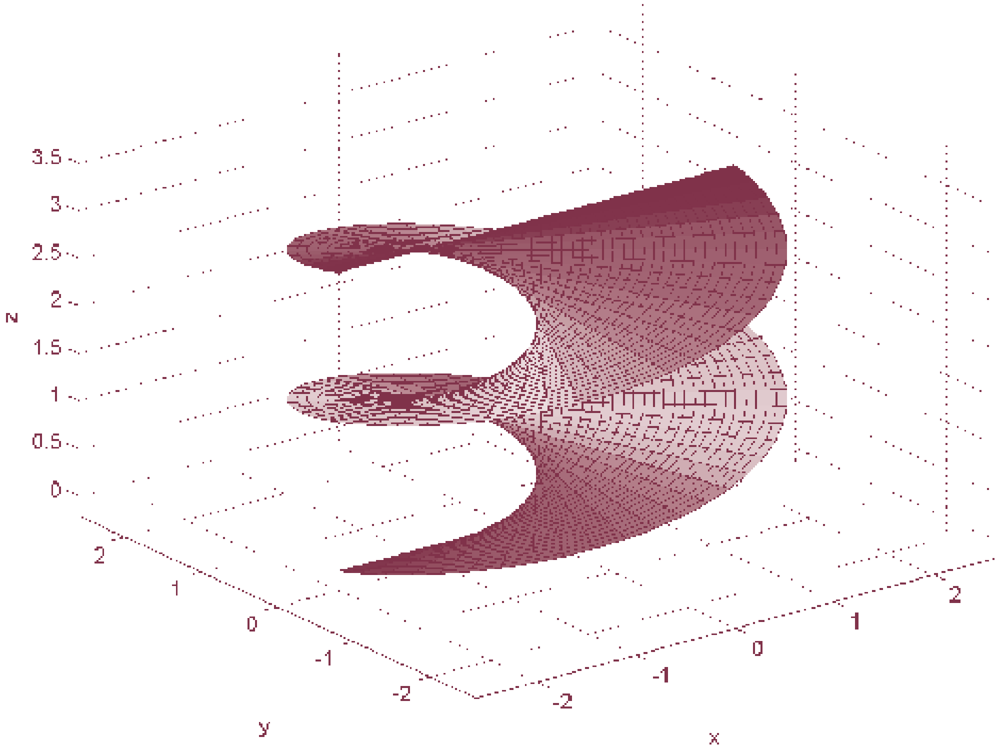
STUDIA UNIVERSITATIS
BABEŞ-BOLYAI



MATHEMATICA

3/2010

Anul LV **2010**

S T U D I A
UNIVERSITATIS BABEŞ-BOLYAI

MATHEMATICA
3

Redacţia: 400084 Cluj-Napoca, str. M. Kogălniceanu nr. 1
Telefon: 405300

SUMAR - CONTENTS - SOMMAIRE

STUDIA UNIV. "BABEŞ–BOLYAI", MATHEMATICA, Volume **LV**, Number 3, September 2010

A NOTE ON DIFFERENTIAL SUPERORDINATIONS USING A MULTIPLIER TRANSFORMATION AND RUSCHEWEYH DERIVATIVE

ALINA ALB LUPAŞ

Dedicated to Professor Grigore Ştefan Sălăgean on his 60^{th} *birthday*

Abstract. In the present paper we define a new operator, by means of convolution product between Ruscheweyh operator and the multiplier transformation $I(m,\lambda,l)$. For functions f belonging to the class $\mathcal{A}_n$ we define the differential operator $IR^m_{\lambda,l}:\mathcal{A}_n\to\mathcal{A}_n$,

$$IR^m_{\lambda,l}f(z):=\left(I(m,\lambda,l)*R^m\right)f(z),$$

where $\mathcal{A}_n=\{f\in\mathcal{H}(U):\ f(z)=z+a_{n+1}z^{n+1}+\ldots,\ z\in U\}$ is the class of normalized analytic functions. We study some differential superordinations regarding the operator $IR^m_{\lambda,l}$.

1. Introduction

Denote by U the unit disc of the complex plane $U=\{z\in\mathbb{C}:|z|<1\}$ and $\mathcal{H}(U)$ the space of holomorphic functions in U.

Let

$$\mathcal{A}(p,n)=\{f\in\mathcal{H}(U):\ f(z)=z^p+\sum_{j=p+n}^{\infty}a_jz^j,\quad z\in U\},$$

with $\mathcal{A}(1,n)=\mathcal{A}_n$ and

$$\mathcal{H}[a,n]=\{f\in\mathcal{H}(U),\ f(z)=a+a_nz^n+a_{n+1}z^{n+1}+\ldots,\ z\in U\}$$

for $a\in\mathbb{C}$ and $p,n\in\mathbb{N}$.

Received by the editors: 26.04.2010.

2000 *Mathematics Subject Classification.* 30C45, 30A20, 34A40.

Key words and phrases. Differential superordination, convex function, best subordinant, differential operator.

If f and g are analytic functions in U, we say that f is superordinate to g, written $g \prec f$, if there is a function w analytic in U, with $w(0) = 0$, $|w(z)| < 1$, for all $z \in U$ such that $g(z) = f(w(z))$ for all $z \in U$. If f is univalent, then $g \prec f$ if and only if $f(0) = g(0)$ and $g(U) \subseteq f(U)$.

Let $\psi : \mathbb{C}^2 \times U \to \mathbb{C}$ and h analytic in U. If p and $\psi\left(p\left(z\right), zp'\left(z\right); z\right)$ are univalent in U and satisfies the (first-order) differential superordination

$$h(z) \prec \psi(p(z), zp'(z); z), \qquad z \in U, \tag{1.1}$$

then p is called a solution of the differential superordination. The analytic function q is called a subordinant of the solutions of the differential superordination, or more simply a subordinant, if $q \prec p$ for all p satisfying (1.1).

An univalent subordinant $\widetilde{q}$ that satisfies $q \prec \widetilde{q}$ for all subordinants q of (1.1) is said to be the best subordinant of (1.1). The best subordinant is unique up to a rotation of U.

Definition 1.1. [7] For $f \in \mathcal{A}(p,n)$, $p, n \in \mathbb{N}$, $m \in \mathbb{N}\cup\{0\}$, $\lambda, l \geq 0$, the operator $I_p\left(m,\lambda,l\right)f(z)$ is defined by the following infinite series

$$I_p\left(m,\lambda,l\right)f(z) := z^p + \sum_{j=p+n}^{\infty}\left(\frac{p+\lambda\left(j-1\right)+l}{p+l}\right)^m a_j z^j.$$

Remark 1.2. It follows from the above definition that

$$I_p\left(0,\lambda,l\right)f(z) = f(z),$$

$$\left(p+l\right)I_p\left(m+1,\lambda,l\right)f(z) = \left[p(1-\lambda)+l\right]I_p\left(m,\lambda,l\right)f(z) + \lambda z\left(I_p\left(m,\lambda,l\right)f(z)\right)',$$

$z \in U$.

Remark 1.3. If $p = 1$, we have $\mathcal{A}(1,n) = \mathcal{A}_n$, $I_1\left(m,\lambda,l\right)f(z) = I\left(m,\lambda,l\right)$ and

$$\left(l+1\right)I\left(m+1,\lambda,l\right)f(z) = \left[l+1-\lambda\right]I\left(m,\lambda,l\right)f(z) + \lambda z\left(I\left(m,\lambda,l\right)f(z)\right)',$$

$z \in U$.

Remark 1.4. If $f \in \mathcal{A}_n$, $f(z) = z + \sum_{j=n+1}^{\infty} a_j z^j$, then

$$I\left(m,\lambda,l\right)f\left(z\right) = z + \sum_{j=n+1}^{\infty}\left(\frac{1+\lambda\left(j-1\right)+l}{l+1}\right)^m a_j z^j,\ z \in U.$$

Remark 1.5. For $l = 0$, $\lambda \geq 0$, the operator $D_{\lambda}^{m} = I(m, \lambda, 0)$ was introduced and studied by Al-Oboudi [6], which is reduced to the Sălăgean differential operator $S^m = I(m, 1, 0)$ [10] for $\lambda = 1$.

Definition 1.6. (Ruscheweyh [9]) For $f \in \mathcal{A}_n$, $m, n \in \mathbb{N}$, the operator R^m is defined by $R^m : \mathcal{A}_n \to \mathcal{A}_n$,

$$\begin{aligned} R^0 f(z) &= f(z) \\ R^1 f(z) &= z f'(z) \\ &\dots \\ (m+1) R^{m+1} f(z) &= z\left(R^m f(z)\right)' + m R^m f(z), \quad z \in U. \end{aligned}$$

Remark 1.7. If $f \in \mathcal{A}_n$, $f(z) = z + \sum_{j=n+1}^{\infty} a_j z^j$, then

$$R^m f(z) = z + \sum_{j=n+1}^{\infty} C_{m+j-1}^{m} a_j z^j, \ z \in U.$$

Definition 1.8. [8] We denote by Q the set of functions that are analytic and injective on $\overline{U} \backslash E(f)$, where $E(f) = \{\zeta \in \partial U : \lim_{z \to \zeta} f(z) = \infty\}$, and $f'(\zeta) \neq 0$ for $\zeta \in \partial U \backslash E(f)$. The subclass of Q for which $f(0) = a$ is denoted by $Q(a)$.

We will use the following lemmas.

Lemma 1.9. *(Miller and Mocanu [8]) Let h be a convex function with $h(0) = a$, and let $\gamma \in \mathbb{C} \backslash \{0\}$ be a complex number with Re $\gamma \geq 0$. If $p \in \mathcal{H}[a, n] \cap Q$, $p(z) + \frac{1}{\gamma} z p'(z)$ is univalent in U and*

$$h(z) \prec p(z) + \frac{1}{\gamma} z p'(z), \qquad z \in U,$$

then

$$q(z) \prec p(z), \qquad z \in U,$$

where

$$q(z) = \frac{\gamma}{n z^{\gamma/n}} \int_0^z h(t) t^{\gamma/n - 1} dt, \ z \in U.$$

The function q is convex and is the best subordinant.

Lemma 1.10. *(Miller and Mocanu [8]) Let q be a convex function in U and let*

$$h(z) = q(z) + \frac{1}{\gamma} z q'(z), \ z \in U,$$

where $Re\ \gamma \geq 0$.

If $p \in \mathcal{H}[a,n] \cap Q$, $p(z) + \frac{1}{\gamma} z p'(z)$ is univalent in U and

$$q(z) + \frac{1}{\gamma} z q'(z) \prec p(z) + \frac{1}{\gamma} z p'(z), \qquad z \in U,$$

then

$$q(z) \prec p(z), \qquad z \in U,$$

where

$$q(z) = \frac{\gamma}{n z^{\gamma/n}} \int_0^z h(t) t^{\gamma/n - 1} dt, \ z \in U.$$

The function q is the best subordinant.

2. Main results

Definition 2.1. ([4]) Let $m, n, \lambda, l \in \mathbb{N}$. Denote by $IR_{\lambda,l}^m$ the operator given by the Hadamard product (the convolution product) of the operator $I(m, \lambda, l)$ and the Ruscheweyh operator R^m, $IR_{\lambda,l}^m : \mathcal{A}_n \to \mathcal{A}_n$,

$$IR_{\lambda,l}^m f(z) = (I(m, \lambda, l) * R^m) f(z).$$

Remark 2.2. If $f \in \mathcal{A}_n$, $f(z) = z + \sum_{j=n+1}^{\infty} a_j z^j$, then

$$IR_{\lambda,l}^m f(z) = z + \sum_{j=n+1}^{\infty} \left(\frac{1 + \lambda(j-1) + l}{l+1} \right)^m C_{m+j-1}^m a_j^2 z^j, \ z \in U.$$

Remark 2.3. For $l = 0$, $\lambda \geq 0$, we obtain the Hadamard product DR_λ^m [1] of the generalized Sălăgean operator D_λ^m and Ruscheweyh operator R^m.

For $l = 0$ and $\lambda = 1$, we obtain the Hadamard product SR^m [5] of the Sălăgean operator S^m and Ruscheweyh operator R^m.

Theorem 2.4. *Let h be a convex function, $h(0) = 1$. Let $m, n, \lambda, l \in \mathbb{N}$, $f \in \mathcal{A}_n$ and suppose that*

$$\frac{l+1}{[\lambda(l - m + 2) - (l+1)] z} \cdot \left[(m+1) IR_{\lambda,l}^{m+1} f(z) - (m-2) IR_{\lambda,l}^m f(z) \right]$$

$$+\left(1-\frac{l+1}{\lambda\left(l-m+2\right)-\left(l+1\right)}\right)-\frac{2\left(l+1\right)\left(m-1\right)-2\lambda m}{\lambda\left(l-m+2\right)-\left(l+1\right)}\int_0^z\frac{IR_{\lambda,l}^m f\left(t\right)-t}{t^2}dt$$

is univalent and $\left(IR_{\lambda,l}^m f\left(z\right)\right)'\in\mathcal{H}\left[1,n\right]\cap Q$. *If*

$$h(z)\prec\frac{l+1}{\left[\lambda\left(l-m+2\right)-\left(l+1\right)\right]z}\left[\left(m+1\right)IR_{\lambda,l}^{m+1}f\left(z\right)-\left(m-2\right)IR_{\lambda,l}^m f\left(z\right)\right]\quad(2.1)$$

$$+\left(1-\frac{l+1}{\lambda\left(l-m+2\right)-\left(l+1\right)}\right)-\frac{2\left(l+1\right)\left(m-1\right)-2\lambda m}{\lambda\left(l-m+2\right)-\left(l+1\right)}\int_0^z\frac{IR_{\lambda,l}^m f\left(t\right)-t}{t^2}dt,$$

$z\in U$, *then*

$$q(z)\prec\left(IR_{\lambda,l}^m f\left(z\right)\right)',\qquad z\in U,$$

where

$$q(z)=\frac{\lambda\left(l-m+2\right)-\left(l+1\right)}{\lambda\left(l+1\right)nz^{\frac{\lambda(l-m+2)-(l+1)}{\lambda(l+1)n}}}\int_0^z h\left(t\right)t^{\frac{\lambda(l-m-nl-n+2)-(l+1)}{\lambda(l+1)n}}dt.$$

The function q *is convex and it is the best subordinant.*

Proof. With notation

$$p\left(z\right)=\left(IR_{\lambda,l}^m f\left(z\right)\right)'=1+\sum_{j=n+1}^{\infty}\left(\frac{1+\lambda\left(j-1\right)+l}{l+1}\right)^m C_{m+j-1}^m j a_j^2 z^{j-1}\text{ and }p\left(0\right)=1,$$

we obtain for $f(z)=z+\sum_{j=n+1}^{\infty}a_j z^j$,

$$p\left(z\right)+zp'\left(z\right)=1+\sum_{j=n+1}^{\infty}\left(\frac{1+\lambda\left(j-1\right)+l}{l+1}\right)^m C_{m+j-1}^m j a_j^2 z^{j-1}$$

$$+\sum_{j=n+1}^{\infty}\left(\frac{1+\lambda\left(j-1\right)+l}{l+1}\right)^m C_{m+j-1}^m j\left(j-1\right)a_j^2 z^{j-1}$$

$$=\frac{1}{z}\left(\frac{m+1}{\lambda}IR_{\lambda,l}^{m+1}f\left(z\right)-\frac{m-2}{\lambda}IR_{\lambda,l}^m f\left(z\right)\right)+\frac{\lambda\left(m-1\right)-\left(l+1\right)}{\lambda\left(l+1\right)}\left(IR_{\lambda,l}^m f\left(z\right)\right)'$$

$$+\left(1-\frac{m-1}{l+1}-\frac{2}{\lambda}\right)-\frac{2\left(l+1\right)\left(m-1\right)-2\lambda m}{\lambda\left(l+1\right)}\int_0^z\frac{IR_{\lambda,l}^m f\left(t\right)-t}{t^2}dt.$$

Therefore

$$p\left(z\right)+\frac{\lambda\left(l+1\right)}{\lambda\left(l-m+2\right)-\left(l+1\right)}zp'\left(z\right)$$

$$=\frac{l+1}{\left[\lambda\left(l-m+2\right)-\left(l+1\right)\right]z}\left[\left(m+1\right)IR_{\lambda,l}^{m+1}f\left(z\right)-\left(m-2\right)IR_{\lambda,l}^m f\left(z\right)\right]$$

$$+\left(1-\frac{l+1}{\lambda\left(l-m+2\right)-\left(l+1\right)}\right)-\frac{2\left(l+1\right)\left(m-1\right)-2\lambda m}{\lambda\left(l-m+2\right)-\left(l+1\right)}\int_0^z\frac{IR_{\lambda,l}^m f\left(t\right)-t}{t^2}dt.$$

Then (2.1) becomes

$$h(z) \prec p(z) + \frac{\lambda(l+1)}{\lambda(l-m+2)-(l+1)} zp'(z), \quad z \in U.$$

By using Lemma 1.9 for $\gamma = 1 - \frac{m-1}{l+1} - \frac{1}{\lambda}$, we have

$$q(z) \prec p(z), \quad z \in U, \quad \text{i.e.} \quad q(z) \prec \left(IR^m_{\lambda,l} f(z)\right)', \quad z \in U,$$

where

$$q(z) = \frac{\lambda(l-m+2)-(l+1)}{\lambda(l+1) n z^{\frac{\lambda(l-m+2)-(l+1)}{\lambda(l+1)n}}} \int_0^z h(t) t^{\frac{\lambda(l-m-nl-n+2)-(l+1)}{\lambda(l+1)n}} dt.$$

The function q is convex and it is the best subordinant. □

Corollary 2.5. [3] *Let h be a convex function, $h(0) = 1$. Let $\lambda \geq 0$, $m, n \in \mathbb{N}$, $f \in \mathcal{A}_n$ and suppose that*

$$\frac{m+1}{(m\lambda+1) z} DR^{m+1}_{\lambda} f(z) - \frac{m(1-\lambda)}{(m\lambda+1) z} DR^m_{\lambda} f(z)$$

is univalent and $(DR^m_{\lambda} f(z))' \in \mathcal{H}[1,n] \cap Q$. If

$$h(z) \prec \frac{m+1}{(m\lambda+1) z} DR^{m+1}_{\lambda} f(z) - \frac{m(1-\lambda)}{(m\lambda+1) z} DR^m_{\lambda} f(z), \quad z \in U, \tag{2.2}$$

then

$$q(z) \prec (DR^m_{\lambda} f(z))', \quad z \in U,$$

where

$$q(z) = \frac{m\lambda+1}{n\lambda z^{\frac{m\lambda+1}{n\lambda}}} \int_0^z h(t) t^{\frac{(m-n)\lambda+1}{n\lambda}} dt.$$

The function q is convex and it is the best subordinant.

Corollary 2.6. [2] *Let h be a convex function, $h(0) = 1$. Let $m, n \in \mathbb{N}$, $f \in \mathcal{A}_n$ and suppose that*

$$\frac{1}{z} SR^{m+1} f(z) + \frac{m}{m+1} z (SR^m f(z))''$$

is univalent and $(SR^m f(z))' \in \mathcal{H}[1,n] \cap Q$. If

$$h(z) \prec \frac{1}{z} SR^{m+1} f(z) + \frac{m}{m+1} z (SR^m f(z))'', \quad z \in U, \tag{2.3}$$

then

$$q(z) \prec (SR^m f(z))', \quad z \in U,$$

where

$$q(z) = \frac{1}{nz^{\frac{1}{n}}} \int_0^z h(t)t^{\frac{1}{n}-1}dt.$$

The function q is convex and it is the best subordinant.

Theorem 2.7. *Let q be convex in U and let h be defined by*

$$h(z) = q(z) + \frac{\lambda(l+1)}{\lambda(l-m+2)-(l+1)} zq'(z),$$

$m, n, \lambda, l \in \mathbb{N}$. If $f \in \mathcal{A}_n$, suppose that

$$\frac{l+1}{[\lambda(l-m+2)-(l+1)]z} \left[(m+1) IR_{\lambda,l}^{m+1} f(z) - (m-2) IR_{\lambda,l}^{m} f(z)\right]$$

$$+\left(1 - \frac{l+1}{\lambda(l-m+2)-(l+1)}\right) - \frac{2(l+1)(m-1)-2\lambda m}{\lambda(l-m+2)-(l+1)} \int_0^z \frac{IR_{\lambda,l}^{m} f(t) - t}{t^2} dt$$

is univalent, $\left(IR_{\lambda,l}^{m} f(z)\right)' \in \mathcal{H}[1,n] \cap Q$ and satisfies the differential superordination

$$h(z) = q(z) + \frac{\lambda(l+1)}{\lambda(l-m+2)-(l+1)} zq'(z) \prec \frac{l+1}{[\lambda(l-m+2)-(l+1)]z} \cdot \tag{2.4}$$

$$\left[(m+1) IR_{\lambda,l}^{m+1} f(z) - (m-2) IR_{\lambda,l}^{m} f(z)\right] + \left(1 - \frac{l+1}{\lambda(l-m+2)-(l+1)}\right)$$

$$-\frac{2(l+1)(m-1)-2\lambda m}{\lambda(l-m+2)-(l+1)} \int_0^z \frac{IR_{\lambda,l}^{m} f(t) - t}{t^2} dt, \qquad z \in U,$$

then

$$q(z) \prec \left(IR_{\lambda,l}^{m} f(z)\right)', \qquad z \in U,$$

where

$$q(z) = \frac{\lambda(l-m+2)-(l+1)}{\lambda(l+1) nz^{\frac{\lambda(l-m+2)-(l+1)}{\lambda(l+1)n}}} \int_0^z h(t) t^{\frac{\lambda(l-m-nl-n+2)-(l+1)}{\lambda(l+1)n}} dt.$$

The function q is the best subordinant.

Proof. Let

$$p(z) = \left(IR_{\lambda,l}^{m} f(z)\right)' = 1 + \sum_{j=n+1}^{\infty} \left(\frac{1+\lambda(j-1)+l}{l+1}\right)^m C_{m+j-1}^{m} j a_j^2 z^{j-1}.$$

Differentiating, we obtain

$$p(z) + zp'(z) = \frac{1}{z}\left(\frac{m+1}{\lambda} IR_{\lambda,l}^{m+1} f(z) - \frac{m-2}{\lambda} IR_{\lambda,l}^{m} f(z)\right)$$

$$+\frac{\lambda(m-1)-(l+1)}{\lambda(l+1)} \left(IR_{\lambda,l}^{m} f(z)\right)' + \left(1 - \frac{m-1}{l+1} - \frac{2}{\lambda}\right)$$

$$-\frac{2(l+1)(m-1)-2\lambda m}{\lambda(l+1)}\int_0^z \frac{IR_{\lambda,l}^m f(t)-t}{t^2}dt$$

and

$$p(z)+\frac{\lambda(l+1)}{\lambda(l-m+2)-(l+1)}zp'(z)=$$

$$\frac{l+1}{[\lambda(l-m+2)-(l+1)]z}\left[(m+1)IR_{\lambda,l}^{m+1}f(z)-(m-2)IR_{\lambda,l}^m f(z)\right]$$

$$+\left(1-\frac{l+1}{\lambda(l-m+2)-(l+1)}\right)$$

$$-\frac{2(l+1)(m-1)-2\lambda m}{\lambda(l-m+2)-(l+1)}\int_0^z \frac{IR_{\lambda,l}^m f(t)-t}{t^2}dt,\ z\in U$$

and (2.4) becomes

$$q(z)+\frac{\lambda(l+1)}{\lambda(l-m+2)-(l+1)}zq'(z)\prec p(z)+\frac{\lambda(l+1)}{\lambda(l-m+2)-(l+1)}zp'(z),\quad z\in U.$$

Using Lemma 1.10 for $\gamma=1-\frac{m-1}{l+1}-\frac{1}{\lambda}$, we have $q(z)\prec p(z)$, $z\in U$, i.e.

$$q(z)=\frac{\lambda(l-m+2)-(l+1)}{\lambda(l+1)nz^{\frac{\lambda(l-m+2)-(l+1)}{\lambda(l+1)n}}}\int_0^z h(t)t^{\frac{\lambda(l-m-nl-n+2)-(l+1)}{\lambda(l+1)n}}dt$$

$$\prec\left(IR_{\lambda,l}^m f(z)\right)',\ \ z\in U,$$

and q is the best subordinant. □

Corollary 2.8. [3] *Let q be convex in U and let h be defined by*

$$h(z)=q(z)+\frac{\lambda}{m\lambda+1}zq'(z),$$

$\lambda\geq 0$, $m,n\in\mathbb{N}$. *If $f\in\mathcal{A}_n$, suppose that*

$$\frac{m+1}{(m\lambda+1)z}DR_\lambda^{m+1}f(z)-\frac{m(1-\lambda)}{(m\lambda+1)z}DR_\lambda^m f(z)$$

is univalent and $(DR_\lambda^m f(z))'\in\mathcal{H}[1,n]\cap Q$ and satisfies the differential superordination

$$h(z)=q(z)+\frac{\lambda}{m\lambda+1}zq'(z)\prec\frac{m+1}{(m\lambda+1)z}DR_\lambda^{m+1}f(z)-\frac{m(1-\lambda)}{(m\lambda+1)z}DR_\lambda^m f(z),\tag{2.5}$$

$z\in U$, then

$$q(z)\prec(DR_\lambda^m f(z))',\qquad z\in U,$$

where

$$q(z) = \frac{m\lambda + 1}{n\lambda z^{\frac{m\lambda+1}{n\lambda}}} \int_0^z h(t) \, t^{\frac{(m-n)\lambda+1}{n\lambda}} dt.$$

The function q is the best subordinant.

Corollary 2.9. [2] *Let q be convex in U and let h be defined by $h(z) = q(z) + zq'(z)$. If $m, n \in \mathbb{N}$, $f \in \mathcal{A}_n$, suppose that $\frac{1}{z} SR^{m+1} f(z) + \frac{m}{m+1} z \left(SR^m f(z)\right)''$ is univalent, $\left(SR^m f(z)\right)' \in \mathcal{H}[1, n] \cap Q$ and satisfies the differential superordination*

$$h(z) = q(z) + zq'(z) \prec \frac{1}{z} SR^{m+1} f(z) + \frac{m}{m+1} z \left(SR^m f(z)\right)'', \qquad z \in U, \tag{2.6}$$

then

$$q(z) \prec \left(SR^m f(z)\right)', \qquad z \in U,$$

where

$$q(z) = \frac{1}{nz^{\frac{1}{n}}} \int_0^z h(t) t^{\frac{1}{n}-1} dt.$$

The function q is the best subordinant.

Theorem 2.10. *Let h be a convex function, $h(0) = 1$. Let $m, n, \lambda, l \in \mathbb{N}$, $f \in \mathcal{A}_n$ and suppose that $\left(IR^m_{\lambda,l} f(z)\right)'$ is univalent and $\frac{IR^m_{\lambda,l} f(z)}{z} \in \mathcal{H}[1, n] \cap Q$. If*

$$h(z) \prec \left(IR^m_{\lambda,l} f(z)\right)', \qquad z \in U, \tag{2.7}$$

then

$$q(z) \prec \frac{IR^m_{\lambda,l} f(z)}{z}, \qquad z \in U,$$

where

$$q(z) = \frac{1}{nz^{\frac{1}{n}}} \int_0^z h(t) t^{\frac{1}{n}-1} dt.$$

The function q is convex and it is the best subordinant.

Proof. Consider

$$p(z) = \frac{IR^m_{\lambda,l} f(z)}{z} = \frac{z + \sum_{j=n+1}^{\infty} \left(\frac{1+\lambda(j-1)+l}{l+1}\right)^m C^m_{m+j-1} a_j^2 z^j}{z}$$
$$= 1 + \sum_{j=n+1}^{\infty} \left(\frac{1 + \lambda(j-1) + l}{l+1}\right)^m C^m_{m+j-1} a_j^2 z^{j-1}.$$

Evidently $p \in \mathcal{H}[1, n]$.

We have $p(z) + zp'(z) = \left(IR^m_{\lambda,l} f(z)\right)'$, $z \in U$.

Then (2.7) becomes

$$h(z) \prec p(z) + zp'(z), \quad z \in U.$$

By using Lemma 1.9 for $\gamma = 1$, we have

$$q(z) \prec p(z), \quad z \in U, \quad \text{i.e.} \quad q(z) \prec \frac{IR^m_{\lambda,l} f(z)}{z}, \quad z \in U,$$

where $q(z) = \frac{1}{nz^{\frac{1}{n}}} \int_0^z h(t) t^{\frac{1}{n}-1} dt$. The function q is convex and it is the best subordinant. □

Corollary 2.11. [3] *Let h be a convex function, $h(0) = 1$. Let $\lambda \geq 0$, $m, n \in \mathbb{N}$, $f \in \mathcal{A}_n$ and suppose that $(DR^m_\lambda f(z))'$ is univalent and $\frac{DR^m_\lambda f(z)}{z} \in \mathcal{H}[1, n] \cap Q$. If*

$$h(z) \prec (DR^m_\lambda f(z))', \quad z \in U, \tag{2.8}$$

then

$$q(z) \prec \frac{DR^m_\lambda f(z)}{z}, \quad z \in U,$$

where

$$q(z) = \frac{1}{nz^{\frac{1}{n}}} \int_0^z h(t) t^{\frac{1}{n}-1} dt.$$

The function q is convex and it is the best subordinant.

Corollary 2.12. [2] *Let h be a convex function, $h(0) = 1$. Let $m, n \in \mathbb{N}$, $f \in \mathcal{A}_n$ and suppose that $(SR^m f(z))'$ is univalent and $\frac{SR^m f(z)}{z} \in \mathcal{H}[1, n] \cap Q$. If*

$$h(z) \prec (SR^m f(z))', \quad z \in U, \tag{2.9}$$

then

$$q(z) \prec \frac{SR^m f(z)}{z}, \quad z \in U,$$

where

$$q(z) = \frac{1}{nz^{\frac{1}{n}}} \int_0^z h(t) t^{\frac{1}{n}-1} dt.$$

The function q is convex and it is the best subordinant.

Theorem 2.13. *Let q be convex in U and let h be defined by $h(z) = q(z) + zq'(z)$. If $m, n, \lambda, l \in \mathbb{N}$, $f \in \mathcal{A}_n$, suppose that $\left(IR_{\lambda,l}^m f(z)\right)'$ is univalent, $\frac{IR_{\lambda,l}^m f(z)}{z} \in \mathcal{H}[1,n] \cap Q$ and satisfies the differential superordination*

$$h(z) = q(z) + zq'(z) \prec \left(IR_{\lambda,l}^m f(z)\right)', \qquad z \in U, \tag{2.10}$$

then

$$q(z) \prec \frac{IR_{\lambda,l}^m f(z)}{z}, \qquad z \in U,$$

where

$$q(z) = \frac{1}{nz^{\frac{1}{n}}} \int_0^z h(t) t^{\frac{1}{n}-1} dt.$$

The function q is the best subordinant.

Proof. Let

$$p(z) = \frac{IR_{\lambda,l}^m f(z)}{z} = \frac{z + \sum_{j=n+1}^{\infty} \left(\frac{1+\lambda(j-1)+l}{l+1}\right)^m C_{m+j-1}^m a_j^2 z^j}{z}$$

$$= 1 + \sum_{j=n+1}^{\infty} \left(\frac{1+\lambda(j-1)+l}{l+1}\right)^m C_{m+j-1}^m a_j^2 z^{j-1}.$$

Evidently $p \in \mathcal{H}[1,n]$.

Differentiating, we obtain $p(z) + zp'(z) = \left(IR_{\lambda,l}^m f(z)\right)'$, $z \in U$ and (2.10) becomes

$$q(z) + zq'(z) \prec p(z) + zp'(z), \quad z \in U.$$

Using Lemma 1.10 for $\gamma = 1$, we have

$$q(z) \prec p(z), z \in U, \quad \text{i.e.} \quad q(z) = \frac{1}{nz^{\frac{1}{n}}} \int_0^z h(t) t^{\frac{1}{n}-1} dt \prec \frac{IR_{\lambda,l}^m f(z)}{z}, z \in U,$$

and q is the best subordinant. □

Corollary 2.14. [3] *Let q be convex in U and let h be defined by $h(z) = q(z) + zq'(z)$. If $\lambda \geq 0$, $m, n \in \mathbb{N}$, $f \in \mathcal{A}_n$, suppose that $(DR_\lambda^m f(z))'$ is univalent,*

$$\frac{DR_\lambda^m f(z)}{z} \in \mathcal{H}[1,n] \cap Q$$

and satisfies the differential superordination

$$h(z) = q(z) + zq'(z) \prec (DR_\lambda^m f(z))', \qquad z \in U, \tag{2.11}$$

then

$$q(z) \prec \frac{DR_{\lambda}^{m} f(z)}{z}, \qquad z \in U,$$

where $q(z) = \frac{1}{nz^{\frac{1}{n}}} \int_0^z h(t) t^{\frac{1}{n}-1} dt$. *The function* q *is the best subordinant.*

Corollary 2.15. [2] *Let* q *be convex in* U *and let* h *be defined by* $h(z) = q(z) + zq'(z)$. *If* $m, n \in \mathbb{N}$, $f \in \mathcal{A}_n$, *suppose that* $(SR^m f(z))'$ *is univalent,* $\frac{SR^m f(z)}{z} \in \mathcal{H}[1,n] \cap Q$ *and satisfies the differential superordination*

$$h(z) = q(z) + zq'(z) \prec (SR^m f(z))', \qquad z \in U, \tag{2.12}$$

then

$$q(z) \prec \frac{SR^m f(z)}{z}, \qquad z \in U,$$

where

$$q(z) = \frac{1}{nz^{\frac{1}{n}}} \int_0^z h(t) t^{\frac{1}{n}-1} dt.$$

The function q *is the best subordinant.*

Theorem 2.16. *Let*

$$h(z) = \frac{1 + (2\beta - 1)z}{1 + z}$$

be a convex function in U, *where* $0 \le \beta < 1$. *Let* $m, n, \lambda, l \in \mathbb{N}$, $f \in \mathcal{A}_n$ *and suppose that* $\left(IR_{\lambda,l}^{m} f(z)\right)'$ *is univalent and*

$$\frac{IR_{\lambda,l}^{m} f(z)}{z} \in \mathcal{H}[1,n] \cap Q.$$

If

$$h(z) \prec \left(IR_{\lambda,l}^{m} f(z)\right)', \qquad z \in U, \tag{2.13}$$

then

$$q(z) \prec \frac{IR_{\lambda,l}^{m} f(z)}{z}, \qquad z \in U,$$

where q *is given by*

$$q(z) = 2\beta - 1 + \frac{2(1-\beta)}{nz^{\frac{1}{n}}} \int_0^z \frac{t^{\frac{1}{n}-1}}{1+t} dt, \;\; z \in U.$$

The function q *is convex and it is the best subordinant.*

Proof. Following the same steps as in the proof of Theorem 2.10 and considering

$$p(z) = \frac{IR_{\lambda,l}^{m} f(z)}{z},$$

the differential superordination (2.13) becomes

$$h(z) = \frac{1 + (2\beta - 1)z}{1 + z} \prec p(z) + zp'(z), \quad z \in U.$$

By using Lemma 1.9 for $\gamma = 1$, we have $q(z) \prec p(z)$, i.e.,

$$q(z) = \frac{1}{nz^{\frac{1}{n}}} \int_0^z h(t)\, t^{\frac{1}{n}-1} dt = \frac{1}{nz^{\frac{1}{n}}} \int_0^z t^{\frac{1}{n}-1} \frac{1 + (2\beta - 1)t}{1 + t} dt$$

$$= 2\beta - 1 + \frac{2(1-\beta)}{nz^{\frac{1}{n}}} \int_0^z \frac{t^{\frac{1}{n}-1}}{1+t} dt \prec \frac{IR_{\lambda,l}^{m} f(z)}{z}, \quad z \in U.$$

The function q is convex and it is the best subordinant. □

Theorem 2.17. *Let h be a convex function, $h(0) = 1$. Let $m, n, \lambda, l \in \mathbb{N}$, $f \in \mathcal{A}_n$ and suppose that $\left(\frac{zIR_{\lambda,l}^{m+1} f(z)}{IR_{\lambda,l}^{m} f(z)}\right)'$ is univalent and $\frac{IR_{\lambda,l}^{m+1} f(z)}{IR_{\lambda,l}^{m} f(z)} \in \mathcal{H}[1, n] \cap Q$. If*

$$h(z) \prec \left(\frac{zIR_{\lambda,l}^{m+1} f(z)}{IR_{\lambda,l}^{m} f(z)}\right)', \qquad z \in U, \tag{2.14}$$

then

$$q(z) \prec \frac{IR_{\lambda,l}^{m+1} f(z)}{IR_{\lambda,l}^{m} f(z)}, \qquad z \in U,$$

where

$$q(z) = \frac{1}{nz^{\frac{1}{n}}} \int_0^z h(t)\, t^{\frac{1}{n}-1} dt.$$

The function q is convex and it is the best subordinant.

Proof. Consider

$$p(z) = \frac{IR_{\lambda,l}^{m+1} f(z)}{IR_{\lambda,l}^{m} f(z)} = \frac{z + \sum_{j=n+1}^{\infty} \left(\frac{1+\lambda(j-1)+l}{l+1}\right)^{m+1} C_{m+j}^{m+1} a_j^2 z^j}{z + \sum_{j=n+1}^{\infty} \left(\frac{1+\lambda(j-1)+l}{l+1}\right)^{m} C_{m+j-1}^{m} a_j^2 z^j}$$

$$= \frac{1 + \sum_{j=n+1}^{\infty} \left(\frac{1+\lambda(j-1)+l}{l+1}\right)^{m+1} C_{m+j}^{m+1} a_j^2 z^{j-1}}{1 + \sum_{j=n+1}^{\infty} \left(\frac{1+\lambda(j-1)+l}{l+1}\right)^{m} C_{m+j-1}^{m} a_j^2 z^{j-1}}.$$

Evidently $p \in \mathcal{H}[1, n]$.

We have

$$p'(z)=\frac{\left(IR_{\lambda,l}^{m+1}f(z)\right)'}{IR_{\lambda,l}^{m}f(z)}-p(z)\cdot\frac{\left(IR_{\lambda,l}^{m}f(z)\right)'}{IR_{\lambda,l}^{m}f(z)}.$$

Then

$$p(z)+zp'(z)=\left(\frac{zIR_{\lambda,l}^{m+1}f(z)}{IR_{\lambda,l}^{m}f(z)}\right)'.$$

Then (2.14) becomes

$$h(z)\prec p(z)+zp'(z),\quad z\in U.$$

By using Lemma 1.9 for $\gamma=1$, we have

$$q(z)\prec p(z),\quad z\in U,\quad \text{i.e.}\quad q(z)\prec\frac{IR_{\lambda,l}^{m+1}f(z)}{IR_{\lambda,l}^{m}f(z)},\quad z\in U,$$

where

$$q(z)=\frac{1}{nz^{\frac{1}{n}}}\int_0^z h(t)\,t^{\frac{1}{n}-1}dt.$$

The function q is convex and it is the best subordinant. □

Corollary 2.18. [3] *Let h be a convex function, $h(0)=1$. Let $\lambda\geq 0$, $m,n\in\mathbb{N}$, $f\in\mathcal{A}_n$ and suppose that $\left(\frac{zDR_{\lambda}^{m+1}f(z)}{DR_{\lambda}^{m}f(z)}\right)'$ is univalent and $\frac{DR_{\lambda}^{m+1}f(z)}{DR_{\lambda}^{m}f(z)}\in\mathcal{H}[1,n]\cap Q$. If*

$$h(z)\prec\left(\frac{zDR_{\lambda}^{m+1}f(z)}{DR_{\lambda}^{m}f(z)}\right)',\quad z\in U, \tag{2.15}$$

then

$$q(z)\prec\frac{DR_{\lambda}^{m+1}f(z)}{DR_{\lambda}^{m}f(z)},\quad z\in U,$$

where

$$q(z)=\frac{1}{nz^{\frac{1}{n}}}\int_0^z h(t)\,t^{\frac{1}{n}-1}dt.$$

The function q is convex and it is the best subordinant.

Corollary 2.19. [2] *Let h be a convex function, $h(0)=1$. Let $m,n\in\mathbb{N}$, $f\in\mathcal{A}_n$ and suppose that $\left(\frac{zSR^{m+1}f(z)}{SR^{m}f(z)}\right)'$ is univalent and $\frac{SR^{m+1}f(z)}{SR^{m}f(z)}\in\mathcal{H}[1,n]\cap Q$. If*

$$h(z)\prec\left(\frac{zSR^{m+1}f(z)}{SR^{m}f(z)}\right)',\quad z\in U, \tag{2.16}$$

then

$$q(z) \prec \frac{SR^{m+1}f(z)}{SR^m f(z)}, \qquad z \in U,$$

where

$$q(z) = \frac{1}{nz^{\frac{1}{n}}} \int_0^z h(t)\, t^{\frac{1}{n}-1} dt.$$

The function q is convex and it is the best subordinant.

Theorem 2.20. *Let q be convex in U and let h be defined by $h(z) = q(z) + zq'(z)$. If $m, n, \lambda, l \in \mathbb{N}$, $f \in \mathcal{A}_n$, suppose that $\left(\frac{zIR^{m+1}_{\lambda,l}f(z)}{IR^m_{\lambda,l}f(z)}\right)'$ is univalent,*

$$\frac{IR^{m+1}_{\lambda,l}f(z)}{IR^m_{\lambda,l}f(z)} \in \mathcal{H}[1,n] \cap Q$$

and satisfies the differential superordination

$$h(z) = q(z) + zq'(z) \prec \left(\frac{zIR^{m+1}_{\lambda,l}f(z)}{IR^m_{\lambda,l}f(z)}\right)', \qquad z \in U, \tag{2.17}$$

then

$$q(z) \prec \frac{IR^{m+1}_{\lambda,l}f(z)}{IR^m_{\lambda,l}f(z)}, \qquad z \in U,$$

where

$$q(z) = \frac{1}{nz^{\frac{1}{n}}} \int_0^z h(t)\, t^{\frac{1}{n}-1} dt.$$

The function q is the best subordinant.

Proof. Let

$$p(z) = \frac{IR^{m+1}_{\lambda,l}f(z)}{IR^m_{\lambda,l}f(z)} = \frac{z + \sum_{j=n+1}^{\infty} \left(\frac{1+\lambda(j-1)+l}{l+1}\right)^{m+1} C^{m+1}_{m+j} a_j^2 z^j}{z + \sum_{j=n+1}^{\infty} \left(\frac{1+\lambda(j-1)+l}{l+1}\right)^{m} C^{m}_{m+j-1} a_j^2 z^j}$$

$$= \frac{1 + \sum_{j=n+1}^{\infty} \left(\frac{1+\lambda(j-1)+l}{l+1}\right)^{m+1} C^{m+1}_{m+j} a_j^2 z^{j-1}}{1 + \sum_{j=n+1}^{\infty} \left(\frac{1+\lambda(j-1)+l}{l+1}\right)^{m} C^{m}_{m+j-1} a_j^2 z^{j-1}}.$$

Evidently $p \in \mathcal{H}[1,n]$.

Differentiating, we obtain

$$p(z) + zp'(z) = \left(\frac{zIR^{m+1}_{\lambda,l}f(z)}{IR^m_{\lambda,l}f(z)}\right)', \; z \in U$$

and (2.17) becomes

$$q(z) + zq'(z) \prec p(z) + zp'(z), \quad z \in U.$$

Using Lemma 1.10 for $\gamma = 1$, we have

$$q(z) \prec p(z), \quad z \in U, \quad \text{i.e.} \quad q(z) = \frac{1}{nz^{\frac{1}{n}}} \int_0^z h(t)\, t^{\frac{1}{n}-1} dt \prec \frac{IR_{\lambda,l}^{m+1} f(z)}{IR_{\lambda,l}^{m} f(z)}, \quad z \in U,$$

and q is the best subordinant. □

Corollary 2.21. [3] *Let q be convex in U and let h be defined by $h(z) = q(z) + zq'(z)$. If $\lambda \geq 0$, $m, n \in \mathbb{N}$, $f \in \mathcal{A}_n$, suppose that $\left(\frac{zDR_{\lambda}^{m+1} f(z)}{DR_{\lambda}^{m} f(z)}\right)'$ is univalent,*

$$\frac{DR_{\lambda}^{m+1} f(z)}{DR_{\lambda}^{m} f(z)} \in \mathcal{H}[1, n] \cap Q$$

and satisfies the differential superordination

$$h(z) = q(z) + zq'(z) \prec \left(\frac{zDR_{\lambda}^{m+1} f(z)}{DR_{\lambda}^{m} f(z)}\right)', \qquad z \in U, \tag{2.18}$$

then

$$q(z) \prec \frac{DR_{\lambda}^{m+1} f(z)}{DR_{\lambda}^{m} f(z)}, \qquad z \in U,$$

where

$$q(z) = \frac{1}{nz^{\frac{1}{n}}} \int_0^z h(t)\, t^{\frac{1}{n}-1} dt.$$

The function q is the best subordinant.

Corollary 2.22. [2] *Let q be convex in U and let h be defined by $h(z) = q(z) + zq'(z)$. If $m, n \in \mathbb{N}$, $f \in \mathcal{A}_n$, suppose that $\left(\frac{zSR^{m+1} f(z)}{SR^{m} f(z)}\right)'$ is univalent, $\frac{SR^{m+1} f(z)}{SR^{m} f(z)} \in \mathcal{H}[1, n] \cap Q$ and satisfies the differential superordination*

$$h(z) = q(z) + zq'(z) \prec \left(\frac{zSR^{m+1} f(z)}{SR^{m} f(z)}\right)', \qquad z \in U, \tag{2.19}$$

then

$$q(z) \prec \frac{SR^{m+1} f(z)}{SR^{m} f(z)}, \qquad z \in U,$$

where

$$q(z) = \frac{1}{nz^{\frac{1}{n}}} \int_0^z h(t)\, t^{\frac{1}{n}-1} dt.$$

The function q is the best subordinant.

Theorem 2.23. *Let*

$$h(z)=\frac{1+(2\beta-1)z}{1+z}$$

be a convex function in U*, where* $0\le\beta<1$*. Let* $m,n,\lambda,l\in\mathbb{N}$*,* $f\in\mathcal{A}_n$ *and suppose that* $\left(\frac{zIR_{\lambda,l}^{m+1}f(z)}{IR_{\lambda,l}^{m}f(z)}\right)'$ *is univalent,* $\frac{IR_{\lambda,l}^{m+1}f(z)}{IR_{\lambda,l}^{m}f(z)}\in\mathcal{H}[1,n]\cap Q$*. If*

$$h(z)\prec\left(\frac{zIR_{\lambda,l}^{m+1}f(z)}{IR_{\lambda,l}^{m}f(z)}\right)',\qquad z\in U,\tag{2.20}$$

then

$$q(z)\prec\frac{IR_{\lambda,l}^{m+1}f(z)}{IR_{\lambda,l}^{m}f(z)},\qquad z\in U,$$

where q *is given by*

$$q(z)=2\beta-1+\frac{2(1-\beta)}{nz^{\frac{1}{n}}}\int_0^z\frac{t^{\frac{1}{n}-1}}{1+t}dt,\ z\in U.$$

The function q *is convex and it is the best subordinant.*

Proof. Following the same steps as in the proof of Theorem 2.17 and considering

$$p(z)=\frac{IR_{\lambda,l}^{m+1}f(z)}{IR_{\lambda,l}^{m}f(z)},$$

the differential superordination (2.20) becomes

$$h(z)=\frac{1+(2\beta-1)z}{1+z}\prec p(z)+zp'(z),\quad z\in U.$$

By using Lemma 1.9 for $\gamma=1$, we have $q(z)\prec p(z)$, i.e.,

$$q(z)=\frac{1}{nz^{\frac{1}{n}}}\int_0^z h(t)\,t^{\frac{1}{n}-1}dt=\frac{1}{nz^{\frac{1}{n}}}\int_0^z t^{\frac{1}{n}-1}\frac{1+(2\beta-1)\,t}{1+t}dt$$

$$=2\beta-1+\frac{2(1-\beta)}{nz^{\frac{1}{n}}}\int_0^z\frac{t^{\frac{1}{n}-1}}{1+t}dt\prec\frac{IR_{\lambda,l}^{m+1}f(z)}{IR_{\lambda,l}^{m}f(z)},\quad z\in U.$$

The function q is convex and it is the best subordinant. ∎

References

[1] Alb Lupaş, A., *A note on differential subordinations using a generalized Sălăgean and Ruscheweyh operators*, Acta Universitatis Apulensis, Proceedings of ICTAMI 2009, Alba Iulia, 29-35.

[2] Alb Lupaş, A., *A note on differential superordinations using Sălăgean and Ruscheweyh operators*, Acta Universitatis Apulensis, **23** (2010) (to appear).

[3] Alb Lupaş, A., *A note on differential superordinations using a generalized Sălăgean and Ruscheweyh operators*, General Mathematics, 2011 (to appear).

[4] Alb Lupaş, A., Breaz, D., *On a certain subclass of analytic functions defined by multiplier transformation*, Proceedings of International Symposium on Analysis and Theory of Functions, Istanbul, 2009 (to appear).

[5] Alb Lupaş, A., Cătaş, A., *A note on differential subordinations using Sălăgean and Ruscheweyh operators*, Romai Journal, **1** (2010) (to appear).

[6] Al-Oboudi, F. M., *On univalent functions defined by a generalized Sălăgean operator*, Ind. J. Math. Math. Sci., **25-28** (2004), 1429-1436.

[7] Cătaş, A., *On certain class of p-valent functions defined by new multiplier transformations*, A. Catas, Proceedings Book of the International Symposium on Geometric Function Theory and Applications, August 20-24, 2007, TC Istanbul Kultur University, Turkey, 241-250.

[8] Miller, S. S., Mocanu, P. T., *Subordinants of Differential Superordinations*, Complex Variables, **48** (2003), no. 10, 815-826.

[9] Ruscheweyh, S., *New criteria for univalent functions*, Proc. Amer. Math. Soc., **49** (1975), 109-115.

[10] Sălăgean, G. S., *Subclasses of univalent functions*, Lecture Notes in Math., Springer Verlag, Berlin, **1013** (1983), 362-372.

Department of Mathematics and Computer Science
University of Oradea
Str. Universităţii nr. 1, 410087 Oradea, Romania
E-mail address: dalb@uoradea.ro

STUDIA UNIV. "BABEŞ–BOLYAI", MATHEMATICA, Volume **LV**, Number 3, September 2010

VARIOUS PROPERTIES OF A CERTAIN CLASS OF MULTIVALENT ANALYTIC FUNCTIONS

MOHAMED K. AOUF, ALI SHAMANDY, ADELA O. MOSTAFA, AND FATMA Z. EL-EMAM

Dedicated to Professor Grigore Ştefan Sălăgean on his 60^{th} birthday

Abstract. By using the techniques of Briot-Bouquet differential subordination, we study various properties and characteristics of the subclass $V_{p,q,s}^{\lambda}(\alpha_1;\beta_1;A,B)$ of multivalent analytic functions.

1. **Introduction**

Let $A(p)$ denote the class of functions of the form:

$$f(z) = z^p + \sum_{k=1}^{\infty} a_{k+p} z^{k+p} \quad (p \in \mathbb{N} = \{1, 2, 3, ...\}), \tag{1.1}$$

which are analytic and p-valent in the open unit disk $U = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$. Let Ω denote the class of bounded analytic functions satisfying $\omega(0) = 0$ and $|\omega(z)| \leq |z|$ for $z \in U$. For functions $f(z) \in A(p)$ given by (1.1) and $g(z) \in A(p)$ defined by $g(z) = z^p + \sum_{k=1}^{\infty} b_{k+p} z^{k+p}$ $(p \in \mathbb{N})$, the Hadamard product (or convolution) of $f(z)$ and $g(z)$ is given by

$$(f * g)(z) = z^p + \sum_{k=1}^{\infty} a_{k+p} b_{k+p} z^{k+p} = (g * f)(z).$$

For given arbitrary numbers A, B $(-1 \leq B < A \leq 1)$, we denote by $P(A, B)$ the class of functions of the form:

$$\varphi(z) = 1 + b_1 z + b_2 z^2 + ..., \tag{1.2}$$

Received by the editors: 01.07.2010.
2000 *Mathematics Subject Classification.* 30C45.
Key words and phrases. Differential subordination, Hadamard product, multivalent functions, Dziok-Srivastava operator.

which are analytic in U and satisfy the following condition:

$$\varphi(z) \prec \frac{1+Az}{1+Bz} \quad (z \in U).$$

(Here the symbol $\prec$ stands for subordination.) The class $P(A,B)$ was investigated by Janowski [11].

For a function $f(z) \in A(p)$ given by (1.1), the generalized Bernardi-Libera-Livingston integral operator $F_{\delta,p}$ is defined by (see [5])

$$\begin{aligned} F_{\delta,p}(f)(z) &= \frac{\delta+p}{z^{\delta}} \int_0^z t^{\delta-1} f(t)dt \\ &= z^p + \sum_{k=1}^{\infty} \left(\frac{\delta+p}{\delta+p+k} \right) a_{k+p} z^{k+p} \quad (\delta > -p; z \in U). \end{aligned} \tag{1.3}$$

It readily follows from (1.3) that $f \in A(p) \Longleftrightarrow F_{\delta,p} \in A(p)$. Furthermore, we have

$$\theta_m(z) = F_{\delta_m,p}(F_{\delta_{m-1},p}...(F_{\delta_1,p}(z)))$$

$$= z^p + \sum_{k=1}^{\infty} \left(\prod_{j=1}^{m} \frac{\delta_j+p}{\delta_j+p+k} \right) a_{k+p} z^{k+p} \quad (\delta_j > -p;\ j = 1,...,m). \tag{1.4}$$

For complex parameters $\alpha_1,...,\alpha_q$ and $\beta_1,...,\beta_s$ ($\beta_j \notin \mathbb{Z}_0^-$; $\mathbb{Z}_0^- = \{0,-1,-2,...\}$; $j = 1,...,s$,) the generalized hypergeometric function ${}_qF_s(z)$ is defined (cf., e.g., [28]) as follows:

$${}_qF_s(z) \equiv {}_q F_s(\alpha_1,...,\alpha_q;\beta_1,...,\beta_s;z) = \sum_{k=0}^{\infty} \frac{(\alpha_1)_k...(\alpha_q)_k}{(\beta_1)_k...(\beta_s)_k} \frac{z^k}{(1)_k} \tag{1.5}$$

$$(q \leq s+1; q, s \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}; z \in U),$$

where $(x)_k$ is the Pochhammer symbol defined (in terms of the Gamma function) by

$$(x)_k = \frac{\Gamma(x+k)}{\Gamma(x)} = \begin{cases} x(x+1)...(x+k-1) & (k \in \mathbb{N} \text{ and } x \in \mathbb{C}) \\ 1 & (k = 0 \text{ and } x \in \mathbb{C}\backslash\{0\}). \end{cases}$$

We note that the series (1.5) converges absolutely for $z \in U$ and hence represents an analytic function in the open unit disk U (see [29]). Corresponding to a function $\mathcal{F}_p(\alpha_1,...,\alpha_q,\beta_1,...,\beta_s;z)$ defined by

$$\mathcal{F}_p(\alpha_1,...,\alpha_q;\beta_1,...,\beta_s;z) = z^p \ {}_qF_s(\alpha_1,...,\alpha_q;\beta_1,...,\beta_s;z),$$

Dziok and Srivastava [6] defined a linear operator $H_p(\alpha_1, ..., \alpha_q; \beta_1, ..., \beta_s) : A(p) \to A(p)$ by the following Hadamard product:

$$H_p(\alpha_1, ..., \alpha_q; \beta_1, ..., \beta_s)f(z) = \mathcal{F}_p(\alpha_1, ..., \alpha_q; \beta_1, ..., \beta_s; z) * f(z),$$

$$(q \leq s + 1; q, s \in \mathbb{N}_0; z \in U).$$

If $f \in A(p)$ is given by (1.1), then we have

$$H_p(\alpha_1, ..., \alpha_q; \beta_1, ..., \beta_s)f(z) = z^p + \sum_{k=1}^{\infty} \Gamma_k a_{k+p} z^{k+p} \ , \tag{1.6}$$

where

$$\Gamma_k = \frac{(\alpha_1)_k ... (\alpha_q)_k}{(\beta_1)_k ... (\beta_s)_k (1)_k} \quad (k \in \mathbb{N}).$$

For convenience, we write

$$H_{p,q,s}(\alpha_1; \beta_1) = H_p(\alpha_1, ..., \alpha_q; \beta_1, ..., \beta_s).$$

It follows from (1.6) that

$$H_{p,2,1}(p, 1; p)f(z) = f(z), \ H_{p,2,1}(p + 1, 1; p)f(z) = \frac{zf^{'}(z)}{p}$$

and

$$\begin{aligned} z(H_{p,q,s}(\alpha_1; \beta_1)f(z))^{'} &= (\beta_1 - 1)H_{p,q,s}(\alpha_1; \beta_1 - 1)f(z) \\ &+ (p + 1 - \beta_1)H_{p,q,s}(\alpha_1; \beta_1)f(z). \end{aligned} \tag{1.7}$$

The linear operator $H_{p,q,s}(\alpha_1; \beta_1)$ includes various other linear operators which were considered in earlier works. In particular, for $f \in A(p)$ we have the following observations:

(i) $H_{1,2,1}(a, b; c)f(z) = I_c^{a,b} f(z)$ $(a, b \in \mathbb{C}; c \notin \mathbb{Z}_0^-)$, where $I_c^{a,b}$ is the linear operator investigated by Hohlov [10];

(ii) $H_{p,2,1}(n + p, 1; 1)f(z) = D^{n+p-1} f(z)$ $(n > -p; p \in \mathbb{N})$, where D^{n+p-1} is the linear operator studied by Goel and Sohi [8]. In the case when $p = 1$, $D^n f(z)$ is the $n - th$ Ruscheweyh derivative of $f(z)$ (see [22]);

(iii) $H_{p,2,1}(\delta+p,1;\delta+p+1)f(z)= F_{\delta,p}(f)(z)$ $(\delta>-p)$, where $F_{\delta,p}$ is the generalized Bernardi–Libera–Livingston integral operator ([5]);

(iv) $H_{p,2,1}(p+1,1;p+1-\mu)f(z)= \Omega_z^{(\mu,p)}f(z)$ $(-\infty<\mu<p+1)$, where $\Omega_z^{(\mu,p)}$ $(-\infty<\mu<p+1)$ is the extended fractional differintegral operator (see [20]), defined by

$$\begin{aligned}\Omega_z^{(\mu,p)}f(z) &= z^p+\sum_{k=1}^{\infty}\frac{\Gamma(k+p+1)\Gamma(p+1-\mu)}{\Gamma(p+1)\Gamma(k+p+1-\mu)}a_{k+p}z^{k+p}\\ &= \frac{\Gamma(p+1-\mu)}{\Gamma(p+1)}z^{\mu}D_z^{\mu}f(z)\quad(-\infty<\mu<p+1),\end{aligned}$$

where $D_z^{\mu}f(z)$ is, respectively, the fractional integral of $f(z)$ of order $-\mu$ when $-\infty<\mu<0$ and the fractional derivative of $f(z)$ of order μ when $0<\mu<p+1$ (see, for details [18], [19] and [20]). The fractional differential operator $\Omega_z^{(\mu,p)}$ with $0\le\mu<1$ was investigated by Srivastava and Aouf [27].

(v) $H_{p,2,1}(a,1;c)f(z)= L_p(a;c)f(z)$ $(a\in\mathbb{R};c\in\mathbb{R}\backslash\mathbb{Z}_0^-)$, where $L_p(a;c)$ is the linear operator studied by Saitoh [24] which yields the operator $L(a;c)f(z)$ introduced by Carlson and Shaffer [3] for $p=1$;

(vi) $H_{1,2,1}(\mu,1;\lambda+1)f(z)= I_{\lambda,\mu}f(z)$ $(\lambda>-1;\mu>0)$, where $I_{\lambda,\mu}$ is the Choi–Saigo–Srivastava operator [5];

(vii) $H_{p,2,1}(p+1,1;n+p)f(z)= I_{n,p}f(z)$ $(n>-p;p\in\mathbb{N})$, where $I_{n,p}$ is the Noor integral operator of $(n+p-1)-th$ order , studied by Liu and Noor [15];

(viii) $H_{p,2,1}(\lambda+p,c;a)f(z)= I_p^{\lambda}(a;c)f(z)$ $(a,c\in\mathbb{R}\backslash\mathbb{Z}_0^-;\lambda>-p)$, where $I_p^{\lambda}(a;c)$ is the Cho–Kwon–Srivastava operator [4].

Now, by making use of the Dziok–Srivastava operator $H_{p,q,s}(\alpha_1;\beta_1)$, we introduce a subclass of functions in $A(p)$ as follows.

Definition 1.1. *A function* $f(z) \in A(p)$ *is said to be in the class* $V_p^\lambda(\alpha_1, ..., \alpha_q; \beta_1, ..., \beta_s; A, B)$ $((\alpha_j > 0;\ j = 1, ..., q),\ (\beta_j \notin \mathbb{Z}_0^-;\ j = 1, ..., s),\ \beta_1 > 1,\ \lambda \geq 0$ *and* $-1 \leq B < A \leq 1)$, *if and only if it satisfies*

$$(1-\lambda)\frac{H_{p,q,s}(\alpha_1;\beta_1)f(z)}{z^p} + \lambda\frac{H_{p,q,s}(\alpha_1;\beta_1-1)f(z)}{z^p} \prec \frac{1+Az}{1+Bz}\ . \tag{1.8}$$

For convenience, we write $V_{p,q,s}^\lambda(\alpha_1;\beta_1;A,B) = V_p^\lambda(\alpha_1,..,\alpha_q;\beta_1,..,\beta_s;A,B)$.

We note that

(i) $V_{1,2,1}^1(2,1;2;1-2\alpha,-1) = R(\alpha)\quad (0 \leq \alpha < 1)$ [7];

(ii) $V_{p,2,1}^1(p+1,1;p+1;1,\frac{1}{M}-1) = S_p(M)\quad (M > \frac{1}{2})$ [26];

(iii) $V_{1,2,1}^1(2,1;2;2\alpha\beta-1,2\beta-1) = R_1(\alpha,\beta)\quad (0 \leq \alpha < 1,\ 0 < \beta \leq 1)$ [16];

(iv) $V_{1,2,1}^1(2,1;2;(2\alpha-1)\beta,\beta) = R(\alpha,\beta)\quad (0 \leq \alpha < 1,\ 0 < \beta \leq 1)$ [12];

(v) $V_{1,2,1}^1(n+2,1;2;A,B) = V_n(A,B)\quad (n > -1)$ [14];

(vi) $V_{1,2,1}^1(n+2,1;2;B+(A-B)(1-\alpha),B) = V_n(A,B,\alpha)\quad (n > -1; 0 \leq \alpha < 1)$ [2];

(vii) $V_{p,2,1}^\lambda(p+1,1,p+1-\mu;\beta(1-(2\alpha/p)),-\beta) = V_p^\lambda(\mu,\alpha,\beta)$; where $V_p^\lambda(\mu,\alpha,\beta)$ denotes the class of functions $f\ (z) \in A(p)$ satisfying the condition:

$$\left|\frac{(1-\lambda)\Omega_z^{(\mu,p)}f(z) + \lambda\Omega_z^{(1+\mu,p)}f(z) - z^p}{(1-\lambda)\Omega_z^{(\mu,p)}f(z) + \lambda\Omega_z^{(1+\mu,p)}f(z) + (1-(2\alpha/p))z^p}\right| < \beta\quad (z \in U),$$

where $0 \leq \mu < 1$, $0 \leq \alpha < p$, $p \in \mathbb{N}$ and $0 < \beta \leq 1$;

(viii) $V_{p,q,s}^\lambda(\alpha_1;\beta_1;1,\frac{1}{M}-1) = V_{p,q,s}^\lambda(\alpha_1;\beta_1;M)\quad (M > \frac{1}{2})$, where $V_{p,q,s}^\lambda(\alpha_1;\beta_1;M)$ denotes the class of functions $f\ (z) \in A(p)$ satisfying the condition:

$$\left|\left[(1-\lambda)\frac{H_{p,q,s}(\alpha_1;\beta_1)f(z)}{z^p} + \lambda\frac{H_{p,q,s}(\alpha_1;\beta_1-1)f(z)}{z^p}\right] - M\right| < M$$

$$(M > \frac{1}{2}; z \in U);$$

(ix) $V_{p,2,1}^1(p+1,1;p+2-\mu;1,\frac{1}{M}-1) = V_p(\mu,M)\quad (M > \frac{1}{2}; -\infty < \mu < p+1)$, where $V_p(\mu,M)$ denotes the class of functions $f\ (z) \in A(p)$ satisfying the condition:

$$\left|\frac{\Omega_z^{(\mu,p)}f(z)}{z^p} - M\right| < M\quad (M > \frac{1}{2}; -\infty < \mu < p+1; z \in U).$$

2. **Preliminaries**

To prove our main results, we need the following lemmas.

Lemma 2.1. [9] *Let the function $h(z)$ be analytic and convex (univalent) in U with $h(0)=1$ and let the function $\phi(z)$ given by (1.2) be analytic in U. If*

$$\phi(z)+\frac{z\phi^{'}(z)}{\gamma}\prec h(z)\quad (\mathrm{Re}(\gamma)\geq 0;\ \gamma\neq 0),$$

then

$$\phi(z)\prec\psi(z)=\frac{\gamma}{z^{\gamma}}\int_{0}^{z}t^{\gamma-1}h(t)dt\prec h(z)\ ,$$

and $\psi(z)$ is the best dominant.

Lemma 2.2. [25] *Let $\Phi(z)$ be analytic in U with*

$$\Phi(0)=1\qquad \mathit{and}\quad \mathrm{Re}(\Phi(z))>\frac{1}{2}\ (z\in U).$$

Then, for any function $F(z)$ analytic in U, $(\Phi*F)(U)$ is contained in the convex hull of $F(U)$.

Lemma 2.3. [29] *For real or complex numbers a,b and c $(c\neq 0,-1,-2,...)$, we have*

$$\int_{0}^{1}t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a}dt=\frac{\Gamma(b)\Gamma(c-b)}{\Gamma(c)}\ {}_2F_1(a,b;c;z)\ (\mathrm{Re}\,(c)>\mathrm{Re}\,(b)>0);\tag{2.1}$$

and

$${}_2F_1(a,b;c;z)=(1-z)^{-a}\ {}_2F_1(a,c-b;c;\frac{z}{z-1}).\tag{2.2}$$

Lemma 2.4. [13] *Let $\omega(z)=\sum\limits_{k=1}^{\infty}d_kz^k\in\Omega$, if ν is a complex number, then*

$$\left|d_2-\nu d_1^2\right|\leq\max\{1,|\upsilon|\}.\tag{2.3}$$

Equation (2.3) may be attend with the functions $\omega(z)=z$ and $\omega(z)=z^2$, respectively, for $|\upsilon|\geq 1$ and $|\upsilon|<1$.

3. Main results

Otherwise unless mention throughout this paper, we assume that $-1 \leq B < A \leq 1$, $\lambda > 0$, $p \in \mathbb{N}$, $\beta_1 > 1$ and $z \in U$.

Theorem 3.1. *Let the function f defined by (1.1) be in the class $V_{p,q,s}^{\lambda}(\alpha_1;\beta_1;A,B)$. Then*

$$\frac{H_{p,q,s}(\alpha_1;\beta_1)f(z)}{z^p} \prec Q(z) \prec \frac{1+Az}{1+Bz} \ , \tag{3.1}$$

where

$$Q(z) = \begin{cases} \frac{A}{B} + \left(1-\frac{A}{B}\right)(1+Bz)^{-1}\ {}_2F_1\left(1,1;\frac{\beta_1-1}{\lambda}+1;\frac{Bz}{Bz+1}\right) & (B \neq 0) \\ 1 + \frac{\beta_1-1}{\beta_1-1+\lambda}Az & (B=0), \end{cases}$$

is the best dominant of (3.1). Furthermore,

$$\mathrm{Re}\left\{\frac{H_{p,q,s}(\alpha_1;\beta_1)f(z)}{z^p}\right\} > \eta(\lambda,\beta_1,A,B), \tag{3.2}$$

where

$$\eta(\lambda,\beta_1,A,B) = \begin{cases} \frac{A}{B} + \left(1-\frac{A}{B}\right)(1-B)^{-1}\ {}_2F_1\left(1,1;\frac{\beta_1-1}{\lambda}+1;\frac{B}{B-1}\right) & (B \neq 0) \\ 1 - \frac{\beta_1-1}{\beta_1-1+\lambda}A & (B=0). \end{cases}$$

The estimate in (3.2) is best possible.

Proof. Setting

$$\phi(z) = \frac{H_{p,q,s}(\alpha_1;\beta_1)f(z)}{z^p} \ . \tag{3.3}$$

Then $\phi(z)$ is of the form (1.2) and is analytic in U. Differentiating (3.3), and using identity (1.7) in the resulting equation, we have

$$\begin{aligned}(1-\lambda)\frac{H_{p,q,s}(\alpha_1;\beta_1)f(z)}{z^p} + \lambda\frac{H_{p,q,s}(\alpha_1;\beta_1-1)f(z)}{z^p} &= \phi(z) + \frac{\lambda z\phi'(z)}{\beta_1-1} \\ &\prec \frac{1+Az}{1+Bz} \ .\end{aligned}$$

Now, by using Lemma 2.1 for $\gamma = \frac{\beta_1 - 1}{\lambda}$, we deduce that

$$\phi(z) \prec Q(z) = \frac{\beta_1 - 1}{\lambda} z^{-\frac{\beta_1 - 1}{\lambda}} \int_0^z t^{\frac{\beta_1 - 1}{\lambda} - 1} (\frac{1 + At}{1 + Bt}) dt$$

$$= \begin{cases} \frac{A}{B} + \left(1 - \frac{A}{B}\right)(1 + Bz)^{-1} {}_2F_1\left(1, 1; \frac{\beta_1 - 1}{\lambda} + 1; \frac{Bz}{Bz+1}\right) & (B \neq 0) \\ 1 + \frac{\beta_1 - 1}{\beta_1 - 1 + \lambda} Az & (B = 0), \end{cases}$$

by change of variables followed by using the identities (2.1) and (2.2) (with $a = 1$, $b = \frac{\beta_1 - 1}{\lambda}$ and $c = b + 1$). This proves the assertion (3.1) of Theorem 3.1. Next, to prove (3.2), it suffices to show that

$$\inf_{|z|<1} \{\mathrm{Re}(Q(z))\} = Q(-1). \tag{3.4}$$

For $|z| \leq r < 1$, we have

$$\mathrm{Re}\left\{\frac{1 + Az}{1 + Bz}\right\} \geq \frac{1 - Ar}{1 - Br}.$$

Setting

$$g(s, z) = \frac{1 + Asz}{1 + Bsz} \text{ and } d\mu(s) = \frac{\beta_1 - 1}{\lambda} s^{\frac{\beta_1 - 1}{\lambda} - 1} ds \quad (0 \leq s \leq 1),$$

we get

$$Q(z) = \int_0^1 g(s, z) d\mu(s),$$

so that

$$\mathrm{Re}\{Q(z)\} \geq \int_0^1 \frac{1 - Asr}{1 - Bsr} d\mu(s) = Q(-r) \quad (|z| \leq r < 1).$$

Letting $r \to 1^-$ in the above inequality, we obtain the assertion (3.4). The result in (3.2) is best possible as the function $Q(z)$ is the best dominant of (3.1). □

Corollary 3.2. *For* $0 < \lambda_2 < \lambda_1$, *we have*

$$V_{p,q,s}^{\lambda_1}(\alpha_1; \beta_1; A, B) \subset V_{p,q,s}^{\lambda_2}(\alpha_1; \beta_1; A, B).$$

Proof. Let $f \in V_{p,q,s}^{\lambda_1}(\alpha_1; \beta_1; A, B)$.

Then by Theorem 3.1, we have $f \in V^{0}_{p,q,s}(\alpha_1;\beta_1;A,B)$. Since

$$\begin{aligned}
&(1-\lambda_2)\frac{H_{p,q,s}(\alpha_1;\beta_1)f(z)}{z^p}+\lambda_2\frac{H_{p,q,s}(\alpha_1;\beta_1-1)f(z)}{z^p}\\
=&\left(1-\frac{\lambda_2}{\lambda_1}\right)\frac{H_{p,q,s}(\alpha_1;\beta_1)f(z)}{z^p}\\
&+\frac{\lambda_2}{\lambda_1}\left\{(1-\lambda_1)\frac{H_{p,q,s}(\alpha_1;\beta_1)f(z)}{z^p}+\lambda_1\frac{H_{p,q,s}(\alpha_1;\beta_1-1)f(z)}{z^p}\right\}\\
\prec&\ \frac{1+Az}{1+Bz}\ ,
\end{aligned}$$

we see that $f \in V^{\lambda_2}_{p,q,s}(\alpha_1;\beta_1;A,B)$. □

Taking $\lambda = s = \alpha_2 = 1$, $q = 2$, $\alpha_1 = p+1$, $\beta_1 = n+p$, $A = 1-\frac{2\alpha}{p}$ and $B = -1$ in Theorem 3.1, we get the following corollary.

Corollary 3.3. *Let the function f given by (1.1)satisfy*

$$\operatorname{Re}\left\{\frac{I_{n-1,p}f(z)}{z^p}\right\} > \frac{\alpha}{p} \quad (0 \le \alpha < p;\ n > -p).$$

Then

$$\operatorname{Re}\left\{\frac{I_{n,p}f(z)}{z^p}\right\} > \frac{\alpha}{p}+\left(1-\frac{\alpha}{p}\right)\left\{{}_2F_1\left(1,1;p+n;\frac{1}{2}\right)-1\right\}.$$

The result is best possible.

Putting $n = 1$ in Corollary 3.3, we have the following corollary.

Corollary 3.4. *If $f \in A(p)$ satisfies*

$$\operatorname{Re}\left\{\frac{f'(z)}{z^{p-1}}\right\} > \alpha \quad (0 \le \alpha < p),$$

then

$$\operatorname{Re}\left\{\frac{f(z)}{z^p}\right\} > \frac{\alpha}{p}+\left(1-\frac{\alpha}{p}\right)\left\{{}_2F_1\left(1,1;p+1;\frac{1}{2}\right)-1\right\}.$$

The result is best possible.

Remark 3.5. The above result improves the corresponding result of Saitoh [23, Corollary 2].

Theorem 3.6. *Let $f(z) \in V^{0}_{p,q,s}(\alpha_1;\beta_1;A,B)$, then the function $F_{\delta,p}$ defined by (1.3) satisfies*

$$\frac{H_{p,q,s}(\alpha_1;\beta_1)F_{\delta,p}(z)}{z^p} \prec q(z) \prec \frac{1+Az}{1+Bz}\ , \tag{3.5}$$

where

$$q(z) = \begin{cases} \frac{A}{B} + \left(1 - \frac{A}{B}\right)(1+Bz)^{-1}\ {}_2F_1\left(1, 1; p+\delta+1; \frac{Bz}{Bz+1}\right) & (B \neq 0) \\ 1 + \frac{p+\delta}{p+\delta+1} Az & (B = 0), \end{cases}$$

and $q(z)$ is the best dominant of (3.5). Furthermore,

$$\mathrm{Re}\left\{\frac{H_{p,q,s}(\alpha_1, \beta_1)F_{\delta,p}(z)}{z^p}\right\} > \xi(\delta, p, A, B) \ , \tag{3.6}$$

where

$$\xi(\delta, p, A, B) = \begin{cases} \frac{A}{B} + \left(1 - \frac{A}{B}\right)(1-B)^{-1}\ {}_2F_1\left(1, 1; p+\delta+1; \frac{B}{B-1}\right) & (B \neq 0) \\ 1 - \frac{p+\delta}{p+\delta+1} A & (B = 0). \end{cases}$$

The estimate in (3.6) is best possible.

Proof. Let

$$\phi(z) = \frac{H_{p,q,s}(\alpha_1, \beta_1)F_{\delta,p}(z)}{z^p} \ . \tag{3.7}$$

Then $\phi(z)$ is analytic in U with $\phi(0) = 1$. Differentiating (3.7) and using the identity

$$z(H_{p,q,s}(\alpha_1; \beta_1)F_{\delta,p}(z))' = (\delta + p)H_{p,q,s}(\alpha_1; \beta_1)f(z) - \delta H_{p,q,s}(\alpha_1; \beta_1)F_{\delta,p}(z) \tag{3.8}$$

in the resulting equation, we obtain

$$\phi(z) + \frac{z\phi'(z)}{\delta + p} = \frac{H_{p,q,s}(\alpha_1; \beta_1)f(z)}{z^p} \prec \frac{1 + Az}{1 + Bz} \quad (z \in U).$$

Now, by using Lemma 2.1 for $\gamma = \delta + p$, we deduce that

$$\phi(z) \prec q(z) = (\delta + p)z^{-(\delta+p)} \int_0^z t^{\delta+p-1} \left(\frac{1 + At}{1 + Bt}\right) dt.$$

The assertions (3.5) and (3.6) can now be deduced on the same lines that used in Theorem 3.1. This completes the proof of Theorem 3.6. □

Taking $A = 1 - \frac{2\alpha}{p}$ $(0 \leq \alpha < p)$ and $B = -1$ in Theorem 3.6, we get the following corollary.

Corollary 3.7. *If $f \in A(p)$ satisfies*

$$\mathrm{Re}\left\{\frac{H_{p,q,s}(\alpha_1; \beta_1)f(z)}{z^p}\right\} > \frac{\alpha}{p} \quad (0 \leq \alpha < p),$$

then

$$\operatorname{Re}\left\{\frac{H_{p,q,s}(\alpha_1;\beta_1)F_{\delta,p}(z)}{z^p}\right\} > \frac{\alpha}{p} + \left(1-\frac{\alpha}{p}\right)\left\{{}_2F_1\left(1,1;p+\delta+1;\frac{1}{2}\right)-1\right\}.$$

The result is best possible.

Taking $s=\alpha_2=1$, $q=2$, $\alpha_1=p+1$ and $\beta_1=p+1-\mu$ $(-\infty<\mu<p+1)$ in Corollary 3.7, we get the following corollary.

Corollary 3.8. *If $f\in A(p)$ satisfies*

$$\operatorname{Re}\left\{\frac{\Omega_z^{(\mu,p)}f(z)}{z^p}\right\} > \frac{\alpha}{p} \quad (0\le\alpha<p;-\infty<\mu<p+1),$$

then

$$\operatorname{Re}\left\{\frac{\Omega_z^{(\mu,p)}F_{\delta,p}(z)}{z^p}\right\} > \frac{\alpha}{p} + \left(1-\frac{\alpha}{p}\right)\left\{{}_2F_1\left(1,1;p+\delta+1;\frac{1}{2}\right)-1\right\}.$$

The result is best possible.

Corollary 3.9. *Under the hypothesis of Corollary 3.7, the function $\theta_m(z)$ defined by (1.4)satisfies*

$$\operatorname{Re}\left\{\frac{H_{p,q,s}(\alpha_1;\beta_1)\theta_m(z)}{z^p}\right\} > \frac{\rho_m}{p},$$

where $\rho_0=\alpha$ and

$$\rho_j=\rho_{j-1}+(p-\rho_{j-1}\ )\left\{{}_2F_1\left(1,1;p+\delta+1;\frac{1}{2}\right)-1\right\} \quad (j=1,2,...,m).$$

The result is best possible.

Taking $s=\alpha_2=1, q=2$, $\alpha_1=p+1$ and $\beta_1=n+p$ $(n>-p)$ in Corollary 3.7, we have the following corollary.

Corollary 3.10. *If $f\ \in A(p)$ satisfies*

$$\operatorname{Re}\left\{\frac{I_{n,p}f(z)}{z^p}\right\} > \frac{\alpha}{p} \quad (0\le\alpha<p),$$

then

$$\operatorname{Re}\left\{\frac{I_{n,p}F_{\delta,p}(z)}{z^p}\right\} > \frac{\alpha}{p} + (1-\frac{\alpha}{p}\ )\left\{{}_2F_1\left(1,1;p+\delta+1;\frac{1}{2}\right)-1\right\}.$$

The result is the best possible.

Putting $n = 0$ in Corollary 3.10, we have the following corollary which in turn improves the corresponding result of Fukui et al. [7] for $p = 1$.

Corollary 3.11. *If $f \in A(p)$ satisfies*

$$\operatorname{Re}\left\{\frac{f'(z)}{z^{p-1}}\right\} > \alpha \quad (0 \le \alpha < p),$$

then

$$\operatorname{Re}\left\{\frac{F'_{\delta,p}(z)}{z^{p-1}}\right\} > \alpha + (p-\alpha)\left\{{}_2F_1\left(1,1;p+\delta+1;\frac{1}{2}\right) - 1\right\}.$$

The result is best possible.

Theorem 3.12. *For $f \in A(p)$, we have*

$$f \in V^0_{p,q,s}(\alpha_1;\beta_1;A,B) \Leftrightarrow F_{\beta_1-p-1,p} \in V^1_{p,q,s}(\alpha_1;\beta_1;A,B).$$

Proof. Using identity (3.8) and

$$z(H_{p,q,s}(\alpha_1;\beta_1)F_{\delta,p}(z))' = (\beta_1-1)H_{p,q,s}(\alpha_1;\beta_1-1)F_{\delta,p}(z)$$
$$+(p+1-\beta_1)H_{p,q,s}(\alpha_1;\beta_1)F_{\delta,p}(z),$$

for $\delta = \beta_1 - p - 1$, we deduce that

$$H_{p,q,s}(\alpha_1;\beta_1)f(z) = H_{p,q,s}(\alpha_1;\beta_1-1)F_{\beta_1-p-1,p}(z)$$

and the assertion of Theorem 3.12 follows by using the definition of the class $V^\lambda_{p,q,s}(\alpha_1;\beta_1;A,B)$. □

Theorem 3.13. *If the function $f(z)$ given by (1.1) belongs to the class $V^\lambda_{p,q,s}(\alpha_1;\beta_1;A,B)$, then*

$$|a_{k+p}| \le \frac{(A-B)(\beta_1-1)_{k+1}(\beta_2)_k...(\beta_s)_k(1)_k}{(\beta_1-1+\lambda k)(\alpha_1)_k...(\alpha_q)_k} \quad (k \ge 1). \tag{3.9}$$

The estimate is sharp.

Proof. Since $f(z) \in V^\lambda_{p,q,s}(\alpha_1;\beta_1;A,B)$, then

$$(1-\lambda)\frac{H_{p,q,s}(\alpha_1;\beta_1)f(z)}{z^p} + \lambda\frac{H_{p,q,s}(\alpha_1;\beta_1-1)f(z)}{z^p} = p(z), \tag{3.10}$$

where $p(z) = 1 + \sum_{k=1}^{\infty} p_k z^k \in P(A,B)$. Substituting the power series expansion of $H_{p,q,s}(\alpha_1;\beta_1)f(z)$, $H_{p,q,s}(\alpha_1;\beta_1-1)f(z)$ and $p(z)$ in (3.10) and equating the coefficients of z^k on the both sides of the resulting equation, we obtain

$$\frac{(\beta_1-1+\lambda k)(\alpha_1)_k...(\alpha_q)_k}{(\beta_1-1)_{k+1}(\beta_2)_k...(\beta_s)_k k!} a_{k+p} = p_k \quad (k \geq 1). \tag{3.11}$$

Using the well-known [1] coefficient estimates

$$|p_k| \leq A - B \qquad (k \geq 1),$$

in (3.11), we get the required result (3.9). The estimate in (3.9) is sharp for the functions $f_k(z)$ defined by

$$(1-\lambda)\frac{H_{p,q,s}(\alpha_1;\beta_1)f_k(z)}{z^p} + \lambda\frac{H_{p,q,s}(\alpha_1;\beta_1-1)f_k(z)}{z^p} = \frac{1+Az^k}{1+Bz^k} \qquad (k \geq 1).$$

Clearly, $f_k(z) \in V^{\lambda}_{p,q,s}(\alpha_1;\beta_1;A,B)$ for each $k \geq 1$. It is easy to see that the functions $f_k(z)$ have the series expansion

$$f_k(z) = z^p + \frac{(A-B)(\beta_1-1)_{k+1}(\beta_2)_k...(\beta_s)_k\ (1)_k}{(\beta_1-1+\lambda k)(\alpha_1)_k...(\alpha_q)_k} z^{k+p} + ...\ ,$$

show that the estimates in (3.9) are sharp. □

Taking $A = \lambda = s = \alpha_2 = 1, q = 2$, $\alpha_1 = p+1$ and $\beta_1 = p+2-\mu$ $(-\infty < \mu < p+1)$, $B = \frac{1}{M} - 1$ $(M > \frac{1}{2})$ in Theorem 3.13, we have the following corollary.

Corollary 3.14. *If the function $f(z)$ given by (1.1) belongs to the class $V_p(\mu, M)$, then*

$$|a_{k+p}| \leq \frac{(2M-1)(p+1-\mu)_k}{M(p+1)_k} \quad (k \geq 1).$$

The estimate is sharp.

Theorem 3.15. *Let f given by (1.1) belongs to the class $V^{\lambda}_{p,q,s}(\alpha_1;\beta_1;A,B)$ and ζ be any complex number. Then*

$$\left|a_{p+2} - \zeta a_{p+1}^2\right| \leq \frac{2(A-B)(\beta_1-1)_3(\beta_2)_2...(\beta_s)_2}{(\beta_1-1+2\lambda)(\alpha_1)_2...(\alpha_q)_2}.$$

$$.\max\left\{1, \left|B + \zeta\frac{(\beta_1-1)_2\ \beta_2...\beta_s(A-B)(\beta_1-1+2\lambda)(\alpha_1+1)...(\alpha_q+1)}{2\alpha_1...\alpha_q\ (\beta_1+1)...(\beta_s+1)(\beta_1-1+\lambda)^2}\right|\right\}. \tag{3.12}$$

The estimate in (3.12) is sharp.

Proof. From (1.8), we deduce that

$$(1-\lambda)\frac{H_{p,q,s}(\alpha_1;\beta_1)f(z)}{z^p}+\lambda\frac{H_{p,q,s}(\alpha_1;\beta_1-1)f(z)}{z^p}-1$$

$$=\left[A-B\left\{(1-\lambda)\frac{H_{p,q,s}(\alpha_1;\beta_1)f(z)}{z^p}+\lambda\frac{H_{p,q,s}(\alpha_1;\beta_1-1)f(z)}{z^p}\right\}\right]\omega(z), \quad (3.13)$$

where $\omega(z)=\sum_{k=1}^{\infty}\omega_k z^k$ is analytic in U and satisfies $|\omega(z)|\le|z|$ for $z\in U$. Substituting the power series expansion of $H_{p,q,s}(\alpha_1;\beta_1)f(z)$, $H_{p,q,s}(\alpha_1;\beta_1-1)f(z)$ and $\omega(z)$ in (3.13), and equating the coefficients of z and z^2, we get

$$a_{p+1}=\frac{(A-B)(\beta_1-1)_2\ \beta_2...\beta_s}{(\beta_1-1+\lambda)\ \alpha_1...\alpha_q}\omega_1, \quad (3.14)$$

$$a_{p+2}=\frac{2(A-B)(\beta_1-1)_3\ (\beta_2)_2...(\beta_s)_2}{(\beta_1-1+2\lambda)\ (\alpha_1)_2...(\alpha_q)_2}(\omega_2-B\omega_1^2). \quad (3.15)$$

From (3.14) and (3.15), we have

$$\left|a_{p+2}-\zeta a_{p+1}^2\right|=\frac{2(A-B)(\beta_1-1)_3(\beta_2)_2...(\beta_s)_2}{(\beta_1-1+2\lambda)(\alpha_1)_2...(\alpha_q)_2}\left|\omega_2-\upsilon\omega_1^2\right|, \quad (3.16)$$

where

$$\upsilon=B+\zeta\frac{(\beta_1-1)_2\ \beta_2...\beta_s(A-B)(\beta_1-1+2\lambda)(\alpha_1+1)...(\alpha_q+1)}{2\alpha_1...\alpha_q\ (\beta_1+1)...(\beta_s+1)(\beta_1-1+\lambda)^2}.$$

Now, by using (2.3) in (3.16), we get the required result. The result (3.12) is sharp as the estimate (2.3) is sharp. □

Taking $A=\lambda=s=\alpha_2=1, q=2$, $\alpha_1=p+1$ and $\beta_1=p+2-\mu$ $(-\infty<\mu<p+1)$, $B=\frac{1}{M}-1$ $(M>\frac{1}{2})$ in Theorem 3.15, we have the following corollary.

Corollary 3.16. *Let f, given by (1.1), belongs to the class $V_p(\mu,M)$, and ζ be any complex number. Then*

$$\left|a_{p+2}-\zeta a_{p+1}^2\right|\le\frac{(2M-1)(p+1-\mu)_2}{M(p+1)_2}.$$

$$.\max\left\{1,\left|\frac{1-M}{M}+\zeta\frac{(2M-1)(p+2)(p+1-\mu)}{M(p+1)(p+2-\mu)}\right|\right\}.$$

The estimate is sharp.

Theorem 3.17. *Let* $f \in V^{\lambda}_{p,q,s}(\alpha_1;\beta_1;A,B)$ *and* $g \in A(p)$ *with* $\mathrm{Re}\{\frac{g(z)}{z^p}\} > \frac{1}{2}$ *for* $z \in U$. *Then the function* $h = f * g$ *belongs to the class* $V^{\lambda}_{p,q,s}(\alpha_1;\beta_1;A,B)$.

Proof. We can write

$$(1-\lambda)\frac{H_{p,q,s}(\alpha_1;\beta_1)h(z)}{z^p} + \lambda\frac{H_{p,q,s}(\alpha_1;\beta_1-1)h(z)}{z^p}$$

$$= \left\{(1-\lambda)\frac{H_{p,q,s}(\alpha_1;\beta_1)f(z)}{z^p} + \lambda\frac{H_{p,q,s}(\alpha_1;\beta_1-1)f(z)}{z^p}\right\} * \frac{g(z)}{z^p}. \tag{3.17}$$

Since $\mathrm{Re}\{\frac{g(z)}{z^p}\} > \frac{1}{2}$ in U and $f \in V^{\lambda}_{p,q,s}(\alpha_1;\beta_1;A,B)$, it follows from (3.17) and Lemma 2.2 that $h \in V^{\lambda}_{p,q,s}(\alpha_1,\beta_1;A,B)$. The proof is completed. □

Corollary 3.18. *Let* $f \in V^{\lambda}_{p,q,s}(\alpha_1,\beta_1;A,B)$ *and* $g \in A(p)$ *satisfy*

$$\mathrm{Re}\left\{(1-\mu)\frac{g(z)}{z^p} + \mu\frac{g'(z)}{pz^{p-1}}\right\} > \frac{3-2\ {}_2F_1(1,1;\frac{p}{\mu}+1;1/2)}{2\left\{2-{}_2F_1(1,1;\frac{p}{\mu}+1;1/2)\right\}} \quad (\mu > 0; z \in U). \tag{3.18}$$

Then $f * g \in V^{\lambda}_{p,q,s}(\alpha_1;\beta_1;A,B)$.

Proof. From Theorem 3.1 (for $q=2$, $s=1$, $\alpha_1=\beta_1=p+1$, $\alpha_2=1$, $\lambda=\mu>0$, $A=\frac{{}_2F_1(1,1;\frac{p}{\mu}+1;1/2)-1}{2-{}_2F_1(1,1;\frac{p}{\mu}+1;1/2)}$, and $B=-1$), condition (3.18) implies

$$\mathrm{Re}\left\{\frac{g(z)}{z^p}\right\} > \frac{1}{2}.$$

Using this, it follows from Theorem 3.17 that $f * g \in V^{\lambda}_{p,q,s}(\alpha_1;\beta_1;A,B)$. □

Theorem 3.19. *If each of the functions* $f(z)$ *given by (1.1) and* $g(z) = z^p + \sum_{k=1}^{\infty} b_{k+p}z^{k+p}$ *belongs to the class* $V^{\lambda}_{p,q,s}(\alpha_1;\beta_1;A,B)$, *then so does the function* $h(z) = (1-\lambda)H_{p,q,s}(\alpha_1;\beta_1)(f*g)(z) + \lambda H_{p,q,s}(\alpha_1;\beta_1-1)(f*g)(z)$.

Proof. Since $f \in V^{\lambda}_{p,q,s}(\alpha_1;\beta_1;A,B)$, it follow by (3.13) that

$$\left|(1-\lambda)\frac{H_{p,q,s}(\alpha_1;\beta_1)f(z)}{z^p} + \lambda\frac{H_{p,q,s}(\alpha_1;\beta_1-1)f(z)}{z^p} - 1\right|$$

$$< \left|A - B\left\{(1-\lambda)\frac{H_{p,q,s}(\alpha_1;\beta_1)f(z)}{z^p} + \lambda\frac{H_{p,q,s}(\alpha_1;\beta_1-1)f(z)}{z^p}\right\}\right|,$$

which is equivalent to

$$\left|(1-\lambda)\frac{H_{p,q,s}(\alpha_1;\beta_1)f(z)}{z^p} + \lambda\frac{H_{p,q,s}(\alpha_1;\beta_1-1)f(z)}{z^p} - \xi\right| < \eta \quad (z \in U), \tag{3.19}$$

where

$$\xi = \frac{1-AB}{1-B^2} \quad \text{and} \quad \eta = \frac{A-B}{1-B^2}.$$

It is known [17] that if $G(z) = \sum_{k=0}^{\infty} g_k z^k$ is analytic in U and $|G(z)| \le E$, then

$$\sum_{k=0}^{\infty} |g_k|^2 \le E^2. \tag{3.20}$$

Applying (3.20) to (3.19), we get

$$(1-\xi)^2 + \sum_{k=1}^{\infty} \left\{ \frac{(\beta_1 - 1 + \lambda k)(\alpha_1)_k ... (\alpha_q)_k}{(\beta_1 - 1)_{k+1}(\beta_2)_k ... (\beta_s)_k (1)_k} \right\}^2 |a_{k+p}|^2 < \eta^2,$$

that is, that

$$\sum_{k=1}^{\infty} \left\{ \frac{(\beta_1 - 1 + \lambda k)(\alpha_1)_k ... (\alpha_q)_k}{(\beta_1 - 1)_{k+1}(\beta_2)_k ... (\beta_s)_k (1)_k} \right\}^2 |a_{k+p}|^2 < \frac{(A-B)^2}{1-B^2}. \tag{3.21}$$

Similarly,

$$\sum_{k=1}^{\infty} \left\{ \frac{(\beta_1 - 1 + \lambda k)(\alpha_1)_k ... (\alpha_q)_k}{(\beta_1 - 1)_{k+1}(\beta_2)_k ... (\beta_s)_k (1)_k} \right\}^2 |b_{k+p}|^2 < \frac{(A-B)^2}{1-B^2}. \tag{3.22}$$

Now, for $|z| = r < 1$, by applying Cauchy-Schwarz inequality, we find that

$$\left| (1-\lambda) \frac{H_{p,q,s}(\alpha_1; \beta_1) h(z)}{z^p} + \lambda \frac{H_{p,q,s}(\alpha_1; \beta_1 - 1) h(z)}{z^p} - \xi \right|^2$$

$$= \left| (1-\xi) + \sum_{k=1}^{\infty} \left\{ \frac{(\beta_1 - 1 + \lambda k)(\alpha_1)_k ... (\alpha_q)_k}{(\beta_1 - 1)_{k+1}(\beta_2)_k ... (\beta_s)_k (1)_k} \right\}^2 a_{k+p} b_{k+p} z^k \right|^2$$

$$\le (1-\xi)^2 + 2(1-\xi) \sum_{k=1}^{\infty} \left\{ \frac{(\beta_1 - 1 + \lambda k)(\alpha_1)_k ... (\alpha_q)_k}{(\beta_1 - 1)_{k+1}(\beta_2)_k ... (\beta_s)_k (1)_k} \right\}^2 |a_{k+p}| \, |b_{k+p}| \, r^k$$

$$+ \left| \sum_{k=1}^{\infty} \left\{ \frac{(\beta_1 - 1 + \lambda k)(\alpha_1)_k ... (\alpha_q)_k}{(\beta_1 - 1)_{k+1}(\beta_2)_k ... (\beta_s)_k (1)_k} \right\}^2 a_{k+p} b_{k+p} z^k \right|^2$$

$$\le (1-\xi)^2 + 2(1-\xi) \left[\sum_{k=1}^{\infty} \left\{ \frac{(\beta_1 - 1 + \lambda k)(\alpha_1)_k ... (\alpha_q)_k}{(\beta_1 - 1)_{k+1}(\beta_2)_k ... (\beta_s)_k (1)_k} \right\}^2 |a_{k+p}|^2 r^k \right]^{1/2} \cdot$$

$$\cdot \left[\sum_{k=1}^{\infty} \left\{ \frac{(\beta_1 - 1 + \lambda k)(\alpha_1)_k ... (\alpha_q)_k}{(\beta_1 - 1)_{k+1}(\beta_2)_k ... (\beta_s)_k (1)_k} \right\}^2 |b_{k+p}|^2 r^k \right]^{1/2}$$

$$+ \left[\sum_{k=1}^{\infty} \left\{ \frac{(\beta_1 - 1 + \lambda k)(\alpha_1)_k ... (\alpha_q)_k}{(\beta_1 - 1)_{k+1}(\beta_2)_k ... (\beta_s)_k (1)_k} \right\}^2 |a_{k+p}|^2 r^k \right].$$

$$\cdot\left[\sum_{k=1}^{\infty}\left\{\frac{(\beta_1-1+\lambda k)(\alpha_1)_k...(\alpha_q)_k}{(\beta_1-1)_{k+1}(\beta_2)_k...(\beta_s)_k(1)_k}\right\}^2|b_{k+p}|^2 r^k\right]$$

$$\leq (1-\xi)^2+2(1-\xi)\left[\sum_{k=1}^{\infty}\left\{\frac{(\beta_1-1+\lambda k)(\alpha_1)_k...(\alpha_q)_k}{(\beta_1-1)_{k+1}(\beta_2)_k...(\beta_s)_k(1)_k}\right\}^2|a_{k+p}|^2\right]^{1/2}\cdot$$

$$\cdot\left[\sum_{k=1}^{\infty}\left\{\frac{(\beta_1-1+\lambda k)(\alpha_1)_k...(\alpha_q)_k}{(\beta_1-1)_{k+1}(\beta_2)_k...(\beta_s)_k(1)_k}\right\}^2|b_{k+p}|^2\right]^{1/2}$$

$$+\left[\sum_{k=1}^{\infty}\left\{\frac{(\beta_1-1+\lambda k)(\alpha_1)_k...(\alpha_q)_k}{(\beta_1-1)_{k+1}(\beta_2)_k...(\beta_s)_k(1)_k}\right\}^2|a_{k+p}|^2\right]\cdot$$

$$\cdot\left[\sum_{k=1}^{\infty}\left\{\frac{(\beta_1-1+\lambda k)(\alpha_1)_k...(\alpha_q)_k}{(\beta_1-1)_{k+1}(\beta_2)_k...(\beta_s)_k(1)_k}\right\}^2|b_{k+p}|^2\right]$$

$$\leq (1-\xi)^2+2(1-\xi)\frac{(A-B)^2}{1-B^2}+\frac{(A-B)^4}{(1-B^2)^2}$$

$$=\left\{\frac{B(A-B)}{1-B^2}\right\}^2+2\frac{B(A-B)^3}{(1-B^2)^2}+\frac{(A-B)^4}{(1-B^2)^2}<\eta^2,$$

by using (3.22) and (3.23).

Thus, again with the aid of (3.20), we have $h\in V^{\lambda}_{p,q,s}(\alpha_1,\beta_1;\,A,B)$. □

Theorem 3.20. *Let* $f\in V^{\lambda}_{p,q,s}(\alpha_1;\beta_1;A,B)$ *and* $S_n(z)=z^p+\sum_{k=1}^{n-1}a_{k+p}z^{k+p}$ $(n\geq 2)$. *Then for* $z\in U$, *we have*

$$\text{Re}\left[\frac{\int_0^z t^{-p}(H_{p,q,s}(\alpha_1;\beta_1)S_n(t))dt}{z}\right]>\eta(\lambda,\beta_1,A,B),$$

where $\eta(\lambda,\beta_1,A,B)$ *is defined as in Theorem 3.1.*

Proof. Singh and Singh [25] proved that

$$\text{Re}\left\{1+\sum_{k=1}^{n-1}\frac{z^k}{k+1}\right\}>\frac{1}{2}\quad(z\in U). \tag{3.23}$$

Writing

$$\frac{\int_0^z t^{-p}(H_{p,q,s}(\alpha_1;\beta_1)S_n(t))dt}{z}=\frac{H_{p,q,s}(\alpha_1;\beta_1)f(z)}{z^p}*\left\{1+\sum_{k=1}^{n-1}\frac{z^k}{k+1}\right\}$$

and making use of (3.23), Theorem 3.1 and Lemma 2.2, the assertion of the theorem follows at once. □

Remark 3.21. *By taking* $q = 2$, $s = 1$, $\alpha_1 = a$ $(a > 0)$, $\alpha_2 = 1$ *and* $\beta_1 = c$ $(c > 1;$ $c \notin \mathbb{Z}_0^-)$ *in our results, we obtain the results obtained by Patel and Sahoo [21].*

Acknowledgements. The authors would like to thank the referee of the paper for his helpful suggestions.

References

[1] Anh, V., *K-fold symmetric starlike univalent functions*, Bull. Austral. Math. Soc., **32** (1985), 419-436.

[2] Aouf, M. K., *On a new criterion for univalent functions of order alpha*, Rend. Mat. Ser. VII, Roma, **11** (1991), 47-59.

[3] Carlson, B. C., Shaffer, D. B., *Starlike and prestarlike hypergeometric functions*, SIAM J. Math. Anal., **15** (1984), 737-745.

[4] Cho, N. E., Kwon, O. H., Srivastava, H. M., *Inclusion and argument properties for certain subclasses of multivalent functions associated with a family of linear operators*, J. Math. Anal. Appl., **292** (2004), 470-483.

[5] Choi, J. H., Saigo, M., Srivastava, H. M., *Some inclusion properties of a certain family of integral operators*, J. Math. Anal. Appl., **276** (2002), 432-445.

[6] Dziok, J., Srivastava, H. M., *Classes of analytic functions associated with the generalized hypergeometric function*, Appl. Math. Comput., **103** (1999), 1-13.

[7] Fukui, S., Kim, J. A., Srivastava, H. M., *On certain subclass of univalent functions by some integral operators*, Math. Japon., **50** (1999), 359-370.

[8] Goel, R. M., Sohi, N. S., *A new criterion for p-valent functions*, Proc. Amer. Math. Soc., **78** (1980), 353-357.

[9] Hallenbeck, D. J., Ruscheweyh, S., *Subordination by convex functions*, Proc. Amer. Math. Soc., **52** (1975), 191-195.

[10] Hohlov, Yu. E., *Operators and operations in the class of univalent functions*, (in Russian), Izv. Vysšh. Učebn. Zaved. Mat., **10** (1978), 83-89.

[11] Janowski, Z., *Some extremal problems for certain families of analytic functions*, Ann. Polon. Math., **28** (1973), 297-326.

[12] Juneja, O. P., Mogra, M. L., *A class of univalent functions,* Bull. Sci. Math., **103** (1979), 435-447.

[13] Keogh, F. R., Merkes, E. P., *A coefficient inequality for certain classes of analytic functions*, Proc. Amer. Math. Soc., **20** (1969), 8-12.

[14] Kumar, V., *On a new criterion for univalent functions,* Demonstratio Math., **17** (1984), no. 4, 875-886.

[15] Liu, J.-L., Noor, K. I., *Some properties of Noor integral operator*, J. Natur. Geom., **21** (2002), 81-90.

[16] Mogra, M. L., *On a class of univalent functions whose derivatives have a positive real part*, Riv. Mat. Univ. Parma, **7** (1981), 163-172.

[17] Nehari, Z., *Conformal Mapping*, McGraw Hill, New York 1952.

[18] Owa, S., *On the distortion theorems. I*, Kyungpook Math. J., **18** (1978), 53-59.

[19] Owa, S., Srivastava, H. M., *Univalent and starlike generalized hypergeometric functions*, Canad. J. Math., **39** (1987), 1057-1077.

[20] Patel, J., Mishra, A. K., *On certain subclasses of multivalent functions associated with an extended fractional differintegral operator*, J. Math. Anal. Appl., **332** (2007), 109-122.

[21] Patel, J., Sahoo, P., *Properties of a certain class of multivalent analytic functions*, Comput. Math. Appl., **46** (2003), 1633-1644.

[22] Ruscheweyh, S., *New criteria for univalent functions*, Proc. Amer. Math. Soc., **49** (1975), 109-115.

[23] Saitoh, H., *Properties of certain analytic functions*, Proc. Japan Acad., **65** (1989), 131-134.

[24] Saitoh, H., *A linear operator and its applications of first order differential subordinations*, Math. Japon., **44** (1996), 31-38.

[25] Singh, R., Singh, S., *Convolution properties of a class of starlike functions*, Proc. Amer. Math. Soc., **106** (1989), 145-152.

[26] Sohi, N. S., *A class of p-valent analytic functions*, Indian J. Pure Appl. Math., **10** (1979), no. 7, 826-834.

[27] Srivastava, H. M., Aouf, M. K., *A certain fractional derivative operator and its applications to a new class of analytic and multivalent functions with negative coefficients. I*, J. Math. Anal. Appl., **171** (1992) 1-13; *II*, J. Math. Anal. Appl., **192** (1995), 673-688.

[28] Srivastava, H. M., Karlsson, P. W., *Multiple Gaussian Hypergeometric Series*, Halsted Press (Ellis Horwood, Chichester), John Wiley and Sons, New York, 1985.

[29] Whittaker, E. T., Watson, G. N., *A course on modern analysis : An introduction to the general theory of infinite processes and of analytic functions; With an account of the principal transcendental functions*, Fourth edition (reprinted), Cambridge University Press, Cambridge, 1927.

Faculty of Science, Mansoura University
Mansoura, 35516, Egypt
E-mail address: mkaouf127@yahoo.com

Faculty of Science, Mansoura University
Mansoura, 35516, Egypt
E-mail address: shamandy16@hotmail.com

Faculty of Science, Mansoura University
Mansoura, 35516, Egypt
E-mail address: adelaeg254@yahoo.com

Faculty of Science, Mansoura University
Mansoura, 35516, Egypt
E-mail address: fatma_elemam@yahoo.com

STUDIA UNIV. "BABEŞ–BOLYAI", MATHEMATICA, Volume **LV**, Number 3, September 2010

ON ORDER OF CONVOLUTION CONSISTENCE OF THE ANALYTIC FUNCTIONS

URSZULA BEDNARZ AND JANUSZ SOKÓŁ

Dedicated to Professor Grigore Ştefan Sălăgean on his 60^{th} *birthday*

Abstract. In this paper we consider the convolution of certain classes of analytic functions. We discuss when it is in a given class. By means of the Sălăgean integral operator we define a constant S which describes a measure of convolution consistence of three classes. We shall examine some special families for which we can determine the order of convolution consistence.

1. **Introduction**

Let $\mathcal{H}$ denote the class of analytic functions in the unit disc $\mathcal{U} = \{z : \ |z| < 1\}$ on the complex plane $\mathbb{C}$. Let $\mathcal{A}$ denote the subclass of $\mathcal{H}$ consisting of functions normalized by $f(0) = 0$, $f'(0) = 1$ and let $\mathcal{S} \subset \mathcal{A}$ denote the class of functions univalent in $\mathcal{U}$. Everywhere in this paper $z \in \mathcal{U}$ unless we make a note. A function f maps $\mathcal{U}$ onto a starlike domain with respect to $w_0 = 0$ if and only if

$$\mathfrak{Re}\left[\frac{zf'(z)}{f(z)}\right] > 0 \quad (z \in \mathcal{U}). \tag{1.1}$$

It is well known that if an analytic function f satisfies (1.1) and $f(0) = 0$, $f'(0) \neq 0$, then f is univalent and starlike in $\mathcal{U}$.

A set E is said to be convex if and only if it is starlike with respect to each of its points, that is if and only if the linear segment joining any two points of E lies

Received by the editors: 25.04.2010.

2000 *Mathematics Subject Classification.* 30C45, 30C50, 30C55.

Key words and phrases. Hadamard product, integral convolution, k-starlike functions, k-uniformly convex functions, order of convolution consistence, Sălăgean integral operator, starlike functions, convex functions, functions with positive real part.

entirely in E. Let f be analytic and univalent in $\mathcal{U}$. Then f maps $\mathcal{U}$ onto a convex domain E if and only if

$$\mathfrak{Re}\left[1+\frac{zf''(z)}{f'(z)}\right]>0 \quad (z\in\mathcal{U}). \tag{1.2}$$

Such a function f is said to be convex in $\mathcal{U}$ (or briefly convex). The set of all functions $f\in\mathcal{A}$ that are starlike univalent in $\mathcal{U}$ will be denoted by $\mathcal{ST}$. The set of all functions $f\in\mathcal{A}$ that are convex univalent in $\mathcal{U}$ by $\mathcal{CV}$. Recall that the Hadamard product or convolution of two power series

$$f(z)=z+\sum_{n=2}^{\infty}a_nz^n \quad\text{and}\quad g(z)=z+\sum_{n=2}^{\infty}b_nz^n,$$

is defined as

$$(f*g)(z)=z+\sum_{n=2}^{\infty}a_nb_nz^n$$

and the integral convolution is defined by

$$(f\otimes g)(z)=z+\sum_{n=2}^{\infty}\frac{a_nb_n}{n}z^n.$$

It is well known [10] that if $f,g\in\mathcal{CV}$, then $f*g\in\mathcal{CV}$ while if $f,g\in\mathcal{ST}$, then $f*g$ may not be in $\mathcal{ST}$ and even may fail to be univalent. To examine deeply this problem let us consider the Sălăgean integral operator (see [12]) $\mathcal{I}^s:\mathcal{A}\to\mathcal{A}$, $s\in\mathbb{R}$, such that

$$\mathcal{I}^sf(z)=\mathcal{I}^s\left(\sum_{n=1}^{\infty}a_nz^n\right)=\sum_{n=1}^{\infty}\frac{a_n}{n^s}z^n.$$

Now, one can ask if exists there a number $s\in\mathbb{R}$ such that

$$\mathcal{I}^s(f*g)\in\mathcal{ST}\quad \forall f,g\in\mathcal{ST}.$$

The answer there is in Theorem 2.1 below. This problem may be consider more generally for other classes of functions when the Sălăgean integral operator is defined on $\mathcal{H}$ as follows

$$\mathcal{I}^s\left(a_0+\sum_{n=1}^{\infty}a_nz^n\right)=a_0+\sum_{n=1}^{\infty}\frac{a_n}{n^s}z^n.$$

Definition 1.1. Let $\mathcal{X}$, $\mathcal{Y}$ and $\mathcal{Z}$ be subsets of $\mathcal{H}$. We say that the three $(\mathcal{X},\mathcal{Y},\mathcal{Z})$ is S–closed under convolution if there exists a number $S = S(\mathcal{X},\mathcal{Y},\mathcal{Z})$ such that

$$\begin{aligned} S(\mathcal{X},\mathcal{Y},\mathcal{Z}) &= \min\{s \in \mathbb{R} : \mathcal{I}^s(f * g) \in \mathcal{Z} \quad \forall f \in \mathcal{X} \ \ \forall g \in \mathcal{Y}\} \qquad (1.3) \\ &= \min\{s \in \mathbb{R} : \mathcal{I}^s(\mathcal{X} * \mathcal{Y}) \subseteq \mathcal{Z}\}, \end{aligned}$$

where $\mathcal{I}^s$ denote the Sălăgean integral operator. The number $S(\mathcal{X},\mathcal{Y},\mathcal{Z})$ is called the order of convolution consistence the three $(\mathcal{X},\mathcal{Y},\mathcal{Z})$. It would be called the Sălăgean number.

2. Main results

We shall examine some special families for which we can determine the order of convolution consistence. First we shall restrict our attention to the classes of starlike and convex functions.

Theorem 2.1. *The order of convolution consistence of the class $\mathcal{ST}$ is equal to 1:*

$$S(\mathcal{ST},\mathcal{ST},\mathcal{ST}) = 1. \qquad (2.1)$$

Proof. It is well known [10] that $\mathcal{ST} \otimes \mathcal{ST} = \mathcal{ST}$ and $\mathcal{I}^1(f * g) = f \otimes g$. Thus if $f, g \in \mathcal{ST}$, then $\mathcal{I}^1(f * g) \in \mathcal{ST}$. This means that $S(\mathcal{ST},\mathcal{ST},\mathcal{ST}) \leq 1$. If we consider the functions $f, g \in \mathcal{ST}$ such that

$$f(z) = g(z) = \frac{z}{(1-z)^2} \quad (z \in \mathcal{U}),$$

then

$$\mathcal{I}^s(f * g) = \sum_{n=1}^{\infty} n^{2-s} z^n.$$

The coefficients of the functions in the class $\mathcal{ST}$ cannot be greater than n. If we want that $n^{2-s} \leq n$, then $s \geq 1$. Therefore we deduce that $S(\mathcal{ST},\mathcal{ST},\mathcal{ST}) = 1$. □

Theorem 2.2. *We have the following orders of convolution consistence*

(i) $S(\mathcal{CV},\mathcal{CV},\mathcal{ST}) = -1$,

(ii) $S(\mathcal{CV},\mathcal{ST},\mathcal{ST}) = 0$,

(iii) $S(\mathcal{ST},\mathcal{ST},\mathcal{CV}) = 2$,

(iv) $S(\mathcal{CV},\mathcal{CV},\mathcal{CV}) = 0$,

(v) $S(\mathcal{CV},\mathcal{ST},\mathcal{CV}) = 1$.

Proof. (i) It is well known [10] that $\mathcal{CV} * \mathcal{ST} = \mathcal{ST}$. Let $f, g \in \mathcal{CV}$. Then $zg' \in \mathcal{ST}$ and $\mathcal{I}^{-1}(f * g)(z) = f(z) * (zg'(z)) \in \mathcal{ST}$, so $S(\mathcal{CV}, \mathcal{CV}, \mathcal{ST}) \leq -1$. If

$$f(z) = g(z) = \frac{z}{1-z} \in \mathcal{CV},$$

then

$$\mathcal{I}^s(f * g) = \sum_{n=1}^{\infty} n^{-s} z^n.$$

Because the coefficients of the functions in the class $\mathcal{ST}$ cannot be greater than n we obtain the condition $n^{-s} \leq n$. Therefore we deduce that $s \geq -1$ and then $S(\mathcal{CV}, \mathcal{CV}, \mathcal{ST}) = -1$.

The proofs of (ii) – (v) run as the proof of (i). □

To find the order of convolution consistence of other classes let us recall the classes of k-uniformly convex and of k-starlike functions:

$$k\text{-}\mathcal{UCV} := \left\{ f \in \mathcal{S} : \mathfrak{Re}\left[1 + \frac{zf''(z)}{f'(z)}\right] > k \left|\frac{zf''(z)}{f'(z)}\right|, \quad (z \in \mathcal{U};\ 0 \leq k < \infty) \right\},$$

$$k\text{-}\mathcal{ST} := \left\{ f \in \mathcal{S} : \mathfrak{Re}\left[\frac{zf'(z)}{f(z)}\right] > k \left|\frac{zf'(z)}{f(z)} - 1\right|, \quad (z \in \mathcal{U};\ 0 \leq k < \infty) \right\}.$$

The class k-$\mathcal{UCV}$ was introduced by Kanas and Wiśniowska [5], where its geometric definition and connections with the conic domains were considered. The class k-$\mathcal{UCV}$ was defined pure geometrically as a subclass of univalent functions, that map each circular arc contained in the unit disk $\mathcal{U}$ with a center $\xi, |\xi| \leq k$ $(0 \leq k < \infty)$, onto a convex arc. The notion of k-uniformly convex function is a natural extension of the classical convexity. Observe that, if $k = 0$ then the center ξ is the origin and the class k-$\mathcal{UCV}$ reduces to the class of convex univalent functions $\mathcal{CV}$. Moreover for $k = 1$ corresponds to the class of uniformly convex functions $\mathcal{UCV}$ introduced by Goodman [2] and studied extensively by Rønning [9] and independently by Ma and Minda [8]. The class k-$\mathcal{ST}$ is related to the class k-$\mathcal{UCV}$ by means of the well-known Alexander equivalence between the usual classes of convex $\mathcal{CV}$ and starlike $\mathcal{ST}$ functions (see also the works [4, 6, 7, 8, 9] for further developments involving each of the classes k-$\mathcal{UCV}$ and k-$\mathcal{ST}$). Moreover, in [1] the authors studied the properties of the integral convolution of the neighborhoods of these classes. To start examine the order of convolution consistence connected with the classes k-$\mathcal{UCV}$ and k-$\mathcal{ST}$ we need recall some basic results about these classes. Let us denote (see [4])

$$P_1(k) = \begin{cases} \frac{8(\arccos k)^2}{\pi^2(1-k^2)} & \text{for} \quad 0 \le k < 1 \\ \frac{8}{\pi^2} & \text{for} \quad k = 1 \\ \frac{\pi^2}{4\sqrt{t}(1+t)(k^2-1)\mathcal{K}^2(t)} & \text{for} \quad k > 1 \end{cases}, \tag{2.2}$$

where $t \in (0,1)$ is determined by $k = \cosh(\pi\mathcal{K}'(t)/[4\mathcal{K}(t)])$, $\mathcal{K}$ is the Legendre's complete Elliptic integral of the first kind

$$\mathcal{K}(t) = \int_0^1 \frac{\mathrm{d}x}{\sqrt{(1-x^2)(1-t^2x^2)}}$$

and $\mathcal{K}'(t) = \mathcal{K}(\sqrt{1-t^2})$ is the complementary integral of $\mathcal{K}(t)$. Let Ω_k be a domain such that $1 \in \Omega_k$ and

$$\partial\Omega_k = \left\{w = u + iv : \quad u^2 = k^2(u-1)^2 + k^2v^2\right\}, \quad 0 \le k < \infty.$$

The domain Ω_k is elliptic for $k > 1$, hyperbolic when $0 < k < 1$, parabolic when $k = 1$, and a right half-plane when $k = 0$. If $\widetilde{p}_\alpha$ is an analytic function with $\widetilde{p}_\alpha(0) = 1$ which maps the unit disc $\mathcal{U}$ conformally onto the region Ω_k, then $P_1(k) = \widetilde{p}_\alpha'(0)$. $P_1(k)$ is strictly decreasing function of the variable k and its values are included in the interval $(0,2]$.

Lemma 2.3. (see [4]) *Let $0 \le k < \infty$ and let $f \in k$-$\mathcal{ST}$ be of the form*

$$f(z) = z + \sum_{n=2}^{\infty} a_n(k)z^n \quad (|z| < 1),$$

then

$$|a_n(k)| \le \frac{(P_1(k))_{(n-1)}}{(n-1)!}, \quad n = 2, 3, \dots ,$$

where $(\lambda)_n$ is the Pochhammer symbol defined by

$$(\lambda)_n = \begin{cases} 1 & (n = 0) \\ \lambda(\lambda+1)\cdot \ldots \cdot(\lambda+n-1) & (n \in \mathbf{N}). \end{cases}$$

For $k = 0$ the estimates are sharp; otherwise only the bound on $|a_2(k)|$ is sharp.

Lemma 2.4. (see [4]) *Let* $0 \le k < \infty$ *and let* $f \in k\text{-}\mathcal{UCV}$ *be of the form*

$$f(z) = z + \sum_{n=2}^{\infty} a_n(k) z^n \quad (|z| < 1),$$

then

$$|a_n(k)| \le \frac{(P_1(k))_{(n-1)}}{n!}, \quad n = 2, 3, \dots ,$$

where $P_1(k)$ *is given in (2.2). For* $k = 0$ *the estimates are sharp; otherwise only the bound on* $|a_2(k)|$ *is sharp.*

Theorem 2.5. *The following inequalities hold true*

(i) $\log_2 P_1(k) \le S(k\text{-}\mathcal{ST}, k\text{-}\mathcal{ST}, k\text{-}\mathcal{ST}) \le 1$,
(ii) $1 + \log_2 P_1(k) \le S(k\text{-}\mathcal{ST}, k\text{-}\mathcal{ST}, k\text{-}\mathcal{UCV}) \le 2$,
(iii) $S(k\text{-}\mathcal{ST}, \mathcal{CV}, k\text{-}\mathcal{UCV}) = 1$,
(iv) $S(k\text{-}\mathcal{ST}, \mathcal{CV}, k\text{-}\mathcal{ST}) = 0$,
(v) $S(k\text{-}\mathcal{UCV}, \mathcal{CV}, k\text{-}\mathcal{UCV}) = 0$,

whenever there exist the above orders of convolution consistence.

Proof. (i) In [4] it was proved that if $f, g \in k\text{-}\mathcal{ST}$ then $f \otimes g \in k\text{-}\mathcal{ST}$ so $\mathcal{I}^1(f * g) = f \otimes g \in k\text{-}\mathcal{ST}$. Therefore $S(k\text{-}\mathcal{ST}, k\text{-}\mathcal{ST}, k\text{-}\mathcal{ST}) \le 1$, whenever it there exist. Suppose that

$$f(z) = g(z) = z \exp \int_0^z \frac{\widetilde{p}_\alpha(t) - 1}{t}\, \mathrm{d}t = z + P_1(k) z^2 + \cdots , \tag{2.3}$$

where $P_1(k)$ is given in (2.2). Then $f, g \in k\text{-}\mathcal{ST}$ and by Lemma 2.3 for the second coefficient we have

$$\mathcal{I}^s(f * g) \in k\text{-}\mathcal{ST} \Rightarrow \frac{P_1(k) P_1(k)}{2^s} \le P_1(k) \Leftrightarrow P_1(k) \le 2^s.$$

Therefore we deduce that $S(k\text{-}\mathcal{ST}, k\text{-}\mathcal{ST}, k\text{-}\mathcal{ST}) \ge \log_2 P_1(k)$. Notice that $P_1(k)$ is strictly decreasing function of the variable k and its values are included in the interval $(0, 2]$.

(ii) This proof runs as the previous proof.

(iii) Let $f \in k\text{-}\mathcal{ST}$ and $g \in \mathcal{CV}$. Then [4] $f \otimes g \in k\text{-}\mathcal{UCV}$ so $\mathcal{I}^1(f * g) \in k\text{-}\mathcal{UCV}$, hence $S(\mathcal{CV}, \mathcal{CV}, \mathcal{ST}) \le 1$. If f is given as in (2.3) and $g(z) = z/(1 - z) \in \mathcal{CV}$, then by Lemma 2.4

$$\mathcal{I}^s(f * g) \in k\text{-}\mathcal{UCV} \Rightarrow \frac{P_1(k)}{2^s} \le \frac{P_1(k)}{2} \Leftrightarrow s \ge 1.$$

Therefore $S(k\text{-}\mathcal{ST}, \mathcal{CV}, k\text{-}\mathcal{UCV}) = 1$

(iv), (v) Those proofs run as the previous proof. $\square$

Lemma 2.6. (see [11]) *Let F and G be in $\mathcal{CV}$. Then*

$$f \prec F \quad and \quad g \prec G \quad \Rightarrow \quad f * g \prec F * G. \tag{2.4}$$

Let us consider for $\alpha < 1$ the class of functions:

$$\mathcal{P}(\alpha) = \{p : zp(z) \in \mathcal{A} \text{ and } \mathfrak{Re}\,[p(z)] > \alpha \text{ for } z \in \mathcal{U}\}.$$

Lemma 2.7. *If $h \in \mathcal{P}(\alpha)$ and $h(z) = 1 + a_1 z + a_2 z^2 + \cdots$, then the function*

$$H(z) = 1 + \sum_{n=1}^{\infty} \frac{a_n}{n} z^n \quad (z \in \mathcal{U}) \tag{2.5}$$

satisfies

$$H(z) \prec 1 - 2(1-\alpha)\log(1-z) \quad (z \in \mathcal{U}) \tag{2.6}$$

and belongs to the class $\mathcal{P}(1 + 2(\alpha - 1)\log 2)$.

Proof. It is well known that the function

$$g(z) = -\log(1-z) = \sum_{n=1}^{\infty} \frac{z^n}{n} \quad (z \in \mathcal{U})$$

belongs to the class $\mathcal{CV}$ of convex univalent functions so $g(z) + 1$ is convex univalent too. Thus as in (2.4) we have

$$\begin{cases} h(z) \prec \frac{1+(1-2\alpha)z}{1-z} \\ g(z) + 1 \prec g(z) + 1 \end{cases} \Rightarrow \quad h(z) * (g(z)+1) \prec \frac{1+(1-2\alpha)z}{1-z} * (g(z)+1),$$

Therefore we can write

$$\begin{aligned} h(z) * (g(z)+1) &= 1 + \sum_{n=1}^{\infty} \frac{a_n}{n} z^n \\ &\prec \frac{1+(1-2\alpha)z}{1-z} * (1 - \log(1-z)) \\ &= \left[1 + 2(1-\alpha)(z + z^2 + \cdots)\right] * (1 - \log(1-z)) \\ &= 1 - 2(1-\alpha)\log(1-z). \end{aligned} \tag{2.7}$$

The function

$$H(z) = \frac{1+(1-2\alpha)z}{1-z} * (1 - \log(1-z)) = 1 - 2(1-\alpha)\log(1-z) \quad (z \in \mathcal{U})$$

is convex univalent as a convolution of convex univalent functions and is typically–real so the geometric properties of the image of $H(\mathcal{U})$ show that

$$\min\{\mathfrak{Re} H(z) : |z| < 1\} = H(-1) = 1 + 2(\alpha - 1)\log 2.$$

Therefore from (2.7) we obtain that $H \in \mathcal{P}(1 + 2(\alpha - 1)\log 2)$. □

Lemma 2.8. [13] *If $a \leq 1$, $b \leq 1$, and $f \in \mathcal{P}(a)$, $g \in \mathcal{P}(b)$ for $z \in \mathcal{U}$, then*

$$\mathfrak{Re}[(f * g)(z)] > c \quad for\ z \in \mathcal{U},$$

were $c = 1 - 2(1 - a)(1 - b)$.

Theorem 2.9. *If $S(\mathcal{P}(\alpha), \mathcal{P}(\alpha), \mathcal{P}(\delta))$ there exists, then*

(i) $\mu \leq S(\mathcal{P}(\alpha), \mathcal{P}(\alpha), \mathcal{P}(\delta)) \leq 1$,
where $\delta = 1 - 4(1 - \alpha)^2 \log 2$, $\mu = -\frac{\log(2\log 2)}{\log 2} = -0.732\ldots$,

(ii) $S(\mathcal{P}(\alpha), \mathcal{P}(\beta), \mathcal{P}(\gamma)) = 0$,
where $\gamma = 1 - 2(1 - \alpha)(1 - \beta)$,

(iii) $1 + \log_2 \frac{(1-\alpha)(1-\beta)}{1-\gamma} \leq S(\mathcal{P}(\alpha), \mathcal{P}(\beta), \mathcal{P}(\gamma)) \leq 0$,
where $\gamma < 1 - 2(1 - \alpha)(1 - \beta)$.

Proof. (i) Let $g \in \mathcal{P}(\alpha)$ and let

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n.$$

Let h, H be given as in Lemma 2.7. Therefore we have $H \in \mathcal{P}(\gamma)$, where

$$\gamma = 1 + 2(\alpha - 1)\log 2.$$

Further, by Lemma 2.8 we have

$$\begin{aligned} \mathcal{I}^1(g * h)(z) &= 1 + \sum_{n=1}^{\infty} \frac{a_n b_n}{n} z^n = g(z) * H(z) \\ &\in \mathcal{P}(1 - 2(1 - \alpha)(1 - \gamma)) \\ &= \mathcal{P}(1 - 4(1 - \alpha)^2 \log 2), \end{aligned} \tag{2.8}$$

so $S(\mathcal{P}(\alpha), \mathcal{P}(\alpha), \mathcal{P}(\delta)) \leq 1$. Suppose that

$$h(z) = g(z) = 1 + 2(1 - \alpha)\sum_{n=1}^{\infty} z^n \in \mathcal{P}(\alpha).$$

It is known that if $1 + a_1 z + \cdots \in \mathcal{P}(\delta)$, then $|a_n| \le 2(1-\delta)$. Therefore, examining the second coefficients we get

$$\mathcal{I}^s(g*h) \in \mathcal{P}(\delta) \Rightarrow \frac{4(1-\alpha)^2}{2^s} \le 2(4(1-\alpha)^2 \log 2) \Leftrightarrow \frac{1}{2\log 2} \le 2^s \Leftrightarrow s > \log_2 \frac{1}{2\log 2}$$

and we can see that $S(\mathcal{P}(\alpha), \mathcal{P}(\alpha), \mathcal{P}(\delta)) \ge \mu$, where $\mu = -\frac{\log(2\log 2)}{\log 2} = -0.732\ldots$.

For the proof of (ii) notice that by Lemma 2.8 if $f \in \mathcal{P}(\alpha)$ and $g \in \mathcal{P}(\beta)$, then $\mathcal{I}^0(f*g) \in \mathcal{P}(\gamma))$. This means that $S(\mathcal{P}(\alpha), \mathcal{P}(\beta), \mathcal{P}(\gamma)) \le 0$. If

$$\begin{aligned} f(z) &= 1 + 2(1-\alpha)\sum_{n=1}^{\infty} z^n \in \mathcal{P}(\alpha) \\ g(z) &= 1 + 2(1-\beta)\sum_{n=1}^{\infty} z^n \in \mathcal{P}(\beta), \end{aligned} \tag{2.9}$$

then for the second coefficient we have

$$\mathcal{I}^s(f*g) \in \mathcal{P}(\gamma) \Rightarrow \frac{4(1-\alpha)(1-\beta)}{2^s} \le 2(1-\gamma) \Leftrightarrow 2^s \ge 1.$$

Therefore we deduce that $S(\mathcal{P}(\alpha), \mathcal{P}(\beta), \mathcal{P}(\gamma)) = 0$.

In order to prove (iii) notice that by Lemma 2.8 if $f \in \mathcal{P}(\alpha)$ and $g \in \mathcal{P}(\beta)$, then

$$\mathcal{I}^0(f*g) \in \mathcal{P}(1-2(1-\alpha)(1-\beta)) \subseteq \mathcal{P}(\gamma).$$

This means that $S(\mathcal{P}(\alpha), \mathcal{P}(\beta), \mathcal{P}(\gamma)) \le 0$. If $f \in \mathcal{P}(\alpha)$ and $g \in \mathcal{P}(\beta)$ are given as in (2.9), then for the second coefficient we have

$$\mathcal{I}^s(f*g) \in \mathcal{P}(\gamma) \Rightarrow \frac{4(1-\alpha)(1-\beta)}{2^s} \le 2(1-\gamma) \Leftrightarrow 2^{s-1} \ge \frac{(1-\alpha)(1-\beta)}{1-\gamma}.$$

Thus we see that

$$1 + \log_2 \frac{(1-\alpha)(1-\beta)}{1-\gamma} \le S(\mathcal{P}(\alpha), \mathcal{P}(\beta), \mathcal{P}(\gamma)).$$

Note that if $\gamma < 1 - 2(1-\alpha)(1-\beta)$, then

$$1 + \log_2 \frac{(1-\alpha)(1-\beta)}{1-\gamma} < 0.$$

$\square$

References

[1] Bednarz, U., Kanas, S., *Stability of the integral convolution of k-uniformly convex and k-starlike functions*, Journal of Appl. Analisis, **10** (2004), no. ,1 105-115.

[2] Goodman, A. W., *On uniformly convex functions, Ann. Polon. Math.*, **56** (1991), 87-92.

[3] Kanas, S., *Stability of convolution and dual sets for the class of k-uniformly convex and k-starlike functions*, Folia Sci. Univ. Tech. Resov., **22** (1998), 51-64.

[4] Kanas, S., Wiśniowska, A., *Conic regions and k-uniform convexity II*, Folia Sci. Univ. Tech. Resov., **22**(1998), 65-78.

[5] Kanas, S., Wiśniowska, A., *Conic regions and k-uniform convexity*, J. Comput. Appl. Math., **105** (1999), 327-336.

[6] Kanas, S., Wiśniowska, A., *Conic regions and k-starlike functions*, Rev. Roumaine Math. Pures Appl., **45** (2000), 647-657.

[7] Kanas, S., Srivastava, H. M., *Linear operators associated with k-uniformly convex functions*, Integral Transform. Spec. Funct., **9** (2000), no. 2, 121-132.

[8] Ma, W., Minda, D., *Uniformly convex functions*, Ann. Polon. Math., **57** (1992), no. 2, 165-175.

[9] Rønning, F., *Uniformly convex functions and a corresponding class of starlike functions*, Proc. Amer. Math. Soc., **118** (1993), 189-196.

[10] Ruscheweyh, S., Sheil-Small, T., *Hadamard product of schlicht functions and the Poyla-Schoenberg conjecture*, Comm. Math. Helv., **48** (1973), 119-135.

[11] Ruscheweyh, S., Stankiewicz, J., *Subordination under convex univalent function*, Bull. Pol. Acad. Sci. Math., **33** (1985), 499-502.

[12] Sălăgean, G. S., *Subclasses of univalent functions.* Complex Analysis, Fifth Romanian Finnish Seminar, Part 1 (Bucharest, 1981), Lecture Notes in Math., 1013, Springer, Berlin, 1983, 362-372.

[13] Stankiewicz, J., Stankiewicz, Z., *Some Applications of the Hadamard Convolution in the Thory of Functions*, Ann. Univ. Mariae Curie-Skłodowska, **40** (1986), 251-265.

Department of Mathematics
Rzeszów University of Technology
ul. W. Pola 2, 35-959 Rzeszów, Poland
E-mail address: `ubednarz@prz.edu.pl`

Department of Mathematics
Rzeszów University of Technology
ul. W. Pola 2, 35-959 Rzeszów, Poland
E-mail address: `jsokol@prz.edu.pl`

STUDIA UNIV. "BABEŞ–BOLYAI", MATHEMATICA, Volume LV, Number 3, September 2010

BERNSTEIN TYPE OPERATORS ON A SQUARE WITH ONE AND TWO CURVED SIDES

PETRU BLAGA, TEODORA CĂTINAŞ AND GHEORGHE COMAN

Dedicated to Professor Grigore Ştefan Sălăgean on his 60^{th} *birthday*

Abstract. We construct some Bernstein-type operators on a square with one and two curved sides, their product and Boolean sum. We study their interpolation properties, the orders of accuracy and the remainders of the corresponding approximation formulas. Finally, we give some numerical examples.

1. Introduction

Approximation operators on polygonal domains with some curved sides have important applications especially in finite element method for differential equations with given boundary conditions and in computer aided geometric design. Such operators were considered in the papers [14], [15], [13], [2], [5], [19]. Lately, such problems were studied in [11], [12] using interpolation operators, and in the papers [7], [8], [9], using Bernstein-type operators.

The aim of this paper is to introduce some Bernstein-type operators on a square with one curved side and, respectively, with two curved sides. We study three main aspects of the constructed operators: the interpolation properties, the orders of accuracy and the remainders of the corresponding approximation formulas.

Using the interpolation properties of such operators, it can be constructed blending function interpolants, which exactly matches function on some sides of a

Received by the editors: 04.05.2010.

2000 *Mathematics Subject Classification.* 41A05, 41A25, 41A80.

Key words and phrases. triangle, curve side, interpolation operators, remainders.

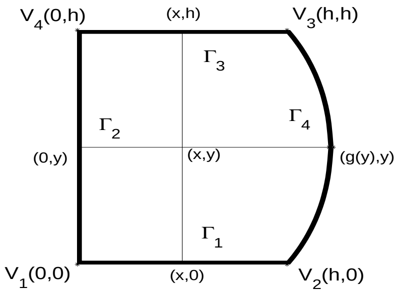


FIGURE 1. The square D_h.

rectangular region. Important applications of these blending functions are in finite element method for differential equations problems with Dirichlet boundary conditions or for construction of surfaces which satisfy some given conditions.

2. Bernstein type operators on a square with one curved side

Let D_h be the square with one curved side having the vertices $V_1 = (0, 0)$, $V_2 = (h, 0)$, $V_3 = (h, h)$ and $V_4 = (0, h)$, three straight sides Γ_1, Γ_2, along the coordinate axes and Γ_3 parallel to axis Ox, and the curved side Γ_4 which is defined by the function g, such that $g(h) = g(0) = h$ (See Figure 1).

2.1. **Univariate operators.** Let F be a real-valued function defined on D_h and $(0, y)$, $(g(y), y)$, respectively, $(x, 0)$, (x, h) be the points in which the parallel lines to the coordinate axes, passing through the point $(x, y) \in D_h$, intersect the sides Γ_2, Γ_4, respectively Γ_1 and Γ_3. We consider the uniform partitions of the intervals $[0, g(y)]$ and $[0, h]$, $y \in [0, h]$:

$$\Delta_m^x = \left\{ \tfrac{i}{m} g(y) \middle| \ i = \overline{0, m} \right\}$$

and

$$\Delta_n^y = \left\{ \tfrac{j}{n} h \middle| \ j = \overline{0, n} \right\}$$

and the Bernstein-type operators B_m^x and B_n^y defined by

$$(B_m^x F)(x, y) = \sum_{i=0}^{m} p_{m,i}(x, y) F\left(\tfrac{i}{m} g(y), y\right), \tag{2.1}$$

with

$$p_{m,i}(x,y) = \binom{m}{i} \left[\frac{x}{g(y)}\right]^i \left[1 - \frac{x}{g(y)}\right]^{m-i}, \tag{2.2}$$

respectively,

$$(B_n^y F)(x,y) = \sum_{j=0}^{n} q_{n,j}(x,y) F\left(x, \tfrac{j}{n}h\right) \tag{2.3}$$

with

$$q_{n,j}(x,y) = \binom{n}{j} \left(\tfrac{y}{h}\right)^j \left(1 - \tfrac{y}{h}\right)^{n-j}.$$

Remark 2.1. In Figures 2 and 3 we plot the points $(\frac{i}{m}g(y), y)$, $i = \overline{0,m}$ and respectively, $(x, \frac{j}{n}h)$, $j = \overline{0,n}$, $x, y \in [0,h]$, for $m = 5$ and $n = 6$.

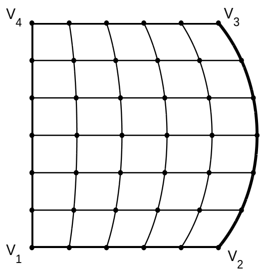


Figure 2. Points $(\frac{i}{m}g(y), y)$, $i = \overline{0,m}$.

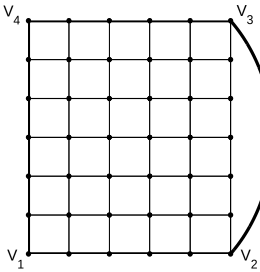


Figure 3. Points $(x, \frac{j}{n}h)$, $j = \overline{0,n}$.

Theorem 2.2. *If F is a real-valued function defined on D_h then:*

(i) $B_m^x F = F$ *on* $\Gamma_2 \cup \Gamma_4$,

(ii) $B_n^y F = F$ *on* $\Gamma_1 \cup \Gamma_3$,

and

(iii) $(B_m^x e_{ij})(x,y) = x^i y^j$, $\quad i = 0,1;\ j \in \mathbb{N}$;

$$(B_m^x e_{2j})(x,y) = \left\{x^2 + \tfrac{x[g(y)-x]}{m}\right\} y^j, \quad j \in \mathbb{N};$$

(iv) $(B_n^y e_{ij})(x,y) = x^i y^j$, $\quad i \in \mathbb{N};\ j = 0,1$;

$$(B_n^y e_{i2})(x,y) = x^i \left[y^2 + \tfrac{y(h-y)}{n}\right], \quad i \in \mathbb{N}.$$

Proof. The interpolation properties (i) and (ii) follow by the relations:

$$p_{m,i}(0,y) = \begin{cases} 1, & \text{for } i = 0, \\ 0, & \text{for } i > 0, \end{cases}$$

and

$$p_{m,i}\left(g(y),y\right)=\begin{cases}0, \text{ for } i<m,\\ 1, \text{ for } i=m,\end{cases}$$

respectively, by

$$q_{n,j}(x,0))=\begin{cases}1, \text{ for } j=0,\\ 0, \text{ for } j>0,\end{cases}$$

and

$$q_{n,j}(x,h)=\begin{cases}0, \text{ for } j<n,\\ 1, \text{ for } j=n.\end{cases}$$

Regarding the properties (iii), we get

$$\left(B_m^x e_{ij}\right)(x,y)=y^j(B_m^x e_{i0})(x,y), \qquad j\in\mathbb{N}$$

and

$$\left(B_m^x e_{00}\right)(x,y)=\sum_{i=0}^{m}p_{m,i}\left(x,y\right)=1,$$

$$\begin{aligned}B_m^x e_{10}(x,y)&=\sum_{i=0}^{m}p_{m,i}\left(x,y\right)\tfrac{i}{m}g(y)\\&=\sum_{i=0}^{m}\binom{m}{i}\left(\tfrac{x}{g(y)}\right)^i\left(1-\tfrac{x}{g(y)}\right)^{m-i}\tfrac{i}{m}g(y)\\&=x\sum_{i=0}^{m-1}\binom{m-1}{i}\left[\tfrac{x}{g(y)}\right]^i\left[1-\tfrac{x}{g(y)}\right]^{m-i-1}\\&=x\left[\frac{x}{g(y)}+1-\frac{x}{g(y)}\right]^{m-1}=x,\end{aligned}$$

$$\begin{aligned}B_m^x e_{20}(x,y)&=\sum_{i=0}^{m}p_{m,i}\left(x,y\right)\left[\tfrac{i}{m}g(y)\right]^2\\&=\sum_{i=0}^{m}\binom{m}{i}\left(\tfrac{x}{g(y)}\right)^i\left(1-\tfrac{x}{g(y)}\right)^{m-i}i^2\left[\tfrac{g(y)}{m}\right]^2\\&=\left[\tfrac{g(y)}{m}\right]^2\sum_{i=0}^{m}\binom{m}{i}i(i-1)\left[\tfrac{x}{g(y)}\right]^i\left[1-\tfrac{x}{g(y)}\right]^{m-i}+x\tfrac{g(y)}{m}\\&=\frac{m-1}{m}x^2+\frac{g(y)}{m}x=x^2+\frac{x[g(y)-x]}{m}\end{aligned}$$

Properties (iv) are proved in the same way. □

Remark 2.3. The interpolation properties of $B_m^x F$ and $B_n^y F$ are illustrated in Figures 4 and 5. The bold sides indicate the interpolation sets.

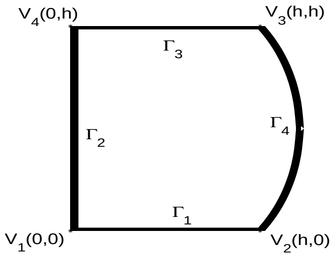


Figure 4. Interpolation domain for $B_m^x F$.

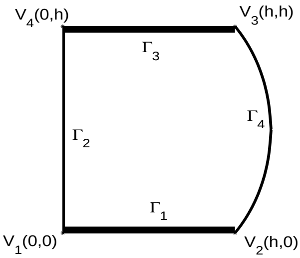


Figure 5. Interpolation domain for $B_y^n F$.

We consider the approximation formula

$$F = B_m^x F + R_m^x F.$$

Theorem 2.4. *If $F(\cdot, y) \in C[0, g(y)]$, $y \in [0, h]$, then*

$$\left| (R_m^x F)(x,y) \right| \leq \left[1 + \frac{g(y)}{2\delta\sqrt{m}} \right] \omega(F(\cdot, y); \delta), \quad y \in [0, h],$$

and

$$\left| (R_m^x F)(x,y) \right| \leq \left(1 + \frac{M}{2\delta\sqrt{m}} \right) \omega(F(\cdot, y); \delta), \quad y \in [0, h],$$

where $\omega(F(\cdot, y); \delta)$ is the modulus of continuity of the function F with regard to the variable x and

$$M = \max_{0 \leq y \leq h} |g(y)|. \tag{2.4}$$

Moreover, if $\delta = 1/\sqrt{m}$ then

$$\left| \left(R_m^x F\right)(x,y) \right| \leq \left(1 + \tfrac{M}{2}\right) \omega(F(\cdot, y); \tfrac{1}{\sqrt{m}}), \quad y \in [0, h]. \tag{2.5}$$

Proof. By $(B_m^x e_{00})(x, y) = 1$, it follows that

$$\left| \left(R_m^x F\right)(x,y) \right| \leq \sum_{i=0}^{m} p_{m,i}(x,y) \left| F(x,y) - F(i\tfrac{g(y)}{m}, y) \right|.$$

Using the inequality

$$\left|F(x,y)-F(i\tfrac{g(y)}{m},y)\right| \leq \left(\tfrac{1}{\delta}\left|x-i\tfrac{g(y)}{m}\right|+1\right)\omega(F(\cdot,y);\delta)$$

one obtains

$$\begin{aligned}|(R_m^x F)(x,y)| &\leq \sum_{i=0}^{m} p_{m,i}(x,y)\left(\tfrac{1}{\delta}\left|x-i\tfrac{g(y)}{m}\right|+1\right)\omega(F(\cdot,y);\delta)\\ &\leq \left[1+\frac{1}{\delta}\Big(\sum_{i=0}^{m} p_{m,i}(x,y)\big(x-i\tfrac{g(y)}{m}\big)^2\Big)^{1/2}\right]\omega(F(\cdot,y);\delta)\\ &= \left[1+\frac{1}{\delta}\sqrt{\tfrac{x(g(y)-x)}{m}}\right]\omega(F(\cdot,y);\delta).\end{aligned}$$

Since,

$$\max_{0\leq x\leq g(y)}\big[x(g(y)-x)\big]=\tfrac{g^2(y)}{4},$$

it follows that

$$\big|(R_m^x F)(x,y)\big| \leq \left[1+\tfrac{g(y)}{2\delta\sqrt{m}}\right]\omega(F(\cdot,y);\delta)\leq \left(1+\tfrac{M}{2\delta\sqrt{m}}\right)\omega(F(\cdot,y);\delta),$$

with M given in (2.4). For $\delta=\frac{1}{\sqrt{m}}$, one obtains (2.5). □

Theorem 2.5. *If $F(\cdot,y)\in C^2[0,g(y)]$ then*

$$(R_m^x F)(x,y)=\frac{x[x-g(y)]}{2m}F^{(2,0)}(\xi,y), \quad \textit{for } \xi\in[0,g(y)],$$

and

$$\big|(R_m^x F)(x,y)\big| \leq \frac{M^2}{8m}M_{20}F$$

where M is given in (2.4) and

$$M_{ij}F=\max_{D_h}\left|F^{(i,j)}(x,y)\right|.$$

Proof. Taking into account that the operator B_m^x reproduces the polynomials of first degree, i.e., $\mathrm{dex}(B_m^x)=1$, by Peano's theorem (see, e.g., [17]), it follows

$$(R_m^x F)(x,y)=\int_0^{g(y)} K_{20}(x,y;s)F^{(2,0)}(s,y)ds,$$

where

$$K_{20}(x,y;s)=(x-s)_+ - \sum_{i=0}^{m} p_{m,i}(x,y)\big(i\tfrac{g(y)}{m}-s\big)_+.$$

For a given $\nu \in \{1,...,m\}$ one denotes by $K_{20}^{\nu}(x,y;\cdot)$ the restriction of the kernel $K_{20}(x,y;\cdot)$ to the interval $\left[(\nu-1)\frac{g(y)}{m}, \nu\frac{g(y)}{m}\right]$, i.e.,

$$K_{20}^{\nu}(x,y;\nu) = (x-s)_{+} - \sum_{i=\nu}^{m} p_{m,i}(x,y)\left(i\frac{g(y)}{m} - s\right),$$

whence,

$$K_{20}^{\nu}(x,y;s) = \begin{cases} x - s - \sum\limits_{i=\nu}^{m} p_{m,i}(x,y)\left(i\frac{g(y)}{m} - s\right), & s < x \\ -\sum\limits_{i=\nu}^{m} p_{m,i}(x,y)\left(i\frac{g(y)}{m} - s\right), & s \geq x. \end{cases}$$

It follows that $K_{20}^{\nu}(x,y;s) \leq 0$, for $s \geq x$. For $s < x$ we have

$$K_{20}^{\nu}(x,y;s) = x - s - \sum_{i=0}^{m} p_{m,i}(x,y)\left[i\frac{g(y)}{m} - s\right] + \sum_{i=0}^{\nu-1} p_{m,i}(x,y)\left[i\frac{g(y)}{m} - s\right].$$

As,

$$\sum_{i=0}^{m} p_{m,i}(x,y)\left[i\frac{g(y)}{m} - s\right] = x - s,$$

it follows that

$$K_{20}^{\nu}(x,y;s) = \sum_{i=0}^{\nu-1} p_{m,i}(x,y)\left[i\frac{g(y)}{m} - s\right] \leq 0.$$

So, $K_{20}^{\nu}(x,y;\cdot) \leq 0$, for any $\nu \in \{1,...,m\}$, i.e., $K_{20}(x,y;s) \leq 0$, for $s \in [0, g(y)]$.

By mean value theorem, one obtains

$$(R_m^x F)(x,y) = F^{(2,0)}(\xi, y)\int_0^{g(y)} K_{20}(x,y;s)ds, \quad 0 \leq \xi \leq g(y).$$

Since,

$$\int_0^{g(y)} K_{20}(x,y;s)ds = \frac{x[x-g(y)]}{2m}$$

and

$$\max_{0\leq x\leq g(y)} \frac{|x[x-g(y)]|}{2m} = \frac{g^2(y)}{8m} \leq \frac{M^2}{8m}, \quad y \in [0,h]$$

the conclusion follows. □

Remark 2.6. Analogous results are obtained for the remainder of the formula

$$F = B_n^y F + R_n^y F,$$

i.e., for $F(x,\cdot)\in C[0,h]$ we have

$$\left|(R_n^y F)(x,y)\right| \le \left(1+\tfrac{h}{2\delta\sqrt{n}}\right)\omega(F(x,\cdot);\delta), \qquad F(x,\cdot)\in C[0,h]$$

and

$$\left|(R_n^y F)(x,y)\right| \le \left(1+\tfrac{h}{2}\right)\omega\left(F(x,\cdot);\tfrac{1}{\sqrt{n}}\right)$$

respectively, for $F(x,\cdot)\in C^2[0,h]$ we have

$$(R_n^y F)(x,y) = \tfrac{y[y-f(x)]}{2n} F^{(0,2)}(x,\eta), \quad \eta\in[0,h]$$

and

$$\left|(R_n^y F)(x,y)\right| \le \frac{h^2}{8n} M_{02}F, \qquad (x,y)\in D_h.$$

2.2. **Product operators.** Let $P_{mn} = B_m^x B_n^y$, respectively, $Q_{nm} = B_n^y B_m^x$ be the products of the operators B_m^x and B_n^y.

We have

$$(P_{mn}F)(x,y) = \sum_{i=0}^{m}\sum_{j=0}^{n} p_{m,i}(x,y)\, q_{n,j}\left(i\tfrac{g(y)}{m}, y\right) F\left(i\tfrac{g(y)}{m}, j\tfrac{h}{n}\right),$$

respectively,

$$(Q_{nm}F)(x,y) = \sum_{i=0}^{m}\sum_{j=0}^{n} p_{m,i}\left(x, j\tfrac{h}{n}\right) q_{n,j}(x,y)\, F\left(\tfrac{i}{m} g(j\frac{h}{n}), j\tfrac{h}{n}\right).$$

Remark 2.7. The nodes of the operators P_{mn}, respectively Q_{nm} are given in Figures 6 and 7 and they are in domain $[0,M]\times[0,h]$, with M given (2.4).

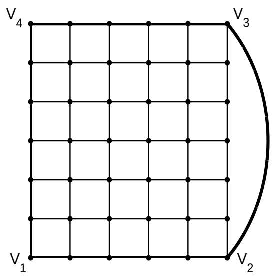


Figure 6. The nodes of P_{mn}.

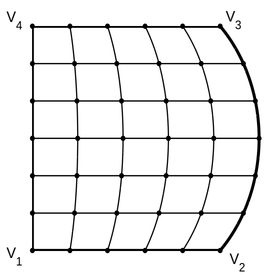


Figure 7. The nodes of Q_{nm}.

Theorem 2.8. *If F is a real-valued function defined on D_h then:*

(i) $(P_{mn}F)(V_i) = F(V_i), \quad i = 1, ..., 4;$

(ii) $(Q_{nm}F)(V_i) = F(V_i), \quad i = 1, ..., 4.$

Proof. The proof follows by a straightforward computation. □

Let us consider now the approximation formula

$$F = P_{mn}F + R^P_{mn}F,$$

where R^P_{mn} is the remainder operator.

Theorem 2.9. *If $F \in C([0, M] \times [0, h])$ then*

$$\left| \left(R^P_{mn}F\right)(x, y)\right| \leq \left(1 + \frac{M}{2} + \frac{h}{2}\right) \omega\left(F; \tfrac{1}{\sqrt{m}}, \tfrac{1}{\sqrt{n}}\right), \quad \text{for } (x, y) \in D_h,$$

where M is given in (2.4).

Proof. We have

$$\begin{aligned}\left|(R^P_{mn}F)(x, y)\right| \leq \Bigg[&\frac{1}{\delta_1}\sum_{i=0}^{m}\sum_{j=0}^{n} p_{m,i}(x, y)q_{n,j}\left(\tfrac{i}{m}g(y), y\right)\left|x - \tfrac{i}{m}g(y)\right| \\ &+ \frac{1}{\delta_2}\sum_{i=0}^{m}\sum_{j=0}^{n} p_{m,i}(x, y)q_{n,j}\left(\tfrac{i}{m}g(y), y\right)\left|y - \tfrac{j}{n}h\right| \\ &+ \sum_{i=0}^{m}\sum_{j=0}^{n} p_{m,i}(x, y)q_{n,j}\left(\tfrac{i}{m}g(y), y\right)\Bigg]\omega(F; \delta_1, \delta_2).\end{aligned}$$

Since,

$$\sum_{i=0}^{m}\sum_{j=0}^{n} p_{m,i}(x, y)q_{n,j}\left(\tfrac{i}{m}g(y), y\right)\left|x - \tfrac{i}{m}g(y)\right| \leq \sqrt{\tfrac{x(g(y)-x)}{m}},$$

$$\sum_{i=0}^{m}\sum_{j=0}^{n} p_{m,i}(x, y)q_{n,j}\left(\tfrac{i}{m}g(y), y\right)\left|y - \tfrac{j}{n}h\right| \leq \sqrt{\tfrac{y(h-y)}{n}},$$

$$\sum_{i=0}^{m}\sum_{j=0}^{n} p_{m,i}(x, y)q_{n,j}\left(\tfrac{i}{m}g(y), y\right) = 1,$$

it follows that

$$\left|(R^P_{mn}F)(x, y)\right| \leq \left(1 + \frac{1}{\delta_1}\sqrt{\tfrac{x[g(y)-x]}{m}} + \frac{1}{\delta_2}\sqrt{\tfrac{y(h-y)}{n}}\right)\omega(F; \delta_1, \delta_2).$$

But

$$x[g(y) - x] \leq \frac{M^2}{4} \text{ and } y[h - y] \leq \frac{h^2}{4},$$

with M given in (2.4), whence

$$\left|(R^P_{mn}F)(x,y)\right| \leq \left(1 + \frac{1}{\delta_1}\frac{M}{2\sqrt{m}} + \frac{1}{\delta_2}\frac{h}{2\sqrt{n}}\right)\omega(F;\delta_1,\delta_2)$$

and

$$\left|(R^P_{mn}F)(x,y)\right| \leq \left(1 + \frac{M}{2} + \frac{h}{2}\right)\omega\left(F;\frac{1}{\sqrt{m}},\frac{1}{\sqrt{n}}\right).$$

□

Remark 2.10. An analogous inequality can be obtained for the error $R^Q_{nm}F = F - Q_{nm}F$.

2.3. **Boolean sum operators.** We consider the Boolean sums of the operators B^x_m and B^y_n, i.e.,

$$S_{mn} := B^x_m \oplus B^y_n = B^x_m + B^y_n - B^x_m B^y_n,$$

respectively,

$$T_{nm} := B^y_n \oplus B^x_m = B^y_n + B^x_m - B^y_n B^x_m.$$

Theorem 2.11. *If F is a real-valued function defined on D_h then*

$$S_{mn}F\,|_{\partial D_h} = F|_{\partial D_h}$$

and

$$T_{nm}F\,|_{\partial D_h} = F|_{\partial D_h}\,. \tag{2.6}$$

Proof. The proof follows by a straightforward computation. □

For the remainder of the Boolean sum approximation formula,

$$F = S_{mn}F + R^S_{mn}F,$$

we have the following result.

Theorem 2.12. *If $F \in C([0,M]\times[0,h])$ then*

$$\begin{aligned}\left|(R^S_{mn}F)(x,y)\right| \leq &(1+\tfrac{M}{2})\omega(F(\cdot,y);\tfrac{1}{\sqrt{m}}) + (1+\tfrac{h}{2})\omega(F(x,\cdot);\tfrac{1}{\sqrt{n}})\\ &+\left(1+\frac{M}{2}+\frac{h}{2}\right)\omega(F;\tfrac{1}{\sqrt{m}},\tfrac{1}{\sqrt{n}}),\ (x,y)\in D_h,\end{aligned}$$

with M given in (2.4).

Proof. The identity

$$F - S_{mn}F = F - B_m^x F + F - B_n^y F - (F - P_{mn}F)$$

implies that

$$\left|(R_{mn}^S F)(x,y)\right| \leq \left|\,(R_m^x F)\,(x,y)\right| + \left|\,(R_n^y F(\,x,y)\right| + \left|(R_{mn}^P F)(x,y)\right|$$

and the conclusion follows. □

Remark 2.13. An analogous inequality can be obtained for the error

$$R_{nm}^T F = F - T_{nm}F.$$

3. Bernstein type operators on a square with two curved sides

Let $\tilde{D}_h$ be the square with the same vertices as in the previous case, $V_1 = (0,0)$, $V_2 = (h,0)$, $V_3 = (h,h)$ and $V_4 = (0,h)$, two straight sides Γ_1, Γ_2, along the coordinate axes and two curved sides $\tilde{\Gamma}_3$ and Γ_4, defined by the function f, with $f(0) = f(h) = h$, respectively by the function g, such that $g(0) = g(h) = h$. (See Figure 8).

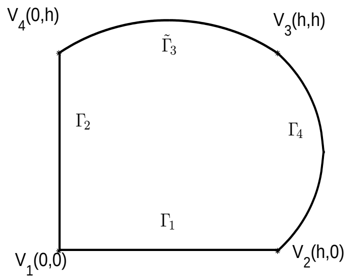


Figure 8. The square $\tilde{D}_h$.

3.1. Univariate operators.

Let F be a real-valued function defined on $\tilde{D}_h$ and $(0,y)$, $(g(y),y)$, respectively, $(x,0)$, $(x,f(x))$ be the points in which the parallel lines to the coordinate axes, passing through the point $(x,y) \in \tilde{D}_h$, intersect the sides Γ_2, Γ_4, respectively Γ_1 and $\tilde{\Gamma}_3$.

For the uniform partitions of the intervals $[0,g(y)]$ and $[0,f(x)]$:

$$\Delta_m^x = \left\{\,\tfrac{i}{m}g(y)\right|\; i = \overline{0,m}\}$$

and

$$\Delta_n^y = \left\{ \tfrac{j}{n} f(x) \middle| \; j = \overline{0, n} \right\}$$

we consider the Bernstein-type operators:

$$\left(B_m^x F\right)(x, y) = \sum_{i=0}^{m} p_{m,i}\,(x, y)\, F\left(\tfrac{i}{m} g(y), y\right), \tag{3.1}$$

as in the previous case, with $p_{m,i}$, $i = \overline{0, m}$, given in (2.2), and

$$\left(\tilde{B}_n^y F\right)(x, y) = \sum_{j=0}^{n} \tilde{q}_{n,j}\,(x, y)\, F\left(x, \tfrac{j}{n} f(x)\right) \tag{3.2}$$

with

$$\tilde{q}_{n,j}\,(x, y) = \binom{n}{j} \left(\tfrac{y}{f(x)}\right)^j \left(1 - \tfrac{y}{f(x)}\right)^{n-j}.$$

Remark 3.1. In Figures 9 and 10 we plot the points $(\frac{i}{m} g(y), y)$, $i = \overline{0, m}$ and respectively, $(x, \frac{j}{n} f(x))$, $j = \overline{0, n}$, for $m = 5$ and $n = 6$.

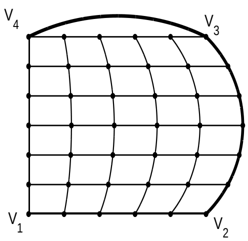


Figure 9. Points $(\frac{i}{m} g(y), y)$, $i = \overline{0, m}$.

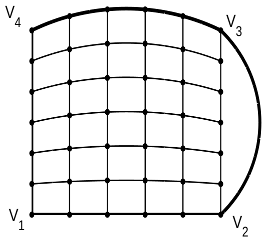


Figure 10. Points $(x, \frac{j}{n} f(x))$, $j = \overline{0, n}$.

Remark 3.2. The operator B_m^x and the remainder of the approximation formula, $R_m^x F = F - B_m^x F$, are studied in Section 2.1.

In a similar way we can prove the following results for $\tilde{B}_n^y$.

Theorem 3.3. *If F is a real-valued function defined on $\tilde{D}_h$ then:*

(i) $\tilde{B}_n^y F = F$ *on* $\Gamma_1 \cup \tilde{\Gamma}_3$,

(ii) $\left(\tilde{B}_n^y e_{ij}\right)(x, y) = x^i y^j, \; i \in \mathbb{N}; j = 0, 1;$

(iii) $\left(\tilde{B}_n^y e_{i2}\right)(x, y) = x^i \left\{ y^2 + \frac{y[f(x) - y]}{n} \right\}, \quad i \in \mathbb{N}.$

Remark 3.4. The interpolation properties of $\tilde{B}_n^y F$ are illustrated in Figure 11.

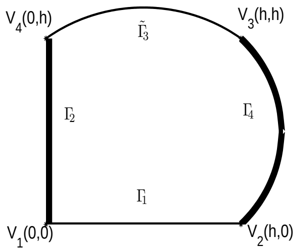


Figure 11. Interpolation domain for $\tilde{B}_n^y F$.

Also, similarly with the results in Section 2.1, we can prove the following results for $\tilde{R}_n^y F = F - \tilde{B}_n^y F$.

Theorem 3.5. *If $F(x,\cdot) \in C[0, f(x)]$, $x \in [0, g(y)]$ then we have*

$$\left|(\tilde{R}_n^y F)(x,y)\right| \leq \left(1 + \tfrac{f(x)}{2\delta\sqrt{n}}\right) \omega(F(x,\cdot);\delta), \qquad F(x,\cdot) \in C[0,h]$$

and

$$\left|(\tilde{R}_n^y F)(x,y)\right| \leq \left(1 + \tfrac{N}{2}\right) \omega\left(F(x,\cdot); \tfrac{1}{\sqrt{n}}\right).$$

If $F(x,\cdot) \in C^2[0, f(x)]$ we have

$$(\tilde{R}_n^y F)(x,y) = \tfrac{y[y-f(x)]}{2n} F^{(0,2)}(x,\eta), \quad \eta \in [0, f(x)]$$

and

$$\left|(\tilde{R}_n^y F)(x,y)\right| \leq \frac{N^2}{8n} M_{02} F, \qquad (x,y) \in \tilde{D}_h,$$

where

$$N = \max_{0 \leq x \leq g(y)} |f(x)|. \tag{3.3}$$

3.2. **Product operators.** Denote by $\tilde{P}_{mn} = B_m^x \tilde{B}_n^y$, respectively, $\tilde{Q}_{nm} = \tilde{B}_n^y B_m^x$ the products of the operators B_m^x and $\tilde{B}_n^y$.

We have

$$\left(\tilde{P}_{mn}F\right)(x,y) = \sum_{i=0}^{m} \sum_{j=0}^{n} p_{m,i}(x,y)\, \tilde{q}_{n,j}\left(\tfrac{i}{m} g(y), y\right) F\left(\tfrac{i}{m} g(y), \tfrac{j}{n} f\left(\frac{i}{m} g(y)\right)\right),$$

respectively,

$$\left(\tilde{Q}_{nm}F\right)(x,y) = \sum_{i=0}^{m} \sum_{j=0}^{n} p_{m,i}\left(x, \tfrac{j}{n} f(x)\right) \tilde{q}_{n,j}(x,y)\, F\left(\tfrac{i}{m} g\left(\tfrac{j}{n} f(x)\right), \tfrac{j}{n} f(x)\right).$$

Remark 3.6. The nodes of the operators $\tilde{P}_{mn}$, respectively $\tilde{Q}_{nm}$ are given in Figures 12 and 13 and they are in domain $[0, M] \times [0, N]$, with M and N given in (2.4) and (3.3).

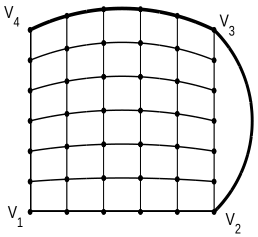


Figure 12. The nodes of $\tilde{P}_{mn}$.

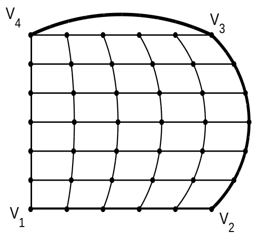


Figure 13. The nodes of $\tilde{Q}_{nm}$.

Theorem 3.7. *If F is a real-valued function defined on $\tilde{D}_h$ we have:*

(i) $(\tilde{P}_{mn}F)(V_i) = F(V_i), \quad i = 1, ..., 4;$

(ii) $(\tilde{Q}_{nm}F)(V_i) = F(V_i), \quad i = 1, ..., 4;$

(iii) $\left|\left(\tilde{R}^P_{mn}F\right)(x, y)\right| \leq (1 + \frac{M}{2} + \frac{N}{2})\omega\left(F; \frac{1}{\sqrt{m}}, \frac{1}{\sqrt{n}}\right)$, *for* $(x, y) \in \tilde{D}_h$, *where* $\tilde{R}^P_{mn}F = F - \tilde{P}_{mn}F$ *and* M *is given in (2.4) and* N *is given in (3.3). An analogous inequality can be obtained for* $\tilde{R}^Q_{nm}F = F - \tilde{Q}_{nm}$.

Proof. The proof is similarly with the proof of Theorem 2.9. □

3.3. **Boolean sum operators.** Let $\tilde{S}_{mn} := B^x_m \oplus \tilde{B}^y_n$ and $\tilde{T}_{nm} = \tilde{B}^y_n \oplus B^x_m$ be the Boolean sum of the operators B^x_m and $\tilde{B}^y_n$.

If F is a real-valued function defined on $\tilde{D}_h$ then we have

$$\tilde{S}_{mn}F\left|_{\partial\tilde{D}_h} = F\right|_{\partial\tilde{D}_h}$$

and

$$\tilde{T}_{nm}F\left|_{\partial\tilde{D}_h} = F\right|_{\partial\tilde{D}_h}.$$

For the remainder $\tilde{R}^S_{mn}F = F - \tilde{S}_{mn}F$ we have:

$$\begin{aligned}\left|(\tilde{R}^S_{mn}F)(x, y)\right| \leq & (1 + \tfrac{M}{2})\omega\left(F(\cdot, y); \tfrac{1}{\sqrt{m}}\right) + (1 + \tfrac{N}{2})\omega\left(F(x, \cdot); \tfrac{1}{\sqrt{n}}\right) \\ & + \left(1 + \frac{M}{2} + \frac{N}{2}\right)\omega\left(F; \tfrac{1}{\sqrt{m}}, \tfrac{1}{\sqrt{n}}\right), \ (x, y) \in \tilde{D}_h,\end{aligned}$$

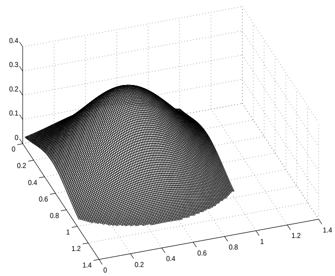
with M given in (2.4) and N given in (3.3).

An analogous result can be obtained for the remainder $\tilde{R}^T_{nm}F = F - \tilde{T}_{nm}F$.

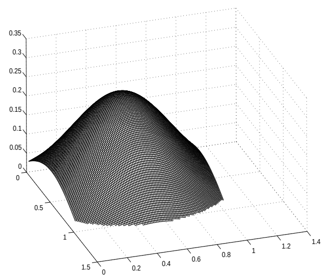
Example 3.8. We consider the function:

$$\text{Gentle:} \quad F(x,y) = \tfrac{1}{3}\exp[-\tfrac{81}{16}\left((x-0.5)^2 + (y-0.5)^2\right)], \tag{3.4}$$

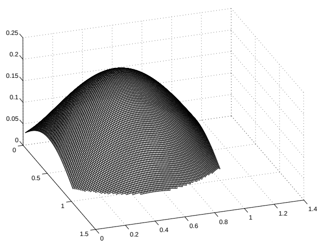
that is generally used in the literature, (see, e.g., [16]). In Figures 14 and 15 we plot the graphs of F, $B^x_m F$, $P_{mn}F$, $S_{mn}F$, $\tilde{P}_{mn}F$, $\tilde{S}_{mn}F$, on D_h, and respectively on $\tilde{D}_h$, considering $h = 1, m = 5$, $n = 6$.



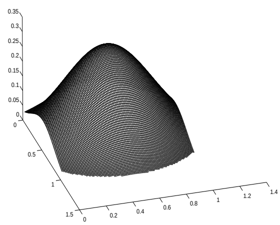
Graph of F on D_h.



Graph of $B^x_m F$ on D_h.

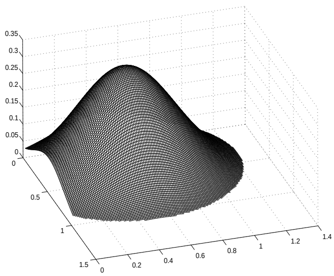


Graph of $P_{mn}F$ on D_h.

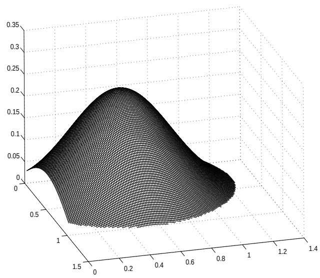


Graph of $S_{mn}F$ on D_h.

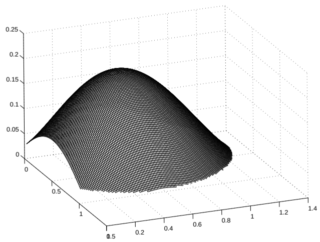
Figure 14. Graphs for domain D_h.



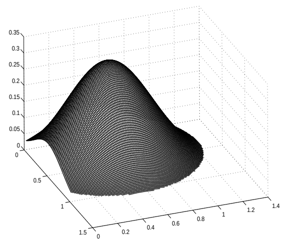
Graph of F on $\tilde{D}_h$.



Graph of $B_m^x F$ on $\tilde{D}_h$.



Graph of $\tilde{P}_{mn}F$ on $\tilde{D}_h$.



Graph of $\tilde{S}_{mn}F$ on $\tilde{D}_h$.

Figure 15. Graphs for domain $\tilde{D}_h$.

References

[1] Barnhill, R. E., Birkhoff, G., Gordon,W. J., *Smooth interpolation in triangle*, J. Approx. Theory, **8** (1983), 114-128.

[2] Barnhill, R. E., Gregory, I. A., *Polynomial interpolation to boundary data on triangles*, Math. Comp., **29** (1975), no. 131, 726-735.

[3] Barnhill, R. E., Mansfield, L., *Error bounds for smooth interpolation in triangles*, J. Approx. Theory, **11** (1974), 306-318.

[4] Barnhill, R. E., Ou, H. S., *Surfaces defined on surfaces*, Computer Aided Geom. Design, **7** (1990), 323-336.

[5] Bernardi, C., *Optimal finite-element interpolation on curved domains,* SIAM J. Numer. Anal., **26** (1989) , no. 5, 1212-1240.

[6] Blaga, P., Coman, G., *Bernstein-type operators on triangle,* Rev. Anal. Numer. Theor. Approx., **38** (2009), no. 1, 9-21.

[7] Blaga, P., Cătinaş, T., Coman, G., *Bernstein-type operators on triangle with one curved side*, submitted.

[8] Blaga, P., Cătinaş, T., Coman, G., *Bernstein-type operators on triangle with all curved sides,* submitted.

[9] Blaga, P., Cătinaş, T., Coman, G., *Bernstein-type operators on tetrahedrons*, Stud. Univ. Babeş-Bolyai Math., **54** (2009), no. 4, 3-19.

[10] Cătinaş, T., Coman, G., *Some interpolation operators on a simplex domain*, Stud. Univ. Babeş-Bolyai Math., **52** (2007), no. 3, 25-34.

[11] Coman, G., Cătinaş, T., *Interpolation operators on a triangle with one curved side,* BIT Numerical Mathematics, **50** (2010), no. 2, 243-267.

[12] Coman, G., Cătinaş, T., *Interpolation operators on a tetrahedron with three curved sides,* Calcolo, **47** (2010), no. 2, 113-128.

[13] Marshall, J. A., McLeod, R., *Curved elements in the finite element method*, Conference on Numer. Sol. Diff. Eq., Lectures Notes in Math., **363** (1974), Springer Verlag, 89-104.

[14] Marshall, J. A., Mitchell, A. R., *An exact boundary tehnique for improved accuracy in the finite element method*, J. Inst. Maths. Applics., 12 (1973), pp. 355-362.

[15] Mitchell, A. R., McLeod, R., *Curved elements in the finite element method*, Conference on Numer. Sol.Diff. Eq., Lectures Notes in Mathematics, 363, 89-104 (1974).

[16] Renka, R. J., Cline, A. K., *A triangle-based C^1 interpolation method*, Rocky Mountain J. Math., **14** (1984), no. 1, 223-237.

[17] A. Sard, *Linear Approximation*, AMS, Providence, RI, (1963).

[18] Stancu, D. D., *Evaluation of the remainder term in approximation formulas by Bernstein polynomials*, Math. Comp., **17** (1963) , 270-278.

[19] Stancu, D. D., *A method for obtaining polynomials of Bernstein type of two variables*, Amer. Math. Monthly, **70** (1963), no. 3, 260-264.

[20] Stancu, D. D., *Approximation of bivariate functions by means of some Bernstein-type operators*, Multivariate approximation (Sympos., Univ. Durham, Durham, 1977), 189-208, Academic Press, London-New York, 1978.

Babeş-Bolyai University
Faculty of Mathematics and Computer Science
Str. M. Kogălniceanu Nr. 1
RO-400084 Cluj-Napoca, Romania
E-mail address: blaga@math.ubbcluj.ro
tcatinas@math.ubbcluj.ro
ghcoman@math.ubbcluj.ro

STUDIA UNIV. "BABEŞ–BOLYAI", MATHEMATICA, Volume **LV**, Number 3, September 2010

CERTAIN CLASS OF λ STARLIKE HARMONIC FUNCTIONS ASSOCIATED WITH A CONVOLUTION STRUCTURE

DANIEL BREAZ, GANGADHARAN MURUGUSUNDARAMOORTHY, AND KALLIYAPAN VIJAYA

Dedicated to Professor Grigore Ştefan Sălăgean on his 60^{th} *birthday*

Abstract. Making use of a convolution structure, we introduce a new class of complex valued harmonic functions which are orientation preserving and univalent in the open unit disc. Among the results presented in this paper include the coefficient bounds, distortion inequality and covering property, extreme points and certain inclusion results for this generalized class of functions.

1. Introduction and preliminaries

A continuous function $f = u+iv$ is a complex- valued harmonic function in a complex domain $\mathcal{G}$ if both u and v are real and harmonic in $\mathcal{G}$. In any simply-connected domain $D \subset \mathcal{G}$, we can write $f = h + \overline{g}$, where h and g are analytic in D. We call h the analytic part and g the co-analytic part of f. A necessary and sufficient condition for f to be locally univalent and orientation preserving in D is that $|h'(z)| > |g'(z)|$ in D (see [2]).

Denote by $\mathcal{H}$ the family of functions

$$f = h + \overline{g} \tag{1.1}$$

which are harmonic, univalent and orientation preserving in the open unit disc $\mathcal{U} = \{z : |z| < 1\}$ so that f is normalized by $f(0) = f'(0) - 1 = 0$. Thus, for $f = h + \overline{g} \in \mathcal{H}$,

Received by the editors: 02.03.2010.

2000 *Mathematics Subject Classification.* 30C45, 30C50.

Key words and phrases. Harmonic univalent functions, distortion bounds, extreme points, convolution, inclusion property.

the functions h and g analytic $\mathcal{U}$ can be expressed in the following forms:

$$h(z) = z + \sum_{n=2}^{\infty} a_n z^n, \;\; g(z) = \sum_{n=1}^{\infty} b_n z^n \;\; (0 \leq b_1 < 1),$$

and $f(z)$ is then given by

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n + \overline{\sum_{n=1}^{\infty} b_n z^n} \quad (0 \leq b_1 < 1). \tag{1.2}$$

We note that the family $\mathcal{H}$ of orientation preserving, normalized harmonic univalent functions reduces to the well known class S of normalized univalent functions if the co-analytic part of f is identically zero, i.e. $g \equiv 0$.

For functions $f \in \mathcal{H}$ given by (1.1) and $F \in \mathcal{H}$ given by

$$F(z) = H(z) + \overline{G(z)} = z + \sum_{n=2}^{\infty} A_n z^n + \overline{\sum_{n=1}^{\infty} B_n z^n}, \tag{1.3}$$

we recall the Hadamard product (or convolution) of f and F by

$$(f * F)(z) = z + \sum_{n=2}^{\infty} a_n A_n z^n + \overline{\sum_{n=1}^{\infty} b_n B_n z^n} \;\; (z \in \mathcal{U}). \tag{1.4}$$

In terms of the Hadamard product (or convolution), we choose F as a fixed function in $\mathcal{H}$ such that $(f * F)(z)$ exists for any $f \in \mathcal{H}$, and for various choices of F we get different linear operators which have been studied in recent past. To illustrate some of these cases which arise from the convolution structure (1.4), we consider the following examples.

(1) If

$$F(z) = z + \sum_{n=2}^{\infty} \sigma_n(\alpha_1)\; z^n + \sum_{n=1}^{\infty} \sigma_n(\alpha_1)\; \overline{z}^n \tag{1.5}$$

and $\sigma_n(\alpha_1)$is defined by

$$\sigma_n(\alpha_1) = \frac{\Theta\Gamma(\alpha_1 + A_1(n-1)) \ldots \Gamma(\alpha_p + A_p(n-1))}{(n-1)!\Gamma(\beta_1 + B_1(n-1)) \ldots \Gamma(\beta_q + B_q(n-1))}\;. \tag{1.6}$$

where Θ is given by

$$\Theta = \left(\prod_{m=0}^{p} \Gamma(\alpha_m)\right)^{-1} \left(\prod_{m=0}^{q} \Gamma(\beta_m)\right) \tag{1.7}$$

and then the convolution (1.4)gives the Wright's generalized hypergeometric function (see [13])

$$ {}_p\Psi_q[(\alpha_1, A_1), \ldots, (\alpha_p, A_p); (\beta_1, B_1), \ldots, (\beta_q, B_q); z] = {}_p\Psi_q[(\alpha_n, A_n)_{1,p}(\beta_n, B_n)_{1,q}; z] $$

is defined by

$$ {}_p\Psi_q[(\alpha_n, A_n)_{1,p}(\beta_n, B_n)_{1,q}; z] $$

$$ = \sum_{n=0}^{\infty} \left\{ \prod_{m=1}^{p} \Gamma(\alpha_m + nA_m) \right\} \left\{ \prod_{m=1}^{q} \Gamma(\beta_m + nB_m) \right\}^{-1} \frac{z^m}{n!} \quad (z \in \mathbb{U}) $$

which was initially studied by Murugusundaramoorthy (see [9]).

(2) If $A_m = 1(m = 1, ..., p)$ and $B_m = 1(m = 1, ..., q)$, then we have the following obvious relationship

$$ F(z) = z + \sum_{n=2}^{\infty} \Gamma_n z^n + \sum_{n=1}^{\infty} \Gamma_n \overline{z}^n, \tag{1.8} $$

where

$$ \Gamma_n = \frac{(\alpha_1)_{n-1} \ldots (\alpha_p)_{n-1}}{(\beta_1)_{n-1} \ldots (\beta_q)_{n-1}} \frac{1}{(n-1)!}, $$

then the convolution (1.4) gives the Dziok–Srivastava operator (see [4]):

$$ \Lambda(\alpha_1, \cdots, \alpha_p; \beta_1, \cdots, \beta_q; z) f(z) \equiv \mathcal{H}_q^p(\alpha_1, \beta_1) f(z), $$

where $\alpha_1, \cdots, \alpha_p$; $\beta_1, \cdots, \beta_q$ are positive real numbers, $p \leq q+1; p, q \in \mathbb{N} \cup \{0\}$, and $(\alpha)_n$ denotes the familiar Pochhammer symbol (or shifted factorial).

Remark 1.1. When $p = 1, q = 1; \alpha_1 = a, \alpha_2 = 1; \beta_1 = c$, then (1.8) corresponds to the operator due to Carlson-Shaffer operator(see [1]) given by

$$ \mathcal{L}(a, c) f(z) := (f * F)(z), $$

where

$$ F(z) := z + \sum_{n=2}^{\infty} \frac{(a)_{n-1}}{(c)_{n-1}} z^n + \sum_{n=1}^{\infty} \frac{(a)_{n-1}}{(c)_{n-1}} \overline{z}^n \quad (c \neq 0, -1, -2, \cdots). \tag{1.9} $$

Remark 1.2. When $p=1, q=0; \alpha_1 = n+1, \alpha_2 = 1; \beta_1 = 1$, then (1.8) yields the Ruscheweyh derivative operator (see [7]) given$D^k f(z) := (f * F)(z)$ where

$$F(z) = z + \sum_{n=2}^{\infty} \begin{pmatrix} k+n-1 \\ n-1 \end{pmatrix} z^n + \sum_{n=1}^{\infty} \begin{pmatrix} k+n-1 \\ n-1 \end{pmatrix} \overline{z}^n. \tag{1.10}$$

which was initially studied by Jahangiri et al.(see [7]).

(3) Lastly, if $\mathcal{D}^l f(z) = f * F$ where

$$F(z) = z + \sum_{n=2}^{\infty} n^l z^n + (-1)^l \sum_{n=1}^{\infty} n^l \overline{z}^n \quad (\, l \geq 0) \,, \tag{1.11}$$

which was initially studied by Jahangiri et al.(see [8]).

For the purpose of this paper, we introduce here a subclass of $\mathcal{H}$ denoted by $S_H(F;\lambda,\gamma)$ which involves the convolution (1.3) and consist of all functions of the form (1.1) satisfying the inequality:

$$\text{Re} \left\{ \frac{z(f(z) * F(z))'}{(1-\lambda)(f(z) * F(z)) + \lambda z (f(z) * F(z))'} \right\} \geq \gamma \tag{1.12}$$

Equivalently

$$\text{Re} \left\{ \frac{z(h(z) * H(z))' - \overline{z(g(z) * G(z))'}}{(1-\lambda)[h(z) * H(z) + \overline{g(z) * G(z)}] + \lambda[z(g(z) * H(z))' - \overline{z(g(z) * G(z))'}]} \right\} \geq \gamma \tag{1.13}$$

where $z \in \mathcal{U}, \; 0 \leq \lambda \leq 1$.

Also denote $\mathcal{T}_{\mathcal{H}}(F;\lambda,\gamma) = S_H(F;\lambda,\gamma) \bigcap \mathcal{T}_{\mathcal{H}}$ where $\mathcal{T}_{\mathcal{H}}$ the subfamily of $\mathcal{H}$ consisting of harmonic functions $f = h + \overline{g}$ of the form

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n + \overline{\sum_{n=1}^{\infty} b_n z^n} \;\; (0 \leq b_1 < 1). \tag{1.14}$$

called the class of harmonic functions with negative coefficients (see [11])

We deem it proper to mention below some of the function classes which emerge from the function class $\mathcal{S}_{\mathcal{H}}(F;\lambda,\gamma)$ defined above. Indeed, we observe that if we specialize the function F by means of (1.5) to (1.11), and denote the corresponding reducible classes of functions of $\mathcal{S}_{\mathcal{H}}(F;\gamma)$, respectively, by $\mathcal{W}_q^p(\lambda,\gamma)$, $\mathcal{G}_q^p(\lambda,\gamma)$ $\mathcal{L}_c^a(\lambda,\gamma)$, $\mathcal{R}(k,\lambda,\gamma)$, $\Omega(\lambda,\gamma)$ and $\mathcal{S}(l,\lambda,\gamma)$.

It is of special interest because for suitable choices of F from (1.6) we can define the following subclasses:

(i) If F is given by (1.5) we have $(f * F)(z) = W_q^p[\alpha_1]f(z)$ hence we define a class $\mathcal{W}_q^p(\lambda, \gamma)$ satisfying the criteria

$$\operatorname{Re}\left\{\frac{z(W_q^p[\alpha_1]f(z))'}{(1-\lambda)W_q^p[\alpha_1]f(z) + \lambda z(W_q^p[\alpha_1]f(z))'}\right\} \geq \gamma$$

where $W_q^p[\alpha_1]$ is the Wright's generalized operator on harmonic functions (see [9]) .

(ii) If F is given by (1.8) we have $(f * F)(z) = H_q^p[\alpha_1]f(z)$ hence we define a class $\mathcal{G}_q^p(\lambda, \gamma)$ satisfying the criteria

$$\operatorname{Re}\left\{\frac{z(H_q^p[\alpha_1]f(z))'}{(1-\lambda)H_q^p[\alpha_1]f(z) + \lambda z(H_q^p[\alpha_1]f(z))'}\right\} \geq \gamma$$

where $H_q^p[\alpha_1]$ is the Dziok - Srivastava operator (see [4]).

(iii) $H_1^2([a, 1; c]) = \mathcal{L}(a, c)f(z)$, hence we define a class $\mathcal{L}_c^a(\lambda, \gamma)$satisfying the criteria

$$\operatorname{Re}\left\{\frac{z\mathcal{L}(a, c)f(z))'}{(1-\lambda)\mathcal{L}(a, c)f(z) + \lambda z(\mathcal{L}(a, c)f(z))'}\right\} \geq \gamma$$

where $\mathcal{L}(a, c)$ is the Carlson - Shaffer operator (see [1]).

(iv) $H_1^2([k + 1, 1; 1]) = D^k f(z)$, hence we define a class $\mathcal{R}(k, \lambda, \gamma)$ satisfying the criteria

$$\operatorname{Re}\left\{\frac{z(D^k f(z))'}{(1-\lambda)D^k f(z) + \lambda z(D^k f(z))'}\right\} \geq \gamma$$

where $D^k f(z)(k > -1)$ is the Ruscheweyh derivative operator (see [10]) (also see [7]).

(v) $H_1^2([2, 1; 2-\mu]) = \Omega_z^\mu f(z)$ we define another class $\Omega(\lambda, \gamma)$ satisfying the condition

$$\operatorname{Re}\left\{\frac{z(\Omega_z^\mu f(z))'}{(1-\lambda)\Omega_z^\mu f(z) + \lambda z(\Omega_z^\mu f(z))'}\right\} \geq \gamma$$

given by

$$\Omega_z^\mu f(z) = \Gamma(2-\mu) z^\mu D_z^\mu f(z); (0 \leq \mu < 1),$$

where Ω_z^μ is the Srivastava-Owa fractional derivative operator (see [12]).

(vi) If F is given by (1.11), we have$D^l f(z) = (f * F)(z)$, hence we define a class $S(l, \lambda, \gamma)$ satisfying the criteria

$$\operatorname{Re}\left\{\frac{z(D^l f(z))'}{(1-\lambda)D^l f(z) + \lambda z(D^l f(z))'}\right\} \geq \gamma$$

where $D^l f(z); (l \in N = 0, 1, 2, 3,)$ is the Sălăgean derivative operator for harmonic functions (see [8]).

Motivated by the earlier works of (see [5, 8, 13]) on the subject of harmonic functions, in this paper we obtain a sufficient coefficient condition for functions f given by (1.2) to be in the class $\mathcal{S}_{\mathcal{H}}(F; \lambda, \gamma)$. It is shown that this coefficient condition is necessary also for functions belonging to the class $\mathcal{T}_{\mathcal{H}}(F; \lambda, \gamma)$. Further, distortion results and extreme points for functions in $\mathcal{T}_{\mathcal{H}}(F; \lambda, \gamma)$ are also obtained.

For the sake of brevity we denote the corresponding coefficient of F as C_n throughout our study unless otherwise stated.

2. **Coefficient bounds**

In our first theorem, we obtain a sufficient coefficient condition for harmonic functions in $\mathcal{S}_{\mathcal{H}}(F; \lambda, \gamma)$.

Theorem 2.1. *Let $f = h + \overline{g}$ be given by (1.2). If*

$$\sum_{n=1}^{\infty} \left[\frac{n - \gamma - \gamma\lambda(n-1)}{1-\gamma} |a_n| + \frac{n + \gamma - \gamma\lambda(n+1)}{1-\gamma} |b_n| \right] C_n \leq 2 \tag{2.1}$$

where $a_1 = 1$ and $0 \leq \gamma < 1$, then $f \in \mathcal{S}_{\mathcal{H}}(F; \lambda, \gamma)$.

Proof. We first show that if (2.1) holds for the coefficients of $f = h + \overline{g}$, the required condition (2.1) is satisfied. From (1.13) we can write

$$\operatorname{Re} \left\{ \frac{z(h(z) * H(z))' - \overline{z(g(z) * G(z))'}}{(1-\lambda)[h(z) * H(z) + \overline{g(z) * G(z)}] + \lambda[z(g(z) * H(z))' - \overline{z(g(z) * G(z))'}]} \right\} \geq \gamma$$

$$= \operatorname{Re} \frac{A(z)}{B(z)} \geq \gamma$$

where

$$A(z) = zh(z) * H(z))' - \overline{z(g(z) * G(z))'} = z + \sum_{n=2}^{\infty} nC_n a_n z^n - \sum_{n=1}^{\infty} nC_n \bar{b}_n \bar{z}^n$$

and

$$B(z) = (1-\lambda)[h(z) * H(z) + \overline{g(z) * G(z)}] + \lambda[z(g(z) * H(z))' - \overline{z(g(z) * G(z))'}]$$

$$= z + \sum_{n=2}^{\infty} (1 - \lambda + n\lambda) C_n a_n z^n + \sum_{n=1}^{\infty} (1 - \lambda - n\lambda) C_n \bar{b}_n \bar{z}^n.$$

Using the fact that Re $\{w\} \geq \gamma$ if and only if $|1-\gamma+w| \geq |1+\gamma-w|$, it suffices to show that

$$|A(z)+(1-\gamma)B(z)| - |A(z)-(1+\gamma)B(z)| \geq 0. \tag{2.2}$$

Substituting for $A(z)$ and $B(z)$ in (2.2), we get

$$|A(z)+(1-\gamma)B(z)| - |A(z)-(1+\gamma)B(z)|$$

$$= |(2-\gamma)z + \sum_{n=2}^{\infty}[(n+1-\gamma)(1-\lambda+n\lambda)]C_n a_n z^n - \sum_{n=1}^{\infty}[n-(1-\gamma)(1-\lambda+n\lambda)]C_n \overline{b}_n\ \overline{z}^n|$$

$$-|-\gamma z + \sum_{n=2}^{\infty}[n-(1+\gamma)(1-\lambda+n\lambda)C_n a_n z^n - \sum_{n=1}^{\infty}[n+(1+\gamma)(1-\lambda+n\lambda)]C_n \overline{b}_n \overline{z}^n|$$

$$\geq (2-\gamma)|z| - \sum_{n=2}^{\infty}[n+(1-\gamma)(1-\lambda+n\lambda)C_n|a_n||z|^n - \sum_{n=1}^{\infty}[n-(1-\gamma)(1-\lambda-n\lambda)]C_n|b_n|\ |z|^n$$

$$-\gamma|z| - \sum_{n=2}^{\infty}[n-(1+\gamma)(1-\lambda+n\lambda)]C_n|a_n|\ |z|^n - \sum_{n=1}^{\infty}[n+(1+\gamma)(1-\lambda-n\lambda)]C_n|b_n|\ |z|^n$$

$$\geq 2(1-\gamma)|z|\left\{2-\sum_{n=1}^{\infty}\left[\frac{n-\gamma-\gamma\lambda(n-1)}{1-\gamma}|a_n| + \frac{n+\gamma-\gamma\lambda(n-1)}{1-\gamma}|b_n|\right]C_n|z|^{n-1}\right\}$$

$$\geq 2(1-\gamma)\left\{2-\sum_{n=1}^{\infty}\left[\frac{n-\gamma-\gamma\lambda(n-1)}{1-\gamma}|a_n| + \frac{n+\gamma-\gamma\lambda(n-1)}{1-\gamma}|b_n|\right]C_n\right\}.$$

The above expression is non negative by (2.1), and so $f \in \mathcal{S}_{\mathcal{H}}(F;\lambda,\gamma)$. □

The harmonic function

$$f(z) = z+\sum_{n=2}^{\infty}\frac{1-\gamma}{[n-\gamma-\gamma\lambda(n-1)]C_n}x_n z^n + \sum_{n=1}^{\infty}\frac{1-\gamma}{[n+\gamma-\gamma\lambda(n-1)]C_n}\overline{y}_n(\overline{z})^n \tag{2.3}$$

where $\sum\limits_{n=2}^{\infty}|x_n| + \sum\limits_{n=1}^{\infty}|y_n| = 1$ shows that the coefficient bound given by (2.1) is sharp. The functions of the form (2.3) are in $\mathcal{S}_{\mathcal{H}}(F;\lambda,\gamma)$ because

$$\sum_{n=1}^{\infty}\left(\frac{[n-\gamma-\gamma\lambda(n-1)]C_n}{1-\gamma}|a_n| + \frac{[n+\gamma-\gamma\lambda(n-1)]C_n}{1-\gamma}|b_n|\right)$$

$$=1+\sum_{n=2}^{\infty}|x_n| + \sum_{n=1}^{\infty}|y_n| = 2.$$

Next theorem establishes that such coefficient bounds cannot be improved further.

Theorem 2.2. *For $a_1 = 1$ and $0 \leq \gamma < 1$, $f = h + \overline{g} \in \mathcal{T}_{\mathcal{H}}(F;\lambda,\gamma)$ if and only if*

$$\sum_{n=1}^{\infty}\left[\frac{n-\gamma-\gamma\lambda(n-1)}{1-\gamma}|a_n| + \frac{n+\gamma-\gamma\lambda(n-1)}{1-\gamma}|b_n|\right]C_n \leq 2. \tag{2.4}$$

Proof. Since $\mathcal{T}_{\mathcal{H}}(F;\lambda,\gamma) \subset \mathcal{S}_{\mathcal{H}}(F;\lambda,\gamma)$, we only need to prove the "only if" part of the theorem. To this end, for functions f of the form (1.14), we notice that the condition

$$\text{Re}\left\{\frac{z(h(z)*H(z))' - \overline{z(g(z)*G(z))'}}{(1-\lambda)[h(z)*H(z) + \overline{g(z)*G(z)}] + \lambda[z(g(z)*H(z))' - \overline{z(g(z)*G(z))'}]}\right\} \geq \gamma$$

Equivalently,

$$\text{Re}\left\{\frac{(1-\gamma)z - \sum\limits_{n=2}^{\infty}[n-\gamma-\gamma\lambda(n-1)]C_n a_n z^n - \sum\limits_{n=1}^{\infty}[n+\gamma-\gamma\lambda(n-1)]C_n\overline{b}_n\overline{z}^n}{z - \sum\limits_{n=2}^{\infty}(1-\lambda+n\lambda)C_n a_n z^n + \sum\limits_{n=1}^{\infty}(1-\lambda-n\lambda)C_n\overline{b}_n\overline{z}^n}\right\} \geq 0.$$

The above required condition must hold for all values of z in U. Upon choosing the values of z on the positive real axis where $0 \leq z = r < 1$, we must have

$$\frac{(1-\gamma) - \sum\limits_{n=2}^{\infty}[n-\gamma-\gamma\lambda(n-1)]C_n a_n r^{n-1} - \sum\limits_{n=1}^{\infty}[n+\gamma-\gamma\lambda(n-1)]C_n b_n r^{n-1}}{1 - \sum\limits_{n=2}^{\infty}(1-\lambda+n\lambda)C_n a_n r^{n-1} + \sum\limits_{n=1}^{\infty}(1-\lambda-n\lambda)C_n b_n r^{n-1}} \geq 0. \tag{2.5}$$

If the condition (2.4) does not hold, then the numerator in (2.5) is negative for r sufficiently close to 1. Hence, there exist $z_0 = r_0$ in (0,1) for which the quotient of (2.5) is negative. This contradicts the required condition for $f \in \mathcal{T}_{\mathcal{H}}(F;\lambda,\gamma)$. This completes the proof of the theorem. □

3. **Distortion bounds and extreme points**

The following theorem gives the distortion bounds for functions in $\mathcal{T}_{\mathcal{H}}(F;\lambda,\gamma)$ which yields a covering result for the class $\mathcal{T}_{\mathcal{H}}(F;\lambda,\gamma)$.

Theorem 3.1. *Let $f \in \mathcal{T}_{\mathcal{H}}(F;\lambda,\gamma)$. Then for $|z| = r < 1$, we have*

$$(1-b_1)r - \frac{1}{C_2}\left(\frac{1-\gamma}{2-\gamma-\gamma\lambda} - \frac{1+\gamma}{2-\gamma-\gamma\lambda}b_1\right)r^2 \leq |f(z)|$$
$$\leq (1+b_1)r + \frac{1}{C_2}\left(\frac{1-\gamma}{2-\gamma-\gamma\lambda} - \frac{1+\gamma}{2-\gamma-\gamma\lambda}b_1\right)r^2.$$

Proof. We only prove the right hand inequality. Taking the absolute value of $f(z)$, we obtain

$$\begin{aligned}|f(z)| &= \left|z+\sum_{n=2}^{\infty}a_nz^n+\sum_{n=1}^{\infty}\overline{b}_n\overline{z}^n\right|\\ &\le (1+b_1)|z|+\sum_{n=2}^{\infty}(a_n+b_n)|z|^n\\ &\le (1+b_1)r+\sum_{n=2}^{\infty}(a_n+b_n)r^2\\ &\le (1+b_1)r+\frac{(1-\gamma)}{(2-\gamma-\gamma\lambda)C_2}\sum_{n=2}^{\infty}\left(\frac{(2-\gamma-\gamma\lambda)C_2}{(1-\gamma)}a_n+\frac{(2-\gamma-\gamma\lambda)C_2}{(1-\gamma)}b_n\right)r^2\\ &\le (1+b_1)r+\frac{(1-\gamma)1}{(2-\gamma-\gamma\lambda)C_2}\left(1-\frac{1+\gamma}{1-\gamma}b_1\right)r^2\\ &\le (1+b_1)r+\frac{1}{C_2}\left(\frac{1-\gamma}{2-\gamma-\gamma\lambda}-\frac{1+\gamma}{2-\gamma-\gamma\lambda}b_1\right)r^2.\end{aligned}$$

The proof of the left hand inequality follows on lines similar to that of the right hand side inequality. □

The covering result follows from the left hand inequality given in Theorem 3.1.

Corollary 3.2. *If $f(z)\in\mathcal{T}_{\mathcal{H}}(F;\lambda,\gamma)$, then*

$$\left\{w:|w|<\frac{2C_2-1-[(1+\lambda)C_2-1]\gamma}{(2-\gamma-\gamma\lambda)C_2}-\frac{2C_2-1-[(1+\lambda)C_2+1]\gamma}{(2-\gamma-\gamma\lambda)C_2}|b_1|\right\}\subset f(U).$$

Proof. Using the left hand inequality of Theorem 3.1 and letting $r\to 1$, we prove that

$$\begin{aligned}&(1-b_1)-\frac{1}{C_2}\left(\frac{1-\gamma}{2-\gamma-\gamma\lambda}-\frac{1+\gamma}{2-\gamma-\gamma\lambda}b_1\right)\\ &=(1-b_1)-\frac{1}{C_2(2-\gamma-\gamma\lambda)}[1-\gamma-(1+\gamma)b_1]\\ &=\frac{(1-b_1)C_2(2-\gamma-\gamma\lambda)-(1-\gamma)+(1+\gamma)b_1}{C_2(2-\gamma-\gamma\lambda)}\\ &=\left\{\frac{2C_2-1-[(1+\lambda)C_2-1]\gamma}{(2-\gamma-\gamma\lambda)C_2}-\frac{2C_2-1-[(1+\lambda)C_2+1]\gamma}{(2-\gamma-\gamma\lambda)C_2}|b_1|\right\}\subset f(U).\end{aligned}$$

□

Next we determine the extreme points of closed convex hulls of $\mathcal{T}_{\mathcal{H}}(F;\lambda,\gamma)$ denoted by $clco\mathcal{T}_{\mathcal{H}}(F;\lambda,\gamma)$.

Theorem 3.3. *A function $f(z)\in\mathcal{T}_{\mathcal{H}}(F;\lambda,\gamma)$ if and only if*

$$f(z)=\sum_{n=1}^{\infty}\left(X_nh_n(z)+Y_ng_n(z)\right)$$

where

$$h_1(z)=z, h_n(z)=z-\frac{1-\gamma}{[n-\gamma-\gamma\lambda(n-1)]C_n}z^n;\quad (n\geq 2),$$
$$g_n(z)=z+\frac{1-\gamma}{[n+\gamma-\gamma\lambda(n-1)]C_n}\overline{z}^n;$$
$$(n\geq 2),\sum_{n=1}^{\infty}(X_n+Y_n)=1,\quad X_n\geq 0 \text{ and } \ Y_n\geq 0.$$

In particular, the extreme points of $\mathcal{T}_{\mathcal{H}}(F;\lambda,\gamma)$ are $\{h_n\}$ and $\{g_n\}$.

Proof. First, we note that for f as in the theorem above, we may write

$$\begin{aligned} f(z) &= \sum_{n=1}^{\infty}\left(X_nh_n(z)+Y_ng_n(z)\right)\\ &= \sum_{n=1}^{\infty}(X_n+Y_n)z-\sum_{n=2}^{\infty}\frac{1-\gamma}{n[n-\gamma-\gamma\lambda(n-1)]C_n}X_nz^n\\ &\quad +\sum_{n=1}^{\infty}\frac{1-\gamma}{[n+\gamma-\gamma\lambda(n-1)]C_n}Y_n\overline{z}^n \end{aligned}$$

Then

$$\begin{aligned} &\sum_{n=2}^{\infty}\frac{n[n-\gamma-\gamma\lambda(n-1)]C_n}{1-\gamma}|a_n|+\sum_{n=1}^{\infty}\frac{n[n+\gamma-\gamma\lambda(n-1)]C_n}{1-\gamma}|b_n|\\ &=\sum_{n=2}^{\infty}X_n+\sum_{n=1}^{\infty}Y_n\\ &=1-X_1\leq 1, \end{aligned}$$

and so $f(z)\in clco\mathcal{T}_{\mathcal{H}}(F;\lambda,\gamma)$.

Conversely, suppose that $f(z)\in clco\mathcal{T}_{\mathcal{H}}(F;\lambda,\gamma)$. Setting

$$X_n=\frac{n[n-\gamma-\gamma\lambda(n-1)]C_n}{1-\gamma}|a_n|,\quad (0\leq X_n\leq 1, n\geq 2)$$

$$Y_n=\frac{n[n+\gamma-\gamma\lambda(n-1)]C_n}{1-\gamma}|b_n|,\quad (0\leq Y_n\leq 1, n\geq 1)$$

and $X_1 = 1 - \sum_{n=2}^{\infty} X_n - \sum_{n=1}^{\infty} Y_n$. Therefore ,$f(z)$ can be rewritten as

$$\begin{aligned} f(z) &= z - \sum_{n=2}^{\infty} a_n z^n + \sum_{n=1}^{\infty} \overline{b}_n \overline{z}^n \\ &= z - \sum_{n=2}^{\infty} \frac{1-\gamma}{[n-\gamma-\gamma\lambda(n-1)]C_n} X_n z^n + \sum_{n=1}^{\infty} \frac{1-\gamma}{[n+\gamma-\gamma\lambda(n-1)]C_n} Y_n \overline{z}^n \\ &= z + \sum_{n=2}^{\infty}(h_n(z)-z)X_n + \sum_{n=1}^{\infty}(g_n(z)-z)Y_n \\ &= z\{1 - \sum_{n=2}^{\infty} X_n - \sum_{n=1}^{\infty} Y_n\} + \sum_{n=2}^{\infty} h_n(z)X_n + \sum_{n=1}^{\infty} g_n(z)Y_n \\ &= \sum_{n=1}^{\infty}(X_n h_n(z) + Y_n g_n(z)), \end{aligned}$$

as required. □

4. **Inclusion results**

Now we show that $\mathcal{T}_{\mathcal{H}}(F;\lambda,\gamma)$ is closed under convex combinations of its member and also closed under the convolution product.

Theorem 4.1. *The family $\mathcal{T}_{\mathcal{H}}(F;\lambda,\gamma)$ is closed under convex combinations.*

Proof. For $i = 1, 2, \dots,$ suppose that $f_i \in \mathcal{T}_{\mathcal{H}}(F;\lambda,\gamma)$ where

$$f_i(z) = z - \sum_{n=2}^{\infty} a_{i,n} z^n + \sum_{n=2}^{\infty} \overline{b}_{i,n} \overline{z}^n.$$

Then, by Theorem 2.2

$$\sum_{n=2}^{\infty} \frac{n[n-\gamma-\gamma\lambda(n-1)]C_n}{(1-\gamma)} a_{i,n} + \sum_{n=1}^{\infty} \frac{n[n+\gamma-\gamma\lambda(n-1)]C_n}{(1-\gamma)} b_{i,n} \le 1. \tag{4.1}$$

For $\sum_{i=1}^{\infty} t_i = 1$, $0 \le t_i \le 1$, the convex combination of f_i may be written as

$$\sum_{i=1}^{\infty} t_i f_i(z) = z - \sum_{n=2}^{\infty} \left(\sum_{i=1}^{\infty} t_i a_{i,n} \right) z^n + \sum_{n=1}^{\infty} \left(\sum_{i=1}^{\infty} t_i \overline{b}_{i,n} \right) \overline{z}^n.$$

Using the inequality (2.4), we obtain

$$\sum_{n=2}^{\infty}\frac{n[n-\gamma-\gamma\lambda(n-1)]C_n}{1-\gamma}\left(\sum_{i=1}^{\infty}t_ia_{i,n}\right)+\sum_{n=1}^{\infty}\frac{n[n+\gamma-\gamma\lambda(n-1)]C_n}{1-\gamma}\left(\sum_{i=1}^{\infty}t_ib_{i,n}\right)$$
$$=\sum_{i=1}^{\infty}t_i\left(\sum_{n=2}^{\infty}\frac{n[n-\gamma-\gamma\lambda(n-1)]C_n}{1-\gamma}a_{i,n}+\sum_{n=1}^{\infty}\frac{n[n+\gamma-\gamma\lambda(n-1)]C_n}{1-\gamma}b_{i,n}\right)$$
$$\leq\sum_{i=1}^{\infty}t_i=1,$$

and therefore $\sum_{i=1}^{\infty}t_if_i\in\mathcal{T}_{\mathcal{H}}(F;\lambda,\gamma)$. □

Now, we will examine the closure properties of the class $\mathcal{T}_{\mathcal{H}}(F;\lambda,\gamma)$ under the generalized Bernardi-Libera-Livingston integral operator$\mathcal{L}_c(f)$ which is defined by

$$\mathcal{L}_c(f)=\frac{c+1}{z^c}\int_0^z t^{c-1}f(t)dt, c>-1.$$

Theorem 4.2. *Let $f(z)\in\mathcal{T}_{\mathcal{H}}(F;\lambda,\gamma)$. Then $\mathcal{L}_c(f(z))\in\mathcal{T}_{\mathcal{H}}(F;\lambda,\gamma)$*

Proof. From the representation of $\mathcal{L}_c(f(z))$, it follows that

$$\mathcal{L}_c(f)=\frac{c+1}{z^c}\int_0^z t^{c-1}\left[h(t)+\overline{g(t)}\right]dt.$$

$$=\frac{c+1}{z^c}\left(\int_0^z t^{c-1}\left(t-\sum_{n=2}^{\infty}a_nt^n\right)dt+\overline{\int_0^z t^{c-1}\left(\sum_{n=1}^{\infty}b_nt^n\right)dt}\right)$$
$$=z-\sum_{n=2}^{\infty}\frac{c+1}{c+n}a_nz^n+\sum_{n=1}^{\infty}\frac{c+1}{c+n}\ b_nz^n.$$

Using the inequality (2.4), we get

$$\sum_{n=1}^{\infty}\left(\frac{n[n-\gamma-\gamma\lambda(n-1)]}{1-\gamma}(\frac{c+1}{c+n}|a_n|)+\frac{n+\gamma-\gamma\lambda(n-1)}{1-\gamma}(\frac{c+1}{c+n}|b_n|)\right)C_n$$
$$\leq\ \sum_{n=1}^{\infty}\left(\frac{n[n-\gamma-\gamma\lambda(n-1)]}{1-\gamma}|a_n|+\frac{n+\gamma-\gamma\lambda(n-1)}{1-\gamma}|b_n|\right)C_n$$
$$\leq\ 2(1-\gamma),\ \text{ since } f(z)\in\mathcal{T}_{\mathcal{H}}(F;\lambda,\gamma).$$

Hence by Theorem 2.2, $\mathcal{L}_c(f(z))\in\mathcal{T}_{\mathcal{H}}(F;\lambda,\gamma)$. □

Concluding remarks. For suitable choices of $F(z)$, as we pointed out the $\mathcal{S}_{\mathcal{H}}(F;\lambda,\gamma)$ contains, various function class defined by linear operators such as the Carlson-Shaffer operator, the Ruscheweyh derivative operator, the Sălăgean operator, the fractional derivative operator, and so on. When $\lambda = 0$ the various results presented in this paper would provide interesting extensions and generalizations of those considered earlier for simpler harmonic function classes(see [7, 8, 9]. The details involved in the derivations of such specializations of the results presented in this paper are fairly straight- forward, hence omitted.

References

[1] Carlson, B. C., Shaffer, S. B., *Starlike and prestarlike hypergeometric functions,* SIAM, J. Math. Anal., **15** (2002), 737-745.

[2] Clunie, J., Sheil-Small, T., *Harmonic univalent functions,* Ann. Acad. Aci. Fenn. Ser. A. I. Math., **9** (1984), 3-25.

[3] Dziok, J., Raina, *Families of analytic functions associated with the Wright's generalized hypergeometric function,* Demonstratio Math., **37** (2004), no. 3, 533-542.

[4] Dziok, J., Srivastava, H. M., *Certain subclasses of analytic functions associated with the generalized hypergeometric function,* Intergral Transform Spec. Funct., **14** (2003), 7-18.

[5] Jahangiri, J. M., Silverman, H., *Harmonic univalent functions with varying arguments,* Internat. J. Appl. Math., **8** (2002), 267-275.

[6] Jahangiri, J. M., *Harmonic functions starlike in the unit disc.,* J. Math. Anal. Appl., **235** (1999), 470-477.

[7] Jahangiri, J. M., Murugusundaramoorthy, G., Vijaya, K., *Starlikeness of Rucheweyh type harmonic univalent functions*, J. Indian Acad. Math., **26** (2004), 191-200.

[8] Jahangiri, J. M., Murugusundaramoorthy, G., Vijaya, K., *Salagean-type harmonic univalent functions,* Southwest J. Pure Appl. Math., **2** (2002), 77-82.

[9] Murugusundaramoorthy, G., Vijaya, K., *A Subclass of harmonic functions associated with Wright hypergeometric functions,* Advanced Stud. Contemp. Math., **18** (2009), no.1, 87-95.

[10] Ruscheweyh, S., *New criteria for univalent functions,* Proc. Amer. Math. Soc., **49** (1975), 109-115.

[11] Silverman, H., *Harmonic univalent functions with negative coefficients,* J. Math. Anal. Appl., **220** (1998), 283-289.

[12] Srivastava, H. M., Owa, S., *Some characterization and distortion theorems involving fractional calculus, generalized hypergeometric functions, Hadamard products, linear operators and certain subclasses of analytic functions,* Nagoya Math. J., **106** (1987), 1-28.

[13] Wright, E. M., *The asymptotic expansion of the generalized hypergeometric function,* Proc. London. Math. Soc., **46** (1946), 389-408.

Corresponding author
Department of Mathematics
"1 Decembrie 1918" University of Alba Iulia
510009, Alba, Romania
E-mail address: dbreaz@uab.ro

School of Science
VIT University
Vellore 632014, India
E-mail address: gmsmoorthy@yahoo.com

School of Science
VIT University
Vellore 632014, India
E-mail address: kvijaya@vit.ac.in

STUDIA UNIV. "BABEŞ–BOLYAI", MATHEMATICA, Volume **LV**, Number 3, September 2010

ONE-SIDED CLEAN RINGS

GRIGORE CĂLUGĂREANU

Dedicated to Professor Grigore Ştefan Sălăgean on his 60^{th} *birthday*

Abstract. Replacing units by one-sided units in the definition of clean rings (and modules), new classes of rings (and modules) are defined and studied, generalizing most of the properties known in the clean case.

1. **Introduction**

For a ring with identity, we denote by $U(R)$ the units, $U_l(R)$ and $U_r(R)$ the left respectively right invertible elements of R (shortly, right-units or left-units), and by $N(R)$ the nilpotent elements.

An element in a ring R is *right (or left) clean* if it is a sum of an idempotent and a right (respectively left) unit. A ring R is *right clean* if all its elements are right clean and it is *left clean* if R^{op} is right clean. Moreover, it is *one-sided clean* if each element is left or right clean. These classes are included in the class of *almost clean* rings considered by McGovern ([8]: every element is a sum of a non-zero divisor and an idempotent) and studied further (in the commutative case) by Ahn and D. D. Anderson ([1]).

Further, a ring R is *weakly right exchange* if for every element $a \in R$ there are two orthogonal idempotents f, f' with $f \in aR$, $f' \in (1-a)R$, such that $f + f' \cong 1$.

In this paper the main results are the following

- *Let* $e^2 = e \in R$ *be such that* eRe *and* $(1-e)R(1-e)$ *are both right clean rings. Then* R *is a right clean ring.*

Received by the editors: 01.03.2010.

2000 *Mathematics Subject Classification.* 16L30, 16U60, 16D60.

Key words and phrases. Right clean, weakly right exchange rings and modules.

- *Any ring $R = U_l(R) \cup U_r(R) \cup N(R)$ is both right and left clean.*
- *Any right clean ring is weakly right exchange.*

  and,
- *A ring R is weakly right exchange if and only if for every $a \in R$ there are elements $b, c \in R$ such that $bab = b$, $c(1-a)c = c$, $ab(1-a)c = 0 = (1-a)cab$.*

Finally results on strongly respectively weakly one-sided clean rings are given.

2. **Right clean rings**

In the sequel we will merely state our results for right clean rings, but most of them have a left or one-sided analogue.

Obviously Dedekind finite (and in particular abelian or commutative) one-sided clean rings are (strongly) clean.

The following is immediate from definitions

Lemma 2.1. *(i) Every homomorphic image of a right clean ring is right clean.*

(ii) A direct product of rings $\prod R_i$ is right clean if and only if each R_i is right clean.

The next result is elementary. We supply a proof for later reference.

Proposition 2.2. *Let A, B be rings, ${}_AC_B$ a bimodule and $R = \begin{bmatrix} A & C \\ 0 & B \end{bmatrix}$. Then R is right clean if and only if A and B are right clean.*

Proof. If R is right clean, the maps $f : R \longrightarrow A$, $f\left(\begin{bmatrix} a & c \\ 0 & b \end{bmatrix}\right) = a$ and $g : R \longrightarrow B$, $g\left(\begin{bmatrix} a & c \\ 0 & b \end{bmatrix}\right) = b$ are ring epimorphisms, and so A, B are right clean by (i), previous Lemma.

Conversely, let $\begin{bmatrix} a & c \\ 0 & b \end{bmatrix} \in R$. Then there are $u_a \in U_l(A)$, $e_a = e_a^2 \in A$ with $a = u_a + e_a$ and a similar decomposition for b. Suppose $v_a u_a = 1 = v_b u_b$. Clearly

$\begin{bmatrix} a & c \\ 0 & b \end{bmatrix} = \begin{bmatrix} u_a & c \\ 0 & u_b \end{bmatrix} + \begin{bmatrix} e_a & 0 \\ 0 & e_b \end{bmatrix}$ where $\begin{bmatrix} e_a & 0 \\ 0 & e_b \end{bmatrix}^2 = \begin{bmatrix} e_a & 0 \\ 0 & e_b \end{bmatrix}$ and $\begin{bmatrix} u_a & c \\ 0 & u_b \end{bmatrix} \in U_l(R)$. Indeed, $\begin{bmatrix} v_a & -v_a c v_b \\ 0 & v_b \end{bmatrix}$ is a left inverse for $\begin{bmatrix} u_a & c \\ 0 & u_b \end{bmatrix}$. □

Remark 2.3. This property fails for one-sided clean rings A and B.

Proposition 2.4. *Let $e^2 = e \in R$ be such that eRe and $(1-e)R(1-e)$ are both right clean rings. Then R is a right clean ring.*

Proof. Using the Pierce decomposition of the ring R, let $\begin{bmatrix} a & x \\ y & b \end{bmatrix} \in R = \begin{bmatrix} eRe & eR(1-e) \\ (1-e)Re & (1-e)R(1-e) \end{bmatrix}$. For $u_1 u = e$ and $a = f + u$ in eRe, $v_1 v = 1-e$ and $b - y u_1 x = g + v$ in $(1-e)R(1-e)$, $\begin{bmatrix} a & x \\ y & b \end{bmatrix}$ decomposes into $\begin{bmatrix} f & 0 \\ 0 & g \end{bmatrix} + \begin{bmatrix} u & x \\ y & v + y u_1 x \end{bmatrix}$ and all we need is a left inverse for the latter. But this is $\begin{bmatrix} e & -u_1 x \\ 0 & 1-e \end{bmatrix} \begin{bmatrix} u_1 & 0 \\ 0 & v_1 \end{bmatrix} \begin{bmatrix} e & 0 \\ -y u_1 & 1-e \end{bmatrix} = \begin{bmatrix} u_1 + u_1 x v_1 y u_1 & -u_1 x v_1 \\ -v_1 y u_1 & v_1 \end{bmatrix}$. □

By induction, we have

Theorem 2.5. *If $1 = e_1 + e_2 + ... + e_n$ in a ring R where e_i are orthogonal idempotents and each $e_i R e_i$ is right clean, then R is right clean.*

Hence

Corollary 2.6. *If R is right clean then so is the matrix ring $\mathcal{M}_n(R)$.*

As in the clean case, we were not able to prove that corner rings (even full) of right (or left or one-sided) clean rings have the same property.

Only recently, classes of rings defined by equalities like: $R = U(R) \cup \mathrm{Id}(R)$ or, $R = U(R) \cup \mathrm{Id}(R) \cup -\mathrm{Id}(R)$ (here $\mathrm{Id}(R)$ denotes the idempotent elements of R), have received a great deal of attention (see [2] and [1] for the commutative case). In a similar vein, examples of right clean rings are provided by the next Proposition.

Proposition 2.7. *Any ring $R = U_l(R) \cup U_r(R) \cup N(R)$ is both right and left clean.*

Proof. We first show that *every right unit is right clean.* Let $a \in U_l(R)$ and $ba = 1$. Then $e = ab$ is an idempotent, so is $1 - e$, and using the decomposition $a = (1 - e) + (a + (e - 1))$ we have to find a left inverse for $a + (e - 1)$. But this is $ebe + (e - 1)$ since $(ebe + (e - 1))(a + (e - 1)) = ebea + ea - a + 0 + 1 - e = 1$ (because $ebea = abbaba = ab = e$).

Coming back to the proof of the Proposition, if $a \in N(R)$ it is well-known that $1 - a = u \in U(R)$ and so $a = 1 - u$ is even strongly clean. If $a \in U_l(R) \cup U_r(R)$ we just use the previous result and its left analogue. □

Remark 2.8. 1) In general $(a + (e - 1))(ebe + (e - 1)) = 1$ fails (equivalently $(e - 1)(b + 1) = 0$).

2) A slightly larger class is suggested by the following example which can be found in David Arnold's 1982 book ([3]): " In the endomorphism ring of a torsion-free strongly indecomposable Abelian group of finite rank, every element is a monomorphism (i.e., a non-zero divisor) or nilpotent".

3) Recently, H. Chen (see [5]) has proved that regular one-sided unit-regular rings are (though he does not consider this notion) exactly one-sided clean. So these are also examples for the notion we deal with.

3. **Right clean modules**

For the sake of completeness we first restate some results given in [4]: let $f, e \in S = \mathrm{End}(M_R)$ with $e^2 = e$, $A = \ker e$ and $B = \mathrm{im} e$.

Proposition 3.1. *$f - e$ is a monomorphism if and only if the restrictions $f|_A$, $(1 - f)|_B$ are monomorphisms and $fA \cap (1 - f)B = 0$.*

$f - e$ is an epimorphism if and only if and $fA + (1 - f)B = M$.

Lemma 3.2. *$f - e$ is a unit in S if and only if the restrictions $f|_A$, $(1 - f)|_B$ are monomorphisms and $fA \oplus (1 - f)B = M$.*

Observe that the (double) restriction (for the domain - we use | and for the codomain - we use $\widetilde{\ }$) $\widetilde{f|_A} : A \longrightarrow fA$ and $\widetilde{(1 - f)|_B} : B \longrightarrow (1 - f)B$ are always onto, so $f|_A$, $(1 - f)|_B$ are monomorphisms if and only if $\widetilde{f|_A}$ and $\widetilde{(1 - f)|_B}$ are isomorphisms. If $fA\cap(1-f)B=0$, then $u=\widetilde{f|_A}\oplus\widetilde{(1-f)|_B}:A\oplus B\longrightarrow fA\oplus(1-f)B$ is an isomorphism too (the codomain sum is direct, but not necessarily equal to M).

Therefore, our analogues are

Lemma 3.3. *Let $f,e\in S=\mathrm{End}(M_R)$ with $e^2=e$, $A=\ker e$ and $B=\mathrm{im} e$. Then $f-e\in U_l(S)$ if and only if the restrictions $f|_A$, $(1-f)|_B$ are monomorphisms, $fA\cap(1-f)B=0$ and the monomorphism $\widetilde{f|_A}\oplus\widetilde{(1-f)|_B}\in S$ has a left inverse in S.*

Proposition 3.4. *An element $f\in\mathrm{End}(M_R)$ is right clean if and only if there is a R-module decomposition $M=A\oplus B$ such that the restrictions $f|_A$, $(1-f)|_B$ are monomorphisms, $fA\cap(1-f)B=0$ and the monomorphism $\widetilde{f|_A}\oplus\widetilde{(1-f)|_B}:M\longrightarrow M$ has a left inverse in $\mathrm{End}(M_R)$.*

Remark 3.5. 1) Due to Theorem 2.5, finite direct sums of right clean modules are right clean.

2) Using Lemma 2.1, if $M_R=A\oplus B$ and $\mathrm{Hom}_R(A,B)=0$, then M is right clean if and only if A, B are right clean.

4. **Weakly exchange rings**

A ring is called (right) *exchange* (or *suitable* in [10]) if for every equation $a+a'=1$ there are idempotents $e\in aR$ and $e'\in a'R$ such that $e+e'=1$.

Since these idempotents are complementary, they must be orthogonal (and commute).

Recall that an idempotent $e\in R$ is *isomorphic* to 1 if and only if there are elements $u,v\in R$ with $vu=1$ and $e=uv$ (equivalently, $eR\cong R$ as right R-modules). If $e\neq 1$, such a ring is not Dedekind finite.

We define *weakly right exchange* rings R by the conditions: for every equation $a+a'=1$ there are two orthogonal idempotents f,f' with $f\in aR$, $f'\in a'R$, such that $f+f'\cong 1$ (obviously, since the idempotents f,f' are orthogonal, their sum is also an idempotent).

According to the above definition, there are elements $u, v \in R$ with $vu = 1$ and $f + f' = uv$.

Remark 4.1. We must require these two idempotents to be orthogonal. Indeed, if we require only $vu = 1$ and $f + f' = uv$ (i.e., $f + f' \cong 1$), then $f + f'$ is an idempotent ($uvuv = uv$) and this implies $f + f' = (f + f')^2 = ff' + f'f + f + f'$ and so only $ff' + f'f = 0$ (so not orthogonal nor commuting).

We can naturally associate with these (orthogonal but not necessarily complementary) idempotents two complementary idempotents, two by two isomorphic, namely vfu and $vf'u$.

1) $vfu + vf'u = v(f + f')u = vuvu = vu = 1$

2) $(vfu)^2 = vfuvfu = vf(f + f')fu = vfu$ (and so is $vf'u$)

3) $vfu \cong f$ and $vf'u \cong f'$: indeed, $vfu = (vfu)^2 = vf.uvfu \cong uvfu.vf = (f + f')f(f + f')f = f$, and similarly, $vf'u \cong f'$.

Remark 4.2. Related to lifting idempotents, since $f \in aR$ and $f' \in (1 - a)R$, all we can check is

$$f - a(f + f') = (1 - a)f - af' \in (a - a^2)R.$$

Obviously, if u is a unit, $f + f' = 1$ and $f - a \in (a - a^2)R$ shows that idempotents can be lifted.

Theorem 4.3. *Any right clean ring is weakly right exchange.*

Proof. If $a = u + e$ with $e^2 = e$ and $vu = 1$ (but not necessarily $uv = 1$), since $(uev)^2 = uevuev = uev$, we consider the idempotent

$$f' = uev.$$

Similarly, $(u(1 - e)v)^2 = u(1 - e)vu(1 - e)v = u(1 - e)v$ and we denote

$$f = u(1 - e)v = uv - uev.$$

Take $b = uv + (1 - a)v = (1 - e)v$ and $c = uv - av = -ev$. Then $ab = f$, $(1 - a)c = f'$ and so $f \in aR$ and $f' \in (1 - a)R$.

Thus $ff' = f'f = 0$ (these idempotents are orthogonal) and the sum $f + f' = uv$ (is an idempotent) isomorphic with 1.

Moreover $vfu = 1 - e$ is idempotent (and f, f' are isomorphic to complementary idempotents: $f \cong 1 - e$, and $f' \cong e$). $\square$

Remark 4.4. In a right clean ring the following is also true:

(a) We have $bf = b$ (i.e., $bab = b$) and $bf' = 0$ and similarly $cf' = c$ (i.e., $c(1-a)c = c$) and $cf = 0$. We also have $f'u = (1-f)u$ and $vf' = v(1-f)$.

(b) As in the clean initial case, $c = b+v$, and $a^2 - a = (a-1+f)u = (a-f')u$, and since this relation cannot be solved for $f - 1 + a$ or for $f' - a$ (in order to obtain $f - 1 + a$ or $f' - a$ in $(a - a^2)R$), idempotents cannot be lifted modulo any right (or left) ideal.

Actually, since $f \in aR$ and $f' \in (1-a)R$, all we can check is

$$f - a(f + f') = (1-a)f - af' \in (a - a^2)R.$$

(c) Obviously, if u is a unit, $f + f' = 1$ and $f - a \in (a - a^2)R$ shows that idempotents can be lifted.

It is well-known that exchange rings were ring theoretic described by Monk (see [9]). Here is the characterization for weakly right exchange rings.

Theorem 4.5. *A ring R is weakly right exchange if and only if for every $a \in R$ there are elements $b, c \in R$ such that $bab = b$, $c(1-a)c = c$, $ab(1-a)c = 0 = (1-a)cab$.*

Proof. If R is weakly right exchange, take orthogonal idempotents $f = at \in aR$ and $f' = (1-a)s \in (1-a)R$. Then $b = tat$ satisfies $bab = b$, $ab = f$ and $c = s(1-a)s$ satisfies $c(1-a)c = c$ and $f' = (1-a)c$. Since f, f' are orthogonal, we also have $ab(1-a)c = 0 = (1-a)ca$ and $(1-ab)(1-a)c + ab = (1-f)f' + f = f + f'$ is (an idempotent) isomorphic to 1.

Conversely, $f = ab$ and $f' = (1-a)c$ are readily checked to be orthogonal idempotents and $f + f' = (1-ab)(1-a)c + ab$ is (an idempotent) isomorphic to 1. $\square$

Similarly (right exchange and left exchange properties are equivalent), an **open problem** remains: are weakly right exchange rings also weakly left exchange?

5. Strongly one-sided clean rings

All the above one-sided clean notions have corresponding strongly versions.

Unlike the strongly clean version, here $ue = eu$ does not imply $u^{-1}e = eu^{-1}$. Therefore R is *strongly right clean* if it is right clean, $ue = eu$ and $ve = ev$.

Proposition 5.1. *Let $e^2 = e \in R$. An element $a \in eRe$ is strongly right clean in R if and only if a is strongly right clean in eRe.*

Proof. First notice that if $a \in eRe$ then $a(1-e) = (1-e)a = 0$ and so $a = ae = ea = eae$.

If $a = g + u$ is strongly right clean in R, then $(g+u)(1-e) = 0$ implies $1-e = -vg(1-e) = -gv(1-e)$ and so (by left multiplication with g) $g(1-e) = 1-e$. Thus (using also $(1-e)a = 0$) $eg = ge$. Therefore $eg = ege = ge$ is an idempotent in eRe. Since a and g commute with e, so is $u = a - g$. Hence $eu = eue = ue$ has eve as left inverse in eRe. Finally, $a = eae = e(g+u)e = ege + eue$ is strongly right clean in eRe.

Conversely, if $a = f + v$ is strongly right clean in eRe with $fv = vf$, $f^2 = f \in eRe$ and $w \in eRe$, $wv = e$ then $a = (a-u) + u$ is strongly right clean in R as $w + (1-e)$ is a left inverse for $u = v + (1-e)$ and $a - u = f + (1-e)$ is idempotent (sum of two orthogonal idempotents). □

Remark 5.2. The converse does not use $ev = ve$ from our definition.

Corollary 5.3. *Corner rings of strongly right clean rings are strongly right clean.*

Further, strongly right clean is not a Morita invariant property. The example given in [11], i.e. the localization $\mathbf{Z}_{(2)}$ can be used in order to disprove: R strongly right clean implies $\mathcal{M}_n(R)$ strongly right clean.

6. Weakly left-clean rings

We can get even closer to almost clean rings by weakening our right clean elements as follows: an element $a \in R$ is *weakly left-clean* if it is the sum of an idempotent e and a left nonzero-divisor (or left cancellable element) u of R, and a ring is *weakly left-clean* if all its elements share this property.

Remark 6.1. For regular rings, right clean and weakly left-clean coincide (Ex. 1.4, [7]).

In this setting, the *weak left-clean modules* are characterized by Proposition 4.4 in [4].

However, since images of non-zero divisors may not be non-zero divisors, properties for such rings are worse, compared with the right clean rings.

Direct products of weakly left-clean rings are weakly left-clean.

Homomorphic images of weakly left-clean rings may not be weakly left-clean.

Thus, (see Lemma 2.1) *if A, B are rings, ${}_AC_B$ a bimodule and* $R = \begin{bmatrix} A & C \\ 0 & B \end{bmatrix}$*, then R weakly left-clean generally does not imply A and B weakly left-clean.*

Nevertheless, the converse is true:

Proposition 6.2. *If A, B are weakly left-clean rings and ${}_AC_B$ is a bimodule then* $R = \begin{bmatrix} A & C \\ 0 & B \end{bmatrix}$ *is also weakly left-clean.*

Proof. With the notations in the proof of Lemma 2.1, if u_a, u_b are left non-zero divisors, so is $\begin{bmatrix} u_a & c \\ 0 & u_b \end{bmatrix}$ in R.

Indeed, it is readily checked that matrices of the type $\begin{bmatrix} x & y \\ 0 & z \end{bmatrix}$ with left non-zero divisors x and z, are left non-zero divisors in R. □

References

[1] Ahn, M.S., Anderson, D.D., *Weakly clean rings and almost clean rings*, Rocky M. Journal of Math., **36 (3)** (2006), 783-798.

[2] Anderson, D.D., Camillo, V.P. *Commutative rings whose elements are a sum of a unit and idempotent*, Comm. Algebra, **30(7)** (2002), 3327-3336.

[3] Arnold, D., *Finite rank torsion free abelian groups and rings*, Lecture Notes in Mathematics, 931. Springer-Verlag, Berlin-New York, 1982.

[4] C amillo, V. P., Khurana, D., Lam, T. Y., Nicholson, W. K., Zhou, Y., *Continuous modules are clean*, J. Algebra, **304(1)** (2006), 94-111.

[5] Chen, H., *Weakly stable conditions for exchange rings*, J. Korean Math. Soc., **44(4)** (2007), 903-913.

[6] Han, J., Nicholson, W. K., *Extensions of clean rings*, Comm. Algebra, **29(6)** (2001), 2589-2595.

[7] Lam, T., *Exercises in Classical Ring Theory*, Problem Books in Mathematics, Springer-Verlag, New York, 1995.

[8] McGovern, W. W., *Clean Semiprime f-Rings with Bounded Inversion*, Comm. Algebra, **31(7)** (2003), 3295-3304.

[9] Monk, G.S., *A characterization of exchange rings*, Proc. Amer. Math. Soc., **35** (1972), 349-353.

[10] Nicholson, W. K., *Lifting idempotents and exchange rings*, Trans. Amer. Math. Soc., **229** (1977), 269-278.

[11] Sanchez Campos, E., *On strongly clean rings*, Unpublished.

Faculty of Mathematics and Computer Science
Babeş-Bolyai University
1 Kogalniceanu Str., 400084 Cluj-Napoca, Romania
E-mail address: calu@math.ubbcluj.ro

STUDIA UNIV. "BABEŞ–BOLYAI", MATHEMATICA, Volume **LV**, Number 3, September 2010

CONVOLUTIONS OF UNIVALENT FUNCTIONS WITH NEGATIVE COEFFICIENTS USING A GENERALIZED SĂLĂGEAN OPERATOR

ADRIANA CĂTAŞ

Dedicated to Professor Grigore Ştefan Sălăgean on his 60^{th} *birthday*

Abstract. The object of this paper is to derive several interesting properties of the class $\mathcal{T}_j(n,\gamma,\alpha,\lambda)$ consisting of analytic and univalent functions with negative coefficients. Integral operators and modified Hadamard products of several functions belonging to the class $\mathcal{T}_j(n,\gamma,\alpha,\lambda)$ are studied.

1. **Introduction and definitions**

Let N denote the set of nonnegative integers $\{0,1,2,\ldots,n,\ldots\}$, $\mathbb{N}^*=\mathbb{N}\setminus\{0\}$ and let N_j, $j\in\mathbb{N}^*$, be the class of functions of the form

$$f(z)=z-\sum_{k=j+1}^{\infty}a_kz^k,\quad a_k\geq 0,\ k\geq j+1, \tag{1.1}$$

which are analytic in the open unit disc $U=\{z\in\mathbb{C}:\ |z|<1\}$.

We define the following generalized Sălăgean operator which has been introduced by Al-Oboudi in [1]

$$D^0f(z)=f(z) \tag{1.2}$$

$$D^1_\lambda f(z)=(1-\lambda)f(z)+\lambda zf'(z)=D_\lambda f(z),\quad \lambda>0 \tag{1.3}$$

$$D^n_\lambda f(z)=D_\lambda(D^{n-1}_\lambda f(z)). \tag{1.4}$$

Received by the editors: 26.04.2010.

2000 *Mathematics Subject Classification.* 30C45.

Key words and phrases. Analytic function, univalent function, generalized Sălăgean operator, negative coefficients, modified Hadamard product, integral operator.

If f is given by (1.1), then (1.2), (1.3) and (1.4) yield to a convolution with the functions

$$\psi(n,\lambda) = z - \sum_{k=j+1}^{\infty} [1+(k-1)\lambda]^n z^k$$

$$D_\lambda^n f(z) = \psi(n,\lambda) * f(z) = z - \sum_{k=j+1}^{\infty} c_k(n,\lambda) z^k$$

where

$$c_k(n,\lambda) = [1+(k-1)\lambda]^n, \quad \lambda \geq 0,\ n = 0,1,2,\dots \tag{1.5}$$

When $\lambda = 1$ we get Sălăgean differential operator [8].

Definition 1.1. [6] Let $\alpha, \gamma \in [0,1)$, $n \in \mathbb{N}$, $j \in \mathbb{N}^*$. A function f belonging to N_j is said to be in the class $\mathcal{T}_j(n,\gamma,\alpha,\lambda)$ if and only if

$$\operatorname{Re} \frac{D_\lambda^{n+1} f(z)/D_\lambda^n f(z)}{\gamma(D_\lambda^{n+1} f(z)/D_\lambda^n f(z)) + 1 - \gamma} > \alpha, \quad z \in U. \tag{1.6}$$

Remark 1.2. The class $\mathcal{T}_j(n,\gamma,\alpha,\lambda)$ is a generalization of the subclasses

i) $\mathcal{T}_1(0,0,\alpha,1) = \mathcal{T}^*(\alpha)$ and $\mathcal{T}_1(1,0,\alpha,1) = C(\alpha)$ defined and studied by Silverman [10] (these classes are the class of starlike functions of order α with negative coefficients and the class of convex functions of order α with negative coefficients respectively);

ii) $\mathcal{T}_j(0,0,\alpha,1)$ and $\mathcal{T}_j(1,0,\alpha,1)$ studied by Chatterjea [4] and Srivastava et al. [11];

iii) $\mathcal{T}_1(n,0,\alpha,1) = \mathcal{T}(n,\alpha)$ studied by Hur and Oh [7];

iv) $\mathcal{T}_1(0,\gamma,\alpha,1) = \mathcal{T}(\gamma,\alpha)$ and $\mathcal{T}_1(1,\gamma,\alpha,1) = C(\gamma,\alpha)$ studied by Altintaş and Owa [2];

v) $\mathcal{T}_1(n,\gamma,\alpha,1)$ studied by Aouf and Cho [3], [5].

Theorem 1.3. [6] *Let the function f be defined by (1.1). Then f belongs to the class $\mathcal{T}_j(n,\gamma,\alpha,\lambda)$ if and only if*

$$\sum_{k=j+1}^{\infty} [1+(k-1)\lambda]^n \{1+(k-1)\lambda - \alpha[1+\gamma(k-1)\lambda]\} a_k \leq 1-\alpha. \tag{1.7}$$

The result is sharp and the extremal functions are

$$f_k(z) = z - \frac{1-\alpha}{[1+(k-1)\lambda]^n\{1+(k-1)\lambda-\alpha[1+\gamma(k-1)\lambda]\}} \cdot z^k \tag{1.8}$$

with $k \geq j+1$.

2. Main results

Let the functions f_i be defined for $i = 1, 2, \ldots, m$, by

$$f_i(z) = z - \sum_{k=j+1}^{\infty} a_{k,i} z^k, \quad a_{k,i} \geq 0,\ j \in \mathbb{N}^*, \quad z \in U. \tag{2.1}$$

Theorem 2.1. *Let the functions* f_i *defined by (2.1) be in the class* $\mathcal{T}_j(n,\gamma,\alpha,\lambda)$, *for every* $i = 1, 2, \ldots, m$. *Then the functions* h *defined by*

$$h(z) = \sum_{i=1}^{m} d_i f_i(z), \quad d_i \geq 0 \tag{2.2}$$

where

$$\sum_{i=1}^{m} d_i = 1, \tag{2.3}$$

is also in the same class $\mathcal{T}_j(n,\gamma,\alpha,\lambda)$.

Proof. According to the definition of h, we can write

$$h(z) = z - \sum_{k=j+1}^{\infty} \left(\sum_{i=1}^{m} d_i a_{k,i} \right) z^k.$$

Further, since f_i are in the class $\mathcal{T}_j(n,\gamma,\alpha,\lambda)$ for every $i = 1, 2, \ldots, m$ we get

$$\sum_{k=j+1}^{\infty} c_k(n,\lambda)\{1+(k-1)\lambda-\alpha[1+\gamma(k-1)\lambda]\} a_{k,i} \leq 1-\alpha,$$

where $c_k(n,\lambda)$ is given by (1.5).

Hence we can see that

$$\sum_{k=j+1}^{\infty} c_k(n,\lambda)\{1+(k-1)\lambda-\alpha[1+\gamma(k-1)\lambda]\} \left(\sum_{i=1}^{m} d_i a_{k,i} \right) =$$

$$= \sum_{i=1}^{m} d_i \left(\sum_{k=j+1}^{\infty} c_k(n,\lambda)\{1+(k-1)\lambda-\alpha[1+\gamma(k-1)\lambda]\} a_{k,i} \right) \leq$$

$$\leq (1-\alpha)\sum_{i=1}^{m} d_i = 1-\alpha,$$

which implies that h is in $\mathcal{T}_j(n,\gamma,\alpha,\lambda)$. □

Theorem 2.2. *Let the function f defined by (1.1) be in the class $\mathcal{T}_j(n,\gamma,\alpha,\lambda)$ and let c be any real number such that $c > -1$. Then the function F defined by*

$$F(z) = \frac{c+1}{z^c}\int_0^z t^{c-1} f(t)dt \tag{2.4}$$

also belongs to the class $\mathcal{T}_j(n,\gamma,\alpha,\lambda)$.

Proof. From the representation (2.4) it follows that

$$F(z) = z - \sum_{k=j+1}^{\infty} b_k z^k$$

where

$$b_k = \left(\frac{c+1}{c+k}\right) a_k.$$

Therefore, we get

$$\sum_{k=j+1}^{\infty} c_k(n,\lambda)\{1+(k-1)\lambda - \alpha[1+\gamma(k-1)\lambda]\} b_k =$$

$$= \sum_{k=j+1}^{\infty} c_k(n,\lambda)\{1+(k-1)\lambda - \alpha[1+\gamma(k-1)\lambda]\}\left(\frac{c+1}{c+k}\right) a_k \leq$$

$$\leq \sum_{k=j+1}^{\infty} c_k(n,\lambda)\{1+(k-1)\lambda - \alpha[1+\gamma(k-1)\lambda]\} a_k \leq 1-\alpha.$$

Hence, by Theorem 1.3, $F \in \mathcal{T}_j(n,\gamma,\alpha,\lambda)$. □

Theorem 2.3. *Let c be a real number such that $c > -1$. If the function F belongs to the class $\mathcal{T}_j(n,\gamma,\alpha,\lambda)$ then the function f defined by (2.4) is univalent in $|z| < R^*$, where*

$$R^* = \inf_k \left[\frac{(c+1)c_k(n,\lambda)\{1+(k-1)\lambda - \alpha[1+\gamma(k-1)k]\}}{(1-\alpha)(c+k)k}\right]^{\frac{1}{k-1}} \tag{2.5}$$

and $c_k(n,\lambda)$ is given by (1.5). The result is sharp.

Proof. Let

$$F(z) = z - \sum_{k=j+1}^{\infty} a_k z^k, \quad a_k \geq 0.$$

It follows from (2.4) that

$$f(z) = \frac{z^{1-c}[z^c F(z)]'}{c+1} = z - \sum_{k=j+1}^{\infty} \left(\frac{c+k}{c+1}\right) a_k z^k.$$

In order to obtain the required result, it is sufficient to show that

$$|f'(z) - 1| < 1 \text{ whenever } |z| < R^*.$$

Now,

$$|f'(z) - 1| \leq \sum_{k=j+1}^{\infty} \frac{k(c+k)}{c+1} a_k |z|^{k-1}.$$

Thus, $|f'(z) - 1| < 1$ if

$$\sum_{k=j+1}^{\infty} \frac{k(c+k)}{c+1} a_k |z|^{k-1} < 1. \tag{2.6}$$

But, from Theorem 1.3 we have

$$\sum_{k=j+1}^{\infty} \frac{c_k(n,\lambda)\{1 + (k-1)\lambda - \alpha[1 + \gamma(k-1)\lambda]\}}{1-\alpha} a_k \leq 1. \tag{2.7}$$

Hence, by using (2.7), (2.6) will be satisfied if

$$\frac{k(c+k)}{c+1} |z|^{k-1} < \frac{c_k(n,\lambda)\{1 + (k-1)\lambda - \alpha[1 + \gamma(k-1)\lambda]\}}{1-\alpha}$$

that is

$$|z| < \left[\frac{(c+1)c_k(n,\lambda)\{1 + (k-1)\lambda - \alpha[1 + \gamma(k-1)\lambda]\}}{(1-\alpha)k(c+k)}\right]^{\frac{1}{k-1}}.$$

Therefore, f is univalent in $|z| < R^*$.

The sharpness follows if we take

$$f_k(z) = z - \frac{(1-\alpha)(c+k)}{(c+1)c_k(n,\lambda)\{1 + (k-1)\lambda - \alpha[1 + \gamma(k-1)\lambda]\}} z^k$$

$k \geq j+1$, $c_k(n,\lambda)$ is given by (1.5). □

Let the functions f_i, $(i = 1, 2)$ be defined by (2.1). The modified Hadamard product of f_1 and f_2 is defined here by

$$f_1 * f_2(z) = z - \sum_{k=j+1}^{\infty} a_{k,1}a_{k,2}z^k. \tag{2.8}$$

Theorem 2.4. *Let the function f_1 defined by (2.1) be in the class $\mathcal{T}_j(n, \gamma, \alpha, \lambda)$ and the function f_2 defined by (2.1) be in the class $\mathcal{T}_j(n, \gamma, \beta, \lambda)$. Then $f_1 * f_2$ belongs to the class $\mathcal{T}_j(n, \gamma, \delta, \lambda)$ where*

$$\delta = \delta(n, \gamma, \alpha, \beta, \lambda) = \tag{2.9}$$

$$= 1 - \frac{j\lambda(1-\gamma)(1-\alpha)(1-\beta)}{(1+j\lambda)^n[1+\lambda j - \alpha(1+\gamma j\lambda)][1+\lambda j - \beta(1+\gamma j\lambda)] - (1+\gamma j\lambda)(1-\alpha)(1-\beta)}.$$

The result is best possible for the functions

$$f_1(z) = z - \frac{1-\alpha}{[1+j\lambda - \alpha(1+j\lambda\gamma)](1+j\lambda)^n} z^{j+1} \tag{2.10}$$

and

$$f_2(z) = z - \frac{1-\beta}{[1+j\lambda - \beta(1+j\lambda\gamma)](1+j\lambda)^n} z^{j+1}. \tag{2.11}$$

Proof. Employing the technique used earlier by Schild and Silverman [9], we need to find the largest δ such that

$$\sum_{k=j+1}^{\infty} \frac{c_k(n,\lambda)\{1+(k-1)\lambda - \delta[1+\gamma(k-1)\lambda]\}}{1-\delta} a_{k,1}a_{k,2} \leq 1.$$

Since

$$\sum_{k=j+1}^{\infty} \frac{c_k(n,\lambda)\{1+(k-1)\lambda - \alpha[1+\gamma(k-1)\lambda]\}}{1-\alpha} a_{k,1} \leq 1 \tag{2.12}$$

and

$$\sum_{k=j+1}^{\infty} \frac{c_k(n,\lambda)\{1+(k-1)\lambda - \beta[1+\gamma(k-1)\lambda]\}}{1-\beta} a_{k,2} \leq 1, \tag{2.13}$$

by the Cauchy-Schwarz inequality, we have

$$\sum_{k=j+1}^{\infty} c_k(n,\lambda)\sqrt{A(\gamma,\alpha,\lambda;k)B(\gamma,\beta,\lambda;k)} \cdot \sqrt{a_{k,1}a_{k,2}} \leq 1 \tag{2.14}$$

where

$$A(\gamma,\alpha,\lambda;k) = \frac{1+(k-1)\lambda - \alpha[1+\gamma(k-1)\lambda]}{1-\alpha}$$

and

$$B(\gamma,\beta,\lambda;k)=\frac{1+(k-1)\lambda-\beta[1+\gamma(k-1)\lambda]}{1-\beta}.$$

Thus it is sufficient to show that

$$\sqrt{a_{k,1}a_{k,2}}\le\frac{(1-\delta)\sqrt{A(\gamma,\alpha,\lambda;k)B(\gamma,\beta,\lambda;k)}}{1+(k-1)\lambda-\delta[1+\gamma(k-1)\lambda]}.$$

Note that

$$\sqrt{a_{k,1}a_{k,2}}\le\frac{1}{c_k(n,\lambda)\sqrt{A(\gamma,\alpha,\lambda,k)B(\gamma,\beta,\lambda,k)}}.$$

Consequently, we need only to prove that

$$\frac{1}{c_k(n,\lambda)\sqrt{A(\gamma,\alpha,\lambda;k)B(\gamma,\beta,\lambda;k)}}\le\frac{(1-\delta)\sqrt{A(\gamma,\alpha,\lambda;k)B(\gamma,\beta,\lambda;k)}}{1+(k-1)\lambda-\delta[1+\gamma(k-1)\lambda]}$$

which is equivalent to

$$\delta\le 1-\frac{\lambda(k-1)(1-\gamma)(1-\alpha)(1-\beta)}{c_k(n,\lambda)E_\alpha(\gamma,\lambda;k)E_\beta(\gamma,\lambda;k)-[1+\gamma(k-1)\lambda](1-\alpha)(1-\beta)}$$

where

$$E_\alpha(\gamma,\lambda;k)=1+(k-1)\lambda-\alpha[1+\gamma(k-1)\lambda] \tag{2.15}$$

and

$$E_\beta(\gamma,\lambda,k)=1+(k-1)\lambda-\beta[1+\gamma(k-1)\lambda]. \tag{2.16}$$

If we denote

$$S(n,\gamma,\alpha,\beta,\lambda;k)= \tag{2.17}$$

$$=1-\frac{\lambda(k-1)(1-\gamma)(1-\alpha)(1-\beta)}{c_k(n,\lambda)E_\alpha(\gamma,\lambda;k)E_\beta(\gamma,\lambda;k)-[1+\gamma(k-1)\lambda](1-\alpha)(1-\beta)}$$

one obtains that $S(n,\gamma,\alpha,\beta,\lambda,k)$ is an increasing function of k, $k\ge j+1$. Letting $k=j+1$ in (2.17), we obtain

$$\delta\le S(n,\gamma,\alpha,\beta,\lambda;j+1).$$

This completes the proof of Theorem 2.4. □

Theorem 2.5. *Let the function* f_i, $(i = 1, 2)$ *defined by (2.1) be in the class* $\mathcal{T}_j(n, \gamma, \alpha, \lambda)$. *Then* $f_1 * f_2(z)$ *belongs to the class* $\mathcal{T}_j(n, \gamma, \beta, \lambda)$ *where*

$$\beta = \beta(n, \gamma, \alpha, \lambda) = \tag{2.18}$$

$$= 1 - \frac{j\lambda(1-\alpha)^2(1-\gamma)}{(1+j\lambda)^n[1+j\lambda-\alpha(1+\gamma j\lambda)]^2-(1-\alpha)^2(1+\gamma j\lambda)}.$$

The result is sharp.

Proof. Employing the technique used earlier by Schild and Silverman [9], we need to find the largest β such that

$$\sum_{k=j+1}^{\infty} c_k(n,\lambda)\{1+(k-1)\lambda-\beta[1+\gamma(k-1)\lambda]\}a_{k,1}a_{k,2} \leq 1-\beta.$$

The proof is the same as in the previous theorem.

Finally, by taking the functions f_i, given by

$$f_i(z) = z - \frac{1-\alpha}{[1+j\lambda-\alpha(1+j\lambda\gamma)](1+j\lambda)^n} z^{j+1}, \quad i = 1, 2 \tag{2.19}$$

we can see that the result is sharp. □

Corollary 2.6. *For* f_1 *and* f_2 *as in Theorem 2.4, the function*

$$h(z) = z - \sum_{k=j+1}^{\infty} \sqrt{a_{k,1}a_{k,2}}\, z^k \tag{2.20}$$

belongs to the class $\mathcal{T}_j(n, \gamma, \alpha, \lambda)$. *The result is sharp.*

Proof. This result follows from the Cauchy-Schwarz inequality. It is sharp for the same function as in Theorem 2.4. □

Corollary 2.7. *Let the functions* f_i, $(i = 1, 2, 3)$ *defined by (2.1) be in the class* $\mathcal{T}_j(n, \gamma, \alpha, \lambda)$. *Then* $f_1 * f_2 * f_3$ *belongs to the class* $\mathcal{T}_j(n, \gamma, \eta, \lambda)$ *where*

$$\eta = \eta(n, \gamma, \alpha, \lambda) = \tag{2.21}$$

$$= 1 - \frac{j\lambda(1-\alpha)^3(1-\gamma)}{(1+j\lambda)^{2n}[1+j\lambda-\alpha(1+\gamma j\lambda)]^3-(1+j\gamma\lambda)(1-\alpha)^3}.$$

The result is best possible for the functions

$$f_i(z) = z - \frac{1-\alpha}{[1+j\lambda-\alpha(1+j\lambda\gamma)](1+j\lambda)^n} z^{j+1}, \quad i = 1, 2, 3. \tag{2.22}$$

Proof. From Theorem 2.5 one obtains that $f_1 * f_2$ belongs to the class $\mathcal{T}_j(n,\gamma,\beta,\lambda)$ where β is given by (2.18). By using Theorem 2.4 we get $f_1 * f_2 * f_3$ belongs to the class $\mathcal{T}_j(n,\gamma,\eta,\lambda)$ where

$$\eta = \eta(n,\gamma,\alpha,\beta,\lambda) =$$
$$= 1 - \frac{j\lambda(1-\gamma)(1-\alpha)(1-\beta)}{(1+j\lambda)^n E_\alpha(\gamma,\lambda;j+1)E_\beta(\gamma,\lambda;j+1) - (1+\gamma j\lambda)(1-\alpha)(1-\beta)}$$

and $E_\alpha(\gamma,\lambda;j+1)$, $E_\beta(\gamma,\lambda;j+1)$ are given as in (2.15) and (2.16).

Hence, Corollary 2.7 follows at once. □

Theorem 2.8. *Let the function f_i, $(i = 1,2)$ defined by (2.1) be in the class $\mathcal{T}_j(n,\gamma,\alpha,\lambda)$. Then the function*

$$h(z) = z - \sum_{k=j+1}^{\infty} (a_{k,1}^2 + a_{k,2}^2) z^k, \tag{2.23}$$

belongs to the class $\mathcal{T}_j(n,\gamma,\eta,\lambda)$ where

$$\eta = \eta(n,\gamma,\alpha,\lambda) = \tag{2.24}$$
$$= 1 - \frac{2j\lambda(1-\alpha)^2(1-\gamma)}{(1+j\lambda)^n[1+j\lambda-\alpha(1+\gamma j\lambda)]^2 - 2(1-\alpha)^2(1+\gamma j\lambda)}.$$

The result is sharp for the functions f_i, $(i = 1,2)$ defined by (2.19).

Proof. By virtue of Theorem 1.3, one obtains

$$\sum_{k=j+1}^{\infty} \left[\frac{c_k(n,\lambda)\{1+(k-1)\lambda - \alpha[1+\gamma\lambda(k-1)]\}}{1-\alpha}\right]^2 a_{k,i}^2 \le$$
$$\le \left[\sum_{k=j+1}^{\infty} \frac{c_k(n,\lambda)\{1+(k-1)\lambda - \alpha[1+\gamma(k-1)\lambda]\}}{1-\alpha} a_{k,i}\right]^2 \le 1, \quad i = 1,2.$$

It follows that

$$\sum_{k=j+1}^{\infty} \frac{1}{2}\left[\frac{c_k(n,\lambda)\{1+(k-1)\lambda - \alpha[1+\gamma(k-1)\lambda]\}}{1-\alpha}\right]^2 (a_{k,1}^2 + a_{k,2}^2) \le 1.$$

Therefore, we need to find the largest η such that

$$\frac{c_k(n,\lambda)\{1+(k-1)\lambda - \eta[1+\gamma(k-1)\lambda]\}}{1-\eta} \le$$
$$\le \frac{1}{2}\left[\frac{c_k(n,\lambda)\{1+(k-1)\lambda - \alpha[1+\gamma(k-1)\lambda]\}}{1-\alpha}\right]^2$$

that is

$$\eta \leq 1 - \frac{2\lambda(1-\alpha)^2(k-1)(1-\gamma)}{c_k(n,\lambda)\{1+(k-1)\lambda-\alpha[1+\gamma(k-1)\lambda]\}^2 - 2(1-\alpha)^2[1+\gamma(k-1)\lambda]}.$$

Since

$$F(n,\gamma,\alpha,\lambda;k) =$$

$$= 1 - \frac{2\lambda(1-\alpha)^2(k-1)(1-\gamma)}{c_k(n,\lambda)\{1+(k-1)\lambda-\alpha[1+\gamma(k-1)\lambda]\}^2 - 2(1-\alpha)^2[1+\gamma(k-1)\lambda]}$$

is an increasing function of k, $(k \geq j+1)$ we get

$$\eta \leq F(n,\gamma,\alpha,\lambda;j+1)$$

and Theorem 2.8 follows at once. □

Theorem 2.9. *Let the function*

$$f_1(z) = z - \sum_{k=j+1}^{\infty} a_{k,1} z^k, \quad a_{k,1} \geq 0$$

be in the class $\mathcal{T}_j(n,\gamma,\alpha,\lambda)$ *and*

$$f_2(z) = z - \sum_{k=j+1}^{\infty} |a_{k,2}| z^k,$$

with $|a_{k,2}| \leq 1$. *Then* $f_1 * f_2$ *belongs to the class* $\mathcal{T}_j(n,\gamma,\alpha,\lambda)$.

Proof. Since

$$\sum_{k=j+1}^{\infty} c_k(n,\lambda)\{1+(k-1)\lambda-\alpha[1+\gamma(k-1)\lambda]\}|a_{k,1}a_{k,2}| =$$

$$= \sum_{k=j+1}^{\infty} c_k(n,\lambda)\{1+(k-1)\lambda-\alpha[1+\gamma(k-1)\lambda]\}a_{k,1}|a_{k,2}| \leq$$

$$\leq \sum_{k=j+1}^{\infty} c_k(n,\lambda)\{1+(k-1)\lambda-\alpha[1+\gamma(k-1)\lambda]\}a_{k,1} \leq 1-\alpha$$

by Theorem 1.3, one obtains that $f_1 * f_2$ belongs to the class $\mathcal{T}_j(n,\gamma,\alpha,\lambda)$. □

References

[1] Al-Oboudi, F. M., *On univalent functions defined by a generalized Sălăgean operator*, Inter. J. of Math. and Mathematical Sci., **27** (2004), 1429-1436.

[2] Altintaş, O., Owa, S., *On subclasses of univalent functions with negative coefficients*, Pusan Kyongnam Math. J., **4**(1988), no. 4, 41-46.

[3] Aouf, M. K., Cho, N. E., *On a certain subclass of analytic functions with negative coefficients*, Turkish J. Math., **22** (1998), no. 1, 15-32.

[4] Chatterjea, S. K., *On starlike functions*, J. Pure Math., **1** (1981), 23-26.

[5] Cho, N. E., Aouf, M. K., *Some applications of fractional calculus operators to a certain subclass of analytic functions with negative coefficients*, Turkish J. Math., **20** (1996), no. 4, 553-562.

[6] Darwish, H. E., *Certain Subclasses of Analytic Functions with Negative Coefficients Defined by Generalized Sălăgean Operator*, Gen. Math. Vol., 15, **4** (2007), 69-82.

[7] Hur, M. D., Oh, G. H., *On certain class of analytic functions with negative coefficients*, Pusan Kyongnam Math. J., **5** (1989), 69-80.

[8] Sălăgean, G. S., *Subclasses of univalent functions*, Lecture Notes in Math., Springer Verlag, **1013** (1983), 362-372.

[9] Schild, A., Silverman, H., *Convolutions of univalent functions with negative coefficients*, Ann. Univ. Mariae Curie-Sklodowska Sect. A., **29** (1975), 99-107.

[10] Silverman, H., *Univalent functions with negative coefficients*, Proc. Amer. Math. Soc., **51** (1975), 109-116.

[11] Srivastava, H. M., Owa, S., Chatterjea, S. K., *A note on certain classes of starlike functions*, Rend. Sem. Mat. Univ. Padova, **77** (1987), 115-124.

University of Oradea
Faculty of Science, Department of Mathematics
1 University Street, 410087, Oradea, Romania
E-mail address: acatas@gmail.com

ON A FRACTIONAL DIFFERENTIAL INCLUSION WITH BOUNDARY CONDITIONS

AURELIAN CERNEA

Abstract. We prove a Filippov type existence theorem for solutions of a fractional differential inclusion defined by a nonconvex set-valued map with Dirichlet boundary conditions. The method consists in application of the contraction principle in the space of selections of the set-valued map instead of the space of solutions.

1. **Introduction**

In this note we study the following problem

$$-D^{\alpha}x(t) \in F(t, x(t)) \quad a.e. \ ([0,1]), \tag{1.1}$$

$$x(0) = x(1) = 0, \tag{1.2}$$

where $\alpha \in (1, 2]$, D^{α} is the standard Riemann-Liouville fractional derivative and $F : I \times \mathbf{R} \to \mathcal{P}(\mathbf{R})$ is a set-valued map.

Differential equations with fractional order have recently proved to be strong tools in the modelling of many physical phenomena; for a complete bibliography on this topic we refer to [23]. As a consequence there was an intensive development of the theory of differential equations of fractional order ([2, 15, 20, 22, 24] etc.).

The study of fractional differential inclusions was initiated by El-Sayed and Ibrahim ([16]). Very recently several qualitative results for fractional differential inclusions were obtained in [3, 18].

Received by the editors: 04.10.2008.

2000 *Mathematics Subject Classification.* 34A60, 47H10.

Key words and phrases. Differential inclusion, fractional derivative, boundary value problem, fixed point, contractive set-valued map.

The present note is motivated by a recent paper of Ouahab ([23]) where several existence results concerning problem (1.1)-(1.2) are obtained. The aim of our paper is to provide an additional existence result for problem (1.1)-(1.2). More exactly, we prove a Filippov type result concerning the existence of solutions to the boundary value problem (1.1)-(1.2). We recall that for a differential inclusion defined by a lipschitzian set-valued map with nonconvex values, Filippov's theorem ([17]) consists in proving the existence of a solution starting from a given "quasi" or "almost" solution. Moreover, the result provides an estimation between the "quasi" solution and the solution obtained.

Our approach is different from the ones in [23] and consists in the application of the set-valued contraction principle in the space of selections of the set-valued map instead of the space of solutions. We note that the idea of applying the set-valued contraction principle due to Covitz and Nadler ([14]) in the space of derivatives of the solutions belongs to Tallos ([19], [25]) and it was already used for similar results obtained for other classes of differential inclusions ([5-13]).

The paper is organized as follows: in Section 2 we present the notations, definitions and the preliminary results to be used in the sequel and in Section 3 we prove the main result.

2. Preliminaries

In this short section we sum up some basic facts that we are going to use later.

Let (X,d) be a metric space and consider a set valued map T on X with nonempty closed values in X. T is said to be a λ -contraction if there exists $0<\lambda<1$ such that:

$$d_H(T(x),T(y))\leq \lambda d(x,y) \quad \forall x,y\in X,$$

where $d_H(.,.)$ denotes the Pompeiu-Hausdorff distance. Recall that the Pompeiu-Hausdorff distance of the closed subsets $A,B\subset X$ is defined by

$$d_H(A,B)=\max\{d^*(A,B),d^*(B,A)\},\quad d^*(A,B)=\sup\{d(a,B);a\in A\},$$

where $d(x,B) = \inf_{y\in B} d(x,y)$.

If X is complete, then every set valued contraction has a fixed point, i.e. a point $z \in X$ such that $z \in T(z)$ ([14]).

We denote by $Fix(T)$ the set of all fixed points of the set-valued map T. Obviously, $Fix(T)$ is closed.

Proposition 2.1. ([21]) *Let X be a complete metric space and suppose that T_1, T_2 are λ-contractions with closed values in X. Then*

$$d_H(Fix(T_1), Fix(T_2)) \le \frac{1}{1-\lambda} \sup_{z\in X} d(T_1(z), T_2(z)).$$

Let $I := [0,1]$, denote by $C(I,\mathbf{R})$ the Banach space of all continuous functions from I to $\mathbf{R}$ and by $L^1(I,\mathbf{R})$ we denote the Banach space of Lebegue integrable functions $u(.) : I \to \mathbf{R}$ endowed with the norm $||u||_1 = \int_0^1 |u(t)|dt$.

Definition 2.2. a) *The fractional integral of order $\alpha > 0$* of a Lebesgue integrable function $f(.) : (0,\infty) \to \mathbf{R}$ is defined by

$$I_0^\alpha f(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s)ds,$$

provided the right-hand side is pointwise defined on $(0,\infty)$ and $\Gamma(.)$ is the (Euler's) Gamma function.

b) *The fractional derivative of order $\alpha > 0$* of a continuous function $f(.) : (0,\infty) \to \mathbf{R}$ is defined by

$$\frac{d^\alpha f(t)}{dt^\alpha} = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_0^t (t-s)^{-\alpha+n-1} f(s)ds,$$

where $n = [\alpha] + 1$, provided the right-hand side is pointwise defined on $(0,\infty)$.

Definition 2.3. A function $x(.) \in C(I,\mathbf{R})$ is called a solution of problem (1.1)-(1.2) if there exists a function $v(.) \in L^1(I,\mathbf{R})$ with $v(t) \in F(t,x(t))$, *a.e.* (I) such that $-D^\alpha x(t) = v(t)$, *a.e.* (I) and conditions (1.2) are satisfied.

We need the following result ([1]).

Lemma 2.4. ([1]) *Let* $f(.) : [0,1] \to \mathbf{R}$ *be continuous. Then* $x(.)$ *is the unique solution of the boundary value problem*

$$D^\alpha x(t) + f(t) = 0 \quad t \in I, \tag{2.1}$$

$$x(0) = x(1) = 0, \tag{2.2}$$

if and only if

$$x(t) = \int_0^1 G(t,s)f(s)ds, \tag{2.3}$$

where

$$G(t,s) := \frac{1}{\Gamma(\alpha)} \begin{cases} [t(1-s)]^{\alpha-1} - (t-s)^{\alpha-1}, & \text{if} \quad 0 \le s < t \le 1, \\ [t(1-s)]^{\alpha-1}, & \text{if} \quad 0 \le t < s \le 1. \end{cases}$$

Note that $|G(t,s)| \le \frac{2}{\Gamma(\alpha)}$ $\forall t, s \in I$.

In the sequel we assume the following conditions on F.

Hypothesis 2.5. i) $F(.,.) : I \times \mathbf{R} \to \mathcal{P}(\mathbf{R})$ *has nonempty closed values and for every* $x \in \mathbf{R}$ $F(.,x)$ *is measurable.*

ii) *There exists* $L(.) \in L^1(I, \mathbf{R})$ *such that for almost all* $t \in I, F(t,\cdot)$ *is L(t)-Lipschitz in the sense that*

$$d_H(F(t,x), F(t,y)) \le L(t)|x-y| \quad \forall \ x, y \in \mathbf{R}$$

and $d(0, F(t,0)) \le L(t)$ *a.e.* (I).

3. The main result

We are able now to prove our main result.

Theorem 3.1. *Assume that Hypothesis 2.5 is satisfied and* $\frac{2}{\Gamma(\alpha)}||L||_1 < 1$. *Let* $y(.) \in C(I, \mathbf{R})$ *be such that there exists* $q(.) \in L^1(I, \mathbf{R})$ *with* $d(-D^\alpha y(t), F(t, y(t))) \le q(t)$, *a.e.* (I), $y(0) = y(1) = 0$.

Then for every $\varepsilon > 0$ *there exists* $x(.)$ *a solution of (1.1)-(1.2) satisfying for all* $t \in I$

$$|x(t) - y(t)| \le \frac{2}{\Gamma(\alpha) - 2||L||_1} \int_0^1 q(t)dt + \varepsilon. \tag{3.1}$$

Proof. For $u(.) \in L^1(I, \mathbf{R})$ define the following set valued maps:

$$M_u(t) = F(t, \int_0^1 G(t,s)u(s)ds), \quad t \in I,$$

$$T(u) = \{\phi(.) \in L^1(I, \mathbf{R}); \quad \phi(t) \in M_u(t) \quad a.e.\ (I)\}.$$

It follows from the definition and Lemma 2.4 that $x(.)$ is a solution of (1.1)-(1.2) if and only if $-D^\alpha x(.)$ is a fixed point of $T(.)$.

We shall prove first that $T(u)$ is nonempty and closed for every $u \in L^1(I, \mathbf{R})$. The fact that the set valued map $M_u(.)$ is measurable is well known. For example the map $t \to \int_0^1 G(t,s)u(s)ds$ can be approximated by step functions and we can apply Theorem III. 40 in [4]. Since the values of F are closed with the measurable selection theorem (Theorem III.6 in [4]) we infer that $M_u(.)$ admits a measurable selection ϕ. One has

$$|\phi(t)| \leq d(0, F(t,0)) + d_H(F(t,0), F(t, \int_0^1 G(t,s)u(s)ds) \leq$$

$$\leq L(t)\left(1 + \frac{2}{\Gamma(\alpha)}\int_0^1 |u(s)|ds\right),$$

which shows that $\phi \in L^1(I, \mathbf{R})$ and $T(u)$ is nonempty.

On the other hand, the set $T(u)$ is also closed. Indeed, if $\phi_n \in T(u)$ and $||\phi_n - \phi||_1 \to 0$ then we can pass to a subsequence ϕ_{n_k} such that $\phi_{n_k}(t) \to \phi(t)$ for a.e. $t \in I$, and we find that $\phi \in T(u)$.

We show next that $T(.)$ is a contraction on $L^1(I, \mathbf{R})$.

Let $u, v \in L^1(I, \mathbf{R})$ be given, $\phi \in T(u)$ and let $\delta > 0$. Consider the following set-valued map:

$$H(t) = M_v(t) \cap \left\{x \in \mathbf{R};\ |\phi(t) - x| \leq L(t)\left|\int_0^1 G(t,s)(u(s) - v(s))ds\right| + \delta\right\}.$$

From Proposition III.4 in [4], $H(.)$ is measurable and from Hypothesis 2.5 ii) $H(.)$ has nonempty closed values. Therefore, there exists $\psi(.)$ a measurable selection of $H(.)$. It follows that $\psi \in T(v)$ and according with the definition of the norm we have

$$||\phi - \psi||_1 = \int_0^1 |\phi(t) - \psi(t)|dt \leq \int_0^1 L(t)\left(\int_0^1 |G(t,s)|.|u(s) - v(s)|ds\right) dt+$$

$$\int_0^1 \delta dt = \int_0^1 \left(\int_0^1 L(t)|G(t,s)|dt\right)|u(s)-v(s)|ds + \delta \leq \frac{2}{\Gamma(\alpha)}||L||_1||u-v||_1 + \delta.$$

Since $\delta > 0$ was chosen arbitrary, we deduce that

$$d(\phi, T(v)) \leq \frac{2}{\Gamma(\alpha)}||L||_1||u-v||_1.$$

Replacing u by v we obtain

$$d_H(T(u), T(v)) \leq \frac{2}{\Gamma(\alpha)}||L||_1||u-v||_1,$$

thus $T(.)$ is a contraction on $L^1(I, \mathbf{R})$.

We consider next the following set-valued maps

$$F_1(t,x) = F(t,x) + q(t)[-1,1], \quad (t,x) \in I \times \mathbf{R},$$

$$M_u^1(t) = F_1(t, \int_0^1 G(t,s)u(s)ds), \quad t \in I, \quad u(.) \in L^1(I, \mathbf{R}),$$

$$T_1(u) = \{\psi(.) \in L^1(I, \mathbf{R}); \quad \psi(t) \in M_u^1(t) \quad a.e.\ (I)\}.$$

Obviously, $F_1(.,.)$ satisfies Hypothesis 2.5.

Repeating the previous step of the proof we obtain that T_1 is also a $\frac{2}{\Gamma(\alpha)}||L||_1$-contraction on $L^1(I, \mathbf{R})$ with closed nonempty values.

We prove next the following estimate

$$d_H(T(u), T_1(u)) \leq \int_0^1 q(t)dt. \tag{3.2}$$

Let $\phi \in T(u), \delta > 0$ and define

$$H_1(t) = M_u^1(t) \cap \{z \in \mathbf{R}; \quad |\phi(t) - z| \leq q(t) + \delta\}.$$

With the same arguments used for the set valued map $H(.)$, we deduce that $H_1(.)$ is measurable with nonempty closed values. Hence let $\psi(.)$ be a measurable selection of $H_1(.)$. It follows that $\psi \in T_1(u)$ and one has

$$||\phi - \psi||_1 = \int_0^1 |\phi(t) - \psi(t)|dt \leq \int_0^1 [q(t) + \delta]dt \leq \int_0^1 q(t) + \delta.$$

Since δ is arbitrary, as above we obtain (3.2).

We apply Proposition 2.1 and we infer that

$$d_H(Fix(T), Fix(T_1)) \leq \frac{1}{1 - \frac{2}{\Gamma(\alpha)}||L||_1} \int_0^1 q(t)dt.$$

Since $-D^\alpha y(.) \in Fix(T_1)$ it follows that there exists $u(.) \in Fix(T)$ such that for any $\varepsilon > 0$

$$|| - D^\alpha y - u||_1 \leq \frac{1}{1 - \frac{2}{\Gamma(\alpha)}||L||_1} \int_0^1 q(t)dt + \frac{\Gamma(\alpha)\varepsilon}{2}.$$

We define $x(t) = \int_0^1 G(t,s)u(s)ds$, $t \in I$ and we have

$$|x(t) - y(t)| \leq \int_0^1 |G(t,s)|.|u(s) + D^\alpha y(s)|ds \leq$$

$$\leq \frac{2}{\Gamma(\alpha)}||u + D^\alpha y||_1 \leq \frac{2}{\Gamma(\alpha) - 2||L||_1}||q||_1 + \varepsilon,$$

which completes the proof. □

Remark 3.2. The assumption in Theorem 3.1 is satisfied, in particular, for $y(.) = 0$ and therefore, via Hypothesis 2.5, with $q(.) = L(.)$. In this case, Theorem 3.1 provides an existence result for problem (1.1)-(1.2) together with a priori bounds for the solution. More precisely, the estimate (3.1) becomes in this case

$$|x(t)| \leq \frac{2||L||_1}{\Gamma(\alpha) - 2||L||_1} + \varepsilon, \quad \forall t \in I \tag{3.3}$$

In [23] among other existence results for problem (1.1)-(1.2) it is obtained in Theorem 4.9 the existence of solutions by applying, as usual in the study of the existence of solutions using fixed points, the contraction principle in the space of solutions. This approach does not allows to obtain an estimate as in (3.3).

On the other hand, in [23], Theorem 6.2, another Filippov type result for problem (1.1)-(1.2) is provided. Its proof follows Filippov's ideas and uses Kuratowsky and Ryll-Nardjewski selection theorem (e.g., [4]). More exactly, if the assumptions in Theorem 3.1 are satisfied then there exists $x(.) \in C(I, \mathbf{R})$ a solution of problem (1.1)-(1.2) such that, for all $t \in I$

$$|x(t) - y(t)| \leq \frac{2}{\Gamma(\alpha)}||q||_1 + \frac{16||q||_1^3}{\Gamma^2(\alpha)(\Gamma(\alpha) - 2||L||_1)}||L||_1. \tag{3.4}$$

We note that in our approach we obtain a "pointwise" estimate from a norm estimate and in general the estimates in (3.1) and (3.4) are not comparable. However, in particular cases the estimate in (3.1) is better that the one in (3.4). If the function $q(.) \in L^1(I, \mathbf{R})$ satisfies $\int_0^1 q(t)dt > \frac{\sqrt{\Gamma(\alpha)}}{2}$, than if we take in (3.1)

$$\varepsilon = \frac{4||q||_1||L||_1(4||q||_1^2 - \Gamma(\alpha))}{\Gamma^2(\alpha)(\Gamma(\alpha) - 2||L||_1)}$$

we obtain (3.4).

References

[1] Bai, Z., Lu, H., *Positive solutions for boundary value problem for functional differential equations*, J. Math. Anal. Appl., **311** (2005), 495-505.

[2] Benchohra, M., Henderson, J., Ntouyas, S. K., Ouahab, A., *Existence results for fractional order functional differential equations with infinite delay*, J. Math. Anal. Appl., **338** (2008), 1340-1350.

[3] Benchohra, M., Henderson, J., Ntouyas, S. K., Ouahab, A., *Existence results for fractional order functional differential inclusions with infinite delay and applications to control theory*, Fract. Calc. Appl. Anal., **11** (2008), 35-56.

[4] Castaing, C., Valadier, M., *Convex Analysis and Measurable Multifunctions*, Springer, Berlin, 1977.

[5] Cernea, A., *A Filippov type existence theorem for an infinite operational differential inclusion*, Stud. Cerc. Mat., **50** (1998), 15-22.

[6] Cernea, A., *Existence for nonconvex integral inclusions via fixed points*, Arch. Math (Brno), **39** (2003), 293-298.

[7] Cernea, A., *An existence theorem for some nonconvex hyperbolic differential inclusions*, Mathematica (Cluj), **45(68)** (2003), 121-126.

[8] Cernea, A., *An existence result for nonlinear integrodifferential inclusions*, Comm. Applied Nonlin. Anal., **14** (2007), 17-24.

[9] Cernea, A., *On the existence of solutions for a higher order differential inclusion without convexity*, Electron. J. Qual. Theory Differ. Equ., **8** (2007), 1-8.

[10] Cernea, A., *On a nonlocal boundary value problem for a second order differential inclusion*, An. Univ. Bucureşti Mat., **56** (2007), 281-288.

[11] Cernea, A., *On the existence of mild solutions of a nonconvex evolution inclusion*, Math. Commun., **13** (2008), 107-114.

[12] Cernea, A., *An existence result for a Fredholm-type integral inclusion*, Fixed Point Theory, **9** (2008), 441-447.

[13] Cernea, A., *An existence result for bilocal problems with mixed boundary conditions*, Bull. Math. Soc. Sci. Math. Roum., **51(99)** (2008), 137-143.

[14] Covitz, H., Nadler jr., S. B., *Multivalued contraction mapping in generalized metric spaces*, Israel J. Math., **8** (1970), 5-11.

[15] El-Sayed, A. M. A., *Nonlinear functional differential equations of arbitrary orders*, Nonlinear Anal., **33** (1998), 181-186.

[16] El-Sayed, A. M. A., Ibrahim, A. G. *Multivalued fractional differential equations of arbitrary orders*, Appl. Math. Comput., **68** (1995), 15-25.

[17] Filippov, A. F., *Classical solutions of differential equations with multivalued right hand side*, SIAM J. Control, **5** (1967), 609-621.

[18] Henderson, J., Ouahab, A., *Fractional functional differential inclusions with finite delay*, Nonlinear Anal., doi:10.1016/j.na.2008.02.111.

[19] Kannai, Z., Tallos, P., *Stability of solution sets of differential inclusions*, Acta Sci. Math. (Szeged), **61** (1995), 197-207.

[20] Kilbas, A., Trujillo, J., *Differential equations of fractional orders: Methods, results and problems II*, Appl. Anal., **81** (2002), 435-493.

[21] Lim, T. C., *On fixed point stability for set-valued contractive mappings with applications to generalized differential equations*, J. Math. Anal. Appl., **110** (1985), 436-441.

[22] Miller, K., Ross, B., *An Introduction to the Fractional Calculus and Differential Equations*, John Wiley, New York, 1993.

[23] Ouahab, A., *Some results for fractional boundary value problem of differential inclusions*, Nonlinear Anal., doi:10.1016/j.na.2007.10.021.

[24] Podlubny, I., *Fractional Differential Equations*, Academic Press, San Diego, 1999.

[25] Tallos, P., *A Filippov-Gronwall type inequality in infinite dimensional space*, Pure Math. Appl., **5** (1994), 355-362.

Faculty of Mathematics and Informatics
University of Bucharest
Academiei 14, Bucharest, Romania
E-mail address: acernea@fmi.unibuc.ro

STUDIA UNIV. "BABEŞ–BOLYAI", MATHEMATICA, Volume **LV**, Number 3, September 2010

SOME PROPERTIES OF A NEW CLASS OF CERTAIN ANALYTIC FUNCTIONS OF COMPLEX ORDER

LUMINIŢA-IOANA COTÎRLĂ

Dedicated to Professor Grigore Ştefan Sălăgean on his 60^{th} *birthday*

Abstract. In this paper we introduce a new class, $\mathcal{F}_n(b, M)$ of certain analytic functions. For this class we determine sufficient condition in terms of coefficients, coefficient estimate, and maximization theorem concerning the coefficients.

1. Introduction and preliminaries

Let A be the class of functions f of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \tag{1.1}$$

which are analytic and univalent in the open unit disk

$$U = \{z \in \mathbb{C} : |z| < 1\}.$$

For n a positive integer and $a \in \mathbb{C}$, let

$$\mathcal{H}[a, n] = \{f \in \mathcal{H} : \ f(z) = a + a_n z^n + \dots\}.$$

We use Ω to denote the class of functions $w(z)$ in U satisfying the conditions $w(0) = 0$ and $|w(z)| < 1$ for $z \in U$.

For a function $f(z)$ in A, we define

$$I^0 f(z) = f(z); \tag{1.2}$$

Received by the editors: 16.12.2009.
2000 *Mathematics Subject Classification.* 30C45.
Key words and phrases. Integral operator, analytic function, complex order.

$$I^1 f(z) = If(z) = \int_0^z f(t)t^{-1}dt; \tag{1.3}$$

and

$$I^n f(z) = I(I^{n-1}f(z)), \quad (z \in U \quad and \quad n \in \mathbb{N} = \{1, 2, 3, ...\}). \tag{1.4}$$

The integral operator I^n was introduced by Sălăgean in [8]. We note that, for a function $f \in A$ of the form (1.1)

$$I^n f(z) = z + \sum_{k=2}^{\infty} k^{-n} a_k z^k, \quad (z \in U, n \in \mathbb{N}).$$

In [1], [2], [3], [4], [7] and others papers, are introduced and studied certain subclasses of analitic functions defined by Sălăgean operator defined in [8]. Recently,in[5], [6] are studied some class of analytic functions defined by the integral operator defined in [8].

With the help of the integral operator I^n, we say that a function $f(z)$ belonging to A is in the class $\mathcal{F}_n(b, M)$ if and only if

$$\left| \frac{1}{b}\left(\frac{I^n f(z)}{I^{n+1}f(z)} - 1\right) + 1 - M \right| < M, \tag{1.5}$$

where $M > \frac{1}{2}$, $z \in U$ and $b \neq 0$ is complex number.

We shall need in this paper the following lemma:

Lemma 1.1. [4] *Let* $w(z) = \sum_{k=1}^{\infty} c_k z^k \in \Omega$ *if* μ *is any complex number, then*

$$|c_2 - \mu c_1^2| \leq max\{1, |\mu|\} \tag{1.6}$$

for any complex μ. *Equality in (1.6) may be attained for the functions* $w(z) = z^2$ *and* $w(z) = z$ *for* $|\mu| < 1$ *and* $|\mu| \geq 1$, *respectively.*

We know from [3] that $f(z) \in H_n(b, M)$ if and only if for $z \in U$

$$\frac{I^n f(z)}{I^{n+1}f(z)} = \frac{1 + [b(1+m) - m]w(z)}{1 - mw(z)},$$

where $m = 1 - \frac{1}{M}$, $(M > \frac{1}{2})$ and $w(z) \in \Omega$.

The purpose of the present paper is to determine sufficient condition in terms of coefficients for function belong to $\mathcal{F}_n(b, M)$, coefficient estimate, and maximization of $|a_3 - \mu a_2^2|$ on the class $\mathcal{F}_n(b, M)$ for complex value of μ.

2. Main results

Theorem 2.1. *Let the function $f(z)$ be defined by (1.1). If*

$$\sum_{k=2}^{\infty}\left\{\left(1-\frac{1}{k}\right)+\left|\frac{b(1+m)}{k}+m(1-\frac{1}{k})\right|\right\}\frac{|a_k|}{k^n}\leq|b(1+m)|, \tag{2.1}$$

holds, then $f(z)$ belongs to $\mathcal{F}_n(b,M)$, where $m=1-\frac{1}{M}$ $\quad(M>\frac{1}{2})$.

Proof. Suppose that the inequality (2.1) holds. Then we have for $z\in U$

$$|I^nf(z)-I^{n+1}f(z)|-|b(1+m)I^{n+1}f(z)+m(I^nf(z)-I^{n+1}f(z)|$$

$$=\left|\sum_{k=2}^{\infty}\frac{1}{k^n}(1-\frac{1}{k})a_kz^k\right|-\left|b(1+m)\left\{z+\sum_{k=2}^{\infty}\frac{a_k}{k^{n+1}}z^k\right\}+m\sum_{k=2}^{\infty}\frac{1}{k^n}(1-\frac{1}{k})a_kz^k\right|$$

$$\leq\sum_{k=2}^{\infty}\frac{1}{k^n}\left(1-\frac{1}{k}\right)|a_k|r^k-\left\{b(1+m)|r-\sum_{k=2}^{\infty}\left|\frac{b(1+m)}{k}+m\left(1-\frac{1}{k}\right)\right|\frac{|a_k|}{k^n}r^k\right\}$$

$$=\sum_{k=2}^{\infty}\frac{1}{k^n}|a_k|r^k\left\{\left(1-\frac{1}{k}\right)+\left|\frac{b(1+m)}{k}+m\left(1-\frac{1}{k}\right)\right|\right\}-|b(1+m)|r.$$

Letting $r\to -1$, then we have

$$|I^nf(z)-I^{n+1}f(z)|-|b(1+m)I^{n+1}f(z)+m(I^nf(z)-I^{n+1}f(z)|$$

$$=\sum_{k=2}^{\infty}\left\{\left(1-\frac{1}{k}\right)+\left|\frac{b(1+m)}{k}+m\left(1-\frac{1}{k}\right)\right|\right\}\frac{1}{k^n}|a_k|r^k-|b(1+m)|\leq 0,$$

by (2.1). Hence, it follows that

$$\left|\frac{\frac{I^nf(z)}{I^{n+1}f(z)}-1}{b(1+m)+m\{\frac{I^nf(z)}{I^{n+1}f(z)}-1\}}\right|<1,\quad z\in U.$$

Letting

$$w(z)=\frac{\frac{I^nf(z)}{I^{n+1}f(z)}-1}{b(1+m)+m\{\frac{I^nf(z)}{I^{n+1}f(z)}-1\}},$$

then $w(0)=0$, $w(z)$ is analytic in $|z|<1$ and $|w(z)|<1$. Hence, we have

$$\frac{I^nf(z)}{I^{n+1}f(z)}=\frac{1+[b(1+m)-m]w(z)}{1-mw(z)},\quad m=1-\frac{1}{M},\quad M>\frac{1}{2},w(z)\in\Omega,$$

and this shows that $f(z)$ belongs to $\mathcal{F}_n(b,M)$.

Theorem 2.2. *Let the function $f(z)$defined by (1.1)be in the class $\mathcal{F}_n(b, M)$, $z \in U$.*
a) For

$$2m\left(1-\frac{1}{k}\right)Re\{b\} > \left(1-\frac{1}{k}\right)^2(1-m)-|b|^2(1+m),$$

let

$$N=\left[\frac{2m\left(1-\frac{1}{k}\right)Re\{b\}}{\left(1-\frac{1}{k}\right)^2(1-m)-|b|^2(1+m)}\right], \quad k=1,3,...,j-1.$$

Then

$$|a_j| \le \frac{1}{\frac{1}{j^n}(1-\frac{1}{j})!}\prod_{k=2}^{j}\left|\frac{b(1+m)}{k}+\left(\frac{k-2}{k}\right)m\right|, \tag{2.2}$$

for $j = 2, 3, ..., N + 2$; and

$$|a_j| \le \frac{1}{\frac{1}{j^n}(1-\frac{1}{j})(N+1)!}\prod_{k=2}^{N+3}\left|\frac{b(1+m)}{k}+\left(\frac{k-2}{k}\right)m\right|, \tag{2.3}$$

for $j > N + 2$.

b) If

$$2m\left(1-\frac{1}{k}\right)Re\{b\} \le \left(1-\frac{1}{k}\right)^2(1-m)-|b|^2(1+m),$$

then

$$|a_j| \le \frac{(1+m)|b|}{\frac{1}{j^n}(1-\frac{1}{j})}, \quad for j \ge 2, \tag{2.4}$$

where $m = 1 - \frac{1}{M}$ $\quad (M > \frac{1}{2})$ and $b \neq 0$ complex.

Proof. Since $f(z) \in \mathcal{F}_n(b, M)$, from

$$\frac{I^n f(z)}{I^{n+1} f(z)} = \frac{1+[b(1+m)-m]w(z)}{1-mw(z)},$$

where $m = 1 - \frac{1}{M}$ $\quad (M > \frac{1}{2})$ and $w(z) \in \Omega$, we have that

$$\sum_{k=2}^{\infty}\frac{1}{k^n}\left(1-\frac{1}{k}\right)a_k z^k = w(z)\left\{z(1+m)b+\sum_{k=2}^{\infty}\frac{1}{k^n}\left[\frac{b(1+m)}{k}+m\left(1-\frac{1}{k}\right)\right]a_k z^k\right\}. \tag{2.5}$$

The equality (2.5) can be written in the form

$$\sum_{k=2}^{j}\frac{1}{k^n}\left(1-\frac{1}{k}\right)a_k z^k+\sum_{k=2}^{\infty}d_k z^k =$$

$$= \left\{ b(1+m)z + \sum_{k=2}^{j-1} \frac{1}{k^n} \left[\frac{b(1+m)}{k} + m\left(1 - \frac{1}{k}\right) \right] a_k z^k \right\} w(z),$$

where d_j's are some appropriates complex numbers. Then since $|w(z)| < 1$, we have

$$\left| \sum_{k=2}^{j} \frac{1}{k^n} \left(1 - \frac{1}{k}\right) a_k z^k + \sum_{k=j+1}^{\infty} d_k z^k \right| \leq \tag{2.6}$$

$$\left| b(1+m)z + \sum_{k=2}^{j-1} \frac{1}{k^n} \left[\frac{b(1+m)}{k} + m\left(1 - \frac{1}{k}\right) \right] a_k z^k \right|.$$

Squaring both sides of (2.6) and integrating round $|z| = r < 1$,we get, after taking the limit with $r \to 1$

$$\frac{1}{j^{2n}}(1 - \frac{1}{j})^2 |a_j|^2 \leq (1+m)^2 |b|^2 + \tag{2.7}$$

$$+ \sum_{k=2}^{j-1} \frac{1}{k^{2n}} \left\{ \left| \frac{b(1+m)}{k} + m\left(1 - \frac{1}{k}\right) \right|^2 - \left(1 - \frac{1}{k}\right)^2 \right\} |a_k|^2.$$

Now there may be following two cases:

(a) Let

$$\frac{2m(k-1)Re\{b\}}{k^2} > \frac{(k-1)^2(1-m)}{k^2} - \frac{(1+m)|b|^2}{k^2}.$$

Suppose that $j \leq n+2$.Then for $j = 2$, (2.7) gives

$$|a_2| \leq (1+m)|b| 2^{n+1}$$

which gives (2.2) for $j = 2$. We establish (2.2), by mathematical induction. Suppose (2.2)is valid for $k = 2, 3, ..., j-1$. Then it follows from (2.7)

$$\frac{1}{j^{2n}} \left(1 - \frac{1}{j}\right)^2 |a_j|^2 \leq$$

$$(1+m)^2|b|^2 + \sum_{k=2}^{j-1} \frac{1}{k^{2n}} \left\{ \left| \frac{b(1+m)}{k} + m\left(1 - \frac{1}{k}\right) \right|^2 - \left(1 - \frac{1}{k}\right)^2 \right\} \times$$

$$\times \frac{1}{\frac{1}{k^{2n}}((1 - \frac{1}{k})!)^2} \prod_{p=2}^{k} \left| \frac{b(1+m)}{k} + m\left(\frac{p+2}{p}\right) \right|^2$$

$$= \frac{1}{((1 - \frac{1}{j})!)^2} \prod_{k=2}^{j} \left| \frac{b(1+m)}{k} + \left(\frac{k-2}{k}\right) m \right|^2.$$

Thus, we get

$$|a_j| \leq \frac{1}{\frac{1}{j^n}(1-\frac{1}{j})!} \prod_{k=2}^{j} \left| \frac{b(1+m)}{k} + \left(\frac{k-2}{k}\right) m \right|,$$

which completes the proof of (2.2). Next, we suppose $j > N+2$. Then (2.7) gives

$$\frac{1}{j^{2n}}(1-\frac{1}{j})^2 |a_j|^2 \leq$$

$$\leq (1+m)^2|b|^2 + \sum_{k=2}^{N+2} \frac{1}{k^{2n}} \left\{ \left| \frac{b(1+m)}{k} + m\left(1-\frac{1}{k}\right) \right|^2 - \left(1-\frac{1}{k}\right)^2 \right\} |a_k|^2 +$$

$$+ \sum_{k=N+3}^{j-1} \frac{1}{k^{2n}} \left\{ \left| \frac{b(1+m)}{k} + m\left(1-\frac{1}{k}\right) \right|^2 - \left(1-\frac{1}{k}\right)^2 \right\} |a_k|^2 \leq$$

$$\leq (1+m)^2|b|^2 + \sum_{k=2}^{N+2} \frac{1}{k^{2n}} \left\{ \left| \frac{b(1+m)}{k} + m\left(1-\frac{1}{k}\right) \right|^2 - \left(1-\frac{1}{k}\right)^2 \right\} |a_k|^2.$$

On substituting upper estimates for $a_2, a_3, ..., a_{N+2}$ obtained above, and simplifying, we obtain (2.3).

(b) Let

$$2m\left(1-\frac{1}{k}\right) Re\{b\} \leq \left(1-\frac{1}{k}\right)^2 (1-m) - (1+m)|b|^2,$$

then it follows from (2.7)

$$\frac{1}{j^{2n}} \left(1-\frac{1}{j}\right)^2 |a_j|^2 \leq (1+m)^2|b|^2, \quad (j \geq 2)$$

which prove (2.4).

Theorem 2.3. *If a function $f(z)$ defined by (1.1) is in the class $\mathcal{F}_n(b, M)$ and μ is any complex number, then*

$$|a_3 - \mu a_2^2| \leq \frac{3^{n+1}}{2} |b(1+m)| max\{1, |d|\} \tag{2.8}$$

where

$$d = \frac{b(1+m)}{2 \cdot 3^{n+1}}[2^{2n+4}\mu - 3^{n+1}] - \frac{m}{2}. \tag{2.9}$$

The result is sharp.

Proof. Since $f(z) \in \mathcal{F}_n(b, M)$, we have

$$w(z) = \frac{I^n f(z) - I^{n+1} f(z)}{[b(1+m)-m]I^{n+1}f(z) + mI^n f(z)} =$$

$$= \frac{\sum_{k=2}^{\infty} \frac{a_k}{k^n} z^{k-1} \left(1 - \frac{1}{k}\right)}{b(1+m) + \sum_{k=2}^{\infty} \frac{a_k}{k^n} z^{k-1} \left[\frac{b(1+m)}{k} + m\left(1 - \frac{1}{k}\right)\right]} =$$

$$= \frac{\sum_{k=2}^{\infty} \frac{a_k}{k^n} z^{k-1} \left(1 - \frac{1}{k}\right)}{b(1+m)} \times \left\{ 1 + \frac{\sum_{k=2}^{\infty} \frac{a_k}{k^n} z^{k-1} \left[\frac{b(1+m)}{k} + m\left(1 - \frac{1}{k}\right)\right]}{b(1+m)} \right\}. \quad (2.10)$$

Now compare the coefficients of z and z^2 on both sides of (2.10). Thus we obtain

$$a_2 = 2^{n+1} b(1+m) c_1 \quad (2.11)$$

and

$$a_3 = \frac{3^{n+1} b(1+m)}{2} \left\{ c_2 + \left[\frac{b(1+m)}{2} + \frac{m}{2}\right] c_1^2 \right\}. \quad (2.12)$$

Hence

$$a_3 - \mu a_2^2 = \frac{3^{n+1}}{2} b(1+m)\{c_2 - c_1^2 d\}, \quad (2.13)$$

where

$$d = \frac{b(1+m)}{2 \cdot 3^{n+1}} [2^{2n+4}\mu - 3^{n+1}] - \frac{m}{2}.$$

Taking modulus both sides in (2.13), we have

$$|a_3 - \mu a_2^2| \leq \frac{3^{n+1}}{2} |b(1+m)| \cdot |c_2 - d c_1^2|. \quad (2.14)$$

Using Lemma 1.1.in (2.14), we have

$$|a_3 - \mu a_2^2| \leq \frac{3^{n+1}}{2} |b(1+m)| max\{1, |d|\}.$$

Finally, the assertion (2.8) of Theorem 2.3. is sharp in view of the fact that the assertion (1.6) of Lemma 1.1 is sharp.

References

[1] Alintas, O., Ozkan, O., Srivastava, H. M., *Majorization by starlike functions of complex order*, Complex Variables Theory Appl., **46** (2001), 207-218.

[2] Al-Oboudi, F. M., *On univalent functions defined by a generalized Sălăgean operator*, Int. J. Math. Sci., **2** (2004), 1429-1436.

[3] Aouf, M. K., Al-Amiri, A. M., *On certain fractional operators for certain subclass of prestarlike functions defined by Sălăgean operator*, J. Fract. Calc., **22** (2002), 47-56.

[4] Aouf, M. K., Darwish, H. E., Attiya, A. A., *On a class of certain analytic functions of complex order*, Indian J.Pure Appl. Math., **32** (2001), 1443-1452.

[5] Cotîrlă, L. I., *Harmonic multivalent functions defined by integral operator*, Studia Univ. Babeş-Bolyai, **1** (2009), 65-74.

[6] Cotîrlă, L. I., *A differential sandwich theorem for analytic functions defined by the integral operator*, Studia Univ. Babeş-Bolyai, **2** (2009), 13-22.

[7] G. Murugusundaramoorthy, H. M. Srivastava, *Neighorhoods of certain classes of analytic functions of complex order*, J. Pure Appl. Math., **52**, Article **24** (2004), 1-8 (electronic).

[8] Sălăgean, G. S. *Subclasses of univalent functions*, Complex Analysis, Fifth Romanian-Finnish Seminar, Part 1 (Bucharest, 1981), Lecture Notes in Math., vol. 1013, Springer, Berlin, 1983, 362-372.

Babeş-Bolyai University
Faculty of Mathematics and Computer Science
400084 Cluj-Napoca, Romania
E-mail address: luminita.cotirla@yahoo.com

STUDIA UNIV. "BABEŞ–BOLYAI", MATHEMATICA, Volume **LV**, Number 3, September 2010

CLASSES OF MEROMORPHIC FUNCTIONS DEFINED BY THE EXTENDED SĂLĂGEAN OPERATOR

JACEK DZIOK

Dedicated to Professor Grigore Ştefan Sălăgean on his 60^{th} birthday

Abstract. In the paper, we define classes of meromorphic functions, in terms of the extended Sălăgean operator. By using Jack's Lemma and the Briot-Bouquet differential subordination we obtain some inclusion relations for defined classes.

1. **Introduction**

Let $\mathcal{A}$ denote the class of functions which are *analytic* in $\mathcal{U} := \mathcal{U}(1)$, where $\mathcal{U}(R) := \{z : |z| < R\}$, $0 < R \leq 1$. By Ω we denote the class of the Schwarz functions, i.e. the class of functions $\omega \in \mathcal{A}$, such that

$$\omega(0) = 0,\ |\omega(z)| < 1 \quad (z \in \mathcal{U})\,.$$

For complex parameters β, γ and functions $h \in \mathcal{A}$, $\omega \in \Omega$, we consider the first-order differential equation of the form

$$q(z) + \frac{zq'(z)}{\beta q(z) + \gamma} = (h \circ \omega)(z), \quad q(0) = h(0) = 1. \tag{1.1}$$

If there exist a function $\omega \in \Omega$, such that the function $q \in \mathcal{A}$ is a solution of the Cauchy problem (1.1) then we write

$$q(z) + \frac{zq'(z)}{\beta q(z) + \gamma} \prec h(z). \tag{1.2}$$

The expression (1.2) is a first-order differential subordination and it is called the Briot-Bouquet differential subordination.

Received by the editors: 01.05.2010.
2000 *Mathematics Subject Classification.* 30C45, 30C55.
Key words and phrases. meromorphic functions, differential subordination, Jack's Lemma, Sălăgean operator.

More general, we say that a function $f \in \mathcal{A}$ is *subordinate* to a function $F \in \mathcal{A}$, and write $f(z) \prec F(z)$, if and only if there exists a function $\omega \in \Omega$, such that

$$f(z) = (F \circ \omega)(z) \quad (z \in \mathcal{U}).$$

Moreover, we say that f is subordinate to F in $\mathcal{U}(R)$, if $f(Rz) \prec F(Rz)$. We shall write

$$f(z) \prec_R F(z)$$

in this case. In particular, if F is univalent in $\mathcal{U}$ we have the following equivalence (cf. [5]):

$$f(z) \prec F(z) \iff f(0) = F(0) \text{ and } f(\mathcal{U}) \subset F(\mathcal{U}).$$

Let $\mathcal{M}$ denote the class of functions f of the form

$$f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n z^n, \tag{1.3}$$

which are analytic in $\mathcal{D} = \mathcal{U} \setminus \{0\}$. By $f * g$ we denote the *Hadamard product* (*or convolution*) of f, $g \in \mathcal{M}$, defined by

$$(f * g)(z) = \left(\sum_{n=-1}^{\infty} a_n z^n \right) * \left(\sum_{n=-1}^{\infty} b_n z^n \right) := \sum_{n=-1}^{\infty} a_n b_n z^n.$$

Let λ, σ be positive real numbers. Motivated by the Sălăgean operator [6] we consider the linear operator $D_\sigma^\lambda : \mathcal{M} \to \mathcal{M}$ defined by

$$D_\sigma^\lambda f(z) = \left(f * h_{\lambda,\sigma} \right)(z),$$

where

$$h_{\lambda,\sigma}(z) = \frac{1}{z} + \sum_{n=1}^{\infty} \left(\frac{n+\sigma+1}{\sigma} \right)^{\lambda} z^n \quad (z \in \mathcal{D}).$$

It is closely related to Cho and Srivastava operator [1] (see also [7]) and the multiplier transformations studied by Flett [3].

For a function $f \in \mathcal{M}$ we have

$$z \left[D_\sigma^\lambda f(z) \right]' = \sigma D_\sigma^{\lambda+1} f(z) - (1+\sigma) D_\sigma^\lambda f(z). \tag{1.4}$$

A function $f \in \mathcal{A}$ of the form

$$f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n \quad (z \in \mathcal{U}(r))$$

is said to be p *-valently starlike* in $\mathcal{U}(r)$ if and only if

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > 0 \quad (z \in \mathcal{U}(r);\ 0 < r \leq 1).$$

Note that all functions p-valently starlike in $\mathcal{U}(r)$ are p-valent in $\mathcal{U}(r)$. In particular we have

$$f(z) \neq 0 \quad (z \in \mathcal{U}(r) \setminus \{0\}).$$

Let h be a function convex in $\mathcal{U}$ with

$$h(0) = 1,\ \text{Re}\, h(z) > 0 \quad (z \in \mathcal{U}) \tag{1.5}$$

and let t be a complex number. We denote by $\mathcal{V}(t, \lambda, \sigma; h)$ the class of functions $f \in \mathcal{M}$ satisfying the following condition:

$$z\left[(1-t)\, D_\sigma^\lambda f(z) + tD_\sigma^{\lambda+1} f(z)\right] \prec h(z), \tag{1.6}$$

in terms of subordination.

Moreover we define the class $\mathcal{W}(t, \lambda, \sigma; h)$ of functions $f \in \mathcal{M}$ satisfying the following condition:

$$\frac{(1-t)\, D_\sigma^{\lambda+1} f(z) + tD_\sigma^{\lambda+2} f(z)}{(1-t)\, D_\sigma^\lambda f(z) + tD_\sigma^{\lambda+1} f(z)} \prec h(z). \tag{1.7}$$

In particular for real constants $A, B,\ -1 \leq A < B \leq 1$, we denote

$$\begin{aligned} \mathcal{V}\ (t, \lambda, \sigma; A, B) &= \mathcal{V}\left(t, \lambda, \sigma; \frac{1+Az}{1+Bz}\right), \\ \mathcal{W}\ (t, \lambda, \sigma; A, B) &= \mathcal{W}\left(t, \lambda, \sigma; \frac{1+Az}{1+Bz}\right). \end{aligned}$$

In the paper we present some inclusion relations for the defined classes.

2. Main results

We shall need some lemmas.

Lemma 2.1. [4] *Let w be a nonconstant function analytic in $\mathcal{U}(r)$ with $w(0) = 0$. If*

$$|w(z_0)| = \max\{|w(z)|\,;\ |z| \le |z_0|\} \quad (z_0 \in \mathcal{U}(r)),$$

then there exists a real number k $(k \ge 1)$, such that

$$z_0 w'(z_0) = kw(z_0).$$

We shall need also a modified result of Eenigenburg, Miller, Mocanu and Reade [2] (see also [5]).

Lemma 2.2. *Let h be a convex function in U, with*

$$\mathrm{Re}[\beta h(z) + \gamma] > 0 \quad (z \in \mathcal{U})$$

If a function q satisfies the Briot-Bouquet differential subordination (1.2) in $\mathcal{U}(R)$, i.e

$$q(z) + \frac{zq'(z)}{\beta q(z) + \gamma} \prec_R h(z),$$

then

$$q(z) \prec_R h(z).$$

Making use of above lemmas, we get the following two theorem.

Theorem 2.3.

$$\mathcal{V}(t, \lambda + m, \sigma; h) \subset \mathcal{V}(t, \lambda, \sigma; h) \quad (m \in \mathbf{N}).$$

Proof. It is clear that it is sufficient to prove the theorem for $m = 1$. Let a function f belong to the class $\mathcal{V}(t, \lambda + 1, \sigma; h)$ or equivalently

$$z\left[(1-t)\, D_\sigma^{\lambda+1} f(z) + tD_\sigma^{\lambda+2} f(z)\right] \prec h(z). \tag{2.1}$$

It is sufficient to verify the condition (1.6). The function

$$q(z) = z\left[(1-t)\, D_\sigma^{\lambda} f(z) + tD_\sigma^{\lambda+1} f(z)\right] \tag{2.2}$$

is analytic in $\mathcal{U}$ and $q(0) = 1$. Taking the derivative of (2.2) we get

$$z\left[(1-t)\, D_\sigma^{\lambda+1} f(z) + tD_\sigma^{\lambda+2} f(z)\right] = q(z) + \frac{zq'(z)}{\sigma} \quad (z \in \mathcal{U}). \tag{2.3}$$

Thus by (2.1) we have

$$q(z)+\frac{zq'(z)}{\sigma}\prec h(z).$$

Lemma 2.2 now yields

$$q(z)\prec h(z).$$

Thus by (2.2) $f\in\mathcal{V}(t,\lambda,\sigma;h)$ and this proves Theorem 2.3. □

Theorem 2.4.

$$\mathcal{W}(t,\lambda{+}m,\sigma;h)\subset\mathcal{W}(t,\lambda,\sigma;h)\quad(m\in\mathbf{N}).$$

Proof. It is clear that it is sufficient to prove the theorem for $m=1$. Let a function f belong to the class $\mathcal{W}(t,\lambda{+}1,\sigma;h)$ or equivalently

$$\frac{(1-t)\,D_\sigma^{\lambda+2}f(z)+tD_\sigma^{\lambda+3}f(z)}{(1-t)\,D_\sigma^{\lambda+1}f(z)+tD_\sigma^{\lambda+2}f(z)}\prec h(z). \tag{2.4}$$

It is sufficient to verify the condition (1.7). If we put

$$R=\sup\left\{r:\ (1-t)\,D_\sigma^{\lambda}f(z)+tD_\sigma^{\lambda+1}f(z)\neq 0,\ 0<|z|<r\right\}, \tag{2.5}$$

then the function

$$q(z)=\frac{(1-t)\,D_\sigma^{\lambda+1}f(z)+tD_\sigma^{\lambda+2}f(z)}{(1-t)\,D_\sigma^{\lambda}f(z)+tD_\sigma^{\lambda+1}f(z)} \tag{2.6}$$

is analytic in $\mathcal{U}(R)$ and $q(0)=1$. Taking the logarithmic derivative of (2.6) and applying (1.4) we get

$$\frac{(1-t)\,D_\sigma^{\lambda+2}f(z)+tD_\sigma^{\lambda+3}f(z)}{(1-t)\,D_\sigma^{\lambda+1}f(z)+tD_\sigma^{\lambda+2}f(z)}=q(z)+\frac{zq'(z)}{\sigma q(z)}\quad(z\in\mathcal{U}(R)). \tag{2.7}$$

Thus by (2.4) we have

$$q(z)+\frac{zq'(z)}{\sigma q(z)}\prec_R h(z).$$

Lemma 2.2 now yields

$$q(z)\prec_R h(z). \tag{2.8}$$

By (2.6) it suffices to verify that $R=1$. Let p be the positive integer such that $p>\sigma$ and let

$$F(z)=z^{p+1}\left[(1-t)\,D_\sigma^{\lambda}f(z)+tD_\sigma^{\lambda+1}f(z)\right]\quad(z\in\mathcal{U}).$$

Then by (1.4), (2.6) and (2.8) we have

$$\frac{zF'(z)}{F(z)}=\sigma\frac{(1-t)\,D_\sigma^{\lambda+1}f(z)+tD_\sigma^{\lambda+2}f(z)}{(1-t)\,D_\sigma^{\lambda}f(z)+tD_\sigma^{\lambda+1}f(z)}+p-\sigma\prec_R\sigma h(z)+p-\sigma.$$

Thus by (1.5) we obtain

$$\mathrm{Re}\frac{zF'(z)}{F(z)}>0 \quad (z\in\mathcal{U}(R)).$$

It means, that F is p-valently starlike in $\mathcal{U}(R)$ and consequently it is p-valent in $\mathcal{U}(R)$. Thus we see that F can not vanish on $|z|=R$ if $R<1$. Hence by (2.5) we have $R=1$ and the proof of Theorem 2.4 is complete. □

Putting $h(z)=\frac{1+Az}{1+Bz}$ in Theorems 2.2 and 2.3 we obtain the following two corollaries:

Corollary 2.5.

$$\mathcal{V}(t,\lambda+m,\sigma;A,B)\subset\mathcal{V}(t,\lambda,\sigma;A,B) \quad (m\in\mathbf{N}).$$

Corollary 2.6.

$$\mathcal{W}(t,\lambda+m,\sigma;A,B)\subset\mathcal{W}(t,\lambda,\sigma;A,B) \quad (m\in\mathbf{N}).$$

Using Lemma 2.1 we show the following sufficient conditions for functions to belong to the class $\mathcal{W}(t,\lambda,\sigma;A,B)$.

Theorem 2.7. *Let σ,λ,A,B be real numbers, and let*

$$\sigma>0,\ \lambda>0,\ -1\le A<B\le 1,\ B-A\ge 2AB. \tag{2.9}$$

If a function $f\in M$ satisfies the inequality

$$\left|\frac{(1-t)\,D_\sigma^{\lambda+2}f(z)+tD_\sigma^{\lambda+3}f(z)}{(1-t)\,D_\sigma^{\lambda+1}f(z)+tD_\sigma^{\lambda+2}f(z)}-1\right|<\frac{(B-A)\,(1+\sigma-\sigma A)-2AB}{\sigma\,(1+B)\,(1-A)} \quad (z\in\mathcal{U}), \tag{2.10}$$

then f belongs to the class $W(t,\lambda,\sigma;A,B)$.

Proof. Let a function f belong to the class $\mathcal{M}$. Putting

$$q(z)=\frac{1+Aw(z)}{1+Bw(z)} \quad (z\in\mathcal{U}(R)) \tag{2.11}$$

in (2.7), we obtain

$$\frac{(1-t)\,D_\sigma^{\lambda+2}f(z)+tD_\sigma^{\lambda+3}f(z)}{(1-t)\,D_\sigma^{\lambda+1}f(z)+tD_\sigma^{\lambda+2}f(z)}=\frac{1+Aw(z)}{1+Bw(z)}+\frac{1}{\sigma}\left(\frac{Azw'(z)}{1+Aw(z)}-\frac{Bzw'(z)}{1+Bw(z)}\right).$$

Consequently, we have

$$F(z)=w(z)\left\{\frac{zw'(z)}{\sigma w(z)}\left(\frac{A}{1+Aw(z)}-\frac{B}{1+Bw(z)}\right)-\frac{B-A}{1+Bw(z)}\right\}, \tag{2.12}$$

where

$$F(z) = \frac{(1-t)\,D_\sigma^{\lambda+2} f(z) + tD_\sigma^{\lambda+3} f(z)}{(1-t)\,D_\sigma^{\lambda+1} f(z) + tD_\sigma^{\lambda+2} f(z)} - 1.$$

By (1.7), (2.6) and (2.11) it is sufficient to verify that w is analytic in $\mathcal{U}$ and

$$|w(z)| < 1 \quad (z \in \mathcal{U}).$$

Now, suppose that there exists a point $z_0 \in \mathcal{U}(R)$, such that

$$|w(z_0)| = 1,\ |w(z)| < 1 \ \ (|z| < |z_0|).$$

Then, applying Lemma 2.1, we can write

$$z_0 w'(z_0) = kw(z_0),\ w(z_0) = e^{i\theta} \quad (k \geq 1).$$

Combining these with (2.12), we obtain

$$\begin{aligned} |F(z_0)| &= \left| \frac{k}{\sigma} \left(\frac{-A}{1+Ae^{i\theta}} + \frac{B}{1+Be^{i\theta}} \right) + \frac{B-A}{1+Be^{i\theta}} \right| \\ &\geq \frac{k}{\sigma} \mathrm{Re} \left(\frac{-A}{1+Ae^{i\theta}} + \frac{B}{1+Be^{i\theta}} \right) + \frac{B-A}{1+B}. \end{aligned}$$

Thus, by (2.9) we have

$$\begin{aligned} |F(z_0)| &\geq \frac{k}{\sigma} \left(\frac{-A}{1-A} + \frac{B}{1+B} \right) + \frac{B-A}{1+B} \\ &\geq \frac{(B-A)\,(1+\sigma-\sigma A) - 2AB}{\sigma\,(1+B)\,(1-A)}. \end{aligned}$$

Since this result contradicts (2.10) we conclude that w is the analytic function in $\mathcal{U}(R)$ and $|w(z)| < 1$ $(z \in \mathcal{U}\,(R))$. Applying the same methods as in the proof of Theorem 2.4 we obtain $R = 1$, which completes the proof of Theorem 2.7. □

Putting $t = 0,\ A = 2\alpha - 1$ and $B = 1$ in Corollaries 2.5 and 2.6 and Theorem 2.7 we obtain following relationships for the operator D_σ^λ .

Corollary 2.8. *Let* $0 \leq \alpha < 1$ *and* $m \in N$. *If a function* $f \in M$ *satisfies the inequality*

$$\mathrm{Re}\left(zD_\sigma^{\lambda+m} f(z)\right) > \alpha \quad (z \in \mathcal{D}),$$

then

$$\mathrm{Re}\left(zD_\sigma^{\lambda} f(z)\right) > \alpha \quad (z \in \mathcal{D}).$$

Corollary 2.9. *Let* $0 \leq \alpha < 1$ *and* $m \in N$. *If a function* $f \in M$ *satisfies the inequality*

$$\mathrm{Re}\left\{\frac{D_\sigma^{\lambda+m+1}f(z)}{D_\sigma^{\lambda+m}f(z)}\right\} > \alpha \quad (z \in \mathcal{D}),$$

then

$$\mathrm{Re}\left\{\frac{D_\sigma^{\lambda+1}f(z)}{D_\sigma^{\lambda}f(z)}\right\} > \alpha \quad (z \in \mathcal{D}).$$

Corollary 2.10. *Let* $0 \leq \alpha \leq 2/3$. *If a function* $f \in M$ *satisfies the inequality*

$$\left|\frac{D_\sigma^{\lambda+2}f(z)}{D_\sigma^{\lambda+1}f(z)} - 1\right| < 1 - \alpha + \frac{2-3\alpha}{2\sigma(1-\alpha)} \quad (z \in \mathcal{D}),$$

then

$$\mathrm{Re}\left\{\frac{D_\sigma^{\lambda+1}f(z)}{D_\sigma^{\lambda}f(z)}\right\} > a \quad (z \in \mathcal{D}).$$

References

[1] Cho, N. E., Srivastava, H. M., *Argument estimates of certain analytic functions defined by a class of multiplier transformations,* Math. Comput. Modelling, **37** (2003), no. 1-2, 39-49.

[2] Eenigenburg, P. J., Miller, S. S., Mocanu, P. T., Reade, M. O., *Second order differential inequalities in the complex plane,* J. Math. Anal. Appl., **65** (1978), 289-305.

[3] Flett, T. M., *The dual of an inequality of Hardy and Littlewood and some related inequalities,* J. Math. Anal. Appl., **38** (1972), 746-765.

[4] Jack, I. S., *Functions starlike and convex of order* α, J. London Math. Soc., **3** (1971), 469-474.

[5] Miller, S. S., Mocanu, P. T., *Differential subordinations: theory and applications.* Series on Monographs and Textbooks in Pure and Applied Mathematics, Vol. 225, Marcal Dekker, New York, 2000.

[6] Sălăgean, G. S., *Subclasses of univalent functions,* Lecture Notes in Math. Springer-Verlag, **1013** (1983), 362-372.

[7] Uralegaddi, B. A., Somanatha, C., *Certain classes of univalent functions,* in Current Topics in Analytic Function Theory, (Edited by H. M. Srivastava and S. Own), 371-374, World Scientific, Singapore, (1992).

Institute of Mathematics
University of Rzeszów
ul. Rejtana 16A, PL-35-310, Rzeszów, Poland
E-mail address: jdziok@univ.rzeszow.pl

STUDIA UNIV. "BABEŞ–BOLYAI", MATHEMATICA, Volume **LV**, Number 3, September 2010

SOME SUBCLASSES OF MEROMORPHICALLY UNIVALENT FUNCTIONS

RABHA M. EL-ASHWAH

Dedicated to Professor Grigore Ştefan Sălăgean on his 60^{th} *birthday*

Abstract. Making use of certain linear operator, we introduce two novel subclasses $\sum_n(A,B,\lambda)$ and $\sum_{p,n}^*(A,B,\lambda)$ of meromorphically univalent functions in the punctured disc U^*. The main object of this paper is to investigate the various important properties and characteristics of these subclasses of meromorphically univalent functions. We extend the familiar concept of neighborhoods of analytic functions to these subclasses of meromorphically univalent functions. We also derive many result for the Hadamard products of functions belonging to the class $\sum_{p,n}^*(\alpha,\beta,\gamma,\lambda)$.

1. Introduction

Let $\sum$ denote the class of functions of the form:

$$f(z)=\frac{1}{z}+\sum_{k=0}^{\infty}a_kz^k. \tag{1.1}$$

which are analytic and univalent in the punctured disc

$$U^*=\{z:z\in\mathbb{C}\text{ and }0<|z|<1\}=U\backslash\{0\}$$

and which have a simple pole at the origin with residue one there. Define a linear operator as follows:

$$D^0f(z)=f(z),$$

Received by the editors: 25.04.2010.

2000 *Mathematics Subject Classification.* 30C45, 33C50.

Key words and phrases. Linear operator, Hadamard product, meromorphically univalent functions, neighborhoods.

$$D^1 f(z)=\frac{1}{z}+\sum_{k=0}^{\infty}(k+2)a_k z^k=\frac{(z^2 f(z))'}{z},$$
$$D^2 f(z)=D(D^1 f(z)),$$

and (in general)

$$D^n f(z)=\frac{1}{z}+\sum_{k=0}^{\infty}(k+2)^n a_k z^k$$
$$=\frac{(z^2 D^{n-1} f(z))'}{z}\ \ (f\in\sum;\ n\in\mathbb{N}=\{1,2,...\}).\qquad(1.2)$$

The linear operator D^n was considered by Uralegaddi and Somanath [15].

Let

$$F_{\lambda,n}(z)=(1-\lambda)D^n f(z)+\lambda z(D^n f(z))'(f\in\sum;n\in\mathbb{N}_0=\mathbb{N}\cup\{0\};\ 0\le\lambda<\frac{1}{2}),\ (1.3)$$

so that, obviously,

$$F_{\lambda,n}(z)=\frac{1-2\lambda}{z}+\sum_{k=0}^{\infty}(k+2)^n[1+\lambda(k-1)]a_k z^k(n\in\mathbb{N}_0;\ 0\le\lambda<\frac{1}{2}),\qquad(1.4)$$

it is easily verified that

$$zF'_{\lambda,n}(z)=F_{\lambda,n+1}(z)-2F_{\lambda,n}(z).\qquad(1.5)$$

For a function $f(z)\in\sum$, we say that f(z) is a member of the class $\sum_n(A,B,\lambda)$ if the function $F_{\lambda,n}(z)$ defined by (1.3) satisfies the inequality:

$$\left|\frac{z^2F'_{\lambda,n}(z)+(1-2\lambda)}{Bz^2F'_{\lambda,n}(z)+(1-2\lambda)A}\right|<1\ \ (z\in U^*),\qquad(1.6)$$

where (and throughout this paper) the parameters A,B,λ,p and n are constrained as follows:

$$-1\le A<B\le 1,\ 0<B\le 1\,,\ 0\le\lambda<\frac{1}{2};p\in\mathbb{N}\ \text{ and } n\in\mathbb{N}_0.\qquad(1.7)$$

Furthermore, we say that a function $f(z)\in\sum^*_{p,n}(A,B,\lambda)$ whenever $f(z)$ is of the form [cf. Equation (1.1)]:

$$f(z)=\frac{1}{z}+\sum_{k=p}^{\infty}|a_k|\,z^k\ \ (k\ge p\,;\,p\in\mathbb{N}).\qquad(1.8)$$

We note that:

(i) $\sum_{p,0}^{*}((2\alpha\gamma-1)\beta,\ (2\gamma-1)\beta,0)=\sum_{p}(\alpha,\beta,\gamma)\,(0\leq\alpha<1\,;\,0<\beta\leq1;\frac{1}{2}\leq\gamma\leq1)$ (Cho et al. [6]);

(ii) $\sum_{1,0}^{*}((2\alpha\gamma-1)\beta,\ (2\gamma-1)\beta,0)=\sum_{1}(\alpha,\beta,\gamma)\,(0\leq\alpha<1\,;\,0<\beta\leq1;\frac{1}{2}\leq\gamma\leq1)$ (Cho et al. [5]);

(iii) $\sum_{1,0}^{*}(-A,-B,0)=\sum_{d}(A,B)(-1\leq B<A\leq1\,;\,-1\leq B<0)$ (Cho [4]);

(iv) $\sum_{p,0}^{*}(B,A,\lambda)=\Omega^{+}(p;0;1,1,A,B,\lambda)=\Omega^{+}(p,A,B,\lambda)$ (Joshi et al. [9]).

Also we note that:

(v) $\sum_{p,n}^{*}((2\alpha\gamma-1)\beta,(2\gamma-1)\beta,\lambda)=\sum_{p,n}^{*}(\alpha,\beta,\gamma,\lambda)$

$$=\left\{f\in\sum\nolimits_{p}^{*}:\left|\frac{z^{2}F_{\lambda,n}^{'}(z)+(1-2\lambda)}{(2\gamma-1)z^{2}F_{\lambda,n}^{'}(z)+(1-2\lambda)(2\gamma\alpha-1)}\right|<\beta,\right.$$

$$\left.(z\in U^{*}\,;\,0\leq\alpha<1;\,0<\beta\leq1;\frac{1}{2}\leq\gamma\leq1;0\leq\lambda<\frac{1}{2}\,;\,n\in\mathbb{N}_{0})\right\};\tag{1.9}$$

(vi) $\sum_{p,n}^{*}((2\alpha\gamma-1)\beta,(2\gamma-1)\beta,0)=\sum_{p,n}^{*}(\alpha,\beta,\gamma)$

$$=\left\{f\in\sum\nolimits_{p}^{*}:\left|\frac{z^{2}(D^{n}f(z))^{'}+1}{(2\gamma-1)z^{2}(D^{n}f(z))^{'}+(2\gamma\alpha-1)}\right|<\beta,\right.$$

$$\left.(z\in U^{*}\,;\,0\leq\alpha<1;0<\beta\leq1;\frac{1}{2}\leq\gamma\leq1;\ n\in\mathbb{N}_{0})\right\}.\tag{1.10}$$

2. Inclusion properties of the class $\sum_{n}(A,B,\lambda)$

We begin by recalling the following result (Jack's lemma), which we shall apply in proving our first theorem.

Lemma 2.1. [8] *Let the (nonconstant) function $w(z)$ be analytic in U with $w(0)=0$. If $|w(z)|$ attains its maximum value on the circle $|z|=r<1$ at a point $z_0\in U$, then*

$$z_{0}w^{'}(z_{0})=\gamma w(z_{0}),\tag{2.1}$$

where γ is a real and $\gamma\geq1$.

Theorem 2.2. *The following inclusion property holds true for the class $\sum_{n}(A,B,\lambda)$*

$$\sum\nolimits_{n+1}(A,B,\lambda)\subset\sum\nolimits_{n}(A,B,\lambda)\,(n\in\mathbb{N}_{0}).\tag{2.2}$$

Proof. Let $f(z) \in \sum_{n+1}(A, B, \lambda)$ and suppose that

$$z^2 F'_{\lambda,n}(z) = -\frac{(1-2\lambda)(1+Aw(z))}{1+Bw(z)}, \tag{2.3}$$

where the function $w(z)$ is either analytic or meromorphic in U, with $w(0) = 0$. Then, by using (1.5) and (2.3), we have

$$z^2 F'_{\lambda,n+1}(z) = -(1-2\lambda)\left[\frac{1+Aw(z)}{1+Bw(z)} + \frac{(A-B)zw'(z)}{(1+Bw(z))^2}\right]. \tag{2.4}$$

We claim that $|w(z)| < 1$ for $z \in U$. Otherwith there exists a point $z_0 \in U$ such that $\underset{|z|\leq|z_0|}{max} |w(z)| = |w(z_0)| = 1$. Applying Jack's lemma, we have $z_0 w'(z_0) = \gamma w(z_0)(\gamma \geq 1)$. Writing $w(z_0) = e^{i\theta}(0 \leq \theta \leq 2\pi)$ and putting $z = z_0$ in (2.4), we get

$$\left|\frac{z_0^2 F'_{\lambda,n+1}(z_0) + (1-2\lambda)}{Bz_0^2 F'_{\lambda,n+1}(z_0) + (1-2\lambda)A}\right|^2 - 1$$

$$= \frac{\left|1+\gamma+Be^{i\theta}\right|^2 - \left|1+B(1-\gamma)e^{i\theta}\right|^2}{\left|1+B(1-\gamma)e^{i\theta}\right|^2}$$

$$= \frac{\gamma^2(1-B^2) + 2\gamma(1+B^2+2B\cos\theta)}{\left|1+B(1-\gamma)e^{i\theta}\right|^2} \geq 0, \tag{2.5}$$

which obviously contradicts our hypothesis that $f(z) \in \sum_{n+1}(A, B, \lambda)$. Thus we must have $|w(z)| < 1$ $(z \in U)$, so from (2.3), we conclude that $f(z) \in \sum_n(A, B, \lambda)$, which evidently completes the proof of Theorem 1.

Theorem 2.3. *Let α be a complex number such that $Re(\alpha) > 0$. If $f(z) \in \sum_n(A, B, \lambda)$, then the function $G_{\lambda,n}(z)$ given by*

$$G_{\lambda,n}(z) = \frac{\alpha}{z^{\alpha+1}} \int_0^z t^\alpha F_{\lambda,n}(t)dt \tag{2.6}$$

is also in the same class $\sum_n(A, B, \lambda)$.

Proof. From (2.6), we have

$$zG'_{\lambda,n}(z) = \alpha F_{\lambda,n}(z) - (\alpha+1)G_{\lambda,n}(z). \tag{2.7}$$

Put

$$z^2 G'_{\lambda,n}(z) = -\frac{(1-2\lambda)(1+Aw(z))}{1+Bw(z)}, \tag{2.8}$$

where $w(z)$ is either analytic or meromorphic in U with $w(0) = 0$. Then , by using (2.7) and (2.8), we have

$$z^2 F^{'}_{\lambda,n}(z) = -(1-2\lambda)\left[\frac{1+Aw(z)}{1+Bw(z)} + \frac{(A-B)zw^{'}(z)}{\alpha(1+Bw(z))^2}\right]. \tag{2.9}$$

The remaining part of the proof is similar to that of Theorem 1 and so is omitted.

3. **Properties of the class** $\sum^{*}_{p,n}(A, B, \lambda)$

Theorem 3.1. *Let* $f(z) \in \sum^{*}_{p}$ *be given by* (1.8)*. Then* $f(z) \in \sum^{*}_{p,n}(A, B, \lambda)$ *if and only if*

$$\sum_{k=p}^{\infty} k(k+2)^n[1+\lambda(k-1)](1+B)\,|a_k| \le (B-A)(1-2\lambda), \tag{3.1}$$

where the parameters A, B, n *and* λ *are constrained as in* (1.7).

Proof. Let $f(z) \in \sum^{*}_{p,n}(A, B, \lambda)$ be given by (1.8). Then , from (1.8) and (1.6), we have

$$\left|\frac{z^2F^{'}_{\lambda,n}(z) + (1-2\lambda)}{Bz^2F^{'}_{\lambda,n}(z) + (1-2\lambda)A}\right|$$

$$= \left|\frac{\sum\limits_{k=p}^{\infty} k(k+2)^n[1+\lambda(k-1)]\,|a_k|\, z^{k+1}}{(B-A)(1-2\lambda) - B\sum\limits_{k=p}^{\infty} k(k+2)^n[1+\lambda(k-1)]\,|a_k|\, z^{k+1}}\right| < 1 \quad (z \in U^*). \tag{3.2}$$

Since $|\mathrm{Re}(z)| \le |z|$ $(z \in \mathbb{C})$, we have

$$\mathrm{Re}\left\{\frac{\sum\limits_{k=p}^{\infty} k(k+2)^n[1+\lambda(k-1)]\,|a_k|\, z^{k+1}}{(B-A)(1-2\lambda) - B\sum\limits_{k=p}^{\infty} k(k+2)^n[1+\lambda(k-1)]\,|a_k|\, z^{k+1}}\right\} < 1. \tag{3.3}$$

Choose values of z on the real axis so that $z^2F^{'}_{\lambda,n}(z)$ is real. Upon clearing the denominator in (3.3) and letting $z \to 1^-$ through real values we obtain (3.1).

In order to prove the converse, we assume that the inequality (3.1) holds true. then, if we let $z \in \partial U$, we find from (1.8) and (3.1) that

$$\left|\frac{z^2F^{'}_{\lambda,n}(z) + (1-2\lambda)}{Bz^2F^{'}_{\lambda,n}(z) + (1-2\lambda)A}\right|$$

$$\leq \frac{\sum\limits_{k=p}^{\infty} k(k+2)^n[1+\lambda(k-1)]\,|a_k|}{(B-A)(1-2\lambda) - B\sum\limits_{k=p}^{\infty} k(k+2)^n[1+\lambda(k-1)]\,|a_k|}$$

$$< 1 (z \in \partial U = \{z : z \in \mathbb{C} \text{ and } \ |z| = 1\}). \tag{3.4}$$

Hence, by the maximum modulus theorem, we have $f(z) \in \sum_{p,n}^{*}(A, B, \lambda)$.

Corollary 3.2. *If the function $f(z)$ defined by* (1.8) *is in the class $\sum_{p,n}^{*}(A, B, \lambda)$, then*

$$|a_k| \leq \frac{(B-A)(1-2\lambda)}{k(k+2)^n[1+\lambda(k-1)](1+B)} \quad (k \geq p; p \in \mathbb{N}; n \in \mathbb{N}_0), \tag{3.5}$$

with equality for the function

$$f(z) = \frac{1}{z} + \frac{(B-A)(1-2\lambda)}{k(k+2)^n[1+\lambda(k-1)](1+B)} z^k \quad (k \geq p; p \in \mathbb{N}; n \in \mathbb{N}_0). \tag{3.6}$$

Putting $A = (2\gamma\alpha - 1)\beta$ and $B = (2\gamma - 1)\beta$ $(0 \leq \alpha < 1, 0 < \beta \leq 1$ and $\frac{1}{2} \leq \gamma \leq 1)$ in Theorem 2.3, we obtain:

Corollary 3.3. *A function $f(z)$ defined by* (1.8) *is in the class $\sum_{p,n}^{*}(\alpha, \beta, \gamma, \lambda)$ if and only if*

$$\sum_{k=p}^{\infty} k(k+2)^n[1+\lambda(k-1)](1+2\beta\gamma-\beta)\,|a_k| \leq 2\beta\gamma(1-2\lambda)(1-\alpha). \tag{3.7}$$

Next we prove the following growth and distortion properties for the class $\sum_{p,n}^{*}(A, B, \lambda)$.

Theorem 3.4. *If a function $f(z)$ defined by* (1.8) *is in the class $\sum_{p,n}^{*}(A, B, \lambda)$, then*

$$\left\{ m! - \frac{(p-1)!(B-A)(1-2\lambda)}{(p-m!)(p+2)^n[1+\lambda(p-1)](1+B)} r^{p+1} \right\} r^{-(m+1)} \leq \left| f^{(m)}(z) \right|$$

$$\leq \left\{ m! + \frac{(p-1)!(B-A)}{(p-m!)(p+2)^n[1+\lambda(p-1)](1+B)} r^{p+1} \right\} r^{-(m+1)} \tag{3.8}$$

$$(0 < |z| = r < 1; p \in \mathbb{N}; m, n \in \mathbb{N}_0; m < p).$$

The result is sharp for the function $f(z)$ given by

$$f(z) = \frac{1}{z} + \frac{(B-A)(1-2\lambda)}{p(p+2)^n[1+\lambda(p-1)](1+B)} z^p \quad (p \in \mathbb{N};\, n \in \mathbb{N}_0). \tag{3.9}$$

Proof. In view of Theorem 2.3, we have

$$\frac{p(p+2)^n[1+\lambda(p-1)](1+B)}{p!}\sum_{k=p}^{\infty}k!\,|a_k| \le \sum_{k=p}^{\infty}k(k+2)^n[1+\lambda(k-1)](1+B)\,|a_k|$$
$$\le (B-A)(1-2\lambda),$$

which yields

$$\sum_{k=p}^{\infty}k!\,|a_k| \le \frac{p!(B-A)(1-2\lambda)}{p(p+2)^n[1+\lambda(p-1)](1+B)} \quad (p\in\mathbb{N};\ n\in\mathbb{N}_0). \tag{3.10}$$

Now, by differentiating both sides of (1.8) m times with respect to z, we have

$$f^{(m)}(z) = (-1)^m m! z^{-(m+1)} + \sum_{k=p}^{\infty}\frac{k!}{(k-m)!}\,|a_k|\,z^{k-m},$$
$$(p\in\mathbb{N}; m,n\in\mathbb{N}_0; m<p), \tag{3.11}$$

and Theorem 3.1 follows easily from (3.10) and (3.11).

Finally, it is easy to see that the bounds in (3.8) are attained for the function $f(z)$ given by (3.9).

By the same way as in the proof given by Cho et al. [5] , we have the radii of meromorphically starlikeness of order $\phi(0\le\phi<1)$ and meromorphically convexity of order $\phi(0\le\phi<1)$ for functions in the class $\sum_{p,n}^{*}(A,B,\lambda)$.

Theorem 3.5. *Let the function $f(z)$ defined by (1.8) be in the class $\sum_{p,n}^{*}(A,B,\lambda)$, then, we have*

(i) $f(z)$ is meromorphically starlike of order $\phi(0\le\phi<1)$ in the disc $|z|<r_1$, that is,

$$\operatorname{Re}\left\{-\frac{zf'(z)}{f(z)}\right\} > \phi \quad (|z|<r_1; 0\le\phi<1), \tag{3.12}$$

where

$$r_1 = \inf_{k\ge p}\left\{\frac{k(k+2)^n[1+\lambda(k-1)](1+B)(1-\phi)}{(B-A)(1-2\lambda)(k+2-\phi)}\right\}^{\frac{1}{k+1}}. \tag{3.13}$$

(ii) $f(z)$ is meromorphically convex of order $\phi(0\le\phi<1)$ in the disc $|z|<r_2$, that is,

$$\operatorname{Re}\left\{-(1+\frac{zf''(z)}{f'(z)})\right\} > \phi \quad (|z|<r_2\,; 0\le\phi<1), \tag{3.14}$$

where

$$r_2 = \inf_{k\geq p}\left\{\frac{(k+2)^n[1+\lambda(k-1)](1+B)(1-\phi)}{(B-A)(1-2\lambda)(k+2-\phi)}\right\}^{\frac{1}{k+1}}. \tag{3.15}$$

Each of these results is sharp for the function $f(z)$ given by (3.6).

4. **Neighborhoods and partial sums**

Following the earlier works (based upon the familiar concept of neighborhoods of analytic functions) by Goodman [7] and Ruscheweyh [13], and (more recently) by Altintas et al. ([1], [2] and [3]) , Liu [10] and Liu and Srivastava ([11] and [12]), we begin by introducing here the $\delta-$ neighborhood of a function $f(z) \in \sum$ of the form (1.1) by means of the definition given below:

$$N_\delta(f) = \left\{ g \in \sum \ : \ g(z) = \frac{1}{z} + \sum_{k=0}^{\infty} b_k z^k \text{ and} \right.$$

$$\sum_{k=0}^{\infty} \frac{k(k+2)^n[1+\lambda(k-1)](1+|B|)}{(B-A)(1-2\lambda)} |a_k - b_k| \leq \delta,$$

$$\left. (-1 \leq A < B \leq 1, 0 \leq \lambda < \frac{1}{2}, \delta > 0, p \in \mathbb{N}, n \in \mathbb{N}_0) \right\}. \tag{4.1}$$

Making use of the definition (4.1), we now prove Theorem 6 below:

Theorem 4.1. *Let the function $f(z)$ defined by* (1.1) *be in the class $\sum_n(A,B,\lambda)$. If $f(z)$ satisfies the following condition:*

$$\frac{f(z)+\epsilon z^{-1}}{1+\epsilon} \in \sum\nolimits_n (A,B,\lambda) \ \ (\epsilon \in \mathbb{C}, |\epsilon| < \delta, \delta > 0),$$

then

$$N_\delta(f) \subset \sum\nolimits_n (A,B,\lambda). \tag{4.3}$$

Proof. It is easily seen from (1.6) that $g(z) \in \sum_n(A,B,\lambda)$ if and only if for any complex number σ with $|\sigma| = 1$,

$$\frac{z^2 G'_{\lambda,n}(z) + (1-2\lambda)}{Bz^2 G'_{\lambda,n}(z) + (1-2\lambda)A} \neq \sigma \quad (z \in U), \tag{4.4}$$

which is equivalent to

$$\frac{(g*h)(z)}{z^{-1}} \neq 0 \ \ (z \in U), \tag{4.5}$$

which, for convenience,

$$h(z) = \frac{1}{z} + \sum_{k=0}^{\infty} c_k z^k$$

$$= \frac{1}{z} + \sum_{k=0}^{\infty} \frac{k(k+2)^n[1+\lambda(k-1)](1-\sigma B)}{\sigma(B-A)(1-2\lambda)} z^k. \quad (4.6)$$

From (4.6), we have

$$|c_k| \le \frac{k(k+2)^n[1+\lambda(k-1)](1+|B|)}{(B-A)(1-2\lambda)} \;\; (0 \le \lambda < \frac{1}{2}\,; n \in \mathbb{N}_0). \quad (4.7)$$

Now , if $f(z) = \frac{1}{z} + \sum\limits_{k=0}^{\infty} a_k z^k \in \sum$ satisfies the condition (4.2), then (4.5) yields

$$\left| \frac{(f*h)(z)}{z^{-1}} \right| \ge \delta \qquad (z \in U; \delta > 0). \quad (4.8)$$

By letting

$$g(z) = \frac{1}{z} + \sum_{k=0}^{\infty} b_k z^k \in N_\delta(f), \quad (4.9)$$

so that

$$\left| \frac{[g(z)-f(z)]*h(z)}{z^{-1}} \right| = \left| \sum_{k=0}^{\infty} (b_k - a_k) c_k z^{k+1} \right|$$

$$\le |z| \sum_{k=0}^{\infty} \frac{k(k+2)^n[1+\lambda(k-1)](1+|B|)}{(B-A)(1-2\lambda)} \, |b_k - a_k|$$

$$< \delta \;\; (z \in U; \delta > 0). \quad (4.10)$$

Thus we have (4.5), and hence also (4.4) for any $\sigma \in \mathbb{C}$ such that $|\sigma| = 1$, which implies that $g(z) \in \sum_n(A, B, \lambda)$. This evidently proves the assertion (4.3) of Theorem 6.

We now define the $\delta-$neighborhood of a function $f(z) \in \sum_p^*$ of the form (1.8) as follows:

$$N_\delta^+(f) = \Big\{ g \in \sum\nolimits_p^* \; : \; g(z) = \frac{1}{z} + \sum_{k=p}^{\infty} |b_k| \, z^k \quad \text{and}$$

$$\sum_{k=0}^{\infty} \frac{k(k+2)^n[1+\lambda(k-1)](1+B)}{(B-A)(1-2\lambda)} \, ||b_k| - |a_k|| \le \delta,$$

$$(-1 \le A < B \le 1\,; \; 0 \le \lambda < \frac{1}{2}; \delta > 0; p \in \mathbb{N}; n \in \mathbb{N}_0) \Big\}. \quad (4.11)$$

Making use of the definition (4.11), we now prove Theorem 3.4 below:

Theorem 4.2. *Let the function $f(z)$ defined by* (1.8) *be in the class* $\sum_{p,n}^{*}(A,B,\lambda)$, $-1 \leq A < B \leq 1, 0 < B \leq 1$, $0 \leq \lambda < \dfrac{1}{2}, p \in \mathbb{N}$ *and* $n \in \mathbb{N}_0$, *then*

$$N_{\delta}^{+}(f) \subset \sum\nolimits_{p,n}^{*}(A,B,\lambda) \ (\delta = \frac{p+1}{p+2}). \tag{4.12}$$

The result is sharp .

Proof. Making use the same method as in the proof of Theorem 6, we can show that [cf. Eq. (4.6)]

$$\begin{aligned} h(z) &= \frac{1}{z} + \sum_{k=p}^{\infty} c_k z^k \\ &= \frac{1}{z} + \sum_{k=0}^{\infty} \frac{k(k+2)^n[1+\lambda(k-1)](1-\sigma B)}{\sigma(B-A)(1-2\lambda)} z^k. \end{aligned} \tag{4.13}$$

Thus under the hypothesis $-1 \leq A < B \leq 1\,, 0 < B \leq 1$, $0 \leq \lambda < \dfrac{1}{2}\,, p \in \mathbb{N}$ and $n \in \mathbb{N}_0$, if $f(z) \in \sum_{p,n+1}^{*}(A,B,\lambda)$ is given by (1.8), we obtain

$$\left|\frac{(f*h)(z)}{z^{-1}}\right| = \left|1 + \sum_{k=p}^{\infty} c_k \left|a_k\right| z^{k+1}\right|$$

$$\geq 1 - \frac{1}{p+2} \sum_{k=p}^{\infty} \frac{k(k+2)^{n+1}[1+\lambda(k-1)](1+B)}{(B-A)(1-2\lambda)} \left|a_k\right|,$$

which in view of Theorem 2.3, yields

$$\left|\frac{(f*h)(z)}{z^{-1}}\right| \geq 1 - \frac{1}{p+2} = \frac{p+1}{p+2} = \delta.$$

The remaing part of the proof of Theorem 3.4 is similar to that of Theorem 6, and we skip the details involved.

To show the sharpness, we consider the functions $f(z)$ and $g(z)$ given by

$$f(z) = \frac{1}{z} + \frac{(B-A)(1-2\lambda)}{p(p+2)^{n+1}[1+\lambda(p-1)](1+B)} z^p \in \sum\nolimits_{p,n+1}^{*}(A,B,\lambda) \tag{4.14}$$

and

$$g(z) = \frac{1}{z} + \left[\frac{(B-A)(1-2\lambda)}{p(p+2)^{n+1}[1+\lambda(p-1)](1+B)} + \frac{(B-A)(1-2\lambda)\delta'}{p(p+2)^n[1+\lambda(p-1)](1+B)}\right] z^p, \tag{4.15}$$

where $\delta^{'} > \delta = \dfrac{p+1}{p+2}$. Clearly, the function $g(z)$ belongs to $N^{+}_{\delta'}(f)$. On the other hand, we find from Theorem 2.3 that g(z) is not in the class $\sum^{*}_{p,n}(A,B,\lambda)$.

Thus the proof of Theorem 3.4 is completed.

Next we prove the following result.

Theorem 4.3. *Let* $f(z) \in \sum$ *be given by* (1.1) *and define the partial sums* $s_1(z)$ *and* $s_m(z)$ *as follows:*

$$s_1(z) = \frac{1}{z} \quad \text{and} \quad s_m(z) = \frac{1}{z} + \sum_{k=0}^{m-2} a_k z^k \ \ (m \in \mathbb{N}\backslash\{1\}). \tag{4.16}$$

Suppose also that

$$\sum_{k=0}^{\infty} d_k \, |a_k| \leq 1 \quad \left(d_k = \frac{k(k+2)^n[1+\lambda(k-1)](1+|B|)}{(B-A)(1-2\lambda)} \right). \tag{4.17}$$

Then we have

(i) $f(z) \in \sum_n(A,B,\lambda)$,

$$(ii) \ \ Re\left\{\frac{f(z)}{s_m(z)}\right\} > 1 - \frac{1}{d_{m-1}} \quad (z \in U; \ m \in \mathbb{N}) \tag{4.18}$$

and

$$(iii) Re\left\{\frac{s_m(z)}{f(z)}\right\} > \frac{d_{m-1}}{1+d_{m-1}} \quad (z \in U; \ m \in \mathbb{N}). \tag{4.19}$$

The estimates in (4.18) *and* (4.19) *are sharp for each* $m \in \mathbb{N}$.

Proof. (i) It is not difficult to see that

$$z^{-1} \in \sum\nolimits_n (A,B,\lambda) \quad (n \in \mathbb{N}_0).$$

Thus, from Theorem 6 and the hypothesis (4.17) of Theorem 3.5, we have

$$N_1(z^{-1}) \subset \sum\nolimits_n (A,B,\lambda) \quad (n \in \mathbb{N}_0), \tag{4.20}$$

which shows that $f(z) \in \sum_n(A,B,\lambda)$ as asserted by Theorem 3.5.

(ii) For the coefficients d_k given by(4.17), it is not difficult to verify that

$$d_{k+1} > d_k > 1 \quad (k \in \mathbb{N}). \tag{4.21}$$

Therefore, we have

$$\sum_{k=0}^{m-2} |a_k| + d_{m-1} \sum_{k=m-1}^{\infty} |a_k| \leq \sum_{k=0}^{\infty} d_k |a_k| \leq 1, \tag{4.22}$$

where we have used the hypothesis (4.17) again.

By setting

$$h_1(z) = d_{m-1} \left\{ \frac{f(z)}{s_m(z)} - \left(1 - \frac{1}{d_{m-1}}\right) \right\} = 1 + \frac{d_{m-1} \sum\limits_{k=m-1}^{\infty} a_k z^{k+1}}{1 + \sum\limits_{k=0}^{m-2} a_k z^{k+1}}, \tag{4.23}$$

and applying (4.22), we find that

$$\left| \frac{h_1(z) - 1}{h_1(z) + 1} \right| \leq \frac{d_{m-1} \sum\limits_{k=m-1}^{\infty} |a_k|}{2 - 2 \sum\limits_{k=0}^{m-2} |a_k| - d_{m-1} \sum\limits_{k=m-1}^{\infty} |a_k|} \leq 1 \quad (z \in U), \tag{4.24}$$

which readily yields the assertion (4.18) of Theorem 3.5. If we take

$$f(z) = \frac{1}{z} - \frac{z^{m-1}}{d_{m-1}}, \tag{4.25}$$

then

$$\frac{f(z)}{s_m} = 1 - \frac{z^m}{d_{m-1}} \to 1 - \frac{1}{d_{m-1}} \quad \text{as } z \to 1^-,$$

which shows that the bound in (4.18) is the best possible for each $n \in \mathbb{N}$.

(iii) Just as in Part (ii) above, if we put

$$h_2(z) = (1 + d_{m-1}) \left(\frac{s_m(z)}{f(z)} - \frac{d_{m-1}}{1 + d_{m-1}} \right)$$

$$= 1 - \frac{(1 + d_{m-1}) \sum\limits_{k=m-1}^{\infty} a_k z^{k+1}}{1 + \sum\limits_{k=0}^{\infty} a_k z^{k+1}}, \tag{4.26}$$

and make use of (4.22), we can deduce that

$$\left| \frac{h_2(z) - 1}{h_2(z) + 1} \right| \leq \frac{(1 + d_{m-1}) \sum\limits_{k=m-1}^{\infty} |a_k|}{2 - 2 \sum\limits_{k=0}^{m-2} |a_k| - (1 - d_{m-1}) \sum\limits_{k=m-1}^{\infty} |a_k|} \leq 1 \quad (z \in U),$$

which leads us immediately to the assertion (4.19) of Theorem 3.5.

The bound in (4.19) is sharp for each $m \in N$, with the extremal function $f(z)$ given by (4.25). The proof of Theorem 3.5 is thus completed.

5. **Convolution properties**

For the functions

$$f_j(z) = \frac{1}{z} + \sum_{k=p}^{\infty} |a_{k,j}|\, z^k \qquad (j = 1, 2; p \in \mathbb{N}), \tag{5.1}$$

we dnote by $(f_1 * f_2)(z)$ the Hadamard product (or convolution) of the functions $f_1(z$ and $f_2(z)$, that is,

$$(f_1 * f_2)(z) = \frac{1}{z} + \sum_{k=p}^{\infty} |a_{k,1}|\, |a_{k,2}|\, z^k. \tag{5.2}$$

Theorem 5.1. *Let the functions* $f_j(z)\,(j = 1, 2)$ *defined by* (5.1) *be in the class* $\sum_{p,n}^{*}(\alpha, \beta, \gamma, \lambda)$. *Then* $(f_1 * f_2)(z) \in \sum_{p,n}^{*}(\delta, \beta, \gamma, \lambda)$, *where*

$$\delta = 1 - \frac{2\beta\gamma(1 - 2\lambda)(1 - \alpha)^2}{p(p + 2)^n[1 + \lambda(p - 1)](1 + 2\beta\gamma - \beta)}. \tag{5.3}$$

The result is sharp for the functions

$$f_j(z) = \frac{1}{z} + \frac{2\beta\gamma(1 - 2\lambda)(1 - \alpha)}{p(p + 2)^n[1 + \lambda(p - 1)](1 + 2\beta\gamma - \beta)} z^p \; (j = 1, 2; p \in \mathbb{N}; n \in \mathbb{N}_0). \tag{5.4}$$

Proof. Employing the technique used earlier by Schild and Silverman [14], we need to find the largest δ such that

$$\sum_{k=p}^{\infty} \frac{k(k + 2)^n[1 + \lambda(k - 1)](1 + 2\beta\gamma - \beta)}{2\beta\gamma(1 - 2\lambda)(1 - \delta)}\, |a_{k,1}|\, |a_{k,2}| \leq 1 \tag{5.5}$$

for $f_j(z) \in \sum_{p,n}^{*}(\alpha, \beta, \gamma, \lambda)(j = 1, 2)$. Since $f_j(z) \in \sum_{p,n}^{*}(\alpha, \beta, \gamma, \lambda)(j = 1, 2)$, we readily see that

$$\sum_{k=p}^{\infty} \frac{k(k + 2)^n[1 + \lambda(k - 1)](1 + 2\beta\gamma - \beta)}{2\beta\gamma(1 - 2\lambda)(1 - \alpha)}\, |a_{k,j}| \leq 1 \quad (j = 1, 2). \tag{5.6}$$

Therefore, by the Cauchy-Schwarz inequality , we obtain

$$\sum_{k=p}^{\infty} \frac{k(k + 2)^n[1 + \lambda(k - 1)](1 + 2\beta\gamma - \beta)}{2\beta\gamma(1 - 2\lambda)(1 - \alpha)} \sqrt{|a_{k,1}|\, |a_{k,2}|} \leq 1\,. \tag{5.7}$$

This implies that we need only to show that

$$\frac{|a_{k,1}|\,|a_{k,2}|}{(1-\delta)} \leq \frac{\sqrt{|a_{k,1}|\,|a_{k,2}|}}{(1-\alpha)} \quad (k \geq p) \tag{5.8}$$

or , equivalently , that

$$\sqrt{|a_{k,1}|\,|a_{k,2}|} \leq \frac{(1-\delta)}{(1-\alpha)} \quad (k \geq p). \tag{5.9}$$

Hence, by the inequality (5.7), it is sufficient to prove that

$$\frac{2\beta\gamma(1-2\lambda)(1-\alpha)}{k(k+2)^n[1+\lambda(k-1)](1+2\beta\gamma-\beta)} \leq \frac{(1-\delta)}{(1-\alpha)} \quad (k \geq p). \tag{5.10}$$

It follows from (5.10) that

$$\delta \leq 1 - \frac{2\beta\gamma(1-2\lambda)(1-\alpha)^2}{k(k+2)^n[1+\lambda(k-1)](1+2\beta\gamma-\beta)} \quad (k \geq p). \tag{5.11}$$

Now, defining the function $\varphi(k)$ by

$$\varphi(k) = 1 - \frac{2\beta\gamma(1-2\lambda)(1-\alpha)^2}{k(k+2)^n[1+\lambda(k-1)](1+2\beta\gamma-\beta)} \quad (k \geq p). \tag{5.12}$$

We see that $\varphi(k)$ is an increasing function of k. Therefore , we conclude that

$$\delta \leq \varphi(p) = 1 - \frac{2\beta\gamma(1-2\lambda)(1-\alpha)^2}{p(p+2)^n[1+\lambda(p-1)](1+2\beta\gamma-\beta)}, \tag{5.13}$$

which evidently completes the proof of Theorem 4.1.

Using arguments similar to those in the proof of Theorem 4.1, we obtain the following result.

Theorem 5.2. *Let the function* $f_1(z)$ *defined by* (5.1) *be in the class* $\sum_{p,n}^{*}(\alpha,\beta,\gamma,\lambda)$. *Suppose also that the function* $f_2(z)$ *defined by (5.1) be in the class* $\sum_{p,n}^{*}(\zeta,\beta,\gamma,\lambda)$. *Then* $(f_1 * f_2)(z) \in \sum_{p,n}^{*}(\xi,\beta,\gamma,\lambda)$, *where*

$$\xi = 1 - \frac{2\beta\gamma(1-2\lambda)(1-\alpha)(1-\zeta)}{p(p+2)^n[1+\lambda(p-1)](1+2\beta\gamma-\beta)}\,. \tag{5.14}$$

The result is sharp for the functions $f_j(z)(j=1,2)$ *given by*

$$f_1(z) = \frac{1}{z} + \frac{2\beta\gamma(1-2\lambda)(1-\alpha)}{p(p+2)^n[1+\lambda(p-1)](1+2\beta\gamma-\beta)} z^p \quad (p \in \mathbb{N}; n \in \mathbb{N}_0), \tag{5.15}$$

and

$$f_2(z) = \frac{1}{z} + \frac{2\beta\gamma(1-2\lambda)(1-\zeta)}{p(p+2)^n[1+\lambda(p-1)](1+2\beta\gamma-\beta)} z^p \quad (p \in \mathbb{N}; n \in \mathbb{N}_0). \tag{5.16}$$

Theorem 5.3. *Let the functions* $f_j(z)(j = 1,2)$ *defined by* (5.1) *be in the class* $\sum_{p,n}^{*}(\alpha,\beta,\gamma,\lambda)$*. Then the function* $h(z)$ *defined by*

$$h(z) = \frac{1}{z} + \sum_{k=p}^{\infty}(|a_{k,1}|^2 + |a_{k,2}|^2)z^k \tag{5.17}$$

belongs to the class $\sum_{p,n}^{*}(\tau,\beta,\gamma,\lambda)$*, where*

$$\tau = 1 - \frac{4\beta\gamma(1-2\lambda)(1-\alpha)^2}{p(p+2)^n[1+\lambda(p-1)](1+2\beta\gamma-\beta)}. \tag{5.18}$$

This result is sharp for the functions $f_j(z)(j = 1,2)$ *given already by (5.4).*

Proof. Noting that

$$\sum_{k=p}^{\infty} \frac{\{k(k+2)^n[1+\lambda(k-1)](1+2\beta\gamma-\beta)\}^2}{[2\beta\gamma(1-2\lambda)(1-\alpha)]^2}\,|a_{k,j}|^2$$

$$\leq (\sum_{k=p}^{\infty} \frac{k(k+2)^n[1+\lambda(k-1)](1+2\beta\gamma-\beta)}{2\beta\gamma(1-2\lambda)(1-\alpha)}\,|a_{k,j}|)^2 \leq 1 \;(j = 1,2), \tag{5.19}$$

for $f_j(z) \in \sum_{p,n}^{*}(\alpha,\beta,\gamma,\lambda)(j = 1,2)$, we have

$$\sum_{k=p}^{\infty} \frac{\{k(k+2)^n[1+\lambda(k-1)](1+2\beta\gamma-\beta)\}^2}{2\,[2\beta\gamma(1-2\lambda)(1-\alpha)]^2}(|a_{k,1}|^2 + |a_{k,2}|^2) \leq 1\,. \tag{5.20}$$

Therefore, we have to find the largest τ such that

$$\frac{1}{(1-\tau)} \leq \frac{k(k+2)^n[1+\lambda(k-1)](1+2\beta\gamma-\beta)}{4\beta\gamma(1-2\lambda)(1-\alpha)^2}\;\;(k \geq p), \tag{5.21}$$

that is, that

$$\tau \leq 1 - \frac{4\beta\gamma(1-2\lambda)(1-\alpha)^2}{k(k+2)^n[1+\lambda(k-1)](1+2\beta\gamma-\beta)}\;(k \geq p). \tag{5.22}$$

Now, defining a function $\Psi(k)$ by

$$\Psi(k) = 1 - \frac{4\beta\gamma(1-2\lambda)(1-\alpha)^2}{k(k+2)^n[1+\lambda(k-1)](1+2\beta\gamma-\beta)}\;(k \geq p). \tag{5.23}$$

We observe that $\Psi(k)$ is an increasing function of k. We thus conclude that

$$\tau \leq \Psi(p) = 1 - \frac{4\beta\gamma(1-2\lambda)(1-\alpha)^2}{p(p+2)^n[1+\lambda(p-1)](1+2\beta\gamma-\beta)}\;, \tag{5.24}$$

which completes the proof of Theorem 4.3.

Putting $n = \lambda = 0$ in Theorem 4.3, we obtain:

Corollary 5.4. *Let the functions* $f_j(z)(j = 1, 2)$ *defined by* (5.1) *be in the class* $\sum_p^*(\alpha, \beta, \gamma)$. *Then the function* $h(z)$ *defined by* (5.17) *belongs to the class* $\sum_p^*(\tau, \beta, \gamma)$, *where*

$$\tau = 1 - \frac{4\beta\gamma(1-\alpha)^2}{p(1+2\beta\gamma-\beta)}. \tag{5.25}$$

The result is sharp.

Remark 5.5. The result obtained by Cho et al. ([5] and [6]) is not correct. The correct result is given by Corollary 3.

References

[1] Altintas, O., Owa, S., *Neighborhoods of certain analytic functions with negative coefficients*, Internat. J. Math. Math. Sci., **19** (1996), 797-800.

[2] Altintas, O., Ozkan, O., Srivastava, H. M., *Neighborhoods of a class of analytic functions with negative coefficients*, Appl. Math. Lett., **13** (2000), no. 3, 63-67.

[3] Altintas, O., Ozkan, O., Srivastava, H. M., *Neighborhoods of a certain family of multivalent functions with negative coefficients*, Comput. Math. Appl., **47** (2004), 1667-1672.

[4] Cho, N. E., *On certain class of meromorphic functions with positive coefficients*, J. Inst. Math. Comput. Sci. (Math. Ser.) **3** (1990), no. 2, 119-125.

[5] Cho, N. E., Lee, S. H., Owa, S., *A class of meromorphic univalent functions with positive coefficients*, Kobe J. Math., **4** (1987), 43-50.

[6] Cho, N. E., Owa, S., Lee, S. H., Altintas, O., *Generalization class of certain meromorphic univalent functions with positive coefficients*, Kyungpook Math. J., **29** (1989), no. 2, 133-139.

[7] Goodman, A. W., *Univalent functions and nonanalytic curves*, Proc. Amer. Math. Soc., **8** (1957), 598-601.

[8] Jack, I. S., *Functions starlike and convex functions of order* α, J. London Math. Soc., **2** (1971), no. 3, 469-474.

[9] Joshi, S. B., Kulkarni, S. R., Srivastava, H. M., *Certain classes of meromorphic functions with positive and missing coefficients*, J. Math. Anal Appl., **193** (1995), 1-14.

[10] Liu, J.-L., *Properties of some families of meromorphically p-valent function*, Math. Japon., **52** (2000), no. 3, 425-434.

[11] Liu, J.-L., Srivastava, H. M., *A linear operator and associated families of meromorphically multivalent functions*, J. Math. Anal. Appl., **259** (2001), 566-581.

[12] Liu, J.-L., Srivastava, H. M., *Subclasses of meromorphically multivalent functions associated with a certain linear operator*, Math. Comput. Modelling, **39** (2004), 35-44.

[13] Ruscheweyh, S., *Neighborhoods of univalent functions*, Proc. Amer. Math. Soc., **81** (1981), 521-527.

[14] Schild, A., Silvarman, H., *Convolution of univalent functions with negative coefficients*, Ann. Univ. Mariae Curie - Sklodowska Sect. A, **29** (1975), 99-107.

[15] Uralegaddi, B. A., Somanatha, C., *Certain classes of meromophic multivlent functions*, Tamkang J. Math., **23**(1992), 223-231.

Department of Mathematics
Faculty of Science
Mansoura University
Mansoura 35516, Egypt
E-mail address: elashwah@mans.edu.eg

STUDIA UNIV. "BABEŞ–BOLYAI", MATHEMATICA, Volume **LV**, Number 3, September 2010

ORDER OF CLOSE-TO-CONVEXITY FOR ANALYTIC FUNCTIONS OF COMPLEX ORDER

BASEM A. FRASIN

Dedicated to Professor Grigore Ştefan Sălăgean on his 60^{th} *birthday*

Abstract. The aim of this paper is to find the order of close-to-convexity for certain analytic functions of complex order.

1. Introduction and definitions

Let $\mathcal{A}$ denote the class of functions of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

which are analytic in the open unit disk $\mathcal{U} = \{z : |z| < 1\}$. A function $f(z)$ in $\mathcal{A}$ is said to be starlike function of complex order $\gamma(\gamma \in \mathbb{C}-\{0\})$, if and only if

$$\operatorname{Re}\left\{1 + \frac{1}{\gamma}\left(\frac{zf'(z)}{f(z)} - 1\right)\right\} > 0, \qquad (z \in \mathcal{U}). \tag{1.2}$$

We denote by $\mathcal{S}(\gamma)$ the class of all such functions. Also, a function $f(z)$ in $\mathcal{A}$ is said to be convex function of complex order $\gamma(\gamma \in \mathbb{C}-\{0\})$, that is, $f \in \mathcal{C}(\gamma)$, if and only if

$$\operatorname{Re}\left\{1 + \frac{1}{\gamma}\frac{zf''(z)}{f'(z)}\right\} > 0, \qquad (z \in \mathcal{U}). \tag{1.3}$$

The class $\mathcal{S}(\gamma)$ was introduced by Nasr and Aouf [7] and the class $\mathcal{C}(\gamma)$ was introduced by Wiatrowski [15] and considered in[6] (see also [5], [10], [13] and [2]).

Received by the editors: 01.03.2010.

2000 *Mathematics Subject Classification.* 30C45.

Key words and phrases. Analytic, starlike and convex functions of complex order, close-to-convex functions.

We note that $f(z) \in \mathcal{C}(\gamma) \Leftrightarrow zf'(z) \in \mathcal{S}(\gamma)$ and $\mathcal{S}(1-\alpha) = \mathcal{S}^*(\alpha)$, $\mathcal{C}(1-\alpha) = \mathcal{C}(\alpha)$ where $\mathcal{S}^*(\alpha)$ and $\mathcal{C}(\alpha)$ denote, respectively, the familiar classes of starlike and convex functions of a real order $\alpha(0 \le \alpha < 1)$ in $\mathcal{U}$ (see, for example, [14]).

A function $f(z)$ in $\mathcal{A}$ is said to be close-to-convex of complex order $\gamma(\gamma \in \mathbb{C}-\{0\})$, and type $\delta \in \mathbb{R}$ if there exists a function $g(z)$ belonging to $\mathcal{S}(\gamma)$ such that

$$\operatorname{Re}\left\{1 + \frac{1}{\gamma}\left(\frac{zf'(z)}{g(z)} - 1\right)\right\} > \delta, \qquad (z \in \mathcal{U}). \tag{1.4}$$

We denote by $\mathcal{K}(\gamma, \delta)$ the subclass of $\mathcal{A}$ consisting of functions which are close-to-convex of complex order γ and type β in $\mathcal{U}$. We note that the class $\mathcal{K}(1, 0)$ is the class of close-to-convex functions introduced by Kaplan [4] and Ozaki [11].

Pfaltzgraff *et al.*[12] have proved that if $f(z)$ in $\mathcal{A}$ satisfies the condition

$$\operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha \qquad (\frac{1}{2} \le \alpha < 1), \tag{1.5}$$

then $f(z)$ in the class $\mathcal{S}$ (and convex in at least one direction in $\mathcal{U}$). Furthermore, Cerebiez-Tarabicka et al. [1] have shown that if $f(z)$ in $\mathcal{A}$ satisfies the condition

$$\operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > -\frac{1}{2} \qquad (\frac{1}{2} \le \alpha < 1), \tag{1.6}$$

then

$$\operatorname{Re}\left(\frac{zf'(z)}{g(z)}\right) > 0, \qquad (z \in \mathcal{U}). \tag{1.7}$$

Recently, Owa [9] proved that if $f(z)$ in $\mathcal{A}$ satisfies the condition

$$\operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > 0 \qquad (z \in \mathcal{U}) \tag{1.8}$$

then

$$\operatorname{Re}\left(\frac{zf'(z)}{g(z)}\right) > \frac{3}{5} \qquad (z \in \mathcal{U}) \tag{1.9}$$

where $g(z) \in \mathcal{S}^*(\alpha/(\alpha+1))$, $\alpha \ge 0$.

Also, Frasin and Oros [3] proved that if the function $f(z)$ in $\mathcal{A}$ satisfies the condition

$$\operatorname{Re}\left(\frac{zf''(z)}{f'(z)} - \beta\right) > 0 \qquad (z \in \mathcal{U}) \tag{1.10}$$

then

$$\operatorname{Re}\left(\frac{zf'(z)}{g(z)}\right) > \frac{1}{2\beta - 1} \qquad (z \in \mathcal{U}) \tag{1.11}$$

where $g(z) \in \mathcal{S}^*$and $1 < \beta \leq 3/2$.

In order to show our results, we shall need the following lemma due to Obradovič *et al.*[8].

Lemma 1.1. *Let $f \in \mathcal{S}(b)$, $b \in \mathbb{C}-\{0\}$, and let $a \in \mathbb{C}-\{0\}$ with $0 < 2ab \leq 1$. Then*

$$Re\left\{\left(\frac{f(z)}{z}\right)^{a}\right\} > 2^{-2ab} \qquad (z \in \mathcal{U}). \tag{1.12}$$

2. Main results

With the aid of Lemma 1.1, we can prove the following result.

Theorem 2.1. *If the functions $f(z)$and $g(z)$ are in $\mathcal{A}$ and satisfies the conditions*

$$Re\left\{1+\frac{1}{b}\left(\frac{zf''(z)}{f'(z)}\right)\right\} > 0 \qquad (z \in \mathcal{U}), \tag{2.1}$$

with $0 < 2a\gamma \leq 1$, $\gamma = b/\ (a+1)$; $a, b \in \mathbb{C}-\{0\}$; $a \neq -1$, and

$$Im\left(\frac{a+1}{b}\right) \leq 0 \ or \ Im\left(\frac{zf'(z)}{g(z)}\right) \leq 0, \tag{2.2}$$

then $f(z)$ belongs to the class $\mathcal{K}(\gamma,\delta)$, where

$$\delta = 1+\left(2^{\frac{-2ab}{a+1}}-1\right) Re\left(\frac{a+1}{b}\right).$$

Proof. If we define $g(z)$ by

$$1+\frac{a+1}{b}\left(\frac{zg'(z)}{g(z)}-1\right) = 1+\frac{1}{b}\left(\frac{zf''(z)}{f'(z)}\right) \tag{2.3}$$

then from the condition (2.1) and (2.3), we have $g(z) \in \mathcal{S}(\gamma)$, with $\gamma = b/\ (a+1)$. It is easy to see that (2.3) implies

$$f'(z) = \left(\frac{g(z)}{z}\right)^{a+1} \tag{2.4}$$

or

$$\frac{zf'(z)}{g(z)} = \left(\frac{g(z)}{z}\right)^{a} \tag{2.5}$$

Applying Lemma 1.1 to $g(z)$, we obtain

$$\begin{aligned}
\mathrm{Re}\left\{1+\frac{a+1}{b}\left(\frac{zf'(z)}{g(z)}-1\right)\right\} &= \mathrm{Re}\left\{1+\frac{a+1}{b}\left(\left(\frac{g(z)}{z}\right)^{a}-1\right)\right\} \\
&= 1+\mathrm{Re}\left(\frac{a+1}{b}\right)\mathrm{Re}\left\{\left(\frac{g(z)}{z}\right)^{a}-1\right\} \\
&- \mathrm{Im}\left(\frac{a+1}{b}\right)\mathrm{Im}\left\{\left(\frac{g(z)}{z}\right)^{a}-1\right\} \\
&\geq 1+\mathrm{Re}\left(\frac{a+1}{b}\right)\mathrm{Re}\left\{\left(\frac{g(z)}{z}\right)^{a}-1\right\} \\
&> 1+\left(2^{-2a\gamma}-1\right)\mathrm{Re}\left(\frac{a+1}{b}\right) \\
&= 1+\left(2^{\frac{-2ab}{a+1}}-1\right)\mathrm{Re}\left(\frac{a+1}{b}\right).
\end{aligned}$$

This completes the proof of Theorem 2.1. □

Letting $a=1$ in Theorem 2.1, we have

Corollary 2.2. *If the function* $f\in\mathcal{C}(b)$ *with* $0<b\leq 2$, *then* $f\in\mathcal{K}(b/2,\delta)$, *where*

$$\delta = 1+\frac{2^{1-b}-2}{b}.$$

Letting $b=1$ in Theorem 2.1, we have

Corollary 2.3. *If the functions* $f(z)$*and* $g(z)$ *are in* $\mathcal{A}$ *and satisfies the conditions*

$$Re\left\{1+\frac{zf''(z)}{f'(z)}\right\}>0 \qquad (z\in\mathcal{U}), \tag{2.6}$$

with $0<2a\gamma\leq 1$, $\gamma=1/\ (a+1)$; $a\in\mathbb{C}-\{0\}$; $a\neq -1$, *and*

$$Im\,(a+1)\leq 0 \text{ or } Im\left(\frac{zf'(z)}{g(z)}\right)\leq 0, \tag{2.7}$$

then $f(z)$ *belongs to the class* $\mathcal{K}(\gamma,\delta)$, *where*

$$\delta = 1+\left(2^{\frac{-2a}{a+1}}-1\right)Re\,(a+1)\,.$$

Letting $b=1$ in Corollary 2.2 or $a=1$ in Corollary 2.3, we have

Corollary 2.4. *Let the functions* $f(z)$*and* $g(z)$ *be in* $\mathcal{A}$ *. If*

$$Re\left\{1+\frac{zf''(z)}{f'(z)}\right\}>0 \qquad (z\in\mathcal{U}), \tag{2.8}$$

then

$$Re\left(\frac{zf'(z)}{g(z)}\right)>\frac{1}{2} \qquad (z\in\mathcal{U}), \tag{2.9}$$

Therefore, if $f(z)$ *is convex in* U *then* $f(z)$ *is close-to-convex of order* $1/2$ *in* $\mathcal{U}$*.*

Letting $b=a+1$ in in Theorem 2.1, we have

Corollary 2.5. *Let the functions* $f(z)$*and* $g(z)$ *be in* $\mathcal{A}$ *. If*

$$Re\left\{1+\frac{1}{a+1}\left(\frac{zf''(z)}{f'(z)}\right)\right\}>0 \qquad (z\in\mathcal{U}), \tag{2.10}$$

where $0<a\leq 1/2$,*then*

$$Re\left\{\frac{zf'(z)}{g(z)}\right\}>\frac{1}{4^a}, \qquad (z\in\mathcal{U}). \tag{2.11}$$

Letting $a=1/2$ in Corollary 2.5, we have

Corollary 2.6. *Let the functions* $f(z)$*and* $g(z)$ *be in* $\mathcal{A}$ *. If*

$$Re\left\{1+\frac{2}{3}\left(\frac{zf''(z)}{f'(z)}\right)\right\}>0 \qquad (z\in\mathcal{U}), \tag{2.12}$$

then

$$Re\left\{\frac{zf'(z)}{g(z)}\right\}>\frac{1}{2}, \qquad (z\in\mathcal{U}), \tag{2.13}$$

That is, $f(z)$ *is close-to-convex of order* $1/2$ *in* $\mathcal{U}$*.*

References

[1] Cerebiez-Tarabicka, K., Godula, J., Zlotkiewics, E., *On a class of Bazilevič functions*, Ann. Univ. mariae Curie-Sklodowska, **33** (1979), 45-57.

[2] Frasin, B. A., *Family of analytic functions of complex order*, Acta Math. Acad. Paed. Ny., **22(2)** (2006), 179-191.

[3] Frasin, B. A., Oros, Gh., *Order of certain classes of analytic and univalent functions using Ruscheweyh derivative*, General Mathematics, **12(2)** (2004), 3-10.

[4] Kaplan, W., *Close-to-convex schlicht functions*, Michigan Math. J., **1** (1952), 169-185.

[5] Murugusundaramoorthy, G., Srivastava, H. M., *Neighborhoods of certain classes of analytic functions of complex order*, J. Inequal. Pure Appl. Math., **5(2)** (2004), Art. 24, 1-8. Online: http://jipam.vu.edu.au/article.php?sid=374]

[6] Nasr, M. A., Aouf, M. K., *On convex functions of complex order*, Mansoura Sc. Bull. Egypt, **9** (1982), 565-582.

[7] Nasr, M. A., Aouf, M. K., *Starlike functions of complex order,* J. Natur. Sci. Math., **25** (1985), 1-12.

[8] Obradovič, M., Aouf, M. K., Owa, S., *On some results for starlike functions of complex order*, Publ. De L'Ins. Math., **46(60)** (1989),79-85.

[9] Owa, S., *The order of close-to-convexity for certain univalent functions*, J. Math. Anal. Appl., **138** (1989), 393-369.

[10] Owa, S., Sălăgean, G. S., *On an open problem of S. Owa*, J. Math. Anal. Appl., **218** (1998), 453-457.

[11] Ozaki, S., *On the theory of multivalent functions*, Sci. Rep. Tokyo Bunrika Daigaku A, **2** (1935), 167-188.

[12] Pfaltzgraff, J. A., Reade, M. O., Umezawa, T., *Sufficient conditions for univalence*, Ann. Fac. Sci. Kinshasa Zaire Sect. Math. Phys., **2** (1976), 94-101.

[13] Srivastava, H. M., Lashin, A. Y., *Some applications of the Briot-Bouquet differential subordination*, J. Inequal. Pure Appl. Math., **6(2)** (2005), Art. 41, 1-7. Online: http://jipam.vu.edu.au/article.php?sid=510.

[14] Srivastava, H. M., Owa, S., (Eds.), *Current Topics in Analytic Function Theory*, World Scientific Publishing Company, Singapore, New Jersey, London and Hong Kong, 1992.

[15] Wiatroski, P., *On the coefficients of some family of holomorphic functions*, Zeszyty Nauk. Uniw. Lődz Nauk. Mat.-Przyrod, **2(39)** (1970), 75-85.

Faculty of Science
Department of Mathematics
Al al-Bayt University, P.O. Box 130095
Mafraq, Jordan
E-mail address: bafrasin@yahoo.com

STUDIA UNIV. "BABEŞ–BOLYAI", MATHEMATICA, Volume **LV**, Number 3, September 2010

STARLIKE FUNCTIONS WITH REGULAR REFRACTION PROPERTY

PETRU T. MOCANU

Dedicated to Professor Grigore Ştefan Sălăgean on his 60^{th} birthday

Abstract. Let $C : z = z(t), t \in [a, b]$, be a smooth Jordan curve of the class C^2 and let f be a complex univalent function of the class C^1 in a domain which contains the curve C together with its interior. Suppose that the origin lies inside of C and $f(0) = 0$. Let $\Gamma = f(C)$ and suppose that Γ is starlike with respect to the origin. Let consider the radius vector $\overrightarrow{R}$ from 0 to a point $w \in \Gamma$ and let $\overrightarrow{N}$ be the outer normal to Γ at the point $w = f[z(t)]$. Let denote by $\omega = (\overrightarrow{N}, \overrightarrow{R})$ the angle between $\overrightarrow{N}$ and $\overrightarrow{R}$ and consider the vector $\overrightarrow{V}$ starting from w, such that $\sin\Psi = \gamma \sin\omega$, where $\Psi = (\overrightarrow{N}, \overrightarrow{V})$ and γ is a positive number. We say that the starlike curve $\Gamma = f(C)$ has the regular refraction property, with index γ, iff the argument of the vector $\overrightarrow{V}$ is an increasing function of $t \in [a, b]$. The concept of regular refraction property was introduced in [2] and developed in [3], [4], [5], [6] and [7]. We mention that this concept is closed to the concept of α-convexity introduced in [1]. In this paper we continue to study this geometric property by introducing the concept of regular refraction interval of a given function. We also give a significant example.

1. Preliminaries

Let f an analytic and univalent function in a domain D and let $C : z = z(t)$, $t \in [a, b]$, be a smooth Jordan curve of the class C^2. Suppose that D contains the curve C together with its interior and that the origin lies inside of C and $f(0) = 0$. Let $\Gamma = f(C)$ and suppose that Γ is starlike with respect to 0.

Received by the editors: 02.03.2010.

2000 *Mathematics Subject Classification.* 30C45.

Key words and phrases. Starlike functions, regular refraction.

Let $\overrightarrow{R}$ be the radius vector from 0 to a point $w \in \Gamma$ and let $\overrightarrow{N}$ be the outer normal to Γ at the point $w = f(z(t))$. Let denote by $\omega = (\overrightarrow{N}, \overrightarrow{R})$ the angle between $\overrightarrow{N}$ and $\overrightarrow{R}$ and let consider the vector $\overrightarrow{V}$ starting from w, such that

$$\sin \Psi = \gamma \sin \omega, \tag{1.1}$$

where $\Psi = (\overrightarrow{N}, \overrightarrow{V})$ and γ is a positive number.

From the optical point of view, we remark that if Γ separates two media of different refraction indices and if $\overrightarrow{R}$ and $\overrightarrow{V}$ are the trajectories of the light in these media (starling from the origin), then (1) is the well -known refraction law.

Definition 1.1. We say that the curve $\Gamma = f(C)$ has the *regular refraction property* with index γ, iff the argument of the vector $\overrightarrow{V} = \overrightarrow{V}(t)$, defined by (1) is an increasing function of $t \in [a, b]$, i.e.

$$\frac{d}{dt} \arg \overrightarrow{V}(t) \geq 0, t \in [a, b]. \tag{1.2}$$

We also say, in this case, that the function f has the *regular refraction property* on $C : z = z(t)$.

Sometimes we are interesting to study the property of regular refraction only on some arcs of the curve C.

2. Main results

If we let $\varphi = \arg f(z)$ and $\chi = \arg \overrightarrow{V}$, then we have

$$\chi = \varphi + \omega - \psi.$$

If $z = z(t)$ and if we denote $\dot{z}, \dot{\chi}, ...$ the derivatives with respect to t, then we have

$$\dot{\chi} = \dot{\varphi} + F\dot{\omega},$$

where

$$F = 1 - \frac{\gamma \cos \omega}{[1 - \gamma^2 + \gamma^2 \cos^2 \omega]^{\frac{1}{2}}} = 1 - \frac{\gamma}{\sqrt{1 + (1 - \gamma^2) \tan^2 \omega}}$$

and $\omega = \arg P$, with $|\sin \omega| \leq \frac{1}{\gamma}$ with

$$P = \frac{\dot{z} f'(z)}{i f(z)}, z = z(t). \tag{2.1}$$

The condition (1.2) becomes

$$\Im\Big[iP + F\frac{\dot{P}}{P}\Big] \geq 0, t \in [a,b], \tag{2.2}$$

where

$$F = 1 - \frac{\gamma\Re P}{[(1-\gamma^2)|P|^2 + \gamma^2(\Re P)^2]^{\frac{1}{2}}}, |\sin\omega| \leq \frac{1}{\gamma}, \tag{2.3}$$

with P given by (2.1).

Hence we deduce the following result.

Theorem 2.1. *The function f has the regular refraction property, with index γ, on the curve $C : z = z(t), t \in [a,b]$, if and only if the inequality (2.2) holds for all $t \in [a,b]$.*

If we let $f(z) \equiv z$, then we have $P = i\frac{\dot{z}}{z}$,

$$F = 1 - \frac{\gamma\Im\frac{\dot{z}}{z}}{[(1-\gamma^2)|\frac{\dot{z}}{z}|^2 + \gamma^2(\Im\frac{\dot{z}}{z})^2]^{\frac{1}{2}}} \tag{2.4}$$

and (2.2) becomes

$$(1-F)\Im\frac{\dot{z}}{z} + F\Im\frac{\ddot{z}}{\dot{z}} \geq 0, z = z(t) \tag{2.5}$$

where F is given by (2.4), with $|\sin\omega| \leq \frac{1}{\gamma}$.

Since the curvature of the curve C at the point $z = z(t)$ is given by

$$k = k(t) = \frac{1}{|\dot{z}|}\Im\frac{\ddot{z}}{\dot{z}},$$

the condition (2.5) can be rewritten as

$$\gamma\Big(\Im\frac{\dot{z}}{z}\Big)^2 + \Big\{\Big[(1-\gamma^2)\Big|\frac{\dot{z}}{z}\Big|^2 + \gamma^2\Big(\Im\frac{\dot{z}}{z}\Big)^2\Big]^{\frac{1}{2}} - \gamma\Im\frac{\dot{z}}{z}\Big\}|\dot{z}|k \geq 0 \tag{2.6}$$

and we deduce

Theorem 2.2. *The curve $C : z = z(t), t \in [a,b]$ has the regular refraction property of index $\gamma \geq 0$ if and only if the inequality (2.6) holds for all $t \in [a,b]$.*

If C is convex then $k \geq 0$ and we deduce the following interesting result.

Corollary 2.3. *If the smooth curve C is convex, then it has the regular refraction property of any index $\gamma \in [0,1]$.*

If we let

$$\Delta = (1-\gamma^2)\Big|\frac{\dot{z}}{z}\Big|^2 + \gamma^2\Big(\Im\frac{\dot{z}}{z}\Big)^2,$$

then Theorem 2.2 can be rewritten as

Theorem 2.4. *The curve $C : z = z(t), t \in [a,b]$ has the regular refraction property of index γ if and only if the following inequalities hold for all $t \in [a,b]$:*

(i) $\Delta \leq 0$;

(ii) $\gamma\Big(\Im\frac{\dot{z}}{z}\Big)^2 + \Big[\sqrt{\Delta} - \gamma\Im\frac{\dot{z}}{z}\Big]\Im\frac{\ddot{z}}{\dot{z}} \geq 0.$

Let f be analytic and univalent in the closed unit disc $\overline{U}$, with $f(0) = 0$ and $f'(0) = 1$. If $C = C_r : re^{it}, t \in [0, 2\pi], 0 < r \leq 1$, then we have

$$P = p(z) = \frac{zf'(z)}{f(z)}.$$

and Theorem 2.1 becomes

Theorem 2.5. *The function f has the regular refraction property of index γ on the circle C_r if and only if*

$$\Re\Big[p(z) + F(z,\gamma)\frac{zp'(z)}{p(z)}\Big] \geq 0, \text{ for } |z| = r, \tag{2.7}$$

where

$$p(z) = \frac{zf'(z)}{f(z)} \tag{2.8}$$

$$F(z,\gamma) = 1 - \frac{\gamma\Re p(z)}{[(1-\gamma^2)|p(z)|^2 + \gamma^2(\Re p(z))^2]^{\frac{1}{2}}} \tag{2.9}$$

and

$$(1-\gamma^2)|p(z)|^2 + \gamma^2(\Re p(z))^2 \geq 0. \tag{2.10}$$

Definition 2.6. We say that the normalized analytic and univalent function f in the unit disc belongs to the class $\mathcal{RP}(\gamma)$, of functions with *regular refraction property* of index γ iff

$$\Re J(f; z, \gamma) \geq 0, \text{ for all } z \in U, \tag{2.11}$$

$$J(f; z, \gamma) = p(z) + F(z,\gamma)\frac{zp'(z)}{p(z)}, \tag{2.12}$$

with p and F given by (2.8), and (2.9) respectively, with condition (2.10).

Let S^* and K be respectively the class of starlike and convex functions in the unit disc.

Also, let $M(\alpha)$ be the class of α-convex functions in U.

It is easy to prove the following main result:

Theorem 2.7. *If $f \in \mathcal{RP}(\gamma), 0 \leq \gamma \leq 1$ then $f \in S^*$.*

Moreover

$$K \subset \mathcal{RP}(\gamma_1) \subset \mathcal{RP}(\gamma_2) \subset S^*, \text{ for } 0 < \gamma_1 < \gamma_2 < 1$$

and

$$K \subset \mathcal{RP}(1-\alpha) \subset M(\alpha), \text{ for } 0 < \alpha < 1.$$

We also have

$$\mathcal{RP}(\gamma_2) \subset \mathcal{RP}(\gamma_1) \subset S^*, \text{ for } 1 < \gamma_1 < \gamma_2.$$

An interesting extremal problem suggested by Theorem 2.7 is the following:

Given the function f, find the largest interval $[\gamma_0, \gamma_1]$, with $\gamma_0 \leq 1 \leq \gamma_1$, such that $f \in \mathcal{RP}(\gamma)$, for all $\gamma \in [\gamma_0, \gamma_1]$. We shall call this interval as *the regular refraction interval* of the function f.

We illustrate this last problem by the following.

Example 2.8. Let

$$f(z) = z \exp\left(\frac{z^n}{2n}\right), z \in \overline{U}.$$

In this case we have

$$p(z) = \frac{1}{2}(2 + z^n) \text{ and } \frac{zp'(z)}{p(z)} = \frac{nz^n}{2 + z^n}.$$

If $z = e^{it}$, then we have

$$\cos nt = x - 1, \text{ with } 0 \leq x \leq 2$$

and

$$|p(z)|^2 = \frac{1}{4}(1 + 4x), \Re p(z) = \frac{1}{2}(1 + x), \Re\frac{zp'(z)}{p(z)} = n\frac{2x - 1}{1 + 4x}.$$

Hence

$$F(z, \gamma) = 1 - \frac{\gamma(1 + x)}{\sqrt{E(x, \gamma)}},$$

where

$$E(x,\gamma) = 1 + 2(2-\gamma^2)x + \gamma^2 x^2.$$

Hence the inequality (2.7) becomes

$$\frac{1}{2}(1+x) + n\Big(1 - \frac{\gamma(1+x)}{\sqrt{E(x,\gamma)}}\Big)\frac{2x-1}{1+4x} \leq 0, \text{ for } 0 \leq x \leq 2. \tag{2.13}$$

We remark that for $\gamma \leq 2$ we have

$$E(x,\gamma) \geq 0, \text{ for } x \in [0,2].$$

For $x = 0$ we have $\frac{2n-1}{2n} \leq \gamma < 2$, and for $x = 2$ we have $\gamma < 1 + \frac{9}{2n}$.

From (2.13) we deduce

$$\frac{1}{\gamma^2} \geq \frac{1}{1+4x}\Big\{x(2-x) + \Big[\frac{2n(2x+x-1)}{4x^2+(4n+5)x+1-2n}\Big]^2\Big\} \equiv \Phi_n(x),$$

with $\frac{1}{2} < x \leq 2$.

For $n = 1$ we have

$$\max_{x\in\left[\frac{1}{2},2\right]} \Phi_1(x) = 0.25059...$$

and we deduce that the regular refraction interval of the function

$$f(z) = z\exp\left(\frac{z}{2}\right)$$

is given by $\left[\frac{1}{2}, 1.9976\cdots\right]$.

For $n = 2$ we have

$$\max_{x\in\left[\frac{1}{2},2\right]} \Phi_2(x) = 0.2934...$$

and we deduce that the regular refraction interval of the function

$$f(z) = z\exp\left(\frac{z^2}{4}\right)$$

is given by $\left[\frac{3}{4}, 1.9123\cdots\right]$.

References

[1] Mocanu, P. T., *Une propriété de convexité géneralisée dans la théorie de la représentation conforme*, Mathematica (Cluj), **11(34)** (1969), 43-50.

[2] Mocanu, P. T., *Conformal mappings and refraction law*, Babeş-Bolyai Univ., Fac of Math., Research Seminars, **2** (1988), 113-116.

[3] Mocanu, P. T., C^1*-maps and refraction law*, Buletinul Ştiinţific al Institutului Politehnic Cluj-Napoca, Seria Matematică Aplicată, Mecanică, **33** (1990), 11-14.

[4] Mocanu, P. T., *Regular refraction property of the ellipse*, Bull. Math. Soc. Sc. Math. Roumaine, **45(93)** (2002), no. 1-2, 29-43.

[5] Mocanu, P. T., *Regular refraction property of the parabolic lens*, Mathematica, **47(70)** (2005), no. 2, 179-181.

[6] Mocanu, P. T., *Starlike curves with regular refraction property*, Bull. Math. Soc. Sc. Math. Roumaine, **48(96)** (2005), no. 3, 303-311.

[7] Mocanu, P. T., Şerb, I., *Regular refraction property for a conic lens*, Bull. Math. Soc. Sc. Math. Roumaine, **50(98)** (2007), no. 1, 75-87.

Babeş-Bolyai University
Faculty of Mathematics and Computer Science
400084 Cluj-Napoca, Romania
E-mail address: pmocanu@math.ubbcluj.ro

STUDIA UNIV. "BABEŞ–BOLYAI", MATHEMATICA, Volume **LV**, Number 3, September 2010

ON ARGUMENT PROPERTY OF CERTAIN ANALYTIC FUNCTIONS

MAMORU NUNOKAWA AND ALPHA MAMADOU BAH

Dedicated to Professor Grigore Ştefan Sălăgean on his 60^{th} birthday

Abstract. In this paper we generalize the results of Libera and McGregor concerning argument property of analytic functions. We use the result in [3] to prove the following:

Theorem. *Let*

$$f(z) = z + \sum_{n=p+K}^{\infty} a_n z^n, \qquad g(z) = z + \sum_{n=p+K}^{\infty} b_n z^n$$

be analytic in Δ, $f(z) \neq 0$ *in* $0 < |z| < 1$, *and suppose that for some* α, β $(0 < \alpha < 1,\ 0 < \beta < 1)$

$$\left| \arg \left(\frac{f'(z)}{g'(z)} \right) \right| < \frac{\pi}{2}\alpha + \mathrm{Tan}^{-1} \frac{2\alpha\beta}{1-\beta^2} - \mathrm{Tan}^{-1} \frac{2\alpha\beta}{(1-\beta^2)\sqrt{1+\alpha^2}}$$

in Δ, *and that* $\dfrac{g'(z)}{zg(z)} \prec \dfrac{1+\beta z}{1-\beta z}$ *where* $\prec$ *means subordination. Then we have*

$$\left| \arg \left(\frac{f(z)}{g(z)} \right) \right| < \frac{\pi}{2}\alpha \quad in \quad \Delta.$$

1. **Introduction**

Let f and g be analytic in the unit disk $\Delta = \{z : |z| < 1\}$ $f(0) = g(0) = 0$, g maps Δ onto a many sheeted domain which is starlike with respect to the origin, and

$$\mathrm{Re} \frac{f'(z)}{g'(z)} > 0 \quad \text{in } \Delta\ .$$

Then Libera [1] proved

$$\mathrm{Re} \frac{f(z)}{g(z)} > 0 \quad \text{in } \Delta\ .$$

Received by the editors: 26.04.2010.

2000 *Mathematics Subject Classification.* 30C45.

Key words and phrases. Univalent functions, Argument property of analytic functions.

The essential ideas of the proof of the above result are the same as given by Sakaguchi [6].

On the other hand, MacGregor [2] proved that for real β,

$$\mathrm{Re}\frac{f'(z)}{g'(z)} > \beta \quad \text{in } \Delta.$$

implies

$$\mathrm{Re}\frac{f(z)}{g(z)} > \beta \quad \text{in } \Delta.$$

Ponnusamy and Karunakaran [4] generalized the above results as the following:

Theorem 1.1. *Let* α *be a complex number satisfying* $Re\alpha > 0$ *and* $\beta < 1$*. Let*

$$f(z) = z^p + \sum_{n=p+K}^{\infty} a_n z^n, \quad g(z) = z^p + \sum_{n=p+K}^{\infty} b_n z^n$$

are analytic in Δ *for* $1 \le p$, $1 \le K$ *and that* g *satisfies*

$$\mathrm{Re}\left(\alpha\frac{g(z)}{g'(z)}\right) > \delta \quad in\ \Delta$$

where

$$0 \le \delta < \frac{\mathrm{Re}\alpha}{p}.$$

If

$$\mathrm{Re}\left\{(1-\alpha)\frac{f(z)}{g(z)} + \alpha\frac{f'(z)}{g'(z)}\right\} > \beta \quad in\ \Delta.$$

Then

$$\mathrm{Re}\frac{f(z)}{g(z)} > \frac{2\beta + K\delta}{2 + K\delta} \quad in\ \Delta.$$

Putting $\alpha = 1$ in Theorem 1.1, it follows that

Corollary 1.2. *If*

$$f(z) = z^p + \sum_{n=p+K}^{\infty} a_n z^n \quad , \quad 1 \le p, \quad 1 \le pK \ \ and \quad g(z) = z^p + \sum_{n=p+K}^{\infty} b_n z^n$$

are analytic in Δ *and* g *satisfies*

$$\mathrm{Re}\frac{g(z)}{zg'(z)} > \delta \quad in\ \Delta$$

where $\quad 0 \le \delta < \frac{1}{p} \quad$ *then for* β *real*

$$\frac{f'(z)}{g'(z)}) > \beta \quad in\ \Delta$$

implies

$$\mathrm{Re}\frac{f(z)}{g(z)} > \frac{2\beta + K\delta}{2 + K\delta} \quad in\ \Delta.$$

For a argument properties of analytic functions, Pommerenke [5] obtained the following result. If f is analytic in Δ and h is convex in Δ and

$$\left|\arg\left(\frac{f'(z)}{h'(z)}\right)\right| < \frac{\alpha\pi}{2} \quad (0 \le \alpha \le 1)$$

then

$$\left|\arg\left(\frac{f(z_2) - f(z_1)}{h(z_2) - h(z_1)}\right)\right| < \frac{\alpha\pi}{2}$$

where

$$|z_1| < 1 \quad \text{and} \quad |z_2| < 1.$$

2. Main theorem

In this short paper, we will obtain a generalization of Libera's result by applying Nunokawa's result [3].

Lemma 2.1. *Let p be analytic in Δ, $p(0) = 1$, $p(z) \neq 0$ in Δ and suppose that there exists a point $z_0 \in \Delta$ such that*

$$|\arg p(z_0)| < \frac{\pi}{2}\alpha \qquad for\ |z| < |z_0|$$

and

$$|\arg p(z_0)| = \frac{\pi}{2}\alpha$$

where $0 < \alpha$.

Then we have

$$\frac{z_0 p'(z_0)}{p(z_0)} = iK\alpha$$

where

$$K \ge \frac{1}{2}\left(a + \frac{1}{a}\right) \quad when \quad \arg p(z_0) = \frac{\pi}{2}\alpha$$

and

$$K \le -\frac{1}{2}\left(a + \frac{1}{a}\right) \quad when \quad \arg p(z_0) = -\frac{\pi}{2}\alpha$$

where

$$\arg p(z_0)^{\frac{1}{\alpha}} = \pm ia \quad and \quad 0 < a$$

Theorem 2.2. *Let*

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

be analytic in $\Delta \quad f(z) \neq 0 \quad in \quad 0 < |z| < 1$,

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n$$

be analytic in Δ *and suppose*

$$|\arg(\frac{f'(z)}{g'(z)})| < \frac{\pi}{2}\alpha + \mathrm{Tan}^{-1}\frac{2\alpha\beta}{1-\beta^2} - \mathrm{Tan}^{-1}\frac{2\alpha\beta}{(1-\beta^2)\sqrt{1+\alpha^2}}$$

in Δ *where* $0 < \alpha < 1, \quad 0 < \beta < 1$ *and*

$$\frac{zg'(z)}{g(z)} \prec \frac{1+\beta z}{1-\beta z}$$

where $\prec$ *means the subordination. Then we have*

$$\left|\arg\left(\frac{f(z)}{g(z)}\right)\right| < \frac{\pi}{2}\alpha \qquad \text{in } \Delta.$$

Proof. Let us put

$$p(z) = \frac{f(z)}{g(z)}, \quad p(0) = 1$$

Then it follows that

$$\frac{f'(z)}{g'(z)} = p(z) + \frac{g(z)}{g'(z)}p'(z)$$

$$= p(z)\left(1 + \frac{g(z)}{zg'(z)}.\frac{zp'(z)}{p(z)}\right).$$

If there exist a point z_0, $|z_0| < 1$ such that

$$|\arg p(z)| < \frac{\pi}{2}\alpha \quad \text{for } |z| < |z_0|$$

and

$$|\arg p(z_0)| = \frac{\pi}{2}\alpha$$

then from Lemma 2.1 we have

$$\frac{z_0 p'(z_0)}{p(z_0)} = iK\alpha.$$

From the hypothesis, we have the image of the circle Δ under the mapping $w = \frac{1+\beta z}{1-\beta z}$ is contained in the circle whose center is $\frac{1+\beta^2}{1-\beta^2}$ with radius $\frac{2\beta}{1-\beta^2}$. Applying the above properties, for the case

$$\arg p(z_0) = \frac{\pi}{2}\alpha,$$

we have

$$\begin{aligned}\arg \frac{f'(z_0)}{g'(z_0} &= \arg p(z_0) + \arg\left(1 + i\alpha K \frac{g(z_0)}{g'(z_0)}\right)\\ &\geq \frac{\pi}{2}\alpha + \mathrm{Tan}^{-1}\delta K - \mathrm{Tan}^{-1}\frac{\delta K}{\sqrt{1+(\rho^2-\delta^2)K^2}}\end{aligned}$$

where $\rho = \frac{\alpha(1+\beta^2)}{1-\beta^2}$, $\delta = \frac{2\alpha\beta}{1-\beta^2}$ and then it follows that

$$\rho^2 - \delta^2 = \alpha^2.$$

Now let us put

$$\mathrm{F(K)} = \mathrm{Tan}^{-1}\delta K - Tan^{-1}\frac{\delta K}{\sqrt{1+(\rho^2-\delta^2)K^2}} \qquad , \quad 1 \leq K.$$

Then we have

$$\begin{aligned}\mathrm{F'(K)} &= \frac{\delta}{1+\delta^2K^2} - \left(\frac{\delta(1-(\rho^2-\delta^2)K^2) - \delta(\rho^2-\delta^2)K^2}{(1+(\rho^2-\delta^2)K^2)^{\frac{3}{2}}}\right)\frac{(1+(\rho^2-\delta^2)K^2)}{1+\rho^2K^2}\\ &= \frac{\delta}{1+\delta^2K^2} - \frac{\delta}{(1+\rho^2K^2)\sqrt{1+(\rho^2-\delta^2)K^2}}\\ &> \frac{\delta}{1+\delta^2K^2} - \frac{\delta}{1+\rho^2K^2}\\ &= \frac{\delta(\rho^2-\delta^2)}{(1+\delta^2K^2)(1+\rho^2K^2)} > 0.\end{aligned}$$

This shows that $F(K)$ takes the minimum value at $K = 1$. Therefore we have

$$\arg\frac{f'(z_0)}{g'(z_0)} \geq \frac{\pi}{2}\alpha + \mathrm{Tan}^{-1}\frac{2\alpha\beta}{1-\beta^2} - \mathrm{Tan}^{-1}\frac{2\alpha\beta}{(1-\beta^2)\sqrt{1+\alpha^2}}$$

This contradicts the hypothesis and for the case $\arg p(z_0) = -\frac{\pi}{2}\alpha$, applying the same method as the above, we have

$$\arg\frac{f'(z_0)}{g'(z_0)} \leq -\left(\frac{\pi}{2}\alpha + Tan^{-1}\frac{2\alpha\beta}{1-\beta^2} - Tan^{-1}\frac{2\alpha\beta}{(1-\beta^2)\sqrt{1+\alpha^2}}\right).$$

This is also contradiction and it completes the proof. □

3. Acknowledgement

The second author is grateful to Yahata Memorial Ikuei Foundation Japan for grant assistance.

References

[1] Libera, R. J., *Some classes for univalent functions*, Proc. Amer. Math. Soc., **16** (1965), 755-758.

[2] MacGregor, T. H., *A subordination for convex function of order* α, J. London Math. Soc., **(2)9** (1975), 530-536.

[3] Nunokawa, M., *On the orrder of strongly starlikeness of strongly convex functions*, Proc. Japan Acad., **69** (1993), no. 7, 234-237.

[4] Ponnusamy, S., Karunakaran, V., *Differential subordination and conformal mappings*, Complex variables, **11** (1989), 79-86.

[5] Pommerenke, Ch., *On close-to-convex analytic functions*, Trans. Amer. Math. Soc., **114(1)** (1965), 176-186.

[6] Sakaguchi, K., *On a certain univalent mappings*, J. Math. Soc. Japan, **11** (1959), 72-75.

Department of Mathematics
University of Gunma, Aramaki, Maebashi
Gunma, 371, Japan

Graduate School of Engineering
Applied Mathematics Department
Hiroshima University
739-8527, Hiroshima Japan
E-mail address: bah@hiroshima-u.ac.jp

STUDIA UNIV. "BABEŞ–BOLYAI", MATHEMATICA, Volume **LV**, Number 3, September 2010

A CONVEXITY PROPERTY FOR AN INTEGRAL OPERATOR F_m

GEORGIA IRINA OROS AND GHEORGHE OROS

Dedicated to Professor Grigore Ştefan Sălăgean on his 60^{th} birthday

Abstract. In this paper we define an integral operator denoted by $F_m(z)$ using the Ruscheweyh derivative of order n applied to the functions $f_i(z) \in \mathcal{A}$, $i = \{1, 2, \ldots, m\}$, $z \in U$. We determine conditions on the functions $R^n f_i(z)$, where R^n is the Ruscheweyh operator (Definition 1.1), in order for $F_m(z)$ to be convex.

1. Introduction and preliminaries

Let U be the unit disk of the complex plane:

$$U = \{z \in \mathbb{C} : \ |z| < 1\}.$$

Let $\mathcal{H}(U)$ be the space of holomorphic functions in U. Also, let

$$A_n = \{f \in \mathcal{H}(U), \ f(z) = z + a_{n+1}z^{n+1} + \ldots, \ z \in U\}$$

with $A_1 = A$ and

$$S = \{f \in A : \ f \text{ is univalent in } U\}.$$

Let

$$K(\alpha) = \left\{ f \in A : \ \mathrm{Re}\, \frac{zf''(z)}{f'(z)} + 1 > \alpha, \ z \in U \right\}$$

denote the class of normalized convex functions of order α, where $0 \le \alpha < 1$,

$$K(0) = K,$$

Received by the editors: 01.03.2010.

2000 *Mathematics Subject Classification.* 30C45, 30A20, 34A40.

Key words and phrases. analytic function, univalent function, integral operator, convex function, starlike function, differential operator.

$$S^*(\alpha) = \left\{ f \in A : \ \mathrm{Re} \, \frac{zf'(z)}{f(z)} > \alpha, \ z \in U \right\}$$

denote the class of starlike functions of order α, with $0 \le \alpha < 1$, $S^*(0) = S^*$.

In the papers [9], [10], F. Ronning introduces two classes of univalent functions denoted by SP and $SP(\alpha, \beta)$, respectively. The class SP consists of those functions $f \in S$ which satisfy the condition

$$\mathrm{Re} \, \frac{zf'(z)}{f(z)} > \left| \frac{zf'(z)}{f(z)} - 1 \right|, \ \text{for all } z \in U. \tag{1.1}$$

The class $SP(\alpha, \beta)$, $\alpha > 0$, $\beta \in [0, 1)$ consists of the functions $f \in S$ which satisfy the condition

$$\left| \frac{zf'(z)}{f(z)} - (\alpha + \beta) \right| \le \mathrm{Re} \, \frac{zf'(z)}{f(z)} + \alpha - \beta, \ \text{for all } z \in U. \tag{1.2}$$

In [12], the authors introduce the class denoted by $SD(\alpha, \beta)$ consisting of the functions $f \in A$ which satisfy the inequality

$$\mathrm{Re} \, \frac{zf'(z)}{f(z)} > \alpha \left| \frac{zf'(z)}{f(z)} - 1 \right| + \beta, \tag{1.3}$$

for $\alpha \ge 0$ and $\beta \in [0, 1)$.

Definition 1.1. (St. Ruscheweyh [11]). For $f \in A$, $n \in \mathbb{N} \cup \{0\}$, let R^n be the operator defined by $R^n : A \to A$

$$R^0 f(z) = f(z)$$

$$(n+1) R^{n+1} f(z) = z[R^n f(z)]' + n R^n f(z), \ z \in U.$$

Remark 1.2. If $f \in A$

$$f(z) = z + \sum_{j=2}^{\infty} a_j z^j$$

then

$$R^n f(z) = z + \sum_{j=1}^{\infty} C_{n+j-1}^n a_j z^j, \quad z \in U,$$

with

$$R^n f(0) = 0 \quad \text{and} \quad [R^n f(0)]' = 1.$$

2. Main results

By using the Ruscheweyh differential operator (Definition 1.1) we introduce the following integral operator.

Definition 2.1. Let $n, m \in \mathbb{N} \cup \{0\}$, $i \in \{1, 2, 3, \ldots, m\}$, $\alpha_i \in \mathbb{R}$ with $\alpha_i \geq 0$,

$$A^m = \underbrace{A \times A \times \cdots \times A}_{m}.$$

We define the integral operator $I : A^m \to A$

$$I(f_1, f_2, \ldots, f_m)(z) = F_m(z) \tag{2.1}$$

$$= \int_0^z \left[\frac{R^n f_1(t)}{t}\right]^{\alpha_1} \cdots \left[\frac{R^n f_m(t)}{t}\right]^{\alpha_m} dt, \quad z \in U$$

where $f_i(z) \in A$, $i \in \{1, 2, 3, \ldots, m\}$ and R^n is the Ruscheweyh differential operator given by Definition 1.1.

Remark 2.2. (i) For $n = 0$, $m = 1$, $\alpha_1 = 1$, $\alpha_2 = \alpha_3 = \cdots = \alpha_m = 0$,

$$R^0 f(z) = f(z) \in A$$

and we obtain Alexander integral operator introduced in 1915 in [1]:

$$I(z) = \int_0^z \frac{f(t)}{t} dt, \quad z \in U.$$

(ii) For $n = 0$, $m = 1$, $\alpha_1 = \alpha \in [0, 1]$, $\alpha_2 = \alpha_3 = \cdots = \alpha_m = 0$, $R^0 f(z) = f(z) \in S$ and we obtain the integral operator

$$I(z) = \int_0^z \left[\frac{f(t)}{t}\right]^{\alpha} dt, \quad z \in U$$

which was studied in several papers such as [6]. For $\alpha \in \mathbb{C}$, $|\alpha| \leq \frac{1}{4}$ the operator was studied in [4], [5] and for $|\alpha| \leq \frac{1}{3}$ in [8].

(iii) For $n = 1$, $m = 1$, $\alpha_1 = \alpha \in \mathbb{C}$, $|\alpha| \leq \frac{1}{4}$, $\alpha_2 = \cdots = \alpha_m = 0$, $R^1 f(z) = z f'(z)$, $z \in U$, $f \in S$, and we obtain the integral operator

$$I(z) = \int_0^z [f'(t)]^{\alpha} dt$$

which was studied in [7].

(iv) For $n = 0$, $m \in \mathbb{N} \cup \{0\}$, $\alpha_i > 0$, $i \in \{1, 2, \dots, m\}$ we obtain the integral operator defined by D. Breaz and N. Breaz in [3] given by

$$F(z) = \int_0^z \left[\frac{f_1(t)}{t}\right]^{\alpha_1} \left[\frac{f_2(t)}{t}\right]^{\alpha_2} \dots \left[\frac{f_m(t)}{t}\right]^{\alpha_m} dt.$$

Property 2.3. *Let $m \in \mathbb{N} \cup \{0\}$, $i \in \{1, 2, \dots, m\}$. If $f_i(z) \in A$ then $F_m(z)$ given by (2.1) belongs to the class A.*

Proof. From (2.1) we have

$$\begin{aligned} F_m(z) &= \int_0^z \left[\frac{R^n f_1(t)}{t}\right]^{\alpha_1} \dots \left[\frac{R^n f_m(t)}{t}\right]^{\alpha_m} dt \\ &= \int_0^z \left[\frac{t + \sum_{k=2}^{\infty} a_{k,1} t^k}{t}\right]^{\alpha_1} \dots \left[\frac{t + \sum_{k=2}^{\infty} a_{k,m} t^k}{t}\right]^{\alpha_m} dt \\ &= \int_0^z \left[1 + \sum_{k=2}^{\infty} a_{k,1} t^{k-1}\right]^{\alpha_1} \dots \left[1 + \sum_{k=2}^{\infty} a_{k,m} t^{k-1}\right]^{\alpha_m} dt \\ &= \int_0^z \left(1 + \sum_{k=2}^{\infty} \gamma_k t\right) dt = t\Big|_0^z + \sum_{k=2}^{\infty} \gamma_k \frac{t^k}{k}\Big|_0^z \\ &= z + \sum_{k=2}^{\infty} \delta_k t^k \in A, \end{aligned}$$

hence $F_m(z) \in A$. □

Definition 2.4. Let $\mathcal{R}(\beta)$ be the subclass of functions $f \in A$ which satisfy the condition

$$\operatorname{Re} \frac{z[R^n f(z)]'}{R^n f(z)} > \beta, \quad 0 \le \beta < 1,\ z \in U. \tag{2.2}$$

Remark 2.5. (i) For $n = 0$, $\mathcal{R}(\beta)$ becomes the class of starlike functions of order β denoted by $S^*(\beta)$.

(ii) For $n = 0$, $\beta = 0$, $\mathcal{R}(\beta)$ becomes $\mathcal{R}(0) = S^*$, the class of starlike functions.

Definition 2.6. Let $K(\beta) \subset A^m = \underbrace{A \times A \times \dots \times A}_{m}$ denote the subclass of functions $(f_1, f_2, \dots, f_m) \in A^m$ which satisfy the condition

$$\operatorname{Re} \left[1 + \frac{zF''_m(z)}{F'_m(z)}\right] > \beta, \quad \beta < 1,\ z \in U, \tag{2.3}$$

where $F_m(z)$ is given by (2.1).

Theorem 2.7. *Let* $n, m \in \mathbb{N} \cup \{0\}$, $i \in \{1, 2, \ldots, m\}$, $\alpha_i \in \mathbb{R}$ *with* $\alpha_i \geq 0$, $\beta_i \in \mathbb{R}$, $0 \leq \beta_i < 1$ *and* $\sum_{i=1}^{m} \alpha_i(\beta_i - 1) \geq -1$. *If* $f_i \in \mathcal{R}(\beta_i)$ *then* $F_m(z) \in K(\delta)$ *where* F_m *is given by (4) and*

$$\delta = 1 + \sum_{i=1}^{m} \alpha_i(\beta_i - 1).$$

Proof. By differentiating (2.1), we obtain

$$F'_m(z) = \left[\frac{R^n f_1(z)}{z}\right]^{\alpha_1} \cdots \left[\frac{R^n f_m(z)}{z}\right]^{\alpha_m}, \quad z \in U. \tag{2.4}$$

Using (2.4) we obtain:

$$\text{Log } F'_m(z) = \alpha_1[\text{Log } R^n f_1(z) - \text{Log } z] + \cdots + \tag{2.5}$$

$$+ \alpha_m[\text{Log } R^n f_m(z) - \text{Log } z], \quad z \in U.$$

By differentiating (2.5) we have

$$\frac{F''_m(z)}{F'_m(z)} = \alpha_1 \left[\frac{(R^n f_1(z))'}{R^n f_1(z)} - \frac{1}{z}\right] + \cdots + \alpha_m \left[\frac{(R^n f_m(z))'}{R^n f_m(z)} - \frac{1}{z}\right] \tag{2.6}$$

and after a short calculation we obtain

$$1 + \frac{zF''_m(z)}{F'_m(z)} = \alpha_1 \frac{z(R^n f_1(z))'}{R^n f_1(z)} + \cdots + \alpha_m \frac{z(R^n f_m(z))'}{R^n f_m(z)} \tag{2.7}$$

$$+1 - (\alpha_1 + \cdots + \alpha_m).$$

Since $f_i \in \mathcal{R}(\beta_i)$ we have

$$\text{Re } \left[1 + \frac{zF''_m(z)}{F'_m(z)}\right] = \alpha_1 \text{Re } \frac{z(R^n f_1(z))'}{R^n f_1(z)} + \cdots + \alpha_m \text{Re } \frac{z(R^n f_m(z))'}{R^n f_m(z)}$$

$$+1 - \sum_{i=1}^{m} \alpha_i > \alpha_1\beta_1 + \cdots + \alpha_m\beta_m + 1 - \sum_{i=1}^{m} \alpha_i$$

$$> \sum_{i=1}^{m} \alpha_i\beta_i + 1 - \sum_{i=1}^{m} \alpha_i > 1 + \sum_{i=1}^{m} \alpha_i(\beta_i - 1).$$

□

If $f_i \in \mathcal{R}(\beta)$ then Theorem 2.7 can be rewritten as the following:

Corollary 2.8. *Let* $n, m \in \mathbb{N} \cup \{0\}$*,* $i \in \{1, 2, \ldots, m\}$*,* $\alpha_i \in \mathbb{R}$ *with* $\alpha_i \geq 0$*,* $\beta \in \mathbb{R}$*,* $\beta < 1$*. If* $f_i \in \mathcal{R}(\beta)$ *then* $F_m(z) \in K(\delta')$ *where*

$$\delta' = 1 + (\beta - 1) \sum_{i=1}^{m} \alpha_i.$$

If $\beta = 0$ and $\sum_{i=1}^{m} \alpha_i = 1$ then Theorem 2.7 can be rewritten as the following:

Corollary 2.9. *Let* $n, m \in \mathbb{N} \cup \{0\}$*,* $i \in \{1, 2, \ldots, m\}$*,* $\alpha_i \in \mathbb{R}$ *with* $\alpha_i \geq 0$ *and* $\sum_{i=1}^{m} \alpha_i = 1$*, and* $\beta = 0$*. If* $f_i \in \mathcal{R}(0)$ *then* $F_m(z) \in K$*.*

Theorem 2.10. *Let* $n, m \in \mathbb{N} \cup \{0\}$*,* $i \in \{1, 2, \ldots, m\}$*,* $\alpha_i \in \mathbb{R}$ *with* $0 \leq \alpha_i < 1$ *and* $\sum_{i=1}^{m} \frac{\alpha_i}{2} < 1$*. If* $R^n f_i \in SP$ *and*

$$\left| \frac{z(R^n f_i(z))'}{R^n f_i(z)} - 1 \right| \geq \frac{1}{2}, \quad z \in U \tag{2.8}$$

then

$$F_m(k) \in K(\omega),$$

where F_m *is given by (4) and*

$$\omega = 1 - \sum_{i=1}^{m} \frac{\alpha_i}{2}.$$

Proof. Since $R^n f_i \in SP$, using (2.7) and (2.8) we have:

$$\begin{aligned}
\operatorname{Re}\left[1 + \frac{zF''_m(z)}{F'_m(z)}\right] &= \alpha_1 \operatorname{Re} \frac{z(R^n f_1(z))'}{R^n f_1(z)} + \cdots + \alpha_m \operatorname{Re} \frac{z(R^n f_m(z))'}{R^n f_m(z)} \\
+1 - \sum_{i=1}^{m} \alpha_i &\geq \alpha_1 \left| \frac{z(R^n f_1(z))'}{R^n f_1(z)} - 1 \right| + \cdots + \alpha_m \left| \frac{z(R^n f_m(z))'}{R^n f_m(z)} - 1 \right| \\
+1 - \sum_{i=1}^{m} \alpha_i &> 1 - \sum_{i=1}^{m} \alpha_i + \sum_{i=1}^{m} \frac{\alpha_i |z|}{2} \\
&= 1 - \sum_{i=1}^{m} \alpha_i \left(1 - \frac{|z|}{2}\right).
\end{aligned}$$

□

If $\sum_{i=1}^{m} \alpha_i = 1$ then Theorem 2.10 can be rewritten as the following:

Corollary 2.11. *Let $n, m \in \mathbb{N} \cup \{0\}$, $i \in \{1, 2, \dots, m\}$, $\alpha_i \in \mathbb{R}$ with $0 \le \alpha_i < 1$ and $\sum_{i=1}^{m} \alpha_i \left(1 - \frac{|z|}{2}\right) = 1$. If $R^n f_i \in SP$ then $F_m(z)$ is a convex function.*

Example 2.12. Let $n \in \mathbb{N} \cup \{0\}$, $m = 2$, $\alpha_1 = \frac{1}{2}$, $\alpha_2 = \frac{1}{3}$,

$$f_1(z) = z + a_2 z^2, \quad f_2(z) = z + b_2 z^2,$$

$$R^n f_1(z) = z + (n+1)a_2 z^2, \quad R^n f_2(z) = z + (n+1)b_2 z^2,$$

where $a_2, b_2 \in \mathbb{C}$, $|a_2| \ge \frac{1}{(n+1)(2-|z|)}$, $|b_2| \ge \frac{1}{(n+1)(2-|z|)}$, $z \in U$.

Evaluate

$$\left| \frac{z[z + (n+1)a_2 z^2]'}{z[1 + (n+1)a_2 z]} - 1 \right| = \left| \frac{(n+1)a_2 z}{1 + (n+1)a_2 z} \right|$$

$$= \sqrt{\frac{(n+1)^2 |a_2|^2 |z|}{(1 + (n+1)|a_2||z|)^2}} = \frac{(n+1)|a_2||z|}{1 + (n+1)|a_2||z|} \ge \frac{1}{2}$$

$$\left| \frac{z[z + (n+1)b_2 z^2]'}{z(1 + (n+1)b_2 z^2]} - 1 \right| = \left| \frac{1 + 2(n+1)b_2 z}{1 + (n+1)b_2 z} - 1 \right| = \left| \frac{(n+1)b_2 z}{1 + (n+1)b_2 z} \right|$$

$$= \sqrt{\frac{(n+1)^2 |b_2|^2 |z|^2}{[1 + (n+1)|b_2||z|]^2}} = \frac{(n+1)|b_2||z|}{1 + (n+1)|b_2||z|} \ge \frac{1}{2}$$

Using Theorem 2.10, we have

$$F_2(z) = \int_0^z [1 + (n+1)a_2 t]^{\frac{1}{2}} [1 + (n+1)b_2 t]^{\frac{1}{3}} dt \in K\left(\frac{1}{2}\right), \quad z \in U.$$

Theorem 2.13. *Let $n, m \in \mathbb{N} \cup \{0\}$, $i \in \{1, 2, \dots, m\}$, $\alpha_i \in \mathbb{R}$ with $\alpha_i \ge 0$, $\lambda \in \mathbb{R}$ with $\lambda > 0$, $\mu \in \mathbb{R}$ with $\mu \in [0, 1)$ and $(\lambda - \mu + 1) \sum_{i=1}^{m} \alpha_i \le 1$. If $R^n f_i \in SP(\lambda, \mu)$ then $F_m \in K(\omega)$, where F_m is given by (4) and*

$$\omega = 1 - (\lambda - \mu + 1) \sum_{i=1}^{m} \alpha_i.$$

Proof. Since $R^n f_i \in SP(\lambda, \mu)$, using (2.7) we have:

$$\text{Re}\left[1 + \frac{zF_m''(z)}{F_m'(z)}\right] = \alpha_1 \text{Re}\, \frac{z(R^n f_1(z))'}{R^n f_1(z)} + \dots + \alpha_m \text{Re}\, \frac{z(R^n f_m(z))'}{R^n f_m(z)}$$

$$+1 - \sum_{i=1}^{m} \alpha_i \ge \alpha_1 \left[\left| \frac{z(R^n f_1(z))'}{R^n f_1(z)} - (\lambda + \mu) \right| - (\lambda - \mu) \right] + \dots +$$

$$+\alpha_m\left[\left|\frac{z(R^nf_m(z))'}{R^nf_m(z)}-(\lambda+\mu)\right|-(\lambda-\mu)\right]+1-\sum_{i=1}^m\alpha_i$$
$$\geq\sum_{i=1}^m\alpha_i\left[\left|\frac{z(R^nf_i(z))'}{R^nf_1(z)}-(\lambda+\mu)\right|\right]-\sum_{i=1}^m\alpha_i(\lambda-\mu)$$
$$+1-\sum_{i=1}^m\alpha_i\geq1-\sum_{i=1}^m\alpha_i(\lambda-\mu)-\sum_{i=1}^m\alpha_i$$
$$=1-\sum_{i=1}^m\alpha_i(1-\mu+1)=1-(\lambda-\mu+1)\sum_{i=1}^m\alpha_i.$$

□

If $\sum_{i=1}^m\alpha_i(\lambda-\mu+1)=1$ then Theorem 2.13 can be rewritten as the following:

Corollary 2.14. *Let $n,m\in\mathbb{N}\cup\{0\}$, $i\in\{1,2,\dots,m\}$, $\alpha_i\in\mathbb{R}$ with $\alpha_i\geq0$, $\lambda>0$, $\mu\in[0,1)$ with $(\lambda-\mu+1)\sum_{i=1}^m\alpha_i=1$. If $R^nf_i\in SP(\lambda,\mu)$ then $F_m(z)\in K$.*

Theorem 2.15. *Let $n,m\in\mathbb{N}\cup\{0\}$, $i\in\{1,2,\dots,m\}$, $\alpha_i\in\mathbb{R}$, $\alpha_i\geq0$, $\gamma\in\mathbb{R}$, $\gamma\geq0$, $\delta\in(0,1)$ with $\left(1-\frac{\gamma}{4}-\delta\right)\sum_{i=1}^m\alpha_i<1$. If $R^nf_i\in SD(\gamma,\delta)$ and*

$$\left|\frac{z(R^nf_i(z))'}{R^nf_i(z)}-1\right|\geq\frac{1}{4},\quad z\in U,\ i\in\{1,2,3,\dots,m\},\tag{2.9}$$

then $F_m\in K(\xi)$, where F_m is given by (4) and

$$\xi=1-\left(1-\frac{\gamma}{4}-\delta\right)\sum_{i=1}^m\alpha_i.$$

Proof. Since $R^nf_i\in SD(\gamma,\delta)$, using (10) and (12), we have

$$\text{Re}\left[1+\frac{zF_m''(z)}{F_m'(z)}\right]=\alpha_1\text{Re}\,\frac{z(R^nf_1(z))'}{R^nf_1(z)}+\dots+\alpha_m\text{Re}\,\frac{z(R^nf_m(z))'}{R^nf_m(z)}+1-\sum_{i=1}^m\alpha_i$$
$$\geq\alpha_1\left[\gamma\left|\frac{z(R^nf_1(z))'}{R^nf_1(z)}-1\right|+\delta\right]+\dots+\alpha_m\left[\gamma\left|\frac{z(R^nf_m(z))'}{R^nf_m(z)}-1\right|+\delta\right]+1-\sum_{i=1}^m\alpha_i$$
$$\geq\gamma\left[\alpha_1\left|\frac{z(R^nf_1(z))'}{R^nf_1(z)}-1\right|+\dots+\alpha_m\left|\frac{z(R^nf_m(z))'}{R^nf_m(z)}-1\right|\right]+\delta\sum_{i=1}^m\alpha_i+1-\sum_{i=1}^m\alpha_i$$
$$\geq\frac{\gamma|z|}{4}\left(\frac{\alpha_1}{4}+\dots+\frac{\alpha_m}{4}\right)+\delta\sum_{i=1}^m\alpha_i+1-\sum_{i=1}^m\alpha_i$$
$$=1-\left(1-\frac{\gamma}{4}-\delta\right)\sum_{i=1}^m\alpha_i.$$

□

If $1-\left(1-\frac{\gamma}{4}-\delta\right)\sum_{i=1}^{m}\alpha_i = 0$, then Theorem 2.15 can be rewritten as the following.

Corollary 2.16. *Let $n, m \in \mathbb{N}\cup\{0\}$, $i \in \{1,2,3,\dots,m\}$, $\alpha_i \in \mathbb{R}$, $\alpha_i \geq 0$, $\gamma \geq 0$, $\delta \in (0,1)$ with $1-\left(1-\frac{\gamma}{4}-\delta\right)\sum_{i=1}^{m}\alpha_i = 0$. If $R^n f_i \in SD(\gamma,\delta)$ and*

$$\left|\frac{z(R^n f_i(z))'}{R^n f_i(z)}-1\right| \geq \frac{1}{4}, \quad z \in U,\ i \in \{1,2,\dots,m\}$$

then $F_m(z)$ is convex.

References

[1] Alexander, I. W., *Functions which map the interior of the unit circle upon simple regions*, Ann. of Math., **17** (1915), 12-22.

[2] Breaz, D., Breaz, N., *Two integral operators*, Studia Univ. Babeş-Bolyai, Mathematica, Cluj-Napoca, **3** (2002), 13-21.

[3] Breaz, D., *A convexity property for an integral operator on the class $S_p(\beta)$*, General Mathematics, **15** (2007), no. 2-3, 177-183.

[4] Kim, J. J., Merkes, E. P., *On a integral of powers of spiral-like function*, Kyungpook. Math. J., **12** (1972), 249-253.

[5] Krzyk, J., Lewandowski, Z., *On the integral of univalent functions*, Bull. Acad. Polon. Sc. Ser. Sci. Math. Astronom. Phys., **11** (1963), 447-458.

[6] Miller, S. S., Mocanu, P. T., Reade, M. O., *Starlike integral operators*, Pacific Journal of Mathematics, **79(1)**(1978), 157-168.

[7] Pascu, N. N., Pescar, V., *On integral operators of Kim-Merkes and Pfaltzgraff*, Mathematica, Cluj-Napoca, **32(55)** (1990), no. 2, 185-192.

[8] Pescar, V., *On some integral operations which preserve the univalence*, Punjab University Journal of Mathematics, **30** (1997), 1-10.

[9] Ronning, F., *Uniformly convex functions and a corresponding class of starlike functions*, Proc. Amer. Math. Soc., **118(1)** (1993), 190-196.

[10] Ronning, F., *Integral representations of bounded starlike functions*, Ann. Polon. Math., **60(3)** (1995), 289-297.

[11] Ruscheweyh, S., *New criteria for univalent functions*, Proc. Amer. Math. Soc., **49** (1975), 109-115.

[12] Sarangi, S. M., *Univalent functions with positive coefficients*, Tamkang J. Math., **25** (1994), 225-230.

Department of Mathematics
University of Oradea
Str. Universităţii, No.1
410087 Oradea, Romania
E-mail address: georgia_oros_ro@yahoo.co.uk

Department of Mathematics
University of Oradea
Str. Universităţii, No.1
410087 Oradea, Romania

STUDIA UNIV. "BABEŞ–BOLYAI", MATHEMATICA, Volume **LV**, Number 3, September 2010

INJECTIVITY CRITERIA FOR C^1 FUNCTIONS DEFINED IN NON-CONVEX DOMAINS

NICOLAE R. PASCU AND MIHAI N. PASCU

Dedicated to Professor Grigore Ştefan Sălăgean on his 60^{th} birthday

Abstract. In the present paper we obtain sufficient conditions for the injectivity of functions of class C^1 defined in type φ convex domains. In particular, we obtain some injectivity criteria for functions of class C^1 defined in some simply and doubly connected domains, and we derive as a corollary the well-known Ozaki-Nunokawa-Krzyz univalence criterion.

1. Preliminaries

We denote by $B(z_0, r) = \{z \in \mathbb{C} : |z - z_0| < r\}$ the open disk centered at $z_0 \in \mathbb{C}$ of radius $r > 0$ and by $U = B(0,1)$ the unit disk in $\mathbb{C}$.

In [4], the authors introduced the *convexity constant* $K(D)$ of a planar domain $D \subset \mathbb{C}$, as follows:

Definition 1.1 ([4])**.** For a domain $D \subset \mathbb{C}$, we define the convexity constant of the domain D by

$$K(D) = \inf_{\substack{a,b \in D \\ a \neq b}} \sup_{\gamma \in \Gamma(a,b;D)} \frac{|a-b|}{l(\gamma)},$$

where $\Gamma(a, b; D)$ is the family of all rectifiable arcs $\gamma \subset D$ with distinct endpoints a and b, and $l(\gamma)$ denotes the length of γ.

The authors showed that in the class of simply connected domains, the convexity constant $K(D)$ characterizes the convexity of the domain D, in the following sense:

Received by the editors: 26.04.2010.

2000 *Mathematics Subject Classification.* 30C55, 30C45, 52A30, 52A10.

Work supported by CNCSIS – UEFISCSU research grant PNII – IDEI 209/2007.

Theorem 1.2 ([4])**.** *The simply connected domain $D \subset \mathbb{C}$ is convex if and only if $K(D) = 1$.*

Given two domains $\Omega \subset D \subset \mathbb{C}$, denote by D_Ω the domain

$$D_\Omega = D - \overline{\Omega} = \left\{ z \in \mathbb{C} : z \in D,\ z \notin \overline{\Omega} \right\} \tag{1.1}$$

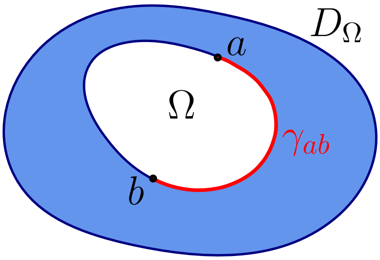


FIGURE 1. The domain $D_\Omega = D - \overline{\Omega}$.

In [4], the authors proposed the following conjecture:

Conjecture 1.3. *If D and Ω are convex domains with $\overline{\Omega} \subset D$, the convexity constant of the domain $D_\Omega = D - \overline{\Omega}$ is given by*

$$K(D_\Omega) = \min_{\substack{a,b \in \partial\Omega \\ a \neq b}} \frac{|a-b|}{l(\gamma_{ab})},$$

where γ_{ab} denotes the shorter of the two arcs of the boundary $\partial\Omega$ with endpoints a and b.

They proved the validity of the above conjecture in the following cases:

1. If $D \subset \mathbb{C}$ is a domain and $\gamma \subset D$ is a Jordan arc which joins two points $z_0 \in D$ and $w_0 \in \partial D$, then $K(D_\gamma) = 0$.
2. If $D \subset \mathbb{C}$ is a convex domain, $z_0 \in D$ and $r > 0$ are chosen such that $\overline{B(z_0, r)} \subset D$, then $K(D_{B(z_0,r)}) = \frac{2}{\pi}$.
3. If D is a convex domain and $z_0 \in D$ and $r > 0$ are chosen such that $\overline{S(z_0, r)} \subset D$, then $K(D_{S(z_0,r)}) = \frac{1}{2}$, where

$$S(z_0, r) = \left\{ z \in \mathbb{C} : |\mathrm{Re}(z - z_0)| < \frac{r}{2}, |\mathrm{Im}(z - z_0)| < \frac{r}{2} \right\}$$

denotes the interior of the square having z_0 as center of symmetry and sides parallel to the coordinate axes, of length equal to r.

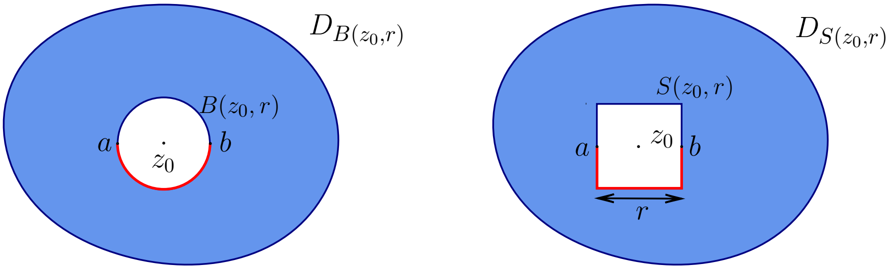


FIGURE 2. The domains $D_{B(z_0 r)} = D - \overline{B\,(z_0 r)}$ and $D_{S(z_0 r)} = D - \overline{S\,(z_0 r)}$.

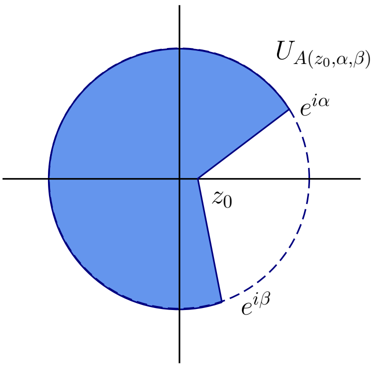


FIGURE 3. The domain $U_{A(z_0,\alpha,\beta)} = U - \overline{A\,(z_0,\alpha,\beta)}$.

4. The convexity constant of the domain $U_{A(z_0,\alpha,\beta)}$ is given by

$$K(U_{A(z_0,\alpha,\beta)}) = \begin{cases} 1, & \text{if } z_0 \in \left[\frac{\cos\frac{\alpha+\beta}{2}}{\cos\frac{\alpha-\beta}{2}}, 1\right) \\ \sin\frac{\arg\left(e^{i\alpha}-z_0\right)+\arg\left(e^{i\beta}-z_0\right)}{2}, & \text{if } z_0 \in \left(-1, \frac{\cos\frac{\alpha+\beta}{2}}{\cos\frac{\alpha-\beta}{2}}\right) \end{cases},$$

where

$$A(z_0,\alpha,\beta) = \{z_0 + re^{i\theta} : r > 0, \ -\arg\left(e^{i\beta} - z_0\right) < \theta < \arg\left(e^{i\alpha} - z_0\right)\}$$

represents the angular region with vertex z_0 and opening angles α and β.

M. O. Reade ([5]) generalized the class of convex planar domains as follows:

Definition 1.4 ([5])**.** Let $\varphi \in [0, \pi)$ be a real number. We say that the domain $D \subset \mathbb{C}$ is a type φ convex domain if for any distinct points $a, b \in D$ there exists $c \in D$ such that the line segments $[a, c], [c, b] \subset D$ and

$$\left| \arg \frac{b-c}{c-a} \right| \leq \varphi. \tag{1.2}$$

The family of type φ convex domains is denoted by C_φ.

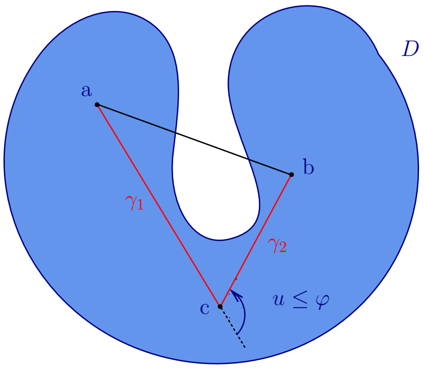


Figure 4. A type φ convex domain D, $\varphi \in [0, \pi)$.

Remark 1.5. Geometrically, condition (1.2) shows that the angle $u = \pi - \widehat{acb}$ is less than or equal to φ (see Figure 4).

It can be shown (see [4]) the following connection between type φ convex domains and the convexity constant:

Lemma 1.6. *If $D \in C_\varphi$ is a type φ convex domain for some $\varphi \in [0, \pi)$, then $K(D) \geq \cos \frac{\varphi}{2}$.*

Remark 1.7. The above lemma shows that if D is a type φ convex domain, the convexity constant of D cannot be too small. In particular, if $D \subset \mathbb{C}$ is a convex domain then it is also a type φ convex domain for $\varphi = 0$, and therefore from the above lemma it follows that $K(D) = 1$.

2. Univalence criteria for functions of class $C^1(D)$

P. T. Mocanu ([2], p. 137) obtained the following univalence criterion for C^1 functions defined in type φ domains:

Theorem 2.1. *Let $D \in C_\varphi$, $\varphi \in [0, \pi)$. If the function $f \in C^1(D)$ satisfies one of the two equivalent conditions*

i) $|\arg f'_\theta(z)| < \frac{\pi - \varphi}{2}$, $z \in D$, *for any* $\theta \in [0, 2\pi)$

ii) $\operatorname{Re} \frac{\partial f(z)}{\partial z} - \left| \operatorname{Im} \frac{\partial f(z)}{\partial z} \right| \tan \frac{\varphi}{2} > \frac{1}{\cos \frac{\varphi}{2}} \left| \frac{\partial f(z)}{\partial \bar{z}} \right|$, $z \in D$,

then the function f is injective in D and $Jf(z) > 0$, $z \in D$.

Using the convexity constant of a domain, we can obtain a similar result as follows:

Theorem 2.2. *Let $f : D \subset U \to \mathbb{C}$ be a C^1 function in the domain $D \in C_\varphi$ for some $\varphi \in [0, \pi)$. If*

$$\left| D_\theta \left(\frac{1}{f(z)} - \frac{1}{z} \right) \right| \leq \cos \frac{\varphi}{2}, \qquad z \in D, \tag{2.1}$$

for all $\theta \in [0, 2\pi)$, where D_θ is the operator defined on C^1 functions by

$$D_\theta f = \frac{\partial f}{\partial z} + e^{-2i\theta} \frac{\partial f}{\partial \bar{z}},$$

then the function f is injective in D.

Proof. Let $a, b \in D$, $a \neq b$ be arbitrarily fixed distinct points.

Since $D \in C_\varphi$, by definition, there exists $c \in D$ such that $\gamma = [a, c] \cup [c, b] \subset D$. Let $\gamma_1(t) = a + t(c - a)$, $t \in [0, 1]$ and $\gamma_2(t) = c + t(b - c)$, $t \in [0, 1]$, be two parametrizations of the line segments $[a, c]$, respectively $[c, b]$.

We have:

$$\begin{aligned}
&\frac{1}{f(c)}-\frac{1}{f(a)}-\left(\frac{1}{c}-\frac{1}{a}\right)= \\
&=\int_0^1 \frac{d}{dt}\left(\frac{1}{f(\gamma_1(t))}-\frac{1}{\gamma_1(t)}\right)dt \\
&=\int_0^1 \frac{\partial}{\partial z}\left(\frac{1}{f(z)}-\frac{1}{z}\right)(\gamma_1(t))\frac{d\gamma_1(t)}{dt}+\frac{\partial}{\partial \bar z}\left(\frac{1}{f(z)}-\frac{1}{z}\right)(\gamma_1(t))\frac{d\overline{\gamma_1(t)}}{dt}dt \\
&=\int_0^1 (c-a)\frac{\partial}{\partial z}\left(\frac{1}{f(z)}-\frac{1}{z}\right)(\gamma_1(t))+\overline{(c-a)}\frac{\partial}{\partial \bar z}\left(\frac{1}{f(z)}-\frac{1}{z}\right)(\gamma_1(t))\,dt \\
&=(c-a)\int_0^1 D_{\theta_1}\left(\frac{1}{f(z)}-\frac{1}{z}\right)(\gamma_1(t))\,dt,
\end{aligned}$$

where $\theta_1=\arg(c-a)$, and similarly

$$\frac{1}{f(b)}-\frac{1}{f(c)}-\left(\frac{1}{b}-\frac{1}{c}\right)=(b-c)\int_0^1 D_{\theta_2}\left(\frac{1}{f(z)}-\frac{1}{z}\right)(\gamma_2(t))\,dt,$$

where $\theta_2=\arg(b-c)$.

We obtain

$$\begin{aligned}
\frac{1}{f(b)}-\frac{1}{f(a)}-\left(\frac{1}{b}-\frac{1}{a}\right)&=\frac{1}{f(b)}-\frac{1}{f(c)}-\left(\frac{1}{b}-\frac{1}{c}\right)+\frac{1}{f(c)}-\frac{1}{f(a)}-\left(\frac{1}{c}-\frac{1}{a}\right) \\
&=(c-a)\int_0^1 D_{\theta_1}\left(\frac{1}{f(z)}-\frac{1}{z}\right)(\gamma_1(t))\,dt+ \\
&\quad(c-b)\int_0^1 D_{\theta_2}\left(\frac{1}{f(z)}-\frac{1}{z}\right)(\gamma_2(t))\,dt,
\end{aligned}$$

and therefore using the hypothesis we have

$$\begin{aligned}
\left|\frac{1}{f(b)}-\frac{1}{f(a)}-\left(\frac{1}{b}-\frac{1}{a}\right)\right|&\le |c-a|\int_0^1\left|D_{\theta_1}\left(\frac{1}{f(z)}-\frac{1}{z}\right)(\gamma_1(t))\right|dt+ \\
&\quad|b-c|\int_0^1\left|D_{\theta_2}\left(\frac{1}{f(z)}-\frac{1}{z}\right)(\gamma_2(t))\right|dt \\
&\le |c-a|\cos\frac{\varphi}{2}+|b-c|\cos\frac{\varphi}{2} \\
&=l(\gamma)\cos\frac{\varphi}{2}.
\end{aligned}$$

If $f(a)=f(b)$, from the above inequality we obtain equivalent

$$\frac{|b-a|}{l(\gamma)}\le |ab|\cos\frac{\varphi}{2},$$

where $\gamma = [a, c] \cup [c, b]$.

Approximating now an arbitrary curve $\gamma \in \Gamma(a, b; D)$ by a polygonal path $\gamma_n = [a_0, c_1] \cup \ldots \cup [c_n, b] \subset D$ and using an argument similar to the previous proof, by passing to the limit we obtain:

$$\frac{|b-a|}{l(\gamma)} \le |ab| \cos\frac{\varphi}{2},$$

for any $\gamma \in \Gamma(a, b; D)$, and therefore

$$\sup_{\gamma\in\Gamma(a,b;D)} \frac{|a-b|}{l(\gamma)} \le |ab| \cos\frac{\varphi}{2},$$

which shows that

$$K(D) = \inf_{a,b\in D} \sup_{\gamma\in\Gamma(a,b;D)} \frac{|a-b|}{l(\gamma)} \le \sup_{\gamma\in\Gamma(a,b;D)} \frac{|a-b|}{l(\gamma)} \le |ab| \cos\frac{\varphi}{2}.$$

Since from Lemma 1.6 we have $K(D) \ge \cos\frac{\varphi}{2} > 0$, we obtain

$$K(D) \le |ab| K(D),$$

which contradicts the hypothesis $a, b \in D \subset U$ (and therefore $|ab| < 1$).

The contradiction shows that the hypothesis $f(a) = f(b)$ is false, and therefore we must have $f(a) \ne f(b)$ for all $a, b \in D$ distinct, which shows that f is injective in D, concluding the proof. □

Following the proof of the above theorem it can be seen that we can replace the right side of (2.1) by the larger constant $K(D)$, thus obtaining the following more general result:

Theorem 2.3. *Let $f : D \subset B(0, R) \to \mathbb{C}$ be a C^1 function in the domain $D \in C_\varphi$ for some $\varphi \in [0, \pi)$. If*

$$\left| D_\theta \left(\frac{1}{f(z)} - \frac{1}{z} \right) \right| \le \frac{K(D)}{R^2}, \qquad z \in D, \tag{2.2}$$

for all $\theta \in [0, 2\pi)$, where D_θ is the operator defined on C^1 functions by

$$D_\theta f = f_z + e^{-2i\theta} f_{\bar{z}},$$

then the function f is injective in D.

Remark 2.4. Using the values of the convexity constants of the domains D_Ω presented in Section 1, from the above theorem we obtain as corollaries sufficient conditions for univalence for functions of class C^1 defined in some simply and doubly connected domains.

Remark 2.5. In the particular case $D = U$, we have $K(U) = 1$, and Theorems 2.2 and 2.3 above become (in the case when $f : U \to \mathbb{C}$ is a normalized analytic function in U) the well-known Ozaki-Nunokawa-Krzyz univalence criterion (see [1], [3]).

References

[1] Krzyz, J. G., *Convolution and quasiconformal extension*, Comment. Math. Helv., **51** (1976), no. 1, 99-104.

[2] Mocanu, P. T., Bulboacă, T., Sălăgean, G., *Teoria geometrică a funcţiilor univalente*, Casa Cărţii de Ştiinţă, Cluj, 1999, 131-140.

[3] Ozaki, S., Nunokawa, M., *The Schwarzian derivative and univalent functions*, Proc. Amer. Math. Soc., **33** (1972), no. 2, 392-394.

[4] Pascu, N. R., Pascu, M. N., *Convexity constant of a domain and applications* (to appear).

[5] Reade, M. O., *On Umezawa's criteria for univalence II*, Math. Soc. Japan, **10** (1958), 255-259.

1100 South Marietta Parkway
Marietta, GA 30060-2896, U.S.A.
E-mail address: npascu@spsu.edu

Faculty of Mathematics and Computer Science
Transilvania University of Braşov
Str. Iuliu Maniu Nr. 50
Braşov 500091, Romania
E-mail address: mihai.pascu@unitbv.ro

ON THE PROPERTIES OF A SUBCLASS OF ANALYTIC FUNCTIONS

DORINA RĂDUCANU

Dedicated to Professor Grigore Ştefan Sălăgean on his 60^{th} *birthday*

Abstract. In this paper we consider a new class of analytic functions defined by a generalized differential operator. Inclusion results, structural formula, coefficient estimates and other properties of this class of functions are obtained.

1. Introduction

Let $\mathcal{A}$ denote the class of functions f of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

which are analytic in the open unit disk $\mathbb{U} := \{z \in \mathbb{C} : |z| < 1\}$.

The Hadamard product or convolution of the functions

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \text{ and } g(z) = z + \sum_{n=2}^{\infty} b_n z^n$$

is given by

$$(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n \ , \ z \in \mathbb{U}.$$

Let $f \in \mathcal{A}$. We consider the following differential operator introduced by Răducanu and Orhan in [7]:

$$D^0_{\lambda\mu} f(z) = f(z)$$

$$D^1_{\lambda\mu} f(z) = D_{\lambda\mu} f(z) = \lambda\mu z^2 f''(z) + (\lambda - \mu) z f'(z) + (1 - \lambda + \mu) f(z)$$

Received by the editors: 26.04.2010.

2000 *Mathematics Subject Classification.* 30C45.

Key words and phrases. Analytic functions, differential operator, structural formula, extreme points.

$$D^m_{\lambda\mu} f(z) = D_{\lambda\mu}\left(D^{m-1}_{\lambda\mu} f(z)\right) \tag{1.2}$$

where $0 \leq \mu \leq \lambda$ and $m \in \mathbb{N} := \{1, 2, \ldots\}$.

If the function f is given by (1.1), then from (1.2) we see that:

$$D^m_{\lambda\mu} f(z) = z + \sum_{n=2}^{\infty} A_n(\lambda, \mu, m) a_n z^n \tag{1.3}$$

where

$$A_n(\lambda, \mu, m) = \left[1 + (\lambda\mu n + \lambda - \mu)(n-1)\right]^m . \tag{1.4}$$

If $\lambda = 1$ and $\mu = 0$, we get Sălăgean differential operator [9] and if $\mu = 0$, we obtain the differential operator defined by Al-Oboudi [1].

From (1.3) it follows that $D^m_{\lambda\mu} f(z)$ can be written in terms of convolution as

$$D^m_{\lambda\mu} f(z) = (f * g)(z) \tag{1.5}$$

where

$$g(z) = z + \sum_{n=2}^{\infty} A_n(\lambda, \mu, m) z^n. \tag{1.6}$$

Definition 1.1. We say that a function $f \in \mathcal{A}$ is in the class $\mathcal{R}^m_{\lambda\mu}(\alpha, \gamma)$ if

$$\Re\left\{(1-\alpha)\frac{D^m_{\lambda\mu} f(z)}{z} + \alpha (D^m_{\lambda\mu} f(z))'\right\} > \gamma\ ,\ z \in \mathbb{U}$$

for $\alpha \geq 0$, $0 \leq \gamma < 1$, $0 \leq \mu \leq \lambda$ and $m \in \mathbb{N}_0 := \{0, 1, 2, \ldots\}$.

Note that:

i. $\mathcal{R}^0_{\lambda\mu}(1, \gamma)$ is the subclass of $\mathcal{A}$ consisting of functions with $\Re f'(z) > \gamma$.
ii. $\mathcal{R}^m_{\lambda 0}(1, \gamma)$ is the class of functions investigated in [1].
iii. $\mathcal{R}^m_{\lambda\mu}(1, \gamma)$ reduces to the class of functions considered in [8].
iv. $\mathcal{R}^0_{\lambda\mu}(\alpha, \gamma)$ is the class of functions studied by G. Chunyi and S. Owa in [4].

The main object of this paper is to present a systematic investigation for the class $\mathcal{R}^m_{\lambda\mu}(\alpha, \gamma)$. In particular, for this class of functions we obtain some inclusion results, structural formula, extreme points and other properties.

2. **Inclusion results**

In order to prove our inclusion results we need the following lemmas.

Lemma 2.1. *([4]) Let $\alpha \geq 0$ and $\gamma \geq 0$. Let $D(z)$ be a starlike function in $\mathbb{U}$ and let $N(z)$ be an analytic function in $\mathbb{U}$ such that $N(0) = D(0) = 0$ and $N'(0) = D'(0) = 1$. If*

$$\Re\left[(1-\alpha)\frac{N(z)}{D(z)} + \alpha\frac{N'(z)}{D'(z)}\right] > \gamma\ ,\ z \in \mathbb{U}$$

then

$$\Re\frac{N(z)}{D(z)} > \gamma\ ,\ z \in \mathbb{U}.$$

Lemma 2.2. *([6]) Let $h(z)$ be a convex function in $\mathbb{U}$ and let $A \geq 0$. Suppose that $B(z)$ and $C(z)$ are analytic in $\mathbb{U}$ with $C(0) = 0$ and*

$$\Re B(z) \geq A + 4\left|\frac{C(z)}{h'(0)}\right|\ ,\ z \in \mathbb{U}.$$

If p is an analytic function , with $p(0) = h(0)$, which satisfies

$$Az^2p''(z) + B(z)zp'(z) + p(z) + C(z) \prec h(z)\ ,\ z \in \mathbb{U}$$

then $p(z) \prec h(z)$, $z \in \mathbb{U}$.

Note that the symbol " $\prec$ " stands for subordination.

Theorem 2.3. *Let $\alpha \geq 0$, $0 \leq \gamma < 1$, $0 \leq \mu \leq \lambda$ and $m \in \mathbb{N}_0$. Then*

$$\mathcal{R}^m_{\lambda\mu}(\alpha,\gamma) \subset \mathcal{R}^m_{\lambda\mu}(0,\gamma).$$

Proof. Suppose $f \in \mathcal{R}^m_{\lambda\mu}(\alpha,\gamma)$. Then, from Definition 1.1, we have

$$\Re\left\{(1-\alpha)\frac{D^m_{\lambda\mu}f(z)}{z} + \alpha(D^m_{\lambda\mu}f(z))'\right\} > \gamma\ ,\ z \in \mathbb{U}.$$

Consider $N(z) = D^m_{\lambda\mu}f(z)$. Making use of (1.3) we have $N(0) = 0$ and $N'(0) = 1$. Let $D(z) = z$. Since $D(z)$ is starlike in $\mathbb{U}$ and $D(0) = 0 = D'(0) - 1$, from Lemma 2.1, we obtain

$$\Re\left\{\frac{D^m_{\lambda\mu}f(z)}{z}\right\} > \gamma\ ,\ z \in \mathbb{U}$$

which implies $f \in \mathcal{R}^m_{\lambda\mu}(0,\gamma)$. Thus $\mathcal{R}^m_{\lambda\mu}(\alpha,\gamma) \subset \mathcal{R}^m_{\lambda\mu}(0,\gamma)$ and the proof of the theorem is completed. □

Theorem 2.4. *Let $0 \le \beta < \alpha$, $0 \le \gamma < 1$, $0 \le \mu \le \lambda$ and $m \in \mathbb{N}_0$. Then*

$$\mathcal{R}^m_{\lambda\mu}(\alpha,\gamma) \subset \mathcal{R}^m_{\lambda\mu}(\beta,\gamma).$$

Proof. If $\beta = 0$, from Theorem 2.3, we have $\mathcal{R}^m_{\lambda\mu}(\alpha,\gamma) \subset \mathcal{R}^m_{\lambda\mu}(0,\gamma)$.
Let $f \in \mathcal{R}^m_{\lambda\mu}(\alpha,\gamma)$ and assume $\beta \neq 0$. Then

$$(1-\beta)\frac{D^m_{\lambda\mu}f(z)}{z} + \beta(D^m_{\lambda\mu}f(z))' =$$

$$\frac{\beta}{\alpha}\left[\left(\frac{\alpha}{\beta}-1\right)\frac{D^m_{\lambda\mu}f(z)}{z} + (1-\alpha)\frac{D^m_{\lambda\mu}f(z)}{z} + \alpha(D^m_{\lambda\mu}f(z))'\right].$$

Since $f \in \mathcal{R}^m_{\lambda\mu}(\alpha,\gamma)$, making use of Definition 1.1 and Theorem 2.3, we obtain

$$\Re\left\{(1-\beta)\frac{D^m_{\lambda\mu}f(z)}{z} + \beta(D^m_{\lambda\mu}f(z))'\right\} =$$

$$\frac{\beta}{\alpha}\left[\left(\frac{\alpha}{\beta}-1\right)\Re\frac{D^m_{\lambda\mu}f(z)}{z} + \Re\left\{(1-\alpha)\frac{D^m_{\lambda\mu}f(z)}{z} + \alpha(D^m_{\lambda\mu}f(z))'\right\}\right]$$

$$> \frac{\beta}{\alpha}\left(\frac{\alpha}{\beta}-1\right)\gamma + \frac{\beta}{\alpha}\gamma = \gamma.$$

It follows that $f \in \mathcal{R}^m_{\lambda\mu}(\beta,\gamma)$ and thus, $\mathcal{R}^m_{\lambda\mu}(\alpha,\gamma) \subset \mathcal{R}^m_{\lambda\mu}(\beta,\gamma)$. □

Another inclusion result is given in the next theorem.

Theorem 2.5. *Let $\alpha \ge 0$, $0 \le \gamma < 1$, $0 \le \mu \le \lambda$ and $m \in \mathbb{N}_0$. Then*

$$\mathcal{R}^{m+1}_{\lambda\mu}(\alpha,\gamma) \subset \mathcal{R}^m_{\lambda\mu}(\alpha,\gamma).$$

Proof. Suppose $f \in \mathcal{R}^{m+1}_{\lambda\mu}(\alpha,\gamma)$. Then

$$\Re\left\{(1-\alpha)\frac{D^{m+1}_{\lambda\mu}f(z)}{z} + \alpha(D^{m+1}_{\lambda\mu}f(z))'\right\} > \gamma$$

which is equivalent to

$$(1-\alpha)\frac{D^{m+1}_{\lambda\mu}f(z)}{z} + \alpha(D^{m+1}_{\lambda\mu}f(z))' \prec h(z)\ ,\ z \in \mathbb{U} \tag{2.1}$$

where

$$h(z) = \frac{1+(1-2\gamma)z}{1-z}\ ,\ z \in \mathbb{U}. \tag{2.2}$$

From (1.2), we have

$$D^{m+1}_{\lambda\mu}f(z) = \lambda\mu z^2[D^m_{\lambda\mu}f(z)]'' + (\lambda-\mu)z[D^m_{\lambda\mu}f(z)]' + (1-\lambda+\mu)D^m_{\lambda\mu}f(z).$$

It follows that

$$R(z) := (1-\alpha)\frac{D_{\lambda\mu}^{m+1}f(z)}{z} + \alpha(D_{\lambda\mu}^{m+1}f(z))'$$
$$= \lambda\mu\left\{(1-\alpha)\frac{z^2(D_{\lambda\mu}^{m}f(z))''}{z} + \alpha[z^2(D_{\lambda\mu}^{m}f(z))'']'\right\}$$
$$+(\lambda-\mu)\left\{(1-\alpha)\frac{z(D_{\lambda\mu}^{m}f(z))'}{z} + \alpha[z(D_{\lambda\mu}^{m}f(z))']'\right\}$$
$$+(1-\lambda+\mu)\left\{(1-\alpha)\frac{D_{\lambda\mu}^{m}f(z)}{z} + \alpha(D_{\lambda\mu}^{m}f(z))'\right\}.$$

Denote

$$p(z) = (1-\alpha)\frac{D_{\lambda\mu}^{m}f(z)}{z} + \alpha(D_{\lambda\mu}^{m}f(z))'\ ,\ z \in \mathbb{U}. \tag{2.3}$$

Simple calculations show that

$$R(z) = \lambda\mu z^2 p''(z) + (2\lambda\mu + \lambda - \mu)zp'(z) + p(z). \tag{2.4}$$

Making use of (2.4), the differential subordination (2.1) becomes

$$\lambda\mu z^2 p''(z) + (2\lambda\mu + \lambda - \mu)zp'(z) + p(z) \prec h(z)\ ,\ z \in \mathbb{U}.$$

It is easy to check that conditions of Lemma 2.2 with $h(z)$ given by (2.2), $p(z)$ given by (2.3), $A = \lambda\mu$, $B(z) \equiv 2\lambda\mu + \lambda - \mu$ and $C(z) \equiv 0$ are satisfied. Thus, we obtain $p(z) \prec h(z)$ which implies that

$$\Re\left\{(1-\alpha)\frac{D_{\lambda\mu}^{m}f(z)}{z} + \alpha(D_{\lambda\mu}^{m}f(z))'\right\} > \gamma\ ,\ z \in \mathbb{U}.$$

Therefore, $f \in \mathcal{R}_{\lambda\mu}^m(\alpha,\gamma)$ and the proof of our theorem is completed. □

3. Structural formula

In this section a structural formula, extreme points, coefficient bounds for functions in $\mathcal{R}_{\lambda\mu}^m(\alpha,\gamma)$ are given.

Theorem 3.1. *A function $f \in \mathcal{A}$ is in the class $\mathcal{R}_{\lambda\mu}^m(\alpha,\gamma)$ if and only if it can be expressed as*

$$f(z) = \left[z + \sum_{n=2}^{\infty}\frac{z^n}{A_n(\lambda,\mu,m)}\right] * \int_{|\zeta|=1}\left[z + 2(1-\gamma)\bar{\zeta}\sum_{n=2}^{\infty}\frac{(\zeta z)^n}{1+(n-1)\alpha}\right]d\mu(\zeta) \tag{3.1}$$

where $\mu(\zeta)$ is the probability measure defined on the unit circle

$$\mathbb{T} = \{\zeta \in \mathbb{C} : |\zeta| = 1\}.$$

Proof. Definition 1.1 implies that $f \in \mathcal{R}^m_{\lambda\mu}(\alpha,\gamma)$ if and only if

$$\frac{(1-\alpha)\frac{D^m_{\lambda\mu}f(z)}{z} + \alpha(D^m_{\lambda\mu}f(z))' - \gamma}{1-\gamma} = p(z)\ ,\ z \in \mathbb{U} \tag{3.2}$$

where $p(z)$ belongs to the class $\mathcal{P}$ consisting of normalized analytic functions which have positive real part in $\mathbb{U}$.

From (3.2) we have

$$(1-\alpha)\frac{D^m_{\lambda\mu}f(z) - \gamma z}{z} + \alpha[(D^m_{\lambda\mu}f(z))' - \gamma] = (1-\gamma)p(z). \tag{3.3}$$

If $\alpha \neq 0$, multiplying both sides of (3.3) by $\frac{1}{\alpha}z^{\frac{1}{\alpha}-1}$, we obtain

$$\left[z^{\frac{1}{\alpha}-1}(D^m_{\lambda\mu}f(z) - \gamma z)\right]' = z^{\frac{1}{\alpha}-1}\frac{1-\gamma}{\alpha}p(z).$$

Using Herglotz expression of functions in the class $\mathcal{P}$, we have

$$\left[z^{\frac{1}{\alpha}-1}(D^m_{\lambda\mu}f(z) - \gamma z)\right]' = z^{\frac{1}{\alpha}-1}\frac{1-\gamma}{\alpha}\int_{|\zeta|=1}\frac{1+\zeta z}{1-\zeta z}d\mu(\zeta).$$

Integrating both sides of this equality we get

$$z^{\frac{1}{\alpha}-1}(D^m_{\lambda\mu}f(z) - \gamma z) = \int_0^z \left[u^{\frac{1}{\alpha}-1}\frac{1-\gamma}{\alpha}\int_{|\zeta|=1}\frac{1+\zeta u}{1-\zeta u}d\mu(\zeta)\right]du$$

which is equivalent to

$$D^m_{\lambda\mu}f(z) = \frac{1}{\alpha}\int_{|\zeta|=1}\left[z^{1-\frac{1}{\alpha}}\int_0^z u^{\frac{1}{\alpha}-1}\frac{1+\zeta u(1-2\gamma)}{1-\zeta u}du\right]d\mu(\zeta).$$

So we have

$$D^m_{\lambda\mu}f(z) = \int_{|\zeta|=1}\left[z + 2(1-\gamma)\bar{\zeta}\sum_{n=2}^{\infty}\frac{(\zeta z)^n}{1+(n-1)\alpha}\right]d\mu(\zeta). \tag{3.4}$$

From (1.5), (1.6) and (3.4) it follows that

$$f(z) = \left[z + \sum_{n=2}^{\infty}\frac{z^n}{A_n(\lambda,\mu,m)}\right] * \int_{|\zeta|=1}\left[z + 2(1-\gamma)\bar{\zeta}\sum_{n=2}^{\infty}\frac{(\zeta z)^n}{1+(n-1)\alpha}\right]d\mu(\zeta).$$

Since this deductive process can be converse, we have proved our theorem. $\square$

Remark 3.2. If $\alpha = 0$, the expression (3.1) is also true and it says that if $f \in \mathcal{A}$ satisfies $\Re \frac{D^m_{\lambda\mu} f(z)}{z} > \gamma$, then f can be expressed as

$$f(z) = \left[z + \sum_{n=2}^{\infty} \frac{z^n}{A_n(\lambda, \mu, m)} \right] * \int_{|\zeta|=1} \left[z + 2(1-\gamma)\bar{\zeta} \sum_{n=2}^{\infty} (\zeta z)^n \right] d\mu(\zeta).$$

Corollary 3.3. *The extreme points of the class $\mathcal{R}^m_{\lambda\mu}(\alpha, \gamma)$ are*

$$f_\zeta(z) = z + 2(1-\gamma)\bar{\zeta} \sum_{n=2}^{\infty} \frac{(\zeta z)^n}{[1 + (n-1)\alpha] A_n(\lambda, \mu, m)} \ , \ z \in \mathbb{U} \ , \ |\zeta| = 1. \tag{3.5}$$

Proof. Denote

$$[D^m_{\lambda\mu} f(z)]_\zeta = z + 2(1-\gamma)\bar{\zeta} \sum_{n=2}^{\infty} \frac{(\zeta z)^n}{1 + (n-1)\alpha}.$$

Then, equality (3.4) can be written as

$$[D^m_{\lambda\mu} f(z)]_\mu = \int_{|\zeta|=1} [D^m_{\lambda\mu} f(z)]_\zeta d\mu(\zeta).$$

Since probability measures $\{\mu\}$ and class $\mathcal{P}$ are one-to-one it follows that the map $\mu \to [D^m_{\lambda\mu} f(z)]_\mu$ is one-to-one and the assertion follows (see [5]). ☐

Making use of Corollary 3.3 we can obtain coefficients bounds for the functions in the class $\mathcal{R}^m_{\lambda\mu}(\alpha, \gamma)$.

Corollary 3.4. *If $f \in \mathcal{R}^m_{\lambda\mu}(\alpha, \gamma)$, then*

$$|a_n| \leq \frac{2(1-\gamma)}{[1 + (n-1)\alpha] A_n(\lambda, \mu, m)} \ , \ n \geq 2.$$

The result is sharp.

Proof. The coefficient bounds are maximized at an extreme point so, the result follows from (3.5). ☐

Corollary 3.5. *If $f \in \mathcal{R}^m_{\lambda\mu}(\alpha, \gamma)$, then for $|z| = r < 1$*

$$|f(z)| \geq r - 2(1-\gamma) r^2 \sum_{n=2}^{\infty} \frac{1}{[1 + (n-1)\alpha] A_n(\lambda, \mu, m)}$$

$$|f(z)| \leq r + 2(1-\gamma) r^2 \sum_{n=2}^{\infty} \frac{1}{[1 + (n-1)\alpha] A_n(\lambda, \mu, m)}$$

and

$$|f'(z)| \geq 1 - 2(1-\gamma)r\sum_{n=2}^{\infty}\frac{n}{[1+(n-1)\alpha]A_n(\lambda,\mu,m)}$$

$$|f'(z)| \leq 1 + 2(1-\gamma)r\sum_{n=2}^{\infty}\frac{n}{[1+(n-1)\alpha]A_n(\lambda,\mu,m)}$$

4. **Convolution property**

In order to prove a convolution property for the class $\mathcal{R}_{\lambda\mu}^{m}(\alpha,\gamma)$ we need the following result.

Lemma 4.1. *([10]) If $p(z)$ is analytic in $\mathbb{U}$, $p(0)=1$ and $\Re p(z) > \frac{1}{2}$, then for any analytic function F in $\mathbb{U}$, the function $F * p$ takes values in the convex hull of $F(\mathbb{U})$.*

Theorem 4.2. *The class $\mathcal{R}_{\lambda\mu}^{m}(\alpha,\gamma)$ is closed under the convolution with a convex function. That is, if $f \in \mathcal{R}_{\lambda\mu}^{m}(\alpha,\gamma)$ and g is convex in $\mathbb{U}$, then $f * g \in \mathcal{R}_{\lambda\mu}^{m}(\alpha,\gamma)$.*

Proof. It is known that, if g is a convex function in $\mathbb{U}$, then

$$\Re\frac{g(z)}{z} > \frac{1}{2}.$$

Suppose that $f \in \mathcal{R}_{\lambda\mu}^{m}(\alpha,\gamma)$. Making use of the convolution properties, we have

$$\Re\left\{(1-\alpha)\frac{D_{\lambda\mu}^{m}(f*g)(z)}{z} + \alpha[D_{\lambda\mu}^{m}(f*g)(z)]'\right\} =$$

$$\Re\left\{\left[(1-\alpha)\frac{D_{\lambda\mu}^{m}f(z)}{z} + \alpha[D_{\lambda\mu}^{m}f(z)]'\right] * \frac{g(z)}{z}\right\}.$$

Using Lemma 4.1, the result follows. □

Corollary 4.3. *The class $\mathcal{R}_{\lambda\mu}^{m}(\alpha,\gamma)$ is invariant under Bernardi integral operator* [3] *defined by*

$$F_c(f)(z) = \frac{1+c}{z^c}\int_0^z t^{c-1}f(t)dt\ ,\ \Re c > 0.$$

Proof. Assume $f \in \mathcal{R}_{\lambda\mu}^{m}(\alpha,\gamma)$. It is easy to check that $F_c(f)(z) = (f*g)(z)$, where

$$g(z) = \sum_{n=1}^{\infty}\frac{1+c}{n+c}z^n.$$

Since the function g is convex (see [2]), the result follows by applying Theorem 4.2. Therefore, $F_c[\mathcal{R}_{\lambda\mu}^{m}(\alpha,\gamma)] \subset \mathcal{R}_{\lambda\mu}^{m}(\alpha,\gamma)$. □

References

[1] Al-Oboudi, F. M., *On univalent functions defined by a generalized Sălăgean operator*, Internat. J. Math. Math. Sci., **27** (2004), 1429-1436.

[2] Barnard, R. W., Kellogg, C., *Applications of convolution operators to problems in univalent function theory*, Michigan Math. J., **27(1)** (1980), 81-94.

[3] Bernardi, S. D., *Convex and starlike functions*, Trans. Amer. Math. Soc., **135** (1969), 429-446.

[4] Chunyi, G., Owa, S., *Certain class of analytic functions in the unit disk*, Kyungpook Math. J., **33(1)** (1993), 13-23.

[5] Hallenbeck, D. J., *Convex hulls and extreme points of some families of univalent functions*, Trans. Amer. Math. Soc., **192** (1974), 285-292.

[6] Miller, S. S., Mocanu, P. T., *Differential Subordinations. Theory and Applications*, Marcel-Dekker, New-York, 2000.

[7] Răducanu, D., Orhan, H., *Subclasses of analytic functions defined by a generalized differential operator*, Int. Journ. Math. Anal., **4(1-4)** (2010), 1-16.

[8] Răducanu, D., *On a subclass of univalent functions defined by a generalized differential operator*, (submitted).

[9] Sălăgean, G. S., *Subclasses of univalent functions*, Complex Analysis 5th Romanian-Finnish Seminar, Part. I (Bucharest, 1981), Lect. Notes Math., **1013**, Springer-Verlag, (1983), 362-372.

[10] Singh, R., Singh, S., *Convolution properties of a class of starlike functions*, Proc. Amer. Math. Soc., **106(1)** (1989), 145-152.

Faculty of Mathematics and Computer Science
"Transilvania" University of Braşov
50091, Iuliu Maniu, 50, Braşov, Romania
E-mail address: dorinaraducanu@yahoo.com

STUDIA UNIV. "BABEŞ–BOLYAI", MATHEMATICA, Volume **LV**, Number 3, September 2010

STRONG DIFFERENTIAL SUBORDINATIONS OBTAINED BY THE MEDIUM OF AN INTEGRAL OPERATOR

ROXANA ŞENDRUŢIU

Dedicated to Professor Grigore Ştefan Sălăgean on his 60^{th} *birthday*

Abstract. The concept of differential subordination was introduced in [2] by S. S. Miller and P. T .Mocanu and developed in [3], and the concept of strong differential subordination was introduced in [1] by J. A. Antonino and S. Romaquera and developed in [4], [5] by Georgia Irina Oros and Gheorghe Oros. In this paper we define the class $S_n^m(\alpha)$, and we study strong differential subordination.

1. **Introduction and preliminaries**

Let U denote the unit disc of the complex plane :

$$U = \{z \in \mathbb{C} : \ |z| < 1\}$$

and

$$\overline{U} = \{z \in \mathbb{C} : \ |z| \leq 1\}.$$

Let $\mathcal{H}(U \times \overline{U})$ denote the class of analytic functions in $U \times \overline{U}$. In [4], the author has defined the class

$$\mathcal{H}\zeta[a,n] = \{f \in \mathcal{H}(U \times \overline{U}) : \ f(z,\zeta) = a + a_n(\zeta)z^n + a_{n+1}(\zeta)z^{n+1} + \cdots, \ z \in U, \zeta \in \overline{U}\}$$

with $a_k(\zeta)$ holomorphic functions in $\overline{U}$, $k \geq n$,

$$\mathcal{H}_n(U) = \{f \in \mathcal{H}\zeta[a,n] : \ f(z,\zeta) \,\text{univalent in U for all}\, \zeta \in \overline{U}\},$$

Received by the editors: 10.05.2010.

2000 *Mathematics Subject Classification.* Primary 30C80, Secondary 30C45, 30A20.

Key words and phrases. Analytic function, differential subordination, strong differential subordination, univalent function, convex function, dominant.

$$\mathcal{A}\zeta_n = \{f \in \mathcal{H}\zeta[a,n]: \ f(z,\zeta) = z + a_2(\zeta)z^2 + \cdots + a_n(\zeta)z^n + \cdots, \ z \in U, \zeta \in \overline{U}\}$$

with $\mathcal{A}\zeta_1 = \mathcal{A}\zeta$,

$$K\zeta = \left\{ f \in \mathcal{H}\zeta[a,n]: \ \mathrm{Re}\,\frac{zf''(z,\zeta)}{f'(z,\zeta)} + 1 > 0, \ z \in U, \text{for all}\, \zeta \in \overline{U} \right\}.$$

Definition 1.1. [4] Let $H(z,\zeta)$, $f(z,\zeta)$ be analytic in $U \times \overline{U}$. The function $f(z,\zeta)$ is said to be strongly subordinate to $H(z,\zeta)$, or $H(z,\zeta)$ is said to be strongly superordinate to $f(z,\zeta)$, if there exists a function ω analytic in U, $\omega(0) = 0$, $|\omega(z)| < 1$, such that $f(z,\zeta) = H[\omega(z),\zeta]$, for all $\zeta \in \overline{U}$. In such a case we write $f(z,\zeta) \prec\prec H(z,\zeta), \quad z \in U, \zeta \in \overline{U}$.

Remark 1.2. (i) If $H(z,\zeta)$ is analytic in $U \times \overline{U}$ and univalent in U for all $\zeta \in \overline{U}$, Definition (1.1) is equivalent to $f(0,\zeta) = H[0,\zeta]$, for all $\zeta \in \overline{U}$ and

$$f(U \times \overline{U}) \subset H(U \times \overline{U}).$$

(ii) If $H(z,\zeta) \equiv H(z)$ and $f(z,\zeta) \equiv f(z)$ then the strong subordination becomes the usual notion of subordination.

Definition 1.3. [6] For $f(z,\zeta) \in \mathcal{A}\zeta_n$, $n \in \mathbb{N}^* \cup \{0\}$, we define the integral operator: $I^n : \mathcal{A}\zeta_n \to \mathcal{A}\zeta_n$

$$\begin{aligned} &I^0 f(z,\zeta) = f(z,\zeta)\\ &I^1 f(z,\zeta) = If(z,\zeta) = \textstyle\int_0^z f(t,\zeta)t^{-1}dt\\ &\dots\\ &I^n f(z,\zeta) = I(I^{n-1}f(z,\zeta)) \qquad (z \in U, \zeta \in \overline{U}). \end{aligned}$$

Property 1.4. *For $f(z,\zeta) \in \mathcal{A}\zeta_n$, $n \in \mathbb{N}^* \cup \{0\}$, with the integral operator $I^n : \mathcal{A}\zeta_n \to \mathcal{A}\zeta_n$ we have:*

$$z[I^{n+1}f(z,\zeta)]' = I^n f(z,\zeta) \qquad (z \in U, \zeta \in \overline{U}).$$

In order to prove the main results we use the following definitions and lemmas, adapted to the class defined in [4]:

Lemma 1.5. [2, 3] *(Miller and Mocanu) Let $h(z,\zeta)$ be a convex function, with $h(0,\zeta) = a$ and let $\gamma \in \mathbb{C}^*$ be a complex number with $\operatorname{Re}\gamma \geq 0$. If $p \in \mathcal{H}\zeta[a,n]$ and*

$$p(z,\zeta) + \frac{1}{\gamma} zp'(z,\zeta) \prec\prec h(z,\zeta)$$

then

$$p(z,\zeta) \prec\prec g(z,\zeta) \prec\prec h(z,\zeta),$$

where

$$g(z,\zeta) = \frac{\gamma}{nz^{\gamma/n}} \int_0^z h(t,\zeta) t^{\frac{\gamma}{n}-1} dt \quad (z \in U, \zeta \in \overline{U}).$$

The function g is convex and is the best (a,n) dominant.

Lemma 1.6. [2, 3] *(Miller and Mocanu) Let $h(z,\zeta)$ be a convex function in U and let*

$$h(z,\zeta) = g(z,\zeta) + n\alpha z g'(z,\zeta), \quad z \in U, \zeta \in \overline{U}$$

where $\alpha > 0$ and n is a positive integer. If

$$p(z,\zeta) = g(0,\zeta) + p_n(\zeta) z^n + p_{n+1}(\zeta) z^{n+1} + \cdots$$

is holomorphic in $U \times \overline{U}$ and

$$p(z,\zeta) + \alpha z p'(z,\zeta) \prec\prec h(z,\zeta),$$

then

$$p(z,\zeta) \prec\prec g(z,\zeta)$$

and this result is sharp.

2. **Main results**

Definition 2.1. Let $\alpha > 1$ and $m, n \in \mathbb{N}$. We denote by $S_n^m(\alpha)$ the set of functions $f \in A\zeta_n$ that satisfy the inequality

$$Re[I^m f(z,\zeta)]' > \alpha, \quad z \in U, \zeta \in \overline{U}.$$

Theorem 2.2. *If $\alpha < 1$, and $m, n \in \mathbb{N}$, then*

$$S_n^m(\alpha) \subset S_n^{m+1}(\delta),$$

where

$$\delta = \delta(\alpha, \zeta, n) = 2\alpha - \zeta + \frac{2(\zeta - \alpha)}{n}\sigma\left(\frac{1}{n}\right)$$

and

$$\sigma(x) = \int_0^1 \frac{t^{x-1}}{1+t}dt. \tag{2.1}$$

Proof. Let $f(z,\zeta) \in S_n^m(\alpha)$. From Definition 2.1 we have

$$Re[I^m f(z,\zeta)]' > \alpha, \qquad z \in U, \zeta \in \overline{U}. \tag{2.2}$$

Using Property 1.4, we have

$$I^m f(z,\zeta) = z[I^{m+1} f(z,\zeta)]', \qquad z \in U, \zeta \in \overline{U}. \tag{2.3}$$

Differentiating (2.3), with respect to z, we obtain

$$[I^m f(z,\zeta)]' = [I^{m+1} f(z,\zeta)]' + z[I^{m+1} f(z,\zeta)]'', \qquad z \in U, \zeta \in \overline{U}. \tag{2.4}$$

We denote by

$$p(z,\zeta) = [I^{m+1} f(z,\zeta)]', \qquad z \in U, \zeta \in \overline{U}, p(0,\zeta) = 1, \zeta \in \overline{U}. \tag{2.5}$$

Using (2.5), the relation (2.3) becomes

$$[I^m f(z,\zeta)]' = p(z,\zeta) + zp'(z,\zeta), \qquad z \in U, \zeta \in \overline{U} \tag{2.6}$$

and replacing in (2.2), we obtain

$$Re[p(z,\zeta) + zp'(z,\zeta)] > \alpha, \qquad z \in U, \zeta \in \overline{U}$$

equivalent to

$$p(z,\zeta) + zp'(z,\zeta) \prec\prec \frac{\zeta + (2\alpha - \zeta)z}{1+z} = h(z,\zeta). \tag{2.7}$$

Using Lemma 1.5, we obtain

$$p(z,\zeta) \prec\prec q(z,\zeta) \prec\prec h(z,\zeta)$$

where

$$q(z,\zeta) = \frac{1}{nz^{\frac{1}{n}}} \int_0^z \frac{\zeta + (2\alpha - \zeta)t}{1+t} t^{\frac{1}{n}-1} dt = 2\alpha - \zeta + \frac{2(\zeta - \alpha)}{n} \sigma(x),$$

where $\sigma(x)$ is given by (2.1). The function $q(z,\zeta)$ is convex and is the best dominant. With $p(z,\zeta) \prec\prec q(z,\zeta)$ and $q(z,\zeta)$ being convex, and the fact that the image of $U \times \overline{U}$ through $g(z,\zeta)$ is symmetric with respect to the real axis, we deduce that

$$Re\ p(z,\zeta) > g(1,\zeta) = 2\alpha - \zeta + \frac{2(\zeta - \alpha)}{n} \sigma(\frac{1}{n}) = \delta(\alpha,\zeta,n) = \delta, \tag{2.8}$$

equivalent to

$$Re[I^{m+1} f(z,\zeta)]' > \delta, \qquad z \in U, \zeta \in \overline{U}. \tag{2.9}$$

Using Definition 2.1 we obtain $f \in S_n^{m+1}(\delta)$. Since $f \in S_n^m(\alpha)$, we obtain that

$$S_n^m(\alpha) \subset S_n^{m+1}(\delta).$$

□

Theorem 2.3. *Let $h(z,\zeta)$ an analytic function from $U \times \overline{U}$, with $h(0,\zeta) = 1$, $h'(0,\zeta) \neq 0$, $\zeta \in \overline{U}$, that satisfies inequality*

$$Re[1 + \frac{zh''(z,\zeta)}{h'(z,\zeta)}] > -\frac{1}{2}.$$

If $f(z,\zeta) \in A\zeta_n$ and verify the strong differential subordination

$$[I^m f(z,\zeta)]' \prec\prec h(z,\zeta), \tag{2.10}$$

then

$$[I^{m+1} f(z,\zeta)]' \prec\prec g(z,\zeta)$$

where

$$g(z,\zeta) = \frac{1}{nz^{\frac{1}{n}}} \int_0^z h(t,\zeta) t^{\frac{1}{n}-1} dt, \qquad z \in U, \zeta \in \overline{U}.$$

The function g is convex and is the best dominant.

Proof. A simple application of the differential subordination technique [1, 2], shows that the function $g(z,\zeta)$ is convex. By using (2.6), the strong differential subordination (2.10) becomes

$$p(z,\zeta)+zp'(z,\zeta)\prec\prec h(z,\zeta). \tag{2.11}$$

Using Lemma 1.5, we have

$$p(z,\zeta)\prec\prec g(z,\zeta)=\frac{1}{nz^{\frac{1}{n}}}\int_0^z h(t,\zeta)t^{\frac{1}{n}-1}dt.$$

Using (2.5), we obtain

$$[I^{m+1}f(z,\zeta)]'\prec\prec \frac{1}{nz^{\frac{1}{n}}}\int_0^z h(t,\zeta)t^{\frac{1}{n}-1}dt.$$

□

Theorem 2.4. *Let $g(z,\zeta)$ be a convex function with $g(0,\zeta)=1$ and suppose that*

$$h(z,\zeta)=g(z,\zeta)+zg'(z,\zeta),\qquad z\in U,\zeta\in\overline{U}.$$

If $f(z,\zeta)\in A\zeta_n$ and verify the strong differential subordination

$$[I^m f(z,\zeta)]'\prec\prec h(z,\zeta), \tag{2.12}$$

then

$$[I^{m+1}f(z,\zeta)]'\prec\prec g(z,\zeta).$$

Proof. By using (2.6), the strong differential subordination (2.12) becomes

$$p(z,\zeta)+zp'(z,\zeta)\prec\prec g(z,\zeta)+zg'(z,\zeta)\equiv h(z,\zeta).$$

Using Lemma 1.6, we have

$$p(z,\zeta)\prec\prec g(z,\zeta)$$

and using the notation (2.5), we obtain

$$[I^{m+1}f(z,\zeta)]'\prec\prec g(z,\zeta).$$

□

Theorem 2.5. *Let $g(z,\zeta)$ be a convex function with $g(0,\zeta) = 1$ and the function $h(z,\zeta)$, given by*

$$h(z,\zeta) = g(z,\zeta) + nzg'(z,\zeta).$$

If $f(z,\zeta) \in A\zeta_n$ and verify the strong differential subordination

$$[I^m f(z,\zeta)]' \prec\prec h(z,\zeta), \tag{2.13}$$

then

$$\frac{I^m f(z,\zeta)}{z} \prec\prec g(z,\zeta).$$

Proof. We denote with

$$p(z,\zeta) = \frac{I^m f(z,\zeta)}{z}, \qquad z \in U, \zeta \in \overline{U}, p(0,\zeta) = 1. \tag{2.14}$$

Using (2.14), we obtain

$$I^m f(z,\zeta) = zp(z,\zeta), \qquad z \in U, \zeta \in \overline{U}. \tag{2.15}$$

Differentiating (2.15), with respect to z, we obtain

$$[I^m f(z,\zeta)]' = p(z,\zeta) + zp'(z,\zeta), \qquad z \in U, \zeta \in \overline{U}. \tag{2.16}$$

Using (2.16), the strong differential subordination (2.13) becomes

$$p(z,\zeta) + zp'(z,\zeta) \prec\prec g(z,\zeta) + nzg'(z,\zeta).$$

Using Lemma 1.6, we have

$$p(z,\zeta) \prec\prec g(z,\zeta), \quad i.e. \quad \frac{I^m f(z,\zeta)}{z} \prec\prec g(z,\zeta).$$

□

Example 2.6. Let $g(z,\zeta)$ be the function

$$g(z,\zeta) = \frac{1 + (2\alpha - \zeta)z}{1 + z}, \qquad z \in U, \zeta \in \overline{U}, g(0,\zeta) = 1, \alpha \in \mathbb{R}, \alpha < 1. \tag{2.17}$$

We verify that $g(z,\zeta)$ is a convex function. Differentiating (2.17), with respect to z, we obtain

$$\text{Re}\left[\frac{zg''(z,\zeta)}{g'(z,\zeta)} + 1\right] = \text{Re}\left[\frac{1-z}{1+z}\right] > 0.$$

From the Theorem (2.4), and using (2.17) we obtain that

$$h(z,\zeta) = g(z,\zeta) + zg'(z,\zeta) = \frac{1+(2\alpha-\zeta)z(2+z)}{(1+z)^2}, \qquad z \in U, \zeta \in \overline{U}. \quad (2.18)$$

For $\alpha = 0$ we obtain

$$h(z,\zeta) = \frac{1-\zeta z(2+z)}{(1+z)^2}.$$

We consider the function

$$g(z,\zeta) = \frac{z - \zeta\frac{z^2}{2}}{1+\frac{z}{2}}.$$

By Theorem (2.4) we obtain that, the strong differential subordination

$$\frac{1-\zeta z-\zeta\frac{z^2}{4}}{(1+\frac{z}{2})^2} \prec\prec \frac{1-\zeta z(2+z)}{(1+z)^2}$$

implies

$$\frac{1-\zeta\frac{z}{2}}{1+\frac{z}{2}} \prec\prec \frac{1-\zeta z}{1+z}.$$

Example 2.7. Let $h(z,\zeta)$ be the function

$$h(z,\zeta) = \frac{\zeta+z}{\zeta-z}, \qquad z \in U, \zeta \in \overline{U}, h(0,\zeta) = 1. \quad (2.19)$$

Let $g(z,\zeta)$ be a convex function with $g(0,\zeta) = 1$ and

$$h(z,\zeta) = g(z,\zeta) + zg'(z,\zeta), \qquad z \in U, \zeta \in \overline{U}.$$

That implies

$$g(z,\zeta) = \frac{1}{z}\int_0^z h(t,\zeta)dt = \frac{1}{z}\int_0^z \frac{\zeta+t}{\zeta-t}dt$$

and

$$g(z,\zeta) = \frac{-2\zeta}{z}log(\zeta-z) + \frac{2\zeta}{z}log(\zeta) - 1.$$

By Theorem (2.4) we obtain that, the strong differential subordination

$$\frac{2\zeta+z}{2\zeta-z} \prec\prec \frac{\zeta+z}{\zeta-z}$$

implies

$$\frac{-4\zeta}{z}log(2\zeta-z) + \frac{4\zeta}{z}log(2\zeta) - 1 \prec\prec \frac{-2\zeta}{z}log(\zeta-z) + \frac{2\zeta}{z}log(\zeta) - 1.$$

References

[1] Antonino, J. A., Romaquera, S., *Strong differential subordination to Briot-Bouquet differential equations*, Journal of Differential Equations, **114** (1994), 101-105.

[2] Miller, S. S., Mocanu, P. T., *Differential subordinations and univalent functions*, Michig. Math. J., **28** (1981), 157-171.

[3] Miller, S. S., Mocanu, P. T., *Differential subordinations.*, Theory and Applications, Pure and Applied Mathematics, Marcel Dekker, Inc., New York, 2000.

[4] Oros, G. I., *On a new strong differential subordination*, (to appear).

[5] Oros, G. I., Oros, Gh., *Strong differential subordination*, Turkish Journal of Mathematics, **32** (2008), 1-11.

[6] Sălăgean, G. S., *Subclasses of univalent functions*, Complex Analysis-Fift Romanian-Finnish Seminar, Part 1 (Bucharest, 1981), 362-372, Lecture Notes in Math., 1013, Springer, Berlin 1983.

Faculty of Environmental Protection
University of Oradea
Str. B-dul Gen. Magheru, No.26
410048 Oradea, Romania
E-mail address: roxana.sendrutiu@gmail.com

STUDIA UNIV. "BABEŞ–BOLYAI", MATHEMATICA, Volume **LV**, Number 3, September 2010

AN APPLICATION OF MILLER AND MOCANU LEMMA

HITOSHI SHIRAISHI AND SHIGEYOSHI OWA

Dedicated to Professor Grigore Ştefan Sălăgean on his 60^{th} *birthday*

Abstract. Let $\mathcal{H}[a,n]$ be the class of functions $f(z) = a + a_n z^n + \dots$ which are analytic in the open unit disk $\mathbb{U}$. For $f(z) \in \mathcal{H}[a,n]$, S. S. Miller and P. T. Mocanu (J. Math. Anal. Appl. **65**(1978), 289-305) have shown Miller and Mocanu lemma which is the generalization of Jack lemma by I. S. Jack (J. London Math. Soc. **3**(1971), 469-474). Applying Miller and Mocanu lemma, an interesting property for $f(z) \in \mathcal{H}[a,n]$ and an example are discussed.

1. Introduction

Let $\mathcal{H}[a,n]$ denote the class of functions $f(z)$ of the form

$$f(z) = a + \sum_{k=n}^{\infty} a_k z^k \qquad (n = 1, 2, 3, \dots)$$

which are analytic in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$, where $a \in \mathbb{C}$. Jack [1] has shown the result for analytic functions $w(z)$ in $\mathbb{U}$ with $w(0) = 0$, which is called Jack's lemma. In 1978, Miller and Mocanu [2] have given the generalization theorem for Jack's lemma, which was called Miller and Mocanu lemma.

Lemma 1.1 (Miller and Mocanu lemma)**.** *Let $f(z) \in \mathcal{H}[a,n]$ with $f(z) \not\equiv a$. If there exists a point $z_0 \in \mathbb{U}$ such that*

$$\max_{|z| \leqq |z_0|} |f(z)| = |f(z_0)|,$$

Received by the editors: 26.04.2010.

2000 *Mathematics Subject Classification.* 30C45.

Key words and phrases. Analytic, univalent, Jack's lemma, Miller and Mocanu lemma.

then

$$\frac{z_0 f'(z_0)}{f(z_0)} = m$$

and

$$\mathrm{Re}\frac{z_0 f''(z_0)}{f'(z_0)} + 1 \geqq m,$$

where m is real and

$$m \geqq n\frac{|f(z_0) - a|^2}{|f(z_0)|^2 - |a|^2} \geqq n\frac{|f(z_0)| - |a|}{|f(z_0)| + |a|}.$$

If $a = 0$, then the above lemma becomes Jack's lemma due to Jack [1].

2. **Main theorem**

Applying Miller and Mocanu lemma, we derive

Theorem 2.1. *Let $f(z) \in \mathcal{H}[a, n]$ with $f(z) \neq 0$ for $z \in \mathbb{U}$. If there exists a point $z_0 \in \mathbb{U}$ such that*

$$\min_{|z| \leqq |z_0|} |f(z)| = |f(z_0)|,$$

then

$$\frac{z_0 f'(z_0)}{f(z_0)} = -m \tag{2.1}$$

and

$$\mathrm{Re}\frac{z_0 f''(z_0)}{f'(z_0)} + 1 \geqq -m, \tag{2.2}$$

where

$$m \geqq n\frac{|a - f(z_0)|^2}{|a|^2 - |f(z_0)|^2} \geqq n\frac{|a| - |f(z_0)|}{|a| + |f(z_0)|}.$$

Proof. We defined the function $g(z)$ by

$$\begin{aligned} g(z) &= \frac{1}{f(z)} \\ &= c + c_n z^n + c_{n+1} z^{n+1} + \dots \qquad \left(c = \frac{1}{a}\right). \end{aligned}$$

Then, $g(z)$ is analytic in $\mathbb{U}$ and $g(0) = c \neq 0$. Furthermore, by the assumtion of the theorem, $|g(z)|$ takes its maximum value at $z = z_0$ in the closed disk $|z| \leqq |z_0|$. It follows from this that

$$|g(z_0)| = \frac{1}{|f(z_0)|} = \frac{1}{\min\limits_{|z| \leqq |z_0|} |f(z)|} = \max_{|z| \leqq |z_0|} |g(z)|.$$

Therefore, applying Lemma 1.1 to $g(z)$, we observe that

$$\frac{z_0 g'(z_0)}{g(z_0)} = -\frac{z_0 f'(z_0)}{f(z_0)} = m$$

which shows (2.1) and

$$\begin{aligned} \mathrm{Re}\frac{z_0 g''(z_0)}{g'(z_0)} + 1 &= \mathrm{Re}\left(\frac{z_0 f''(z_0)}{f'(z_0)} - 2\frac{z_0 f'(z_0)}{f(z_0)}\right) + 1 \\ &= \mathrm{Re}\frac{z_0 f''(z_0)}{f'(z_0)} + 2m + 1 \\ &\geqq m \end{aligned}$$

which implies (2.2), where

$$m \geqq n\frac{|g(z_0) - c|^2}{|g(z_0)|^2 - |c|^2} = n\frac{|a - f(z_0)|^2}{|a|^2 - |f(z_0)|^2} \geqq n\frac{|a| - |f(z_0)|}{|a| + |f(z_0)|}.$$

This completes the assertion of Theorem 2.1. □

Example 2.2. Let us consider the function $f(z)$ given by

$$\begin{aligned} f(z) &= \frac{a + \left(e^{i\arg(a)} - a\right) z^n}{1 - z^n} \\ &= a + e^{i\arg(a)} z^n + e^{i\arg(a)} z^{2n} + \dots \qquad (z \in \mathbb{U}) \end{aligned}$$

for some complex number a with $|a| > \frac{1}{2}$.

Then, $f(z)$ maps the disk $\mathbb{U}_r = \{z : |z| < r \leqq 1\}$ onto the domain

$$\left| f(z) - \left(a + \frac{e^{i\arg(a)} r^{2n}}{1 - r^{2n}}\right)\right| \leqq \frac{r^n}{1 - r^{2n}}.$$

Thus, we know that there exists a point $z_0 = re^{i\frac{\pi}{n}} \in \mathbb{U}$ such that

$$\min_{|z| \leqq |z_0|} |f(z)| = |f(z_0)| = |a| - \frac{r^n}{1 - r^{2n}}.$$

For such a point z_0, we obtain that

$$\frac{z_0 f'(z_0)}{f(z_0)} = -\frac{nr^n}{(1+r^n)(|a|-(1-|a|)r^n)} = -m$$

where

$$m = \frac{nr^n}{(1+r^n)(|a|-(1-|a|)r^n)} > 0.$$

Therefore, we get that

$$\mathrm{Re}\frac{z_0 f''(z_0)}{f'(z_0)} + 1 = n\frac{1-r^n}{1+r^n} > 0 > -m.$$

Furthermore, we obtain that

$$n\frac{|a-f(z_0)|^2}{|a|^2-|f(z_0)|^2} = \frac{nr^n}{2|a|+(2|a|-1)r^n} = \frac{nr^n}{2\left(|a|-(1-|a|)r^n+\dfrac{1}{2}r^n\right)} < m.$$

Putting a with a real number in Example 2.2, we get Example 2.3.

Example 2.3. Let us consider the function

$$\begin{aligned} f(z) &= \frac{a+(1-a)z^n}{1-z^n} \\ &= a + z^n + z^{2n} + \dots \qquad (z \in \mathbb{U}) \end{aligned}$$

for $a > \dfrac{1}{2}$. Then, it follows that the function $f(z)$ maps the disk $\mathbb{U}_r$ onto the domain

$$\left| f(z) - \left(a + \frac{r^{2n}}{1-r^{2n}}\right)\right| \leqq \frac{r^n}{1-r^{2n}}.$$

Thus, there exists a point $z_0 = re^{i\frac{\pi}{n}} \in \mathbb{U}$ such that

$$\min_{|z| \leqq |z_0|} |f(z)| = |f(z_0)| = a - \frac{r^n}{1-r^{2n}}.$$

For such a point z_0, we obtain

$$\frac{z_0 f'(z_0)}{f(z_0)} = -\frac{nr^n}{(1+r^n)(a-(1-a)r^n)} = -m$$

where

$$m = \frac{nr^n}{(1+r^n)(a-(1-a)r^n)} > 0.$$

Therefore, we see that

$$\mathrm{Re}\frac{z_0 f''(z_0)}{f'(z_0)} + 1 = n\frac{1-r^n}{1+r^n} > 0 > -m.$$

Moreover, we have that

$$n\frac{|a-f(z_0)|^2}{|a|^2-|f(z_0)|^2} = \frac{nr^n}{2a+(2a-1)r^n} = \frac{nr^n}{2\left(a-(1-a)r^n+\dfrac{1}{2}r^n\right)} < m.$$

References

[1] Jack, I. S., *Functions starlike and convex of order α*, J. London Math. Soc., **3** (1971), 469-474.

[2] Miller, S. S., Mocanu, P. T., *Second-order differential inequalities in the complex plane*, J. Math. Anal. Appl., **65** (1978), 289-305.

Department of Mathematics, Kinki University
Higashi-Osaka, Osaka 577-8502, Japan
E-mail address: shiraishi@math.kindai.ac.jp

Department of Mathematics, Kinki University
Higashi-Osaka, Osaka 577-8502, Japan
E-mail address: owa@math.kindai.ac.jp

STUDIA UNIV. "BABEŞ–BOLYAI", MATHEMATICA, Volume **LV**, Number 3, September 2010

DUALITY FOR HADAMARD PRODUCTS APPLIED TO CERTAIN CONDITION FOR α-STARLIKENESS

JANUSZ SOKÓŁ

Dedicated to Professor Grigore Ştefan Sălăgean on his 60^{th} birthday

Abstract. Let $\mathcal{P}(\alpha,\beta)$, $\alpha > 0$, $\beta < 1$, denote the class of all analytic functions f in the unit disc with the normalization $f(0) = 1$, $f'(0) = 1$ and satisfying the condition

$$\mathfrak{Re}[e^{i\varphi}(f'(z) + \frac{1}{\alpha}zf''(z) - \beta)] > 0, \quad |z| < 1$$

for some $\varphi \in \mathbb{R}$. In this paper we find conditions on α, β so that $\mathcal{P}(\alpha,\beta) \subseteq \mathcal{S}^*(\mu)$, where $\mu < 1$ is given and $\mathcal{S}^*(\mu)$ denote the class of starlike function of order μ. We take advantage of the Ruscheweh's Duality theory.

1. Introduction

Let $\mathcal{H}$ denote the class of analytic functions in the open unit disc

$$U = \{z : \ |z| < 1\}$$

of the complex plane $\mathbb{C}$. Everywhere in this paper $z \in U$ unless we make a note. We say that $f \in \mathcal{H}$ is convex when $f(U)$ is a convex set. Let $\mathcal{A}$ denote the subclass of $\mathcal{H}$ consisting of functions normalized by $f(0) = 0$, $f'(0) = 1$. For $\mu < 1$, by $\mathcal{S}^*(\mu)$ we denote the well known subclass of $\mathcal{A}$ consisting of starlike function of order μ. As is well known

$$\mathcal{S}^*(\mu) = \left\{ f \in \mathcal{A} : \ \mathfrak{Re}\left[\frac{zf'(z)}{f(z)}\right] > \mu \text{ for } z \in \ U \right\}$$

Received by the editors: 26.04.2010.

2000 *Mathematics Subject Classification.* 30C45, 33C05.

Key words and phrases. Duality principle, dual set, Hadamard product, Starlike functions, convex functions, functions with positive real part, Gaussian hypergeometric function.

$\mathcal{S}^*(0) = \mathcal{S}^*$ is the class of starlike functions which map U onto a starlike domain with respect to the origin. For $\alpha > 0$ and $\beta < 1$ given, define

$$\mathcal{P}(\alpha,\beta) = \left\{ f \in \mathcal{A} : \exists \varphi \in \mathbb{R} \;\text{ s. t. }\; \mathfrak{Re}\left[e^{i\varphi}\left(f'(z) + \frac{1}{\alpha} z f''(z) - \beta \right)\right] > 0, z \in U \right\}.$$

In the geometric theory of function, a variety of sufficient conditions for starlikeness have been considered. We refer to the monographs [4], [5] for details. In the present work we tray to find conditions on α, β so that $\mathcal{P}(\alpha,\beta) \subseteq S^*(\mu)$, where $\mu < 1$ is given. If f and g are analytic in U with $f(z) = a_0 + a_1 z + a_2 z^2 + \dots$ and $g(z) = b_0 + b_1 z + b_2 z^2 + \dots$ then the Hadamard product (or convolution) of f and g is defined by

$$(f * g)(z) = a_0 b_0 + a_1 b_1 z + a_2 b_2 z^2 + \dots .$$

The convolution has the algebraic properties of ordinary multiplication. In convolution theory, the concept of duality is central. For a set

$$V \subseteq \mathcal{A}_0 = \left\{ g : g(z) = \frac{f(z)}{z}, f \in \mathcal{A} \right\}$$

the dual set V^* is defined as

$$V^* = \{ g \in \mathcal{A}_0 : (f * g)(z) \neq 0 \;\text{ for all }\; f \in V, z \in U \}.$$

In this paper we use the powerful method of duality principle in geometric function theory developed by Ruscheweyh [8]. The basic results of Ruscheweyh's duality theory one can find in the book [9]. The duality principle states that, under certain conditions on V, the range of a continuous linear functional on V equals the range of the same linear functional on $(V^*)^* = V^{**}$. This is a useful information since in many cases of interest V^{**} is much larger than V. Then by investigating the small set we can get results about the large set. One such pair of the sets is described in the theorem below.

Theorem 1.1. *Let*

$$V_\beta = \left\{ \beta + \frac{(1-\beta)(1+xz)}{1+yz} : \quad |x| = |y| = 1 \right\}, \beta \in \mathbb{R}, \beta \neq 1.$$

Then

$$V_\beta^{**} = \left\{ g \in \mathcal{A}_0 : \exists \varphi \in \mathbb{R} \;\text{ such that }\; \mathfrak{Re}\left[e^{i\varphi}\left(g(z) - \beta \right)\right] > 0, z \in U \right\}.$$

Theorem 1.1 with $\beta = 0$ one can find in [9, p. 22]. Notice that if $h \in V_\beta$, $h(z) = \beta + (1-\beta)\frac{1+xz}{1+yz}$ with $|x| = |y| = 1$, $\beta \in \mathbb{R}$, $\beta \neq 1$, then

$$h(z) = 1 + (1-\beta)\left(1 - \frac{x}{y}\right)\frac{yz}{1-yz} = 1 + (1-\beta)(1-e^{i\psi})\sum_{k=1}^{\infty}(yz)^k \tag{1.1}$$

for some $\psi \in \mathbb{R}$. A subset $V \subseteq \mathcal{A}_0$ is said to be complete if it has the following property:

$$f \in V \Rightarrow f(xz) \in V \ \forall |x| \leq 1.$$

Theorem 1.2. *(Duality principle, see* [8]*) Let $V \subseteq \mathcal{A}_0$ be compact and complete. If λ is a continuous linear functional on $\mathcal{H}$, then*

$$\lambda(V) = \lambda\left(V^{**}\right), \quad \overline{co}(V) = \overline{co}\left(V^{**}\right).$$

The sets V_β and V_β^{**} in Theorem 1.1 are compact and complete. The following Theorem 1.3 one can find in [9, p. 23] and in [10].

Theorem 1.3. *(see* [10]*) Let $f \in \mathcal{A}$. Then f belongs to the class $\mathcal{S}^*(\mu)$ of starlike function of order μ if and only if*

$$\frac{f(z)}{z} * \frac{1 + \frac{\varepsilon+2\mu-1}{2(1-\mu)}z}{(1-z^2)} \neq 0 \quad \forall\ |\varepsilon| = 1, \quad \forall\ z \in U.$$

2. Main results

Theorem 2.1. *Suppose that $\alpha > 0$, $\beta < 1$, $\mu < 1$. Then $\mathcal{P}(\alpha, \beta) \subseteq \mathcal{S}^*(\mu)$ if and only if*

$$\mathfrak{Re}\left[H(\varepsilon; z)\right] > -\frac{1-\mu}{1-\beta} \quad \forall |\varepsilon| = 1, \quad \forall z \in U, \tag{2.1}$$

where

$$H(\varepsilon; z) = \alpha \sum_{k=1}^{\infty} \frac{k(1+\varepsilon) + 2(1-\mu)}{(k+1)(k+\alpha)} z^k. \tag{2.2}$$

Proof. Let a function f be in the class $\mathcal{P}(\alpha, \beta)$. If we denote $f'(z) + \frac{z}{\alpha}f''(z) = g_\alpha(z)$, then we have $g_\alpha \in V_\beta^{**}$. If $f(z) = \sum\limits_{k=1}^{\infty} a_k z^k$, $a_1 = 1$, then

$$f'(z) + \frac{z}{\alpha}f''(z) = \sum_{k=1}^{\infty} \frac{k(k-1+\alpha)}{\alpha} a_k z^{k-1} = g_\alpha(z)$$

so

$$\frac{f(z)}{z} = \sum_{k=1}^{\infty} a_k z^{k-1} = g_\alpha(z) * \sum_{k=1}^{\infty} \frac{\alpha z^{k-1}}{k(k-1+\alpha)},$$

and we obtain one-to-one correspondence between $\mathcal{P}(\alpha,\beta)$ and V_β^{**}. Thus, by Theorem 1.3, $\mathcal{P}(\alpha,\beta) \subseteq \mathcal{S}^*(\mu)$ if and only if

$$g_\alpha(z) * \sum_{k=1}^{\infty} \frac{\alpha z^{k-1}}{k(k-1+\alpha)} * \frac{1+\frac{\varepsilon+2\mu-1}{2(1-\mu)}z}{(1-z)^2} \neq 0 \quad \forall g_\alpha \in V_\beta^{**},\ \forall\ |\varepsilon|=1, \forall\ z \in U. \quad (2.3)$$

Let us consider for $z \in U$ the continuous linear functional $\lambda_z : \mathcal{A}_0 \to \mathbb{C}$, such that

$$\lambda_z(h) := h(z) * \sum_{k=1}^{\infty} \frac{\alpha z^{k-1}}{k(k-1+\alpha)} * \frac{1+\frac{\varepsilon+2\mu-1}{2(1-\mu)}z}{(1-z)^2},$$

By Duality principle we have $\lambda_z(V) = \lambda_z(V_\beta^{**})$. Therefore (2.3) holds if and only if

$$\left[1+(1-\beta)(1-e^{i\psi})\sum_{k=1}^{\infty} z^k\right] * \left[1+\sum_{k=1}^{\infty} \frac{\alpha z^k}{(k+1)(k+\alpha)}\right] * \left[\frac{1+\frac{\varepsilon+2\mu-1}{2(1-\mu)}z}{(1-z)^2}\right] \neq 0 \quad (2.4)$$

for all $\psi \in \mathbb{R}$, $|\varepsilon| = 1$, $z \in U$. Using the properties of convolution we can reformulate (2.4) as

$$\alpha \sum_{k=1}^{\infty} \frac{k(1+\varepsilon)+2(1-\mu)}{(k+1)(k+\alpha)} z^k \neq -\frac{2(1-\mu)}{(1-e^{i\psi})(1-\beta)}. \quad (2.5)$$

For $\psi \in \mathbb{R}$ the quantity on the right site of (2.5) takes its values on the line $\mathfrak{Re}\, w = -\frac{1-\mu}{1-\beta}$ so (2.5) is equivalent to (2.1) . □

Starlikeness of functions in $\mathcal{P}(\alpha,\beta)$ has been investigated. For example we have the reformulated version from [3].

Theorem 2.2. *(see [3]) If $f \in \mathcal{P}(\alpha,\beta)$ and $\alpha \leq 3$ and $\beta(\alpha)$ be given by*

$$\frac{\beta(\alpha)}{1-\beta(\alpha)} = \alpha \int_0^1 \frac{t^{\alpha-1}(t-1)}{t+1}\, \mathrm{d}t,$$

then $f \in \mathcal{S}^*(0)$ and the value of $\beta(\alpha)$ is sharp.

Note that Fournier and Ruscheweyh introduced in [3] the integral transform

$$V_\lambda : \mathcal{A} \to \mathcal{A}$$

such that

$$V_\lambda(f)(z) = \int_0^1 \lambda(t) \frac{f(tz)}{t}\, \mathrm{d}t,$$

where $\lambda(t)$is real valued integrable function satisfying the normalizing condition

$$\int_0^1 \lambda(t)\ \mathrm{d}t = 1.$$

This operator was introduced mainly to find conditions on $\lambda(t)$ and β so that $V_\lambda(f)$ maps $\mathcal{P}(\alpha,\beta)$ into $S^*(0)$, when $\alpha \to \infty$. Recently Balasubramanian, Ponnusamy and Prabhakaran in [2] and Ponnusamy and Rønning in [7] extended this considerations to find conditions on $\lambda(t)$ and β such that $V_\lambda(f)$ is starlike of order μ, $(0 \le \mu \le 1/2)$ when $f \in \mathcal{P}(\alpha,\beta)$. For convexity of this integral transform see [1].

While Theorem 2.1 precisely answers when $\mathcal{P}(\alpha,\beta) \subseteq \mathcal{S}^*(\mu)$ it is difficult to answer when the condition (2.1) is satisfied in general. It seems that $\mathfrak{Re} H(\varepsilon; z)$ attains its minimum at $z = -1$ and $\varepsilon = 1$ but it is hard to show.

Conjecture 2.3. *Let f be given by (2.2). Then*

$$\min\{\mathfrak{Re} H(\varepsilon; z) : |\varepsilon| = 1,\ |z| < 1\} = H(1; -1).$$

In [11] we apply the general theory of differential subordinations to obtain several weaker but simple sufficient conditions for μ-starlikeness while Owa and Sălăgean in [6] considered a sufficient condition and a necessary condition for starlikeness of complex order of functions with negative coefficients. One can expressed the function $H(\varepsilon; z)$ in terms of the Gaussian hypergeometric function

$$_2F_1(a,b,c;z) \quad = \quad \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k k!} z^k,$$

where $(x)_k$ denotes the Pochhammer symbol defined by

$$(x)_k = x(x+1)(x+2)\cdots(x+k-1) \ \text{ for } \ k \in \mathbb{N} \ \text{ and } \ (x)_0 = 1.$$

Then for $\alpha \neq 1$ we have

$$\begin{aligned} H(\varepsilon; z) &= \alpha \sum_{k=1}^{\infty} \frac{k(1+\varepsilon) + 2(1-\mu)}{(k+1)(k+\alpha)} z^k \\ &= \frac{\alpha(\varepsilon + 2\mu - 1)}{1-\alpha} \sum_{k=1}^{\infty} \frac{z^k}{k+1} + \frac{2(1-\mu) - \alpha(\varepsilon+1)}{1-\alpha} \sum_{k=1}^{\infty} \frac{\alpha z^k}{k+\alpha} \end{aligned}$$

$$= \frac{\alpha(\varepsilon+2\mu-1)\left[{}_2F_1(1,1,2;z)-1\right]+\left[2(1-\mu)-\alpha(\varepsilon+1)\right]\left[{}_2F_1(1,\alpha,\alpha+1;z)-1\right]}{1-\alpha}$$

$$= 2(\mu-1)+\frac{\alpha(\varepsilon+2\mu-1)}{1-\alpha}{}_2F_1(1,1,2;z)+\frac{2(1-\mu)-\alpha(\varepsilon+1)}{1-\alpha}{}_2F_1(1,\alpha,\alpha+1;z)$$

$$= 2(\mu-1)+\frac{\alpha(\varepsilon+2\mu-1)}{1-\alpha}\frac{1}{z}\ln\frac{1}{1-z}+\frac{2(1-\mu)-\alpha(\varepsilon+1)}{1-\alpha}{}_2F_1(1,\alpha,\alpha+1;z).$$

We can rewrite the inequality (2.1) in the form

$$\frac{1-\mu}{\alpha(1-\beta)}+\mathfrak{Re}\left[\sum_{k=1}^{\infty}\frac{kz^k}{(k+1)(k+\alpha)}\right]+2(1-\mu)\mathfrak{Re}\left[\sum_{k=1}^{\infty}\frac{z^k}{(k+1)(k+\alpha)}\right] \tag{2.6}$$

$$> \mathfrak{Re}\left[-\varepsilon\sum_{k=1}^{\infty}\frac{kz^k}{(k+1)(k+\alpha)}\right] \quad \forall|\varepsilon|=1,\quad \forall z\in U,$$

thus we can see that (2.6) is satisfied when

$$\frac{1-\mu}{\alpha(1-\beta)}+\mathfrak{Re}\left[\sum_{k=1}^{\infty}\frac{kz^k}{(k+1)(k+\alpha)}\right]+2(1-\mu)\mathfrak{Re}\left[\sum_{k=1}^{\infty}\frac{z^k}{(k+1)(k+\alpha)}\right] \tag{2.7}$$

$$> \left|\sum_{k=1}^{\infty}\frac{kz^k}{(k+1)(k+\alpha)}\right| \quad \forall z\in U.$$

Conjecture 2.4. *Let the function G be given by*

$$G(z)=2(1+\alpha)\sum_{k=1}^{\infty}\frac{1}{(k+1)(k+\alpha)}z^k$$

Then the function $zG'(z)$ is a convex function when $-1<\alpha$.

Note tat it is known that G is a convex while zG' is a starlike function. With this notation (2.7) becomes

$$\frac{2(1+\alpha)(1-\mu)}{\alpha(1-\beta)}+\mathfrak{Re}zG'(z)+2(1-\mu)\mathfrak{Re}G(z)>|zG'(z)|\forall z\in U. \tag{2.8}$$

If Conjecture 2.4 is true, then we have $G'(-1)<\mathfrak{Re}G'(z)<G'(1)$ so for (2.8)it suffices that

$$\frac{1-\mu}{\alpha(1-\beta)}+\sum_{k=1}^{\infty}\frac{k(-1)^k}{(k+1)(k+\alpha)}+2(1-\mu)\sum_{k=1}^{\infty}\frac{(-1)^k}{(k+1)(k+\alpha)} \tag{2.9}$$

$$> \sum_{k=1}^{\infty}\frac{k}{(k+1)(k+\alpha)}.$$

While (2.9) is not a necessary for (2.8) it still remains hard to verify.

References

[1] Balasubramanian, R., Ponnusamy, M., Prabhakaran, D. J., *Convexity of integral transforms and function spaces*, Integral Transforms and Special Functions, **18(1)** (2007), 1-14.

[2] Balasubramanian, R., Ponnusamy, M., Prabhakaran, D. J., *On extremal problems related to integral trnsforms of a class of analytic functions*, J. Math. Anal. Appl., **336** (2007), 542-555.

[3] Fournier, R., Ruscheweyh, S., *On two extremal problems related to univalent functions*, Rocky Mountain I. Math., **24(2)** (1994), 529-538.

[4] Goodman, A. W., *Univalent Functions*, vol. I and II, Mariner Publishing Co., Tampa, Florida (1983).

[5] Miller, S. S., Mocanu, P. T., *Differential subordinations: theory and applications*, Series of Monographs and Textbooks in Pure and Applied Mathematics, Vol. 225, Marcel Dekker Inc., New York / Basel 2000.

[6] Owa, S., Sălăgean, G. S., *On an open problem of S. Owa*, J. Math. Anal. Appl., **218(2)** (1998), 453-457.

[7] Ponnusamy, S., Rønning, F., *Integral transforms of a class of analytic functions*, Complex Var. Elliptic. Eq., **53(5)** (2008), 423-434.

[8] Ruscheweyh, S., *Duality for Hadamard products with applications to extremal problems for functions regular in the unit disc*, Trans. Amer. Math Soc., **210** (1975), 63-74.

[9] Ruscheweyh, S., *Convolutions in Geometric Function Theory*, Sem. Math. Sup. 83, Press Univ. Montreal 1982.

[10] Silverman, H., Silvia, E. M., Telage, D., *Convolutions conditions for convexity starlikeness and spiral-likeness*, Math. Z., **162(2)** (1978), 125-130.

[11] Sokół, J., *On a condition for alpha-starlikeness*, J. Math. Anal. Appl., **352** (2009), 696-701.

Department of Mathematics
Rzeszów University of Technology
ul. W. Pola 2, 35-959 Rzeszów, Poland
E-mail address: jsokol@prz.edu.pl

STUDIA UNIV. "BABEŞ–BOLYAI", MATHEMATICA, Volume **LV**, Number 3, September 2010

SOME STRONG DIFFERENTIAL SUBORDINATIONS OBTAINED BY SĂLĂGEAN DIFFERENTIAL OPERATOR

ADELA OLIMPIA TĂUT

Dedicated to Professor Grigore Ştefan Sălăgean on his 60^{th} *birthday*

Abstract. S. S. Miller and P. T. Mocanu introduced the notion of differential superordination as a dual concept of differential subordination . The notion of strong differential subordination was introduced by J. A. Antonino and S. Romaguera. By using the Sălăgean differential operator we introduce a class of holomorphic functions denoted by $S_n^m(\alpha)$, and obtain some strong subordinations results.

1. Introduction and preliminaries

Denote by U the unit disc of the complex plane,

$$U = \{z \in \mathbb{C};\ |z| < 1\} \tag{1.1}$$

$$\overline{U} = \{z \in \mathbb{C};\ |z| \leq 1\} \tag{1.2}$$

the closed unit disc of the complex plane.

In the paper [3], Georgia I. Oros defined the classes $\mathcal{H}(U \times \overline{U})$ denote the class of analytic functions in $U \times \overline{U}$,

$$A_\zeta^* = \{f \in \mathcal{H}(U \times \overline{U}) \mid f(z,\zeta) = z + a_2(\zeta)z^2 + \dots,\ z \in U,\ \zeta \in \overline{U}\}, \tag{1.3}$$

$$A_{n\zeta}^* = \{f \in \mathcal{H}(U \times \overline{U}) \mid f(z,\zeta) = z + a_{n+1}(\zeta)z^{n+1} + \dots,\ z \in U,\ \zeta \in \overline{U}\}, \tag{1.4}$$

Received by the editors: 01.03.2010.

2000 *Mathematics Subject Classification.* 30C80, 30C45, 30A20.

Key words and phrases. differential superordination, strong differential superordination, univalent function, subordinant, best subordinant, differential operator.

for $n = 1$, $A^*_{n\zeta} = A^*_\zeta$, with $a_k(\zeta)$ holomorphic functions in $\overline{U}$, $k \geq 2$,

$$\mathcal{H}^*[a, n, \zeta] = \{f \in \mathcal{H}(U \times \overline{U}) \mid f(z, \zeta) = a + a_n(\zeta)z^n + a_{n+1}(\zeta)z^{n+1} + \dots,\ z \in U,\ \zeta \in \overline{U}\} \tag{1.5}$$

where $a_k(\zeta)$ holomorphic functions in $\overline{U}$, $k \geq n$, and let

$$\mathcal{H}_u(U) = \{f \in \mathcal{H}^*[a, n, \zeta] \mid f(z, \zeta) \text{ univalent in } U \text{ for all } \zeta \in \overline{U}\}, \tag{1.6}$$

$$K^* = \left\{ f \in \mathcal{H}^*[a, n, \zeta] \mid \mathrm{Re}\, \frac{zf''(z, \zeta)}{f'(z, \zeta)} + 1 > 0,\ z \in U \text{ for all } \zeta \in \overline{U} \right\} \tag{1.7}$$

the class of convex functions,

$$S^* = \left\{ f \in \mathcal{H}^*[a, n, \zeta] \mid \mathrm{Re}\, \frac{zf'(z, \zeta)}{f(z, \zeta)} > 0,\ z \in U \text{ for all } \zeta \in \overline{U} \right\} \tag{1.8}$$

the class of starlike functions.

Definition 1.1. [4] Let $f(z, \zeta)$, $H(z, \zeta)$ analytic in $U \times \overline{U}$. The function $f(z, \zeta)$ is said to be strongly subordinate to $H(z, \zeta)$, or $H(z, \zeta)$ is said to be strongly superordinate to $f(z, \zeta)$, if there exists a function w analytic in U, with $w(0) = 0$, and $|w(z)| < 1$ such that $f(z, \zeta) = H(w(z), \zeta)$ for all $\zeta \in \overline{U}$. In such a case we write $f(z, \zeta) \prec\prec H(z, \zeta)$, $z \in U$, $\zeta \in \overline{U}$.

Remark 1.2. [4] (i) Since $f(z, \zeta)$ is analytic in $U \times \overline{U}$, for all $\zeta \in \overline{U}$ and univalent in U, for all $\zeta \in \overline{U}$, Definition 1.1 is equivalent to $f(0, \zeta) = H(0, \zeta)$ for all $\zeta \in \overline{U}$ and $f(U \times \overline{U}) \subset H(U \times \overline{U})$.

(ii) If $H(z, \zeta) \equiv H(z)$ and $f(z, \zeta) \equiv f(z)$ then strong subordination becomes usual notion of subordination.

Lemma 1.3. [2, page 71] *Let $h(z, \zeta)$ be a convex function with $h(0, \zeta) = a$ for every $\zeta \in \overline{U}$ and let $\gamma \in \mathbb{C}^*$ be a complex number with* $\mathrm{Re}\, \gamma \geq 0$. *If $p \in \mathcal{H}^*[a, n, \zeta]$ and*

$$p(z, \zeta) + \frac{1}{\gamma} zp'(z, \zeta) \prec\prec h(z, \zeta) \tag{1.9}$$

then $p(z, \zeta) \prec\prec q(z, \zeta) \prec\prec h(z, \zeta)$ where

$$g(z, \zeta) = \frac{\gamma}{nz^{\gamma/n}} \int_0^z h(t, \zeta) t^{(\gamma/n)-1} dt. \tag{1.10}$$

The function $g(z, \zeta)$ is convex and is the best dominant.

Lemma 1.4. [1] *Let $g(z,\zeta)$ be a convex function in U ,for all $\zeta \in \overline{U}$ and let*

$$h(z,\zeta) = g(z,\zeta) + n\alpha g'(z,\zeta), \tag{1.11}$$

where $\alpha > 0$ and n is a positive integer. If

$$p(z,\zeta) = g(0,\zeta) + p_n(\zeta)z^n + \dots$$

is holomorphic in U, for all $\zeta \in \overline{U}$ and

$$p(z,\zeta) + \alpha z p'(z,\zeta) \prec\prec h(z,\zeta) \tag{1.12}$$

then

$$p(z,\zeta) \prec\prec g(z,\zeta) \tag{1.13}$$

and this result is sharp.

Definition 1.5. [5] For $f \in A^*_\zeta$, $n \in \mathbb{N}^* \cup \{0\}$, the operator $S^n f$ is defined by

$$S^n : A^*_\zeta \to A^*_\zeta$$

$$S^0 f(z,\zeta) = f(z,\zeta)$$

$$S^1 f(z,\zeta) = z f'(z,\zeta)$$

$$\dots$$

$$S^{n+1} f(z,\zeta) = z[S^n f(z,\zeta)]', \; z \in U, \; \zeta \in \overline{U}.$$

Remark 1.6. If $f \in A^*_\zeta$,

$$f(z,\zeta) = z + \sum_{j=2}^{\infty} a_j(\zeta) z^j$$

then

$$S^n f(z,\zeta) = z + \sum_{j=2}^{\infty} j^n a_j(\zeta) z^j, \quad z \in U, \; \zeta \in \overline{U}.$$

2. Main results

Definition 2.1. If $\alpha < 1$ and $m, n \in \mathbb{N}$, let $S_m^n(\alpha)$ denote the class of functions $f \in A_{n\zeta}^*$ which satisfy the inequality

$$\text{Re } [S^m f(z,\zeta)]' > \alpha. \tag{2.1}$$

Theorem 2.2. *If $\alpha < 1$ and $m, n \in \mathbb{N}$, then*

$$S_n^{m+1}(\alpha) \subset S_n^m(\delta) \tag{2.2}$$

where

$$\begin{gathered} \delta = \delta(\alpha, n, m) = (2\alpha - 1) + 1 - (2\alpha - 1)\frac{1}{n}\beta\left(\frac{1}{n}\right), \\ \beta(x) = \int_0^1 \frac{t^{x-1}}{1+t} dt. \end{gathered} \tag{2.3}$$

Proof. Let $f \in S_n^{m+1}(\alpha)$. By using the properties of the operator $S^m f(z,\zeta)$, we have

$$S^{m+1} f(z,\zeta) = z[S^m f(z,\zeta)]',\ z \in U,\ \zeta \in \overline{U}. \tag{2.4}$$

Differentiating (2.4) we obtain

$$[S^{m+1} f(z,\zeta)]' = [S^m f(z,\zeta)]' + z[S^m f(z,\zeta)]'', \quad z \in U,\ \zeta \in \overline{U}. \tag{2.5}$$

If we let $p(z,\zeta) = [S^m f(z,\zeta)]'$, then

$$p'(z,\zeta) = [S^m f(z,\zeta)]''$$

and (2.5) becomes

$$[S^{m+1} f(z,\zeta)]' = p(z,\zeta) + zp'(z,\zeta). \tag{2.6}$$

Since $f \in S_n^{m+1}(\alpha)$, by using Definition 2.1, we have

$$\text{Re } [p(z,\zeta) + zp'(z,\zeta)] > \alpha \tag{2.7}$$

which is equivalent to

$$p(z,\zeta) + zp'(z,\zeta) \prec\prec \frac{1 + (2\alpha - 1)z}{1+z} \equiv h(z,\zeta). \tag{2.8}$$

By using Lemma 1.3, we have

$$p(z,\zeta) \prec\prec g(z,\zeta) \prec\prec h(z,\zeta) \tag{2.9}$$

where

$$g(z,\zeta) = \frac{1}{nz^{1/n}} \int_0^z \frac{1-(2\alpha-1)t}{1+t} t^{(1/n)-1} dt. \tag{2.10}$$

The function $g(z,\zeta)$ is convex and is the best dominant.

From $p(z,\zeta) \prec\prec g(z,\zeta)$, it results that

$$\operatorname{Re} p(z,\zeta) > \delta = g(1,\zeta) = \delta(\alpha,n,m) \tag{2.11}$$

where

$$g(1,\zeta) = \frac{1}{n} \int_0^1 t^{\frac{1}{n}-1} \cdot \frac{1+(2\alpha-1)t}{1+t} dt \tag{2.12}$$

$$= \frac{1}{n} \int_0^1 t^{\frac{1}{n}-1} \cdot \frac{1+(2\alpha-1)t+(2\alpha-1)-(2\alpha-1)}{1+t} dt$$

$$= \frac{1}{n} \int_0^1 t^{\frac{1}{n}-1} \left[\frac{(2\alpha-1)(t+1)}{1+t} + \frac{1-2\alpha+1}{1+t}\right] dt$$

$$= (2\alpha-1)\frac{1}{n} \int_0^1 t^{\frac{1}{n}-1} dt + \frac{1}{n} \int_0^1 t^{\frac{1}{n}-1} \cdot \frac{1-(2\alpha-1)}{1+t} dt$$

$$= (2\alpha-1)\frac{1}{n} \cdot \frac{t^{\frac{1}{n}}}{\frac{1}{n}}\Big|_0^1 + \frac{1-(2\alpha-1)}{n} \int_0^1 \frac{t^{\frac{1}{n}-1}}{1+t} dt$$

$$= (2\alpha-1) + \frac{1-(2\alpha-1)}{n} \beta\left(\frac{1}{n}\right) \tag{2.13}$$

from which we deduce that $S_n^{m+1}(\alpha) \subset S_n^m(\delta)$. $\square$

Theorem 2.3. *Let $g(z,\zeta)$ be a convex function $g(0,\zeta)=1$ and let $h(z,\zeta)$ be a function such that*

$$h(z,\zeta) = g(z,\zeta) + zg'(z,\zeta). \tag{2.14}$$

If $f \in A^*_{n\zeta}$ and verifies the strong differential subordination

$$[S^{m+1} f(z,\zeta)]' \prec\prec h(z,\zeta) \tag{2.15}$$

then

$$[S^m f(z,\zeta)]' \prec\prec g(z,\zeta). \tag{2.16}$$

Proof. From

$$S^{m+1}f(z,\zeta) = z[S^m f(z,\zeta)]' \tag{2.17}$$

we obtain

$$[S^{m+1}f(z,\zeta)]' = [S^m f(z,\zeta)]' + z[S^m f(z,\zeta)]''. \tag{2.18}$$

If we let $p(z,\zeta) = [S^m f(z,\zeta)]'$, then we obtain

$$[S^{m+1}f(z,\zeta)]' = p(z,\zeta) + zp'(z,\zeta) \tag{2.19}$$

and (2.15) becomes

$$p(z,\zeta) + zp'(z,\zeta) \prec\prec g(z,\zeta) + zg'(z,\zeta) \equiv h(z,\zeta). \tag{2.20}$$

Using Lemma 1.4, we have

$$p(z,\zeta) \prec\prec g(z,\zeta), \text{ i.e., } S^m f(z,\zeta) \prec\prec g(z,\zeta). \tag{2.21}$$

□

Theorem 2.4. *Let $h \in \mathcal{H}^*[a,n,\zeta]$, with $h(0,\zeta) = 1$, $h'(0,\zeta) \neq 0$ which verifies the inequality*

$$\text{Re}\left[1 + \frac{zh''(z,\zeta)}{h'(z,\zeta)}\right] > -\frac{1}{2(m+1)}, \quad m \geq 0. \tag{2.22}$$

If $f \in A^*_{n\zeta}$ and verifies the strong differential subordination

$$[S^{m+1}f(z,\zeta)]' \prec\prec h(z,\zeta), \quad z \in U \tag{2.23}$$

then

$$[S^m f(z,\zeta)]' \prec\prec g(z,\zeta), \tag{2.24}$$

where

$$g(z,\zeta) = \frac{1}{nz^{1/n}} \int_0^z t^{(1/n)-1} h(t,\zeta)dt. \tag{2.25}$$

The function g is convex and is the best dominant.

Proof. From

$$S^{m+1}f(z,\zeta) = z[S^m f(z,\zeta)]' \tag{2.26}$$

we obtain

$$[S^{m+1}f(z,\zeta)]' = [S^m f(z,\zeta]' + z[S^m f(z,\zeta)]''. \tag{2.27}$$

If we let $p(z,\zeta) = [S^m f(z,\zeta)]'$, then we obtain

$$[S^{m+1} f(z,\zeta)]' = p(z,\zeta) + zp'(z,\zeta) \tag{2.28}$$

and (2.23) becomes

$$p(z,\zeta) + zp'(z,\zeta) \prec\prec h(z,\zeta). \tag{2.29}$$

By using Lemma 1.3 we have

$$p(z,\zeta) \prec\prec g(z,\zeta) = \frac{1}{nz^{1/n}} \int_0^z h(t,\zeta) t^{\frac{1}{n}-1} dt. \tag{2.30}$$

□

Theorem 2.5. *Let $g(z,\zeta)$ be a convex function with $g(0,\zeta) = 1$ and*

$$h(z,\zeta) = g(z,\zeta) + zg'(z,\zeta). \tag{2.31}$$

If $f \in A^*_{n\zeta}$ and verifies the differential subordination

$$[S^m f(z,\zeta)]' \prec\prec h(z,\zeta), \quad z \in U,\ \zeta \in \overline{U} \tag{2.32}$$

then

$$\frac{S^m f(z,\zeta)}{z} \prec\prec g(z,\zeta). \tag{2.33}$$

Proof. We let

$$p(z,\zeta) = \frac{S^m f(z,\zeta)}{z}, \quad z \in U,\ \zeta \in \overline{U},$$

we obtain

$$S^m f(z,\zeta) = zp(z,\zeta). \tag{2.34}$$

By differentiating, we obtain

$$[S^m f(z,\zeta)]' = p(z,\zeta) + zp'(z,\zeta), \quad z \in U,\ \zeta \in \overline{U}. \tag{2.35}$$

Then (2.32) becomes

$$p(z,\zeta) + zp'(z,\zeta) \prec\prec h(z,\zeta) = g(z,\zeta) + zg'(z,\zeta). \tag{2.36}$$

Using Lemma 1.4 we have

$$p(z,\zeta) \prec\prec g(z,\zeta).$$

□

References

[1] Miller, S. S., Mocanu, P. T., *On some classes of first-order differential subordinations*, Michigan Math. J., **32** (1985), no. 2, 185-195.

[2] Miller, S. S., Mocanu, P. T., *Differential Subordinations. Theory and Applications*, Monographs and Textbooks in Pure and Applied Mathematics, vol. 225, Marcel Dekker, New York, 2000.

[3] Oros, G. I., *On a new strong differential subordination* (to appear).

[4] Oros, G. I., Oros, Gh. *Strong differential subordination*, Turkish Journal of Mathematics, **33** (2009), 249-257.

[5] Sălăgean, G. S., *Subclasses of univalent functions*, Complex Analysis, Fift Romanian Finish Seminar, Part 1 (Bucharest, 1981), 362-372, Lecture Notes in Math., 1013, Springer, Berlin, 1983.

University of Oradea
Faculty of Environmental Protection
General Magheru Street
Oradea, Romania
E-mail address: adela_taut@yahoo.com

ON STARLIKENESS OF A CLASS OF INTEGRAL OPERATORS FOR MEROMORPHIC STARLIKE FUNCTIONS

ALINA TOTOI

Dedicated to Professor Grigore Ştefan Sălăgean on his 60^{th} *birthday*

Abstract. Let M_0 be the class of meromorphic functions in $\dot{U}$ of the form $g(z) = \frac{1}{z} + \alpha_0 + \alpha_1 z + \cdots$, $z \in \dot{U}$. For $\Phi, \varphi \in H[1,1]$, $\Phi(z)\varphi(z) \neq 0$, $z \in U$, $\alpha, \beta, \gamma, \delta \in \mathbb{C}$ with $\beta \neq 0$ and $g \in M_0$, we consider the integral operator $J^{\Phi,\varphi}_{\alpha,\beta,\gamma,\delta} : K \subset M_0 \to M_0$ defined by

$$J^{\Phi,\varphi}_{\alpha,\beta,\gamma,\delta}(g)(z) = \left[\frac{\gamma - \beta}{z^\gamma \Phi(z)} \int_0^z g^\alpha(t)\varphi(t)t^{\delta-1}dt\right]^{\frac{1}{\beta}}, \ z \in \dot{U}.$$

The first result of this paper gives us the conditions for which $J^{\Phi,\varphi}_{\alpha,\beta,\gamma,\delta}$ will be well-defined. Furthermore, we study the properties of a function $G = J_{\beta,\gamma}(g)$, where $J_{\beta,\gamma} = J^{1,1}_{\beta,\beta,\gamma,\gamma}$, when $g \in M_0^*(\alpha, \delta)$. For the second result we consider $\beta < 0$, $\gamma - \beta > 0$, $\alpha \in [\alpha_0, 1)$, where $\alpha_0 = \max\left\{\frac{\beta+\gamma+1}{2\beta}, \frac{\gamma}{\beta}\right\}$ and we find the order of starlikeness of the class $J_{\beta,\gamma}(M_0^*(\alpha))$. For the third result we consider $0 \leq \alpha < 1$, $0 < \beta < \gamma$ and we find some conditions for α, β, γ and $\delta = \delta(\alpha, \beta, \gamma)$ such that

$$J_{\beta,\gamma}[M_0^*(\alpha) \cap K_{\beta,\gamma}] \subset M_0^*(\delta).$$

1. Introduction and preliminaries

Let $U = \{z \in \mathbb{C} : |z| < 1\}$ be the unit disc in the complex plane, $\dot{U} = U \setminus \{0\}$ and $H(U) = \{f : U \to \mathbb{C} : f \text{ is holomorphic in } U\}$.

We will also use the following notations:

$$H[a, n] = \{f \in H(U) : f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \ldots\} \text{ for } a \in \mathbb{C},\ n \in \mathbb{N}^*,$$

$$A_n = \{f \in H(U) : f(z) = z + a_{n+1} z^{n+1} + a_{n+2} z^{n+2} + \ldots\},\ n \in \mathbb{N}^*,$$

Received by the editors: 26.04.2010.

2000 *Mathematics Subject Classification.* 30C45.

Key words and phrases. meromorphic starlike functions, integral operators.

and for $n = 1$ we denote A_1 by A and this set is called *the class of analytic functions normalized at the origin.*

Let S^* be the class of normalized starlike functions on U, i.e.

$$S^* = \left\{ f \in A : \operatorname{Re} \frac{zf'(z)}{f(z)} > 0,\ z \in U \right\}.$$

We denote by M_0 the class of meromorphic functions in $\dot{U}$ of the form

$$g(z) = \frac{1}{z} + \alpha_0 + \alpha_1 z + \cdots,\ z \in \dot{U}.$$

Let

$$M_0^* = \left\{ g \in M_0 : \operatorname{Re} \left[-\frac{zg'(z)}{g(z)} \right] > 0,\ z \in U \right\}$$

be called the class of meromorphic starlike functions in $\dot{U}$.

We note that if f is a normalized starlike function in U, then the function $g = \frac{1}{f}$ belongs to the class M_0^*.

For $\alpha < 1$, $\delta > 1$ let

$$M_0^*(\alpha) = \left\{ g \in M_0 : \operatorname{Re} \left[-\frac{zg'(z)}{g(z)} \right] > \alpha,\ z \in U \right\},$$

$$M_0^*(\alpha, \delta) = \left\{ g \in M_0 : \alpha < \operatorname{Re} \left[-\frac{zg'(z)}{g(z)} \right] < \delta,\ z \in U \right\}.$$

Definition 1.1. [3, p.4], [4, p.45] Let $f, g \in H(U)$. We say that the function f is subordinate to the function g, and we denote this by $f(z) \prec g(z)$, if there is a function $w \in H(U)$, with $w(0) = 0$ and $|w(z)| < 1$, $z \in U$, such that

$$f(z) = g[w(z)],\ z \in U.$$

Remark 1.2. If $f(z) \prec g(z)$, then $f(0) = g(0)$ and $f(U) \subseteq g(U)$.

Theorem 1.3. [3, p.4], [4, p.46] *Let $f, g \in H(U)$ and let g be a univalent function in U. Then $f(z) \prec g(z)$ if and only if $f(0) = g(0)$ and $f(U) \subseteq g(U)$.*

Definition 1.4. [3, p. 46], [4, p.228] Let $c \in \mathbb{C}$ with $\operatorname{Re} c > 0$ and $n \in \mathbb{N}^*$. We consider

$$C_n = C_n(c) = \frac{n}{\operatorname{Re} c} \left[|c| \sqrt{1 + \frac{2 \operatorname{Re} c}{n}} + \operatorname{Im} c \right].$$

If the univalent function $R : U \to \mathbb{C}$ is given by $R(z) = \dfrac{2C_n z}{1-z^2}$, then we will denote by $R_{c,n}$ the "Open Door" function, defined as

$$R_{c,n}(z) = R\left(\frac{z+b}{1+\bar{b}z}\right) = 2C_n \frac{(z+b)(1+\bar{b}z)}{(1+\bar{b}z)^2-(z+b)^2}\ ,$$

where $b = R^{-1}(c)$.

Theorem 1.5. [3, Theorem 2.5c.] *Let $\Phi, \varphi \in H[1,n]$ with $\Phi(z) \neq 0, \varphi(z) \neq 0$, for $z \in U$. Let $\alpha, \beta, \gamma, \delta \in \mathbb{C}$ with $\beta \neq 0$, $\alpha + \delta = \beta + \gamma$ and $\operatorname{Re}(\alpha+\delta) > 0$. Let the function $f(z) = z + a_{n+1}z^{n+1} + \cdots \in A_n$ and suppose that*

$$\alpha\frac{zf'(z)}{f(z)} + \frac{z\varphi'(z)}{\varphi(z)} + \delta \prec R_{\alpha+\delta,n}(z).$$

If $F = I^{\Phi,\varphi}_{\alpha,\beta,\gamma,\delta}(f)$ is defined by

$$F(z) = I^{\Phi,\varphi}_{\alpha,\beta,\gamma,\delta}(f)(z) = \left[\frac{\beta+\gamma}{z^{\gamma}\Phi(z)}\int_0^z f^{\alpha}(t)\varphi(t)t^{\delta-1}dt\right]^{\frac{1}{\beta}}, \tag{1.1}$$

then $F \in A_n$ with $\dfrac{F(z)}{z} \neq 0$, $z \in U$, and

$$\operatorname{Re}\left[\beta\frac{zF'(z)}{F(z)} + \frac{z\Phi'(z)}{\Phi(z)} + \gamma\right] > 0,\ z \in U.$$

All powers in (1.1) are principal ones.

Lemma 1.6. [3, Theorem 2.3i.], [4, p.209] *Let $\psi : \mathbb{C}^2 \times U \to \mathbb{C}$ be a function that satisfies the condition*

$$\operatorname{Re}\psi(\rho i, \sigma; z) \leq 0\,, \tag{1.2}$$

when $\rho, \sigma \in \mathbb{R}, \sigma \leq -\dfrac{n}{2}(1+\rho^2)$, $z \in U, n \geq 1$.

If $p \in H[1,n]$ and

$$\operatorname{Re}\psi(p(z), zp'(z); z) > 0, \quad z \in U,$$

then

$$\operatorname{Re} p(z) > 0, \quad z \in U.$$

Theorem 1.7. [3, Theorem 3.2a.], [4, p.247] *Let $\beta, \gamma \in \mathbb{C}$, $\beta \neq 0$ and let h be a convex function on U such that $\operatorname{Re}[\beta h(z) + \gamma] > 0$, $z \in U$. If $p \in H[h(0), n]$ and*

$$p(z) + \frac{zp'(z)}{\beta p(z) + \gamma} \prec h(z),$$

then $p(z) \prec h(z)$.

Theorem 1.8. [5], [4, p.299]**(the order of starlikeness of the class $I_{\beta,\gamma}(S^*(\alpha))$)** *Let $\beta > 0$, $\gamma + \beta > 0$ and consider the integral operator $I_{\beta,\gamma}$ defined by*

$$I_{\beta,\gamma}(f)(z) = \left[\frac{\gamma+\beta}{z^\gamma}\int_0^z t^{\gamma-1} f^\beta(t)dt\right]^{\frac{1}{\beta}}.$$

If $\alpha \in [\alpha_0, 1)$ where $\alpha_0 = \max\left\{\frac{\beta-\gamma-1}{2\beta}, -\frac{\gamma}{\beta}\right\}$, then the order of starlikeness of the class $I_{\beta,\gamma}(S^*(\alpha))$ is given by

$$\delta(\alpha;\beta,\gamma) = \frac{1}{\beta}\left[\frac{\gamma+\beta}{{}_2F_1(1, 2\beta(1-\alpha), \gamma+1+\beta; \frac{1}{2})} - \gamma\right],$$

where ${}_2F_1$ represents the hypergeometric function.

2. Main results

Let $\Phi, \varphi \in H[1,1]$ with $\Phi(z)\varphi(z) \neq 0$, $z \in U$ and let $\alpha, \beta, \gamma, \delta \in \mathbb{C}$ with $\beta \neq 0$. The first result of this section is a corollary of Theorem 1.5 and gives us the conditions for which the integral operator $J^{\Phi,\varphi}_{\alpha,\beta,\gamma,\delta} : K \subset M_0 \to M_0$,

$$J^{\Phi,\varphi}_{\alpha,\beta,\gamma,\delta}(g)(z) = \left[\frac{\gamma-\beta}{z^\gamma \Phi(z)}\int_0^z g^\alpha(t)\varphi(t)t^{\delta-1}dt\right]^{\frac{1}{\beta}},$$

is well-defined.

Theorem 2.1. *Let $\Phi, \varphi \in H[1,1]$ with $\Phi(z)\varphi(z) \neq 0$, $z \in U$. Let $\alpha, \beta, \gamma, \delta \in \mathbb{C}$ with $\beta \neq 0$, $\alpha + \gamma = \beta + \delta$ and $\mathrm{Re}\,(\gamma - \beta) > 0$. If $g \in M_0$ and*

$$\alpha\frac{zg'(z)}{g(z)} + \frac{z\varphi'(z)}{\varphi(z)} + \delta \prec R_{\delta-\alpha,1}(z), \tag{2.1}$$

then

$$G(z) = J^{\Phi,\varphi}_{\alpha,\beta,\gamma,\delta}(g)(z) = \left[\frac{\gamma-\beta}{z^\gamma \Phi(z)}\int_0^z g^\alpha(t)\varphi(t)t^{\delta-1}dt\right]^{\frac{1}{\beta}} \in M_0,$$

with $zG(z) \neq 0$, $z \in U$, and

$$\mathrm{Re}\left[\beta\frac{zG'(z)}{G(z)} + \frac{z\Phi'(z)}{\Phi(z)} + \gamma\right] > 0,\ z \in U.$$

All powers are chosen as principal ones.

Proof. We denote $\alpha_1 = -\alpha$, $\beta_1 = -\beta$, so we have $\gamma+\beta_1 = \delta+\alpha_1$, and $\mathrm{Re}\,(\gamma+\beta_1) > 0$. We remark that from (2.1) we have $zg(z) \neq 0$, $z \in U$.

We know that $g \in M_0$ with $zg(z) \neq 0$, $z \in U$, if and only if $f = \dfrac{1}{g} \in A_1$ with $\dfrac{f(z)}{z} \neq 0$, $z \in U$. It is also easy to see that $\dfrac{zg'(z)}{g(z)} = -\dfrac{zf'(z)}{f(z)}$, $z \in U$.

Using these new notations we obtain

$$\alpha_1 \frac{zf'(z)}{f(z)} + \frac{z\varphi'(z)}{\varphi(z)} + \delta \prec R_{\delta+\alpha_1,1}(z),\ z \in U,$$

and applying Theorem 1.5 we have

$$F(z) = I^{\Phi,\varphi}_{\alpha_1,\beta_1,\gamma,\delta}(f)(z) = \left[\frac{\beta_1+\gamma}{z^{\gamma}\Phi(z)} \int_0^z f^{\alpha_1}(t)\varphi(t)t^{\delta-1}dt\right]^{\frac{1}{\beta_1}} \in A_1,$$

with $\dfrac{F(z)}{z} \neq 0$, $z \in U$, and

$$\mathrm{Re}\left[\beta_1 \frac{zF'(z)}{F(z)} + \frac{z\Phi'(z)}{\Phi(z)} + \gamma\right] > 0,\ z \in U.$$

Therefore, we have $G(z) = \dfrac{1}{F(z)} \in M_0$ with $zG(z) \neq 0$ and, because

$$\frac{zG'(z)}{G(z)} = -\frac{zF'(z)}{F(z)},\ z \in U,$$

we also have

$$\mathrm{Re}\left[\beta \frac{zG'(z)}{G(z)} + \frac{z\Phi'(z)}{\Phi(z)} + \gamma\right] > 0,\ z \in U.$$

□

We next consider a special case of Theorem 2.1. If we let $\Phi \equiv \varphi \equiv 1$, $\alpha = \beta$, $\gamma = \delta$ and if we use the notation $J_{\beta,\gamma}$ instead of $J^{1,1}_{\beta,\beta,\gamma,\gamma}$, we obtain:

Corollary 2.2. *Let $\beta, \gamma \in \mathbb{C}$ with $\beta \neq 0$ and $\mathrm{Re}\,(\gamma-\beta) > 0$. If $g \in M_0$ and*

$$\beta \frac{zg'(z)}{g(z)} + \gamma \prec R_{\gamma-\beta,1}(z),$$

then

$$G(z) = J_{\beta,\gamma}(g)(z) = \left[\frac{\gamma-\beta}{z^{\gamma}} \int_0^z g^{\beta}(t)t^{\gamma-1}dt\right]^{\frac{1}{\beta}} \in M_0, \tag{2.2}$$

with $zG(z) \neq 0$, $z \in U$, and

$$\mathrm{Re}\left[\beta \frac{zG'(z)}{G(z)} + \gamma\right] > 0,\ z \in U.$$

Remark 2.3. 1. Let us define the classes $K_{\beta,\gamma}$ as

$$K_{\beta,\gamma} = \left\{ g \in M_0 : \gamma + \beta \frac{zg'(z)}{g(z)} \prec R_{\gamma-\beta,1}(z),\, z \in U \right\}.$$

From Corollary 2.2, we have $J_{\beta,\gamma} : K_{\beta,\gamma} \to M_0$ with $zJ_{\beta,\gamma}(g)(z) \neq 0$, $z \in U$, and

$$\mathrm{Re}\left[\gamma + \beta \frac{zJ'_{\beta,\gamma}(g)(z)}{J_{\beta,\gamma}(g)(z)}\right] > 0,\, z \in U.$$

2. We denote

$$\tilde{K}_{\beta,\gamma} = \left\{ g \in M_0 : \mathrm{Re}\left[\gamma + \beta \frac{zg'(z)}{g(z)}\right] > 0,\, z \in U \right\}.$$

Using the above corollary we have $J_{\beta,\gamma}(K_{\beta,\gamma}) \subset \tilde{K}_{\beta,\gamma}$, so $J_{\beta,\gamma}(\tilde{K}_{\beta,\gamma}) \subset \tilde{K}_{\beta,\gamma}$, where $\beta,\, \gamma \in \mathbb{C}$ with $\beta \neq 0$ and $\mathrm{Re}\,(\gamma - \beta) > 0$.

3. Let $\beta < 0$, $\gamma \in \mathbb{C}$ with $\mathrm{Re}\,\gamma > \beta$ and $\dfrac{\mathrm{Re}\,\gamma}{\beta} \leq \alpha < 1$. Then, from $J_{\beta,\gamma}(\tilde{K}_{\beta,\gamma}) \subset \tilde{K}_{\beta,\gamma}$, we deduce $J_{\beta,\gamma}(M_0^*(\alpha)) \subset M_0^*\left(\dfrac{\mathrm{Re}\,\gamma}{\beta}\right)$.

It's easy to see that from

$$G(z) = \left[\frac{\gamma - \beta}{z^\gamma} \int_0^z t^{\gamma-1} g^\beta(t)dt\right]^{\frac{1}{\beta}},\, z \in \dot{U},$$

we obtain

$$p(z) + \frac{zp'(z)}{\gamma - \beta p(z)} = -\frac{zg'(z)}{g(z)}, \quad \text{where} \quad p(z) = -\frac{zG'(z)}{G(z)},\, z \in U. \tag{2.3}$$

Next we will study the properties of the image of a function $g \in M_0^*(\alpha, \delta)$ through the integral operator $J_{\beta,\gamma}$ defined by (2.2).

Theorem 2.4. *Let* $\beta > 0$, $\gamma \in \mathbb{C}$ *and* $0 \leq \alpha < 1 < \delta \leq \dfrac{\mathrm{Re}\,\gamma}{\beta}$.
If $g \in M_0^*(\alpha, \delta)$, *then* $G = J_{\beta,\gamma}(g) \in M_0^*(\alpha, \delta)$.

Proof. We know that $g \in M_0^*(\alpha, \delta)$ is equivalent to

$$\alpha < \mathrm{Re}\left[-\frac{zg'(z)}{g(z)}\right] < \delta,\, z \in U,$$

so,

$$\mathrm{Re}\,\gamma - \beta\delta < \mathrm{Re}\left[\gamma + \beta\frac{zg'(z)}{g(z)}\right] < \mathrm{Re}\,\gamma - \beta\alpha,\, z \in U, \quad \text{when} \quad \beta > 0.$$

Because $\delta \leq \dfrac{\mathrm{Re}\,\gamma}{\beta}$ we get $\mathrm{Re}\left[\gamma + \beta\dfrac{zg'(z)}{g(z)}\right] > 0$, $z \in U$, and using Corollary 2.2, we obtain that $G = J_{\beta,\gamma}(g) \in M_0$, $zG(z) \neq 0$, $z \in U$, and $\mathrm{Re}\left[\gamma + \beta\dfrac{zG'(z)}{G(z)}\right] > 0$, $z \in U$.

From (2.3) we know that

$$p(z) + \frac{zp'(z)}{\gamma - \beta p(z)} = -\frac{zg'(z)}{g(z)}, \quad \text{where} \quad p(z) = -\frac{zG'(z)}{G(z)}.$$

Since $G \in M_0$ with $zG(z) \neq 0$, $z \in U$, we have $p(z) = -\dfrac{zG'(z)}{G(z)} \in H[1,1]$.
It's not difficult to see that there is a convex function q on U such that $q(U) = \{z \in \mathbb{C} : \alpha < \operatorname{Re} z < \delta\}$ and $q(0) = 1$, so

$$g \in M_0^*(\alpha, \delta) \Rightarrow -\frac{zg'(z)}{g(z)} \prec q(z).$$

Now we have

$$p(z) + \frac{zp'(z)}{\gamma - \beta p(z)} \prec q(z)\,, \text{with } q \text{ convex on } U,\ q(0) = 1.$$

We want to apply Theorem 1.7 to the above differential subordination, so we need to see that $\operatorname{Re}[\gamma - \beta q(z)] > 0$, $z \in U$.
Since $\beta > 0$, we obtain from $\alpha < \operatorname{Re} q(z) < \delta$, $z \in U$, that

$$\operatorname{Re}\gamma - \beta\delta < \operatorname{Re}[\gamma - \beta q(z)] < \operatorname{Re}\gamma - \beta\alpha,\ z \in U.$$

Because $\delta \leq \dfrac{\operatorname{Re}\gamma}{\beta}$ we have $\operatorname{Re}[\gamma - \beta q(z)] > 0$, $z \in U$, and using Theorem 1.7 we obtain $p(z) \prec q(z)$, which is equivalent to

$$-\frac{zG'(z)}{G(z)} \prec q(z),\ z \in U. \tag{2.4}$$

Since $G \in M_0$, we get from (2.4) that $G \in M_0^*(\alpha, \delta)$. □

Taking $\beta = 1$ in the above theorem we obtain:

Corollary 2.5. *Let $\gamma \in \mathbb{C}$ and $0 \leq \alpha < 1 < \delta \leq \operatorname{Re}\gamma$. If $g \in M_0^*(\alpha, \delta)$, then*

$$G = J_{1,\gamma}(g) = \frac{\gamma - 1}{z^\gamma} \int_0^z t^{\gamma-1} g(t)dt \in M_0^*(\alpha, \delta).$$

Theorem 2.6. *Let $\beta < 0$, $\gamma \in \mathbb{C}$ and $\dfrac{\operatorname{Re}\gamma}{\beta} \leq \alpha < 1 < \delta$.*
If $g \in M_0^*(\alpha, \delta)$, then $G = J_{\beta,\gamma} \in M_0^*(\alpha, \delta)$.

Proof. From Remark 2.3 item 3., we have $J_{\beta,\gamma}(M_0^*(\alpha)) \subset M_0^*\left(\dfrac{\operatorname{Re}\gamma}{\beta}\right)$, hence $G = J_{\beta,\gamma}(g) \in M_0^*\left(\dfrac{\operatorname{Re}\gamma}{\beta}\right)$. Since $G \in M_0^*\left(\dfrac{\operatorname{Re}\gamma}{\beta}\right)$, we have $G \in M_0$ and $zG(z) \neq 0$, $z \in U$, so $-\dfrac{zG'(z)}{G(z)} \in H[1,1]$.

Because $g \in M_0^*(\alpha, \delta)$ and

$$p(z) + \frac{zp'(z)}{\gamma - \beta p(z)} = -\frac{zg'(z)}{g(z)}, \quad \text{where} \quad p(z) = -\frac{zG'(z)}{G(z)},$$

we will use the same idea as at the proof of Theorem 2.4. So, we have to see that $\operatorname{Re}[\gamma - \beta q(z)] > 0$, $z \in U$, where q is convex on U, $q(0) = 1$, $q(U) = \{z \in \mathbb{C} : \alpha < \operatorname{Re} z < \delta\}$.

From $\operatorname{Re} q(z) > \alpha$, $z \in U$, we obtain $\operatorname{Re}\gamma - \beta \operatorname{Re} q(z) > \operatorname{Re}\gamma - \alpha\beta \geq 0$, $z \in U$, when $\alpha \geq \frac{\operatorname{Re}\gamma}{\beta}$, $\beta < 0$.

Applying Theorem 1.7 to the differential subordination

$$p(z) + \frac{zp'(z)}{\gamma - \beta p(z)} \prec q(z),\ z \in U,$$

we obtain $p(z) \prec q(z)$, which is equivalent to

$$-\frac{zG'(z)}{G(z)} \prec q(z),\ z \in U. \tag{2.5}$$

Since $G \in M_0$, we get from (2.5) that $G \in M_0^*(\alpha, \delta)$. □

Remark 2.7. If we consider $\delta \to \infty$ in the above theorem, we obtain that for $\beta < 0$, $\gamma \in \mathbb{C}$, $\beta < \operatorname{Re}\gamma$ and $\frac{\operatorname{Re}\gamma}{\beta} \leq \alpha < 1$,

$$g \in M_0^*(\alpha) \Rightarrow G = J_{\beta,\gamma}(g) \in M_0^*(\alpha).$$

Definition 2.8. For a given number $\alpha \in \left[\frac{\operatorname{Re}\gamma}{\beta}, 1\right)$, where $\beta < 0$, $\gamma \in \mathbb{C}$, $\beta < \operatorname{Re}\gamma$, we define the order of starlikeness of the class $J_{\beta,\gamma}(M_0^*(\alpha))$ as the biggest number $\mu = \mu(\alpha; \beta, \gamma)$ such that $J_{\beta,\gamma}(M_0^*(\alpha)) \subset M_0^*(\mu)$.

Theorem 2.9. (the order of starlikeness of the class $J_{\beta,\gamma}(M_0^*(\alpha))$) *Let $\beta < 0, \gamma - \beta > 0$ and let $J_{\beta,\gamma}$ be given by (2.2). If $\alpha \in [\alpha_0, 1)$, where $\alpha_0 = \max\left\{\frac{\beta + \gamma + 1}{2\beta}, \frac{\gamma}{\beta}\right\}$, then the order of starlikeness of the class $J_{\beta,\gamma}(M_0^*(\alpha))$ is given by*

$$\mu(\alpha; \beta, \gamma) = -\frac{1}{\beta}\left[\frac{\gamma - \beta}{{}_2F_1(1, 2\beta(\alpha - 1), \gamma + 1 - \beta; \frac{1}{2})} - \gamma\right],$$

where ${}_2F_1$ represents the hypergeometric function.

Proof. We know that if $g \in M_0$ with $zg(z) \neq 0$, $z \in U$, then $\frac{1}{g} \in A$.
It's not difficult to see that

$$J_{\beta,\gamma}(g) = \frac{1}{I_{-\beta,\gamma}\left(\frac{1}{g}\right)}, \ \beta < 0, \ g \in M_0^*(\alpha).$$

Using the fact that $g \in M_0^*(\alpha)$ is equivalent to $\frac{1}{g} \in S^*(\alpha)$, we obtain from Theorem 1.8 that

$$I_{-\beta,\gamma}(S^*(\alpha)) \subset S^*(\delta(\alpha; -\beta, \gamma)),$$

so

$$J_{\beta,\gamma}(M_0^*(\alpha)) \subset M_0^*(\delta(\alpha; -\beta, \gamma)).$$

It's easy to prove that $\delta(\alpha; -\beta, \gamma)$ is the largest number μ such that $J_{\beta,\gamma}(M_0^*(\alpha)) \subset M_0^*(\mu)$, so the order of starlikeness of the class $J_{\beta,\gamma}(M_0^*(\alpha)))$ is $\mu(\alpha; \beta, \gamma) = \delta(\alpha; -\beta, \gamma)$. □

Further we will find some conditions for α, β, γ and $\delta = \delta(\alpha, \beta, \gamma)$ such that

$$J_{\beta,\gamma}[M_0^*(\alpha) \cap K_{\beta,\gamma}] \subset M_0^*(\delta).$$

Theorem 2.10. *Let $0 \leq \alpha < 1$ and $0 < \beta < \gamma$. Let's denote*

$$\beta_1(\alpha, \gamma) = \frac{2\sqrt{2\gamma(\alpha-1)^2 + \alpha} - \alpha - 1}{2(\alpha-1)^2},$$

$$\delta_1(\alpha, \beta, \gamma) = \frac{2\alpha\beta + 2\gamma + 1 - \sqrt{(1 + 2\alpha\beta - 2\gamma)^2 + 8(\gamma - \beta)}}{4\beta},$$

$$\delta_2(\alpha, \beta, \gamma) = \frac{2\alpha\beta + 2\beta + 1 - \sqrt{(1 + 2\alpha\beta - 2\beta)^2 + 8(\beta - \gamma)}}{4\beta}.$$

If $\gamma > \frac{1}{8}$ and $\beta < \beta_1(\alpha, \gamma)$, then $J_{\beta,\gamma}[M_0^*(\alpha) \cap K_{\beta,\gamma}] \subset M_0^*(\delta_1(\alpha, \beta, \gamma))$.
If $\gamma \leq \frac{1}{8}$ or $\begin{cases} \gamma > \frac{1}{8} \\ \beta \geq \beta_1(\alpha, \gamma) \end{cases}$, then $J_{\beta,\gamma}[M_0^*(\alpha) \cap K_{\beta,\gamma}] \subset M_0^*(\delta(\alpha, \beta, \gamma))$, where

$$\delta(\alpha, \beta, \gamma) = \min\{\delta_1(\alpha, \beta, \gamma), \delta_2(\alpha, \beta, \gamma)\}. \tag{2.6}$$

The operator $J_{\beta,\gamma}$ is defined by (2.2).

Proof. We remark that $\beta_1(\alpha, \gamma)$ is a real number and it is the greatest root for the equation

$$\Delta_2 = (1 + 2\alpha\beta - 2\beta)^2 + 8(\beta - \gamma) = 4(\alpha-1)^2\beta^2 + 4\beta(\alpha+1) + 1 - 8\gamma = 0,$$

hence $\Delta_2 \geq 0$, when $\beta \geq \beta_1(\alpha, \gamma)$.

It's not difficult to see that

$$\beta_1(\alpha, \gamma) \geq 0 \Leftrightarrow (8\gamma - 1)(\alpha - 1)^2 \geq 0 \Leftrightarrow \gamma \geq \frac{1}{8}.$$

We next verify that the number $\delta_1(\alpha, \beta, \gamma)$ is less than 1. It's obvious that $\delta_1(\alpha, \beta, \gamma)$ is a real number since $\gamma - \beta > 0$. Further we will use the notation δ_1 instead of $\delta_1(\alpha, \beta, \gamma)$.

We have $\delta_1 < 1$ if and only if

$$2\alpha\beta + 2\gamma + 1 - 4\beta < \sqrt{(1 + 2\alpha\beta - 2\gamma)^2 + 8(\gamma - \beta)}. \tag{2.7}$$

If $2\alpha\beta + 2\gamma + 1 - 4\beta < 0$ then the inequality (2.7) is fulfilled.
If $2\alpha\beta + 2\gamma + 1 - 4\beta \geq 0$, we use the square of the inequality (2.7) and after a simple computation, we obtain that (2.7) is equivalent to $(\beta - \gamma)(1 - \alpha) < 0$ which is true for $\beta < \gamma$ and $\alpha \in [0, 1)$. Thus, we have $\delta_1 < 1$.
Since $g \in K_{\beta,\gamma}$, with $\beta < \gamma$, we have from Corollary 2.2 that $zG(z) = zJ_{\beta,\gamma}(g)(z) \neq 0$, $z \in U$. Now let us put

$$-\frac{zG'(z)}{G(z)} = (1 - \delta)p(z) + \delta, \; z \in U, \tag{2.8}$$

where $p \in H(U)$ with $p(0) = 1$ and $\delta < 1$. We remark that the function p also depends on δ.
Using (2.8) and the logarithmic differential for (2.2), we obtain

$$-\frac{zg'(z)}{g(z)} - \alpha = (1 - \delta)p(z) + \delta - \alpha + \frac{(1 - \delta)zp'(z)}{\gamma - \beta\delta - (1 - \delta)\beta p(z)}, \; z \in U.$$

Let us denote

$$\psi(p(z), zp'(z); z) = (1 - \delta)p(z) + \delta - \alpha + \frac{(1 - \delta)zp'(z)}{\gamma - \beta\delta - (1 - \delta)\beta p(z)}, \; z \in U.$$

Since $g \in M_0^*(\alpha)$, we have $\mathrm{Re}\left[-\frac{zg'(z)}{g(z)}\right] > \alpha$, so

$$\mathrm{Re}\,\psi(p(z), zp'(z); z) > 0, \; z \in U.$$

To be able to use Lemma 1.6 we need to verify the condition (1.2) for $n = 1$.
For $\rho \in \mathbb{R}$, $z \in U$ and $\sigma \leq -\frac{1}{2}(1 + \rho^2)$, we have

$$\mathrm{Re}\,\psi(i\rho, \sigma; z) = \delta - \alpha + (1 - \delta)\sigma \mathrm{Re}\,\frac{1}{\gamma - \beta\delta - (1 - \delta)\beta\rho i} = \tag{2.9}$$

$$= \delta - \alpha + \frac{(\gamma - \beta\delta)(1-\delta)\sigma}{(\gamma - \beta\delta)^2 + (1-\delta)^2\beta^2\rho^2}.$$

Because $(\gamma - \beta\delta)(1-\delta) > 0$ and $\sigma \leq -\frac{1}{2}(1+\rho^2)$, we obtain from (2.9) that

$$\operatorname{Re}\psi(i\rho, \sigma; z) \leq \delta - \alpha - \frac{(\gamma - \beta\delta)(1-\delta)}{2[(\gamma - \beta\delta)^2 + (1-\delta)^2\beta^2\rho^2]}.$$

Thus,

$$\operatorname{Re}\psi(i\rho, \sigma; z) \leq -\frac{1}{D}(A + B\rho^2),\ \rho \in \mathbb{R},$$

where

$$A = (\gamma - \beta\delta)[2\beta\delta^2 - (1 + 2\gamma + 2\alpha\beta)\delta + 2\alpha\gamma + 1],$$

$$B = (1-\delta)[2\beta^2\delta^2 - \beta(1 + 2\beta + 2\alpha\beta)\delta + 2\alpha\beta^2 + \gamma],$$

$$D = 2[(\gamma - \beta\delta)^2 + (1-\delta)^2\beta^2\rho^2] > 0.$$

If $\gamma > \frac{1}{8}$ and $0 < \beta < \beta_1(\alpha, \gamma)$, then $\Delta_2 < 0$, so $B > 0$ for every $\delta \in \mathbb{R}$. Moreover, since $\beta > 0$, we have $A \geq 0$ when $\delta \leq \delta_1(\alpha, \beta, \gamma)$. Hence, the condition (1.2) is satisfied for $\delta \leq \delta_1(\alpha, \beta, \gamma) < 1$ and applying Lemma 1.6 we obtain $\operatorname{Re} p(z) > 0$, $z \in U$, when $\delta \leq \delta_1(\alpha, \beta, \gamma)$.

From (2.8) and $\operatorname{Re} p(z) > 0$, $z \in U$, when $\delta \leq \delta_1(\alpha, \beta, \gamma)$, we get $G \in M_0^*(\delta_1(\alpha, \beta, \gamma))$.

If $\gamma \leq \frac{1}{8}$ or $\begin{cases} \gamma > \frac{1}{8} \\ \beta \geq \beta_1(\alpha, \gamma) \end{cases}$ and $\delta \leq \delta(\alpha, \beta, \gamma)$, where $\delta(\alpha, \beta, \gamma)$ is given by (2.6), then $A \geq 0$ and $B \geq 0$, therefore the condition (1.2) is satisfied. Applying Lemma 1.6 we obtain $\operatorname{Re} p(z) > 0$, $z \in U$, for all $\delta \leq \delta(\alpha, \beta, \gamma)$, so $G \in M_0^*(\delta(\alpha, \beta, \gamma))$. □

We see that if we consider, in the above theorem, the condition $zG(z) = zJ_{\alpha,\beta}(g)(z) \neq 0$, $z \in U$, we get:

Theorem 2.11. *Let $0 \leq \alpha < 1$, $0 < \beta < \gamma$, $g \in M_0^*(\alpha)$ and $G(z) = J_{\alpha,\beta}(g)(z)$, where the operator $J_{\beta,\gamma}$ is defined by (2.2). Suppose that $zG(z) \neq 0$, $z \in U$. Let's denote*

$$\beta_1(\alpha, \gamma) = \frac{2\sqrt{2\gamma(\alpha-1)^2 + \alpha} - \alpha - 1}{2(\alpha-1)^2},$$

$$\delta_1(\alpha, \beta, \gamma) = \frac{2\alpha\beta + 2\gamma + 1 - \sqrt{(1 + 2\alpha\beta - 2\gamma)^2 + 8(\gamma - \beta)}}{4\beta},$$

$$\delta_2(\alpha, \beta, \gamma) = \frac{2\alpha\beta + 2\beta + 1 - \sqrt{(1 + 2\alpha\beta - 2\beta)^2 + 8(\beta - \gamma)}}{4\beta}.$$

If $\gamma > \dfrac{1}{8}$ *and* $\beta < \beta_1(\alpha, \gamma)$*, then* $G \in M_0^*(\delta_1(\alpha, \beta, \gamma))$.

If $\gamma \leq \dfrac{1}{8}$ *or* $\begin{cases} \gamma > \dfrac{1}{8} \\ \beta \geq \beta_1(\alpha, \gamma) \end{cases}$*, then* $G \in M_0^*(\delta(\alpha, \beta, \gamma))$*, where*

$$\delta(\alpha, \beta, \gamma) = \min\{\delta_1(\alpha, \beta, \gamma), \delta_2(\alpha, \beta, \gamma)\}.$$

The properties of the integral operator $J_{1,\gamma}$, were studied by many authors in different papers, from which we remember [1], [2], [6], [7], [8].

References

[1] Bajpai, S. K., *A note on a class of meromorphic univalent functions*, Rev. Roum. Math. Pures Appl., **22** (1977), 295-297.

[2] Goel, R. M., Sohi, N. S., *On a class of meromorphic functions*, Glas. Mat. Ser. III, **17(37)** (1981), 19-28.

[3] Miller, S. S., Mocanu, P. T., *Differential subordinations. Theory and applications*, Marcel Dekker Inc. New York, Basel, 2000.

[4] Mocanu, P. T., Bulboacă, T., Sălăgean, G. S., *The geometric theory of univalent functions*, (in Romanian), Casa Cărţii de Ştiinţă, Cluj-Napoca, 2006.

[5] Mocanu, P. T., Ripeanu, D., Şerb, I., *The order of starlikeness of certain integral operators*, Mathematica (Cluj), **23(46)** (1981), no. 2, 225-230.

[6] Mocanu, P. T., Sălăgean, G. S., *Integral operators and meromorphic starlike functions*, Mathematica (Cluj), **32(55)** (1990), no. 2, 147-152.

[7] Reddy, T. R., Juneja, O. P., *Integral operators on a class of meromorphic functions*, C. R. Acad. Bulgare Sci., **40** (1987), 21-23.

[8] Sălăgean, G. S., *Meromorphic starlike univalent functions*, Babeş Bolyai Univ., Fac. Math. and Phys. Res. Sem., Itinerant Seminar on Functional Equations, Approximations and Convexity, Preprint **7** (1986), 261-266.

Department of Mathematics, Faculty of Science
University "Lucian Blaga" Sibiu, Romania
E-mail address: totoialina@yahoo.com

STUDIA UNIV. "BABEŞ–BOLYAI", MATHEMATICA, Volume **LV**, Number 3, September 2010

ASYMPTOTIC BEHAVIOR OF INTERMEDIATE POINTS IN CERTAIN MEAN VALUE THEOREMS. II

TIBERIU TRIF

Dedicated to Professor Grigore Ştefan Sălăgean on his 60^{th} *birthday*

Abstract. The paper deals with the asymptotic behavior of the intermediate points in the mean value theorems for integrals as the involved interval shrinks to zero.

1. **Introduction**

Especially in the last two decades a great deal of work has been done in connection with the asymptotic behavior of intermediate points in certain mean value theorems (see, for instance, [1], [2], [3], [5], [9], [12], [13], [14]). The investigations in this direction started with the paper by Azpeitia [3], dealing with the asymptotic behavior of the intermediate point in the Lagrange-Taylor mean value theorem. A significant step forward was realized by Abel [1], who obtained a complete asymptotic expansion of the intermediate point in the Lagrange-Taylor mean value theorem when the length of the involved interval approaches zero. Later, following Abel's method of proof, similar complete asymptotic expansions have been obtained by several authors for other mean value theorems (Abel and Ivan [2] for the differential mean value theorem of divided differences, Xu, Cui and Hu [13] for the differential mean value theorem of divided differences with repetitions, Trif [12] for the Pawlikowska mean value theorem).

The purpose of the present paper is to continue our investigations started in [12]. But unlike the paper [12], here we deal with the asymptotic behavior of the

Received by the editors: 02.03.2010.

2000 *Mathematics Subject Classification.* 26A06.

Key words and phrases. mean value theorems for integrals, asymptotic approximations.

intermediate points in the mean value theorems for integrals as the involved interval shrinks to zero. For the reader's convenience we recall first the two mean value theorems for integrals.

Theorem 1.1 (first mean value theorem for integrals)**.** *If $f : [a,b] \to \mathbb{R}$ is a continuous function and $g : [a,b] \to [0,\infty)$ is a nonnegative Riemann integrable function, then there is a number $c \in [a,b]$ such that*

$$\int_a^b f(t)g(t)dt = f(c)\int_a^b g(t)dt.$$

Corollary 1.2. *If $f : [a,b] \to \mathbb{R}$ is a continuous function, then there is a number $c \in [a,b]$ such that*

$$\int_a^b f(t)dt = f(c)(b-a).$$

Theorem 1.3 (second mean value theorem for integrals)**.** *If $f : [a,b] \to \mathbb{R}$ is monotone and $g : [a,b] \to \mathbb{R}$ is Riemann integrable on $[a,b]$, then there is a number $c \in [a,b]$ such that*

$$\int_a^b f(t)g(t)dt = f(a)\int_a^c g(t)dt + f(b)\int_c^b g(t)dt.$$

The second mean value theorem for integrals is instrumental in theories like trigonometric series or Laplace transforms (see [8] for a proof and [11] for an interesting application of Theorem 1.3).

If $x \in (a,b)$, then Theorem 1.1, Corollary 1.2 and Theorem 1.3 applied to the interval $[a,x]$ instead of $[a,b]$ yield the existence of numbers $c_x \in [a,b]$ as functions of x on (a,b) such that

$$\int_a^x f(t)g(t)dt = f(c_x)\int_a^x g(t)dt, \tag{1.1}$$

$$\int_a^x f(t)dt = f(c_x)(x-a), \tag{1.2}$$

and

$$\int_a^x f(t)g(t)dt = f(a)\int_a^{c_x} g(t)dt + f(x)\int_{c_x}^x g(t)dt, \tag{1.3}$$

respectively.

Zhang [14, Theorem 4] proved that the point c_x in (1.2) satisfies

$$\lim_{x\to a}\frac{c_x - a}{x-a} = \frac{1}{\sqrt[n]{n+1}}, \tag{1.4}$$

provided that f is continuous on $[a,b]$ and n times differentiable at a with $f^{(j)}(a)=0$ $(1\le j\le n-1)$ and $f^{(n)}(a)\ne 0$. In the special case when $n=1$, an earlier result obtained by Jacobson [7] is recovered.

In section 2 of our paper we obtain a formula which is similar to (1.4), but involves the asymptotic behavior of the point c_x in the mean value formula (1.1). The asymptotic behavior of the point c_x in the mean value formula (1.3) is investigated in section 3.

2. Asymptotic behavior of the intermediate point in the first mean value theorem for integrals

In the proofs of the main results in this and the next section we need the following

Lemma 2.1. *If p is a nonnegative integer and $\omega:[a,b]\to\mathbb{R}$ is a continuous function such that $\omega(t)\to 0$ as $t\searrow a$, then*

$$\int_a^x \omega(t)(t-a)^p dt = o((x-a)^{p+1}) \qquad (x\searrow a).$$

Proof. Indeed, for every $x\in(a,b)$ by Theorem 1.1 there exists $c_x\in[a,x]$ such that

$$\int_a^x \omega(t)(t-a)^p dt = \omega(c_x)\int_a^x (t-a)^p dt = \frac{\omega(c_x)}{p+1}(x-a)^{p+1}.$$

Since $\omega(c_x)\to 0$ as $x\searrow a$, we obtain the conclusion. □

Theorem 2.2. *Suppose that $f,g:[a,b]\to\mathbb{R}$ are two functions satisfying the following conditions:*

(i) *f is continuous on $[a,b]$ and there is a positive integer n such that f is n times differentiable at a with $f^{(j)}(a)=0$ for $1\le j\le n-1$ and $f^{(n)}(a)\ne 0$;*

(ii) *g is nonnegative, Riemann integrable on $[a,b]$ and there is a nonnegative integer k such that g is k times differentiable at a with $g^{(j)}(a)=0$ for $0\le j\le k-1$ and $g^{(k)}(a)\ne 0$.*

Then the point c_x *in* (1.1) *satisfies*

$$\lim_{x\to a}\frac{c_x-a}{x-a}=\sqrt[n]{\frac{k+1}{n+k+1}}\,. \tag{2.1}$$

Proof. Without loosing the generality we may assume that $f(a)=0$. Indeed, otherwise we replace f by the function $t\in[a,b]\mapsto f(t)-f(a)$. Note that if c_x satisfies (1.1), then c_x satisfies also

$$\int_a^x (f(t)-f(a))g(t)dt=(f(c_x)-f(a))\int_a^x g(t)dt.$$

By the Taylor expansions of f and g we have

$$\begin{aligned} f(t) &= \frac{f^{(n)}(a)}{n!}\,(t-a)^n+\omega(t)(t-a)^n,\\ g(t) &= \frac{g^{(k)}(a)}{k!}\,(t-a)^k+\varepsilon(t)(t-a)^k, \end{aligned}$$

where ω and ε are continuous functions on $[a,b]$ satisfying $\omega(t)\to 0$ and $\varepsilon(t)\to 0$ as $t\searrow a$. Therefore we have

$$f(t)g(t)=\frac{f^{(n)}(a)g^{(k)}(a)}{n!\,k!}\,(t-a)^{n+k}+\gamma(t)(t-a)^{n+k},$$

where γ is continuous on $[a,b]$ and $\gamma(t)\to 0$ as $t\searrow a$. By Lemma 2.1 we deduce that

$$\int_a^x f(t)g(t)dt=\frac{f^{(n)}(a)g^{(k)}(a)}{n!\,k!\,(n+k+1)}\,(x-a)^{n+k+1}+o((x-a)^{n+k+1}) \tag{2.2}$$

as $x\searrow a$. By Lemma 2.1 we have also

$$\int_a^x g(t)dt=\frac{g^{(k)}(a)}{(k+1)!}\,(x-a)^{k+1}+o((x-a)^{k+1})\qquad (x\searrow a).$$

Since

$$f(c_x)=\frac{f^{(n)}(a)}{n!}\,(c_x-a)^n+\omega(c_x)(c_x-a)^n$$

and $0\le c_x-a\le x-a$, it follows that

$$f(c_x)\int_a^x g(t)dt=\frac{f^{(n)}(a)g^{(k)}(a)}{n!\,(k+1)!}\,(x-a)^{k+1}(c_x-a)^n+o((x-a)^{n+k+1}) \tag{2.3}$$

as $x \searrow a$. By (1.1), (2.2) and (2.3) we conclude that

$$\begin{aligned}\frac{f^{(n)}(a)g^{(k)}(a)}{n!\,(k+1)!}\,(x-a)^{k+1}(c_x-a)^n \\ = \frac{f^{(n)}(a)g^{(k)}(a)}{n!\,k!\,(n+k+1)}\,(x-a)^{n+k+1} + o((x-a)^{n+k+1}) \qquad (x \searrow a).\end{aligned}$$

Multiplying both sides by $n!\,(k+1)!(x-a)^{-(n+k+1)}/(f^{(n)}(a)g^{(k)}(a))$ we get

$$\left(\frac{c_x-a}{x-a}\right)^n = \frac{k+1}{n+k+1} + o(1) \qquad (x \searrow a),$$

whence the conclusion (2.1). □

Note that if $g(t) = 1$ for all $t \in [a,b]$, then (ii) is satisfied for $k = 0$. In this case (1.1) becomes (1.2) and (2.1) becomes (1.4), i.e., we recover Zhang's result mentioned in the introduction as a special case of Theorem 2.2.

3. Asymptotic behavior of the intermediate point in the second mean value theorem for integrals

Theorem 3.1. *Suppose that $f, g : [a,b] \to \mathbb{R}$ are two functions satisfying the following conditions:*

(i) *f is monotone and there is a positive integer n such that f is n times differentiable at a with $f^{(j)}(a) = 0$ for $1 \le j \le n-1$ and $f^{(n)}(a) \neq 0$;*

(ii) *g is Riemann integrable on $[a,b]$ and there is a nonnegative integer k such that g is k times differentiable at a with $g^{(j)}(a) = 0$ for $0 \le j \le k-1$ and $g^{(k)}(a) \neq 0$.*

Then the point c_x in (1.3) satisfies

$$\lim_{x\to a} \frac{c_x-a}{x-a} = \sqrt[k+1]{\frac{n}{n+k+1}}\,.$$

Proof. Note that (1.3) is equivalent to

$$\int_a^x (f(t)-f(a))g(t)dt = (f(x)-f(a))\int_{c_x}^x g(t)dt.$$

So, without loosing the generality we may assume that $f(a)=0$ (otherwise we replace f by the function $t\in[a,b]\mapsto f(t)-f(a)$). Under the assumption that $f(a)=0$ equality (1.3) becomes

$$\int_a^x f(t)g(t)dt = f(x)\int_{c_x}^x g(t)dt. \tag{3.1}$$

By using the Taylor expansions of f and g and proceeding as in the proof of Theorem 2.2 we deduce that (2.2) holds and that

$$\begin{aligned} f(x)\int_{c_x}^x g(t)dt &= \frac{f^{(n)}(a)g^{(k)}(a)}{n!\,(k+1)!}(x-a)^n\left[(x-a)^{k+1}-(c_x-a)^{k+1}\right] \\ &\quad + o((x-a)^{n+k+1}) \qquad (x\searrow a). \end{aligned} \tag{3.2}$$

By (3.1), (2.2) and (3.2) we conclude that

$$\begin{aligned} &\frac{f^{(n)}(a)g^{(k)}(a)}{n!\,(k+1)!}(x-a)^n\left[(x-a)^{k+1}-(c_x-a)^{k+1}\right] \\ &= \frac{f^{(n)}(a)g^{(k)}(a)}{n!\,k!\,(n+k+1)}(x-a)^{n+k+1}+o((x-a)^{n+k+1}) \qquad (x\searrow a). \end{aligned}$$

Multiplying both sides by $n!\,(k+1)!(x-a)^{-(n+k+1)}/(f^{(n)}(a)g^{(k)}(a))$ we get

$$1-\left(\frac{c_x-a}{x-a}\right)^{k+1} = \frac{k+1}{n+k+1}+o(1) \qquad (x\searrow a),$$

whence the conclusion. □

References

[1] Abel, U., *On the Lagrange remainder of the Taylor formula*, Amer. Math. Monthly, **110** (2003), 627-633.

[2] Abel, U., Ivan, M., *The differential mean value of divided differences*, J. Math. Anal. Appl., **325** (2007), 560-570.

[3] Azpeitia, A. G., *On the Lagrange remainder of the Taylor formula*, Amer. Math. Monthly, **89** (1982), 311-312.

[4] Duca, D. I., *Properties of the intermediate point from the Taylor's theorem*, Math. Inequal. Appl., **12** (2009), 763-771.

[5] Duca, D. I., Pop, O., *On the intermediate point in Cauchy's mean-value theorem*, Math. Inequal. Appl., **9** (2006), 375-389.

[6] Duca, D. I., Pop, O. T., *Concerning the intermediate point in the mean value theorem*, Math. Inequal. Appl., **12** (2009), 499-512.

[7] Jacobson, B., *On the mean value theorem for integrals*, Amer Math. Monthly, **89** (1982), 300-301.

[8] Porter, M. B., *The second mean value theorem for summable functions*, Bull. Amer. Math. Soc., **29**(1923), no. 9, 399-400.

[9] Powers, R. C., Riedel, T., Sahoo, P. K., *Limit properties of differential mean values*, J. Math. Anal. Appl., **227** (1998), 216-226.

[10] Sahoo, P. K., Riedel, T., *Mean Value Theorems and Functional Equations*, World Scientific, River Edge, NJ, 1998.

[11] Stark, E. L., *Application of a mean value theorem for integrals to series summation*, Amer. Math. Monthly, **85** (1978), 481-483.

[12] Trif, T., *Asymptotic behavior of intermediate points in certain mean value theorems*, J. Math. Inequal., **2** (2008), 151-161.

[13] Xu, A., Cui F., and Hu, Z., *Asymptotic behavior of intermediate points in the differential mean value theorem of divided differences with repetitions*, J. Math. Anal. Appl., **365** (2010), 358-362.

[14] Zhang, B., *A note on the mean value theorem for integrals*, Amer. Math. Monthly, **104** (1997), 561-562.

Babeş-Bolyai University
Faculty of Mathematics and Computer Science
Str. Kogălniceanu No. 1
RO-400084 Cluj-Napoca, Romania
E-mail address: `ttrif@math.ubbcluj.ro`