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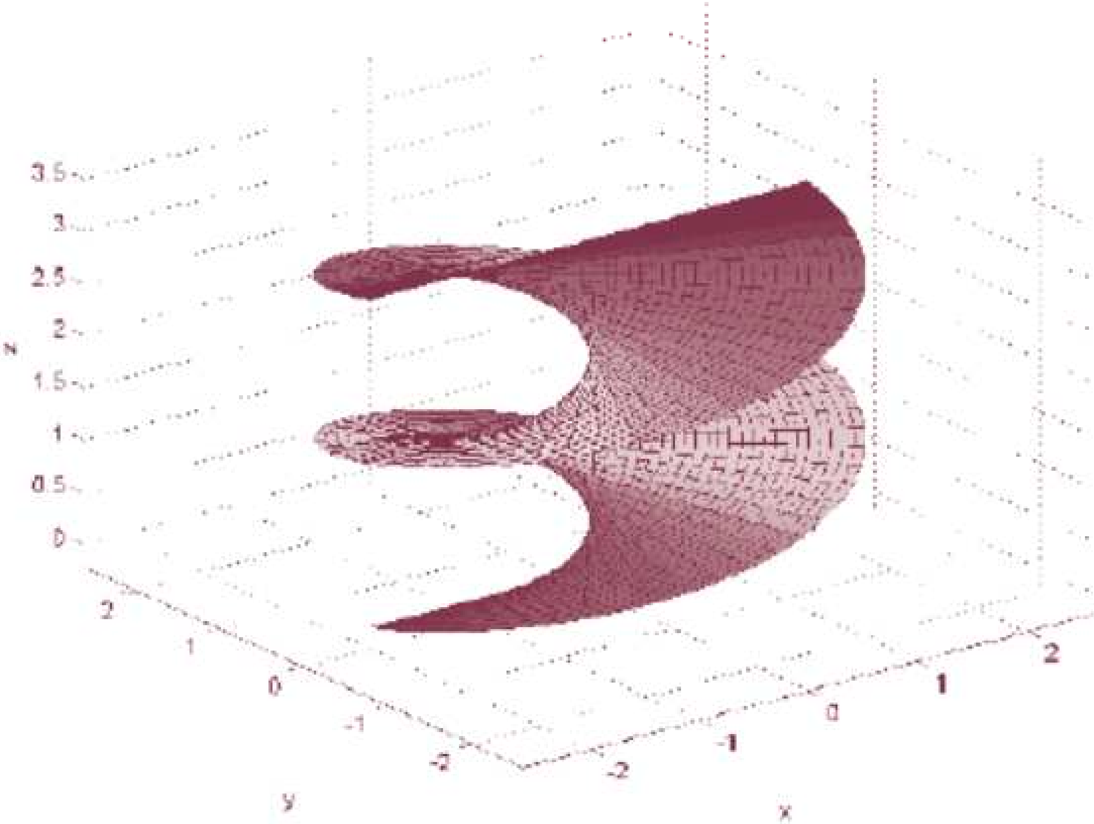

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SOME RESULTS ON TRANSFORMATIONS GROUPS OF N-LINEAR CONNECTIONS IN THE 2-TANGENT BUNDLE

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Abstract. In the present paper we study the transformations for the coefficients of an N-linear connection (definition 1.1) on the tangent bundle of order two, T^2M, by a transformation of a nonlinear connection in T^2M. We prove that the set $\mathcal{T}$ of these transformations together with composition of mappings isn't a group. But we give some groups of $\mathcal{T}$, which keep invariant a part of components of the local coefficients of an N-linear connection. We also determine the transformation laws of the torsion and curvature tensor fields, with respect to the transformations of the group $\mathcal{T}_N \subset \mathcal{T}$.

1. **The N-and JN-linear connections on tangent bundle of order two**

Let M be a real C^∞-manifold with n dimensions and (T^2M, π, M) its 2-tangent bundle, [1]. The local coordinates on 3n-dimensional manifold T^2M are denoted by $(x^i, y^{(1)i}, y^{(2)i}) = (x, y^{(1)}, y^{(2)}) = u$, $(i = 1, 2, ...n)$.

Let $\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial y^{(1)i}}, \frac{\partial}{\partial y^{(2)i}}\right)$ be the natural basis of the tangent space TT^2M at the point $u \in T^2M$ and let us consider the natural 2-tangent structure on T^2M, $J : \chi(T^2M) \to \chi(T^2M)$ given by

$$J\left(\frac{\partial}{\partial x^i}\right) = \frac{\partial}{\partial y^{(1)i}},\ J\left(\frac{\partial}{\partial y^{(1)i}}\right) = \frac{\partial}{\partial y^{(2)i}}, J\left(\frac{\partial}{\partial y^{(2)i}}\right) = 0. \qquad (1.1)$$

We denote with N a nonlinear connection on T^2M with the local coefficients $(\underset{1}{N}{}^i{}_j, \underset{2}{N}{}^i{}_j)$ $(i, j = 1, 2, ..., n)$, [7], [8].

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Hence, the tangent space of T^2M in the point $u \in T^2M$ is given by the direct sum of the linear vector spaces:

$$T_nT^2M = N_0(u) \oplus N_1(u) \oplus V_2(u),\ \forall u \in T^2M. \tag{1.2}$$

An adapted basis to the direct decomposition (1.2) is given by

$$\left\{ \frac{\delta}{\delta x^i}, \frac{\delta}{\delta y^{(1)i}}, \frac{\delta}{\delta y^{(2)i}} \right\}, \tag{1.3}$$

where:

$$\begin{aligned} &\tfrac{\delta}{\delta x^i} = \tfrac{\partial}{\partial x^i} - \underset{1}{N}{}^j{}_i \tfrac{\partial}{\partial y^{(1)j}} - \underset{2}{N}{}^j{}_i \tfrac{\partial}{\partial y^{(2)j}}, \\ &\tfrac{\delta}{\delta y^{(1)i}} = \tfrac{\partial}{\partial y^{(1)i}} - \underset{1}{N}{}^j{}_i \tfrac{\partial}{\partial y^{(2)j}}, \\ &\tfrac{\delta}{\delta y^{(2)i}} = \tfrac{\partial}{\partial y^{(2)i}}. \end{aligned} \tag{1.4}$$

Let us consider the dual basis of (1.3):

$$\{dx^i, \delta y^{(1)i}, \delta y^{(2)i}\}, \tag{1.5}$$

where

$$\begin{aligned} &\delta x^i = dx^i, \\ &\delta y^{(1)i} = dy^{(1)i} + \underset{1}{N}{}^i{}_j dx^j, \\ &\delta y^{(2)i} = dy^{(2)i} + \underset{1}{N}{}^i{}_j dy^{(1)j} + (\underset{2}{N}{}^i{}_j + \underset{1}{N}{}^i{}_m \underset{1}{N}{}^m{}_j) dx^j. \end{aligned} \tag{1.6}$$

Definition 1.1. *([1]-[3]) A linear connection D on T^2M, $D : \chi(T^2M) \times \chi(T^2M) \to \chi(T^2M)$ is called an N-linear connection on T^2M if it preserves by parallelism the horizontal and vertical distributions N_0, N_1 and V_2 on T^2M.*

An N-linear connection D on T^2M is characterized by its coefficients in the adapted basis (1.3) in the form:

$$\begin{aligned} &D_{\frac{\delta}{\delta x^k}} \tfrac{\delta}{\delta x^j} = \underset{(00)}{L}{}^i{}_{jk} \tfrac{\delta}{\delta x^i},\ D_{\frac{\delta}{\delta x^k}} \tfrac{\delta}{\delta y^{(1)j}} = \underset{(10)}{L}{}^i{}_{jk} \tfrac{\delta}{\delta y^{(1)i}},\ D_{\frac{\delta}{\delta x^k}} \tfrac{\partial}{\partial y^{(2)j}} = \underset{(20)}{L}{}^i{}_{jk} \tfrac{\partial}{\partial y^{(2)i}}, \\ &D_{\frac{\delta}{\delta y^{(1)k}}} \tfrac{\delta}{\delta x^j} = \underset{(01)}{C}{}^i{}_{jk} \tfrac{\delta}{\delta x^i}, D_{\frac{\delta}{\delta y^{(1)k}}} \tfrac{\delta}{\delta y^{(1)j}} = \underset{(11)}{C}{}^i{}_{jk} \tfrac{\delta}{\delta y^{(1)i}}, D_{\frac{\delta}{\delta y^{(1)k}}} \tfrac{\partial}{\partial y^{(2)j}} = \underset{(21)}{C}{}^i{}_{jk} \tfrac{\partial}{\partial y^{(2)i}}, \\ &D_{\frac{\partial}{\partial y^{(2)k}}} \tfrac{\delta}{\delta x^j} = \underset{(02)}{C}{}^i{}_{jk} \tfrac{\delta}{\delta x^i}, D_{\frac{\partial}{\partial y^{(2)k}}} \tfrac{\delta}{\delta y^{(1)j}} = \underset{(12)}{C}{}^i{}_{jk} \tfrac{\delta}{\delta y^{(1)i}}, D_{\frac{\partial}{\partial y^{(2)k}}} \tfrac{\partial}{\partial y^{(2)j}} = \underset{(22)}{C}{}^i{}_{jk} \tfrac{\partial}{\partial y^{(2)i}}. \end{aligned} \tag{1.7}$$

The system of nine functions

$$D\Gamma(N) = (\underset{(00)}{L^i}_{jk}, \underset{(10)}{L^i}_{jk}, \underset{(20)}{L^i}_{jk}, \underset{(01)}{C^i}_{jk}, \underset{(11)}{C^i}_{jk}, \underset{(21)}{C^i}_{jk}, \underset{(02)}{C^i}_{jk}, \underset{(12)}{C^i}_{jk}, \underset{(22)}{C^i}_{jk}), \tag{1.8}$$

are called the **coefficients** of the N-linear connection D.

Generally, an N-linear connection $D\Gamma(N)$ on T^2M is not compatible with the natural 2-tangent structure J given by (1.1).

Definition 1.2. *An N-linear connection D on T^2M is called JN-linear connection if it is absolute parallel with respect to D:*

$$D_X J = 0, \ \ \forall X \in \chi(T^2M). \tag{1.9}$$

Theorem 1.1. (Gh. Atanasiu, [1]) *A JN-linear connection on T^2M is characterized by the coefficients $JD\Gamma(N)$ given by (1.8), where*

$$\begin{aligned} \underset{(00)}{L^i}_{jk} &= \underset{(10)}{L^i}_{jk} = \underset{(20)}{L^i}_{jk} \ (= L^i{}_{jk}), \\ \underset{(01)}{C^i}_{jk} &= \underset{(11)}{C^i}_{jk} = \underset{(21)}{C^i}_{jk} \ (= \underset{(1)}{C^i}_{jk}), \\ \underset{(02)}{C^i}_{jk} &= \underset{(12)}{C^i}_{jk} = \underset{(22)}{C^i}_{jk} \ (= \underset{(2)}{C^i}_{jk}). \end{aligned} \tag{1.10}$$

It results that a $JD\Gamma(N)$- linear connection on T^2M has three essentially coefficients:

$$JD\Gamma(N) = (L^i{}_{jk}, \underset{(1)}{C^i}_{jk}, \underset{(2)}{C^i}_{jk}). \tag{1.11}$$

Obvious, the geometrical theory on 2-tangent bundle (T^2M, π, M) with the N- linear connection [1]-[3], [15], generalize on that with the JN-linear connection (cf.with R. Miron and Gh. Atanasiu [5]-[8]; see, also M. Purcaru [12], [13]).

In the following we use the N-linear connections, only.

2. The set of transformations of N-linear connections

Let $D\Gamma(N)$ be an N-linear connection on T^2M with the coefficients given by (1.8). If $\bar{N}$ is another nonlinear connection on T^2M with the coefficients $(\underset{1}{\bar{N}^i}_j, \underset{2}{\bar{N}^i}_j)$, $(i, j = 1, 2, ..., n)$, then there exists the uniquely determined tensor fields $\underset{\beta}{A^i}_j, \in \tau_1^1(T^2M)$ such that:

$$\underset{\beta}{\bar{N}^i}_j = \underset{\beta}{N^i}_j - \underset{\beta}{A^i}_j, \quad (\beta = 1, 2). \tag{2.1}$$

Conversely, if $\underset{\beta}{N^i}_j$and $\underset{\beta}{A^i}_j$, are fixed, then $\underset{\beta}{\bar{N}^i}_j$, $(\beta = 1, 2)$, given by (2.1) is a nonlinear connection.

Let us suppose that the mapping $N \to \bar{N}$ is given by (2.1) and we denote:

$$\frac{\bar{\delta}}{\delta x^i} = \frac{\partial}{\partial x^i} - \underset{1}{\bar{N}^j}_i \frac{\partial}{\partial y^{(1)j}} - \underset{2}{\bar{N}^j}_i \frac{\partial}{\partial y^{(2)j}}, \frac{\bar{\delta}}{\delta y^{(1)i}} = \frac{\partial}{\partial y^{(1)i}} - \underset{1}{\bar{N}^j}_i \frac{\partial}{\partial y^{(2)j}}, \frac{\bar{\delta}}{\delta y^{(2)i}} = \frac{\partial}{\partial y^{(2)i}}.$$

It follows first of all that the transformations (2.1) preserve the coefficients $\underset{(\alpha 2)}{C^i}_{jk}$, $(\alpha = 0, 1, 2)$.

Taking in account the fact that:

$$\frac{\bar{\delta}}{\delta x^i} = \frac{\delta}{\delta x^i} + \underset{1}{A^j}_i \frac{\partial}{\partial y^{(1)j}} + \underset{2}{A^j}_i \frac{\partial}{\partial y^{(2)j}}, \quad \frac{\bar{\delta}}{\delta y^{(1)i}} = \frac{\delta}{\delta y^{(1)i}} + \underset{1}{A^j}_i \frac{\partial}{\partial y^{(2)j}}, \frac{\bar{\delta}}{\delta y^{(2)i}} = \frac{\delta}{\delta y^{(2)i}},$$

it follows:

$$D_{\frac{\bar{\delta}}{\delta x^k}} \frac{\bar{\delta}}{\delta y^{(2)j}} = D_{\frac{\bar{\delta}}{\delta x^k}} \frac{\partial}{\partial y^{(2)j}} = \underset{(20)}{\bar{L}^i}_{jk} \frac{\partial}{\partial y^{(2)i}} = D_{(\frac{\delta}{\delta x^k} + \underset{1}{A^l}_k \frac{\partial}{\partial y^{(1)l}} + \underset{2}{A^l}_k \frac{\partial}{\partial y^{(2)l}})} \frac{\partial}{\partial y^{(2)j}} =$$

$$= D_{\frac{\delta}{\delta x^k}} \frac{\partial}{\partial y^{(2)j}} + \underset{1}{A^l}_k D_{(\frac{\delta}{\delta y^{(1)l}} + \underset{1}{N^m}_l \frac{\partial}{\partial y^{(2)m}})} \frac{\partial}{\partial y^{(2)j}} + \underset{2}{A^l}_k D_{\frac{\partial}{\partial y^{(2)l}}} \frac{\partial}{\partial y^{(2)j}} =$$

$$= \underset{(20)}{L^i}_{jk} \frac{\partial}{\partial y^{(2)i}} + \underset{1}{A^l}_k \underset{(21)}{C^i}_{jl} \frac{\partial}{\partial y^{(2)i}} + \underset{1}{A^l}_k \underset{1}{N^m}_l \underset{(22)}{C^i}_{jm} \frac{\partial}{\partial y^{(2)i}} + \underset{2}{A^l}_k \underset{(22)}{C^i}_{jl} \frac{\partial}{\partial y^{(2)i}} =$$

$$= (\underset{(20)}{L^i}_{jk} + \underset{1}{A^l}_k \underset{(21)}{C^i}_{jl} + \underset{1}{A^l}_k \underset{1}{N^m}_l \underset{(22)}{C^i}_{jm} + \underset{2}{A^l}_k \underset{(22)}{C^i}_{jl}) \frac{\partial}{\partial y^{(2)i}}.$$

$$D_{\frac{\bar{\delta}}{\delta y^{(1)k}}} \frac{\bar{\delta}}{\delta y^{(2)j}} = D_{\frac{\bar{\delta}}{\delta y^{(1)k}}} \frac{\partial}{\partial y^{(2)j}} = \underset{(21)}{\bar{C}^i}_{jk} \frac{\partial}{\partial y^{(2)i}} = D_{(\frac{\delta}{\delta y^{(1)k}} + \underset{1}{A^l}_k \frac{\partial}{\partial y^{(2)l}})} \frac{\partial}{\partial y^{(2)j}} =$$

$$= D_{\frac{\delta}{\delta y^{(1)k}}} \frac{\partial}{\partial y^{(2)j}} + \underset{1}{A^l}_k D_{\frac{\partial}{\partial y^{(2)l}}} \frac{\partial}{\partial y^{(2)j}} = \underset{(21)}{C^i}_{jk} \frac{\partial}{\partial y^{(2)i}} + \underset{1}{A^l}_k \underset{(22)}{C^i}_{jl} \frac{\partial}{\partial y^{(2)i}} =$$

$$= (\underset{(21)}{C}{}^i{}_{jk} + \underset{1}{A}{}^l{}_k \underset{(22)}{C}{}^i{}_{jl})\frac{\partial}{\partial y^{(2)i}}.$$

Therefore the change we are looking for is:

$$\begin{cases} \underset{(\alpha 0)}{\bar{L}}{}^i{}_{jk} = \underset{(\alpha 0)}{L}{}^i{}_{jk} + \underset{1}{A}{}^l{}_k \underset{(\alpha 1)}{C}{}^i{}_{jl} + \underset{1}{A}{}^l{}_k \underset{1}{N}{}^m{}_l \underset{(\alpha 2)}{C}{}^i{}_{jm} + \underset{2}{A}{}^l{}_k \underset{(\alpha 2)}{C}{}^i{}_{jl}, \\ \underset{(\alpha 1)}{\bar{C}}{}^i{}_{jk} = \underset{(\alpha 1)}{C}{}^i{}_{jk} + \underset{1}{A}{}^l{}_k \underset{(\alpha 2)}{C}{}^i{}_{jl}, \\ \underset{(\alpha 2)}{\bar{C}}{}^i{}_{jk} = \underset{(\alpha 2)}{C}{}^i{}_{jk}, (\alpha = 0, 1, 2). \end{cases} \tag{2.2}$$

So, we have proved:

Proposition 2.1. *The transformation (2.1) of nonlinear connections imply the transformations (2.2) for the coefficients $D\Gamma(N)$ of the N-linear connection D.*

Theorem 2.1. *Let N and $\bar{N}$ be two nonlinear connections, with the coefficients $(\underset{1}{N}{}^i{}_j, \underset{2}{N}{}^i{}_j), (\underset{1}{\bar{N}}{}^i{}_j, \underset{2}{\bar{N}}{}^i{}_j)$-respectively. If*

$$D\Gamma(N) = (\underset{(00)}{L}{}^i{}_{jk}, \underset{(10)}{L}{}^i{}_{jk}, \underset{(20)}{L}{}^i{}_{jk}, \underset{(01)}{C}{}^i{}_{jk}, \underset{(11)}{C}{}^i{}_{jk}, \underset{(21)}{C}{}^i{}_{jk}, \underset{(02)}{C}{}^i{}_{jk}, \underset{(12)}{C}{}^i{}_{jk}, \underset{(22)}{C}{}^i{}_{jk})$$

and

$$D\bar{\Gamma}(\bar{N}) = (\underset{(00)}{\bar{L}}{}^i{}_{jk}, \underset{(10)}{\bar{L}}{}^i{}_{jk}, \underset{(20)}{\bar{L}}{}^i{}_{jk}, \underset{(01)}{\bar{C}}{}^i{}_{jk}, \underset{(11)}{\bar{C}}{}^i{}_{jk}, \underset{(21)}{\bar{C}}{}^i{}_{jk}, \underset{(02)}{\bar{C}}{}^i{}_{jk}, \underset{(12)}{\bar{C}}{}^i{}_{jk}, \underset{(22)}{\bar{C}}{}^i{}_{jk})$$

are two N-, respectively $\bar{N}$-linear connections on the differentiable manifold T^2M, then there exists only one system of tensor fields

$$(\underset{1}{A}{}^i{}_j, \underset{2}{A}{}^i{}_j, \underset{(00)}{B}{}^i{}_{jk}, \underset{(10)}{B}{}^i{}_{jk}, \underset{(20)}{B}{}^i{}_{jk}, \underset{(01)}{D}{}^i{}_{jk}, \underset{(11)}{D}{}^i{}_{jk}, \underset{(21)}{D}{}^i{}_{jk}, \underset{(02)}{D}{}^i{}_{jk}, \underset{(12)}{D}{}^i{}_{jk}, \underset{(22)}{D}{}^i{}_{jk}),$$

such that:

$$\begin{cases} \underset{\beta}{\bar{N}}{}^i{}_j = \underset{\beta}{N}{}^i{}_j - \underset{\beta}{A}{}^i{}_j, \\ \underset{(\alpha 0)}{\bar{L}}{}^i{}_{jk} = \underset{(\alpha 0)}{L}{}^i{}_{jk} + \underset{1}{A}{}^l{}_k \underset{(\alpha 1)}{C}{}^i{}_{jl} + \underset{1}{A}{}^l{}_k \underset{1}{N}{}^m{}_l \underset{(\alpha 2)}{C}{}^i{}_{jm} + \underset{2}{A}{}^l{}_k \underset{(\alpha 2)}{C}{}^i{}_{jl} - \underset{(\alpha 0)}{B}{}^i{}_{jk}, \\ \underset{(\alpha 1)}{\bar{C}}{}^i{}_{jk} = \underset{(\alpha 1)}{C}{}^i{}_{jk} + \underset{1}{A}{}^l{}_k \underset{(\alpha 2)}{C}{}^i{}_{jl} - \underset{(\alpha 1)}{D}{}^i{}_{jk}, \\ \underset{(\alpha 2)}{\bar{C}}{}^i{}_{jk} = \underset{(\alpha 2)}{C}{}^i{}_{jk} - \underset{(\alpha 2)}{D}{}^i{}_{jk}, (\alpha = 0, 1, 2;\ \beta = 1, 2). \end{cases} \tag{2.3}$$

Proof. The first equality (2.3) determines uniquely the tensor fields $\underset{\beta}{A}{}^i{}_j, (\beta = 1, 2)$. Since $\underset{(01)}{C}{}^i{}_{jk}, \underset{(11)}{C}{}^i{}_{jk}, \underset{(21)}{C}{}^i{}_{jk}, \underset{(02)}{C}{}^i{}_{jk}, \underset{(12)}{C}{}^i{}_{jk}$ *and* $\underset{(22)}{C}{}^i{}_{jk}$ are tensor fields, the second equation (2.3) determines uniquely the tensor fields $\underset{(00)}{B}{}^i{}_{jk}$, $\underset{(10)}{B}{}^i{}_{jk}$ *and* $\underset{(20)}{B}{}^i{}_{jk}$. Similarly the third and the fourth equation (2.3) determine the tensor fields $\underset{(01)}{D}{}^i{}_{jk}$, $\underset{(11)}{D}{}^i{}_{jk}$, $\underset{(21)}{D}{}^i{}_{jk}$ and $\underset{(02)}{D}{}^i{}_{jk}$, $\underset{(12)}{D}{}^i{}_{jk}$, $\underset{(22)}{D}{}^i{}_{jk}$, respectively.

Conversely, we have

Theorem 2.2. *If*

$$D\Gamma(N) = (\underset{(00)}{L}{}^i{}_{jk}, \underset{(10)}{L}{}^i{}_{jk}, \underset{(20)}{L}{}^i{}_{jk}, \underset{(01)}{C}{}^i{}_{jk}, \underset{(11)}{C}{}^i{}_{jk}, \underset{(21)}{C}{}^i{}_{jk}, \underset{(02)}{C}{}^i{}_{jk}, \underset{(12)}{C}{}^i{}_{jk}, \underset{(22)}{C}{}^i{}_{jk})$$

are local coefficients of an N-linear connection D and

$$(\underset{1}{A}{}^i{}_j, \underset{2}{A}{}^i{}_j, \underset{(00)}{B}{}^i{}_{jk}, \underset{(10)}{B}{}^i{}_{jk}, \underset{(20)}{B}{}^i{}_{jk}, \underset{(01)}{D}{}^i{}_{jk}, \underset{(11)}{D}{}^i{}_{jk}, \underset{(21)}{D}{}^i{}_{jk}, \underset{(02)}{D}{}^i{}_{jk}, \underset{(12)}{D}{}^i{}_{jk}, \underset{(22)}{D}{}^i{}_{jk})$$

is a system of tensor fields, then:

$$D\bar{\Gamma}(\bar{N}) = (\underset{(00)}{\bar{L}}{}^i{}_{jk}, \underset{(10)}{\bar{L}}{}^i{}_{jk}, \underset{(20)}{\bar{L}}{}^i{}_{jk}, \underset{(01)}{\bar{C}}{}^i{}_{jk}, \underset{(11)}{\bar{C}}{}^i{}_{jk}, \underset{(21)}{\bar{C}}{}^i{}_{jk}, \underset{(02)}{\bar{C}}{}^i{}_{jk}, \underset{(12)}{\bar{C}}{}^i{}_{jk}, \underset{(22)}{\bar{C}}{}^i{}_{jk})$$

given by (2.3) are local coefficients of an $\bar{N}$-linear connections $\bar{D}$.

The system of tensor fields

$$(\underset{1}{A}{}^i{}_j, \underset{2}{A}{}^i{}_j, \underset{(00)}{B}{}^i{}_{jk}, \underset{(10)}{B}{}^i{}_{jk}, \underset{(20)}{B}{}^i{}_{jk}, \underset{(01)}{D}{}^i{}_{jk}, \underset{(11)}{D}{}^i{}_{jk}, \underset{(21)}{D}{}^i{}_{jk}, \underset{(02)}{D}{}^i{}_{jk}, \underset{(12)}{D}{}^i{}_{jk}, \underset{(22)}{D}{}^i{}_{jk})$$

are called the **difference** tensor fields of $D\Gamma(N)$ to $D\bar{\Gamma}(\bar{N})$ and the mapping $D\Gamma(N) \to D\bar{\Gamma}(\bar{N})$ given by (2.3) is called **the transformation** of N-linear connection to $\bar{N}$-linear connection, and it is noted by:

$$t(\underset{1}{A}{}^i{}_j, \underset{2}{A}{}^i{}_j, \underset{(00)}{B}{}^i{}_{jk}, \underset{(10)}{B}{}^i{}_{jk}, \underset{(20)}{B}{}^i{}_{jk}, \underset{(01)}{D}{}^i{}_{jk}, \underset{(11)}{D}{}^i{}_{jk}, \underset{(21)}{D}{}^i{}_{jk}, \underset{(02)}{D}{}^i{}_{jk}, \underset{(12)}{D}{}^i{}_{jk}, \underset{(22)}{D}{}^i{}_{jk}).$$

Theorem 2.3. *The set $\mathcal{T}$ of the transformations of N-linear connections to $\bar{N}$- linear connections, together with the composition of mappings isn't a group.*

Proof. Let

$$t(\underset{1}{A}{}^i{}_j, \underset{2}{A}{}^i{}_j, \underset{(00)}{B}{}^i{}_{jk}, \underset{(10)}{B}{}^i{}_{jk}, \underset{(20)}{B}{}^i{}_{jk}, \underset{(01)}{D}{}^i{}_{jk}, \underset{(11)}{D}{}^i{}_{jk}, \underset{(21)}{D}{}^i{}_{jk}, \underset{(02)}{D}{}^i{}_{jk}, \underset{(12)}{D}{}^i{}_{jk}, \underset{(22)}{D}{}^i{}_{jk}) : D\Gamma(N) \to D\bar{\Gamma}(\bar{N})$$

and

$$t(\underset{1}{\bar{A}}{}^i{}_j, \underset{2}{\bar{A}}{}^i{}_j, \underset{(00)}{\bar{B}}{}^i{}_{jk}, \underset{(10)}{\bar{B}}{}^i{}_{jk}, \underset{(20)}{\bar{B}}{}^i{}_{jk}, \underset{(01)}{\bar{D}}{}^i{}_{jk}, \underset{(11)}{\bar{D}}{}^i{}_{jk}, \underset{(21)}{\bar{D}}{}^i{}_{jk}, \underset{(02)}{\bar{D}}{}^i{}_{jk}, \underset{(12)}{\bar{D}}{}^i{}_{jk}, \underset{(22)}{\bar{D}}{}^i{}_{jk}) : D\bar{\Gamma}(\bar{N}) \to D\bar{\bar{\Gamma}}(\bar{\bar{N}})$$

be two transformations from $\mathcal{T}$, given by (2.3).

From (2.3) we have:

$$\underset{\beta}{\bar{\bar{N}}}{}^i{}_j = \underset{\beta}{N}{}^i{}_j - (\underset{\beta}{A}{}^i{}_j + \underset{\beta}{\bar{A}}{}^i{}_j), (\beta = 1, 2).$$

We obtain:

$$\begin{cases} \underset{(\alpha 0)}{\bar{\bar{L}}}{}^i{}_{jk} = \underset{(\alpha 0)}{L}{}^i{}_{jk} + \underset{(\alpha 1)}{C}{}^i{}_{jl}\,(\underset{1}{A}{}^l{}_k + \underset{1}{\bar{A}}{}^l{}_k) + \underset{(\alpha 2)}{C}{}^i{}_{jm}\,\underset{1}{N}{}^m{}_l(\underset{1}{A}{}^l{}_k + \underset{1}{\bar{A}}{}^l{}_k) + \\ \quad + \underset{(\alpha 2)}{C}{}^i{}_{jl}\,(\underset{2}{A}{}^l{}_k + \underset{2}{\bar{A}}{}^l{}_k) + (\underset{(\alpha 2)}{C}{}^i{}_{jm} + \underset{(\alpha 2)}{D}{}^i{}_{jm})\underset{1}{A}{}^m{}_l \underset{1}{\bar{A}}{}^l{}_k - \\ \quad - (\underset{(\alpha 1)}{D}{}^i{}_{jl}\,\underset{1}{\bar{A}}{}^l{}_k + \underset{(\alpha 2)}{D}{}^i{}_{jm}\,\underset{1}{N}{}^m{}_l \underset{1}{A}{}^l{}_k + \underset{(\alpha 2)}{C}{}^i{}_{jm}\,\underset{(1)}{A}{}^m{}_l\,\underset{1}{\bar{A}}{}^l{}_k + \\ \quad + \underset{(\alpha 2)}{D}{}^i{}_{jl}\,\underset{(2)}{\bar{A}}{}^l{}_k - (\underset{(\alpha 0)}{B}{}^i{}_{jk} + \underset{(\alpha 0)}{\bar{B}}{}^i{}_{jk}), \\ \underset{(\alpha 1)}{\bar{\bar{C}}}{}^i{}_{jk} = \underset{(\alpha 1)}{C}{}^i{}_{jk} + \underset{(\alpha 2)}{C}{}^i{}_{jl}\,(\underset{1}{A}{}^l{}_k + \underset{1}{\bar{A}}{}^l{}_k) - (\underset{(\alpha 2)}{D}{}^i{}_{jl}\,\underset{1}{\bar{A}}{}^l{}_k + \underset{(\alpha 1)}{D}{}^i{}_{jk} + \underset{(\alpha 1)}{\bar{D}}{}^i{}_{jk}), \\ \underset{(\alpha 2)}{\bar{\bar{C}}}{}^i{}_{jk} = \underset{(\alpha 2)}{C}{}^i{}_{jk} - (\underset{(\alpha 2)}{D}{}^i{}_{jk} + \underset{(\alpha 2)}{\bar{D}}{}^i{}_{jk}), (\alpha = 0, 1, 2). \end{cases} \tag{2.4}$$

So $\underset{(\alpha 0)}{\bar{\bar{L}}}{}^i{}_{jk}$, $(\alpha = 0, 1, 2)$ hasn't the form (2.3). Result that the mapping of two transformations from $\mathcal{T}$, isn't a transformation from $\mathcal{T}$, so $\mathcal{T}$, together with the composition of mappings isn't a group.

Remark 2.1. If we consider $\underset{\beta}{A}{}^i{}_j = 0, (\beta = 1, 2)$, in (2.3) we obtain the set $\mathcal{T}_N$ of transformations of N-linear connections, having the same nonlinear connection N:

$$\mathcal{T}_N = \{t(0, 0, \underset{(00)}{B}{}^i{}_{jk}, \underset{(10)}{B}{}^i{}_{jk}, \underset{(20)}{B}{}^i{}_{jk}, \underset{(01)}{D}{}^i{}_{jk}, \underset{(11)}{D}{}^i{}_{jk}, \underset{(21)}{D}{}^i{}_{jk}, \underset{(02)}{D}{}^i{}_{jk}, \underset{(12)}{D}{}^i{}_{jk}, \underset{(22)}{D}{}^i{}_{jk}) \in \mathcal{T}\}.$$

We have:

Theorem 2.4. *The set $\mathcal{T}_N$ of the transformations of N-linear connections to N-linear connections, together with the composition of mappings is a group. This group, $\mathcal{T}_N$, acts effectively and transitivelly on the set of N-linear connections.*

Proof. Let

$$t(0,0,\underset{(00)}{B^i}_{jk},\underset{(10)}{B^i}_{jk},\underset{(20)}{B^i}_{jk},\underset{(01)}{D^i}_{jk},\underset{(11)}{D^i}_{jk},\underset{(21)}{D^i}_{jk},\underset{(02)}{D^i}_{jk},\underset{(12)}{D^i}_{jk},\underset{(22)}{D^i}_{jk}):D\Gamma(N)\to D\bar{\Gamma}(N)$$

be a transformation from $\mathcal{T}_N$ given by (2.5):

$$\begin{cases} \underset{\beta}{\bar{N}^i}_j = \underset{\beta}{N^i}_j, \\ \underset{(\alpha 0)}{\bar{L}^i}_{jk} = \underset{(\alpha 0)}{L^i}_{jk} - \underset{(\alpha 0)}{B^i}_{jk}, \\ \underset{(\alpha 1)}{\bar{C}^i}_{jk} = \underset{(\alpha 1)}{C^i}_{jk} - \underset{(\alpha 1)}{D^i}_{jk}, \\ \underset{(\alpha 2)}{\bar{C}^i}_{jk} = \underset{(\alpha 2)}{C^i}_{jk} - \underset{(\alpha 2)}{D^i}_{jk}, (\alpha = 0,1,2;\ \beta = 1,2). \end{cases} \tag{2.5}$$

The composition of two transformations from $\mathcal{T}_N$ is a transformation from $\mathcal{T}_N$, given by:

$$t(0,0,\underset{(00)}{\bar{B}^i}_{jk},\underset{(10)}{\bar{B}^i}_{jk},\underset{(20)}{\bar{B}^i}_{jk},\underset{(01)}{\bar{D}^i}_{jk},\underset{(11)}{\bar{D}^i}_{jk},\underset{(21)}{\bar{D}^i}_{jk},\underset{(02)}{\bar{D}^i}_{jk},\underset{(12)}{\bar{D}^i}_{jk},\underset{(22)}{\bar{D}^i}_{jk})$$

$$\circ t(0,0,\underset{(00)}{B^i}_{jk},\underset{(10)}{B^i}_{jk},\underset{(20)}{B^i}_{jk},\underset{(01)}{D^i}_{jk},\underset{(11)}{D^i}_{jk},\underset{(21)}{D^i}_{jk},\underset{(02)}{D^i}_{jk},\underset{(12)}{D^i}_{jk},\underset{(22)}{D^i}_{jk})$$

$$= t(0,0,\underset{(00)}{B^i}_{jk} + \underset{(00)}{\bar{B}^i}_{jk},\underset{(10)}{B^i}_{jk} + \underset{(10)}{\bar{B}^i}_{jk},\underset{(20)}{B^i}_{jk} + \underset{(20)}{\bar{B}^i}_{jk},\underset{(01)}{D^i}_{jk} + \underset{(01)}{\bar{D}^i}_{jk},\underset{(11)}{D^i}_{jk} + \underset{(11)}{\bar{D}^i}_{jk},$$

$$\underset{(21)}{D^i}_{jk} + \underset{(21)}{\bar{D}^i}_{jk},\underset{(02)}{D^i}_{jk} + \underset{(02)}{\bar{D}^i}_{jk},\underset{(12)}{D^i}_{jk} + \underset{(12)}{\bar{D}^i}_{jk},\underset{(22)}{D^i}_{jk} + \underset{(22)}{\bar{D}^i}_{jk}).$$

The inverse of a transformation from $\mathcal{T}_N$ is the transformation:

$$t(0,0,-\underset{(00)}{B^i}_{jk},-\underset{(10)}{B^i}_{jk},-\underset{(20)}{B^i}_{jk},-\underset{(01)}{D^i}_{jk},-\underset{(11)}{D^i}_{jk},-\underset{(21)}{D^i}_{jk},-\underset{(02)}{D^i}_{jk},-\underset{(12)}{D^i}_{jk},-\underset{(22)}{D^i}_{jk}):$$

$$D\Gamma(N)\to D\bar{\Gamma}(N).$$

The transformation (2.5) preserves all the N-linear connections D if

$$\underset{(\alpha 0)}{B^i}_{jk} = \underset{(\alpha 0)}{D^i}_{jk} = 0, (\alpha = 0,1,2).$$

Therefore $\mathcal{T}_N$ acts effectively on the set of N-linear connections. From the Theorem 2.1. results that $\mathcal{T}_N$ acts transitively on this set. □

Let us consider:

$$\mathcal{T}_{NL} = \{t(0,0,0,0,0, \underset{(01)}{D^i{}_{jk}}, \underset{(11)}{D^i{}_{jk}}, \underset{(21)}{D^i{}_{jk}}, \underset{(02)}{D^i{}_{jk}}, \underset{(12)}{D^i{}_{jk}}, \underset{(22)}{D^i{}_{jk}}) \in \mathcal{T}_N\},$$

$$\mathcal{T}_{N\underset{(1)}{C}} = \{t(0,0, \underset{(00)}{B^i{}_{jk}}, \underset{(10)}{B^i{}_{jk}}, \underset{(20)}{B^i{}_{jk}}, 0,0,0, \underset{(02)}{D^i{}_{jk}}, \underset{(12)}{D^i{}_{jk}}, \underset{(22)}{D^i{}_{jk}}) \in \mathcal{T}_N\},$$

$$\mathcal{T}_{N\underset{(2)}{C}} = \{t(0,0, \underset{(00)}{B^i{}_{jk}}, \underset{(10)}{B^i{}_{jk}}, \underset{(20)}{B^i{}_{jk}}, \underset{(01)}{D^i{}_{jk}}, \underset{(11)}{D^i{}_{jk}}, \underset{(21)}{D^i{}_{jk}}, 0,0,0) \in \mathcal{T}_N\},$$

$$\mathcal{T}_{N\underset{(1)}{C}\underset{(2)}{C}} = \{t(0,0, \underset{(00)}{B^i{}_{jk}}, \underset{(10)}{B^i{}_{jk}}, \underset{(20)}{B^i{}_{jk}}, 0,0,0,0,0,0) \in \mathcal{T}_N\}.$$

Proposition 2.2. *$\mathcal{T}_{NL}, \mathcal{T}_{N\underset{(1)}{C}}, \mathcal{T}_{N\underset{(2)}{C}}$ and $\mathcal{T}_{N\underset{(1)}{C}\underset{(2)}{C}}$ are abelian subgroups of $\mathcal{T}_N$.*

Proposition 2.3. *The group $\mathcal{T}_N$ preserves the nonlinear connection N; $\mathcal{T}_{NL}$ preserves the nonlinear connection N and the components $\underset{(\alpha 0)}{L}$, $(\alpha = 0,1,2)$ of the local coefficients $D\Gamma(N)$; $\mathcal{T}_{N\underset{(1)}{C}}$ preserves the nonlinear connection N and the components $\underset{(\alpha 1)}{C}$, $(\alpha = 0,1,2)$ of the local coefficients $D\Gamma(N)$; $\mathcal{T}_{N\underset{(2)}{C}}$ preserves the nonlinear connection N and the components $\underset{(\alpha 2)}{C}$, $(\alpha = 0,1,2)$ of the local coefficients $D\Gamma(N)$ and $\mathcal{T}_{N\underset{(1)}{C}\underset{(2)}{C}}$ preserves the nonlinear connection N and the components $\underset{(\alpha 1)}{C}$ and $\underset{(\alpha 2)}{C}$, $(\alpha = 0,1,2)$ of the local coefficients $D\Gamma(N)$.*

3. The transformations of the d-tensors of torsion and curvature in $\mathcal{T}_N$

In the following, we shall study the Abelian group $\mathcal{T}_N$. Its elements are the transformations $t : D\Gamma(N) \to D\bar{\Gamma}(N)$ given by

$$\begin{cases} \underset{\beta}{\bar{N}^i{}_j} = \underset{\beta}{N^i{}_j}, \\ \underset{(\alpha 0)}{\bar{L}^i{}_{jk}} = \underset{(\alpha 0)}{L^i{}_{jk}} - \underset{(\alpha 0)}{B^i{}_{jk}}, \\ \underset{(\alpha 1)}{\bar{C}^i{}_{jk}} = \underset{(\alpha 1)}{C^i{}_{jk}} - \underset{(\alpha 1)}{D^i{}_{jk}}, \\ \underset{(\alpha 2)}{\bar{C}^i{}_{jk}} = \underset{(\alpha 2)}{C^i{}_{jk}} - \underset{(\alpha 2)}{D^i{}_{jk}}, (\alpha = 0,1,2;\ \beta = 1,2). \end{cases} \tag{3.1}$$

Firstly, we shall study the transformations of the d-tensors of torsion of $D\Gamma(N)$ (see, (7.2) and (7.5), [1]). We obtain:

Proposition 3.1. *The transformations of the Abelian group $\mathcal{T}_N$, given by (3.1) lead to the transformations of the d-tensors of torsion in the following way:*

$$\underset{(0\beta)}{\bar{R}^i{}_{jk}} = \underset{(0\beta)}{R^i{}_{jk}} \;, \tag{3.2}$$

$$\underset{(0)}{\overset{\alpha}{\bar{T}}{}^i{}_{jk}} = \underset{(0)}{\overset{\alpha}{T}{}^i{}_{jk}} + (\underset{(\alpha 0)}{B^i{}_{kj}} - \underset{(\alpha 0)}{B^i{}_{jk}}), \tag{3.3}$$

$$\underset{(\beta)}{\overset{\alpha}{\bar{S}}{}^i{}_{jk}} = \underset{(\beta)}{\overset{\alpha}{S}{}^i{}_{jk}} + (\underset{(\alpha\beta)}{D^i{}_{kj}} - \underset{(\alpha\beta)}{D^i{}_{jk}}), \tag{3.4}$$

$$\underset{(21)}{\bar{Q}^i{}_{jk}} = \underset{(21)}{Q^i{}_{jk}} - \underset{(12)}{D^i{}_{jk}} \;, \tag{3.5}$$

$$\underset{(22)}{\overset{\alpha}{\bar{Q}}{}^i{}_{jk}} = \underset{(22)}{\overset{\alpha}{Q}{}^i{}_{jk}} + \underset{(\alpha 1)}{D^i{}_{kj}} \;, \tag{3.6}$$

$$\underset{(12)}{\bar{S}^i{}_{jk}} = \underset{(12)}{S^i{}_{jk}} \;, \tag{3.7}$$

$$\underset{(\beta\beta)}{\overset{\alpha}{\bar{P}}{}^i{}_{jk}} = \underset{(\beta\beta)}{\overset{\alpha}{P}{}^i{}_{jk}} + \underset{(\alpha 0)}{B^i{}_{kj}} \;, \tag{3.8}$$

$$\underset{(\beta 0)}{\bar{P}^i{}_{jk}} = \underset{(\beta 0)}{P^i{}_{jk}} - \underset{(0\beta)}{D^i{}_{jk}} \;, \tag{3.9}$$

$$\underset{(12)}{\bar{P}^i{}_{jk}} = \underset{(12)}{P^i{}_{jk}} \;, \tag{3.10}$$

$$\underset{(21)}{\bar{P}^i{}_{jk}} = \underset{(21)}{P^i{}_{jk}} \;, \quad (\alpha = 0,1,2;\ \beta = 1,2). \tag{3.11}$$

Now, we shall study the transformations of the d-tensors of curvature of $D\Gamma(N)$ (see, (7.11),[1]). We get:

Proposition 3.2. *The transformations of the Abelian group $\mathcal{T}_N$, given by (3.1) lead to the transformations of the d-tensors of curvature in the following way:*

$$\underset{(0\alpha)}{\bar{R}_h{}^i{}_{jk}} = \underset{(0\alpha)}{R_h{}^i{}_{jk}} - \underset{(\alpha 1)}{D^i{}_{hs}} \underset{(01)}{R^s{}_{jk}} - \underset{(\alpha 2)}{D^i{}_{hs}} \underset{(02)}{R^s{}_{jk}} - \underset{(\alpha 0)}{B^i{}_{hs}} \underset{(0)}{\overset{\alpha}{T}{}^s{}_{jk}} + \tag{3.12}$$

$$+\mathcal{A}_{jk}\{-\underset{(\alpha 0)}{B^i{}_{hj|\alpha k}} + \underset{(\alpha 0)}{B^s{}_{hj}} \underset{(\alpha 0)}{B^i{}_{sk}}\},$$

$$\underset{(1\alpha)}{\bar{P}_h{}^i{}_{jk}} = \underset{(1\alpha)}{P_h{}^i{}_{jk}} - \underset{(\alpha 1)}{D^i{}_{hs}} \underset{(11)}{\overset{\alpha}{P}{}^s{}_{jk}} - \underset{(\alpha 2)}{D^i{}_{hs}} \underset{(12)}{P^s{}_{jk}} - \underset{(\alpha 0)}{B^i{}_{hs}} \underset{(\alpha 1)}{C^s{}_{jk}} + \tag{3.13}$$

$$+ \underset{(\alpha 0)}{L}{}^{s}{}_{kj} \underset{(\alpha 1)}{D}{}^{i}{}_{hs} - \underset{(\alpha 0)}{B}{}^{i}{}_{hj} \overset{(1)}{|}_{\alpha k} + \underset{(\alpha 1)}{D}{}^{i}{}_{hk|\alpha j} + \underset{(\alpha 0)}{B}{}^{s}{}_{hj} \underset{(\alpha 1)}{D}{}^{i}{}_{sk} -$$

$$- \underset{(\alpha 1)}{D}{}^{s}{}_{hk} \underset{(\alpha 0)}{B}{}^{i}{}_{sj} + \underset{(\alpha 1)}{C}{}^{i}{}_{hs} \underset{(\alpha 0)}{B}{}^{s}{}_{kj} - \underset{(\alpha 1)}{D}{}^{i}{}_{hs} \underset{(\alpha 0)}{B}{}^{s}{}_{kj} \,,$$

$$\underset{(2\alpha)}{\bar{P}}{}_{h}{}^{i}{}_{jk} = \underset{(2\alpha)}{P}{}_{h}{}^{i}{}_{jk} - \underset{(\alpha 1)}{D}{}^{i}{}_{hs} \underset{(21)}{P}{}^{s}{}_{jk} - \underset{(\alpha 2)}{D}{}^{i}{}_{hs} \underset{(22)}{\overset{\alpha}{P}}{}^{s}{}_{jk} - \underset{(\alpha 0)}{B}{}^{i}{}_{hs} \underset{(\alpha 2)}{C}{}^{s}{}_{jk} + \tag{3.14}$$

$$+ \underset{(\alpha 0)}{L}{}^{s}{}_{kj} \underset{(\alpha 2)}{D}{}^{i}{}_{hs} - \underset{(\alpha 0)}{B}{}^{i}{}_{hj} \overset{(2)}{|}_{\alpha k} + \underset{(\alpha 2)}{D}{}^{i}{}_{hk|\alpha j} + \underset{(\alpha 0)}{B}{}^{s}{}_{hj} \underset{(\alpha 2)}{D}{}^{i}{}_{sk} -$$

$$- \underset{(\alpha 2)}{D}{}^{s}{}_{hk} \underset{(\alpha 0)}{B}{}^{i}{}_{sj} + \underset{(\alpha 2)}{C}{}^{i}{}_{hs} \underset{(\alpha 0)}{B}{}^{s}{}_{kj} - \underset{(\alpha 2)}{D}{}^{i}{}_{hs} \underset{(\alpha 0)}{B}{}^{s}{}_{kj} \,,$$

$$\underset{(2\alpha)}{\bar{Q}}{}_{h}{}^{i}{}_{jk} = \underset{(2\alpha)}{Q}{}_{h}{}^{i}{}_{jk} - \underset{(\alpha 2)}{C}{}^{s}{}_{jk} \underset{(\alpha 1)}{D}{}^{i}{}_{hs} + \underset{(\alpha 1)}{C}{}^{s}{}_{kj} \underset{(\alpha 2)}{D}{}^{i}{}_{hs} - \underset{(\alpha 1)}{D}{}^{i}{}_{hj} \overset{(2)}{|}_{\alpha k} + \tag{3.15}$$

$$+ \underset{(\alpha 2)}{D}{}^{i}{}_{hk} \overset{(1)}{|}_{\alpha j} + \underset{(\alpha 1)}{D}{}^{s}{}_{hj} \underset{(\alpha 2)}{D}{}^{i}{}_{sk} - \underset{(\alpha 2)}{D}{}^{s}{}_{hk} \underset{(\alpha 1)}{D}{}^{i}{}_{sj} - \underset{(\alpha 2)}{D}{}^{i}{}_{hs} \underset{(21)}{P}{}^{s}{}_{jk} \,,$$

$$\underset{(\beta\alpha)}{\bar{S}}{}_{h}{}^{i}{}_{jk} = \underset{(\beta\alpha)}{S}{}_{h}{}^{i}{}_{jk} - \underset{(\alpha\beta)}{D}{}^{i}{}_{hs} \underset{(\beta)}{\overset{\alpha}{S}}{}^{s}{}_{jk} + \mathcal{A}_{jk}\{- \underset{(\alpha\beta)}{D}{}^{i}{}_{hj} \overset{(\beta)}{|}_{\alpha k} + \tag{3.16}$$

$$+ \underset{(\alpha\beta)}{D}{}^{s}{}_{hj} \underset{(\alpha\beta)}{D}{}^{i}{}_{sk} \} - \underset{(\alpha 2)}{D}{}^{i}{}_{hs} \underset{(\beta 2)}{R}{}^{s}{}_{jk} \,, \quad (\alpha = 0,1,2;\ \beta = 1,2),$$

where $\mathcal{A}_{ij}$ denotes the alternate summation.

We shall consider the tensor fields:

$$\underset{(0\alpha)}{\mathbb{K}}{}_{h}{}^{i}{}_{jk} = \underset{(0\alpha)}{R}{}_{h}{}^{i}{}_{jk} - \underset{(\alpha 1)}{C}{}^{i}{}_{hs} \underset{(01)}{R}{}^{s}{}_{jk} - \underset{(\alpha 2)}{C}{}^{i}{}_{hs} \underset{(02)}{R}{}^{s}{}_{jk} \,, \tag{3.17}$$

$$\underset{(1\alpha)}{\mathbb{P}}{}_{h}{}^{i}{}_{jk} = \mathcal{A}_{jk}\{ \underset{(1\alpha)}{P}{}_{h}{}^{i}{}_{jk} - \underset{(\alpha 1)}{C}{}^{i}{}_{hs} \frac{\delta \underset{1}{N}{}^{s}_{j}}{\delta y^{(1)k}} - \underset{(\alpha 2)}{C}{}^{i}{}_{hs} (\underset{1}{N}{}^{s}{}_{m} \frac{\delta \underset{1}{N}{}^{m}_{j}}{\delta y^{(1)k}} + \tag{3.18}$$

$$+ \frac{\delta \underset{2}{N}{}^{s}_{j}}{\delta y^{(1)k}} - \frac{\delta \underset{2}{N}{}^{s}_{k}}{\delta y^{(1)j}})\},$$

$$\underset{(2\alpha)}{\mathbb{P}}{}_{h}{}^{i}{}_{jk} = \mathcal{A}_{jk}\{ \underset{(2\alpha)}{P}{}_{h}{}^{i}{}_{jk} - \underset{(\alpha 1)}{C}{}^{i}{}_{hs} \frac{\partial \underset{1}{N}{}^{s}_{j}}{\delta y^{(2)k}} - \underset{(\alpha 2)}{C}{}^{i}{}_{hs} (\underset{1}{N}{}^{s}{}_{m} \frac{\partial \underset{1}{N}{}^{m}_{j}}{\delta y^{(2)k}} + + \frac{\partial \underset{2}{N}{}^{s}_{j}}{\delta y^{(2)k}})\}, \tag{3.19}$$

$$\underset{(2\alpha)}{\mathbb{Q}}{}_{h}{}^{i}{}_{jk} = \mathcal{A}_{jk}\{ \underset{(2\alpha)}{Q}{}_{h}{}^{i}{}_{jk} + \underset{(\alpha 2)}{C}{}^{i}{}_{hs} \frac{\partial \underset{1}{N}{}^{s}{}_{j}}{\delta y^{(2)k}} \}, \tag{3.20}$$

$$\underset{(\beta\alpha)}{\mathbb{S}}{}_{h}{}^{i}{}_{jk} = \underset{(\beta\alpha)}{S}{}_{h}{}^{i}{}_{jk} - \underset{(\alpha 2)}{C}{}^{i}{}_{hs} \underset{(\beta 2)}{R}{}^{s}{}_{jk} \,, (\alpha = 0,1,2;\ \beta = 1,2). \tag{3.21}$$

Proposition 3.3. *By a transformation of the Abelian group* $\mathcal{T}_N$*, given by (3.1), the tensor fields:* $\underset{(0\alpha)}{\mathbb{K}}{}_{h\ jk}^{\ i}$, $\underset{(1\alpha)}{\mathbb{P}}{}_{h\ jk}^{\ i}$, $\underset{(2\alpha)}{\mathbb{P}}{}_{h\ jk}^{\ i}$, $\underset{(2\alpha)}{\mathbb{Q}}{}_{h\ jk}^{\ i}$ *and* $\underset{(\beta\alpha)}{\mathbb{S}}{}_{h\ jk}^{\ i}$, $(\alpha = 0, 1, 2;\ \beta = 1, 2)$ *are transformed according to the following laws:*

$$\underset{(0\alpha)}{\bar{\mathbb{K}}}{}_{h\ jk}^{\ i} = \underset{(0\alpha)}{\mathbb{K}}{}_{h\ jk}^{\ i} - \underset{(\alpha 0)}{B^i}{}_{hs} \underset{(0)}{\overset{\alpha}{T}}{}^{s}_{\ jk} + \mathcal{A}_{jk}\{-\underset{(\alpha 0)}{B^i}{}_{hj|\alpha k} + \underset{(\alpha 0)}{B^s}{}_{kj} \underset{(\alpha 0)}{B^i}{}_{sk}\}, \tag{3.22}$$

$$\underset{(\beta\alpha)}{\bar{\mathbb{P}}}{}_{h\ jk}^{\ i} = \underset{(\beta\alpha)}{\mathbb{P}}{}_{h\ jk}^{\ i} - 2\underset{(\alpha\beta)}{D^i}{}_{hs} \underset{(0)}{\overset{\alpha}{T}}{}^{s}_{\ jk} - \underset{(\alpha 0)}{B^i}{}_{hs} \underset{(\beta)}{\overset{\alpha}{S}}{}^{s}_{\ jk} + \mathcal{A}_{jk}\{-\underset{(\alpha 0)}{B^i}{}_{hj} \overset{(\beta)}{|}_{\alpha k} - \tag{3.23}$$
$$-\underset{(\alpha\beta)}{D^i}{}_{hj|\alpha k} + \underset{(\alpha 0)}{B^s}{}_{hj} \underset{(\alpha\beta)}{D^i}{}_{sk} + \underset{(\alpha\beta)}{D^s}{}_{hj} \underset{(\alpha 0)}{B^i}{}_{sk} + \underset{(\alpha\beta)}{D^i}{}_{hs} \underset{(\alpha 0)}{B^s}{}_{jk} - \underset{(\alpha\beta)}{C^i}{}_{hs} \underset{(\alpha 0)}{B^s}{}_{jk}\},$$

$$\underset{(2\alpha)}{\bar{\mathbb{Q}}}{}_{h\ jk}^{\ i} = \underset{(2\alpha)}{\mathbb{Q}}{}_{h\ jk}^{\ i} + \underset{(2)}{\overset{\alpha}{S}}{}^{s}_{\ jk} \underset{(\alpha 1)}{D^i}{}_{hs} - \underset{(1)}{\overset{\alpha}{S}}{}^{s}_{\ jk} \underset{(\alpha 2)}{D^i}{}_{hs} \tag{3.24}$$
$$+\mathcal{A}_{jk}\{\underset{(\alpha 1)}{D^i}{}_{hj} \overset{(2)}{|}_{\alpha k} + \underset{(\alpha 2)}{D^i}{}_{hk} \overset{(1)}{|}_{\alpha j} + + \underset{(\alpha 1)}{D^s}{}_{hj} \underset{(\alpha 2)}{D^i}{}_{sk} - \underset{(\alpha 2)}{D^s}{}_{hk} \underset{(\alpha 1)}{D^i}{}_{sj}\},$$

$$\underset{(\beta\alpha)}{\bar{\mathbb{S}}}{}_{h\ jk}^{\ i} = \underset{(\beta\alpha)}{\mathbb{S}}{}_{h\ jk}^{\ i} - \underset{(\alpha\beta)}{D^i}{}_{hs} \underset{(\beta)}{\overset{\alpha}{S}}{}^{s}_{\ jk} + \mathcal{A}_{jk}\{-\underset{(\alpha\beta)}{D^i}{}_{hj} \overset{(\beta)}{|}_{\alpha k} + \underset{(\alpha\beta)}{D^s}{}_{hj} \underset{(\alpha\beta)}{D^i}{}_{sk}\}, \tag{3.25}$$

$(\alpha = 0, 1, 2;\ \beta = 1, 2)$.

The transformations for the coefficients of an N-linear connection on the tangent bundle of order two, T^2M, by a transformation of a nonlinear connection in T^2M, together with the transformation laws of the torsion and curvature tensor fields, with respect to the transformations of the group $\mathcal{T}_N$, given in the present paper are necessary for the study of a important subgroup of the group $\mathcal{T}_N$: the group of transformations of the metric semi-symmetric N-linear connections in T^2M, $\overset{ms}{\mathcal{T}}_N$. This study is in our attention.

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EXPONENTIAL INSTABILITY OF SKEW-EVOLUTION SEMIFLOWS IN BANACH SPACES

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Abstract. This paper emphasizes a couple of characterizations for the exponential instability property of skew-evolution semiflows in Banach spaces, defined by means of evolution semiflows and evolution cocycles. Some Datko type results for this asymptotic behavior are proved. There is provided a unified treatment for the uniform case.

1. Introduction

The study of the asymptotic behaviors of skew-product semiflows, that has witnessed lately an impressive development, has been used in the theory of evolution equations in infinite dimensional spaces. It was essential that the theory was approached from point of view of asymptotic properties for the evolution semigroup associated to the skew-product semiflows. Some results on the instability of skew-product flows can be found in [2].

A particular concept of skew-evolution semiflow introduced by us in [3] is considered to be more interesting for the study of evolution equations connected to the theory of evolution operators. Some asymptotic behaviors for skew-evolution semiflows have been presented in [4].

In this paper we emphasize the property of exponential instability for skew-evolution semiflows defined by means of evolution semiflows and evolution cocycles.

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We remark that Theorem 3.1 and Theorem 3.2 of this paper are approaches for the uniform exponential instability property, extending Theorem 11 from [1] concerning the case of the property of uniform exponential stability. The skew-evolution semiflows considered in this paper are not necessary strongly continuous.

2. **Notations and definitions. Preliminary results**

Let us consider X a metric space, V a Banach space, $\mathcal{B}(V)$ the space of all bounded operators from V into itself. We denote $\mathcal{T} = \left\{(t,t_0) \in \mathbb{R}_+^2 : t \geq t_0\right\}$, respectively $Y = X \times V$ and the norm of vectors on V and operators on $\mathcal{B}(V)$ is denoted by $\|\cdot\|$. Let I be the identity operator on V.

Definition 2.1. A mapping $\varphi : \mathcal{T} \times X \to X$ is called *evolution semiflow* on X if it satisfies the following properties

(s_1) $\varphi(t,t,x) = x,\ \forall (t,x) \in \mathbb{R}_+ \times X$

(s_2) $\varphi(t,s,\varphi(s,t_0,x)) = \varphi(t,t_0,x),\ \forall (t,s),(s,t_0) \in \mathcal{T},\ \forall x \in X.$

Definition 2.2. A mapping $\Phi : \mathcal{T} \times X \to \mathcal{B}(V)$ that satisfies the following properties

(c_1) $\Phi(t,t,x) = I,\ \forall t \geq 0,\ \forall x \in X$

(c_2) $\Phi(t,s,\varphi(s,t_0,x))\Phi(s,t_0,x) = \Phi(t,t_0,x),\ \forall (t,s),(s,t_0) \in \mathcal{T},\ \forall x \in X$

(c_3) there exists a nondecreasing function $f : \mathbb{R}_+ \to \mathbb{R}_+^*$ such that

$$\|\Phi(t,t_0,x)\| \leq f(t-t_0),\ \forall t \geq t_0 \geq 0,\ \forall x \in X \tag{2.1}$$

is called *evolution cocycle* over the evolution semiflow φ.

Definition 2.3. A function $\xi : \mathcal{T} \times Y \to Y$ defined by

$$\xi(t,s,x,v) = (\varphi(t,s,x), \Phi(t,s,x)v),\ \forall (t,s,x,v) \in \mathcal{T} \times Y \tag{2.2}$$

where Φ is an evolution cocycle over the evolution semiflow φ, is called *skew-evolution semiflow* on Y.

Example 2.1. Let $f : \mathbb{R} \to \mathbb{R}_+$ be a function which is nondecreasing on $(-\infty,0)$ and decreasing on $(0,\infty)$ such that there exist

$$\lim_{t\to\pm\infty} f(t) = l > 0.$$

We consider the metric space

$$\mathcal{C}(\mathbb{R},\mathbb{R}) = \{h : \mathbb{R} \to \mathbb{R} \mid h \text{ continuous}\},$$

with the topology of uniform convergence on compact subsets of $\mathbb{R}$.

Let X be the closure in $\mathcal{C}(\mathbb{R},\mathbb{R})$ of the set of all functions $f_t,\ t \in \mathbb{R}$, where $f_t(\tau) = f(t+\tau),\ \forall \tau \in \mathbb{R}$. Then X is a metric space and the mapping

$$\varphi : \mathcal{T} \times X \to X,\ \varphi(t,s,x) = x_{t-s}$$

is an evolution semiflow on X.

Let $V = \mathbb{R}^2$ be a Banach space with the norm $\|(v_1,v_2)\| = |v_1| + |v_2|$. The mapping $\Phi : \mathcal{T} \times X \to \mathcal{B}(V)$ given by

$$\Phi(t,s,x)(v_1,v_2) = \left(e^{\alpha_1 \int_s^t x(\tau-s)d\tau} v_1, e^{\alpha_2 \int_s^t x(\tau-s)d\tau} v_2\right)$$

where $(\alpha_1,\alpha_2) \in \mathbb{R}^2$ is fixed, is an evolution cocycle and $\xi = (\varphi,\Phi)$ is a skew-evolution semiflow on Y.

We introduce a particular class of skew-evolution semiflows in the next

Definition 2.4. A skew-evolution semiflow $\xi = (\varphi,\Phi)$ has *uniform exponential decay* if there exist $N > 1$ and $\omega > 0$ such that

$$\|\Phi(s,t_0,x)v\| \le N e^{\omega(t-s)} \|\Phi(t,t_0,x)v\|,\ \forall t \ge s \ge t_0 \ge 0,\ \forall (x,v) \in Y. \tag{2.3}$$

A characterization of the uniform exponential decay is given by

Proposition 2.1. *The skew-evolution semiflow $\xi = (\varphi,\Phi)$ has uniform exponential decay if and only if there exists a decreasing function $g : [0,\infty) \to (0,\infty)$ with the properties $\lim\limits_{t\to\infty} g(t) = 0$ and*

$$\|\Phi(t,t_0,x)v\| \ge g(t-t_0)\|v\|,\ \forall t \ge t_0 \ge 0,\ \forall (x,v) \in Y.$$

Proof. Necessity. It is a simple verification.

Sufficiency. According to the property of function g, there exists a constant $\lambda > 0$ such that $g(\lambda) < 1$. For all $t \ge t_0 \ge 0$, there exist $n \in \mathbb{N}$ and $r \in [0,\lambda)$ such that

$$t - t_0 = n\lambda + r.$$

Following inequalities

$$\|\Phi(t,t_0,x)v\| \geq g(r)\,\|\Phi(t_0+n\lambda,t_0,x)v\| \geq ... \geq g(\lambda)^{n+1}\,\|v\| \geq N_1e^{-\omega(t-t_0)}\,\|v\|$$

hold for all $(x,v)\in Y$, where we have denoted

$$N_1 = g(\lambda) \text{ and } \omega = -\lambda^{-1}\ln g(\lambda).$$

The property of uniform exponential decay for ξ is thus obvious. □

Definition 2.5. A skew-evolution semiflow $\xi = (\varphi, \Phi)$ is said to be *uniformly instable* if there exists $N > 1$ such that

$$\|\Phi(s,t_0,x)v\| \leq N\,\|\Phi(t,t_0,x)v\|\,,\ \forall t\geq s\geq t_0\geq 0,\ \forall (x,v)\in Y.$$

We can obtain a characterization of the former property as in the next

Proposition 2.2. *A skew-evolution semiflow $\xi = (\varphi, \Phi)$ with uniform exponential decay is uniformly instable if there exists $M > 1$ such that*

$$M\,\|\Phi(t,t_0,x)v\| \geq \|\Phi(s,t_0,x)v\|\,,\ \forall t\geq s+1 > s\geq t_0\geq 0,\ \forall (x,v)\in Y.$$

Proof. Let us consider a function g is given as in Proposition 2.1. Then there exists $\lambda > 1$ such that $g(\lambda) < 1$.

Let $s \geq 0$. For all $t\in [s, s+1)$, by the same result, we obtain following inequalities

$$\|\Phi(t,t_0,x)v\| \geq g(t-s)\,\|\Phi(s,t_0,x)v\| \geq g(\lambda)\,\|\Phi(s,t_0,x)v\|\,.$$

Hence, if we denote

$$N = \max\left\{M, g(\lambda)^{-1}\right\} > 1,$$

the property of uniform instability for ξ has been proved. □

Definition 2.6. A skew-evolution semiflow $\xi = (\varphi, \Phi)$ is called *uniformly exponentially instable* if there exist $N > 1$ and $\nu > 0$ with the property

$$\|\Phi(s,t_0,x)v\| \leq Ne^{-\nu(t-s)}\,\|\Phi(t,t_0,x)v\|\,,\ \forall t\geq s\geq t_0\geq 0,\ \forall (x,v)\in Y. \quad (2.4)$$

Example 2.2. We consider the metric space X and an evolution semiflow on X defined as in Example 2.1.

Let $V = \mathbb{R}$. We consider $\Phi : \mathcal{T} \times X \to \mathcal{B}(V)$ given by

$$\Phi(t, t_0, x)v = e^{\int_{t_0}^{t} x(\tau - t_0)d\tau} v$$

which is an evolution cocycle. Then the skew-evolution semiflow $\xi = (\varphi, \Phi)$ is uniformly exponentially instable with $N = 1$ and $\nu = l > 0$.

The following result are characterizations for the property of uniform exponential instability, by means of other asymptotic properties and, also, of a special class of skew-evolution semiflow.

Proposition 2.3. *A skew-evolution semiflow $\xi = (\varphi, \Phi)$ with uniform exponential decay is uniformly exponentially instable if and only if there exists a decreasing function $h : [0, \infty) \to (0, \infty)$ with property $\lim_{t\to\infty} h(t) = 0$ such that*

$$\|v\| \leq h(t - t_0) \|\Phi(t, t_0, x)v\|, \ \forall t \geq t_0 \geq 0, \ \forall (x, v) \in Y.$$

Proof. It is similar with the proof of Proposition 2.1. □

Proposition 2.4. *A skew-evolution semiflow $\xi = (\varphi, \Phi)$ has uniform exponential decay if and only if there exists a constant $\alpha > 0$ such that the a skew-evolution semiflow $\xi_{-\alpha} = (\varphi, \Phi_{-\alpha})$, where $\Phi_{-\alpha}(t, t_0, x) = e^{\alpha(t-t_0)}\Phi(t, t_0, x)$, $(t, t_0) \in \mathcal{T}$, $x \in X$, is uniformly exponentially instable.*

Proof. Necessity. If ξ has uniform exponential decay then there exist $M \geq 1$ and $\omega > 0$ such that

$$e^{-\omega(t-s)} \|\Phi(s, t_0, x)v\| \leq M \|\Phi(t, t_0, x)v\|, \ \forall t \geq s \geq t_0 \geq 0, \ \forall (x, v) \in \mathcal{Y}.$$

We consider $\alpha = 2\omega > 0$ and we obtain

$$e^{\omega(t-s)} \|\Phi_{-\alpha}(s, t_0, x)v\| \leq M \|\Phi_{-\alpha}(t, t_0, x)v\|$$

for all $t \geq s \geq t_0 \geq 0$ and all $(x, v) \in \mathcal{Y}$, which shows that $\xi_{-\alpha}$ is uniformly exponentially instable.

Suficiency. If there exists $\alpha > 0$ such that $\xi_{-\alpha}$ is uniformly exponentially instable, then there exist $N > 1$ and $\beta > 0$ such that

$$N \left\| \Phi_{-\alpha}(t, t_0, x)v \right\| = N e^{\alpha(t-t_0)} \left\| \Phi(t, t_0, x)v \right\| \geq$$

$$\geq e^{\beta(t-s)} e^{\alpha(s-t_0)} \left\| \Phi(s, t_0, x)v \right\| = e^{\beta(t-s)} \left\| \Phi_{-\alpha}(s, t_0, x)v \right\| .$$

It follows that

$$N \left\| \Phi(t, t_0, x)v \right\| \geq e^{(\beta-\alpha)(t-s)} \left\| \Phi(s, t_0, x)v \right\| \geq e^{-\nu(t-s)} \left\| \Phi(s, t_0, x)v \right\|$$

for all $t \geq s \geq t_0 \geq 0$ and all $(x, v) \in \mathcal{Y}$, where we have denoted

$$\nu = \begin{cases} \alpha - \beta, & \text{if } \alpha > \beta \\ 1, & \text{if } \alpha \leq \beta \end{cases}$$

Hence, the uniform exponential decay for ξ is proven. □

A connection between the asymptotic behaviors of skew-evolution semiflows presented in Definition 2.4, Definition 2.6 and Definition 2.5 is given by

Remark 2.1. *The property of uniform exponential instability of a skew-evolution semiflow implies the uniform instability and, further, the uniform exponential decay.*

3. The main results

In this section we will give two characterizations for the property of uniform exponential instability in the case of a particular class of skew-evolution semiflows introduced by the following

Definition 3.1. A skew-evolution semiflow $\xi = (\varphi, \Phi)$ is called *strongly measurable* if the mapping $t \mapsto \left\| \Phi(t, t_0, x)v \right\|$ is measurable on $[t_0, \infty)$, for all $(t_0, x, v) \in \mathbb{R}_+ \times Y$.

Theorem 3.1. *A strongly measurable skew-evolution semiflow $\xi = (\varphi, \Phi)$ is uniformly exponentially instable if and only if it is uniformly instable and there exists $M \geq 1$ such that*

$$\int_{t_0}^{t} \left\| \Phi(s, t_0, x)v \right\| ds \leq M \left\| \Phi(t, t_0, x)v \right\| , \ \forall t \geq t_0 \geq 0, \ \forall (x, v) \in Y. \qquad (3.1)$$

Proof. Necessity. Let ξ be uniformly exponentially instable. According to Remark 2.1, as ξ is also uniformly instable, there exist $N > 1$ and $\nu > 0$ such that

$$\int_{t_0}^{t} \|\Phi(s, t_0, x)v\| \, ds \le N \|\Phi(t, t_0, x)v\| \int_{t_0}^{t} e^{-\nu(t-s)} ds \le M \|\Phi(t, t_0, x)v\|$$

for all $t \ge t_0 \ge 0$ and all $(x, v) \in Y$, where we have denoted $M = N\nu^{-1}$.

Sufficiency. As ξ is uniformly instable, there exists $N > 1$ such that following inequality holds

$$\|v\| \le N \|\Phi(\tau, t_0, x)v\|, \forall \tau \ge t_0 \ge 0$$

and, further, by hypothesis, there exists $M \ge 1$ such that

$$(t - s) \|\Phi(s, t_0, x)v\| \le N \int_{s}^{t} \|\Phi(\tau, t_0, x)v\| \, d\tau \le MN \|\Phi(t, t_0, x)v\|$$

for all $t \ge s \ge t_0 \ge 0$ and all $(x, v) \in Y$.

It follows that

$$\|v\| \le \frac{MN}{(t - t_0 + 1)} \|\Phi(t, t_0, x)v\|, \ \forall t \ge t_0 \ge 0, \ \forall (x, v) \in Y.$$

As by Remark 2.1 ξ has also exponential decay, then, according to Proposition 2.3, the property of uniformly exponentially instability for ξ is obtained. □

Theorem 3.2. *A strongly measurable skew-evolution semiflow $\xi = (\varphi, \Phi)$ is uniformly exponentially instable if and only if it has uniform exponential decay and there exists $M \ge 1$ such that relation* (3.1) *hold.*

Proof. Let function g be given as in Proposition 2.1.

Following relations hold for all $t \ge s + 1 > s \ge t_0 \ge 0$ and all $(x, v) \in Y$

$$\|\Phi(s, t_0, x)v\| \int_{0}^{1} g(\tau) d\tau = \int_{s}^{s+1} g(u - s) \|\Phi(s, t_0, x)v\| \, du \le$$

$$\le \int_{s}^{s+1} \|\Phi(u, t_0, x)v\| \, du \le \int_{t_0}^{t} \|\Phi(u, t_0, x)v\| \, du \le M \|\Phi(t, t_0, x)v\|.$$

The property of uniform instability for ξ is obtained by Proposition 2.2, where we have considered

$$N = \int_{0}^{1} g(\tau) d\tau,$$

Further, by Theorem 3.1 the proof is concluded. □

As a conclusion we obtain the next

Corollary 3.1. *Let ξ be a skew-evolution semiflow for which there exists $M \geq 1$ such that relation* (3.1) *hold. Following properties are equivalent:*

(i) ξ has uniform exponential decay;

(ii) ξ is uniformly instable;

(iii) ξ is uniformly exponentially instable.

Proof. It is obtained according to Theorem 3.1, Theorem 3.2 and Remark 2.1. □

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A MONOTONY METHOD IN QUASISTATIC PROCESSES FOR VISCOPLASTIC MATERIALS

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Abstract. In this paper, we study a quasistatic problem for semi-linear rate-type viscoplastic models with two parameters χ, θ; χ may be interpreted as the absolute temperature or an internal state variable. The existence and uniqueness of the solution is proved using monotony arguments followed by a Cauchy-Lipschitz technique.

1. **Introduction**

Throughout the paper, Ω is a bounded in $IR^N(N = 1, 2, 3)$ with a smooth boundary $\partial\Omega = \Gamma$ and Γ_1 is an open subset of Γ such that $meas\Gamma_1 > 0$. We denote $\Gamma_2 = \Gamma\ - \overline{\Gamma}_1$. Let ν be the outward unit normal vector on Γ and S_N the set of second order symmetric tensors on IR^N. Let T be a real positive constant.LET us the mixed problem.

$$\dot{\sigma} = \mathcal{E}(\varepsilon(\dot{u}), \theta, \chi) + F(\sigma, \varepsilon(u), \theta) \quad \text{in } \Omega \times (0, T) \tag{1}$$

$$Div\ \sigma + f = 0 \quad \text{in } \Omega \times (0, T) \tag{2}$$

$$u = g \quad \text{on } on\Gamma_1 \times (0, T) \tag{3}$$

$$\sigma\nu = h \quad \text{on } \Gamma_2 \times (0, T) \tag{4}$$

$$u(0) = u_0, \quad \sigma(0) = \sigma_0 \quad \text{in } \Omega \tag{5}$$

in which the unknowns are the displacement function $u : \Omega \times [0, T] \to R^N$, the stress function $\sigma : \Omega \times [0, T] \to S_N$ This problem represents a quasistatic problem for rate-type models of the form (1) in with ε is a nonlinear function depending on

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$\varepsilon(\dot{u})$, θ and χ ,are parameters and $\varepsilon(u) : \Omega \times [0,T] \to S_N$ is the small strain tensor (i.e. $\varepsilon(u) = \frac{1}{2}\nabla u + \nabla^t u$). In (1) $\mathcal{E}$ and F are given constitutive function .

In (2) $Div\ \sigma$ represent the divergence of vector valued function σ and f represents the given body force, g and h are the given bounded data and, finally, u_0, σ_0 are the initial data.

In the case when ε depends only on χ, existence and uniqueness results for problems of the form (1)-(5) was obtained by Sofonea (1991) reducing the studied problem to an ordinary differential equation in a Hilbert space. In the case when $\mathcal{E}$ is a nonlinear function depending only on $\varepsilon(\dot{u})$ and χ existence and uniqueness results for problems of the form (1)-(5) was obtained by Djabi (1993) using monotony arguments followed by a Cauchy-Lipschitz technique.

The purpose of this paper is to give a now proof for the existence and uniqueness of the solution for the problem (1)-(5) there based only on monotony arguments followed by a Cauchy-Lipschitz technique (theorem 3.1).

2. Notations and preliminaries

Everywhere in this paper we utilize the following notations: ".“ the inner product on the spaces $\mathbb{R}^N$, $\mathbb{R}^M$ and S_N and $|\cdot|$ are the Euclidean norms on these spaces.

$$H = \{\ v = (v_i) \mid v_i \in L^2(\Omega),\ i = \overline{1,N}\ \},$$

$$H_1 = \{\ v = (v_i) \mid v_i \in H^1(\Omega),\ i = \overline{1,N}\ \},$$

$$\mathcal{H} = \{\ \tau = (\tau_{ij}) \mid \tau_{ij} = \tau_{ji} \in L^2(\Omega),\ i,j = \overline{1,N}\ \},$$

$$\mathcal{H}_1 = \{\ \tau = (\tau_{ij}) \mid Div\ \tau \in H\ \},$$

$$Y = \{\ \kappa = (\kappa_i) \mid \kappa_i \in L^2(\Omega),\ i = \overline{1,M}\ \}.$$

The spaces H, H_1, $\mathcal{H}$, $\mathcal{H}_1$ and Y are real Hilbert spaces endowed with the canonical inner products denoted by $< \cdot,\cdot >_H$, $< \cdot,\cdot >_{H_1}$, $< \cdot,\cdot >_{\mathcal{H}}$, $< \cdot,\cdot >_{\mathcal{H}_1}$ and $< \cdot,\cdot >_Y$ respectively.

Let $H_\Gamma = [H^{\frac{1}{2}}(\Gamma)]^N$ and $\gamma : H_1 \to H_\Gamma$ be the trace map. We denote by

$$V = \{ \ u \in H_1 \mid \gamma u = 0 \text{ on } \Gamma_1 \ \}$$

and let E be the subspace of H_Γ defined by

$$E = \gamma(V) = \{ \ \xi \in H_\Gamma \mid \xi = 0 \text{ on } \Gamma_1 \}. \tag{6}$$

Let $H'_\Gamma = [H^{-\frac{1}{2}}(\Gamma)]^N$ be the strong dual of the space H_Γ and let $< \cdot, \cdot >$ denote the duality between H'_Γ and H_Γ. If $\tau \in \mathcal{H}_1$ there exists an element $\gamma_\nu \tau \in H'_\Gamma$ such that

$$< \gamma_\nu \tau, \gamma v > = < \tau, \varepsilon(v) >_{\mathcal{H}} + \ < Div\ \tau, v >_H \ \text{ for all } v \in H_1. \tag{7}$$

By $\tau\nu$ we shall understand the element of E' (the strong dual of E) that is the projection of $\gamma_\nu \tau$ on E.

Let us now denote by $\mathcal{V}$ the following subspace of $\mathcal{H}_1$.

$$\mathcal{V} = \{ \ \tau \in \mathcal{H}_1 \mid \ Div\ \tau = 0 \text{ in } \Omega, \ \ \tau\nu = 0 \text{ on } \ \Gamma_2 \ \}$$

Using (7), it may be proved that $\varepsilon(V)$ is the orthogonal complement of $\mathcal{V}$ in $\mathcal{H}$, hence

$$< \tau, \varepsilon(v) >_{\mathcal{H}} = 0, \ \text{ for all } v \in V, \ \tau \in \mathcal{V}. \tag{8}$$

Finally, for every real Hilbert space X we denote by $|\cdot|_X$ the norm on X and by $C^j(0,T,X)(j = 0,1)$ the spaces defined as follows:

$$C^0(0,T,X) = \{ z : [0,T] \to X \mid z \text{ is continuous } \}.$$

Let us recall that if $C^j(0,T,X)$ are real Banach spaces endowed with the norms

$$C^1(0,T,X) = \{ z : [0,T] \to X \mid \text{there exists } \dot{z} \text{ the derivate of } z \text{ and } \dot{z} \in C^0(0,T,X) \}.$$

$$|z|_{0,T,X} = \max_{t \in [0,T]} |z(t)|_X \tag{9}$$

and

$$|z|_{1,T,X} = |z|_{0,T,X} + |\dot{z}|_{0,T,X}$$

respectively.

Let us recall that if K is a convex closed non empty set of X and $P : X \to K$ is the projector map on K, we have

$$y = Px \text{ if only if } y \in K \text{ and } < y - x,\ z - x >_X \geq 0 \text{ for all } z \in K. \tag{10}$$

3. **An existence and uniqueness result**

In the study of the problem (1)-(5), we consider the following assumptions:

$$\begin{cases} \mathcal{E} : \Omega \times S_N \times L^2(\Omega)^p \times L^2(\Omega)^M \to S_N \text{ and} \\ \text{(a) there exists } \ m > 0 \text{ such that} \\ < \mathcal{E}(\varepsilon_1, \theta, \chi) - \mathcal{E}(\varepsilon_2, \theta, \chi), \varepsilon_1 - \varepsilon_2 > \geq \\ \geq m|\varepsilon_1 - \varepsilon_2|^2 \text{ for all } \varepsilon_1, \varepsilon_2 \in S_N, , \theta \in L^2(\Omega)^p, \chi \in L^2(\Omega)^M a.e. \text{ in } \Omega, \\ \text{(b) there exists } L' > 0 \text{ such that} \\ |\mathcal{E}(\varepsilon_1, \theta_1, \chi_1) - \mathcal{E}(\varepsilon_2, \theta_2, \chi_2)| \leq L'|\varepsilon_1 - \varepsilon_2| + |\theta_1 - \theta_2| + |\chi_1 - \chi_2| \\ \text{for all } \varepsilon_1, \varepsilon_2 \in S_N, a.e. \text{ in } \Omega, \\ \text{(c) } x \to \mathcal{E}(x, \varepsilon, \theta, \chi) \text{ is a measurable function with respect to} \\ \text{the lebesgue measure in } \Omega \text{ for all } \varepsilon \in \mathrm{S}_N \ , \\ \text{(d) } x \to \mathcal{E}(x, 0, 0, 0) \in \mathcal{H} \end{cases} \tag{11}$$

$$\begin{cases} F : \Omega \times S_N \times S_N \times L^2(\Omega)^p \times L^2(\Omega)^M \to S_N \text{ and} \\ \text{a) there exists } L > 0 \text{such that} \\ \qquad |F(x, \sigma_1, \varepsilon_1, \theta_1, \chi_1) - F(x, \sigma_2, \varepsilon_2, \theta_2, \chi_2)| \leq \\ \quad \leq \ L\left(|\sigma_1 - \sigma_2| + |\varepsilon_1 - \varepsilon_2| + |\theta_1 - \theta_2| + |\chi_1 - \chi_2|\right) \\ \text{(b) } x \to F(x, \sigma, \varepsilon, \theta, \chi) \text{ is a measurable function with respect to} \\ \text{the Lebesgue measure on } \Omega, \text{ for all } \sigma, \varepsilon \in S_N, \kappa \in \mathbb{R}^M, \theta \in \mathbb{R}^P, \\ \text{(c) } x \to F(x, 0, 0, 0, 0) \in \mathcal{H}. \end{cases} \tag{12}$$

$$f \ \in C^1\left(0, T, H\right), \qquad g \in \left(0, T, H_\Gamma\right), \qquad h \in C^1\left(0, T, E'\right) \tag{13}$$

$$u_0 \in H_1, \qquad \sigma_0 \in \mathcal{H}_1 \tag{14}$$

$$Div\ \sigma_0 + f\left(0\right) = 0 \text{ in } \Omega, \quad u_0 = g\left(0\right) \text{ on } \Gamma_1, \quad \sigma_0 \nu = h\left(0\right) \text{ on } \Gamma_2. \tag{15}$$

$$\theta \in C^0\left(0, T, L^2\left(\Omega\right)^P\right) . \chi \in C^0\left(0, T, L^2\left(\Omega\right)^M\right) \tag{16}$$

The main result of this section is as follows.

Theorem 3.1. Let (11)-(16) hold. Then there exists a unique solution $u \in C^1(0,T,H_1), \sigma \in C^1(0,T,\mathcal{H}_1)$ of the problem (1)-(5). In order to prove theorem 3.1, we need some preliminaries.

Let $\tilde{u} \in C^1(0,T,H_1), \tilde{\sigma} \in C^1(0,T,\mathcal{H}_1)$be two functions such that

$$Div\ \tilde{\sigma} + f = 0 \text{ in } \Omega \times (0,T) \tag{17}$$

$$\tilde{u} = g \text{ on } \Gamma_1 \times (0,T) \tag{18}$$

$$\tilde{\sigma}\nu = h \text{ on } \Gamma_2 \times (0,T) \tag{19}$$

(the existence of this couple follows from (13) and the properties of the trace maps).

Considering the functions defined by

$$\bar{u} = u - \tilde{u}, \quad \bar{\sigma} = \sigma - \tilde{\sigma}, \tag{20}$$

$$\bar{u}_0 = u_0 - \tilde{u}_0, \quad \bar{\sigma}_0 = \sigma_0 - \tilde{\sigma}_0, \tag{21}$$

it easy to see that the triplet $(u,\sigma) \in C^1(0,T,H \times \mathcal{H}_1)$is a solution of the problem (1)-(5) if and only if

$$(\bar{u},\bar{\sigma}) \in C^1(0,T,V \times \mathcal{V}) \tag{22}$$

$$\dot{\sigma} = \mathcal{E}(\varepsilon(\dot{u}) + \varepsilon(\dot{\tilde{u}}),\theta,\chi) + F(\bar{\sigma}+\tilde{\sigma},\varepsilon(\bar{u})+\varepsilon(\tilde{u}),\theta,\chi) - \dot{\tilde{\sigma}} \text{ in } \Omega \times (0,T) \tag{23}$$

$$\bar{u}(0) = \bar{u}_0, \quad \bar{\sigma}(0) = \bar{\sigma}_0 \text{ in } \Omega \tag{24}$$

hence we may write (22)-(24) in the form

$$\dot{y}(t) = \mathcal{G}(\theta(t),\chi(t),x(t),y(t),\dot{x}(t)) \tag{25}$$

$$x(0) = x_0, \quad y(0) = y_0 \tag{26}$$

In which the unknowns are the function $x : [0.T] \to X$ and $y : [0.T] \to Y$ $\mathcal{G}$:$L^2(\Omega)^p \times L^2(\Omega)^M \times X \times Y \times H \to H$ is a nonlinear operator, and $X : [0.T] \to L^2(\Omega)^M, \theta : [0.T] \to L^2(\Omega)^p$ are parameters, where H is a real Hilbert space, X, Y, are two orthogonal subspaces of H such that $\mathsf{H} = X \oplus Y$ and $L^2(\Omega)^M, L^2(\Omega)^p$, are real normed space.

Hence (22)-(24) may be written in the form (25)-(26) where

$$y = \bar{\sigma},\ x = \varepsilon(\bar{u}), \dot{x} = \varepsilon\left(\dot{\bar{u}}\right)$$

and replacing the spaces $\varepsilon(V), \mathcal{V}, \mathcal{H}$, by X, Y, H respectively.

For resolving the problem (22)-(24), we consider the product Hilbert space $Z = \varepsilon(V) \times V$ which $H = \varepsilon(V) \oplus V$,and the problem $\mathcal{G}$ defined by

$$\mathcal{G} : L^2(\Omega)^p \times L^2(\Omega)^M \times \varepsilon(V) \times v \times H \to H$$

$$\mathcal{G}(\theta, \chi, x, y, q) = \mathcal{E}\left(q + \varepsilon(\dot{\tilde{u}}), \dot{\theta}(t), \chi(t))\right) + F(y + \tilde{\sigma}(t), x + \varepsilon(\tilde{u}), \theta(t), \chi(t)) - \dot{\tilde{\sigma}}(t) \quad (27)$$

We have the following result.

Lemma 3.1. Let $\theta(t) \in L^2(\Omega)^P$, $\chi(t) \in L^2(\Omega)^M$ $x \in X, y \in Y$ and $t \in [0.T]$. Then there exists a unique element $z = (\varepsilon(v), \tau) \in Z$ such that

$$\tau = \mathcal{G}(\theta, \chi, x, y, \varepsilon(v)) \quad (28)$$

Proof. The uniqueness part is a consequence of (11); indeed, if

$$z_1 = (\varepsilon(v_1), \tau_1), \quad z_2 = (\varepsilon(v_2), \tau_2) \in Z$$

are such that

$$\tau_1 = \mathcal{G}(\theta, \chi, x, y, \varepsilon(v_1))$$

$$\tau_2 = \mathcal{G}(\theta, \chi, x, y, \varepsilon(v_2)),$$

using (11-a) we have

$$\langle \tau_1 - \tau_2, \varepsilon(v_1) - \varepsilon(v_2) \rangle_{\mathcal{H}} =$$

$$\left\langle \mathcal{E}(\varepsilon(v_1) + \varepsilon(\dot{\tilde{u}}(t)), \theta(t), \chi(t))) - \mathcal{E}(\varepsilon(v_2) + \varepsilon(\dot{\tilde{u}}(t)), \theta(t), \chi(t)), \varepsilon(v_1) - \varepsilon(v_2) \right\rangle_{\mathcal{H}}$$

$$\geq m \left| \varepsilon(v_1) - \varepsilon(v_2) \right|_{\mathcal{H}}$$

Using now the orthogonality in H of $(\tau_1 - \tau_2) \in \mathcal{V}$ and $(\varepsilon(v_1) - \varepsilon(v_2)) \in \varepsilon(V)$,we deduce that $\varepsilon(v_1) = \varepsilon(v_2)$, which implies $\tau_1 = \tau_2$.

For the existence part,let us consider the operator $S : \varepsilon(V) \to \varepsilon(V)$ given by $S = P \circ \mathcal{G}$, where P is the projector map $\varepsilon(V)$.

Using now the hypothesis $\mathcal{E}$, F and the properties of the projectors, we can prove for θ, χ, x, y fixed, the following inequalities:

$$\begin{cases} \langle S(\theta, \chi, x, y, q_1) - S(\theta, \chi, x, y, q_2), q_1 - q_2 \rangle_{\mathcal{H}} \geq \\ \geq \langle \mathcal{G}(\theta, \chi, x, y, q_1) - \mathcal{G}(\theta, \chi, x, y, q_2), q_1 - q_2 \rangle_{\mathcal{H}} \geq \\ \geq m \, |q_1 - q_2|^2_{\mathcal{H}} . \end{cases} \tag{29}$$

Moreover, from (11), (12), and the properties of the projectors, we get

$$\begin{cases} |S(\theta, \chi, x, y, q_1) - S(\theta, \chi, x, y, q_2)|_{\mathcal{H}} \leq \\ \leq |\mathcal{G}(\theta, \chi, x, y, q_1) - \mathcal{G}(\theta, \chi, x, y, q_2)|_{\mathcal{H}} \leq \\ \leq L' \, |q_1 - q_2|^2_{\mathcal{H}} . \end{cases} \tag{30}$$

Hence $S(\theta, x, y, .) : \varepsilon(V) \to \varepsilon(V)$ is a strongly monotone Lipschitz operator. Using now Browder's surjectivity theorem we get that there exists $\varepsilon(v) \in \varepsilon(V)$such that $S(\theta, \chi, x, y, \varepsilon(v)) = 0_{\varepsilon(V)}$. It results that the element $\mathcal{G}(\theta, \chi, x, y, \varepsilon(v))$ belongs to $\mathcal{V}$ and we finish the proof using $z = (\varepsilon(v), \tau)$ where

$$\tau = \mathcal{G}(\theta, \chi, x, y, \varepsilon(v)).$$

The previous lemma allows to consider the operator $B : L^2(\Omega)^P \times L^2(\Omega)^M \times Z \to Z$ defined as follows:

$$\begin{cases} B(\theta, \chi, \omega) = z \\ \omega = (x, y), z = (\varepsilon(v), \tau) \\ \tau = \mathcal{G}(\theta, \chi, x, y, \varepsilon(v)). \end{cases} \tag{31}$$

Moreover we have

Lemma 3.2. For all $\theta \in L^2(\Omega)^P$ and $\chi \in L^2(\Omega)^M$ $\omega_1, \omega_2 \in Z$, the operator $L^2(\Omega)^P \times L^2(\Omega)^M \times Z \to Z$ is continuous and there exists $C > 0$ such that

$$|B(\theta, \chi, \omega_1) - B(\theta, \chi, \omega_2)|_Z \leq C|\omega_1 - \omega_2|_Z \tag{32}$$

for all $\theta \in L^2(\Omega)^P$ and $\chi \in L^2(\Omega)^M \omega_1, \omega_2 \in Z$.

Proof. Let $\theta_i \in L^2(\Omega)^P$, $\omega_i = (x_i, y_i) \in Z$ and

$$z_i = (\varepsilon(v_i), \tau_i) = B(\theta_i, \chi_i, \omega_i) \ , \ i = 1, 2.$$

Using (32)

$$\tau_i = \mathcal{G}(\theta_i, \chi_i, x_i, y_i, \varepsilon(v_i))), \quad i = 1, 2 \tag{33}$$

which implies

$$S(\theta_i, \chi_i, x_i, y_i, \varepsilon(v_i)) = 0_{\varepsilon(V)}\ ,\ i = 1, 2. \tag{34}$$

Using the hypothesis on $\mathcal{E}$, F, and the properties of the projectors, we get:

$$m|\varepsilon(v_1) - \varepsilon(v_1)_2|^2_{\mathcal{H}} \leq < S(\theta_1, \chi_1, x_1, y_1, \varepsilon(v_1))$$

$$-S(\theta_1, \chi_1, x_1, y_1, \varepsilon(v_2)), \varepsilon(v_1) - \varepsilon(v_2) >_{\mathcal{H}}$$

$$= < S(\theta_2, \chi_2, x_2, y_2, \varepsilon(v_2)) - S(\theta_1, \chi_1, x_1, y_1, \varepsilon(v_2)), \varepsilon(v_1) - \varepsilon(v_2) >^2_{\mathcal{H}} \leq$$

$$\leq |\mathcal{G}(\theta_2, \chi_2, x_2, y_2, \varepsilon(v_2)) - \mathcal{G}(\theta_1, \chi_1, x_1, y_1, \varepsilon(v_2))|_{\mathcal{H}} \times |\varepsilon(v_1) - \varepsilon(v_2)|^2_{\mathcal{H}}$$

which implies

$$|\varepsilon(v_1) - \varepsilon(v_2)|_{\mathcal{H}} \leq \frac{1}{m} \times |\mathcal{G}(\theta_2, \chi_2, x_2, y_2, \varepsilon(v_2)) - \mathcal{G}(\theta_1, \chi_1, x_1, y_1, \varepsilon(v_2))|_{\mathcal{H}}\ . \tag{35}$$

Using now (12), (34) we get

$$\begin{cases} |\tau_1 - \tau_2|_{\mathcal{H}} \leq L'|\varepsilon(v_1) - \varepsilon(v_2|_{\mathcal{H}} + \\ |\mathcal{G}(\theta_1, \chi_1, x_1, y_1, \varepsilon(v_2)) - \mathcal{G}(\theta_2, \chi_2, x_2, y_2, \varepsilon(v_2)|_{\mathcal{H}} \end{cases} \tag{36}$$

Hence by (36) it result

$$\begin{cases} |\tau_1 - \tau_2|_{\mathcal{H}} \leq \\ \leq (\frac{L'}{m} + 1)|\mathcal{G}(\theta_1, \chi_1, x_1, y_1, \varepsilon(v_2)) - \mathcal{G}(\theta_2, \chi_2, x_2, y_2, \varepsilon(v_2)|_{\mathcal{H}} \end{cases} \tag{37}$$

Using now (11)-(12)(27) and the fact that $\bar{\sigma}, \dot{\bar{\sigma}}$ are continuous, we get that

$$|\mathcal{G}(\theta_1, \chi_1, x_1, y_1, \varepsilon(v_2)) - \mathcal{G}(\theta_2, \chi_2, x_2, y_2, \varepsilon(v_2)|_{\mathcal{H}} \to 0$$

When $\theta_1 \to \theta_2, in\ L^2(\Omega)^P x_1 \to x_2\ in\ X,\ y_1 \to y_2\ in\ Y$ it follows that B is continuous operator. Taking $\theta_1 = \theta_2$ and X_1=X_2 from (37) we get (33).

Proof of theorem 3.1. Let $A : [0.T] \times Z \to Z$ and z_0 be defined by:

$$\{A(t, z) = B(\theta(t), \chi(t), z) \text{ for all t} \in [0.T]\ and\ z \in Z \tag{38}$$

$$z_0 = (x_0, y_0) = \varepsilon((u_0), \bar{\sigma}_0)\,.$$

Using the definition of operator B, we get that

$$x = \varepsilon(\dot{\bar{u}}) \in C^1(0, T, \varepsilon(V)) \in C^1(0, T, Z'), y = \bar{\sigma} \in C^1(0, T, \mathcal{V})$$

is solution to (22)-(24), if and only

$$\dot{z} = (\dot{x}, \dot{y}) = A(\theta, z(t)) \text{ for all } t \in [0.T] \tag{39}$$

$$z(0) = z_0 \tag{40}$$

In order to study the problem (39)-(40), let us remark that, by lemma 3.2, A is a continuous operator and

$$|A(t, z_1) - A(t, z_2)|_Z \leq C|z_1 - z_2|_Z, \text{ for all } t \in [0.T] \text{ and } z_1, z_2 \in Z.$$

Moreover, by (14), (38), $\tilde{u} \in C^1(0, T, H_1)$ and $\tilde{\sigma} \in C^1(0, T, \mathcal{H}_1)$

We get z_0 belongs to Z and by lemma 3.2 and the classical Cauchy-Lipschitz theorem we have that $z \in C^1(0, T, Z)$ and the proof of theorem 3.1 is complete.

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RADIAL SOLUTIONS FOR SOME CLASSES OF ELLIPTIC BOUNDARY VALUE PROBLEMS

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Abstract. The aim of this paper is to present some existence and localization results of radial solutions for elliptic equations and systems on an annulus Ω of $\mathbb{R}^N$ $(N \geq 1)$ of radii a and b with $0 < a < b$. The main tool is Schauder's fixed point theorem.

1. Introduction

In this paper, we are concerned with the existence of radial solutions and their localization in a ball, for the elliptic boundary value problem

$$\begin{cases} -\Delta u = f\left(|x|, u\right) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{1.1}$$

and the elliptic system

$$\begin{cases} -\Delta u = g\left(|x|, u, v\right) & \text{in } \Omega \\ -\Delta v = h\left(|x|, u, v\right) & \text{in } \Omega \\ u = v = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.2}$$

Here Ω is an annulus of $\mathbb{R}^N$ $(N \geq 1)$ of radii a and b with $0 < a < b$, $|x|$ is the Euclidean norm in $\mathbb{R}^N$, and $f : [a, b] \times \mathbb{R} \longrightarrow \mathbb{R}$ and $g,\ h : [a, b] \times \mathbb{R}^2 \longrightarrow \mathbb{R}$ are continuous functions. By a solution of problem (1.1) we mean a function $u \in C^1\left(\overline{\Omega}, \mathbb{R}\right)$ which satisfies (1.1) in the sense of distributions. A solution to problem (1.2) is a vector-valued function $(u, v) \in C^1\left(\overline{\Omega}, \mathbb{R}^2\right) := C^1\left(\overline{\Omega}, \mathbb{R}\right) \times C^1\left(\overline{\Omega}, \mathbb{R}\right)$ satisfying (1.2) in the sense of distributions.

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Radial solutions for elliptic boundary value problems has been discussed extensively in the literature; see [1], [2], [3], [5], [6] and the references therein.

In [2], using fixed point theorems of cone expansion/compression type, the lower and upper solution method and degree arguments, the authors study existence, non-existence and multiplicity of positive radial solutions.

The same authors in [3] deal with a class of second-order elliptic problems on a ball with non-homogeneous boundary condition. They obtain via a fixed point theorem the existence of at least three positive radial solutions.

The study of existence of positive radial solutions to a singular semilinear elliptic equation was investigated in [5]. Throughout, their nonlinearity is allowed to change sign and the singularity may occur.

In this paper, some existence and localization results of radial solutions for elliptic equations and systems on an annulus are presented. Our main tool in proving the existence of solutions to problems (1.1) and (1.2) is Schauder's fixed point theorem [4], [7].

2. **Existence result for Problem (1.1)**

Theorem 2.1. *Assume that for some $R > 0$, one of the following hypotheses is satisfied:*

($\mathcal{H}1$)

$$|f(t,y)| \leq \alpha(t)F(y), \text{ for all } t \in [a,b] \text{ and } y \in \mathbb{R},$$

where the functions $\alpha \in L^1([a,b],\mathbb{R}_+)$ and $F \in C(\mathbb{R},\mathbb{R}_+)$ satisfy

$$|\alpha|_{L^1} \max_{|y|\leq R} F(y) \leq \frac{R}{b-a}\left(\frac{a}{b}\right)^{N-1};$$

($\mathcal{H}2$)

$$|f(t,y)| \leq F(t,|y|), \text{ for all } t \in [a,b] \text{ and } y \in \mathbb{R},$$

for some $F \in C([a,b]\times\mathbb{R}_+,\mathbb{R}_+)$ nondecreasing with respect to its last variable, and with

$$\int_a^b F(s,R)\,ds \le \frac{R}{b-a}\left(\frac{a}{b}\right)^{N-1}.$$

Then the boundary value problem (1.1) *has at least one radial solution with*

$$\|u\|_0 = \sup_{a\le|x|\le b} |u(x)| \le R.$$

Proof. For $v \in C([a,b],\mathbb{R})$, let $u = Tv$ be the solution of

$$\begin{cases} -(r^{N-1}u')' = r^{N-1}f\left(r, v(r)\right), & a<r<b \\ u(a)=u(b)=0 \end{cases}$$

where we have replaced $|x|$ by r. Then $T : C([a,b],\mathbb{R}) \longrightarrow C([a,b],\mathbb{R})$ is completely continuous and the fixed points of T are solutions of problem (1.1). One can write the expression of T as

$$Tu(r) = \int_a^r \left[\frac{1}{s^{N-1}}\int_s^\theta \tau^{N-1} f\left(\tau, u(\tau)\right)\, d\tau\right] ds$$

where θ is such that $\|u\|_0 = |u(\theta)|$.

Consider the closed ball:

$$B = \{u \in C([a,b],\mathbb{R}) :\ \|u\|_0 \le R\}$$

where R is as in Assumptions $(\mathcal{H}1)$, $(\mathcal{H}2)$ and check that $T(B) \subset B$.

(a) Assume $(\mathcal{H}1)$. For any $u \in B$ and $r \in [a,b]$, we have

$$\begin{aligned} |Tu(r)| &\le \int_a^b \left[\frac{1}{s^{N-1}}\int_a^b \tau^{N-1}|f\left(\tau, u(\tau)\right)|\, d\tau\right] ds \\ &\le \left(\frac{b}{a}\right)^{N-1}(b-a)\int_a^b |f\left(s,u(s)\right)|\,ds \\ &\le \left(\frac{b}{a}\right)^{N-1}(b-a)\int_a^b \alpha(s)F(u(s))\,ds \\ &\le \left(\frac{b}{a}\right)^{N-1}(b-a)|\alpha|_{L^1}\max_{|y|\le R}F(y) \\ &\le R. \end{aligned}$$

Passing to the supremum, we obtain

$$\|Tu\|_0 \le R.$$

(b) When $(\mathcal{H}2)$ holds, then

$$\begin{aligned} |Tu(r)| &\leq \int_a^b \left[\frac{1}{s^{N-1}} \int_a^b \tau^{N-1} |f(\tau, u(\tau))| \, d\tau \right] ds \\ &\leq \left(\frac{b}{a}\right)^{N-1} (b-a) \int_a^b |f(s, u(s))| \, ds \\ &\leq \left(\frac{b}{a}\right)^{N-1} (b-a) \int_a^b F(s, |u(s)|) \, ds \\ &\leq \left(\frac{b}{a}\right)^{N-1} (b-a) \int_a^b F(s, R) \, ds \\ &\leq R. \end{aligned}$$

Passing to the supremum, we obtain

$$\|Tu\|_0 \leq R.$$

Therefore, in both cases, the operator T maps the ball B into itself, ending the proof of our claim. Since T is completely continuous, the conclusion of Theorem 2.1 follows from Schauder's fixed point theorem. □

3. **Existence results for Problem (1.2)**

In this section, we are concerned with the existence and localization of radial solutions to the Dirichlet problem (1.2) for elliptic systems.

Theorem 3.1. *Assume that for some $R > 0$ one of the following hypotheses is satisfied:*

$(\mathcal{H}3)$

$$|g(t, y, z)| \leq \beta(t) G(y, z), \text{ for all } t \in [a, b] \text{ and } y, z \in \mathbb{R}$$

and

$$|h(t, y, z)| \leq \gamma(t) H(y, z), \text{ for all } t \in [a, b] \text{ and } y, z \in \mathbb{R}$$

for some functions $\beta, \gamma \in L^1([a, b], \mathbb{R}_+)$ and $G, H \in C(\mathbb{R}^2, \mathbb{R}_+)$ with

$$|\beta|_{L^1} \max_{|y|, |z| \leq R} G(y, z) \leq \frac{R}{b-a} \left(\frac{a}{b}\right)^{N-1}$$

and

$$|\gamma|_{L^1} \max_{|y|,|z| \leq R} H(y,z) \leq \frac{R}{b-a}\left(\frac{a}{b}\right)^{N-1};$$

$(\mathcal{H}4)$

$$|g(t,y,z)| \leq G(t,|y|,|z|), \textit{for all } t \in [a,b] \textit{ and } y,z \in \mathbb{R}$$

and

$$|h(t,y,z)| \leq H(t,|y|,|z|), \textit{ for all } t \in [a,b] \textit{ and } y,z \in \mathbb{R}$$

for some functions $G, H \in C([a,b] \times \mathbb{R}_+^2, \mathbb{R}_+)$ *nondecreasing with respect to the last two arguments, and with*

$$\int_a^b G(s,R,R)\,ds \leq \frac{R}{b-a}\left(\frac{a}{b}\right)^{N-1}$$

$$\int_a^b H(s,R,R)\,ds \leq \frac{R}{b-a}\left(\frac{a}{b}\right)^{N-1}.$$

Then the boundary value problem (1.2) *has at least one radial solution* (u,v) *with*

$$\|u\|_0 \leq R \textit{ and } \|v\|_0 \leq R.$$

Proof. We shall apply Schauder's fixed point theorem in the space $C([a,b],\mathbb{R}^2)$ endowed with the norm $\|(.,.)\|_0$ given by

$$\|(u,v)\|_0 = \|u\|_0 + \|v\|_0.$$

For $(\overline{u},\overline{v}) \in C([a,b],\mathbb{R}^2)$, let $(u,v) = T(\overline{u},\overline{v})$ be the solution of

$$\begin{cases} -(r^{N-1}u')' = r^{N-1} g\left(r, (\overline{u}(r), \overline{v}(r))\right), & a < r < b \\ -(r^{N-1}v')' = r^{N-1} h\left(r, (\overline{u}(r), \overline{v}(r))\right), & a < r < b \\ u(a) = u(b) = v(a) = v(b) = 0. \end{cases}$$

Then $T : C([a,b],\mathbb{R}^2) \longrightarrow C([a,b],\mathbb{R}^2)$ is completely continuous and the fixed points of T are solutions of problem (1.2). One can write the expression of T as $T = (T_1, T_1)$, where

$$T_1(u,v)(r) = \int_a^r \left[\frac{1}{s^{N-1}} \int_s^{\theta_1} \tau^{N-1} g\left(\tau, u(\tau), v(\tau)\right)\, d\tau\right] ds,$$

$$T_2(u,v)(r) = \int_a^r \left[\frac{1}{s^{N-1}} \int_s^{\theta_2} \tau^{N-1} h\left(\tau, u(\tau), v(\tau)\right)\, d\tau\right] ds$$

and θ_1, θ_2 are such that $\|u\|_0 = |u(\theta_1)|$ and $\|v\|_0 = |v(\theta_2)|$.

Consider the closed, bounded and convex subset of $C([a,b],\mathbb{R}^2):$

$$B = \left\{(u,v) \in C([a,b],\mathbb{R}^2) : \; \|u\|_0 \leq R, \|v\|_0 \leq R\right\},$$

where R is as in Assumptions $(\mathcal{H}3)$, $(\mathcal{H}4)$, and check that $T(B) \subset B$.

(a) Assume $(\mathcal{H}3)$. For any $(u,v) \in B$ and $r \in [a,b]$, we have

$$\begin{aligned}
|T_1(u,v)(r)| &\leq \int_a^b \left[\frac{1}{s^{N-1}} \int_a^b \tau^{N-1} |g(\tau, u(\tau), v(\tau))|\, d\tau\right] ds \\
&\leq \left(\frac{b}{a}\right)^{N-1} (b-a) \int_a^b |g(\tau, u(\tau), v(\tau))|\, ds \\
&\leq \left(\frac{b}{a}\right)^{N-1} (b-a) \int_a^b \beta(s) G(u(s), v(s))\, ds \\
&\leq \left(\frac{b}{a}\right)^{N-1} (b-a) |\beta|_{L^1} \max_{|y|,|z| \leq R} G(y,z) \\
&\leq R.
\end{aligned}$$

Passing to the supremum, we obtain

$$\|T_1(u,v)\|_0 \leq R.$$

Also

$$\begin{aligned}
|T_2(u,v)(r)| &\leq \int_a^b \left[\frac{1}{s^{N-1}} \int_a^b \tau^{N-1} |h(\tau, u(\tau), v(\tau))|\, d\tau\right] ds \\
&\leq \left(\frac{b}{a}\right)^{N-1} (b-a) \int_a^b |h(\tau, u(\tau), v(\tau))|\, ds \\
&\leq \left(\frac{b}{a}\right)^{N-1} (b-a) \int_a^b \gamma(s) H(u(s), v(s))\, ds \\
&\leq \left(\frac{b}{a}\right)^{N-1} (b-a) |\gamma|_{L^1} \max_{|y|,|z| \leq R} H(y,z) \\
&\leq R.
\end{aligned}$$

Hence

$$\|T_2(u,v)\|_0 \leq R.$$

Therefore, the operator T maps the ball B into itself.

(b) Assume ($\mathcal{H}4$). Then

$$\begin{aligned} |T_1(u,v)(r)| &\leq \int_a^b \left[\frac{1}{s^{N-1}}\int_a^b \tau^{N-1}|g\left(\tau,u(\tau),v(\tau)\right)|\,d\tau\right] ds \\ &\leq \left(\frac{b}{a}\right)^{N-1}(b-a)\int_a^b |g\left(s,u(s),v(\tau)\right)|\,ds \\ &\leq \left(\frac{b}{a}\right)^{N-1}(b-a)\int_a^b G(s,|u(s)|,|v(s)|)\,ds \\ &\leq \left(\frac{b}{a}\right)^{N-1}(b-a)\int_a^b G(s,R,R)\,ds \\ &\leq R. \end{aligned}$$

Hence

$$\|T_1(u,v)\|_0 \leq R.$$

Also

$$\begin{aligned} |T_2(u,v)(r)| &\leq \int_a^b \left[\frac{1}{s^{N-1}}\int_a^b \tau^{N-1}|h\left(\tau,u(\tau),v(\tau)\right)|\,d\tau\right] ds \\ &\leq \left(\frac{b}{a}\right)^{N-1}(b-a)\int_a^b |h\left(s,u(s),v(s)\right)|\,ds \\ &\leq \left(\frac{b}{a}\right)^{N-1}(b-a)\int_a^b H(s,|u(s)|,|v(s)|)\,ds \\ &\leq \left(\frac{b}{a}\right)^{N-1}(b-a)\int_a^b H(s,R,R)\,ds \\ &\leq R. \end{aligned}$$

Then

$$\|T_2(u,v)\|_0 \leq R.$$

Therefore, in both cases, the operator T maps the set B into itself, ending the proof of our claim. Since T is completely continuous, the conclusion of Theorem 3.1 follows from Schauder's fixed point theorem. □

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HOMOMORPHISMS BETWEEN JC^*-ALGEBRAS

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Abstract. It is shown that every almost linear mapping $h : \mathcal{A} \to \mathcal{B}$ of a JC^*-algebra $\mathcal{A}$ into a JC^*-algebra $\mathcal{B}$ is a homomorphism when $h(2^n u \circ y) = h(2^n u) \circ h(y)$ for all unitaries $u \in \mathcal{A}$, all $y \in \mathcal{A}$ and all $n \in \mathbb{Z}$, and that every almost linear continuous mapping $h : \mathcal{A} \to \mathcal{B}$ of a JC^*-algebra $\mathcal{A}$ of real rank zero to a JC^*-algebra $\mathcal{B}$ is a homomorphism when $h(2^n u \circ y) = h(2^n u) \circ h(y)$ for all $u \in \{v \in \mathcal{A} \mid v = v^*, ||v|| = 1, v \text{ is invertible}\}$, all $y \in \mathcal{A}$ and all $n \in \mathbb{Z}$. We moreover prove the Hyers-Ulam stability of homomorphisms in JC^*-algebras. This concept of stability of mappings was introduced for the first time by Th.M. Rassias in his paper [On the stability of the linear mapping in Banach spaces, Proc. Amer. Math. Soc. **72** (1978), 297-300].

1. **Introduction**

Let X and Y be Banach spaces with norms $|| \cdot ||$ and $\| \cdot \|$, respectively. Consider $f : X \to Y$ to be a mapping such that $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in X$. Th.M. Rassias [26] introduced the following inequality, that is known as *Cauchy-Rassias inequality* : Assume that there exist constants $\epsilon \geq 0$ and $p \in [0, 1)$ such that

$$\|f(x + y) - f(x) - f(y)\| \leq \epsilon(||x||^p + ||y||^p)$$

for all $x, y \in X$. Th.M. Rassias [26] showed that there exists a unique $\mathbb{R}$-linear mapping $T : X \to Y$ such that

$$\|f(x) - T(x)\| \leq \frac{2\epsilon}{2 - 2^p} ||x||^p$$

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for all $x \in X$. This inequality has provided a lot of influence in the development of what is called *generalized Hyers-Ulam stability* of functional equations. Beginning around the year 1980 the topic of approximate homomorphisms, or the stability of the equation of homomorphism, was studied by a number of mathematicians. Găvruta [7] generalized the Rassias' result in the following form: Let G be an abelian group and Y a Banach space. Denote by $\varphi : G \times G \to [0, \infty)$ a function such that

$$\widetilde{\varphi}(x, y) = \sum_{k=0}^{\infty} \frac{1}{2^k} \varphi(2^k x, 2^k y) < \infty$$

for all $x, y \in G$. Suppose that $f : G \to Y$ is a mapping satisfying

$$\|f(x + y) - f(x) - f(y)\| \le \varphi(x, y)$$

for all $x, y \in G$. Then there exists a unique additive mapping $T : G \to Y$ such that

$$\|f(x) - T(x)\| \le \frac{1}{2} \widetilde{\varphi}(x, x)$$

for all $x \in G$. C. Park [15] applied the Găvruta's result to linear functional equations in Banach modules over a C^*-algebra. Various functional equations have been investigated by several authors ([1], [3]-[6], [8]-[12], [16]-[25], [27]-[32]).

Throughout this paper, let $\mathcal{A}$ be a JC^*-algebra with norm $||\cdot||$ and unit e, and $\mathcal{B}$ a JC^*-algebra with norm $\|\cdot\|$ and unit e'. Let $\mathcal{U}(\mathcal{A}) = \{u \in \mathcal{A} \mid uu^* = u^*u = e\}$, $\mathcal{A}_{sa} = \{x \in \mathcal{A} \mid x = x^*\}$, and $I_1(\mathcal{A}_{sa}) = \{v \in \mathcal{A}_{sa} \mid ||v|| = 1, v \text{ is invertible}\}$.

Using the stability methods of linear mappings, we prove that every almost linear mapping $h : \mathcal{A} \to \mathcal{B}$ is a homomorphism when $h(2^n u \circ y) = h(2^n u) \circ h(y)$ for all $u \in \mathcal{U}(\mathcal{A})$, all $y \in \mathcal{A}$ and all $n \in \mathbb{Z}$, and that for a JC^*-algebra $\mathcal{A}$ of real rank zero (see [2]), every almost linear continuous mapping $h : \mathcal{A} \to \mathcal{B}$ is a homomorphism when $h(2^n u \circ y) = h(2^n u) \circ h(y)$ for all $u \in I_1(\mathcal{A}_{sa})$, all $y \in \mathcal{A}$ and all $n \in \mathbb{Z}$. We moreover prove the Hyers-Ulam stability of homomorphisms in JC^*-algebras.

2. **Homomorphisms between JC^*-algebras**

The original motivation to introduce the class of nonassociative algebras known as Jordan algebras came from quantum mechanics (see [33]). Let $\mathcal{H}$ be a complex Hilbert space, regarded as the "state space" of a quantum mechanical system. Let $\mathcal{L}(\mathcal{H})$ be the real vector space of all bounded self-adjoint linear operators on $\mathcal{H}$, interpreted as the (bounded) *observables* of the system. In 1932, Jordan observed that $\mathcal{L}(\mathcal{H})$ is a (nonassociative) algebra via the *anticommutator product* $x \circ y := \frac{xy+yx}{2}$. A commutative algebra X with product $x \circ y$ is called a *Jordan algebra* if $x^2 \circ (x \circ y) = x \circ (x^2 \circ y)$ holds.

A complex Jordan algebra $\mathcal{C}$ with product $x \circ y$ and involution $x \mapsto x^*$ is called a JB^*-algebra if $\mathcal{C}$ carries a Banach space norm $\| \cdot \|$ satisfying $\|x \circ y\| \le \|x\| \cdot \|y\|$ and $\|\{xx^*x\}\| = \|x\|^3$. Here $\{xy^*z\} := x \circ (y^* \circ z) - y^* \circ (z \circ x) + z \circ (x \circ y^*)$ denotes the *Jordan triple product* of $x, y, z \in \mathcal{C}$. A unital Jordan C^*-subalgebra of a C^*-algebra, endowed with the anticommutator product, is called a *JC^*-algebra* (see [23]-[25], [33]).

We investigate homomorphisms between JC^*-algebras.

Theorem 1. *Let $h : \mathcal{A} \to \mathcal{B}$ be a mapping satisfying $h(0) = 0$ and $h(2^n u \circ y) = h(2^n u) \circ h(y)$ for all $u \in \mathcal{U}(\mathcal{A})$, all $y \in \mathcal{A}$ and all $n \in \mathbb{Z}$, for which there exists a function $\varphi : \mathcal{A} \times \mathcal{A} \to [0, \infty)$ such that*

$$\widetilde{\varphi}(x, y) := \sum_{j=0}^{\infty} \frac{1}{2^j} \varphi(2^j x, 2^j y) < \infty, \tag{2.1}$$

$$\|h(\mu x + \mu y) - \mu h(x) - \mu h(y)\| \le \varphi(x, y) \tag{2.2}$$

for all $\mu \in \mathbb{T}^1 := \{\lambda \in \mathbb{C} \mid |\lambda| = 1\}$ and all $x, y \in \mathcal{A}$. Assume that

$$\lim_{n\to\infty} \frac{h(2^n e)}{2^n} = e'. \tag{2.3}$$

Then the mapping $h : \mathcal{A} \to \mathcal{B}$ is a homomorphism.

Proof. Put $\mu = 1 \in \mathbb{T}^1$. It follows from Găvruta Theorem [7] that there exists a unique additive mapping $H : \mathcal{A} \to \mathcal{B}$ such that

$$\|h(x) - H(x)\| \le \frac{1}{2} \widetilde{\varphi}(x, x) \tag{2.4}$$

for all $x \in \mathcal{A}$. The additive mapping $H : \mathcal{A} \to \mathcal{B}$ is given by

$$H(x) = \lim_{n\to\infty} \frac{1}{2^n} h(2^n x)$$

for all $x \in \mathcal{A}$.

By the assumption, for each $\mu \in \mathbb{T}^1$,

$$\|h(2^n \mu x) - 2\mu h(2^{n-1}x)\| \leq \varphi(2^{n-1}x, 2^{n-1}x)$$

for all $x \in \mathcal{A}$. One can show that

$$\|\mu h(2^n x) - 2\mu h(2^{n-1}x)\| \leq |\mu| \cdot \|h(2^n x) - 2h(2^{n-1}x)\| \leq \varphi(2^{n-1}x, 2^{n-1}x)$$

for all $\mu \in \mathbb{T}^1$ and all $x \in \mathcal{A}$. So

$$\begin{aligned}\|h(2^n \mu x) - \mu h(2^n x)\| \leq & \|h(2^n \mu x) - 2\mu h(2^{n-1}x)\| + \|2\mu h(2^{n-1}x) - \mu h(2^n x)\| \\ \leq & \varphi(2^{n-1}x, 2^{n-1}x) + \varphi(2^{n-1}x, 2^{n-1}x)\end{aligned}$$

for all $\mu \in \mathbb{T}^1$ and all $x \in \mathcal{A}$. Thus $2^{-n}\|h(2^n \mu x) - \mu h(2^n x)\| \to 0$ as $n \to \infty$ for all $\mu \in \mathbb{T}^1$ and all $x \in \mathcal{A}$. Hence

$$H(\mu x) = \lim_{n\to\infty} \frac{h(2^n \mu x)}{2^n} = \lim_{n\to\infty} \frac{\mu h(2^n x)}{2^n} = \mu H(x) \tag{2.5}$$

for all $\mu \in \mathbb{T}^1$ and all $x \in \mathcal{A}$.

Now let $\lambda \in \mathbb{C}$ $(\lambda \neq 0)$ and M an integer greater than $4|\lambda|$. Then $|\frac{\lambda}{M}| < \frac{1}{4} < 1 - \frac{2}{3} = \frac{1}{3}$. By [13], Theorem 1, there exist three elements $\mu_1, \mu_2, \mu_3 \in \mathbb{T}^1$ such that $3\frac{\lambda}{M} = \mu_1 + \mu_2 + \mu_3$. So by (2.5)

$$\begin{aligned}H(\lambda x) &= H(\frac{M}{3} \cdot 3\frac{\lambda}{M}x) = M \cdot H(\frac{1}{3} \cdot 3\frac{\lambda}{M}x) = \frac{M}{3}H(3\frac{\lambda}{M}x) \\ &= \frac{M}{3}H(\mu_1 x + \mu_2 x + \mu_3 x) = \frac{M}{3}(H(\mu_1 x) + H(\mu_2 x) + H(\mu_3 x)) \\ &= \frac{M}{3}(\mu_1 + \mu_2 + \mu_3)H(x) = \frac{M}{3} \cdot 3\frac{\lambda}{M}H(x) \\ &= \lambda H(x)\end{aligned}$$

for all $x \in \mathcal{A}$. Hence

$$H(\zeta x + \eta y) = H(\zeta x) + H(\eta y) = \zeta H(x) + \eta H(y)$$

for all $\zeta, \eta \in \mathbb{C}(\zeta, \eta \neq 0)$ and all $x, y \in \mathcal{A}$. We have that $H(0x) = 0 = 0H(x)$ for all $x \in \mathcal{A}$. So the unique additive mapping $H : \mathcal{A} \to \mathcal{B}$ is a $\mathbb{C}$-linear mapping.

Since $h(2^n u \circ y) = h(2^n u) \circ h(y)$ for all $u \in \mathcal{U}(\mathcal{A})$, all $y \in \mathcal{A}$ and all $n \in \mathbb{Z}$,

$$H(u \circ y) = \lim_{n\to\infty} \frac{1}{2^n} h(2^n u \circ y) = \lim_{n\to\infty} \frac{1}{2^n} h(2^n u) \circ h(y) = H(u) \circ h(y) \tag{2.6}$$

for all $u \in \mathcal{U}(\mathcal{A})$ and all $y \in \mathcal{A}$. By the additivity of H and (2.6),

$$2^n H(u \circ y) = H(2^n u \circ y) = H(u \circ (2^n y)) = H(u) \circ h(2^n y)$$

for all $u \in \mathcal{U}(\mathcal{A})$ and all $y \in \mathcal{A}$. Hence

$$H(u \circ y) = \frac{1}{2^n} H(u) \circ h(2^n y) = H(u) \circ \frac{1}{2^n} h(2^n y) \tag{2.7}$$

for all $u \in \mathcal{U}(\mathcal{A})$ and all $y \in \mathcal{A}$. Taking the limit in (2.7) as $n \to \infty$, we obtain

$$H(u \circ y) = H(u) \circ H(y) \tag{2.8}$$

for all $u \in \mathcal{U}(\mathcal{A})$ and all $y \in \mathcal{A}$. Since H is $\mathbb{C}$-linear and each $x \in \mathcal{A}$ is a finite linear combination of unitary elements (see [14], Theorem 4.1.7), i.e., $x = \sum_{j=1}^m \lambda_j u_j$ $(\lambda_j \in \mathbb{C}, u_j \in \mathcal{U}(\mathcal{A}))$, it follows from (2.8) that

$$\begin{aligned} H(x \circ y) &= H(\sum_{j=1}^m \lambda_j u_j \circ y) = \sum_{j=1}^m \lambda_j H(u_j \circ y) \\ &= \sum_{j=1}^m \lambda_j H(u_j) \circ H(y) = H(\sum_{j=1}^m \lambda_j u_j) \circ H(y) \\ &= H(x) \circ H(y) \end{aligned}$$

for all $x, y \in \mathcal{A}$.

By (2.3) and (2.6),

$$H(y) = H(e \circ y) = H(e) \circ h(y) = e' \circ h(y) = h(y)$$

for all $y \in \mathcal{A}$.

Therefore, the mapping $h : \mathcal{A} \to \mathcal{B}$ is a homomorphism, as desired. □

Corollary 2. *Let $h : \mathcal{A} \to \mathcal{B}$ be a mapping satisfying $h(0) = 0$ and $h(2^n u \circ y) = h(2^n u) \circ h(y)$ for all $u \in \mathcal{U}(\mathcal{A})$, all $y \in \mathcal{A}$ and all $n \in \mathbb{Z}$, for which there exist constants $\theta \geq 0$ and $p \in [0, 1)$ such that*

$$\|h(\mu x + \mu y) - \mu h(x) - \mu h(y)\| \leq \theta(||x||^p + ||y||^p)$$

for all $\mu \in \mathbb{T}^1$ *and all* $x, y \in \mathcal{A}$. *Assume that* $\lim_{n\to\infty} \frac{h(2^n e)}{2^n} = e'$. *Then the mapping* $h : \mathcal{A} \to \mathcal{B}$ *is a homomorphism.*

Proof. Define $\varphi(x, y) = \theta(||x||^p + ||y||^p)$ and apply Theorem 1. □

Theorem 3. *Let* $h : \mathcal{A} \to \mathcal{B}$ *be a mapping satisfying* $h(0) = 0$ *and* $h(2^n u \circ y) = h(2^n u) \circ h(y)$ *for all* $u \in \mathcal{U}(\mathcal{A})$, *all* $y \in \mathcal{A}$ *and all* $n \in \mathbb{Z}$, *for which there exists a function* $\varphi : \mathcal{A} \times \mathcal{A} \to [0, \infty)$ *satisfying* (2.1) *and* (2.3) *such that*

$$\|h(\mu x + \mu y) - \mu h(x) - \mu h(y)\| \leq \varphi(x, y) \tag{2.9}$$

for $\mu = 1, i$ *and all* $x, y \in \mathcal{A}$. *If* $h(tx)$ *is continuous in* $t \in \mathbb{R}$ *for each fixed* $x \in \mathcal{A}$, *then the mapping* $h : \mathcal{A} \to \mathcal{B}$ *is a homomorphism.*

Proof. Put $\mu = 1$ in (2.9). By the same reasoning as in the proof of Theorem 1, there exists a unique additive mapping $H : \mathcal{A} \to \mathcal{B}$ satisfying (2.4).

By the same reasoning as in the proof of [26], Theorem, the additive mapping $H : \mathcal{A} \to \mathcal{B}$ is $\mathbb{R}$-linear.

Put $\mu = i$ in (2.9). By the same method as in the proof of Theorem 1, one can obtain that

$$H(ix) = \lim_{n\to\infty} \frac{h(2^n ix)}{2^n} = \lim_{n\to\infty} \frac{ih(2^n x)}{2^n} = iH(x)$$

for all $x \in \mathcal{A}$.

For each element $\lambda \in \mathbb{C}$, $\lambda = s + it$, where $s, t \in \mathbb{R}$. So

$$\begin{aligned} H(\lambda x) &= H(sx + itx) = sH(x) + tH(ix) = sH(x) + itH(x) = (s + it)H(x) \\ &= \lambda H(x) \end{aligned}$$

for all $\lambda \in \mathbb{C}$ and all $x \in \mathcal{A}$. Thus

$$H(\zeta x + \eta y) = H(\zeta x) + H(\eta y) = \zeta H(x) + \eta H(y)$$

for all $\zeta, \eta \in \mathbb{C}$, and all $x, y \in \mathcal{A}$. Hence the additive mapping $H : \mathcal{A} \to \mathcal{B}$ is $\mathbb{C}$-linear.

The rest of the proof is the same as in the proof of Theorem 1. □

From now on, assume that $\mathcal{A}$ is a JC^*-algebra of real rank zero, where "real rank zero" means that the set of invertible self-adjoint elements is dense in the set of self-adjoint elements (see [2]).

Now we investigate continuous homomorphisms between JC^*-algebras.

Theorem 4. *Let $h : \mathcal{A} \to \mathcal{B}$ be a continuous mapping satisfying $h(0) = 0$ and $h(2^n u \circ y) = h(2^n u) \circ h(y)$ for all $u \in I_1(\mathcal{A}_{sa})$, all $y \in \mathcal{A}$ and all $n \in \mathbb{Z}$, for which there exists a function $\varphi : \mathcal{A} \times \mathcal{A} \to [0, \infty)$ satisfying* (2.1)*,* (2.2) *and* (2.3)*. Then the mapping $h : \mathcal{A} \to \mathcal{B}$ is a homomorphism.*

Proof. By the same reasoning as in the proof of Theorem 1, there exists a unique $\mathbb{C}$-linear mapping $H : \mathcal{A} \to \mathcal{B}$ satisfying (2.4).

Since $h(2^n u \circ y) = h(2^n u) \circ h(y)$ for all $u \in I_1(\mathcal{A}_{sa})$, all $y \in \mathcal{A}$ and all $n \in \mathbb{Z}$,

$$H(u \circ y) = \lim_{n\to\infty} \frac{1}{2^n} h(2^n u \circ y) = \lim_{n\to\infty} \frac{1}{2^n} h(2^n u) \circ h(y) = H(u) \circ h(y) \tag{2.10}$$

for all $u \in I_1(\mathcal{A}_{sa})$ and all $y \in \mathcal{A}$. By the additivity of H and (2.10),

$$2^n H(u \circ y) = H(2^n u \circ y) = H(u \circ (2^n y)) = H(u) \circ h(2^n y)$$

for all $u \in I_1(\mathcal{A}_{sa})$ and all $y \in \mathcal{A}$. Hence

$$H(u \circ y) = \frac{1}{2^n} H(u) \circ h(2^n y) = H(u) \circ \frac{1}{2^n} h(2^n y) \tag{2.11}$$

for all $u \in I_1(\mathcal{A}_{sa})$ and all $y \in \mathcal{A}$. Taking the limit in (2.11) as $n \to \infty$, we obtain

$$H(u \circ y) = H(u) \circ H(y) \tag{2.12}$$

for all $u \in I_1(\mathcal{A}_{sa})$ and all $y \in \mathcal{A}$.

By (2.3) and (2.10),

$$H(y) = H(e \circ y) = H(e) \circ h(y) = e' \circ h(y) = h(y)$$

for all $y \in \mathcal{A}$. So $H : \mathcal{A} \to \mathcal{B}$ is continuous. But by the assumption that $\mathcal{A}$ has real rank zero, it is easy to show that $I_1(\mathcal{A}_{sa})$ is dense in $\{x \in A_{sa} \mid ||x|| = 1\}$. So for each $w \in \{z \in \mathcal{A}_{sa} \mid ||z|| = 1\}$, there is a sequence $\{\kappa_j\}$ such that $\kappa_j \to w$ as $j \to \infty$

and $\kappa_j \in I_1(\mathcal{A}_{sa})$. Since $H : \mathcal{A} \to \mathcal{B}$ is continuous, it follows from (2.12) that

$$\begin{aligned} H(w \circ y) &= H(\lim_{j\to\infty} \kappa_j \circ y) = \lim_{j\to\infty} H(\kappa_j \circ y) \\ &= \lim_{j\to\infty} H(\kappa_j) \circ H(y) = H(\lim_{j\to\infty} \kappa_j) \circ H(y) \\ &= H(w) \circ H(y) \end{aligned} \tag{2.13}$$

for all $w \in \{z \in \mathcal{A}_{sa} \mid ||z|| = 1\}$ and all $y \in \mathcal{A}$.

For each $x \in \mathcal{A}$, $x = \frac{x+x^*}{2} + i\frac{x-x^*}{2i}$, where $x_1 := \frac{x+x^*}{2}$ and $x_2 := \frac{x-x^*}{2i}$ are self-adjoint.

First, consider the case that $x_1 \neq 0, x_2 \neq 0$. Since $H : \mathcal{A} \to \mathcal{B}$ is $\mathbb{C}$-linear, it follows from (2.13) that

$$\begin{aligned} H(x \circ y) &= H(x_1 \circ y + ix_2 \circ y) = H(||x_1||\frac{x_1}{||x_1||} \circ y + i||x_2||\frac{x_2}{||x_2||} \circ y) \\ &= ||x_1||H(\frac{x_1}{||x_1||} \circ y) + i||x_2||H(\frac{x_2}{||x_2||} \circ y) \\ &= ||x_1||H(\frac{x_1}{||x_1||}) \circ H(y) + i||x_2||H(\frac{x_2}{||x_2||}) \circ H(y) \\ &= \{H(||x_1||\frac{x_1}{||x_1||}) + iH(||x_2||\frac{x_2}{||x_2||})\} \circ H(y) = H(x_1 + ix_2) \circ H(y) \\ &= H(x) \circ H(y) \end{aligned}$$

for all $y \in \mathcal{A}$.

Next, consider the case that $x_1 \neq 0, x_2 = 0$. Since $H : \mathcal{A} \to \mathcal{B}$ is $\mathbb{C}$-linear, it follows from (2.13) that

$$\begin{aligned} H(x \circ y) &= H(x_1 \circ y) = H(||x_1||\frac{x_1}{||x_1||} \circ y) = ||x_1||H(\frac{x_1}{||x_1||} \circ y) \\ &= ||x_1||H(\frac{x_1}{||x_1||}) \circ H(y) = H(||x_1||\frac{x_1}{||x_1||}) \circ H(y) = H(x_1) \circ H(y) \\ &= H(x) \circ H(y) \end{aligned}$$

for all $y \in \mathcal{A}$.

Finally, consider the case that $x_1 = 0, x_2 \neq 0$. Since $H : \mathcal{A} \to \mathcal{B}$ is $\mathbb{C}$-linear, it follows from (2.13) that

$$\begin{aligned} H(x \circ y) &= H(ix_2 \circ y) = H(i||x_2||\frac{x_2}{||x_2||} \circ y) = i||x_2||H(\frac{x_2}{||x_2||} \circ y) \\ &= i||x_2||H(\frac{x_2}{||x_2||}) \circ H(y) = H(i||x_2||\frac{x_2}{||x_2||}) \circ H(y) = H(ix_2) \circ H(y) \\ &= H(x) \circ H(y) \end{aligned}$$

for all $y \in \mathcal{A}$. Hence

$$H(x \circ y) = H(x) \circ H(y)$$

for all $x, y \in \mathcal{A}$.

Therefore, the mapping $h : \mathcal{A} \to \mathcal{B}$ is a homomorphism, as desired. □

Corollary 5. *Let $h : \mathcal{A} \to \mathcal{B}$ be a continuous mapping satisfying $h(0) = 0$ and $h(2^n u \circ y) = h(2^n u) \circ h(y)$ for all $u \in I_1(\mathcal{A}_{sa})$, all $y \in \mathcal{A}$ and all $n \in \mathbb{Z}$, for which there exist constants $\theta \geq 0$ and $p \in [0, 1)$ such that*

$$\|h(\mu x + \mu y) - \mu h(x) - \mu h(y)\| \leq \theta(||x||^p + ||y||^p)$$

for all $\mu \in \mathbb{T}^1$ and all $x, y \in \mathcal{A}$. Assume that $\lim_{n\to\infty} \frac{h(2^n e)}{2^n} = e'$. Then the mapping $h : \mathcal{A} \to \mathcal{B}$ is a homomorphism.

Proof. Define $\varphi(x, y) = \theta(||x||^p + ||y||^p)$ and apply Theorem 4. □

Theorem 6. *Let $h : \mathcal{A} \to \mathcal{B}$ be a continuous mapping satisfying $h(0) = 0$ and $h(2^n u \circ y) = h(2^n u) \circ h(y)$ for all $u \in I_1(\mathcal{A}_{sa})$, all $y \in \mathcal{A}$ and all $n \in \mathbb{Z}$, for which there exists a function $\varphi : \mathcal{A} \times \mathcal{A} \to [0, \infty)$ satisfying (2.1), (2.3) and (2.9). Then the mapping $h : \mathcal{A} \to \mathcal{B}$ is a homomorphism.*

Proof. By the same reasoning as in the proof of Theorem 3, there exists a unique $\mathbb{C}$-linear mapping $H : \mathcal{A} \to \mathcal{B}$ satisfying (2.4).

The rest of the proof is the same as in the proofs of Theorems 1 and 4. □

3. Stability of homomorphisms in JC^*-algebras

In this section, we prove the generalized Hyers-Ulam stability of homomorphisms in JC^*-algebras.

Theorem 7. *Let $h : \mathcal{A} \to \mathcal{B}$ be a mapping with $h(0) = 0$ for which there exists a function $\varphi : \mathcal{A}^4 \to [0, \infty)$ such that*

$$\widetilde{\varphi}(x, y, z, w) := \sum_{j=0}^{\infty} 2^{-j} \varphi(2^j x, 2^j y, 2^j z, 2^j w) < \infty, \tag{3.1}$$

$$\|h(\mu x + \mu y + z \circ w) - \mu h(x) - \mu h(y) - h(z) \circ h(w)\| \le \varphi(x, y, z, w) \tag{3.2}$$

for all $\mu \in \mathbb{T}^1$ and all $x, y, z, w \in \mathcal{A}$. Then there exists a unique homomorphism $H : \mathcal{A} \to \mathcal{B}$ such that

$$\|h(x) - H(x)\| \le \frac{1}{2} \widetilde{\varphi}(x, x, 0, 0) \tag{3.3}$$

for all $x \in \mathcal{A}$.

Proof. Put $z = w = 0$ in (3.2). By the same reasoning as in the proof of Theorem 1, there exists a unique $\mathbb{C}$-linear mapping $H : \mathcal{A} \to \mathcal{B}$ satisfying (3.3). The $\mathbb{C}$-linear mapping $H : \mathcal{A} \to \mathcal{B}$ is given by

$$H(x) = \lim_{n\to\infty} \frac{1}{2^n} h(2^n x) \tag{3.4}$$

for all $x \in \mathcal{A}$.

Let $x = y = 0$ in (3.2). Then we get

$$\|h(z \circ w) - h(z) \circ h(w)\| \le \varphi(0, 0, z, w)$$

for all $z, w \in \mathcal{A}$. Since

$$\frac{1}{2^{2n}} \varphi(0, 0, 2^n z, 2^n w) \le \frac{1}{2^n} \varphi(0, 0, 2^n z, 2^n w),$$

$$\begin{aligned} \frac{1}{2^{2n}} \|h(2^n z \circ 2^n w) - h(2^n z) \circ h(2^n w)\| &\le \frac{1}{2^{2n}} \varphi(0, 0, 2^n z, 2^n w) \\ &\le \frac{1}{2^n} \varphi(0, 0, 2^n z, 2^n w) \end{aligned}$$

for all $z, w \in \mathcal{A}$. By (3.1) and (3.5),

$$\begin{aligned} H(z \circ w) &= \lim_{n\to\infty} \frac{h(2^{2n} z \circ w)}{2^{2n}} = \lim_{n\to\infty} \frac{h(2^n z \circ 2^n w)}{2^n \cdot 2^n} \\ &= \lim_{n\to\infty} \left(\frac{h(2^n z)}{2^n} \circ \frac{h(2^n w)}{2^n}\right) = \lim_{n\to\infty} \frac{h(2^n z)}{2^n} \circ \lim_{n\to\infty} \frac{h(2^n w)}{2^n} \\ &= H(z) \circ H(w) \end{aligned}$$

for all $z, w \in \mathcal{A}$. Hence the $\mathbb{C}$-linear mapping $H : \mathcal{A} \to \mathcal{B}$ is a homomorphism satisfying (3.3), as desired. □

Corollary 8. *Let $h : \mathcal{A} \to \mathcal{B}$ be a mapping with $h(0) = 0$ for which there exist constants $\theta \geq 0$ and $p \in [0, 1)$ such that*

$$\|h(\mu x + \mu y + z \circ w) - \mu h(x) - \mu h(y) - h(z) \circ h(w)\| \\ \leq \theta(\|x\|^p + \|y\|^p + \|z\|^p + \|w\|^p)$$

for all $\mu \in \mathbb{T}^1$ and all $x, y, z, w \in \mathcal{A}$. Then there exists a unique homomorphism $H : \mathcal{A} \to \mathcal{B}$ such that

$$\|h(x) - H(x)\| \leq \frac{2\theta}{2 - 2^p} ||x||^p$$

for all $x \in \mathcal{A}$.

Proof. Define $\varphi(x, y, z, w) = \theta(\|x\|^p + \|y\|^p + \|z\|^p + \|w\|^p)$ and apply Theorem 7. □

Theorem 9. *Let $h : \mathcal{A} \to \mathcal{B}$ be a mapping with $h(0) = 0$ for which there exists a function $\varphi : \mathcal{A}^4 \to [0, \infty)$ satisfying (3.1) such that*

$$\|h(\mu x + \mu y + z \circ w) - \mu h(x) - \mu h(y) - h(z) \circ h(w)\| \leq \varphi(x, y, z, w)$$

for $\mu = 1, i$ and all $x, y, z, w \in \mathcal{A}$. If $h(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in \mathcal{A}$, then there exists a unique homomorphism $H : \mathcal{A} \to \mathcal{B}$ satisfying (3.3).

Proof. By the same reasoning as in the proof of Theorem 3, there exists a unique $\mathbb{C}$-linear mapping $H : \mathcal{A} \to \mathcal{B}$ satisfying (3.3).

The rest of the proof is the same as in the proofs of Theorems 1 and 7. □

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ON SOME INTEGRAL OPERATORS WHICH PRESERVE THE UNIVALENCE

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Abstract. We study some integral operators and determine conditions for the univalence of these integral operators.

1. Introduction

Let A be the class of the functions $f(z)$ which are analytic in the open unit disc $U = \{z \in C : |z| < 1\}$ and $f(0) = f'(0) - 1 = 0$.

We denote by S the subclass of A consisting of functions $f(z) \in A$ which are univalent in U.

In this paper, we consider the integral operators

$$H_\beta(z) = \left\{\beta \int_0^z [h(u)]^{\beta-1}\, du\right\}^{\frac{1}{\beta}} \tag{1}$$

and

$$G_\beta(z) = \left\{\beta \int_0^z u\,[g(u)]^{\beta-2}\, du\right\}^{\frac{1}{\beta}} \tag{2}$$

for $h(z) \in S$, $g(z) \in S$ and $\beta \in C$.

2. Preliminary results

To discuss our integral operators, we need the following theorem.

Theorem 2.1 [1]. *Let α be a complex number,* $\operatorname{Re}\alpha > 0$ *and $f \in A$. If*

$$\frac{1-|z|^{2\operatorname{Re}\alpha}}{\operatorname{Re}\alpha}\left|\frac{zf''(z)}{f'(z)}\right| \leq 1, \tag{3}$$

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for all $z \in U$*, then for any complex number* β, $\operatorname{Re}\beta \geq \operatorname{Re}\alpha$ *the function*

$$F_\beta(z) = \left[\beta \int_0^z u^{\beta-1} f'(u)du\right]^{\frac{1}{\beta}} \tag{4}$$

is in the class S.

3. **Main results**

Theorem 3.1. *Let* α, β *be complex numbers and the function* $h \in S$,

$$h(z) = z + a_2 z^2 + \dots.$$

If

$$\operatorname{Re}\,\beta \geq \operatorname{Re}\,\alpha > 0 \tag{j_1}$$

$$|\beta - 1| \leq \frac{\operatorname{Re}\,\alpha}{4} \quad \textit{for} \operatorname{Re}\,\alpha \in (0,1) \tag{j_2}$$

or

$$|\beta - 1| \leq \frac{1}{4} \quad \textit{for} \operatorname{Re}\,\alpha \in [1,\infty) \tag{j_3}$$

then the function

$$H_\beta(z) = \left\{\beta \int_0^z [h(u)]^{\beta-1}\, du\right\}^{\frac{1}{\beta}} \tag{5}$$

belongs to the class S.

Proof. From (5) we have

$$H_\beta(z) = \left\{\beta \int_0^z u^{\beta-1} \left[\frac{h(u)}{u}\right]^{\beta-1} du\right\}^{\frac{1}{\beta}} \tag{6}$$

The function $h(z)$ is regular and univalent, hence $\frac{h(z)}{z} \neq 0$ for all $z \in U$. We can choose the regular brach of the function $\left[\frac{h(z)}{z}\right]^{\beta-1}$, which is equal to 1 at the origin.

Let us consider the regular function in U, given by

$$f(z) = \int_0^z \left[\frac{h(u)}{u}\right]^{\beta-1} du. \tag{7}$$

Because $h \in S$, we obtain

$$\left|\frac{z\,h'(z)}{h(z)}\right| \leq \frac{1+|z|}{1-|z|} \tag{8}$$

for all $z \in U$.

We have

$$\left|\frac{z\,f''(z)}{f'(z)}\right| = |\beta - 1|\left|\frac{z\,h'(z)}{h(z)} - 1\right| \le |\beta - 1|\frac{2}{1-|z|}. \tag{9}$$

Now, we consider the cases

$i_1)$ Re $\alpha \ge 1$.

We observe that the function $p : [1, \infty) \to \mathbb{R}$,

$$p(x) = \frac{1 - a^{2x}}{x} \quad (0 < a < 1) \tag{10}$$

is a decreasing function, and that, if we take $a = |z|$, $z \in U$, then

$$\frac{1 - |z|^{2\mathrm{Re}\ \alpha}}{\mathrm{Re}\ \alpha} \le 1 - |z|^2 \tag{11}$$

for all $z \in U$.

From (11) and (9) we obtain

$$\frac{1 - |z|^{2\mathrm{Re}\ \alpha}}{\mathrm{Re}\ \alpha}\left|\frac{zf''(z)}{f'(z)}\right| \le 4\,|\beta - 1|. \tag{12}$$

From (12) and (j_3), we have

$$\frac{1 - |z|^{2\mathrm{Re}\ \alpha}}{\mathrm{Re}\ \alpha}\left|\frac{zf''(z)}{f'(z)}\right| \le 1 \tag{13}$$

for all $z \in U$.

$i_2)$ $0 <$ Re $\alpha < 1$.

The function $v : (0, 1) \to \mathbb{R}$,

$$v(x) = 1 - a^{2x} \quad (0 < a < 1) \tag{14}$$

is a increasing function and for $a = |z|$, $z \in U$, we obtain

$$1 - |z|^{2\,\mathrm{Re}\ \alpha} \le 1 - |z|^2, \quad z \in U \tag{15}$$

for all $z \in U$.

From (9) and (15), we have

$$\frac{1 - |z|^{2\mathrm{Re}\ \alpha}}{\mathrm{Re}\ \alpha}\left|\frac{zf''(z)}{f'(z)}\right| \le \frac{4\,|\beta - 1|}{\mathrm{Re}\ \alpha} \tag{16}$$

for all $z \in U$.

Using the condition (j_2) and (16) we get

$$\frac{1-|z|^{2\text{Re }\alpha}}{\text{Re }\alpha}\left|\frac{zf''(z)}{f'(z)}\right| \leq 1 \tag{17}$$

for all $z \in U$.

Because $f'(z) = \left[\frac{h(z)}{z}\right]^{\beta-1}$, from Theorem 2.1 it results that the function $H_\beta(z)$ is regular and univalent in U.

Theorem 3.2. *Let α, β be complex numbers and the function $g \in S$, $g(z) = z + a_2z^2 + \dots$.*

If

$$\text{Re }\beta \geq \text{Re }\alpha > 0 \tag{p_1}$$

and

$$|\beta - 2| \leq \frac{\text{Re }\alpha}{4} \quad \textit{for } \text{Re }\alpha \in (0,1) \tag{p_2}$$

or

$$|\beta - 2| \leq \frac{1}{4} \quad \textit{for } \text{Re }\alpha \in [1,\infty) \tag{p_3}$$

then the function

$$G_\beta(z) = \left\{\beta \int_0^z u\,[g(u)]^{\beta-2}\, du\right\}^{\frac{1}{\beta}} \tag{18}$$

is in the class S.

Proof. We observe that

$$G_\beta(z) = \left\{\beta \int_0^z u^{\beta-1}\left[\frac{g(u)}{u}\right]^{\beta-2} du\right\}^{\frac{1}{\beta}} \tag{19}$$

We consider the regular function in U

$$\left[f(z) = \int_0^z \frac{g(u)}{u}\right]^{\beta-2} du$$

and by the same reasoning with a view to the Theorem 3.1. we conclude that the function $G_\beta(z)$ is in the class S in the conditions (p_1), (p_2) and (p_3).

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A CUTTING PLANE APPROACH TO SOLVE THE RAILWAY TRAVELING SALESMAN PROBLEM

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Abstract. We consider the Railway Traveling Salesman Problem. We show that this problem can be reduced to a variant of the generalized traveling salesman problem, defined on an undirected graph $G = (V, E)$ with the nodes partitioned into clusters, which consists in finding a minimum cost cycle spanning a subset of nodes with the property that exactly two nodes are chosen from each cluster. We describe an exact exponential time algorithm for the problem, as well we present two mixed integer programming models of the problem. Based on one of this models proposed, we present an efficient solution procedure based on a cutting plane algorithm. Extensive computational results for instances taken from the railroad company of the Netherlands Nederlandse Spoorwegen and involving graphs with up to 2182 nodes and 38650 edges are reported.

1. Introduction

Assume that a salesman traveling with railways wishes to visit a certain number of cities. The salesman has a limited budget and the goal is to establish a schedule that allows him to visit all the cities and returning to the starting city at the total minimum cost, taking into consideration that when arrived at a station he/she has to spend some time for his affairs and then to continue his journey to another city with the first available train. We shall refer to this problem as the *Railway Traveling Salesman problem*, denoted (RTSP).

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The salesman is aware of the time schedule and is therefore able to construct the corresponding time-expanded graph $G = (V, E)$, see [1]. In that graph, every event (arrival or departure of a train) at a station corresponds to a node and edges between nodes represent either elementary connections between two events (i.e. served by a train that does not stop in-between), or waiting within a station. Nodes representing time events belonging to the same station (city) will be referred to as nodes within the same cluster, and the total number of clusters equals the total number of stations p that the salesman has to visit. There are two types of edges: inter-cluster edges (corresponding to elementary connections between the stations) and intra-cluster edges (corresponding to waiting in a station for some later connection). With this graph at hand the salesman can associate costs to its edges according to the cost measure he/she wants to minimize. Consequently, the RTSP reduces in finding a Hamiltonian tour H of the minimum cost in the subgraph of G induced by S, where $S \subseteq V$ such that S contains exactly two nodes from every cluster. This leads to a variant of the so-called *generalized traveling salesman problem* (GTSP).

The generalized traveling salesman problem, introduced by Laporte and Nobert [5] and by Noon and Bean [6] is defined on a complete undirected graph G whose nodes are partitioned into a number of subsets (clusters) and whose edges have a nonnegative cost. The GTSP asks for finding a minimum-cost Hamiltonian tour H in the subgraph of G induced by S, where $S \subseteq V$ such that S contains *at least* one node from each cluster.

A different version of the problem called E-GTSP arises when imposing the additional constraint that *exactly* one node from each cluster must be visited.

Both problems GTSP and E-GTSP are *NP*-hard, as they reduce to traveling salesman problem when each cluster consists of exactly one node.

The GTSP has several applications to location and telecommunication problems. More information on these problems and their applications can be found in Fischetti, Salazar and Toth [1, 2], Laporte, Asef-Vaziri and Sriskandarajah [3], Laporte, Mercure and Nobert [4]. It is worth to mention that Fischetti, Salazar and

Toth [2] solved the GMST problem to optimality for graphs with up to 442 nodes using a branch-and-cut algorithm.

In this paper, we introduce the (above mentioned) variant of the GTSP, called the 2-GTSP, which, given a graph G with non-negative edge costs, asks for finding a minimum cost Hamiltonian tour H of G spanning a subset of nodes that includes exactly two nodes from each cluster and exactly one edge from each cluster. Clearly, a solution to 2-GTSP is a solution to the railway traveling salesman problem.

The aim of this paper is to provide an exact algorithm for the 2-GTSP as well as two integer programming formulations of the problem and an efficient cutting plane algorithm.

2. **Definition and Complexity of the 2-GTSP**

Let $G = (V, E)$ be an n-node undirected graph whose edges are associated with non-negative costs. We will assume w.l.o.g. that G is a complete graph (if there is no edge between two nodes, we can add it with an infinite cost). Let $V_1, ..., V_p$ be a partition of V into p subsets called *clusters* (i.e. $V = V_1 \cup V_2 \cup ... \cup V_p$ and $V_l \cap V_k = \emptyset$ for all $l, k \in \{1, ..., p\}$). We denote the cost of an edge $e = \{i, j\} \in E$ by c_{ij} or by $c(i, j)$. Let $e = \{i, j\}$ be an edge with $i \in V_l$ and $j \in V_k$. If $l \neq k$ the e is called an *inter-cluster* edge; otherwise e is called an *intra-cluster* edge.

The *2-generalized traveling salesman problem* (2-GTSP) asks for finding a minimum-cost tour H spanning a subset of nodes such that H contains exactly two nodes from each cluster V_i, $i \in \{1, ..., p\}$. The problem involved two related decisions:

- choosing a node subset $S \subseteq V$, such that $|S \cap V_k| = 2$, for all $k = 1, ..., p$.
- finding a minimum cost Hamiltonian cycle in the subgraph of G induced by S.

We will call such a cycle a *2-Hamiltonian tour*. An example of a 2-Hamiltonian tour for a graph with the nodes partitioned into 6 clusters is presented in the next figure.

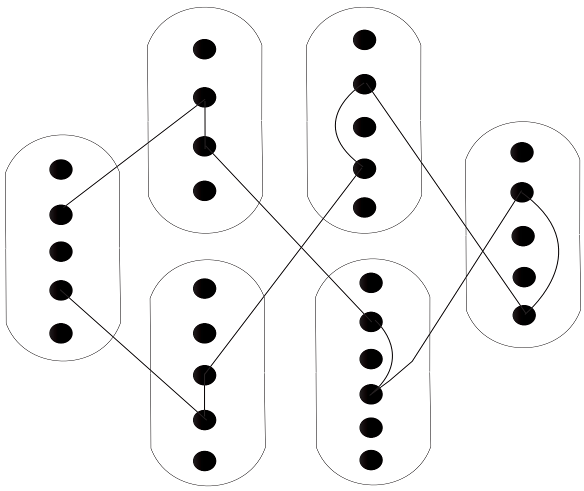

FIGURE 1. Example of a 2-Hamiltonian tour

As we already mentioned, both problems GTSP and E-GTSP are *NP*-hard, as they reduce to traveling salesman problem when each cluster consists of exactly one node. Consequently, the 2-GSTP is also an NP-hard problem.

3. An Exact Algorithm for the 2-GTSP

In this section, we present an algorithm that finds an exact solution to the 2-GTSP.

Given a sequence $(V_{k_1}, ..., V_{k_p})$ in which the clusters are visited, we want to find the best feasible 2-Hamiltonian tour H^* (w.r.t cost minimization), visiting the clusters according to the given sequence. This can be done in polynomial time, by solving $|V_{k_1}|$ shortest path problems as we will describe below.

We construct a layered network, denoted by LN, having $p + 1$ layers corresponding to the clusters $V_{k_1}, ..., V_{k_p}$ and in addition we duplicate the cluster V_{k_1}. The layered network contains all the nodes of G plus some extra nodes v' for each $v \in V_{k_1}$. There is an arc (i, j) for each $i \in V_{k_l}$ and $j \in V_{k_{l+1}}$ $(l = 1, ..., p - 1)$, having the cost c_{ij} and an arc (i, h), $i, h \in V_{k_l}$, $(l = 2, ..., p)$ having cost c_{ih}. Moreover, there is an arc (i, j') for each $i \in V_{k_p}$ and $j' \in V_{k_1}$ having cost $c_{ij'}$.

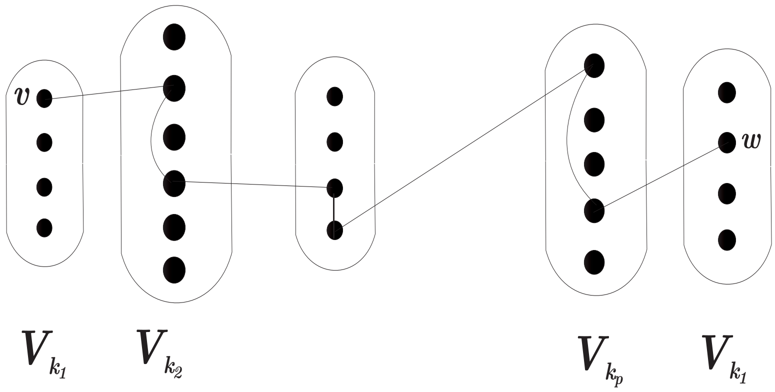

FIGURE 2. Example showing a 2-Hamiltonian tour in the constructed layered network LN

For any given $v \in V_{k_1}$, we consider paths from v to w', $w' \in V_{k_1}$, that visits exactly two nodes from each cluster $V_{k_2}, ..., V_{k_p}$, hence it gives a feasible 2-Hamiltonian tour.

Conversely, every 2-Hamiltonian tour visiting the clusters according to the sequence $(V_{k_1}, ..., V_{k_p})$ corresponds to a path in the layered network from a certain node $v \in V_{k_1}$ to $w' \in V_{k_1}$.

Therefore, it follows that the best (w.r.t cost minimization) 2-Hamiltonian tour H^* visiting the clusters in a given sequence can be found by determining all the shortest paths from each $v \in V_{k_1}$ to each $w' \in V_{k_1}$ with the property that visits exactly two nodes and one edge each from clusters $(V_{k_2}, ..., V_{k_p})$.

The overall time complexity is then $|V_{k_1}|O(m + n \log n)$, i.e. $O(nm + nlogn)$ in the worst case. We can reduce the time by choosing $|V_{k_1}|$ as the cluster with minimum cardinality.

Notice that the above procedure leads to an $O((p-1)!(nm + nlogn))$ time exact algorithm for the 2-GTSP, obtained by trying all the $(p-1)!$ possible cluster sequences. So, we have established the following result:

Theorem 1. *The above procedure provides an exact solution to the 2-GSTP in* $O((p-1)!(nm+nlogn))$ *time, where* n *is the number of nodes,* m *is the number of edges and* p *is the number of clusters in the input graph.*

Clearly, the algorithm presented, is an exponential time algorithm unless the number of clusters p is fixed.

4. **Integer Programming Formulations of the 2-GTSP**

In this section, we present two different integer programming formulations of the 2-GTSP.

In order to formulate the 2-GTSP as an integer program, we introduce the binary variables:

$$x_e = x_{ij} = \begin{cases} 1 & \text{if the edge } e = \{i,j\} \in E \\ & \text{is included in the selected subgraph} \\ & \\ 0 & \text{otherwise,} \end{cases}$$

$$z_i = \begin{cases} 1 & \text{if the node } i \text{ is included in the selected subgraph} \\ & \\ 0 & \text{otherwise.} \end{cases}$$

A feasible solution to the 2-GTSP can be seen as a cycle free subgraph with $2p-1$ edges connecting all the clusters such that exactly two nodes are selected from each cluster.

For $F \subseteq E$ and $S \subseteq V$, let $E(S) = \{e = \{i,j\} \in E \mid i,j \in S\}$, $x(F) = \sum_{e \in F} x_e$ and $z(S) = \sum_{i \in S} z_i$. Also, let $x(V_k, V_k) = \sum_{i,j \in V_k, i<j} x_{ij}$, where $k \in \{1, ..., p\}$.

The 2-GTSP can be formulated as the following 0-1 integer programming problem:

$$
\begin{aligned}
\min \quad & \sum_{e \in E} c_e x_e & & \\
s.t. \quad & z(V_k) = 2, & \forall\, k \in \{1, ..., p\} & \quad (1)\\
& x(\delta(i)) = 2z_i, & \forall\, i \in V \setminus V_1 & \quad (2)\\
& x(E) = 2p - 1 & & \quad (3)\\
& x(V_k, V_k) = 1, & \forall\, k \in \{2, ..., p\} & \quad (4)\\
& x(E(S)) \leq 2r - 1, & \forall\, S = \cup_{i=1}^{r} V_{l_i}, 2 \leq r \leq p-1 & \quad (5)\\
& x_e \in \{0,1\}, & \forall\, e \in E & \quad (6)\\
& z_i \in \{0,1\}, & \forall\, i \in V. & \quad (7)
\end{aligned}
$$

where for $i \in V \setminus V_1$, the set, denoted by $\delta(i)$, is defined as

$$\delta(i) = \{e = \{i, j\} \in E \mid j \in V\}.$$

In the above formulation, constraint (1) guarantee that from every cluster we select exactly two nodes, constraints (2) require that the number of edges incident with a node i to be either 2 (if node i is visited) or 0 otherwise, constraint (3) guarantees that the selected subgraph has $2p-1$ edges, constraints (4) guarantee that from every cluster we select (except the starting cluster) we select one edge and finally constraints (5) eliminate all the cycles connecting at most $p-1$ clusters.

Replacing the subtour elimination constraints (5) by connectivity constraints, we result in the so-called *generalized cut-set formulation*:

$$
\begin{aligned}
\min \quad & \sum_{e \in E} c_e x_e \\
s.t. \quad & (1), (3) - (5) \text{ and} \\
& x(\delta(S)) \geq 2(z_i + z_j - 1), \quad \forall\, S \subset V, \text{with } 2 \leq |S| \leq p-1 \\
& \qquad\qquad\qquad \text{and } \forall\, i \in S, j \in V \setminus S. \qquad (8)
\end{aligned}
$$

where for $S \subseteq V$, the *cut-set*, denoted by $\delta(S)$, is defined as

$$\delta(S) = \{e = \{i, j\} \in E \mid i \in S, j \notin S\}.$$

In the above formulation, constraints (8) are the connectivity constraints saying that each cut separating two visited nodes (i and j) must be crossed at least twice.

In the addition to the constraints that appear in the previous formulations, we consider also the following constraints specific to the railway traveling salesman problem:

$$t_d z_d - t_a z_a \geq t_k, \quad \forall\, a, d \in V_k,\ 2 \leq k \leq p. \tag{9}$$

The above constraints are saying that the difference between the departure and arrival times has to be at least a specified time period t_h (depending on the city), this means that the traveling salesman has to stay in each city for some time to finish his business. If the difference is too small, the salesman may fail to solve his business, on the other hand, if the difference is too large, the waiting time at the station will be inconvenient.

The disadvantage of the described integer programming formulations is their exponential number of constraints (we have to choose subsets of V, constraints (5) and (8)). These constraints can be omitted and then can be generated as needed by a *separation algorithm*: one can start without constraints (5), solve the corresponding relaxation, then generate subtour inequalities that are violated by the current solution. The separation algorithm for subtour constraints is based on network flow techniques, for further details see [2].

5. **Solution procedure and computational results**

We used the following *cutting plane algorithm* in order to solve the 2-GTSP:

1. Let the integer programming (IP) formulation consists of the constraints (1)-(4),(6),(7) and (9).

2. Solve the IP and assume that the optimal solution consists of r subtours: $S_1, ..., S_r$.
3. If $r = 1$, then STOP; the solution is optimal to the 2-GTSP. Otherwise, add to the IP formulation the corresponding constraints that eliminate the cycles $S_1, ..., S_r$ and go to Step 2.

The algorithm was written in C and for each instance we have created the corresponding integer program, which we solved it with CPLEX 6.5.

Test data for our algorithm are real networks from the Dutch railroad company *Nederlandse Spoorwegen.*

The first three data sets contains the Intercity train connections among the larger cities in the Netherlands, stopping only at the main train stations, and thus are considered faster than the normal trains. These trains operate at least every half an hour. The second real-world data set, contains the schedules of the Intercity trains and regional trains. The regional trains connect the cities in only one region, including some main stations, while trains stop at each intermediate station between two main ones.

Some characteristics of the graphs that were used for the real-world data sets and the computational results obtained using the cutting plane algorithm are displayed in the next table:

Table: Computational results for solving the RTSP

Pb. name	*No. stations*	*No. nodes*	*No. edges*	*LB/OPT*	*Sol. time*
NS1 (IC)	5	394	4240	100	14.08 s
NS2 (IC)	7	674	9754	100	64.57 s
NS3 (IC)	9	926	16271	100	206.52 s
NS4 (IC+IR)	12	1470	23850	100	39.30 min
NS5 (IC+IR)	12	1586	27383	100	72.28 min
NS6 (IC+IR)	15	1722	28200	100	1.05 h
NS7 (IC+IR)	15	1946	34450	100	5.45 h
NS8 (IC+IR)	18	2182	38650	100	4.52 h

The first four columns in the table give the name of the problem and the size of the problem: the number of stations, the number of nodes and the number of edges. The next two columns describe the cutting plane procedure and contain: the lower bounds obtained as a percentage of the optimal value of the RTSP (LB/OPT) and the computational times (CPU) for solving the RTSP to optimality.

6. **Conclusions**

We considered the Railway Traveling Salesman Problem (RTSP), which consists in finding a minimum cost tour for a salesman traveling with railways and wishing to visit a certain number of cities. We showed that the RTSP can be reduced to a variant of the Generalized Traveling Salesman problem.

Based on one of the integer programming formulations that we proposed, we present an efficient solution procedure based on a cutting plane algorithm. Computational results for real networks from the Dutch railroad company *Nederlandse Spoorwegen* and involving graphs with up to 2182 nodes and 38650 edges are reported.

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TRANSIENT LAMINAR FREE CONVECTION FROM A VERTICAL CONE WITH NON-UNIFORM SURFACE HEAT FLUX

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Abstract. In this paper, transient laminar free convection from an incompressible viscous fluid past a vertical cone with non-uniform surface heat flux $q_w(x) = x^m$ varying as a power function of the distance from the apex of the cone ($x = 0$) is presented. Here m is the exponent in power law variation of the surface heat flux. The dimensionless governing equations of the flow that are unsteady, coupled and non-linear partial differential equations are solved by an efficient, accurate and unconditionally stable finite difference scheme of Crank-Nicolson type. The velocity and temperature fields have been studied for various parameters such as Prandtl number Pr and the exponent m. The local as well as average skin friction and Nusselt number are also presented graphically and discussed in details. The present results are compared with available results from the open literature and are found to be in very good agreement.

1. **Introduction**

Natural convection flows under the influence of gravitational force have been investigated most extensively because they occur frequently in nature as well as in science and engineering applications. When a heated surface is in contact with the fluid, the result of temperature difference causes buoyancy force, which induces the natural convection. Recently heat flux applications are widely using in industries, engineering and science fields. Heat flux sensors can be used in industrial measurement and control systems. Examples of few applications are detection fouling (Boiler

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Fouling Sensor), monitoring of furnaces (Blast Furnace Monitoring/General Furnace Monitoring) and flare monitoring. Use of heat flux sensors can lead to improvements in efficiency, system safety and modeling.

Several authors have developed similarity solutions for two-dimensional axi-symmetrical problems for natural convection laminar flow over vertical cone in steady state. Merk and Prins [14, 15] developed the general relation for similar solutions on iso-thermal axi-symmetric forms and they showed that the vertical cone has such a solution in steady state. Further, Hossain et al. [10] have discussed the effects of transpiration velocity on laminar free convection boundary layer flow from a vertical non-isothermal cone and concluded, increase in temperature gradient the velocity as well as the surface temperature decreases. Ramanaiah et al. [25] discussed free convection about a permeable cone and a cylinder subjected to radiation boundary condition. Alamgir [1] has investigated the overall heat transfer in laminar natural convection from vertical cones using the integral method. Pop et al. [20] have studied the compressibility effects in laminar free convection from a vertical cone. Recently, Pop et al. [22] analyzed the steady laminar mixed convection boundary-layer flow over a vertical isothermal cone for fluids of any Pr for the both cases of buoyancy assisting and buoyancy opposing flow conditions. The resulting non-similarity boundary-layer equations are solved numerically using the Keller-box scheme for fluids of any Pr from very small to extremely large values ($0.001 \leq Pr \leq 10000$). Takhar et al. [27] discussed the effect of thermo physical quantities on the free convection flow of gases over an isothermal vertical cone in steady-state flow, in which thermal conductivity, dynamic viscosity and specific heat at constant pressure were to be assumed a power law variation with absolute temperature. They concluded that the heat transfer increases with suction and decreases with injection.

Recently theoretical studies on laminar free convection flow over an axisymmetric body have received wide attention especially in case of uniform and non-uniform surface heat flux distributions. Similarity solutions for the laminar free convection from a right circular cone with prescribed uniform heat flux conditions for $Pr = 0.72$, 1, 2, 4, 6, 8, 10 and 100 and were reported by Lin [13] and expressions for both wall skin friction and wall temperature distributions at $Pr \to \infty$ were presented. Na et al. [17, 18] studied the non-similar solutions of the problems for transverse curvature effects of the natural convection flow over a slender frustum of a cone. Later, Na et al. [19] investigated the laminar natural convection flow over a frustum of a cone without transverse curvature effects. In the above investigations the constant wall temperature as well as the constant wall heat flux was considered. The effects of the amplitude of the wavy surfaces associated with natural convection over a vertical frustum of a cone with constant wall temperature or constant wall heat flux was studied by Pop et al. [21]. Gorla et al. [24] presented numerical solution for laminar free convection of power-law fluids past a vertical frustum of a cone without transverse curvature effect (i.e. large cone angles when the boundary layer thickness is small compared with the local radius of the cone).

Further, Pop et al. [23] focused the theoretical study on the effects of suction or injection on steady free convection from a vertical cone with uniform surface heat flux condition. Kumari et al. [12] studied free convection from vertical rotating cone with uniform wall heat flux. Hasan et al. [8] analyzed double diffusion effects in free convection under flux condition along a vertical cone. Hossain et al. [9, 11] studied the non-similarity solutions for the free convection from a vertical permeable cone with non-uniform surface heat flux and the problem of laminar natural convective flow and heat transfer from a vertical circular cone immersed in a thermally stratified medium with either a uniform surface temperature or a uniform surface heat flux. Using a finite difference method, a series solution method and asymptotic solution method, the solutions have been obtained for the non-similarity boundary layer equations.

Many investigations have been done for free convection past a vertical cone/frustum of cone in porous media. Yih [29, 30] studied in saturated porous media combined heat and mass transfer effects over a full cone with uniform wall temperature/concentration or heat/mass flux and for truncated cone with non-uniform wall temperature/variable wall concentration or variable heat/variable mass flux. Recently Chamkha et al. [3] studied the problem of combined heat and mass transfer by natural convection over a permeable cone embedded in a uniform porous medium in the presence of an external magnetic field and internal heat generation or absorption effects with the cone surface is maintained at either constant temperature or concentration or uniform heat and mass fluxes. Groşan et al. [7] considering the boundary conditions either for a variable wall temperature or variable heat flux studied the similarity solutions for the problem of steady free convection over a heated vertical cone embedded in a porous medium saturated with a non-Newtonian power-law fluid driven by internal heat generation. Wang et al. [28] studied the steady laminar forced convection of micropolar fluids past two-dimensional or axisymmetric bodies with porous walls and different thermal boundary conditions (i.e. constant wall temperature/constant wall heat flux). Further, solutions of the transient free convection flow problems over moving vertical plates and cylinders as well as inclined plates have been obtained by Soundalgekar et al. [26], Muthucumaraswamy et al. [16] and Ganesan et al. [6, 4, 5] using finite difference method.

The present investigation, namely, the transient free convection from a vertical cone with non-uniform surface heat flux has not received any attention. Hence, the present work is considered to deal with transient free convection over a vertical cone with non-uniform surface heat flux. The governing boundary layer equations are solved by an implicit finite-difference scheme of Crank-Nicolson type with various parameters Pr and m. In order to check the accuracy of our numerical results, the present results are compared with the available results of Hossain and Paul [9] for non-uniform surface heat flux and Lin [13] for uniform heat flux and are found to be in excellent agreement.

2. **Mathematical analysis**

We consider the axisymmetric transient laminar free convection of a viscous and incompressible fluid of uniform ambient temperature T'_∞ past a vertical cone with non-uniform surface heat flux. It is assumed that the viscous dissipation effects are negligible. It is assumed that initially ($t' \leq 0$), the cone surface and the surrounding fluid that are at rest. Then at time $t' > 0$, it is assumed that heat is supplied from cone surface to the fluid at the rate $q_w(x) = x^m$ and it is maintained at this value with m being a constant. The co-ordinate system is chosen (as shown in Fig.1) such that x measures the distance along the surface of the cone from the apex ($x = 0$) and y measures the distance normally outward, respectively. Here, ϕ is the semi vertical angle of the cone and r is the local radius of the cone. The fluid properties are assumed to be constant except for density variations, which induce buoyancy force term in the momentum equation. The governing boundary layer equations of continuity, momentum and energy under Boussinesq approximation with the viscous dissipation effect neglected are as follows:

- continuity

$$\frac{\partial}{\partial x}(ru) + \frac{\partial}{\partial y}(ru) = 0, \tag{1}$$

- momentum

$$\frac{\partial u}{\partial t'} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = g\beta(T' - T'_\infty)\cos\phi + \nu\frac{\partial^2 u}{\partial y^2}, \tag{2}$$

- energy

$$\frac{\partial T'}{\partial t'} + u\frac{\partial T'}{\partial x} + v\frac{\partial T'}{\partial y} = \alpha\frac{\partial^2 T'}{\partial y^2}. \tag{3}$$

The initial and boundary conditions are

$$\begin{aligned}
&t' \leq 0: \ u = 0,\ v = 0,\ T' = T'_\infty \text{ for all } x \text{ and } y,\\
&t' > 0: \ u = 0,\ v = 0,\ \frac{\partial T'}{\partial y} = \frac{-q_w(x)}{k} \text{ at } y = 0,\\
&u = 0,\ T' = T'_\infty \text{ at } x = 0,\\
&u \to 0,\ T' \to T'_\infty \text{ as } y \to \infty
\end{aligned} \tag{4}$$

where u and v are the velocity components along $x-$ and $y-$ axes, T' is the fluid temperature, t' is the time, g is the acceleration due to gravity, r is the local radius of the cone, k is the thermal conductivity of the fluid, α is the thermal diffusivity, β is the thermal expansion coefficient, semi-vertical angle of the cone and ν is the kinematic viscosity.

The physical quantities of interest are the local skin friction τ_x and the local Nusselt number Nu_x which are given, respectively, by

$$\tau_x = \mu \left(\frac{\partial u}{\partial y}\right)_{y=0}, \quad Nu_x = \frac{x}{(T'_w - T'_\infty)} \left(-\frac{\partial T'}{\partial y}\right)_{y=0} \tag{5}$$

where μ is the dynamic viscosity. Also, the average skin friction $\overline{\tau}_L$ and the average heat transfer coefficient $\overline{h}$ over the cone surface are given by

$$\overline{\tau}_L = \frac{2\mu}{L^2} \int_0^L x \left(\frac{\partial u}{\partial y}\right)_{y=0} dx, \quad \overline{h} = \frac{2k}{L^2} \int_0^L \frac{x}{(T'_w - T'_\infty)} \left(-\frac{\partial T'}{\partial y}\right)_{y=0} dx \tag{6}$$

The average Nusselt number is then given by

$$\overline{Nu}_L = \frac{L\overline{h}}{k} = \frac{2}{L} \int_0^L \frac{x}{(T'_w - T'_\infty)} \left(-\frac{\partial T'}{\partial y}\right)_{y=0} dx \tag{7}$$

Further, we introduce the following non-dimensional variables:

$$\begin{gathered} X = \frac{x}{L},\ Y = \frac{y}{L} Gr^{1/5},\ t = \left(\frac{\nu}{L^2} Gr^{2/5}\right) t',\ R = \frac{r}{L}, \\ U = \left(\frac{L}{\nu} Gr^{-2/5}\right) u,\ V = \left(\frac{L}{\nu} Gr^{-1/5}\right) v,\ T = \frac{(T' - T'_\infty)}{(q_w(L)L/k)} Gr_L^{1/5}, \end{gathered} \tag{8}$$

where $Gr_L = g\beta(q_w L/k)L^4 \cos\phi/\nu^2$ is the Grashof number based on the reference length L, $Pr = \nu/\alpha$ is the Prandtl number and $r = x \sin\phi$. Equations (1), (2) and (3) can then be written in the following non-dimensional form:

$$\frac{\partial}{\partial X}(RU) + \frac{\partial}{\partial Y}(RV) = 0, \tag{9}$$

$$\frac{\partial U}{\partial t} + U\frac{\partial U}{\partial X} + V\frac{\partial U}{\partial Y} = T + \frac{\partial^2 U}{\partial Y^2}, \tag{10}$$

$$\frac{\partial T}{\partial t} + U\frac{\partial T}{\partial X} + V\frac{\partial T}{\partial Y} = \frac{1}{Pr}\frac{\partial^2 T}{\partial Y^2}, \tag{11}$$

where Pr is the Prandtl number and R is the dimensionless radius of the cone. The corresponding non-dimensional initial and boundary conditions (4) become

$$\begin{array}{lllll} t \le 0: & U = 0, & V = 0, & T = 0 \text{ for all } X \text{ and } Y & \\ t > 0: & U = 0, & V = 0, & \dfrac{\partial T}{\partial Y} = -X^m & \text{at } Y = 0 \\ & U = 0, & & T = 0 & \text{at } X = 0 \\ & U \to 0, & & T \to 0 & \text{as } Y \to \infty \end{array} \tag{12}$$

The local non-dimensional skin friction τ_X and the local Nusselt number Nu_X given by (5) become

$$\tau_X = Gr_L^{3/5}\left(\frac{\partial U}{\partial Y}\right)_{Y=0}, \quad Nu_X = \frac{Gr_L^{1/5}}{T_{Y=0}}X^{m+1} \tag{13}$$

Also, the non-dimensional average skin-friction $\overline{\tau}$ and the average Nusselt number $\overline{Nu}$ are reduced to

$$\overline{\tau} = 2Gr_L^{3/5}\int_0^1 X\left(\frac{\partial U}{\partial Y}\right)_{Y=0} dX, \quad \overline{Nu} = 2Gr_L^{1/5}\int_0^1 \frac{X^{m+1}}{(T)_{Y=0}}dX. \tag{14}$$

3. **Solution procedure**

The unsteady, non-linear, coupled and partial differential Equations (9), (10) and (11) with the initial and boundary conditions (12) are solved by employing a finite-difference scheme of Crank-Nicolson type. The finite-difference scheme of dimensionless governing equations is reduced to tri-diagonal system of equations and is solved by Thomas algorithm as discussed in Carnahan et al. [2]. The region of integration is considered as a rectangle with $X_{max} = 1$ and $Y_{max} = 26$ where Y_{max} corresponds to Y_∞, which lies very well out side both the momentum and thermal boundary layers. The maximum of Y was chosen as 26, after some preliminary investigation so that the last two boundary conditions of (12) are satisfied within the tolerance limit 10^{-5}. After experimenting with a few sets of mesh sizes, the mesh sizes have been fixed as $\Delta X = 0.05$, $\Delta Y = 0.05$ with time step $\Delta t = 0.01$. The scheme is unconditionally stable. The local truncation error is $O(\Delta t^2 + \Delta Y^2 + \Delta X)$ and it tends to zero as Δt, ΔY and ΔX tend to zero. Hence, the scheme is compatible. Stability and compatibility ensure the convergence.

4. **Results and discussion**

In order to prove the accuracy of our numerical results, the present results in steady state at $X = 1.0$ are compared with available similarity solutions in literature. The velocity and temperature profiles of cone with uniform surface heat flux when $Pr = 0.72$ are displayed in Fig.2 and the numerical values of local skin-friction τ_X and local Nusselt number Nu_X, for different values of Prandtl number shown in Table

1 are compared with similarity solutions of Lin [13] in steady state using a suitable transformation (i.e. $Y = (20/9)^{1/5}\eta$, $T = (20/9)^{1/5}[-\theta(0)]$, $U = (20/9)^{1/5} f'(\eta)$, $\tau_X = (20/9) f''(0)$), where η is the similarity variable, $f'(\eta)$ is the velocity profile and $f''(0)$ is the reduced skin friction, which are defined in [13]. In addition, the local skin-friction τ_X and the local Nusselt number Nu_X for different values of Prandtl number when heat flux gradient $m = 0.5$ at $X = 1.0$ in steady state are compared with the non-similarity results of Hossain and Paul [9] in Table 2 given as $F_0''(0)$. It is observed that the results are in good agreement with each other. We also noticed that the present results agree well with those of Pop and Watanabe [23] (see Table 1)

In Figs.3-6, transient velocity and temperature profiles are shown at $X = 1.0$, with various parameters Pr and m. The value of t with star $(*)$ symbol denotes the time taken to reach the steady-state flow. In Figs.3 and 4, transient velocity and temperature profiles are plotted for various values of Pr and $m = 0.25$. Increasing Pr means that the viscous force increases and thermal diffusivity reduces, which causes a reduction in the velocity and temperature, as expected. It is also noticed that the time taken to reach steady-state flow increases and thermal boundary layer thickness reduces with increasing Pr. Further, it is clear seen from Fig.3 that the momentum boundary layer thickness increases with the increase of Pr from unity. In Figs.5 and 6, transient velocity and temperature profiles are shown for various values of m with $Pr = 1.0$. Impulsive forces are reduced along the surface of the cone near the apex for increasing values of m (i.e. the gradient of heat flux along the cone near the apex reduces with the increasing values of m). Due to this, the difference between temporal maximum values and steady-state values reduces with increasing m. It is also observed that increasing in m reduces the velocity as well as temperature and takes more time to reach steady-state.

The study of the effects of the parameters on local as well as the average skin-friction, and the rate of heat transfer is more important in heat transfer problems. The derivatives involved in Eqs. (13) and (14) are obtained using five-points approximation formula and then the integrals are evaluated using Newton-Cotes closed integration formula. The variation of the local skin-friction τ_X and the local Nusselt number

Nu_X in the transient period at various positions on the surface of the cone ($X = 0.25$ and 1.0) for different values of m, are shown in Figs.7 and 8. It is observed from Fig.7 that the local skin-friction decreases with increasing m and the effect of m over the local skin-friction τ_X is more near the apex of the cone and reduces gradually with increasing the distance along the surface of the cone from the apex. From Fig.8, it is noticed that near the apex, local Nusselt number Nu_X reduces with increasing m, but that trend is slowly changed and reversed as distance increases along the surface from apex. The variation of the local skin-friction τ_X and the local Nusselt number Nu_X in the transient regime is displayed in Figs.9 and 10 for different values of Pr and at various positions on the surface of the cone ($X = 0.25$ and 1.0). It is clear from these figures that the local skin frictions τ_X reduces and the local Nusselt number increases with the increasing Pr, these effects gradually increase in the transient period with increasing the distance along the cone surface from the apex. The influence of m on average skin-friction $\overline{\tau}$ is more when m is reduced as it can be seen in Fig.11. Finally, Fig.12 displays the influence of Pr and m on the average Nusselt number $\overline{Nu}$ in the transient period. This shows that there is no significant influence of m over the average Nusselt number. Average Nusselt number $\overline{Nu}$ increases with increasing Pr.

5. **Conclusions**

A numerical study has been carried out for the transient laminar free convection from a vertical cone subjected to a non-uniform surface heat flux. The dimensionless governing boundary layer equations are solved numerically using an implicit finite-difference method of Crank-Nicolson type. Present results are compared with available results from the literature and are found to be in good agreement. The following conclusions are made:

1. The time taken to reach steady-state increases with increasing Pr or m.
2. The difference between temporal maximum values and steady state values (for both velocity and temperature) becomes less when Pr or m increases.
3. The influence of m over the local skin friction τ_X is large near the apex of the cone and that reduces slowly with increasing distance from it.

4. In transient period, the local Nusselt number reduces with increasing m near the apex but that trend is changed and reversed as the distance increases from it.
5. The influence of Pr on the local skin-friction τ_X and the local Nusselt number Nu_X increases along the surface from the apex.
6. The average skin-friction $\overline{\tau}$ decreases with increasing m and the effect of m on average Nusselt number $\overline{Nu}$ is almost negligible.

Table 1. Comparison of steady state local skin-friction and temperature values at $X = 1.0$ with those of Lin [13] for uniform surface heat flux

	Temperature			Local skin friction		
	Lin [13]		Present results	Lin [13]		Present results
Pr	$-\theta(0)$	$-\left(\frac{20}{9}\right)^{1/5}\theta(0)$	T	$f''(0)$	$\left(\frac{20}{9}\right)^{2/5}f''(0)$	τ_X
0.72	1.52278[1]	1.7864	1.7714	0.22930[1]	1.224	1.2105
1	1.39174	1.6327	1.6182	0.78446	1.0797	1.0669
2	1.16209	1.3633	1.3499	0.60252	0.8293	0.8182
4	0.98095	1.1508	1.1385	0.46307	0.6373	0.6275
6	0.89195	1.0464	1.0344	0.39688	0.5462	0.5371
8	0.83497	0.9796	0.9677	0.35563	0.4895	0.4808
10	0.79388	0.9314	0.9196	0.32655	0.4494	0.4411
100	0.48372	0.5675	0.5531	0.13371	0.184	0.1778

[1] Values taken from Pop and Watanabe [23] when suction/injection is zero.

Table 2. Comparison of steady state local skin-friction and local Nusselt number values at $X = 1.0$ with those of Hossain and Paul [9] for different values of Pr when $m = 0.5$

	Local skin-friction		Local Nusselt number	
	Results [9]	Present results	Results [9]	Present results
Pr	$F_0''(0)$	$\tau_X/Gr_L^{3/5}$	$1/\phi_0(0)$	$Nu_X/Gr_L^{1/5}$
0.01	5.13457	5.1388	0.14633	0.1463
0.05	2.93993	2.9352	0.26212	0.2634
0.1	2.29051	2.2853	0.33174	0.3332

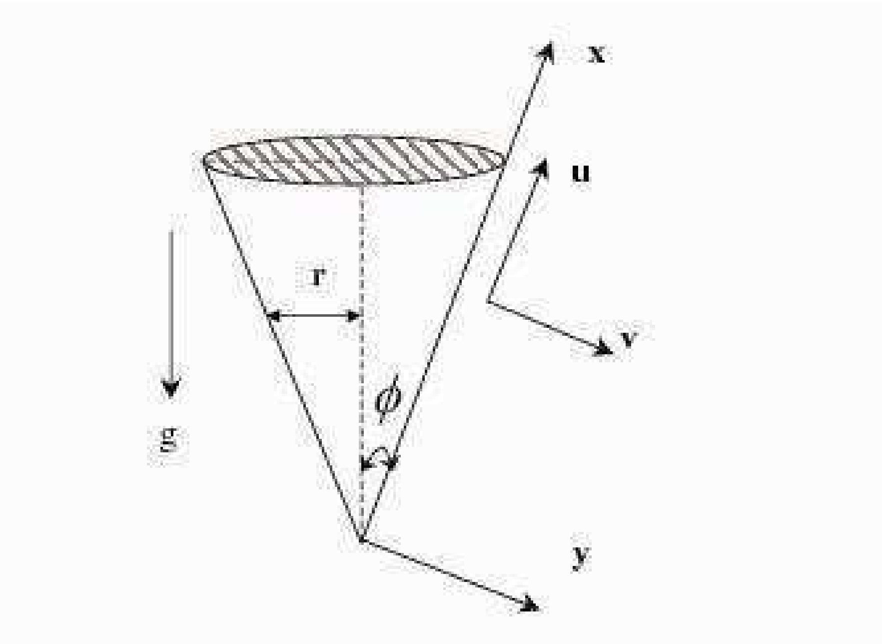

FIGURE 1. Physical model and co-ordinate system

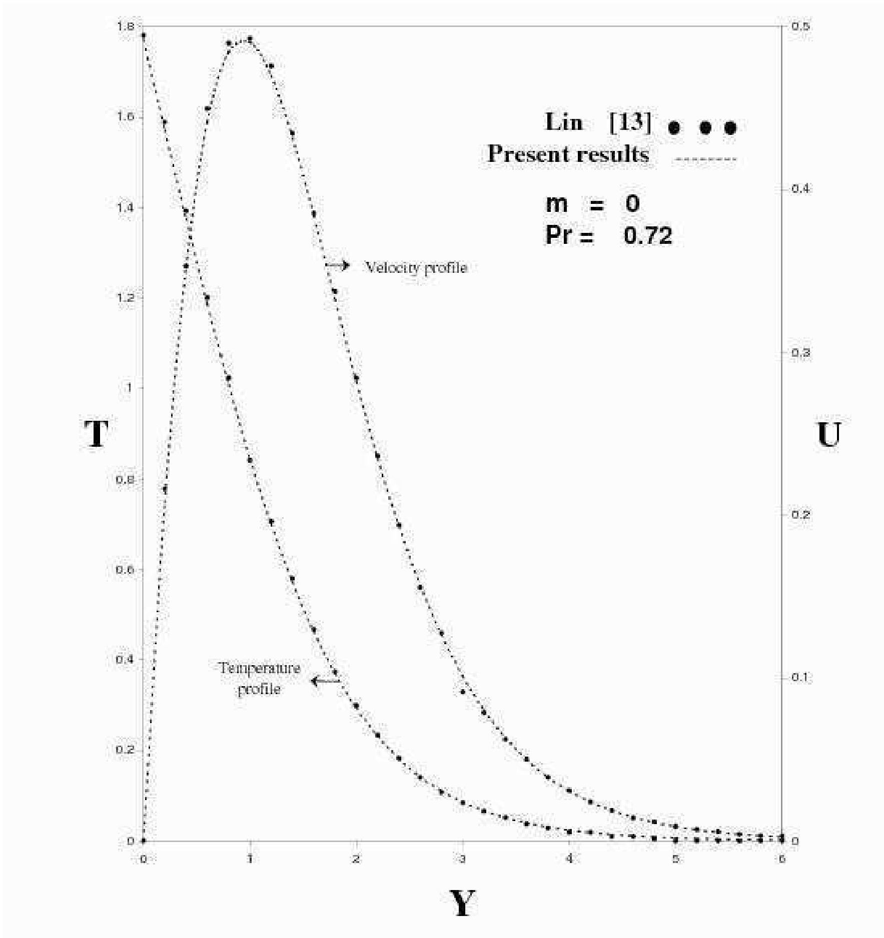

FIGURE 2. Comparison of steady state temperature and velocity profiles at $X = 1.0$

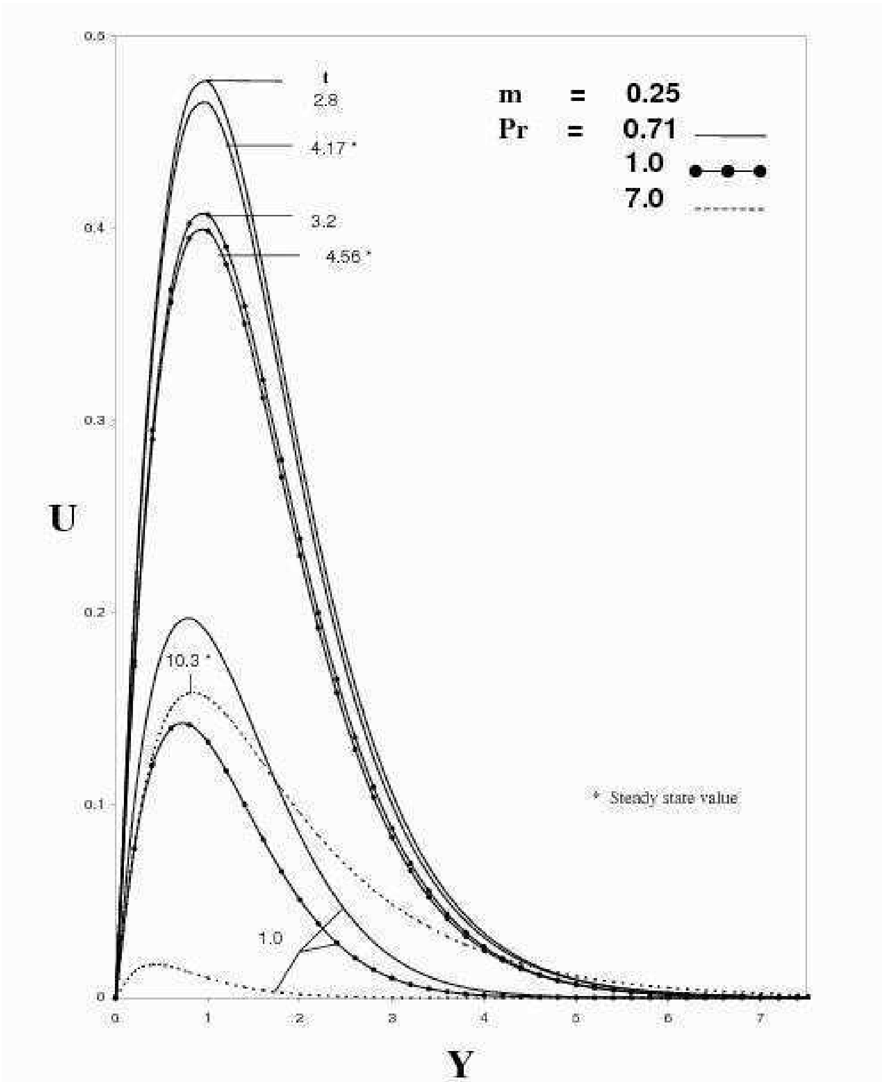

FIGURE 3. Transient velocity profiles at $X = 1.0$ for different values of Pr

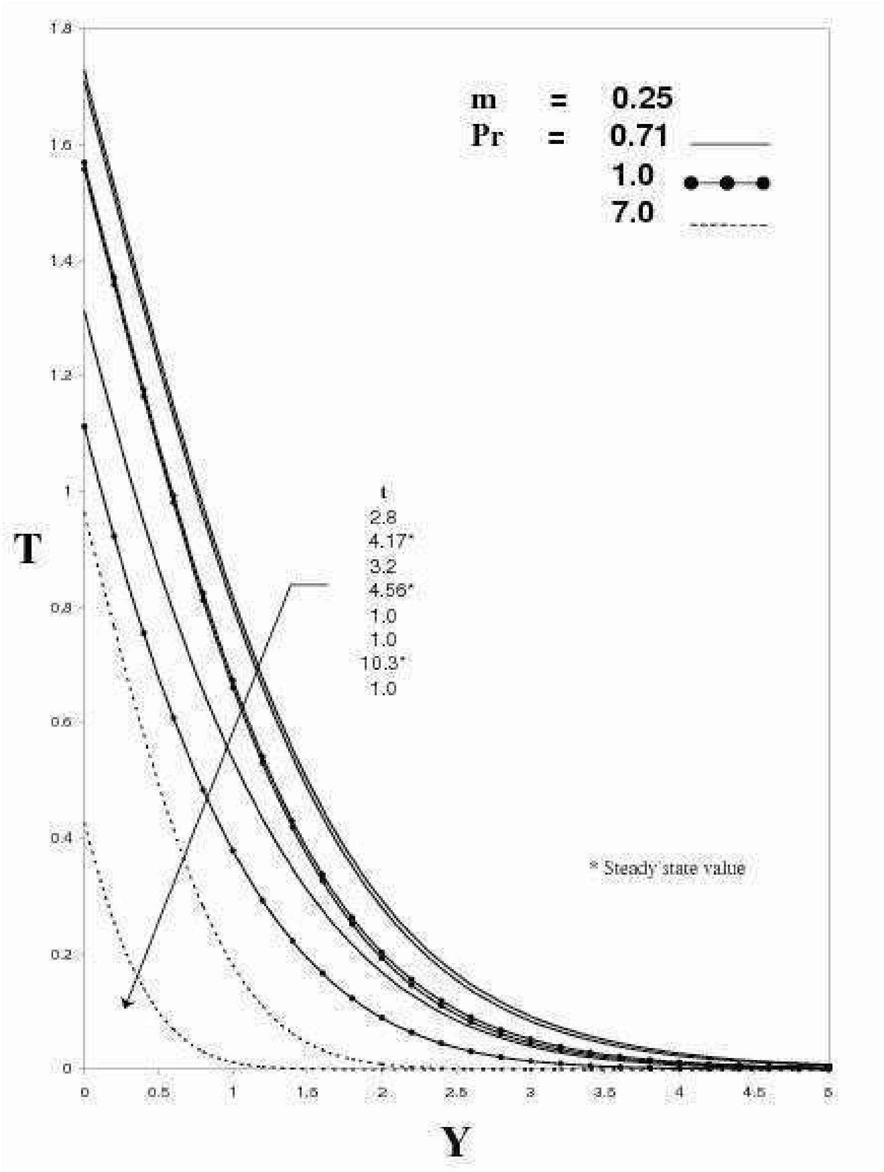

FIGURE 4. Transient temperature profiles at $X = 1.0$ for different values of Pr

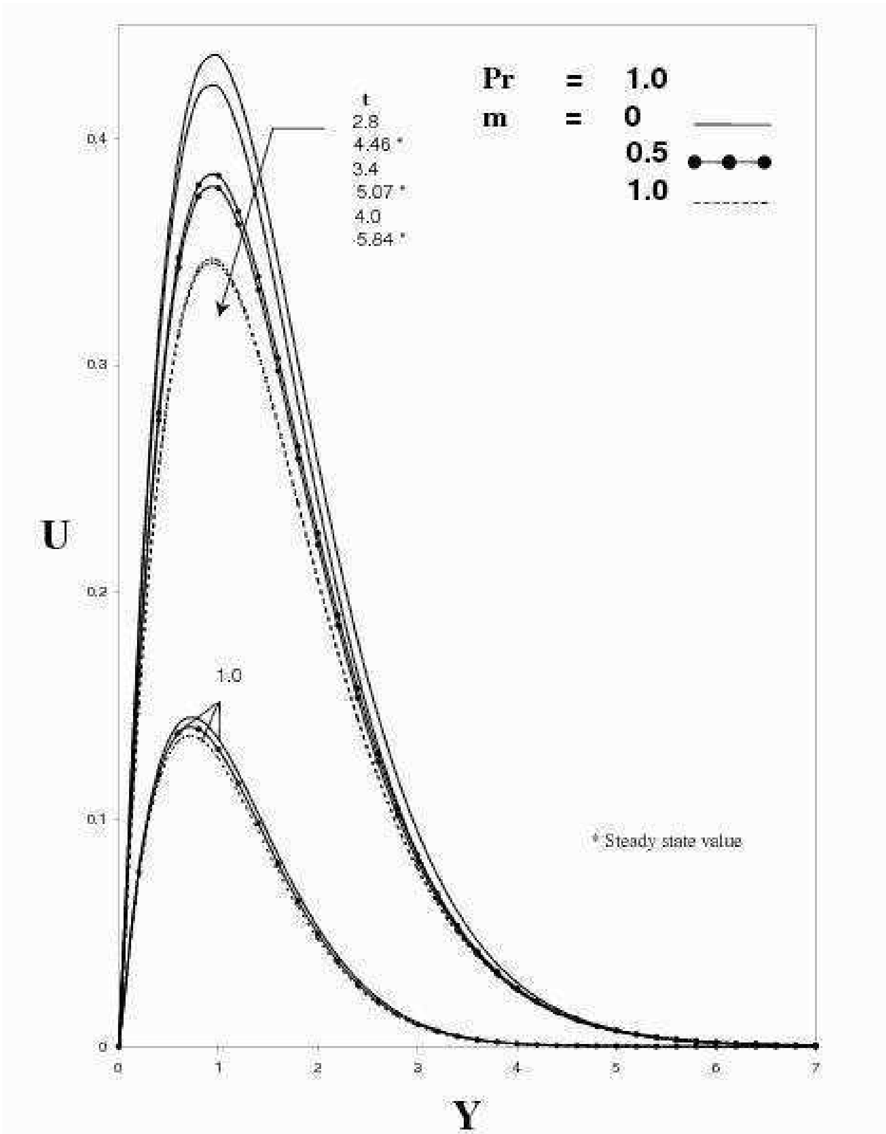

FIGURE 5. Transient velocity profiles at $X = 1.0$ for different values of m

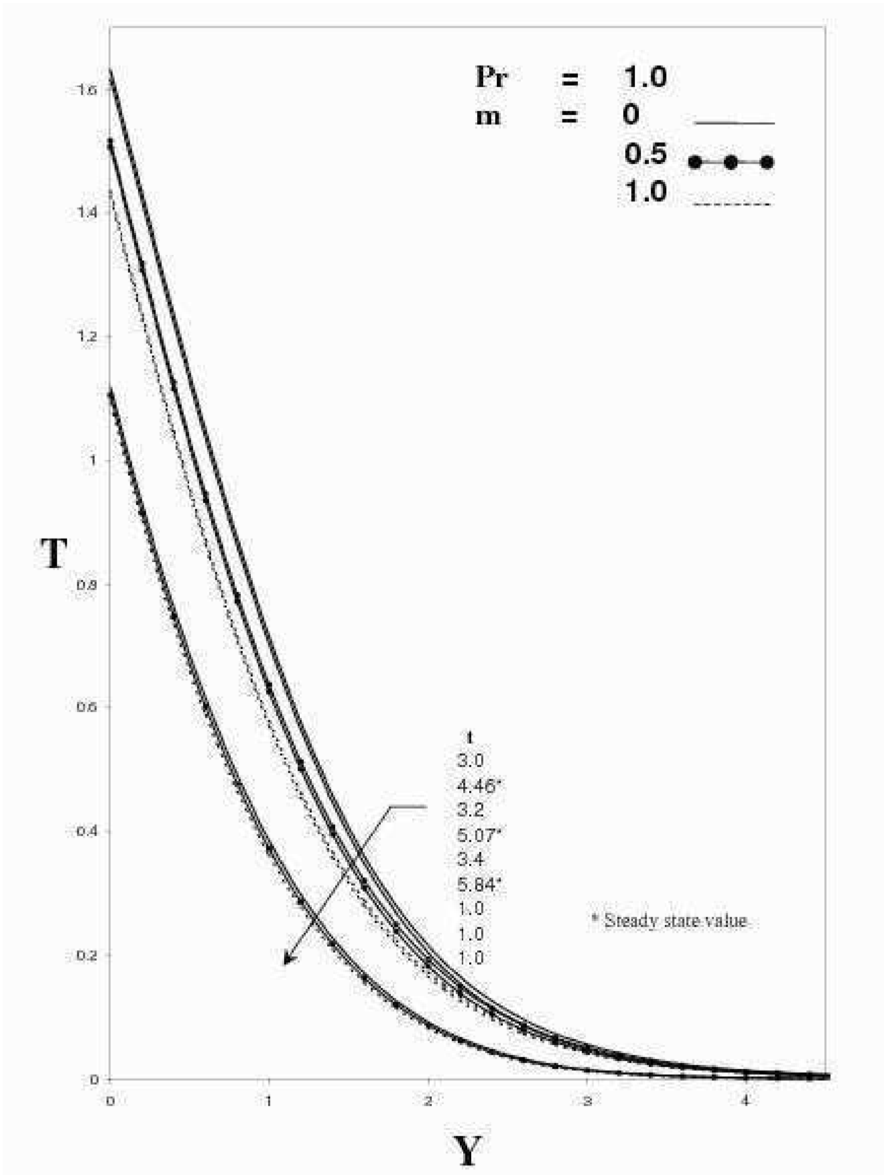

FIGURE 6. Transient temperature profiles at $X = 1.0$ for different values of m

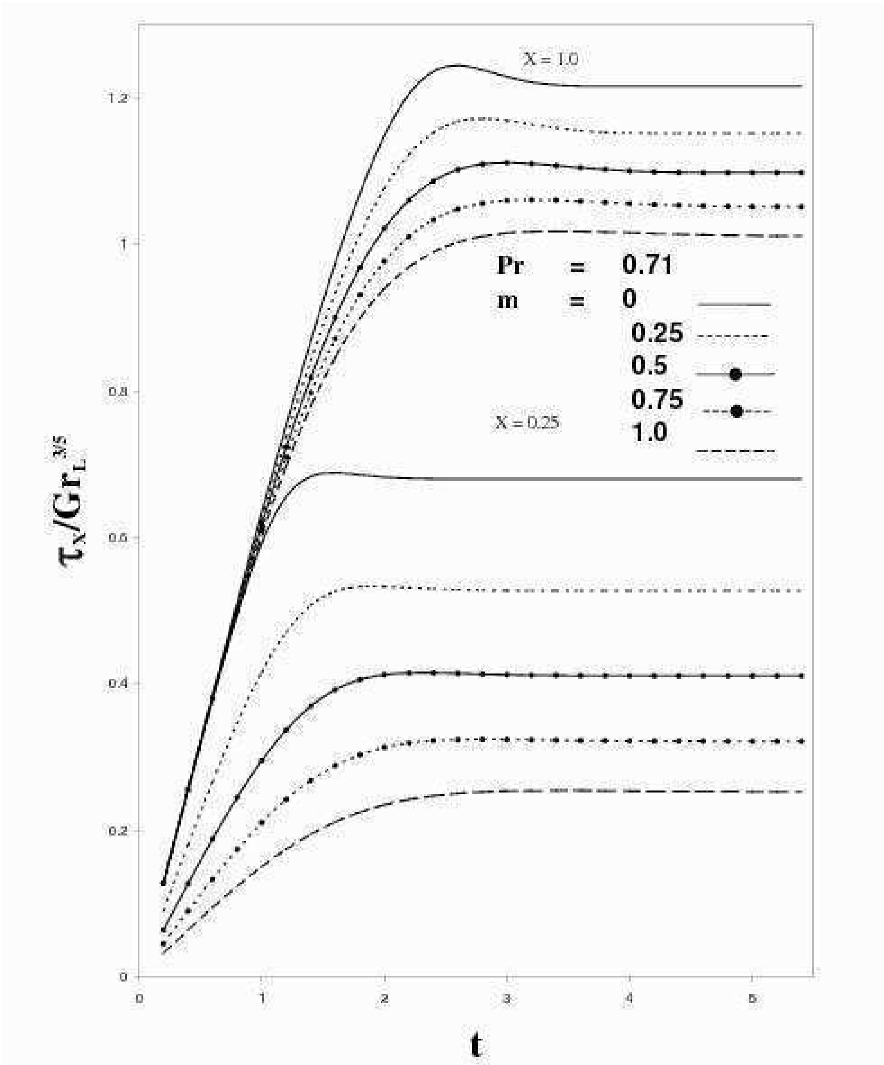

FIGURE 7. Local skin friction at $X = 0.25$ and 1.0 for different values of m in transient period

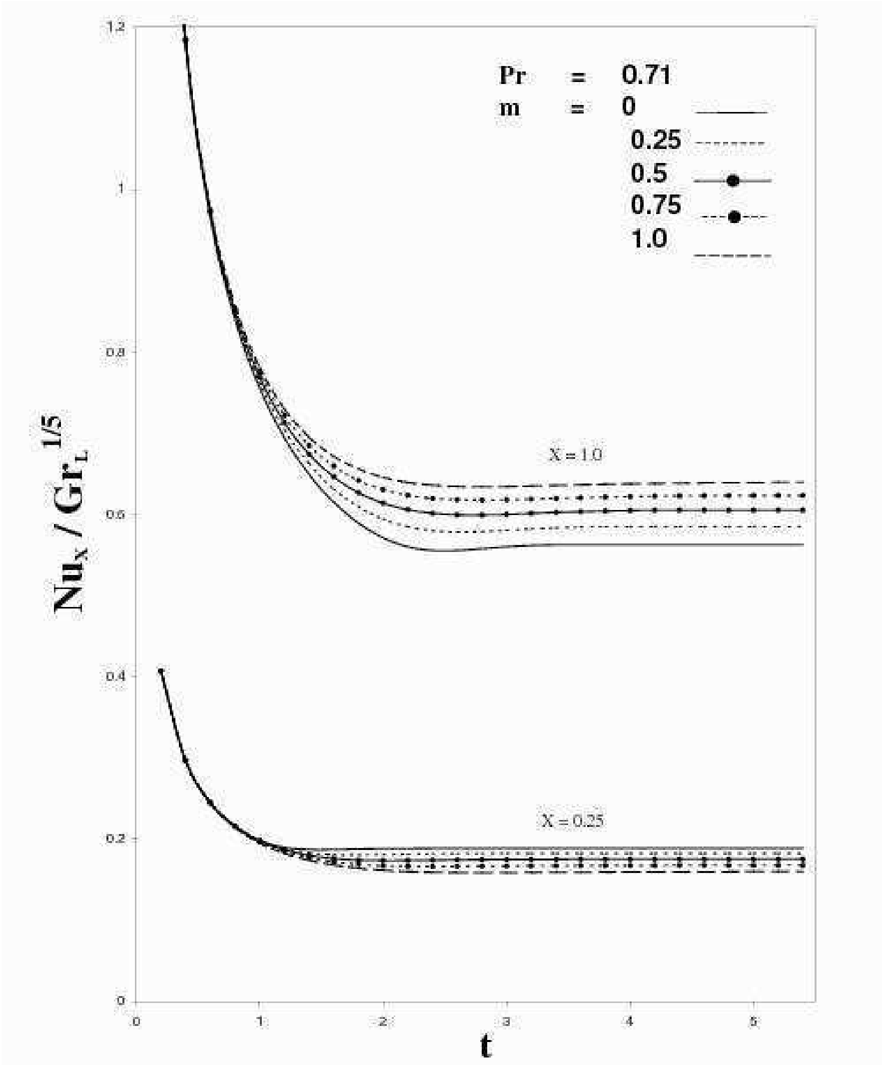

FIGURE 8. Local Nusselt number at $X = 0.25$ and 1.0 for different values of m in transient period

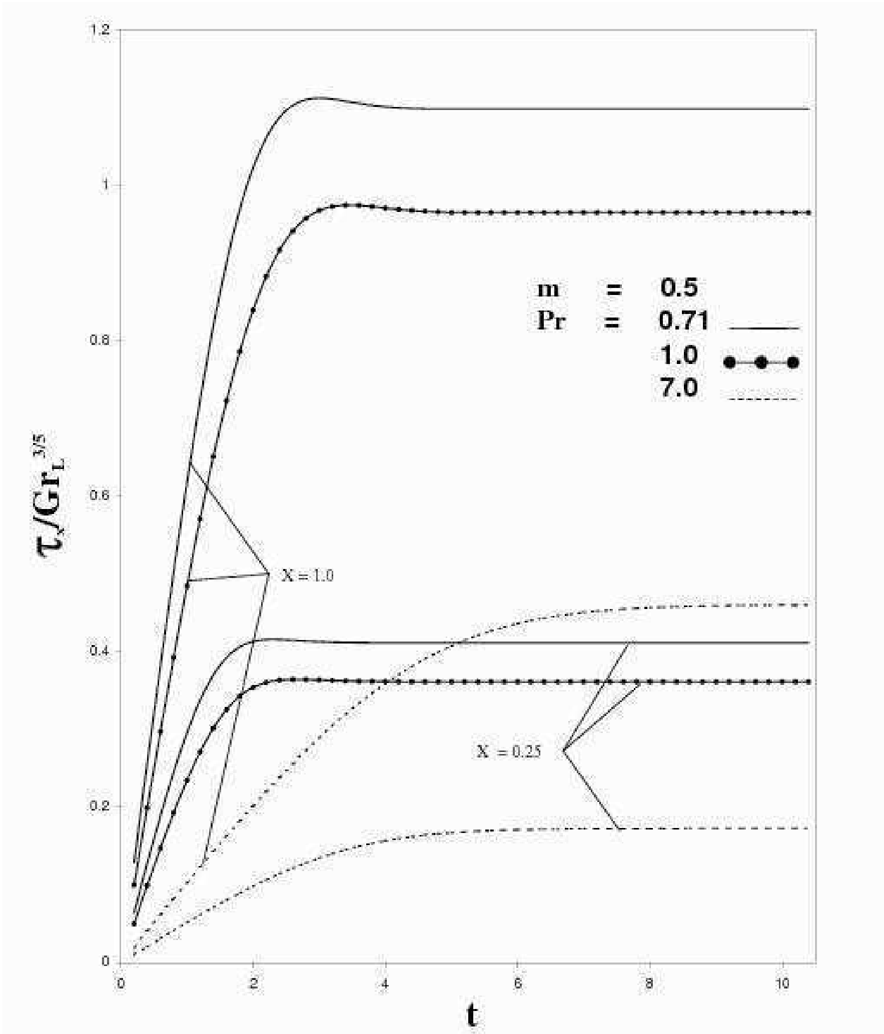

FIGURE 9. Local skin friction at $X = 0.25$ for different values of Pr in transient period

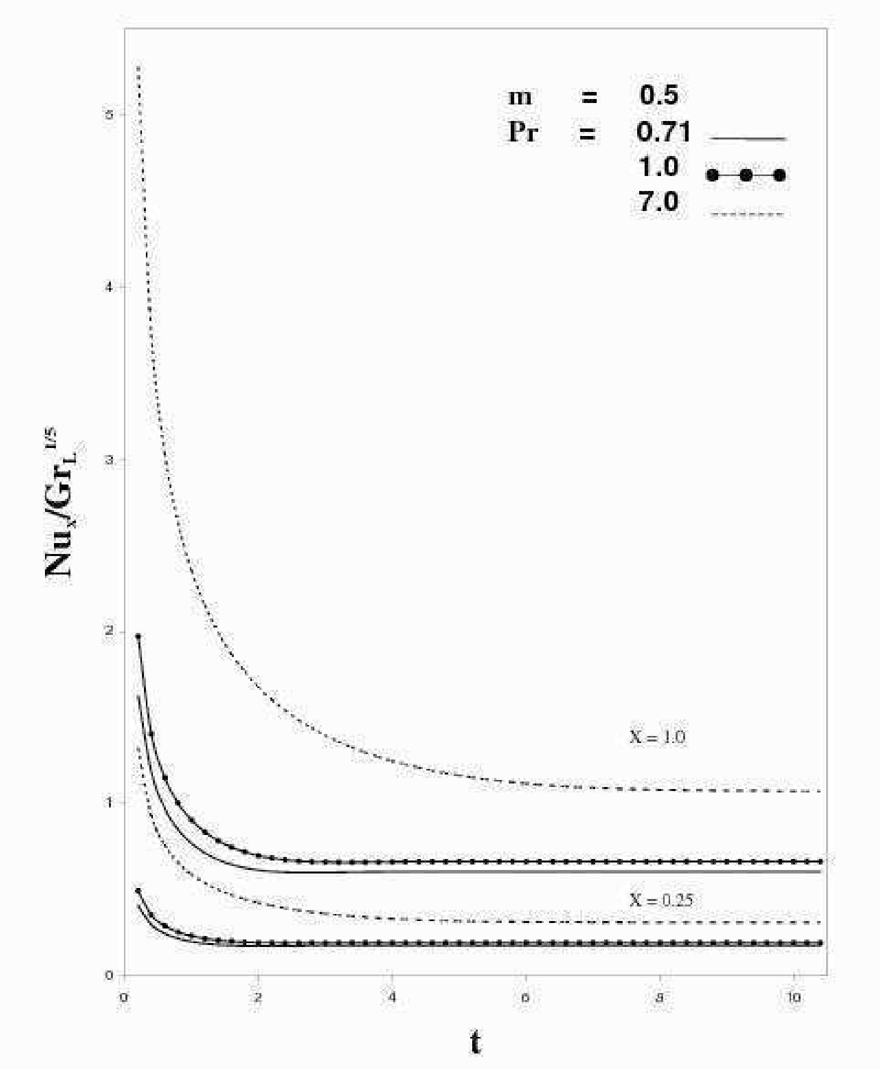

FIGURE 10. Local Nusselt number at $X = 0.25$ and 1.0 for different values of Pr in transient period

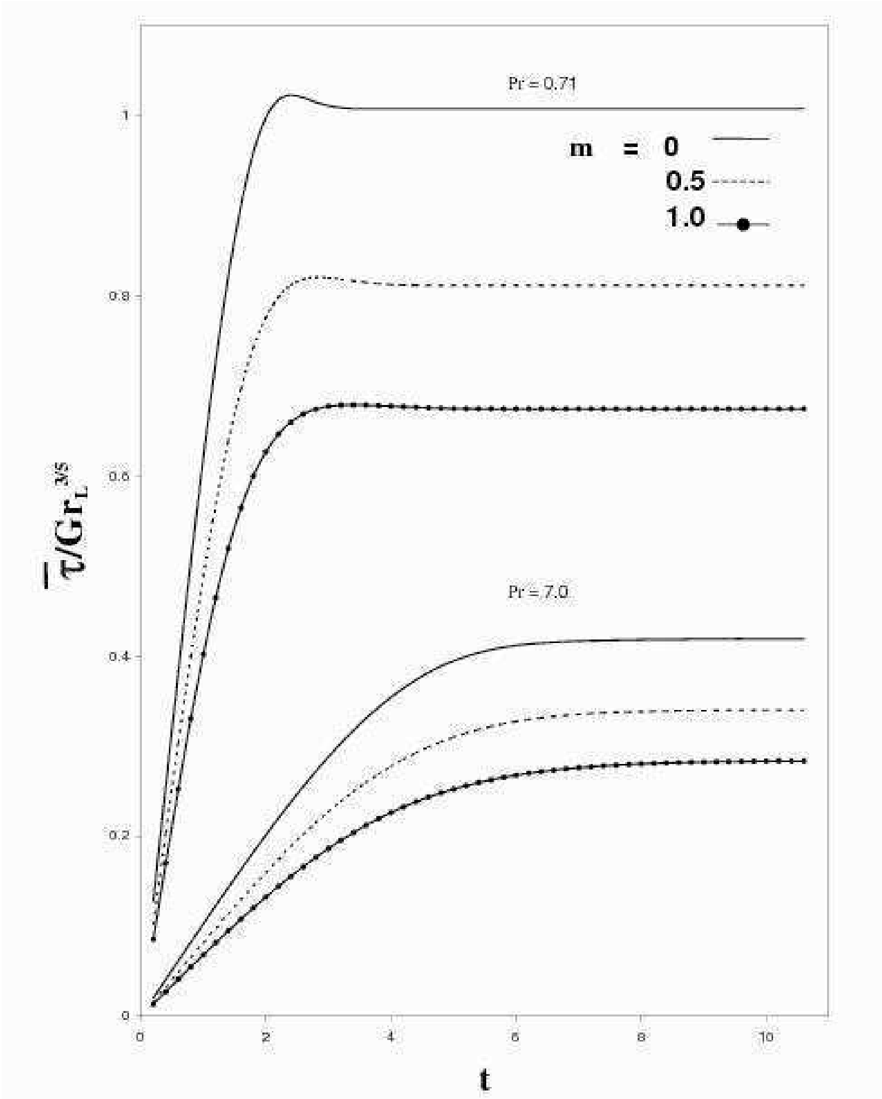

FIGURE 11. Average skin friction for different values of Pr and m in transient period

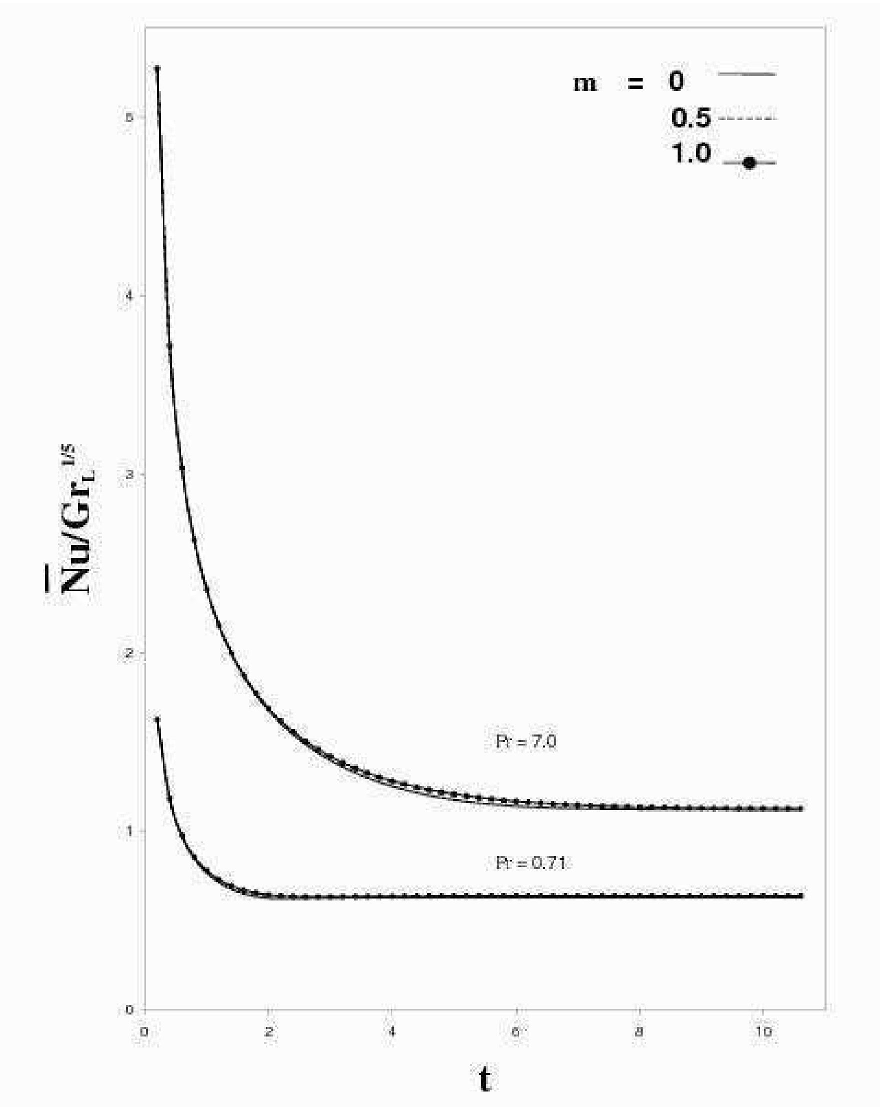

FIGURE 12. Average Nusselt number for different values of Pr and m in transient period

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ON THE LIPSCHITZ EXTENSION CONSTANT FOR A COMPLEX-VALUED LIPSCHITZ FUNCTION

ALEXANDRU ROŞOIU AND DRAGOŞ FRĂŢILĂ

Abstract. In order to show that the Lipschitz constant for the extension of a complex-valued Lipschitz function cannot generally be 1, one can use the following example (see *Lipschitz Algebras*, by N. Weaver, World Scientific, Singapore, 1999, p. 18, Example 1.5.7): Let $X = \{e, p_1, p_2, p_3\}$ be a metric space such that $d(p_i, p_j) = 1$, for all distinct $i, j \in \{1, 2, 3\}$ and $d(e, p_i) = \frac{1}{2}$, for all $i \in \{1, 2, 3\}$ and let $X_0 = \{p_1, p_2, p_3\}$ be a subset of X. An isometric map of X_0 into $\mathbb{C}$ can be extended to X with an increase in the Lipschitz constant of at least $\frac{2}{\sqrt{3}}$, this constant being attained for the function that takes e to the circumcenter of the triangle formed by the points $f(p_i)$, for all $i \in \{1, 2, 3\}$. The purpose of this article is to show that we can loosen somewhat the conditions imposed on d, namely we show that considering a metric space $X = \{e, p_1, p_2, p_3\}$ such that $d(e, p_i) + d(e, p_j) = d(p_i, p_j)$, for all distinct $i, j \in \{1, 2, 3\}$, the above increase in the Lipschitz constant for the extended Lipschitz function is preserved.

1. **Introduction**

The problem of the extension of a Lipschitz function is a central one in the theory of Lipschitz functions. There are a lot of results in this direction (see for example [1-17]).

One of the main problems which is not completely answered in Lipschitz analysis is the following:

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Given two metric spaces X and Y and a subset X_0 of X under what conditions can we extend a Lipschitz function $f_0 : X_0 \to Y$ to a Lipschitz function $f : X \to Y$ with only a multiplicative loss in the Lipschitz constant and such that $f/X_0 = f_0$?

The problem is of particular interest especially when we take Y to be $\mathbb{R}$ or $\mathbb{C}$ (the so called scalar-valued Lipschitz functions). Under this hypothesis one fundamental theorem that we will now state works. (See [18] for more details.)

Theorem. If we take X to be a metric space and X_0 a subset of X then:

i) For any $f_0 : X_0 \to \mathbb{R}$ there exists $f : X \to \mathbb{R}$ such that $f/X_0 = f_0$ and $L(f) = L(f_0)$;

ii) For any $f_0 : X_0 \to \mathbb{C}$ there exists $f : X \to \mathbb{C}$ such that $f/X_0 = f_0$ and $L(f) \leq \sqrt{2} \cdot L(f_0)$, where by $L(f)$ we denoted the Lipschitz constant of f.

As one can see, in the real case one can extent the Lipschitz function to the whole space with no multiplicative loss in the Lipschitz constant whatsoever, whereas in the complex case we can only say that $L(f) \leq \sqrt{2} \cdot L(f_0)$. What is then the best constant that we can use for this inequality?

One step in this direction was taken by Kirszbraun when he proved the following

Theorem. If X is a subspace of $\mathbb{R}^n$ (for some $n \in \mathbb{N}^*$) equipped with the inherited Euclidean metric then the function $f_0 : X_0 \to \mathbb{C}$ can be extended to all of X without increasing its Lipschitz number. (See [7])

Could the constant we search for be 1? As we shall see in the following example (taken from [18, p. 18]) the answer is no. (The reason for which the constant is 1 in Kirszbraun's theorem has to do with the fact that in this particular case the space X is Euclidean.)

Example. Let $X = \{e, p_1, p_2, p_3\}$ be a four element set and let d be a distance on X such that $d(p_i, p_j) = 1$, for all distinct $i, j \in \{1, 2, 3\}$ and $d(e, p_i) = \frac{1}{2}$, for all $i \in \{1, 2, 3\}$. Now let $X_0 = \{p_1, p_2, p_3\}$ and $f_0 : X_0 \to \mathbb{C}$ be an isometric map. f_0 therefore takes the points p_1, p_2, p_3 to the vertices of an equilateral triangle. The Lipschitz extension of f_0 to the whole of X with the smallest Lipschitz constant will be the one taking e to the center of the triangle as one can easily see. In this case the constant we search for is $\frac{2}{\sqrt{3}}$.

2. **Main Result**

Could this constant still work for a more general setting? The answer is yes. As in the above example let $X = \{r,\ a,\ b,\ c\}$ be a four element set and let d be a distance on X such that $d(r,a) + d(r,b) = d(a,b)$ and the analogues. Given an isometric map f_0 taking a, b, c to the points A, B, C of the complex plane we claim that we can extend it to X such that $L(f) \leq \frac{2}{\sqrt{3}}$. Let us notice that by taking $d(a,b) = d(b,c) = d(c,a) = 1$ and $d(r,a) = d(r,b) = d(r,c) = \frac{1}{2}$ we get the example already mentioned.

Let us now restate the problem. For an extension f of f_0 to X to exist it is necessary and sufficient that there exists a value for $f(r)$ and a constant k such that $|f(r) - f(x)| \leq k \cdot d(r,x)$, for all $x \in X_0$. Or, from another point of view, it is necessary and sufficient that there exists a constant k such that by expanding the discs $D(f(a), d(r,a))$, $D(f(b), d(r,b))$, $D(f(c), d(r,c))$ by a factor of k there intersection will not be empty. From the main theorem we stated before it is clear that such a value of k exists and $k \leq \sqrt{2}$. It is also quite obvious that the smallest constant for these circles does exist and is attained when the circles have exactly one point in common. If one could prove that this happens for a constant of at most $\frac{2}{\sqrt{3}}$ then one would get that this is the smallest possible constant when we pass from X_0 to X.

Given that $AB = d(a,b) = d(r,a) + d(r,b)$ we can see that the circles $C(A, d(r,a))$ and $C(B, d(r,b))$ are tangent. The same goes for the other pairs. For briefness we will take $d(r,a) = r_A$, $d(r,b) = r_B$, $d(r,c) = r_C$. Let A', B' and C' be the points where the three tangent circles touch each other. Let also M be the point of intersection for the lines AA', BB', CC' and B" be the intersection point of the segment BM with the circle $C(B, r_B)$. If one can prove for example that $\frac{\|BM\|}{\|BB"\|} \leq \frac{2}{\sqrt{3}}$ then one would get that M belongs to the disc $D(B, \frac{2}{\sqrt{3}} \cdot r_B)$. By doing the same for A and C one obtains that the point M belongs to all the three discs $D(A, \frac{2}{\sqrt{3}} \cdot r_A)$,

$D(B, \frac{2}{\sqrt{3}} \cdot r_B)$, $D(C, \frac{2}{\sqrt{3}} \cdot r_C)$ and therefore the minimum constant we search for will be less than or equal to $\frac{2}{\sqrt{3}}$.

Let us now prove that indeed $\frac{\|BM\|}{\|BB"\|} \leq \frac{2}{\sqrt{3}}$. Consider the origin of the plane to be B and let the Ox axis be the one containing C. One can easily see that $\overrightarrow{BA} = \left\|\overrightarrow{BA}\right\| \cos B \cdot \overrightarrow{i} + \left\|\overrightarrow{BA}\right\| \sin B \cdot \overrightarrow{j}$. (Here we denote by $\overrightarrow{i}$ and $\overrightarrow{j}$ the unity vectors of the axes Ox and Oy respectively.)

Given that $\frac{B'A}{B'C} = \frac{r_A}{r_C}$, we have

$$\overrightarrow{BB'} = \frac{r_C}{r_A + r_C} \cdot \overrightarrow{BA} + \frac{r_A}{r_A + r_C} \cdot \overrightarrow{BC} =$$

$$= (\frac{r_C(r_A + r_B)\cos B}{r_A + r_C} + \frac{r_A(r_B + r_C)}{r_A + r_C}) \cdot \overrightarrow{i} + \frac{r_C(r_A + r_B)\sin B}{r_A + r_C} \cdot \overrightarrow{j}.$$

By applying Menelaos' Theorem to the triangle CBB' and the line $A-M-A'$ we obtain the equality:

$$\frac{A'B}{A'C} \cdot \frac{AC}{AB'} \cdot \frac{MB'}{MB} = 1.$$

Expressing $\frac{BM}{B'M}$ from here we get

$$\frac{BM}{B'M} = \frac{r_B}{r_C} \cdot \frac{r_A + r_C}{r_A} = \frac{r_B(r_A + r_C)}{r_A r_C}.$$

Or equivalently

$$\frac{BM}{BB'} = \frac{r_B(r_A + r_C)}{\sum r_A r_B}.$$

Given the expression of $\overrightarrow{BB'}$ above we have

$$\overrightarrow{BM} = (\frac{r_B r_C(r_A + r_B)\cos B}{\sum r_A r_B} + \frac{r_A r_B(r_B + r_C)}{\sum r_A r_B}) \cdot \overrightarrow{i} + \frac{r_B r_C(r_A + r_B)\sin B}{\sum r_A r_B} \cdot \overrightarrow{j}.$$

From here

$$\frac{\|BM\|}{\|BB"\|} = \frac{\sqrt{\frac{[r_B r_C(r_A + r_B)\cos B + r_A r_B(r_B + r_C)]^2 + [r_B r_C(r_A + r_B)\sin B]^2}{[\sum r_A r_B]^2}}}{r_B},$$

or

$$\frac{\|BM\|}{\|BB"\|} = \frac{\sqrt{[r_B r_C(r_A + r_B)\cos B + r_A r_B(r_B + r_C)]^2 + [r_B r_C(r_A + r_B)\sin B]^2}}{r_B \sum r_A r_B}.$$

After simplifying both in the numerator and denominator by r_B we get

$$\frac{\|BM\|}{\|BB"\|} = \frac{\sqrt{\left[r_C(r_A+r_B)\cos B + r_A(r_B+r_C)\right]^2 + \left[r_C(r_A+r_B)\sin B\right]^2}}{\sum r_A r_B},$$

that is, we have to prove the inequality

$$\frac{\sqrt{{r_C}^2(r_A+r_B)^2 + {r_A}^2(r_B+r_C)^2 + 2r_Ar_C(r_A+r_B)(r_B+r_C)\cos B}}{\sum r_A r_B} \le \frac{2}{\sqrt{3}}.$$

From the cosine theorem in triangle ABC one gets that

$$2(r_A+r_B)(r_B+r_C)\cos B = (r_A+r_B)^2 + (r_B+r_C)^2 - (r_A+r_C)^2.$$

By replacing this in the above expression, the inequality becomes

$$\frac{\sqrt{{r_C}^2(r_A+r_B)^2 + {r_A}^2(r_B+r_C)^2 + r_Ar_C(r_A+r_B)^2 + r_Ar_C(r_B+r_C)^2 - r_Ar_C(r_A+r_C)^2}}{\sum r_A r_B} \le \frac{2}{\sqrt{3}}.$$

Squaring we get

$$\frac{(r_A+r_B)^2 r_C(r_A+r_C) + (r_B+r_C)^2 r_A(r_A+r_C) - r_Ar_C(r_A+r_C)^2}{(\sum r_A r_B)^2} \le \frac{4}{3},$$

or

$$\frac{(r_A+r_C)({r_B}^2(r_A+r_C) + 4r_Ar_Br_C)}{(\sum r_A r_B)^2} \le \frac{4}{3}.$$

Factoring out we get

$$3{r_B}^2(r_A+r_C)^2 + 12r_Ar_Br_C(r_A+r_C) \le 4{r_B}^2(r_A+r_C)^2 + 4{r_A}^2{r_C}^2 + 8r_Ar_Br_C(r_A+r_C),$$

or

$$4r_Ar_Br_C(r_A+r_C) \le {r_B}^2(r_A+r_C)^2 + 4{r_A}^2{r_C}^2,$$

which is nothing more than a trivial case of AM-GM inequality. Notice also that the equality is attained for

$$r_B = \frac{2r_Ar_C}{r_A+r_C}.$$

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HIGHER-ORDER LINEARLY IMPLICIT ONE-STEP METHODS FOR THREE-DIMENSIONAL INCOMPRESSIBLE NAVIER-STOKES EQUATIONS

IOAN TELEAGA AND JENS LANG

Abstract. In this work higher-order methods for integrating the three-dimensional incompressible Navier-Stokes equations are proposed. The numerical solution is achieved by using linearly implicit one-step methods up to third order in time coupled with up to third order stable finite element discretizations in space. These orders of convergence are demonstrated by comparing the numerical solution with exact Navier-Stokes solutions. Finally, we present benchmark computations for flow around a cylinder.

1. **Introduction**

Laminar incompressible flows play an important role in natural and industrial processes. For this type of flows the governing equations are the well known Navier-Stokes equations. Let $[0,T]\times\Omega$, $\Omega\subset\mathbb{R}^3$, be the computational domain, then the incompressible Navier-Stokes equations for viscous flows are

$$\begin{aligned} \partial_t U + (U\cdot\nabla)U + \nabla P - \nabla\cdot(2\nu\, S(U)) &= f, && \text{in } (0,T]\times\Omega \\ \nabla\cdot U &= 0, && \text{in } (0,T]\times\Omega \\ U &= U_b, && \text{on } (0,T]\times\partial\Omega \\ U &= U_0, && \text{in } \{0\}\times\Omega \\ \textstyle\int_\Omega P\,dx &= 0, && \text{in } [0,T] \end{aligned} \tag{1}$$

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where ν is the viscosity of the fluid, and $U = (u, v, w)^T$, P, $S = (\nabla U + \nabla U^T)/2$ represent the velocity field, the pressure, and the stress tensor. The initial data U_0 and boundary data U_b have to be chosen such that system (1) describes a well posed problem.

Over the last years, there has been considerable development of numerical methods for solving numerically this set of equations [1]. The challenge nowadays consists in combining accuracy of the numerical solution and efficiency of the whole numerical algorithm.

This report extends the numerical methods based on linearly implicit one-step methods coupled with stabilized finite element discretizations in space presented in [6, 7] to three-dimensional incompressible Navier-Stokes equations. All of these methods are implemented in the finite element code KARDOS [5]. For this paper we will select two time integrators ROS2 and ROS3PL already included in the above mentioned code. ROS2 is an L-stable Rosenbrock solver of order 2 which is second order consistent for any approximation of the Jacobian matrix, and ROS3PL is an L-stable stiffly accurate Rosenbrock solver of order 3 which has no order reduction for PDEs with complex boundary conditions. It improves ROS3P [9] which is only A-stable. For a comparison of time-discretization and linearization approaches for the two-dimensional incompressible Navier-Stokes equations we refer to [4].

The basic solution algorithm contained in the KARDOS code also serves as a good foundation for developing new codes with other capabilities. For example, a scalar transport equation for the density can be easily added to investigate buoyancy driven flows. Additionally, these approaches can be improved using adaptive strategies based on a posteriori error estimates.

An outline of this paper is as follows. In Section 2 we recall the general discretization of the incompressible Navier-Stokes equations (1) according to our setting. Section 3 contains convergence studies and numerical results for a benchmark flow around a cylinder [11]. Finally, conclusions are presented in Section 4.

2. Discretization of the equations

Firstly, system (1) is discretized in time employing linearly implicit one-step methods to achieve higher-order temporal discretizations by working the Jacobian matrix directly into the integration formula [10, 2].

Let τ_n be a variable time step. Then an s-stage linearly implicit time integrator of Rosenbrock type applied to (1) reads as follows:

$$\begin{aligned} \tfrac{U_{ni}}{\gamma\tau_n} + (U_n\cdot\nabla)U_{ni} &+ (U_{ni}\cdot\nabla)U_n - \nabla\cdot(2\nu S(U_{ni})) + \nabla P_{ni} \\ &= f(t_i) - (U_i\cdot\nabla)U_i + \nabla\cdot(2\nu S(U_i)) - \nabla P_i \\ &\quad - \textstyle\sum_{j=1}^{i-1}\frac{c_{ij}}{\tau_n}U_{nj} + \tau_n\gamma_i\partial_t f(t_i), \\ -\nabla\cdot U_{ni} &= \nabla\cdot U_i, \end{aligned} \tag{2}$$

with $i = 1, ..., s$ and the internal values are given by

$$t_i = t_n + \alpha_i\tau_n, \quad U_i = U_n + \sum_{j=1}^{i-1} a_{ij}U_{nj}, \quad P_i = P_n + \sum_{j=1}^{i-1} a_{ij}P_{nj}.$$

The new solution (U_{n+1},P_{n+1}) at time $t_{n+1} = t_n + \tau_n$ is computed by

$$U_{n+1} = U_n + \sum_{j=1}^{s} m_jU_{nj}, \quad P_{n+1} = P_n + \sum_{j=1}^{s} m_jP_{nj}, \tag{3}$$

where the coefficients a_{ij}, c_{ij}, γ_i, α_i, and m_j are chosen such that they satisfy certain consistency conditions.

To estimate the error in time we make use of an embedding strategy. By replacing the coefficients m_j in (3) with different coefficients $\hat{m}_j$, a new solution $(\hat{U}_{n+1}, \hat{P}_{n+1})$ of inferior order, that is, order 1 for ROS2 and order 2 for ROS3PL. The difference

$$\delta_{n+1} \quad := \quad ||(U_{n+1}, P_{n+1}) - (\hat{U}_{n+1}, \hat{P}_{n+1})||,$$

can be used as a step size control. A new step size with respect to a desired user tolerance TOL_t is selected by

$$\tau_{n+1} \quad = \quad C\frac{\tau_n}{\tau_{n-1}}\left(\frac{\delta_n\, TOL_t}{\delta_{n+1}\,\delta_{n+1}}\right)^{1/p}\tau_n, \tag{4}$$

where C represents a safety factor and is set to 0.95, and p denotes the order of the method used. Further details are given in [6, 8].

We describe now the derivation of the discrete equations obtained by a finite element discretization. Let $\mathcal{T}_h$ be an unstructured finite element mesh where the elements are tetrahedra, and S_h^q be the associated finite dimensional space with $q = 1, 2$ consisting of all continuous functions which are polynomials of order q on each tetrahedron $T \in \mathcal{T}_h$. In this way, the finite element approximation $U_{ni}^h \in S_h^q$ of the intermediate values U_{ni} in (2) has to satisfy the following equations

$$\left(\frac{U_{ni}^h}{\gamma\tau_n}, \varphi\right) + \left((U_n^h \cdot \nabla)U_{ni}^h, \varphi\right) + \left((U_{ni}^h \cdot \nabla)U_n^h, \varphi\right) + \left(\nabla P_{ni}^h, \varphi\right)$$
$$- \left(\nabla \cdot \left(2\nu S(U_{ni}^h)\right), \varphi\right) = \left(F^h(t_i, U_i^h, P_i^h), \varphi\right),$$
$$-(\nabla \cdot U_{ni}^h, \varphi) = (\nabla \cdot U_i^h, \varphi), \ \forall \varphi \in S_h^q, \ i = 1, ..., s, \tag{5}$$

where $F^h(t_i, U_i^h, P_i^h)$ is the right hand side of the first equation in (2).

Equal-order finite element functions for all unknown components are used. However, in this case the Babuska-Brezzi conditions is not satisfied, resulting in spurious pressure modes in the discrete solution. A way to get a stable discretization is to relax the incompressibility constraint in (5) as follows

$$- \left(\delta^h \, \nabla \cdot \left(\frac{U_{ni}^h}{\gamma\tau_n} + (U_n^h \cdot \nabla)U_{ni}^h + (U_{ni}^h \cdot \nabla)U_n^h - \nabla \cdot \left(2\nu S(U_{ni}^h)\right) + \nabla P_{ni}^h\right), \varphi\right)$$
$$- \left(\nabla \cdot U_{ni}^h, \varphi\right) = \left(\nabla \cdot U_i^h, \varphi\right) - (\nabla \cdot F^h(t_i, U_i^h, P_i^h), \varphi), \ \forall \varphi \in S_h^q, \ i = 1, ..., s,$$

with

$$\delta^h \quad = \quad c\,\frac{h_e}{2|u_e|}\frac{Re_e}{\sqrt{1 + Re_e^2}}, \quad Re_e = \frac{\rho_0 h_e u_e}{\nu}, \quad c = 0.4,$$

where u_e and h_e are a global reference velocity and the diameter of the n-dimensional ball which is area-equivalent to an element $T \in \mathcal{T}_h$, respectively. Although, the incompressibility equation looks now rather complicated, to the version in (5) just the divergence of the discrete equation (2) times a local factor δ^h has been added.

For each unknown pair (U_{ni}^h, P_{ni}^h), $i = 1, .., s$, a linear system with one and the same stiffness matrix has to be solved. Then, the new solution is updated using

(3). ROS2 requires two internal stages for each time step, while ROS3PL requires four stages. Rosenbrock methods offer several structural advantages. From an efficiency point of view, the most important advantage is that no nonlinear systems have to be solved (which can be sometimes cumbersome). Moreover, there is no problem to construct Rosenbrock methods with optimum linear stability properties and no order reduction for stiff equations. Because of their one-step nature, they are easy to implement.

3. **Numerical Results**

In this section we first present convergence studies of our numerical methods using academic three-dimensional test problems with known solutions. Two examples address the temporal and spatial convergence properties of the proposed numerical methods. It will be shown that ROS2 and ROS3PL have second and third order accuracy in time, respectively. Next, convergence rates for spatial discretizations are presented. The last example shows results for the benchmark flow around a cylinder defined in [11].

3.1. **An example to test the time discretization error.**

The first example allows us to check the order of the time discretization. On the integration domain $[0,T]\times\Omega$, $T=1$, and $\Omega=(0,1)^3$, we consider (1) with the exact solution

$$\begin{aligned} u &= y^2\exp(-t),\\ v &= z^2\exp(-t),\\ w &= x^2\exp(-t),\\ p &= (x+y+z)\exp(-t). \end{aligned}$$

The right-hand side f, the initial condition U_0 and the boundary condition U_b are chosen accordingly. The viscosity is set to 10^{-3}. The simulations are performed with quadratic elements on a fixed uniform mesh with 70400 tetrahedra. In this way, for any time t the temporal error will dominate the spatial error. The computations were

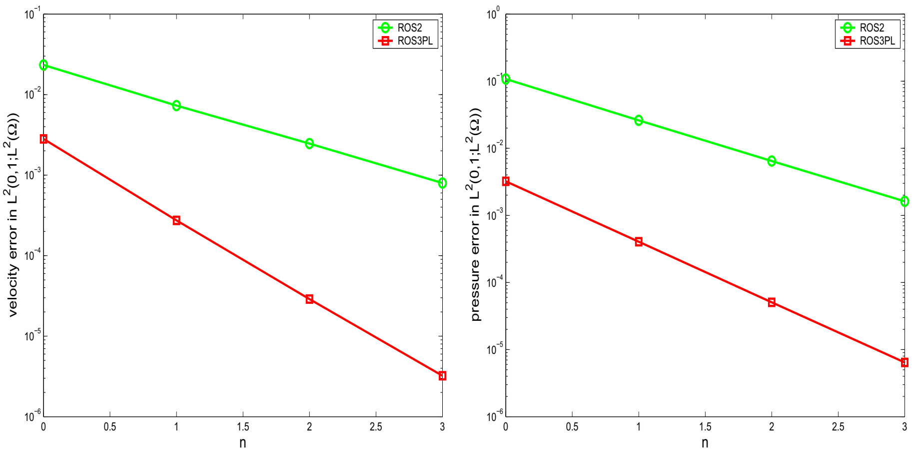

FIGURE 1. Time errors for ROS2 and ROS3PL, where n corresponds to $\tau = 0.5 \times 2^{-n}$.

done for fixed time steps $\tau = 0.5 \times 2^{-n}$, $n = 0, 1, 2, 3$.

We study the velocity and pressure errors in the norm $L^2(0,1;L^2(\Omega))$, i.e.,

$$||U - U^h||_{L^2(0,1;L^2(\Omega))} = \left(\int_0^1 ||(U - U^h)(t)||^2_{L^2(\Omega)} dt \right)^{1/2},$$

$$||P - P^h||_{L^2(0,1;L^2(\Omega))} = \left(\int_0^1 ||(P - P^h)(t)||^2_{L^2(\Omega)} dt \right)^{1/2}.$$

Figure 1 presents convergence results for both velocity (left) and pressure (right) components. ROS2 shows second order accuracy, while ROS3PL shows third order accuracy in all components. The best velocity and pressure error in $L^2(0,1;L^2(\Omega))$ has been obtained by the ROS3PL scheme. ROS2 induces for large time steps larger errors in both velocity and pressure components. For long time computations ROS3PL is more efficient with respect to CPU time.

3.2. **An example to test the spatial discretization error.** The following exact three-dimensional test solution to the incompressible Navier-Stokes equations has been proposed in [13]:

$$\begin{aligned} u &= \sin(mx)\cos(ly)\cos(nz)\exp(-t\nu), \\ v &= -\frac{m+n}{l}\cos(mx)\sin(ly)\cos(nz)\exp(-t\nu), \\ w &= \cos(mx)\cos(ly)\sin(nz)\exp(-t\nu), \end{aligned} \tag{6}$$

where m, n, l define the wave numbers along all three directions. The pressure is determined by assuming no force in the y- direction, that is,

$$\begin{aligned} p &= -\frac{(m+n)\nu}{l^2}\cos(mx)\cos(ly)\cos(nz)\exp(-t\nu) \\ &\quad +\frac{m(m+n)}{4l^2}\sin^2(mx)\cos(2ly)\cos^2(nz)\exp(-2t\nu) \\ &\quad +\frac{(m+n)^2}{4l^2}\cos^2(mx)\cos(2ly)\cos^2(nz)\exp(-2t\nu) \\ &\quad +\frac{m(m+n)}{4l^2}\cos^2(mx)\cos(2ly)\sin^2(nz)\exp(-2t\nu). \end{aligned} \tag{7}$$

The other forces are determined such that (6)-(7) form an exact solution to the Navier-Stokes equations. For the sake of simplicity, the computational domain is the unit cube, and we set $m = n = l = 1$, and viscosity $\nu = 1$. To test the convergence error in space with linear elements we apply the ROS2 time solver. The results for this combination are presented in Table 1. As expected, this numerical scheme preserves second-order accuracy in space at the final time $T = 1$.

Grid level		L^2-norm (OOC)	Max-norm (OOC)
$h = 1/4$	u	3.38e-03	7.47e-03
	v	5.78e-03	1.52e-02
	w	2.78e-03	7.61e-03
$h = 1/8$	u	8.35e-04 (2.01)	2.05e-03 (1.86)
	v	1.44e-03 (2.00)	4.06e-03 (1.90)
	w	7.13e-04 (1.96)	1.99e-03 (1.93)
$h = 1/16$	u	2.06e-04 (2.01)	5.58e-04 (1.87)
	v	3.57e-04 (2.01)	1.08e-03 (1.91)
	w	1.78e-04 (1.99)	5.54e-04 (1.84)

Table 1. Error-norms and numerically observed order of convergence (OOC) for ROS2 with linear elements at time $T = 1$.

Grid level		L^2-norm (OOC)	Max-norm (OOC)
$h = 1/4$	u	5.85e-05	2.77e-04
	v	1.22e-04	5.42e-04
	w	6.28e-05	3.09e-04
$h = 1/8$	u	7.76e-06 (2.91)	3.96e-05 (2.80)
	v	1.57e-05 (2.95)	7.95e-05 (2.76)
	w	8.18e-06 (2.94)	4.18e-05 (2.88)
$h = 1/16$	u	1.01e-06 (2.94)	5.37e-06 (2.88)
	v	2.04e-06 (2.94)	1.13e-05 (2.81)
	w	1.07e-06 (2.93)	5.52e-06 (2.92)

Table 2. Error-norms and numerically observed order of convergence (OOC) for ROS3PL with quadratic elements at $T = 1$.

The simulation was done with a fixed time step $\tau = 10^{-2}$. Indeed, time errors induced by ROS2 were settled down. For the test with quadratic elements we have chosen ROS3PL time solver. Table 2 shows results for this numerical scheme. The observed order of convergence in space is nearly three. Here the time step used was $\tau = 10^{-1}$. In engineering computations it is very important to have robust solvers which allow large time steps. In this sense ROS3PL behaves adequately and will be used in our further turbulence research.

3.3. **Flow around a cylinder.** This benchmark problem has been defined within the DFG high priority research program "Flow simulation with high-performance computers".

Figure 2 shows the considered computational domain where $L = 2.25$ m, $H = 0.41$ m, and the diameter of the cylinder is $D = 0.1$ m. The purpose of this benchmark is to numerically evaluate

- the drag force, i.e., $C_D = \int_S \left(\rho\nu \frac{\partial U_t}{\partial n} n_y - P n_x\right) dS$
- the lift force, i.e., $C_L = -\int_S \left(\rho\nu \frac{\partial U_t}{\partial n} n_x + P n_y\right) dS$
- the pressure difference $\Delta P(t) = P(0.45, 0.20, 0.205) - P(0.55, 0.20, 0.205)$

where S, $n = (n_x, n_y, 0)^T$, U_t, $t = (n_y, -n_x, 0)$ represent the cylinder surface, the normal vector on S, the tangential velocity on S, and the tangent vector respectively. Further, the viscosity of the fluid is $\nu = 0.001\ m^2/s$, and the density is $\rho = 1\ kg/m^3$. Then the drag and lift coefficients are defined to be

$$c_D = \frac{2C_D}{\rho \overline{u}^2 DH}, \quad c_L = \frac{2C_L}{\rho \overline{u}^2 DH},$$

where $\overline{u} = 4u(0, H/2, H/2, t)$ represents the characteristic velocity. Next, for our computations we take the following two benchmark cases from [11]:

- Case 1 (steady): The inflow boundary condition is

  $$u(0, y, z) = 16 u_m y z (H - y)(H - z)/H^4,\ v = w = 0$$

  with $u_m = 0.45 m/s$. The corresponding Reynolds number is 20 based on u_m, D, and ν.

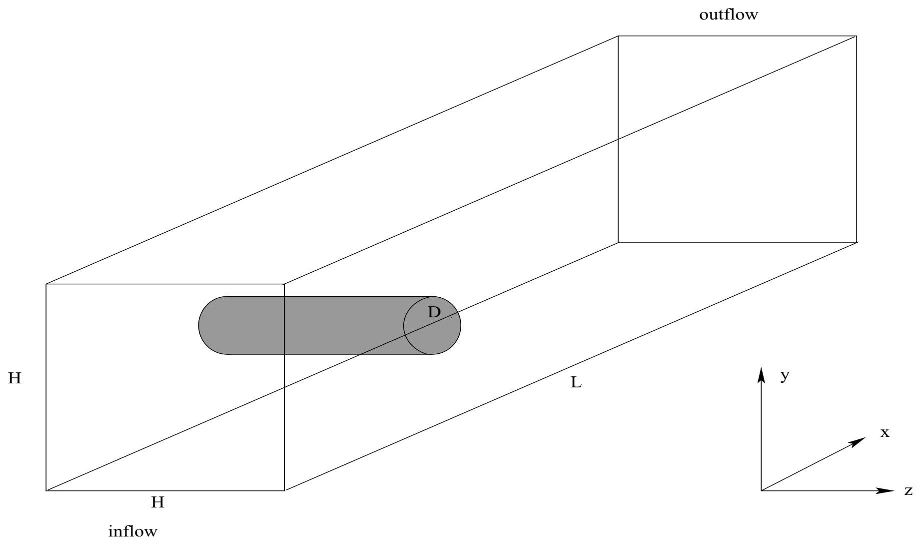

FIGURE 2. The computational geometry.

- Case 2 (unsteady): The inflow boundary condition is

$$u(0, y, z) = 16u_m yz(H - y)(H - z)sin(\pi t/8)/H^4,\ v = w = 0$$

  with $u_m = 2.25m/s$. The simulation time is $0 \leq t \leq 8\,s$. The corresponding Reynolds number is 100 based on u_m, D, and ν.

A complete description of these cases can be found in [11].

The mesh used for these tests consists of 14148 points and is presented in Figure 3. Using less grid points we were unable to obtain the drag and lift coefficient values even for the first test case. The discretization near the cylinder has a crucial importance. Two boundary layers around the cylinder have been used to ensure a proper resolution. The minimum grid spacing between the cylinder and the first layer was set to 0.005, while the grid spacing between the second layer and the cylinder was set to 0.01. Moreover, each cross-section of the cylinder in z- direction has been resolved with 32 points. We will restrict ourselves to linear elements. A comparison of the drag and lift coefficients and pressure difference at steady state with benchmark results from [11] and [3] is shown in Table 3. Both ROS2 and ROS3PL time schemes produce good results.

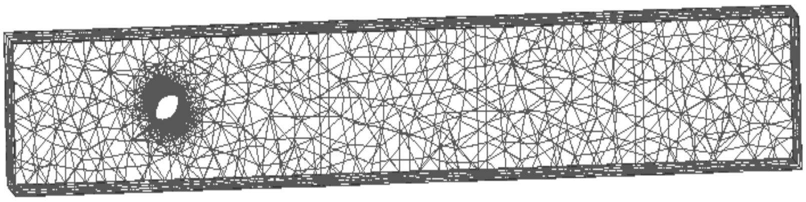

FIGURE 3. The mesh used

	ROS2	ROS3PL	Ref. [3]	Benchmark [11]
c_D	6.1199	6.1199	6.1853	6.0500 - 6.2500
c_L	0.0195	0.0195	0.009400	0.0080 - 0.0100
Δp	0.1725	0.1725	0.1707	0.1650 - 0.1750

Table 3. Comparison of drag and lift coefficient and pressure difference for Case 1.

	ROS2	ROS3PL	Benchmark [11]
c_D	3.1617	3.1558	3.2000 - 3.3000
c_L	0.0120	0.0110	0.0020 - 0.0040
Δp	-0.1191	-0.1183	-0.0900 - -0.1100

Table 4. Comparison of drag and lift coefficient and pressure difference for Case 2.

In Table 4 we present results for the second test case. Here, using less grid points as the authors in [11] we obtain similar coefficients with small variations. For all these cases the simulation was run with adaptive time steps according to equation (4). Although the drag coefficient is relatively simple to obtain, the lift coefficient is very sensitive to the mesh near the cylinder. For more accurate results one need to construct meshes with a better resolution of the cylinder region.

The linear systems arising from every time-stage are solved with the BiCGStab algorithm [12] with ILU as a preconditioner.

4. **Conclusions**

In this paper we have presented numerical methods based on linearly implicit time schemes of Rosenbrock type and stabilized finite elements in space to numerically solve laminar flow problems described through the three-dimensional incompressible Navier-Stokes equations. All these methods have been included in our adaptive finite element code KARDOS. The numerical examples studied in Section 3 clearly reveal second and third order of accuracy in space and time for our schemes. More specifically, applied to laminar flow problems with known smooth solutions, ROS2 and ROS3PL show their theoretical time order two and three, respectively. In these cases, ROS3PL performs more efficiently with respect to computing time. Our stabilization technique in space allows us to use equal-order finite elements for velocity and pressure components. We have found that combined with a Rosenbrock solver of suitable order, linear and quadratic Lagrange elements yield second and third order of spatial accuracy measured in the L^2- and maximum norm. From further practical experiences, our approach appears to provide a promising starting point for the development of efficient numerical solvers for more complex, turbulent flows. In our future work we are extending our code KARDOS to study and validate subgrid-scale models in the context of large eddy simulations for various physical applications.

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BOOK REVIEWS

Advanced Courses in Mathematical Analysis, II, Proceedings of the Second International School, M. V. Velasco and A. Rodríguez-Palacios (Editors), World Scientific Publishers, London - Singapore 2007, xi+213 pp, ISBN:13-978-981-256-652-2 and 10-981-256-652-X.

The volume contains the written versions of the talks and lectures delivered at the Second International Conference on Mathematical Analysis in Andalucía, which was held in Granada from 20 to 24 September, 2004. The first conference took place in Cadiz in September 2003, and its proceeding were published also with World Scientific in 2004.

The aim of the course was to bring together different research groups working in mathematical analysis and to provide the young researchers of these groups with access to the most advanced lines of research.

The present volume contains 11 papers covering a variety of topics from analysis - survey papers, contributed papers and historical surveys as well.

There are three papers of historical nature: F. Bombal, *Alexander Grothendieck's work on functional analysis,* (presenting not only the outstanding contributions of Grothendieck to tensor products and their applications to Banach space theory, but also some aspects of his unusual life and philosophy); L. Narici, *On the Hahn-Banach theorem,* B. Rubio Segovia, *Tribute to Miguel de Guzmán: Reflections on mathematical education centered on the mathematical analysis.* Miguel de Guzmán, one of the leading analysts of Spain, was scheduled to deliver a talk at the conference, but passed away shortly before the holding of the conference.

The paper by R. M. Aron, *Linearity in non-linear situations,* is concerned with the question whether some peculiar classes of functions (e.g, continuous and nowhere differentiable) contain some infinite dimensional linear subspaces - a property called lineability. The paper of Manuel Valdivia, the dean of the main speakers, *On certain spaces of holomorphic functions,* contains some related results (appearing for the first time in print), but concerning spaces of holomorphic functions.

The following papers survey results in the respective topics and present new contributions and open problems: J. Duoadiketxea, *The Hardy-Littlewood maximal function and some of its variants,* Gilles Godefroy, *Linear dynamics* (is concerned with hypercyclic operators, focussing on some recent results obtained by S. Grivaux and F. Bayart), Nigel J. Kalton, *Greedy algorithms and bases from the point of view of Banach space theory* (discusses some recent results about greedy, quasi-greedy and almost-greedy bases in Banach spaces obtained by the author, Konyagin, Temlyakov, Wojtaszczyk, a.o.), Michael M. Neumann, *Spectral properties of Cesáro-like operators* (the fine spectrum of Cesàro-like operators on Hardy spaces and on weighted Bergman space) and Joan Verdera, *Classical potential theory and analytic capacity* (reports on the spectacular solutions given in 1998 by G. David to an old problem of Vitushkin (1967) and by the author in 2003 to a problem by J. Garnett (1972)).

The last paper in the volume, containing mainly previously unpublished results, F. Zó and H. H. Cuenya, *Best approxiamtion ion small regions - A general approach,* proposes a unified approach, via monotone norms and Orlicz spaces, to some problems in local approximation theory by polynomials of Taylor type.

The volume presents interests for mathematicians desiring to get first-hand information on some recent results and trends in various domains of mathematical analysis.

I. V. Şerb

David Bachman, *A Geometric Approach to Differential Forms*, Birkhäuser, Boston - Basel - Berlin, 2007, xii+568 pp, ISBN: 978-3-7643-8146-2.

The integration of differential forms and Stokes' theorem are among the most difficult parts of the multivariable integral calculus. The main difficulty consists in understanding the connection between the algebraic and analytic machinery of differential forms and their geometric support. In concrete applications in $\mathbb{R}^2$ and $\mathbb{R}^3$ one draws pictures and one thinks geometrically, while in higher dimensions abstract calculations must be used. In many cases it is hard to realize how the abstract notions in the calculus of the differential forms look like in our drawings.

The aim of the present book is to reverse the situation - one starts with intuitive geometrical considerations in the spaces $\mathbb{R}^2$ and $\mathbb{R}^3$, and extending them to higher dimensions. As the author mentions in the Preface, the motivation for writing such a book comes from his experience with the abstract algebraic approach to differential forms in the book on differential topology by Guillemin and Pollack and the geometric approach in V. Arnold's book on mathematical methods of classical mechanics.

The book starts with a review of basic results in the multivariable calculus - vectors, functions, multiple integrals partial derivatives and gradients, in the first chapter, and an introduction to parametrization in the second one. In the third chapter the integral of the function $f(x, y) = y^2$ on a semicircle is calculated for two parametrizations with different results. Based on this clever example, the author explains why the differential forms must enter on the stage, a general theory being given in Chapter 4, including computations (addition, multiplication).

The core of the book is formed by Chapters 5. *Differential forms,* 6. *Differentiation of forms,* and 7. *Stokes' theorem,* while Chapter 8 is devoted to applications - Maxwell's equations, foliations and contact structures.

Chapter 9. *Manifolds,* is an introduction to more advanced topics - differential forms on manifolds, culminating with a short presentation of De Rham cohomology.

Non-linear forms defining surface area and arc length are treated in an appendix.

Based on some courses taught by the author at Calyfornia Polythechnic State University, San Luis Obispo and Pitzer College, the book is well written, in a pedestrian and pleasant style, with a lot of illuminating pictures. I consider it a good (and necessary) companion to more advanced books on the subject. In fact, a preliminary knowledge of algebraic theory of differential forms is an advantage in reading this book.

It can be used for a third semester course in calculus, or a sophomore level in Vector Calculus, and parts of it, for advanced undergraduate or graduate courses.

Tiberiu Trif

A. Borel and L. Ji, *Compactifications of Symmetric and Locally Symmetric Spaces*, Birkhäuser (Mathematics: Theory and Applications), 2006, Hardback, 476 pp., ISBN-10: 0-8176-3247-6, ISBN-13: 978-0-8176-3247-2.

Symmetric and locally symmetric spaces occur, as the authors of this monograph emphasize from the very beginning, in many branches of modern mathematics. In many situations, they are noncompact. As it usually happens, it is a lot easier to deal with *compact* spaces, therefore there were elaborated several ways to compactify them. This is, to my knowledge, the first serious attempt to present in a unified manner, the most important compactification methods known so far.

The authors point out that there are, essentially, three types of compactifications:

1. compact spaces that contain a symmetric space as an open dense subset;
2. compact smooth analytic manifolds containing a disjoint union of finitely many (but al least two!) symmetric spaces as an open dense subspace and
3. compact spaces containing a locally symmetric space as an open dense subset.

Clearly, the first and the last compactifications are the usual ones (in the sense of the point set topology), but the compactifications of type 2 are also important in many applications.

The book is, roughly, structured according to this classification of compactifications. Thus, after a short introductions, including, also, historical material, the first part of the book is devoted to compactifications of type 1 (more specifically, the compactifications of Riemannian symmetric spaces), the second part is concern with the smooth compactifications of semisimple symmetric spaces, while the last part is dedicated to compactifications of locally symmetric spaces.

In all the cases, as I've already pointed out, there are discussed particular compactification methods, but there are also made effort to unify different approaches.

This is a highly technical book, addressed only to researchers or advanced graduate students, as the prerequisite are rather demanding: semisimple Lie groups, algebraic geometry, algebraic groups, etc. Much of the material is taken from authors publications, some of it is here for the first time in a monograph. Both author are well known experts in the field. Actually, the first author (Armand Borel) who, unfortunately, passed away before the publication of the book, was one of the finest mathematician of the twentieth century and this is, in a way, his scientific testament.

The book is, definitely, a very valuable addition to the literature on compactification theory for symmetric and locally symmetric spaces, and it will soon become an indispensable reference for anyone working in the field.

Paul Blaga

Albrecht Pietsch, ***History of Banach Spaces and Linear Operators,*** Birkhäuser Verlag AG, Boston - Basel - Berlin, 2007, xxiii + 855 pp, ISBN: 10: 0-8176-4367-2 and 13:978-0-8176-4367-6.

This book is a welcome and waited addition to the existent books on the history of functional analysis. In fact there were two such books - A. F. Monna, *Functional analysis in historical perspective,* Oetshoek, Utrecht, 1973, and J Dieudonné, *History of functional analysis,* North-Holland, Amsterdam, 1981 - both of them covering the period up to 1950, so that a book dealing with the modern developments in Banach space theory was strongly required, a difficult task taken and brilliantly accomplished by the author of the present book. Due to broadening of the subject and the explosion of results, writing a book about functional analysis is almost impossible, so the author restricted to Banach spaces and bounded linear operators acting on them, fields in which he was actively involved over the last 50 years - he received his M.Sc. in 1958, exactly when a new era started in Banach space theory. In fact the author divides the development of Banach spaces in seven periods: 1900-1920 - **the prenatal period** (Fredholm, Hilbert, Riesz-Fischer theorem); 1920 -**the birth**, marked by Banach's thesis; 1920 - 1932 - **the youth** (the principles of uniform boundedness, closed graph and open mapping, Hahn-Banach theorem); 1932 - **the maturity** marked by the publication of Banach's monograph *Théorie des opérations linéaires*; 1932 - 1958 - **post-Banach period** (interrupted by Holocaust and World War II); 1958 - **classical books** (Dunford - Schwartz, vol. I, Hille-Phillips, Taylor), **midlife crisis** and **big bang** (Grothendieck's resumé, Mazur's school in Warszawa, Dvoretzky's theorem), and **the modern period** from 1958 on.

A crucial event in the last years was the publication of the *Handbook of the geometry of Banach spaces,* edited by W. B. Johnson and J. Lindenstrauss, Elsevier, Amsterdam, vol. I (2001), vol. II (2003), concerned almost exclusively with the present-day situation in Banach space theory. The author considers that his book may be regarded as a historical companion of these volumes.

The exposition is divided into seven chapters: 1. *The birth of Banach spaces,* 2. *Historical roots and basic results,* 3. *Topological concepts - weak topologies,* 4. *Classical Banach spaces,* 5. *Basic results from the post-Banach period* (analysis in Banach spaces, spectral theory, convexity and extreme points, geometry of the unit ball, bases, tensor products and approximation properties), 6. *Modern Banach* *space theory* (geometry of Banach spaces, s-numbers and operator ideals (author's specialty), eigenvalue distributions, interpolation theory, function spaces, probability on Banach spaces), 7. *Miscellaneous topics* (modern techniques - probabilistic and combinatorial, counterexamples, Banach spaces and axiomatic set theory).

The last chapter of the book, 8. *Mathematics is made by mathematicians,* starts with a tribute to mathematicians victims of the terror - killed in wars or murdered by totalitarian regimes - especially from the former Soviet Union and Poland. Then one presents some important schools as well as short biographies of some prominent contributors to the development of the Banach space theory or who have written influential books. As the author mentions in the Preface his main concern is not *Who proved a theorem?* nor to rank mathematicians, but rather to answer the question *Why and how was a theorem proved?* For this reason precise definitions and statements are formulated and, in some cases, even proofs are included.

The book is very well organized - a huge bibliography (approximatively 2600 items and 4600 quotations), an index name, a chronological index, a notion index.

The book is written in a very pleasant style, combining erudition and precision with anecdotes about mathematicians, witty remarks and pertinent comments by the author, making the reading instructive and entertaining as well.

Professor Pietsch made a great service to the mathematical community by writing this book.

S. Cobzaş

S. Dragomir and G. Tomassini, *Differential Geometry and Analysis on CR Manifolds*, Birkhäuser (Progress in Mathematics, 246), 2006, Hardback, 487 pages, ISBN-10: 0-8176-4388-5, ISBN-13: 978-0-8176-4388-1.

A CR-manifold M^m is, essentially, a real manifold endowed with a special rank n complex subbundle of its complexified tangent bundle. n and $k = m - 2n$ are called, respectively, the CR-dimension and CR-codimension of the CR-manifold M. A very important class of examples (and the original motivation) is that of real hypersurfaces of complex manifolds. Obviously, such a hypersurface has an *odd* real dimension, therefore it cannot carry a complex structure, but it inherits, nevertheless, a CR-structure, of codimension 1, from the ambient complex manifold. More generally, all the codimension 1 CR-manifolds are called of hypersurface type.

This monograph, written by two of the active researchers in the field and including much original material, treats some of the moderns aspects of the very interesting field, lying at the intersection between complex analysis, differential geometry and partial differential equations.

Much of the book is devoted to CR-manifolds of hypersurface type, which are most interesting from the geometrical point of view, since they carry significant geometrical structure (connections and metric). Thus, the first chapter of the book is devoted to the so-called pseudo-Hermitian geometry, namely the geometry of CR-manifolds endowed with a canonical connection, introduced by Tanaka and Webster in that late seventieth. The second chapter deals with another essential geometric object associated to a CR-manifold, the Fefferman metric, a Lorentzian metric (which doesn't live on the manifold itself, but it is intimately related to it). The remaining part of the book investigates different objects and problems that can be formulated starting from the basic geometric structure, by analogy with the classical Riemannian geometry: the CR Yamabe problem, pseudoharmonic maps, pseudo-Einsteinian manifolds, psudo-Hermitian immersions, spectral geometry, Yang-Mills fields on CR-manifolds, quasiconformal mappings.

Much of the material of the book is for the first time present in the monograph literature and, which is, perhaps, even more important, the book has a distinct geometric flavor, unlike many of the recent books on this topic, which were written mainly by analysts and treated especially problems from the realm of complex analysis of several variables or partial differential equations. We have, finally, a monograph trying to do justice to both parts, insisting, however (as I said) more on geometry than on analysis. Let me also say that the description of the geometrical part is very technical and can only be done mostly with the tools of the analyst.

The book is very well written and, although this is, clearly, a research monograph, can be read with real benefit also by advanced graduate students and PhD students with interests in all the three fields mentioned above: differential geometry, complex functions of several variables and PDE.

I couldn't help noticing the impressive literature list (449 titles), giving an idea of the documentation work lying behind this excellent book.

Paul Blaga

Pavel Drábek and Jaroslav Milota, ***Methods of Nonlinear Analysis - Applications to Differential Equations,*** Birkhäuser Advanced Texts, Birkhäuser Verlag AG, Basel - Boston - Berlin, 2007, xii+568 pp, ISBN: 978-3-7643-8146-2.

This introductory course contains the basic results and methods in nonlinear analysis, with applications to boundary value problems for ordinary and partial differential equations. To avoid technicalities and make the text accessible to beginners, some of the assertions and examples are not treated in the most general known form, but rather in typical situations containing the essential features of the problem. In fact, the book is written at two levels - the basic material contained in the seven chapters of the book and the appendices, containing more advanced topics. The basic material can be read independently, while the appendices, following some sections in the main text and written in a smaller font size, depend on the basic material.

The first chapter 1. *Preliminaries,* contains some results from linear algebra and normed linear spaces, including a presentation of basic of L^p- and Sobolev spaces. Some results are given with proofs. Chapter 2. *Properties of linear and nonlinear operators*, is concerned with the basic principles of functional analysis in normed spaces and some properties of linear compact operators (Schauder compactness theorem and Riesz spectral theory). The part on nonlinear operators deals with Banach contraction principle, Browder fixed point theorem for nonexpansive mappings in Hilbert space, Edelstein fixed point theorem. Chapter 3. *Abstract and differential calculus,* contains the basic results on Riemann integral for Banach space valued functions, Bochner integral and Dunford functional calculus, differential calculus in normed spaces, and Newton method in an appendix.

The inverse function theorem (local and global), the implicit function theorem, local structure of differentiable mappings and bifurcation, the rank theorem, Morse theorem, are treated in the fourth chapter *Local properties of differentiable mappings.* Some more refined results, as differentiable manifolds and vector fields, differential forms and Poincaré theorem, integration on manifolds and Stokes theorem, Brouwer degree with applications to Borsuk-Ulam antipodal theorem and Jordan separation theorem, are included in the appendices to this chapter.

Chapter 5. *Topological and montonicity methods,* presents Brouwer and Schauder fixed point theorems, topological degree, and some results on monotone operators. The appendices contain some fixed point theorems (involving measures of noncompactness) for noncompact operators, Rabinowitz global bifurcation theorem (Izé's proof), topological degree for generalized monotone operators (following Browder and Skrypnik), Krein-Rutman theorem.

Chapter 6. *Variational methods,* is devoted to the basic methods of the variational calculus - local and global extrema, Lagrange multipliers, the Mountain Pass and Saddle Point Theorems, Ritz method and, in appendices, Krasnoselski potential bifurcation theorem, Ekeland variational principle with applications, Lusternik-Schirelman category method, Rabinowitz linking theorem.

The last chapter, 7. *Boundary value problems for partial differential equations,* contains applications of the results and methods developed in the previous chapters to boundary value problems for partial differential equations - classical and weak solutions as well.

There are a lot of examples and exercises spread through the book as well as a lot of explanatory footnotes.

The book ends with some tabular synthesis material - summaries of methods and of typical applications, a comparison of bifurcation results, a list of symbols, an index and a bibliography of 137 titles.

The book is well written, in an accessible and clear style and well organized. It can be used for graduate courses in nonlinear analysis or for self-study.

Damian Trif