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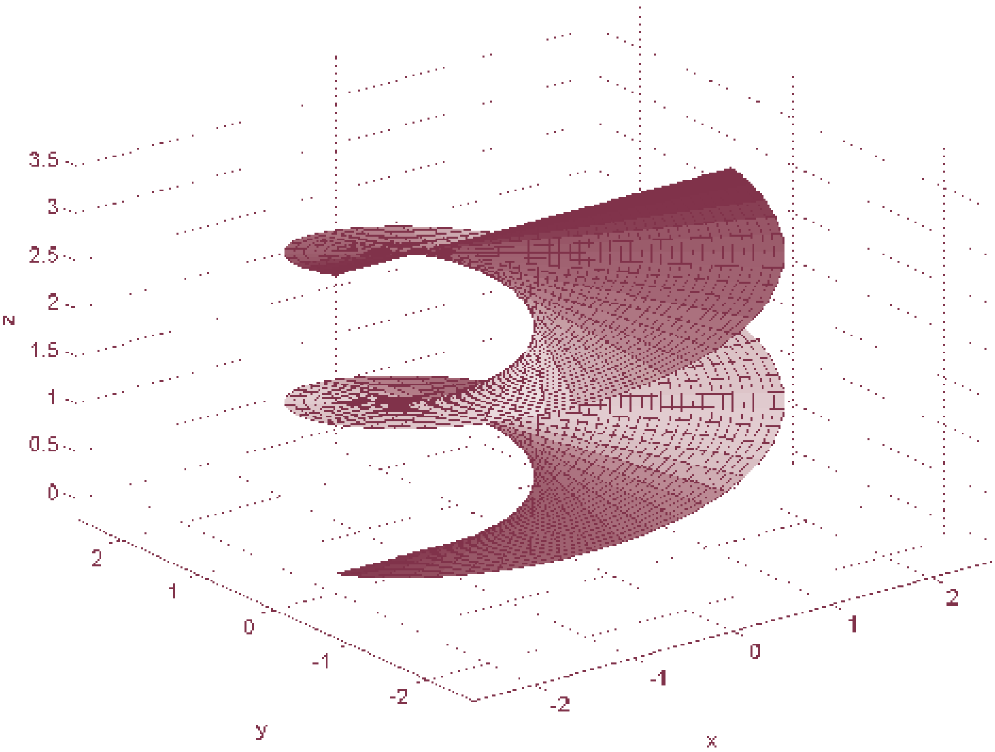

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STATISTICAL APPROXIMATION BY DOUBLE PICARD SINGULAR INTEGRAL OPERATORS

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Abstract. We first construct a sequence of double smooth Picard singular integral operators which do not have to be positive in general. After giving some useful estimates, we mainly show that it is possible to approximate a function by these operators in statistical sense even though they do not obey the positivity condition of the statistical Korovkin theory.

1. **Introduction**

In the classical Korovkin theory, the positivity condition of linear operators and the validity of their (ordinary) limits are crucial points in approximating a function by these operators (see [1, 22]). However, there are many approximation operators that do not have to be positive, such as Picard, Poisson-Cauchy and Gauss-Weierstrass singular integral operators (see, e.g., [2, 3, 4, 8, 9, 10, 19]). Furthermore, using the concept of statistical convergence from the summability theory which is a weaker method than the usual convergence, it is possible to approximate (in statistical sense) a function by means of a sequence of positive linear operators although the limit of the sequence fails (see, e.g., [5, 6, 11, 12, 13, 14, 15, 16, 23, 24]).

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The aim of the present paper is to construct a sequence of linear operators that are not necessarily positive and to investigate its statistical approximation properties. Hence, we demonstrate that it is possible to find some statistical approximation operators that are not in general positive.

This paper is organized as follows. In the first section we recall some definitions and set the main notation used in the paper, while, in the second section, we construct the double smooth Picard singular integral operators which do not have to be positive. In the third section, we give some useful estimates on these operators. In the forth section, we obtain some statistical approximation theorems for our operators. The last section of the paper is devoted to the concluding remarks and discussion.

Let $A := [a_{jn}]$, $j, n = 1, 2, ...$, be an infinite summability matrix and assume that, for a given sequence $x = (x_n)_{n\in\mathbb{N}}$, the series $\sum_{n=1}^{\infty} a_{jn}x_n$ converges for every $j \in \mathbb{N}$. Then, by the A-transform of x, we mean the sequence $Ax = ((Ax)_j)_{j\in\mathbb{N}}$ such that, for every $j \in \mathbb{N}$,

$$(Ax)_j := \sum_{n=1}^{\infty} a_{jn}x_n.$$

A summability matrix A is said to be regular (see [20]) if for every $x = (x_n)_{n\in\mathbb{N}}$ for which $\lim_{n\to\infty} x_n = L$ we get $\lim_{j\to\infty} (Ax)_j = L$. Now, fix a non-negative regular summability matrix A. In [18] Freedman and Sember introduced a convergence method, the so-called A-statistical convergence, as in the following way. A given sequence $x = (x_n)_{n\in\mathbb{N}}$ is said to be A-statistically convergent to L if, for every $\varepsilon > 0$,

$$\lim_{j\to\infty} \sum_{n\,:\,|x_n - L|\geq\varepsilon} a_{nj} = 0.$$

This limit is denoted by $st_A - \lim_n x_n = L$.

Observe that if $A = C_1 = [c_{jn}]$, the Cesáro matrix of order one defined to be $c_{jn} = 1/j$ if $1 \leq n \leq j$, and $c_{jn} = 0$ otherwise, then C_1-statistical convergence coincides with the concept of statistical convergence, which was first introduced by Fast [17]. In this case, we use the notation $st - \lim$ instead of $st_{C_1} - \lim$ (see Section 5 for this situation). Notice that every convergent sequence is A-statistically convergent to the same value for any non-negative regular matrix A, however, its converse is not always true. Actually, Kolk [21] proved that A-statistical convergence is stronger than (usual) convergence if $A = [a_{jn}]$ is any nonnegative regular summability matrix satisfying the condition $\lim_j \max_n \{a_{jn}\} = 0$. Not all properties of convergent sequences hold true for A-statistical convergence (or statistical convergence). For instance, although it is well-known that a subsequence of a convergent sequence is convergent, this is not always true for A-statistical convergence. Another example is that every convergent sequence must be bounded, however it does not need to be bounded of an A-statistically convergent sequence. Of course, with these properties, the usage of A-statistical convergence in the approximation theory provides us many advantages.

2. Construction of the operators

Throughout the paper, for $r \in \mathbb{N}$ and $m \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$, we use

$$\alpha_{j,r}^{[m]} := \begin{cases} (-1)^{r-j} \binom{r}{j} j^{-m} & \text{if } j = 1, 2, ..., r, \\ 1 - \sum\limits_{j=1}^{r} (-1)^{r-j} \binom{r}{j} j^{-m} & \text{if } j = 0. \end{cases} \tag{2.1}$$

and

$$\delta_{k,r}^{[m]} := \sum_{j=1}^{r} \alpha_{j,r}^{[m]} j^k, \quad k = 1, 2, ..., m \in \mathbb{N}. \tag{2.2}$$

Then observe that

$$\sum_{j=0}^{r} \alpha_{j,r}^{[m]} = 1 \tag{2.3}$$

and

$$-\sum_{j=1}^{r} (-1)^{r-j} \binom{r}{j} = (-1)^r \binom{r}{0}. \tag{2.4}$$

We now define the double smooth Picard singular integral operators as follows:

$$P_{r,n}^{[m]}(f; x, y) = \frac{1}{2\pi\xi_n^2} \sum_{j=0}^{r} \alpha_{j,r}^{[m]} \left(\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x + sj, y + tj)\, e^{-(\sqrt{s^2+t^2})/\xi_n} ds dt \right), \tag{2.5}$$

where $(x, y) \in \mathbb{R}^2$, $n, r \in \mathbb{N}$, $m \in \mathbb{N}_0$, $f : \mathbb{R}^2 \to \mathbb{R}$ is a Lebesgue measurable function, and also $(\xi_n)_{n \in \mathbb{N}}$ is a bounded sequence of positive real numbers.

Remark 2.1. The operators $P_{r,n}^{[m]}$ are not in general positive. For example, consider the function $\varphi(u,v) = u^2 + v^2$ and also take $r = 2$, $m = 3$, $x = y = 0$. Observe that $\varphi \geq 0$, however

$$\begin{aligned}
P_{2,n}^{[3]}(\varphi;0,0) &= \frac{1}{2\pi\xi_n^2}\left(\sum_{j=1}^{2} j^2\alpha_{j,2}^{[3]}\right)\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(s^2+t^2\right)e^{-(\sqrt{s^2+t^2})/\xi_n}dsdt \\
&= \frac{2}{\pi\xi_n^2}\left(\alpha_{1,2}^{[3]} + 4\alpha_{2,2}^{[3]}\right)\int_{0}^{\infty}\int_{0}^{\infty}\left(s^2+t^2\right)e^{-(\sqrt{s^2+t^2})/\xi_n}dsdt \\
&= \frac{2}{\pi\xi_n^2}\left(-2+\frac{1}{2}\right)\int_{0}^{\pi/2}\int_{0}^{\infty} e^{-\rho/\xi_n}\rho^3 d\rho d\theta \\
&= -9\xi_n^2 < 0.
\end{aligned}$$

Lemma 2.1. *The operators* $P_{r,n}^{[m]}$ *given by* (2.5) *preserve the constant functions in two variables.*

Proof. Let $f(x,y) = C$, where C is any real constant. By (2.1) and (2.3), we get, for every $r, n \in \mathbb{N}$ and $m \in \mathbb{N}_0$, that

$$\begin{aligned}
P_{r,n}^{[m]}(C;x,y) &= \frac{C}{2\pi\xi_n^2}\sum_{j=0}^{r}\alpha_{j,r}^{[m]}\left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(\sqrt{s^2+t^2})/\xi_n}dsdt\right) \\
&= \frac{C}{2\pi\xi_n^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(\sqrt{s^2+t^2})/\xi_n}dsdt \\
&= \frac{2C}{\pi\xi_n^2}\int_{0}^{\infty}\int_{0}^{\infty} e^{-(\sqrt{s^2+t^2})/\xi_n}dsdt \\
&= \frac{2C}{\pi\xi_n^2}\int_{0}^{\pi/2}\int_{0}^{\infty} e^{-\rho/\xi_n}\rho d\rho d\theta \\
&= \frac{C}{\xi_n^2}\int_{0}^{\infty} e^{-\rho/\xi_n}\rho d\rho \\
&= C,
\end{aligned}$$

which completes the proof. ◻

Lemma 2.2. *Let $k \in \mathbb{N}_0$. Then, it holds, for each $\ell = 0, 1, ..., k$ and for every $n \in \mathbb{N}$, that*

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} s^{k-\ell}t^{\ell}e^{-(\sqrt{s^2+t^2})/\xi_n}dsdt = \begin{cases} 0 & \text{if } k \text{ is odd} \\ 2B\left(\frac{k-\ell+1}{2}, \frac{\ell+1}{2}\right)\xi_n^{k+2}(k+1)! & \text{if } k \text{ is even} \end{cases}$$

where $B(a, b)$ denotes the Beta function.

Proof. It is clear that if k is odd, then the integrand is a odd function with respect to s and t; and hence the above integral is zero. Also, if k is even, then the integrand is an even function with respect to s and t. So, we may write that

$$\begin{aligned} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} s^{k-\ell}t^{\ell}e^{-(\sqrt{s^2+t^2})/\xi_n}dsdt &= 4\int_{0}^{\infty}\int_{0}^{\infty} s^{k-\ell}t^{\ell}e^{-(\sqrt{s^2+t^2})/\xi_n}dsdt \\ &= 4\left(\int_0^{\pi/2}(\cos\theta)^{k-\ell}(\sin\theta)^{\ell}\,d\theta\right)\left(\int_0^{\infty}\rho^{k+1}e^{-\rho/\xi_n}d\rho\right) \\ &= 2B\left(\frac{k-\ell+1}{2}, \frac{\ell+1}{2}\right)\xi_n^{k+2}(k+1)! \end{aligned}$$

whence the result. □

3. Estimates for the operators (2.5)

Let $f \in C_B(\mathbb{R}^2)$, the space of all bounded and continuous functions on $\mathbb{R}^2$. Then, the rth (double) modulus of smoothness of f is given by (see, e.g., [7])

$$\omega_r(f; h) := \sup_{\sqrt{u^2+v^2}\le h} \left\|\Delta_{u,v}^r(f)\right\| < \infty, \quad h > 0, \tag{3.1}$$

where $\|\cdot\|$ is the sup-norm and

$$\Delta_{u,v}^r\left(f(x, y)\right) = \sum_{j=0}^{r}(-1)^{r-j}\binom{r}{j}f(x + ju, y + jv). \tag{3.2}$$

Let $m \in \mathbb{N}$. By $C^{(m)}\left(\mathbb{R}^2\right)$ we mean the space of functions having m times continuous partial derivatives with respect to the variables x and y. Assume now that a function $f \in C^{(m)}\left(\mathbb{R}^2\right)$ satisfies the condition

$$\left\|\frac{\partial^m f(\cdot,\cdot)}{\partial^{m-\ell}x\partial^{\ell}y}\right\| := \sup_{(x,y)\in\mathbb{R}^2}\left|\frac{\partial^m f(x, y)}{\partial^{m-\ell}x\partial^{\ell}y}\right| < \infty \tag{3.3}$$

for every $\ell = 0, 1, ..., m$. Then, we consider the function

$$G_{x,y}^{[m]}(s,t) := \frac{1}{(m-1)!} \sum_{j=0}^{r} \binom{r}{j} \int_0^1 (1-w)^{m-1}$$

$$\times \left\{ \sum_{\ell=0}^{m} \binom{m}{m-\ell} \left| \frac{\partial^m f(x+jsw, y+jtw)}{\partial^{m-\ell} x \partial^{\ell} y} \right| \right\} dw \tag{3.4}$$

for $m \in \mathbb{N}$ and $(x,y), (s,t) \in \mathbb{R}^2$. Notice that the condition (3.3) implies that $G_{x,y}^{[m]}(s,t)$ is well-defined for each fixed $m \in \mathbb{N}$.

We first estimate the case of $m \in \mathbb{N}$ in (2.5).

Theorem 3.1. *Let $m \in \mathbb{N}$ and $f \in C^{(m)}\left(\mathbb{R}^2\right)$ for which (3.3) holds. Then, for the operators $P_{r,n}^{[m]}$, we have*

$$\Bigg| P_{r,n}^{[m]}(f;x,y) - f(x,y) - \frac{1}{\pi} \sum_{i=1}^{[m/2]} (2i+1) \delta_{2i,r}^{[m]} \xi_n^{2i}$$

$$\times \left\{ \sum_{\ell=0}^{2i} \binom{2i}{2i-\ell} \frac{\partial^{2i} f(x,y)}{\partial^{2i-\ell} x \partial^{\ell} y} B\left(\frac{2i-\ell+1}{2}, \frac{\ell+1}{2} \right) \right\} \Bigg|$$

$$\leq \frac{1}{2\pi \xi_n^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} G_{x,y}^{[m]}(s,t) (|s|^m + |t|^m) e^{-(\sqrt{s^2+t^2})/\xi_n} ds dt. \tag{3.5}$$

The sums in the left hand side of (3.5) collapse when $m = 1$.

Proof. Let $(x,y) \in \mathbb{R}^2$ be fixed. By Taylor's formula, we may write that

$$\begin{aligned} f(x+js, y+jt) &= \sum_{k=0}^{m-1} \frac{j^k}{k!} \sum_{\ell=0}^{k} \binom{k}{k-\ell} s^{k-\ell} t^{\ell} \frac{\partial^k f(x,y)}{\partial^{k-\ell} x \partial^{\ell} y} \\ &\quad + \frac{j^m}{(m-1)!} \int_0^1 (1-w)^{m-1} \left\{ \sum_{\ell=0}^{m} \binom{m}{m-\ell} s^{m-\ell} t^{\ell} \right. \\ &\quad \times \left. \frac{\partial^m f(x+jsw, y+jtw)}{\partial^{m-\ell} x \partial^{\ell} y} \right\} dw, \end{aligned}$$

which implies that

$$\begin{aligned} f(x+js,y+jt)-f(x,y) &= \sum_{k=1}^{m}\frac{j^k}{k!}\sum_{\ell=0}^{k}\binom{k}{k-\ell}s^{k-\ell}t^{\ell}\frac{\partial^k f(x,y)}{\partial^{k-\ell}x\partial^{\ell}y} \\ &\quad -\frac{j^m}{(m-1)!}\int_0^1 (1-w)^{m-1} \\ &\quad \times\left\{\sum_{\ell=0}^{m}\binom{m}{m-\ell}s^{m-\ell}t^{\ell}\frac{\partial^m f(x,y)}{\partial^{m-\ell}x\partial^{\ell}y}\right\}dw \\ &\quad +\frac{j^m}{(m-1)!}\int_0^1 (1-w)^{m-1}\left\{\sum_{\ell=0}^{m}\binom{m}{m-\ell}s^{m-\ell}t^{\ell}\right. \\ &\quad \times\left.\frac{\partial^m f(x+jsw,y+jtw)}{\partial^{m-\ell}x\partial^{\ell}y}\right\}dw. \end{aligned}$$

Now multiplying both sides of the above equality by $\alpha_{j,r}^{[m]}$ and summing up from 0 to r we obtain

$$\begin{aligned} \sum_{j=0}^{r}\alpha_{j,r}^{[m]}\left(f(x+js,y+jt)-f(x,y)\right) &= \sum_{k=1}^{m}\frac{\delta_{k,r}^{[m]}}{k!}\sum_{\ell=0}^{k}\binom{k}{k-\ell}s^{k-\ell}t^{\ell}\frac{\partial^k f(x,y)}{\partial^{k-\ell}x\partial^{\ell}y} \\ &\quad +\frac{1}{(m-1)!}\int_0^1 (1-w)^{m-1}\varphi_{s,t}^{[m]}(w)dw, \end{aligned}$$

where

$$\begin{aligned} \varphi_{x,y}^{[m]}(w;s,t) &= \sum_{j=0}^{r}\alpha_{j,r}^{[m]}j^m\left\{\sum_{\ell=0}^{m}\binom{m}{m-\ell}s^{m-\ell}t^{\ell}\frac{\partial^m f(x+jsw,y+jtw)}{\partial^{m-\ell}x\partial^{\ell}y}\right\} \\ &\quad -\delta_{m,r}^{[m]}\sum_{\ell=0}^{m}\binom{m}{m-\ell}s^{m-\ell}t^{\ell}\frac{\partial^m f(x,y)}{\partial^{m-\ell}x\partial^{\ell}y}. \end{aligned}$$

We first estimate $\varphi_{x,y}^{[m]}(w;s,t)$. Using (2.1), (2.2) and (2.4), we have

$$\begin{aligned} \varphi_{x,y}^{[m]}(w;s,t) = \sum_{j=1}^{r}(-1)^{r-j}\binom{r}{j}\left\{\sum_{\ell=0}^{m}\binom{m}{m-\ell}s^{m-\ell}t^{\ell}\frac{\partial^m f(x+jsw,y+jtw)}{\partial^{m-\ell}x\partial^{\ell}y}\right\} \\ -\sum_{j=1}^{r}(-1)^{r-j}\binom{r}{j}\left\{\sum_{\ell=0}^{m}\binom{m}{m-\ell}s^{m-\ell}t^{\ell}\frac{\partial^m f(x,y)}{\partial^{m-\ell}x\partial^{\ell}y}\right\} \end{aligned}$$

$$= \sum_{j=1}^{r}(-1)^{r-j}\binom{r}{j}\left\{\sum_{\ell=0}^{m}\binom{m}{m-\ell}s^{m-\ell}t^{\ell}\frac{\partial^m f(x+jsw,y+jtw)}{\partial^{m-\ell}x\partial^{\ell}y}\right\}$$

$$+(-1)^{r}\binom{r}{0}\left\{\sum_{\ell=0}^{m}\binom{m}{m-\ell}s^{m-\ell}t^{\ell}\frac{\partial^m f(x,y)}{\partial^{m-\ell}x\partial^{\ell}y}\right\}$$

$$= \sum_{j=0}^{r}(-1)^{r-j}\binom{r}{j}\left\{\sum_{\ell=0}^{m}\binom{m}{m-\ell}s^{m-\ell}t^{\ell}\frac{\partial^m f(x+jsw,y+jtw)}{\partial^{m-\ell}x\partial^{\ell}y}\right\}.$$

In this case, we see that

$$\left|\varphi_{x,y}^{[m]}(w;s,t)\right| \le \left(|s|^m+|t|^m\right)\sum_{j=0}^{r}\binom{r}{j}\left\{\sum_{\ell=0}^{m}\binom{m}{m-\ell}\left|\frac{\partial^m f(x+jsw,y+jtw)}{\partial^{m-\ell}x\partial^{\ell}y}\right|\right\}. \tag{3.6}$$

After integration and some simple calculations, and also using Lemma 2.1, we obtain, for every $n\in\mathbb{N}$, that

$$\begin{aligned} P_{r,n}^{[m]}(f;x,y)-f(x,y) &= \frac{1}{2\pi\xi_n^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left\{\sum_{j=0}^{r}\alpha_{j,r}^{[m]}\left(f\left(x+sj,y+tj\right)-f(x,y)\right)\right\} \\ &\quad \times e^{-(\sqrt{s^2+t^2})/\xi_n}dsdt \\ &= \frac{1}{2\pi\xi_n^2}\sum_{k=1}^{m}\frac{\delta_{k,r}^{[m]}}{k!}\sum_{\ell=0}^{k}\binom{k}{k-\ell}\frac{\partial^k f(x,y)}{\partial^{k-\ell}x\partial^{\ell}y} \\ &\quad \times\left\{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}s^{k-\ell}t^{\ell}e^{-(\sqrt{s^2+t^2})/\xi_n}dsdt\right\} \\ &\quad +R_n^{[m]}(x,y) \end{aligned}$$

where

$$R_n^{[m]}(x,y) := \frac{1}{2\pi\xi_n^2(m-1)!}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\int_0^1(1-w)^{m-1}\varphi_{x,y}^{[m]}(w;s,t)dw\right)e^{-(\sqrt{s^2+t^2})/\xi_n}dsdt.$$

By (3.4) and (3.6), it is clear that

$$\left|R_n^{[m]}(x,y)\right| \le \frac{1}{2\pi\xi_n^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}G_{x,y}^{[m]}(s,t)\left(|s|^m+|t|^m\right)e^{-(\sqrt{s^2+t^2})/\xi_n}dsdt.$$

Then, combining these results with Lemma 2.2, we immediately get (3.5). The proof is completed. □

The next estimate answers the case of $m=0$ in (2.5).

Theorem 3.2. *Let $f\in C_B\left(\mathbb{R}^2\right)$. Then, we have*

$$\left|P_{r,n}^{[0]}(f;x,y)-f(x,y)\right|\leq\frac{2}{\pi\xi_n^2}\int_0^\infty\int_0^\infty\omega_r\left(f;\sqrt{s^2+t^2}\right)e^{-(\sqrt{s^2+t^2})/\xi_n}dsdt. \tag{3.7}$$

Proof. Taking $m=0$ in (2.1) we observe that

$$\begin{aligned}P_{r,n}^{[0]}(f;x,y)-f(x,y)&=\frac{1}{2\pi\xi_n^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left\{\sum_{j=1}^{r}\alpha_{j,r}^{[0]}\left(f\left(x+sj,y+tj\right)-f(x,y)\right)\right\}\\&\quad\times e^{-(\sqrt{s^2+t^2})/\xi_n}dsdt\\&=\frac{1}{2\pi\xi_n^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left\{\sum_{j=1}^{r}(-1)^{r-j}\binom{r}{j}\left(f\left(x+sj,y+tj\right)-f(x,y)\right)\right\}\\&\quad\times e^{-(\sqrt{s^2+t^2})/\xi_n}dsdt\\&=\frac{1}{2\pi\xi_n^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left\{\sum_{j=1}^{r}(-1)^{r-j}\binom{r}{j}f\left(x+sj,y+tj\right)\right.\\&\quad\left.+\left(-\sum_{j=1}^{r}(-1)^{r-j}\binom{r}{j}\right)f(x,y)\right\}e^{-(\sqrt{s^2+t^2})/\xi_n}dsdt.\end{aligned}$$

Now using (2.4) we have

$$\begin{aligned}P_{r,n}^{[0]}(f;x,y)-f(x,y)&=\frac{1}{2\pi\xi_n^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left\{\sum_{j=1}^{r}(-1)^{r-j}\binom{r}{j}f\left(x+sj,y+tj\right)\right.\\&\quad\left.+(-1)^r\binom{r}{0}f(x,y)\right\}e^{-(\sqrt{s^2+t^2})/\xi_n}dsdt\\&=\frac{1}{2\pi\xi_n^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left\{\sum_{j=0}^{r}(-1)^{r-j}\binom{r}{j}f\left(x+sj,y+tj\right)\right\}\\&\quad\times e^{-(\sqrt{s^2+t^2})/\xi_n}dsdt,\end{aligned}$$

and hence, by (3.2),

$$P_{r,n}^{[0]}(f;x,y)-f(x,y)=\frac{1}{2\pi\xi_n^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\Delta_{s,t}^r\left(f(x,y)\right)e^{-(\sqrt{s^2+t^2})/\xi_n}dsdt.$$

Therefore, we obtain from (3.1) that

$$\begin{aligned}\left|P_{r,n}^{[0]}(f;x,y)-f(x,y)\right| &\leq \frac{1}{2\pi\xi_n^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left|\Delta_{s,t}^{r}\left(f(x,y)\right)\right|e^{-(\sqrt{s^2+t^2})/\xi_n}dsdt\\ &\leq \frac{1}{2\pi\xi_n^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\omega_r\left(f;\sqrt{s^2+t^2}\right)e^{-(\sqrt{s^2+t^2})/\xi_n}dsdt\\ &= \frac{2}{\pi\xi_n^2}\int_{0}^{\infty}\int_{0}^{\infty}\omega_r\left(f;\sqrt{s^2+t^2}\right)e^{-(\sqrt{s^2+t^2})/\xi_n}dsdt\end{aligned}$$

which completes the proof. □

4. **Statistical approximation by the operators (2.5)**

We first get the following statistical approximation theorem for the operators (2.5) in case of $m \in \mathbb{N}$.

Theorem 4.1. *Let $A = [a_{jn}]$ be a non-negative regular summability matrix, and let $(\xi_n)_{n\in\mathbb{N}}$ be a bounded sequence of positive real numbers for which*

$$st_A - \lim_n \xi_n = 0 \tag{4.1}$$

holds. Then, for each fixed $m \in \mathbb{N}$ and for all $f \in C^{(m)}\left(\mathbb{R}^2\right)$ satisfying (3.3), *we have*

$$st_A - \lim_n \left\|P_{r,n}^{[m]}(f) - f\right\| = 0. \tag{4.2}$$

Proof. Let $m \in \mathbb{N}$ be fixed. Then, we obtain from the hypothesis and (3.5) that

$$\begin{aligned}\left\|P_{r,n}^{[m]}(f)-f\right\| &\leq \sum_{i=1}^{[m/2]}(2i+1)K_i\delta_{2i,r}^{[m]}\xi_n^{2i}\\ &\quad+\frac{1}{2\pi\xi_n^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left\|G_{x,y}^{[m]}(s,t)\right\|(|s|^m+|t|^m)e^{-(\sqrt{s^2+t^2})/\xi_n}dsdt,\end{aligned}$$

where

$$K_i := \frac{1}{\pi}\sum_{\ell=0}^{2i}\binom{2i}{2i-\ell}\left\|\frac{\partial^{2i}f(\cdot,\cdot)}{\partial^{2i-\ell}x\partial^{\ell}y}\right\|B\left(\frac{2i-\ell+1}{2},\frac{\ell+1}{2}\right)$$

for $i = 1, ..., \left[\frac{m}{2}\right]$. By (3.4) we get that

$$\begin{aligned}\left\|G_{x,y}^{[m]}(s,t)\right\| &\leq \frac{2^r}{(m-1)!}\left(\sum_{\ell=0}^{m}\binom{m}{m-\ell}\left\|\frac{\partial^m f(\cdot,\cdot)}{\partial^{m-\ell}x\partial^\ell y}\right\|\right)\int_0^1 (1-w)^{m-1}dw \\ &= \frac{2^r}{m!}\sum_{\ell=0}^{m}\binom{m}{m-\ell}\left\|\frac{\partial^m f(\cdot,\cdot)}{\partial^{m-\ell}x\partial^\ell y}\right\|,\end{aligned}$$

thus we obtain

$$\begin{aligned}\left\|P_{r,n}^{[m]}(f)-f\right\| &\leq \sum_{i=1}^{[m/2]}(2i+1)K_i\delta_{2i,r}^{[m]}\xi_n^{2i} \\ &\quad +\frac{2^{r+1}}{\pi m!\xi_n^2}\left(\sum_{\ell=0}^{m}\binom{m}{m-\ell}\left\|\frac{\partial^m f(\cdot,\cdot)}{\partial^{m-\ell}x\partial^\ell y}\right\|\right) \\ &\quad \times\int_0^\infty\int_0^\infty (s^m+t^m)e^{-(\sqrt{s^2+t^2})/\xi_n}dsdt.\end{aligned}$$

Then, we have

$$\begin{aligned}\left\|P_{r,n}^{[m]}(f)-f\right\| &\leq \sum_{i=1}^{[m/2]}(2i+1)K_i\delta_{2i,r}^{[m]}\xi_n^{2i} \\ &\quad +L_m\int_0^\infty\int_0^\infty (s^m+t^m)e^{-(\sqrt{s^2+t^2})/\xi_n}dsdt \\ &= \sum_{i=1}^{[m/2]}(2i+1)K_i\delta_{2i,r}^{[m]}\xi_n^{2i} \\ &\quad +L_m\int_0^{\pi/2}\int_0^\infty (\cos^m\theta+\sin^m\theta)\rho^{m+1}e^{-\rho/\xi_n}d\rho d\theta,\end{aligned}$$

where

$$L_m := \frac{2^{r+1}}{\pi m!\xi_n^2}\left(\sum_{\ell=0}^{m}\binom{m}{m-\ell}\left\|\frac{\partial^m f(\cdot,\cdot)}{\partial^{m-\ell}x\partial^\ell y}\right\|\right).$$

After some simple calculations, we see that

$$\left\|P_{r,n}^{[m]}(f)-f\right\| \leq \sum_{i=1}^{[m/2]}(2i+1)K_i\delta_{2i,r}^{[m]}\xi_n^{2i} + L_m\xi_n^{m+2}(m+1)!U_m,$$

where

$$U_m := \int_0^{\pi/2} (\cos^m \theta + \sin^m \theta) d\theta = B\left(\frac{m+1}{2}, \frac{1}{2}\right),$$

which yields

$$\left\| P_{r,n}^{[m]}(f) - f \right\| \le S_m \left\{ \xi_n^{m+2} + \sum_{i=1}^{[m/2]} \xi_n^{2i} \right\}, \tag{4.3}$$

where

$$S_m := (m+1)! U_m L_m + \max_{i=1,2,\dots,[m/2]} \left\{ (2i+1) K_i \delta_{2i,r}^{[m]} \right\}.$$

Now for a given $\varepsilon > 0$, define the following sets:

$$\begin{aligned} D &: = \left\{ n \in \mathbb{N} : \left\| P_{r,n}^{[m]}(f) - f \right\| \ge \varepsilon \right\}, \\ D_i &: = \left\{ n \in \mathbb{N} : \xi_n^{2i} \ge \frac{\varepsilon}{(1+[m/2])\, S_m} \right\},\ i = 1, ..., \left[\frac{m}{2}\right], \\ D_{1+[m/2]} &: = \left\{ n \in \mathbb{N} : \xi_n^{m+2} \ge \frac{\varepsilon}{(1+[m/2])\, S_m} \right\}. \end{aligned}$$

Then, the inequality (4.3) gives that

$$D \subseteq \bigcup_{i=1}^{1+[m/2]} D_i,$$

and hence, for every $j \in \mathbb{N}$,

$$\sum_{n \in D} a_{jn} \le \sum_{i=1}^{1+[m/2]} \sum_{n \in D_i} a_{jn}.$$

Now taking limit as $j \to \infty$ in the both sides of the above inequality and using the hypothesis (4.1), we obtain that

$$\lim_j \sum_{n \in D} a_{jn} = 0,$$

which implies (4.2). So, the proof is completed. □

Finally, we investigate the statistical approximation properties of the operators (2.5) when $m = 0$. We need the following result.

Lemma 4.1. *Let* $A = [a_{jn}]$ *be a non-negative regular summability matrix, and let* $(\xi_n)_{n\in\mathbb{N}}$ *be a bounded sequence of positive real numbers for which* (4.1) *holds. Then, for every* $f \in C_B\left(\mathbb{R}^2\right)$, *we have*

$$st_A - \lim_n \omega_r\left(f;\xi_n\right) = 0. \tag{4.4}$$

Proof. By the right-continuity of $\omega_r\left(f;\cdot\right)$ at zero, we may write that, for a given $\varepsilon > 0$, there exists a $\delta > 0$ such that $\omega_r\left(f;h\right) < \varepsilon$ whenever $0 < h < \delta$. Hence, $\omega_r\left(f;h\right) \geq \varepsilon$ implies that $h \geq \delta$. Now replacing h by ξ_n, for every $\varepsilon > 0$, we see that

$$\{n : \omega_r\left(f;\xi_n\right) \geq \varepsilon\} \subseteq \{n : \xi_n \geq \delta\},$$

which guarantees that, for each $j \in \mathbb{N}$,

$$\sum_{n:\omega_r(f;\xi_n)\geq\varepsilon} a_{jn} \leq \sum_{n:\xi_n\geq\delta} a_{jn}.$$

Also, by (4.1), we get

$$\lim_j \sum_{n:\xi_n\geq\delta} a_{jn} = 0,$$

which implies

$$\lim_j \sum_{n:\omega_r(f;\xi_n)\geq\varepsilon} a_{jn} = 0.$$

So, the proof is completed. □

Theorem 4.2. *Let* $A = [a_{jn}]$ *be a non-negative regular summability matrix, and let* $(\xi_n)_{n\in\mathbb{N}}$ *be a bounded sequence of positive real numbers for which* (4.1) *holds. Then, for all* $f \in C_B\left(\mathbb{R}^2\right)$, *we have*

$$st_A - \lim_n \left\| P_{r,n}^{[0]}(f) - f \right\| = 0. \tag{4.5}$$

Proof. By (3.7), we can write

$$\left\| P_{r,n}^{[0]}(f) - f \right\| \leq \frac{2}{\pi\xi_n^2} \int_0^\infty \int_0^\infty \omega_r\left(f; \sqrt{s^2+t^2}\right) e^{-\left(\sqrt{s^2+t^2}\right)/\xi_n} ds dt.$$

Now using the fact that $\omega_r(f;\lambda u) \leq (1+\lambda)^r \omega_r(f;u)$, $\lambda, u > 0$, we get

$$\begin{aligned}
\left\|P_{r,n}^{[0]}(f)-f\right\| &\leq \frac{2}{\pi\xi_n^2}\int_0^\infty\int_0^\infty \omega_r\left(f;\xi_n\frac{\sqrt{s^2+t^2}}{\xi_n}\right)e^{-(\sqrt{s^2+t^2})/\xi_n}dsdt \\
&\leq \frac{2\omega_r(f;\xi_n)}{\pi\xi_n^2}\int_0^\infty\int_0^\infty\left(1+\frac{\sqrt{s^2+t^2}}{\xi_n}\right)^r e^{-(\sqrt{s^2+t^2})/\xi_n}dsdt \\
&= \frac{2\omega_r(f;\xi_n)}{\pi\xi_n^2}\int_0^{\pi/2}\int_0^\infty\left(1+\frac{\rho}{\xi_n}\right)^r \rho e^{-\rho/\xi_n}d\rho d\theta \\
&= \omega_r(f;\xi_n)\int_0^\infty (1+u)^r ue^{-u}du \\
&\leq \omega_r(f;\xi_n)\int_0^\infty (1+u)^{r+1} e^{-u}du \\
&= \left(\sum_{k=0}^{r+1}\binom{r+1}{k}k!\right)\omega_r(f;\xi_n),
\end{aligned}$$

and hence

$$\left\|P_{r,n}^{[0]}(f)-f\right\| \leq K_r\omega_r(f;\xi_n), \tag{4.6}$$

where

$$K_r := \sum_{k=0}^{r+1}\binom{r+1}{k}k!.$$

Then, from (4.6), for a given $\varepsilon > 0$, we observe that

$$\left\{n\in\mathbb{N}:\left\|P_{r,n}^{[0]}(f)-f\right\|\geq\varepsilon\right\}\subseteq\left\{n\in\mathbb{N}:\omega_r(f;\xi_n)\geq\frac{\varepsilon}{K_r}\right\},$$

which implies that

$$\sum_{n:\left\|P_{r,n}^{[0]}(f)-f\right\|\geq\varepsilon} a_{jn} \leq \sum_{n:\omega_r(f;\xi_n)\geq\varepsilon/K_r} a_{jn} \tag{4.7}$$

holds for every $j\in\mathbb{N}$. Now, taking limit as $j\to\infty$ in the both sides of inequality (4.7) and also using (4.4), we obtain that

$$\lim_j \sum_{n:\left\|P_{r,n}^{[0]}(f)-f\right\|\geq\varepsilon} a_{jn} = 0,$$

which means (4.5). Hence, the proof is completed. □

5. Concluding remarks

In this section, we give some special cases of our results obtained in the previous sections.

Taking $A = C_1$, the Cesáro matrix of order one, and also combining Theorems 4.1 and 4.2, we immediately get the following result.

Corollary 5.1. *Let $(\xi_n)_{n\in\mathbb{N}}$ be a bounded sequence of positive real numbers for which*

$$st - \lim_n \xi_n = 0$$

holds. Then, for each fixed $m \in \mathbb{N}_0$ and for all $f \in C^{(m)}\left(\mathbb{R}^2\right)$ satisfying (3.3), *we have*

$$st - \lim_n \left\| P_{r,n}^{[m]}(f) - f \right\| = 0.$$

Furthermore, choosing $A = I$, the identity matrix, in Theorems 4.1 and 4.2, we have the next approximation theorems with the usual convergence.

Corollary 5.2. *Let $(\xi_n)_{n\in\mathbb{N}}$ be a bounded sequence of positive real numbers for which*

$$\lim_n \xi_n = 0$$

holds. Then, for each fixed $m \in \mathbb{N}_0$ and for all $f \in C^{(m)}\left(\mathbb{R}^2\right)$ satisfying (3.3), *the sequence $\left\{P_{r,n}^{[m]}(f)\right\}$ is uniformly convergent to f on $\mathbb{R}^2$.*

Now we define a special sequence $(\xi_n)_{n\in\mathbb{N}}$ as follows:

$$\xi_n := \begin{cases} 1, & \text{if } n = k^2,\ k = 1, 2, ... \\ \frac{1}{n}, & \text{otherwise.} \end{cases} \tag{5.1}$$

Then, observe that $st - \lim_n \xi_n = 0$. In this case, taking $A = C_1$, we obtain from Corollary 5.1 (or, Theorems 4.1 and 4.2) that

$$st - \lim_n \left\| P_{r,n}^{[m]}(f) - f \right\| = 0$$

holds for each $m \in \mathbb{N}_0$ and for all $f \in C^{(m)}\left(\mathbb{R}^2\right)$ satisfying (3.3). However, since the sequence $(\xi_n)_{n\in\mathbb{N}}$ given by (5.1) is non-convergent, the classical approximation to a function f by the operators $P_{r,n}^{[m]}(f)$ is impossible.

Notice that Theorems 4.1, 4.2 and Corollary 5.1 are also valid when $\lim \xi_n = 0$ because every convergent sequence is A-statistically convergent, and so statistically convergent. But, as in the above example, our theorems still work although $(\xi_n)_{n\in\mathbb{N}}$ is non-convergent. Therefore, this non-trivial example clearly demonstrates the power of our statistical approximation method in Theorems 4.1 and 4.2 with respect to Corollary 5.2.

In the end, we should remark that, so far, almost all statistical approximation results have dealt with positive linear operators. Of course, in this case, one has the following natural problem:

- Can we use the concept of A-statistical convergence in the approximation by non-positive approximation operators?

The same question was also asked as an open problem by Duman et. al. in [13]. With this paper we find affirmative answers to this problem by using the double smooth Picard singular integral operators given by (2.5). However, some similar arguments may be valid for other non-positive operators. Thus, in the future studies, it would be very interesting to improve such structures.

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SOME APPLICATIONS OF SALAGEAN INTEGRAL OPERATOR

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Abstract. In this paper we introduce and study some new subclasses of starlike, convex, close-to-convex and quasi-convex functions defined by Salagean integral operator. Inclusion relations are established and integral operator $L_c(f)(c \in N = \{1, 2, ...\})$ is also discussed for these subclasses.

1. **Introduction**

Let A denote the class of functions of the form:

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \tag{1.1}$$

which are analytic in the unit disc $U = \{z : |z| < 1\}$. Also let S denote the subclass of A consisting of univalent functions in U. A function $f(z) \in S$ is called starlike of order $\gamma, 0 \leq \gamma < 1$, if and only if

$$\text{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \gamma \ \ (z \in U) \ . \tag{1.2}$$

We denote by $S^*(\gamma)$ the class of all functions in S which are starlike of order γ in U.

A function $f(z) \in S$ is called convex of order $\gamma, 0 \leq \gamma < 1$, in U if and only if

$$\text{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \gamma \ \ (z \in U) \ . \tag{1.3}$$

We denote by $C(\gamma)$ the class of all functions in S which are convex of order γ in U.

It follows from (1.2) and (1.3) that:

$$f(z) \in C(\gamma) \qquad \text{if and only if} \qquad zf'(z) \in S^*(\gamma) \ . \tag{1.4}$$

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The classes $S^*(\gamma)$ and $C(\gamma)$ was introduced by Robertson [12].

Let $f(z) \in A$, and $g(z) \in S^*(\gamma)$. Then $f(z) \in K(\beta, \gamma)$ if and only if

$$\operatorname{Re}\left\{\frac{zf^{'}(z)}{g(z)}\right\} > \beta \ \ (z \in U)\,, \tag{1.5}$$

where $0 \leq \beta < 1$ and $0 \leq \gamma < 1$. Such functions are called close-to-convex functions of order β and type γ. The class $K(\beta, \gamma)$ was introduced by Libera [4].

A function $f(z) \in A$ is called quasi-convex of order β and type γ if there exists a function $g(z) \in C(\gamma)$ such that

$$\operatorname{Re}\left\{\frac{(zf^{'}(z))^{'}}{g^{'}(z)}\right\} > \beta \ \ (z \in U)\,, \tag{1.6}$$

where $0 \leq \beta < 1$ and $0 \leq \gamma < 1$. We denote this class by $K^*(\beta, \gamma)$. The class $K^*(\beta, \gamma)$ was introduced by Noor [10].

It follows from (1.5) and (1.6) that:

$$f(z) \in K^*(\beta, \gamma) \qquad \text{if and only if} \qquad zf^{'}(z) \in K(\beta, \gamma)\ . \tag{1.7}$$

For a function $f(z) \in A$, we define the integral operator $I^n f(z), n \in N_0 = N \cup \{0\}$, where $N = \{1, 2, ...\}$, by

$$I^0 f(z) = f(z)\ , \tag{1.8}$$

$$I^1 f(z) = If(z) = \int_0^z f(t)t^{-1}dt\ \ , \tag{1.9}$$

and

$$I^n f(z) = I(I^{n-1}f(z))\ \ . \tag{1.10}$$

It is easy to see that:

$$I^n f(z) = z + \sum_{k=2}^{\infty} \frac{a_k}{k^n}\, z^k \ \ (n \in N_0)\ , \tag{1.11}$$

and

$$z(I^n f(z))^{'} = I^{n-1}f(z)\ . \tag{1.12}$$

The integral operator $I^n f(z)$ $(f \in A)$ was introduced by Salagean [13] and studied by Aouf et al. [1]. We call the operator I^n by Salagean integral operator.

Using the operator I^n, we now introduce the following classes:

$$S_n^*(\gamma) = \{f \in A : I^n f \in S^*(\gamma)\} \ ,$$

$$C_n(\gamma) = \{f \in A : \ I^n f \in C(\gamma)\} \ ,$$

$$K_n(\beta,\gamma) = \{f \in A : I^n f \in K(\beta,\gamma)\} \ ,$$

and

$$K_n^*(\beta,\gamma) = \{f \in A : I^n f \in K^*(\beta,\gamma)\} \ .$$

In this paper, we shall establish inclusion relation for these classes and integral operator $L_c(f)(c \in N)$ is also discussed for these classes. In [11], Noor introduced and studied some classes defined by Ruscheweyh derivatives and in [6] Liu studied some classes defined by the one-parameter family of integral operator $I^\sigma f(z)(\sigma > 0, f \in A)$.

2. **Inclusion relations**

We shall need the following lemma.

Lemma 2.1. [8], [9] *Let $\varphi(u,v)$ be a complex function, $\phi : D \to C, D \subset C \times C$, and let $u = u_1 + iu_2, v = v_1 + iv_2$. Suppose that $\varphi(u,v)$ satisfies the following conditions:*

(i) $\varphi(u,v)$ is continuous in D;

(ii) $(1,0) \in D$ and $\operatorname{Re}\{\varphi(1,0)\} > 0$;

(iii) $\operatorname{Re}\{\varphi(iu_2,v_1)\} \leq 0$ for all $(iu_2,v_1) \in D$ and such that $v_1 \leq -\frac{1}{2}(1+u_2^2)$.

Let $h(z) = 1 + c_1 z + c_2 z^2 + ...$ be analytic in U, such that $(h(z), zh'(z)) \in D$ for all $z \in U$. If $\operatorname{Re}\{\varphi(h(z), zh'(z))\} > 0 \, (z \in U)$, then $\operatorname{Re}\{h(z)\} > 0$ for $z \in U$.

Theorem 2.1. $S_n^*(\gamma) \subset S_{n+1}^*(\gamma)(0 \leq \gamma < 1, n \in N_0)$.

Proof. Let $f(z) \in S_n^*(\gamma)$ and set

$$\frac{z(I^{n+1}f(z))'}{I^{n+1}f(z)} = \gamma + (1-\gamma)h(z) \, , \tag{2.1}$$

where $h(z) = 1 + h_1 z + h_2 z^2 + ...$ Using the identity (1.12), we have

$$\frac{I^n f(z)}{I^{n+1}f(z)} = \gamma + (1-\gamma)h(z) \, . \tag{2.2}$$

Differentiating (2.2) with respect to z logarithmically, we obtain

$$\begin{aligned}\frac{z(I^n f(z))'}{I^n f(z)} &= \frac{z(I^{n+1} f(z))'}{I^{n+1} f(z)} + \frac{(1-\gamma)zh'(z)}{\gamma+(1-\gamma)h(z)} \\ &= \gamma+(1-\gamma)h(z)+\frac{(1-\gamma)zh'(z)}{\gamma+(1-\gamma)h(z)} \ ,\end{aligned}$$

or

$$\frac{z(I^n f(z))'}{I^n f(z)} - \gamma = (1-\gamma)h(z)+\frac{(1-\gamma)zh'(z)}{\gamma+(1-\gamma)h(z)} \ . \tag{2.3}$$

Taking $h(z) = u = u_1 + iu_2$ and $zh'(z) = v = v_1 + iv_2$, we define the function $\varphi(u, v)$ by:

$$\varphi(u,v) = (1-\gamma)u + \frac{(1-\gamma)v}{\gamma+(1-\gamma)u} \ . \tag{2.4}$$

Then it follows from (2.4) that

(i) $\varphi(u, v)$ is continuous in $D = (C - \left\{\frac{\gamma}{\gamma-1}\right\}) \times C$;

(ii) $(1, 0) \in D$ and $\text{Re}\{\varphi(1, 0)\} = 1 - \gamma > 0$;

(iii) for all $(iu_2, v_1) \in D$ such that $v_1 \leq -\frac{1}{2}(1 + u_2^2)$,

$$\begin{aligned}\text{Re}\{\varphi(iu_2,v_1)\} &= \text{Re}\left\{\frac{(1-\gamma)v_1}{\gamma+(1-\gamma)iu_2}\right\} \\ &= \frac{\gamma(1-\gamma)v_1}{\gamma^2+(1-\gamma)^2u_2^2} \\ &\leq -\frac{\gamma(1-\gamma)(1+u_2^2)}{2[\gamma^2+(1-\gamma)^2u_2^2]} < 0 \ ,\end{aligned}$$

for $0 \leq \gamma < 1$. Therefore, the function $\varphi(u, v)$ satisfies the conditions in Lemma. It follows from the fact that if $\text{Re}\{\varphi(h(z), zh'(z))\} > 0, z \in U$, then $\text{Re}\{h(z)\} > 0$ for $z \in U$, that is, if $f(z) \in S_n^*(\gamma)$ then $f(z) \in S_{n+1}^*(\gamma)$. This completes the proof of Theorem 2.1. □

We next prove:

Theorem 2.2. $C_n(\gamma) \subset C_{n+1}(\gamma) (0 \leq \gamma < 1, n \in N_0)$.

Proof. $f \in C_n(\gamma) \Leftrightarrow I^n f \in C(\gamma) \Leftrightarrow z(I^n f)' \in S^*(\gamma) \Leftrightarrow I^n(zf') \in S^*(\gamma) \Leftrightarrow zf' \in S_n^*(\gamma) \Rightarrow zf' \in S_{n+1}^*(\gamma) \Leftrightarrow I^{n+1}(zf') \in S^*(\gamma) \Leftrightarrow z(I^{n+1}f)' \in S^*(\gamma) \Leftrightarrow I^{n+1}f \in C(\gamma) \Leftrightarrow f \in C_{n+1}(\gamma)$.

This completes the proof of Theorem 2.2. □

Theorem 2.3. $K_n(\beta,\gamma) \subset K_{n+1}(\beta,\gamma)(0 \le \gamma < 1, 0 \le \beta < 1, n \in N_0)$.

Proof. Let $f(z) \in K_n(\beta,\gamma)$. Then there exists a function $k(z) \in S^*(\gamma)$ such that

$$\text{Re}\left\{\frac{z(I^n f(z))'}{k(z)}\right\} > \beta \ \ (z \in U) \ .$$

Taking the function $g(z)$ which satisfies $I^n g(z) = k(z)$, we have $g(z) \in S_n^*(\gamma)$ and

$$\text{Re}\left\{\frac{z(I^n f(z))'}{I^n g(z)}\right\} > \beta \ \ (z \in U) \ . \tag{2.5}$$

Now put

$$\frac{z(I^{n+1} f(z))'}{I^{n+1} g(z)} - \beta = (1-\beta)h(z) \ , \tag{2.6}$$

where $h(z) = 1 + c_1 z + c_2 z^2 + ...$ Using (1.12) we have

$$\frac{z(I^n f(z))'}{I^n g(z)} = \frac{I^n(zf'(z))}{I^n g(z)} = \frac{z(I^{n+1}(zf'(z)))'}{z(I^{n+1}g(z))'}$$

$$= \frac{\dfrac{z(I^{n+1}(zf'(z))'}{I^{n+1}g(z)}}{\dfrac{z(I^{n+1}g(z))'}{I^{n+1}g(z)}} \ . \tag{2.7}$$

Since $g(z) \in S_n^*(\gamma)$ and $S_n^*(\gamma) \subset S_{n+1}^*(\gamma)$, we let $\dfrac{z(I^{n+1}g(z))'}{I^{n+1}g(z)} = \gamma + (1-\gamma)H(z)$, where $\text{Re}\, H(z) > 0 (z \in U)$. Thus (2.7) can be written as

$$\frac{z(I^n f(z))'}{I^n g(z)} = \frac{\dfrac{z(I^{n+1}(zf'(z))'}{I^{n+1}g(z)}}{\gamma + (1-\gamma)H(z)} \ . \tag{2.8}$$

Consider

$$z(I^{n+1} f(z))' = I^{n+1} g(z)[\beta + (1-\beta)h(z)] \ . \tag{2.9}$$

Differentiating both sides of (2.9), we have

$$\frac{z(I^{n+1}(zf'(z)))'}{I^{n+1}g(z)} = (1-\beta)zh'(z) + [\beta + (1-\beta)h(z)] \ . \ [\gamma + (1-\gamma)H(z)] \, . \tag{2.10}$$

Using (2.10) and (2.8), we have

$$\frac{z(I^n f(z))'}{I^n g(z)} - \beta = (1-\beta)h(z) + \frac{(1-\beta)zh'(z)}{\gamma + (1-\gamma)H(z)} \ . \tag{2.11}$$

Taking $u = h(z) = u_1 + iu_2, v = zh'(z) = v_1 + iv_2$ in (2.11), we form the function $\Psi(u, v)$ as follows:

$$\Psi(u, v) = (1-\beta)u + \frac{(1-\beta)v}{\gamma + (1-\gamma)H(z)} \ . \tag{2.12}$$

It is clear that the function $\Psi(u, v)$ defined in $D = C \times C$ by (2.12) satisfies conditions (i) and (ii) of Lemma easily. To verify condition (iii), we proceed as follows:

$$\operatorname{Re}\Psi(iu_2, v_1) = \frac{(1-\beta)v_1[\gamma + (1-\gamma)h_1(x, y)]}{[\gamma + (1-\gamma)h_1(x, y)]^2 + [(1-\gamma)h_2(x, y)]^2} \ ,$$

where $H(z) = h_1(x, y) + ih_2(x, y), h_1(x, y)$ and $h_2(x, y)$ being the functions of x and y and $\operatorname{Re} H(z) = h_1(x, y) > 0$. By putting $v_1 \leq -\frac{1}{2}(1 + u_2^2)$, we obtain

$$\operatorname{Re}\Psi(iu_2, v_1) \leq -\frac{(1-\beta)(1+u_2^2)[\gamma + (1-\gamma)h_1(x, y)]}{2\{[\gamma + (1-\gamma)h_1(x, y)]^2 + [(1-\gamma)h_2(x, y)]^2\}} < 0 \ .$$

Hence $\operatorname{Re} h(z) > 0 (z \in U)$ and $f(z) \in K_{n+1}(\beta, \gamma)$. The proof of Theorem 2.3 is complete. □

Using the same method as in Theorem 2.3 with the fact that $f(z) \in K_n^*(\beta, \gamma) \Leftrightarrow zf'(z) \in K_n(\beta, \gamma)$, we can deduce from Theorem 2.3 the following:

Theorem 2.4. $K_n^*(\beta, \gamma) \subset K_{n+1}^*(\beta, \gamma) (0 \leq \beta, \gamma < 1, n \in N_0)$.

3. Integral operator

For $c > -1$ and $f(z) \in A$, we recall here the generalized Bernardi-Libera-Livingston integral operator as:

$$L_c(f) = \frac{c+1}{z^c} \int_0^z t^{c-1} f(t) dt \ . \tag{3.1}$$

The operator $L_c(f)$ when $c \in N$ was studied by Bernardi [2]. For $c = 1, L_1(f)$ was investigated ealier by Libera [5] and Livingston [7].

The following theorems deal with the generalized Bernardi-Libera-Livingston integral operator $L_c(f)$ defined by (3.1).

Theorem 3.1. *Let $c > -\gamma$. If $f(z) \in S_n^*(\gamma)$, then $L_c(f) \in S_n^*(\gamma)$.*

Proof. From (3.1), we have

$$z(I^nL_c(f))^{'} = (c+1)I^nf(z) - cI^nL_c(f)\ . \tag{3.2}$$

Set

$$\frac{z(I^nL_c(f))^{'}}{I^nL_c(f)} = \frac{1+(1-2\gamma)w(z)}{1-w(z)}\ , \tag{3.3}$$

where $w(z)$ is analytic or meromorphic in $U, w(0) = 0$. Using (3.2) and (3.3) we get

$$\frac{I^nf(z)}{I^nL_c(f)} = \frac{c+1+(1-c-2\gamma)w(z)}{(c+1)(1-w(z))}\ . \tag{3.4}$$

Differentiating (3.4) with respect to z logarithmically, we obtain

$$\frac{z(I^nf(z))^{'}}{I^nf(z)} = \frac{1+(1-2\gamma)w(z)}{1-w(z)} + \frac{zw^{'}(z)}{1-w(z)} + \frac{(1-c-2\gamma)zw^{'}(z)}{1+c+(1-c-2\gamma)w(z)}\ . \tag{3.5}$$

Now we claim that $|w(z)| < 1 (z \in U)$. Otherwise, there exists a point $z_0 \in U$ such that $\max\limits_{|z|\leq|z_0|}|w(z)| = |w(z_0)| = 1$. Then by Jack's lemma [3], we have $z_0w^{'}(z_0) = kw(z_0)(k \geq 1)$.

Putting $z = z_0$ and $w(z_0) = e^{i\theta}$ in (3.5), we have

$$\mathrm{Re}\left\{\frac{1+(1-2\gamma)w(z_0)}{1-w(z_0)}\right\} = \mathrm{Re}\left\{(1-\gamma)\frac{1+w(z_0)}{1-w(z_0)} + \gamma\right\} = \gamma\ ,$$

and

$$\mathrm{Re}\left\{\frac{z_0(I^nf(z_0))^{'}}{I^nf(z_0)} - \gamma\right\} = \mathrm{Re}\left\{\frac{2(1-\gamma)ke^{i\theta}}{(1-e^{i\theta})[1+c+(1-c-2\gamma)e^{i\theta}]}\right\}$$

$$= 2k(1-\gamma)\mathrm{Re}\left\{\frac{(e^{i\theta}-1)\left[1+c+(1-c-2\gamma)e^{-i\theta}\right]}{2(1-\cos\theta)\left[(1+c)^2+2(1+c)(1-c-2\gamma)\cos\theta+(1-c-2\gamma)^2\right]}\right\}$$

$$= \frac{-2k(1-\gamma)(c+\gamma)}{(1+c)^2+2(1+c)(1-c-2\gamma)\cos\theta+(1-c-2\gamma)^2} \leq 0\ ,$$

which contradicts the hypothesis that $f(z) \in S_n^*(\gamma)$. Hence $|w(z)| < 1$ for $z \in U$, and it follows from (3.3) that $L_c(f) \in S_n^*(\gamma)$. The proof of Theorem 3.1 is complete. □

Theorem 3.2. *Let $c > -\gamma$. If $f(z) \in C_n(\gamma)$, then $L_c(f) \in C_n(\gamma)$.*

Proof. $f \in C_n(\gamma) \Leftrightarrow zf^{'} \in S_n^*(\gamma) \Rightarrow L_c(zf^{'}) \in S_n^*(\gamma) \Leftrightarrow z(L_cf)^{'} \in S_n^*(\gamma) \Leftrightarrow L_c(f) \in C_n(\gamma)$. □

Theorem 3.3. *Let $c > -\gamma$. If $f(z) \in K_n(\beta,\gamma)$, then $L_c(f) \in K_n(\beta,\gamma)$.*

Proof. Let $f(z) \in K_n(\beta,\gamma)$. Then, by definition, there exists a function $g(z) \in S_n^*(\gamma)$ such that

$$\operatorname{Re}\left\{\frac{z(I^n f(z))^{'}}{I^n g(z)}\right\} > \beta \ (z \in U) \ .$$

Put

$$\frac{z(I^n L_c(f))^{'}}{I^n L_c(g)} - \beta = (1-\beta)h(z) \ , \tag{3.6}$$

where $h(z) = 1 + c_1 z + c_2 z^2 + \ldots$. From (3.2), we have

$$\begin{aligned}\frac{z(I^n f(z))^{'}}{I^n g(z)} &= \frac{I^n(zf^{'}(z))}{I^n g(z)} \\ &= \frac{z(I^n L_c(zf^{'}))^{'} + cI^n L_c(zf^{'})}{z(I^n L_c(g))^{'} + cI^n L_c(g)}\end{aligned}$$

$$= \frac{\dfrac{z(I^n L_c(zf^{'}))^{'}}{I^n L_c(g)} + \dfrac{cI^n L_c(zf^{'})}{I^n L_c(g)}}{\dfrac{z(I^n L_c(g))^{'}}{I^n L_c(g)} + c} \ . \tag{3.7}$$

Since $g(z) \in S_n^*(\gamma)$, then from Theorem 3.1, we have $L_c(g) \in S_n^*(\gamma)$. Let

$$\frac{z(I^n L_c(g))^{'}}{I^n L_c(g)} = \gamma + (1-\gamma)H(z) \ ,$$

where $\operatorname{Re} H(z) > 0 (z \in U)$. Using (3.7), we have

$$\frac{z(I^n f(z))^{'}}{I^n g(z)} = \frac{\dfrac{z(I^n L_c(zf^{'}))^{'}}{I^n L_c(g)} + c[(1-\beta)h(z) + \beta]}{\gamma + c + (1-\gamma)H(z)} \ . \tag{3.8}$$

Also, (3.6) can be written as

$$z(I^n L_c(f))^{'} = I^n L_c(g)[\beta + (1-\beta)h(z)] \ . \tag{3.9}$$

Differentiating both sides of (3.9), we have

$$z\left\{z(I^n L_c(f))^{'}\right\}^{'} = z(I^n L_c(g))^{'}[\beta + (1-\beta)h(z)] + (1-\beta)zh^{'}(z)I^n L_c(g) \ ,$$

or

$$\frac{z\left\{z(I^n L_c(f))^{'}\right\}^{'}}{I^n L_c(g)} = \frac{z(I^n L_c(zf^{'}))^{'}}{I^n L_c(g)}$$

$$= (1-\beta)zh^{'}(z) + [\beta + (1-\beta)h(z)]\,[\gamma + (1-\gamma)H(z)] \ .$$

From (3.8), we have

$$\frac{z(I^n f(z))'}{I^n g(z)} - \beta = (1-\beta)h(z) + \frac{(1-\beta)zh'(z)}{\gamma + c + (1-\gamma)H(z)} \ . \tag{3.10}$$

We form the function $\Psi(u,v)$ by taking $u = h(z)$ and $v = zh'(z)$ in (3.10) as:

$$\Psi(u,v) = (1-\beta)u + \frac{(1-\beta)v}{\gamma + c + (1-\gamma)H(z)} \ . \tag{3.11}$$

It is clear that the function $\Psi(u,v)$ defined by (3.11) satisfies the conditions (i), (ii) and (iii) of Lemma 2.1. Thus we have $I_n(f(z)) \in K_n(\beta,\gamma)$. The proof of Theorem 3.3 is complete. □

Similarly, we can prove:

Theorem 3.4. *Let $c > -\gamma$. If $f(z) \in K_n^*(\beta,\gamma)$, then $I_n(f(z)) \in K_n^*(\beta,\gamma)$.*

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A SPECIAL DIFFERENTIAL SUPERORDINATION IN THE COMPLEX PLANE

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Abstract. In this paper some differential superordinations, by applying the integral operator, are introduced.

1. **Introduction and preliminaries**

Denote by U the unit disc of the complex plane:

$$U = \{z \in \mathbb{C} :\mid z \mid< 1\} ,$$

for $0 < r < 1$, we let

$$U_r = \{z \in \mathbb{C} : |z| < r\} ,$$

and

$$\dot{U} = U \setminus \{0\} .$$

Let $\mathcal{H}(U)$ be the space of holomorphic functions in U.

For $a \in \mathbb{C}$ and $n \in \mathbb{N}^*$ we let:

$$\mathcal{H}[a, n] = \{f \in \mathcal{H}(U),\ f(z) = a + a_n z^n + \dots,\ z \in U\}$$

and

$$A_n = \left\{f \in \mathcal{H}(U),\ f(z) = z + a_{n+1} z^{n+1} + \dots,\ z \in U\right\}$$

with $A = A_1$.

Let f and F be members of $\mathcal{H}(U)$. The function f is said to be subordinate to F, or F is said to be superordinate to f, if there exists a function w analytic in

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U, with $w(0) = 0$ and $|w(z)| < 1$, and such that $f(z) = F(w(z))$; in such a case we write $f(z) \prec F(z)$. If F is univalent, then $f(z) \prec F(z)$ if and only if $f(0) = F(0)$ and $f(U) \subset F(U)$.

Let Ω be any set in the complex plane $\mathbb{C}$, let p be analytic in the unit disk U and let $\psi(\gamma, s, t; z) : \mathbb{C}^3 \times U \to \mathbb{C}$.

In a series of articles the S.S. Miller and P.T. Mocanu, D.J. Hallenbeck and S. Ruscheweyh have determined properties of functions p that satisfy the differential subordination

$$\{\psi(p(z), zp'(z), z^2p''(z); z) |\ z \in U\} \subset \Omega.$$

In this article we consider the dual problem of determining properties of functions p that satisfy the differential superordination

$$\Omega \subset \{\psi(p(z), zp'(z), z^2p''(z); z) |\ z \in U\}.$$

The following definitions, comments and lemmas have been presented in [5].

Definition 1.1. *Let $\varphi : \mathbb{C}^2 \times U \to \mathbb{C}$ and let h be analytic in U. If p and $\varphi(p(z), zp'(z); z)$ are univalent in U and satisfy the (first-order) differential superordination*

$$h(z) \prec \varphi(p(z), zp'(z); z) \tag{1.1}$$

then p is called a solution of the differential superordination. An analytic function q is called a subordinant of the solutions of the differential superordination, or more simply a subordinant if $q \prec p$ for all p satisfying (1.1). A univalent subordinant $\widetilde{q}$ that satisfies $q \prec \widetilde{q}$ for all subordinants q of (1.1) is said to be the best subordinant.

Note that the best subordinant is unique up to a rotation of U.

For Ω a set in $\mathbb{C}$, with φ and p as given in Definition 1.1, suppose (1.1) is replaced by

$$\Omega \subset \{\varphi(p(z), zp'(z); z) |\ z \in U\}. \tag{1.2}$$

Although this more general situation is a “differential containment”, the condition in (1.2) will also be referred to as a differential superordination, and the definitions of solution, subordinant and best dominant as given above can be extend to this generalization.

Before obtaining some of the main results we need to introduce a class of univalent functions defined on the unit disc that have some nice boundary properties.

Definition 1.2. *We denote by Q the set of functions f that are analytic and injective on $\overline{U} \setminus E(f)$, where*

$$E(f) = \left\{ \zeta \in \partial U : \lim_{z \to \zeta} f(z) = \infty \right\}$$

and are such that $f'(\zeta) \neq 0$ for $\zeta \in \partial U \setminus E(f)$.

The subclass of Q for which $f(0) = a$ is denoted by $Q(a)$.

Definition 1.3. *Let Ω be a set in $\mathbb{C}$ and $q \in \mathcal{H}[a,n]$ with $q'(z) \neq 0$. The class of admissible functions $\Phi_n[\Omega, q]$, consist of those functions $\varphi : \mathbb{C}^2 \times \overline{U} \to \mathbb{C}$ that satisfy the admissibility condition*

$$\varphi\left(q(z), \frac{zq'(z)}{m}; \zeta\right) \in \Omega \tag{1.3}$$

where $z \in U$, $\zeta \in \partial U$ and $m \geq n \geq 1$.

In order to prove the new results we shall use the following lemmas:

Lemma 1.4. [5] *Let h be convex in U, with $h(0) = a$, $\gamma \neq 0$ and $\mathrm{Re}\gamma \geq 0$. If $p \in \mathcal{H}[a,n] \cap Q$ and $p(z) + \frac{zp'(z)}{\gamma}$ is univalent in U with*

$$h(z) \prec p(z) + \frac{zp'(z)}{\gamma}$$

then

$$q(z) \prec p(z)$$

where

$$q(z) = \frac{\gamma}{nz^{\frac{\gamma}{n}}} \int_0^z h(t) t^{\frac{\gamma}{n}-1} dt$$

The function q is convex and is the best subordinant.

Lemma 1.5. [5] *Let q be convex in U and let h be defined by*

$$h(z) = q(z) + \frac{zq'(z)}{\gamma}, \quad z \in U,$$

with $\gamma \neq 0$, $\operatorname{Re} \gamma \geq 0$. *If* $p \in \mathcal{H}[a,n] \cap Q$, $p(z) + \dfrac{zp'(z)}{\gamma}$ *is univalent in* U, *and*

$$q(z) + \frac{zq'(z)}{\gamma} \prec p(z) + \frac{zp'(z)}{\gamma}, \quad z \in U$$

then

$$q(z) \prec p(z),$$

where

$$q(z) = \frac{\gamma}{nz^{\frac{\gamma}{n}}} \int_0^z h(t) t^{\frac{\gamma}{n}-1} dt$$

The function q is the best subordinant.

Definition 1.6. [6] *For $f \in A_n$ and $m \geq 0$, $m \in \mathbb{N}$, the operator $I^m f$ is defined by*

$$I^0 f(z) = f(z)$$

$$I^1 f(z) = If(z) = \int_0^z f(t) t^{-1} dt$$

$$I^m f(z) = I\left[I^{m-1} f(z)\right], z \in U.$$

Remark 1.7. *If we denote $l(z) = -\log(1-z)$, then*

$$I^m f(z) = [\underbrace{(l * l * \cdots * l)}_{n-times} * f](z), f \in \mathcal{H}(U), f(0) = 0$$

By " $*$ " we denote the Hadamard product or convolution (i.e. if $f(z) = \sum_{j=0}^{\infty} a_j z^j$, $g(z) = \sum_{j=0}^{\infty} b_j z^j$ *then* $(f * g)(z) = \sum_{j=0}^{\infty} a_j b_j z^j$*).*

Remark 1.8. $I^m f(z) = \int_0^z \int_0^{t_m} \dots \int_0^{t_2} \frac{f(t_1)}{t_1 t_2 \dots t_m} dt_1 dt_2 \dots dt_m$

Remark 1.9. $D^m I^m f(z) = I^m D^m f(z) = f(z)$, $f \in \mathcal{H}(U)$, $f(0) = 0$ *, where $D^m f$ is the Sălăgean differential operator.*

2. Main results

Theorem 2.1. *Let $h \in \mathcal{H}(U)$ be a convex function in U, with $h(0) = 1$ and $f \in A_n$, $n \in \mathbb{N}^*$.*

Assume that $[I^m f(z)]'$ is univalent with $\left[I^{m+1} f(z)\right]' \in \mathcal{H}[1, n] \cap Q$.

If

$$h(z) \prec [I^m f(z)]', z \in U \tag{2.1}$$

then

$$q(z) \prec [I^{m+1} f(z)]', z \in U \tag{2.2}$$

where

$$q(z) = \frac{1}{nz^{\frac{1}{n}}} \int_0^z h(t)\, t^{\frac{1}{n}-1} dt.$$

The function q is convex and is the best subordinant.

Proof. Let $f \in A_n$. By using the properties of the integral operator we have

$$I^m f(z) = z \left[I^{m+1} f(z)\right]', \ z \in U. \tag{2.3}$$

Differentiating (2.3), we obtain

$$[I^m f(z)]' = [I^{m+1} f(z)]' + z[I^{m+1} f(z)]'', \quad z \in U. \tag{2.4}$$

If we denote $p(z) = [I^{m+1} f(z)]'$, $z \in U$, then (2.4) becomes

$$[I^m f(z)]' = p(z) + zp'(z), \quad z \in U.$$

By using Lemma 1.4 for $\gamma = 1$ we deduce that

$$q(z) \prec p(z) = [I^{m+1} f(z)]'$$

where

$$q(z) = \frac{1}{nz^{\frac{1}{n}}} \int_0^z h(t) t^{\frac{1}{n}-1} dt.$$

Moreover the function q is the best subordinant. □

As a corollary of Theorem 2.1, we have the next corollary.

Corollary 2.2. *Let $h \in \mathcal{H}(U)$ be a convex function in U, with $h(0) = 1$ and $f \in A$.*

Assume that $[I^m f(z)]'$ is univalent with $[I^{m+1} f(z)]' \in \mathcal{H}[1, 1] \cap Q$.

$$h(z) \prec [I^m f(z)]',\ \ z \in U \tag{2.5}$$

then

$$q(z) \prec [I^{m+1} f(z)]', z \in U \tag{2.6}$$

where

$$q(z) = \frac{1}{z}\int_0^z h(t)dt.$$

The function q is convex and is the best subordinant.

Theorem 2.3. *Let $h \in \mathcal{H}(U)$ a convex function in U, with $h(0) = 1$ and $f \in A_n$.*

Assume that $[I^m f(z)]'$ is univalent with $\frac{I^m f(z)}{z} \in \mathcal{H}[1, n] \cap Q$.

If

$$h(z) \prec [I^m f(z)]',\ \ z \in U \tag{2.7}$$

then

$$q(z) \prec \frac{I^m f(z)}{z},\ \ z \in U \tag{2.8}$$

where

$$q(z) = \frac{1}{nz^{\frac{1}{n}}}\int_0^z h\,(t)\,t^{\frac{1}{n}-1}dt.$$

The function q is convex and is the best subordinant.

Proof. If we denote

$$p(z) = \frac{I^m f(z)}{z} \tag{2.9}$$

then

$$I^m f(z) = zp(z). \tag{2.10}$$

Differentiating (2.10) we obtain

$$[I^m f(z)]' = p(z) + zp'(z).$$

By using Lemma 1.4 we have

$$q(z) \prec p(z)$$

where

$$q(z) = \frac{1}{nz^{\frac{1}{n}}} \int_0^z h(t)\, t^{\frac{1}{n}-1} dt.$$

Also, the function q is the best subordinant. □

For $n = 1$ we have the following corollary.

Corollary 2.4. *Let $h \in \mathcal{H}(U)$ a convex function in U, with $h(0) = 1$ and $f \in A$. Assume that $[I^m f(z)]'$ is univalent with $\frac{I^m f(z)}{z} \in \mathcal{H}[1,1] \cap Q$.*

If

$$h(z) \prec [I^m f(z)]', \ z \in U \tag{2.11}$$

then

$$q(z) =\prec \frac{I^m f(z)}{z}, \ z \in U \tag{2.12}$$

where

$$q(z) = \frac{1}{z} \int_0^z h(t)dt.$$

The function q is convex and is the best subordinant.

Theorem 2.5. *Let q be a convex function in U and h defined by*

$$h(z) = q(z) + zq'(z), \ z \in U.$$

If $f \in A_n$, $[I^{m+1}]'$ is univalent in U, $[I^{m+1} f(z)]' \in \mathcal{H}[1,n] \cap Q$ and

$$h(z) \prec [I^{m+1} f(z)]' \tag{2.13}$$

then

$$q(z) \prec [I^{m+1} f(z)]' \tag{2.14}$$

where

$$q(z) = \frac{1}{nz^{\frac{1}{n}}} \int_0^z h(t) t^{\frac{1}{n}-1} dt.$$

The function q is the best subordinant.

Proof. Let $f \in A_n$. By using the properties of integral operator we have

$$[I^m f(z)]' = [I^{m+1} f(z)]' + z[I^{m+1} f(z)]'', z \in U. \tag{2.15}$$

If we denote

$$p(z) = [I^{m+1} f(z)]'$$

in (2.15) we obtain

$$[I^m f(z)]' = p(z) + zp'(z), z \in U. \tag{2.16}$$

By using Lemma 1.5 we obtain:

$$q(z) \prec [I^{m+1} f(z)]'$$

where

$$q(z) = \frac{1}{nz^{\frac{1}{n}}} \int_0^z h(t) t^{\frac{1}{n}-1} dt.$$

Moreover this is the best subordinant. □

For $n = 1$, we have the following corollary.

Corollary 2.6. *Let q convex in U and h defined by*

$$h(z) = q(z) + zq'(z), \;\; z \in U.$$

If $f \in A$, $[I^{m+1} f(z)]'$ is univalent in U, $[I^{m+1} f(z)]' \in \mathcal{H}[1,1] \cap Q$ and

$$h(z) = q(z) + zq'(z) \prec [I^m f(z)]' \tag{2.17}$$

then

$$q(z) \prec [I^{m+1} f(z)]' \tag{2.18}$$

where

$$q(z) = \frac{1}{z} \int_0^z h(t) dt.$$

Theorem 2.7. *Let q a convex function in U and h defined by*

$$h(z) = q(z) + zq'(z).$$

If $f \in A_n$, $[I^m f(z)]'$ is univalent in U, $\frac{I^m f(z)}{z} \in \mathcal{H}[1,n] \cap Q$ and

$$h(z) \prec [I^m f(z)]' \tag{2.19}$$

then

$$q(z) \prec \frac{I^m f(z)}{z} \tag{2.20}$$

where

$$q(z) = \frac{1}{nz^{\frac{1}{n}}} \int_0^z h(t) t^{\frac{1}{n}-1} dt.$$

The function q is convex and is the best subordinant.

Proof. If

$$p(z) = \frac{I^m f(z)}{z}$$

then

$$[I^m f(z)]' = p(z) + zp'(z)$$

From relation (2.19), we have

$$q(z) + zq'(z) \prec p(z) + zp'(z).$$

By using Lemma 1.5 we obtain

$$q(z) \prec \frac{I^m f(z)}{z}$$

where

$$q(z) = \frac{1}{nz^{\frac{1}{n}}} \int_0^z h(t) t^{\frac{1}{n}-1} dt$$

and it is the best subordinant. □

As a corollary of this theorem, we have:

Corollary 2.8. *Let q a convex function in U and h defined by*

$$h(z) = q(z) + zq'(z).$$

If $f \in A$, with $[I^m f(z)]'$ univalent in U, $\frac{I^m f(z)}{z} \in \mathcal{H}[1,1] \cap Q$ and

$$h(z) \prec [I^m f(z)]' \tag{2.21}$$

then

$$q(z) \prec \frac{I^m f(z)}{z} \tag{2.22}$$

where

$$q(z) = \frac{1}{z} \int_0^z h(t) dt.$$

The function q is the best subordinant.

Remark 2.9. *For the special case of the function*

$$h(z) = \frac{1 + (2\alpha - 1)z}{1 + z}$$

this results were obtained in [1].

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ON THE $\delta(\varepsilon)$-STABLE OF COMPOSED RANDOM VARIABLES

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Abstract. Let ξ be a random variable (r.v.) with the characteristic function $\varphi(t)$ and ν be a r.v. with the generating function $a(z)$, ν is independent of ξ. It is known (see [1]) that the composed r.v. η of ξ and ν (denote by $\eta =< \nu, \xi >$) is the r.v. having the characteristic function $\psi(t) = a[\varphi(t)]$. The r.v. ν is called to be the first component of η and ξ is called to be the second component of η. In this paper, we shall investigate the changes of the distribution function of the composed r.v. η if we have the small changes of the distribution function of the first component ν or the second component ξ of η.

1. **Introduction**

Let ξ be a random variable (r.v.) with the characteristic function $\varphi(t)$ and the distribution function $F(x)$. Let ν be a r.v. independent of ξ and has the generating function $a(z)$ with the distribution function $A(x)$. It is known (see [1]) that the composed r.v. of ν and ξ is denote by

$$\eta =< \nu, \xi > \tag{1.1}$$

and has the characteristic function

$$\psi(t) = a[\varphi(t)]. \tag{1.2}$$

The r.v. ν is called to be the first component and the r.v. ξ is called to be the second component of the r.v. η.

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Example 1.1. Let us consider the integer valued nonegative r.v.:

$$\eta = \sum_{k=1}^{\nu} \xi_k \tag{1.3}$$

where $\xi_1, \xi_2, ...$ are i.i.d random variables have the same the distribution function with r.v. ξ, ν is a positive value r.v., independent of all ξ_k $(k = 1, 2, ...)$, η is composed r.v. of ν and ξ and $\eta =< \nu, \xi >$.

In many practical problems, we always meet this composed random variable (special in queuing theory - see [7]) where ν is assumed having Poisson law and ξ has the Exponential law. But, in practice, we also know best that ν has only a distribution function which arrives at Poisson law or ξ has a distribution function which arrives at Exponential law. Our question is the following: If we have the small changes of the distribution function of ν or ξ, whether the distribution function of $\eta =< \nu, \xi >$ shall has the small changes or not?

The composed r.v. η is called to be stable if the small changes in the distribution function of ν or ξ lead to the small changes in the distribution function of η.

More detail we have the following definitions:

Definition 1.1. Suppose that $\Psi(x)$ and $\psi(t)$ are the distribution function and characteristic function of η , $A_\varepsilon(x)$ and $a_\varepsilon(z)$ are the distribution function and the generating function of ν_ε such that

$$\rho(A; A_\varepsilon) = \sup_{x \in R} |A(x) - A_\varepsilon(x)| < \varepsilon$$

(for some sufficiently small positive number ε).

Put $\Psi^1_\varepsilon(x)$ be the distribution of the composed r.v. $< \nu_\varepsilon; \xi >$. The composed r.v. η is called to be $\delta_1(\varepsilon)$-stable on the first component with metric $\rho(.,.)$ if and only if

$$\rho(\Psi; \Psi^1_\varepsilon) \leqslant \delta_1(\varepsilon) \qquad (\delta_1(\varepsilon) \to 0 \quad when \quad \varepsilon \to 0).$$

Definition 1.2. Suppose that $F_\varepsilon(x)$ and $\varphi_\varepsilon(t)$ are the distribution function and the characteristic function of ξ_ε such that $\rho(F_\varepsilon; F) < \varepsilon$ (for some sufficiently small positive number ε) and $\Psi^2_\varepsilon(x)$ is distribution function with the characteristic function $\psi^2_\varepsilon(t)$ of the composed r.v. $< \nu; \xi_\varepsilon >$.

The composed r.v. η is called to be $\delta_2(\varepsilon)$-stable on the second component with metric $\rho(.,.)$ if and only if $\rho(\Psi;\Psi^2_\varepsilon) \leqslant \delta_2(\varepsilon) \qquad (\delta_2(\varepsilon) \to 0 \quad \textit{when} \quad \varepsilon \to 0)$.

Remark 1.1. In some following stability theorems, metric $\rho(.,.)$ may be changed by metric $\lambda_0(.,.)$ (See [6])

$$\lambda_0(\Psi;\Psi^2_\varepsilon) = \sup_{t\in R} |\psi(t) - \psi^2_\varepsilon(t)|.$$

2. Stability Theorems

Theorem 2.1. *If the first component of the composed r.v. η has the generating function $a(z)$ which satisfies the following condition:*

$$|a(z_1) - a(z_2)| \leqslant K|z_2 - z_1|, \tag{2.1}$$

for all complex numbers $z_1, z_2, |z_1| \leqslant 1, |z_2| \leqslant 1$ and K is a constant, then η shall be $K\varepsilon$-stable on the second component with metric $\lambda_0(.,.)$.

Proof. According to the hypothesis $\lambda_0(F, F_\varepsilon) < \varepsilon$,

$$|\varphi(t) - \varphi_\varepsilon(t)| < \varepsilon, \quad \forall t$$

so that

$$|\psi(t) - \psi^2_\varepsilon(t)| = |a[\varphi(t)] - a[\varphi_\varepsilon(t)]| \leqslant K|\varphi(t) - \varphi_\varepsilon(t)| \leqslant K\varepsilon$$

for all t. That means

$$\lambda_0(\Psi;\Psi^2_\varepsilon) \leqslant K\varepsilon. \tag{2.2}$$

Example 2.1. If ν is the r.v. having the Poisson law with parameter $\lambda > 0$ and $\varphi_1(t)$ is the characteristic function of the r.v. ξ having exponential law with parameter $\theta > 0$ then the composed r.v. $\eta =< \nu; \xi >$ shall be $e^{4\lambda}\varepsilon$-stable on the second component with metric $\lambda_0(.,.)$ (where $e^{4\lambda}\varepsilon$ is a constant).

Example 2.2. If ν is r.v. having the binomial distribution function with the parameters p, n and ξ has the exponential distribution function with parameter $\theta > 0$ then $\eta =< \nu; \xi >$ shall be $np(1+2p)^{n-1}\varepsilon$-stable on the second component with metric $\lambda_0(.,.)$ (where $np(1+2p)^{n-1}\varepsilon$ is a constant).

Example 2.3. If ν is r.v. having the geometric distribution function with the parameters p $(p = 1 - q)$ and ξ has the exponential distribution function then $\eta =< \nu; \xi >$ shall be $\frac{q}{p}\varepsilon$-stable on the second component with metric $\lambda_0(.,.)$ (where $\frac{q}{p}\varepsilon$ is a constant).

All above examples are immediate from Theorem 2.1 since the corresponding generating functions clearly satisfy the condition (2.1). Indeed, for instance, to show Example 2.3, let $a_3(z)$ be the generating function of geometric law, i.e.:

$$a_3(z) = p[1 - qz]^{-1}.$$

For any complex numbers z_1, z_2 satisfying $|z_1| \leqslant 1, |z_2| \leqslant 1$; we have the following estimation:

$$|a_3(z_1) - a_3(z_2)| = |\frac{p}{1 - qz_1} - \frac{p}{1 - qz_2}| \leqslant \frac{pq|z_1 - z_2|}{|1 - qz_1||1 - qz_2|}$$

Notice that

$$|1 - qz_1| \geq |1 - q|z_1|| \geq 1 - q, \quad for \quad all \quad |z_1| \leqslant 1$$

$$|1 - qz_2| \geq |1 - q|z_2|| \geq 1 - q, \quad for \quad all \quad |z_2| \leqslant 1$$

It follows that

$$|a_3(z_1) - a_3(z_2)| \leqslant \frac{pq|z_1 - z_2|}{(1 - q)^2}.$$

Thus $a_3(z)$ satisfies the condition (2.1) with the constant $K = \dfrac{pq}{(1 - q)^2}$.

Theorem 2.2. (See [2]) *Suppose $\eta =< \nu, \xi >$, ν has the distribution function $A(x)$ such that*

$$\mu_A^\alpha = \int_0^{+\infty} z^\alpha dA(z) < +\infty, \quad \forall \alpha : 0 < \alpha < 1$$

and ξ has the stable law with the characteristic function:

$$\varphi(t) = \exp\{i\mu t - c|t|^\alpha (1 + i\beta \frac{t}{|t|} \omega(|t|; \alpha))\}, \tag{2.3}$$

where c, μ, α, β are real numbers, $c \geq 0, |\beta| \leqslant 1$ and

$$1 < \alpha_1 \leqslant \alpha \leqslant 2; \quad \omega(|t|; \alpha) = tg\frac{\alpha\pi}{2}. \tag{2.4}$$

For every $\varepsilon-$ sufficiently small positive number is given, such that

$$\varepsilon < (\frac{\pi}{3c_2})^3, c_1 = (c + |\beta||tg\frac{\alpha_1\pi}{2}| + |\mu|) \tag{2.5}$$

η shall be $K_1\varepsilon^{1/6}$-stable on the first component with metric $\rho(.,.)$.

Theorem 2.3. *Assume that ν has any distribution function $A(z)$ which has moment $\mu_A = \int_0^\infty zdA(z) < +\infty$, ξ has the stable law with the characteristic function satisfying condition (2.3), (2.4). Then, the composed r.v. $\eta =< \nu, \xi >$ shall be $K_2(\varepsilon)^{1/8}$-stable on the second component with metric $\rho(.,.)$ for some ε is sufficiently small number satisfying condition (2.5).*

Lemma 2.1. Let a be a complex number, $a = \rho e^{i\theta}$, such that

$$|\theta| \leqslant \frac{\pi}{3}, 0 \leqslant \rho \leqslant 1. \tag{2.6}$$

Then we always have following estimation

$$|a^t - 1| \leqslant \frac{\sqrt{14}t|a-1|}{(1-|a-1|)} \quad \text{for every} \quad t > 0,\ t \in R. \tag{2.7}$$

Proof. Since $a = \rho(cos\theta + isin\theta)$, it follows that

$$|a^t - 1|^2 = (\rho^t cost\theta - 1)^2 + (\rho^t sint\theta)^2. \tag{2.8}$$

We also have $(\rho^t cost\theta - 1) = (\rho^t - 1)cost\theta + (cost\theta - 1)$.

Notice that $|1 - cosx| \leqslant |x|$ for all $x \in R$, thus

$$|\rho^t cost\theta - 1| \leqslant |\rho^t - 1| + |t\theta|.$$

On the other hand, since $|sinu| \leqslant |u|$ for all u$\in R$, from (2.8) we shall have

$$|a^t - 1|^2 \leqslant 2|\rho^t - 1|^2 + 2t^2\theta^2 + \rho^{2t}(t\theta)^2. \tag{2.9}$$

We can see $|a-1|^2 = (\rho cos\theta - 1)^2 + \rho^2 sin^2\theta$. It follows that

$$|\rho sin\theta| \leqslant |a-1|. \tag{2.10}$$

Further more

$$||a| - 1| \leqslant |a-1| \Rightarrow |\rho - 1| \leqslant |a-1|.$$

Hence

$$|\rho - 1| \geq -|a-1| \Rightarrow \rho \geq 1 - |a-1|. \tag{2.11}$$

From (2.10) we obtain $|sin\theta| \leqslant \dfrac{|a-1|}{\rho} \leqslant \dfrac{|a-1|}{1-|a-1|}$.

For every $\theta, |\theta| \leqslant \dfrac{\pi}{3}$, we always have inequality: $|sin\theta| \geq \dfrac{|\theta|}{2}$. So, from (2.)

$$|\theta| \leqslant \frac{2|a-1|}{1-|a-1|}.$$

From (2.9) and (2.11)

$$|a^t - 1|^2 \leqslant 2|\rho^t - 1|^2 + \frac{8t^2|a-1|^2}{(1-|a-1|)^2} + 4\frac{\rho^{2t}t^2|a-1|^2}{(1-|a-1|)^2}. \tag{2.12}$$

For all $t \geq 0$, the following inequality holds

$$1 - \rho^t \leqslant \frac{t(1-\rho)}{\rho}.$$

Notice $|1-\rho| = |1-|a|| \leqslant |a-1|$. We have

$$|1-\rho^t| \leqslant \frac{t|a-1|}{\rho}. \tag{2.13}$$

Hence by (2.12) and (2.13), we shall get: $|a^t - 1|^2 \leqslant \dfrac{14t^2|a-1|^2}{(1-|a-1|)^2}$.

Proof of theorem 2.3. At first, we shall estimate $|\psi(t) - \psi_\varepsilon(t)|$ for all $t, |t| \leqslant T(\varepsilon)$ where $T(\varepsilon) \to \infty$ when $\varepsilon \to 0$. At last, using Esseen's inequality (see [4]) we shall have the conclusion. Throughout the proof, we shall denote by $c_1, c_2, .., c_{14}, c_{15}$ are constants independent of ε. At first, we have:

$$|\psi(t) - \psi_\varepsilon(t)| = |a[\varphi(t)] - a[\varphi_\varepsilon(t)]| = |\int_0^{+\infty}[\varphi^z(t) - \varphi_\varepsilon^z(t)]dA(z)|$$

$$\leqslant |\int_1^{+\infty}[\varphi^z(t) - \varphi_\varepsilon^z(t)]dA(z)| + |\int_0^1[\varphi^z(t) - \varphi_\varepsilon^z(t)]dA(z)| = J_1 + J_2. \tag{2.14}$$

Consider J_1: Using the Lagrange-formula of the function $[\varphi(t)]^z$ (for $|z| \geq 1$), we get

$$|\varphi^z(t) - \varphi_\varepsilon^z(t)| = z|\tilde{\varphi}(t)|^{z-1}|\varphi(t) - \varphi_\varepsilon(t)|, \tag{2.15}$$

where $\tilde{\varphi}(t)$ is a complex number satisfying the condition $|\tilde{\varphi}(t)| \leqslant \max\{|\varphi(t)|; |\varphi_\varepsilon(t)|\}$. Notice that:

$$|\tilde{\varphi}(t)|^{z-1} \leqslant |\tilde{\varphi}(t)| \leqslant 1 \quad for \ \ all \quad z : 2 \leqslant z < +\infty$$

and

$$|\tilde{\varphi}(t)|^{z-1} \leqslant |\tilde{\varphi}(t)|^{0} = 1 \quad for \;\; all \quad z : 1 \leqslant z < 2,$$

$$i.e., \quad |\tilde{\varphi}(t)|^{z-1} \leqslant 1 \quad for \;\; all \quad z : 1 \leqslant z < +\infty. \tag{2.16}$$

We shall have

$$|\varphi(t) - \varphi_{\varepsilon}(t)| = |\int_{-\infty}^{+\infty} e^{itx} d[F(x) - F_{\varepsilon}(x)]|.$$

For some $N = N(\varepsilon)$ (it also be chosen later), we also have the following estimation:

$$|\varphi(t) - \varphi_{\varepsilon}(t)| = |\int_{-N}^{+N} e^{itx} d[F(x) - F_{\varepsilon}(x)]| + 2\int_{N}^{+\infty} d[F(x) + F_{\varepsilon}(x)]$$

$$\leqslant |[F(x) - F_{\varepsilon}(x)]||_{-N}^{N} + |\int_{-N}^{N} [F(x) - F_{\varepsilon}(x)] d(e^{itx})| + 2\int_{N}^{+\infty} |\frac{x}{N}| d[F(x) + F_{\varepsilon}(x)]$$

$$\leqslant 2\varepsilon + \int_{-N}^{N} \varepsilon |t| dx + 2.\frac{\mu_F + \mu_{F_{\varepsilon}}}{N(\varepsilon)}.$$

(where $\mu_F = \int_{-\infty}^{+\infty} |x| dF(x) < +\infty$ and $\mu_{F_{\varepsilon}} = \int_{-\infty}^{+\infty} |x| dF_{\varepsilon}(x) < +\infty$). Now, for all t, $|t| \leqslant T(\varepsilon)$ (where $T(\varepsilon) \to \infty$ when $\varepsilon \to 0$, $T(\varepsilon)$ will be chosen later) we always have

$$|\varphi(t) - \varphi_{\varepsilon}(t)| \leqslant 2\varepsilon + 2N(\varepsilon)T(\varepsilon)\varepsilon + 2.\frac{\mu_F + \mu_{F_{\varepsilon}}}{N(\varepsilon)}. \tag{2.17}$$

Now, consider J_2. Using the Lagrange-formula of the function $[\varphi(t)]^z$ for all $z, 0 \leqslant z \leqslant 1$ at $\varphi_{\varepsilon}(t)$, we get

$$|\varphi^{z}(t) - \varphi_{\varepsilon}^{z}(t)| = \frac{z}{|\tilde{\tilde{\varphi}}(t)|^{1-z}} |\varphi(t) - \varphi_{\varepsilon}(t)|. \tag{2.18}$$

For every ε- satisfying condition (2.5) we shall choose $T(\varepsilon)$ such that

$$\min\{|\varphi(t)|; |\varphi_{\varepsilon}(t)|\} \geq c_4 \varepsilon^{1/2} \geq |\varphi(t) - \varphi_{\varepsilon}(t)| \text{ for all } t, \; |t| \leqslant T(\varepsilon),$$

(where c_4 is a constant independent of ε).

Because $\varphi(t)$ is the characteristic function of stable law satisfying condition (2.3), so we have the following estimations:

$$|\ln \varphi(t)| \leqslant |\mu||t| + |t|^{\alpha}(c + c|\beta||tg\frac{\alpha_1 \pi}{2}|) \leqslant |\mu||t| + c_2|t|^{\alpha} \leqslant T^{\alpha}(\varepsilon).$$

Thus , $\quad |\varphi(t)| = |e^{\ln \varphi(t)}| \geq e^{-|\ln \varphi(t)|} \geq e^{-c_2 T^{\alpha}(\varepsilon)}.$

If we choose:

$$T(\varepsilon) = [\frac{1}{c_2} \ln \frac{1}{\varepsilon^{1/8}}]^{\frac{1}{\alpha}} \qquad (T(\varepsilon) \to \infty \ \ when \ \ \varepsilon \to 0). \tag{2.19}$$

Then $c_2 T^{\alpha}(\varepsilon) \leqslant \ln \dfrac{1}{\varepsilon^{1/8}}$ (for all $\alpha > 1$) and $|\varphi(t)| \geq e^{-c_2 T^{\alpha}(\varepsilon)} \geq \varepsilon^{1/8}$. Now we shall choose $N(\varepsilon) = \varepsilon^{-1/2}$ $(N(\varepsilon) \to +\infty)$ when $\varepsilon \to 0$. Thus

$$2\varepsilon T(\varepsilon) N(\varepsilon) \leqslant \frac{2}{c_2^{1/\alpha}} \ln \frac{1}{\varepsilon^{1/8}} . \varepsilon^{1/2} \leqslant c_3 \varepsilon^{3/8}. \tag{2.20}$$

Put

$$c_0(\varepsilon) = 2\varepsilon + 2\varepsilon T(\varepsilon) N(\varepsilon) + 2\Big(\frac{\mu_F + \mu_{F_\varepsilon}}{N(\varepsilon)}\Big).$$

We shall have the estimation

$$c_0(\varepsilon) \leqslant 2\varepsilon + c_3 \varepsilon^{3/8} + 2(\mu_F + \mu_{F_\varepsilon}) \varepsilon^{1/2} \leqslant c_4 \varepsilon^{1/2}. \tag{2.21}$$

That means, the condition:

$$c_4 \varepsilon^{1/2} \geq |\varphi(t) - \varphi_\varepsilon(t)| \tag{2.22}$$

shall be satisfied for every t, $|t| \leqslant T(\varepsilon)$.

Notice that, from $|\varphi(t) - \varphi_\varepsilon(t)| \leqslant c_4 \varepsilon^{1/2}$ we always have

$$||\varphi(t)| - |\varphi_\varepsilon(t)|| \leqslant |\varphi(t) - \varphi_\varepsilon(t)| \quad (See [5]).$$

So

$$|\varphi(t)| - |\varphi_\varepsilon(t)| \leqslant |\varphi(t) - \varphi_\varepsilon(t)| \leqslant c_4 \varepsilon^{1/2}$$

and

$$|\varphi_\varepsilon(t)| \geq |\varphi(t)| - c_4 \varepsilon^{1/2} \geq \varepsilon^{1/8} - c_4 \varepsilon^{1/2} \geq c_5 \varepsilon^{1/8}.$$

That also means, the estimation $\min\{|\varphi(t)|; |\varphi_\varepsilon(t)|\} \geq c_4 \varepsilon^{1/2}$ shall be satisfied.

On the other hand, for every complex number z_3 which belong to the interval joining z_1 and z_2 we have only two cases:

1) $|z_3| \geq \min\{|z_1|; |z_2|\}$

2) $|z_3| \geq \sqrt{\max\{|z_1|^2; |z_2|^2\} - \dfrac{|z_1 - z_2|^2}{2}}$.

Therefore

$$\tilde{\tilde{\varphi}}(t) \geq \min\{|\varphi(t)|; |\varphi_\varepsilon(t)|\} \geq c_5 \varepsilon^{1/8}$$

or

$$|\tilde{\tilde{\varphi}}(t)| \geq \sqrt{c_5^2\varepsilon^{2/8} - \frac{c_4\varepsilon^{2/4}}{2}} \geq c_6\varepsilon^{1/8}$$

i.e., $|\tilde{\tilde{\varphi}}(t)| \geq c_6\varepsilon^{1/8}$ in both above cases. Besides that, we always have,

$$|\tilde{\tilde{\varphi}}(t)|^{1-z} \geq |\tilde{\tilde{\varphi}}(t)| \qquad \text{for all complex number } z, 0 \leqslant |z| \leqslant 1. \tag{2.23}$$

Taking into account (2.18), (2.20), (2.23) we shall get

$$J_2 = |\int_0^1 |\varphi^z(t) - \varphi_\varepsilon^z(t)|dA(z)| \leqslant \int_0^1 |z|\frac{|\varphi(t) - \varphi_\varepsilon(t)|}{|\tilde{\tilde{\varphi}}(t)|}dA(z) \leqslant c_7\varepsilon^{3/8}. \tag{2.24}$$

Combine (2.14), (2.16), (2.17), (2.24) we can see that

$$J_1 + J_2 \leqslant \mu_A c_0(\varepsilon) + c_7\varepsilon^{3/8} \leqslant c_8\varepsilon^{3/8}. \tag{2.25}$$

Thus, for all t, $|t| \leqslant T(\varepsilon)$ (which is chosen from (2.19)) we always have the estimation

$$|\psi(t) - \psi_\varepsilon(t)| \leqslant c_8\varepsilon^{3/8}. \tag{2.26}$$

Now we shall choose $\delta = \delta(\varepsilon)$ be a positive number ($\delta(\varepsilon) \to 0$ when $\varepsilon \to 0$) such that

$$\max\{|\arg[\varphi(t)]|; |\arg[\varphi_\varepsilon(t)]|\} \leqslant \frac{\pi}{3}, \qquad \forall t, |t| \leqslant \delta(\varepsilon). \tag{2.27}$$

We always have

$$\int_{-T(\varepsilon)}^{T(\varepsilon)} |\frac{\psi(t) - \psi_\varepsilon(t)}{t}|dt \leqslant \int_{-\delta(\varepsilon)}^{\delta(\varepsilon)} |\frac{\psi(t) - \psi_\varepsilon(t)}{t}|dt + \int_{\delta(\varepsilon) \leqslant |t| \leqslant T(\varepsilon)} |\frac{\psi(t) - \psi_\varepsilon(t)}{t}|dt.$$

Consider $|\psi(t) - \psi_\varepsilon(t)|$ on $|t| \leqslant \delta(\varepsilon)$, we have

$$|\psi(t) - \psi_\varepsilon(t)| \leqslant \int_0^{+\infty} |\varphi^z(t) - 1|dA(z) + \int_0^{+\infty} |\varphi_\varepsilon^z(t) - 1|dA(z). \tag{2.28}$$

In $|t| \leqslant \delta(\varepsilon)$, with $\delta(\varepsilon)$ is chosen from the condition (2.27), the condition (2.6) of Lemma 2.1 shall be satisfied (with $a = \varphi(t)$), we shall use Lemma 2.1 and we have the following estimations

$$|\varphi^z(t) - 1| \leqslant \frac{\sqrt{14}z|\varphi(t) - 1|}{(1 - |\varphi(t) - 1|)}$$

and,

$$|\varphi_\varepsilon^z(t) - 1| \leqslant \frac{\sqrt{14}z|\varphi_\varepsilon(t) - 1|}{(1 - |\varphi_\varepsilon(t) - 1|)}. \tag{2.29}$$

for all complex numbers z.

Notice that, for all t:

$$|e^{itx} - 1| = |(\cos tx - 1)^2 + \sin^2 tx| = 2\sin\frac{tx}{2} \leqslant |t||x|. \tag{2.30}$$

In $|t| \leqslant \delta(\varepsilon)$ with $\delta(\varepsilon) \to 0$ when $\varepsilon \to 0$, so we always have

$$|\varphi(t) - 1| \leqslant \frac{1}{2}, \quad |\varphi_\varepsilon(t) - 1| \leqslant \frac{1}{2},$$

and therefore, from (2.7)

$$\int_0^{+\infty} |\varphi_\varepsilon^z(t) - 1| dA(z) \leqslant \int_0^{+\infty} \frac{\sqrt{14}|\varphi(t) - 1|}{(1 - \frac{1}{2})} dA(z) \leqslant c_9|t|.$$

Similarly,

$$\int_0^{+\infty} |\varphi_\varepsilon^z(t) - 1| dA(z) \leqslant c_{10}|t|.$$

That means

$$\int_{-\delta(\varepsilon)}^{\delta(\varepsilon)} |\frac{\psi(t) - \psi_\varepsilon(t)}{t}| dt \leqslant c_{11}\delta(\varepsilon). \tag{2.31}$$

Now, if we choose

$$\delta(\varepsilon) = \frac{1}{c_2}\varepsilon^{1/4} \ln\frac{1}{\varepsilon^{1/8}} \quad (\delta(\varepsilon) \to 0 \text{ when } \varepsilon \to 0 \tag{2.32}$$

with ε satisfying (2.5) , then we have

$$\int_{-\delta(\varepsilon)}^{\delta(\varepsilon)} |\frac{\psi(t) - \psi_\varepsilon(t)}{t}| dt \leqslant c_{11}\delta(\varepsilon) \leqslant c_{13}\varepsilon^{1/4}\frac{1}{\varepsilon^{1/8}} \leqslant c_{14}\varepsilon^{1/8}. \tag{2.33}$$

On the other hand, from (2.19) and (2.26)

$$\int_{\delta(\varepsilon)\leqslant|t|\leqslant T(\varepsilon)} |\frac{\psi(t) - \psi_\varepsilon(t)}{t}| dt \leqslant c_8\varepsilon^{3/8} \int_{\delta(\varepsilon)}^{T(\varepsilon)} \frac{1}{t} dt = c_8\varepsilon^{3/8} \ln\frac{T(\varepsilon)}{\delta(\varepsilon)}, \tag{2.34}$$

and notice that

$$c_8\varepsilon^{3/8} \ln\frac{T(\varepsilon)}{\delta(\varepsilon)} \leq c_8\varepsilon^{3/8} \ln\frac{T^\alpha(\varepsilon)}{\delta(\varepsilon)} \leq c_8\varepsilon^{3/8}\frac{1}{\varepsilon^{1/4}} = c_8\varepsilon^{1/8}. \tag{2.35}$$

With $T(\varepsilon)$ and $\delta(\varepsilon)$ chosen from conditions (2.19) and (2.32), we shall have:

$$\int_{\delta(\varepsilon)\leqslant|t|\leqslant T(\varepsilon)} |\frac{\psi(t) - \psi_\varepsilon(t)}{t}| dt \leqslant c_8\varepsilon^{3/8} \ln\frac{T(\varepsilon)}{\delta(\varepsilon)} \leqslant c_8\varepsilon^{1/8}. \tag{2.36}$$

By using Esseen's inequality (see [4]) and combine (2.33) with (2.36) we can conclude that

$$\rho(\Psi;\Psi_\varepsilon) \leqslant c_{14}\varepsilon^{1/8} + c_8\varepsilon^{1/8} \leqslant K_2\varepsilon^{1/8},$$

where K_2 is a constant independent of ε. This completes the proof of Theorem 2.3.

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SINGULARITY OF A BOUNDARY VALUE PROBLEM OF THE ELASTICITY EQUATIONS IN A POLYHEDRON

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Abstract. In this work we study the regularity of a boundary value problem governed by the Lamé equations in a cylindrical domain. By studying the longitudinal displacement singularity along an edge and the perpendicular displacement singularity to the same edge, we arrive to describe the behavior singular of solutions of the *Lamé* equations in a polyhedron.

1. **Introduction**

Let Ω be homogeneous, elastic and isotropic medium occupying a bounded domain in $\mathbb{R}^2$, limited by straight polygonal boundary Γ which is supposed to be regular, $\Gamma = \bigcup_{j=1}^{N} \Gamma_j$, $\Gamma_i \cap \Gamma_j = \varnothing, \forall i \neq j$, where $\Gamma_j =]S_j, S_{j+1}[$, and S_j are the different corners of Ω. $\omega_j, 0 < \omega_j \leq 2\pi$, $j = 0, ..., N$ represent the opening of the angle that makes Γ_j and Γ_{j+1} toward the interior of Ω, η^j and τ^j represent the unit outward normal vector and the tangent vector on Γ_j, respectively.

L is the *Lamé* operator defined by:

$$Lu = \mu \Delta u + (\lambda + \mu) \nabla .divu,$$

where u, f represent the displacement vector, and external forces density respectively. $\Sigma(u)$ is the stress tensor given by *Hook*'s law using *Lamé* coefficients λ and μ ($\lambda > 0$ and $\lambda + \mu \geq 0$)

$$\Sigma(u) = (\sigma_{ij}(u))_{ij}, \text{ where } \sigma_{ij}(u) = 2\mu\varepsilon_{ij}(u) + \lambda tr(\varepsilon(u))\delta_{ij},$$

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where δ_{ij} is the *Kronecker* symbol and $\varepsilon_{ij}(u) = \frac{1}{2}(\partial_i x_j + \partial_j x_i)$ is the linearized tensor of deformation. We will suppose $\nu_0 = \frac{1}{2-\nu}$, where ν designates the *Poisson* coefficient such as $0 < \nu < \frac{1}{2}$.

In the case of a polyhedron, we consider a domain Q of $\mathbb{R}^3$, limited by straight polyhedral boundary Σ. It is considered a particular edge, denoted A, of Σ. It is assumed to fix ideas that A is carried by the axis $z^{'}Oz$, the adjacent faces Γ_0 and Γ_ω are carried by the plans $\{y = 0\}$ and $\{y = ax\}$, respectively. The dihedral so definite has for measure ω toward the interior of Q.

It is indispensable to signal that the results that will be demonstrated in this work are not verified to the corners neighborhood. That's why, we fix an opened interval I, whose closure is interior to A. Besides we fix a neighborhood U of the origin O in $Q \cap \{z = 0\}$, such as $\overline{U} \times \overline{I}$ doesn't have any corners of Q. $\eta^{'} = (\eta_1, \eta_2, \eta_3)^t = (\eta, \eta_3)^t$ and $\tau^{'} = (\tau_1, \tau_2, \tau_3)^t = (\tau, \tau_3)^t$ represent the unit outward normal vector and the tangent vector on Σ respectively.

We consider the corresponding cylinder $Q = \Omega \times \mathbb{R}$ which has an edge along $z^{'}Oz$.

For $f \in L^2(Q)^3$, the problem considered here consists of finding the displacement field $u : \Omega \longrightarrow \mathbb{R}^3$, if possible in $H^2(Q)^3$, satisfying:

$$(P)\begin{cases} Lu + f = 0 \text{ in } Q \\ \left(u.\eta^{'}, \left(\Sigma(u).\eta^{'}\right).\tau^{'}\right) = 0, \text{ on } \Sigma \end{cases},$$

Or equivalent variational form:

$$(P_V)\begin{cases} \text{Find } u \in V \text{ such as} \\ a(u,v) = \ell(v), \text{ for all } v \in V \end{cases}$$

where

$$\begin{aligned} a(u,v) &= \sum_{i,j=1}^{3} \int_Q \sigma_{ij}(u)\varepsilon_{ij}(v)dx, \ \ell(v) = \sum_{i=1}^{3} \int_Q f_i v_i dx, \\ V &= \left\{ v \in H^1(Q)^3;\ u.\eta = 0, \text{ in } \Sigma \right\} \end{aligned}$$

It is assumed that u, therefore as f, is to bounded support in the direction of z.

To describe the behavior of u along an edge, it is necessary to introduce, as in P. Grisvard [5], the following three convolution kernels, in z :

$$\begin{aligned} K_{\lambda,\mu,r}(r,z) &= \frac{r\sqrt{1+\nu}}{\pi\left[r^2+(1+\nu)z^2\right]}, \\ K_{\lambda,\mu,\theta}(r,z) &= \frac{r}{\pi\left[r^2+z^2\right]}, \\ K_{\lambda,\mu,z}(r,z) &= \frac{r\sqrt{1+\nu}}{\pi\left[(1+\nu)r^2+z^2\right]} \end{aligned}$$

1.1. **Singular solutions.** In B. Benabderrahmane [1] and P. Grisvard [8], there was found that the solutions of the problem (P), (in the case $f=0$) are characterized by the following transcendent equation (1.1) :

$$\sin^2\alpha\omega = \sin^2\omega,\ \alpha \neq 0, \neq \pm 1 \tag{1.1}$$

where $\operatorname{Re}\alpha \in \left]0,1\right[$.

It is easy to verify that the solutions of the transcendent equation (1.1) are given by

$$\alpha_\ell = \frac{\ell\pi}{\omega} \pm 1,\ \ell \in \mathbb{N}^*.$$

Besides they are simple if $\omega \neq \frac{k\pi}{2}$, $k \in \mathbb{Z}^*$, else they are double. By the simple calculations we find that:

$*$ If $\omega < \frac{\pi}{2}$, then $u \in H^2(\Omega)^2$;

$*$ If $\omega = \frac{\pi}{2}, \pi$, it was a simple poles $\alpha = 0, \pm 1$;

$*$ If $\omega = \frac{3\pi}{2}$, then $\alpha = \frac{1}{3}$ is a double root.

In the other cases, there is only one simple real root when $\omega \in \left]\pi, \frac{3\pi}{2}\right[\bigcup \left]\frac{3\pi}{2}, 2\pi\right[$; and no solution when $\omega \in \left]\frac{\pi}{2}, \pi\right[$.

It is known in B. Benabderrahmane [2] that there are linearly independent functions S_α and $S'_\alpha \in V$, such as S_α, $S'_\alpha \notin H^2(\Omega)^2$ and LS_α, $LS'_\alpha \in L^2(\Omega)^2$ and as the *Lamé* operator is an isomorphism of

$$Sp\left(H^2(\Omega)^2, S_\alpha, S_\alpha^{'}\right) \cap V \text{ on } L^2(\Omega)^2,$$

where the *Sp* symbol designates the vector space generated by elements that are contained in parentheses that follow. These functions are given explicitly, in B. Benabderrahmane [2], by $S_\alpha(r,\theta) = r^\alpha \Psi_\alpha(\theta)$ such as

$$\Psi_\alpha(\theta) = \begin{cases} [(\rho_0 - \rho_1)\sin(\alpha+1)\omega - 2\rho_1 \sin(\alpha-1)\omega]\cos\alpha\theta + \\ (\rho_0+\rho_1)\sin(\alpha+1)\omega\cos(\alpha-2)\theta, \\ [(-\rho_1+\rho_0)\sin(\alpha+1)\omega - 2\rho_1\sin(\alpha-1)\omega]\sin\alpha\theta \\ -(\rho_0+\rho_1)\sin(\alpha+1)\omega\sin(\alpha-2)\theta \end{cases} \tag{1.2}$$

where $\rho_0 = \nu_0(\alpha-1) - 2$, $\rho_0 = \nu_0(\alpha+1) + 2$.

2. **Singularity in a polyhedron**

The behavior of the singular solutions of *Lamé* equations in a polyhedron is described by the following theorem:

Theorem 2.1. *Let $\omega < 2\pi$, $u \in V$. For $f \in L^2(Q)^3$, there are functions $C_\alpha, C_\alpha^{'}, C_{\alpha'}$ and $C_{\alpha'}^{'}$ such as $C_\alpha, C_\alpha^{'} \in H^{1-\alpha}(\mathbb{R})$, $C_{\alpha'}, C_{\alpha'}^{'} \in H^{1-\alpha^{'}}(\mathbb{R})$ verifying*

$$\begin{cases} u_r - \sum\limits_{\alpha,\ 0<\operatorname{Re}\alpha<1} (K_{\lambda,\mu,r}(r,z) * C_\alpha)\, r^\alpha \Psi_{\alpha,r}(\theta) - \\ - \sum\limits_{\alpha,\ 0<\operatorname{Re}\alpha<1} \left(K_{\lambda,\mu,r}(r,z) * C_\alpha^{'}\right) r^\alpha \Phi_{\alpha,r}(\theta) \end{cases} \in H^2(U\times\mathbb{R}) \tag{1.3}$$

$$\begin{cases} u_\theta - \sum\limits_{\alpha,\ 0<\operatorname{Re}\alpha<1} (K_{\lambda,\mu,\theta}(r,z) * C_\alpha)\, r^\alpha \Psi_{\alpha,\theta}(\theta) - \\ \sum\limits_{\alpha,\ 0<\operatorname{Re}\alpha<1} \left(K_{\lambda,\mu,\theta}(r,z) * C_\alpha^{'}\right) r^\alpha \Phi_{\alpha,\theta}(\theta) \end{cases} \in H^2(U\times\mathbb{R}) \tag{1.4}$$

$$\begin{cases} u_3 - \sum\limits_{\alpha',\ 0<\operatorname{Re}\alpha'<1} (K_{\lambda,\mu,z}(r,z) * C_{\alpha'})\, r^\alpha \Psi_{\alpha'}(\theta) - \\ \sum\limits_{\alpha',\ 0<\operatorname{Re}\alpha'<1} \left(K_{\lambda,\mu,z}(r,z) * C_{\alpha'}^{'}\right) r^\alpha \Phi_{\alpha'}(\theta) \end{cases} \in H^2(U\times\mathbb{R}) \tag{1.5}$$

where the functions

$$\Psi_\alpha(\theta) = (\Psi_{\alpha,r}(\theta), \Psi_{\alpha,\theta}(\theta))$$

are given by (1.2) *and*

$$\Phi_\alpha = \frac{\partial \Psi_\alpha(\theta)}{\partial \alpha} = \left[\log r \Psi_\alpha(\theta) + \frac{\partial}{\partial r}\Psi_\alpha(\theta)\right].$$

The functions $\Psi_{\alpha,r}(\theta), \Psi_{\alpha,\theta}(\theta)$ represents the radial part, angular part of $\Psi_\alpha(\theta)$, respectively. The functions $\Psi_{\alpha'}(\theta)$ are the first singular functions of the *Laplace* operator in a polygon.

The first sums in (1.3) and (1.4) are extended to all $\alpha; \operatorname{Re}\alpha \in \left]0,1\right[$ simple roots of the equation (1.1), while the second sums are extended to all the double roots of the same equation. In (1.5), the first sums are extended to all α' simple roots of the corresponding transcendent equation to the *Laplace* operator with the boundary conditions associated and the second sums are extended to all α' double roots of the same equation.

The symbol $*$ represents the convolution in relation to z. The Indices r, θ and z in the relations $(1.3), (1.4)$ and (1.5) are, respectively, the radial component, angular and longitudinal vector by using cylindrical coordinates.

For more details, we are given the similar of the Theorem 2.1, in the following cases:

- Case of simple roots such as $0 < \operatorname{Re}\alpha < 1$;
- Case of double roots such as $0 < \operatorname{Re}\alpha < 1$;
- Case of the fissure $(\omega = 2\pi)$.

Theorem 2.2. *We assume that* $\omega \in \left]\pi, \frac{3\pi}{2}\right[\bigcup \left]\frac{3\pi}{2}, 2\pi\right[$. *Let* $u \in V$ *be a variational solution, is to bounded support in the direction of* z. *For all* $f \in L^2(Q)^3$, *there are functions* C *and* C_α *such as*

$$C \in H^{1-\frac{\pi}{\omega}}(\mathbb{R}), C_\alpha \in H^{1-\alpha}(\mathbb{R}) \text{ and}$$

$$\begin{cases} u_r - \sum\limits_{\alpha,\ 0<\alpha<1} (K_{\lambda,\mu,r}(r,z) * C_\alpha) r^\alpha \Psi_{\alpha,r}(\theta) \in H^2(U \times \mathbb{R}) \\ u_\theta - \sum\limits_{\alpha,\ 0<\alpha<1} (K_{\lambda,\mu,\theta}(r,z) * C_\alpha) r^\alpha \Psi_{\alpha,\theta}(\theta) \in H^2(U \times \mathbb{R}) \\ u_3 - (K_{\lambda,\mu,z}(r,z) * C) r^{\frac{\pi}{\omega}} \cos\left(\frac{\pi}{\omega}\theta\right) \in H^2(U \times \mathbb{R}) \end{cases}$$

where $\alpha = \frac{\ell\pi}{\omega} \pm 1$, $\ell \in \mathbb{N}^*$ *are the simple roots of the equation* (1.1).

For $\omega = \frac{3\pi}{2}$, $\alpha = \frac{1}{3}$ is a double root of the equation (1.1). Therefore, it is necessary to modify the result of the Theorem 2.2 as follows: there are two constants C and $C^{'}$ such as

$$C \in H^{\frac{2}{3}}\left(\mathbb{R}\right), C^{'} \in H^{\frac{1}{3}}\left(\mathbb{R}\right) \text{ and}$$

$$\begin{cases} u_r - \left(K_{\lambda,\mu,r}\left(r,z\right) * C\right) r^{\frac{1}{3}} \Phi_{\frac{1}{3},r}\left(\theta\right) \in H^2\left(U \times \mathbb{R}\right) \\ u_\theta - \left(K_{\lambda,\mu,\theta}\left(r,z\right) * C\right) r^{\frac{1}{3}} \Phi_{\frac{1}{3},\theta}\left(\theta\right) \in H^2\left(U \times \mathbb{R}\right) \\ u_3 - \left(K_{\lambda,\mu,z}\left(r,z\right) * C^{'}\right) r^{\frac{2}{3}} \cos\left(\frac{2\theta}{3}\right) \in H^2\left(U \times \mathbb{R}\right) \end{cases}$$

In the case $\omega = 2\pi$, we obtain the existence of the functions C and $C^{'}$ of $H^{\frac{1}{2}}\left(\mathbb{R}\right)$ such as

$$\begin{cases} u_r - \left(K_{\lambda,\mu,r}\left(r,z\right) * C\right) \sqrt{r} \Phi_{\frac{1}{2},r}\left(\theta\right) \in H^2\left(U \times \mathbb{R}\right) \\ u_\theta - \left(K_{\lambda,\mu,\theta}\left(r,z\right) * C\right) \sqrt{r} \Phi_{\frac{1}{2},\theta}\left(\theta\right) \in H^2\left(U \times \mathbb{R}\right) \\ u_3 - \left(K_{\lambda,\mu,z}\left(r,z\right) * C^{'}\right) \sqrt{r} \cos\left(\frac{\theta}{2}\right) \in H^2\left(U \times \mathbb{R}\right). \end{cases}$$

The demonstration is essentially based on the study of the following points:

- Decompose every problem in plane part, u and u_θ, and in longitudinal part, u_3.
- Study of the longitudinal displacement singularity along an edge.
- Study of the perpendicular displacement singularity along an edge.

2.1. **Problem decomposition.** We start by studying the *Lamé* solutions in the tridimensional domain $Q = \Omega \times \mathbb{R}$, who present an edge along $z^{'}Oz$.

For $f \in L^2\left(Q\right)^3$, let $u \in V$ be a variational solution of $\left(P\right)$, then we have

$$a(u,v) = \ell(v), \text{ where}$$

$$a(u,v) = \sum_{i,j=1}^{3} \int_Q \sigma_{ij}(u)\varepsilon_{ij}(v) dx_1 dx_2 dx_3, \ \ell(v) = \sum_{i=1}^{3} \int_Q f_i v_i dx_1 dx_2 dx_3.$$

The invariance of the problems in relation to z implies the following partial regularity result:

Lemma 2.1. *We have*

$$\frac{\partial^2 u}{\partial x \partial z}, \ \frac{\partial^2 u}{\partial y \partial z} \ \textit{and} \ \frac{\partial^2 u}{\partial z^2} \in L^2\left(Q\right)^3.$$

Let's decompose the fields u and f to the plane components and longitudinal component by posing:

$$u = (v, u_3)^t \text{ and } f = (g, f_3)^t,$$

where v and g are vector fields of dimension 2 (also depend of z).

We will use the following notations:

- $\triangle_2$: Laplace in dimension 2, (variables x_1, x_2).
- ∇_2 : *Gradient* in dimension 2, .(variables x_1, x_2).
- Div_2 : *Divergence* in dimension 2, (variables x_1, x_2).

Using these notations the Lamé equations in dimension 3 become

$$\begin{cases} \mu\left(\triangle_2 v + \dfrac{\partial^2 v}{\partial z^2}\right) + (\lambda+\mu)\nabla_2\left(Div_2 v + \dfrac{\partial u_3}{\partial z}\right) = g \\ \mu\left(\triangle_2 u_3 + \dfrac{\partial^2 u_3}{\partial z^2}\right) + (\lambda+\mu)\dfrac{\partial}{\partial z}\left(Div_2 v + \dfrac{\partial u_3}{\partial z}\right) = f_3. \end{cases}$$

Thanks to Lemma 2.1, it can see that

$$\begin{cases} \mu\left(\triangle_2 v + \dfrac{\partial^2 v}{\partial z^2}\right) + (\lambda+\mu)\nabla_2\, Div_2 v = g - (\lambda+\mu)\nabla_2\left(\dfrac{\partial u_3}{\partial z}\right) \in L^2(Q)^3 \\ \mu\left(\triangle_2 u_3 + \dfrac{\partial^2 u_3}{\partial z^2}\right) + (\lambda+\mu)\dfrac{\partial^2 u_3}{\partial z^2} = f_3 - (\lambda+\mu)\dfrac{\partial}{\partial z}(Div_2 v) \in L^2(Q)^3. \end{cases} \tag{1.6}$$

This formulation has the advantage to decouple v and u_3. The left member in the first equations in (1.6) concerns the plane components of u, while the right member concerns the longitudinal component.

2.2. **Study of the boundary conditions.** It is assumed that

$$\eta' = (\eta_1, \eta_2, \eta_3)^t = (\eta, \eta_3)^t \text{ and } \tau' = (\tau_1, \tau_2, \tau_3)^t = (\tau, \tau_3)^t.$$

The condition $u.\eta' = 0$ becomes $u_3\eta_3 = -v.\eta$. As $\eta_3 = 0$ and $\tau_3 = 1$ then

$$u.\eta' = 0 \Leftrightarrow v.\eta = 0 \text{ (no condition on } u_3)$$

Concerning the condition on $\left(\sum(u).\eta'\right)$, we set $u = (v,0) + (0,0,u_3)$. Using the relations $\sigma_{ij}(u) = 2\mu\varepsilon_{ij}(u) + \lambda tr(\varepsilon(u))\delta_{ij}$, $i,j = 1,2,3$, it results

$$\sigma_{11}(v,0) = (\lambda+\mu)\frac{\partial u_1}{\partial x} + \lambda\frac{\partial u_2}{\partial y},\ \sigma_{12}(v,0) = \sigma_{21}(v,0) = \mu\left(\frac{\partial u_1}{\partial y} + \frac{\partial u_2}{\partial x}\right),$$

$$\sigma_{13}(v,0)=\sigma_{31}(v,0)=\mu\frac{\partial u_1}{\partial z},\ \sigma_{23}(v,0)=\sigma_{32}(v,0)=\mu\frac{\partial u_2}{\partial z},$$

$$\sigma_{22}(v,0)=(\lambda+\mu)\frac{\partial u_2}{\partial y}+\lambda\frac{\partial u_1}{\partial x},\ \sigma_{33}(v,0)=\mu\left(\frac{\partial u_1}{\partial x}+\frac{\partial u_2}{\partial y}\right),$$

$$\sigma_{11}(0,0,u_3)=\sigma_{22}(0,0,u_3)=\lambda\frac{\partial u_3}{\partial z},\sigma_{12}(0,0,u_3)=\sigma_{21}(0,0,u_3)=0,$$

$$\sigma_{13}(0,0,u_3)=\sigma_{31}(0,0,u_3)=\lambda\frac{\partial u_3}{\partial x},\ \sigma_{23}(0,0,u_3)=\sigma_{32}(0,0,u_3)=\lambda\frac{\partial u_3}{\partial y},$$

$$\sigma_{33}(0,0,u_3)=(\lambda+\mu)\frac{\partial u_3}{\partial z}.$$

Using the fact that $\eta_3=0$ and $\tau_3=1$, these last relations involve

$$\left\{\begin{array}{c}(\sigma(v,0).\eta).\tau=(\sigma(v).\eta).\tau+\mu\frac{\partial}{\partial z}(u_3\eta_3)=(\sigma(v).\eta).\tau\\(\sigma(0,0,u_3).\eta).\tau=\lambda\frac{\partial u_3}{\partial\eta}.\end{array}\right.$$

Therefore, we have the conditions that must be verified by each components of $u=(v,u_3)$ for the considered boundary conditions :

$$u.\eta^{'}=0\Leftrightarrow v.\eta=0 \text{ and no condition on } u_3$$

$$\left(\sum(u).\eta^{'}\right).\tau^{'}=0\Leftrightarrow\left(\sum(v).\eta\right).\tau=-\lambda\frac{\partial u_3}{\partial\eta}=0.$$

2.3. **Study of the longitudinal displacement along an edge.** In (1.6) the second equation is none other than the *Laplace* equation in Q, using a change of scale in z. By posing

$$z=\sqrt{\frac{\mu}{\lambda+2\mu}}z^{'},$$

we obtain

$$\mu\triangle_2 u_3+(\lambda+2\mu)\frac{\partial}{\partial z}\left(Div_2 v+\frac{\partial u_3}{\partial z}\right)=\mu\triangle_2 u_3+\mu\frac{\partial^2 u_3}{\partial(z^{'})^2}=\mu\triangle u_3.$$

This result attached to the results of the preceding paragraph permits us, for the longitudinal displacement part, to deduce the following problem:

$$(P_1)\left\{\begin{array}{l}\Delta u_3=f_3 \text{ in } Q\\ \dfrac{\partial u_3}{\partial\eta}=h \text{ on } \Sigma,\end{array}\right.$$

where $h\in H^{-\frac{1}{2}}(U\times\mathbb{R})$, thanks to Lemma 2.1.

The study of this problem is already made by P. Grisvard [10]. The application of results of P. Grisvard [9], concerning the *Laplace* equations, gives after change of scale in z the following decomposition of u_3 :

$$u_3 - \sum_{\alpha,\ 0<\mathrm{Re}\,\alpha<1} \left(K_{\lambda,\mu,z}\left(r,z\right) * C\right) r^{\alpha} \Psi_{\alpha}\left(\theta\right) \in H^2\left(Q\right),$$

where $C \in H^{1-\alpha}(\mathbb{R})$ and the functions $\Psi_{\alpha}(\theta)$ are the first singular functions of the problem (P_1), which are given, see P.Grisvard [8], by $\Psi_{\alpha}(\theta) = \cos\alpha\theta$ where $K_{\lambda,\mu,z}(r,z)$ represents the kernel of the *Laplace* operator. This establishes the part of the Theorem 2.2 that concerns the longitudinal part u_3.

2.4. **Study of the perpendicular displacement singularity along an edge.** We analyze the behavior of v from the first equation of (1.6) :

$$\mu\left(\triangle_2 v + \frac{\partial^2 v}{\partial z^2}\right) + (\lambda+\mu)\,\nabla_2\, Div_2 v = g - (\lambda+\mu)\,\nabla_2\left(\frac{\partial u_3}{\partial z}\right) \in L^2\left(Q\right)^3.$$

To simplify we note h the second member of this equation. Using the partial *Fourier* transformation in z, we see that the previous equation amounts to the following problem which is governed by the *Lamé* system resolving:

$$L\widehat{v} - \mu\zeta^2\widehat{v} = \widehat{h}.$$

Concerning the boundary conditions, as we can see that the conditions remain unaltered, we will be able to have the same conditions but non homogeneous. However by subtracting v to a field $u \in H^2\left(Q\right)^2$ verifying the same conditions to limits that v, consequently the field $w = v - u$ verifies the homogeneous conditions. To simplify the notations, we will note this field again by v.

The uniqueness of the variational solution implies that $\widehat{v} \in D_L$ where

$$D_L = \left\{u \in sp\left(H^2\left(\Omega\right)^2, S_\alpha, S_\alpha^{'}\right);\ \left(u.\eta^{'}, \left(\Sigma(u).\eta^{'}\right).\tau^{'}\right) = 0,\ \text{on } \Sigma\right\}.$$

Therefore

$$\widehat{v} = \widehat{v}_R + \sum_{\alpha,\ 0<\mathrm{Re}\,\alpha<1} \widehat{C}_\alpha \Im_\alpha$$

where $\widehat{v}_R \in H^2(Q)^2$ and $\widehat{C}_\alpha \in \mathbb{R}$, for all $\zeta \in \mathbb{R}$. Moreover, according B. Benabderrahmane [2], we have the following inequalities:

$$\begin{cases} \zeta^2 \left\|\widehat{v}_R\right\|_{L^2(Q)^2} + \zeta \left\|\widehat{v}_R\right\|_{H^1(Q)^2} + \left\|\widehat{v}_R\right\|_{H^2(Q)^2} \le C \left\|\widehat{h}\right\|_{L^2(Q)^2} \\ \sum_{\alpha,\ 0<\operatorname{Re}\alpha<1} \left|\widehat{C}_\alpha\right| |\zeta|^{1-\alpha} \le C \left\|\widehat{h}\right\|_{L^2(Q)^2}. \end{cases}$$

From where it comes that $\widehat{v} \in H^2(Q)^2$ and $\widehat{C}_\alpha \in H^{1-\alpha}(\mathbb{R})$. Besides the following decomposition:

$$\widehat{v} = \widehat{v}_R + \sum_{\alpha,\ 0<\operatorname{Re}\alpha<1} \widehat{C}_\alpha \Im_\alpha,$$

which is equivalent by proceeding the inverse *Fourier* transformation, taking into account the fact that $\widehat{f * g} = \widehat{f}.\widehat{g}$, to

$$\begin{cases} v_r = (v_R)_r + \sum_{\alpha,\ 0<\operatorname{Re}\alpha<1} \left(K_{\lambda,\mu,r}(r,z) * C_\alpha\right)(S_\alpha)_r \\ v_\theta = (v_R)_\theta + \sum_{\alpha,\ 0<\operatorname{Re}\alpha<1} \left(K_{\lambda,\mu,\theta}(r,z) * C_\alpha\right)(S_\alpha)_\theta \end{cases}$$

because

$$K_{\lambda,\mu,r}(r,\zeta) = e^{\frac{-r|\zeta|}{\sqrt{1+\nu}}} \text{ and } K_{\lambda,\mu,\theta}(r,\zeta) = e^{-r|\zeta|}$$

and by definition

$$(\Im_\alpha)_r = e^{\frac{-r|\zeta|}{\sqrt{1+\nu}}} (S_\alpha)_r \text{ and } (\Im_\alpha)_\theta = e^{-r|\zeta|} (S_\alpha)_\theta.$$

This establishes the first inequality of the Theorem 2.2.

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SIMPSON, NEWTON AND GAUSS TYPE INEQUALITIES

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Abstract. In this paper using the Simpson's quadrature formula, the Newton quadrature formula and the Gauss quadrature formula, we present new inequalities between means.

1. **Introduction**

This papers deals with the comparison of means. If s and t are two real parameters and a and b are positive numbers, then we may consider the following two families of means:

- the *Gini means*,

$$G_{s,t}(a,b)=\begin{cases}\left(\frac{a^s+b^s}{a^t+b^t}\right)^{1/(s-t)}, & \text{if } s\neq t\\ \exp\left(\frac{a^s\log a+b^s\log b}{a^s+b^s}\right), & \text{if } s=t\end{cases};$$

- the *Stolarski means*,

$$S_{s,t}(a,b)=\begin{cases}\left(\frac{t(a^s-b^s)}{s(a^t-b^t)}\right)^{1/(s-t)}, & \text{if } (s-t)\,st\neq 0,\ a\neq b\\ \exp\left(-\frac{1}{s}+\frac{a^s\log a-b^s\log b}{a^s-b^s}\right), & \text{if } s=t\neq 0,\ a\neq b\\ \left(\frac{a^s-b^s}{s(\log a-\log b)}\right)^{1/s}, & \text{if } s\neq 0,\ t=0,\ a\neq b\\ \sqrt{ab}, & \text{if } s=t=0\\ a, & \text{if } a=b.\end{cases}$$

Some particular cases are important in themselves.

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$G_{s,0}(a,b)$ coincides with the *Hölder mean* of order $s > 0$,

$$A_s(a,b) = \left(\frac{a^s + b^s}{2}\right)^{1/s} = \left(\frac{s}{b^s - a^s}\int_a^b x^{2s-1}dx\right)^{1/s}$$

($A_1(a,b)$ is precisely the *arithmetic mean* of a and b, also denoted $A(a,b)$).

$G_{0,0}(a,b)$ coincides with the *geometric mean,*

$$G(a,b) = \sqrt{ab} = \left(\frac{1}{b-a}\int_a^b \frac{1}{x^2}dx\right)^{-1/2};$$

$S_{1,0}(a,b)$ coincides with the *logarithmic mean,*

$$L(a,b) = \frac{b-a}{\ln b - \ln a} = \left(\frac{1}{b-a}\int_a^b \frac{dx}{x}\right)^{-1}$$

while $S_{1,1}(a,b)$ coincides with the *identric mean,*

$$I(a,b) = \frac{1}{e}\left(\frac{b^b}{a^a}\right)^{\frac{1}{b-a}} = \exp\left(\frac{1}{b-a}\int_a^b \ln x dx\right).$$

We will be concerned with the problem of comparing the different means. Our approach is based on certain inequalities satisfied by the 4-convex functions. Recall that in the differentiable case these are precisely those 4-time differentiable functions f such that $f^{(4)}(x) \geq 0$ for all x.

Lemma 1.1. *If $f \in C^4([a,b])$ and $f^{(4)} \geq 0$, then the mean value of f,*

$$M(f) = \frac{1}{b-a}\int_a^b f(x)\,dx,$$

does not exceed any of the following three sums:

i) $\frac{1}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right];$

ii) $\frac{1}{8}\left[f(a) + 3f\left(\frac{2a+b}{3}\right) + 3f\left(\frac{a+2b}{3}\right) + f(b)\right];$

iii) $\left[f\left(\frac{a+b}{2} - \frac{b-a}{6}\sqrt{3}\right) + f\left(\frac{a+b}{2} + \frac{b-a}{6}\sqrt{3}\right)\right].$

Proof. According to Simpson's quadrature formula,

$$\frac{1}{b-a}\int_a^b f(x)\,dx = \frac{1}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right] - \frac{(b-a)^4}{2880}f^{(4)}(\xi_1),$$

for some $\xi_1 \in (a,b)$, whence *i*). The cases *ii*) and *iii*) are motivated by the Newton quadrature formula,

$$\frac{1}{b-a}\int_a^b f(x)\,dx = \frac{1}{8}\left[f(a) + 3f\left(\frac{2a+b}{3}\right) + 3f\left(\frac{a+2b}{3}\right) + f(b)\right] - \frac{(b-a)^4}{648} f^{(4)}(\xi_2),$$

and respectively by the Gauss quadrature formula

$$\frac{1}{b-a}\int_a^b f(x)\,dx = \frac{1}{2}\left[f\left(\frac{a+b}{2} - \frac{b-a}{6}\sqrt{3}\right) + f\left(\frac{a+b}{2} + \frac{b-a}{6}\sqrt{3}\right)\right] + \frac{(b-a)^4}{4320} f^{(4)}(\xi_3),$$

where ξ_2 and ξ_3 are suitable points in (a,b). □

2. Applications

Theorem 2.1. *If $a, b > 0$ then holds the following inequality*

$$G^2(a,b) \geq \frac{6a^2b^2(a+b)^2}{(a^2+b^2)(a+b)^2 + 16a^2b^2}$$

or, in an equivalent form,

$$A\left(a^2, b^2\right) A^2(a,b) + 2G^4(a,b) \geq 3G^2(a,b)\,A^2(a,b).$$

Proof. In Lemma 1.1, we take $f(x) = \frac{1}{x^2}$, from which $f^{(4)}(x) = \frac{120}{x^6} > 0$, therefore

$$\frac{1}{G^2(a,b)} = \frac{1}{b-a}\int_a^b \frac{1}{x^2}dx \leq \frac{1}{6}\left(\frac{1}{a^2} + \frac{16}{(a+b)^2} + \frac{1}{b^2}\right).$$

After calculus we obtain:

$$G^2(a,b) \geq \frac{6a^2b^2(a+b)^2}{(a^2+b^2)(a+b)^2 + 16a^2b^2},$$

that is,

$$A\left(a^2, b^2\right) A^2(a,b) + 2G^4(a,b) \geq 3G^2(a,b)\,A^2(a,b).$$

□

Theorem 2.2. *If $a, b, t > 0$ then the following inequality holds*

$$G_t^2(a,b) \geq \frac{(b^t - a^t)(ab(a+b))^{t+1}}{t(b-a)\left((a^{t+1}+b^{t+1})(a+b)^{t+1} + 2^{t+3}(ab)^{t+1}\right)}$$

or, in an equivalent form,

$$A\left(a^{t+1}, b^{t+1}\right) A^{t+1}(a+b) + 2G^{2t+2}(a,b) \geq \frac{3(b^t - a^t)}{t(b-a)} \cdot \frac{G^{2t+2}(a,b)}{G_t^2(a,b)} \cdot A^{t+1}(a,b).$$

Proof. In Lemma 1.1, we take $f(x) = \frac{1}{x^{t+1}}$, from which $f^{(4)}(x) > 0$ and so the proof follows easily. □

Theorem 2.3. *If $a, b > 0$ then the following inequality holds*

$$I^6(a,b) \geq ab\left(\frac{a+b}{2}\right)^4$$

or, in an equivalent form,

$$I(a,b) \geq G^{1/3}(a,b) A^{2/3}(a,b).$$

Proof. In Lemma 1.1, we take $f(x) = \ln x$ for which $f^{(4)}(x) < 0$, therefore $I(a,b) = \exp\left(\frac{1}{b-a}\int_a^b \ln x dx\right)$

$\geq \exp\frac{1}{6}\left(\ln a + 4\ln\left(\frac{a+b}{2}\right) + \ln b\right) = \sqrt[6]{ab\left(\frac{a+b}{2}\right)^4}$. □

Exercise 2.1. *If $a, b > 0$ then*

$$\frac{A(a,b)}{L(a,b)} \geq 1 + \frac{2}{3}\ln\frac{A(a,b)}{G(a,b)}.$$

Proof. From the definitions of identric and logarithmic mean, we have

$$\ln I(a,b) = \frac{a}{L(a,b)} + \ln b - 1$$

and

$$\ln I(a,b) = \frac{b}{L(a,b)} + \ln a - 1.$$

After addition, we obtain:

$$\frac{a+b}{L(a,b)} + \ln ab - 2 = 2\ln I(a,b)$$

or, equivalently,

$$\frac{A(a,b)}{L(a,b)} + \ln G(a,b) - 1 = \ln I(a,b). \quad (2.1)$$

Using the statement of the Theorem 2.3 we obtain:

$$\frac{A(a,b)}{L(a,b)} + \ln G(a,b) - 1 \geq \ln \left(G^2(a,b) A^4(a,b)\right)^{\frac{1}{6}}.$$

□

Theorem 2.4. *If $a, b > 0$ then the following inequality holds:*

$$L(a,b) \geq \frac{(a+b)^2 + 8ab}{6ab(a+b)}$$

or, in an equivalent form,

$$3L(a,b) \geq \frac{A(a,b)}{G^2(a,b)} + \frac{2}{A(a,b)}.$$

Proof. In Lemma 1.1 we take $f(x) = \frac{1}{x}$ for which $f^{(4)}(x) > 0$, therefore

$$\frac{1}{L(a,b)} = \frac{1}{b-a} \int_a^b \frac{dx}{x} \leq \frac{1}{6}\left(\frac{1}{a} + \frac{8}{a+b} + \frac{1}{b}\right)$$

or, equivalently,

$$L(a,b) \geq \frac{6ab(a+b)}{(a+b)^2 + 8ab}.$$

□

Theorem 2.5. *If $a, b > 0$ and $t \in \left(-\infty, \frac{1}{2}\right] \cup \left[1, \frac{3}{2}\right] \cup [2, +\infty)$, then*

$$A_t^t(a,b) \leq \frac{t(b-a)\left(2^{2t-1}\left(a^{2t-1} + b^{2t-1}\right) + 4(a+b)^{2t-1}\right)}{3 \cdot 2^{2t}\left(b^t - a^t\right)}$$

or, in an equivalent form,

$$A_t^t(a,b) \leq \frac{t(b-a)}{3\left(b^t - a^t\right)}\left(A\left(a^{2t-1}, b^{2t-1}\right) + 2A^{2t-1}(a,b)\right).$$

If $t \in \left(\frac{1}{2}, 1\right) \cup \left(\frac{3}{2}, 2\right)$, then the reverse inequality holds.

Proof. In Lemma 1.1 we take $f(x) = x^{2t-1}$ for which

$$f^{(4)}(x) = (2t-1)(2t-2)(2t-3)(2t-4)x^{2t-5}.$$

If $t \in \left(-\infty, \frac{1}{2}\right] \cup \left[1, \frac{3}{2}\right] \cup [2, +\infty)$, then

$$f^{(4)}(x) > 0$$

and

$$A_t^t(a,b) = \frac{t}{b^t - a^t}\int_a^b x^{2t-1}dx \le \frac{t(b-a)}{6(b^t-a^t)}\left(a^{2t-1} + 4\left(\frac{a+b}{2}\right)^{2t-1} + b^{2t-1}\right)$$

and the proof continues in an easy manner. □

3. Newton Type Inequalities

Theorem 3.1. *If $a, b > 0$ then the following inequality holds*

$$G^2(a,b) \ge \frac{8a^2b^2(2a+b)^2(a+2b)^2}{(a^2+b^2)(2a+b)^2(a+2b)^2 + 27a^2b^2(5a^2+8ab+5b^2)}$$

or, in an equivalent form,

$$16A\left(a^2,b^2\right)A(2a,b)A(a,2b) + 27G^4(a,b)\left(5A\left(a^2,b^2\right) + 4G^4(a,b)\right)$$
$$\ge 64G^2(a,b)A(2a,b)A(a,2b).$$

Proof. In Lemma 1.1 we take $f(x) = \frac{1}{x^2}$ for which $f^{(4)}(x) > 0$, therefore

$$\frac{1}{G^2(a,b)} = \frac{1}{b-a}\int_a^b \frac{dx}{x^2} \le \frac{1}{8}\left(\frac{1}{a^2} + \frac{27}{(2a+b)^2} + \frac{27}{(a+2b)^2} + \frac{1}{b^2}\right).$$

□

Theorem 3.2. *If $a, b, t > 0$ then $G_t^2(a,b)$ is greater or equal to*

$$\frac{8\left(b^t-a^t\right)(ab)^{t+1}(2a+b)^{t+1}(a+2b)^{t+1}}{t(b-a)\left(\left(a^{t+1}+b^{t+1}\right)(2a+b)^{t+1}(a+2b)^{t+1} + 3^{t+2}(ab)^{t+1}(2a+b)^{t+1} + (a+2b)^{t+1}\right)}.$$

Proof. In Lemma 1.1 ii) we take $f(x) = \frac{1}{x^{t+1}}$ for which $f^{(4)}(x) > 0$ and so on. □

Theorem 3.3. *If $a, b > 0$ then the following inequality holds*

$$I^8(a,b) \ge ab\left(\frac{2a+b}{3}\right)^3\left(\frac{a+2b}{3}\right)^3.$$

Proof. In Lemma 1.1 ii), we take $f(x) = \ln x$ for where $f^{(4)}(x) < 0$, therefore

$$\begin{aligned}I(a,b) &= \exp\left(\frac{1}{b-a}\int_a^b \ln x dx\right)\\ &\geq \exp\left(\frac{1}{8}\left(\ln a + 3\ln\frac{2a+b}{3} + 3\ln\frac{a+2b}{3} + \ln b\right)\right)\\ &= \left(ab\left(\frac{2a+b}{3}\right)^3\left(\frac{a+2b}{3}\right)^3\right)^{\frac{1}{8}}.\end{aligned}$$

□

Exercise 3.1. *If* $a, b > 0$ *then*

$$\frac{A(a,b)}{L(a,b)} \geq 1 + \ln\left(\left(\frac{2}{3}\right)^6 \frac{A^{\frac{3}{8}}(2a,b)\, A^{\frac{3}{8}}(a,2b)}{G^{\frac{3}{4}}(a,b)}\right).$$

Proof. Using (2.1) and the Theorem 3.3 we obtain

$$\frac{A(a,b)}{L(a,b)} + \ln G(a,b) - 1 \geq \ln\left(ab\left(\frac{2a+b}{3}\right)^3\left(\frac{a+2b}{3}\right)^3\right)^{\frac{1}{8}}$$

and the proof follows easily. □

Theorem 3.4. *If* $a, b > 0$ *then the following inequality holds:*

$$L(a,b) \geq \frac{4ab(2a+b)(a+2b)}{(a+b)(a^2+16ab+b^2)}.$$

Proof. In Lemma 1.1 ii) we take $f(x) = \frac{1}{x}$ for which $f^{(4)}(x) > 0$, therefore

$$\frac{1}{L(a,b)} = \frac{1}{b-a}\int_a^b \frac{dx}{x} \leq \frac{1}{8}\left(\frac{1}{a} + \frac{9}{2a+b} + \frac{9}{a+2b} + \frac{1}{b}\right)$$

and so on. □

Theorem 3.5. *If* $a, b > 0$ *and* $t \in \left(-\infty, \frac{1}{2}\right] \cup \left[1, \frac{3}{2}\right] \cup [2, +\infty)$, *then*

$$A_t^t(a,b) \leq \frac{t(b-a)\left(3^{2t-1}\left(a^{2t-1}+b^{2t-1}\right) + 3(2a+b)^{2t-1} + 3(a+2b)^{2t-1}\right)}{8 \cdot 3^{2t-1}\left(b^t - a^t\right)}.$$

If $t \in \left(\frac{1}{2}, 1\right) \cup \left(\frac{3}{2}, 2\right)$, *then the reverse inequality holds true.*

Proof. In Lemma 1.1 ii) we take $f(x) = x^{2t-1}$ for which $f^{(4)}(x) > 0$, for $t \in \left(-\infty, \frac{1}{2}\right] \cup \left[1, \frac{3}{2}\right] \cup [2, +\infty)$, therefore

$$\begin{aligned} A_t^t(a,b) &= \frac{t}{b^t - a^t}\int_a^b x^{2t-1}dx \\ &\leq \frac{t(b-a)}{8(b^t-a^t)}\left(a^{2t-1} + 3\left(\frac{2a+b}{3}\right)^{2t-1} + 3\left(\frac{a+2b}{3}\right)^{2t-1} + b^{2t-1}\right) \end{aligned}$$

and the proof follows. □

4. **Gauss Type Inequalities**

Theorem 4.1. *If $a, b > 0$ then*

$$G^2(a,b) \leq \frac{\left(a^2+4ab+b^2\right)^2}{12\left(a^2+ab+b^2\right)}.$$

Proof. In Lemma 1.1 iii) we take $f(x) = \frac{1}{x^2}$ for which $f^{(4)}(x) > 0$, therefore

$$\begin{aligned} \frac{1}{G^2(a,b)} &= \frac{1}{b-a}\int_a^b \frac{dx}{x^2} \\ &\geq \frac{1}{2}\left(\frac{1}{\left(\frac{a+b}{2} - \frac{(b-a)\sqrt{3}}{6}\right)^2} + \frac{1}{\left(\frac{a+b}{2} + \frac{(b-a)\sqrt{3}}{6}\right)^2}\right) \\ &= \frac{12\left(a^2+ab+b^2\right)}{\left(a^2+4ab+b^2\right)^2}. \end{aligned}$$

□

Theorem 4.2. *If $a, b, t > 0$ then $G_t^2(a,b)$ does not exceeds*

$$\frac{2\left(b^t-a^t\right)\left(a^2+4ab+b^2\right)^{t+1}}{t(b-a)\left(\left(\left(3+\sqrt{3}\right)a + \left(3-\sqrt{3}\right)b\right)^{t+1} + \left(\left(3-\sqrt{3}\right)a + \left(3+\sqrt{3}\right)b\right)^{t+1}\right)}.$$

Proof. In Lemma 1.1 iii) we take $f(x) = \frac{1}{x^{t+1}}$ for which $f^{(4)}(x) > 0$, therefore

$$\frac{1}{G_t^2(a,b)} = \frac{t}{b^t-a^t}\int_a^b \frac{dx}{x^{t+1}}$$

$$\geq \frac{t(b-a)6^{t+1}}{2(b^t-a^t)}\left(\frac{1}{\left(\left(3+\sqrt{3}\right)a+\left(3-\sqrt{3}\right)b\right)^{t+1}} + \frac{1}{\left(\left(3-\sqrt{3}\right)a+\left(3+\sqrt{3}\right)b\right)^{t+1}}\right)$$

and the proof just follows. □

Theorem 4.3. *If* $a, b > 0$ *then*

$$I^2(a,b) \le \frac{a^2+4ab+b^2}{6}.$$

Proof. In Lemma 1.1 iii) we take $f(x) = \ln x$ for which $f^{(4)}(x) < 0$, therefore

$$\begin{aligned} I(a,b) &= \exp\left(\frac{1}{b-a}\int_a^b \ln x dx\right) \\ &\le \exp\left(\frac{1}{2}\left(\ln\left(\frac{a+b}{2} - \frac{(b-a)\sqrt{3}}{6}\right) + \ln\left(\frac{a+b}{2} + \frac{(b-a)\sqrt{3}}{6}\right)\right)\right) \\ &= \sqrt{\frac{a^2+4ab+b^2}{6}}. \end{aligned}$$

□

Exercise 4.1. *If* $a, b > 0$ *then*

$$\frac{A(a,b)}{L(a,b)} \le 1 + \frac{1}{2}\ln\left(\frac{1}{3} + \frac{2A^2(a,b)}{3G^2(a,b)}\right).$$

Proof. Using (2.1) and Theorem 4.3 we obtain the desired result. □

Theorem 4.4. *If* $a, b > 0$ *then*

$$L(a,b) \le \frac{2\left(a^2+4ab+b^2\right)}{3(a+b)}.$$

Proof. In Lemma 1.1 iii) we take $f(x) = \frac{1}{x}$ for which $f^{(4)}(x) > 0$, therefore

$$\begin{aligned} \frac{1}{L(a,b)} &= \frac{1}{b-a}\int_a^b \frac{dx}{x} \\ &\ge \frac{1}{2}\left(\frac{1}{\frac{a+b}{2} - \frac{(b-a)\sqrt{3}}{6}} + \frac{1}{\frac{a+b}{2} + \frac{(b-a)\sqrt{3}}{6}}\right) \\ &= \frac{3(a+b)}{2\left(a^2+4ab+b^2\right)}. \end{aligned}$$

□

Theorem 4.5. *If* $a, b > 0$ *and* $t \in \left(-\infty, \frac{1}{2}\right] \cup \left[1, \frac{3}{2}\right] \cup [2, +\infty)$, *then*

$$\frac{t(b-a)}{2\cdot 6^{2t+1}(b^t - a^t)}\left(\left(\left(3+\sqrt{3}\right)a + \left(3-\sqrt{3}\right)b\right)^{2t+1} + \left(\left(3-\sqrt{3}\right)a + \left(3+\sqrt{3}\right)b\right)^{2t+1}\right)$$

does not exceeds $A_t^t(a,b)$.

If $t \in \left(\frac{1}{2}, 1\right) \cup \left(\frac{3}{2}, 2\right)$, *then the reverse inequality holds true.*

Proof. In Lemma 1.1 iii) we take $f(x) = x^{2t-1}$ for which $f^{(4)}(x) > 0$, if $t \in \left(-\infty, \frac{1}{2}\right] \cup \left[1, \frac{3}{2}\right] \cup [2, +\infty)$, therefore

$$A_t^t(a,b) = \frac{t}{b^t - a^t} \int_a^b x^{2t-1} dx$$
$$\geq \frac{t(b-a)}{2(b^t - a^t)} \left(\left(\frac{\left(3+\sqrt{3}\right)a + \left(3-\sqrt{3}\right)b}{6} \right)^{2t+1} + \left(\frac{\left(3-\sqrt{3}\right)a + \left(3+\sqrt{3}\right)b}{6} \right)^{2t+1} \right).$$

□

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DUALITIES INDUCED BY RIGHT ADJOINT CONTRAVARIANT FUNCTORS

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Abstract. We characterize some dualities which are induced by pairs of contravariant functors which are adjoint on the right.

1. **Introduction**

Let $\mathcal{A}$ and $\mathcal{B}$ be abelian categories and $\mathrm{F}:\mathcal{A}\leftrightarrows\mathcal{B}:\mathrm{G}$ a pair of contravariant functors which are adjoint on the right. Then the natural transformation

$$\eta_{X,Y}:\mathrm{Hom}_{\mathcal{A}}(X,\mathrm{G}(Y))\to\mathrm{Hom}_{\mathcal{B}}(Y,\mathrm{F}(X)),$$

which corresponds to this duality, induces two natural transformations

$$\delta:1_{\mathcal{A}}\to\mathrm{GF},\ \delta_X=\eta^{-1}_{X,\mathrm{F}(X)}(1_{\mathrm{F}(X)})\text{ and }\zeta:1_{\mathcal{B}}\to\mathrm{FG},\ \zeta_Y=\eta^{-1}_{\mathrm{G}(Y),Y}(1_{\mathrm{G}(Y)}).$$

An object X is called δ(respectively ζ)-*reflexive* if δ_X (respectively ζ_X) is an isomorphism. We will denote by Refl_δ (respectively Refl_ζ) the classes of F-reflexive (respectively G-reflexive) objects. A main topic is the study of dualities induced by F and G between some full subcategories of $\mathcal{A}$ and $\mathcal{B}$. The restrictions of F and G to the classes of reflexive objects induce a duality $\mathrm{F}:\mathrm{Refl}_\delta\leftrightarrows\mathrm{Refl}_\zeta:\mathrm{G}$. Moreover, if $\mathrm{F}:\mathcal{C}\leftrightarrows\mathcal{D}:\mathrm{G}$ is a duality then $\mathcal{C}\subseteq\mathrm{Refl}_\delta$ and $\mathcal{D}\subseteq\mathrm{Refl}_\zeta$ (see [11]).

We also fix an δ-reflexive object Q, and, following ideas from [4], we will call the triple $\mathfrak{D}=(Q,\mathrm{F},\mathrm{G})$ a *pointed pair of right adjoint contravariant functors*. Let $V=\mathrm{F}(Q)$. Then $\mathrm{add}(Q)\subseteq\mathrm{Refl}_\delta$ and $\mathrm{add}(V)\subseteq\mathrm{Refl}_\zeta$ (recall that $\mathrm{add}(X)$ denotes

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the class of all summands of finite direct sums of copies of X). We will denote by Faith_δ (Faith_ζ) the classes of all objects $X \in \mathcal{A}$ ($X \in \mathcal{B}$) such that δ_X (ζ_X) is a monomorphism, and we will call they *δ-faithful* (respectively *ζ-faithful*) objects. We recall that the natural transformations δ and ζ satisfy the identities

$$\mathrm{F}(\delta_X) \circ \zeta_{\mathrm{F}(X)} = 1_{\mathrm{F}(X)} \text{ for all } X \in \mathcal{A}$$

and

$$\mathrm{G}(\zeta_Y) \circ \delta_{\mathrm{G}(Y)} = 1_{\mathrm{G}(Y)} \text{ for all } Y \in \mathcal{B},$$

hence $\mathrm{F}(\mathcal{A}) \subseteq \mathrm{Faith}_\zeta$ and $\mathrm{G}(\mathcal{B}) \subseteq \mathrm{Faith}_\delta$.

Example 1.1. The typical example of such functors is the following: Let R and S be unital rings and Q an S-R-bimodule. Then the contravariant functors $\Delta = \mathrm{Hom}_R(-,Q) : \text{Mod-}R \to S\text{-Mod}$ and $\Delta' = \mathrm{Hom}_S(-,Q) : S\text{-Mod} \to \text{Mod-}R$ are right adjoint. If S is the endomorphism ring of Q then (Q, Δ, Δ') is a pointed pair of right adjoint contravariant functors.

The study of dualities induced by this pair of functors is an important topic in Module Theory. The starting point was the papers [8] and [1]. During the time this topic developed important concepts as (f)-cotilting and costar module (see [6] and [10] for complete surveys on the subjects).

Another important example was exhibited by Castaño-Iglesias in [3].

Example 1.2. Let G be a group. If $R = \bigoplus_{x\in G} R_x$ and $S = \bigoplus_{x\in G} {}_xS$ are two G-graded unital rings, we will denote by $\mathrm{Mod}_{\mathrm{gr}}\text{-}R$ (respectively, by $S\text{-}\mathrm{Mod}_{\mathrm{gr}}$) the category of all G-graded unital right R- (respectively, left S-) modules (see [9]).

If $Q, M \in \mathrm{Mod}_{\mathrm{gr}}\text{-}R$ we consider the G-graded abelian group $\mathrm{HOM}_R(M,Q)$ whose homogeneous component in x is

$${}_x\mathrm{HOM}_R(M,Q) = \{f \in \mathrm{Hom}_R(M,Q) \mid f(M_y) \subseteq Q_{xy}, \text{ for all } y \in G\}.$$

We note that $\mathrm{HOM}_R(Q,Q) = \mathrm{END}_R(Q)$ has a canonical structure of G-graded unital ring. If $M, N \in S\text{-}\mathrm{Mod}_{\mathrm{gr}}$ we consider the G-graded abelian group $\mathrm{HOM}_S(M,Q)$

whose homogeneous component in x is

$$\mathrm{HOM}_S(M,Q)_x = \{f \in \mathrm{Hom}_R(M,Q) \mid f({}_yM) \subseteq_{yx} Q, \text{ for all } y \in G\}.$$

Then we have a pair of contravariant functors

$$\mathrm{H}_R^{\mathrm{gr}} = \mathrm{HOM}_R(-,Q_R) : \mathrm{Mod}_{\mathrm{gr}}\text{-}R \leftrightarrows S\text{-}\mathrm{Mod}_{\mathrm{gr}} : \mathrm{HOM}_S(-,{}_SQ) = {}_S\mathrm{H}^{\mathrm{gr}}.$$

If $Q \in \mathrm{Mod}_{\mathrm{gr}}\text{-}R$ and $S = \mathrm{END}_R(Q)$, then $(Q, \mathrm{H}_R^{\mathrm{gr}}, {}_S\mathrm{H}^{\mathrm{gr}})$ is a pointed pair of right adjoint contravariant functors.

In this note we continue a approach initiated by Castagño in [3]. In this paper the author generalizes the notion of costar module introduced by Colby and Fuller in [5] to Grothendieck categories. We continue this kind of study, generalizing a duality exhibited in [2, Theorem 2.8] to abelian categories.

2. **Right pointed pairs of contravariant functors**

In the following $\mathfrak{D}$ will denote a pointed pair of right adjoint contravariant functors $(Q, \mathrm{F}, \mathrm{G})$ between the abelian categories $\mathcal{A}$ and $\mathcal{B}$.

Lemma 2.1. *Let $\mathfrak{D}$ be a pointed pair of right adjoint contravariant functors. If*

$$(\sharp)\ 0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$$

is an exact sequence in $\mathcal{A}$ then the unique homomorphism α, for which the diagram with exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \longrightarrow & 0 \\
 & & \alpha\big\downarrow & & \delta_Y\big\downarrow & & \delta_Z\big\downarrow & & \\
0 & \longrightarrow & \mathrm{G}(\mathrm{Im}(\mathrm{F}(f))) & \longrightarrow & \mathrm{GF}(Y) & \xrightarrow{\mathrm{GF}(g)} & \mathrm{GF}(Z) & &
\end{array}$$

is commutative, is given by the formula $\alpha = \mathrm{G}(j) \circ \delta_X$, where $j : \mathrm{Im}(\mathrm{F}(f)) \to \mathrm{F}(X)$ is the inclusion map.

Proof. The existence of α comes from the universal property of the kernel. Moreover, α is unique.

Let $\mathrm{F}(f) = j \circ p$ be the canonical decomposition of $\mathrm{F}(f)$. Since ($\sharp$) is an exact sequence, the sequence

$$0 \to \mathrm{F}(Z) \overset{\mathrm{F}(g)}{\to} \mathrm{F}(Y) \overset{p}{\to} \mathrm{Im}(\mathrm{F}(f)) \to 0$$

is exact, hence the sequence

$$0 \to \mathrm{G}(\mathrm{Im}(\mathrm{F}(f))) \overset{\mathrm{G}(p)}{\to} \mathrm{GF}(Y) \overset{\mathrm{GF}(g)}{\to} \mathrm{GF}(Z)$$

is also exact.

If we denote $\mathrm{G}(j) \circ \delta_X$ by α we have: $\mathrm{G}(p) \circ \alpha = \mathrm{G}(p) \circ \mathrm{G}(j) \circ \delta_X = \mathrm{G}(j \circ p) \circ \delta_X = \mathrm{GF}(f) \circ \delta_X = \delta_Y \circ f$ hence the following diagram is commutative with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\delta_Y} & & \downarrow{\scriptstyle\delta_Z} & & \\ 0 & \longrightarrow & \mathrm{G}(\mathrm{Im}(\mathrm{F}(f))) & \xrightarrow{\mathrm{G}(p)} & \mathrm{GF}(Y) & \xrightarrow{\mathrm{GF}(g)} & \mathrm{GF}(Z) & & \end{array}$$

□

The following result is a version for [3, Lemma 2.2] and [3, Proposition 2.3].

Lemma 2.2. *Let $\mathfrak{D}$ be a pointed pair of right adjoint contravariant functors.*

a) *An object $X \in \mathcal{A}$ is δ-faithful and $\mathrm{F}(X) \in \mathrm{gen}(V)$ if and only if there exists a monomorphism $f : X \to Q^n$ such that $\mathrm{F}(f)$ is an epimorphism.*

b) F *is exact with respect an exact sequence $0 \to Y \overset{f}{\to} X \overset{g}{\to} Z \to 0$ with $X \in \mathrm{Refl}_\delta$ and $Z \in \mathrm{Faith}_\delta$ if and only if $\mathrm{Im}(\mathrm{F}(f)) \in \mathrm{Refl}_\zeta$.*

Proof. a) Suppose that $X \in \mathrm{Faith}_\delta$ and there exists an epimorphism $V^n \overset{p}{\to} \mathrm{F}(X) \to 0$. Applying the functor G we obtain an monomorphism $\mathrm{G}(p) : \mathrm{GF}(X) \to \mathrm{G}(V^n) \cong Q^n$. Let $f = \mathrm{G}(p) \circ \delta_X$. Then $\mathrm{F}(f) = \mathrm{F}(\delta_X) \circ \mathrm{FG}(p)$. Then

$$\mathrm{F}(f) \circ \zeta_{V^n} = \mathrm{F}(\delta_X) \circ \mathrm{FG}(p) \circ \zeta_{V^n} = p,$$

hence $\mathrm{F}(f)$ is an epimorphism.

The converse implication is obvious.

b) Let $0 \to Y \xrightarrow{f} X \xrightarrow{g} Z \to 0$ be an exact sequence in $\mathcal{A}$ such that $X \in \mathrm{Refl}_\delta$ and $Z \in \mathrm{Faith}_\delta$. Let $\mathrm{F}(f) = j \circ p$ be the canonical factorization of $\mathrm{F}(f)$. Since ζ is a natural transformation we have the identity $\zeta_{\mathrm{F}(Y)} \circ j = \mathrm{FG}(j) \circ \zeta_{\mathrm{Im}(\mathrm{F}(f))}$.

By Lemma 2.1, the following diagram is commutative with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Y & \xrightarrow{f} & X & \xrightarrow{g} & Z & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \delta_X} & & \downarrow{\scriptstyle \delta_Z} & & \\ 0 & \longrightarrow & \mathrm{G}(\mathrm{Im}(\mathrm{F}(f))) & \xrightarrow{\mathrm{G}(p)} & \mathrm{GF}(X) & \xrightarrow{\mathrm{GF}(g)} & \mathrm{GF}(Z) & & \end{array}$$

where $\alpha = \mathrm{G}(j) \circ \delta_Y$.

Since X is F-reflexive and Z is δ-faithful it follows, from Snake Lemma, that α is an isomorphism hence $\mathrm{F}(\alpha)$ is an isomorphism.

We have $j = 1_{\mathrm{F}(Y)} \circ j = \mathrm{F}(\delta_Y) \circ \zeta_{\mathrm{F}(Y)} \circ j = \mathrm{F}(\delta_Y) \circ \mathrm{FG}(j) \circ \zeta_{\mathrm{Im}(\mathrm{F}(f))} = \mathrm{F}(\alpha) \circ \zeta_{\mathrm{Im}(\mathrm{F}(f))}$.

Since j is a monomorphism, $j = \mathrm{F}(\alpha) \circ \zeta_{\mathrm{Im}(\mathrm{F}(f))}$ and $\mathrm{F}(\alpha)$ is an isomorphism we conclude that $\mathrm{F}(f)$ is an epimorphism if and only if $\mathrm{Im}(\mathrm{F}(f)) \in \mathrm{Refl}_\zeta$. □

Theorem 2.3. *The following are equivalent for a pair* $\mathfrak{D}$*:*

a) $\mathrm{F} : \mathrm{cog}(Q) \rightleftarrows \mathrm{pres}(V) \cap \mathrm{Faith}_\zeta : \mathrm{G}$ *is a duality;*

b) i) $\mathrm{cog}(Q) = \mathrm{cop}(Q)$*;*

   ii) F *is exact with respect exact sequences* $0 \to X \to Q^n \to Y \to 0$ *with* $Y \in \mathrm{cog}(Q)$*.*

Proof.

a)⇒b) Let $X \in \mathrm{cog}(Q)$. From a) we have $\mathrm{F}(X) \in \mathrm{pres}(V)$ and $X \in \mathrm{Refl}_\delta$. Then there exists an exact sequence $V^m \to V^n \to \mathrm{F}(X) \to 0$ and hence the sequence $0 \to \mathrm{GF}(X) \to \mathrm{G}(V^n) \to \mathrm{G}(V^m)$ is exact. It follows that $0 \to X \to Q^n \to Q^m$ is exact which shows that $X \in \mathrm{cop}(Q)$, hence $\mathrm{cog}(Q) = \mathrm{cop}(Q)$.

Let $0 \to X \xrightarrow{f} Q^n \xrightarrow{g} Y \to 0$ be an exact sequence with $Y \in \mathrm{cog}(Q)$. Since $\mathrm{F}(Y) \in \mathrm{pres}(V)$ and $0 \to \mathrm{F}(Y) \xrightarrow{\mathrm{F}(g)} \mathrm{F}(Q^n) \xrightarrow{\mathrm{F}(f)} \mathrm{Im}(\mathrm{F}(f)) \to 0$ is exact we obtain that $\mathrm{Im}(\mathrm{F}(f)) \in \mathrm{pres}(V)$. But $\mathrm{Im}(\mathrm{F}(f)) \in \mathrm{Faith}_\zeta$ because $\mathrm{F}(X) \in \mathrm{Faith}_\zeta$. So

$\mathrm{Im}(\mathrm{F}(f)) \in \mathrm{pres}(V) \cap \mathrm{Faith}_\zeta \subseteq \mathrm{Refl}_\zeta$. By Lemma 2.2, the sequence

$$0 \to \mathrm{F}(Y) \stackrel{\mathrm{F}(g)}{\to} \mathrm{F}(Q^n) \stackrel{\mathrm{F}(f)}{\to} \mathrm{F}(X) \to 0$$

is exact.

b)⇒a) Let $X \in \mathrm{cog}(Q)$. There exists an exact sequence $0 \to X \stackrel{f_1}{\to} Q^{m_1} \stackrel{f_2}{\to} Q^{m_2}$ hence the sequence $0 \to X \stackrel{f_1}{\to} Q^{m_1} \stackrel{f_2}{\to} Y \to 0$ is exact with $Y \in \mathrm{cog}(Q)$, where Y is $\mathrm{Im}f_2$. By ii) we obtain that the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X & \xrightarrow{f_1} & Q^{m_1} & \xrightarrow{f_2} & Y & \longrightarrow & 0 \\ & & \delta_X \downarrow & & \delta_{Q^{m_1}} \downarrow & & \delta_Y \downarrow & & \\ 0 & \longrightarrow & \mathrm{GF}(X) & \xrightarrow{\mathrm{GF}(f_1)} & \mathrm{GF}(Q^{m_1}) & \xrightarrow{\mathrm{GF}(f_2)} & \mathrm{GF}(Y) & & \end{array}$$

is commutative with exact sequences. Moreover, all vertical arrows are monomorphisms and $\delta_{Q^{m_1}}$ is an isomorphism. From the Snake Lemma we obtain δ_X is an isomorphism, hence $X \in \mathrm{Refl}_\delta$. Therefore $\mathrm{cog}(Q) \subseteq \mathrm{Refl}_\delta$.

Since $Y \in \mathrm{cog}(Q) = \mathrm{cop}(Q)$ there exists an exact sequence

$$0 \to Y \stackrel{g_1}{\to} Q^{n_1} \stackrel{g_2}{\to} Q^{n_2},$$

hence the sequence $0 \to Y \stackrel{g_1}{\to} Q^{n_1} \stackrel{g_2}{\to} Z \to 0$ is exact with $Z \in \mathrm{cog}(Q)$, where Z is $\mathrm{Im}g_2$. The sequences

$$0 \longrightarrow \mathrm{F}(Y) \xrightarrow{\mathrm{F}(f_2)} \mathrm{F}(Q^{m_1}) \xrightarrow{\mathrm{F}(f_1)} \mathrm{F}(X) \longrightarrow 0$$

and

$$0 \longrightarrow \mathrm{F}(Z) \xrightarrow{\mathrm{F}(g_2)} \mathrm{F}(Q^{n_1}) \xrightarrow{\mathrm{F}(g_1)} \mathrm{F}(Y) \longrightarrow 0$$

are exact. Then the sequence

$$V^{n_1} \xrightarrow{\mathrm{F}(g_1 f_2)} V^{m_1} \xrightarrow{\mathrm{F}(f_1)} \mathrm{F}(X) \longrightarrow 0$$

is exact, hence $\mathrm{F}(X) \in \mathrm{pres}(V)$. But $\mathrm{F}(X) \in \mathrm{Faith}_\zeta$, so $\mathrm{F}(X) \in \mathrm{pres}(V) \cap \mathrm{Faith}_\zeta$. Therefore $\mathrm{F} : \mathrm{cog}(Q) \to \mathrm{pres}(V) \cap \mathrm{Faith}_\zeta$ is well-defined.

Let $A \in \mathrm{pres}(V) \cap \mathrm{Faith}_\zeta$. There is an exact sequence $V^m \stackrel{f}{\to} V^n \stackrel{g}{\to} A \to 0$, and applying G we obtain that the sequence $0 \to \mathrm{G}(A) \to Q^n \to Q^m$ is exact. Then $\mathrm{G}(A) \in \mathrm{cog}(Q)$. Therefore G is well defined.

Since the sequence $V^m \xrightarrow{f} V^n \xrightarrow{g} A \to 0$ is exact we have that the sequence

$$0 \to \mathrm{G}(A) \xrightarrow{\mathrm{G}(g)} \mathrm{G}(V^n) \xrightarrow{\mathrm{G}(f)} \mathrm{Im}(\mathrm{G}(f)) \to 0$$

is exact with $\mathrm{Im}(\mathrm{G}(f)) \in \mathrm{cog}(Q)$. From b)ii) we have that the sequence

$$0 \to \mathrm{F}(\mathrm{Im}(\mathrm{G}(f))) \xrightarrow{\mathrm{FG}(f)} \mathrm{FG}(V^n) \xrightarrow{\mathrm{FG}(g)} \mathrm{FG}(A) \to 0$$

is exact.

In the commutative diagram

$$\begin{array}{ccc} V^n & \xrightarrow{\zeta_{V^n}} & \mathrm{FG}(V^n) \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle \mathrm{FG}(g)} \\ A & \xrightarrow{\zeta_A} & \mathrm{FG}(A) \end{array}$$

$\mathrm{FG}(g)$ and ζ_{V^n} are epimorphisms, so ζ_A is an epimorphisms. Since ζ_A is a monomorphism ($A \in \mathrm{Faith}_\zeta$) we obtain that $A \in \mathrm{Refl}_\zeta$. Therefore $\mathrm{pres}(V) \cap \mathrm{Faith}_\zeta \subseteq \mathrm{Refl}_\zeta$. $\square$

Suppose that $\mathfrak{D} = (Q, \Delta, \Delta')$ is the (classical) pointed pair of right adjoint contravariant functors from Example 1.1. By [2, Theorem 3.4] it satisfies the equivalent conditions from Theorem 2.3 if and only if the conditions:

i) Δ is exact with respect exact sequences $0 \to X \to Q^n \to Y \to 0$ if and only if $Y \in \mathrm{cog}(Q)$,
ii) $\mathrm{F}(\mathrm{cog}(Q)) \subseteq \mathrm{gen}(V)$.

are satisfied. In the proof of this result it is used the fact that, in this particular setting, the class $\mathrm{Pres}(V) \cap \mathrm{Faith}_\zeta$ is closed with respect kernels of epimorphisms.

The property i) and the closure with respect kernels of epimorphisms are not valid for the general case, as it is showed in the next example.

Example 2.4. Let p be a prime integer, $\mathbb{J}_p$ the ring of p-adic integers and $\mathbb{Z}(p^\infty) \cong \mathbb{Z}_p/\mathbb{Z}$, where $\mathbb{Z}_p = \{\frac{m}{p^k} \mid m, k \in \mathbb{Z},\ k \geq 0\} \leq \mathbb{Q}$. Observe that $\mathbb{J}_p$ is the endomorphism ring of the $\mathbb{Z}$-module $\mathbb{Z}(p^\infty)$. Moreover $\mathbb{Z}(p^\infty)$ is injective as a $\mathbb{Z}$-module, and also as a $\mathbb{J}_p$-module (see [7]).

We consider the functors

$$\mathrm{F} = \mathrm{Hom}_{\mathbb{J}_p}(-, \mathbb{Z}(p^\infty)) : \mathbb{J}_p\text{-Mod} \leftrightarrows \text{Mod-}\mathbb{Z} : \mathrm{Hom}_{\mathbb{Z}}(-, \mathbb{Z}(p^\infty)) = \mathrm{G}.$$

If $Q = \mathbb{J}_p$ then $V = \mathrm{F}(Q) = \mathbb{Z}(p^\infty)$, and it is not hard to see that Q is F-reflexive.

If $K \in \mathrm{cog}(Q)$ then it is a finitely generated torsion-free $\mathbb{J}_p$-module, hence it is free. Then $\mathrm{cog}(Q) = \{\mathbb{J}_p^n \mid n \in \mathbb{N}\} = \mathrm{cop}(Q)$. Moreover, the $\mathbb{J}_p$-module $\mathbb{Z}(p^\infty)$ is injective, hence $(Q, \mathrm{F}, \mathrm{G})$ satisfies the condition b) in Theorem 2.3.

However,

1. there exists an exact sequence

$$(\star)\ 0 \to p\mathbb{J}_p \to \mathbb{J}_p \to \mathbb{J}_p/p\mathbb{J}_p \to 0$$

   such that F is exact with respect $(\star)$, but $\mathbb{J}_p/p\mathbb{J}_p \notin \mathrm{cog}(Q)$.

2. since we have an exact

$$0 \to \mathbb{Z}(p) \to \mathbb{Z}(p^\infty) \to \mathbb{Z}(p^\infty) \to 0,$$

   the class $\mathrm{Pres}(V) \cap \mathrm{Faith}_\zeta$ is not closed with respect kernels of epimorphisms.

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ON THE STABILITY OF DISTRIBUTIONS OF THE COMPOSED RANDOM VARIABLE BASED ON THE STABILITY OF THE SOLUTION OF THE DIFFERENTIAL EQUATIONS FOR CHARACTERISTIC FUNCTIONS

PHAM VAN CHUNG

Abstract. In this paper we give some conditions for the stability of the distribution functions of composed random variables by considering the stability of the solutions of differential equations for characteristic functions.

1. **Introduction**

We consider a random variable (r.v.)

$$\eta = \sum_{k=1}^{\nu} \xi_k, \tag{1.1}$$

where $\xi_1, \xi_2, ...$ are i.i.d. random variables possessing the same distribution function $F(x)$ with the corresponding characteristic function $\varphi(t)$, ν is a positive valued r.v. independent of all ξ_k $(k = 1, 2, ...)$ and has the moment generating function $a(z)$.

The r.v. η is called the composed random variable of ξ_j, η has the characteristic function defined by

$$\psi(t) = a[\varphi(t)].$$

In [2], [3] and [4], we obtained the following results.

1. Suppose that ν follows the Poisson law with the parameter λ and ξ follows the exponential law with the parameter θ. If the statistic T_1 is zero-regression w.r.t

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the statistic λ_1 (T_1 and λ_1 were showed in [2]), then the characteristic function $\psi_1(t)$ of η satisfies the following equation

$$3[\psi_1^{''}(t)]^2\psi_1^2(t) - 2\psi_1^{'}(t)\psi_1^{'''}(t)\psi_1^2(t) - [\psi_1^{'}(t)]^4 = 0 \tag{1.2}$$

where $\psi_1(0) = 0; \psi_1^{'}(0) = i\lambda\theta; \psi_1^{''}(0) = -\lambda\theta^2(2 + \lambda)$.

2. Assume that ν follows the Poisson law with the parameter λ, ξ follows the negative binomial distribution function with the parameters p and q. If the statistic T_2 is zero-regression with the statistic λ_1 (T_2 and λ_1 were showed in [2]) then the characteristic function $\psi_2(t)$ of η satisfies the following equation

$$[\psi_2^{'}(t)]^4 + 2\psi_2^{'''}(t)\psi_2^{'}(t)\psi_2^2(t) - 3[\psi_2^{''}(t)]^2\psi_2^2(t) - \psi_2^{'}(t)\psi_2^2(t) = 0, \tag{1.3}$$

where $\psi_2(0) = 1; \psi_2^{'}(0) = i\lambda\dfrac{q}{p}; \psi_2^{''}(0) = -\dfrac{\lambda^2q^2}{p^2} - \dfrac{2\lambda q^2}{p^2}$.

3. If ν follows the Poisson law with the paramater λ, ξ follows the Normal law $N(0, 1)$ and if the statistic T_3 is zero-regression with the statistic λ_1 (T_3 and λ_1 were showed in [2]) then the characteristic function $\psi_3(t)$ of η satisfies the following equation

$$\begin{gathered}\psi_3^{(4)}(t)\psi_3^{''}(t)\psi_3^4(t) - \psi_3^{(4)}(t)[\psi_3^{'}(t)]^2\psi_3^3(t) + 2\psi_3^{'}(t)\psi_3^{''}(t)\psi_3^{'''}(t)\psi_3^3(t)\\ -3[\psi_3^{''}(t)]^2\psi_3^3(t) + 6\psi_3^{''}(t)[\psi_3^{'}(t)]^2\psi_3^2(t) + 6\psi_3^{''}(t)[\psi_3^{'}(t)]^4\psi_3(t)\\ +2\psi_3^{''}(t) + 2[\psi_3^{''}(t)]^2\psi_3^4(t) - [\psi_3^{'}(t)]^4\psi_3^2(t) - [\psi_3^{'''}(t)]^2\psi_3^2(t)\\ -2\psi_3^{'}(t)\psi_3^{''}(t)\psi_3^4(t) = 0,\end{gathered}$$

where

$$\psi_3(0) = 1; \psi_3^{'}(0) = 0; \psi_3^{''}(0) = -\lambda; \psi_3^{'''}(0) = 0. \tag{1.4}$$

4. If ν follows the binomial law with the parameters p and q, ξ follows the exponential law with the parameter θ, and if the statistic T_4 is zero-regression with the statistic λ_1 (T_4 and λ_1 were showed in [2]) then the characteristic function $\psi_4(t)$ of η satisfies the following equation

$$3n^2[\psi_4^{''}(t)]^2\psi_4^2(t) - 2n^2\psi_4^{'}(t)\psi_4^{'''}(t)\psi_4^2(t) - (n^2 - 1)[\psi_4^{'}(t)]^4 = 0, \tag{1.5}$$

where $\psi_4(0) = 1; \psi_4^{'}(0) = inp\theta; \psi_4^{''}(0) = -n^2\theta^2p^2 - n\theta^2p^2$.

5. If ν follows the negative binomial law with the parameters p and q, ξ follows the exponential law with the parameter θ, and if the statistic T_5 is zero-regression with the statistic λ_1 (T_5 and λ_1 were showed in [2]) then the characteristic function $\psi_5(t)$ of η satisfies the following equation

$$3[\psi_5^{''}(t)]^2 - 2\psi_5^{'}(t)\psi_5^{'''}(t) = 0, \tag{1.6}$$

where $\psi_5(0) = 1; \psi_5^{'}(0) = i\theta\frac{q}{p}; \psi_5^{''}(0) = -\frac{2\theta^2 q}{p^2}$.

6. If ν follows geometric law with the parameters α and β, ξ follows the exponential law with the parameter θ and if the statistic T_6 is zero-regression with the statistic λ_1 (T_6 and λ_1 were showed in [4]) then the characteristic function $\psi_6(t)$ of η satisfies the following equation

$$3[\psi_6^{''}(t)]^2 - 2\psi_6^{'}(t)\psi_6^{'''}(t) = 0, \tag{1.7}$$

where $\psi_6(0) = 1; \psi_6^{'}(0) = i\frac{\theta}{\alpha}; \psi_6^{''}(0) = -2(\frac{\theta}{\alpha})^2$.

7. If ν follows the Geometric law with the parameters α and β, ξ follows the negative binomial law with the parameters p and q, and if the statistic T_7 is zero-regression with the statistic λ_1 (T_7 and λ_1 were showed in [4]) then the characteristic function $\psi_7(t)$ of η satisfies the equation

$$\{[\psi_7^{''}(t)]^2 - \psi_7^{'}(t)\psi_7^{'''}(t)\}\psi_7^2(t) + 2[\psi_7^{'}(t)]^2\psi_7^{''}(t)\psi_7(t) - 2[\psi_7^{'}(t)]^4 = 0, \tag{1.8}$$

where $\psi_7(0) = 1; \psi_7^{'}(0) = \frac{iq}{p\alpha}; \psi_7^{''} = -\frac{q(1-\beta p + q)}{\alpha^2 p^2}$.

In [2] and [4] we have considered also the stability of the composed random variables and we showed that if the condition that the statistics T_i ($i = 1, 5, 6, 7$) are zero-regression with the statistic λ_1 (T_i and λ_1 were showed in [2], [4]) is replaced by the condition that $T_i (i = 1, 5, 6, 7)$ are ϵ-zero regression with the statistic λ_1 (for some small enough number ϵ) then the characteristic functions $\psi_i(t)$ of η have to satisfy the differential equations which have the same left sides of the differential equations (1.2), (1.3),...,(1.8) but their right sides are functions $r_i(t)$ which are small enough for all t.

Let us consider the following differential equations

$$F(\psi(t), \psi^{'}(t), \psi^{''}(t), ..., \psi^{(n)}(t)) = 0 \tag{1.9}$$

and

$$F(\psi(t), \psi^{'}(t), \psi^{''}(t), ..., \psi^{(n)}(t)) = r(t), \tag{1.10}$$

where $r(t) = \overline{r(t)}, r(0) = 0, |r(t)| \leqslant \epsilon$ (for some small enough number ϵ). If the function F in equations (1.9) and (1.10) satisfies the condition $\frac{\partial F}{\partial \psi^{(n)}} \neq 0$ then from (1.9) we can represent $\psi(t)$ in the form

$$\psi^{(n)}(t) = f[\psi(t), \psi^{'}(t), ..., \psi^{(n-1)}(t)], \tag{1.11}$$

where the solution $\psi_\epsilon(t)$ of equation (1.10) can be represented in the form

$$\psi_\epsilon^{(n)}(t) = f[\psi_\epsilon(t), \psi_\epsilon^{'}(t), ..., \psi_\epsilon^{(n-1)}(t)] + a(t), \tag{1.12}$$

where $|a(t)| < \epsilon$.

A problem arisen is that under which condition imposing on the function $f(x_1, x_2, ..., x_n)$, the solution of the differential equation (1.9), is stable in the following sense: there exist $T = T(\epsilon)$ such that $T(\epsilon) \to \infty$ when $\epsilon \to 0$, and $\delta = \delta(\epsilon)$, such that $\delta(\epsilon) \to 0$ when $\epsilon \to 0$ such that

$$|\psi_\epsilon(t) - \psi(t)| < C\delta(\epsilon), \text{for all } \ t, |t| \leqslant T(\epsilon),$$

where C is a constant independent of ϵ.

2. **Stability theorem of the solution of the differential equations**

Let us consider the differential equations (1.9) and (1.10) with solutions satisfying the equations (1.11) and (1.12).

Theorem 2.1. *If the function $f(x_1, x_2, ..., x_n)$ is continuous, differentiable in variables and satisfies the Lipschitz's condition, that means there exists a positive constant N, such that*

$$|f(x_1, x_2, ..., x_n) - f(y_1, y_2, ..., y_n)| \leqslant N \sum_{i=1}^{n} |x_i - y_i|$$

for all $(x_1, x_2, ..., x_n) \in R^n$ *and* $(y_1, y_2, ..., y_n) \in R^n$ *and if* $\psi(t)$ *is a bounded function and satisfies the conditions:*

$$\exists M \in \mathbb{R}^1, 0 < M < +\infty, |\psi^{(k)}(t)| < M, \textit{for all } \ k = 1, 2, ..., n; \ \ \textit{for all } \ t,$$

then, for every small enough positive number ϵ*, there exists a positive number* $T = T(\epsilon)$*,* $T(\epsilon) \to \infty$ *when* $\epsilon \to 0$ *and a positive number* $\delta, 0 < \delta < 1$*, such that*

$$|\psi_\epsilon(t) - \psi(t)| < C\epsilon^{1-\delta}, \textit{for all } \ t, |t| \leqslant T(\epsilon),$$

where C *is a constant independent of* ϵ*.*

Lemma 2.1. *Suppose that all eigenvalues of a constant matrix* A *have negative real parts, then there exist constants* $\alpha > 0$ *and* $\beta > 0$*, such that*

$$||e^{At}|| \leqslant \beta e^{-\alpha t}. \tag{2.1}$$

where $||\cdot||$ *denotes the norm in the space of the square matrics and* $e^{At} = \sum_{k=0}^{\infty} \dfrac{(At)^k}{k!}$*.*

Lemma 2.2. *Suppose that* $u(t)$ *and* $f(t)$ *are integrable nonegative real functions on* $[t_0, t_0 + T]$ *and* $K(t, s)$ *is a nonegative real function, bounded on* $[t_0, t_0 + T]$*.*

If the following inequality holds:

$$u(t) \leqslant f(t) + \int_{t_0}^{t} K(t, s)u(s)ds, \tag{2.2}$$

then

$$u(t) \leqslant h(t), \textit{for all } \ t, \ \ t_0 \leqslant t \leqslant t_0 + T, \tag{2.3}$$

where $h(t)$ *is the solution of the equation*

$$h(t) = f(t) + \int_{t_0}^{t} K(t, s)h(s)ds. \tag{2.4}$$

Proof of the theorem 2.1. At first, we consider $t \geq 0$, (the case $t \leqslant 0$ is carried out similarly).

Putting $x_1 = \psi(t), x_2 = \psi^{'}(t), ..., x_n = \psi^{(n-1)}(t)$, then the differential equation (1.11) can be written in the form

$$\begin{cases} \dfrac{dx_1}{dt} = x_2 \\ \dfrac{dx_2}{dt} = x_3 \\ \cdots \\ \dfrac{dx_n}{dt} = f(x_1, x_2, ..., x_n). \end{cases} \tag{2.6}$$

Let us denote $X = (x_1, x_2, ..., x_n)^T \in \mathbb{R}^n$

$$A = \begin{bmatrix} -n & 1 & 1 & \cdots & 1 \\ 1 & -n & 1 & \cdots & 1 \\ . & . & . & \cdots & . \\ 1 & 1 & 1 & \cdots & -n \end{bmatrix}$$

$$G(X) = \begin{bmatrix} +nx_1 - x_3 & \cdots - x_n \\ -x_1 + nx_2 & \cdots - x_n \\ .. & \cdots \\ -x_1 - x_2 & \cdots + nx_{n-1} \\ -x_1 - x_2 & \cdots - x_{n-1} + nx_n + f(x_1, x_2, ..., x_n) \end{bmatrix}.$$

Then the differential equation (2.6) reduces to the equation:

$$\frac{dX}{dt} = AX + G(X). \tag{2.7}$$

By a similar way, the differential equation (1.10) can be rewritten as follows

$$\frac{dY}{dt} = AY + G(Y) + a(t), \tag{2.8}$$

where $Y = (y_1, y_2, ..., y_n)^T \in \mathbb{R}^n, y_1 = \psi_\epsilon(t), ..., y_n = \psi_\epsilon^{(n-1)}(t)$ and $a(t)$ is given in (1.12).

Since $f(x_1, x_2, ..., x_n)$ is continuous and differentiable function in variables and satisfies the Lipshitz condition, there exists a positive constant l, such that

$$||G(X) - G(Y)|| \leqslant l||X - Y|| \text{ for all } X, Y \in \mathbb{R}^n. \tag{2.9}$$

On the other hand, we have

$$det(A - \lambda E) = (\lambda + 1)(\lambda + n + 1)^{n-1}, \tag{2.10}$$

so, the eigenvalues of matrix A are

$$\lambda_1 = -1, \lambda_2 = -(n+1) = \lambda_3 = ... = \lambda_n.$$

We see that the eigenvalues of matrix A have negative real parts.

According to the Lemma 2.1, there exists constants $\alpha, \beta > 0$, such that

$$||e^{At}|| \leqslant \beta e^{-\alpha t}. \tag{2.11}$$

From (2.7) and (2.8) we get

$$X(t) = e^{At}X(0) + \int_0^t e^{A(t-s)}G[X(s)]ds \tag{2.12}$$

$$Y(t) = e^{At}Y(0) + \int_0^t e^{A(t-s)}G[Y(s)]ds + \int_0^t e^{A(t-s)}a(s)ds. \tag{2.13}$$

Since $X(0) = Y(0)$,

$$||X(t) - Y(t)|| \leqslant \int_0^t ||e^{A(t-s)}||.||G[X(s)] - G[Y(s)]||ds + \int_0^t ||e^{A(t-s)}||.|a(s)|ds.$$

Using the estimations $(2.9), (2.11)$ and by (1.12) we have

$$||X(t) - Y(t)|| \leqslant \beta e^{-\alpha t} \int_0^t le^{\alpha s}||X(s) - Y(s)||ds + \beta e^{-\alpha t}\epsilon \int_0^t e^{\alpha s}ds.$$

Hence

$$||X(t) - Y(t)||e^{\alpha t} \leqslant \beta\epsilon \int_0^t e^{\alpha s}ds + \int_0^t \beta l||X(s) - Y(s)||e^{\alpha s}ds. \tag{2.14}$$

If we put $||X(t) - Y(t)||e^{\alpha t} = u(t), f(t) = \beta\epsilon \int_0^t e^{\alpha s}ds, K(s,t) = l\beta$ and $t_0 = 0$, and by the lemma 2.2, we have the following estimation

$$u(t) \leqslant f(t) + \int_0^t \beta l u(s)ds. \tag{2.15}$$

It follows from the Lemma 2.2 that

$$u(t) \leqslant \psi(t),$$

where $\psi(t)$ is the solution of equation

$$\psi(t) = f(t) + \int_0^t \beta l \psi(s) ds.$$

Therefore we have

$$\begin{aligned}\psi(t) &= e^{\int_0^t \beta l ds}[f(0) + \int_0^t f^{'}(s) e^{-\int_0^s \beta l ds} ds] \\ &= e^{\beta l t} \int_0^t \beta \epsilon e^{\alpha s - \beta l s} ds \\ &= \frac{\beta \epsilon}{\alpha - \beta l}(e^{\alpha t} - e^{\beta l t}).\end{aligned}$$

So we obtain

$$||X(t) - Y(t)|| \leqslant \frac{\beta}{\alpha - \beta l} \epsilon (1 - e^{\beta l t - \alpha t}). \tag{2.16}$$

If $\alpha - \beta l > 0$ then $||X(t) - Y(t)|| \leqslant \dfrac{\beta}{\alpha - \beta l}\epsilon$ for all t.

If $\alpha - \beta l < 0$, then

$$\frac{\beta}{\alpha - \beta l} \epsilon (1 - e^{\beta l t - \alpha t}) \leqslant \frac{\beta}{|\alpha - \beta l|} \epsilon e^{(\beta l - \alpha) t}.$$

Now if we choose $T(\epsilon) = \dfrac{1}{\beta l - \alpha} \ln(\dfrac{1}{\epsilon})^{\delta}$, where $0 < \delta < 1$, then

$$T(\epsilon) \to \infty, \quad \text{when} \quad \epsilon \to 0.$$

So, for all t, $0 < t \leqslant T(\epsilon)$ we get the estimation:

$$||X(t) - Y(t)|| \leqslant \frac{\beta}{|\alpha - \beta l|} \epsilon^{1-\delta} = C \epsilon^{1-\delta},$$

where C is a constant independent of ϵ.

3. **Stability theorems for the distribution of the composed random variable**

Let us consider the composed random variable η in (1.1)

$$\eta = \sum_{k=1}^{\nu} \xi_k.$$

Suppose that $(X_1, X_2, ..., X_n)$ is n independent observations on η and that the absolute moments $E(|\eta|^k)$ for $k = 1, 2, 3, 4$ are finite.

We put

$$\lambda_k = \sum_{i=1}^{n} X_i^k \quad (k = 1, 2, 3, 4).$$

- $T_1 = A_1\lambda_4 + 3B_1\lambda_2^2 + 2C_1\lambda_1\lambda_3 + 6\lambda_2\lambda_1^2 - \lambda_1^4$, where

$$A_1 = n(5-n); B_1 = n^2 - 5n + 5; C_1 = (n^2 - 5n + 10) \tag{3.1}$$

- $T_2 = A_2\lambda_4 + 3B_2\lambda_2^2 - C_2\lambda_1\lambda_3 + 6\lambda_2\lambda_1^2 - \lambda_1^4 + (n-2)(n-3)(\lambda_1^2 - \lambda_2)$

$$A_2 = n(5-n); B_2 = n^2 - 5n + 5; C_2 = n^2 - 5n + 10 \tag{3.2}$$

- $T_3 = A_3\lambda_6 + B_3\lambda_3\lambda_1 + C_3\lambda_4\lambda_2 + E_3\lambda_1 + F_3\lambda_3 + G_3\lambda_1\lambda_2\lambda_3 + 2\lambda_1^6$

$$+(n-4)(n-5)[M_3\lambda_4 + N_3\lambda_1\lambda_2 + P_3\lambda_1^2 + \lambda_1\lambda_2 - \lambda_1^4]$$

$$+H_3\lambda_3\lambda_1^2 + K_3\lambda_2\lambda_1^2 + L_3\lambda_2\lambda_1^4,$$

where

$$A_3 = -4n(n-1)(n-2); B_3 = 24(n^2 - 3n + 2)$$

$$C_3 = n^4 - 6n^3 - 5n^2 - 60n - 120$$

$$D_3 = -n^3 + 6n^2 - 65n + 1; F_3 = 3(-n^3 + 10n^2 - 35n + 40)$$

$$E_3 = -n^4 + 12n^2 - 35n + 20; G_3 = 5(n^2 - 3n + 5), L_3 = -6n$$

$$M_3 = n(5-n); N_3 = 2(-n^2 + 5n - 13); P_3 = 3(n^2 - 5n + 7) \tag{3.3}$$

- $T_4 = A_4\lambda_4 + 3B_4\lambda_2^2 + 2C_4\lambda_1\lambda_3 + 6\lambda_2\lambda_1^2 - \lambda_1^4$, where

$$A_4 = \frac{-n^4 + 5n^3 - 6}{n^2 - 1}; B_4 = \frac{n^4 - 5n^3 + 5n^2 + 1}{n^2 - 1}; C_4 = \frac{-n^4 + 5n^3 - 10n^2 + 4}{n^2 - 1} \tag{3.4}$$

- $T_5 = 3\lambda_2^2 - 2\lambda_1\lambda_3 - \lambda_4$ (3.5)
- $T_6 = 3\lambda_2^2 - 2\lambda_1\lambda_3 - \lambda_4$ (3.6)
- $T_7 = A_7\lambda_3\lambda_1 + B_7\lambda_2^2 + C_7\lambda_1^2\lambda_2 + H_7\lambda_4 + 2\lambda_1^4$, where

$$A_7 = n^2 - n + 10; B_7 = -n^2 + 7n - 6; C_7 = -2(n+3); H_7 = -4n \tag{3.7}$$

(notice that the statistics T_1, T_5 are considered in [2] and the statistics T_6, T_7 are considered in [4]).

Definition 3.1. *Let X and Y be two random variables with $EY < \infty$. Y is said to be ϵ- zero regression with respect to X if*

$$|E(Y/X)| \leqslant \epsilon. \tag{3.8}$$

Definition 3.2. *The composed r.v. η with the distribution $\Psi_\epsilon(x))$ is called r.v. with the $\epsilon-$ approximate distribution function $\Psi_0(t)$ if $\lambda(\Psi_\epsilon; \Psi_0) \leqslant \epsilon$, where metric $\lambda(.;.)$ is defined as follows*

$$\lambda(\boldsymbol{\Psi}_\epsilon; \boldsymbol{\Psi}_{\mathbf{0}}) = \min_{\mathbf{T}>\mathbf{0}} \quad \max\{\max_{|\mathbf{t}|\leqslant \mathbf{T}} \frac{\mathbf{1}}{\mathbf{2}}|(\psi_\epsilon(\mathbf{t}) - \psi_{\mathbf{0}}(\mathbf{t}))|; \frac{\mathbf{1}}{\mathbf{T}}\}, \tag{3.9}$$

where $\psi_0(t), \psi_\epsilon(t)$ are the characteristic functions corresponding to the distributions $\Psi_0(x), \Psi_\epsilon(x)$ respectively.

Now we obtain the following stability theorems:

Theorem 3.1. *If the statistic $T_i (i = 1, 2, 3, 4, 5, 6, 7)$ is ϵ-zero regression with respect to the λ_1 then the $\psi_{i\epsilon}(i = 1, 2, 3, 4, 5, 6, 7)$ satisfies the following differential equations with the same left sides as in (1.2), (1.3), (1.4), (1.5), (1.6), (1.7) but their right sides are the following functions:*

$$\frac{r(t)}{i^4(n-1)(n-2)(n-3)n}, \quad for (1.2) \tag{3.10}$$

where $r(0) = 0, r(t) = \overline{r(-t)}, |r(t)| \leqslant \epsilon \quad \forall t,$

$$\frac{r(t)}{i^4(n-1)(n-2)(n-3)n}, \quad for (1.3) \tag{3.11}$$

where $r(0) = 0, r(t) = \overline{r(-t)}, |r(t)| \leqslant \epsilon \quad \forall t,$

$$\frac{r(t)}{i^4 n(n-1)(n-2)(n-3)(n-4)(n-5)}, \quad for (1.4) \tag{3.12}$$

where $r(0) = 0, r(t) = \overline{r(-t)}, |r(t)| \leqslant \epsilon \quad \forall t,$

$$\frac{(n+1)r(t)}{(n-2)(n-3)n}, \quad for (1.5) \tag{3.13}$$

where $r(0) = 0, r(t) = \overline{r(-t)}, |r(t)| \leqslant \epsilon \quad \forall t,$

$$\frac{r(t)}{i^4(n-1)n}, \quad for (1.6) \tag{3.14}$$

where $r(0) = 0, r(t) = \overline{r(-t)}, |r(t)| \leqslant \epsilon \quad \forall t,$

$$\frac{r(t)}{i^4(n-1)n}, \quad for(1.7) \tag{3.15}$$

where $r(0) = 0, r(t) = \overline{r(-t)}, |r(t)| \leqslant \epsilon \quad \forall t,$

$$\frac{r(t)}{i^4 n(n-1)(n-2)(n-3)}, \quad for(1.8) \tag{3.16}$$

where $r(0) = 0, r(t) = \overline{r(-t)}, |r(t)| \leqslant \epsilon \quad \forall t.$

Theorem 3.2. *Let* $\Psi_i(x)(i = 1, 2, 4, 5, 6, 7)$ *be distribution functions with respect to the characteristic functions* $\psi_i(t)$ *of the composed ramdom variable where* $\psi_i(t)$ *satisfy the differential equation* (1.2), (1.3), (1.5), (1.6), (1.7), (1.8).

Suppose that $\Psi_{i\epsilon}(x)$ *is distribution of the composed random variable corresponding to the characteristic function* $\psi_{i\epsilon}(t)$*, which is the solution of the equation with the same left side of the equation* (1.2), (1.3), (1.5), (1.6), (1.7), (1.8), *respectively, but with the right side defined by* (3.10), (3.11), (3.13), (3.14), (3.15), (3.16), *respectively, then the distribution function of composed random variable* $\Psi_{i\epsilon}(x)$ *is* $C\gamma(\epsilon)$*-approximate* $\Psi_i(x)$ *respectively, where* $\gamma(\epsilon) = \max\{\epsilon^{1-\delta}, \dfrac{1}{\delta \ln(\frac{1}{\epsilon})}\}$ *for* ϵ *is small enough positive number and* C *is a constant independent of* ϵ, $(0 < \delta < 1)$.

The proof of the theorem 3.1 is carried out similarly as in proof of the theorem in [4] by using definition 3.1.

The conclusion of the theorem 3.2 follows directly from the above definition 3.2 and Theorem 2.1 and with notice that $f(x_1, x_2, ..., x_n)$ in (1.11) and (1.12) to be continuous and piecewise smooth on its domain, therefore it satisfies the Lipschitz's condition.

Notice that the differential equation with the left side (1.4) and right side (3.12) does not satisfy the condition (1.19), therefore the theorem 3.2 is not valid for this case.

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COVERING SUBGROUPS IN FINITE PRIMITIVE π-SOLVABLE GROUPS

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Abstract. Let π be an arbitrary set of primes and let X be a π-closed Schunck class. The paper deals with the study of X-covering subgroups in finite primitive π-solvable groups, connecting them with complements, stabilizers and X-maximal subgroups. Some characterization theorems for X-covering subgroups in finite primitive π-solvable groups by means of complements of appropriate minimal normal subgroups, by means of stabilizers and by means of some X-maximal subgroups are given.

1. **Preliminaries**

All groups considered in this paper are finite. Let π be a set of primes and π' the complement to π in the set of all primes.

We first remind some definitions and theorems which will be useful for our considerations.

Definition 1.1. a) Let G be a group, M and N two normal subgroups of G such that $N \subseteq M$. The factor M/N is called a *chief factor* of G if M/N is a minimal normal subgroup of G/N.

b) A group G is said to be π-*solvable* if every chief factor of G is either a solvable π-group or a π'-group. In particular, for π the set of all primes we obtain the notion of solvable group.

Definition 1.2. a) Let G be a group and W a subgroup of G. We define

$$core_G W = \cap\{W^g \mid g \in G\},$$

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where $W^g = g^{-1}Wg$.

b) W is a *stabilizer* of G if W is a maximal subgroup of G and $core_G W = 1$.

c) A group G is said to be *primitive* if there exists a stabilizer W of G.

In the formation theory are well-known the following notions:

Definition 1.3. a) A class X of groups is a *homomorph* if X is closed under homomorphisms, i.e. if $G \in X$ and N is a normal subgroup of G, then $G/N \in X$.

b) A homomorph X is a *Schunck class* if X is primitively closed, i.e. if any group G, all of whose primitive factor groups are in X, is itself in X.

Definition 1.4. a) A class X of groups is called π-*closed* if

$$G/O_{\pi'}(G) \in X \ \Rightarrow \ G \in X,$$

where $O_{\pi'}(G)$ denotes the largest normal π'-subgroup of G.

b) We shall call π-*homomorph*, respectively π-*Schunck class*, a π-closed homomorph, respectively a π-closed Schunck class.

Definition 1.5. Let X be a class of groups, G a group and H a subgroup of G.

a) H is an X-*maximal subgroup* of G if:

(i) $H \in X$;

(ii) $H \leq H^* \leq G,\ H^* \in X \ \Rightarrow \ H = H^*$.

b) H is an X-*covering subgroup* of G if:

(i) $H \in X$;

(ii) $H \leq K \leq G,\ K_0 \trianglelefteq K,\ K/K_0 \in X \ \Rightarrow \ K = HK_0$.

Remark 1.6. If X is a class of groups, G is a group and H is an X-*covering subgroup* of G, then H is X-maximal in G.

The following results will be used in the paper:

Theorem 1.7. ([1]) *A solvable minimal normal subgroup of a finite group is abelian.*

Theorem 1.8. ([2], [3]) *Let G be a primitive π-solvable group. If G has a minimal normal subgroup which is a solvable π-group, then G has one and only one minimal normal subgroup.*

Theorem 1.9. ([3]) *If G is a primitive π-solvable group, $V < G$, such that there exists a minimal normal subgroup M of G which is a solvable π-group and $MV = G$, then V is a stabilizer of G.*

Theorem 1.10. ([5]) *Let X be a π-homomorph. The following conditions are equivalent:*

(1) X is a Schunck class;

(2) if G is a π-solvable group, $G \notin X$ and N is a minimal normal subgroup of G such that $G/N \in X$, then N has a complement in G;

(3) any π-solvable group G has X-covering subgroups.

Theorem 1.11. ([5]) *Let X be a π-Schunck class, G a π-solvable group, $G \notin X$, N a minimal normal subgroup of G such that $G/N \in X$ and H an X-covering subgroup of G. Then H is a complement of N in G, i.e. $G = HN$ and $H \cap N = 1$.*

Theorem 1.12. ([5]) *If X is a π-Schunck class, G is a π-solvable group, $G \notin X$ and N is a minimal normal subgroup of G such that $G/N \in X$, then:*

a) N has a complement H in G;

b) H is X-maximal in G;

c) H is conjugate to any X-maximal subgroup S of G with $NS = G$.

2. On stabilizers in finite primitive π-solvable groups

Lemma 2.1. *If G is a group and W a stabilizer of G, then:*

a) for any normal subgroup $K \neq 1$ of G, we have $KW = G$;

b) for any minimal normal subgroup M of G, we have $MW = G$.

Proof. a) Let $K \neq 1$ be a normal subgroup of G. Since W is maximal in G and $W \leq KW \leq G$, we have $KW = W$ or $KW = G$. Suppose that $KW = W$. It follows that $K \leq W$ and so $K^g \leq W^g$ for any $g \in G$. But K being normal in G, $K^g = K$ for any $g \in G$. Then $K \leq W^g$ for any $g \in G$, hence $K \leq core_G W = 1$. So $K = 1$, in contradiction to our hypothesis. So $KW = G$.

b) Follows immediately from a). $\square$

Theorem 2.2. *Let G be a π-solvable group, W a stabilizer of G and M a minimal normal subgroup of G such that M is a solvable π-group. Then W is a complement of M in G, i.e. $MW = G$ and $M \cap W = 1$.*

Proof. $MW = G$ follows from Lemma 2.1. Let us now prove that $M \cap W = 1$. Since M is normal in G and $W \leq G$, we have that $M \cap W$ is normal in W. By 1.7, M is abelian. In order to prove that $M \cap W$ is normal in G, consider $g \in G$ and $m \in M \cap W$. Since $G = MW$, we have $g = nw$, where $n \in M$ and $w \in W$. So

$$g^{-1}mg = (nw)^{-1}m(nw) = w^{-1}n^{-1}mnw = w^{-1}n^{-1}nmw = w^{-1}mw \in M \cap W,$$

where we used that M is abelian and that $M \cap W \trianglelefteq W$. It follows that $M \cap W$ is normal in G. From this and from the fact that M is a minimal normal subgroup of G, we deduce that $M \cap W = 1$ or $M \cap W = M$. But $M \cap W = M$ leads to $M \subseteq W$, hence $G = MW = W$, in contradiction with the hypothesis that W is a stabilizer of G. So $M \cap W = 1$. □

Theorem 2.3. *Let G be a primitive π-solvable group such that there exists a minimal normal subgroup M of G, M solvable π-group. Let $W < G$. The following two conditions are equivalent:*

(1) W is a stabilizer of G;

(2) $MW = G$.

Proof. By 1.8, M is the unique minimal normal subgroup of G.

(1) $\Rightarrow$ (2): Let W be a stabilizer of G. Applying 2.2, we obtain that $MW = G$.

(2) $\Rightarrow$ (1): Let $MW = G$. Then, by 1.9, W is a stabilizer of G. □

3. Covering subgroups and complements in finite primitive π-solvable groups

In preparation for the main result of this section, we first prove a lemma.

Lemma 3.1. *Let X be a π-homomorph, G a π-solvable group, $G \notin X$ and N a minimal normal subgroup of G such that $G/N \in X$. Then:*

a) N is a solvable π-group;

b) N is abelian.

Proof. a) Since G is a π-solvable group and N is a minimal normal subgroup of G, we conclude that N is either a solvable π-group or a π'-group. Suppose that N is a π'-group. Then $N \leq O_{\pi'}(G) \leq G$, hence

$$G/O_{\pi'}(G) \simeq (G/N)/(O_{\pi'}(G)/N).$$

But $G/N \in X$. Then by the above isomorphism and X being a homomorph, $G/O_{\pi'}(G) \in X$. It follows by the π-closure of X that $G \in X$, a contradiction. This shows that N is a solvable π-group.

b) We apply 1.7 and a) and obtain that N is abelian. □

Theorem 3.2. *Let X be a π-Schunck class, G a finite primitive π-solvable group, $G \notin X$, N a minimal normal subgroup of G such that $G/N \in X$ and let $H \leq G$. The following two conditions are equivalent:*

(1) H is an X-covering subgroup of G;

(2) H is a complement of N in G, i.e. $HN = G$ and $H \cap N = 1$.

Proof. (1) $\Rightarrow$ (2): Let H be an X-covering subgroup of G. By applying 1.11, H is a complement of N in G.

(2) $\Rightarrow$ (1): Let H be a complement of N in G (according to 1.10, H exists), i.e. we have $HN = G$ and $H \cap N = 1$. By lemma 3.1, N is a solvable π-group, hence N is abelian. We will prove that H is an X-covering subgroup of G by verifying conditions (i) and (ii) from 1.5.b).

(i) $H \in X$. Indeed, we have:

$$H \simeq H/1 = H/H \cap N \simeq HN/N = G/N \in X.$$

(ii) Let $H \leq K \leq G$, $K_0 \trianglelefteq K$, $K/K_0 \in X$. We prove that $K = HK_0$. For this, we first prove that H is a maximal subgroup of G. Indeed, $H \neq G$ (since $H \in X$ and $G \notin X$) and let now $H \leq H^* < G$. In order to show that $H = H^*$, suppose $H < H^*$. Then there exists an element $h^* \in H^* \setminus H \subset G = HN$ and so

$$h^* = hn, \text{ with } h \in H, \ n \in N$$

hence

$$n = h^{-1}h^* \in H^* \cap N.$$

Let us show that $H^* \cap N = 1$. For this, we notice that from $N \trianglelefteq G$ and $H^* \leq G$ follows that $H^* \cap N \trianglelefteq H^*$. Furthermore, $H^* \cap N \trianglelefteq G$, since for any $g \in G$ and any $n \in H^* \cap N$, we have that $g^{-1}ng \in H^* \cap N$, as we show below:

$$g \in G = HN = H^*N = NH^* \;\Rightarrow\; g = mh^*,\; m \in N,\; h^* \in H^*$$

$$\Rightarrow\; g^{-1}ng = (mh^*)^{-1}n(mh^*) = (h^*)^{-1}m^{-1}nmh^*$$

$$= (h^*)^{-1}m^{-1}mnh^* = (h^*)^{-1}nh^* \in H^* \cap N,$$

where we used that N is abelian and that $H^* \cap N \trianglelefteq H^*$. So $H^* \cap N \trianglelefteq G$. But N is a minimal normal subgroup of G, hence $H^* \cap N = 1$ or $H^* \cap N = N$. Suppose $H^* \cap N = N$. Then $N \subseteq H^*$, hence $G = H^*N = H^*$, a contradiction. It follows that $H^* \cap N = 1$. Then

$$n = h^{-1}h^* \in H^* \cap N = 1 \;\Rightarrow\; n = 1 \;\Rightarrow\; h^{-1}h^* = 1 \;\Rightarrow\; h = h^* \in H^* \setminus H,$$

in contradiction with $h \in H$. It follows that H is a maximal subgroup of G. Hence from $H \leq K \leq G$, we have only two possibilities: $K = H$ or $K = G$.

If $K = H$, the hypotheses of (ii) become $H \leq H \leq G$, $K_0 \trianglelefteq H$, $H/K_0 \in X$ and clearly $K = H = HK_0$.

If $K = G$, the hypotheses of (ii) become $H \leq G \leq G$, $K_0 \trianglelefteq G$, $G/K_0 \in X$. We have to prove that $G = HK_0$. Observe that $K_0 \neq 1$. Indeed, supposing that $K_0 = 1$, we have $G \simeq G/1 = G/K_0 \in X$, a contradiction with $G \notin X$. Furthermore, by 1.8, N is the unique minimal normal subgroup of G. Hence for $K_0 \trianglelefteq G$, $K_0 \neq 1$ follows that $N \subseteq K_0$. So $G = HN \subseteq HK_0$, which leads to $K = G = HK_0$. □

Theorems 1.12 and 3.2 have the following consequence:

Corollary 3.3. *Let X be a π-Schunck class, G a finite primitive π-solvable group, $G \notin X$ and N a minimal normal subgroup of G such that $G/N \in X$. Then:*

a) N has a complement H in G;

b) H is an X-covering subgroup of G;

c) H is X-maximal in G;

d) H is conjugate to any X-maximal subgroup S of G with $SN = G$;

e) conditions a) and b) are equivalent.

4. Covering subgroups and stabilizers in finite primitive π-solvable groups

In this section we will establish a connection between covering subgroups and stabilizers in finite primitive π-solvable groups.

Theorem 4.1. *Let X be a π-Schunck class, G a finite primitive π-solvable group, $G \notin X$, N a minimal normal subgroup of G such that $G/N \in X$ and let $H \leq G$. The following two conditions are equivalent:*

(1) H is an X-covering subgroup of G;

(2) H is a stabilizer of G.

Proof. By lemma 3.1, N is a solvable π-group.

(1) $\Rightarrow$ (2): Let H be an X-covering subgroup of G. Then $H \in X$. This implies $H \neq G$, since $G \notin X$. Applying Theorem 3.2, we obtain that $HN = G$. This and $H < G$ show that we are in the hypotheses of Theorem 1.9. It follows that H is a stabilizer of G.

(2) $\Rightarrow$ (1): Let H be a stabilizer of G. Then, by 2.2, H is a complement of N in G. Now by applying Theorem 3.2, we conclude that H is an X-covering subgroup of G. □

Theorems 3.2 and 4.1 have the following corollary:

Corollary 4.2. *Let X be a π-Schunck class, G a finite primitive π-solvable group, $G \notin X$, N a minimal normal subgroup of G such that $G/N \in X$ and let $H \leq G$. The following three conditions are equivalent:*

(1) H is an X-covering subgroup of G;

(2) H is a complement of N in G;

(3) H is a stabilizer of G.

5. X-maximal subgroups and complements in finite π-solvable groups

In this last section of the paper, we show that there is a connection between some particular X-maximal subgroups and the complements of some special minimal normal subgroups in finite π-solvable groups. This connection allows us to characterize the X-covering subgroups in finite primitive π-solvable groups by means of these particular X-maximal subgroups.

Theorem 5.1. *Let X be a π-Schunck class, G a finite π-solvable group, $G \notin X$ and let N be a minimal normal subgroup of G such that $G/N \in X$. Then:*

a) N has a complement H in G; furthermore, H is X-maximal in G and H is conjugate to any X-maximal subgroup S of G with $SN = G$;

b) the following two conditions on $H \leq G$ are equivalent:

(i) H is an X-maximal subgroup of G such that $HN = G$;

(ii) H is a complement of N in G;

c) any two complements H_1 and H_2 of N in G are conjugate in G.

Proof. a) Immediately follows from Theorem 1.12.

b) (i) $\Rightarrow$ (ii): Let H be X-maximal in G such that $HN = G$. We have to prove that $H \cap N = 1$. Observe first that $H \neq G$, since $H \in X$ and $G \notin X$. From $H \leq G$ and $N \trianglelefteq G$ follows that $H \cap N \trianglelefteq H$. Lemma 3.1 implies that N is abelian. Let us now prove that $H \cap N$ is normal in G. Let $g \in G$ and $n \in H \cap N$. Then:

$$g \in G = HN = NH \ \Rightarrow \ g = mh, \text{ where } m \in N,\ h \in H,$$

hence

$$g^{-1}ng = (mh)^{-1}n(mh) = h^{-1}m^{-1}nmh$$

$$= h^{-1}m^{-1}mnh = h^{-1}nh \in H \cap N,$$

where we used that $H \cap N \trianglelefteq H$. In order to prove that $H \cap N = 1$, we consider the normal subgroup $H \cap N$ of G and observe that $H \cap N \subseteq N$, where N is a minimal normal subgroup of G. It follows that $H \cap N = 1$ or $H \cap N = N$. If we suppose that $H \cap N = N$, we obtain $N \subseteq H$, hence $G = HN = H$, in contradiction with $H \neq G$. So $H \cap N = 1$.

(ii) $\Rightarrow$ (i): Let H be a complement of N in G. Hence, by 1.12, H is X-maximal in G. Obviously $HN = G$, H being a complement of N in G.

c) Let H_1 and H_2 be two complements of N in G. Applying b) to H_2, we obtain that H_2 is X-maximal in G and $H_2N = G$. But H_1 is a complement of N in G. Now applying Theorem 1.12.c) it follows that H_1 is conjugate with H_2 in G. □

From Theorem 5.1.b) and Corollary 4.2 follows:

Corollary 5.2. *Let X be a π-Schunck class, G a finite primitive π-solvable group, $G \notin X$, N a minimal normal subgroup of G such that $G/N \in X$ and let $H \leq G$. The following four conditions are equivalent:*

(1) H is a complement of N in G;

(2) H is X-maximal in G and $HN = G$;

(3) H is an X-covering subgroup of G;

(4) H is a stabilizer of G.

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SET-VALUED APPROXIMATION OF MULTIFUNCTIONS

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Abstract. This survey paper introduces several results on approximation of multifunctions with convex and non-convex values. We consider multifunctions having at least nonempty and compact values in $\mathbb{R}^n$. The convex case (when the multifunctions have convex values) is closer to the point-to-point case. The non-convex case (the values of the multifunctions are not longer assumed to be convex) is more challenging. In the convex case we present results on the Bernstein approximation, the Stone-Weierstrass approximation theorem, and the Korovkin-type approximation. In the non-convex case we present results on linear operators on multifunctions based on a metric linear combination of ordered sets, metric piecewise linear approximations of multifunctions, and approximation by metric Bernstein, Schoenberg, and interpolation operators. The present survey paper was introduced at University of Duisburg–Essen located in Duisburg while the author was a visiting scientist under a grant of "Center of Excellence for Applications of Mathematics" supported by DAAD. The author expresses his gratitude to Prof. H. Gonska for his invitation and warm hospitality in Duisburg. The author also appreciates the valuable comments and remarks of Mr. Michael Wozniczka from the same University.

1. Introduction

The aim of the paper is to introduce some older and newer results on approximation of multifunctions with convex and non-convex values.

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Let X, Y be nonempty sets and $\mathrm{P}(Y) = \mathcal{P}(Y) \setminus \{\emptyset\}$ be the collection of nonempty subsets (parts) of Y. A *multifunction, set-valued function*, or *correspondence* is an ordinary map $F : X \to \mathrm{P}(Y)$ (sometimes denoted as $F : X \rightrightarrows Y$), see [2], [19].

Suppose that Y is a real vector space. The so-called "convex case" refers to the case when the images $F(x)$ are convex, for all $x \in X$. Otherwise we say that the "non-convex case" is in force. We just mention that in the cases connected to mathematical economics (Arrow-Debreau economical equilibrium, etc.) for some multifunctions the empty set belongs to the range of F.

We will immediately see why convexity plays such a crucial role here.

The tentative plan consists

- of substituting numbers by sets (which seems to be all right, although the point corresponding to a real number is at least convex and compact);
- and consequently of substituting the operations on numbers by some operations on sets.

The *Minkowski sum* of two non-empty sets (in $\mathbb{R}^n$ or in a vector space) is defined by

$$K + L = \{x + y \mid x \in K, \quad y \in L\}.$$

Nice property: if K and L are singletons, their Minkowski sum is exactly the usual addition of numbers or vectors. Although $\{0\}$ is the identity for addition of sets, i.e., $K + \{0\} = K$, generally no additive inverse exists $(K + X = \{0\}$ cannot be solved for X unless K is reduced to a point). Moreover,

$$K + X = K + Y \not\Longrightarrow X = Y.$$

Multiplication of a set by a scalar is defined by

$$\alpha K = \{\alpha x \mid x \in K\}.$$

For $K = \{0, 1\}$, we have $K + K = \{0, 1, 2\}$ whereas $2K = \{0, 2\}$. It is hard to accept that $K + K \neq 2K$. Thus the distributive law $\alpha K + \beta K = (\alpha + \beta)K$ generally fails to hold. However, if K is convex, then

$$\alpha K + \beta K = (\alpha + \beta)K, \text{ for } \alpha, \beta \geq 0. \tag{1.1}$$

A generalization of (1.1) can be proved by induction, namely, a set K is said to be convex if and only if

$$\alpha_1, \ldots, \alpha_N \geq 0, \quad N \geq 2 \implies \alpha_1 K + \cdots + \alpha_N K = (\alpha_1 + \cdots + \alpha_N)K. \tag{1.2}$$

Equality (1.1) suggests that the class of convex-valued multifunctions might be an appropriate setting in which we might begin considering set-valued approximation problems.

Subtraction is not well defined and is generally impossible.

$X_1 + \cdots + X_N$ generally "gets bigger" as N increases.

Let $\mathbb{K}$ be the *collection of nonempty and compact subsets* of $\mathbb{R}^n$.

We begin by posing the question: Is it possible to *approximate* a multifunction $F : [0,1] \rightrightarrows \mathbb{R}^n$ by a "simpler" one? More concrete, by a *linear* approximant of the form

$$\sum_{j=0}^{N} \varphi_j K_j = \varphi_0 K_0 + \cdots + \varphi_N K_N,$$

where the K_j 's are fixed elements in $\mathbb{K}$ and the φ_j 's are scalar-valued maps defined on $[0,1]$.

Recall that

- the *Bernstein operator* for $f \in C[0,1]$ is

$$\mathrm{B}_N(f,x) = \sum_{k=0}^{N} p_{N,k}(x) f\left(k/N\right), \tag{1.3}$$

where

$$p_{N,k}(x) = \binom{N}{k} x^k (1-x)^{N-k}, \quad x \in [0,1], \tag{1.4}$$

- the *Lagrange interpolation* formula for $f : [a,b] \to \mathbb{R}$ is

$$a = x_0 < x_1 < \cdots < x_{N-1} < x_N = b, \quad L_N(x) = \sum_{k=0}^{N} l_k(x) f(x_k),$$

where

$$l_k(x) = \frac{(x - x_0)\dots(x - x_{k-1})(x - x_{k+1})\dots(x - x_N)}{(x_k - x_0)\dots(x_k - x_{k-1})(x_k - x_{k+1})\dots(x_k - x_N)}, \tag{1.5}$$

• the *Hermite-Fejér polynomial* $H_N(f, x)$ of a function $f : [-1, 1] \to \mathbb{R}$, based on the zeros

$$x_k = x_k^{(N)} = \cos\frac{(2k-1)\pi}{2N}, \quad k = 1, 2, \dots, N, \tag{1.6}$$

of the Chebyshev polynomial $T_N(x) = \cos(N \arccos x)$, is

$$q_k(x) = \left(\frac{T_N(x)}{N(x - x_k)}\right)^2 (1 - x x_k), \quad H_N(x) = \sum_{k=1}^{N} q_k(x) f(x_k), \quad x \in [-1, 1],$$

• and a (univariate, polynomial) *spline* $S : [a, b] \to \mathbb{R}$ is a piecewise polynomial function, that is, it consists of polynomial pieces $P_i : [x_i, x_{i+1}[\to \mathbb{R}$, where $a = x_0 < x_1 < \dots < x_N = b$, $i = 0, 1, \dots, N - 1$, such that

$$\begin{cases} S(x) = P_0(x), & x \in [x_0, x_1[\,, \\ S(x) = P_1(x), & x \in [x_1, x_2[\,, \\ \dots \\ S(x) = P_{N-1}(x), & x \in [x_{N-1}, x_N], \end{cases}$$

and at x_i the two pieces P_{i-1} and P_i share common derivative values.

"*Polynomials are wonderful even after they are cut into pieces, but the cutting must be done with care. One way of doing the cutting leads to the so-called* spline *functions.*" Isaac Jacob Schoenberg (Galatzi 1903 - Madison, WI, 1990) penned these prophetic words in 1964.

Along this paper by $\square$ we denote the end of a proof and by $\triangle$ the end of a remark or an example. We mention some notations that appear along our paper. $\mathbb{B}$ denotes the closed unit ball in $\mathbb{R}^n$, $\mathbb{K}$ the family of nonempty and compact subsets in $\mathbb{R}^n$, $\mathbb{K}_c$ the collection of elements of $\mathbb{K}$ that are also convex, H the Hausdorff-Pompeiu metric on $\mathbb{K}$, $\langle \cdot, \cdot \rangle$ is the inner product on $\mathbb{R}^n$, $\mathbb{S}$ is the unit sphere in $\mathbb{R}^n$, $\sigma(\cdot, \cdot)$ the support function, $\mathbb{C}[\mathbb{K}]$ and $\mathbb{C}[\mathbb{K}_c]$ the spaces of continuous functions on

$[0,1]$ into $\mathbb{K}$ and $\mathbb{K}_c$, respectively, B_n the Banach space of continuous functions defined on the unit sphere in $\mathbb{R}^n$, and $\mathrm{B}_N(\cdot,\cdot)$ the Bernstein operator.

Some results of R. Vitale in [28] are introduced in the sequel.

The *Hausdorff-Pompeiu metric* can be introduced on $\mathbb{K}$ in several ways, one of these being as follows

$$\mathrm{H}(K_1,K_2)=\min\{\varepsilon>0 \mid K_1\subset K_2+\varepsilon\mathbb{B},\quad K_2\subset K_1+\varepsilon\mathbb{B}\} \tag{1.7}$$

where $\mathbb{B}$ is the closed unit ball in $\mathbb{R}^n$. We note that

$$\mathrm{H}(K_1,K_2)<+\infty,\quad \forall\, K_1,K_2\in\mathbb{K}. \tag{1.8}$$

Thus $(\mathbb{K},\mathrm{H})$ is a complete, separable, and locally compact metric space. Define

$$\|K\|=\mathrm{H}(\{0\},K),$$

the "norm" of $K\in\mathbb{K}$.

Proposition 1.1. *Let $A,B\in\mathbb{K}$ and α be a real number. Then*

$$\mathrm{H}(\alpha A,\alpha B)=|\alpha|\,\mathrm{H}(A,B). \tag{1.9}$$

Proof. If $\alpha=0$, we have $\mathrm{H}(\alpha A,\alpha B)=\mathrm{H}(\{0\},\{0\})=0=0\cdot\mathrm{H}(A,B)$. If $\alpha>0$, we successively have

$$\begin{aligned}\mathrm{H}(\alpha A,\alpha B)&=\min\{\varepsilon>0 \mid \alpha A\subset\alpha B+\varepsilon\mathbb{B},\ \alpha B\subset\alpha A+\varepsilon\mathbb{B}\}\\ &=\min\{\varepsilon>0 \mid A\subset B+(\varepsilon/\alpha)\mathbb{B},\ B\subset A+(\varepsilon/\alpha)\mathbb{B}\}\\ &=\alpha\ \min\{\varepsilon/\alpha>0 \mid A\subset B+(\varepsilon/\alpha)\mathbb{B},\ B\subset A+(\varepsilon/\alpha)\mathbb{B}\}=\alpha\,\mathrm{H}(A,B).\end{aligned}$$

If $\alpha<0$, then consider $\beta=-\alpha$ and it follows

$$\begin{aligned}\mathrm{H}(\alpha A,\alpha B)&=\min\{\varepsilon>0 \mid -\beta A\subset-\beta B+\varepsilon\mathbb{B},\ -\beta B\subset-\beta A+\varepsilon\mathbb{B}\}\\ &=\min\{\varepsilon>0 \mid \beta A\subset\beta B+\varepsilon\mathbb{B},\ \beta B\subset\beta A+\varepsilon\mathbb{B}\}=\beta\,\mathrm{H}(A,B)=|\alpha|\,\mathrm{H}(A,B).\end{aligned}$$

□

2. The convex case

2.1. $\mathbb{K}_c$ **.** We denote by $\mathbb{K}_c$ the collection of elements of $\mathbb{K}$ which are also convex. Then $\mathbb{K}_c$ is closed under

- Minkowski addition,
- Minkowski multiplication with scalars,
- the distributive property (1.1),
- $\mathbb{K}_c$ inherits its metric from $\mathbb{K}$ as a closed, separable and locally compact subspace.

Given an element $K \in \mathbb{K}$, we often form its *convex hull* denoted as $\mathrm{conv}K$, which obviously lies in $\mathbb{K}_c$. The map

$$\mathbb{K} \ni K \mapsto \mathrm{conv}K \in \mathbb{K}_c$$

is continuous in respect to the Hausdorff-Pompeiu metric since

$$\mathrm{H}(\mathrm{conv}A, \mathrm{conv}B) \leq \mathrm{H}(A, B), \quad \forall\, A, B \in \mathbb{K}, \tag{2.1}$$

and additionally satisfies $\mathrm{conv}(\alpha K_1 + \beta K_2) = \alpha\, \mathrm{conv}K_1 + \beta\, \mathrm{conv}K_2$, for all $\alpha, \beta \geq 0$.

2.1.1. *Support function.* To each $K \in \mathbb{K}$ we associate its *support function* given by

$$\sigma(p, K) = \max\{\langle p, k\rangle \mid k \in K\}, \quad p \in \mathbb{S}, \tag{2.2}$$

where $\langle \cdot, \cdot \rangle$ is the inner product on $\mathbb{R}^n$ and $\mathbb{S}$ is the unit sphere in $\mathbb{R}^n$.

A set $K \in \mathbb{K}_c$ and a point not in K can always be separated by some hyperplane and this leads to the useful equivalence

$$K_1 \subset K_2 \Longleftrightarrow \sigma(p, K_1) \leq \sigma(p, K_2,), \quad \forall\, p \tag{2.3}$$

and the consequent uniqueness of support functions, namely

$$K_1 = K_2 \Longleftrightarrow \sigma(p, K_1) = \sigma(p, K_2), \quad \forall\, p.$$

Proposition 2.1. *As a function of p, the support function is Lipschitz (and thus continuous), that is*

$$|\sigma(p_1, K) - \sigma(p_2, K)| \leq \|p_1 - p_2\|\, \|K\|.$$

Obviously, we may use and we will do so, the map

$$\mathbb{K}_c \ni K \mapsto \sigma(\cdot, K)$$

to embed $\mathbb{K}_c$ in the Banach space B_n of continuous functions defined on the unit sphere in $\mathbb{R}^n$.

Proposition 2.2. ([17]) *The following properties hold:*

$$\sigma(p, \mathbb{B}) = 1, \quad \forall\, p \in \mathbb{S}, \tag{2.4}$$

$$\sigma(\cdot, \alpha K) = \alpha\sigma(\cdot, K), \quad \alpha \geq 0, \tag{2.5}$$

$$\sigma(\cdot, K_1 + K_2) = \sigma(\cdot, K_1) + \sigma(\cdot, K_2), \tag{2.6}$$

$$\mathrm{H}(K_1, K_2) = \|\sigma_1 - \sigma_2\|, \text{ (uniform norm)}, \tag{2.7}$$

$$\|K\| = \|\sigma(\cdot, K)\|, \tag{2.8}$$

where at (2.7) *we mean* $\sigma_1 = \sigma_1(\cdot, K_1)$ *and* $\sigma_2 = \sigma_2(\cdot, K_2)$.

Proof. The support function of the closed unit ball $\mathbb{B}$ is identically 1 since by the Cauchy inequality we have that $\langle p, b\rangle \leq 1$, for all $p \in \mathbb{S}$, and $b \in \mathbb{B}$ and on the other hand each $p \in \mathbb{R}^n$ with $\|p\| = 1$ also belongs to $\mathbb{B}$, so $\langle p, p\rangle = 1$. Thus (2.4) follows.

Let us see how (2.7) comes about. Since the support function of the closed unit ball $\mathbb{B}$ is identically 1, so that (2.5) and (2.6) imply $\sigma(p, K_2+\varepsilon\mathbb{B}) = \sigma(p, K_2)+\varepsilon$. Together with (2.3) this yields, for all p,

$$K_1 \subset K_2 + \varepsilon\mathbb{B} \iff \sigma(p, K_1) \leq \sigma(p, K_2) + \varepsilon \iff \sigma(p, K_1) - \sigma(p, K_2) \leq \varepsilon.$$

Similarly, for all p,

$$K_2 \subset K_1 + \varepsilon\mathbb{B} \iff \sigma(p, K_2) \leq \sigma(p, K_1) + \varepsilon \iff \sigma(p, K_2) - \sigma(p, K_1) \leq \varepsilon.$$

For both inclusions to hold, we have to have

$$|\sigma(p, K_1) - \sigma(p, K_2)| \leq \varepsilon, \quad \forall\, p. \tag{2.9}$$

The infimum of all $\varepsilon > 0$ satisfying (2.9) is at once $\mathrm{H}(K_1, K_2)$ and $\|\sigma_1 - \sigma_2\|$. In particular, $K_2 = \{0\}$ implies $\|K\| = \|\sigma(\cdot, K)\|$. $\square$

Corollary 2.3. *For any* $p_1, p_2 \in \mathbb{S}$ *and* $K_1, K_2 \in \mathbb{K}$, *we have that*

$$|\sigma(p_1, K_1) - \sigma(p_2, K_2)| \leq \|p_1 - p_2\| \, \|K_1\| + \mathrm{H}(K_1, K_2).$$

$\mathbb{C}[\mathbb{K}]$ and $\mathbb{C}[\mathbb{K}_c]$ denote the spaces of continuous functions on $[0,1]$ into $\mathbb{K}$ and $\mathbb{K}_c$, respectively. Given a map $F \in \mathbb{C}[\mathbb{K}]$, we denote its *norm* by

$$\mathrm{H}(F) = \sup_{x \in [0,1]} \{\|F(x)\|\}$$

and define the related metric by

$$\mathrm{H}(F, G) = \sup_{x \in [0,1]} \{\mathrm{H}(F(x), G(x))\}. \tag{2.10}$$

2.2. **Bernstein approximation.** Given a multifunction F defined on $[0,1]$, we define the Nth *Bernstein approximant* to be

$$\mathrm{B}_N(F, x) = \sum_{k=0}^{N} p_{N,k}(x) F\left(k/N\right), \tag{2.11}$$

where the $p_{N,k}(\cdot)$ polynomials are given by (1.4). The addition and multiplication in the right-hand of (2.11) are understood in the Minkowski sense. It is clear that this map necessarily lies in $\mathbb{C}[\mathbb{K}]$ and, indeed, in $\mathbb{C}[\mathbb{K}_c]$ if $F \in \mathbb{C}[\mathbb{K}_c]$.

Theorem 2.4. *Let* $F \in \mathbb{C}[\mathbb{K}_c]$. *Then* $\mathrm{B}_N(F, \cdot)$ *converges uniformly to* F, *i. e.,* $\mathrm{H}(F, \mathrm{B}_N(F,\cdot)) \xrightarrow[N\to\infty]{u} 0$, *where* $\xrightarrow{u}$ *denotes the uniform convergence.*

Proof. We use the Banach space embedding

$$\begin{array}{ccc}
\mathrm{B}_N(F,\cdot) & \xrightarrow[\mathrm{H}(\cdot,\cdot)]{u\,?} & F \\
\downarrow & & \uparrow \\
\sigma(\cdot, \mathrm{B}_N(F,\cdot)) & \xrightarrow[\|\cdot\|_\infty]{u} & \sigma(\cdot, F).
\end{array}$$

The above diagram has to be read as follows. We ask if the sequence $(\mathrm{B}_N(F,\cdot))_N$ converges uniformly to F in respect to the Hausdorff-Pompeiu metric. The answer is given by embedding the sequence $(\mathrm{B}_N(F,\cdot))_N$ into the Banach space of continuous functions on $\mathbb{S}$ by the support function, then checking the uniform convergence of the sequence $(\sigma(\mathrm{B}_N(F,\cdot)))_N$ toward $\sigma(\cdot, F)$, and then returning to F.

Then $F \in \mathbb{C}[\mathbb{K}_c]$ is equivalent to the continuity of the map from $[0,1]$ into B_n given by $x \mapsto \sigma(\cdot, F(x))$.

A Bernstein approximant of F corresponds to the map

$$[0,1] \ni x \mapsto \sum_{k=0}^{N} p_{N,k}(x)\sigma(\cdot, F(k/N)).$$

Hence it is enough to show the uniform convergence (in B_n) of the latter maps to $x \mapsto \sigma(\cdot, F(x))$. Indeed

$$\begin{aligned} \mathrm{H}(F(\cdot), \mathrm{B}_N(F,\cdot)) &\overset{(2.10)}{=} \sup_{x\in[0,1]} \{\mathrm{H}(F(x), \mathrm{B}_N(F,x))\} \\ &\overset{(2.7)}{=} \sup_{x\in[0,1]} \{\|\sigma(\cdot, F(x)) - \sigma(\cdot, \mathrm{B}_N(F,x))\|\} \\ &\overset{(2.5)}{=} \sup_{x\in[0,1]} \{\|\sigma(\cdot, F(x)) - \mathrm{B}_N\sigma(\cdot, F(x))\|\} \xrightarrow[N\to\infty]{\text{uniformly in } x} 0, \end{aligned}$$

by a result of T. Popoviciu, [6, pp. 109–111], or [20, pp. 155–160]. □

We turn to the case when $F \in \mathbb{C}[\mathbb{K}]$ does not necessarily have convex values. This does not preclude forming $\mathrm{B}_N(F,\cdot)$.

Remark 2.5. If $K = \{0,1\}$, then $\mathrm{conv}K = [0,1]$,

$$(1/N)\sum_{k=0}^{N} K = (1/N)(K + K + \cdots + K) = \{0, 1/N, 2/N, \ldots, 1\}$$

and

$$\mathrm{H}((1/N)\sum_{k=0}^{N} K, \, \mathrm{conv}K) \to 0 \text{ as } N \to \infty,$$

since if $K_1 = \{0, 1/N, 2/N, \ldots, 1\}$ and $K_2 = \mathrm{conv}K$, then

$$\begin{aligned} \mathrm{H}(K_1, K_2) &= \min\{\varepsilon > 0 \mid K_1 \subset K_2 + \varepsilon[-1,1], \quad K_2 \subset K_1 + \varepsilon[-1,1]\} \\ &= \min\{\varepsilon > 0 \mid K_2 \subset K_1 + \varepsilon[-1,1]\} = 1/(2N). \quad \triangle \end{aligned}$$

An uncomfortable situation is revealed by the following result and its consequences.

Theorem 2.6. (Shapley-Folkman-Starr, [25]) *If* $K_1, \ldots, K_N \in \mathbb{K}$, *then*

$$\mathrm{H}(K_1 + \cdots + K_N, \, \mathrm{conv}(K_1 + \cdots + K_N)) \leq \sqrt{n} \max_{1\leq i\leq N} \|K_i\|. \tag{2.12}$$

The special case $K_1 = \cdots = K_N = K$ leads to

Corollary 2.7. *If* $K_1 = K_2 = \cdots = K_N = K \in \mathbb{K}$, *then*

$$\mathrm{H}(K + \cdots + K, \mathrm{conv}(K + \cdots + K)) \overset{(2.12)}{\leq} \sqrt{n}\,\|K\|. \tag{2.13}$$

Corollary 2.8.

$$\mathrm{H}\left((K + \cdots + K)/N,\ (1/N)\mathrm{conv}(K + \cdots + K)\right) \overset{(1.9)}{\leq} (\sqrt{n}/N)\|K\| \xrightarrow[N\to\infty]{} 0.$$

It comes another uncomfortable case.

Proposition 2.9. *A set* $K \in \mathbb{K}$ *is infinitely divisible for Minkowski sums, i. e., admits the following representation*

$$K = L_N + \cdots + L_N, \ \textit{for all } N \geq 2, \tag{2.14}$$

if and only if K *is convex.*

Proof. *Sufficiency.* Suppose that K is a convex set. Then a representation of the form (2.14) exists by taking $L_N = (1/N)K$ and applying (1.2).
Necessity. From (2.14) it follows that $\|L_N\| \leq \|K\|/N$. Then Corollary 2.7 and the previous inequality imply that

$$\mathrm{H}(K, \mathrm{conv}K) \leq \sqrt{n}\,\|L_N\| \leq \sqrt{n}\,\|K\|/N.$$

□

A more exact variant of the Shapley-Folkman-Starr theorem is valid. For a set $K \in \mathbb{K}$, define its *diameter* respectively, *radius* by $\mathrm{diam}\,K = \max_{x,y\in K}\|x - y\|$ and $\mathrm{rad}\,K = (1/2)\mathrm{diam}\,K$.

Theorem 2.10. (Shapley-Folkman-Starr, [18, p. 407]) *If* $K_1, \ldots, K_N$ *are compact subsets of* $\mathbb{R}^n$, *then*

$$\mathrm{H}(K_1 + \cdots + K_N,\ \mathrm{conv}(K_1 + \cdots + K_N)) \leq \sqrt{n} \max_{1\leq i\leq N} \mathrm{rad}\,K_i.$$

Another result of the same sort is mentioned below.

Lemma 2.11. (Shapley-Folkman-Starr, [18, p. 407]) *If* $K_1, \ldots, K_N$ *are compact subsets of* $\mathbb{R}^n$, *then*

$$\mathrm{H}(K_1 + \cdots + K_N,\, \mathrm{conv}(K_1 + \cdots + K_N))^2 \leq \sum_{i=1}^{N} (\mathrm{rad}\, K_i)^2.$$

Theorem 2.12. *Let* $F \in \mathbb{C}[\mathbb{K}]$. *Then in any interval* $[\varepsilon, 1-\varepsilon]$, $0 < \varepsilon < 1/2$, $\mathrm{B}_N(F,\cdot)$ *converges uniformly to* $\mathrm{conv}F$ *(here* $(\mathrm{conv}F)(t) = \mathrm{conv}F(t)$, $t \in [0,1]$*).*

Proof. With

$$\mathrm{B}_N(F,x) = \sum_{k=0}^{N} p_{N,k}(x) F\left(k/N\right),$$

we identify $K_k = p_{N,k}(x)F(k/N)$ in (2.12). Now

$$\|K_k\| \leq \|F(k/N)\|\, |p_{N,k}(x)| \leq \mathrm{H}(F) \sup\{\, p_{N,k}(x) \mid \varepsilon \leq x \leq 1-\varepsilon,\ k = 0, 1, \ldots, N\}.$$

The indicated supremum is shown to be $O(N^{-1/2})$ [1], so by the Shapley-Folkman-Starr theorem one has that

$$\mathrm{H}(\mathrm{B}_N(F,x), \mathrm{B}_N(\mathrm{conv}F, x)) \leq \mathrm{H}(F)\, O(N^{-1/2})\, n^{1/2}.$$

Theorem 2.4 and the triangle inequality yields

$$\begin{aligned} \mathrm{H}(\mathrm{conv}F, \mathrm{B}_N(F,\cdot)) &\leq \mathrm{H}(\mathrm{B}_N(F,\cdot), \mathrm{B}_N(\mathrm{conv}F,\cdot)) \\ &+ \mathrm{H}(\mathrm{B}_N(\mathrm{conv}F,\cdot), \mathrm{conv}F) \xrightarrow[N\to\infty]{u} 0. \end{aligned}$$

□

Remark 2.13. The result cannot be extended to the full interval since

- at each endpoint $x = 0,\ 1$, $\mathrm{B}_N(F,x) = F(x)$ independent of N;
- the $O(N^{-1/2})$ bound breaks down at the endpoints. △

Below there are some properties which follow directly from the support function embedding and properties of Bernstein approximant in the real-valued case.

Proposition 2.14. *Given* $F : [0,1] \to \mathbb{K}_c$ *a set-valued mapping.*

[1] It follows from the limit $\lim_{N\to\infty} \sqrt{Nx(1-x)}\, \max_{0\leq k\leq N} p_{N,k} = 1/\sqrt{2\pi}$, [13, Secs. 11, 12]

(a) *Suppose* $K_1, K_2 \in \mathbb{K}_c$. *Then*

$$K_1 \subset F(x) \subset K_2,\ \textit{for all } x \implies K_1 \subset \mathrm{B}_N(F,x) \subset K_2,\ \textit{for all } x.$$

In particular, $\cap_{t\in[0,1]}F(t) \subset \mathrm{B}_N(F,x) \subset \mathrm{conv}(\cup_{t\in[0,1]}F(t))$, *for all* x.

(b) $F(s) \subset (\supseteq)\, F(t)$, *for all* s,t *with* $0 \le s \le t \le 1 \implies \mathrm{B}_N(F,s) \subset (\supseteq)\, \mathrm{B}_N(F,t)$, *for all* s,t *with* $0 \le s \le t \le 1$.

(c) $F\left((s+t)/2\right) \subset (\supseteq)\, (1/2)(F(s)+F(t))$, *for all* s,t, *implies that*

$$\mathrm{B}_N\left(F,(s+t)/2\right) \subset (\supseteq)\, (1/2)(\mathrm{B}_N(F,s)+\mathrm{B}_N(F,t)),\ \textit{for all } s,t.$$

(d) *Let* $G : [0,1] \to \mathbb{K}_c$ *be a set-valued mapping. Suppose that* $F(x) \cap G(x) = \emptyset$, *for all* x. *Then, for* N *sufficiently large,*

$$\mathrm{B}_N(F,x) \cap \mathrm{B}_N(G,x) = \emptyset,\ \textit{for all } x.$$

Proof. (a) For the first claim we successively have

$$\begin{aligned}
K_1 \subset F(x) \subset K_2,\ \forall x &\implies \sigma(p,K_1) \le \sigma(p,F(x)) \le \sigma(p,K_2),\ \forall\, p,x \\
&\implies \sigma(p,K_1) \le \sigma(p,F(k/N)) \le \sigma(p,K_2), \quad \forall\, p,k \\
\implies \binom{N}{k}x^k(1-x)^{N-k}\sigma(p,K_1) &\le \binom{N}{k}x^k(1-x)^{N-k}\sigma(p,F(k/N)) \\
&\le \binom{N}{k}x^k(1-x)^{N-k}\sigma(p,K_2),\ \forall\, p,x,k \\
\overset{\sum_k}{\implies} \sigma(p,K_1) &\le \sigma(p,\sum_{k=0}^{N}\binom{N}{k}x^k(1-x)^{N-k}F(k/N)) \le \sigma(p,K_2),\ \forall\, p,x \\
&\implies K_1 \subset \mathrm{B}_N(F,x) \subset K_2, \quad \forall\, x \in [0,1].
\end{aligned}$$

For the second claim we successively have

$$\begin{aligned}
&F(k/N) = F(k/n) = F(k/N) \\
&\implies \cap_{t\in[0,1]}F(t) \subset F(k/N) \subset \cup_{t\in[0,1]}F(t) \subset \mathrm{conv}\left(\cup_t F(t)\right) \\
&\implies \cap_{t\in[0,1]}F(t) \subset \mathrm{B}_N(F,x) \subset \mathrm{conv}\left(\cup_{t\in[0,1]}F(t)\right), \quad \forall\, x \in [0,1].
\end{aligned}$$

(d) Choose an arbitrary but fixed $x \in [0,1]$. Since F and G are of compact values, there exists an ε so that

$$0 < 3\varepsilon = \min_{u \in F(x),\, v \in G(x)} \|u - v\|.$$

Denote $A = \operatorname{cl}(F(x) + \varepsilon\mathbb{B})$ and $B = \operatorname{cl}(G(x) + \varepsilon\mathbb{B})$, where cl stands for the closure of a set. We have that $A \cap B = \emptyset$. Since $\mathrm{B}_N(F,x)$ and $\mathrm{B}_N(G,x)$ converge uniformly to $F(x)$, respectively $G(x)$, from a given rank N_0 we have that $\mathrm{B}_N(F,x) \subset A$ and $\mathrm{B}_N(G,x) \subset B$, for all $N > N_0$. Therefore $\mathrm{B}_N(F,x)$ and $\mathrm{B}_N(G,x)$ are disjoint sets for all $N > N_0$. □

Remark 2.15. (a), (b), and (c) in Proposition 2.14 can be obtained from a more general result introduced in [31]. These considerations are presented in the sequel. △

Definition 2.16. *Let L be an operator defined on the linear space $\mathbb{R}^{[0,1]}$ (of real-valued functions defined on $[0,1]$ with the usual operations) having values in $\mathbb{R}^{[0,1]}$. An operator $\mathcal{L}$ defined on the set $\mathbb{K}_c^{[0,1]}$ (of functions defined on $[0,1]$ with values in $\mathbb{K}_c$) having values in $\mathbb{K}_c^{[0,1]}$ and satisfying*

$$L(\sigma(p, F(\cdot)), x) = \sigma(p, \mathcal{L}(F, x)) \tag{2.15}$$

for all $F \in \mathbb{K}_c^{[0,1]}$, $x \in [0,1]$, and $p \in \mathbb{S}$, where $\mathbb{S}$ denotes the unit sphere in $\mathbb{R}^n$, is said to be a set-valued equivalent *of L.*

Example 2.17. Let $L : \mathbb{R}^{[0,1]} \to \mathbb{R}^{[0,1]}$ be an operator of the form

$$L : \mathbb{R}^{[0,1]} \ni f \mapsto \sum_{i=0}^{N} f(\xi_i)\alpha_i \in \mathbb{R}^{[0,1]} \tag{2.16}$$

with abscissae $\xi_i \in [0,1]$ and fundamental functions $\alpha_i \in \mathbb{R}_{\geq 0}^{[0,1]}$ such that $\sum_{i=0}^{N} \alpha_i = 1$. By definition, L is discretely defined, positive, linear, and exact for constant functions. The operator $\mathcal{L} : \mathbb{K}_c^{[0,1]} \to \mathbb{K}_c^{[0,1]}$, specified by

$$\mathcal{L} : \mathbb{K}_c^{[0,1]} \ni F \mapsto \sum_{i=0}^{N} F(\xi_i)\alpha_i \in \mathbb{K}_c^{[0,1]}, \tag{2.17}$$

is a set-valued equivalent of L reproducing constant functions in $\mathbb{K}_c^{[0,1]}$. △

Definition 2.18. *Let* $L : \mathbb{R}^{[0,1]} \to \mathbb{R}^{[0,1]}$ *,* $f \in \mathbb{R}^{[0,1]}$ *, and* $k_1, k_2 \in \mathbb{R}$ *. The operator* L *is said to preserve*

(a) lower bounds *if* $k_1 \le f \Longrightarrow k_1 \le Lf$ *, i.e., for all* $x \in [0,1]$, $k_1 \le f(x)$ *implies* $k_1 \le L(f,x)$, *where* k_1 *is independent of* x *,*

(b) upper bounds *if* $f \le k_2 \Longrightarrow Lf \le k_2$ *,*

(c) bounds *if it preserves lower and upper bounds,*

(d) monotonicity *if* $f(x) \le f(y)$, *for all* $0 \le x \le y \le 1$, *implies*

$$L(f,x) \le L(f,y), \quad \forall\, 0 \le x \le y \le 1,$$

and

(e) midconvexity *if for all* $x, y \in [0,1]$ $f\left((x+y)/2\right) \le (f(x)+f(y))/2$ *implies*

$$L\left(f, (x+y)/2\right) \le (L(f,x) + L(f,y))/2.$$

Similarly, we agree upon

Definition 2.19. *Let* $\mathcal{L} : \mathbb{K}_c^{[0,1]} \to \mathbb{K}_c^{[0,1]}$ *,* $F \in \mathbb{K}_c^{[0,1]}$ *, and* $K_1, K_2 \in \mathbb{K}_c$ *. The operator* $\mathcal{L}$ *is said to preserve*

(a) lower bounds *if* $K_1 \subseteq F \Longrightarrow K_1 \subseteq \mathcal{L}F$ *, i.e., for all* $x \in [0,1]$, $K_1 \subseteq F(x) \Longrightarrow K_1 \subseteq \mathcal{L}(F,x)$, *where* K_1 *is independent of* x *,*

(b) upper bounds *if* $F \subseteq K_2 \Longrightarrow \mathcal{L}F \subseteq K_2$ *,*

(c) bounds *if it preserves lower and upper bounds,*

(d) monotonicity *if for all* $0 \le x \le y \le 1$, $F(x) \subseteq F(y) \Longrightarrow \mathcal{L}(F,x) \subseteq \mathcal{L}(F,y)$, *and*

(e) midconvexity *if for all* $x, y \in [0,1]$,

$$F\left((x+y)/2\right) \subseteq (F(x)+F(y))/2 \Longrightarrow \mathcal{L}\left(F, (x+y)/2\right) \subseteq (\mathcal{L}(F,x) + \mathcal{L}(F,y))/2.$$

Proposition 2.20. (Inheritance, [31]) *A set-valued equivalent* $\mathcal{L} : \mathbb{K}_c^{[0,1]} \to \mathbb{K}_c^{[0,1]}$ *of an operator* $L : \mathbb{R}^{[0,1]} \to \mathbb{R}^{[0,1]}$ *preserves*

(a) *(lower, upper) bounds,*

(b) *monotonicity, and*

(c) *midconvexity,*

respectively, if L *possesses the corresponding property.*

Proof. Let $F, G \in \mathbb{K}_c^{[0,1]}$ and $K_1 \in \mathbb{K}_c$.

(a) If L preserves lower bounds, for all $p \in \mathbb{S},\ x \in [0,1]$, we successively have

$$K_1 \subseteq F(x) \Longrightarrow \sigma(p, K_1) \leq \sigma(p, F(x)) \Longrightarrow \sigma(p, K_1) \leq L(\sigma(p, F(\cdot)), x)$$
$$\Longrightarrow \sigma(p, K_1) \leq \sigma(p, \mathcal{L}(F, x)) \Longrightarrow K_1 \subseteq \mathcal{L}(F, x).$$

The preservation of upper bounds by $\mathcal{L}$ follows analogously if L preserves upper bounds.

(b) If L is monotonicity preserving, for all $p \in \mathbb{S},\ 0 \leq x \leq y \leq 1$, it holds

$$F(x) \subseteq F(y) \Longrightarrow \sigma(p, F(x)) \leq \sigma(p, F(y)) \Longrightarrow L(\sigma(p, F(\cdot)), x) \leq L(\sigma(p, F(\cdot)), y)$$
$$\Longrightarrow \sigma(p, \mathcal{L}(F, x)) \leq \sigma(p, \mathcal{L}(F, y)) \Longrightarrow \mathcal{L}(F, x) \subseteq \mathcal{L}(F, y)1.$$

(c) For midconvex L and for all $p \in \mathbb{S},\ x, y \in [0,1]$ we successively have

$$F\left(\frac{x+y}{2}\right) \subseteq \frac{F(x)+F(y)}{2} \Longrightarrow \sigma\left(p, F\left(\frac{x+y}{2}\right)\right) \leq \sigma\left(p, \frac{F(x)+F(y)}{2}\right)$$
$$\Longrightarrow \sigma\left(p, F\left(\frac{x+y}{2}\right)\right) \leq \frac{\sigma(p, F(x)) + \sigma(p, F(y))}{2}$$
$$\Longrightarrow L\left(\sigma\left(p, F(\cdot)\right), \frac{x+y}{2}\right) \leq \frac{L(\sigma(p, F(\cdot)), x) + L(\sigma(p, F(\cdot)), y)}{2}$$
$$\Longrightarrow \sigma\left(p, \mathcal{L}\left(F, \frac{x+y}{2}\right)\right) \leq \frac{\sigma(p, \mathcal{L}(F, x)) + \sigma(p, \mathcal{L}(F, y))}{2}$$
$$\Longrightarrow \sigma\left(p, \mathcal{L}\left(F, \frac{x+y}{2}\right)\right) \leq \sigma\left(p, \frac{\mathcal{L}(F, x) + \mathcal{L}(F, y)}{2}\right)$$
$$\Longrightarrow \mathcal{L}\left(F, \frac{x+y}{2}\right) \subseteq \frac{\mathcal{L}(F, x) + \mathcal{L}(F, y)}{2}.$$

□

Remark 2.21. For bounds preserving operators $\mathcal{L} : \mathbb{K}_c^{[0,1]} \to \mathbb{K}_c^{[0,1]}$ and arbitrary $F \in \mathbb{K}_c^{[0,1]}$, we have

$$\bigcap_{t \in [0,1]} F(t) \subseteq \mathcal{L}F \subseteq \operatorname{conv}\left(\bigcup_{t \in [0,1]} F(t)\right)$$

since

$$\bigcap_{t\in[0,1]} F(t) \subseteq F \subseteq \bigcup_{t\in[0,1]} F(t) \subseteq \operatorname{conv}\left(\bigcup_{t\in[0,1]} F(t)\right). \quad \triangle$$

Example 2.22. The the Nth Bernstein operator $\mathrm{B}_N : \mathbb{K}_c^{[0,1]} \to \mathbb{K}_c^{[0,1]}$ as given in (2.11) preserves bounds, monotonicity, and convexity. $\triangle$

2.3. **Stone-Weierstrass approximation theorem.** The approximation theorem as originally discovered by K. Weierstrass is as follows:

Theorem 2.23. (Weierstrass) *Suppose f is a continuous complex-valued function defined on the real interval $[a,b]$. For every $\varepsilon > 0$, there exists a polynomial function p over $\mathbb{C}$ such that for all $x \in [a,b]$, we have $|f(x) - p(x)| < \varepsilon$, or equivalently, the supremum norm $\|f-p\| < \varepsilon$.*

If f is real-valued, the polynomial function can be taken over $\mathbb{R}$.

A constructive proof of this theorem for f real-valued using Bernstein polynomials can be found in many books, see [6], [20].

An *associative algebra* A over a field F is defined to be a vector space A over F together with an F-bilinear multiplication $A \times A \to A$ (where the image of (x,y) is written as xy) such that the associative law holds:

- $(xy)z = x(yz)$ for all x, y and $z \in A$.

The bilinearity of the multiplication is expressed as

- $(x+y)z = xz + yz$ for all x, y and $z \in A$,
- $x(y+z) = xy + xz$ for all x, y and $z \in A$,
- $\alpha(xy) = (\alpha x)y = x(\alpha y)$ for all $x, y \in A$ and $\alpha \in F$.

The set $C[a,b]$ of continuous real-valued functions on $[a,b]$, together with the supremum norm $\|f\| = \sup_{x\in[a,b]} |f(x)|$, is a Banach algebra, (i. e., an associative algebra and a Banach space such that $\|fg\| \leq \|f\| \cdot \|g\|$ for all f, g, [24, Chapter I]). The set of all polynomial functions forms a subalgebra of $C[a,b]$ (i. e., a vector subspace of C[a,b] that is closed under multiplication of functions), and the content of the Weierstrass approximation theorem is that this subalgebra is dense in C[a,b].

Theorem 2.24. (Stone-Weierstrass, the $\mathbb{R}$ version) *Suppose X is a compact Hausdorff space and A is a subalgebra of $C(X,\mathbb{R})$ which contains a non-zero constant function. Then A is dense in $C(X,\mathbb{R})$ if and only if it separates points.*

The Stone-Weierstrass Theorem 2.24 implies the Weierstrass Theorem 2.23 since the polynomials on $[a,b]$ form a subalgebra of $C[a,b]$ which contains the constants and separates points.

2.4. **Korovkin-type approximation results.** Recall that $\mathbb{C}[\mathbb{K}]$ and $\mathbb{C}[\mathbb{K}_c]$ denote the spaces of continuous functions on $[0,1]$ into $\mathbb{K}$ and $\mathbb{K}_c$, respectively.

Proposition 2.25. *For F and G multifunctions defined on $[0,1]$ with values in $\mathbb{K}_c$, we have*

$$F \subset G \quad (F(x) \subset G(x),\ \forall\, x) \implies \mathrm{B}_N(F,\cdot) \subset \mathrm{B}_N(G,\cdot).$$

Proof. Successively we have

$$\begin{aligned} F(k/N) \subset G(k/N) &\overset{(2.3)}{\Longrightarrow} \sigma(p, F(k/N)) \le \sigma(p, G(k/N)), \quad \forall\, p \in \mathbb{S} \\ &\overset{(2.5)}{\Longrightarrow} \sigma(p, p_{N,k}(x)F(k/N)) \le \sigma(p, p_{N,k}(x)G(k/N)), \\ &\qquad \forall\, p \in \mathbb{S},\ x \in [0,1],\ k = 0,1,\ldots,N, \\ &\overset{(2.6)}{\Longrightarrow} \sigma(p, \mathrm{B}_N(F,x)) \le \sigma(p, \mathrm{B}_N(G,x)), \quad \forall\, p \in \mathbb{S},\ x \in [0,1], \\ &\overset{(2.3)}{\Longrightarrow} \mathrm{B}_N(F,x) \subset \mathrm{B}_N(G,x), \quad \forall\, x \in [0,1]. \end{aligned}$$

□

A map $T : \mathbb{C}[\mathbb{K}_c] \to \mathbb{C}[\mathbb{K}_c]$ is said to be $\mathbb{K}_c$-*linear* if

$$T(\alpha F + \beta G) = \alpha\, TF + \beta\, TG, \quad \forall\, \alpha, \beta \ge 0,\ F, G \in \mathbb{C}[\mathbb{K}_c],$$

and $\mathbb{K}_c$-*positive (monotone)* if

$$F \subset G \implies TF \subset TG, \quad \forall\, F, G \in \mathbb{C}[\mathbb{K}_c].$$

Remark 2.26. The Bernstein polynomial $\mathrm{B}_N(\cdot,\cdot)$ is an example of such a map (operator) T. △

Theorem 2.27. *Let* $(T_\nu)_\nu$ *be a sequence of* $\mathbb{K}_c$ *-linear and* $\mathbb{K}_c$ *-positive maps. In order to have*

$$T_\nu F \to F \text{ for each } F \in \mathbb{C}[\mathbb{K}_c],$$

it is necessary and sufficient that

(a) $T_\nu F^{(i)} \to F^{(i)}, \ \ i = 0, 1, 2$ *where* $F^{(i)}(x) = x^i \mathbb{B}$,

(b) $\sup\{\mathrm{H}(T_\nu F, F) \mid F(x) = K, \ \|K\| = 1\} \to 0.$

Proof. *Necessity.* (a) is obvious. Suppose that (b) does not hold. Then there exists an $\varepsilon > 0$ and a subsequence (K_{ν_j}) of (K_ν) such that $\mathrm{H}(T_{\nu_j} K_{\nu_j}, K_{\nu_j}) \geq \varepsilon$ $(F_{\nu_j}(x) = K_{\nu_j})$. Local compactness of $\mathbb{K}_c$ and the uniform normalization $\|K_{\nu_j}\| = 1$ assure the existence of a convergent subsequence of the (K_{ν_j}). Without loss of generality denote this new subsequence again as (K_ν) and suppose that $K_\nu \to K_\infty$. Then by the triangle inequality

$$\mathrm{H}(T_\nu K_\nu, K_\nu) \leq \mathrm{H}(T_\nu K_\nu, T_\nu K_\infty) + \mathrm{H}(T_\nu K_\infty, K_\infty) + \mathrm{H}(K_\infty, K_\nu).$$

Now $\varepsilon_\nu = \mathrm{H}(K_\infty, K_\nu) \to 0$. The twin inclusions $K_\nu \subset K_\infty + \varepsilon_\nu \mathbb{B}$ and $K_\infty \subset K_\nu + \varepsilon_\nu \mathbb{B}$ together with the properties of T_ν, imply

$$T_\nu K_\nu \subset T_\nu K_\infty + \varepsilon_\nu T_\nu \mathbb{B}, \quad T_\nu K_\infty \subset T_\nu K_\nu + \varepsilon_\nu T_\nu \mathbb{B},$$

so that $\mathrm{H}(T_\nu K_\nu, T_\nu K_\infty) \leq \varepsilon_\nu \mathrm{H}(T_\nu \mathbb{B}) \to 0$. Hence $\lim \mathrm{H}(T_\nu K_\infty, K_\infty) \geq \varepsilon$, but this violates our assumption.

Sufficiency. It is rather long in [28]. An easier path is supplied by Theorem 2.32 that follows. □

Remark 2.28. Theorem 2.27 and Remark 2.26 imply Theorem 2.4. △

Now we introduce a result in [15] that

- generalizes Theorem 2.27,
- allows transferring a Korovkin system from the single-valued to the multivalued case.

Mathematically the growth function is modeled by a multifunction F associating a compact convex subset of $\mathbb{R}^n$ for every value $x \in [0, 1]$. We need a couple of special functions: for a given $K \in \mathbb{K}_c$, K will denote the constant function $F(x) = K$, while $x\mathbb{B}$ and $x^2\mathbb{B}$ denote the multifunctions $F(x) = x\mathbb{B}$ and $F(x) = x^2\mathbb{B}$, respectively.

Let X and Y be metric spaces and $\mathcal{F}$ a family of functions from X into Y. The family $\mathcal{F}$ is said to be *equicontinuous* at a point $x_0 \in X$, if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that $\rho(f(x_0), f(x)) < \varepsilon$ for all $f \in \mathcal{F}$ and $x \in X$ such that $\rho(x_0, x) < \delta$.

Let X be a compact Hausdorff space, $C(X)$ the Banach space of real-valued continuous functions on X. We consider a set $M \subset C(X)$ of "test functions", and we denote by $\mathrm{span}(M)$ the linear subspace of $C(X)$ spanned by M. The *Korovkin closure* $\mathcal{K}(M)$ is the set of all functions $f \in C(X)$ which satisfies the following property:

For every equicontinuous net (T_α) of positive linear operators on $C(X)$ one has:

If $T_\alpha(g) \to g$ for all $g \in M$, then $T_\alpha(f) \to f$.
One says that M is a *Korovkin system* for $C(X)$ if $\mathcal{K}(M) = C(X)$.

Remark 2.29. (a) If the constant function 1 belongs to M, the hypothesis of equicontinuity is superfluous. Indeed,

$$|T_\alpha(f) - T_\alpha(g)| = |fT_\alpha(1) - gT_\alpha(1)| = |f - g| \cdot T_\alpha(1) \to |f - g|.$$

(b) For compact metric spaces X, the net (T_α) is equivalent to a sequence (T_n).

(c) On the unit interval $X = [0, 1]$, the polynomials $p_0 = 1$, $p_1 = x$, and $p_2 = x^2$ form a Korovkin system. $\triangle$

Let X and Y be compact Hausdorff spaces. There are natural embeddings of $C(X)$ and $C(Y)$ into $C(X \times Y)$. Indeed, every function $f : X \to \mathbb{R}$ may be considered as a function from $X \times Y$ into $\mathbb{R}$ not depending on the second variable and, likewise, for functions on Y.

Lemma 2.30. *If M_1 is a Korovkin system for $C(X)$ and M_2 is a Korovkin system for $C(Y)$, then $M = M_1 \cup M_2$ is a Korovkin system for $C(X \times Y)$.*

For an arbitrary compact Hausdorff space X we denote

$$\mathcal{C} = C(X, \mathbb{K}_c)$$

the set of all continuous multifunctions F defined on X with values $F(x)$ in the set $\mathbb{K}_c$ of nonempty convex, and compact subsets of $\mathbb{R}^n$.

An operator $T : \mathcal{C} \to \mathcal{C}$ is called *linear* if

$$T(F+G) = TF + TG, \quad T(\alpha F) = \alpha TF, \quad \forall\, F, G \in \mathcal{C},\ \alpha \geq 0.$$

It is said to be *monotone* if

$$F \subset G \implies TF \subset TG.$$

As in the real case, we say that a set $\mathcal{M} \subset \mathcal{C}$ of test functions is a *Korovkin closure* for $\mathcal{C}$ if

For every equicontinuous net (T_α) of monotone linear operators on $\mathcal{C}$ one has:

If $T_\alpha(G) \to G$ for all $G \in \mathcal{M}$, then $T_\alpha(F) \to F$ for all $F \in \mathcal{K}(\mathcal{M})$.

One says that $\mathcal{M}$ is a *Korovkin system* for $\mathcal{C}$ if $\mathcal{K}(\mathcal{M}) = \mathcal{C}$.

Remark 2.31. (a) If the constant function $\mathbb{B}$ belongs to $\mathcal{M}$, the hypothesis of equicontinuity of the net (T_α) is superfluous since it follows from $T_\alpha(\mathbb{B}) \to \mathbb{B}$.

(b) For compact metric spaces X, the net (T_α) is equivalent to a sequence (T_n). $\triangle$

Theorem 2.32. *Let X be a compact Hausdorff space, and $\mathbb{B}$ the unit ball for an arbitrary norm in $\mathbb{R}^n$. If M is a Korovkin system of nonnegative functions for $C(X)$, then*

$$\mathcal{M} = \{x \mapsto f(x)\mathbb{B} \mid f \in M\} \cup \{\textit{all constant functions}\}$$

is a Korovkin system for $\mathcal{C}$.

Proof. Let Y be the dual unit sphere of $\mathbb{B} = \{x \in \mathbb{R}^n \mid \|x\| \leq 1\}$, that is, the set of linear functionals y on $\mathbb{R}^n$ such that

$$\|y\| = \sup\{y(x) \mid x \in \mathbb{B}\} = 1.$$

Topologically Y is homeomorphic to the Euclidean sphere $\mathbb{S}$.

With every $K \in \mathbb{K}_c$ we associate the support function

$$\sigma(\cdot, K) : Y \to \mathbb{R}, \quad \sigma(p, K) = \max\{\langle p, x\rangle \mid x \in K\}.$$

We summarize several properties of the support function.

(a) Since $\sigma(\cdot, K)$ is sublinear on $\mathbb{R}^n$, it is continuous, Proposition 2.1.

(b) $\sigma(y, \mathbb{B}) = 1$ for all $y \in Y$, this is (2.4).

(c) $\sigma(\cdot, K_1 + K_2) = \sigma(\cdot, K_1) + \sigma(\cdot, K_2)$, $\sigma(\cdot, \alpha K) = \alpha\sigma(\cdot, K)$, $\alpha \geq 0$, these equalities are (2.6) and (2.5), respectively.

(d) $\sup\{\sigma(\cdot, K_1), \sigma(\cdot, K_2)\} = \sigma(\cdot, K)$, where $K = \operatorname{conv}\{K_1 \cup K_2\}$.

(e) $K_1 \subset K_2 + \varepsilon\mathbb{B} \iff \sigma(\cdot, K_1) \leq \sigma(\cdot, K_2) + \varepsilon$. In particular $K_1 \subset K_2 \iff \sigma(\cdot, K_1) \leq \sigma(\cdot, K_2)$. See (2.3).

(f) Equality (2.7) takes place, i. e., $\mathrm{H}(K_1, K_2) = \|\sigma_1 - \sigma_2\|$, (uniform norm).

Thus the mapping

$$\mathbb{K}_c \ni K \ \mapsto \ \sigma(\cdot, K)$$

is a linear isometric order embedding of $\mathbb{K}_c$ into $C(Y)$. The linear subspace

$$L = \{\sigma(\cdot, K_1) - \sigma(\cdot, K_2) \mid K_1, K_2 \in \mathbb{K}_c\}$$

is a vector lattice by (d), containing 1 by (b). As clearly L separates the points, L is dense in $C(Y)$ by the Stone-Weierstrass theorem.

The embedding $\sigma : \mathbb{K}_c \to C(Y)$ yields a linear isometric order embedding

$$i : \mathcal{C}\,(= C(X, \mathbb{K}_c)) \ \to \ C(X, C(Y))$$

given by $i(F)(x) = \sigma(\cdot, F(x))$ for all $F \in \mathcal{C}$ and $x \in X$. Combining with the isomorphism

$$j : C(X, C(Y)) \to C(X \times Y)$$

given by $j(f)(x, y) = (f(x))(y)$ for all $f \in C(X, C(Y))$ and all $(x, y) \in X \times Y$, we obtain a linear order embedding

$$\kappa : \mathcal{C} \to C(X \times Y).$$

The image of κ generates a dense vector sublattice of $C(X \times Y)$ and contains the constant function 1. Thus, every monotone linear operator T on $\mathcal{C}$ extends uniquely to a positive linear operator $\overline{T}$ on $C(X \times Y)$. For an equicontinuous family (T_α) of monotone linear operators on $\mathcal{C}$, the family $(\overline{T}_\alpha)$ of extensions is equicontinuous on $C(X \times Y)$.

It remains to prove that $\kappa(\mathcal{M})$ is a Korovkin system for $C(X \times Y)$. One easily checks that under κ

$$x \to f(x)\mathbb{B} \text{ goes to } (x, y) \to f(x),$$

the constant function

$$x \to K \text{ goes to } (x, y) \to \sigma(y, K).$$

As M is a Korovkin system for X and the functions $\sigma(\cdot, C)$ for $C \in \mathbb{K}_c$ generate a dense linear subspace of $C(Y)$, Lemma 2.30 allows one to conclude that $\kappa(\mathcal{M})$ is a Korovkin system for $C(X \times Y)$. □

Theorem 2.33. *Let X be a compact Hausdorff space, and $\mathbb{B}$ the unit ball for an arbitrary norm in $\mathbb{R}^n$. If M is a Korovkin system of nonnegative functions for $C(X)$, then*

$$\mathcal{M} = \{x \mapsto f(x)\mathbb{B} \mid f \in M\} \cup \{\mathbb{B}, e_1, \dots, e_n\},$$

where $\{e_1, \dots, e_n\}$ is the canonical basis in $\mathbb{R}^n$, is a Korovkin system for $\mathcal{C} = C(X, \mathbb{K}_c)$.

We recall just [11] and [23] (the latter is not new but useful) for the convex case.

It seems that the work with Minkowski operations on sets pushes us considering only convex sets in order to get substantial results. Naturally arises the question: what is happening in the non-convex case, i. e., do exist methods allowing us to get satisfactory deep results in the non-convex case? We will see that the answer is positive.

Hereafter we review some results on the non-convex case, mainly in [10] but also [9] and [11].

3. **The non-convex case**

3.1. **Preliminaries.** Recall that $\mathbb{K}$ is the collection of all compact nonempty subsets of $\mathbb{R}^n$. This section follows [10].

We introduce the following notions.

- The *linear Minkowski combination* of two nonempty sets A and B in $\mathbb{R}^n$ is defined (as we already saw) as $\lambda A + \mu B = \{\lambda a + \mu b \mid a \in A,\ b \in B\}$, with $\lambda, \mu \in \mathbb{R}$.

- The *Euclidean distance from a point* $a \in \mathbb{R}^n$ *to a set* $B \in \mathbb{K}$ is defined as

$$\mathrm{d}(a, B) = \inf_{b \in B} \|a - b\| \overset{\mathbb{K}}{=} \min_{b \in B} \|a - b\|,$$

where $\|\cdot\|$ is the Euclidean norm in $\mathbb{R}^n$.

- The *Hausdorff(-Pompeiu) distance between two sets* $A, B \in \mathbb{K}$ is defined by

$$\begin{aligned} \mathrm{H}(A, B) &= \max\left\{\max_{a \in A} \mathrm{d}(a, B),\ \max_{b \in B} \mathrm{d}(b, A)\right\} \\ &= \max\left\{\max_{a \in A}\min_{b \in B} \|a - b\|,\ \max_{b \in B}\min_{a \in A} \|a - b\|\right\} \\ &\overset{(1.7)}{=} \min\{\varepsilon > 0 \mid A \subset B + \varepsilon\mathbb{B},\quad B \subset A + \varepsilon\mathbb{B}\}. \end{aligned} \tag{3.1}$$

We already saw that $\|A\| = \mathrm{H}(\{0\}, A)$.

- The *set of all projections of* $a \in \mathbb{R}^n$ *into a set* $B \in \mathbb{K}$ is

$$\Pi_B(a) = \{b \in B \mid \|a - b\| = \mathrm{d}(a, B)\}.$$

- For $A, B \in \mathbb{K}$ and $0 \le t \le 1$, the *one sided* t*-weighted metric average of* A *and* B (in this is order) is

$$M(A, t, B) = \bigcup_{a \in A} \{ta + (1 - t)\Pi(A, B)\} \tag{3.2}$$

and the t*-weighted metric average of* A *and* B is

$$A \oplus_t B = \{ta + (1 - t)b \mid (a, b) \in \Pi(A, B)\} \tag{3.3}$$

with

$$\Pi(A,B) = \{(a,b) \in A \times B \mid a \in \Pi_A(b) \ \text{ or } \ b \in \Pi_B(a)\}.$$

Proposition 3.1. *The graph of the mapping* $[0,1] \ni t \mapsto C(t) = A \oplus_t B$ *is the union of the graphs of* $(1-t)a + tb$ *for* $(a,b) \in \Pi(A,B)$.

Proposition 3.2. *The one sided metric average and the metric average, for all* $A, B \in \mathbb{K}$ *and* $0 \leq s, t \leq 1$, *have the following metric properties:*

$$A \oplus_t B = M(A,t,B) \cup M(B,1-t,A), \tag{3.4}$$

$$M(M(A,t,B),s,B) = M(A,ts,B), \tag{3.5}$$

$$M(A \cap B,t,B) = A \cap B \subset M(B,s,A), \tag{3.6}$$

$$A \oplus_t B = (A \cap B) \cup M(A \setminus B,t,B) \cup M(B \setminus A,1-t,A), \tag{3.7}$$

$$A \oplus_1 B = C(1) = A, \tag{3.8}$$

$$A \oplus_0 B = C(0) = B, \tag{3.9}$$

$$A \oplus_t B = C(t) \in \mathbb{K}, \tag{3.10}$$

$$A \oplus_t B = B \oplus_{1-t} A, \tag{3.11}$$

$$A \oplus_t A = A, \tag{3.12}$$

$$A \cap B \subset A \oplus_t B \subset \ tA + (1-t)B \subset \operatorname{conv}(A \cup B), \tag{3.13}$$

$$\mathrm{H}(A \oplus_t B,\ A \oplus_s B) = \mathrm{H}(C(t), C(s)) = |t-s|\mathrm{H}(A,B), \tag{3.14}$$

$$\mathrm{H}(A \oplus_t B,\ A) = (1-t)\mathrm{H}(A,B), \tag{3.15}$$

$$\mathrm{H}(A \oplus_t B,\ B) = t \cdot \mathrm{H}(A,B), \tag{3.16}$$

$$\text{if } B \text{ is a convex superset of } A, \text{ then for } 0 \leq t \leq s \leq 1, \tag{3.17}$$

$$A \subset A \oplus_s B \subset A \oplus_t B \subset B.$$

Proof. Let us see where these relations come from.

Equality (3.4) follows from (3.2) and (3.3).

From

$$M(M(A,t,B),s,B) = \{s(ta+(1-t)b)+(1-s)b \mid a \in A, \quad b \in \Pi_B(a)\}$$
$$= \{(ts)a+(1-(ts))b \mid a \in A, \quad b \in \Pi_B(s)\} = M(A,ts,B),$$

(3.5) follows.

Equalities (3.6) and (3.7) follow from the definitions (3.2) and (3.3).

Since

$$A \oplus_1 B = \{1 \cdot a + 0 \cdot b \mid (a,b) \in \Pi(A,B)\} = \{a \mid b \in \Pi_B(a),\ a \in A\},$$

(3.8) follows.

Equality (3.9) follows similarly.

For $A, B \in \mathbb{K}$, $A \oplus_t B$ is nonempty and compact, that is (3.10).

Equalities (3.11) and (3.12) are obvious.

Inclusions in (3.13) immediately follow.

We follow [1] and consider an arbitrary element $x = ta+(1-t)b \in C(t)$ with $a \in \Pi_A(b)$ or $b \in \Pi_B(a)$, then $y = sa+(1-s)b \in C(s)$. Since $\|x-y\| = |t-s|\cdot\|a-b\|$, we have that $\mathrm{H}(C(t),C(s)) \le |t-s|\mathrm{H}(A,B)$. To prove the reverse inequality without loss of generality we admit that $0 \le s < t \le 1$. Then

$$\mathrm{H}(A,B) = \mathrm{H}(C(0),C(1)) \le \mathrm{H}(C(0),C(s)) + \mathrm{H}(C(s),C(t)) + \mathrm{H}(C(t),C(1))$$
$$\le s\mathrm{H}(A,B) + (t-s)\mathrm{H}(A,B) + (1-t)\mathrm{H}(A,B) = H(A,B).$$

Now we conclude that equality (3.14) is true.

We have

$$(1-t)\mathrm{H}(A,B) \overset{(3.14)}{=} \mathrm{H}(A \oplus_t B,\ A \oplus_1 B) \overset{(3.8)}{=} \mathrm{H}(A \oplus_t B,\ A)$$

and (3.15) follows.

In order to get (3.16) we have

$$t \cdot \mathrm{H}(A,B) \overset{(3.14)}{=} \mathrm{H}(A \oplus_t B,\ A \oplus_0 B) \overset{(3.9)}{=} \mathrm{H}(A \oplus_t B,\ B).$$

We recall the proof of (3.17) in [8]. Obviously, we have that since $A \subset B$, $A = A \cap B \subset B$. By (3.4),

$$M(A, t, B) = M(A \cap B, t, B) \subset M(B, 1 - t, A)$$

and therefore $A \oplus_t B = M(B, 1 - t, A)$. Hence, by (3.13) and the convexity of B,

$$A = A \cap B \subset M(B, 1 - t, A) = A \oplus_t B \subset \mathrm{conv}(A \cup B) = B.$$

it remains to prove that $M(B, 1 - s, A) \subset M(B, 1 - t, A)$. By the convexity of B, for each $b \in B$ and $a \in \Pi_A(b)$, the whole segment $[a, ta + (1 - t)b]$ is a subset of $M(B, 1 - t, A)$. Since, for $s \geq t$, $[a, sa + (1 - s)b] \subset [a, ta + (1 - t)b]$, the conclusion follows. □

• The *modulus of continuity of* $f : [a, b] \to X$ with images in a metric space (X, ρ) is

$$\omega_{[a,b]}(f, \delta) = \sup\{\rho(f(x), f(y)) \mid |x - y| \leq \delta, \ x, y \in [a, b]\}, \quad \delta > 0. \tag{3.18}$$

Hereafter X is either $\mathbb{R}^n$ or $\mathbb{K}$, and ρ is either the Euclidean distance or the Hausdorff-Pompeiu distance, respectively. A property of the modulus is

$$\omega_{[a,b]}(f, \lambda\delta) \leq \lceil \lambda \rceil \omega_{[a,b]}(f, \delta), \tag{3.19}$$

[5, Problem 6. II, p. 38], where $\lceil \cdot \rceil$ is the ceiling function.

• By *Lip*$([a, b], \mathcal{L})$ we denote the set of all Lipschitz functions $f : [a, b] \to X$ satisfying $\rho(f(x), f(y)) \leq \mathcal{L}|x - y|, \quad x, y \in [a, b]$, where $\mathcal{L}$ is a constant independent of x and y.

• The *variation* of a function $f : [a, b] \to X$ on a partition $\chi = \{a = x_0 < \cdots < x_N = b \mid x_i \in [a, b], i = 0, \ldots, N\}$ is defined by $V(f, \chi) = \sum_{k=1}^{N} \rho(f(x_k), f(x_{k-1}))$. The *total variation* of f on $[a, b]$ is $\bigvee_a^b(f) = \sup_\chi V(f, \chi)$. It is said that f is of *bounded variation* if $\bigvee_a^b(f) < \infty$. In this case we define the function

$$v_f(x) = \bigvee_a^x(f), \quad x \in [a, b]. \tag{3.20}$$

Obviously, v_f is nondecreasing. If f is also continuous, then v_f is continuous as well. For the sake of completeness we recall the next statement.

Proposition 3.3. *A function* $f : [a, b] \to X$ *is continuous and of bounded variation on* $[a, b]$ *if and only if* v_f *is a continuous function on* $[a, b]$.

It holds

$$\rho(f(x), f(y)) \leq \overset{y}{\underset{x}{\vee}}(f) = v_f(y) - v_f(x), \text{ for } x < y. \tag{3.21}$$

From (3.21) it follows that

$$\omega_{[a,b]}(f, \delta) \leq \omega_{[a,b]}(v_f, \delta). \tag{3.22}$$

- By $CBV[a, b]$ is denoted the set of all functions which are continuous and of bounded variation on $[a, b]$.
- For a multifunction $F : [a, b] \to \mathbb{K}$ any single-valued function $f : [a, b] \to \mathbb{R}^n$ with $f(x) \in F(x)$, for all $x \in [a, b]$ is said to be a *selection* of F, e. g. [29], [14], [30], [19, Chapter 2], and [21, Chapter 2].

A set of selections of F, let it be $\{f^\alpha \mid \alpha \in \mathcal{A}\}$, is said to be a *representation* of F if $F(x) = \{f^\alpha(x) \mid \alpha \in \mathcal{A}\}$, $\forall x \in [a, b]$. This is expressed by writing $F = \{f^\alpha \mid \alpha \in \mathcal{A}\}$. Note that such a representation always exists thanks to the axiom of choice.

A concrete **motivation** for the study of metric average is the reconstruction of a 3D object from a set of its 2D cross-section, [26], with applications in tomography, microscopy, and computer vision. An algorithm for the computation of the metric average of two simple polygons is introduced in [16].

3.2. **Linear operators on multifunctions based on a metric linear combination of ordered sets.** A new operation on a finite number of ordered sets is introduced. Using this operation a new adaptation of linear operators to multifunctions is presented.

Definition 3.4. *Let* $(A_0, A_1, \ldots, A_N)$ *be a finite sequence of nonempty compact sets. A vector* $(a_0, a_1, \ldots, a_N)$ *with* $a_i \in A_i$, $i = 0, \ldots, N$, *for which there exists* j, $0 \leq j \leq N$ *such that*

$$a_{i-1} \in \Pi_{A_{i-1}}(a_i), \quad 1 \leq i \leq j \ \textit{and} \ a_{i+1} \in \Pi_{A_{i+1}}(a_i), \quad j \leq i \leq N-1$$

*is called a **metric chain** of* $(A_0,\ldots,A_N)$.

Thus each element of each set A_i, $i=0,\ldots,N$ generates at least one metric chain. The collection of all metric chains of $(A_0,\ldots,A_N)$ is denoted by $CH(A_0,\ldots,A_N)$. The set $CH(A_0,\ldots,A_N)$ depends on the order of the sets A_i, $i=0,\ldots,N$.

Definition 3.5. *A **metric linear combination** of a sequence of sets* $A_0,\ldots,A_N$ *with coefficients* $\lambda_0,\ldots,\lambda_N\in\mathbb{R}$, *is*

$$\bigoplus_{k=0}^{N}\lambda_k A_k=\left\{\sum_{k=0}^{N}\lambda_k a_k \mid (a_0,\ldots,a_N)\in CH(A_0,\ldots,A_N)\right\}. \tag{3.23}$$

Since for two sets $CH(A,B)=\Pi(A,B)$, in the special case $N=1$ and $\lambda_0,\lambda_1\in[0,1]$, $\lambda_0+\lambda_1=1$, the metric linear combination is the metric average, [1].

Proposition 3.6. *Several properties of the metric linear combinations are introduced below.*

(i) $\bigoplus_{k=0}^{N}\lambda_k A_k=\bigoplus_{k=0}^{N}\lambda_{N-k}A_{N-k}$,

(ii) $\bigoplus_{k=0}^{N}\lambda_k A=\left(\sum_{k=0}^{N}\lambda_k\right)A$,

(iii) $\bigoplus_{k=0}^{N}\lambda A_k=\lambda\left(\bigoplus_{k=0}^{N}1\cdot A_k\right)$,

(iv) *For* $\lambda_0,\ldots,\lambda_N$ *such that* $\sum_{k=0}^{N}\lambda_k=1$, $\bigoplus_{k=0}^{N}\lambda_k A=A$.

Proof. (i) We remark that

$$(a_0,\ldots,a_N)\in CH(A_0,\ldots,A_N)\iff (a_N,\ldots,a_0)\in CH(A_N,\ldots,A_0).$$

Then

$$\begin{aligned}\bigoplus_{k=0}^{N}\lambda_k A_k&=\left\{\sum_{k=0}^{N}\lambda_k a_k \mid (a_0,\ldots,a_N)\in CH(A_0,\ldots,A_N)\right\}\\&=\left\{\sum_{k=0}^{N}\lambda_{N-k}a_{N-k} \mid (a_N,\ldots,a_0)\in CH(A_N,\ldots,A_0)\right\}=\bigoplus_{k=0}^{N}\lambda_{N-k}A_{N-k}.\end{aligned}$$

(ii) Since $CH(A,\ldots,A)=A$, the property follows.

(iii) is obvious.

(iv) follows from (ii). □

Remarks 3.7. • Now we can define the *metric sum* of two sets by

$$A_0 \oplus A_1 = \oplus_{k=0}^{1} 1 \cdot A_k.$$

This operation is commutative by property (i) in Proposition 3.6 and it is not associative.

• Similarly we can define *metric subtraction* between two sets by

$$A_0 \ominus A_1 = \oplus_{k=0}^{1} \lambda_k A_k$$

with $\lambda_0 = 1$ and $\lambda_1 = -1$. Then from (ii) in Proposition 3.6 it follows that

$$A \ominus A = \{0\}. \tag{3.24}$$

In spite of the previous result, the operation $A \ominus B$ does not have the usual properties of subtraction as follows from the example

$$A = [0,1], \quad B = \{0,1\} \implies A \ominus B = B \ominus A = \{-1, 0, 1\}.$$

• With the operation defined by (3.23), the class of sample based linear operators for real-valued functions, namely such defined by

$$A_\chi(f, x) = \sum_{k=0}^{N} c_k(x) f(x_k) \tag{3.25}$$

can be adapted to set-valued functions. △

Definition 3.8. *Let* $F : [a,b] \to \mathbb{K}$, $\{a = x_0, x_1, \ldots, x_N = b\} \subset [a,b]$ *and let* $\{F(x_k),\, k = 0, \ldots, N\}$ *be samples of* F *at* χ. *For* A_χ *of the form* (3.25) *it is defined a* ***metric linear operator*** A_χ^M *on* F *by*

$$(A_\chi^M F)(x) = A_\chi^M(F, x) = \bigoplus_{k=0}^{N} c_k(x) F(x_k). \tag{3.26}$$

This operator is said to be the ***metric analogue*** *of* (3.25).

Remark 3.9. Due to property (ii) in Proposition 3.6, the metric analogue of a linear operator which preserves constants, preserves constant multifunctions, too. Indeed, for a nonzero constant c we have

$$c = \sum_{k=0}^{N} c_k(x)\, c \implies \sum_{k=0}^{N} c_k(x) = 1 \overset{(3.25)}{\implies} K = \bigoplus_{k=0}^{N} c_k(x)\, K.$$

The analogue of (ii) in Proposition 3.6 does not hold for Minkowski linear combinations with some negative coefficients, even for convex sets. This is one reason why only positive operators, based on Minkowski sum, were applied to multifunctions, [23]. $\triangle$

The analysis of the approximation properties of $A_\chi^M F$ is based on properties of the metric piecewise linear approximation operator. These are studied in the next subsection.

3.3. **Metric piecewise linear approximations of multifunctions.** From now on

- $F : [a, b] \to \mathbb{K}$, $\{F_k = F(x_k)\}_{k=0}^N$, where $a = x_0 < x_1 < \cdots < x_N = b$ and $\chi = (x_0, \ldots, x_N)$ denotes a partition of $[a, b]$,
- $CH = CH(F_0, \ldots, F_N)$, $\delta_k = x_{k+1} - x_k$, $k = 0, \ldots, N-1$,
- $\delta_{max} = \max\{\delta_k \mid 0 \le k \le N-1\}$, $\delta_{min} = \min\{\delta_k \mid 0 \le k \le N-1\}$.

In case of a uniform partition, we have $\delta_{max} = \delta_{min} = h = (b-a)/N$ and denote such a partition by χ_N.

Definition 3.10. *The* ***metric piecewise linear approximation*** *to a multifunction F at a partition χ is*

$$S_\chi^M(F, x) = \{\lambda_k(x) f_k + (1 - \lambda_k(x)) f_{k+1} \mid (f_0, \ldots, f_N) \in CH\},\ x \in [x_k, x_{k+1}],$$

where

$$\lambda_k(x) = (x_{k+1} - x)/(x_{k+1} - x_k). \tag{3.27}$$

$\lambda_k(\cdot)$ in (3.27) was proposed in [1].

By construction, the set valued function $S_\chi^M F$ has a representation in terms of selections

$$S_\chi^M F = \{s(\chi, \varphi) \mid \varphi \in CH(F_0, \ldots, F_N)\}, \tag{3.28}$$

where $s(\chi, \varphi)$ is a piecewise linear single-valued function interpolating the data (x_k, f_k), $k = 0, \ldots, N$ with $\varphi = (f_0, \ldots, f_N)$.

Recall the *piecewise linear interpolant based on metric average*, introduced in [1], is

$$S_\chi^{MA}(F,x) = F_k \oplus_{\lambda_k(x)} F_{k+1}, \quad x \in [x_k, x_{k+1}]$$

with $\lambda_k(x)$ defined by (3.27).

Lemma 3.11. *For a multifunction $F : [a,b] \to \mathbb{K}$ the metric piecewise linear approximation and the piecewise linear interpolant based on the metric average coincide, that is*

$$S_\chi^{MA} F = S_\chi^{M} F \tag{3.29}$$

and

$$\mathrm{H}(F(x), S_\chi^{MA}(F,x)) \le 2\omega_{[a,b]}(F, \delta_{max}), \quad x \in [a,b]. \tag{3.30}$$

Proof. To prove (3.29) we first show that $(S_\chi^{MA} F)(x) \subset (S_\chi^{M} F)(x)$ for any $x \in [a,b]$, and then show the reverse inclusion.

For a fixed $x \in [x_k, x_{k+1}]$ and any $y \in S_\chi^{MA}(F,x)$, one has $y = \lambda_k(x) f_k + (1-\lambda_k(x)) f_{k+1}$ for some $(f_k, f_{k+1}) \in \Pi(F_k, F_{k+1})$. Thus there exists a metric chain $\varphi = (f_0, \ldots, f_k, \;\; f_{k+1}, \ldots, f_N)$, $\varphi \in CH$, such that $y = s(\chi, \varphi)(x)$.

We now show the reverse inclusion, namely $(S_\chi^{M} F)(x) \subset (S_\chi^{MA} F)(x)$. It is obvious that, for any $x \in [a,b]$ and any $\varphi \in CH$, $s(\chi,\varphi)(x) \in (S_\chi^{MA} F)(x)$.

To prove (3.30) we use (3.29), (3.15), and the triangle inequality for the Hausdorff-Pompeiu metric, and obtain for $x \in [x_k, x_{k+1}]$,

$$\begin{aligned} &\mathrm{H}(F(x), S_\chi^{M}(F,x)) \overset{(3.29)}{=} \mathrm{H}(F(x), S_\chi^{MA}(F,x)) \\ \le\, &\mathrm{H}(F(x), F(x_k)) + \mathrm{H}(F(x_k), S_\chi^{MA}(F,x)) = \mathrm{H}(F(x), F(x_k)) + \mathrm{H}(F_k, F_k \oplus_{\lambda_k(x)} F_{k+1}) \\ &\overset{(3.15)}{=} \mathrm{H}(F(x), F(x_k)) + (1-\lambda_k)\mathrm{H}(F_k, F_{k+1}) \le 2\omega_{[a,b]}(F, \delta_k). \end{aligned}$$

□

Next we show that $S_\chi^{M} F$ and its piecewise linear selections (3.28) "inherit" some continuity properties of a continuous multifunction F.

Lemma 3.12. *Let* $F \in Lip([a,b]),\mathcal{L})$ *and let* χ *be a partition of* $[a,b]$. *Then the metric piecewise linear approximation satisfies*

$$S_\chi^M F \in Lip([a,b],\mathcal{L}).$$

Proof. Since by (3.29) we have that $S_\chi^{MA} F = S_\chi^M F$, we use the piecewise linear interpolant based on metric average instead of the metric piecewise linear approximation.

Suppose that $x, y \in [x_k, x_{k+1}]$. Then

$$\mathrm{H}(S_\chi^{MA}(F,x), S_\chi^{MA}(F,y)) = \mathrm{H}(F_k \oplus_{\lambda_k(x)} F_{k+1}, F_k \oplus_{\lambda_i(y)} F_{k+1})$$
$$\overset{(3.14)}{=} |\lambda_k(x) - \lambda_k(y)|\mathrm{H}(F_k, F_{k+1}) \le |\lambda_k(x) - \lambda_k(y)|\mathcal{L}(x_{k+1} - x_k) = \mathcal{L}|x-y|.$$

Now let $x \in [x_j, x_{j+1}]$ and $y \in [x_k, x_{k+1}]$, where $0 \le j < k \le N-1$. Using the triangle inequality, (3.14), and the Lipschitz continuity of F, we get

$$\mathrm{H}(S_\chi^{MA}(F,x), S_\chi^{MA}(F,y)) \le \frac{x_{j+1}-x}{x_{j+1}-x_j}\mathrm{H}(F_j, F_{j+1}) + \mathrm{H}(F_{j+1}, F_k)$$
$$+\frac{y-x_k}{x_{k+1}-x_k}\mathrm{H}(F_k, F_{k+1}) \le \mathcal{L}(x_{j+1} - x + x_k - x_{j+1} + y - x_k) \le \mathcal{L}|y-x|.$$

□

Corollary 3.13. *Under the conditions of Lemma 3.12 and for any* $s(\chi,\varphi)$ *in* (3.28),

$$s(\chi,\varphi) \in Lip([a,b],\mathcal{L}).$$

Proof. The proof of this corollary is similar to the proof of the previous lemma and uses the observation that

$$|s(\chi,\varphi)(x_k) - s(\chi,\varphi)(x_{k+1})| \le \mathrm{H}(S_\chi^M(F,x_k), S_\chi^M(F,x_{k+1})), \quad k = 0,\dots,N-1. \tag{3.31}$$

□

Now we consider the case when F is a general continuous multifunction. It follows a statement concerning the so-called "global smoothness preservation".

Lemma 3.14. *Let* $F : [a,b] \to \mathbb{K}$ *be a continuous multifunction. Then for any partition* χ *of* $[a,b]$ *the modulus of continuity to the metric piecewise linear approximation satisfies*

$$\omega_{[a,b]}(S_\chi^M F, \delta) \le 4\omega_{[a,b]}(F, \delta). \tag{3.32}$$

Proof. By definition, for any $\delta > 0$,

$$\omega_{[a,b]}(S_\chi^M F, \delta) = \sup\{\mathrm{H}(S_\chi^M(F,x), S_\chi^M(F,y)) \mid |x-y| \le \delta,\ x,y \in [a,b]\}$$
$$\overset{(3.29)}{=} \omega_{[a,b]}(S_\chi^{MA} F, \delta) = \sup\{\mathrm{H}(S_\chi^{MA}(F,x), S_\chi^{MA}(F,y))|\ |x-y| \le \delta,\ x,y \in [a,b]\}.$$

In this case $x, y \in [x_j, x_{j+1}]$, $|x-y| \le \delta$, the claim of the lemma is obtained using (3.14) and (3.19). Indeed

$$\mathrm{H}(S_\chi^{MA}(F,x), S_\chi^{MA}(F,y)) = \mathrm{H}(F_j \oplus_{\lambda_j(x)} F_{j+1}, F_j \oplus_{\lambda_j(y)} F_{j+1})$$
$$= (|x-y|/\delta_j)\mathrm{H}(F_j, F_{j+1}) \le (|x-y|)/\delta_j)\omega_{[a,b]}(F, \delta_j) \tag{3.33}$$
$$\le (|x-y|)/\delta_j)(1+\delta_j/\delta)\omega_{[a,b]}(F,\delta) \le ((|x-y|)/\delta_j + (|x-y|)/\delta)\,\omega_{[a,b]}(F,\delta),$$

which implies (3.32).

Now, let $x \in [x_j, x_{j+1}]$, $y \in [x_k, x_{k+1}]$, $0 \le j < k \le N-1$, and $|x-y| \le \delta$. By the triangle inequality,

$$\mathrm{H}(S_\chi^{MA}(F,x), S_\chi^{MA}(F,y)) \le \mathrm{H}(S_\chi^{MA}(F,x), S_\chi^{MA}(F,x_{j+1})) \tag{3.34}$$
$$+\mathrm{H}(S_\chi^{MA}(F,x_{j+1}), S_\chi^{MA}(F,x_k)) + \mathrm{H}(S_\chi^{MA}(F,x_k), S_\chi^{MA}(F,y)),$$

while by the interpolation property of $S_\chi^{MA} F$ and since $|x_k - x_{j+1}| \le \delta$, we have

$$\mathrm{H}(S_\chi^{MA}(F,x_{j+1}), S_\chi^{MA}(F,x_k)) \le \omega_{[a,b]}(F,\delta). \tag{3.35}$$

Applying (3.33) and (3.35) to (3.34) we obtain

$$\mathrm{H}(S_\chi^{MA}(F,x), S_\chi^{MA}(F,y)) \tag{3.36}$$
$$= ((x_{j+1}-x)/\delta_j + (x_{j+1}-x)/\delta + 1 + (y-x_k)/\delta_k + (y-x_k)/\delta)\,\omega_{[a,b]}(F,\delta)$$
$$\le (3 + (x_{j+1} - x + y - x_k)/\delta_j)\,\omega_{[a,b]}(F,\delta) \le 4\omega_{[a,b]}(F,\delta).$$

Hence we also have (3.32) in the second situation. □

Corollary 3.15. *For any* $s(\chi,\varphi)$ *in* (3.28) *and any* $x,y\in[x_j,x_{j+1}]$, $0\le j\le N-1$, $|x-y|\le\delta$,

$$|s(\chi,\varphi)(x)-s(\chi,\varphi)(y)|\le(|x-y|/\delta_j+|x-y|/\delta)\,\omega_{[a,b]}(F,\delta). \tag{3.37}$$

For $|x-y|\le\delta\le\delta_{\min}$ *and* $x,y\in[x_j,x_{j+2}]$, $j=0,\dots,N-2$,

$$\omega_{[a,b]}(s(\chi,\varphi),\delta)\le(2/\delta)|x-y|\,\omega_{[a,b]}(F,\delta)\le 2\omega_{[a,b]}(F,\delta). \tag{3.38}$$

Proof. One has

$$\begin{aligned}|s(\chi,\varphi)(x)-s(\chi,\varphi)(y)|&\le \mathrm{H}(S^M(F,x),S^M(F,y))\\ &\overset{(3.33)}{\le}(|x-y|/\delta_j+|x-y|/\delta)\,\omega_{[a,b]}(F,\delta)\end{aligned}$$

and so (3.37) follows.

We establish inequality (3.38). Then

$$\begin{aligned}&\omega_{[a,b]}(s(\chi,\varphi),\delta)\\ &=\sup\{|(s(\chi,\varphi),x)-(s(\chi,\varphi),y)|\mid |x-y|\le\delta\le\delta_{\min},\ x,y\in[x_j,x_{j+2}]\}\\ &\le\sup\{\mathrm{H}(S^M(F,x),S^M(F,y))\mid |x-y|\le\delta\le\delta_{\min},\ x,y\in[x_j,x_{j+2}]\}\\ &\le 2\omega_{[a,b]}(F,\delta).\end{aligned}$$

□

Lemma 3.16. *Let* $F\in CBV([a,b])$*. Then for any* $s(\chi,\varphi)$ *in* (3.28),

$$\omega_{[a,b]}(s(\chi,\varphi),\delta)\le 3\omega_{[a,b]}(F,\delta)+\omega_{[a,b]}(v_F,\delta)\le 4\,\omega_{[a,b]}(v_F,\delta).$$

Proof. Denote $s=s(\chi,\varphi)$. For a given $\delta>0$, let $x\in[x_j,x_{j+1}]$, $y\in[x_k,x_{k+1}]$, $0\le j\le k\le N-1$, such that $|x-y|\le\delta$. Then

$$|s(x)-s(y)|\le|s(x)-s(x_{j+1})|+\sum_{l=j+1}^{k-1}|s(x_{l+1})-s(x_l)|+|s(y)-s(x_k)|.$$

By (3.37), (3.28), (3.31), and by the definition of $S_\chi^M F$, we get

$$|s(x) - s(y)| \leq ((x_{j+1} - x)/\delta_j + (x_{j+1} - x)/\delta)\, \omega_{[a,b]}(F, \delta)$$
$$+ \sum_{l=j+1}^{k-1} \mathrm{H}(F(x_{l+1}), F(x_l)) + ((y - x_k)/\delta_k + (y - x_k)/\delta)\, \omega_{[a,b]}(F, \delta).$$

Since $(x_{j+1} - x)/\delta + (y - x_k)/\delta < 1$, by the definition of the bounded variation of F and by (3.22), we obtain

$$|s(x) - s(y)| \leq 3\omega_{[a,b]}(F, \delta) + \bigvee_{x_{j+1}}^{x_k}(F) \leq 3\omega_{[a,b]}(F, \delta) + \omega_{[a,b]}(v_F, \delta) \leq 4\omega_{[a,b]}(F, \delta).$$

Taking the supremum over $|x - y| \leq \delta$, the proof ends. □

3.4. **Approximation by metric linear operators.** We will use the metric piecewise approximation to obtain error estimates for metric linear operators.

Let $A_\chi^M F$ be defined by (3.26), namely

$$A_\chi^M F(x) = A_\chi^M(F, x) = \bigoplus_{k=0}^{N} c_k(x) F(x_k)$$

and $S_\chi^M F$ be a metric piecewise linear approximation as defined in Subsection 3.3. By Definition 3.10, namely by

$$S_\chi^M(F, x) = \{\lambda_k(x) f_k + (1 - \lambda_k(x)) f_{k+1} \mid (f_0, \dots, f_N) \in CH\},\ x \in [x_k, x_{k+1}],$$

we get that

$$A_\chi^M F \equiv A_\chi^M(S_\chi^M F). \tag{3.39}$$

Moreover, by (3.25), (3.26), and (3.28),

$$A_\chi^M(S_\chi^M F) = \{A_\chi s(\chi, \varphi) \mid \varphi \in CH(F_0, \dots, F_N)\}. \tag{3.40}$$

The metric analogues of linear operators of the form (3.25), approximate certain classes of set-valued functions. By (3.39) and (3.40) the approximation results depend on the way A_χ approximates piecewise linear real-valued functions.

Theorem 3.17. *Let* A_χ *be of the form* (3.25). *Then for a continuous multifunction* $F:[a,b]\to\mathbb{K}$ *one has*

$$\mathrm{H}(A_\chi^M(F,x),F(x))\le 2\omega_{[a,b]}(F,\delta_{\max})+\sup_{\varphi\in CH}|A_\chi(s(\chi,\varphi),x)-s(\chi,\varphi)(x)|. \quad (3.41)$$

Proof. By the triangle inequality and by (3.39),

$$\mathrm{H}(A_\chi^M(F,x),F(x))\le \mathrm{H}(A_\chi^M(S_\chi^M F,x),S_\chi^M(F,x))+\mathrm{H}(S_\chi^M(F,x),F(x)),$$

while by (3.40)

$$\mathrm{H}(A_\chi^M(S_\chi^M(F,x)),S_\chi^M(F,x))\le \sup_{\varphi\in CH}|A_\chi(s(\chi,\varphi),x)-s(\chi,\varphi)(x)|.$$

This inequality together with (3.30), that is,

$$\mathrm{H}(F(x),S_\chi^{MA}(F,x))\le 2\omega_{[a,b]}(F,\delta_{\max}),\quad x\in[a,b],$$

completes the proof. □

Assume that $g:[a,b]\times[0,\infty[\,\to[0,\infty[$ is a continuous real-valued function, nondecreasing in the second argument, satisfying $g(x,0)=0$, and $\mathcal{S}_\chi$ denotes the set of piecewise linear continuous single-valued functions, with values in $\mathbb{R}^n$ and knots at χ.

Corollary 3.18. *Let* $F\in Lip([a,b],\mathcal{L})$ *and let* A_χ *be of the form* (3.25), *satisfying*

$$|A_\chi(s,x)-s(x)|\le C\cdot\mathcal{L}\cdot g(x,\delta_{\max}),\quad s\in\mathcal{S}_\chi\cap Lip([a,b],\mathcal{L}).$$

Then

$$\mathrm{H}(A_\chi^M(F,x),F(x))\le 2\mathcal{L}\delta_{\max}+C\cdot\mathcal{L}\cdot g(x,\delta_{\max}). \quad (3.42)$$

Corollary 3.19. *Let* $F\in CBV[a,b]$ *and let* A_χ *be of the form* (3.25), *satisfying*

$$|A_\chi(s,x)-s(x)|\le C\omega_{[a,b]}(s,g(x,\delta_{\max})),\quad s\in\mathcal{S}_\chi. \quad (3.43)$$

Then

$$\mathrm{H}(A_\chi^M(F,x),F(x))\le 2\,\omega_{[a,b]}(F,\delta_{\max})+4C\omega_{[a,b]}(v_F,g(x,\delta_{\max})). \quad (3.44)$$

For continuous set-valued functions which are not of bounded variation there are some limited results only for uniform partitions, see [10].

Corollary 3.20. *Let* $F : [a, b] \to \mathbb{K}(\mathbb{R}^m)$ *be continuous, and let* A_N *be a linear operator of the form* (3.25) *defined on a uniform partition* χ_N *with* $h = (b-a)/N$, *satisfying*

$$|A_N(s, x) - s(x)| \leq Cg(x, \omega_{[a,b]}(s, h)), \quad s \in \mathcal{S}_\chi. \tag{3.45}$$

Then

$$\mathrm{H}(A^M_\chi(F, x), F(x)) \leq 2\omega_{[a,b]}(F, h) + Cg(x, 2\omega_{[a,b]}(F, h)). \tag{3.46}$$

3.5. **Examples.**

3.5.1. *Metric Bernstein operators.* We recall the Bernstein operator $\mathrm{B}_N(f, x)$ in (1.3). It is known [7, Chapter 10] that there is a constant C independent of f such that

$$|f(x) - \mathrm{B}_N(f, x)| \leq C \cdot \omega_{[0,1]}(f, \sqrt{x(1-x)/N}\,). \tag{3.47}$$

The classical Bernstein operator for $F : [0, 1] \to \mathbb{K}$ with sums of numbers replaced by Minkowski sums of sets is given by (2.11). We have shown by Theorem 2.12 that for $x \in]0, 1[$ the limit of $\mathrm{B}_N(F)(x)$ when $N \to \infty$ is $\mathrm{conv}F(x)$, therefore these operators cannot approximate multifunctions with general images.

Definition 3.21. *For* $F : [0, 1] \to \mathbb{K}$ *the* ***metric Bernstein operator*** *is*

$$\begin{aligned}\mathrm{B}^M_N(F, x) &= \bigoplus_{k=0}^{N} \binom{N}{k} x^k (1-x)^{N-k} F\left(\frac{k}{N}\right) \\ &= \left\{ \sum_{k=0}^{N} \binom{N}{k} x^k (1-x)^{N-k} f_k \mid (f_0, \ldots, f_N) \in CH \right\}\end{aligned}$$

where $CH = CH(F(0), F(1/N), \ldots, F(1))$.

Corollary 3.22. *Let* $F \in Lip([0, 1], \mathcal{L})$, *then*

$$\mathrm{H}(\mathrm{B}^M_N(F, x), F(x)) \leq 2\mathcal{L}/N + C\mathcal{L}\sqrt{x(1-x)/N}.$$

Proof. Apply Corollary 3.18 with $A^M_\chi(F) = \mathrm{B}^M_N(F)$ and (3.47) and the conclusion follows. ☐

Corollary 3.23. *Let* $F \in CBV[0, 1]$, *then*

$$\mathrm{H}(\mathrm{B}^M_N(F, x), F(x)) \leq 2\omega_{[0,1]}(F, 1/N) + 4C\omega_{[0,1]}(v_F, \sqrt{x(1-x)/N}\,).$$

Proof. Apply Corollary 3.19 with $A_{\chi}^{M}(F) = \mathrm{B}_{N}^{M}(F)$ and (3.47). □

Since (3.45) does not hold for these operators, Corollary 3.20 cannot be applied.

3.5.2. *Metric Schoenberg operators.* For a uniform partition χ_N, the "classical" set-valued analogues of the Schoenberg spline operators for $F : [0,1] \to \mathbb{K}$ is

$$S_{m,N}(F,x) = \sum_{k=0}^{N} F(k/N) b_m(Nx-k), \tag{3.48}$$

where $b_m(x)$ is the (normalized) B-spline of order m (degree $m-1$) with integer knots and support $[0,m]$, and where the linear combination is in Minkowski sense. In [28] by an example it is shown that operators (3.48) with $m=2$ and $N \to \infty$ cannot approximate F with general compact images in any point of $[0,1] \setminus \chi_N$.

Definition 3.24. *The* ***metric Schoenberg operator*** *of order* m *for a multifunction* $F : [0,1] \to \mathbb{K}$ *and a uniform partition* χ_N *is defined by*

$$S_{m,N}^{M}(F,x) = \bigoplus_{k=0}^{N} b_m(Nx-k)F(k/N) = \left\{ \sum_{k=0}^{N} b_m(Nx-k) f_k \mid (f_0,\dots,f_N) \in CH \right\},$$

where $CH = CH(F(0), F(1/N), \dots, F(1))$.

The estimate below for the single valued case may be found in [5, p. 167]

$$|S_{m,N}f - f| \le \lfloor (m+1)/2 \rfloor \, \omega_{[0,1]}(f, 1/N) \ \text{ on } \ [(m-1)/N \, , 1] \tag{3.49}$$

where $\lfloor \cdot \rfloor$ is the floor function.

Corollary 3.25. *Let* F *be a continuous multifunction defined on* $[0,1]$. *Then*

$$\mathrm{H}(S_{m,N}^{M}(F,x), F(x)) \le 2\,(1 + \lfloor (m+1)/2 \rfloor)\, \omega_{[0,1]}(F, 1/N), \ x \in [(m-1)/N, 1] \, .$$

Proof. By Corollary 3.20 and by (3.49) we have

$$\begin{aligned} \mathrm{H}(S_{m,N}^{M}(F,x), F(x)) &\le 2\omega_{[0,1]}(F, 1/N) + \lfloor (m+1)/2 \rfloor \, \omega_{[0,1]}(f, 1/N) \\ &\overset{(3.37)}{\le} 2\,(1 + \lfloor (m+1)/2 \rfloor)\, \omega_{[0,1]}(F, 1/N), \quad x \in [(m-1)/N, 1] \, . \end{aligned}$$

□

If a function f is Lipschitz on $[0,1]$ of rank $\mathcal{L}$, then from (3.49) it follows

$$|S_{m,N}f - f| \le \lfloor (m+1)/2 \rfloor \, \mathcal{L}/N. \tag{3.50}$$

Corollary 3.26. *For* $F \in Lip([0,1],\mathcal{L})$,

$$\mathrm{H}(S^M_{m,N}(F,x), F(x)) \le (2 + \lfloor (m+1)/2 \rfloor) \, \frac{\mathcal{L}}{N}, \quad x \in [(m-1)/N\,, 1]\,.$$

Proof. By Corollary 3.18 and by (3.50) one has

$$\mathrm{H}(S^M_{m,N}(F,x), F(x)) \le 2\mathcal{L}/N + \lfloor (m+1)/2 \rfloor \, \mathcal{L}/N,$$

from where the conclusion follows. □

3.5.3. *Metric Polynomial Interpolants.*

Definition 3.27. (i) *Let* (x_k, A_k) *be given, where* $x_0 < x_1 < \cdots < x_N$ *are real numbers and* $A_k \in \mathbb{K}$, $k = 0, \ldots, N$ *are sets. The* ***metric polynomial interpolant*** *of these data is*

$$\bigoplus_{k=0}^{N} l_k A_k,$$

with l_k *defined by* (1.5).

(ii) *For* $F : [a,b] \to \mathbb{K}$, *the* ***metric polynomial interpolation operator*** *at the partition* χ *of* $[a,b]$, *is given by*

$$P^M_\chi(F,x) = \bigoplus_{k=0}^{N} l_k(x) F(x_k) = \left\{ \sum_{k=0}^{N} l_k(x) f_k \mid (f_0, f_1, \ldots, f_N) \in CH(F_0, \ldots, F_N) \right\},$$

with $F_k = F(x_k)$, $k = 0, \ldots, N$.

Let the interpolation points χ be the roots of the Chebyshev polynomial of degree $N+1$ on $[-1,1]$. It is known that (see, e.g., [22]) $\sum_{k=0}^{N} |\, l_k(x)| \le C \ln N$. Here and below C stands for a generic constant.

For a real-valued function f,

$$|f - \sum_{k=0}^{N} l_k(x) f(x_i)| \le (1 + \sum_{k=0}^{N} |l_k(x)|) E_N(f),$$

with $E_N(f)$ the error of the best approximation by polynomials of degree N on $[-1,1]$. Since $E_N(f) \leq C\omega_{[-1,1]}(f,1/N)$, [7, (1.3) in Chap. 7], we obtain for a Lipschitz function f

$$\left|f - \sum_{k=0}^{N} l_k(x) f(x_k)\right| \leq C \ln N/N \xrightarrow{N\to\infty} 0. \tag{3.51}$$

When adapting these interpolation operators to Lipschitz multifunctions, by Theorem 3.17 we get

Corollary 3.28. *For $F \in Lip([0,1],\mathcal{L})$, and let the points χ be the roots of the Chebyshev polynomial of degree $N+1$ on this interval, then*

$$\mathrm{H}(P_\chi^M(F,x), F(x)) \leq 2\mathcal{L}\delta_{\max} + C\ln N/N = O\left(\ln N/N\right).$$

The last equality follows from the fact that $\delta_{\max} \leq \pi/N$ for N large enough.

Remark 3.29. In [3] and [4] the family of nonempty convex and compact subsets in $\mathbb{R}^n$ is used for similar goals but in a different framework. $\triangle$

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A MULTI-STEP ITERATIVE METHOD FOR APPROXIMATING COMMON FIXED POINTS OF PRESIĆ-RUS TYPE OPERATORS ON METRIC SPACES

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Abstract. The existence of coincidence points and common fixed points for mappings satisfying a Presić type condition in metric spaces is proved. A multi-step iterative method for constructing the common fixed points and its rate of convergence are also provided. This is a generalization of several fixed point and common fixed point results in literature.

1. **Introduction**

One of the most important results in fixed point theory is the contraction mapping principle of Banach, not only for its applications but also due to its position as a central point for a remarkable number of generalizations that appeared along the time, on various and sometimes very different directions (see for example [8], [11]). The present paper deals with two of these directions, aiming to establish a new general result at their point of intersection.

One direction, the first under our attention, was opened in 1965 by S. Presić [7] who proved the existence and uniqueness of fixed points for operators satisfying a special type of contraction condition, also providing a so-called multi-step iteration method for approximating the fixed points. In the sequel we shall consider (X, d) a metric space. Presić' condition generalizes Banach's contraction condition, namely

$$d(f(x), f(y)) \leq \alpha d(x, , y),$$

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for any $x, y \in X$, where $f : X \to X$ an operator and $\alpha \in [0, 1)$ a constant, by considering instead

$$d(f(x_0, x_1, \ldots, x_{k-1}), f(x_1, x_2, \ldots, x_k)) \le \sum_{i=1}^{k} \alpha_i \cdot d(x_{i-1}, x_i),$$

for any $x_0, x_1, \ldots, x_k \in X$, where k a positive integer, $f : X^k \to X$ an operator and $\alpha_1, \ldots, \alpha_k \in \mathbb{R}_+$ constants such that $\sum\limits_{i=1}^{k} \alpha_i < 1$.

Several general Presić type results followed in literature, see for example the papers due to M.R. Taskovic [13], M. Şerban [12], our paper [6] and I.A. Rus [10, 9], in the latter of which the following result is proved:

Theorem 1.1 (I.A. Rus [9], 1981)**.** *Let (X, d) be a complete metric space, k a positive integer, $\varphi : \mathbb{R}_+^k \to \mathbb{R}_+$ with the properties:*

a) $\varphi(r) \le \varphi(s)$, *for* $r, s \in \mathbb{R}_+^k$, $r \le s$*;*

b) $\varphi(r, r, \ldots, r) < r$, *for* $r \in \mathbb{R}_+, r > 0$*;*

c) φ *continuous;*

d) $\sum\limits_{i=0}^{\infty} \varphi^i(r) < \infty$*;*

e) $\varphi(r, 0, \ldots, 0) + \varphi(0, r, 0, \ldots, 0) + \ldots + \varphi(0, \ldots, 0, r) \le \varphi(r, r \ldots, r)$, *for any* $r \in \mathbb{R}_+$,

and $f : X^k \to X$ an operator such that:

$$d(f(x_0, \ldots, x_{k-1}), f(x_1, \ldots, x_k)) \le \varphi(d(x_0, x_1), d(x_1, x_2), \ldots, d(x_{k-1}, x_k)),$$

for any $x_0, x_1, \ldots, x_k \in X$.

Then:

ι) *there exists a unique $x^* \in X$ solution of the equation*

$$x = f(x, x, \ldots, x);$$

ιι) *the sequence $\{x_n\}_{n\ge0}$, with $x_0, x_1, \ldots, x_{k-1} \in X$ and*

$$x_n = f(x_{n-k}, x_{n-k+1}, \ldots, x_{n-1}), \text{for } n \ge k,$$

converges to x^*;

ιιι) the rate of convergence for $\{x_n\}_{n\geq 0}$ *is given by*

$$d(x_n, x^*) \leq k\sum_{i=0}^{\infty}\varphi^{\left[\frac{n+i}{k}\right]}(d_0, \ldots, d_0),$$

where $d_0 = \max\{d(x_0, x_1), d(x_1, x_2), \ldots, d(x_{k-1}, x_k)\}$.

A generalization of the contraction mapping principle on a different direction was given in 1976 by G. Jungck [4], regarding common fixed points of commuting mappings. A recent result on the existence of common fixed points has been proved by M. Abbas and G. Jungck [1] in a cone metric space setting.

Theorem 1.2 (M. Abbas, G. Jungck [1], 2008). *Let* (X, d) *be a cone metric space,* P *a normal cone with normal constant* K *and* $f, g : X \to X$ *two operators satisfying*

$$d(f(x), f(y)) \leq kd(g(x), g(y)), \text{ for all } x, y \in X, \tag{1.1}$$

where $k \in [0, 1)$ *is a constant. If the range of* g *contains the range of* f *and* $g(X)$ *is a complete subspace of* X*, then* f *and* g *have a unique coincidence point in* X*. If in addition* f *and* g *are weakly compatible, then they have a unique common fixed point in* X*.*

Remark 1.1. For the condition (1.1) see also K. Goebel [3].

Remark 1.2. In practical situations, the condition "$g(X)$ complete metric subspace" is too restrictive. It suffices to make sure there exists a complete metric subspace $Y \subset X$ such that $f(X) \subset Y \subset g(X)$, which also implies $f(X) \subseteq g(X)$. This is what we shall require in this paper.

The main aim of this paper is to establish new common fixed point results situated at the intersection of the two aforementioned directions, thus generalizing Theorems 1.1, 1.2 and several other subsequent results.

2. Preliminaries

We begin by recalling some concepts used in [4, 5, 1] and several related papers.

Definition 2.1 ([4]). *Let X be nonempty set and $f, g : X \to X$ two operators.*

If

$$f(p) = g(p)$$

*for some $p \in X$, then p is called a **coincidence point** of f and g, while $s = f(p) = g(p)$ is a **coincidence value** for them.*

If

$$f(p) = g(p) = p$$

*for some $p \in X$, then p is called a **common fixed point** of f and g.*

Remark 2.1. We shall denote by

$$C(f, g) = \{p \in X \,|\, f(p) = g(p)\}$$

the set of all coincidence points of f and g.

Obviously, the following hold:

a) $F_f \cap F_g \subset C(f, g)$;

b) $F_f \cap C(f, g) = F_g \cap C(f, g) = F_f \cap F_g$.

Definition 2.2 ([5]). *Let X be a nonempty set and $f, g : X \to X$. The operators f and g are said to be **weakly compatible** if they commute at their coincidence points, namely if*

$$f(g(p)) = g(f(p)),$$

for any p a coincidence point of f and g.

Lemma 2.1. *Let X be a nonempty set and $f, g : X \to X$. If f and g are weakly compatible, then*

$$C(f, g) \in I(f) \cap I(g),$$

i.e., $C(f, g)$ is an invariant set for both f and g.

Proof. Let $p \in C(f, g)$. We shall prove that $f(p), g(p) \in C(f, g)$, as well.

By definition,

$$f(p) = g(p) = q \in X. \tag{2.1}$$

As f and g are weakly compatible, we have:

$$f(g(p)) = g(f(p)),$$

which by (2.1) yields

$$f(q) = g(q),$$

so $q = f(p) = g(p) \in C(f, g)$. Thus $C(f, g) \in I(f) \cap I(g)$. □

Using this Lemma, the proof of the following Lemma is immediate.

Lemma 2.2 ([1]). *Let X be a nonempty set and $f, g : X \to X$ two weakly compatible operators.*

If they have a unique coincidence value $x^* = f(p) = g(p)$, for some $p \in X$, then x^* is their unique common fixed point.

Remark 2.2. For any operator $f : X^n \to X$, n a positive integer, we can define its **associate operator** $F : X \to X$ by

$$F(x) = f(x, \dots, x), x \in X.$$

Obviously, $x \in X$ is a fixed point of $f : X^k \to X$, i.e., $x = f(x, \dots, x)$, if and only if it is a fixed point of its associate operator F, in the sense of the classical definition. For details see for example [10].

Based on this remark, we can extend the previous definitions for the case $f : X^k \to X$, k a positive integer.

Definition 2.3. *Let X be a nonempty set, k a positive integer and $f : X^k \to X$, $g : X \to X$ two operators.*

*An element $p \in X$ is called a **coincidence point** of f and g if it is a coincidence point for F and g.*

*Similarly, $s \in X$ is a **coincidence value** of f and g if it is a coincidence value for F and g.*

*An element $p \in X$ is a **common fixed point** of f and g if it is a common fixed point of F and g.*

Definition 2.4. *Let X be a nonempty set, k a positive integer and $f : X^k \to X$, $g : X \to X$. The operators f and g are said to be **weakly compatible** if F and g are weakly compatible.*

The following result is a generalization of Lemma 1.4 in [1], included above as Lemma 2.2.

Lemma 2.3. *Let X be a nonempty set, k a positive integer and $f : X^k \to X, g : X \to X$ two weakly compatible operators.*

If f and g have a unique coincidence value $x^* = f(p, \ldots, p) = g(p)$, then x^* is the unique common fixed point of f and g.

Proof. As f and g are weakly compatible, F and g are also weakly compatible. The proof follows by Lemma 2.2. □

3. The main result

The main result of the paper is the following theorem which unifies two generalizations of the contraction mapping principle of Banach, namely the Presić type result due to I.A. Rus [9] and the common fixed point result due to M. Abbas and G. Jungck [1], in metric spaces. It also provides a *multi-step iteration method* for effectively determining the coincidence values/common fixed points of the operators referred.

Theorem 3.1. *Let (X, d) be a metric space, k a positive integer, $\varphi : \mathbb{R}_+^k \to \mathbb{R}_+$ with the properties $a) - e)$ in Theorem 1.1 and $f : X^k \to X$, $g : X \to X$ two operators for which there exists a complete subspace $Y \subseteq X$ such that $f(X^k) \subseteq Y \subseteq g(X)$ and*

$$d(f(x_0, \ldots, x_{k-1}), f(x_1, \ldots, x_k)) \leq \tag{cPR}$$

$$\leq \varphi(d(g(x_0), g(x_1)), \ldots, d(g(x_{k-1}), g(x_k))),$$

for any $x_0, \ldots, x_k \in X$.
Then:

1) *f and g have a unique coincidence value, say x^*, in X;*

2) *the sequence* $\{g(x_n)\}_{n\geq 0}$ *defined by* $x_0, \ldots, x_{k-1} \in X$ *and*

$$g(x_n) = f(x_{n-k}, \ldots, x_{n-1}), n \geq k, \tag{3.1}$$

converges to x^**, with a rate estimated by*

$$d(g(x_n), x^*) \leq k \sum_{i=0}^{\infty} \varphi^{\left[\frac{n+i}{k}\right]}(d_0, \ldots, d_0), \tag{3.2}$$

where $d_0 = \max\{d(g(x_0), g(x_1)), \ldots, d(g(x_{k-1}), g(x_k))\}$*;*

3) *if in addition* f *and* g *are weakly compatible, then* x^* *is their unique common fixed point.*

Proof. Let $x_0, x_1, \ldots, x_{k-1} \in X$.
Then $f(x_0, \ldots, x_{k-1}) \in f(X^k) \subset g(X)$, so there exists $x_k \in X$ such that

$$f(x_0, \ldots, x_{k-1}) = g(x_k).$$

Further on, $f(x_1, \ldots, x_k) \in f(X^k) \subset g(X)$, so there exists $x_{k+1} \in X$ such that

$$f(x_1, \ldots, x_k) = g(x_{k+1}).$$

In this manner we construct the sequence $\{g(x_n)\}_{n\geq 0}$ such that

$$g(x_n) = f(x_{n-k}, \ldots, x_{n-1}), n \geq k. \tag{3.3}$$

We should remark now that, due to construction,

$$\{g(x_n)\}_{n\geq 0} \subseteq f(X^k) \subseteq Y \subseteq g(X). \tag{3.4}$$

We denote

$$d_0 = \max\{d(g(x_0), g(x_1)), \ldots, d(g(x_{k-1}), g(x_k))\} \tag{3.5}$$

and this is positive, assuming $g(x_0), \ldots, g(x_k)$ are not all equal (otherwise one can easily choose another convenient initial point x_j, $j \in \{0, \ldots, k-1\}$).

The following estimations hold then, by hypothesis:

$$\begin{aligned}
&d(g(x_k), g(x_{k+1})) = d(f(x_0, \ldots, x_{k-1}), f(x_1, \ldots, x_k)) \le \\
&\quad \le \varphi(d(g(x_0), g(x_1)), \ldots, d(g(x_{k-1}), g(x_k))) \le \\
&\quad \le \varphi(d_0, \ldots, d_0) < d_0 \\
&d(g(x_{k+1}), g(x_{k+2})) = d(f(x_1, \ldots, x_k), f(x_2, \ldots, x_{k+1})) \le \\
&\quad \le \varphi(d(g(x_1), g(x_2)), \ldots, d(g(x_k), g(x_{k+1}))) \le \\
&\quad \le \varphi(d_0, \ldots, d_0, \varphi(d_0, \ldots, d_0)) \le \varphi(d_0, \ldots, d_0) < d_0 \\
&\vdots \\
&d(g(x_{2k-1}), g(x_{2k})) = d(f(x_{k-1}, \ldots, x_{2k-2}), f(x_k, \ldots, x_{2k-1})) \le \\
&\quad \le \varphi(d(g(x_{k-1}), g(x_k)), \ldots, d(g(x_{2k-2}), g(x_{2k-1}))) \le \\
&\quad \le \varphi(d_0, \varphi(d_0, \ldots, d_0), \ldots, \varphi(d_0, \ldots, d_0)) \le \varphi(d_0, \ldots, d_0) < d_0 \\
&d(g(x_{2k}), g(x_{2k+1})) = d(f(x_k, \ldots, x_{2k-1}), f(x_{k+1}, \ldots, x_{2k})) \le \\
&\quad \le \varphi(d(g(x_k), g(x_{k+1})), \ldots, d(g(x_{2k-1}), g(x_{2k}))) \le \\
&\quad \le \varphi(\varphi(d_0, \ldots, d_0), \varphi(d_0, \ldots, d_0), \ldots, \varphi(d_0, \ldots, d_0)) = \\
&\quad = \varphi^2(d_0, \ldots, d_0) \le \varphi(d_0, \ldots, d_0) < d_0 \\
&d(g(x_{2k+1}), g(x_{2k+2})) \le \cdots \le \varphi^2(d_0, \ldots, d_0, \varphi(d_0, \ldots, d_0)) \le \\
&\quad \le \varphi^2(d_0, \ldots, d_0) < d_0
\end{aligned}$$

and so on

$$d(g(x_n), g(x_{n+1})) \le \varphi^{\left[\frac{n}{k}\right]}(d_0, \ldots, d_0), n \ge k. \tag{3.6}$$

Thus, for some integer $p \ge 1$, we obtain:

$$d(g(x_n), g(x_{n+p})) \le \varphi^{\left[\frac{n}{k}\right]}(d_0, \ldots, d_0) + \cdots + \varphi^{\left[\frac{n+p-1}{k}\right]}(d_0, \ldots, d_0), n \ge 0. \tag{3.7}$$

By denoting

$$l = \left[\frac{n}{k}\right] \text{ and } m = \left[\frac{n+p-1}{k}\right], \tag{3.8}$$

we have that $m \geq l$. Besides, the above relation (3.7) implies further estimation

$$d(g(x_n), g(x_{n+p})) \leq \underbrace{\varphi^l(d_0, \ldots, d_0) + \cdots + \varphi^l(d_0, \ldots, d_0)}_{k \text{ times}} +$$
$$+ \underbrace{\varphi^{l+1}(d_0, \ldots, d_0) + \cdots + \varphi^{l+1}(d_0, \ldots, d_0)}_{k \text{ times}} +$$
$$+ \cdots + \underbrace{\varphi^m(d_0, \ldots, d_0) + \cdots + \varphi^m(d_0, \ldots, d_0)}_{k \text{ times}},$$

so

$$d(g(x_n), g(x_{n+p})) \leq k \sum_{i=l}^{m} \varphi^i(d_0, \ldots, d_0), n \geq 0, p \geq 1. \tag{3.9}$$

Denoting $S_n = \sum_{i=0}^{m} \varphi^i(d_0, \ldots, d_0)$ we have that

$$\sum_{i=l}^{m} \varphi^i(d_0, \ldots, d_0) = S_m - S_{l-1}, m \geq l.$$

As, for $s \in \mathbb{R}_+^k$, $\sum_{i=0}^{\infty} \varphi^i(s) < +\infty$ from assumption $d)$ upon φ, there exists

$$S = \lim_{n \to \infty} S_n.$$

Considering (3.8) it follows that

$$\lim_{l \to \infty} \sum_{i=l}^{m} \varphi^i(d_0, \ldots, d_0) = S - S = 0,$$

and, in view of (3.9), $d(g(x_n), g(x_{n+p})) \to 0$, as $n \to \infty$. This means that $\{g(x_n)\}_{n \geq 0}$ is a Cauchy sequence contained, by (3.4), in the complete metric subspace Y, so there exists $x^* \in g(X)$ such that

$$x^* = \lim_{n \to \infty} g(x_n).$$

Consequently, since by (3.4) $Y \subseteq g(X)$, there exists $r \in X$ such that

$$g(r) = x^* = \lim_{n \to \infty} g(x_n). \tag{3.10}$$

Next we shall prove that $f(r,\dots,r)=x^*$ as well. In this respect we estimate

$$d(g(x_{n+1}),f(r,\dots,r))=d(f(x_{n-k+1},\dots,x_n),f(r,\dots,r))\le$$
$$\le d(f(x_{n-k+1},\dots,x_n),f(x_{n-k+2},\dots,x_n,r)+$$
$$+d(f(x_{n-k+2},\dots,x_n,r),f(x_{n-k+3},\dots,x_n,r,r))+$$
$$+\dots+$$
$$+d(f(x_n,r,\dots,r),f(r,\dots,r)),$$

which by (cPR) becomes

$$d(g(x_{n+1}),f(r,\dots,r))\le \tag{3.11}$$
$$\le \varphi(d(g(x_{n-k+1}),g(x_{n-k+2})),\dots,d(g(x_n),g(r)))+$$
$$+\varphi(d(g(x_{n-k+2}),g(x_{n-k+3})),\dots,d(g(x_n),g(r)),d(g(r),g(r)))+$$
$$+\dots+\varphi(d(g(x_n),g(r)),d(g(r),g(r)),\dots,d(g(r),g(r))).$$

Now by assumption $d)$ on φ and (3.6), it follows that

$$d(g(x_n),g(x_{n+1}))\to 0 \text{ as } n\to\infty,$$

while, by the continuity of the distance and (3.10),

$$d(g(x_n),g(r))=d(g(x_n),x^*)\to 0 \text{ as } n\to\infty.$$

Using the previous results, by letting $n\to\infty$ in (3.11), we get that $d(f(r,\dots,r),x^*)\le 0$, i.e.,

$$d(f(r,\dots,r),x^*)=0$$

So, in view of (3.10,) $f(r,\dots,r)=x^*=g(r)$ holds, i.e., r is a coincidence point of f and g, while x^* is a coincidence value for them.
In order to prove the uniqueness of x^*, we suppose there would be $q\in X$ such that

$$f(q,\dots,q)=g(q)\ne x^*.$$

Then

$$\begin{aligned}
d(g(r), g(q)) &= d(f(r, \dots, r), f(q, \dots, q)) \leq \\
&\leq d(f(r, \dots, r), f(r, \dots, r, q)) + \dots + \\
&\quad + d(f(r, q, \dots, q), f(q, \dots, q)) \leq \\
&\leq \varphi(d(g(r), g(r)), \dots, d(g(r), g(r)), d(g(r), g(q))) + \dots + \\
&\quad + \varphi(d(g(r), g(q)), d(g(q), g(q)), \dots, d(g(q), g(q))) = \\
&= \varphi(0, \dots, 0, d(g(r), g(q))) + \dots + \varphi(d(g(r), g(q)), 0, \dots, 0) \leq \\
&\leq \varphi(d(g(r), g(q)), \dots, d(g(r), g(q))).
\end{aligned}$$

Supposing $g(r) \neq g(q)$, by hypothesis $e)$ on φ it would follow that

$$d(g(r), g(q)) < d(g(r), g(q)), \tag{3.12}$$

which is obviously a contradiction. This proves the uniqueness of the coincidence value x^*. In case f and g are also weakly compatible, by Lemma 2.3 this guarantees the existence and uniqueness of their common fixed point, which is actually the coincidence value here denoted by x^*.

The estimation (3.2) follows immediately from (3.7), by letting $p \to \infty$. □

Remark 3.1. 1) Theorem 3.1 above reduces to the result due to I.A. Rus [9] for $g = 1_X$.

2) For $g = 1_X$ and $\varphi : \mathbb{R}_+^k \to \mathbb{R}_+$, $\varphi(r_1, \dots, r_k) = \sum_{i=1}^{k} \alpha_i r_i$, with constants $\alpha_1, \dots, \alpha_k \in \mathbb{R}_+$ and $\sum_{i=1}^{k} \alpha_i < 1$, the result due to S. Presić [7] is obtained.

3) For $k = 1$ and $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$, $\varphi(r) = r$, the result of M. Abbas and G. Jungck [1] in metric spaces is obtained, in a slightly more general version, as we replaced the condition "$g(X)$ a complete metric subspace" by the less restrictive and more practical one "there exists a complete subspace $Y \subseteq X$ such that $f(X^k) \subseteq Y \subseteq g(X)$".

4. An extension of the main result

While the great majority of the common fixed point results in literature deal with the case when both f and g are self-operators on X, the above result offers information about coincidence and common fixed points of two operators, one of them defined on the Cartesian product X^k, $f : X^k \to X$, where k is a positive integer, and the second one a self-operator on X, $g : X \to X$.

Our aim in this section is to establish common fixed point theorems for $f : X^k \to X$ and $g : X^l \to X$, with k and l positive integers. In this respect it is necessary to start with some definitions, which extend the corresponding ones in the previous section.

Definition 4.1. *Let X be a metric space, k, l positive integers and $f : X^k \to X$, $g : X^l \to X$ two operators.*

An element $p \in X$ is called a ***coincidence point*** *of f and g if it is a coincidence point of F and G, where $F, G : X \to X$ are their associate operators, see Remark 2.2.*

An element $s \in X$ is a ***coincidence value*** *of f and g if it is a coincidence value of F and G.*

An element $p \in X$ is a ***common fixed point*** *of f and g if it is a common fixed point of F and G.*

Definition 4.2. *Let (X, d) be a metric space, k, l positive integers and $f : X^k \to X$, $g : X^l \to X$. The operators f and g are said to be* ***weakly compatible*** *if F and G are weakly compatible.*

In these terms we state now the following result, which extends Theorem 3.1.

Theorem 4.1. *Let (X, d) be a metric space, k and l positive integers, $\varphi : \mathbb{R}_+^k \to \mathbb{R}_+$ with the properties $a) - e)$ in Theorem 1.1 and $f : X^k \to X$, $g : X^l \to X$ two operators such that f and G satisfy the conditions in Theorem 3.1, where G is the operator associated to g.*

Then:

1) *f and g have a unique coincidence value, say x^*, in X;*

2) *the sequence* $\{G(z_n)\}_{n\geq 0}$ *defined by* $z_0 \in X$ *and*

$$G(z_n) = f(z_{n-1}, \ldots, z_{n-1}), n \geq 1, \tag{4.1}$$

converges to x^**;*

3) *the sequence* $\{G(x_n)\}_{n\geq 0}$ *defined by* $x_0, \ldots, x_{k-1} \in X$ *and*

$$G(x_n) = f(x_{n-k}, \ldots, x_{n-1}), n \geq k, \tag{4.2}$$

converges to x^* *as well, with a rate estimated by*

$$d(G(x_n), x^*) \leq k \sum_{i=0}^{\infty} \varphi^{\left[\frac{n+i}{k}\right]}(D_0, \ldots, D_0), \tag{4.3}$$

where $D_0 = \max\{d(G(x_0), G(x_1)), \ldots, d(G(x_{k-1}), G(x_k))\}$*;*

4) *if in addition* f *and* g *are weakly compatible, then* x^* *is their unique common fixed point.*

Proof. Considering the definitions given in the current section of this paper, all the conclusions follow by applying Theorem 3.1 for $f : X^k \to X$ and $G : X \to X$. $\square$

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HEAT TRANSFER IN AXISYMMETRIC STAGNATION FLOW ON THIN CYLINDERS

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Abstract. The Navier-Stokes and energy equations for the steady laminar incompressible flow past a row of circular cylinders at constant temperature are solved numerically. Prime attention was focused on how heat transfer characteristics are affected by variation of Reynolds number. The study was limited to Reynolds number ranging from 1 to 100 and the Prandtl number has been fixed to a value equal to 1. For different values of above parameters streamlines, isotherms and the local Nusselt number has been determined and are shown on several graphs.

1. **Introduction**

Heat transfer from bodies of different geometries is one of the important problems that has received much attention due to its engineering applications, such as: heat transfer from rotating machinery, spinning projectiles, cooling of electronic devices, design of heat exchangers, theory of hot wire anemometer. According to the literature, the available information in these areas is limited to a few special cases. A general method for solving such problems is not easy available, not only due to the mathematical difficulties involved, but also due to the wide range of body shapes as well as the different characteristics of the velocity and thermal fields.

References related to this topic can be found in the books by: Bejan (1995), Postelnicu and Pop (1999), Pop and Ingham (2001), Khor and Pop (2005), White (2008). On the other hand the problem of the heat transfer between a circular cylinder

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and its surrounding stream of a viscous fluid is of great interest. A large number of experimental papers on this problem are available in the literature, see Jain and Goel (1976). It is well known that numerical solution of Navier-Stokes equations for fluid flow problems may give reliable information in a case when experimental measurements are difficult. This view is well supported by the numerical studies made by Collins and Dennis (1973), Ingham (1984), Nam (1990), etc. To our best knowledge, flow and the heat transfer through an array of cylinders has not been to much study. However, the natural convective heat transfer from a pair of horizontal cylinder of the same temperature placed one above the other in a vertical plane has been theoretically studied by Yuncu and Batta (1994).

In the present paper the problem of cooling by forced convection of a row of circular cylinders is numerically studied. Namely, flow and heat transfer characteristics are determined for different values of the Reynolds number, keeping the Prandtl number constant ($Pr = 1$). The value of the Reynolds number is consider to be in the range $1 \leq Re \leq 100$ and the viscous dissipation is negligible small.

2. **Basic Equations**

Consider the steady two-dimensional forced convection flow over a circular cylinders' row of radius R placed in a viscous fluid of ambient temperature T_∞ and velocity U_∞ (see Figure 1). It is assumed that the distance between the cylinders is $2R$ and that the temperature of the cylinders is constant T_w ($T_w > T_\infty$). The mathematical model is given by continuity, Navier-Stokes and energy equations:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{2.1}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \tag{2.2}$$

$$u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) \tag{2.3}$$

$$\rho c_P\left(u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y}\right) = k\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right) \tag{2.4}$$

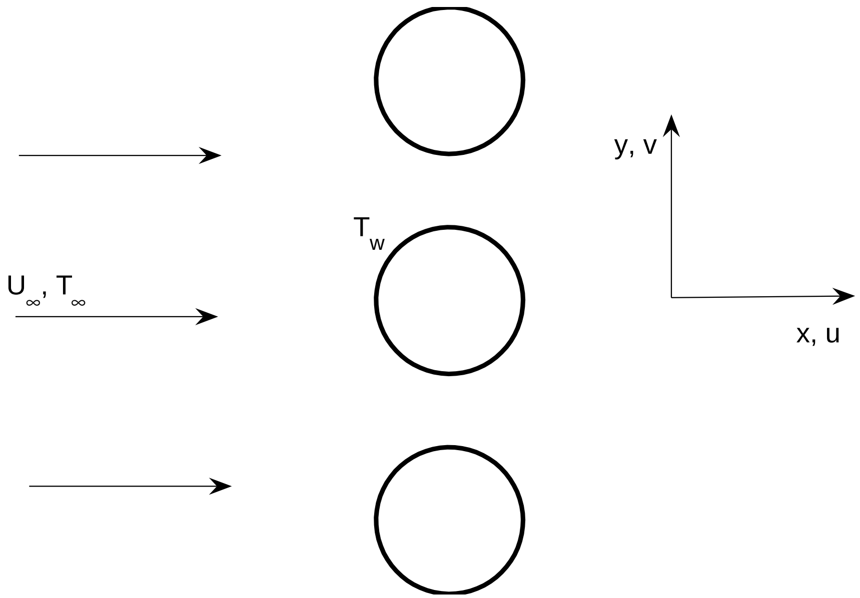

FIGURE 1. Geometry of the problem and the co-ordinate system

where x and y are the Cartesian co-ordinate along the horizontal and vertical direction, respectively, u and v are the velocity components along x and y -axes, p is the pressure, T is the temperature, k is the thermal diffusivity of the viscous fluid, ρ is the fluid density and c_p is the specific heat at constant pressure.

Further we use the following dimensionless variables for co-ordinate, velocity components and pressure:

$$X = \frac{x}{R}, \quad Y = \frac{y}{R}, \quad U = \frac{u}{U_\infty}, \quad V = \frac{v}{U_\infty}, \quad P = \frac{p - p_\infty}{\rho U_\infty^2} \tag{2.5}$$

Using (2.5) in Eqs.(2.1)-(2.4) we obtain:

$$\frac{\partial U}{\partial X} + \frac{\partial V}{\partial Y} = 0 \tag{2.6}$$

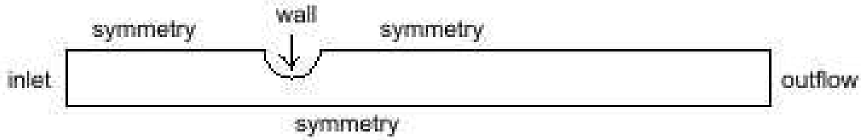

FIGURE 2. Domain of integration and boundary condition type

$$U\frac{\partial U}{\partial X}+V\frac{\partial U}{\partial Y}=-\frac{\partial P}{\partial X}+\frac{1}{Re}\left(\frac{\partial^2 U}{\partial X^2}+\frac{\partial^2 U}{\partial Y^2}\right) \tag{2.7}$$

$$U\frac{\partial V}{\partial X}+V\frac{\partial V}{\partial Y}=-\frac{\partial P}{\partial Y}+\frac{1}{Re}\left(\frac{\partial^2 V}{\partial X^2}+\frac{\partial^2 V}{\partial Y^2}\right) \tag{2.8}$$

$$\left(U\frac{\partial T}{\partial X}+V\frac{\partial T}{\partial Y}\right)=\frac{1}{RePr}\left(\frac{\partial^2 T}{\partial X^2}+\frac{\partial^2 T}{\partial Y^2}\right) \tag{2.9}$$

where $Re = U_\infty R/\nu$ is the Reynolds number and $Pr = \nu/\alpha$ is the Prandtl number. Due to the software used to solve the model the temperature is mentained in the dimensional form. Due to the symmetry of the problem we consider a small domain to integrate equations (2.7)-(2.9), see Figure 2. The boundary conditions for the new domain can be expressed as:

$$-inlet: U_\infty = 1, T = T_\infty \tag{2.10}$$

$$-wall: U = 0, V = 0, T = T_w \tag{2.11}$$

$$-symmetry: \frac{\partial U}{\partial Y}=0, \frac{\partial V}{\partial Y}=0, \frac{\partial T}{\partial Y}=0 \tag{2.12}$$

$$-outflow: \frac{\partial V}{\partial Y}=0 \tag{2.13}$$

A quantity of interest is the local Nusselt number which express the ratio of the convective and conductive heat transfer. Using the energetic balance on the plate we deduce the convection heat transfer coefficient, h:

$$-k\left[\frac{\partial T}{\partial y}\right]_{y=0} = h(T_w - T_{fluid}) \tag{2.14}$$

and using the definition of the local Nusselt number, $Nu_w = \frac{hR}{k}$, one obtain:

$$Nu_w = \frac{-\frac{\partial T}{\partial \mathbf{n}}}{T_w - T_{fluid}} \tag{2.15}$$

where $\mathbf{n}$ is the outer normal vector to the cylindrical wall and T_{fluid} is the temperature of the fluid in the vicinity of the wall.

3. Results and Discussions

The full Navier-Stokes and energy equations (2.6) - (2.9), with the corresponding boundary conditions (2.10) - (2.13) were numerically solved using FLUENT. The model is solved for $T_\infty = 281$ K and $T_w = 400$ K. We use the following discretization: standard for pressure, SIMPLE for pressure-velocity coupling and power-law for momentum and energy equations.The stop residual were $1e-4$ for continuity and velocity while for energy the value $1e-6$ was used. Also the underrelaxation method has been used, the underrelaxation factor was 0.3 for the pressure and 0.7 for the momentum equation.

To examine the effect of the Reynolds number the streamlines, isotherms and local Nusselt number are presented in Figures 3 to 11. It is noticed that for larger values of the Reynolds number the vortex region increases, see Figures 3, 5 7 and 9. Further we notice that the maximum value of the streamline function increases with the increase of the Reynolds number.

On the other hand Figures 4, 6, 8 and 10 display the distribution of the temperature field for different values of the Reynolds number. These figures indicate that the temperature decrease with the increasing of the Reynolds number. Therefore, the cooling of the cylinders is more efficient for large values of the Reynolds number.

The variation of the local Nusselt number around the cylinder is shown in Figure 11 for several values of the Reynolds number. In this figure $\theta = 0^o$ correspond to the region of the forward stagnation point of the cylinder while $\theta = 180^o$ correspond

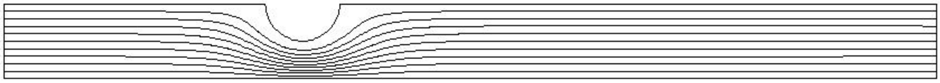

FIGURE 3. Streamlines for $Re = 1$, $\psi_{max} = 2.000003$

FIGURE 4. Isotherms for $Re = 1$

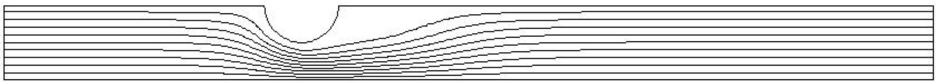

FIGURE 5. Streamlines for $Re = 10$, $\psi_{max} = 2.029994$

to the region of the rear stagnation point, respectively. It can be seen that the local Nusselt number sharply increases as the value of the Reynolds number increases, and then gradually decreases with the increases of the angle θ. In addition we notice that for $Re = 50$ and $Re = 100$ the graphs of the local Nusselt numbers change their shapes due to recirculation of the fluid.

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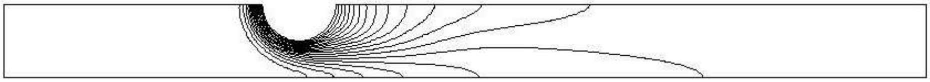

Figure 6. Isotherms for $Re = 10$

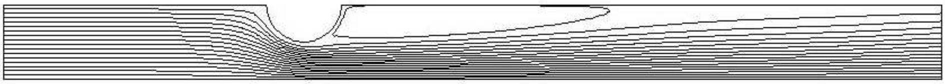

Figure 7. Streamlines for $Re = 50$, $\psi_{max} = 2.122836$

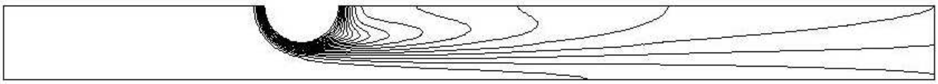

Figure 8. Isotherms for $Re = 50$

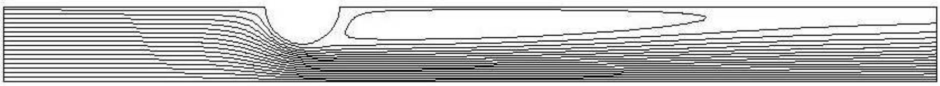

Figure 9. Streamlines for $Re = 100$, $\psi_{max} = 2.155566$

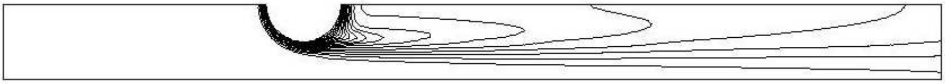

Figure 10. Isotherms for $Re = 100$

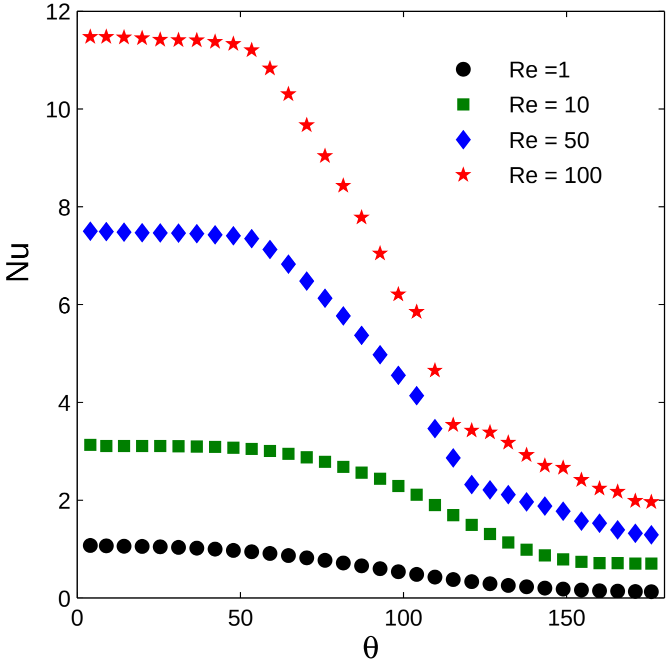

FIGURE 11. Variation of the local Nusselt number, Nu, for different values of the Reynold number Re

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