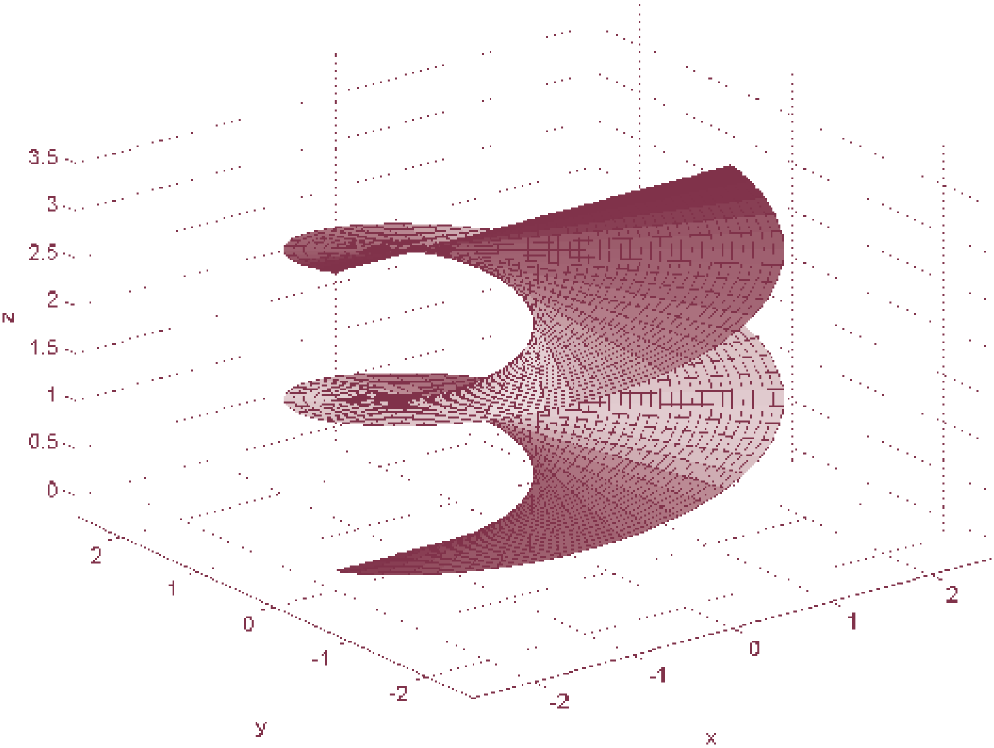
STUDIA UNIVERSITATIS
BABEŞ-BOLYAI



MATHEMATICA

3/2011

YEAR **(LVI) 2011**
MONTH **SEPTEMBER**
ISSUE **3**

S T U D I A
UNIVERSITATIS BABEŞ-BOLYAI

MATHEMATICA
3

Redacţia: 400084 Cluj-Napoca, str. M. Kogălniceanu nr. 1
Telefon: 405300

CONTENTS

Stud. Univ. Babeş-Bolyai Math. 56(2011), No. 3, 3–6

Professor Francesco Altomare at his 60^{th} anniversary

If there is a God, he is a great mathematician (Paul Dirac)

Between September $22^{\text{nd}} - 24^{\text{th}}$, 2011, the International Conference *Recent Developments in Functional Analysis and Approximation Theory* was held at Lecce, Italy. It was organized in collaboration by members of the Universities of Salento, Bari and Basilicata being devoted to some significant aspects of contemporary mathematical research on Functional Analysis, Operator Theory and Approximation Theory including the applications of these fields in other areas such as partial differential equations, integral equations and numerical analysis. Behind this scientific activity there was an emotional manifestation namely the celebration of Francesco Altomare's 60^{th} birthday. The present note is intended to pay tribute to the man and professor Francesco Altomare, pointing out his contribution to the mathematical community.

Biographical notes. Francesco Altomare was born on May 18^{th}, 1951, in Giovinazzo, a charming small town on the Adriatic coast. Growing up close to the sea, he learned to love the beauty and the fascination of the Infinite so far in mathematics. He married Raffaella Bavaro who is 58 years old and nowdays teaches Economical Geography at secondary school. They have two children: Bianca Maria (1983) and Gianluigi (1986). The former got a PhD in Greek Philology in 2010 at the University of Bari and now she is spending a study stay in Paris supported by a post-doc fellowship. The latter is completing his university studies on Pharmacology at the University of Parma.

Career. Francesco Altomare graduated in mathematics from the University of Bari (1975). In time, he has covered all levels of professional career: senior research fellow at the Institute of Mathematical Analysis of the University of Bari (1975-1978), assistant professor at the Faculty of Sciences of the University of Bari (1978-1985), associate professor at the same institution (1985-1987). Since 1987 he was promoted professor at the Faculty of Sciences of the University of Basilicata (Potenza). From 1990 he has held a professorship at the University of Bari where he is currently employed. In the past years professor Altomare held many leadership positions: director of the

Institute of Mathematics at the University of Basilicata (1987-1990), director of the Graduate School in Mathematics at the University of Bari (1993-1995), head of the Interuniversity Department of Mathematics of the University and the Polytechnic of Bari (1997-1999), coordinator of the PhD School in Mathematics of the University of Bari (1999-2003). Under his guidance, the following students received a PhD in mathematics: Sabrina Diomede, Mirella Cappelletti Montano, Rachida Amiar, Vita Leonessa, Sabina Milella, Graziana Musceo. Their present day scientific activity hallmarks the impress of professor Altomare.

Research areas. Albert-Szent Gyorgyi, a Hungarian biochemist who obtained the Nobel Prize for Medicine in 1937, said: "Research is to see what everybody else can see and to think what nobody else has thought." At a close look at professor Altomare's activity we can identify three major scientific research directions.

i) Real and Functional Analysis - Choquet representation theory, Choquet boundaries, continuous function spaces, function algebras and Banach algebras, locally convex vector lattices, positive linear forms and applications to abstract Potential Theory and Harmonic Analysis.

ii) Operator Theory - positive operators, semigroups of operators, differential operators and applications to evolution equations.

iii) Approximation Theory - Korovkin-type approximation theory, positive approximation processes, approximation of semigroups by means of positive operators.

His main achievements are concerned with general methods of construction of positive approximation processes by means of selections of Borel measures and a new method to investigate qualitative properties of positive operator semigroups as well as of solutions of evolution equations by means of positive operators.

Further on, we briefly certify his outstanding scientific activity and its recognition.

Academic prestige. The main results of the above mentioned researches are documented in about 80 papers published in scientific journals, conference proceedings and special issues. We do not intend to present here this list of publications. Consulting the *MathSciNet* database it can be easily identified. Instead, we want to emphasize the following monograph written jointly with Michele Campiti

Korovkin-type Approximation Theory and its Applications, de Gruyter Studies in Mathematics, 17, Walter de Gruyter & Co., Berlin, 1994, xii + 627 pp **MR** 95g:41001

that serves as a landmark for many mathematicians who are grounded in this research area. In this monograph it is presented a modern and comprehensive exposition of the Korovkin-type theorems and some of their applications, by following ingenious new paths that, other than to add new results, allows to synthesize in a well-organized logical exposition the main results of about six hundred articles on the subject. The monograph also well emphasizes one of the main peculiarities of Francesco Altomare, namely to be able to put the mathematical problems of his concern, in the right general perspective and to use (sometimes, to create) general tools that can be useful to better understand the problems as well as other related aspects. As a matter of fact, searching on *MathSciNet* we found that, so far, this book has been cited 111 times. Moreover, until now, Altomare's papers have been cited 246 times by 90 authors in the MR Citation Database.

The activities carried out as visiting professor and as invited speaker at several international meetings are other expressions of his value as a researcher.

Francesco Altomare was a research visitor at the Universities of Paris VI (1980 and 1981), Tübingen (1983) and Münster (1985). In 1985 he also awarded a NATO research grant. On 2004 he spent a research period at the Mathematical Institute of Oberwolfach under the RiP program. He was invited to deliver lectures and postgraduate short courses and to develop joint researches at several Italian and foreign universities such as Napoli, Lecce, Cosenza, Potenza, Roma, Milano, Bologna, Trieste, Salerno, Palermo Perugia, Sofia, Annaba, Erlangen, Passau, Valencia, Praga, Paseky, Siegen, Vienna.

He also attended about fifty international meetings as invited speaker. In addition to those which took place in many Italian cities, we mention recent ones from abroad: Kaohsiung (Taiwan, 2000), Vienna (Austria, 2000), Blaubeuren (Germany, 2001), Cluj-Napoca (Romania, 2002, 2006, 2010), Piteşti (Romania, 2003), Witten-Bommerholz (Germany, 2004), Eger (Hungary, 2005), Kitakyushu and Osaka (Japan, 2006), Ubeda (Spain, 2007, 2010).

Since 2004 F. Altomare has been the founding Editor-in-Chief at *Mediterranean Journal of Mathematics*, a well-reputed international mathematical journal issued by the Department of Mathematics of the University of Bari and published by Birkhäuser Verlag - Basel.

Also, his name is included in the Editorial Board of the following journals: *Conferenze del Seminario di Matematica dell'Università di Bari* (from 1990 to 2003), *Revue d'Analyse Numérique et de Théorie de l'Approximation* (since 1998), *Mathematical Reports* (since 2000), *Journal of Interdisciplinary Mathematics* (since 2004), *Journal of Applied Functional Analysis* (since 2004), *Numerical Functional Analysis and Optimization* (since 2008), *Bollettino dell'Unione Matematica Italiana* (since 2008), *Studia Universitatis Babes-Bolyai, Mathematica* (since 2009), *The Journal of the Indian Academy of Mathematics* (since 2009).

But above all, Altomare's name is forever associated with the international conferences FAAT (Functional Analysis and Approximation Theory) held in Acquafredda di Maratea (Potenza). Six editions took place in 1989, 1992, 1996, 2000, 2004, 2009, respectively. Under Altomare's wand and with the help of his collaborators, for 20 years these meetings have brought together hundreds of mathematicians from all over the world in the fields of

Functional Analysis, Operator Theory, Approximation Theory and have accumulated over 300 papers published in *Supplemento ai Rendiconti del Circolo Matematico di Palermo*. F. Altomare was co-editor of the corresponding Proceedings. Practically, these conferences have marked two decades of scientific work of the mathematicians who have investigated the mentioned areas.

Returning to the conference in Lecce, in a short speech professor Altomare revealed the secret of his success: a permanent support in family life and the sacrifice made by someone who has created optimal conditions to complete his scientific work. With a tear in the corner of his eyes he pronounced a name: Raffaella - his wife.

Those 36 years of scientific activity and a lifetime cannot be condensed in enough words on four pages, so, at this point, we limit ourselves to wishing professor Francesco Altomare health and creative strenght. May he crop the scientific seeds planted by himself.

Octavian Agratini

Stud. Univ. Babeş-Bolyai Math. 56(2011), No. 3, 7–17

Periodic solutions in totally nonlinear difference equations with functional delay

Abdelouaheb Ardjouni and Ahcene Djoudi

Abstract. We use the modification of Krasnoselskii's fixed point theorem due to T. A. Burton ([1] Theorem 3) to show that the totally nonlinear difference equation with functional delay

$$\triangle x(t) = -a(t)\,x^3(t+1) + G\left(t, x^3(t), x^3(t - g(t))\right), \ \forall t \in \mathbb{Z},$$

has periodic solutions. We invert this equation to construct a sum of a compact map and a large contraction which is suitable for applying Krasnoselskii-Burton theorem. Finally, an example is given to illustrate our result.

Mathematics Subject Classification (2010): 39A10, 39A12.

Keywords: Fixed point, large contraction, periodic solutions, totally nonlinear delay difference equations.

1. Introduction

In this paper, we are interested in the analysis of qualitative theory of periodic solutions of difference equation. Motivated by the papers [3], [5]-[7] and the references therein, we consider the following totally nonlinear difference equation with functional delay

$$\triangle x(t) = -a(t)\,x^3(t+1) + G\left(t, x^3(t), x^3(t - g(t))\right), \ \forall t \in \mathbb{Z}, \qquad (1.1)$$

where

$$G : \mathbb{Z} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R},$$

with $\mathbb{Z}$ is the set of integers and $\mathbb{R}$ is the set of real numbers. Throughout this paper $\triangle$ denotes the forward difference operator $\triangle x(t) = x(t+1) - x(t)$ for any sequence $\{x(t),\ t \in \mathbb{Z}\}$. For more on the calculus of difference equations, we refer the reader to [4]. The equation (1.1) is totally nonlinear and we have to add a linear term to both sides of the equation. Although the added term destroys a contraction already present but it will be replaced it with the so called large contraction which is suitable in the fixed point theory. Our purpose here is to use a modification of Krasnoselskii's fixed point theorem due T. A. Burton (see [1] Theorem 3) to show the existence of periodic solutions for equation (1.1). To apply this variant of Krasnoselskii's fixed point theorem we have to invert equation (1.1) to construct two mappings; one is large contraction and the other is compact. For details on Krasnoselskii's theorem we refer the reader to [8]. In Section 2, we present the inversion of difference equations (1.1) and the modification of Krasnoselskii's fixed point theorem. We present our main results on periodicity in Section 3 and at the end we provide an example to illustrate this work.

2. Inversion of the equation

Let T be an integer such that $T \geq 1$. Define

$$C_T = \{\varphi \in C(\mathbb{Z}, \mathbb{R}) : \varphi(t+T) = \varphi(t)\}$$

where $C(\mathbb{Z}, \mathbb{R})$ is the space of all real valued functions. Then $(C_T, \|.\|)$ is a Banach space with the maximum norm

$$\|\varphi\| = \max_{t \in [0, T-1]} |\varphi(t)|.$$

In this paper we assume the periodicity conditions

$$a(t+T) = a(t), \; g(t+T) = g(t), \; g(t) \geq g^* > 0, \tag{2.1}$$

for some constant g^*. Also, we assume that

$$a(t) > 0. \tag{2.2}$$

We also require that $G(t, x, y)$ is periodic in t and Lipschitz continuous in x and y. That is

$$G(t+T, x, y) = G(t, x, y), \tag{2.3}$$

and there are positive constants k_1, k_2 such that

$$|G(t, x, y) - G(t, z, w)| \leq k_1 |x - z| + k_2 |y - w|, \text{ for } x, y, z, w \in \mathbb{R}. \tag{2.4}$$

Lemma 2.1. *Suppose (2.1) and (2.3) hold. If $x \in C_T$, then x is a solution of equation (1.1) if and only if*

$$\begin{aligned} x(t) = {} & \left(1 - \prod_{s=t-T}^{t-1} (1 + a(s))^{-1}\right)^{-1} \\ & \times \left[\sum_{r=t-T}^{t-1} a(r) \left(x(r+1) - x^3(r+1)\right) \prod_{s=r}^{t-1} (1 + a(s))^{-1} \right. \\ & \left. + \sum_{r=t-T}^{t-1} G\left(r, x^3(r), x^3(r - g(r))\right) \prod_{s=r}^{t-1} (1 + a(s))^{-1}\right]. \end{aligned} \tag{2.5}$$

Proof. Let $x \in C_T$ be a solution of (1.1). First we write this equation as

$$\triangle x(t) + a(t)x(t+1) = a(t)x(t+1) - a(t)x^3(t+1) + G\left(t, x^3(t), x^3(t-g(t))\right).$$

Multiplying both sides of the above equation by $\prod_{s=0}^{t-1}(1+a(s))$ and then summing from $t-T$ to $t-1$ to obtain

$$\sum_{r=t-T}^{t-1} \triangle\left[\prod_{s=0}^{r-1}(1+a(s))x(r)\right] = \sum_{r=t-T}^{t-1}\left[a(r)\left\{x(r+1) - x^3(r+1)\right\} + G\left(r, x^3(r), x^3(r-g(r))\right)\right]\prod_{s=0}^{r-1}(1+a(s)).$$

As a consequence, we arrive at

$$\prod_{s=0}^{t-1}(1+a(s))x(t) - \prod_{s=0}^{t-T-1}(1+a(s))x(t-T) = \sum_{r=t-T}^{t-1}\left[a(r)\left\{x(r+1) - x^3(r+1)\right\} + G\left(r, x^3(r), x^3(r-g(r))\right)\right]\prod_{s=0}^{r-1}(1+a(s)).$$

Now, the lemma follows by dividing both sides of the above equation by $\prod_{s=0}^{t-1}(1+a(s))$ and using the fact that $x(t) = x(t-T)$. □

In the analysis, we employ a fixed point theorem in which the notion of a large contraction is required as one of the sufficient conditions. First, we give the following definition which can be found in [1] or [2].

Definition 2.2. *(***Large Contraction***) Let (M,d) be a metric space and $B : M \to M$. B is said to be a large contraction if $\phi, \varphi \in M$, with $\phi \neq \varphi$ then $d(B\phi, B\varphi) \leq d(\phi, \varphi)$ and if for all $\epsilon > 0$, there exists a $\delta < 1$ such that*

$$[\phi, \varphi \in M, d(\phi, \varphi) \geq \epsilon] \Rightarrow d(B\phi, B\varphi) \leq \delta d(\phi, \varphi).$$

The next theorem, which constitutes a basis for our main result, is a reformulated version of Krasnoselskii's fixed point theorem due to T. A. Burton (see [1], [2]).

Theorem 2.3. *(***Krasnoselskii-Burton***) Let M be a bounded convex nonempty subset of a Banach space $(\mathbb{B}, \|.\|)$. Suppose that A and B map M into $\mathbb{B}$ such that*

i. $x, y \in M$, implies $Ax + By \in M$;

ii. A is continuous and AM is contained in a compact subset of M;
iii. B is a large contraction mapping.
Then there exists $z \in M$ with $z = Az + Bz$.

We will use this theorem to prove the existence of periodic solutions for equation (1.1). We begin with the following proposition.

Proposition 2.4. *If $\|.\|$ is the maximum norm,*

$$M = \left\{ \varphi \in C(\mathbb{Z}, \mathbb{R}) : \|\varphi\| \leq \sqrt{3}/3 \right\},$$

and $(B\varphi)(t) = \varphi(t+1) - \varphi^3(t+1)$, then B is a large contraction of the set M.

Proof. For each $t \in \mathbb{Z}$ we have for the real functions φ, ψ

$$\begin{aligned} &|(B\varphi)(t) - (B\psi)(t)| \\ &= |\varphi(t+1) - \psi(t+1)| \\ &\times \left|1 - \left(\varphi^2(t+1) + \varphi(t+1)\psi(t+1) + \psi^2(t+1)\right)\right|. \end{aligned}$$

On the other hand,

$$\begin{aligned} |\varphi(t+1) - \psi(t+1)|^2 &= \varphi^2(t+1) - 2\varphi(t+1)\psi(t+1) + \psi^2(t+1) \\ &\leq 2\left(\varphi^2(t+1) + \psi^2(t+1)\right). \end{aligned}$$

Using $\varphi^2(t+1) + \psi^2(t+1) < 1$ we have

$$\begin{aligned} &|(B\varphi)(t) - (B\psi)(t)| \\ &\leq |\varphi(t+1) - \psi(t+1)| \\ &\times \left[1 - \left(\varphi^2(t+1) + \psi^2(t+1)\right) + |\varphi(t+1)\psi(t+1)|\right] \\ &\leq |\varphi(t+1) - \psi(t+1)| \\ &\times \left[1 - \left(\varphi^2(t+1) + \psi^2(t+1)\right) + \frac{\varphi^2(t+1) + \psi^2(t+1)}{2}\right] \\ &\leq |\varphi(t+1) - \psi(t+1)| \left[1 - \frac{\varphi^2(t+1) + \psi^2(t+1)}{2}\right] \\ &\leq \|\varphi - \psi\|. \end{aligned}$$

Consequently we get

$$\|B\varphi - B\psi\| \leq \|\varphi - \psi\|.$$

Thus B is a large pointwise contraction. But B is still a large contraction for the maximum norm. To show this, let $\epsilon \in (0,1)$ be given and let $\varphi, \psi \in M$ with $\|\varphi - \psi\| \geq \epsilon$.

a) Suppose that for some t we have

$$\epsilon/2 \leq |\varphi(t+1) - \psi(t+1)|.$$

Then

$$(\epsilon/2)^2 \leq |\varphi(t+1) - \psi(t+1)|^2 \leq 2\left(\varphi^2(t+1) + \psi^2(t+1)\right),$$

that is

$$\varphi^2 (t+1) + \psi^2 (t+1) \geq \epsilon^2/8.$$

For all such t we have

$$\begin{aligned} |(B\varphi)(t) - (B\psi)(t)| &\leq |\varphi(t+1) - \psi(t+1)| \left[1 - \frac{\epsilon^2}{16}\right] \\ &\leq \left[1 - \frac{\epsilon^2}{16}\right] \|\varphi - \psi\| . \end{aligned}$$

b) Suppose that for some t we have

$$|\varphi(t+1) - \psi(t+1)| \leq \epsilon/2,$$

then

$$|(B\varphi)(t) - (B\psi)(t)| \leq |\varphi(t+1) - \psi(t+1)| \leq (1/2) \|\varphi - \psi\| .$$

So, for all t we have

$$|(B\varphi)(t) - (B\psi)(t)| \leq \max\left\{1/2, 1 - \frac{\epsilon^2}{16}\right\} \|\varphi - \psi\| .$$

Hence, for each $\epsilon > 0$, if $\delta = \max\left\{1/2, 1 - \frac{\epsilon^2}{16}\right\} < 1$, then

$$\|B\varphi - B\psi\| \leq \delta \|\varphi - \psi\| . \qquad \square$$

3. Existence of periodic solutions

To apply Theorem 2.3, we need to define a Banach space $\mathbb{B}$, a bounded convex subset M of $\mathbb{B}$ and construct two mappings, one is a large contraction and the other is compact. So, we let $(\mathbb{B}, \|.\|) = (C_T, \|.\|)$ and $M = \{\varphi \in \mathbb{B} \mid \|\varphi\| \leq L\}$, where $L = \sqrt{3}/3$. We express equation (2.5) as

$$\varphi(t) = (B\varphi)(t) + (A\varphi)(t) := (H\varphi)(t),$$

where $A, B : M \to \mathbb{B}$ are defined by

$$\begin{aligned} (A\varphi)(t) = &\left(1 - \prod_{s=t-T}^{t-1} (1 + a(s))^{-1}\right)^{-1} \\ &\times \sum_{r=t-T}^{t-1} G\left(r, \varphi^3(r), \varphi^3(r - g(r))\right) \prod_{s=r}^{t-1} (1 + a(s))^{-1}, \end{aligned} \tag{3.1}$$

and

$$\begin{aligned} (B\varphi)(t) = &\left(1 - \prod_{s=t-T}^{t-1} (1 + a(s))^{-1}\right)^{-1} \\ &\times \sum_{r=t-T}^{t-1} a(r) \left(\varphi(r+1) - \varphi^3(r+1)\right) \prod_{s=r}^{t-1} (1 + a(s))^{-1} . \end{aligned} \tag{3.2}$$

We suppose an additional condition, there is $J \geq 3$ with

$$J\left((k_1+k_2)L^3+|G(t,0,0)|\right)\leq La(t),\ \forall t\in\mathbb{Z}. \tag{3.3}$$

We shall prove that the mapping H has a fixed point which solves (1.1).

Lemma 3.1. *For A defined in (3.1), suppose that (2.1)–(2.4) and (3.3) hold. Then $A : M \to M$ is continuous in the maximum norm and maps M into a compact subset of M.*

Proof. We first show that $A : M \to M$.
Let $\varphi \in M$. Evaluate (3.1) at $t+T$.

$$\begin{aligned}(A\varphi)(t+T) &= \left(1-\prod_{s=t}^{t+T-1}(1+a(s))^{-1}\right)^{-1} \\ &\quad\times\sum_{r=t}^{t+T-1}G\left(r,\varphi^3(r),\varphi^3(r-g(r))\right)\prod_{s=r}^{t+T-1}(1+a(s))^{-1}.\end{aligned}$$

Let $j=r-T$, then

$$\begin{aligned}(A\varphi)(t+T) &= \left(1-\prod_{s=t}^{t+T-1}(1+a(s))^{-1}\right)^{-1} \\ &\quad\times\sum_{j=t-T}^{t-1}G\left(j+T,\varphi^3(j+T),\varphi^3(j+T-g(j+T))\right) \\ &\quad\times\prod_{s=j+T}^{t+T-1}(1+a(s))^{-1}.\end{aligned}$$

Now let $k=s-t$, then

$$\begin{aligned}(A\varphi)(t+T) &= \left(1-\prod_{k=t-T}^{t-1}(1+a(k))^{-1}\right)^{-1} \\ &\quad\times\sum_{j=t-T}^{t-1}G\left(j,\varphi^3(j),\varphi^3(j-g(j))\right)\prod_{k=j}^{t-1}(1+a(k))^{-1} \\ &= (A\varphi)(t).\end{aligned}$$

That is, $A : C_T \to C_T$.
In view of (2.4) we arrive at

$$\begin{aligned}|G(t,x,y)| &= |G(t,x,y)-G(t,0,0)+G(t,0,0)| \\ &\leq |G(t,x,y)-G(t,0,0)|+|G(t,0,0)| \\ &\leq k_1\|x\|+k_2\|y\|+|G(t,0,0)|.\end{aligned}$$

Note that from (2.2), we have $1-\prod_{s=t-T}^{t-1}(1+a(s))^{-1}>0$. So, for any $\varphi\in M$, we have

$$\begin{aligned}|(A\varphi)(t)| &\le \left(1-\prod_{s=t-T}^{t-1}(1+a(s))^{-1}\right)^{-1}\\ &\times\sum_{r=t-T}^{t-1}\left|G\left(r,\varphi^3(r),\varphi^3(r-g(r))\right)\right|\prod_{s=r}^{t-1}(1+a(s))^{-1}\\ &\le \left(1-\prod_{s=t-T}^{t-1}(1+a(s))^{-1}\right)^{-1}\\ &\times\sum_{r=t-T}^{t-1}\left((k_1+k_2)L^3+|G(r,0,0)|\right)\prod_{s=r}^{t-1}(1+a(s))^{-1}\\ &\le \left(1-\prod_{s=t-T}^{t-1}(1+a(s))^{-1}\right)^{-1}\sum_{r=t-T}^{t-1}\frac{La(r)}{J}\prod_{s=r}^{t-1}(1+a(s))^{-1}\\ &=\frac{L}{J}<L.\end{aligned}$$

Thus $A\varphi\in M$.
Consequently, we have $A:M\to M$.

We show that A is continuous in the maximum norm. Let $\varphi,\psi\in M$, and let

$$\alpha=\left(1-\prod_{s=t-T}^{t-1}(1+a(s))^{-1}\right)^{-1}.$$

Note that from (2.2), we have $\max_{r\in[t-T,t-1]}\prod_{s=r}^{t-1}(1+a(s))^{-1}\le 1$. So,

$$\begin{aligned}|(A\varphi)(t)-(A\psi)(t)| &\le \left(1-\prod_{s=t-T}^{t-1}(1+a(s))^{-1}\right)^{-1}\\ &\times\sum_{r=t-T}^{t-1}\left|G\left(r,\varphi^3(r),\varphi^3(r-g(r))\right)\right.\\ &\left.-G\left(r,\varphi^3(r),\varphi^3(r-g(r))\right)\right|\prod_{s=r}^{t-1}(1+a(s))^{-1}\\ &\le (k_1+k_2)\left\|\varphi^3-\psi^3\right\|\\ &\times\left(1-\prod_{s=t-T}^{t-1}(1+a(s))^{-1}\right)^{-1}\sum_{r=t-T}^{t-1}\prod_{s=r}^{t-1}(1+a(s))^{-1}\\ &\le 3(k_1+k_2)T\alpha L^2\|\varphi-\psi\|.\end{aligned}$$

Let $\epsilon > 0$ be arbitrary. Define $\eta = \epsilon / K$ with $K = 3\left(k_1 + k_2\right) T \alpha L^2$, where k_1 and k_2 are given by (2.4). Then, for $\|\varphi - \psi\| < \eta$ we obtain

$$\|A\varphi - A\psi\| \leq K\|\varphi - \psi\| < \epsilon.$$

This proves that A is continuous.

Next, we show that A maps bounded subsets into compact sets. As M is bounded and A is continuous, then AM is a subset of $\mathbb{R}^T$ which is bounded. Thus AM is contained in a compact subset of M. Therefore A is continuous in M and AM is contained in a compact subset of M. □

Lemma 3.2. *Let B be defined by (3.2) and suppose that (2.1)–(2.2) hold. Then $B : M \to M$ is a large contraction.*

Proof. We first show that $B : M \to M$.
Let $\varphi \in M$. Evaluate (3.2) at $t + T$.

$$\begin{aligned}(B\varphi)(t+T) = &\left(1 - \prod_{s=t}^{t+T-1} \left(1 + a(s)\right)^{-1}\right)^{-1} \\ &\times \sum_{r=t}^{t+T-1} a(r)\left(\varphi(r+1) - \varphi^3(r+1)\right) \prod_{s=r}^{t+T-1}\left(1 + a(s)\right)^{-1}.\end{aligned}$$

Let $j = r - T$, then

$$\begin{aligned}(B\varphi)(t+T) = &\left(1 - \prod_{s=t}^{t+T-1} \left(1 + a(s)\right)^{-1}\right)^{-1} \\ &\times \sum_{j=t-T}^{t-1} a(j+T)\left(\varphi(j+T+1) - \varphi^3(j+T+1)\right) \\ &\times \prod_{s=j+T}^{t+T-1}\left(1 + a(s)\right)^{-1}.\end{aligned}$$

Now let $k = s - t$, then

$$\begin{aligned}(B\varphi)(t+T) = &\left(1 - \prod_{k=t-T}^{t-1} \left(1 + a(k)\right)^{-1}\right)^{-1} \\ &\times \sum_{j=t-T}^{t-1} a(j)\left(\varphi(j+1) - \varphi^3(j+1)\right) \prod_{k=j}^{t-1}\left(1 + a(k)\right)^{-1} \\ &= (B\varphi)(t).\end{aligned}$$

That is, $B : C_T \to C_T$.

Note that from (2.2), we have $1-\prod_{s=t-T}^{t-1}(1+a(s))^{-1}>0$. So, for any $\varphi\in M$, we have

$$\begin{aligned}|(B\varphi)(t)| &\leq \left(1-\prod_{s=t-T}^{t-1}(1+a(s))^{-1}\right)^{-1} \\ &\times \sum_{r=t-T}^{t-1} a(r)\left|\varphi(r+1)-\varphi^3(r+1)\right| \prod_{s=r}^{t-1}(1+a(s))^{-1} \\ &\leq \left(1-\prod_{s=t-T}^{t-1}(1+a(s))^{-1}\right)^{-1} \\ &\times \sum_{r=t-T}^{t-1} a(r)\left\|\varphi-\varphi^3\right\| \prod_{s=r}^{t-1}(1+a(s))^{-1} \\ &= \left\|\varphi-\varphi^3\right\|.\end{aligned}$$

Since $\|\varphi\|\leq L$, we have $\left\|\varphi-\varphi^3\right\|\leq\left(2\sqrt{3}\right)/9<L$. So, for any $\varphi\in M$, we have

$$\|B\varphi\|<L.$$

Thus $B\varphi\in M$. Consequently, we have $B:M\to M$.

It remains to show that B is large contraction in the maximum norm. From the proof of Proposition 2.4 we have for $\varphi,\psi\in M$, with $\varphi\neq\psi$

$$\begin{aligned}|(B\varphi)(t)-(B\psi)(t)| &\leq \left(1-\prod_{s=t-T}^{t-1}(1+a(s))^{-1}\right)^{-1} \\ &\times \sum_{r=t-T}^{t-1} a(r)\|\varphi-\psi\| \prod_{s=r}^{t-1}(1+a(s))^{-1} \\ &= \|\varphi-\psi\|.\end{aligned}$$

Then $\|B\varphi-B\psi\|\leq\|\varphi-\psi\|$. Thus B is a large pointwise contraction. But B is still a large contraction for the maximum norm. To show this, let $\epsilon\in(0,1)$ be given and let $\varphi,\psi\in M$ with $\|\varphi-\psi\|\geq\varepsilon$. From the proof of the Proposition 2.4 we have found $\delta<1$ such that

$$\begin{aligned}|(B\varphi)(t)-(B\psi)(t)| &\leq \left(1-\prod_{s=t-T}^{t-1}(1+a(s))^{-1}\right)^{-1} \\ &\times \sum_{r=t-T}^{t-1} a(r)\,\delta\,\|\varphi-\psi\| \prod_{s=r}^{t-1}(1+a(s))^{-1} \\ &= \delta\,\|\varphi-\psi\|.\end{aligned}$$

Then $\|B\varphi-B\psi\|\leq\delta\,\|\varphi-\psi\|$. Consequently, B is a large contraction. □

Theorem 3.3. *Let $(C_T, \|.\|)$ be the Banach space of T-periodic real valued functions and $M = \{\varphi \in C_T \mid \|\varphi\| \leq L\}$, where $L = \sqrt{3}/3$. Suppose (2.1)–(2.4) and (3.3) hold. Then equation (1.1) has a T-periodic solution φ in the subset M.*

Proof. By Lemma 3.1, $A : M \to M$ is continuous and AM is contained in a compact set. Also, from Lemma 3.2, the mapping $B : M \to M$ is a large contraction. Moreover, if $\varphi, \psi \in M$, we see that

$$\|A\varphi + B\psi\| \leq \|A\varphi\| + \|B\psi\| \leq L/J + \left(2\sqrt{3}\right)/9 \leq L.$$

Thus $A\varphi + B\psi \in M$.
Clearly, all the hypotheses of Krasnoselskii-Burton Theorem 2.3 are satisfied. Thus there exists a fixed point $\varphi \in M$ such that $\varphi = A\varphi + B\varphi$. Hence the equation (1.1) has a T-periodic solution which lies in M. □

Example 3.4. We consider the totally nonlinear difference equation with functional delay

$$\triangle x(t) = -8x^3(t+1) + \sin\left(x^3(t)\right) + \cos\left(x^3(t - g(t))\right), \ t \in \mathbb{Z}, \tag{3.4}$$

where

$$g(t+T) = g(t).$$

So, we have

$$a(t) = 8, \ G\left(t, x^3(t), x^3(t-g(t))\right) = \sin\left(x^3(t)\right) + \cos\left(x^3(t-g(t))\right).$$

Clearly, $G(t,x,y)$ is periodic in t Lipschitz continuous in x and y. That is

$$G(t+T, x, y) = G(t, x, y),$$

and

$$\begin{aligned} |G(t,x,y) - G(t,z,w)| &= |\sin(x) - \sin(z) + \cos(y) - \cos(w)| \\ &\leq |\sin(x) - \sin(z)| + |\cos(y) - \cos(w)| \\ &\leq |x - z| + |y - w|. \end{aligned}$$

Note that if $J = 3$ we have

$$\begin{aligned} J\left((k_1 + k_2) L^3 + |G(t,0,0)|\right) &= 3\left(2\left(\sqrt{3}/3\right)^3 + 1\right) \\ &\leq \left(\sqrt{3}/3\right) 8 \\ &= La(t), \ \forall t \in \mathbb{Z}. \end{aligned}$$

Define $M = \{\varphi \in C_T \mid \|\varphi\| \leq L\}$, where $L = \sqrt{3}/3$. Then the difference (3.4) has a T-periodic solution in M, by Theorem 2.

Acknowledgments

The authors would like to thank the anonymous referee for his/her valuable remarks.

References

[1] Burton, T.A., *Liapunov functionals, fixed points and stability by Krasnoselskii's theorem. Nonlinear Stud*, **9**(2002), no. 2, 181-190.

[2] Burton, T.A., *Stability by Fixed Point Theory for Functional Differential Equations*, Dover, New York, 2006.

[3] Deham, H., Djoudi, A., *Periodic solutions for nonlinear differential equation with functional delay*, Georgian Mathematical Journal, **15**(2008), no. 4, 635-642.

[4] Kelly, W.G., Peterson, A.C., *Difference Equations: An Introduction with Applications*, Academic Press, 2001.

[5] Maroun, M.R., Raffoul, Y.N., *Periodic solutions in nonlinear neutral difference equations with functional delay*, J. Korean Math. Soc., **42**(2005), no. 2, 255-268.

[6] Raffoul, Y.N., *Stability and periodicity in discrete delay equations*, J. Math. Anal. Appl., **324**(2006), no. 2, 1356-1362.

[7] Raffoul, Y.N., *Periodicity in general delay non-linear difference equations using fixed point theory*, J. Difference Equ. Appl., **10**(2004), no. 13-15, 1229-1242.

[8] Smart, D.R., *Fixed point theorems, Cambridge Tracts in Mathematics*, no. 66, Cambridge University Press, London-New York, 1974.

Abdelouaheb Ardjouni
Laboratory of Applied Mathematics (LMA)
University of Annaba, Department of Mathematics
P.O.Box 12, Annaba 23000
Algeria
e-mail: abd_ardjouni@yahoo.fr

Ahcene Djoudi
Laboratory of Applied Mathematics (LMA)
University of Annaba, Department of Mathematics
P.O.Box 12, Annaba 23000
Algeria
e-mail: adjoudi@yahoo.com

Stud. Univ. Babeş-Bolyai Math. 56(2011), No. 3, 19–28

Perov's fixed point theorem for multivalued mappings in generalized Kasahara spaces

Alexandru-Darius Filip

Abstract. In this paper we give some corresponding results to Perov's fixed point theorem which was given in a complete generalized metric space. Our results will be given in a more general space, the so called generalized Kasahara space. We will also use the case of multivalued operators and give some fixed point results for multivalued Kannan, Reich and Caristi operators.

Mathematics Subject Classification (2010): 47H10, 54H25.

Keywords: Fixed point, generalized Kasahara space, multivalued mapping, matrix convergent to zero, sequence of successive approximations.

1. Introduction and preliminaries

The classical Banach contraction principle was extended for contractive maps on spaces endowed with vector-valued metrics by Perov in 1964 (see [5]).We recall some notions regarding Perov's result.
Let X be a nonempty set and $m \in \mathbb{N}$, $m \geq 1$. A mapping $d : X \times X \to \mathbb{R}^m$ is called a vector-valued metric on X if the following statements are satisfied for all $x, y, z \in X$:

$d_1)$ $d(x, y) \geq 0_m$, where $0_m := (0, 0, \ldots, 0) \in \mathbb{R}^m$;
$d_2)$ $d(x, y) = 0_m \Rightarrow x = y$;
$d_3)$ $d(x, y) = d(y, x)$;
$d_4)$ $d(x, y) \leq d(x, z) + d(z, y)$.

We mention that if $\alpha, \beta \in \mathbb{R}^m$, $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_m)$, $\beta = (\beta_1, \beta_2, \ldots, \beta_m)$ and $c \in \mathbb{R}$, then by $\alpha \leq \beta$ (respectively $\alpha < \beta$), we mean that $\alpha_i \leq \beta_i$ (respectively $\alpha_i < \beta_i$), for all $i = \overline{1, m}$ and by $\alpha \leq c$ we mean that $\alpha_i \leq c$, for all $i = \overline{1, m}$.

A set X equipped with a vector-valued metric d is called a generalized metric space. We will denote such a space with (X, d). For generalized metric spaces, the notions of convergent sequence, Cauchy sequence, completeness, open subset and closed subset are similar to those for usual metric spaces.

Throughout this paper we denote by $\mathcal{M}_{m,m}(\mathbb{R}_+)$ the set of all $m \times m$ matrices with positive elements, by Θ the zero $m \times m$ matrix and by I_m the identity $m \times m$ matrix. If $A \in \mathcal{M}_{m,m}(\mathbb{R}_+)$, then the symbol A^τ stands for the transpose matrix of A. Notice also that, for the sake of simplicity, we will make an identification between row and column vectors in $\mathbb{R}^m$.

A matrix $A \in \mathcal{M}_{m,m}(\mathbb{R}_+)$ is said to be convergent to zero if and only if $A^n \to \Theta$ as $n \to \infty$ (see [11]). Regarding this class of matrices we have the following classical result in matrix analysis (see [1](Lemma 3.3.1, page 55), [6], [7](page 37), [11](page 12). More considerations can be found in [10].

Theorem 1.1. *Let $A \in \mathcal{M}_{m,m}(\mathbb{R}_+)$. The following statements are equivalent:*

i) $A^n \to \Theta$, *as* $n \to \infty$;
ii) *the eigenvalues of A lies in the open unit disc, i.e., $|\lambda| < 1$, for all $\lambda \in \mathbb{C}$ with $det(A - \lambda I_m) = 0$;*
iii) *the matrix $I_m - A$ is non-singular and*

$$(I_m - A)^{-1} = I_m + A + A^2 + \ldots + A^n + \ldots;$$

iv) *the matrix $(I_m - A)$ is non-singular and $(I_m - A)^{-1}$ has nonnegative elements;*
v) *the matrices Aq and $q^\tau A$ converges to zero for each $q \in \mathbb{R}^m$.*

The main result for self contractions on generalized metric spaces is Perov's fixed point theorem (see [5]):

Theorem 1.2 (A.I. Perov). *Let (X, d) be a complete generalized metric space and the mapping $f : X \to X$ with the property that there exists a matrix $A \in \mathcal{M}_{m,m}(\mathbb{R}_+)$ such that $d(f(x), f(y)) \leq Ad(x, y)$, for all $x, y \in X$. If A is a matrix convergent to zero, then*

p_1) *there exists a unique $x^* \in X$ such that $x^* = f(x^*)$, i.e., the mapping f has a unique fixed point;*
p_2) *the sequence of successive approximations $(x_n)_{n\in\mathbb{N}} \subset X$, $x_n = f^n(x_0)$ is convergent and it has the limit x^*, for all $x_0 \in X$;*
p_3) $d(x_n, x^*) \leq A^n(I_m - A)^{-1}d(x_0, x_1)$, *for all* $n \in \mathbb{N}$;
p_4) *if $g : X \to X$ satisfies the condition $d(f(x), g(x)) \leq \eta$, for all $x \in X$ and $\eta \in \mathbb{R}^m$, then by considering the sequence $(y_n)_{n\in\mathbb{N}} \subset X$, $y_n = g^n(x_0)$ one has $d(y_n, x^*) \leq (I_m - A)^{-1}\eta + A^n(I_m - A)^{-1}d(x_0, x_1)$, for all $n \in \mathbb{N}$.*

In this paper we give some corresponding results to Perov fixed point theorem. We will use the multivalued operators and we will adapt Perov's result to the context of generalized Kasahara spaces. In order to do this, we recall the following notions and results:

Definition 1.3 (see [8]). *Let X be a nonempty set, $\to$ be an L-space structure on X, $(G, +, \leq, \xrightarrow{G})$ be an L-space ordered semigroup with unity, 0 be the least element in $(G, \leq)$ and $d_G : X \times X \to G$ be an operator. The triple $(X, \to, d_G)$ is called a generalized Kasahara space if and only if the following compatibility* *condition between* $\to$ *and* d_G *holds:*

$$\text{for all } (x_n)_{n\in\mathbb{N}} \subset X \text{ with } \sum_{n\in\mathbb{N}} d(x_n, x_{n+1}) < +\infty$$
$$\Rightarrow (x_n)_{n\in\mathbb{N}} \text{ is convergent in } (X, \to).$$

Example 1.4. Let $\rho : X \times X \to \mathbb{R}^m_+$ be a generalized complete metric on a set X. Let $x_0 \in X$ and $\lambda \in \mathbb{R}^m_+$ with $\lambda \neq 0$. Let $d_\lambda : X \times X \to \mathbb{R}^m_+$ be defined by

$$d_\lambda(x,y) = \begin{cases} \rho(x,y) & , \text{ if } x \neq x_0 \text{ and } y \neq x_0, \\ \lambda & , \text{ if } x = x_0 \text{ or } y = x_0. \end{cases}$$

Then $(X, \overset{\rho}{\to}, d_\lambda)$ is a generalized Kasahara space.

In [3], S. Kasahara gives a useful tool which is used in proving the uniqueness of a fixed point.

Lemma 1.5. *Let* $(X, \to, d_G)$ *be a generalized Kasahara space. Then*

$$\text{for all } x, y \in X \text{ with } d_G(x,y) = d_G(y,x) = 0 \Rightarrow x = y.$$

For more considerations on generalized Kasahara spaces, see [8] and the references therein.

Through this paper, we consider $G = \mathbb{R}^m$. The functional d_G will be denoted by d, which is not necessary a metric on X. In other words, we will consider the generalized Kasahara space $(X, \to, d)$ where $d : X \times X \to \mathbb{R}^m_+$ is a functional.

Finally, in the above setting, for a multivalued operator $F : X \multimap X$, we shall use the following notations:

m_1) $P(X) := \{Y \subset X \mid Y \neq \emptyset\}$, so $F : X \to P(X)$;

m_2) $Fix(F) := \{x^* \in X \mid x^* \in F(x^*)\}$, the set of all fixed points for F. For simplicity, we will use the notation Fx instead of $F(x)$, where $x \in X$;

m_3) $Graph(F) = \{(x,y) \in X \times X \mid y \in Fx\}$, the graph of F. We say that F has closed graph, if and only if $Graph(F)$ is closed in $X \times X$ with respect to $\to$, i.e., if $(x_n)_{n\in\mathbb{N}} \subset X$ and $y_n \in Fx_n$, for all $n \in \mathbb{N}$ with $x_n \to x^* \in X$, as $n \to \infty$ and if $y_n \to y^*$, as $n \to \infty$ then $y^* \in Fx^*$.

2. Main results

Theorem 2.1. *Let* $(X, \to, d)$ *be a generalized Kasahara space and* $F : X \to P(X)$ *be a multivalued operator. We assume that:*

i) *there exists* $A \in \mathcal{M}_{m,m}(\mathbb{R}_+)$ *and for all* $x, y \in X$ *and* $u \in Fx$*, there exists* $v \in Fy$ *such that*

$$d(u,v) \leq A d(x,y);$$

ii) $Graph(F)$ *is closed in* $X \times X$ *with respect to* $\to$.

If A converges to zero, then $Fix(F) \neq \emptyset$. If, in addition, $(I_m - A)$ is non-singular, $(I_m - A)^{-1} \in \mathcal{M}_{m\times m}(\mathbb{R}_+)$ and

$$\max\{d(u,v) \mid u \in Fx,\ v \in Fy\} \leq Ad(x,y),\ \textit{for all } x,y \in X$$

then F has a unique fixed point in X.

Proof. Let $x_0 \in X$ and $x_1 \in Fx_0$. If $x_1 = x_0$ then $x_0 \in Fix(F)$. We assume that $x_1 \neq x_0$. Then by $i)$ there exists $x_2 \in Fx_1$ such that

$$d(x_1,x_2) \leq Ad(x_0,x_1).$$

Since $x_2 \in Fx_1$, if $x_2 = x_1$ then $x_1 \in Fix(F)$. If we consider $x_2 \neq x_1$ then there exists $x_3 \in Fx_2$ such that

$$d(x_2,x_3) \leq Ad(x_1,x_2) \leq A^2 d(x_0,x_1).$$

By induction, we construct the sequence of successive approximations for F starting from $(x_0,x_1) \in Graph(F)$. This sequence has the following properties:

1°) $x_{n+1} \in Fx_n$, for all $n \in \mathbb{N}$;
2°) $d(x_n,x_{n+1}) \leq A^n d(x_0,x_1)$, for all $n \in \mathbb{N}$.

Next, we have the following estimation:

$$\sum_{n\in\mathbb{N}} d(x_n,x_{n+1}) \leq \sum_{n\in\mathbb{N}} A^n d(x_0,x_1) = (I_m - A)^{-1} d(x_0,x_1) < +\infty.$$

Since $(X,\to,d)$ is a generalized Kasahara space, the sequence $(x_n)_{n\in\mathbb{N}}$ is convergent in X with respect to $\to$. Hence there exists $x^* \in X$ such that $x_n \to x^*$ as $n \to \infty$. On the other hand, F has closed graph, so $x^* \in Fix(F)$.

We prove now the uniqueness of the fixed point x^*.

Let $x^*, y^* \in Fix(F)$ such that $x^* \neq y^*$. Since $x^* \in Fx^*$ and $y^* \in Fy^*$, we get that

$$d(x^*,y^*) \leq \max_{\substack{u\in Fx^*\\ v\in Fy^*}} d(u,v) \leq Ad(x^*,y^*) \Leftrightarrow (I_m - A)d(x^*,y^*) \leq 0_m.$$

Since $I_m - A$ is a non-singular matrix and $(I_m - A)^{-1}$ has non-negative elements, it follows that $d(x^*,y^*) = 0_m$. By the same way of proof, we get that $d(y^*,x^*) = 0_m$. By Lemma 1.5, we obtain $x^* = y^*$. □

Remark 2.2. Let X be a nonempty set and $\rho : X \times X \to \mathbb{R}^m_+$ be a complete generalized metric on X. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in X and let $x \in X$. Then

$$x_n \overset{\rho}{\to} x \ \Leftrightarrow\ \rho(x_n,x) \to 0_m, \text{ as } n \to \infty.$$

We have the following Maia type result:

Corollary 2.3. *Let X be a nonempty set and $\rho : X \times X \to \mathbb{R}^m_+$ be a complete generalized metric on X. Let $d : X \times X \to \mathbb{R}^m_+$ be a functional and $F : X \to P(X)$ be a multivalued operator. We assume that*

i) *there exists* $A \in \mathcal{M}_{m,m}(\mathbb{R}_+)$ *and for all* $x, y \in X$ *and* $u \in Fx$, *there exists* $v \in Fy$ *such that*

$$d(u,v) \leq Ad(x,y);$$

ii) $Graph(F)$ *is closed in* $X \times X$ *with respect to* $\xrightarrow{\rho}$;

iii) *there exists* $c > 0$ *such that* $\rho(x,y) \leq c \cdot d(x,y)$.

Then the following statements hold:

1) *if* A *converges to zero, then* $Fix(F) \neq \emptyset$. *If, in addition,* $(I_m - A)$ *is non-singular,* $(I_m - A)^{-1} \in \mathcal{M}_{m\times m}(\mathbb{R}_+)$ *and*

$$\max\{d(u,v) \mid u \in Fx,\ v \in Fy\} \leq Ad(x,y),\ \textit{for all } x, y \in X$$

then F *has a unique fixed point in* X.

2) $\rho(x_n, x^*) \leq c \cdot A^n (I_m - A)^{-1} d(x_0, x_1)$, *for all* $n \in \mathbb{N}$, *where* $x^* \in Fix(F)$ *and* $(x_n)_{n\in\mathbb{N}}$ *is the sequence of successive approximations for* F *starting from* $(x_0, x_1) \in Graph(F)$.

Proof. By $i)$ and by following the proof of Theorem 2.1, we can construct a sequence $(x_n)_{n\in\mathbb{N}}$ of successive approximations for F starting from $(x_0, x_1) \in Graph(F)$ such that $x_{n+1} \in Fx_n$ and $d(x_n, x_{n+1}) \leq A^n d(x_0, x_1)$, for all $n \in \mathbb{N}$. By $iii)$ there exists $c > 0$ such that

$$\rho(x_n, x_{n+1}) \leq c \cdot d(x_n, x_{n+1}) \leq c \cdot A^n d(x_0, x_1),\ \text{for all } n \in \mathbb{N}.$$

Now let $p \in \mathbb{N}$, $p > 0$. Since ρ is a metric, we have that

$$\begin{aligned} \rho(x_n, x_{n+p}) &\leq \rho(x_n, x_{n+1}) + \rho(x_{n+1}, x_{n+2}) + \ldots + \rho(x_{n+p-1}, x_{n+p}) \\ &\leq c \cdot A^n d(x_0, x_1) + c \cdot A^{n+1} d(x_0, x_1) + \ldots + c \cdot A^{n+p-1} d(x_0, x_1). \end{aligned}$$

Thus, for all $n, p \in \mathbb{N}$ with $p > 0$, the following estimation holds

$$\rho(x_n, x_{n+p}) \leq c \cdot A^n (I_m + A + \ldots + A^{p-1}) d(x_0, x_1). \tag{2.1}$$

By letting $n \to \infty$, we get that $\rho(x_n, x_{n+p}) \to 0_m$, so $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in the complete generalized metric space (X, ρ). Therefore $(x_n)_{n\in\mathbb{N}}$ is convergent in (X, ρ), so there exists $x^* \in X$ such that $x_n \xrightarrow{\rho} x$.

By $ii)$ it follows that $x^* \in Fix(F)$. The uniqueness of the fixed point x^* follows from Theorem 2.1.

By letting $p \to \infty$ in (2.1), we get the estimation mentioned in the conclusion 2) of the corollary. □

Corollary 2.4. *Let* $(X, \to, d)$ *be a generalized Kasahara space where* d *satisfies* $d(x,x) = 0_m$, *for all* $x \in X$. *Let* $F : X \to P(X)$ *be a multivalued operator. We assume that:*

i) *there exists* $A \in \mathcal{M}_{m,m}(\mathbb{R}_+)$, $B \in \mathcal{M}_{m,m}(\mathbb{R})$ *and for all* $x, y \in X$ *and* $u \in Fx$, *there exists* $v \in Fv$ *such that*

$$d(u,v) \leq Ad(x,y) + Bd(y,u);$$

ii) $Graph(F)$ *is closed in* $X \times X$ *with respect to* $\to$.

If A *converges to zero, then* F *has at least one fixed point in* X.

Proof. Let $x_0 \in X$ and $x_1 \in Fx_0$. If $x_1 = x_0$ then $x_0 \in Fix(F)$. We assume that $x_1 \neq x_0$. Then by $i)$ there exists $x_2 \in Fx_1$ such that

$$d(x_1, x_2) \leq Ad(x_0, x_1) + Bd(x_1, x_1) = Ad(x_0, x_1).$$

By following the proof of Theorem 2.1, the conclusion follows. □

As an application of the previous results we present an existence theorem for a semi-linear inclusion systems.

Theorem 2.5. *Let $\varphi, \psi : [0,1]^2 \to]0, \frac{1}{2}]$ be two functions and $F_1, F_2 : [0,1]^2 \to P([0,1])$ be two multivalued operators defined as follows:*

$$F_1(x_1, x_2) = \left[\varphi(x_1, x_2), \frac{1}{2} + \varphi(x_1, x_2)\right] \text{ and}$$
$$F_2(x_1, x_2) = \left[\psi(x_1, x_2), \frac{1}{2} + \psi(x_1, x_2)\right].$$

We assume that for each $(x_1, x_2), (y_1, y_2) \in [0,1]^2$ and each $u_1 \in F_1(x_1, x_2)$, $u_2 \in F_2(x_1, x_2)$, there exist $v_1 \in F_1(y_1, y_2)$ and $v_2 \in F_2(y_1, y_2)$ such that

$$|u_1 - v_1| \leq a|x_1 - y_1| + b|x_2 - y_2|,$$
$$|u_2 - v_2| \leq c|x_1 - y_1| + d|x_2 - y_2|,$$

for all $a, b, c, d \in \mathbb{R}_+$ with $|a + d \pm \sqrt{(a-d)^2 + 4bc}| < 2$.

Then the system

$$\begin{cases} x_1 \in F_1(x_1, x_2) \\ x_2 \in F_2(x_1, x_2), \end{cases} \tag{2.2}$$

has at least one solution in $[0,1]^2$.

Proof. Let $F := (F_1, F_2) : [0,1]^2 \to P([0,1]^2)$. Then the system (2.2) can be represented as a fixed point problem of the form

$$x \in Fx, \text{ where } x = (x_1, x_2) \in [0,1]^2.$$

We consider the generalized Kasahara space $([0,1]^2, \xrightarrow{\rho_e}, d)$ where:

i) $\rho_e : [0,1]^2 \times [0,1]^2 \to \mathbb{R}^2_+$ is defined by

$$\rho_e(x, y) = (|x_1 - y_1|, |x_2 - y_2|),$$

for all $x = (x_1, x_2), y = (y_1, y_2) \in [0,1]^2$;

ii) $d : [0,1]^2 \times [0,1]^2 \to \mathbb{R}^2_+$ is defined by

$$d(x, y) = \begin{cases} \rho_e(x, y) & , \ x \neq \theta \text{ and } y \neq \theta \\ (1, 1) & , \ x = \theta \text{ or } y = \theta \end{cases},$$

for all $x = (x_1, x_2), y = (y_1, y_2) \in [0,1]^2$, where $\theta = (0, 0)$.

For each $x = (x_1, x_2), y = (y_1, y_2) \in [0,1]^2$ and $u = (u_1, u_2) \in Fx$, there exists $v = (v_1, v_2) \in Fy$ such that

$$d(u,v) \leq Ad(x,y),$$

where $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is a matrix convergent to zero, having its eigenvalues in the open unit disc.

Since $Graph(F)$ is closed in $[0,1]^2$ w.r.t. $\xrightarrow{\rho_e}$, Theorem 2.1 holds. □

Remark 2.6. Some examples of matrix convergent to zero are:

a) any matrix $A = \begin{pmatrix} a & a \\ b & b \end{pmatrix}$, where $a, b \in \mathbb{R}_+$ and $a + b < 1$;

b) any matrix $A = \begin{pmatrix} a & b \\ a & b \end{pmatrix}$, where $a, b \in \mathbb{R}_+$ and $a + b < 1$;

c) any matrix $A = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$, where $a, b, c \in \mathbb{R}_+$ and $\max\{a, c\} < 1$;

In what follows, we present some results regarding the fixed points for multivalued Kannan and Reich operators. For our proofs, we will need the following result:

Lemma 2.7. *Let $A = (a_{ij})_{i,j=\overline{1,m}} \in \mathcal{M}_{m,m}(\mathbb{R}_+)$ be a triangular matrix with*

$$\max\left\{a_{ii} \mid i = \overline{1,m}\right\} < \frac{1}{2}.$$

Then the matrix $\Lambda = (I_m - A)^{-1}A$ is convergent to zero.

Proof. Suppose that $A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ 0 & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & a_{mm} \end{pmatrix} \in \mathcal{M}_{m,m}(\mathbb{R}_+)$. Then the eigenvalues of Λ are $\lambda_i = \frac{a_{ii}}{1-a_{ii}}$, for all $i = \overline{1,m}$. Since all of the eigenvalues of Λ are in the open unit disc, the conclusion follows from Theorem 1.1. □

A result for multivalued Kannan operators is presented bellow:

Theorem 2.8. *Let $(X, \to, d)$ be a generalized Kasahara space and $F : X \to P(X)$ be a multivalued operator. We assume that:*

i) *there exists $A = (a_{ij})_{i,j=\overline{1,m}} \in \mathcal{M}_{m,m}(\mathbb{R}_+)$ a triangular matrix such that $\max\limits_{i=\overline{1,m}} a_{ii} < \frac{1}{2}$ and for all $x, y \in X$ and $u \in Fx$, there exists $v \in Fy$ such that*

$$d(u,v) \leq A[d(x,u) + d(y,v)];$$

ii) *$Graph(F)$ is closed in $X \times X$ with respect to $\to$.*

Then F has at least one fixed point in X.

Proof. Let $x_0 \in X$ and $x_1 \in Fx_0$. If $x_1 = x_0$, then we already have a fixed point for F ($x_0 \in Fix(F)$). Assuming that $x_1 \neq x_0$, then by *i*), there exists $x_2 \in Fx_1$ such that

$$d(x_1, x_2) \leq A[d(x_0, x_1) + d(x_1, x_2)] \Leftrightarrow d(x_1, x_2) \leq (I_m - A)^{-1} A d(x_0, x_1).$$

We denote $\Lambda = (I_m - A)^{-1}A$ and we have

$$d(x_1, x_2) \leq \Lambda d(x_0, x_1).$$

By taking into account Lemma 2.7 and by following the proof of Theorem 2.1, replacing A with Λ, the conclusion follows. □

Next we present a result regarding the fixed points for the multivalued operators of Reich type:

Theorem 2.9. *Let $(X, \to, d)$ be a generalized Kasahara space and $F : X \to P(X)$ be a multivalued operator. We assume that:*

i) *there exist $A = (a_{ij})_{i,j=\overline{1,m}}$, $B = (b_{ij})_{i,j=\overline{1,m}}$, $C = (c_{ij})_{i,j=\overline{1,m}} \in \mathcal{M}_{m,m}(\mathbb{R}_+)$, where*

   1) *C is a triangular matrix with $\max\limits_{i=\overline{1,m}} c_{ii} < \frac{1}{2}$*

   2) *$A + B \leq C$, i.e., $a_{ij} + b_{ij} \leq c_{ij}$, for all $i, j = \overline{1,m}$*

   and for all $x, y \in X$ and $u \in Fx$, there exists $v \in Fy$ such that

$$d(u, v) \leq Ad(x, y) + Bd(x, u) + Cd(y, v);$$

ii) *$Graph(F)$ is closed in $X \times X$ with respect to $\to$.*

Then F has at least one fixed point in X.

Proof. Let $x_0 \in X$ and $x_1 \in Fx_0$. If $x_1 = x_0$, then we already have a fixed point for F ($x_0 \in Fix(F)$). Assuming that $x_1 \neq x_0$, then by *i*), there exists $x_2 \in Fx_1$ such that

$$d(x_1, x_2) \leq Ad(x_0, x_1) + Bd(x_0, x_1) + Cd(x_1, x_2)$$
$$\Leftrightarrow d(x_1, x_2) \leq (I_m - C)^{-1}(A + B)d(x_0, x_1) \leq (I_m - C)^{-1} C d(x_0, x_1).$$

We denote $\Lambda = (I_m - C)^{-1}C$. By taking into account Lemma 2.7 and by following the proof of Theorem 2.1, replacing A with Λ, the conclusion follows. □

Some other fixed point results can be established for the multivalued Caristi operators:

Definition 2.10. *Let $(X, \to, d)$ be a generalized Kasahara space and $F : X \to P(X)$ be a multivalued operator. Let $\varphi : X \to \mathbb{R}^m_+$ be a functional. We say that F is a multivalued Caristi operator if for all $x \in X$, there exists $y \in Fx$ such that*

$$d(x, y) \leq \varphi(x) - \varphi(y).$$

For more considerations on multivalued Caristi operators see [4] and [2].

Theorem 2.11. *Let $(X, \to, d)$ be a generalized Kasahara space and $F : X \to P(X)$ be a multivalued Caristi operator, having closed graph with respect to $\to$. Then F has at least one fixed point in X.*

Proof. Let $x_0 \in X$. Then there exists $x_1 \in Fx_0$. If $x_1 = x_0$ then $x_0 \in Fix(F)$ and the proof is complete. If $x_1 \neq x_0$ then

$$d(x_0, x_1) \leq \varphi(x_0) - \varphi(x_1).$$

Since $x_1 \in Fx_0$, there exists $x_2 \in Fx_1$. If $x_2 = x_1$ then $x_1 \in Fix(F)$ and the proof is complete. If $x_2 \neq x_1$ then

$$d(x_1, x_2) \leq \varphi(x_1) - \varphi(x_2).$$

By induction, there exists $x_{n+1} \in Fx_n$ such that

$$d(x_n, x_{n+1}) \leq \varphi(x_n) - \varphi(x_{n+1}), \text{ for all } n \in \mathbb{N}.$$

We have the following estimations

$$\sum_{n\in\mathbb{N}} d(x_n, x_{n+1}) \leq \varphi(x_0) - \varphi(x_{n+1}) \leq \varphi(x_0) < +\infty.$$

Since $(X, \to, d)$ is a Kasahara space, the sequence $(x_n)_{n\in\mathbb{N}}$ is convergent in $(X, \to)$. So there exists $x^* \in X$ such that $x_n \to x^*$, as $n \to \infty$.

Since $Graph(F)$ is closed, $x^* \in Fix(F)$. □

By taking into account the Remark 2.2, we have the following result:

Corollary 2.12. *Let X be a nonempty set and $\rho : X \times X \to \mathbb{R}^m_+$ be a complete generalized metric on X. Let $d : X \times X \to \mathbb{R}^m_+$ be a functional. Let $\varphi : X \to \mathbb{R}^m_+$ be a functional.*

Let $F : X \to P(X)$ be a multivalued operator such that

i) *$Graph(F)$ is closed in $X \times X$ with respect to $\xrightarrow{\rho}$;*
ii) *for all $x \in X$, there exists $y \in Fx$ such that $d(x, y) \leq \varphi(x) - \varphi(y)$;*
iii) *there exists $c > 0$ such that $\rho(x, y) \leq c \cdot d(x, y)$.*

Then F has at least one fixed point in X.

Proof. By *ii*) and the proof of the Theorem 2.11, there exists a sequence $(x_n)_{n\in\mathbb{N}}$ in X such that

1) $x_{n+1} \in Fx_n$, for all $n \in \mathbb{N}$;
2) $d(x_n, x_{n+1}) \leq \varphi(x_n) - \varphi(x_{n+1})$, for all $n \in \mathbb{N}$.

By *iii*) there exists $c > 0$ such that

$$\rho(x_n, x_{n+1}) \leq c \cdot d(x_n, x_{n+1}) \leq c \cdot (\varphi(x_n) - \varphi(x_{n+1})), \text{ for all } n \in \mathbb{N}.$$

We will prove that the series $\sum_{n\in\mathbb{N}} \rho(x_n, x_{n+1})$ is convergent. For this purpose, we need to show that the sequence of its partial sums is convergent in $\mathbb{R}^m_+$.

Denote by $s_n = \sum_{k=0}^{n} \rho(x_k, x_{k+1})$. Then $s_{n+1} - s_n = \rho(x_{n+1}, x_{n+2}) \geq 0$, for each $n \in \mathbb{N}$. Moreover $s_n \leq \sum_{k=0}^{n} \left[c\varphi(x_k) - c\varphi(x_{k+1})\right] \leq c\varphi(x_0)$. Hence $(s_n)_{n\in\mathbb{N}}$ is upper bounded and increasing in $\mathbb{R}^m_+$. So the sequence $(s_n)_{n\in\mathbb{N}}$ is convergent in $\mathbb{R}^m_+$. It follows that the sequence $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence and, from the completeness of the metric space (X, ρ), convergent to a certain element $x^* \in X$. The conclusion follows from i). □

For more considerations on multivalued Kannan, Reich and Caristi operators, see [9] and the references therein.

Acknowledgement. The author wish to thank for the financial support provided from programs co-financed by The Sectoral Operational Programme Human Resources Development, Contract POSDRU 6/1.5/S/3-"Doctoral studies: through science towards society".

References

[1] Allaire, G., *Numerical Linear Algebra*, Springer, New York, 2008.

[2] Aubin, J.-P., Siegel, J., *Fixed points and stationary points of dissipative multivalued maps*, Proc. Amer. Math. Soc., **78**(1980), 391-398.

[3] Kasahara, S., *On some generalizations of the Banach contraction theorem*, Publ. RIMS, Kyoto Univ., **12**(1976), 427-437.

[4] Mizoguchi, N., Takahashi, W., *Fixed point theorems for multivalued mappings on complete metric spaces*, J. Math. Anal. Appl., **141**(1989), 177-188.

[5] Perov, A.I., *On Cauchy problem for a system of ordinary differential equations*, Pviblizhen, Met. Reshen. Differ. Uvavn., **2**(1964), 115-134.

[6] Precup, R., *The role of the matrices that are convergent to zero in the study of semilinear operator systems*, Math. & Computer Modeling, **49**(2009), 703-708.

[7] Rus, I.A., *Principles and Applications of the Fixed Point Theory*, (in Romanian), Ed. Dacia, 1979.

[8] Rus, I.A., *Kasahara spaces*, Sci. Math. Jpn., **72**(2010), no. 1, 101-110.

[9] Rus, I. A., Petruşel, A., Petruşel, G., *Fixed Point Theory*, Cluj Univ. Press, Cluj-Napoca, 2008.

[10] Turinici, M., *Finite dimensional vector contractions and their fixed points*, Studia Univ. Babeş-Bolyai Math., **35**(1990), no. 1, 30-42.

[11] Varga, R.S., *Matrix Iterative Analysis*, Springer, Berlin, 2000.

Alexandru-Darius Filip
"Babeş-Bolyai" University
Faculty of Mathematics and Computer Sciences
1, Kogălniceanu Street
400084 Cluj-Napoca
Romania
e-mail: `darius.filip@econ.ubbcluj.ro`

Stud. Univ. Babeş-Bolyai Math. 56(2011), No. 3, 29–44

Some multivalent functions with negative coefficients defined by using a certain fractional derivative operator

Mohamed K. Aouf

Abstract. In this paper we investigate the various important properties and characteristics of the subclasses $S_n(p, q, \alpha, \lambda)$ and $C_n(p, q, \alpha, \lambda)$ of multivalent functions with negative coefficients defined by using a certain operator of fractional derivatives. We also derive many results for the modified Hadamard products of functions belonging to the classes $S_n(p, q, \alpha, \lambda)$ and $C_n(p, q, \alpha, \lambda)$. Finally, several applications involving an integral operator and certain fractional calculus operators are also considered.

Mathematics Subject Classification (2010): 30C45.

Keywords: Multivalent functions, differential operator, modified-Hadamard product, fractional calculus.

1. Introduction

Let $T(n, p)$ denote the class of functions of the form :

$$f(z) = z^p - \sum_{k=n+p}^{\infty} a_k z^k \quad (a_k \geq 0; p, n \in N = \{1, 2, ...\}), \tag{1.1}$$

which are analytic and p-valent in the open unit disc $U = \{z : |z| < 1\}$. A function $f(z) \in T(n, p)$ is said to be p-valently starlike of order α if it satisfies the inequality :

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha \quad (z \in U;\ 0 \leq \alpha < p; p \in N). \tag{1.2}$$

We denote by $T_n^*(p, \alpha)$ the class of all p-valently starlike functions of order α. Also a function $f(z) \in T(n, p)$ is said to be p-valently convex of order α if it satisfies the inequality:

$$\operatorname{Re}\left\{1+\frac{zf''(z)}{f'(z)}\right\} > \alpha \quad (z \in U;\ 0 \le \alpha < p; p \in N). \tag{1.3}$$

We denote by $C_n(p,\alpha)$ the class of all p-valently convex functions of order α. We note that (see for example Duren [4] and Goodman [5])

$$f(z) \in C_n(p,\alpha) \Longleftrightarrow \frac{zf'(z)}{p} \in T_n^*(p,\alpha) \quad (0 \le \alpha < p; p \in N). \tag{1.4}$$

The classes $T_n^*(p,\alpha)$ and $C_n(p,\alpha)$ are studied by Owa [12].

Various operators of fractional calculus (that is, fractional integral and fractional derivative) have been studied in the literature rather extensively (cf., e.g. [3], [10], [15] and [16]) see also the various references cited therein). For our present investigations, we recall the following definitions.

Definition 1.1. (Fractional Integral Operator). *The fractional integral operator of order λ is defined, for a function $f(z)$, by*

$$D_z^{-\lambda} f(z) = \frac{1}{\Gamma(\lambda)} \int_0^z \frac{f(\zeta)}{(z-\zeta)^{1-\lambda}} d\zeta \quad (\lambda > 0), \tag{1.5}$$

where $f(z)$ is an analytic function in a simply-connected region of the z-plane containg the origin, and the multiplicity of $(z-\zeta)^{\lambda-1}$ is removed by requiring $\log(z-\zeta)$ to be real when $z-\zeta > 0$.

Definition 1.2. (Fractional Derivative Operator). *The fractional derivative of order λ is defined, for a function $f(z)$, by*

$$D_z^{\lambda} f(z) = \frac{1}{\Gamma(1-\lambda)} \frac{d}{dz} \int_0^z \frac{f(\zeta)}{(z-\zeta)^{\lambda}} d\zeta \quad (0 \le \lambda < 1), \tag{1.6}$$

where $f(z)$ is constrained, and the multiplicity of $(z-\zeta)^{-\lambda}$ is removed, as in Definition 1.1.

Definition 1.3. (Extended Fractional Derivative Operator). *Under the hypotheses of Definition 1.2, the fractional derivative of order $n+\lambda$ is defined, for a function $f(z)$, by*

$$D_z^{n+\lambda} f(z) = \frac{d^n}{dz^n} D_z^{\lambda} f(z) \quad (0 \le \lambda < 1; n \in N_0 = N \cup \{0\}). \tag{1.7}$$

Srivastava and Aouf [15] defined and studied the operator :

$$\Omega_z^{(\lambda,p)} f(z) = \frac{\Gamma(p+1-\lambda)}{\Gamma(p+1)} z^{\lambda} D_z^{\lambda} f(z) \quad (0 \le \lambda \le 1; p \in N). \tag{1.8}$$

For each $f(z) \in T(n,p)$, we have

$$\text{(i)} \qquad \Omega_z^{(\lambda,p)} f(z) = z^p - \sum_{k=n+p}^{\infty} \frac{\Gamma(k+1)\Gamma(p+1-\lambda)}{\Gamma(p+1)\Gamma(k+1-\lambda)} a_k z^k, \tag{1.9}$$

$$\text{(ii)} \qquad \left(\Omega_z^{(\lambda,p)} f(z)\right)^{(q)} = \delta(p,q) z^{p-q} - \sum_{k=n+p}^{\infty} \frac{\Gamma(k+1)\Gamma(p+1-\lambda)}{\Gamma(p+1)\Gamma(k+1-\lambda)} \delta(k,q) a_k z^{k-q}$$

$$(q \in N_0 = N \cup \{0\}) \tag{1.10}$$

where

$$\delta(p,q) = \begin{cases} 1 & (q = 0) \\ p(p-1)...(p-q+1) & (q \neq 0), \end{cases} \tag{1.11}$$

and

$$\text{(iii)} \qquad \Omega_z^{(0,p)} f(z) = f(z) \quad \text{and} \quad \Omega_z^{(1,p)} f(z) = \frac{zf'(z)}{p}. \tag{1.12}$$

In this paper we investigate various interesting properties and characteristics of functions belonging to two subclasses $S_n(p,q,\alpha,\lambda)$ and $C_n(p,q,\alpha,\lambda)$ of the class $T(n,p)$, which consist (respectively) of p-valently starlike and p-valently convex functions of order $\alpha\,(0 \le \alpha < p; p \in N)$. Indeed we have

$$S_n(p,q,\alpha,\lambda) = \left\{ f(z) \in T(n,p) : \text{Re}\left(\frac{z\left(\Omega_z^{(\lambda,p)} f(z)\right)^{(1+q)}}{\left(\Omega_z^{(\lambda,p)} f(z)\right)^{(q)}} \right) > \alpha \right.$$

$$\left. (z \in U; 0 \le \alpha < p-q; p,n \in N; q \in N_0; p > q) \right\} \tag{1.13}$$

and

$$C_n(p,q,\alpha,\lambda) = \left\{ f(z) \in T(n,p) : \text{Re}\left(1 + \frac{z\left(\Omega_z^{(\lambda,p)} f(z)\right)^{(2+q)}}{\left(\Omega_z^{(\lambda,p)} f(z)\right)^{(1+q)}} \right) > \alpha \right.$$

$$\left. (z \in U; 0 \le \alpha < p-q; p,n \in N; q \in N_0; p > q) \right\}. \tag{1.14}$$

We note that, by specializing the parameters n, p, q, α and λ, we obtain the following subclasses studied by various authors :

(i) $S_n(p,q,\alpha,0) = S_n(p,q,\alpha)$ and $C_n(p,q,\alpha,0) = C_n(p,q,\alpha)$ (Chen et al. [2]);

(ii) $S_n(p,0,\alpha,0)$ $(0 \le \alpha < p; p,n \in N)$

$$= \begin{cases} T_n^*(p,\alpha) & \text{(Owa [12])} \\ T_\alpha(p,n) & \text{(Yamakawa [19])}; \end{cases}$$

(iii) $S_n(p,0,\alpha,1) = C_n(p,0,\alpha,0)$ $(0 \le \alpha < p; p,n \in N)$

$$= \begin{cases} C_n(p,\alpha) & \text{(Owa [12])} \\ CT_\alpha(p,n) & \text{(Yamakawa [19])}; \end{cases}$$

(iv) $S_1(p,0,\alpha,0) = T^*(p,\alpha)$ and $S_n(1,0,\alpha,1) = C_1\ (p,0,\alpha,0) = C(p,\alpha)$, $(0 \le \alpha < p; p \in N)$ (Owa [11] and Salagean et al. [13]);

(v) $S_1(p,0,\alpha,\beta) = S^*(p,\alpha,\beta)$ and $C_1(p,0,\alpha,\beta) = C^*\ (p,\alpha,\beta)$ $(0 \le \alpha < p;$ $p \in N;\ 0 \le \beta < 1)$ (Hossen [7]);

(vi) $S_1(1,0,\alpha,\beta) = T^*(\alpha,\beta)$ and $C_1(1,0,\alpha,\beta) = C(\alpha,\beta)$ $(0 \le \alpha < 1;\ 0 < \beta \le 1)$ (Gupta and Jain [6]);

(vii) $S_n(1,0,\alpha,1) = T_\alpha(n)$ and $C_n(1,0,\alpha,1) = C_\alpha(n)\,(0 \le \alpha < 1;\, n \in N)$ (Srivastava et al. [18]);

In our present paper, we shall make use of the familiar operator $J_{c,p}$ defined by (cf. [1], [8] and [9]; see also [17])

$$(J_{c,p}f)(z) = \frac{c+p}{z^c}\int_0^z t^{c-1}f(t)dt\,. \tag{1.15}$$

$$(f(z) \in T(n,p); c > -p; p \in N)$$

as well as the fractional calculus operator D_z^μ for which it is well known that (see, for details, [10] and [15]; see also Section 6 below)

$$D_z^\mu\{z^\rho\} = \frac{\Gamma(\rho+1)}{\Gamma(\rho+1-\mu)}z^{\rho-\mu} \quad (\rho > -1;\, \mu \in R) \tag{1.16}$$

in terms of Gamma functions.

2. Coefficients estimates

Theorem 2.1. *Let the function $f(z)$ be defined by (1.1).*

Then $f(z) \in S_n(p,q,\alpha,\lambda)$ if and only if

$$\sum_{k=n+p}^{\infty} \frac{(k-q-\alpha)\Gamma(k+1)\Gamma(p+1-\lambda)}{\Gamma(p+1)\Gamma(k+1-\lambda)}\delta(k,q)a_k \le (p-q-\alpha)\delta(p,q)\,, \tag{2.1}$$

where $\delta(p,q)$ is given by (1.9).

Proof. Assume that the inequality (2.1) holds true. Thus we find that

$$\left| \frac{z\left(\Omega_z^{(\lambda,p)}f(z)\right)^{(1+q)}}{\left(\Omega_z^{(\lambda,p)}f(z)\right)^{(q)}} - (p-q) \right|$$

$$\le \frac{\sum_{k=n+p}^{\infty} \frac{(k-p)\Gamma(k+1)\Gamma(p+1-\lambda)}{\Gamma(p+1)\Gamma(k+1-\lambda)}\delta(k,q)a_k\,|z|^{k-p}}{\delta(p,q) - \sum_{k=n+p}^{\infty} \frac{\Gamma(k+1)\Gamma(p+1-\lambda)}{\Gamma(p+1)\Gamma(k+1-\lambda)}\delta(k,q)a_k\,|z|^{k-p}}$$

$$\le \frac{\sum_{k=n+p}^{\infty} \frac{(k-p)\Gamma(k+1)\Gamma(p+1-\lambda)}{\Gamma(p+1)\Gamma(k+1-\lambda)}\delta(k,q)a_k}{\delta(p,q) - \sum_{k=n+p}^{\infty} \frac{\Gamma(k+1)\Gamma(p+1-\lambda)}{\Gamma(p+1)\Gamma(k+1-\lambda)}\delta(k,q)a_k}$$

$$\le p-q-\alpha\,.$$

This shows that the values of the function

$$\Phi(z) = \frac{z\left(\Omega_z^{(\lambda,p)}f(z)\right)^{(1+q)}}{\left(\Omega_z^{(\lambda,p)}f(z)\right)^{(q)}} \tag{2.2}$$

lie in a circle which is centered at $w = (p-q)$ and whose radius is $(p-q-\alpha)$. Hence $f(z)$ satisfies the condition (1.11).

Conversely, assume that the function $f(z)$ defined by (1.1) is in the class $S_n(p, q, \alpha, \lambda)$. Then we have

$$\operatorname{Re}\left\{ \frac{z\left(\Omega_z^{(\lambda,p)} f(z)\right)^{(1+q)}}{\left(\Omega_z^{(\lambda,p)} f(z)\right)^{(q)}} \right\}$$

$$= \operatorname{Re}\left\{ \frac{(p-q)\delta(p,q) - \sum_{k=n+p}^{\infty} \frac{(k-q)\Gamma(k+1)\Gamma(p+1-\lambda)}{\Gamma(p+1)\Gamma(k+1-\lambda)} \delta(k,q) a_k z^{k-p}}{\delta(p,q) - \sum_{k=n+p}^{\infty} \frac{\Gamma(k+1)\Gamma(p+1-\lambda)}{\Gamma(p+1)\Gamma(k+1-\lambda)} \delta(k,q) a_k z^{k-p}} \right\} > \alpha \tag{2.3}$$

for some $\alpha\,(0 \leq \alpha < p-q)$, $0 \leq \lambda \leq 1$, $p, n \in N$, $q \in N_0$, $p > q$ and $z \in U$. Choose values of z on the real axis so that $\Phi(z)$ given by (2.2) is real. Upon clearing the denominator in (2.3) and letting $z \to 1^-$ through real values, we can see that

$$(p-q)\delta(p,q) - \sum_{k=n+p}^{\infty} \frac{(k-q)\Gamma(k+1)\Gamma(p+1-\lambda)}{\Gamma(p+1)\Gamma(k+1-\lambda)} \delta(k,q) a_k$$

$$\geq \alpha \left(\delta(p,q) - \sum_{k=n+p}^{\infty} \frac{\Gamma(k+1)\Gamma(p+1-\lambda)}{\Gamma(p+1)\Gamma(k+1-\lambda)} \delta(k,q) a_k \right). \tag{2.4}$$

Thus we have the inequality (2.1). □

Corollary 2.2. *Let the function $f(z)$ defined by (1.1) be in the class $S_n(p, q, \alpha, \lambda)$. Then*

$$a_k \leq \frac{(p-q-\alpha)\delta(p,q)\Gamma(p+1)\Gamma(k+1-\lambda)}{(k-q-\alpha)\delta(k,q)\Gamma(k+1)\Gamma(p+1-\lambda)}$$

$$(k \geq n+p; p, n \in N; q \in N_0; p > q). \tag{2.5}$$

The result is sharp for the function $f(z)$ given by

$$f(z) = z^p - \frac{(p-q-\alpha)\delta(p,q)\Gamma(p+1)\Gamma(k+1-\lambda)}{(k-q-\alpha)\delta(k,q)\Gamma(k+1)\Gamma(p+1-\lambda)} z^k$$

$$(k \geq n+p; p, n \in N; q \in N_0; p > q). \tag{2.6}$$

Theorem 2.3. *Let the function $f(z)$ defined by (1.1). Then $f(z) \in C_n(p, q, \alpha, \lambda)$ if and only if*

$$\sum_{k=n+p}^{\infty} \frac{(k-q-\alpha)\Gamma(k+1)\Gamma(p+1-\lambda)}{(p-q-\alpha)\Gamma(p+1)\Gamma(k+1-\lambda)} \delta(k, q+1) a_k \leq (p-q-\alpha)\delta(p, q+1). \tag{2.7}$$

Corollary 2.4. *Let the function $f(z)$ defined by (1.1) be in the class $C_n(p,q,\alpha,\lambda)$. Then*

$$a_k \leq \frac{(p-q-\alpha)\delta(p,q+1)\Gamma(p+1)\Gamma(k+1-\lambda)}{(k-q-\alpha)\delta(k,q+1)\Gamma(k+1)\Gamma(p+1-\lambda)}$$
$$(k \geq n+p; p,n \in N; q \in N_0; p > q)\,. \tag{2.8}$$

The result is sharp for the function $f(z)$ given by

$$f(z) = z^p - \frac{(p-q-\alpha)\delta(p,q+1)\Gamma(p+1)\Gamma(k+1-\lambda)}{(k-q-\alpha)\delta(k,q+1)\Gamma(k+1)\Gamma(p+1-\lambda)} z^k$$
$$(k \geq n+p; p,n \in N; q \in N_0; p > q)\,. \tag{2.9}$$

3. Distortion theorems

Theorem 3.1. *If a function $f(z)$ defined by (1.1) is in the class $S_n(p,q,\alpha,\lambda)$, then*

$$\left\{\frac{p!}{(p-j)!} - \frac{(p-q-\alpha)\delta(p,q)(n+p-q)!\Gamma(p+1)\Gamma(n+p+1-\lambda)}{(n+p-q-\alpha)(n+p-j)!\Gamma(n+p+1)\Gamma(p+1-\lambda)}|z|^n\right\}|z|^{p-j}$$
$$\leq \left|f^{(j)}(z)\right| \leq \left\{\frac{p!}{(p-j)!} + \frac{(p-q-\alpha)\delta(p,q)(n+p-q)!\Gamma(p+1)\Gamma(n+p+1-\lambda)}{(n+p-q-\alpha)(n+p-j)!\Gamma(n+p+1)\Gamma(p+1-\lambda)}|z|^n\right\}|z|^{p-j}$$
$$(z \in U; 0 \leq \alpha < p-q;\ p,n \in N;\ q,j \in N_0;\ p > \max\{q,j\})\,. \tag{3.1}$$

The result is sharp for the function $f(z)$ given by

$$f(z) = z^p - \frac{(p-q-\alpha)\delta(p,q)\Gamma(p+1)\Gamma(n+p+1-\lambda)}{(n+p-q-\alpha)\delta(n+p,q)\Gamma(n+p+1)\Gamma(p+1-\lambda)} z^{n+p}\,(p,n \in N)\,. \tag{3.2}$$

Proof. In view of Theorem 2.1, we have

$$\frac{(n+p-q-\alpha)\delta(n+p,q)\Gamma(n+p+1)\Gamma(p+1-\lambda)}{(p-q-\alpha)\delta(p,q)(n+p)!\Gamma(p+1)\Gamma(n+p+1-\lambda)} \sum_{k=n+p}^{\infty} k!a_k$$
$$\leq \sum_{k=n+p}^{\infty} \frac{(k-q-\alpha)\delta(k,q)\Gamma(k+1)\Gamma(p+1-\lambda)}{(p-q-\alpha)\delta(p,q)\Gamma(p+1)\Gamma(k+1-\lambda)} a_k \leq 1$$

which readily yields

$$\sum_{k=n+p}^{\infty} k!a_k \leq \frac{(p-q-\alpha)\delta(p,q)(n+p-q)!\Gamma(p+1)\Gamma(n+p+1-\lambda)}{(n+p-q-\alpha)\Gamma(n+p+1)\Gamma(p+1-\lambda)}\,. \tag{3.3}$$

Now, by differentiating both sides of (1.1) j times, we obtain

$$f^{(j)}(z) = \frac{p!}{(p-j)!} z^{p-j} - \sum_{k=n+p}^{\infty} \frac{k!}{(k-j)!} a_k z^{k-j} \tag{3.4}$$
$$(k \geq n+p; p,n \in N; q,j \in N_0 = N \cup \{0\}; p > \max\{q,j\})\,.$$

Theorem 3.1 follows readily from (3.3) and (3.4).

Finally, it is easy to see that the bounds in (3.1) are attained for the function $f(z)$ given by (3.2). □

Theorem 3.2. *If a function $f(z)$ defined by (1.1) is in the class $C_n(p,q,\alpha,\lambda)$, then*

$$\left\{\frac{1}{(p-j)!} - \frac{(p-q-\alpha)(n+p-q-1)!\Gamma(p+1)\Gamma(n+p+1-\lambda)}{(n+p-q-\alpha)(p-q-1)!(n+p-j)!\Gamma(n+p+1)\Gamma(p+1-\lambda)}|z|^n\right\} p!\,|z|^{p-j}$$

$$\leq \left|f^{(j)}(z)\right|$$

$$\leq \left\{\frac{1}{(p-j)!} + \frac{(p-q-\alpha)(n+p-q-1)!\Gamma(p+1)\Gamma(n+p+1-\lambda)}{(n+p-q-\alpha)(p-q-1)!(n+p-j)!\Gamma(n+p+1)\Gamma(p+1-\lambda)}|z|^n\right\} p!\,|z|^{p-j}$$

$$(z \in U; 0 \leq \alpha < p-q;\ p,n \in N;\ q,j \in N_0;\ p > \max\{q,j\}). \tag{3.5}$$

The result is sharp for the function $f(z)$ given by

$$f(z) = z^p - \frac{(p-q-\alpha)\delta(p,q+1)}{(n+p-q-\alpha)\delta(n+p,q+1)} z^{n+p} \quad (p,n \in N; q \in N_0; p > q). \tag{3.6}$$

Radii of close-to-convexity, starlikeness and convexity

Theorem 3.3. *Let the function $f(z)$ defined by (1.1) be in the class $S_n(p,q,\alpha,\lambda)$, then*

(i) $f(z)$ is p-valently close-to-convex of order $\varphi\,(0 \leq \varphi < p)$ in $|z| < r_1$, where

$$r_1 = \inf_k \left\{\frac{(k-q-\alpha)\delta(k,q)\Gamma(k+1)\Gamma(p+1-\lambda)}{(p-q-\alpha)\delta(p,q)\Gamma(p+1)\Gamma(k+1-\lambda)}\left(\frac{p-\varphi}{k}\right)\right\}^{\frac{1}{k-p}}$$

$$(k \geq n+p; p,n \in N; q \in N_0; p > q), \tag{4.1}$$

(ii) $f(z)$ is p-valently starlike of order $\varphi\,(0 \leq \varphi < p)$ in $|z| < r_2$, where

$$r_2 = \inf_k \left\{\frac{(k-q-\alpha)\delta(k,q)\Gamma(k+1)\Gamma(p+1-\lambda)}{(p-q-\alpha)\delta(p,q)\Gamma(p+1)\Gamma(k+1-\lambda)}\left(\frac{p-\varphi}{k-\varphi}\right)\right\}^{\frac{1}{k-p}}$$

$$(k \geq n+p; p,n \in N; q \in N_0; p > q), \tag{4.2}$$

(iii) $f(z)$ is p-valently convex of order $\varphi\,(0 \leq \varphi < p)$ in $|z| < r_3$, where

$$r_3 = \inf_k \left\{\frac{(k-q-\alpha)\delta(k,q)\Gamma(k+1)\Gamma(p+1-\lambda)}{(p-q-\alpha)\delta(p,q)\Gamma(p+1)\Gamma(k+1-\lambda)} \cdot \frac{p(p-\varphi)}{k(k-\varphi)}\right\}^{\frac{1}{k-p}}$$

$$(k \geq n+p; p,n \in N; q \in N_0; p > q). \tag{4.3}$$

Each of these results is sharp for the function $f(z)$ given by (2.6).

Proof. It is sufficient to show that

$$\left|\frac{f'(z)}{z^{p-1}} - p\right| \leq p - \varphi \quad (|z| < r_1; 0 \leq \varphi < p; p \in N), \tag{4.4}$$

$$\left|\frac{zf'(z)}{f(z)} - p\right| \leq p - \varphi \quad (|z| < r_2; 0 \leq \varphi < p; p \in N), \tag{4.5}$$

and that

$$\left|1+\frac{zf''(z)}{f'(z)}-p\right| \le p-\varphi \quad (|z|<r_3; 0\le\varphi<p; p\in N) \tag{4.6}$$

for a function $f(z) \in S_n(p,q,\alpha,\lambda)$, where r_1, r_2 and r_3 are defined by (4.1), (4.2) and (4.3), respectively. □

4. Modified Hadamard products

For the functions $f_\nu(z)\,(\nu=1,2)$ given by

$$f_\nu(z)=z^p-\sum_{k=n+p}^{\infty} a_{k,\nu} z^k \quad (a_{k,\nu}\ge 0; \nu=1,2) \tag{5.1}$$

we denote that $(f_1 \otimes f_2)(z)$ the modified Hadamard product (or convolution) of the functions $f_1(z)$ and $f_2(z)$, defined by

$$(f_1\otimes f_2)(z)=z^p-\sum_{k=n+p}^{\infty} a_{k,1}.a_{k,2} z^k\,. \tag{5.2}$$

Theorem 4.1. *Let the functions $f_\nu(z)\,(\nu=1,2)$ defined by (5.1) be in the class $S_n(p,q,\alpha,\lambda)$. Then $(f_1\otimes f_2)(z)\in S_n(p,q,\gamma,\lambda)$ where*

$$\gamma=(p-q)-$$

$$\frac{n(p-q-\alpha)^2\delta(p,q)\Gamma(p+1)\Gamma(n+p+1-\lambda)}{(n+p-q-\alpha)^2\delta(n+p,q)\Gamma(n+p+1)\Gamma(p+1-\lambda)-(p-q-\alpha)^2\delta(p,q)\Gamma(p+1)\Gamma(n+p+1-\lambda)}\,. \tag{5.3}$$

The result is sharp for the functions $f_\nu(z)\,(\nu=1,2)$ given by

$$f_\nu(z)=z^p-\frac{(p-q-\alpha)\delta(p,q)\Gamma(p+1)\Gamma(n+p+1-\lambda)}{(n+p-q-\alpha)\delta(n+p,q)\Gamma(n+p+1)\Gamma(p+1-\lambda)} z^{n+p}$$

$$(p,n\in N; \nu=1,2)\,. \tag{5.4}$$

Proof. Emploing the technique used earlier by Schild and Silverman [14], we need to find the largest γ such that

$$\sum_{k=n+p}^{\infty}\frac{(k-q-\gamma)\delta(k,q)\Gamma(k+1)\Gamma(k+1)\Gamma(p+1-\lambda)}{(p-q-\gamma)\delta(p,q)\Gamma(p+1)\Gamma(k+1-\lambda)} a_{k,1}; a_{k,2}\le 1$$

$$(f_\nu(z)\in S_n(p,\alpha,\beta,\lambda) \quad (\nu=1,2))\,. \tag{5.5}$$

Since $f_\nu(z)\in S_n(p,\alpha,\beta,\lambda)$ $(\nu=1,2)$, we readily see that

$$\sum_{k=n+p}^{\infty}\frac{(k-q-\alpha)\delta(k,q)\Gamma(k+1)\Gamma(p+1-\lambda)}{(p-q-\alpha)\delta(p,q)\Gamma(p+1)\Gamma(k+1-\lambda)} a_{k,\nu}\le 1 \;\; (\nu=1,2). \tag{5.6}$$

Therefore, by the Cauchy-Schwarz inequality, we obtain

$$\sum_{k=n+p}^{\infty}\frac{(k-q-\alpha)\delta(k,q)\Gamma(k+1)\Gamma(p+1-\lambda)}{(p-q-\gamma)\delta(p,q)\Gamma(p+1)\Gamma(k+1-\lambda)} \sqrt{a_{k,1}; a_{k,2}}\le 1\,. \tag{5.7}$$

Thus we only need to show that

$$\frac{(k-q-\gamma)}{(p-q-\gamma)}a_{k,1}.a_{k,2} \leq \frac{(k-q-\alpha)}{(p-q-\alpha)}\sqrt{a_{k,1}.a_{k,2}}\ (k \geq n+p; p, n \in N), \quad (5.8)$$

or, equivalently, that

$$\sqrt{a_{k,1}.a_{k,2}} \leq \frac{(p-q-\gamma)(k-q-\alpha)}{(p-\alpha)(k-q-\gamma)} \quad (k \geq n+p; p, n \in N). \quad (5.9)$$

Hence, in light of the inequality (5.7), it is sufficient to prove that

$$\frac{(p-q-\alpha)\delta(p,q)\Gamma(p+1)\Gamma(k+1-\lambda)}{(k-q-\alpha)\delta(k,q)\Gamma(k+1)\Gamma(p+1-\lambda)} \leq$$

$$\frac{(p-\gamma)(k-q-\alpha)}{(p-\alpha)(k-q-\gamma)} \quad (k \geq n+p; p, n \in N). \quad (5.10)$$

It follows from (5.10) that

$$\gamma \leq (p-q)-$$

$$\frac{(k-p)(p-q-\alpha)^2\delta(p,q)\Gamma(p+1)\Gamma(k+1-\lambda)}{(k-q-\alpha)^2\delta(k,q)\Gamma(k+1)\Gamma(p+1-\lambda)-(p-q-\alpha)^2\delta(p,q)\Gamma(p+1)\Gamma(k+1-\lambda)}$$

$$(k \geq n+p;\ p, n \in N;\ q \in N_0;\ p > q). \quad (5.11)$$

Now, defining the function $\theta(k)$ by

$$\theta(k) = (p-q)-$$

$$\frac{(k-p)(p-q-\alpha)^2\delta(p,q)\Gamma(p+1)\Gamma(k+1-\lambda)}{(k-q-\alpha)^2\delta(k,q)\Gamma(k+1)\Gamma(p+1-\lambda)-(p-q-\alpha)^2\delta(p,q)\Gamma(p+1)\Gamma(k+1-\lambda)}$$

$$(k \geq n+p;\ p, n \in N;\ q \in N_0;\ p > q), \quad (5.12)$$

we see that $\theta(k)$ is an increasing function of k. Therefore, we conclude that

$$\gamma \leq \theta(n+p) = (p-q)-$$

$$\frac{n(p-q-\alpha)^2\delta(p,q)\Gamma(p+1)\Gamma(n+p+1-\lambda)}{(n+p-q-\alpha)^2\delta(n+p,q)\Gamma(n+p+1)\Gamma(p+1-\lambda)-(p-q-\alpha)^2\delta(p,q)\Gamma(p+1)\Gamma(n+p+1-\lambda)} \quad (5.13)$$

which evidently completes the proof of Theorem 4.1. □

Putting $\lambda = 0$ in Theorem 4.1, we obtain

Corollary 4.2. *Let the functions* $f_\nu(z)\,(\nu = 1, 2)$ *defined by (5.1) be in the class* $S_n(p, q, \alpha)$*. Then* $(f_1 \otimes f_2)(z) \in S_n(p, q, \gamma)$*, where*

$$\gamma = (p-q) - \frac{n(p-q-\alpha)^2\delta(p,q)}{(n+p-q-\alpha)^2\delta(n+p,q)-(p-q-\alpha)^2\delta(p,q)}. \quad (5.14)$$

The result is sharp.

Remark 4.3. We note that the result obtained by Chen et al. [2, Theorem 5] is not correct. The correct result is given by (5.14).

Using arguments similar to those in the proof of Theorem 4.1, we obtain the following results.

Theorem 4.4. *Let the functions $f_\nu(z)\,(\nu = 1,2)$ defined by (5.1) be in the class $C_n(p,q,\alpha,\lambda)$. Then $(f_1 \otimes f_2)(z) \in C_n(p,q,\gamma,\lambda)$ where*

$$\gamma = (p-q)- \frac{n(p-q-\alpha)^2\delta(p,q+1)\Gamma(p+1)\Gamma(n+p+1-\lambda)}{(n+p-q-\alpha)^2\delta(n+p,q+1)\Gamma(n+p+1)\Gamma(p+1-\lambda)-(p-q-\alpha)^2\delta(p,q+1)\Gamma(p+1)\Gamma(n+p+1-\lambda)}. \tag{5.15}$$

The result is sharp for the functions $f_\nu(z)\,(\nu = 1,2)$ given by

$$f(z) = z^p - \frac{(p-q-\alpha)\delta(p,q+1)\Gamma(p+1)\Gamma(n+p+1-\lambda)}{(n+p-q-\alpha)\delta(n+p,q+1)\Gamma(n+p+1)\Gamma(p+1-\lambda)} z^{n+p}$$
$$(p,n \in N;\ q \in N_0;\ p > q;\ \nu = 1,2)\,. \tag{5.16}$$

Putting $\lambda = 0$ in Theorem 4.4, we obtain

Corollary 4.5. *Let the functions $f_\nu(z)\,(\nu = 1,2)$ defined by (5.1) be in the class $C_n(p,q,\alpha)$. Then $(f_1 \otimes f_2)(z) \in C_n(p,q,\gamma)$, where*

$$\gamma = (p-q) - \frac{n(p-q-\alpha)^2\delta(p,q+1)}{(n+p-q-\alpha)^2\delta(n+p,q+1)-(p-q-\alpha)^2\delta(p,q+1)}. \tag{5.17}$$

The result is sharp.

Remark 4.6. We note that the result obtained by Chen et al. [2, Theorem 6] is not correct. The correct result is given by (5.17).

Theorem 4.7. *Let the function $f_1(z)$ defined by (5.1) be in the class $S_n(p,q,\alpha,\lambda)$. Suppose also that the function $f_2(z)$ defined by (5.2) be in the class $S_n(p,q,\gamma,\lambda)$. Then $(f_1 \otimes f_2)(z) \in S_n(p,q,\zeta,\lambda)$, where $\zeta = (p-q)-$*

$$\frac{n(p-q-\alpha)(p-q-\gamma)\delta(p,q)\Gamma(p+1)\Gamma(n+p+1-\lambda)}{(n+p-q-\alpha)(n+p-q-\gamma)\delta(n+p,q)\Gamma(n+p+1)\Gamma(p+1-\lambda)-\Omega}.$$
$$(\Omega = (p-q-\alpha)(p-q-\gamma)\delta(p,q)\Gamma(p+1)\Gamma(n+p+1-\lambda))\,. \tag{5.18}$$

The result is sharp for the functions $f_\nu(z)\,(\nu = 1,2)$ given by

$$f_1(z) = z^p - \frac{(p-q-\alpha)\delta(p,q)\Gamma(p+1)\Gamma(n+p+1-\lambda)}{(n+p-q-\alpha)\delta(n+p,q)\Gamma(n+p+1)\Gamma(p+1-\lambda)} z^{n+p} \quad (p,n \in N) \tag{5.19}$$

and

$$f_2(z) = z^p - \frac{(p-q-\gamma)\delta(p,q)\Gamma(p+1)\Gamma(n+p+1-\lambda)}{(n+p-q-\gamma)\delta(n+p,q)\Gamma(n+p+1)\Gamma(p+1-\lambda)} z^{n+p} \quad (p,n \in N)\,. \tag{5.20}$$

Theorem 4.8. *Let the functions $f_\nu(z)\,(\nu = 1,2)$ defined by (5.1) be in the class $S_n(p,q,\alpha,\lambda)$. Then the function*

$$h(z) = z^p - \sum_{k=n+p}^{\infty} \left(a_{k,1}^2 + a_{k,2}^2\right) z^k \tag{5.21}$$

belongs to the class $S_n(p,q,\xi,\lambda)$, where

$$\xi = (p-q)- \frac{2n(p-q-\alpha)^2\delta(p,q)\Gamma(p+1)\Gamma(n+p+1-\lambda)}{(n+p-q-\alpha)^2\delta(n+p,q)\Gamma(n+p+1)\Gamma(p+1-\lambda)-2(p-q-\alpha)^2\delta(p,q)\Gamma(p+1)\Gamma(n+p+1-\lambda)}. \tag{5.22}$$

The result is sharp for the functions $f_\nu(z)\,(\nu = 1,2)$ defined by (5.4).

Theorem 4.9. *Let the functions* $f_\nu(z)\,(\nu = 1,2)$ *defined by (5.1) be in the class* $C_n(p,q,\alpha,\lambda)$. *Then the function* $h(z)$ *defined by (5.21) belongs to the class* $C_n(p,q,\eta,\lambda)$, *where*

$$\eta = (p-q)- \frac{2n(p-q-\alpha)^2\delta(p,q+1)\Gamma(p+1)\Gamma(n+p+1-\lambda)}{(n+p-q-\alpha)^2\delta(n+p,q+1)\Gamma(n+p+1)\Gamma(p+1-\lambda)-2(p-q-\alpha)^2\delta(p,q+1)\Gamma(p+1)\Gamma(n+p+1-\lambda)}. \tag{5.23}$$

The result is sharp for the functions $f_\nu(z)\,(\nu = 1,2)$ *given by (5.16).*

5. Applications of fractional calculus

In this section, we shall investigate the growth and distortion properties of functions in the classes $S_n(p,q,\alpha,\beta)$ and $C_n(p,q,\alpha,\beta)$, involving the operators $J_{c,p}$ and D_z^μ. In order to derive our results, we need the following lemma given by Chen et al. [3].

Lemma 5.1. (see Chen et al. [3]). *Let the function* $f(z)$ *defined by (1.1). Then*

$$D_z^\mu\left\{(J_{c,p})(z)\right\} = \frac{\Gamma(p+1)}{\Gamma(p+1-\mu}z^{p-\mu} - \sum_{k=n+p}^{\infty}\frac{(c+p)\Gamma(k+1)}{(c+k)\Gamma(k+1-\mu)}a_k z^{k-\mu}$$

$$(\mu \in R; c > -p; p,n \in N) \tag{6.1}$$

and

$$J_{c,p}(D_z^\mu\{f(z)\}) = \frac{(c+p)\Gamma(p+1)}{(c+p-\mu)\Gamma(p+1-\mu)}z^{p-\mu} - \sum_{k=n+p}^{\infty}\frac{(c+p)\Gamma(k+1)}{(c+k-\mu)\Gamma(k+1-\mu)}a_k z^{k-\mu}$$

$$(\mu \in R; c > -p; p,n \in N), \tag{6.2}$$

provided that no zeros appear in the denominators in (6.1) and (6.2).

Theorem 5.2. *Let the function* $f(z)$ *defined by (1.1) be in the class* $S_n(p,q,\alpha,\lambda)$. *Then*

$$\left|D_z^{-\mu}\left\{(J_{c,p}f)(z)\right\}\right| \geq \left\{\frac{\Gamma(p+1)}{\Gamma(p+1+\mu)} - \frac{(c+p)(p-q-\alpha)\delta(p,q)\Gamma(p+1)\Gamma(n+p+1-\lambda)}{(c+n+p)(n+p-q-\alpha)\delta(n+p,q)\Gamma(n+p+1+\mu)\Gamma(p+1-\lambda)}|z|^n\right\}|z|^{p+\mu}$$

$$(z \in U; 0 \leq \alpha < p-q; 0 \leq \lambda \leq 1; \mu > 0; c > -p; p,n \in N; q \in N_0; p > q) \tag{6.3}$$

and

$$\left|D_z^{-\mu}\left\{(J_{c,p}f)(z)\right\}\right| \leq \left\{\frac{\Gamma(p+1)}{\Gamma(p+1+\mu)} + \frac{(c+p)(p-q-\alpha)\delta(p,q)\Gamma(p+1)\Gamma(n+p+1-\lambda)}{(c+n+p)(n+p-q-\alpha)\delta(n+p,q)\Gamma(n+p+1+\mu)\Gamma(p+1-\lambda)}|z|^n\right\}|z|^{p+\mu}$$

$$(z \in U; 0 \leq \alpha < p-q; 0 \leq \lambda \leq 1; \mu > 0; c > -p; p,n \in N; q \in N_0; p > q). \tag{6.4}$$

Each of the assertions (6.3) and (6.4) is sharp.

Proof. In view of Theorem 2.1, we have

$$\frac{(n+p-q-\alpha)\delta(n+p,q)\Gamma(n+p+1)\Gamma(p+1-\lambda)}{(p-q-\alpha)\delta(p,q)\Gamma(p+1)\Gamma(n+p+1-\lambda)}\sum_{k=n+p}^{\infty}a_k$$

$$\leq \sum_{k=n}^{\infty}\frac{(k-q-\alpha)\delta(k,q)\Gamma(k+1)\Gamma(p+1-\lambda)}{(p-q-\alpha)\delta(p,q)\Gamma(p+1)\Gamma(k+1-\lambda)}a_k \leq 1\,,$$

which readily yields

$$\sum_{k=n+p}^{\infty}a_k \leq \frac{(p-q-\alpha)\delta(p,q)\Gamma(p+1)\Gamma(n+p+1-\lambda)}{(n+p-q-\alpha)\delta(n+p,q)\Gamma(n+p+1)\Gamma(p+1-\lambda)}\,. \tag{6.5}$$

Consider the function $F(z)$ defined in U by

$$\begin{aligned} F(z) &= \frac{\Gamma(p+1+\mu)}{\Gamma(p+1)}z^{-\mu}D_z^{-\mu}\{(J_{c,p}f)(z)\} \\ &= z^p-\sum_{k=n+p}^{\infty}\frac{(c+p)\Gamma(k+1)\Gamma(p+1+\mu)}{(c+k)\Gamma(p+1)\Gamma(k+1+\mu)}a_kz^k \\ &= z^p-\sum_{k=n+p}^{\infty}\Phi(k)a_kz^k \qquad (z\in U)\,, \end{aligned}$$

where

$$\Phi(k)=\frac{(c+p)\Gamma(k+1)\Gamma(p+1+\mu)}{(c+k)\Gamma(p+1)\Gamma(k+1+\mu)} \quad (k\geq n+p;p,n\in N;\mu>0)\,. \tag{6.6}$$

Since $\Phi(k)$ is a decreasing function of k when $\mu>0$, we get

$$0<\Phi(k)\leq\Phi(n+p)=\frac{(c+p)\Gamma(n+p+1)\Gamma(p+1+\mu)}{(c+n+p)\Gamma(p+1)\Gamma(n+p+1+\mu)}$$

$$(c>-p;p,n\in N;\mu>0)\,. \tag{6.7}$$

Thus, by using (6.5) and (6.7), we deduce that

$$|F(z)|\geq|z|^p-\Phi(n+p)\,|z|^{n+p}\sum_{k=n+p}^{\infty}a_k\geq|z|^p-$$

$$\frac{(c+p)(p-q-\alpha)\delta(p,q)\Gamma(p+1+\mu)\Gamma(n+p+1-\lambda)}{(c+n+p)(n+p-q-\alpha)\delta(n+p,q)\Gamma(n+p+1+\mu)\Gamma(p+1-\lambda)}\,|z|^{n+p} \quad (z\in U)$$

and

$$|F(z)|\leq|z|^p+\Phi(n+p)\,|z|^{n+p}\sum_{k=n+p}^{\infty}a_k\leq|z|^p+$$

$$\frac{(c+p)(p-q-\alpha)\delta(p,q)\Gamma(p+1+\mu)\Gamma(n+p+1-\lambda)}{(c+n+p)(n+p-q-\alpha)\delta(n+p,q)\Gamma(n+p+1+\mu)\Gamma(p+1-\lambda)}\,|z|^{n+p} \quad (z\in U),$$

which yield the inequalities (6.3) and (6.4) of Theorem 5.2. The equalities in (6.3) and (6.4) are attained for the function $f(z)$ given by

$$D_z^{-\mu}\{(J_{c,p}f)(z)\}\leq\left\{\frac{\Gamma(p+1)}{\Gamma(p+1+\mu)}-\right.$$

$$\left.\frac{(c+p)(p-q-\alpha)\delta(p,q)\Gamma(p+1)\Gamma(n+p+1-\lambda)}{(c+n+p)(n+p-q-\alpha)\delta(n+p,q)\Gamma(n+p+1)\Gamma(p+1-\lambda)}z^n\right\} z^{p+\mu} \tag{6.8}$$

or, equivalently, by

$$(J_{c,p}f)(z) = z^p - \frac{(c+p)(p-q-\alpha)\delta(p,q)\Gamma(p+1)\Gamma(n+p+1-\lambda)}{(c+n+p)(n+p-q-\alpha)\delta(n+p,q)\Gamma(n+p+1)\Gamma(p+1-\lambda)}z^{n+p}. \tag{6.9}$$

Thus we complete the proof of Theorem 5.2. □

Theorem 5.3. *Let the function $f(z)$ defined by (1.1) be in the class $S_n(p,q,\alpha,\lambda)$. Then*

$$|D_z^\mu\{(J_{c,p}f)(z)\}| \geq \left\{\frac{\Gamma(p+1)}{\Gamma(p+1-\mu)} - \frac{(c+p)(p-q-\alpha)\delta(p,q)\Gamma(p+1)\Gamma(n+p+1-\lambda)}{(c+n+p)(n+p-q-\alpha)\delta(n+p,q)\Gamma(n+p+1-\mu)\Gamma(p+1-\lambda)}|z|^n\right\}|z|^{p-\mu}$$

$$(z \in U; 0 \leq \alpha < p-q; 0 \leq \lambda \leq 1; 0 \leq \mu < 1; c > -p; p,n \in N; q \in N_0; p > q) \tag{6.10}$$

and

$$|D_z^\mu\{(J_{c,p}f)(z)\}| \leq \left\{\frac{\Gamma(p+1)}{\Gamma(p+1-\mu)} + \frac{(c+p)(p-q-\alpha)\delta(p,q)\Gamma(p+1)\Gamma(n+p+1-\lambda)}{(c+n+p)(n+p-q-\alpha)\delta(n+p,q)\Gamma(n+p+1-\mu)\Gamma(p+1-\lambda)}|z|^n\right\}|z|^{p-\mu}$$

$$(z \in U; 0 \leq \alpha < p-q; 0 \leq \lambda \leq 1; 0 \leq \mu < 1; c > -p; p,n \in N; q \in N_0; p > q). \tag{6.11}$$

Each of the assertions (6.10) and (6.11) is sharp.

Proof. It follows from Theorem 2.1, that

$$\sum_{k=n+p}^{\infty} ka_k \leq \frac{(p-q-\alpha)\delta(p,q)\Gamma(p+1)\Gamma(n+p+1-\lambda)}{(n+p-q-\alpha)\delta(n+p,q)\Gamma(n+p)\Gamma(p+1-\lambda)}. \tag{6.12}$$

We consider the function $H(z)$ defined in U by

$$\begin{aligned} H(z) &= \frac{\Gamma(p+1-\mu)}{\Gamma(p+1)} z^\mu D_z^\mu\{(J_{c,p}f)(z)\} \\ &= z^p - \sum_{k=n+p}^{\infty} \Psi(k) k a_k z^k \qquad (z \in U), \end{aligned}$$

where, for convience,

$$\Psi(k) = \frac{(c+p)\Gamma(k)\Gamma(p+1-\mu)}{(c+k)\Gamma(p+1)\Gamma(k+1-\mu)} \quad (k \geq n+p; p,n \in N; 0 \leq \mu < 1).$$

Since $\Psi(k)$ is a decreasing function of k when $\mu < 1$, we find that

$$0 < \Psi(k) \leq \Psi(n+p) = \frac{(c+p)\Gamma(n+p)\Gamma(p+1-\mu)}{(c+n+p)\Gamma(p+1)\Gamma(n+p+1-\mu)}$$

$$(c > -p; p,n \in N; 0 \leq \mu < 1). \tag{6.13}$$

Consequently, with the aid of (6.12) and (6.13), we find that

$$|H(z)| \geq |z|^p - \Psi(n+p)\,|z|^{n+p} \sum_{k=n+p}^{\infty} ka_k \geq |z|^p -$$

$$\frac{(c+p)(p-q-\alpha)\delta(p,q)\Gamma(p+1-\mu)\Gamma(n+p+1-\lambda)}{(c+n+p)(n+p-q-\alpha)\delta(n+p,q)\Gamma(n+p+1-\mu)\Gamma(p+1-\lambda)}\,|z|^{n+p} \quad (z \in U)$$

and

$$|H(z)| \leq |z|^p + \Psi(n+p)\,|z|^{n+p} \sum_{k=n+p}^{\infty} ka_k \leq |z|^p +$$

$$\frac{(c+p)(p-q-\alpha)\delta(p,q)\Gamma(p+1-\mu)\Gamma(n+p+1-\lambda)}{(c+n+p)(n+p-q-\alpha)\delta(n+p,q)\Gamma(n+p+1-\mu)\Gamma(p+1-\lambda)}\,|z|^{n+p} \quad (z \in U)$$

which yield the inequalities (6.10) and (6.11) of Theorem 5.3. The equalities in (6.10) and (6.11) are attained for the function $f(z)$ given by

$$D_z^{\mu}\{(J_{c,p}f)(z)\} = \left\{\frac{\Gamma(p+1)}{\Gamma(p+1-\mu)} - \right.$$

$$\left.\frac{(c+p)(p-q-\alpha)\delta(p,q)\Gamma(p+1)\Gamma(n+p+1-\lambda)}{(c+n+p)(n+p-q-\alpha)\delta(n+p,q)\Gamma(n+p+1-\mu)\Gamma(p+1-\lambda)}\,z^n\right\} z^{p-\mu} \tag{6.14}$$

or for the function $(J_{c,p}f)(z)$ given by (6.9). The proof of Theorem 5.3 is thus completed. □

Theorem 5.4. *Let the function $f(z)$ defined by (1.1) be in the class $C_n(p,q,\alpha,\lambda)$. Then for $z \in U$; $0 \leq \alpha < p-q$; $0 \leq \lambda \leq 1$; $\mu > 0$; $c > 0$; $p, n \in N$; $q \in N_0$ and $p > q$, we have*

$$\left|D_z^{-\mu}\{(J_{c,p}f)(z)\}\right| \geq \left\{\frac{\Gamma(p+1)}{\Gamma(p+1+\mu)} - \right.$$

$$\left.\frac{(c+p)(p-q-\alpha)\delta(p,q+1)\Gamma(p+1)\Gamma(n+p+1-\lambda)}{(c+n+p)(n+p-q-\alpha)\delta(n+p,q+1)\Gamma(n+p+1+\mu)\Gamma(p+1-\lambda)}\,|z|^n\right\} |z|^{p-\mu} \tag{6.15}$$

and

$$\left|D_z^{-\mu}\{(J_{c,p}f)(z)\}\right| \leq \left\{\frac{\Gamma(p+1)}{\Gamma(p+1-\mu)} + \right.$$

$$\left.\frac{(c+p)(p-q-\alpha)\delta(p,q+1)\Gamma(p+1)\Gamma(n+p+1-\lambda)}{(c+n+p)(n+p-q-\alpha)\delta(n+p,q+1)\Gamma(n+p+1+\mu)\Gamma(p+1-\lambda)}\,|z|^n\right\} |z|^{p-\mu}. \tag{6.16}$$

Also for $z \in U$; $0 \leq \alpha < p-q$; $0 \leq \lambda \leq 1$; $0 \leq \mu < 1$; $c > -p$; $p, n \in N$; $q \in N_0$ and $p > q$, we have

$$\left|D_z^{\mu}\{(J_{c,p}f)(z)\}\right| \geq \left\{\frac{\Gamma(p+1)}{\Gamma(p+1-\mu)} - \right.$$

$$\left.\frac{(c+p)(p-q-\alpha)\delta(p,q+1)\Gamma(p+1)\Gamma(n+p+1-\lambda)}{(c+n+p)(n+p-q-\alpha)\delta(n+p,q+1)\Gamma(n+p+1-\mu)\Gamma(p+1-\lambda)}\,|z|^n\right\} |z|^{p-\mu} \tag{6.17}$$

and

$$\left|D_z^{\mu}\{(J_{c,p}f)(z)\}\right| \leq \left\{\frac{\Gamma(p+1)}{\Gamma(p+1-\mu)} + \right.$$

$$\left.\frac{(c+p)(p-q-\alpha)\delta(p,q+1)\Gamma(p+1)\Gamma(n+p+1-\lambda)}{(c+n+p)(n+p-q-\alpha)\delta(n+p,q+1)\Gamma(n+p+1-\mu)\Gamma(p+1-\lambda)}\,|z|^n\right\} |z|^{p-\mu}. \tag{6.18}$$

The equalities in (6.15), (6.16), (6.17) and (6.18) are attained for the function $f(z)$ given by

$$(J_{c,p}f)(z) = z^p - \frac{(c+p)(p-q-\alpha)\delta(p,q+1)\Gamma(p+1)\Gamma(n+p+1-\lambda)}{(c+n+p)(n+p-q-\alpha)\delta(n+p,q+1)\Gamma(n+p+1)\Gamma(p+1-\lambda)} z^{n+p}. \quad (6.19)$$

Remark 5.5. Putting $\lambda = 0$ in Theorems 5.2, 5.3, and 5.4, we obtain the corresponding results for the classes $S_n(p,q,\alpha)$ and $C_n(p,q,\alpha)$, respectively.

Acknowledgement. The author is thankful to the referee for his comments and suggestions.

References

[1] Bernardi, S.D., *Convex and starlike univalent functions*, Trans. Amer. Math. Soc., **135**(1969), 429-446.

[2] Chen, M.-P., Irmak, H., Srivastava, H.M., *Some multivalent functions with negative coefficients, defined by using differential operator,* Pan Amer. Math. J., **6**(1996), no. 2, 55-64.

[3] Chen, M.-P., Irmak, H., Srivastava, H.M., *Some families of multivalently analytic functions with negative coefficients*, J. Math. Anal. Appl., **214**(1997), 674-490.

[4] Duren, P.L., *Univalent Functions*, Grundlehen der Mathematischen Wissenschaften 259, Springer-Verlag, New York, Berlin, Heidelberg, and Tokoyo, 1983.

[5] Goodman, A.W., *Univalent Functions*, Vol. I and II, Polygonal Publishing House, Washington, New Jersey, 1983.

[6] Gupta, V.P., Jain, P.K.,*Certain classes of univalent functions with negative coefficients*, Bull. Austral. Math. Soc., **14**(1976), 409-416.

[7] Hossen, H.M., *Ouasi-Hadamard product of certain p-valent functions,* Demonstratio Math., **33**(2000), no. 2, 177-281.

[8] Libera, R.J., *Some classes of regular univalent functions*, Proc. Amer. Math. Soc., **16**(1969), 755-758.

[9] Livingston, A.E., *On the radius of univalence of certain analytic functions*, Proc. Amer. Math. Soc., **17**(1966), 352-357.

[10] Owa, S., *On the distortion theorems. I*, Kyungpook Math. J., **18**(1978), 55-59.

[11] Owa, S., *On certain classes of p-valent functions with negative coefficients,* Simon Stevin, **59**(1985), 385-402.

[12] Owa, S., *The quasi-Hadamard products of certain analytic functions*, in: H. M. Srivastava and S. Owa (Eds.), Current Topics in Analytic Function Theory, World Scientific Publishing Company, Singapore, New Jersey, Lnodon, and Hong Kong, 1992, 234-251.

[13] Sălăgean, G.S., Hossen, H.M., Aouf, M.K., *On certain classes of p-valent functions with negative coefficients. II*, Studia Univ. Babes-Bolyai, **69**(2004), no. 1, 77-85.

[14] Schild, A., Silverman, H., *Convolultions, of univalent functions with negative coefficients*, Ann. Univ. Mariae-Curie Sklodowska Sect. A, **29**(1975), 99-107.

[15] Srivastava, H.M., Aouf, M.K., *A certain fractional derivative operator and its applications to a new class of analytic and multivalent functions with negative coefficients. I and II*, J. Math. Anal. Appl., **171**(1992), 1-13; ibid. **192**(1995), 973-688.

[16] Srivastava, H.M., Owa, S. (Eds.), *Univalent Functions, Fractional Calculus, and Their Applivations,* Halsted Press (Ellis Horwood Limited, Chichester), John Wiley and Sons, New York, Chichester, Brisbane and Toronto, 1989.

[17] Srivastava, H.M., Owa, S. (Eds.), *Current Topics in Analytic Function Theory*, World Scientific Publishing Company, Singapore, New Jersey, London and Hong Kong, 1992.

[18] Srivastava, H.M., Owa, S., Chatterjea, S.K., *A note on certain classes of starlike functions*, Rend. Sem. Mat. Univ. Padova, **77**(1987), 115-124.

[19] Yamakawa, R., *Certain subclasses of p-valently starlike functions with negative coefficients*, in: H.M. Srivastava and S. Owa (Eds.), Current Topics in Analytic Function Theory, World Scientific Publishing Company, Singapore, New Jersey, London and Hong Kong, 1992, 393-402.

Mohamed K. Aouf
Faculty of Science
Mansoura University
Mansoura 35516, Egypt
e-mail: `mkaouf127@yahoo.com`

Stud. Univ. Babeş-Bolyai Math. 56(2011), No. 3, 45–51

Differential subordinations obtained by using Al-Oboudi and Ruscheweyh operators

Oana Crişan

Abstract. We introduce the operator $\mathcal{D}^n_{\lambda\delta} f$ using the Al-Oboudi and Ruscheweyh operators and we investigate several differential subordinations that generalize previous results.

Mathematics Subject Classification (2010): 30C45.

Keywords: Analytic function, differential operator, subordination, convex function.

1. Introduction

Let $\mathcal{H}(U)$ denote the class of analytic functions in the unit disc

$$U = \{z \in \mathbb{C} \ : \ |z| < 1\}.$$

For $a \in \mathbb{C}$ and $m \in \mathbb{N} = \{1, 2, \dots\}$, let

$$\mathcal{H}[a, m] = \{f \in \mathcal{H}(U) \ : \ f(z) = a + a_m z^m + \cdots, \ z \in U\}$$

and

$$\mathcal{A}_m = \left\{f \in \mathcal{H}(U) \ : \ f(z) = z + a_{m+1} z^{m+1} + \cdots, \ z \in U\right\},$$

with $\mathcal{A}_1 = \mathcal{A}$.

Let f and g be members of $\mathcal{H}(U)$. The function f is said to be subordinate to g if there exists a function w analytic in U, with $w(0) = 0$ and $|w(z)| < 1, \ z \in U$, such that $f(z) = g(w(z)), \ z \in U$. In this case, we write $f \prec g$ or $f(z) \prec g(z), \ z \in U$. If the function g is univalent in U, then $f \prec g$ if and only if $f(0) = g(0)$ and $f(U) \subseteq g(U)$.

Let $\Psi : \mathbb{C}^3 \times U \to \mathbb{C}$ and let h be univalent in U. If p is analytic in U and satisfies the second-order differential subordination

$$\Psi(p(z), \, zp'(z), \, z^2 p''(z); \, z) \prec h(z), \quad z \in U, \tag{1.1}$$

then p is called a solution of the differential subordination. A univalent function q is called a dominant of the solution of the differential subordination, or more simply a dominant, if $p \prec q$ for all p satisfying (1.1). A dominant $\tilde{q}$ that satisfies $\tilde{q} \prec q$ for all dominants q of (1.1) is said to be the best dominant of (1.1).

In order to prove our main results we shall need the following lemmas.

Lemma 1.1 ([2, p. 71]). *Let h be a convex function with $h(0) = a$ and let $\gamma \in \mathbb{C}^*$ be a complex number such that $Re\gamma \geq 0$. If $p \in \mathcal{H}[a, n]$ and*

$$p(z) + \frac{1}{\gamma} zp'(z) \prec h(z), \quad z \in U$$

then

$$p(z) \prec q(z) \prec h(z), \quad z \in U,$$

where

$$q(z) = \frac{\gamma}{nz^{\gamma/n}} \int_0^z h(t)t^{\gamma/n-1} dt, \quad z \in U.$$

The function q is convex and the best dominant.

Lemma 1.2 ([3, p. 419]). *Let r be a convex function in U and let*

$$h(z) = r(z) + n\alpha zr'(z), \quad z \in U,$$

where $\alpha > 0$ and $n \in \mathbb{N}$. If

$$p(z) = r(0) + p_n z^n + p_{n+1} z^{n+1} + \cdots, \quad z \in U$$

is holomorphic in U and

$$p(z) + \alpha zp'(z) \prec h(z), \quad z \in U,$$

then

$$p(z) \prec r(z), \quad z \in U,$$

and this result is sharp.

Definition 1.3 ([1, p. 1429]). *For a function $f \in \mathcal{A}$, $\delta \geq 0$ and $n \in \mathbb{N} \cup \{0\}$, the Al-Oboudi differential operator $D_\delta^n f$ is defined by*

$$\begin{aligned} D^0 f(z) &= f(z), \\ D_\delta^1 f(z) &= (1-\delta)f(z) + \delta zf'(z) = D_\delta f(z), \\ D_\delta^n f(z) &= D_\delta \left(D_\delta^{n-1} f(z)\right), \quad z \in U. \end{aligned} \tag{1.2}$$

Remark 1.4. D_δ^n is a linear operator and for $f \in \mathcal{A}$,

$$f(z) = z + \sum_{j=2}^{\infty} a_j z^j,$$

we have

$$D_\delta^n f(z) = z + \sum_{j=2}^{\infty} [1 + (j-1)\delta]^n a_j z^j, \quad z \in U \tag{1.3}$$

and

$$\left(D_\delta^{n+1} f(z)\right)' = \left(D_\delta^n f(z)\right)' + \delta z \left(D_\delta^n f(z)\right)'', \quad z \in U. \tag{1.4}$$

Also, when $\delta = 1$, we obtain the Sălăgean differential operator ([6, p. 363]).

Definition 1.5 ([5, p. 110]). *For a function* $f \in \mathcal{A}$ *and* $n \in \mathbb{N} \cup \{0\}$, *the Ruscheweyh differential operator* $R^n f$ *is defined by*

$$R^n f(z) = \frac{z}{(1-z)^{n+1}} * f(z) = \frac{z}{n!}[z^{n-1} f(z)]^{(n)}, \quad z \in U, \tag{1.5}$$

where $*$ *stands for the Hadamard product or convolution.*

Remark 1.6. If $f \in \mathcal{A}$,

$$f(z) = z + \sum_{j=2}^{\infty} a_j z^j,$$

we have

$$R^0 f(z) = f(z),$$

$$R^1 f(z) = z f'(z),$$

$$(n+1) R^{n+1} f(z) = n R^n f(z) + z \left(R^n f(z)\right)', \tag{1.6}$$

$$R^n f(z) = z + \sum_{j=2}^{\infty} C_{n+j-1}^{n} a_j z^j, \quad z \in U. \tag{1.7}$$

Definition 1.7. *Let* $n \in \mathbb{N} \cup \{0\}$, $\delta \geq 0$ *and* $\lambda \geq 0$ *with* $\delta \neq (\lambda - 1)/\lambda$. *For* $f \in \mathcal{A}$, *let* $\mathcal{D}_{\lambda\delta}^{n} f$ *denote the operator defined by* $\mathcal{D}_{\lambda\delta}^{n} : \mathcal{A} \to \mathcal{A}$,

$$\mathcal{D}_{\lambda\delta}^{n} f(z) = \frac{1}{1 - \lambda + \lambda\delta}[(1-\lambda) D_{\delta}^{n} f(z) + \lambda\delta R^n f(z)], \quad z \in U, \tag{1.8}$$

where the operators $D_{\delta}^{n} f$ *and* $R^n f$ *are given by Definition* 1.3 *and Definition* 1.5, *respectively.*

Remark 1.8. When $\lambda = 0$ in (1.8), we get the Al-Oboudi differential operator, and when $\lambda = 1$ we obtain the Ruscheweyh differential operator.

Also, for $n = 0$, we have

$$\mathcal{D}_{\lambda\delta}^{0} f(z) = \frac{1}{1 - \lambda + \lambda\delta}[(1-\lambda) D_{\delta}^{0} f(z) + \lambda\delta R^0 f(z)] = f(z), \quad z \in U.$$

Remark 1.9. $\mathcal{D}_{\lambda\delta}^{n}$ is a linear operator and for $f \in \mathcal{A}$,

$$f(z) = z + \sum_{j=2}^{\infty} a_j z^j,$$

by using (1.3) and (1.7), we have

$$\mathcal{D}_{\lambda\delta}^{n} f(z) = z + \frac{1}{1 - \lambda + \lambda\delta} \sum_{j=2}^{\infty} \left[(1-\lambda)\left(1 + (j-1)\delta\right)^n + \lambda\delta C_{n+j-1}^{n}\right] a_j z^j, \tag{1.9}$$

$z \in U$.

2. Main results

Theorem 2.1. *If* $0 \le \alpha < 1$, $f \in \mathcal{A}_m$ *and*

$$\mathrm{Re}\left[\left(\mathcal{D}_{\lambda\delta}^{n+1} f(z)\right)' + \frac{\lambda\delta z(\delta n + \delta - 1)\left(R^n f(z)\right)''}{(1-\lambda+\lambda\delta)(n+1)}\right] > \alpha\,, \quad z \in U \tag{2.1}$$

then

$$\mathrm{Re}\,\left(\mathcal{D}_{\lambda\delta}^{n} f(z)\right)' > \gamma\,, \quad z \in U,$$

where

$$\gamma = \gamma(\alpha) = 2\alpha - 1 \,+\, \frac{2(1-\alpha)}{\delta m}\beta\left(\frac{1}{\delta m}\right)$$

and

$$\beta(x) = \int_0^1 \frac{t^{x-1}}{1+t}dt\,.$$

Proof. Let $f \in \mathcal{A}_m$,

$$f(z) = z + \sum_{j=m+1}^{\infty} a_j z^j, \quad z \in U.$$

If

$$h(z) = \frac{1+(2\alpha-1)z}{1+z}\,, \quad z \in U,$$

then (2.1) is equivalent to

$$\left(\mathcal{D}_{\lambda\delta}^{n+1} f(z)\right)' + \frac{\lambda\delta z(\delta n + \delta - 1)\left(R^n f(z)\right)''}{(1-\lambda+\lambda\delta)(n+1)} \prec h(z), \quad z \in U. \tag{2.2}$$

Using the properties of $\mathcal{D}_{\lambda\delta}^n f$, $D_\delta^n f$ and $R^n f$, we obtain

$$\begin{aligned}
&\left(\mathcal{D}_{\lambda\delta}^{n+1} f(z)\right)' + \frac{\lambda\delta z(\delta n + \delta - 1)\left(R^n f(z)\right)''}{(1-\lambda+\lambda\delta)(n+1)} \\
=\ & \frac{\left[(1-\lambda)D_\delta^{n+1} f(z) + \lambda\delta R^{n+1} f(z)\right]'}{1-\lambda+\lambda\delta} + \frac{\lambda\delta z(\delta n + \delta - 1)\left(R^n f(z)\right)''}{(1-\lambda+\lambda\delta)(n+1)} \\
=\ & \frac{1-\lambda}{1-\lambda+\lambda\delta}\left[\left(D_\delta^n f(z)\right)' + \delta z\left(D_\delta^n f(z)\right)''\right] \\
& + \frac{\lambda\delta}{1-\lambda+\lambda\delta}\left[\frac{z\left(R^n f(z)\right)' + nR^n f(z)}{n+1}\right]' + \frac{\lambda\delta z(\delta n + \delta - 1)\left(R^n f(z)\right)''}{(1-\lambda+\lambda\delta)(n+1)} \\
=\ & \frac{1-\lambda}{1-\lambda+\lambda\delta}\left(D_\delta^n f(z)\right)' + \frac{(1-\lambda)\delta}{1-\lambda+\lambda\delta} z\left(D_\delta^n f(z)\right)'' \\
& + \frac{\delta\lambda\left[\left(R^n f(z)\right)' + z\left(R^n f(z)\right)'' + n\left(R^n f(z)\right)'\right]}{(1-\lambda+\lambda\delta)(n+1)}
\end{aligned}$$

$$
\begin{aligned}
& + \frac{\lambda\delta z(\delta n+\delta-1)\left(R^n f(z)\right)''}{(1-\lambda+\lambda\delta)(n+1)} \\
= \quad & \frac{1}{1-\lambda+\lambda\delta}\left[(1-\lambda)\left(D_\delta^n f(z)\right)' + \lambda\delta\left(R^n f(z)\right)'\right] \\
& + \delta z \frac{1}{1-\lambda+\lambda\delta}\left[(1-\lambda)\left(D_\delta^n f(z)\right)'' + \lambda\delta\left(R^n f(z)\right)''\right] \\
= \quad & \left(\mathcal{D}_{\lambda\delta}^n f(z)\right)' + \delta z\left(\mathcal{D}_{\lambda\delta}^n f(z)\right)'', \quad z\in U. \qquad (2.3)
\end{aligned}
$$

Then, from (2.2) and (2.3), we have

$$\left(\mathcal{D}_{\lambda\delta}^n f(z)\right)' + \delta z\left(\mathcal{D}_{\lambda\delta}^n f(z)\right)'' \prec h(z), \quad z\in U. \qquad (2.4)$$

Let

$$p(z) = \left(\mathcal{D}_{\lambda\delta}^n f(z)\right)', \quad z\in U. \qquad (2.5)$$

In view of (1.9), we get

$$
\begin{aligned}
p(z) &= 1 + \frac{1}{1-\lambda+\lambda\delta}\sum_{j=m+1}^{\infty}\left[(1-\lambda)\left(1+(j-1)\delta\right)^n + \lambda\delta C_{n+j-1}^n\right] j a_j z^{j-1} \\
&= 1 + b_m z^m + b_{m+1} z^{m+1} + \dots, \quad z\in U.
\end{aligned}
$$

and from (2.4), we obtain

$$p(z) + \delta z p'(z) \prec h(z), \quad z\in U. \qquad (2.6)$$

By applying now Lemma 1.1, we have

$$p(z) \prec q(z) \prec h(z), \quad z\in U,$$

where

$$
\begin{aligned}
q(z) \quad &= \quad \frac{1}{\delta m z^{1/\delta m}}\int_0^z h(t) t^{\frac{1}{\delta m}-1} dt \\
&= \quad \frac{1}{\delta m z^{1/\delta m}}\int_0^z \left[2\alpha - 1 + 2(1-\alpha)\frac{1}{1+t}\right] t^{\frac{1}{\delta m}-1} dt \\
&= \quad \frac{2\alpha-1}{\delta m z^{1/\delta m}}\int_0^z t^{\frac{1}{\delta m}-1} dt \; + \; \frac{2(1-\alpha)}{\delta m z^{1/\delta m}}\int_0^z \frac{t^{\frac{1}{\delta m}-1}}{1+t} dt \\
&= \quad 2\alpha - 1 \; + \; \frac{2(1-\alpha)}{\delta m z^{1/\delta m}}\int_0^z \frac{t^{\frac{1}{\delta m}-1}}{1+t} dt, \quad z\in U.
\end{aligned}
$$

The function q is convex, it is the best dominant and because $q(U)$ is symmetric with respect to the real axis, we get

$$\mathrm{Re}\,\left(\mathcal{D}_{\lambda\delta}^n f(z)\right)' = \mathrm{Re}\,p(z) > \mathrm{Re}\,q(1) = \gamma(\alpha) = 2\alpha - 1 \; + \; \frac{2(1-\alpha)}{\delta m}\beta\left(\frac{1}{\delta m}\right).$$

□

Example 2.2. If $f\in\mathcal{A}$, $n=1$, $\lambda = 1/2$, $\delta = 1$ and $\alpha = 1/2$, then $\gamma(\alpha) = \ln 2$ and the inequality

$$\mathrm{Re}\,\left[f'(z) + 3zf''(z) + z^2 f'''(z)\right] > \frac{1}{2}, \quad z\in U,$$

implies that

$$\text{Re}\,[f'(z) + zf''(z)] > \ln 2, \quad z \in U.$$

Theorem 2.3. *Let $m \in \mathbb{N}$, $\delta > 0$ and let r be a convex function with $r(0) = 1$ and h a function such that*

$$h(z) = r(z) + m\delta zr'(z)\,, \quad z \in U.$$

If $f \in \mathcal{A}_m$, then the following subordination

$$\left(\mathcal{D}_{\lambda\delta}^{n+1} f(z)\right)' + \frac{\lambda\delta z(\delta n + \delta - 1)\left(R^n f(z)\right)''}{(1 - \lambda + \lambda\delta)(n+1)} \prec h(z) = r(z) + m\delta zr'(z)\,, \quad z \in U \tag{2.7}$$

implies that

$$(\mathcal{D}_{\lambda\delta}^n f(z))' \prec r(z)\,, \quad z \in U,$$

and the result is sharp.

Proof. By using (2.3) and (2.5), the subordination (2.7) is equivalent to

$$p(z) + \delta zp'(z) \prec h(z) = r(z) + m\delta zr'(z)\,, \quad z \in U.$$

Hence, from Lemma 1.2, we conclude that

$$p(z) \prec r(z)\,, \quad z \in U,$$

that is,

$$(\mathcal{D}_{\lambda\delta}^n f(z))' \prec r(z)\,, \quad z \in U,$$

and the result is sharp. □

Theorem 2.4. *Let $m \in \mathbb{N}$ and let r be a convex function with $r(0) = 1$ and h a function such that*

$$h(z) = r(z) + mzr'(z)\,, \quad z \in U.$$

If $f \in \mathcal{A}_m$, then the following subordination

$$(\mathcal{D}_{\lambda\delta}^n f(z))' \prec h(z) = r(z) + mzr'(z)\,, \quad z \in U \tag{2.8}$$

implies that

$$\frac{\mathcal{D}_{\lambda\delta}^n f(z)}{z} \prec r(z)\,, \quad z \in U,$$

and the result is sharp.

Proof. Let

$$p(z) = \frac{\mathcal{D}_{\lambda\delta}^n f(z)}{z}\,, \quad z \in U. \tag{2.9}$$

Differentiating (2.9), we have

$$(\mathcal{D}_{\lambda\delta}^n f(z))' = p(z) + zp'(z)\,, \quad z \in U,$$

and consequently, (2.8) becomes

$$p(z) + zp'(z) \prec h(z) = r(z) + mzr'(z)\,, \quad z \in U.$$

Hence, by applying Lemma 1.2, we conclude that

$$p(z) \prec r(z)\,, \quad z \in U,$$

that is,

$$\frac{\mathcal{D}^n_{\lambda\delta} f(z)}{z} \prec r(z)\,, \quad z \in U,$$

and the result is sharp. □

Remark 2.5. For $m = 1$ and $\delta = 1$, the above theorems were obtained by G. I. Oros and G. Oros in [4].

References

[1] Al-Oboudi, F.M., *On univalent functions defined by a generalized Sălăgean operator*, Int. J. Math. Math. Sci., **27**(2004), 1429-1436.

[2] Miller, S.S., Mocanu, P.T., *Differential subordinations: Theory and Applications*, Marcel Dekker Inc., New York, Basel, 2000.

[3] Mocanu, P.T., Bulboacă, T., Sălăgean, G.Ş., *Geometric Theory of Univalent Functions*, Casa Cărţii de Ştiinţă, Cluj-Napoca, 2006 (in Romanian).

[4] Oros, G.I., Oros, G., *Differential subordinations obtained by using generalized Sălăgean and Ruscheweyh operators*, Acta Univ. Apul., **14**(2007), 129-140.

[5] Ruscheweyh, S., *New criteria for univalent functions*, Proc. Amer. Math. Soc., **49**(1975), 109-115.

[6] Sălăgean, G.Ş., *Subclasses of univalent functions*, Lecture Notes in Math., Springer Verlag, Berlin, **1013**(1983), 362-372.

Oana Crişan
"Babeş-Bolyai" University
Faculty of Mathematics and Computer Sciences
1, Kogălniceanu Street
400084 Cluj-Napoca, Romania
e-mail: oanacrisan310@yahoo.com

Stud. Univ. Babeş-Bolyai Math. 56(2011), No. 3, 53–60

Properties of certain analytic functions defined by a linear operator

Elsayed A. Elrifai, Hanan E. Darwish and Abdusalam R. Ahmed

Abstract. In this paper, we study and investigate starlikeness and convexity of a class of multivalent functions defined by a linear operator $L_{p,k}(a,c)f(z)$. As a consequence, a number of sufficient conditions for starlikeness and convexity of analytic functions are also obtained.

Mathematics Subject Classification (2010): 30C45.

Keywords: Multivalent function, starlike function, convex function, linear operator.

1. Introduction

Let $A(p,k)(p,k \in N = \{1,2,3,....\})$ be the class of functions of the form

$$f(z) = z^p + \sum_{m=k}^{\infty} a_{p+m} z^{p+m} \tag{1.1}$$

which are analytic in the unit disk $U = \{z : |z| < 1\}$. We denote $A(p,1) = A_p$ and $A(1,1) = A$.

A function $f(z) \in A(p,k)$ is said to be p-valent starlike of order α $(0 \le \alpha < p)$ in U if

$$\text{Re}\left(\frac{zf'(z)}{f(z)}\right) > \alpha, \quad z \in U.$$

We denote by $S_p^*(\alpha)$, the class of all such functions.

A function $f(z) \in A(p,k)$ is said to be p-valent convex of order α $(0 \le \alpha < p)$ in U if

$$\text{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha, \quad z \in U.$$

Let $K_p(\alpha)$ denote the class of all those functions $f \in A(p,k)$, which are multivalently convex of order α in U. Note that $S_1^*(\alpha)$ and $K_1(\alpha)$ are, respectively, the usual classes of univalent starlike functions of order α and univalent convex functions of order $\alpha, 0 \le \alpha < 1$, and will be denoted here by $S^*(\alpha)$ and $K(\alpha)$, respectively. We shall use S^* and K to denote $S^*(0)$ and $K(0)$, respectively, which are the classes of univalent starlike (w.r.t the origin) and univalent convex functions. These classes considered also by S. Singh et. al. [6].
The class $A(p,k)$ is closed under the Hadamard product (or convolution)

$$\begin{aligned} f(z) * g(z) &= (f*g)(z) = z^p + \sum_{m=k}^{\infty} a_{p+m} b_{p+m} z^{p+m} \\ &= (g*f)(z) \qquad (z \in U), \end{aligned}$$

where

$$f(z) = z^p + \sum_{m=k}^{\infty} a_{p+m} z^{p+m}, \quad g(z) = z^p + \sum_{m=k}^{\infty} b_{p+m} z^{p+m}.$$

Let the function $\varphi_{p,k}(a,c)$ be defined by

$$\varphi_{p,k}(a,c;z) = z^p + \sum_{m=k}^{\infty} \frac{(a)_m}{(c)_m} z^{p+m} \quad (z \in U), \tag{1.2}$$

where $c \neq 0, -1, -2, ..., (\lambda)_0 = 1$ and $(\lambda)_m = \lambda(\lambda+1)...(\lambda+m-1)$ for $m \in N$.

Carlson and Shaffer [2] defined a convolution operator on A by

$$L(a,c)f(z) = \varphi_{1,1}(a,c) * f(z) \quad (f(z) \in A). \tag{1.3}$$

Similarly Xu and Aouf [1] define a linear operator $L_{p,k}(a,c)$ on $A(p,k)$ by

$$\begin{aligned} L_{p,k}(a,c)f(z) &= \varphi_{p,k}(a,c) * f(z) \quad (f(z) \in A(p,k)) \\ &= \left(z^p + \sum_{m=k}^{\infty} \frac{(a)_m}{(c)_m} z^{p+m} \right) * \left(z^p + \sum_{m=k}^{\infty} a_{p+m} z^{p+m} \right) \\ &= z^p + \sum_{m=k}^{\infty} \frac{(a)_m}{(c)_m} a_{p+m} z^{p+m}. \end{aligned} \tag{1.4}$$

It is easily seen from (1.4) that

$$z(L_{p,k}(a,c)f(z))' = aL_{p,k}(a+1,c)f(z) - (a-p)L_{p,k}(a,c)f(z). \tag{1.5}$$

Clearly $L_{p,k}(a,c)$ maps $A(p,k)$ into itself and $L_{p,k}(c,c)$ is identity.
If $a \neq 0, -1, -2, ...,$ then $L_{p,k}(a,c)$ has an inverse $L_{p,k}(c,a)$.
We note that

$$L_{p,k}(p+1,p)f(z) = \frac{zf'(z)}{p}.$$

For a real number $\lambda > -p$, we get

$$L_{p,\lambda}(\lambda+p,\lambda+p+1)f(z) = J_{p,\lambda}f(z) = \frac{\lambda+p}{z^{\lambda}}\int_0^z t^{\lambda-1}f(t)dt \qquad (1.6)$$

where $J_{p,\lambda}$ the generalized Libera integral operator (see [4]), and

$$L_{p,k}(\lambda+p,1)f(z) = D^{\lambda+p-1}f(z)$$

where $D^{\lambda+p-1}$ the generalized Ruscheweyh derivative (see [5]).

A function $f(z) \in A(p,k)$ is said to be in the class $S_{p,k}(\alpha,a,c)$ for all z in U if it satisfies

$$\mathrm{Re}\left[\frac{L_{p,k}(a+1,c)f(z)}{L_{p,k}(a,c)f(z)}\right] > \frac{\alpha}{p}\ , \qquad (1.7)$$

for some $\alpha(0 \le \alpha < p, p \in N)$. We note that $S_{p,k}(\alpha,p,p)$ is the usual class $S_p^*(\alpha)$ of p-valent starlike functions of order α.

In the present paper, our aim is to determine sufficient conditions for a function $f \in A(p,k)$ to be a member of the class $S_{p,k}(\alpha,a,c)$. As a consequence of our main result we get a number of sufficient conditions for starlikeness and convexity of analytic functions.

2. Main result

To prove our result, we shall make use of the famous Jack's Lemma which we state below.

Lemma 2.1. (Jack [3]). *Suppose $w(z)$ be a nonconstant analytic function in U with $w(0)=0$. If $|w(z)|$ attains its maximum value at a point $z_0 \in U$ on the circle $|z| = r < 1$, then $z_0 w'(z_0) = m w(z_0)$, where m is a real and $m \ge 1$.*

We now state and prove our main result.

Theorem 2.2. *If $f(z) \in A(p,k)$ satisfies*

$$\left|\frac{L_{p,k}(a+1,c)f(z)}{L_{p,k}(a,c)f(z)} - 1\right|^{\gamma}\left|\frac{L_{p,k}(a+2,c)f(z)}{L_{p,k}(a+1,c)f(z)} - 1\right|^{\beta} < M_p(k,a,c,\alpha,\beta,\gamma) \qquad (2.1)$$

$(z \in U)$, for some $\alpha(0 \le \alpha < p), \beta(\beta \ge 0)$ $\gamma \ge 0$ and $\beta+\gamma > 0$, then $f(z) \in S_p(k,a,c,\alpha)$, and

$$M_p(k,a,c,\alpha,\beta,\gamma) = \begin{cases} (1-\frac{\alpha}{p})^{\gamma}(1-\frac{\alpha}{p}+\frac{1}{2a})^{\beta}, & 0 \le \alpha < \frac{p}{2} \\ (1-\frac{\alpha}{p})^{\gamma+\beta}(1+\frac{1}{a})^{\beta}, & \frac{p}{2} \le \alpha < p. \end{cases}$$

Proof. Case (i). Let $0 \le \alpha < \frac{p}{2}$. Writing $\frac{\alpha}{p} = \mu$, we see that $0 \le \mu \le \frac{1}{2}$. Define a function $w(z)$ as:

$$\frac{L_{p,k}(a+1,c)f(z)}{L_{p,k}(a,c)f(z)} = \frac{1+(1-2\mu)w(z)}{1-w(z)}\ , \quad z \in U. \qquad (2.2)$$

Then $w(z)$ is analytic in U, $w(0) = 0$ and $w(z) \neq 1$ in U. By a simple computation, we obtain from (2.2),

$$\frac{z(L_{p,k}(a+1,c)f(z))^{'}}{L_{p,k}(a+1,c)f(z)} - \frac{z(L_{p,k}(a,c)f(z))^{'}}{L_{p,k}(a,c)f(z)} = \frac{2(1-\mu)zw^{'}(z)}{(1-w(z))(1+(1-2\mu)w(z))} \tag{2.3}$$

and from (1.5) we get

$$\frac{L_{p,k}(a+2,c)f(z)}{L_{p,k}(a+1,c)f(z)} = \frac{1+(1-2\mu)w(z)}{1-w(z)} + \frac{2(1-\mu)zw^{'}(z)}{a(1-w(z))(1+(1-2\mu)w(z))}.$$

Thus, we have

$$\left|\frac{L_{p,k}(a+1,c)f(z)}{L_{p,k}(a,c)f(z)} - 1\right|^{\gamma} \left|\frac{L_{p,k}(a+2,c)f(z)}{L_{p,k}(a+1,c)f(z)} - 1\right|^{\beta}$$

$$= \left|\frac{2(1-\mu)w(z)}{1-w(z)}\right|^{\gamma+\beta} \left|1 + \frac{zw^{'}(z)}{aw(z)(1+(1-2\mu))w(z))}\right|^{\beta}.$$

Suppose that there exists a point $z_0 \in U$ such that

$$\max_{|z|\leq|z_0|} |w(z)| = |w(z_0)| = 1,$$

then by Lemma 2.1, we have $w(z_0) = e^{i\theta}, 0 < \theta \leq 2\pi$ and

$$z_0 w^{'}(z_0) = mw(z_0), \ m \geq 1.$$

Therefore, we have

$$\left|\frac{L_{p,k}(a+1,c)f(z_0)}{L_{p,k}(a,c)f(z_0)} - 1\right|^{\gamma} \left|\frac{L_{p,k}(a+2,c)f(z_0)}{L_{p,k}(a+1,c)f(z_0)} - 1\right|^{\beta}$$

$$= \left|\frac{2(1-\mu)w(z_0)}{1-w(z_0)}\right|^{\gamma+\beta} \left|1 + \frac{m}{a(1+(1-2\mu)w(z_0))}\right|^{\beta}$$

$$= \frac{2^{\gamma+\beta}(1-\mu)^{\gamma+\beta}}{|1-e^{i\theta}|^{\beta+\gamma}} \left|1 + \frac{m}{a(1+(1-2\mu)e^{i\theta})}\right|^{\beta}$$

$$\geq (1-\mu)^{\gamma+\beta}\left(1+\frac{m}{2a(1-\mu)}\right)^{\beta} \geq (1-\mu)^{\beta+\gamma}\left(1+\frac{1}{2a(1-\mu)}\right)^{\beta}$$

$$= (1-\mu)^{\gamma}\left(1-\mu+\frac{1}{2a}\right)^{\beta}$$

which contradicts (2.1) for $0 \leq \alpha \leq \frac{p}{2}$. Therefore, we must have $|w(z)| < 1$ for all $z \in U$, and hence $f(z) \in S_p(k, a, c, \alpha)$.

Case (ii). When $\frac{p}{2} \leq \alpha < p$. In this case, we must have $\frac{1}{2} \leq \mu < 1$, where $\mu = \frac{\alpha}{p}$. Let w be defined by

$$\frac{L_{p,k}(a+1,c)f(z_0)}{L_{p,k}(a,c)f(z)} = \frac{\mu}{\mu-(1-\mu)w(z)}, \qquad z \in U,$$

where $w(z) \neq \frac{\mu}{1-\mu}$ in U. Then $w(z)$ is analytic in U and $w(0)=0$. Proceeding as in Case (i) and using identity (1.5), we obtain

$$\left|\frac{L_{p,k}(a+1,c)f(z)}{L_{p,k}(a,c)f(z)}-1\right|^{\gamma}\left|\frac{L_{p,k}(a+2,c)f(z)}{L_{p,k}(a+1,c)f(z)}-1\right|^{\beta}$$

$$=\left|\frac{(1-\mu)w(z)}{\mu-(1-\mu)w(z)}\right|^{\gamma}\left|\frac{(1-\mu)w(z)}{\mu-(1-\mu)w(z)}+\frac{(1-\mu)zw^{'}(z)}{a(\mu-(1-\mu)w(z))}\right|^{\beta}$$

$$=\left|\frac{1-\mu}{\mu-(1-\mu)w(z)}\right|^{\gamma+\beta}|w(z)|^{\gamma}\left|w(z)+\frac{zw^{'}(z)}{a}\right|^{\beta}.$$

Suppose that there exists a point $z_0 \in U$ such that $\max\limits_{|z|\leq|z_0|}|w(z)|=|w(z_0)|=1$, then by Lemma 2.1, we obtain $w(z_0)=e^{i\theta}$ and $z_0w^{'}(z_0)=mw(z_0), m\geq 1$. Therefore

$$\left|\frac{L_{p,k}(a+1,c)f(z_0)}{L_{p,k}(a,c)f(z_0)}-1\right|^{\gamma}\left|\frac{L_{p,k}(a+2,c)f(z_0)}{L_{p,k}(a+1,c)f(z_0)}-1\right|^{\beta}$$

$$=\frac{(1-\mu)^{\gamma+\beta}\left(1+\frac{m}{a}\right)^{\gamma+\beta}}{|\mu-(1-\mu)e^{i\theta}|^{\gamma+\beta}}\geq\left(1-\frac{\alpha}{p}\right)^{\gamma+\beta}\left(1+\frac{1}{a}\right)^{\beta}$$

which contradicts (2.1) for $\frac{p}{2}\leq\alpha<p$. Therefore, we must have $|w(z)|<1$ for all $z\in U$, and hence $f(z)\in S_p(k,a,c,\alpha)$. This completes the proof of our theorem.

3. Deductions

For $p=1$, Theorem 2.2 reduces to the following results:

Corollary 3.1. *If, for all $z\in U$, a function $f(z)\in A$ satisfies*

$$\left|\frac{L(a+1,c)f(z)}{L(a,c)f(z)}-1\right|^{\gamma}\left|\frac{L(a+2,c)f(z)}{L(a+1,c)f(z)}-1\right|^{\beta}$$

$$<\begin{cases}(1-\alpha)^{\gamma}(1-\alpha+\frac{1}{2a})^{\beta}, & 0\leq\alpha<\frac{1}{2}\\ (1-\alpha)^{\gamma+\beta}(1+\frac{1}{a})^{\beta} & \frac{1}{2}\leq\alpha<1,\end{cases}$$

for some real $\alpha(0\leq\alpha<1),\beta\geq 0$ and $\gamma\geq 0$ with $\beta+\gamma>0$, then $f(z)\in S_1(k,a,c,\alpha)$.

For $\dot{\gamma}=0$ and $\beta=1$ in Theorem 2.2, we obtain

Corollary 3.2. *If, for all $z\in U$, a function $f(z)\in A(p,k)$ satisfies*

$$\left|\frac{L(a+2,c)f(z)}{L(a+1,c)f(z)}-1\right|<\begin{cases}1-\frac{\alpha}{p}+\frac{1}{2a}, & 0\leq\alpha<\frac{p}{2}\\ (1-\frac{\alpha}{p})(1+\frac{1}{a}) & \frac{p}{2}\leq\alpha<p,\end{cases}$$

then

$$\left(\frac{L_{p,k}(a+1,c)f(z)}{L_{p,k}(a,c)f(z)}\right)>\frac{\alpha}{p},\quad (z\in U).$$

Setting $p=a=c=1$ in Theorem 2.2, we obtain the following result:

Corollary 3.3. *If $f(z) \in A$ satisfies*

$$\left|\frac{zf'(z)}{f(z)}-1\right|^{\gamma}\left|\frac{zf''(z)}{f'(z)}\right|^{\beta}<2^{2\beta}(1-\alpha)^{\gamma+\beta}, 0\leq\alpha<1 (z\in U)$$

for some $\beta \geq 0$ and $\gamma \geq 0$ with $\beta+\gamma>0$, then

$$f(z)\in S^*(\alpha).$$

Setting $\alpha = 0$ in Corollary 3.1, we obtain the following criterion for starlikeness:

Corollary 3.4. *For some non-negative real numbers β and γ with $\beta+\gamma>0$, if $f(z) \in A$ satisfies*

$$\left|\frac{zf'(z)}{f(z)}-1\right|^{\gamma}\left|\frac{zf''(z)}{f'(z)}\right|^{\beta}<2^{2\beta} \qquad (z\in U),$$

then $f(z)\in S^*$.

In particular, for $\beta = 1$ and $\gamma = 1$, we obtain the following interesting criterion for starlikeness:

Corollary 3.5. *If $f(z) \in A$ satisfies*

$$\left|\frac{zf'(z)}{f(z)}-1\right|\left|\frac{zf''(z)}{2f'(z)}\right|<2^{2} \qquad (z\in U),$$

then $f(z)\in S^*$.

Setting $a = c = p$ in Theorem 2.2, we obtain the following sufficient condition for a function $f(z) \in A(p,k)$ to be a p-valent starlike function of order α.

Corollary 3.6. *For all $z\in U$, if $f(z) \in A(p,k)$ satisfies the following condition*

$$\left|\frac{zf'(z)}{pf(z)}-1\right|^{\gamma}\left(\frac{p}{p+1}\right)^{\beta}\left|\frac{1}{p}\left(1+\frac{zf''(z)}{f'(z)}\right)-1\right|^{\beta}$$

$$<\begin{cases}\left(1-\frac{\alpha}{p}\right)^{\gamma}\left(1-\frac{\alpha}{p}+\frac{1}{2p}\right)^{\beta}, & 0\leq\alpha\leq\frac{p}{2}\\ \left(1-\frac{\alpha}{p}\right)^{\gamma+\beta}\left(1+\frac{1}{p}\right)^{\beta}, & \frac{p}{2}\leq\alpha<0,\end{cases}$$

for some real numbers α,β and γ with $0\leq\alpha<p, \beta\geq0, \gamma\geq0, \beta+\gamma>0$ then $f(z)\in S_p^*(\alpha)$.

The substition $p = 1$ in Corollary 3.6, yields the following result:

Corollary 3.7. *If $f(z) \in A$ satisfies*

$$\left|\frac{zf'(z)}{f(z)}-1\right|^{\gamma}\left|\frac{zf''(z)}{f'(z)}\right|^{\beta}<\begin{cases}(1-\alpha)^{\gamma}(3-2\alpha)^{\beta}, & 0\leq\alpha<\frac{1}{2}\\ (1-\alpha)^{\gamma+\beta}(4)^{\beta}, & \frac{1}{2}\leq\alpha<1\end{cases}$$

where $z \in U$ and α,β,γ are real numbers with $0\leq\alpha<1$, $\beta\geq0$, $\gamma\geq0$, $\beta+\gamma>0$, then $f(z)\in S^*(\alpha)$.

In particular, writing $\beta = 1, \gamma = 1$ and $\alpha = 0$ in Corollary 3.7, we obtain the following result:

Corollary 3.8. *If* $f(z) \in A$ *satisfies*

$$\left| \frac{zf''(z)}{f'(z)} \left(\frac{zf'(z)}{f(z)} - 1 \right) \right| < 3, \quad z \in U$$

then $f \in S^*$.

Taking $a = p+1, c = p$ in Theorem 2.2, we get the following interesting criterion for convexity of multivalent functions:

Corollary 3.9. *If, for all* $z \in U$, *a function* $f(z) \in A(p,k)$ *satisfies*

$$\left| \frac{L_{p,k}(p+2,p)f(z)}{L_{p,k}(p+1,p)f(z)} - 1 \right|^{\gamma} \left| \frac{L_{p,k}(p+3,p)f(z)}{L_{p,k}(p+2,p)f(z)} - 1 \right|^{\beta}$$

$$= \left(\frac{p}{p+1} \right)^{\gamma} \left| \frac{1}{p} \left(1 + \frac{zf''(z)}{f'(z)} \right) - 1 \right|^{\gamma} \left| \frac{z^2 f'''(z) + 6zf'' + 6f'}{(p+2)\left(zf''(z) + 2f'(z)\right)} - 1 \right|^{\beta}$$

$$< \begin{cases} (1-\frac{\alpha}{p})^{\gamma} \left(1 - \frac{\alpha}{p} + \frac{1}{2p} \right)^{\beta}, & 0 \leq \alpha < \frac{p}{2} \\ (1-\frac{\alpha}{p})^{\gamma+\beta} (1+\frac{1}{p})^{\beta}, & \frac{p}{2} \leq \alpha < p, \end{cases}$$

for some real numbers α, β *and* γ *with* $0 \leq \alpha < p, \beta \geq 0, \gamma \geq 0, \beta + \gamma > 0$, *then* $f(z) \in K_p(\alpha)$.

Taking $p = 1$ in Corollary 3.9, we obtain the following sufficient condition for convexity of univalent functions.

Corollary 3.10. *For some non-negative real numbers* α, β *and* γ *with* $\beta + \gamma > 0$ *and* $\alpha < 1$, *if* $f(z)$ *satisfies*

$$\left| \frac{zf''(z)}{f'(z)} \right|^{\gamma} \left| \frac{z^2 f'''(z) + 6z^2 f''(z) + 6f'(z)}{zf''(z) + 2f'(z)} - 1 \right|^{\beta}$$

$$< \begin{cases} (2)^{\gamma} (1-\alpha)^{\gamma} \left(\frac{9}{2} - 3\alpha \right)^{\beta}, & 0 \leq \alpha < \frac{1}{2} \\ (2)^{\gamma} (1-\alpha)^{\gamma+\beta} (6)^{\beta}, & \frac{1}{2} \leq \alpha < 1, \end{cases}$$

for all $z \in U$, *then* $f \in K(\alpha)$.

In particular, writing $\beta = 1, \gamma = 1$, and $\alpha = 0$ in Corollary 3.10, we obtain the following sufficient condition for convexity of analytic functions:

Corollary 3.11. *If* $f \in A$ *satisfies*

$$\left| \frac{zf''(z)}{f'(z)} \left(\frac{z^2 f'''(z) + 5z^2 f''(z) + 4f'(z)}{zf''(z) + 2f'(z)} \right) \right| < 9, \quad (z \in U)$$

then $f(z) \in K$.

References

[1] Xu, N., Aouf, M.K., *Properties of certain analytic multivalent functions defined by a linear operator*, Comput. Math. Appl., **58**(2009), 1169-1179.

[2] Carlson, B.C., Sharer, D.B., *Starlike and prestarlike hypergeometric functions*, SIAM J. Math. Anal., **15**(1984), 737-745.

[3] Jack, I.S., *Functions starlike and convex of order α*, J. London Math. Soc., **3**(1971), 469-474.

[4] Libera, R.J., *Some classes of regular univalent functions*, Proc. Amer. Math. Soc., **16**(1965), 653-758.

[5] Ruscheweyh, S., *New criteria for univalent functions*, Proc. Amer. Math. Soc., **49**(1975), no. 1, 109-115.

[6] Singh, S., Gupta, S., Singh, S., *On a class of multivalent functions defined by a multipilier transformation*, Math. Bechnk, **60**(2008), 87-94.

Elsayed A. Elrifai
Department of Mathematics, Faculty of Science
University of Mansoura, Mansoura Egypt
e-mail: Rifai@mans.edu.eg

Hanan E. Darwish
Department of Mathematics, Faculty of Science
University of Mansoura, Mansoura Egypt
e-mail: Darwish333@yahoo.com

Abdusalam R. Ahmed
Department of Mathematics, Faculty of Science
University of Mansoura, Mansoura Egypt
e-mail: Abdusalam5056@yahoo.com

Stud. Univ. Babeş-Bolyai Math. 56(2011), No. 3, 61–72

Subclasses of analytic functions associated with Fox-Wright's generalized hypergeometric functions based on Hilbert space operator

Gangadharan Murugusundaramoorthy and Thomas Rosy

Abstract. In this paper, we define a generalized class of starlike functions which are based upon some convolution operators on Hilbert space involving the Fox-Wright generalization of the classical hypergeometric pF_q function (with p numerator and q denominator parameters). The various results presented in this paper include (for example) normed coefficient inequalities and estimates, distortion theorems, and the radii of convexity and starlikeness for each of the analytic function classes which are investigated here. Also we obtain modified Hadamard product and integral means results.

Mathematics Subject Classification (2010): 30C45.

Keywords: Analytic, univalent, starlikeness, convexity, Hadamard product (convolution) hypergeometric functions.

1. Introduction

Let A denote the class of functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \tag{1.1}$$

which are analytic and univalent in the open disc $U = \{z : z \in \mathcal{C};\ |z| < 1\}$. For functions $f \in A$ given by (1.1) and $g \in A$ given by $g(z) = z + \sum\limits_{k=2}^{\infty} b_k z^k$, we define the Hadamard product (or Convolution) of f and g by

$$(f * g)(z) = z + \sum_{k=2}^{\infty} a_k b_k z^k, \quad z \in U. \tag{1.2}$$

For positive real parameters $\alpha_1, A_1 \ldots, \alpha_l, A_l$ and $\beta_1, B_1 \ldots, \beta_m, B_m$, $(l, m \in N = 1, 2, 3, ...)$ such that

$$1 + \sum_{k=1}^{m} B_k - \sum_{k=1}^{l} A_k \geq 0, \quad z \in U, \tag{1.3}$$

the Wright generalized hypergeometric function [15]

$$\begin{aligned} &{}_l\Psi_m[(\alpha_1, A_1), \ldots, (\alpha_l, A_l); (\beta_1, B_1), \ldots, (\beta_m, B_m); z] \\ &={}_l\ \Psi_m[(\alpha_t, A_t)_{1,l}(\beta_t, B_t)_{1,m}; z] \end{aligned}$$

is defined by

$${}_l\Psi_m[(\alpha_t, A_t)_{1,l}(\beta_t, B_t)_{1,m}; z] = \sum_{k=0}^{\infty} \{\prod_{t=0}^{l} \Gamma(\alpha_t + kA_t\}\{\prod_{t=0}^{m} \Gamma(\beta_t + kB_t\}^{-1} \frac{z^k}{k!},$$

$z \in U$.
If $A_t = 1 (t = 1, 2, ..., l)$ and $B_t = 1 (t = 1, 2, ..., m)$ we have the relationship:

$$\begin{aligned} \Omega_l\Psi_m[(\alpha_t, 1)_{1,l}(\beta_t, 1)_{1,m}; z] &\equiv {}_lF_m(\alpha_1, \ldots \alpha_l; \beta_1, \ldots, \beta_m; z) \\ &= \sum_{k=0}^{\infty} \frac{(\alpha_1)_k \ldots (\alpha_l)_k}{(\beta_1)_k \ldots (\beta_m)_k} \frac{z^k}{k!} \end{aligned} \tag{1.4}$$

$(l \leq m + 1;\ l, m \in N_0 = N \cup \{0\}; z \in U)$ is the generalized hypergeometric function (see for details [6]), where N denotes the set of all positive integers and $(\alpha)_n$ is the Pochhammer symbol and

$$\Omega = \left(\prod_{t=0}^{l} \Gamma(\alpha_t)\right)^{-1} \left(\prod_{t=0}^{m} \Gamma(\beta_t)\right). \tag{1.5}$$

By using the generalized hypergeometric function, Dziok and Srivastava [6] introduced a linear operator which was subsequently extended by Dziok and Raina [5] by using the Fox-Wright generalized hypergeometric function.
Let $\mathcal{W}[(\alpha_t, A_t)_{1,l}; (\beta_t, B_t)_{1,m}] : A \to A$ be a linear operator defined by

$$\mathcal{W}[(\alpha_t, A_t)_{1,p}; (\beta_t, B_t)_{1,q}]f(z) \quad := \quad z\ {}_l\phi_m[(\alpha_t, A_t)_{1,l}; (\beta_t, B_t)_{1,m}; z] * f(z)$$

We observe that, for $f(z)$ of the form(1.1), we have

$$\mathcal{W}[(\alpha_t, A_t)_{1,l}; (\beta_t, B_t)_{1,m}]f(z) = z + \sum_{k=2}^{\infty} \sigma_k(\alpha_1)\ a_k z^k \tag{1.6}$$

where $\sigma_k(\alpha_1)$ is defined by

$$\sigma_k(\alpha_1) = \frac{\Omega\Gamma(\alpha_1 + A_1(k-1)) \ldots \Gamma(\alpha_l + A_l(k-1))}{(k-1)!\Gamma(\beta_1 + B_1(k-1)) \ldots \Gamma(\beta_m + B_m(k-1))}. \tag{1.7}$$

For convenience, we adopt the contracted notation $\mathcal{W}[\alpha_1]f(z)$ to represent the following:

$$\mathcal{W}[\alpha_1]f(z) = \mathcal{W}[(\alpha_1, A_1), \ldots, (\alpha_l, A_l); (\beta_1, B_1), \ldots, (\beta_m, B_m)]f(z) \tag{1.8}$$

throughout the sequel. The linear operator $\mathcal{W}[\alpha_1]f(z)$ contains the Dziok-Srivastava operator (see [6]), and as its various special cases contain such linear operators as the Hohlov operator, Carlson-Shaffer operator [3], Ruscheweyh derivative operator [14], generalized Bernardi-Libera-Livingston operator and fractional derivative operator [9]. Details and references about these operators can be found in [5] and [6].

Let $\mathcal{H}$ be a complex Hilbert space and let $\mathcal{L}(\mathcal{H})$ denote the algebra of all bounded linear operators on H. For a complex-valued function $f$analytic in a domain $\mathbb{E}$ of the complex z-plane containing the spectrum $\sigma(\mathbb{P})$ of the bounded linear operator$\mathbb{P}$, let $f(\mathbb{P})$ denote the operator on H defined by [[2], p. 568]

$$f(\mathbb{P}) = \frac{1}{2\pi i}\int_{\mathcal{C}} (z\mathbb{I} - \mathbb{P})^{-1} f(z)dz, . \tag{1.9}$$

where $\mathbb{I}$ is the identity operator on $\mathcal{H}$ and $\mathcal{C}$ is a positively-oriented simple rectifiable closed contour containing the spectrum $\sigma(\mathbb{P})$ in the interior domain. The operator $f(\mathbb{P})$ can also be defined by the following series:

$$f(\mathbb{P}) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}\mathbb{P}^n$$

which converges in the normed topology (cf. [4]).

We introduced a new subclass of analytic functions with negative coefficients and discuss some interesting properties of this generalized function class.

For $0 \leq \alpha < 1$ and $0 < \beta \leq 1$, we let $\mathcal{W}(\alpha, \beta)$ be the subclass of A consisting of functions of the form (1.1) and satisfying the inequality

$$\left\| \frac{\mathcal{J}_\lambda(\mathbb{P}) - 1}{\mathcal{J}(\mathbb{P}) - (2\alpha - 1)} \right\| < \beta \tag{1.10}$$

where

$$\mathcal{J}_\lambda(\mathbb{P}) = (1-\lambda)\frac{\mathcal{W}[\alpha_1]f(\mathbb{P})}{\mathbb{P}} + \lambda(\mathcal{W}[\alpha_1]f(\mathbb{P}))', \tag{1.11}$$

$0 < \gamma \leq 1$, $\mathcal{W}[\alpha_1]f(z)$ is given by (1.8).

We further let $\mathcal{W}(\alpha, \beta) = \mathcal{WT}(\alpha, \beta) \cap T$, where

$$T \; := \left\{ f \in A : f(z) = z - \sum_{k=2}^{\infty} a_k z^k, \quad a_k \geq 0; z \in U \right\} \tag{1.12}$$

is a subclass of A introduced and studied by Silverman [10].

In the following section we obtain coefficient estimates and extreme points for the class $\mathcal{WT}(\lambda, \alpha, \beta,)$.

2. Coefficient bounds

Theorem 2.1. *Let the function f be defined by (1.12). Then $f \in \mathcal{WT}(\lambda, \alpha, \beta,)$ if and only if*

$$\sum_{k=2}^{\infty}(1 + \lambda(k-1))[1+\beta]\sigma_k(\alpha_1)a_k \leq 2\beta(1-\alpha). \tag{2.1}$$

The result is sharp for the function

$$f(z) = z - \frac{2\beta(1-\alpha)}{(1+\lambda(k-1))[1+\beta]\Omega\sigma_k(\alpha_1)} z^k, \quad k \geq 2. \tag{2.2}$$

Proof. Suppose f satisfies (2.1). Then for $\|z\| = \mathbb{P} = r\mathbb{I}$,

$$\begin{aligned}
&\|\mathcal{J}_\lambda(\mathbb{P}) - 1\| - \beta \|\mathcal{J}_\lambda(\mathbb{P}) + 1 - 2\alpha\| \\
&= \left\| -\sum_{k=2}^{\infty} [1+\lambda(k-1)]\sigma_k(\alpha_1) a_k \mathbb{P}^{k-1} \right\| \\
&-\beta \left\| 2(1-\alpha) - \sum_{k=2}^{\infty} [1+\lambda(k-1)]\sigma_k(\alpha_1) a_k \mathbb{P}^{k-1} \right\| \\
&\leq \sum_{k=2}^{\infty} [1+\lambda(k-1)] a_k \ \sigma_k(\alpha_1) r^{k-1} - 2\beta(1-\alpha) \\
&+ \sum_{k=2}^{\infty} [1+\lambda(k-1)]\beta\sigma_k(\alpha_1) a_k r^{k-1} \\
&= \sum_{k=2}^{\infty} [1+\lambda(k-1)][1+\beta]\sigma_k(\alpha_1) a_k - 2\beta(1-\alpha) \leq 0, \qquad \text{by (2.1)}.
\end{aligned}$$

Hence, by maximum modulus theorem and (1.10), $f \in \mathcal{WT}(\alpha, \beta)$.
To prove the converse, assume that

$$\left\| \frac{\mathcal{J}_\lambda(\mathbb{P}) - 1}{\mathcal{J}_\lambda(\mathbb{P}) + 1 - 2\alpha} \right\| = \left\| \frac{-\sum_{k=2}^{\infty} [1+\lambda(k-1)]\sigma_k(\alpha_1) a_k \mathbb{P}^{k-1}}{2(1-\alpha) - \sum_{k=2}^{\infty} [1+\lambda(k-1)]\sigma_k(\alpha_1) a_k \mathbb{P}^{k-1}} \right\|$$

$$\leq \beta, \quad z \in U.$$

Putting $\mathbb{P} = r\mathbb{I}(0 < r < 1$, and upon letting $r \to 1-$, yields the assertion (2.1) of Theorem 2.1. □

Corollary 2.2. *If $f(z)$ of the form (1.12) is in $\mathcal{WT}(\lambda, \alpha, \beta)$, then*

$$a_k \leq \frac{2\beta(1-\alpha)}{(1+k\lambda-\lambda)[1+\beta]\sigma_k(\alpha_1)}, \quad k \geq 2, \tag{2.3}$$

with equality only for functions of the form (2.2).

Theorem 2.3. (Extreme Points) *Let*

$$f_1(z) = z \quad \text{and}$$

$$f_k(z) = z - \frac{2\beta(1-\alpha)}{[1+\lambda(k-1)][1+\beta]\sigma_k(\alpha_1)} z^k, \ \ k \geq 2, \tag{2.4}$$

for $0 \leq \alpha < 1$, $0 < \beta \leq 1$, $\lambda \geq 0$. Then $f(z)$ is in the class $\mathcal{WT}(\alpha, \beta)$ if and only if it can be expressed in the form

$$f(z) = \sum_{k=1}^{\infty} \mu_k f_k(z), \tag{2.5}$$

where $\mu_k \geq 0$ *and* $\sum\limits_{k=1}^{\infty} \mu_k = 1.$

Proof. Suppose $f(z)$ can be written as in (2.5). Then

$$f(z) \quad = \quad z - \sum_{k=2}^{\infty} \mu_k \frac{2\beta(1-\alpha)}{[1+\lambda(k-1)][1+\beta]\sigma_k(\alpha_1)} z^k.$$

Now,

$$\sum_{k=2}^{\infty} \frac{[1+\lambda(k-1)][1+\beta]\sigma_k(\alpha_1)}{2\beta(1-\alpha)} \mu_k \frac{2\beta(1-\alpha)}{[1+\lambda(k-1)][1+\beta]\sigma_k(\alpha_1)}$$

$$= \sum_{k=2}^{\infty} \mu_k = 1 - \mu_1 \leq 1.$$

Thus $f \in \mathcal{WT}(\alpha, \beta)$. Conversely, let us have $f \in \mathcal{WT}(\alpha, \beta)$. Then by using (2.3), we set

$$\mu_k = \frac{[1+\lambda(k-1)][1+\beta]\sigma_k(\alpha_1)}{2\beta(1-\alpha)} a_k, \quad k \geq 2$$

and

$$\mu_1 = 1 - \sum_{k=2}^{\infty} \mu_k.$$

Then we have

$$f(z) = \sum_{k=1}^{\infty} \mu_k f_k(z)$$

and hence this completes the proof of Theorem 2.3. □

3. Distortion bounds

In this section we obtain distortion bounds for the class $\mathcal{WT}(\alpha, \beta)$.

Theorem 3.1. *If* $f \in \mathcal{WT}(\alpha, \beta)$, *then*

$$r - \frac{2\beta(1-\alpha)}{(1+\lambda)[1+\beta]\sigma_2(\alpha_1)} r^2 \leq \|f(\mathbb{P})\| \ \leq r + \frac{2\beta(1-\alpha)}{(1+\lambda)[1+\beta]\sigma_2(\alpha_1)} r^2 \quad (3.1)$$

$$1 - \frac{4\beta(1-\alpha)}{(1+\lambda)[1+\beta]\sigma_2(\alpha_1)} r \leq \|f'(\mathbb{P})\| \ \leq 1 + \frac{4\beta(1-\alpha)}{(1+\lambda)[1+\beta]\sigma_2(\alpha_1)} r, \quad (3.2)$$

$(\mathbb{P} = r(0 < r < 1))$. *The bounds in (*3.1*) and (*3.2*) are sharp, since the equalities are attained by the function*

$$f(z) = z - \frac{2\beta(1-\alpha)}{(1+\lambda)[1+\beta]\sigma_2(\alpha_1)} z^2 \quad z = \pm r. \quad (3.3)$$

Proof. In the view of Theorem 2.1, we have

$$\sum_{k=2}^{\infty} a_k \leq \frac{2\beta(1-\alpha)}{(1+\lambda)[1+\beta]\sigma_2(\alpha_1)} \tag{3.4}$$

Using (1.12) and (3.4), we obtain

$$\begin{aligned} \|\mathbb{P}\| - \|\mathbb{P}\|^2 \sum_{k=2}^{\infty} a_k \quad &\leq \quad \|f(\mathbb{P})\| \ \leq \ \|\mathbb{P}\| + \|\mathbb{P}\|^2 \sum_{k=2}^{\infty} a_k \\ r - r^2 \frac{2\beta(1-\alpha)}{(1+\lambda)[1+\beta]\sigma_2(\alpha_1)} \quad &\leq \quad \|f(\mathbb{P})\| \ \leq \ r + r^2 \frac{2\beta(1-\alpha)}{(1+\lambda)[1+\beta]\sigma_2(\alpha_1)} \end{aligned} \tag{3.5}$$

Hence (3.1) follows from (3.5).

Further, since

$$\sum_{k=2}^{\infty} k a_k \leq \frac{4\beta(1-\alpha)}{(1+\lambda)[1+\beta]\sigma_2(\alpha_1)}.$$

Hence (3.2) follows from

$$1 - r\sum_{k=2}^{\infty} k a_k \leq \|f'(\mathbb{P})\| \ \leq \ 1 + r\sum_{k=2}^{\infty} k a_k.$$

□

4. Radius of starlikeness and convexity

The radii of close-to-convexity, starlikeness and convexity for the class $\mathcal{WT}(\alpha,\beta)$ are given in this section.

Theorem 4.1. *Let the function $f(z)$ defined by (1.12) belongs to the class $\mathcal{WT}(\alpha,\beta)$. Then $f(z)$ is close-to-convex of order δ $(0 \leq \delta < 1)$ in the disc $\|\mathbb{P}\| < r_1$, where*

$$r_1 := \left[\frac{(1-\delta)[1+\lambda(k-1)][1+\beta]\ \sigma_k(\alpha_1)}{2k\beta(1-\alpha)}\right]^{\frac{1}{k-1}} \quad (k \geq 2). \tag{4.1}$$

The result is sharp, with extremal function $f(z)$ given by (2.4).

Proof. Given $f \in T$ and f is close-to-convex of order δ, we have

$$\|f'(\mathbb{P}) - 1\| < 1 - \delta. \tag{4.2}$$

For the left hand side of (4.2) we have

$$\|f'(\mathbb{P}) - 1\| \leq \sum_{k=2}^{\infty} k a_k \|\mathbb{P}\|^{k-1}.$$

The last expression is less than $1-\delta$ if

$$\sum_{k=2}^{\infty} \frac{k}{1-\delta} a_k \|\mathbb{P}\|^{k-1} < 1.$$

Using the fact, that $f \in \mathcal{WT}(\alpha, \beta)$ if and only if

$$\sum_{k=2}^{\infty} \frac{[1+\lambda(k-1)][1+\beta]a_k \ \sigma_k(\alpha_1)}{2\beta(1-\alpha)} \leq 1.$$

We can say (4.2) is true if

$$\frac{k}{1-\delta}\|\mathbb{P}\|^{k-1} \leq \frac{[1+\lambda(k-1)][1+\beta]\sigma_k(\alpha_1)}{2\beta(1-\alpha)}.$$

Or, equivalently,

$$\|\mathbb{P}\|^{k-1} = r^{k-1} < \left[\frac{(1-\delta)[1+\lambda(k-1)][1+\beta] \ \sigma_k(\alpha_1)}{2k\beta(1-\alpha)}\right]$$

which completes the proof. □

Theorem 4.2. *Let $f \in \mathcal{WT}(\alpha, \beta)$. Then*

1. *f is starlike of order $\delta (0 \leq \delta < 1)$ in the disc $|z| < r_2$; that is, $Re\left\{\frac{zf'(z)}{f(z)}\right\} > \delta$, $(\|\mathbb{P}\| < r_2 \ ; \ 0 \leq \delta < 1)$, where*

$$r_2 = \inf_{k \geq 2}\left\{\frac{(1-\delta)[1+\lambda(k-1)][1+\beta] \ \sigma_k(\alpha_1)}{2\beta(1-\alpha)(k-\delta)}\right\}^{\frac{1}{k-1}}.$$

2. *f is convex of order δ $(0 \leq \delta < 1)$ in the disc $|z| < r_3$, that is $Re\left\{1+\frac{zf''(z)}{f'(z)}\right\} > \delta$, $(\|\mathbb{P}\| < r_3; 0 \leq \delta < 1)$, where*

$$r_3 = \inf_{k \geq 2}\left\{\frac{(1-\delta)[1+\lambda(k-1)][1+\beta] \ \sigma_k(\alpha_1)}{2\beta(1-\alpha)k(k-\delta)}\right\}^{\frac{1}{k-1}}.$$

Each of these results are sharp for the extremal function $f(z)$ given by (2.4).

Proof. Given $f \in T$ and f is starlike of order δ, we have

$$\left\|\frac{\mathbb{P}f'(\mathbb{P})}{f(\mathbb{P})} - 1\right\| < 1-\delta \quad (\mathbb{P} = r_2\mathbb{I}(0 < r_1 < 1)). \tag{4.3}$$

For the left hand side of (4.3) we have

$$\left\|\frac{\mathbb{P}f'(\mathbb{P})}{f(\mathbb{P})} - 1\right\| \leq \frac{\sum_{k=2}^{\infty}(k-1)a_k \ \|\mathbb{P}\|^{k-1}}{1-\sum_{k=2}^{\infty} a_k \ \|\mathbb{P}\|^{k-1}}.$$

The last expression is less than $1-\delta$ if

$$\sum_{k=2}^{\infty}\frac{k-\delta}{1-\delta}a_k \ \|\mathbb{P}\|^{k-1} < 1.$$

Using the fact, that $f \in \mathcal{WT}(\alpha, \beta)$ if and only if

$$\sum_{k=2}^{\infty} \frac{[1+\lambda(k-1)][1+\beta]a_k \ \sigma_k(\alpha_1)}{2\beta(1-\alpha)} < 1.$$

We can say that (4.3) is true if

$$\frac{k-\delta}{1-\delta}\|\mathbb{P}\|^{k-1} < \frac{[1+\lambda(k-1)][1+\beta]\ \sigma_k(\alpha_1)}{2\beta(1-\alpha)}.$$

Or, equivalently,

$$\|\mathbb{P}\|^{k-1} = r_2 < \frac{(1-\delta)[1+\lambda(k-1)][1+\beta]\ \sigma_k(\alpha_1)}{2\beta(1-\alpha)(k-\delta)},$$

which yields the starlikeness of the family.

(ii) Using the fact that f is convex if and only if zf' is starlike, we can prove (ii), similar the to proof of (i). □

5. Modified Hadamard products

Let the functions $f_j(z)(j = 1, 2)$ be defined by (1.12). The modified Hadamard product of $f_1(z)$ and $f_2(z)$ is defined by

$$(f_1 * f_2)(z) \ = \ z - \sum_{n=2}^{\infty} a_{k,1}a_{k,2}\ z^k.$$

Using the techniques of Schild and Silverman [13], we prove the following results.

Theorem 5.1. *For functions $f_j(z)(j = 1, 2)$ defined by (1.12), let $f_1 \in \mathcal{WT}(\alpha, \beta)$ and $f_2 \in \mathcal{WT}(\gamma, \beta)$. Then $(f_1 * f_2) \in\in \mathcal{WT}(\xi, \beta)$ where*

$$\xi = 1 - \frac{2\beta(1-\alpha)(1-\gamma)}{\sigma_2(\alpha_1)}, \tag{5.1}$$

where $\sigma_k(\alpha_1)$ is given by (1.7).

Proof. In view of Theorem 2.1, it suffice to prove that

$$\sum_{k=2}^{\infty} \frac{(1+\lambda(k-1))[1+\beta]\sigma_k(\alpha_1)}{2\beta(1-\alpha)(1-\xi)} a_{k,1}a_{k,2} \le 1, \ \ (0 \le \xi < 1)$$

where ξ is defined by (5.1). On the other hand, under the hypothesis, it follows from (2.1) and the Cauchy's-Schwarz inequality that

$$\sum_{k=2}^{\infty} \frac{(1+\lambda(k-1))[1+\beta]\sigma_k(\alpha_1)}{2\beta\sqrt{(1-\alpha)(1-\gamma)}} \sqrt{a_{k,1}a_{k,2}} \ \le 1. \tag{5.2}$$

We need to find the largest ξ such that

$$\begin{aligned} &\sum_{k=2}^{\infty} \frac{(1+\lambda(k-1))[1+\beta]\sigma_k(\alpha_1)}{2\beta(1-\xi)(1-\alpha)} a_{k,1}a_{k,2} \\ \le \ &\sum_{k=2}^{\infty} \frac{(1+\lambda(k-1))[1+\beta]\sigma_k(\alpha_1)}{\sqrt{(1-\alpha)(1-\gamma)}} \sqrt{a_{k,1}a_{k,2}} \end{aligned}$$

or, equivalently that

$$\sqrt{a_{k,1}a_{k,2}} \ \ \le \ \ \frac{1-\xi}{\sqrt{(1-\alpha)((1-\gamma))}}, \ \ (k \ge 2).$$

By view of (5.2) it is sufficient to find the largest ξ such that

$$2\beta\sqrt{(1-\alpha)(1-\gamma)}(\sigma_n(\alpha_1))^{-1} \leq \frac{1-\xi}{\sqrt{(1-\alpha)((1-\gamma))}}$$

which yields

$$\xi = 1 - \frac{2\beta(1-\alpha)(1-\gamma)}{\sigma_k(\alpha_1)} \text{ for } k \geq 2 \tag{5.3}$$

is an increasing function of k and letting $k=2$ in (5.3), we have

$$\xi = 1 - \frac{2\beta(1-\alpha)(1-\gamma)}{\sigma_2(\alpha_1)}$$

where $\sigma_2(\alpha_1)$ is given by (1.7). □

Theorem 5.2. *Let the function $f(z)$ defined by (1.12) be in the class $\mathcal{WT}(\alpha,\beta)$. Also let $g(z) = z - \sum\limits_{k=2}^{\infty} b_k z^k$ for $|b_k| \leq 1$. Then $(f*g) \in \mathcal{WT}(\alpha,\beta)$.*

Proof. Since

$$\begin{aligned}
& \sum_{k=2}^{\infty}(1+\lambda(k-1))[1+\beta]\sigma_k(\alpha_1)|a_k b_k| \\
\leq \; & \sum_{k=2}^{\infty}(1+\lambda(k-1))[1+\beta]\sigma_k(\alpha_1)a_k|b_k| \\
\leq \; & \sum_{k=2}^{\infty}(1+\lambda(k-1))[1+\beta]\sigma_k(\alpha_1)a_k \\
\leq \; & 2\beta(1-\alpha)
\end{aligned}$$

it follows that $(f*g) \in \mathcal{WT}(\alpha,\beta)$, by the view of Theorem 2.1. □

Theorem 5.3. *Let the functions $f_j(z)(j=1,2)$ defined by (1.12) be in the class $\in \mathcal{WT}(\alpha,\beta)$. Then the function $h(z)$ defined by $h(z) = z - \sum\limits_{k=2}^{\infty}(a_{n,1}^2 + a_{n,2}^2)z^n$ is in the class $\in \mathcal{WT}(\xi,\beta)$, where*

$$\xi = 1 - \frac{4\beta(1-\alpha)^2}{\sigma_2(\alpha_1)(1+\lambda)(1+\beta)}$$

and $\sigma_2(\alpha_1)$ is given by (1.7).

Proof. By virtue of Theorem 2.1, it is sufficient to prove that

$$\sum_{k=2}^{\infty}\frac{(1+\lambda(k-1))[1+\beta]\sigma_k(\alpha_1)}{2\beta(1-\xi)}(a_{n,1}^2 + a_{n,2}^2) \leq 1, \tag{5.4}$$

where $f_j \in \mathcal{WT}(\xi,\beta)$ we find from (2.1) and Theorem 2.1, that

$$\sum_{k=2}^{\infty}\left[\frac{(1+\lambda(k-1))[1+\beta]\sigma_k(\alpha_1)}{2\beta(1-\alpha)}\right]^2 a_{n,j}^2 \leq \sum_{k=2}^{\infty}\left[\frac{(1+\lambda(k-1))[1+\beta]\sigma_k(\alpha_1)}{2\beta(1-\alpha)}a_{n,j}\right]^2, \tag{5.5}$$

which yields

$$\sum_{k=2}^{\infty}\frac{1}{2}\left[\frac{(1+\lambda(k-1))[1+\beta]\sigma_k(\alpha_1)}{2\beta(1-\alpha)}\right]^2 (a_{n,1}^2+a_{n,2}^2) \leq 1. \tag{5.6}$$

On comparing (5.5) and (5.6), it is easily seen that the inequality (5.4) will be satisfied if

$$\frac{(1+\lambda(k-1))[1+\beta]\sigma_k(\alpha_1)}{2\beta(1-\xi)} \leq \frac{1}{2}\left[\frac{(1+\lambda(k-1))[1+\beta]\sigma_k(\alpha_1)}{2\beta(1-\alpha)}\right]^2, \quad \text{for } k \geq 2.$$

That is an increasing function of k $(k \geq 2)$. Taking $k=2$ in (5.7), we have

$$\xi = 1 - \frac{4\beta(1-\alpha)^2}{\sigma_k(\alpha_1)(1+\lambda)(1+\beta)} \tag{5.7}$$

which completes the proof. □

6. Integral means inequalities

Lemma 6.1. [8] *If the functions f and g are analytic in Δ with $g \prec f$, then for $\kappa > 0$, and $0 < r < 1$,*

$$\int_0^{2\pi} \left|g(re^{i\theta})\right|^{\kappa} d\theta \leq \int_0^{2\pi} \left|f(re^{i\theta})\right|^{\kappa} d\theta. \tag{6.1}$$

In [10], Silverman found that the function $f_2(z) = z - \frac{z^2}{2}$ is often extremal over the family T. He applied this function to resolve his integral means inequality, conjectured in [11] and settled in [12], that

$$\int_0^{2\pi} \left|f(re^{i\theta})\right|^{\kappa} d\theta \leq \int_0^{2\pi} \left|f_2(re^{i\theta})\right|^{\kappa} d\theta,$$

for all $f \in T$, $\kappa > 0$ and $0 < r < 1$. In [12], he also proved his conjecture for the subclasses of starlike functions of order α and convex functions of order α.

In this section, we obtain integral means inequalities for the functions in the family $\mathcal{WT}(\alpha,\beta)$.

Applying Lemma 6.1, Theorem 2.1 and Theorem 2.3, we prove the following result.

Theorem 6.2. *Suppose* $f(z) \in \mathcal{WT}(\alpha, \beta)$ *and* $f_2(z)$ *is defined by*

$$f_2(z) = z - \frac{2\beta(1-\alpha)}{[1+\lambda][1+\beta]\sigma_2(\alpha_1)} z^2.$$

Then for $z = re^{i\theta}$, $0 < r < 1$, *we have*

$$\int_0^{2\pi} \|f(z)\|^{\kappa}\, d\theta \leq \int_0^{2\pi} \|f_2(z)\|^{\kappa}\, d\theta. \tag{6.2}$$

Proof. For $f(z) = z - \sum_{k=2}^{\infty} a_k z^n$, (6.2) is equivalent to proving that

$$\int_0^{2\pi} \left\| 1 - \sum_{k=2}^{\infty} a_k z^{n-1} \right\|^{\kappa} d\theta \leq \int_0^{2\pi} \left\| 1 - \frac{2\beta(1-\alpha)}{[1+\lambda][1+\beta]\sigma_2(\alpha_1)} z \right\|^{\kappa} d\theta.$$

By Lemma 6.1, it suffices to show that

$$1 - \sum_{k=2}^{\infty} a_k \|\mathbb{P}\|^{n-1} \prec 1 - \frac{2\beta(1-\alpha)}{[1+\lambda][1+\beta]\sigma_2(\alpha_1)} \|\mathbb{P}\|.$$

Setting

$$1 - \sum_{k=2}^{\infty} a_k \|\mathbb{P}\|^{n-1} = 1 - \frac{2\beta(1-\alpha)}{[1+\lambda][1+\beta]\sigma_2(\alpha_1)} w(z), \tag{6.3}$$

and using (2.1), we obtain

$$\begin{aligned}
\|w(z)\| &= \left\| \sum_{k=2}^{\infty} \frac{2\beta(1-\alpha)}{(1+k\lambda-\lambda)[1+\beta]\sigma_k(\alpha_1)} a_k z^{n-1} \right\| \\
&\leq \|\mathbb{P}\| \sum_{k=2}^{\infty} \frac{2\beta(1-\alpha)}{(1+k\lambda-\lambda)[1+\beta]\sigma_k(\alpha_1)} |a_k| \\
&\leq \|\mathbb{P}\|.
\end{aligned}$$

This completes the proof. □

Remark 6.3. In view of the relationship (1.4) the linear operator (1.6) and by setting $A_t = 1(t = 1, ..., l)$ and $B_t = 1(t = 1, ..., m)$ and specific choices of parameters l, m, α_1, β_1 the various results presented in this paper would provide interesting extensions and generalizations of those considered earlier for simpler analytic function classes.

References

[1] Altintas, O., *A subclass of analytic functions with negative coefficients,* Bull. Sci-Eng. Hacettepe Univ., **19**(1990), 15-24.

[2] Bao, G.J., Ling, Y., *On Some classes of analytic functions with negative coefficients,* J. Harbin Inst. Tech., **23**(1991), 100-103.

[3] Carlson, B.C., Shaffer, D.B., *Starlike and prestarlike hypergeometric functions,* SIAM J. Math. Anal., **15**(1984), 737-745.

[4] Dunford, N., Schwartz, J.T., *Linear Operators. Part I: General Theory (Reprinted from the 1958 original)*, A Wiley-Interscience Publication, John Wiley and Sons, New York, 1988.

[5] Dziok, J., Raina, R.K., *Families of analytic functions associated with the Wright generalized hypergeometric function,* Demonstratio Math., **37**(2004), no. 3, 533-542.

[6] Dziok, J., Srivastava, H.M., *Classes of analytic functions associated with the generalized hypergeometric function,* Appl. Math. Comput. **103**(1999), no. 1, 1-13.

[7] Gupta, V.P., Jain, P.K., *Certain classes of univalent functions with negative coefficients-II,* Bull. Austral. Math. Soc., **15**(1976), 467-473.

[8] Littlewood, J. E., *On inequalities in theory of functions,* Proc. London Math. Soc., **23**(1925), 481-519.

[9] Srivastava, H.M., Owa, S., *Some characterization and distortion theorems involving fractional calculus, generalized hypergeometric functions, Hadamard products, linear operators and certain subclasses of analytic functions,* Nagoya Math. J., **106**(1987), 1-28.

[10] Silverman, H., *Univalent functions with negative coefficients,* Proc. Amer. Math. Soc., **51**(1975), 109-116.

[11] Silverman, H., *A survey with open problems on univalent functions whose coefficients are negative,* Rocky Mt. J. Math., **21**(1991), 1099-1125.

[12] Silverman, H., *Integral means for univalent functions with negative coefficients,* Houston J. Math., **23**(1997), 169-174.

[13] Schild, A., Silverman, H., *Convolution of univalent functions with negative coefficients,* Annales Univ. Mariae Curie, **29**(1975), 99-107.

[14] Ruscheweyh, S., New criteria for univalent functions, Proc. Amer. Math. Soc., **49**(1975), 109-115.

[15] Wright, E.M., *The asymptotic expansion of the generalized hypergeometric function,* Proc. London. Math. Soc., **46**(1946), 389-408.

[16] Yaguchi, T., Sekine, T., Saitoh, H., Owa, S., Nunokawa, M., Fukui, S., *A generalization class of certain subclasses of analytic functions with negative coefficients,* Proc. Inst. Natur. Sci. Nihon Univ., **25**(1990), 67-80.

Gangadharan Murugusundaramoorthy
School of Science and Humanities, VIT University
Vellore - 632014, India
e-mail: gmsmoorthy@yahoo.com

Thomas Rosy
Department of Mathematics, Madras Christian College
Chennai - 600059, India
e-mail: thomas.rosy@gmail.com

Stud. Univ. Babeş-Bolyai Math. 56(2011), No. 3, 73–84

Subclasses of analytic functions involving a family of integral operators

Zhi-Gang Wang, Feng-Hua Wen and Qing-Guo Li

Abstract. In the present paper, we introduce and investigate some new subclasses of analytic functions associated with a family of generalized Srivastava-Attiya operator. Such results as subordination and superordination properties, inclusion relationships, integral-preserving properties and convolution properties are proved. Several sandwich-type results are also derived.

Mathematics Subject Classification (2010): Primary 30C45; Secondary 30C80.

Keywords: Analytic functions, Hadamard product (or convolution), subordination and superordination between analytic functions, generalized Srivastava-Attiya operator.

1. Introduction

Let $\mathcal{A}$ denote the class of functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \tag{1.1}$$

which are *analytic* in the *open* unit disk

$$\mathbb{U} := \{z : \ z \in \mathbb{C} \quad \text{and} \quad |z| < 1\}.$$

Let $\mathcal{H}(\mathbb{U})$ be the linear space of all analytic functions in $\mathbb{U}$. For a positive integer number n and $a \in \mathbb{C}$, we let

$$\mathcal{H}[a, n] := \left\{\mathfrak{f} \in \mathcal{H}(\mathbb{U}) : \quad \mathfrak{f}(z) = a + a_n z^n + a_{n+1} z^{n+1} + \cdots \right\}.$$

Let $f,\ g \in \mathcal{A}$, where f is given by (1.1) and g is defined by

$$g(z) = z + \sum_{k=2}^{\infty} b_k z^k.$$

Then the Hadamard product (or convolution) $f * g$ of the functions f and g is defined by

$$(f * g)(z) := z^p + \sum_{k=2}^{\infty} a_k b_k z^k =: (g * f)(z).$$

For two functions f and g, analytic in $\mathbb{U}$, we say that the function f is subordinate to g in $\mathbb{U}$, and write

$$f(z) \prec g(z) \quad (z \in \mathbb{U}),$$

if there exists a Schwarz function ω, which is analytic in $\mathbb{U}$ with

$$\omega(0) = 0 \quad \text{and} \quad |\omega(z)| < 1 \quad (z \in \mathbb{U})$$

such that

$$f(z) = g\big(\omega(z)\big) \quad (z \in \mathbb{U}).$$

Indeed, it is known that

$$f(z) \prec g(z) \quad (z \in \mathbb{U}) \Longrightarrow f(0) = g(0) \quad \text{and} \quad f(\mathbb{U}) \subset g(\mathbb{U}).$$

Furthermore, if the function g is univalent in $\mathbb{U}$, then we have the following equivalence:

$$f(z) \prec g(z) \quad (z \in \mathbb{U}) \Longleftrightarrow f(0) = g(0) \quad \text{and} \quad f(\mathbb{U}) \subset g(\mathbb{U}).$$

In the following we recall a general Hurwitz-Lerch Zeta function $\Phi(z, s, a)$ defined by (cf., e.g., [18, p. 121 et sep.])

$$\Phi(z, s, a) := \sum_{k=0}^{\infty} \frac{z^k}{(k + a)^s}$$

$$(a \in \mathbb{C} \setminus \mathbb{Z}_0^-;\ s \in \mathbb{C} \ \text{ when } \ |z| < 1;\ \Re(s) > 1 \ \text{ when } \ |z| = 1),$$

where, as usual,

$$\mathbb{Z}_0^- := \mathbb{Z} \setminus \mathbb{N} \quad (\mathbb{Z} := \{0, \pm 1, \pm 2, \ldots\}; \quad \mathbb{N} := \{1, 2, 3, \ldots\}).$$

Several interesting properties and characteristics of the Hurwitz-Lerch Zeta function $\Phi(z, s, a)$ can be found in the recent investigations by (for example) Choi and Srivastava [3], Ferreira and López [5], Garg et al. [6], Lin et al. [7], Luo and Srivastava [10], Wen and Liu [19], Wen and Yang [20] and others.

Recently, Srivastava and Attiya [17] (see also Răducanu and Srivastava [14], Liu [9], Prajapat and Goyal [13]) introduced and investigated the linear operator:

$$\mathcal{J}_{s,\ b}(f) :\ \mathcal{A} \longrightarrow \mathcal{A}$$

defined, in terms of the Hadamard product (or convolution), by

$$\mathcal{J}_{s,\ b}f(z) := G_{s,\ b}(z) * f(z) \quad (z \in \mathbb{U};\ b \in \mathbb{C} \setminus \mathbb{Z}_0^-;\ s \in \mathbb{C};\ f \in \mathcal{A}), \tag{1.2}$$

where, for convenience,

$$G_{s,\ b}(z) := (1 + b)^s [\Phi(z, s, b) - b^{-s}] \quad (z \in \mathbb{U}). \tag{1.3}$$

It is easy to observe from (1.2) and (1.3) that

$$\mathcal{J}_{s,\ b}f(z) = z + \sum_{k=2}^{\infty} \left(\frac{1+b}{k+b}\right)^{s} a_k z^k.$$

Motivated essentially by the above-mentioned Srivastava-Attiya operator, Al-Shaqsi and Darus [1] (see also Darus and Al-Shaqsi [4]) introduced and investigated the following integral operator:

$$\mathcal{J}_{s,\ b}^{\lambda,\ \mu} f(z) = z + \sum_{k=2}^{\infty} \left(\frac{1+b}{k+b}\right)^{s} \frac{\lambda!(k+\mu-2)!}{(\mu-2)!(k+\lambda-1)!} a_k z^k \quad (z \in \mathbb{U}), \tag{1.4}$$

where (and throughout this paper unless otherwise mentioned) the parameters $s,\ b,\ \lambda$ and μ are constrained as follows:

$$s \in \mathbb{C}; \quad b \in \mathbb{C} \setminus \mathbb{Z}_0^- \quad \lambda > -1 \quad \text{and} \quad \mu > 0.$$

We note that $\mathcal{J}_{s,b}^{1,2}$ is the Srivastava-Attiya operator, $\mathcal{J}_{0,b}^{\lambda,\mu}$ is the well-known Choi-Saigo- Srivastava operator (see [2]).

It is easily verified from (1.4) that

$$z\left(\mathcal{J}_{s,\ b}^{\lambda,\ \mu} f\right)'(z) = \mu \mathcal{J}_{s,\ b}^{\lambda,\ \mu+1} f(z) - (\mu-1)\mathcal{J}_{s,\ b}^{\lambda,\ \mu} f(z), \tag{1.5}$$

$$z\left(\mathcal{J}_{s,\ b}^{\lambda+1,\ \mu} f\right)'(z) = (\lambda+1)\mathcal{J}_{s,\ b}^{\lambda,\ \mu} f(z) - \lambda \mathcal{J}_{s,\ b}^{\lambda+1,\ \mu} f(z), \tag{1.6}$$

and

$$z\left(\mathcal{J}_{s+1,\ b}^{\lambda,\ \mu} f\right)'(z) = (b+1)\mathcal{J}_{s,\ b}^{\lambda,\ \mu} f(z) - b\mathcal{J}_{s+1,\ b}^{\lambda,\ \mu} f(z). \tag{1.7}$$

By making use of the subordination between analytic functions and the operator $\mathcal{J}_{s,\ b}^{\lambda,\ \mu}$, we now introduce the following subclasses of analytic functions.

Definition 1.1. A function $f \in \mathcal{A}$ is said to be in the class $\mathcal{F}_{s,\ b}^{\lambda,\ \mu}(\alpha;\phi)$ if it satisfies the subordination condition

$$(1-\alpha)\frac{\mathcal{J}_{s,\ b}^{\lambda,\ \mu} f(z)}{z} + \alpha \frac{\mathcal{J}_{s,\ b}^{\lambda,\ \mu+1} f(z)}{z} \prec \phi(z) \quad (z \in \mathbb{U};\ \alpha \in \mathbb{C};\ \phi \in \mathcal{P}). \tag{1.8}$$

Definition 1.2. A function $f \in \mathcal{A}$ is said to be in the class $\mathcal{G}_{s,\ b}^{\lambda,\ \mu}(\alpha;\phi)$ if it satisfies the subordination condition

$$(1-\alpha)\frac{\mathcal{J}_{s,\ b}^{\lambda+1,\ \mu} f(z)}{z} + \alpha \frac{\mathcal{J}_{s,\ b}^{\lambda,\ \mu} f(z)}{z} \prec \phi(z) \quad (z \in \mathbb{U};\ \alpha \in \mathbb{C};\ \phi \in \mathcal{P}). \tag{1.9}$$

Definition 1.3. A function $f \in \mathcal{A}$ is said to be in the class $\mathcal{H}_{s,\ b}^{\lambda,\ \mu}(\alpha;\phi)$ if it satisfies the subordination condition

$$(1-\alpha)\frac{\mathcal{J}_{s+1,\ b}^{\lambda,\ \mu} f(z)}{z} + \alpha \frac{\mathcal{J}_{s,\ b}^{\lambda,\ \mu} f(z)}{z} \prec \phi(z) \quad (z \in \mathbb{U};\ \alpha \in \mathbb{C};\ \phi \in \mathcal{P}). \tag{1.10}$$

In the present paper, we aim at proving some subordination and superordination properties, inclusion relationships, integral-preserving properties and convolution properties associated with the operator $\mathcal{J}_{s,\ b}^{\lambda,\ \mu}$. Several sandwich-type results involving this operator are also derived.

2. Preliminary results

In order to prove our main results, we need the following lemmas.

Lemma 2.1. ([11]) *Let the function Ω be analytic and convex (univalent) in $\mathbb{U}$ with $\Omega(0) = 1$. Suppose also that the function Θ given by*

$$\Theta(z) = 1 + c_n z^n + c_{n+1} z^{n+1} + \cdots$$

is analytic in $\mathbb{U}$. If

$$\Theta(z) + \frac{z\Theta'(z)}{\zeta} \prec \Omega(z) \quad (\Re(\zeta) > 0;\ \zeta \neq 0;\ z \in \mathbb{U}), \tag{2.1}$$

then

$$\Theta(z) \prec \chi(z) = \frac{\zeta}{n} z^{-\frac{\zeta}{n}} \int_0^z t^{\frac{\zeta}{n}-1} h(t)dt \prec \Omega(z) \quad (z \in \mathbb{U}),$$

and χ is the best dominant of (2.1).

Denote by Q the set of all functions f that are analytic and injective on $\overline{\mathbb{U}} - E(f)$, where

$$E(f) = \left\{ \varepsilon \in \partial\mathbb{U} : \lim_{z \to \varepsilon} f(z) = \infty \right\},$$

and such that $f'(\varepsilon) \neq 0$ for $\varepsilon \in \partial\mathbb{U} - E(f)$.

Lemma 2.2. ([12]) *Let q be convex univalent in $\mathbb{U}$ and $\kappa \in \mathbb{C}$. Further assume that $\Re(\overline{\kappa}) > 0$. If*

$$p \in \mathcal{H}[q(0), 1] \cap Q,$$

and $p + \kappa z p'$ is univalent in $\mathbb{U}$, then

$$q(z) + \kappa z q'(z) \prec p(z) + \kappa z p'(z)$$

implies $q \prec p$, and q is the best subdominant.

Lemma 2.3. ([15]) *Let q be a convex univalent function in $\mathbb{U}$ and let $\sigma,\ \eta \in \mathbb{C}$ with*

$$\Re\left(1 + \frac{zq''(z)}{q'(z)}\right) > \max\left\{0,\ -\Re\left(\frac{\sigma}{\eta}\right)\right\}.$$

If p is analytic in $\mathbb{U}$ and

$$\sigma p(z) + \eta z p'(z) \prec \sigma q(z) + \eta z q'(z),$$

then $p \prec q$, and q is the best dominant.

Lemma 2.4. ([16]) *Let the function Υ be analytic in $\mathbb{U}$ with*

$$\Upsilon(0) = 1 \quad \text{and} \quad \Re(\Upsilon(z)) > \frac{1}{2} \quad (z \in \mathbb{U}).$$

Then, for any function Ψ analytic in $\mathbb{U}$, $(\Upsilon * \Psi)(\mathbb{U})$ is contained in the convex hull of $\Psi(\mathbb{U})$.

3. Properties of the function class $\mathcal{F}_{s,\ b}^{\lambda,\ \mu}(\alpha;\phi)$

We begin by proving our first subordination property given by Theorem 3.1 below.

Theorem 3.1. *Let* $f \in \mathcal{F}_{s,\ b}^{\lambda,\ \mu}(\alpha;\phi)$ *with* $\Re(\alpha) > 0$. *Then*

$$\frac{\mathcal{J}_{s,\ b}^{\lambda,\ \mu} f(z)}{z} \prec \frac{\mu}{\alpha} z^{-\frac{\mu}{\alpha}} \int_0^z t^{\frac{\mu}{\alpha}-1}\phi(t)dt \prec \phi(z) \quad (z \in \mathbb{U}). \tag{3.1}$$

Proof. Let $f \in \mathcal{F}_{s,\ b}^{\lambda,\ \mu}(\alpha;\phi)$ and suppose that

$$h(z) := \frac{\mathcal{J}_{s,\ b}^{\lambda,\ \mu} f(z)}{z} \quad (z \in \mathbb{U}). \tag{3.2}$$

Then h is analytic in $\mathbb{U}$. Combining (1.5), (1.8) and (3.2), we easily find that

$$h(z)+\frac{\alpha}{\mu}zh'(z) = (1-\alpha)\frac{\mathcal{J}_{s,\ b}^{\lambda,\ \mu} f(z)}{z} + \alpha\frac{\mathcal{J}_{s,\ b}^{\lambda,\ \mu+1} f(z)}{z} \prec \phi(z) \quad (z \in \mathbb{U}). \tag{3.3}$$

Therefore, an application of Lemma 2.1 for $n = 1$ to (3.3) yields the assertion of Theorem 3.1. □

By virtue of Theorem 3.1, we easily get the following inclusion relationship.

Corollary 3.2. *Let* $\Re(\alpha) > 0$. *Then* $\mathcal{F}_{s,\ b}^{\lambda,\ \mu}(\alpha;\phi) \subset \mathcal{F}_{s,\ b}^{\lambda,\ \mu}(0;\phi)$.

Theorem 3.3. *Let* $\alpha_2 > \alpha_1 \geqq 0$. *Then* $\mathcal{F}_{s,\ b}^{\lambda,\ \mu}(\alpha_2;\phi) \subset \mathcal{F}_{s,\ b}^{\lambda,\ \mu}(\alpha_1;\phi)$.

Proof. Suppose that $f \in \mathcal{F}_{s,\ b}^{\lambda,\ \mu}(\alpha_2;\phi)$. It follows that

$$(1-\alpha_2)\frac{\mathcal{J}_{s,\ b}^{\lambda,\ \mu}}{z} + \alpha_2\frac{\mathcal{J}_{s,\ b}^{\lambda,\ \mu+1}}{z} \prec \phi(z) \quad (z \in \mathbb{U}). \tag{3.4}$$

Since

$$0 \leqq \frac{\alpha_1}{\alpha_2} < 1$$

and the function ϕ is convex and univalent in $\mathbb{U}$, we deduce from (3.1) and (3.4) that

$$\begin{aligned}(1-\alpha_1)&\frac{\mathcal{J}_{s,\ b}^{\lambda,\ \mu} f(z)}{z} + \alpha_1\frac{\mathcal{J}_{s,\ b}^{\lambda,\ \mu+1} f(z)}{z} \\ &= \frac{\alpha_1}{\alpha_2}\left[(1-\alpha_1)\frac{\mathcal{J}_{s,\ b}^{\lambda,\ \mu} f(z)}{z} + \alpha_1\frac{\mathcal{J}_{s,\ b}^{\lambda,\ \mu+1} f(z)}{z}\right] + \left(1-\frac{\alpha_1}{\alpha_2}\right)\frac{\mathcal{J}_{s,\ b}^{\lambda,\ \mu} f(z)}{z} \\ &\prec \phi(z) \quad (z \in \mathbb{U}),\end{aligned}$$

which implies that $f \in \mathcal{F}_{s,\ b}^{\lambda,\ \mu}(\alpha_1;\phi)$. The proof of Theorem 3.3 is evidently completed. □

Theorem 3.4. *Let $f \in \mathcal{F}_{s,\ b}^{\lambda,\ \mu}(\alpha;\phi)$. If the integral operator F is defined by*

$$F(z) := \frac{\nu+1}{z^{\nu}} \int_0^z t^{\nu-1} f(t)dt \quad (z \in \mathbb{U};\ \nu > -1), \tag{3.5}$$

then

$$\frac{\mathcal{J}_{s,\ b}^{\lambda,\ \mu} F(z)}{z} \prec \phi(z) \quad (z \in \mathbb{U}). \tag{3.6}$$

Proof. Let $f \in \mathcal{F}_{s,\ b}^{\lambda,\ \mu}(\alpha;\ \phi)$. Suppose also that

$$G(z) := \frac{\mathcal{J}_{s,\ b}^{\lambda,\ \mu} F(z)}{z} \quad (z \in \mathbb{U}). \tag{3.7}$$

From (3.5), we deduce that

$$z\left(\mathcal{J}_{s,\ b}^{\lambda,\ \mu} F\right)'(z) + \nu \mathcal{J}_{s,\ b}^{\lambda,\ \mu} F(z) = (\nu+1)\mathcal{J}_{s,\ b}^{\lambda,\ \mu} f(z). \tag{3.8}$$

Combining (3.1), (3.7) and (3.8), we easily get

$$G(z) + \frac{1}{\nu+1} zG'(z) = \frac{\mathcal{J}_{s,\ b}^{\lambda,\ \mu} f(z)}{z} \prec \phi(z) \quad (z \in \mathbb{U}). \tag{3.9}$$

Thus, by Lemma 2.1 and (3.9), we conclude that the assertion (3.6) of Theorem 3.4 holds. □

Theorem 3.5. *Let $f \in \mathcal{F}_{s,\ b}^{\lambda,\ \mu}(\alpha;\phi)$ and $g \in \mathcal{A}$ with $\Re\left(\frac{g(z)}{z}\right) > \frac{1}{2}$. Then*

$$(f * g)(z) \in \mathcal{F}_{s,\ b}^{\lambda,\ \mu}(\alpha;\phi).$$

Proof. Let $f \in \mathcal{F}_{s,\ b}^{\lambda,\ \mu}(\eta;\phi)$ and $g \in \mathcal{A}$ with $\Re\left(\frac{g(z)}{z}\right) > \frac{1}{2}$. Suppose also that

$$H(z) := (1-\alpha)\frac{\mathcal{J}_{s,\ b}^{\lambda,\ \mu} f(z)}{z} + \alpha \frac{\mathcal{J}_{s,\ b}^{\lambda,\ \mu+1} f(z)}{z} \prec \phi(z) \quad (z \in \mathbb{U}). \tag{3.10}$$

It follows from (3.10) that

$$(1-\alpha)\frac{\mathcal{J}_{s,\ b}^{\lambda,\ \mu}(f*g)(z)}{z} + \alpha \frac{\mathcal{J}_{s,\ b}^{\lambda,\ \mu+1}(f*g)(z)}{z} = H(z) * \frac{g(z)}{z} \quad (z \in \mathbb{U}). \tag{3.11}$$

Since the function ϕ is convex and univalent in $\mathbb{U}$, by virtue of (3.10), (3.11) and Lemma 2.2, we conclude that

$$(1-\alpha)\frac{\mathcal{J}_{s,\ b}^{\lambda,\ \mu}(f*g)(z)}{z} + \alpha \frac{\mathcal{J}_{s,\ b}^{\lambda,\ \mu+1}(f*g)(z)}{z} \prec \phi(z) \quad (z \in \mathbb{U}), \tag{3.12}$$

which implies that the assertion of Theorem 3.5 holds. □

Theorem 3.6. *Let q_1 be univalent in $\mathbb{U}$ and $\Re(\alpha) > 0$. Suppose also that q_1 satisfies*

$$\Re\left(1 + \frac{zq_1''(z)}{q_1'(z)}\right) > \max\left\{0, -\Re\left(\frac{\mu}{\alpha}\right)\right\}. \tag{3.13}$$

If $f \in \mathcal{A}$ satisfies the subordination

$$(1-\alpha)\frac{\mathcal{J}_{s,\ b}^{\lambda,\ \mu} f(z)}{z} + \alpha\frac{\mathcal{J}_{s,\ b}^{\lambda,\ \mu+1} f(z)}{z} \prec q_1(z) + \frac{\alpha}{\mu} z q_1'(z), \tag{3.14}$$

then

$$\frac{\mathcal{J}_{s,\ b}^{\lambda,\ \mu} f(z)}{z} \prec q_1'(z),$$

and q_1 is the best dominant.

Proof. Let the function h be defined by (3.2). We know that (3.3) holds. Combining (3.3) and (3.14), we find that

$$h(z) + \frac{\alpha}{\mu} z h'(z) \prec q_1(z) + \frac{\alpha}{\mu} z q_1'(z). \tag{3.15}$$

By Lemma 2.3 and (3.15), we readily get the assertion of Theorem 3.6. □

If f is subordinate to $\mathcal{F}$, then $\mathcal{F}$ is superordinate to f. We now derive the following superordination result for the class $\mathcal{F}_{s,\ b}^{\lambda,\ \mu}(\alpha; \phi)$.

Theorem 3.7. *Let q_2 be convex univalent in $\mathbb{U}$, $\alpha \in \mathbb{C}$ with $\Re(\alpha) > 0$. Also let*

$$\frac{\mathcal{J}_{s,\ b}^{\lambda,\ \mu} f(z)}{z} \in \mathcal{H}[q_2(0), 1] \cap Q$$

and

$$(1-\alpha)\frac{\mathcal{J}_{s,\ b}^{\lambda,\ \mu} f(z)}{z} + \alpha\frac{\mathcal{J}_{s,\ b}^{\lambda,\ \mu+1} f(z)}{z}$$

be univalent in $\mathbb{U}$. If

$$q_2(z) + \frac{\alpha}{\mu} z q_2'(z) \prec (1-\alpha)\frac{\mathcal{J}_{s,\ b}^{\lambda,\ \mu} f(z)}{z} + \alpha\frac{\mathcal{J}_{s,\ b}^{\lambda,\ \mu+1} f(z)}{z},$$

then

$$q_2(z) \prec \frac{\mathcal{J}_{s,\ b}^{\lambda,\ \mu} f(z)}{z},$$

and q_2 is the best subdominant.

Proof. Let the function h be defined by (3.2). Then

$$q_2(z) + \frac{\alpha}{\mu} z q_2'(z) \prec (1-\alpha)\frac{\mathcal{J}_{s,\ b}^{\lambda,\ \mu} f(z)}{z} + \alpha\frac{\mathcal{J}_{s,\ b}^{\lambda,\ \mu+1} f(z)}{z} = h(z) + \frac{\alpha}{\mu} z h'(z).$$

An application of Lemma 2.4 yields the desired assertion of Theorem 3.7. □

Combining the above results of subordination and superordination, we easily get the following "sandwich-type result".

Theorem 3.8. *Let q_3 be convex univalent and q_4 be univalent in $\mathbb{U}$, $\alpha \in \mathbb{C}$ with $\Re(\alpha) > 0$. Suppose also that q_4 satisfies*

$$\Re\left(1 + \frac{z q_4''(z)}{q_4'(z)}\right) > \max\left\{0, -\Re\left(\frac{\mu}{\alpha}\right)\right\}.$$

If

$$0 \neq \frac{\mathcal{J}_{s,\ b}^{\lambda,\ \mu} f(z)}{z} \in \mathcal{H}[q_3(0), 1] \cap Q,$$

and

$$(1-\alpha)\frac{\mathcal{J}_{s,\ b}^{\lambda,\ \mu} f(z)}{z} + \alpha \frac{\mathcal{J}_{s,\ b}^{\lambda,\ \mu+1} f(z)}{z}$$

is univalent in $\mathbb{U}$, *also*

$$q_3(z) + \frac{\alpha}{\mu} z q_3'(z) \prec (1-\alpha)\frac{\mathcal{J}_{s,\ b}^{\lambda,\ \mu} f(z)}{z} + \alpha \frac{\mathcal{J}_{s,\ b}^{\lambda,\ \mu+1} f(z)}{z} \prec q_4(z) + \frac{\alpha}{\mu} z q_4'(z),$$

then

$$q_3(z) \prec \frac{\mathcal{J}_{s,\ b}^{\lambda,\ \mu} f(z)}{z} \prec q_4(z),$$

and q_3 *and* q_4 *are, respectively, the best subordinant and the best dominant.*

4. Properties of the function classes $\mathcal{G}_{s,\ b}^{\lambda,\ \mu}(\alpha;\phi)$ and $\mathcal{H}_{s,\ b}^{\lambda,\ \mu}(\alpha;\phi)$

By means of (1.6) and (1.7), and by similarly applying the methods used in the proofs of Theorems 3.1–3.8, respectively, we easily get the following properties for the function classes $\mathcal{G}_{s,\ b}^{\lambda,\ \mu}(\alpha;\phi)$ and $\mathcal{H}_{s,\ b}^{\lambda,\ \mu}(\alpha;\phi)$. Here we choose to omit the details involved.

Corollary 4.1. *Let* $f \in \mathcal{G}_{s,\ b}^{\lambda,\ \mu}(\alpha;\phi)$ *with* $\Re(\alpha) > 0$. *Then*

$$\frac{\mathcal{J}_{s,\ b}^{\lambda+1,\ \mu} f(z)}{z} \prec \frac{\lambda+1}{\alpha} z^{-\frac{\lambda+1}{\alpha}} \int_0^z t^{\frac{\lambda+1}{\alpha}-1}\phi(t)dt \prec \phi(z) \quad (z \in \mathbb{U}).$$

Corollary 4.2. *Let* $\alpha_2 > \alpha_1 \geqq 0$. *Then* $\mathcal{G}_{s,\ b}^{\lambda,\ \mu}(\alpha_2;\phi) \subset \mathcal{G}_{s,\ b}^{\lambda,\ \mu}(\alpha_1;\phi)$.

Corollary 4.3. *Let* $f \in \mathcal{G}_{s,\ b}^{\lambda,\ \mu}(\alpha;\phi)$. *If the integral operator* F *is defined by* (3.5), *then*

$$\frac{\mathcal{J}_{s,\ b}^{\lambda+1,\ \mu} F(z)}{z} \prec \phi(z) \quad (z \in \mathbb{U}).$$

Corollary 4.4. *Let* $f \in \mathcal{G}_{s,\ b}^{\lambda,\ \mu}(\alpha;\phi)$ *and* $g \in \mathcal{A}$ *with* $\Re\left(\frac{g(z)}{z}\right) > \frac{1}{2}$. *Then*

$$(f * g)(z) \in \mathcal{G}_{s,\ b}^{\lambda,\ \mu}(\alpha;\phi).$$

Corollary 4.5. *Let* q_5 *be univalent in* $\mathbb{U}$ *and* $\Re(\alpha) > 0$. *Suppose also that* q_5 *satisfies*

$$\Re\left(1 + \frac{z q_5''(z)}{q_5'(z)}\right) > \max\left\{0, -\Re\left(\frac{\lambda+1}{\alpha}\right)\right\}.$$

If $f \in \mathcal{A}$ *satisfies the subordination*

$$(1-\alpha)\frac{\mathcal{J}_{s,\ b}^{\lambda+1,\ \mu} f(z)}{z} + \alpha \frac{\mathcal{J}_{s,\ b}^{\lambda,\ \mu} f(z)}{z} \prec q_5(z) + \frac{\alpha}{\lambda+1} z q_5'(z),$$

then

$$\frac{\mathcal{J}_{s,\ b}^{\lambda+1,\ \mu} f(z)}{z} \prec q_5'(z),$$

and q_5 *is the best dominant.*

Corollary 4.6. *Let* q_6 *be convex univalent in* $\mathbb{U}$, $\alpha \in \mathbb{C}$ *with* $\Re(\alpha) > 0$. *Also let*

$$\frac{\mathcal{J}_{s,\ b}^{\lambda+1,\ \mu} f(z)}{z} \in \mathcal{H}[q_6(0), 1] \cap Q$$

and

$$(1-\alpha)\frac{\mathcal{J}_{s,\ b}^{\lambda+1,\ \mu} f(z)}{z} + \alpha\frac{\mathcal{J}_{s,\ b}^{\lambda,\ \mu} f(z)}{z}$$

be univalent in $\mathbb{U}$. *If*

$$q_6(z) + \frac{\alpha}{\lambda+1} z q_6'(z) \prec (1-\alpha)\frac{\mathcal{J}_{s,\ b}^{\lambda+1,\ \mu} f(z)}{z} + \alpha\frac{\mathcal{J}_{s,\ b}^{\lambda,\ \mu} f(z)}{z},$$

then

$$q_6(z) \prec \frac{\mathcal{J}_{s,\ b}^{\lambda+1,\ \mu} f(z)}{z},$$

and q_6 *is the best subdominant.*

Corollary 4.7. *Let* q_7 *be convex univalent and* q_8 *be univalent in* $\mathbb{U}$, $\alpha \in \mathbb{C}$ *with* $\Re(\alpha) > 0$. *Suppose also that* q_8 *satisfies*

$$\Re\left(1 + \frac{z q_8''(z)}{q_8'(z)}\right) > \max\left\{0, -\Re\left(\frac{\lambda+1}{\alpha}\right)\right\}.$$

If

$$0 \neq \frac{\mathcal{J}_{s,\ b}^{\lambda+1,\ \mu} f(z)}{z} \in \mathcal{H}[q_7(0), 1] \cap Q,$$

and

$$(1-\alpha)\frac{\mathcal{J}_{s,\ b}^{\lambda+1,\ \mu} f(z)}{z} + \alpha\frac{\mathcal{J}_{s,\ b}^{\lambda,\ \mu} f(z)}{z}$$

is univalent in $\mathbb{U}$, *also*

$$q_7(z) + \frac{\alpha}{\lambda+1} z q_7'(z) \prec (1-\alpha)\frac{\mathcal{J}_{s,\ b}^{\lambda+1,\ \mu} f(z)}{z} + \alpha\frac{\mathcal{J}_{s,\ b}^{\lambda,\ \mu} f(z)}{z} \prec q_8(z) + \frac{\alpha}{\lambda+1} z q_8'(z),$$

then

$$q_7(z) \prec \frac{\mathcal{J}_{s,\ b}^{\lambda+1,\ \mu} f(z)}{z} \prec q_8(z),$$

and q_7 *and* q_8 *are, respectively, the best subordinant and the best dominant.*

Corollary 4.8. *Let* $f \in \mathcal{H}_{s,\ b}^{\lambda,\ \mu}(\alpha;\phi)$ *with* $\Re(\alpha) > 0$. *Then*

$$\frac{\mathcal{J}_{s+1,\ b}^{\lambda,\ \mu} f(z)}{z} \prec \frac{b+1}{\alpha} z^{-\frac{b+1}{\alpha}} \int_0^z t^{\frac{b+1}{\alpha}-1}\phi(t)dt \prec \phi(z) \quad (z \in \mathbb{U}).$$

Corollary 4.9. *Let* $\alpha_2 > \alpha_1 \geqq 0$. *Then* $\mathcal{H}_{s,\ b}^{\lambda,\ \mu}(\alpha_2;\phi) \subset \mathcal{H}_{s,\ b}^{\lambda,\ \mu}(\alpha_1;\phi)$.

Corollary 4.10. *Let $f \in \mathcal{H}_{s,\ b}^{\lambda,\ \mu}(\alpha;\phi)$. If the integral operator F is defined by* (3.5)*, then*

$$\frac{\mathcal{J}_{s+1,\ b}^{\lambda,\ \mu}F(z)}{z} \prec \phi(z) \quad (z \in \mathbb{U}).$$

Corollary 4.11. *Let $f \in \mathcal{H}_{s,\ b}^{\lambda,\ \mu}(\alpha;\phi)$ and $g \in \mathcal{A}$ with $\Re\left(\frac{g(z)}{z}\right) > \frac{1}{2}$. Then*

$$(f * g)(z) \in \mathcal{H}_{s,\ b}^{\lambda,\ \mu}(\alpha;\phi).$$

Corollary 4.12. *Let q_9 be univalent in $\mathbb{U}$ and $\Re(\alpha) > 0$. Suppose also that q_9 satisfies*

$$\Re\left(1 + \frac{zq_9''(z)}{q_9'(z)}\right) > \max\left\{0, -\Re\left(\frac{b+1}{\alpha}\right)\right\}.$$

If $f \in \mathcal{A}$ satisfies the subordination

$$(1-\alpha)\frac{\mathcal{J}_{s+1,\ b}^{\lambda,\ \mu}f(z)}{z} + \alpha\frac{\mathcal{J}_{s,\ b}^{\lambda,\ \mu}f(z)}{z} \prec q_9(z) + \frac{\alpha}{b+1}zq_9'(z),$$

then

$$\frac{\mathcal{J}_{s+1,\ b}^{\lambda,\ \mu}f(z)}{z} \prec q_9'(z),$$

and q_9 is the best dominant.

Corollary 4.13. *Let q_{10} be convex univalent in $\mathbb{U}$, $\alpha \in \mathbb{C}$ with $\Re(\alpha) > 0$. Also let*

$$\frac{\mathcal{J}_{s+1,\ b}^{\lambda,\ \mu}f(z)}{z} \in \mathcal{H}[q_{10}(0), 1] \cap Q$$

and

$$(1-\alpha)\frac{\mathcal{J}_{s+1,\ b}^{\lambda,\ \mu}f(z)}{z} + \alpha\frac{\mathcal{J}_{s,\ b}^{\lambda,\ \mu}f(z)}{z}$$

be univalent in $\mathbb{U}$. If

$$q_{10}(z) + \frac{\alpha}{b+1}zq_{10}'(z) \prec (1-\alpha)\frac{\mathcal{J}_{s+1,\ b}^{\lambda,\ \mu}f(z)}{z} + \alpha\frac{\mathcal{J}_{s,\ b}^{\lambda,\ \mu}f(z)}{z},$$

then

$$q_{10}(z) \prec \frac{\mathcal{J}_{s+1,\ b}^{\lambda,\ \mu}f(z)}{z},$$

and q_{10} is the best subdominant.

Corollary 4.14. *Let q_{11} be convex univalent and q_{12} be univalent in $\mathbb{U}$, $\alpha \in \mathbb{C}$ with $\Re(\alpha) > 0$. Suppose also that q_{12} satisfies*

$$\Re\left(1 + \frac{zq_{12}''(z)}{q_{12}'(z)}\right) > \max\left\{0, -\Re\left(\frac{b+1}{\alpha}\right)\right\}.$$

If

$$0 \neq \frac{\mathcal{J}_{s+1,\ b}^{\lambda,\ \mu}f(z)}{z} \in \mathcal{H}[q_{11}(0), 1] \cap Q,$$

and

$$(1-\alpha)\frac{\mathcal{J}_{s+1,\ b}^{\lambda,\ \mu}f(z)}{z}+\alpha\frac{\mathcal{J}_{s,\ b}^{\lambda,\ \mu}f(z)}{z}$$

is univalent in $\mathbb{U}$*, also*

$$q_{11}(z)+\frac{\alpha}{b+1}zq_{11}'(z)\prec(1-\alpha)\frac{\mathcal{J}_{s+1,\ b}^{\lambda,\ \mu}f(z)}{z}+\alpha\frac{\mathcal{J}_{s,\ b}^{\lambda,\ \mu}f(z)}{z}$$
$$\prec q_{12}(z)+\frac{\alpha}{b+1}zq_{12}'(z),$$

then

$$q_{11}(z)\prec\frac{\mathcal{J}_{s+1,\ b}^{\lambda,\ \mu}f(z)}{z}\prec q_{12}(z),$$

and q_{11} *and* q_{12} *are, respectively, the best subordinant and the best dominant.*

Acknowledgements. The present investigation was supported by the *Natural Science Foundation of China* under Grant 70971013, the *Natural Science Foundation of Hunan Province* under Grant 09JJ1010, the *Key Project of Chinese Ministry of Education* under Grant 211118, the *Excellent Youth Foundation of Educational Committee of Hunan Province* under Grant 10B002, and the *Open Fund Project of Key Research Institute of Philosophies and Social Sciences in Hunan Universities* under Grant 11FEFM02 of the People's Republic of China.

References

[1] Al-Shaqsi, K., Darus, M., *On certain subclasses of analytic functions defined by a multiplier transformation with two parameters*, Appl. Math. Sci., **3**(2009), 1799-1810.

[2] Choi, J.H., Saigo, M., Srivastava, H.M., *Some inclusion properties of a certain family of integral operators*, J. Math. Anal. Appl., **276**(2002), 432-445.

[3] Choi, J., Srivastava, H.M., *Certain families of series associated with the Hurwitz-Lerch Zeta function*, Appl. Math. Comput., **170**(2005), 399-409.

[4] Darus, M., Al-Shaqsi, K., *On subordinations for certain analytic functions associated with generalized integral operator,* Lobachevskii J. Math., **29**(2008), 90-97.

[5] Ferreira, C., López, J.L., *Asymptotic expansions of the Hurwitz-Lerch Zeta function*, J. Math. Anal. Appl., **298**(2004), 210-224.

[6] Garg, M., Jain, K., Srivastava, H.M., *Some relationships between the generalized Apostol-Bernoulli polynomials and Hurwitz-Lerch Zeta functions*, Integral Transforms Spec. Funct., **17**(2006), 803-815.

[7] Lin, S.D., Srivastava, H.M., *Some families of the Hurwitz-Lerch Zeta functions and associated fractional derivative and other integral representations*, Appl. Math. Comput., **154**(2004), 725-733.

[8] Lin, S.D., Srivastava, H.M., Wang, P.Y., *Some expansion formulas for a class of generalized Hurwitz-Lerch Zeta functions,* Integral Transforms Spec. Funct., **17**(2006), 817-827.

[9] Liu, J.L., *Subordinations for certain multivalent analytic functions associated with the generalized Srivastava-Attiya operator,* Integral Transforms Spec. Funct., **19**(2008), 893-901.

[10] Luo, Q.M., Srivastava, H.M., *Some generalizations of the Apostol-Bernoulli and Apostol-Euler polynomials*, J. Math. Anal. Appl., **308**(2005), 290-302.

[11] Miller, S.S., Mocanu, P.T., *Differential Subordination: Theory and Applications*, in: Series in Pure and Applied Mathematics, Vol. 225, Marcel Dekker, New York, 2000.

[12] Miller, S.S., Mocanu, P.T., *Subordinats of differential superordinations*, Complex Var. Theory Appl., **48**(2003), 815-826.

[13] Prajapat, J.K., Goyal, S.P., *Applications of Srivastava-Attiya operator to the classes of strongly starlike and strongly convex functions*, J. Math. Inequal., **3**(2009), 129-137.

[14] Răducanu, D., Srivastava, H.M., *A new class of analytic functions defined by means of a convolution operator involving the Hurwitz-Lerch Zeta function*, Integral Transforms Spec. Funct., **18**(2007), 933-943.

[15] Shanmugam, T.N., Ravichandran, V., Sivasubramanian, S., *Differential sandwich theorems for subclasses of analytic functions,* Aust. J. Math. Anal. Appl., **3**(2006), Article 8, pp. 1-11, (electronic).

[16] Singh, R., Singh, S., *Convolution properties of a class of starlike functions*, Proc. Amer. Math. Soc., **106**(1989), 145-152.

[17] Srivastava, H.M., Attiya, A.A., *An integral operator associated with the Hurwitz-Lerch Zeta function and differential subordination*, Integral Transforms Spec. Funct., **18**(2007), 207-216.

[18] Srivastava, H.M., Choi, J., *Series associated with the Zeta and related functions*, Dordrecht, Boston, London: Kluwer Academic Publishers, 2001.

[19] Wen, F., Liu, Z., *A Copula-based correlation measure and its application in Chinese stock market*, Int. J. Inform. Tech. Decision Making, **8**(2009), 1-15.

[20] Wen, F., Yang, X., *Skewness of return distribution and coeffcient of risk premium*, J. Systems Sci. Complexity, **22**(2009), 360-371.

Zhi-Gang Wang
School of Mathematics and Statistics
Anyang Normal University
Anyang 455002, Henan, People's Republic of China
e-mail: zhigwang@163.com

Feng-Hua Wen
School of Econometrics and Management
Changsha University of Science and Technology
Changsha 410114, Hunan, People's Republic of China

Qing-Guo Li
School of Mathematics and Econometrics
Hunan University
Changsha 410082, Hunan, People's Republic of China

Stud. Univ. Babeş-Bolyai Math. 56(2011), No. 3, 85–93

On a class of pseudo-parallel submanifolds in Kenmotsu space forms

Maria Cîrnu

Abstract. In this article we prove that pseudo-parallel normal anti-invariant submanifolds in Kenmotsu space forms are always semi-parallel.

Mathematics Subject Classification (2010): 53C25, 53C42.

Keywords: Kenmotsu and Sasaki space form, pseudo-parallel and semi-parallel submanifold, normal anti-invariant submanifold, Legendre submanifold.

1. Introduction

In 2008, [2], F. Dillen, J. Van der Veken and L. Vrancken proved that Lagrange pseudo-parallel submanifolds of complex space forms are always semi-parallel.

In this paper we prove that a n-dimensional pseudo-parallel and normal anti-invariant submanifold M in a $(2n+1)$-dimensional Kenmotsu space form $\widetilde{M}(c)$ is always semi-parallel. We also prove that this is not generally true for pseudo-parallel Legendre submanifolds in Sasaki space forms.

Now, we remember some necessary useful notions and results for our next considerations.
Let $\widetilde{M}$ be a C^{∞}–differentiable, $(2n+1)$–dimensional almost contact manifold with the almost contact metric structure (F, ξ, η, g), where F is a $(1,1)$ tensor field, η is a 1-form, g is a Riemannian metric on $\widetilde{M}$, ξ is the Reeb vector field, all these tensors satisfying the following conditions :

$$F^2 = -I + \eta \otimes \xi; \quad \eta(\xi) = 1; \quad g(FX, FY) = g(X, Y) - \eta(X)\eta(Y) \quad (1.1)$$

for all X, Y in $\chi(\widetilde{M})$.

Let M be a submanifold of $\widetilde{M}$. We consider ∇ the Levi-Civita connection induced by $\widetilde{\nabla}$ on M, $\nabla^{\perp}$ the connection in the normal bundle $T^{\perp}(M)$, h the second fundamental form on M and $A_{\vec{n}}$ the Weingarten operator. The well-known Gauss–Weingarten formulas on M are:

$$\widetilde{\nabla}_X Y = \nabla_X Y + h(X,Y); \qquad \widetilde{\nabla}_X \vec{n} = -A_{\vec{n}} X + \nabla^{\perp}_X \vec{n} \tag{1.2}$$

for X, Y in $\chi(M)$ and $\vec{n}$ in $\chi^{\perp}(M)$.

We consider the Sasaki form Ω on $\widetilde{M}$, given by $\Omega(X,Y) = g(X, FY)$. Also, denote by N_F the Nijenhius tensor of F. It is known that $\widetilde{M}$ is a Sasaki manifold if and only if

$$d\eta = \Omega; \qquad N^{(1)} = N_F + 2d\eta \otimes \xi = 0$$

or equivalently

$$(\widetilde{\nabla}_X F)Y = g(X,Y)\xi - \eta(Y)X. \tag{1.3}$$

An almost normal contact manifold $\widetilde{M}$ is a Kenmotsu manifold if and only if

$$d\eta = 0; \quad d\Omega = 2\eta \wedge \Omega.$$

It is also known that, similar to the characterization (1.3) of Sasaki manifolds, $\widetilde{M}$ is a Kenmotsu manifold if and only if

$$(\widetilde{\nabla}_X F)Y = -\eta(Y)FX - g(X, FY)\xi \tag{1.4}$$

for all X, Y in $\chi(\widetilde{M})$.

From [3] and [5], we have the following expressions of the curvature tensor in Sasaki and Kenomotsu space forms :

$$\begin{aligned}\widetilde{R}(X,Y)Z = {} & \frac{c+3(-1)^{i+1}}{4}\left[g(Y,Z)X - g(X,Z)Y\right] + \frac{c-(-1)^{i+1}}{4}[\eta(X)\eta(Z)Y \\ & - \eta(Y)\eta(Z)X + g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi + \Omega(X,Z)FY \\ & - \Omega(Y,Z)FX + 2\Omega(X,Y)FZ],\end{aligned} \tag{1.5}$$

where $i = -1$ for Sasakian case and $i = 1$ for Kenmotsu case.

In the case of a $(2n+1)$-dimensional contact manifold $\widetilde{M}$, the contact distribution $\mathcal{D} = \ker \eta$ is totally non integrabile and the maximal dimension of its integral submanifolds M (called *the integral submanifolds of the contact manifold* $\widetilde{M}$) is n. A maximal integral submanifold M of a contact manifold $\widetilde{M}$ is a *Legendre* submanifold. Moreover, it is well known that an integral submanifold M of a contact manifold $\widetilde{M}$ is characterized by any of

(i) $\eta = 0, \quad d\eta = 0$;
(ii) $FX \in \chi^{\perp}(M)$ for all X in $\chi(M)$.

Another properties valid on these submanifolds in the case of Sasaki manifolds and useful for our considerations are given in [7] by

Proposition 1.1. *Let M be an integral submanifold of a $(2n+1)$-dimensional Sasaki manifold $\widetilde{M}$, $n \geq 1$. Then:*

(i) $A_\xi = 0$;
(ii) $A_{FX}Y = A_{FY}X$;
(iii) $A_{FY}X = -[Fh(X,Y)]^T$;
(iv) $\nabla^{\perp}_X(FY) = g(X,Y)\xi + F\nabla_X Y + [Fh(X,Y)]^{\perp}$;

(v) $\nabla^{\perp}_X \xi = -FX$ *for all* X, Y *in* $\chi(M)$.

In the case of Kenmotsu manifolds, N. Papaghiuc, [6], introduced the following

Definition 1.2. *A submanifold* M *of a Kenmotsu manifold* $\widetilde{M}$ *is a normal semi-invariant submanifold if* ξ *is normal to* M *and* M *has two distributions* D *and* $D^{\perp}$*, called the invariant, respectively, the anti-invariant distribution of* M *so that*

(i) $T_xM = D_x \oplus D_x^{\perp} \oplus < \xi_x >$;
(ii) D_x, $D_x^{\perp}$, $< \xi_x >$ *are othogonal;*
(iii) $FD_x \subseteq D_x$; $FD_x^{\perp} \subseteq T_x^{\perp}$,

for all $x \in M$.
If $D = 0$ *then* M *is a normal anti-invariant submanifold of* $\widetilde{M}$ *and if* $D^{\perp} = 0$ *then* M *is a invariant submanifold of* $\widetilde{M}$.

Also, from [6], we have the following result

Proposition 1.3. *If* M *is a normal anti-invariant submanifold of a Kenmotsu manifold* $\widetilde{M}$*, then*

(i) $A_{FX}Y = A_{FY}X$*, for all* $X, Y \in D^{\perp}$;
(ii) $A_{\xi}Z = -Z$ *and* $\nabla^{\perp}_Z \xi = 0$*, for all* $Z \in \chi(M)$.

2. Pseudo-parallel submanifolds in Kenmotsu and Sasaki space forms

Proposition 2.1. *If* M *is a* m*-dimensional, normal anti-invariant submanifold of a* $(2n+1)$*-dimensional Kenmotsu manifold* $\widetilde{M}(c)$*, then* $m \le n$.

Proof. For $x \in M$ we have $T_x\widetilde{M} = T_xM \oplus T_x^{\perp}M$ and $\dim FT_xM = \dim T_xM = m$. Moreover, because M is normal anti-invariant we have $FT_xM \subseteq T_x^{\perp}M$; $FT_xM \perp < \xi_x >$ and then

$$\dim T_x^{\perp}M \ge \dim FT_xM + \dim < \xi_x >= m + 1.$$

Now,

$$2m \le m + \dim T_x^{\perp}M - 1 = \dim T_xM + \dim T_x^{\perp}M - 1 = \dim T_x\widetilde{M} - 1 = 2n$$

and then $m \le n$. □

Recall that a submanifold M of the Riemannian manifold $\widetilde{M}$ is *semi-parallel* if

$$(\widetilde{R} \cdot h)(X, Y, V, W) = 0 \tag{2.1}$$

where

$$\begin{aligned}(\widetilde{R} \cdot h)(X, Y, V, W) &= R^{\perp}(X, Y)h(V, W) - h(R(X, Y)V, W) \\ &- h(V, R(X, Y)W)\end{aligned}$$

for all X, Y, Z, W in $\chi(M)$. Here R is the curvature tensor of M and $R^{\perp}$ is the normal component of the curvature tensor $\widetilde{R}$ of $\widetilde{M}$ on M.
M is *pseudo-parallel* if

$$(\widetilde{R} \cdot h)(X, Y, V, W) + \Phi \cdot Q(g, h)(X, Y, V, W) = 0, \tag{2.2}$$

where Φ is a differential function on $\widetilde{M}$ and

$$Q(g, h)(X, Y, V, W) = h((X \wedge Y)V, W) + h(V, (X \wedge Y)W),$$
$$(X \wedge Y)V = g(Y, V)X - g(X, V)Y$$

for all X, Y, V, W in $\chi(M)$.

Let $\widetilde{M}(c)$ be a Kenmotsu space form with $\dim \widetilde{M}(c) = 2n+1$ and M be a n-dimensional normal anti-invariant submanifold. We consider $\{X_1, ..., X_n\}$ a local orthonormal basis in $\chi(M)$ and $\{\xi, FX_1, ..., FX_n\}$ a local orthonormal basis in $\chi^{\perp}(M)$.
Because M is normal anti-invariant manifold and taking into account (1.1) and (1.4) we have:

$$g(FX, FY) = g(X, Y);\ \widetilde{\nabla}_X(FY) = F\widetilde{\nabla}_X Y;\ F\widetilde{R}(X, Y)Z = \widetilde{R}(X, Y)FZ \tag{2.3}$$

for all X, Y, Z in $\chi(M)$. Because $Fh(X, Y)$ belongs to $\chi(M)$ and taking into account (1.2) and (2.3), we obtain

$$\nabla^{\perp}_X(FY) = F\nabla_X Y; \qquad -A_{FY}X = Fh(X, Y). \tag{2.4}$$

From (1.1) and Proposition 1.3 we have

$$h(X, Y) = FA_{FY}X - g(X, Y)\xi = FA_{FX}Y - g(X, Y)\xi. \tag{2.5}$$

We define the 3-form $C(X, Y, Z) = g(h(X, Y), FZ)$ for all X, Y, Z in $\chi(M)$. From the symmetry of h and taking into account Proposition 1.3 and (2.5), it follows that C is a totally symmetric 3-form.

From (1.5), the Codazzi equation and the fact that M is normal and anti-invariant, we have

$$\widetilde{R}(X, Y)Z = \frac{c-3}{4}[g(Y, Z)X - g(X, Z)Y] \tag{2.6}$$

and

$$R(X, Y)Z = \frac{c-3}{4}[g(Y, Z)X - g(X, Z)Y] + A_{h(Y,Z)}X - A_{h(X,Z)}Y.$$

But from (2.5) and Proposition 1.3, we obtain

$$A_{h(X,Z)}Y = A_{FY}A_{FX}Z + g(X, Z)Y$$

and then

$$R(X, Y)Z = \frac{c+1}{4}[g(Y, Z)X - g(X, Z)Y] + [A_{FX}, A_{FY}]Z. \tag{2.7}$$

Moreover, from (2.4) we have:

$$R^{\perp}(X, Y)FZ = FR(X, Y)Z \tag{2.8}$$

for all X, Y, Z in $\chi(M)$.

Now, we give the main result of this article.

Theorem 2.2. *Any n-dimensional pseudo-parallel normal anti-invariant submanifold M of a $(2n+1)$-dimensional Kenmotsu space form $\widetilde{M}(c)$, with $n \geq 1$, is semi-parallel.*

Proof. We have

$$\begin{aligned} g((\widetilde{R}\cdot h)(X,Y,V,W),FZ) &= g(R^{\perp}(X,Y)h(V,W),FZ) \\ &- g(h(R(X,Y)V,W),FZ) \\ &- g(h(V,R(X,Y)W),FZ) \end{aligned}$$

for X, Y, V, W in $\chi(M)$. Denote by

$$T_1 = g(R^{\perp}(X,Y)h(V,W),FZ) \quad T_2 = g(h(R(X,Y)V,W),FZ)$$

$$T_3 = g(h(V,R(X,Y)W),FZ).$$

Because the 3-form C is totally symmetric, it follows that T_2 is symmetric in Z and W. From (2.5), Proposition 1.3, (2.7), (1.1) and (2.8), we obtain:

$$\begin{aligned} T_1 &= g(R^{\perp}(X,Y)h(V,W),FZ) = g(R^{\perp}(X,Y)FA_{FV}W,FZ) \\ &= \frac{c+1}{4}[g(X,Y)g(h(Y,W),FV) - g(Y,Z)g(h(X,W),FV)] \\ &+ g([A_{FX},A_{FY}]A_{FV}W,Z), \end{aligned}$$

$$\begin{aligned} T_3 &= g(h(V,R(X,Y)W),FZ) = g(h(V,Z),FR(X,Y)W) \\ &= \frac{c+1}{4}[g(Y,W)g(h(X,Z),FV) - g(h(Y,Z),FV)g(X,W)] \\ &+ g(A_{FV}Z,[A_{FX},A_{FY}]W). \end{aligned}$$

Also,

$$T_1 - T_3 = T_4 + T_5$$

where

$$\begin{aligned} T_4 &= \frac{c+1}{4}[g(h(Y,W),FV)g(X,Y) - g(h(X,W),FV)g(Y,Z) \\ &- g(h(X,Z),FV)g(Y,W) + g(h(Y,Z),FV)g(X,W)] \end{aligned}$$

is symmetric in W and Z and

$$T_5 = g([A_{FX},A_{FY}]A_{FV}W,Z) - g(A_{FV}Z,[A_{FX},A_{FY}]W).$$

On the other hand, from the symmetry of h we have

$$g(A_{FV}Z,[A_{FX},A_{FY}]W) = -g([A_{FX},A_{FY}]A_{FV}Z,W).$$

From this we deduce that

$$T_5 = g([A_{FX},A_{FY}]A_{FV}Z,W)) + g([A_{FX},A_{FY}]A_{FV}W,Z)$$

is symmetric in W and Z and $g((\widetilde{R} \cdot h)(X,Y,V,W), FZ)$ is symmetric in W and Z. Because M is pseudo-parallel it follows that $g(Q(g,h)(X,Y,V,W), FZ)$ is symmetric in W and Z or equivalently

$$g(Y,W)g(h(V,X),FZ) - g(X,W)g(h(V,Y),FZ)$$
$$= g(Y,Z)g(h(V,X),FW) - g(X,Z)g(h(V,Y),FW).$$

Taking $X = W = V$, $Z, Y \perp X$ in this relation, we obtain

$$-g(X,X)g(h(Y,Z),FX) = g(Y,Z)g(h(X,X),FX). \qquad (2.9)$$

Let x be in M and $S = \{V \in T_pM | g(V,V) = 1\}$ – the unit sphere and $f : S \to \mathcal{F}(M)$, where $f(V) = g(h(V,V),FV)$ for all V in S. Because f is a continue function on S, it results that f attains its maximum in a vector field X_0, tangent to the submanifold in x.

Let $\{e_1, ..., e_{n-1}, X_0\}$ be a local orthonormal basis in $\chi(M)$. Taking $Y = Z = X_0$ and $X = e_i$ in (2.9), we have:

$$g(h(X_0,X_0),Fe_i) = -f(e_i), \quad i = 1, ..., n-1.$$

and for $Y = Z = e_i$ and $X = X_0$

$$g(h(e_i,e_i),FX_0) = -f(X_0), \quad i = 1, ..., n-1.$$

$\{\xi, Fe_1, ..., Fe_{n-1}, FX_0\}$ is a local orthonormal basis in $\chi^{\perp}(M)$ and

$$h(e_i,e_i) = -f(X_0)FX_0 - \xi - \sum_{j=1}^{n-1} f(e_j)Fe_j$$

$$h(X_0,X_0) = f(X_0)FX_0 - \xi - \sum_{j=1}^{n-1} f(e_j)Fe_j.$$

From these last two equalities we obtain

$$h(e_i,e_i) = h(X_0,X_0) - 2f(X_0)FX_0, \quad h(X_0,X_0) = f(X_0)FX_0 - \xi. \quad (2.10)$$

and $f(e_i) = 0$ for $i = 1...n-1$. From (2.10) we have $g(h(X_0,X_0),FV) = 0$ for all $V \perp X_0$, V in S. Moreover, (2.10) and (2.5) implies that

$$FA_{FX_0}X_0 = f(X_0)FX_0 \quad \text{or} \quad -A_{FX_0}X_0 = -f(X_0)X_0$$

and then

$$A_{FX_0}X_0 = \lambda_1 X_0; \quad \lambda_1 = f(X_0). \qquad (2.11)$$

Putting $X = X_0$ and $Y \perp X_0$ in (2.9), we obtain

$$-g(X_0,X_0)g(h(Y,Z),FX_0) = g(Y,Z)g(h(X_0,X_0),FX_0)$$

and then

$$A_{FX_0}Y = -\lambda_1 Y. \qquad (2.12)$$

For $Y \perp X_0$, $X = Y$, $Y = Z = X_0$ in (2.9) we have:

$$-g(Y,Y)g(h(X_0,X_0),FY) = g(X_0,X_0)g(h(Y,Y),FY)$$

or

$$g(h(Y,Y),FY) = 0.$$

Using the totally symmetry of the 3-form C and the last equality, we have

$$g(h(Y,Z),FW)=0$$

for all $Y,Z,W\perp X_0$, Y,Z,W in $\chi(M)$. From (2.11) and (2.12) we have

$$h(X_0,X_0)=\lambda_1 FX_0+\xi, \qquad h(X_0,Y)=-\lambda_1 FY \tag{2.13}$$

for $Y\perp X_0$. Taking $X=X_0$ and $Z,Y\perp X_0$ in (2.9) we have:

$$h(Y,Z)=-\lambda_1 g(Y,Z)FX_0. \tag{2.14}$$

Taking $Z=Y$, $Z\perp X_0$ and Z an unitary vector field in (2.14), we obtain

$$A_{FY}Y=-\lambda_1 X_0. \tag{2.15}$$

If $\lambda_1=0$ then h vanishes at x. We suppose that $\lambda_1\neq 0$. For $n>2$, we consider two othonormal vector fields Y and Z, so that $Y,Z\perp X_0$. Then

$$R(X_0,Y)Y=(\frac{c+1}{4}-2\lambda_1^2)X_0,$$

and

$$R(Y,Z)Z=(\frac{c+1}{4}+\lambda_1^2)Y.$$

Because M is a pseudo-parallel manifold, we have

$$(\widetilde{R}\cdot h)(X_0,Y,Y,Y)+\Phi(x)Q(g,h)(X_0,Y,Y,Y)=0.$$

where

$$(\widetilde{R}\cdot h)(X_0,Y,Y,Y)=3\lambda_1(\frac{c+1}{4}-2\lambda_1^2)FY,$$

$$(Q\cdot h)(X_0,Y,Y,Y)=-2\lambda_1 FY.$$

From these last three equalities we have:

$$\Phi(x)=\frac{3(\frac{c+1}{4}-2\lambda_1^2)}{2}. \tag{2.16}$$

Also, we have:

$$(\widetilde{R}\cdot h)(X_0,Y,Y,Z)+\Phi(x)Q(g,h)(X_0,Y,Y,Z)=0.$$

But

$$(\widetilde{R}\cdot h)(X_0,Y,Y,Z)=\lambda_1(\frac{c+1}{4}-2\lambda_1^2)FZ$$

$$Q(g,h)(X_0,Y,Y,Z)=-\lambda_1 FZ.$$

From these last three equalities we deduce

$$\Phi(x)=\frac{c+1}{4}-2\lambda_1^2. \tag{2.17}$$

From (2.17) and (2.16) we obtain $\Phi(x)=0$, that is M is semi-parallel. □

Now, let M be a Legendre submanifold in a Sasaki space form $\widetilde{M}(c)$. Taking into account (1.3) and the fact that M is a Legendre submanifold, we have

$$F\widetilde{\nabla}_X Y = \widetilde{\nabla}_X(FY) - g(X,Y)\xi$$

$$F\widetilde{R}(X,Y)Z = \widetilde{R}(X,Y)FZ + g(Y,Z)FX - g(X,Z)FY$$

for all X, Y, Z in $\chi(M)$.

Because M is a Legendre submanifold, using (1.2) and (1.3) we obtain:

$$h(X,Y) = FA_{FY}X; \quad \nabla^{\perp}_X FY = F\nabla_X Y + g(X,Y)\xi \tag{2.18}$$

for X, Y, Z in $\chi(M)$. We also obtain that the 3-form C is totally symmetric for Legendre pseudo-parallel submanifolds in Sasaki space forms. Moreover, from (1.5) we have

$$\widetilde{R}(X,Y)Z = \frac{c+3}{4}[g(Y,Z)X - g(X,Z)Y]$$

and

$$R^{\perp}(X,Y)FZ = FR(X,Y)Z - g(Y,Z)FX + g(X,Z)FY$$

for all X, Y, Z in $\chi(M)$.

We define the tensor field

$$\theta(X,Y,Z,V,W) = g(h(X,V),FZ)g(Y,W) - g(h(Y,V),FZ)g(X,W) \tag{2.19}$$

for X, Y, Z, W in $\chi(M)$. Then θ is anti-symmetric in X and Y.

The submanifold M has *axial semi-symmetry* if θ is symmetric in Z and W.

Proposition 2.3. *Let M be a Legendre pseudo-parallel submanifold in the Sasaki space form $\widetilde{M}(c)$ so that M has axial semi-symmetry. Then, for each $x \in M$, there is $X_0 \in T_pM$, X_0 a unit vector field and $\lambda_1 \in \mathcal{F}(M)$ so that:*

$$A_{FX_0}X_0 = \lambda_1 X_0; \qquad c = 1 + 8\lambda_1^2.$$

From Proposition 2.3, we observe that the Sasaki space form $\widetilde{M}(c)$ has Legendre pseudo-parallel submanifolds only if the λ_1 is constant.

References

[1] Blair, D.E., *Contact Manifolds in Riemannian Geometry*, Lecture Notes in Math., Spring-Verlag, **509**(1976).

[2] Dillen, F., Van der Veken, J., Vranken, L., *Pseudo-parallel submanifolds are semi-parallel*, The SAO/Nasa Astrophysics Data System, Math. DC, 2008.

[3] Kenmotsu, K., *A class of almost contact Riemannian manifolds*, Tôhoku Math. J., **24**(1976), 93-103.

[4] Kobayashi, S., Nomizu, K., *Foundations of differential geometry*, Interscience Publ., **I**(1963), **II**(1969).

[5] Ogiue, K., *On almost contact manifolds admitting axiom of planes or axiom of free mobility*, Kodai Math. Sem. Rep., **16**(1964), 223–232.

[6] Papaghiuc, N., *Semi-invariant submanifolds in a Kenmotsu manifold*, Rend. di. Math., **4**(1983), 607-622.

[7] Pitiş, Gh., *On parallel submanifolds of a Sasakian space form*, Rendinconti di Mathematica, **VII**(1989), 103–111.

[8] Pitiş, Gh., *Geometry on Kenmotsu manifolds*, Editura Transilvania Braşov, 2007.

[9] Sular, S., Ozgur, C., *Pseodoparallel Submanifolds of Kenmotsu Manifolds*, Workshop on CR and Sasakian Geometry, 2009, Luxembourg.

[10] Yildiz, A., Murathan, C., Arslan, K., Ezentaş, R., *Totally real pseudo-parallel submanifolds of Sasakian space forms*, Monatsh. Math. **151**(2007), 247–256.

Maria Cîrnu
Negru-Vodă College
50, Stadionului Street
905800, Negru-Vodă
Constanţa, Romania
e-mail: maria.cirnu@yahoo.com

Stud. Univ. Babeş-Bolyai Math. 56(2011), No. 3, 95–104

Four-dimensional matrix transformation and rate of A-statistical convergence of Bögel-type continuous functions

Fadime Dirik and Kamil Demirci

Abstract. The purpose of this paper is to investigate the effects of four-dimensional summability matrix methods on the A-statistical approximation of sequences of positive linear operators defined on the space of all real valued Bögel-type continuous functions on a compact subset of the real line. Furthermore, we study the rates of A-statistical convergence in our approximation.

Mathematics Subject Classification (2010): 41A25, 41A36.

Keywords: The Korovkin theorem, B-continuous functions, rates of A-statistical convergence for double sequences, regularity for double sequences.

1. Introduction

In order to improve the classical Korovkin theory, the space of Bögel-type continuous (or, simply, B-continuous) functions instead of the classical one has been used in [2, 3, 4, 5]. Recall that the concept of B-continuity was first introduced in 1934 by Bögel [6] (see also [7, 8]). On the other hand, this Korovkin theory has also been generalized via the concept of statistical convergence (see, for instance, [11, 12]). It is well-known that every convergent sequence (in the usual sense) is statistically convergent but its converse is not always true. Also, statistical convergent sequences do not need to be bounded. With these properties, the usage of the statistical convergence in the approximation theory leads us to more powerful results than the classical aspects.

We now recall some basic definitions and notations used in the paper.

A double sequence

$$x = \{x_{m,n}\}, \quad m, n \in \mathbb{N},$$

is convergent in Pringsheim's sense if, for every $\varepsilon > 0$, there exists $N = N(\varepsilon) \in \mathbb{N}$ such that $|x_{m,n} - L| < \varepsilon$ whenever $m, n > N$. Then, L is called the Pringsheim limit of x and is denoted by $P - \lim x = L$ (see [19]). In this case, we say that $x = \{x_{m,n}\}$ is "P-convergent to L". Also, if there exists a positive number M such that $|x_{m,n}| \leq M$ for all $(m,n) \in \mathbb{N}^2 = \mathbb{N} \times \mathbb{N}$, then $x = \{x_{m,n}\}$ is said to be bounded. Note that in contrast to the case for single sequences, a convergent double sequence not to be bounded.

Now let

$$A = [a_{j,k,m,n}], \quad j, k, m, n \in \mathbb{N},$$

be a four-dimensional summability matrix. For a given double sequence $x = \{x_{m,n}\}$, the A-transform of x, denoted by $Ax := \{(Ax)_{j,k}\}$, is given by

$$(Ax)_{j,k} = \sum_{(m,n)\in\mathbb{N}^2} a_{j,k,m,n} x_{m,n}, \quad j, k \in \mathbb{N},$$

provided the double series converges in Pringsheim's sense for every $(j,k) \in \mathbb{N}^2$. In summability theory, a two-dimensional matrix transformation is said to be regular if it maps every convergent sequence into a convergent sequence with the same limit. The well-known characterization for two-dimensional matrix transformations is known as Silverman-Toeplitz conditions (see, for instance, [16]). In 1926, Robison [20] presented a four-dimensional analog of the regularity by considering an additional assumption of boundedness. This assumption was made because a double P-convergent sequence is not necessarily bounded. The definition and the characterization of regularity for four-dimensional matrices is known as Robison-Hamilton conditions, or briefly, RH-regularity (see, [15, 20]).

Recall that a four dimensional matrix $A = [a_{j,k,m,n}]$ is said to be RH-regular, if it maps every bounded P-convergent sequence into a P-convergent sequence with the same P-limit. The Robison-Hamilton conditions state that a four dimensional matrix $A = [a_{j,k,m,n}]$ is RH-regular if and only if

(i) $P - \lim_{j,k} a_{j,k,m,n} = 0$ for each $(m,n) \in \mathbb{N}^2$,

(ii) $P - \lim_{j,k} \sum_{(m,n)\in\mathbb{N}^2} a_{j,k,m,n} = 1$,

(iii) $P - \lim_{j,k} \sum_{m\in\mathbb{N}} |a_{j,k,m,n}| = 0$ for each $n \in \mathbb{N}$,

(iv) $P - \lim_{j,k} \sum_{n\in\mathbb{N}} |a_{j,k,m,n}| = 0$ for each $m \in \mathbb{N}$,

(v) $\sum_{(m,n)\in\mathbb{N}^2} |a_{j,k,m,n}|$ is $P-$convergent for each $(j,k) \in \mathbb{N}^2$,

(vi) there exist finite positive integers A and B such that

$$\sum_{m,n>B} |a_{j,k,m,n}| < A$$

holds for every $(j,k) \in \mathbb{N}^2$.

Now let $A = [a_{j,k,m,n}]$ be a non-negative RH-regular summability matrix, and let $K \subset \mathbb{N}^2$. Then, a real double sequence $x = \{x_{m,n}\}$ is said to be

A-statistically convergent to a number L if, for every $\varepsilon > 0$,

$$P - \lim_{j,k} \sum_{(m,n)\in K(\varepsilon)} a_{j,k,m,n} = 0,$$

where

$$K(\varepsilon) := \{(m,n) \in \mathbb{N}^2 : |x_{m,n} - L| \geq \varepsilon\}.$$

In this case, we write $st_A^{(2)} - \lim x_{m,n} = L$. Observe that, a P-convergent double sequence is A-statistically convergent to the same value but the converse does not hold. For example, consider the double sequence $x = \{x_{m,n}\}$ given by

$$x_{m,n} = \begin{cases} mn, & \text{if } m \text{ and } n \text{ are squares,} \\ 1, & \text{otherwise.} \end{cases}$$

We should note that if we take $A = C(1,1)$, which is the double Cesáro matrix, then $C(1,1)$-statistical convergence coincides with the notion of statistical convergence for a double sequence, which was introduced in [17, 18]. Finally, if we replace the matrix A by the identity matrix for four-dimensional matrices, then A-statistical convergence reduces to the Pringsheim convergence.

In most investigations the approximated functions are assumed to be continuous. However, the considered approximation processes are often meaningful for a bigger class of functions, namely for so-called $B-$continuous functions introduced by Bögel [6, 7, 8].

The definition of $B-$continuous was introduced by Bögel as follows:

Let X and Y be compact subsets of the real numbers, and let $D = X \times Y$. Then, a function $f : D \to \mathbb{R}$ is called $B-$continuous at a point $(x,y) \in D$ if, for every $\varepsilon > 0$, there exists a positive number $\delta = \delta(\varepsilon)$ such that

$$|\Delta_{x,y}[f(u,v)]| < \varepsilon,$$

for any $(u,v) \in D$ with $|u - x| < \delta$ and $|v - y| < \delta$, where the symbol $\Delta_{x,y}[f(u,v)]$ denotes the mixed difference of f defined by

$$\Delta_{x,y}[f(u,v)] = f(u,v) - f(u,y) - f(x,v) + f(x,y).$$

By $C_b(D)$ we denote the space of all B-continuous functions on D. Recall that $C(D)$ and $B(D)$ denote the space of all continuous (in the usual sense) functions and the space of all bounded functions on D, respectively. Then, notice that $C(D) \subset C_b(D)$. Moreover, one can find an unbounded $B-$continuous function, which follows from the fact that, for any function of the type $f(u,v) = g(u) + h(v)$, we have $\Delta_{x,y}[f(u,v)] = 0$ for all $(x,y),(u,v) \in D$.

The usual supremum norm on the spaces $B(D)$ is given by

$$\|f\| := \sup_{(x,y)\in D} |f(x,y)| \quad \text{for } f \in B(D).$$

Throughout the paper, for fixed $(x,y) \in D$ and $f \in C_b(D)$, we use the function $F_{x,y}$ defined as follows:

$$F_{x,y}(u,v) = f(u,y) + f(x,v) - f(u,v) \qquad \text{for } (u,v) \in D. \tag{1.1}$$

Since

$$\Delta_{x,y}\left[F_{x,y}(u,v)\right]=-\Delta_{x,y}\left[f(u,v)\right]$$

holds for all (x,y), $(u,v)\in D$, the $B-$continuity of f implies the $B-$continuity of $F_{x,y}$ for every fixed $(x,y)\in D$. We also use the following test functions

$$e_0(u,v)=1,\ e_1(u,v)=u,\ e_2(u,v)=v \text{ and } e_3(u,v)=u^2+v^2.$$

With this terminology the authors [14] proved the following theorem, which corresponds to the A-statistical formulation of the problem above studied by Badea et. al. [3].

Theorem 1.1. [14] *Let $\{L_{m,n}\}$ be a double sequence of positive linear operators acting from $C_b(D)$ into $B(D)$, and let $A=[a_{j,k,m,n}]$ be a non-negative RH−regular summability matrix method. Assume that the following conditions hold:*

$$\delta_A^{(2)}\left\{(m,n)\in\mathbb{N}^2 : L_{m,n}(e_0;x,y)=e_0(x,y)\quad \text{for all } (x,y)\in D\right\}=1$$

and

$$st_A^{(2)}-\lim_{m,n}\|L_{m,n}(e_i)-e_i\|=0\quad \text{for } i=1,2,3.$$

Then, for all $f\in C_b(D)$, we have

$$st_A^{(2)}-\lim_{m,n}\|L_{m,n}(F_{x,y})-f\|=0,$$

where $F_{x,y}$ is given by (1.1).

The aim of the present paper is to compute the rates of A-statistical approximation in Theorem 1.1 with the help of mixed modulus of smoothness.

2. Rate of A-statistical convergence

Various ways of defining rates of convergence in the A-statistical sense for two-dimensional summability matrix were introduced in [10]. In a similar way, for four-dimensional summability matrix, defining rates of convergence in the A-statistical sense introduced in [13]. In this section, we compute the corresponding rates of A-statistical convergence in Theorem 1.1 by means of four different ways.

Definition 2.1. [13] *Let $A=[a_{j,k,m,n}]$ be a non-negative RH-regular summability matrix and let $\{\alpha_{m,n}\}$ be a positive non-increasing double sequence. A double sequence $x=\{x_{m,n}\}$ is A-statistical convergent to a number L with the rate of $o(\alpha_{m,n})$, if for every $\varepsilon>0$,*

$$P-\lim_{j,k\to\infty}\frac{1}{\alpha_{j,k}}\sum_{(m,n)\in K(\varepsilon)}a_{j,k,m,n}=0,$$

where

$$K(\varepsilon):=\left\{(m,n)\in\mathbb{N}^2 : |x_{m,n}-L|\geq\varepsilon\right\}.$$

In this case, it is denoted by

$$x_{m,n} - L = st_A^{(2)} - o(\alpha_{m,n}) \quad as \quad m,n \to \infty.$$

Definition 2.2. [13] *Let $A = [a_{j,k,m,n}]$ and $\{\alpha_{m,n}\}$ be the same as in Definition 2.1. Then, a double sequence $x = \{x_{m,n}\}$ is A-statistical bounded with the rate of $O(\alpha_{m,n})$ if for every $\varepsilon > 0$,*

$$\sup_{j,k} \frac{1}{\alpha_{j,k}} \sum_{(m,n) \in L(\varepsilon)} a_{j,k,m,n} < \infty,$$

where

$$L(\varepsilon) := \left\{(m,n) \in \mathbb{N}^2 : |x_{m,n}| \geq \varepsilon \right\}.$$

In this case, it is denoted by

$$x_{m,n} = st_A^{(2)} - O(\alpha_{m,n}) \quad as \quad m,n \to \infty.$$

Definition 2.3. [13] *Let $A = [a_{j,k,m,n}]$ and $\{\alpha_{m,n}\}$ be the same as in Definition 2.1. Then, a double sequence $x = \{x_{m,n}\}$ is A-statistical convergent to a number L with the rate of $o_{m,n}(\alpha_{m,n})$ if for every $\varepsilon > 0$,*

$$P - \lim_{j,k \to \infty} \sum_{(m,n) \in M(\varepsilon)} a_{j,k,m,n} = 0,$$

where

$$M(\varepsilon) := \left\{(m,n) \in \mathbb{N}^2 : |x_{m,n} - L| \geq \varepsilon \ \alpha_{m,n} \right\}.$$

In this case, it is denoted by

$$x_{m,n} - L = st_A^{(2)} - o_{m,n}(\alpha_{m,n}) \quad as \quad m,n \to \infty.$$

Definition 2.4. [13] *Let $A = [a_{j,k,m,n}]$ and $\{\alpha_{m,n}\}$ be the same as in Definition 2.1. Then, a double sequence $x = \{x_{m,n}\}$ is A-statistical bounded with the rate of $O_{m,n}(\alpha_{m,n})$ if for every $\varepsilon > 0$,*

$$P - \lim_{j,k} \sum_{(m,n) \in N(\varepsilon)} a_{j,k,m,n} = 0,$$

where

$$N(\varepsilon) := \left\{(m,n) \in \mathbb{N}^2 : |x_{m,n}| \geq \varepsilon \ \alpha_{m,n} \right\}.$$

In this case, it is denoted by

$$x_{m,n} = st_A^{(2)} - O_{m,n}(\alpha_{m,n}) \quad as \quad m,n \to \infty.$$

We see from the above statements that, in Definitions 2.1 and 2.2 the rate sequence $\{\alpha_{m,n}\}$ directly effects the entries of the matrix $A = [a_{j,k,m,n}]$ although, according to Definitions 2.3 and 2.4, the rate is more controlled by the terms of the sequence $x = \{x_{m,n}\}$.

Using these definitions we have the following auxiliary result [13].

Lemma 2.5. [13] *Let $\{x_{m,n}\}$ and $\{y_{m,n}\}$ be double sequences. Assume that let $A = [a_{j,k,m,n}]$ be a non-negative RH-regular summability matrix and let $\{\alpha_{m,n}\}$ and $\{\beta_{m,n}\}$ be positive non-increasing sequences. If $x_{m,n} - L_1 = st_A^{(2)} - o(\alpha_{m,n})$ and $y_{m,n} - L_2 = st_A^{(2)} - o(\beta_{m,n})$, then we have* *(i)* $(x_{m,n}-L_1)\mp(y_{m,n}-L_2)=st_A^{(2)}-o(\gamma_{m,n})$ *as* $m,n\to\infty$ *, where* $\gamma_{m,n}:=\max\{\alpha_{m,n},\beta_{m,n}\}$ *for each* $(m,n)\in\mathbb{N}^2$,

(ii) $\lambda(x_{m,n}-L_1)=st_A^{(2)}-o(\alpha_{m,n})$ *as* $m,n\to\infty$ *for any real number* λ.

Furthermore, similar conclusions hold with the symbol "o" replaced by "O".

The above result can easily be modified to obtain the following result similarly.

Lemma 2.6. [13] *Let* $\{x_{m,n}\}$ *and* $\{y_{m,n}\}$ *be double sequences. Assume that* $A=[a_{j,k,m,n}]$ *is a non-negative RH-regular summability matrix and let* $\{\alpha_{m,n}\}$ *and* $\{\beta_{m,n}\}$ *be positive non-increasing sequences. If* $x_{m,n}-L_1=st_A^{(2)}-o_{m,n}(\alpha_{m,n})$ *and* $y_{m,n}-L_2=st_A^{(2)}-o_{m,n}(\beta_{m,n})$*, then we have*

(i) $(x_{m,n}-L_1)\mp(y_{m,n}-L_2)=st_A^{(2)}-o_{m,n}(\gamma_{m,n})$ *as* $m,n\to\infty$ *, where* $\gamma_{m,n}:=\max\{\alpha_{m,n},\beta_{m,n}\}$ *for each* $(m,n)\in\mathbb{N}^2$,

(ii) $\lambda(x_{m,n}-L_1)=st_A^{(2)}-o_{m,n}(\alpha_{m,n})$ *as* $m,n\to\infty$ *for any real number* λ.

Furthermore, similar conclusions hold with the symbol "$o_{m,n}$*" replaced by "*$O_{m,n}$*".*

Now we recall the concept of mixed modulus of smoothness. For $f\in C_b(D)$, the *mixed modulus of smoothness* of f, denoted by $\omega_{mixed}(f;\delta_1,\delta_2)$, is defined to be

$$\omega_{mixed}(f;\delta_1,\delta_2)=\sup\{|\Delta_{x,y}[f(u,v)]|:|u-x|\le\delta_1,\ \ |v-y|\le\delta_2\}$$

for $\delta_1,\delta_2>0$. In order to obtain our result, we will make use of the elementary inequality

$$\omega_{mixed}(f;\lambda_1\delta_1,\lambda_2\delta_2)\le(1+\lambda_1)(1+\lambda_2)\,\omega_{mixed}(f;\delta_1,\delta_2)$$

for $\lambda_1,\lambda_2>0$. The modulus ω_{mixed} has been used by several authors in the framework of "Boolean sum type" approximation (see, for example, [9]). Elementary properties of ω_{mixed} can be found in [21] (see also [1]) and in particular for the case of B-continuous functions in [2].

Then we have the following result.

Theorem 2.7. *Let* $\{L_{m,n}\}$ *be a sequence of positive linear operators acting from* $C_b(D)$ *into* $B(D)$ *and let* $A=[a_{j,k,m,n}]$ *be a non-negative RH−regular summability matrix. Let* $\{\alpha_{m,n}\}$ *and* $\{\beta_{m,n}\}$ *be a positive non-increasing double sequence. Assume that the following conditions hold:*

$$P-\lim_{j,k\to\infty}\frac{1}{\alpha_{j,k}}\sum_{(m,n)\in K}a_{j,k,m,n}=1, \tag{2.1}$$

where $K=\left\{(m,n)\in\mathbb{N}^2:L_{m,n}(e_0;x,y)=1 \text{ for all } (x,y)\in D\right\}$ *; and*

$$\omega_{mixed}(f;\gamma_{m,n},\delta_{m,n})=st_A^{(2)}-o(\beta_{m,n}) \quad as \quad m,n\to\infty, \tag{2.2}$$

where $\gamma_{m,n}:=\sqrt{\|L_{m,n}(\varphi)\|}$ *and* $\delta_{m,n}:=\sqrt{\|L_{m,n}(\Psi)\|}$ *with* $\varphi(u,v)=(u-x)^2$, $\Psi(u,v)=(v-y)^2$*. Then we have, for all* $f\in C_b(D)$,

$$\|L_{m,n}(F_{x,y})-f\|=st_A^{(2)}-o(c_{m,n}) \quad as \quad m,n\to\infty,$$

where $F_{x,y}$ is given by (1.1) and $c_{m,n} := \max\{\alpha_{m,n}, \beta_{m,n}\}$ for each $(m,n) \in \mathbb{N}^2$. Furthermore, similar results hold when the symbol "o" is replaced by "O".

Proof. Let $(x,y) \in D$ and $f \in C_b(D)$ be fixed. It follows from (2.1) that

$$P - \lim_{j,k\to\infty} \frac{1}{\alpha_{j,k}} \sum_{(m,n)\in\mathbb{N}^2\setminus K} a_{j,k,m,n} = 0. \tag{2.3}$$

Also, since

$$\Delta_{x,y}\left[F_{x,y}(u,v)\right] = -\Delta_{x,y}\left[f(u,v)\right],$$

we observe that

$$L_{m,n}\left(F_{x,y};x,y\right) - f(x,y) = L_{m,n}\left(\Delta_{x,y}\left[F_{x,y}(u,v)\right];x,y\right)$$

holds for all $(m,n) \in K$. Then, using the properties of ω_{mixed} we obtain

$$\begin{aligned} \left|\Delta_{x,y}\left[F_{x,y}(u,v)\right]\right| &\leq \omega_{mixed}\left(f;|u-x|,|v-y|\right) \\ &\leq \left(1+\frac{1}{\delta_1}|u-x|\right)\left(1+\frac{1}{\delta_2}|v-y|\right) \\ &\times\omega_{mixed}\left(f;\delta_1,\delta_2\right). \end{aligned} \tag{2.4}$$

Hence, using the monotonicity and the linearity of the operators $L_{m,n}$, for all $(m,n) \in K$, it follows from (2.4) that

$$\begin{aligned} & \left|L_{m,n}(F_{x,y};x,y) - f(x,y)\right| \\ = & \left|L_{m,n}\left(\Delta_{x,y}\left[F_{x,y}(u,v)\right];x,y\right)\right| \\ \leq & L_{m,n}\left(\left|\Delta_{x,y}\left[F_{x,y}(u,v)\right]\right|;x,y\right) \\ \leq & L_{m,n}\left(\left(1+\frac{1}{\delta_1}|u-x|\right)\left(1+\frac{1}{\delta_2}|v-y|\right);x,y\right)\omega_{mixed}\left(f;\delta_1,\delta_2\right) \\ = & \left\{1+\frac{1}{\delta_1}L_{m,n}\left(|u-x|;x,y\right)+\frac{1}{\delta_2}L_{m,n}\left(|v-y|;x,y\right)\right. \\ & \left.\frac{1}{\delta_1\delta_2}L_{m,n}\left(|u-x|.|v-y|;x,y\right)\right\}\omega_{mixed}\left(f;\delta_1,\delta_2\right). \end{aligned}$$

Using the Cauchy-Schwarz inequality, we have

$$\begin{aligned} & \left|L_{m,n}(F_{x,y};x,y) - f(x,y)\right| \\ & \leq \left\{1+\frac{1}{\delta_1}\sqrt{L_{m,n}\left(\varphi;x,y\right)}+\frac{1}{\delta_2}\sqrt{L_{m,n}\left(\Psi;x,y\right)}\right. \\ & \left.\frac{1}{\delta_1\delta_2}\sqrt{L_{m,n}\left(\varphi;x,y\right)}\sqrt{L_{m,n}\left(\Psi;x,y\right)}\right\}\omega_{mixed}\left(f;\delta_1,\delta_2\right) \end{aligned} \tag{2.5}$$

for all $(m,n) \in K$. Taking supremum over $(x,y) \in D$ on the both-sides of inequality (2.5) we obtain, for all $(m,n) \in K$, that

$$\left\|L_{m,n}(F_{x,y}) - f\right\| \leq 4\omega_{mixed}\left(f;\gamma_{m,n},\delta_{m,n}\right) \tag{2.6}$$

where $\delta_1 := \gamma_{m,n} := \sqrt{\|L_{m,n}(\varphi)\|}$ and $\delta_2 := \delta_{m,n} := \sqrt{\|L_{m,n}(\Psi)\|}$. Now, given $\varepsilon > 0$, define the following sets:

$$\begin{aligned} U &: = \left\{(m,n) \in \mathbb{N}^2 : \left\|L_{m,n}(F_{x,y}) - f\right\| \geq \varepsilon\right\}, \\ U_1 &: = \left\{(m,n) \in \mathbb{N}^2 : \omega_{mixed}\left(f;\gamma_{m,n},\delta_{m,n}\right) \geq \frac{\varepsilon}{4}\right\}. \end{aligned}$$

Hence, it follows from (2.6) that

$$U \cap K \subseteq U_1 \cap K,$$

which gives, for all $(j,k) \in \mathbb{N}^2$,

$$\begin{aligned} \frac{1}{c_{j,k}} \sum_{(m,n)\in U\cap K} a_{j,k,m,n} &\leq \frac{1}{c_{j,k}} \sum_{(m,n)\in U_1\cap K} a_{j,k,m,n} \\ &\leq \frac{1}{c_{j,k}} \sum_{(m,n)\in U_1} a_{j,k,m,n} \\ &\leq \frac{1}{\beta_{j,k}} \sum_{(m,n)\in U_1} a_{j,k,m,n}. \end{aligned} \tag{2.7}$$

where $c_{m,n} = \max\{\alpha_{m,n}, \beta_{m,n}\}$. Letting $j,k \to \infty$ (in any manner) in (2.7) and from (2.2), we conclude that

$$P - \lim_{j,k\to\infty} \frac{1}{c_{j,k}} \sum_{(m,n)\in U\cap K} a_{j,k,m,n} = 0. \tag{2.8}$$

Furthermore, we use the inequality

$$\begin{aligned} \sum_{(m,n)\in U} a_{j,k,m,n} &= \sum_{(m,n)\in U\cap K} a_{j,k,m,n} + \sum_{(m,n)\in U\cap(\mathbb{N}^2\setminus K)} a_{j,k,m,n} \\ &\leq \sum_{(m,n)\in U\cap K} a_{j,k,m,n} + \sum_{(m,n)\in \mathbb{N}^2\setminus K} a_{j,k,m,n} \end{aligned}$$

which gives,

$$\frac{1}{c_{j,k}} \sum_{(m,n)\in U} a_{j,k,m,n} \leq \frac{1}{c_{j,k}} \sum_{(m,n)\in U\cap K} a_{j,k,m,n} + \frac{1}{\alpha_{j,k}} \sum_{(m,n)\in \mathbb{N}^2\setminus K} a_{j,k,m,n}. \tag{2.9}$$

Letting $j,k \to \infty$ (in any manner) in (2.9) and from (2.8) and (2.3), we conclude that

$$P - \lim_{j,k\to\infty} \frac{1}{c_{j,k}} \sum_{(m,n)\in U} a_{j,k,m,n} = 0.$$

The proof is completed. □

The following similar result holds.

Theorem 2.8. *Let $\{L_{m,n}\}$ be a sequence of positive linear operators acting from $C_b(D)$ into $B(D)$ and let $A = [a_{j,k,m,n}]$ be a non-negative RH−regular summability matrix. Let $\{\alpha_{m,n}\}$ and $\{\beta_{m,n}\}$ be a positive non-increasing double sequence. Assume that the following conditions holds:*

$$P - \lim_{j,k\to\infty} \sum_{(m,n)\in K} a_{j,k,m,n} = 1, \tag{2.10}$$

where $K = \left\{(m,n) \in \mathbb{N}^2 : L_{m,n}(e_0; x, y) = 1 \text{ for all } (x,y) \in \mathbb{R}^2\right\}$; and

$$\omega_{mixed}(f; \gamma_{m,n}, \delta_{m,n}) = st_A^{(2)} - o_{m,n}(\beta_{m,n}) \quad as \quad m,n \to \infty, \tag{2.11}$$

where $\gamma_{m,n} := \sqrt{\|L_{m,n}(\varphi)\|}$ *and* $\delta_{m,n} := \sqrt{\|L_{m,n}(\Psi)\|}$ *with* $\varphi(u,v) = (u-x)^2$, $\Psi(u,v) = (v-y)^2$. *Then we have, for all* $f \in C_b(D)$,

$$\|L_{m,n}(F_{x,y}) - f\| = st_A^{(2)} - o_{m,n}(\beta_{m,n}) \quad as \quad m,n \to \infty,$$

where $F_{x,y}$ *is given by* (1.1)*. Similar results hold when little "*$o_{m,n}$*" is replaced by capital "*$O_{m,n}$*".*

3. Concluding remarks

1) Specializing the sequences $\{\alpha_{m,n}\}$ and $\{\beta_{m,n}\}$ in Theorem 2.7 or Theorem 2.8 we can easily get Theorem 1.1. Thus, Theorem 2.7 gives us the rates of $A-$statistical convergence of the operators $L_{m,n}$ from $C_b(D)$ into $B(D)$.

2) Replacing the matrix A by a double identity matrix and taking $\alpha_{m,n} = \beta_{m,n} = 1$ for all $m,n \in \mathbb{N}$, we get the ordinary rate of convergence of the operators $L_{m,n}$.

References

[1] Anastassiou, G.A., Gal, S.G., *Approximation Theory: Moduli of Continuity and Global Smoothness Preservation*, Birkhäuser, Boston, 2000.

[2] Badea, I., *Modulul de continuitate în sens Bögel şi unele aplicaţii în apoximarea printr-un operator Benstein*, Studia Univ. Babeş-Bolyai, Ser. Math. Mech., **18**(1973), 69-78.

[3] Badea, C., Badea, I., Gonska, H.H., *A test function and approximation by pseudopolynomials*, Bull. Austral. Math. Soc., **34**(1986), 53-64.

[4] Badea, C., Badea, I., Cottin, C., *A Korovkin-type theorems for generalizations of Boolean sum operators and approximation by trigonometric pseudopolynomials*, Anal. Numér. Théor. Approx., **17**(1988), 7-17.

[5] Badea, C. and Cottin, C., *Korovkin-type theorems for generalized Boolean sum operators*, Approximation theory (Kecskemét, 1990), 51-68, Colloq. Math. Soc. János Bolyai, 58, North-Holland, Amsterdam, 1991.

[6] Bögel, K., *Mehrdimensionale Differentiation von Funktionen mehrerer Veränderlicher*, J. Reine Angew. Math., **170**(1934), 197-217.

[7] Bögel, K., *Über mehrdimensionale Differentiation, Integration und beschränkte Variation*, J. Reine Angew. Math., **173**(1935), 5-29.

[8] Bögel, K., *Über die mehrdimensionale Differentiation*, Jahresber. Deutsch. Mat.-Verein., **65**(1962), 45-71.

[9] Cottin, C., *Approximation by bounded pseudo-polynomials, in "Function Spaces" (ed. J. Musielak et al.)*, Teubner-Texte zur Mathematik., **120**(1991), 152-160.

[10] Duman, O., Khan, M.K., Orhan, C., *A-statistical convergence of approximating operators*, Math. Inequal. Appl., **6**(2003), 689-699.

[11] Duman, O., Erkuş, E., Gupta, V., *Statistical rates on the multivariate approximation theory*, Math. Comput. Modelling, **44**(2006), 763-770.

[12] Erkuş, E., Duman, O. and Srivastava, H.M., *Statistical approximation of certain positive linear operators constructed by means of the Chan-Chyan-Srivastava polynomials*, Appl. Math. Comput., **182** (2006), 213-222.

[13] Dirik, F., Demirci, K., *Four-dimensional matrix transformation and rate of A-statistical convergence of continuous functions*, Comput. Math. Appl., **59**(2010), 2976-2981.

[14] Dirik, F., Duman, O., Demirci, K., *Approximation in statistical sense to B-continuous functions by positive linear operators*, Studia Sci. Math. Hungarica DOI 10.1556/SScMath.2009.1129.

[15] Hamilton, H.J., *Transformations of multiple sequences*, Duke Math. J., **2**(1936), 29-60.

[16] Hardy, G.H., *Divergent Series*, Oxford Univ. Press, London, 1949.

[17] Moricz, F., *Statistical convergence of multiple sequences*, Arch. Math. (Basel), **81**(2004), 82-89.

[18] Mursaleen, Edely, O.H.H., *Statistical convergence of double sequences*, J. Math. Anal. Appl., **288**(2003), 223-231.

[19] Pringsheim, A., *Zur Theorie der zweifach unendlichen Zahlenfolgen*, Math. Ann., **53** (1900), 289-321.

[20] Robison, G.M., *Divergent double sequences and series*, Amer. Math. Soc. Transl., **28**(1926), 50-73.

[21] Schumaker, L.L., *Spline Functions: Basic Theory*, John Wiley & Sons, New York, 1981.

Fadime Dirik
Sinop University, Faculty of Arts and Sciences
Department of Mathematics
TR-57000, Sinop, Turkey
e-mail: fdirik@sinop.edu.tr

Kamil Demirci
Sinop University, Faculty of Arts and Sciences
Department of Mathematics
TR-57000, Sinop, Turkey
e-mail: kamild@sinop.edu.tr

Stud. Univ. Babeş-Bolyai Math. 56(2011), No. 3, 105–112

Solution of a nonlinear system of second kind Lagrange's equations by fixed-point method

Ljubomir Georgiev and Konstantin Kostov

Abstract. The effect of forces acting upon a ferromagnetic rotational ellipsoid located in a homogeneous rotating magnetic field is considered. Lagrange's equations of the second kind connecting the motion parameters of a particle with torques acting upon it are composed. A non-homogeneous nonlinear autonomous system of second-order differential equations is obtained. That system is not solvable by quadrature. A solution by fixed-point method is proposed in this paper.

Mathematics Subject Classification (2010): 74H20, 74H25, 47H10, 58C30.

Keywords: Fixed point, Lagrange's equations, differential equations.

1. Introduction

The principle of rotating magnetic field is applied in designing machines that intensify some technological processes like milling, emulsifying, mixing, etc. Ferromagnetic working particles are placed in the so-called active volume of the machine where they are driven by the field and exert a force-applying effect upon the treated material. It is characteristic for their motion that due to frequent collisions these particles are always in transition mode, i. e. the angle between the field vector and the longitudinal axis of the working particle changes. It can be assumed that after each collision there emerges a motion of new initial conditions. Calculating precisely the technological effect obtained requires good knowledge of the law of motion at arbitrary initial conditions. Our goal is to establish the existence of unique solution of the initial value problem for the corresponding system of two nonlinear second-order differential equations. We take advantage of the fixed point method to do this. At the end of this paper we present a sequence of successive analytical approximations of the solution, which belongs to a suitable subset of the space $C([0,\infty))$.

2. Physical model

A ferromagnetic rotational ellipsoid, formed by the rotation of an ellipse of axes $2a$ and $2b$ around its long axis of length $2a$ and located in a homogeneous rotating magnetic field of flux-density modulus $\overrightarrow{B}_0$, is considered. In this case the ellipsoid is homogeneously magnetized, which makes possible the analytical determination of its electromagnetic torque.

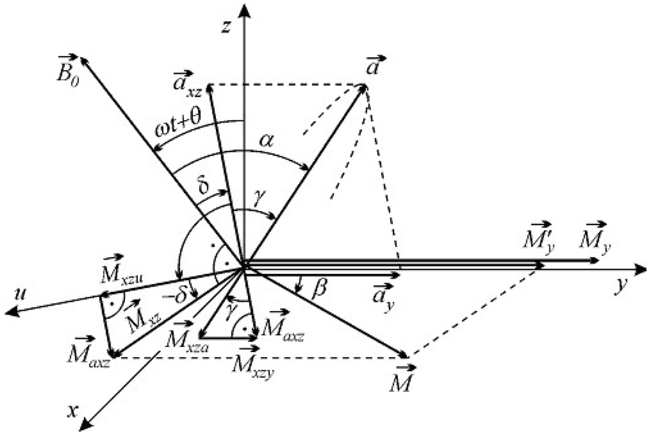


Fig. 1

Fig. 1 shows a layout of the particle, magnetic flux density and respective torque $\overrightarrow{M}$. The denotations ([3]) are as follows: $\overrightarrow{a}$ is a vector applied to the center of the ellipsoid and directed along its axis. It shows the spatial position of the particle considered ($\left|\overrightarrow{a}\right|$ is equal to the long (rotational) half-axis a of the ellipsoid). α is the smaller angle between vectors $\overrightarrow{B}_0$ and $\overrightarrow{a}$, ω is the angular velocity of the rotating magnetic field, $\omega t+\theta$ is the angle between the axis z and field vector $\overrightarrow{B}_0$, γ is the angle formed between the plane xOz and vector $\overrightarrow{a}$, $\overrightarrow{a}_{xz}$ is a vector component of $\overrightarrow{a}$ (its projection onto the plane xOz), δ is the angle between $\overrightarrow{B}_0$ and $\overrightarrow{a}_{xz}$. Denoted $\alpha, \gamma, \omega t+\theta$ and δ are oriented angles between vectors or between vectors and axes.

The synchronous reactive torque is determined in [3]:

$$M = -K{B_0}^2 \sin 2\alpha = -M_0 \sin 2\alpha \tag{2.1}$$

Vector $\overrightarrow{M}$ is perpendicular to the plane defined by $2\overrightarrow{a}$ and $\overrightarrow{B}_0$ and it is of the same direction as that of $\overrightarrow{a'} \times \overrightarrow{B}_0$. Here, $\overrightarrow{a'}$ is the vector along the ellipsoid's long axis, which makes with $\overrightarrow{B}_0$ an angle smaller than $\frac{\pi}{2}$.

Angles γ and δ are selected as generalized coordinates, defining uniquely the spatial position of the ellipsoid. An additional axis u lying in the plane

xOz and being at a positive angle $\frac{\pi}{2}$ with respect to $\vec{a}_{xz}$ is introduced. The synchronous torque $\vec{M}$ is decomposed along the axes y, u and $\vec{a}_{xz}$, ([5]): $\vec{M}=\vec{M}_y + \vec{M}_u + \vec{M}_a$, where $M_y = -M_0 \sin 2\delta, M_u = M_{xzu} = -M_0 \cos^2 \delta \sin 2\gamma$, and $M_a = M_{xza} = -M_0 \sin 2\delta \sin \gamma$ are scalar components of $\vec{M}$ along the respective axes.

The kinetic energy of the ellipsoid has the form:

$$T = \frac{1}{2} J \left(\omega + \dot{\delta} \right)^2 + \frac{1}{2} J \dot{\gamma}^2 + \frac{1}{2} J_a \dot{\varphi}^2, \tag{2.2}$$

where J_a is the inertia torque of the rotational ellipsoid with respect to the axis $2a$, J is the inertia torque of the rotational ellipsoid with respect to the axis $2b$ that goes through its center of gravity and is perpendicular to $\vec{a}$, $\dot{\varphi}$ is the angular velocity of the ellipsoid in its rotation around the axis $2a$. Therefore, φ is the third generalized coordinate. Let us read φ from the line, which is located further away from the plane xOz and in which the plane formed by $\vec{a}$ and $\vec{a}_{xz}$ crosses the ellipsoid at the initial time point $t = 0$. We assume the positive direction should be determined by the right-hand screw rule with axis $\vec{a}$.

A torque defined by currents acts on the ellipsoid as well. Its average value ([5]) is

$$\vec{M}_e = -\vec{j}\, M_\gamma \cos^3 \gamma \dot{\delta}, \tag{2.3}$$

where $M_\gamma = \dfrac{\pi \mu_r^2 \sigma B_0^2 d^4 l}{256(1 + \mu_r N_l)^2 (1 + k^4 d^4/256)}$ is the current torque for $\gamma = 0$ at $\dot{\delta} = 1\ rad/s$. For a cylinder of determined size and given magnetic permeability $M_\gamma = const$.

The torque M_e acts along the axis y and exhibits itself only when there is a difference between the angular velocities of the field and particle along the axis y. The negative sign indicates that this torque opposes the change in the angle δ. Besides the driving torques considered so far, there is also a hysteresis torque that will be neglected for we consider a particle made of soft-magnetic material and because of considerations related to its shape ([4]). There exist resisting torques as well, resulting from the friction forces. Due to the small size of the ellipsoid the linear velocities are of low values, and we can consequently assume that frictional torques are proportional to the first power of the respective angular velocity. Correspondingly, the proportionality factors for motions along δ and along γ are equal to each other for in both cases the rotation is realized around the axis $2b$ of the ellipsoid. Having in mind that torques M_u, M_y, and M_a act along the direction of generalized coordinates γ, δ, and φ, respectively, we compose the following system of

Lagrange's differential equations of the second order ([6]):

$$\left|\begin{array}{l} \dfrac{d}{dt}\dfrac{\partial T}{\partial \dot{\gamma}} - \dfrac{\partial T}{\partial \gamma} = M_u - k_1\dot{\gamma} \\[2ex] \dfrac{d}{dt}\dfrac{\partial T}{\partial \dot{\delta}} - \dfrac{\partial T}{\partial \delta} = M_y - M_\gamma \dot{\delta}\cos^3\gamma - k_1(\omega + \dot{\delta}) \\[2ex] \dfrac{d}{dt}\dfrac{\partial T}{\partial \dot{\varphi}} - \dfrac{\partial T}{\partial \varphi} = M_a - k_2\dot{\varphi}. \end{array}\right.$$

Here, $k_1\dot{\gamma}, k_1(\omega + \dot{\delta})$ and $k_2\dot{\varphi}$ are frictional torques in tracing out the respective angles, and k_1 and k_2 are proportionality factors. The negative signs before the frictional torques indicate that they are inversely proportional to respective angular velocities. We replace with the torques derived above and obtain

$$\left|\begin{array}{l} J\ddot{\gamma} = -M_0\cos^2\delta\sin 2\gamma - k_1\dot{\gamma} \\[2ex] J\ddot{\delta} = -M_0\sin 2\delta - M_\gamma\dot{\delta}\cos^3\gamma - k_1\dot{\delta} - k_1\omega \\[2ex] J_a\ddot{\varphi} = -M_0\sin 2\delta\sin\gamma - k_2\dot{\varphi}. \end{array}\right. \tag{2.4}$$

The Lagrange's equations (2.4) describe the motion of a rotational ellipsoid placed in general position in a homogeneous magnetic field, rotating with constant angular velocity, for every time instant. This is a non-homogeneous nonlinear autonomous system of differential equations of second order. The system is unsolvable by quadrature. We notice that the first two equations are independent of the third one. In addition, the latter does not contribute to solving the formulated problem as rotation around the axis $\overrightarrow{a}$ does not exert any technological effect.

3. Mathematical model

Let us consider the system composed by the first two equations, assuming that the current torque is much smaller than the synchronous one, which means we can neglect it. We obtain:

$$\left|\begin{array}{l} J\ddot{\gamma} = -M_0\cos^2\delta\sin 2\gamma - k_1\dot{\gamma} \\[2ex] J\ddot{\delta} = -M_0\sin 2\delta - k_1\dot{\delta} - k_1\omega. \end{array}\right. \tag{3.1}$$

The system (3.1) is unsolvable by quadrature, too. We seek a solution by means of contraction mapping principle ([1], [2], [7]) for it.

We denote by $M = \dfrac{M_0}{J} > 0, k = \dfrac{k_1}{J} > 0$ and obtain the system

$$\left\{\begin{array}{l} \ddot{\gamma} = -M\cos^2\delta\sin 2\gamma - k\dot{\gamma} \\ \ddot{\delta} = -M\sin 2\delta - k\dot{\delta} - k\omega. \end{array}\right. \tag{3.2}$$

where $\gamma = \gamma(t), \delta = \delta(t), t \in [0, \infty)$, with the corresponding initial conditions at $t = 0$.

Consider the second equation of the system (3.2).

If δ is a solution of

$$\ddot{\delta} = -M \sin 2\delta - k\dot{\delta} - k\omega \tag{3.3}$$

then $(\ddot{\delta}(s) + k\dot{\delta}(s))e^{ks} = -(M \sin 2\delta(s) + k\omega)e^{ks}$. After integrating along s from 0 to τ, we obtain: $\dot{\delta}(\tau) = (\omega + \dot{\delta}(0))e^{-k\tau} - \omega - Me^{-k\tau} \int_0^\tau e^{ks} \sin 2\delta(s)ds$ and integrating once again along τ from 0 to t, we obtain:

$$\delta(t) = \delta(0) + \frac{\omega + \dot{\delta}(0)}{k}(1 - e^{-kt}) - \omega t - M \int_0^t \int_0^\tau e^{-k(\tau-s)} \sin 2\delta(s)dsd\tau =$$
$$= \delta(0) + \frac{\omega + \dot{\delta}(0)}{k}(1 - e^{-kt}) - \omega t - M \int_0^t \left(\int_s^t e^{-k(\tau-s)}d\tau\right) \sin 2\delta(s)ds =$$
$$= \delta(0) + \frac{\omega + \dot{\delta}(0)}{k}(1 - e^{-kt}) - \omega t - \frac{M}{k} \int_0^t \left(1 - e^{-k(t-s)}\right) \sin 2\delta(s)ds,$$

which means that

$$\delta(t) = G(\delta)(t), \ \forall t \geq 0, \tag{3.4}$$

where the operator G is defined on a suitable subset $\mathbf{B}$ of the space of the functions continuous in $[0, \infty)$:

$$G(f)(t) = \delta(0) + \frac{\omega + \dot{\delta}(0)}{k}(1 - e^{-kt}) - \omega t - \frac{M}{k} \int_0^t (1 - e^{-k(t-s)}) \sin 2f(s)ds, t \geq 0. \tag{3.5}$$

If δ is a continuous solution of (3.4) then

$$\dot{\delta} = (\omega + \dot{\delta}(0))e^{-kt} - \omega - \frac{M}{k} \sin 2\delta(t) + \frac{M}{k} \frac{d}{dt}\left(e^{-kt} \int_0^t e^{ks} \sin 2\delta(s)ds\right) =$$
$$= (\omega + \dot{\delta}(0))e^{-kt} - \omega - M \int_0^t e^{-k(t-s)} \sin 2\delta(s)ds;$$
$$\ddot{\delta} = -k(\omega + \dot{\delta}(0))e^{-kt} + Mk \int_0^t e^{-k(t-s)} \sin 2\delta(s)ds - M \sin 2\delta(t) =$$
$$= -M \sin 2\delta(t) - k\dot{\delta} - k\omega.$$

In other words, δ is a twice-differentiable function satisfying (3.3).

By means of analogous transformations on the first of equations from (3.2) we reduce the system (3.2) to the following one:

$$\begin{cases} \gamma(t) = F_\delta(\gamma)(t), \ \forall t \geq 0 \\ \delta(t) = G(\delta)(t), \quad \forall t \geq 0 \end{cases}, \tag{3.6}$$

where G is defined as (3.5) and for any fixed function $f \in \mathbf{B}$ the operator F is defined on the same set $\mathbf{B}$ as follows: $F(g) = F_f(g)$, and for any $t \geq 0$:

$$F_f(g)(t) = \gamma(0) + \frac{\dot{\gamma}(0)}{k}(1 - e^{-kt}) - \frac{M}{k} \int_0^t (1 - e^{-k(t-s)}) \cos^2 f(s) \sin 2g(s)ds. \tag{3.7}$$

Remark 3.1. *One can try to use various kinds of schemes to find numerical approximations of the solution of the system* (3.1). *For example, one can seek the approximations with some methods such as Runge-Kutta methods (or Euler's method, or Newton's method) for the corresponding system:*

$$\begin{cases} \dot{x}(t) = F(x(t)), \quad 0 < t < T_0 \\ x(0) = x_0, \end{cases}$$

where $x(t) = (x_1(t), x_2(t), x_3(t), x_4(t))^T$,
$F(x) = (F_1(x), F_2(x), F_3(x), F_4(x))^T$; $x_1 = \delta, x_2 = \gamma, x_3 = \dot{\delta}, x_4 = \dot{\gamma}$;
$F_1 = x_3, F_2 = x_4, F_3 = -M sin 2x_1 - kx_3 - k\omega, F_4 = -M cos^2 x_1 sin 2x_2 - kx_4$.

But does the last nonlinear system have a solution and whether, if the system has a solution, it is only one?

We look for global solution of the system (3.6) (resp. of the system (3.1)).

In what follows we give a proof (by means of fixed point method) that there exists a unique solution.

Define the set $\mathbf{B} : \mathbf{B} = \left\{ h \in C([0, \infty)) : |h(t)| \le Ce^{\lambda t}, \ \forall t \ge 0 \right\}$ with constants $\lambda > m$, $m = \max\left\{ \frac{3M}{k}, \ \omega + \frac{M}{k} \right\} = \max\left\{ \frac{3M_0}{k_1}, \ \omega + \frac{M_0}{k_1} \right\}$,

and

$$C = |\delta(0)| + \frac{\left|\omega + \dot{\delta}(0)\right|}{k} + |\gamma(0)| + \frac{|\dot{\gamma}(0)|}{k} + \frac{1}{2}.$$

Norm in $\mathbf{B}$ is introduced as follows:

$$\|f\|_B = \sup\left\{ e^{-\lambda t} |f(t)| : \ t \ge 0 \right\}, \ f \in \mathbf{B},$$

and with the corresponding metrics: $d(f, \overline{f}) = \left\|f - \overline{f}\right\|_B$ $\left(f, \overline{f} \in \mathbf{B}\right)$ the set $\mathbf{B}$ becomes a complete metric space.

Define the product space $E = \mathbf{B} \times \mathbf{B}$ with a norm:

$$\|(g, f)\| = \|g\|_B + \|f\|_B \, .$$

With the corresponding metrics $d\left((g, f), (\overline{g}, \overline{f})\right) = \|g - \overline{g}\|_B + \left\|f - \overline{f}\right\|_B$, E becomes a Banach space.

Define on E the operator $T : T((g, f)) = (F_f(g), \ G(f)), \ (g, f) \in E$.

It has the following properties: $T((g, f)) \in E, \ \ \forall (g, f) \in E$.

Indeed, $G(f), F_f(g)$ are continuous functions in $[0, \infty)$;

$$e^{-\lambda t} |G(f)(t)| \le e^{-\lambda t} |a(t)| + \frac{M}{k} te^{-\lambda t} \le |\delta(0)| + \frac{\left|\omega + \dot{\delta}(0)\right|}{k} + \left(\omega + \frac{M}{k}\right) \cdot \frac{1}{\lambda e},$$

consequently $e^{-\lambda t} |G(f)(t)| \le |\delta(0)| + \frac{\left|\omega + \dot{\delta}(0)\right|}{k} + \frac{1}{e} < C$

$\left(a(t) = \delta(0) + \frac{\omega + \dot{\delta}(0)}{k}(1 - e^{-kt}) - \omega t, \ t \ge 0\right)$;

$e^{-\lambda t} |F_f(g)(t)| \le |\gamma(0)| + \frac{|\dot{\gamma}(0)|}{k} + \frac{M}{k} \cdot \frac{1}{\lambda e} \le |\gamma(0)| + \frac{|\dot{\gamma}(0)|}{k} + \frac{1}{e} < C.$

Moreover,

$$d\left(T((g,f)),T((\overline{g},\overline{f}))\right)=\left\|F_f(g)-F_{\overline{f}}(\overline{g})\right\|_B+\left\|G(f)-G(\overline{f})\right\|_B=$$
$$=\frac{M}{k}\sup_{t\geq 0}\left\{e^{-\lambda t}\int_0^t(1-e^{-k(t-s)})[\cos^2 f(s)\sin 2g(s)-\cos^2\overline{f}(s)\sin 2\overline{g}(s)]ds\right\}+$$
$$+\frac{M}{k}\sup_{t\geq 0}\left\{e^{-\lambda t}\int_0^t(1-e^{-k(t-s)})[\sin(2f(s))-\sin(2\overline{f}(s))]ds\right\}\leq$$
$$\leq\frac{M}{k}\sup_{t\geq 0}\left\{e^{-\lambda t}\int_0^t\left|2\sin(g(s)-\overline{g}(s))\cos^2(f(s))\cos(g(s)+\overline{g}(s))\right|ds\right\}+$$
$$+\frac{M}{k}\sup_{t\geq 0}\left\{e^{-\lambda t}\int_0^t\left|\sin(f(s)-\overline{f}(s))\sin(2\overline{g}(s))\sin(f(s)+\overline{f}(s))\right|ds\right\}+$$
$$+\frac{M}{k}\sup_{t\geq 0}\left\{e^{-\lambda t}\int_0^t(1-e^{-k(t-s)})\left|2\sin(f(s)-\overline{f}(s))\cos(f(s)+\overline{f}(s))\right|ds\right\}\leq$$
$$\leq\frac{2M}{k}\sup_{t\geq 0}\left\{e^{-\lambda t}\int_0^t\left|g(s)-\overline{g}(s)\right|ds\right\}+\frac{3M}{k}\sup_{t\geq 0}\left\{e^{-\lambda t}\int_0^t\left|f(s)-\overline{f}(s)\right|ds\right\},$$

from where we obtain:

$$d\left(T((g,f)),T((\overline{g},\overline{f}))\right)\leq$$
$$\leq\frac{M}{k}\left(2\left\|g-\overline{g}\right\|_B+3\left\|f-\overline{f}\right\|_B\right)\sup_{t\geq 0}\left\{e^{-\lambda t}\int_0^t e^{\lambda s}ds\right\}\leq$$
$$\leq\frac{M}{k\lambda}\left(2\left\|g-\overline{g}\right\|_B+3\left\|f-\overline{f}\right\|_B\right).$$

Therefore $d\left(T((g,f)),T((\overline{g},\overline{f}))\right)\leq\beta d\left((g,f),(\overline{g},\overline{f})\right)$,

i. e. T is a contraction operator on E with Lipschitz constant $\beta=\dfrac{3M}{k\lambda}<1$.

In view of contraction mapping principle T has a unique fixed point on E, which allows us making the following conclusion:

4. Conclusion

The system (3.1) has a unique solution (γ,δ) the coordinate functions of which belong to the set $\mathbf{B}$. The solution can be obtained as the limit (in $\mathbf{B}\times\mathbf{B}$) of the sequence of successive approximations $\{(g_n,f_n)\}_{n=0}^{\infty}$:

$$g_0(t)=\gamma(0)+\frac{\dot{\gamma}(0)}{k}\left(1-e^{-kt}\right);$$
$$f_0(t)=\delta(0)+\frac{\omega+\dot{\delta}(0)}{k}\left(1-e^{-kt}\right),\ \forall t\geq 0\quad((g_0,f_0)\in E);$$
$$f_n(t)=f_0(t)-\omega t-\frac{M}{k}\int_0^t\left(1-e^{-k(t-s)}\right)\sin[2f_{n-1}(s)]ds, n=1,2,...$$
$$g_n(t)=g_0(t)-\frac{M}{k}\int_0^t\left(1-e^{-k(t-s)}\right)\cos^2[f_{n-1}(s)]\sin[2g_{n-1}(s)]ds, n=1,2,...$$

As we have already shown, the limit of $\{(g_n,f_n)\}_{n=0}^{\infty}$ in E is the unique fixed point of the operator T. In particular, the limit of $\{f_n\}_{n=0}^{\infty}$ in $\mathbf{B}$ is the function δ and therefore, the limit of $\{g_n\}_{n=0}^{\infty}$ in $\mathbf{B}$ is the function γ, that is the unique fixed point of T is the ordered pair (γ, δ), which is the unique solution (in E) of the system $(5'')$, and respectively – of the system (3.1).

References

[1] Angelov, V.G., *Fixed point theorems in uniform spaces and applications*, Czechoslovak Math. Journal, **37**(112)(1987), 19-33.

[2] Angelov, V.G., *Fixed points of densifying mappings in locally convex spaces and applications*, J. Inst. Math. Computer Sci., Calcutta, **2**(1989), 22-39.

[3] Kostov, K., *Rotational ferromagnetic ellipsoid in rotating magnetic field* (Bulgarian), Annual of UMG "St. I. Rilski", **47**, Part III (2004), Sofia, 105-107.

[4] Kostov, K., *Magnetic Field, Parameters and Electromagnetic Calculation of a Vortex Machine* (Bulgarian), PhD thesis, UMG "St I. Rilski", Sofia, 2007, p. 185.

[5] Kostov, K., Pulev, S., Trichkov, K., *Analysis of the motion of a rotational ferromagnetic ellipsoid placed in rotating magnetic field* (Bulgarian), in Annual of UMG "St. I. Rilski", **48**, Part III (2005), Sofia, 79-84.

[6] Pisarev, A., Paraskov, Ts., Bachvarov, S., *Lectures on Theoretical Mechanics* (Bulgarian), Sofia, Tekhnika, 1988, p. 501.

[7] Rus, I. A., *Metrical Fixed Point Theorems*, Cluj-Napoca, 1979.

Ljubomir Georgiev
UMG "St. I. Rilski",
Department of Mathematics
1700 Sofia,
Bulgaria
e-mail: lubo_62@mgu.bg

Konstantin Kostov
UMG "St. I. Rilski",
Department of Electrotechiques
1700 Sofia,
Bulgaria
e-mail: costovs@yahoo.com

Stud. Univ. Babeş-Bolyai Math. 56(2011), No. 3, 113–125

On the nonlocal initial value problem for first order differential systems

Octavia Nica and Radu Precup

Abstract. The aim of the is to study the existence of solutions of initial value problems for nonlinear first order differential systems with nonlocal conditions. The proof will rely on the Perov, Schauder and Leray-Schauder fixed point principles which are applied to a nonlinear integral operator splitted into two parts, one of Fredholm type for the subinterval containing the points involved by the nonlocal condition, and an another one of Volterra type for the rest of the interval. The novelty in this paper is that this approach is combined with the technique that uses convergent to zero matrices and vector norms.

Mathematics Subject Classification (2010): 34A34, 34A12, 45G10.

Keywords: Nonlinear differential system, nonlocal initial condition, fixed point theorem, vector norm, matrix convergent to zero.

1. Introduction

In this paper we deal with the nonlocal initial value problem for the first order differential system

$$\begin{cases} x'(t) = f(t, x(t), y(t)) & \\ y'(t) = g(t, x(t), y(t)) & \text{(a.e. on } [0,1]) \\ x(0) + \sum_{k=1}^{m} a_k x(t_k) = 0 & \\ y(0) + \sum_{k=1}^{m} \widetilde{a}_k y(t_k) = 0. & \end{cases} \tag{1.1}$$

Here $f, g : [0,1] \times \mathbf{R}^2 \to \mathbf{R}$ are Carathéodory functions, t_k are given points with $0 \leq t_1 \leq t_2 \leq \ ... \ \leq t_m < 1$ and a_k, $\widetilde{a}_k$ are real numbers with

$$1 + \sum_{k=1}^{m} a_k \neq 0 \text{ and } 1 + \sum_{k=1}^{m} \widetilde{a}_k \neq 0.$$

Notice that the nonhomogeneous nonlocal initial conditions

$$\begin{cases} x(0) + \sum_{k=1}^{m} a_k x(t_k) = x_0 \\ y(0) + \sum_{k=1}^{m} \widetilde{a}_k y(t_k) = y_0 \end{cases}$$

can always be reduced to the homogeneous ones (with $x_0 = y_0 = 0$) by the change of variables $x_1(t) := x(t) - a\ x_0$ and $y_2(t) := y(t) - \widetilde{a}\ y_0$, where

$$a = \left(1 + \sum_{k=1}^{m} a_k\right)^{-1} \text{ and } \widetilde{a} = \left(1 + \sum_{k=1}^{m} \widetilde{a}_k\right)^{-1}.$$

According to [2], Problem (1.1) is equivalent to the following integral system in $C[0,1]^2$:

$$\begin{cases} x(t) = -a\sum_{k=1}^{m} a_k \int_0^{t_k} f(s, x(s), y(s))\, ds + \int_0^t f(s, x(s), y(s))\, ds \\ y(t) = -\widetilde{a}\sum_{k=1}^{m} \widetilde{a}_k \int_0^{t_k} g(s, x(s), y(s))\, ds + \int_0^t g(s, x(s), y(s))\, ds. \end{cases}$$

This can be viewed as a fixed point problem in $C[0,1]^2$ for the completely continuous operator $T = (T_1, T_2)$, $T : C[0,1]^2 \to C[0,1]^2$, where T_1 and T_2 are given by

$$T_1(x,y)(t) = -a\sum_{k=1}^{m} a_k \int_0^{t_k} f(s, x(s), y(s))\, ds + \int_0^t f(s, x(s), y(s))\, ds,$$

$$T_2(x,y)(t) = -\widetilde{a}\sum_{k=1}^{m} \widetilde{a}_k \int_0^{t_k} g(s, x(s), y(s))\, ds + \int_0^t g(s, x(s), y(s))\, ds.$$

Operators T_1 and T_2 appear as sums of two integral operators, one of Fredholm type, whose values depend only on the restrictions of functions to $[0, t_m]$, and the other one, a Volterra type operator depending on the restrictions to $[t_m, 1]$, as this was pointed out in [3]. Thus, T_1 can be rewritten as $T_1 = T_{F_1} + T_{V_1}$, where

$$T_{F_1}(x,y)(t) = \begin{cases} -a\sum_{k=1}^{m} a_k \int_0^{t_k} f(s, x(s), y(s))\, ds + \int_0^t f(s, x(s), y(s))\, ds, \\ \qquad \text{if } t < t_m \\ -a\sum_{k=1}^{m} a_k \int_0^{t_k} f(s, x(s), y(s))\, ds + \int_0^{t_m} f(s, x(s), y(s))\, ds, \\ \qquad \text{if } t \geq t_m \end{cases}$$

and

$$T_{V_1}(x,y)(t)=\begin{cases}0, & \text{if } \ t<t_m\\ \displaystyle\int_{t_m}^{t} f\left(s,x\left(s\right),y(s)\right)ds, & \text{if } \ t\geq t_m.\end{cases}$$

Similarly, $T_2 = T_{F_2} + T_{V_2}$, where

$$T_{F_2}(x,y)(t)=\begin{cases}-\widetilde{a}\displaystyle\sum_{k=1}^{m}\widetilde{a}_k\int_0^{t_k} g\left(s,x\left(s\right),y(s)\right)ds+\int_0^{t} g\left(s,x\left(s\right),y(s)\right)ds, \\ \qquad \text{if } t<t_m\\ -\widetilde{a}\displaystyle\sum_{k=1}^{m}\widetilde{a}_k\int_0^{t_k} g\left(s,x\left(s\right),y(s)\right)ds+\int_0^{t_m} g\left(s,x\left(s\right),y(s)\right)ds, \\ \qquad \text{if } t\geq t_m\end{cases}$$

and

$$T_{V_2}(x,y)(t)=\begin{cases}0, & \text{if } \ t<t_m\\ \displaystyle\int_{t_m}^{t} g\left(s,x\left(s\right),y(s)\right)ds, & \text{if } \ t\geq t_m.\end{cases}$$

This allows us to split the growth condition on the nonlinear terms $f(t,x,y)$ and $g(t,x,y)$ into two parts, one for $t\in[0,t_m]$ and another one for $t\in[t_m,1]$, in a such way that one reobtains the classical growth when $t_m=0$, that is for the local initial condition $x(0)=0$. In what follows, the notation $|x|_{C[a,b]}$ stands for the max-norm on $C[a,b]$

$$|x|_{C[a,b]}=\max\nolimits_{t\in[a,b]}|x(t)|,$$

while $\|x\|_{C[a,b]}$ denotes the Bielecki norm

$$\|x\|_{C[a,b]}=\left|x\left(t\right)e^{-\theta(t-a)}\right|_{C[a,b]}$$

for some suitable $\theta>0$.

Nonlocal initial value problems were extensively discussed in the literature by different methods (see for example [2], [3], [5], [6], [8], [10]). The results in the present paper extend to systems those established for equations in [3]. The method could be adapted to treat systems of evolution equations as this was done for equations in [4].

In the next section three different fixed point principles will be used in order to prove the existence of solutions for the semilinear problem, namely the fixed point theorems of Perov, Schauder and Leray-Schauder (see [10]). In all three cases a key role will be played by the so called convergent to zero matrices. A square matrix M with nonnegative elements is said to be *convergent to zero* if

$$M^k\to 0 \quad \text{as} \quad k\to\infty.$$

It is known that the property of being convergent to zero is equivalent to each of the following three conditions (for details see [10], [11]):

(a) $I - M$ is nonsingular and $(I - M)^{-1} = I + M + M^2 + ...$ (where I stands for the unit matrix of the same order as M);

(b) the eigenvalues of M are located inside the unit disc of the complex plane;

(c) $I - M$ is nonsingular and $(I - M)^{-1}$ has nonnegative elements.

The following lemma, whose proof is immediate from characterization (b) of convergent to zero matrices, will be used in the sequel:

Lemma 1.1. *If A is a square matrix that converges to zero and the elements of an other square matrix B are small enough, then $A + B$ also converges to zero.*

We finish this introductory section by recalling (see [1], [10]) three fundamental results which will be used in the next sections. Let X be a nonempty set. By a *vector-valued metric* on X we mean a mapping $d : X \times X \to \mathbf{R}^n_+$ such that

(i) $d(u, v) \geq 0$ for all $u, v \in X$ and if $d(u, v) = 0$ then $u = v$;
(ii) $d(u, v) = d(v, u)$ for all $u, v \in X$;
(iii) $d(u, v) \leq d(u, w) + d(w, v)$ for all $u, v, w \in X$.

Here, if $x, y \in \mathbf{R}^n$, $x = (x_1, x_2, ..., x_n)$, $y = (y_1, y_2, ..., y_n)$, by $x \leq y$ we mean $x_i \leq y_i$ for $i = 1, 2, ..., n$. We call the pair (X, d) a *generalized metric space.* For such a space convergence and completeness are similar to those in usual metric spaces.

An operator $T : X \to X$ is said to be *contractive* (with respect to the vector-valued metric d on X) if there exists a convergent to zero matrix M such that

$$d(T(u), T(v)) \leq M d(u, v) \quad \text{for all } u, v \in X.$$

Theorem 1.2 (Perov). *Let (X, d) be a complete generalized metric space and $T : X \to X$ a contractive operator with Lipschitz matrix M. Then T has a unique fixed point u^* and for each $u_0 \in X$ we have*

$$d(T^k(u_0), u^*) \leq M^k (I - M)^{-1} d(u_0, T(u_0)) \ \textit{for all } k \in \mathbf{N}.$$

Theorem 1.3 (Schauder). *Let X be a Banach space, $D \subset X$ a nonempty closed bounded convex set and $T : D \to D$ a completely continuous operator (i.e., T is continuous and $T(D)$ is relatively compact). Then T has at least one fixed point.*

Theorem 1.4 (Leray–Schauder). *Let $(X, ||\,.\,||)$ be a Banach space, $R > 0$, $\overline{B}_R(0; X) = \{u \in X : \|u\| \leq R\}$ and $T : \overline{B}_R(0; X) \to X$ a completely continuous operator. If $||u|| < R$ for every solution u of the equation $u = \lambda T(u)$ and any $\lambda \in (0, 1)$, then T has at least one fixed point.*

Throughout the paper we shall assume that the following conditions are satisfied:

(H1) $1 + \sum_{k=1}^{m} a_k \neq 0$ and $1 + \sum_{k=1}^{m} \widetilde{a}_k \neq 0$.

(H2) $f, g : [0,1]\times\mathbf{R}^2 \to \mathbf{R}$ is such that $f(., x, y), g(., x, y)$ are measurable for each $(x, y) \in \mathbf{R}^2$ and $f(t, ., .), g(t, ., .)$ are continuous for almost all $t \in [0, 1]$.

2. Nonlinearities with the Lipschitz property. Application of Perov's fixed point theorem

Here we show that the existence of solutions of problem (1.1) follows from Perov's fixed point theorem in case that f, g satisfy Lipschitz conditions in x and y :

$$|f(t,x,y) - f(t,\overline{x},\overline{y})| \leq \begin{cases} b_1\,|x-\overline{x}| + \widetilde{b}_1\,|y-\overline{y}| & \text{if } t\in[0,t_m] \\ c_1\,|x-\overline{x}| + \widetilde{c}_1\,|y-\overline{y}| & \text{if } t\in[t_m,1]\,, \end{cases} \tag{2.1}$$

$$|g(t,x,y) - g(t,\overline{x},\overline{y})| \leq \begin{cases} B_1\,|x-\overline{x}| + \widetilde{B}_1\,|y-\overline{y}| & \text{if } t\in[0,t_m] \\ C_1\,|x-\overline{x}| + \widetilde{C}_1\,|y-\overline{y}| & \text{if } t\in[t_m,1] \end{cases} \tag{2.2}$$

for all $x, y, \overline{x}, \overline{y} \in \mathbf{R}$.

Theorem 2.1. *If f, g satisfy the Lipschitz conditions* (2.1), (2.2) *and the matrix*

$$M_0 := \begin{bmatrix} b_1 t_m A_1 & \widetilde{b}_1 t_m A_1 \\ B_1 t_m A_2 & \widetilde{B}_1 t_m A_2 \end{bmatrix} \tag{2.3}$$

converges to zero, then problem (1.1) *has a unique solution.*

Proof. We shall apply Perov's fixed point theorem in $C\,[0,1]^2$ endowed with the vector norm $\|.\|$ defined by

$$\|u\| = (\|x\|\,, \|y\|)$$

for $u = (x, y)$, where for $z \in C\,[0, 1]$, we let

$$\|z\| = \max\left\{|z|_{C[0,t_m]}\,, \|z\|_{C[t_m,1]}\right\}.$$

We have to prove that T is contractive, more exactly that

$$\|T(u) - T(\overline{u})\| \leq M_\theta\,\|u-\overline{u}\|$$

for all $u = (x, y), \overline{u} = (\overline{x}, \overline{y}) \in C\,[0,1]^2$ and some matrix M_θ converging to zero. To this end, let $u = (x, y), \overline{u} = (\overline{x}, \overline{y})$ be any elements of $C\,[0,1]^2$. For $t \in [0, t_m]$, if we denote

$$A_1 := 1 + |a| \sum_{k=1}^{m} |a_k|\,,$$

we have

$$\begin{aligned}
&\left|T_1(x,y)(t)-T_1(\overline{x},\overline{y})(t)\right| \\
= &\left|-a\sum_{k=1}^{m}a_k\int_0^{t_k} f\left(s,x\left(s\right),y(s)\right)ds+\int_0^t f\left(s,x\left(s\right),y(s)\right)ds\right. \\
&\left.+a\sum_{k=1}^{m}a_k\int_0^{t_k} f\left(s,\overline{x}\left(s\right),\overline{y}(s)\right)ds-\int_0^t f\left(s,\overline{x}\left(s\right),\overline{y}(s)\right)ds\right| \\
\leq\ & A_1\int_0^{t_m}\left|f\left(s,x\left(s\right),y(s)\right)-f\left(s,\overline{x}\left(s\right),\overline{y}(s)\right)\right|ds \\
\leq\ & b_1t_mA_1\left|x-\overline{x}\right|_{C[0,t_m]}+\widetilde{b}_1t_mA_1\left|y-\overline{y}\right|_{C[0,t_m]}.
\end{aligned}$$

Taking the supremum, we obtain that

$$\left|T_1(x,y)-T_1(\overline{x},\overline{y})\right|_{C[0,t_m]}\leq b_1t_mA_1\left|x-\overline{x}\right|_{C[0,t_m]}+\widetilde{b}_1t_mA_1\left|y-\overline{y}\right|_{C[0,t_m]}. \tag{2.4}$$

For $t\in[t_m,1]$ and any $\theta>0$, we have

$$\begin{aligned}
&\left|T_1(x,y)(t)-T_1(\overline{x},\overline{y})(t)\right| \\
= &\left|-a\sum_{k=1}^{m}a_k\int_0^{t_k} f\left(s,x\left(s\right),y(s)\right)ds+\int_0^t f\left(s,x\left(s\right),y(s)\right)ds\right. \\
&\left.+a\sum_{k=1}^{m}a_k\int_0^{t_k} f\left(s,\overline{x}\left(s\right),\overline{y}(s)\right)ds-\int_0^t f\left(s,\overline{x}\left(s\right),\overline{y}(s)\right)ds\right| \\
\leq\ & b_1t_mA_1\left|x-\overline{x}\right|_{C[0,t_m]}+\widetilde{b}_1t_mA_1\left|y-\overline{y}\right|_{C[0,t_m]} \\
&+\int_{t_m}^t\left(c_1\left|x(s)x-\overline{x}(s)\right|+\widetilde{c}_1\left|y(s)-\overline{y}(s)\right|\ \right)ds.
\end{aligned}$$

The last integral can be further estimated as follows:

$$\begin{aligned}
&\int_{t_m}^t\left(c_1\left|x(s)x-\overline{x}(s)\right|+\widetilde{c}_1\left|y(s)-\overline{y}(s)\right|\ \right)ds \\
= &\ c_1\int_{t_m}^t\left|x(s)-\overline{x}(s)\right|\cdot e^{-\theta(s-t_m)}\cdot e^{\theta(s-t_m)}ds \\
&+\widetilde{c}_1\int_{t_m}^t\left|y(s)-\overline{y}(s)\right|\cdot e^{-\theta(s-t_m)}\cdot e^{\theta(s-t_m)}ds \\
\leq\ & \frac{c_1}{\theta}e^{\theta(t-t_m)}\left\|x-\overline{x}\right\|_{C[t_m,1]}+\frac{\widetilde{c}_1}{\theta}e^{\theta(t-t_m)}\left\|y-\overline{y}\right\|_{C[t_m,1]}.
\end{aligned}$$

Thus

$$\begin{aligned}
&\left|T_1(x,y)(t)-T_1(\overline{x},\overline{y})(t)\right| \\
\leq\ & b_1t_mA_1\left|x-\overline{x}\right|_{C[0,t_m]}+\widetilde{b}_1t_mA_1\left|y-\overline{y}\right|_{C[0,t_m]} \\
&+\frac{c_1}{\theta}e^{\theta(t-t_m)}\left\|x-\overline{x}\right\|_{C[t_m,1]}+\frac{\widetilde{c}_1}{\theta}e^{\theta(t-t_m)}\left\|y-\overline{y}\right\|_{C[t_m,1]}.
\end{aligned}$$

Dividing by $e^{\theta(t-t_m)}$ and taking the supremum when $t \in [t_m, 1]$, we obtain

$$\begin{aligned} & \|T_1(x,y) - T_1(\overline{x},\overline{y})\|_{C[t_m,1]} \quad (2.5) \\ \leq\ & b_1 t_m A_1 \left|x - \overline{x}\right|_{C[0,t_m]} + \widetilde{b}_1 t_m A_1 \left|y - \overline{y}\right|_{C[0,t_m]} \\ \leq\ & \frac{c_1}{\theta} \left\|x - \overline{x}\right\|_{C[t_m,1]} + \frac{\widetilde{c}_1}{\theta} \left\|y - \overline{y}\right\|_{C[t_m,1]} . \end{aligned}$$

Now (2.4) and (2.5) imply that

$$\|T_1(x,y) - T_1(\overline{x},\overline{y})\| \leq \left(b_1 t_m A_1 + \frac{c_1}{\theta}\right) \|x - \overline{x}\| + \left(\widetilde{b}_1 t_m A_1 + \frac{\widetilde{c}_1}{\theta}\right) \|y - \overline{y}\| . \tag{2.6}$$

Similarly

$$\|T_2(x,y) - T_2(\overline{x},\overline{y})\| \leq \left(B_1 t_m A_2 + \frac{C_1}{\theta}\right) \|x - \overline{x}\| + \left(\widetilde{B}_1 t_m A_2 + \frac{\widetilde{C}_1}{\theta}\right) \|y - \overline{y}\| , \tag{2.7}$$

where

$$A_2 = 1 + |\widetilde{a}| \sum_{k=1}^{m} |\widetilde{a}_k| .$$

Using the vector norm we can put both inequalities (2.6), (2.7) under the vector inequality

$$\|T(u) - T(\overline{u})\| \leq M_\theta \|u - \overline{u}\| ,$$

where

$$M_\theta = \begin{bmatrix} b_1 t_m A_1 + \frac{c_1}{\theta} & \widetilde{b}_1 t_m A_1 + \frac{\widetilde{c}_1}{\theta} \\ B_1 t_m A_2 + \frac{C_1}{\theta} & \widetilde{B}_1 t_m A_2 + \frac{\widetilde{C}_1}{\theta} \end{bmatrix} . \tag{2.8}$$

Clearly matrix M_θ can be represented as $M_\theta = M_0 + M_1$, where

$$M_1 = \begin{bmatrix} \frac{c_1}{\theta} & \frac{\widetilde{c}_1}{\theta} \\ \frac{C_1}{\theta} & \frac{\widetilde{C}_1}{\theta} \end{bmatrix} .$$

Since M_0 is assumed to be convergent to zero, from Lemma 1.1 we have that M_θ also converges to zero for large enough $\theta > 0$. The result follows now from Perov's fixed point theorem. □

3. Nonlinearities with growth at most linear. Application of Schauder's fixed point theorem

Here we show that the existence of solutions of problem (1.1) follows from Schauder's fixed point theorem in case that f, g satisfy instead of the Lipschitz condition the more relaxed condition of at most linear growth:

$$|f(t,x,y)| \leq \begin{cases} b_1 |x| + \widetilde{b}_1 |y| + d_1 & \text{if } t \in [0, t_m] \\ c_1 |x| + \widetilde{c}_1 |y| + d_2 & \text{if } t \in [t_m, 1] , \end{cases} \tag{3.1}$$

$$|g(t,x,y)| \leq \begin{cases} B_1 |x| + \widetilde{B}_1 |y| + D_1 & \text{if } t \in [0,t_m] \\ C_1 |x| + \widetilde{C}_1 |y| + D_2 & \text{if } t \in [t_m,1]. \end{cases} \tag{3.2}$$

Theorem 3.1. *If* f, g *satisfy* (3.1), (3.2) *and the matrix* (2.3) *converges to zero, then problem* (1.1) *has at least one solution.*

Proof. In order to apply Schauder's fixed point theorem, we look for a nonempty, bounded, closed and convex subset B of $C[0,1]^2$ so that $T(B) \subset B$. Let x, y be any elements of $C[0,1]$.
For $t \in [0,t_m]$, we have

$$\begin{aligned} |T_1(x,y)(t)| &= \left| -a\sum_{k=1}^{m} a_k \int_0^{t_k} f(s,x(s),y(s))\,ds + \int_0^t f(s,x(s),y(s))\,ds \right| \\ &\leq A_1 \int_0^{t_m} |f(s,x(s),y(s))|\,ds \\ &\leq b_1 t_m A_1 |x|_{C[0,t_m]} + \widetilde{b}_1 t_m A_1 |y|_{C[0,t_m]} + d_1 t_m A_1. \end{aligned}$$

Taking the supremum, we obtain that

$$|T_1(x,y)|_{C[0,t_m]} \leq b_1 t_m A_1 |x|_{C[0,t_m]} + \widetilde{b}_1 t_m A_1 |y|_{C[0,t_m]}. \tag{3.3}$$

For $t \in [t_m,1]$ and any $\theta > 0$, we have

$$\begin{aligned} |T_1(x,y)(t)| &= \left| -a\sum_{k=1}^{m} a_k \int_0^{t_k} f(s,x(s),y(s))\,ds + \int_0^t f(s,x(s),y(s))\,ds \right| \\ &\leq b_1 t_m A_1 |x|_{C[0,t_m]} + \widetilde{b}_1 t_m A_1 |y|_{C[0,t_m]} + d_1 t_m A_1 \\ &+ \int_{t_m}^t (c_1 |x(s)| + \widetilde{c}_1 |y(s)| + d_2)\,ds \\ &\leq b_1 t_m A_1 |x|_{C[0,t_m]} + \widetilde{b}_1 t_m A_1 |y|_{C[0,t_m]} + d_1 t_m A_1 + (1-t_m) d_2 \\ &+ c_1 \int_{t_m}^t |x(s)| \cdot e^{-\theta(s-t_m)} \cdot e^{\theta(s-t_m)} ds \\ &+ \widetilde{c}_1 \int_{t_m}^t |y(s)| \cdot e^{-\theta(s-t_m)} \cdot e^{\theta(s-t_m)} ds \\ &\leq b_1 t_m A_1 |x|_{C[0,t_m]} + \widetilde{b}_1 t_m A_1 |y|_{C[0,t_m]} + c_0 \\ &+ \frac{c_1}{\theta} e^{\theta(t-t_m)} \|x\|_{C[t_m,1]} + \frac{\widetilde{c}_1}{\theta} e^{\theta(t-t_m)} \|y\|_{C[t_m,1]}, \end{aligned}$$

where $c_0 = d_1 t_m A_1 + (1-t_m) d_2$. Dividing by $e^{\theta(t-t_m)}$ and taking the supremum, it follows that

$$\begin{aligned} \|T_1(x,y)\|_{C[t_m,1]} &\leq b_1 t_m A_1 |x|_{C[0,t_m]} + \widetilde{b}_1 t_m A_1 |y|_{C[0,t_m]} \\ &\quad + \frac{c_1}{\theta} \|x\|_{C[t_m,1]} + \frac{\widetilde{c}_1}{\theta} \|y\|_{C[t_m,1]} + c_0. \end{aligned} \tag{3.4}$$

Clearly (3.3), (3.4) give

$$\|T_1(x,y)\| \le \left(b_1 t_m A_1 + \frac{c_1}{\theta}\right)\|x\| + \left(\widetilde{b}_1 t_m A_1 + \frac{\widetilde{c}_1}{\theta}\right)\|y\| + \widetilde{c}_0, \tag{3.5}$$

where $\widetilde{c}_0 = \max\{d_1 t_m A_1, c_0\}$. Similarly

$$\|T_2(x,y)\| \le \left(B_1 t_m A_2 + \frac{C_1}{\theta}\right)\|x\| + \left(\widetilde{B}_1 t_m A_2 + \frac{\widetilde{C}_1}{\theta}\right)\|y\| + \widetilde{C}_0, \tag{3.6}$$

with $\widetilde{C}_0 = \max\{D_1 t_m A_2, C_0\}$. Now (3.5), (3.6) can be put together as

$$\begin{bmatrix} \|T_1(x,y)\| \\ \|T_2(x,y)\| \end{bmatrix} \le M_\theta \begin{bmatrix} \|x\| \\ \|y\| \end{bmatrix} + \begin{bmatrix} \widetilde{c}_0 \\ \widetilde{C}_0 \end{bmatrix},$$

where matrix M_θ is given by (2.8) and converges to zero for large enough $\theta > 0$. Next we look for two positive numbers R_1, R_2 such that if $\|x\| \le R_1, \|y\| \le R_2$, then $\|T_1(x,y)\| \le R_1$, $\|T_2(x,y)\| \le R_2$. To this end it is sufficient that

$$\begin{cases} \left(b_1 t_m A_1 + \frac{c_1}{\theta}\right) R_1 + \left(\widetilde{b}_1 t_m A_1 + \frac{\widetilde{c}_1}{\theta}\right) R_2 + \widetilde{c}_0 \le R_1 \\ \left(B_1 t_m A_2 + \frac{C_1}{\theta}\right) R_1 + \left(\widetilde{B}_1 t_m A_2 + \frac{\widetilde{C}_1}{\theta}\right) R_2 + \widetilde{C}_0 \le R_2, \end{cases} \tag{3.7}$$

or equivalently

$$M_\theta \begin{bmatrix} R_1 \\ R_2 \end{bmatrix} + \begin{bmatrix} \widetilde{c}_0 \\ \widetilde{C}_0 \end{bmatrix} \le \begin{bmatrix} R_1 \\ R_2 \end{bmatrix},$$

whence

$$\begin{bmatrix} R_1 \\ R_2 \end{bmatrix} \ge (I - M_\theta)^{-1} \begin{bmatrix} \widetilde{c}_0 \\ \widetilde{C}_0 \end{bmatrix}.$$

Notice that $I - M_\theta$ is invertible and its inverse $(I - M_\theta)^{-1}$ has nonnegative elements since M_θ converges to zero. Thus, if

$$B = \left\{(x,y) \in C[0,1]^2 : \|x\| \le R_1, \|y\| \le R_2\right\},$$

then $T(B) \subset B$ and Schauder's fixed point theorem can be applied. □

4. More general nonlinearities. Application of the Leray-Schauder principle

We now consider that nonlinearities f, g satisfy more general growth conditions, namely:

$$|f(t,u)| \le \begin{cases} \omega_1(t, |u|_e) & \text{if } t \in [0, t_m] \\ \alpha(t)\beta_1(|u|_e), & \text{if } t \in [t_m, 1], \end{cases} \tag{4.1}$$

$$|g(t,u)| \le \begin{cases} \omega_2(t, |u|_e) & \text{if } \ t \in [0, t_m] \\ \alpha(t)\beta_2(|u|_e) & \text{if } \ t \in [t_m, 1], \end{cases} \tag{4.2}$$

for all $u=(x,y)\in\mathbf{R}^2$, where by $|u|_e$ we mean the Euclidean norm in $\mathbf{R}^2$. Here ω_1,ω_2 are Carathéodory functions on $[0,t_m]\times\mathbf{R}_+$, nondecreasing in their second argument, $\alpha\in L^1[t_m,1]$, while $\beta_1,\beta_2:\mathbf{R}_+\to\mathbf{R}_+$ are nondecreasing and $1/\beta_1, 1/\beta_2\in L^1_{loc}(\mathbf{R}_+)$.

Theorem 4.1. *Assume that conditions* (4.1), (4.2) *hold. In addition assume that there exists a positive number* R_0 *such that for* $\rho=(\rho_1,\rho_2)\in(0,\infty)^2$

$$\begin{cases} \frac{1}{\rho_1}\int_0^{t_m}\omega_1(t,|\rho|_e)dt\geq\dfrac{1}{A_1} \\ \frac{1}{\rho_2}\int_0^{t_m}\omega_2(t,|\rho|_e)dt\geq\dfrac{1}{A_2}\end{cases} \quad \textit{implies} \quad |\rho|_e\leq R_0 \tag{4.3}$$

and

$$\int_{R^*}^{\infty}\frac{d\tau}{\beta_1(\tau)+\beta_2(\tau)}>\int_{t_m}^1\alpha(s)ds, \tag{4.4}$$

where $R^*=\left[\left(A_1\int_0^{t_m}\omega_1(t,R_0)dt\right)^2+\left(A_2\int_0^{t_m}\omega_2(t,R_0)dt\right)^2\right]^{1/2}$.
Then problem (1.1) *has at least one solution.*

Proof. The result will follow from the Leray-Schauder fixed point theorem once we have proved the boundedness of the set of all solutions to equations $u=\lambda T(u)$, for $\lambda\in[0,1]$. Let $u=(x,y)$ be such a solution. Then, for $t\in[0,t_m]$, we have

$$\begin{aligned} |x(t)| &= |\lambda T_1(x,y)(t)| \\ &= \lambda\left|-a\sum_{k=1}^m a_k\int_0^{t_k}f(s,x(s),y(s))\,ds+\int_0^t f(s,x(s),y(s))\,ds\right| \\ &\leq \left(1+|a|\sum_{k=1}^m|a_k|\right)\int_0^{t_m}|f(s,x(s),y(s))|\,ds \\ &= A_1\int_0^{t_m}|f(s,u(s))|\,ds \\ &\leq A_1\int_0^{t_m}\omega_1(s,|u(s)|_e)ds. \end{aligned} \tag{4.5}$$

Similarly

$$|y(t)|\leq A_2\int_0^{t_m}\omega_2(s,|u(s)|_e)ds. \tag{4.6}$$

Let $\rho_1=|x|_{C[0,t_m]}$, $\rho_2=|y|_{C[0,t_m]}$. Then from (4.5), (4.6), we deduce

$$\begin{cases}\rho_1\leq A_1\int_0^{t_m}\omega_1(t,|\rho|_e)dt \\ \rho_1\leq A_1\int_0^{t_m}\omega_1(t,|\rho|_e)dt.\end{cases}$$

This by (4.3) guarantees

$$|\rho|_e \leq R_0. \tag{4.7}$$

Next we let $t \in [t_m, 1]$. Then

$$\begin{aligned} |x(t)| &= |\lambda T_1(x,y)(t)| \\ &= \lambda \left| -a \sum_{k=1}^{m} a_k \int_0^{t_k} f(s, x(s), y(s))\, ds + \int_0^t f(s, x(s), y(s))\, ds \right| \\ &\leq A_1 \int_0^{t_m} \omega_1(t, R_0) dt + \int_{t_m}^t |f(s, x(s), y(s))|\, ds \\ &\leq A_1 \int_0^{t_m} \omega_1(t, R_0) dt + \int_{t_m}^t \alpha(s) \beta_1(|u(s)|_e) ds \\ &= : \phi_1(t) \end{aligned}$$

and similarly

$$\begin{aligned} |y(t)| &\leq A_1 \int_0^{t_m} \omega_2(t, R_0) dt + \int_{t_m}^t \alpha(s) \beta_2(|u(s)|_e) ds \\ &= : \phi_2(t). \end{aligned}$$

Denote $\psi(t) := \left(\phi_1^2(t) + \phi_2^2(t)\right)^{1/2}$. Then

$$\begin{cases} \phi_1'(t) = \alpha(t)\beta_1(|u(t)|_e) \leq \alpha(t)\beta_1(\psi(t)) \\ \phi_2'(t) = \alpha(t)\beta_2(|u(t)|_e) \leq \alpha(t)\beta_2(\psi(t))). \end{cases} \tag{4.8}$$

Consequently

$$\begin{aligned} \psi'(t) &= \frac{\phi_1(t)\phi_1'(t) + \phi_2(t)\phi_2'(t)}{\psi(t)} \\ &\leq \alpha(t) \cdot \frac{\phi_1(t)}{\psi(t)} \cdot \beta_1(\psi(t)) + \alpha(t) \cdot \frac{\phi_2(t)}{\psi(t)} \cdot \beta_2(\psi(t)) \\ &\leq \alpha(t) \left[\beta_1(\psi(t)) + \beta_2(\psi(t))\right]. \end{aligned}$$

It follows that

$$\int_{t_m}^t \frac{\psi'(s)}{\beta_1(\psi(s)) + \beta_2(\psi(s))} ds \leq \int_{t_m}^t \alpha(s) ds.$$

Furthermore, also using (4.4) we obtain

$$\int_{\psi(t_m)}^{\psi(t)} \frac{d\tau}{\beta_1(\tau) + \beta_2(\tau))} \leq \int_{t_m}^t \alpha(s) ds \leq \int_{t_m}^1 \alpha(s) ds < \int_{R^*}^{\infty} \frac{d\tau}{\beta_1(\tau) + \beta_2(\tau)}. \tag{4.9}$$

Note that $\psi(t_m) = R^*$. Then from (4.9) it follows that there exists R_1 such that

$$\psi(t) \leq R_1$$

for all $t \in [t_m, 1]$. Then $|x(t)| \leq R_1$ and $|y(t)| \leq R_1$, for all $t \in [t_m, 1]$, whence

$$|x|_{C[t_m,1]} \leq R_1, \quad |y|_{C[t_m,1]} \leq R_1. \tag{4.10}$$

Let $R=\max\{R_0,R_1\}$. From (4.7), (4.10) we have $|x|_{C[0,1]}\leq R$ and $|y|_{C[0,1]}\leq R$. □

Remark 4.2. If $\omega_1(t,\tau)=\alpha_0(t)\beta_0(\tau)$, then the first inequality in (4.3) implies that $\beta_0(\tau)\leq c\tau+c'$ for all $\tau\in R_+$ and some constants c and c', i.e. the growth of β_0 is at most linear. However, β_1 may have a superlinear growth. Thus we may say that under the assumptions of Theorem 4.1, the growth of $f(t,u)$ in u is at most linear for $t\in[0,t_m]$ and can be superlinear for $t\in[t_m,1]$. The same can be said about $g(t,u)$.

In particular, when $t_m=0$, that is when problem (1.1) becomes the classical local initial value problem

$$\begin{cases} x'=f(t,x,y) \\ y'=g(t,x,y) \quad \text{(a.e. } t\in[0,1]) \\ x(0)=y(0)=0, \end{cases} \tag{4.11}$$

our assumptions reduce to the classical conditions (see [7], [9]) and Theorem 4.1 gives the following result:

Corollary 4.3. *Assume that*

$$\begin{aligned} |f(t,u)| &\leq \alpha(t)\beta_1(|u|_e), \\ |g(t,u)| &\leq \alpha(t)\beta_2(|u|_e) \end{aligned}$$

for $t\in[0,1]$ *and* $u\in\mathbf{R}^2$, *where* $\alpha\in L^1[0,1]$, *while* $\beta_1,\beta_2:\mathbf{R}_+\to\mathbf{R}_+$ *are nondecreasing and* $1/\beta_1, 1/\beta_2\in L^1_{loc}(\mathbf{R}_+)$. *In addition assume that*

$$\int_0^\infty \frac{d\tau}{\beta_1(\tau)+\beta_2(\tau)}>\int_0^1\alpha(s)ds.$$

Then, problem (4.11) *has at least one solution.*

Acknowledgements. This work was possible with the financial support of the Sectoral Operational Programme for Human Resources Development 2007-2013, co-financed by the European Social Fund, under the project number POSDRU/107/1.5/S/76841 with the title "Modern Doctoral Studies: Internationalization and Interdisciplinarity".

References

[1] Agarwal, R.P., Meehan, M., O'Regan, D., *Fixed Point Theory and Applications*, Cambridge University Press, Cambridge, 2001.

[2] Boucherif, A., *Differential equations with nonlocal boundary conditions*, Nonlinear Anal., **47**(2001), 2419-2430.

[3] Boucherif, A., Precup, R., *On the nonlocal initial value problem for first order differential equations,* Fixed Point Theory, **4**(2003), 205-212.

[4] Boucherif, A., Precup, R., *Semilinear evolution equations with nonlocal initial conditions*, Dynamic Systems Appl., **16**(2007), 507-516.

[5] Byszewski, L., *Theorems about the existence and uniqueness of solutions of a semilinear evolution nonlocal Cauchy problem*, J. Math. Anal. Appl., **162**(1991), 494-505.

[6] Byszewski, L., Lakshmikantham, V., *Theorem about the existence and uniqueness of a solution of a nonlocal abstract Cauchy problem in a Banach space*, Appl. Anal., **40**(1990), 11-19.

[7] Frigon, M., *Application de la théorie de la transversalite topologique a des problemes non linearies pour des equations differentielles ordinaires*, Dissertationes Math. 296, PWN, Warsawa, 1990.

[8] Ntouyas, S.K., Tsamatos, P.Ch., *Global existence for semilinear evolution equations with nonlocal conditions,* J. Math. Anal. Appl., **210**(1997), 679-687.

[9] O'Regan, D., Precup, R., *Theorems of Leray-Schauder Type and Applications*, Gordon and Breach, Amsterdam, 2001.

[10] Precup, R., *Methods in Nonlinear Integral Equations*, Kluwer, Dordrecht, 2002.

[11] Precup, R., *The role of matrices that are convergent to zero in the study of semilinear operator systems*, Math. Comp. Modelling, **49**(2009), 703-708.

Octavia Nica
Department of Mathematics, Babeş-Bolyai University
400084 Cluj-Napoca, Romania
e-mail: `octavia.nica@math.ubbcluj.ro`

Radu Precup
Department of Mathematics, Babeş-Bolyai University
400084 Cluj-Napoca, Romania
e-mail: `r.precup@math.ubbcluj.ro`

Stud. Univ. Babeş-Bolyai Math. 56(2011), No. 3, 127–134

Bezier blending surfaces on astroid

Marius Birou

Abstract. In this article we construct Bezier surfaces on a domain bounded by an astroid using the univariate polynomial Bernstein operator. We study the monotonicity and we give conditions of convexity in some directions for the constructed surfaces. Also, we give conditions for obtaining hyperbolic, parabolic and elliptic surfaces.

Mathematics Subject Classification (2010): 65D10, 41A10.

Keywords: Bezier surfaces, monotonicity, convexity.

1. Introduction

The surfaces of blending type have been introduced by Coons in [5]. They have the property of matching some given curves. In some previous papers [1, 2, 3, 4, 9] there were constructed the blending surfaces with the support on the border of a rectangular, triangular or circular domain and having a fixed height in a point from the domain. In this paper we obtained the Bezier surfaces which stay on an astroid. We construct the surfaces using the univariate Bernstein operator. The obtained surfaces are defined on a domain bounded by an astroid, they stay on the border of the domain and have a fixed height in the center of the domain. Instead of the control points from the case of classical Bezier surfaces we use a curves network (one of the curves from network is reduced to a point). We study the monotonicity and the convexity using the first and the second directional derivatives respectively (like in [7, 8, 10]).

These surfaces can be used in civil engineering (as roofs for buildings) or in Computer Aided Geometric Design (CAGD). For roof surfaces the maximal stress acts in the parabolical points (see [3, 4, 9, 11]). It is preferable to avoid having the parabolic points among the points of other type (hyperbolic, elliptic). We give conditions for obtaining the surfaces of hyperbolic, parabolic or elliptic type.

2. Construction of the surfaces

Let $n \in \mathbb{N}$, $n \geq 2$ and $h_i, h \in \mathbb{R}$, $i = 1, ..., n-1$ be such that

$$0 = h_n < h_{n-1} < ... < h_1 < h_0 = h \tag{2.1}$$

and let $f : [0,1] \to \mathbb{R}$ be a function with the properties

$$\begin{array}{l} f(0) = h, \\ f(\frac{j}{n}) = h_j, \; j = 1, ..., n-1, \\ f(1) = 0. \end{array} \tag{2.2}$$

Let B_n the univariate Bernstein operator on the interval $[0,1]$,

$$(B_n f)(y) = \sum_{j=0}^{n} b_{jn}(y) f(\frac{j}{n}),$$

where the functions b_{jn} are given by formula

$$b_{jn}(y) = \binom{n}{j} y^j (1-y)^{n-j}, \; for \; j = 0, ..., n.$$

Taking into account (2.2), we obtain

$$(B_n f)(y) = b_{0n}(y)h + \sum_{j=1}^{n-1} b_{jn}(y) h_j. \tag{2.3}$$

The function in (2.3) has the properties

$$(B_n f)(0) = h, \; (B_n f)(1) = 0.$$

Let $D = \{(X,Y) \in \mathbb{R}^2 : \; X^{\frac{2}{3}} + Y^{\frac{2}{3}} \leq 1\}$ be a domain in the XOY plane (the domain bounded by the astroid $X^{\frac{2}{3}} + Y^{\frac{2}{3}} = 1$).

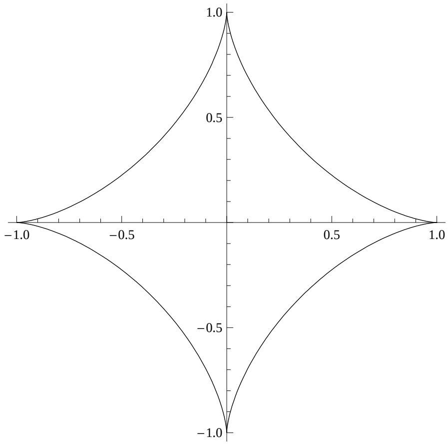


FIGURE 1. The astroid

If we make the substitution $y = \left(X^{\frac{2}{3}} + Y^{\frac{2}{3}}\right)^{\alpha}$, $\alpha > 0$ in (2.3), we obtain the surfaces

$$F(X,Y) := (B_n f)\left(\left(X^{\frac{2}{3}} + Y^{\frac{2}{3}}\right)^{\alpha}\right) = \tag{2.4}$$

$$= b_{0n}\left(\left(X^{\frac{2}{3}} + Y^{\frac{2}{3}}\right)^{\alpha}\right) h + \sum_{j=1}^{n-1} b_{jn}\left(\left(X^{\frac{2}{3}} + Y^{\frac{2}{3}}\right)^{\alpha}\right) h_j,\ (X,Y) \in D.$$

The surfaces (2.4) have the properties

$$F|_{\partial D} = 0,$$

$$F(0,0) = h.$$

It follows that the surfaces F match the astroid $X^{\frac{2}{3}} + Y^{\frac{2}{3}} = 1$, $Z = 0$ (the surfaces stay on the border of domain D) and the height of the surfaces in the point $(0,0)$ is h.

We can give a parametrical representation for these surfaces

$$\begin{cases} X = u\cos^3 v, \\ Y = u\sin^3 v, \\ Z = b_{0n}(u^{\frac{2\alpha}{3}})h + \sum\limits_{j=1}^{n-1} b_{jn}(u^{\frac{2\alpha}{3}})h_j \end{cases} \quad u \in [0,1], \quad v \in [0, 2\pi].$$

Next sections, we study the monotonicity and the convexity using the directional derivative of the first and the second order respectively. The domain D is not convex but it is a star convex set with respect to the point $(0,0)$. We will use directions that pass by the point $(0,0)$. Also, some results about the type of the points of the surfaces F on the domain $D \setminus D_1$, where $D_1 = \{(x,0), x \in [-1,1]\} \cup \{(0,y), y \in [-1,1]\}$, are given.

3. Monotonicity of the surfaces

We denote

$$\Delta_1 h_j = h_{j+1} - h_j,\ j = 0, ..., n-1.$$

We recall that a bivariate function G is increasing (decreasing) in the direction $d = (d_1, d_2) \in \mathbb{R}^2$ if

$$G(X + \lambda d_1, Y + \lambda d_2) \geq (\leq) G(X,Y), \tag{3.1}$$

for every $(X,Y) \in A \subset \mathbb{R}^2$ and every $\lambda > 0$ such that $(X + \lambda d_1, Y + \lambda d_2) \in A$. The first order directional derivative in the direction $d = (d_1, d_2)$ of a C^1 function G is

$$D_d G = d_1 G_X + d_2 G_Y.$$

The conditions (3.1) are equivalent to

$$D_d G \geq 0 (\leq 0),\ \textit{on } A.$$

Next theorem gives conditions for the monotonicity in some directions of the surfaces F.

Theorem 3.1. *If* $\left(d_1X^{-\frac{1}{3}}+d_2Y^{-\frac{1}{3}}\right)<0$ (>0) *on* $D\setminus D_1$*, then* F *is increasing (decreasing) in the direction* (d_1,d_2) *on* $D\setminus D_1$*, where* (d_1,d_2) *is a direction that pass by the point* $(0,0)$*.*

Proof. Let $(X,Y)\in D\setminus D_1$ and (d_1,d_2) a direction that pass by the point $(0,0)$. Using some results from [6], it follows that the first partial derivatives of the function F are given by

$$F_X(X,Y)=\frac{2n\alpha X^{-\frac{1}{3}}}{3}\left(X^{\frac{2}{3}}+Y^{\frac{2}{3}}\right)^{\alpha-1}\sum_{j=0}^{n-1}b_{j,n-1}\left(\left(X^{\frac{2}{3}}+Y^{\frac{2}{3}}\right)^{\alpha}\right)\Delta_1h_j,$$

$$F_Y(X,Y)=\frac{2n\alpha Y^{-\frac{1}{3}}}{3}\left(X^{\frac{2}{3}}+Y^{\frac{2}{3}}\right)^{\alpha-1}\sum_{j=0}^{n-1}b_{j,n-1}\left(\left(X^{\frac{2}{3}}+Y^{\frac{2}{3}}\right)^{\alpha}\right)\Delta_1h_j.$$

If we compute the first order directional derivative of the function F in the direction $d=(d_1,d_2)$, we obtain

$$D_dF(X,Y)=$$

$$=\frac{2n\alpha}{3}\left(d_1X^{-\frac{1}{3}}+d_2Y^{-\frac{1}{3}}\right)\left(X^{\frac{2}{3}}+Y^{\frac{2}{3}}\right)^{\alpha-1}\sum_{j=0}^{n-1}b_{j,n-1}\left(\left(X^{\frac{2}{3}}+Y^{\frac{2}{3}}\right)^{\alpha}\right)D_1h_j.$$

Taking into account (2.1), the condition $d_1X^{-\frac{1}{3}}+d_2Y^{-\frac{1}{3}}<0$ (>0) on $D\setminus D_1$ implies $D_dF>0$ (<0) on $D\setminus D_1$, and the theorem is proved. □

4. Convexity and type of the surfaces

We denote

$$\Delta_2h_{j+1}=h_{j+2}-2h_{j+1}+h_j,\ j=0,...,n-2.$$

We recall that a bivariate C^2 function G is convex (concave) in the the direction $d=(d_1,d_2)\in\mathbb{R}^2$ if and only if $D_d^2G\geq 0$ (≤ 0) on $A\subset\mathbb{R}^2$, where D_d^2G is the second order directional derivative in direction $d=(d_1,d_2)$ of the function G,

$$D_d^2G=d_1^2G_{XX}+2d_1d_2G_{XY}+d_2^2G_{YY}.$$

We give sufficient conditions for convexity in some directions of the surfaces F.

Theorem 4.1. *If* $\alpha\in(0,1]$ *and* $\Delta_2h_j\geq 0$, $j=0,...,n-2$*, then the function* F *is convex in the direction* (d_1,d_2) *on* $D\setminus D_1$*, where* (d_1,d_2) *is a direction that pass by the point* $(0,0)$ *.*

Proof. Let $(X,Y)\in D\setminus D_1$ and (d_1,d_2) a direction that pass by the point $(0,0)$. Taking into account results from [6], the second order derivatives of the function F are

$$F_{XX}(X,Y)= \tag{4.1}$$

$$=\frac{4X^{-\frac{2}{3}}n(n-1)\alpha^2}{9}\left(X^{\frac{2}{3}}+Y^{\frac{2}{3}}\right)^{2\alpha-2}\sum_{j=0}^{n-2}b_{j,n-2}\left(\left(X^{\frac{2}{3}}+Y^{\frac{2}{3}}\right)^{\alpha}\right)\Delta_2h_j+$$

$$+\left(\frac{4X^{-\frac{2}{3}}\alpha(\alpha-1)}{9}\left(X^{\frac{2}{3}}+Y^{\frac{2}{3}}\right)^{\alpha-2}-\frac{2X^{-\frac{4}{3}}\alpha}{9}\left(X^{\frac{2}{3}}+Y^{\frac{2}{3}}\right)^{\alpha-1}\right)\times$$

$$\times n\sum_{j=0}^{n-1}b_{j,n-1}\left(\left(X^{\frac{2}{3}}+Y^{\frac{2}{3}}\right)^{\alpha}\right)\Delta_1 h_j,$$

$$F_{XY}(X,Y)=\frac{4X^{-\frac{1}{3}}Y^{-\frac{1}{3}}\alpha^2}{9}\left(X^{\frac{2}{3}}+Y^{\frac{2}{3}}\right)^{2\alpha-2}\times \tag{4.2}$$

$$\times n(n-1)\sum_{j=0}^{n-2}b_{j,n-2}\left(\left(X^{\frac{2}{3}}+Y^{\frac{2}{3}}\right)^{\alpha}\right)\Delta_2 h_j+$$

$$+\frac{4X^{-\frac{1}{3}}Y^{-\frac{1}{3}}n\alpha(\alpha-1)}{9}\left(X^{\frac{2}{3}}+Y^{\frac{2}{3}}\right)^{\alpha-2}\sum_{j=0}^{n-1}b_{j,n-1}\left(\left(X^{\frac{2}{3}}+Y^{\frac{2}{3}}\right)^{\alpha}\right)\Delta_1 h_j,$$

$$F_{YY}(X,Y)= \tag{4.3}$$

$$=\frac{4X^{-\frac{2}{3}}n(n-1)\alpha^2}{9}\left(X^{\frac{2}{3}}+Y^{\frac{2}{3}}\right)^{2\alpha-2}\sum_{j=0}^{n-2}b_{j,n-2}\left(\left(X^{\frac{2}{3}}+Y^{\frac{2}{3}}\right)^{\alpha}\right)\Delta_2 h_j+$$

$$+\left(\frac{4Y^{-\frac{2}{3}}\alpha(\alpha-1)}{9}\left(X^{\frac{2}{3}}+Y^{\frac{2}{3}}\right)^{\alpha-2}-\frac{2Y^{-\frac{4}{3}}\alpha}{9}\left(X^{\frac{2}{3}}+Y^{\frac{2}{3}}\right)^{\alpha-1}\right)\times$$

$$\times n\sum_{j=0}^{n-1}b_{j,n-1}\left(\left(X^{\frac{2}{3}}+Y^{\frac{2}{3}}\right)^{\alpha}\right)\Delta_1 h_j.$$

If we compute the second order directional derivative in the direction $d=(d_1,d_2)$ of the function F, we obtain

$$D_d^2F(X,Y)=\frac{4\alpha^2}{9}\left(d_1X^{-\frac{1}{3}}+d_2Y^{-\frac{1}{3}}\right)^2\left(X^{\frac{2}{3}}+Y^{\frac{2}{3}}\right)^{2\alpha-2}\times$$

$$\times n(n-1)\sum_{j=0}^{n-2}b_{j,n-2}\left(\left(X^{\frac{2}{3}}+Y^{\frac{2}{3}}\right)^{\alpha}\right)\Delta_2 h_j+$$

$$+\frac{4\alpha(\alpha-1)}{9}\left(d_1X^{-\frac{1}{3}}+d_2Y^{-\frac{1}{3}}\right)^2\left(X^{\frac{2}{3}}+Y^{\frac{2}{3}}\right)^{\alpha-2}\times$$

$$\times n\sum_{j=0}^{n-1}b_{j,n-1}\left(\left(X^{\frac{2}{3}}+Y^{\frac{2}{3}}\right)^{\alpha}\right)\Delta_1 h_j-$$

$$-\frac{2n\alpha}{9}\left(d_1^2X^{-\frac{4}{3}}+d_2^2Y^{-\frac{4}{3}}\right)\left(X^{\frac{2}{3}}+Y^{\frac{2}{3}}\right)^{\alpha-1}\sum_{j=0}^{n-1}b_{j,n-1}\left(\left(X^{\frac{2}{3}}+Y^{\frac{2}{3}}\right)^{\alpha}\right)\Delta_1 h_j.$$

From $\alpha\in(0,1]$, $\Delta_2 h_j\geq 0$, $j=0,...,n-2$ and the condition (2.1), it follows $D_d^2F\geq 0$ on $D\setminus D_1$. Thus, the conclusion of the theorem holds. $\square$

We recall that a point of a surface $Z = G(X,Y)$, $(X,Y) \in A \subset R^2$ is parabolic point if $PG(X,Y) = 0$, where

$$PG(X,Y) = G_{XX}(X,Y)G_{YY}(X,Y) - (G_{XY}(X,Y))^2. \qquad (4.4)$$

If we have $PG(X,Y) < 0$ (> 0) the point (X,Y) is called hyperbolic point (elliptic point). The surface G is called of parabolic (hyperbolic, elliptic) type if all the points of the surface are parabolic (hyperbolic, elliptic).

The following theorem gives conditions for obtaining the surfaces F of different types on $D \setminus D_1$.

Theorem 4.2. *We have:*

1. *If* $\alpha \in \left(0, \frac{3}{2}\right)$ *and* $\Delta_2 h_j \geq 0$, $j = 0, ..., n-2$, *then the surfaces* F *are of elliptic type on* $D \setminus D_1$.
2. *If* $\alpha = \frac{3}{2}$ *and* $\Delta_2 h_j \geq 0$, $j = 0, ..., n-2$ *and there exists* $j_0 \in \{0, ..., n-2\}$ *such that* $\Delta_2 h_{j_0} \neq 0$, *then the surfaces* F *are of elliptic type on* $D \setminus D_1$.
3. *If* $\alpha = \frac{3}{2}$ *and* $\Delta_2 h_j = 0$, $j = 0, ..., n-2$, *then the surfaces* F *are of parabolic type on* $D \setminus D_1$.
4. *If* $\alpha = \frac{3}{2}$ *and* $\Delta_2 h_j \leq 0$, $j = 0, ..., n-2$ *and there exists* $j_0 \in \{0, ..., n-2\}$ *such that* $\Delta_2 h_{j_0} \neq 0$, *then the surfaces* F *are of hyperbolic type on* $D \setminus D_1$.
5. *If* $\alpha \in \left(\frac{3}{2}, \infty\right)$ *and* $\Delta_2 h_j \leq 0$, $j = 0, ..., n-2$, *then the surfaces* F *are of hyperbolic type on* $D \setminus D_1$.

Proof. Let $(X,Y) \in D \setminus D_1$. From (4.4) and (4.1)-(4.3) we obtain

$$PF(X,Y) =$$

$$= -\frac{4\alpha^2(2\alpha-3)\left(X^{\frac{2}{3}}+Y^{\frac{2}{3}}\right)^{2\alpha-2}}{81X^{\frac{4}{3}}Y^{\frac{4}{3}}}\left(n\sum_{j=0}^{n-1} b_{j,n-1}\left(\left(X^{\frac{2}{3}}+Y^{\frac{2}{3}}\right)^{\alpha}\right)\Delta_1 h_j\right)^2 +$$

$$-\frac{8\alpha^3\left(X^{\frac{2}{3}}+Y^{\frac{2}{3}}\right)^{3\alpha-2}}{81X^{\frac{4}{3}}Y^{\frac{4}{3}}}\left(n\sum_{j=0}^{n-1} b_{j,n-1}\left(\left(X^{\frac{2}{3}}+Y^{\frac{2}{3}}\right)^{\alpha}\right)\Delta_1 h_j\right)\times$$

$$\times\left(n(n-1)\sum_{j=0}^{n-2} b_{j,n-2}\left(\left(X^{\frac{2}{3}}+Y^{\frac{2}{3}}\right)^{\alpha}\right)\Delta_2 h_j\right),$$

The conclusions of the theorem follow using the condition (2.1). □

Remark 4.3. The conditions from Theorem 4.1 and Theorem 4.2 depend only the parameters h_j (i.e. they depend only on the control network).

We have plotted the surface F for $n = 3$.

In Figure 2.a we take $h = h_0 = 3$, $h_1 = 1.5$, $h_2 = 0.5$, $h_3 = 0$ and $\alpha = 1$; we have $\Delta_2 h_j > 0$, $j = 0, 1$. The surface is of elliptic type.

In Figure 2.b we take $h = h_0 = 3$, $h_1 = 1.5$, $h_2 = 0.5$, $h_3 = 0$ and $\alpha = 1.5$; we have $\Delta_2 h_j > 0$, $j = 0, 1$. The surface is of elliptic type.

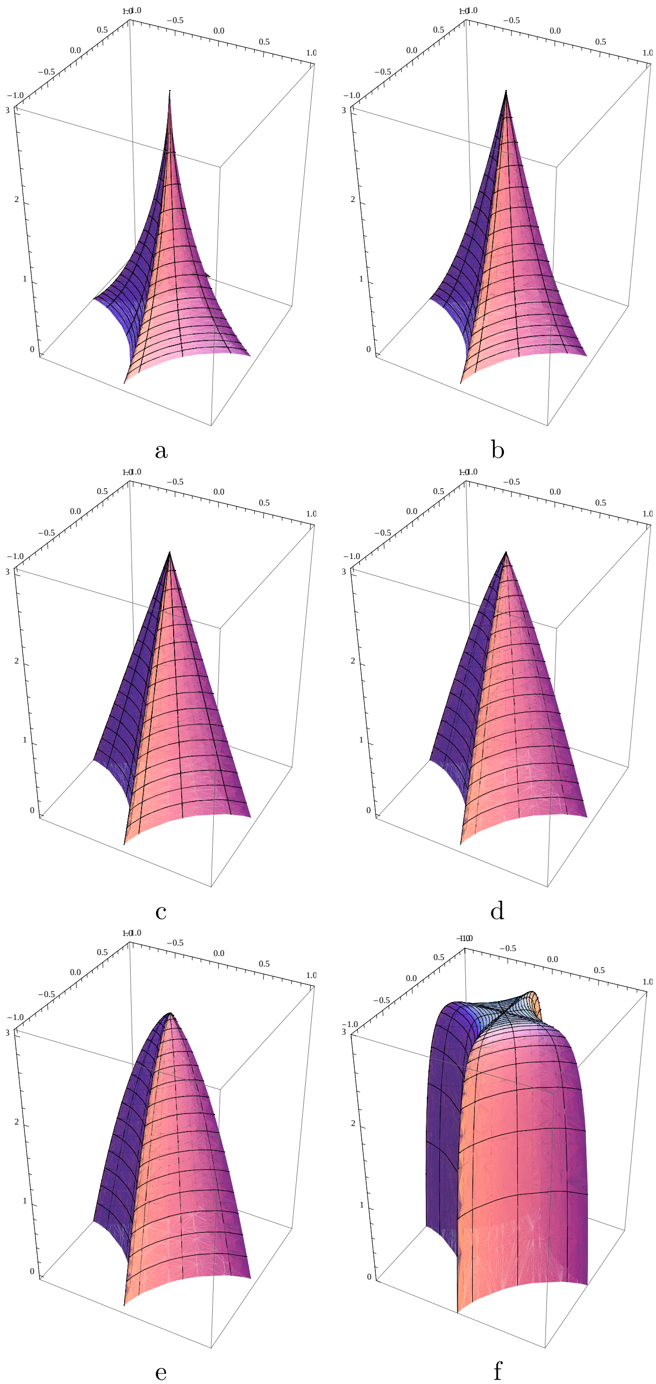


FIGURE 2. The surface F for $n = 3$.

In Figure 2.c we take $h = h_0 = 3$, $h_1 = 2$, $h_2 = 1$, $h_3 = 0$ and $\alpha = 1.5$; we have $\Delta_2 h_j = 0$, $j = 0, 1$. The surface is of parabolic type.

In Figure 2.d we take $h = h_0 = 3$, $h_1 = 2.3$, $h_2 = 1.2$, $h_3 = 0$ and $\alpha = 1.5$; we have $\Delta_2 h_j < 0$, $j = 0, 1$. The surface is of hyperbolic type.

In Figure 2.e we take $h = h_0 = 3$, $h_1 = 2.3$, $h_2 = 1.2$, $h_3 = 0$ and $\alpha = 2$; we have $\Delta_2 h_j < 0$, $j = 0, 1$. The surface is of hyperbolic type.

In Figure 2.f we take $h = h_0 = 3$, $h_1 = 2.3$, $h_2 = 1.2$, $h_3 = 0$ and $\alpha = 10$; we have $\Delta_2 h_j < 0$, $j = 0, 1$. The surface is of hyperbolic type.

References

[1] Birou, M., *Blending surfaces on circular domains generated by Hermite interpolation*, Studia Univ. Babeş-Bolyai Math., **55**(2010), no. 4, 185-192.

[2] Birou, M., *Blending surfaces on circular domains generated by Birkhoff interpolation*, Autom. Comput. Appl. Math, **18**(2009), no 1, 97-106.

[3] Coman, Gh., Gânscă, I., Tâmbulea, L., *Some new roof-surfaces generated by blending interpolation technique*, Studia Univ. Babeş-Bolyai Math., **36**(1991), no. 1, 119-130.

[4] Coman, Gh., Gânscă, I., Tâmbulea, L., *Surfaces generated by blending interpolation*, Studia Univ. Babeş-Bolyai Math., **38**(1993), no. 3, 39-48.

[5] Coons, S.A., *Surface for computer aided design of space forms*, Project MAC, Design Div., Dep. of Mech. Engineering, MIT, 1964.

[6] Farin, G., *Curves and surfaces for Computer Aided Geometric*, Design. Academic Press, Inc Harcourt Brace Jovanovich, Publishers, Boston, San Diego, New York, London, Sidney, Tokyo, Toronto, 1990.

[7] Floater, M.S., *Shape Properties of Tensor-Product Bernstein Polynomials*, in Shape preserving representations in Computer-Aided Geometric Design, J. M. Pena (ed.), Nova Science Publications, New York, 1999, 149-162.

[8] Floater, M.S., Pena, J.M., *Tensor-product Monotonicity Preservation*, Advances in Comp. Math., **9**(1998), 353-362.

[9] Gânscă, I., Coman, Gh., *Aproximare blending cu aplicaţii in construcţii*, Bul. Stiint. Inst. Politehn. Cluj-Napoca, Ser. Electrotehn. Energet. Inform., **24**(1981), 35-40.

[10] Goodman, T.N.T., *Shape preserving representations, in Mathematical methods in Computer Aided Geometric Design*, T. Lyche and L.L. Schumaker (eds.), Academic Press, New York, 1989, 333-357.

[11] Mihailescu, M., Horvath, I., *Velaroidal shells for covering universal industrial halls*, Acta Tech. Acad. Hungaricae, **85**(1977), 135-145.

Marius Birou
Technical University of Cluj Napoca
Department of Mathematics
28, Memorandumului Street
400114 Cluj-Napoca,Romania
e-mail: marius.birou@math.utcluj.ro

Stud. Univ. Babeş-Bolyai Math. 56(2011), No. 3, 135–146

Uniform weighted approximation by positive linear operators

Adrian Holhoş

Abstract. We characterize the functions defined on a weighted space, which are uniformly approximated by a sequence of positive linear operators and we obtain the range of the weights which can be used for uniform approximation. We, also, obtain an estimation of the remainder in terms of the usual modulus of continuity. We give particular results for the Szász-Mirakjan and Baskakov operators.

Mathematics Subject Classification (2010): 41A36, 41A25.

Keywords: Positive linear operators, weighted spaces, rate of approximation, modulus of continuity.

1. Introduction

Let $I \subseteq \mathbb{R}$ be a noncompact interval and let $\rho\colon I \to [1,\infty)$ be an increasing and differentiable function called weight. Let $B_\rho(I)$ be the space of all functions $f\colon I \to \mathbb{R}$ such that $|f(x)| \le M \cdot \rho(x)$, for every $x \in I$, where $M > 0$ is a constant depending on f and ρ, but independent of x. The space $B_\rho(I)$ is called weighted space and it is a Banach space endowed with the ρ-norm

$$\|f\|_\rho = \sup_{x \in I} \frac{|f(x)|}{\rho(x)}.$$

Let $C_\rho(I) = C(I) \cap B_\rho(I)$ be the subspace of $B_\rho(I)$ containing continuous functions.

Let $(A_n)_{n\ge 1}$ be a sequence of positive linear operators defined on the weighted space $C_\rho(I)$. It is known (see [13]) that A_n maps $C_\rho(I)$ onto $B_\rho(I)$ if and only if $A_n\rho \in B_\rho(I)$.

In the paper [7], the authors present some ideas related to the approximation of functions in weighted spaces and enounced some unsolved problems in weighted approximation theory. Three such problemms are:

1. Let $\mathcal{F}$ be a linear subspace of $\mathbb{R}^I$ and $A_n\colon \mathcal{F} \to C(I)$ a sequence of positive linear operators. For which weights ρ, does A_n map $C_\rho(I) \cap \mathcal{F}$ onto $C_\rho(I)$

with uniformly bounded norms?
2. For which functions $f \in C_\rho(I)$ do we have $\|A_n - f\|_\rho \to 0$, as $n \to \infty$?
3. Which moduli of smoothness are appropriate for weighted approximation?

Some ideas to solve these problems and some partial solutions were given in the article already mentioned. In this paper, we give some answers to these three problems. For a given sequence of positive linear operators, A_n, we characterize those functions f belonging to $C_\rho(I)$ such that $\|A_n - f\|_\rho \to 0$ and obtain all the weights ρ for which this uniform convergence in the ρ-norm is true. We, also, obtain an estimation of the remainder $A_n f - f$, in terms of the modulus of continuity of the function f. As applications, we give some results related to the Szász-Mirakjan and Baskakov operators.

We will use the following modulus of continuity

$$\omega_\varphi(f,\delta) = \sup_{\substack{t,x\in I \\ |\varphi(t)-\varphi(x)|\le\delta}} |f(t) - f(x)|,$$

for all $f \in B(I)$, where $\varphi\colon I \to J$, $(J \subset \mathbb{R})$, is a differentiable bijective function, with $\varphi'(x) > 0$, for all $x \in I$. This modulus is a particular case of the general modulus

$$\omega_d(f,\delta) = \sup\{\, |f(t) - f(x)| \;:\; t, x \in X,\ d(t,x) \le \delta \,\},$$

where f is a bounded function defined on X and (X, d) is a compact metric space. For details related to this general modulus of continuity see [8], [15] and [20]. The particular modulus $\omega_\varphi(f,\delta)$ is obtained for the metric $d(t,x) = |\varphi(t) - \varphi(x)|$ and has the following properties (see [17], for example)

Proposition 1.1. *Let $f \in B(I)$ and $\delta > 0$.*

(i) $\omega_\varphi(f,\delta) = \omega(f \circ \varphi^{-1}, \delta)$, where ω is the usual modulus of continuity.

(ii) Let $(\delta_n)_{n\ge1}$ be sequence of positive real numbers converging to 0. Then $f \circ \varphi^{-1}$ is uniformly continuous on J if and only if $\omega_\varphi(f,\delta_n) \to 0$.

(iii) $|f(t) - f(x)| \le \left(1 + \dfrac{|\varphi(t) - \varphi(x)|}{\delta}\right)\omega_\varphi(f,\delta)$, for every $t, x \in I$.

2. Main result

Theorem 2.1. *Let $A_n : C_\rho(I) \to B_\rho(I)$ be positive linear operators reproducing constant functions and satisfying the conditions*

$$\sup_{x\in I} A_n(|\varphi(t) - \varphi(x)|, x) = a_n \to 0, \quad (n \to \infty) \tag{2.1}$$

$$\sup_{x\in I} \frac{A_n(|\rho(t) - \rho(x)|, x)}{\rho(x)} = b_n \to 0. \quad (n \to \infty) \tag{2.2}$$

If $A_n(f,x)$ is continuously differentiable and there is a constant $K(f,\rho,n)$ such that

$$\frac{|(A_n f)'(x)|}{\varphi'(x)} \le K(f,\rho,n)\cdot\rho(x), \quad \text{for every } x \in I, \tag{2.3}$$

and ρ and φ are such that there exists a constant $\alpha > 0$

$$\frac{\rho'(x)}{\varphi'(x)} \leq \alpha \cdot \rho(x), \quad \text{for every } x \in I, \tag{2.4}$$

then, the following statements are equivalent

(*i*) $\|A_n f - f\|_\rho \to 0$ *as* $n \to \infty$.

(*ii*) $\frac{f}{\rho} \circ \varphi^{-1}$ *is uniformly continuous on* J.

Furthermore, we have

$$\|A_n f - f\|_\rho \leq b_n \cdot \|f\|_\rho + 2 \cdot \omega_\varphi \left(\frac{f}{\rho}, a_n \right), \quad \text{for every } n \geq 1. \tag{2.5}$$

Proof. Let us prove that (ii) implies (i). Using the inequality (ii) of Proposition 1.1, we obtain for a function $f \in C_\rho(I)$

$$\begin{aligned} |f(t) - f(x)| &\leq \frac{|f(t)|}{\rho(t)} \cdot |\rho(t) - \rho(x)| + \rho(x) \cdot \left| \frac{f(t)}{\rho(t)} - \frac{f(x)}{\rho(x)} \right| \\ &\leq \|f\|_\rho \cdot |\rho(t) - \rho(x)| + \rho(x) \cdot \left(1 + \frac{|\varphi(t) - \varphi(x)|}{\delta} \right) \omega_\varphi \left(\frac{f}{\rho}, \delta \right). \end{aligned}$$

Applying the positive linear operators A_n to the last inequality, we obtain

$$\begin{aligned} \frac{|A_n(f,x) - f(x)|}{\rho(x)} \leq \|f\|_\rho \cdot \frac{A_n(|\rho(t) - \rho(x)|, x)}{\rho(x)} \\ + \left(1 + \frac{A_n(|\varphi(t) - \varphi(x)|, x)}{\delta_n} \right) \omega_\varphi \left(\frac{f}{\rho}, \delta_n \right), \end{aligned}$$

which proves the relation (2.5). Because $a_n \to 0$ and $\frac{f}{\rho} \circ \varphi^{-1}$ is uniformly continuous on J we deduce that $\omega_\varphi \left(\frac{f}{\rho}, a_n \right) \to 0$ as $n \to \infty$. Because $b_n \to 0$ we obtain that $\|A_n f - f\|_\rho \to 0$.

Now, let us prove that (i) implies (ii). Because of the relation

$$\begin{aligned} \omega_\varphi \left(\frac{f}{\rho}, \delta_n \right) &\leq \omega_\varphi \left(\frac{f - A_n f}{\rho}, \delta_n \right) + \omega_\varphi \left(\frac{A_n f}{\rho}, \delta_n \right) \\ &\leq \|f - A_n f\|_\rho + \omega_\varphi \left(\frac{A_n f}{\rho}, \delta_n \right) \end{aligned}$$

it remains to prove that $\omega_\varphi \left(\frac{A_n f}{\rho}, \delta_n \right) \to 0$.

Applying the Cauchy mean value theorem, there is c between $x \in I$ and $t \in I$, such that

$$\varphi'(c) \left[\frac{A_n(f,t)}{\rho(t)} - \frac{A_n(f,x)}{\rho(x)} \right] = \left(\frac{A_n f}{\rho} \right)' (c) \cdot [\varphi(t) - \varphi(x)].$$

We have

$$\omega_\varphi \left(\frac{A_n f}{\rho}, \delta_n \right) = \sup_{\substack{t,x \in I \\ |\varphi(t) - \varphi(x)| \leq \delta_n}} \left| \frac{A_n(f,t)}{\rho(t)} - \frac{A_n(f,x)}{\rho(x)} \right| \leq \left\| \frac{1}{\varphi'} \cdot \left(\frac{A_n f}{\rho} \right)' \right\| \cdot \delta_n,$$

which implies $\omega_\varphi\left(\frac{A_n f}{\rho}, \delta_n\right) \to 0$, for a suitable choice of the sequence $\delta_n \to 0$, if

$$\left\|\frac{1}{\varphi'} \cdot \left(\frac{A_n f}{\rho}\right)'\right\| = \sup_{x\in I}\left|\frac{1}{\varphi'(x)} \cdot \left(\frac{A_n f}{\rho}\right)'(x)\right| < \infty.$$

But, for every $f \in C_\rho(I)$ and for every $n \geq 1$

$$\left\|\frac{1}{\varphi'} \cdot \left(\frac{A_n f}{\rho}\right)'\right\| = \left\|\frac{1}{\varphi'} \cdot \left(\frac{(A_n f)'}{\rho} - \frac{A_n f \cdot \rho'}{\rho^2}\right)\right\|$$
$$\leq \left\|\frac{(A_n f)'}{\rho \cdot \varphi'}\right\| + \|f\|_\rho \cdot \|A_n \rho\|_\rho \cdot \left\|\frac{\rho'}{\rho \cdot \varphi'}\right\| < \infty,$$

because of the relations (2.3) and (2.4). So, the theorem is proved. □

Remark 2.2. For $\rho(x) = 1$, the result of Theorem 2.1 was obtained by Totik [23], by de la Cal and Cárcamo [10] and by myself [18].

Remark 2.3. The function φ is close connected with the given sequence of positive linear operators. It can be obtained in the following manner: we choose the function θ such that

$$\theta'(x)\sqrt{A_n((t-x)^2, x)} \leq K_n,$$

where K_n is a constant not depending on x, and such that θ verifies the conditions (2.3) and (2.4). Then, by the argument of the implication $(i) \Rightarrow (ii)$ from the Theorem 2.1, we obtain that $\frac{f}{\rho} \circ \theta^{-1}$ is uniformly continuous. But, in most of the cases, θ^{-1} has a complicate form and the relation (2.1) is difficult to prove. To overcome this, we consider φ such that $\theta \circ \varphi^{-1}$ is uniformly continuous. So, we get that $\frac{f}{\rho} \circ \varphi^{-1}$ is a uniformly continuous function.

Remark 2.4. The relation (2.4) gives us the connection between the function φ and the weight ρ. We must have

$$\rho(x) \leq M e^{\alpha \cdot \varphi(x)}, \quad \text{for every } x \in I, \tag{2.6}$$

where $M, \alpha > 0$ are constants independent of x. So, we have obtained the range of the weights ρ, for which Theorem 2.1 is valid.

Remark 2.5. The maximal class of weights is $\rho(x) = e^{\alpha\varphi(x)}$. In order to prove the result of the Theorem for this weight, we prove first the inequality $A_n(\rho, x) \leq C_\alpha \rho(x)$, for every $x \in I$ and for every $\alpha > 0$, where $C_\alpha > 0$ is a constant independent of x. Then, we prove the relation

$$\lim_{n\to\infty} \sup_{x\in I} A_n(|\varphi(t) - \varphi(x)|^2, x) = 0. \tag{2.7}$$

Using the Cauchy-Schwarz inequality for positive linear operators we get that the sequence

$$a_n = \sup_{x\in I} A_n(|\varphi(t) - \varphi(x)|, x) \leq \sup_{x\in I} \sqrt{A_n(|\varphi(t) - \varphi(x)|^2, x)}$$

is convergent to 0. Using the Geometric-Logarithmic-Arithmetic Mean Inequality (see [19, p. 40])

$$\sqrt{u\cdot v}\le \frac{u-v}{\ln u-\ln v}<\frac{u+v}{2},\quad 0<v<u, \tag{2.8}$$

we obtain

$$\left|e^{\alpha\varphi(t)}-e^{\alpha\varphi(x)}\right|\le \frac{e^{\alpha\varphi(t)}+e^{\alpha\varphi(x)}}{2}\cdot\alpha\left|\varphi(t)-\varphi(x)\right|,\quad t,x\in I,$$

and

$$\begin{aligned}
b_n&=\sup_{x\in I}\frac{A_n(|\rho(t)-\rho(x)|,x)}{\rho(x)}\\
&\le \sup_{x\in I}\frac{\alpha}{2}\frac{\sqrt{A_n((\rho(t)+\rho(x))^2,x)}}{\rho(x)}\left(A_n(|\varphi(t)-\varphi(x)|^2,x)\right)^{\frac12}\\
&\le \sup_{x\in I}\frac{\alpha}{2}\left(\frac{S_n(\rho^2(t),x)}{\rho^2(x)}+2\frac{S_n(\rho,x)}{\rho(x)}+1\right)^{\frac12}\left(A_n(|\varphi(t)-\varphi(x)|^2,x)\right)^{\frac12}\\
&\le \frac{\alpha}{2}\sqrt{C_{2\alpha}+2C_\alpha+1}\cdot\sup_{x\in I}\left(A_n(|\varphi(t)-\varphi(x)|^2,x)\right)^{\frac12}.
\end{aligned}$$

If (2.7) is true, then $(b_n)_{n\in\mathbb{N}}$ converges to 0. To obtain the result of the Theorem 2.1 it remains to prove (2.3).

3. Applications

Lemma 3.1. *For every $\alpha>0$ and $\rho(x)=e^{\alpha\sqrt{x}}$, the Szász-Mirakjan operators defined by*

$$S_nf(x)=e^{-nx}\sum_{k=0}^{\infty}\frac{(nx)^k}{k!}f\left(\frac{k}{n}\right),\quad x\in[0,\infty),$$

map $C_\rho[0,\infty)$ onto $C_\rho[0,\infty)$.

Proof. Let us notice that $S_n(\rho,x)$ exists for every $x\ge 0$. This is true because

$$S_n(\rho,x)=e^{-nx}\sum_{k=0}^{\infty}\frac{(nx)^k}{k!}e^{\alpha\frac{\sqrt{k}}{\sqrt{n}}}\le e^{-nx}\sum_{k=0}^{\infty}\frac{(nx)^k}{k!}e^{\alpha\frac{k}{\sqrt{n}}}=e^{nxe^{\frac{\alpha}{\sqrt{n}}}-nx}.$$

Because S_nf converges uniformly to f on $[0,1]$ (see [1], for example), we have $S_n(\rho,x)\le C_{1,\alpha}\cdot\rho(x)$, for every $x\in[0,1]$. Let us prove that $S_n(\rho,x)\le C_{2,\alpha}\cdot\rho(x)$, for every $x\ge 1$.

$$\begin{aligned}
\frac{S_n(e^{\alpha\sqrt{t}},x)}{e^{\alpha\sqrt{x}}}&=e^{-nx}\sum_{k>nx}\frac{(nx)^k}{k!}e^{\alpha\left(\sqrt{\frac{k}{n}}-\sqrt{x}\right)}+e^{-nx}\sum_{k\le nx}\frac{(nx)^k}{k!}\frac{e^{\alpha\sqrt{\frac{k}{n}}}}{e^{\alpha\sqrt{x}}}\\
&\le e^{-nx}\sum_{k>nx}\frac{(nx)^k}{k!}e^{\frac{\alpha}{\sqrt{x}}\left(\frac{k}{n}-x\right)}+e^{-nx}\sum_{k\le nx}\frac{(nx)^k}{k!}\\
&\le e^{nx\left(e^{\frac{\alpha}{n\sqrt{x}}}-1\right)-\alpha\frac{x}{\sqrt{x}}}+1.
\end{aligned}$$

Using the inequality $e^t - 1 \leq te^t$, we obtain for every $x \geq 1$ and every $n \geq 1$ that

$$\frac{S_n(e^{\alpha\sqrt{t}}, x)}{e^{\alpha\sqrt{x}}} \leq e^{nx\frac{\alpha}{n\sqrt{x}}e^{\frac{\alpha}{n\sqrt{x}}} - \frac{\alpha x}{\sqrt{x}}} + 1 \leq e^{\alpha\sqrt{x}\frac{\alpha}{n\sqrt{x}}e^{\frac{\alpha}{n\sqrt{x}}}} + 1 \leq e^{\alpha^2 e^{\alpha}} + 1.$$

We have proved that

$$\|S_n\rho\|_\rho = \sup_{x\geq 0} \frac{S_n(\rho, x)}{\rho(x)} \leq C_\alpha, \tag{3.1}$$

where $C_\alpha > 0$ is a constant dependent of α, but independent of n. □

Corollary 3.2. *For a number $\alpha > 0$ and $\rho(x) = e^{\alpha\sqrt{x}}$, the Szász-Mirakjan operators $S_n\colon C_\rho[0,\infty) \to C_\rho[0,\infty)$ have the property that*

$$\|S_n f - f\|_\rho \to 0, \text{ as } n \to \infty$$

if and only if, the function

$$f(x^2)e^{-\alpha x} \text{ is uniformly continuous on } [0,\infty).$$

Moreover, for $f \in C_\rho[0,\infty)$ we have

$$\|S_n f - f\|_\rho \leq \|f\|_\rho \cdot \frac{\alpha C}{\sqrt{n}} + 2 \cdot \omega\left(f(t^2)e^{-\alpha t}, \frac{1}{\sqrt{n}}\right), \quad \text{for every } n \geq 1,$$

where $C = \sup_{n\in\mathbb{N}} \frac{1}{2}\sqrt{\|S_n\rho^2\|_{\rho^2} + 2\|S_n\rho\|_\rho + 1}$ is a constant depending only on α.

Proof. We set $\varphi(x) = \sqrt{x}$. The function $\rho(x) = e^{\alpha\sqrt{x}}$ verifies the relation (2.4) with equality.

We have the relations $S_n(1,x) = 1$ and $S_n((t-x)^2, x) = x/n$ (see [1], for example). We prove now the relation (2.7).

$$\begin{aligned}\sup_{x\geq 0} S_n(|\varphi(t) - \varphi(x)|^2, x) &= \sup_{x\geq 0} S_n\left(\frac{|t-x|^2}{\left(\sqrt{t}+\sqrt{x}\right)^2}, x\right)\\ &\leq \sup_{x\geq 0}\frac{S_n(|t-x|^2, x)}{x} = \frac{1}{n}.\end{aligned}$$

For a function $f \in C_\rho(I)$ the derivative $(S_n f)'(x)$ fulfills:

$$\begin{aligned}|(S_n f)'(x)| &= \left|\frac{n}{x}\sum_{k=0}^{\infty} f\left(\frac{k}{n}\right)\left(\frac{k}{n} - x\right)e^{-nx}\frac{(nx)^k}{k!}\right|\\ &\leq \frac{n}{x}\|f\|_\rho S_n(\rho(t)|t-x|, x) \leq \sqrt{C_{2\alpha}}\,\|f\|_\rho\,\rho(x)\frac{\sqrt{n}}{\sqrt{x}},\end{aligned}$$

because

$$S_n(\rho(t)|t-x|, x) \leq \sqrt{S_n(\rho^2(t), x)} \cdot \sqrt{S_n((t-x)^2, x)} \leq \sqrt{C_{2\alpha}}\rho(x)\sqrt{\frac{x}{n}}.$$

We obtain

$$\frac{|(S_n f)'(x)|}{\varphi'(x)} \leq C_{f,n,\alpha} \cdot \rho(x), \quad \text{for every } x \geq 0,$$

so, the relation (2.3) is proved. □

Remark 3.3. The result from Corollary 3.2 for the limit case, α=0, was obtained in [21], [23], [10] and [18].

Remark 3.4. In [16], it was proved that $S_n(f,x)$ exists for every function f with the property $f(x) = \mathcal{O}(e^{\alpha x \ln x})$, $\alpha > 0$ and moreover, $S_n f$ converges uniformly to f on compact subsets of the interval $[0,\infty)$. In [5], Becker studied the global approximation of functions using Szász-Mirakjan operators for the polynomial weight $\rho(x) = 1 + x^N$, $N \in \mathbb{N}$. Becker, Kucharsky and Nessel [6] studied the global approximation for the exponential weight $\rho(x) = e^{\beta x}$. But because

$$\sup_{x\geq 0} \frac{S_n(e^{\beta t},x)}{e^{\beta x}} = \sup_{x\geq 0} e^{nx(e^{\frac{\beta}{n}}-1)-\beta x} = +\infty,$$

they obtain results only for the space $C(\eta) = \cap_{\beta>\eta} C_\beta$, where C_β is C_ρ for $\rho = e^{\beta x}$. It is also mentioned, that for any $f \in C_\beta$ we have $S_n f \in C_\gamma$, for $\gamma > \beta$ and for $n > \beta/\ln(\gamma/\beta)$. Ditzian [11], also, gives some inverse theorems for exponential spaces. In [2], Amanov obtained that the condition

$$\sup_{x\geq 0} \frac{\rho(x+\sqrt{x})}{\rho(x)} < \infty$$

upon the weight ρ, is necessary and sufficient for the uniform boundedness of the norms of the operators $S_n\colon C_\rho[0,\infty) \to C_\rho[0,\infty)$. He mentions that this condition implies the inequality

$$\rho(x) \leq e^{\alpha\sqrt{1+x}}, \quad x \geq 0.$$

He, also, gives a characterization of the functions f which are uniformly approximated by $S_n f$ in the ρ-norm, using a weighted second order modulus of smoothness.

The fact that $\rho(x) = \mathcal{O}(e^{\alpha\sqrt{x}})$ is the maximal class of weights for which S_n maps $C_\rho[0,\infty)$ into $C_\rho[0,\infty)$ can be proved by the following argument: we take $\rho(x) = e^{\alpha\Phi(x)}$, $\alpha > 0$, where $\Phi(x)$ is a strictly increasing differentiable function with the properties that

$$\lim_{x\to\infty} \Phi(x) = \infty \quad \text{and} \quad \lim_{x\to\infty} \Phi'(x)\sqrt{x} = \infty, \tag{3.2}$$

and we prove that $\|S_n\rho\|_\rho$ is not finite for all $\alpha > 0$. From condition (3.2), by l'Hospital's rule, $\lim_{x\to\infty} \Phi(x)/\sqrt{x} = \infty$, so, there are $M > 1/\alpha$ and $x_0 \geq 1$ such that $\Phi(x) > M\sqrt{x}$, for $x \geq x_0$. We obtain for $x \geq x_0$

$$\left(e^{\alpha\Phi(x)}\right)'' = e^{\alpha\Phi(x)}\left[\alpha\Phi''(x) + [\alpha\Phi'(x)]^2\right] > e^{\alpha\Phi(x)}\left(-\frac{\alpha M}{4x^{\frac{3}{2}}} + \frac{(\alpha M)^2}{4x}\right) > 0,$$

so, $e^{\alpha\Phi}$ is convex on $[x_0,\infty)$. We can redefine Φ (if it is necessary), such that $e^{\alpha\Phi}$ is a convex function on $[0,\infty)$. By a result of Cheney and Sharma [9], we deduce that $S_n(\rho,x) \geq \rho(x)$. Suppose that

$$\lim_{x\to\infty} \frac{S_n(\rho,x)}{\rho(x)} = L_\alpha < \infty, \quad \text{for every } \alpha > 0. \tag{3.3}$$

Because $S_n\rho \geq \rho$, we obtain $L_\alpha \geq 1$. But, using l'Hospital's rule, we have

$$\begin{aligned} L_\alpha = \lim_{x\to\infty} \frac{S_n(e^{\alpha\Phi(t)},x)}{e^{\alpha\Phi(x)}} &= \lim_{x\to\infty} \frac{(S_n e^{\alpha\Phi})'(x)}{\alpha\Phi'(x)e^{\alpha\Phi(x)}} \\ &\leq \lim_{x\to\infty} \frac{\frac{n}{x}\sqrt{S_n((t-x)^2,x)}}{\alpha\Phi'(x)} \cdot \frac{\sqrt{S_n(e^{2\alpha\Phi(t)},x)}}{e^{\alpha\Phi(x)}} \\ &\leq \lim_{x\to\infty} \frac{1}{\alpha\sqrt{n}\Phi'(x)\sqrt{x}} \cdot \sqrt{\frac{S_n(e^{2\alpha\Phi(t)},x)}{e^{2\alpha\Phi(x)}}} \\ &= 0 \cdot \sqrt{L_{2\alpha}} = 0, \end{aligned}$$

which is a contradiction with $L_\alpha \geq 1$.

Lemma 3.5. *For every $\alpha > 0$ and $\rho(x) = (1+x)^\alpha$, the Baskakov operators defined by*

$$V_n f(x) = \sum_{k=0}^{\infty} \binom{n+k-1}{k} \frac{x^k}{(1+x)^{n+k}} f\left(\frac{k}{n}\right), \quad x \geq 0,$$

map $C_\rho[0,\infty)$ onto $C_\rho[0,\infty)$.

Proof. In [5], Becker proves that $V_n(1+t^N,x) \leq C_1(1+x^N)$, for every $x \geq 0$ and every $N \in \mathbb{N}$. We deduce that

$$V_n((1+t)^m,x) \leq C_2 \cdot V_n(1+t^m,x) \leq C_3(1+x^m) \leq C_3(1+x)^m,$$

for every $x \geq 0$ and every $m \in \mathbb{N}$. We prove, now, that for $\beta \in [0,1)$ we have $V_n((1+t)^\beta,x) \leq C_4(1+x)^\beta$. Using the inequality $\ln(1+t) \leq t$, for $t > -1$, we obtain

$$\begin{aligned} \frac{V_n((1+t)^\beta,x)}{(1+x)^\beta} &= \sum_{k=0}^{\infty} \binom{n+k-1}{k} \frac{x^k}{(1+x)^{n+k}} e^{\beta\left[\ln\left(1+\frac{k}{n}\right)-\ln(1+x)\right]} \\ &= \sum_{k=0}^{\infty} \binom{n+k-1}{k} \frac{x^k}{(1+x)^{n+k}} e^{\beta\ln\left(1+\frac{\frac{k}{n}-x}{1+x}\right)} \\ &\leq \sum_{k=0}^{\infty} \binom{n+k-1}{k} \frac{x^k}{(1+x)^{n+k}} e^{\beta\frac{\frac{k}{n}-x}{1+x}} \\ &= \left(1-x\left(e^{\frac{\beta}{n(1+x)}}-1\right)\right)^{-n} \cdot e^{\frac{-\beta x}{1+x}}. \end{aligned}$$

The last expression is well-defined for every $x \geq 0$ and every $n \geq 1$, because

$$1 - x\left(e^{\frac{\beta}{n(1+x)}}-1\right) \geq 1 - \lim_{x\to\infty} x\left(e^{\frac{\beta}{n(1+x)}}-1\right) = 1 - \frac{\beta}{n} > 0.$$

Because of the inequality

$$\sup_{x\geq 0}\left[\left(1-x\left(e^{\frac{\beta}{n(1+x)}}-1\right)\right)^{-n} \cdot e^{\frac{-\beta x}{1+x}}\right] \leq \left(1-\frac{\beta}{n}\right)^{-n} \cdot 1 \leq \frac{1}{1-\beta},$$

we deduce that $V_n((1+t)^\beta,x) \leq C_4(1+x)^\beta$, for every $x \geq 0$, where C_4 is a constant not depending on x and n.

For $\alpha > 0$, we choose $m = [2\alpha] \in \mathbb{N}$ and $\beta = 2\alpha - m \in [0,1)$. Using Cauchy-Schwarz inequality, we obtain

$$\begin{aligned} V_n((1+t)^\alpha, x) &= V_n((1+t)^{\frac{m}{2}} \cdot (1+t)^{\frac{\beta}{2}}, x) \\ &\le \sqrt{V_n((1+t)^m, x) \cdot V_n((1+t)^\beta, x)} \\ &\le \sqrt{C_3(1+x)^m \cdot C_4(1+x)^\beta} = C_5(1+x)^\alpha, \end{aligned}$$

which proves that $V_n\rho \in C_\rho[0,\infty)$. □

Corollary 3.6. *For a real number $\alpha > 0$ and for $\rho(x) = (1+x)^\alpha$ the Baskakov operators $V_n\colon C_\rho[0,\infty) \to C_\rho[0,\infty)$ have the property that*

$$\|V_n f - f\|_\rho \to 0, \ \ \text{as } n \to \infty$$

if and only if

$$f(e^x - 1)e^{-\alpha x}, \ \text{is uniformly continuous on } [0,\infty).$$

Moreover, for $f \in C_\rho[0,\infty)$ and for $n \ge 2$, we have

$$\|V_n f - f\|_\rho \le \|f\|_\rho \cdot \frac{\alpha C}{\sqrt{n-1}} + 2 \cdot \omega\left(f(e^t - 1)e^{-\alpha t}, \frac{1}{\sqrt{n-1}}\right).$$

Proof. Setting $\varphi(x) = \ln(1+x)$, the function $\rho(x) = (1+x)^\alpha$ verifies the relation (2.4) with equality.

We have the relations $V_n(1,x) = 1$, $V_n(t,x) = x$ and $V_n((t-x)^2, x) = x(1+x)/n$ (see [1], for example). We prove now the relation (2.7). Using the inequality (2.8) we have

$$\begin{aligned} |\varphi(t) - \varphi(x)|^2 &= |\ln(1+t) - \ln(1+x)|^2 \\ &\le \frac{|t-x|^2}{(1+t)(1+x)} = \left|\sqrt{\frac{1+t}{1+x}} - \sqrt{\frac{1+x}{1+t}}\right|^2 \end{aligned}$$

and using the fact that

$$\begin{aligned} V_n\left(\frac{1}{1+t}, x\right) &= \sum_{k=0}^{\infty} \binom{n+k-1}{k} \frac{x^k}{(1+x)^{n+k}} \cdot \frac{n}{n+k} \\ &\le \frac{n}{(n-1)(1+x)} \sum_{k=0}^{\infty} \binom{n+k-2}{k} \frac{x^k}{(1+x)^{n-1+k}} \\ &= \frac{n}{(n-1)(1+x)} \end{aligned}$$

we obtain

$$\begin{aligned} V_n(|\varphi(x) - \varphi(t)|^2, x) &\le \frac{V_n(1+t,x)}{1+x} - 2V_n(1,x) + (1+x)V_n\left(\frac{1}{1+t}, x\right) \\ &\le 1 - 2 + \frac{n}{n-1} = \frac{1}{n-1}, \ \ \text{for } \ n \ge 2. \end{aligned}$$

The derivative $(V_n f)'(x)$ verifies the relation

$$|(V_n f)'(x)| = \left| \frac{n}{x(1+x)} \sum_{k=0}^{\infty} f\left(\frac{k}{n}\right)\left(x - \frac{k}{n}\right)\binom{n+k-1}{k}\frac{x^k}{(1+x)^{n+k}} \right|$$
$$\leq \frac{n}{x(1+x)} \|f\|_\rho \cdot V_n(\rho(t)|t-x|, x) \leq C_1 \|f\|_\rho \rho(x) \frac{\sqrt{n}}{\sqrt{x(1+x)}},$$

because

$$V_n(\rho(t)|t-x|, x) \leq \sqrt{V_n(\rho^2(t), x)} \cdot \sqrt{V_n((t-x)^2, x)} \leq C_1 \rho(x) \sqrt{\frac{x(x+1)}{n}}.$$

We obtain

$$\frac{|(V_n f)'(x)|}{\theta'(x)} \leq C_2 \rho(x), \quad \text{for every } x \geq 0,$$

where $\theta(x) = \ln\left(x + \frac{1}{2} + \sqrt{x(1+x)}\right)$. The inequality

$$\frac{\rho'(x)}{\theta'(x)} = \alpha(1+x)^{\alpha-1}\sqrt{x(1+x)} \leq \alpha \cdot \rho(x), \quad x \geq 0,$$

proves the relation (2.4) for the function θ instead of φ. Using the fact that

$$(\theta \circ \varphi^{-1})(x) = \ln\left(e^x - \frac{1}{2} + \sqrt{(e^x - 1)e^x}\right)$$

is a uniformly continuous function on $[0, \infty)$ (this is true, because it is a continuous function with the property that $(\theta \circ \varphi^{-1})(x) - x$ has finite limit at infinity) and using Theorem 2.1, the Remarks 2.3 and 2.5, the proof of the corollary is complete. □

Remark 3.7. The result of the Corollary 3.6 for the limit case, $\alpha = 0$, was obtained in [22], [23], [10] and [18].

Remark 3.8. Becker [5] studied the global approximation of functions from the polynomial weighted space and remarked that "polynomial growth is the frame best suited for global results for the Baskakov operators". The reason is that for the exponential weight $\rho(x) = e^{\beta x}$, the series $V_n(\rho, x)$ exists only for $x < (e^{\frac{\beta}{n}} - 1)^{-1}$. Nevertheless, Ditzian [11] gave some inverse results for functions with exponential growth.

References

[1] Altomare, F., Campiti, M., *Korovkin-Type Approximation Theory and its Applications*, De Gruyter Series Studies in Mathematics, Vol. 17, Walter de Gruyter & Co., Berlin, New York, 1994.

[2] Amanov, N.T., *On the weighted approximation by Szász-Mirakjan operators*, Anal. Math., **18**(1992), 167–184.

[3] Boyanov, B.D., Veselinov, V.M., *A note on the approximation of functions in an infinite interval by linear positive operators*, Bull. Math. de la Soc. Sci. Math. de la R.S. de Roumanie, **14**(62)(1970), no. 1, 9-13.

[4] Bustamante, J., Morales de la Cruz, L., *Korovkin Type Theorems for Weighted Approximation*, Int. Journal of Math. Analysis, **1**(2007), no. 26, 1273-1283.

[5] Becker, M., *Global Approximation Theorems for Szász-Mirakjan and Baskakov Operators in Polynomial Weight Spaces*, Indiana Univ. Math. J., **27**(1978), no. 1, 127–142.

[6] Becker, M., Kucharski, D., Nessel, R.J., *Global approximation for the Szasz-Mirakjan operators in exponential weight spaces*, Linear Spaces and Approximation (Proc. Conf. Math. Res. Inst. Oberwolfach, 1977), Internat. Ser. Numer. Math., Birkhäuser Verlag, Basel, **40**(1978), 318–333.

[7] Bustamante, J., Morales de la Cruz, L., *Positive linear operators and continuous functions on unbounded intervals*, Jaen J. Approx., **1**(2009), no. 2, 145–173.

[8] Cheney, E.W., *Introduction to Approximation Theory*, AMS Chelsea Publishing, 1966.

[9] Cheney, E.W., Sharma, A., *Bernstein power series*, Canad. J. Math., **16**(1964), no. 2, 241–252.

[10] de la Cal, J., Cárcamo, J., *On uniform approximation by some classical Bernstein-type operators*, J. Math. Anal. Appl., **279**(2003), 625–638.

[11] Ditzian, Z., *On global inverse theorems of Szász and Baskakov operators*, Canad. J. Math., **31**(1979), no. 2, 255–263.

[12] Ersan, S., Doğru, O., *Statistical approximation properties of q-Bleimann, Butzer and Hahn operators*, Math. Comput. Modelling, **49**(2009), 1595-1606.

[13] Gadjiev, A.D., *Theorems of Korovkin type*, Mat. Zametki., **20**(5)(1976), 781–786, (in Russian), Math. Notes, **20**(5)(1976), 996–998.

[14] Gadjiev, A.D., Çakar, Ö., *On uniform approximation by Bleimann, Butzer and Hahn operators on all positive semiaxis*, Trans. Natl. Acad. Sci. Azerb. Ser. Phys.-Tech. Math. Sci., **19**(1999), no. 5, 21–26.

[15] Gonska, H.H., *On approximation in spaces of continuous functions*, Bull. Austral. Math. Soc., **28**(1983), 411–432.

[16] Hermann, T., *Approximation of unbounded functions on unbounded interval*, Acta Math. Hungar., **29**(3-4)(1977), 393–398.

[17] Holhoş, A., *The rate of convergence of positive linear operators in weighted spaces*, Automat. Comput. Appl. Math., **17**(2008), no. 2, 239–246.

[18] Holhoş, A., *Uniform Approximation by Positive Linear Operators on Noncompact Intervals*, Automat. Comput. Appl. Math., **18**(2009), no. 1, 121–132.

[19] Mitrinović, D.S., Pečarić, J., Fink, A.M., *Classical and New Inequalities in Analysis*, Kluwer Academic Publishers, Dordrecht, 1993.

[20] Nishishiraho, T., *Convergence of positive linear approximation processes*, Tohoku Math. J., **35**(1983), 441–458.

[21] Totik, V., *Uniform approximation by Szász-Mirakjan type operators*, Acta Math. Hungar., **41**(3-4)(1983), 291–307.

[22] Totik, V., *Uniform approximation by Baskakov and Meyer-König and Zeller operators*, Period. Math. Hungar., **14**(3-4)(1983), 209–228.

[23] Totik, V., *Uniform approximation by positive operators on infinite intervals*, Anal. Math., **10**(1984), 163–182.

Adrian Holhoş
Technical University of Cluj-Napoca
28, Memorandumului Street
400114 Cluj-Napoca, Romania
e-mail: adrian.holhos@math.utcluj.ro

Stud. Univ. Babeş-Bolyai Math. 56(2011), No. 3, 147–160

Some new properties of Generalized Bernstein polynomials

Donatella Occorsio

Abstract. Let $B_m(f)$ be the Bernstein polynomial of degree m. The Generalized Bernstein polynomials

$$B_{m,\lambda}(f,x) = \sum_{i=1}^{\infty}(-1)^{i+1}\binom{\lambda}{i}B_m^i(f;x), \lambda \in \mathbb{R}^+$$

were introduced in [13]. In the present paper some of their properties are revisited and some applications are presented. Indeed, the stability and the convergence of a quadrature rule on equally spaced knots is studied and a class of curves depending on the shape parameter λ, including both Bézier and Lagrange curves, is introduced.

Mathematics Subject Classification (2010): 41A10, 65D30, 65D17.

Keywords: Linear operators, approximation by polynomials, numerical integration, Computer Aided Geometric Design.

1. Introduction

The operator $B_{m,\lambda}$, introduced and studied in [13], is defined as

$$B_{m,\lambda} = \sum_{i=1}^{\infty}(-1)^{i+1}\binom{\lambda}{i}B_m^i, \quad \lambda \in \mathbb{R}^+,$$

where $B_m^i = B_m(B_m^{i-1})$, and B_m is the Bernstein operator. $B_{m,\lambda}$ is a linear operator, not always positive, that maps bounded functions into polynomials of degree at most m. The sequence $\{B_{m,\lambda}(f)\}_m$ has the property of improving the order of convergence when the smoothness of the function increases (see [11, 14]). For instance, assuming $f \in C^{(2[\lambda])}([0,1]), \lambda \geq 1$, we have $|f - B_{m,\lambda}(f)| = O\left(\frac{1}{m^\lambda}\right)$. In this sense, the sequence $\{B_{m,\lambda}(f)\}_m$ produces a significant enhancement with respect to the behavior of the ordinary Bernstein sequence.

Moreover the sequence $\{B_{m,\lambda}(f)\}_m$ includes both Bernstein polynomials ($\lambda = 1$) and, as limit case, the Lagrange interpolating polynomial on equally spaced knots ($\lambda \to \infty$). In spite of these mentioned properties, the expression derived in [13] in the monomial basis $(1, x, \ldots, x^m)$ is no easy for the computation and, in addition, produces instability in the polynomial evaluation.

In the present paper we first express $B_{m,\lambda}(f)$ as the Bernstein polynomial of a function g, suitable related to f. Therefore, the evaluation of $B_m(g)$ can be performed by de Casteljau scheme, which is a stable algorithm. Moreover, using $B_{m,\lambda}(f) = B_m(g)$, we can revisit some proofs, like, for instance, the property of mapping bounded functions into polynomials. In order to exploit the above mentioned "good" properties, we consider two applications. The first is the approximation of integrals $\int_0^1 f(x)dx$, obtained by replacing the function f with $B_{m,\lambda}(f)$. By this way, it is derived a simple quadrature rule that we prove to be stable and convergent and whose order of accuracy as faster decays as smoother is the integrand function f. Such kind of formulas can be of interest since there are not so many polynomial quadrature rules involving equally spaced points and having a "good" behavior of the error.

The second application deals with the employment of $\{B_{m,\lambda}\}_\lambda$ in CAGD (Computer Aided Geometric Design), by considering a possible generalization of the well-known Bézier curves. Given a control polygon

$$\mathbf{P} = [\mathbf{P}_0, \ldots, \mathbf{P}_m], \ \mathbf{P}_j \in \mathbb{R}^2,$$

we call the curves of parametric equations

$$\mathbf{B}_{m,\lambda}[\mathbf{P}_0, \ldots, \mathbf{P}_m](t) = \sum_{j=0}^{m} p_{m,j}^{(\lambda)}(t)\mathbf{P}_j, \quad 0 \le t \le 1,$$

Generalized Bézier curves. Curves in this class change continuously their shape, "bridging" the Bézier curve $\mathbf{B}_m[\mathbf{P}_0, \ldots, \mathbf{P}_m]$ to the Lagrange interpolating curve $\mathbf{L}_m[\mathbf{P}_0, \ldots, \mathbf{P}_m]$. Some generalization in this sense where introduced and studied in [2], [3], [4], [15] (see also [9], [16]).

The outline of this paper is as follows. Section 2 contains the new vector expression and some properties deducible from this. In Section 3 are stated the announced applications, equipped with some numerical and graphical tests. Finally, Section 4 will contain the proofs of the main results.

2. The $B_{m,\lambda}(f)$ polynomials

For any continuous function f on the unit interval $[0, 1]$ ($f \in C^0([0, 1])$), let $B_m(f)$ be the $m-$th Bernstein polynomial

$$B_m(f; x) = \sum_{k=0}^{m} p_{m,k}(x) f\left(\frac{k}{m}\right), \quad p_{m,k}(x) = \binom{m}{k} x^k (1-x)^{m-k}. \qquad (2.1)$$

Denoting by $B_m^i(f) = B_m(B_m^{i-1}(f)), B_m^0(f) = f$ the i-th iterate of the Bernstein polynomial, in [13] (see also [1], [8], [12]) the authors introduced and studied the following linear combination of $B_m^i(f)$,

$$B_{m,\lambda}(f,x) = \sum_{i=1}^{\infty}(-1)^{i+1}\binom{\lambda}{i}B_m^i(f;x), \lambda \in \mathbb{R}^+. \tag{2.2}$$

For any fixed λ, $\{B_{m,\lambda}(f)\}_m$, will be called sequence of *generalized Bernstein polynomials of parameter* λ. For $\lambda = 1$, $B_{m,\lambda} = B_m$. The special case $\lambda \in \mathbf{N}$ was studied in [12]. Here we will consider the case $\lambda \geq 1$. An expression of the polynomial $B_{m,\lambda}(f)$ is

$$B_{m,\lambda}(f;x) = \sum_{j=0}^{m} p_{m,j}^{(\lambda)}(x) f\left(\frac{j}{m}\right), \quad 0 \leq x \leq 1, \tag{2.3}$$

where

$$p_{m,j}^{(\lambda)}(x) = \sum_{i=1}^{\infty}\binom{\lambda}{i}(-1)^{i-1}B_m^{i-1}(p_{m,i};x). \tag{2.4}$$

Since by (2.3) the evaluation of $B_{m,\lambda}$ is not feasible, first we derive a vectorial form of the basis $\{p_{m,k}^{(\lambda)}\}_{k=0}^m$, by which for any function f, the polynomial $B_{m,\lambda}(f)$ coincides with the Bernstein polynomial $B_m(g)$, g being a function related to f.

Theorem 2.1. *Assume* $\lambda \geq 1$. *Setting*

$$\mathbf{p}_m^{(\lambda)}(x) = [p_{m,0}^{(\lambda)}(x), p_{m,1}^{(\lambda)}(x), \ldots, p_{m,m}^{(\lambda)}(x)]^T,$$

and

$$\mathbf{p}_m(x) = [p_{m,0}(x), \ldots, p_{m,m}(x)]^T,$$

one has

$$\mathbf{p}_m^{(\lambda)}(x)^T = \mathbf{p}_m(x)^T C_{m,\lambda}, \tag{2.5}$$

where

$$C_{m,\lambda} = A^{-1}[I-(I-A)^{\lambda}] = [I-(I-A)^{\lambda}]A^{-1} \in \mathbb{R}^{(m+1)\times(m+1)}, \tag{2.6}$$

$$(A)_{i,j} = p_{m,j}(t_i), \quad i = 0,1,\ldots,m, j = 1,2,\ldots,m \tag{2.7}$$

$t_i = i/m, i = 0,1,\ldots,m$, *and* I *is the identity matrix of order* $(m+1)$. *Then, for any* $f \in C^0([0,1])$, *setting*

$$\mathbf{f}_m = [f_0, f_1, \ldots, f_m]^T, \quad f_i = f(t_i), \tag{2.8}$$

the polynomial $B_{m,\lambda}(f)$ *can be represented in the following form*

$$B_{m,\lambda}(f;x) = \mathbf{p}_m(x)^T C_{m,\lambda}\mathbf{f}_m. \tag{2.9}$$

In the case $\lambda = k \in \mathbb{N}$, the matrix $C_{m,\lambda}$ is given by

$$\begin{aligned} C_{m,k} &= [I + (I-A) + (I-A)^2 + \ldots (I-A)^{k-1}] \\ &= A^{-1}[I-(I-A)^k] \end{aligned} \tag{2.10}$$

and the polynomial $B_{m,\lambda}(f)$ is directly computed by using a very simple algorithm, as the expression in (2.10) suggests. However, when λ is not an integer, the matrix series in (2.2) can be obtained by an equivalent finite process. To do this, we need the following definition of matrix function on the spectrum (see for instance [10]).

Definition 2.2. *Let B a real matrix of order n and suppose that $\xi_1, \xi_2, \dots, \xi_s$ are the distinct eigenvalues of B of algebraic multiplicity $n_1, n_2, \dots, n_s$, respectively. Let f be defined on the spectrum of B. Then $f(B) := H_n(f; B)$, where $H_n(f)$ is the Hermite interpolating polynomial of degree less than n that satisfies the interpolation conditions*

$$H_n(f^{(j)}; \xi_i) = f^{(j)}(\xi_i), \quad j = 0, 1, \dots, n_i - 1, \quad i = 1, 2, \dots, s.$$

Denote by $\Delta_h f(x) = f(x+h) - f(x)$ the forward difference of the function f and shift $h \in \mathbb{R}$, and let be $\Delta_h^i = \Delta_h^{i-1}(\Delta_h)$. About the eigenvalues of the matrix A we prove:

Proposition 2.3. *The eigenvalues $\{\xi_{m,i}\}_{i=0}^m$ of the matrix A are*

$$\xi_{m,0} = \xi_{m,1} = 1, \quad \xi_{m,i} = \prod_{j=1}^{i-1} \left(1 - \frac{j}{m}\right) = \binom{m}{i} \Delta_{\frac{1}{m}}^i e_i(0), \quad i = 2, \dots, m, \tag{2.11}$$

with $e_k(x) = x^k, k \in \mathbb{N}$. Therefore, denoting by $\{\mu_i\}_{i=1}^m$ the eigenvalues of $C_{m,\lambda}$,

$$\mu_{m,0} = \mu_{m,1} = 1, \quad \mu_{m,i} = \frac{1 - (1 - \xi_{m,i})^\lambda}{\xi_{m,i}}, \quad i \geq 2. \tag{2.12}$$

For any set of knots $x_1, x_2, \dots, x_i$, the so-called *divided differences* of a given function f are defined recursively by

$$[x_1; f] = f(x_1),$$

$$[x_1, ..., x_k; f] = \frac{[x_2, ..., x_k; f] - [x_1, ..., x_{k-1}; f]}{x_k - x_1}, \quad \text{if } x_k \neq x_{k-1}$$

and, if $f^{(i-1)}(x_1)$ exists,

$$[x_1, x_2, \dots, x_i; f] = \frac{f^{(i-1)}(x_1)}{(i-1)!}, \quad \text{if } x_1 = x_2 = \dots = x_i, \quad i \geq 2.$$

Then, by using Proposition 2.3 and Definition 2.2, we can deduce

Corollary 2.4. *Assume $\lambda \geq 1$. Setting*

$$\sigma(x) = [1 - (1-x)^\lambda] x^{-1},$$

we have

$$C_{m,\lambda} = I\sigma(\xi_{m,0}) + \sum_{j=1}^m [\xi_{m,0}, \xi_{m,1}, \dots, \xi_{m,j}; \sigma] \prod_{k=0}^{j-1} (A - \xi_{m,k} I) =: \rho(A). \tag{2.13}$$

Therefore

$$B_{m,\lambda}(f; x) = \mathbf{p}_m(x)^T \rho(A) \mathbf{f}_m, \tag{2.14}$$

Remark 2.5. By the previous result it follows that $B_{m,\lambda}(f)$ can be considered as the $m-$th Bernstein polynomial of the function g such that

$$g_k := g(t_k) = [\rho(A)\mathbf{f}_m]_k, \quad k = 0, 1, \ldots, m,$$

i.e.

$$B_m(g; x) = B_{m,\lambda}(f; x) = \mathbf{p}_m(x)^T \mathbf{g}_m,$$

where

$$\mathbf{g}_m := [g_0, g_1, \ldots, g_m]^T. \tag{2.15}$$

As a consequence, we can now compute the polynomial $B_{m,\lambda}(f)$ by using the de Casteljau recursive scheme.

Remark 2.6. Let us denote by $L_m(f)$ the Lagrange polynomial interpolating f at the equally spaced knots $t_j, \quad j = 0, 1, \ldots, m$, i.e.

$$L_m(f; x) = \sum_{j=0}^{m} l_{m,j}(x) f(t_j) = \mathbf{l}_m(x)^T \mathbf{f}_m,$$

where

$$l_{m,j}(x) = \prod_{j \neq i=1}^{m} \frac{(x - t_i)}{(t_j - t_i)}, \quad \mathbf{l}_m(x) = [l_{m,0}(x), l_{m,1}(x), \ldots, l_{m,m}(x)]^T,$$

and $\mathbf{f}_m$ is defined in (2.8). By (2.9) and using $\mathbf{p}_m(x)^T A^{-1} = \mathbf{l}_m(x)^T$ [15], it follows

$$B_{m,\lambda}(f; x) = L_m(h; x) = \mathbf{l}_m(x)^T \mathbf{h}_m \tag{2.16}$$

where

$$\mathbf{h}_m := [h_0, h_1, \ldots, h_m]^T = C_{m,\lambda}\mathbf{f}_m, \quad h_i = h(t_i), \quad i = 0, 1, \ldots, m, \tag{2.17}$$

i.e. $B_{m,\lambda}(f)$ is also the Lagrange polynomial interpolating the function h at the equally spaced knots $t_j, \quad j = 0, 1, \ldots, m$.

As consequence of (2.16), it is very easy to revisit the proof of the next result obtained in [13]:

For any m,

$$\lim_{\lambda \to \infty} B_{m,\lambda}(f; x) = L_m(f; x), \quad \forall f \in C^0([0, 1]), \tag{2.18}$$

uniformly in $x \in [0, 1]$. Indeed, it immediately follows by (2.16), (2.17) and

$$\lim_{\lambda \to \infty} C_{m,\lambda} = A^{-1}. \tag{2.19}$$

Relation (2.18) allows to say that the sequence $\{B_{m,\lambda}\}_\lambda$ links continuously the Bernstein operator to the Lagrange one.

In the next Proposition we derive another representation of $B_{m,\lambda}(f)$ by means of the finite difference of the function f at the point 0. This expression generalizes the well-known relation

$$B_m(f; x) = \sum_{k=0}^{m} \binom{m}{k} x^k \Delta^k_{\frac{1}{m}} f(0), \tag{2.20}$$

and it is useful to determine the closed expression of $B_{m,\lambda}(e_k), \quad k = 1, 2, \ldots,$ being $e_k(x) = x^k, k \in \mathbb{N}$.

Theorem 2.5. *Assume $\lambda \geq 1$. Let M be the upper triangular matrix of elements $(M)_{i,j} = \binom{m}{i}\Delta^i_{\frac{1}{m}} e_j(0), \quad i = 0, 1, \ldots, m, j = 0, 1, \ldots, i$, and define*

$$M_{m,\lambda} = M^{-1}[I - (I - M)^\lambda] = [I - (I - M)^\lambda]M^{-1} \in \mathbb{R}^{(m+1)\times(m+1)}. \quad (2.21)$$

For any $f \in C^0([0, 1])$, setting

$$\mathbf{d}_m = [f(0), m\Delta_{\frac{1}{m}} f(0), \ldots, \binom{m}{k}\Delta^k_{\frac{1}{m}} f(0), ..., \Delta^m_{\frac{1}{m}} f(0)]^T,$$

the polynomial $B_{m,\lambda}(f)$ can be represented in the following form

$$B_{m,\lambda}(f; x) = \mathbf{x}^T M_{m,\lambda}\mathbf{d}_m. \quad (2.22)$$

and also

$$B_{m,\lambda}(f; x) = \mathbf{x}^T \rho(M)\mathbf{d}_m, \quad (2.23)$$

where

$$\rho(M) = I\sigma(\xi_{m,0}) + \sum_{j=1}^{m}[\xi_{m,0}, \xi_{m,1}, \ldots, \xi_{m,j}; \sigma] \prod_{k=0}^{j-1}(M - \xi_{m,k}I). \quad (2.24)$$

$\sigma(x) = [1 - (1 - x)^\lambda]x^{-1}$ *and* $\xi_{m,i} = \binom{m}{i}\Delta^i_{\frac{1}{m}} e_i(0)$.

Remark 2.6. In view of (2.23), we derive

$$B_{m,\lambda}(e_k; x) = \mathbf{x}^T \tilde{\rho}(A)_k \tilde{\mathbf{d}}_k \quad (2.25)$$

where $\tilde{\rho}(A)_k \in \mathbb{R}^{(m+1)\times k}$ is the matrix formed by the first k columns of $\rho(A)$ and $\tilde{\mathbf{d}}_k \in \mathbb{R}^k$ is the vector formed by the first k components of $\mathbf{d}$.

Remark 2.7. Denoting by $\mathbf{V_m} := \mathbf{V_m}(t_0, t_1, \ldots, t_m)$ the Vandermonde matrix w.r.t the knots $t_0, t_1, \ldots, t_m$, i.e. $(\mathbf{V}_m)_{i,j} = t_i^j, \quad i = 0, 1, \ldots, m, j = 0, 1, \ldots, m$, we get

$$M_{m,\lambda} = \mathbf{V}_m^{-1} C_{m,\lambda}\mathbf{V}_m \quad (2.26)$$

which easily follows by combining $\mathbf{V}_m\mathbf{d}_m = \mathbf{f}_m$ and $\mathbf{x}^T = \mathbf{p}_m^T\mathbf{V}_m$.

We conclude this section, giving some details about the computation of polynomials $B_{m,\lambda}(f)$. Since the polynomial $B_{m,\lambda}(f; x)$ is also the Bernstein polynomial of the function $g = C_{m\lambda}\mathbf{f}$, it can be computed by using the de Casteljau algorithm w.r.t. g. The algorithm is numerically stable and requires m^2 long operations, for any $x \in [0, 1]$. Since A is a centrosymmetric matrix (i.e. $a_{i,j} = a_{m-i,m-j}, i, j = 0, 1, 2, \ldots, m$), we deduce that its construction can be performed in $\left[\frac{m+1}{2}\right]^3$ long operations. Let us distinguish between the case λ integer or not. If $\lambda = k \in \mathbb{N}$, by (2.10), the global cost to construct $C_{m,k}$ is $(k-1)\left[\frac{m+1}{2}\right]^3$. A significant reduction is obtained by choosing $k = 2^p$, whereas, by using

$$C_{m,2^p} = C_{m,2^{p-1}} + (I - A)^{2^{p-1}} C_{m,2^{p-1}},$$

the computational effort is almost $m^3 \log_2 k$. (see [15].)
In the general case $\lambda \in \mathbb{R}^+$, we have to use (2.13) and the global cost for compute $C_{m,\lambda}$ increases, requiring almost $(m-2)m^3/2 \sim m^4/2$. Even though the computation of $C_{m,\lambda}$ requires the major computational effort, for fixed values of m and λ its construction can be performed only once.

3. Two applications

In this section we discuss two different applications.

3.1. A quadrature rule on equally spaced knots

As we have said, quadrature rules involving equally spaced points and having a "good" behavior of the error can be of interest. Indeed, the Newton-Cotes rules present catastrophic instability, since they are based on interpolation processes on equally spaced knots. About the error of composite rules, like Trapezoidal or Simpson rule, they suffer from saturation phenomena, and the error decays like $O(\frac{1}{m^2})$ and $O(\frac{1}{m^4})$, respectively. Here we revisit the following quadrature rule suggested in [12],

$$\int_0^1 f(x)dx = \int_0^1 B_{m,k}(f;x)dx + R_m^k(f) =: \Sigma_m(f) + R_m^k(f), \tag{3.1}$$

where $\lambda = k \in \mathbb{N}$.
Since for any $j = 0, 1, \ldots, m$

$$\int_0^1 p_{m,j}(x)dx = \frac{1}{m+1},$$

by (2.9) and (2.10), we derive

$$\Sigma_m(f) = \frac{1}{m+1} \sum_{j=0}^{m} \left(\sum_{i=0}^{m} (C_{m,k})_{i,j} \right) f(t_j) := \sum_{j=0}^{m} D_j^{(k)} f(t_j). \tag{3.2}$$

Now we prove that the rule is numerically stable and convergent and that for smooth functions the rate of convergence improves as the parameter k increases.

Theorem 3.1. *With the notation used in (3.1)-(3.2),*

$$\sup_m \sum_{j=0}^{m} |D_j^{(k)}| < \infty, \tag{3.3}$$

and for any $f \in C^{2k}([0,1]), k \geq 2, 2k < m$

$$|R_m^k(f)| \leq \frac{\mathcal{C}}{m^k} \left(\|f\|_\infty + \|f^{(2k)}\|_\infty \right), \tag{3.4}$$

where $\mathcal{C}$ is a positive constant independent of f and m.

Now we show the performance of the method by some numerical tests. In the tables for each degree m and for the specified values ok k, we report the values obtained in computing the quadrature sum (3.2) in 16−digits precision, comparing also with the results obtained by using the composite Trapezoidal and Simpson rules. For these rules the value of m represents the number of function evaluations.

Example 3.2.

$$\int_0^1 \frac{\arctan(x)}{(1+x^2)^3}dx$$

In this example the exact value is 0.1713839674246280. Here $f \in C^\infty([0,1])$. The apparent slow convergence depends on the "fast" increasing values of the seminorm $\|f^{(2k)}\|_\infty$. For instance $\|f^{(16)}\|_\infty \sim 1.2 \times 10^{15}$.

m	$k=5$	$k=8$	$k=16$
4	0.17	0.17	0.17
8	0.171	0.171	0.17138
16	0.17138	0.171383	0.171383
32	0.17138396	0.17138395	0.17138396
64	0.171383967	0.1713839674	0.17138396742
128	0.1713839674	0.171383967424	**0.171383967424462**
256	0.171383967424	**0.171383967424628**	
512	**0.171383967424628**		

m	Trapezoidal	m	Simpson
256	0.17138	64	0.173839
512	0.171383	128	0.17383967
4096	0.17138396	256	0.173839674
16384	0.171383967	512	0.1738396742
131072	0.1713839674	1024	0.17383967424
1048576	0.171383967424	2048	0.1713839674246
4194304	**0.17138396742462**	4096	**0.171383967424628**

Example 3.3.

$$\int_0^1 \frac{(1-x)^{5\pi}}{1+x^3}dx$$

In this example the exact value is 0.0597973223176919. Since the function $f \in C^{15}([0,1])$, in view of the Theorem 3.1, the error behaves like $O\left(\frac{1}{m^7}\right)$. As we can see the machine precision is attained for $m = 1024, \quad k = 7$, whereas according to the estimate (3.4) and taking into account the high value of the seminorm $\|f^{(14)}\|_\infty \sim 1.5 \times 10^{15}$, we can expect only 5 exact digits. We remark that the order of convergence improves even though k exceeds the maximum value assuring estimate (3.4).

m	$k=7$	$k=16$	$k=32$
32	0.059	0.059797	0.0597973
64	0.059797	0.059797322	0.0597973223
128	0.059797322	0.059797322317	**0.0597973223176919**
256	0.05979732231	**0.0597973223176919**	
512	0.05979732231769		
1024	**0.0597973223176919**		

m	Trapezoidal	m	Simpson
256	0.059	64	0.05979
1024	0.05979	128	0.0597973
2048	0.059797	512	0.059797322
16384	0.05979732	1024	0.0597973223
131072	0.0597973223	4096	0.059797322317
2097152	0.171383967424	8192	0.05979732231769
3600000	**0.0597973223176**	32768	**0.059797322317691**

As can be observed, the number of function's evaluation required w.r.t. Trapezoidal and Simpson rules is drastically reduced. This aspect can justify the high computational cost needed for the construction of $C_{m,k}$ in (3.2).

3.2. Generalized Bézier curves

Finally we want to show some properties of the parametric curves based on $B_{m,\lambda}$ operator and that in some sense generalize the classical Bézier curves. Such a kind of curves were introduced and studied in [15] in the special case $\lambda \in \mathbb{N}$.

The class of Polya curves represent, for instance, a family of polynomial curves which generalizes Bézier and Lagrange curves (see [2],[3], [4]).

Definition 3.4. *Let* $\mathbf{P} = [\mathbf{P}_0, \ldots, \mathbf{P}_m]^T$,$\mathbf{P}_j \in \mathbb{R}^2$ *be a given control polygon. Curves of parametric equations*

$$\mathbf{B}_{m,\lambda}[\mathbf{P}_0, \ldots, \mathbf{P}_m](t) = \sum_{j=0}^{m} p_{m,j}^{(\lambda)}(t)\mathbf{P}_j, \quad 0 \le t \le 1, \lambda \in \mathbb{R}^+, \tag{3.5}$$

with blending functions $p_{m,j}^{(\lambda)}$ *given in (*2.4*), will be called GB(*λ*) curves. In particular the curve of equation (*3.5*) reduces to Bézier curve for* $\lambda = 1$

$$\mathbf{B}_{m}[\mathbf{P}_0, \ldots, \mathbf{P}_m](t) = \sum_{j=0}^{m} p_{m,j}(t)\mathbf{P}_j, \quad 0 \le t \le 1, \tag{3.6}$$

while, for $\lambda \to \infty$*, (*3.5*) represents the Lagrange curve of the same control polygon* $\mathbf{P}$.

The flexible parameter λ is used in order to model different shapes w.r.t the same control polygon $\mathbf{P}$, obtaining as extreme cases the Bézier curve and the Lagrange interpolating curve. In this sense λ is a "shape parameter".

It is known (see [7]) that relevant geometric properties of parametric curves descend from corresponding properties of the blending functions $\{p_{m,k}^{(\lambda)}\}$. We now collect some properties satisfied by GB(λ) curves.

- **Coordinate system independence**
  GB(λ) curves will not change if the coordinate system is changed, since
  $$\sum_{j=0}^{m} p_{m,j}^{(\lambda)}(x) = 1.$$
  Indeed, this is proved taking into account that the sum of the elements of each row of $C_{m,\lambda}$ is equal to 1.
- **Smoothness**
  GB(λ) are polynomial curves.
- **Endpoint Interpolation** Indeed,
  $$\mathbf{B}_{m,\lambda}[\mathbf{P}_0,\dots,\mathbf{P}_m](0) = \mathbf{P}_0, \quad \mathbf{B}_{m,\lambda}[\mathbf{P}_0,\dots,\mathbf{P}_m](1) = \mathbf{P}_m,$$
  since $B_{m,\lambda}(f;0) = f(0), \quad B_{m,\lambda}(f;1) = f(1)$ [13].
- **Symmetry**
  Curves are symmetric if they do not change under a reverse reordering of the control points sequence, i.e. if and only if
  $$\mathbf{B}_{m,\lambda}[\mathbf{P}_0,\dots,\mathbf{P}_m](t) = \mathbf{B}_{m,\lambda}[\mathbf{P}_m,\dots,\mathbf{P}_0](1-t),$$
  which holds taking into account
  $$p_{m,j}^{(\lambda)}(x) = p_{m,m-j}^{(\lambda)}(1-x), \quad j = 0,\dots,m. \tag{3.7}$$
- **Preservation of points and lines**
  This is equivalent to $\sum_{j=0}^{m} p_{m,j}^{(\lambda)}(x) = 1, \quad \sum_{k=0}^{m} k p_{m,k}^{(\lambda)}(x) = mx$. The first relation is equivalent to the coordinate system independence, while the second holds in view of [13]
  $$B_{m,\lambda}(e_1;t) = e_1(t), \quad e_1(t) = t.$$
- **Nondegeneracy**
  The curve cannot collapse to a single point, and this is implied from the linear independence of the blending functions $\{p_{m,k}^{(\lambda)}\}$.
- **Numerical stability**
  Since GB(λ) are the Bézier curves of the polygon
  $$\mathbf{T}^{\lambda} := \rho(\mathbf{A})[\mathbf{P_0},\dots,\mathbf{P_m}], \tag{3.8}$$
  the rendering algorithm is essentially the de Casteljau recursive scheme applied to the new control polygon $\mathbf{T}^{\lambda}$.

Moreover, GB(λ) curves satisfy all the properties of the Bézier curves w.r.t. the new control polygon $\mathbf{T}^{\lambda}$.

We conclude proposing two graphical examples showed in Figures 1 and 2. Here, for two given control polygons of 5 and 9 vertices, respectively, the curves GB(λ) are rendered for different shape parameter values.

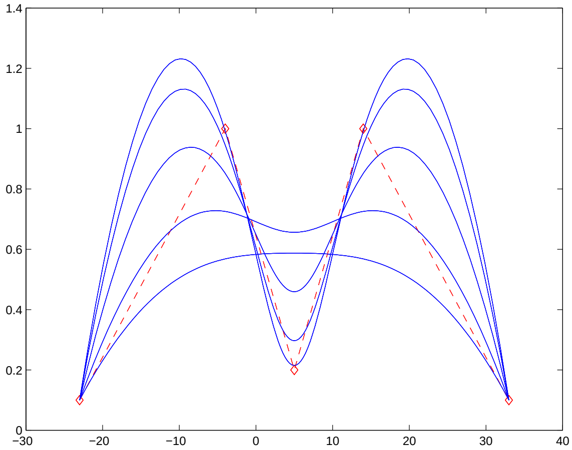


FIGURE 1. $\lambda = 1, 3, 7.1, 11, 19.9$

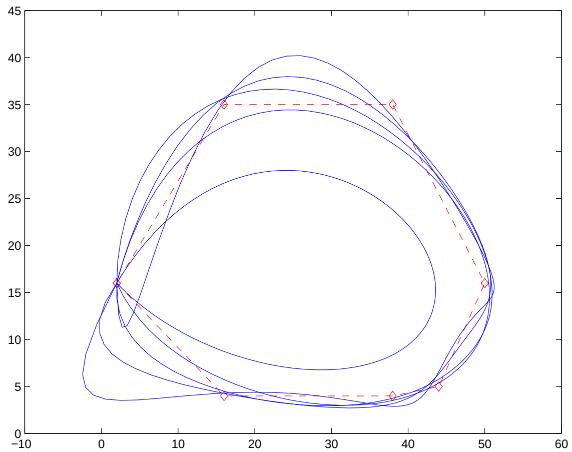


FIGURE 2. $\lambda = 1, 3, 8.9, 40, 120$

4. The proofs

Proof of Proposition 2.3. It is known that [5]

$$B_m(q_i; x) = \xi_{m,i} q_i(x), \quad m \geq i, \quad q_i \in \mathbb{P}_i. \tag{4.1}$$

i.e., $\xi_{m,i}$ in (2.11) are the eigenvalues of the operator B_m and q_i, $i = 0, \ldots, m$ are the corresponding eigenfunctions. Setting

$$\mathbf{p}_m(x) = [p_{m,0}(x), \ldots, p_{m,m}(x)]^T, \quad \mathbf{q}_m(x) = [q_0(x), \ldots, q_m(x)]^T,$$

$$\gamma_i = [q_i(0), q_i(1/m), \ldots, q_i(1)]^T,$$

$$\Gamma = [\gamma_0, \gamma_1, \ldots, \gamma_m], \quad \Psi = diag[\xi_{m,0}, \xi_{m,1}, \ldots, \xi_{m,m}],$$

(4.1) can be rewritten as

$$\mathbf{p}_m(x)^T\gamma_i = q_i(x)\xi_{m,i}, \quad i=0,\ldots,m,$$

that is

$$\mathbf{p}_m(x)^T\Gamma = \mathbf{q}_m(x)^T\Psi,$$

and evaluating at $x = t_0, t_1, \ldots, t_m$, it follows

$$A\Gamma = \Gamma\Psi,$$

where A is the matrix in (2.7). Since $\{q_i\}_{i=0}^m$ is a basis for the space $\mathbb{P}_m$, Γ is nonsingular, and by

$$A = \Gamma\Psi\Gamma^{-1}, \tag{4.2}$$

the proposition follows. □

In order to prove Theorem 2.1, we need the following

Theorem 4.1. [10, Th. 3, p.328] *Let B a real matrix of order n and suppose that $z_1, z_2, \ldots, z_s$ are the distinct eigenvalues of B of algebraic multiplicity $n_1, n_2, \ldots, n_s$, respectively. Let the function $f(z)$ have a Taylor series about $z_0 \in \mathbb{R}$*

$$f(z) = \sum_{\nu=0}^{\infty} \alpha_\nu (z-z_0)^\nu$$

with radius of convergence r. Then the function $f(B)$ is defined and is given by

$$f(B) = \sum_{\nu=0}^{\infty} \alpha_\nu (B - z_0 I)^\nu$$

if and only if the distinct eigenvalues of A satisfy one of the following conditions:

1. $|z_k - z_0| < r$;
2. $|z_k - z_0| = r$ *and the series for $f^{(n_k-1)}(z)$ is convergent at the point $z = z_k, \quad 1 \le k \le s$.*

Proof of Theorem 2.1. We recall the following representation given in [15]

$$B_m^i(f;x) = \mathbf{p}_m^T A^{i-1}\mathbf{f}_m. \tag{4.3}$$

Therefore (2.3) becomes

$$B_{m,\lambda}(f;x) = \mathbf{p}_m(x)^T \sum_{i=1}^{\infty} (-1)^{i+1}\binom{\lambda}{i} A^{i-1}\mathbf{f}_m. \tag{4.4}$$

Denoting by $\{\varphi_{m,i} := 1-\xi_{m,i}\}_{i=0}^m$ the eigenvalues of $I-A$, by Proposition 2.3 it follows $0 \le \varphi_{m,i} < 1, \quad i=1,2,\ldots,m$ and

$$\sum_{i=0}^{\infty} (-1)^i \binom{\lambda}{i} A^i = (I-A)^\lambda, \quad \lambda \ge 1. \tag{4.5}$$

The proposition is completely proved combining last relation with (4.4). □

Proof of Theorem 2.5. First we prove

$$B_m^i(f;x) = \mathbf{x}^T M^{i-1}\mathbf{d}_m. \tag{4.6}$$

For $i = 1$ (4.6) holds, in view of (2.20). Assume that (4.6) holds for i. By using (2.20)

$$B_m^{i+1}(f;x) = B_m(B_m^i(f);x) = \sum_{l=0}^{m} B_m(e_l;x) \sum_{k=0}^{m} M_{l,k}^{i-1} \binom{m}{k} \Delta_{\frac{1}{m}}^{k} f(0)$$
$$= \sum_{l=0}^{m} \sum_{j=0}^{m} x^j \binom{m}{j} \Delta_{\frac{1}{m}}^{j} e_l(0) \sum_{k=0}^{m} M_{l,k}^{i-1} \binom{m}{k} \Delta_{\frac{1}{m}}^{k} f(0)$$
$$= \sum_{j=0}^{m} x^j \sum_{k=0}^{m} \binom{m}{k} \Delta_{\frac{1}{m}}^{k} f(0) \sum_{l=0}^{m} M_{i,l} M_{l,k}^{i-1} = \mathbf{x}^T M^i \mathbf{d}_m.$$

By induction (4.6) is true for every i. Following the same arguments used in the proof of Theorem 2.1, under the assumption $\lambda \geq 1$, we get

$$B_{m,\lambda}(f,x) = \mathbf{x}^T \sum_{i=1}^{\infty} (-1)^{i+1} \binom{\lambda}{i} M^{i-1} \mathbf{d}_m = \mathbf{x}^T M^{-1} [I - (I-M)^{\lambda}] \mathbf{d}_m. \quad \square$$

Proof of Theorem 3.1. In order to prove (3.3), we start from

$$\sum_{j=0}^{m} |D_j^{(k)}| = \frac{1}{m+1} \sum_{j=0}^{m} \left| \sum_{i=0}^{m} (C_{m,k})_{i,j} \right| \leq \max_{0 \leq i \leq m} \sum_{j=0}^{m} |(C_{m,k})_{i,j}| = \|C_{m,k}\|_{\infty}$$

and by (2.10),

$$\sum_{i=0}^{m} |D_j^{(k)}| \leq \|C_{m,k}\|_{\infty} \leq \|I\|_{\infty} + \|I - A\|_{\infty} + \|I - A\|_{\infty}^2 + \cdots + \|I - A\|_{\infty}^{k-1}$$
$$\leq 1 + 2 + \cdots + 2^{k-1} = 2^k - 1, \text{ since } \|A\|_{\infty} = 1.$$

To prove (3.4), we use [12]

$$\|f - B_{m,k}(f)\|_{\infty} \leq m^{-k} \sum_{\nu=0}^{2k} b_{\nu} \|f^{(\nu)}\|_{\infty}$$

(see also [17]) where b_{ν} are positive constants independent of f. Therefore, since [6, p.310, Lemma 2.1]

$$\sum_{\nu=0}^{2k} b_{\nu} \|f^{(\nu)}\|_{\infty} \leq \mathcal{C}(\|f\|_{\infty} + \|f^{(2k)}\|_{\infty}),$$

(3.4) follows. $\square$

Acknowledgement. I am grateful to professor Mastroianni for his helpful suggestions. Moreover, I thank the referee for his useful remarks.

References

[1] Agrawal, P.N., Kasana, H.S., *On iterative combinations of Bernstein polynomials*, Demonstr. Math., **17**(1984), 777–783.

[2] Barry, P.J., Goldman, R.N., *Interpolation and Approximation of Curves and Surfaces Using Polya Polynomials*, Graphical Models and Image Processing, **53**(1991), 137–148.

[3] Barry, P.J., Goldman, R.N., *What is the natural generalization of a Bézier curve*, Math. Meth. in C.A.G.D., Lyche T. and Shumaker L. eds, Academic Press, 1989, 71–85.

[4] Barry, P.J., Goldman, R.N., *Shape parameter deletion for Pólya curves,* Numerical Algorithms, **1**(1991), no. 2, 121–137.

[5] Calugareanu, G., *On operators of S. N. Bernstein. Spectra of operators* , Gaz. Math. (A), **71**(1966), 448–451.

[6] Ditzian, Z., *On interpolation of $L_p[a,b]$ and weighted Sobolev spaces,* Pacific Journal of Mathematics, **90**(1980), no. 2, 307–323.

[7] Goldman, R.N., *Pólya urn's model and Computer Aided Geometric Design*, Siam J. Alg. Disc. Meth., **6**(1985), 1–28.

[8] Keliski, R.P., Rivlin, T.J., *Iterates of Bernstein polynomials,* Pacific Journal of Mathematics, **21**(1967), no. 3, 511–520.

[9] Kocic, Lj.M., Occorsio, D., Simoncelli, A.C., *Comparison between two generalizations of Bézier curves,* Facta Universitatis - Ser. Math. Inform., **12**(1997), 217–232.

[10] Lancaster, P., Tismenetsky, M., *The Theory of Matrices,* Second Edition: With Applications (Computer Science and Scientific Computing), 1985.

[11] Lorentz, G.G., *Bernstein polynomials*, Toronto, University of Toronto Press, Series: Mathematical expositions, no. 8, 1953.

[12] Mastroianni, G., Occorsio, M.R., *Una generalizzazione dell'operatore di Bernstein*, Rend. dell'Accad. di Scienze Fis. e Mat. Napoli (Serie IV), **44**(1977), 151–169.

[13] Mastroianni, G., Occorsio, M.R., *A new operator of Bernstein type*, L'Analyse Numerique et la Theory de l'Approximation (Cluj-Napoca), **16**(1987), 55–63.

[14] Micchelli, C., *The saturation class and iterates of Bernstein polynomials*, J. Approx. Th., **8**(1973), 1–18.

[15] Occorsio, D., Simoncelli, A.C., *How to go from Bézier to Lagrange curves by means of generalized Bézier curves*, Facta Univ. Ser. Math. Inform. (Niš), **11**(1996), 101–111.

[16] Occorsio, D., Simoncelli, A.C., *Generalized Stancu-Polya Curves*, Revue d'Analyse Numerique et de Theorie de l'Approximation, **XXVII**(1998), no. 1, 127–146.

[17] Vladislav, T., Raşa, I., *Analiză Numerică. Aproximare, problema lui Cauchy abstractă, proiectori Altomare*, Editura Technică, Colecţia Universitaria, 1999.

Donatella Occorsio
Dipartimento di Matematica ed Informatica
Università degli Studi della Basilicata
Via dell'Ateneo Lucano 10, 85100 Potenza, Italy
e-mail: `donatella.occorsio@unibas.it`

Stud. Univ. Babeş-Bolyai Math. 56(2011), No. 3, 161–168

Book reviews

Barry Simon, Szegö's Theorem and Its Descendants – Spectral Theory for L^2 Perturbations of Orthogonal Polynomials, Princeton University Press, Princeton and Oxford 2011, x + 650 pp. ISBN-13: 978-0-691-14704-8.

This volume, dealing with orthogonal polynomials on the real line (OPRL), can be considered as complementary to the monumental two volume treatise of the author on orthogonal polynomials on the unit circle $\mathbb{D}$ (OPUC), published by the American Mathematical Society, AMS Colloquium Series, volumes 54.1 and 54.2, Providence RI, 2005. As the author does mention in the Preface, although there are some inevitable overlap between them (mainly in Chapters 2 and 3), the present one is concentrated on topics not contained there. The focus is on sum rules for OPRL, but some results existing at the time when the OPUC volumes were written but of which the author was not aware, are also included. In fact, as remarked Szegö, using the transformations $E : \mathbb{D} \to \mathbb{C} \cup \{\infty\},\ E(z) = z + z^{-1},$ and $Q = E|_{\partial\mathbb{D}},\ Q(e^{i\theta}) = 2\sin\theta,$ some results on OPUC can be translated to OPRL.

The main goal of the book is to emphasize the deep connections between spectral theory and topics from classical analysis related to OPRL. The author calls a *gem of spectral theory* a theorem putting in one-to-one correspondence a class of spectral data with a class of objects.

The general framework is that of orthogonal polynomials $(P_n)_{n=0}^{\infty}$ with respect to a finite positive measure $d\rho$ on $\mathbb{R}$ having finite moments: $\int |x^n| d\rho(x) < \infty$ for all n. The measure $d\rho$ is called trivial if supp $(d\rho)$ is finite (equivalently, $L^2(\partial\mathbb{D}, d\rho)$ is finite dimensional), and nontrivial otherwise. If $d\rho$ is nontrivial, then $0 < \int |P(x)| d\rho(x) < \infty$ for every polynomial P. The OPRL satisfy a recursion relation $xP_n(x) = P_{n+1}(x) + b_{n+1}P_n(x) + a_n^2 P_{n-1}(x)$, with $P_{-1}(x) \equiv 0$. The first gems of spectral theory presented in the book are the Blumenthal-Weil theorem asserting that for an infinite Jacobi matrix J with coefficients satisfying the conditions (1): $a_n \to 1$ and $b_n \to 0$, the essential spectrum of J is $\sigma_{\text{ess}}(J) = [-2, 2]$, and the converse, the theorem of Denisov and Rakhmanov, asserting that, under some supplementary hypotheses, the equality $\sigma_{\text{ess}}(J) = [-2, 2]$ implies the conditions (1).

Szegö's theorem, published in 1915, alluded to in the title of the book, solves positively a conjecture of Pólya asserting that $\lim_{n\to\infty} D_n(w)^{1/n} = \exp\left(\int \log(w(\theta)\frac{d\theta}{2\pi}\right),$ where $w > 0$ is a weight function and $D_n(w)$ is the determinant of the Toeplitz matrix of order n associated with the moments

$c_k = \int e^{-ik\theta} w(\theta) \frac{d\theta}{2\pi}$, $k \geq 0$. In fact, Szegö proved a stronger result, namely that D_{n+1}/D_n has as limit the written quantity. A detailed proof of this theorem, some extensions and detours are given in Chapter 2, while in Chapter 3, *The Killip-Simon theorem: Szegö theorem for OPRL*, one gives a proof of Szegö theorem for OPRL whose essential support is $[-2, 2]$. Matrix orthogonal polynomials on the real line are discussed in Chapter 4 and periodic OPRL in Chapter 5. Chapter 6, *Toda flows and symplectic structures,* is concerned with the close relations between periodic Jacobi matrices and Toda lattices of dynamical systems. Chapter 7, *Right limits,* contains some results needed for the proofs in Chapter 8 of Szegö and Killip-Simon theorems for periodic OPRL, and in Chapter 9 of Szegö theorem for finite gap OPRL. The last chapter of the book, Chapter 10, *A.C. spectrum for Bethe-Cayley trees,* is concerned with the sum rules in the study of perturbed Laplacians, called Bethe lattices by physicists and Cayley trees by mathematicians. The author adopted a mixed terminology.

The book is clearly written and very well organized - each section ends with a paragraph called *Remarks and Historical Notes,* containing references to bibliography as well as some pertinent and cute remarks of the author. The bibliography counts 465 items with specifications of the pages where each one is cited in the text.

The author is a reputed specialist in the area, most of his contributions and of his students appearing here for the first time in book form.

Presenting a modern approach to some classical problems, relating classical analysis and spectral theory, but with some problems in physics as well, this book together with the AMS volumes on orthogonal polynomials on the unit circle, will become standard references in the field and an invaluable source for further research.

Mirela Kohr

A. Ya. Helemskii, Quantum Functional Analysis. A Non-Coordinate Approach, University Lecture Series, Vol. 56, American Mathematical Society, Providence, Rhode Island 2010, xvii+241 pp, ISBN:978-0-8218-5254-5

The term "quantum space" used in the book is synonym to that of "abstract operator space". As the author explains in the Introduction the aim of the present book is to introduce the "pedestrian" reader to this fascinating area of investigation, being based on the difficulties he encountered when reading the classical texts of the "founding fathers" of the theory - E. G. Effros, G. Pisier, V. Paulsen, U. Haagerup. a.o. The term "quantum" or "non-commutative" means that at an early stage, in some crucial definitions, some commutative objects, functions or scalars, are replaced by "non-commutative" ones, meaning matrices or operators. In the case of a linear space E, written as $\mathbb{C} \otimes E$, the scalars $\mathbb{C}$ are replaced by some "good" operator algebras, and the usual norm by a "quantum norm" (synonym for "operator space structure"). In the present book as good algebras one takes the algebra $\mathcal{F}(L)$ of bounded finite rank operators on a fixed separable Hilbert space L or the algebra $\mathcal{B}(L)$ of bounded operators on L. A natural core notion is that of completely bounded map which makes the whole machinery to work properly. In this way a far reaching generalization of classical functional analysis is obtained and, at a same time, it led to spectacular solutions of some long standing problems in operator theory and in other areas, to quote only the negative solution given by Pisier in 1997 (see, G. Pisier, *Similarity problem and completely bounded maps*, 2nd, Expanded Edition, Lect. Notes. Math. vol. 1618, Springer, Berlin 2001) to the problem of the similarity of polynomially bounded operators, posed by Halmos in his famous paper "Ten problems in Hilbert space", Bulletin AMS 76 (1970).

Although, as the author does mention, in essence a matter of test, the non-coordinate approach (i.e., based on operators) adopted in this book has some advantages over the matrix (coordinate) based approach, or at least it can be an alternative for the presentation given in most of the books on operator spaces.

Some fundamental results of the theory such as Ruan's representation theorem (every abstract operator space can be realized as a concrete operator space), the Hahn-Banach type theorem of Arveson and Wittstock and the decomposition theorem of Paulsen are enounced in Chapter 0. *Three basic definitions and three principal theorems.* Their proofs, rather long, deep and intricate, requiring considerable effort from the reader are postponed to Part III. *Principal theorems revisited in earnest,* of the book.

The presentation is made axiomatically, based on Ruan's axioms, allowing a quick access to some fundamental constructions (as quantum tensor products and duality theory) as well as the presentation of some illuminating examples. This is done in Part I. *The beginning: Spaces and operators,* containing the chapters 1. *Preparing the stage*, 2. *Abstract operator (=quantum) spaces*, 3. *Completely bounded operators*, 4. *The completion of abstract operator spaces*, and Part II. *Bilinear operators, tensor products and duality*, with the chapters 5. *Strongly and weakly completely bounded bilinear operators*, 6. *New preparations: Classical tensor products*, 7. *Quantum tensor products*, and 8. *Quantum duality.*

The author is well-known for his results on the homology of Banach algebras (see the book, A. Ya. Helemskii, *The homology of Banach and topological algebras*, Kluwer A.P., Dordrecht, 1989), as well as for his expository texts as, for instance, *Lectures and exercises on functional analysis*, AMS , Providence, RI, 2005.

Written in a didactic manner, the book contains a very clear presentation of the basic results of quantum functional analysis, accessible to a reader with few experience in the area - the prerequisites are basic functional analysis (for instance, as exposed by ordinary print in author's book mentioned above), some results on modules and bimodules over an algebra and some basic facts on C^*-algebras (in fact, this last field can be avoided by considering them as the algebra of all bounded operators on a Hilbert space).

S. Cobzaş

Peter Kosmol and Dieter Müller-Wichards, Optimization in Function Spaces - with stability considerations in Orlicz spaces, Series in Nonlinear Analysis and Applications, Vol. 13, Walter de Gruyter, Berlin - New York, 2011, xiv + 377 pages, ISBN: 978-3-11-025020-6, e-ISBN 978-3-11-025021-3, ISSN 0941-813X.

The book is concerned with convex optimization in Banach spaces, with emphasis on Orlicz spaces.

The first two chapters, 1. *Approximation in Orlicz spaces,* and 2. *Polya algorithms in Orlicz spaces,* deal with Haar subspaces and Chebyshev alternation theorem in $C(T)$, their extensions to Orlicz spaces and with Polya algorithm for discrete Chebyshev approximation - convergence and stability.

Chapter 3. *Convex sets and convex functions,* contains a fairly complete presentation of basic results about convex functions, including continuity, differentiability (Gâteaux and Fréchet), Fenchel-Moreau duality with applications to optimization problems (existence, characterization, Lagrange multipliers). The fact that the Gâteaux differential of a continuous convex function is demi-continuous is used later (in Chapter 8) to prove that a reflexive and Gâteaux smooth Orlicz space is Fréchet smooth.

The fourth chapter contains an overview of some numerical methods for non-linear and optimization problems (secant and Newton-type methods), a detailed treatment being given in an other monograph by P. Kosmol, *Methoden zur numerischen Behandlung nichtlinearer Gleichungen und Optimierungaufgaben,* B. G. Teubner Studienbücher, Stuttgart, 1993, 2nd ed.

The main tools used in Chapter 5, *Stability and two-stage optimization problems*, are some uniform boundedness results for families of convex functions and convex operators, proved by the first author, and extending the well-known Banach-Steinhaus principle.

Orlicz spaces are studied in chapters 6. *Orlicz spaces*, 7. *Orlicz norms and duality*, and 8. *Differentiability and convexity in Orlicz spaces.* This study includes the Orlicz spaces L^{Φ} and M^{Φ} equipped with Luxemburg or Orlicz norms, duality, reflexivity, as well as geometric properties of Orlicz spaces - rotundity and smoothness, Efimov-Stechkin property, with applications to best approximation and optimization, Tikhonov regularization, Ritz method and greedy algorithms.

In the last chapter of the book, 9. *Variational calculus*, one considers minimization with respect to both the state variables x and $\dot{x}$ in some minimization problems. The fundamental theorem of the Calculus of Variations (the Euler-Lagrange equation) is obtained by using some quadratic supplements making the Lagrangian convex in the vicinity of the solution, avoiding in this way the usage of field theory and of Hamilton-Jacobi equations. As application, a detailed treatment of the isoperimetric problem, called by the authors the Dido problem, is included.

The authors are authoritative voices in the area, known for their papers and books (e.g., P. Kosmol, *Optimierung und Approximation*, de Gruyter, Berlin-New York, 2010, 2nd ed, and the book quoted above).

Written in a clear and accessible manner (only mathematical analysis, linear algebra and familiarity with measure theory and functional analysis is required), the book can serve as a base for second half undergraduate or master courses on linear and nonlinear functional analysis, dealing with themes as convex functions and optimization, Orlicz spaces and their geometry, variational calculus.

W. W. Breckner

Siu-Ah Ng, Nonstandard methods in functional analysis - lectures and notes, World Scientific, London - Singapore - Beijing, xxii + 316 pages, ISBN: 13 978-981-4287-54-8 and 10 981-4287-54-7.

The nonstandard analysis has its origins in the 60s in the work of Abraham Robinson in his attempt to put on a rigorous basis Leibniz's differential calculus based on infinitesimal quantities (called monads). The construction, based on techniques from model theory for the first order logic, was presented for the first time in book form by A. Robinson, *Non-standard analysis*, North Holland 1966. Although, at the beginning, the idea was to present nonstandard proofs of known results, soon new results were obtained by nonstandard methods, the most striking being the solution to the invariant space problem for polynomially compact operators obtained by A. R. Bernstein and A. Robinson in 1966. In the same year P. Halmos gave a standard proof to this result. A presentation of the result is given in the book by M. Davis, *Applied nonstandard analysis*, J. Wiley 1977. Some spectacular results in measure theory with applications to probability theory, based on nonstandard methods, were obtained in 1975 by P. Loeb. The methods of nonstandard analysis require some preliminary effort from the newcomer, for which, as remarked A. Uspenski in the authoritative Preface written for the Russian edition (Mir, Moskva 1982) of the above mentioned book of M. Davis, some nonstandard reasonings can look as strange as "*a description of the endocrine systems of griffons and unicorns or of the chemical reactions between the philosophical stone and phlogiston*".

The aim of the present book is to disprove this impression and to show that, once acquainted, the methods of nonstandard analysis lead to more transparent and intuitive proofs of known results in functional analysis, and to new results as well.

The first chapter contains an overview of the basic methods and tools of nonstandard analysis (extensions and ultraproduct techniques) with applications to elementary calculus, topology and measure theory. In the second chapter, *Banach spaces,* the author passes to the presentation of basic results on normed spaces - nonstandard hulls, linear operators, Hahn-Banach theorem, weak compactness, reflexivity. Two topics that fit very well with nonstandard methods are finite representability and superreflexivity. The exposure continues in Chapter 3 with a presentation of basic results on Banach algebras. The last chapter of the book, Chapter 4, *Selected research topics*, is concerned with fixed points, noncommutative Loeb measures, Hilbert space-valued integrals.

The book is a good introduction to nonstandard methods in functional analysis and can serve as a base text for master courses or for self-study.

S. Cobzaş

Kenneth Kuttler, Calculus - Theory and applications,
World Scientific, London - Singapore - Beijing, 2011
Volume I, xii + 480 pages, ISBN: 13 978-981-4329-69-9 (pbk) and 10 981-4329-69-X (pbk).
Volume II, xii + 410 pages, ISBN: 13 978-981-4329-70-5 (pbk) and 10 981-4329-70-3(pbk).

This is a comprehensive course on Calculus of functions of one and of several variables. To make the book self-contained (as much as possible), the author has included in the first chapter of the first volume, *A short review of the precalculus*, some supplementary material, mainly from linear algebra and geometry (some trigonometry is included as well). This volume contains also the elements of calculus of functions of one variable - limits, continuity, derivatives, antiderivatives, some elementary differential equations, the Riemann integral and infinite series.

Chapters 12, *Fundamentals*, and 13, *Vector products*, are devoted to vector calculus in $\mathbb{R}^n$ including the dot and the cross products. The last chapter of the first volume, 13, S*ome curvilinear coordinate systems*, beside some results on curvilinear coordinates (as, e.g., graphs and area in polar coordinates) contains also an exposition of Kepler's laws on the planetary motion, completed in Appendix B with a presentation of Newton's laws of motion. Appendix A is devoted to some results in plane geometry and trigonometry. This topic is considered again in Appendix F of the second volume with the study of Christoffel symbols, curl and cross products in curvilinear coordinates

The second volume is devoted to the calculus of functions of several variables. As the linear algebra is the skeleton of the n-dimensional calculus and at the same time furnishes a lot of useful tools, the first three chapters 1, *Matrices and linear transformations*, 2, *Determinants*, and 3, *Spectral theory* (a study of eigenvalues and eigenvectors), are devoted to this topic, completed in Appendix A, *The mathematical theory of determinants*, with rigorous proofs of the properties of determinants and applications to the diagonalization of matrices.

Chapter 4, *Vector valued functions*, is concerned with limits and continuity properties. A special chapter (Chapter 5) is devoted to vector functions of one real variable (derivatives and integrals) with applications in Chapter 6, *Motion on a space curve*, to spatial curves and their geometry.

The differential calculus of functions of several variables is developed in Chapters 7, *Functions of many variables*, and 8, *The derivatives of functions of many variables*, with applications, in Chapter 9, *Optimization*, to local and conditioned extrema. This study is completed in Appendix B with a proof of the implicit function theorem with applications to local structure of differentiable functions.

The basic notions and tools of the Riemann integral in $\mathbb{R}^n$ are given in chapters 10, *The Riemann integral* and 11, *The integral in other coordinates*, at an informal level, the rigorous proofs (including a proof of the change of variables formula) being postponed to Appendices C, *The theory of Riemann integral*, and D, *Volumes in higher dimensions* (the function Gamma and the volume of balls in $\mathbb{R}^n$).

The integral on two dimensional surfaces in $\mathbb{R}^3$ is treated in Chapter 12, while Chapter 13 is concerned with the calculus of vector fields. Other physical applications (as, e.g., the Coriolis acceleration of the rotating earth) are given in Appendix E.

The book is written in a very didactic manner reflecting the teaching experience of the author - one starts with examples and particular cases before presenting the general case and rigorous proofs (most being given in appendices).

There are a lot of interesting concrete examples (some of them included in the exercises) from physics, mechanics, astronomy, economics, some of them presented in an entertaining amazing way. Each chapter ends with a set of well chosen exercises, many having answers or hints at the end of each volume. The most challenging are marked by *. A web page, *http://www.byu.edu/ klkuttle/CalculusMaterial*, contains supplementary material and routine exercises.

The author is well known for his books on calculus and linear algebra, from which *Modern Calculus*, CRC Press, Boca Raton 1998, contains more advanced topics.

These volumes, written in a live and accessible but at the same time rigorous style, can be used for basic courses in calculus (or linear algebra), at beginning or advanced levels.

Tiberiu Trif

I. Meghea, Ekeland variational principle: with generalizations and variants, Old City Publishing, Philadelphia and Éditions des Archives Contemporaines, Paris, 2009, iv+524 pp. ISBN: 978-1-933153-08-7; 978-2-914610-96-4

The variational principle discovered by Ivar Ekeland in 1972 (called in what follows EkVP) turned out to be a powerful and versatile tool in many branches of mathematics – Banach space geometry, optimization, economics, etc. This was masterly illustrated by Ekeland in the survey paper from the Bulletin of the American Mathematical Society 39 (2002), no. 2, pp. 207265, and proved by subsequent developments. In fact, EkVP is a paradigm of a maximality principle used by E. Bishop and R. Phelps in the proof of their famous result on the density of support functionals and is also related to the Brezis-Browder maximality principle. Nowadays there are a lot of variational principles originating from EkVP: smooth variational principles, vector variational principles, perturbed variational principles. After more than 30 years since the discovery of EkVP, the present monograph demonstrates that this principle could be considered as the landmark of modern variational calculus.

The first chapter, I. *Ekeland variational principle*, is devoted to the presentation of the original Ekeland variational principle in complete metric spaces and of some equivalent results – the drop and flower petal theorems, the Bishop-Phelps theorem, Caristi-Kirk fixed point theorem. Applications to minimax-type theorems in Banach space and on Finsler manifolds and to Clarke's subdifferential calculus for locally Lipschitz functions are included. This chapter contains also some extensions of EkVP – vector variants of EkVP, EkVP in locally convex spaces, in uniform spaces and in probabilistic metric spaces.

The second chapter II. *Smooth variational principles*, is concerned with Borwein-Preiss and Deville-Godefroy-Zizler smooth variational principles and Ghoussoub-Maurey perturbed variational principle. The chapter ends with a variational principle, proved by Yongxin Li and Shuzhong Shi (2000), unifying and generalizing both EkVP and Borwein-Preiss variational principle. As it is shown in the book EkVP cannot be recovered in its full generality from Borwein-Preiss variational principle.

Beside the bibliography referred to in the main text, the book contains also an Additional Bibliography with brief presentations of some results appeared after the book was written or which do not fit the general context.

Collecting the essential results on variational principles and presenting them in a coherent and didactic way (there are a lot of improvements of the results taken from various sources and corrections of the proofs, belonging to the author), the book is a very useful reference for researchers in this area as well as for those interested in applications. By the detailed presentation of the subject the book can be used also by the newcomers or as a support for an advanced course in nonlinear analysis.

S. Cobzaş