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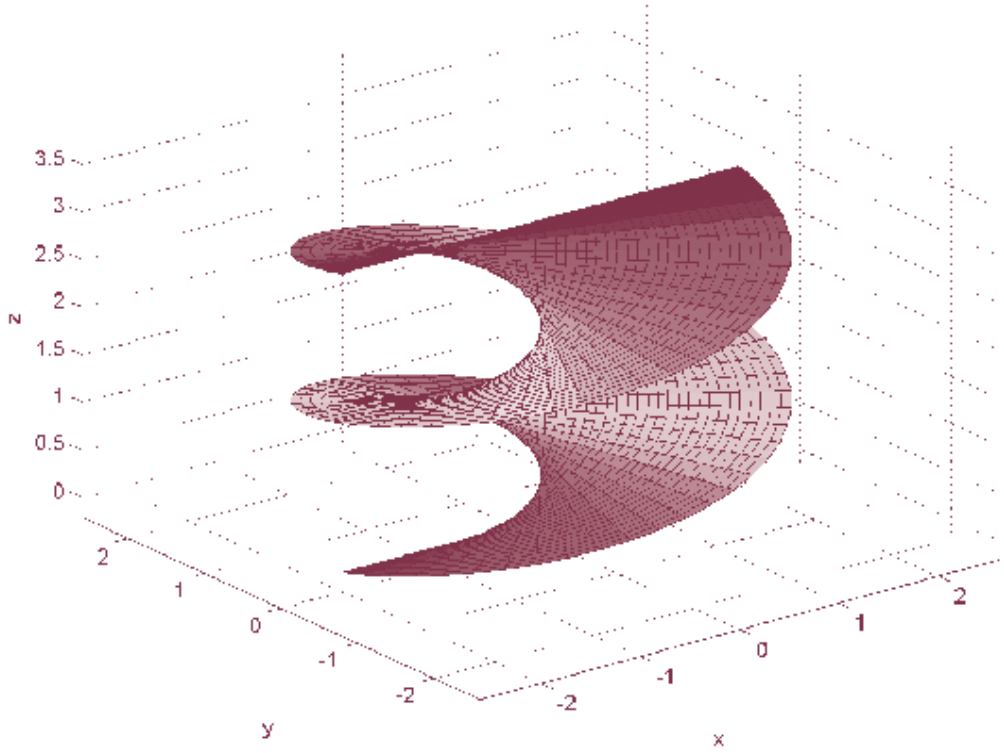

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OPTIMAL DYNAMIC PORTFOLIOS UNDER A TAIL CONDITIONAL EXPECTATION CONSTRAINT

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Abstract. We consider a portfolio problem when a tail conditional expectation constraint is imposed. The financial market is composed of n risky assets driven by geometric Brownian motion and one risk-free asset. The tail conditional expectation is derived, re-calculated at short intervals of time and imposed continuously. The method of Lagrange multipliers is combined with the Hamilton-Jacobi-Bellman equation to insert the constraint into the resolution framework. A numerical method is applied to obtain an approximate solution to the problem. We find that the imposition of the tail conditional expectation constraint when risky assets evolve following a log-normal distribution, curbs investment in the risky assets and increases consumption.

1. Introduction

In recent years particular stress has been laid on the substitution of variance as a risk measure in the standard Markowitz [11] (1952) mean-variance problem. Since it makes no distinction between positive and negative deviations from the mean, variance is a good measure of risk only for distributions that are (approximately) symmetric around the mean such as the normal distribution or more generally, elliptical distributions (see e.g., McNeil, Frey and Embrechts [12] (2004)). However, in most cases such as in portfolios containing options, we are dealing with wealth distributions

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that are highly skewed. It is thus more reasonable to consider asymmetric risk measures since individuals are typically loss averse. In this regard, Value-at-Risk (VaR) has emerged as the industry standard as regulatory authorities enforced the use of VaR which is a downside risk measure (see, e.g., Jorion [9] (1997)).

Despite its widespread acceptance, VaR is known to possess unappealing features. Artzner et al. [3] (1999) proposed an axiomatic foundation for risk measures, by identifying four properties that a reasonable risk measure should satisfy and providing a characterization of the risk measures satisfying these properties, which they called coherent risk measures. Tail conditional expectation (TCE) is one of such so-called coherent risk measures (see Rockafellar and Uryasev [14] (2002)). Going by these axioms, VaR is not coherent.

Our focus in this paper is the dynamic portfolio and consumption choice of a trader subject to a risk limit specified in terms of TCE. Yiu [15] (2004) has successfully controlled risky investment by imposing VaR as a dynamic constraint, with a model that applies the VaR constraint over time and emphasizes the repeated re-calculations of the VaR like in practice. He expresses the belief that other risk measures imposed in the same way will achieve similar results. We close that gap here by experimenting with the TCE constraint and extending the utility maximization to cover consumption and terminal wealth. This problem has not yet received adequate attention in the existing literature. We show through numerical simulations by applying an algorithm similar to that in Yiu [15] (2004) that the introduction of a TCE constraint reduces investment in risky assets and increases consumption.

The rest of this paper is structured as follows. In section 2, we model the financial market and describe the portfolio dynamic. Section 3 derives the Value-at-Risk and tail conditional expectation constraints, while section 4 makes precise the optimal control problem to be solved. Section 5 develops the solution of the problem by using the Lagrange technique to combine the Hamilton-Jacobi-Bellman (HJB) equation and the TCE constraint. In section 6, a numerical algorithm is presented to obtain an approximate solution to the TCE-constrained problem. Section 7 presents simulations and section 8 concludes the paper.

2. **The model**

We consider a standard Black-Scholes type market (see, e.g., Korn [10] (1997) for relevant definitions) consisting of one riskless bond and n risky stocks. The financial market is continuous-time with a finite time horizon [0,T].

Uncertainty in the financial market is modeled by a probability space $(\Omega, \mathcal{F}, \mathbf{P})$, equipped with a filtration that is a non-decreasing family $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}$ of sub-σ-fields of $\mathcal{F}$

$$\mathcal{F}_s \subseteq \mathcal{F}_t \subseteq \mathcal{F} \quad \forall \quad 0 \leq s < t < \infty.$$

It is assumed throughout this paper that all inequalities as well as equalities hold **P**- almost surely. Moreover, it is assumed that all stated processes are well defined without giving any regularity conditions ensuring this. The riskfree rate $r = r_t$ of the riskless asset (bond) S^0 is supposed to evolve according to

$$dS_t^0 = rS_t^0 dt, \ S_0^0 = s. \tag{1}$$

For the risky assets (stocks), for which the prices will be denoted by $S_t = (S_t^1, \ldots, S_t^n)$ for some $n \in \mathbb{N}$, the basic evolution model is that of a log-normal diffusion process.

$$\frac{dS_t^i}{S_t^i} = \mu^i dt + \sum_{j=1}^{k} \sigma^{ij} dW_t^j \tag{2}$$

$$S_0^i = s^i, \ i = 1, \ldots, n \ \forall \ t \in [0, T],$$

where, for some $k \in \mathbb{N}$, $W_t = [W_t^1, \ldots, W_t^k]'$, with the symbol $(')$ standing for transpose, is a k-dimensional Wiener process, i.e., a vector of k independent one-dimensional Wiener processes.

The n-vector $\mu = \mu_t = (\mu_t^1, \ldots, \mu_t^n)'$, contains the expected instantaneous rates of return and the $n\times k$-matrix $\sigma = \sigma_t = \sigma_t^{ij}$, $(i = 1, \ldots, n, \ j = 1, \ldots, k)$ measures the instantaneous sensitivities of the risky asset prices with respect to exogenous shocks so that the $(n \times n)$-matrix $\sigma\sigma'$ contains the variance and covariance rates of instantaneous rates of return. μ and σ must be adapted to the information filtration $\mathbb{F} = (\mathcal{F}_t)$.

An agent invests according to an investment strategy that can be described by the $(n+1)$-dimensional, $\mathcal{F}_t$-predictable process

$$x_t = (x_t^0, x_t^1, \ldots, x_t^n), \tag{3}$$

where $x_t^i,\ (i = 1, \ldots, n)$ denotes the number of shares of asset i held in the portfolio at time t ($i = 0$ refers to the bond). The process x describes an investor's portfolio as carried forward through time. The value of the investor's wealth at time t is then

$$V_t^x = x_t^0 S_t^0 + \sum_{i=1}^{n} x_t^i S_t^i, \tag{4}$$

where $x_t^i S_t^i$ represents the amount invested in asset i at time t.
Equivalently, one may consider the vector

$$\theta_t = (\theta_t^1, \ldots, \theta_t^n),$$

where

$$\theta_t^i = \frac{x_t^i S_t^i}{V_t^x}, \quad (i = 1, \ldots, n)$$

denotes the fraction of wealth invested in the risky asset i at time t.

Let therefore θ_t^i be the proportion of the investor's wealth in the risky security i at time t, for $i = 1 \ldots n$, with the remainder $1 - \sum_{j=1}^{n} \theta_t^i$ invested in the risk-free asset. Let also c_t be the instantaneous consumption rate. It is assumed that $\theta_t^1, \ldots, \theta_t^n$ and c_t are admissible and $\mathcal{F}_t$- adapted control processes. That is, θ_t^i and c_t are non-anticipative functions that satisfy the condition of bounded variation $\int_0^T \sum_{1=1}^{n} (\theta_t^i)^2 < \infty$ and $\int_0^T c_t^2 < \infty$ respectively, for an investment time horizon $T < \infty$. The corresponding portfolio value process reads

$$dV_t^\theta = V_t^\theta \left[\left(1 - \sum_{i=1}^{n} \theta_t^i \right) \frac{dS_t^0}{S_t^0} + \sum_{i=1}^{n} \theta_t^i \frac{dS_t^i}{S_t^i} \right] - c_t dt,\ V_t^\theta = v$$

$$= V_t^\theta \left[\left(r + \sum_{i=1}^{n} \theta_t^i (\mu^i - r) \right) dt + \sum_{i=1}^{n} \sum_{j=1}^{k} \theta_t^i \sigma^{i,j} dW_t^j \right] - c_t dt,\ V_t^\theta = v. \tag{5}$$

To have a better exposition, we adopt a matrix expression: denote $\sigma = [\sigma^{i,j}]$, $\theta_t = [\theta_t^1 \ldots \theta_t^n]'$, $\mu - r = [\mu^1 - r \ldots \mu^n - r]'$ and $W_t = [W_t^1 \ldots W_t^k]'$, so that σ is an

$n \times k$ matrix, $\mu - r$ and θ_t are n-dimensional column vectors and W_t is a k-dimensional column vector. Hence equation (5) can be rewritten as

$$dV_t^\theta = V_t^\theta \left[(r_t + \theta_t'(\mu_t - r))\, dt + \theta_t' \sigma_t dW_t \right] - c_t dt,\ V_t^\theta = v. \tag{6}$$

Thus,

$$V_t^\theta = \left(1 - \sum_{i=1}^{n} \theta_t^i \right) V_t^\theta + \sum_{i=1}^{n} \theta_t^i V_t^\theta. \tag{7}$$

We have adopted an incomplete market asset pricing setting of He and Pearson [7] (1991). To eliminate redundant assets, we assume that σ is of full row rank- that is, $\sigma\sigma'$ is an invertible matrix.

3. The tail conditional expectation (TCE) constraint

Here we start by defining Value-at-risk since the subsequent definition of tail conditional expectation will depend on it.

Definition 1. ***(Value-at-Risk)***
Given some probability level $\alpha \in (0,1)$, the time t wealth benchmark Υ_t and horizon Δt, the Value-at-Risk of time t wealth V_t at the confidence level $(1-\alpha)$, denoted VaR_t^α, is given by the smallest number L such that the probability that the loss $G_{t+\Delta t} := \Upsilon_{t+\Delta t} - V_{t+\Delta t}$ exceeds L is no larger than α.

$$VaR_t^\alpha = \inf \{ L \geq 0 : \mathbf{P}(G_{t+\Delta t} \geq L | \mathcal{F}_t) \leq \alpha \} := (Q_t^\alpha)^-, \tag{8}$$

where

$$Q_t^\alpha = \sup \left\{ L \in \mathbb{R} : \mathbf{P}((V_{t+\Delta t}^\theta - \Upsilon_{t+\Delta t}) \leq L | \mathcal{F}_t) \leq \alpha \right\} \tag{9}$$

is the quantile of the projected wealth surplus at the horizon $t + \Delta t$ and $x^- = \max[0, -x]$.

Thus $VaR_t^\alpha = 0$ for $Q_t^\alpha > 0$. VaR_t^α is therefore the loss of wealth with respect to a benchmark $\Upsilon_{t+\Delta t}$ at the horizon Δt which could be exceeded only with a small conditional probability α if the current portfolio θ_t were kept unchanged. Typical values for the probability level α are $\alpha = 0.05$ or $\alpha = 0.01$. In market risk management the time horizon Δt is usually one or ten days.

Proposition 1. ***(Computation of Value-at-Risk)***
We have

$$VaR_t^\alpha = (Q_t^\alpha)^- = \Big(V_t^\theta \exp\Big[\Phi^{-1}(\alpha)\|\theta_t'\sigma\|\sqrt{\Delta t} + \left(\theta_t'(\mu - r) + r - \frac{c_t}{V_t^\theta} - \frac{1}{2}\|\theta_t'\sigma\|^2\right)\Delta t\Big] - \Upsilon_{t+\Delta t}\Big)^-, \quad (10)$$

where $\Phi(\cdot)$ *and* $\Phi^{-1}(\cdot)$ *denote the normal distribution and the inverse distribution functions respectively, and* $\|\cdot\|$. *stands for norm.*

Proof. The distribution of wealth at time $t + \Delta t$ is approached by

$$V_{t+\Delta t}^\theta = V_t^\theta \exp\left[(\theta_t'(\mu - r) + r - \frac{c_t}{V_t^\theta} - \frac{1}{2}\|\theta_t'\sigma\|^2)\Delta t + \theta_t'\sigma(W_{t+\Delta t} - W_t)\right], \quad (11)$$

This follows immediately from (6) and Itô's Lemma (see Korn [10] (1997)), if we consider that given a portfolio $\{\theta_t, c_t\}$ and the associated portfolio value V_t at time t, the random variable $V_{t+\Delta t}(V_t, t)$ would be the future value of the portfolio at time $t + \Delta t$ with the portfolio weights being kept constant between time t and time $t + \Delta t$.

In accordance with expression (9) on the definition of VaR_t^α, we have

$$\mathbf{P}\left((V_{t+\Delta t}^\theta - \Upsilon_{t+\Delta t}) \leq L|\mathcal{F}_t\right)$$

$$= \mathbf{P}\left(V_t^\theta \exp\left[(\theta_t'(\mu - r) + r - \frac{c_t}{V_t^\theta} - \frac{1}{2}\|\theta_t'\sigma\|^2)\Delta t + \theta_t'\sigma(W_{t+\Delta t} - W_t)\right] - \Upsilon_{t+\Delta t} \leq L|\mathcal{F}_t\right)$$

$$= \mathbf{P}\left(\exp\left[(\theta_t'(\mu - r) + r - \frac{c_t}{V_t^\theta} - \frac{1}{2}\|\theta_t'\sigma\|^2)\Delta t + \theta_t'\sigma(W_{t+\Delta t} - W_t)\right] \leq \frac{L + \Upsilon_{t+\Delta t}}{V_t^\theta}|\mathcal{F}_t\right)$$

$$= \mathbf{P}\left(\frac{\theta_t'\sigma(W_{t+\Delta t} - W_t)}{\|\theta_t'\sigma\|\sqrt{\Delta t}} \right.$$
$$\left. \leq \frac{\ln\left(\frac{L+\Upsilon_{t+\Delta t}}{V_t^\theta}\right) - \left(\theta_t'(\mu - r) + r - \frac{c_t}{V_t^\theta} - \frac{1}{2}\|\theta_t'\sigma\|^2\right)\Delta t}{\|\theta_t'\sigma\|\sqrt{\Delta t}} | \mathcal{F}_t \right)$$

$$= \Phi\left(\frac{\ln\left(\frac{L+\Upsilon_{t+\Delta t}}{V_t^\theta}\right) - \left(\theta_t'(\mu - r) + r - \frac{c_t}{V_t^\theta} - \frac{1}{2}\|\theta_t'\sigma\|^2\right)\Delta t}{\|\theta_t'\sigma\|\sqrt{\Delta t}}\right),$$

where $\Phi(\cdot)$ is the cumulative distribution function of a standard normal random variable, given that the random variable $\theta_t'\sigma(W_{t+\Delta t} - W_t)$ is conditionally normally distributed with zero mean and variance $\|\theta_t'\sigma\|^2\Delta t$. Thus,

$$\mathbf{P}\left((V_{t+\Delta t}^\theta - \Upsilon_{t+\Delta t}) \leq L | \mathcal{F}_t\right) \leq \alpha$$
$$\iff \Phi\left(\frac{\ln\left(\frac{L+\Upsilon_{t+\Delta t}}{V_t^\theta}\right) - \left(\theta_t'(\mu - r) + r - \frac{c_t}{V_t^\theta} - \frac{1}{2}\|\theta_t'\sigma\|^2\right)\Delta t}{\|\theta_t'\sigma\|\sqrt{\Delta t}}\right) \leq \alpha$$
$$\iff \ln\left(\frac{L + \Upsilon_{t+\Delta t}}{V_t^\theta}\right) \leq \Phi^{-1}(\alpha)\|\theta_t'\sigma\|\sqrt{\Delta t} + \left(\theta_t'(\mu - r) + r - \frac{c_t}{V_t^\theta} - \frac{1}{2}\|\theta_t'\sigma\|^2\right)\Delta t$$
$$\iff L \leq V_t^\theta \exp\left[\Phi^{-1}(\alpha)\|\theta_t'\sigma\|\sqrt{\Delta t} + \left(\theta_t'(\mu - r) + r - \frac{c_t}{V_t^\theta} - \frac{1}{2}\|\theta_t'\sigma\|^2\right)\Delta t\right]$$
$$- \Upsilon_{t+\Delta t},$$

which implies

$$Q_t^\alpha = V_t^\theta \exp\left[\Phi^{-1}(\alpha)\|\theta_t'\sigma\|\sqrt{\Delta t} + \left(\theta_t'(\mu - r) + r - \frac{c_t}{V_t^\theta} - \frac{1}{2}\|\theta_t'\sigma\|^2\right)\Delta t\right] - \Upsilon_{t+\Delta t}.$$

Therefore,

$$VaR_t^\alpha = (Q_t^\alpha)^- = -V_t^\theta \exp\left[\Phi^{-1}(\alpha)\|\theta_t'\sigma\|\sqrt{\Delta t}\right.$$
$$\left. + \left(\theta_t'(\mu - r) + r - \frac{c_t}{V_t^\theta} - \frac{1}{2}\|\theta_t'\sigma\|^2\right)\Delta t\right] + \Upsilon_{t+\Delta t}.$$

□

Tail conditional expectation is closely related to the Value-at-Risk concept, but overcomes some of the conceptual deficiencies of Value-at-Risk (cf. Rockafellar

and Uryasev [14] (2002)). In particular, it is a coherent risk measure (see Artzner et al. [2] (1997)).

Definition 2. ***(Tail conditional expectation)***
Consider distribution of the loss $G_{t+\Delta t} := \Upsilon_{t+\Delta t} - V_{t+\Delta t}$ *represented by a continuous distribution function* $F_{G_{t+\Delta t}}$ *with* $\int_{\mathbb{R}} |G_{t+\Delta t}| dF(G_{t+\Delta t}) < \infty$. *Then the* TCE_t^α *at confidence level* $(1-\alpha)$ *is defined as*

$$TCE_t^\alpha = \mathbb{E}_t \left\{ (\Upsilon_{t+\Delta t} - V^\theta_{t+\Delta t}) \geq VaR_t^\alpha | \mathcal{F}_t \right\}.$$

$$TCE_t^\alpha = \frac{\mathbb{E}_t \left\{ (\Upsilon_{t+\Delta t} - V^\theta_{t+\Delta t}) I((\Upsilon_{t+\Delta t} - V_{t+\Delta t}) \geq -Q_t^\alpha) | \mathcal{F}_t \right\}^+}{\alpha},$$

where $I(A)$ *is the indicator function of the set* A *and* $x^+ = \max[0, x]$.

In other words, the tail conditional expectation of wealth V_t at time t is the conditional expected value of the loss exceeding $(Q_t^\alpha)^-$. Again, given the log-normal distribution of asset returns, the TCE_t^α can be explicitly computed as can be seen in the following proposition.

Proposition 2. ***(Computation of tail conditional expectation)***
We have

$$TCE_t^\alpha = \frac{\alpha \Upsilon_{t+\Delta t} - V_t \left[\exp\left((\theta_t'(\mu - r) + r - \frac{c_t}{V_t^\theta}) \Delta t \right) \Phi\left(\Phi^{-1}(\alpha) - \|\theta_t' \sigma\| \sqrt{\Delta t} \right) \right]}{\alpha}.$$

where $\Phi(\cdot)$ *and* $\Phi^{-1}(\cdot)$ *denote the normal distribution and the inverse distribution functions.*

Proof.

$$\mathbb{E} \left\{ (\Upsilon_{t+\Delta t} - V^\theta_{t+\Delta t}) I(\Upsilon_{t+\Delta t} - V_{t+\Delta t} \geq -Q_t^\alpha) | \mathcal{F}_t \right\}$$

$$= \mathbb{E} \Bigg\{ \Bigg(\Upsilon_{t+\Delta t} - V_t^\theta \exp \Bigg[(\theta_t'(\mu - r) + r - \frac{c_t}{V_t^\theta} - \frac{1}{2} \|\theta_t' \sigma\|^2) \Delta t + \theta_t' \sigma (W_{t+\Delta t} - W_t) \Bigg] \Bigg) I(\Upsilon_{t+\Delta t} - V_{t+\Delta t} \geq -Q_t^\alpha) | \mathcal{F}_t \Bigg\} \quad (12)$$

The argument of the indicator function is evaluated as follows

$$\begin{aligned}\Upsilon_{t+\Delta t} - V_t^\theta \exp\left[(\theta_t'(\mu - r) + r - \frac{c_t}{V_t^\theta} - \frac{1}{2}\|\theta_t'\sigma\|^2)\Delta t + \theta_t'\sigma(W_{t+\Delta t} - W_t)\right] \\ \geq -V_t^\theta \exp\left[\Phi^{-1}(\alpha)\|\theta_t'\sigma\|\sqrt{\Delta t} + \left(\theta_t'(\mu - r) + r - \frac{c_t}{V_t^\theta} - \frac{1}{2}\|\theta_t'\sigma\|^2\right)\Delta t\right] \\ + \Upsilon_{t+\Delta t}\end{aligned}$$

$$\begin{aligned}\Rightarrow -V_t^\theta \exp\left[\theta_t'\sigma(W_{t+\Delta t} - W_t)\right] \geq -V_t^\theta \exp\left[\Phi^{-1}(\alpha)\|\theta_t'\sigma\|\sqrt{\Delta t}\right] \\ = \frac{\theta_t'\sigma(W_{t+\Delta t} - W_t)}{\|\theta_t'\sigma\|\sqrt{\Delta t}} \leq \Phi^{-1}(\alpha).\end{aligned}$$

Therefore (12) becomes

$$\begin{aligned}\mathbb{E}\left\{\left(\Upsilon_{t+\Delta t} - V_t \exp\left[(\theta_t'(\mu - r) + r - \frac{c_t}{V_t} - \frac{1}{2}\|\theta_t'\sigma\|^2)\Delta t\right.\right.\right. \\ \left.\left.\left. + \theta_t'\sigma(W_{t+\Delta t} - W_t)\right]\right) I\left(\frac{\theta_t'\sigma(W_{t+\Delta t} - W_t)}{\|\theta_t'\sigma\|\sqrt{\Delta t}} \leq \Phi^{-1}(\alpha)\right) |\mathcal{F}_t\right\}\end{aligned}$$

$$\begin{aligned}= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\Phi^{-1}(\alpha)} \left(\Upsilon_{t+\Delta t} - V_t^\theta \exp\left[(\theta_t'(\mu - r) + r - \frac{c_t}{V_t^\theta}\right.\right. \\ \left.\left. - \frac{1}{2}\|\theta_t'\sigma\|^2)\Delta t + \theta_t'\sigma x\sqrt{\Delta t}\right]\right) \exp\frac{-x}{2} dx\end{aligned}$$

$$\begin{aligned}= \alpha\Upsilon_{t+\Delta t} - V_t^\theta \left[\exp((\theta_t'(\mu - r) + r - \frac{c_t}{V_t^\theta})\Delta t)\right. \\ \left. \cdot \int_{-\infty}^{\Phi^{-1}(\alpha)} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{(x - \|\theta_t'\sigma\|\sqrt{\Delta t})^2}{2}\right) dx\right]\end{aligned}$$

We calculate the integral by change of variables and obtain

$$= \alpha\Upsilon_{t+\Delta t} - V_t^\theta \left[\exp\left((\theta_t'(\mu - r) + r - \frac{c_t}{V_t^\theta})\Delta t\right) \Phi\left(\Phi^{-1}(\alpha) - \|\theta_t'\sigma\|\sqrt{\Delta t}\right)\right].$$

Dividing by α, we obtain the Tail Conditional Expectation as

$$TCE_t^\alpha = \frac{\alpha\Upsilon_{t+\Delta t} - V_t^\theta \left[\exp\left((\theta_t'(\mu - r) + r - \frac{c_t}{V_t^\theta})\Delta t\right) \Phi\left(\Phi^{-1}(\alpha) - \|\theta_t'\sigma\|\sqrt{\Delta t}\right)\right]}{\alpha}.$$

□

4. **Problem statement**

We seek the optimal asset and consumption allocation that maximizes (over all allowable $\{\theta_t, c_t\}$) the expected utility of discounted terminal wealth at time T and consumption over the entire horizon $[0, T]$, for a risk averse investor who limits his risk by imposing an upper bound on the TCE.

In mathematical terms the final optimal control problem with TCE constraint is

$$\max_{\{\theta,\ c\}\in A(v)} \mathbb{E}_{0,V_0}\left\{\int_0^T e^{-\rho s}U^1(c_s,s)ds + e^{-\rho T}U^2(V_T,T)\right\}, \tag{13}$$

subject to the wealth dynamics

$$dV_t^\theta = \left[V_t^\theta(\theta_t'(\mu - r) + r)\right]dt - c_t dt + V_t^\theta \theta_t'\sigma dW_t,\ V_0^\theta = v$$

and the TCE constraint

$$\frac{1}{\alpha}\left(\alpha\Upsilon_{t+\Delta t} - V_t^\theta\left[\exp\left((\theta_t'(\mu - r) + r - \frac{c_t}{V_t^\theta})\Delta t\right)\right.\right.$$
$$\left.\left.\cdot\Phi\left(\Phi^{-1}(\alpha) - \|\theta_t'\sigma\|\sqrt{\Delta t}\right)\right]\right) \le \varepsilon,\ \forall\ t \in [0, T), \tag{14}$$

where $\mathbb{E}$ denotes the expectation operator, given $V_t^\theta = v$ (and given the chosen consumption and investment strategies), U^1 and U^2 are twice differentiable, increasing, concave utility functions (CRRA), ε is an upper bound on TCE and $\rho > 0$ is the rate at which consumption and terminal wealth are discounted. Furthermore, we let

$$U(x) = U^1(x) = U^2(x) = \frac{x^{1-\gamma}}{1-\gamma},$$

where $\gamma \in (0, \infty)\backslash\{1\}$.

5. **Optimality conditions**

In applying the dynamic programming approach we solve the HJB equation associated with the utility maximization problem (13). From Fleming and Rishel [6]

(1975) we have that the corresponding HJB equation is given by

$$\rho J(v,t) = \sup_{c_t \geq 0,\ \theta_t \in \mathbb{R}^n} \{U(c_t) + J_t(v,t) + J_v(v,t)\left(v[\theta_t'(\mu - r) + r] - c_t\right) + \frac{1}{2} J_{vv}(v,t) v^2 \theta_t' \sigma\sigma' \theta_t \Big\}, \quad (15)$$

subject to the terminal condition

$$J(v,T) = U(v),$$

where J, the value function is given by

$$J(v,t) = \max_{\{\theta,\ c\} \in A(v)} \mathbb{E}_{t,V_t} \left\{ \int_t^T e^{-\rho s} U(c_s, s) ds + e^{-\rho T} U(V_T, T) \right\}, \quad (16)$$

where subscripts on J denote partial derivatives and $V_t^\theta = v$, the wealth realization at time t.

In solving the HJB equation (15), the static optimization problem

$$\max_{c_t \geq 0,\ \theta_t \in \mathbb{R}^n} \left\{ U(c_t) + J_v(v,t)\left(v[\theta_t'(\mu - r) + r] - c_t\right) + \frac{1}{2} J_{vv}(v,t) v^2 \theta_t' \sigma\sigma' \theta_t \right\}, \quad (17)$$

subject to the TCE constraint (14) can be tackled separately to reduce the HJB equation (15) to a nonlinear partial differential equation of J only.

Introducing the Lagrange function $\mathcal{L}(\cdot)$ as

$$\begin{aligned} \mathcal{L}\left(\theta(v,t), c(v,t), \lambda(v,t)\right) &= J_v(v,t)\left(v\left[\theta_t'(\mu - r) + r - c_t\right]\right) \\ &+ \frac{1}{2} v^2 \|\theta_t' \sigma\|^2 J_{vv}(v,t) + U(c_t) - \lambda(v,t)\left(\alpha \Upsilon_{t+\Delta t}\right. \\ &- v\left[\exp\left((\theta_t'(\mu - r) + r - \frac{c_t}{v})\Delta t\right) \cdot \Phi\left(\Phi^{-1}(\alpha) - \|\theta_t' \sigma\| \sqrt{\Delta t}\right)\right] - \varepsilon_1\Big), \quad (18) \end{aligned}$$

where λ is the Lagrange multiplier and $\varepsilon_1 = \varepsilon \cdot \alpha$ and the first-order necessary conditions with respect to θ, c and λ respectively of the static optimization problem (18)

are given by

$$vJ_v(\mu - r) + \frac{1}{2}J_{vv}v^2\sigma\sigma'\theta_t + \lambda(v,t)v\left[(\mu - r)\Delta t \exp\left((\theta_t'(\mu - r) + r - \frac{c_t}{v})\Delta t\right)\right.$$
$$\cdot\Phi\left(\Phi^{-1}(\alpha) - \|\theta_t'\sigma\|\sqrt{\Delta t}\right) - \exp\left((\theta_t'(\mu - r) + r - \frac{c_t}{v})\Delta t\right) \cdot \frac{\sqrt{\Delta t}}{2}\frac{\sigma\sigma'\theta_t}{\|\theta_t'\sigma\|}$$
$$\left.\cdot\frac{1}{\sqrt{2\pi}}\exp\left[-\frac{1}{2}(\Phi^{-1}(\alpha) - \|\theta_t'\sigma\|\sqrt{\Delta t})^2\right]\right] = 0, \quad (19)$$

whereby $\pi \approx 3.14159$ and we have applied the product law of differentiation and the fundamental theorem of calculus in deriving the latter first-order derivative.

$$U_c(c_t) + \lambda(v,t)\Delta t \cdot \Phi\left(\Phi^{-1}(\alpha) - \|\theta_t^{'}\sigma\|\sqrt{\Delta t}\right)$$
$$\cdot\left[\exp\left((\theta_t'(\mu - r) + r - \frac{c_t}{v})\Delta t\right) \cdot \Phi\left(\Phi^{-1}(\alpha) - \|\theta_t^{'}\sigma\|\sqrt{\Delta t}\right)\right] = J_v(v,t), \quad (20)$$

where U_c is the first-order derivative of U with respect to c and

$$H(v,t) = \alpha\Upsilon_{t+\Delta t} + v\left[\exp\left((\theta_t'(\mu - r) + r - \frac{c_t}{v})\Delta t\right)\right.$$
$$\left.\cdot\Phi\left(\Phi^{-1}(\alpha) - \|\theta_t'\sigma\|\sqrt{\Delta t}\right)\right] + \varepsilon_1 = 0, \quad (21)$$

while the complimentary slackness condition is given as

$$\lambda(v,t)H(v,t) = 0, \quad (22)$$

$$\lambda(v,t) \geq 0.$$

Simultaneous resolution of these first-order conditions yields the optimal solutions θ^{opt}, c^{opt} and λ^{opt}. Substituting these into (15) gives the partial differential equation

$$-\rho J(v,t) + \frac{(c^{opt}(v,t))^{1-\gamma}}{1-\gamma} + J_t(v,t) + J_v(v,t)\left(v[(\theta^{opt}(v,t))'(\mu - r) + r]\right.$$
$$\left.-c^{opt}(v,t)\right) + \frac{1}{2}J_{vv}(v,t)v^2(\theta^{opt}(v,t))'\sigma\sigma'(\theta^{opt}(v,t)) = 0, \quad (23)$$

with terminal condition

$$J(v,T) = \frac{v^{1-\gamma}}{1-\gamma},$$

which can then be solved for the optimal value function $J^{opt}(v,t)$. Because of the non-linearity in θ^{opt} and c^{opt}, the first-order conditions together with the HJB equation

are a non-linear system so the stochastic differential equation (23) has no analytic solution and numerical methods such as Newton's method or Sequential Quadratic Programming (SQP)(see, e.g., Nocedal and Wright [13] (1999)) are required to solve for $\theta^{opt}(v,t)$, $c^{opt}(v,t)$, $\lambda^{opt}(v,t)$ and $J^{opt}(v,t)$ iteratively.

6. **Numerical method**

We use an iterative algorithm similar to that of Yiu [15] (2004) which yields a $\mathcal{C}^{2,1}$ approximation $\widehat{J}$ of the exact solution J. $\{\widehat{\theta}_t, \widehat{c}_t\}$ is the investment strategy related to $\widehat{J}$.

When the optimal solution strictly satisfies the TCE constraint (14), the Lagrange multiplier $\lambda(v,t)$ is zero. If the constraint is active, the multiplier is positive.

First, we divide the domain of resolution into a grid of $n_v \times n_t$ mesh points. Iterations are indexed by k.

1. For each point (t,v), with $t \in [0, \Delta t, \ldots, n_t \Delta t]$ and $v \in [0, \Delta v, \ldots, n_v \Delta v]$, we compute the value function $\widehat{J}^{k=0} = J(v,t)$ and the optimal strategy $\{\theta_t^{opt}, c_t^{opt}\}$ of the unconstrained problem. All Lagrange multipliers are set to zero, $\lambda_{t,v}^{k=0} = 0$. This solution is the starting point of the algorithm.
2. For all points of the grid, the constraint is checked. If the constraint is not active ($TCE_t^\alpha < \varepsilon$), the multiplier is zero $\lambda_{t,v}^{k+1} = 0$ and $\{\theta_t^{k+1}, c_t^{k+1}\}$ is the solution of a similar equation to that of the unconstrained case.

$$\lambda_{t,v}^{k+1} = 0,$$

$$\theta_t^{k+1} = -\frac{\widehat{J}_v}{v\widehat{J}_{vv}}(\mu - r)(\sigma^T\sigma)^{-1},$$

$$\widehat{U}_c(c_t^{k+1}) = \widehat{J}_v.$$

If the VaR_t^α constraint is active, ($VaR_t^\alpha \geq \varepsilon$), we solve a nonlinear system in $\lambda_{t,v}^{k+1}$, $\widehat{\theta}_t^{j+1}$ and $\widehat{c}_t^{j+1}$. This nonlinear system is composed of the first-order necessary conditions of the static optimization problem

(18). That system is numerically solved by the sequential quadratic programming method (see Nocedal and Wright [13] (1999)).

3. The last stage consists in the calculation of the value function $\widehat{J}^{k+1}$ according to the investment/consumption strategy $\{\widehat{\theta}_t^{k+1}, \widehat{c}_t^{k+1}\}$ as detailed below this algorithm.
4. Return to step 2 with $k = k+1$ until the error at time t from wealth level v, $\epsilon_{t,v}$, satisfies $|\epsilon_{t,v}| < 1 \cdot e^{-5}$, where

$$\epsilon_{t,v} = \hat{J}_t - \rho\hat{J}(v,t) + \hat{J}_v\left(v[(\hat{\theta}_t^{opt})'(\mu - r) + r] - c_t^{opt}\right) + \frac{1}{2}v^2\|(\hat{\theta}_t^{\ opt})'\sigma\|^2\hat{J}_{vv} + U(c_t^{opt}).$$

For the numerical solution of the partial differential equation (23) to obtain the value function we use the trial function

$$J(v,t) = f(t)\frac{v^{1-\gamma}}{1-\gamma}, \qquad f(T) = 1,$$

such that

$$\begin{aligned} J_t &= f'(t)\frac{1}{1-\gamma}v^{1-\gamma} \\ J_v &= f(t)v^{-\gamma} \\ J_{vv} &= -\gamma f(t)v^{-(\gamma+1)}. \end{aligned}$$

Substituting these partials in (23) and dividing by $v^{1-\gamma}$, after some tedious computation, we obtain the ordinary differential equation

$$f'(t) = -\kappa(\theta^{opt}(v,t), c^{opt}(v,t), v)f(t) - B(c^{opt}(v,t), v), \tag{24}$$

whereby

$$\kappa(\theta^{opt}(v,t), c^{opt}(v,t), v) = (1-\gamma)\left(\frac{-\rho}{1-\gamma} + (\theta^{opt}(v,t))'(\mu - r) - c^{opt}(v,t)v^{-1} - \frac{1}{2}v^2(\theta^{opt}(v,t))'\sigma\sigma'(\theta^{opt}(v,t))\right)$$

and

$$B(c^{opt}(v,t), v) = (c^{opt}(v,t))^{1-\gamma}v^{\gamma-1},$$

with terminal condition

$$f(T) = 1.$$

The function f in equation (24) is computed numerically by the Euler-Cauchy method (See Isaacson and Keller [8] (1994)).

We have implemented the above algorithm to illustrate the optimal portfolio of the preceding section with examples. To this end, we have written a program in MATLAB 7.0 to carry out the procedure and run it on a personal computer with an Intel Pentium IV processor. We assume that $n = 2$. That is, the market is composed of two risky stocks and a risk-free bond. Table 1 shows the parameters for the portfolio optimization problem and the underlying Black-Scholes model of the financial market. We achieve convergence in 300 seconds after three iterations.

Parameter	**Value**
Stock (S^1)	$\mu = 4\%$, $\sigma^{11} = 5\%$, $\sigma^{12} = 5\%$
Stock (S^2)	$\mu = 6\%$, $\sigma^{21} = 5\%$, $\sigma^{22} = 20\%$
Bond (S^0)	$r = 3\%$
Investment horizon	$t \in [0, 1]$
State of wealth	$v \in [0, 20]$
Shortfall probability	$\alpha = 1\%$
Value-at-Risk horizon	$\Delta t = \frac{1}{48} \approx 7$ days
No. of wealth mesh points	$N_v = 81$
Mesh size for wealth	$\Delta v = \frac{20}{80} = 0.25$
Utility function	$U(x) = \frac{x^{1-\gamma}}{1-\gamma}$, $\gamma = 0.9$

TABLE 1. Parameters for the consumption and investment portfolio optimization problem.

Wealth benchmark, Υ_t	**Bound**, ε
conditional expectation	0.3
Money market	1.0

TABLE 2. Bounds and benchmarks for the TCE-constrained problem.

7. **Simulations**

We consider the tail conditional expectation of the wealth surplus $V_t - \Upsilon_{t+\Delta t}$ with respect to the benchmark $\Upsilon_{t+\Delta t}$ such that it satisfies

$$TCE_t^{\alpha}(V_{t+\Delta t} - \Upsilon_{t+\Delta t}) \leq \varepsilon,$$

where ε comes from table 2. That is, the TCE is re-evaluated at each discrete time step (TCE horizon) Δt and kept below the upper bound ε, by making use of conditioning information. Figures 1 and 2 show in the right panel the amount of wealth invested in the risky assets with and without the TCE constraint, plotted against the possible wealth realization at different times. The left panel shows the value function.

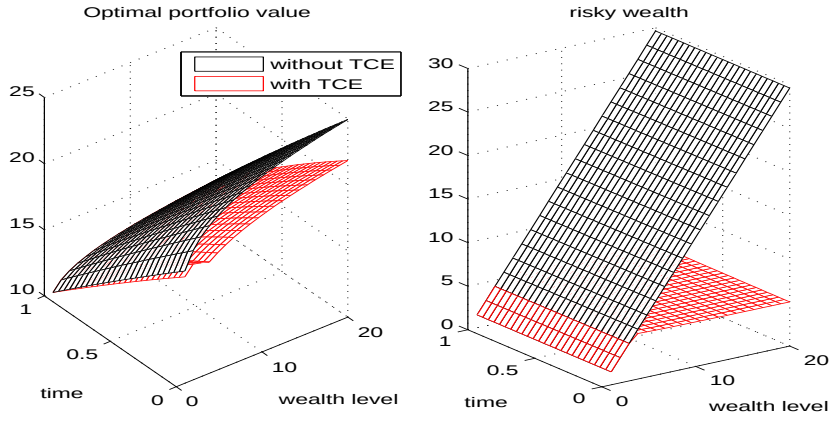

FIGURE 1. TCE when benchmark is the conditional expected wealth plotted against wealth at various times of the investment horizon. In red, TCE $\leq \varepsilon = 0.3$.

In Figure 1 the shortfall benchmark is taken to be the conditional expected wealth $\Upsilon_{t+\Delta t} = \mathbb{E}_t\{V_{t+\Delta t}\}$, given as

$$\Upsilon_{t+\Delta t} = \mathbb{E}_t\{V_{t+\Delta t}\} = V_t \exp\left[\left(\theta_t'(\mu - r) + r - \frac{c_t}{V_t}\right)\Delta t\right], \tag{25}$$

while in Figure 2 it is the investment in the risk-free bond $\Upsilon_{t+\Delta t} = V_t e^{r\Delta t}$.

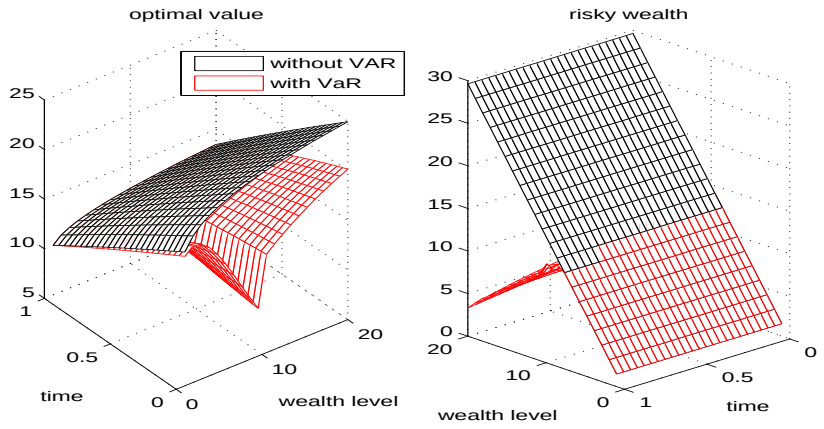

FIGURE 2. Effect of the TCE constraint when benchmark is investment in the bond.

As can be observed from the images, as the wealth level increases, so does the investment in risky assets. This results from the property of constant relative risk aversion of the utility function. A good control over the investment in the risky assets has been achieved and the proportions invested in the risky assets are reduced in order to fulfil the TCE constraint. In particular, when the constraint is not active, the optimal portfolio follows the unconstrained solution; as the portfolio value increases, the TCE constraint becomes active and allocates less to the risky assets. Figure 3 reveals to us that the local minimum (around wealth level 10) observed in the left panel of Figure 2 comes as a result of a sudden increase in the consumption rate once the constraint becomes active. The left panel of Figure 1 suggests that this increase in consumption is more subtle when we take as wealth benchmark, the conditional expected wealth.

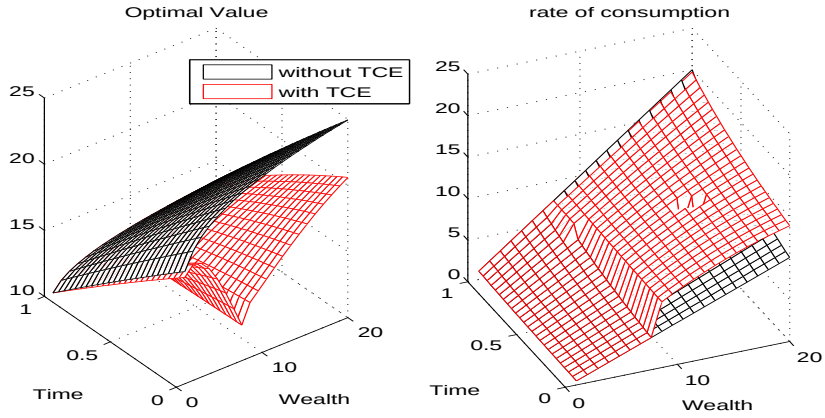

FIGURE 3. Effect of the TCE constraint on consumption when benchmark is investment in the bond.

The value function of the constrained problem is identical to that of the unconstrained one when the Lagrange multipliers are null, whereas it is inferior when the constraint is active.

8. **Concluding remarks**

Using a CRRA utility function, we have investigated how a bound imposed on TCE affects the optimal portfolio choice and consumption. In so doing, we have used dynamic wealth benchmarks - conditional expected wealth and investment in riskless stocks, whereby the TCE was re-evaluated at short intervals along the investment horizon. We deduce from our observations that the constraint reduces risky investment. Moreover, part of the wealth hitherto invested in risky assets is diverted to consumption when the constraint is tight.

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DIFFERENT TYPES OF PARALLEL APPROACHES FOR SOLVING NONLINEAR EQUATIONS BY NUMERICAL METHODS

IOANA CHIOREAN, DIANA BÂGIU, AND SEVER GROZE

Abstract. The purpose of this paper is to give and to compare different parallel approaches to some numerical methods for solving nonlinear equations on $\mathbb{R}$, such as the secant method and the Dekker's method.

1. Introduction

The solving of a nonlinear equation is a main part of Numerical Analysis. Lots of methods are known and many papers dealing with them were written. E.g., see [6] for the secant method, [1] for Dekker's method, [7] for Steffensen's method, all these methods are considered classical ones. In order to improve the rate of convergence, several other methods have been constructed, e.g. Brent's method (see [2]), Halley's method (see [8]), and so on.

All these methods have many serial implementations, used by different software packages.

Due to the requirement of improving the speed-up of execution, several parallel approaches of these methods were also developed. For instance, for the secant method, in [3], [4], for Steffensen's method, in [5], where the technique of nonlinear multisplitting is used.

In this paper we compare some possible parallel approaches for these methods, by using parallelism at the data level, or at the execution level.

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2. **A parallel approach for the secant method**

Assuming that $f \in C^2[a,b]$ and that there exists a number $x^* \in [a,b]$ with $f(x^*) = 0$, and $f'(x^*) \neq 0$, the well known secant formula for approximating the root x^* can be written as follows:

$$x_{k+1} = x_k - \frac{x_k - x_{k-1}}{f(x_k) - f(x_{k-1})} f(x_k) \tag{1}$$

for $k = 1, 2, \ldots$, with x_0 and x_1 two initial approximations.

Remark 1. The proof of the corresponding theorem can be found in [6].

Remark 2. The chord method is derived from (1), if we keep all the time one fixed initial approximation.

In order to make (1) appropriate for parallel calculus, and observing that it is a nonlinear recurrence relation, we have to linearize it. The technique is the following (see [4]): at every step $k = 1, 2, \ldots$ we denote by

$$y_k = \frac{f(x_1) - f(x_0)}{f(x_1)} \cdot \frac{f(x_2) - f(x_1)}{f(x_2)} \cdot \ldots \cdot \frac{f(x_k) - f(x_{k-1})}{f(x_k)} \tag{2}$$

and consider $y_0 = 1$.

Then, multiplying (1) by (2), at every step $k = 1, 2, \ldots$, we get the following relation:

$$x_{k+1} \cdot y_k = x_k \cdot y_k - (x_k - x_{k-1}) \cdot y_{k-1} \tag{3}$$

which is a linear recurrence relation with two terms.

Rearranging the terms, (3) becomes:

$$x_{k+1} = x_k \left(1 - \frac{y_{k-1}}{y_k}\right) + x_{k-1} \cdot \frac{y_{k-1}}{y_k} \tag{4}$$

which can be written in matricial form as:

$$\begin{bmatrix} x_{k+1} \\ x_k \end{bmatrix} = \begin{bmatrix} 1 - \dfrac{y_{k-1}}{y_k} & \dfrac{y_{k-1}}{y_k} \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_k \\ x_{k-1} \end{bmatrix} \tag{5}$$

or, denoting by $z_k = \begin{bmatrix} x_{k+1} \\ x_k \end{bmatrix}$ and by M_k the above matrix, we have

$$z_k = M_k \cdot z_{k-1}, \quad k = 1, 2, \ldots \tag{6}$$

In order to get the best approximation, x_n, according with a given error, we have to compute the matrices product

$$M_1 \cdot M_2 \cdot \ldots \cdot M_n$$

which can be performed in $\log_2 n$ steps on a binary tree connectivity. So, the execution time is not of order $O(n)$, but of order $O(\log_2 n)$.

Remark. In the case of the secant method, we gave a parallel approach at the data level.

3. **Dekker's method**

By combining the secant method with the bisection method (see [1]), Dekker's method is obtained.

Supposing the same hypothesis for the function f and the existence of a real solution, x^*, of the equation $f(x) = 0$ in the interval $[a, b]$, by Dekker's method, two provisional approximated values of the solution are computed, at every step $k = 1, 2, \ldots$:

- the first is given by the secant method:

$$s = b_k - \frac{b_k - b_{k-1}}{f(b_k) - f(b_{k-1})} \cdot f(b_k),$$

  where b_k and b_{k-1} are the current guesses

- the second one is given by the bisection method:

$$m = \frac{a_k + b_k}{2},$$

  where a_k is a point such as $f(a_k)$ and $f(b_k)$ have opposite signs, so the interval $[a_k, b_k]$ contains the solution. Furthermore, $|f(b_k)|$ should be less than or equal to $|f(a_k)|$, so that b_k is a better guess for the unknown than a_k.

Remark 1. $b_{-1} = a_0$.

Remark 2. If the result of the secant method, s, lies between b_k and m, then it becomes the next iterate ($b_{k+1} = s$), else the midpoint is used ($b_{k+1} = m$).

Then, the value of a_{k+1} is chosen such that $f(a_{k+1})$ and $f(b_{k+1})$ have opposite signs. If $f(a_k)$ and $f(b_{k+1})$ have opposite signs, then $a_{k+1} = a_k$, otherwise $a_{k+1} = b_k$.

Finally, if $|f(a_{k+1})| < |f(b_{k+1})|$, then a_{k+1} is a better guess for the solution than b_{k+1}, and hence the values of a_{k+1} and b_{k+1} are exchanged.

3.1. A parallel approach

In this case, due to the fact that at every step $k = 1, 2, \ldots,$ the current approximation is used in computing two values (s and m), another parallel approach may be more appropriate. If we consider two processors "slaves" and one processor "master", we have the following "master-slave" type of execution at every step k:

{Processor Master}
Repeat
Send Message-to-Slave (k, a_k, b_k, ind);
Get Message-from-Slave (s, m);
Makes-Comparison;
Until End-Condition;
{Processor-Slave}
Get Message-from-Master (k, a_k, b_k, ind);
If $ind = 1$ then Compute (s);
Send Message-to-Master (s);
else Compute (m);
Send Message-to-Master (m);
End

Remark 1. The variable "*ind*" stands for the identification number of each slave processor.

Remark 2. In the case of Dekker's method, we gave a parallel approach at the execution level.

Conclusions

In this paper we have presented different types of parallel approaches for two numerical methods for solving nonlinear equations. These methods were chosen because their expressions are appropriate for our purpose. Of course, other methods may be considered, with the same remarks. E.g., the bisection method requires an execution type of parallelism, Steffensen's method can be parallelized at the data level, and so on.

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LACUNARY STRONGLY ALMOST SUMMABLE SEQUENCES

MIKAIL ET AND AYSEGÜL GÖKHAN

Abstract. The purpose of this paper is to introduce the concepts of $_q-$ lacunary strongly almost convergence with respect to a modulus function and $_q-$lacunary almost statistical convergence. We establish some connections between $q-$ lacunary strongly almost convergence and $q-$ lacunary almost statistical convergence. It is also shown that if a sequence is $q-$lacunary strongly almost convergent with respect to a modulus function then it is $q-$lacunary almost statistically convergent.

1. Introduction

Let w denote the set of all real sequences $x = (x_n)$. By ℓ_∞ and c, we denote respectively the Banach space of bounded and the Banach space of convergent sequences $x = (x_n)$, both normed by $\|x\| = \sup_n |x_n|$. A linear functional $\mathcal{L}$ on ℓ_∞ is said to be a Banach limit [1] if it has the properties

i) $\mathcal{L}(x) \geq 0$ if $x \geq 0$ (i.e. $x_n \geq 0$ for all n),

ii) $\mathcal{L}(e) = 1$, where $e = (1, 1, \ldots)$,

iii) $\mathcal{L}(Dx) = \mathcal{L}(x)$,

where the shift operator D is defined by $(Dx_n) = (x_{n+1})$.

Let $\mathfrak{B}$ be the set of all Banach limits on ℓ_∞. A sequence x is said to be almost convergent to a number L if $\mathcal{L}(x) = L$ for all $\mathcal{L} \in \mathfrak{B}$. Lorentz [11] has shown that x is almost convergent to L if and only if

$$t_{km} = t_{km}(x) = \frac{x_m + x_{m+1} + \ldots + x_{m+k}}{k+1} \to L \text{ as } k \to \infty, \text{ uniformly in } m.$$

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Key words and phrases. statistical convergence, lacunary sequence, modulus function, almost convergence.

Let $\hat{c}$ denote the set of all almost convergent sequences. Maddox [12] and (independently) Freedman et al. [8] have defined x to be strongly almost convergent to a number L if

$$\frac{1}{k+1}\sum_{i=0}^{k}|x_{i+m}-L|\to 0 \text{ as } k\to\infty, \text{ uniformly in } m.$$

Let $[\hat{c}]$ denote the set of all strongly almost convergent sequences. It is easy to see that $[\hat{c}]\subset\hat{c}\subset\ell_\infty$. Das and Sahoo [5] defined the sequence space

$$[w(p)]=\left\{x\in w:\frac{1}{n+1}\sum_{k=0}^{n}|t_{km}(x-L)|^{p_k}\to 0 \text{ as } n\to\infty, \text{ uniformly in } m\right\}$$

and investigated some of its properties.

The notion of statistical convergence was introduced by Fast [7] and Schoenberg [24] independently. Later on it was further investigated from sequence space point of view and linked with summability theory by Başarır [2], Fridy [9], Maddox [15], Nuray and Savaş [18], Tripathy ([20],[21]) and Salat [23]. Recently, statistical convergence has been studied by various authors ($cf.$ [3], [16], [17]).

The statistical convergence is depended on the density of subsets of $\mathbb{N}$, the set of natural numbers. A subset E of $\mathbb{N}$ is said to have density $\delta(E)$ if

$$\delta(E)=\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\chi_E(k) \text{ exists},$$

where χ_E is the characteristic function of E.

A sequence $x\in w$ is said to be statistically convergent to L if for every $\varepsilon>0$, $\delta(\{k\in\mathbb{N}:|x_k-L|\geq\varepsilon\})=0$. In this case we write stat-lim $x_k=L$.

Let $\theta=(k_r)$ be the sequence of positive integers such that $k_0=0$, $0<k_r<k_{r+1}$ and $h_r=k_r-k_{r-1}\to\infty$ as $r\to\infty$. Then θ is called a lacunary sequence. The intervals determined by θ will be denoted by $I_r=(k_{r-1},k_r]$ and the ratio k_r/k_{r-1} will be denoted by η_r.

Lacunary sequences have been studied in [4], [8], [10], [19].

We recall that a modulus f is a function from [0,∞) to [0,∞) such that

i) $f(x)=0$ if and only if $x=0$,

ii) $f(x+y)\leq f(x)+f(y)$ for $x,y\geq 0$,

iii) f is increasing,

iv) f is continuous from the right at 0.

It follows that f must be continuous everywhere on $[0,\infty)$. A modulus may be unbounded or bounded. Ruckle [22] and Maddox [13] used a modulus f to construct some sequence spaces.

A sequence space E is said to be solid (or normal) if $(\alpha_k x_k) \in E$ whenever $(x_k) \in E$ for all sequences (α_k) of scalars with $|\alpha_k| \leq 1$ for all $k \in \mathbb{N}$.

2. **Definitions and Preliminaries**

Let f be a modulus function, $p = (p_k)$ be a sequence of positive real numbers and X be a seminormed space over the field $\mathbb{C}$ of complex numbers with the seminorm q. $w(X)$ denotes the space of all sequences $x = (x_k)$, where $x_k \in X$. We define the following sequence spaces:

$$(w,\theta,f,p,q) = \{x \in w(X) : \lim_r \frac{1}{h_r} \sum_{k\in I_r} [f(q(t_{km}(x-L)))]^{p_k} = 0,$$

$$\text{uniformly in } m, \text{for some } L\},$$

$$(w,\theta,f,p,q)_0 = \{x \in w(X) : \lim_r \frac{1}{h_r} \sum_{k\in I_r} [f(q(t_{km}(x)))]^{p_k} = 0, \text{ uniformly in } m\},$$

$$(w,\theta,f,p,q)_\infty = \{x \in w(X) : \sup_{r,m} \frac{1}{h_r} \sum_{k\in I_r} [f(q(t_{km}(x)))]^{p_k} < \infty\}.$$

Throughout the paper Z denotes 0, 1 or ∞. We get the following sequence spaces from the above sequence spaces on giving particular values to θ, f and p.

i) If $p_k = 1$ for all $k \in \mathbb{N}$, then we shall write $(w,\theta,f,q)_Z$ instead of $(w,\theta,f,p,q)_Z$.

If $x \in (w,\theta,f,q)$ we say that x is $q-$lacunary almost strongly convergent with respect to the modulus function f.

ii) Taking $p_k = 1$ for all $k \in \mathbb{N}$ and $f(x) = x$, we denote the above sequence spaces by $(w,\theta,q)_Z$.

iii) In the case $\theta = (2^r)$, then we shall denote the above sequence spaces by $(w,f,p,q)_Z$.

Theorem 2.1 Let f be a modulus function, then $(w,\theta,f,p,q)_0 \subset (w,\theta,f,p,q) \subset (w,\theta,f,p,q)_\infty$.

Proof. The first inclusion is obvious. We establish the second inclusion. Let $x \in (w,\theta,f,p,q)$. By definition of f we have

$$\frac{1}{h_r}\sum_{k\in I_r}\left[f\left(q\left(t_{km}\left(x\right)\right)\right)\right]^{p_k} = \frac{1}{h_r}\sum_{k\in I_r}\left[f\left(q\left(t_{km}\left(x-L+L\right)\right)\right)\right]^{p_k}$$

$$\leq C\frac{1}{h_r}\sum_{k\in I_r}\left[f\left(q\left(t_{km}\left(x-L\right)\right)\right)\right]^{p_k} + C\frac{1}{h_r}\sum_{k\in I_r}\left[f\left(q\left(L\right)\right)\right]^{p_k}.$$

There exists a positive integer K_L such that $q(L) \leq K_L$. Hence we have

$$\frac{1}{h_r}\sum_{k\in I_r}\left[f\left(q\left(t_{km}\left(x\right)\right)\right)\right]^{p_k} \leq C\frac{1}{h_r}\sum_{k\in I_r}\left[f\left(q\left(t_{km}\left(x-L\right)\right)\right)\right]^{p_k} + \frac{C}{h_r}\left[K_L f(1)\right]^H h_r,$$

where $\sup_k p_k = H$ and $C = \max(1, 2^{H-1})$. Since $x \in (w,\theta,f,p,q)$, we have $x \in (w,\theta,f,p,q)_\infty$ and this completes the proof.

The following theorem can be proved using the same technique of Theorem 2.1 of Et [6], therefore we give without proof.

Theorem 2.2 Let the sequence (p_k) be bounded, then $(w,\theta,f,p,q)_Z$ are linear spaces over the set of complex numbers.

The proof of the following results are easy and thus omitted.

Theorem 2.3 Let f, f_1, f_2 be modulus function. For any two sequences $p = (p_k)$ and $t = (t_k)$ of strictly positive real numbers and any two seminorms q_1 , q_2 we have

i) $(w,\theta,f_1,q)_Z \subset (w,\theta,f\circ f_1,q)_Z$,

ii) $(w,\theta,f_1,p,q)_Z \cap (w,\theta,f_2,p,q)_Z \subset (w,\theta,f_1+f_2,p,q)_Z$,

iii) $(w,\theta,f,p,q_1)_Z \cap (w,\theta,f,p,q_2) \subset (w,\theta,f,p,q_1+q_2)$,

iv) If q_1 is stronger than q_2 then $(w,\theta,f,p,q_1)_Z \subset (w,\theta,f,p,q_2)_Z$,

v) If q_1 equivalent to q_2 then $(w,\theta,f,p,q_1)_Z = (w,\theta,f,p,q_2)_Z$,

vi) $(w,\theta,f,p,q)_Z \cap (w,\theta,f,t,q)_Z \neq \emptyset$.

The following result is a consequence of Theorem 2.3 (i).

Proposition 2.4 Let f be a modulus function. Then $(w,\theta,q)_Z \subset (w,\theta,f,q)_Z$.

Theorem 2.5 Let f be a modulus function, if $\lim \frac{f(t)}{t} = \beta > 0$, then $(w,\theta,q) = (w,\theta,f,q)$.

Proof. By Proposition 2.4, we need only show that $(w,\theta,f,q) \subset (w,\theta,q)$. Let $\beta > 0$ and
$x \in (w,\theta,f,q)$. Since $\beta > 0$, we have $f(t) \geq \beta t$ for all $t \geq 0$. Hence we have

$$\frac{1}{h_r}\sum_{k\in I_r} f\left(q\left(t_{km}\left(x-L\right)\right)\right) \geq \frac{\beta}{h_r}\sum_{k\in I_r} q\left(t_{km}\left(x-L\right)\right).$$

Therefore we have $x \in (w,\theta,q)$.

Theorem 2.6 Let $0 < p_k \leq t_k$ and $\left(\frac{t_k}{p_k}\right)$ be bounded, then $(w,\theta,f,t,q)_Z \subset (w,\theta,f,p,q)_Z$.

Proof. If we take $w_{km} = \left[f\left(q\left(t_{km}\left(x\right)\right)\right)\right]^{t_k}$ for all k,m and using the same technique of Theorem 5 of Maddox [14], it is easy to prove this Theorem.

Theorem 2.7 The sequence spaces $(w,\theta,f,p,q)_0$ and $(w,\theta,f,p,q)_\infty$ are not solid.

Proof. We give the proof only for $(w,\theta,f,p,q)_0$. For this let $p_k = 1$ for all $k \in \mathbb{N}$, $\theta = (2^r)$, $f(x) = x$ and $q(x) = |x|$. Consider the sequence $x_k = (-1)^k$ for all $k \in \mathbb{N}$ and (α_k) be defined as $\alpha_k = (-1)^k$ for all $k \in \mathbb{N}$. Then $(x_k) \in (w,\theta,f,p,q)_0$ but $(\alpha_k x_k) \notin (w,\theta,f,p,q)_0$. Hence $(w,\theta,f,p,q)_0$ is not solid.

Theorem 2.8 Let $\theta = (k_r)$ be a lacunary sequence. If $1 < \liminf_r \eta_r \leq \limsup_r \eta_r < \infty$ then for any modulus function f, we have $(w,f,p,q)_0 = (w,\theta,f,p,q)_0$.

Proof. Suppose $\liminf_r \eta_r > 1$ then there exist $\delta > 0$ such that $\eta_r = \left(\frac{k_r}{k_{r-1}}\right) \geq 1+\delta$ for all $r \geq 1$. Then for $x \in (w,f,p,q)_0$, we write

$$\frac{1}{h_r}\sum_{k\in I_r}\left[f\left(q\left(t_{km}\left(x\right)\right)\right)\right]^{p_k} = \frac{1}{h_r}\sum_{k=1}^{k_r}\left[f\left(q\left(t_{km}\left(x\right)\right)\right)\right]^{p_k} - \frac{1}{h_r}\sum_{k=1}^{k_{r-1}}\left[f\left(q\left(t_{km}\left(x\right)\right)\right)\right]^{p_k}$$

$$= \frac{k_r}{h_r}\left(k_r^{-1}\sum_{k=1}^{k_r}\left[f\left(q\left(t_{km}\left(x\right)\right)\right)\right]^{p_k}\right) - \frac{k_{r-1}}{h_r}\left(k_{r-1}^{-1}\sum_{k=1}^{k_{r-1}}\left[f\left(q\left(t_{km}\left(x\right)\right)\right)\right]^{p_k}\right).$$

Since $h_r = k_r - k_{r-1}$, we have

$$\frac{k_r}{h_r} \leq \frac{1+\delta}{\delta}$$

and

$$\frac{k_{r-1}}{h_r} \leq \frac{1}{\delta}.$$

The terms $k_r^{-1}\sum_{k=1}^{k_r}\left[f\left(q\left(t_{km}\left(x\right)\right)\right)\right]^{p_k}$ and $k_{r-1}^{-1}\sum_{k=1}^{k_{r-1}}\left[f\left(q\left(t_{km}\left(x\right)\right)\right)\right]^{p_k}$ both converge to zero, uniformly in m and it follows that

$$\frac{1}{h_r}\sum_{k\in I_r}\left[f\left(q\left(t_{km}\left(x\right)\right)\right)\right]^{p_k} \to 0,$$

as $r \to \infty$ uniformly in m, that is, $x \in \left(w, \theta, f, p, q\right)_0$.

If $\limsup_r \eta_r < \infty$, there exists $B > 0$ such that $\eta_r < B$ for all $r \geq 1$. Let $x \in \left(w, \theta, f, p, q\right)_0$ and $\varepsilon > 0$ be given. Then there exits $R > 0$ such that for every $j \geq R$ and all m

$$A_j = \frac{1}{h_j}\sum_{k\in I_j}\left[f\left(q\left(t_{km}\left(x\right)\right)\right)\right]^{p_k} < \varepsilon.$$

We can also find $K > 0$ such that $A_j < K$ for all $j = 1, 2, \ldots$. Now let t be any integer with $k_{r-1} < t \leq k_r$, where $r > R$. Then

$$t^{-1}\sum_{k=1}^{t}\left[f\left(q\left(t_{km}\left(x\right)\right)\right)\right]^{p_k} \leq k_{r-1}^{-1}\sum_{k=1}^{k_r}\left[f\left(q\left(t_{km}\left(x\right)\right)\right)\right]^{p_k}$$

$$= k_{r-1}^{-1}\sum_{k\in I_1}\left[f\left(q\left(t_{km}\left(x\right)\right)\right)\right]^{p_k} + k_{r-1}^{-1}\sum_{k\in I_2}\left[f\left(q\left(t_{km}\left(x\right)\right)\right)\right]^{p_k} +$$

$$\ldots + k_{r-1}^{-1}\sum_{k\in I_r}\left[f\left(q\left(t_{km}\left(x\right)\right)\right)\right]^{p_k}$$

$$= \frac{k_1}{k_{r-1}}k_1^{-1}\sum_{k\in I_1}\left[f\left(q\left(t_{km}\left(x\right)\right)\right)\right]^{p_k} + \frac{k_2-k_1}{k_{r-1}}\left(k_2-k_1\right)^{-1}\sum_{k\in I_2}\left[f\left(q\left(t_{km}\left(x\right)\right)\right)\right]^{p_k}$$

$$+\ldots + \frac{k_R-k_{R-1}}{k_{r-1}}\left(k_R-k_{R-1}\right)^{-1}\sum_{k\in I_R}\left[f\left(q\left(t_{km}\left(x\right)\right)\right)\right]^{p_k} +$$

$$\ldots + \frac{k_r-k_{r-1}}{k_{r-1}}\left(k_r-k_{r-1}\right)^{-1}\sum_{k\in I_r}\left[f\left(q\left(t_{km}\left(x\right)\right)\right)\right]^{p_k}$$

$$= \frac{k_1}{k_{r-1}}A_1 + \frac{k_2-k_1}{k_{r-1}}A_2 + \ldots + \frac{k_R-k_{R-1}}{k_{r-1}}A_R + \frac{k_{R+1}-k_R}{k_{r-1}}A_{R+1} + \ldots + \frac{k_r-k_{r-1}}{k_{r-1}}A_r$$

$$\leq \left(\sup_{j\geq 1} A_j\right)\frac{k_R}{k_{r-1}} + \left(\sup_{j\geq R} A_j\right)\frac{k_r-k_R}{k_{r-1}} \leq K\frac{k_R}{k_{r-1}} + \varepsilon B.$$

Since $k_{r-1} \to \infty$ as $t \to \infty$, it follows that $t^{-1}\sum_{k=1}^{t}\left[f\left(q\left(t_{km}\left(x\right)\right)\right)\right]^{p_k} \to 0$ uniformly in m and consequently $x \in \left(w, f, p, q\right)_0$.

3. $q-$ lacunary almost statistical convergence

In this section we give some relations between $q-$lacunary almost statistical convergence and $q-$lacunary strongly almost convergence with respect to the modulus functions f.

Definition 3.1 ([3]) Let θ be a lacunary sequence, then the sequence $x = (x_k)$ is said to be $q-$lacunary almost statistically convergent to the number L provided that for every $\varepsilon > 0$,

$$\lim_r \frac{1}{h_r}\left|\left\{k \in I_r : q\left(t_{km}\left(x - L\right)\right) \geq \varepsilon\right\}\right| = 0, \text{ uniformly in } m.$$

In this case we write $\left[S_\theta\right]_q - \lim x = L$ or $x_k \to L\left(\left[S_\theta\right]_q\right)$ and we define

$$\left[S_\theta\right]_q = \left\{x \in w\left(X\right) : \ \left[S_\theta\right]_q - \lim x = L, \text{ for some } L\right\}.$$

In the case $\theta = (2^r)$, we shall write $\left[S\right]_q$ instead of $\left[S_\theta\right]_q$.

Theorem 3.2 Let f be a modulus function and $0 < h = \inf_k p_k \leq p_k \leq \sup_k p_k = H < \infty$. Then $\left(w, \theta, f, p, q\right) \subset \left[S_\theta\right]_q$.

Proof. Let $x \in \left(w, \theta, f, p, q\right)$ and $\varepsilon > 0$ be given. Then

$$\frac{1}{h_r}\sum_{k\in I_r}\left[f\left(q\left(t_{km}\left(x-L\right)\right)\right)\right]^{p_k} \geq \frac{1}{h_r}\sum_{\substack{k\in I_r \\ q(t_{km}(x-L))\geq\varepsilon}}\left[f\left(q\left(t_{km}\left(x-L\right)\right)\right)\right]^{p_k}$$

$$\begin{aligned}
&\geq \frac{1}{h_r} \sum_{\substack{k\in I_r \\ q(t_{km}(x-L))\geq\varepsilon}} [f(\varepsilon)]^{p_k} \\
&\geq \frac{1}{h_r} \sum_{\substack{k\in I_r \\ q(t_{km}(x-L))\geq\varepsilon}} \min\left([f(\varepsilon)]^h, [f(\varepsilon)]^H\right) \\
&\geq \frac{1}{h_r} \left|\{k\in I_r : q(t_{km}(x-L))\geq\varepsilon\}\right| \min\left([f(\varepsilon)]^h, [f(\varepsilon)]^H\right).
\end{aligned}$$

Hence $x \in [S_\theta]_q$.

Theorem 3.3 Let f be bounded and $0 < h = \inf_k p_k \leq p_k \leq \sup_k p_k = H < \infty$. Then $[S_\theta]_q \subset (w, \theta, f, p, q)$.

Proof. Suppose that f is bounded. Then there exists an integer K such that $f(t) < K$, for all $t \geq 0$. Then

$$\frac{1}{h_r}\sum_{k\in I_r}[f(q(t_{km}(x-L)))]^{p_k} = \frac{1}{h_r}\sum_{\substack{k\in I_r \\ q(t_{km}(x-L))\geq\varepsilon}}[f(q(t_{km}(x-L)))]^{p_k}$$

$$+\frac{1}{h_r}\sum_{\substack{k\in I_r \\ q(t_{km}(x-L))<\varepsilon}}[f(q(t_{km}(x-L)))]^{p_k}$$

$$\leq \frac{1}{h_r}\sum_{\substack{k\in I_r \\ q(t_{km}(x-L))\geq\varepsilon}} \max\left(K^h, K^H\right) + \frac{1}{h_r}\sum_{\substack{k\in I_r \\ q(t_{km}(x-L))<\varepsilon}}[f(\varepsilon)]^{p_k}$$

$$\leq \max\left(K^h, K^H\right)\frac{1}{h_r}\left|\{k\in I_r : q(t_{km}(x-L))\geq\varepsilon\}\right| + \max\left([f(\varepsilon)]^h,\ [f(\varepsilon)]^H\right).$$

Hence $x \in (w, \theta, f, p, q)$.

Theorem 3.4 $[S_\theta]_q = (w, \theta, f, p, q)$ if and only if f is bounded.

Proof. Let f be bounded. By the Theorem 3.2 and Theorem 3.3, we have $[S_\theta]_q = (w, \theta, f, p, q)$.

Conversely, suppose that f is unbounded. Then there exists a positive sequence (t_n) with $f(t_n) = n^2$, $n = 1, 2, \cdots$. If we choose

$$x_k = \begin{cases} t_n, & k = n^2,\ n = 1, 2, \ldots \\ 0, & \text{otherwise} \end{cases}$$

then we have

$$\frac{1}{n} \left|\{k \leq n : |x_k| \geq \varepsilon\}\right| \leq \frac{\sqrt{n}}{n} \to 0,\ n \to \infty.$$

Hence $x_k \to 0([S_\theta]_q)$ for $t_{0m}(x) = x_m$, $\theta = (2^r)$ and $q(x) = |x|$, but $x \notin (w, \theta, f, q)$. This contradicts to $[S_\theta]_q = (w, \theta, f, p, q)$.

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A STOCHASTIC MODEL FOR THE GROWTH OF CANCER TUMORS

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Abstract. In this paper we study a stochastic model for the behavior of cancer tumors, described by a stochastic differential equation with multiplicative noise term. We consider that the number of tumor cells is influenced by the drug therapy and by random perturbations. We study the existence of the solution process, as well as its behavior in the framework of stochastic inclusion problems and random dynamical systems (long time behavior). Computer simulations are also given.

1. Introduction

Different types of mathematical models of cancer progression and treatment have already been constructed. They simulate important elements of the complex process of tumor growth and response to the therapy, the effects and interactions between tumor cells and immune cells. For example, there are many papers written on the subject of optimal control for mathematical models in cancer chemotherapy, such as J.M. Murray [17], K.R. Fister and J.C. Panetta [11], L.G. de Pillis and A.E. Radunskaya [8], [9], L. G. Hanin, S. T. Rachev and A. Yu. Yakovlev [13] etc. In the last years, stochastic growth models for cancer cells were developed, we mention the papers of W.Y. Tan and C.W. Chen [20], N. Komarova [15], G. Albano and V. Giorno [1], L. Ferrante, S. Bompadre, L. Possati and L. Leone [10], A. Boondirek,

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Y. Lenbury, J. Wong-Ekkabut, W. Triampo, I.M. Tang, P. Picha [3]. Also stochastic optimal control problems in chemotherapy were investigated by A.J. Coldman and J.M. Murray [6].

Following the models developed by G. W. Swan [19] and continued by W. Krabs [16] we complete their results by studying a *growth model for tumor cells under the influence of random perturbations*. We especially study a growth model with multiplicative noise term for which we investigate the existence of the solution (Section 2). We consider that the size of the tumor is controlled by the function which is the drug dose and rewrite our control problem as a differential inclusion problem (Section 3). Furthermore we investigate the long time behavior of our model in the framework of random dynamical systems (Section 4). At the end we also give some computer simulations of the solutions and the solution-tube for different possible functions for the drug exposure (Section 5). This article is the starting point for further research on stochastic control problems in cancer growth models.

2. **The stochastic model**

We denote by $p(t)$ the number of cancerous tumor cells at time $t > 0$. In the book of G.W. Swan [19] the following model for the number of tumor cells in the absence of drugs is studied: $dp(t) = \lambda \ln\left(\frac{\mu}{p(t)}\right) p(t)dt,\ p(0) = p_0 > 0$, where $\lambda, \mu > 0$ are parameters. In [19] the following controlled cancer tumor growth model under influence of drugs is given

$$dp(t) = \left(\lambda \ln\left(\frac{\mu}{p(t)}\right) - G(v(t))\right) p(t)dt,\ p(0) = p_0 > 0, \tag{1}$$

where $v(t) > 0$ is the *dose of the drug* at time t, $G(v(t))$ is the *destroying rate* per tumor cell and time unit. W. Krabs uses in [16] the following monotone increasing and bounded function for G:

$$G(v) = \frac{k_1 v}{k_2 + v}, \tag{2}$$

where $k_1, k_2 > 0$ are constants. The optimal control problem of finding the control function $v > 0$ for which the drug exposure on the body is minimal is studied in [16]

$$\begin{cases} \int_0^T v(t) \to \min \\ p \text{ is a solution of (1) for } t \in [0,T] \\ p(T) = p_T, \end{cases}$$

where the values $T > 0$ and $p_T \in (0, \mu)$ are given. It is showed that the optimal control v has the form $v(t) = \sqrt{\frac{k_1 k_2}{D}} e^{\lambda t} - k_2,\ t \in [0,T]$, where D is a parameter. If $k_1 > D\, k_2$, then it is assured that $v(t) > 0$ for all $[0,T]$.

The aim of our paper is to generalize the model (1) to the stochastic case: Let $\left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P\right)$ be a filtered probability space and let $(W(t))_{t\geq 0}$ be a standard Wiener process adapted to the filtration $(\mathcal{F}_t)_{t\geq 0}$. We perturb (1) by a multiplicative noise term and consider the following stochastic differential equation with stochastic Itô integral

$$p(t) = p_0 + \int_0^t \left(\lambda \ln\left(\frac{\mu}{p(s)}\right) - G(v(s))\right) p(s)ds + \sigma \int_0^t p(s)dW(s), t \geq 0, \qquad (3)$$

where $p_0 > 0$ and $\sigma \in \mathbb{R}$ is a parameter. We assume that $G, v : \mathbb{R}_+ \times \Omega \to \mathbb{R}_+$ are processes that are measurable, adapted to the filtration $(\mathcal{F}_t)_{t\geq 0}$ and are a.s. locally bounded.

Theorem 1. *Equation* (3) *has a unique solution which has the following explicit form*

$$p(t) = (p_0)^{e^{-\lambda t}} \mu^{1-e^{-\lambda t}} \exp\left\{ -\int_0^t e^{\lambda(s-t)} G(v(s))ds + \sigma \int_0^t e^{\lambda(s-t)} dW(s) \right\} \qquad (4)$$

for a.e. $\omega \in \Omega$ *and all* $t \geq 0$.

Proof. We consider two geometric Brownian motions (see [14, pg. 349]) starting at $x_0 = 1$ given by $B(t) = \exp\{\sigma W(t)\}$ and $\beta(t) = \exp\{-\sigma W(t)\}$, $t \geq 0$, which are the

solutions of the following linear equations

$$B(t) = 1 + \frac{\sigma^2}{2}\int_0^t B(s)ds + \sigma\int_0^t B(s)dW(s),\ t \geq 0, \tag{5}$$

$$\beta(t) = 1 + \frac{\sigma^2}{2}\int_0^t \beta(s)ds - \sigma\int_0^t \beta(s)dW(s),\ t \geq 0. \tag{6}$$

First, we prove that the solution of (3) is *unique*: Let $(p(t))_{t\geq 0}$ be a solution of (3). Applying the Itô formula for $Z := p \cdot \beta$ (see [14, Theorem 3.6]) we obtain from (3) and (6) that

$$Z(t) = p_0 + \int_0^t \left(\lambda \ln\left(\frac{\mu}{Z(s)}\right) - G(v(s) - \lambda\sigma W(s))\right) Z(s)ds,\ t \geq 0. \tag{7}$$

We denote $Y := \ln Z$, then $(Y(t))_{t\geq 0}$ satisfies the equation

$$Y(t) = \ln(p_0) + \int_0^t \left(\lambda \ln(\mu) - G(v(s)) - \lambda\sigma W(s) - \lambda Y(s)\right) ds,\ t \geq 0. \tag{8}$$

Obviously, the solution of (8) is unique (it is linear in Y), hence the solution of (3) must also be unique. Now, we prove the *existence of the solution* of (3). Note, that the solution of (8) has the explicit form

$$Y(t) = e^{-\lambda t}\ln(p_0) + \int_0^t e^{\lambda(s-t)}\left(\lambda \ln(\mu) - G(v(s)) - \lambda\sigma W(s)\right) ds,\ t \geq 0.$$

Then, $Z(t) = \exp\{Y(t)\}$ satisfies equation (7). By using the Itô formula for $Z \cdot B$ (see [14, Theorem 3.6]) we obtain from (7) and (5) that $Z \cdot B$ is a solution of (3). From the uniqueness of the solution of (3) it follows that

$$\begin{aligned} p(t) &= Z(t)B(t) = \exp\left\{Y(t) + \sigma W(t)\right\} \\ &= \exp\left\{e^{-\lambda t}\ln(p_0) + \int_0^t e^{\lambda(s-t)}\left(\lambda \ln(\mu) - G(v(s)) - \lambda\sigma W(s)\right) ds + \sigma W(t)\right\}. \end{aligned}$$

By calculations we get that the explicit form for p is given in (4). □

Remark 1. We introduce the set $\Omega^* \subset \Omega$ with $P(\Omega^*) = 1$ such that for all $\omega \in \Omega^*$ it hold:

- $e^{\lambda t}W(t) = \int_0^t e^{\lambda s}dW(s) + \lambda \int_0^t e^{\lambda s}W(s)ds$ for all $t \geq 0$;
- W has sublinear growth at $\pm\infty$, i.e. $\lim_{t\to\pm\infty} \dfrac{W(t)}{t} = 0$ for all $\omega \in \Omega^*$;
- the *Ornstein-Uhlenbeck process*

$$O(t) = \sigma \int_0^t e^{\lambda(s-t)}dW(s) \quad \text{for all } t \geq 0, \tag{9}$$

is well defined. ◊

Remark 2. Without loss of generality, we can say that (4) holds for all $\omega \in \Omega^*$. Since in the expression (4) appears the *Ornstein-Uhlenbeck process* $\Big(O(t)\Big)_{t\geq 0}$ which is a zero-mean Gaussian process with variance $\nu(t) = \mathrm{Var}(O(t)) = \dfrac{\sigma^2}{2\lambda}(1 - e^{-2\lambda t})$, we can compute the expected number of tumor cells at time $t > 0$ by

$$E(p(t)) = (p_0)^{e^{-\lambda t}} \mu^{1-e^{-\lambda t}} E\left(\exp\left\{-\int_0^t e^{\lambda(s-t)}G(v(s))ds + O(t)\right\}\right).$$

If, G and v are independent of the process W, then

$$E(p(t)) = (p_0)^{e^{-\lambda t}} \mu^{1-e^{-\lambda t}} E\left(\exp\left\{-\int_0^t e^{\lambda(s-t)}G(v(s))ds\right\}\right) E(\exp\{O(t)\}).$$

But, $E(\exp\{O(t)\}) = \exp\left\{\dfrac{\nu(t)}{2}\right\}$, then in this case we obtain

$$E(p(t)) = (p_0)^{e^{-\lambda t}} \mu^{1-e^{-\lambda t}} \exp\left\{\frac{\sigma^2}{4\lambda}(1 - e^{-2\lambda t})\right\} E\left(\exp\left\{-\int_0^t e^{\lambda(s-t)}G(v(s))ds\right\}\right).$$

Moreover, if G and v do not depend on ω, then

$$E(p(t)) = (p_0)^{e^{-\lambda t}} \mu^{1-e^{-\lambda t}} \exp\left\{-\int_0^t e^{\lambda(s-t)}G(v(s))ds + \frac{\sigma^2}{4\lambda}(1 - e^{-2\lambda t})\right\},$$

while the variance is given by

$$\mathrm{Var}(p(t)) = (p_0)^{2e^{-\lambda t}} \mu^{2(1-e^{-\lambda t})} \exp\left\{ -2\int_0^t e^{\lambda(s-t)} G(v(s))ds + \nu(t) \right\} (\exp\{\nu(t)\} - 1).$$

$\Diamond$

3. **Random and stochastic differential inclusions**

We want to investigate (3) in the framework of differential inclusions (DIs), which are roughly speaking given by corresponding set valued differential equations.

Notations: Let (X, d) be a complete metric space.

- We denote by $\mathcal{K}(X)$ the set of all nonempty compact and convex subsets of X.
- In the set valued setting we use an appropriate concept for distance, namely the *Hausdorff semi metric* $d_H^*(\cdot,\cdot)$ and the *Hausdorff metric* $d_H(\cdot,\cdot)$. The Hausdorff semi metric for $A,\ B \subset X$ is given by $d_H^*(A,B) = \sup_{a\in A} \inf_{b\in B} d(a,b)$. Note, that $d_H^*(\cdot,\cdot)$ is only a semi metric, because in general $d_H^*(A,B) \neq d_H^*(B,A)$. We obtain the full metric by $d_H(A,B) := \max\{d_H^*(A,B), d_H^*(B,A)\}$.
- For $A,\ B \subset X$ and $\alpha \in \mathbb{R}$ we define $A + \alpha B := \{a + \alpha b \mid a \in A, b \in B\}$.
- For $x \in X$ and $\varepsilon > 0$ we denote by $B_\varepsilon(x) := \{y \in X | d(x,y) < \varepsilon\}$, the ε-ball for x.

Such as sets are characterized by their elements, set valued mappings are characterized by *selections*.

Definition 1. *Let $F : \mathbb{R}^+ \times \mathbb{R} \mapsto \mathcal{K}(\mathbb{R})$. A* selection *is a scalar valued mapping $f : \mathbb{R}^+ \times X \to X$ with $f(t,\cdot) \in F(t,\cdot)$ for a.e. $t \in [0,T]$.*

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. If we introduce DIs driven by random or stochastic processes over $(\Omega, \mathcal{F}, \mathbb{P})$, then we obtain *random differential inclusions* (RDI) and also *stochastic differential inclusions* (SDI) of Itô type having the form

$$\frac{d\varphi(t)}{dt} \in F(\theta_t\omega, \varphi(t)),\ t \geq 0, \quad \varphi(0) = x_0 \in \mathbb{R} \quad \text{(RDI)} \tag{10}$$

where θ is a metric dynamical system (see Definition 2 in Section 4) and

$$d\varphi(t) \in F(t,\varphi(t))dt + g(\varphi(t))dW,\ t \geq 0, \quad \varphi(0) = x_0 \in \mathbb{R} \quad \text{(SDI)} \tag{11}$$

respectively. Equation (11) is the symbolic notation for

$$\varphi(t) \in x_0 + \int_0^t F(\varphi(s))\,ds + \int_0^t g(\varphi(s))dW(s), \tag{12}$$

where the first integral is the so-called Aumann integral, defined as the set of the form

$$\int_0^t F(s,\cdot)\,ds = \left\{ \int_0^t f(s,\cdot)\,ds \mid f \in \mathcal{I}(F) \text{ for } t \in [0,T] \right\}$$

with the space of selectors

$$\begin{aligned} \mathcal{I}(F) \quad := \quad & \Big\{ f : [0,T] \times \mathbb{R} \mapsto \mathbb{R} \mid f(\cdot,x) \in L_1[0,T]\, \forall x \in \mathbb{R}, \\ & f(t,\cdot) \in F(t,\cdot) \text{ for a. e. } t \in [0,T] \Big\} \end{aligned}$$

and the second integral in (12) is a stochastic integral of Itô type.

We can interpret (3) as a SDI by writing, for example,

$$d\varphi(t) \in F(\varphi(t))dt + g(\varphi(t))dW(t), \quad t \geq 0, \quad \varphi(0) = x_0 > 0, \tag{13}$$

where $F : \mathbb{R} \mapsto \mathcal{K}(\mathbb{R})$ is the set valued mapping given by

$$F(\varphi(t)) := \left(\lambda \ln \left(\frac{\mu}{\varphi(t)} \right) - [0,\rho] \right) \varphi(t) \tag{14}$$

$g(\varphi(t)) := \sigma\varphi(t)$ and $\rho > 0$ is a parameter.

We replaced $G(v)$ by the set $[0,\rho]$. In our model $v(t) > 0$ denotes the dose of the drug at time t, while $G(v) > 0$ denotes the destroying rate of the cancer cells. It seems reasonable that G has to be a monotone increasing and bounded function (see [16]). In the special case mentioned in (2) we have $\lim_{v\to\infty} G(v) = \lim_{v\to\infty} \dfrac{k_1\, v}{k_2 + v} = k_1$. Therefore, we can take, for example, $\rho := k_1$.

Real therapy protocols are somehow periodic, drugs are given in periodic time intervals, and then a while no drugs are given, in order to allow the physical body of the

patient to recover after the drug exposure. The following type of control for the set valued mapping for the SDI (13) takes this fact better into account:

$$F(\varphi(t)) := \left(\lambda \ln\left(\frac{\mu}{\varphi(t)}\right) - [0,\rho]\left(\frac{1+\operatorname{sgn}(\sin(\alpha t+\beta))}{2}\right)\right)\varphi(t) \tag{15}$$

with α, $\beta \in \mathbb{R}$ are the parameter for the velocity and shifting of the protocol and $\operatorname{sgn}(x) = -1$, if $x < 0$ and $\operatorname{sgn}(x) = 1$, if $x \geq 0$.
Another further generalization is to consider the parameter ρ (i.e. the maximal destroy rate) as a stationary stochastic process $(\omega, t) \to \rho(\theta_t\omega)$.

4. **Random dynamical systems**

We give now a brief introduction into the theory of random dynamical systems. A complete survey can be found in [2]. Random dynamical systems are dynamical systems under random influences. Formally, they are given by two ingredients: a model for the underlying noise (*the metric dynamical system*) and a model, which describes the dynamics under the influence of that noise (*the cocycle*).

Definition 2. *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. A* metric dynamical system *(MDS) $\theta : \mathbb{R} \times \Omega \mapsto \Omega$ is a $(\mathcal{B}(\mathbb{R}) \otimes \mathcal{F}, \mathcal{F})$-measurable flow that fulfills the group property $\theta_0 = \mathrm{id}$, $\theta_{t+s} = \theta_t \circ \theta_s$ for all $s, t \in \mathbb{R}$. Moreover, we suppose that $(\theta_t)_{t\in\mathbb{R}}$ is continuous, i.e. $(t, \omega) \mapsto \theta_t\omega$ is continuous, and it is measure preserving, i.e. $\theta_t\mathbb{P} = \mathbb{P}$, for all $t \in \mathbb{R}$.*

Example 1. A well-known example of a MDS, which appears if we deal with stochastic differential equations, is the following: Let $(W_t)_{t\in\mathbb{R}}$ be a 1-dimensional two-sided standard *Wiener process* over the *canonical Wiener space* $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$, where $\tilde{\Omega} = \{\omega \in C(\mathbb{R}, \mathbb{R}) : \omega(0) = 0\}$, $\tilde{\mathcal{F}}$ is the Borel σ-algebra of $\tilde{\Omega}$ and $\tilde{\mathbb{P}}$ is the Wiener measure. Then, the *Wiener shift* $\theta_t\omega(\cdot) = \omega(\cdot + t) - \omega(t)$ defines a MDS. ♢

From now on let θ be an MDS over the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Let (X, d) be a complete metric space.

Definition 3. *We call (ϕ, θ) a* random dynamical system *(RDS) if $\phi : \mathbb{R}^+ \times \Omega \times X \mapsto X$ is a $(\mathcal{B}(\mathbb{R}^+) \otimes \mathcal{F} \otimes \mathcal{B}(X), \mathcal{B}(X))$-measurable mapping*

and satisfies for all $s,t\in\mathbb{R}^+$, $\omega\in\Omega$ *and* $x\in X$ *the* perfect cocycle property

$$\phi(0,\omega,x)=x,\quad \phi(t+s,\omega,x)=\phi\big(t,\theta_s\omega,\phi(s,\omega,x)\big).$$

An RDS can be generated for example by *random differential equations* or *stochastic differential equations*. An overview of typical generators of RDSs can be found in [2] and [5].

Definition 4. *A random variable* x^* *is called a* random fixed point *for a random dynamical system* (ϕ,θ), *if* $\phi(t,\omega,x^*(\omega))=x^*(\theta_t\omega)$ *for all* $\omega\in\Omega$ *and* $t\in\mathbb{R}^+$. *We say that a random fixed point* x^* *is* stable, *if it satisfies the pullback convergence relation* $\lim\limits_{t\to\infty}\phi(t,\theta_{-t}\omega,x)=x^*(\omega)$ *for all* $x\in X$ *and all* $\omega\in\Omega$.

The concept of pullback convergence was introduced in the 1990s by Crauel and Flandoli [7], Flandoli and Schmalfuß [12], and Schmalfuß [18].

We study now the long time behavior of the solution p of equation (3) for different types of control functions v:

I. We consider the probability space $(\Omega,\mathcal{F},\mathbb{P})$ to be the Wiener space $(\tilde{\Omega},\tilde{\mathcal{F}},\tilde{\mathbb{P}})$ and the MDS θ the Wiener shift given in Example 1. Let $X=\mathbb{R}^+$ be the phase space. We define for all $x\in\mathbb{R}^+$ and all $t\geq 0$ the function

$$\phi(t,\omega,x)=x^{e^{-\lambda t}}\mu^{1-e^{-\lambda t}}\exp\left\{-\int_0^t e^{\lambda(s-t)}G(v(s))ds+\sigma\int_0^t e^{\lambda(s-t)}dW(s)\right\},\ \forall\omega\in\Omega^* \tag{16}$$

$$\phi(t,\omega,x)=x,\ \forall\omega\in\Omega\setminus\Omega^*,$$

where $\Omega^*\subset\Omega$ is the set of measure 1 that satisfies the properties from Remark 1.

Assume that $G\circ v$ is a strictly stationary process, i.e. $G\circ v(s,\theta_t\omega)=G\circ v(s+t,\omega)$ for all $s,t\in\mathbb{R},\omega\in\Omega$, which is also bounded $G\circ v(t,\omega)<M$ for all $t\geq 0$ and a.e. $\omega\in\Omega$.

One can check by calculations that in our case (ϕ,θ) is a RDS over $(\Omega,\mathcal{F},\mathbb{P})$ (the

Wiener space). In these calculations it is essential that the MDS θ is measure preserving with respect to P.

Example 2. Stationary processes $G \circ v$ are obtained for example when G is a non-random continuous function $G : \mathbb{R} \to \mathbb{R}^+$ which is continuous and $v(t,\omega) := z(\theta_t\omega)$ $t \in \mathbb{R}, \omega \in \tilde{\Omega}$, where z is a positive random variable. In our simulations we take $v(t,\omega) := \exp\{-z^*(\theta_t\omega)\}$, where $z^*(\omega) = -\sigma\lambda \int_{-\infty}^{0} e^{\lambda s} W(s)ds$ is the random fixed point of the *Ornstein-Uhlenbeck equation* $dO(t) = \lambda O(t)dt + \sigma dW(t)$, $t \geq 0$. For G we take the function given in (2). $\Diamond$

Theorem 2. *The solution of equation* (3) *has the following long time behavior: for each* $x \geq 0$ *it holds*

$$\lim_{t\to\infty} \phi(t, \theta_{-t}\omega, x) = \mu \exp\Big\{ - \int_{-\infty}^{0} e^{\lambda s} \Big(G(v(s)) + \sigma\lambda W(s) \Big) ds \Big\} \tag{17}$$

for all $\omega \in \Omega^*$.

Proof. For $\omega \in \Omega^*$ we take $\omega \mapsto \theta_{-t}\omega$ in (16) and analyze the expressions occurring in the formula. We have

$$\begin{aligned} \lim_{t\to\infty} \int_0^t e^{\lambda(s-t)} G(v(s, \theta_{-t}\omega))ds &= \lim_{t\to\infty} \int_{-t}^{0} e^{\lambda s} G(v(s+t, \theta_{-t}\omega)) \\ &= \int_{-\infty}^{0} e^{\lambda s} G(v(s,\omega))ds. \end{aligned}$$

For the expression containing the Ornstein-Uhlenbeck process we compute as follows: Using the notation introduced in (9) and the Wiener shift operator θ given in Example 1, we obtain $e^{-\lambda t} W(-t) = \dfrac{1}{\sigma} O(t, \theta_{-t}\omega) + \lambda \displaystyle\int_{-t}^{0} e^{\lambda s} W(s)ds$ for all $t \geq 0, \omega \in \Omega^*$. We take into consideration that the process W has sublinear growth (see Remark 1), therefore,

$$\lim_{t\to\infty} O(t, \theta_{-t}\omega) = -\sigma\lambda \int_{-\infty}^{0} e^{\lambda s} W(s)ds, \text{ for all } \omega \in \Omega^*. \tag{18}$$

The integral from the right-hand side of the above relation exists. Finally, we take $\omega \mapsto \theta_{-t}\omega$ in (16), then $t \to \infty$ and use (18) to get (17). $\square$

By calculations it is easy to prove that by the above result we obtained the *stable random fixed point* for (ϕ, θ)

$$x^*(\omega) = \begin{cases} \mu \exp\Big\{ - \displaystyle\int_{-\infty}^{0} e^{\lambda s}\Big(G(v(s)) + \sigma\lambda W(s)\Big)ds\Big\} \text{ for } \omega \in \Omega^* \\ \\ \mu \text{ for } \omega \in \Omega \setminus \Omega^* \end{cases}$$

which acts as a *random attractor* for our RDS. This means that other solutions are attracted by this random fixed point. Moreover, any ε-neighborhood $\overline{B_\varepsilon(x^*)}$, $\varepsilon > 0$ absorbs any other solution and any bounded solution set in finite time. Note, for every time t we have a finite well defined random variable.

Theorem 3. *The solutions of the SDI* (13) *satisfies the following property* $\phi_\rho(t, \omega, x) \leq \varphi(t, \omega, x) \leq \phi_0(t, \omega, x)$ *for all* $x > 0, \omega \in \Omega^*$, *where* ϕ_0 *and* ϕ_ρ *are the cocycles corresponding to* $G \circ v \equiv 0$ *and* $G \circ v \equiv \rho$, *respectively.*

Proof. The solution φ of (13) exists, since for each selection f of F (defined in (14)) the corresponding stochastic differential equation admits a solution $\varphi(t, \omega, x)$ given in (16) by $\phi(t, \omega, x)$, for which $G(v(\cdot)) \in [0, \rho]$. The stated inequalities follow from the fact that for each selection for the SDI we have in fact $G(v(\cdot)) \in [0, \rho]$. □

Remark 3. We see from this theorem that the solution tube for the SDI (13) is delimited by the two "extreme" solutions, namely ϕ_ρ and ϕ_0. Analogously we get that the set of random fixed points corresponding to the SDI (13) is delimited by the two random fixed points x_0^* and x_ρ^*, corresponding to $G \circ v \equiv 0$ and $G \circ v \equiv \rho$.

II. Now we consider that our equation is driven not only by the underlying noise term $\omega(t) = W(t)\ \omega \in \tilde{\Omega}$ but also by nonrandom control functions $v \in C(\mathbb{R}, \mathbb{R}^+)$ (note, that v is the dose drug in the cancer growth model). In this case, the theory of RDS is embedded into the theory of *non-autonomous dynamical systems.*
Let $\hat{\Omega}$ be a nonempty set of elements. For each $t \in \mathbb{R}$ we consider $\hat{\theta}_t : \mathbb{R} \times \hat{\Omega} \mapsto \hat{\Omega}$ satisfying the group property $\hat{\theta}_0 = id$, $\hat{\theta}_{t+s} = \hat{\theta}_t \circ \hat{\theta}_s$ for all $s, t \in \mathbb{R}$.

Definition 5. *We call* $(\hat{\phi}, \hat{\theta})$ *a* non-autonomous dynamical system *(NDS), if* $\hat{\phi} : \mathbb{R}^+ \times \hat{\Omega} \times X \mapsto X$ *satisfies for all* $s, t \in \mathbb{R}^+$, $\hat{\omega} \in \hat{\Omega}$ *and* $x \in X$ *the* cocycle property $\hat{\phi}(0, \hat{\omega}, x) = x$, $\hat{\phi}(t + s, \hat{\omega}, x) = \hat{\phi}\big(t, \hat{\theta}_s\hat{\omega}, \hat{\phi}(s, \hat{\omega}, x)\big)$.

Let $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ be the Wiener space and θ the Wiener shift MDS given in Example 1. Let $\tilde{\Omega}^*$ be the set of measure 1 which satisfies the properties from Remark 1.

For our problem (3) we consider the NDS: $\hat{\Omega} := \tilde{\Omega} \times C(\mathbb{R}, \mathbb{R}^+)$

$$\hat{\theta}_t(\omega, v)(\cdot) = (\theta_t\omega(\cdot), v(\cdot + t)) \text{ for all } \ (\omega, v) \in \tilde{\Omega} \times C(\mathbb{R}, \mathbb{R}^+),$$

and for $x \in \mathbb{R}^+$ the cocycle is given for each $(\omega, v) \in \tilde{\Omega}^* \times C(\mathbb{R}, \mathbb{R}^+)$ by

$$\hat{\phi}(t, (\omega, v), x) = x^{e^{-\lambda t}} \mu^{1-e^{-\lambda t}} \exp\left\{ -\int_0^t e^{\lambda(s-t)} G(v(s)) ds + \sigma \int_0^t e^{\lambda(s-t)} dW(s) \right\}, \quad (19)$$

while for $(\omega, v) \in (\tilde{\Omega} \setminus \tilde{\Omega}^*) \times C(\mathbb{R}, \mathbb{R}^+)$ by $\hat{\phi}(t, (\omega, v), x) = x$.

One can check by calculations that in our case $(\hat{\phi}, \hat{\theta})$ is a NDS over $\hat{\Omega}$. In these calculations it is essential that the MDS θ is measure preserving with respect to $\tilde{P}$.

Theorem 4. *If* $G, v \in C(\mathbb{R}, \mathbb{R}^+)$ *and* G *is bounded, then the solution of equation* (3) *has the following long time behavior: for each* $x \geq 0$ *it holds*

$$\lim_{t\to\infty} \hat{\phi}(t, \hat{\theta}_{-t}(\omega, v), p_0) = \mu \exp\Big\{ -\int_{-\infty}^0 e^{\lambda s} \Big(G(v(s)) + \sigma\lambda W(s) \Big) ds \Big\}$$

for all $(\omega, v) \in \tilde{\Omega}^* \times C(\mathbb{R}, \mathbb{R}^+)$.

The proof is similar to the proof of Theorem 2.

This result shows, that there exists a *stable random fixed point* for the NDS $(\hat{\phi}, \hat{\theta})$

$$x^*((\omega, v)) = \begin{cases} \mu \exp\left\{ -\displaystyle\int_{-\infty}^0 e^{\lambda s} \left(G(v(s)) + \sigma\lambda W(s) \right) ds \right\} \\ \qquad \text{for } (\omega, v) \in \tilde{\Omega}^* \times C(\mathbb{R}, \mathbb{R}^+) \\ \\ \mu \text{ for } (\omega, v) \in (\tilde{\Omega} \setminus \tilde{\Omega}^*) \times C(\mathbb{R}, \mathbb{R}^+). \end{cases}$$

There is a strong relation between a large set of DIs and set valued dynamical systems. Like differential equations often generate dynamical systems, DIs generate set valued

dynamical systems. That is, we can use the numerical methods for the approximation of dynamical systems also for DIs by taking the set valued nature of the inclusions into account. We can interpret (13) also as a *set valued random dynamical system*. If we replace the cocycle mapping ϕ with a set valued nonempty compact and convex cocycle mapping Φ (see [4]) we can define set valued random dynamical systems.

Definition 6. *We call* (Φ, θ) *a* set valued random dynamical system *(SVRDS) if* $\Phi : \mathbb{R}^+ \times \Omega \times X \mapsto \mathcal{K}(X)$ *is measurable and satisfies for all* $s, t \in \mathbb{R}^+$, $\omega \in \Omega$ *and* $x \in X$ *the* perfect set valued cocycle property $\Phi(0, \omega, x) = \{x\}$, $\Phi(t + s, \omega, x) = \Phi(t, \theta_s\omega, \Phi(s, \omega, x))$ $\forall s, t \in \mathbb{R}^+$.

In addition, we make also the following assumptions on Φ: We assume the *continuity in time* i.e. $\lim_{t \to s} d_H(\Phi(t, \omega, x), \Phi(s, \omega, x)) = 0$ $\forall$ $\omega \in \Omega$ and *upper semi-continuity* in parameter and initial value i.e. for $x, y \in \mathbb{R}$ we have

$$\lim_{x \to y, \omega_1 \to \omega_2} d_H^*(\Phi(t, \omega_1, x), \Phi(t, \omega_2, y)) = 0$$

uniformly in t, where t belongs to any compact interval from $[0, \infty)$ and for all $\omega \in \Omega$. A *trajectory* of a SVRDS is a single valued mapping $\phi : \mathbb{R}^+ \mapsto \mathbb{R}$ which for all $\omega \in \Omega$ satisfies $\phi(\omega, t) \in \Phi(t - s, \theta_s\omega, \phi(\omega, s))$, where $0 \leq s \leq t$.
As mentioned before SVRDS are generated for example by RDI (10) or SDI (11). Of course in this cases the trajectories of the SVRDS correspond to selections of the inclusion. The SDI (13) with (14) generates a SVRDS, while the SDI (13) with (15) generates set valued non-autonomous dynamical system.

5. **Simulations**

In this section we want to give some numerical results for (3). Note that our model is qualitative and not quantitative, the values given on the axes are not realistic. However it is possible to scale the model to any desired situation.

In our simulations we will use the parameters $k_1 = 1$, $k_2 = 1$, $\lambda = 1$, $\mu = 1$. In Figure 1 we can see the stochastic model without control functions. We used three different initial conditions. The existence of the random fixed point x_0^*, which is the

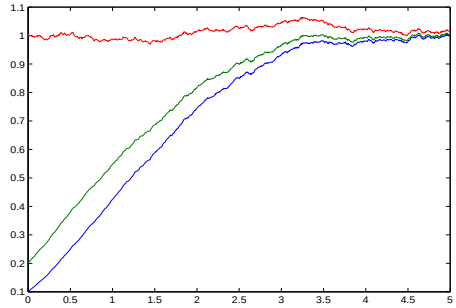

FIGURE 1. Simulation of three initial conditions for the control free system ($G \circ v = 0$).

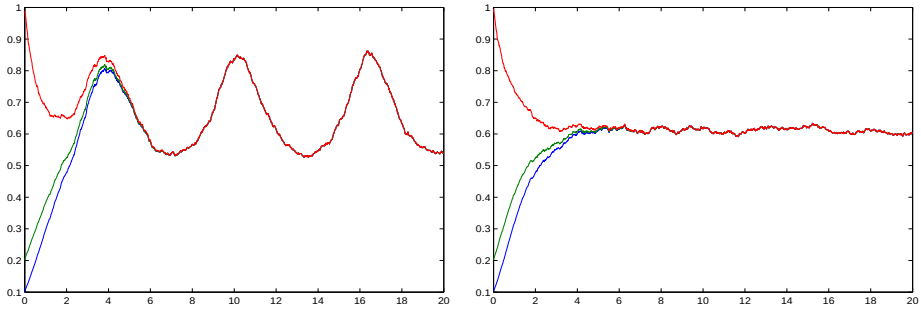

FIGURE 2. Simulations with different controls.

bound for the maximal tumor size is well visible.

In Figure 2 we have simulated our stochastic model for different control functions. We used $v(t) = 1 + \cos(t)$ (nonrandom case) and $v(t) = e^{-z^*(t)}$ (stationary process). Also in these cases the random fixed point exists. However such controls are only of theoretical interest but they support the results of the theory given in Section 2.

From the theoretical results it is clear that we have to use drugs with a high enough destroying rate for the cancer cells. Our simulations support this assertion. Let us interpret our model as an SDI (13) with (14). The approximations in Figure 3 show the reachable set of the inclusion for different maximal destroying rates with the same selection strategy. Of course the simulation takes into account that we can use the maximal destroy rate for all t, which is obviously not possible because this destroy

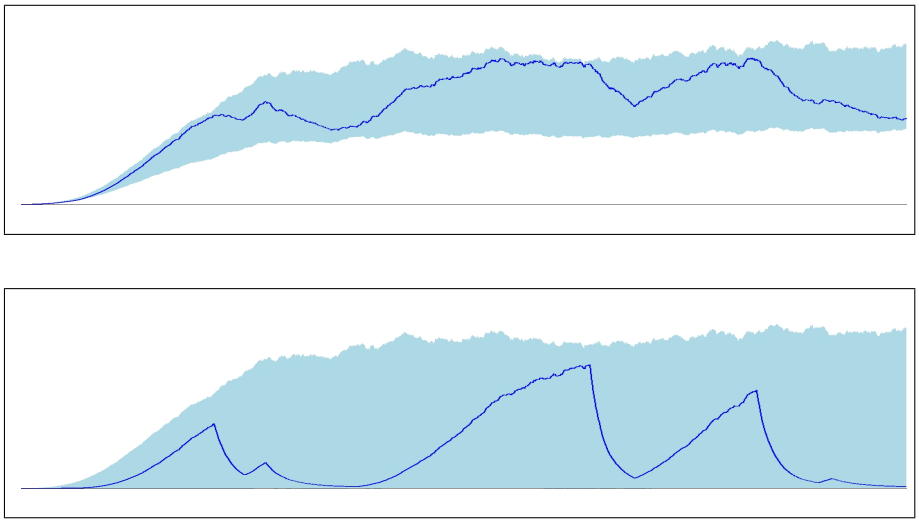

FIGURE 3. Simulations for different maximal destroy rates for the SDI (13) with (14).

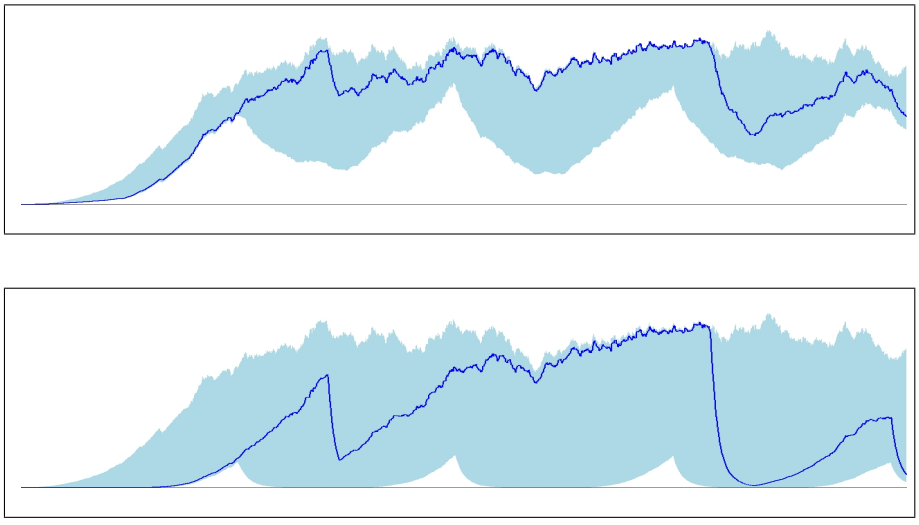

FIGURE 4. Simulations for different maximal destroy rates for the SDI (13) with (15).

rate damages also good cells. However the inclusion for (14) includes this very optimistic but unrealistic case.

It is clear from real world examples that realistic controls have to be some kind of periodic functions, because the drugs are given in intervals. It is not possible to control the concentration of the drugs for every time t and it is surely not possible to shorten the interval arbitrary.

We get a more appropriate model, if we use (15) in the SDI (13). Note, we have made the assumption that v has to be some kind of periodic function, where the time span for the therapy and the time span for rest has the same size. Also in this case we need a high enough destroying rate to get a successful procedure. In the time span, where we do not use drugs the tumor is again growing but the patient has the chance to improve his health for the new therapy session. The simulations are shown in Figure 4.

We point out that we are not experts in the topic of real healing procedures. The mathematical strategies used here are probably not realistic. But it is easily possible to extend these ideas to other more appropriate strategies.

However, the numerical results imply that it seems not possible to use a gentle procedure for the drug disposal. To get a successful procedure it seems to be necessary to use an aggressive strategy depending of the strength and health of the patient.

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A MIXED MONTE CARLO AND QUASI-MONTE CARLO METHOD WITH APPLICATIONS TO MATHEMATICAL FINANCE

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Abstract. In this paper, we apply a mixed Monte Carlo and Quasi-Monte Carlo method, which we proposed in a previous paper, to problems from mathematical finance. We estimate the value of an European Call option and of an Asian option using our mixed method, under different horizont times. We assume that the stock price of the underlaying asset $S = S(t)$ is driven by a Lévy process $L(t)$. We compare our estimates with the estimates obtained by using the Monte Carlo and Quasi-Monte Carlo methods. Numerical results show that a considerable improvement can be achieved by using the mixed method.

1. Introduction

The valuation of financial derivatives is one of the most important problems from mathematical finance. The risk-neutral price of such a derivative can be expressed in terms of a risk-neutral expectation of a random payoff. In some cases, the expectation is explicitly computable, such as the Black & Scholes formula for pricing call options on assets modelled by a geometric Brownian motion. However, if we consider an Asian option, there exists no longer closed form expressions for the price, and therefore numerical methods are involved. This is the case, even if we consider call options written on assets with non-normal returns. Among these methods, Monte Carlo (MC) and Quasi-Monte Carlo (QMC) methods play an increasingly important role.

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One of the first applications of the MC method in this field appeared in Boyle [2], who used simulation to estimate the value of a standard European option. Applications of the QMC method to option pricing problems can be found in [15] and [12].

Barndorff-Nielsen [1] proposed to model the log returns of asset prices by using the normal inverse Gaussian (NIG) distribution. This family of non-normal distributions has proven to fit the semi-heavily tails observed in financial time series of various kinds extremely well (see Rydberg [21] or Eberlein and Keller [7]). The time dynamics of the asset prices are modelled by an exponential Lévy process. To price such derivatives, even simple call and put options, we need to consider the numerical evaluation of the expectation. Raible [18] has considered a Fourier method to evaluate call and put options. Alternatives to this method are the MC method or the QMC method. The QMC method has been applied with succes in financial applications by many authors (see [8]), and has strong convergence properties. Majority of the work done on applying these simulation techniques to financial problems was in direction where one needs to simulate from the normal distribution. One exception is Kainhofer (see [13]), who proposes a QMC algorithm for NIG variables, based on a technique proposed by Hlawka and Mück (see [11]) to generate low-discrepancy sequences for general distributions.

In a recent paper [19], we proposed a mixed MC and QMC method for estimating an s-dimensional integral I and we defined a new hybrid sequence that we called the H-mixed sequence. Other authors who combine the ideas from MC and QMC methods in estimating multidimensional integrals are G. Ökten (see [16]) and N. Roşca (see [20]). Using these sequences, we defined a new estimator and proved a central limit theorem for this estimator. In this paper, we apply our mixed method to practical problems from financial mathematics. First, we remember the theoretical background of our method and give some important results. Then, we apply the H-mixed sequence to valuation of an European Call option and compare the effectiveness of it with that of pseudorandom and low-discrepancy sequences. At the end, we apply the mixed method to a more difficult problem from finance, namely the

valuation of Asian options. We also compare numerically our method with the MC and QMC methods.

2. H-mixed sequences

Let us consider the problem of estimating integrals of the form

$$I = \int_{[0,1]^s} f(x) dH(x), \tag{1}$$

where $f : [0,1]^s \to \mathbb{R}$ is the function we want to integrate and $H : \mathbb{R}^s \to [0,1]$ is a distribution function on $[0,1]^s$. In the continuous case, the integral I can be rewritten as

$$I = \int_{[0,1]^s} f(x)h(x)dx,$$

where h is the density function corresponding to the distribution function H.

In the MC method (see [22]), the integral I is estimated by sums of the form

$$\hat{I}_N = \frac{1}{N} \sum_{k=1}^{N} f(x_k),$$

where $x_k = (x_k^{(1)}, \ldots, x_k^{(s)})$, $k \geq 1$, are independent identically distributed random points on $[0,1]^s$, with the common density function h.

In the QMC method (see [22]), the integral I is approximated by sums of the form $\frac{1}{N} \sum_{k=1}^{N} f(x_k)$, where $(x_k)_{k \geq 1}$ is a H-distributed low-discrepancy sequence on $[0,1]^s$.

Next, the notions of discrepancy and marginal distributions are introduced.

Definition 1 (H-discrepancy)**.** *Consider an s-dimensional continuous distribution on $[0,1]^s$, with distribution function H. Let λ_H be the probability measure induced by H. Let $P = (x_1, \ldots, x_N)$ be a sequence of points in $[0,1]^s$. The H-discrepancy of sequence P is defined as*

$$D_{N,H}(P) = \sup_{J \subseteq [0,1]^s} \left| \frac{1}{N} A_N(J, P) - \lambda_H(J) \right|,$$

where the supremum is calculated over all subintervals $J = \prod_{i=1}^{s}[a_i, b_i] \subseteq [0,1]^s$*;* $A_N(J,P)$ *counts the number of elements of the set* $(x_1, \ldots, x_N)$*, falling into the interval* J*, i.e.*

$$A_N(J,P) = \sum_{k=1}^{N} 1_J(x_k).$$

1_J *is the characteristic function of* J*.*

The sequence P *is called* H*-distributed if* $D_{N,H}(P) \to 0$ *as* $N \to \infty$*.*

The H*-distributed sequence* P *is said to be a low-discrepancy sequence if* $D_{N,H}(P) = \mathcal{O}\big((\log N)^s/N\big)$*.*

The non-uniform Koksma-Hlawka inequality ([3]) gives an upper bound for the approximation error in QMC integration, when H-distributed low-discrepancy sequences are used.

Theorem 2 (non-uniform Koksma-Hlawka inequality)**.** *Let* $f : [0,1]^s \to \mathbb{R}$ *be a function of bounded variation in the sense of Hardy and Krause and* $(x_1, \ldots, x_N)$ *be a sequence of points in* $[0,1]^s$*. Consider an* s*-dimensional continuous distribution on* $[0,1]^s$*, with distribution function* H*. Then, for any* $N > 0$

$$\left| \int_{[0,1]^s} f(x)dH(x) - \frac{1}{N}\sum_{k=1}^{N} f(x_k) \right| \leq V_{HK}(f) D_{N,H}(x_1, \ldots, x_N), \tag{2}$$

where $V_{HK}(f)$ *is the variation of* f *in the sense of Hardy and Krause.*

Definition 3. *Consider an* s*-dimensional continuous distribution on* $[0,1]^s$*, with density function* h *and distribution function* H*. For a point* $u = \big(u^{(1)}, \ldots, u^{(s)}\big) \in [0,1]^s$*, the marginal density functions* h_l*,* $l = 1, \ldots, s$*, are defined by*

$$h_l\big(u^{(l)}\big) = \underbrace{\int \ldots \int}_{[0,1]^{s-1}} h\big(t^{(1)}, \ldots, t^{(l-1)}, u^{(l)}, t^{(l+1)}, \ldots t^{(s)}\big) dt^{(1)} \ldots dt^{(l-1)} dt^{(l+1)} \ldots dt^{(s)},$$

and the marginal distribution functions H_l*,* $l = 1, \ldots, s$*, are defined by*

$$H_l\big(u^{(l)}\big) = \int_0^{u^{(l)}} h_l(t)dt.$$

We consider s-dimensional continuous distributions on $[0,1]^s$, with independent marginals, i.e.,

$$H(u) = \prod_{l=1}^{s} H_l(u^{(l)}),\ \forall u = (u^{(1)}, \ldots, u^{(s)}) \in [0,1]^s.$$

This can be expressed, using the marginal density functions, as follows:

$$h(u) = \prod_{l=1}^{s} h_l(u^{(l)}),\ \forall u = (u^{(1)}, \ldots, u^{(s)}) \in [0,1]^s.$$

Consider an integer $0 < d < s$. Using the marginal density functions, we construct the following density functions on $[0,1]^d$ and $[0,1]^{s-d}$, respectively:

$$h_q(u) = \prod_{l=1}^{d} h_l(u^{(l)}),\ \forall u = (u^{(1)}, \ldots, u^{(d)}) \in [0,1]^d,$$

and

$$h_X(u) = \prod_{l=d+1}^{s} h_l(u^{(l)}),\ \forall u = (u^{(d+1)}, \ldots, u^{(s)}) \in [0,1]^{s-d}.$$

The corresponding distribution functions are

$$H_q(u) = \int_0^{u^{(1)}} \cdots \int_0^{u^{(d)}} h_q\big(t^{(1)}, \ldots, t^{(d)}\big) dt^{(1)} \ldots dt^{(d)},$$
$$u = (u^{(1)}, \ldots, u^{(d)}) \in [0,1]^d, \tag{3}$$

and

$$H_X(u) = \int_0^{u^{(d+1)}} \cdots \int_0^{u^{(s)}} h_X\big(t^{(d+1)}, \ldots, t^{(s)}\big) dt^{(d+1)} \ldots dt^{(s)},$$
$$u = (u^{(d+1)}, \ldots, u^{(s)}) \in [0,1]^{s-d}. \tag{4}$$

Next, we introduce the new notion of a H-mixed sequence.

Definition 4 (H-mixed sequence)**.** *([19])*

Consider an s-dimensional continuous distribution on $[0,1]^s$, with distribution function H and independent marginals H_l, $l = 1, \ldots, s$. Let H_q and H_X be the distribution functions defined in (3) and (4), respectively.

Let $(q_k)_{k\geq 1}$ be a H_q-distributed low-discrepancy sequence on $[0,1]^d$, with $q_k = (q_k^{(1)}, \ldots, q_k^{(d)})$, and X_k, $k \geq 1$, be independent and identically distributed random vectors on $[0,1]^{s-d}$, with distribution function H_X, where $X_k = (X_k^{(d+1)}, \ldots, X_k^{(s)})$.

A sequence $(m_k)_{k\geq 1}$, with the general term given by

$$m_k = (q_k, X_k), \quad k \geq 1, \tag{5}$$

is called a H-mixed sequence on $[0,1]^s$.

Remark 5. *For an interval $J = \prod_{l=1}^{s}[a_l, b_l] \subseteq [0,1]^s$, we define the subintervals $J' = \prod_{l=1}^{d}[a_l, b_l] \subseteq [0,1]^d$ and $J'' = \prod_{l=d+1}^{s}[a_l, b_l] \subseteq [0,1]^{s-d}$ (i.e. $J = J' \times J''$).*

Let $(m_k)_{k\geq 1}$ be a H-mixed sequence on $[0,1]^s$, with the general term given by (5).

Based on definitions (1) and (4), the H-discrepancy of the sequence $(m_1, \ldots, m_N)$ can be expressed as

$$D_{N,H}(m_1, \ldots, m_N) = \sup_{J \subseteq [0,1]^s} \left| \frac{1}{N} \sum_{k=1}^{N} 1_J(m_k) - \prod_{l=1}^{s} [H_l(b_l) - H_l(a_l)] \right|,$$

and the H_q-discrepancy of the sequence $(q_1, \ldots, q_N)$ is given by

$$D_{N,H_q}(q_1, \ldots, q_N) = \sup_{J' \subseteq [0,1]^d} \left| \frac{1}{N} \sum_{k=1}^{N} 1_{J'}(q_k) - \prod_{l=1}^{d} [H_l(b_l) - H_l(a_l)] \right|.$$

The following result gives a probabilistic error bound for the H-mixed sequences.

Theorem 6. *([19]) If $(m_k)_{k\geq 1} = (q_k, X_k)_{k\geq 1}$ is a H-mixed sequence, then $\forall \varepsilon > 0$ we have*

$$P\Big(D_{N,H}(m_1, \ldots, m_N) \leq \varepsilon + D_{N,H_q}(q_1, \ldots, q_N)\Big) \geq 1 - \frac{1}{\varepsilon^2}\frac{1}{4N}\Big(D_{N,H_q}(q_1, \ldots, q_N) + 1\Big). \tag{6}$$

In order to estimate general integrals of the form (1), we introduce the following estimator.

Definition 7. *([19]) Let $(m_k)_{k\geq 1} = (q_k, X_k)_{k\geq 1}$ be an s-dimensional H-mixed sequence, introduced by us in Definition 4, with $q_k = (q_k^{(1)}, \ldots, q_k^{(d)})$ and $X_k = (X_k^{(d+1)}, \ldots, X_k^{(s)})$. We define the following estimator for the integral I:*

$$\theta_m = \frac{1}{N} \sum_{k=1}^{N} f(m_k). \tag{7}$$

We consider the independent random variables:

$$Y_k = f(m_k) = f(q_k^{(1)}, \ldots, q_k^{(d)}, X_k^{(d+1)}, \ldots, X_k^{(s)}), \quad k \geq 1. \tag{8}$$

We denote the expectation of Y_k by

$$E(Y_k) = \mu_k, \tag{9}$$

and the variance of Y_k by

$$Var(Y_k) = \sigma_k^2. \tag{10}$$

We assume that

$$0 < \sigma_k^2 < \infty, \tag{11}$$

and we denote

$$0 < \sigma_{(N)}^2 = \sigma_1^2 + \ldots + \sigma_N^2 < \infty. \tag{12}$$

In what follows, we give and prove an important result, concerning the estimator (7) introduced previously by us.

Theorem 8. *Let $(m_k)_{k\geq 1} = (q_k, X_k)_{k\geq 1}$ be an s-dimensional H-mixed sequence, defined in (5). We assume that the integrant f is bounded on $[0,1]^s$ and that the function*

$$g(x^{(1)}, \ldots, x^{(d)}) = \int_{[0,1]^{s-d}} f(x^{(1)}, \ldots, x^{(s)}) \Big(\prod_{l=d+1}^{s} h_l(x^{(l)}) \Big) dx^{(d+1)} \cdot \ldots \cdot dx^{(s)},$$

is of bounded variation in the sense of Hardy and Krause. Then, the estimator θ_m, defined in relation (7), is asymptotically unbiased i.e.,

$$E(\theta_m) \to I, \text{ as } N \to \infty.$$

Proof. As $(q_k)_{k\geq 1}$, with $q_k = (q_k^{(1)}, \ldots, q_k^{(d)})$, is a H_q-distributed low-discrepancy sequence on $[0,1]^d$, it follows that

$$D_{N,H_q}(q_1, \ldots, q_N) \to 0, \text{ as } N \to \infty. \tag{13}$$

Using this and the fact that function g is of bounded variation in the sense of Hardy and Krause, it follows from Koksma-Hlawka inequality (2) that

$$\frac{1}{N}\sum_{k=1}^{N} g(q_k^{(1)},\dots,q_k^{(d)}) \longrightarrow \int_{[0,1]^d} g(x^{(1)},\dots,x^{(d)})\Big(\prod_{l=1}^{d} h_l(x^{(l)})\Big)dx^{(1)}\dots dx^{(d)}$$

$$= \int_{[0,1]^d}\Big[\int_{[0,1]^{s-d}} f(x^{(1)},\dots,x^{(s)})\Big(\prod_{l=d+1}^{s} h_l(x^{(l)})\Big)dx^{(d+1)}\cdot\ldots\cdot dx^{(s)}\Big]$$

$$\cdot\Big(\prod_{l=1}^{d} h_l(x^{(l)})\Big)dx^{(1)}\dots dx^{(d)}$$

$$= \int_{[0,1]^s} f(x^{(1)},\dots,x^{(s)})\Big(\prod_{l=1}^{s} h_l(x^{(l)})\Big)dx^{(1)}\cdot\ldots\cdot dx^{(s)} = I,\ as\ N\to\infty.$$

The expectation of our estimator is

$$E(\theta_m) = E\left(\frac{1}{N}\sum_{k=1}^{N} f(m_k)\right) = \frac{1}{N}\sum_{k=1}^{N} E(f(q_k^{(1)},\dots,q_k^{(d)},X_k^{(d+1)},\dots,X_k^{(s)}))$$

$$= \frac{1}{N}\sum_{k=1}^{N}\int_{[0,1]^{s-d}} f(q_k^{(1)},\dots,q_k^{(d)},x^{(d+1)},\dots,x^{(s)})\Big(\prod_{l=d+1}^{s} h_l(x^{(l)})\Big)dx^{(d+1)}\cdot\ldots\cdot dx^{(s)}$$

$$= \frac{1}{N}\sum_{k=1}^{N} g(q_k^{(1)},\dots,q_k^{(d)}).$$

Hence, we get in the end that

$$\lim_{N\to\infty} E(\theta_m) = I.$$

□

We call the method of estimating the integral I, based on the estimator θ_m, defined in (7), the mixed method.

Proposition 9. *([19]) Let $(m_k)_{k\geq 1} = (q_k, X_k)_{k\geq 1}$ be an s-dimensional H-mixed sequence. We assume that f is bounded on $[0,1]^s$ and that the functions*

$$f_1(x^{(1)},\dots,x^{(d)}) = \int_{[0,1]^{s-d}} (f(x^{(1)},\dots,x^{(s)}))^2\Big(\prod_{l=d+1}^{s} h_l(x^{(l)})\Big)dx^{(d+1)}\cdot\ldots\cdot dx^{(s)},$$

$$f_2(x^{(1)},\dots,x^{(d)}) = \Big[\int_{[0,1]^{s-d}} f(x^{(1)},\dots,x^{(s)})\Big(\prod_{l=d+1}^{s} h_l(x^{(l)})\Big)dx^{(d+1)}\cdot\ldots\cdot dx^{(s)}\Big]^2$$

are of bounded variation in the sense of Hardy and Krause. Then, we have

$$\frac{\sigma_{(N)}^2}{N} \longrightarrow L, \ as \ N \longrightarrow \infty,$$

where

$$L = \int_{[0,1]^s} (f(x^{(1)}, \ldots, x^{(s)}))^2 \Big(\prod_{l=1}^{s} h_l(x^{(l)}) \Big) dx^{(1)} \ldots dx^{(s)} -$$

$$- \int_{[0,1]^d} \Big[\int_{[0,1]^{s-d}} f(x^{(1)}, \ldots, x^{(s)}) \Big(\prod_{l=d+1}^{s} h_l(x^{(l)}) \Big) dx^{(d+1)} \cdot \ldots \cdot dx^{(s)} \Big]^2$$

$$\cdot \Big(\prod_{l=1}^{d} h_l(x^{(l)}) \Big) dx^{(1)} \ldots dx^{(d)}.$$

Another important result regarding the estimator defined before is recalled next.

Theorem 10. *([19]) In the same hypothesis as in Proposition 9 and, in addition, assuming that $L \neq 0$, we have*

a)
$$Y_{(N)} = \frac{\sum_{k=1}^{N} Y_k - \sum_{k=1}^{N} \mu_k}{\sigma_{(N)}} \longrightarrow Y, \quad as \ N \to \infty, \tag{14}$$
where the random variable Y has the standard normal distribution.

b) *If we denote the crude Monte Carlo estimator for the integral (1) by θ_{MC}, then*
$$Var(\theta_m) \leq Var(\theta_{MC}), \tag{15}$$
meaning that, by using our estimator, we obtain asymptotically a smaller variance than by using the classical Monte Carlo method.

3. **Application to finance: European options**

In this section, we apply our mixed method to a problem from mathematical finance. The general setting of the problem is presented next. We consider the situation where the stock price of the underlaying asset $S = S(t)$ is driven by a Lévy process $L(t)$,

$$S(t) = S(0)e^{L(t)}. \tag{16}$$

Lévy processes can be characterized by the distribution of the random variable $L(1)$. This distribution can be hyperbolic (see [7]), normal inverse gaussian (NIG), variance-gamma (see [14]), or Meixner distribution.

According to the fundamental theory of asset pricing (see [5]), the risk-neutral price of an option, $C(0)$, is given by

$$C(0) = e^{-rT} E^Q(C_T(S)), \tag{17}$$

where $C_T(S)$ is the so-called payoff of the derivative, which coincides with its value at expiration or exercise time T, and Q is an equivalent martingale measure. In this paper, we are mostly concerned with exponential NIG-Lévy processes, meaning that $L(t)$ has independent increments, distributed according to a NIG distribution. For a detailed discussion of the NIG distribution and the corresponding Lévy process, we refer to Barndorff-Nielsen [1] and Rydberg [21]. In the situation of exponential NIG-Lévy models, we have an incomplete market, leading to a continuum of equivalent martingale measures Q, which can be used for risk-neutral pricing. Here, we choose the approach of Raible [18] and consider the measure obtained by Esscher transform method. This approach is so-called structure preserving, in the sense that the distribution of $L(1)$ remains in the class of NIG distributions.

In the following, we consider the evaluation of so-called European Call options, which have to be valued by simulation. The risk-neutral price of such an option is

$$C(0) = e^{-rT} E^Q(\max\{S(T) - K, 0\}) = e^{-rT} E^Q((S(T) - K)_+), \tag{18}$$

where the constant K is called the strike price. If we replace the stock price by (16), we obtain

$$C(0) = e^{-rT} E^Q((S(0)e^{L(T)} - K)_+). \tag{19}$$

From practice, we know that the evaluation of the stock price $S(t)$ is made at discrete times $0 = t_0 < t_1 < t_2 < \ldots < t_s = T$. For simplicity, we focus on regular time intervals, $\Delta t = t_i - t_{i-1}$. We note that

$$X_i = L(t_i) - L(t_{i-1}) = L(t_{i-1} + \Delta t) - L(t_{i-1}) \sim L(\Delta t), \quad i = 1, \ldots, s,$$

are independent and identically distributed NIG random variables with the same distribution as $L(t_1)$. Dropping the discounted factor from the risk-neutral option price, we get the expected payoff under the Esscher transform measure of the European Call option

$$E^Q((S(0)e^{L(T)} - K)_+) = E((S(0)e^{\sum_{i=1}^{s} X_i} - K)_+), \tag{20}$$

Our purpose is to evaluate the expected payoff (20). For this, we rewrite the expectation (20) as a multidimensional integral on $\mathbb{R}^s$

$$I = \int_{\mathbb{R}^s} \underbrace{\left(S(0)e^{\sum_{i=1}^{s} x^{(i)}} - K\right)_+}_{E(x)} dG(x) = \int_{\mathbb{R}^s} E(x)dG(x), \tag{21}$$

where $G(x) = \prod_{i=1}^{s} G_i(x^{(i)}),\ \forall x = (x^{(1)}, \ldots, x^{(s)}) \in \mathbb{R}^s$, and $G_i(x^{(i)})$ denotes the distribution function of the so-called log returns induced by $L(t_1)$, with the corresponding density function $g_i(x^{(i)})$. These log increments are independent and NIG distributed, having a common probability density

$$f_{NIG}(x; \mu, \beta, \alpha, \delta) = \frac{\alpha}{\pi} \exp\left(\delta\sqrt{\alpha^2 - \beta^2} + \beta(x - \mu)\right) \frac{\delta K_1(\alpha\sqrt{\delta^2 + (x-\mu)^2})}{\sqrt{\delta^2 + (x-\mu)^2}} \tag{22}$$

where $K_1(x)$ denotes the modified Bessel function of third type of order 1 (see [17]).

In order to approximate the integral (21), we have to transform it to an integral on $[0,1]^s$. We can do this using an integral transformation, as follows.

We first consider the family of independent double-exponential distributions with the same parameter $\lambda > 0$, having the cumulative distribution functions $G_{\lambda,i} : \mathbb{R} \to [0,1],\ i = 1, \ldots, s$,

$$G_{\lambda,i}(x) = \begin{cases} \frac{1}{2}e^{\lambda x} & , x < 0 \\ 1 - \frac{1}{2}e^{-\lambda x} & , x \geq 0, \end{cases} \tag{23}$$

and the inverses $G_{\lambda,i}^{-1} : [0,1] \to \mathbb{R},\ i = 1, \ldots, s$, given by

$$G_{\lambda,i}^{-1}(x) = \begin{cases} \frac{1}{\lambda}\log(2x) & , x \leq \frac{1}{2} \\ -\frac{1}{\lambda}\log(2 - 2x) & , x > \frac{1}{2}. \end{cases} \tag{24}$$

Next, we consider the substitutions $x^{(i)} = G_{\lambda,i}^{-1}(1 - y^{(i)}),\ i = 1, \ldots, s$, and then take $y^{(i)} = 1 - z^{(i)},\ i = 1, \ldots, s$.

The integral (21) becomes

$$I = \int_{[0,1]^s} \underbrace{\left(S(0)e^{\sum_{i=1}^{s} G_{\lambda,i}^{-1}(z^{(i)})} - K\right)_+}_{f(z)} dH(z) = \int_{[0,1]^s} f(z)dH(z), \tag{25}$$

where $H : [0,1]^s \to [0,1]$, defined by

$$H(z) = \prod_{i=1}^{s} (G_i \circ G_{\lambda,i}^{-1})(z^{(i)}),\ \forall z = (z^{(1)}, \ldots, z^{(s)}) \in [0,1]^s, \tag{26}$$

is a distribution function on $[0,1]^s$, with independent marginals $H_i = G_i \circ G_{\lambda,i}^{-1}$, $i = 1, \ldots, s$.

In the following, we compare numerically our mixed method with the MC and QMC methods. As a measure of comparison, we will use the absolute errors produced by these three methods in the approximation of the integral (25).

The MC estimate is defined as follows:

$$\theta_{MC} = \frac{1}{N} \sum_{k=1}^{N} f(x_k^{(1)}, \ldots, x_k^{(s)}), \tag{27}$$

where $x_k = (x_k^{(1)}, \ldots, x_k^{(s)})$, $k \geq 1$, are independent identically distributed random points on $[0,1]^s$, with the common distribution function H defined in (26).

In order to generate such a point x_k, we proceed as follows. We first generate a random point $\omega_k = (\omega_k^{(1)}, \ldots, \omega_k^{(s)})$, where $\omega_k^{(i)}$ is a point uniformly distributed on $[0,1]$, for each $i = 1, \ldots, s$. Then, for each component $\omega_k^{(i)}$, $i = 1, \ldots, s$, we apply the inversion method (see [4] and [6]), and obtain that $H_i^{-1}(\omega_k^{(i)}) = (G_{\lambda,i} \circ G_i^{-1})(\omega_k^{(i)})$ is a point with the distribution function H_i. As the s-dimensional distribution with the distribution function H has independent marginals, it follows that $x_k = ((G_{\lambda,1} \circ G_1^{-1})(\omega_k^{(1)}), \ldots, (G_{\lambda,s} \circ G_s^{-1})(\omega_k^{(s)}))$ is a point on $[0,1]^s$, with the distribution function H. As we can see, in order to generate non-uniform random points on $[0,1]^s$, with distribution function H, we need to know the inverse of the distribution function of a NIG distributed random variable or, at least an approximation of it. As the inverse function is not explicitly known, an approximation of it is needed in our simulations. In order to obtain an approximation of the inverse, we use the Matlab

function "niginv" as implemented by R. Werner, based on a method proposed by K. Prause in his Ph.D. dissertation [17].

The QMC estimate is defined as follows:

$$\theta_{QMC} = \frac{1}{N}\sum_{k=1}^{N} f(x_k^{(1)}, \ldots, x_k^{(s)}), \tag{28}$$

where $x = (x_k)_{k\geq 1}$ is a H-distributed low-discrepancy sequence on $[0,1]^s$, with $x_k = (x_k^{(1)}, \ldots, x_k^{(s)})$, $k \geq 1$.

In order to generate such a sequence, we apply a method proposed by Hlawka and Mück in [11]. In their method, they create directly H-distributed low-discrepancy sequences, where H can be any distribution function on $[0,1]^s$, with density function h, which can be factored into a product of independent, one-dimensional densities. The method is based on the following theoretical result.

Theorem 11. *([10]) Consider an s-dimensional continuous distribution on $[0,1]^s$, with distribution function H and density function $h(u) = \prod_{j=1}^{s} h_j(u^{(j)})$, $\forall u = \left(u^{(1)}, \ldots, u^{(s)}\right) \in [0,1]^s$. Assume that $h_j(t) \neq 0$, for almost every $t \in [0,1]$ and for all $j = 1, \ldots, s$. Furthermore, assume that h_j, $j = 1, \ldots, s$, are continuous on $[0,1]$. Denote by $M_f = \sup_{u\in[0,1]^s} f(u)$. Let $\omega = (\omega_1, \ldots, \omega_N)$ be a sequence in $[0,1]^s$. Generate the sequence $x = (x_1, \ldots, x_N)$, with*

$$x_k^{(j)} = \frac{1}{N}\sum_{r=1}^{N}\left[1 + \omega_k^{(j)} - H_j\left(\omega_r^{(j)}\right)\right] = \frac{1}{N}\sum_{r=1}^{N} 1_{[0,\omega_k^{(j)}]}\left(H_j\left(\omega_r^{(j)}\right)\right), \tag{29}$$

for all $k = 1, \ldots, N$ and all $j = 1, \ldots, s$, where $[a]$ denotes the integer part of a. Then the generated sequence x has a H-discrepancy of

$$D_{N,H}(x_1, \ldots, x_N) \leq (2 + 6sM_f) D_N(\omega_1, \ldots, \omega_N).$$

As our distribution function H can be factored into independent marginals, and has the support on $[0,1]^s$, we can apply directly the above theorem, to generate H-distributed low-discrepancy sequences. During our experiments, we employed as low-discrepancy sequences $\omega = (\omega_k)_{k\geq 1}$ on $[0,1]^s$, the Halton sequences (see [9]).

All points constructed by the Hlawka-Mück method are of the form i/N, $i = 0, \ldots, N$, in particular some elements of the sequence $x = (x_1, \ldots, x_N)$ might

assume a value of 0 or 1. A value of 1 is a singularity of the function $f(x)$, due to the logarithm from the definition of $G^{-1}_{\lambda,i}(x)$, which becomes unbounded if $x = 1$. Hence, the sequence constructed with Hlawka-Mück method is not directly suited for unbounded problems. To overcome this problem, Kainhofer (see [13]) suggests to define a new sequence, in which the value 1 is replaced by $1/N$, where N is the number of points in the set. This slight modification of the sequence is shown to have a minor influence, as the transformed set does not loose its low-discrepancy and can be used for QMC integration.

The estimate proposed by us earlier is:

$$\theta_m = \frac{1}{N}\sum_{k=1}^{N} f(q_k^{(1)}, \ldots, q_k^{(d)}, X_k^{(d+1)}, \ldots, X_k^{(s)}), \tag{30}$$

where $(q_k, X_k)_{k\geq 1}$ is an s-dimensional *H-mixed sequence* on $[0,1]^s$.

In order to obtain such a H-mixed sequence, we first construct the H_q-distributed low-discrepancy sequence $(q_k)_{k\geq 1}$ on $[0,1]^d$, using the Hlawka-Mück method (the distribution function H_q was defined in (3)). Next, we generate the independent and identically distributed random points x_k, $k \geq 1$ on $[0,1]^{s-d}$, with the common distribution function H_X, using the inversion method (the distribution function H_X was defined in (4)). Finally, we concatenate q_k and x_k for each $k \geq 1$, and get our H-mixed sequence on $[0,1]^s$.

In our experiments, we used as low-discrepancy sequences on $[0,1]^d$, for the generation of H-mixed sequences, the Halton sequences (see [9]).

We suppose that the parameters of the NIG-distributed log-returns under the equivalent martingale measure given by the Esscher transform are given by

$$\mu = 0.00079 * 5,\ \beta = -15.1977,\ \alpha = 136.29,\ \delta = 0.0059 * 5, \tag{31}$$

and they are the same as in Kainhofer (see [13]). We observe that these parameters are relevant for daily observed stock price log-returns (see [21]). As the class of NIG distributions is closed under convolution, we can derive weekly stock prices by using a factor of 5 for the parameters μ and δ. We suppose further that the initial stock price is $S(0) = 100$ and the risk-free annual interest rate is $r = 3.75\%$.

The option is sampled at weekly time intervals. We also let the option to have maturities of 12 and 20 weeks. Hence, our problem is a 12 and 20-dimensional integral, respectively, over the payoff function.

We are going to compare the three estimates in terms of their absolute error, where the "exact" option price is obtained as the average of 10 MC simulations, with $N = 100000$ for the initial integral (21).

In our tests we have considered the following dimensions of the transformed integral (25) on $[0,1]^s$: $s = 12,\ 20$. The MC and H-mixed estimates are the mean values of 10 independent runs, while the QMC estimate is the result of a single run. The results are presented in two tables, each table containing the number of samples N, which varies from 5000 to 8500 with a step of 500, and the absolute error of the three estimates.

N	Absolute error MC	Absolute error QMC	Absolute error Mixed Method
5000	0.014731	0.012385	0.007676
5500	0.004485	0.016085	0.003780
6000	0.009268	0.011866	0.001892
6500	0.020887	0.014721	0.002547
7000	0.027395	0.014732	0.008411
7500	0.006316	0.012404	0.017385
8000	0.015027	0.010519	0.012538
8500	0.009207	0.010140	0.007248

Table 1: European Call Option. Case $d = 4$ and $s = 12$.

The numerical results for $s = 12$ and $d = 4$ are presented in Table 1. The results produced by our H-mixed sequence are much better than the ones obtained by using pseudorandom or low-discrepancy sequences, in almost all situations.

N	Absolute error MC	Absolute error QMC	Absolute error Mixed Method
5000	0.006381	0.049304	0.004311
5500	0.030035	0.039222	0.008018
6000	0.017018	0.042674	0.019373
6500	0.004735	0.041674	0.012044
7000	0.023534	0.038581	0.013131
7500	0.020561	0.030509	0.001833
8000	0.028440	0.027873	0.008792
8500	0.012737	0.032972	0.007264

Table 2: European Call Option. Case $d = 7$ and $s = 20$.

To increase the difficulty of the problem, we increase the dimension of the integral to $s = 20$. Table 2 displays the results we get for $s = 20$ and $d = 7$. From this simulations, we see again that the H-mixed sequence outperforms both the pseudorandom and low-discrepancy sequences, for almost all sample sizes N. The absolute error produced by our H-mixed sequence is smaller than the one produced by the low-discrepancy sequence, in all situations.

As a general conclusion for this option pricing problem, we can say that by the use of H-mixed sequences, we obtain increasing advantages over the classical pseudo-random and low-discrepancy sequences, for relatively high dimensions and moderate sample sizes.

4. **Application to finance: Asian options**

In this section, we consider an Asian option pricing problem. We compare numerically our mixed method with the MC and QMC methods, when they are applied to so-called (discrete sampled) Asian options driven by the asset dynamics $S(t)$, as defined in (16). The general setting remains the same as in the previous section, but the payoff function is changed. The payoff of an Asian call option is defined as

$$C_T(S) = \Big(\frac{1}{s}\sum_{i=1}^{s} S(t_i) - K\Big)_+ = \max\Big\{\frac{1}{s}\sum_{i=1}^{s} S(t_i) - K, 0\Big\}, \tag{32}$$

with $0 = t_0 < t_1 < t_2 < \ldots < t_s = T$. The constant $K \geq 0$ is called the strike price. Hence, we get the following integration problem:

$$I = \int_{\mathbb{R}^s} \underbrace{\Big(\frac{S(0)}{s} \sum_{i=1}^{s} e^{\sum_{j=1}^{i} x^{(j)}} - K \Big)_+}_{A(x)} dG(x) = \int_{\mathbb{R}^s} A(x) dG(x), \tag{33}$$

where $G(x) = \prod_{i=1}^{s} G_i(x^{(i)})$, $\forall x = (x^{(1)}, \ldots, x^{(s)}) \in \mathbb{R}^s$, and $G_i(x^{(i)})$ denotes the distribution function of the so-called log returns induced by $L(t_1)$, with the corresponding density function $g_i(x^{(i)})$. These log increments are independent and NIG distributed, having the common density function defined in (22).

In order to approximate the integral (33), we have to transform it to an integral on $[0,1]^s$. We can do this in a similar way as we did for European Call options, in the previous section. In the end, we get the following integration problem on $[0,1]^s$:

$$I = \int_{[0,1]^s} \underbrace{\Big(\frac{S(0)}{s} \sum_{i=1}^{s} e^{\sum_{j=1}^{i} G_{\lambda,i}^{-1}(z^{(j)})} - K \Big)_+}_{f(z)} dH(z) = \int_{[0,1]^s} f(z) dH(z), \tag{34}$$

where $H : [0,1]^s \to [0,1]$, defined by

$$H(z) = \prod_{i=1}^{s} (G_i \circ G_{\lambda,i}^{-1})(z^{(i)}), \ \forall z = (z^{(1)}, \ldots, z^{(s)}) \in [0,1]^s, \tag{35}$$

is a distribution function on $[0,1]^s$, with independent marginals $H_i = G_i \circ G_{\lambda,i}^{-1}$, $i = 1, \ldots, s$.

Next, we compare numerically our estimator θ_m, with the estimators obtained using the MC and QMC methods. All three estimators θ_m, θ_{MC} and θ_{QMC}, and the corresponding sequences are defined in the previous section. The function $f(z)$ is defined in relation (34). As a measure of comparison, we will use the absolute errors produced by these three methods, in the approximation of the integral (34).

We suppose that the parameters of the NIG-distributed log-returns under the equivalent martingale measure given by the Esscher transform are the same as in (31). We assume that the initial stock price is $S(0) = 100$, and the risk-free annual interest rate is $r = 3.75\%$.

For our mixed method and QMC estimate, we use a Halton sequence as low-discrepancy sequence on $[0,1]^s$. The Asian call option is sampled weekly. We also let the option to have maturities of 12 and 20 weeks. Hence, our problem is a 12 and 20-dimensional integral, respectively, over the payoff function.

We are going to compare the three estimates in terms of their absolute error, where the "true" price is obtained as the average of 10 MC simulations, with $N = 100000$. The MC and H-mixed estimates are the mean values of 10 independent runs, while the QMC estimate is the result of a single run. The results are presented in two tables, each table containing the number of samples N, which varies from 4000 to 7000 with a step size of 500, and the absolute error of the three estimates.

N	Absolute error MC	Absolute error QMC	Absolute error Mixed Method
4000	0.004833	0.000723	0.000690
4500	0.003060	0.001083	0.000977
5000	0.001095	0.000380	0.001653
5500	0.000293	0.000618	0.000599
6000	0.011389	0.001482	0.000898
6500	0.001733	0.003187	0.000218
7000	0.008720	0.001582	0.000047

Table 3: Asian Option. Case $d = 4$ and $s = 12$.

In Table 3 we present the results obtained for $s = 12$ and $d = 4$. The H-mixed sequence gives excellent estimates for almost all N, clearly dominating both the pseudorandom and low-discrepancy sequences.

N	Absolute error MC	Absolute error QMC	Absolute error Mixed Method
4000	0.036868	0.014541	0.004318
4500	0.007101	0.011003	0.016654
5000	0.002396	0.009723	0.012004
5500	0.016070	0.008897	0.000818
6000	0.004666	0.008920	0.000003
6500	0.017100	0.009541	0.003007
7000	0.010705	0.009437	0.002776

Table 4: Asian Option. Case $d = 7$ and $s = 20$.

The estimates presented in Table 4 are the results of the simulations for a higher dimensional problem, with $s = 20$ and $d = 7$. Again, the H-mixed method outperforms the conventional MC and QMC methods, in almost all situations.

We can conclude that our mixed method can give considerable improvements over the MC and QMC methods, in estimating high dimensional integrals, which we encounter in problems from financial mathematics, such as valuation of Asian options and European options.

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ON THE TIME-DEPENDENT MOTION OF A VISCOUS INCOMPRESSIBLE FLUID THROUGH A TUBE WITH COMPLIANT WALLS

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Abstract. In this note we study the flow of a viscous, incompressible fluid through an elastic cylinder which is very long when compared to its diameter. The fluid flows due to a given small time-dependent pressure drop between the inflow and the outflow boundary. This creeping flow is modeled by the Stokes equations for a viscous, incompressible flow, while Navier's equations for an elastic membrane describe the behavior of the flexible tube. We show existence and uniqueness of the solution for the system consisting of these equations and the corresponding boundary conditions.

1. Introduction

Fluid-structure interaction problems arise in many practical applications, like in aerospace, naval engineering, biomechanics and biomedical engineering (see e.g., [8], [4], [5]). A main issue in this context is haemodynamics.

The cardiovascular system is a very complex system, having a great variety of blood vessels, from large arteries through medium caliber vessels to capillaries. The blood flow is thus a very complicated phenomenon and blood itself is a fluid not easy to describe mathematically. Unless for the very tiny capillaries, it may be regarded as a continuum [2]) and (although Nonnewtonian) as Newtonian and incompressible, excepting some pathological situations [10], [12], [15].

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The problem we study is the following: a viscous incompressible fluid flows through an elastic tube which is very long when compared to its radius. The flow is driven by the difference of the pressures at both ends of the tube. The stress on the fluid depends on the displacement of the flexible wall; this in turn depends on the stresses exerted by the fluid on the interface between the two media. The only stress acting on the structure is supposed to come from the fluid. Fluid and solid mechanics are coupled through the wall position and the traction exerted by the fluid on the tube wall. This scenario can be seen as describing e.g., the blood flow through a segment of a smaller artery.

Our aim is to prove the existence of a unique solution of the above coupled problem. In [13] this has been done in the stationary case for the three-dimensional problem of a fluid flow through an elastic cylindrical tube with thickness and periodic conditions at the ends of the cylinder and in [14] for the full Navier-Stokes equations for the fluid and the nonlinear Navier-Lamé equations for the elastic structure with more general boundary conditions. For the two-dimensional case, when the equations of the fluid were coupled with the ones of an elastic beam we refer to [6]. Another model for a steady-state slow flow in a collapsible tube is studied numerically in [7], where geometrically nonlinear shell theory is used to accurately model the behavior of the tube wall, however by further simplifying the equations for the fluid. Instationary fluid-structure interaction problems (when the fluid domain has moving boundaries depending on time) are considered for instance in [11] and [9] for the two-dimensional case, where the flexible wall is modeled by the equations of an elastic beam or in [3] for the three-dimensional case of a fluid interacting with a structure having a finite number of elastic modes.

2. **The mathematical model**

The fluid is considered viscous, incompressible, unsteady and axisymmetric. We suppose that the pressure drop between the inflow and the outflow ends of the tube is small and that the viscous effects of blood are strongly predominant when

compared to the inertial ones. The flexible structure is a thin, long cylinder with very small thickness (an elastic cylindrical membrane). We thus model the fluid by the Stokes equations without time derivatives (a creeping flow) and the flexible tube by Navier's equations for a cylindrical elastic membrane. This seems to be a good model for blood flow in small arteries [5].

We denote by Ω the following domain:

$$\Omega := \{\mathbf{x} \in \mathbf{R}^3 \ : \ \mathbf{x} = (r\cos\theta, r\sin\theta, z),\ 0 \le r \le R,\ 0 \le z \le L\}, \tag{1}$$

where R and L are the radius, respectively the length of the cylinder.

We denote by S the lateral surface (elastic wall) of the cylinder and suppose its evolution is described by Navier's equations [12]:

$$\rho_w h \frac{\partial^2 u_r}{\partial t^2} = kGh\frac{\partial^2 u_r}{\partial z^2} - \frac{Eh}{1-\zeta^2}\left(\frac{\zeta}{R}\frac{\partial u_z}{\partial z} + \frac{u_r}{R^2}\right) + \Phi_r \text{ in } S \times (0,T) \tag{2}$$

$$\rho_w h \frac{\partial^2 u_z}{\partial t^2} = \frac{Eh}{1-\zeta^2}\left(\frac{\zeta}{R}\frac{\partial u_r}{\partial z} + \frac{\partial^2 u_z}{\partial z^2}\right) + \Phi_z \text{ in } S \times (0,T). \tag{3}$$

The unknown variables u_r and u_z represent the radial, respectively longitudinal displacement in the local frame of reference (cylindrical coordinates) (r, θ, z), h is the wall thickness, R is the arterial reference radius at rest, k is the Timoshenko shear correction factor, G is the shear modulus, E the Young modulus of elasticity, ζ the Poisson ratio ($\zeta = \frac{1}{2}$ for an incompressible material), ρ_w is the arterial wall volumetric mass. $\mathbf{\Phi} = (\Phi_r, \Phi_z)^t$ is the forcing term due to the external forces, included the stress coming from the fluid ($\mathbf{\Phi}$ depends on the velocity $\mathbf{v}$ and the pressure p of the fluid, that is of the blood).

Note that this model is based on a Lagrangian description of the motion of the elastic wall. It is referred to a material domain $\Omega(0)$, corresponding to the rest position where $u_r = u_z = 0$.

We also need initial and boundary conditions for the system (2), (3). For the former we consider the rest position and assume that initially there is no deformation of the elastic membrane. Since (2) and (3) are of second order in time, we also need

a condition at rest for the time derivatives of the displacements:

$$u_r(0) = u_z(0) = 0, \quad \frac{\partial u_r}{\partial t}(0) = \frac{\partial u_z}{\partial t}(0) = 0 \text{ on } S := \{r = R\} \times (0, L). \tag{4}$$

We further consider the ends of the elastic membrane fixed and take as corresponding boundary conditions the following:

$$u_r = u_z = 0 \text{ for } z = 0 \text{ and } u_r = u_z = 0 \text{ for } z = L, \ \forall t \in \mathbf{R}_+. \tag{5}$$

We come now to the equations modeling the fluid flow. Initially, the elastic tube is filled with fluid and the whole system is in equilibrium. This is the reference state. The pressure drop between the inflow and the outflow gives rise to a deviation from the reference state. If we assume that the acceleration of the fluid is small relatively to the predominant viscous effects (creeping flow), we can write for the fluid the following Stokes equations in cylindrical coordinates (we assume rotational symmetry and thus neglect the circumferential component of the velocity):

$$-\nu\left(\frac{\partial^2 v_r}{\partial r^2} + \frac{1}{r}\frac{\partial v_r}{\partial r} - \frac{1}{r^2}v_r + \frac{\partial^2 v_r}{\partial z^2}\right) + \frac{\partial p}{\partial r} = 0 \text{ in } \Omega \times (0, T) \tag{6}$$

$$-\nu\left(\frac{\partial^2 v_z}{\partial r^2} + \frac{1}{r}\frac{\partial v_z}{\partial r} + \frac{\partial^2 v_z}{\partial z^2}\right) + \frac{\partial p}{\partial z} = 0 \text{ in } \Omega \times (0, T) \tag{7}$$

$$\frac{v_r}{r} + \frac{\partial v_r}{\partial r} + \frac{\partial v_z}{\partial z} = 0 \text{ in } \Omega \times (0, T). \tag{8}$$

Here v_r, v_z are the radial, respectively the longitudinal component of the fluid velocity, ν is the viscosity of the fluid and p is the pressure. Equation (8) represents the incompressibility condition div $\mathbf{v} = 0$, written in cylindrical coordinates.

We also need initial and boundary conditions for this system. We assume the initial velocity zero:

$$\mathbf{v} = 0 \text{ in } \Omega \times \{0\} \tag{9}$$

and take the following boundary conditions at the inflow and outflow:

$$v_r = 0, \ p = 0 \text{ on } (\partial\Omega \cap \{z = 0\}) \times (0, T) \tag{10}$$

$$v_r = 0, \ p = P(t) \text{ on } (\partial\Omega \cap \{z = L\}) \times (0, T), \tag{11}$$

where $P(t)$ is the pressure drop driving the fluid.

The following condition, imposed on the rest of the fluid boundary, is a coupling condition and it ensures the continuity of the velocity field:

$$\mathbf{v} = \frac{\partial \mathbf{u}}{\partial t} \text{ on } S \times (0, T). \tag{12}$$

There is one more coupling condition to be satisfied, namely the continuity of stresses. This means that the forcing term on the elastic structure is due to the stresses exerted by the fluid (and possibly by external terms due, for instance, to surrounding organs or muscle tissue, which, however, we neglect here). Thus, the forcing term $\mathbf{\Phi}$ in (2), (3) takes the form:

$$\mathbf{\Phi} = -(p\mathbf{I} - 2\nu\mathbf{e}(\mathbf{v})) \cdot \mathbf{e}_r \text{ on } S \times (0, T). \tag{13}$$

Here $\mathbf{e}(\mathbf{v}) := \frac{1}{2}(\nabla\mathbf{v} + (\nabla\mathbf{v})^t)$ is the strain tensor (the symmetrized gradient of the velocity).

Remark 2.1.

- The boundary conditions for the structure considered in (5) have been taken homogeneous just for the sake of simplicity. More natural boundary conditions at the outflow should not be zero, since the ends of the elastic structure ($z = 0$ and $z = L$) are tipically "artificial boundaries" (just like the inflow and outflow ends of the fluid) and one should choose them in order not to perturb the numerics.
- We take the pressure drop $P(t)$ in (11) as being as regular as we need in all our further considerations.
- Here we consider the case of a fixed fluid-structure interface. It is known in general that the movement of a solid body implies rigid body motions and displacements caused by the stresses and strains induced in the solid body by the loads coming from the fluid which interacts with the structure. If these displacements are small enough, then one may assume that the interface is stationary, i.e. it does not move in time (unlikely for large displacements). However, even if the displacements are small, the velocity

of deformation is not, therefore we have to take condition (12) instead of a homogeneous Dirichlet type condition for the velocity at the interface.

Thus, the problem can be stated as follows:

Problem 2.2. *Determine a solution* $(\mathbf{u}, \mathbf{v})$ *of the system (2), (3), (6)-(8) in* $S \times \Omega$, *with the initial conditions (4) and (9) and the boundary conditions (5), (10), (11) and (12), where the force* $\mathbf{\Phi}$ *in the equations for the elastic structure is given by the fluid stresses as in (13).*

Our aim is to prove the existence of a unique solution to the coupled problem. This will be done with the aid of Galerkin approximations.

3. **Weak formulation and main result**

In this section we give the weak formulation of the coupled problem and state the main result.

Let us define the space of test functions by:

$$\mathbf{\Psi} := \{\psi \in \mathbf{H}^1(\Omega) \ : \ \psi_r, \psi_z \in H^1(0,L), \ \text{div } \psi = 0 \text{ in } \Omega \text{ and}$$

$$\psi_r(r,0) = \psi_r(r,L) = \psi_z(R,L) = \psi_z(R,0) = 0 \text{ for } r \in [0,R]\} \tag{14}$$

Definition 3.1. $(\mathbf{v}, \mathbf{u}) \in H^1(0,T;\mathbf{H}^1(\Omega)) \times L^2(0,T;\mathbf{H}^1(0,L))$ *with* $\mathbf{u}' \in L^2(0,T;$ $\mathbf{L}^2(0,L))$ *and* $\mathbf{u}'' \in L^2(0,T;\mathbf{H}^{-1}(0,L))$ *is called a weak solution of Problem 2.2 if for all* $\psi \in \mathbf{\Psi}$ *the following variational formulation is satisfied in the sense of distributions (in* $\mathcal{D}'(0,T)$*):*

$$R\rho_w h \frac{d^2}{dt^2} \int_0^L (u_r\psi_r + u_z\psi_z)dz + R\int_0^L \Bigg[kGh \frac{\partial u_r}{\partial z}\frac{\partial \psi_r}{\partial z}$$

$$+ \frac{Eh}{1-\zeta^2}\left(\frac{\zeta}{R}\frac{\partial u_z}{\partial z}\psi_r + \frac{u_r}{R^2}\psi_r + \frac{\partial u_z}{\partial z}\frac{\partial \psi_z}{\partial z} - \frac{\zeta}{R}\frac{\partial u_r}{\partial z}\psi_z\right)\Bigg] dz \tag{15}$$

$$+ 2\nu \int_\Omega \mathbf{e}(\mathbf{v}) : \mathbf{e}(\psi) r dr dz = -\int_0^R P(t)\psi_z r dr$$

and

$$\mathbf{u} = \frac{\partial \mathbf{u}}{\partial t} = 0 \text{ on } S \times \{0\} \text{ and } \mathbf{v} = 0 \text{ in } \Omega \times \{0\}. \tag{16}$$

(15) has been obtained by testing in (2) and (3) with ψ_r, respectively ψ_z and in (6), (7) with ψ, integrating by parts on the corresponding domains, using equation (8) and conditions (5), (10), (11), (13) and summing up the equations resulted after the testing.

We now can state the main result.

Theorem 3.1. *There exists a unique weak solution of Problem 2.2.*

4. **Proof of the existence**

4.1. **Galerkin Approximations.** The proof is based on the method of Galerkin, that is we build a weak solution of the problem by first constructing solutions of certain finite dimensional approximations and then passing to limits. We therefore take the functions $\mathbf{w}_k = \mathbf{w}_k(r,z)$ $(k = 1, 2, ...)$ such that

$$\{\mathbf{w}_k\}_{k=1,...,\infty} \text{ is a basis of } \mathbf{\Psi}. \tag{17}$$

In particular, we take $\{\mathbf{w}_k\}_k$ to be the complete set of eigenfunctions of the eigenvalue problem

$$\mathbf{w} \in \mathbf{\Psi},\ (\nabla \mathbf{w}, \nabla \boldsymbol{\psi})_{(0,L)} + (\mathbf{e}(\mathbf{w}), \mathbf{e}(\boldsymbol{\psi}))_\Omega = \lambda[(\mathbf{w}, \boldsymbol{\psi})_{(0,L)} + (\mathbf{w}, \boldsymbol{\psi})_\Omega],\ \boldsymbol{\psi} \in \mathbf{\Psi};$$

we also assume that $\{\mathbf{w}_k\}_k$ is orthonormalized w.r.t. the $\mathbf{H}^1_{0,ends}(\Omega \cup S)$-inner product[1] $(\nabla \cdot, \nabla \cdot)_{(0,L)} + (\mathbf{e}(\cdot), \mathbf{e}(\cdot))_\Omega$. Also observe that $\{\mathbf{w}_k\}_k$ is orthogonal w.r.t. the $\mathbf{L}^2$-inner product in the right hand side of the equation above.

Fix a positive integer m and write

$$\mathbf{v}_m(t) := \sum_{k=1}^{m} c_{km}(t)\mathbf{w}_k, \tag{18}$$

where the coefficients $c_{km}(t),\ k = 1, ..., m,\ 0 \le t \le T$ are intended to satisfy

$$c_{km}(0) = 0,\ k = 1, ..., m. \tag{19}$$

[1] We have denoted by $\mathbf{H}^1_{0,ends}$ the functions which are in $\mathbf{H}^1$ and vanish at the ends of the cylinder.

This would be the approximation of the fluid's velocity. By

$$u_{m,r}(t) := \sum_{k=1}^{m} \alpha_{km}(t) w_{k,r} \text{ and } u_{m,z}(t) := \sum_{k=1}^{m} \alpha_{km}(t) w_{k,z} \tag{20}$$

we construct (with the same basis $\{\mathbf{w}_k\}_{k=1,...}$) an approximation of the displacement of the elastic membrane. The coefficients of these approximations should satisfy (by the continuity of velocities on $S \times (0,T)$) the equation

$$\alpha_{km}(t) = \int_0^t c_{km}(s) ds. \tag{21}$$

Observe that in virtue of (18), (20) and (21) we may write

$$\frac{\partial u_{m,r}}{\partial t} = \sum_{k=1}^{m} c_{km}(t) w_{k,r} \text{ in } S \times (0,T) \tag{22}$$

$$\frac{\partial u_{m,z}}{\partial t} = \sum_{k=1}^{m} c_{km}(t) w_{k,z} \text{ in } S \times (0,T). \tag{23}$$

By (19), the coefficients $\alpha_{km}(t),\ k = 1, ..., m,\ 0 \le t \le T$ satisfy

$$\alpha_{km}(0) = 0 \text{ and } \alpha'_{km}(0) = 0,\ k = 1, ..., m. \tag{24}$$

The Galerkin approximation corresponding to (15) writes ($0 \le t \le T,\ k = 1, ..., m$):

$$R\rho_w h(\mathbf{u}''_m(t), \mathbf{w}_k)_{(0,L)} + C[u_{m,r}, \mathbf{w}_k; t] + D[u_{m,z}, \mathbf{w}_k; t] + B[\mathbf{v}_m, \mathbf{w}_k; t] = -\int_0^R P(t) w_{k,z} r dr, \tag{25}$$

where

$$B[\mathbf{v}, \mathbf{w}; t] := 2\nu \int_\Omega \mathbf{e}(\mathbf{v}) : \mathbf{e}(\mathbf{w}) r dr dz,$$

$$C[u_r, \mathbf{w}; t] := R \int_0^L \left[kGh \frac{\partial u_r}{\partial z} \frac{\partial w_r}{\partial z} + \frac{Eh}{1-\zeta^2} \left(\frac{u_r}{R^2} w_r - \frac{\zeta}{R} \frac{\partial u_r}{\partial z} w_z \right) \right] dz$$

and

$$D[u_z, \mathbf{w}; t] := R \int_0^L \frac{Eh}{1-\zeta^2} \left(\frac{\zeta}{R} \frac{\partial u_z}{\partial z} w_r + \frac{\partial u_z}{\partial z} \frac{\partial w_z}{\partial z} \right) dz.$$

Further (use (17)),

$$B[\mathbf{v}_m, \mathbf{w}_k; t] + C[u_{m,r}, \mathbf{w}_k; t] + D[u_{m,z}, \mathbf{w}_k; t] = \sum_{l=1}^{m} \beta^{kl} \alpha_{lm}(t),$$

where

$$\beta^{kl} := B[\mathbf{w}_l, \mathbf{w}_k] + C[w_{l,r}, \mathbf{w}_k] + D[w_{l,z}, \mathbf{w}_k],\ k, l = 1, ..., m.$$

Consequently, (25) becomes the following linear system of ODEs:

$$\alpha''_{km}(t) + \sum_{l=1}^{m} \beta^{kl} \alpha_{lm}(t) = P^k(t)\ (0 \le t \le T,\ k = 1, ..., m), \tag{26}$$

where $P^k(t) := -\int_0^R P(t) w_{k,z} r dr$.

The system is subject to the initial conditions (24). By the standard theory for ordinary differential equations (remember that $P(t)$ is regular enough, see Remark 2.1), there exists a unique function $\alpha_m(t) = (\alpha_{1m}(t), ..., \alpha_{mm}(t))$ in C^2, satisfying (24) and solving (26) for $0 \le t \le T$.

4.2. **Energy Estimates.** We intend to pass to the limit with $m \to \infty$ and for this we need some estimates that should be uniform in m.

Theorem 4.1. *There exists a constant $C > 0$ such that*

$$\sup_{0 \le t \le T} \left(||\mathbf{u}'_m(t)||^2_{\mathbf{L}^2(0,L)} + ||\mathbf{u}_m||^2_{\mathbf{H}^1(0,L)} \right) + ||\mathbf{u}''_m(t)||^2_{L^2(0,T;\mathbf{H}^{-1}(0,L))}$$

$$+ ||\mathbf{v}_m||^2_{L^2(0,T;\mathbf{H}^1(\Omega))} \le C(1 + ||P||^2_{L^2(0,T)}) \tag{27}$$

The constant C depends only on Ω, T, R, G, h, k, ζ, ν and ρ_w.

Proof. Multiply (25) by $c_{km}(t)$. By summing up after $k = 1, ..., m$ and taking into account (18), (22), (23), we get:

$$R\rho_w h \int_0^L \frac{\partial^2 \mathbf{u}_m(t)}{\partial t^2} \frac{\partial \mathbf{u}_m(t)}{\partial t} dz + R \int_0^L kGh \frac{\partial u_{m,r}(t)}{\partial z} \frac{\partial^2 u_{m,r}(t)}{\partial t \partial z} dz$$

$$+ R \int_0^L \frac{Eh}{1 - \zeta^2} \left((\frac{\zeta}{R} \frac{\partial u_{m,z}(t)}{\partial z} + \frac{u_{m,r}(t)}{R^2}) \frac{\partial u_{m,r}(t)}{\partial t} + \frac{\partial u_{m,z}(t)}{\partial z} \frac{\partial^2 u_{m,z}(t)}{\partial t \partial z} \right.$$

$$-\frac{\zeta}{R}\frac{\partial u_{m,r}(t)}{\partial z}\frac{\partial u_{m,z}(t)}{\partial t}\Big)dz+2\nu\int_\Omega \mathbf{e}(\mathbf{v}_m(t)):\mathbf{e}(\mathbf{v}_m(t))rdrdz$$
$$=-\int_0^R P(t)v_{m,z}(t,r,L)rdr.$$

Let us have a closer look at the term whose coefficient is $R\frac{Eh}{1-\zeta^2}$. If we perform a partial integration on the last term of it, use (5) to get rid of the boundary terms and rearrange what we get, it takes the form:

$$R\frac{Eh}{1-\zeta^2}\cdot\frac{1}{2}\frac{d}{dt}\int_0^L\Big[(1-\zeta)\Big(\Big(\frac{\partial u_{m,z}}{\partial z}\Big)^2+\Big(\frac{u_{m,r}}{R}\Big)^2\Big)+\zeta\Big(\frac{u_{m,r}}{R}+\frac{\partial u_{m,z}}{\partial z}\Big)^2\Big]dz.$$

Thus, the above identity becomes:

$$\frac{R}{2}\frac{d}{dt}\Big[\rho_w h||\mathbf{u}_m'(t)||^2_{L^2(0,L)}+kGh||u_{m,r}(t)||^2_{H^1(0,L)}$$
$$+\frac{Eh}{1-\zeta^2}\Big(\zeta||\frac{u_{m,r}(t)}{R}+\frac{\partial u_{m,z}(t)}{\partial z}||^2_{L^2(0,L)}+(1-\zeta)\Big(||\frac{u_{m,r}(t)}{R}||^2_{L^2(0,L)}$$
$$+||u_{m,z}(t)||^2_{H^1(0,L)}\Big)\Big)\Big]+2\nu|\mathbf{e}(\mathbf{v}_m)|^2_{L^2(\Omega)}=-\int_0^R P(t)v_{m,z}(t,r,L)rdr.$$

The right hand side above may be majorized as follows:

$$-\int_0^R P(t)v_{m,z}(t,r,L)rdr\le|\int_0^R P(t)v_{m,z}(t,r)rdr|\le\frac{1}{L}|P(t)|\int_\Omega|v_{m,z}(t)|$$
$$\le\frac{\delta}{L^2}|P(t)|^2+\frac{1}{\delta}|v_m(t)|^2_{L^2(\Omega)}\le\frac{\delta}{L^2}|P(t)|^2+\frac{1}{\delta}||v_m(t)||^2_{H^1(\Omega)}.$$

We use this estimation and Korn's inequality (see, for instance, [1]) in the identity above to obtain:

$$\frac{R}{2}\frac{d}{dt}\Big[\rho_w h||\mathbf{u}_m'(t)||^2_{L^2(0,L)}+kGh||u_{m,r}(t)||^2_{H^1(0,L)}$$
$$+\frac{Eh}{1-\zeta^2}\Big(\zeta||\frac{u_{m,r}(t)}{R}+\frac{\partial u_{m,z}(t)}{\partial z}||^2_{L^2(0,L)}+(1-\zeta)\Big(||\frac{u_{m,r}(t)}{R}||^2_{L^2(0,L)}$$
$$+||u_{m,z}(t)||^2_{H^1(0,L)}\Big)\Big)\Big]+C_1||\mathbf{v}_m||^2_{H^1(\Omega)}\le\frac{\delta}{L^2}|P(t)|^2. \tag{28}$$

Here $C_1>0$ is a constant depending on ν, δ and the constant in Korn's inequality.

We now integrate (28) from 0 to t ($t > 0$) and use the initial conditions (4), in order to get the following:

$$\frac{R}{2}\Bigg[\rho_w h||\mathbf{u}'_m(t)||^2_{L^2(0,L)} + kGh||u_{m,r}(t)||^2_{H^1(0,L)}$$

$$+\frac{Eh}{1-\zeta^2}\Big(\zeta||\frac{u_{m,r}(t)}{R} + \frac{\partial u_{m,z}(t)}{\partial z}||^2_{L^2(0,L)} + (1-\zeta)\Big(||\frac{u_{m,r}(t)}{R}||^2_{L^2(0,L)}$$

$$+||u_{m,z}(t)||^2_{H^1(0,L)}\Big)\Big)\Bigg] + C_1\int_0^t ||\mathbf{v}_m(s)||^2_{H^1(\Omega)}ds \leq \frac{\delta}{L^2}||P||^2_{L^2(0,T)}. \tag{29}$$

From (29) it follows that:

$$\sup{}_{0\leq t\leq T}\left(||\mathbf{u}'_m(t)||^2_{L^2(0,L)} + ||\mathbf{u}_m||^2_{H^1(0,L)}\right) \leq C_2||P||^2_{L^2(0,T)}, \tag{30}$$

where $0 < C2 := \frac{\delta}{L^2}\frac{2}{R}\cdot(\min\{\rho_w h,\ kGh,\ \frac{Eh}{1+\zeta}\})^{-1}$.

Now integrate (28) from 0 to T, use again (4) and obtain:

$$\int_0^T ||\mathbf{v}_m(t)||^2_{H^1(\Omega)}dt \leq C_3||P||^2_{L^2(0,T)}, \tag{31}$$

where $0 < C_3 := \frac{\delta}{L^2 C_1}$.

In order to obtain (27), we still need some estimate for the second derivative in time of $\mathbf{u}_m$. In order to do that, let us fix any $\xi \in \mathbf{\Psi}$ with $||\xi||_{\mathbf{H}^1} \leq 1$ and write $\xi = \xi_1 + \xi_2$, where $\xi_1 \in$ span $\{\mathbf{w}_k\}_{k=1,...,m}$ and $(\xi_2, \mathbf{w}_k) = 0$ $(k = 1, ..., m)$.

Notice that

$$||\xi_1||_{\mathbf{H}^1} = ||\xi - \xi_2||_{\mathbf{H}^1} \leq ||\xi||_{\mathbf{H}^1} + ||\xi_2||_{\mathbf{H}^1} \leq 1.$$

We also consider that the only nonzero component of ξ is the radial one: $\xi = \xi_1 + \xi_2 = (\xi_{1,r} + \xi_{2,r})\mathbf{e}_r + (\xi_{1,z} + \xi_{2,z})\mathbf{e}_z = (\xi_{1,r} + \xi_{2,r})\mathbf{e}_r$.

Now we test in (25) with ξ subject to the above conditions and obtain (remember (20):

$$R\rho_w h(u''_{m,r}(t), \xi_{1,r})_{(0,L)} + C[u_{m,r}, \xi_1; t] + D[u_{m,z}, \xi_1; t] + B[\mathbf{v}_m, \xi_1; t] = 0. \tag{32}$$

Here

$$B[\mathbf{v}_m, \xi_1; t] = 2\nu \int_\Omega \mathbf{e}(\mathbf{v}_m) : \mathbf{e}(\xi_1) r dr dz,$$

thus

$$|B[\mathbf{v}_m, \xi_1; t]| \leq \text{const } ||\mathbf{v}_m||_{H^1(\Omega)} ||\xi_1||_{H^1(\Omega)} \leq \text{const } ||\mathbf{v}_m||_{H^1(\Omega)}.$$

Further,

$$|C[u_{m,r}, \xi_1; t]| \leq \text{const } (1 + ||u_{m,r}||_{L^2(0,L)})$$

and

$$|D[u_{m,z}, \xi_1; t]| \leq \text{const } ||\mathbf{u}_m||_{H^1(0,L)}.$$

By using these estimates, (32), (30) and (31), it follows:

$$|(u''_{m,r}, \xi_{1,r})| \leq \text{const } (||\mathbf{v}_m||_{H^1(\Omega)} + 1 + ||u_{m,r}||_{L^2(0,L)}).$$

Thus,

$$\int_0^T ||u''_{m,r}(t)||^2_{H^{-1}(0,L)} dt \leq C_4, \tag{33}$$

where the constant $C_4 > 0$ depends on C_2 and C_3.

Now we consider that the only nonzero component of ξ is the longitudinal one. Testing under the above conditions in (25) with ξ leads to:

$$R\rho_w h(u''_{m,z}(t), \xi_{1,z})_{(0,L)} + C[u_{m,r}, \xi_1; t] + D[u_{m,z}, \xi_1; t] + B[\mathbf{v}_m, \xi_1; t] = 0. \tag{34}$$

Analogously as above, it follows that

$$\int_0^T ||u''_{m,z}(t)||^2_{H^{-1}(0,L)} dt \leq C_5(1 + ||P||^2_{L^\infty(0,T)}), \tag{35}$$

where, again, the constant $C_5 > 0$ depends on C_2 and C_3.

(27) follows now from (30), (31), (33) and (35), where the constant C may be taken as $\sum_{i=2}^{5} C_i$. □

4.3. **Existence of a Weak Solution.** We now pass to limits (for $m \to \infty$) in our Galerkin approximations.

The estimate (27) implies that:

$$(\mathbf{u}_m)_m \text{ is bounded in } L^2(0,T;\mathbf{H}^1(0,L)) \tag{36}$$

$$(\mathbf{u}'_m)_m \text{ is bounded in } L^2(0,T;\mathbf{L}^2(0,L)) \tag{37}$$

$$(\mathbf{u}''_m)_m \text{ is bounded in } L^2(0,T;\mathbf{H}^{-1}(0,L)) \tag{38}$$

and

$$(\mathbf{v}_m)_m \text{ is bounded in } L^2(0,T;\mathbf{H}^1(\Omega)). \tag{39}$$

Consequently, there exist some sequences $(\mathbf{u}_{m_k})_k \subset (\mathbf{u}_m)_m$ and $(\mathbf{v}_{m_k})_k \subset (\mathbf{v}_m)_m$ and the functions $\mathbf{u} \in L^2(0,T;\mathbf{H}^1(0,L))$ with $\mathbf{u}' \in L^2(0,T;\mathbf{L}^2(0,L))$, $\mathbf{u}'' \in L^2(0,T;$
$H^{-1}(0,L))$, $\mathbf{v} \in L^2(0,T;\mathbf{H}^1(\Omega))$ such that

$$\mathbf{u}_{m_k} \overset{k\to\infty}{\rightharpoonup} \mathbf{u} \text{ in } L^2(0,T;\mathbf{H}^1(0,L)) \tag{40}$$

$$\mathbf{u}'_{m_k} \overset{k\to\infty}{\rightharpoonup} \mathbf{u}' \text{ in } L^2(0,T;\mathbf{L}^2(0,L)) \tag{41}$$

$$\mathbf{u}''_{m_k} \overset{k\to\infty}{\rightharpoonup} \mathbf{u}'' \text{ in } L^2(0,T;\mathbf{H}^{-1}(0,L)) \tag{42}$$

and

$$\mathbf{v}_{m_k} \overset{k\to\infty}{\rightharpoonup} \mathbf{v} \text{ in } L^2(0,T;\mathbf{H}^1(\Omega)). \tag{43}$$

We now fix an integer N and choose a function $\varphi \in C^1(0,T;\mathbf{\Psi})$ of the form

$$\varphi(t) := \sum_{k=1}^{N} \alpha_k(t)\mathbf{w}_k, \tag{44}$$

where $\{\alpha_k\}_{k=\overline{1,N}}$ are smooth functions. We choose N such that $N \le m$, multiply (25) by $\alpha_k(t)$, sum after $k = 1, ..., N$ and integrate by parts to obtain:

$$R\rho_w h \int_0^T (\mathbf{u}''_m(t), \varphi(t))_{(0,L)} dt + \int_0^T \{C[u_{m,r}, \varphi; t] + D[u_{m,z}, \varphi; t]$$

$$+B[\mathbf{v}_m, \varphi; t]\}dt = -\int_0^T \int_0^R P(t)\varphi_z r dr dt. \tag{45}$$

We may now pass to the limit in the above identity, in virtue of (40), (41), (42) and (43) (set $m = m_k$) and obtain:

$$R\rho_w h \int_0^T (\mathbf{u}''(t), \varphi(t))_{(0,L)} dt + \int_0^T \{C[u_r, \varphi; t] + D[u_z, \varphi; t]$$

$$+B[\mathbf{v}, \varphi; t]\} dt = -\int_0^T \int_0^R P(t)\varphi_z r dr dt. \tag{46}$$

Note that (46) holds for all functions $\varphi \in L^2(0, T; \mathbf{\Psi})$, since functions of the form (44) are dense in this space. It also follows from (46) that

$$R\rho_w h(\mathbf{u}'', \varphi)_{(0,L)} + C[u_r, \varphi; t] + D[u_z, \varphi; t] + B[\mathbf{v}, \varphi; t] = -\int_0^R P(t)\varphi_z r dr$$

for all $\varphi \in \mathbf{\Psi}$ and a.e. $0 \leq t \leq T$.

Also notice that $\mathbf{u} \in C([0, T]; \mathbf{L}^2(0, L))$ and $\mathbf{u}' \in C([0, T]; \mathbf{H}^{-1}(0, L))$.

We still have to verify that

$$\mathbf{u}(0) = 0, \ \mathbf{u}'(0) = 0 \text{ in } S \tag{47}$$

and

$$\mathbf{v}(0) = 0 \text{ in } \Omega. \tag{48}$$

We therefore choose any function $\varphi \in C^2([0, T]; \mathbf{\Psi})$, with $\varphi(T) = \varphi'(T) = 0$. We then integrate by parts twice in time in (46) to obtain

$$R\rho_w h \int_0^T (\mathbf{u}(t), \varphi''(t))_{(0,L)} dt + \int_0^T \{C[u_r, \varphi; t] + D[u_z, \varphi; t] + B[\mathbf{v}, \varphi; t]\} dt$$

$$= -\int_0^T \int_0^R P(t)\varphi_z r dr dt - (\mathbf{u}(0), \varphi'(0))_{(0,L)} + (\mathbf{u}'(0), \varphi(0))_{(0,L)}. \tag{49}$$

Analogously, we deduce from (45) that

$$R\rho_w h \int_0^T (\mathbf{u}_m, \varphi'')_{(0,L)} dt + \int_0^T \{C[u_{m,r}, \varphi; t] + D[u_{m,z}, \varphi; t] + B[\mathbf{v}_m, \varphi; t]\} dt$$

$$= -\int_0^T \int_0^R P(t)\varphi_z r dr dt - (\mathbf{u}_m(0), \varphi'(0))_{(0,L)} + (\mathbf{u}'_m(0), \varphi(0))_{(0,L)}.$$

We set again $m = m_k$ and deduce from (24), (40), (41), (42) and (43) that

$$R\rho_w h \int_0^T (\mathbf{u}_m(t), \varphi''(t))_{(0,L)} dt + \int_0^T \{C[u_{m,r}, \varphi; t] + D[u_{m,z}, \varphi; t]$$

$$+B[\mathbf{v}_m, \varphi; t]\} dt = -\int_0^T \int_0^R P(t)\varphi_z r dr dt. \tag{50}$$

Compare now the identities (49) and (50) to deduce (47), since $\varphi(0)$, $\varphi'(0)$ are arbitrary.

We now intend to verify (48). For this we need some estimate on $\mathbf{v}'_m$. We therefore differentiate in (25) with respect to time and get:

$$R\rho_w h(\mathbf{v}''_m(t), \mathbf{w}_k)_{(0,L)} + C[v_{m,r}(t), \mathbf{w}_k; t] + D[v_{m,z}(t), \mathbf{w}_k; t]$$

$$+B[\mathbf{v}'_m(t), \mathbf{w}_k; t] = -\int_0^R P'(t) w_{k,z} r dr. \tag{51}$$

Multiply (51) with $c'_{km}(t)$ and sum after $k = 1, ..., m$. It follows:

$$R\rho_w h \frac{1}{2}\frac{d}{dt}|\mathbf{v}_m(t)|^2_{L^2(0,L)} + C[v_{m,r}, \mathbf{v}'_m; t] + D[v_{m,z}, \mathbf{v}'_m; t]$$

$$+2\nu|\mathbf{e}(\mathbf{v}'_m(t))|^2_{L^2(\Omega)} = -\int_0^R P'(t) v'_{m,z} r dr. \tag{52}$$

The right hand side in (52) can be majorized in the following way:

$$-\int_0^R P'(t) v'_{m,z} r dr \leq \gamma |P'(t)|^2 + \frac{1}{\gamma}|\mathbf{v}'_m|^2_{L^2(\Omega)},$$

where γ is a positive constant.

Applying again Korn's inequality and using Gronwall's inequality it follows from (52) that

$$\mathbf{v}'_m \text{ is bounded in } L^2(0, T; \mathbf{L}^2(\Omega)).$$

Integrate in (52) from 0 to T and using the boundedness of $\mathbf{v}'_m$ in $L^2(0,T;\mathbf{L}^2(\Omega))$, we obtain that

$$\mathbf{v}'_m \text{ is bounded in } L^2(0,T;\mathbf{H}^1(\Omega)).$$

Consequently, there exists a subsequence $(\mathbf{v}_{m_k})_k$ of $(\mathbf{v}_m)_m$ and $\mathbf{v}' \in L^2(0,T;\mathbf{H}^1(\Omega))$ with

$$\mathbf{v}'_{m_k} \overset{k\to\infty}{\rightharpoonup} \mathbf{v}' \text{ in } L^2(0,T;\mathbf{H}^1(\Omega)). \tag{53}$$

Multiply (25) by $\alpha'_k(t)$, sum after $k = 1, ..., m$, use the assumptions on φ and integrate by parts with respect to time the term with $B[.,.;t]$ to get:

$$R\rho_w h \int_0^T (\mathbf{u}''_m(t), \varphi'(t))_{(0,L)} dt + \int_0^T [C[u_{m,r}, \varphi'; t] + D[u_{m,z}, \varphi'; t]] dt$$

$$- \int_0^T B[\mathbf{v}'_m, \varphi; t] dt = - \int_0^T \int_0^R P(t) \varphi'_z r dr dt. \tag{54}$$

We may now pass to the limit (take $m_k = m$) in (54), in virtue of the weak convergences obtained so far. It follows that:

$$R\rho_w h \int_0^T (\mathbf{u}''(t), \varphi'(t))_{(0,L)} dt + \int_0^T [C[u_r, \varphi'; t] + D[u_z, \varphi'; t]] dt$$

$$- \int_0^T B[\mathbf{v}', \varphi; t] dt = - \int_0^T \int_0^R P(t) \varphi'_z r dr dt. \tag{55}$$

If we pass to the limit in (54) before integrating by parts the term with $B[.,.;t]$, we obtain (doing the integration by parts *afterwards*):

$$R\rho_w h \int_0^T (\mathbf{u}''(t), \varphi'(t))_{(0,L)} dt + \int_0^T [C[u_r, \varphi'; t] + D[u_z, \varphi'; t]] dt$$

$$- \int_0^T B[\mathbf{v}', \varphi; t] dt - B[\mathbf{v}(0), \varphi(0); 0] = - \int_0^T \int_0^R P(t) \varphi'_z r dr dt. \tag{56}$$

Now, by comparing (55) and (56), since $\varphi(0)$ is arbitrary, we obtain (48). Consequently, $(\mathbf{u}, \mathbf{v})$ is a weak solution of Problem 2.2, corresponding to the weak formulation (15), (16). □

4.4. **Proof of the uniqueness.** In this section we prove the uniqueness of the weak solution found in the previous section. For this it suffices to show that the only weak solution of Problem 2.2 with $P(t) \equiv 0$ is

$$(\mathbf{u}, \mathbf{v}) \equiv \mathbf{0}. \tag{57}$$

Fix $0 \leq s \leq T$ and take

$$\boldsymbol{\zeta}(t) := \begin{cases} \int\limits_t^s \mathbf{v}(\tau)d\tau & \text{if} \quad 0 \leq t \leq s \\ 0 & \text{if} \quad s \leq t \leq T \end{cases}. \tag{58}$$

Observe that

$$\boldsymbol{\zeta}'(t) = -\mathbf{v}(t),$$

thus on $S \times (0,T)$ also $\boldsymbol{\zeta}(t) = -\mathbf{u}(t)$.

Then from the regularity properties of $\mathbf{v}$ and $\mathbf{u}$ it follows that $\boldsymbol{\zeta}(t) \in \mathbf{H}^1(\Omega)$, $\forall\ 0 \leq t \leq T$ with $\zeta_r(t),\ \zeta_z(t) \in H^1(0,L),\ \zeta_r(R,L) = \zeta_r(R,0) = 0,\ \zeta_z(R,L) = \zeta_z(R,0) = 0$ and thus we can write (see (25)):

$$R\rho_w h \int\limits_0^s (\mathbf{u}''(t), \boldsymbol{\zeta}(t))_{(0,L)} dt + \int\limits_0^s \{C[u_r, \boldsymbol{\zeta}; t] + D[u_z, \boldsymbol{\zeta}; t] + B[\mathbf{v}, \boldsymbol{\zeta}; t]\} dt = 0.$$

Integrate by parts with respect to time and use (4) and (58) to write:

$$-R\rho_w h \int\limits_0^s (\mathbf{u}'(t), \boldsymbol{\zeta}'(t))_{(0,L)} dt + \int\limits_0^s B[\mathbf{v}, \boldsymbol{\zeta}; t] dt$$

$$= -\int\limits_0^s \{C[u_r, \boldsymbol{\zeta}; t] + D[u_z, \boldsymbol{\zeta}; t]\} dt.$$

We have

$$R\rho_w h \int\limits_0^s |\mathbf{u}'(t)|^2_{(0,L)} dt - \nu \frac{d}{dt} \int\limits_0^s |\mathbf{e}(\boldsymbol{\zeta}(t))|^2_{L^2(\Omega)} \leq C \int\limits_0^s ||\boldsymbol{\zeta}(t)||^2_{\mathbf{H}^1(\Omega)} dt,$$

thus (use again Korn's inequality)

$$\nu |\boldsymbol{\zeta}(0)|^2_{\mathbf{H}^1(\Omega)} \leq C \int\limits_0^s ||\boldsymbol{\zeta}(t)||^2_{\mathbf{H}^1(\Omega)} dt. \tag{59}$$

We now define $\mathbf{g}(t) := \int_0^t \mathbf{v}(\tau)d\tau$, $0 \leq t \leq T$. Then note that $\boldsymbol{\zeta}(0) = \mathbf{g}(s)$ and $\boldsymbol{\zeta}(t) = \mathbf{g}(s) - \mathbf{g}(t)$. Consequently, we deduce from (59) that

$$(1 - C(\nu)s)||\mathbf{g}(s)||^2_{\mathbf{H}^1(\Omega)} \leq C(\nu) \int_0^s ||\mathbf{g}(t)||^2_{\mathbf{H}^1(\Omega)} dt$$

and we choose $0 < T_1$ small enough ($0 \leq T_1 \leq \frac{1}{2C(\nu)}$).

Then for $0 \leq s \leq T_1$ we have

$$||\mathbf{g}(s)||^2_{\mathbf{H}^1(\Omega)} \leq C(\nu, T_1) \int_0^s ||\mathbf{g}(t)||^2_{\mathbf{H}^1(\Omega)} dt.$$

Applying the integral form of Gronwall's inequality, it follows that $\mathbf{g} \equiv \mathbf{0}$, thus $\boldsymbol{\zeta} \equiv \mathbf{0}$ and so $\mathbf{v} \equiv \mathbf{0}$ on $\Omega \times [0, T_1]$ and $\mathbf{u} \equiv \mathbf{0}$ on $S \times [0, T_1]$.

We apply the same argument on the intervals $[T_1, 2T_1]$, $[2T_1, 3T_1]$, etc. to eventually obtain that $(\mathbf{u}, \mathbf{v}) \equiv \mathbf{0}$. □

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ON THE FRACTIONAL STOCHASTIC FILTERING

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Abstract. The aim of this note is to introduce an approximate approach to fractional filtering problems, where either the signal process or observation process, or both are perturbed by a fractional noise. Approximate filtering equations are established and the true filtering is considered as the limit case of approximate filterings.

1. Introduction

It is known that fractional Brownian motion (fBm) was introduced firstly by B. Mandelbrot and Van Nees. This is a centered Gaussian process $B^H = \{B_t^H,\ t \geq 0\}$ with covariance

$$E(B_s^H B_t^H) = \frac{1}{2}\big(s^{2H} + t^{2H} - |t-s|^{2H}\big) \tag{1.1}$$

where H is called the Hurst parameter, $0 < H < 1$

In the case where $H = \frac{1}{2}$, $E(B_s^{1/2} B_t^{1/2}) = \frac{1}{2}(s+t-|t-s|)$ we have an ordinary standard Brownian motion. The fractional Brownian motion is in general neither a martingale nor a Markov process. In contrary, it exhibits a long-range dependence. Some approaches to fractional stochastic calculus have been introduced by L. Coutin, L. Decreusefond, W. Dai, C. Heyde, Lin, A.S. Üstünel, D. Feyel, de La Pradelle, T. Duncan, B. Duncan (refer for example to [1, 2, 3])

A fractional Brownian motion has been considered also by C. Ciesielski and al. as a special sequence of random functions in some Orlicz-Besov space [4].

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Stochastic filtering problems in fractional stochastic framework were studied by some authors. The main obstacle in the study of these problems is the fact that the signal process or the observation process is not driven by a martingale and powerful tools of martingale theory can not be applied as in traditional stochastic filtering theory. Some attempts have been made by L. Decreusefond, A.A. Üstünel to overcome this difficulty by using the Malliavin Calculus [3].

In a fractional filtering problem, the state (or signal) process is some stochastic process X_t, while the observation Y_t is given by a fractional process of the form:

$$Y_t \ = \ \int_0^t h_s ds + B_t^H \tag{1.2}$$

where B_t^H is a fractional Brownian motion with a Hurst parameter H such that $0 < H < \frac{1}{2}$ and $h_t = h(X_t)$ is some process of finite energy, i.e.

$$E \int_0^\infty h_s^2 ds < \infty.$$

As shown in [6], the fBm $B^H = (B_t^H,\ t \geq 0)$ has the following representation

$$B_t^H = \frac{1}{\Gamma(1-\alpha)} \left\{ Z_t + \int_0^t (t-s)^\alpha dW_s \right\}, \tag{1.3}$$

where $\{W_s,\ s \in \mathbb{R}\}$ is a standard Brownian motion, $\alpha = H - \frac{1}{2} \in \left(-\frac{1}{2}, \frac{1}{2} \right)$ and $Z_t = \int_{-\infty}^0 \left[(t-s)^\alpha - (-s)^\alpha \right] dW_s$. Since the process Z_t has absolutely continuous trajectories, it suffices to consider the term

$$B_t = \int_0^t (t-s)^\alpha dW_s. \tag{1.4}$$

In fact, B_t is a fractional Brownian motion of the Liouville form.

In our filtering problems, we consider the observation process Y_t of the form

$$Y_t = \int_0^t h_s ds + B_t \,, \tag{1.5}$$

with B_t defined by (1.4), where h_t is some continuous process, $h_t = h(X_t)$.

Since $B_t = \int_0^t (t-s)^\alpha dW_s$ can be approximated by a semimartingale B_t^ε as shown below, our filtering problem can be considered as the limit case of following

filtering problems when $\varepsilon \to 0$:

$$\begin{aligned} &\text{Signal process:} && X_t\ , \\ &\text{Observation process:} && Y_t^\varepsilon = \int_0^t h_s ds + B_t^\varepsilon, \end{aligned} \tag{1.6}$$

where B_t^ε is some semimartingale for every $\varepsilon > 0$.

2. L^2-approximation for B_t

Let B_t^H be the fractional noise in the observation process Y_t in (1.5) and W_t the corresponding Brownian motion in its representation (1.3). Suppose that $0 < \alpha < \frac{1}{2}$, where $\alpha = H - \frac{1}{2}$.

Define

$$B_t = \int_0^t (t-s)^\alpha dW_s\,, \tag{2.1}$$

and

$$B_t^\varepsilon = \int_0^t (t-s+\varepsilon)^\alpha dW_s\,, \tag{2.2}$$

for every $\varepsilon > 0$.

The Ito stochastic differential of B_t^ε is then:

$$dB_t^\varepsilon = \Big(\int_0^t \alpha(t-s+\varepsilon)^{\alpha-1} dW_s \Big) dt + \varepsilon^\alpha dW_t\,. \tag{2.3}$$

Indeed by applying of the stochastic theorem of Fubini, we have

$$\begin{aligned} \int_0^t \int_0^s (s-u+\varepsilon)^{\alpha-1} dW_s ds &= \int_0^t [\int_u^s (s-u+\varepsilon)^{\alpha-1} ds] dW_u \\ &= \frac{1}{\alpha}[\int_0^t (t-u+\varepsilon)^\alpha dW_u - \varepsilon^\alpha W_t] \\ &= \frac{1}{\alpha}[B_t^\varepsilon - \varepsilon^\alpha W_t]. \end{aligned}$$

Therefore

$$B_t^\varepsilon = \alpha \int_0^t [\int_0^s (s-u+\varepsilon)^{\alpha-1} dW_s] dt + \varepsilon^\alpha W_t\,,$$

or

$$B_t^\varepsilon = \int_0^t \alpha \varphi_s^\varepsilon ds + \varepsilon^\alpha W_t\,, \tag{2.4}$$

or equivalently,

$$dB_t^\varepsilon = \alpha \varphi_t^\varepsilon ds + \varepsilon^\alpha dW_t\,, \tag{2.5}$$

where

$$\varphi_t^\varepsilon = \int_0^t (s-u+\varepsilon)^{\alpha-1} dW_s\,,$$

so B_t^ε is a semimartingale.

We would like to recall here a fundamental result [8] on the L^2- convergence of semimartingales B_t^ε to the fractional process B_t as $\varepsilon \to 0$. Basing on this result we introduce an approximate approach to fractional filtering problems.

Theorem 2.1. *B_t^ε converges to B_t in $L^2(\Omega)$ when ε tends to 0. This convergence is uniform with respect to $t \in [0,T]$.*

3. **Fractional filtering for a general signal process**

We consider first a filtering problem where the signal process is a general stochastic process $(X_t, t \geq 0)$ with $E|X_t| < \infty$ for every $t > 0$ and the observation process Y_t is given by

$$Y_t = \int_0^t h_s ds + B_t\ ,\ 0 \leq t \leq T, \tag{3.1}$$

where B_t is the fractional process given by (1.4) and $h_t = h(X_t)$ is a continuous process with

$$E\int_0^t h_s^2 ds < \infty\,. \tag{3.2}$$

Now for every $\varepsilon > 0$ we establish a new filtering problem (an 'approximate' one), where the signal process is $(X_t, 0 \leq t \leq T)$, $E|X_t|^2 < \infty$ and the observation process is

$$Y_t^\varepsilon = \int_0^t h_s ds + B_t^\varepsilon\ ,\ 0 \leq t \leq T \tag{3.3}$$

where B_t^ε is given by (2.2), and T is some positive real number.

From now on, we take $\varepsilon = \frac{1}{n}$ and put

$\mathcal{F}_t = \mathcal{F}_t^Y$: σ-algebra generated by $(Y_s, s \leq t)$

$\mathcal{F}_t^{(n)} = \mathcal{F}_t^{Y^{1/n}}$: σ-algebra generated by $(Y_s^{1/n}, s \leq t)$.

Define the filter π_t of (X_t) based on observations (Y_t) as the following conditional expectation

$\pi_t(X) = E(X|\mathcal{F}_t)$, or more general

$\pi_t(f) = E(f(X)|\mathcal{F}_t), f$ is any continuous and bounded function on $\mathbb{R} : f \in C_b(\mathbb{R})$.

Denote also by $\pi_t^{(n)}$ the filter of X based on observation $Y_t^{1/n}$:

$$\pi_t^{(n)}(X) = E(X|\mathcal{F}_t^{Y^{1/n}})$$

and

$$\pi_t^{(n)}(f) = E(f(X)|\mathcal{F}_t^{Y^{1/n}}), f \in C_b(\mathbb{R}).$$

Theorem 3.1. *The filter $\pi_t^{(n)}$ converges to π_t in $L^2(\Omega, \mathcal{F}, P)$ as $n \to \infty$.*

Proof. Consider two observations

$$Y_t^{1/n} = \int_0^t h_s ds + B_t^{1/n} \tag{3.4}$$

and

$$Z_t^{1/n} = \int_0^t h_s ds + B_{t+\frac{1}{n}}, \tag{3.5}$$

where $B_t^{1/n} = \int_0^t (t + \frac{1}{n} - s)^\alpha dW_s$ and $B_{t+\frac{1}{n}} = \int_0^{t+\frac{1}{n}} (t + \frac{1}{n} - s)^\alpha dW_s$.

We observe that

$$E|Y_t^{1/n} - Z_t^{1/n}|^2 = E|B_t^{1/n} - B_{t+\frac{1}{n}}|^2 = E|\int_t^{t+\frac{1}{n}} (t + \frac{1}{n} - s)^\alpha dW_s|^2$$

$$= \int_t^{t+\frac{1}{n}} (t + \frac{1}{n} - s)^{2\alpha} ds = \frac{1}{2\alpha+1} \frac{1}{n^{2\alpha+1}} \to 0 \ (n \to \infty) \tag{3.6}$$

where the last equality of (3.6) holds by virtue of the Itô isometry.

Now it follows from the convergence

$$\|Y_t^{1/n} - Z_t^{1/n}\|_{L^2} \to 0 \text{ as } n \to \infty$$

that

$$\|E(X_t|Y_t^{1/n}) - E(X_t|Z_t^{1/n})\|_{L^2} \to 0 \text{ as } n \to \infty \text{ (refer to [5])}$$

or more general

$$\|E(X_t|\mathcal{F}_t^{Y^{1/n}}) - E(X_t|\mathcal{F}_t^{Z^{1/n}})\|_{L^2} \to 0 \text{ as } n \to \infty\,,$$

and for $f \in C_b(\mathbb{R})$:

$$\|E\big(f(X_t)|\mathcal{F}_t^{Y^{1/n}}\big) - E\big(f(X_t)|\mathcal{F}_t^{Z^{1/n}}\big)\|_{L^2} \to 0 \text{ as } n \to \infty\,. \tag{3.7}$$

Since the family of σ-algebras $\mathcal{F}_t^{Z^{1/n}}$ is non-increasing such that $\bigcap\limits_n \mathcal{F}_t^{Z^{1/n}} = \mathcal{F}_t^Y$ then it follows from a Levy theorem on the convergence of conditional expectations that (refer to [5] or to [7]):

$$E\big(f(X_t)|\mathcal{F}_t^{Z^{1/n}}\big) \xrightarrow{L^2} E\big(f(X_t)|\mathcal{F}_t^Y\big) \to 0 \text{ as } n \to \infty\,. \tag{3.8}$$

By combining (3.7) and (3.8) and using the Minkowski inequality we have

$$E\big(f(X_t)|\mathcal{F}_t^{Y^{1/n}}\big) \xrightarrow{L^2} E\big(f(X_t)|\mathcal{F}_t^Y\big) \to 0 \text{ as } n \to \infty\,, \tag{3.9}$$

or $\pi_t^{(n)} \to \pi_t$ in L^2 as $n \to \infty$ by notation of filters. □

4. **Fractional filtering for a semimartingale**

In this section we consider a filtering problem where the signal process is a semimartingale

$$X_t = X_0 + \int_0^t H_s ds + V_t\,, \tag{4.1}$$

where V_t is a Brownian motion and H_t is a stochastic process such that

$$E\int_0^t H_s^2 ds < \infty\,, \tag{4.2}$$

and the observation is the fractional process

$$Y_t = \int_0^t h(X_s) ds + B_t\,, \tag{4.3}$$

where B_t is a fractional Brownian motion defined as in (1.4) such that the corresponding Brownian motion W_t in this expression is independent of V_t, and that

$$E\int_0^t h^2(X_s) ds < \infty\,. \tag{4.4}$$

As in Section II we can consider the 'approximate' filtering problem:

Signal process:

$$X_t = X_0 + \int_0^t H_s ds + V_t \,. \tag{4.5}$$

Observation process:

$$Y_t^{1/n} = \int_0^t h_s ds + B_t^{1/n} \quad , \tag{4.6}$$

where $B_t^{1/n}$ is given by (2.2), and $h_t = h(X_t)$.

Replacing B_t^{ε} in (4.6) for $\varepsilon = \frac{1}{n}$ by its expression in (2.4) we have:

$$Y_t^{1/n} = \int_0^t h_s ds + \alpha \int_0^t \varphi_s^{1/n} ds + \frac{1}{n^\alpha} W_t,\ 0 \le t \le T \ , \tag{4.7}$$

where $\varphi_t^{1/n} = \int_0^t (t - s + \frac{1}{n})^{\alpha-1} dW_s \,.$

Put $\bar{h}_s = h_s + \alpha \varphi_s^{1/n}$, then (4.7) becomes:

$$Y_t^{1/n} = \int_0^t \bar{h}_s ds + \frac{1}{n^\alpha} W_t \ , 0 \le t \le T \ , \tag{4.8}$$

So $Y_t^{1/n}$ is a $\mathcal{F}_t^W$- semimartingale. Notice that

$$\bar{h}_s^2 \le 2(h_s^2 + \alpha^2 (\varphi_s^{1/n})^2) \,,$$

$$E\bar{h}_s^2 \le 2Eh_s^2 + \alpha^2 E(\varphi_s^{1/n})^2 \,,$$

But by the Ito isometry, we see that

$$E(\varphi_s^{1/n})^2 = E[(\int_0^t (t - s + \frac{1}{n})^{\alpha-1} dW_s)^2]$$

$$= \int_0^t (t - s + \frac{1}{n})^{2\alpha-2} ds \le \int_0^T (t - s + \frac{1}{n})^{2\alpha-2} ds < \infty.$$

It follows from Fubini's theorem that

$$E \int_0^t \bar{h}_s^2 ds < \infty \tag{4.9}$$

Now define the innovation process:

$$\nu_t^{1/n} = Y_t^{1/n} - \int_0^t \pi_s^{(n)}(\bar{h}) ds \tag{4.10}$$

then $\nu_t^{1/n}$ is a $\mathcal{F}_t^{Y^{1/n}}$- martingale.

Now we are in the position to write down the FKK (Fujisaki - Kallianpur - Kunita) equation for the filtering problem (4.1)- (4.3):

$$\pi_t^{(n)}(f) = \pi_0^{(n)}(f) + \int_0^t \pi_s^{(n)}(f(H))ds + \int_0^t [\pi_s^{(n)}(f(X)\bar{h}) - \pi_s^{(n)}(f(X))\pi_s^{(n)}(\bar{h})]d\nu_s^{1/n} \quad , \tag{4.11}$$

where $f \in C_b(\mathbb{R})$ and $\pi_0^{(n)}(f) = E(f(X_0)|\mathcal{F}_0^{Y^{1/n}})$.

Notice that from (4.5) we have

$$\begin{aligned} E|X_t| &\leq E|X_0| + E|\int_0^t H_s ds| + E|V_t| \\ &\leq E|X_0| + T.E\int_0^t H_s^2 ds < \infty. \end{aligned}$$

then by the Levy theorem we can see that $L^2\text{-}\lim_{n\to\infty} \pi_t^{(n)}$ exists as in proof of Theorem 3.1. Now we can state:

Theorem 4.1. *The filter* $\pi_t(f(X)) = E(f(X_t)|\mathcal{F}_t^Y) = L^2\text{-}\lim_{n\to\infty} \pi_t^{(n)}(f)$ *exists, where* $\pi_t^{(n)}(f)$ *satisfies the equation (4.11).*

5. **General fractional filtering**

Suppose now that the signal process $(X_t,\ 0 \leq t \leq T)$ and the observation process $(Y_t,\ 0 \leq t \leq T)$ are fractional processes given by

$$X_t = X_0 + \int_0^t H_s ds + B_t^{(1)} \quad , \ E|X_t| < \infty, \tag{5.1}$$

$$Y_t = \int_0^t h_s ds + B_t^{(2)} \quad , \tag{5.2}$$

where

$$B_t^{(1)} = \int_0^t (t-s)^{\beta} dU_s \quad , \tag{5.3}$$

$$B_t^{(2)} = \int_0^t (t-s)^{\alpha} dW_s \quad , \tag{5.4}$$

U_s and W_t are two independent standard Brownian motions, $\beta = H_1 - \frac{1}{2},\ \alpha = H_2 - \frac{1}{2}$, H_1 and H_2 are two Hurst parameters and $0 < \alpha,\ \beta < \frac{1}{2}$.

H_t and h_t are $\mathcal{F}_t$-adapted process, $h_t = h(X_t)$ with continuous function $h(.)$, such that

$$E\int_0^t H_s^2 ds < \infty \quad , \tag{5.5}$$

$$E\int_0^t h_s^2 ds < \infty \quad , \tag{5.6}$$

As before we consider an "approximate model" for the filtering problem (5.1)-(5.6) as follows:

Signal:

$$X_t^{1/n} = X_0 + \int_0^t H_s ds + B_t^{(1)1/n},\ 0 \le t \le T \quad , \tag{5.7}$$

Observation:

$$Y_t^{1/n} = \int_0^t h_s ds + B_t^{(2)1/n},\ 0 \le t \le T \quad , \tag{5.8}$$

where $h_t = h(X_t^{1/n})$ and

$$B_t^{(1)1/n} = \int_0^t (t - s + \frac{1}{n})^{\beta} dU_s$$

and

$$B_t^{(2)1/n} = \int_0^t (t - s + \frac{1}{n})^{\alpha} dW_s \, . \tag{5.9}$$

The filter $\pi_t^{(n)}$ for the problem (5.7)-(5.8) is defined as

$$\pi_t^{(n)}(f) = E[f(X_t^{1/n})|\mathcal{F}_t^{Y_t^{1/n}}], \quad f \in C_b(\mathbb{R}) \tag{5.10}$$

and we will verify if the filter π_t for the original problem (5.1)-(5.6) can be defined as a L^2-limit of $\pi_t^{(n)}$ as $n \to \infty$.

We need the following lemma (refer to [7]).

Lemma 5.1. *Let (X_n) be a sequence of random variables such that for every n, $|X_n| \le Y$, where Y is integrable. If $(\mathcal{F}_n)$ is an increasing (resp. decreasing) sequence of σ-algebras, then $E[X_n|\mathcal{F}_n]$ converges a.s to $E[X|\mathcal{F}]$, where $\mathcal{F} = \sigma(\cup\mathcal{F}_n)$ (resp. $\mathcal{F} = \cap\mathcal{F}_n$)*

Proof. Take $\varepsilon > 0$ and put

$$A = \inf_{k \geq m} X_n, \ B = \sup_{k \geq m} X_n \tag{5.11}$$

where m is chosen such that

$$E[B - A] < \varepsilon \ . \tag{5.12}$$

Then for $n \geq m$ we have

$$E[A|\mathcal{F}_n] \leq E[X_n|\mathcal{F}_n] \leq E[B|\mathcal{F}_n]. \tag{5.13}$$

The left and right- hand sides of (5.13) are martingales converging a.s. to $E(A|\mathcal{F})$ and $E(B|\mathcal{F})$ respectively. We have

$$E(A|\mathcal{F}) \leq \underline{lim}\ E(X_n|\mathcal{F}_n) \leq \overline{lim}\ E(X_n|\mathcal{F}_n) \leq E(B|\mathcal{F}) \ , \tag{5.14}$$

and

$$E(A|\mathcal{F}) \leq E(X|\mathcal{F}) \leq E(B|\mathcal{F}) \ , \tag{5.15}$$

It follows that

$$E[\underline{lim}\ E(X_n|\mathcal{F}_n) - \overline{lim}\ E(X_n|\mathcal{F}_n)] \leq \varepsilon \ ,$$

hence $E(X_n|\mathcal{F}_n)$ converges a.s. and the limit is $E(X|\mathcal{F})$.

Remark. The Lemma 5.1 still holds if we replace the a.s. convergence by the L^2- convergence (refer to [5]).

Theorem 5.1. *Under the conditions given by (5.1)-(5.8) the filter* $\pi_t(f) = E[f(X_t)|\mathcal{F}_t^Y]$ *is determined by*

$$\pi_t(f) = L^2\text{-}\lim\ \pi_t^{(n)}(f), \ f \in C_b(\mathbb{R})$$

where $\pi_t^{(n)}$ *satisfies the following filtering equation*

$$\pi_t^{(n)}(f) = \pi_0^{(n)}(f) + \int_0^t \pi_s^{(n)}(f(\bar{H}))ds + \int_0^t [\pi_s^{(n)}(f(X)\bar{h}) - \pi_s^{(n)}(f(X))\pi_s^{(n)}(\bar{h})]d\nu_s^{1/n} \ , \tag{5.16}$$

and

$$\bar{H}_t = H_t + \beta\psi_t^{1/n} \ , \textit{where } \psi_t^{1/n} = \int_0^t (t - s + \frac{1}{n})^{\beta} dU_t \, , \tag{5.17}$$

$$\bar{h}_t = h_t + \alpha\varphi_t^{1/n} \ \ , \textit{where } \ \varphi_t^{1/n} = \int_0^t (t - s + \frac{1}{n})^{\alpha} dW_t \, , \tag{5.18}$$

$$\nu_t^{1/n} = Y_t^{1/n} - \int_0^t \pi_s^{(n)}(\bar{h})ds, \tag{5.19}$$

H_t and h_t satisfy conditions (5.5) and (5.6) and $X_t^{1/n}$ and $Y_t^{1/n}$ are defined by (5.7), (5.8) and (5.9) for $0 \le t \le T$.

Proof. It follows from the definition (5.7) for the process $X_t^{1/n}$ and from Theorem 2.1 that $X_t^{1/n} \to X_t$ in $L^2(\Omega, \mathcal{F}, P)$ as $n \to \infty$.

As for $Y_t^{1/n}$ defined by (5.8) we can see that

$$Y_t^{1/n} - Y_t = \int_0^t [h(X_s^{1/n}) - h(X_s)]ds + B_t^{(2)1/n} - B_t^{(2)}\,, \tag{5.20}$$

where $h : R \to R$ is a continuous function by assumption, then the L^2-convergence of $B_t^{(2)1/n}$ and $X_t^{1/n}$ respectively to $B_t^{(2)}$ and X_t implies that of $Y_t^{1/n}$ to Y_t.

Now by a calculation as in the proof of Theorem 4.1 we have

$$X_t^{1/n} = X_0 \int_0^t \bar{H}_s ds + \frac{1}{n^\beta} U_t\,, \tag{5.21}$$

$$Y_t^{1/n} = \int_0^t \bar{h}_s ds + \frac{1}{n^\alpha} W_t\,, \tag{5.22}$$

where

$$\bar{H}_s = H_s + \beta\psi_t^{1/n}, \quad \psi_t^{1/n} = \int_0^t (t - s + \frac{1}{n})^\beta dU_s\,,$$

$$\bar{h}_s = h_s + \alpha\varphi_t^{1/n}, \quad \varphi_t^{1/n} = \int_0^t (t - s + \frac{1}{n})^\alpha dW_s\,.$$

By the Ito isometry we can see that:

$$\int_0^t E\bar{H}_s^2 ds < \infty \text{ and } \int_0^t E\bar{h}_s^2 ds < \infty\,. \tag{5.23}$$

Then we can write the FKK filtering equation for the approximate model (5.21)-(5.22)-(5.23) as in (5.16), where $\nu_t^{1/n}$ is the innovation process

$$\nu_t^{1/n} = Y_t^{1/n} - \int_0^t \pi_s^{(n)}(\bar{h})ds.$$

Here $\pi_t^{(n)}(f) = E(f(X_t^{1/n})|\mathcal{F}_t^{Y_t^{1/n}})$.

Because $X_t^{1/n} \to X_t$ and $Y_t^{1/n} \to Y_t$ in L^2 and also

$$\|E\big(f(X_t^{1/n})|\mathcal{F}_t^{B^{(2)1/n}}\big) - E\big(f(X_t^{1/n})|\mathcal{F}_t^{Z^{1/n}}\big)\|_{L^2} \to 0 \text{ as } n \to \infty\,, \tag{5.24}$$

where $Z^{1/n} = \int_0^t h_s ds + B^{(2)}_{t+\frac{1}{n}}$, so an analogous assertion to the proof of Theorem 3.1 says that $\pi_t(f) = L^2\text{-}\lim_{n\to\infty} \pi_t^{(n)}(f)$ exists where $\pi_t^{(n)}$ satisfies the FKK filtering equation (5.16). □

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THE WEIGHTED SPLINE QUASI-INTERPOLANT OPERATORS

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Abstract. A new quasi-interpolant operator starting from the operator described by Sablonnière [1], [7] is presented here. The operator is a linear combination of some linear functionals and normalized B-spline functions. If Sablonièr uses the arithmetic mean of the consecutive given points, the linear functionals presented here use the mesh points chosen as the weighted arithmetic mean of given points from the interval $[a, b]$. The article describes the way of computing the quadratic and cubic weighted spline quasi-interpolant operators and underlines the good numerical approximation of these new operators using implemented Matlab functions. The fact that the cubic weighted spline quasi-intepolant operators are a completion of the cubic spline quasi-interpolant operators offering a better approximation, but only among some intervals, is proven in the last section of the paper.

1. Introduction

The general construction of quasi-interpolants, which were first developed by Carl de Boor and G. J. Fix [2] and generalized later by Lyche and Schumaker [4], starts from the following problem. Given a function f, the basic problem of spline approximation is to determine B-spline coefficients $(c_i)_{i=1}^n$ such that $Pf = \sum_{i=1}^n c_i N_{i,k}$ is a reasonable approximation to f. Let assume that f is defined on an interval $I = [a, b]$, and select the space of splines of order $k + 1$, $S_{k+1}(\Delta, I)$, $\Delta : a = x_1 < x_2 < ... < x_n = b$ defined on I (i.e., so that $\overline{\Delta} : x_{-k+1} = x_{-k+2} = ... = x_{-1} = x_0 = a, b = x_{n+1} = x_{n+2} = ... = x_{n+k}$). To emphasize the dependence on f,

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the coefficient c_i is written $c_i = \mu_i f$, with μ_i some linear functionals. Thus, a quasi-interpolant spline is an approximation operator obtained as a linear combination of functions with finite support (B-splines $N_{i,k}$) $Qf = \sum_{i=1}^{n+1} \mu_i(f) N_i$. There are known some different types of these linear functionals μ_i such as: differential type ($\mu_i(f)$ is a linear combination of values of derivatives of f) or discrete type (combination of discrete values of f). This paper treats the case of the coefficients of discrete type where the combination is formed by the weighted mean values of the given points from Δ weighted by the values of the function f.

2. **The most important features**

This section concerns upon the construction of the weighted quasi-interpolant operators. Two methods of obtaining the new operators are presented here: the first one involves the non-recurrent expressions of the normalized quadratic and cubic B-splines and the second one presents the exact formulations of the coefficients of the weighted spline quasi-interpolant.

2.1. **The construction of the weighted spline quasi-interpolant operators using the normalized B-spline expressions.**

Let $n, k > 0$ with $n \geq k + 3$ be known integers and let $f \in C^k(I)$ be a function with the known values $f(x_i), x_i \in \Delta$ such as $f(x_i) + f(x_{i-1}) \neq 0$. We choose $n + 1$ weighted values points

$$t_s := \frac{x_{s-1} f(x_{s-1}) + x_s f(x_s)}{f(x_{s-1}) + f(x_s)}, s = 1, ..., n + 1. \tag{2.1}$$

It is obvious that $t_1 = a$ and $t_{n+1} = b$.

Definition 1. *Let $x \in [x_l, x_{l+1}] \subset [a, b]$, for $l = 1, ..., n - 1$, $a, b \in R$. We define the spline quasi-interpolant of degree k and order $k + 1$*

$$Q_k f(x) = \sum_{i=1}^{k+1} \mu_i^{\{l\}}(f) \cdot N_{-k-1+i+l}(x), \tag{2.2}$$

$N_i := N_{i,k+1}$, with the functionals $\mu_i^{\{l\}}$,

$$\mu_i^{\{l\}}(f) = \sum_{j=1}^{k+1} a_{i,j}^{\{l\}} \cdot f(t_{i+j-1}), i = 1, ..., k + 1 \tag{2.3}$$

where the mid points t_s are defined in (2.1) and $a_{i,j}^{\{l\}}$ are coefficients which depend on l, $\forall l = 1, ..., , n-1$.

We denote the quasi-interpolant operator with weighted values by QI_w.

Problem 1. *Construct an algorithm for weighted spline quasi-interpolant operator and implemet a corespondent routine using the Matlab application.*

`Step I.` Find the coefficients $c_{i,l-r}$, $i, r = 0, ..., k$ of the normalized B-spline functions $N_{l-r,k+1}, r = 0, ..., k$ from the expression $N_{l-r,k+1}(x) = c_{k,l-r}x^k + c_{k-1,l-r}$ x^{k-1} $+... + c_{0,l-r}$. These coefficients are obtained from the well known recurrence formula (see, for example, [6])

$$N_{j,1} = \begin{cases} 1, & x \in [x_j, x_{j+1}); \\ 0, & else \end{cases}$$

$$N_{j,k+1} = \frac{(x - x_j)N_{j,k}(x)}{x_{j+k} - x_j} + \frac{(x_{j+k+1} - x)N_{j+1,k}(x)}{x_{j+1+k} - x_{j+1}}, j = -k+1, ..., n-1. \quad (2.4)$$

It is known from [8] that for any $x \in [x_l, x_{l+1}), l \in \{1, ..., n-1\}$, there are only $k+1$ nonzero B-splines $N_{j,k+1}$, $j = l-k, ..., l$. Thus, an explicit non-recurrent expression for the spline coefficients can be deduced from the relations (2.4) in the cases of quadratic ($k = 2$) [11] and cubic ($k = 3$) [10] B-splines.

$$N_{l,k+1} = \frac{\sum_{i=0}^{k}(-1)^i \binom{k}{i} x_l^i}{\prod_{j=2}^{k}(x_{l+j} - x_l)} M_{l,1} \cdot x^{k-i}, k \geq 2,$$

$$N_{l-1,k+1} = \sum_{p=1}^{k} \frac{\sum_{s=0}^{k}(-1)^{s+1}(x_{l+p}P_{s,p} + P_{s+1,p})M_{l,1} \cdot x^{k-s}}{\prod_{j=1,p+j\leq k}^{k-1}(x_{l+k-j} - x_{l-1}) \prod_{j=1,p+j>k}^{k-1}(x_{l+k-j+1} - x_l)}, k \geq 2,$$

$$N_{l-2,k+1} = \sum_{p=1}^{k} \frac{\sum_{s=0}^{k}(-1)^{s}(x_{l+p-k}R_{s,p} + R_{s+1,p})M_{l,1} \cdot x^{k-s}}{\prod_{j=1,p+j\leq k}^{k-1}(x_{l+1} - x_{l-k+j+p}) \prod_{j=1,p+j>k}^{k-1}(x_{l+2} - x_{l+1-j})}, k = 3,$$

and finally

$$N_{l-k,k+1} = \frac{\sum_{i=0}^{k}(-1)^{i+k} \binom{k}{i} x_{l+1}^i}{\prod_{j=2}^{k}(x_{l+1} - x_{l-j+1})} M_{l,1} \cdot x^{k-i}, k \geq 2,$$

where $P_{s,p} = \sum_{j=0}^{s} \binom{p-1}{s-j-1} \binom{k-p}{j} x_l^{s-j-1} x_{l-1}^j$ and $R_{s,p} = \sum_{j=0}^{s} \binom{p-1}{s-j-1} \binom{k-p}{j} \cdot$ $x_{l+2}^{s-j-1} x_{l+1}^j$. We take $P_{0,p} = P_{k+1,p} := 0, \forall p = 1, ..., k$ and $\binom{k}{i} := 0$ if $i < 0$ or $i > k$ and we observe that $P_{1,p} = R_{1,p} = 1$, $\forall p = 1, ..., k$.

`Step II.` Find the coefficients $a_{i,j}^{\{l\}}$ of the linear functionals (2.3) as the solution of the system obtained by applying the conditions of exactness of the quasi-interpolant operator in the space of polynomial of degree at most k. So, the conditions of the exactness of Q_k operator in the set of polynomials of degree at most k, $Q_k p = p, p \in P_k$, leads to the identities $Q_k(e_i) = e_i$ where $e_i(x) = x^i, i = 0, ..., k$. Rearranging after the powers of x and equalizing both sides we obtain a system with $(k+1)(k+1)$ equations and $(k+1)^2$ unknowns, $A \cdot C = B$, where A is the matrix of the unknown coefficients $a_{i,j}^{\{l\}}, i = 1, ..., k+1$, $j = 1, ..., k+1$, C is the matrix of the coefficients of the normalized B-spline functions and the values ξ_j, and B is the line matrix of the unity vector $u_i, i = 1, ..., k+1$ as presented below. The matrix C is of the form

$$C = \left[\begin{array}{l|l|l|l} c_{l-k} & c_{l-k}X_1 & ... & c_{l-k}X_1^k \\ c_{l-k+1} & c_{l-k+1}X_2 & ... & c_{l-k+1}X_2^k \\ ... & & & \\ c_l & c_l X_{k+1} & ... & c_l X_{k+1}^k \end{array}\right], \tag{2.5}$$

where c_{l-k} is the square bloc of the coefficients of the B-spline $N_{l-k,k+1}$

$$c_{l-k} = \begin{array}{llll} c_{k,l-k} & c_{k-1,l-k} & ... & c_{0,l-k} \\ ... & & & \\ c_{k,l-k} & c_{k-1,l-k} & ... & c_{0,l-k} \end{array}.$$

The X_i, $i = 1, ..., k+1$ vectors are defined as $X_i^j = [t_i^j, t_{i+1}^j, ..., t_{i+k}^j]^t$, $i = 1, ..., k+1$ for $j = 0, ..., k$. Maintaining the above notations, vector B is defined as $B = [u_{k+1} \,|\, u_k \,|\, ... \,|\, u_1]$ where u_i is a vector with 1 on the position of i, $i = 1, ..., k+1$, and the rest of k elements are zero. We can observe that for every value of $x \in [x_l, x_{l+1}], l = 1, ..., n-1$, it is necessary to solve $n-1$ systems with $(k+1)^2$ equations. In order to obtain the compatibility of the systems and the uniqueness of the solution, we impose the knots condition

$$n \geq (k+1) + 2.$$

`Step III.` Compute the values of the linear functionals $\mu_i^{\{l\}}, \forall x \in [x_l, x_{l+1})$.

`Step IV.` Compute the values of the weighted spline quasi-interpolant operator $Q_k(f)$, $\forall x \in [x_l, x_{l+1})$.

In order to solve this system, we have implemented an algorithm and by using the Matlab application we will obtain these practical results.

Problem 2. *Implement a routine which calculates*

1. *the joint values* t_j, $j = 1, ..., n+1$;

2. *the coefficients of the normalized B-spline functions* N_j, $j = -k+l, ..., l$, $l = 1, ..., n-1$;

3. *the values of the functionals* $\mu_i^{\{l\}}(f)$, $i = 1, ..., k+1$ *for some given functions* f

4. $Q_k(f)(x)$, $k \in \{2, 3\}$, *the value of the quasi-interpolant operator* Q_k *for a given number* $x \in (a, b)$

5. *the values of the quasi-interpolant operator* Q_k *for all the equidistant numbers* $x \in [a, b]$ *with step* 0.1.

The algorithm calculates the coefficients and the values of the quadratic and cubic weighted spline quasi-interpolant Q_k for any partition X of the interval $I = [a, b]$ and any $x \in (a, b)$.

`Input:`

a) X, the vector of the extended partition $\overline{\Delta}$;

b) $x \in (a, b)$;

c) f, a function which may be chosen from the set of functions

$\{ax^2 + bx + c, a/(b + cx), a \cdot (sin(bx))^c, a \cdot (cos(bx))^c, (ae^{b \cdot x})^c, (ax^2 + 1)/(bx + c)\}$.

`Output:`

i) CC, the vector of the coefficients of the quasi-interpolant Q_k;

ii) $Q_k(f)(x)$, the value of the quasi-interpolant operator Q_k for a given number $x \in (a, b)$;

iii) q, the vector values of the quasi-interpolant operator Q_k for all the equidistant numbers $x \in [a, b]$ with the step 0.1.

`Step 1:` The computation of the matrix denoted with n of order $k+1$ of the B-spline $N_j, j = l-k, ..., l$ coefficients $c_{k-s,l-r}$, $s = 0, ..., k, r = 0, ..., k, l = 1, ..., n-1, k \in \{2, 3\}$

using the non-recurrent expressions of the coefficients from the equalities mentioned above.

`Step 2:` The elements of the vector T of the $t_j, j = 1, ..., n+1$ from (2.1) ;

`Step 3:` The construction of the matrix C of the form (2.5);

`Step 4:` The construction of the vector B of the form $B = [u_{k+1} \mid u_k \mid ... \mid u_1]$;

`Step 5:` The computation of the solution of the matrix equation $A \cdot C = B$, A as $A = B \cdot C^{-1}$;

`Step 6:` The computation of the values of the functionals $\mu_i^{\{l\}}, i = l, ..., k+1$. (It is not necessary to compute all the values $\mu_i^{\{l\}}$, $i = 1, ...n-1, n \geq k$, because only $N_{l-k}, ..., N_l$ are nonzero).

In what follows, we will exemplify our results on the non polynomial case.

Example 1. *(Numerical results)*
Let $I = [0, 1.25]$, $\Delta : 0 < 0.25 < 0.5 < 0.75 < 1 < 1.25$ *and* $0.1 := x \in [x_1, x_2] := [0, 0.25]$. *For a given function, say* $f(x) = (x^2+1)/(x+1)$, *find the value of the cubic quasi-interpolant operator* Q.

Solution: Applying the Matlab function

`[Q]=coef_w_cubic(X,x,a,b,c,functia)`,

the vector of the extended partition of the cubic (order=4, degree=3) B-spline function is X = [0 0 0 0 0.25 0.5 0.75 1 1.25 1.25 1.25 1.25] and the coefficients of the function are $a = 1, b = 1, c = 1$. The Matlab function

`coef_w_cubic(X,0.1,1,1,1,'(a*x.^2+1)/(b*x+c)')`

generates the following results:

1. the joint values t_j, $j = 1, ..., 7$ are

$T =$ 0 0.1149 0.3738 0.6293 0.8821 1.1331 1.2500;

2. the coefficients of the normalized B-spline functions N_j, $j = -2, ..., 1$ are contained in the following matrix

$n =$

N_1	10.6667	0	0	0
N_0	-58.6667	24.0000	0	0
N_{-1}	112.0000	-72.0000	12.0000	0
N_{-2}	-64.0000	48.0000	-12.0000	1.0000

,

from where we can construct the expressions of the B-spline functions:

$N_{1,4} = 10.(6)x^3$

$N_{0,4} = -58.(6)x^3 + 24x^2$

$N_{-1,4} = 112x^3 - 72x^2 + 12x$

$N_{-2,4} = -64x^3 + 48x^2 - 12x + 1$;

3. the values of the functionals $\mu_i^{\{1\}}(f)$, $i = 1, ..., 4$ are the elements of the vector

$CC = 1.0000 \quad 0.9193 \quad 0.8302 \quad 0.8216$;

4. the value of the quasi-interpolant operator Q_3 is

$Q = 0.9195$

Remark 1. *The value of the function f for $x = 0.1$ is* 0.9182 *which means that the weighted spline quasi-interpolant operator offers a good approximation.*

Furthermore, for computing the values of the quasi-interpolant operator Q_3 for all the equidistant numbers $x \in [a, b]$ with the step 0.1, we implemented another Matlab function

$$[q] = table\_val\_Q\_cubic(T, X, a, b, c, functia).$$

Thus, for $T = [0 \quad 1.25]$,

$q = 1.0000 \;\; 0.9195 \;\; 0.8679 \;\; 0.8400 \;\; 0.8400 \;\; 0.8313 \;\; 0.8484 \;\; 0.8753 \quad 0.9128 \;\; 0.9539$
$1.0000 \;\; 1.0525.$

Remark 2. *We have also implemented a function*

$$[tabel] = final(X, T, a, b, c, functia)$$

to compare the values of the given function with the values of the weighted cubic spline quasi interpolant operator and the values of the cubic spline quasi-interpolant operator described by Sablonière. Thus, applying

$$final(X, [0 \;\; 1.25], 1, 1, 1, '(a * x.^2 + 1)/(b * x + c)')$$

we get

tabel =

x	0	0.1	0.2	0.3	0.4	0.5	0.6
f	1.000	0.9182	0.8667	0.8385	0.8286	0.8333	0.8500
$Q_3(f)$	1.0000	0.9203	0.8675	0.8378	0.8275	0.8325	0.8494
$Q_{w3}(f)$	1.0000	0.9195	0.8679	0.8400	0.8298	0.8313	0.8484
$Q-f$	0.0000	0.0022	0.0008	-0.0006	-0.0011	-0.0008	-0.0006
Q_w-f	0.0000	0.0013	0.0012	0.0016	0.0012	-0.0020	-0.0016

x	0.7	0.8	0.9	1	1.1
f	0.8765	0.9111	0.9526	1.0000	1.0524
$Q_3(f)$	0.8760	0.9108	0.9524	1.0000	1.0526
$Q_{w3}(f)$	0.8753	0.9128	0.9539	1.0000	1.0525
$Q-f$	-0.0004	-0.0003	-0.0002	-0.0000	0.0002
Q_w-f	-0.0012	0.0017	0.0012	0.0000	0.0001

Analyzing the errors expressed in the last two rows of the table, we can notice that the operators described by Sablonière are better approximations than the operators presented in this paper, with the exception of the edged values. Making further investigations, the weighted cubic quasi-interpolant operators are more convenient for approximation of this function in values contained on the interval $[0; 0.17]\cup[1; 1.24]$. *This better approximation can be visualized in the graphical error representation, Fig.1, where the errors generated by quasi-interpolated operators with mean values are larger than the errors generated by quasi-interpolated operators with weighted values.*

2.2. **The construction of the weighted cubic spline quasi-interpolant operators which does not require the normalized B-spline expressions.** As we could see, the construction of the weighted spline quasi-interpolant operators requires the solution of linear systems. To avoid this volume of computation, following the idea presented in [7], we can generate the exact expressions of the coefficients $a_{i,j}^{\{l\}}$, $\forall i, j = 1, ...k+1$ and $\forall l = 1, ..., n-1$.

Let $\overline{\Delta}$ be the extended partition of the interval $[a, b]$. We recall the definition of Greville's points (mentioned in [5] and [9])

$$\xi_j = \frac{x_{j+1} + x_{j+2} + ... + x_{j+k}}{k}, j = -k+1, ..., n-1,$$

$$\xi_j^{(2)} = \frac{x_{j+1}x_{j+2} + x_{j+1}x_{j+3} + ... + x_{j+k-1}x_{j+k}}{\binom{k}{2}}, j = -k+1, ..., n-1,$$

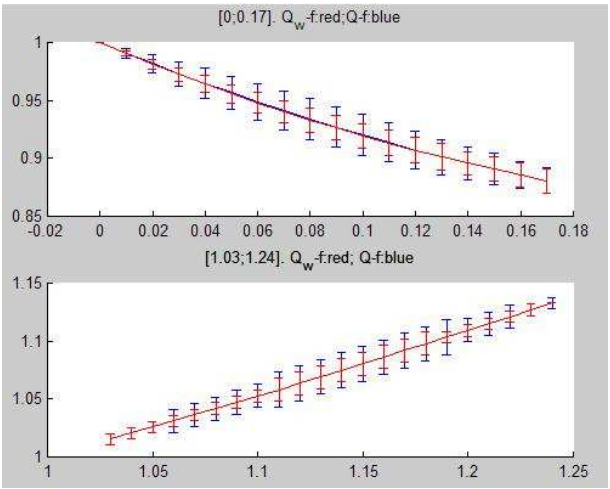

FIGURE 1. The graphical error representation

$$\xi_j^{(3)} = \frac{x_{j+1}x_{j+2}x_{j+3} + x_{j+1}x_{j+2}x_{j+4} + ... + x_{j+k-2}x_{j+k-1}x_{j+k}}{\binom{k}{3}}, j = -k+1, ..., n-1.$$

Theorem 1. *For $k = 3$, the exact expressions of the coefficients $a_{i,j}^{\{l\}}$ of the linear functionals $\mu_i^{\{l\}}$ from (2.3) are given by the formula*

$$a_{i,j}^{\{l\}} = (-1)^{j+1} \frac{P_{i,j} - \xi_{-4+i+l} \cdot (SP)_{i,j} + \xi_{-4+i+l}^{(2)} \cdot S_{i,j} - \xi_{-4+i+l}^{(3)}}{\prod_{i \le s < p \le i+k}^{(s=i+j-1) \vee (p=i+j-1)} (t_p - t_s)} \tag{2.6}$$

where P, SP and S are respectively the product of all the elements t where t_{i+j-1} is omitted, SP denotes the sum of all combinations of the products of two elements t from P and, finally, S is the sum of all elements from P:

$$P_{i,j} := \frac{\prod_{s=i}^{i+k} t_s}{t_{i+j-1}},$$

$$(SP)_{i,j} := \sum_{i \le s < p \le i+k} t_s t_p - \sum_{i \le s < p \le i+k}^{(s=i+j-1) \vee (p=i+j-1)} t_s t_p,$$

$$S_{i,j} := \sum_{s=i}^{i+k} t_s - t_{i+j-1}.$$

Proof. We begin by imposing the conditions of exactness for the quasi-interpolant operators Q_k in the set of polynomials of degree at most k, $Q_k p = p, p \in P_k$ which

lead to the identities $Q_k(e_i) = e_i$ where $e_i(x) = x^i, i = 0, ..., k$. The e_i functions can be rewrites using Marsden's equalities [5]

$$x^j = \sum_{i=1}^{k+1} \xi_{-4+i+l}^{(j)} N_{-4+i+l}(x), j = 0, ..., k,$$

with $\xi_i^{(0)} := 1$ and $\xi_i^{(1)} := \xi_i$. Equalizing the coefficients $a_{i,j}^{\{l\}}$ from the equations $(Q_k e_i)(x) = e_i(x), i = 0, ..., k$ we obtain a system with $(k+1)^2$ equations and $(k+1)^2$ unknows for every $l \in \{1, ..., n-1\}$. The system being a separable variables one, we can rearrange the equations obtaining $(k+1)$ systems with $(k+1)$ equations by the form $\sum_{j=1}^{k+1} a_{i,j}^{\{l\}} t_j^s = \xi_{-4+i+l}^{(s)}$, for fixed $i \in \{1, ..., k+1\}$ and $s = 0, ..., k$ with $\xi_i^{(0)} := 1$ and $\xi_i^{(1)} := \xi_i$. The determinant of these systems is Vandermonde determinant, thus the computation is quite simple and each solution $a_{i,j}$ of the systems can be generalized by the (2.6). □

3. **The evaluation of the error**

In this section a comparison of the norm of the cubic quasi-interpolant operator with weighted values (QI_w) and the norm of the cubic quasi-interpolant operator with mean values (QI) is presented. In general, it is difficult to minimize the true norm of the operators. In order to avoid this direct minimization we use the idea of Sablonniére [1] of the minimization problem:
Let $Qf = \sum_i \mu_i(f) N_i$ be the general form of the spline quasi-interpolant of f, with $\mu_i(f) = \sum_i a_i f(x_i)$. Find $a_i \in R^n$ solution of the problem

$$\|a_i^*\|_1 = min \{\|a_i\|_1, a_i \in R^n, V_i a_i = b_i\}$$

where $\|Q\|_\infty \leq \sum_i |\mu_i(f)| N_i \leq max_i |\mu_i(f)| \leq max_i \|a_i\|_1$. The notation V_i denotes the Vandermonde matrix.
Thus, the minimization of the norm $\|Q\|_\infty$ reduces to the operate with the coefficients a_i.

It is known from [1] that the norm of QI is not uniformly bounded independent of the partition.

Theorem 2. [1] *For the cubic spline quasi-interpolant operators Q_3 let the linear functionals be $\mu_i(f) = a_i f(x_{i-1}) + b_i f(x_i) + c_i f(x_{i+1})$. If there exists $r > 0$ such that the partition satisfies $\frac{1}{r} \leq \frac{h_{i+1}}{h_i} \leq r$, $i \in Z$, where $h_i = x_i - x_{i-1}$, than we obtain the following upper bounds $|a_i|, |c_i| \leq \frac{1}{3}\frac{r^2}{1+r}$, $|b_i| \leq \frac{1}{3}(1+r)^2$, from which $\|Q_3\|_\infty \leq \frac{1}{3}\left((1+r)^2 + \frac{2r^2}{1+r}\right)$.*

Thus, in the case of uniform partition, $r = 1$, the upper bound of the norm is ≈ 1.66.

The next result states that the cubic weighted spline quasi-interpolant operators QI_w for the uniform partition case also have the upper bound less than 1.66.

Theorem 3. *Let $I = [a, b]$ be an interval with the uniform partition Δ and Q_3 the cubic weighted spline quasi-interpolant operator given by*

$$Q_3 f(x) = \sum_{i=1}^{n+3} \mu_i(f) N_i(x)$$

with linear functionals

$$\mu_i(f) := a_i f(t_{i-1}) + b_i f(t_i) + c_i f(t_{i+1}),$$

$$\mu_1(f) = f(a), \mu_{n+3}(f) = f(b)$$

and the points t_i defined as in (2.1). For $f \in C[a, b]$ smooth enough we have

$$\|Q_3\|_\infty \leq 1.66.$$

Proof. We define the auxiliary points

$$t_i := (1 - m)x_i + m x_{i-1}, \tag{3.1}$$

$i = 1, ..., n+3, m \in [0, 1]$ which are a generalization of the points t_i taken as mean values of $x_i \in \Delta$ and taken as weighted values of $x_i \in \Delta$ and $f(x_i)$.

The idea of the proof is to express the coefficients a_i, b_i, c_i depending only on the parameter m.

To compute these coefficients, we follow the same idea presented in Subsection 2.2. Thus, after imposing the conditions of the exactness $Q_3(e_s) = e_s$, $s = 0, ..., 3$ and

after using Marsden's equalities, rearranging after the powers of t_i, $i = 1, ..., n+3$ we get a 3×3 system for every i, from which the coefficients are

$$a_i = \frac{t_i t_{i+1} - \xi_i^{(1)}(t_i + t_{i+1}) + \xi_i^{(2)}}{(t_i - t_{i-1})(t_{i+1} - t_{i-1})}, \tag{3.2}$$

$$b_i = -\frac{t_{i+1} t_{i-1} - \xi_i^{(1)}(t_{i+1} + t_{i-1}) + \xi_i^{(2)}}{(t_i - t_{i-1})(t_{i+1} - t_i)},$$

$$c_i = \frac{t_{i-1} t_i - \xi_i^{(1)}(t_i + t_{i-1}) + \xi_i^{(2)}}{(t_{i+1} - t_i)(t_{i+1} - t_{i-1})},$$

$i = 1, ..., n+2$. It is obvious that $a_i + b_i + c_i = 1$, $\forall i = 1, ..., n+3$.

These expressions are easily computable when the relation (3.1) is used, $t_i = (1-m)x_i + mx_{i-1}$. Thus, we get

$$a_i = \frac{3m^2 + 9m + 5}{6}, b_i = -m^2 - 4m - \frac{8}{3}, c_i = \frac{3m^2 + 15m + 17}{6}$$

and again $a_i + b_i + c_i = 1$, $\forall i = 1, ..., n+3$. Now from the fact that $\|Q_3\|_\infty \leq |a_i| + |b_i| + |c_i|$ [1], $\forall i = 1, ..., n+3$ and using the Matlab application to evaluate this expression for every $m \in [0, 1]$, we have that $\|Q_3\|_\infty \leq 1.66$. □

It is well known ([12], chapter 5) that for any subinterval $I_i = [x_{i-1}, x_i]$, $i = 1, ..., n$ and for any function f, $\|f - Q_k f\|_{\infty, I_i} \leq (1 + \|Q_k\|_\infty) d_{\infty, I_i}(f, \Pi_k)$ where the distance of f to polynomials is defined by $d_{\infty, I_i}(f, \Pi_d) = inf\{\|f - p\|_{\infty,, I_i}, p \in \Pi_k\}$, $\|f - p\|_{\infty,, I_i} = max_{x \in I_i} |f(x) - p(x)|$. Therefore, for $f \in C^4(I)$ the error estimated is

$$\|f - Q_3 f\|_{\infty, I_i} \leq 2.66 \cdot d_{\infty, I_i}(f, \Pi_3)$$

for $i = 1, ..., n$.

From these theoretical arguments and numerical computations, an approach between these two quasi-interpolant operators QI_w and QI can be observed. The QI_w operators complete the QI operators because they can provide better approximations on some subintervals of I.

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BOOK REVIEWS

Joe Diestel, Jan H. Fourie and Johan Swart, *The metric theory of tensor products - Grothendieck's Résumé revisited*, American Mathematical Society, Providence, Rhode Island 2008,x+278 pp, ISBN:978-0-8218-4440-3

More that fifty years ago Alexander Grothendieck published his famous *Résumé de la théorie metriques des produits tensoriels topologiques,* in Boll. Soc. Mat. Sao Paolo **8**(1953/1956), 1–79, revitalizing the interest in Banach space theory and tracing the way for future investigation. Among the great ideas the genius of Grothendieck (guided by "his inborn compass", as is said in the Preface) isolated, were: the study of isomorphic invariants of special Banach spaces by comparing them with other Banach spaces via the bounded linear operators between them, the importance of the nature and location of finite dimensional subspaces (the local theory of Banach spaces), the use of diagram chasing to catch the essential isomorphic characteristics of Banach spaces. In spite of the wealth of ideas contained in Grothendieck's paper, "the infamous Résumé" (as it is called called in the Preface) is very hard to read and to find (practically no proofs are given and the Sao Paolo journal is a bibliographical rarity), so that its results are not generally known even to experts in Banach space theory. In a paper published in 1968 in the Polish journal Studia Mathematica, J. Lindenstrauss and A. Pelczynski demystified the ideas of the Résumé by getting rid of their tensor product formulations, giving a new proof to the Fundamental Grothendieck's Inequality used in the study of operator ideals and opening a new fertile and rich period in the study of Banach spaces, lasting in lethargy after the glorious time of Banach and his school from the thirties.

The present book is devoted to the presentation of the fundamental ideas from the Résumé by using mathematical tools available at the time of its writing. Some parts, were subsequent developments might shorten some arguments are presented in Notes and Remarks. Also, three appendices B. *The Blaschke selection principle and compact convex sets in finite dimensional Banach spaces,* C. *A short introduction to Banach lattices,* and D. *Stonean spaces and injectivity,* present, this time in a modern language, the main tools available to Grothendieck. The first appendix, A. *The problems of the Résumé*, discusses the solutions to the open problems from the Résumé starting with the most famous of them - the approximation problem, solved negatively by Per Enflo in 1973. An Epilogue contains a brief discussion on some recent results in the theory of operator space theory (or noncommutative Banach space theory) as developed by Effros, Ruan, Blecher, Paulsen, Pisier, a.o.

The basic results of the Résumé are presented in the four chapters of the main text: 1. *Basics of tensor norms,* 2. *The role of $C(K)$-spaces and L^1-spaces,* 3. $\otimes$-*norms related to Hilbert space*, and 4. *The fundamental theorem and its consequences* (containing a proof of Grothendieck's Fundamental Inequality).

Written in an alive and entertaining style, but with detailed and rigorous proofs, the book makes available to a large audience the treasure of fundamental ideas contained in the Résumé, a landmark in the development of functional analysis.

The book can be used for advanced courses on Banach spaces, or for self-study.

S. Cobzaş

Victor G. Zvyagin and Dmitry A. Vorotnikov, *Topological approximation methods for evolutionary problems of nonlinear hydrodynamics,* Walter de Gruyter, Berlin 2008, xii+ 230 pp, ISBN 978-3-11-020222-9, ISSN 0941-813X

There are several methods to solve the evolutionary problems of fluid dynamics as the Faedo-Galerkin method, the iteration method, the method of evolutionary equations, and others. The authors of the present book propose another approach, based on the interpretation of the initial-boundary value problem as an operator equation in some appropriate function space. Usually the maps involved in this equation do not posses good operator properties, so that one approximates the initial equation by smoothing the nonlinear terms, or adding terms of higher order with a small parameter, allowing the study of this approximating equation in spaces with more suitable topological properties and the use of various discretization method with guaranteed convergence. The final step consists in passing to limit in the approximating equation, by letting the parameters to tend to 0 to find a solution of the original equation (usually in a topology weaker than that of the spaces where the approximating equation was studied).

In order to make the book as self-contained as possible, the authors have included (mostly with full proofs) the basic results on Sobolev function spaces, degree theory and operator equations. This is done in Chapters 2. *Basic function spaces.Embedding and compactness theorems*, 3. *Operator equations in Banach spaces* (including a section on Leray-Schauder degree), and 4. *Attractors for evolutionary equations in Banach spaces.*

The preliminary material from rheology, required for the understanding of the considered models, is presented (from a mathematician's point of view) in Chapter 1. *Non-Newtonian flows.*

Chapters 5. *Strong solutions for equations of motion of viscoelastic medium*, 6. *Weak solutions for equations of motion of viscoelastic medium*, and 7. *The regularized Jeffreys model*, are dedicated to the application of the developed methods to the equations describing the motion of viscoelastic media. Since the problem of the existence of global strong solutions is open in the general case, after presenting some particular cases of the existence of strong solutions in the fifth chapter, the authors concentrate on in the rest of the book on the existence of the weak solutions.

The book is clearly written, in a didactic manner, providing the reader with a good mathematical introduction to the operator methods for the solution of initial-boundary value problems for the equations of viscoelastic fluid mechanics.

The book will be useful both for mathematicians interested in nonlinear operator equations as well as for those working in fluid mechanics.

Mirela Kohr

J.W.P. Hirschfeld, G. Korchmáros, F. Torres,*Algebraic Curves over a Finite Field*, Princeton University Press (Princeton Series in Applied Mathematics), 2008, Hardback, 696 pp., ISBN: 978-0-691-09679-7.

The theory of algebraic curves over finite fields has become of great importance, both for its own sake and for the many applications that it has in number theory, finite geometry, coding theory and cryptography as well. Being given the enormous progress in this subject, an encyclopedia-like book devoted to it, like the book published by Professors Hirschfeld, Korchmáros and Torres, is welcome in the mathematical community.

The book is a self-contained introduction to this subject. It contains a huge amount of material, consisting of elementary, classical results, but also of current research topics. The exposition is divided into three parts.

The first part deals with the general theory of algebraic curves over an algebraically closed field of arbitrary characteristic. One defines first a plane algebraic curve as the zero locus of a (homogeneous) polynomial and then one develops the theory in a classical, geometric way. All the important projective invariants (namely the degree, inflexion, k-fold point, ordinary singularity, intersection number, bitangents) are introduced and studied, starting from the very beginning of the book.

Because many problems on curves can be reduced to investigating their intersection, the intersection number plays a central role. In Chapter 2, using elimination theory, its usual definition is extended in such a way that a sort of Bezout's theorem could work for intersections of plane curves, not only for the intersection of a plane curve with a line.

In working with curves their singularity is always important. One could also look for methods to eliminate some kinds of singularities of a certain curve (unfortunately projective transformations are not enough). This is done in the third chapter, where, among other useful results concerning singularities of curves, it is proved that every plane curve can be transformed by locally quadratic transformations to one with only ordinary singularities. In this sort of analysis the notion of the branch of a plane curve is essential. Chapter 4 treats the theory of branches, using formal power series. An idea that turns out to be useful in many contexts is that a plane curve needs to be considered as the set of its branches rather than the set of its points.

In the fifth chapter there are studied the effects of birational transformations on plane curves and the birational invariants of such curves, particularly their genus. Other birational invariants are the order and dimension of linear series, extensively studied in the next chapter based on the idea of adjoint curves. Here we come across

to the Riemann-Roch theorem, which, besides giving an alternative definition of the genus, has several applications in algebraic geometry.

Even if we use birational transformations, an irreducible plane curve cannot always be transformed into a non-singular plane curve. In order to accomplish this space curves must be considered (a space curve is the image of an irreducible plane curve under a birational transformation), which are studied in Chapter 7. In this chapter the theory of non-classical curves is also presented.

The second part of the book is the central one. It develops extensively the theory of algebraic curves defined on the algebraic closure of a finite field. In chapter eight there are laid down the foundations of this particular topic. The important Stöhr-Voloch theorem is presented, followed by an elementary proof of the Stöhr-Voloch Bound for non-classical plane curves. The latter provides an accurate estimate of the number of F_q-rational branches for large families of curves.

In the following chapter it it deduced the famous Hasse-Weil Bound from the Riemann hypothesis for function fields over finite fields. There are also discussed some far-reaching consequences of the Hasse-Weil Theorem concerning curves over finite fields.

The third part contains several advanced results on curves over finite fields and on automorphism groups of curves. The major result is the finiteness of the K-automorphism group of the function field of irreducible plane curves of genus greater than one. It also collects the most important families of curves over finite fields. We could mention the maximal curves (curves for which the Hasse-Weil upper bound is attained), which are naturally used in algebraic-geometry codes. In the last chapter there are presented some applications of curves in coding theory and in the combinatorics of finite projective spaces.

The book is clearly written. Besides its 13 chapters, it contains an appendix, presenting the necessary background on field theory and group theory. The book is very well documented: the bibliography has an impressive number of 520 titles, giving a rough idea of the breadth of the subject and of the enormous documenting work done by the authors.

The publishing of this book written by professors Hirschfeld, Korchmáros and Torres is certainly a welcome and waited event. I am sure that every mathematician (graduate student, professor or researcher) working in the subject of algebraic curves over finite fields finds it indispensable.

Daniel Arnold Moldovan

Luca Capogna, Donatella Danielli, Scott D. Pauls and Jeremy T. Tyson, *An Introduction to the Heisenberg Group and the Sub-Riemannian Isoperimetric Problem*, Progress in Mathematics (series editors: H. Bass, J. Oesterlé, A. Weinstein), vol 259, Birkhäuser Verlag, Basel-Boston-Berlin, 2007, 223 pp; ISBN-13: 978-3-7643-8132-5, e-ISBN-13: 978-3-7643-8133-2.

Sub-Riemannian (also known as Carnot-Carathéodory) spaces are spaces whose metric structure may be viewed as a constrained geometry, where motion is

possible only along a given set of directions, changing from point to point. They play a central role in the general program of analysis on metric spaces, while simultaneously continuing to figure prominently in applications from other scientific disciplines ranging from robotic control and planning problems to MRI function, to new models of neurobiological visual processing and digital image reconstruction.

The book is divided in nine chapters.

The first chapter, *The isoperimetric Problem in Euclidean Space*, contains a short presentation of the isoperimetric problem and its solution in Euclidean space, indicating a few proofs for the sharp isoperimetric inequality in the plane arising from diverse areas such as complex analysis, differential geometry, geometric measure theory, nonlinear evolution PDEs (curvature flow), and integral geometry.

The second chapter, *The Heisenberg group and Sub-Riemannian Geometry* is concerned with the presentation of the Heisenberg group, the sub-Riemannian structure of it, and the Riemannian approximants to Heisenberg and Carnot groups.

Chapter 3, *Applications of Heisenberg group*, contains a selection of pure and applied mathematical models which feature aspects of Heisenberg geometry: CR geometry, Gromov hyperbolic spaces, jet spaces, path planning for nonholonomic motion, and the functional structure of the mammalian visual cortex.

The fourth chapter, *Horizontal Geometry of Submanifolds*, discusses invariance of the Sub-Riemannian metric with respect to Riemannian extension, the second fundamental form and horizontal geometry of hypersurfaces in $\mathbb{H}^N$.

Chapter 5, *Sobolev and* **BV** *Spaces*, contains the sub-Riemannian Green's formula and the fundamental solutions of the Heisenberg Laplacian, and embedding theorems for the Sobolev and BV-spaces, in particular Sobolev-Gagliardo-Nirenberg inequality, and the compactness of the embedding $BV \hookrightarrow L^1$ on the John domains.

In Chapter 6, *Geometric Measure Theory and Geometric Function Theory*, there are presented area and co-area formulas, Pansu-Rademacher theorem, first variation of the perimeter and the quasiconformal mapping on $\mathbb{H}$.

Chapters 7 and 8 are ample study of isoperimetric inequality in $\mathbb{H}$ and isoperimetric profile of $\mathbb{H}$. These chapters contain the isoperimetric inequality in Hadamard manifold, Pansu's proof of the isoperimetric inequality in $\mathbb{H}$, Pansu's conjecture, C^2 and convex isoperimetric profile in $\mathbb{H}$. Also, here is presented the Riemannian approximation approach, the horizontal mean curvature, the isoperimetric problem in the Grushin plane and the classification of symmetric CMC surfaces in $\mathbb{H}^n$.

The last chapter, Chapter 9, *Best Constants for other Geometric Inequalities on the Heisenberg group*, contains the L^2-Sobolev embedding theorem, Moser-Trudinger and Hardy inequalities.

The book is very well written and it is a nice introduction to the theory of sub-Riemannian differential geometry and geometric analysis in the Heisenberg group. I warmly recommend the book to researchers in sub-Riemannian geometry, and to those interested in PDEs, calculus of variations and its applications.

Csaba Varga

Vasile Staicu (Editor), ***Differential Equations, Chaos and Variational Problems***, Progress in Nonlinear Differential Equations and Their Applications Vol. 75, Birkhäuser, Basel, 2007, ISBN 978-3-7643-8481-4.

The book under review is a collection of original papers and state-of-the-art contributions written by leading mathematicians in honor of Arrigo Cellina and James A. Yorke on the occasion of their 65^{th} anniversary and introduced at the Conference Views on ODEs (VODE2006), June 21 - 24, 2006, Aveiro, Portugal. Arrigo Cellina and James A. Yorke were born in the same day of August, 3^{rd} 1941. Their outstanding contributions deeply influenced the scientific developments of many younger mathematicians. A short presentation of their lives and work is contained in the Editorial Introduction. The volume contains 32 contributed papers by distinguished mathematicians from all over the world covering topics related to the work of Cellina and Yorke - differential equations, delay-differential equations, variational problems , differential inclusions, Young measures, control theory, dynamical systems, chaotic systems and their relations with physical systems. Among the contributors (some of them co-workers of the celebrated) we mention: P. Agarwal, Z. Arstein, J.-P. Aubin, A. Bessan, H. Frankowska, A. Cellina, F. Clarke, C. Corduneanu, J.Mahwin, B. S. Mordukhovich, J. Myjak, D. O'Reagan, N. S. Papageorgiu, B. Ricceri.

Covering a lot of research areas, both pure and applied, this collection of wonderful papers will be of interest to a large audience, including mathematicians, physicists and engineers. No doubtably that it will be included in many libraries all over the world.

Marian Mureşan

Luigi Ambrosio, Nicola Gigli and Giuseppe Savaré, ***Gradient Flows in Metric Spaces and in the Space of Probability Measures***, Lectures in Mathematics, ETH Zürich, 2nd Edition, Birkhäuser Verlag, Basel-Boston-Berlin, 2008, vii+334 pp; ISBN: 978-3-7643-8721-1, e-ISBN3: 978-3-7643-8722-8.

The present book is formed of two parts: I. *Gradient flow in metric spaces,* and II. *Gradient flow in the space of probability measures.* Apparently independent, the inclusion of them in the same book is motivated by the fact that the space of probability measures, treated in the second part of the book, is one of the main field of application of the general theory of analysis in metric spaces. The last years were marked by an intense research activity in this field, one of the leading schools being that from the Scuola Normale Superiore di Pisa headed by Luigi Ambrosio, one of the authors of the present book.

The first part contains four chapters 1. *Curves and gradients in metric spaces*, 2. *Existence of curves of maximal slope*, 3. *Proofs of the convergence theorems*, and 4. *Generation of contraction semigroups.*

The main idea used in the first part is that of maximal slope of a curve in a complete (or at least Polish) metric space $(\mathcal{S}, d)$, meaning an absolutely continuous mapping $v : (a, b) \to \mathcal{S}$ such that $d(v(s), v(t)) \leq \int_s^t m(r)dr,\ \forall s, t,\ a < s, t < b,$ for some $m \in L^p(a, b)$, where (a, b) is an interval in $\mathbb{R}$ (possibly unbounded). One proves that there exists a minimal function $m = |v'|$, called the maximal slope of v, given by $|v'|(t) = \lim_{s\to t} d(v(s), v(t))/|s - t|$, $\mathcal{L}^1$-a.e. $t \in (a, b)$. In the case when $\mathcal{S}$ is a Banach or Hilbert space, this allows to extend some results on Fréchet differentiable curves. The main result of this part is the convergence of an Euler type discretization method for finding a curve ϕ of maximal slope such that $u_0 \in D(\phi)$ and $u(0+) = u_0$. The convergence theorem is enounced in the second chapter, while Chapter 3 is dedicated to the long and delicate proof of this theorem.

The second part of the book is concerned with spaces of probability measures endowed with the Kantorovich-Rubinstein-Wasserstein distnce, called here briefly the Wasserstein distance, one of the main illustrating realizations of the theory developed in the first part. This theory is closely related to the optimal transportation problem presented in Chapter 6. *The optimal transportation problem.* The first chapter of the second part, Chapter 5. *Preliminary notions on measure theory,* contains a survey, mostly without proofs, on the measure theory on separable metric spaces. The rest of the chapters of this part are headed as follows: 7. *The Wasserstein distance and its behaviour along geodesics*; 8. *A. C. curves and the continuity equation*; 9. *Convex functionals on* $\mathcal{P}_p(X)$; 10. *Metric slope and subdifferential calculus in* $\mathcal{P}_p(X)$.

The book is based on a NachDiplom course taught by the first author at ETH Zürich in the fall of 2001, but the material was substantially enlarged by the contributions of the second and the third authors, mainly in what concerns the error estimates in the first part and the generalized convexity properties in the second part.

This second edition of the book generally agrees with the first one, modulo some corrections and an updated bibliography. By the detailed presentation of the subject the book can be used as a textbook, but by some results never published elsewhere it is a research book as well.

Radu Precup

T.V. Panchapagesan, ***The Bartle-Dunford-Schwartz integral,*** Monografie Matematyczne (New Series), Vol. 69, Birkhäuser Verlag, Boston-Basel-Berlin, 2008, xv+301 pp, ISBN: 978-3-7643-8601-6 and e-ISBN: 978-3-7643-2431-5

The present book is concerned with the integration of scalar functions (real or complex) with respect to vector measures taking values in a Banach or, more generally, in a locally convex Hausdorff space (lcHs for short). The author develops a theory of integration for vector measures defined on more general structures than σ-algebras - usually δ- or σ-rings. The theory has its origins in a famous paper by Grothendieck (Canadian Math. Bull. **5** (1953), 129-173) where he showed that there is a bijection between the weakly compact linear operators $u : C(K) \to F$, K a compact Hausdorff space and F a complete lcHs, and the σ-additive F-valued vector measures, but he did not develop any integration theory to represent these operators. This was done in

1955 by Bartle, Dunford and Schwartz who developed a theory of integration for σ-additive Banach-valued measures and used it to represent weakly compact operators $u : C(K) \to X$, K a compact Hausdorff space and X a Banach space. To honor them the author calls this type of integral the BDS-integral.

About fifteen years later, Lewis developed a Pettis weak type integral of scalar functions with respect to a σ-additive vector measure $\mathbf{m}$ with range in a lcHs X. Since this kind of integral was considered also by Kluwanek, the author call it the KL-integral. Lewis proved that if $\mathbf{m}$ is defined on a σ-algebra and X is Banach, then the BDS and KL integrals agree. The author fills in some essential details lacking from Lewis' proof.

Other important spaces considered in the book are the space $\mathcal{K}(T)$ of all continuous functions with compact support defined on a locally compact Hausdorff space T, equipped with the inductive limit locally convex topology, the space $C_c(T)$ of all continuous functions with compact support with the supremum norm $\|\cdot\|_T$, and its completion $(C_0(T), \|\cdot\|_T)$ of all functions vanishing at infinity. For a lcHs X, a linear continuous operator $u : \mathcal{K}(T) \to X$ is called a Radon operator. If further, $u : C_c(T) \to X$ is continuous and its extension to $(C_0(T), \|\cdot\|_T)$ is weakly compact, then u is called a weakly compact bounded Radon operator. A representation theory for these type of operators, as well as for another class of operators, called prolongable, in the case of a quasicomplete lcHs X, is developed in Chapters 5 and 6, dedicated to integration on locally compact Hausdorff spaces.

The first chapter, 1. *Preliminaries,* has an expository character, while chapters 2. *Basic properties of the Bartle-Dunfors-Schwartz integral,* and 3. *$\mathcal{L}_p$-spaces,* $1 \leq p \leq \infty$, are devoted to integration of Banach-valued measures defined on δ- or σ-rings.

The main concern of the last chapter of the book, 7. *Complements to the Thomas theory,* is to extend the integration and representation results obtained by E. Thomas, Ann. Inst. Fourier (Grenoble), **20** (1970), 55-191, from the real space $\mathcal{K}(T, \mathbb{R})$ to the complex one $\mathcal{K}(T)$.

The book contains very general results about the integration of scalar functions with respect to measures with values in locally convex spaces with applications to the representation of weakly compact operators on spaces of continuous functions, completing the program initiated by A. Grothendieck. The author contributed essentially to this domain and his results are incorporated in the book.

The book is a worthy working tool for researchers in functional analysis, interested in vector measures and operator theory.

V. Anisiu