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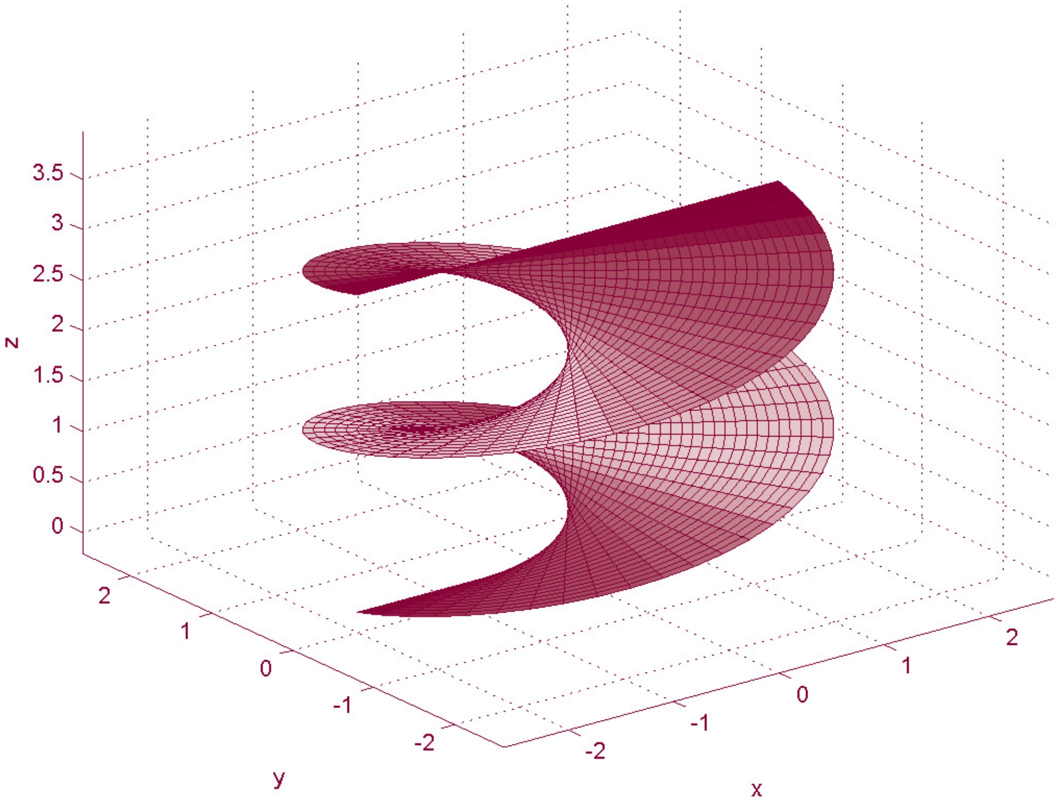

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A NEW SEQUENCE SPACE DEFINED BY A MODULUS

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Abstract. The idea of difference sequence spaces was introduced by Kızmaz [8] and this concept was generalized by Et and Çolak [6]. In this paper we define the space $\ell(\Delta^m, f, p, q, s)$ on a seminormed complex linear space by using modulus function and we give various properties and some inclusion relations on this space. Furthermore we study some of its properties, solidity, symmetricity etc.

1. **Introduction**

Let w denote the space of all sequences, and let ℓ_∞, c and c_0 be the linear spaces of bounded, convergent and null sequences $x = (x_k)$ with complex terms, respectively, normed by $\|x\|_\infty = \sup_k |x_k|$, where $k \in \mathbb{N} = \{1, 2, 3, ...\}$, the set of positive integers. Kızmaz [8] defined the sequence spaces

$$X(\Delta) = \{x = (x_k) : (\Delta x_k) \in X\}$$

for $X = \ell_\infty$, c and c_0, where $\Delta x = (\Delta x_k) = (x_k - x_{k+1})$.

The sequence spaces $\ell_\infty(\Delta^m)$, $c(\Delta^m)$, $c_0(\Delta^m)$ have been introduced by Et and Çolak [6]. These sequence spaces are BK spaces (Banach coordinate spaces) with norm

$$\|x\|_\Delta = \sum_{i=0}^{m} |x_i| + \|\Delta^m x\|_\infty ,$$

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where $m \in \mathbb{N}, \Delta^0 x = (x_k)$, $\Delta x = (x_k - x_{k+1})$, $\Delta^m x = (\Delta^{m-1} x_k - \Delta^{m-1} x_{k+1})$, and so

$$\Delta^m x_k = \sum_{v=0}^{m} (-1)^v \binom{m}{v} x_{k+v}.$$

The operators

$$\Delta^m, \sum\nolimits^m : w \to w$$

are defined by

$$\left(\Delta^1 x\right)_k = \Delta^1 x_k = x_k - x_{k+1}, \qquad \left(\sum\nolimits^1 x\right)_k = \sum_{j=1}^{k-1} x_j, \qquad (k = 0, 1, \ldots),$$

$$\Delta^m = \Delta^1 \circ \Delta^{m-1}, \qquad \sum\nolimits^m = \sum\nolimits^1 \circ \sum\nolimits^{m-1} \qquad (m \geq 2)$$

and

$$\sum\nolimits^m \circ \Delta^m = \Delta^m \circ \sum\nolimits^m = id$$

the identity on w (see [10]).

It is trivial that the generalized difference operator Δ^m is a linear operator. Recently, spectral properties of the difference operator were given by Malafosse [9], Altay and Başar [1].

Subsequently difference sequence spaces have been studied by various authors: (Çolak, Et and Malkowsky [4], Et [5], Mursaleen [8]).

The notion of a modulus function was introduced by Nakano [14] in 1953. We recall that a modulus f is a function from $[0, \infty)$ to $[0, \infty)$ such that

i) $f(x) = 0$ if and only if $x = 0$,

ii) $f(x + y) \leq f(x) + f(y)$, for all $x \geq 0$, $y \geq 0$,

iii) f is increasing,

iv) f is continuous from the right at 0.

It follows that f must be continuous on $[0, \infty)$. A modulus may be bounded or unbounded. For example, $f(x) = x^p$, $(0 < p \leq 1)$ is unbounded and $f(x) = \frac{x}{1+x}$ is bounded. Maddox [12] and Ruckle [16] used a modulus function to construct some sequence spaces.

After then some sequence spaces, defined by a modulus function, were introduced and studied by Bilgin [3], Pehlivan and Fisher [15], Waszak [17], Bhardwaj [2] and many others.

Proposition 1.1. Let f be a modulus and let $0 < \delta < 1$. Then for each $x \geq \delta$ we have $f\ (x) \leq 2f\,(1)\,\delta^{-1}x$, [15].

Let $p = (p_k)$ be a sequence of strictly positive real numbers. Let X be a seminormed space over the field $\mathbb{C}$ of complex numbers with the seminorm q. We introduce the following set of X - valued sequences

$$\ell(\Delta^m, f, p, q, s) = \left\{ x = (x_k) : x_k \in X,\ \sum_{k=1}^{\infty} k^{-s}[f(q(\Delta^m x_k))]^{p_k} < \infty\,,\ s \geq 0 \right\}$$

where f is a modulus. For different seminormed spaces X we get different sequence spaces $\ell(\Delta^m, f, p, q, s)$. Throughout the paper without writing X we use the notation $\ell(\Delta^m, f, p, q, s)$ for any but the same seminormed space X, unless otherwise indicated.

The following inequality will be used throughout this paper.

Let $p = (p_k)$ be a sequence of strictly positive real numbers with $0 < p_k \leq \sup_k p_k = H < \infty$. Then for $a_k, b_k \in \mathbb{C}$, we have

$$|a_k + b_k|^{p_k} \leq C\,\{|a_k|^{p_k} + |b_k|^{p_k}\}\,, \tag{1}$$

where $C = \max\,(1, 2^{H-1})$ (see for instance [11]).

The set $\ell(\Delta^m, f, p, q, s)$ is not a subset of ℓ_∞ for $m \geq 2$, in case $X = \mathbb{C}$ or $X = \mathbb{R}$, the set of real numbers. For this let $X = \mathbb{C}$, $s = 0$, $f\,(x) = x$, $q\,(x) = |x|$ and $p_k = 1$ for all $k \in \mathbb{N}$. If $x_k = k$ for all $k \in \mathbb{N}$, then $(x_k) \in \ell(\Delta^m, f, p, q, s)$ and $(x_k) \notin \ell_\infty$.

Definition 1.2. Let X be a sequence space. Then X is called

a) Solid (or normal) if $(\alpha_k x_k) \in X$ whenever $(x_k) \in X$ for all sequences (α_k) of scalars with $|\alpha_k| \leq 1$,

b) Symmetric if $(x_k) \in X$ implies $\left(x_{\pi(k)}\right) \in X$, where $\pi\,(k)$ is a permutation of $\mathbb{N}$ [7].

Definition 1.3. Let p, q be seminorms on a vector space X. Then p is said to be stronger than q if whenever (x_n) is a sequence such that $p(x_n) \to 0$, then also $q(x_n) \to 0$. If each one stronger then the other one, then p and q are said to be equivalent [18].

Lemma 1.4. Let p and q be seminorms on a linear space X. Then p is stronger than q if and only if there exists a constant $M \geq 0$ such that $q(x) \leq Mp(x)$ for all $x \in X$ [18].

2. Main results

In this section we will give some results on the sequence space $\ell(\Delta^m, f, p, q, s)$, those characterize the structure of the space $\ell(\Delta^m, f, p, q, s)$.

Theorem 2.1. The sequence space $\ell(\Delta^m, f, p, q, s)$ is a linear space over $\mathbb{C}$.

Proof. Let $x, y \in \ell(\Delta^m, f, p, q, s)$. For $\lambda, \mu \in \mathbb{C}$, there exist positive integers M_λ and N_μ such that $|\lambda| \leq M_\lambda$ and $|\mu| \leq N_\mu$. Since f is subadditive, q is a seminorm and Δ^m is linear

$$\sum_{k=1}^{\infty} k^{-s}[f(q(\Delta^m(\lambda x_k + \mu y_k)))]^{p_k} \leq \sum_{k=1}^{\infty} k^{-s}[f(|\lambda|\, q(\Delta^m x_k)) + f(|\mu|\, q(\Delta^m y_k))]^{p_k}$$

$$\leq C\,(M_\lambda)^H \sum_{k=1}^{\infty} k^{-s}[f(q(\Delta^m x_k)]^{p_k} + C\,(N_\mu)^H \sum_{k=1}^{\infty} k^{-s}[f(q(\Delta^m y_k)]^{p_k} < \infty.$$

This proves that $\ell(\Delta^m, f, p, q, s)$ is a linear space.

Theorem 2.2. $\ell(\Delta^m, f, p, q, s)$ is a paranormed space (not totally paranormed), paranormed by

$$g_\Delta(x) = \left\{ \sum_{k=1}^{\infty} k^{-s}\left[f\left(q(\Delta^m x_k)\right)\right]^{p_k} \right\}^{\frac{1}{M}}$$

where $H = \sup p_k < \infty$ and $M = \max(1, H)$.

Proof. Clearly $g_\Delta(\theta) = 0$ and $g_\Delta(x) = g_\Delta(-x)$, where $\theta = (\theta, \theta, \theta, ...)$ and is the zero of X.

It also follows from (1), Minkowski's inequality and the definition of f that g_Δ is subadditive. Now for a complex number λ, by inequality

$$|\lambda|^{p_k} \leq \max\left(1, |\lambda|^H\right)$$

and the definition of modulus f, we have

$$\begin{aligned} g_\Delta(\lambda x) &= \left(\sum_{k=1}^{\infty} k^{-s}\left[f\left(q\left(\lambda\Delta^m x_k\right)\right)\right]^{p_k}\right)^{\frac{1}{M}} \\ &\leq \left(1+[|\lambda|]\right)^{\frac{H}{M}} . g_\Delta(x) \end{aligned}$$

where $[|\lambda|]$ denotes the integer part of λ, hence $\lambda \to 0$, $x \to \theta$ imply $\lambda x \to \theta$ and also $x \to \theta$, λ fixed imply $\lambda x \to \theta$.

Now suppose $\lambda_n \to 0$ and x is a fixed point in $\ell\left(\Delta^m, f, p, q, s\right)$. Given $\varepsilon > 0$, let K be such that

$$\sum_{k=K+1}^{\infty} k^{-s}\left[f\left(q\left(\Delta^m x_k\right)\right)\right]^{p_k} < \left(\frac{\varepsilon}{2}\right)^M .$$

Hence we have

$$\left(\sum_{k=K+1}^{\infty} k^{-s}\left[f\left(q(\Delta^m x_k)\right)\right]^{p_k}\right)^{\frac{1}{M}} < \frac{\varepsilon}{2}.$$

Since f is continuous on $[0, \infty)$

$$h(t) = \sum_{k=1}^{K} k^{-s}\left[f\left(q((\Delta^m\left(tx_k\right)))\right)\right]^{p_k}$$

is continuous at 0. Therefore, there exists $0 < \delta < 1$ such that $|\lambda_n| < \delta$ implies

$$\left(\sum_{k=1}^{K} k^{-s}\left[f\left(q(\lambda_n\Delta^m x_k)\right)\right]^{p_k}\right) < \frac{\varepsilon}{2}.$$

for $n > N$. Hence

$$\left(\sum_{k=1}^{\infty} k^{-s}\left[f\left(q(\lambda_n\Delta^m x_k)\right)\right]^{p_k}\right)^{\frac{1}{M}} < \varepsilon$$

for $n > N$. Therefore $g(\lambda x) \to 0$ as $\lambda \to 0$.

Theorem 2.3. Let f, f_1 and f_2 be modulus functions, q, q_1 and q_2 seminorms and $s, s_1, s_2 \geq 0$ real numbers.

i) If $s > 1$, then $\ell\left(\Delta^m, f_1, p, q, s\right) \subseteq \ell\left(\Delta^m, f \circ f_1, p, q, s\right)$,

ii) $\ell\left(\Delta^m, f_1, p, q, s\right) \cap \ell\left(\Delta^m, f_2, p, q, s\right) \subseteq \ell\left(\Delta^m, f_1 + f_2, p, q, s\right)$,

iii) $\ell\left(\Delta^m, f, p, q_1, s\right) \cap \ell\left(\Delta^m, f, p, q_2, s\right) \subseteq \ell\left(\Delta^m, f, p, q_1 + q_2, s\right)$,

iv) If q_1 is stronger than q_2 then $\ell\left(\Delta^m, f, p, q_1, s\right) \subseteq \ell\left(\Delta^m, f, p, q_2, s\right)$,

v) If $s_1 \leq s_2$, then $\ell\left(\Delta^m, f, p, q, s_1\right) \subseteq \ell\left(\Delta^m, f, p, q, s_2\right)$.

Proof. i) Let $x_k \in \ell\left(\Delta^m, f_1, p, q, s\right)$. Let $\varepsilon > 0$ and choose δ with $0 < \delta < 1$ such that $f(t) < \varepsilon$ for $0 \leq t \leq \delta$. Write $t_k = f_1(q(\Delta^m x_k))$ and consider

$$\sum_{k=1}^{\infty} k^{-s}[f(t_k)]^{p_k} = \sum_{1} k^{-s}[f(t_k)]^{p_k} + \sum_{2} k^{-s}[f(t_k)]^{p_k}$$

where the first summation is over $t_k \leq \delta$ and the second over $t_k > \delta$. Since f is continuous, we have

$$\sum_{1} k^{-s}[f(t_k)]^{p_k} < \max\left(1, \varepsilon^H\right) \sum_{k=1}^{\infty} k^{-s} \tag{2}$$

and for $t_k > \delta$ we use the fact that

$$t_k < \frac{t_k}{\delta} < 1 + \left[\left|\frac{t_k}{\delta}\right|\right].$$

By the definition of f we have for $t_k > \delta$,

$$f(t_k) \leq f(1)[1 + \left(\frac{t_k}{\delta}\right)] \leq 2f(1)\frac{t_k}{\delta}$$

$$\sum_{2} k^{-s}[f(t_k)]^{p_k} \leq \max\left(1, \left(\frac{2f(1)}{\delta}\right)^H\right) \sum_{k=1}^{\infty} k^{-s}[t_k]^{p_k} < \infty. \tag{3}$$

By (2) and (3) we have $\ell\left(\Delta^m, f_1, p, q, s\right) \subseteq \ell\left(\Delta^m, f \circ f_1, p, q, s\right)$.

ii) Let $x = x_k \in \ell\left(\Delta^m, f_1, p, q, s\right) \cap \ell\left(\Delta^m, f_2, p, q, s\right)$. Then using (1) it can be shown that $x_k \in \ell\left(\Delta^m, f_1 + f_2, p, q, s\right)$. Hence $\ell\left(\Delta^m, f_1, p, q, s\right) \cap \ell\left(\Delta^m, f_2, p, q, s\right) \subseteq \ell\left(\Delta^m, f_1 + f_2, p, q, s\right)$.

iii) The proof of (iii) is similar to the proof of (ii) by using the inequality

$$k^{-s}\left[f(q_1 + q_2)\left(\Delta^m x_k\right)\right]^{p_k} \leq Ck^{-s}\left[f\left(q_1\left(\Delta^m x_k\right)\right)\right]^{p_k} + Ck^{-s}\left[f\left(q_2\left(\Delta^m x_k\right)\right)\right]^{p_k}$$

where $C = \max\left(1, 2^{H-1}\right)$.

(iv) and (v) follows easily.

We get the following sequence spaces from $\ell(\Delta^m, f, p, q, s)$ by choosing some of the special $p, f,$ and s :
For $f(x) = x$ we get

$$\ell(\Delta^m, p, q, s) = \left\{ x \in w(X) : \sum_{k=1}^{\infty} k^{-s} [q(\Delta^m x_k)]^{p_k} < \infty \,,\; s \geq 0 \right\};$$

for $p_k = 1$, for all k, we get

$$\ell(\Delta^m, f, q, s) = \left\{ x \in w(X) : \sum_{k=1}^{\infty} k^{-s} [f\,(q(\Delta^m x_k))] < \infty \,,\; s \geq 0 \right\};$$

for $s = 0$ we get

$$\ell(\Delta^m, f, p, q) = \left\{ x \in w(X) : \sum_{k=1}^{\infty} [f\,(q(\Delta^m x_k))]^{p_k} < \infty \right\};$$

for $f(x) = x$ and $s = 0$ we get

$$\ell(\Delta^m, p, q) = \left\{ x \in w(X) : \sum_{k=1}^{\infty} [q(\Delta^m x_k)]^{p_k} < \infty \right\};$$

for $p_k = 1$, for all k, and $s = 0$ we get

$$\ell(\Delta^m, f, q) = \left\{ x \in w(X) : \sum_{k=1}^{\infty} f\,(q(\Delta^m x_k)) < \infty \right\};$$

for $f(x) = x$, $p_k = 1$, for all k, and $s = 0$ we have

$$\ell(\Delta^m, q) = \left\{ x \in w(X) : \sum_{k=1}^{\infty} q(\Delta^m x_k) < \infty \right\}.$$

Corallary 2.4. i) If $s > 1$ then for any modulus f we have

$$\ell(\Delta^m, p, q, s) \subseteq \ell\,(\Delta^m, f, p, q, s)\,,$$

ii) If q_1 and q_2 are equivalent seminorms then

$$\ell(\Delta^m, f, p, q_1, s) = \ell\,(\Delta^m, f, p, q_2, s)\,,$$

iii) $\ell(\Delta^m, f, p, q) \subseteq \ell\,(\Delta^m, f, p, q, s)\,,$

iv) $\ell(\Delta^m, p, q) \subseteq \ell\,(\Delta^m, p, q, s)\,,$

v) $\ell(\Delta^m, f, q) \subseteq \ell\,(\Delta^m, f, q, s)\,.$

Proof. i) If $f_1(t) = t$ in Theorem 2.3 (i), then the result follows easily.

ii) It follows from Theorem 2.3 (iv).

iii) If we take $s_1 = 0$ and $s_2 = s$ in Theorem 2.3 (vi), then we get $\ell(\Delta^m, f, p, q) \subseteq \ell\left(\Delta^m, f, p, q, s\right)$.

iv) If we take $s_1 = 0$, $s_2 = s$, and $f(t) = t$ in Theorem 2.3 (vi), then we get $\ell(\Delta^m, p, q) \subseteq \ell\left(\Delta^m, p, q, s\right)$.

v) If we take $s_1 = 0$, $s_2 = s$, and $p_k = 1$ for all k, in Theorem 2.3 (vi) then $\ell(\Delta^m, f, q) \subseteq \ell\left(\Delta^m, f, q, s\right)$.

Theorem 2.5. $\ell(\Delta^{m-1}, f, q, s) \subset \ell(\Delta^m, f, q, s)$ for $m \geq 1$ and the inclusion is strict. In general $\ell(\Delta^i, f, q, s) \subset \ell(\Delta^m, f, q, s)$ for all $i = 1, 2, 3, ..., m-1$ and the inclusions are strict.

Proof. Let $x \in \ell(\Delta^{m-1}, f, q, s)$. Then we have

$$\sum_{k=1}^{\infty} k^{-s} f\left(q(\Delta^{m-1} x_k)\right) < \infty. \tag{4}$$

Since $(k+1)^{-s} < k^{-s} \leq 2^s (k+1)^{-s}$ for all $k \in \mathbb{N}$ we get the following inequality

$$k^{-s} f\left(q(\Delta^{m-1} x_{k+1})\right) \leq 2^s (k+1)^{-s} f\left(q(\Delta^{m-1} x_{k+1})\right). \tag{5}$$

(4) and (5) together imply that

$$\sum_{k=1}^{\infty} k^{-s} f\left(q(\Delta^{m-1} x_{k+1})\right) < \infty. \tag{6}$$

Since f is increasing, $f(x+y) \leq f(x) + f(y)$ and q is a seminorm, from (4) and (6) we get

$$\begin{aligned} \sum_{k=1}^{\infty} k^{-s} f\left(q(\Delta^m x_k)\right) &= \sum_{k=1}^{\infty} k^{-s} f\left(q(\Delta^{m-1} x_k - \Delta^{m-1} x_{k+1})\right) \\ &\leq \sum_{k=1}^{\infty} k^{-s} f\left(q(\Delta^{m-1} x_k)\right) + \sum_{k=1}^{\infty} k^{-s} f\left(q(\Delta^{m-1} x_{k+1})\right) \text{ ¡ } \infty. \end{aligned}$$

Thus $\ell(\Delta^{m-1}, f, q, s) \subset \ell(\Delta^m, f, q, s)$.

In general $\ell(\Delta^i, f, q, s) \subset \ell(\Delta^m, f, q, s)$ for $i = 1, 2, 3, ..., m-1$ and the inclusions are strict. For this consider the following example.

Example 2.1. Let $X = \mathbb{C}$, $f(x) = x$, $q(x) = |x|$, $s = 0$. Consider the sequence $(x_k) = \left(k^{m-1}\right)$. Then $(x_k) \in \ell(\Delta^m, f, q, s)$ but $(x_k) \notin \ell(\Delta^{m-1}, f, q, s)$, since $\Delta^m x_k = 0$ and $\Delta^{m-1} x_k = (-1)^{m-1}(m-1)!$ for all $k \in \mathbb{N}$.

Theorem 2.6. The sequence space $\ell\left(\Delta^m, f, p, q, s\right)$ is not solid.

Proof. To show that the space is not solid in general, consider the following example.

Example 2.2. Let $X = \mathbb{C}$, $f(x) = x$, $q(x) = |x|$, $m = 2$, $s = 0$ and $p_k = 1$ for all $k \in \mathbb{N}$. Then $x = (x_k) = (k) \in \ell\left(\Delta^m, f, p, q, s\right)$ but $\alpha x = (\alpha_k x_k) \notin \ell\left(\Delta^m, f, p, q, s\right)$ for $\alpha_k = (-1)^k$ for all $k \in \mathbb{N}$. Hence $\ell\left(\Delta^m, f, p, q, s\right)$ is not solid.

Theorem 2.7. i) Let $0 < t_k \leq r_k < \infty$ for each $k \in \mathbb{N}$. Then

$$\ell\left(\Delta^m, f, t, q\right) \subseteq \ell(\Delta^m, f, r, q),$$

ii) $\ell\left(\Delta^m, f, q\right) \subseteq \ell(\Delta^m, f, q, s)$,

iii) $\ell\left(\Delta^m, f, t, q\right) \subseteq \ell(\Delta^m, f, t, q, s)$.

Proof. i) If $x \in \ell\left(\Delta^m, f, t, q\right)$ then, for all sufficiently large k,

$$\left[f\left(q\left(\Delta^m x_k\right)\right)\right]^{t_k} \leq 1$$

and so

$$\left[f\left(q\left(\Delta^m x_k\right)\right)\right]^{r_k} \leq \left[f\left(q\left(\Delta^m x_k\right)\right)\right]^{t_k}.$$

This completes the proof.

The proof of *(ii)* and *(iii)* is trivial.

Theorem 2.8. i) If $0 < p_k \leq 1$ for each $k \in \mathbb{N}$, then $\ell\left(\Delta^m, f, p, q\right) \subseteq \ell(\Delta^m, f, q)$,

ii) If $p_k \geq 1$ for all $k \in \mathbb{N}$, then $\ell(\Delta^m, f, q) \subseteq \ell\left(\Delta^m, f, p, q\right)$.

Proof. i) If we take $p_k = t_k$ and $r_k = 1$ for all $k \in \mathbb{N}$, in Theorem 2.7 (i), then

$$\ell\left(\Delta^m, f, p, q\right) \subseteq \ell(\Delta^m, f, q).$$

ii) If we take $p_k = r_k$ and $t_k = 1$ for all $k \in \mathbb{N}$, in Theorem 2.7 (i), then

$$\ell(\Delta^m, f, q) \subseteq \ell\left(\Delta^m, f, p, q\right).$$

Theorem 2.9. The sequence space $\ell\left(\Delta^m, f, p, q, s\right)$ is not symmetric.

Proof. To show that the space is not symmetric, consider the following example.

Example 2.3. Let $X = \mathbb{C}$, $f(x) = x$, $q = |x|$, $s = 0$ and $p_k = 1$ for all $k \in \mathbb{N}$. Then the sequence $(x_k) = (k)$ belongs to $\ell(\Delta^m, f, p, q, s)$. Let (y_k) be a rearrangement of (x_k) , which is defined as follows:

$$y_k = (x_1, x_2, x_4, x_3, x_9, x_5, x_{16}, x_6, x_{25}, x_7, x_{36}, x_8, x_{49}, x_{10}, ...)$$

then the sequence (y_k) does not belong to $\ell(\Delta^m, f, p, q, s)$.

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A SURROGATE DUAL ALGORITHM FOR QUASICONVEX QUADRATIC PROBLEMS

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Abstract. The purpose of this paper is to solve, via a surrogate dual method, a quadratic program where the objectif function is not explicitly given. We apply our study to quasiconvex quadratic programs.

1. **Introduction**

In general a quadratic optimization problem can be formulated as:

$$\min\{Q\left(x\right) = \frac{1}{2}\ x^T Hx\ +\ c^T x : Ax \leq b, x \geq 0\} \tag{1}$$

where H is a symmetric $n \times n$ matrix, $c \in \mathbb{R}^n$, A is a $m \times n$ matrix and $b \in \mathbb{R}^m$. The computational cost for solving such a problem depends on the properties of the matrix H and the dimensions m and n. The convex quadratic problem (i.e. when H is positive semidefinite) is often not more difficult to solve than a linear problem. The non convex case is more difficult, stationary points and local minimums which are not global minimums may exist [15]. In this paper, we are interested in the same quadratic programs (1) with only quasiconvex objective. Historically, the first criteria on the quasiconvex and pseudoconvex quadratic functions were given by Martos [11], Cottle and Ferland [1]. As we will see in the second section, these authors characterize this class of nonconvex quadratic functions with a finite number of conditions, contrary to the classical definitions. Furthermore, Ferland [6] and Schaible [12] independently obtained a characterization of quasiconvex and pseudoconvex quadratic functions on arbitrary solid convex sets. In mathematical programming, the pseudoconvexity of

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the objective is more wished than the quasiconvexity owing to the fact that the conditions of optimality of Karuch-Kuhn-Tucker (K-K-T) are necessary and sufficient to ensure the global minimum of the problem. But by weakening the pseudoconvexity assumptions to the quasiconvex case, these conditions become only necessary, and give only critical points.The third section is devoted to the surrogate duality [3], which is more adapted to quasiconvex programming than Lagrangian duality [8]. Indeed often we obtain an non empty duality gap. In the situation when the surrogate dual can be explicitly computing (for example Q is strictly convex), this gave rise to interesting numerical treatment[14]; but in the general case the objective function is expressed only in implicit form. Our aim is to give a surrogate dual method in this difficult situation. By taking as a starting point the paper of Dyer [5], we present in the fourth section, an algorithm based on the cutting planes method, well adapted to solve a problem of type (1) with quasiconvex objective function. An example is solved via this algorithm within a small number of iterations.

2. **Quasiconvex and pseudoconvex quadratic functions**

In this section, we present criteria in terms of eigenvalues and eigenvectors of the quasi-convex and pseudo-convex quadratic functions defined on a solid convex set, and especially on the positive orthant $\mathbb{R}^n_+$. We note by $intC$ the interior of the set C.

2.1. **Definitions.** We consider the quadratic function

$$Q(x) = \frac{1}{2}\, x^T H x \; + \; c^T x$$

$$H = (h_{ij})_{i,j=1,..,n} \;\; , \; H \textit{ symmetric}, \;\; c = (c_i)_{i=1,..,n}$$

and let $C \subset \mathbb{R}^n$ denote a solid convex set, i.e., $intC \neq \varnothing$.

Definition 2.1. The quadratic function Q is said to be quasiconvex [2] on C if,

$$\forall x, y \in C,\; \forall \lambda \in \left]0,1\right[, \quad Q\left((1-\lambda)\, x + \lambda y\right) \leq \max\left(Q\left(x\right), Q\left(y\right)\right). \tag{2}$$

Equivalently, this means that the lower-level sets

$$L_\alpha(Q) = \{x \in C : Q(x) \le \alpha\}$$

are convex $\forall \alpha \in \mathbb{R}$ [2]. in the smooth case, which is the situation here (Q is quadratic), definition 2.1 becomes

$$\forall x, y \in C, \quad Q(y) \le Q(x) \quad \Longrightarrow \; (y - x)^T \nabla Q(x) \le 0. \tag{3}$$

Definition 2.2. Q is pseudoconvex [10] if,

$$\forall x, y \in C, \quad (y - x)^T \nabla Q(x) \ge 0 \Longrightarrow \; Q(y) \ge Q(x) \tag{4}$$

Definition 2.3. Q is said to be strictly pseudoconvex if,

$$\forall x, y \in C, x \ne y, \quad (y - x)^T \nabla Q(x) \ge 0 \Longrightarrow \; Q(y) > Q(x). \tag{5}$$

It is easy to show that strict pseudoconvexity implies pseudoconvexity, and pseudoconvexity implies quasiconvexity. On the other hand the opposite is not always true. A quasi-convex function (resp. pseudo-convex, strictly quasi-convex) which is not convex is called merely quasi-convex (resp. pseudo-convex, strictly quasi-convex).

2.2. **Finite criteria for a solid convex set.** Denote by $H^\dagger$ the Moore-Penrose pseudoinverse matrix of H, and denote by the triple $In(H) = (\mu_+(H), \mu_-(H), \mu_0(H))$ the inertia of the matrix H, where $\mu_+(H)$, $\mu_-(H)$ and $\mu_0(H)$ denote respectively the numbers of positive, negative and null eigenvalues of H. There exist a $n \times n$ diagonal matrix D and $n \times n$ matrix P such that $H = P^t H P$, $P^t P = I$ and let (d_i) where $i = 1, ..., n$ the i-th diagonal entry of D. We denote by $U = \{y : \langle Dy, y\rangle \le 0\}$ and by T the set $T = P^t U$. It is known that the quadratic function is convex if and only if $\mu_-(H) = 0$. we look at the merely quasiconvex and pseudoconvex case. The caracterization of generalized convex quadratic functions in terms of spectral properties is given by the following theorem

Theorem 2.1.[4] *A nonconvex quadratic function*

$$Q(x) = \frac{1}{2}\, x^T H x \; + \; c^T x$$

is quasiconvex (resp. pseudoconvex) on a solid convex $C \subset \mathbb{R}^n$ if and only if *(i) H has one and only one negative eigenvalue, i.e.,* $\mu_-(H) = 1$*;*

(ii) $c \in H(\mathbb{R}^n)$*;*

(iii) $C - H^\dagger c \subset T$ *or* $C - H^\dagger c \subset -T$ *(* $C - H^\dagger c \subset intT$ *or* $C - H^\dagger c \subset -intT$*).*

It is also seen in [4] that T and $-T$ (resp. $intT$ and -$intT$) are the maximal domains of quasiconvexity (pseudoconvexity) of Q. the algorithm presented in section 4 is applied to the problem (1) with general quasiconvex objective Q and constraint included in the maximal set of quasiconvexity. The fact that the constraints in Problem (1) below to the posif orthant more attention is given to the case $C = \mathbb{R}^n_+$.

2.3. **Finite criteria for nonnegative orthant.** We give below criteria for quasiconvex and pseudoconvex quadratic functions defined on $\mathbb{R}^n_+$ making the definitions (2), (3), (4) and (5) much more practical, this criteria can be derived by specializing the general result in theorem (2.1). We note that a quadratic function is quasiconvex on $\mathbb{R}^n$ if and only if it is convex on $\mathbb{R}^n$, and contrary to the convex functions the quasiconvex functions can be quasiconvex on a convex subset of $\mathbb{R}^n$ without being it on all the space $\mathbb{R}^n$.

Theorem 2.2. [11] and [1] *The quadratic function Q is merely quasiconvex (esp. merely pseudoconvex)on* $\mathbb{R}^n_+$ *(on* $int\mathbb{R}^n_+$*) if and only if*

(i) $H \leq 0$*; i.e.* $h_{ij} \leq 0\ \forall i, j = 1, ..., n.$

(ii) $c \leq 0$*; i.e.* $c_i \leq 0\ \ \forall i = 1, ..., n.$

(iii) H has exactly one and only one eigenvalue, i.e.,$\mu_-(H) = 1$*;*

(iv) $c^T H^\dagger c \leq 0.$

Remark 2.1. We note that the condition (iv) of theorem (2.1) imposes that the component c_k of the vector c is necessarily equal to 0 if the line h_k of the matrix H is null. Furthermore, if H is nonsingular, then Q is strictly pseudo-convex if and only if (i), (ii), (iii) and (iv) are checked, this last condition can be replaced by the condition $c^T H^{-1} c \leq 0$. With true statement, if the quadratic function Q is quasi-convex on $\mathbb{R}^n_+$, and if we suppose moreover that $c \neq 0$, then Q is always pseudo-convex on $\mathbb{R}^n_+ - \{0\}$. This result can be found in [1].

Exemple 2.1. Consider the function

$$Q_1(x) = -\frac{1}{2}(x_1 + x_2)^2 - x_1 - x_2$$

where $H_1 = \begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix}$, et $c_1 = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$. The two eigenvectors of H_1 are $\lambda_1 = -2$ et $\lambda_2 = 0$. $Q_1$is not convex ($H_1$is not positive semidefinite), we can remark that the vector $\begin{pmatrix} -1 \\ 0 \end{pmatrix}$ satisfies the condition (iv) of theorem (2.2), then Q_1 is merely quasiconvex on $\mathbb{R}^2_+$, and with remark (2.1) Q is also merely pseudo-convex on $\mathbb{R}^2_+ - \{0\}$.

Pseudo-convexity is wished in mathematical programming, since the conditions of optimality of K-K-T become necessary and sufficient. This makes it possible to solve our problem with the various algorithms using the system of K-K-T (method of Lemke, methods of interior points...). Problems appear when the function Q is merely quasiconvex, in such a situation the algorithm of the section 4 can be registered.

It is significant to also announce that the conditions (i) and (ii) of theorem (2.2) are not restrictive, because if we want to solve a problem of minimization with objective Q, (iii) and (iv) are checked but $(h_{ij} \geq 0 \forall i, j = 1, ..., n)$ and $(c_i \geq 0 \forall i = 1, ..., n)$ on a compact polyhedral. Thus we will have to solve the following problem:

$$\min \left\{ Q(x) : Ax - b = 0, x \in \mathbb{R}^n_+ \right\} = -\max \left\{ -Q(x) : Ax - b = 0, x \in \mathbb{R}^n_+ \right\}$$

then we have, a maximization problem of a quasiconvex function, where the solution is characterized by the following proposition:

Proposition 2.2. *Let C be a polyhedral compact set of $\mathbb{R}^n$, and $f : \mathbb{R}^n \to \mathbb{R}$ a continuous quasiconvex function on C. Consider the problem to maximize f on C. An optimal solution $\widetilde{x}$ to the problem then exists, where $\widetilde{x}$ is an extreme point of C.*

Proof. f attains its maximum at $\widetilde{x} \in C$. Let $x_1, x_2, ..., x_k$ the extreme points of C, assumes that $f(\widetilde{x}) > f(x_j)$ for all $j = 1, ..., k$. By definition $\widetilde{x} = \sum_{j=1}^k \lambda_j x_j$ where $\sum_{j=1}^k \lambda_j = 1$ and $\lambda_j \geq 0$ for $j = 1, ..., k$.

Since $f(\widetilde{x}) > f(x_j)$ for every j, then

$$f(\widetilde{x}) > \max_{1\leq\ j\ \leq k} f(x_j) = \alpha$$

or f is quasiconvex, then

$$f(\widetilde{x}) = f\left(\sum_{j=1}^{k} \lambda_j x_j\right) \leq \max_{1\leq\ j\leq\ k} f(x_j) = \alpha$$

hence the contradiction, so there exists necessarily $j_0 \in \{1,..,k\}$ such that $f(\widetilde{x}) = f(x_{j_0})$. □

3. Surrogate duality for the quasiconvex programming

Return now to our problem (1), for every u belonging to a compact X set in $\mathbb{R}_+^m$, we define:

$$X(u) = \left\{x \in X : u^\top (Ax - b) \leq 0\right\} \tag{6}$$

and the dual function

$$s(u) = \min\{Q(x) : x \in X(u)\}. \tag{7}$$

then the problem

$$(SP) \qquad s^* = \sup\left\{s(u) : u \in \mathbb{R}_+^m\right\} \tag{8}$$

is called the surrogate dual problem associated with the primal problem (1). it is clear that $s(tu) = s(u)$ $\forall\ u \in X$ et $\forall t > 0$. This property simplifies the formulation of the problem (SP) which can be rewritten as:

$$(SP) \qquad s^* = \sup\left\{s(u) : u \in \mathbb{R}_+^m, \|u\|_1 = 1\right\}$$

where $\|\ \ \|_1$, is the norm 1 of $\mathbb{R}^m$, the problem (SP) becomes:

$$(SP) \qquad s^* = \sup\{s(u) : u \in \triangle\} \tag{9}$$

where $\triangle = \left\{u \in \mathbb{R}_+^m : \sum_{i=1}^{m} u_i = 1\right\}$ is the simplex of $\mathbb{R}_+^m$.

The following result is a deduction of two theorems. The first is due to Luenberger [9] and the second to Greenberg and Pierskalla [8].

Proposition 3.1. *The function s is continuous and quasiconcave (i.e. -s is quasiconvex) on the simplex $\triangle$.*

If we note by $v(P)$ the value of the primal problem, we check the weak duality easily ($s^* \leq v(P)$). Luemberger [9] has shown that if $v(P)$ is finite then, there exists $\widetilde{u} \in \triangle$ such that

$$v(P) = s^* = s(\widetilde{u}) = \max\{s(u) : u \in \triangle\}. \tag{10}$$

The fundamental reason for choosing the surrogate duality is that it produces a strong duality (the duality gap $v(P) - s^*$=0), this is due of course to the historical result of Luemberger. In addition, we can always associate the Lagrangian dual problem to (1)

$$(LDP) \qquad L^* = \sup\{\min\{Q(x) + \lambda^\top (Ax - b) : x \in \mathbb{R}_+^n\} : \lambda \in \mathbb{R}_+^m\}$$

It is important to notice that the objective function of our problem is not necessarily pseudo-convex, from where the possibility of having a non nulle Lagrangean duality gap ($v(P) - L^* \neq 0$). In addition, if we manage to calculate by a means or another a multiplier of Lagrange, this last can be a good point of initialization for the algorithm to present in the preceding section. In the article of Dyer [5] we find the proposition quoted below which makes in evidence what we have just said.

Proposition 3.2. *If $\overline{\lambda}$ is a Lagrange multiplier, and $\overline{\overline{\lambda}} = \frac{\overline{\lambda}}{\|\overline{\lambda}\|_1}$. then, we have always*

$$s\left(\overline{\overline{\lambda}}\right) \geq L^*$$

moreover exactly one of the situations below holds:

(i) $s\left(\overline{\overline{\lambda}}\right) \geq L^*$.

(ii) $s\left(\overline{\overline{\lambda}}\right) = L^*$ but every neighbourhood of $\overline{\overline{\lambda}}$ in $\triangle$, contains a point u such that $s(u) > L^*$.

(iii) $s\left(\overline{\overline{\lambda}}\right) = L^* = s^*$.

For that follows we consider the set

$$G(\alpha) = A(L_\alpha(Q)) - b = \left\{g = Ax - b : Q(x) \leq \alpha, \forall x \in \mathbb{R}_+^n\right\}$$

and it's polar set

$$G^\oplus(\alpha) = \left\{u \in \triangle : g^\top u \geq 0, \forall g \in G(\alpha)\right\}$$

$$= \left\{u \in \triangle : (Ax - b)^\top u \geq 0, Q(x) \leq \alpha, \forall x \in \mathbb{R}_+^n\right\}$$

these two sets will be fundamental for the characterization of the solution $\overline{u}$ of $s(.)$.

Proposition 3.3. *s^* is the minimum number α such that $intG^{\oplus}(\alpha) = \varnothing$.*

Proof. If $\alpha < s^*$, then there exists u such that $s(u) > \alpha$. Which is equivalent to $X(u) \cap L_\alpha(Q) = \varnothing$, which is again true if and only if $g^T u > 0$ for all $g \in G(\alpha)$, but this is equivalent to say that $u \in intG^{\oplus}(\alpha)$, we conclude that if $\alpha < s^*$ then $intG^{\oplus}(\alpha) \neq \varnothing$. If now $\alpha \geq s^*$, we get necessarily for all $u \in intG^{\oplus}(\alpha)$, $s(u) > \alpha \geq s^*$, which is impossible, hence $intG^{\oplus}(\alpha) = \varnothing$. □

4. **An algorithm for a quasiconvex quadratic problem**

The method of resolution suggested here is a dual method, it is a question of finding the point $\widetilde{u}$ which solves the surrogate dual problem, and which will give the value of the primal problem thus s^* and a solution $x(\widetilde{u})$, if it is feasible it is the optimal solution of the primal problem. When the quadratic function Q is strictly convex (i.e., H is positive definite), for the following problem

$$\min\{Q(x) = \frac{1}{2}\, x^\top Hx \; + \; c^\top x : Ax \leq b, x \in \mathbb{R}^n\}$$

we can calculate explicitly the dual function $s(.)$, which can be formulated as

$$s(u) = \frac{1}{2}\frac{\left(u^\top \left(AH^{-1}c + b\right)\right)^2}{u^\top AH^{-1}A^\top u} - \frac{1}{2}c^T H^{-1}c$$

see [14] for more detail. Unfortunately, it is not the case for problem(1) with Q only quasiconvex.

The algorithm described below gives to each iteration k the point s_k the element of the sequence $(s_k)_k$ which will have to converge towards the optimal value s^*, each point s_k, is equal to $s(u_k)$ if

$$X\left(u^k\right) \cap L_{s_{k-1}}(Q) = \varnothing \tag{11}$$

else take the value s_{k-1}.

The formula (11) lead us to the resolution of the problem with a single constraint

$$(NLP)_k \qquad s(u_k) = \min\left\{Q(x) : (u^k)^\top (Ax - b) \leq 0, x \in \mathbb{R}^n_+\right\}. \tag{12}$$

The point $u_k \in intU_k$, this set will have the property to contain $intG^{\oplus}(u_k)$ at each iteration k, considering the proposition (3.3) the algorithm will stop at the first k such that $intU_k = \varnothing$, and this is true if the radius r_k of U_k becomes negative.

The set $U_k = U_{k-1} \cap \left\{u \in \triangle : u^{\top}(Ax^k - b) \geq 0\right\}$, where x_k is the optimal solution of $(NLP)_k$, if this last admits a solution in this step of the iteration k, and in this case, like noted above, s_{k-1} is increased with the value $s_k = s\left(u^k\right) = Q\left(x^k\right)$. Otherwise x^k is any feasible solution of $(NLP)_k$. In each iteration k we add a cutting plane, defined by the hyperplane $H^k = \left\{u \in \mathbb{R}^n : (u)^{\top} A\left(x^k - b\right) = 0\right\}$.

Let us note by g^k the vector $Ax^k - b$. The Euclidean distance between the point u of U_k and its border is equal to $r_k(u) = \dfrac{u^{\top} g^k}{\gamma_k}$, where $\gamma_k = \sqrt{(g^k)^{\top} g^k - \frac{1}{m}(e^{\top} g^k)^2}$ and $e^{\top} = (1, ..., 1)$.

The radius of U_k is given by

$$r_k^* = \max\left\{r_k(u) : u \in U_k\right\},$$

we can check that the $intU_k \neq \varnothing$ if and only if $r_k^* > 0$.

It is not difficult to see that

$$(LP)_k \qquad r_k^* = \max\left\{r : u^{\top} g^k - \gamma_k r \geq 0, , u \in \triangle\right\} \tag{13}$$

the problem $(LP)_k$ is linear, it is considered at each iteration k and its resolution by a classical method such as the simplex method will give the solution $\left(\overline{u}^k, r_k^*\right)$. At each iteration k the choice of u^{k+1} depends on a parameter of convergence $\theta \in]0,1]$ fixed at the beginning, the number $\alpha_k \in]0,1]$ calculated at each iteration k, the point $\overline{u}^k$solution of the linear problem $(LP)_k$ and the point u^k who should not belong to the the interior of U_k in the iteration k. The point u^{k+1} must be sufficiently distant from the boundary of U_k, then for any boundary point u, $u_{k+1} = \theta \overline{u}_k + (1-\theta)\, u \in intU_k$ but for $\overline{u}^k$ and u^k it is easy to find a boundary point u of U_k, let us choose it as

$$\alpha_k \overline{u}^k + (1-\alpha_k)\, u^k \tag{14}$$

where $\alpha \in [0, 1[$ is given by

$$\alpha_k = \frac{-(u^k)^T g^k}{(\overline{u}^k) g^k - (u^k)^T g^k} \tag{15}$$

we replace (15) in (14), the point u^{k+1} can be taken as

$$u^{k+1} = (1 - \beta_k)\, \overline{u}^k + \beta_k u^k \quad whith \quad \beta_k = (1 - \alpha_k)\,(1 - \theta) \tag{16}$$

and will have the property to belong to the $intU_k$, and if the parameter θ is quite selected the continuity of the dual function $s\,(.)$ will give an accepted variation from the point u^k to the point u^{k+1} which will ensure a growth moderated towards the optimal value s^*. We give the steps of the algorithm at each iteration k and the convergence result.

The Algorithm

■ step 0:

$k = 1, 0 < \theta \leq 1$ let $\varepsilon > 0$ the tolerance, and a given u_1.

■ step 1:

Resolution of the nonlinear problem $(NLP)_k$

$$(NLP)_k : \quad s\left(u^k\right) = \min \left\{ Q\,(x) \;: \left(u^k\right)^\top (Ax - b) \leq 0, x \in IR^n_+ \right\}$$

♦ If $(NLP)_k$ has a solution x^k and if $s\left(u^k\right) \geq s_{k-1}$

$$s_k = s\left(u^k\right) = Q(x^k)$$

♦ else consider any feasible solution x^k of $(NLP)_k$ and put

$$s_k = s_{k-1}$$

compute

$$g^k, \quad \gamma_k, \quad \beta_k$$

■ step 2:

Resolution of the linear problem $(PL)_k$

$$(PL)_k : \; r^*_k = \max \left\{ r : \; \sum_{i=1}^{n} u_i g^l_i \; - \gamma_k r \geq 0, \; l = 1, ..., k, \; \sum_{i=1}^{m} u_i = 1, \quad u \geq 0 \right\}$$

♦ if $r^*_k < \varepsilon$!then stop.

⧫ Else consider the solution $\left(r_k^*, \overline{u}^k\right)$ of $(PL)_k$, and compute the vector of the simplex $\triangle$

$$u^{k+1} = (1 - \beta_k)\,\overline{u}^k + \beta_k u^k$$

■ step 3:

$k = k + 1$. Go to step 1.

The convergence result is presented in the following proposition

Proposition 4.1. *The sequence of points $(s_k)_k$ generated by the algorithm will become stationary and take the value s^* from a certain rank, or $\lim_{k\to\infty} s_k = s^*$.*

Proof. By construction the sequence $(s_k)_k$ is nondecreasing and is majored by s^* where $s_k \le s^* \forall k$, thus either it becomes stationary starting from a certain rank, or it converges towards a limit. The nondecreasing of the sequence $(s_k)_k$ gives the following inclusion $\left\{x^l\right\} \subseteq L_{s_k}(Q)\ \ \forall l \le k$, this leads us to say that $G^{\oplus}(s_k) \subseteq U_k\ \forall k$.

If $r_k \le 0$ for a certain k then, $intU_k = \varnothing$ and consequently $intG^{\oplus}(s_k) = \varnothing$. But the proposition (3.3) implies that $s_k \ge s^* \forall k$, thus necessarily $s_k = s^*$. Let us show now that if s_k converges to a limit $\widetilde{s}$ then necessarily $\widetilde{s} = s^*$. To be done let us show initially that r_k tends inevitably to 0. Let us suppose that $r_k > 0\ \forall k$. At the iteration k, $u^l \notin intU_k$ for all $l < k$, since for all x^l we have $intU_k \subseteq \left\{u : u^T g^l > 0\right\}$ and $u_l^T g^k \le 0$, and hence the Euclidean distance between u_k and u_l , $\|u_k - u_l\| \ge r_k^*$. The sequence $\left(u^k\right)_k$ admits a value of adherence since it is contained in the simplex $\triangle$, the sequence $(r_k)_k$ is convergent towards a limit since it is nonincreasing and lowerbounded by 0, then $\forall \eta > 0\ \ \exists N \in \mathbb{N}$ such that for all $k > l \ge N$, we have $\|u_k - u_l\| < \eta$, from where $r_k < \eta$, this shows that $\lim_{k\to\infty} r_k = 0$.

Let us suppose now that $\lim_{k\to\infty} s_k = \widetilde{s} < s^*$, then $s_k \le \widetilde{s}$ for any k, we deduce that $G^{\oplus}(\widetilde{s}) \subseteq U_k\ \forall k$, but from the proposition (3.3) we deduced that $intG^{\oplus}(\widetilde{s}) \ne \varnothing$, and hence this set contains a point $\widehat{u}$ of distance $\widehat{r} > 0$ from the boundary of $intG^{\oplus}(\widetilde{s})$, and there will be $r_k^* \ge \widehat{r}\ \forall k$, which gives that $\lim_{k\to\infty} r_k^* \ge \widehat{r} > 0$, this contradiction show that $\lim_{k\to\infty} s_k = s^*$. □

Example. The algorithm given above can be applied to general quasiconvex programming, but for illustration we consider the counterexample of Martos given in

[11], where some (not all) primal convex quadratic algorithms fail to solve it.

$$\min\{Q_2(x) = 1/2\ x^T H_2 x : A_2(x) \leq b; x = (x_1, x_2) \geq 0\}$$

where

$$H_2 = \begin{bmatrix} -1 & -2 & -7 \\ -2 & 0 & 0 \\ -7 & 0 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix}, \quad b_2 = \begin{bmatrix} 16 \\ 12 \end{bmatrix}$$

the optimal solution of this problem is $(5, 0, 6)$, and -222.5 is the optimum value.

The convergence parameter θ is set equal to 0.25, for this example a relatively small value would not work better, we take for the starting point u^1 the center of the simplex $(1/2, 1/2)$, at each iteration the quantities α_k, β_k are as defined in (15) and (16). The implementation is proposed in the Matlab environment, at each iteration we use the two functions of Matlab *quadprog* and *linprog* for the problem $(NLP)_k$ and $(LP)_k$ respectively. The following table gives the evolution of the sequence $(s_k)_k$ for this example.

iteration k	u_k	s_k	x^k	g^k	r_k
1	(1/2,1/2)	-256.11	(7.84,0,4.12)	(3.79,3.79)	0.71
2	(0.62,0.37)	-232.34	(6.29,0,4.82)	(1.41,-2.35)	0.53
3	(0.63,0.36)	-224.53	(5.52,0,5.41)	(0.46,-1.18)	0.19
4	(0.65,0.34)	-222.55	(5.08,0,5.89)	(0.06,-0.21)	0.08
5	(0.66,0.33)	-222.50	(5.01,0,6.02)	(0.04,-5.98)	0.00

after five iterations we get $s_4 \simeq -222.5$, the corresponding surrogate multiplier $u_4 = (0.66, 0.33)$ and the solution $x^4 \simeq (5, 0, 6)$.

Conclusion. The computing experiences that we have done for several examples with general quasiconvex programming, shows that if we get at hand a good subroutine to solve at each iteration the problem $(NLP)_k$ with a single constraint this algorithm converges to the optimal value, it is the case in non linear quadratic programming, which is explains our choice. The question of how we can compute a global minimum of a nonlinear program is always very difficult, but in this context we get at least a tool that lead's to the optimal value.

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ON UNIVALENT FUNCTIONS DEFINED BY A GENERALIZED SĂLĂGEAN OPERATOR

ADRIANA CĂTAŞ

Abstract. The object of this paper is to obtain some inclusion relations regarding a new class, denoted by $S^m(\lambda,\alpha)$, using the generalized Sălăgean operator.

1. Introduction

We define the class of normalized analytic functions $\mathcal{A}_n$ as

$$\mathcal{A}_n = \{f \in \mathcal{H}(U):\ f(z) = z + a_{n+1}z^{n+1} + a_{n+2}z^{n+2} + \dots\}, \tag{1.1}$$

$n \in \mathbb{N}^* = \{1, 2, \dots\}$, with $\mathcal{A}_1 = \mathcal{A}$.

F.M. Al-Oboudi in [1] defined, for a function in $\mathcal{A}_n$, the following differential operator:

$$D^0 f(z) = f(z) \tag{1.2}$$

$$D^1_\lambda f(z) = D_\lambda f(z) = (1-\lambda)f(z) + \lambda z f'(z) \tag{1.3}$$

$$D^m_\lambda f(z) = D_\lambda(D^{m-1}_\lambda f(z)), \quad \lambda > 0. \tag{1.4}$$

When $\lambda = 1$, we get the Sălăgean operator [5].

If f and g are analytic functions in U, then we say that f is subordinate to g, written $f \prec g$, or $f(z) \prec g(z)$, if there is a function w analytic in U with $w(0) = 0$, $|w(z)| < 1$, for all $z \in U$ such that $f(z) = g[w(z)]$ for $z \in U$. If g is univalent, then $f \prec g$ if and only if $f(0) = g(0)$ and $f(U) \subset g(U)$.

To prove the main results we will need the following lemmas.

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Lemma 1.1. (Hallenbeck and Ruschweyh [2]) *Let h be convex in U with $h(0) = a$, $\gamma \neq 0$ and* Re $\gamma \geq 0$. *If $p \in \mathcal{H}[a, n]$ and*

$$p(z) + \frac{zp'(z)}{\gamma} \prec h(z)$$

then

$$p(z) \prec q(z) \prec h(z)$$

where

$$q(z) = \frac{\gamma}{nz^{\gamma/n}} \int_0^z h(t)t^{\frac{\gamma}{n}-1}dt.$$

The function q is convex and is the best (a, n)-dominant.

Lemma 1.2. (Miller and Mocanu [3]) *Let q be a convex function in U and let*

$$h(z) = q(z) + n\alpha zq'(z)$$

where $\alpha > 0$ and n is a positive integer. If $p \in \mathcal{H}(U)$ with

$$p(z) = q(0) + p_n z^n + \dots$$

and

$$p(z) + \alpha zp'(z) \prec h(z)$$

then

$$p(z) \prec q(z)$$

and this result is sharp.

2. **Main results**

Definition 2.1. Let $f \in \mathcal{A}$. We say that the function f is in the class $S^m(\lambda, \alpha)$, $\lambda > 0$, $\alpha \in [0, 1)$, $m \in \mathbb{N}$, if f satisfies the condition

$$\text{Re } [D_\lambda^m f(z)]' > \alpha, \quad z \in U. \tag{2.1}$$

Theorem 2.1. *If $\alpha \in [0,1)$ and $m \in \mathbb{N}$ then*

$$S^{m+1}(\lambda,\alpha) \subset S^m(\lambda,\delta) \tag{2.2}$$

where

$$\delta = \delta(\lambda,\alpha) = 2\alpha - 1 + 2(1-\alpha)\frac{1}{\lambda}\beta\left(\frac{1}{\lambda}\right) \tag{2.3}$$

β being the Beta function

$$\beta(x) = \int_0^1 \frac{t^{x-1}}{t+1}dt. \tag{2.4}$$

Proof. Let $f \in S^{m+1}(\lambda,\alpha)$. By using the properties of the operator D_λ^m, we have

$$D_\lambda^{m+1}f(z) = (1-\lambda)D_\lambda^m f(z) + \lambda z(D_\lambda^m f(z))' \tag{2.5}$$

If we denote by

$$p(z) = (D_\lambda^m f(z))' \tag{2.6}$$

where

$$p(z) = 1 + p_1 z^1 + p_2 z^2 + \ldots, \quad p(z) \in \mathcal{H}[1,1],$$

then after a short computation we get

$$(D_\lambda^{m+1}f(z))' = p(z) + \lambda z p'(z), \quad z \in U. \tag{2.7}$$

Since $f \in S^{m+1}(\lambda,\alpha)$, from Definition 2.1 we have

$$\text{Re }(D_\lambda^{m+1}f(z))' > \alpha, \quad z \in U.$$

Using (2.7) we get

$$\text{Re }(p(z) + \lambda z p'(z)) > \alpha$$

which is equivalent to

$$p(z) + \lambda z p'(z) \prec \frac{1+(2\alpha-1)z}{1+z} \equiv h(z). \tag{2.8}$$

From Lemma 1.1, with $\gamma = \frac{1}{\lambda}$, we have

$$p(z) \prec q(z) \prec h(z),$$

where

$$q(z) = \frac{1}{\lambda z^{1/\lambda}} \int_0^z \frac{1+(2\alpha-1)t}{1+t} t^{(1/\lambda)-1} dt.$$

The function q is convex and is the best $(1,1)$-dominant.

Since

$$(D_\lambda^m f(z))' \prec 2\alpha - 1 + \frac{2(1-\alpha)}{\lambda} \cdot \frac{1}{z^{1/\lambda}} \int_0^z \frac{t^{(1/\lambda)-1}}{t+1} dt$$

it results that

$$\text{Re } (D_\lambda^m f(z))' > q(1) = \delta \tag{2.9}$$

where

$$\delta = \delta(\lambda, \alpha) = 2\alpha - 1 + \frac{2(1-\alpha)}{\lambda} \beta\left(\frac{1}{\lambda}\right) \tag{2.10}$$

$$\beta\left(\frac{1}{\lambda}\right) = \int_0^1 \frac{t^{(1/\lambda)-1}}{t+1} dt. \tag{2.11}$$

From (2.9) we deduce that $f \in S^m(\lambda, \delta)$ and the proof of the theorem is complete. □

Theorem 2.2. *Let $q(z)$ be a convex function, $q(0) = 1$, and let h be a function such that*

$$h(z) = q(z) + \lambda z q'(z), \quad \lambda > 0. \tag{2.12}$$

If $f \in \mathcal{A}$ and verifies the differential subordination

$$(D_\lambda^{m+1} f(z))' \prec h(z) \tag{2.13}$$

then

$$(D_\lambda^m f(z))' \prec q(z) \tag{2.14}$$

and the result is sharp.

Proof. From (2.7) and (2.13) we obtain

$$p(z) + \lambda z p'(z) \prec q(z) + \lambda z q'(z) \equiv h(z) \tag{2.15}$$

then, by using Lemma 1.2 we get

$$p(z) \prec q(z)$$

or

$$(D_\lambda^m f(z))' \prec q(z), \quad z \in U$$

and this result is sharp. □

Theorem 2.3. *Let q be a convex function with $q(0) = 1$ and let h be a function of the form*

$$h(z) = q(z) + zq'z(z), \quad \lambda > 0,\ z \in U. \tag{2.16}$$

If $f \in \mathcal{A}$ verifies the differential subordination

$$(D_\lambda^m f(z))' \prec h(z), \quad z \in U \tag{2.17}$$

then

$$\frac{D_\lambda^m f(z)}{z} \prec q(z) \tag{2.18}$$

and this result is sharp.

Proof. If we let

$$p(z) = \frac{D_\lambda^m f(z)}{z}, \quad z \in U$$

then we obtain

$$(D_\lambda^m f(z))' = p(z) + zp'(z), \quad z \in U.$$

The subordination (2.17) becomes

$$p(z) + zp'(z) \prec q(z) + zq'(z)$$

and from Lemma 1.2 we have (2.18). The result is sharp. □

Remark 2.1. For $\lambda = 1$ these results were obtained in [4].

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QUASI-INTERPOLATORY AND INTERPOLATORY SPLINE OPERATORS: SOME APPLICATIONS

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Abstract. In this paper we consider quasi-interpolatory spline operators that satisfy some interpolation conditions. We give some applications of these operators constructing approximating integral operators and numerically solving Volterra integral equations of the second kind. We prove convergence results for the constructed methods and we perform numerical examples and comparisons with other spline methods.

1. Introduction

It is known that quasi-interpolatory operators play a main role in the approximation of data and functions, in the numerical solution of integrals or, more in general, of integral equations. Interpolatory operators also are very important in function approximation theory and there exists a wide literature on such two class of operators. In the last years, in [11], a method for constructing a quasi-interpolatory operator with interpolation properties, has been presented giving a general convergence theorem and in [7] a new class of operators, which are refinable, quasi-interpolatory and that satisfy some interpolation conditions has been studied. In this paper we consider a quasi-interpolatory spline operator that satisfies some interpolation conditions (qi-i operator) and we propose some its applications; for example, we construct a collocation method for solving a second kind linear Volterra integral equation

$$f(x) = g(x) + \int_0^x k(x,s)f(s)ds, \quad x \in [0,X] \tag{1.1}$$

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with $k(x,s) = k(x-s)$ and $k \in C(0,X] \cap L_1[0,X]$. The integral equation (1.1) has a unique solution $f \in C[0,X]$ if g is a continuous function in $[0,X]$, but the derivatives of this solution can be unbounded at $x = 0$; then graded grids used in the partition of $[0,X]$ reflect the possible singular behaviour of the derivative of the exact solution near $x = 0$. For example, in [2] and in [3], a collocation method using graded meshes and piecewise polynomials, for weakly singular Volterra integral equations, has been considered. In [6] and in [8] collocation methods based on spline functions have been studied for numerically solving (1.1). In [8] a method based on projector splines has been used in a suitable, first subinterval of $[0,X]$ combined with a Simpson's rule in the last part of $[0,X]$. In [6] nodal splines that are quasi-interpolatory and interpolatory (with the number of interpolation points that increases when the number knots increases) has been considered for numerically solving (1.1). The collocation method of this paper, based on qi-i spline operators of order $m \geq 2$, has several good properties as a low computational complexity and good performance when the solution of (1.1) is a continuous function. The results obtained are comparable with those obtained by using nodal splines or projector splines. Moreover, with the collocation method of this paper, we can opportunely choose the interpolation points of the qi-i spline, that can be different from the partition knots and we can obtain directly by a linear system, the value of f on such points without a successive evaluation of the approximation of f (as required, instead, if we use only projector splines). The approximate solution error obtained will converges to zero at the same rate as the quasi-interpolatory spline error.

This paper is organized as follows. In Section 2 we give definition and properties of qi-i spline operators on graded meshes and we give convergence results. In Section 3 we define an approximating integral operator based on qi-i spline operators and we analize its main properties and convergence. In Section 4 we describe a collocation method for Volterra integral equation of second kind based on the approximating integral operator of Section 3 and we give convergence results. In Section 5 we give some numerical results and comparisons with rules based on projector splines and on nodal splines.

2. Qi-i spline operators

We give the definition and the main properties of qi-i spline operators.

Let $s \geq 0$ be a given positive integer and consider the partition of $[a, b]$

$$\Delta_s := \{a = y_0 < y_1 < \ldots < y_s < y_{s+1} = b\} \tag{2.1}$$

in $s+1$ subintervals $[y_k, y_{k+1})$, with $h_k = y_{k+1} - y_k$, $k = 0, 1, ..., s$. We shall assume that $\max\limits_{0 \leq k \leq s} h_k \to 0$ as $s \to \infty$.

We say that the sequence of partitions $\{\Delta_s,\ s = 1, 2, ...\}$ is locally uniform ($l.u.$) if there exists a constant $R \geq 1$ such that

$$\frac{1}{R} \leq \frac{y_{i+1} - y_i}{y_{j+1} - y_j} \leq R,\ j = i \pm 1,\ \forall i.$$

We consider the sequence of partitions Δ_s obtained by using graded meshes (see for example [2]) of the form

$$y_i = \left(\frac{i}{s+1}\right)^r \cdot (b - a) + a,\ 0 \leq i \leq s+1,\ r \geq 1. \tag{2.2}$$

In [6] has been proved that the sequence $\{\Delta_s\}$ is $l.u.$. Let m be a given positive integer and $n = m + s$; we denote by Δ_s^e the extended partition of Δ_s defined as

$$\Delta_s^e := \{a = x_1 = ... = x_m < x_{m+1} < \ldots < x_{m+s} < x_{n+1} = ... = x_{n+m} = b\}$$

where $x_i = y_0$, $x_{n+i} = y_{s+1}$, $i = 1, ..., m$, $x_{m+j} = y_j$, $j = 1, ..., s$.

We denote by $I\!P_l$ the set of polynomials of degree $\leq l$. The space of polynomial splines of order m with simple knots $y_1, y_2, ..., y_s$ and $S_m(\Delta_s) \subset C^{m-2}[a, b]$ is defined by:

$$S_m(\Delta_s) := \left\{ \begin{array}{l} s : s(x) = s_k(x) \in I\!P_{m-1},\ x \in [y_k, y_{k+1}),\ k = 0, 1, ..., s; \\ D^j s_{k-1}(y_k) = D^j s_k(y_k),\ j = 0, 1, ..., m-2,\ k = 1, 2, ..., s. \end{array} \right\}. \tag{2.3}$$

The set of normalized B-splines of order m, B_{im} , $i = 1, 2, ..., n$, constitutes a basis for $S_m(\Delta_s)$ [10].

We define the following quasi-interpolatory and interpolatory operator applied to a function $f \in C[a, b]$ ([11], [7]):

$$T_n f := Q_n f + U f - U Q_n f \tag{2.4}$$

where Q_n is the quasi-interpolating operator defined as

$$Q_n f(x) := \sum_{i=1}^{n} (\lambda_i f) B_{im}(x) = \sum_{i=1}^{n} [\sum_{j=1}^{m} v_{ij} f(\tau_{ij})] B_{im}(x) \tag{2.5}$$

with $x \in [a, b]$,

$$v_{ij} = \sum_{r=j}^{m} \frac{\alpha_{ir}}{\prod\limits_{s=1, s\neq j}^{r} (\tau_{ij} - \tau_{is})}, \; r = 2, ..., m$$

and

$$\alpha_{ir} = \frac{(m-r)!}{(m-1)!} \sum \prod_{l=1}^{r-1} (x_{v_l} - \tau_{il})$$

where the sum is extended over all choices of distinct $v_1, ..., v_{r-1}$ from $i+1, ..., i+m-1$ and is set equal to 1 when $r = 1$; the τ_{ij} are m distinct points opportunely chosen in $[x_i, x_{i+m}]$, $i = 1, ..., n$ [4]. A possible distribution for $\{\tau_{ij}\}$ is the following

$$\tau_{ij} = x_{\rho_i} + j \frac{x_{\rho_i+1} - x_{\rho_i}}{k}, \; k = \begin{cases} m, & \text{if } \rho_i \neq n \\ m+1, & \text{if } \rho_i = n \end{cases}, \; i = 1, ..., n, \; j = 1, ..., m$$

with $[x_{\rho_i}, x_{\rho_i+1}] \subseteq [x_i, x_{i+m}]$, $m \leq \rho_i \leq n$.

The interpolating operator U is defined as:

$$Uf(x) := \sum_{k=1}^{l} c_k(f) \overline{B}_{km}(x) = \sum_{k=1}^{l} [\sum_{h=1}^{l} \overline{b}_{kh}^{-1} f(t_h)] \overline{B}_{km}(x) \tag{2.6}$$

where $\overline{B}_{km}(x)$, $k = 1, ..., l$ are normalized B-splines constituting a basis for the spline space $S_m(\Delta_{\overline{s}})$, $\overline{s} = l - m$; l is a fixed integer and t_k, $k = 1, ..., l$ are l distinct interpolation points with $t_k \in (\overline{x}_k, \overline{x}_{k+m})$ where $\overline{x}_k$, $\overline{x}_{k+m}$ belong to the extended partition $\Delta_{\overline{s}}^{e}$. The coefficients c_k have been obtained by imposing the interpolation conditions $Uf(t_h) = f(t_h)$, $h = 1, ..., l$ and $\overline{b}_{kh}^{-1}$, h, $k = 1, ..., l$ denote the coefficients of the inverse matrix $\overline{B}_{\underline{t}}^{-1}$ of $\overline{B}(\underline{t}) = \begin{bmatrix} \overline{B}_{1m}(t_1) & ... & \overline{B}_{lm}(t_1) \\ \vdots & & \vdots \\ \overline{B}_{1m}(t_l) & ... & \overline{B}_{lm}(t_l) \end{bmatrix}$.

Remark 2.1. *We observe that the inverse matrix $\overline{B}_{\underline{t}}^{-1}$ exists because (theorem 4.63 in* [10]*), choosing the distinct interpolation points t_k in $(\overline{x}_k, \overline{x}_{k+m})$, $k = 1, ..., l$, we obtain $\overline{B}(\underline{t})$ not singular.*

Then we can write

$$UQ_nf(x) := \sum_{k=1}^{l}\sum_{h=1}^{l}\overline{b}_{kh}^{-1}Q_nf(t_h)\overline{B}_{km}(x). \tag{2.7}$$

By using (2.5), (2.6) and (2.7), the operator (2.4) can be written in the form:

$$\begin{aligned} T_nf(x) \quad &:= \sum_{i=1}^{n}\sum_{j=1}^{m} v_{ij}f(\tau_{ij})B_{im}(x) \\ &+ \sum_{k=1}^{l}\sum_{h=1}^{l}\overline{b}_{kh}^{-1}\left[f(t_h) - Q_nf(t_h)\right]\overline{B}_{km}(x). \end{aligned} \tag{2.8}$$

The operator (2.8) is quasi-interpolatory and interpolatory on the knots t_k, $k = 1, ..., l$ [7]. In fact: $T_nf(t_k) := Q_nf(t_k)+Uf(t_k)-UQ_nf(t_k) = Q_nf(t_k)+f(t_k)-Q_nf(t_k) = f(t_k)$ and $T_np(x) := Q_np(x)+Up(x)-UQ_np(x) = p(x)+Up(x)-Up(x) = p(x)$, where $p(x)$ is a polynomial of degree less or equal to $m-1$.

We observe that we can use a vectorial notation for the operator (2.8) that will be very useful in the following Sections. We set the following column vectors:

$\underline{t}= [t_1, ..., t_l]^T \in R^l$,

$\underline{\tau}= [\tau_{11}, ..., \tau_{1m}, ..., \tau_{n1}, ..., \tau_{nm}]^T \in R^{n\cdot m}$ and

$\underline{\xi}= [\underline{\tau}; \underline{t}] \in R^{n\cdot m+l}$.

We will use the notation: $f(\underline{z}) = [f(z_1), ..., f(z_r)]^T$, where $\underline{z}= [z_1, ..., z_r]^T$ is a column vector with the elements belonging to $[a, b]$ and we will indicate $B_{im}(x)$ with $B_i(x)$, $\overline{B}_{im}(x)$ with $\overline{B}_i(x)$, where $x \in [a, b]$. If we denote with:

$$B(x) = \begin{bmatrix} B_1(x), & ..., & B_n(x) \end{bmatrix},\ \overline{B}(x) = \begin{bmatrix} \overline{B}_1(x), & ..., & \overline{B}_l(x) \end{bmatrix}, \tag{2.9}$$

$$B(\underline{t}) = \begin{bmatrix} B_1(t_1) & ... & B_n(t_1) \\ \vdots & & \vdots \\ B_1(t_l) & ... & B_n(t_l) \end{bmatrix},\ \overline{B}(\underline{t}) = \begin{bmatrix} \overline{B}_1(t_1) & ... & \overline{B}_l(t_1) \\ \vdots & & \vdots \\ \overline{B}_1(t_l) & ... & \overline{B}_l(t_l) \end{bmatrix}, \tag{2.10}$$

$$V = \begin{bmatrix} v_{11} & \dots & v_{1m} & 0 & \dots & 0 & \dots & 0 & \dots & 0 \\ 0 & \dots & 0 & v_{21} & \dots & v_{2m} & \dots & 0 & \dots & 0 \\ \vdots & & \vdots & \vdots & & \vdots & \ddots & \vdots & & \vdots \\ 0 & \dots & 0 & 0 & \dots & 0 & \dots & v_{n1} & \dots & v_{nm} \end{bmatrix} \tag{2.11}$$

then we can write

$$Q_n f(x) = B(x)Vf(\underline{\tau}), \tag{2.12}$$

$$Uf(x) = \overline{B}(x)\overline{B}_{\underline{t}}^{-1} f(\underline{t}), \tag{2.13}$$

$$UQ_n f(x) = \overline{B}(x)\overline{B}_{\underline{t}}^{-1} B(\underline{t})Vf(\underline{\tau}) \tag{2.14}$$

and

$$T_n f(x) = M_n(x) f(\underline{\xi}) \tag{2.15}$$

where, considering (2.4), (2.12), (2.13) and (2.14) $M_n(x)$ is the row vector:

$$M_n(x) = \left[\ \left[B(x) - \overline{B}(x)\overline{B}_{\underline{t}}^{-1} B(\underline{t})\right] V, \quad \overline{B}(x)\overline{B}_{\underline{t}}^{-1} \ \right] \in R^{n \cdot m + l}.$$

By (2.15), we can see that T_n is a linear operator. We recall that the norm of a bounded operator $F : C[0, X] \to C[0, X]$ can be defined as

$$\|F\| = \sup_{\|h\| \leq 1} \|Fh\| .$$

The following proposition holds:

Proposition 2.1. *The operator T_n in (2.15) is a bounded operator for all n in $[a, b]$ and $\forall f \in C[a, b]$.*

Proof. T_n is a linear operator and so it suffices to prove that $\forall f \in C[a, b]$ and $\forall n$, exists a constant α such that

$$\|T_n f\|_\infty \leq \alpha \|f\|_\infty$$

where $\|g\|_\infty = \max_{x \in [a,b]} |g(x)|$, $g \in C[a, b]$.

By (2.4) we can write

$$|T_n f| \leq |Q_n f| + |Uf| + |UQ_n f| ;$$

in [10] (Theorem 6.22) has been proved that $Q_n f$ is bounded; by (2.5), (2.6), (2.7) it easy also to get

$$|Q_n f(x)| \leq \|V\|_\infty \|f\|_\infty , \tag{2.16}$$

$$|Uf(x)| \leq \left\|\overline{B}_{\underline{t}}^{-1}\right\|_\infty \|f\|_\infty , \tag{2.17}$$

$$|UQ_n f| \leq \|V\|_\infty \left\|\overline{B}_{\underline{t}}^{-1}\right\|_\infty \|f\|_\infty \tag{2.18}$$

with $\|V\|_\infty = \max_{1\le i\le n} \sum_{j=1}^{m} |v_{ij}| \le mD$ for all n ([4]) and D independent from n, $\left\|\overline{B}_{\underline{t}}^{-1}\right\|_\infty = \max_{1\le k\le n} \sum_{h=1}^{l} \left|\overline{b}_{kh}^{-1}\right|$ bounded because $\overline{B}_{\underline{t}}^{-1}$ is independent from n. The thesis follows, by setting $\alpha = \|V\|_\infty + \left\|\overline{B}_{\underline{t}}^{-1}\right\|_\infty + \|V\|_\infty \left\|\overline{B}_{\underline{t}}^{-1}\right\|_\infty$. □

We can give, now, the following

Theorem 2.1. *Let $f \in C[a,b]$. For the qi-i spline operator (2.15), the following relation holds*

$$\|f - T_n f\|_\infty \le C_1 \|f - Q_n f\|_\infty \tag{2.19}$$

where $C_1 = 1 + \left\|\overline{B}_{\underline{t}}^{-1}\right\|_\infty$ *and*

$$\lim_{n\to\infty} \|f - T_n f\|_\infty = 0. \tag{2.20}$$

Proof. Considering that we can write $T_n f(x) = Q_n f(x) + U(f - Q_n f)(x)$ and (2.17) holds, the proof is similar to the proof of Theorem 4.1 in [7].

By Lemma 3.3 in [4], (2.20) follows. □

3. An approximating integral operator

Let $[a,b] \equiv [0,X]$ and K the following integral operator:

$$Kf(x) := \int_0^x k(x,s) f(s) ds, \; k \in C(0,X] \cap L_1[0,X]; \tag{3.1}$$

we consider the approximating operator KT_n

$$KT_n f(x) := \int_0^x k(x,s) T_n f(s) ds \tag{3.2}$$

that, by (2.4) and (2.8) we can write

$$\begin{aligned} KT_n f(x) &= KQ_n f(x) + KUf(x) - KUQ_n f(x) \\ &= \sum_{i=1}^{n}\sum_{j=1}^{m} v_{ij} f(\tau_{ij}) KB_i(x) \\ &+ \sum_{k=1}^{l}\sum_{h=1}^{l} \overline{b}_{kh}^{-1} \left[f(t_h) - Q_n f(t_h)\right] K\overline{B}_k(x) \end{aligned} \tag{3.3}$$

where

$$KB_i(x) = \int_0^x k(x,s) B_i(s) ds, \; i = 1, ..., n$$

and

$$K\overline{B}_k(x) = \int_0^x k(x,s)\overline{B}_k(s)ds, \ k = 1, ..., l.$$

By using the same vectorial notation used for the T_n operator, we can set:

$$\begin{aligned} KB(x) &= \begin{bmatrix} KB_1(x), & ..., & KB_n(x) \end{bmatrix} = \\ &= \begin{bmatrix} \int_0^x k(x,s)B_1(s)ds, & ..., & \int_0^x k(x,s)B_n(s)ds \end{bmatrix}, \end{aligned} \tag{3.4}$$

$$\begin{aligned} K\overline{B}(x) &= \begin{bmatrix} K\overline{B}_1(x), & ..., & K\overline{B}_l(x) \end{bmatrix} = \\ &= \begin{bmatrix} \int_0^x k(x,s)\overline{B}_1(s)ds, & ..., & \int_0^x k(x,s)\overline{B}_l(s)ds \end{bmatrix}; \end{aligned} \tag{3.5}$$

then

$$\begin{cases} KQ_nf(x) & = KB(x)Vf(\underline{\tau}) \\ \\ KUf(x) & = K\overline{B}(x)\overline{B}_{\underline{t}}^{-1}f(\underline{t}) \\ \\ KUQ_nf(x) & = K\overline{B}(x)\overline{B}_{\underline{t}}^{-1}B(\underline{t})Vf(\underline{\tau}) \\ \end{cases} \tag{3.6}$$

and

$$KT_nf(x) = A_n(x)f(\underline{\xi}) \tag{3.7}$$

where $A_n(x)$ is the row vector

$$A_n(x) = \begin{bmatrix} [KB(x) - K\overline{B}(x)\overline{B}_{\underline{t}}^{-1}B(\underline{t})]V, & K\overline{B}(x)\overline{B}_{\underline{t}}^{-1} \end{bmatrix} \in R^{n\cdot m+l}.$$

We observe that, by (3.7), the operator KT_n is a linear operator. Now we can define

$$\widetilde{k}(x,s) = \begin{cases} k(x,s) & \text{if } 0 \le s \le x, \\ \\ 0 & \text{if } s > x; \end{cases} \tag{3.8}$$

if $\widetilde{k}(x,s)$ satisfies:

$$\begin{aligned}
&1)\ \widetilde{k}(x,s) \text{ is Riemann integrable in the variable } s\in[0,X],\\
&2)\ \lim_{x'\to x}\int_0^X \left|\widetilde{k}(x',s)-\widetilde{k}(x,s)\right| ds = 0,\ x',\ x\in[0,X],\\
&3)\ \max_{x\in[0,X]}\int_0^X \left|\widetilde{k}(x,s)\right| ds < \infty
\end{aligned} \tag{3.9}$$

then we can say that $\int_0^X \left|\widetilde{k}(x,s)\right| ds = \int_0^x |k(x,s)|\, ds$, is a compact operator on $C[0,X]$.

Proposition 3.1. *Let KT_n be the operator (3.7) and the hypoteses (3.9) hold. Then KT_n is a bounded operator for all n, on $[0,X]$.*

Proof. KT_n is a linear operator and so it suffices to prove that $\forall f\in C[0,X]$ and $\forall n$, a constant β exists such that

$$\|KT_nf\|_\infty \le \beta\,\|f\|_\infty\,.$$

By (3.3), we can write

$$|KT_nf| \le |KQ_nf| + |KUf| + |KUQ_nf|\,.$$

$\max_{x\in[0,X]}\int_0^x |k(x,s)|\, ds \le L$ because (3.9) holds; recalling that $\forall\ x\in[0,X]$ and $\forall n$ we have $\sum_{i=1}^n B_i(x) = \sum_{i=1}^l \overline{B}_i(x) = 1$, we obtain

$$\begin{aligned}
|KQ_nf(x)| &= \left|\sum_{i=1}^n\sum_{j=1}^m v_{ij}f(\tau_{ij})KB_i(x)\right| \le \|f\|_\infty \sum_{i=1}^n |KB_i(x)| \sum_{j=1}^m |v_{ij}|\\
&\le \|f\|_\infty\|V\|_\infty \int_0^x |k(x,s)| \sum_{i=1}^n |B_i(s)|\, ds \le L\,\|V\|_\infty\|f\|_\infty\,,
\end{aligned}$$

$$\begin{aligned}
|KUf(x)| &\le \left|\sum_{k=1}^l\sum_{h=1}^l \overline{b}_{kh}^{-1} f(t_h)K\overline{B}_k(x)\right| \le \|f\|_\infty \sum_{k=1}^l \left|K\overline{B}_k(x)\right| \sum_{h=1}^l \left|\overline{b}_{kh}^{-1}\right|\\
&\le \|f\|_\infty \left\|\overline{B}_{\underline{t}}^{-1}\right\|_\infty \int_0^x |k(x,s)| \sum_{k=1}^l \left|\overline{B}_k(s)\right| ds \le L\left\|\overline{B}_{\underline{t}}^{-1}\right\|_\infty \|f\|_\infty
\end{aligned}$$

and

$$|KUQ_nf| \le L\,\|V\|_\infty \left\|\overline{B}_{\underline{t}}^{-1}\right\|_\infty \|f\|_\infty$$

with $\|V\|_\infty$ and $\left\|\overline{B}_{\underline{t}}^{-1}\right\|_\infty$ bounded (see proof of Proposition 2.1). The thesis follows, by setting $\beta = L\alpha = L(\|V\|_\infty + \left\|\overline{B}^{-1}(\underline{t})\right\|_\infty + \left\|\overline{B}^{-1}(\underline{t})\right\|_\infty \|V\|_\infty)$. □

Theorem 3.1. *Let $f \in C[0,X]$ and $k(x,s)$ such that (3.9) holds. Then*

$$\|K(f - T_n f)\|_\infty \leq C_2 \|f - T_n f\|_\infty, \ \forall\ n \tag{3.10}$$

where $C_2 = \max\limits_{x\in[0,X]} \int_0^x |k(x,s)|\, ds$ and

$$\lim_{n\to\infty} \|K(f - T_n f)\|_\infty = 0. \tag{3.11}$$

Proof. $\|K(f - T_n f)\|_\infty = \max\limits_{x\in[0,X]} \left|\int_0^x k(x,s)(f(s) - T_n f(s))ds\right|$
$\leq \|f - T_n f\|_\infty \max\limits_{x\in[0,X]} \int_0^x |k(x,s)|\, ds$
and by (3.9) we have $\max\limits_{x\in[0,X]} \int_0^x |k(x,s)|\, ds < \infty$.

By Theorem 2.1 we have that $\|f - T_n f\|_\infty \to 0$ as $n \to \infty$ and by (3.10), (3.11) follows. □

4. A collocation method and convergence results

Consider the equation (1.1); substituting there $T_n f$ for f in the integral, we obtain

$$f(x) = g(x) + \int_0^x k(x,s) T_n f(s) ds + r_n(x) \tag{4.1}$$

with

$$r_n(x) = \int_0^x k(x,s)\,(f(s) - T_n f(s))\,ds = K(f(x) - T_n f(x)) \tag{4.2}$$

the residual term obtained approximating f by $T_n f$ in the integral. If we do not consider the error term, the (4.1) becomes

$$f_n(x) = g(x) + K T_n f_n(x), \tag{4.3}$$

or equivalently

$$\begin{aligned} f_n(x) &= g(x) + \sum_{i=1}^{n}\sum_{j=1}^{m} v_{ij} f_n(\tau_{ij}) K B_i(x) \\ &+ \sum_{k=1}^{l}\sum_{h=1}^{l} \overline{b}_{kh}^{-1} \left[f_n(t_h) - Q_n f_n(t_h)\right] K \overline{B}_k(x) \end{aligned} \tag{4.4}$$

if we collocate the equation (4.3) in the vector $\underline{\xi}$ defined in Section 2, considering (3.7), we obtain the linear system

$$(Id - A_n(\underline{\xi}))f_n(\underline{\xi}) = g(\underline{\xi})$$

where Id is the identity matrix of order $nm + l$ and $A_n(\underline{\xi}) = [A_n(\xi_1), ..., A_n(\xi_{nm+l})] \in R^{(n \cdot m+l)\times(n \cdot m+l)}$.

When we have solved the linear system just written, the value $f_n(x)$, $x \in [0, X]$, can be recovered by (4.4). Substracting (4.3) from (4.1) we obtain

$$f(x) - f_n(x) = KT_n(f(x) - f_n(x)) + r_n(x),$$

from which, considering also (4.2)

$$(I - KT_n)\,(f(x) - f_n(x)) = K(f(x) - T_n f(x)). \tag{4.5}$$

We can prove now, the following proposition:

Proposition 4.1. *Let $I - KT_n$ the operator in (4.5) and $k(x, s)$ such that (3.9) holds. For all n sufficiently large, $n \geq n_0$ with n_0 an integer > 0, the operator $(I - KT_n)^{-1}$ exists and*

$$\sup_{n \geq n_0} \left\| (I - KT_n)^{-1} \right\| \leq L < \infty.$$

Proof. By Proposition 3.1, KT_n is a bounded operator. We observe that the operators K in (3.1) and KT_n in (3.2) can be written as

$$\widetilde{K}f(x) := \int_0^X \widetilde{k}(x, s)f(s)ds,$$

$$\widetilde{K}T_n f(x) := \int_0^X \widetilde{k}(x, s)T_n f(s)ds$$

with $\widetilde{k}(x, s)$ defined in (3.8). Then $I - KT_n = I - \widetilde{K}T_n$. Following the proof of the Theorem 1. in [6] we can write

$$I - \widetilde{K}T_n = (I - \widetilde{K})\left[I - (I - \widetilde{K})^{-1}(\widetilde{K}T_n - \widetilde{K})\right].$$

Considering that T_n and $\widetilde{K}T_n$ are bounded operators and taking in account (3.11), the proof is similar to that one in [6] (theorem 1). □

Theorem 4.1. *We consider the equation (1.1) . Let* $f \in C[0,X]$ *and* $k(x,s)$ *such that (3.9) holds. Then* f_n *in (4.3) exists and is unique* $\forall\ n \geq n_0$ *where* n_0 *is an integer* > 0*; moreover* f_n *converges uniformly to the solution* f *that is*

$$\lim_{n\to\infty} \|f_n - f\|_\infty = 0$$

and there results

$$\|f_n - f\|_\infty \leq C_3 \|f - T_n f\|_\infty$$

where $C_3 = C_2 \sup\limits_{n\geq n_0} \left\|(I - KT_n)^{-1}\right\|$ *and* $C_2 = \max\limits_{x\in[0,X]} \int_0^x |k(x,s)|\, ds.$

Proof. By using (4.5), Proposition 4.1 and Theorem 3.1 the thesis follows. □

5. **Numerical applications and comparisons**

We consider now some numerical results obtained by applying our collocation method to (1.1). The results have been compared with those obtained by a collocation method based on projector splines and with those proposed in [6].

We have considered the following equations of type (1.1):

$$\begin{cases} f(x) = & \sqrt{x} + \frac{1}{2}\pi x - \int_0^x (x-s)^{-\frac{1}{2}} f(s) ds, \ x \in [0,1], \\ \\ f(x) = & \sqrt{x} \qquad \text{is the solution.} \end{cases} \tag{5.1}$$

and

$$\begin{cases} f(x) = & \frac{1}{\sqrt{1+x}} + \frac{\pi}{8} - \frac{1}{4}\arcsin\frac{1-x}{1+x} - \frac{1}{4}\int_0^x (x-s)^{-\frac{1}{2}} f(s) ds, \ x \in [0,1], \\ \\ f(x) = & \frac{1}{\sqrt{1+x}} \qquad \text{is the solution.} \end{cases} \tag{5.2}$$

From Table 1 to Table 4 we have indicated with $\mathbf{e}_N$, $\mathbf{e}_P$ and $\mathbf{e}_{QI-I}$, the absolute error evaluated in $\mathbf{x}$, respectively obtained by the method in [6], by the collocation method that use the projector splines and by our method that in these examples takes the value $l = 10$ for the interpolatory spline. The methods use graded partitions of the form (2.2) with $r = 1$ and $r = 2$.

In Tables 1, 2 and 3 the values of x, of n and of m are chosen as in the examples in [6] in order to compare the results.

The results in Table 4 obtained with $r = 2$ for (5.2) and not shown in [6], confirm the theorical convergence results of our method.

Table 1

f(x)=$\sqrt{x}$, g(x)=$\sqrt{x}+\frac{1}{2}\pi$x, k=λ(x-s)$^{-1/2}$, λ=-1

r = 1, m = 3									
	n = 11			**n = 21**			**n = 41**		
x	$\mathbf{e}_N$	$\mathbf{e}_P$	$\mathbf{e}_{QI-I}$	$\mathbf{e}_N$	$\mathbf{e}_P$	$\mathbf{e}_{QI-I}$	$\mathbf{e}_N$	$\mathbf{e}_P$	$\mathbf{e}_{QI-I}$
0	-	0	0	-	0	0	-	0	0
0.01	5.8e-2	6.0e-3	5.2e-4	4.3e-2	2.2e-3	3.1e-4	2.4e-2	5.9e-4	2.6e-5
0.51	9.9e-4	1.6e-4	7.1e-5	3.3e-4	5.0e-5	6.3e-6	1.1e-4	1.7e-5	8.7e-6
1	4.2e-4	7.5e-5	3.5e-5	1.4e-4	2.3e-5	3.6e-6	4.9e-5	7.7e-6	3.7e-6

Table 2

f(x)=$\sqrt{x}$, g(x)=$\sqrt{x}+\frac{1}{2}\pi$x, k=λ(x-s)$^{-1/2}$, λ=-1

r = 2, m = 3									
	n = 11			**n = 21**			**n = 41**		
x	$\mathbf{e}_N$	$\mathbf{e}_P$	$\mathbf{e}_{QI-I}$	$\mathbf{e}_N$	$\mathbf{e}_P$	$\mathbf{e}_{QI-I}$	$\mathbf{e}_N$	$\mathbf{e}_P$	$\mathbf{e}_{QI-I}$
0	-	0	**0**	-	0	0	-	0	0
0.01	1.7e-3	7.1e-5	1.1e-4	2.8e-4	9.9e-5	1.4e-5	3.0e-5	2.3e-5	6.4e-6
0.51	2.3e-5	1.7e-4	7.7e-6	9.4e-6	2.1e-5	4.6e-7	2.2e-6	2.8e-6	1.8e-8
1	1.2e-4	6.0e-5	9.2e-6	1.2e-5	8.4e-6	6.1e-7	2.4e-6	1.2e-6	2.1e-7

Table 3

f(x)=$\frac{1}{\sqrt{1+x}}$, g(x)=$\frac{1}{\sqrt{1+x}}+\frac{\pi}{8}$-$\frac{1}{4}$ arcsin($\frac{1-x}{1+x}$),k=λ(x-s)$^{-1/2}$, λ=-$\frac{1}{4}$

r = 1, m = 4									
	n = 11			**n = 21**			**n = 41**		
x	$\mathbf{e}_N$	$\mathbf{e}_P$	$\mathbf{e}_{QI-I}$	$\mathbf{e}_N$	$\mathbf{e}_P$	$\mathbf{e}_{QI-I}$	$\mathbf{e}_N$	$\mathbf{e}_P$	$\mathbf{e}_{QI-I}$
0	-	0	0	-	0	0	-	0	0
0.1	1.0e-6	3.9e-7	6.4e-8	3.7e-8	1.5e-8	1.3e-9	1.5e-9	8.3e-10	9.7e-10
0.4	3.0e-7	4.3e-8	2.1e-9	1.4e-8	1.6e-8	5.1e-9	4.7e-10	1.2e-9	4.8e-10
1	2.1e-7	1.0e-7	1.1e-8	8.0e-9	9.2e-9	8.0e-10	9.5e-10	6.5e-10	1.4e-10

Table 4

f(x)=$\frac{1}{\sqrt{1+x}}$, g(x)=$\frac{1}{\sqrt{1+x}}+\frac{\pi}{8}$-$\frac{1}{4}$ arcsin($\frac{1-x}{1+x}$),k=λ(x-s)$^{-1/2}$, λ=-$\frac{1}{4}$

r = 2, m = 4						
	n = 11		**n = 21**		**n = 41**	
x	$\mathbf{e}_P$	$\mathbf{e}_{QI-I}$	$\mathbf{e}_P$	$\mathbf{e}_{QI-I}$	$\mathbf{e}_P$	$\mathbf{e}_{QI-I}$
0	0	0	0	0	0	0
0.01	1.9e-9	2.7e-10	2.7e-12	4.7e-12	9.8e-13	3.2e-13
0.1	1.0e-7	2.5e-8	4.0e-9	1.3e-9	1.7e-10	1.4e-11
0.4	9.5e-7	7.4e-8	3.8e-8	5.1e-9	1.8e-9	2.2e-10
0.51	1.4e-6	2.3e-7	5.0e-8	1.0e-8	2.3e-9	3.9e-10
1	1.3e-6	7.7e-8	5.8e-8	8.0e-10	3.2e-9	1.6e-10

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ON RANDOM FIXED POINTS IN RANDOM CONVEX STRUCTURES

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Abstract. In this paper, we present some random fixed point theorems in random convex structures.

1. **Introduction and preliminaries**

Random fixed point theory has received much attention for the last two decades, since the publication of the paper by Bharucha-Reid [2]. Also random best approximation attracted authors after the papers by Sehgal and Singh [15], Papageorgiou [13], Lin [11], and Beg et al. [1].

On the other hand, in the past years, because of practical necessities, the attempts of generalizing the notion of convexity introduced by J. Von Neumann and O. Morgenstern [12], M. Stone [16] were brought up-to-date by S.P. Gudder [5]. Consequently, Gudder (1979) introduced the notion of convex structure and of F-convex set with applications in quantum mechanics, colour vision and petroleum engineering. Subsequently, fixed point theorems for nonexpansive mappings using the convex structures introduced by Gudder was proved by Petrusel [14] and later by Ganguly and Jadhav [6] for approximation theorems.

Again, away from this, Takahashi [17] also introduced a notion of convexity in metric spaces and presented fixed point theorems for nonexpansive mappings. This motivated Guay et al. [7] to discuss the results on convex metric spaces. These

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works alongwith those on random approximations motivated Beg et al. [3] to present random fixed point theorems and related results in random convex metric spaces.

It is a need for further research to study a relationship between convex structures introduced by Takahashi [17] and Gudder [5] respectively. In this vein, we are presenting random fixed point theorems in random convex structures, following Gudder [5], Petrusel [14], Beg and Shahzad [3].

Before we present our theorems, we will introduce some basic preliminaries.

Let (Ω, Σ) be a measurable space, (X, d) a metric space, 2^X the family of all subsets of X, K(X) family of all nonempty compact subsets of X and CB(X) family of all nonempty closed bounded subsets of X.

A mapping $T : \Omega \to 2^X$ is called measurable if for any open subset C of X,

$$T^{-1}(C) = \{\omega \in \Omega : T(\omega) \cap C \neq \phi\} \in \Sigma.$$

A mapping $\xi : \Omega \to$X is said to be a measurable selector of T if ξ is measurable and for any $\omega \in \Omega,\ \xi(\omega) \in T(\omega)$.

A mapping $f : \Omega \times X \to$ X is called a random operator if for any $x \in X, f(.,x)$ is measurable.

A measurable mapping $\xi : \Omega \to$ X is called a random fixed point of a random multivalued (single valued) operator $T : \Omega \times X \to CB(X)(f : \Omega \times X \to X)$ if for every $\omega \in \Omega,\ \xi(\omega) \in T(\omega,\ \xi(\omega))\ (\xi(\omega) = f(\omega,\ \xi(\omega)))$.

A random operator $T : \Omega \times X \to CB(X)$ is called Lipschitzian if

$$H(T(\omega,\ x),\ T(\omega,\ y)) \leq L(\omega)\, d(x,\ y)$$

for any $x, y \in X$and $\omega \in \Omega$, where $L : \Omega \to [0, \infty)$ is a measurable map and H is the Pompeiu-Hausdorff metric on CB(X), induced by the metric d. When $L(\omega) < 1$, $(L(\omega) = 1)$ for each $\omega \in \Omega$, T is called contraction (nonexpansive).

We present, for the convenience of readers, the following definitions which also appear in Petrusel [14].

Definition 1.1. *Let X be a set and* $F : [0, 1] \times X \times X \to X$ *a mapping. Then the pair (X, F) forms a convex prestructure.*

Definition 1.2. *Let (X, F) be a convex prestructure. If F satisfies the following conditions:*

1. $F(\lambda, x, F(\mu, y, z)) = F(\lambda+(1-\lambda)\mu, F(\lambda(\lambda+(1-\lambda)\mu)^{-1}, x, y), z)$ *for every* $\lambda, \mu \in [0,1]$ *with* $\lambda + (1-\lambda)\mu \neq 0$ *and* $x, y, z \in X$.
2. $F(\lambda, x, x) = x$ *for any* $x \in X$ *and* $\lambda \in [0,1]$, *then (X, F) forms a semi-convex structure.*

   If (X, F) is a semi-convex structure, then $F(1, x, y) = x$ *for any* $x, y \in X$.

Definition 1.3. *A semi-convex structure (X, F) is said to form a convex structure if F also satisfies the conditions:*

1. $F(\lambda, x, y) = F(1-\lambda, y, x)$ *for every* $\lambda \in [0,1]$, $x, y \in X$
2. *If* $F(\lambda, x, y) = F(\lambda, x, z)$ *for some* $\lambda \neq 0$, $x \in X$, *then* $y = z$.

Definition 1.4. *Let (X, F) be a semi-convex structure. A subset Y of X is called F - semi-starshaped if there exists a* $p \in Y$, *so that for any* $x \in Y$ *and*

$$\lambda \in [0,1],\ F(\lambda, x, p) \in Y.$$

Definition 1.5. *Let (X,F) be a convex structure. A subset Y of X is called:*

1. *F - starshaped if there exists a* $p \in Y$, *so that for any* $x \in Y$ *and*

$$\lambda \in [0,1],\ F(\lambda, x, p) \in Y.$$

2. *F - convex if for any* $u, v \in Y$ *and* $\lambda \in [0,1]$, *we have* $F(\lambda, u, v) \in Y$.

For $F(\lambda, u, v) = \lambda u + (1-\lambda)v$, we obtain the known notions of starshaped and convexity from linear spaces.

Petruşel [14] noted with an example that a set can be a F - semi convex structure without being a convex structure. So, it follows that the results on fixed point theory and on best approximation theory obtained for semi-convex and semi-starshaped structures will be more general than those on F - convex structure.

Definition 1.6 (Random Semi-Convex Structure)**.** *Let* $F : \Omega \times X \times X \times [0,1] \to X$ *be a mapping having the following properties:*

1. *For each* $\omega \in \Omega$, $F(\omega, ., ., .)$ *is a semi-convex structure on X,*

2. *For each* $x, y \in X,\ \lambda \in [0,1],\ F(., x, y, \lambda)$ *is measurable.*

The mapping F is called a random semi-convex structure on X.

Example 1 [5]. The mapping $F : [0,1] \times R_+^* \times R_+^* \to R_+^*$ given by

$$F(\lambda, u, v) = u^\lambda . v^{1-\lambda}$$

together with the set of strict positive real numbers form a convex structure.

Example 2 [14]. The mapping $F : [0,1] \times R \times R \to R$ given by

$$F(\lambda, u, v) = [\lambda u^{2k} + (1-\lambda) v^{2k}]^{1/2k}, k \in N^*$$

together with the set of real numbers form a semi-convex structure without being a convex structure.

2. **Main results**

Theorem 2.1. *Let X be a separable random Banach space with semi-convex structure F, where the mapping* $F : \Omega \times X \times X \times [0,1] \to X$ *satisfies the following conditions:*

1. *F is* ϕ *- contractive relative to the second argument, i.e., there exists a mapping* $\phi : [0,1[\to [0,1[$ *so that:*

$$||F(\omega, x, p, \lambda) - F(\omega, y, p, \lambda)|| \leq \phi(\lambda).||x - y||,$$

for any $x,\ y,\ p \in X$ *and* $\lambda \in [0,1[$ *and* $\omega \in \Omega$.

2. *F is continuous relative to the first argument.*

Let Y be a compact and F - semi-starshaped subset of X and the mapping $T : \Omega \times Y \to Y$ *be nonexpansive random operator. Then T has a random fixed point.*

Proof. Choose $p \in Y$ so that for any $u \in Y$ and $\lambda \in [0,1[$, we have $F(\omega, u, p, \lambda) \in Y$ for each $\omega \in \Omega$. Let $\{K_n\}$ be a sequence of measurable mappings $K_n : \Omega \to (0,1)$ and $K_n(\omega) \to 1$ as $n \to \infty$.

Define the random operator $T_n : \Omega \times Y \to Y$ by

$$T_n(\omega, x) = F(\omega, T(\omega, x), p, K_n(\omega))$$

T_n is, because of F - semi-starshaped of Y, well defined. The operator T_n is a contraction Indeed

$$||T_n(\omega, x_1) - T_n(\omega, x_2)|| = ||F(\omega, T(\omega, x_1), p, K_n(\omega)) - F(\omega, T(\omega, x_2), p, K_n(\omega))||$$

$$\leq \phi(K_n(\omega))||T(\omega, x_1) - T(\omega, x_2)||$$

for all $x, y \in Y$ and $\omega \in \Omega$. By Hans [8], T_n has a unique random fixed point ξ_n.

For each n, define $G_n : \Omega \to K(X)$ by $G_n(w) = Cl\{\xi_i(\omega) : i \geq n\}$.

Define $G : \Omega \to K(X)$ by $G(\omega) = \bigcap_{n=1}^{\infty} G_n(\omega)$. Since G is measurable (see Himmelberg [9], Theorem 4.1), by Kuratowski and Ryll-Nardzewski theorem in [10] we have that G has a measurable selector ξ. Because Y is compact, $\{\xi_n(\omega)\}$ has a subsequence $\{\xi_{n_j}(\omega)\}$ converging to $\xi(\omega)$. By the continuity of T and F, $T(\omega, \xi_{n_j}(\omega))$ converges to $T(\omega, \xi(\omega))$. Thus, $T(\omega, \xi(\omega)) = \xi(\omega)$ for each $\omega \in \Omega$.

Next we have the following:

Theorem 2.2. *Let X be a separable random Banach space with a semi-convex structure F,where the mapping $F : \Omega \times X \times X \times [0, 1] \to X$ satisfies the conditions:*

1. *F is ϕ - contractive relative to the second argument.*
2. *F is continuous relative to the first argument.*

Let Y be a weakly compact and F - semi-starshaped subset of X and the mapping $T : \Omega \times Y \to Y$ be nonexpansive and weakly continuous mapping. In these conditions the mapping T has a random fixed point.

Proof. As in Theorem 2.1, define $\{K_n\}$ and the random operator T_n. As before, each Tn is a contraction mapping on Y. Since the weak topology of X is Hausdorff and Y is weakly compact,we have that Y is weakly closed and therefore, strongly closed (See Dotson, Theorem 2 [4]). Hence Y is a complete metric space (with the norm topology of the Banach space X). By Hans [8], T_n has a unique random fixed point $\xi_n \in Y$. By the Eberlein-Smulian [4]theorem, Y is weakly sequentially compact. Thus there is a subsequence $\{\xi_n(\omega)\}$ such that $\xi_{n_j}(\omega) \xrightarrow{\omega} \xi(\omega) \in Y$ (denotes weak convergence). Since T is weakly continuous and F - continuous, we have

$$T(\omega, \xi_{n_j}(\omega)) \xrightarrow{\omega} T(\omega, \xi(\omega))$$

Thus, $T(\omega,\ \xi(\omega)) = \xi(\omega)$. for each $\omega \in \Omega$.

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MONOTONE INTERPOLANT BUILT WITH SLOPES OBTAINED BY LINEAR COMBINATION

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Abstract. Slopes needed to obtain a monotone piecewise cubic Hermite interpolant are constructed. These slopes are obtained local as linear combination of the slopes of the line segments joining the data.

The most used methods to construct a monotone interpolant to monotone data is to insert new points between two adjacent knots, respectively to give the slopes needed to build the piecewise interpolant. The paper of Fritsch-Carlson [3] refers to necessary and sufficient condition to obtain a monotone cubic interpolant. There is also discussed a nonlocal algorithm to built the adequate slopes. We use the domain given there and we propose a local method to compute the slopes necessary to built a monotone piecewise cubic interpolant.

Let $\pi : x_1 < x_2 < ... < x_n$ be a partition of the interval $I = [x_1, x_n]$. Let $\{f_i : i = 1, ..., n\}$ be a given set of monotone data values at the partition points (knots): $f_i \leq f_{i+1}$ or $f_i \geq f_{i+1}$, $i = 1, ..., n-1$. The goal is to construct a monotone piecewise cubic function $p \in C^1(I)$ that interpolate the given data. In each subinterval $[x_i, x_{i+1}]$ the function p is the cubic Hermite interpolant that interpolates the points $(x_i, f_i), (x_{i+1}, f_{i+1})$ and with the endslopes d_i, d_{i+1} which will be determined later. Let $\Delta_i = (f_{i+1} - f_i)/h_i$ be the slope of the line segment joining the data $(x_i, f_i), (x_{i+1}, f_{i+1})$ where $h_i = x_{i+1} - x_i$. Let $\alpha = \frac{d_i}{\Delta_i}, \beta = \frac{d_{i+1}}{\Delta_i}$ be the ratios of the endpoint derivates to the slope of the secant line.

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In [3] it was proved that the piecewise cubic interpolant is monotone on each $[x_i, x_{i+1}]$ if and only if:

$$(\alpha, \beta) \in \mathcal{M} \tag{1}$$

where the monotonicity region $\mathcal{M}$ is depicted in Figure 1.

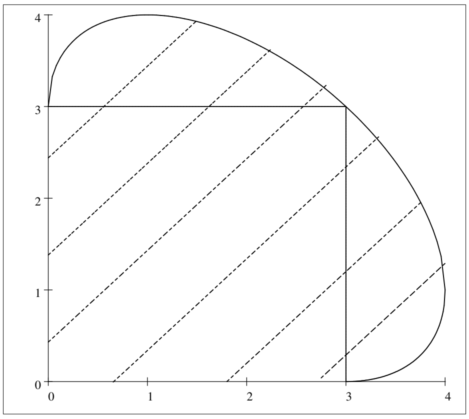

FIGURE 1. The region $\mathcal{M}$ (dashed) with the square $\mathcal{S} = [0,3] \times [0,3]$ inside

As domain we use a subregion of $\mathcal{M}$ bounded by the four lines $\alpha = 0, 3$ and $\beta = 0, 3$:

$$\mathcal{S} = [0,3] \times [0,3] .$$

We build the slopes d_i as a linear combination of the adjacent Δ_{i-1}, Δ_i :

$$d_i = (1 - \lambda_i)\, \Delta_{i-1} + \lambda_i \Delta_i,\; i = 2, ..., n-1. \tag{2}$$

Such a linear combination was also proposed by Akima in [1] with

$$\lambda_i = \frac{|\Delta_{i-1} - \Delta_{i-2}|}{|\Delta_{i+1} - \Delta_i| + |\Delta_{i-1} - \Delta_{i-2}|},\; i = 3, ..., n-2$$

but this method fails to preserve everywhere the monotonicity. Another local method proposed in [4] use the harmonic mean of the Δ_{i-1}, Δ_i.

We search the admissible values of the parameter λ_i according to relation (1), such that:

$$\left(\frac{d_i}{\Delta_{i-1}}, \frac{d_i}{\Delta_i}\right) \in [0, c] \times [0, c] \tag{3}$$

with $c \in [0, 3]$. The value $c = 0$, discussed also in [6], produce a slightly flat interpolant.

The condition (3) is equivalent with the following two inequalities:

$$0 \leq \frac{(1-\lambda_i)\,\Delta_{i-1} + \lambda_i \Delta_i}{\Delta_{i-1}} \leq c \tag{4}$$

$$0 \leq \frac{(1-\lambda_i)\,\Delta_{i-1} + \lambda_i \Delta_i}{\Delta_i} \leq c. \tag{5}$$

From (4) and (5) we obtain:

$$-\Delta_{i-1} \leq \lambda_i\,(\Delta_i - \Delta_{i-1}) \leq (c-1)\,\Delta_{i-1} \tag{6}$$

$$-\Delta_{i-1} \leq \lambda_i\,(\Delta_i - \Delta_{i-1}) \leq c\Delta_i - \Delta_{i-1} \tag{7}$$

If $\Delta_i - \Delta_{i-1} \neq 0$ the admissible interval for λ_i becomes:

$$-\frac{\Delta_{i-1}}{\Delta_i - \Delta_{i-1}} \leq \lambda_i \leq \frac{(c-1)\,\Delta_{i-1}}{\Delta_i - \Delta_{i-1}}, \text{ if } \Delta_i - \Delta_{i-1} > 0, \tag{8}$$

$$\frac{c\Delta_i - \Delta_{i-1}}{\Delta_i - \Delta_{i-1}} \leq \lambda_i \leq -\frac{\Delta_{i-1}}{\Delta_i - \Delta_{i-1}}, \text{ if } \Delta_i - \Delta_{i-1} < 0. \tag{9}$$

If $\Delta_i - \Delta_{i-1} = 0$, then λ_i have no influence on $d_i : d_i = \Delta_i$.

For $\lambda_i = -\frac{\Delta_{i-1}}{\Delta_i - \Delta_{i-1}}$ the slope $d_i = 0$ and, although this value is admissible, the interpolant becomes flat. It seems reasonable to impose that the slope $d_i \geq \min\{\Delta_{i-1}, \Delta_i\}$. That's mean:

$$0 \leq \lambda_i, \text{ if } \Delta_i - \Delta_{i-1} > 0,$$

$$\lambda_i \leq 1, \text{ if } \Delta_i - \Delta_{i-1} < 0.$$

So, we restrict the relations (8) and (9) to:

$$0 \leq \lambda_i \leq \frac{(c-1)\,\Delta_{i-1}}{\Delta_i - \Delta_{i-1}}, \text{ if } \Delta_i - \Delta_{i-1} > 0, \tag{10}$$

$$\frac{c\Delta_i - \Delta_{i-1}}{\Delta_i - \Delta_{i-1}} \leq \lambda_i \leq 1, \text{ if } \Delta_i - \Delta_{i-1} < 0. \tag{11}$$

The inequalities (10),(11) are consistent if $0 \leq \frac{(c-1)\Delta_{i-1}}{\Delta_i - \Delta_{i-1}}$ and $\frac{c\Delta_i - \Delta_{i-1}}{\Delta_i - \Delta_{i-1}} \leq 1$, which are equivalent with $c \geq 1$. So we impose:

$$c \in [1, 3] .$$

To fix the value of λ_i in the admissible interval given in (10),(11) we use a convex combination between the ends of these intervals:

$$\lambda_i = \begin{cases} (1-w_i)\,0 + w_i \frac{(c-1)\Delta_{i-1}}{\Delta_i - \Delta_{i-1}}, & \text{if } \Delta_i - \Delta_{i-1} > 0, \\ (1-v_i) + v_i \frac{c\Delta_i - \Delta_{i-1}}{\Delta_i - \Delta_{i-1}}, & \text{if } \Delta_i - \Delta_{i-1} < 0 \end{cases}$$

equivalent with

$$\lambda_i = \begin{cases} \frac{\Delta_{i-1}}{\Delta_i - \Delta_{i-1}} w_i \,(c-1) \;, & \text{if } \Delta_i - \Delta_{i-1} > 0, \\ \frac{1}{\Delta_i - \Delta_{i-1}} \left((1 + (c-1)\, v_i)\, \Delta_i - \Delta_{i-1} \right), & \text{if } \Delta_i - \Delta_{i-1} < 0. \end{cases}$$

Then from (2) follows for the slopes:

$$d_i = \begin{cases} (1 + (c-1)\, w_i)\, \Delta_{i-1}, & \text{if } \Delta_i - \Delta_{i-1} \geq 0, \\ (1 + (c-1)\, v_i)\, \Delta_i \;, & \text{if } \Delta_i - \Delta_{i-1} < 0, \end{cases} \tag{12}$$

We would like that the value d_i depends not only on the slope of line segment but also on the relative spacing of x_i and f_i-values. For this reason we use the length of the line segments (in $||\cdot||_1$ norm)

$$l_i = |x_{i+1} - x_i| + |f_{i+1} - f_i|$$

and we choose the weights w_i, v_i as follow:

$$w_i \;=\; \left(1 - \frac{\Delta_{i-1}}{\Delta_i}\right) \frac{1}{1 + \frac{l_{i-1}}{l_i}} \in [0, 1] , \tag{13}$$

$$v_i \;=\; \left(1 - \frac{\Delta_i}{\Delta_{i-1}}\right) \frac{1}{1 + \frac{l_i}{l_{i-1}}} \in [0, 1] . \tag{14}$$

The proposed values are based on the following idea:

- if Δ_i is close to Δ_{i-1} then naturally d_i must be also close to this value; the first therm in (13),(14) care about this because $\left(1 - \frac{\Delta_{i-1}}{\Delta_i}\right) \simeq 0$ ($\left(1 - \frac{\Delta_i}{\Delta_{i-1}}\right) \simeq 0$) so $d_i \simeq \Delta_{i-1} \simeq \Delta_i$.

- if Δ_i is not close to Δ_{i-1} ($\Delta_i \gg \Delta_{i-1}$, or $\Delta_i \ll \Delta_{i-1}$) then the slope d_i must be close to the value Δ_{i-1} if $l_{i-1} > l_i$, respectively close to Δ_i if $l_{i-1} < l_i$. The second therm in (13),(14) have the function to meet this requirement.

The slopes at end points are computed using a formula for numerical differentiation (the three-point formula), but inside of the admissible values. This values coresponds to the slopes of the parabola built on three consecutive points.

The rate of convergence of the derivative is in general $O(h)$, but for $c = 2$ and uniformly spaced data, the rate becomes $O(h^2)$.

Theorem 1. *Let* $(x_i)_{i=1}^n$ *a uniformly spaced data* $x_{i+1} - x_i = h,\ i = 1, ..., n-1,$ *and let* $f \in C^3[a,b]$ *be a monotone incresing function with:*

$$f_i = f(x_i).$$

Then for $c = 2$ *the values (12) gives* $O(h^2)$ *approximation to* $f'(x_i)$ *:*

$$f'(x_i) - d_i = O(h^2).$$

Proof. If $\Delta_i - \Delta_{i-1} \geq 0$, then $d_i = (1 + w_i)\Delta_{i-1}$, where $\Delta_{i-1} = \frac{f_i - f_{i-1}}{x_i - x_{i-1}}$, so using a Taylor formula we get:

$$f_{i-1} = f(x_i - h) = f(x_i) - hf'(x_i) + \frac{h^2 f''(x_i)}{2} - \frac{h^3 f'''(\xi_i)}{6},\ \xi_i \in (x_{i-1}, x_i)$$

consequently

$$d_i = (1 + w_i)\frac{1}{h}\left(hf'(x_i) - \frac{h^2 f''(x_i)}{2} + \frac{h^3 f'''(\xi_i)}{6}\right).$$

To compute w_i we use also the expansion:

$$f_{i+1} = f(x_i + h) = f(x_i) + hf'(x_i) + \frac{h^2 f''(x_i)}{2} + \frac{h^3 f'''(\theta_i)}{6},\ \theta_i \in (x_i, x_{i+1}).$$

So we obtain for the difference:

$$f'(x_i) - d_i = \frac{E}{3\left(12\left(f'_i + 1\right) + \left(f'''_i + f'''_{i+1}\right)h^2\right)\left(6f'_i + 3f''_i h + f'''_i h^2\right)} h^2 \tag{15}$$

with

$$E=-f_i'''^2 f_{i+1}''' h^4+3 f_i'''\ f_i''\left(f_i'''-2 f_{i+1}'''\right) h^3+$$
$$+3\left(f_{i+1}'''^2-3 f_i' f_i''' f_{i+1}'''-3 f_i''' f_{i+1}'''-f_i' f_i'''^2+3\ f_i''^2\left(2 f_i'''-f_{i+1}'''\right)\right) h^2+$$
$$+9\ f_i''\left(3 f_i'''-5 f_{i+1}'''+f_i' f_i'''-3 f_i' f_{i+1}'''+3\ f_i''^2\right) h-$$
$$-18\left(f_i'\left(f_i'+1\right)\left(f_i'''+f_{i+1}'''\right)-3\ f_i''^2\left(f_i'+2\right)\right)$$

where

$$f_i'=f'\left(x_i\right),\ f_i''=f''\left(x_i\right),\ f_i'''=f'''\left(\xi_i\right),\ f_{i+1}'''=f'''\left(\theta_i\right).$$

The case $\Delta_i-\Delta_{i-1}<0$ can be treated in the same manner and we obtain:

$$f'\left(x_i\right)-d_i=\frac{F}{3\left(12\left(f_i'+1\right)+\left(f_i'''+f_{i+1}'''\right) h^2\right)\left(-6 f_i'+3 f_i'' h-f_i''' h^2\right)} h^2$$

with

$$\begin{aligned}
F\ =\ & -f_i'''^2 f_{i+1}''' h^4+3 f_i'''\ f_i''\left(f_i'''-2 f_{i+1}'''\right) h^3+\\
&+3\left(f_{i+1}'''^2-3 f_i' f_i''' f_{i+1}'''-3 f_i''' f_{i+1}'''-f_i' f_{i+1}'''^2+3\ f_i''^2\left(2 f_i'''-f_{i+1}'''\right)\right) h^2+\\
&9\ f_i''\left(3 f_i'''-5 f_{i+1}'''+f_i' f_i'''-3 f_i' f_{i+1}'''+3\ f_i''^2\right) h+\\
&-18\left(f_i'\left(f_i'+1\right)\left(f_i'''+f_{i+1}'''\right)-3\ f_i''^2\left(f_i'+2\right)\right).
\end{aligned}$$

□

Corollary 2. *If $c=2$ the cubic Hermite interpolant with slopes (12) gives an $O\left(h^3\right)$ approximation to f for uniformly spaced data.*

For the particular value $c=2$ the slope d_i fulfill another (reasonable) properties, namely it's value don't break through the maximum between Δ_{i-1} and Δ_i.

Proposition 3. *If $c=2$ the slopes d_i given in (12) satisfy:*

$$\min\left\{\Delta_{i-1},\Delta_i\right\}\leq d_i\leq\max\left\{\Delta_{i-1},\Delta_i\right\}. \tag{16}$$

Proof. The inequality:

$$\min\left\{\Delta_{i-1},\Delta_i\right\}\leq d_i$$

was already used.

Admit now that $\Delta_i - \Delta_{i-1} \geq 0$, then we must prove that:

$$d_i \leq \Delta_i$$

equivalent with:

$$(1 + w_i)\,\Delta_{i-1} \leq \Delta_i.$$

Substituting (13) it follows:

$$\left(1 - \frac{\Delta_{i-1}}{\Delta_i}\right) \frac{1}{1 + \frac{l_{i-1}}{l_i}} \leq \frac{\Delta_i}{\Delta_{i-1}} - 1$$

equivalent with:

$$\frac{1}{1 + \frac{l_{i-1}}{l_i}} \leq \frac{\Delta_i}{\Delta_{i-1}}$$

which is true because the left side is lower, while the right side is greater than 1.

The case $\Delta_i - \Delta_{i-1} \leq 0$ can be treated similarly. □

Remark 4. *The property (16) hold for* $c \in [1, 2]$.

As example we use the data from [1]:

x_i	0	2	3	5	6	8	9	11	12	14	15
f_i	10	10	10	10	10	10	10.5	15	50	60	85

The cubic Hermite interpolant for $c = 2$ respectively for $c = 3$ are represented in Figure 2.

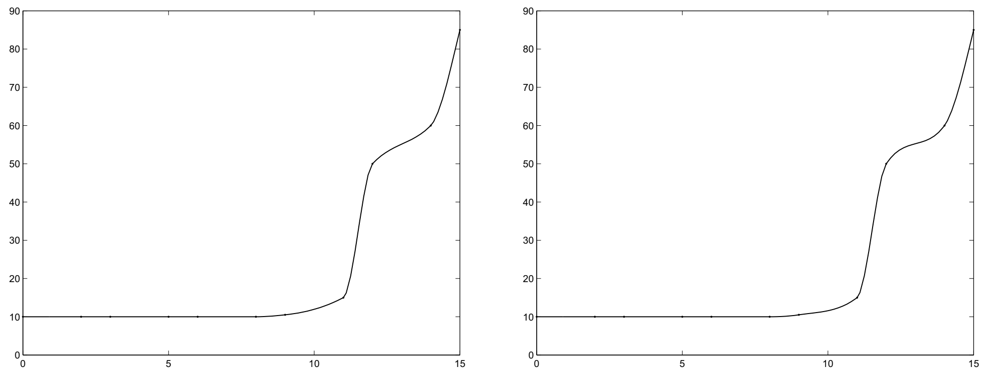

FIGURE 2. The monotone interpolant for $c = 2$ (left) and $c = 3$ (right)

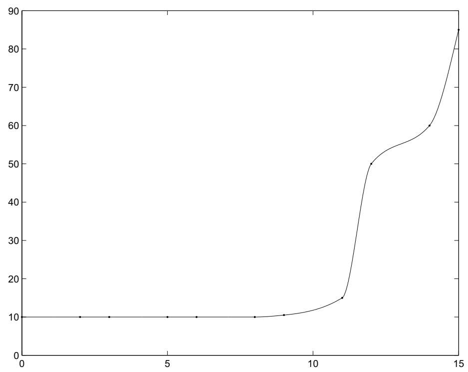

FIGURE 3. The piecewise cubic Hermite interpolating polynomial-pchip

By comparison we have represented in Figure 3 the cubic interpolant using the MATLAB's specialized function *pchip*. Those slopes d_i are computed using a weighted average of Δ_{i-1}, Δ_i.

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ON A LIMIT THEOREM FOR FREELY INDEPENDENT RANDOM VARIABLES

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Abstract. A direct proof of Voiculescu's addition theorem for freely independent real-valued random variables, using resolvents of self-adjoint operators, is given. The addition theorem leads to a central limit theorem for freely independent, identically distributed random variables of finite variance is given.

1. **Introduction**

The concept of independent random variables lies at the heart of classical probability. Via independent sequences it leads to the Gauss and Poisson distribution. Classical, commutative independence of random variables amounts to a factorisation property of probability spaces.

At the opposite, non-comutative extreme Voiculescu discovered in 1983 the notion of *free independence* of random variables, which corresponds to a *free* product of von Neumann algebras [3]. He showed that this notion leads naturally to analogues of the Gauss and Poisson distributions, very different in form from the classical ones [3] and [5]. For instance the free analoque of the Gauss curve is a semi-ellipse.

In this paper we consider the addition problem: Which is the probability distribution μ of the sum X_1+X_2 of two freely independent random varibles, given the distribution μ_1 and μ_2 of the summands? This problem was solved by Voiculescu in 1986 for the case of bounded, not necessarily self-adjoint random variables, relying on the existence of all the moments of the probability distributions μ_1 and μ_2 ([4]). Later this problem

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was solve by Hans Massen in 1992 for the case of self-adjoint random variables with finite variance. The result is an explicit calculation procedure for the free convolution product of two probability distributions. In this procedure a central role is played by the Cauchy transform $G(z)$ of a distribution μ, which equals the expectation of the resolvent of the associated operator X. If we take X self-adjoint, μ is a probability measure on $\mathbb{R}$ and we may write:

$$G(z) := \int_{-\infty}^{\infty} \frac{\mu \, \mathrm{d}x}{z-x} = E\left((z-X)^{-1}\right)$$

This formula points at a direct way to find the free convolution product of μ_1 and μ_2. This article consists of four sections. The first contains some preliminaries on free independence. In the second we gather some facts about Cauchy transforms. In three section it is shown that $F_1 \otimes F_2 = E\left((z-(\overline{X_1+X_2}))^{-1}\right)^{-1}$, where X_1 and X_2 are freely independent random variables with distributions μ_1 and μ_2 respectively, and the bar denotes operator closure. The last section contains the central limit theorem.

2. **Free independence of random variables**

By a random variable we shall mean a self-adjoint operator X on a Hibert space $\mathcal{H}$ in which a particular unit vector ξ has been singled out. Via the functional calculus of spectral theory such an operator determines an embedding ι_X of the commutative C^*-algebra $C(\overline{\mathbb{R}})$ of continuous functions on the one-point compactification $\overline{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$ of $\mathbb{R}$ to be bounded operators on $\mathcal{H}$:

$$\iota_X(f) = f(X)$$

We shall consider the spectral measure μ of X, which is determined by

$$< \xi, \iota_X(f)\xi >= \int_{-\infty}^{\infty} f(x)\mu \, \mathrm{d}x \qquad \left(f \in C(\overline{\mathbb{R}})\right)$$

as the *probability distribution* of X and we shall think of $< \xi, \iota_X(f)\xi >$ as the *expectation value* of the (complex-valued) random variable $f(X)$, which is a bounded normal operator on $\mathcal{H}$.

Definition 2.1. The random variables X_1 and X_2 on $(\mathcal{H}, \xi)$ are said to be *freely independent* if for all $n \in \mathbb{N}$ and all alternating sequences $i_1, i_2, ..., i_n$ such that $i_1 \neq i_2 \neq i_3 \neq ... \neq i_n$ and for all $f_k \in C(\overline{\mathbb{R}})$, $k = \overline{1, n}$ one has

$$< \xi, f_k(X_{i_k})\xi >= 0 \implies < \xi, f_1(X_{i_1}) f_2(X_{i_2}) ... f_n(X_{i_n})\xi >= 0$$

3. **The reciprocal Cauchy transform**

We consider the expectation values of functions $f \in C(\overline{\mathbb{R}})$ of the form

$$f(x) = \frac{1}{z - x} \ , \qquad (\Im(z) > 0)$$

In particular they play a key role in the addition of freely independent random variables.

For the complex plane $\mathbb{C}$ denote $\mathbb{C}^+ = \{z \in \mathbb{C} : \Im(z) > 0\}$ the upper half-plane, $\mathbb{C}^- = \{z \in \mathbb{C} : \Im(z) < 0\}$ the lower half-plane. If μ is a finite positive measure on $\mathbb{R}$, then its Cauchy transform

$$G(z) := \int_{-\infty}^{\infty} \frac{\mu \, \mathrm{d}x}{z - x} \ , \qquad (\Im(z) > 0) \ ,$$

is a holomorphic function ($G : \mathbb{C}^+ \to \mathbb{C}^+$) with the property

$$\limsup_{y \to \infty} y \, |G(iy)| < \infty \tag{1}$$

Conversely every holomorphic function $\mathbb{C}^+ \to \mathbb{C}^-$ with this property is the Cauchy transform of some finite positive measure on $\mathbb{R}$, and the lim sup in (1) equals $\mu(\mathbb{R})$. The inverse correspondence is given by Stieltjes' inversion formula:

$$\mu(B) = -\frac{1}{\pi} \lim_{\epsilon \downarrow 0} \int_B \Im(G(x + i\epsilon) \, \mathrm{d}x$$

valid for all Borel sets $B \in \mathbb{R}$ for which $\mu(\partial B) = 0$ ([1]).

We shall be mainly interested in the *reciprocal Cauchy transform*

$$F(z) = \frac{1}{G(z)}$$

The coresponding classes of reciprocal Cauchy transforms of probability measures with finite variance and zero mean will be denoted by $\mathcal{F}_0^2$.
The next proposition characterises the class $\mathcal{F}_0^2$.

Proposition 3.1. [2] *Let F be a holomorphic function $G : \mathbb{C}^+ \to \mathbb{C}^+$. Then the following statements are equivalent:*

(a): *F is the reciprocal Cauchy transform of a probability measure on $\mathbb{R}$ with finite variance and zero mean:*

$$\int_{-\infty}^{\infty} x^2 \mu \,\mathrm{d}x < \infty \qquad \textit{and} \qquad \int_{-\infty}^{\infty} x\mu \,\mathrm{d}x = 0\ ;$$

(b): *There exists a finite positive measure ρ on $\mathbb{R}$ such that for all $z \in \mathbb{C}^+$:*

$$F(z) = z + \int_{-\infty}^{\infty} \frac{\rho \,\mathrm{d}x}{x - z}\ ;$$

(c): *There exists a positive number C such that for all $z \in \mathbb{C}^+$:*

$$|F(z) - z| \leq \frac{C}{\Im(z)}$$

Moreover, the variance σ^2 of μ in (a), the total weight $\rho(\mathbb{R})$ of ρ in (b) and the (smallest possible) constant C in (c) are all equal.

Proof. For the proof it is useful to introduce the function $C_F : (0, \infty) \to \mathbb{C}$

$$y \longmapsto y^2 \left(\frac{1}{F(iy)} - \frac{1}{iy} \right) = \frac{iy}{F(iy)} \left(F(iy) - iy \right)$$

In case F is the reciprocal Cauchy transform of some probability measure μ on $\mathbb{R}$, the limiting behaviour of $C_F(y)$ as $y \to \infty$ gives information on the integrals $\int x^2 \mu \,\mathrm{d}x$ and $\int x\mu \,\mathrm{d}x$. Indeed one has

$$C_F(y) = y^2 \int_{-\infty}^{\infty} \left(\frac{1}{iy - 1} - \frac{1}{iy} \right) \mu \,\mathrm{d}x = \int_{-\infty}^{\infty} \frac{-xy^2 + ix^2 y}{x^2 + y^2} \,\mu \,\mathrm{d}x$$

The function $y \mapsto \Im(C_F(y))$ is nondecreasing and

$$\sup_{y>0} y\,\Im(C_F(y)) = \lim_{y\to\infty} y\,\Im(C_F(y)) \tag{2}$$
$$= \lim_{y\to\infty} \int_{-\infty}^{\infty} \frac{y^2}{x^2+y^2}\, x^2\mu\,\mathrm{d}x = \int_{-\infty}^{\infty} x^2\mu\,\mathrm{d}x < \infty$$

On the other side, by the dominated convergence theorem,

$$\int_{-\infty}^{\infty} x\mu\,\mathrm{d}x = \lim_{y\to\infty} \int_{-\infty}^{\infty} \frac{y^2}{x^2+y^2}\, x\mu\,\mathrm{d}x = -\lim_{y\to\infty} \Re(C_F(y)) \tag{3}$$

(a)⇒(b). If $F \in \mathcal{F}_0^2$, then by (2) and (3) both the real and the imaginary part of $C_F(y)$ tends to zero as $y \to \infty$. How $C_F(y) = \frac{iy}{F(iy)}(F(iy) - iy))$, then $iC_F(y) = \frac{iy^2}{F(iy)} - y$ and $|C_F(y)| = y\left|\frac{iy}{F(iy)} - 1\right|$. But $\lim_{y\to\infty} C_F(y) = 0$, it follows that

$$\lim_{y\to\infty} \frac{F(iy)}{iy} = 1$$

Therefore

$$\sigma^2 = \lim_{y\to\infty} y\Im(C_F(y)) = \lim_{y\to\infty} y\,|C_F(y)| \tag{4}$$
$$= \lim_{y\to\infty} y\left|\frac{iy}{F(iy)}\right| |F(iy) - iy| = \lim_{y\to\infty} y\,|F(iy) - iy| < \infty$$

This condition says that the function $z \mapsto F(z) - z$ satisfies (1) and is therefore the Cauchy transform of some finite positive measure ρ on $\mathbb{R}$ with $\rho(\mathbb{R}) = \sigma^2$. This proves (b).

(b)⇒(c). If F is of the form (b), then

$$|F(z) - z| = \left|\int_{-\infty}^{\infty} \frac{\rho\,\mathrm{d}x}{x - z}\right| \le \int_{-\infty}^{\infty} \frac{\rho\,\mathrm{d}x}{|z - x|} \le \frac{\rho(\mathbb{R})}{\Im(z)} \tag{5}$$

where C it may be equal with $\rho(\mathbb{R})$, whatever is $z \in \mathbb{C}^+$.

(c)⇒(a). Since $F : \mathbb{C}^+ \to \mathbb{C}^+$ is holomorphic, it can written in Nevanlinna's integral form [1]:

$$F(z) = a + bz + \int_{-\infty}^{\infty} \frac{1 + xz}{x - z}\,\tau\,\mathrm{d}x \tag{6}$$

where $a, b \in \mathbb{R}$ with $b \geq 0$ and τ is a finite positive measure. Putting $z = iy$, $y > 0$, we find that

$$\begin{aligned} y\,\Im(F(iy) - iy) &= y\,\Im\left(a + iby + \int_{-\infty}^{\infty} \frac{1 + ixy}{x - iy}\,\tau\,\mathrm{d}x - iy\right) \\ &= y^2\left[(b-1) + \int_{-\infty}^{\infty} \frac{x^2+1}{x^2+y^2}\,\tau\,\mathrm{d}x\right] \end{aligned}$$

As $y \to \infty$, the integral tends to zero. By the assumptiom (c), the whole expression must remain bounded, which can be the case if $b = 1$. But then by (6), F must increase the imaginary part:

$$\Im(F(z)) \leq \Im(z)$$

Moreover, (c) implies that $F(z)$ and z can be brought arbitrarily close together, so by [2], proposition 2.1 F is the reciprocal Cauchy transform of some probability measure μ on $\mathbb{R}$.

Again by (c) this measure μ must have the properties

$$\int_{-\infty}^{\infty} x^2\mu\,\mathrm{d}x \leq \lim_{y\to\infty} \sup y\,|C_F(y)| = \lim_{y\to\infty} \sup y\,|F(iy) - iy| \leq y\,\frac{C}{\Im(iy)} = C$$

and

$$\int_{-\infty}^{\infty} x\mu\,\mathrm{d}x = -\lim_{y\to\infty} \Re(C_F(y)) = 0$$

The fact that

$$\sigma^2 \geq \rho(R) \geq C \geq \sigma^2$$

is clear from the above; this these three numbers must be equal. □

We now presents one lemma about invertibility of reciprocal Cauchy transforms of measures and certain related functions, to be called φ-functions. The lemma act in opposite directions; from reciprocal Cauchy transforms of probability measures to φ-functions and vice versa.

Lemma 3.1. [2] *Let $C > 0$ and let $\varphi : \mathbb{C}^+ \to \mathbb{C}^-$ be analytic with*

$$|\varphi(z)| \leq \frac{C}{\Im(z)}$$

Then the function $K : \mathbb{C}^+ \to \mathbb{C}^+$, $K(u) = u + \varphi(u)$ takes every value in $\mathbb{C}^+$ precisely once. The inverse $K^{-1} : \mathbb{C}^+ \to \mathbb{C}^+$ thus defined is of class $\mathcal{F}_0^2$ with variance $\sigma^2 \leq C$.

4. The addition theorem

We now formulate the main theorem of this section, namely the addition theorem.

Theorem 4.1. [2] *Let X_1 and X_2 be freely independent random variables on some Hilbert space $\mathcal{H}$ with distinguished vector ξ, cyclic for X_1 and X_2. Suppose that X_1 and X_2 have distributions μ_1 and μ_2 with variances σ_1^2 and σ_2^2. Then the closure of the operator*

$$X = X_1 + X_2$$

defined on Dom $(X_1) \cap$ Dom (X_2) *is self-adjoint and its probability distribution μ on $(\mathcal{H}, \xi)$ is given by*

$$\mu = \mu_1 \otimes \mu_2$$

where $\otimes$ is the free convolution product.
In particular in the region $\left\{z \in \mathbb{C} \mid \Im(z) > 2\sqrt{\sigma_1^2 + \sigma_2^2}\right\}$ the φ-functions related to μ, μ_1 and μ_2 satisfy

$$\varphi = \varphi_1 + \varphi_2$$

The proof of this theorem is given in [2] where show that $< \xi, (z - \overline{X})^{-1}\xi >^{-1} = (F_1 \otimes F_2)(z)$ for all $z \in \mathbb{C}^+$.

5. A free limit theorem

In this section, we prove that sums of large numbers of frelly independent random variables of finite variance tend to certain distribution different to semiellipse distribution. The semiellipse distribution was first encountered by Wigner [6] when a studying spectra of large random matrices. The distribution obtained by author is defined by:

$$b_\sigma(x) = \frac{\sigma^2}{\pi(x^2+\sigma^4)}$$

where the graphics representation is in figure 1 for $\sigma_i = 1, 4, 10, 25, 50, 100$, $i = \overline{1,6}$. We remark that $b_\sigma(x)$ is the Cauchy distribution $Cau(0, \sigma^2)$.

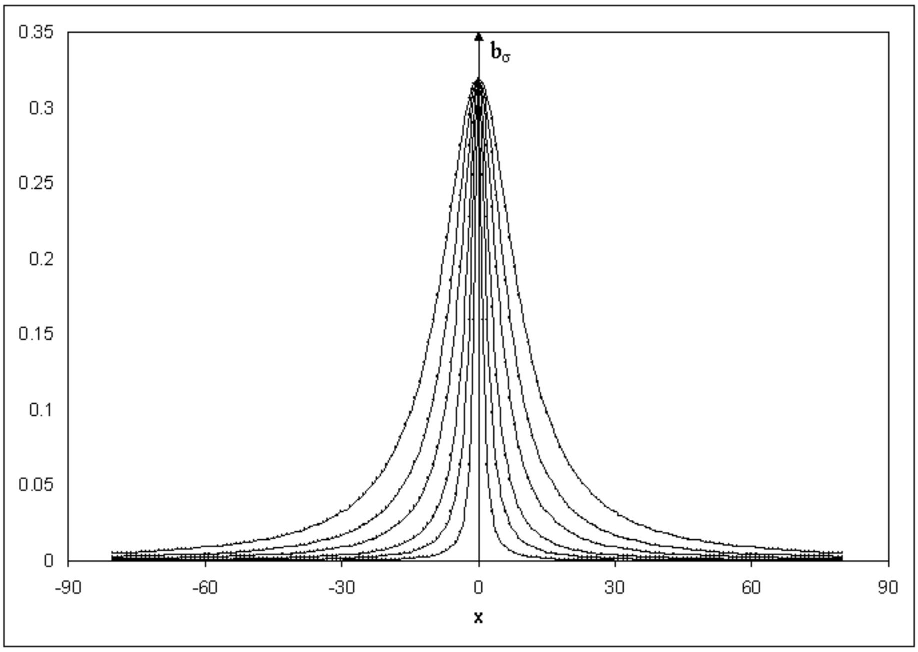

FIGURE 1. The graphics representation of distribution b_σ

Lemma 5.1. *The distribution b_σ has the following φ-function:*

$$\varphi(u) = -i\sigma^2 \tag{7}$$

Proof. We know that the inverse of the function $K_\sigma : \mathbb{C}^+ \to \mathbb{C}^+$, $K_\sigma(u) = u - i\sigma^2$ is the function $F_\sigma \in \mathcal{F}_0^2$. This is

$$F_\sigma : \mathbb{C}^+ \to \mathbb{C}^+,\ F_\sigma(z) = z + i\sigma^2$$

But this is the reciprocal Cauchy transform of b_σ by Stieltjes' inversion formula

$$\lim_{\epsilon\searrow 0} -\frac{1}{\pi}\Im\left(\frac{1}{F(x+i\epsilon)}\right) = b_\sigma(x)$$

Indeed:

$$\begin{aligned}\lim_{\epsilon\searrow 0} -\frac{1}{\pi}\Im\left(\frac{1}{F(x+i\epsilon)}\right) &= \lim_{\epsilon\searrow 0} -\frac{1}{\pi}\Im\left(\frac{1}{x+i(\epsilon+\sigma^2)}\right)\\ &= \lim_{\epsilon\searrow 0} -\frac{1}{\pi}\Im\left(\frac{x-i(\epsilon+\sigma^2)}{x^2+(\epsilon+\sigma^2)^2}\right)\\ &= \frac{1}{\pi}\cdot\frac{\sigma^2}{x^2+\sigma^4}\end{aligned}$$

□

We now formulate the free central limit theorem. We denote by $D_\lambda\mu$ its dilation by a factor λ for a probability measure μ on $\mathbb{R}$:

$$D_\lambda\mu(A) = \mu(\lambda^{-1}A)\ ,\ (A\subset\mathbb{R}\text{ measurable})$$

Theorem 5.1. *Let μ be a probability measure on $\mathbb{R}$ with mean 0 and variance σ^2, and for $n\in\mathbb{N}^*$ let*

$$\mu_n = \underbrace{D_{1/n}\mu\otimes ...\otimes D_{1/n}\mu}_{n-\ times}$$

Then

$$\lim_{n\to\infty}\mu_n = b_\sigma$$

Proof. Let F, $\widetilde{F}_n$ and F_n denote the reciprocal Cauchy transforms of μ, $D_n\mu$ and μ_n respectively. Denote the associated φ-functions by φ, $\widetilde{\varphi}_n$ and φ_n. Let as in the proof of lemma 5.1, F_σ denote the reciprocal Cauchy transform of b_σ. By the continuity theorem 2.5 in [2] it suffices to show that for some $M>0$ and all $z\in\mathbb{C}_M^+$:

$$\lim_{n\to\infty} F_n(z) = F_\sigma(z)$$

or is equivalent with

$$\lim_{n\to\infty} K_\sigma\circ F_n(z) = z \tag{8}$$

Now, fix $z \in \mathbb{C}_M^+$ and put $u_n = F_n(z)$ and $z_n = \widetilde{F}_n^{-1}(u_n)$. Then $z - u_n = \varphi_n(u_n)$ and $z_n - u_n = \widetilde{\varphi}_n(u_n)$. Hence by an n-fold application of the addition theorem 4.1,

$$z - u_n = n(z_n - u_n)$$

Note that also

$$|z - u_n| \leq \frac{\sigma^2}{M} \ , \ \Im(u_n) > M$$

with respect to lemma 3.1.

By the property $F_{D_\lambda \mu}(z) = \lambda F(\lambda^{-1} z)$ and the integral reprezentation of F in accord to proposition 3.1,(b), we have:

$$\begin{aligned} z - u_n &= n(z_n - u_n) = n(z_n - \widetilde{F}_n(z_n)) \\ &= n(z_n - n^{-1} F(nz_n)) = nz_n - F(nz_n) \\ &= \int_{-\infty}^{+\infty} \frac{\rho \, \mathrm{d}x}{nz_n - x} \end{aligned}$$

Hence

$$\begin{aligned} |z - K_\sigma \circ F_n(z)| &= |z - K_\sigma(u_n)| = \left| z - u_n + i\sigma^2 \right| \\ &= \int_{-\infty}^{+\infty} \left| \frac{1}{nz_n - x} + i\sigma^2 \right| \rho \, \mathrm{d}x \end{aligned}$$

The integrand on the right hand side is uniformly bounded and tends to zero pointwise as n tends to infinity. ☐

Remark 5.1. First note that every φ-function goes like $-i\sigma^2$ high above the real line. Indeed we have $z = F^{-1}(u) \approx u$ and

$$\varphi(u) = K(u) - u = F^{-1}(u) - u = \underbrace{z - F(z)}_{\varphi(z)} \approx -i\sigma^2$$

Now, due to the scaling law $\varphi_{D_\lambda \mu}(u) = \lambda \varphi(\lambda^{-1} u)$ and by proposition 3.1 we obtain

$$\varphi_n(u) = n\widetilde{\varphi}_n(u) = n\varphi_{D_{\frac{1}{n}}\mu}(u) = n \cdot \frac{1}{n} \ \varphi(nu) \to -i\sigma^2 \ , \ (n \to \infty)$$

In [3], the author to use in place of b_σ Cauchy distribution, the distribution defined by

$$b_\sigma(x) = \begin{cases} \frac{1}{2\sqrt{2}\pi x}\sqrt{\sqrt{1+16x^2\sigma^4}-1} & \text{if} \quad x > 0 \\ 0 & \text{if} \quad x \leq 0 \end{cases}$$

where the dilation of probability measure has a factor $\lambda = n$.

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THE DIRAC EQUATION AND THE NONCOMMUTATIVE HARMONIC OSCILATOR

LAURIAN PIŞCORAN

Abstract. In this paper we analyzed the Dirac equation using the noncommutative harmonic oscillator. Also we analyzed some particular wave functions cases using this noncommutative operator.

1. **Introduction**

The wave function $\psi(t,x)$ describes the probability distribution in time and space of an particle.

In general the Dirac equation (see [1]), is given by :

$$i\hbar\frac{\partial}{\partial t}\psi(t,x) = H_0\psi(t,x) \tag{$*$}$$

Here H_0 represents a differential operator, which is for instance, in the two-dimensional case:

$$H_0 = -i\hbar c\left(\sigma_1\frac{\partial}{\partial x_1} + \sigma_2\frac{\partial}{\partial x_2}\right) + \sigma_3 mc^2.$$

Here $\sigma_1, \sigma_2, \sigma_3$ represent the Pauli matrices, $\hbar$ is the Planck constant and m is the mass of the particle.

The non-commutative harmonic oscilator $Q(x, \partial_x)$ is defined to be the second-order ordinary differential operator:

$$Q(x,\partial_x) = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}\left(-\frac{\partial_x^2}{2} + \frac{x^2}{2}\right) + \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\left(x\partial_x + \frac{1}{2}\right)$$

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$$= \begin{pmatrix} -\frac{\alpha\partial_x^2}{2} + \alpha\frac{x^2}{2} & -x\partial_x - \frac{1}{2} \\ x\partial_x + \frac{1}{2} & -\beta\frac{\partial_x^2}{2} - \beta\frac{x^2}{2} \end{pmatrix}$$

where α, β are two constants, $\alpha, \beta > 0$.

If we change the operator H_0 with $Q(x, \partial_x)$ one obtain a new equation:

$$i\hbar\frac{\partial}{\partial t}\psi(t,x) = Q(x,\partial_x)\psi(t,x) \qquad (**)$$

Let's call this equation the "*noncommutative Dirac equation*" . In this paper we will analyze this new equation.

2. **Main result**

Theorem 2.1. *For a free particle the noncommutative Dirac equation is:*

$$\hbar\frac{\partial}{\partial t}\psi(t,x) = -\left(x\partial_x + \frac{1}{2}\right)\psi(t,x)$$

Proof. We know that the noncommutative harmonic oscilator is:

$$Q(x,\partial_x) = \begin{pmatrix} -\frac{\alpha\partial_x^2}{2} + \alpha\frac{x^2}{2} & -x\partial_x - \frac{1}{2} \\ x\partial_x + \frac{1}{2} & -\beta\frac{\partial_x^2}{2} - \beta\frac{x^2}{2} \end{pmatrix},$$

then the *noncommutative Dirac equation* is:

$$i\hbar\frac{\partial}{\partial t}\psi(t,x) = \begin{pmatrix} -\frac{\alpha\partial_x^2}{2} + \alpha\frac{x^2}{2} & -x\partial_x - \frac{1}{2} \\ x\partial_x + \frac{1}{2} & -\beta\frac{\partial_x^2}{2} - \beta\frac{x^2}{2} \end{pmatrix}\psi(t,x).$$

Using the matricial representation for the complex numbers,

$$i = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

one obtains:

$$\begin{pmatrix} 0 & \hbar\frac{\partial\psi(t,x)}{\partial t} \\ -\hbar\frac{\partial\psi(t,x)}{\partial t} & 0 \end{pmatrix} = \begin{pmatrix} \left(-\frac{\alpha\partial_x^2}{2} + \alpha\frac{x^2}{2}\right)\psi(t,x) & \left(-x\partial_x - \frac{1}{2}\right)\psi(t,x) \\ \left(x\partial_x + \frac{1}{2}\right)\psi(t,x) & \left(-\beta\frac{\partial_x^2}{2} - \beta\frac{x^2}{2}\right)\psi(t,x) \end{pmatrix}$$

Identifying, one obtains: $\hbar\frac{\partial}{\partial t}\psi(t,x) = -\left(x\partial_x + \frac{1}{2}\right)\psi(t,x)$, so the theorem is proved. □

Corollary 2.2. *If the wave function is* $\psi(t,x)=\varphi(x)e^{-\frac{iEt}{\hbar}}$, *where* E *represents the total energy, using the noncommutative Dirac equation, one obtains the total energy:* $E=\frac{1}{\hbar}\left(xp-\frac{i\hbar}{2}\right)$.

Proof.

$$\hbar\frac{\partial}{\partial t}\left(\varphi(x)e^{-\frac{iEt}{\hbar}}\right)=-\left(x\partial_x+\frac{1}{2}\right)\varphi(x)e^{-\frac{iEt}{\hbar}}\Rightarrow$$

$$\hbar\varphi(x)e^{-\frac{iEt}{\hbar}}\left(\frac{-iE}{\hbar}\right)=-\left(x\partial_x+\frac{1}{2}\right)\varphi(x)e^{-\frac{iEt}{\hbar}}\Rightarrow E=\frac{1}{i}\left(x\partial_x+\frac{1}{2}\right).$$

But, using the Schrödinger equation from quantum physics, the impulse is:

$$p=-i\hbar\frac{\partial}{\partial x},$$

so, one obtains:

$$E=\frac{1}{i^2\hbar}\left(i\hbar x\frac{\partial}{\partial x}+\frac{i\hbar}{2}\right)=\frac{1}{\hbar}\left(xp-\frac{i\hbar}{2}\right).$$ □

Using this expression for total energy, for the wave function, one obtains:

$$\psi(t,x)=\varphi(x)e^{-\left(\frac{it}{\hbar}\frac{1}{\hbar}\left(xp-\frac{i\hbar}{2}\right)\right)}=\varphi(x)e^{-\frac{it}{\hbar^2}xp-\frac{t}{2\hbar}}.$$

Corollary 2.3. *If we consider a plane wave function:* $\psi(t,x)=ce^{\left(-\frac{i}{\hbar}px-\frac{iEt}{\hbar}\right)}$, *using noncommutative Dirac equation, one obtains the total energy:*

$$E=\frac{1}{\hbar}(px+\frac{\hbar}{2i}).$$

Proof.

$$\hbar\frac{\partial}{\partial t}\left(ce^{\left(\frac{ipx}{\hbar}-\frac{iEt}{\hbar}\right)}\right)=-\left(x\partial_x+\frac{1}{2}\right)ce^{\left(\frac{ipx}{\hbar}-\frac{iEt}{\hbar}\right)}\Rightarrow$$

$$\hbar ce^{\left(\frac{ipx}{\hbar}-\frac{iEt}{\hbar}\right)}\left(-\frac{iE}{\hbar}\right)=-\left(x\partial_x+\frac{1}{2}\right)ce^{\left(\frac{ipx}{\hbar}-\frac{iEt}{\hbar}\right)}.$$

So, finally, we obtain:

$$iE=x\partial_x+\frac{1}{2}\Rightarrow E=\frac{1}{i}\left(x\partial_x+\frac{1}{2}\right)=\frac{1}{\hbar}\left(-\frac{i\hbar}{i^2}x\partial_x+\frac{\hbar}{2i}\right)=\frac{1}{\hbar}(px+\frac{\hbar}{2i}).$$ □

If we replace this expression of the total energy in the wave function, we get:

$$\psi(x,t)=ce^{\frac{i}{\hbar}px-\frac{it}{\hbar}\left(\frac{1}{\hbar}\left(px+\frac{\hbar}{2i}\right)\right)}=ce^{\frac{i}{\hbar}\left(px-\frac{tpx}{\hbar}-\frac{t}{2i}\right)}.$$

Every fermion also has an antifermion. An antiparticle was observed for the first time in 1933, but the idea had been introduced theoretically by Dirac in 1928. We start from the assumption that a particle in free space is described by the de Broglie wave function:

$$\psi(t,x) = N\exp[i(px - Et)/\hbar],$$

with frequency $\nu = \frac{E}{\hbar}$ and wavelenght $\lambda = \frac{\hbar}{p}$. Working nonrelativisticaly the relationship between momentum and energy is : $E = \frac{p^2}{2m}$ and substituting operators one obtains Schrödinger equation:

$$i\hbar\frac{\partial}{\partial t}\psi(t,x) = -\frac{\hbar}{2m}\bigtriangledown^2\psi(t,x).$$

Relativistically:

$$E^2 = p^2c^2 + m^2c^4,$$

and again substituting operators, we have:

$$-\hbar^2\frac{\partial^2}{\partial t^2}\psi(t,x) = -\hbar^2c^2\bigtriangledown^2\psi(t,x) + m^2c^4\psi(t,x).$$

This equation is called the Klein-Gordon equation. The solutions of this equation are:

$$\psi(x,t) = N\exp[i(px - Et)/\hbar].$$

Corollary 2.4 *If we consider the deBroglie wave function:*

$$\psi(x,t) = N\exp[i(px - Et)/\hbar],$$

using the noncommutative Dirac equation (**), one obtains the total energy:

$$E = \frac{1}{\hbar}(px + \frac{\hbar}{2i}).$$

Proof. From noncommutative Dirac equation (**), one obtains:

$$\hbar\frac{\partial}{\partial t}(Ne^{-\frac{i(px-Et)}{\hbar}}) = -\left(x\partial_x + \frac{1}{2}\right)Ne^{-\frac{i(px-Et)}{\hbar}} \Rightarrow iE = x\partial_x + \frac{1}{2},$$

so, finally, we obtain:

$$E = \frac{1}{i}\left(x\partial_x + \frac{1}{2}\right) = \frac{1}{\hbar}\left(px + \frac{\hbar}{2i}\right). \qquad \square$$

Then the deBroglie wave function becomes:

$$\psi(x,t) = N\exp[i(px - \frac{1}{i}\left(x\partial_x + \frac{1}{2}\right)t)/\hbar].$$

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THE GENERALIZATION OF VORONOVSKAJA'S THEOREM FOR A CLASS OF BIVARIATE OPERATORS

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Abstract. In this paper we generalize Voronovakaja's theorem and we give an approximation property for a class of bivariate operators and then, through particular cases, we obtain statements verified by the bivariate operators of Bernstein, Schurer, Durrmeyer, Kantorovich, Stancu and Bleimann, Butzer and Hahn.

1. **Introduction**

In this section, we recall some notions and results which we will use in this article. Let $\mathbb{N}$ be the set of positive integers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

For $m \in \mathbb{N}$, let $B_m : C([0,1]) \to C([0,1])$ the Bernstein operators, defined for any function $f \in C([0,1])$ by

$$(B_m f)(x) = \sum_{k=0}^{n} p_{m,k}(x) f\left(\frac{k}{m}\right), \tag{1.1}$$

where $p_{m,k}(x)$ are the fundamental polynomials of Bernstein, defined as follows

$$p_{m,k}(x) = \binom{m}{k} x^k (1-x)^{m-k}, \tag{1.2}$$

for any $x \in [0,1]$ and any $k \in \{0,1,\dots,m\}$.

Let $p \in \mathbb{N}_0$. For $m \in \mathbb{N}$, F. Schurer (see [15]) introduced and studied in 1962, the operators $\widetilde{B}_{m,p} : C([0,1+p]) \to C([0,1])$, named Bernstein-Schurer operators,

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defined for any function $f \in C([0,1+p])$ by

$$\left(\widetilde{B}_{m,p}f\right)(x) = \sum_{k=0}^{m+p} \widetilde{p}_{m,k}(x) f\left(\frac{k}{m}\right), \tag{1.3}$$

where $\widetilde{p}_{m,k}(x)$ denotes the fundamental Bernstein-Schurer polynomials, defined as follows

$$\widetilde{p}_{m,k}(x) = \binom{m+p}{k} x^k (1-x)^{m+p-k} = p_{m+p,k}(x) \tag{1.4}$$

for any $x \in [0,1]$ and any $k \in \{0,1,\ldots,m+p\}$.

For $m \in \mathbb{N}$ let the operators $M_n : L_1([0,1]) \to C([0,1])$ defined for any function $f \in L_1([0,1])$ by

$$(M_m f)(x) = (m+1) \sum_{k=0}^{m} p_{m,k}(x) \int_0^1 p_{m,k}(t) f(t) dt, \tag{1.5}$$

for any $x \in [0,1]$.

These operators were introduced in 1967 by J. L. Durrmeyer in [7] and were studied in 1981 by M. M. Derriennic in [5].

For $m \in \mathbb{N}$ let the operator $K_m : L_1([0,1]) \to C([0,1])$ defined for any function $f \in L_1([0,1])$ by

$$(K_m f)(x) = (m+1) \sum_{k=0}^{m} p_{m,k}(x) \int_{\frac{k}{m+1}}^{\frac{k+1}{m+1}} f(t) dt, \tag{1.6}$$

for any $x \in [0,1]$.

The operators K_m, where $m \in \mathbb{N}$, are named Kantorovich operators, introduced and studied in 1930 by L. V. Kantorovich (see [8]).

For the following construction see [11].

Define the natural number m_0 by

$$m_0 = \begin{cases} \max\{1, -[\beta]\}, & \text{if} \quad \beta \in \mathbb{R} \backslash \mathbb{Z} \\ \max\{1, 1-\beta\}, & \text{if} \quad \beta \in \mathbb{Z}. \end{cases} \tag{1.7}$$

For the real number β, we have that

$$m + \beta \geq \gamma_\beta \tag{1.8}$$

for any natural number m, $m \geq m_0$, where

$$\gamma_\beta = m_0 + \beta = \begin{cases} \max\big\{1+\beta, \{\beta\}\big\}, & \text{if} \quad \beta \in \mathbb{R}\setminus\mathbb{Z} \\ \max\{1+\beta, 1\}, & \text{if} \quad \beta \in \mathbb{Z}. \end{cases} \tag{1.9}$$

For the real numbers α, β, $\alpha \geq 0$, we note

$$\mu^{(\alpha,\beta)} = \begin{cases} 1, & \text{if} \quad \alpha \leq \beta \\ 1 + \dfrac{\alpha-\beta}{\gamma_\beta}, & \text{if} \quad \alpha > \beta. \end{cases} \tag{1.10}$$

For the real numbers α and β, $\alpha \geq 0$, we have that $1 \leq \mu^{(\alpha,\beta)}$ and

$$0 \leq \frac{k+\alpha}{m+\beta} \leq \mu^{(\alpha,\beta)} \tag{1.11}$$

for any natural number m, $m \geq m_0$ and for any $k \in \{0, 1, \ldots, m\}$.

For the real numbers α and β, $\alpha \geq 0$, m_0 and $\mu^{(\alpha,\beta)}$ defined by (1.7) - (1.10), let the operators $P_m^{(\alpha,\beta)} : C\big([0, \mu^{(\alpha,\beta)}]\big) \to C\big([0,1]\big)$, defined for any function $f \in C\big([0, \mu^{(\alpha,\beta)}]\big)$ by

$$\big(P_m^{(\alpha,\beta)} f\big)(x) = \sum_{k=0}^{m} p_{m,k}(x) f\left(\frac{k+\alpha}{m+\beta}\right), \tag{1.12}$$

for any natural number $m, m \geq m_0$ and for any $x \in [0,1]$.

These operators are named Stancu operators, introduced and studied in 1969 by D. D. Stancu in the paper [16]. In [16], the domain of definition of the Stancu operators is $C([0,1])$ and the numbers α and β verify the condition $0 \leq \alpha \leq \beta$.

In 1980, G. Bleimann, P. L. Butzer and L. Hahn introduced in [4] a sequence of linear positive operators $(L_m)_{m\geq 1}$, $L_m : C_B([0,\infty)) \to C_B([0,\infty))$, defined for any function $f \in C_B([0,\infty))$ by

$$(L_m f)(x) = \frac{1}{(1+x)^m} \sum_{k=0}^{m} \binom{m}{k} x^k f\left(\frac{k}{m+1-k}\right), \tag{1.13}$$

for any $x \in [0,\infty)$ and any $m \in \mathbb{N}$, where $C_B([0,\infty)) = \{f \,|\, f : [0,\infty) \to \mathbb{R},\ f$ bounded and continuous on $[0,\infty)\}$.

Let $I_1, I_2 \subset \mathbb{R}$ be given intervals and $f : I_1 \times I_2 \to \mathbb{R}$ be a bounded function.

The function $\omega_{total}(f;\cdot,*):[0,\infty)\times[0,\infty)\to\mathbb{R}$, defined for any $(\delta_1,\delta_2)\in[0,\infty)\times[0,\infty)$ by

$$\omega_{total}(f;\delta_1,\delta_2)=\sup\left\{|f(x,y)-f(x',y')|:(x,y),(x',y')\in I_1\times I_2,\right.\\ \left.|x-x'|\le\delta_1,\ |y-y'|\le\delta_2\right\} \tag{1.14}$$

is called the first order modulus of smoothness of function f or total modulus of continuity of function f (see [18]).

The first order modulus of smoothness for bivariate functions has properties similar to the properties of the first order modulus of smoothness for univariate functions. Some of them are contained in Lemma 1.1.

Lemma 1.1. *The first order modulus of smoothness for bounded function $f:I_1\times I_2\to\mathbb{R}$ has the following properties:*

(i) $\omega_{total}(f;\delta_1,\delta_2)\le\omega_{total}(f;\delta_1',\delta_2')$ for any (δ_1,δ_2), $(\delta_1',\delta_2')\in[0,\infty)\times[0,\infty)$ such that $\delta_1\le\delta_1'$ and $\delta_2\le\delta_2'$;

(ii) $\omega_{total}(f;|t-x|,|\tau-y|)\le\left(1+\delta_1^{-2}(t-x)^2\right)\left(1+\delta_2^{-2}(\tau-y)^2\right)\omega_{total}(f;\delta_1,\delta_2)$ for any $(\delta_1,\delta_2)\in(0,\infty)\times(0,\infty)$ and any (t,τ), $(x,y)\in I_1\times I_2$.

For some further informations on this measure of smoothness see for example [18].

2. **Preliminaries**

Let $I,J\subset\mathbb{R}$ intervals with $I\cap J\ne\emptyset$. For $m\in\mathbb{N}$ we consider the functions $p^*_{m,k}:J\to\mathbb{R}$ with the property that $p^*_{m,k}(x)\ge 0$ for any $x\in J$, $k\in\{0,1,\dots,m\}$ and the linear positive functionals $A_{m,k}:E(I)\to\mathbb{R}$, $k\in\{0,1,\dots,m\}$.

Definition 2.1. Let $m\in\mathbb{N}$. Define the operator $L^*_m:E(I)\to F(J)$ by

$$\left(L^*_m f\right)(x)=\sum_{k=0}^{m}p^*_{m,k}(x)A_{m,k}(f) \tag{2.1}$$

for any function $f\in E(I)$ and any $x\in J$, where $E(I)$ and $F(J)$ are subsets of the set of real functions defined on I, respectively on J.

Proposition 2.1. *The operators $(L^*_m)_{m\ge1}$ are linear and positive on $E(I\cap J)$.*

Proof. The proof follows immediately. □

Definition 2.2. Let $m \in \mathbb{N}$. For $i \in \mathbb{N}_0$ define $T^*_{m,i}$ by

$$\left(T^*_{m,i} L^*_m\right)(x) = m^i \left(L_m \psi^i_x\right)(x) = m^i \sum_{k=0}^{m} p^*_{m,k}(x) A_{m,k}\left(\psi^i_x\right) \tag{2.2}$$

for any $x \in I \cap J$, where for $x \in I$, $\psi_x : I \to \mathbb{R}$, $\psi_x(t) = t - x$ for any $t \in I$.

In the following, let $s \in \mathbb{N}_0$, s even. We suppose that the operators $(L^*_m)_{m \geq 1}$ verify the conditions: there exists the smallest $\alpha_j \in [0, \infty)$ so that

$$\lim_{m \to \infty} \frac{\left(T^*_{m,j} L^*_m\right)(x)}{m^{\alpha_j}} = B_j(x) \in \mathbb{R} \tag{2.3}$$

for any $x \in I \cap J$, $j \in \{0, 2, 4, \ldots, s+2\}$ and

$$\begin{cases} \alpha_{s-2l} + \alpha_{2l} - \alpha_s \leq 0 \\ \alpha_{s-2l+2} + \alpha_{2l} - \alpha_s - 2 < 0 \\ \alpha_{s-2l+2} + \alpha_{2l+2} - \alpha_s - 4 < 0 \end{cases} \tag{2.4}$$

where $l \in \left\{0, 1, 2, \ldots, \frac{s}{2}\right\}$.

Remark 2.1. From the first relation from (2.4), for $l = 0$ it results that $\alpha_0 = 0$.

Now, we construct with the $(L^*_m)_{m \geq 1}$ operators the bivariate operators of L^*-type.

For $m, n \in \mathbb{N}$, let the linear positive functionals $A_{m,n,k,j} : E(I \times I) \to \mathbb{R}$ with the property

$$A_{m,n,k,j}\left((\cdot - x)^i (* - y)^l\right) = A_{m,k}\left((\cdot - x)^i\right) A_{n,j}\left((* - y)^l\right) \tag{2.5}$$

for any $k \in \{0, 1, \ldots, m\}$, $j \in \{0, 1, \ldots, n\}$, $i, l \in \{0, 1, \ldots, s\}$, $x, y \in I$, where " $\cdot$ " and " $*$ " stand for the first and second variable.

Definition 2.3. Let $m, n \in \mathbb{N}$. The operator $L^*_{m,n} : E(I \times I) \to F(J \times J)$ defined for any function $f \in E(I \times I)$ and any $(x, y) \in J \times J$ by

$$\left(L^*_{m,n} f\right)(x, y) = \sum_{k=0}^{m} \sum_{j=0}^{n} p^*_{m,k}(x) p^*_{n,j}(y) A_{m,n,k,j}(f) \tag{2.6}$$

is named the bivariate operator of L^*-type.

Proposition 2.2. *The operators* $\left(L^*_{m,n}\right)_{m,n\geq 1}$ *are linear and positive on* $E\left((I\times I)\cap(J\times J)\right)$.

Proof. The proof follows immediately. □

In the following we consider that

$$\left(T^*_{m,0}L^*_m\right)(x)=1 \tag{2.7}$$

for any $x\in I\cap J$, any $m\in\mathbb{N}$.

3. Main results

Theorem 3.1. *Let* $I_1, I_2\subset\mathbb{R}$ *be intervals,* $(a,b)\in I_1\times I_2$, $n\in\mathbb{N}_0$ *and the function* $f: I_1\times I_2\to\mathbb{R}$, f *admits partial derivatives of order* n *continuous in a neighborhood* V *of the point* (a,b). *According to Taylor's expansion theorem for the function* f *around* (a,b), *for* $(x,y)\in V$ *we have*

$$f(x,y)=\sum_{k=0}^{n}\frac{1}{k!}\left(\frac{\partial}{\partial x}(x-a)+\frac{\partial}{\partial y}(y-b)\right)^k f(a,b)+ \tag{3.1}$$
$$+\rho^n(x,y)\mu(x-a,y-b)$$

where

$$\left(\frac{\partial}{\partial x}(x-a)+\frac{\partial}{\partial y}(y-b)\right)^k f(a,b)= \tag{3.2}$$
$$=\sum_{i=0}^{k}\binom{k}{i}\frac{\partial^k f}{\partial x^{k-i}\partial y^i}(a,b)(x-a)^{k-i}(y-b)^i,$$

$k\in\{0,1,\dots,n\}$, μ *is a bounded function with* $\lim\limits_{(x,y)\to(a,b)}\mu(x-a,y-b)=0$ *and*

$$\rho(x,y)=\sqrt{(x-a)^2+(y-b)^2}. \tag{3.3}$$

Then

$$|\mu(x-a,y-b)|\leq\frac{1}{n!}\sum_{i=0}^{n}\binom{n}{i}\omega_{total}\left(\frac{\partial^n f}{\partial x^{n-i}\partial y^i};|x-a|,|y-b|\right) \tag{3.4}$$

and for any $\delta_1, \delta_2 > 0$*, we have*

$$|\mu(x-a, y-b)| \leq \tag{3.5}$$
$$\leq \frac{1}{n!}\left(1+\delta_1^{-2}(x-a)^2\right)\left(1+\delta_2^{-2}(y-b)^2\right)\sum_{i=0}^{n}\binom{n}{i}\omega_{total}\left(\frac{\partial^n f}{\partial x^{n-i}\partial y^i};\delta_1,\delta_2\right).$$

Proof. If $n = 0$, it is verified immediately. Let $n \in \mathbb{N}$. According to Taylor's expansion theorem with the Lagrange's remainder, we have

$$f(x,y) = \sum_{k=0}^{n-1}\frac{1}{k!}\left(\frac{\partial}{\partial x}(x-a)+\frac{\partial}{\partial y}(y-b)\right)^k f(a,b)+ \tag{3.6}$$
$$+\frac{1}{n!}\left(\frac{\partial}{\partial x}(x-a)+\frac{\partial}{\partial y}(y-b)\right)^n f(\xi,\eta)$$

where $(\xi,\eta) \in V$, (ξ,η) is on the interval determined by the points (a,b) and (x,y) and

$$|\xi-a| \leq |x-a|,\ |\eta-b| \leq |y-b|. \tag{3.7}$$

From (3.1) and (3.6) it results that

$$\mu(x-a,y-b) = \frac{1}{n!}\frac{1}{\rho^n(x,y)}\left[\left(\frac{\partial}{\partial x}(x-a)+\frac{\partial}{\partial y}(y-b)\right)^n f(\xi,\eta)-\right.$$
$$\left.-\left(\frac{\partial}{\partial x}(x-a)+\frac{\partial}{\partial y}(y-b)\right)^n f(a,b)\right] =$$
$$=\frac{1}{n!}\frac{1}{\rho^n(x,y)}\sum_{i=0}^{n}\binom{n}{i}\left[\frac{\partial^n f}{\partial x^{n-i}\partial y^i}(\xi,\eta)-\right.$$
$$\left.-\frac{\partial^n f}{\partial x^{n-i}\partial y^i}(a,b)\right](x-a)^{n-i}(y-b)^i.$$

Because $|x-a| \leq \rho(x,y)$ and $|y-b| \leq \rho(x,y)$, the relation above becomes

$$|\mu(x-a,y-b)| \leq \frac{1}{n!}\sum_{i=0}^{n}\binom{n}{i}\left|\frac{\partial^n f}{\partial x^{n-i}\partial y^i}(\xi,\eta)-\frac{\partial^n f}{\partial x^{n-i}\partial y^i}(a,b)\right|\cdot$$
$$\cdot\frac{|x-a|^{n-i}}{\rho^{n-i}(x,y)}\frac{|y-b|^i}{\rho^i(x,y)},$$

from where

$$|\mu(x-a,y-b)| \leq \frac{1}{n!}\sum_{i=0}^{n}\binom{n}{i}\left|\frac{\partial^n f}{\partial x^{n-i}\partial y^i}(\xi,\eta)-\frac{\partial^n f}{\partial x^{n-i}\partial y^i}(a,b)\right|. \tag{3.8}$$

Taking (3.7) into account, from (3.8) we have that

$$|\mu(x-a,y-b)| \leq \frac{1}{n!}\sum_{i=0}^{n}\binom{n}{i}\sup\left\{\left|\frac{\partial^n f}{\partial x^{n-i}\partial y^i}(u,v)-\frac{\partial^n f}{\partial x^{n-i}\partial y^i}(u',v')\right|: \right.$$
$$\left. |u-u'|\leq|x-a|,\ |v-v'|\leq|y-b|\right\},$$

from where we obtain the relation (3.4).

From (3.4) taking Lemma 1.2 into account, we obtain the relation (3.5). □

In the following we consider the construction from Preliminaries.

Theorem 3.2. *Let $f: I\times I\to\mathbb{R}$ be a bivariate function.*

If $(x,y)\in(I\times I)\cap(J\times J)$ and f admits partial derivatives of order s continuous in a neighborhood of the point (x,y), then

$$\lim_{m\to\infty} m^{s-\alpha_s}\Bigg[\left(L^*_{m,m}f\right)(x,y)- \tag{3.9}$$
$$-\sum_{i=0}^{s}\frac{1}{m^i i!}\sum_{l=0}^{i}\binom{i}{l}\frac{\partial^i f}{\partial t^{i-l}\partial\tau^l}(x,y)\left(T^*_{m,i-l}L^*_m\right)(x)\left(T^*_{m,l}L^*_m\right)(y)\Bigg]=0.$$

If f admits partial derivatives of order s continuous on $(I\times I)\cap(J\times J)$ and there exists an interval $K\subset I\cap J$ such that there exist $m(s)\in\mathbb{N}$ and $k_{2l}\in\mathbb{R}$ depending on K, so that for any $m\in\mathbb{N}$, $m\geq m(s)$ and for any $x\in K$ we have

$$\frac{\left(T^*_{m,2l}L^*_m\right)(x)}{m^{\alpha_{2l}}}\leq k_{2l} \tag{3.10}$$

where $l \in \left\{0, 1, \ldots, \frac{s}{2}+1\right\}$, *then the convergence given in* (3.9) *is uniform on* $K \times K$ *and*

$$m^{s-\alpha_s}\Bigg| \left(L^*_{m,m} f\right)(x, y) - \tag{3.11}$$

$$-\sum_{i=0}^{s} \frac{1}{m^i i!} \sum_{l=0}^{i} \binom{i}{l} \frac{\partial^i f}{\partial t^{i-l} \partial \tau^l}(x, y) \left(T^*_{m,i-l} L^*_m\right)(x) \left(T^*_{m,l} L^*_m\right)(y)\Bigg| \leq$$

$$\leq \frac{1}{s!} \sum_{l=0}^{\frac{s}{2}} \binom{\frac{s}{2}}{l} (k_{2l} + k_{2l+2})(k_{s-2l} + k_{s-2l+2}) \cdot$$

$$\cdot \sum_{i=0}^{s} \binom{s}{i} \omega_{total}\left(\frac{\partial^s f}{\partial t^{s-i} \partial \tau^i}; \frac{1}{\sqrt{m^{\beta_s}}}, \frac{1}{\sqrt{m^{\beta_s}}}\right)$$

for any $(x, y) \in (K \times K)$, *any natural number* m, $m \geq m(s)$, *where*

$$\beta_s = -\max\left\{\alpha_{s-2l+2} + \alpha_{2l} - \alpha_s - 2, \frac{1}{2}(\alpha_{s-2l+2} + \alpha_{2l+2} - \alpha_s - 4) : l \in \left\{0, 1, \ldots, \frac{s}{2}\right\}\right\}.$$

Proof. Let $m, n \in \mathbb{N}$. According to Taylor's theorem for the function f around (x, y), we have

$$f(t, \tau) = \sum_{i=0}^{s} \frac{1}{i!} \left(\frac{\partial}{\partial t}(t - x) + \frac{\partial}{\partial \tau}(\tau - y)\right)^i f(x, y) + \rho^s(t, \tau)\mu(t - x, \tau - y),$$

from where

$$f(t, \tau) = \sum_{i=0}^{s} \frac{1}{i!} \sum_{l=0}^{i} \binom{i}{l} \frac{\partial^i f}{\partial t^{i-l} \partial \tau^l}(x, y)(t - x)^{i-l}(\tau - y)^l + \tag{3.12}$$

$$+ \rho^s(t, \tau)\mu(t - x, \tau - y),$$

where μ is a bounded function and $\lim\limits_{(t,\tau)\to(x,y)} \mu(t - x, \tau - y) = 0$.

Because $A_{m,n,k,j}$ is linear positive functional and verifies (2.5), from (3.12) we have

$$A_{m,n,k,j}(f) = \sum_{i=0}^{s} \frac{1}{i!} \sum_{l=0}^{i} \binom{i}{l} \frac{\partial^i f}{\partial t^{i-l} \partial \tau^l}(x, y) A_{m,k}\left((\cdot - x)^{i-l}\right) A_{n,j}\left((* - y)^l\right) +$$

$$+ A_{m,n,k,j}\left(\rho^s(\cdot, *)\mu_{xy}\right),$$

where $\mu_{xy} : (I \times I) \cap (J \times J) \to \mathbb{R}$, $\mu_{xy}(t, \tau) = \mu(t - x, \tau - y)$ for any $(t, \tau) \in (I \times I) \cap (J \times J)$.

Multiplying by $p^*_{m,k}(x)p^*_{n,j}(y)$ and summing after k, j, where $k\in\{0,1,\dots,m\}$, $j\in\{0,1,\dots,n\}$, we obtain

$$\begin{aligned}&\left(L^*_{m,n}f\right)(x,y)=\\&=\sum_{i=0}^{s}\frac{1}{i!}\sum_{l=0}^{i}\binom{i}{l}\frac{\partial^i f}{\partial t^{i-l}\partial\tau^l}(x,y)\frac{1}{m^{i-l}}\,\frac{1}{n^l}\left(T^*_{m,i-l}L^*_m\right)(x)\left(T^*_{n,l}L^*_n\right)(y)+\\&+\sum_{k=0}^{m}\sum_{j=0}^{n}p^*_{m,k}(x)p^*_{n,j}(y)A_{m,n,k,j}\left(\rho^s(\cdot\,,*)\mu_{xy}\right),\end{aligned}$$

from which

$$\begin{aligned}&m^{s-\alpha_s}\Bigg[\left(L^*_{m,m}f\right)(x,y)-\\&-\sum_{i=0}^{s}\frac{1}{m^i i!}\sum_{l=0}^{i}\binom{i}{l}\frac{\partial^i f}{\partial t^{i-l}\partial\tau^l}(x,y)\left(T^*_{m,i-l}L^*_m\right)(x)\left(T^*_{m,l}L^*_m\right)(y)\Bigg]=\\&=(R_{m,m}f)(x,y),\end{aligned}\tag{3.13}$$

where

$$(R_{m,m}f)(x,y)=m^{s-\alpha_s}\sum_{k=0}^{m}\sum_{j=0}^{m}p^*_{m,k}(x)p^*_{m,j}(y)A_{m,m,k,j}\left(\rho^s(\cdot\,,*)\mu_{xy}\right).\tag{3.14}$$

Then

$$|(R_{m,m}f)(x,y)|\le m^{s-\alpha_s}\sum_{k=0}^{m}\sum_{j=0}^{m}p^*_{m,k}(x)p^*_{m,j}(y)\left|A_{m,m,k,j}\left(\rho^s(\cdot\,,*)\mu_{xy}\right)\right|,$$

from where

$$\begin{aligned}&|(R_{m,m}f)(x,y)|\le\\&\le m^{s-\alpha_s}\sum_{k=0}^{m}\sum_{j=0}^{m}p^*_{m,k}(x)p^*_{m,j}(y)A_{m,m,k,j}\left(\rho^s(\cdot\,,*)|\mu_{xy}|\right).\end{aligned}\tag{3.15}$$

According to the relation (3.5), for any $\delta_1, \delta_2 > 0$ and for any $(t,\tau) \in (I \times I) \cap (J \times j)$, we have that

$$|\mu_{xy}(t,\tau)| = |\mu(t-x,\tau-y)| \leq$$
$$\leq \frac{1}{s!}\left(1+\delta_1^{-2}(t-x)^2+\delta_2^{-2}(\tau-y)^2+\delta_1^{-2}\delta_2^{-2}(t-x)^2(\tau-y)^2\right)\cdot$$
$$\cdot\sum_{i=0}^{s}\binom{s}{i}\omega_{total}\left(\frac{\partial^s f}{\partial t^{s-i}\partial\tau^i};\delta_1,\delta_2\right)$$

and taking $\rho^s(t,\tau)=\sum_{l=0}^{\frac{s}{2}}\binom{\frac{s}{2}}{l}(t-x)^{s-2l}(\tau-y)^{2l}$ into account, (3.16) results

$$A_{m,m,k,j}\left(\rho^s(\cdot\,,*)|\mu_{xy}|\right) \leq \frac{1}{s!}\sum_{l=0}^{\frac{s}{2}}\binom{\frac{s}{2}}{l}\Big[A_{m,k}\left(\psi_x^{s-2l}\right)A_{m,j}\left(\psi_y^{2l}\right)+ \tag{3.16}$$
$$+\delta_1^{-2}A_{m,k}\left(\psi_x^{s-2l+2}\right)A_{m,j}\left(\psi_y^{2l}\right)+\delta_2^{-2}A_{m,k}\left(\psi_x^{s-2l}\right)A_{m,j}\left(\psi_y^{2l+2}\right)+$$
$$+\delta_1^{-2}\delta_2^{-2}A_{m,k}\left(\psi_x^{s-2l+2}\right)A_{m,j}\left(\psi_y^{2l+2}\right)\Big]\cdot$$
$$\cdot\sum_{i=0}^{s}\binom{s}{i}\omega_{total}\left(\frac{\partial^s f}{\partial t^{s-i}\partial\tau^i};\delta_1,\delta_2\right).$$

From (3.15) and (3.16), it results that

$$|(R_{m,m}f)(x,y)| \leq$$
$$\leq \frac{1}{s!}\,m^{s-\alpha_s}\sum_{l=0}^{\frac{s}{2}}\binom{\frac{s}{2}}{l}\sum_{k=0}^{m}\sum_{j=0}^{m}p^*_{m,k}(x)p^*_{m,j}(y)\Big[A_{m,k}\left(\psi_x^{s-2l}\right)A_{m,j}\left(\psi_y^{2l}\right)+$$
$$+\delta_1^{-2}A_{m,k}\left(\psi_x^{s-2l+2}\right)A_{m,j}\left(\psi_y^{2l}\right)+\delta_2^{-2}A_{m,k}\left(\psi_x^{s-2l}\right)A_{m,j}\left(\psi_y^{2l+2}\right)+$$
$$+\delta_1^{-2}\delta_2^{-2}A_{m,k}\left(\psi_x^{s-2l+2}\right)A_{m,j}\left(\psi_y^{2l+2}\right)\Big]\sum_{i=0}^{s}\binom{s}{i}\omega_{total}\left(\frac{\partial^s f}{\partial t^{s-i}\partial\tau^i};\delta_1,\delta_2\right),$$

or

$$|(R_{m,m}f)(x,y)| \leq \frac{1}{s!}\, m^{s-\alpha_s} \sum_{l=0}^{\frac{s}{2}} \binom{\frac{s}{2}}{l} \left[\frac{\left(T^*_{m,s-2l}L^*_m\right)(x)}{m^{s-2l}} \, \frac{\left(T^*_{m,2l}L^*_m\right)(y)}{m^{2l}} + \right.$$

$$+ \delta_1^{-2} \frac{\left(T^*_{m,s-2l+2}L^*_m\right)(x)}{m^{s-2l+2}} \, \frac{\left(T^*_{m,2l}L^*_m\right)(y)}{m^{2l}} +$$

$$+ \delta_2^{-2} \frac{\left(T^*_{m,s-2l}L^*_m\right)(x)}{m^{s-2l}} \, \frac{\left(T^*_{m,2l+2}L^*_m\right)(y)}{m^{2l+2}} +$$

$$\left. + \delta_1^{-2}\delta_2^{-2} \frac{\left(T^*_{m,s-2l+2}L^*_m\right)(x)}{m^{s-2l+2}} \, \frac{\left(T^*_{m,2l+2}L^*_m\right)(y)}{m^{2l+2}} \right] \cdot$$

$$\cdot \sum_{i=0}^{s} \binom{s}{i} \omega_{total} \left(\frac{\partial^s f}{\partial t^{s-i} \partial \tau^i}\, ; \delta_1, \delta_2 \right),$$

so

$$|(R_{m,m}f)(x,y)| \leq$$

$$\leq \frac{1}{s!} \sum_{l=0}^{\frac{s}{2}} \binom{\frac{s}{2}}{l} \left[\frac{\left(T^*_{m,s-2l}L^*_m\right)(x)}{m^{\alpha_{s-2l}}} \, \frac{\left(T^*_{m,2l}L^*_m\right)(y)}{m^{\alpha_{2l}}} \, m^{\alpha_{s-2l}+\alpha_{2l}-\alpha_s} + \right.$$

$$+ \delta_1^{-2} \frac{\left(T^*_{m,s-2l+2}L^*_m\right)(x)}{m^{\alpha_{s-2l+2}}} \, \frac{\left(T^*_{m,2l}L^*_m\right)(y)}{m^{\alpha_{2l}}} \, m^{\alpha_{s-2l+2}+\alpha_{2l}-\alpha_s-2} +$$

$$+ \delta_2^{-2} \frac{\left(T^*_{m,s-2l}L^*_m\right)(x)}{m^{\alpha_{s-2l}}} \, \frac{\left(T^*_{m,2l+2}L^*_m\right)(y)}{m^{\alpha_{2l+2}}} \, m^{\alpha_{s-2l}+\alpha_{2l+2}-\alpha_s-2} +$$

$$\left. + \delta_1^{-2}\delta_2^{-2} \frac{\left(T^*_{m,s-2l+2}L^*_m\right)(x)}{m^{\alpha_{s-2l+2}}} \, \frac{\left(T^*_{m,2l+2}L^*_m\right)(y)}{m^{\alpha_{2l+2}}} \, m^{\alpha_{s-2l+2}+\alpha_{2l+2}-\alpha_s-4} \right] \cdot$$

$$\cdot \sum_{i=0}^{s} \binom{s}{i} \omega_{total} \left(\frac{\partial^s f}{\partial t^{s-i} \partial \tau^i}\, ; \delta_1, \delta_2 \right).$$

We have

$$\beta_s \leq -(\alpha_{s-2l+2} + \alpha_{2l} - \alpha_s - 2),$$

$$\beta_s \leq -\frac{1}{2}(\alpha_{s-2l+2} + \alpha_{2l+2} - \alpha_s - 4)$$

for any $l \in \left\{0, 1, \ldots, \frac{s}{2}\right\}$, from where $\beta_s + \alpha_{s-2l+2} + \alpha_{2l} - \alpha_s - 2 \leq 0$, $2\beta_s + \alpha_{s-2l+2} + \alpha_{2l+2} - \alpha_s - 4 \leq 0$, for any $l \in \left\{0, 1, \ldots, \frac{s}{2}\right\}$. Replacing l with $l+1$ in the relation $\beta_s + \alpha_{s-2l+2} + \alpha_{2l} - \alpha_s - 2 \leq 0$, we have $\beta_s + \alpha_{s-2l} + \alpha_{2l+2} - \alpha_s - 2 \leq 0$. From the first inequality from (2.4) and from the inequalities above, we have

$$m^{\alpha_{s-2l}+\alpha_{2l}-\alpha_s} \leq 1, \qquad m^{\beta_s+\alpha_{s-2l+2}+\alpha_{2l}-\alpha_s-2} \leq 1,$$
$$m^{\beta_s+\alpha_{s-2l}+\alpha_{2l+2}-\alpha_s-2} \leq 1, \quad m^{2\beta_s+\alpha_{s-2l+2}+\alpha_{2l+2}-\alpha_s-4} \leq 1,$$

where $l \in \left\{0, 1, \ldots, \frac{s}{2}\right\}$.

Considering $\delta_1 = \delta_2 = \dfrac{1}{\sqrt{m^{\beta_s}}}$, we have

$$\begin{aligned} |(R_{m,m}f)(x,y)| \leq \frac{1}{s!} \sum_{l=0}^{\frac{s}{2}} \binom{\frac{s}{2}}{l} \Bigg[& \frac{\left(T^*_{m,s-2l}L^*_m\right)(x)}{m^{\alpha_{s-2l}}} \frac{\left(T^*_{m,2l}L^*_m\right)(y)}{m^{\alpha_{2l}}} + \\ & + \frac{\left(T^*_{m,s-2l+2}L^*_m\right)(x)}{m^{\alpha_{s-2l+2}}} \frac{\left(T^*_{m,2l}L^*_m\right)(y)}{m^{\alpha_{2l}}} + \\ & + \frac{\left(T^*_{m,s-2l}L^*_m\right)(x)}{m^{\alpha_{s-2l}}} \frac{\left(T^*_{m,2l+2}L^*_m\right)(y)}{m^{\alpha_{2l+2}}} + \\ & + \frac{\left(T^*_{m,s-2l+2}L^*_m\right)(x)}{m^{\alpha_{s-2l+2}}} \frac{\left(T^*_{m,2l+2}L^*_m\right)(y)}{m^{\alpha_{2l+2}}} \Bigg] \cdot \\ & \cdot \sum_{i=0}^{s} \binom{s}{i} \omega_{total} \left(\frac{\partial^s f}{\partial t^{s-i} \partial \tau^i}; \frac{1}{\sqrt{m^{\beta_s}}}, \frac{1}{\sqrt{m^{\beta_s}}} \right). \end{aligned} \tag{3.17}$$

Taking (2.3) into account and considering the fact that

$$\lim_{m\to\infty} \omega_{total} \left(\frac{\partial^s f}{\partial t^{s-i} \partial \tau^i}; \frac{1}{\sqrt{m^{\beta_s}}}, \frac{1}{\sqrt{m^{\beta_s}}} \right) = \omega_{total} \left(\frac{\partial^s f}{\partial t^{s-i} \partial \tau^i}; 0, 0 \right) = 0,$$

$i \in \{0, 1, \ldots, s\}$, from (3.17) we have that

$$\lim_{m\to\infty} (R_{m,m}f)(x,y) = 0. \tag{3.18}$$

From (3.13) and (3.18), (3.9) follows.

If in addition (3.10) takes place, then (3.17) becomes

$$|(R_{m,m}f)(x,y)| \leq \frac{1}{s!}\sum_{l=0}^{\frac{s}{2}}\binom{\frac{s}{2}}{l}(k_{2l}+k_{2l+2})(k_{s-2l}+k_{s-2l+2})\cdot \tag{3.19}$$

$$\cdot\sum_{i=0}^{s}\binom{s}{i}\omega_{total}\left(\frac{\partial^s f}{\partial t^{s-i}\partial\tau^i};\frac{1}{\sqrt{m^{\beta_s}}},\frac{1}{\sqrt{m^{\beta_s}}}\right)$$

for any $m \in \mathbb{N}$, $m \geq m(s)$ and for any $(x,y) \in K \times K$, from which, the convergence from (3.9) is uniform on $K \times K$. From (3.13) and (3.19), (3.11) follows. □

Corollary 3.1. *Let $f : I \times I \to \mathbb{R}$ be a bivariate function.*

If $(x,y) \in (I \times I) \cap (J \times J)$ and f is continuous in (x,y), then

$$\lim_{m\to\infty}\left(L^*_{m,m}f\right)(x,y) = f(x,y). \tag{3.20}$$

If f is continuous on $(I \times I) \cap (J \times J)$, and there exists an interval $K \subset I \cap J$ such that there exist $m(0) \in \mathbb{N}$ and $k_2 \in \mathbb{R}$ depending on K so that for any $m \in \mathbb{N}$, $m \geq m(0)$ and any $x \in K$ we have that

$$\frac{\left(T^*_{m,2}L^*_m\right)(x)}{m^{\alpha_2}} \leq k_2, \tag{3.21}$$

then the convergence given in (3.20) is uniform on $K \times K$ and

$$\left|\left(L^*_{m,m}f\right)(x,y) - f(x,y)\right| \leq (1+k_2)^2\omega_{total}\left(f;\frac{1}{\sqrt{m^{2-\alpha_2}}},\frac{1}{\sqrt{m^{2-\alpha_2}}}\right), \tag{3.22}$$

for any $(x,y) \in K \times K$, any $m \in \mathbb{N}$, $m \geq m(0)$.

Proof. It results from Theorem 3.2 for $s = 0$ and one verifies immediately that $\beta_0 = 2 - \alpha_2$, $k_0 = 1$. □

In the Application 3.1 - 3.4, we consider that $p^*_{m,k} = p_{m,k}$, $m \in \mathbb{N}$ and $k \in \{0,1,\ldots,m\}$. By particularization and applying Theorem 3.2 and Corollary 3.1, we give convergence and approximation theorem for some bivariate operators. In all applications we give the convergence theorem for $s = 2$ and the approximation theorem for $s = 0$. In every application we have $\alpha_2 = 1$ and $k_0 = 1$.

Application 3.1. We consider $I = J = K = [0,1]$ and for any $m \in \mathbb{N}$, let the functionals $A_{m,k} : C([0,1]) \to \mathbb{R}$, $A_{m,k}(f) = f\left(\frac{k}{m}\right)$, for any $f \in C([0,1])$, $k \in \{0,1,\dots,m\}$. In this application, we obtain the Bernstein operators.

We have that

$$\left(T^*_{m,i}B_m\right)(x) = m^i \sum_{k=0}^{m} p_{m,k}(x)\left(\frac{k}{m} - x\right)^i = T_{m,i}(x), \tag{3.23}$$

$x \in [0,1]$, $m \in \mathbb{N}$, $i \in \mathbb{N}_0$,

$$B_j(x) = [x(1-x)]^{\left[\frac{j}{2}\right]}(a_j x + b_j), \tag{3.24}$$

$$\alpha_j = \left[\frac{j}{2}\right], \tag{3.25}$$

$j \in \mathbb{N}_0$, $x \in [0,1]$,

$$a_j = \begin{cases} 0, & \text{if } j \text{ is even or } j = 1 \\ -(j-1)!!\displaystyle\sum_{k=1}^{\left[\frac{j}{2}\right]} \frac{(2k-1)!!}{(2k-2)!!}, & \text{if } j \text{ is odd, } j \geq 3, \end{cases} \tag{3.26}$$

$$b_j = \begin{cases} 1, & \text{if } j = 0 \\ 0, & \text{if } j = 1 \\ (j-1)!!, & \text{if } j \text{ is even, } j \geq 2 \\ \dfrac{1}{2}(j-1)!!\displaystyle\sum_{k=1}^{\left[\frac{j}{2}\right]} \frac{(2k-1)!!}{(2k-2)!!}, & \text{if } j \text{ is odd, } j \geq 3, \end{cases} \tag{3.27}$$

and

$$k_{2l} = \left(\frac{1}{4}\right)^l b_{2l} + 1, \tag{3.28}$$

$l \in \mathbb{N}_0$ (see [9] and [12]).

Let $m, n \in \mathbb{N}$. The operator $B_{m,n} : C([0,1]\times[0,1]) \to C([0,1]\times[0,1])$ defined for any function $f \in C([0,1]\times[0,1])$ and any $(x,y) \in [0,1]\times[0,1]$ by

$$(B_{m,n}f)(x,y) = \sum_{k=0}^{m}\sum_{j=0}^{n} p_{m,k}(x)p_{n,j}(y) f\left(\frac{k}{m}, \frac{j}{n}\right) \tag{3.29}$$

is named the bivariate operator of Bernstein type.

On verify immediately that the condition (2.4), (3.10) and (3.22) take place and then Theorem 3.2 holds for the bivariate operators of Bernstein type.

We have that $T_{m,0}(x)=1$, $T_{m,1}(x)=0$, $T_{m,2}(x)=mx(1-x)$, $m\in\mathbb{N}$, $x\in[0,1]$ and then $k_2=\dfrac{1}{4}$ and $k_4=\dfrac{19}{16}$.

Theorem 3.3. *Let $f:[0,1]\times[0,1]\to\mathbb{R}$ be a bivariate function.*

(i) If $(x,y)\in[0,1]\times[0,1]$ and f admits partial derivatives of second order continuous in a neighborhood of the point (x,y), then

$$\lim_{m\to\infty} m\left[(B_{m,m}f)(x,y)-f(x,y)\right]= \tag{3.30}$$
$$=\frac{x(1-x)}{2}\,f''_{x^2}(x,y)+\frac{y(1-y)}{2}\,f''_{y^2}(x,y).$$

If f admits partial derivatives of second order continuous on $[0,1]\times[0,1]$, then the convergence given in (3.30) is uniform on $[0,1]\times[0,1]$.

(ii) If f is continuous on $[0,1]\times[0,1]$, then

$$|(B_{m,m}f)(x,y)-f(x,y)|\le\frac{25}{16}\,\omega_{total}\left(f;\frac{1}{\sqrt{m}},\frac{1}{\sqrt{m}}\right) \tag{3.31}$$

for any $(x,y)\in[0,1]\times[0,1]$, any $m\in\mathbb{N}$.

Application 3.2. We consider $I=J=K=[0,1]$. For any $m\in\mathbb{N}$, let the functionals $A_{m,k}:L_1([0,1])\to\mathbb{R}$, $A_{m,k}(f)=(m+1)\int\limits_0^1 p_{m,k}(t)f(t)dt$, for any $f\in L_1([0,1])$, $k\in\{0,1,\dots,m\}$. In this case, we obtain the Durrmeyer operators.

We have that

$$\left(T^*_{m,i}M_m\right)(x)=(-1)^i m^i(m+1)\sum_{k=0}^{m}p_{m,k}(x)\int\limits_0^1 p_{m,k}(t)(x-t)^i dt, \tag{3.32}$$

$x\in[0,1]$, $m\in\mathbb{N}$, $i\in\mathbb{N}_0$,

$$\alpha_j=\left[\frac{j}{2}\right], \tag{3.33}$$

$$B_j(x)=\begin{cases}\dfrac{j!}{\left(\frac{j}{2}\right)!}[x(1-x)]^{\frac{j}{2}}, & \text{if } j \text{ is even}\\[2ex] -\dfrac{(j+1)!}{2\left(\frac{j-1}{2}\right)!}(1-2x)[x(1-x)]^{\frac{j-1}{2}}, & \text{if } j \text{ is odd}\end{cases} \tag{3.34}$$

$j \in \mathbb{N}_0$, $x \in [0,1]$, and in the same way from Application 3.1

$$k_{2l} = \left(\frac{1}{4}\right)^l \frac{(2l)!}{l!} + 1, \tag{3.35}$$

$l \in \mathbb{N}_0$ (see [5] and [12]).

Let $m, n \in \mathbb{N}$. The operator $M_{m,n} : L_1([0,1] \times [0,1]) \to C([0,1] \times [0,1])$ defined for any function $f \in L_1([0,1] \times [0,1])$ and any $(x,y) \in [0,1] \times [0,1]$ by

$$(M_{m,n}f)(x,y) = \tag{3.36}$$

$$= (m+1)(n+1) \sum_{k=0}^{m} \sum_{j=0}^{n} p_{m,k}(x) p_{n,j}(y) \int_0^1 p_{m,k}(t) p_{n,j}(s) f(t,s) dt ds$$

is named the bivariate operator of Durrmeyer type.

The Theorem 3.2 holds for these operators.

We have

$$\left(T^*_{m,0} M_m\right)(x) = 1,$$

$$\left(T^*_{m,1} M_m\right)(x) = \frac{m(1-2x)}{m+2},$$

$$\left(T^*_{m,2} M_m\right)(x) = m^2 \frac{2(m-3)x(1-x)+2}{(m+2)(m+3)}, \quad m \in \mathbb{N},$$

$$B_0(x) = 1, \quad B_1(x) = 1-2x, \quad B_2(x) = 2x(1-x), \quad x \in [0,1],$$

$$k_2 = \frac{3}{2} \quad \text{and} \quad k_4 = \frac{7}{4}$$

(see [12]).

Theorem 3.4. *Let $f : [0,1] \times [0,1] \to \mathbb{R}$ be a bivariate function.*

(i) If $(x,y) \in [0,1] \times [0,1]$ and f admits partial derivatives of second order continuous in a neighborhood of the point (x,y), then

$$\lim_{m \to \infty} m\left[(M_{m,m}f)(x,y) - f(x,y)\right] = (1-2x) f'_x(x,y) + \tag{3.37}$$

$$+ (1-2y) f'_y(x,y) + x(1-x) f''_{x^2}(x,y) + y(1-y) f''_{y^2}(x,y).$$

If f admits partial derivatives of second order continuous on $[0,1] \times [0,1]$, then the convergence given in (3.37) is uniform on $[0,1] \times [0,1]$.

(ii) If f is continuous on $[0,1]\times[0,1]$, then

$$|(M_{m,m}f)(x,y)-f(x,y)|\leq\frac{25}{4}\,\omega_{total}\left(f;\frac{1}{\sqrt{m}},\frac{1}{\sqrt{m}}\right),\tag{3.38}$$

for any $(x,y)\in[0,1]\times[0,1]$, any natural number m, $m\geq3$.

Application 3.3. We consider $I=J=K=[0,1]$. For any $m\in\mathbb{N}$, let the functionals $A_{m,k}:L_1([0,1])\to\mathbb{R}$,

$$A_{m,k}(f)=(m+1)\int_{\frac{k}{m+1}}^{\frac{k+1}{m+1}}f(t)dt,$$

for any $f\in L_1([0,1])$, $k\in\{0,1,\ldots,m\}$. In this case, we obtain the Kantorovich operators.

We have

$$\left(T^*_{m,0}K_m\right)(x)=1,$$

$$\left(T^*_{m,1}K_m\right)=\frac{m}{2(m+1)}(1-2x),$$

$$\left(T^*_{m,2}K_m\right)(x)=\left(\frac{m}{m+1}\right)^2\frac{(1-x)^3+x^3+3mx(1-x)}{3},\quad m\in\mathbb{N},\quad x\in[0,1],$$

$$B_0(x)=1,\quad B_1(x)=\frac{1-2x}{2},\quad B_2(x)=x(1-x),\quad x\in[0,1],$$

$k_2=1$ and $k_4=\dfrac{3}{2}$ (see [12]).

Let $m,n\in\mathbb{N}$. The operator $K_{m,n}:L_1([0,1]\times[0,1])\to C([0,1]\times[0,1])$ defined for any function $f\in L_1([0,1]\times[0,1])$ and any $(x,y)\in[0,1]\times[0,1]$ by

$$(K_{m,n}f)(x,y)=\tag{3.39}$$
$$=(m+1)(n+1)\sum_{k=0}^{n}\sum_{j=0}^{n}p_{m,k}(x)p_{n,j}(y)\int_{\frac{k}{m+1}}^{\frac{k+1}{m+1}}\int_{\frac{j}{n+1}}^{\frac{j+1}{n+1}}f(t,s)dtds$$

is named the bivariate operator of Kantorovich type.

Theorem 3.5. *Let $f : [0,1] \times [0,1] \to \mathbb{R}$ be a bivariate function.*

(i) If $(x,y) \in [0,1] \times [0,1]$ and f admits partial derivatives of second order continuous in a neighborhood of the point (x,y), then

$$\lim_{m\to\infty} m\left[(K_{m,m}f)(x,y) - f(x,y)\right] = \frac{1-2x}{2} f'_x(x,y) + \tag{3.40}$$
$$+ \frac{1-2y}{2} f'_y(x,y) + \frac{x(1-x)}{2} f''_{x^2}(x,y) + \frac{y(1-y)}{2} f''_{y^2}(x,y).$$

If f admits partial derivatives of second order continuous on $[0,1] \times [0,1]$, then the convergence in (3.40) is uniform on $[0,1] \times [0,1]$.

(ii) If f is continuous on $[0,1] \times [0,1]$, then

$$|(K_{m,m}f)(x,y) - f(x,y)| \leq 4\omega_{total}\left(f; \frac{1}{\sqrt{m}}, \frac{1}{\sqrt{m}}\right), \tag{3.41}$$

for any $(x,y) \in [0,1] \times [0,1]$, any natural number m, $m \geq 3$.

Application 3.4. Let $I = [0, \mu^{(\alpha,\beta)}]$, $J = K = [0,1]$ (see (1.7)-(1.10)).
For any $m \in \mathbb{N}$, $m \geq m_0$, let the functionals $A_{m,k} : C([0,\mu^{(\alpha,\beta)}]) \to \mathbb{R}$,

$$A_{m,k}(f) = f\left(\frac{k+\alpha}{m+\beta}\right),$$

for any $f \in C([0,\mu^{(\alpha,\beta)}])$, $k \in \{0,1,\ldots,m\}$. In this case, we obtain the Stancu operators.

We have that

$$\left(T^*_{m,0}P_m^{(\alpha,\beta)}\right)(x) = 1,$$

$$\left(T^*_{m,1}P_m^{(\alpha,\beta)}\right)(x) = \frac{m(\alpha-\beta x)}{m+\beta},$$

$$\left(T^*_{m,2}P_m^{(\alpha,\beta)}\right)(x) = \frac{m^2[mx(1-x)+(\alpha-\beta x)^2]}{(m+\beta)^2}, \quad m \in \mathbb{N}, \quad m \geq m_0,$$

$$B_0(x) = 1, \quad B_1(x) = \alpha - \beta x, \quad B_2(x) = x(1-x), \quad x \in [0,1].$$

There exists a natural number $m(0)$ such that $\dfrac{\left(T^*_{m,2}P^{(\alpha,\beta)}\right)(x)}{m} \leq \dfrac{5}{4} = k_2$ for any natural number m, $m \geq m(0)$, any $x \in [0,1]$ and $k_4 = 1$ (see [13]).

For the real numbers $\alpha_1, \alpha_2, \beta_1, \beta_2$ with $\alpha_1 \geq 0$ and $\alpha_2 \geq 0$, m_1, m_2, $\mu^{(\alpha_1,\beta_1)}$ and $\mu^{(\alpha_2,\beta_2)}$ are defined through

$$m_i = \begin{cases} \max\{1, -[\beta_i]\}, & \text{if} \quad \beta_i \in \mathbb{R}\backslash\mathbb{Z} \\ \max\{1, 1-\beta_i\}, & \text{if} \quad \beta_i \in \mathbb{Z} \end{cases}, \tag{3.42}$$

$$\gamma_{\beta_i} = m_i + \beta_i = \begin{cases} \max\left\{1+\beta_i, \{\beta_i\}\right\}, & \text{if} \quad \beta_i \in \mathbb{R}\backslash\mathbb{Z} \\ \max\{1+\beta_i, 1\}, & \text{if} \quad \beta_i \in \mathbb{Z} \end{cases}, \tag{3.43}$$

$$\mu^{(\alpha_i,\beta_i)} = \begin{cases} 1, & \text{if} \quad \alpha_i \leq \beta_i \\ 1 + \dfrac{\alpha_i - \beta_i}{\gamma_{\beta_i}}, & \text{if} \quad \alpha_i > \beta_i \end{cases}, \tag{3.44}$$

where $i \in \{1, 2\}$.

Let the bivariate operators $P_{m,n}^{(\alpha_1,\beta_1)(\alpha_2,\beta_2)} : C([0, \mu^{(\alpha_1,\beta_1)}] \times [0, \mu^{(\alpha_2,\beta_2)}]) \to C([0,1] \times [0,1])$ defined for any function $f \in C([0, \mu^{(\alpha_1,\beta_1)}] \times [0, \mu^{(\alpha_2,\beta_2)}])$ by

$$\left(P_{m,n}^{(\alpha_1,\beta_1)(\alpha_2,\beta_2)} f\right)(x,y) = \sum_{k=0}^{m} \sum_{j=0}^{n} p_{m,k}(x) p_{n,j}(y) f\left(\frac{k+\alpha_1}{m+\beta_1}, \frac{j+\alpha_2}{n+\beta_2}\right), \tag{3.45}$$

for any $(x,y) \in [0,1] \times [0,1]$ and any natural numbers m, n, $m \geq m_1$ and $n \geq m_2$.

These operators are named the bivariate operators of Stancu type.

Theorem 3.6. *Let $f : [0, \mu^{(\alpha_1,\beta_1)}] \times [0, \mu^{(\alpha_2,\beta_2)}] \to \mathbb{R}$ be a bivariate function.*

(i) If $(x,y) \in [0,1] \times [0,1]$ and f admits partial derivatives of second order continuous in a neighborhood of the point (x,y), then

$$\begin{aligned} &\lim_{m\to\infty} m\left[\left(P_{m,m}^{(\alpha_1,\beta_1)(\alpha_2,\beta_2)} f\right)(x,y) - f(x,y)\right] = \\ &= (\alpha_1 - \beta_1 x) f'_x(x,y) + (\alpha_2 - \beta_2 y) f'_y(x,y) + \\ &+ \frac{x(1-x)}{2} f''_{x^2}(x,y) + \frac{y(1-y)}{2} f''_{y^2}(x,y). \end{aligned} \tag{3.46}$$

If f admits partial derivatives of second order continuous on $[0,1] \times [0,1]$, then the convergence given in (3.46) is uniform on $[0,1] \times [0,1]$.

(ii) If f is continuous on $[0,1] \times [0,1]$, then

$$\left|\left(P_{m,m}^{(\alpha_1,\beta_1)(\alpha_2,\beta_2)} f\right)(x,y) - f(x,y)\right| \leq \frac{81}{16}\, \omega_{total}\left(f; \frac{1}{\sqrt{m}}, \frac{1}{\sqrt{m}}\right), \tag{3.47}$$

for any $(x, y) \in [0, 1] \times [0, 1]$, *any natural number* m, $m \geq m(0)$.

For the particular case from this applications (see [13]), we obtain the Voronovskaja's type theorem and approximation theorem for the bivariate operator of Bernstein, Schurer and Schurer-Stancu.

Application 3.5. In this application, $I = J = [0, \infty)$ and for any $m \in \mathbb{N}$, we consider $p^*_{m,k}(x) = \binom{m}{k} \dfrac{x^k}{(1+x)^m}$ for any $x \in [0, \infty)$, $k \in \{0, 1, \ldots, m\}$, the functionals $A_{m,k} : C_B([0, \infty)) \to \mathbb{R}$, $A_{m,k}(f) = f\left(\dfrac{k}{m+1-k}\right)$ defined for any $f \in C_B([0, \infty))$ and $k \in \{0, 1, \ldots, m\}$. We obtain the Bleimann, Butzer and Hahn operators.

We have

$$\left(T^*_{m,0} L_m\right)(x) = 1,$$

$$\left(T^*_{m,1} L_m\right)(x) = -mx\left(\frac{x}{1+x}\right)^m, \quad m \in \mathbb{N},$$

$$B_0(x) = 1, \quad B_1(x) = 0, \quad B_2(x) = x(1+x)^2, \quad x \in [0, \infty),$$

$$k_2 = 4b(1+b)^2,$$

where $K = [0, b]$, $b > 0$ and $m(0) = 24(1+b)$ (see [14]).

Let $m, n \in \mathbb{N}$. The operator $L_{m,n} : C_B([0, \infty) \times [0, \infty)) \to C_B([0, \infty) \times [0, \infty))$ defined for any function $f \in C_B([0, \infty) \times [0, \infty))$ and any $(x, y) \in [0, \infty) \times [0, \infty)$ by

$$(L_{m,n} f)(x, y) = \tag{3.48}$$
$$= \frac{1}{(1+x)^m (1+y)^n} \sum_{k=0}^{m} \sum_{j=0}^{n} \binom{m}{k} \binom{n}{j} f\left(\frac{k}{m+1-k}, \frac{j}{n+1-j}\right)$$

is named the bivariate operator of Bleimann-Butzer-Hahn type.

Theorem 3.7. *Let* $f : [0, \infty) \times [0, \infty) \to \mathbb{R}$ *be a bivariate function.*

(i) If $(x, y) \in [0, \infty) \times [0, \infty)$ *and* f *admits partial derivatives of second order continuous in a neighborhood of the point* (x, y), *then*

$$\lim_{m \to \infty} m \left[(L_{m,m} f)(x, y) - f(x, y)\right] = \tag{3.49}$$
$$= \frac{x(1+x)^2}{2} f''_{x^2}(x, y) + \frac{y(1+y)^2}{2} f''_{y^2}(x, y).$$

(ii) If f is continuous on $[0,\infty)\times[0,\infty)$ and $b>0$, then

$$|(L_{m,m}f)(x,y)-f(x,y)|\leq \tag{3.50}$$
$$\leq\left[1+8b(1+b)^2+16b^2(1+b)^4\right]\omega_{total}\left(f;\frac{1}{\sqrt{m}},\frac{1}{\sqrt{m}}\right),$$

for any $(x,y)\in[0,b]$, any natural number m, $m\geq 24(1+b)$.

Remark 3.1. From the Theorem 3.2 - 3.7, for (ii) results the uniform convergence of the bivariate operators.

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ON α-CONVEX ANALYTIC FUNCTIONS DEFINED BY GENERALIZED RUSCHEWEYH DERIVATIVES OPERATOR

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Abstract. In this paper we introduce a class of alpha-convex functions by using the generalised Ruscheweyh derivative operator. We study properties of this class and give a theorem about the image of a function from this class through the Bernardi integral operator.

1. Introduction

Let $\mathcal{H} = \mathcal{H}(U)$ denote the class of functions analytic in $U = \{z \in \mathbb{C} : |z| < 1\}$. For n a positive integer and $a \in \mathbb{C}$, let

$$\mathcal{H}[a, n] = \{f \in \mathcal{H} : f(z) = a + a_n z^n + ...\}.$$

We also consider the class

$$\mathcal{A} = \left\{f \in \mathcal{H} : f(z) = z + a_2 z^2 + ...\right\}.$$

We denote by $\mathcal{Q}$ the set of functions f that are analytic and injective on $\overline{U} \setminus E(f)$, where

$$E(f) = \left\{\zeta \in \partial U : \lim_{z \to \zeta} f(z) = \infty\right\},$$

and are such that $f'(\zeta) \neq 0$ for $\zeta \in \partial U \setminus E(f)$.

Since the functions considered in this paper and conditions on them are defined uniformly in the unit disk U, we shall omit the requirement "$z \in U$".

We use the terms of subordination and superordination, so we review here those definitions. Let $f, F \in \mathcal{H}$. The function f is said to be *subordinate* to F, or

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F is said to be *superordinate* to f, if there exists a function w analytic in U, with $w(0) = 0$ and $|w(z)| < 1$, and such that $f(z) = F(w(z))$. In such a case we write $f \prec F$ or $f(z) \prec F(z)$. If F is univalent, then $f \prec F$ if and only if $f(0) = F(0)$ and $f(U) \subset F(U)$.

Let $\psi : \mathbb{C}^3 \times U \to \mathbb{C}$, let h be a univalent function in U and $q \in \mathcal{Q}$. In [7], the authors considered the problem of determining conditions on admissible functions ψ such that

$$\psi\left(p(z), zp'(z), z^2p''(z); z\right) \prec h(z) \tag{1}$$

implies $p(z) \prec q(z)$, for all functions $p \in \mathcal{H}[a,n]$ that satisfy the differential subordination (1). Moreover, they found conditions such that the function q is the "smallest" function with this property, called the best dominant of the subordination (1).

Let $\varphi : \mathbb{C}^3 \times \overline{U} \to \mathbb{C}$, let $h \in \mathcal{H}$ and $q \in \mathcal{H}[a,n]$. Recently, in [8], the authors studied the dual problem and determined conditions on φ such that

$$h(z) \prec \varphi\left(p(z), zp'(z), z^2p''(z); z\right) \tag{2}$$

implies $q(z) \prec p(z)$, for all functions $p \in \mathcal{Q}$ that satisfy the above differential superordination. Moreover, they found conditions such that the function q is the "largest" function with this property, called the best subordinant of the superodination (2).

In the present paper we shall also need a recent generalization of the Ruscheweyh derivatives. This was introduced in the paper [3].

Let $f \in \mathcal{A}$, $\lambda \geq 0$ and $m \in \mathbb{R}$, $m > -1$, then we consider

$$\mathcal{D}_\lambda^m f(z) = \frac{z}{(1-z)^{m+1}} * \mathcal{D}_\lambda f(z), \; z \in U,$$

where $\mathcal{D}_\lambda f(z) = (1-\lambda) f(z) + \lambda z f'(z)$, $z \in U$.

If $f \in \mathcal{A}$, $f(z) = z + \sum\limits_{n=2}^{\infty} a_n z^n$, $z \in U$, we obtain the power series expansion of the form

$$\mathcal{D}_\lambda^m f(z) = z + \sum_{n=2}^{\infty} [1 + (n-1)\lambda] \frac{(m+1)_{n-1}}{(1)_{n-1}} a_n z^n, \; z \in U,$$

where $(a)_n$ is the Pochhammer symbol, given by

$$(a)_n := \frac{\Gamma(a+n)}{\Gamma(a)} = \begin{cases} 1, & \text{for } n = 0 \\ a(a+1)(a+2)\dots(a+n-1), & \text{for } n \in \mathbb{N}^*. \end{cases}$$

In the case $m \in \mathbb{N}$, we have

$$\mathcal{D}_\lambda^m f(z) = \frac{z\left(z^{m-1}\mathcal{D}_\lambda f(z)\right)^{(m)}}{m!},\ z \in U,$$

and for $\lambda = 0$ we obtain the m-th Ruscheweyh derivative introduced in [12], $\mathcal{D}_0^m = \mathcal{D}^m$.

We next introduce the two classes of α-convex functions by using the generalized Ruscheweyh derivatives.

Definition 1.1. *Let q be a univalent function in U, with $q(0) = 1$ and such that $D = q(U)$ is a convex domain from the right half-plane. We consider $\alpha \in [0,1]$, $\lambda \geq 0$ and $m \in \mathbb{N}^*$. The function $f \in \mathcal{A}$ is said to be in the class*

(i) $M_\alpha(m,\lambda,q)$, *if*

$$J(\alpha,m,\lambda,f;z) = (1-\alpha)\frac{z\left(\mathcal{D}_\lambda^m f(z)\right)'}{\mathcal{D}_\lambda^m f(z)} + \alpha\frac{\left(z\left(\mathcal{D}_\lambda^m f(z)\right)'\right)'}{\left(\mathcal{D}_\lambda^m f(z)\right)'} \prec q(z),$$

for $z \in U$, or, equivalently,

$$J(\alpha,m,\lambda,f;z) = (1-\alpha)\frac{z\left(\mathcal{D}_\lambda^m f(z)\right)'}{\mathcal{D}_\lambda^m f(z)} + \alpha\left(1 + \frac{z\left(\mathcal{D}_\lambda^m f(z)\right)''}{\left(\mathcal{D}_\lambda^m f(z)\right)'}\right) \prec q(z).$$

(ii) $\overline{M}_\alpha(m,\lambda,q)$, *if*

$$q(z) \prec J(\alpha,m,\lambda,f;z).$$

Subclasses of $M_\alpha(m,\lambda,q)$ were studied by several authors, out of which we mention

$$M_0(0,0,q) = S^*(q),$$

$$M_\alpha(0,0,q) = M_\alpha(q),$$

$$M_0(0,0,q_\gamma) = S^*(\gamma),\ \text{where } q_\gamma(z) = \frac{1+(1-2\gamma)z}{1-z},\ 0 \leq \gamma < 1,$$

$$M_\alpha(0,0,\varphi) = M_\alpha,\ \text{for } \varphi(z) = \frac{1+z}{1-z},$$

$$M_0(m,0,\varphi) = R_n,$$

$$M_0(m,0,q_\gamma) = R_n(\gamma).$$

The class $S^*(q)$ was introduced by W. Ma and D. Minda in [5], the class $M_\alpha(q)$ was studied by V. Ravichandran and M. Darus in [11], M_α is the class of α-convex functions introduced by P.T. Mocanu in [10], R_n is the class defined by R. Singh and S. Singh in [13], and $R_n(\gamma)$ makes the object of the papers of O.P. Ahuja, [1] and [2].

We shall use the following notations

$$S^*_{m,\lambda}(q) = \left\{ f \in \mathcal{A} : \frac{z\left(\mathcal{D}^m_\lambda f(z)\right)'}{\mathcal{D}^m_\lambda f(z)} \prec q(z), z \in U \right\}$$

and

$$\overline{S}^*_{m,\lambda}(q) = \left\{ f \in \mathcal{A} : q(z) \prec \frac{z\left(\mathcal{D}^m_\lambda f(z)\right)'}{\mathcal{D}^m_\lambda f(z)}, z \in U \right\}.$$

2. **Preliminaries**

In our present investigation we shall need the folllowing results concerning Briot-Bouquet differential subordinations, and generalizations of Briot-Bouquet differential subordinations and superordinations.

Theorem 2.1 ([4]). *Let $\beta, \gamma \in \mathbb{C}$, $\beta \neq 0$ and consider the convex function h, such that*

$$\operatorname{Re}[\beta h(z) + \gamma] > 0,\ z \in U.$$

If $p \in \mathcal{H}[h(0), n]$, then

$$p(z) + \frac{zp'(z)}{\beta p(z) + \gamma} \prec h(z) \Rightarrow p(z) \prec h(z).$$

Theorem 2.2 ([6]). *Let q be a univalent function in U and consider θ and φ to be analytic functions in a domain $D \supset q(U)$, such that $\varphi(w) \neq 0$, for all $w \in q(U)$. We denote by $Q(z) = zq'(z) \cdot \varphi[q(z)]$, $h(z) = \theta[q(z)] + Q(z)$ and assume that*

(i) *h is convex, or*

(ii) *Q is starlike.*

We further suppose that

(iii) $\operatorname{Re} \dfrac{zh'(z)}{Q(z)} = \operatorname{Re}\left[\dfrac{\theta'[q(z)]}{\varphi[q(z)]} + \dfrac{zQ'(z)}{Q(z)}\right] > 0.$

If p is an analytic function in U, with $p(0) = q(0)$, $p(U) \subseteq D$ and such that

$$\theta[p(z)] + zp'(z)\varphi[p(z)] \prec \theta[q(z)] + zq'(z)\varphi[q(z)] = h(z)$$

then

$$p(z) \prec q(z)$$

and q is the best dominant.

Theorem 2.3 ([9])**.** *Let θ, φ be analytic functions in a domain D and consider q a univalent function in U, such that $q(0) = a$, $q(U) \subset D$. We define $Q(z) = zq'(z) \cdot \varphi[q(z)]$, $h(z) = \theta[q(z)] + Q(z)$ and suppose that*

(i) $\operatorname{Re}\left[\dfrac{\theta'[q(z)]}{\varphi[q(z)]}\right] > 0$ *and*

(ii) *Q is starlike.*

If $p \in \mathcal{H}[a,1] \cap \mathcal{Q}$, $p(U) \subset D$ and $\theta[p(z)] + zp'(z) \cdot \varphi[p(z)]$ is univalent in U, then

$$\theta[q(z)] + zq'(z) \cdot \varphi[q(z)] \prec \theta[p(z)] + zp'(z) \cdot \varphi[p(z)] \Rightarrow q(z) \prec p(z)$$

and q is the best subordinant.

3. Main results

Theorem 3.1. *Let $\alpha \in [0,1]$. Then $f \in M_\alpha(m, \lambda, q)$ if and only if the function g defined by*

$$g(z) = \mathcal{D}_\lambda^m f(z) \left[\frac{z(\mathcal{D}_\lambda^m f(z))'}{\mathcal{D}_\lambda^m f(z)}\right]^\alpha, \ z \in U$$

belongs to $S_{m,\lambda}^*(q)$. The branch of the power function is chosen such that

$$\left.\left[\frac{z(\mathcal{D}_\lambda^m f(z))'}{\mathcal{D}_\lambda^m f(z)}\right]^\alpha\right|_{z=0} = 1.$$

Proof. We calculate the logarithmic derivative of g and obtain

$$\frac{zg'(z)}{g(z)} = \frac{z(\mathcal{D}_\lambda^m f(z))'}{\mathcal{D}_\lambda^m f(z)} + \alpha\left[1 + \frac{z(\mathcal{D}_\lambda^m f(z))''}{(\mathcal{D}_\lambda^m f(z))'} - \frac{z(\mathcal{D}_\lambda^m f(z))'}{\mathcal{D}_\lambda^m f(z)}\right],$$

or

$$\frac{zg'(z)}{g(z)} = J(\alpha, m, \lambda, f; z).$$

The equivalence from the hypothesis is immediately verified. $\square$

Theorem 3.2. *If the function f belongs to the class $M_\alpha(m,\lambda,q)$, for a given $\alpha \in (0,1]$, then $f \in S^*_{m,\lambda}(q)$.*

Proof. We define the function p to be given by

$$p(z) = \frac{z\left(\mathcal{D}_\lambda^m f(z)\right)'}{\mathcal{D}_\lambda^m f(z)}.$$

The logarithmic derivative of p is

$$\frac{zp'(z)}{p(z)} = 1 + \frac{z\left(\mathcal{D}_\lambda^m f(z)\right)''}{\left(\mathcal{D}_\lambda^m f(z)\right)'} - \frac{z\left(\mathcal{D}_\lambda^m f(z)\right)'}{\mathcal{D}_\lambda^m f(z)},$$

thus

$$p(z) + \alpha\frac{zp'(z)}{p(z)} = J(\alpha, m, \lambda, f; z).$$

Because $f \in M_\alpha(m,\lambda,q)$, we get

$$p(z) + \alpha\frac{zp'(z)}{p(z)} \prec q(z).$$

The function q was supposed to be convex and we also assumed that the image $q(U)$ is in the right half-plane. We have $\alpha \in (0,1]$, and therefore

$$\operatorname{Re}\left[\frac{1}{\alpha}q(z)\right] > 0, z \in U.$$

By applying Theorem 2.1 for $\beta = \dfrac{1}{\alpha}$ and $\gamma = 0$ we conclude that $p(z) \prec q(z)$, and thus $f \in S^*_{m,\lambda}(q)$. □

Let a be a complex number such that $\operatorname{Re} a > 0$ and $f \in \mathcal{A}$. We also consider the Bernardi integral operator given by

$$F(f)(z) = \frac{1+a}{z^a}\int_0^z f(t)\, t^{a-1}dt. \tag{3}$$

Theorem 3.3. *If $f \in M_\alpha(m,\lambda,q)$, then $F \in S^*_{m,\lambda}(q)$.*

Proof. We calculate the derivative of F from the relation (3) and obtain

$$(1+a) f(z) = aF(z) + zF'(z). \tag{4}$$

Then we apply the generalized Ruscheweyh derivatives operator to both terms in (4), and we get

$$(1+a)\mathcal{D}_\lambda^m f(z) = a\mathcal{D}_\lambda^m F(z) + \mathcal{D}_\lambda^m\left(zF'(z)\right). \tag{5}$$

If the analytic function f has a Taylor series expansion of the form

$$f(z) = z + \sum_{n=2}^{n} a_n z^n,$$

then

$$F(z) = z + \sum_{n=2}^{n} b_n z^n,$$

where $b_n = \dfrac{1+a}{a+n} a_n$, for all $n \geq 2$. We have

$$\mathcal{D}_\lambda^m (zF'(z)) = z + \sum_{n=2}^{\infty} [1 + \lambda(n-1)] C(m,n) n b_n z^n.$$

By applying the generalized Ruscheweyh derivatives operator to F, we obtain

$$\mathcal{D}_\lambda^m F(z) = z + \sum_{n=2}^{\infty} [1 + \lambda(n-1)] C(m,n) b_n z^n,$$

and from here we conclude that

$$z(\mathcal{D}_\lambda^m F(z))' = z + \sum_{n=2}^{\infty} [1 + \lambda(n-1)] C(m,n) n b_n z^n.$$

Therefore the following equality

$$\mathcal{D}_\lambda^m (zF'(z)) = z(\mathcal{D}_\lambda^m F(z))'$$

is satisfied. The equation (5) becomes

$$(1+a)\mathcal{D}_\lambda^m f(z) = a\mathcal{D}_\lambda^m F(z) + z(\mathcal{D}_\lambda^m F(z))'. \tag{6}$$

We calculate the derivative of both terms in (6), we multiply with z and obtain

$$(1+a) z(\mathcal{D}_\lambda^m f(z))' = (a+1) z(\mathcal{D}_\lambda^m F(z))' + z^2 (\mathcal{D}_\lambda^m F(z))''. \tag{7}$$

We divide the identity (7) to the relation (6) and have

$$\frac{z(\mathcal{D}_\lambda^m f(z))'}{\mathcal{D}_\lambda^m f(z)} = \frac{z(\mathcal{D}_\lambda^m F(z))'}{\mathcal{D}_\lambda^m F(z)} \frac{a + 1 + \dfrac{z(\mathcal{D}_\lambda^m F(z))''}{(\mathcal{D}_\lambda^m F(z))'}}{a + \dfrac{z(\mathcal{D}_\lambda^m F(z))'}{\mathcal{D}_\lambda^m F(z)}},$$

or, by using the notation

$$P(z) := \frac{z(\mathcal{D}_\lambda^m F(z))'}{\mathcal{D}_\lambda^m F(z)} \text{ and } p(z) := \frac{z(\mathcal{D}_\lambda^m f(z))'}{\mathcal{D}_\lambda^m f(z)},$$

$$p(z)=P(z)\frac{a+\dfrac{zP'(z)}{P(z)}+P(z)}{a+P(z)}=P(z)+\frac{zP'(z)}{a+P(z)},$$

Because $f\in M_{\alpha}(m,\lambda,q)$, by applying Theorem cat o fi, we get $f\in S^{*}_{m,\lambda}(q)$, or

$$p(z)=\frac{z\left(\mathcal{D}^{m}_{\lambda}f(z)\right)'}{\mathcal{D}^{m}_{\lambda}f(z)}\prec q(z).$$

The subordination

$$P(z)+\frac{zP'(z)}{a+P(z)}\prec q(z)$$

holds. Because q is a convex function and $\operatorname{Re}[a+q(z)]>0$, from Theorem 2.1 with $\beta=1$ and $\gamma=a$ we can conclude that

$$P(z)\prec q(z),$$

or

$$\frac{z\left(\mathcal{D}^{m}_{\lambda}F(z)\right)'}{\mathcal{D}^{m}_{\lambda}F(z)}\prec q(z),$$

and thus $F\in S^{*}_{m,\lambda}(q)$. □

Theorem 3.4. *Let q be a convex function in$U$, with $q(0)=1$ and $\operatorname{Re}q(z)>0$. We consider $Q(z)=\alpha\dfrac{zq'(z)}{q(z)}$ and $h(z)=q(z)+Q(z)$, $z\in U$. If Q is a convex function in U and $f\in M_{\alpha}(m,\lambda,h)$ for an $\alpha\in(0,1]$, then $f\in S^{*}_{m,\lambda}(q)$.*

Proof. We choose the functions θ and φ to be $\theta(w)=w$, $\varphi(w)=\dfrac{\alpha}{w}$ and notice that the hypothesis of Theorem 2.2 are satisfied. It follows that $\dfrac{z\left(\mathcal{D}^{m}_{\lambda}f(z)\right)'}{\mathcal{D}^{m}_{\lambda}f(z)}\prec q(z)$ and q is the best dominant. Therefore $f\in S^{*}_{m,\lambda}(q)$. □

Theorem 3.5. *Let q be a convex function in U, with $q(0)=1$ and $\operatorname{Re}q(z)>0$. We consider $Q(z)=\alpha\dfrac{zq'(z)}{q(z)}$ and $h(z)=q(z)+Q(z)$, $z\in U$. If Q is convex in U, f belongs to the class $\overline{M}_{\alpha}(m,\lambda,h)$ for an $\alpha\in(0,1]$, $\dfrac{z\left(\mathcal{D}^{m}_{\lambda}f(z)\right)'}{\mathcal{D}^{m}_{\lambda}f(z)}\in\mathcal{H}[1,1]\cap\mathcal{Q}$ and $J(\alpha,m,\lambda,f;z)$ is univalent in U, then $f\in\overline{S}^{*}_{m,\lambda}(q)$.*

Proof. We choose $\theta(w)=w$, $\varphi(w)=\dfrac{\alpha}{w}$ and notice that the conditions of theorem 2.3 are satisfied. It follows that $q(z)\prec\dfrac{z\left(\mathcal{D}^{m}_{\lambda}f(z)\right)'}{\mathcal{D}^{m}_{\lambda}f(z)}$ and q is the best subordinant. Therefore $f\in\overline{S}^{*}_{m,\lambda}(q)$. □

Corollary 3.6. *For $k=1,2$, let q_k be two convex functions in U, with $q_k(0)=1$ and $\operatorname{Re} q_k(z)>0$. We consider $Q_k(z)=\alpha\dfrac{zq_k'(z)}{q_k(z)}$ and $h_k(z)=q_k(z)+Q_k(z)$, $z\in U$. If Q_k are convex in U, $f\in\overline{M}_\alpha(m,\lambda,h_1)\cap M_\alpha(m,\lambda,h_2)$ for an $\alpha\in(0,1]$, $\dfrac{z\left(\mathcal{D}_\lambda^m f(z)\right)'}{\mathcal{D}_\lambda^m f(z)}\in\mathcal{H}[1,1]\cap\mathcal{Q}$ and $J(\alpha,m,\lambda,f;z)$ is univalent in U, then $f\in\overline{S}^*_{m,\lambda}(q_1)\cap S^*_{m,\lambda}(q_2)$.*

We will give an example by taking $q_1(z)=1+\beta z$, $\beta\in\mathbb{C}^*$, $|\beta|\le 1$ and $q_2(z)=1+z$, $z\in U$. The functions $Q_1(z)=\dfrac{\beta z}{1+\beta z}$, $Q_2(z)=\dfrac{z}{1+z}$, $z\in U$ are convex in this case, and $h_1(z)=1+\beta z+\dfrac{\beta z}{1+\beta z}$, $h_2(z)=1+z+\dfrac{z}{1+z}$, $z\in U$ are also convex and have positive real part.

Example 3.7. *Let $\beta\in\mathbb{C}^*$, $|\beta|\le 1$, and $f\in\mathcal{A}$ such that*

$$1+\beta z+\frac{\beta z}{1+\beta z}\prec J(\alpha,m,\lambda,f;z)\prec 1+z+\frac{z}{1+z},\ z\in U.$$

Then

$$1+\beta z\prec\frac{z\left(\mathcal{D}_\lambda^m f(z)\right)'}{\mathcal{D}_\lambda^m f(z)}\prec 1+z,\ z\in U.$$

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HEAT TRANSFER IN AXISYMMETRIC STAGNATION FLOW ON A THIN CYLINDER

CORNELIA REVNIC, TEODOR GROŞAN, AND IOAN POP

Abstract. The steady axisymetric stagnation flow and heat transfer on a thin infinite cylinder of radius a is studied in this paper. Both cases of constant wall temperature and constant wall heat flux are considered. Using similarity variables the governing partial differential equations are transformed into ordinary differential equations. The resulting set of two equations is solved numerically using Runge-Kutta method combined with a shooting technique. For the special case of the Reynolds number $Re >> 1$ (boundary layer approximation), we obtained an asymptotic solution which include the Hiemenz solution. The present results are compared in some particular cases with existing results from the open literature and with the asymptotic approximation, and we found a very good agreement. It is shown that the Nusselt number and the skin friction increase and the boundary layer thickness decreases with the increase of the Reynolds number. Some graphs for the velocity and temperature profiles are presented. Also, tables with values related to the skin friction and Nusselt number are given.

1. **Introduction**

The two-dimensional orthogonal stagnation-point flow of a viscous fluid impinging on a flat wall is a very interesting problem in the history of fluid mechanics. This flow appears in virtually all flow fields of engineering and scientific interest. Hiemenz [1] was the first who derived an exact solution of the Navier-Stokes equations

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which describes the steady forced convection flow directed perpendicular (orthogonal) to an infinite flat plate. Homann [2] studied the axisymmetric stagnation flow, also against a plate, and Howarth [3] and Davey [4] extended the results to unsymmetric cases. Later, Wang [5] presented an exact solution for the steady axisymmetric stagnation-point flow on an infinite thin circular cylinder. Gorla [6] has then considered the steady boundary layer heat transfer in an axisymmetric stagnation-point flow on an infinite thin circular cylinder. Both the cases of constant wall temperature and constant wall heat flux at the surface of the cylinder were considered. Numerical results for the velocity and temperature profiles as well as for the local Nusselt number were obtained when the Reynolds number is relatively small. Further, Gorla [7] has investigated the unsteady fluid dynamic characteristics of an axisymmetric stagnation point flow on a circular cylinder performing an harmonic motion in its own plane. Also, Gorla [8] has investigated the final approach to steady state in an axisymmetric stagnation-point flow on a thin circular cylinder.

The aim of this paper is to extend the paper by Gorla [6] on heat transfer in axisymmetric stagnation point flow on a thin infinite circular cylinder to the case when the Reynolds number is large.

2. Basic equations

Consider the steady-state flow and heat transfer at an axisymmetric stagnation point on a thin circular cylinder of radius a placed in a viscous and incompressible fluid of ambient uniform temperature T_∞, as shown in Fig. 1. The flow is axisymmetric about z- axis and also symmetric to the $z = 0$ plane. It is assumed that both the temperature of the surface of the cylinder T_w or the heat flux from the surface of the cylinder q_w are constants. Under these assumptions, the basic equations in cylindrical co-ordinates (r, z) are:

Continuity

$$\frac{1}{r}\frac{\partial(ru)}{\partial r} + \frac{\partial w}{\partial z} = 0 \tag{1}$$

Navier Stokes

$$u\frac{\partial u}{\partial r}+w\frac{\partial u}{\partial z}=-\frac{1}{\rho}\frac{\partial p}{\partial r}+\upsilon\left(\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\frac{\partial^2 u}{\partial z^2}-\frac{u}{r^2}\right) \tag{2}$$

$$u\frac{\partial w}{\partial r}+w\frac{\partial w}{\partial z}=-\frac{1}{\rho}\frac{\partial p}{\partial z}+\nu\left(\frac{\partial^2 w}{\partial r^2}+\frac{1}{r}\frac{\partial w}{\partial r}+\frac{\partial^2 w}{\partial z^2}\right) \tag{3}$$

Energy

$$u\frac{\partial T}{\partial r}+w\frac{\partial T}{\partial z}=\alpha\left(\frac{\partial^2 T}{\partial r^2}+\frac{1}{r}\frac{\partial T}{\partial r}+\frac{\partial^2 T}{\partial z^2}\right) \tag{4}$$

subject to the boundary conditions of these equations

$$\begin{aligned} &r=a: \quad u=w=0 \\ &\qquad T=T_w(\text{CWT})\ or\ \frac{\partial T}{\partial r}=-\frac{q_w}{k}(\text{CWHF}) \\ &r\longrightarrow\infty: \quad u=-A\left(r-\frac{a^2}{r}\right), w=2Az \\ &\qquad T=T_\infty \end{aligned} \tag{5}$$

Here u and v are the velocity components along $r-$ and $z-$ axes, T is the fluid temperature, p is the pressure, ρ is the density, α is the thermal expansion coefficient, ν is the kinematic viscosity and A is a given constant.

In order to solve Eqs. (1) - (4), we introduce the following similarity variables

$$\begin{aligned} &u=-Aa\eta^{-1/2}f(\eta),\ w=2Af'(\eta)z,\ \eta=\left(\frac{r}{a}\right)^2, \\ &\theta(\eta)=\frac{T-T_\infty}{T_w-T_\infty}(\text{CWT}),\quad \theta(\eta)=\frac{2(T-T_\infty)}{(aq_w/k)}(\text{CWHF}) \end{aligned} \tag{6}$$

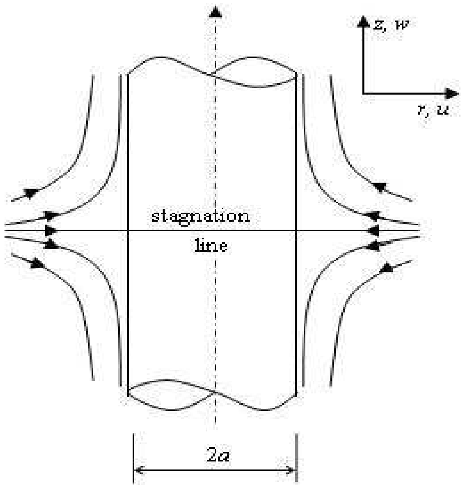

FIGURE 1. The coordinate axis

Substituting (6) into Eqs. (2) and (4), we get the following ordinary differential equations

$$\eta f''' + f'' + \mathrm{Re}(1 + ff'' - f'^2) = 0 \tag{7}$$

$$(\eta\theta')' + \mathrm{Pr}\,\mathrm{Re}\, f\ \theta' = 0 \tag{8}$$

subject to the boundary conditions (5) which become

$$f(1) = 0,\ f'(1) = 0,\ f'(\infty) = 1 \tag{9}$$

$$\theta(1) = 1,\ \theta(\infty) = 0\ \text{(CWT)}$$

$$\theta'(1) = -1,\ \theta(\infty) = 0\ \text{(CWHF)}$$

where Re is the Reynolds number and Pr is the Prandtl number which are defined

$$\mathrm{Re} = \frac{Aa^2}{2\nu}, \mathrm{Pr} = \frac{\nu}{\alpha} \tag{10}$$

The physical quantities of interest in this problem are the skin friction coefficient C_f, the Nusselt numbers for the wall constant temperature case Nu and for the constant wall heat flux case Nu^*. It is easily to show that these quantities can be expressed as

$$\mathrm{Re}\, C_f = -f''(1),\ \ Nu = -2\theta'(1)\ (\mathrm{CWT}),\ \ Nu^* = \frac{2}{\theta(1)}\ (\mathrm{CHF}) \tag{11}$$

Case $Re >> 1$

We consider now the boundary layer approximation ($Re >> 1$) of the problem under consideration. In this respect, we introduce the following new variables:

$$\xi = \mathrm{Re}^{1/2}(\eta - 1),\ \ f(\eta) = \mathrm{Re}^{-1/2}\, F(\xi), \tag{12}$$

$$\theta(\eta) = \Theta(\xi)\ (CWT),\ \ \theta(\eta) = \mathrm{Re}^{-1/2}\, \Theta(\xi)\ (\mathrm{CHF})$$

Substituting (12) into Eqs. (7) and (8), we obtain:

$$\left(1 + \mathrm{Re}^{-1/2}\, \xi\right) F''' + 1 + FF'' - F'^2 + \mathrm{Re}^{-1/2}\, F'' = 0 \tag{13}$$

$$\left(1 + \mathrm{Re}^{-1/2}\, \xi\right) \Theta'' + \mathrm{Pr}\, F\Theta' + \mathrm{Re}^{-1/2}\, \Theta' = 0 \tag{14}$$

along with the boundary conditions

$$F(0) = 0, F'(0) = 0, F'(\infty) = 1 \tag{15}$$

$$\Theta(0) = 1, \Theta(\infty) = 0 (CWT)$$

$$\Theta'(0) = -1, \Theta_0(\infty) = 0 (CWHF)$$

We notice that for $Re \longrightarrow \infty$, that corresponds to the boundary layer approximation, Eq. (13) - (15) reduce to the Hiemenz equations that describe the stagnation point flow on a plate, see Hiemenz [1]. Equations (13) - (15) were solved analytically using the following series expansions:

$$F = F_0 + \mathrm{Re}^{-1/2}\, F_1 + \mathrm{Re}^{-1}\, F_2 + ... \tag{16}$$

$$\Theta = \Theta_0 + \mathrm{Re}^{-1/2}\, \Theta_1 + \mathrm{Re}^{-1}\, \Theta_2 + ...$$

Substituting (16) into (13) - (15), we get the following three sets of equations:

first order approximation:

$$F_0''' + F_0 F_0'' - F_0'^2 + 1 = 0 \tag{17}$$

$$\Theta_0'' + \Pr F_0 \Theta_0' = 0$$

$$F_0(0) = 0,\ F_0'(0) = 0,\ F_0'(\infty) = 1$$

$$\Theta_0(0) = 1,\ \Theta_0(\infty) = 0 \text{ (CWT)}$$

$$\Theta_0'(0) = -1,\ \Theta_0(\infty) = 0 \text{ (CWHF)}$$

second order approximation:

$$F_1''' + F_0 F_1'' - 2F_0' F_1' + F_0'' F_1 + F_0'' + \xi F_0''' = 0 \tag{18}$$

$$\Theta_1'' + \Pr(F_0 \Theta_1' + F_1 \Theta_0') + \Theta_0' + \xi \Theta_0'' = 0$$

$$F_1(0) = 0,\ F_1'(0) = 0,\ F_1'(\infty) = 0$$

$$\Theta_1(0) = 0,\ \Theta_1(\infty) = 0 \text{ (CWT)}$$

$$\Theta_1'(0) = 0,\ \Theta_1(\infty) = 0 \text{ (CWHF)}$$

third order approximation:

$$F_2''' + F_0 F_2'' - 2F_0' F_2' + F_0'' F_2 + F_1'' + F_1'' F_1 - F_1'^2 + \xi F_1''' = 0 \tag{19}$$

$$\Theta_2'' + \Pr(F_0 \Theta_2' + F_1 \Theta_1' + F_2 \Theta_0') + \Theta_1' + \xi \Theta_1'' = 0$$

$$F_2(0) = 0,\ F_2'(0) = 0,\ F_2'(\infty) = 0$$

$$\Theta_2(0) = 0,\ \Theta_2(\infty) = 0 \text{ (CWT)}$$

$$\Theta_2'(0) = 0,\ \Theta_2(\infty) = 0 \text{ (CWHF)}$$

3. **Results and discussions**

Equations (7) - (8) subject to boundary conditions (9) were solved numerically for different values of the Prandtl number ($Pr = 0.01,\ 0.1,\ 1,\ 10,\ 100$) and some values of Reynolds number, $Re = 0.01,\ 0.1,\ 0.2,\ 1,\ 10,\ 20,\ 50,\ 100$ using Runge-Kutta method combined with a shooting technique. Some values related to the Nusselt numbers and skin friction are given in Table 1 for $Pr = 7$. Results reported by Wang

[5] are also included in this table. It is seen that there is a very good agreement between the present results and those reported by Wang [5]. We are, therefore, confident that our results are very accurate. The validity of the results are also illustrated in Figs. 2 to 4.

Figures 5 to 9 show the dimensionless velocity and temperature profiles for different values of the Reynolds and Prandtl numbers. Thus, it is seen that for a fixed value of the Parndtl number, the velocity profiles increase with the increase of the Reynolds number. However, the temperature profiles decrease with increase of the Reynolds number in the both cases of constant wall temperature and constant heat flux from the plate, respectively, see Figs. 5 to 7. Further, Figs. 8 and 9 show that for the both cases of constant wall temperature and constant heat flux from the plate, temperature profiles decreases with the Parndtl number when the Reynolds number is fixed. As expected the thickness of the temperature boundary layer decreases when the Parndtl number increases.

Finally, Figs. 10 and 11 show the variation of the Nusselt number with the Parndtl number in both cases of constant wall temperature and constant heat flux from the surface for a fixed value of the Reynolds number. The increase of the Nusselt number with the Reynolds number is in agreement with the results given in Table 1.

η_∞	Re	f"(1)	$\theta'(1)$	$\theta(1)$
320	0.01	0.313605	-0.320451	3.120599
80	0.1	0.615487	-0.615504	1.624684
35	0.2	0.786053	- 0.780247	1.281645
		0.78605*		
11	1	1.484185	- 1.450720	0.689313
		1.484185*		
3.5	10	4.162922	- 4.013979	0.249129
		4.16292*		
2	20	5.779734	- 5.560052	0.179855
1.75	50	8.985168	- 8.624974	0.115942
1.5	100	12.596429	-12.077699	0.082797

Wang[5]

Table 1. Values of the skin friction, $f''(1)$, Nusselt numbers, ($\theta'(1)$ for constant temperature case and $\theta(1)$ for the constant wall heat flux case), and boundary layer thickness, η_∞, for Prandtl number, $Pr = 7$ and different values of the Reynolds number, Re.

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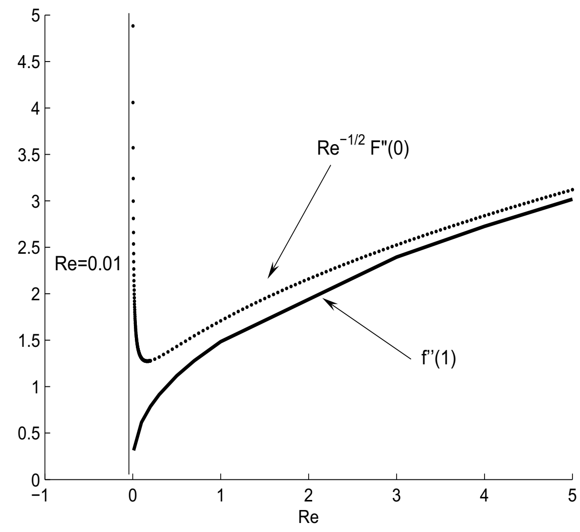

FIGURE 2. Validity range of the asymptotic approximation for velocity in the case $Re >> 1$.

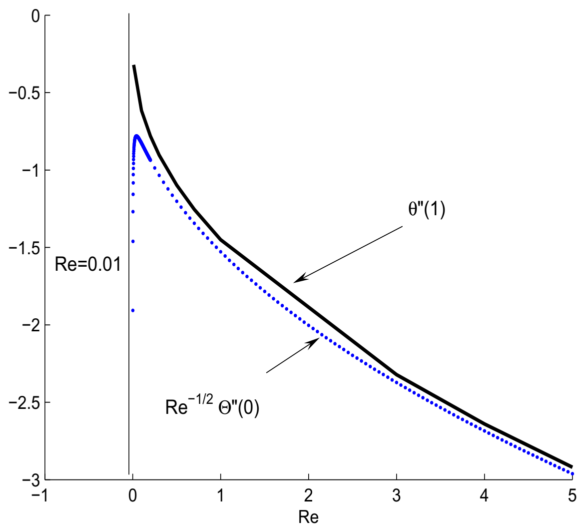

FIGURE 3. Validity range of the asymptotic approximation for temperature (CWT) in the case $Re >> 1$.

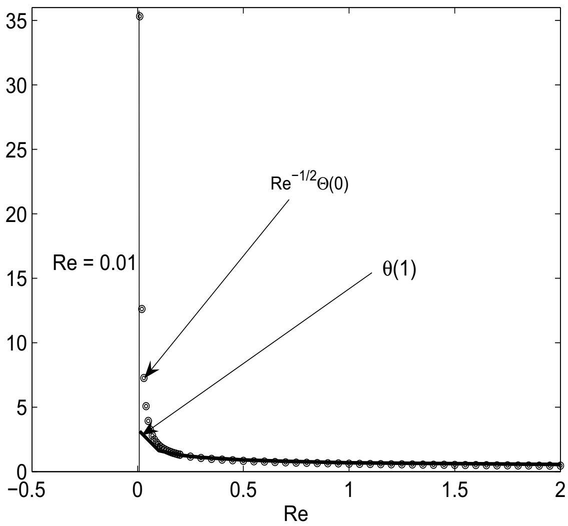

FIGURE 4. Validity range of the asymptotic approximation for temperature (CWHF) in the case $Re >> 1$.

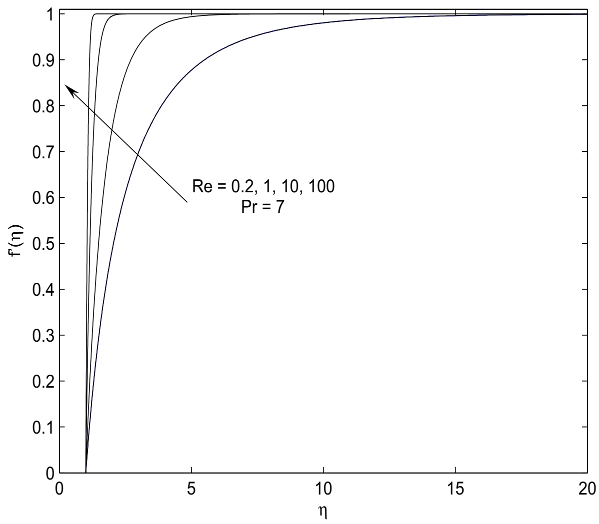

FIGURE 5. Dimensionless velocity profiles for $Pr = 7$ and $Re = 0.2, 1, 10, 100$.

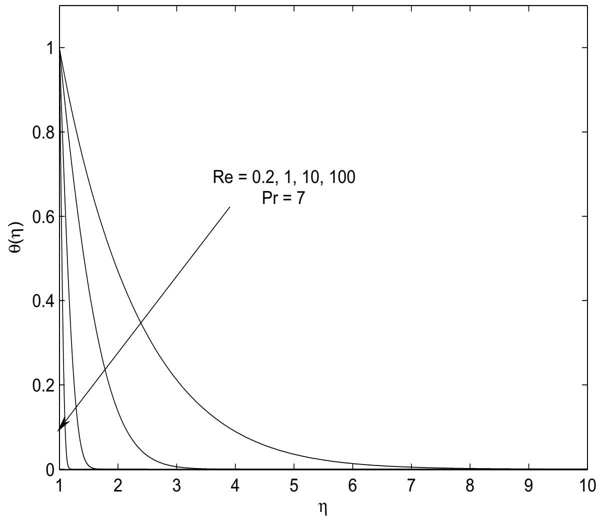

FIGURE 6. Dimensionless temperature profiles for $Pr = 7$ and $Re = 0.2, 1, 10, 100$ in the constant wall temperature case.

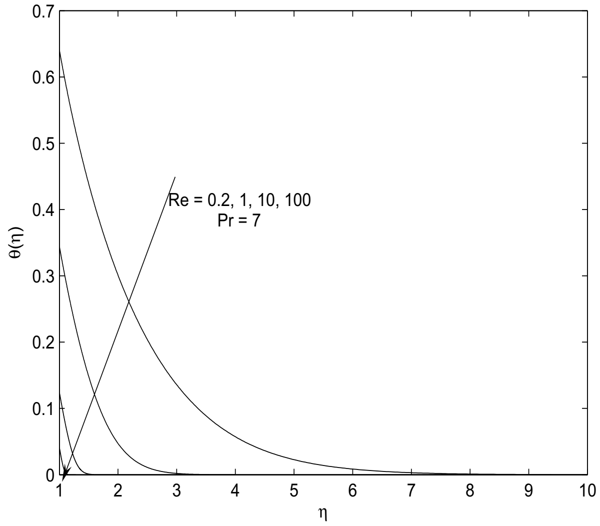

FIGURE 7. Dimensionless temperature profiles for $Pr = 7$ and $Re = 0.2, 1, 10, 100$ in the constant wall heat flux case.

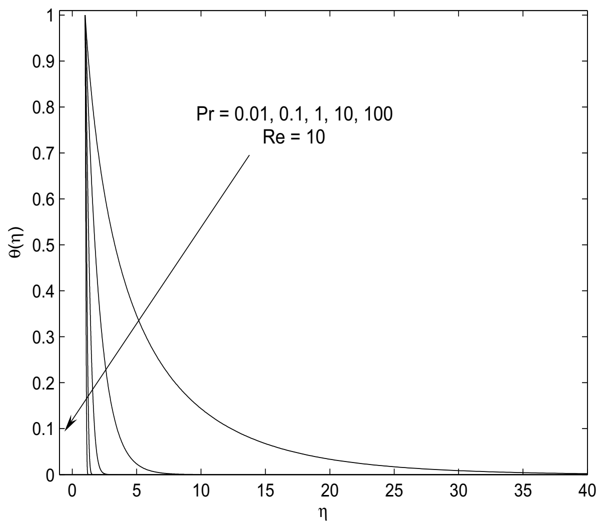

FIGURE 8. Dimensionless temperature profiles for $Pr = 0.01$, 0.1, 1, 10, 100 and $Re = 10$ for the constant wall temperature case.

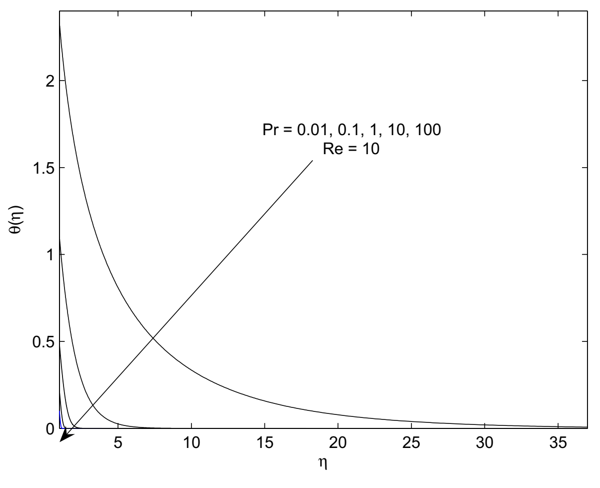

FIGURE 9. Dimensionless temperature profiles for $Pr = 0.01$, 0.1, 1, 10, 100 and $Re = 10$ for the constant wall heat flux case.

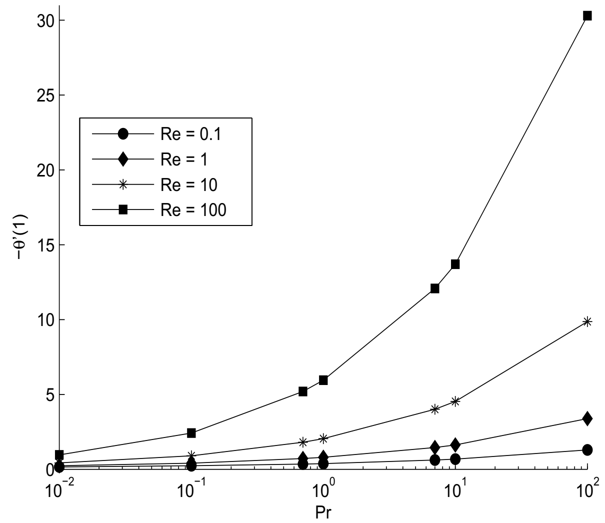

FIGURE 10. Variation of the Nusselt number with Prandtl number for $Re = 0.1, 1, 10, 100$ in the case of constant wall temperature.

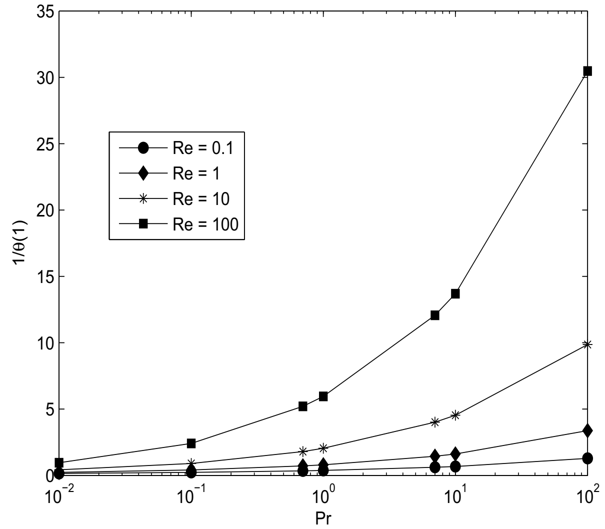

FIGURE 11. Variation of Nusselt number with Prandtl number for $Re = 0.1, 1, 10, 100$ in the constant wall heat flux case.

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BOOK REVIEWS

S.V. Emelyanov, S.K. Korovin, N.A. Bobylev, A.V. Bulatov, ***Homotopy of Extremal Problems***, Walter de Gruyter, Berlin - New York, 2007, 303 pp, ISBN 978-1-11-018942-1

The idea of homotopy appears in many branches of mathematics such as algebraic topology, differential topology, nonlinear analysis, variational calculus etc. Its importance comes from the invariance of some powerful tools, such as the *induced homomorphism* at the level of various homology/cohomology groups, the *degree*, the *Conley index* etc., on corresponding homotopy classes of maps or homotopically deformed spaces.

This book deals with the homotopy method applied in variational calculus and is structured in five chapters as follows:

The first chapter presents some classical facts on certain spaces of functions and their various topologies as well as on some special operators/functionals such as linear, nonlinear, monotone and potential operators as well as Lipschitzian and convex functionals. Among these facts we mention the presence of some necessary and sufficient conditions on a functional in order that either one of its critical points is a local minimizer or the functional itself is convex, strictly convex or strongly convex.

The second chapter starts with a sufficient condition, in finite dimensional context on a continuously differentiable deformation in order for a local minimizer of the initial function to be deformed into a local minimizer of the final one. This type of results are called *deformation principles for minimizers* and they are present all along the book. As a consequence one gets a sufficient condition, in terms of gradients, on two continuously differentiable functions of finitely many variables in order for a local minimizer of one of them to be a local minimizer of the other one. These type of results are then extended to the class of lipschitzian functions, in which case the role of the gradients is played by the generalized gradients, and even to the class of continuous functions. The chapter ends with a proof of the Hopf theorem on self maps of the N-sphere of zero degree, which are proved to be homotopic to each other, and with the Parusinski theorem. The last one concerns the gradient vector fields on the N-ball which are nondegenerate on the $(N-1)$-sphere and homotopic in the class of continuous vector fields, which are proved to be gradient homotopic.

The third chapter deals with problems similar to those treated in the second chapter, but in infinite dimensional setting.

The fourth chapter starts with some elementary facts on flows and then defines the Conley index of a set which is invariant with respect to a flow. The Conley index is proved to be invariant under a deformation of the flow and the initial invariant set. The isolated critical points of a differentiable function of finitely many variables are proved to be invariant sets with respect to the gradient flow and the Conley index of such a point is explicitly computed. Eventually, the Conley index is defined and studied in infinite dimensional context as well.

The fifth chapter is devoted to applications of the homotopy invariance of minimizers and of the Conley index. Among them we mention some deformation theorems and invariance of the global minimizers for the classical nonlinear programming problems, multicriteria problems, week minimizers problems, optimal control problems and bifurcation points problems. Stability of solutions of ordinary differential equations, focused on stability of gradient systems, stability of Hamiltonian systems and stability of dynamical systems, are also treated in this chapter.

The book is very well written and combines the power of homotopy methods with results coming from functional analysis, differential equations, variational calculus and other mathematical fields, either to prove some well known facts or to get relatively recent results.

It is useful for researchers in variational calculus and/or optimization desiring to be acquainted with the powerful tools of homotopy theory as well as for those working in homotopy theory, looking for applications.

Cornel Pintea

Beata Randrianatoanina and Narcisse Randrianatoanina (Editors) ***Banach Spaces and their Applications in Analysis*** - In Honor of Nigel Kalton's 60th Birhday, Walter de Gruyter • Berlin • New York, 2007, ix + 453 pp, ISBN: 978-3-11-019449-4

In recent years a lot of problems in analysis, apparently far from the theory of Banach spaces, were solved using Banach space methods. The aim of this conference was to bring together specialists who have been involved in these developments to honor the 60th birthday of Nigel Kalton. An excellent survey on Kalton's influential work in functional analysis and its applications is given in the introductory paper by Gilles Godefroy. It deals with quasi-Banach spaces and p-normed spaces (called *The Kalton zone*: $0 < p < 1$), non-linear geometry (mainly Lipschitz), isometric theory, interpolation and twisted sums, multipliers in spaces of vector functions. Although impressive, this survey covers only a part of the fundamental contributions Professor Kalton made in various areas of analysis.

The topics of the conference were:

1. Nonlinear theory (Lipschitz classification of Banach and metric spaces);
2. Isomorphism theory of Banach spaces (including connections with combinatorics and set theory);
3. Algebraic and homological methods in Banach spaces;
4. Approximation theory and algorithms in Banach spaces (greedy approximation, interpolation, abstract approximation theory);
5. Functional calculus and applications to partial differential equations.

The Conference was attended by over 160 mathematicians from around the world who delivered 15 plenary talk and 105 talks in specialized sessions. The present Proceedings reflect this situation - they contain 11 papers by plenary speakers and 18 specialized papers. In the following we shall mention some of them.

Concerning the first topic there are a survey paper by J. Lindenstrauss, D. Preis anf J. Tišer on the differentiability of Lipschitz functions on Banach spaces (a book dedicated to this topic is announced), J. Duda and O. Maleva (metric differentiability), A. Kaminska and A. M. Parrish (q-concavity and q-convexity in Lorentz spaces), R. Ni (fixed points of Φ-contractive mappings), T. Oikhberg (the Daugavet property), O. Brezhneva and A. Tretyakov (implicit function theorem for nonregular mappings in Banach spaces). Some papers dealing with the second theme , isompomorphic theory of Banach spaces, are those by V. Ferenczi and C. Rosendal (complexity and homogeneity in Banach spaces), E. Odell, Th. Schlumprecht, A. Zsák (a new infinite game in Banach spaces), G. Androulakis and F. Sanacory (equivalent norms on Hilbert space), M. Gonzales and M. Wójtowicz (quotients of $\ell_1(\Gamma)$), J. Talponen (asymptotically transitive Banach spaces).

Some approximation problems in Banach space setting are treated in the papers of Y. Brudnyi (multivariate functions of bounded variation), V. Temlyakov (greedy approximation in Banach spaces), P. Bandyopadhyay, B.-L. Lin and T. S. S. R. K. Rao (ball proximity in Banach spaces), R. Vershynin (numerical algorithms in asymptotic convex geometry).

There are some papers dealing with analysis of vector functions as, for instance, J. van Neerven, M. Veraar and Lutz Weis (stochastic integrability in UMD Banach spaces), M. D. Acosta, L. A. Morales (boundaries of spaces of holomorphic functions), T. Hytönen (a probabilistic Littlewood-Paley theory in Banach spaces).

Emphasizing connections between seemingly distant areas of analysis and illustrating the power and versatility of applications of Banach space theory, the volume will be of great interest to researchers in various domains of mathematics, especially to those interested in Banach space methods.

I. V. Şerb

Cédric Villani, *Topics in Optimal Transportation*, American Mathematical Society, Graduate Studies in Mathematics, Volume 58, Providence, Rhode Island 2003, ISBN:0-8218-3312-X

The mass transportation problem (MTP) as posed initially in 1871 by Gaspard Monge in his paper *Mémoire sur la théorie des déblais et des remblais*, consists in finding an optimal volume-preserving map between two sets X, Y of equal volume. The optimality is evaluated by a cost function $c(x, y)$ representing the cost per unit mass for transporting from $x \in X$ to $y \in Y$, and one asks to minimize $I[T] = \int_X c(x, T(x))d\mu(x)$ over all transportation plans T. The functional $I[T]$ is nonlinear in the transportation plan T and the set of admissible transportation plans is a nonconvex set, explaining the difficulty of this problem. A solution in the case $c(x, y) = |x - y|$ considered by Monge for - the Euclidean distance, was given only in 1979 by Sudakov in a 178 pages paper published as a volume of Trudy of the Steklov Institute. Recently some inaccuracies in Sudakov's paper were fixed by Alberti, Kircheim and Preis.

In 1942 L. V. Kantorovich proposed a new approach to the problem asking for the minimization of the functional $I[\pi] = \int_{X\times Y} c(x,y)d\pi(x,y)$ for $\pi \in \Pi(\mu,\nu)$. Here X, Y are Polish spaces (i.e., complete metrizable topological spaces), μ,ν regular probability measures on X and Y respectively, and $\Pi(\mu,\nu)$ denotes the set of all probability measures on $X \times Y$ with marginals μ,ν. In this way the nonlinear original Monge problem becomes a linear optimization problem over a convex sets of probability measures, allowing the use of the tools of linear programming and leading to the famous Kantorovich-Rubinshtein duality theorem. For this reason Kantorovich MTP is easy to solve that the original Monge MTP. At the same time it can be considered as a relaxation of Monge problems. It is worth to mention that Kantorovich contributions to the related problem of optimal allocation of resources earned him, jointly with Koopmans, the 1975 Nobel prize in economy.

It turned out that the MTP is a prototype for a class of problems arising in various fields as functional analysis, probability and statistics, linear and stochastic programming, differential geometry, with numerous applications to fluid mechanics, quantum physics and other domains. At the same time the solution of MTP requires tools, methods and results from these domains, explaining its beauty and the great appeal of MTP for mathematicians of various specialties.

The present book, based on a graduate course taught by the author at the Georgia Tech in the fall of 1999, is a carefully written introduction to various aspects of MTP.

The basic theory is developed in Chapters 1. *The Kantorovich duality*, 2. *Geometry of optimal transportation*, 4. *The Monge-Ampère equation*, and 7. *The metric side of the optimal transportation.* This part must be read by every graduate students to be acquainted with the basic results and tools of the theory. Here the proofs are given in detail, excepting Chapter 4 where the waste and difficult subject of regularity for fully nonlinear elliptic equations is only sketched. Chapter 3. *Brenier's polar factorization theorem*, present some of the motivations from fluid mechanics which led Brebier to his polar factorization theorem proved in 1987. As the author mention in the Preface, this give rise to a revival in the study of MTP "paving the way to a beautiful interplay between differential equations, fluid mechanics, geometry, probability theory and functional analysis". Chapter 5. *Displacement interpolation and displacement convexity,* is concerned with these two important notions, introduced by MvCann in 1994 and some applications.

In Chapter 6. *Geometric and Gaussian inequalities*, the author explains how mass transportation provides powerful tools to study some functional inequalities with geometric content, having as prototype the isoperimetric inequality - the Brunn-Minkowski inequality, the inequality of Prékopa-Leindler, Gaussian inequalities.

Chapters 8. *A differential point of view on optimal transportation*, and 9. *Entropy production and transportation*, are more advanced requiring some basic notions in partial differential equations and functional analysis.

There are a lot of exercises disseminated over the text and the last chapter, 10. *Problems,* gathers longer problems taken from recent research papers.

The book is clearly written and well organized and can be warmly recommended as an introductory text to this multidisciplinary area of research, both pure and applied - the mass transportation problem.

S. Cobzaş

György Darvas, ***Symmetry***, Cultural-Historical and Ontological Aspects of Science-Arts Relations; the Natural and Man-made World in an Interdisciplinary Approach, translated from the Hungarian by David Robert Evans 2007, XI, 508 pp. 420 illus., 66 in color., Softcover ISBN: 978-3-7643-7554-6, Birkhäuser 2007

As its subtitle shows ("Cultural-historical and ontological aspects of science-arts relations. The natural and man-made world in an interdisciplinary approach"), the book "Symmetry" by Darvas György is a wonderful voyage through different sciences and arts all connected by the universal concept of symmetry.

Symmetry (and the lack of it) is a fundamental phenomenon in physics, chemistry, mathematics, biology, psychology, architecture and all kind of arts, creating interesting interferences between these seemingly different subjects.

The book contains 15 chapters the first 4 introducing the basic notions and definitions related to symmetry and outlining its historical evolution. The rest of the chapters present the most typical applications of different appearances of symmetries in the sciences and the humanities. It is important to note the ontological ordering of these chapters: starting from the self-organization of the matter and the inanimate nature, through the formation of organic matter we end up investigating the human creativity. We also emphasize the huge number (350) of pictures and illustration making things much more accessible.

The book avoids difficult mathematical formalisms, however exceeds the limits of popular science being formulated at a university level. In this way it is highly recommended for every student and scientist interested in interdisciplinary interactions.

Cs. Szántó

A. Bensoussan, G. Da Prato, M. C. Delfour, S. K. Mitter, ***Representation and Control of Infinite Dimensional Systems***, Birkhäuser, Boston, 2007, 2nd ed., XXVI + 575 p. 5 illus., Series: Systems & Control: Foundations & Applications, ISBN 978-0-8176-4461-1

This reorganized, revised, and expanded edition is originated in a two-volume set: Representation and Control of Infinite Dimensional Systems (vol. I), Birkhäuser, Basel, 1992, 315 p., Series: Systems & Control: Foundations & Applications, ISBN 3-7643-3641-2 and Representation and Control of Infinite Dimensional Systems (vol. II), Birkhäuser, Boston, 1993, 372 p., Series: Systems & Control: Foundations & Applications, ISBN 978-0-8176-3642-5.

Since the publications of the two volumes in 1992-93 more sophisticated mathematical tools and approaches have been introduced in the field and a whole range of challenging applications appeared from new phenomenological, technological, and design developments. The two volumes have been recognized as key references in the field, hence a revised and corrected edition became desirable.

As the authors state in the Introduction to the book "the primary concern of this book is the control of linear infinite dimensional systems", systems whose state space is infinite dimensional and its evolution is typically described a linear differential equation, linear functional equation or linear integral equation.

Now we introduce the main parts of this impressive book.

Introduction. Part I. Finite dimensional linear control of dynamical systems. Control of linear differential systems. Controllability, observability, duality, stabilizability and detectability. Optimal control. Finite time horizon and infinite time horizon. Dissipative systems. Linear quadratic two-person zero-sum differential games.

Part II. Representation of infinite dimensional linear control dynamical systems. Semi-groups of operators and interpolation. Variational theory of parabolic systems. Semigroup methods for systems with unbounded control and observation operators. State space theory of differential systems with delays.

Part III. Qualitative properties of linear control dynamical systems. Controllability and observability for a class of infinite dimensional systems.

Part IV. Quadratic optimal control: finite time horizon. Bounded control operators: control inside the domain. Unbounded control operators: parabolic equations with control on the boundary. Unbounded control operators: hyperbolic equations with control on the boundary.

Part V. Quadratic optimal control: infinite time horizon. Bounded control operators: control inside the domain. Unbounded control operators: parabolic equations with control on the boundary. Unbounded control operators: hyperbolic equations with control on the boundary.

An isomorphism result is given in the Appendix A. Each part of the book is completed by important comments and/or references.

We mention some new material and original features of the second edition:

- Part I on finite dimensional controlled dynamical systems contains new material: an expanded chapter on the control of linear systems including a glimpse into H-infinity theory and dissipative systems, and a new chapter on linear quadratic two-person zero-sum differential games.
- A unique chapter on semigroup theory and interpolation of linear operators brings together advanced concepts and techniques that are usually treated independently.
- The material on delay systems and structural operators is not available elsewhere in book form.

Control of infinite dimensional systems has a wide range and growing number of challenging applications. This book is a key reference for anyone working on these applications, which arise from new phenomenological studies, new technological developments, and more stringent design requirements. It will be useful for mathematicians, graduate students, and engineers interested in the field and in the underlying conceptual ideas of systems and control.

This book represents a remarkable contribution to the development of this scientific field very useful for mathematicians, theoretical engineers, and, in general, for all the scientists interested in control of infinite dimensional systems.

The book ends with an extensively list of references and a useful index of notions and symbols.

We can state doubtless that in front of us there is a masterpiece on the topic of representation and control of infinite dimensional systems. Certainly this book will be included in many libraries all over the world.

Marian Mureşan

Dorothee D. Haroshke and Hans Trienbel, *Distributions, Sobolev Spaces, Elliptic Equations,* EMS Textbooks in Mathematics, European Mathematical Society, Zürich 2008, ix+294 pp, ISBN: 978-3-03719-042-5.

The book is based on two-semester courses taught several times over a period of ten years by the authors to graduate students and PhD students at the Friedrich Schiller University in Jena. Its aim is to give a gentle introduction to the basic results and techniques of the L_2 theory of elliptic differential operators of second order on bounded domains in $\mathbb{R}^n$. The prerequisites are calculus, measure theory and basic elements of functional analysis.

The book starts with the classical Laplace-Poisson equations and harmonic functions. The basic properties of distributions, including Fourier transform, are treated in the second chapter.

Chapters 3. *Sobolev spaces on* $\mathbb{R}^n$ *and* $\mathbb{R}^n_+$, and 4. *Sobolev spaces on domains,* constitute a self-contained introduction to the basic properties of Sobolev spaces - embeddings, extensions, traces.

The fifth chapter, *Elliptic operators in* L_2, is concerned with the L_2 theory of general elliptic operators on bounded domains Ω in $\mathbb{R}^n$, having as leading model the Laplacian studied in the first chapter. This study concerns: a priori estimates, homogeneous boundary problems, inhomogeneous boundary problems, smoothness theory, Green functions and Sobolev embeddings, degenerate elliptic operators. Chapter 6. *Spectral theory in Hilbert spaces and Banach spaces*, is a short introduction to spectral theory of self-adjoint operators in Hilbert space, approximation numbers, entropy numbers. This machinery is applied in the seventh chapter, *Compact embeddings, spectral theory of elliptic operators*, to the study of distribution of the eigenvalues and of the associated eigenelements of the self-adjoint operator $Au = -\sum_{j,k=1}^{n} \frac{\partial}{\partial x_j}\big(a_{j,k}(x)\frac{\partial u}{\partial x_k}\big) - a(x)u,\ \mathrm{dom}A = W^2_{2,0}(\Omega)$.

The book ends with six appendices: A. *Domains, basic spaces, and integral formulae,* B. *Orthonormal bases of trigonometric functions,* C. *Operator theory,* D. *Some integral inequalities,* E. *Function spaces,* collecting the basic notions and results used in the main text, or presenting more general function spaces (Appendix E), references to which were made in the Notes from the end of the chapters. A thorough and detailed presentation of these spaces is given in the recent books of the second-named author: *Theory of Function Spaces II,* Birkhäuser 1992, and *Theory of Function Spaces III,* Birkhäuser 2006.

Written by two leading experts in the area and including their teaching experience, the book is of great use for students and mathematicians looking for an accessible introduction to function spaces and partial differential equations. After its reading, more advanced and difficult texts on similar topics can be successfully approached with less effort.

S. Cobzaş

William Byers, ***How Mathematicians Think Using Ambiguity, Contradiction, and Paradox to Create Mathematics,*** Princeton University Press, 415 pages, ISBN-13:978-0-691-12738-5.

There are very much number of paper on the nature of mathematical thinking, on how mathematicians create mathematics. Here are some basic books on this direction:

- J. Hadamard, The Psychology of Invention in the Mathematical Field, Princeton University Press, 1949.
- H. Poincaré, Science and Hypothesis, Dover, New York, 1952.
- H. Weyl, Philosophy of Mathematics and Natural Science, Princeton University Press, 1949.
- P. Sergescu, Gândirea matematică (The mathematical thinking), Ed. Ardealul, Cluj, 1928 (Romanian).
- A. Froda, Eroare şi paradox în matematică, (Error and paradox in mathematics) Ed. Enciclopedică Română, Bucureşti, 1971 (Romanian).
- M. Kline, Mathematical Thought from Ancient to Modern Times, Oxford University Press, 1972.
- J. Dieudonné, Mathématique vides et mathématique significatives, Luxemburg, 1976.
- I. Lakatos, Proofs of Refutations, Cambridge University Press, 1976.
- R.L. Wilder, Mathematics as a Cultural System, Pergamon Press, New York, 1981.
- S. Mac Lane, Mathematics: Form and Function, Springer, New York, 1986.
- R. Penrose, The Emperor's New Mind, Oxford University Press, 1989 (Romanian translation: Ed. Tehnică, 1996).
- B. Heinz, Die Innenwelt der Mathematik, Springer, 2000.
- R. Hersch (Ed.), 18 Unconventional Essays on the Nature of Mathematics, Springer, 2005.

Byers's book provides a novel approach to many questions such as:

- Is mathematics objectively true?

- Is mathematics discovered and/or invented?
- Where does mathematical creativity come from?
- Is mathematical thought algorithmic in nature?

The book is divided into three sections: The light of ambiguity (Ambiguity in Mathematics, The Contradictory in Mathematics, Paradoxes and Mathematics: Infinity and the Real Numbers), The light as idea (The Idea as an Organizing Principle, Ideas, Logic and Paradox, Great Ideas) and The light and the eye of the beholder (The Truth of Mathematics, Conclusion: Is Mathematics Algorithmic or Creative?).

Well-organized and carefully written the present book is very useful to all who are interested in "How Mathematicians Think"! A related question could be: "Do mathematicians really think?"

Ioan A. Rus