in addition, the fact that the present value of any amount at time T will be that amount discounted by multiplying by $e^{-r(T-t)}$ (for more details, see [5]), we may write the value V^m of the derivative security at time-step $m\delta t$ as the expected value of the security at time-step $(m+1)\delta t$ discounted by the risk-free interest rate r:

$$V^m = E(e^{-r\delta t} \cdot V^{m+1}) \tag{3.2}$$

Remark. Relation (3.2) is another way of interpreting the Black-Scholes formula.

3.3. How does a binomial method work

In a binomial method, we first build a tree of possible values of asset prices and their probabilities, given an initial asset price, then use this tree to determine the possible asset prices at expiry. The possible values of the security at expiry can then be calculated and, by working back, according with (3.2), the security can be valued.

In order to build up the tree of possible asset prices, we start at the current time $t = 0$. We assume that at this time we know the asset price, S_0^0. Then, at next time-step, δt, there are two possible asset prices: $S_1^1 = uS_0^0$ and $S_0^1 = vS_0^0$. At the following time-step, $2\delta t$, there are three possible asset prices: $S_2^2 = u^2S_0^0$, $S_1^2 = uvS_0^0$ and $S_0^2 = v^2S_0^0$. At the third time-step, $3\delta t$, the possible values are: $S_3^3 = u^3S_0^0$, $S_2^3 = u^2vS_0^0$, $S_1^3 = uv^2S_0^0$ and $S_0^3 = v^3S_0^0$, and so on.

At the m-th time-steps, $m\delta t$, there are $m+1$ possible values of the asset price,

$$S_n^m = u^n \cdot v^{m-n} \cdot S_0^0, \quad n = 0, 1, \ldots, m \tag{3.3}$$

Remark. In (3.3), S_n^m denotes the n-th possible value S at time-step $m\delta t$, whereas v^n and u^n denote v and u raised to the n-th power.

At the final time-step, $M\delta t$, we have $M+1$ possible values of the underlying asset, and we know all of them.

4. **Valuing the Option**

In what follows, we suppose that we know the payoff function for our derivative security and that it depends only on the values of the underlying asset at expiry. Then, we are able to value the option at expiry, i.e. time-step $M\delta t$. For example, for a call option, we find that

$$V_n^M = \max(S_n^m - E, 0), \quad n = 0, 1, \dots, M \tag{4.1}$$

where E is the exercise price and V_n^M denotes the n-th possible value of the call at time-step M.

Then, we can find the expected value of the derivative security at the time-step prior to expiry, $(M-1)\delta t$, and for possible asset price S_n^{M-1}, $n = 0, 1, \dots, M-1$, since we know the probability of an asset priced at S_n^{M-1} moving to S_{n+1}^M during a time-step is p, and the probability of it moving to S_n^M is $(1-p)$. Using the risk-neutral argument, we can calculate the value of the security at each possible asset price for the time-step $(M-1)$. Then, for $(M-2)$, and so on, back to time-step 0. This gives us the value of our option at the current time.

5. **The Case of European Option**

Let V_n^m denotes the value of the option at time-step $m\delta t$ and asset price S_n^m (where $0 \le n \le m$). According with (3.2), we calculate the expected value of the option at time-step $m\delta t$ from the values at time-step $(m+1)\delta t$ and discount in order to obtain the present value using the risk-free interest rate, r:

$$e^{r\delta t} \cdot V_n^m = p \cdot V_{n+1}^{m+1} + (1-p) \cdot V_n^{m+1} \tag{5.1}$$

which gives:

$$V_n^m = e^{-r\delta t}(p \cdot V_{n+1}^{m+1} + (1-p) \cdot V_n^{m+1}) \tag{5.2}$$

for every $n = 0, 1, \dots, m$.

As we know the value of V_n^M, $n = 0, 1, \dots, M$ from the payoff function, as in (4.1), we can, recursively, determine the values V_n^m for each $n = 0, 1, \dots, m$, for $m < M$ to arrive at the current value of the option, V_0^0.

As in [5], the computation (5.2) may be permorned step by step, in M steps, to get the value V_0^0. We give another possible computation, based on the following theorem:

Theorem 1. *The value of the option at time-step m, $0 \le m \le M$, V_n^m, for every $0 \le n \le m$ can be calculated using only the values at expiry time, V_n^M, $0 \le n \le m$, according with the formula:*

$$C_n^m = \sum_{n=0}^{m} A_n \cdot V_n^M,$$

for every $0 \le m \le M$, where A_n, $0 \le n \le m$ are the binomial coefficients of $(\alpha+\beta)^m$, where $\alpha = e^{-r\delta t}p$ and $\beta = e^{-r\delta t}(1-p)$.

Proof. Using the notation α and β for the coefficients in (5.2), we have

$$V_n^m = \alpha V_{n+1}^{m+1} + \beta V_n^{m+1}, \tag{5.3}$$

for fixed m, $(m < M)$ and $0 \le n \le m$, or, in matriceal form:

$$\begin{bmatrix} V_0^m \\ V_1^m \\ \vdots \\ V_m^m \end{bmatrix} = \alpha \begin{bmatrix} V_1^{m+1} \\ V_2^{m+1} \\ \vdots \\ V_{m+1}^{m+1} \end{bmatrix} + \beta \begin{bmatrix} V_0^{m+1} \\ V_1^{m+1} \\ \vdots \\ V_m^{m+1} \end{bmatrix} \tag{5.4}$$

Knowing the values V_n^M, $n = 0, 1, \dots, M$, we may compute the value V_n^{M-1}:

$$V_n^{M-1} = \alpha V_{n+1}^M + \beta V_n^M, \quad n = 0, 1, \dots, M-1.$$

Then, at the step $(M-2)$, we get:

$$V_n^{M-2} = \alpha^2 V_{n+2}^M + \alpha\beta V_{n+1}^M + \beta^2 V_n^M, \quad n = 0, \dots, M-2$$

and

$$V_n^{M-3} = \alpha^3 V_{n+3}^M + \alpha^2\beta V_{n+2}^M + \alpha\beta^2 V_{n+1}^M + \beta^3 V_n^m, \quad n = 0, \dots, M-3$$

and so on, finally:

$$V_0^0 = \sum_{i=0}^{M} A_i \cdot V_i^M$$

where A_i are the binomial coefficients of $(\alpha+\beta)^M$. □

6. **Conclusions**

This method of computing the value of an option is more economical from time and memory space point of view than a serial computation made step by step, according with the step-time m. Our result indicates the resemblance of the binomial method with the finite-differences way of computation. The speed of computation can also be reduced by parallel calculus.

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BERNSTEIN-STANCU OPERATORS

VOICHIŢA ADRIANA CLECIU

Dedicated to Professor D. D. Stancu on his 80^{th} birthday

Abstract. The purpose of this paper is to investigate the modifications operators $C_n : Y \to \Pi_n$

$$(C_n f)(x) = \sum_{k=0}^{n} \frac{k!}{n^k} \binom{n}{k} \mathbf{m}_{k,n} \left[0, \frac{1}{n}, ..., \frac{k}{n}; f\right] x^k, f \in Y,$$

where the real numbers $(m_{k,n})_{k=0}^{\infty}$ are selected in order to preserve some important properties of Bernstein operators. For $\mathbf{m}_{j,n} = \frac{(a_n)_j}{j!}, a_n \in (0,1]$ we obtain Bernstein-Stancu operators

$$\left(\overline{C}_n f\right)(x) = \sum_{k=0}^{n} \frac{(a_n)_k}{n^k} \binom{n}{k} \left[0, \frac{1}{n}, ..., \frac{k}{n}; f\right] x^k, \ f \in Y$$

and we study some of their properties.

1. Introduction

Let Π_n be the linear space of all real polynomials of degree $\leq$ n and denote by Y the linear space of all functions $[0,1] \to \mathbb{R}$.

Consider the sequence of Bernstein operators $B_n : Y \to \Pi_n$ where

$$(B_n f)(x) = \sum_{k=0}^{n} b_{n,k}(x) f\left(\frac{k}{n}\right), \ b_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k} \quad , f \in Y.$$

Because for $j \in \{0, 1, ..., n\}$

$$\frac{1}{j!} \frac{d^j (B_n f)(x)}{dx^j} = \binom{n}{j} \frac{j!}{n^j} \sum_{k=0}^{n-j} b_{n-j,k}(x) \left[\frac{k}{n}, \frac{k+1}{n}, ..., \frac{k+j}{n}; f\right],$$

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the following well-known formula holds

$$(B_n f)(x) = \sum_{k=0}^{n} \frac{k!}{n^k} \binom{n}{k} \left[0, \frac{1}{n}, ..., \frac{k}{n}; f\right] x^k. \tag{1}$$

Starting with (1), we investigate the following modifications $C_n : Y \to \Pi_n$

$$(C_n f)(x) = \sum_{k=0}^{n} \frac{k!}{n^k} \binom{n}{k} \mathbf{m}_{k,n} \left[0, \frac{1}{n}, ..., \frac{k}{n}; f\right] x^k, f \in Y, \tag{2}$$

where the real numbers $(m_{k,n})_{k=0}^{\infty}$ are selected in order to preserve some important properties of Bernstein operators. Observe that from (2)

$$\begin{cases} C_n e_0 = \mathbf{m}_{0,n} \\ C_n e_1 = \mathbf{m}_{1,n} e_1 \\ C_n e_2 = \mathbf{m}_{2,n} e_2 + \dfrac{e_1}{n} (\mathbf{m}_{1,n} - \mathbf{m}_{2,n} e_1) \\ (C_n \Omega_{2,x})(x) = (\mathbf{m}_{2,n} - 2\mathbf{m}_{1,n} + \mathbf{m}_{0,n}) x^2 + \dfrac{x}{n} (\mathbf{m}_{1,n} - \mathbf{m}_{2,n} x), \end{cases} \tag{3}$$

where $e_j(t) = t^j$ and $\Omega_{2,x} = (t-x)^2$. In the following, we shall consider that $\mathbf{m}_{0,n} = 1$.

The following problem arises to emphasize numbers $\mathbf{m}_{k,n}$, $k \in \mathbb{N}$, for which the linear transformations $(C_n)_{n=1}^{\infty}$ are *positive operators* and moreover

$$\lim_{n\to\infty} \mathbf{m}_{1,n} = 1 \qquad and \qquad \lim_{n\to\infty} \mathbf{m}_{2,n} = 1.$$

Denote by Π_s the set of all real polynomial functions of exact degree s.

Lemma 1. *If* $p \in \Pi_s$ *and* $\mathbf{m}_{s,n} \neq 0$, *then* $C_n p \in \Pi_s, n \geq s$.

Proof. Use the fact that

$$\left[0, \frac{1}{n}, ..., \frac{k}{n}; f\right] = \begin{cases} 0, k > j \\ 1, k = j \end{cases}.$$

Therefore, if $p(x) = a_0 x^s + ... + a_{s-1} x + a_s, a_0 \neq 0$, then from (2) one finds

$$(C_n p)(x) = b_0 x^s + ... + b_s,$$

with

$$b_s := \frac{s!}{n^s} \binom{n}{s} \mathbf{m}_{s,n}, b_0 \neq 0.$$

□

Lemma 2. *If C_n is a positive operator with $\mathbf{m}_{0,n} = 1$, then $\mathbf{m}_{1,n} \in [0,1]$ and*

$$0 \leq \mathbf{m}_{1,n} - \mathbf{m}_{2,n} \leq \frac{n}{n-1} \left(1 - \mathbf{m}_{1,n}\right).$$

Proof. For the proof it is enough to observe that

$$0 \leq e_1 (t) \leq 1, t \in [0,1]$$

implies

$$0 \leq \left(C_n e_1\right)(x) = \mathbf{m}_{1,n} x \leq \mathbf{m}_{0,n} = 1, \forall x \in [0,1],$$

that is $\mathbf{m}_{1,n} \in [0,1]$. Further, from $t(1-t) \geq 0, \forall t \in [0,1]$, we have

$$\left(C_n e_1\right)(x) - \left(C_n e_2\right)(x) \geq 0$$

for any x from $[0,1]$, that is $\mathbf{m}_{2,n} x \leq \mathbf{m}_{1,n}$. To complete the proof it is sufficient to use the fact that $\left(C_n \Omega_{2,x}\right)(x)$ must be non-negative on $[0,1]$. □

Lemma 3. *Suppose that C_n is a positive operator with $\mathbf{m}_{0,n} = 1$.*

1) If $\mathbf{m}_{2,n} = 1$, then $\mathbf{m}_{1,n} = 1$;

2) If $\mathbf{m}_{1,n} = 1$, then $\mathbf{m}_{2,n} = 1$.

Proof. Use Lemma 2. □

Lemma 4. *Suppose that $f : [0,1] \to \mathbb{R}$ is convex on $[0,1]$. If C_n is a positive operator with $\mathbf{m}_{0,n} = 1$, then*

$$f\left(m_{1,n} x\right) \leq \left(C_n f\right)(x), \forall x \in [0,1].$$

Proof. It is known that for a convex function $f : [0,1] \to \mathbb{R}$ and a linear positive operator $T : Y \to Y$, we have

$$f\left(\left(T e_1\right)(x)\right) \leq (T f)(x), \ \forall x \in [0,1] \qquad \text{(see [7] and [8]).}$$ □

Lemma 5. *Suppose that $\left(C_n\right)_{n=1}^{\infty}$ are positive operators with $\mathbf{m}_{0,n} = 1$. If*

$$\lim_{n \to \infty} \mathbf{m}_{1,n} = 1,$$

then

$$\lim_{n \to \infty} \mathbf{m}_{2,n} = 1.$$

Proof. From $0 \leq \mathbf{m}_{1,n} - \mathbf{m}_{2,n} \leq \frac{n}{n-1}(1 - \mathbf{m}_{1,n})$, see Lemma 2. ∎

Further we consider the uniform norm $\|g\| := \max_{x\in[0,1]} |g(t)|$.

Lemma 6. *Suppose that* $\mathbf{m}_{0,n} = 1$, $\forall n \in \mathbb{N}^*$. *If* $(C_n)_{n=1}^{\infty}$ *is a sequence of positive operators, then* $\lim_{n\to\infty} \mathbf{m}_{1,n} = 1$ *implies*

$$\lim_{n\to\infty} \|f - C_n f\| = 0, \quad \forall f \in C[0,1].$$

2. **The Bernstein form of the operator C_n**

Theorem 7. *Suppose that C_n is defined as in (2). Then for* $f : [0,1] \to \mathbb{R}$

$$(C_n f)(x) = \sum_{k=0}^{n} b_{n,k}(x)\, C_{k,n}[f], \tag{4}$$

with

$$C_{k,n}[f] = \sum_{j=0}^{k} \binom{k}{j} f\left(\frac{j}{n}\right) \sum_{\nu=0}^{k-j} (-1)^{\nu} \binom{k-j}{\nu} \mathbf{m}_{\nu+j,n}.$$

Proof. Observe that

$$\left[0, \frac{1}{n}, ..., \frac{k}{n}; f\right] = \frac{k!}{n^k} \sum_{\nu=0}^{k} (-1)^{k-\nu} \binom{k}{\nu} f\left(\frac{\nu}{n}\right).$$

From (2)

$$(C_n f)(x) = \sum_{k=0}^{n} A_k x^k,$$

with

$$A_k := \mathbf{m}_{k,n} \binom{n}{k} \sum_{\nu=0}^{k} (-1)^{k-\nu} \binom{k}{\nu} f\left(\frac{\nu}{n}\right).$$

Further, using the rule

$$\sum_{k=0}^{n} C_k \sum_{j=k}^{n} D_{k,j} = \sum_{k=0}^{n} \sum_{j=0}^{k} C_j D_{j,k},$$

we get (see [9])

$$(C_n f)(x) = \sum_{k=0}^{n} A_k x^k \left((1-x)+x\right)^{n-k} = \sum_{k=0}^{n} A_k \sum_{j=k}^{n} \binom{n-k}{j-k} x^j (1-x)^{n-j} =$$

$$= \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} C_{k,n}[f],$$

where

$$C_{k,n}[f] := \sum_{j=0}^{k} A_j \frac{(n-j)!k!}{(k-j)!n!}.$$

Therefore

$$C_{k,n}[f] := \sum_{j=0}^{k} \binom{k}{j} \mathbf{m}_{j,n} \sum_{\nu=0}^{j} (-1)^{\nu-j} \binom{j}{\nu} f\left(\frac{\nu}{n}\right) =$$

$$= \sum_{j=0}^{k} \binom{k}{j} f\left(\frac{j}{n}\right) \sum_{\nu=0}^{k-j} (-1)^{\nu} \binom{k-j}{\nu} \mathbf{m}_{\nu+j,n}$$

and we conclude with (4). □

3. Bernstein - Stancu operators: the case $\mathbf{m}_{j,n} = \frac{(a_n)_j}{j!}, a_n \in (0,1]$

Further, for $k \in \mathbb{N}$, $z \in \mathbb{C}$, let $(z)_0 = 1$ and $(z)_k = z(z+1)\ldots(z+k-1)$.

Then the operator C_n from (2), denoted further by $\overline{C}_n$, becomes

$$\left(\overline{C}_n f\right)(x) = \sum_{k=0}^{n} \frac{(a_n)_k}{n^k} \binom{n}{k} \left[0, \frac{1}{n}, \ldots, \frac{k}{n}; f\right] x^k, \ \ f \in Y. \tag{5}$$

Lemma 8. *Assume that* $\overline{C}_n$ *is a positive operator, i.e.* $a_n \in (0,1]$. *Then*

$$\begin{cases} \left(\overline{C_n} e_0\right)(x) = 1 \\ \left(\overline{C_n} e_1\right)(x) = a_n x = x - (1-a_n)x \\ \left(\overline{C_n} e_2\right)(x) = x^2 + \dfrac{x(1-x)}{n} a_n + \dfrac{1-a_n}{2}\left(\frac{a_n}{n} - (2+a_n)\right) x^2 \\ \left(\overline{C_n} \Omega_{2,x}\right)(x) = \dfrac{x(1-x)}{n} a_n + x^2(1-a_n)\left(\dfrac{2-a_n}{2} + \dfrac{a_n}{2n}\right). \end{cases} \tag{6}$$

Moreover

$$\left|\left(\overline{C_n} \Omega_{2,x}\right)(x)\right| \le \frac{a_n}{4n} + (1-a_n), \quad \forall x \in [0,1]. \tag{7}$$

Proof. The above assertions follow using (3):

$$
\begin{aligned}
&\left(\overline{C_n}e_0\right)(x) = \frac{(a_n)_0}{0!} = \\
&\left(\overline{C_n}e_1\right)(x) = \frac{(a_n)_1}{1!}e_1 = a_n e_1 = a_n x = x - (1-a_n)\, x \\
&\left(\overline{C_n}e_2\right)(x) = \frac{(a_n)_2}{2!}e_2 + \frac{e_1}{n}\left(\frac{(a_n)_1}{1!} - \frac{(a_n)_2}{2!}e_1\right) = \\
&= \frac{a_n\,(a_n+1)}{2}e_2 + \frac{a_n}{n}e_1\left(1 - \frac{a_n+1}{2}e_1\right) = \\
&= \frac{a_n\,(a_n+1)}{2}x^2 + \frac{a_n}{n}x\left(1 - \frac{a_n+1}{2}x\right) = \\
&= \frac{x(1-x)}{n}a_n + x^2\left(\frac{a_n^2+a_n}{2} + \frac{a_n-a_n^2}{2n}\right) = \\
&= \frac{x(1-x)}{n}a_n + x^2 + x^2\left(\frac{(a_n+2)\,(a_n-1)}{2} + \frac{a_n\,(1-a_n)}{2n}\right) = \\
&= x^2 + \frac{x(1-x)}{n}a_n + \frac{1-a_n}{2}\left(\frac{a_n}{n} - (2+a_n)\right)x^2
\end{aligned}
$$

and

$$
\begin{aligned}
&\left(\overline{C_n}\Omega_{2,x}\right)(x) = \left(\frac{(a_n)_2}{2!} - 2\frac{(a_n)_1}{1!} + 1\right)x^2 + \frac{x}{n}\left(\frac{(a_n)_1}{1!} - \frac{(a_n)_2}{2!}x\right) = \\
&= \left(\frac{a_n\,(a_n+1)}{2} - 2a_n + 1\right)x^2 + \frac{x}{n}\left(a_n - \frac{a_n\,(a_n+1)}{2}x\right) = \\
&= \left(\frac{a_n\,(a_n-3)}{2} + 1\right)x^2 + \frac{a_n}{n}x\left(1 - \frac{a_n+1}{2}x\right) \\
&= \frac{a_n}{n}x + x^2\left(-\frac{a_n}{n} + \frac{a_n^2-3a_n+2}{2} + \frac{a_n-a_n^2}{2n}\right) = \\
&= \frac{x(1-x)}{n}a_n + x^2\,(1-a_n)\left(\frac{2-a_n}{2} + \frac{a_n}{2n}\right).
\end{aligned}
$$

□

Lemma 9. *Assume that $\overline{C}_n$ is a positive operator, i.e. $a_n \in (0,1]$. Then*

$$
\begin{aligned}
\left(\overline{C_n}e_3\right)(x) &= \frac{(a_n)_3}{n^3}\begin{pmatrix} n \\ 3 \end{pmatrix}x^3 + \frac{3\,(a_n)_2}{n^3}\begin{pmatrix} n \\ 2 \end{pmatrix}x^2 + \frac{a_n}{n^2}x = \\
&= x^3 + \frac{3(n-1)}{2n^2}\,(a_n)_2\,x^2 + \frac{2-3n}{6n^2}\,(a_n)_3\,x^3 + \frac{a_n}{n^2}x + \frac{(a_n)_3-6}{6}x^3
\end{aligned}
$$

$$\left(\overline{C_n}e_4\right)(x)=\frac{(a_n)_4}{n^4}\begin{pmatrix}n\\4\end{pmatrix}x^4+\frac{6\,(a_n)_3}{n^4}\begin{pmatrix}n\\3\end{pmatrix}x^3+\frac{7\,(a_n)_2}{n4}\begin{pmatrix}n\\2\end{pmatrix}x^2+\frac{a_n}{n^3}x=$$

$$=\ x^4+\frac{(n-1)\,(n-2)}{n^3}\,(a_n)_3\,x^3-\frac{6n^2-11n+6}{24n^3}\,(a_n)_4\,x^4+\frac{a_n}{n^3}x+$$

$$+\ \frac{7(n-1)}{2n^3}\,(a_n)_2\,x^2-\frac{(1-a_n)\left(a_n^3+7a_n^2+18a_n+24\right)}{24}x^4$$

$$\left(\overline{C_n}\Omega_{4,x}\right)(x)=\left[\frac{(a_n)_4}{n^4}\begin{pmatrix}n\\4\end{pmatrix}-\frac{4\,(a_n)_3}{n^3}\begin{pmatrix}n\\3\end{pmatrix}+\frac{6\,(a_n)_2}{n^2}\begin{pmatrix}n\\2\end{pmatrix}-4a_n+1\right]x^4+$$

$$+\left[\frac{6\,(a_n)_3}{n^4}\begin{pmatrix}n\\3\end{pmatrix}-\frac{12\,(a_n)_2}{n^3}\begin{pmatrix}n\\2\end{pmatrix}+\frac{6a_n}{n}\right]x^3+$$

$$+\left[\frac{7\,(a_n)_2}{n^4}\begin{pmatrix}n\\2\end{pmatrix}-\frac{4a_n}{n^2}\right]x^2+\frac{a_n}{n^3}x$$

$$=-\left(e_4\,(x)-\left(\overline{C_n}e_4\right)(x)\right)+4x\left(e_3\,(x)-\left(\overline{C_n}e_3\right)(x)\right)-$$

$$-\,6x^2\left(e_2\,(x)-\left(\overline{C_n}e_2\right)(x)\right)+4x^3\left(e_1\,(x)-\left(\overline{C_n}e_1\right)(x)\right)$$

Proof. Using (5) we have:

$$\left(\overline{C_n}e_3\right)(x)=\frac{a_n}{n}\begin{pmatrix}n\\1\end{pmatrix}\left[0,\frac{1}{n};e_3\right]x+\frac{(a_n)_2}{n^2}\begin{pmatrix}n\\2\end{pmatrix}\left[0,\frac{1}{n},\frac{2}{n};e_3\right]x^2+$$

$$+\frac{(a_n)_3}{n^3}\begin{pmatrix}n\\3\end{pmatrix}\left[0,\frac{1}{n},\frac{2}{n},\frac{3}{n};e_3\right]x^3=\frac{(a_n)_3}{n^3}\begin{pmatrix}n\\3\end{pmatrix}x^3+$$

$$+\frac{3\,(a_n)_2}{n^3}\begin{pmatrix}n\\2\end{pmatrix}x^2+\frac{a_n}{n^2}x=\frac{a_n}{n^2}x+\frac{3(n-1)}{2n^2}(a_n)_2x^2+\frac{(n-1)(n-2)}{6n^2}\,(a_n)_3\,x^3=$$

$$=x^3+\frac{3(n-1)}{2n^2}\,(a_n)_2\,x^2+\frac{2-3n}{6n^2}\,(a_n)_3\,x^3+\frac{a_n}{n^2}x+\frac{(a_n)_3-6}{6}x^3$$

$$\left(\overline{C_n}e_4\right)(x)=\frac{a_n}{n}\begin{pmatrix}n\\1\end{pmatrix}\left[0,\frac{1}{n};e_4\right]x+\frac{(a_n)_2}{n^2}\begin{pmatrix}n\\2\end{pmatrix}\left[0,\frac{1}{n},\frac{2}{n};e_4\right]x^2+$$

$$+\frac{(a_n)_3}{n^3}\begin{pmatrix}n\\3\end{pmatrix}\left[0,\frac{1}{n},\frac{2}{n},\frac{3}{n};e_4\right]x^3+\frac{(a_n)_4}{n^4}\begin{pmatrix}n\\4\end{pmatrix}\left[0,\frac{1}{n},\frac{2}{n},\frac{3}{n},\frac{4}{n};e_4\right]x^4$$

$$= \frac{(a_n)_4}{n^4}\begin{pmatrix} n \\ 4 \end{pmatrix}x^4 + \frac{6\,(a_n)_3}{n^4}\begin{pmatrix} n \\ 3 \end{pmatrix}x^3 + \frac{7\,(a_n)_2}{n4}\begin{pmatrix} n \\ 2 \end{pmatrix}x^2 + \frac{a_n}{n^3}x$$

$$= \frac{a_n}{n^3}x + \frac{7(n-1)}{2n^3}(a_n)_2 x^2 + \frac{(n-1)(n-2)}{n^3}(a_n)_3\,x^3 + \frac{(n-1)(n-2)(n-3)}{24n^3}(a_n)_4\,x^4$$

$$= x^4 + \frac{(n-1)\,(n-2)}{n^3}(a_n)_3\,x^3 - \frac{6n^2-11n+6}{24n^3}(a_n)_4\,x^4 + \frac{a_n}{n^3}x + \frac{7(n-1)}{2n^3}(a_n)_2\,x^2$$

$$-\frac{(1-a_n)\left(a_n^3+7a_n^2+18a_n+24\right)}{24}x^4.$$

We use the fact that

$$\left(\overline{C_n}\Omega_{4,x}\right)(x) = \left(\overline{C_n}e_4\right)(x) - 4x\left(\overline{C_n}e_3\right)(x) + 6x^2\left(\overline{C_n}e_2\right)(x) - 4x^3\left(\overline{C_n}e_1\right)(x) + x^4$$

to obtain the above assertions. □

Theorem 10. *The linear operator $\overline{C}_n$ from (5) may be written in the Bernstein basis in the form*

$$\left(\overline{C}_n f\right)(x) = \sum_{k=0}^{n} b_{n,k}(x)\,\overline{C}_{k,n}[f]\,, \tag{8}$$

with

$$\overline{C}_{k,n}[f] = \frac{1}{k!}\sum_{j=0}^{k}\binom{k}{j} f\left(\frac{j}{n}\right)(a_n)_j\,(1-a_n)_{k-j}$$

Proof. Let us find a convenient form of the coefficients $\overline{C}_{k,n}[f]$ from (4). In our case we have

$$\overline{C}_{k,n}[f] = \sum_{j=0}^{k}\binom{k}{j} f\left(\frac{j}{n}\right)\sum_{\nu=0}^{k-j}(-1)^{\nu}\binom{k-j}{\nu}\frac{(a_n)_{\nu+j}}{(\nu+j)!} =$$

$$= \sum_{j=0}^{k}\binom{k}{j} f\left(\frac{j}{n}\right)\frac{(a_n)_j}{j!}\sum_{\nu=0}^{k-j}\frac{(-k+j)_{\nu}\,(j+a_n)_{\nu}}{(j+1)_{\nu}\,\nu!} =$$

$$= \sum_{j=0}^{k}\binom{k}{j} f\left(\frac{j}{n}\right)\frac{(a_n)_j}{(j)!}\cdot{}_2F_1\left(\,-k+j, j+a_n; j+1; 1\right).$$

Because ${}_2F_1\left(\,-m, b; c; 1\right) = \dfrac{(c-b)_m}{(c)_m}$ for $m \in \mathbb{N}^*$, we have

$$\overline{C}_{k,n}[f] = \sum_{j=0}^{k}\binom{k}{j} f\left(\frac{j}{n}\right)\frac{(a_n)_j\,(1-a_n)_{k-j}}{j!\,(j+1)_{k-j}},$$

in other words

$$\overline{C}_{k,n}\left[f\right]=\frac{1}{k!}\sum_{j=0}^{k}\binom{k}{j}f\left(\frac{j}{n}\right)\left(a_n\right)_j\left(1-a_n\right)_{k-j}.$$

When $a_n\in(0,1)$, it is clear that $f\geq 0$ on $[0,1]$ implies $\overline{C}_{k,n}\left[f\right]\geq 0$, that is $\overline{C}_n$ is a linear positive operator. □

For $g:[0,1]\to\mathbb{R}$ the Stancu operators $S_k:g\to S_kg$, $k\in\mathbb{N}$, are defined as $\left(S_0^{<b>}g\right)(x)=g(0)$ and for $k\in\{1,2,...\}$ (see [17], [18] and [4]):

$$\left(S_0^{<b>}g\right)(x)=\frac{1}{(b)_k}\sum_{j=0}^{k}\binom{k}{j}(bx)_j\,(b-bx)_{k-j}\,g\left(\frac{j}{k}\right),\quad x\in[0,1],$$

where $b\in[0,1]$ is a parameter. Observe that $\overline{C}_0f=\overline{C}_{0,0}\left[f\right]:=f(0)$ and

$$\left(S_k^{<1>}g\right)(a_n)=\frac{1}{k!}\sum_{j=0}^{k}\binom{k}{j}(a_n)_j\,(1-a_n)_{k-j}\,g\left(\frac{j}{k}\cdot\frac{k}{n}\right),\ k\geq 1.$$

Therefore,

$$\overline{C}_{k,n}\left[f\right]=\left(S_k^{<1>}g_{n,k}^{<f>}\right)(a_n)$$

with

$$g_{n,k}^{<f>}(t)=f\left(t\frac{k}{n}\right),\ k\geq 1.$$

Definition 11. *The linear transformations* $\overline{C}_{k,n}:Y\to\mathbb{R}$, $k\in\{0,1,2,...,n\}$, $n\in\mathbb{N}^*$, *are the Stancu functionals. When* $a_n\in(0,1)$, *the linear positive transformations* $\overline{C}_n:Y\to\Pi_n$, $n\in\mathbb{N}^*$, *are called* ***Bernstein-Stancu operators****.*

Using the Chu-Vandermonde identity

$$\sum_{k=0}^{n}\binom{n}{k}(a)_k\,(b)_{n-k}=(a+b)_n$$

it is possible to find the images of Stancu functionals $\overline{C}_{k,n}$ at some monomials.

Next we use the following proposition

Lemma 12. *(A.Lupas [9], pag. 205). Let n be fixed, $1 \leq s \leq n$, and $\|b_{n,k}\|$ be the Bernstein basis. Suppose that A is a linear mapping defined on the algebra of polynomials such that $\Pi_{s-1} \subseteq Ker\,(A)$. If*

$$p(x) = \sum_{k=0}^{n} a_k b_{n,k}(x),$$

then

$$A(p) = \sum_{j=0}^{n-s} A\left(\psi_{j,s}\right) \Delta^s a_j,$$

where

$$\Delta^s a_j = \sum_{\nu=0}^{s} (-1)^{s-\nu} \binom{s}{\nu} a_{j+\nu}$$

and

$$\psi_{j,s}\left(x\right) = \binom{n}{s+j} x^{s+j}_2 F_1\left(\, -n+s+j, j+1; s+j+1; x\right) =$$

$$= s\binom{n}{s} \int_0^x \left(x-y\right)^{s-1} b_{n-s,j}(y)dy.$$

Moreover,

$$\frac{1}{s!} \cdot \frac{d^s}{dx^s} \psi_{j,s}\left(x\right) = \binom{n}{s} b_{n-s,j}(x).$$

Using this proposition one can prove:

Theorem 13. *Let $a_n \in (0,1)$ and*

$$I_{n,j,\nu} = \int_0^1 t^{j-1+a_n}\left(1-t\right)^{-a_n} b_{n-j,\nu}(xt)dt.$$

Then

$$\frac{d^j}{dx^j}\left(\overline{C}_n f\right)(x) = \binom{n}{j} \frac{\left(j!\right)^2}{n^j} \cdot \frac{\sin\left(\pi a_n\right)}{\pi} \sum_{\nu=0}^{n-j} \left[\frac{j}{n}, \frac{j+1}{n}, ..., \frac{j+\nu}{n}; f\right] I_{n,j,\nu}.$$

Because the integrals $I_{n,j,\nu}$, $j \in \{0,1,2,...,n\}$, are positive it follows:

Corollary 14. *Let $j,n \in \mathbb{N}^*$, $0 \leq j \leq n-2$. The operator $\overline{C}_n$ preserves the convexity of order j.*

The asymptotic behavior of the sequence $\left(\overline{C}_n\right)_{n=1}^{\infty}$ on a certain subspaces of $C[-1,1]$ is given in the following proposition:

Theorem 15. *Suppose $x_0 \in [0,1]$ and $f''(x_0)$ exists. If $a_n \in (0,1)$, $\lim_{n\to\infty} a_n = 1$ and $L := \lim_{n\to\infty} n(1-a_n)$ exists, then*

$$\lim_{n\to\infty} n\left[f(x_0) - \left(\overline{C}_n f\right)(x_0)\right] = -\frac{x(1-x)}{2} f''(x_0) + \left[x_0 f'(x_0) - \frac{x_0^2}{4} f''(x_0)\right] L.$$

Proof. We apply a version of a general proposition given by R. G. Mamedov (see [7]). More precisely, let $\varphi : \mathbb{N} \to \mathbb{R}$, $\lim_{n\to\infty} \varphi(n) = \infty$, such that

$$\lim_{n\to\infty} \varphi(n)\left[e_k(x_0) - \left(\overline{C}_n e_k\right)(x_0)\right] = r_k(x_0),$$

for $k \in \{1,2\}$.

In our case

$$\begin{aligned}
& n\left[e_1(x_0) - \left(\overline{C}_n e_1\right)(x_0)\right] = n(1-a_n)x_0 \\
& n\left[e_2(x_0) - \left(\overline{C}_n e_2\right)(x_0)\right] = -x_0(1-x_0)a_n - \\
& -\frac{a_n(1-a_n)x_0^2}{2} + \frac{n(1-a_n)(2-a_n)x_0^2}{2} \\
& n\left[e_3(x_0) - \left(\overline{C}_n e_3\right)(x_0)\right] = \frac{3(1-n)}{2n}(a_n)_2 x_0^2 + \\
& +\frac{3n-2}{6n}(a_n)_3 x_0^3 - \frac{a_n}{n}x_0 - n\frac{(a_n)_3 - 6}{6}x_0^3 = \\
& = \frac{3(1-n)}{2n}(a_n)_2 x_0^2 + \frac{3n-2}{6n}(a_n)_3 x_0^3 - \\
& -\frac{a_n}{n}x_0 + \frac{n(1-a_n)(a_n^2+4a_n+6)}{6}x_0^3 \\
& n\left[e_4(x_0) - \left(\overline{C}_n e_4\right)(x_0)\right] = -\frac{(n-1)(n-2)}{n^2}(a_n)_3 x_0^3 + \\
& +\frac{6n^2-11n+6}{24n^2}(a_n)_4 x_0^4 + \frac{a_n}{n^2}x_0 + \frac{7(n-1)}{2n^2}(a_n)_2 x_0^2 - \\
& -\frac{n(1-a_n)\left(a_n^3+7a_n^2+18a_n+24\right)}{24}x_0^4.
\end{aligned}$$

Therefore

$$\begin{aligned}
& r_1(x_0) = Lx_0, \\
& r_2(x_0) = -x_0(1-x_0) + \frac{3L}{2}x_0^2, \\
& r_3(x_0) = -3x_0^2(1-x_0) + \frac{11L}{6}x_0^3, \\
& r_4(x_0) = -6x_0^3(1-x_0) + \frac{25L}{12}x_0^4.
\end{aligned}$$

If $\Omega_{4,x} = (t-x)^4$, then

$$\begin{array}{l} n\left(\overline{C_n}\Omega_{4,x}\right)(x) = -n\left(e_4(x) - \left(\overline{C_n}e_4\right)(x)\right) + 4nx\left(e_3(x) - \left(\overline{C_n}e_3\right)(x)\right) - \\ -6nx^2\left(e_2(x) - \left(\overline{C_n}e_2\right)(x)\right) + 4nx^3\left(e_1(x) - \left(\overline{C_n}e_1\right)(x)\right) \end{array} \Rightarrow$$

$$\lim_{n\to\infty} n\left(\overline{C}_n\Omega_{4,x}\right)(x) = -r_4(x) + 4xr_3(x) - 6x^2r_2(x) + 4x^3r_1(x) = \frac{Lx^4}{4}$$

and

$$\lim_{n\to\infty} \varphi(n)\left(\overline{C}_n\Omega_{4,x_0}\right)(x_0) = 0,$$

then

$$\lim_{n\to\infty} \varphi(n)\left[f(x_0) - \left(\overline{C}_n f\right)(x_0)\right] = \left[f'(x_0) - x_0 f''(x_0)\right] r_1(x_0) + \frac{r_2(x_0)}{2} f''(x_0) \quad (*)$$

and from $(*)$ we complete the proof. □

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FIRST ORDER ITERATIVE FUNCTIONAL-DIFFERENTIAL EQUATION WITH PARAMETER

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Dedicated to Professor D. D. Stancu on his 80^{th} *birthday*

Abstract. We consider the following first order iterative functional-differential equation with parameter

$$x'(t) = f(t, x(t), x(x(t))) + \lambda, \quad t \in [a, b];$$
$$x(t) = \varphi(t), \quad a_1 \le t \le a,$$
$$x(t) = \psi(t), \quad b \le t \le b_1.$$

Using the Schauder's fixed point theorem we first establish an existence theorem, then by means of the contraction principle state an existence and uniqueness theorem, and after that a data dependence result. Finally, we give some examples which illustrate our results.

1. Introduction

Although many works on functional-differential equation exist (see for example J. K. Hale and S. Verduyn Lunel [9], V. Kalmanovskii and A. Myshkis [10] and T. A. Burton [3] and the references therein), there are a few on iterative functional-differential equations ([2], [4], [5], [8], [12], [13], [16], [17], [19]).

In this paper we consider the following problem:

$$x'(t) = f(t, x(t), x(x(t))) + \lambda, \quad t \in [a, b]; \tag{1.1}$$

$$x|_{[a_1,a]} = \varphi, \qquad x|_{[b,b_1]} = \psi. \tag{1.2}$$

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where

(C_1) $a, b, a_1, b_1 \in \mathbb{R}$, $a_1 \le a < b \le b_1$;

(C_2) $f \in C([a,b] \times [a_1, b_1]^2, \mathbb{R})$;

(C_3) $\varphi \in C([a_1, a], [a_1, b_1])$ and $\psi \in C([b, b_1], [a_1, b_1])$;

The problem is to determine the pair (x, λ),

$$x \in C([a_1, b_1], [a_1, b_1]) \cap C^1([a, b], [a_1, b_1]), \quad \lambda \in \mathbb{R},$$

which satisfies (1.1)+(1.2).

In this paper, using the Schauder's fixed point theorem we first establish an existence theorem, then by means of the contraction principle state an existence and uniqueness theorem, and after that a data dependence result. Finally, we take an example to illustrate our results.

2. **Existence**

We begin our considerations with some remarks.

Let (x, λ) be a solution of the problem (1.1)+(1.2). Then this problem is equivalent with the following fixed point equation

$$x(t) = \begin{cases} \varphi(t), & t \in [a_1, a], \\ \varphi(a) + \displaystyle\int_a^t f(s, x(s), x(x(s)))\,\mathrm{d}s + \lambda(t - a), & t \in [a, b], \\ \psi(t), & t \in [b, b_1]. \end{cases} \tag{2.3}$$

From the condition of continuity of x in $t = b$, we have that

$$\lambda = \frac{\psi(b) - \varphi(a)}{b - a} - \frac{1}{b - a} \int_a^b f(s, x(s), x(x(s)))\,\mathrm{d}s. \tag{2.4}$$

Now we consider the operator

$$A : C([a_1, b_1], [a_1, b_1]) \to C([a_1, b_1], \mathbb{R}),$$

where

$$A(x)(t) := \begin{cases} \varphi(t), & t \in [a_1, a], \\ \varphi(a) + \dfrac{t-a}{b-a}(\psi(b) - \varphi(a)) - \dfrac{t-a}{b-a}\displaystyle\int_a^b f(s, x(s), x(x(s)))\,\mathrm{d}s + \\ \qquad + \displaystyle\int_a^t f(s, x(s), x(x(s)))\,\mathrm{d}s, & t \in [a, b], \\ \psi(t), & t \in [b, b_1]. \end{cases} \tag{2.5}$$

It is clear that (x, λ) is a solution of the problem (1.1)+(1.2) iff x is a fixed point of the operator A and λ is given by (2.4).

So, the problem is to study the fixed point equation

$$x = A(x).$$

We have

Theorem 2.1. *We suppose that*

(i) the conditions $(\mathrm{C}_1) - (\mathrm{C}_3)$ *are satisfied;*

(ii) $m_f \in \mathbb{R}$ *and* $M_f \in \mathbb{R}$ *are such that* $m_f \leq f(t, u_1, u_2) \leq M_f,\ \forall\, t \in [a, b],\ u_i \in [a_1, b_1],\ i = 1, 2,$ *and we have:*

$$a_1 \leq \min(\varphi(a), \psi(b)) - \max(0, M_f(b-a)) + \min(0, m_f(b-a)),$$

and

$$\max(\varphi(a), \psi(b)) - \min(0, m_f(b-a)) + \max(0, M_f(b-a)) \leq b_1.$$

Then the problem $(1.1) + (1.2)$ *has in* $C([a_1, b_1], [a_1, b_1])$ *at least a solution.*

Proof. In what follow we consider on $C([a_1, b_1], \mathbb{R})$ the Chebyshev norm, $||\cdot||_C$.

Condition (ii) assures that the set $C([a_1, b_1], [a_1, b_1])$ is an invariant subset for the operator A, that is, we have

$$A(C([a_1, b_1], [a_1, b_1])) \subset C([a_1, b_1], [a_1, b_1]).$$

Indeed, for $t \in [a_1, a] \cup [b, b_1]$, we have $A(x)(t) \in [a_1, b_1]$. Furthermore, we we obtain

$$a_1 \leq A(x)(t) \leq b_1,\ \forall t \in [a, b],$$

if and only if

$$a_1 \leq \min_{t\in[a,b]} A(x)(t) \tag{2.6}$$

and

$$\max_{t\in[a,b]} A(x)(t) \leq b_1 \tag{2.7}$$

hold. Since

$$\min_{t\in[a,b]} A(x)(t) = \min\left(\varphi(a), \psi(b)\right) - \max\left(0, M_f(b-a)\right) + \min\left(0, m_f(b-a)\right),$$

respectively

$$\max_{t\in[a,b]} A(x)(t) = \max\left(\varphi(a), \psi(b)\right) - \min\left(0, m_f(b-a)\right) + \max\left(0, M_f(b-a)\right),$$

the requirements (2.6) and (2.7) are equivalent with the conditions appearing in (ii).

So, in the above conditions we have a selfmapping operator

$$A : C([a_1, b_1], [a_1, b_1]) \to C([a_1, b_1], [a_1, b_1]).$$

It is clear that A is completely continuous and the set $C([a_1, b_1], [a_1, b_1]) \subseteq C([a_1, b_1], \mathbb{R})$ is a bounded convex closed subset of the Banach space $(C([a_1, b_1], \mathbb{R}), \| \cdot \|_C)$. By Schauder's fixed point theorem the operator A has at least a fixed point. □

3. **Existence and uniqueness**

Let $L > 0$, and introduce the following notation:

$$C_L([a_1, b_1], [a_1, b_1]) := \{x \in C([a_1, b_1], [a_1, b_1]) |\ |x(t_1) - x(t_2)| \leq L|t_1 - t_2|,$$
$$\forall\ t_1, t_2 \in [a_1, b_1]\}.$$

Remark that $C_L([a_1, b_1], [a_1, b_1]) \subset (C([a_1, b_1], \mathbb{R}), \| \cdot \|_C)$ is a complete metric space.

We have

Theorem 3.1. *We suppose that*

(i) *the conditions* $(C_1) - (C_3)$ *are satisfied;*

(ii) *there exists* $L_f > 0$ *such that:*

$$|f(t,u_1,u_2) - f(t,v_1,v_2)| \leq L_f\left(|u_1 - v_1| + |u_2 - v_2|\right),$$

for all $t \in [a,b]$, $u_i, v_i \in [a_1,b_1]$, $i = 1,2$;

(iii) $\varphi \in C_L([a_1,a],[a_1,b_1])$, $\psi \in C_L([b,b_1],[a_1,b_1])$;

(iv) $m_f, M_f \in \mathbb{R}$ *are such that* $m_f \leq f(t,u_1,u_2) \leq M_f$, $\forall\, t \in [a,b]$, $u_i \in [a_1,b_1]$, $i = 1,2$, *and we have:*

$$a_1 \leq \min\left(\varphi(a),\psi(b)\right) - \max\left(0, M_f(b-a)\right) + \min\left(0, m_f(b-a)\right),$$

and

$$\max\left(\varphi(a),\psi(b)\right) - \min\left(0, m_f(b-a)\right) + \max\left(0, M_f(b-a)\right) \leq b_1;$$

(v) $2\max\{|m_f|,|M_f|\} + \left|\dfrac{\psi(b)-\varphi(a)}{b-a}\right| \leq L$;

(vi) $2L_f(L+2)(b-a) < 1$.

Then the problem (1.1)+(1.2) *has in* $C_L([a_1,b_1],[a_1,b_1])$ *a unique solution. Moreover, if we denote by* (x^*,λ^*) *the unique solution of the Cauchy problem, then it can be determined by*

$$x^* = \lim_{n\to\infty} A^n(x), \text{ for all } x \in X,$$

and

$$\lambda^* = \frac{\psi(b)-\varphi(a)}{b-a} - \frac{1}{b-a}\int_a^b f(s, x^*(s), x^*(x^*(s)))\, ds.$$

Proof. Consider the operator $A : C_L([a_1,b_1],[a_1,b_1]) \to C([a_1,b_1],\mathbb{R})$ given by (2.5).

Conditions (iii) and (iv) imply that $C_L([a_1,b_1],[a_1,b_1])$ is an invariant subset for A. Indeed, from the Theorem 2.1 we have

$$a_1 \leq A(x)(t) \leq b_1,\ x(t) \in [a_1,b_1]$$

for all $t \in [a_1,b_1]$.

Now, consider $t_1,t_2 \in [a_1,a]$. Then,

$$|A(x)(t_1) - A(x)(t_2)| = |\varphi(t_1) - \varphi(t_2)| \leq L|t_1 - t_2|,$$

as $\varphi \in C_L([a_1, a], [a_1, b_1])$, due to (iii).

Similarly, for $t_1, t_2 \in [b, b_1]$

$$|A(x)(t_1) - A(x)(t_2)| = |\psi(t_1) - \psi(t_2)| \leq L|t_1 - t_2|,$$

that follows from (iii), too.

On the other hand, if $t_1, t_2 \in [a, b]$, we have,

$$\begin{aligned}
&|A(x)(t_1) - A(x)(t_2)| = \\
&= \left|\varphi(a) + \frac{t_1 - a}{b - a}(\psi(b) - \varphi(a)) - \frac{t_1 - a}{b - a}\int_a^b f(s, x(s), x(x(s)))\,\mathrm{d}s \right. \\
&+ \int_a^{t_1} f(s, x(s), x(x(s)))\,\mathrm{d}s - \varphi(a) - \frac{t_2 - a}{b - a}(\psi(b) - \varphi(a)) \\
&\left. + \frac{t_2 - a}{b - a}\int_a^b f(s, x(s), x(x(s)))\,\mathrm{d}s - \int_a^{t_2} f(s, x(s), x(x(s)))\,\mathrm{d}s \right| \\
&= \left|\frac{t_1 - t_2}{b - a}[\psi(b) - \varphi(a)] - \frac{t_1 - t_2}{b - a}\int_a^b f(s, x(s), x(x(s)))\,\mathrm{d}s - \int_{t_1}^{t_2} f(s, x(s), x(x(s)))\,\mathrm{d}s\right| \\
&\leq |t_1 - t_2| \left[\left|\frac{\psi(b) - \varphi(a)}{b - a}\right| + 2\max\{|m_f|, |M_f|\}\right] \leq L|t_1 - t_2|.
\end{aligned}$$

Therefore, due to (v), the operator A is L-Lipschitz and, consequently, it is an invariant operator on the space $C_L([a_1, b_1], [a_1, b_1])$.

From the condition (v) it follows that A is an L_A-contraction with

$$L_A := 2L_f(L + 2)(b - a).$$

Indeed, for all $t \in [a_1, a] \cup [b, b_1]$, we have $|A(x_1)(t) - A(x_2)(t)| = 0$.

Moreover, for $t \in [a,b]$ we get

$$
\begin{aligned}
&|A(x_1)(t) - A(x_2)(t)| \leq \\
\leq &\left| \frac{t-a}{b-a} \int_a^b [f(s,x_1(s),x_1(x_1(s))) - f(s,x_2(s),x_2(x_2(s)))] \, \mathrm{ds} \right| + \\
+ &\left| \int_a^t [f(s,x_1(s),x_1(x_1(s))) - f(s,x_2(s),x_2(x_2(s)))] \, \mathrm{ds} \right| \leq \\
\leq &\max_{t\in[a,b]} \left| \frac{t-a}{b-a} \right| \cdot L_f \int_a^b (|x_1(s) - x_2(s)| + |x_1(x_1(s)) - x_2(x_2(s))|) \, \mathrm{ds} + \\
+ &L_f \int_a^t (|x_1(s) - x_2(s)| + |x_1(x_1(s)) - x_2(x_2(s))|) \, \mathrm{ds} \leq \\
\leq &L_f \left[(b-a)||x_1 - x_2||_C + \int_a^b |x_1(x_1(s)) - x_1(x_2(s)) + x_1(x_2(s)) - x_2(x_2(s))| \mathrm{ds} \right] + \\
+ &L_f \left[(t-a)||x_1 - x_2||_C + \int_a^t |x_1(x_1(s)) - x_1(x_2(s)) + x_1(x_2(s)) - x_2(x_2(s))| \mathrm{ds} \right] \leq \\
\leq &2L_f(b-a) \left(||x_1 - x_2||_C + L||x_1 - x_2||_C + ||x_1 - x_2||_C \right) = \\
= &2L_f(L+2)(b-a)||x_1 - x_2||_C.
\end{aligned}
$$

By the contraction principle the operator A has a unique fixed point, that is the problem (1.1) + (1.2) has in $C_L([a_1,b_1],[a_1,b_1])$ a unique solution (x^*, λ^*).

Obviously, x^* can be determined by

$$x^* = \lim_{n\to\infty} A^n(x), \text{ for all } x \in X,$$

and, from (2.4) we get

$$\lambda^* = \frac{\psi(b) - \varphi(a)}{b-a} - \frac{1}{b-a} \int_a^b f(s, x^*(s), x^*(x^*(s))) \, \mathrm{d}s.$$

□

4. **Data dependence**

Consider the following two problems

$$\begin{cases} x'(t) = f_1(t, x(t), x(x(t))) + \lambda_1, & t \in [a, b] \\ x(t) = \varphi_1(t), & t \in [a_1, a] \\ x(t) = \psi_1(t), & t \in [b, b_1] \end{cases} \tag{4.8}$$

and

$$\begin{cases} x'(t) = f_2(t, x(t), x(x(t))) + \lambda_2, & t \in [a, b] \\ x(t) = \varphi_2(t), & t \in [a_1, a] \\ x(t) = \psi_2(t), & t \in [b, b_1] \end{cases} \tag{4.9}$$

Let f_i, φ_i and ψ_i, $i = 1, 2$ be as in the Theorem 3.1.

Consider the operators $A_1, A_2 : C_L([a_1, b_1], [a_1, b_1]) \to C_L([a_1, b_1], [a_1, b_1])$ given by

$$A_i(x)(t) := \begin{cases} \varphi_i(t), & t \in [a_1, a], \\ \varphi_i(a) + \dfrac{t-a}{b-a}(\psi_i(b) - \varphi_i(a)) - \dfrac{t-a}{b-a}\displaystyle\int_a^b f_i(s, x(s), x(x(s)))\,\mathrm{d}s + \\ \qquad + \displaystyle\int_a^t f_i(s, x(s), x(x(s)))\,\mathrm{d}s, & t \in [a, b], \\ \psi_i(t), & t \in [b, b_1], \end{cases} \tag{4.10}$$

$i = 1, 2$.

Thus, these operators are contractions. Denote by x_1^*, x_2^* their unique fixed points.

We have

Theorem 4.1. *Suppose we are in the conditions of the Theorem 3.1, and, moreover*

(i) *there exists* η_1 *such that*

$$|\varphi_1(t) - \varphi_2(t)| \le \eta_1, \quad \forall t \in [a_1, a],$$

and

$$|\psi_1(t) - \psi_2(t)| \le \eta_1, \quad \forall t \in [b, b_1];$$

(ii) *there exists* $\eta_2 > 0$ *such that*

$$|f_1(t, u_1, u_2) - f_2(t, u_1, u_2)| \leq \eta_2, \ \forall \ t \in [a, b], \ \forall \ u_i \in [a_1, b_1], \ i = 1, 2.$$

Then

$$||x_1^* - x_2^*||_C \leq \frac{3\eta_1 + 2(b-a)\eta_2}{1 - 2L_f(L+2)(b-a)},$$

and

$$|\lambda_1^* - \lambda_2^*| \leq \frac{2\eta_1}{b-a} + \eta_2,$$

where $L_f = \max(L_{f_1}, L_{f_2})$*, and* (x_i^*, λ_i^*)*,* $i = 1, 2$ *are the solutions of the corresponding problems* (4.8)*,* (4.9)*.*

Proof. It is easy to see that for $t \in [a_1, a] \cup [b, b_1]$ we have

$$\|A_1(x) - A_2(x)\|_C \leq \eta_1.$$

On the other hand, for $t \in [a, b]$, we obtain

$$\begin{aligned}
|A_1(x)(t) - A_2(x)(t)| = \Bigg| &\varphi_1(a) - \varphi_2(a) + \frac{t-a}{b-a}\left[\psi_1(b) - \psi_2(b) - (\varphi_1(a) - \varphi_2(a))\right] - \\
&- \frac{t-a}{b-a}\int_a^b [f_1(s, x(s), x(x(s))) - f_2(s, x(s), x(x(s)))]\, \mathrm{d}s + \\
&+ \int_a^t [f_1(s, x(s), x(x(s))) - f_2(s, x(s), x(x(s)))]\, \mathrm{d}s \Bigg| \leq \\
&\leq |\varphi_1(a) - \varphi_2(a)| + \frac{t-a}{b-a}\left[|\psi_1(b) - \psi_2(b)| + |\varphi_1(a) - \varphi_2(a)|\right] + \\
&+ \frac{t-a}{b-a}\int_a^b |f_1(s, x(s), x(x(s))) - f_2(s, x(s), x(x(s)))|\, \mathrm{d}s + \\
&+ \int_a^t |f_1(s, x(s), x(x(s))) - f_2(s, x(s), x(x(s)))|\, \mathrm{d}s \leq \\
&\leq \eta_1 + \max_{t \in [a,b]} \frac{t-a}{b-a} \cdot [2\eta_1 + \eta_2(b-a)] + \eta_2 \cdot \max_{t \in [a,b]}(t-a) = \\
&= 3\eta_1 + 2(b-a)\eta_2
\end{aligned}$$

So, we have

$$\|A_1(x) - A_2(x)\|_C \leq 3\eta_1 + 2(b-a)\eta_2, \ \forall \ x \in C_L([a_1, b_1], [a_1, b_1]).$$

Consequently, from the data dependence theorem we obtain

$$\|x_1^* - x_2^*\|_C \le \frac{3\eta_1 + 2(b-a)\eta_2}{1 - 2L_f(L+2)(b-a)}.$$

Moreover, we get

$$|\lambda_1^* - \lambda_2^*| =$$
$$= \left| \frac{\psi_1(b) - \varphi_1(a)}{b-a} - \frac{1}{b-a}\int_a^b f_1(s, x(s), x(x(s)))\,\mathrm{d}s - \frac{\psi_2(b) - \varphi_2(a)}{b-a} + \right.$$
$$\left. + \frac{1}{b-a}\int_a^b f_2(s, x(s), x(x(s)))\,\mathrm{d}s \right| \le$$
$$\le \frac{1}{b-a}\left[|\psi_1(b) - \psi_2(b)| + |\varphi_1(a) - \varphi_2(a)| + \right.$$
$$\left. + \int_a^b |f_1(s, x(s), x(x(s))) - f_2(s, x(s), x(x(s)))|\,\mathrm{d}s \right] \le$$
$$\le \frac{1}{b-a}[\eta_1 + \eta_1 + \eta_2(b-a)] = \frac{2\eta_1}{b-a} + \eta_2,$$

and the proof is complete. □

5. **Examples**

Consider the following problem:

$$x'(t) = \mu x(x(t)) + \lambda; \quad t \in [0,1], \ \ \mu \in \mathbb{R}_+^*, \ \ \lambda \in \mathbb{R} \tag{5.11}$$

$$x|_{[-h,0]} = 0; \quad x|_{[1,1+h]} = 1, \quad h \in \mathbb{R}_+^* \tag{5.12}$$

with $x \in C([-h, 1+h], [-h, 1+h]) \cap C^1([0,1], [-h, 1+h])$.

We have

Proposition 5.1. *We suppose that*

$$\mu \le \frac{h}{1+2h}.$$

Then the problem (5.11) *has in* $C([-h, 1+h], [-h, 1+h])$ *at least a solution.*

Proof. First of all notice that accordingly to the Theorem 2.1 we have $a = 0$, $b = 1$, $\psi(b) = 1, \varphi(a) = 0$ and $f(t, u_1, u_2) = \mu u_2$. Moreover, $a_1 = -h$ and $b_1 = 1 + h$ can be taken. Therefore, from the relation

$$m_f \leq f(t, u_1, u_2) \leq M_f,\ \forall t \in [0,1], \forall u_1, u_2 \in [-h, 1+h],$$

we can choose $m_f = -h\mu$ and $M_f = (1+h)\mu$.

For these data it can be easily verified that the conditions (ii) from the Theorem 2.1 are equivalent with the relation

$$\mu \leq \frac{h}{1+2h},$$

consequently we have the proof. □

Let $L > 0$ and consider the complete metric space $C_L([-h, h+1], [-h, h+1])$ with the Chebyshev norm $\|\cdot\|_C$.

Another result reads as follows.

Proposition 5.2. *Consider the problem* (5.11). *We suppose that*

(i) $\mu \leq \frac{h}{1+2h}$;

(ii) $2(1+h)\mu + 1 \leq L$

(iii) $2\mu(L+2) < 1$

Then the problem (5.11) *has in* $C_L([-h, h+1], [-h, h+1])$ *a unique solution.*

Proof. Observe that the Lipschitz constant for the function $f(t, u_1, u_2) = \mu u_2$ is $L_f = \mu$.

By a common check in the conditions of Theorem 3.1 we can make sure that

$$2\max\{|m_f|, |M_f|\} + \left|\frac{\psi(b) - \varphi(a)}{b-a}\right| \leq L \Longleftrightarrow 2(1+h)\mu + 1 \leq L,$$

and

$$2L_f(L+2)(b-a) < 1 \Longleftrightarrow 2\mu(L+2) < 1.$$

Therefore, by Theorem 3.1 we have the proof. □

Now take the following problems

$$x'(t) = \mu_1 x(x(t)) + \lambda; \quad t \in [0,1],\ \ \mu_1 \in \mathbb{R}_+^*,\ \ \lambda \in \mathbb{R} \tag{5.13}$$

$$x|_{[-h,0]} = \varphi_1; \quad x|_{[1,1+h]} = \psi_1, \quad h \in \mathbb{R}_+^* \tag{5.14}$$

$$x'(t) = \mu_2 x(x(t)) + \lambda; \quad t \in [0,1], \ \ \mu_2 \in \mathbb{R}_+^*, \ \ \lambda \in \mathbb{R} \tag{5.15}$$

$$x|_{[-h,0]} = \varphi_2; \quad x|_{[1,1+h]} = \psi_2, \quad h \in \mathbb{R}_+^*. \tag{5.16}$$

Suppose that we have satisfied the following assumptions

(H_1) $\varphi_i \in C_L([-h,0],[-h,1+h])$, $\psi_i \in C_L([1,1+h],[-h,1+h])$, such that $\varphi_i(0) = 0$, $\psi_i(1) = 1$, $i = 1,2$;

(H_2) we are in the conditions of Proposition 5.2 for both of the problems (5.13) and (5.15).

Let (x_1^*, λ_1^*) be the unique solution of the problem (5.13) and (x_2^*, λ_2^*) the unique solution of the problem (5.15). We are looking for an estimation for $\|x_1^* - x_2^*\|_C$.

Then, build upon Theorem 4.1, by a common substitution one can make sure that we have

Proposition 5.3. *Consider the problems* (5.13), (5.15) *and suppose the requirements* $H_1 - H_2$ *hold. Additionally,*

(i) *there exists* η_1 *such that*

$$|\varphi_1(t) - \varphi_2(t)| \leq \eta_1, \quad \forall t \in [-h,0],$$

$$|\psi_1(t) - \psi_2(t)| \leq \eta_1, \quad \forall t \in [1,1+h];$$

(ii) *there exists* $\eta_2 > 0$ *such that*

$$|\mu_1 - \mu_2| \cdot |u_2| \leq \eta_2, \ \forall \ t \in [0,1], \ \forall \ u_2 \in [-h,1+h].$$

Then

$$\|x_1^* - x_2^*\|_C \leq \frac{3\eta_1 + 2\eta_2}{1 - 2(L+2) \cdot \max\{\mu_1, \mu_2\}},$$

and

$$|\lambda_1^* - \lambda_2^*| \leq 2\eta_1 + \eta_2.$$

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ON BERNSTEIN-STANCU TYPE OPERATORS

I. GAVREA

Dedicated to Professor D. D. Stancu on his 80^{th} *birthday*

Abstract. D.D. Stancu defined in [5] a class of approximation operators depending of two non-negative parameters α and β, $0 \leq \alpha \leq \beta$. We consider here another class of Bernstein-Stancu type operators.

1. **Introduction**

Let f be a continuous functions, $f : [0,1] \to \mathbb{R}$. For every natural number n we denote by $B_n f$ Bernstein's polynomial of degree n,

$$(B_n f)(x) = \sum_{k=0}^{n} p_{n,k}(x) f\left(\frac{k}{n}\right),$$

where

$$p_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}, \quad k = 0, 1, \ldots n.$$

In 1968 D.D. Stancu introduced in [5] a linear positive operator depending on two non-negative parameters α and β satisfying the condition $0 \leq \alpha \leq \beta$.

For every continuous function f and for every $n \in \mathbb{N}$ the polynomial $P_n^{(\alpha,\beta)} f$ defined in [5] is given by

$$(P_n^{(\alpha,\beta)} f)(x) = \sum_{k=0}^{n} p_{n,k}(x) f\left(\frac{k+\alpha}{n+\beta}\right).$$

Note that for $\alpha = \beta = 0$ the Bernstein-Stancu operators become the classical Bernstein operators B_n. In [2] were introduced the following linear operators A_n :

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$C[0,1] \to \Pi_n$, defined as

$$A_n(f)(x) = \sum_{k=0}^{n} p_{n,k}(x) T_{n,k} f \tag{1.1}$$

where $T_{n,k} : C[0,1] \to \mathbb{R}$ are positive linear functionals with the property that $T_{n,k}e_0 = 1$ for $k = 0, 1, \ldots, n$ and $e_i(t) = t^i$, $i \in \mathbb{N}$.

So, for $T_{n,k}f = f\left(\frac{k}{n}\right)$ we obtain Bernstein's polynomial of degree n and for

$$T_{n,k} f = f\left(\frac{k+\alpha}{n+\beta}\right)$$

where $0 \le \alpha \le \beta$ the operator A_n becomes Bernstein-Stancu operator $P_n^{(\alpha,\beta)}$.

In [4] C. Mortici and I. Oancea defined a new class of operators of Bernstein-Stancu type operators. They considered the non-nonegative real numbers $\alpha_{n,k}$, $\beta_{n,k}$ so that

$$\alpha_{n,k} \le \beta_{n,k}.$$

They define an approximation operator denoted by

$$P_n^{(A,B)} : C[0,1] \to C[0,1]$$

with the formula

$$(P_n^{(A,B)} f)(x) = \sum_{k=0}^{n} p_{n,k}(x) f\left(\frac{k+\alpha_{n,k}}{m+\beta_{n,k}}\right).$$

In [4] the following theorem was proved:

Theorem 1.1. *Given the infinite dimensional lower triangular matrices*

$$A = \begin{pmatrix} \alpha_{00} & 0 & \ldots & & \\ \alpha_{10} & \alpha_{11} & 0 & \ldots & \\ \alpha_{20} & \alpha_{21} & \alpha_{22} & 0 & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \end{pmatrix}$$

and

$$B = \begin{pmatrix} \beta_{00} & 0 & \dots & & \\ \beta_{10} & \beta_{11} & 0 & \dots & \\ \beta_{20} & \beta_{21} & \beta_{22} & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots \end{pmatrix}$$

with the following properties:

a) $0 \le \alpha_{n,k} \le \beta_{n,k}$ *for every non-negative integers* n *and* $k \le n$

b) $\alpha_{n,k} \in [a,b]$, $\beta_{n,k} \in [c,d]$ *for every non-negative integers* n *and* k, $k \le n$ *and for some non-negative real numbers* $0 \le a < b$ *and* $0 \le c < d$. *Then for every continuous function* $f \in C[0,1]$, *we have*

$$\lim_{m\to\infty} P_n^{(\alpha,\beta)} f = f, \text{ uniformly on } [0,1].$$

In the following, by the definition, an operator of the form (1.1), where

$$T_{n,k}f = f(x_{k,n}), \quad k \le n, \quad k, n \in \mathbb{N}$$

is an operator of the Bernstein-Stancu type.

2. Main results

First we characterize the Bernstein-Stancu operators which transform the polynomial of degree one into the polynomials of degree one.

Theorem 2.1. *Let* $A_n : C[0,1] \to C[0,1]$ *an operator of the Bernstein-Stancu type.*

Then

$$x_{k,n} = \alpha_n \frac{k}{n} + \beta_n, \quad k \le n$$

where α_n, β_n *are positive numbers such that*

$$\alpha_n + \beta_n \le 1.$$

Proof. By the definition of the operator A_n of the Bernstein-Stancu type we have

$$A_n(e_0)(x) = \sum_{k=0}^{n} p_{n,k}(x) = 1.$$

Let us suppose that

$$A_n(e_1)(x) = \alpha_n x + \beta_n.$$

From the equality

$$\sum_{k=0}^{n} p_{n,k}(x)\frac{k}{n} = x$$

we get

$$\sum_{k=0}^{n} p_{n,k}(x)x_{k,n} = \sum_{k=0}^{n} p_{n,k}(x)\left(\alpha_n\frac{k}{n} + \beta_n\right). \tag{2.1}$$

Because the set $\{p_{n,k}\}_{k\in\{0,1,\dots,n\}}$ forms a basis in Π_n we get

$$x_{k,n} = \alpha_n\frac{k}{n} + \beta_n.$$

By the condition $x_{k,n} \in [0,1]$, $0 \le k \le n$, $k,n \in \mathbb{N}$ we obtain

$$\alpha_n, \beta_n \ge 0 \text{ and } \alpha_n + \beta_n \le 1.$$

Remark. There exist operators of the Bernstein-Stancu type which don't transform polynomials of degree one into the polynomials of the same degree.

An interesting operator of Bernstein-Stancu type, which maps e_2 into e_2 is the following:

$$B_n^*(f)(x) = \sum_{k=0}^{n} p_{n,k} f\left(\sqrt{\frac{k(k-1)}{n(n-1)}}\right), \quad n \in \mathbb{N},\ n > 1. \tag{2.2}$$

For the operator B_n^* verifies the following relations:

$$B_n^*(e_0) = e_0$$

$$B_n^*(e_2) = e_2$$

$$\frac{nx-1}{n-1} - \frac{1}{n}p_{n,1}(x) \le B_n(e_1)(x) \le x.$$

The following result describes the fact that $(A_n)_{n\in\mathbb{N}}$ given by (1.1) is a positive linear approximation process.

Theorem 2.2. *Let $(A_n)_{n\in\mathbb{N}}$ be defined as in (1.1) and $f \in C[0,1]$. Then*

$$\lim_{n\to\infty} \|f - A_n f\|_\infty = 0 \tag{2.3}$$

if and only if

$$\lim_{n\to\infty} \|\Delta_n\|_\infty = 0 \tag{2.4}$$

where

$$\Delta_n(x) := \sum_{k=0}^{n} p_{n,k}(x) T_{n,k}\left(\cdot - \frac{k}{n}\right)^2. \tag{2.5}$$

Proof. ($\Rightarrow$): For the validity of (2.4) it is sufficient to verify the assumption of Popoviciu-Bohman-Korovkin theorem. We first notice that

$$|\Delta_n(x)| = \left|\sum_{k=0}^{n} p_{n,k}(x) T_{n,k}(e_2) - 2\sum_{k=0}^{n} p_{n,k}(x)\frac{k}{n} T_{n,k}(e_1) + x^2 + \frac{x(1-x)}{n}\right| \tag{2.6}$$

and if for all $f \in C[0,1]$

$$\lim_{n\to\infty} \|f - A_n f\|_\infty = 0,$$

we get

$$\lim_{n\to\infty} \sum_{k=0}^{n} p_{n,k}(x) T_{n,k}(e_2) = \lim_{n\to\infty} A_n(e_2)(x) = x^2$$

and

$$\lim_{n\to\infty} \left\{\sum_{k=0}^{n} p_{n,k}(x)\frac{k}{n} T_{n,k}(e_1) - x^2\right\} = \lim_{n\to\infty} \sum_{k=0}^{n} p_{n,k}(x)\frac{k}{n}\{T_{n,k}(e_1) - x\}.$$

Now, we can estimate

$$\left|\sum_{k=0}^{n} p_{n,k}(x)\frac{k}{n}\{T_{n,k}(e_1) - x\}\right| \le \sum_{k=0}^{n} p_{n,k}(x) T_{n,k}(|e_1 - x|) \le \sqrt{A_n(\cdot - x)^2(x)}.$$

From this and (2.6) it follows that

$$|\Delta_n(x)| \le |A_n(e_2)(x) - x^2| + 2\sqrt{A_n(1-x)^2(x)} + \frac{x(1-x)}{n}$$

and therefore one obtains

$$\lim_{n\to\infty} \|\Delta_n\|_\infty = 0.$$

($\Leftarrow$): Suppose now that (2.4) holds with the following two estimates

$$|A_n(e_1)(x) - x| \le \sqrt{\Delta_n(x)}$$

and

$$|A_n(e_2)(x) - x^2| = \left|\sum_{k=0}^{n} p_{n,k}(x) T_{n,k}\left(\cdot - \frac{k}{n}\right)\left(\cdot + \frac{k}{n}\right) + \frac{x(1-x)}{n}\right|$$
$$\leq 2\sum_{k=0}^{n} p_{n,k}(x) T_{n,k}\left(\left|\cdot - \frac{k}{n}\right|\right) + \frac{x(1-x)}{n}$$
$$\leq 2\sqrt{\Delta_n(x)} + \frac{x(1-x)}{n}$$

and finishes the proof of this theorem.

Remarks. 1. Theorem 2.2 can be find in [2].

2. Theorem 1.1 ([4]) follows from the following estimate:

$$\Delta_n(x) = \sum_{k=0}^{n} p_{n,k}(x)\left(\frac{k+\alpha_{n,k}}{n+\beta_{n,k}} - \frac{k}{n}\right)^2$$
$$= \sum_{k=0}^{n} p_{n,k}(x) \frac{(n\alpha_{n,k} - k\beta_{n,k})^2}{n^2(n+\beta_{n,k})^2}$$
$$\leq \sum_{k=0}^{n} p_{n,k}(x) \frac{(b+d)^2}{(n+a)^2} = \frac{(b+d)^2}{(n+a)^2}$$

Theorem 2.3. *Let A_n be an operator of the form (1.1) such that*

$$A_n e_1 = \alpha_n e_1 + \beta_n.$$

We denote by L_n the operator of Bernstein-Stancu type given by

$$(L_n f)(x) = \sum_{k=0}^{n} p_{n,k}(x) f\left(\alpha_n \frac{k}{n} + \beta_n\right).$$

Then, for all $x \in [0,1]$ and for all convex functions f we have

$$f(\alpha_n x + \beta_n) \leq (L_n f)(x) \leq A_n(f)(x).$$

Moreover, if f is a strict convex function and $L_n(f)(x_0) = A_n(f(x_0))$ for some $x_0 \in (0,1)$, if and only if $L_n = A_n$.

Proof. Because $(p_{n,k})_{k=0,n}$ is a basis in Π_n by the condition

$$A_n e_1 = \alpha_n e_1 + \beta_n$$

we obtain that

$$T_{n,k}e_1 = \alpha_n \frac{k}{n} + \beta_n.$$

Let f be a convex function. From Jensen's inequality we have

$$T_{n,k}(f) \geq f(T_{n,k}(e_1)) = f\left(\alpha_n \frac{k}{n} + \beta_n\right) \tag{2.7}$$

By (2.7) we get

$$\sum_{k=0}^{n} p_{n,k}(x) T_{n,k}(f) \geq \sum_{k=0}^{n} p_{n,k}(x) f\left(\alpha_n \frac{k}{n} + \beta_n\right) \geq f(\alpha_n x + \beta_n)$$

or

$$A_n(f)(x) \geq (L_n f)(x) \geq f(\alpha_n x + \beta_n).$$

Let us suppose that

$$L_n(f)(x_0) = A_n(f)(x_0). \tag{2.8}$$

The equality (2.8) can be written as:

$$\sum_{k=0}^{n} p_{n,k}(x_0) \left(T_{n,k}(f) - f\left(\alpha_n \frac{k}{n} + \beta_n\right)\right) = 0.$$

Because

$$p_{n,k}(x_0) \geq 0, \quad k = 0, 1, \ldots, n$$

follows that

$$T_{n,k}(f) - f\left(\alpha_n \frac{k}{n} + \beta_n\right) = 0, \quad k = 0, 1, \ldots, n. \tag{2.9}$$

It is known the following result [3]:

Let A be a linear positive functional, $A : C[0,1] \to \mathbb{R}$. Then, there exists the distinct points $\xi_1, \xi_2 \in [0,1]$ such that

$$A(f) - f(a_1) = [a_2^2 - a_1^2] \left[\xi_1, \frac{\xi - 1 + \xi_2}{2}, \xi_2; f\right] \tag{2.10}$$

where $a_i = A(e_i)$, $i \in \mathbb{N}$.

By (2.9) and (2.10) we obtain

$$(T_{n,k}(e_2)) - T_{n,k}^2(e_1) = 0, \quad k = 0, 1, \ldots, n. \tag{2.11}$$

From (2.11) we get

$$T_{n,k}(f) = f(T_{n,k}(e_1)) = f\left(\alpha_n \frac{k}{n} + \beta_n\right), \quad k = 0, 1, \ldots, n$$

for every continuous function f.

This finished the proof.

Remark. This extremal relation for the Bernstein-Stancu operators was considered in [1] in particular case when $f = e_2$.

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MEAN CONVERGENCE OF FOURIER SUMS ON UNBOUNDED INTERVALS

G. MASTROIANNI AND D. OCCORSIO

Dedicated to Professor D. D. Stancu on his 80^{th} *birthday*

Abstract. In this paper we consider the approximation of functions by suitable "truncated" Fourier Sums in the generalized Freud and Laguerre systems. We prove necessary and sufficient conditions for the uniform boundedness in L_p weighted spaces.

1. **Introduction**

Let be $W_{\alpha,\beta}(x) =: W_\alpha(x) = |x|^\alpha e^{-|x|^\beta}, x \in \mathbb{R}, \alpha > -1, \beta > 1$ a generalized Freud weight and denote by $\{p_m(W_\alpha)\}_m$ the corresponding sequence of orthonormal polynomials with positive leading coefficients, i.e.

$$p_m(W_\alpha, x) = \gamma_m(W_\alpha)x^m + \dots, \quad \gamma_m(W_\alpha) > 0, \quad m = 0, 1, \dots.$$

These polynomials introduced and studied in [3](see also [4], [5]) are a generalization of Sonin-Markov polynomials. Let be $S_m(W_\alpha, f)$ the $m-$th partial Fourier sum of a measurable function f in the system $\{p_m(W_\alpha)\}_m$, i.e.

$$S_m(W_\alpha, f, x) = \sum_{k=0}^{m} c_k p_k(W_\alpha, x), \quad c_k = \int_{\mathbb{R}} f(t)p_k(W_\alpha, t)W_\alpha(t)dt.$$

For $\alpha = 0$, the boundedness in weighted L_p spaces of $S_m(W_\alpha, f, x)$ holds only for a "small" range of p (see [2]). To be more precise, in [2] the authors proved the bound

$$\|S_m(W_0(x), f,)\sqrt{W_0}\|_p \le \mathcal{C}\|f\sqrt{W_0}\|_p \quad (1)$$

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for $\frac{4}{3} < p < 4$ and $\beta = 2, 4, 6...$, while for $p \geq 4$ and $p \leq \frac{4}{3}$ estimate of kind (1) cannot always hold. In the same paper [2] the authors, in order to extend the range of p, modify the weight in the norm obtaining, under suitable assumptions on b, B, β, not homogenous estimates of the kind

$$\|S_m(W_0(x), f)\sqrt{W_0}(1+|x|)^b\|_p \leq \mathcal{C}\|f\sqrt{W_0}(1+|x|)^B\|_p, \quad 1 < p < \infty. \tag{2}$$

In the case $\alpha = 0$ and $\beta = 2$ (Hermite polynomials) estimates of types (1) and (2) were already proved in [12] (see also [1]).
Let be $U_\gamma(x) = |x|^\gamma e^{-\frac{|x|^\beta}{2}}, x \in \mathbb{R}, \gamma > -\frac{1}{p}$. Denote by $a_m = a_m(W_\alpha)$ the Mhaskar-Rahmanov-Saff number (M-R-S number) with respect to W_α and by $\Delta_{m,\theta}$ the characteristic function of the segment $A_m = [-\theta a_m, \theta a_m]$, with $0 < \theta < 1$. In this paper, we will prove inequalities of kind

$$\|S_m(W_\alpha, f\Delta_{m,\theta})U_\gamma\Delta_{m,\theta}\|_p \leq \mathcal{C}\|fU_\gamma\Delta_{m,\theta}\|_p, \tag{3}$$

with $1 < p < \infty$, under certain conditions on α and γ which are necessary and sufficient. Since we prove also that, for $m \to \infty$, the norm $\|[f - \Delta_{m,\theta}S_m(W_\alpha, \Delta_{m,\theta}f)]U_\gamma\|_p$ converges to zero essentially like the error of the best approximation in $L^p_{U_\gamma}$, then in order to approximate a function $f \in L^p_{U_\gamma}$ (see (7) for the definition) the sequence $\{\Delta_{m,\theta}S_m(W_\alpha, f\Delta_{m,\theta})\}_m$ is simpler and more convenient than the ordinary Fourier sum.

An inequality of type (3) has been proved in [12], in the special case of the Hermite weight. The proof in [12] requires a precise estimate of the difference $|p_{m+1}(x) - p_{m-1}(x)|$ where $p_m(x)$ is the $m-$th Hermite polynomial. This estimate for weights W_α is not available in the literature and, on the other hand, it isn't required in our proof. The case $p = 1$ is also considered when the functions are in the Calderon-Zygmund spaces.

As consequence of estimate (3) we derive the analogous one for Fourier sums in the system of orthogonal polynomials w.r.t generalized Laguerre weights $w_\alpha(x) = x^\alpha e^{-x^\beta/2}$, $x \geq 0, \alpha > -1, \beta > \frac{1}{2}$.

The plan of the paper is the following: next section contains some basic facts necessary to introduce the main results given in section 3. Section 4 contains all the proofs.

2. **Preliminary**

In the sequel $\mathcal{C}$ denotes a positive constant which can be different in different formulas. Moreover we write $\mathcal{C} \neq \mathcal{C}(a, b, ..)$ when the constant $\mathcal{C}$ is independent of $a, b, ..$

Let be $U_\gamma(x) = |x|^\gamma e^{-\frac{|x|^\beta}{2}}, \gamma > -\frac{1}{p}, \beta > 1$ and denote by $\overline{a}_m = \overline{a}_m(U_\gamma)$ the M-R-S number w.r.t. U_γ. The following "infinite-finite range inequality" holds [3]

$$\|P_m U_\gamma\|_{L_p(\mathbb{R})} \leq \mathcal{C}\|P_m U_\gamma\|_{L_p(|x| \leq \overline{a}_m(1-\mathcal{C}m^{-2/3})}.$$

We remark that $\overline{a}_m = \overline{a}_m(U_\gamma)$ can be expressed as [7]

$$\overline{a}_m = m^{\frac{1}{\beta}}\mathcal{C}(\beta, \gamma), \tag{4}$$

where the positive constant $\mathcal{C}(\beta, \gamma)$ will not be used in the sequel (analogously for $a_m = a_m(W_\alpha)$). Moreover we recall the following inequalities [7]:

$$\|P_m U_\gamma\|_{L_p(|x| \geq \overline{a}_m(1+\delta))} \leq \mathcal{C}_1 e^{-\mathcal{C}_2 m}\|P_m U_\gamma\|_{L_p(-\overline{a}_m, \overline{a}_m)} \tag{5}$$

and

$$\|P_m U_\gamma\|_{L_p(\mathbb{R})} \leq \mathcal{C}\|P_m U_\gamma\|_{L_p(\frac{\overline{a}_m}{m} \leq |x| \leq \overline{a}_m)}, \tag{6}$$

where $\delta > 0$ is fixed and the constants $\mathcal{C}, \mathcal{C}_1, \mathcal{C}_2$ are independent of m and P_m.
For $1 \leq p < \infty$ define the space

$$L^p_{U_\gamma} = \left\{ f : \left(\int_{-\infty}^{\infty} |f(x) U_\gamma(x)|^p dx \right)^{\frac{1}{p}} < \infty \right\} \tag{7}$$

and denote by

$$E_m(f)_{U_\gamma, p} = \inf_{P \in \mathbb{P}_m} \|(f - P)U_\gamma\|_p \tag{8}$$

the error of the best approximation in $L^p_{U_\gamma}$.

For a fixed real θ with $0 < \theta < 1$ we shall denote by $\overline{\Delta}_{m,\theta}$ the characteristic function of $D_m = (-\theta \overline{a}_m, \theta \overline{a}_m), \overline{a}_m = \overline{a}_m(U_\gamma)$. Next Proposition is useful for our goals.

Proposition 2.1. *Let $f \in L^p_{U_\gamma}$ and $1 \le p \le \infty$. For m sufficiently large (say $m > m_0$) we have*

$$\|f(1-\overline{\Delta}_{m,\theta})U_\gamma\|_p \le \mathcal{C}_1 \left(E_M(f)_{U_\gamma,p} + e^{-\mathcal{C}_2 m} \|fU_\gamma\|_p \right), \tag{9}$$

where $M = \left[m \left(\frac{\theta}{1+\theta} \right)^\beta \right]$ [1] *and the constants $\mathcal{C}, \mathcal{C}_1, C_2$ are independent on m and f.*

By (9) we get

$$\|fU_\gamma\|_p \le \mathcal{C} \left(E_M(f)_{U_\gamma,p} + \|f\overline{\Delta}_{m,\theta}U_\gamma\|_p \right). \tag{10}$$

Then, by virtue of Proposition 2.1 we will go to consider the behaviour of the sequence $\{\Delta_{m,\theta} S_m(W_\alpha, \Delta_{m,\theta} f)\}_m$ instead of $\{S_m(W_\alpha, f)\}_m$, where here and in the sequel $\Delta_{m,\theta}$ is the characteristic function of $[-\theta a_m, \theta a_m]$, with $a_m = a_m(W_\alpha) < \overline{a}_m(U_\gamma)$.

3. **Main results**

Now we are able to state the next two Theorems.

Theorem 3.1. *Let be $U_\gamma(x) = |x|^\gamma e^{-|x|^\beta/2}, \gamma > -\frac{1}{p}, \beta > 1$, $1 < p < \infty$ and $f \in L^p_{U_\gamma}$. Then, there exists a constant $\mathcal{C} \ne \mathcal{C}(m,f)$ such that*

$$\|S_m(W_\alpha, f\Delta_{m,\theta})U_\gamma \Delta_{m,\theta}\|_p \le \mathcal{C} \|fU_\gamma \Delta_{m,\theta}\|_p, \tag{11}$$

if and only if

$$-\frac{1}{p} < \gamma - \frac{\alpha}{2} < \frac{1}{q}, \quad q = \frac{p}{p-1}. \tag{12}$$

Moreover, if (12) holds, it results also

$$\|[f - \Delta_{m,\theta} S_m(W_\alpha, \Delta_{m,\theta} f)]U_\gamma\|_p \le \mathcal{C} \left(E_M(f)_{U_\gamma,p} + e^{-\mathcal{C}_1 m} \|fU_\gamma\|_p \right) \tag{13}$$

with $\mathcal{C} \ne \mathcal{C}(m,f)$, $\mathcal{C}_1 \ne \mathcal{C}_1(m,f)$.

Setting

$$\log^+ f(x) = \log\left(max(1, f(x))\right),$$

we prove

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[1] $[a]$ denotes the largest integer smaller than or equal to $a \in \mathbb{R}^+$

Theorem 3.2. *Let be $U_\gamma(x) = |x|^\gamma e^{-|x|^\beta/2}, \gamma > -1, \beta > 1$, and let be f such that $\int_{\mathbb{R}} |f(x)U_\gamma(x)| \log^+ |f(x)| dx < \infty$. If it results*

$$-1 < \gamma - \frac{\alpha}{2} < 0 \tag{14}$$

then

$$\|S_m(W_\alpha, f\Delta_{m,\theta})U_\gamma\Delta_{m,\theta}\|_1 \le \mathcal{C} + \mathcal{C}\int_{\mathbb{R}} |f(x)U_\gamma(x)| \left[1 + \log^+ |f(x)| + \log^+ |x|\right] dx, \tag{15}$$

where $\mathcal{C} \neq \mathcal{C}(m, f)$.

Theorem 3.2 can be useful to prove the convergence of some product integration rules. We state now some inequalities that can be useful in different contests. Assuming (12) true with p belonging to the right hand mentioned intervals, the following inequalities hold

$$\|S_m(W_\alpha, f)U_\gamma\Delta_{m,\theta}\|_p \le \mathcal{C}\|fU_\gamma\|_p, \quad 1 < p < 4 \tag{16}$$

$$\|S_m(W_\alpha, f\Delta_{m,\theta})U_\gamma\|_p \le \mathcal{C}\|fU_\gamma\Delta_{m,\theta}\|_p, \quad p > \frac{4}{3} \tag{17}$$

$$\|S_m(W_\alpha, f)U_\gamma\|_p \le \mathcal{C}\|fU_\gamma\|_p, \quad \frac{4}{3} < p < 4 \tag{18}$$

$$\|S_m(W_\alpha, f)U_\gamma\|_p \le \mathcal{C}m^{\frac{1}{3}}\|fU_\gamma\|_p, \quad p \in \left(1, \frac{4}{3}\right) \cup (4, \infty) \tag{19}$$

with $\mathcal{C} \neq \mathcal{C}(m, f)$.

For $\beta = 2$ Theorem 3.1 and inequalities (16)-(19) were proved in [6]. Estimates of $E_m(f)_{U_\gamma,p}$ can be found in [7] and [8].

Now we want to show an useful consequence of the previous results. Let $w_\alpha(x) = x^\alpha e^{-x^\beta}, x > 0, \alpha > -1, \beta > \frac{1}{2}$ be a generalized Laguerre weight and let $\{p_m(w_\alpha)\}_m$ be the corresponding sequence of orthonormal polynomials with positive leading coefficients. With $u_\gamma(x) = x^\gamma e^{-x^\beta/2}, \gamma > -\frac{1}{p}, \beta > \frac{1}{2}$, let $L^p_{u_\gamma}$, $1 < p < \infty$, be the set of measurable functions with norm

$$\|fu_\gamma\|_p = \left(\int_0^\infty |f(x)u_\gamma(x)|^p dx\right)^{\frac{1}{p}} < \infty$$

and denote by $S_m(w_\alpha, f)$ the $m-$th Fourier sum of $f \in L^p_{u_\gamma}$, i.e.

$$S_m(w_\alpha, f, x) = \sum_{k=0}^{m} c_k p_k(w_\alpha, x), \quad c_k = \int_0^\infty f(t)p_k(w_\alpha, t)w_\alpha(t)dt.$$

The theorems that we are going to establish are a direct consequence of Theorems 3.1-3.2. To introduce these results, let $a_m = a_m(w_\alpha)$ the M-R-S number with respect to w_α and for $\theta \in (0,1)$ let be $\chi_{m,\theta}$ the characteristic function of $[0, \theta a_m]$. We have

Theorem 3.3. *Let* $u_\gamma(x) = x^\gamma e^{-x^\beta/2}, \gamma > -\frac{1}{p}, \beta > \frac{1}{2}$, $f \in L^p_{u_\gamma}$ *and* $1 < p < \infty$. *Then there exists a constant* $\mathcal{C} \neq \mathcal{C}(m, f)$ *such that*

$$\|S_m(w_\alpha, f\chi_{m,\theta})u_\gamma \chi_{m,\theta}\|_p \leq \mathcal{C}\|fu_\gamma \chi_{m,\theta}\|_p, \tag{20}$$

if and only if

$$\frac{v^\gamma}{\sqrt{v^\alpha \varphi}} \in L_p(0,1) \quad \textit{and} \quad \sqrt{\frac{v^\alpha}{\varphi}}\frac{1}{v^\gamma} \in L_q(0,1), \tag{21}$$

where $\frac{1}{p} + \frac{1}{q} = 1$, $v^r = x^r$, *and* $\varphi(x) = \sqrt{x}$.

Moreover, if (21) holds, it results also

$$\|[f - \chi_{m,\theta} S_m(w_\alpha, \chi_{m,\theta} f)]u_\gamma\|_p \leq \mathcal{C}\left(E_M(f)_{u_\gamma,p} + e^{-\mathcal{C}_1 m}\|fu_\gamma\|_p\right) \tag{22}$$

with $\mathcal{C} \neq \mathcal{C}(m, f)$, $\mathcal{C}_1 \neq \mathcal{C}_1(m, f)$.

Theorem 3.4. *Let* $u_\gamma(x) = x^\gamma e^{-x^\beta/2}, \gamma > -1, \beta > \frac{1}{2}$, *and let be* $\int_0^\infty |f(x)u_\gamma(x)| \log^+ |f(x)|dx < \infty$. *If it results*

$$\frac{v^\gamma}{\sqrt{v^\alpha \varphi}} \in L_1(0,1) \quad \frac{\sqrt{v^\alpha}}{v^\gamma \sqrt{\varphi}} \in L_\infty(0,1), \tag{23}$$

then

$$\|S_m(w_\alpha, f\chi_{m,\theta})u_\gamma \chi_{m,\theta}\|_1 \leq \mathcal{C} + \mathcal{C}\int_0^\infty |f(x)u_\gamma(x)| \left[1 + \log^+|f(x)| + \log^+ x\right] dx, \tag{24}$$

where $\mathcal{C} \neq \mathcal{C}(m, f)$, $v^r = x^r$, *and* $\varphi(x) = \sqrt{x}$.

The case $\beta = 1$ in the Theorem 3.3 was proved in [9].

The following inequalities

$$\|S_m(w_\alpha, f)u_\gamma \chi_{m,\theta}\|_p \leq \mathcal{C}\|fu_\gamma\|_p, \quad 1 < p < 4 \tag{25}$$

$$\|S_m(w_\alpha, f\chi_{m,\theta})u_\gamma\|_p \leq \mathcal{C}\|fu_\gamma \chi_{m,\theta}\|_p, \quad p > \frac{4}{3} \tag{26}$$

$$\|S_m(w_\alpha, f)u_\gamma\|_p \leq \mathcal{C}\|fu_\gamma\|_p, \quad \frac{4}{3} < p < 4 \tag{27}$$

$$\|S_m(w_\alpha, f)u_\gamma\|_p \leq \mathcal{C}m^{\frac{1}{3}}\|fu_\gamma\|_p, \quad p \in (1,\infty)\backslash(\frac{4}{3}, 4) \tag{28}$$

are true with $\mathcal{C} \neq \mathcal{C}(m, f)$, and assuming (21) true with p belonging to the indicated intervals.

The case $\beta = 1$ in Theorem 3.3 and in the inequalities (25)-(28) was just proved in [9].

4. Proofs

4.1. Proof of Proposition 2.1.

We have:

$$\begin{aligned} \|f(1-\overline{\Delta}_{m,\theta})U_\gamma\|_p &= \|fU_\gamma\|_{L_p(|x|\geq\theta\overline{a}_m)} \\ &\leq \|[f-P_M]U_\gamma\|_p + \|P_M U_\gamma\|_{L_p(|x|\geq\theta\overline{a}_m)}, \end{aligned}$$

$$M = \left[m \left(\frac{\theta}{1+\theta} \right)^\beta \right],$$

where P_M is the best approximation polynomial of $f \in L^p_{U_\gamma}$. Since (5)

$$\begin{aligned} \|f(1-\overline{\Delta}_{m,\theta})U_\gamma\|_p &\leq E_M(f)_{U_\gamma,p} + \mathcal{C}e^{-\mathcal{C}_2 m}\|P_M U_\gamma\|_p \\ &\leq \mathcal{C}_1(E_M(f)_{U_\gamma,p} + e^{-\mathcal{C}_2 m}\|fU_\gamma\|_p), \end{aligned}$$

i.e. the Proposition is proved. □

In the sequel we need some inequalities about the polynomials $p_m(W_\alpha)$.

In [3, Th. 1.8, p. 16] the authors proved

$$|p_m(W_\alpha, x)|\sqrt{W_\alpha(x)} \leq \frac{\mathcal{C}}{\sqrt{a_m}\sqrt[4]{\left|1-\frac{|x|}{a_m}\right| + m^{-\frac{2}{3}}}}, \qquad \frac{a_m}{m} \leq |x| \leq a_m.$$

from which, for a fixed θ, with $0 < \theta < 1$, we can deduce

$$|p_m(w_\alpha, x)|\sqrt{w_\alpha(x)} \leq \mathcal{C}\frac{1}{\sqrt{a_m}}, \qquad \frac{a_m}{m} \leq |x| \leq \theta a_m. \tag{29}$$

Denote by x_d a zero of $p_m(W_\alpha)$ closest to x, by $l_{m,d}$ the $d-$th fundamental Lagrange polynomial based on the zeros of $p_m(W_\alpha)$, and recall the following Erdös-Turán estimate [4]

$$\frac{l^2_{m,d}(x)W_\alpha(x)}{W_\alpha(x_d)} + \frac{l^2_{m,d+1}(x)W_\alpha(x)}{W_\alpha(x_d)} > 1. \tag{30}$$

Denoted by $\lambda_m(W_\alpha, x)$ the m-th Christoffel function $\quad m = 1, 2, \dots,$

$$\lambda_m(W_\alpha; x) = \left[\sum_{k=0}^{m-1} p_k^2(W_\alpha; x)\right]^{-1},$$

in [3] the authors proved

$$\frac{1}{\mathcal{C}}\varphi_m(x) \le \frac{\lambda_m(W_\alpha, x)}{\left(|x| + \frac{a_m}{m}\right)^\alpha e^{-|x|^\beta}} \le \mathcal{C}\varphi_m(x), \tag{31}$$

where

$$\varphi_m(x) = \frac{a_m}{m}\frac{1}{\sqrt{\left|1 - \frac{|x|}{a_m}\right| + m^{-\frac{1}{3}}}}, \quad |x| \le a_m.$$

Combining (30) and (31) we deduce

$$\frac{l_{m,d}^2(x) W_\alpha(x)}{W_\alpha(x_d)} \sim 1. \tag{32}$$

Since from [3, p.16-17], for $|x_d| \le \theta a_m$,

$$W_\alpha(x_d) {p'_m}^2(W_\alpha, x_d) \sim \frac{1}{\Delta^2 x_d}, \quad |\Delta x_d| = |x_{d\pm1} - x_d|,$$

we deduce

$$|p_m(w_\alpha, x)|\sqrt{W_\alpha(x)}\sqrt{a_m} \sim \left|\frac{x - x_d}{x_d - x_{d\pm1}}\right|, \quad \frac{a_m}{m} \le |x| \le \theta a_m. \quad \Box \tag{33}$$

The following proposition will be useful in the sequel.

Proposition 4.1. *Let be* $W_\alpha(x) = v^\alpha(x) e^{-|x|^\beta}$, $v^\alpha(x) = |x|^\alpha$ *and* $U_\rho(x) = v^\rho(x) e^{-\frac{|x|^\beta}{2}}$, $v^\rho(x) = |x|^\rho$. *For a fixed* $0 < \theta < 1$, $1 \le p < \infty$ *and* $\rho - \frac{\alpha}{2} > -\frac{1}{p}$, *we have*

$$\|p_m(W_\alpha) U_\rho\|_{L_p[-\theta a_m, \theta a_m]} \ge \frac{C}{\sqrt{a_m}} \left\|\frac{v^\rho}{\sqrt{v^\alpha}}\right\|_{L^p(-1,1)}, \tag{34}$$

where $\mathcal{C}$ *is independent of* m.

Proof. Let $\delta > 0$ be "small". Define $\delta_k = \frac{\delta}{4}\Delta x_k = \frac{\delta}{4}(x_{k+1} - x_k)$, and $I_m = \bigcup_{1 \le k \le m} ([x_k - \delta_k, x_k + \delta_k])$. To prove (34), set $CI_m = [-1, 1] \backslash I_m$. By (33) we get

$$|p_m(W_\alpha, x)| U_\rho(x) \ge C \frac{|x|^{\rho - \frac{\alpha}{2}}}{\sqrt{a_m}}, \quad x \in CI_m,$$

and consequently

$$\|p_m(W_\alpha)\sigma\|_{L^p[-a_m\theta,a_m\theta]} \geq \frac{C}{\sqrt{a_m}} \left\| \frac{v^\rho}{\sqrt{v^\alpha}} \right\|_{L^p(CI_m)}.$$

Since the measure of I_m is bounded by δ, for a suitable δ, we conclude

$$\|p_m(W_\alpha)U_\rho\|_{L^p[-a_m\theta,a_m\theta]} \geq \frac{C}{\sqrt{a_m}} \left\| \frac{v^\rho}{\sqrt{v^\alpha}} \right\|_{L^p([-1,1])}. \qquad \square$$

In order to prove next theorem, we recall the following expression for $S_m(W_\alpha, f)$

$$\begin{aligned} S_m(W_\alpha, f, x) &= \frac{\gamma_{m-1}(W_\alpha)}{\gamma_m(W_\alpha)} \{p_m(W_\alpha, x)H(f\Delta_{m,\theta}p_{m-1}(W_\alpha)W_\alpha; x) \\ &+ p_{m-1}(W_\alpha, x)H(f\Delta_{m,\theta}p_m(W_\alpha)W_\alpha; x)\}, \end{aligned} \tag{35}$$

where

$$H(g,t) = \int_{\mathbb{R}} \frac{g(x)}{x-t} dx$$

is the Hilbert transform of g in $\mathbb{R}$, and [3]

$$\frac{\gamma_{m-1}(W_\alpha)}{\gamma_m(W_\alpha)} \sim a_m(W_\alpha). \tag{36}$$

4.2. **Proof of Theorem 3.1.** By (6) we have

$$\|S_m(W_\alpha, f\Delta_{m,\theta})U_\gamma\Delta_{m,\theta}\|_p \leq \mathcal{C}\|S_m(W_\alpha, f\Delta_{m,\theta})U_\gamma\Delta_{m,\theta}\|_{L_p(C_m)},$$

$$C_m = \{x : \mathcal{C}\frac{a_m}{m} \leq |x| \leq \theta a_m\},$$

Taking into account (35) and (36)

$$\begin{aligned} &\|S_m(W_\alpha, f\Delta_{m,\theta})U_\gamma\Delta_{m,\theta}\|_p \\ &\leq a_m \left(\int_{C_m} |p_m(W_\alpha, t)H(p_{m-1}(W_\alpha)W_\alpha f\Delta_{m,\theta}; t)U_\gamma(t)|^p\, dt \right)^{\frac{1}{p}} \\ &+a_m \left(\int_{C_m} |p_{m-1}(W_\alpha, t)H(p_m(W_\alpha)W_\alpha f\Delta_{m,\theta}; t)U_\gamma(t)|^p\, dt \right)^{\frac{1}{p}} = B_1 + B_2 \end{aligned} \tag{37}$$

Using (29)

$$B_1 \leq \mathcal{C}\sqrt{a_m} \left(\int_{C_m} |t|^{\gamma-\frac{\alpha}{2}} \left| \int_{C_m} \frac{p_{m-1}(W_\alpha, x)f(x)\Delta_{m,\theta}(x)W_\alpha(x)}{x-t} dx \right|^p dt \right)^{\frac{1}{p}}$$

By the changes of variables $x = a_m y$, $t = a_m z$, we get

$$B_1 \leq \mathcal{C}a_m^{\frac{1}{2}+\gamma-\frac{\alpha}{2}+\frac{1}{p}} \left(\int_{\tilde{C}_m} |z|^{\gamma-\frac{\alpha}{2}} \left| \int_{\tilde{C}_m} \frac{(p_{m-1}(W_\alpha)f\Delta_{m,\theta}W_\alpha)(a_m y)}{y-z} dy \right|^p dz \right)^{\frac{1}{p}}$$

where

$$\tilde{C}_m = [-1,1] \backslash \left[-\frac{\mathcal{C}}{m}, \frac{\mathcal{C}}{m} \right]$$

Under the assumptions (12), $|z|^{\gamma - \frac{\alpha}{2}}$ is an A_p weight and therefore, recalling a result in [13] (see also [11, p.57 and 313-314]) about the boundedness of the Hilbert transform in $[-1,1]$, we have

$$B_1 \le \mathcal{C} a_m^{\frac{1}{2}+\gamma-\frac{\alpha}{2}+\frac{1}{p}} \left(\int_{-1}^{1} |z|^{\gamma-\frac{\alpha}{2}} \left| (p_{m-1}(W_\alpha) f \Delta_{m,\theta} W_\alpha)(a_m z) \right|^p dz \right)^{\frac{1}{p}}.$$

So, by the change of variable $a_m z = x$, we have

$$B_1 \le \mathcal{C} a_m^{\frac{1}{2}} \left(\int_{-a_m}^{a_m} |x|^{\gamma-\frac{\alpha}{2}} \left| (p_{m-1}(W_\alpha, x) f(x) \Delta_{m,\theta}(x) W_\alpha(x) \right|^p dx \right)^{\frac{1}{p}}$$

and using again (29)

$$B_1 \le \mathcal{C} \left(\int_{\mathbb{R}} |f(x) \Delta_{m,\theta}(x) U_\gamma(x)|^p dx \right)^{\frac{1}{p}}. \tag{38}$$

By similar arguments used to bound B_1, we get

$$B_2 \le \mathcal{C} \left(\int_{\mathbb{R}} |f(x) \Delta_{m,\theta}(x) U_\gamma(x)|^p dx \right)^{\frac{1}{p}}. \tag{39}$$

Combining (38),(39) with (37), (11) follows.

Now we prove (11) implies (12). Let be

$$C_m = \left\{ x : \mathcal{C} \frac{a_m}{m} \le |x| \le \theta a_m \right\}, \quad C_{m-1} = \left\{ x : \mathcal{C} \frac{a_{m-1}}{m} \le |x| \le \theta a_{m-1} \right\},$$

and let $\Delta_{m,\theta}$, $\Delta_{m-1,\theta}$ the corresponding characteristic functions. Setting $\tilde{f} = f\Delta_{m-1,\theta}$, we have

$$\begin{aligned} &\|[S_m(W_\alpha, \tilde{f}\Delta_{m,\theta}) - S_{m-1}(W_\alpha, \tilde{f}\Delta_{m,\theta})] U_\gamma \Delta_{m,\theta}\|_p \\ &= \left| \int_{\mathbb{R}} \tilde{f}(x) \Delta_{m,\theta}(x) p_m(W_\alpha, x) W_\alpha(x) dx \right| \|\Delta_{m,\theta} p_m(W_\alpha) U_\gamma\|_p. \end{aligned}$$

In view of (11) for $1 < p < \infty$

$$\left| \int_{\mathbb{R}} \tilde{f}(x) \Delta_{m,\theta}(x) p_m(W_\alpha, x) W_\alpha(x) dx \right| \|\Delta_{m,\theta} p_m(W_\alpha) U_\gamma\|_p \le 2 \|f U_\gamma\|_p.$$

Then

$$\|\Delta_{m,\theta} p_m(W_\alpha) U_\gamma\|_p \sup_{||h||_q=1} \left| \int_{\mathbb{R}} \tilde{h}(x) \Delta_{m,\theta}(x) p_m(W_\alpha, x) \frac{W_\alpha(x)}{U_\gamma(x)} dx \right| \le 2\mathcal{C}$$

and also

$$\|\Delta_{m,\theta}p_m(W_\alpha)U_\gamma\|_p \cdot \|\Delta_{m,\theta}p_m(W_\alpha)\frac{W_\alpha}{U_\gamma}\|_q \le 2\mathcal{C}.$$

Using then Proposition 4.1

$$\frac{1}{a_m}\left(\int_{-1}^{1}|x|^{(\gamma-\frac{\alpha}{2})p}dx\right)^{\frac{1}{p}}\left(\int_{-1}^{1}|x|^{(\frac{\alpha}{2}-\gamma)q}dx\right)^{\frac{1}{q}} \le 2\mathcal{C},$$

by which conditions in (12) follow.

Now we prove (13). Let $P \in \mathbb{P}_M$, with $M = \left[m\left(\frac{\theta}{1+\theta}\right)^{\beta}\right]$, the polynomial of best approximation of f in $L^p_{U_\gamma}$. By

$$\begin{aligned}
&\|[f-\Delta_{m,\theta}S_m(W_\alpha, f\Delta_{m,\theta})]U_\gamma\|_p\\
&\le \|(1-\Delta_{m,\theta})fU_\gamma\|_p + \|[f-S_m(W_\alpha, f\Delta_{m,\theta})]U_\gamma\Delta_{m,\theta}\|_p\\
&\le \|(1-\Delta_{m,\theta})fU_\gamma\|_p + \|(f-P)\Delta_{m,\theta}U_\gamma\|_p \qquad (40)\\
&+\|S_m(W_\alpha,(f-P)\Delta_{m,\theta})\Delta_{m,\theta}U_\gamma\|_p\\
&+\|S_m(W_\alpha,P(1-\Delta_{m,\theta})\Delta_{m,\theta}U_\gamma\|_p\\
&=: I_1+I_2+I_3+I_4. \qquad (41)
\end{aligned}$$

Using Proposition 2.1,

$$I_1+I_2 \le \mathcal{C}_1\left(E_M(f)_{U_\gamma,p} + e^{-\mathcal{C}_2 m}\|fU_\gamma\|_p\right)$$

and by (11)

$$I_3 \le \mathcal{C}\|(f-P)\Delta_{m,\theta}U_\gamma\|_p \le \mathcal{C}E_M(f)_{U_\gamma,p}.$$

To estimate I_4 we use (19)

$$I_4 \le \mathcal{C}m^{\frac{1}{3}}|P(1-\Delta_{m,\theta})U_\gamma\|_p,$$

and by (5), we have

$$I_4 \le \mathcal{C}m^{\frac{1}{3}}e^{-\mathcal{C}_1 m}\|P\Delta_{m,\theta}U_\gamma\|_p.$$

Therefore

$$\|[f-\Delta_{m,\theta}S_m(W_\alpha, f\Delta_{m,\theta})]U_\gamma\|_p \le \mathcal{C}[E_M(f)_{U_\gamma,p} + e^{-Am}\|fU_\gamma\|_p],$$

that is (13) follows. □

4.3. **Proof of Theorem 3.2.** Using (6)

$$\|S_m(W_\alpha, f\Delta_{m,\theta})U_\gamma\Delta_{m,\theta}\|_1 \le \mathcal{C}\|S_m(W_\alpha, f\Delta_{m,\theta})U_\gamma\Delta_{m,\theta}\|_{L_1(C_m)},$$

$$C_m = \{x : \mathcal{C}\frac{a_m}{m} \le |x| \le \theta a_m\}, \tag{42}$$

and setting $g = sgn(S_m(W_\alpha, f\Delta_{m,\theta}))$,

$$\|S_m(W_\alpha, f\Delta_{m,\theta})U_\gamma\Delta_{m,\theta}\|_1 \le \mathcal{C}\int_{C_m} S_m(W_\alpha, f\Delta_{m,\theta}, x)g(x)U_\gamma(x)dx. \tag{43}$$

By (35) and (36)

$$\begin{aligned}
&\|S_m(W_\alpha, f\Delta_{m,\theta})U\Delta_{m,\theta}\|_1 \\
&\le \mathcal{C}\left[a_m\int_{C_m} |p_m(W_\alpha, x)H(f\Delta_{m,\theta}p_{m-1}(W_\alpha)W_\alpha; x)|\, U_\gamma(x)dx\right. \\
&\left.+\, a_m\int_{C_m} |p_{m-1}(W_\alpha, x)H(f\Delta_{m,\theta}p_m(W_\alpha)W_\alpha; x)|\, U_\gamma(x)dx\right] \\
&=: A_1 + A_2.
\end{aligned} \tag{44}$$

First we bound A_1. By (29)

$$A_1 \le \mathcal{C}\sqrt{a_m}\int_{C_m} |x|^{\gamma-\frac{\alpha}{2}}\, |H(f\Delta_{m,\theta}p_m(W_\alpha)W_\alpha; x)|\, dx \le \mathcal{C}\int_{\mathbb{R}} |x|^{\gamma-\frac{\alpha}{2}}\, |H(G_m; x)|\, dx$$

where $G_m = \sqrt{a_m}f\Delta_{m,\theta}p_m(W_\alpha)W_\alpha$. Here we recall the following inequality due to Muckenhoupt in [12, Lemma 9, p.440]:

$$\begin{aligned}
&\int_{\mathbb{R}} \left(\frac{|x|}{1+|x|}\right)^r (1+|x|)^s \left|\int_{\mathbb{R}} \frac{g(y)}{x-y}dy\right| dx \\
&\le\ \mathcal{C} + \mathcal{C}\int_{\mathbb{R}} |g(x)\left(\frac{|x|}{1+|x|}\right)^R (1+|x|)^S(1+\log^+|g(x)| + \log^+|x|)dx
\end{aligned}$$

under the assumptions $r > -1, s < 0, R \le 0, S \ge -1, r \ge R, s \le S$ and $f\log^+ f \in L_1$.

Using previous result with $r = R = \gamma - \frac{\alpha}{2} = s = S$, under the assumption $0 < \gamma - \frac{\alpha}{2} < 1$ and taking into account $|G_m(x)| \le \mathcal{C}|f(x)|\sqrt{W_\alpha(x)}$, we have

$$A_1 \le \mathcal{C} + \mathcal{C}\int_{\mathbb{R}} |f(x)U_\gamma(x)|\left\{1 + \log^+|f(x)| + \log^+|x|\right\} dx. \tag{45}$$

Similarly we obtain

$$A_2 \le \mathcal{C} + \mathcal{C}\int_{\mathbb{R}} |f(x)U_\gamma(x)|\left\{1 + \log^+|f(x)| + \log^+|x|\right\} dx. \tag{46}$$

Combining (45), (46) with (44), the Theorem follows. □

To prove Theorems 3.3 and 3.4 we need some relations between generalized Freud and generalized Laguerre polynomials and then we apply the previous estimates about Fourier Sums with respect to generalized Freud weights.

Setting $\tilde{W}_\alpha(x) = |x|^{2\alpha+1}e^{-|x|^{2\beta}}$ and $\tilde{U}_\gamma(x) = |x|^{2\gamma+\frac{1}{p}}e^{-|x|^\beta}$, for the orthogonal polynomials we have

$$p_{2m}(\tilde{W}_\alpha, x) = p_m(w_\alpha, x^2) \tag{47}$$

Moreover, assuming F be an even extension in $\mathbb{R}$ of f defined on $(0,\infty)$, the following relation holds

$$S_{2m}(\tilde{W}_\alpha, F, x) = S_m(w_\alpha, f, x^2). \tag{48}$$

Denoted by $\tilde{\chi}_{2m,\theta}$ the characteristic function of $\tilde{C}_{2m} = [-\theta a_{2m}(\tilde{W}_\alpha)^2, \theta a_{2m}(\tilde{W}_\alpha)^2]$, from (48) easily follows

$$\|S_{2m}(\tilde{W}_\alpha, F\tilde{\chi}_{2m,\theta})\tilde{\Delta}_{2m,\theta}\tilde{U}_\gamma\|_p = \|S_m(w_\alpha, f\chi_{m,\theta})u_\gamma\chi_{m,\theta}\|_p \tag{49}$$

4.4. **Proof of Theorem 3.3.** Let F be an even extension in $\mathbb{R}$ of f defined on $(0,\infty)$. Using Theorem 3.1 we have

$$\|S_{2m}(\tilde{W}_\alpha, F\tilde{\Delta}_{2m,\theta})\tilde{\Delta}_{2m,\theta}\tilde{U}_\gamma\|_p \le \mathcal{C}\|F\tilde{U}_\gamma\tilde{\Delta}_{2m,\theta}\|_p, \tag{50}$$

if and only if

$$\gamma - \frac{\alpha}{2} + \frac{1}{4} < \frac{1}{q} \quad and \quad \gamma - \frac{\alpha}{2} - \frac{1}{4} > -\frac{1}{p},$$

which are equivalent to (21).

By (49), and using $\|F\tilde{U}_\gamma\tilde{\Delta}_{2m,\theta}\|_p = \|fu_\gamma\Delta_{m,\theta}\|_p$, with $a_m(w_\alpha) = a_{2m}^2(\tilde{W}_\alpha$, the first part of the Theorem follows.

To prove (22), we premit a Proposition which is the equivalent in $\mathbb{R}^+$ of the Proposition 2.1.

Proposition 4.2. *Let $f \in L^p_{u_\gamma}$ and $1 \le p < \infty$. For m sufficiently large (say $m > m_0$) we have*

$$\|f(1-\chi_{m,\theta})u_\gamma\|_p \le \mathcal{C}_1\left(E_M(f)_{u_\gamma,p} + e^{-\mathcal{C}_2 m}\|fu_\gamma\|_p\right), \tag{51}$$

where $M = \left[m\left(\frac{\theta}{1+\theta}\right)^\beta\right]$ and the constants $\mathcal{C}, \mathcal{C}_1, C_2$ are independent on m and f.

Now we prove (22). Let $P \in \mathbb{P}_M$, with $M = \left[m \left(\frac{\theta}{1+\theta} \right)^{\beta} \right]$, the polynomial of best approximation of f in $L^p_{u_\gamma}$. By

$$\begin{aligned}
\|[f - \chi_{m,\theta} S_m(w_\alpha, f\chi_{m,\theta})]u_\gamma\|_p &\le \|(1- \chi_{m,\theta}) f u_\gamma\|_p + \|[f - S_m(w_\alpha, f\chi_{m,\theta})]u_\gamma \chi_{m,\theta}\|_p \\
&\le \|(1 - \chi_{m,\theta}) f u_\gamma\|_p + \|(f - P)\chi_{m,\theta} u_\gamma\|_p \\
&+ \|S_m(w_\alpha, (f - P)\chi_{m,\theta})\chi_{m,\theta} u_\gamma\|_p \\
&+ \|S_m(w_\alpha, P(1 - \chi_{m,\theta})\chi_{m,\theta} u_\gamma\|_p \\
&=: I'_1 + I'_2 + I'_3 + I'_4.
\end{aligned}$$

Estimate (22) follows using Proposition 4.2,(20) and (28).□

We omit the proof of Theorem 3.4 since it follows by arguments similar to those used in the proof of Theorem 3.3.

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POLYNOMIAL APPROXIMATION ON THE REAL SEMIAXIS WITH GENERALIZED LAGUERRE WEIGHTS

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Dedicated to Professor D. D. Stancu on his 80^{th} *birthday*

Abstract. We present a complete collection of results dealing with the polynomial approximation of functions on $(0, +\infty)$.

1. Introduction

This paper is dedicated to the approximation of functions which are defined on $(0, +\infty)$, have singularities in the origin and increase exponentially for $x \to +\infty$. Therefore, it is natural to consider weighted approximation with the generalized Laguerre weight $w_\alpha(x) = x^\alpha e^{-x^\beta}$. We first prove the main polynomial inequalities: "infinite-finite" range inequalities, Remez-type inequalities, Markov-Bernstein and Nikolski inequalities. In Section 2 we introduce a new modulus of continuity, the equivalent $K-$functional and some function spaces. With these tools we prove the Jackson theorem, the Stechkin inequality and estimate the derivatives of the polynomial of best approximation (or "near best approximant" polynomial). We will also prove an embedding theorem between functional spaces. In Section 5, generalizing analogous results proved in [10], we will study the behaviour of Fourier sums and Lagrange polynomials. This paper can be considered as a survey on the topic. However, all the results are new and cover the ones available in the literature.

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2. **Polynomial inequalities**

In this context the main idea is to prove polynomial inequalities with exponential weights on unbounded intervals by using well known polynomial inequalities (eventually with weight) on bounded intervals. To this end the main gradients are the "infinite-finite range inequality" and the approximation of weight by polynomials on a finite interval.

In our case, the weight $w_\alpha(x) = w_{\alpha\beta}(x) = x^\alpha e^{-x^\beta}$ is related, by a quadratic transformation, to the generalized Freud weight $u(x) = |x|^{2\alpha+1}e^{-x^{2\beta}}$.

The Mhaskar-Rakhmanov-Saff number $\bar{a}_m(u)$, related to the weight u, is [9]: $\bar{a}_m(u) \sim m^{1/2\beta}$ where the constant in "$\sim$" depends on α and β and does not depend on m. Then for the weight w_α we have

$$a_m(w) = \bar{a}_{2m}(u)^2 \sim m^{1/\beta} \tag{2.1}$$

and, for an arbitrary polynomial P_m, the following inequalities easily follow:

$$\left(\int_0^\infty |P_m(x)w_{\alpha\beta}(x)|^p dx\right)^{1/p} \le C\left(\int_{\Gamma_m} |P_m(x)w_{\alpha\beta}(x)|^p dx\right)^{1/p}, \tag{2.2}$$

$$\left(\int_{a_m(1+\delta)}^{+\infty} |P_m(x)w_{\alpha\beta}(x)|^p dx\right)^{1/p} \le Ce^{-Am}\left(\int_0^{+\infty} |P_m(x)w_{\alpha\beta}(x)|^p dx\right)^{1/p} \tag{2.3}$$

where $\Gamma_m = [0, a_m(1-k/m^{2/3})]$ (k =const), $p \in (0,+\infty]$, $\beta > \frac{1}{2}$, $\alpha > -\frac{1}{p}$ if $p < +\infty$ and $\alpha \ge 0$ if $p = +\infty$; the constants A and C are independent of m and p and A depend on $\delta > 0$. Then, as a consequence of some results in [5], [11], with $x \in [0, Aa_m]$, $A \ge 1$ fixed, there exist polynomials Q_m such that $Q_m(x) \sim e^{-x^\beta}$ and

$$\frac{\sqrt{a_m}}{m}|\sqrt{x}Q_m'(x)| \le Ce^{-x^\beta}, \tag{2.4}$$

where C and the constants in "$\sim$" are independent of x. Therefore, by using (2.2) and (2.4) and a linear transformation in $[0,1)$, polynomial inequalities of Bernstein-type, Remez and Shur can be deduced by analogous inequalities in $[0,1]$ with Jacobi weights x^α.

The next theorems can be proved by using the previous considerations.

With $A > 0$ $0 < t_1 < \ldots < t_r < a_m$ fixed, we put

$$A_m = \left[A\frac{a_m}{m^2}, a_m\left(1 - \frac{A}{m^{2/3}}\right)\right] \setminus \left(\bigcup_{i=1}^{r}\left[t_i - A\frac{\sqrt{a_m}}{m}, t_i + A\frac{\sqrt{a_m}}{m}\right]\right)$$

where m is sufficiently large ($m > m_0$), $r \geq 0$. Let us specify that if $r = 0$ then $A_m = \left[A\frac{a_m}{m^2}, a_m\left(1 - \frac{A}{m^{2/3}}\right)\right]$.

Theorem 2.1. *Let $A, t_1, \ldots, t_r$ be as in the previous definition and $0 < p \leq +\infty$. Then, for each polynomial P_m, there exists a constant $C = C(A)$, independent of m, p and P_m, such that*

$$\left(\int_0^{+\infty} |(P_m w_{\alpha\beta})(x)|^p dx\right)^{1/p} \leq C\left(\int_{A_m} |(P_m w_{\alpha\beta})(x)|^p dx\right)^{1/p}. \tag{2.5}$$

Theorem 2.2. *For each polynomial P_m and $0 < p \leq +\infty$ we have*

$$\left(\int_0^{+\infty} |P'_m(x)\sqrt{x} w_{\alpha\beta}(x)|^p dx\right)^{1/p} \leq C\frac{m}{\sqrt{a_m}}\left(\int_0^{+\infty} |P_m(x) w_{\alpha\beta}(x)|^p dx\right)^{1/p} \tag{2.6}$$

and

$$\left(\int_0^{+\infty} |P'_m(x) w_{\alpha\beta}(x)|^p dx\right)^{1/p} \leq C\left(\frac{m}{\sqrt{a_m}}\right)^2\left(\int_0^{+\infty} |P_m(x) w_{\alpha\beta}(x)|^p dx\right)^{1/p} \tag{2.7}$$

with $C \neq C(m, p, P_m)$.

As in the Markoff-Bernstein inequalities, we have two versions of Nikolski inequality.

Theorem 2.3. *Let $P_m \in \mathbb{P}_m$ be an arbitrary polynomial and $1 \leq q < p \leq +\infty$. Then there exists a constant K, independent of m, p, q and P_m such that, for $\alpha \geq 0$ if $p = +\infty$ and $\alpha > -\frac{1}{p}$ if $p < +\infty$, we have*

$$\|P_m w_{\alpha\beta}\varphi^{\frac{1}{q}}\|_p \leq K\left(\frac{m}{\sqrt{a_m}}\right)^{\frac{1}{q}-\frac{1}{p}}\|P_m w_{\alpha\beta}\|_q, \tag{2.8}$$

$$\|P_m w_{\alpha\beta}\|_p \leq K\left(\frac{m}{\sqrt{a_m}}\right)^{\frac{2}{q}-\frac{2}{p}}\|P_m w_{\alpha\beta}\|_q, \tag{2.9}$$

where $\varphi(x) = \sqrt{x}$.

Proof. We first suppose $\alpha \geq 0$ and prove (2.8) with $p = +\infty$ and $1 \leq q < +\infty$. Set $I_x = [x, x + \Delta_m(x)]$, where $x \geq 0$, $\Delta_m(x) = \frac{\sqrt{a_m}}{m}\sqrt{x}$.
From the relation

$$\int_{I_x} P_m(t)dt = P_m(x)\Delta_m(x) + \int_{I_x} P'_m(t)\,(x + \Delta_m(x) - t)\,dt,$$

(by using Hölder inequality for $q > 1$) we get for $q \geq 1$:

$$|P_m(x)\varphi(x)^{\frac{1}{q}}| \leq \left(\frac{m}{\sqrt{a_m}}\right)^{1/q}\left[\left(\int_{I_x}|P_m(t)|^q\,dt\right)^{1/q} + \frac{\sqrt{a_m}}{m}\left(\int_{I_x}|P'_m(t)\varphi(t)|^q\,dt\right)^{1/q}\right]. \tag{2.10}$$

Since $w_{\alpha\beta}(x) \sim w_{\alpha\beta}(t)$ for $t \in I_x$, $\alpha \geq 0$ it also holds

$$\begin{aligned}\left|P_m(x)w_{\alpha\beta}(x)\varphi(x)^{1/q}\right| &\leq C\left(\frac{m}{\sqrt{a_m}}\right)^{1/q}\left[\left(\int_{I_x}|P_m(t)w_{\alpha\beta}(t)|^q\,dt\right)^{1/q}\right. &(2.11)\\ &+ \left.\frac{\sqrt{a_m}}{m}\left(\int_{I_x}|P'_m(t)\varphi(t)w_{\alpha\beta}(t)|^q\,dt\right)^{1/q}\right].\end{aligned}$$

By extending the integrals to $(0, +\infty)$ and by using Bernstein inequality we deduce:

$$\left\|P_m w_{\alpha\beta}\varphi^{\frac{1}{q}}\right\|_\infty \leq K\left(\frac{m}{\sqrt{a_m}}\right)^{1/q}\|P_m w_{\alpha\beta}\|_q. \tag{2.12}$$

Moreover, using (2.5) with $r = 0$ and $A = 1$, one has

$$\begin{aligned}\|P_m w_{\alpha\beta}\|_\infty &\leq C\left\|P_m w_{\alpha\beta}\varphi^{1/q}\varphi^{-1/q}\right\|_{L^\infty([\frac{a_m}{m^2},\infty))}\\ &\leq C\left(\frac{m}{\sqrt{a_m}}\right)^{1/q}\left\|P_m w_{\alpha\beta}\varphi^{1/q}\right\|_\infty.\end{aligned}$$

Then from (2.12) it follows

$$\|P_m w_{\alpha\beta}\|_\infty \leq K\left(\frac{m}{\sqrt{a_m}}\right)^{2/q}\|P_m w_{\alpha\beta}\|_q. \tag{2.13}$$

Then (2.8) and (2.9) are true with $\alpha \geq 0$, $p = +\infty$, $1 \leq q < +\infty$.

When $\alpha \geq 0$ and $1 \leq q < p < +\infty$, then to prove (2.9), we write

$$\begin{aligned} \|P_m w_{\alpha\beta}\|_p^p &= \left\| |P_m w_{\alpha\beta}|^{p-q} |P_m w_{\alpha\beta}|^q \right\|_1 \\ &\leq \|P_m w_{\alpha\beta}\|_\infty^{p-q} \int_0^{+\infty} |P_m w_{\alpha\beta}|^q (x) dx \leq \\ &\leq K^{p-q} \left(\frac{m}{\sqrt{a_m}} \right)^{(p-q)\frac{2}{q}} \|P_m w_{\alpha\beta}\|_q^{p-q} \|P_m w_{\alpha\beta}\|_q^q , \end{aligned}$$

from which

$$\|P_m w_{\alpha\beta}\|_p \leq K \left(\frac{m}{\sqrt{a_m}} \right)^{2\left(\frac{1}{q} - \frac{1}{p}\right)} \|P_m w_{\alpha\beta}\|_q$$

i.e. (2.9) with $\alpha \geq 0$. In an analogous way we can prove(2.8).

Let us suppose now $1 \leq q < p < +\infty$ and $-\frac{1}{p} < \alpha < 0$. From Theorem 2.1 we get

$$\|P_m w_{\alpha\beta}\|_p \sim \|P_m w_{\alpha\beta}\|_{L^p\left(\frac{a_m}{m^2}, a_m\right)} .$$

In the interval $\left[\frac{a_m}{m^2}, a_m\right]$ we can construct a polynomial Q_{lm} (with l a fixed integer) for which it holds $Q_{lm} \sim x^\alpha$ (see [8] in $[-1, 1]$) and

$$\|P_m w_{0\beta}\|_p \sim \|(P_m Q_{lm}) w_{0\beta}\|_{L^p\left(\frac{a_m}{m^2}, a_m\right)} \leq C \|(P_m Q_{lm}) w_{0\beta}\|_p .$$

Then we can use (2.9) with $\alpha = 0$, $P_m Q_{lm}$ instead of P_m and, finally, Theorem 2.1 to replace Q_{lm} by x^α.

Relation (2.8) can be proved in a similar way and the proof is complete. □

3. **Function spaces, modulus of continuity and K-functionals**

With $w_{\alpha\beta}(x) = x^\alpha e^{-x^\beta}$ and $1 \leq p < +\infty$ we denote by $L^p_{w_{\alpha\beta}}$ the set of all measurable functions such that

$$\|f w_{\alpha\beta}\|_p^p = \int_0^{+\infty} |f w_{\alpha\beta}|^p (x) dx < +\infty, \quad \alpha > -\frac{1}{p}.$$

If $p = +\infty$ we define

$$L^\infty_{w_{\alpha\beta}} = \{ f \in C^0[(0, +\infty)] : \lim_{x \to 0, x \to +\infty} (f w_{\alpha\beta})(x) = 0 \}, \quad \alpha > 0$$

and

$$L^\infty_{w_{0\beta}} = \{ f \in C^0[[0, +\infty)] : \lim_{x \to +\infty} (f w_{\alpha\beta})(x) = 0 \},$$

where $C^0(A)$ is the set of all continuous functions in $A \subseteq [0,+\infty)$.

For more regular functions we introduce the Sobolev-type space

$$W_r^p = W_r^p(w_{\alpha\beta}) = \{f \in L^p_{w_{\alpha\beta}} : f^{(r-1)} \in AC((0,+\infty)) \text{ and } \|f^{(r)}\varphi^r w_{\alpha\beta}\|_p < +\infty\}$$

where $r \geq 1$, $1 \leq p \leq +\infty$, $\varphi(x) = \sqrt{x}$ and $AC(A)$ is the set of absolutely continuous functions in $A \subseteq [0,+\infty)$.

In order to define in $L^p_{w_{\alpha\beta}}$ a modulus of smoothness, for every $h > 0$ we introduce the quantity $h^* = \frac{1}{h^{\frac{2}{2\beta-1}}}$, $\beta > \frac{1}{2}$ and the segment $I_{rh} = [8r^2h^2, Ah^*]$ where A is a fixed positive constant.

Then, following [3] (see also [1]), we define

$$\Omega^r_\varphi(f,t)_{w_{\alpha\beta},p} = \sup_{0<h\leq t} \|(\Delta^r_{h\varphi} f)w_{\alpha\beta}\|_{L^p(I_{rh})} \tag{3.1}$$

as the main part of the modulus of continuity, where $r \geq 1$, $1 \leq p \leq +\infty$, $\Delta^r_{h\varphi} f(x) = \sum_{k=0}^{r} (-1)^k \begin{pmatrix} r \\ k \end{pmatrix} f(x+(r-k)h\sqrt{x})$. The complete modulus of continuity ω^r_φ is defined by

$$\begin{aligned} \omega^r_\varphi(f,t)_{w_{\alpha\beta},p} &= \inf_{P\in\mathbb{P}_{r-1}} \|(f-P)w_{\alpha\beta}\|_{L^p([0,8r^2t^2])} + \Omega^r_\varphi(f,t)_{w_{\alpha\beta},p} + \\ &+ \inf_{P\in\mathbb{P}_{r-1}} \|(f-P)w_{\alpha\beta}\|_{L^p(At^*,\infty)}. \end{aligned} \tag{3.2}$$

Connected with the modulus of continuity ω^r_φ is the K-functional

$$K(f,t^r)_{\omega_{\alpha\beta},p} = \inf_{g\in W^p_r} \{\|(f-g)w_{\alpha\beta}\|_p + t^r\|g^{(r)}\varphi^r w_{\alpha\beta}\|_p\} \tag{3.3}$$

where $r \geq 1$ and $1 \leq p \leq +\infty$, $0 < t < 1$.

In some contexts it is useful to define the main part of the previous K- functional

$$\tilde{K}(f,t^r)_{\omega_{\alpha\beta},p} = \sup_{0<h\leq t} \inf_{g\in W^p_r} \{\|(f-g)w_{\alpha\beta}\|_{L^p(I_{rh})} + h^r\|g^{(r)}\varphi^r w_{\alpha\beta}\|_{L^p(I_{rh})}\} \tag{3.4}$$

In fact the following theorem holds

Theorem 3.1. *Let $f \in L^p_{w_{\alpha\beta}}$ and $1 \leq p \leq +\infty$. Then, as $t \to 0$, we have*

$$\omega^r_\varphi(f,t)_{w_{\alpha\beta},p} \sim K(f,t^r)_{w_{\alpha\beta},p} \tag{3.5}$$

and

$$\Omega^r_\varphi(f,t)_{w_{\alpha\beta},p} \sim \tilde{K}(f,t^r)_{w_{\alpha\beta},p} \tag{3.6}$$

where the constants in "$\sim$" are independent of f and t.

The proof of this theorem is similar to the proof in [1] and later we will prove some crucial steps.

It is useful to observe that, by (3.6) and (3.4) with $g = f$, it follows

$$\Omega_\varphi^r(f,t)_{w_{\alpha\beta},p} \le C \inf_{0<h\le t} h^r \|f^{(r)}\varphi^r w_{\alpha\beta}\|_{L^p(I_{rh})};$$

this last relation allows us to evaluate the main part of the modulus of continuity of differentiable functions in $(0,+\infty)$. For example, for $f(x) = |\log x|$ we have $\Omega_\varphi^r(f,t)_{w_{\alpha\beta},1} \sim t^{2+2\alpha}$.

Now, as in the case of periodic functions or of functions defined on finite intervals, we can define the Besov-type spaces $B_{sq}^p(w_{\alpha\beta})$ by means of modulus of continuity. To this end, with $1 \le p \le +\infty$, we introduce the seminorms

$$\|f\|_{p,q,s} = \begin{cases} \left(\displaystyle\int_0^{1/k} \left[\frac{\omega_\varphi^k(f,t)_{w_{\alpha\beta},p}}{t^{s+1/q}}\right]^q dt\right)^{1/q}, & 1 \le q < +\infty, \quad k > s \\ \displaystyle\sup_{t>0} \frac{\omega_\varphi^k(f,t)_{w_{\alpha\beta},p}}{t^s}, & q = +\infty, \quad k > s \end{cases} \tag{3.7}$$

and define

$$B_{sq}^p = B_{sq}^p(w_{\alpha\beta}) = \{f \in L_{w_{\alpha\beta}}^p : \|f\|_{p,q,s} < +\infty\}$$

equipped with the norm $\|f\|_{B_{sq}^p(w_{\alpha\beta})} = \|fw_{\alpha\beta}\|_p + \|f\|_{p,q,s}$. Here we cannot study these spaces in details. In the next section we will prove some embedding theorems and will characterize the Besov spaces by the error of the best approximation.

4. **Polynomial approximation**

For each function $f \in L_{w_{\alpha\beta}}^p$ with $1 \le p \le +\infty$, $\beta > \frac{1}{2}$, $\alpha > -\frac{1}{p}$ if $p < +\infty$ and $\alpha \ge 0$ if $p = +\infty$, we define, as usual,the error of best approximation

$$E_m(f)_{w_{\alpha\beta},p} = \inf_{P\in\mathbb{P}_{m-1}} \|(f-P)w_{\alpha\beta}\|_p.$$

In this section we will estimate $E_m(f)_{w_{\alpha\beta},p}$ by means of the modulus of continuity and will characterize the classes functions defined in the previous section.

In order to establish a Jakson theorem it is necessary the following

Proposition 4.1. *For each function $f \in W_1^p(w_{\alpha\beta})$, $1 \le p \le +\infty$, we have*

$$E_m(f)_{w_{\alpha\beta},p} \le C\frac{\sqrt{a_m}}{m}\|f'\varphi w_{\alpha\beta}\|_p, \tag{4.1}$$

where $\varphi(x) = \sqrt{x}$, $C \ne C(m, f)$ and $a_m \sim m^{1/\beta}$.

Proof. We first prove that the condition

$$\left(\int_0^{+\infty} |f'(x)e^{-x^\beta}|^p dx\right)^{1/p} < +\infty \tag{4.2}$$

implies the estimate

$$E_m(f)_{w_{\alpha\beta},p} \le C\frac{\sqrt{a_m}}{m}\left(\int_0^{+\infty}\left|f'(x)\left(x+\frac{a_m}{m^2}\right)^{\alpha+\frac{1}{2}}e^{-x^\beta}\right|^p dx\right)^{1/p}. \tag{4.3}$$

To this end, let $1 \le p < +\infty$, $u(x) = |x|^{2\alpha+1/p}e^{-x^{2\beta}}$, $g(x) = f(x^2)$, $x \in \mathbb{R}$ and p_{2m} the best approximation of g. By using Theorem 2.1 in [9] we have

$$\begin{aligned} A &:= \left(\int_{-\infty}^{+\infty} |(g(x) - p_{2m}(x))u(x)|^p dx\right)^{1/p} \le \\ &\le C\frac{\bar{a}_{2m}}{2m}\left(\int_{-\infty}^{+\infty}\left|g'(x)\left(|x|+\frac{\bar{a}_{2m}}{2m}\right)^{2\alpha+\frac{1}{p}}e^{-x^{2\beta}}\right|^p dx\right)^{1/p} =: B \end{aligned}$$

where $\bar{a}_{2m} = \bar{a}_{2m}(u) \sim m^{\frac{1}{2\beta}}$ is the M-R-S number related to the weight u and as we first observed $\bar{a}_{2m} \sim \sqrt{a_m(w_{\alpha\beta})}$. Then a change of variables in A and B leads to (4.3).

Now we suppose $f \in W_1^p(w_{\alpha\beta})$ and we introduce the function

$$f_m(x) = \begin{cases} f\left(\frac{a_m}{m^2}\right) & x \in \left[0, \frac{a_m}{m^2}\right] \\ f(x) & x \ge \frac{a_m}{m^2} \end{cases}.$$

Obviously the condition $\|f_m' e^{-x^\beta}\|_p < +\infty$ is satisfied, (4.3) can be used and we easily deduce

$$E_m(f_m)_{w_{\alpha\beta},p} \le C\frac{\sqrt{a_m}}{m}\|f'\varphi w_{\alpha\beta}\|_{L^p\left(\left[\frac{a_m}{m^2},\infty\right)\right)}. \tag{4.4}$$

Then, since $E_m(f)_{w_{\alpha\beta},p} \le \|(f - f_m)w_{\alpha\beta}\|_p + E_m(f_m)_{w_{\alpha\beta},p}$, we have to estimate only the $L^p_{w_{\alpha\beta}}$-norm of $f - f_m$.

To this end, we put $x_0 = \frac{a_m}{m^2}$ and get

$$\|(f - f_m)w_{\alpha\beta}\|_p = \left(\int_0^{x_0} |[f(x) - f(x_0)]w_{\alpha\beta}(x)|^p dx\right)^{1/p}$$

$$= \left(\int_0^{x_0} \left|\int_0^{x_0} (t-x)_+^0 f'(t) w_{\alpha\beta}(x) dt\right|^p dx\right)^{1/p}$$

$$\leq \int_0^{x_0} |f'(t)| \left(\int_0^{x_0} (t-x)_+^0 w_{\alpha\beta}^p(x) dx\right)^{1/p} dt$$

$$= \int_0^{x_0} |f'(t)| \left(\int_0^t w_{\alpha\beta}^p(x) dx\right)^{1/p} dt \sim \int_0^{x_0} |f'(t)| t^{\alpha+\frac{1}{p}} e^{-t^\beta} dt \leq$$

$$\leq C\|f'\varphi w_{\alpha\beta}\|_{L^p((0,x_0))} \left(\int_0^{x_0} t^{q(1/p-1/2)} dt\right)^{1/q} \sim \frac{\sqrt{a_m}}{m} \|f'\varphi w_{\alpha\beta}\|_{L^p\left(0,\frac{a_m}{m^2}\right)},$$

which, with (4.4), proves (4.1) when $1 \leq p < +\infty$. The case $p = +\infty$ is similar and (4.1) is proved.

By iterating (4.1) we have, for each $g \in W_r^p(w_{\alpha\beta})$, the estimate

$$E_m(g)_{w_{\alpha\beta},p} \leq C \left(\frac{\sqrt{a_m}}{m}\right)^r \|g^{(r)}\varphi^r w_{\alpha\beta}\|_p, \quad C \neq C(m,f),$$

from which, by using the K-functional and its equivalence with ω_φ^r, the Jackson theorem follows. □

Theorem 4.2. *For all $f \in L^p_{w_{\alpha\beta}}$, $1 \leq p \leq +\infty$ and $r < m$ we have*

$$E_m(f)_{w_{\alpha\beta}} \leq C\omega_\varphi^r \left(f, \frac{\sqrt{a_m}}{m}\right)_{w_{\alpha\beta},p}, \quad C \neq C(f,m). \tag{4.5}$$

By using the K-functional and the Bernstein inequality, in a usual way we obtain the Stechkin inequality formulated in the following theorem

Theorem 4.3. *For each $f \in L^p_{w_{\alpha\beta}}, 1 \leq p \leq +\infty$, and an arbitrary integer $r \geq 1$ we have:*

$$\omega_\varphi^r \left(f, \frac{\sqrt{a_m}}{m}\right)_{w_{\alpha\beta},p} \leq C \left(\frac{\sqrt{a_m}}{m}\right)^r \sum_{k=0}^m \left(\frac{1+k}{\sqrt{a_k}}\right)^r \frac{E_k(f)_{w_{\alpha\beta},p}}{1+k} \tag{4.6}$$

with $C = C(r)$ independent of m and f.

By proceeding as in [1], Lemma 3.5 (see also [3], p. 94-95) it is not difficult to show that, setting

$$\widetilde{E}_m(f)_{w_{\alpha\beta},p} = \inf_{P_m} \|(f - P_m) w_{\alpha\beta}\|_{L^p\left(\frac{a_m}{m^2}, a_m\right)}, \quad 1 \leq p \leq +\infty,$$

if $t^{-1}\Omega_\varphi^r(f,t)_{w_{\alpha\beta},p} \in L^1$, it results

$$\widetilde{E}_m(f)_{w_{\alpha\beta},p} \leq C\Omega_\varphi^r \left(f, \frac{\sqrt{a_m}}{m}\right)_{w_{\alpha\beta},p}. \tag{4.7}$$

From this last result the next theorem easily follows.

Theorem 4.4. *For each function $f \in L^p_{w_{\alpha\beta}}$, $1 \le p \le +\infty$, we have*

$$E_m(f)_{w_{\alpha\beta},p} \le C \int_0^{\frac{\sqrt{a_m}}{m}} \frac{\Omega^k_\varphi(f,t)_{w_{\alpha\beta},p}}{t} dt \tag{4.8}$$

where $C \ne C(m,f)$ and $k < m$.

Recall that the main part of the modulus Ω^k_φ is smaller than ω^k_φ and generally the two moduli are not equivalent. Moreover if, for some p, $\Omega^k_\varphi(f,t)_{w_{\alpha\beta},p} \sim t^\lambda$, $0 < \lambda < k$, then by (4.8), we have $E_m(f)_{w_{\alpha\beta},p} \sim \left(\frac{\sqrt{a_m}}{m}\right)^\lambda$ and, by using (4.6), also $\omega^k_\varphi(f,t)_{w_{\alpha\beta},p} \sim t^\lambda$. Then for these classes of functions the two moduli are equivalent. By using Jackson and Stechkin inequalities we can represent the seminorms of the Besov spaces in (3.7) by means of the error of best approximation (see, for instance, [3]). In fact, for $1 \le p \le +\infty$, the following equivalences hold:

$$\|f\|_{pqs} \sim \left(\sum_{k=1}^{+\infty} k^{\left(1-\frac{1}{2\beta}\right)sq-1} E_k(f)^q_{w_{\alpha\beta},p}\right)^{1/q}, \quad 1 \le q < +\infty$$

$$\|f\|_{pqs} \sim \sup_{m\ge1} m^{\left(1-\frac{1}{2\beta}\right)s} E_m(f)_{w_{\alpha\beta},p}, \quad q = +\infty.$$

The next theorem is useful in more contexts.

Theorem 4.5. *For each $f \in L^p_{w_{\alpha\beta}}, 1 \le p \le +\infty$, we have*

$$\omega^r_\varphi\left(f, \frac{\sqrt{a_m}}{m}\right)_{w_{\alpha\beta},p} \sim \inf_{P\in \mathbb{P}_m} \left\{\|(f-P)w_{\alpha\beta}\|_p + \left(\frac{\sqrt{a_m}}{m}\right)^r \|P^{(r)}\varphi^r w_{\alpha\beta}\|_p\right\} \tag{4.9}$$

where the constants in "$\sim$" are independent of m and f.

A consequence of formula (4.9) is the useful inequality

$$\left(\frac{\sqrt{a_m}}{m}\right)^r \left\|P_m^{(r)}\varphi^r w_{\alpha\beta}\right\|_p \le C\omega^r_\varphi\left(f, \frac{\sqrt{a_m}}{m}\right)_{w_{\alpha\beta},p}, \tag{4.10}$$

being P_m the polynomial of quasi best approximation, i.e.

$$\|(f-P_m)\,w_{\alpha\beta}\|_p \le CE_m(f)_{w_{\alpha\beta},p}.$$

For the proof of Theorem 4.5 the reader can use the same tool in [1] with some small changes.

Now we will show some embedding theorems which connect different function norms and moduli of smoothness. For different classes of functions the reader can consult [2].

In the sequel, to simplify the notations, we will set $w = w_{\alpha\beta}$ with $\alpha \geq 0$.

Theorem 4.6. *Let $f \in L^p_w$, $1 \leq p < +\infty$ and let us assume that the condition*

$$\int_0^1 \frac{\Omega^r_\varphi(f,t)_{w,p}}{t^{1+1/p}} dt < +\infty \tag{4.11}$$

is satisfied. Then f is a continuous function in any interval $[a, +\infty)$, $a > 0$.

Moreover, if, with $\widetilde{w} = w/\varphi^{1/p}$, and

$$\int_0^1 \frac{\Omega^r_\varphi(f,t)_{\widetilde{w},p}}{t^{1+1/p}} dt < +\infty \tag{4.12}$$

then we have

$$\left.\begin{array}{c} E_m(f)_{w,\infty} \\ \Omega^r_\varphi\left(f, \frac{\sqrt{a_m}}{m}\right)_{w,\infty} \end{array}\right\} \leq C \int_0^{\frac{\sqrt{a_m}}{m}} \frac{\Omega^r_\varphi(f,t)_{\widetilde{w},p}}{t^{1+1/p}} dt \tag{4.13}$$

and

$$\|fw\|_\infty \leq C \left(\|f\widetilde{w}\|_p + \int_0^1 \frac{\Omega^r_\varphi(f,t)_{\widetilde{w},p}}{t^{1+1/p}} dt \right). \tag{4.14}$$

Finally (4.12) *implies* (4.13) *and* (4.14) *with w in place of $\widetilde{w}$ and $\frac{2}{p}$ in place of $\frac{1}{p}$. Here the positive constants C are independent of m, t and f.*

Proof. In virtue of (4.8), (4.11) implies, for $1 \leq p < +\infty$, $\lim_m E_m(f)_{w,p} = 0$. Therefore, if P_m denotes the polynomial of best approximation (or quasi best approximation) in L^p_w, the equality

$$w(f - P_m) = \sum_{k=0}^{+\infty} (P_{2^{k+1}m} - P_{2^k m})\, w \tag{4.15}$$

is true a.e. in $(0, +\infty)$. If we prove that the series uniformly converges on each half-line $[a, +\infty)$, $a > 0$, then the equality holds everywhere in $[a, +\infty)$ and f is continuous.

Now, by using (2.8), with $p = +\infty$ and $q = p$, one has

$$\|(P_{2^{k+1}m} - P_{2^k m})\, w\|_{L^\infty([a,+\infty))} \leq a^{-\frac{1}{2p}} \left\| (P_{2^{k+1}m} - P_{2^k m})\, w\varphi^{\frac{1}{p}} \right\|_{L^\infty((a,+\infty))}$$

$$\leq a^{-\frac{1}{2p}} K \left(\frac{2^{k+1}m}{\sqrt{a_{2^{k+1}m}}} \right)^{1/p} \|(P_{2^{k+1}m} - P_{2^k m})\, w\|_p$$

$$\leq a^{-\frac{1}{2p}} KC \left(\frac{2^{k+1}m}{\sqrt{a_{2^{k+1}m}}} \right)^{1/p} \|(P_{2^{k+1}m} - P_{2^k m})\, w\|_{L^p(I_{mk})}$$

having used (2.5) in the last inequality and setting $I_{mk} = \left[\frac{a_{2^{k+1}m}}{(2^{k+1}m)^2}, a_{2^{k+1}m} \right]$. Consequently one has, for (4.7),

$$\begin{aligned} \|(P_{2^{k+1}m} - P_{2^k m})\, w\|_{L^\infty([a,+\infty))} &\leq C \left(\frac{2^k m}{\sqrt{a_{2^k m}}} \right)^{1/p} \widetilde{E}_{2^k m}(f)_{w,p} \\ &\leq C \left(\frac{2^k m}{\sqrt{a_{2^k m}}} \right)^{1/p} \Omega_\varphi^r \left(f, \frac{\sqrt{a_{2^k m}}}{2^k m} \right)_{w,p} \end{aligned}$$

and

$$\begin{aligned} \sum_{k=0}^{+\infty} \|(P_{2^{k+1}m} - P_{2^k m})\, w\|_{L^\infty([a,+\infty))} &\leq C \sum_{k=0}^{+\infty} \left(\frac{2^k m}{\sqrt{a_{2^k m}}} \right)^{1/p} \Omega_\varphi^r \left(f, \frac{\sqrt{a_{2^k m}}}{2^k m} \right)_{w,p} \\ &\leq C \int_0^{\frac{\sqrt{a_m}}{m}} \frac{\Omega_\varphi^r(f,t)_{w,p}}{t^{1+1/p}} dt < +\infty. \end{aligned}$$

Then the series in (4.15) absolutely and uniformly converges and the equality in (4.15) is true everywhere in $[a,+\infty)$.

To prove the first relation of (4.13) we use (2.8) in an equivalent form and with the previous notations we obtain

$$\begin{aligned} \|(P_{2^{k+1}m} - P_{2^k m})\, w\|_\infty &\leq K \left(\frac{2^{k+1}m}{\sqrt{a_{2^{k+1}m}}} \right)^{1/p} \left\| (P_{2^{k+1}m} - P_{2^k m})\, w/\varphi^{1/p} \right\|_p \\ &\leq K \left(\frac{2^{k+1}m}{\sqrt{a_{2^{k+1}m}}} \right)^{1/p} \widetilde{E}_{2^k m}(f)_{\tilde{w},p} \\ &\leq C \left(\frac{2^{k+1}m}{\sqrt{a_{2^{k+1}m}}} \right)^{1/p} \Omega_\varphi^r \left(f, \frac{\sqrt{a_{2^k m}}}{2^k m} \right)_{\tilde{w},p}. \end{aligned}$$

It follows

$$\begin{aligned} \|(f - P_m)w\|_\infty &\leq \lim_k \|(P_{2^{k+1}m} - P_{2^k m})\, w\|_\infty = \lim_k \left\| \sum_{i=0}^{k} (P_{2^{i+1}m} - P_{2^i m})\, w \right\|_\infty \\ &\leq \sum_{i=0}^{+\infty} \|(P_{2^{i+1}m} - P_{2^i m})\, w\|_\infty \leq C \int_0^{\frac{\sqrt{a_m}}{m}} \frac{\Omega_\varphi^r(f,t)_{\tilde{w},p}}{t^{1+1/p}} dt. \end{aligned}$$

To prove the second estimate in (4.13) we observe that, with P_m as the polynomial of best approximation in $L^p_{\widetilde{w}}$, we have

$$\begin{aligned}\Omega_\varphi^r\left(f,\frac{\sqrt{a_m}}{m}\right)_{w,\infty} &\leq C\left[\|(f-P_m)w\|_\infty+\left(\frac{\sqrt{a_m}}{m}\right)^r\left\|P_m^{(r)}\varphi^r w\right\|_\infty\right]\\ &\leq C\left[E_m(f)_{w,\infty}+\left(\frac{\sqrt{a_m}}{m}\right)^r\left\|P_m^{(r)}\varphi^r \widetilde{w}\right\|_p\left(\frac{m}{\sqrt{a_m}}\right)^{1/p}\right].\end{aligned}$$

Now for the first term let us use the first estimate of (4.13). The second term, by proceeding as in [3], p.99-100 (see also [1]) is dominated by

$$C\left(\frac{m}{\sqrt{a_m}}\right)^{1/p}\int_0^{\frac{\sqrt{a_m}}{m}}\frac{\Omega_\varphi^r(f,t)_{\widetilde{w},p}}{t}dt. \leq C\int_0^{\frac{\sqrt{a_m}}{m}}\frac{\Omega_\varphi^r(f,t)_{\widetilde{w},p}}{t^{1+1/p}}dt.$$

Then the second estimate in (4.13) follows.
Finally to prove (4.14) we write

$$\|fw\|_\infty \leq \|(f-P_1)w\|_\infty+\|P_1 w\|_\infty$$

with P_1 as best approximation in $L^p_{\widetilde{w}}$. Since

$$\|P_1 w\|_\infty \leq KP_1\widetilde{w}\|_p \leq 2K\|f\widetilde{w}\|_p,$$

for the first term we use the first estimate of (4.13) with $m=1$.
To show the last part of the theorem we proceed as in the proof of (4.13), using inequality (2.9) in place of (2.8). □

5. **Fourier Sum and Lagrange Polynomial**

The approximation of functions by means of their Fourier sums in the system $\{p_m(w_\alpha)\}_m$, where $p_m(w_\alpha,x)=\gamma_m x^m+\cdots,\gamma_m>0$, and

$$\int_0^{+\infty}p_m(w_\alpha,x)p_n(w_\alpha,x)w_\alpha(x)dx=\delta_{mn},$$

is useful in different contexts. Moreover, the weighted Lagrange interpolation based on the zeros of $p_m(w_\alpha,x)$ is useful in numerous problems of numerical analysis, too. We will consider these two approximation processes in the space L^p_u, where $u(x)=x^\gamma e^{-\frac{x^\beta}{2}}$ and $1\leq p\leq+\infty$.

5.1. **Fourier Sums.** For $f \in L^p_u$, the m-th Fourier sum $S_m(w_\alpha, f)$ is defined as follows

$$S_m(w_\alpha, f) = \sum_{k=0}^{m-1} c_k p_k(w_\alpha),$$

where

$$c_k = \int_0^{+\infty} f(t) p_k(w_\alpha, t) w_\alpha(t) dt.$$

Analogously to the cases of Laguerre, Hermite and Freud polynomials (see [10]) the uniform boundedness of $S_m(w_\alpha)$ in L^p_u holds true for $p \in \left(\frac{4}{3}, 4\right)$ and then for a restricted class of functions. This fact leads to modify the polynomial $S_m(w_\alpha, f)$ following a procedure used in [7][6][10] that we will briefly illustrate. Let $a_m := a_m(u)$ be the M-R-S number related to the weight u. Let $\theta \in (0,1), M = \left\lfloor \frac{m\theta}{1+\theta} \right\rfloor \sim m$ and let $\bar{\Delta}_{\theta m}$ be the characteristic function of the segment $[0, \theta a_m]$. Then, using (2.3) with u in place of $w_{\alpha\beta}$, for every $f \in L^p_u$, we get

$$\|f(1-\bar{\Delta}_{\theta m})u\|_p \le \mathcal{C}\left(E_M(f)_{u,p} + e^{-Am}\|fu\|_p\right) \tag{5.1}$$

and

$$\|fu\|_p \le \mathcal{C}\left(\|f\bar{\Delta}_{\theta m}u\|_p + E_M(f)_{u,p}\right), \tag{5.2}$$

where $1 \le p \le +\infty$ and $E_M(f)_{u,p}$ is the error of best approximation of f in $\mathbb{P}_M$. Therefore, by (5.2), it is sufficient to approximate the function f in the more restricted interval $[0, \theta a_m]$ or, equivalently, to replace $\{S_m(w_\alpha, f)\}_m$ with the sequence $\{\Delta_{\theta m} S_m(w_\alpha, f\Delta_{\theta m})\}_m$, where $a_m = a_m(w_\alpha)$ and $\Delta_{\theta m}$ is the characteristic function of $[0, \theta a_m]$ with $\theta \in (0,1)$ arbitrary. The theorems that follow show that this procedure is convenient.

Theorem 5.1. *Let $u \in L^p$ with $1 < p < +\infty$. Then, for every $f \in L^p_u$ there exists a constant $\mathcal{C} \ne \mathcal{C}(m, f)$ such that*

$$\|S_m(w_\alpha, \Delta_{\theta m} f)\Delta_{\theta m} u\|_p \le \mathcal{C}\|f\Delta_{\theta m} u\|_p \tag{5.3}$$

if and only if

$$\frac{v^\gamma}{\sqrt{v^\alpha \varphi}} \in L^p(0,1) \quad \textit{and} \quad \sqrt{\frac{v^\alpha}{\varphi}}\frac{1}{v^\gamma} \in L^q(0,1), \tag{5.4}$$

where $v^\rho(x) = x^\rho, \varphi(x) = \sqrt{x}$ *and* $p^{-1} + q^{-1} = 1$. *Moreover, under the conditions* (5.4), (5.3) *is equivalent to*

$$\|[f - \Delta_{\theta m} S_m(w_\alpha, \Delta_{\theta m} f)]u\|_p \leq \mathcal{C}\left(E_M(f)_{u,p} + e^{-Am}\|fu\|_p\right), \tag{5.5}$$

where A *and* $\mathcal{C}$ *are positive constant independent of* m *and* f.

As an example, if $f \in W_r^p(u), r \geq 1$, and (5.4) holds true, we have

$$\|[f - \Delta_{\theta m} S_m(w_\alpha, \Delta_{\theta m} f)]u\|_p \leq \mathcal{C}\left(\frac{\sqrt{a_m}}{m}\right)^r \|f\|_{W_r^p(u)},$$

i.e. the error of best approximation of functions belonging to $W_r^p(u)$. If $w_\alpha(x) = x^\alpha e^{-x}$ and $u(x) = x^\gamma e^{-\frac{x}{2}}$ (Laguerre case), then Theorem 5.1 is equivalent to Theorem 2.2 in [10]. Moreover, as in the Laguerre case, if (5.4) holds true with $1 < p < 4$ then we get the estimate

$$\|S_m(w_\alpha, \Delta_{\theta m} f)\Delta_{\theta m} u\|_p \leq \mathcal{C}\|f\Delta_{\theta m} u\|_p \tag{5.6}$$

and if (5.4) holds true with $p > \frac{4}{3}$ then it results

$$\|S_m(w_\alpha, f)\Delta_{\theta m} u\|_p \leq \mathcal{C}\|fu\|_p. \tag{5.7}$$

Moreover, we have

$$\|S_m(w_\alpha, f)u\|_p \leq \mathcal{C}\|fu\|_p, \tag{5.8}$$

$$\|S_m(w_\alpha, f)u\|_p \leq \mathcal{C}\begin{cases} m^{\frac{1}{3}}\|fu\|_p \\ \\ \|fu(1 + \cdot^3)\|_p \end{cases} \tag{5.9}$$

if (5.4) is satisfied with $p \in \left(\frac{4}{3}, 4\right)$ or $p \in (1, +\infty) \setminus \left[\frac{4}{3}, 4\right]$ respectively. The cases $p = 1$ or $p = +\infty$ are considered in the following theorems.

Theorem 5.2. *Let* f *be such that*

$$\int_0^{+\infty} |f(x)u(x)| \log^+ |f(x)| < +\infty,$$

with

$$\log^+ |z| = \begin{cases} 0 & \textit{if } |z| \leq 1 \\ \\ \log|z| & \textit{if } z > 1. \end{cases}$$

If

$$\frac{v^\gamma}{\sqrt{v^\alpha \varphi}} \in L^1 \quad \textit{and} \quad \sqrt{\frac{v^\alpha}{\varphi}} \frac{1}{v^\gamma} \in L^\infty, \quad v^\rho(x) = x^\rho, \quad \varphi(x) = \sqrt{x}, \tag{5.10}$$

then we have

$$\|S_m(w_\alpha, \Delta_{\theta m} f)u\Delta_{\theta m}\|_1 \le \mathcal{C}\left[1 + \int_0^{+\infty} |fu|(x)(1 + \log^+|f(x)| + \log^+ x)dx\right],$$

with $\mathcal{C} \ne \mathcal{C}(m, f)$.

Theorem 5.3. *Let* $f \in L_u^\infty, u(x) = x^\gamma e^{-\frac{x^\beta}{2}}, \beta > \frac{1}{2}, \gamma \ge 0$. *If* $\frac{\alpha}{2} + \frac{1}{4} \le \gamma \le \frac{\alpha}{2} + \frac{3}{4}$, *then we have*

$$\|S_m(w_\alpha, \Delta_{\theta m} f)u\Delta_{\theta m}\|_\infty \le \mathcal{C}\|f\Delta_{\theta m} u\|_\infty(\log m),$$

where $\mathcal{C} \ne \mathcal{C}(m, f)$.

Theorems 5.1 and 5.2 and estimates (5.6)-(5.9) have been proved in [10]. Theorem 5.3 has been proved in [6].

5.2. **Lagrange interpolation.** If f is a continuous function in $(0, +\infty)$ then the Lagrange polynomial interpolating f on the zeros $x_1 < x_2 < \cdots < x_m$ of $p_m(w_\alpha)$ is defined as

$$L_m(w_\alpha, f, x) = \sum_{i=1}^m l_i(x) f(x_i), \quad l_i(x) = \frac{p_m(w_\alpha, x)}{p'_m(w_\alpha, x_k)(x - x_k)}.$$

In the sequel we will consider the behaviour of $L_m(w_\alpha, f)$ in L_u^p with $u(x) = x^\gamma e^{-\frac{x^\beta}{2}}$. Analogously to the Fourier sums, the behaviour of $L_m(w_\alpha, f)$ in L_u^p is "poor", i.e. it can be used with good results only for a restricted class of functions. For example, if $p = +\infty$ and $f \in L_u^\infty$ with $\gamma \ge 0$, then for every choice of α and γ,

$$\|L_m(w_\alpha)\| := \sup_{\|fu\|_\infty = 1} \|L_m(w_\alpha, f)u\|_\infty > \mathcal{C} m^\rho,$$

with $\rho > 0$ and $\mathcal{C} \ne \mathcal{C}(f, m)$. Then, as for the Fourier sums, we modify the Lagrange polynomial. To this end, we introduce the following notations. Let

$$x_j = \min_{k=1,\dots,m} \{x_k \ : \ x_k \ge \theta a_m\},$$

where $\theta \in (0, 1)$ and $a_m = a_m(w_\alpha), m$ sufficiently large. With

$$\Psi(x) = \begin{cases} 0 & \text{if } x \le 0 \\ 1 & \text{if } x \ge 1 \end{cases} \quad \text{and} \quad \Psi_j(x) = \Psi\left(\frac{x - x_j}{x_j - x_{j-1}}\right),$$

define the truncated function $f_j := \Phi_j f$, where $\Phi_j = 1 - \Psi_j$. By definition, we deduce that f_j has the same smoothness as f and

$$f_j(x) = \begin{cases} f(x) & \text{if } x \in [0, x_j] \\ 0 & \text{if } x \in [x_{j+1}, +\infty). \end{cases}$$

Now, letting $\theta_1 \in (\theta, 1)$ and denoting by $\Delta_{\theta_1} := \Delta_{\theta_1 m}$ the characteristic function of $[0, \theta_1 a_m]$, we consider the behaviour of the sequence $\{\Delta_{\theta_1} L_m(w_\alpha, f_j)\}_m$ in $L^p_u, u(x) = x^\gamma e^{-\frac{x^\beta}{2}}, 1 < p \le +\infty$.

Theorem 5.4. *If the parameters α and γ of the weights w_α and u satisfy*

$$\frac{\alpha}{2} + \frac{1}{4} \le \gamma \le \frac{\alpha}{2} + \frac{5}{4}, \quad \gamma \ge 0,$$

then

$$\|\Delta_{\theta_1} L_m(w_\alpha, f_j)u\|_\infty \le \mathcal{C}\|f_j u\|_\infty (\log m),$$

with $\mathcal{C} \ne \mathcal{C}(m, f)$.

The following lemma will be useful in the sequel, but it can be used in more contexts too.

Lemma 5.5. *Let $0 < \theta < \theta_1 < 1, 1 \le p < +\infty$ and $\Delta x_k = x_{k+1} - x_k$. Then, for an arbitrary polynomial $P \in P_{ml}$ (l fixed integer), we have*

$$\sum_{k=1}^{j} \Delta x_k |Pu|^p(x_k) \le \mathcal{C} \int_{x_1}^{\theta_1 a_m} |Pu|^p(x)dx,$$

with $\mathcal{C} \ne \mathcal{C}(m, p, P)$.

In order to simplify the notations, from now on we let $v^\rho(x) = x^\rho$.

Theorem 5.6. *Let $1 < p < +\infty$ and assume that*

$$\frac{v^\gamma}{\sqrt{v^\alpha \varphi}} \in L^p \quad \textit{and} \quad \frac{\sqrt{v^\alpha \varphi}}{v^\gamma} \in L^q, \quad \varphi(x) = \sqrt{x}, \quad q = \frac{p}{p-1}. \tag{5.11}$$

Then, for every $f \in C^0(0, +\infty)$, we have

$$\|L_m(w_\alpha, f_j)u\Delta_{\theta_1}\|_p \le \mathcal{C} \sum_{k=1}^{j} \Delta x_k |fu|^p(x_k), \tag{5.12}$$

with $\mathcal{C} \ne \mathcal{C}(m, f)$.

The following lemma estimates the right-hand side of (5.12) in terms of the main part of the modulus of smoothness.

Lemma 5.7. *For every function f belonging to $C^0(0,+\infty)$ we have*

$$\left(\sum_{k=1}^{j} \Delta x_k |fu|^p(x_k)\right)^{\frac{1}{p}} \le \mathcal{C}\left[\|fu\|_{L^p(0,x_j)} + \left(\frac{\sqrt{a_m}}{m}\right)^{\frac{1}{p}} \int_0^{\frac{\sqrt{a_m}}{m}} \frac{\Omega_\varphi^r(f,t)_{u,p}}{t^{1+\frac{1}{p}}} dt\right],$$

with $r < m$ and $\mathcal{C} \ne \mathcal{C}(m,f)$.

Now we can state the following

Theorem 5.8. *Under the assumptions of Theorem* 5.6*, for every continuous function in* $(0,+\infty)$*, we have*

$$\|[f - \Delta_{\theta_1} L_m(w_\alpha, f_j)]u\|_p \le \mathcal{C}\left[\left(\frac{\sqrt{a_m}}{m}\right)^{\frac{1}{p}} \int_0^{\frac{\sqrt{a_m}}{m}} \frac{\Omega_\varphi^r(f,t)_{u,p}}{t^{1+\frac{1}{p}}} dt + e^{-Am}\|fu\|_{L^p}\right],$$

where the constants A and $\mathcal{C}$ are independent of m and f.

As an example, for every $f \in W_r^p(u)$, we have

$$\|[f - \Delta_{\theta_1} L_m(w_\alpha, f_j)]u\|_p \le \mathcal{C}\left(\frac{\sqrt{a_m}}{m}\right)^r \|f\|_{W_r^p(u)},$$

that is the error of best approximation in $W_r^p(u)$.

6. **Proofs**

We first state two propositions whose proofs are easy.

Proposition 6.1. *Let $x \in [(2rh)^2, h^*]$, with $h^* = \frac{1}{h^{\frac{2}{2\beta-1}}}, \beta > \frac{1}{2}$, and $y \in [x - rh\sqrt{x}, x + rh\sqrt{x}]$. Then it results:*

$$w_{\alpha\beta}(x) \sim w_{\alpha\beta}(y),$$

where the constant in "$\sim$" are independent of x and h.

Proposition 6.2. *Let $z > 0$ be such that $w_{\alpha\beta}(x) = x^\alpha e^{-x^\beta}, \beta > \frac{1}{2}$ is a non-decreasing function in $[z,+\infty]$. Then, for every $f \in W_r^p(w_{\alpha\beta})$, with $r \ge 1$ and $1 \le p \le +\infty$,*

$$\left(\int_z^{+\infty} \left|w_{\alpha\beta}(x)\int_z^x (x-u)^{r-1} f^{(r)}(u)du\right|^p dx\right)^{\frac{1}{p}} \le \frac{\mathcal{C}}{(z^{\beta-\frac{1}{2}})^r}\|f^{(r)}\varphi^r w_{\alpha\beta}\|_p,$$

with $\mathcal{C} \ne \mathcal{C}(f,z,p)$.

Proof of Theorem 3.1. We first point out the main steps of the proof. In order to prove (3.6), constructing a suitable function $G_h \in W_r^p(w_{\alpha\beta})$, we state the inequality

$$\tilde{K}(f,t^r)_{w_{\alpha\beta},p} \le \mathcal{C}\Omega_\varphi^r(f,t)_{w_{\alpha\beta},p}. \tag{6.13}$$

Let $t_0 < 8r^2h^2 \leq t_1 < t_2 < \cdots < t_j \leq h^* < t_{j+1}, h > 0$, be a system of knots such that $t_{i+1} - t_i \sim h\sqrt{t_i}, i = 0, \ldots, j$. With $\Psi \in C^\infty(\mathbb{R})$ a non-decreasing function such that

$$\Psi_(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ 1 & \text{if } x \geq 1, \end{cases}$$

and with $y_k = \frac{t_k+t_{k+1}}{2}$, define the functions $\Psi_k(x) = \Psi\left(\frac{x-y_k}{t_{k+1}-y_k}\right)$, where $k = 1, 2, \ldots, j$ and $\Psi_0(x) = 0 = \Psi_{j+1}(x)$. With

$$f_\tau(x) = r^r \int_0^{\frac{1}{r}} \cdots \int_0^{\frac{1}{r}} \left(\sum_{l=1}^{r} (-1)^{l+1} \binom{r}{l} f(x + l\tau(u_1 + \cdots + u_r)) \right) du_1 \ldots du_r$$

and

$$F_{hk}(x) = \frac{2}{h} \int_{\frac{h}{2}} h f_{\tau\varphi(t_k)}(x) d\tau,$$

we introduce the function

$$G_h(x) = \sum_{k=1}^{j} F_{hk}(x) \Psi_{k-1}(x)(1 - \Psi_k(x)). \tag{6.14}$$

After that, in order to prove the inequalities

$$\left.\begin{array}{r} \|(f - G_h) w_{\alpha\beta}\|_{L^p(8r^2h^2, h^*)} \\ h^r \|G_h^{(r)} \varphi^r w_{\alpha\beta}\|_{L^p(8r^2h^2, h^*)} \end{array}\right\} \leq \mathcal{C} \|w_{\alpha\beta} \overrightarrow{\Delta}_{h\varphi} f\|_{L^p(8r^2h^2, Ah^*)},$$

for some constant A, it is sufficient to repeat word for word [3], p. 194-197, with some simplifications due to the forward difference $\overrightarrow{\Delta}_{h\varphi}$ appearing in the definition of the modulus Ω_φ^r. Thus (3.6) follows. In order to prove the inverse inequality of (3.6), we now prove that for every $g \in W_r^p(w_{\alpha\beta})$

$$\|w_{\alpha\beta} \overrightarrow{\Delta}_{h\varphi} f\|_{L^p(8r^2h^2, h^*)}$$

$$\leq \mathcal{C} \left\{ \|(f - g) w_{\alpha\beta}\|_{L^p(8r^2h^2, Ah^*)} + h^r \|g^{(r)} \varphi^r w_{\alpha\beta}\|_{L^p(8r^2h^2, Ah^*)} \right\},$$

with $A = 1 + rh^{\frac{2\beta}{2\beta-1}}$. In fact, we have

$$|w_{\alpha\beta}(x)(\overrightarrow{\Delta}_{h\varphi} f)(x)| \leq \sum_{k=0}^{r} \binom{r}{k} |f - g|(x + (r - k)h\sqrt{x}) w_{\alpha\beta}(x) + |w_{\alpha\beta}(x)(\overrightarrow{\Delta}_{h\varphi} g)(x)|.$$

Now, x and $x+(r-k)h\sqrt{x}$ belong to $[8r^2h^2, Ah^*]$ and $|x-(x+(r-k)h\sqrt{x})| \le rh\sqrt{x}$. Thus, by Proposition 6.1, $w_{\alpha\beta}(x) \le \mathcal{C} w_{\alpha\beta}(x+(r-k)h\sqrt{x})$ and

$$\begin{aligned} \|w_{\alpha\beta}\overrightarrow{\Delta}_{h\varphi}(f-g)\|_{L^p(8r^2h^2,h^*)} &\le \mathcal{C}\sum_{k=0}^{r}\binom{r}{k}\|(f-g)w_{\alpha\beta}(\cdot+(r-k)h\sqrt{\cdot})\|_{L^p(8r^2h^2,h^*)} \\ &\le \mathcal{C}2^r\|(f-g)w_{\alpha\beta}\|_{L^p(8r^2h^2,Ah^*)}, \end{aligned}$$

making the change of variable $u = x+(r-k)h\sqrt{x}$ and using $\left|\frac{du}{dx}\right| \le 2$. Moreover, since

$$\begin{aligned} \overrightarrow{\Delta}^r_h g(x) &= r!h^r\int_0^1\int_0^{t_1}\cdots\int_0^{t_{r-1}} g^{(r)}(x+h(t_1+\cdots+t_r))dt_1\ldots dt_r \\ &=: r!h^r\int_{T_r} g^{(r)}(x+h\tau)dT_r, \end{aligned}$$

with $\tau = t_1+\cdots+t_r < r$ and $T_r = [0,1]\times[0,t_1]\times\cdots\times[0,t_r]$, we can write

$$w_{\alpha\beta}(x)\overrightarrow{\Delta}^r_{h\varphi}g(x) = r!(h\varphi)^r\int_{T_r} g^{(r)}(x+h\tau\sqrt{x})w_{\alpha\beta}(x)dT_r.$$

Consequently, by Proposition 6.1, we have

$$\begin{aligned} \|w_{\alpha\beta}\overrightarrow{\Delta}^r_{h\varphi}g\|_{L^p(8r^2h^2,h^*)} &\le \mathcal{C}r!h^r\left(\int_{8r^2h^2}^{h^*}\left|\int_{T_r} g^{(r)}(x+h\tau\sqrt{x})w_{\alpha\beta}(x)dT_r\right|^p\right)^{\frac{1}{p}} \\ &\le \mathcal{C}r!h^r\int_{T_r}\left(\int_{8r^2h^2}^{h^*}\left|g^{(r)}\varphi^r w_{\alpha\beta}\right|^p(x+h\tau\sqrt{x})dx\right)^{\frac{1}{p}}dT_r \\ &\le \mathcal{C}h^r\left(\int_{8r^2h^2}^{Ah^*}\left|g^{(r)}\varphi^r w_{\alpha\beta}\right|^p(u)du\right)^{\frac{1}{p}}, \end{aligned}$$

being $\int_{T_r} dT_r = \frac{1}{r!}$. Then the equivalence (3.6) easily follows. Now we prove equivalence (3.5), i.e.

$$\omega^r_\varphi(f,t)_{w_{\alpha\beta},p} \sim K(f,t^r)_{w_{\alpha\beta},p}.$$

In order to prove

$$\omega^r_\varphi(f,t)_{w_{\alpha\beta},p} \le \mathcal{C}K(f,t^r)_{w_{\alpha\beta},p},$$

since

$$\Omega^r_\varphi(f,t)_{w_{\alpha\beta},p} \le \mathcal{C}K(f,t^r)_{w_{\alpha\beta},p}, \quad 1\le p\le+\infty,$$

holds true, it remains to prove that the first and third terms in the definition of ω_φ^r are dominated by the $K-$functional. About the first term, in [1], p. 200, we proved that, with $u_\alpha = x^\alpha e^{-x}$,

$$\inf_{q_r\in\mathbb{P}_r} \|(f-q_r)u_\alpha\|_{L^p(0,8r^2t^2)} \le \mathcal{C}\|(f-g)u_\alpha\|_{L^p(0,8r^2t^2)} + t^r\|g^{(r)}\varphi^r u_\alpha\|_{L^p(0,8r^2t^2)}$$

and then, since $e^{-x}\sim e^{-x^\beta}\sim 1$ for $x\in[0,(2rh)^2]$, we can replace u_α with $w_{\alpha\beta}$ in the above norms. About the third term, we have

$$\begin{aligned}\inf_{q_{r-1}\in\mathbb{P}_{r-1}} \|(f-q_{r-1})w_{\alpha\beta}\|_{L^p(t^*,+\infty)} &\le \|(f-g)w_{\alpha\beta}\|_{L^p(t^*,+\infty)}\\ &\quad+\|(g-T_{r-1})w_{\alpha\beta}\|_{L^p(t^*,+\infty)},\end{aligned}$$

where $g\in W_r^p(w_{\alpha\beta})$ is arbitrary and T_{r-1} is the Taylor polynomial of g with initial point t^*. Consequently

$$\|(g-T_{r-1})w_{\alpha\beta}\|_{L^p(t^*,+\infty)} = \left(\int_{t^*}^{+\infty}\left|w_{\alpha\beta}(x)\int_{t^*}^{x}(x-u)^{r-1}g^{(r-1)}(u)du\right|^p dx\right)^{\frac{1}{p}}.$$

Then, using Proposition 6.2 with $z=t^*$ and $f=g$, the right-hand side of the above equality is dominated by $\frac{\mathcal{C}}{\left[(t^*)^{\frac{2\beta-1}{2}}\right]^r}\|g^{(r)}\varphi^r w_{\alpha\beta}\|_{L^p(t^*,+\infty)}$. By definition $t^*=\frac{1}{t^{\frac{2}{2\beta-1}}}$, i.e. $\frac{1}{\left[(t^*)^{\frac{2\beta-1}{2}}\right]^r}=t^r$, and the inequality

$$\omega_\varphi^r(f,t)_{w_{\alpha\beta},p}\le \mathcal{C}K(f,t^*)_{w_{\alpha\beta},p}$$

follows. In order to prove the inverse inequality, recall that for two suitable polynomials p_1 and p_2 belonging to $\mathbb{P}_{r-1}$,

$$\begin{aligned}\|(f-p_1)w_{\alpha\beta}\|_{L^p(0,8r^2t^2)}+t^r\|p_1^{(r)}\varphi^{r-1}w_{\alpha\beta}\|_{L^p(0,8r^2t^2)}&\le \omega_\varphi^r(f,t)_{w_{\alpha\beta},p}\\ \|(f-p_2)w_{\alpha\beta}\|_{L^p(t^*-1,+\infty)}+t^r\|p_2^{(r)}\varphi^{r-1}w_{\alpha\beta}\|_{L^p(t^*-1,+\infty)}&\le \omega_\varphi^r(f,t)_{w_{\alpha\beta},p}\end{aligned}$$

as previously proved. Moreover, for the function $G_t(x)$ defined in (6.14), the inequality

$$\begin{aligned}\|(f-G_t)w_{\alpha\beta}\|_{L^p(8r^2t^2,h^*)}+t^r\|G_t^{(r)}\varphi^r w_{\alpha\beta}\|_{L^p(8r^2t^2,h^*)}&\le \mathcal{C}\Omega_\varphi^r(f,t)_{w_{\alpha\beta},p}\\ &\le \mathcal{C}\omega_\varphi^r(f,t)_{w_{\alpha\beta},p}\end{aligned}$$

holds. Now, with $x_1 = 4r^2t^2, x_2 = 8r^2t^2, x_3 = t^* - 1, x_4 = t^*$, consider the function

$$\begin{aligned}\Gamma_t(x) &= \left(1 - \Psi\left(\frac{x - x_1}{x_2 - x_1}\right)\right) p_1(x) + \Psi\left(\frac{x - x_1}{x_2 - x_1}\right)\left(1 - \Psi\left(\frac{x - x_3}{x_4 - x_3}\right)\right) G_t(x) \\ &+ \Psi\left(\frac{x - x_3}{x_4 - x_3}\right) p_2(x).\end{aligned}$$

Obviously $\Gamma_t \in W_r^p$ and it is not difficult to verify the inequality

$$\|(f - \Gamma_t)w_{\alpha\beta}\|_p + t^r \|\Gamma_t^{(r)} \varphi^r w_{\alpha\beta}\|_{L^p(8r^2t^2, h^*)} \le \mathcal{C}\omega_\varphi^r(f, t)_{w_{\alpha\beta}, p}.$$

Thus the proof of the theorem is complete. □

In order to prove the theorems on interpolation, we recall some basic facts on the orthonormal polynomials $\{p_m(w_\alpha)\}_m$. The zeros of $p_m(w_\alpha)$ are located as follows:

$$\mathcal{C}\frac{a_m}{m^2} \le x_1 < \cdots < x_m \le a_m\left(1 - \frac{\mathcal{C}}{m^{\frac{2}{3}}}\right).$$

Moreover,

$$\Delta x_k = x_{k+1} - x_k \sim \frac{\sqrt{a_m}}{m}\sqrt{x_k}\frac{1}{\sqrt{1 - \frac{x_k}{a_m} + \frac{1}{m^{\frac{2}{3}}}}},$$

where $a_m = a_m(w_\alpha)$ and $\mathcal{C}$ is a positive constant independent of m. The following estimates are useful:

$$|p_m(w_\alpha, x)\sqrt{w_\alpha(x)}| \le \frac{\mathcal{C}}{\sqrt[4]{a_m x}\sqrt[4]{\left|1 - \frac{x}{a_m}\right| + \frac{1}{m^{\frac{2}{3}}}}},$$

where $\mathcal{C}\frac{a_m}{m^2} \le x \le \mathcal{C}a_m(1 + m^{-\frac{2}{3}})$ and $\mathcal{C} \ne \mathcal{C}(m, x)$, and

$$\frac{1}{|p_m'(w_\alpha, x_k)\sqrt{w_\alpha(x_k)}|} \sim \sqrt[4]{x_k a_m}\Delta x_k \sqrt{1 - \frac{x_k}{a_m} + \frac{1}{m^{\frac{2}{3}}}}, \quad k = 1, \ldots, m,$$

where the constants in " $\sim$ " are independent of m and k. The above estimates can be found in [5] or can be directly obtained by [4].

Proof of Theorem 5.4. Since

$$u(x)L_m(w_\alpha, f_j, x) = \sum_{i=1}^{j} u(x)\frac{l_i(x)}{u(x_i)}(fu)(x_i)$$

and, denoting by x_d a knot closest to x, it results $\left|u(x)\frac{l_d(x)}{u(x_d)}\right| \sim 1$, for $x \in [0, x_j]$. Then we have

$$|u(x)L_m(w_\alpha, f_j, x)| \le \mathcal{C}\|fu\|_{L^\infty([0,x_j])} \left(1 + \sum_{\substack{i=1 \\ i\neq d}}^{j} \frac{u(x)}{u(x_i)}|l_i(x)|\right). \tag{6.15}$$

Using the previous estimates and a Remez-type inequality, we get

$$\frac{|u(x)p_m(w_\alpha, x)|}{|p'_m(w_\alpha, x_i)u(x_i)|} \le \mathcal{C}\left(\frac{x}{x_i}\right)^{\gamma-\frac{\alpha}{2}-\frac{1}{4}} \frac{\Delta x_i}{|x - x_i|},$$

where $i = 1, 2, \ldots, j, i \neq d$, and $x \in \left[\frac{a_m}{m^2}, x_j\right]$. Then, under the assumptions of α and γ, the sum in (6.15) is dominated by $\log m$ and the theorem follows. $\square$

Here we omit the proofs of Lemmas 5.5 and 5.7 and the proofs of Theorems 5.6 and 5.8, being completely similar to the proofs of Lemmas 2.5 and 2.7 and Theorems 2.6 and 2.8 in [10] respectively.

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THE ORTHOGONAL PRINCIPLE AND CONDITIONAL DENSITIES

ION MIHOC AND CRISTINA IOANA FĂTU

Dedicated to Professor D. D. Stancu on his 80^{th} birthday

Abstract. Let $X, Y \in L^2(\Omega, K, P)$ be a pair of random variables, where $L^2(\Omega, K, P)$ is the space of random variables with finite second moments. If we suppose that X is an observable random variable but Y is not, than we wish to estimate the unobservable component Y from the knowledge of observations of X. Thus, if $g = g(x)$ is a Borel function and if the random variable $g(X)$ is an estimator of Y, then $e = E\{[Y - g(X)]^2\}$ is the mean -square error of this estimator. Also, if $\widehat{g}(X)$ is an optimal estimator (in the mean-square sense) of Y, then we have the following relation $e_{\min} = e(Y, \widehat{g}(X)) = E\{[Y - \widehat{g}(X)]^2\} = \inf_g E\{[Y - g(X)]^2\}$, where inf is taken over all Borel functions $g = g(x)$. In this paper we shall present some results relative to the mean-square estimation, conditional expectations and conditional densities.

1. Convergence in the mean-square

Let (Ω, K, P) be a probability space and $\mathcal{F}(\Omega, K, P)$ the family of all random variables defined on (Ω, K, P). Let

$$L^p = L^p(\Omega, K, P) = \{X \in \mathcal{F}(\Omega, K, P) \mid E(|X|^p) < \infty\}, p \in \mathbb{N}^* \tag{1.1}$$

be the set of random variables with finite moments of order p, that is

$$\beta_p = E(|X|^p) = \int_{\mathbb{R}} |x|^p \, dF(x) < \infty, p \in \mathbb{N}^*, \tag{1.2}$$

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where

$$F(x) = P(X < x), x \in \mathbb{R} \tag{1.3}$$

is the distribution function of the random variable X.

This set $L^p(\Omega, K, P)$ represent a linear space. An important role among the spaces $L^p = L^p(\Omega, K, P), p \geq 1$, is played by the space $L^2 = L^2(\Omega, K, P)$, the space of random variables with finite second moments.

Definition 1.1. If $X, Y \in L^2(\Omega, K, P)$, then the distance in mean square between X and Y, denoted by $d_2(X, Y)$, is defined by the equality

$$d_2(X, Y) = \|X - Y\| = [E(|X - Y|^2)]^{1/2}. \tag{1.4}$$

Remark 1.1. It is easy to verify that $d_2(X, Y)$ represents a semi-metric on the linear space L^2.

Definition 1.2. If $(X, X_n, n \geq 1) \subset L^2(\Omega, K, P)$, then about the sequence $(X_n)_{n \in \mathbb{N}^*}$ is said to converge to X in mean square (converge in L^2) if

$$\begin{aligned} \underset{n\to\infty}{lim} d_2(X_n, X) = \underset{n\to\infty}{lim} E(|X_n - X|^2)^{1/2} = \\ = \underset{n\to\infty}{lim} E(|X_n - X|^2) = 0. \end{aligned} \tag{1.5}$$

We write $l.i.m.X_n = X$ *or* $X_n \overset{m.p.}{\longrightarrow} X, n \to \infty$, and call X the limit in the mean (or mean square limit) of X_n.

Remark 1.2. If $X \in L^2(\Omega, K, P)$, then

$$Var(X) = E[(X - m)^2] = E[|X - m|^2] = \|X - m\|^2 = d_2^2(X, m),$$

where $m = E(X)$.

Consider two random variables X and Y. Suppose that only X can be observed. If X and Y are correlated, we may expect that knowing the value of X allows us to make some inference about the value of the unobserved variable Y. In this case an interesting problem, namely that of estimating one random variable with another or one random vector with another. If we consider any function $\widehat{X} = g(X)$ on X, then that is called an estimator for Y.

Definition 1.3. We say that a function $X^* = g^*(X)$ on X is best estimator in the mean-square sense if

$$E\{[Y - X^*]^2\} = E\{[Y - g^*(X)]^2\} = \inf_g E\{[Y - g(X)]^2\}. \tag{1.6}$$

If $X \in L^2(\Omega, K, P)$ then a very simple but basic problem consists in: find a constant a (i.e. the constant random variable $a, a \in L^2(\Omega, K, P)$) such that the mean-square error

$$e = e(X; a) = E[(X - a)^2] = \int_{\mathbb{R}} (x - a)^2 dF(x) =$$
$$= \|X - a\|^2 = d_2^2(X, a) \tag{1.7}$$

is minimum.

Evidently, the solution of a such problem is the following: if $a = E(X)$ then the mean-square error is minimum and we have

$$\min_{a \in \mathbb{R}} E[(X - a)^2] = Var(X).$$

Theorem 1.1. ([1]) (The orthogonality principle) Let X, Y be two random variables such that $E(X) = 0$, $E(Y) = 0$ and $\widehat{X}$ a new random variable, $\widehat{X} \in L^2(\Omega, K, P)$, defined as

$$\widehat{X} = g(X) = a_0 X, \; a_0 \in R. \tag{1.8}$$

The real constant a_0 that minimize the mean-square error

$$E[(Y - \widehat{X})^2] = E[(Y - a_0 X)^2] \tag{1.9}$$

is such that the random variable $Y - a_0 X$ is orthogonal to X; that is,

$$E[(Y - a_0 X)X] = 0 \tag{1.10}$$

and the minimum mean-square error is given by

$$e_{min}(Y, \widehat{X}) = e_{\min} = E[(Y - a_0 X)Y], \tag{1.11}$$

where

$$a_0 = \frac{E(XY)}{E(X^2)} = \frac{cov(X, Y)}{\sigma_1^2} \tag{1.12}$$

2. **General mean-square estimation**

Let us now remove the constraints of linear estimator and consider the more general problem of estimating Y with a (possibly nonlinear) function of X. For this, we recall the notion of inner (scalar) product.

Thus, if X and $Y \in L^2\ (\Omega, K, P)$, we put

$$(X, Y) = E(XY). \tag{2.1}$$

It is clear that if $X, Y, Z \in L^2\ (\Omega, K, P)$ then

$$\begin{cases} (aX + bY, Z) & = & a(X, Z) + b(Y, Z), \quad a, b \in \mathbb{R}, \\ (X, X) & \geq & 0, \\ (X, X) & = & 0 \Longleftrightarrow X = 0, \text{a.s.} \end{cases} \tag{2.2}$$

Consequently (X, Y) is a scalar product. The space $L^2\ (\Omega, K, P)$ is complete with respect to the norm

$$\| X \| = (X, X)^{1/2} \tag{2.3}$$

induced by this scalar product. In accordance with the terminology of functional analysis, a space with the scalar product (2.1) is a Hilbert space.

Hibert space methods are extensively used in the probability theory to study proprieties that depend only on the first two moments of random variables.

In the next, we want to estimate the random variable Y by a suitable function $g(X)$ of X so that the mean-square estimation error

$$e = e(Y, g(X)) = E\left\{[(Y - g(X)]^2\right\} = \iint_{\mathbb{R}^2} [y - g(x)]^2 f(x, y) dx dy \tag{2.4}$$

is minimum.

Theorem 2.1. ([3]) Let $\widehat{X}$ be a random variable defined as a nonlinear function of X, namely

$$\widehat{X} = g(X) \tag{2.5}$$

where $g(x)$ represents the value of this random variable $g(X)$ in the point x, $x \in D_x = \{x \in R \mid f(x) > 0\}$.Then, the minimum value of the mean-square error, namely,

$$e_{\min} = e_{\min}(Y, \widehat{X}) = E\left\{[(Y - E(Y \mid X)]^2\right\} \tag{2.6}$$

is obtained if

$$g(X) = E(Y \mid X), \tag{2.7}$$

where

$$\begin{aligned} E(Y \mid X = x) = E(Y \mid x) = \\ = \int_{-\infty}^{\infty} y f(y \mid x) dy \end{aligned} \tag{2.8}$$

is the random variable defined by the conditional expectation of Y with respect to X.

Definition 2.1. We say that the estimator (the nonlinear function)

$$\widehat{X} = g(X) = E(Y \mid X) \tag{2.9}$$

is best (optimal) in the mean-square sense for the unknown random variable Y if

$$\begin{aligned} e_{\min}(Y, \widehat{X}) = \min_{g(X)} E\left\{[(Y - g(X)]^2\right\} = \\ = E\left\{[(Y - E(Y \mid X)]^2\right\}. \end{aligned} \tag{2.10}$$

Lemma 2.1. ([1]) If X and Y are two independent random variable, then

$$E(Y \mid X) = E(Y). \tag{2.11}$$

Corollary 2.1. If X, Y are two independent random variables then the best mean-square estimator of Y in terms of X is $E(Y)$. Thus knowledge of X does not help in the estimation of Y.

3. Conditional expectation and conditional densities

We assume that the random vector (X, Y) have the bivariate normal distribution with the probability density function

$$f(x,y) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-r^2}} e^{-\frac{1}{2(1-r^2)}\left[\left(\frac{x-m_1}{\sigma_1}\right)^2 - \frac{2r(x-m_1)(y-m_2)}{\sigma_1\sigma_2} + \left(\frac{y-m_2}{\sigma_2}\right)^2\right]}, \tag{3.1}$$

where:

$$m_1 = E(X) \in \mathbb{R}, m_2 = E(Y) \in \mathbb{R}, \sigma_1^2 = Var(X) > 0, \sigma_2^2 = Var(Y) > 0, \tag{3.1a}$$

$$r = r(X,Y) = \frac{cov(X,Y)}{\sigma_1\sigma_2}, \quad r \in (-1,1), \tag{3.2}$$

r being the correlation coefficient between X and Y.

First, we will recall some very important definitions and proprieties for a such normal distribution.

Lemma 3.1. If two jointly normal random variable X and Y are uncorrelated, that is, $cov(X,Y) = 0 = r(x,y)$, then they are independent and we have

$$f(x,y) = f(x; m_1, \sigma_1^2) f(y; m_2, \sigma_2), \tag{3.3}$$

where

$$f(x; m_1, \sigma_1^2) = \frac{1}{\sqrt{2\pi}\sigma_1} e^{-\frac{1}{2}\left(\frac{x-m_1}{\sigma_1}\right)^2}, \; f(y; m_2, \sigma_2^2) = \frac{1}{\sqrt{2\pi}\sigma_2} e^{-\frac{1}{2}\left(\frac{y-m_2}{\sigma_2}\right)^2} \tag{3.3a}$$

are the marginal probability density functions for the components X and Y of the normal random vector (X, Y).

Lemma 3.2. If (X, Y) is a random vector with the bivariate normal probability density function (3.1), then for the conditional random variable $(Y \mid X)$, for example, the probability density function, denoted by $f(y \mid x)$, has the form

$$f(y \mid x) = \frac{1}{\sqrt{2\pi(1-r^2)}\sigma_2} e^{-\frac{1}{2\sigma_2^2(1-r^2)}\left[y - \left(m_2 + r\frac{\sigma_2}{\sigma_1}(x-m_1)\right)\right]^2}, \tag{3.4}$$

This conditional probability density function (3.4) may be obtained using the well-bred method which have in view the following relations

$$f(y \mid x) = \frac{f(x,y)}{f(x)}, f(x) > 0, \; f(x) = f(x; m_1, \sigma_1^2) = \int_{-\infty}^{\infty} f(x,y)dy. \tag{3.5}$$

In the next, we shall recover this conditional probability density function using the orthogonality principle.

Theorem 3.1. Let (X, Y) be a normal random vector which is characterized by the relations (3.1), (3.1a) and (3.2). If

$$\overset{o}{X} = X - m_1, \overset{o}{Y} = Y - m_2, \tag{3.6}$$

are the deviation random variables and U is a new random variable which is defined as

$$U = \overset{o}{Y} - c_0 \overset{o}{X},\ where\ c_0 \in \mathbb{R} - \{0\}, \tag{3.7}$$

then the orthogonality principle implies the conditional density function (3.4), which corresponds to the conditional random variable $(\overset{o}{Y} \mid \overset{o}{X})$, and more we have the following relation

$$f(y \mid x) = f(u), \tag{3.8}$$

where $f(u)$ is the probability density function that corresponds to U.

Proof. Indeed, because

$$\begin{cases} E(\overset{o}{X}) = m_{\underset{x}{0}} = 0, & Var(\overset{o}{X}) = \sigma^2_{\underset{x}{0}} = Var(X) = \sigma_1^2, \\ E(\overset{o}{Y}) = m_{\underset{y}{0}} = 0, & Var(\overset{o}{Y}) = \sigma^2_{\underset{y}{0}} = Var(Y) = \sigma_2^2, \end{cases} \tag{3.9}$$

and

$$cov(\overset{o}{X}, \overset{o}{Y}) = E(\overset{o}{X}\overset{o}{Y}) = E[(X - m_1)(Y - m_2)] = cov(X, Y) = r\sigma_1\sigma_2, \tag{3.10}$$

then we obtain

$$E(U) = m_U = 0. \tag{3.11}$$

Also, for the variance of the random variable U, we obtain

$$\begin{aligned} Var(U) = \sigma_U^2 = E\left\{[U - E(U)]^2\right\} = E(U^2) = \\ = E\{[\overset{o}{Y} - c_0\overset{o}{X}]^2\} = \\ = \sigma_2^2 - 2c_0 cov(X, Y) + c_0^2\sigma_1^2 = \\ = \sigma_2^2 - 2c_0 r\sigma_1\sigma_2 + c_0^2\sigma_1^2, \end{aligned}$$

The value of the constant c_0 will be determined using the orthogonality principle, namely: the random variables U and $\overset{\circ}{X}$ to be orthogonal. This condition implies the following relation

$$E(U\overset{o}{X}) = E\left[(\overset{o}{Y} - c_0\overset{o}{X}) \mid \overset{o}{X})\right] = 0, \tag{3.12}$$

and, more, the constant c_0 must to minimize the mean-square error

$$e = E[(\overset{o}{Y} - c_0\overset{o}{X})^2], \tag{3.13}$$

that is,

$$e_{\min} = E[(\overset{o}{Y} - c_0\overset{o}{X})\overset{o}{Y}]. \tag{3.14}$$

Indeed, using (1.12) we obtain the following value

$$c_0 = \frac{E(\overset{o}{X}\overset{o}{Y})}{E(\overset{o}{X}^2)} = r\frac{\sigma_2}{\sigma_1}, \tag{3.15}$$

if we have in view the relations (3.9) and (3.10).

Also, from (3.12), we obtain

$$cov(U,\overset{o}{X}) = E(U\overset{o}{X}) = 0, \rho(U,\overset{o}{X}) = 0, \tag{3.16}$$

where $\rho(U,\overset{o}{X})$ represents the correlation coefficient between the random variables U and $\overset{o}{X}$.

Because the random variables U and $\overset{o}{X}$ are normal distributed with $\rho(U,\overset{o}{X}) = 0$ then, using the Lemma 3.1, it follows that these random variables are independent and their joint probability density function, denoted by $f(\overset{0}{x},u)$, has the form

$$f(\overset{o}{x},u) = f(\overset{o}{x})f(u), \tag{3.17}$$

where $f(\overset{o}{x})$ is the probability density function for the random variable $\overset{o}{X}$, that is,

$$f(\overset{o}{x}) = \frac{1}{\sqrt{2\pi}\sigma_{\overset{o}{x}}}e^{-\frac{1}{2}\left[\frac{\overset{o}{x}-m_{\overset{o}{x}}}{\sigma_{\overset{0}{x}}}\right]^2} = \frac{1}{\sqrt{2\pi}\sigma_1}e^{-\frac{1}{2}\left(\frac{x-m_1}{\sigma_1}\right)^2} =$$

$$= f(x;m_1\sigma_1^2),\ x\in\mathbb{R}, \tag{3.18}$$

if we have in view the relations (3.6) and (3.9).

Also, for the probability density function $f(u)$, we obtain the following forms

$$\begin{aligned} f(u) &= \frac{1}{\sqrt{2\pi}\sigma_u} e^{-\frac{1}{2}\left[\frac{u-m_u}{\sigma_u}\right]^2} = \\ &= \frac{1}{\sigma_2\sqrt{2\pi(1-r^2)}} e^{-\frac{1}{2(1-r^2)}\left(\frac{u}{\sigma_2}\right)^2} = \\ &= \frac{1}{\sigma_2\sqrt{2\pi(1-r^2)}} e^{-\frac{1}{2\sigma_2^2(1-r^2)}\left\{y-\left[m_2+r\frac{\sigma_2}{\sigma_1}(x-m_1)\right]\right\}^2} \\ &= \frac{1}{\sigma_2\sqrt{2\pi(1-r^2)}} e^{-\frac{1}{2\sigma_2^2(1-r^2)}(y-m_{y|x})^2}, \end{aligned} \tag{3.19}$$

if we have in view the relations (3.6) and (3.11) as well as the fact that the values of the random variable $U = \overset{o}{Y} - c_0\overset{o}{X}$ can be express as

$$u = \overset{o}{y} - c_0\overset{o}{x} = y - [m_2 + r\frac{\sigma_2}{\sigma_1}(x - m_1)] = y - m_{Y|x}. \tag{3.20}$$

Therefore, the form (3.1a) of the probability density function $f(u)$, together with the relation (3.4), give us just the relation (3.8), that is, we obtain the following equality

$$f(u) = f(y \mid x) = \frac{1}{\sigma_2\sqrt{2\pi(1-r^2)}} e^{-\frac{1}{2\sigma_2^2(1-r^2)}(y-m_{y|x})^2}. \tag{3.21}$$

Utilizing the forms (3.18) and (3.21) of the probability density functions $f(\overset{o}{x})$ and $f(u)$, from the relation (3.17), we obtain the following expressions

$$f(\overset{o}{x}, u) = f(\overset{o}{x})f(u) = f(x; m_1\sigma_1^2)f(y \mid x) = \tag{3.22}$$

$$\begin{aligned} &= \left[\frac{1}{\sqrt{2\pi}\sigma_1} e^{-\frac{1}{2}\frac{(x-m_1)^2}{\sigma_1^2}}\right] \left[\frac{1}{\sigma_2\sqrt{2\pi(1-r^2)}} e^{-\frac{1}{2\sigma_2^2(1-r^2)}\left\{y-\left[m_2+r\frac{\sigma_2}{\sigma_1}(x-m_1)\right]\right\}^2}\right] = \\ &= \left[\frac{1}{\sqrt{2\pi}\sigma_1} e^{-\frac{1}{2}\frac{(x-m_1)^2}{\sigma_1^2}}\right] \left[\frac{1}{\sigma_2\sqrt{2\pi(1-r^2)}} e^{-\frac{1}{2(1-r^2)}\left[\frac{y-m_2}{\sigma_2}-r\frac{x-m_1}{\sigma_1}\right]^2}\right] = \\ &= \frac{1}{2\pi\sigma_1\sigma_2} e^{-\frac{1}{2(1-r^2)}\left[\frac{(x-m_1)^2}{\sigma_1^2}-2r\frac{(x-m_1)(y-m_2)}{\sigma_1\sigma_2}+\frac{(y-m_2)^2}{\sigma_2^2}\right]^2} = f(x, y), \end{aligned}$$

and, hence, it follows the equality

$$f(x, y) = f(x)f(y \mid x). \tag{3.23}$$

In the next, we must to prove that the minimum of the mean-square error, specified in the relation (3.14), can be obtained if the constant c_0 has the value (3.15).

But, in the beginning, we recall some definitions and some properties of the conditional means.

Lemma 3.3. ([1]) The conditional mean $E(. \mid X)$ is a linear operator, that is,

$$E(cY + dZ \mid X) = cE(Y \mid X) + dE(Z \mid X), c, d \in R. \tag{3.24}$$

Definition 3.1. If (X, Y) is a bivariate random vector with the probability density function $f(x, y)$ and $Z = g(X, Y)$ is a new random variable which is a function of the random variables X and Y, then the conditional mean of the random variable $Z = g(X, Y)$, given $X = x$, is defined as

$$E[g(X, Y) \mid X = x] = \int_{-\infty}^{\infty} g(x, y) f(y \mid X = x) dy, \tag{3.25}$$

for any $x \in D_x = \{x \in R \mid f(x) > 0\}$.

Lemma 3.4. ([1]) If the random variable Z has the form

$$Z = g(X, Y) = g_1(X) g_2(Y), \tag{3.26}$$

then we have the following relation

$$E[g_1(X) g_2(Y) \mid X] = g_1(X) E[g_2(Y) \mid X]. \tag{3.27}$$

Lemma 3.5. ([1]) If X is a random variable and c is a real constant, then

$$E[c \mid X] = c. \tag{3.28}$$

Now, we can return to the our problem, namely to prove that the minimum of the mean-square error, specified in the relation (3.14), can be obtained if the constant c_0 has the value (3.15).

Thus, because the random variables $U = \overset{0}{Y} - c_0\overset{0}{X}$ and $\overset{0}{X}$ are independent, then from (3.12) and Lemma 2.1, we obtain

$$E(U\overset{0}{X}) = E[(\overset{0}{Y} - c_0\overset{0}{X}) \mid \overset{0}{X}] =$$
$$= E[(\overset{0}{Y} - c_0\overset{0}{X})] =$$
$$= E(\overset{0}{Y}) - c_0E(\overset{0}{X}) = 0,$$

that is, we have the following equality

$$E(U\overset{o}{X}) = E(\overset{o}{Y}) - c_0E(\overset{o}{X}) = 0. \qquad (3.29)$$

On the other hand, in accordance with the Lemma 3.4, (respectively, in accordance with the relation (3.26)) and the Lemma 3.5, where $g_1(X) = c_0\overset{o}{X}$ and $g_2(Y) = 1$, we obtain

$$E[c_0\overset{o}{X} \mid \overset{o}{X} = \overset{o}{x}] = c_0\overset{o}{X}\underbrace{E[1 \mid \overset{o}{X} = \overset{o}{x}]}_{=1} = c_0\overset{o}{X}, \qquad (3.30)$$

for any $\overset{o}{x} = x - m_1, x \in \mathbb{R}$

This last relation, together with the Lemma 3.3 give us the possibility to rewritten the conditional mean $E(U\overset{o}{X}) = E[(\overset{o}{Y} - c_0\overset{o}{X}) \mid \overset{o}{X}]$ in an useful form

$$E(U\overset{o}{X}) = E[(\overset{o}{Y} - c_0\overset{o}{X}) \mid \overset{o}{X}] =$$
$$= E(\overset{o}{Y} \mid \overset{o}{X}) - E(c_0\overset{o}{X} \mid \overset{o}{X}) =$$
$$= E(\overset{o}{Y} \mid \overset{o}{X}) - c_0\overset{o}{X},$$

that is,

$$E(U\overset{o}{X}) = E(\overset{o}{Y} \mid \overset{o}{X}) - c_0\overset{o}{X}. \qquad (3.31)$$

From (3.29) and (3.31), we obtain the random variable

$$E(\overset{o}{Y} \mid \overset{o}{X}) = c_0\overset{o}{X} = r\frac{\sigma_2}{\sigma_1}\overset{o}{X}, \qquad (3.32)$$

which has the real values of the form

$$E(\overset{o}{Y} \mid \overset{o}{X} = \overset{o}{x}) = r\frac{\sigma_2}{\sigma_1}\overset{o}{x}, \text{ for any } \overset{o}{x} = x - m_1, x \in \mathbb{R}. \qquad (3.32a)$$

The conditional variance of the random variable $(\overset{o}{Y} \mid \overset{o}{X})$ can be express as

$$\begin{aligned} Var(\overset{o}{Y} \mid \overset{o}{X}) = \sigma^2_{\overset{o}{Y}|\overset{o}{X}} = \\ = E\{[\overset{o}{Y} - E(\overset{o}{Y} \mid \overset{o}{X})]]^2 \mid \overset{o}{X}\} = \\ = E[(\overset{o}{Y} - c_0\overset{o}{X})^2 \mid \overset{o}{X}], \end{aligned} \tag{3.33}$$

and, evidently, it is a random variable which has the real values of the form

$$Var(\overset{o}{Y} \mid \overset{o}{X} = \overset{o}{x}) = E[(\overset{o}{Y} - c_0\overset{o}{X})^2 \mid \overset{o}{X} = \overset{o}{x}],\ for\ any \overset{o}{x} = x - m_1, x \in \mathbb{R}. \tag{3.33a}$$

Because the random variables $U = \overset{o}{Y} - c_0\overset{o}{X}$ and $\overset{o}{X}$ are independent then, evidently, it follows that and the random variable $U^2 = (\overset{o}{Y} - c_0\overset{o}{X})^2$ and $\overset{o}{X}$ are independent. Then, from (3.36), we obtain

$$Var(\overset{o}{Y} \mid \overset{o}{X}) = E[(\overset{o}{Y} - c_0\overset{o}{X})^2 \mid \overset{o}{X}] = \tag{3.34}$$

$$\begin{aligned} &= E[(\overset{o}{Y} - c_0\overset{o}{X})^2] = \\ &= E[(\overset{o}{Y} - c_0\overset{o}{X})\overset{o}{Y} + c_0(\overset{o}{Y} - c_0\overset{o}{X})\overset{o}{X}] = \\ &= E[(\overset{o}{Y} - c_0\overset{o}{X})\overset{o}{Y}] + c_0\underbrace{E[(\overset{o}{Y} - c_0\overset{o}{X})\overset{o}{X}]}_{=0 \text{ (see, (3.14))}} = \\ &= \underbrace{E[(\overset{o}{Y} - c_0\overset{o}{X})\overset{o}{Y}]}_{\text{(see, (3.16))}} = \\ &= e_{\min} = e_{\min}(\overset{o}{Y}, \overset{o}{X}) = \\ &= E(\overset{o}{Y^2}) - 2c_0E(\overset{o}{Y}\overset{o}{X}) + c_0^2E(\overset{o}{X^2}) = \\ &= \sigma_2^2 - r\frac{\sigma_2}{\sigma_1}r\sigma_1\sigma_2 = \sigma_2^2(1 - r^2). \end{aligned} \tag{3.34a}$$

Therefore, the conditional variance of the deviation random variable $\overset{o}{Y}$, given $\overset{o}{X}$, represents just the minimum mean-square error.

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LOGARITHMIC MODIFICATION OF THE JACOBI WEIGHT FUNCTION

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Dedicated to Professor D. D. Stancu on his 80^{th} birthday

Abstract. In this paper we are interested in a logarithmic modification of the Jacobi weight function, i.e., we study the following moment functional $\mathcal{L}^{\alpha,\beta}(p) = \int_{-1}^{1} p(x)(1-x)^\alpha(1+x)^\beta \log(1-x^2)\,dx$, $p \in \mathcal{P}$, where $\alpha, \beta > -1$ and $\mathcal{P}$ is the space of all algebraic polynomials. We give the recurrence relations for the modified moments $\mu_k = \mathcal{L}^{\alpha,\beta}(q_k)$, $k \in \mathbb{N}_0$, in the cases when q_k is a sequence of monic Chebyshev polynomials of the first and second kind. In particular, when $\alpha = \beta = \ell - 1/2$, $\ell \in \mathbb{N}_0$, we derive explicit formulae for the modified moments. As an application of these modified moments, the numerical construction of coefficients in the three-term recurrence relation for polynomials orthogonal with respect the functional $\mathcal{L}^{\alpha,\beta}$ and the corresponding Gaussian quadratures are presented.

1. **Introduction**

We consider the moment functional

$$\mathcal{L}^{\alpha,\beta}(p) = \int_{-1}^{1} p(x)(1-x)^\alpha(1+x)^\beta \log(1-x^2)\,dx, \quad \alpha, \beta > -1, \tag{1.1}$$

on the space of all algebraic polynomials $\mathcal{P}$. In (1.1) we recognize the Jacobi weight function $w^{\alpha,\beta}(x) = (1-x)^\alpha(1+x)^\beta$. Recently, in [1], there appeared an interest in a construction of numerical methods for integration of an integral which appears in the moment functional $\mathcal{L}^{\pm 1/2, \pm 1/2}$.

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In this paper we give a stable numerical procedure which can be used for the construction of polynomials orthogonal with respect to the moment functional $\mathcal{L}^{\alpha,\beta}$, as well as a stable numerical method for the corresponding quadrature rules for computing the mentioned integrals (Section 3).

These procedures are enabled by finding the recurrence relations for modified moments $\mathcal{L}^{\alpha,\beta}(q_k)$, $k \in \mathbb{N}_0$, with respect to the polynomial sequences $q_k = T_k$ and $q_k = U_k$, where T_k and U_k are the monic Chebyshev polynomials of the first and second kind, respectively (Section 2). In the case when $\alpha = \beta = \ell - 1/2$, $\ell \in \mathbb{N}_0$, we derive explicit formulae for these modified moments. The procedure for finding modified moments are loosely connected with an earlier work of Piessens (see [8]).

2. Modified moments

First we introduce the modified moments of the Jacobi weight function with respect to the monic Chebyshev polynomials T_n and U_n, $n \in \mathbb{N}_0$, of the first and the second kind, respectively. We use the following notation

$$m_n^{\alpha,\beta} = \int_{-1}^{1} w^{\alpha,\beta}(x) T_n(x)\, dx \tag{2.2}$$

and

$$e_n^{\alpha,\beta} = \int_{-1}^{1} w^{\alpha,\beta}(x) U_n(x)\, dx. \tag{2.3}$$

We have the following lemma.

Lemma 2.1. *Modified moments of the Jacobi weight, given in* (2.2) *and* (2.3)*, satisfy the following recurrence relations*

$$m_{n+1}^{\alpha,\beta} = \frac{\beta-\alpha}{n+\alpha+\beta+2} m_n^{\alpha,\beta} + \frac{1}{4}\frac{n-\alpha-\beta-2}{n+\alpha+\beta+2}(1+\delta_{n-1,0}) m_{n-1}^{\alpha,\beta}, \quad n \in \mathbb{N},$$

and

$$e_{n+1}^{\alpha,\beta} = \frac{\beta-\alpha}{n+\alpha+\beta+2} e_n^{\alpha,\beta} + \frac{1}{4}\frac{n-\alpha-\beta}{n+\alpha+\beta+2} e_{n-1}^{\alpha,\beta}, \quad n \in \mathbb{N}.$$

In both cases initial conditions are the same

$$\begin{aligned} m_0^{\alpha,\beta} &= e_0^{\alpha,\beta} = 2^{\alpha+\beta+1} B(1+\alpha, 1+\beta), \\ m_1^{\alpha,\beta} &= e_1^{\alpha,\beta} = 2^{\alpha+\beta+1} \frac{\beta-\alpha}{\alpha+\beta+2} B(1+\alpha, 1+\beta). \end{aligned}$$

Proof. We are going to need the following identity (see [2, p. 142])

$$(1-x^2)T_k'(x) = -kT_{k+1}(x) + \frac{k(1+\delta_{k-1,0})}{4}T_{k-1}(x), \quad k \in \mathbb{N},$$

satisfied by the monic Chebyshev polynomials of the first kind. Here, $\delta_{k,m}$ is the Kronecker's delta. Integrating this identity with respect to the Jacobi weight, and using an integration by parts, we have

$$\begin{aligned} -km_{k+1}^{\alpha,\beta} \quad + \quad & \frac{k(1+\delta_{k-1,0})}{4}m_{k-1}^{\alpha,\beta} = \int_{-1}^{1}(1-x^2)w^{\alpha,\beta}(x)T_k'(x)dx \\ = \quad & -\int_{-1}^{1}(-x(\alpha+\beta+2)+\beta-\alpha)w^{\alpha,\beta}(x)T_k(x)dx \\ = \quad & -(\beta-\alpha)m_k^{\alpha,\beta} + (\alpha+\beta+2)\left(m_{k+1}^{\alpha,\beta} + \frac{1+\delta_{k-1,0}}{4}m_{k-1}^{\alpha,\beta}\right), \end{aligned}$$

where we used the three-term recurrence relation for the monic Chebyshev polynomials of the first kind

$$T_{k+1}(x) = xT_k(x) - \frac{1+\delta_{k-1,0}}{4}T_{k-1}(x).$$

Similarly, for the monic Chebyshev polynomials of the second kind we have the following identity (see [2, p. 144])

$$(1-x^2)U_k'(x) = -kU_{k+1}(x) + \frac{k+2}{4}U_{k-1}(x), \quad k \in \mathbb{N}.$$

Integrating this identity with respect to the Jacobi weight we get

$$\begin{aligned} -ke_{k+1}^{\alpha,\beta} \quad + \quad & \frac{k+1}{4}e_{k-1}^{\alpha,\beta} = \int_{-1}^{1}(1-x^2)w^{\alpha,\beta}(x)U_k'(x)dx \\ = \quad & -\int_{-1}^{1}(-x(\alpha+\beta+2)+\beta-\alpha)w^{\alpha,\beta}(x)U_k(x)dx \\ = \quad & -(\beta-\alpha)e_k^{\alpha,\beta} + (\alpha+\beta+2)\left(e_{k+1}^{\alpha,\beta} + \frac{1}{4}e_{k-1}^{\alpha,\beta}\right), \end{aligned}$$

where we used the three-term recurrence relation for the monic Chebyshev polynomials of the second kind

$$U_{n+1}(x) = xU_n(x) - \frac{1}{4}U_{n-1}(x).$$

Regarding the initial conditions, $m_0^{\alpha,\beta}$ and $m_1^{\alpha,\beta}$ are first two moments of the Jacobi weight function. □

Now, for the functional $\mathcal{L}^{\alpha,\beta}$ given by (1.1), we introduce the modified moments with respect to the monic Chebyshev polynomials in the following forms

$$\mu_n^{\alpha,\beta} = \int_{-1}^{1} w^{\alpha,\beta}(x)\log(1-x^2)T_n(x)dx, \quad n \in \mathbb{N}_0, \tag{2.4}$$

and

$$\eta_n^{\alpha,\beta} = \int_{-1}^{1} w^{\alpha,\beta}(x)\log(1-x^2)U_n(x)dx, \quad n \in \mathbb{N}_0, \tag{2.5}$$

where T_n and U_n, $n \in \mathbb{N}$, are sequences of the monic Chebyshev polynomials of the first and second kind, respectively.

Theorem 2.1. *The sequences of the modified moments $\mu_n^{\alpha,\beta}$ and $\eta_n^{\alpha,\beta}$, $n \in \mathbb{N}$, satisfy the following recurrence relations*

$$\begin{aligned} \mu_{n+1}^{\alpha,\beta} &= \frac{\beta-\alpha}{n+\alpha+\beta+2}\mu_n^{\alpha,\beta} + \frac{1+\delta_{n-1,0}}{4}\frac{n-\alpha-\beta-2}{n+\alpha+\beta+2}\mu_{n-1}^{\alpha,\beta} \\ &\qquad -\frac{2}{n+\alpha+\beta+2}\left(m_{n+1}^{\alpha,\beta} + \frac{1+\delta_{n-1,0}}{4}m_{n-1}^{\alpha,\beta}\right), \\ \mu_1^{\alpha,\beta} &= \frac{\beta-\alpha}{\alpha+\beta+2}\mu_0^{\alpha,\beta} - \frac{2}{\alpha+\beta+2}m_1^{\alpha,\beta} \end{aligned} \tag{2.6}$$

and

$$\begin{aligned} \eta_{n+1}^{\alpha,\beta} &= \frac{\beta-\alpha}{n+\alpha+\beta+2}\eta_n^{\alpha,\beta} + \frac{1}{4}\frac{n-\alpha-\beta-2}{n+\alpha+\beta+2}\eta_{n-1}^{\alpha,\beta} \\ &\qquad -\frac{2}{n+\alpha+\beta+2}\left(e_{n+1}^{\alpha,\beta} + \frac{1}{4}e_{n-1}^{\alpha,\beta}\right), \\ \eta_1^{\alpha,\beta} &= \frac{\beta-\alpha}{\alpha+\beta+2}\eta_0^{\alpha,\beta} - \frac{2}{\alpha+\beta+2}e_1^{\alpha,\beta}. \end{aligned} \tag{2.7}$$

Proof. Using the same identities as in the proof of Lemma 2.1, we have

$$\begin{aligned}
-k\mu_{k+1}^{\alpha,\beta} \quad &+ \quad \frac{k(1+\delta_{k-1,0})}{4}\mu_{k-1}^{\alpha,\beta} = \int_{-1}^{1}(1-x^2)w^{\alpha,\beta}(x)\log(1-x^2)T_k'(x)\,dx \\
&= \quad -\int_{-1}^{1}[-x(\alpha+\beta+2)+\beta-\alpha]w^{\alpha,\beta}(x)\log(1-x^2)T_k(x)\,dx \\
&\quad\ +2\int_{-1}^{1}w^{\alpha,\beta}(x)xT_k(x)\,dx \\
&= \quad (\alpha+\beta+2)\left(\mu_{k+1}^{\alpha+\beta}+\frac{1+\delta_{k-1,0}}{4}\mu_{k-1}^{\alpha,\beta}\right)-(\beta-\alpha)\mu_k^{\alpha,\beta} \\
&\quad\ +2\left(m_{k+1}^{\alpha,\beta}+\frac{1+\delta_{k-1,0}}{4}m_{k-1}^{\alpha,\beta}\right).
\end{aligned}$$

Similarly, for the monic Chebyshev polynomials of the second kind we have

$$\begin{aligned}
-k\eta_{k+1}^{\alpha,\beta} \quad &+ \quad \frac{k+2}{4}\eta_{k-1}^{\alpha,\beta} = \int_{-1}^{1}(1-x^2)w^{\alpha,\beta}(x)\log(1-x^2)U_k'(x)\,dx \\
&= \quad -\int_{-1}^{1}[-x(\alpha+\beta+2)+\beta-\alpha]w^{\alpha,\beta}(x)\log(1-x^2)U_k(x)\,dx \\
&\quad\ +2\int_{-1}^{1}w^{\alpha,\beta}(x)xU_k(x)\,dx \\
&= \quad (\alpha+\beta+2)\left(\eta_{k+1}^{\alpha+\beta}+\frac{1}{4}\eta_{k-1}^{\alpha,\beta}\right)-(\beta-\alpha)\eta_k^{\alpha,\beta}+2\left(e_{k+1}^{\alpha,\beta}+\frac{1}{4}e_{k-1}^{\alpha,\beta}\right),
\end{aligned}$$

which gives (2.7).

For the first moment and the Chebyshev polynomials of the first kind, we have

$$\begin{aligned}
(\beta-\alpha)\mu_0^{\alpha,\beta}-(\alpha+\beta+2)\mu_1^{\alpha,\beta} &= \int_{-1}^{1}(\beta-\alpha-x(\alpha+\beta+2))w^{\alpha,\beta}(x)\log(1-x^2)dx \\
&= \int_{-1}^{1}(w^{\alpha+1,\beta+1}(x))'\log(1-x^2)dx = 2\int_{-1}^{1}w^{\alpha,\beta}(x)xdx = 2m_1^{\alpha,\beta},
\end{aligned}$$

and the proof transfers verbatim to the case of the Chebyshev polynomials of the second kind. □

Especially for $\alpha = \beta = -1/2$, it was shown (see [1], [3]) that

$$\mu_n^{-1/2,-1/2} = \int_{-1}^{1} \frac{\log(1-x^2)}{\sqrt{1-x^2}} T_n(x)dx = \begin{cases} -2\pi \log 2, & n = 0, \\ -\dfrac{\pi}{2^{n-1}n}, & n \neq 0. \end{cases} \tag{2.8}$$

Actually, the explicit expressions can be given in the case $\alpha = \beta = \ell - 1/2$, $\ell \in \mathbb{N}_0$.

Theorem 2.2. *If $\alpha = \beta = \ell - 1/2$, $\ell \in \mathbb{N}_0$, then*

$$\begin{aligned} \mu_n^{\ell-1/2,\ell-1/2} &= \frac{2^{1-n-2\ell}}{1+\delta_{n,0}} \left[\sum_{k=0}^{\ell-1} (-1)^{\ell-k} \binom{2\ell}{k} \left(\frac{1+\delta_{2(\ell-k)+n,0}}{2^{1-2(\ell-k)-n}} \mu_{2(\ell-k)+n}^{-1/2,-1/2} \right.\right. \\ &+ \left.\left. \frac{1+\delta_{|2(\ell-k)-n|,0}}{2^{1-|2(\ell-k)-n|}} \mu_{|2(\ell-k)-n|}^{-1/2,-1/2} \right) + \binom{2\ell}{\ell} \frac{1+\delta_{n,0}}{2^{1-n}} \mu_n^{-1/2,-1/2} \right], \end{aligned}$$

and if $\ell \in \mathbb{N}$ we have

$$\begin{aligned} \eta_n^{\ell-1/2,\ell-1/2} = \frac{1}{2^{n+2\ell-1}} \sum_{k=0}^{\ell-1} (-1)^{\ell+k-1} \binom{2\ell-1}{k} & \left(\frac{1+\delta_{|n-2(\ell-k)+2|,0}}{2^{1-|n-2(\ell-k)+2|}} \mu_{|n-2(\ell-k)+2|}^{-1/2,-1/2} \right. \\ & \left. - \frac{1+\delta_{n+2(\ell-k),0}}{2^{1-n-2(\ell-k)}} \mu_{n+2(\ell-k)}^{-1/2,-1/2} \right), \end{aligned}$$

for $n \in \mathbb{N}_0$.

Proof. In order to prove these formulas we interpret the equation (2.8) into the following form

$$\begin{aligned} \mu_n^{-1/2,-1/2} &= \frac{1}{2^{n-2}(1+\delta_{n,0})} \int_0^{\pi} \sin^{2(-1/2)+1} \phi \log \sin \phi \cos n\phi \, d\phi \\ &= \frac{1}{2^{n-2}(1+\delta_{n,0})} \int_0^{\pi} \log \sin \phi \cos n\phi \, d\phi, \end{aligned}$$

which can be obtained from the previous using the substitution $x = \cos\phi$. With the same substitution, we get

$$\begin{aligned} \mu_n^{\ell-1/2,\ell-1/2} &= \frac{2^{2-n}}{1+\delta_{n,0}} \int_0^{\pi} \sin^{2(\ell-1/2)+1} \phi \log \sin \phi \cos n\phi \, d\phi \\ &= \frac{2^{2-n}}{1+\delta_{n,0}} \int_{-1}^{1} \log \sin \phi \cos n\phi \frac{1}{2^{2\ell}} \left[\sum_{k=0}^{\ell-1} 2 \binom{2n}{k} \cos \nu\phi + \binom{2n}{n} \right] d\phi \\ &= \frac{2^{1-n-2\ell}}{1+\delta_{n,0}} \left[\sum_{k=0}^{\ell-1} (-1)^{\ell-k} \binom{2\ell}{k} \left(\frac{1+\delta_{\nu+n,0}}{2^{1-\nu-n}} \mu_{\nu+n}^{-1/2,-1/2} \right.\right. \\ &+ \left.\left. \frac{1+\delta_{|\nu-n|,0}}{2^{1-|\nu-n|}} \mu_{|\nu-n|}^{-1/2,-1/2} \right) + \binom{2\ell}{\ell} \frac{1+\delta_{n,0}}{2^{1-n}} \mu_n^{-1/2,-1/2} \right], \end{aligned}$$

and also

$$\eta_n^{\ell-1/2,\ell-1/2} = \frac{1}{2^{n-1}}\int_0^\pi \sin^{2(\ell-1/2)}\phi \log\sin\phi \sin(n+1)\phi\, d\phi$$

$$= \frac{1}{2^{n+2\ell-2}}\int_0^\pi \log\sin\phi\sin(n+1)\phi\left[\sum_{k=0}^{\ell-1}(-1)^{\ell+k-1}\binom{2\ell-1}{k}\sin(\nu-1)\phi\right]d\phi$$

$$= \frac{1}{2^{n+2\ell-1}}\sum_{k=0}^{\ell-1}(-1)^{\ell+k-1}\binom{2\ell-1}{k}\left(\frac{1+\delta_{|n-\nu+2|,0}}{2^{1-|n-\nu+2|}}\mu_{|n-\nu+2|}^{-1/2,-1/2} - \frac{1+\delta_{n+\nu,0}}{2^{1-n-\nu}}\mu_{n+\nu}^{-1/2,-1/2}\right),$$

where $\nu = 2(\ell-k)$. In the previous derivations we used identities

$$2^{n-1}(1+\delta_{n,0})T_n(\cos\phi) = \cos n\phi, \quad 2^nU_n(\cos\phi) = \frac{\sin(n+1)\phi}{\sin\phi}, \quad n\in\mathbb{N}_0$$

(see [2, pp. 140-145]). □

3. **Numerical construction**

The monic polynomials $\pi_k(x)$, $k\in\mathbb{N}_0$, orthogonal with respect to the functional $\mathcal{L}^{\alpha,\beta}$ given by (1.1), satisfy the three-term recurrence relation

$$\begin{aligned}\pi_{k+1}(x) &= (x-\alpha_k)\pi_k(x) - \beta_k\pi_{k-1}(x), \quad k=0,1,\ldots, \\ \pi_0(x) &= 1, \quad \pi_{-1}(x)=0,\end{aligned} \tag{3.9}$$

with $\alpha_k\in\mathbb{R}$ and $\beta_k>0$. Let $\mu_k = \mathcal{L}^{\alpha,\beta}(x^k)$, $k\in\mathbb{N}_0$, be the corresponding moments. The first $2n$ moments $\mu_0,\mu_1,\ldots,\mu_{2n-1}$ uniquely determine the first n recurrence coefficients $\alpha_k=\alpha_k(\mathcal{L}^{\alpha,\beta})$ and $\beta_k=\beta_k(\mathcal{L}^{\alpha,\beta})$, $k=0,1,\ldots,n-1$, in (3.9). However, the corresponding map

$$[\mu_0\ \mu_1\ \mu_2\ \ldots\ \mu_{2n-1}]^{\mathrm{T}} \mapsto [\alpha_0\ \beta_0\ \alpha_1\ \beta_1\ \ldots\ \alpha_{n-1}\ \beta_{n-1}]^{\mathrm{T}}$$

is severely ill-conditioned when n is large. Namely, this map is very sensitive with respect to small perturbations in moment information (the first $2n$ moments). An analysis of such maps in details can be found in the recent book of Gautschi [4, Chapter 2].

For the numerical construction of the coefficients α_k and β_k in (3.9), for $k \le n-1$, we use the modified Chebyshev algorithm (see [6], [2, pp. 112-115], [4, pp. 76-78]). In fact, it is a generalization from ordinary to modified moments of an algorithm due to Chebyshev. Thus, instead of ordinary moments μ_k, $k = 0, 1, \ldots, 2n-1$, we use the so-called *modified moments* $M_k = \mathcal{L}^{\alpha,\beta}(q_k)$, where $\{q_k(x)\}_{k\in\mathbb{N}_0}$ $(\deg q_k(x) = k)$ is a given system of polynomials chosen to be close in some sense to the desired orthogonal polynomials $\{\pi_k\}_{k\in\mathbb{N}_0}$. Then, the corresponding map

$$[M_0\ M_1\ M_2\ \ldots\ M_{2n-1}]^{\mathrm{T}} \mapsto [\alpha_0\ \beta_0\ \alpha_1\ \beta_1\ \ldots\ \alpha_{n-1}\ \beta_{n-1}]^{\mathrm{T}},$$

can become remarkably well-conditioned, especially for measures supported on a finite interval as is our case.

We suppose that the polynomials q_k are also monic and satisfy a three-term recurrence relation

$$q_{k+1}(x) = (x - a_k)q_k(x) - b_k q_{k-1}(x), \quad k = 0, 1, \ldots,$$

where $q_{-1}(x) = 0$ and $q_0(x) = 1$, with given coefficients $a_n \in \mathbb{R}$ and $b_k \ge 0$. In the case $a_k = b_k = 0$, we have the monomials $q_k(x) = x^k$, and m_k reduce to the ordinary moments μ_k $(k \in \mathbb{N}_0)$.

Following Gautschi [4, pp. 76-78], we introduce the "mixed moments"

$$\sigma_{k,i} = \mathcal{L}^{\alpha,\beta}(\pi_k(x)q_i(x)), \quad k, i \ge -1. \tag{3.10}$$

Here, $\sigma_{0,i} = M_i$, $\sigma_{-1,i} = 0$ and, because of orthogonality, $\sigma_{k,i} = 0$ for $k > i$. Also, we take $\sigma_{0,0} = M_0 =: \beta_0$.

Starting with

$$\alpha_0 = a_0 + \frac{M_1}{M_0}, \quad \beta_0 = M_0,$$

the mixed moments (3.10) and the recursive coefficients α_k and β_k can be generated, for $k = 1, \ldots, n-1$, by

$$\sigma_{k,i} = \sigma_{k-1,i+1} - (\alpha_{k-1} - a_i)\sigma_{k-1,i} - \beta_{k-1}\sigma_{k-2,i} + b_i\sigma_{k-1,i-1}, \quad i = k, \ldots, 2n-k-1,$$

and

$$\alpha_k = a_k + \frac{\sigma_{k,k+1}}{\sigma_{k,k}} - \frac{\sigma_{k-1,k}}{\sigma_{k-1,k-1}}, \quad \beta_k = \frac{\sigma_{k,k}}{\sigma_{k-1,k-1}}.$$

k	$\alpha = \beta = -1/2$	$\alpha = \beta = 1/2$	$\alpha = -\beta = 1/2$	
0	−.435517218060720426(1)	−.606789763508705511	−.860673760222240852	−.435517218060720426(1)
1	.860673760222240851	.573587431195261228	.527113772343850128	.119914438687149513
2	.464736588514111009(−1)	.114461454116468030	−.323288979995422790	.222297722364958557
3	.437750434111890820	.391111576176891121	.252285050300644864	.225398401276919416
4	.138826525876246256	.161829438079109437	−.195426513976275507	.240891954171019052
5	.357255952055384943	.340206037337296702	.167009536268718202	.240001913598555923
6	.173196501068578500	.184754218459783853	−.140316170540163753	.245508439602626979
7	.325070388999947606	.316285630106789601	.124992947351188716	.244627540142048162
8	.191292682755600884	.198234729047124326	−.109507820167391439	.247327535832805290
9	.307742549214515765	.302391236700917694	.999144211328118150(−1)	.246657779751021059
10	.202476815568105879	.207103795178056862	−.898095844978158012(−1)	.248227380967697643
11	.296913379675872664	.293312118423073353	.832344255494742355(−1)	.247724032418561568
12	.210077692873599117	.213380332231185856	−.761239117883410348(−1)	.248737809281559934
13	.289504244019526891	.286914939447949928	.713341949682585194(−1)	.248351917023193687
14	.215580744479691408	.218055524644931387	−.660604287327404782(−1)	.249055199532678131
15	.284115953377671865	.282164324339524842	.624147238273606532(−1)	.248752298245571638
16	.219749600512164189	.221672621153416525	−.583483509979231300(−1)	.249266028855671693
17	.280020972151339655	.278497059977941169	.554798452588170684(−1)	.249023034384907651
18	.223017214719124101	.224554192048122145	−.522493851724157680(−1)	.249413227082215333
19	.276803577220537913	.275580500898679617	.499330667579243568(−1)	.249214543935549266
20	.225647434140755260	.226903820710798416	−.473051538327220806(−1)	.249520079948034282
21	.274208974543520496	.273205540289618306	.453952986211696832(−1)	.249354937727147130
22	.227810241668448622	.228856321932349324	−.432159773487912543(−1)	.249600111094014629
23	.272072299281140578	.271234133616303370	.416140361779129718(−1)	.249460889421264035
24	.229620097438390398	.230504511774221198	−.397776456662248980(−1)	.249661613626413084
25	.270282157440446096	.269571475997677721	.384145741637719542(−1)	.249542794493506471
26	.231156901833905767	.231914377880235599	−.368461961856252644(−1)	.249709903829893082
27	.268760574065860863	.268150301959223885	.356721760118687286(−1)	.249607405942853795
28	.232478125947355157	.233134118216279428	−.343172170827991634(−1)	.249748517822303506
29	.267451335299078592	.266921562814687583	.332953957694100852(−1)	.249659264304746552
30	.233626167045277497	.234199756690383015	−.321131355841375670(−1)	.249779882175776878
31	.266312892274520582	.265848650925761851	.312156637119852377(−1)	.249701513312081518
32	.234632987213776614	.235138758136264563	−.301751136273002383(−1)	.249805707166477196
33	.265313871763564801	.264903690385864234	.293805522152342180(−1)	.249736385250931688
34	.235523138170630016	.235972427512013477	−.284577133254071905(−1)	.249827226457090460
35	.264430140837420667	.264065080137475243	.277492886748764822(−1)	.249765500444632627
36	.236315791462598761	.236717544989121286	−.269252872049970366(−1)	.249845348133479749
37	.263642832194118322	.263315822453103279	.262896866353189157(−1)	.249790057958688233
38	.237026135817784364	.237387506514949344	−.255494758068136499(−1)	.249860752688816095
39	.262936982321762994	.262642358706969972	.249759982534029898(−1)	.249810960433893287

TABLE 3.1. Three-term recurrence coefficients for the linear functionals $\mathcal{L}^{-1/2,-1/2}$, $\mathcal{L}^{1/2,1/2}$ and $\mathcal{L}^{1/2,-1/2}$

Using Theorem 2.1 and Lemma 2.1, we calculate the modified moments of the functionals $\mathcal{L}^{\alpha,\beta}$. Using an implementation of the modified Chebyshev algorithm given in [7] we can construct the three-term recurrence coefficients of the monic polynomials π_k orthogonal with respect to $\mathcal{L}^{\alpha,\beta}$. In Table 3.1 we present the coefficients β_k for $k \le 39$ for polynomials orthogonal with respect the functionals $\mathcal{L}^{-1/2,-1/2}$ (second column) and $\mathcal{L}^{1/2,1/2}$ (third column). Numbers in parenthesis indicate decimal exponents. Note that $\alpha_k = 0$, $k \in \mathbb{N}_0$, due to the symmetry of the weights. Also, we give the coefficients α_k and β_k, $k \le 39$ (columns four and five in the same table), for polynomials orthogonal with respect to the linear functional $\mathcal{L}^{1/2,-1/2}$. For the computation of the integral $\mu_0^{\alpha,\beta}$ which is needed to start the computation according to recurrence relations given in Theorem 2.1, we refer to [5].

We report that computations are completely numerically stable, i.e., using this algorithm the precision of results are practically the same as the precision of the input data.

Finally, we are in the position to give an example. We consider the computation of the integral

$$\begin{aligned} I &= \int_{-1}^{1} \sqrt{\frac{1-x}{1+x}} \frac{4}{1+4x^2} \log(1-x^2)\, dx \qquad (3.11) \\ &= -4.1546445827604700896215341366830791816 4\ldots\ . \end{aligned}$$

The construction of Gaussian quadrature rules for the linear functional $\mathcal{L}^{1/2,-1/2}$ can be performed numerically stable using Q-algorithm (see [9]) with three-term recurrence coefficients given in Table 3.1. Table 3.2 holds relative errors of the application of Gaussian quadrature rules with 10, 20, 30 and 40 points, as we inspect the convergence is evident.

n	10	20	30	40
rel. err.	1(−5)	5(−10)	3(−14)	m.p.

TABLE 3.2. Relative error in the computation of the integral (3.11), using Gaussian quadrature rules with n nodes

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$\mathcal{A}$-SUMMABILITY AND APPROXIMATION OF CONTINUOUS PERIODIC FUNCTIONS

CRISTINA RADU

Dedicated to Professor D. D. Stancu on his 80^{th} birthday

Abstract. The aim of this paper is to present a generalization of the classical Korovkin approximation theorem by using a matrix summability method, for sequences of positive linear operators defined on the space of all real-valued continuous and 2π-periodic functions. This approach is motivated by the works of O. Duman [4] and C. Orhan, Ö.G. Atlihan [1].

1. **Introduction**

One of the most recently studied subject in approximation theory is the approximation of continuous function by linear positive operators using A-statistical convergence or a matrix summability method ([1], [3], [5], [7]).

In this paper, following [1], we will give a Korovkin type approximation theorem for a sequence of positive linear operators defined on the space of all real-valued continuous and 2π-periodic functions via $\mathcal{A}$-summability. Particular cases are also punctuated.

First of all, we recall some notation and definitions used in this paper.

Let $\mathcal{A} := (A^n)_{n\geq 1}$, $A^n = (a^n_{kj})_{k,j\in\mathbb{N}}$ be a sequence of infinite non-negative real matrices.

For a sequence of real numbers, $x = (x_j)_{j\in\mathbb{N}}$, the double sequence

$$\mathcal{A}x := \{(Ax)^n_k : \ k, n \in \mathbb{N}\}$$

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defined by $(Ax)_k^n := \sum_{j=1}^{\infty} a_{kj}^n x_j$ is called the $\mathcal{A}$-transform of x whenever the series converges for all k and n. A sequence x is said to be $\mathcal{A}$-summable to a real number L if $\mathcal{A}x$ converges to L as k tends to infinity uniformly in n (see [2]).

We denote by $C_{2\pi}(\mathbb{R})$ the space of all 2π-periodic and continuous functions on $\mathbb{R}$. Endowed with the norm $\|\cdot\|_{2\pi}$ this space is a Banach space, where

$$\|f\|_{2\pi} := \sup_{t\in\mathbb{R}} |f(t)|, \quad f \in C_{2\pi}(\mathbb{R}).$$

We also have to recall the classical Bohman-Korovkin theorem.

Theorem A. *If $\{L_j\}$ is a sequence of positive linear operators acting from $C_{2\pi}(\mathbb{R})$ into $C_{2\pi}(\mathbb{R})$ such that*

$$\lim_{j\to\infty} \|L_j f_i - f_i\|_{2\pi} = 0 \quad (i = 1, 2, 3),$$

where $f_1(t) = 1$, $f_2(t) = \cos t$, $f_3(t) = \sin t$ for all $t \in \mathbb{R}$, then, for all $f \in C_{2\pi}(\mathbb{R})$ we have

$$\lim_{j\to\infty} \|L_j f - f\|_{2\pi} = 0.$$

Recently, the statistical analog of Theorem A has been studied by O. Duman [4]. It will be read as follows.

Theorem B. *Let $A = (a_{kj})$ be a non-negative regular summability matrix, and let $\{L_j\}$ be a sequence of positive linear operators mapping $C_{2\pi}(\mathbb{R})$ into $C_{2\pi}(\mathbb{R})$. Then, for all $f \in C_{2\pi}(\mathbb{R})$,*

$$st_A - \lim_{j\to\infty} \|L_j f - f\|_{2\pi} = 0$$

if and only if

$$st_A - \lim_{j\to\infty} \|L_j f_i - f_i\|_{2\pi} = 0 \quad (i = 1, 2, 3),$$

where $f_1(t) = 1$, $f_2(t) = \cos t$, $f_3(t) = \sin t$ for all $t \in \mathbb{R}$.

2. A Korovkin type theorem

Theorem 2.1. *Let $\mathcal{A} = (A^n)_{n\geq 1}$ be a sequence of infinite non-negative real matrices such that*

$$\sup_{n,k} \sum_{j=1}^{\infty} a_{kj}^n < \infty \tag{2.1}$$

and let $\{L_j\}$ be a sequence of positive linear operators mapping $C_{2\pi}(\mathbb{R})$ into $C_{2\pi}(\mathbb{R})$.

Then, for all $f \in C_{2\pi}(\mathbb{R})$ we have

$$\lim_{k\to\infty} \sum_{j=1}^{\infty} a_{kj}^n \|L_j f - f\|_{2\pi} = 0, \tag{2.2}$$

uniformly in n if and only if

$$\lim_{k\to\infty} \sum_{j=1}^{\infty} a_{kj}^n \|L_j f_i - f_i\|_{2\pi} = 0 \quad (i = 1, 2, 3), \tag{2.3}$$

uniformly in n, where $f_1(t) = 1,\ f_2(t) = \cos t,\ f_3(t) = \sin t$ for all $t \in \mathbb{R}$.

Proof. Since f_i $(i = 1, 2, 3)$ belong to $C_{2\pi}(\mathbb{R})$, the implication (2.2) $\Rightarrow$ (2.3) is obvious.

Now, assume that (2.3) holds. Let $f \in C_{2\pi}(\mathbb{R})$ and let I be a closed subinterval of length 2π of $\mathbb{R}$. Fix $x \in I$. By the continuity of f at x, it follows that for any $\varepsilon > 0$ there exists a number $\delta > 0$ such that

$$|f(t) - f(x)| < \varepsilon \text{ for all } t \text{ satisfying } |t - x| < \delta. \tag{2.4}$$

By the boundedness of f follows

$$|f(t) - f(x)| \leq 2\|f\|_{2\pi} \text{ for all } t \in \mathbb{R}. \tag{2.5}$$

Further on, we consider the subinterval $(x - \delta, 2\pi + x - \delta]$ of length 2π. We show that

$$|f(t) - f(x)| < \varepsilon + \frac{2\|f\|_{2\pi}}{\sin^2 \frac{\delta}{2}} \psi(t) \text{ holds for all } t \in (x - \delta, 2\pi + x - \delta], \tag{2.6}$$

where $\psi(t) := \sin^2 \left(\frac{t - x}{2}\right)$.

To prove (2.6) we examine two cases.

Case 1. Let $t \in (x-\delta, x+\delta)$. In this case we get $|t-x| < \delta$ and the relation (2.6) follows by (2.4).

Case 2. Let $t \in [x+\delta, 2\pi+x-\delta]$. In this case we have $\delta \le t-x \le 2\pi-\delta$ and $\delta \in (0,\pi]$. We get

$$\sin^2\frac{\delta}{2} \le \sin^2\left(\frac{t-x}{2}\right) \le \sin^2\left(\pi-\frac{\delta}{2}\right), \tag{2.7}$$

for all $\delta \in (0,\pi]$ and $t \in [x+\delta, 2\pi+x-\delta]$.

Then, from (2.5) and (2.7) we obtain

$$|f(t)-f(x)| \le \frac{2\|f\|_{2\pi}}{\sin^2\frac{\delta}{2}}\psi(t) \text{ for all } t \in [x+\delta, 2\pi+x-\delta].$$

Since the function $f \in C_{2\pi}(\mathbb{R})$ is 2π-periodic, the inequality (2.6) holds for all $t \in \mathbb{R}$.

Now, applying the operator L_j, we get

$$\begin{aligned}
|L_j(f;x)-f(x)| &\le L_j(|f-f(x)|;x) + |f(x)||L_j(f_1;x)-f_1(x)| \\
&< L_j\left(\varepsilon + \frac{2\|f\|_{2\pi}}{\sin^2\frac{\delta}{2}}\psi;x\right) + \|f\|_{2\pi}|L_j(f_1;x)-f_1(x)| \\
&= \varepsilon L_j(f_1;x) + \frac{2\|f\|_{2\pi}}{\sin^2\frac{\delta}{2}}L_j(\psi;x) + \|f\|_{2\pi}|L_j(f_1;x)-f_1(x)| \\
&\le \varepsilon + (\varepsilon+\|f\|_{2\pi})|L_j(f_1;x)-f_1(x)| + \frac{2\|f\|_{2\pi}}{\sin^2\frac{\delta}{2}}L_j(\psi;x).
\end{aligned}$$

Since

$$L_j(\psi;x) \le \frac{1}{2}\{|L_j(f_1;x)-f_1(x)|+|\cos x||L_j(f_2;x)-f_2(x)|+|\sin x||L_j(f_3;x)-f_3(x)|\}, \tag{2.8}$$

(see [8], Theorem 4) we obtain

$$|L_j(f;x)-f(x)| < \varepsilon + \left(\varepsilon + \|f\|_{2\pi} + \frac{\|f\|_{2\pi}}{\sin^2 \dfrac{\delta}{2}}\right)\Big\{|L_j(f_1;x)-f_1(x)|$$
$$+ |L_j(f_2;x)-f_2(x)| + |L_j(f_3;x)-f_3(x)|\Big\}$$
$$\leq \varepsilon + K\{\|L_jf_1 - f_1\|_{2\pi} + \|L_jf_2 - f_2\|_{2\pi} + \|L_jf_3 - f_3\|_{2\pi}\},$$

where

$$K := \varepsilon + \|f\|_{2\pi} + \frac{\|f\|_{2\pi}}{\sin^2 \dfrac{\delta}{2}}.$$

Taking supremum over x, for all $j \in \mathbb{N}$ we obtain

$$\|L_jf - f\|_{2\pi} \leq \varepsilon + K\{\|L_jf_1 - f_1\|_{2\pi} + \|L_jf_2 - f_2\|_{2\pi} + \|L_jf_3 - f_3\|_{2\pi}\}.$$

Consequently, we get

$$\sum_{j=1}^{\infty} a_{kj}^n \|L_jf - f\|_{2\pi} \leq \varepsilon \sum_{j=1}^{\infty} a_{kj}^n + K \sum_{j=1}^{\infty} a_{kj}^n \|L_jf_1 - f_1\|_{2\pi}$$
$$+K \sum_{j=1}^{\infty} a_{kj}^n \|L_jf_2 - f_2\|_{2\pi} + K \sum_{j=1}^{\infty} a_{kj}^n \|L_jf_3 - f_3\|_{2\pi}.$$

By taking limit as $k \to \infty$ and by using (2.1), (2.3) we obtain the desired result. □

Using the concept of A-statistical convergence, O. Duman and E. Erkuş [6] obtained a Korovkin type approximation theorem by positive linear operators defined on $C_{2\pi}(\mathbb{R}^m)$, the space of all real-valued continuous and 2π-periodic functions on $\mathbb{R}^m$ ($m \in \mathbb{N}$) endowed with the norm $\|\cdot\|_{2\pi}$ of the uniform convergence. The same result stands for $\mathcal{A}$-summability.

Theorem 2.2. *Let $\mathcal{A} = (A^n)_{n\geq 1}$ be a sequence of infinite non-negative real matrices such that*

$$\sup_{n,k} \sum_{j=1}^{\infty} a_{kj}^n < \infty$$

and let $\{L_j\}$ be a sequence of positive linear operators mapping $C_{2\pi}(\mathbb{R}^m)$ into $C_{2\pi}(\mathbb{R}^m)$.

Then, for all $f \in C_{2\pi}(\mathbb{R}^m)$ *we have*

$$\lim_{k\to\infty}\sum_{j=1}^{\infty} a_{kj}^n \|L_j f - f\|_{2\pi} = 0,$$

uniformly in n*, if and only if*

$$\lim_{k\to\infty}\sum_{j=1}^{\infty} a_{kj}^n \|L_j f_p - f_p\|_{2\pi} = 0 \quad (p = 1, 2, \ldots, (2m+1)),$$

uniformly in n*, where* $f_1(t_1, t_2, \ldots, t_m) = 1$*,* $f_p(t_1, t_2, \ldots, t_m) = \cos t_{p-1}$ $(p = 2, 3, \ldots, m+1)$*,* $f_q(t_1, t_2, \ldots, t_m) = \sin t_{q-m-1}$ $(q = m+2, \ldots, 2m+1)$*.*

3. **Particular cases**

Taking $A^n = I$, I being the identity matrix, Theorem 2.1 reduces to Theorem A.

If $A^n = A$, for some matrix A, then $\mathcal{A}$-summability is the ordinary matrix summability by A.

Note that statistical convergence is a regular summability method. Considering Theorem B and our Theorem 2.1 we obtain the next result.

Corollary 3.1. *Let* $\mathcal{A} = (A^n)_{n\in\mathbb{N}}$ *be a sequence of non-negative regular summability matrices and let* $\{L_j\}$ *be a sequence of positive linear operators mapping* $C_{2\pi}(\mathbb{R})$ *into* $C_{2\pi}(\mathbb{R})$*.*

Then, for all $f \in C_{2\pi}(\mathbb{R})$ *we have*

$$st_{A_n} - \lim_{j\to\infty} \|L_j f - f\|_{2\pi} = 0, \text{ uniformly in } n$$

if and only if

$$st_{A_n} - \lim_{j\to\infty} \|L_j f_i - f_i\|_{2\pi} = 0 \quad (i = 1, 2, 3), \text{ uniformly in } n,$$

where $f_1(t) = 1$*,* $f_2(t) = \cos t$*,* $f_3(t) = \sin t$ *for all* $t \in \mathbb{R}$*.*

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BOOK REVIEWS

Function Spaces, Krzysztof Jarosz (Editor), Contemporary Mathematics, Vol. 435, v+394 pp, American Mathematical Society, Providence, Rhode Island 2007, (ISSN: 0271-4132; v. 435), ISBN: 978-0-8218-4061-0.

Starting with 1990 a Conference on Function Spaces was held each fourth year at the Southern Illinois University Edwardsville. The volumes of the first two conferences were published with Marcel Dekker in *Lecture Notes in Pure and Applied Mathematics*, while the Proceedings of the last three conferences were published by AMS in the series *Contemporary Mathematics*, as volumes 232, 328 and 435 (the present one).

The Fifth Conference which took place from May 16 to May 20, 2006, was attended by 120 participants from 25 countries. The lectures covered a broad range of topics related to the general notion of "function space" - Banach algebras, C^*-algebras, spaces and algebras of continuous, differentiable or analytic functions (scalar and vector as well), geometry of Banach spaces. The main purpose of the Conference was to bring together mathematicians, working in the same domains or in related ones, to share opinions and ideas about the topics they are interested in. For this reason, the lectures have a general informal character, being addressed to non-experts, the survey papers and the papers containing new results as well.

The present volume contains 33 papers covering topics as Young-Fenchel transform and some characteristics of Banach spaces (Ya. I. Alber), Hardy spaces and operators acting on them (O. Balsco, D. P. Blecher, L. E. Labushagne, N. Arcozzi, R. Rochberg, E. Sawyer), spaces of bad (e.g., nowhere differentiable) functions (R. M. Aron et al), cohomology of Banach algebras and the geometry of Banach spaces (A. Blanco, N. Grobnaek), the Kadison-Singer theorem (P. G. Casazza, D. Edidin), strongly proximinal subspaces (S. Dutta, D. Narayana), uniform algebras (G. Bulancea, J. F. Feinstein, M. J. Heath, S. Lambert, A, Luttman, T. Tonev), various questions on Orlicz spaces (A. Kaminska, Y. Raynaud, A. Yu. Karlovich, M. Gonzales, B. Sari, M. Wojtowicz), the moment problem (D. Atanasiu, F. H. Szafraniec), algebras of continuous functions - Stone-Wierstrass type theorems, surjections which preserve spectrum (D. Honma, J. Kauppi), composition operators on spaces of analytic functions ((J. S. Manhas), quasi-similar operators with the same essential spectrum (T. L. Miller, V. G. Miller, M. M. Neumann), joint spectrum (A. Soltysiak), spectral isometries (M. Mathieu, C. Ruddy), complex Banach manifolds (I. Patyi), algebraic equations in C^*-algebras (T. Miura, D. Honma), Takesaki duality ((K. Watanabe), algebras of analytic functions and polynomials on Banach spaces (A. Zagorodnyuk).

Surveying or presenting new results in various areas of analysis related to function spaces, the present volume appeals to a large audience, first of all people working in this domain, but also researchers in related areas who want to be informed about results and methods in this field.

S. Cobzaş

Peter M. Gruber, ***Convex and Discrete Geometry,*** Springer-Verlag, Berlin-Heidelberg, 2007, Grundlehren der mathematischen Wissenschaften, Volume 336, xiii+578 pp, ISBN 978-3-540-71132-2.

The aim of the present book is to give an overview of basic methods and results of convex analysis and discrete geometry and their applications. The general idea of the book is that there are a plenty of beautiful and deep classical results and challenging problems in the domain, which are still in the focus of current research. Some of the problems as, for instance, the isoperimetric problems, the Platonic solids, the volumes of pyramids, have their roots in antiquity, while the modern research in convex geometry concerns local theory of Banach spaces, best and random approximation, surfaces and curvature measures, tilings and packings. Their solution requires tools and methods from various fields of mathematics, as Fourier analysis, probability theory, combinatorics, topology, and, in turn, the results from convex and discrete geometry are very useful in many domain of mathematics.

The book is divided into four parts: *Convex Functions, Convex Bodies, Convex Polytopes* and *Geometry of Numbers.*

The first part presents the basic properties of convex functions of one variable (Chapter 1) and of several variables (Chapter 2): continuity properties and differentiability properties, the highlight being the proof of Alexandrov's theorem on a.e. second-order differentiability of convex functions. Among applications we mention: the use of convex functions in proving various inequalities, the characterization of gamma function by Bohr and Mollerup, and a sufficient condition in the calculus of variation due to Courant and Hilbert.

The second part is devoted to the study of convex bodies, simple to define, but "which possess a surprisingly rich structure", according to a quotation from Ball's book on convex geometry, Cambridge U.P., 1997. The author present the basic properties of convex bodies in the Euclidean space E^d - combinatorial properties (the theorems of Carátheodory, Helly and Radon), boundary structure, extremal points (including Krein-Milman theorem), mixed volumes and Brun-Minkowski inequality, symmetrization, intrinsic metrics, approximation of convex bodies, simplices and Choquet's theorem, Baire category methods in convexity (many, in the sense of Baire category, convex bodies have good rotundity and smoothness properties). Some nice applications to this results are included - Hartogs' theorem on power series in $\mathbb{C}^d$, Lyapunov's convexity theorem, Pontryagin's maximum principle, Birkhoff's theorem on doubly stochastic matrices.

Although convex polytopes, as particular case of convex bodies, are freely used in the second part, their systematic study is done in the third part. Here, after the formal definitions and some elementary properties, one studies the combinatorial theory of polytopes (Euler's formula or, more correctly, Descartes-Euler - "the first important event in algebraic topology", according to a quotation from the fundamental treatise on topology by Alexandrov and Hopf), volumes of polytopes and Hilbert's third problem, the theorems of Alexandrov, Minkowski and Lindelöf, lattice polytopes, Newton polytopes. This part ends with an introduction to linear optimization, including simplex algorithm and a presentation of Khachiyan's polynomial ellipsoid algorithm. Applications are given to irreducibility criteria for polynomials and the Minding-Bernstein theorem on the number of zeros of systems of polynomial equations.

The last part of the book, *Geometry of Numbers,* is concerned with the interplay between the group theoretic notion of lattice in E^d and the geometric concept of convex set - the lattices represent periodicity, while the convex sets the geometry. This field was baptized "Geometry of numbers" by Hermann Minkowski who made breakthrough contributions to the area, some of of them being included in the book, as Minkowski's fundamental theorem and Minkowski-Hlawka theorem giving upper, respectively lower, bounds for the density of lattice packings. There are strong connections with the geometric theory of positive quadratic forms. Among the topics included in this part we do mention: the study of the density of tiling and packing with convex bodies, including the solution by Hales (Annals of Mathematics, 2005) of Kepler's famous conjecture on ball packing, optimum quantization, Koebe's representation theorem for planar graphs. The applications deal with Diophantine approximation, error correcting codes, numerical integration, and an algorithmic approach to Riemann mapping theorem.

There a lot of historical detours in the book as well as pertinent comments of the author about various questions. Some results are given two or three proofs, each shedding a new light on the problem and having its own beauty and originality. A large bibliography of 1052 titles, each of them being referred to in the text, tries to cover all the facets of the subject, from its origins to the present day state.

The author is a well-known specialist in the area with important contributions. Beside numerous research papers, he is the co-editor of two outstanding volumes - *Convexity and its Applications,* Birkhäuser 1983, and *Handbook of Convex Geometry,* A,B, North-Holland 1993, (both with J. M. Wills), as well as the co-author of a book, *Geometry of Numbers,* North-Holland, 1987 (with C. G. Lekkerkerker).

Since the problems in convex and discrete geometry are easy to formulate (and understand) but hardly to solve, the included material and the clear and pleasant style of presentation, make the book accessible to a large audience, including graduate students, teachers and researchers in various areas of mathematics.

S. Cobzaş

J. Kollár, ***Lectures on Resolution of Singularities***, Princeton University Press (Annals of Mathematics Studies, 166), 2007, Paperback, 208 pages, ISBN-10: 0-691-12923-1, ISBN-13: 978-0-12923-5.

Resolution of singularities is one of the most venerable topics in algebraic geometry. We may say that it was, in a way, born before the algebraic geometry, as we know it today, existed.

The essence of the theory is easy to explain. If we consider an arbitrary algebraic variety, it usually has singular points and the variety is difficult to study because of these points. It is, however, possible to parameterize any variety by a smooth variety (without singular points) and many properties of the parameterizing variety ar similar to the original one. The process of finding a parameterizing smooth variety of an arbitrary variety is called *the resolution of singularities* for the given variety.

The first resolution was given by Newton, for curves in the complex plane. The resolution of algebraic surfaces was given at the beginning of the twentieth century, by different authors, while Zariski, in 1944, solved the problem for 3-folds. It was only in 1964 that Hironaka, in a 218 pages paper, managed to settled the general case (for varieties over a field of characteristic zero).

As one can readily guess, the proof of Hironaka is extremely complicated and, until recently, there wasn't any manageable proof available. In the last decade, however, it was given a new, different and much easier proof, accessible even for graduate students. It is the aim of the book, written by one of the most respected experts in algebraic geometry, and based on a course given in 2004/2005 at the Princeton University, to provide an introduction to the resolution of singularities and, in particular, to expose this new proof.

The first chapter of the book is devoted to the resolution of curves and there are given as many as thirteen (!) different proofs of the existence of resolutions. Many of the proof, as the author himself emphasize, are so elementary that they can be given in a first course of algebraic geometry.

The second chapter is concerned with the resolution of surfaces. More elaborate methods are needed here and, again, most of them are specific to this particular case and cannot be easily extended to the general case.

The third (and last) chapter deals with the general case. The new proof is presented and then there are discusses a lot of examples. It is to be noticed that this new proof is given on thirty pages. It is not short, of course, but if we compare it to the original one, we can appreciate the improvement.

The book is written in a very pedagogical manner, with many examples. Many proofs are given in an algorithmic manner. It is, probably, the first really comprehensive textbook in the resolution of singularities, one of the most important topics of algebraic geometry, as mentioned earlier. Of course, many advanced topics are not touched and the proofs refer only to the characteristic zero case, but this make the proof even more useful for graduate students, which are, usually, ont prepared to attack directly the general case. Otherwise, I think it provides a fairly complete picture of resolutions for varieties. Beside the proof of the general theorem, I particularly liked the discussion of the low dimensional cases, many of them of importance for the early history of algebraic geometry.

The prerequisites for this book include, in my opinion, a first course in algebraic geometry and in algebra. As I mentioned earlier, some of the proofs can be even discussed *within* a first course in algebraic geometry. The book will be an invaluable tool not only for graduate student, but also for algebraic geometers. Mathematicians working in different fields will also enjoy the clarity of the exposition and the wealth of ideas included. This will become, I'm sure, as it happened to most books in this series, one of the classics of modern mathematics.

Paul Blaga

Mathematical aspects of Nonlinear Dispersive Equations, Jean Bourgain, Carlos E. Kenig & S. Klainerman (Editors), Annals of Mathematics Studies No. 163, Princeton University Press, Princeton and Oxford 2007, vii + 300 pp., ISBN 13: 978-0-691-12955-6 and 10: 0-691-12955-X.

These are the written versions of a number of lectures delivered at the CMI/IAS Workshop on mathematical aspects of nonlinear PDEs, in the spring of 2004 at the Institute for Advanced Study in Princeton. The workshop is a conclusion of a year-long program at IAS about this topic, leading to significant progress and to the broadening of the subject. At least two important breakthroughs were obtained - the first one is the understanding of the blowup mechanism for critical focusing Schrödinger equation, and the other is a proof of global existence and scattering for the 3D quintic equation for general smooth data. In both cases, hard analysis, in addition to the more geometric approach, turned to play a key role in energy estimates.

The volume contains 12 papers (called chapters), some of them of expository nature (as, e.g., that by W. Schlag on dispersive estimates for Schrödinger operators), describing the state of the art and research directions, while the others are contributed papers, both kinds being fully original accounts. The papers concentrate on new developments on Scrödinger operators, nonlinear Scgrödinger and wave equations, hyperbolic conservation laws, Euler and Navier-Stokes equations.

Among the contributors we mention Jean Bourgain (two papers, one with W.-M. Wang), A. Bressan, H. K. Jensen, H. Brezis, M. Marcus, P. Gérard, N. Tzvetkov, P. Constantin, A. D. Ionescu, B. Nikolaenko, Terence Tao.

The volume contains valuable contributions to the area of nonlinear PDEs, making it indispensable for all researchers interested in partial differential equations and their applications.

Radu Precup