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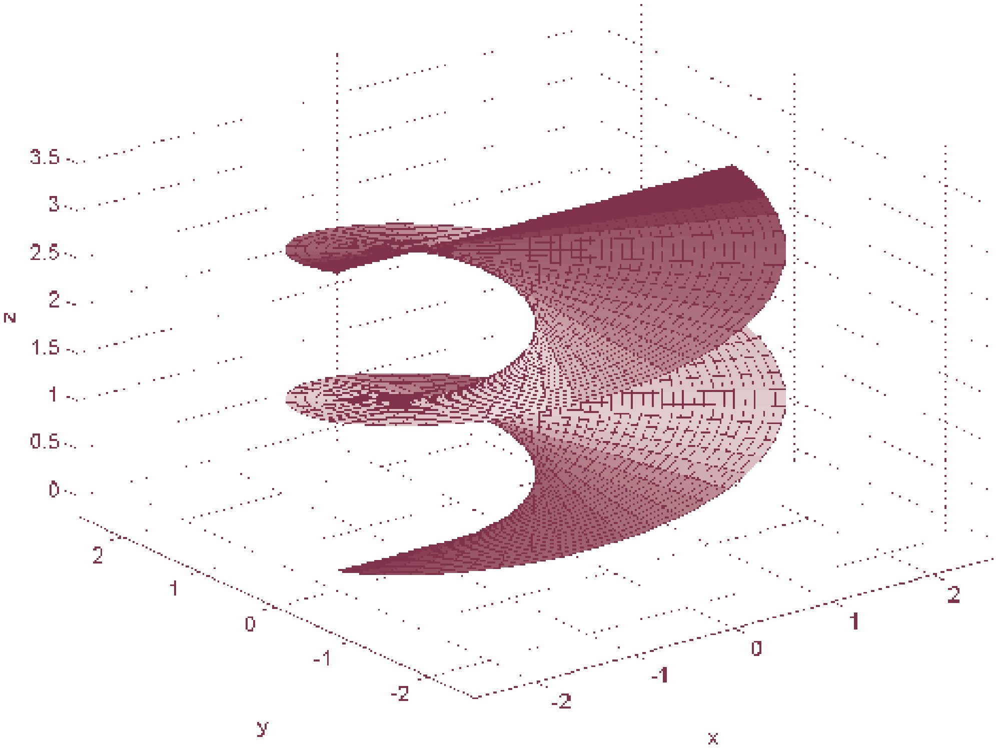

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CONTENTS

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$\mathcal{N}$-structures applied to associative-$\mathcal{I}$-ideals in IS-algebras

Ali H. Handam

Abstract. In this paper the notion of $\mathcal{N}$-$\mathcal{I}$-ideals and $\mathcal{N}$-associative $\mathcal{I}$-ideals in IS-algebra is introduced, as well as some of their properties are investigated. The relations between $\mathcal{N}$-$\mathcal{I}$-ideals and $\mathcal{N}$-associative $\mathcal{I}$-ideals are discussed. A characterization of $\mathcal{N}$-associative $\mathcal{I}$-ideals is provided.

Mathematics Subject Classification (2010): 06F35, 03G25.

Keywords: IS-algebras, $\mathcal{N}$-structure, $\mathcal{N}$- $\mathcal{I}$-ideal, $\mathcal{N}$-associative $\mathcal{I}$-ideal.

1. Introduction

Imai and Iséki [1] in 1966 introduced the notion of a BCK-algebra. In the same year, Iséki [2] introduced BCI-algebras as a super class of the class of BCK-algebras. In 1993, Jun et al. [3] introduced a new class of algebras related to BCI-algebras and semigroups, called a BCI-semigroup/BCI-monoid/BCI-group. In 1998, for the convenience of study, Jun et al. [8] renamed the BCI-semigroup (respectively, BCI-monoid and BCI-group) as the IS-algebra (respectively, IM-algebra and IG-algebra) and studied further properties of these algebras (see [7]).

A (crisp) set A in a universe X can be defined in the form of its characteristic function $\mu_A : X \to \{0, 1\}$ yielding the value 1 for elements belonging to the set A and the value 0 for elements excluded from the set A. So far most of the generalization of the crisp set have been conducted on the unit interval $[0, 1]$ and they are consistent with the asymmetry observation. In other words, the generalization of the crisp set to fuzzy sets relied on spreading positive information that fit the crisp point $\{1\}$ into the interval $[0, 1]$. Because no negative meaning of information is suggested, we now feel a need to deal with negative information. To do so, we also feel a need to supply mathematical tool. To attain such object, Jun et al. [5] introduced a new function which is called negative-valued function, and constructed $\mathcal{N}$-structures. They applied $\mathcal{N}$-structures to BCK/BCI-algebras, and discussed $\mathcal{N}$-subalgebras and $\mathcal{N}$-ideals in BCK/BCI-algebras. Jun et al. [6] considered closed ideals in BCH-algebras based on

$\mathcal{N}$-structures. Jun et al. [4] introduced the notion of a (created) $\mathcal{N}$-ideal of subtraction algebras, and investigated several characterizations of $\mathcal{N}$-ideals.

In this paper, we introduced the notion of $\mathcal{N}$-$\mathcal{I}$-ideals and $\mathcal{N}$-associative $\mathcal{I}$-ideals in IS-algebras, and studied several related properties.

2. Basic results on IS-algebras

The following necessary elementary aspects of IS-algebras will be used throughout this paper.

By a BCI-algebra we mean an algebra $(X, *, 0)$ of type $(2, 0)$ satisfying the following axioms: for every $x, y, z \in X$ [2],
(I) $((x * y) * (x * z)) * (z * y) = 0$,
(II) $(x * (x * y)) * y = 0$,
(III) $x * x = 0$,
(IV) $x * y = 0$ and $y * x = 0$ imply $x = y$.
A BCI-algebra X satisfying $0 \leq x$ for all $x \in X$ is called a BCK-algebra. In any BCI-algebra X one can define a partial order "$\preceq$" by putting $x \preceq y$ if and only if $x * y = 0$.
A BCI-algebra X has the following properties for any $x, y, z \in X$ [2]:
$(A1)$ $x * 0 = x$,
$(A2)$ $(x * y) * z = (x * z) * y$,
$(A3)$ $x \preceq y$ implies that $(x * z) \preceq (y * z)$ and $(z * y) \preceq (z * x)$,
$(A4)$ $(x * z) * (y * z) \preceq x * y$,
$(A5)$ $x * (x * (x * y)) = x * y$,
$(A6)$ $0 * (x * y) = (0 * x) * (0 * y)$,
$(A7)$ $0 * (0 * ((x * z) * (y * z))) = (0 * y) * (0 * x)$.

A non-empty subset I of a BCI-algebra X is called an ideal of X if $(S1)$: $0 \in I$, $(S2)$: $x * y \in I$ and $y \in I$ imply that $x \in I$. A non-empty subset I of X is called a associative ideal of X if it satisfies $(S1)$ and $(S3)$: $((x * y) * z) \in I$, $(y * z) \in I$ imply that $x \in I$.

Definition 2.1. [8]. *An IS-algebra is a non-empty set X with two binary operations "$*$" and "$\cdot$" and constant 0 satisfying the axioms*

$(B1)$ $(X, *, 0)$ is a BCI-algebra,
$(B2)$ $(X, \cdot)$ is a semigroup,
$(B3)$ the operation "$\cdot$" is distributive (on both sides) over the operation "$*$", that is,

$$x \cdot (y * z) = (x \cdot y) * (x \cdot z) \text{ and } (x * y) \cdot z = (x \cdot z) * (y \cdot z) \text{ for all } x, y, z \in X.$$

Note that, the IS-algebra is a generalization of the ring (see [8]).

Proposition 2.2. [3]. *Let X be an IS-algebra. Then we have*

(1) $0 \cdot x = x \cdot 0 = 0$,
(2) $x \preceq y$ implies that $x \cdot z \preceq y \cdot z$ and $z \cdot x \preceq z \cdot y$, for all $x, y, z \in X$.

Definition 2.3. [8]. *A non-empty subset A of an IS-algebra X is called a left (resp. right) $\mathcal{I}$-ideal of X if*
(1) *$x \cdot a \in A$ (resp. $a \cdot x \in A$) whenever $x \in X$ and $a \in A$,*

(2) for any $x, y \in X$, $x * y \in A$ and $y \in A$ imply that $x \in A$.
Both a left and right $\mathcal{I}$-ideal is called $\mathcal{I}$-ideal.

Definition 2.4. [9]. *A non-empty subset A of an IS-algebra X is called a left (resp. right) associative $\mathcal{I}$-ideal of X if*
(1) *$x \cdot a \in A$ (resp. $a \cdot x \in A$) whenever $x \in X$ and $a \in A$,*
(2) *for any $x, y, z \in X$, $(x * y) * z \in A$ and $y * z \in A$ imply that $x \in A$.*

3. $\mathcal{N}$-associative $\mathcal{I}$-ideals

Denote by $\mathcal{F}(X, [-1, 0])$ the collection of functions from a set X to $[-1, 0]$. We say that, an element of $\mathcal{F}(X, [-1, 0])$ is a negative-valued function from X to $[-1, 0]$ (briefly, $\mathcal{N}$-function on X). By an $\mathcal{N}$-structure we mean an ordered pair (X, ξ), where ξ is an $\mathcal{N}$-function on X. In what follows, let X be an IS-algebra and ξ an $\mathcal{N}$-function on X unless otherwise specified.

Definition 3.1. *Let X be an IS-algebra. An $\mathcal{N}$-structure (X, ξ) is called a left $\mathcal{N}$-$\mathcal{I}$-ideal (resp. a right $\mathcal{N}$-$\mathcal{I}$-ideal) of X if*
$(C1)$ $(\xi(xy) \leq \xi(y))$ *(resp. $\xi(xy) \leq \xi(x)$) for all $x, y \in X$;*
$(C2)$ $\xi(x) \leq \max\{\xi(x * y), \xi(y)\}$ *for all $x, y \in X$.*

An $\mathcal{N}$-structure (X, ξ) is called an $\mathcal{N}$-$\mathcal{I}$-ideal of X if it is both a left $\mathcal{N}$-$\mathcal{I}$-ideal and a right $\mathcal{N}$-$\mathcal{I}$-ideal of X.

Definition 3.2. *Let X be an IS-algebra. An $\mathcal{N}$-structure (X, ξ) is called a left $\mathcal{N}$-associative $\mathcal{I}$-ideal (resp. a right $\mathcal{N}$-associative $\mathcal{I}$-ideal) of X if it satisfies $(C1)$ and*
$(C3)$ $\xi(x) \leq max\{\xi((x * y) * z), \xi(y * z)\}$ *for all $x, y, z \in X$.*

An $\mathcal{N}$-structure (X, ξ) is called an $\mathcal{N}$-associative $\mathcal{I}$-ideal of X if it is both a left $\mathcal{N}$-associative $\mathcal{I}$-ideal and a right $\mathcal{N}$-associative $\mathcal{I}$-ideal of X.

Example 3.3. Consider an IS-algebra $X = \{0, a, b, c\}$ with Cayley tables as follows:

$*$	0	a	b	c
0	0	0	b	b
a	a	0	c	b
b	b	b	0	0
c	c	b	a	0

$\cdot$	0	a	b	c
0	0	0	0	0
a	0	a	0	a
b	0	0	b	b
c	0	a	b	c

(1) Let (X, ξ) be an $\mathcal{N}$-structure in which ξ is given by

$$\xi = \begin{pmatrix} 0 & a & b & c \\ t_0 & t_1 & t_0 & t_1 \end{pmatrix}, \text{ where } t_0 < t_1 \text{ in } [-1, 0].$$

Then (X, ξ) is an $\mathcal{N}$-$\mathcal{I}$-ideal of X.
(2) Let (X, ζ) be an $\mathcal{N}$-structure in which ζ is given by

$$\zeta = \begin{pmatrix} 0 & a & b & c \\ t_0 & t_0 & t_1 & t_1 \end{pmatrix}, \text{ where } t_0 < t_1 \text{ in } [-1, 0].$$

Then (X, ζ) is an $\mathcal{N}$-associative $\mathcal{I}$-ideal of X.

Proposition 3.4. *Every left (resp. right) $\mathcal{N}$-associative $\mathcal{I}$-ideal (X, ξ) satisfies the following inequality:*

$$(\forall x \in X)\ (\xi(0) \leq \xi(x)) \tag{3.1}$$

Theorem 3.5. *Every left (resp. right) $\mathcal{N}$-associative $\mathcal{I}$-ideal is a left (resp. right) $\mathcal{N}$-$\mathcal{I}$-ideal.*

Proof. Let (X, ξ) be a left (resp. right) $\mathcal{N}$-associative $\mathcal{I}$-ideal of X. Then, $\xi(xy) \leq \xi(y)$ (resp. $\xi(xy) \leq \xi(x)$) for all $x, y \in X$. Now, let $z = 0$ in $(C3)$, we have $\xi(x) \leq max\{\xi((x*y)*0), \xi(y*0)\}$ for all $x, y \in X$. So, $\xi(x) \leq max\{\xi((x*y)), \xi(y)\}$. Therefore, (X, ξ) is a left (resp. right) $\mathcal{N}$-$\mathcal{I}$-ideal of X. □

The next example shows that the converse of Theorem 3.5 is not always true.

Example 3.6. Consider the $\mathcal{N}$-$\mathcal{I}$-ideal (X, ξ) given in Example 3.3. By routine calculations, it is easy to check that (X, ξ) is not an $\mathcal{N}$-associative $\mathcal{I}$-ideal of X.

Proposition 3.7. *Every left (resp. right) $\mathcal{N}$-associative $\mathcal{I}$-ideal (X, ξ) satisfies the following inequality:*

$$(\forall x, y \in X)\ (\xi(x) \leq \xi((x*y)*y)) \tag{3.2}$$

Proof. Let (X, ξ) be a left (resp. right) $\mathcal{N}$-associative $\mathcal{I}$-ideal of X. If we let $z := y$ in $(C3)$, then we have $\xi(x) \leq max\{\xi((x*y)*y), \xi(y*y)\}$ for all $x, y \in X$. Using 3.1 and (III), it follows that, $\xi(x) \leq \xi((x*y)*y)$ for all $x, y \in X$. □

Proposition 3.8. *If (X, ξ) is a left (resp. right) $\mathcal{N}$-associative $\mathcal{I}$-ideal of X, then*

$$(\forall x, y \in X)\ (x \preceq y \Rightarrow \xi(x) \leq \xi(y)) \tag{3.3}$$

Proof. Let $x, y \in X$ be such that $x \preceq y$. If we let $z := 0$ in $(C3)$, then we have $\xi(x) \leq max\{\xi((x*y)*0), \xi(y*0)\}$ for all $x, y \in X$. Since, $x \preceq y$ implies $x*y = 0$, $\xi(x) \leq max\{\xi(0*0), \xi(y*0)\}$. It follows from axiom (III) and $(A1)$ that $\xi(x) \leq \xi(y)$. □

Proposition 3.9. *Let (X, ξ) be a left (resp. right) $\mathcal{N}$- $\mathcal{I}$-ideal of X. Then, $x*y \preceq z$ implies $\xi(x) \leq max\{\xi(z), \xi(y)\}$ for all $x, y, z \in X$.*

Theorem 3.10. *Let (X, ξ) be a left (resp. right) $\mathcal{N}$-associative $\mathcal{I}$-ideal of X. Then, for any $x, y, z \in X$,*

(i) $x*y \preceq z$ implies $\xi(x) \leq \xi(y*z)$.
(ii) $\xi(x) \leq \xi(0*x)$.
(iii) $\xi((x \cdot y)*(x \cdot z)) \leq \xi(y*z)$ (resp. $\xi((x \cdot z)*(y \cdot z)) \leq \xi(x*y)$).

Proof. (i) Suppose that (X,ξ) is a left (resp. right) $\mathcal{N}$-associative $\mathcal{I}$-ideal of X, by $(C3)$ we have $\xi(x) \leq max\{\xi((x*y)*w),\xi(y*w)\}$ for all $x,y,w \in X$. Since, $x*y \preceq z$ implies $(x*y)*w \preceq z*w$, by (3.3), it follows that $\xi((x*y)*w) \leq \xi(z*w)$. Hence, $\xi(x) \leq max\{(\xi(z*w),\xi(y*w)\}$. If we let $w = z$, then we have, $\xi(x) \leq max\{(\xi(0),\xi(y*z)\} = \xi(y*z)$.
(ii) Let $z = x*y$ in $(C3)$, then

$$\xi(x) \leq max\{\xi(0),\xi(y*(x*y)\} = \xi(y*(x*y) \tag{3.4}$$

If we let $y = 0$ in (3.4), then we obtain also

$$\begin{aligned} \xi(x) &\leq \xi(0*(x*0)) \\ &= \xi(0*x) && \text{by } (A1) \end{aligned}$$

(iii) It follows directly from $(B3)$ and $(C1)$. □

Definition 3.11. [5]. *Let (X,ξ) and (X,ζ) be two $\mathcal{N}$-structures.*

(1) The union, $\xi \cup \zeta$ of ξ and ζ is defined by $(\xi \cup \zeta)(x) = \max\{\xi(x),\zeta(x)\}$ for all $x \in X$.
(2) The intersection, $\xi \cap \zeta$ of ξ and ζ is defined by $(\xi \cap \zeta)(x) = \min\{\xi(x),\zeta(x)\}$ for all $x \in X$.

Obviously, $(X,\xi \cup \zeta)$ and $(X,\xi \cap \zeta)$ are $\mathcal{N}$-structures which are called the union and the intersection of (X,ξ) and (X,ζ), respectively.

Proposition 3.12. *If (X,ξ) and (X,ζ) are left (resp. right) $\mathcal{N}$-associative $\mathcal{I}$-ideals of X, then the union $(X,\xi \cup \zeta)$ is a left (resp. right) $\mathcal{N}$-associative $\mathcal{I}$-ideal of X.*

Now, we give an example to show that the intersection of two $\mathcal{N}$-$\mathcal{I}$-ideals may not be an $\mathcal{N}$-$\mathcal{I}$-ideal.

Example 3.13. Consider the two $\mathcal{N}$-$\mathcal{I}$-ideals (X,ξ) and (X,ζ) given in Example 3.3. The intersection $\xi \cap \zeta$ is given by

$$\xi \cap \zeta = \begin{pmatrix} 0 & a & b & c \\ t_0 & t_0 & t_0 & t_1 \end{pmatrix}, \text{ where } t_0 < t_1 \text{ in } [-1,0].$$

$\xi \cap \zeta$ is not an $\mathcal{N}$-$\mathcal{I}$-ideal of X, since $(\xi \cap \zeta)(c) = t_1 \nleq \max\{(\xi \cap \zeta)(c*b),(\xi \cap \zeta)(b)\} = t_0$.

For any $\mathcal{N}$-function ξ on X and $t \in [-1,0)$, define the set $\mathcal{C}(\xi,t)$ as

$$\mathcal{C}(\xi,t) = \{x \in X \mid \xi(x) \leq t\}.$$

Theorem 3.14. *An $\mathcal{N}$-structure (X,ξ) is a left (resp. right) $\mathcal{N}$-associative $\mathcal{I}$-ideal of X if and only if every non-empty set $\mathcal{C}(\xi,t)$ is a left (resp. right) associative $\mathcal{I}$-ideal of X for all $t \in [-1,0)$.*

Proof. Assume that (X,ξ) is a left (resp. right) $\mathcal{N}$-associative $\mathcal{I}$-ideal of X and let $t \in [-1,0)$ be such that $\mathcal{C}(\xi,t) \neq \emptyset$. Let $x \in X$ and $a \in \mathcal{C}(\xi,t)$. Then, $\xi(a) \leq t$. It follows from $(C1)$ that $\xi(x \cdot a) \leq \xi(a) \leq t$ (resp. $\xi(a \cdot x) \leq \xi(a) \leq t$). Hence, $x \cdot a \in \mathcal{C}(\xi,t)$ (resp. $a \cdot x \in \mathcal{C}(\xi,t)$). Now, let $(x*y)*z \in \mathcal{C}(\xi,t)$ and $(y*z) \in \mathcal{C}(\xi,t)$. Then, $\xi((x*y)*z) \leq t$

and $\xi(y*z) \leq t$. Using $(C3)$ we obtain, $\xi(x) \leq max\{\xi((x*y)*z), \xi(y*z)\} \leq t$. Thus $x \in \mathcal{C}(\xi, t)$. Therefore, $\mathcal{C}(\xi, t)$ is a left (resp. right) associative $\mathcal{I}$-ideal of X for all $t \in [-1, 0)$.
Conversely, suppose that every non-empty set $\mathcal{C}(\xi, t)$ is a left (resp. right) associative $\mathcal{I}$-ideal of X for all $t \in [-1, 0)$. If there are $a, b \in X$ such that $\xi(a \cdot b) > \xi(b)$ (resp. $\xi(a \cdot b) > \xi(a)$), then, $\xi(a \cdot b) > t_0 \geq \xi(b)$ (resp. $\xi(a \cdot b) > t_0 \geq \xi(a)$) for some $t_0 \in [-1, 0)$. Hence, $b \in \mathcal{C}(\xi, t_0)$ (resp. $a \in \mathcal{C}(\xi, t_0)$) and $a \cdot b \notin \mathcal{C}(\xi, t_0)$. This is a contradiction. Thus, $\xi(x \cdot y) \leq \xi(y)$ (resp. $\xi(x \cdot y) \leq \xi(x)$) for all $x, y \in X$. Now, assume that there exist $a, b, c \in X$ such that $\xi(a) > max\{\xi((a*b)*c), \xi(b*c)\}$. Then, $\xi(a) > t_1 \geq \max\{\xi((a*b)*c), \xi(b*c)\}$ for some $t_1 \in [-1, 0)$. Hence, $(a*b)*c, b*c \in \mathcal{C}(\xi, t_1)$ and $a \notin \mathcal{C}(\xi, t_1)$, which is a contradiction. Therefore, (X, ξ) is a left (resp. right) $\mathcal{N}$-associative $\mathcal{I}$-ideal of X. □

Theorem 3.15. *Let A be a left (resp. right) associative $\mathcal{I}$-ideal of X and let (X, ξ) be an $\mathcal{N}$-structure in X defined by*

$$\xi(x) = \begin{cases} t_0 & \text{if x} \in A \\ t_1 & \text{otherwise} \end{cases},$$

where $t_0 < t_1$ in $[-1, 0]$. Then, the $\mathcal{N}$-structure (X, ξ) is a left (resp. right) $\mathcal{N}$-associative $\mathcal{I}$-ideal of X.

Proof. It follows directly from Theorem 3.14. □

For any $\mathcal{N}$-structure (X, ξ) and any element $w \in X$, consider the set

$$\mathcal{D}_w := \{x \in X \mid \xi(x) \leq \xi(w)\}.$$

Then, $\mathcal{D}_w$ is non-empty subset of X.

Theorem 3.16. *If an $\mathcal{N}$-structure (X, ξ) is a left (resp. right) $\mathcal{N}$-associative $\mathcal{I}$-ideal of X, then $\mathcal{D}_w$ is a left (resp. right) associative $\mathcal{I}$-ideal of X for all $w \in X$.*

Proof. Let $a \in \mathcal{D}_w$ and $x \in X$. Then, $\xi(a) \leq \xi(w)$. By $(C1)$ it follows that $\xi(x \cdot a) \leq \xi(a) \leq \xi(w)$ (resp. $\xi(a \cdot x) \leq \xi(a) \leq \xi(w)$). Hence $x \cdot a \in \mathcal{D}_w$ (resp. $a \cdot x \in \mathcal{D}_w$). Now, let $x, y, z \in X$ be such that $(x*y)*z \in \mathcal{D}_w$ and $y*z \in \mathcal{D}_w$. Then, $\xi((x*y)*z) \leq \xi(w)$ and $\xi(y*z) \leq \xi(w)$. By $(C3)$ it follows that $\xi(x) \leq max\{\xi((x*y)*z), \xi(y*z)\} \leq \xi(w)$. Hence, $x \in \mathcal{D}_w$. Therefore, $\mathcal{D}_w$ is a left (resp. right) associative $\mathcal{I}$-ideal of X for all $w \in X$. □

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On skew group algebras and symmetric algebras

Constantin Cosmin Todea

Abstract. We identify and define a class of algebras which we call inv-symm algebras and prove that are principally symmetric. Two important examples are given, and we prove that the skew group algebra associated to these algebras remains inv-symm.

Mathematics Subject Classification (2010): 16SXX.

Keywords: inverse semigroup, symmetric algebra, skew group algebra.

1. Inv-symm algebras

Following [2] we recall the concept of an inverse semigroup and we use basic results without comments. A semigroup $(S, \cdot)$ is *inverse* if for any $s \in S$ there is a unique $\widehat{s}$ (named inverse) such that $s \cdot \widehat{s} \cdot s = s$ and $\widehat{s} \cdot s \cdot \widehat{s} = \widehat{s}$. By [2, 1.1, Theorem 3], if $(S, \cdot)$ is inverse then all idempotents of S commutes and we have $\widehat{\widehat{s}} = s$ and $\widehat{s \cdot t} = \widehat{t} \cdot \widehat{s}$ for any $s \in S$. We denote usually by k a commutative ring and by A a k-algebra. If B is a subset of A with $0 \notin B$, we denote by $B^\sharp$ the set $B \cup \{0\}$ and by $\mathrm{Idemp}(B)$ the set of all idempotents of B. The following definition is suggested by the ideas from [3] and by methods used to prove that the group algebra is a symmetric algebra.

Definition 1.1. *A k-algebra A is inv-symm if there is a finite k-basis B such that:*

(1) *$(B^\sharp, \cdot)$ is an inverse semigroup.*
(2) *For $t, s \in B$ we have $t \cdot s \neq 0$ if and only if $s \cdot \widehat{s} = \widehat{t} \cdot t$.*

Example 1.2. If $A = kG$ is the group algebra over a finite group G then the finite set $B = G$ is a k-basis which satisfies conditions from Definition 1.1. We have in this case $\widehat{s} = s^{-1}$, $t \cdot s \neq 0$ and $s \cdot \widehat{s} = \widehat{t} \cdot t$ for any $t, s \in B$.

Example 1.3. If $A = \mathrm{End}_k(M)$, where M is a kG-lattice (that is a finitely generated, free k-module with a G-stable finite basis X), then $B = \{b_{x,y} \mid x, y \in X\}$ with $b_{x,y} : M \to M$, $b_{x,y}(z) = x$ if $z = y$, and $b_{x,y}(z) = 0$ if $z \neq y$, satisfies the conditions from 1.1. It requires some computation to verify that $b_{x,y} \circ b_{x_1,y_1} = 0$ if $y \neq x_1$, and $b_{x,y} \circ b_{x_1,y_1} = b_{x,y_1}$ if $y = x_1$. We have that $b_{x,y} \in \mathrm{Idemp}(B)$ if and only if $x = y$.

Remark 1.4. Moreover the above two examples are also G-algebras with G-stable basis. This suggest that we can define a class of symmetric G-algebras and to analyze the skew group algebra in this case.

Lemma 1.5. *Let A be an inv-symm k-algebra with basis B satisfying Definition* 1.1 *and $t, s \in B$. The following statements are true:*

a) *For $0 \in B^\sharp$ we have $\widehat{0} = 0$ and $s \in B$ if and only if $\widehat{s} \in B$.*

b) *For all $s \in B$ we have $s \cdot \widehat{s} \in \mathrm{Idemp}(B)$ and $\widehat{s} \cdot s \in \mathrm{Idemp}(B)$. Particularly $\mathrm{Idemp}(B) \neq \emptyset$.*

c) *If $t \cdot s \neq 0$ and $t \cdot s \in \mathrm{Idemp}(B)$ then $t = \widehat{s}$.*

Proof. a) For 0 is easy to check. Let $s \in B$, then there is a unique $\widehat{s} \in B^\sharp$ with the properties of the inverse element. Suppose that $\widehat{s} = 0$ then $\widehat{\widehat{s}} = \widehat{0}$, which gives $s = 0$, a contradiction.

b) For $s \in B$ we have $\widehat{s} \in B^\sharp$ such that $s \cdot \widehat{s} \cdot s = s$ and $\widehat{s} \cdot s \cdot \widehat{s} = \widehat{s}$. Now $s \cdot \widehat{s} \in B$ (since if $s \cdot \widehat{s} = 0 \Rightarrow s = 0 \notin B$) and $(s \cdot \widehat{s}) \cdot (s \cdot \widehat{s}) = (s \cdot \widehat{s} \cdot s) \cdot \widehat{s} = s \cdot \widehat{s}$.

c) Suppose that $t \cdot s \neq 0$ and $t \cdot s \in \mathrm{Idemp}(B)$. Then $s \cdot \widehat{s} = \widehat{t} \cdot t$ and $t \cdot s \cdot t \cdot s = t \cdot s$. We multiply the last relation with $\widehat{s}$ on the right and obtain

$$t \cdot s \cdot t \cdot s \cdot \widehat{s} = t \cdot s \cdot \widehat{s} \Rightarrow t \cdot s \cdot t \cdot \widehat{t} \cdot t = t \cdot \widehat{t} \cdot t \Rightarrow t \cdot s \cdot t = t.$$

Similarly we obtain $s \cdot t \cdot s = t$, thus $t = \widehat{s}$.

□

From [1] we recall the definition of a symmetric algebra. A k-algebra A is called *symmetric* if it is finitely generated and projective as k-module and there is $\tau : A \to k$ a central form (that is k-linear map with $\tau(a \cdot a') = \tau(a' \cdot a)$ for all $a, a' \in A$), which induces an isomorphism of $A - A$-bimodules

$$\widehat{\tau} : A \to A^*, \ \widehat{\tau}(a)(b) = \tau(a \cdot b),$$

where $a, b \in A$ and A^* is the k-dual. τ is called *symmetric form* of A and A is *principally symmetric* if τ is onto.

Theorem 1.6. *If A is an inv-symm k-algebra then A is principally symmetric. In particular it is symmetric.*

Proof. By Definition 1.1 A is a finitely generated k-module and free, thus projective. We define the following k-linear form on the basis B

$$\tau_B : A \to k, \quad \tau_B(s) = \begin{cases} 1_k, s \in \mathrm{Idemp}(B) \\ 0, \ s \notin \mathrm{Idemp}(B) \end{cases}$$

From Lemma 1.5, b) it follows that τ_B is not the zero map and τ_B is a k-linear form. We prove that it is a central form, that is $\tau_B(s \cdot t) = \tau_B(t \cdot s)$ where $t, s \in B$, by considering the cases:

- If $t \cdot s \neq 0$ and $t \cdot s \in \mathrm{Idemp}(B)$, by Lemma 1.5, c) it follows that $\widehat{s} = t$ and then

$$\tau_B(s \cdot \widehat{s}) = 1_k = \tau_B(\widehat{s} \cdot s).$$

- If $t \cdot s \neq 0$ and $t \cdot s \in B \setminus \mathrm{Idemp}(B)$ then $\tau_B(t \cdot s) = 0$. Now, if $s \cdot t \neq 0$ and $s \cdot t \in \mathrm{Idemp}(B)$ by Lemma 1.5, c) we get that $s = \widehat{t}$, which is a contradiction with

$\tau_B(t \cdot s) = 0$. So we have two possibilities: $s \cdot t = 0$, or $s \cdot t \neq 0$ and $s \cdot t \notin \mathrm{Idemp}(B)$. In both subcases $\tau_B(s \cdot t) = 0$.

- If $t \cdot s = 0$ then $\tau_B(t \cdot s) = 0$, and the same analyze to the second case gives us equality.

τ_B induces the following $A-A$-bimodule homomorphism $\widehat{\tau_B} : A \to A^*$ defined by

$$\widehat{\tau_B}(t)(s) = \tau_B(t \cdot s)$$

for any $t, s \in B$.

First we prove that $\widehat{\tau_B}$ is injective. Let $t_1, t_2 \in B$ such that $\tau_B(t_1 \cdot s) = \tau_B(t_2 \cdot s)$ for any $s \in B$. We choose $s = \widehat{t_1}$ and obtain that $\tau_B(t_2 \cdot \widehat{t_1}) = 1_k$. It follows that $t_2 \cdot \widehat{t_1} \neq 0$ and $t_2 \cdot \widehat{t_1} \in \mathrm{Idemp}(B)$. By Lemma 1.5, c) we obtain that $t_2 = \widehat{\widehat{t_1}} = t_1$.

For surjectivity let $\lambda \in A^*$ and define $a = \sum_{t \in B} \lambda(t) \cdot \widehat{t} \in A$. Then for $s \in B$

$$\widehat{\tau_B}(a)(s) = \sum_{t \in B} \lambda(t) \tau_B(\widehat{t} \cdot s).$$

Since $\tau_B(\widehat{t} \cdot s) = 1_k$ if and only if $s = t$ we obtain that

$$\widehat{\tau_B}(a)(s) = \lambda(s) \cdot \tau_B(\widehat{s} \cdot s) = \lambda(s).$$

This concludes the proof. □

2. Skew group algebras

In this section we will investigate the skew group algebra associated to a G-algebra which is an inv-symm algebra, where G is a finite group. The Remark 1.4 is the starting point of the next definition.

Definition 2.1. *A G-algebra A is called G-inv-symm if it is inv-symm, with the basis B (from Definition* 1.1*) G-stable.*

It is easy to show, using Theorem 1.6, that any G-inv-symm algebra is G-permutation and principally symmetric. If A is a G-algebra we denote the action of an $g \in G$ on $a \in A$ by ${}^g a$.

Theorem 2.2. *Let G be a finite group and A a G-algebra. If A is G-inv-symm then the skew group algebra, denoted $A \star G$, is inv-symm. In particular it is principally symmetric.*

Proof. We remind the definition of a skew group algebra. The skew group algebra $A \star G$ is the free A-module of basis

$$\{a \star g \mid a \in A, g \in G\},$$

where $a \star g$ is a notation and the product is given by

$$(a \star g)(b \star h) = a \cdot {}^g b \star gh.$$

Since B is the k-basis of A it is easy to check that the set

$$B \star G = \{s \star g \mid s \in B, g \in G\}$$

is a k-basis of the skew group algebra. Moreover it is a finite semigroup with zero, with the product defined above, since B is G-stable. Next we verify the conditions from Definition 1.1:

(1). We prove that the inverse of $s\star g\in B\star G$ is the element

$$\widehat{s\star g}={}^{g^{-1}}\widehat{s}\star g^{-1}\in B\star G.$$

We have

$$(s\star g)({}^{g^{-1}}\widehat{s}\star g^{-1})(s\star g)=(s\cdot\widehat{s}\star 1_G)(s\star g)=s\cdot\widehat{s}\cdot{}^{1_G}s\star g=s\star g.$$

Similarly we prove the other statement. Suppose now that there is $t\star h\in B\star G$ such that $(s\star g)(t\star h)(s\star g)=s\star g$. Then we have that

$$(s\cdot{}^{g}t\star gh)(s\star g)=s\star g\Rightarrow s\cdot{}^{g}t\cdot{}^{gh}s\star ghg=s\star g.$$

We have that $h=g^{-1}$ and $t={}^{g^{-1}}\widehat{s}$, thus it is unique.

(2). Let $s\star g, t\star h\in B\star G$. We have that $(t\star h)(s\star g)\neq 0$ if and only if $t\cdot{}^{h}s\neq 0$. We also have that

$$(s\star g)({}^{g^{-1}}\widehat{s}\star g^{-1})=({}^{h^{-1}}\widehat{t}\star h^{-1})(t\star h)\Leftrightarrow s\cdot\widehat{s}\star g={}^{h^{-1}}\widehat{t}\cdot{}^{h^{-1}}t\star 1_G\Leftrightarrow$$

$$s\cdot\widehat{s}={}^{h^{-1}}(\widehat{t}\cdot t)\Leftrightarrow{}^{h}s\cdot{}^{h}\widehat{s}=\widehat{t}\cdot t.$$

But since A is G-inv-symm the last condition is equivalent to $t\cdot{}^{h}s\neq 0$, by Definition 1.1, statement(2). $\square$

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On the existence of solutions for a class of fractional differential equations

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Abstract. In this paper, we study the existence and uniqueness of solutions to the Cauchy problem for nonautonomous fractional differential equations involving Caputo derivative in Banach spaces. Definition for the solution in the Carathéodory sense and fundamental lemma are introduced. Some sufficient conditions for the existence and uniqueness of solutions are established by means of fractional calculus, Hölder inequality via fixed point theorem under some weak conditions.

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1. Introduction

Fractional differential equations have gained considerable importance due to their application in various sciences, such as physics, mechanics, chemistry, engineering, etc. One can see the monographs of Diethelm [2], Miller and Ross [3], Kilbas et al. [4], Lakshmikantham et al. [5], Podlubny [6]. In survey, Agarwal et al. [7, 8] establish sufficient conditions for the existence and uniqueness of solutions for various classes of initial and boundary value problem for fractional differential equations and inclusions involving the Caputo derivative in finite and involving the Riemann-Liouville derivative in infinite dimensional spaces. Very recently, a lot of papers have been devoted to fractional differential equations and optimal controls in Banach spaces [9, 10, 11, 12, 13, 14, 15, 16].

In this paper, we reconsider the following Cauchy problem for nonautonomous fractional differential equations

$$\begin{cases} {}^cD^q u(t) = A(t)u(t) + f(t, u(t)),\ t \in J = [0, T],\ T > 0, \\ u(0) = u_0, \end{cases} \tag{1.1}$$

in a Banach space X, where ${}^cD^q$ is the Caputo fractional derivative of order $q \in (0, 1)$, $\{A(t), t \in J\}$ is a family of linear bounded operators in X, the function $t \to A(t)$ is continuous in the uniform operator topology, $f : J \times X \to X$ is Lebesgue measurable with respect to t and satisfies some assumptions that will be specified later.

A pioneering work on the existence of solutions for this kind of Cauchy problems has been studied by Balachandran and Park [9] in the case of $f : J \times X \to X$ is continuous and satisfies uniformly Lipschitz condition. In the present paper, we revisit this interesting problem and introduce a definition for solution of the system (1.1) in the Carathéodory sense and establish the existence and uniqueness of solutions for the system (1.1) under some weak conditions.

To prove our main results, we apply the classical fixed point theory including Krasnoselskii's fixed point theorem and Banach contraction principle via fractional calculus and Hölder inequality. Compared with the results appeared in [9], there are at least two differences: (i) assumptions on f are more general and easy to check; (ii) a definition for solution in the Carathéodory sense is established; (iii) two new existence results of solution in the Carathéodory sense are given.

2. Preliminaries

In this section, we introduce notations, definitions, and preliminary facts. Throughout this paper, $(X, \|\cdot\|)$ will be a Banach spaces. Let $C(J, X)$ be the Banach space of all continuous functions from J into X with the norm $\|u\|_C := \sup\{\|u(t)\| : t \in J\}$ for $u \in C(J, X)$.

Let us recall the following known definitions. For more details see [4].

Definition 2.1. *The fractional integral of order γ with the lower limit zero for a function f is defined as*

$$I^\gamma f(t) = \frac{1}{\Gamma(\gamma)} \int_0^t \frac{f(s)}{(t-s)^{1-\gamma}} ds, \ t > 0, \ \gamma > 0,$$

provided the right side is point-wise defined on $[0, \infty)$, where $\Gamma(\cdot)$ is the gamma function.

Definition 2.2. *The Riemann-Liouville derivative of order γ with the lower limit zero for a function $f : [0, \infty) \to R$ can be written as*

$$^L D^\gamma f(t) = \frac{1}{\Gamma(n-\gamma)} \frac{d^n}{dt^n} \int_0^t \frac{f(s)}{(t-s)^{\gamma+1-n}} ds, \ t > 0, \ n-1 < \gamma < n.$$

Definition 2.3. *The Caputo derivative of order γ for a function $f : [0, \infty) \to R$ can be written as*

$$^c D^\gamma f(t) = \ ^L D^\gamma \left(f(t) - \sum_{k=0}^{n-1} \frac{t^k}{k!} f^{(k)}(0) \right), \ t > 0, \ n-1 < \gamma < n.$$

Remark 2.4. (i) If $f(t) \in C^n[0, \infty)$, then

$$^c D^\gamma f(t) = \frac{1}{\Gamma(n-\gamma)} \int_0^t \frac{f^{(n)}(s)}{(t-s)^{\gamma+1-n}} ds = I^{n-\gamma} f^{(n)}(t), \ t > 0, \ n-1 < \gamma < n.$$

(ii) The Caputo derivative of a constant is equal to zero.

(iii) If f is an abstract function with values in X, then integrals which appear in Definitions 2.1 and 2.2 are taken in Bochner's sense.

For measurable functions $m : J \to R$, define the norm

$$\|m\|_{L^p(J)} = \begin{cases} \left(\int_J |m(t)|^p dt \right)^{\frac{1}{p}}, \ 1 \le p < \infty, \\ \inf_{\mu(\bar{J})=0} \{ \sup_{t \in J - \bar{J}} |m(t)| \}, \ p = \infty, \end{cases}$$

where $\mu(\bar{J})$ is the Lebesgue measure on $\bar{J}$. Let $L^p(J, R)$ be the Banach space of all Lebesgue measurable functions $m : J \to R$ with $\|m\|_{L^p(J)} < \infty$.

Lemma 2.5. *(Lemma 2.1, [17]) For all $\beta > 0$ and $\vartheta > -1$,*

$$\int_0^t (t-s)^{\beta-1} s^{\vartheta} ds = C(\beta, \vartheta) t^{\beta+\vartheta}$$

where $C(\beta, \vartheta) = \frac{\Gamma(\beta)\Gamma(\vartheta+1)}{\Gamma(\beta+\vartheta+1)}$.

Theorem 2.6. *(Krasnoselskii fixed point theorem) Let $\mathfrak{B}$ be a closed convex and nonempty subsets of X. Suppose that $\mathcal{L}$ and $\mathcal{N}$ are in general nonlinear operators which map $\mathfrak{B}$ into X such that*
(i) $\mathcal{L}x + \mathcal{N}y \in \mathfrak{B}$ whenever $x, y \in \mathfrak{B}$;
(ii) $\mathcal{L}$ is a contraction mapping;
(iii) $\mathcal{N}$ is compact and continuous.
Then there exists a $z \in \mathfrak{B}$ such that $z = \mathcal{L}z + \mathcal{N}z$.

3. Main results

In this section, we discuss the existence of solution for the system (1.1) by means of fixed point theorems.

We make the following assumptions:

[H1]: For any $u \in X$, $f(t, u)$ is Lebesgue measurable with respect to t on J.

[H2]: For any $t \in J$, $f(t, u)$ is continuous with respect to u on X.

[H3]: There exist a $q_1 \in (0, q)$ and a function $h(t) \in L^{\frac{1}{q_1}}(J, R^+) := L^{\frac{1}{q_1}}(J)$, such that $\|f(t, u)\| \le h(t)$, for arbitrary $(t, u) \in J \times X$.

[H4]: For every $t \in J$, the set $K = \big\{(t-s)^{q-1} f(s, u(s)) : u \in C(J, X), s \in [0, t]\big\}$ is relatively compact.

Now, let us introduce the definition of a solution of the system (1.1).

Definition 3.1. *A function $u \in C(J, X)$ is called a solution of the system (1.1) on J if*
(i) the function $u(t)$ is absolutely continuous on J,
(ii) $u(0) = u_0$, and
(iii) u satisfies the equation in the system (1.1).

For brevity, let

$$H = \|h\|_{L^{\frac{1}{q_1}}(J)}, \ \|A(t)\| \le M, \ \beta = \frac{q-1}{1-q_1} \in (-1, 0).$$

By Definition 2.1–2.3, using the same method in Theorem 3.2 of [1], we obtain the following lemma immediately.

Lemma 3.2. *Let the hypothesis [H1]–[H3] hold. A function $u \in C(J, X)$ is a solution of the fractional integral equation*

$$u(t) = u_0 + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} A(s)u(s)ds + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s, u(s))ds, \quad (3.1)$$

if and only if u is a solution of the system (1.1).

Now, we are ready to present and prove our main results.

Theorem 3.3. *Assume that [H1]–[H4] hold. If the following condition*

$$\Omega_{M,T,q} = \frac{MT^q}{\Gamma(q+1)} < 1 \quad (3.2)$$

holds, then the system (1.1) has at least one solution.

Proof. Choose

$$r \geq \frac{\frac{HT^{(1+\beta)(1-q_1)}}{\Gamma(q)(1+\beta)^{1-q_1}} + \frac{M\|u_0\|T^q}{\Gamma(q+1)}}{1 - \frac{MT^q}{\Gamma(q+1)}}, \quad (3.3)$$

and define the set

$$C_r = \{u \in C(J, X) : \|u - u_0\| \leq r\}.$$

By Lemma 3.2, the system (1.1) is equivalent to the following fractional integral equation

$$u(t) = u_0 + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} A(s)u(s)ds + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s, u(s))ds.$$

Now we define two operators P and Q on C_r as follows:

$$(Pu)(t) = \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s, u(s))ds,$$

and

$$(Qu)(t) = u_0 + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} A(s)u(s)ds.$$

Therefore, the existence of a solution of the system (1.1) is equivalent to that the operator $P + Q$ has a fixed point on C_r. The proof is divided into three steps.

Step 1: For all $u, v \in C_r$, $Pu + Qv \in C_r$.

For every pair $u, v \in C_r$ and any $\delta > 0$, by using Hölder inequality, we get

$$\begin{aligned}
& \|(Pu+Qv)(t+\delta)-(Pu+Qv)(t)\| \\
\leq\ & \frac{1}{\Gamma(q)}\int_0^t [(t-s)^{q-1}-(t+\delta-s)^{q-1}]h(s)ds \\
& +\frac{1}{\Gamma(q)}\int_t^{t+\delta}(t+\delta-s)^{q-1}h(s)ds \\
& +\frac{1}{\Gamma(q)}\int_0^t [(t-s)^{q-1}-(t+\delta-s)^{q-1}]M\|v(s)\|ds \\
& +\frac{1}{\Gamma(q)}\int_t^{t+\delta}(t+\delta-s)^{q-1}M\|v(s)\|ds \\
\leq\ & \frac{1}{\Gamma(q)}\left(\int_0^t [(t-s)^{q-1}-(t+\delta-s)^{q-1}]^{\frac{1}{1-q_1}}ds\right)^{1-q_1}\left(\int_0^t (h(s))^{\frac{1}{q_1}}ds\right)^{q_1} \\
& +\frac{1}{\Gamma(q)}\left(\int_t^{t+\delta} [(t+\delta-s)^{q-1}]^{\frac{1}{1-q_1}}ds\right)^{1-q_1}\left(\int_t^{t+\delta} (h(s))^{\frac{1}{q_1}}ds\right)^{q_1} \\
& +\frac{M(\|u_0\|+r)}{\Gamma(q)}\int_0^t (t-s)^{q-1}-(t+\delta-s)^{q-1}ds \\
& +\frac{M(\|u_0\|+r)}{\Gamma(q)}\int_t^{t+\delta}(t+\delta-s)^{q-1}ds \\
\leq\ & \frac{H}{\Gamma(q)}\left(\int_0^t (t-s)^{\beta}-(t+\delta-s)^{\beta}ds\right)^{1-q_1} \\
& +\frac{H}{\Gamma(q)}\left(\int_t^{t+\delta}(t+\delta-s)^{\beta}ds\right)^{1-q_1} \\
& +\frac{M(\|u_0\|+r)}{\Gamma(q)}\int_0^t (t-s)^{q-1}-(t+\delta-s)^{q-1}ds \\
& +\frac{M(\|u_0\|+r)}{\Gamma(q)}\int_t^{t+\delta}(t+\delta-s)^{q-1}ds \\
\leq\ & \frac{H}{\Gamma(q)(1+\beta)^{1-q_1}}\left(t^{1+\beta}-(t+\delta)^{1+\beta}+\delta^{1+\beta}\right)^{1-q_1} \\
& +\frac{H}{\Gamma(q)(1+\beta)^{1-q_1}}\delta^{(1+\beta)(1-q_1)} \\
& +\frac{M(\|u_0\|+r)}{\Gamma(q+1)}(t^q-(t+\delta)^q+\delta^q)+\frac{M(\|u_0\|+r)}{\Gamma(q+1)}\delta^q \\
\leq\ & \frac{2H}{\Gamma(q)(1+\beta)^{1-q_1}}\delta^{(1+\beta)(1-q_1)}+\frac{2M(\|u_0\|+r)}{\Gamma(q+1)}\delta^q.
\end{aligned}$$

As $\delta \to 0$, the right-hand side of the above inequality tends to zero.
Therefore $Pu + Qv \in C(J, X)$.

Moreover, for all $t \in J$, we get

$$\begin{aligned}
&\|(Pu)(t)+(Qv)(t)-u_0\| \\
\leq\ & \frac{H}{\Gamma(q)}\left(\int_0^t (t-s)^{\beta}ds\right)^{1-q_1} + \frac{M(\|u_0\|+r)}{\Gamma(q)}\int_0^t (t-s)^{q-1}ds \\
\leq\ & \frac{HT^{(1+\beta)(1-q_1)}}{\Gamma(q)(1+\beta)^{1-q_1}} + \frac{M(\|u_0\|+r)T^q}{\Gamma(q+1)} \\
\leq\ & \left[\frac{HT^{(1+\beta)(1-q_1)}}{\Gamma(q)(1+\beta)^{1-q_1}} + \frac{M\|u_0\|T^q}{\Gamma(q+1)}\right] + \frac{MT^q}{\Gamma(q+1)}r \\
\leq\ & r,
\end{aligned}$$

which implies that $Pu+Qv \in C_r$.

Step 2: Q is a contraction operator.

For arbitrary $u,v \in C_r$, we have

$$\begin{aligned}
\|Qu-Qv\| &\leq \frac{M}{\Gamma(q)}\left(\int_0^t (t-s)^{q-1}ds\right)\|u-v\|_C \\
&\leq \frac{MT^q}{\Gamma(q+1)}\|u-v\|_C,
\end{aligned}$$

which implies that

$$\|Qu-Qv\|_C \leq \Omega_{M,T,q}\|u-v\|_C.$$

From the condition (3.2), we know that Q is a contraction operator.

Step 3: We show that P is a complete continuous operator.

For that, let $\{u_n\}$ be a sequence of C_r such that $u_n \to u$ in C_r. Then, $f(s,u_n(s)) \to f(s,u(s))$ as $n \to \infty$ due to the hypotheses [H2].

Now, for all $t \in J$, we have

$$\|(Pu_n)(t)-(Pu)(t)\| \leq \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}\|f(s,u_n(s))-f(s,u(s))\|ds.$$

On the one other hand using [H3], we get for each $t \in J$,

$$\|f(s,u_n(s))-f(s,u(s))\| \leq 2h(s) \in L^{\frac{1}{q_1}}(J).$$

On the other hand, using the fact that the functions $s \to 2h(s)(t-s)^{q-1}$ is integrable on J, by means of the Lebesgue Dominated Convergence Theorem yields

$$\int_0^t (t-s)^{q-1}\|f(s,u_n(s))-f(s,u(s))\|ds \to 0.$$

Thus, $Pu_n \to Pu$ as $n \to \infty$ which implies that P is continuous.

Let $\{u_n\}$ be a sequence on C_r then

$$(Pu_n)(t) = \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}f(s,u_n(s))ds,$$

for all $t \in J$, using Hölder inequality, we have

$$\begin{aligned} \|(Pu_n)(t)\| &\leq \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} h(s) ds \\ &\leq \frac{HT^{(1+\beta)(1-q_1)}}{\Gamma(q)(1+\beta)^{1-q_1}}. \end{aligned}$$

This yields that the sequence $\{Pu_n\}$ is uniformly bounded.

Now, we need to prove that $\{Pu_n\}$ be equicontinuous.

For $0 \leq t_1 < t_2 \leq T$, we get

$$\begin{aligned} & \|(Pu_n)(t_2) - (Pu_n)(t_1)\| \\ \leq & \frac{1}{\Gamma(q)} \int_0^{t_1} [(t_1-s)^{q-1} - (t_2-s)^{q-1}] h(s) ds \\ & + \frac{1}{\Gamma(q)} \int_{t_1}^{t_2} (t_2-s)^{q-1} h(s) ds \\ \leq & \frac{1}{\Gamma(q)} \left(\int_0^{t_1} [(t_1-s)^{q-1} - (t_2-s)^{q-1}]^{\frac{1}{1-q_1}} ds \right)^{1-q_1} \left(\int_0^{t_1} (h(s))^{\frac{1}{q_1}} ds \right)^{q_1} \\ & + \frac{1}{\Gamma(q)} \left(\int_{t_1}^{t_2} [(t_2-s)^{q-1}]^{\frac{1}{1-q_1}} ds \right)^{1-q_1} \left(\int_{t_1}^{t_2} (h(s))^{\frac{1}{q_1}} ds \right)^{q_1} \\ \leq & \frac{H}{\Gamma(q)} \left(\int_0^{t_1} (t_1-s)^{\beta} - (t_2-s)^{\beta} ds \right)^{1-q_1} \\ & + \frac{H}{\Gamma(q)} \left(\int_{t_1}^{t_2} (t_2-s)^{\beta} ds \right)^{1-q_1} \\ \leq & \frac{H}{\Gamma(q)(1+\beta)^{1-q_1}} \left(t_1^{1+\beta} - t_2^{1+\beta} + (t_2-t_1)^{1+\beta} \right)^{1-q_1} \\ & + \frac{H}{\Gamma(q)(1+\beta)^{1-q_1}} (t_2-t_1)^{(1+\beta)(1-q_1)} \\ \leq & \frac{2H}{\Gamma(q)(1+\beta)^{1-q_1}} (t_2-t_1)^{(1+\beta)(1-q_1)}. \end{aligned}$$

As $t_2 \to t_1$, the right-hand side of the above inequality tends to zero. Therefore $\{Pu_n\}$ is equicontinuous.

In view of the condition [H4] and Mazur Lemma, we know that $\overline{conv}K$ is compact.

For any $t^* \in J$,

$$\begin{aligned} (Pu_n)(t^*) &= \frac{1}{\Gamma(q)} \lim_{k\to\infty} \sum_{i=1}^{k} \frac{t^*}{k} (t^* - \frac{it^*}{k})^{q-1} f(\frac{it^*}{k}, u_n(\frac{it^*}{k})) \\ &= \frac{t^*}{\Gamma(q)} \zeta_n, \end{aligned}$$

where

$$\zeta_n = \lim_{k\to\infty}\sum_{i=1}^{k}\frac{1}{k}(t^*-\frac{it^*}{k})^{q-1}f(\frac{it^*}{k},u_n(\frac{it^*}{k})).$$

Since $\overline{conv}K$ is convex and compact, we know that $\zeta_n \in \overline{conv}K$. Hence, for any $t^* \in J$, the set $\{Pu_n\}$ $(n = 1, 2, \cdots)$ is relatively compact. From Ascoli-Arzela theorem every $\{Pu_n(t)\}$ contains a uniformly convergent subsequence $\{Pu_{n_k}(t)\}$ $(k = 1, 2, \cdots)$ on J. Thus, the set $\{Pu : u \in C_r\}$ is relatively compact.

Therefore, the continuity of P and relatively compactness of the set $\{Pu : u \in C_r\}$ implies that P is a completely continuous operator. By Krasnoselskii's fixed point theorem, we get that $P + Q$ has a fixed point in C_r. Then system (1.1) has a solution on $t \in J$, and this completes the proof. □

Now we assume the following hypotheses:

[H5]: There exist a $q_2 \in [0, q)$ and a real-valued function $\mu(t) \in L^{\frac{1}{q_2}}(J)$ such that

$$\|f(t,u)-f(t,v)\| \le \mu(t)\|u-v\|, \text{ for all } u,v\in X,\ t\in J.$$

[H6]: Let

$$\Phi_{K,M,T,q,q_2} = \frac{KT^{(1+\beta')(1-q_2)}}{\Gamma(q)(1+\beta')^{1-q_2}} + \frac{MT^q}{\Gamma(q+1)} < 1$$

where $K = \|\mu\|_{L^{\frac{1}{q_2}}(J)}$, $\beta' = \frac{q-1}{1-q_2} \in (-1, 0)$.

Theorem 3.4. *Assume that [H1]–[H3], [H5]–[H6] hold. Then the system (1.1) has a unique solution.*

Proof. We define a operator F by

$$(Fu)(t) = u_0 + \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}A(s)u(s)ds + \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}f(s,u(s))ds.$$

Therefore, the existence of a solution of the system (1.1) is equivalent to that the operator F has a fixed point in C_r, where r is given in (3.3).

We can show that $F(C_r) \subseteq C_r$. In fact, for any $u, v \in C_r$, by using Hölder inequality we get

$$\begin{aligned}\|(Fu)(t)-(Fv)(t)\| &\le \frac{K\|u-v\|_C}{\Gamma(q)}\left(\int_0^t (t-s)^{\beta'}ds\right)^{1-q_2} \\ &\quad + \frac{M}{\Gamma(q)}\left(\int_0^t (t-s)^{q-1}ds\right)\|u-v\|_C \\ &\le \left[\frac{KT^{(1+\beta')(1-q_2)}}{\Gamma(q)(1+\beta')^{1-q_2}} + \frac{MT^q}{\Gamma(q+1)}\right]\|u-v\|_C.\end{aligned}$$

Hence,

$$\|Fu-Fv\|_C \le \Phi_{K,M,T,q,q_2}\|u-v\|_C.$$

In view of [H6], by applying the Banach contraction mapping principle we know that the operator F has a unique fixed point in C_r. Therefore, the system (1.1) has a unique solution. The proof is completed. □

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Some properties of certain class of multivalent analytic functions

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Abstract. In this paper we introduce a certain general class $\Phi_p^{\beta}(a, c, A, B)$ ($\beta \geq 0$, $a > 0$, $c > 0$, $-1 \leq B < A \leq 1$, $p \in N = \{1, 2, ...\}$) of multivalent analytic functions in the open unit disc $U = \{z : |z| < 1\}$ involving the linear operator $L_p(a, c)$. The aim of the present paper is to investigate various properties and characteristics of this class by using the techniques of Briot-Bouquet differential subordination. Also we obtain coefficient estimates and maximization theorem concerning the coefficients.

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1. Introduction

Let $A(p)$ denote the class of functions of the form:

$$f(z) = z^p + \sum_{k=1}^{\infty} a_{p+k} z^{p+k} \qquad (p \in N = \{1, 2, ...\}), \tag{1.1}$$

which are analytic and p-valent in the open unit disc $U = \{z : |z| < 1\}$. Let Ω denotes the class of bounded analytic functions $w(0) = 0$ and $|w(z)| \leq |z|$ for $z \in U$. If f and g are analytic in U, we say that f subordinate to g, written symbolically as follows:

$$f \prec g \quad (z \in U) \text{ or } f(z) \prec g(z),$$

if there exists a Schwarz function w, which (by definition) is analytic in U with $w(0) = 0$ and $|w(z)| < 1$ $(z \in U)$ such that $f(z) = g(w(z))$ $(z \in U)$. In particular, if the function $g(z)$ is univalent in U, then we have the following equivalence (cf., e.g., [5], [18]; see also [19, p. 4]):

$$f(z) \prec g(z) \Leftrightarrow f(0) = g(0) \text{ and } f(U) \subset g(U).$$

For functions $f(z) \in A(p)$, given by (1.1), and $g(z) \in A(p)$ given by

$$g(z) = z^p + \sum_{k=1}^{\infty} b_{p+k} z^{p+k} \qquad (p \in N), \tag{1.2}$$

then the Hadamard product (or convolution) of $f(z)$ and $g(z)$ is defined by

$$(f * g)(z) = z^p + \sum_{k=1}^{\infty} a_{p+k} b_{p+k} z^{p+k} = (g * f)(z) \, . \tag{1.3}$$

We now define the function $\varphi_p(a, c; z)$ by

$$\varphi_p(a, c; z) = z^p + \sum_{k=1}^{\infty} \frac{(a)_k}{(c)_k} z^{p+k} \ \left(z \in U; a \in R; c \in R \backslash Z_0^- : Z_0^- = \{0, -1, -2, ...\}\right), \tag{1.4}$$

where $(\lambda)_\nu$ denoted the Pochhammer symbol defined (for $\lambda, \nu \in C$ and in terms of the Gamma function) by

$$(\lambda)_\nu = \frac{\Gamma(\lambda + \nu)}{\Gamma(\lambda)} = \begin{cases} 1 & (\nu = 0; \lambda \in C \backslash \{0\}), \\ \lambda(\lambda + 1)...(\lambda + n - 1) & (\nu \in N; \lambda \in C). \end{cases} \tag{1.5}$$

With the aid of the function $\varphi_p(a, c; z)$ defined by (1.4), we consider a function $\varphi_p^+(a, c; z)$ given by the following convolution:

$$\varphi_p(a, c; z) * \varphi_p^+(a, c; z) = \frac{z^p}{(1 - z)^{\lambda + p}} \quad (\lambda > -p \ ; \ z \in U), \tag{1.6}$$

which yields the following family of linear operator $I_p^\lambda(a, c)$:

$$I_p^\lambda(a, c) f(z) = \varphi_p^+(a, c; z) * f(z) \qquad (a, c \in R \backslash Z_0^-; \lambda > -p; z \in U) \, . \tag{1.7}$$

For a function $f(z) \in A(p)$, given by (1.1), it is easily seen from (1.6) that

$$I_p^\lambda(a, c) f(z) = z^p + \sum_{k=1}^{\infty} \frac{(c)_k (\lambda + p)_k}{(a)_k (1)_k} a_{p+k} z^{p+k} \qquad (z \in U) \, . \tag{1.8}$$

It is readily verified from the definition (1.8) that

$$z \left(I_p^\lambda(a, c) f(z)\right)' = (a - 1) I_p^\lambda(a - 1, c) f(z) + (p + 1 - a) I_p^\lambda(a, c) f(z) \, . \tag{1.9}$$

The operator $I_p^\lambda(a, c)$ was recently introduced by Cho et al. [6].

We observe also that:

(i) $I_p^1(p + 1, 1) f(z) = f(z)$ and $I_p^1(p, 1) f(z) = \frac{z f'(z)}{p}$;

(ii) $I_p^n(a, a) f(z) = D^{n+p-1} f(z)$ $(n > -p)$, where $D^{n+p-1} f(z)$ is the $(n+p-1)-th$ order Ruscheweyh derivative of a function $f(z) \in A(p)$ (see Kumar and Shukla [15]);

(iii) $I_p^\delta(\delta + p + 1, 1) f(z) = F_{\delta,p}(f)(z)$ $(\delta > -p)$, where $F_{\delta,p}(f)(z)$ is the generalized Bernardi-Livingston operator (see [7]), defined by

$$F_{\delta,p}(f)(z) = \frac{\delta + p}{z^\delta} \int_0^z t^{\delta - 1} f(t) dt = z^p + \sum_{k=1}^{\infty} \left(\frac{\delta + p}{\delta + p + k}\right) a_{p+k} z^{p+k} (\delta > -p; p \in N); \tag{1.10}$$

(iv) $I_p^1(n+p,1)f(z) = I_{n,p}f(z)$ $(n > -p)$, where the operator $I_{n,p}$ is the $(n+p-1)-th$ Noor operator, considered by Liu and Noor [16];

(v) $I_p^1(p+1-\mu,1)f(z) = \Omega_z^{(\mu,p)}f(z)(-\infty < \mu < p+1)$, where $\Omega_z^{(\mu,p)}$ $(-\infty < \mu < p+1)$ is the extended fractional differential integral operator (see [26]), defined by

$$\Omega_z^{(\mu,p)}f(z) = z^p + \sum_{k=1}^{\infty}\frac{\Gamma(k+p+1)\Gamma(p+1-\mu)}{\Gamma(p+1)\Gamma(k+p+1-\mu)}a_{p+k}z^{p+k}$$
$$= \frac{\Gamma(p+1-\mu)}{\Gamma(p+1)}z^{\mu}D_z^{\mu}f(z) \quad (-\infty < \mu < p+1; z \in U), \tag{1.11}$$

where $D_z^{\mu}f(z)$ is, respectively, the fractional integral of $f(z)$ of order $-\mu$ when $-\infty < \mu < 0$ and the fractional derivative of $f(z)$ of order μ when $0 \le \mu < p+1$ (see, for details [23], [25] and [26]). The fractional differential operator $\Omega_z^{(\mu,p)}$ with $0 \le \mu < 1$ was investigated by Srivastava and Aouf [29].

Making use of the operator $I_p^{\lambda}(a,c)$, we now introduce a subclass of $A(p)$ as follows:

Definition 1.1. *A function $f(z) \in A(p)$ is said to be in the class $\Phi_p^{\beta}(\lambda,a,c,A,B)$ $(\beta > 0,\ a,c \in R\backslash Z_0^-,\ a > 1;\ \lambda > -p,\ p \in N,\ -1 \le B < A \le 1)$ if and only if it satisfies*

$$(1-\beta)\frac{I_p^{\lambda}(a,c)f(z)}{z^p} + \beta\frac{I_p^{\lambda}(a-1,c)f(z)}{z^p} \prec \frac{1+Az}{1+Bz} \qquad (z \in U). \tag{1.12}$$

By specializing the parameters β,λ,a,c,A and B, we obtain the following subclasses of analytic functions studied by various authors:

(i) $\Phi_p^1(1,p+1,1,1,\frac{1}{M}-1) = S_p(M)$ $(M > \frac{1}{2})$ *(Sohi [28]);*

(ii) $\Phi_p^1(1,p+1,1,\beta[B+(A-B)(p-\alpha)],\beta B) = S_p(\alpha,\beta,A,B)$, $0 \le \alpha < p$, $p \in N$, $0 < \beta \le 1$ *(see Aouf [2]);*

(iii) $\Phi_p^1(1,p+1,1,[B+(A-B)(p-\alpha)],B) = S_p(A,B,\alpha)$, $0 \le \alpha < p$, $p \in N$ *(see Aouf and Chen [4]);*

(iv) $\Phi_1^1(1,2,1,1,\frac{1}{M}-1) = R(M)$ $(M > \frac{1}{2})$ *(see Goel [9]);*

(v) $\Phi_1^1(1,2,1,2\alpha\beta-1,2\beta-1) = R_1(\alpha,\beta)$ $(0 \le \alpha < 1,\ 0 < \beta \le 1)$ *(see Mogra [20]);*

(vi) $\Phi_1^1(1,2,1,(1-2\alpha)\beta,-\beta) = R(\alpha,\beta)$ $(0 \le \alpha < 1,\ 0 < \beta \le 1)$ *(see Juneja and Mogra [12]);*

(vii) $\Phi_p^1(1,2,1,(1-2\alpha)\beta,-\beta) = S_p(\alpha,\beta)$ $(0 \le \alpha < 1,\ 0 < \beta \le 1)$ *(see Owa [24]);*

(viii) $\Phi_1^1(n+1,a,a-1,A,B) = V_n(A,B)$ $(n \in N_0 = N \cup \{0\})$ *(see Kumar [14]);*

(ix) $\Phi_1^1(n+1,a,a-1,[B+(A-B)(1-\alpha)],B) = V_n(A,B,\alpha)$ $(n \in N_0,\ 0 \le \alpha < 1)$ *(see Aouf [3]);*

(x) $\Phi_p^{\beta}(\lambda,a,c,1,\frac{1}{M}-1) = \Phi_p^{\beta}[\lambda,a,c,M]$ $(M > \frac{1}{2})$, *where* $\Phi_p^{\beta}[\lambda,a,c,M]$ *denotes the class of functions* $f(z) \in A(p)$ *satisfying the condition:*

$$\left|\left[(1-\beta)\frac{I_p^\lambda(a,c)f(z)}{z^p}+\beta\frac{I_p^\lambda(a-1,c)f(z)}{z^p}\right]-M\right|<M \quad \left(M>\frac{1}{2}; z\in U\right);$$

(xi) $\Phi_p^1(1,p+1-\mu,1,1,\frac{1}{M}-1)=\Phi_p[\mu,M]$ $(M>\frac{1}{2},-\infty<\mu<p)$, *where* $\Phi_p[\mu,M]$ *denotes the class of functions* $f(z)\in A(p)$ *satisfying the condition:*

$$\left|\frac{\Omega_z^{(\mu,1+p)}f(z)}{z^p}-M\right|<M \quad \left(M>\frac{1}{2}; -\infty<\mu<p; z\in U\right).$$

2. Preliminaries

To establish our main results, we shall need the following lemmas.

Lemma 2.1. [11] *Let* h *be a convex (univalent) in* U *with* $h(0)=1$ *and let the function* φ *given by*

$$\varphi(z)=1+d_1z+d_2z^2+..., \tag{2.1}$$

is analytic in U. *If*

$$\varphi(z)+\frac{1}{\gamma}z\varphi^{'}(z)\prec h(z) \quad (z\in U), \tag{2.2}$$

where $\gamma\neq 0$ *and* $\mathrm{Re}(\gamma)\geq 0$, *then*

$$\varphi(z)\prec\psi(z)=\frac{\gamma}{z^\gamma}\int_0^z t^{\gamma-1}h(t)dt\prec h(z) \qquad (z\in U),$$

and ψ *is the best dominant of (2.2).*

Lemma 2.2. [27] *Let* $\Phi(z)$ *be analytic in* U *with*

$$\Phi(0)=1 \text{ and } \mathrm{Re}\{\Phi(z)\}>\frac{1}{2} \quad (z\in U).$$

Then, for any $F(z)$ *analytic in* U, *the set* $(\Phi * F)(U)$ *is contained in the convex hull of* $F(U)$, *i.e.,* $(\Phi * F)U\subset co\ F(U)$.

For complex numbers a,b and $c(c\neq 0,-1,-2,...)$, the Gaussian hypergeometric function is defined by

$${}_2F_1(a,b;c;z)=1+\frac{a.b}{c}\frac{z}{1!}+\frac{a(a+1)b(b+1)}{c(c+1)}\frac{z^2}{2!}+..., \ z\in U. \tag{2.3}$$

We note that the above series converges absolutely for $z\in U$ and hence represents an analytic function in U (see, for details, [30, Chapter 14]).

Each of the identities (asserted by Lemmas below) is well-known (cf., e.g., [30, Chapter 14]).

Lemma 2.3. [30] *For complex numbers a, b and $c\,(c \neq 0, -1, -2, ...)$, the next equalities hold:*

$$\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a}dt = \frac{\Gamma(b)\Gamma(c-b)}{\Gamma(c)}\ {}_2F_1(a, b; c; z), \tag{2.4}$$

$$(\mathrm{Re}(c) > \mathrm{Re}(b) > 0),$$

$${}_2F_1(a, b; c; z)\ = (1-z)^{-a}\ {}_2F_1(a, c-b; c; \frac{z}{z-1}); \tag{2.5}$$

and

$$(b+1)\ {}_2F_1(1, b; b+1; z) = (b+1) + bz\ \ {}_2F_1(1, b+1; b+2; z)\ . \tag{2.6}$$

Lemma 2.4. [13] *Let $w(z) = \sum_{k=1}^{\infty} d_k z^k \in \Omega$, if ν is any complex number, then*

$$\left|d_2 - \nu d_1^2\right| \leq \max\{1, |\nu|\}\,. \tag{2.7}$$

Equality may be attained with the functions $w(z) = z^2$ and $w(z) = z$.

3. Main results

Unless otherwise mentioned, we assume throughout of this paper that $\beta > 0$, $a, c \in R\backslash Z_0^-$, $\lambda > -p$, $p \in N$ and $-1 \leq B < A \leq 1$.

Theorem 3.1. *Let the function f defined by (1.1) be in the class $\Phi_p^{\beta}(\lambda, a, c, A, B)$. Then*

$$\frac{I_p^{\lambda}(a,c)f(z)}{z^p} \prec Q(z) \prec \frac{1+Az}{1+Bz} \qquad (z \in U), \tag{3.1}$$

where the function $Q(z)$ given by

$$Q(z) = \begin{cases} \dfrac{A}{B} + (1-\dfrac{A}{B})(1+Bz)^{-1}{}_2F_1(1, 1, \dfrac{a-1}{\beta}+1, \dfrac{Bz}{Bz+1})\ , & B \neq 0, \\ 1 + \dfrac{a-1}{a-1+\beta}Az\ , & B = 0, \end{cases}$$

is the best dominant of (3.1). Furthermore,

$$\mathrm{Re}\left\{\frac{I_p^{\lambda}(a,c)f(z)}{z^p}\right\} > \eta(\beta, a, A, B) \qquad (z \in U), \tag{3.2}$$

where

$$\eta(\beta, a, A, B) = \begin{cases} \dfrac{A}{B} + (1-\dfrac{A}{B})(1-B)^{-1}{}_2F_1(1, 1, \dfrac{a-1}{\beta}+1, \dfrac{B}{B-1})\ , & B \neq 0, \\ 1 - \dfrac{a-1}{a-1+\beta}A\ , & B = 0. \end{cases}$$

The estimate in (3.2) is the best possible.

Proof. Consider the function $\varphi(z)$ defined by

$$\varphi(z)=\frac{I_p^\lambda(a,c)f(z)}{z^p}\qquad(z\in U)\ . \tag{3.3}$$

Then $\varphi(z)$ is of the form (2.1) and is analytic in U. Differentiating (3.3) logarithmically with respect to z and using the identity (1.9) in the resulting equation, we obtain

$$(1-\beta)\frac{I_p^\lambda(a,c)f(z)}{z^p}+\beta\frac{I_p^\lambda(a-1,c)f(z)}{z^p}=\varphi(z)+\frac{z\varphi'(z)}{(a-1)/\beta}\prec\frac{1+Az}{1+Bz}\qquad(z\in U).$$

Now, by using Lemma 2.1 for $\gamma=\frac{a-1}{\beta}$, we deduce that

$$\frac{I_p^\lambda(a,c)f(z)}{z^p}\prec Q(z)=\frac{a-1}{\beta}z^{\frac{1-a}{\beta}}\int_0^z t^{\frac{a-1}{\beta}-1}\left(\frac{1+At}{1+Bt}\right)dt$$

$$=\begin{cases}\frac{A}{B}+(1-\frac{A}{B})(1+Bz)^{-1}{}_2F_1(1,1,\frac{a-1}{\beta}+1;\frac{Bz}{Bz+1})\ , & B\neq 0,\\ 1+\frac{a-1}{a-1+\beta}Az\ , & B=0,\end{cases}$$

by change of variables followed by use of the identities (2.4), (2.5) and (2.6) (with $a=1,\ c=b+1,\ b=\frac{a-1}{\beta}$). This proves the assertion (3.1) of Theorem 3.1.

Next, in order to prove the assertion (3.2) of Theorem 3.1, it suffices to show that

$$\inf_{|z|<1}\{\mathrm{Re}(Q(z)\}=Q(-1)\ . \tag{3.4}$$

Indeed we have, for $|z|\le r<1$,

$$\mathrm{Re}\left(\frac{1+Az}{1+Bz}\right)\ge\frac{1-Ar}{1-Br}\ .$$

Upon setting

$$g(s,z)=\frac{1+Asz}{1+Bsz}\quad\text{and}\quad d\nu(s)=\left(\frac{a-1}{\beta}\right)s^{\frac{a-1}{\beta}}ds\ (0\le s\le 1)\ ,$$

which is a positive measure on the closed interval $[0,1]$, we get

$$Q(z)=\int_0^1 g(s,z)d\nu(s)\ ,$$

so that

$$\mathrm{Re}\left\{Q(z)\right\}\ge\int_0^1\left(\frac{1-Asr}{1-Bsr}\right)dv(s)=Q(-r)\quad(|z|\le r<1).$$

Letting $r\to 1^-$ in the above inequality, we obtain the assertion (3.2) of Theorem 3.1. Finally, the estimate in (3.2) is the best possible as the function $Q(z)$ is the best dominant of (3.1).

Corollary 3.2. *For* $0 < \beta_2 < \beta_1$, *we have*

$$\Phi_p^{\beta_1}(\lambda, a, c, A, B) \subset \Phi_p^{\beta_2}(\lambda, a, c, A, B) \ .$$

Proof. Let $f \in \Phi_p^{\beta_1}(\lambda, a, c, A, B)$. Then by Theorem 3.1, we have

$$\frac{I_p^{\lambda}(a,c)f(z)}{z^p} \prec \frac{1+Az}{1+Bz} \qquad (z \in U).$$

Since

$$\begin{aligned} &(1-\beta_2)\frac{I_p^{\lambda}(a,c)f(z)}{z^p} + \beta_2\frac{I_p^{\lambda}(a-1,c)f(z)}{z^p} \\ = &\left(1-\frac{\beta_2}{\beta_1}\right)\frac{I_p^{\lambda}(a,c)f(z)}{z^p} + \frac{\beta_2}{\beta_1}\left\{(1-\beta_1)\frac{I_p^{\lambda}(a,c)f(z)}{z^p} + \beta_1\frac{I_p^{\lambda}(a-1,c)f(z)}{z^p}\right\} \\ \prec &\frac{1+Az}{1+Bz} \qquad (z \in U) \ , \end{aligned}$$

we see that $f \in \Phi_p^{\beta_2}(\lambda, a, c, A, B)$. This proves Corollary 3.2.

Taking $\beta = c = 1,\ a = \delta + p + 1\ (\delta > -p),\ \lambda = \delta,\ A = 1 - \frac{2\alpha}{p}\ (0 \le \alpha < p)$ and $B = -1$ in Theorem 3.1, we obtain the the following corollary.

Corollary 3.3. *If* $f \in A(p)$ *satisfies*

$$\operatorname{Re}\left\{\frac{f(z)}{z^p}\right\} > \frac{\alpha}{p} \qquad (0 \le \alpha < p; z \in U) \ ,$$

then the function $F_{\delta,p}(f)(z)$ *defined by (1.10) satisfies*

$$\operatorname{Re}\left\{\frac{F_{\delta,p}(f)(z)}{z^p}\right\} > \frac{\alpha}{p} + \left(1-\frac{\alpha}{p}\right)\left[{}_2F_1(1,1;p+\delta+1;\frac{1}{2}) - 1\right] \qquad (z \in U) \ .$$

The result is the best possible.

Remark 3.4. We note that Corollary 3.3 improves the corresponding result obtained by Obradovic [22] for $p = 1$.

Taking $\lambda = \beta = c = 1$, $a = p + 1 - \mu$, $-\infty < \mu < p$, $A = 1 - \frac{2\alpha}{p}$ $(0 \le \alpha < p)$ $B = -1$ in Theorem 3.1, we obtain the following corollary.

Corollary 3.5. *Let the function* $f(z)$ *given by (1.1) satisfy*

$$\operatorname{Re}\left\{\frac{\Omega_z^{(1+\mu,p)}f(z)}{z^p}\right\} > \frac{\alpha}{p} \quad (-\infty < \mu < p; 0 \le \alpha < p; p \in N; z \in U) \ .$$

Then

$$\operatorname{Re}\left\{\frac{\Omega_z^{(\mu,p)}f(z)}{z^p}\right\} > \frac{\alpha}{p} + \left(1-\frac{\alpha}{p}\right)\left[{}_2F_1(1,1;p+1-\mu;\frac{1}{2}) - 1\right] \quad (z \in U) \ .$$

The result is the best possible.

Taking $\mu = 0$ in Corollary 3.5, we obtain the following corollary.

Corollary 3.6. *Let the function $f(z)$ given by (1.1) satisfy*

$$\operatorname{Re}\left\{\frac{f'(z)}{z^{p-1}}\right\} > \alpha \quad (0 \le \alpha < p; z \in U)\ ,$$

Then

$$\operatorname{Re}\left\{\frac{f(z)}{z^p}\right\} > \frac{\alpha}{p} + \left(1-\frac{\alpha}{p}\right)\left[{}_2F_1(1,1;p+1;\frac{1}{2}) - 1\right] \quad (z \in U)\ .$$

The result is the best possible.

Remark 3.7. We note that Corollary 3.6 improves the corresponding result obtained by Lee and Owa [17, Theorem 1] with $n = 1$.

Remark 3.8. If $f \in A(p)$ *satisfies* $\operatorname{Re}\left\{f'(z)/z^{p-1}\right\} > \alpha$ $(0 \le \alpha < p; z \in U)$, *then with the aid of Corollaries 2 and 4, we deduce that*

$$\operatorname{Re}\left\{\frac{F_{\delta,p}(f)(z)}{z^p}\right\} > \frac{\alpha}{p} + \left(1-\frac{\alpha}{p}\right)\left[\left({}_2F_1(1,1;p+1;\frac{1}{2}) - 1\right)\right.$$
$$\left. + \left({}_2F_1(1,1;p+\delta+1;\frac{1}{2}) - 1\right)\left(2 - \left({}_2F_1(1,1;p+1;\frac{1}{2})\right)\right)\right],$$

which improve the result of Fukui et al. [8] for $p = 1$.

Corollary 3.9. *Let the function $f(z)$ given by (1.1) satisfy*

$$\operatorname{Re}\left\{\frac{I_p^n(n-1,n)f(z)}{z^p}\right\} > \frac{\alpha}{p} \quad (0 \le \alpha < p; z \in U)\ ,$$

Then

$$\operatorname{Re}\left\{\frac{D^{n+p-1}f(z)}{z^p}\right\} > \frac{\alpha}{p} + \left(1-\frac{\alpha}{p}\right)\left[{}_2F_1(1,1;n;\frac{1}{2}) - 1\right] \quad (z \in U)\ .$$

The result is the best possible.

Theorem 3.10. *Let $f(z) \in \Phi_p^0(\lambda, a, c, A, B)$ and let the function $F_{\delta,p}(f)(z)$ defined by (1.10). Then*

$$\frac{I_p^\lambda(a,c)F_{\delta,p}(f)(z)}{z^p} \prec q(z) \prec \frac{1+Az}{1+Bz}\ , \tag{3.5}$$

where the function $q(z)$ given by

$$q(z) = \begin{cases} \dfrac{A}{B} + (1-\dfrac{A}{B})(1+Bz)^{-1}{}_2F_1(1,1,p+\delta+1;\dfrac{Bz}{Bz+1})\ , & B \ne 0 \\ 1 + \dfrac{p+\delta}{p+\delta+1}Az\ , & B = 0. \end{cases}$$

is the best dominant of (3.5). Furthermore,

$$\operatorname{Re}\left\{\frac{I_p^\lambda(a,c)F_{\delta,p}(f)(z)}{z^p}\right\} > \zeta^* \quad (z \in U)\ , \tag{3.6}$$

where

$$\zeta^* = \begin{cases} \frac{A}{B} + (1-\frac{A}{B})(1-B)^{-1}{}_2F_1(1,1;p+\delta+1;\frac{B}{B-1}) \ , & B \neq 0 \ , \\ 1 - \frac{p+\delta}{p+\delta+1}A \ , & B = 0 \ . \end{cases}$$

The estimate in (3.6) is the best possible.

Proof. From (1.10) it follows that

$$z\left(I_p^\lambda(a,c)F_{\delta,p}(f)(z)\right)' = (\delta+p)I_p^\lambda(a,c)f(z) - \delta I_p^\lambda(a,c)F_{\delta,p}(f)(z) \ . \tag{3.7}$$

By setting

$$\varphi(z) = \frac{I_p^\lambda(a,c)F_{\delta,p}(f)(z)}{z^p} \qquad (z \in U) \ , \tag{3.8}$$

we note that $\varphi(z)$ is of the form (2.1) and is analytic in U. Differentiating (3.8) with respect to z and using the identity (3.7) in the resulting equation, we get

$$\varphi(z) + \frac{z\varphi'(z)}{\delta+p} = \frac{I_p^\lambda(a,c)f(z)}{z^p} \prec \frac{1+Az}{1+Bz} \quad (z \in U) \ ,$$

which with the aid of Lemma 2.1 with $\gamma = \delta + p$, yields

$$\frac{I_p^\lambda(a,c)F_{\delta,p}(f)(z)}{z^p} \prec q(z) = (\delta+p)z^{-(\delta+p)} \int_0^z t^{\delta+p-1}\left(\frac{1+At}{1+Bt}\right) dt \ . \tag{3.9}$$

Now the remaining part of Theorem 3.10 follows by employing the techniques that we used in proving Theorem 3.1 above.

Taking $A = 1 - \frac{2\alpha}{p}$ $(0 \le \alpha < p)$ and $B = -1$ in Theorem 3.10, we obtain the following corollary.

Corollary 3.11. *If $f \in A(p)$ satisfies*

$$\mathrm{Re}\left\{\frac{I_p^\lambda(a,c)f(z)}{z^p}\right\} > \frac{\alpha}{p} \quad (0 \le \alpha < p; z \in U) \ ,$$

then

$$\mathrm{Re}\left\{\frac{I_p^\lambda(a,c)F_{\delta,p}(f)(z)}{z^p}\right\} > \frac{\alpha}{p} + \left(1 - \frac{\alpha}{p}\right)\left\{{}_2F_1(1,1;p+\delta+1;\frac{1}{2}) - 1\right\} \quad (z \in U) \ .$$

The result is the best possible.

Taking $\lambda = c = 1$ and $a = p$ in Corollary 3.11, we get the following corollary which in turn improves the corresponding result of Fukui et al. [8] for $p = 1$.

Corollary 3.12. *If $f \in A(p)$ satisfies*

$$\mathrm{Re}\left\{\frac{f'(z)}{z^{p-1}}\right\} > \alpha \quad (0 \le \alpha < p; z \in U) \ ,$$

then

$$\mathrm{Re}\left\{\frac{F'_{\delta,p}(f)(z)}{z^{p-1}}\right\} > \alpha + (p-\alpha)\left\{{}_2F_1(1,1;p+\delta+1;\frac{1}{2})-1\right\} \quad (z \in U)\ .$$

The result is the best possible.

Taking $\lambda = c = 1$ and $a = p+1-\mu$ $(-\infty < \mu < p+1, p \in N)$ in Corollary 3.11, we obtain the following corollary.

Corollary 3.13. *If $f(z) \in A(p)$ satisfies*

$$\mathrm{Re}\left\{\frac{\Omega_z^{(\mu,p)} f(z)}{z^p}\right\} > \frac{\alpha}{p} \quad (0 \le \alpha < p; -\infty < \mu < p+1; p \in N; z \in U)\ ,$$

then

$$\mathrm{Re}\left\{\frac{\Omega_z^{(\mu,p)} F_{\delta,p}(f)(z)}{z^p}\right\} > \frac{\alpha}{p} + \left(1-\frac{\alpha}{p}\right)\left\{{}_2F_1(1,1;p+\delta+1;\frac{1}{2})-1\right\} \quad (z \in U)\ .$$

The result is the best possible.

Theorem 3.14. *We have*

$$f \in \Phi_p^0(a,c,A,B) \Leftrightarrow F_{a-p-1}(f)(z) \in \Phi_p^1(a,c,A,B)$$

Proof. Using the identity (3.7) and

$$z\left(I_p^\lambda(a,c)F_{\delta,p}(f)(z)\right)' = (a-1)I_p^\lambda(a-1,c)F_{\delta,p}(f)(z) + (p+1-a)I_p^\lambda(a,c)F_{\delta,p}(f)(z)\ ,$$

for $\delta = a-p-1$, we deduce that

$$I_p^\lambda(a,c)f(z) = I_p^\lambda(a-1,c)F_{a-p-1}(f)(z)$$

and the assertion of Theorem 3.14 follows by using the definition of the class $\Phi_p^\beta(a,c,A,B)$.

Theorem 3.15. *If f, given by (1.1), belongs to the class $\Phi_p^\beta(a,c,A,B)$, then*

$$|a_{p+k}| \le \frac{(A-B)(a-1)_{k+1}}{(a-1+\beta k)(c)_k}\frac{(1)_k}{(\lambda+p)_k} \quad (k \ge 1)\ . \tag{3.10}$$

The result is sharp.

Proof. Since $f \in \Phi_p^\beta(a,c,A,B)$, we have

$$(1-\beta)\frac{I_p^\lambda(a,c)f(z)}{z^p} + \beta\frac{I_p^\lambda(a-1,c)f(z)}{z^p} = p(z)\ , \tag{3.11}$$

where $p(z) = 1 + \sum_{k=1}^{\infty} p_k z^k \in P(A,B)$. Substituting the power series expansion of $I_p^\lambda(a,c)f(z)$, $I_p^\lambda(a-1,c)f(z)$ and $p(z)$ in (3.11) and equating the coefficients of z^k on both sides of the resulting equation, we obtain

$$\frac{(a-1+\beta k)(\lambda+k)_k}{(a-1)_{k+1}}\frac{(c)_k}{(1)_k}a_{p+k} = p_k \quad (k \ge 1)\ . \tag{3.12}$$

Using the well-known [1] coefficient estimates

$$|p_k| \leq (A-B) \quad (k \geq 1)$$

in (3.12), we get the required estimate (3.10).

In order to establish the sharpness of (3.10), consider the functions $f_k(z)$ defined by

$$(1-\beta)\frac{I_p^\lambda(a,c)f(z)}{z^p} + \beta\frac{I_p^\lambda(a-1,c)f(z)}{z^p} = \frac{1+Az^k}{1+Bz^k} \quad (k \geq 1) \ .$$

Clearly, $f_k(z) \in \Phi_p^\beta(\lambda, a, c, A, B)$ for each $k \geq 1$. It is easy to see that the functions $f_k(z)$ have the expansion

$$f_k(z) = z^p + \frac{(A-B)(a-1)_{k+1}}{(a-1+\beta k)(\lambda+p)_k}\frac{(1)_k}{(c)_k}z^{p+k} + ...$$

showing that the estimates in (3.10) are sharp.

Taking $\beta = \lambda = c = A = 1,\ a = p+1-\mu,\ -\infty < \mu < p$ and $B = \frac{1}{M} - 1$ $(M > \frac{1}{2})$ in Theorem 3.15, we obtain the following corollary.

Corollary 3.16. *If f, given by (1.1), belongs to the class $\Phi_p[\mu, M]$, then*

$$|a_{p+k}| \leq \frac{\left(\frac{2M-1}{M}\right)(p-\mu)_k}{(p+1)_k} \quad (k \geq 1) \ .$$

The result is sharp.

Theorem 3.17. *Let f, given by (1.1), belongs to the class $\Phi_p^\beta(\lambda, a, c, A, B)$ and ζ is any complex number. Then*

$$\left|a_{p+2} - \zeta a_{p+1}^2\right| \leq \frac{(A-B)(a-1)_3(1)_2}{(c)_2(\lambda+p)_2(a-1+2\beta)} \max\left\{1 \ , \right.$$

$$\left.\left|B + \zeta\frac{(A-B)(a-1)_2(\lambda+p+1)(c+1)(a-1+2\beta)}{2c(a+1)(\lambda+p)(a-1+\beta)^2}\right|\right\} \ . \tag{3.13}$$

The result is sharp.

Proof. From (1.12), we have

$$(1-\beta)\frac{I_p^\lambda(a,c)f(z)}{z^p} + \beta\frac{I_p^\lambda(a-1,c)f(z)}{z^p} - 1$$

$$= \left[A - B\left\{(1-\beta)\frac{I_p^\lambda(a,c)f(z)}{z^p} + \beta\frac{I_p^\lambda(a-1,c)f(z)}{z^p}\right\}\right] w(z), \tag{3.14}$$

where

$$w(z) = \sum_{k=1}^{\infty} d_k z^k \in \Omega.$$

Substituting the power series expansion of $I_p^\lambda(a,c)f(z),\ I_p^\lambda(a-1,c)f(z)$ and $w(z)$ in (3.14), and equating the coefficients of z and z^2 we obtain

$$a_{p+1} = \frac{(A-B)(a-1)_2}{(a-1+\beta)(c)(\lambda+p)}d_1 \tag{3.15}$$

and

$$a_{p+2}=\frac{2(A-B)(a-1)_3}{(a-1+2\beta)(c)_2(\lambda+p)_2}(d_2-Bd_1^2)\ . \tag{3.16}$$

Using (2.7), (3.15) and (3.16), we get:

$$\left|a_{p+2}-\zeta a_{p+1}^2\right|=\frac{(A-B)(a-1)_3}{(c)_2(\lambda+p)_2(a-1+2\beta)}\left|d_2-\nu d_1^2\right|\ ,$$

where

$$\nu=B+\zeta\frac{(A-B)(a-1+2\beta)(c+1)(\lambda+p+1)(a-1)_2}{2c(a+1)(a-1+\beta)^2(\lambda+p)}$$

Since (2.7) is sharp, (3.13) is also sharp.

Taking $\beta=\lambda=c=A=1,\ a=p+1-\mu\ (-\infty<\mu<p)$ and $B=\frac{1}{M}-1\ (M>\frac{1}{2})$ in Theorem 3.17, we obtain the following corollary.

Corollary 3.18. *If f, given by (1.1), belongs to the class $\Phi_p[\mu,M]$, then*

$$\left|a_{p+2}-\zeta a_{p+1}^2\right|\leq\frac{\left(\frac{2M-1}{M}\right)(p-\mu)_3}{(1+p)_2(p+2-\mu)}\max\left\{1,\left|\frac{1}{M}-1+\zeta\frac{\left(\frac{2M-1}{M}\right)(p-\mu)(p+2)}{(p+1-\mu)(p+1)}\right\}\right|\ .$$

The result is sharp.

Theorem 3.19. *Let $f\in\Phi_p^{\beta}(a,c,A,B)$ and $g\in A(p)$ with $\operatorname{Re}\left(\frac{g(z)}{z^p}\right)>\frac{1}{2}$ $(z\in U)$. Then $h=f*g\in\Phi_p^{\beta}(a,c,A,B)$.*

Proof. We have

$$\begin{aligned}&(1-\beta)\frac{I_p^{\lambda}(a,c)h(z)}{z^p}+\beta\frac{I_p^{\lambda}(a-1,c)h(z)}{z^p}\\&=\left\{(1-\beta)\frac{I_p^{\lambda}(a,c)f(z)}{z^p}+\beta\frac{I_p^{\lambda}(a-1,c)f(z)}{z^p}\right\}*\frac{g(z)}{z^p}\quad(z\in U).\end{aligned} \tag{3.17}$$

Since $\operatorname{Re}\left\{\frac{g(z)}{z^p}\right\}>\frac{1}{2}$ $(z\in U)$ and the function $\frac{1+Az}{1+Bz}$ is convex (univalent) in U, it follows from (3.17) and Lemma 2.2 that $h(z)=(f*g)(z)\in\Phi_p^{\beta}(a,c,A,B)$. This completes the proof of Theorem 3.19.

Corollary 3.20. *Let $f\in\Phi_p^{\beta}(a,c,A,B)$ and $g(z)\in A(p)$ satisfy*

$$\operatorname{Re}\left\{(1-\mu)\frac{g(z)}{z^p}+\mu\frac{g'(z)}{pz^{p-1}}\right\}>\frac{3-2\ {}_2F_1(1,1;\frac{p}{\mu}+1;\frac{1}{2})}{2\left[2-\ {}_2F_1(1,1;\frac{p}{\mu}+1;\frac{1}{2})\right]},\ (\mu>0;\ z\in U). \tag{3.18}$$

Then $f*g\in\Phi_p^{\beta}(a,c,A,B)$.

Proof. From Theorem 3.1 (for $a = p+1$, $c = 1$, $\beta = \mu > 0$, $A = \dfrac{{}_2F_1(1,1;\frac{p}{\mu}+1;\frac{1}{2})-1}{2-\ {}_2F_1(1,1;\frac{p}{\mu}+1;\frac{1}{2})}$ and $B = -1$), condition (3.18) implies

$$\operatorname{Re}\left\{\frac{g(z)}{z^p}\right\} > \frac{1}{2} \qquad (z \in U)\ .$$

Using this, it follows from Theorem 3.19, that $(f * g)(z) \in \Phi_p^\beta(a, c, A, B)$.

Theorem 3.21. *If each of the functions $f(z)$ given by (1.1) and*

$$g(z) = z^p + \sum_{k=1}^{\infty} b_{p+k} z^{p+k}$$

belongs to the class $\Phi_p^\beta(\lambda, a, c, A, B)$, then so does the function

$$h(z) = (1-\beta) I_p^\lambda(a,c)(f*g)(z) + \beta I_p^\lambda(a-1,c)(f*g)(z)\ .$$

Proof. Since $f \in \Phi_p^\beta(a, c, A, B)$, it follows from (3.14) that

$$\left|(1-\beta)\frac{I_p^\lambda(a,c)f(z)}{z^p} + \beta\frac{I_p^\lambda(a-1,c)f(z)}{z^p} - 1\right|$$
$$< \quad \left|A - B\left\{(1-\beta)\frac{I_p^\lambda(a,c)f(z)}{z^p} + \beta\frac{I_p^\lambda(a-1,c)f(z)}{z^p}\right\}\right|\ ,$$

which is equivalent to

$$\left|(1-\beta)\frac{I_p^\lambda(a,c)f(z)}{z^p} + \beta\frac{I_p^\lambda(a-1,c)f(z)}{z^p} - \xi\right| < \eta \quad (z \in U)\ , \tag{3.19}$$

where $\xi = \dfrac{1-AB}{1-B^2}$ and $\eta = \dfrac{A-B}{1-B^2}$. It is known [21] that $H(z) = \sum_{k=0}^{\infty} h_k z^k$ is analytic in U and $|H(z)| \le M$, then

$$\sum_{k=0}^{\infty} |h_k|^2 \le M^2\ . \tag{3.20}$$

Applying (3.18) to (3.19), we get

$$(1-\xi)^2 + \sum_{k=1}^{\infty}\left\{\frac{(a-1+\beta k)(c)_k(\lambda+p)_k}{(a-1)_{k+1}(1)_k}\right\}^2 |a_{p+k}|^2 \le \eta^2\ ,$$

that is, that

$$\sum_{k=1}^{\infty}\left\{\frac{(a-1+\beta k)(c)_k(\lambda+k)_k}{(a-1)_{k+1}(1)_k}\right\}^2 |a_{p+k}|^2 \le \frac{(A-B)^2}{1-B^2}\ . \tag{3.21}$$

Similarly, we have

$$\sum_{k=1}^{\infty}\left\{\frac{(a-1+\beta k)(c)_k(\lambda+k)_k}{(a-1)_{k+1}(1)_k}\right\}^2 |b_{p+k}|^2 \le \frac{(A-B)^2}{1-B^2}\ . \tag{3.22}$$

Now, for $|z| = r < 1$, by applying Cauchy-Schwarz inequality, we find that

$$\left|(1-\beta)\frac{I_p^\lambda(a,c)h(z)}{z^p} + \beta\frac{I_p^\lambda(a-1,c)h(z)}{z^p} - \xi\right|^2$$

$$= \left|(1-\xi) + \sum_{k=1}^{\infty}\left\{\frac{(a-1+\beta k)(c)_k(\lambda+p)_k}{(a-1)_{k+1}(1)_k}\right\}^2 a_{p+k}b_{p+k}z^k\right|^2$$

$$\leq (1-\xi)^2 + 2(1-\xi)\sum_{k=1}^{\infty}\left\{\frac{(a-1+\beta k)(c)_k(\lambda+p)_k}{(a-1)_{k+1}(1)_k}\right\}^2 |a_{p+k}|\,|b_{p+k}|\,r^k$$

$$+ \left|\sum_{k=1}^{\infty}\left\{\frac{(a-1+\beta k)(c)_k(\lambda+p)_k}{(a-1)_{k+1}(1)_k}\right\}^2 a_{p+k}b_{p+k}z^k\right|^2$$

$$\leq (1-\xi)^2 + 2(1-\xi)\left[\sum_{k=1}^{\infty}\left\{\frac{(a-1+\beta k)(c)_k(\lambda+p)_k}{(a-1)_{k+1}(1)_k}\right\}^2 |a_{p+k}|^2 r^k\right]^{\frac{1}{2}} \cdot$$

$$\cdot\left[\sum_{k=1}^{\infty}\left\{\frac{(a-1+\beta k)(c)_k(\lambda+p)_k}{(a-1)_{k+1}(1)_k}\right\}^2 |b_{p+k}|^2 r^k\right]^{\frac{1}{2}} +$$

$$\left[\sum_{k=1}^{\infty}\left\{\frac{(a-1+\beta k)(c)_k(\lambda+p)_k}{(a-1)_{k+1}(1)_k}\right\}^2 |a_{p+k}|^2 r^k\right] \cdot$$

$$\cdot\left[\sum_{k=1}^{\infty}\left\{\frac{(a-1+\beta k)(c)_k(\lambda+p)_k}{(a-1)_{k+1}(1)_k}\right\}^2 |b_{p+k}|^2 r^k\right]$$

$$\leq (1-\xi)^2 + 2(1-\xi)\left[\sum_{k=1}^{\infty}\left\{\frac{(a-1+\beta k)(c)_k(\lambda+p)_k}{(a-1)_{k+1}(1)_k}\right\}^2 |a_{p+k}|^2\right]^{\frac{1}{2}} \cdot$$

$$\cdot\left[\sum_{k=1}^{\infty}\left\{\frac{(a-1+\beta k)(c)_k(\lambda+p)_k}{(a-1)_{k+1}(1)_k}\right\}^2 |b_{p+k}|^2\right]^{\frac{1}{2}} +$$

$$\left[\sum_{k=1}^{\infty}\left\{\frac{(a-1+\beta k)(c)_k(\lambda+p)_k}{(a-1)_{k+1}(1)_k}\right\}^2 |a_{p+k}|^2\right] \cdot$$

$$\cdot\left[\sum_{k=1}^{\infty}\left\{\frac{(a-1+\beta k)(c)_k(\lambda+p)_k}{(a-1)_{k+1}(1)_k}\right\}^2 |b_{p+k}|^2\right]$$

$$\leq (1-\xi)^2 + 2(1-\xi)\frac{(A-B)^2}{1-B^2} + \frac{(A-B)^4}{(1-B^2)^2}$$

$$= \left\{\frac{B(A-B)}{1-B^2}\right\}^2 + 2\frac{B(A-B)^3}{(1-B^2)^2} + \frac{(A-B)^4}{(1-B^2)^2} = \frac{A^2(A-B)^2}{(1-B^2)^2} < \eta^2,$$

by using (3.21) and (3.22).
Thus, again with the aid of (3.20), we have $h \in \Phi_p^\beta(\lambda, a, c, A, B)$.

Theorem 3.22. *Let* $f \in \Phi_p^{\beta}(\lambda, a, c, A, B)$ $(\beta > 0)$ *and*

$$S_n(z) = z^p + \sum_{k=1}^{n-1} a_{p+k} z^{p+k} \ (n \geq 2).$$

Then for $z \in U$*, we have*

$$\mathrm{Re} \left\{ \frac{\int_0^z t^{-p}(I_p^{\lambda}(a,c)S_n(t))dt}{z} \right\} > \eta(\beta, a, A, B) \ ,$$

where $\eta(\beta, a, A, B)$ *is defined as in Theorem 3.1.*

Proof. Singh and Singh [27] prove that

$$\mathrm{Re} \left\{ 1 + \sum_{k=1}^{n-1} \frac{z^k}{k+1} \right\} > \frac{1}{2} \quad (z \in U) \ . \tag{3.23}$$

Writing

$$\frac{\int_0^z t^{-p} I_p^{\lambda}(a,c)S_n(t)dt}{z} = \frac{I_p^{\lambda}(a,c)f(z)}{z^p} * \left\{ 1 + \sum_{k=1}^{n-1} \frac{z^k}{k+1} \right\}$$

and making use of (3.23), Theorem 3.1 and Lemma 2.2, the assertion of Theorem 3.22 follows at once.

Taking $\beta = \lambda = c = 1,\ a = p+1,\ A = 1 - \frac{2\alpha}{p}$ $(0 \leq \alpha < p)$ and $B = -1$ in Theorem 3.22, we obtain the following corollary.

Corollary 3.23. *Let* $f \in A(p)$ *satisfies* $\mathrm{Re}\left\{\frac{f'(z)}{z^{p-1}}\right\} > \alpha$ $(0 \leq \alpha < p)$ *in* U*, then*

$$\mathrm{Re} \left[\frac{\int_0^z t^{-p} S_n(t)dt}{z} \right] > \frac{\alpha}{p} + \left(1 - \frac{\alpha}{p}\right) \left\{ {}_2F_1\left(1, 1; p+1; \frac{1}{2}\right) - 1 \right\} \quad (z \in U) \ .$$

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On analytic functions with generalized bounded variation

Ningegowda Ravikumar and Satyanarayana Latha

Abstract. In this paper we study a class introduced by Bhargava and Nanjunda Rao which unifies a number of classes studied previously by Mocanu and others. This class includes several known classes of analytic functions such as convex and starlike functions of order β, α-convex functions, functions with bounded boundary rotation, bounded radius rotation and bounded Mocanu variation. Several interesting properties like inclusion results, arclength problem, coefficient bounds and distortion results of this class are discussed.

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1. First section (Introduction)

Let $\mathcal{A}$ denote the class of analytic functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

in the unit disc $\mathcal{E} = \{z; |z| < 1\}$. Let P designate the class of functions p which are analytic, have positive real part in $\mathcal{E}$ and satisfy $p(0) = 1$. Let M_k denote the class of real-valued functions $\mu(t)$ of bounded variation on $[0, 2\pi]$ which satisfy the conditions,

$$\int_0^{2\pi} d\mu(t) = 2, and \quad \int_0^{2\pi} |d\mu(t)| \leq k. \tag{1.2}$$

M_2 is clearly the class of nondecreasing functions on $[0, 2\pi]$ satisfying

$$\int_0^{2\pi} d\mu(t) = 2.$$

If $\mu(t) \in M_k$ with $k > 2$ we can write $\mu(t) = \alpha(t) - \beta(t)$ where $\alpha(t)$ and $\beta(t)$ are both nondecreasing functions on $[0, 2\pi]$ and satisfy

$$\int_0^{2\pi} d\alpha(t) \leq \frac{k}{2} + 1, and \qquad \int_0^{2\pi} d\beta(t) \leq \frac{k}{2} - 1. \tag{1.3}$$

Let P_k denote the class of functions p analytic in $\mathcal{E}$ such that $p(0) = 1$, $z = re^{i\theta}$. and having the representation

$$p(z) = \frac{1}{2}\int_0^{2\pi} \frac{1 + ze^{-it}}{1 - ze^{-it}} d\mu(t), \tag{1.4}$$

where $\mu(t) \in M_k$. This class has been studied by Pinchuk [5] .

Clearly $P_2 = P$. We can write for $p(z) \in P_k$ as

$$p(z) = \frac{1}{2}\left(\frac{k}{2} + 1\right) P_1(z) - \frac{1}{2}\left(\frac{k}{2} - 1\right) P_2(z)$$

where $P_1, P_2 \in P$.

Definition 1.1. *Let $f \in \mathcal{A}$ with $\frac{f(z)f'(z)}{z} \neq 0$ in $\mathcal{E}$, and let*

$$J_f = J_f(\alpha, b, c) = (1 - \alpha)\left[1 - \frac{1}{c} + \frac{z}{c}\frac{f'(z)}{f(z)}\right] + \alpha\left[1 + \frac{z}{b}\frac{f''(z)}{f'(z)}\right]$$

where α, $b \neq 0\, and\, c \neq 0$ are complex numbers.

Let $B_k(\alpha, b, c)$ be the class of all fuctions f in $\mathcal{E}$, such that if $J_f \in P_k$ for $z \in \mathcal{E}, k \geq 2$.

This class is a particular case of the class studied earlier by Bhargava and Nanjunda Rao [1] which unifies and generalizes various classes studied earlier by Robertson [6], Moulis [3], Pinchuk [5], Padmanabhan and Parvatham [4], and Khalida Inayat Noor and Ali Muhammad [2].

For $f, g \in \mathcal{A}$ given by $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$, and $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$ the Hardmard product is given by $(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n$.

Let Γ denote the Gamma function of Euler and $G(l, m, n; z)$ be the analytic function for z in $\mathcal{E}$ defined by

$$G(l, m, n; z) = \frac{\Gamma(n)}{\Gamma(l)\Gamma(n - l)}\int_0^1 u^{l-1}(1 - u)^{n-l-1}(1 - zu)^{-n} du, \tag{1.5}$$

where $\Re\{l\} > 0,\ and, \Re\{l - n\} > 0$. Also we define

$$N(\alpha, b, c, r) = r\left[G\left(\frac{2b}{\alpha c}, M, M + 1, r\right)\right]^{\frac{1}{M}}. \tag{1.6}$$

and

$$f_\theta(\alpha, b, c, z) = \left[M\int_0^z t^{M-1}(1 - e^{i\theta}t)^{\frac{-2b}{\alpha}} dt\right]^{\frac{1}{M}} \tag{1.7}$$

where $M = 1 + \frac{(1-\alpha)b}{\alpha c}$, $\alpha \neq 0$, $0 \leq \theta \leq 2\pi$.

2. Second section

We use the following lemmas to prove the main results.

Lemma 2.1. *Let p be analytic in $\mathcal{E}$ and $p(0) = 1$, then $\alpha \geq 0, z \in \mathcal{E}$, $\left(p + \frac{\alpha z p'}{p}\right) \in P_k$ implies $p \in P_k$.*

Lemma 2.2. [7] *Let $f \in \mathcal{A}$ with $\frac{f(z)f'(z)}{z} \neq 0$ in $\mathcal{E}$, then f is univalent in $\mathcal{E}$ if and only if for $0 \leq \theta_1 < \theta_2 \leq 2\pi$ and $0 < r < 1$, we have*

$$\int_{\theta_1}^{\theta_2} \left[\Re\left\{1 + z\frac{f''(z)}{f'(z)} + (\beta - 1)z\frac{f'(z)}{f(z)}\right\} - \alpha\Im z\frac{f'(z)}{f(z)}\right] d\theta > -\pi$$

with $z = re^{i\theta}$, $\beta > 0$ and α real.

Theorem 2.3. *$f \in B_k(\alpha, b, c)$, $\alpha \neq 0, b \neq 0, c \neq 0$, if and only if there is a function $g \in B_k(0, b, 1) = R_k$ such that*

$$f(z) = \left[M \int_0^z t^{M-1} \left(\frac{g(t)}{t}\right)^{\frac{b}{\alpha}} dt\right]^{\frac{1}{M}}, \tag{2.1}$$

where $M = 1 + \frac{(1-\alpha)b}{\alpha c}$.

Proof. Using (2.1) we get,

$$(1 - \alpha)\frac{z}{c}\frac{f'(z)}{f(z)} + \frac{\alpha}{b}z\frac{f''(z)}{f'(z)} = \frac{1 - \alpha}{c} + z\frac{g'(z)}{g(z)} - 1$$

$$(1 - \alpha)\left[1 - \frac{1}{c} + \frac{z}{c}\frac{f'(z)}{f(z)}\right] + \alpha\left[1 + \frac{z}{b}\frac{f''(z)}{f'(z)}\right] = z\frac{g'(z)}{g(z)}$$

If J_f belongs to P_k, so does the left hand side and conversely. □

Putting $c = 1$ and $b = 1 - \beta$ in above Theorem we get the following corollary.

Corollary 2.4. [2] *$f \in B_k(\alpha, \beta)$, $\alpha \neq 0$, if and only if there is a function $g \in B_k(0, \beta) = R_k$ such that*

$$f(z) = \left[M \int_0^z t^{M-1} \left(\frac{g(t)}{t}\right)^{\frac{1-\beta}{\alpha}} dt\right]^{\frac{1}{M}},$$

where $M = 1 + \frac{(1-\alpha)(1-\beta)}{\alpha}$.

Theorem 2.5. *Let $f \in B_k(\alpha, b, c)$ then the function*

$$g(z) = z\left(\frac{f(z)}{z}\right)^{\frac{1-\alpha}{c}} (f'(z))^{\frac{\alpha}{b}} \tag{2.2}$$

belongs to R_k for all $z \in \mathcal{E}$.

Proof. Logarithmic differentiation of (2.2) yields

$$z\frac{g'(z)}{g(z)} = 1 + \frac{(1-\alpha)}{c}z\frac{f'(z)}{f(z)} - \frac{(1-\alpha)}{c} + \frac{\alpha}{b}z\frac{f''(z)}{f'(z)}$$
$$= (1-\alpha)\left[1 - \frac{1}{c} + \frac{z}{c}\frac{f'(z)}{f(z)}\right] + \alpha\left[1 + \frac{z}{b}\frac{f''(z)}{f'(z)}\right]$$

Since $f \in B_k(\alpha, b, c)$ the result follows. □

For the parametric values $c = 1$ and $b = 1 - \beta$ we get the following result.

Corollary 2.6. [2] *Let $f \in B_k(\alpha, \beta)$ then the function*

$$g(z) = z\left(\frac{f(z)}{z}\right)^{1-\alpha}(f'(z))^{\frac{\alpha}{1-\beta}}$$

belongs to R_k for all $z \in \mathcal{E}$.

Remark 2.7. The above Theorem can also be obtained as a particular case of Theorem 3.1 by Bhargava and Nanjunda Rao [1].

Theorem 2.8. $B_k(\alpha, b, c) \subset R_k$, *for $\alpha > 0, b \neq 0$.*

Proof. Let $\frac{zf'(z)}{f(z)} = p(z)$, p analytic in $\mathcal{E}$, with $p(0) = 1$. Now

$$\frac{1}{b}\left\{(1-\alpha)b\left[1 - \frac{1}{c} + \frac{z}{c}\frac{f'(z)}{f(z)}\right] + \alpha\left[b + \frac{zf''(z)}{f'(z)}\right]\right\}$$
$$= \frac{\alpha}{b}\left\{\frac{(1-\alpha)}{\alpha c}b[(c-1) + p(z)] + \alpha\left[b + \frac{zf''(z)}{f'(z)}\right]\right\}$$
$$= \frac{\alpha}{b}\left\{(M-1)[(c-1) + p(z)] + \left[b + \frac{zp'(z)}{p(z)} + p(z) - 1\right]\right\}$$
$$= \frac{\alpha}{b}\left[Mp(z) + \frac{zp'(z)}{p(z)} + (M-1)(c-1) + (b-1)\right]$$
$$= \frac{\alpha}{b}\left[M\left\{p(z) + \frac{1}{M}\frac{zp'(z)}{p(z)}\right\} + (M-1)(c-1) + (b-1)\right] \in P_k.$$

Therefore $\left\{p(z) + \frac{1}{M}\frac{zp'(z)}{p(z)}\right\} \in P_k$, and by using Lemma 2.1. it follows that $p \in P_k$, $z \in \mathcal{E}$. This proves that $f \in R_k$. □

Corollary 2.9. [2] $B_k(\alpha, \beta) \subset R_k$, *for $\alpha > 0, 0 \leq \beta < 1$.*

Theorem 2.10. i. $B_k(\alpha, b, c) \subset B_{k_1}(\alpha_1, b, c)$, $0 < \alpha \leq \alpha_1$, *and* $k_1 = k\left(\frac{2\alpha_1 - \alpha}{\alpha}\right)$.
ii. $B_k(\alpha, b, c) \subset B_k(\alpha_1, b, c)$, $0 \leq \alpha_1 < \alpha$.

Proof. (i) Let $f \in B_k(\alpha, b, c)$ then

$$\frac{1}{b}\left\{(1-\alpha_1)b\left[1 - \frac{1}{c} + \frac{z}{c}\frac{f'(z)}{f(z)}\right] + \alpha_1\left[b + \frac{zf''(z)}{f'(z)}\right]\right\}$$
$$= \frac{\alpha_1}{\alpha}\left\{(1-\alpha)b\left[1 - \frac{1}{c} + \frac{z}{c}\frac{f'(z)}{f(z)}\right] + \alpha\left[b + \frac{zf''(z)}{f'(z)}\right]\right\} - \frac{(\alpha_1 - \alpha)}{\alpha}b\left[1 - \frac{1}{c} + \frac{z}{c}\frac{f'(z)}{f(z)}\right]$$

$$= b\left[\frac{\alpha_1}{\alpha}h_1(z) - \frac{(\alpha_1-\alpha)}{\alpha}h_2(z)\right], \quad h_1, h_2 \in \ P_k. \tag{2.3}$$

by using Definition 1.1 and Theorem 2.8. From (2.3) it follows that

$$\int_0^{2\pi} |\Re J_f| d\theta \leq \left[\frac{\alpha_1}{\alpha} + \frac{(\alpha_1-\alpha)}{\alpha}\right] k\pi = \left(\frac{2\alpha_1-\alpha}{\alpha}\right) k\pi.$$

(ii) Let $f \in \ B_k(\alpha, b, c)$. Then

$$(1-\alpha_1)\left[1 - \frac{1}{c} + \frac{z}{c}\frac{f'(z)}{f(z)}\right] + \alpha_1\left[1 + \frac{z}{b}\frac{f''(z)}{f'(z)}\right]$$

$$= \left(1-\frac{\alpha_1}{\alpha}\right)\left[1 - \frac{1}{c} + \frac{z}{c}\frac{f'(z)}{f(z)}\right] + \frac{\alpha_1}{\alpha}\left\{(1-\alpha)\left[1 - \frac{1}{c} + \frac{z}{c}\frac{f'(z)}{f(z)}\right] + \alpha\left[1 + \frac{z}{b}\frac{f''(z)}{f'(z)}\right]\right\}$$

$= \left(1-\frac{\alpha_1}{\alpha}\right) H_1(z) + \frac{\alpha_1}{\alpha} H_2(z)$, $\quad H_1, H_2 \in P_k$, $z \in \mathcal{E}$, since P_k is a convex set. Therefore $f \in B_k(\alpha_1, b, c)$, for $z \in \mathcal{E}$. □

Corollary 2.11. [2]

i. $B_k(\alpha, \beta) \subset B_{k_1}(\alpha_1, \beta)$, $0 < \alpha \leq \alpha_1$, *and* $k_1 = k\left(\frac{2\alpha_1-\alpha}{\alpha}\right)$
ii. $B_k(\alpha, \beta) \subset B_k(\alpha_1, \beta)$, $0 \leq \alpha_1 < \alpha$.

Theorem 2.12. *Let* $f \in \ B_k(\alpha, b, c)$. *Then* f *is univalent in* $\mathcal{E}$ *for* $k \leq \frac{2(3\alpha c - bc + 2b - 2b\alpha)}{bc}$.

Proof. Since $f \in \ B_k(\alpha, b, c)$, also we have $z = re^{i\theta}$, $0 \leq r < 1$, $0 \leq \theta_1 < \theta_2 \leq 2\pi$

$$\int_{\theta_1}^{\theta_2} \Re\left\{\frac{(1-\alpha)}{\alpha c}\left[c - 1 + \frac{zf'(z)}{f(z)}\right] + \frac{1}{b}\left[1 + \frac{zf''(z)}{f'(z)}\right]\right\} d\theta \geq -\left(\frac{k}{2} - 1\right)\frac{\pi}{\alpha} - \left(\frac{b-1}{b}\right) 2\pi$$

$$\int_{\theta_1}^{\theta_2} \Re\left\{\left[1 + \frac{zf''(z)}{f'(z)}\right] + \left[\frac{b(1-\alpha)}{\alpha c} - 1\right]\frac{zf'(z)}{f(z)}\right\} d\theta$$

$$\geq -\left[\left(\frac{k}{2} - 1\right)\frac{b}{\alpha} + 2(b-1) + \frac{2(1-\alpha)(c-1)b}{\alpha c}\right]\pi$$

by using Lemma 2.2, that f is univalent in $\mathcal{E}$ if $k \leq \frac{2(3\alpha c - bc + 2b - 2b\alpha)}{bc}$. □

Corollary 2.13. *Let* $f \in \ B_k(\alpha, \beta)$. *Then* f *is univalent in* $\mathcal{E}$ *for* $k \leq \frac{2(\alpha + 2\alpha\beta - \beta + 1)}{(1-\beta)}$.

Theorem 2.14. *Let* $f \in \ B_k(\alpha, b, c)$, $\alpha > 0$ *and* $L_r(f)$ *denote the length of the curve* $C = f(re^{i\theta})$, $0 \leq \theta \leq 2\pi$ *and* $N(r) = \max_{0\leq\theta\leq 2\pi} |f(re^{i\theta})|$, *then for* $0 < r < 1$,

$$L_r(f) \leq \frac{N(r)b}{\alpha}\left\{k + \frac{(\alpha-1)}{c}[(c-1)2 + k] + \frac{\alpha}{b}(1-b)2\right\}\pi, \quad \alpha > 0.$$

Proof. We have, $z = re^{i\theta}$

$$L_r(f) = \int_0^{2\pi} |zf'(z)| d\theta = \int_0^{2\pi} zf'(z) e^{-iarg(zf'(z))} d\theta.$$

On integration we get,

$$L_r(f) = \int_0^{2\pi} f(z) e^{-iarg(zf'(z))} \Re\left\{\frac{(zf'(z))'}{f'(z)}\right\} d\theta$$

$$\leq \frac{N(r)b}{\alpha}\int_0^{2\pi}\left|\Re J_f + (\alpha-1)\left[1-\frac{1}{c}+\frac{z}{c}\frac{f'(z)}{f(z)}\right]-\alpha+\frac{\alpha}{b}\right|d\theta$$
$$\leq \frac{N(r)b}{\alpha}\left\{k\pi+(\alpha-1)\left[\left(1-\frac{1}{c}\right)2\pi+\frac{\pi k}{c}\right]+\alpha(\frac{1}{b}-1)2\pi\right\}$$
$$\leq \frac{N(r)b}{\alpha}\left\{k+\frac{(\alpha-1)}{c}[(c-1)2+k]+\frac{\alpha}{b}(1-b)2\right\}\pi.$$

□

Corollary 2.15. [2] *Let $f\in B_k(\alpha,\beta)$, $\alpha>2$ and $L_r(f)$ denote the length of the curve $C=f(re^{i\theta})$, $0\leq\theta\leq 2\pi$ and $N(r)=\max_{0\leq\theta\leq 2\pi}|f(re^{i\theta})|$, then for $0<r<1$,*

$$L_r(f)\leq (1-\beta)N(r)\left[k+\frac{2\beta}{1-\beta}\right]\pi,\quad \alpha>0.$$

Theorem 2.16. *Let f given by (1.1) belongs to $B_k(\alpha,b,c)$ for $\alpha\geq 0$. Then for $n\geq 2$, $na_n=O(1)N\left(\frac{n-1}{n}\right)$, where $O(1)$ is a constant depending on α,b,c,k only.*

Proof. We have,

$$na_n=\frac{1}{2\pi r^n}\int_0^{2\pi} zf'(z)e^{-in\theta}d\theta,\quad z=re^{i\theta}$$
$$na_n\leq\frac{1}{2\pi r^n}\int_0^{2\pi}|zf'(z)|d\theta=\frac{1}{2\pi r^n}L_r(f).$$

By using Theorem 2.14 and $r=\frac{n-1}{n}$, we get the required result. □

Corollary 2.17. [2] *Let f given by (1.1) belongs to $B_k(\alpha,\beta)$ for $\alpha\geq 0$. Then for $n\geq 2$, $na_n=O(1)N\left(\frac{n-1}{n}\right)$, where $O(1)$ is a constant depending on α,β,k only.*

Theorem 2.18. *Let $f\in B_2(\alpha,b,c)$, $\alpha\neq 0, b\neq 0, c\neq 0$ and $|z|=r$ $(0<r<1)$. Then*

$$(i)\ N(\alpha,b,c,-r)\leq |f(z)|\leq N(\alpha,b,c,r),\quad for\,\alpha>0.$$
$$(ii)\ N(\alpha,b,c,r)\leq |f(z)|\leq N(\alpha,b,c,-r),\quad for\,\alpha<0.$$

This result is sharp and equality occurs, for the function $f_\theta(\alpha,b,c,z)$ defined by (1.7), with suitably chosen θ.

Proof. We consider $\alpha>0$. From Theorem 2.3, certifies the existence of $f\in B_2(\alpha,b,c)$ if and only if there exists a $g\in R_2=S^*$ such that

$$f(z)=\left[M\int_0^z t^{M-1}\left(\frac{g(t)}{t}\right)^{\frac{b}{\alpha}}dt\right]^{\frac{1}{M}},\quad \text{where}\, M=1+\frac{(1-\alpha)b}{\alpha c}.\tag{2.4}$$

Taking $z=r$, $t=\rho e^{i\theta}$ and integrating , we get from(2.4),

$$f(r)=\left[Me^{i\theta M}\int_0^r\rho^{M-1}\left(\frac{g(\rho)}{\rho}\right)d\rho\right]^{\frac{1}{M}}.\tag{2.5}$$

Since g is starlike , we have

$$\frac{\rho}{(1+\rho)^2}\leq |g(t)|\leq\frac{\rho}{(1-\rho)^2}.\tag{2.6}$$

Using (2.6) in (2.5), we get

$$|f(r)|^M \leq M \int_0^r \rho^{M-1}(1-\rho)^{\frac{-2b}{\alpha}} d\rho = Mr^M \int_0^1 u^{M-1}(1-ru)^{\frac{-2b}{\alpha}} du. \qquad (2.7)$$

Therefore $|f(r)| \leq N(\alpha, b, c, r), \quad \alpha > 0.$

It remains only to prove that the left-hand inequality. We consider the straight line Γ^* joining 0 to $f(z) = Re^{i\phi}$. Γ^* is the image of a Jordan arc γ in $\mathcal{E}$ connecting 0 and $z = re^{i\theta}$. If z_0 is a point on the circumference $|z| = r$ such that

$$|f(z_0)| = \min_{0 \leq \theta \leq 2\pi} |f(re^{i\theta})|.$$

Using (2.5) and (2.6), we get

$$|f(z_0)|^M \geq M \int_0^r \rho^{M-1}(1+\rho)^{\frac{-2b}{\alpha}} d\rho = Mr^M \int_0^1 u^{M-1}(1+ru)^{\frac{-2b}{\alpha}} du.$$

$$|f(z)| \geq N(\alpha, b, c, -r), \quad \alpha > 0.$$

Proof of (*ii*) is analogous to proof of (*i*). □

Corollary 2.19. [2] *Let* $f \in B_2(\alpha, \beta)$, $\alpha \neq 0, 0 < \beta < 1$ *and* $|z| = r$ $(0 < r < 1)$. *Then*

$$(i)\ N(\alpha, \beta, -r) \leq |f(z)| \leq N(\alpha, \beta, r), for\, \alpha > 0.$$

$$(ii)\ N(\alpha, \beta, r) \leq |f(z)| \leq N(\alpha, \beta, -r), for\, \alpha < 0.$$

This result is sharp and equality occurs, for the function $f_\theta(\alpha, b, c, z)$ *defined by* (1.7), *with suitably chosen* θ.

Remark 2.20. The above Theorem can be obtained as a particular case of Corollary 3.2 by Bhargava and Nanjunda Rao [1].

Theorem 2.21. *Let* $f \in B_2(\alpha, 1, c)$, $\alpha > 0$. *Then, for* $|z| = r$ $(0 < r < 1)$, *we have*

$$\frac{r + |\alpha - 1|(1+r)^2 N(\alpha, 1, c, -r)}{\alpha r(1+r)^2} \leq |f'(z)| \leq \frac{r + |\alpha - 1|(1-r)^2 N(\alpha, 1, c, -r)}{\alpha r(1-r)^2}.$$

This result is sharp.

Theorem 2.22. *Let* $f \in B_2(\alpha, b, c)$, $\alpha \neq 0$, $b \neq 0$. *and be given by* (1.1). *Then*

$$|a_2| \leq \frac{2b}{|(1-\alpha)b + 2\alpha c|}.$$

Proof. By using Theorem 2.18, we have

$$N(\alpha, b, c, r) = r + \frac{2b}{(1-\alpha)b + 2\alpha c} r^2 + O(r^3),$$

and

$$|f(r)| = r + a_2 r^2 + O(r^3).$$

Therefore, we have

$$a_2 \leq \frac{2b}{(1-\alpha)b + 2\alpha c} \quad (\alpha > 0).$$ □

Corollary 2.23. [2] *Let* $f \in B_2(\alpha,\beta)$,$\alpha \neq 0,\ 0<\beta<1$. *and be given by* (1.1). *Then*

$$|a_2| \leq \frac{2(1-\beta)}{|(1-\alpha)(1-\beta+2\alpha|}.$$

Remark 2.24. The above Theorem can be obtained as a particular case of Corollary 3.1 by Bhargava and Nanjunda Rao [1].

Theorem 2.25. *Let* $f \in B_k(1,b,c)$. *Then, with* $|z|=r,\ r_1=\frac{1-r}{1+r}$ *we have*

$$\frac{2^{m-1}}{l}[G(l,m,n,-1)-r^l{}_1G(l,m,n,-r_1)] \leq |f(z)|$$

$$\leq \frac{2^{m-1}}{l}[G(l,m,n,-1)-r_1^{-l}G(l,m,n,-r_1^{-1})]$$

where $l=\left(\frac{k}{2}-1\right)b+1, \quad m=2(1-b), \quad n=\left(\frac{k}{2}-1\right)b+2.$

Proof. Since $f \in B_k(1,b,c)$. we have from (2.4)

$$f'(z)=\left(\frac{g(z)}{z}\right)^b, \quad g \in R_k.$$

Since $g \in R_k$

$$\frac{(1-|z|)^{\frac{k}{2}-1}}{(1+|z|)^{\frac{k}{2}+1}} \leq |g(z)| \leq \frac{(1+|z|)^{\frac{k}{2}-1}}{(1-|z|)^{\frac{k}{2}+1}}.$$

Therefore, we have

$$|f'(z)| \geq \frac{(1-|z|)^{\left(\frac{k}{2}-1\right)b}}{(1+|z|)^{\left(\frac{k}{2}+1\right)b}}.$$

Let d_r denote the radius of the largest Schlicht disc centered at the origin contained in the image $|z|<r$ under $f(z)$.

$$d_r=|f(z_0)|=\int_c |f'(z)||dz| \geq \int_c \frac{(1-|z|)^{\left(\frac{k}{2}-1\right)b}}{(1+|z|)^{\left(\frac{k}{2}+1\right)b}}|dz| \geq \int_0^{|z|} \frac{(1-s)^{\left(\frac{k}{2}-1\right)b}}{(1+s)^{\left(\frac{k}{2}+1\right)b}}ds$$

$$=\int_0^{|z|}\left[\frac{1-s}{1+s}\right]^{\left(\frac{k}{2}+1\right)b}\frac{ds}{(1+s)^{2b}}$$

Replacing $\frac{1-s}{1+s}=t$ we get

$$\geq \frac{-2}{4^b}\int_1^{\frac{1-|z|}{1+|z|}} t^{\left(\frac{k}{2}-1\right)b}(1+t)^{2b-2}dt$$

$$=-2^{1-2b}\int_0^{\frac{1-r}{1+r}} t^{\left(\frac{k}{2}-1\right)b}(1+t)^{2(b-1)}dt+2^{1-2b}\int_1^0 t^{\left(\frac{k}{2}-1\right)b}(1+t)^{2(b-1)}dt=I_1+I_2.$$

Taking $\frac{1-r}{1+r}=r_1,\ t=r_1u$, we have

$$I_1=-2^{1-2b}r_1^l\int_0^1 u^{\left(\frac{k}{2}-1\right)b}(1+r_1u)^{2(b-1)}du$$

using (1.5) we obtain,

$$I_1 = r_1^l \left(\frac{-2^{m-1}}{l} \right) G(l, m, n, -r_1),$$

where $l = \left(\frac{k}{2} - 1\right) b + 1, \quad m = 2(1 - b),\ n = \left(\frac{k}{2} - 1\right) b + 2.$

$$I_2 = 2^{1-2b} \int_0^1 t^{\left(\frac{k}{2}-1\right)b} (1+t)^{2(b-1)} dt = \left(\frac{2^{m-1}}{l} \right) G(l, m, n, -1),$$

where $l = \left(\frac{k}{2} - 1\right) b + 1, \quad m = 2(1 - b), \quad n = \left(\frac{k}{2} - 1\right) b + 2.$

Therefore

$$|f(z)| \geq \left(\frac{2^{m-1}}{l} \right) G(l, m, n, -1) - r_1^l \left(\frac{2^{m-1}}{l} \right) G(l, m, n, -r_1).$$

On the other hand we have

$$|f'(z)| \leq \frac{(1+|z|)^{\left(\frac{k}{2}-1\right)b}}{(1-|z|)^{\left(\frac{k}{2}+1\right)b}}.$$

Therefore

$$\begin{aligned} |f(z)| &\leq \int_0^{|z|} \frac{(1-s)^{\left(\frac{k}{2}-1\right)b}}{(1+s)^{\left(\frac{k}{2}+1\right)b}} ds \leq -2^{1-2b} \int_1^{\frac{1-|z|}{1+|z|}} \zeta^{\left(\frac{k}{2}-1\right)b} (1+\zeta)^{2(b-1)} d\zeta \\ &= \frac{2^{m-1}}{l} [G(l, m, n, -1) - r_1^{-l} G(l, m, n, -r_1^{-1})], \end{aligned}$$

where $l = \left(\frac{k}{2} - 1\right) b + 1, \quad m = 2(1 - b), \quad n = \left(\frac{k}{2} - 1\right) b + 2.$ □

Corollary 2.26. *Let $f \in B_k(1, \beta)$. Then, with $|z| = r,\ r_1 = \frac{1-r}{1+r}$ we have*

$$\begin{aligned} &\frac{2^{m-1}}{l} [G(l, m, n, -1) - r_1^l G(l, m, n, -r_1)] \leq |f(z)| \\ &\leq \frac{2^{m-1}}{l} [G(l, m, n, -1) - r_1^{-l} G(l, m, n, -r_1^{-1})] \end{aligned}$$

where $l = \left(\frac{k}{2} - 1\right) (1 - \beta) + 1, \quad m = 2\beta, \quad n = \left(\frac{k}{2} - 1\right) (1 - \beta) + 2.$

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A note on universally prestarlike functions

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Abstract. Universally prestarlike functions of order $\alpha \leq 1$ in the slit domain $\Lambda = \mathcal{C} \setminus [1, \infty)$ have been recently introduced by S. Ruscheweyh. This notion generalizes the corresponding one for functions in the unit disk Δ (and other circular domains in $\mathcal{C}$). In this paper, we discuss the universally prestarlike functions defined through fractional derivatives.

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1. Introduction

Let $H(\Omega)$ denote the set of all analytic functions defined in a domain Ω. For domain Ω containing the origin $H_0(\Omega)$ stands for the set of all function $f \in H(\Omega)$ with $f(0) = 1$. We also use the notation $H_1(\Omega) = \{zf : f \in H_0(\Omega)\}$. In the special case when Ω is the open unit disk $\Delta = \{z \in \mathcal{C} : |z| < 1\}$, we use the abbreviation H, H_0 and H_1 respectively for $H(\Omega), H_0(\Omega)$ and $H_1(\Omega)$. A function $f \in H_1$ is called starlike of order α with $(0 \leq \alpha < 1)$ satisfying the inequality

$$\Re\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha \qquad (z \in \Delta) \tag{1.1}$$

and the set of all such functions is denoted by S_α. The convolution or Hadamard Product of two functions

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \text{ and } g(z) = \sum_{n=0}^{\infty} b_n z^n$$

is defined as

$$(f * g)(z) = \sum_{n=0}^{\infty} a_n b_n z^n.$$

A function $f \in H_1$ is called prestarlike of order α (with $\alpha \leq 1$) if

$$\frac{z}{(1-z)^{2-2\alpha}} * f(z) \in S_\alpha. \tag{1.2}$$

The set of all such functions is denoted by $\mathcal{R}_\alpha$. (see [4]) The notion of prestarlike functions has been extended from the unit disk to other disk and half planes containing the origin. Let Ω be one such disk or half plane.Then there are two unique parameters $\gamma \in \mathcal{C} \setminus \{0\}$ and $\rho \in [0,1]$ such that

$$\Omega_{\gamma,\rho} = \{w_{\gamma,\rho}(z) : z \in \Delta\} \tag{1.3}$$

where,

$$w_{\gamma,\rho}(z) = \frac{\gamma z}{1-\rho z}. \tag{1.4}$$

Note that $1 \notin \Omega_{\gamma,\rho}$ if and only if $|\gamma + \rho| \leq 1$.

Definition 1.1. *(see* [2], [3], [4]*) Let $\alpha \leq 1$, and $\Omega = \Omega_{\gamma,\rho}$ for some admissible pair (γ, ρ). A function $f \in H_1(\Omega_{\gamma,\rho})$ is called prestarlike of order α in $\Omega_{\gamma,\rho}$ if*

$$f_{\gamma,\rho}(z) = \frac{1}{\gamma} f(w_{\gamma,\rho}(z)) \in \mathcal{R}_\alpha \tag{1.5}$$

The set of all such functions f is denoted by $\mathcal{R}_\alpha(\Omega)$.

Let Λ be the slit domain $\mathcal{C} \setminus [1, \infty)$(the slit being along the positive real axis).

Definition 1.2. *(see* [2], [3], [4]*) Let $\alpha \leq 1$. A function $f \in H_1(\Lambda)$ is called universally prestarlike of order α if and only if f is prestarlike of order α in all sets $\Omega_{\gamma,\rho}$ with $|\gamma + \rho| \leq 1$. The set of all such functions is denoted by $\mathcal{R}^u_\alpha$.*

Definition 1.3. *(see* [4]*) Let $\phi(z)$ be an analytic function with positive real part on Δ, which satisfies $\phi(0) = 1$, $\phi'(0) > 0$ and which maps the unit disc Δ onto a region starlike with respect to 1 and symmetric with respect to the real axis. Then the class $\mathcal{R}^u_\alpha(\phi)$ consists of all analytic function $f \in H_1(\Lambda)$ satisfying*

$$\frac{D^{3-2\alpha} f}{D^{2-2\alpha} f} \prec \phi(z) \tag{1.6}$$

where, $(D^\beta f)(z) = \frac{z}{(1-z)^\beta} \star f$, for $\beta \geq 0$ and $\prec$ denotes the subordination.

In particular, for $\beta = n \in \mathrm{N}$, we have $D^{n+1} f = \frac{z}{n!}(z^{n-1} f)^{(n)}$.

Remark 1.4. We let $\mathcal{R}^u_\alpha(A, B)$ denote the class $\mathcal{R}^u_\alpha(\phi)$ where

$$\phi(z) = \frac{1 + Az}{1 + Bz} \; (-1 \leq B < A \leq 1).$$

For suitable choices of A,B,α the class $\mathcal{R}^u_\alpha(A, B)$ reduces to several well known classes of functions. $\mathcal{R}^u_{\frac{1}{2}}(1, -1)$ is the class S^* of starlike univalent functions.

Lemma 1.5. *(see* [1]*) If $P_1(z) = 1 + c_1 z + c_2 z^2 + \ldots$ is an analytic function with positive real part in Δ, then*

$$|c_2 - v c_1^2| \leq \begin{cases} -4v + 2, & v \leq 0 \\ 2, & 0 \leq v \leq 1 \\ 4v + 2, & v \geq 1 \end{cases}$$

when $v < 0$, or $v > 1$, the equality holds if and only if $P_1(z)$ is $\frac{1+z}{1-z}$ or one of its rotations. When $0 < v < 1$, then the equality holds if and only if $P_1(z)$is $\frac{1+z^2}{1-z^2}$ or one of its rotations. If $v = 0$, the equality holds if and only if $P_1(z) = \left(\frac{1}{2} + \frac{\lambda}{2}\right) \frac{1+z}{1-z} + \left(\frac{1}{2} - \frac{\lambda}{2}\right) \frac{1-z}{1+z}$, $0 \le \lambda \le 1$ or one of its rotations. If $v = 1$, the equality holds if and only if $P_1(z)$ is the reciprocal of one of the function for which the equality holds in the case of $v = 0$. Also the above upper bound can be improved as follows when $0 < v < 1$

$$|c_2 - vc_1^2| + v|c_1|^2 \le 2 \qquad (0 < v \le \frac{1}{2}) \tag{1.7}$$

$$|c_2 - vc_1^2| + (1 - v)|c_1|^2 \le 2 \quad (\frac{1}{2} < v \le 1). \tag{1.8}$$

Lemma 1.6. *(see [5]) If $P_1(z) = 1 + c_1z + c_2z^2 + \ldots$ is an analytic function with positive real part in Δ, then $|c_2 - vc_1^2| \le 2max\{1, |2v - 1|\}$ the inequality is sharp for the function $P_1(z) = \frac{1+z}{1-z}$.*

Remark 1.7. Let

$$F(z) = \sum_{k=0}^{\infty} a_k z^k = \int_0^1 \frac{d\mu(t)}{1 - tz}$$

where

$$a_k = \int_0^1 t^k d\mu(t),$$

$\mu(t)$ is a probability measure on $[0, 1]$. Let T denote the set of all such functions F which are analytic in the slit domain Λ.

To Prove our main result we need the following definition.

Definition 1.8. *Let f be analytic in a simply connected region of the z-plane containing the origin. The fractional derivative of f of order λ is defined by*

$$D_z^{\lambda} f(z) := \frac{1}{\Gamma(1 - \lambda)} \frac{d}{dz} \int_0^z \frac{f(\zeta)}{(z - \zeta)^{\lambda}} d\zeta \qquad (0 < \lambda < 1) \tag{1.9}$$

where the multiplicity of $(z - \zeta)^{\lambda}$ is removed by requiring that $log(z - \zeta)$ is real for $z - \zeta > 0$. Using the above definition and its known extensions involving fractional derivatives and fractional integrals, Owa and Srivastava introduced the operator Ω^{λ} : $\mathcal{A} \to \mathcal{A}$ for λ any positive real number $\neq 2, 3, 4, \ldots$ defined by

$$(\Omega^{\lambda} f)(z) = \Gamma(2 - \lambda) z^{\lambda} D_z^{\lambda} f(z) \tag{1.10}$$

and $\mathcal{A} = H_1(\Delta)$. The class $(\mathcal{R}_{\alpha}^u)^{\lambda}(\phi)$ consists of function $f \in \mathcal{A}$ for which $\Omega^{\lambda} f \in (\mathcal{R}_{\alpha}^u)(\phi)$. Note that $(\mathcal{R}_{\alpha}^u)^{\lambda}(\phi)$ is the special case of the class $(\mathcal{R}_{\alpha}^u)^g(\phi)$ when

$$g(z) = z + \sum_{n=2}^{\infty} \frac{\Gamma(n + 1)\Gamma(2 - \lambda)}{\Gamma(n + 1 - \lambda)} z^n \tag{1.11}$$

Let

$$g(z) = z + \sum_{n=2}^{\infty} g_n z^n \ (g_n > 0),$$

g be analytic in Δ *and* $f * g \neq 0$. *Since*

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in (\mathcal{R}_\alpha^u)^g(\phi)$$

if and only if

$$(f * g)(z) = z + \sum_{n=2}^{\infty} g_n a_n z^n \in (\mathcal{R}_\alpha^u)(\phi), \tag{1.12}$$

we obtain the coefficient estimate for functions in the class $(\mathcal{R}_\alpha^u)^g(\phi)$, *from the corresponding estimate for functions in the class* $(\mathcal{R}_\alpha^u)(\phi)$

2. Main Result

Theorem 2.1. *Let the function* ϕ *given by* $\phi(z) = 1 + B_1 z + B_2 z^2 + \ldots$. *If*

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in (\mathcal{R}_\alpha^u)^g(\phi),$$

then

$$|a_3 - \mu a_2^2| \leq \begin{cases} \frac{1}{g_3(3-2\alpha)} \left(B_2 + B_1^2(2-2\alpha) + \frac{(3-2\alpha)\mu g_3 B_1^2}{g_2^2} \right), & \mu \leq \sigma_1 \\ \frac{B_1}{g_3(3-2\alpha)}, & \sigma_1 \leq \mu \leq \sigma_2 \\ \frac{1}{g_3(3-2\alpha)} \left(-B_2 - B_1^2(2-2\alpha) + \frac{(3-2\alpha)\mu g_3 B_1^2}{g_2^2} \right), & \mu \geq \sigma_2, \end{cases}$$

where

$$\sigma_1 = \frac{g_2^2}{g_3} \left[\frac{(B_2 - B_1) + (2-2\alpha)B_1^2}{(3-2\alpha)B_1^2} \right], \tag{2.1}$$

$$\sigma_2 = \frac{g_2^2}{g_3} \left[\frac{(B_2 + B_1) + (2-2\alpha)B_1^2}{(3-2\alpha)B_1^2} \right] \tag{2.2}$$

the result is sharp.

Proof. If $f * g \in \mathcal{R}_\alpha^u$, then there is a schwartz function $w(z)$, analytic in Δ with $w(0) = 0$ and $|w(z)| < 1$ in Δ such that $\frac{D^{3-2\alpha}(f*g)}{D^{2-2\alpha}(f*g)} = \phi(w(z))$. Define the function $P_1(z)$ by,

$$P_1(z) = \frac{1 + w(z)}{1 - w(z)} = 1 + c_1 z + c_2 z^2 + \ldots$$

Since $w(z)$ is a schwartz function, we see that $ReP_1(z) > 0$ and $P_1(0) = 1$. Define the function

$$P(z) = \frac{D^{3-2\alpha}(f * g)}{D^{2-2\alpha}(f * g)} = 1 + b_1 z + b_2 z^2 + \ldots \tag{2.3}$$

Therefore,

$$P(z) = \phi\left(\frac{P_1(z) - 1}{P_1(z) + 1} \right).$$

Now,

$$\frac{P_1(z)-1}{P_1(z)+1} = \frac{c_1z + c_2z^2 + \dots}{2 + c_1z + c_2z^2 + \dots}$$
$$= \frac{1}{2}\left[c_1z + [c_2 - \frac{c_1^2}{2}]z^2 + [c_3 - c_1c_2 + \frac{c_1^3}{4}z^3] + \dots\right]$$

Hence upon simplification, we get,

$$P(z) = 1 + \frac{B_1c_1z}{2} + \left[\frac{B_1}{2}\left(c_2 - \frac{c_1^2}{2}\right) + \frac{B_2c_1^2}{4}\right]z^2 + \dots \tag{2.4}$$

Therefore,

$$1 + b_1z + b_2z^2 + \dots = 1 + \frac{B_1c_1z}{2} + \left[\frac{B_1}{2}\left(c_2 - \frac{c_1^2}{2}\right) + \frac{B_2c_1^2}{4}\right]z^2 + \dots \tag{2.5}$$

Equating the like coefficients we get,

$$b_1 = \frac{B_1c_1}{2} \tag{2.6}$$

$$b_2 = \frac{B_1}{2}\left(c_2 - \frac{c_1^2}{2}\right) + \frac{B_2c_1^2}{4} \tag{2.7}$$

Therefore, from the equation (2.3) we have

$$1 + A_1z + A_2z^2 + \dots = 1 + b_1z + b_2z^2 + \dots \tag{2.8}$$

where,

$$A_1 = [\mathcal{C}'(\alpha,2)a_2g_2 - \mathcal{C}(\alpha,2)a_2g_2]$$

$$A_2 = \left[\mathcal{C}'(\alpha,3)a_3g - \mathcal{C}(\alpha,2)\mathcal{C}'(\alpha,2)a_2^2 - \mathcal{C}(\alpha,3)a_3 + (\mathcal{C}(\alpha,2)a_2)^2\right],$$

$$\mathcal{C}(\alpha,n) = \frac{\prod_{k=2}^{n}(k-2\alpha)}{(n-1)!},\ \mathcal{C}'(\alpha,n) = \frac{\prod_{k=2}^{n}(k+1-2\alpha)}{(n-1)!},$$

$$b_n = \int_0^1 t^n d\mu(t)$$

for $n = 2, 3, \dots$ and $\mu(t)$ a probability measure on $[0,1]$.
Equating the coefficients of z and z^2 respectively and simplifying we get,

$$a_2 = \frac{b_1}{g_2}; \quad a_3 = \frac{b_2 + (2-2\alpha)b_1^2}{g_3(3-2\alpha)}. \tag{2.9}$$

Applying the equations(2.6) and (2.7) in(2.9) , we get,

$$a_2 = \frac{B_1c_1}{2g_2}; \quad a_3 = \frac{1}{g_3(3-2\alpha)}\left[\frac{B_1}{2}\left(c_2 - \frac{c_1^2}{2}\right) + \frac{B_2c_1^2}{4} + (2-2\alpha)\frac{B_1^2c_1^2}{4}\right].$$

Now,

$$\begin{aligned} a_3-\mu a_2^2 &= \frac{1}{g_3(3-2\alpha)}\left[\frac{B_1}{2}\left(c_2-\frac{c_1^2}{2}\right)+\frac{B_2c_1^2}{4}+(2-2\alpha)\frac{B_1^2c_1^2}{4}\right]-\mu\frac{B_1^2c_1^2}{4g_2^2} \\ &= \frac{1}{g_3(3-2\alpha)}\frac{B_1}{2}\left[c_2-c_1^2\left(\frac{1}{2}-\frac{B_2}{2B_1}-(2-2\alpha)\frac{B_1}{2}+(3-2\alpha)\mu\frac{g_3B_1}{2g_2^2}\right)\right] \\ &= \frac{B_1}{2g_3(3-2\alpha)}\left[c_2-c_1^2v\right] \end{aligned}$$

where,

$$v=\left[\frac{1}{2}-\frac{B_2}{2B_1}-(2-2\alpha)\frac{B_1}{2}+(3-2\alpha)\mu\frac{g_3B_1}{2g_2^2}\right] \tag{2.10}$$

Now an application of lemma (1.5) (see [1]) yields the inequalities stated in the theorem under the respective conditions. For the sharpness of the results in the above theorem we have the following:

1. If $\mu=\sigma_1$, then the equality holds in the lemma (1.1) if and only if
$$P_1(z)=\left(\frac{1}{2}+\frac{\lambda}{2}\right)\frac{1+z}{1-z}+\left(\frac{1}{2}-\frac{\lambda}{2}\right)\frac{1-z}{1+z}\quad 0\le\lambda\le 1$$
or one of its rotations.
2. If $\mu=\sigma_2$, then
$$\frac{1}{P_1(z)}=\frac{1}{\left(\frac{1}{2}+\frac{\lambda}{2}\right)\frac{1+z}{1-z}+\left(\frac{1}{2}-\frac{\lambda}{2}\right)\frac{1-z}{1+z}}.$$
3. If $\sigma_1<\mu<\sigma_2$ $P_1(z)=\frac{1+\lambda z^2}{1-\lambda z^2}$.

To show that the bounds are sharp, we define the function $K_\alpha^{\phi_n}$ $(n=2,3,\ldots)$ by

$$\frac{D^{3-2\alpha}K_\alpha^{\phi_n}}{D^{3-2\alpha}K_\alpha^{\phi_n}}=\phi(z^{n-1}) \tag{2.11}$$

$K_\alpha^{\phi_n}(0)=0$, $(K_\alpha^{\phi_n})'(0)=1$ and function F_α^λ and G_α^λ $(0\le\lambda\le1)$ by

$$\frac{\left(D^{3-2\alpha}F_\alpha^\lambda\right)(z)}{\left(D^{2-2\alpha}F_\alpha^\lambda\right)(z)}=\phi\left(\frac{z(z+\lambda)}{1+\lambda z}\right) \tag{2.12}$$

$F_\alpha^\lambda(0)=0$, $(F_\alpha^\lambda)'(0)=1$ and similarly

$$\frac{\left(D^{3-2\alpha}G_\alpha^\lambda\right)(z)}{\left(D^{2-2\alpha}G_\alpha^\lambda\right)(z)}=\phi\left(\frac{z(z+\lambda)}{1+\lambda z}\right) \tag{2.13}$$

$G_\alpha^\lambda(0)=0$, $(G_\alpha^\lambda)'(0)=1$. Clearly, the functions $K_\alpha^{\phi_n}, F_\alpha^\lambda, G_\alpha^\lambda\in\mathcal{R}_\alpha^u$. Also we write $K_\alpha^\phi:=K_\alpha^{\phi_2}$. If $\mu<\sigma_1$ or $\mu<\sigma_2$, then the equality holds if and only if f is K_α^ϕ or one of its rotations. When $\sigma_1<\mu<\sigma_2$, then the equality holds if and only if f is $K_\alpha^{\phi_3}$ or one of its rotations. If $\mu=\sigma_1$, then the equality holds if and only if f is F_α^λ or one of its rotations If $\mu=\sigma_2$ then the equality holds if and only if f is G_α^λ or one of its rotations. Hence the result. □

Corollary 2.2. *If $g(z)=\frac{z}{1-z}\in\mathcal{R}_0^u$ in Theorem 2.1 we get our earlier result viz., Theorem 3.1 of (see [7]).*

Corollary 2.3. *Taking*

$$g(z) = (\Omega^\lambda f)(z) = z + \sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(2-\lambda)}{\Gamma(n+1-\lambda)} a_n z^n,$$

$(f * g)$ denotes the fractional derivative of f and hence if

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in (\mathcal{R}_\alpha^u)^g(\phi) \tag{2.14}$$

then,

$$|a_3 - \mu a_2^2| \leq \begin{cases} \frac{(3-\lambda)(2-\lambda)}{6(3-2\alpha)} \left(B_2 + B_1^2(2-2\alpha) + \frac{(3-2\alpha)\mu g_3 B_1^2}{g_2^2} \right), & \mu \leq \sigma_1 \\ \frac{(3-\lambda)(2-\lambda)B_1}{6(3-2\alpha)}, & \sigma_1 \leq \mu \leq \sigma_2 \\ \frac{(3-\lambda)(2-\lambda)}{6(3-2\alpha)} \left(-B_2 - B_1^2(2-2\alpha) + \frac{(3-2\alpha)\mu g_3 B_1^2}{g_2^2} \right), & \mu \geq \sigma_2, \end{cases}$$

where,

$$\sigma_1 = \frac{2(3-\lambda)}{3(2-\lambda)} \left[\frac{(B_2 - B_1) + (2-2\alpha)B_1^2}{(3-2\alpha)B_1^2} \right], \tag{2.15}$$

$$\sigma_2 = \frac{2(3-\lambda)}{3(2-\lambda)} \left[\frac{(B_2 + B_1) + (2-2\alpha)B_1^2}{(3-2\alpha)B_1^2} \right] \tag{2.16}$$

the result is sharp.

Proof. This corollary follows from the observations

$$g_2 := \frac{\Gamma(3)\Gamma(2-\lambda)}{\Gamma(3-\lambda)} = \frac{2}{2-\lambda} \tag{2.17}$$

and

$$g_3 := \frac{\Gamma(4)\Gamma(2-\lambda)}{\Gamma(4-\lambda)} = \frac{6}{(2-\lambda)(3-\lambda)}. \tag{2.18}$$

□

Corollary 2.4. *Taking*

$$g(z) = z + \sum_{n=2}^{\infty} n^m z^n, \quad m \in \mathcal{N}_o = \{0\} \cup \mathcal{N},$$

$(f * g)$ denotes the Sălăgean derivative of f (see [6]*) and hence if*

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in (\mathcal{R}_\alpha^u)^g(\phi)$$

then,

$$|a_3 - \mu a_2^2| \leq \begin{cases} \frac{1}{3^m(3-2\alpha)} \left(B_2 + B_1^2(2-2\alpha) + \frac{3^m(3-2\alpha)\mu B_1^2}{2^{2m}} \right), & \mu \leq \sigma_1 \\ \frac{B_1}{3^m(3-2\alpha)}, & \sigma_1 \leq \mu \leq \sigma_2 \\ \frac{1}{3^m(3-2\alpha)} \left(-B_2 - B_1^2(2-2\alpha) + \frac{3^m(3-2\alpha)\mu B_1^2}{2^{2m}} \right), & \mu \geq \sigma_2, \end{cases}$$

where,

$$\sigma_1 = \frac{2^{2m}}{3^m}\left[\frac{(B_2 - B_1) + (2-2\alpha)B_1^2}{(3-2\alpha)B_1^2}\right], \tag{2.19}$$

$$\sigma_2 = \frac{2^{2m}}{3^m}\left[\frac{(B_2 + B_1) + (2-2\alpha)B_1^2}{(3-2\alpha)B_1^2}\right] \tag{2.20}$$

the result is sharp.

Proof. This corollary follows from the observations $g_2 = 2^m$ and $g_3 = 3^m$. □

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A nonsmooth sublinear elliptic problem in $\mathbb{R}^N$ with perturbations

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Abstract. We study a differential inclusion problem in $\mathbb{R}^{\mathbb{N}}$ involving the p-Laplace operator and a $(p-1)$-sublinear term, $p > N > 1$. Our main result is a multiplicity theorem; we also show the non-sensitivity of our problem with respect to small perturbations.

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1. Introduction

Very recently, Kristály, Marzantowicz and Varga (see [5]) studied a quasilinear differential inclusion problem in $\mathbb{R}^{\mathbb{N}}$ involving a suitable sublinear term. The aim of the present paper is to show that under the same assumptions, a more precise conclusion can be concluded by exploiting a recent result of Iannizzotto (see [3]). To be more precise, we recall the assumptions and the relevant results from [5].

Let $p > 2$ and $F : \mathbb{R} \to \mathbb{R}$ be a locally Lipschitz function such that

$(\tilde{\mathbf{F}}1)$ $\lim\limits_{t\to 0} \dfrac{\max\{|\xi| : \xi \in \partial F(t)\}}{|t|^{p-1}} = 0;$

$(\tilde{\mathbf{F}}2)$ $\limsup\limits_{|t|\to+\infty} \dfrac{F(t)}{|t|^p} \leq 0;$

$(\tilde{\mathbf{F}}3)$ There exists $\tilde{t} \in \mathbb{R}$ such that $F(\tilde{t}) > 0$, and $F(0) = 0$.

Here and in the sequel, ∂ stands for the generalized gradient of a locally Lipschitz function; see for details Section 2. We consider the differential inclusion problem

$$(\tilde{P}_{\lambda,\mu}) \qquad \begin{cases} -\triangle_p u + |u|^{p-2}u \in \lambda\alpha(x)\partial F(u(x)) + \mu\beta(x)\partial G(u(x)) & \text{on } \mathbb{R}^N, \\ u(x) \to 0 \text{ as } |x| \to \infty, \end{cases}$$

where $p > N \geq 2$, the numbers λ, μ are positive, and $G : \mathbb{R} \to \mathbb{R}$ is any locally Lipschitz function. Furthermore, we assume that $\beta \in L^1(\mathbb{R}^N)$ is any function, and

$(\tilde{\alpha})$ $\alpha \in L^1(\mathbb{R}^N) \cap L^\infty_{\text{loc}}(\mathbb{R}^N)$, $\alpha \geq 0$, and $\sup_{R>0} \operatorname{essinf}_{|x|\leq R}\alpha(x) > 0$.

The functional space where the solutions of $(\tilde{P}_{\lambda,\mu})$ are sought is the usual Sobolev space $W^{1,p}(\mathbb{R}^N)$, endowed with its standard norm

$$\|u\| = \left(\int_{\mathbb{R}^N} |\nabla u(x)|^p + \int_{\mathbb{R}^N} |u(x)|^p\right)^{1/p}.$$

The main application in Kristály, Marzantowicz and Varga [5] is as follows.

Theorem A. *Assume that $p > N \geq 2$. Let $\alpha, \beta \in L^1(\mathbb{R}^N)$ be two radial functions, α fulfilling $(\tilde{\alpha})$, and let $F, G : \mathbb{R} \to \mathbb{R}$ be two locally Lipschitz functions, F satisfying the conditions $(\tilde{\mathbf{F}}1)$-$(\tilde{\mathbf{F}}3)$. Then there exists a non-degenerate compact interval $[a, b] \subset]0, +\infty[$ and a number $\tilde{r} > 0$, such that for every $\lambda \in [a, b]$ there exists $\mu_0 \in]0, \lambda + 1]$ such that for each $\mu \in [0, \mu_0]$, the problem $(\tilde{P}_{\lambda,\mu})$ has at least three distinct, radially symmetric solutions with L^∞-norms less than $\tilde{r}$.*

To be more precise, (weak) solutions for $(\tilde{P}_{\lambda,\mu})$ are in the following sense: We say that $u \in W^{1,p}(\mathbb{R}^N)$ is a solution of problem $(\tilde{P}_{\lambda,\mu})$, if there exist $\xi_F(x) \in \partial F(u(x))$ and $\xi_G(x) \in \partial G(u(x))$ for a. e. $x \in \mathbb{R}^N$ such that for all $v \in W^{1,p}(\mathbb{R}^N)$ we have

$$\int_{\mathbb{R}^N} (|\nabla u|^{p-2} \nabla u \nabla v + |u|^{p-2} uv) dx = \lambda \int_{\mathbb{R}^N} \alpha(x) \xi_F v dx + \mu \int_{\mathbb{R}^N} \beta(x) \xi_G v dx. \tag{1.1}$$

Our main result reads as follows:

Theorem 1.1. *Assume that $p > N \geq 2$. Let $\alpha \in L^1(\mathbb{R}^N)$ be a radial function fulfilling $(\tilde{\alpha})$, and let $F : \mathbb{R} \to \mathbb{R}$ be a locally Lipschitz function satisfying the conditions $(\tilde{\mathbf{F}}1)$-$(\tilde{\mathbf{F}}3)$. Then there exists $\lambda_0 > 0$ such that for each non-degenerate compact interval $[a, b] \subset]\lambda_0, +\infty[$ there exists a number $r > 0$ with the following property: for every $\lambda \in [a, b]$, every radially symmetric function $\beta \in L^1(\mathbb{R}^N)$ and every locally Lipschitz function $G : \mathbb{R} \to \mathbb{R}$, there exists $\delta > 0$ such that for each $\mu \in [0, \delta]$, the problem $(\tilde{P}_{\lambda,\mu})$ has at least three distinct, radially symmetric solutions with L^∞-norms less than r.*

Remark 1.2. (a) Note that since $p > N$, any element $u \in W^{1,p}(\mathbb{R}^N)$ is homoclinic, i.e., $u(x) \to 0$ as $|x| \to \infty$. This is a consequence of Morrey's embedding theorem.

(b) The terms in the right hand side of (1.1) are well-defined. Indeed, due to Morrey's embedding theorem, i.e., $W^{1,p}(\mathbb{R}^N) \hookrightarrow L^\infty(\mathbb{R}^N)$ is continuous ($p > N$), we have $u \in L^\infty(\mathbb{R}^N)$. Thus, there exists a compact interval $I_u \subset \mathbb{R}$ such that $u(x) \in I_u$ for a.e. $x \in \mathbb{R}^N$. Since the set-valued mapping ∂F is upper-semicontinuous, the set $\partial F(I_u) \subset \mathbb{R}$ is bounded; let $C_F = \sup |\partial F(I_u)|$. Therefore,

$$\left| \int_{\mathbb{R}^N} \alpha(x) \xi_F v dx \right| \leq C_F \|\alpha\|_{L^1} \|v\|_\infty < \infty.$$

Similar argument holds for the function G.

(c) Note that no hypothesis on the growth of G is assumed; therefore, the last term in $(\tilde{P}_{\lambda,\mu})$ may have an arbitrary growth.

The paper is organized as follows. In Section 2 we recall some basic elements from the theory of locally Lipschitz functions, a recent non-smooth three critical points result of Ricceri-type proved by Iannizzotto [3], and a compactness embedding theorem. In Section 3 we prove Theorem 1.1.

2. Preliminaries

2.1. Locally Lipschitz functions

Let $(X, \|\cdot\|)$ be a real Banach space and X^* its dual. A function $h : X \to \mathbb{R}$ is called locally Lipschitz if each point $u \in X$ possesses a neighborhood U_u of u such that

$$|h(u_1) - h(u_2)| \le L\|u_1 - u_2\|, \ \forall u_1, u_2 \in U_u,$$

for a constant $L > 0$ depending on U_u. The generalized gradient of h at $u \in X$ is defined as being the subset of X^*

$$\partial h(u) = \{x^* \in X^* : \langle x^*, z\rangle \le h^0(u; z) \ \text{ for all } \ z \in X\},$$

which is nonempty, convex and w^*-compact, where $\langle \cdot, \cdot\rangle$ is the duality pairing between X^* and X, $h^0(u; z)$ being the generalized directional derivative of h at the point $u \in X$ along the direction $z \in X$, namely

$$h^0(u; z) = \limsup_{\substack{w \to u \\ t \to 0^+}} \frac{h(w + tz) - h(w)}{t},$$

see [2]. Moreover, $h^0(u; z) = \max\{\langle x^*, z\rangle : x^* \in \partial h(u)\}, \ \forall z \in X$. It is easy to verify that $(-h)^0(u; z) = h^0(u; -z)$, and for locally Lipschitz functions $h_1, h_2 : X \to \mathbb{R}$ one has

$$(h_1 + h_2)^0(u; z) \le h_1^0(u; z) + h_2^0(u; z), \ \forall u, z \in X,$$

and

$$\partial(h_1 + h_2)(u) \subseteq \partial h_1(u) + \partial h_2(u).$$

The Lebourg's mean value theorem says that for every $u, v \in X$ there exist $\theta \in]0, 1[$ and $x_\theta^* \in \partial h(\theta u + (1 - \theta)v)$ such that $h(u) - h(v) = \langle x_\theta^*, u - v\rangle$. If h_2 is continuously Gâteaux differentiable, then $\partial h_2(u) = h_2'(u)$; $h_2^0(u; z)$ coincides with the directional derivative $h_2'(u; z)$ and the above inequality reduces to $(h_1 + h_2)^0(u; z) = h_1^0(u; z) + h_2'(u; z), \ \forall u, z \in X$.

A point $u \in X$ is a *critical point of* h if $0 \in \partial h(u)$, i.e. $h^0(u, w) \ge 0, \ \forall w \in X$, see [1]. We define $\lambda_h(u) = \inf\{\|x^*\| : x^* \in \partial h(u)\}$. Of course, this infimum is attained, since $\partial h(u)$ is w^*-compact.

2.2. A nonsmooth Ricceri-type critical point theorem

We recall a non-smooth version of a Ricceri-type (see [7]) three critical point theorem proved by Iannizzotto [3]. Before to do that, we need a notion: let X be a Banach space; a functional $I_1 : X \to \mathbb{R}$ is of type (N) if $I_1(u) = \varphi(\|u\|)$ for every $u \in X$, where $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous differentiable, convex, increasing mapping with $\varphi(0) = \varphi'(0) = 0$.

Theorem 2.1. [3, Corollary 7] *Let X be a separable and reflexive real Banach space with uniformly convex topological dual X^*, let $I_1 : X \to \mathbb{R}$ be functional of type (N), $I_2 : X \to \mathbb{R}$ be a locally Lipschitz functional with compact derivative such that $I_2(u_0) = 0$. Setting the numbers*

$$\tau = \max\left\{0, \limsup_{\|u\|\to\infty} \frac{I_2(u)}{I_1(u)}, \limsup_{u\to 0} \frac{I_2(u)}{I_1(u)}\right\}, \tag{2.1}$$

$$\chi = \sup_{I_1(u)>0} \frac{I_2(u)}{I_1(u)}, \tag{2.2}$$

assume that $\tau < \chi$.

Then, for each compact interval $[a, b] \subset (1/\chi, 1/\tau)$ (with the conventions $1/0 = \infty$ and $1/\infty = 0$) there exists $\kappa > 0$ with the following property: for every $\lambda \in [a, b]$ and every locally Lipschitz functional $I_3 : X \to \mathbb{R}$ with compact derivative, there exists $\delta > 0$ such that for each $\mu \in [0, \delta]$, the inclusion

$$0 \in \partial I_1(u) - \lambda\partial I_2(u) - \mu\partial I_3(u)$$

admits at least three solutions in X having norm less than κ.

2.3. Embeddings

The embedding $W^{1,p}(\mathbb{R}^N) \hookrightarrow L^\infty(\mathbb{R}^N)$ is continuous (due to Morrey's theorem $(p > N)$) but it is not compact. As usual, we may overcome this gap by introducing the subspace of radially symmetric functions of $W^{1,p}(\mathbb{R}^N)$. The action of the orthogonal group $O(N)$ on $W^{1,p}(\mathbb{R}^N)$ can be defined by

$$(gu)(x) = u(g^{-1}x),$$

for every $g \in O(N)$, $u \in W^{1,p}(\mathbb{R}^N)$, $x \in \mathbb{R}^N$. It is clear that this compact group acts linearly and isometrically; in particular $\|gu\| = \|u\|$ for every $g \in O(N)$ and $u \in W^{1,p}(\mathbb{R}^N)$. The subspace of radially symmetric functions of $W^{1,p}(\mathbb{R}^N)$ is defined by

$$W^{1,p}_{\mathrm{rad}}(\mathbb{R}^N) = \{u \in W^{1,p}(\mathbb{R}^N) : gu = u \text{ for all } g \in O(N)\}.$$

Proposition 2.2. [6] *The embedding $W^{1,p}_{\mathrm{rad}}(\mathbb{R}^N) \hookrightarrow L^\infty(\mathbb{R}^N)$ is compact whenever $2 \le N < p < \infty$.*

3. Proof of Theorem 1.1

Let $I_1 : W^{1,p}(\mathbb{R}^N) \to \mathbb{R}$ be defined by

$$I_1(u) = \frac{1}{p}\|u\|^p,$$

and let $I_2, I_3 : L^\infty(\mathbb{R}^N) \to \mathbb{R}$ be

$$I_2(u) = \int_{\mathbb{R}^N} \alpha(x)F(u(x))dx \quad \text{and} \quad I_3(u) = \int_{\mathbb{R}^N} \beta(x)G(u(x))dx.$$

Since $\alpha, \beta \in L^1(\mathbb{R}^N)$, the functionals I_2, I_3 are well-defined and locally Lipschitz, see Clarke [2, p. 79-81]. Moreover, we have

$$\partial I_1(u) \subseteq \int_{\mathbb{R}^N} \alpha(x)\partial F(u(x))dx, \quad \partial I_2(u) \subseteq \int_{\mathbb{R}^N} \beta(x)\partial G(u(x))dx.$$

The energy functional $\mathcal{E}_{\lambda,\mu} : W^{1,p}(\mathbb{R}^N) \to \mathbb{R}$ associated to problem $(\tilde{P}_{\lambda,\mu})$, is given by

$$\mathcal{E}_{\lambda,\mu}(u) = I_1(u) - \lambda I_2(u) - \mu I_3(u), \quad u \in W^{1,p}(\mathbb{R}^N).$$

It is clear that the critical points of the functional $\mathcal{E}_{\lambda,\mu}$ are solutions of the problem $(\tilde{P}_{\lambda,\mu})$ in the sense of relation (1.1).

Since α, β are radially symmetric, then $\mathcal{E}_{\lambda,\mu}$ is $O(N)$-invariant, i.e. $\mathcal{E}_{\lambda,\mu}(gu) = \mathcal{E}_{\lambda,\mu}(u)$ for every $g \in O(N)$ and $u \in W^{1,p}(\mathbb{R}^N)$. Therefore, we may apply a non-smooth version of the *principle of symmetric criticality,* proved by Krawcewicz-Marzantowicz [4], whose form in our setting is as follows.

Proposition 3.1. *Any critical point of* $\mathcal{E}^{\text{rad}}_{\lambda,\mu} = \mathcal{E}_{\lambda,\mu}|_{W^{1,p}_{\text{rad}}(\mathbb{R}^N)}$ *will be also a critical point of* $\mathcal{E}_{\lambda,\mu}$.

Therefore, it remains to find critical point for the functional $\mathcal{E}^{\text{rad}}_{\lambda,\mu}$; here, we will check the assumptions of Theorem 2.1 with the choice $X = W^{1,p}_{\text{rad}}(\mathbb{R}^N)$.

It is standard that X is a reflexive, separable Banach space with uniformly convex topological dual X^*. The functional I_1 is of type (N) on X since $I_1(u) = \varphi(\|u\|)$ where $\varphi(s) = \frac{s^p}{p}$, $s \geq 0$.

Proposition 3.2. ∂I_2 *is compact on* $X = W^{1,p}_{\text{rad}}(\mathbb{R}^N)$.

Proof. Let $\{u_n\}$ be a bounded sequence in X and let $u_n^* \in \partial I_2(u_n)$. It is clear that u_n^* is also bounded in X^* by exploiting Remark 1.2 (b) and hypothesis $(\tilde{\alpha})$. Thus, up to a subsequence, we may assume that $u_n^* \to u^*$ weakly in X^* for some $u^* \in X^*$. By contradiction, let us assume that $\|u_n^* - u^*\|_* > M, \ \ \forall n \in \mathbb{N}$, for some $M > 0$. In particular, there exists $v_n \in X$ with $\|v_n\| \leq 1$ such that

$$(u_n^* - u^*)(v_n) > M.$$

Once again, up to a subsequence, we may suppose that $v_n \to v$ weakly in X for some $v \in X$. Now, applying Proposition 2.2, we may also assume that

$$\|v_n - v\|_{L^\infty} \to 0.$$

Combining the above facts, we obtain that

$$\begin{aligned} M < (u_n^* - u^*)(v_n) &= (u_n^* - u^*)(v) + u_n^*(v_n - v) + u^*(v - v_n) \\ &\leq (u_n^* - u^*)(v) + C\|v_n - v\|_{L^\infty} + u^*(v - v_n) \end{aligned}$$

for some $C > 0$. Since all the terms from the right hand side tend to 0, we get a contradiction. ☐

Proposition 3.3. $\lim\limits_{u\to 0} \frac{I_2(u)}{I_1(u)} = 0.$

Proof. Due to ($\tilde{\mathbf{F}}1$), for every $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that

$$|\xi| \leq \varepsilon |t|^{p-1}, \quad \forall t \in [-\delta(\varepsilon), \delta(\varepsilon)], \ \forall \xi \in \partial F(t). \tag{3.1}$$

For any $0 < t \leq \frac{1}{p}\left(\frac{\delta(\varepsilon)}{c_\infty}\right)^p$ define the set

$$S_t = \{\ u \in W^{1,p}_{\text{rad}}(\mathbb{R}^N) : \|u\|^p < pt\},$$

where $c_\infty > 0$ denotes the best constant in the embedding $W^{1,p}(\mathbb{R}^N) \hookrightarrow L^\infty(\mathbb{R}^N)$.

Note that $u \in S_t$ implies that $\|u\|_\infty \leq \delta(\varepsilon)$; indeed, we have $\|u\|_\infty \leq c_\infty \|u\| < c_\infty (pt)^{1/p} \leq \delta(\varepsilon)$. Fix $u \in S_t$; for a.e. $x \in \mathbb{R}^N$, Lebourg's mean value theorem and (3.1) imply the existence of $\xi_x \in \partial F(\theta_x u(x))$ for some $0 < \theta_x < 1$ such that

$$|F(u(x))| = |F(u(x)) - F(0)| = |\xi_x u(x)| \leq \varepsilon |u(x)|^p.$$

Consequently, for every $u \in S_t$ we have

$$\begin{aligned} |I_2(u)| &= |\int_{\mathbb{R}^N} \alpha(x) F(u(x)) dx| \leq \varepsilon \int_{\mathbb{R}^N} \alpha(x) |u(x)|^p dx \\ &\leq \varepsilon \|\alpha\|_{L^1} \|u\|_\infty^p \leq \varepsilon \|\alpha\|_{L^1} c_\infty^p \|u\|^p. \end{aligned}$$

Therefore, for every $u \in S_t \setminus \{0\}$ with $0 < t \leq \frac{1}{p}\left(\frac{\delta(\varepsilon)}{c_\infty}\right)^p$ we have

$$0 \leq \frac{|I_2(u)|}{I_1(u)} \leq \varepsilon \|\alpha\|_{L^1} c_\infty^p p.$$

Since $\varepsilon > 0$ is arbitrary, we obtain the required limit. □

Proposition 3.4. $\limsup_{\|u\| \to \infty} \frac{I_2(u)}{I_1(u)} \leq 0$.

Proof. By ($\tilde{\mathbf{F}}2$), for every $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that

$$F(t) \leq \varepsilon |t|^p, \quad \forall |t| \in [\delta(\varepsilon), \infty[. \tag{3.2}$$

Consequently, for every $u \in W^{1,p}_{\text{rad}}(\mathbb{R}^N)$ we have

$$\begin{aligned} I_2(u) &= \int_{\mathbb{R}^N} \alpha(x) F(u(x)) dx \\ &= \int_{\{x \in \mathbb{R}^N : |u(x)| > \delta(\varepsilon)\}} \alpha(x) F(u(x)) dx + \int_{\{x \in \mathbb{R}^N : |u(x)| \leq \delta(\varepsilon)\}} \alpha(x) F(u(x)) dx \\ &\leq \varepsilon \int_{\{x \in \mathbb{R}^N : |u(x)| > \delta(\varepsilon)\}} \alpha(x) |u(x)|^p dx + \max_{|t| \leq \delta(\varepsilon)} |F(t)| \int_{\{x \in \mathbb{R}^N : |u(x)| \leq \delta(\varepsilon)\}} \alpha(x) dx \\ &\leq \varepsilon \|\alpha\|_{L^1} c_\infty^p \|u\|^p + \|\alpha\|_{L^1} \max_{|t| \leq \delta(\varepsilon)} |F(t)|. \end{aligned}$$

Therefore, for every $u \in W^{1,p}_{\text{rad}}(\mathbb{R}^N) \setminus \{0\}$, we have

$$\frac{I_2(u)}{I_1(u)} \leq \varepsilon p \|\alpha\|_{L^1} c_\infty^p + p \|\alpha\|_{L^1} \max_{|t| \leq \delta(\varepsilon)} |F(t)| \|u\|^{-p}.$$

Once $\|u\| \to \infty$, the claim is proved, taking into account that $\varepsilon > 0$ is arbitrary. □

Due to hypothesis ($\tilde{\alpha}$), one can fix $R > 0$ such that $\alpha_R = \text{essinf}_{|x|\leq R}\alpha(x) > 0$. For $\sigma \in]0,1[$ define the function

$$w_\sigma(x) = \begin{cases} 0, & \text{if} \quad x \in \mathbb{R}^N \setminus B_N(0,R); \\ \tilde{t}, & \text{if} \quad x \in B_N(0,\sigma R); \\ \frac{\tilde{t}}{R(1-\sigma)}(R - |x|), & \text{if} \quad x \in B_N(0,R) \setminus B_N(0,\sigma R), \end{cases}$$

where $B_N(0,r)$ denotes the $N-$dimensional open ball with center 0 and radius $r > 0$, and $\tilde{t}$ comes from ($\tilde{\mathbf{F}}3$). Since $\alpha \in L^\infty_{\text{loc}}(\mathbb{R}^N)$, then $M(\alpha, R) = \sup_{x\in B_N(0,R)} \alpha(x) < \infty$. A simple estimate shows that

$$I_2(w_\sigma) \geq \omega_N R^N[\alpha_R F(\tilde{t})\sigma^N - M(\alpha,R) \max_{|t|\leq|\tilde{t}|} |F(t)|(1-\sigma^N)].$$

When $\sigma \to 1$, the right hand side is strictly positive; choosing σ_0 close enough to 1, for $u_0 = w_{\sigma_0}$ we have $I_2(u_0) > 0$.

Proof of Theorem 1.1. It remains to combine Theorem 2.1 with Propositions 3.1-3.4. The definitions of the number τ and χ, see relations (2.2)-(2.1), show that $\tau = 0$ and

$$\lambda_0 := \chi^{-1} = \inf_{I_2(u)>0} \frac{I_1(u)}{I_2(u)}$$

is well-defined, positive which is the number appearing in the statement of Theorem 1.1. □

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Non-isomorphic contact structures on the torus T^3

Saad Aggoun

Abstract. In this paper, we prove the existence of infinitely many number non-isomorphic contact structures on the torus T^3. Moreover, this structures are explicitly given by $\omega_n = \cos n\theta_3 d\theta_1 + \sin n\theta_3 d\theta_2, (n \in \mathbb{N})$.

Mathematics Subject Classification (2010): 37J55, 53D10, 53D17, 53D35.

Keywords: Contact structures, Reeb field, Poisson brackets.

1. Introduction

In the acts of Colloquium of Brussels in 1958, P. Libermann [3] addressed the study of the automorphisms of the contact structures on a differentiable manifold M. She has proved that these automorphisms correspond bijectively to functions on this manifold. This allows to transport the Lie algebra structure on the vector space $F(M)$ of the functions on M. We obtain, for two given functions $f, g \in F(M)$, a Poisson bracket $[f, g]$ that depends of the contact form ω. The study of the infinite dimensional Lie algebras obtained is far from being advanced. Thus, in 1973 A. Lichnerowicz [4] who hoped to distinguish the contact structures by their Lie algebras, has given a series of results that are all however of general caracter. Some works that have appeared after have emphasis on the similarities of these algebras. In 1979, R. Lutz [7] has proved the existence of infinitely many non-isomorphic contact structures on the sphere S^3. In 1989, as reported by R. Lutz [7] himself, I have opened in my thesis [1] new perspectives in the other direction by studying the sub-algebras of finite dimension of these algebras. We know that if two contact structures $[\omega_1]$ and $[\omega_2]$ are isomorphic then their Lie algebras (of infinite dimension of course) $A([\omega_1])$ and $A([\omega_2])$ are also isomorphic.

Given an n-dimensional smooth manifold M, and a point $p \in M$, a contact element of M with contact point p is an $(n-1)$-dimensional linear subspace of the tangent space to M at p. A contact contact element can be given by the zeros of a 1-form on the tangent space to M at p. However, if a contact element is given by the zeros of a1-form ω, then it will also be given by the zeros of $\lambda\omega$ where $\lambda \neq 0$. thus

$\{\lambda\omega : \lambda \neq 0\}$ all give the same contact element. It follows that the space of all contact elemnts of M can be identified with a quotient of the cotangent bundle PT^*M, where $PT^*M = T^*M/\mathcal{R}$, where, for $\omega_i \in T_p^*M$, $\omega_1\mathcal{R}\omega_2$ iff there exists $\lambda \neq 0 : \omega_1 = \lambda\omega_2$.

A contact structure on an odd dimensional manifold M, of dimension $2k+1$, is a smooth distribution of contact elements, denoted by ξ, which is generic at each point. The genericity condition is that ξ is non-integrable.

Assume that we have a smooth distribution of contact elements ξ given locally by a differential 1-form α; i.e. a smooth section of the cotangent bundle. The non-integrability condition can be given explicitly as $\alpha \wedge (d\alpha)^k \neq 0$.

Notice that if ξ is given by the differential 1-form α, then the same distribution is given locally by $\beta = f\alpha$, where f is a non-zero smooth function. If ξ is co-orientable then α is defined globally.

If α is a contact form for a given contact structure, the Reeb vector field R can be defined as the unique element of the kernel of $d\alpha$ such that $\alpha(R) = 1$.

For more details, we can consult the references $[5, 6, 8]$.

2. The main result

The main result is contained in the following theorem:

Theorem 2.1. *On the torus T^3 the contact structures defined by the contact forms $\omega_n = \cos n\theta_3 d\theta_1 + \sin n\theta_3 d\theta_2$, $(n \in \mathbb{N})$ are non-isomorphic.*

To establish this result, we need the following lemma.

Lemma 2.2. *Let f a $C^\infty-$function on the torus T^3 and R_n the Reeb field of ω_n defined by*

$$R_n = \cos n\theta_3 \frac{\partial}{\partial\theta_1} + \sin n\theta_3 \frac{\partial}{\partial\theta_2}.$$

If $R_n(f) = 0$, then f depends only on θ_3.

Proof. $R_n(f) = 0$ means that f is constant along the integral curves of R_n whose equations are:

$$\begin{aligned} \frac{d\theta_1}{dt} &= \cos n\theta_3, \\ \frac{d\theta_2}{dt} &= \sin n\theta_3, \\ \frac{d\theta_3}{dt} &= 0. \end{aligned}$$

So, we have

$$\begin{aligned} \theta_1 &= t\cos nk_3 + k_1, \\ \theta_2 &= \sin nk_3 + k_2, \\ \theta_3 &= k_3, \end{aligned}$$

where k_1, k_2 and k_3 are real constants.

When $\tan k_3$ is irrational, the trajectories are dense on a torus T^2, so by continuity f is constant on this torus. Hence, we get $\frac{\partial f}{\partial \theta_1} = \frac{\partial f}{\partial \theta_2} = 0$ for θ_1, θ_2 arbitrary and θ_3 in a dense subset of the circle. It follow that f is constant with respect to θ_1 and θ_2. This completes the proof of the lemma. □

Proof of the theorem. It suffices to prove that the structures $[\omega_1]$ and $[\omega_2]$ are non-isomorphic.

From [1] we recall that the Poisson brackets associated to $[\omega_1]$ and $[\omega_2]$ are given respectively by:

$$\begin{aligned}
[f,g]_1 &= \left(f\frac{\partial g}{\partial \theta_1} - g\frac{\partial f}{\partial \theta_1} + \frac{\partial f}{\partial \theta_3}\frac{\partial g}{\partial \theta_2} - \frac{\partial f}{\partial \theta_2}\frac{\partial g}{\partial \theta_3} \right) \cos\theta_3 \\
&+ \left(f\frac{\partial g}{\partial \theta_2} - g\frac{\partial f}{\partial \theta_2} + \frac{\partial f}{\partial \theta_1}\frac{\partial g}{\partial \theta_3} - \frac{\partial f}{\partial \theta_3}\frac{\partial g}{\partial \theta_1} \right) \sin\theta_3, \\
[f,g]_2 &= \left(f\frac{\partial g}{\partial \theta_1} - g\frac{\partial f}{\partial \theta_1} + \frac{1}{2}\frac{\partial f}{\partial \theta_3}\frac{\partial g}{\partial \theta_2} - \frac{1}{2}\frac{\partial f}{\partial \theta_2}\frac{\partial g}{\partial \theta_3} \right) \cos 2\theta_3 \\
&+ \left(f\frac{\partial g}{\partial \theta_2} - g\frac{\partial f}{\partial \theta_2} + \frac{1}{2}\frac{\partial f}{\partial \theta_1}\frac{\partial g}{\partial \theta_3} - \frac{1}{2}\frac{\partial f}{\partial \theta_3}\frac{\partial g}{\partial \theta_1} \right) \sin 2\theta_3.
\end{aligned}$$

Suppose that $[\omega_1]$ and $[\omega_2]$ are isomorphic that is $F^*\omega_1 = \lambda\omega_2$, where λ is a function on T^3 without zeros and F be this diffeomorphism defined from T^3 into T^3 by:

$$F(\theta_1,\theta_2,\theta_3) = (u(\theta_1,\theta_2,\theta_3), v(\theta_1,\theta_2,\theta_3), w(\theta_1,\theta_2,\theta_3)).$$

We obtain the two equations

$$\frac{\partial u}{\partial \theta_1}\cos w + \frac{\partial v}{\partial \theta_1}\sin w = \lambda \cos 2\theta_3. \tag{2.1}$$

$$\frac{\partial u}{\partial \theta_2}\cos w + \frac{\partial v}{\partial \theta_2}\sin w = \lambda \sin 2\theta_3. \tag{2.2}$$

Let $\Phi(\theta_1,\theta_2,\theta_3) = \cos\theta_3$, $\Psi(\theta_1,\theta_2,\theta_3) = \cos\theta_1$ and $\Omega(\theta_1,\theta_2,\theta_3) = -\sin\theta_1$. Thus we have $[\Phi,\Psi]_1 = \Omega$, $[\Psi,\Omega]_1 = \Phi$ and $[\Omega,\Phi]_1 = -\Psi$.

Then Φ, Ψ and Ω generate a three dimensiononal sub-algebra of $A\ [\omega_1]$ isomorphic to $SL_2(\mathbb{R})$ and consequently, we deduce that the functions $\Phi\circ F$, $\Psi\circ F$ and $\Omega\circ F$ generate a three dimensional sub-algebra of $A\ [\omega_2]$ isomorphic to $SL_2(\mathbb{R})$.
Thus, we have by analogy

$$\begin{aligned}
[\cos w, \cos u]_2 &= -\sin u, \\
[\cos u, -\sin u]_2 &= \cos w, \\
[-\sin u, \cos w]_2 &= -\cos u.
\end{aligned}$$

From this equations, it follows that

$$\frac{\partial u}{\partial \theta_1}\cos 2\theta_3 + \frac{\partial u}{\partial \theta_2}\sin 2\theta_3 = -\cos w. \tag{2.3}$$

If $\Phi(\theta_1,\theta_2,\theta_3) = \sin\theta_3$, $\Psi(\theta_1,\theta_2,\theta_3) = \cos\theta_2$ and $\Omega(\theta_1,\theta_2,\theta_3) = -\sin\theta_2$.

We obtain similarly

$$\frac{\partial v}{\partial \theta_1}\cos 2\theta_3 + \frac{\partial v}{\partial \theta_2}\sin 2\theta_3 = -\sin w. \tag{2.4}$$

We take now

$$\Phi(\theta_1, \theta_2, \theta_3) = 1 \text{ and } \Psi(\theta_1, \theta_2, \theta_3) = -\cos\theta_3,$$

we get

$$\frac{\partial\left(\cos w\right)}{\partial \theta_1}\cos 2\theta_3 + \frac{\partial\left(\cos w\right)}{\partial \theta_2}\sin 2\theta_3 = 0. \tag{2.5}$$

From (5) and lemma 2, it follows that the function $\cos w$ and consequently the function w depend only on θ_3.
Differentiating (3) and (4) with respect to θ_1 and θ_2, we get after taking into account the form of Reeb field R_n the four equations

$$R_2\left(\frac{\partial u}{\partial \theta_1}\right) = R_2\left(\frac{\partial u}{\partial \theta_2}\right) = R_2\left(\frac{\partial v}{\partial \theta_1}\right) = R_2\left(\frac{\partial v}{\partial \theta_2}\right) = 0,$$

from those, we deduce that the functions $\frac{\partial u}{\partial \theta_1}, \frac{\partial u}{\partial \theta_2}, \frac{\partial v}{\partial \theta_1}$ and $\frac{\partial v}{\partial \theta_2}$ depend only on θ_3.
The diffeomorphism F can now be completly caracterized in the following way :

$$u\left(\theta_1, \theta_2, \theta_3\right) = \theta_1\alpha_1(\theta_3) + \theta_2\beta_1(\theta_3) + \gamma_1(\theta_3),$$

$$v\left(\theta_1, \theta_2, \theta_3\right) = \theta_1\alpha_2(\theta_3) + \theta_2\beta_2(\theta_3) + \gamma_2(\theta_3),$$

$$w\left(\theta_1, \theta_2, \theta_3\right) = \gamma_3(\theta_3),$$

where the functions $\alpha_i, \beta_i, \gamma_j$, $i = 1, 2$ and $j = 1, 2, 3$ are defined on the torus T^3.
So F is a diffeomorphism iff the functions α_i and β_i take only integer values and subject to the condition

$$\alpha_1\beta_2 - \alpha_2\beta_1 = \pm 1.$$

We return now to the equations (1) and (2),we obtain

$$(\alpha_1 - \beta_2)\sin\left(w + 2\theta_3\right) - (\alpha_1 + \beta_2)\sin\left(w - 2\theta_3\right)$$

$$+ (\alpha_2 - \beta_1)\cos\left(w - 2\theta_3\right) - (\alpha_2 + \beta_1)\cos\left(w + 2\theta_3\right) = 0.$$

Thus if $w = \pm 2\theta_3$, F is not invertible. In the contrary case, the quantities $\sin\left(w + 2\theta_3\right), \sin\left(w - 2\theta_3\right), \cos\left(w - 2\theta_3\right)$ and $\cos\left(w + 2\theta_3\right)$ are linearly independant, so $\alpha_i = \beta_i = 0$.
In all cases this diffeomorphism do not exist and the contact structures $[\omega_1]$ and $[\omega_2]$ are not isomorphic.
Consequently, there are infinitely many non-isomorphic contact structues $[\omega_n]$ on the torus T^3 given by

$$\omega_n = \cos n\theta_3 d\theta_1 + \sin n\theta_3 d\theta_2, \left(n \in \mathbb{N}\right).$$

This completes the proof of the theorem. □

3. Conclusion

The technics used in this work to find non-isomorphic contact structures can be extended to the sphere S^3 in a first steep and may be to other manifolds suitably choosen. It is also interesting to find the group of diffeomorphisms that leaves the contact structure invariante.

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Strongly almost summable sequence spaces in 2-normed spaces defined by ideal convergence and an Orlicz function

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Abstract. In this paper we introduce some certain new sequence spaces via ideal convergence and an Orlicz function in 2-normed spaces and examine some properties of the resulting these spaces.

Mathematics Subject Classification (2010): 40A99, 40A05.

Keywords: 2-normed space, Orlicz function.

1. Introduction

Let X be a non-empty set, then a family of sets $I \subset 2^X$ (the class of all subsets of X) is called an ideal if and only if for each $A, B \in I$, we have $A \cup B \in I$ and for each $A \in I$ and each $B \subset A$, we have $B \in I$. A non-empty family of sets $F \subset 2^X$ is a filter on X if and only if $\varnothing \notin F$, for each $A, B \in F$, we have $A \cap B \in F$ and each $A \in F$ and each $A \subset B$, we have $B \in F$. An ideal I is called non-trivial ideal if $I \neq \varnothing$ and $X \notin I$. Clearly $I \subset 2^X$ is a non-trivial ideal if and only if $F = F(I) = \{X/A : A \in I\}$ is a filter on X. A non-trivial ideal $I \subset 2^X$ is called admissible if and only if $\{\{x\} : x \in X\} \subset I$. Further details on ideals of 2^X can be found in Kostyrko, et.al [3]. The notion was further investigated by Salat, et.al [4], Tripathy and Hazarika $[13-15]$, Tripathy and Mahanta [16] and others.

Recall in $[5, 7]$ that an Orlicz function M is continuous, convex, nondecreasing function define for $x > 0$ such that $M(0) = 0$ and $M(x) > 0$. If convexity of Orlicz function is replaced by $M(x+y) \leq M(x)+M(y)$ then this function is called the modulus function and characterized by Ruckle [6]. An Orlicz function M is said to satisfy Δ_2-*condition* for all values u, if there exists $K > 0$ such that $M(2u) \leq KM(u), u \geq 0$. Subsequently, the notion of Orlicz function was used to defined sequence spaces by Altin et al [8], Tripathy and Mahanta [9], Et et al [10], Tripathy et al [11], Tripathy and Sarma [12] and many others.

Lemma. *Let M be an Orlicz function which satisfies Δ_2-condition and let* $0<\delta<1$. *Then for each $t\geq\delta$, we have $M(t)<K\delta^{-1}M(2)$ for some constant $K>0$.*

A sequence space X is said to be solid or normal if $(\alpha_k x_k)\in X$, and for all sequences $\alpha=(\alpha_k)$ of scalars with $|\alpha_k|\leq 1$ for all $k\in\mathbb{N}$.

Let X be a real vector space of dimension d, where $2\leq d<\infty$. A 2-norm on X is a function $\|.,.\|:X\times X\to R$ which satisfies;

(i) $\|x,y\|=0$ if and only if x and y are linearly dependent,

(ii) $\|x,y\|=\|y,x\|$,

(iii) $\|\alpha x,y\|=|\alpha|\,\|x,y\|\,,\alpha\in\mathbb{R}$,

(iv) $\|x,y+z\|\leq\|x,y\|+\|x,z\|$.

The pair $(X,\|.,.\|)$ is called a 2-normed space [2]. As an example of a 2-normed space we may take $X=\mathbb{R}$ being equipped with the 2-norm $\|x,y\|$ =*the area of paralelogram spanned by the vectors x and y*, which may be given explicitly by the formula

$$\|x,y\|_E=abs\left(\begin{vmatrix}x_{11} & x_{12}\\ y_{11} & y_{12}\end{vmatrix}\right).$$

2. Main results

In this section we introduce the notion of different types of I-convergent sequences.

Let I be an ideal of $2^{\mathbb{N}}$, M be an Orlicz function, $p=(p_k)$ be a bounded sequence of strictly positive real numbers and $(X,\|,.,\|)$ be an 2-normed space. Further $w(2-X)$ denotes X-valued sequence space. Now, we define the following sequence spaces:

$$\widehat{w}^I[M,p,\|,.,\|]_o=\left\{\begin{array}{c}x=(x_k)\in w(2-X):\forall\varepsilon>0,\left\{n\in\mathbb{N}:\frac{1}{n}\sum_{k=1}^{n}\left[M\left(\left\|\frac{t_{km}(x)}{\rho},z\right\|\right)\right]^{p_k}\geq\varepsilon\right\}\in I\\ \text{for some }\rho>0,\ m\in\mathbb{N}\text{ and each }z\in X\end{array}\right\},$$

$$\widehat{w}^I[M,p,\|,.,\|]=\left\{\begin{array}{c}x=(x_k)\in w(2-X):\forall\varepsilon>0,\left\{n\in\mathbb{N}:\frac{1}{n}\sum_{k=1}^{n}\left[M\left(\left\|\frac{t_{km}(x)}{\rho},z\right\|\right)\right]^{p_k}\geq\varepsilon\right\}\in I\\ \text{for some }\rho>0,L\in X,m\in\mathbb{N}\text{ and each }z\in X\end{array}\right\},$$

$$\widehat{w}^I[M,p,\|,.,\|]_\infty=\left\{\begin{array}{c}x=(x_k)\in w(2-X):\exists K>0,\left\{n\in\mathbb{N}:\frac{1}{n}\sum_{k=1}^{n}\left[M\left(\left\|\frac{t_{km}(x)}{\rho},z\right\|\right)\right]^{p_k}\geq K\right\}\in I\\ \text{for some }\rho>0,m\in\mathbb{N}\text{ and each }z\in X\end{array}\right\}$$

and

$$\widehat{w}[M,p,\|,.,\|]_\infty=\left\{\begin{array}{c}x=(x_k)\in w(2-X):\exists K>0,\frac{1}{n}\sum_{k=1}^{n}\left[M\left(\left\|\frac{t_{km}(x)}{\rho},z\right\|\right)\right]^{p_k}\leq K\\ \text{for some }\rho>0,m\in\mathbb{N}\text{ and each }z\in X\end{array}\right\},$$

where

$$t_{km}(x) = t_{km}(x_k) = \frac{1}{k+1}\sum_{i=0}^{k} x_{i+m}, m \in \mathbb{N}.$$

If $p_k = 1$ for all $k \in \mathbb{N}$, we denote

$$\widehat{w}^I[M, p, \|, ., \|] = \widehat{w}^I[M, \|, ., \|], \widehat{w}^I[M, p, \|, ., \|]_o = \widehat{w}^I[M, \|, ., \|]_o,$$

$$\widehat{w}^I[M, p, \|, ., \|]_\infty = \widehat{w}^I[M, \|, ., \|]_\infty$$

and

$$\widehat{w}[M, p, \|, ., \|]_\infty = \widehat{w}[M, \|, ., \|]_\infty$$

respectively.

If for $k = 0$, we get $t_{km}(x) = x_m$ for all $m \in \mathbb{N}$. We denote these three classes of sequences as $w^I[M, p, \|, ., \|], w^I[M, p, \|, ., \|]_o, w^I[M, p, \|, ., \|]_\infty$ and $w[M, p, \|, ., \|]_\infty$ respectively.

The following well-known inequality will be used for establishing some results of this article. If $0 \le \inf_k p_k (= h) \le p_k \le \sup_k (= H) < \infty$, $D = \max\left(1, 2^{H-1}\right)$, then

$$|x_k + y_k|^{p_k} \le D\{|x_k|^{p_k} + |y_k|^{p_k}\}$$

for all $k \in \mathbb{N}$ and $x_k, y_k \in \mathbb{C}$. Also $|x_k|^{p_k} \le \max\left(1, |x_k|^H\right)$ for all $x_k \in \mathbb{C}$.

Theorem 2.1. *The sets $\widehat{w}^I[M, p, \|, ., \|]_o, \widehat{w}^I[M, p, \|, ., \|]$ and $\widehat{w}^I[M, p, \|, ., \|]_\infty$ are linear spaces over the complex field $\mathbb{C}$.*

Proof. We will prove only for $\widehat{w}^I[M, p, \|, ., \|]_o$ and the others can be proved similarly. Let $x, y \in \widehat{w}^I[M, p, \|, ., \|]_o$ and $\alpha, \beta \in \mathbb{C}$. Then

$$\left\{n \in \mathbb{N} : \frac{1}{n}\sum_{k=1}^{n}\left[M\left(\left\|\frac{t_{km}(x)}{\rho_1}, z\right\|\right)\right]^{p_k} \ge \frac{\varepsilon}{2}\right\} \in I, \text{ for some } \rho_1 > 0$$

and

$$\left\{n \in \mathbb{N} : \frac{1}{n}\sum_{k=1}^{n}\left[M\left(\left\|\frac{t_{km}(y)}{\rho_2}, z\right\|\right)\right]^{p_k} \ge \frac{\varepsilon}{2}\right\} \in I, \text{ for some } \rho_2 > 0\ .$$

for all $m \in \mathbb{N}$. Since $\|, ., \|$ is a 2-norm and M is an Orlicz function, the following inequality holds:

$$\frac{1}{n}\sum_{k=1}^{n}\left[M\left(\left\|\frac{t_{km}(\alpha x + \beta y)}{|\alpha|\rho_1 + |\beta|\rho_2}, z\right\|\right)\right]^{p_k}$$

$$\le \frac{D}{n}\sum_{k=1}^{n}\left[\frac{|\alpha|}{|\alpha|\rho_1 + |\beta|\rho_2}M\left(\left\|\frac{t_{km}(x)}{\rho_1}, z\right\|\right)\right]^{p_k}$$

$$+ \frac{D}{n}\sum_{k=1}^{n}\left[\frac{|\beta|}{|\alpha|\rho_1 + |\beta|\rho_2}M\left(\left\|\frac{t_{km}(y)}{\rho_2}, z\right\|\right)\right]^{p_k}$$

$$\le \frac{D}{n}\sum_{k=1}^{n}\left[M\left(\left\|\frac{t_{km}(x)}{\rho_1}, z\right\|\right)\right]^{p_k} + \frac{D}{n}\sum_{k=1}^{n}\left[M\left(\left\|\frac{t_{km}(y)}{\rho_2}, z\right\|\right)\right]^{p_k}$$

for all $m \in \mathbb{N}$. From the above inequality we get

$$\left\{ n \in \mathbb{N} : \frac{1}{n} \sum_{k=1}^{n} \left[M\left(\left\| \frac{t_{km}(\alpha x + \beta y)}{|\alpha| \rho_1 + |\beta| \rho_2}, z \right\| \right) \right]^{p_k} \geq \varepsilon \right\}$$

$$\subset \left\{ n \in \mathbb{N} : \frac{DA}{n} \sum_{k=1}^{n} \left[M\left(\left\| \frac{t_{km}(x)}{\rho_1}, z \right\| \right) \right]^{p_k} \geq \frac{\varepsilon}{2} \right\}$$
$$\cup \left\{ n \in \mathbb{N} : \frac{DA}{n} \sum_{k=1}^{n} \left[M\left(\left\| \frac{t_{km}(y)}{\rho_2}, z \right\| \right) \right]^{p_k} \geq \frac{\varepsilon}{2} \right\}.$$

Two sets on the right hand side belong to I and this completes the proof.

It is also easy verify that the space $\widehat{w}[M, p, \|, ., \|]_\infty$ is also a linear space.

Theorem 2.2. *For fixed $n \in \mathbb{N}$, $\widehat{w}[M, p, \|, ., \|]_\infty$ paranormed space with respect to the paranorm defined by*

$$g(x) = \inf \left\{ \begin{array}{c} \rho^{\frac{p_n}{H}} > 0 : \left(\sup_{n,m} \frac{1}{n} \sum_{k=1}^{n} \left[M\left(\left\| \frac{t_{km}(x)}{\rho}, z \right\| \right) \right]^{p_k} \right)^{\frac{1}{H}} \leq 1, \\ for\ each\ z \in X \end{array} \right\}.$$

Proof. $g(\theta) = 0$ and $g(-x) = g(x)$ are easy to prove, so we omit them. Let us take $x, y \in \widehat{w}[M, p, \|, ., \|]_\infty$. Let

$$A(x) = \left\{ \rho > 0 : \sup_{n,m} \frac{1}{n} \sum_{k=1}^{n} \left[M\left(\left\| \frac{t_{km}(x)}{\rho}, z \right\| \right) \right]^{p_k} \leq 1, \forall z \in X \right\}$$

and

$$A(y) = \left\{ \rho > 0 : \sup_{n,m} \frac{1}{n} \sum_{k=1}^{n} \left[M\left(\left\| \frac{t_{km}(y)}{\rho}, z \right\| \right) \right]^{p_k} \leq 1, \forall z \in X \right\}.$$

Let $\rho_1 \in A(x)$ and $\rho_2 \in A(y)$. If $\rho = \rho_1 + \rho_2$, then we have

$$\sup_{n,m} \frac{1}{n} \sum_{k=1}^{n} \left[M\left(\left\| \frac{t_{km}(x+y)}{\rho}, z \right\| \right) \right]$$

$$\leq \frac{\rho_1}{\rho_1 + \rho_2} \sup_{n,m} \frac{1}{n} \sum_{k=1}^{n} \left[M\left(\left\| \frac{t_{km}(x)}{\rho_1}, z \right\| \right) \right]$$
$$+ \frac{\rho_2}{\rho_1 + \rho_2} \sup_{n,m} \frac{1}{n} \sum_{k=1}^{n} \left[M\left(\left\| \frac{t_{km}(y)}{\rho_1}, z \right\| \right) \right].$$

Thus

$$\sup_{n,m} \frac{1}{n} \sum_{k=1}^{n} \left[M\left(\left\| \frac{t_{km}(x+y)}{\rho_1 + \rho_2}, z \right\| \right) \right]^{p_k} \leq 1$$

and

$$g(x+y) = \inf \left\{ (\rho_1 + \rho_2)^{\frac{p_n}{H}} > 0 : \rho_1 \in A(x)\ \ and\ \rho_2 \in A(y) \right\}$$

$$\leq \inf\left\{(\rho_1)^{\frac{p_n}{H}} > 0 : \rho_1 \in A(x)\right\} + \inf\left\{(\rho_2)^{\frac{p_n}{H}} > 0 : \rho_2 \in A(y)\right\}$$
$$= g(x) + g(y).$$

Now, let $\lambda_k \to \lambda$, where $\lambda_k, \lambda \in \mathbb{C}$ and $g(x_k^u - x_k) \to 0$ as $u \to \infty$. We have to show that $g(\lambda_k x_k^u - \lambda x_k) \to 0$ as $u \to \infty$. Let

$$A(x^u) = \left\{\rho_u > 0 : \sup_{n,m} \frac{1}{n} \sum_{k=1}^{n} \left[M\left(\left\|\frac{t_{km}(x_k^u)}{\rho_u}, z\right\|\right)\right]^{p_{k,l}} \leq 1, \forall z \in X\right\}$$

and

$$A(x^u - x) = \left\{\rho_u^{\imath} > 0 : \sup_{n,m} \frac{1}{n} \sum_{k=1}^{n} \left[M\left(\left\|\frac{t_{km}(x^u - x)}{\rho_u^{\imath}}, z\right\|\right)\right]^{p_k} \leq 1, \forall z \in X\right\}.$$

If $\rho_u \in A(x^u)$ and $\rho_u^{\imath} \in A(x^u - x)$ then we observe that

$$M\left(\left\|\frac{t_{km}(\lambda_k x_k^u - \lambda x_k)}{\rho_u |\lambda_k - \lambda| + \rho_u^{\imath} |\lambda|}, z\right\|\right)$$
$$\leq M\left(\left\|\frac{t_{km}(\lambda_k x_k^u - \lambda x_k^u)}{\rho_u |\lambda_k - \lambda| + \rho_u^{\imath} |\lambda|}, z\right\| + \left\|\frac{t_{km}(\lambda x_k^u - \lambda x_k)}{\rho_u |\lambda_k - \lambda| + \rho_u^{\imath} |\lambda|}, z\right\|\right)$$
$$\leq \frac{\rho_u |\lambda_k - \lambda|}{\rho_u |\lambda_k - \lambda| + \rho_u^{\imath} |\lambda|} M\left(\left\|\frac{t_{km}(x_k^u)}{\rho_u}, z\right\|\right)$$
$$+ \frac{\rho_u^{\imath} |\lambda|}{\rho_u |\lambda_k - \lambda| + \rho_u^{\imath} |\lambda|} M\left(\left\|\frac{t_{km}(x_k^u - x_k)}{\rho_u^{\imath}}, z\right\|\right).$$

From this inequality, it follows that

$$\left[M\left(\left\|\frac{t_{km}(\lambda_k x_k^u - \lambda x_k)}{\rho_u |\lambda_k - \lambda| + \rho_u^{\imath} |\lambda|}, z\right\|\right)\right]^{p_k} \leq 1$$

and consequently

$$g(\lambda_k x_k^u - \lambda x_k) = \inf\left\{(\rho_u |\lambda_k - \lambda| + \rho_u^{\imath} |\lambda|)^{\frac{p_n}{H}} > 0 : \rho_u \in A(x^u) \text{ and } \rho_u^{\imath} \in A(x^u - x)\right\}$$
$$\leq (|\lambda_k - \lambda|)^{\frac{p_n}{H}} \inf\left\{(\rho_u)^{\frac{p_n}{H}} > 0 : \rho_u \in A(x^u)\right\}$$
$$+ (|\lambda|)^{\frac{p_n}{H}} \inf\left\{(\rho_u^{\imath})^{\frac{p_n}{H}} > 0 : \rho_u^{\imath} \in A(x^u - x)\right\}$$
$$\leq \max\left\{|\lambda|, (|\lambda|)^{\frac{p_n}{H}}\right\} g(x_k^u - x_k).$$

Hence by our assumption the right hand side tends to zero as $u \to \infty$. This completes the proof.

Theorem 2.3. *Let M, M_1 and M_2 be Orlicz functions. Then we have*

(i) $\widehat{w}^I[M_1, p, \|., ., \|]_o \subset \widehat{w}^I[MoM_1, p, \|., ., \|]_o$ *provided that $p = (p_k)$ is such that* $h > 0$.

(ii) $\widehat{w}^I[M_1, p, \|., ., \|]_o \cap \widehat{w}^I[M_2, p, \|., ., \|]_o \subset \widehat{w}^I[M_1 + M_2, p, \|., ., \|]_o$.

Proof. (i). For given $\varepsilon > 0$, we first choose $\varepsilon_o > 0$ such that $\max\left\{\varepsilon_o^H, \varepsilon_o^{H_o}\right\} < \varepsilon$. Now using the continuity of M, choose $0 < \delta < 1$ such that $0 < t < \delta$ implies $M(t) < \varepsilon_o$.

Let $x \in \widehat{w}^I\left[M_1, p, \|, ., \|\right]_o$. Now from the definition of the space $\widehat{w}^I\left[M_1, p, \|, ., \|\right]_o$, for some $\rho > 0$

$$A(\delta) = \left\{ n \in \mathbb{N} : \frac{1}{n} \sum_{k=1}^{n} \left[M_1 \left(\left\| \frac{t_{km}(x)}{\rho}, z \right\| \right) \right]^{p_k} \geq \delta^H \right\} \in I, \ m \in \mathbb{N}$$

Thus if $n \notin A(\delta)$ then

$$\frac{1}{n} \sum_{k=1}^{n} \left[M_1 \left(\left\| \frac{t_{km}(x)}{\rho}, z \right\| \right) \right]^{p_k} < \delta^H$$

$$\Rightarrow \sum_{k=1}^{n} \left[M_1 \left(\left\| \frac{t_{km}(x)}{\rho}, z \right\| \right) \right]^{p_k} < n\delta^H,$$

$$\Rightarrow \left[M_1 \left(\left\| \frac{t_{km}(x)}{\rho}, z \right\| \right) \right]^{p_k} < \delta^H \text{ for all } k, m = 1, 2, ...,$$

$$\Rightarrow M_1 \left(\left\| \frac{t_{km}(x)}{\rho}, z \right\| \right) < \delta \text{ for all } k, m = 1, 2, ...$$

Hence from above inequality and using continuity of M, we must have

$$M \left(M_1 \left(\left\| \frac{t_{km}(x)}{\rho}, z \right\| \right) \right) < \varepsilon_o \text{ for all } k, m = 1, 2, ...$$

which consequently implies that

$$\sum_{k=1}^{n} \left[M \left(M_1 \left(\left\| \frac{t_{km}(x)}{\rho}, z \right\| \right) \right) \right]^{p_k} < n \max\left\{ \varepsilon_o^H, \varepsilon_o^{H_o} \right\} < n\varepsilon, \ m = 1, 2, ...,$$

$$\Rightarrow \frac{1}{n} \sum_{k=1}^{n} \left[M \left(M_1 \left(\left\| \frac{t_{km}(x)}{\rho}, z \right\| \right) \right) \right]^{p_k} < \varepsilon, \ m = 1, 2, ...$$

This shows that

$$\left\{ n \in \mathbb{N} : \frac{1}{n} \sum_{k=1}^{n} \left[M \left(M_1 \left(\left\| \frac{t_{km}(x)}{\rho}, z \right\| \right) \right) \right]^{p_k} \geq \varepsilon \right\} \subset A(\delta)$$

and so belongs to I. This completes the proof.

(ii) Let $x \in \widehat{w}^I\left[M_1, p, \|, ., \|\right]_o \cap \widehat{w}^I\left[M_2, p, \|, ., \|\right]_o$. Then the fact that

$$\frac{1}{n} \left[(M_1 + M_2) \left(\left\| \frac{t_{km}(x)}{\rho}, z \right\| \right) \right]^{p_k}$$

$$\leq \frac{D}{n} \left[M_1 \left(\left\| \frac{t_{km}(x)}{\rho}, z \right\| \right) \right]^{p_k} + \frac{D}{n} \left[M_2 \left(\left\| \frac{t_{km}(x)}{\rho}, z \right\| \right) \right]^{p_k}$$

gives us the result.

Theorem 2.4. *(i) If* $0 < h \leq p_k < 1$, *then* $\widehat{w}^I\left[M, p, \|, ., \|\right]_o \subset \widehat{w}^I\left[M, \|, ., \|\right]_o$.

(ii) If $1 \leq p_k \leq H < \infty$, *then* $\widehat{w}^I\left[M, \|, ., \|\right]_o \subset \widehat{w}^I\left[M, p, \|, ., \|\right]_o$.

(iii) If $0 < p_k < q_k < \infty$ *and* $\frac{q_k}{p_k}$ *is bounded, then*

$$\widehat{w}^I\left[M, p, \|, ., \|\right]_o \subset \widehat{w}^I\left[M, q, \|, ., \|\right]_o.$$

Proof. The proof is standard, so we omit it.

Theorem 2.5. *The sequence spaces* $\widehat{w}^I [M,p,\|,.,\|]_o$, $\widehat{w}^I [M,p,\|,.,\|]$, $\widehat{w}^I [M,p,\|,.,\|]_\infty$ *and* $\widehat{w} [M,p,\|,.,\|]_\infty$ *are solid.*

Proof. We give the proof for only $\widehat{w}^I [M,p,\|,.,\|]_o$. The others can be proved similarly. Let $x \in \widehat{w}^I [M_1,p,\|,.,\|]_o$ and $\alpha = (\alpha_k)$ be a sequence of scalars such that $|\alpha_k| \leq 1$ for all $k \in \mathbb{N}$. Then we have

$$\left\{ n \in \mathbb{N} : \frac{1}{n} \sum_{k=1}^{n} \left[M \left(\left\| \frac{t_{km}(\alpha_k x_k)}{\rho}, z \right\| \right) \right]^{p_k} \leq \varepsilon \right\}$$

$$\subset \left\{ n \in \mathbb{N} : \frac{T}{n} \sum_{k=1}^{n} \left[M \left(\left\| \frac{t_{km}(x_k)}{\rho}, z \right\| \right) \right]^{p_k} \leq \varepsilon \right\} \in I,$$

where $T = \sup_k \left\{ 1, |\alpha_k|^H \right\}$. Hence $\alpha x \in \widehat{w}^I [M_1,p,\|,.,\|]_o$ for all sequences $\alpha = (\alpha_k)$ with $|\alpha_k| \leq 1$ for all $k \in \mathbb{N}$ whenever $x \in \widehat{w}^I [M_1,p,\|,.,\|]_o$.

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On the Conjecture of Cao, Gonska and Kacsó

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Abstract. We consider the question if lower estimates in terms of the second order Ditzian-Totik modulus are possible, when we measure the pointwise approximation of continuous function by Bernstein operator. In this case we confirm the conjecture made by Cao, Gonska and Kacsó. To prove this we first establish sharp upper and lower bounds for pointwise approximation of the function $g(x) = x\ln(x) + (1-x)\ln(1-x)$, $x \in [0,1]$ by Bernstein operator.

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Keywords: Bernstein operator, lower bounds, Ditzian-Totik moduli of smoothness.

1. Introduction

In [6] Cao, Gonska and Kacsó formulated the following

Conjecture 1.1. *Let $T_n : C[a,b] \to C[a,b]$ be a sequence of linear operators and $\varepsilon_n > 0$, $\lim\limits_{n\to\infty} \varepsilon_n = 0$, $\varphi(x) = \varphi(x)_{[a,b]} = \sqrt{(x-a)(b-x)}$, and $0 \le \beta < \lambda \le 1$ fixed. If for every $f \in C[a,b]$ one has*

$$|T_n(f,x) - f(x)| \le C(f)\omega_2^{\varphi^\lambda}\left(f; \varepsilon_n \varphi^{1-\lambda}(x)\right), \tag{1.1}$$

then lower pointwise estimates

$$c(f)\omega_2^{\varphi^\beta}\left(f; \varepsilon_n \varphi^{1-\lambda}(x)\right) \le |T_n(f,x) - f(x)|,\ f \in C[a,b], \tag{1.2}$$

do not hold in general.

The case $\beta = 0$ was already solved by the same authors in Theorem 3.1 in [5]. The aim of this note is to confirm conjecture above for the case when T_n is replaced by the Bernstein operator B_n. Instead of T_n we consider further only the classical Bernstein operator B_n applied to a continuous on $[0,1]$ function $f(x)$ and defined by

$$B_n(f;x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right) \cdot \binom{n}{k} x^k (1-x)^{n-k},\ x \in [0,1]. \tag{1.3}$$

By usual translation all considerations over the interval $[0,1]$ could be transformed into the interval $[a,b]$. Let us define the function

$$g(x) = x\ln x + (1-x)\ln(1-x),\ x \in (0,1) \tag{1.4}$$

and $g(0) = g(1) = 0$. This function was studied and used by many authors in different problems in approximation theory - see [1,2,5,6,9,10,11,12,13,14]. For example the function $g(x)$ was used to establish Theorem 3.1 in [5]. Also g was studied to obtain direct pointwise estimates for approximation of a continuous function by linear positive operator L in [13-Lemma 3.2]. V.Maier considered the function g to establish the saturation order of Kantorovich operator (see [11,12] and Ch. 10 in [3]). The first uniform estimate for approximation of $g(x)$ by B_n was given by Berens and Lorentz in [1]:

$$B_n(g,x) - g(x) \le \frac{7}{n},\ \text{ for all }\ x \in [0,1].$$

Different problems in approximation and learning theory, connected with approximation of g by B_n are also studied in [2]. The problem to evaluate in a pointwise form the remainder term

$$R_n(g,x) := B_n(g,x) - g(x),\ x \in [0,1] \tag{1.5}$$

was formulated by the author in [14] as open problem during the fifth Romanian-German seminar on approximation theory, held in Sibiu, Romania in 2002. More precisely, we propose to find (best) bounds of the type

$$k_1 \cdot \frac{x^{\alpha_1}(1-x)^{\alpha_2}}{n^{\beta}} \le R_n(g,x) \le K_2 \cdot \frac{x^{a_1}(1-x)^{a_2}}{n^{b}}, \tag{1.6}$$

for every $x \in [0,1]$, where k_1, K_1 are positive numbers, independent of x and n. Some days after the conference prof. A.Lupaş sent to me the proof of inequality (1.6) with $\alpha_1 = \alpha_2 = \beta = 1$, $k_1 = \frac{1}{2}$ and $a_1 = a_2 = b = \frac{1}{2}$, $K_2 = \sqrt{2}$, i.e.

Theorem 1.2. (see [10]) *For all $x \in [0,1]$ the following holds true*

$$\frac{x(1-x)}{2n} \le R_n(g,x) \le \sqrt{2} \cdot \sqrt{\frac{x(1-x)}{n}}. \tag{1.7}$$

Our first statement is motivated by the result of Lupaş and considerations, made in [5,6,13]. We prove that the values of $\alpha_1 = \alpha_2 = 1$ and $a_1 = a_2 = \frac{1}{2}$ in (1.7) are optimal , namely

Theorem 1.3. *It is not possible to find $a_1 > \frac{1}{2}$, or $a_2 > \frac{1}{2}$, or $\alpha_1 < 1$, or $\alpha_2 < 1$, such that*

$$k_1 \cdot \frac{x^{\alpha_1}(1-x)^{\alpha_2}}{n} \le R_n(g,x) \le K_2 \cdot \frac{x^{a_1}(1-x)^{a_2}}{\sqrt{n}}, \tag{1.8}$$

holds true for all $x \in [0,1]$ with some positive numbers k_1, K_2, independent of x and n.

Our next result confirms the conjecture of Cao, Gonska, Kacsó in [6] and states the following

Theorem 1.4. *Let* $\varphi(x) = \sqrt{x(1-x)}$, $x \in [0,1]$ *and* $0 \le \beta < \lambda \le 1$ *be fixed. For the function* $g(x)$, *defined in* (1.4) *one has*

$$|B_n(g,x) - g(x)| \le C(g)\omega_2^{\varphi^\lambda}\left(g; \frac{1}{\sqrt{n}}\varphi^{1-\lambda}(x)\right), \tag{1.9}$$

but the lower pointwise estimate

$$c(g)\omega_2^{\varphi^\beta}\left(g; \frac{1}{\sqrt{n}}\varphi^{1-\lambda}(x)\right) \le |B_n(g,x) - g(x)|, \tag{1.10}$$

is not valid.

In Section 2 we give the proof of Theorem 1.3. In Section 3 we establish the proof of Theorem 1.4.

2. Proof of Theorem 1.3

Proof. Due to symmetry it is enough to consider in (1.8) only $x \in [0, \frac{1}{2}]$ and to study the possible values of the parameters α_1 and a_1. It is easy to compute that

$$g''(x) = \frac{1}{x(1-x)},\ x \in (0,1), \tag{2.1}$$

i.e. g is a convex function on $[0,1]$. Therefore

$$B_n(g,x) \ge g(x), \text{for all}\, x \in [0,1]. \tag{2.2}$$

If $S_n(g,x)$ is the piecewise linear interpolant for g at the points $0, \frac{1}{n}, \ldots, 1$, then

$$\begin{aligned} B_n(S_n g, x) &= B_n(g,x), \\ B_n(S_n g, x) &\ge S_n(g,x), \end{aligned} \tag{2.3}$$

due to the fact that $S_n g$ is also convex function. Consequently from (2.2)-(2.3) we get

$$B_n(g,x) - g(x) \ge S_n(g,x) - g(x). \tag{2.4}$$

First let us consider the r.h.s. of (1.8). We suppose that (1.8) holds with $a_1 > \frac{1}{2}$. Then from (2.4) it follows that

$$S_n(g,x) - g(x) \le K_2 \cdot \frac{x^{a_1}(1-x)^{a_2}}{\sqrt{n}},\ x \in [0, \frac{1}{2}]. \tag{2.5}$$

We compute for $0 \le x \le \frac{1}{n}$ that

$$\begin{aligned} S_n(g,x) &= nx \cdot g(\tfrac{1}{n}) = nx\left[\tfrac{1}{n}\ln(\tfrac{1}{n}) + (1-\tfrac{1}{n})\ln(1-\tfrac{1}{n})\right] \\ &= x\left[\ln(\tfrac{1}{n}) + (n-1)\ln(1-\tfrac{1}{n})\right]. \end{aligned} \tag{2.6}$$

Also we verify that for $x \in [0, \frac{1}{2}]$,

$$g(x) = x\ln x + (1-x)\ln(1-x) \le x\ln x. \tag{2.7}$$

Consequently (2.5) and (2.7) yield

$$x\left[-\ln n + (n-1)\ln(1-\frac{1}{n})\right] - x\ln x \le K_2 \cdot \frac{x^{a_1}(1-x)^{a_2}}{\sqrt{n}} \tag{2.8}$$

for $0 < x \le \frac{1}{n}$. Therefore

$$-x\ln(nx) + x\left[(n-1)\ln(1-\frac{1}{n})\right] \le K_2 \cdot \frac{x^{a_1}(1-x)^{a_2}}{\sqrt{n}} \tag{2.9}$$

Hence we get

$$x\ln\left[(1-\frac{1}{n})^{n-1}\cdot\frac{1}{nx}\right] \le K_2 \cdot \frac{x^{a_1}(1-x)^{a_2}}{\sqrt{n}}.$$

Consequently

$$x^{1-a_1}\cdot\sqrt{n} \le \frac{K_2\cdot(1-x)^{a_2}}{\ln\left[(1-\frac{1}{n})^{n-1}\cdot\frac{1}{nx}\right]}. \tag{2.10}$$

We set $x = \frac{1}{2en}$ in (2.10) and take $n \to \infty$. Then we arrive at

$$+\infty = \lim_{n\to\infty} n^{a_1-1+\frac{1}{2}} \le \frac{K_2}{\ln 2}, \tag{2.11}$$

when $a_1 > \frac{1}{2}$, which is a contradiction.

To study the best possible value of α_1 in (1.8) we may use the following estimate, proved firstly by Cao in 1964 for all continuous functions, and in particular for $g(x)$-see [4]:

$$|B_n(g,x) - g(x)| \le C\omega_2(g, \sqrt{\frac{x(1-x)}{n}}). \tag{2.12}$$

This nice estimate can not help us to establish the impossibility of the first inequality in (1.8). We suppose that (1.8) holds with $\alpha_1 < 1$. It is easy to observe that

$$R_n(g,x) \le |g(x)|. \tag{2.13}$$

Then we would have

$$k_1\frac{(1-x)^{\alpha_2}}{n} \le x^{-\alpha_1}|g(x)|,$$

which for $x \to 0$ gives

$$\frac{k_1}{n} \le 0$$

a contradiction. The proof of Theorem 1.3 is completed. □

3. Proof of Theorem 1.4

Proof. We recall the definition of the moduli $\omega_2^{\varphi^\lambda}$, $0 \le \lambda \le 1$, which is in complete analogy to those of $\omega_2(f,\cdot)$, $(\lambda = 0)$ and $\omega_2^{\varphi}(f,\cdot)$, $(\lambda = 1)$, (see [8], Chap.2):

$$\omega_2^{\varphi^\lambda}(f,t) = \sup_{0\le h\le t}\|\Delta^2_{h\varphi^\lambda} f\|_\infty, \tag{3.1}$$

where

$$\Delta^2_{h\varphi^\lambda} f(x) := \begin{cases} f(x-h\varphi^\lambda(x)) - 2f(x) + f(x+h\varphi^\lambda(x)), \\ \text{if } [x-h\varphi^\lambda(x), x+h\varphi^\lambda(x)] \subseteq [0,1]; \\ 0, \text{ otherwise.} \end{cases} \tag{3.2}$$

The direct pointwise estimate (1.9) was proved by Ditzian in [7] for all continuous functions, defined in $[0,1]$ and in particular it holds for $g(x)$ too. We suppose that (1.10) holds true. Setting $x=\frac{1}{2}$ in (3.2) we obtain

$$\Delta^2_{h\varphi^\beta} g\left(\frac{1}{2}\right) = h^2\cdot\varphi^{2\beta}(\frac{1}{2})\cdot g''(\xi)\geq h^2\cdot(\frac{1}{2})^{2\beta}\cdot\frac{1}{\frac{1}{2}(1-\frac{1}{2})} = h^2\cdot 2^{2(1-\beta)}.$$

Hence by

$$t := \frac{1}{\sqrt{n}}\varphi^{1-\lambda}(x),\ x\in[0,1] - \text{fixed}$$

it follows

$$\omega_2^{\varphi^\beta}(g,t)\geq t^2\cdot 2^{2(1-\beta)} = \frac{1}{n}(x(1-x))^{1-\lambda}\cdot 2^{2(1-\beta)}. \tag{3.3}$$

From our supposition and (3.3) we get

$$c(g)\cdot 2^{2(1-\beta)}\cdot\frac{x^{1-\lambda}(1-x)^{1-\lambda}}{n}\leq |B_n(g,x)-g(x)| \tag{3.4}$$

for $0\leq\beta<\lambda\leq 1$. It is clear that for $\lambda=1$ (3.4) is not possible, because due to (1.7) it would lead to

$$c(g)\cdot 2^{2(1-\beta)}\leq\sqrt{2}\cdot\sqrt{\frac{x(1-x)}{n}},\ \text{for all}\, x\in[0,1], \tag{3.5}$$

which is a contradiction.

Consequently for $0\leq\beta<\lambda<1$ (3.4) would imply, that (1.8) is valid with $\alpha_1 = 1-\lambda<1$, which contradicts the statement of Theorem 1.3. Thus the proof of Theorem 1.4 is completed. □

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Kantorovich type q-Bernstein-Stancu operators

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Abstract. In this paper, we construct a Kantorovich type generalization of q-Bernstein-Stancu operators by means of the Riemann type q-integral. We investigate some approximation properties and also establish a local approximation theorem for these operators.

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1. Introduction

Let $q > 0$ be a fixed real number. For any nonnegative integer n, the q-integer $[n]_q$ and the q-factorial $[n]_q!$ are respectively defined by (see [2])

$$[n]_q = \begin{cases} \dfrac{1-q^n}{1-q} & \text{if } q \neq 1 \\ n & \text{if } q = 1 \end{cases},$$

and

$$[n]_q! = \begin{cases} [1]_q[2]_q \cdots [n]_q & \text{if } n \geq 1 \\ 1 & \text{if } n = 0 \end{cases}.$$

For the integers $n \geq k \geq 0$, the q-binomial coefficients are defined by

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[k]_q![n-k]_q!}.$$

Now suppose that $0 < a < b$, $0 < q < 1$ and f is a real-valued function. The q-Jackson integral of f over the interval $[0, b]$ and a general interval $[a, b]$ are defined by (see [11])

$$\int_0^b f(x)d_qx = (1-q)b\sum_{j=0}^{\infty} f\left(bq^j\right)q^j$$

and

$$\int_a^b f(x)d_qx = \int_0^b f(x)d_qx - \int_0^a f(x)d_qx,$$

respectively, provided the series converge.

It is clear that q-Jackson integral of f over an interval $[a,b]$ contains two infinite sums, so some problems are encountered in deriving the q-analogues of some well-known integral inequalities which are used to compute order of approximation of linear positive operators containing q-Jackson integral. To solve this problem Marinković et.al. (see [12]) defined the Riemann type q-integral as

$$R_q(f;a,b) = \int_a^b f(x)d_q^R x = (1-q)(b-a)\sum_{j=0}^{\infty} f\left(a+(b-a)q^j\right)q^j$$

which contains only points within the interval of integral.
Dalmanoğlu and Doğru [4] proved that Riemann type q-integral is a linear positive operator and satisfies the Hölder inequality

$$R_q(|fg|;a,b) \le \left(R_q(|f|^{m_1};a,b)\right)^{\frac{1}{m_1}} \left(R_q(|g|^{m_2};a,b)\right)^{\frac{1}{m_2}}$$

where $\frac{1}{m_1}+\frac{1}{m_2}=1$.

In 2009 Nowak [13], for $f \in C[0,1]$, $q>0$, $\alpha \ge 0$ and each $n \in \mathbb{N}$ defined the q-Bernstein-Stancu operators

$$B_n^{q,\alpha}(f;x) = \sum_{k=0}^{n} P_{n,k}^{q,\alpha}(x) f\left(\frac{[k]_q}{[n]_q}\right), \quad x \in [0,1] \tag{1.1}$$

with

$$P_{n,k}^{q,\alpha}(x) = \begin{bmatrix} n \\ k \end{bmatrix}_q \frac{\prod_{i=0}^{k-1}(x+\alpha[i]_q)\prod_{s=0}^{n-k-1}(1-q^s x+\alpha[s]_q)}{\prod_{i=0}^{n-1}(1+\alpha[i]_q)} \tag{1.2}$$

and investigated Korovkin type approximation properties of these operators. Note that in (1.2) an empty product is taken to be equal to 1. In [10], the authors studied the rate of convergence and proved a Voronovskaya type theorem for the operator defined by (1.1). After that Agratini [1] introduced some estimates for the rate of convergence to the sequence $B_n^{q,\alpha}(f;x)$ by means of the modulus of continuity and Lipschitz type maximal function and also explored a probabilistic approach.

It is clear that for $\alpha=0$, $B_n^{q,\alpha}(f;x)$ reduces to q-Bernstein polynomials defined by Phillips [15]

$$B_{n,q}(f;x) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q x^k \prod_{s=0}^{n-k-1}(1-q^s x) f\left(\frac{[k]_q}{[n]_q}\right), \quad x \in [0,1].$$

For $q=1$, $B_n^{q,\alpha}(f;x)$ turns out to be the Bernstein- Stancu polynomials proposed by Stancu in [16]

$$S_n(f;x) = \sum_{k=0}^{n} \binom{n}{k} \frac{\prod_{i=0}^{k-1}(x+\alpha i)\prod_{s=0}^{n-k-1}(1-x+\alpha s)}{\prod_{i=0}^{n-1}(1+\alpha i)} f\left(\frac{k}{n}\right), \quad x \in [0,1].$$

For $\alpha = 0$ and $q = 1$, $B_n^{q,\alpha}(f;x)$ represents the classical Bernstein polynomials given by

$$B_n(f;x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right)\binom{n}{k}x^k(1-x)^{n-k}, \quad x \in [0,1].$$

The following identities hold [13]

$$B_n^{q,\alpha}(1;x) = 1 \tag{1.3}$$

$$B_n^{q,\alpha}(t;x) = x \tag{1.4}$$

$$B_n^{q,\alpha}(t^2;x) = \frac{1}{1+\alpha}\left(x(x+\alpha) + \frac{x(1-x)}{[n]_q}\right) \tag{1.5}$$

for all $x \in [0,1]$ and $n \in \mathbb{N}$.

Generalization of Bernstein polynomials based on q-integers was studied by a number of authors. We now mention some papers related to integral modification of the q-Bernstein polynomials. Gupta [7] constructed Durrmeyer type modification of the q-Bernstein polynomials by means of the q-Jackson integral and studied their some approximation properties. Thereafter, Finta and Gupta [6] obtained some local and global direct results and also established a simultaneous approximation theorem for these operators. In [8], Gupta and Heping defined another Durrmeyer type q-Bernstein polynomials and obtained some approximation properties of such operators. Later in [9], Gupta and Finta proved some direct local and global approximation theorems for the operators given in [8]. Dalmanoğlu [3] presented Kantorovich type q-Bernstein polynomials via q-Jackson integral and investigated their approximation properties and the rate of convergence. Very recently, by introducing the following Kantorovich type generalization of q-Bernstein polynomials by means of the q-Riemann type integral

$$B_n^*(f;q;x) = [n+1]_q \sum_{k=0}^{n} q^{-k} \begin{bmatrix} n \\ k \end{bmatrix}_q x^k \prod_{s=0}^{n-k-1}(1-q^s x) \int_{\frac{[k]_q}{[n+1]_q}}^{\frac{[k+1]_q}{[n+1]_q}} f(t)d_q^R t, \tag{1.6}$$

where $x \in [0,1]$ and $0 < q < 1$, Dalmanoğlu and Doğru [4] studied statistical Korovkin type approximation properties of these operators. The authors derived the formulas

$$B_n^*(1;q;x) = 1,$$

$$B_n^*(t;q;x) = \frac{2q}{1+q}\frac{[n]_q}{[n+1]_q}x + \frac{1}{1+q}\frac{1}{[n+1]_q},$$

$$\begin{aligned} B_n^*(t^2;q;x) = &\left(\frac{q^2}{1+q} + \frac{3q^4}{(1+q)(1+q+q^2)}\right)\frac{[n]_q[n-1]_q}{[n+1]_q^2}x^2 \\ &+ \left(1 + \frac{2q}{1+q} + \frac{q^2-1}{1+q+q^2}\right)\frac{[n]_q}{[n+1]_q^2}x + \frac{1}{1+q+q^2}\frac{1}{[n+1]_q^2}. \end{aligned}$$

In this paper, for $f \in C[0,1]$, $0 < q < 1$ and each $n \in \mathbb{N}$, we consider the Kantorovich type generalization of the q-Bernstein Stancu operators defined by (1.1) with the help of the Riemann type q-integral as follows:

$$B_n^{\alpha}(f;q;x)=\sum_{k=0}^{n}P_{n,k}^{q,\alpha}(x)\frac{[n+1]_q}{q^k}\int_{\frac{[k]_q}{[n+1]_q}}^{\frac{[k+1]_q}{[n+1]_q}}f(t)d_q^Rt,\quad x\in[0,1] \tag{1.7}$$

where $P_{n,k}^{q,\alpha}(x)$ is given by (1.2).

In the case $\alpha=0$ the operator $B_n^{\alpha}(f;q;x)$ turns into the operator $B_n^{*}(f;q;x)$ defined by (1.6).

2. Estimation of moments

Lemma 2.1. *Let m be a nonnegative integer. Then we have*

$$\begin{aligned}I_{n,k}(t^m)&:=\int_{\frac{[k]_q}{[n+1]_q}}^{\frac{[k+1]_q}{[n+1]_q}}t^m d_q^R t\\&=\frac{q^k}{[n+1]_q}\left(\frac{[n]_q}{[n+1]_q}\right)^m\sum_{l=0}^{m}\binom{m}{l}\left(\frac{[k]_q}{[n]_q}\right)^{m-l}C_{m,l}(q,n),\end{aligned}$$

where

$$C_{m,l}(q,n)=\frac{1}{([n]_q)^l}\sum_{s=0}^{m-l}\binom{m-l}{s}(-1)^s\frac{(1-q)^s}{[l+s+1]_q}.$$

Proof. By definition of Riemann type q-integral and Binomial formula, we get

$$\begin{aligned}&I_{n,k}(t^m)\\&=(1-q)\frac{q^k}{[n+1]_q}\sum_{j=0}^{\infty}\left(\frac{[k]_q}{[n+1]_q}+\frac{q^k}{[n+1]_q}q^j\right)^m q^j\\&=(1-q)\frac{q^k}{([n+1]_q)^{m+1}}\sum_{j=0}^{\infty}\left([k]_q+(1-(1-q)[k]_q)\,q^j\right)^m q^j\\&=(1-q)\frac{q^k}{([n+1]_q)^{m+1}}\sum_{i=0}^{m}\sum_{j=0}^{\infty}(q^j)^{i+1}\binom{m}{i}(1-(1-q)[k]_q)^i([k]_q)^{m-i}.\end{aligned}$$

Using the following fact

$$\sum_{j=0}^{\infty}(q^j)^{i+1}=\frac{1}{1-q^{i+1}}=\frac{1}{(1-q)}\frac{1}{[i+1]_q}$$

and Binomial formula again, we can write

$$
\begin{aligned}
&I_{n,k}(t^m)\\
=&\frac{q^k}{([n+1]_q)^{m+1}}\sum_{i=0}^{m}\binom{m}{i}\left(1-(1-q)[k]_q\right)^i\frac{([k]_q)^{m-i}}{[i+1]_q}\\
=&\frac{q^k}{([n+1]_q)^{m+1}}\sum_{i=0}^{m}\binom{m}{i}\frac{([k]_q)^{m-i}}{[i+1]_q}\sum_{l=0}^{i}\binom{i}{l}(-1)^{i-l}\left((1-q)[k]_q\right)^{i-l}\\
=&\frac{q^k}{([n+1]_q)^{m+1}}\sum_{i=0}^{m}\binom{m}{i}\frac{1}{[i+1]_q}\sum_{l=0}^{i}\binom{i}{l}([k]_q)^{m-l}(-1)^{i-l}(1-q)^{i-l}\\
=&\frac{q^k}{([n+1]_q)^{m+1}}\sum_{l=0}^{m}\sum_{i=l}^{m}\binom{m}{i}\binom{i}{l}([k]_q)^{m-l}(-1)^{i-l}\frac{(1-q)^{i-l}}{[i+1]_q}\\
=&\frac{q^k}{([n+1]_q)^{m+1}}\sum_{l=0}^{m}\sum_{i=l}^{m}\frac{m!}{(m-i)!}\frac{1}{l!(i-l)!}([k]_q)^{m-l}(-1)^{i-l}\frac{(1-q)^{i-l}}{[i+1]_q}\\
=&\frac{q^k}{([n+1]_q)^{m+1}}\sum_{l=0}^{m}\sum_{s=0}^{m-l}\frac{m!}{(m-l-s)!}\frac{1}{l!s!}([k]_q)^{m-l}(-1)^{s}\frac{(1-q)^{s}}{[l+s+1]_q}\\
=&\frac{q^k}{([n+1]_q)^{m+1}}\sum_{l=0}^{m}\sum_{s=0}^{m-l}\frac{m!}{l!(m-l)!}\frac{(m-l)!}{s!(m-l-s)!}([k]_q)^{m-l}\\
&\times(-1)^{s}\frac{(1-q)^{s}}{[l+s+1]_q}\\
=&\frac{q^k}{([n+1]_q)^{m+1}}\sum_{l=0}^{m}\sum_{s=0}^{m-l}\binom{m}{l}\binom{m-l}{s}([k]_q)^{m-l}(-1)^{s}\frac{(1-q)^{s}}{[l+s+1]_q}\\
=&\frac{q^k}{([n+1]_q)}\left(\frac{[n]_q}{[n+1]_q}\right)^m\sum_{l=0}^{m}\sum_{s=0}^{m-l}\binom{m}{l}\binom{m-l}{s}\left(\frac{[k]_q}{[n]_q}\right)^{m-l}\frac{1}{([n]_q)^l}\\
&\times(-1)^{s}\frac{(1-q)^{s}}{[l+s+1]_q}\\
=&\frac{q^k}{([n+1]_q)}\left(\frac{[n]_q}{[n+1]_q}\right)^m\sum_{l=0}^{m}\binom{m}{l}\left(\frac{[k]_q}{[n]_q}\right)^{m-l}\frac{1}{([n]_q)^l}\sum_{s=0}^{m-l}\binom{m-l}{s}\\
&\times(-1)^{s}\frac{(1-q)^{s}}{[l+s+1]_q}.
\end{aligned}
$$

Thus, if we take

$$
\frac{1}{([n]_q)^l}\sum_{s=0}^{m-l}\binom{m-l}{s}(-1)^s\frac{(1-q)^s}{[l+s+1]_q}=C_{m,l}(q,n),
$$

then the proof is completed. □

In the light of Lemma 2.1, we can state the following lemma.

Lemma 2.2. *Let m be a nonnegative integer. Then for the operator $B_n^{\alpha}(f;q;x)$ defined by* (1.7)*, we have*

$$B_n^{\alpha}(t^m;q;x)=\left(\frac{[n]_q}{[n+1]_q}\right)^m\sum_{l=0}^{m}\binom{m}{l}C_{m,l}(q,n)B_n^{q,\alpha}(t^{m-l};x),$$

where $B_n^{q,\alpha}$ is given by (1.1) and $C_{m,l}(q,n)$ is defined as in Lemma 2.1.

Proof. Indeed, by using Lemma 2.1 we can write

$$\begin{aligned}B_n^{\alpha}(t^m;q;x)&=\sum_{k=0}^{n}P_{n,k}^{q,\alpha}(x)\frac{[n+1]_q}{q^k}I_{n,k}(t^m)\\&=\left(\frac{[n]_q}{[n+1]_q}\right)^m\sum_{k=0}^{n}P_{n,k}^{q,\alpha}(x)\sum_{l=0}^{m}\binom{m}{l}\left(\frac{[k]_q}{[n]_q}\right)^{m-l}C_{m,l}(q,n)\\&=\left(\frac{[n]_q}{[n+1]_q}\right)^m\sum_{l=0}^{m}\binom{m}{l}C_{m,l}(q,n)\sum_{k=0}^{n}P_{n,k}^{q,\alpha}(x)\left(\frac{[k]_q}{[n]_q}\right)^{m-l}\\&=\left(\frac{[n]_q}{[n+1]_q}\right)^m\sum_{l=0}^{m}\binom{m}{l}C_{m,l}(q,n)B_n^{q,\alpha}(t^{m-l};x).\end{aligned}$$

□

Corollary 2.3. *The operator $B_n^{\alpha}(f;q;x)$ defined by* (1.7) *satisfies*

$$B_n^{\alpha}(1;q;x)=1 \tag{2.1}$$

$$B_n^{\alpha}(t;q;x)=\frac{2q}{1+q}\frac{[n]_q}{[n+1]_q}x+\frac{1}{1+q}\frac{1}{[n+1]_q} \tag{2.2}$$

$$\begin{aligned}&B_n^{\alpha}(t^2;q;x)\\&=\frac{1}{1+\alpha}\frac{4q^4+q^3+q^2}{(1+q)(1+q+q^2)}\frac{[n]_q[n-1]_q}{[n+1]_q^2}x^2\\&\quad+\left\{\frac{\alpha}{1+\alpha}\frac{4q^3+q^2+q}{(1+q)(1+q+q^2)}\frac{[n]_q^2}{[n+1]_q^2}+\left(\frac{1}{1+\alpha}\frac{4q^3+q^2+q}{(1+q)(1+q+q^2)}\right.\right.\\&\quad\left.\left.+\frac{4q^2+2q}{(1+q)(1+q+q^2)}\right)\frac{[n]_q}{[n+1]_q^2}\right\}x+\frac{1}{1+q+q^2}\frac{1}{[n+1]_q^2}.\end{aligned} \tag{2.3}$$

With the help of Lemma 2.2 and identities (1.3)- (1.5) it can be easily proved. So we omit it.

Lemma 2.4. *For the operator $B_n^\alpha(f;q;x)$ defined by (1.7), we have*

$$B_n^\alpha((t-x)^2;q;x) \le \left(\frac{1}{1+\alpha}\frac{4q^2}{(1+q)^2}\frac{[n]_q^2}{[n+1]_q^2} - \frac{4q}{1+q}\frac{[n]_q}{[n+1]_q} + 1\right)x^2 + \left(\frac{\alpha}{1+\alpha}\frac{[n]_q^2}{[n+1]_q^2} + \frac{1}{1+\alpha}\frac{1}{[n+1]_q}\right)x + \frac{1}{1+q+q^2}\frac{1}{[n+1]_q^2}. \tag{2.4}$$

Proof. From the linearity of B_n^α and the equalities (2.1)- (2.3), we may write

$$\begin{aligned} &B_n^\alpha((t-x)^2;q;x) \\ &= \left\{\frac{1}{1+\alpha}\frac{4q^4+q^3+q^2}{(1+q)(1+q+q^2)}\frac{[n]_q[n-1]_q}{[n+1]_q^2} - \frac{4q}{1+q}\frac{[n]_q}{[n+1]_q} + 1\right\}x^2 \\ &+ \left\{\frac{\alpha}{1+\alpha}\frac{4q^3+q^2+q}{(1+q)(1+q+q^2)}\frac{[n]_q^2}{[n+1]_q^2}\right. \\ &+ \left(\frac{1}{1+\alpha}\frac{4q^3+q^2+q}{(1+q)(1+q+q^2)} + \frac{4q^2+2q}{(1+q)(1+q+q^2)}\right)\frac{[n]_q}{[n+1]_q^2} \\ &\left.- \frac{2}{1+q}\frac{1}{[n+1]_q}\right\}x + \frac{1}{1+q+q^2}\frac{1}{[n+1]^2}. \end{aligned} \tag{2.5}$$

In [4], for $0<q<1$ and $n\in\mathbb{N}$ it was showed that

$$\frac{q^2}{1+q} + \frac{3q^4}{(1+q)(1+q+q^2)} = \frac{4q^4+q^3+q^2}{(1+q)(1+q+q^2)} < \frac{4q^2}{(1+q)^2}.$$

Since $[n-1]_q < [n]_q$ this leads to

$$\left(\frac{4q^4+q^3+q^2}{(1+q)(1+q+q^2)}\right)\frac{[n]_q[n-1]_q}{[n+1]_q^2} < \frac{4q^2}{(1+q)^2}\frac{[n]_q^2}{[n+1]_q^2}. \tag{2.6}$$

On the other hand, for $0<q<1$ we have

$$\frac{4q^3+q^2+q}{(1+q)(1+q+q^2)} - 1 = \frac{(3q^2+2q+1)(q-1)}{(1+q)(1+q+q^2)} < 0$$

which gives

$$\frac{4q^3+q^2+q}{(1+q)(1+q+q^2)} < 1. \tag{2.7}$$

Hence using (2.6), (2.7) and the inequality $\frac{[n]_q}{[n+1]_q^2} < \frac{1}{[n+1]_q}$ into (2.5), one gets

$$\begin{aligned}
&B_n^\alpha((t-x)^2;q;x) \\
&\leq \left(\frac{1}{1+\alpha}\frac{4q^2}{(1+q)^2}\frac{[n]_q^2}{[n+1]_q^2} - \frac{4q}{1+q}\frac{[n]_q}{[n+1]_q} + 1\right)x^2 \\
&\quad + \left\{\frac{\alpha}{1+\alpha}\frac{[n]_q^2}{[n+1]_q^2} + \left(\frac{1}{1+\alpha} + \frac{4q^2+2q}{(1+q)(1+q+q^2)} - \frac{2}{1+q}\right)\frac{1}{[n+1]_q}\right\}x \\
&\quad + \frac{1}{1+q+q^2}\frac{1}{[n+1]^2}.
\end{aligned}$$

Finally, for $0 < q < 1$ by means of the fact

$$\frac{4q^2+2q}{(1+q)(1+q+q^2)} - \frac{2}{1+q} = \frac{2(q^2-1)}{(1+q)(1+q+q^2)} < 0$$

we get

$$\begin{aligned}
B_n^\alpha((t-x)^2;q;x) \leq &\left(\frac{1}{1+\alpha}\frac{4q^2}{(1+q)^2}\frac{[n]_q^2}{[n+1]_q^2} - \frac{4q}{1+q}\frac{[n]_q}{[n+1]_q} + 1\right)x^2 \\
&+ \left(\frac{\alpha}{1+\alpha}\frac{[n]_q^2}{[n+1]_q^2} + \frac{1}{1+\alpha}\frac{1}{[n+1]_q}\right)x \\
&+ \frac{1}{1+q+q^2}\frac{1}{[n+1]_q^2}
\end{aligned}$$

which is the required result. □

3. Main results

In this part, we study some approximation properties of the operator $B_n^\alpha(f;q;x)$ defined by (1.7).

Theorem 3.1. *Let $q = q_n \in (0,1)$ and $\alpha = \alpha_n \geq 0$ such that $\lim_{n\to\infty} q_n = 1$ and $\lim_{n\to\infty} \alpha_n = 0$. Then for each $f \in C[0,1]$, $B_n^{\alpha_n}(f;q_n;x)$ converges uniformly to f on $[0,1]$.*

Proof. By the Bohman-Korovkin Theorem it is sufficient to show that

$$\lim_{n\to\infty} ||B_n^{\alpha_n}(t^m;q_n;x) - x^m||_{C[0,1]} = 0, \quad m = 0,1,2.$$

By (2.1), it is clear that

$$\lim_{n\to\infty} ||B_n^{\alpha_n}(1;q_n;x) - 1||_{C[0,1]} = 0$$

Since $B_n^{\alpha_n}(t;q_n;x) = B_n^*(t;q_n;x)$, where B_n^* is defined by (1.6), from the formula (22) in [4] we have

$$||B_n^{\alpha_n}(t;q_n;x) - x||_{C[0,1]} \leq \frac{1-q_n}{1+q_n} + \frac{3}{1+q_n}\frac{1}{[n+1]_{q_n}}$$

which implies that

$$\lim_{n\to\infty} ||B_n^{\alpha_n}(t;q_n;x) - x||_{C[0,1]} = 0.$$

Now using (2.3), (2.7) and the inequality $\frac{[n]_{q_n}}{[n+1]_{q_n}^2} < \frac{1}{[n+1]_{q_n}}$ we get

$$\begin{aligned}
&\left|B_n^{\alpha_n}(t^2;q_n;x) - x^2\right| \\
\leq &\left|\frac{1}{1+\alpha_n}\frac{4q_n^4+q_n^3+q_n^2}{(1+q_n)(1+q_n+q_n^2)}\frac{[n]_{q_n}[n-1]_{q_n}}{[n+1]_{q_n}^2} - 1\right| x^2 \\
&+ \left\{\frac{\alpha_n}{1+\alpha_n}\frac{4q_n^3+q_n^2+q_n}{(1+q_n)(1+q_n+q_n^2)}\frac{[n]_{q_n}^2}{[n+1]_{q_n}^2}\right. \\
&\left.+\left(\frac{1}{1+\alpha_n}\frac{4q_n^3+q_n^2+q_n}{(1+q_n)(1+q_n+q_n^2)} + \frac{4q_n^2+2q_n}{(1+q_n)(1+q_n+q_n^2)}\right)\frac{[n]_{q_n}}{[n+1]_{q_n}^2}\right\}x \\
&+ \frac{1}{1+q_n+q_n^2}\frac{1}{[n+1]_{q_n}^2} \\
\leq &\left|\frac{1}{1+\alpha_n}\frac{4q_n^4+q_n^3+q_n^2}{(1+q_n)(1+q_n+q_n^2)}\frac{[n]_{q_n}[n-1]_{q_n}}{[n+1]_{q_n}^2} - 1\right| x^2 \\
&+ \left\{\frac{\alpha_n}{1+\alpha_n}\frac{[n]_{q_n}^2}{[n+1]_{q_n}^2} + \left(\frac{1}{1+\alpha_n} + \frac{4q_n^2+2q_n}{(1+q_n)(1+q_n+q_n^2)}\right)\frac{1}{[n+1]_{q_n}}\right\}x \\
&+ \frac{1}{1+q_n+q_n^2}\frac{1}{[n+1]_{q_n}^2}.
\end{aligned} \tag{3.1}$$

Since (see [4]),

$$\frac{[n]_{q_n}[n-1]_{q_n}}{[n+1]_{q_n}^2} = \frac{1}{q_n^3}\left(1 - \frac{2+q_n}{[n+1]_{q_n}} + \frac{1+q_n}{[n+1]_{q_n}^2}\right)$$

the inequality (3.1) takes the form

$$\begin{aligned}
&\left|B_n^{\alpha_n}(t^2;q_n;x) - x^2\right| \\
\leq &\left\{\left|\frac{1}{1+\alpha_n}\frac{4q_n^2+q_n+1}{q_n(1+q_n)(1+q_n+q_n^2)} - 1\right|\right. \\
&\left.+ \frac{1}{1+\alpha_n}\frac{4q_n^2+q_n+1}{q_n(1+q_n)(1+q_n+q_n^2)}\left|\frac{1+q_n}{[n+1]_{q_n}^2} - \frac{2+q_n}{[n+1]_{q_n}}\right|\right\}x^2 \\
&+ \left\{\frac{\alpha_n}{1+\alpha_n}\frac{[n]_{q_n}^2}{[n+1]_{q_n}^2} + \left(\frac{1}{1+\alpha_n} + \frac{4q_n^2+2q_n}{(1+q_n)(1+q_n+q_n^2)}\right)\frac{1}{[n+1]_{q_n}}\right\}x \\
&+ \frac{1}{1+q_n+q_n^2}\frac{1}{[n+1]_{q_n}^2}.
\end{aligned} \tag{3.2}$$

Taking maximum of both sides of (3.2) on $[0,1]$, we find

$$\begin{aligned}
&\left\| B_n^{\alpha_n}(t^2;q_n;x)-x^2 \right\|_{C[0,1]} \\
\leq &\left| \frac{1}{1+\alpha_n} \frac{4q_n^2+q_n+1}{q_n(1+q_n)(1+q_n+q_n^2)} - 1 \right| \\
&+ \frac{1}{1+\alpha_n} \frac{4q_n^2+q_n+1}{q_n(1+q_n)(1+q_n+q_n^2)} \left| \frac{1+q_n}{[n+1]_{q_n}^2} - \frac{2+q_n}{[n+1]_{q_n}} \right| \\
&+ \frac{\alpha_n}{1+\alpha_n} \frac{[n]_{q_n}^2}{[n+1]_{q_n}^2} + \left(\frac{1}{1+\alpha_n} + \frac{4q_n^2+2q_n}{(1+q_n)(1+q_n+q_n^2)} \right) \frac{1}{[n+1]_{q_n}} \\
&+ \frac{1}{1+q_n+q_n^2} \frac{1}{[n+1]_{q_n}^2}
\end{aligned}$$

which yields

$$\lim_{n\to\infty} ||B_n^{\alpha_n}(t^2;q_n;x)-x^2||_{C[0,1]} = 0.$$

Thus the proof is completed. □

Remark 3.2. If we choose $q_n = \frac{n}{n+1}$, it is easily seen that $\lim_{n\to\infty} q_n = 1$ and $\lim_{n\to\infty} q_n^n = e^{-1}$. Hence we guarantee that $\lim_{n\to\infty} [n]_{q_n} = \infty$. Since $[n+1]_{q_n} = q_n[n]_{q_n} + 1$ and $\frac{[n]_{q_n}}{[n+1]_{q_n}} = \frac{1}{q_n + \frac{1}{[n]_{q_n}}}$ we have $\lim_{n\to\infty} \dfrac{1}{[n+1]_{q_n}} = 0$ and $\lim_{n\to\infty} \dfrac{[n]_{q_n}}{[n+1]_{q_n}} = 1$.

For $q \in (0,1)$ it is obvious that $\lim_{n\to\infty} [n]_q = \dfrac{1}{1-q}$. In order to reach to convergence results of the operator $B_n^{\alpha_n}$ we take a sequence $q_n \in (0,1)$ such that $\lim_{n\to\infty} q_n = 1$. So we get that $\lim_{n\to\infty} [n]_{q_n} = \infty$.
By the above explanation, Remark 3.2 provides an example that such a sequence can always be found.

Next, we compute the approximation order of the operator $B_n^{\alpha}(f;q;x)$ in terms of the elements of the usual Lipschitz class.

Let $f \in C[0,1]$, $M > 0$ and $0 < \beta \leq 1$. We recall that f belongs to the class $Lip_M(\beta)$ if the inequality

$$|f(t) - f(x)| \leq M \left| t-x \right|^{\beta}; x,t \in [0,1]$$

holds.

Theorem 3.3. *Let $q = q_n \in (0,1)$ and $\alpha = \alpha_n \geq 0$ such that $\lim_{n\to\infty} q_n = 1$ and $\lim_{n\to\infty} \alpha_n = 0$. Then for each $f \in Lip_M(\beta)$ we have*

$$||B_n^{\alpha_n}(f;q_n;x) - f(x)||_{C[0,1]} \leq M\delta_n^{\beta}$$

where

$$\delta_n = \Bigg\{ \left(\frac{1}{1+\alpha_n} \frac{4q_n^2}{(1+q_n)^2} + \frac{\alpha_n}{1+\alpha_n} \right) \frac{[n]_{q_n}^2}{[n+1]_{q_n}^2} - \frac{4q_n}{1+q_n} \frac{[n]_{q_n}}{[n+1]_{q_n}} + \frac{1}{1+\alpha_n} \frac{1}{[n+1]_{q_n}} + \frac{1}{1+q_n+q_n^2} \frac{1}{[n+1]_{q_n}^2} + 1 \Bigg\}^{\frac{1}{2}}.$$

Proof. By the monotonicity of $B_n^{\alpha_n}$, we can write

$$|B_n^{\alpha_n}(f;q_n;x) - f(x)| \le B_n^{\alpha_n}(|f(t)-f(x)|;q_n;x) \le M \sum_{k=0}^{n} P_{n,k}^{q_n,\alpha_n}(x) \frac{[n+1]_{q_n}}{q_n^k} \int_{\frac{[k]_{q_n}}{[n+1]_{q_n}}}^{\frac{[k+1]_{q_n}}{[n+1]_{q_n}}} |t-x|^{\beta} \, d_{q_n}^R t.$$

On the other hand, by using the Hölder inequality for the Riemann type q-integral with $m_1 = \frac{2}{\beta}$ and $m_2 = \frac{2}{2-\beta}$, we have

$$|B_n^{\alpha_n}(f;q_n;x) - f(x)| \le M \sum_{k=0}^{n} P_{n,k}^{q_n,\alpha_n}(x) \left(\frac{[n+1]_{q_n}}{q_n^k} \int_{\frac{[k]_{q_n}}{[n+1]_{q_n}}}^{\frac{[k+1]_{q_n}}{[n+1]_{q_n}}} (t-x)^2 d_{q_n}^R t \right)^{\frac{\beta}{2}}.$$

Now applying the Hölder inequality for the sum with $p_1 = \frac{2}{\beta}$ and $p_2 = \frac{2}{2-\beta}$ and taking into consideration (1.3) and (2.4), one may write

$$\begin{aligned} |B_n^{\alpha_n}(f;q_n;x) - f(x)| &\le M \left(\sum_{k=0}^{n} P_{n,k}^{q_n,\alpha_n}(x) \frac{[n+1]_{q_n}}{q_n^k} \int_{\frac{[k]_{q_n}}{[n+1]_{q_n}}}^{\frac{[k+1]_{q_n}}{[n+1]_{q_n}}} (t-x)^2 d_{q_n}^R t \right)^{\frac{\beta}{2}} \left(\sum_{k=0}^{n} P_{n,k}^{q_n,\alpha_n}(x) \right)^{\frac{2-\beta}{2}} \\ &= M \left(B_n^{\alpha_n}((t-x)^2;q_n;x) \right)^{\frac{\beta}{2}} \left(B_n^{q_n,\alpha_n}(1;x) \right)^{\frac{2-\beta}{2}} \\ &\le M \Bigg\{ \left(\frac{1}{1+\alpha_n} \frac{4q_n^2}{(1+q_n)^2} \frac{[n]_{q_n}^2}{[n+1]_{q_n}^2} - \frac{4q_n}{1+q_n} \frac{[n]_{q_n}}{[n+1]_{q_n}} + 1 \right) x^2 \\ &\quad + \left(\frac{\alpha_n}{1+\alpha_n} \frac{[n]_{q_n}^2}{[n+1]_{q_n}^2} + \frac{1}{1+\alpha_n} \frac{1}{[n+1]_{q_n}} \right) x + \frac{1}{1+q_n+q_n^2} \frac{1}{[n+1]_{q_n}^2} \Bigg\}^{\frac{\beta}{2}}. \end{aligned}$$

This implies that

$$\begin{aligned} \|B_n^{\alpha_n}(f;q_n;x) - f(x)\|_{C[0,1]} \le M \Bigg\{ & \left(\frac{1}{1+\alpha_n} \frac{4q_n^2}{(1+q_n)^2} + \frac{\alpha_n}{1+\alpha_n} \right) \frac{[n]_{q_n}^2}{[n+1]_{q_n}^2} \\ & - \frac{4q_n}{1+q_n} \frac{[n]_{q_n}}{[n+1]_{q_n}} + \frac{1}{1+\alpha_n} \frac{1}{[n+1]_{q_n}} \\ & + \frac{1}{1+q_n+q_n^2} \frac{1}{[n+1]_{q_n}^2} + 1 \Bigg\}^{\frac{\beta}{2}}. \end{aligned}$$

Hence if we choose $\delta := \delta_n$, then we arrive at the desired result. $\square$

Finally, we establish a local approximation theorem for the operator $B_n^\alpha(f;q;x)$ defined by (1.7).

Let $W^2 = \{g \in C[0,1] : g', g'' \in C[0,1]\}$. For any $\delta > 0$, Peetre's K-functional is defined by

$$K_2(f;\delta) = \inf_{g\in W^2} \{\|f-g\| + \delta \|g''\|\}$$

where $\|.\|$ is the uniform norm on $C[0,1]$ (see [14]). From ([5], p.177, Theorem 2.4) there exists an absolute constant $C > 0$ such that

$$K_2(f;\delta) \le C\omega_2(f;\sqrt{\delta}), \tag{3.3}$$

where the second order modulus of smoothness of $f \in C[0,1]$ is denoted by

$$\omega_2(f;\sqrt{\delta}) = \sup_{0<h\le\sqrt{\delta}} \sup_{x,x+2h\in[0,1]} |f(x+2h) - 2f(x+h) + f(x)|.$$

The usual modulus of continuity of $f \in C[0,1]$ is defined by

$$\omega(f;\sqrt{\delta}) = \sup_{0<h\le\sqrt{\delta}} \sup_{x,x+h\in[0,1]} |f(x+h) - f(x)|.$$

Now consider the following operator

$$L_n(f;q;x) = B_n^\alpha(f;q;x) - f\left(\frac{2q}{1+q}\frac{[n]_q}{[n+1]_q}x + \frac{1}{1+q}\frac{1}{[n+1]_q}\right) + f(x) \tag{3.4}$$

for $f \in C[0,1]$.

Lemma 3.4. *Let $g \in W^2$. Then we have*

$$\begin{aligned}|L_n(g;q;x) - g(x)| \le &\left\{ \left(\frac{2+\alpha}{1+\alpha}\frac{4q^2}{(1+q)^2}\frac{[n]_q^2}{[n+1]_q^2} - \frac{8q}{1+q}\frac{[n]_q}{[n+1]_q} + 2\right)x^2 \right.\\ &+ \left(\frac{\alpha}{1+\alpha}\frac{[n]_q^2}{[n+1]_q^2} + \frac{1}{1+\alpha}\frac{1}{[n+1]_q}\right)x \\ &\left.+ \frac{2q^2+3q+2}{(1+q+q^2)(1+q)^2}\frac{1}{[n+1]_q^2}\right]\right\} \|g''\|.\end{aligned} \tag{3.5}$$

Proof. From (3.4), (2.1) and (2.2) it is immediately seen that

$$\begin{aligned}L_n(t-x;q;x) =& B_n^\alpha(t-x;q;x) - \left(\frac{2q}{1+q}\frac{[n]_q}{[n+1]_q}x + \frac{1}{1+q}\frac{1}{[n+1]_q} - x\right)\\ =& B_n^\alpha(t;q;x) - xB_n^\alpha(1;q;x)\\ &- \left(\frac{2q}{1+q}\frac{[n]_q}{[n+1]_q}x + \frac{1}{1+q}\frac{1}{[n+1]_q} - x\right)\\ =&0.\end{aligned} \tag{3.6}$$

For $x \in [0,1]$ and $g \in W^2$, using the Taylor formula

$$g(t) - g(x) = (t-x)g'(x) + \int_x^t (t-u)g''(u)du$$

and (3.6) we have

$$\begin{aligned}
&L_n(g;q;x)-g(x)\\
=&g'(x)L_n(t-x;q;x)+L_n\left(\int_x^t (t-u)g''(u)du;q;x\right)\\
=&L_n\left(\int_x^t (t-u)g''(u)du;q;x\right)\\
=&B_n^\alpha\left(\int_x^t (t-u)g''(u)du;q;x\right)\\
&-\int_x^{\frac{2q}{1+q}\frac{[n]_q}{[n+1]_q}x+\frac{1}{1+q}\frac{1}{[n+1]_q}}\left(\frac{2q}{1+q}\frac{[n]_q}{[n+1]_q}x+\frac{1}{1+q}\frac{1}{[n+1]_q}-u\right)g''(u)du.
\end{aligned}$$

By means of the monotonicity of B_n^α this gives

$$\begin{aligned}
&|L_n(g;q;x)-g(x)|\\
\leq&B_n^\alpha\left(\left|\int_x^t (t-u)g''(u)du\right|;q;x\right)\\
+&\left|\int_x^{\frac{2q}{1+q}\frac{[n]_q}{[n+1]_q}x+\frac{1}{1+q}\frac{1}{[n+1]_q}}\left(\frac{2q}{1+q}\frac{[n]_q}{[n+1]_q}x+\frac{1}{1+q}\frac{1}{[n+1]_q}-u\right)g''(u)du\right|.
\end{aligned}\tag{3.7}$$

On the other hand, it is clear that

$$\left|\int_x^t (t-u)g''(u)du\right|\leq (t-x)^2\left\|g''\right\|.\tag{3.8}$$

Now let

$$I:=\int_x^{\frac{2q}{1+q}\frac{[n]_q}{[n+1]_q}x+\frac{1}{1+q}\frac{1}{[n+1]_q}}\left(\frac{2q}{1+q}\frac{[n]_q}{[n+1]_q}x+\frac{1}{1+q}\frac{1}{[n+1]_q}-u\right)g''(u)du.$$

Then we may write

$$\begin{aligned}
I\leq&\left[\left(\frac{2q}{1+q}\frac{[n]_q}{[n+1]_q}-1\right)x+\frac{1}{1+q}\frac{1}{[n+1]_q}\right]^2\left\|g''\right\|\\
=&\left\{\left(\frac{2q}{1+q}\frac{[n]_q}{[n+1]_q}-1\right)^2x^2+\left(\frac{4q}{(1+q)^2}\frac{[n]_q}{[n+1]_q^2}-\frac{2}{1+q}\frac{1}{[n+1]_q}\right)x\right.\\
&\left.+\frac{1}{(1+q)^2}\frac{1}{[n+1]_q^2}\right\}\left\|g''\right\|.
\end{aligned}$$

Use of the facts $\frac{[n]_q}{[n+1]_q^2} < \frac{1}{[n+1]_q}$ and for $0 < q < 1$, $\frac{4q}{(1+q)^2} - \frac{2}{1+q} = \frac{2(q-1)}{(1+q)^2} < 0$ yields

$$\begin{aligned} I \leq & \left\{ \left(\frac{2q}{1+q} \frac{[n]_q}{[n+1]_q} - 1 \right)^2 x^2 + \left(\frac{4q}{(1+q)^2} - \frac{2}{1+q} \right) \frac{1}{[n+1]_q} x \right. \\ & \left. + \frac{1}{(1+q)^2} \frac{1}{[n+1]_q^2} \right\} \|g''\| \\ \leq & \left\{ \left(\frac{2q}{1+q} \frac{[n]_q}{[n+1]_q} - 1 \right)^2 x^2 + \frac{1}{(1+q)^2} \frac{1}{[n+1]_q^2} \right\} \|g''\| \\ = & \left\{ \left(\frac{4q^2}{(1+q)^2} \frac{[n]_q^2}{[n+1]_q^2} - \frac{4q}{1+q} \frac{[n]_q}{[n+1]_q} + 1 \right) x^2 + \frac{1}{(1+q)^2} \frac{1}{[n+1]_q^2} \right\} \|g''\| . \end{aligned} \tag{3.9}$$

Substituting (3.8) and (3.9) into (3.7), we have

$$\begin{aligned} & |L_n(g;q;x) - g(x)| \\ \leq & \left\{ B_n^\alpha \left((t-x)^2; q; x \right) + \left(\frac{4q^2}{(1+q)^2} \frac{[n]_q^2}{[n+1]_q^2} - \frac{4q}{1+q} \frac{[n]_q}{[n+1]_q} + 1 \right) x^2 \right. \\ & \left. + \frac{1}{(1+q)^2} \frac{1}{[n+1]_q^2} \right\} \|g''\| . \end{aligned} \tag{3.10}$$

Using (2.4), from (3.10) it follows that

$$\begin{aligned} |L_n(g;q;x) - g(x)| \leq & \left\{ \left(\frac{2+\alpha}{1+\alpha} \frac{4q^2}{(1+q)^2} \frac{[n]_q^2}{[n+1]_q^2} - \frac{8q}{1+q} \frac{[n]_q}{[n+1]_q} + 2 \right) x^2 \right. \\ & + \left(\frac{\alpha}{1+\alpha} \frac{[n]_q^2}{[n+1]_q^2} + \frac{1}{1+\alpha} \frac{1}{[n+1]_q} \right) x \\ & \left. + \frac{2q^2+3q+2}{(1+q+q^2)(1+q)^2} \frac{1}{[n+1]_q^2} \Bigg] \right\} \|g''\| . \end{aligned}$$

This completes the proof. □

Theorem 3.5. *Let $f \in C[0,1]$. Then for each $x \in [0,1]$ we have*

$$\begin{aligned} |B_n^\alpha(f;q;x) - f(x)| \leq & C\omega_2(f; \sqrt{\delta_n(x)}) \\ & + \omega \left(f; \left| \left(\frac{2q}{1+q} \frac{[n]_q}{[n+1]_q} - 1 \right) x + \frac{1}{1+q} \frac{1}{[n+1]_q} \right| \right), \end{aligned}$$

where

$$\delta_n(x) = \left(\frac{2+\alpha}{1+\alpha}\frac{4q^2}{(1+q)^2}\frac{[n]_q^2}{[n+1]_q^2} - \frac{8q}{1+q}\frac{[n]_q}{[n+1]_q} + 2\right)x^2 + \left(\frac{\alpha}{1+\alpha}\frac{[n]_q^2}{[n+1]_q^2} + \frac{1}{1+\alpha}\frac{1}{[n+1]_q}\right)x + \frac{2+3q+2q^2}{(1+q+q^2)(1+q)^2}\frac{1}{[n+1]_q^2}$$

and C is a positive constant.

Proof. From (3.4), we have

$$|L_n(f;q;x)| \le |B_n^\alpha(f;q;x)| + 2\|f\| \le \|f\| B_n^\alpha(1;q;x) + 2\|f\| = 3\|f\| \tag{3.11}$$

and

$$B_n^\alpha(f;q;x) - f(x) = L_n(f-g;q;x) - (f-g)(x) + L_n(g;q;x) - g(x) + f\left(\frac{2q}{1+q}\frac{[n]_q}{[n+1]_q}x + \frac{1}{1+q}\frac{1}{[n+1]_q}\right) - f(x).$$

In the light of (3.5) and (3.11), this equality implies that

$$\begin{aligned}
&|B_n^\alpha(f;q;x) - f(x)| \\
\le &|L_n(f-g;q;x)| + |(f-g)(x)| + |L_n(g;q;x) - g(x)| \\
&+ \left|f\left(\frac{2q}{1+q}\frac{[n]_q}{[n+1]_q}x + \frac{1}{1+q}\frac{1}{[n+1]_q}\right) - f(x)\right| \\
\le &4\|f-g\| + |L_n(g;q;x) - g(x)| \\
&+ \omega\left(f; \left|\left(\frac{2q}{1+q}\frac{[n]_q}{[n+1]_q} - 1\right)x + \frac{1}{1+q}\frac{1}{[n+1]_q}\right|\right) \\
\le &4\|f-g\| + \left\{\left(\frac{2+\alpha}{1+\alpha}\frac{4q^2}{(1+q)^2}\frac{[n]_q^2}{[n+1]_q^2} - \frac{8q}{1+q}\frac{[n]_q}{[n+1]_q} + 2\right)x^2\right. \\
&+ \left(\frac{\alpha}{1+\alpha}\frac{[n]_q^2}{[n+1]_q^2} + \frac{1}{1+\alpha}\frac{1}{[n+1]_q}\right)x \\
&\left.+ \frac{2q^2+3q+2}{(1+q+q^2)(1+q)^2}\frac{1}{[n+1]_q^2}\right]\Bigg\}\|g''\| \\
&+ \omega\left(f; \left|\left(\frac{2q}{1+q}\frac{[n]_q}{[n+1]_q} - 1\right)x + \frac{1}{1+q}\frac{1}{[n+1]_q}\right|\right) \\
= &4\|f-g\| + \delta_n(x)\|g''\| + \omega\left(f; \left|\left(\frac{2q}{1+q}\frac{[n]_q}{[n+1]_q} - 1\right)x + \frac{1}{1+q}\frac{1}{[n+1]_q}\right|\right).
\end{aligned}$$

Hence taking infimum on the right-hand side over all $g \in W^2$ and considering (3.3), we get

$$\begin{aligned}
&|B_n^{\alpha}(f;q;x) - f(x)| \\
\leq & 4K_n(f;\delta_n(x)) + \omega\left(f; \left|\left(\frac{2q}{1+q}\frac{[n]_q}{[n+1]_q} - 1\right)x + \frac{1}{1+q}\frac{1}{[n+1]_q}\right|\right) \\
\leq & C\omega_2(f;\sqrt{\delta_n(x)}) + \omega\left(f; \left|\left(\frac{2q}{1+q}\frac{[n]_q}{[n+1]_q} - 1\right)x + \frac{1}{1+q}\frac{1}{[n+1]_q}\right|\right)
\end{aligned}$$

which is the desired result. □

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Note on a property of the Banach spaces

Nuno C. Freire and Maria Fernanda Veiga

Abstract. We show that we may consider a partial ordering $\leq$ in an infinite dimensional Banach space $(X, \|.\|)$, which we obtain through any normed Hamel base of the space, such that $(X, \|.\|, \leq)$ is a Banach lattice.

Mathematics Subject Classification (2010): 46B20, 46B30.

Keywords: Order, norm, lattice.

1. Introduction

Why trying to see, concerning a Banach space X, whether there exists or not a partial ordering in X that is compatible with the topology? The particular geometric properties of Banach lattices and, the contrast concerning the continuity properties of the coodinate linear functionals associated either to a Schauder basis or to a Hamel base in a Banach space ([2], Chapter 4 and [3]), we decided to consider these matters altogether. We prove in Theorem 3.1 that $(X, \|.\|)$ being an infinite dimensional real Banach space and the normed vectors x_α $(\alpha \in \mathcal{A})$ detemining a Hamel base $\mathcal{H}$ of X, we may consider a partial order $\leq_\mathcal{H}$ in X such that the triple $(X, \|.\|_\mathcal{H}, \leq_\mathcal{H})$ is a Banach lattice where $\|.\|_\mathcal{H}$ is an equivalent norm to $\|.\|$ in X. In the Preliminaries, paragraph 2., we briefly set the notations. We consider real Banach spaces X and we say that a linear isomorphism which is a homeomorphism between two topological vector spaces is a linear homeomorphism ([4], II.1, p. 53 in a definition). Also in [4], we can find the algebraic Hamel base of a vector space X not reducing to $\{0\}$ namely (p. 42), $\mathcal{H} = \{x_\alpha : \alpha \in \mathcal{A}\}$ is Hamel base of X if $\mathcal{H}$ is an infinite linearly independent set which spans X, as we consider in paragraph 2.

2. Preliminaries

In what follows we consider a real Banach space $(X, \|.\|)$. Recall that $(X, \leq)$ is a Riesz space through a partial order $\leq$ in X if and only if $\leq$ is compatible with the linear stucture that is, $x + z \leq y + z$ whenever $x \leq y$, $x, y, z \in X$, we have that $\alpha x \geq 0$ for each $x \geq 0$, $\alpha \geq 0$ where $x \in X$ and α is a scalar and, further, there exist $x \vee y =$

$\sup\{x,y\}, x \wedge y = \inf\{x,y\}$ for each $x, y \in X$. .We write $(X, \|.\|, \leq)$ meaning that $(X, \|.\|)$ is a Banach space, $(X, \leq)$ is a Riesz space and $\|x\| \leq \|y\|$ whenever $| x | \leq | y |$ so that $(X, \|.\|, \leq)$ (or just X) is a Banach lattice. Here, we put $| x | = x \vee (-x)$ We write $x^+ = x \vee 0$, $x^- = x \wedge 0$. We see easily that $x^- = (-x) \vee 0 = -(x \wedge 0)$.More generally, $x \wedge y = -((-x) \vee (-y))$. We have that $x = x^+ - x^-, | x | = x^+ + x^-$. Notice that $x \vee y = x + y - ((-x) \vee (-y)) = (x^+ - x^-) + (y^+ - y^-) - ((-x) \vee (-y)) + y - y = (x^+ + y^+ - x^- - y^-) - ((y - x) \vee 0) - y$ ([2], Theorem 1.1.1. i), ii), p. 3) hence for $\leq$ a partial order compatible with the linear structure of X, X is a Riesz space provided that x^+ exists for each x in X.

Definition 2.1. *(Following* [4]*) For $\mathcal{A}$ a nonempty set of indices, we say that the family (λ_α) in $\mathbf{R}^{\mathcal{A}}$ is summable, $\sum_{\mathcal{A}} \lambda_\alpha = s$ if it holds that $\left|\sum_{\alpha \in A} \lambda_\alpha - s\right| \leq \varepsilon$ for each finite superset A of some set $A_\varepsilon \in \mathcal{F}(\mathcal{A})$, the class of all nonempty finite subsets of $\mathcal{A}$, $\varepsilon > 0$ a priori given. The family (λ_α) is said to be absolutely summable if $(|\lambda_\alpha|)$ is a summable family.*

Notation 2.2. We let $l_{\mathcal{F}}(\mathcal{A}) = \{(\lambda_\alpha) \in \mathbf{R}^{\mathcal{A}} : \lambda_\alpha = 0$ for all $\alpha \notin A$ and some $A \in \mathcal{F}(\mathcal{A})\}$.

Notation 2.3. We write $l_1(\mathcal{A})$ for the Banach space determined by the absolutely summable families (λ_α) equipped with the norm $\|(\lambda_\alpha)\|_1 = \sum_{\mathcal{A}} |\lambda_\alpha|$.

Remark 2.4. The space $l_1(\mathcal{A})$ is a Banach lattice when equipped with the partial ordering $(\lambda_\alpha) \leq (\mu_\alpha)$ if and only if $\lambda_\alpha \leq \mu_\alpha$ $(\alpha \in \mathcal{A})$. $l_1(\mathcal{A})$ is the completion of $((l_F(\mathcal{A}).\|.\|_1)$.

Proof. This follows from [4]. The partial ordering is extended the obvious way. ☐

Letting $\{x_\alpha : \alpha \in \mathcal{A}\}$ be a normed Hamel base of X, $\|x_\alpha\| = 1$, $\alpha \in \mathcal{A}$, putting $\sum_A s_\alpha x_\alpha \prec_{\mathcal{H}} \sum_A t_\alpha x_\alpha$ if and only if $s_\alpha \leq t_\alpha$ ($\alpha \in \mathcal{A}$, the finite sms are understood)), we have that $(X, \prec_{\mathcal{H}})$ is a Riesz space. Notice that the linear operator $T(\lambda_\alpha) = \sum \lambda_\alpha x_\alpha$ on $l_{\mathcal{F}}(\mathcal{A})$ to $(X, \|.\|, \prec_{\mathcal{H}})$ is injective, continous of norm 1. We may consider the linear homeomorphism $(\widetilde{T}/K) : (l_1(\mathcal{A})/K, \|. : l_1(\mathcal{A})/K\|) \to (X, \|.\|)$, $\widetilde{T}$ for the linear extension to $l_1(\mathcal{A})$ of T, where $K = Ker(\widetilde{T})$.

3. The results

Following [1], $(X, \|.\|, \leq)$ being a Banach lattice we say that a subspace Y of X has the solid property if $x \in Y$ whenever $| x | \leq | y |$ and $y \in Y$. Y being closed, we then may consider the partial ordering $[x] \preceq [y]$ in the quotient X/Y if and only if $y - x \in P$ where $P = \cup\{\pi(x) : x \geq 0\}, \pi(x) = [x]$, π for the canonical map. Clearly that $\preceq$ is compatible with the linear structure Also $[x]^+ = [x^+], (X/Y, \preceq)$ is a Riesz space such that $[x] \vee [y] = [x \vee y], [x] \wedge [y] = [x \wedge y]$ and $[| x |] = | [x] |$ ([1], 14G, p. 13). We have that $[0] \preceq [x]$ if and only if for each $v \in [x]$ there is some $w \in [0], w \leq x$ hence also $[x] \preceq [y]$ if and only if for each $v \in [y]$, there is some $w \in [x]$ such that $w \leq v$. It follows that $| [x] | \preceq | [y] |$ imples that for each $v \in [y]$ there exists $w \in [x], | w | \leq | v |$ hence $\|[x] : X/Y\| \leq \|[y] : X/Y\|$ and $(Y/X, \preceq)$ is a Banach lattice. We see easily that $K = Ker(\widetilde{T})$ as above in the Preliminaries is a closed subspace of $l_1(A)$ having the solid property, hence $(l_1(\mathcal{A})/K, \leq)$ is a Banach lattice where we keep denoting the ordering in the quotient by the same symbol $\leq$.

Clearly that $\theta : (E, \|. : E\|, \leq_E) \to (F, \|. : F\|)$ being a linear homeomorphism between Banach spaces such that E is a Banach lattice, putting $\theta(a) \leq_\theta \theta(b)$ if and only if $a \leq_E b$ in E we obtain that $(F, \leq_\theta)$ is a Riesz space. We have that $\theta(a \vee b) = \theta(a) \vee \theta(b)$ and, more generally, θ preserves the lattice operations. Further, if we put $\|\theta(a)\|_\theta = \|a : E\|$ for $\theta(a) \in F$ we have that $(F, \|.\|_\theta)$ is a Banach space and it follows from the open mapping theorem that the norms $\|. : F\|, \|.\|_\theta$ are equivalent in F. Also for $\mid \theta(a) \mid \leq_\theta \mid \theta(b) \mid$ we find that $\mid a \mid \leq_E \mid b \mid$ hence $\|a : E\| \leq \|b : E\|$, $\|\theta(a)\|_\theta \leq \|\theta(b)\|_\theta$, we obtain that $(F, \|.\|_\theta, \leq_\theta)$ is a Banach lattice.

Denoting $\theta = \widetilde{T}/K : (l_1(\mathcal{A})/K, \|. : l_1(\mathcal{A})/K\|) \to (X, \|.\|)$ in the above sense (we have that each $x \in X$ is a unique image $\theta\left[(\lambda_\alpha(x))\right], (\lambda_\alpha(x)) \in l_1(\mathcal{A})$) we have

Theorem 3.1. *The elements* $\theta\left[(\lambda_\alpha(x))\right] = x$ *determine the Banach space* $(X, \|.\|_\theta)$ *where the norm* $\|.\|_\theta$ *is equivalent to the original norm of* X.

Proof. This follows from above. □

Corollary 3.2. *Given an infinite dimensional real Banach space* $(X, \|.\|)$ *and a normed Hamel base* $\mathcal{H} = \{x_\alpha : \alpha \in A\}$ *of* X*, there exist an equivalent norm* $\|.\|_\mathcal{H}$ *in* X *and a partial ordering* $\leq_\mathcal{H}$ *in* X *associated to* $\mathcal{H}$ *such that the triple* $(X, \|.\|_\mathcal{H}, \leq_\mathcal{H})$ *is a Banach lattice.*

Proof. This follows from above theorem where we denote $\|.\|_\mathcal{H} = \|.\|_\theta$, $\leq_\mathcal{H} = \leq_\theta$ following the above definition. □

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Weighted composition operators on weighted Lorentz-Karamata spaces

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Abstract. In this paper, a characterization of the non-singular measurable transformations T from X into itself and complex-valued measurable functions u on X inducing weighted composition operators is obtained and subsequently their compactness and closedness of the range on the weighted Lorentz-Karamata spaces $L^w_{p,q;b}(X, \Sigma, \mu)$ are completely identified where (X, Σ, μ) is a $\sigma-$finite measure space and $1 < p \leq \infty$, $1 \leq q \leq \infty$.

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Keywords: Weighted Lorentz Karamata space, weighted composition operator, Multiplication operator.

1. Introduction

A new generalization of Lebesgue, Lorentz, Zygmund, Lorentz-Zygmund and generalized Lorentz-Zygmund spaces was studied by J.S.Neves in [13]. By using the Karamata Theory, he introduced Lorentz-Karamata (simply LK) spaces and gave Bessel and Riesz potentials and emmedings of these spaces. In that paper, he studied the LK spaces $L_{p,q;b}(R, \mu)$ where $p, q \in (0, \infty]$, b is a slowly varying function on $(0, \infty)$ and (R, μ) is a measure space. These spaces give the generalized Lorentz-Zygmund spaces $L_{p,q;\alpha_1,\ldots\alpha_m}(R)$, Lorentz-Zygmund spaces $L^{p,q}(\log L)^{\alpha}(R)$, Zygmund spaces $L^p(\log L)^{\alpha}(R)$ (introduced in [3,16]), Lorentz spaces $L^{p,q}(R)$ and Lebesgue spaces $L^p(R)$ under convenient choices of slowly varying functions.

In [5,13], it is proved that LK spaces $L_{p,q;b}(R, \mu)$ endowed with a convenient norm, is a rearrangment-invariant Banach function spaces with associate spaces $L_{p',q';b^{-1}}(R, \mu)$ if (R, μ) is a resonant measure space, $p \in (1, \infty)$ and $q \in [1, \infty]$. Also it is showed that when $p \in (1, \infty)$ and $q \in [1, \infty)$, LK spaces have absolutely continuous norm.

2. Preliminaries

Throughout the paper (X, Σ, μ) will stand for a σ-finite measure space. We will use weight function w, i.e. a measurable, locally bounded function on X, satisfying $w(x) \geq 1$ for all $x \in X$ and χ_A for characteristic function of a set A. For any two non-negative expressions (i.e. functions or functionals), A and B, the symbol $A \precsim B$ means that $A \leq cB$, for some positive constant c independent of the variables in the expressions A and B. If $A \precsim B$ and $B \precsim A$, we write $A \approx B$ and say that A and B are equivalent. Certain well-known terms such as Banach function space, rearrangement invariant Banach function space, associate space, absolutely continuous norm, etc. will be used frequently in the sequel without their definitions. However, the reader may be found their definitions e.g., in [3,5,8,13] and [16].

A positive measurable function L, defined on some neighborhood of infinity, is said to be slowly varying if, for every $s > 0$,

$$\frac{L(st)}{L(t)} \to 1 \qquad (t \to +\infty). \tag{2.1}$$

These functions were introduced by Karamata [10] (see also [14] for more information). Also another definition for slowly varying functions can be found in [13] such as:

Definition 2.1. *A positive and Lebesgue measurable function b is said to be slowly varying (s.v.) on $(0,\infty)$ in the sense of Karamata if, for each $\varepsilon > 0$, $t^{\varepsilon} b(t)$ is equivalent to a non-decreasing function and $t^{-\varepsilon} b(t)$ is equivalent to a non-increasing function on $(0,\infty)$.*

The detailed study of Karamata Theory, properties and examples of slowly varying functions can be found in [5,10,14] and [16,Chap.V, p.186]. For example, let $m \in \mathbb{N}$ and $\alpha = (\alpha_1, ..., \alpha_m) \in \mathbb{R}^m$. If we denote by ϑ_α^m the real function defined by

$$\vartheta_\alpha^m(t) = \prod_{i=1}^{m} l_i^{\alpha_i}(t) \quad \text{for all } t \in (0,\infty)$$

where $l_1, ..., l_m$ are positive functions defined on $(0,\infty)$ by

$$l_1(t) = 1 + |\log t|, \; l_i(t) = 1 + \log l_{i-1}(t), \; i \geq 2, \; m \geq 2,$$

then the following functions are s.v. on $[1,\infty)$:

1. $b(t) = \vartheta_\alpha^m(t)$ with $m \in \mathbb{N}$ and $\alpha \in \mathbb{R}^m$;
2. $b(t) = \exp(\log^\alpha t)$ with $0 < \alpha < 1$;
3. $b(t) = \exp(l_m^\alpha(t))$ with $0 < \alpha < 1$, $m \in \mathbb{N}$;
4. $b(t) = l_m(t)$ with $m \in \mathbb{N}$.

Given a s.v. function b on $(0,\infty)$, we denote by γ_b the positive function defined by

$$\gamma_b(t) = b\left(\max\left\{t, \frac{1}{t}\right\}\right) \qquad \text{for all } t > 0.$$

It is known that any slowly varying function b on $(0,\infty)$ is equivalent to a slowly varying continuous function $\widetilde{b}$ on $(0,\infty)$. Consequently, without loss of generality, we assume that all slowly varying functions in question are continuous functions in $(0,\infty)$ [6]. We shall need the following property of s.v. functions, for which we refer to [13, Lemma 3.1].

Lemma 2.2. *Let b be a slowly varying function on $(0,\infty)$.*
(i) Let $r \in \mathbb{R}$. Then b^r is a slowly varying function on $(0,\infty)$ and $\gamma_b^r(t) = \gamma_{b^r}(t)$ for all $t > 0$.
(ii) Given positive numbers ε and κ, $\gamma_b(\kappa t) \approx \gamma_b(t)$, i.e., there are positive constants c_ε and C_ε such that

$$c_\varepsilon \min\{\kappa^{-\varepsilon}, \kappa^{\varepsilon}\}\gamma_b(t) \le \gamma_b(\kappa t) \le C_\varepsilon \max\{\kappa^{-\varepsilon}, \kappa^{\varepsilon}\}\gamma_b(t) \tag{2.2}$$

for all $t > 0$.
(iii) Let $\alpha > 0$. Then

$$\int_0^t \tau^{\alpha-1}\gamma_b(\tau)d\tau \approx t^\alpha \gamma_b(t) \quad \text{and} \quad \int_t^\infty \tau^{-\alpha-1}\gamma_b(\tau)d\tau \approx t^{-\alpha}\gamma_b(t) \tag{2.3}$$

for all $t > 0$.

Now, let us take the measure as $wd\mu$. Let f be a complex-valued measurable function defined on a σ-finite measure space $(X, \Sigma, wd\mu)$. Then the distribution function of f is defined as

$$\mu_{f,w}(s) = w\{x \in X : |f(x)| > s\} = \int\limits_{\{x\in X:\ |f(x)|>s\}} w(x)\,d\mu(x), \ s \ge 0. \tag{2.4}$$

The nonnegative rearrangement of f is given by

$$f_w^*(t) = \inf\{s > 0 : \mu_{f,w}(s) \le t\} = \sup\{s > 0 : \mu_{f,w}(s) > t\}, \ t \ge 0 \tag{2.5}$$

where we assume that $\inf \phi = \infty$ and $\sup \phi = 0$. Also the average(maximal) function of f on $(0,\infty)$ is given by

$$f_w^{**}(t) = \frac{1}{t}\int_0^t f_w^*(s)\,ds. \tag{2.6}$$

Note that $\lambda_{f,w}(\cdot), f_w^*(\cdot)$ and $f_w^{**}(\cdot)$ are nonincreasing and right continuous functions.

Definition 2.3. *Let $p, q \in (0,\infty]$ and let b be a slowly varying function on $(0,\infty)$. The weighted Lorentz-Karamata (WLK) space $L^w_{p,q;b}(X, \Sigma, wd\mu)$ is defined to be the set of all functions such that*

$$\|f\|^w_{p,q;b} := \left\| t^{\frac{1}{p}-\frac{1}{q}}\gamma_b(t)\, f_w^{**}(t)\right\|_{q;(0,\infty)} \tag{2.7}$$

is finite. Here $\|\cdot\|_{q;(0,\infty)}$ stands for the usual L_q (quasi-) norm over the interval $(0,\infty)$.

After this point, for the convenience, we will use $L^w_{p,q;b}(X)$ for $L^w_{p,q;b}(X, \Sigma, wd\mu)$. It is easy to show that (by the same arguments in [5,Theorem 3.4.41], [13]) the WLK spaces $L^w_{p,q;b}(X)$ endowed with a convenient norm (2.7), is a rearrangment-invariant Banach function spaces and have absolutely continuous norm when $p \in (1,\infty)$ and $q \in [1,\infty)$. It is clear that, for $0 < p < \infty$, the WLK space $L^w_{p,q;b}(X)$ contains the characteristic function of every measurable subset of X with finite measure and hence, by linearity, every $wd\mu-$simple function. In this case, with a little thought, it is easy to see that the set of simple functions is dense in the WLK space as the WLK spaces have absolutely continuous norm for $p \in (1,\infty)$ and $q \in [1,\infty)$.

Let $T : X \to X$ be a measurable ($T^{-1}(E) \in \Sigma$, for any $E \in \Sigma$) and non-singular transformation ($w\left(T^{-1}(E)\right) = 0$ whenever $w(E) = 0$) and u a complex-valued function defined on X. We define a linear transformation $W = W_{u,T}$ on the WLK space $L^w_{p,q;b}(X)$ into the linear space of all complex-valued measurable functions by

$$W_{u,T}(f)(x) = u(T(x))\, f(T(x)) \tag{2.8}$$

for all $x \in X$ and $f \in L^w_{p,q;b}(X)$. If W is bounded with range in $L^w_{p,q;b}(X)$, then it is called a *weighted composition operator* on $L^w_{p,q;b}(X)$. If $u \equiv 1$, then $W \equiv C_T : f \to f \circ T$ is called a *composition operator* induced by T. If T is the identity mapping, then $W \equiv M_u : f \to u \cdot f$ is a *multiplication operator* induced by u. The study of these operators acting on Lebesgue and Lorentz spaces has been made in [4,9,15] and [1,2,11,12], respectively.

In the next part of this paper, we will characterize the boundedness, compactness and closedness of the range of the weighted composition operators on WLK spaces $L^w_{p,q;b}(X)$ for $1 < p \le \infty$, $1 \le q \le \infty$.

3. Results

Theorem 3.1. *Let $(X, \Sigma, wd\mu)$ be a σ-finite measure space and $u : X \to \mathbb{C}$ a measurable function. Let $T : X \to X$ be a non-singular measurable transformation such that the Radon-Nikodym derivative $f_T = wd\mu\left(T^{-1}\right)/wd\mu$ is in $L^\infty(\mu)$. Then*

$$W_{u,T} : f \to u \circ T \cdot f \circ T \tag{3.1}$$

is bounded on $L^w_{p,q;b}(X)$, $1 < p \le \infty$, $1 \le q \le \infty$ if $u \in L^\infty(\mu)$.

Proof. Suppose that $\|f_T\|_\infty = k$. The distribution function of

$$Wf = W_{u,T}(f) = u \circ T \cdot f \circ T$$

is found that

$$\begin{aligned} \mu_{Wf,w}(s) &= w\{x \in X : \ |u(T(x))\, f(T(x))| > s\} \\ &= \int\limits_{\{x\in X:\ |u(T(x))f(T(x))|>s\}} w(x)\, d\mu(x) \\ &= wT^{-1}\{x \in X : \ |u(x)\, f(x)| > s\} \\ &\le wT^{-1}\{x \in X : \|u\|_\infty\ |f(x)| > s\} \\ &\le kw\{x \in X : \|u\|_\infty\ |f(x)| > s\} = k\mu_{\|u\|_\infty\ f,w}(s). \end{aligned} \tag{3.2}$$

Hence for each $t \ge 0$, by (3.2) we get

$$\left\{s > 0 : \mu_{\|u\|_\infty\ f,w}(s) \le \frac{t}{k}\right\} \subseteq \{s > 0 : \mu_{Wf,w}(s) \le t\} \tag{3.3}$$

and

$$\begin{aligned}(Wf)^*_w(t) &= \inf\{s>0 : \mu_{Wf,w}(s)\le t\}\\ &\le \inf\left\{s>0 : \mu_{\|u\|_\infty\ f,w}(s)\le \frac{t}{k}\right\}\\ &= \inf\left\{s>0 : w\{x\in X : \|u\|_\infty\ |f(x)|>s\}\le \frac{t}{k}\right\} \qquad (3.4)\\ &= \|u\|_\infty f^*_w\left(\frac{t}{k}\right).\end{aligned}$$

Also, we write that $(Wf)^{**}_w(t)\le \|u\|_\infty f^{**}_w\left(\frac{t}{k}\right)$ by (3.4). Therefore,

$$\begin{aligned}\|Wf\|^w_{p,q;b} &= \left\|t^{\frac{1}{p}-\frac{1}{q}}\gamma_b(t)(Wf)^{**}_w(t)\right\|_{q;(0,\infty)}\\ &\le \left\|t^{\frac{1}{p}-\frac{1}{q}}\gamma_b(t)\|u\|_\infty f^{**}_w\left(\frac{t}{k}\right)\right\|_{q;(0,\infty)}\\ &\lesssim \|u\|_\infty k^{\frac{1}{p}}\left\|t^{\frac{1}{p}-\frac{1}{q}}\gamma_b(t) f^{**}_w(t)\right\|_{q;(0,\infty)} = k^{\frac{1}{p}}\|u\|_\infty \|f\|^w_{p,q;b} \qquad (3.5)\end{aligned}$$

can be written by (2.2). Consequently, W is a bounded operator on $L^w_{p,q;b}(X)$ with $1<p\le\infty$, $1\le q\le\infty$ and $\|W\|\lesssim k^{\frac{1}{p}}\|u\|_\infty$ by (3.5). □

Remark 3.2. The above theorem is also valid for $u\in L^\infty\left(w\left(T^{-1}\right)\right)$, i.e.

$$u\circ T\in L^\infty(\mu).$$

Theorem 3.3. *Let u be a complex-valued measurable function and $T : X\to X$ be a non-singular measurable transformation such that $T(E_\varepsilon)\subseteq E_\varepsilon$ for all $\varepsilon>0$, where $E_\varepsilon=\{x\in X : |u(x)|>\varepsilon\}$. If $W_{u,T}$ is bounded on $L^w_{p,q;b}(X)$, $1<p\le\infty$, $1\le q\le\infty$, then $u\in L^\infty(\mu)$.*

Proof. Let us assume that $u\notin L^\infty(\mu)$. Then the set $E_n=\{x\in X : |u(x)|>n\}$ has a positive measure for all $n\in\mathbb{N}$. Since $T(E_n)\subseteq E_n$ or equivalently $\chi_{E_n}\le\chi_{T^{-1}(E_n)}$, we write that

$$\begin{aligned}\{x\in X : |\chi_{E_n}(x)|>s\} &\subseteq \left\{x\in X : \left|\chi_{T^{-1}(E_n)}(x)\right|>s\right\}\\ &\subseteq \left\{x\in X : \left|u(T(x))\chi_{T^{-1}(E_n)}(x)\right|>ns\right\} \qquad (3.6)\end{aligned}$$

and so

$$\begin{aligned}(W\chi_{E_n})^*_w(t) &= \inf\left\{s>0 : \mu_{W\chi_{E_n},w}(s)\le t\right\}\\ &= \inf\{s>0 : w\{x\in X :\ |W\chi_{E_n}(x)|>s\}\le t\}\\ &= \inf\{s>0 : w\{x\in X :\ |u(T(x))\chi_{E_n}(T(x))|>s\}\le t\} \qquad (3.7)\\ &= n\inf\left\{s>0 : w\left\{x\in X :\ \left|u(T(x))\chi_{T^{-1}(E_n)}(x)\right|>ns\right\}\le t\right\}\\ &\ge n\inf\{s>0 : w\{x\in X :\ |\chi_{E_n}(x)|>s\}\le t\} = n(\chi_{E_n})^*_w(t).\end{aligned}$$

Thus we have $(W\chi_{E_n})^{**}_w(t)\ge n(\chi_{E_n})^{**}_w(t)$ for all $t>0$ by (3.7). This gives us the contradiction that $\|W\chi_{E_n}\|^w_{p,q;b}\ge n\|\chi_{E_n}\|^w_{p,q;b}$. □

If we combine Theorem 3.1 and Theorem 3.3, then we have the following theorem.

Theorem 3.4. *Let u be a complex-valued measurable function and $T : X \to X$ be a non-singular measurable transformation such that the Radon-Nikodym derivative $f_T = wd\mu\left(T^{-1}\right)/wd\mu$ is in $L^{\infty}(\mu)$ and $T(E_{\varepsilon}) \subseteq E_{\varepsilon}$ for all $\varepsilon > 0$, where $E_{\varepsilon} = \{x \in X : |u(x)| > \varepsilon\}$. Then $W_{u,T}$ is bounded on $L^{w}_{p,q;b}(X)$, $1 < p \leq \infty$, $1 \leq q \leq \infty$ if and only if $u \in L^{\infty}(\mu)$.*

Now, we are ready to discuss the compactness and the closed range of the weighted composition operator $W = W_{u,T} : f \to u \circ T \cdot f \circ T$ on the WLK spaces $L^{w}_{p,q;b}(X)$, $1 < p \leq \infty$, $1 \leq q \leq \infty$. Let $T : X \to X$ be a non-singular measurable transformation with the Radon-Nikodym derivative $f_T = wd\mu\left(T^{-1}\right)/wd\mu$. If $f_T \in L^{\infty}(\mu)$ with $\|f_T\|_{\infty} = k$, then we get

$$\begin{aligned}(Wf)^{*}_{w}(kt) &= \inf\{s > 0 : \mu_{Wf,w}(s) \leq kt\}\\ &= \inf\{s > 0 : w\{x \in X : |u(T(x))f(T(x))| > s\} \leq kt\}\\ &= \inf\left\{s > 0 : wT^{-1}\{x \in X : |(u \cdot f)(x)| > s\} \leq kt\right\}\\ &\leq \inf\{s > 0 : w\{x \in X : |(u \cdot f)(x)| > s\} \leq t\} = (M_u f)^{*}_{w}(t) \qquad (3.8)\end{aligned}$$

and similarly $(Wf)^{**}_{w}(kt) \leq (M_u f)^{**}_{w}(t)$ for all $f \in L^{w}_{p,q;b}(X)$ and $t > 0$. Therefore, by (2.2), we obtain

$$\begin{aligned}\|Wf\|^{w}_{p,q;b} &= \left\| z^{\frac{1}{p}-\frac{1}{q}}\gamma_b(z)(Wf)^{**}_{w}(z)\right\|_{q;(0,\infty)}\\ &= \left\|(kt)^{\frac{1}{p}-\frac{1}{q}}\gamma_b(kt)(Wf)^{**}_{w}(kt)\right\|_{q;(0,\infty)} \qquad (3.9)\\ &\lesssim k^{\frac{1}{p}}\left\|t^{\frac{1}{p}-\frac{1}{q}}\gamma_b(t)(M_u f)^{**}_{w}(t)\right\|_{q;(0,\infty)} = k^{\frac{1}{p}}\|M_u f\|^{w}_{p,q;b}.\end{aligned}$$

Now, if f_T is bounded away from zero on S, i.e. $f_T > \delta$ almost everywhere for some $\delta > 0$, then

$$w\left(T^{-1}(E)\right) = \int_E f_T w d\mu \geq \delta w(E) \qquad (3.10)$$

for all $E \in \Sigma$, $E \subseteq S$, where $S = \{x : u(x) \neq 0\}$. Therefore, we have

$$\|Wf\|^{w}_{p,q;b} \geq \delta^{\frac{1}{p}}\|M_u f\|^{w}_{p,q;b}. \qquad (3.11)$$

Hence for each $f \in L^{w}_{p,q;b}(X)$, $1 < p \leq \infty$, $1 \leq q \leq \infty$, we have

$$\|Wf\|^{w}_{p,q;b} \approx \|M_u f\|^{w}_{p,q;b} \qquad (3.12)$$

whenever $f_T \in L^{\infty}(\mu)$ and bounded away from zero. By [7, Theorem 2.4] and (3.12), we can write the following theorem:

Theorem 3.5. *Let $T : X \to X$ be a non-singular measurable transformation such that $f_T \in L^{\infty}(\mu)$ and is bounded away from zero. Let u be a complex-valued measurable function and $W_{u,T}$ is bounded on the WLK space $L^{w}_{p,q;b}(X)$, $1 < p \leq \infty$, $1 \leq q \leq \infty$. Then the followings are equivalent:*
(i) $W_{u,T}$ is compact,

(ii) M_u *is compact,*

(iii) $L^w_{p,q;b}(u,\varepsilon)$ *are finite dimensional for each* $\varepsilon > 0$, *where*

$$L^w_{p,q;b}(u,\varepsilon) = \left\{ f\chi_{(u,\varepsilon)} : f \in L^w_{p,q;b}(X) \right\} \text{ and } (u,\varepsilon) = \{x \in X : |u(x)| \geq \varepsilon\}.$$

We know that $W_{u,T} = C_T M_u$ and $wd\mu$ is atomic. Therefore, if we use [11, Theorem 3.1] for $W_{u,T}$ on the WLK space $L^w_{p,q;b}(X)$, $1 < p \leq \infty$, $1 \leq q \leq \infty$, then get the following theorem:

Theorem 3.6. *Let* $T : X \to X$ *be a non-singular measurable transformation such that* $f_T \in L^\infty(\mu)$ *and* u *be a complex-valued measurable function with* $u \in L^\infty(\mu)$. *Let* $\{A_n\}_{n\in\mathbb{N}}$ *be all the atoms of* X *with* $w(A_n) > 0$ *for all* $n \in \mathbb{N}$. *Then* $W_{u,T}$ *is compact on the* WLK *space* $L^w_{p,q;b}(X)$, $1 < p \leq \infty$, $1 \leq q \leq \infty$ *if* $wd\mu$ *is purely atomic and*

$$c_n = \frac{w\left(T^{-1}(A_n)\right)}{w(A_n)} \to 0.$$

Theorem 3.7. *If* $wd\mu$ *is non-atomic and* $W_{u,T}$ *is bounded on the* WLK *space* $L^w_{p,q;b}(X)$, $1 < p \leq \infty$, $1 \leq q \leq \infty$, *then* $W_{u,T}$ *is compact if and only if* $u \cdot f_T = 0$ *almost everywhere.*

Proof. Let us assume that $W = W_{u,T}$ is compact. If $u \cdot f_T \neq 0$ a.e., then there exist $c \geq 1$, such that the set

$$E = \left\{ x \in X : |u(x)| \text{ and } f_T(x) > \frac{1}{c} \right\} \tag{3.13}$$

has positive measure. Since $wd\mu$ is non- atomic, we can find a decreasing sequence $\{E_n\}_{n\in\mathbb{N}}$ of measurable subsets of E such that $w(E_n) = \frac{a}{2^n}$, $0 < a < w(E)$. Now, if we construct a sequence such that $e_n = \frac{\chi_{E_n}}{\|\chi_{E_n}\|^w_{p,q;b}}$, then it is easy to see that $\{e_n\}_{n\in\mathbb{N}}$ is bounded in $L^w_{p,q;b}(X)$. For $m, n \in \mathbb{N}$, let $m = 2n$. Then we have

$$\begin{aligned}
&(We_n - We_m)^*_w\left(\frac{t}{c}\right)\\
&= \inf\left\{ s > 0 : \mu_{We_n - We_m, w}(s) \leq \frac{t}{c} \right\}\\
&= \inf\left\{ s > 0 : w\{x \in X : |u(T(x))\, e_n(T(x)) - u(T(x))\, e_m(T(x))| > s\} \leq \frac{t}{c} \right\}\\
&= \inf\left\{ s > 0 : wT^{-1}\{z \in E_n : |u(z)|\, |e_n(z) - e_m(z)| > s\} \leq \frac{t}{c} \right\}\\
&\geq \inf\{ s > 0 : w\{z \in E_n : |e_n(z) - e_m(z)| > sc\} \leq t\}\\
&= \frac{1}{c}\inf\{ s > 0 : w\{z \in E_n : |e_n(z) - e_m(z)| > s\} \leq t\}\\
&\geq \frac{1}{c}\inf\{ s > 0 : w\{z \in E_n \backslash E_m : |e_n(z) - e_m(z)| > s\} \leq t\}
\end{aligned}$$

for all $t \geq 0$. This gives us that

$$(We_n - We_m)^*_w\left(\frac{t}{c}\right) \geq \frac{\left(\chi_{E_n \backslash E_m}\right)^*_w(t)}{c\,\|\chi_{E_n}\|^w_{p,q;b}} \tag{3.14}$$

and so

$$\|We_n - We_m\|^w_{p,q;b} \gtrsim \frac{1}{c^2}\left(\frac{w(E_n \backslash E_m)}{w(E_n)}\right)^{\frac{1}{p}} \geq \varepsilon \tag{3.15}$$

for some $\varepsilon > 0$ and large values of n by (ii) and (iii) of Lemma 2.2. Thus the sequence $\{We_n\}_{n\in\mathbb{N}}$ doesn't admit a convergent subsequence which conradicts the compactness of W. Hence $u \cdot f_T = 0$ a.e.
The converse of the proof is obvious. □

Theorem 3.8. *Let $T : X \to X$ be a non-singular measurable transformation with f_T in $L^\infty(\mu)$ and bounded away from zero. Let u be a complex-valued measurable function such that $W_{u,T}$ is bounded on the WLK space $L^w_{p,q;b}(X)$, $1 < p \leq \infty$, $1 \leq q \leq \infty$. Then $W_{u,T}$ has closed range if and only if there exists a $\delta > 0$ such that $|u(x)| \geq \delta$ a.e. on the support of u.*

Proof. Suppose that $W = W_{u,T}$ has closed range. Therefore there exists an $\varepsilon > 0$ such that $\|Wf\|^w_{p,q;b} \geq \varepsilon \|f\|^w_{p,q;b}$ for all $f \in L^w_{p,q;b}(S)$ where S is the support of u and $L^w_{p,q;b}(S) = \left\{f\chi_S : f \in L^w_{p,q;b}(X)\right\}$. Now, let us choose $\delta > 0$ such that $k^{\frac{1}{p}}\delta < \varepsilon$ where $k = \|f_T\|_\infty$. Assume that the set $E = \{x \in X : |u(x)| < \delta\}$ has positive measure, i.e. $0 < w(E) < \infty$. Then $\chi_E \in L^w_{p,q;b}(S)$ and

$$\begin{aligned}\|W\chi_E\|^w_{p,q;b} &\lesssim k^{\frac{1}{p}}\|u\cdot\chi_E\|^w_{p,q;b} \leq k^{\frac{1}{p}}\delta\|\chi_E\|^w_{p,q;b} \\ &< \varepsilon\|\chi_E\|^w_{p,q;b}\end{aligned}$$

by (3.9). This conradiction says that $|u(x)| \geq \delta$ a.e. on the support of u.
Conversely, assume that there exists a $\delta > 0$ such that $|u(x)| \geq \delta$ a.e. on S. Since f_T is bounded away from zero, we can write that $f_T > m$ for some $m > 0$. By using this fact and (3.11), we get

$$\|Wf\|^w_{p,q;b} \geq m^{\frac{1}{p}}\|u\cdot f\|^w_{p,q;b} \geq m^{\frac{1}{p}}\delta\|f\|^w_{p,q;b} \tag{3.16}$$

for all $f \in L^w_{p,q;b}(S)$. Therefore W has closed range being $ker(W) = L^w_{p,q;b}(X\backslash S)$. □

Corollary 3.9. *If $T^{-1}(E_\varepsilon) \subseteq E_\varepsilon$ for each $\varepsilon > 0$ and $W_{u,T}$ has closed range, then $|u(x)| \geq \delta$ a.e. on S, the support of u for some $\delta > 0$.*

Using the equivalence (3.12) and [1, Theorem 4.1], we can say the following theorem:

Theorem 3.10. *Let $T : X \to X$ be a non-singular measurable transformation such that $f_T \in L^\infty(\mu)$ and is bounded away from zero. Let u be a complex-valued measurable function such that $W_{u,T}$ is bounded on the WLK space $L^w_{p,q;b}(X)$, $1 < p \leq \infty$, $1 \leq q \leq \infty$. Then the followings are equivalent:*
(i) $W_{u,T}$ has closed range,
(ii) M_u has closed range,
(iii) $|u(x)| \geq \delta$ a.e. for some $\delta > 0$ on S, the support of u.

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On transformations groups of $N-$linear connections on the dual bundle of $k-$tangent bundle

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Abstract. In the present paper we study the transformations for the coefficients of an $N-$linear connection on dual bundle of $k-$tangent bundle, $T^{*k}M$, by a transformation of a nonlinear connection on $T^{*k}M$. We prove that the set $\mathcal{T}$ of these transformations together with the composition of mappings isn't a group. But we give some groups of transformations of $\mathcal{T}$, which keep invariant a part of components of the local coefficients of an $N-$linear connection.

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1. Introduction

The notion of Hamilton space was introduced by Acad. R. Miron in [7] , [8] . The Hamilton spaces appear as dual via Legendre transformation, of the Lagrange spaces.

The differential geometry of the dual bundle of $k-$osculator bundle was introduced and studied by Acad. R. Miron [13] .

The importance of Lagrange and Hamilton geometries consists in the fact that the variational problems for important Lagrangians or Hamiltonians have numerous applications in various fields, as: Mathematics, Mecanics, Theoretical Physics, Theory of Dynamical Systems, Optimal Control, Biology, Economy etc.

In the present section we keep the general setting from Acad. R. Miron [13] , and subsequently we recall only some needed notions. For more details see [13] .

Let M be a real $n-$dimensional C^∞ $-$manifold and let $\left(T^{*k}M, \pi^{*k}, M\right)$, $(k \geq 2)$, $k \in \mathbb{N})$ be the dual bundle of $k-$osculator bundle (or $k-$cotangent bundle), where the total space is:

$$T^{*k}M = T^{*k-1}M \times T^{*}M. \tag{1.1}$$

Let $\left(x^i, y^{(1)i}, ..., y^{(k-1)i}, p_i\right), (i = 1, ..., n)$, be the local coordinates of a point $u = \left(x, y^{(1)}, ..., y^{(k-1)}, p\right) \in T^{*k}M$ in a local chart on $T^{*k}M$.

The change of coordinates on the manifold $T^{*k}M$ is:

$$\begin{cases} \tilde{x}^i = \tilde{x}^i\left(x^1, ..., x^n\right), \det\left(\frac{\partial \tilde{x}^i}{\partial x^j}\right) \neq 0, \\ \tilde{y}^{(1)i} = \frac{\partial \tilde{x}^i}{\partial x^j} y^{(1)j}, \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots \\ (k-1)\,\tilde{y}^{(k-1)i} = \frac{\partial \tilde{y}^{(k-2)i}}{\partial x^j} y^{(1)j} + ... + (k-1)\,\frac{\partial \tilde{y}^{(k-2)i}}{\partial y^{(k-2)j}} y^{(k-1)j}, \\ \tilde{p}_i = \frac{\partial x^j}{\partial \tilde{x}^i} p_j, \end{cases} \tag{1.2}$$

where the following relations hold:

$$\frac{\partial \tilde{y}^{(\alpha)i}}{\partial x^j} = \frac{\partial \tilde{y}^{(\alpha+1)i}}{\partial y^{(1)j}} = ... = \frac{\partial \tilde{y}^{(k-1)i}}{\partial y^{(k-1-\alpha)j}}, \left(\alpha = 0, ..., k-2; y^{(0)} = x\right). \tag{1.3}$$

$T^{*k}M$ is a real differential manifold of dimension $(k+1)\,n$.

With respect to (1.1) the natural basis of the vector space $T_u\left(T^{*k}M\right)$ at the point $u \in T^{*k}M$:

$$\left\{ \left.\frac{\partial}{\partial x^i}\right|_u, \left.\frac{\partial}{\partial y^{(1)i}}\right|_u, ..., \left.\frac{\partial}{\partial y^{(k-1)i}}\right|_u, \left.\frac{\partial}{\partial p_i}\right|_u \right\} \tag{1.4}$$

is transformed as follows:

$$\begin{cases} \left.\frac{\partial}{\partial x^i}\right|_u = \frac{\partial \tilde{x}^j}{\partial x^i} \left.\frac{\partial}{\partial \tilde{x}^j}\right|_u + \frac{\partial \tilde{y}^{(1)j}}{\partial x^i} \left.\frac{\partial}{\partial \tilde{y}^{(1)j}}\right|_u + ... + \frac{\partial \tilde{y}^{(k-1)j}}{\partial x^i} \left.\frac{\partial}{\partial \tilde{y}^{(k-1)j}}\right|_u + \frac{\partial \tilde{p}_j}{\partial x^i} \left.\frac{\partial}{\partial \tilde{p}_j}\right|_u, \\ \left.\frac{\partial}{\partial y^{(1)i}}\right|_u = \frac{\partial \tilde{y}^{(1)j}}{\partial y^{(1)i}} \left.\frac{\partial}{\partial \tilde{y}^{(1)j}}\right|_u + ... + \frac{\partial \tilde{y}^{(k-1)j}}{\partial y^{(1)i}} \left.\frac{\partial}{\partial \tilde{y}^{(k-1)j}}\right|_u, \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ \left.\frac{\partial}{\partial y^{(k-1)i}}\right|_u = \frac{\partial \tilde{y}^{(k-1)j}}{\partial y^{(k-1)i}} \left.\frac{\partial}{\partial \tilde{y}^{(k-1)j}}\right|_u, \\ \left.\frac{\partial}{\partial p_i}\right|_u = \frac{\partial x^j}{\partial \tilde{x}^i} \left.\frac{\partial}{\partial \tilde{p}_j}\right|_u, \end{cases} \tag{1.5}$$

the conditions (1.3) being satisfied.

The null section $0 : M \to T^{*k}M$ of the projection π^{*k} is defined by $0\,(x) \in M \to (x, 0, ..., 0) \in T^{*k}M$. We denote $\widetilde{T^{*k}M} = T^{*k}M \setminus \{0\}$.

Let us consider the tangent bundle of the differentiable manifold $T^{*k}M$ $\left(TT^{*k}M, d\pi^{*k}, T^{*k}M\right)$, where $d\pi^{*k}$ is the canonical projection and the vertical distribution $V : u \in T^{*k}M \to V\,(u) \in T_uT^{*k}M$, locally generated by the vector fields: $\left\{\frac{\partial}{\partial y^{(1)i}}, ..., \frac{\partial}{\partial y^{(k-1)i}}, \frac{\partial}{\partial p_i}\right\}$ at every point $u \in T^{*k}M$.

The following $\mathcal{F}\left(T^{*k}M\right) -$ linear mapping:

$$J : \chi\left(T^{*k}M\right) \to \chi\left(T^{*k}M\right),$$

defined by:

$$J\left(\frac{\partial}{\partial x^i}\right) = \frac{\partial}{\partial y^{(1)i}}, J\left(\frac{\partial}{\partial y^{(1)i}}\right) = \frac{\partial}{\partial y^{(2)i}}, ..., J\left(\frac{\partial}{\partial y^{(k-2)i}}\right) =$$

$$= \frac{\partial}{\partial y^{(k-1)i}}, J\left(\frac{\partial}{\partial y^{(k-1)i}}\right) = 0, J\left(\frac{\partial}{\partial p_i}\right) = 0, \tag{1.6}$$

at every point $u \in \widetilde{T^{*k}M}$ is a tangent structure on $T^{*k}M$.

We denote with N a nonlinear connection on the manifold $T^{*k}M$, with the coefficients:

$$\left(\underset{(1)}{N}{}^{j}{}_{i}\left(x, y^{(1)}, ..., y^{(k-1)}, p\right), ..., \underset{(k-1)}{N}{}^{j}{}_{i}\left(x, y^{(1)}, ..., y^{(k-1)}, p\right),\right.$$

$$\left.N_{ij}\left(x, y^{(1)}, ..., y^{(k-1)}, p\right)\right), (i, j = 1, 2, ..., n).$$

The tangent space of $T^{*k}M$ in the point $u \in T^{*k}M$ is given by the direct sum of vector spaces:

$$T_u\left(T^{*k}M\right) = N_{0,u} \oplus N_{1,u} \oplus ... \oplus N_{k-2,u} \oplus V_{k-1,u} \oplus W_{k,u}, \ \forall u \in T^{*k}M \tag{1.7}$$

A local adapted basis to the direct decomposition (1.7) is given by:

$$\left\{\frac{\delta}{\delta x^i}, \frac{\delta}{\delta y^{(1)i}}, ..., \frac{\delta}{\delta y^{(k-1)i}}, \frac{\delta}{\delta p_i}\right\}, (i = 1, 2, ..., n), \tag{1.8}$$

where:

$$\begin{cases} \frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - \underset{(1)}{N}{}^{j}{}_{i}\frac{\partial}{\partial y^{(1)j}} - ... - \underset{(k-1)}{N}{}^{j}{}_{i}\frac{\partial}{\partial y^{(k-1)j}} + N_{ij}\frac{\partial}{\partial p_j}, \\ \frac{\delta}{\delta y^{(1)i}} = \frac{\partial}{\partial y^{(1)i}} - \underset{(1)}{N}{}^{j}{}_{i}\frac{\partial}{\partial y^{(2)j}} - ... - \underset{(k-2)}{N}{}^{j}{}_{i}\frac{\partial}{\partial y^{(k-1)j}}, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \frac{\delta}{\delta y^{(k-1)i}} = \frac{\partial}{\partial y^{(k-1)i}}, \\ \frac{\delta}{\delta p_i} = \frac{\partial}{\partial p_i}. \end{cases} \tag{1.9}$$

Under a change of local coordinates on $T^{*k}M$, the vector fields of the adapted basis transform by the rule:

$$\frac{\delta}{\delta x^i} = \frac{\partial \tilde{x}^j}{\partial x^i}\frac{\delta}{\delta \tilde{x}^j}, \frac{\delta}{\delta y^{(1)i}} = \frac{\partial \tilde{x}^j}{\partial x^i}\frac{\delta}{\delta \tilde{y}^{(1)j}}, ..., \frac{\delta}{\delta y^{(k-1)i}} = \frac{\partial \tilde{x}^j}{\partial x^i}\frac{\delta}{\delta \tilde{y}^{(k-1)j}}, \frac{\delta}{\delta p_i} = \frac{\delta x^j}{\delta \tilde{x}^i}\frac{\delta}{\delta \tilde{p}_j}. \tag{1.10}$$

The dual basis of the adapted basis (1.8) is given by:

$$\left\{\delta x^i, \delta y^{(1)i}, ..., \delta y^{(k-1)i}, \delta p_i\right\}, \tag{1.11}$$

where:

$$\begin{cases} dx^i = \delta x^i, \\ dy^{(1)i} = \delta y^{(1)i} - \underset{(1)}{N}{}^{i}{}_{j}\delta x^j, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ dy^{(k-1)i} = \delta y^{(k-1)i} - \underset{(1)}{N}{}^{i}{}_{j}\delta y^{(k-2)j} - ... - \underset{(k-2)}{N}{}^{i}{}_{j}\delta y^{(1)j} - \underset{(k-1)}{N}{}^{i}{}_{j}\delta x^j, \\ dp_i = \delta p_i + N_{ji}\delta x^j. \end{cases} \tag{1.12}$$

With respect to (1.2) the covector fields (1.11) are transformed by the rules:

$$\begin{matrix} \delta\tilde{x}^i = \frac{\partial \tilde{x}^i}{\partial x^j}\delta x^j, \delta\tilde{y}^{(1)i} = \frac{\partial \tilde{x}^i}{\partial x^j}\delta y^{(1)j}, ..., \delta\tilde{y}^{(k-1)i} = \frac{\partial \tilde{x}^i}{\partial x^j}\delta y^{(k-1)j}, \\ \delta\tilde{p}_i = \frac{\partial x^j}{\partial \tilde{x}^i}\delta p_j. \end{matrix} \tag{1.13}$$

Let D be an $N-$linear connection on $T^{*k}M$, with the local coefficients in the adapted basis (1.8) :

$$D\Gamma\left(N\right)=\left(H^{i}{}_{jh},\underset{(\alpha)}{C}{}^{i}{}_{jh},C_{i}^{jh}\right),\left(\alpha=1,...,k-1\right). \quad (1.14)$$

An $N-$linear connection D is uniquely represented in the adapted basis in the following form:

$$\begin{array}{l} D_{\frac{\delta}{\delta x^{j}}}\frac{\delta}{\delta x^{i}}=H^{s}{}_{ij}\frac{\delta}{\delta x^{s}},D_{\frac{\delta}{\delta x^{j}}}\frac{\delta}{\delta y^{(\alpha)i}}=H^{s}{}_{ij}\frac{\delta}{\delta y^{(\alpha)s}},\left(\alpha=1,...,k-1\right),\\ D_{\frac{\delta}{\delta x^{j}}}\frac{\delta}{\delta p_{i}}=-H_{sj}^{i}\frac{\delta}{\delta p_{s}},\\ D_{\frac{\delta}{\delta y^{(\alpha)j}}}\frac{\delta}{\delta x^{i}}=\underset{(\alpha)}{C}{}^{s}{}_{ij}\frac{\delta}{\delta x^{s}},D_{\frac{\delta}{\delta y^{(\alpha)j}}}\frac{\delta}{\delta y^{(\beta)i}}=\underset{(\alpha)}{C}{}^{s}{}_{ij}\frac{\delta}{\delta y^{(\beta)s}},\\ D_{\frac{\delta}{\delta y^{(\alpha)j}}}\frac{\delta}{\delta p_{i}}=-\underset{(\alpha)}{C}{}^{i}{}_{sj}\frac{\delta}{\delta p_{s}},\left(\alpha,\beta=1,...,k-1\right),\\ D_{\frac{\delta}{\delta p_{j}}}\frac{\delta}{\delta x^{i}}=C_{i}^{js}\frac{\delta}{\delta x^{s}},D_{\frac{\delta}{\delta p_{j}}}\frac{\delta}{\delta y^{(\alpha)i}}=C_{i}^{js}\frac{\delta}{\delta y^{(\alpha)s}},\left(\alpha=1,...,k-1\right),\\ D_{\frac{\delta}{\delta p_{j}}}\frac{\delta}{\delta p_{i}}=-C_{s}{}^{ij}\frac{\delta}{\delta p_{s}}. \end{array} \quad (1.15)$$

2. The set of the transformations of $N-$linear connections

Let $\bar{N}$ be another nonlinear connection on $T^{*k}M$, with the local coefficients

$$\left(\underset{(1)}{\bar{N}}{}^{j}{}_{i}\left(x,y^{(1)},...,y^{(k-1)},p\right),...,\underset{(k-1)}{\bar{N}}{}^{j}{}_{i}\left(x,y^{(1)},...,y^{(k-1)},p\right),\ N_{ij}\left(x,y^{(1)},...,\ y^{(k-1)},p\right)\right)$$

$(i,j=1,2,...,n)$.

Then there exists the uniquely determined tensor fields

$$\underset{(\alpha)}{A}{}^{j}{}_{i}\in\tau_{1}^{1}\left(T^{*k}M\right),\left(\alpha=1,...,k-1\right)$$

and $A_{ij}\in\tau_{2}^{0}\left(T^{*k}M\right)$, such that:

$$\begin{cases} \underset{(\alpha)}{\bar{N}}{}^{i}{}_{j}=\underset{(\alpha)}{N}{}^{i}{}_{j}-\underset{(\alpha)}{A}{}^{i}{}_{j},\left(\alpha=1,2,...,k-1\right),\\ \bar{N}_{ij}=N_{ij}-A_{ij},\left(i,j=1,2,...,n\right). \end{cases} \quad (2.1)$$

Conversely, if $\underset{(\alpha)}{N}{}^{i}{}_{j}$ and $\underset{(\alpha)}{A}{}^{i}{}_{j},(\alpha=1,2,...,k-1)$, respectively N_{ij} and A_{ij} are given, then $\underset{(\alpha)}{\bar{N}}{}^{i}{}_{j},\left(\alpha=1,2,...,k-1\right)$, respectively $\bar{N}_{ij}$, given by (2.1) are the coefficients of a nonlinear connection.

Theorem 2.1. *Let N and $\bar{N}$ be two nonlinear connections on $T^{*k}M$, $(k\geq 2,\ k\in\mathbb{N})$ with local coefficients:*

$$\left(\underset{(1)}{N}{}^{j}{}_{i}\left(x,y^{(1)},...,y^{(k-1)},p\right),...,\underset{(k-1)}{N}{}^{j}{}_{i}\left(x,y^{(1)},...,y^{(k-1)},p\right),N_{ij}\left(x,y^{(1)},...,y^{(k-1)},p\right)\right),$$

$$\left(\underset{(1)}{\bar{N}}{}^{j}{}_{i}\left(x,y^{(1)},...,y^{(k-1)},p\right),...,\underset{(k-1)}{\bar{N}}{}^{j}{}_{i}\left(x,y^{(1)},...,y^{(k-1)},p\right),\overline{N}_{ij}\left(x,y^{(1)},...,y^{(k-1)},p\right)\right),$$

$\left(i,j=1,2,...,n\right)$, *respectively.*

If D is an $N-$linear connection on $T^{*k}M$, with local coefficients

$$D\Gamma(N)=\left(H^{i}{}_{jh},\underset{(\alpha)}{C}{}^{i}{}_{jh},C_{i}{}^{jh}\right),\ (\alpha=1,...,k-1),$$

then the transformation: $N\longrightarrow\bar{N}$, given by (2.1) *of nonlinear connections implies for the coefficients*

$$D\Gamma\left(\bar{N}\right)=\left(\bar{H}^{i}{}_{jh},\underset{(\alpha)}{\bar{C}}{}^{i}{}_{jh},\bar{C}_{i}{}^{jh}\right),\ (\alpha=1,...,k-1)$$

of the $\bar{N}-$linear connection D, the relations (2.2), *that is the transformation: $D\Gamma(N)\rightarrow D\Gamma\left(\overline{N}\right)$ is given by:*

$$\begin{cases}
\bar{H}^{i}{}_{sj}=H^{i}{}_{sj}+\underset{(1)}{A}{}^{m}{}_{j}\Bigg[\underset{(1)}{C}{}^{i}{}_{sm}+\underset{(1)}{N}{}^{l}{}_{m}\underset{(2)}{C}{}^{i}{}_{sl}+...+\underset{(k-2)}{N}{}^{l}{}_{m}\underset{(k-1)}{C}{}^{i}{}_{sl}+\underset{(1)}{N}{}^{l}{}_{m}\underset{(1)}{N}{}^{t}{}_{l}\underset{(3)}{C}{}^{i}{}_{st}+\\
...+\left(\underset{(1)}{N}{}^{l}{}_{m}\underset{(k-3)}{N}{}^{t}{}_{l}+...+\underset{(k-3)}{N}{}^{l}{}_{m}\underset{(1)}{N}{}^{t}{}_{l}\right)\underset{(k-1)}{C}{}^{i}{}_{st}+...+\underbrace{\underset{(1)}{N}...\underset{(1)}{N}}_{(k-2)}\underset{(k-1)}{C}\Bigg]+\\
+\underset{(2)}{A}{}^{m}{}_{j}\Bigg[\underset{(2)}{C}{}^{i}{}_{sm}+\underset{(1)}{N}{}^{l}{}_{m}\underset{(3)}{C}{}^{i}{}_{sl}+...+\underset{(k-3)}{N}{}^{l}{}_{m}\underset{(k-1)}{C}{}^{i}{}_{sl}+...+\underbrace{\underset{(1)}{N}...\underset{(1)}{N}}_{(k-3)}\underset{(k-1)}{C}\Bigg]+\\
+...+\underset{(k-2)}{A}{}^{m}{}_{j}\left(\underset{(k-2)}{C}{}^{i}{}_{sm}+\underset{(1)}{N}{}^{l}{}_{m}\underset{(k-1)}{C}{}^{i}{}_{sl}\right)+\underset{(k-1)}{A}{}^{m}{}_{j}\underset{(k-1)}{C}{}^{i}{}_{sm}-A_{jm}C_{s}{}^{im},\\
\underset{(1)}{\overline{C}}{}^{i}{}_{sj}=\underset{(1)}{C}{}^{i}{}_{sj}+\underset{(1)}{A}{}^{m}{}_{j}\Bigg[\underset{(2)}{C}{}^{i}{}_{sm}+\underset{(1)}{N}{}^{r}{}_{m}\underset{(3)}{C}{}^{i}{}_{sr}+...+\underset{(k-3)}{N}{}^{r}{}_{m}\underset{(k-1)}{C}{}^{i}{}_{sr}+...+\\
+\underbrace{\underset{(1)}{N}\cdot...\cdot\underset{(1)}{N}}_{(k-3)}\underset{(k-1)}{C}\Bigg]+...+\underset{(k-3)}{A}{}^{m}{}_{j}\left[\underset{(k-2)}{C}{}^{i}{}_{sm}+\underset{(1)}{N}{}^{r}{}_{m}\underset{(k-1)}{C}{}^{i}{}_{sr}\right]+\underset{(k-2)}{A}{}^{m}{}_{j}\underset{(k-1)}{C}{}^{i}{}_{sm},\\
\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\\
\underset{(k-2)}{\overline{C}}{}^{i}{}_{sj}=\underset{(k-2)}{C}{}^{i}{}_{sj}+\underset{(1)}{A}{}^{l}{}_{j}\underset{(k-1)}{C}{}^{i}{}_{sl},\\
\underset{(k-1)}{\bar{C}}{}^{i}{}_{sj}=\underset{(k-1)}{C}{}^{i}{}_{sj},\\
\bar{C}_{s}{}^{ij}=C_{s}{}^{ij},\\
\underset{(1)}{A}{}^{h}{}_{i|j}=0,\\
A_{ih|j}=0,(i,j,h=1,2...,n),
\end{cases}\tag{2.2}$$

where $_|$ denotes the $h-$covariant derivative with respect to $D\Gamma(N)$.

Proof. It follows first of all that the transformations (2.1) preserve the coefficients $\underset{(k-1)}{C}{}^{h}{}_{ij}$, C_i^{jh}.

Using the relations (1.9), (1.15) and (2.1) we obtain:

$$\begin{cases} \frac{\bar{\delta}}{\delta x^i} = \frac{\delta}{\delta x^i} + \underset{(1)}{A}{}^{j}{}_{i}\frac{\partial}{\partial y^{(1)j}} + ... + \underset{(k-1)}{A}{}^{j}{}_{i}\frac{\partial}{\partial y^{(k-1)j}} - A_{ij}\frac{\partial}{\partial p_j}, \\ \frac{\bar{\delta}}{\delta y^{(1)i}} = \frac{\delta}{\delta y^{(1)i}} + \underset{(1)}{A}{}^{j}{}_{i}\frac{\partial}{\partial y^{(2)j}} + ... + \underset{(k-2)}{A}{}^{j}{}_{i}\frac{\partial}{\partial y^{(k-1)j}}, \\ \text{..} \\ \frac{\bar{\delta}}{\delta y^{(k-1)i}} = \frac{\delta}{\delta y^{(k-1)i}}, \\ \frac{\bar{\delta}}{\delta p_i} = \frac{\delta}{\delta p_i}. \end{cases} \tag{2.3}$$

Using (1.15), (2.3) and (1.9) we get:

$$D_{\frac{\bar{\delta}}{\delta x^j}}\frac{\bar{\delta}}{\delta y^{(k-1)i}} = \bar{H}^{s}{}_{ij}\frac{\bar{\delta}}{\delta y^{(k-1)s}} = \bar{H}^{s}{}_{ij}\frac{\bar{\delta}}{\delta y^{(k-1)s}}.$$

$$D_{\frac{\bar{\delta}}{\delta x^j}}\frac{\bar{\delta}}{\delta y^{(k-1)i}} = D_{\left(\frac{\delta}{\delta x^j} + \underset{(1)}{A}{}^{l}{}_{j}\frac{\partial}{\partial y^{(1)l}} + \underset{(2)}{A}{}^{l}{}_{j}\frac{\partial}{\partial y^{(2)l}} + ... + \underset{(k-1)}{A}{}^{l}{}_{j}\frac{\partial}{\partial y^{(k-1)l}} - A_{jl}\frac{\partial}{\partial p_l}\right)}\frac{\delta}{\delta y^{(k-1)i}} =$$

$$= H^{s}{}_{ij}\frac{\delta}{\delta y^{(k-1)s}} + \left(\underset{(1)}{A}{}^{l}{}_{j}\underset{(1)}{C}{}^{s}{}_{il}\frac{\delta}{\delta y^{(k-1)s}} + \underset{(2)}{A}{}^{l}{}_{j}\underset{(2)}{C}{}^{s}{}_{il}\frac{\delta}{\delta y^{(k-1)s}} + ...\right.$$

$$\left. ... + \underset{(k-1)}{A}{}^{l}{}_{j}\underset{(k-1)}{C}{}^{s}{}_{il}\frac{\delta}{\delta y^{(k-1)s}} - A_{jl}C_i^{ls}\frac{\delta}{\delta y^{(k-1)s}}\right) +$$

$$+\underset{(1)}{A}{}^{l}{}_{j}\underset{(1)}{N}{}^{r}{}_{l}D_{\left(\frac{\delta}{\delta y^{(2)r}} + \underset{(1)}{N}{}^{s}{}_{r}\frac{\partial}{\partial y^{(3)s}} + ... + \underset{(k-3)}{N}{}^{s}{}_{r}\frac{\delta}{\delta y^{(k-1)s}}\right)}\frac{\delta}{\delta y^{(k-1)i}} +$$

$$+\underset{(1)}{A}{}^{l}{}_{j}\underset{(2)}{N}{}^{r}{}_{l}D_{\left(\frac{\delta}{\delta y^{(3)r}} + \underset{(1)}{N}{}^{s}{}_{r}\frac{\partial}{\partial y^{(4)s}} + ... + \underset{(k-4)}{N}{}^{s}{}_{r}\frac{\delta}{\delta y^{(k-1)s}}\right)}\frac{\delta}{\delta y^{(k-1)i}} + ...$$

$$+\underset{(1)}{A}{}^{l}{}_{j}\underset{(k-2)}{N}{}^{r}{}_{l}\underset{(k-1)}{C}{}^{s}{}_{ir}\frac{\delta}{\delta y^{(k-1)s}} + \underset{(2)}{A}{}^{l}{}_{j}\underset{(1)}{N}{}^{r}{}_{l}D_{\left(\frac{\delta}{\delta y^{(3)r}} + \underset{(1)}{N}{}^{s}{}_{r}\frac{\partial}{\partial y^{(4)s}} + ... + \underset{(k-4)}{N}{}^{s}{}_{r}\frac{\delta}{\delta y^{(k-1)s}}\right)}\frac{\delta}{\delta y^{(k-1)i}} +$$

$$+ ... + \underset{(2)}{A}{}^{l}{}_{j}\underset{(k-3)}{N}{}^{r}{}_{l}\underset{(k-1)}{C}{}^{s}{}_{ir}\frac{\delta}{\delta y^{(k-1)s}} + ... =$$

$$= \left(H^{s}{}_{ij} + \underset{(1)}{A}{}^{l}{}_{j}\underset{(1)}{C}{}^{s}{}_{il} + \underset{(2)}{A}{}^{l}{}_{j}\underset{(2)}{C}{}^{s}{}_{il} + ... + \underset{(k-1)}{A}{}^{l}{}_{j}\underset{(k-1)}{C}{}^{s}{}_{il} - A_{jl}C_i^{ls}\right)\frac{\delta}{\delta y^{(k-1)s}} +$$

$$+ \left(\underset{(1)}{A}{}^{l}{}_{j}\underset{(1)}{N}{}^{r}{}_{l}\underset{(2)}{C}{}^{s}{}_{ir}\frac{\delta}{\delta y^{(k-1)s}} + \underset{(1)}{A}{}^{l}{}_{j}\underset{(2)}{N}{}^{r}{}_{l}\underset{(3)}{C}{}^{s}{}_{ir}\frac{\delta}{\delta y^{(k-1)s}} + ... + \underset{(1)}{A}{}^{l}{}_{j}\underset{(k-2)}{N}{}^{r}{}_{l}\underset{(k-1)}{C}{}^{s}{}_{ir}\right.$$

$$\left.\frac{\delta}{\delta y^{(k-1)s}}\right) + \left(\underset{(1)}{A}{}^{l}{}_{j}\underset{(1)}{N}{}^{r}{}_{l}\underset{(1)}{N}{}^{s}{}_{r}D_{\frac{\partial}{\partial y^{(3)s}}}\frac{\delta}{\delta y^{(k-1)i}} + ... + \underset{(1)}{A}{}^{l}{}_{j}\underset{(1)}{N}{}^{r}{}_{l}\underset{(k-3)}{N}{}^{s}{}_{r}\underset{(k-1)}{C}{}^{m}{}_{is}\right.$$

$$\left.\frac{\delta}{\delta y^{(k-1)m}}\right) + \left(\underset{(1)}{A}{}^{l}{}_{j}\underset{(2)}{N}{}^{r}{}_{l}\underset{(1)}{N}{}^{s}{}_{r}D_{\frac{\partial}{\partial y^{(4)s}}}\frac{\delta}{\delta y^{(k-1)i}} + ... + \underset{(1)}{A}{}^{l}{}_{j}\underset{(2)}{N}{}^{r}{}_{l}\underset{(k-4)}{N}{}^{s}{}_{r}\underset{(k-1)}{C}{}^{m}{}_{is}\right.$$

$$\frac{\delta}{\delta y^{(k-1)m}}\Big) + ... + \Big(\underset{(2)}{A}^{l}{}_{j}\underset{(1)}{N}^{r}{}_{l}\underset{(3)}{C}^{s}{}_{ir}\frac{\delta}{\delta y^{(k-1)s}} + ... + \underset{(2)}{A}^{l}{}_{j}\underset{(k-3)}{N}^{r}{}_{l}\underset{(k-1)}{C}^{s}{}_{ir}\frac{\delta}{\delta y^{(k-1)s}}\Big) +$$

$$+\Big(\underset{(2)}{A}^{l}{}_{j}\underset{(1)}{N}^{r}{}_{l}\underset{(1)}{N}^{s}{}_{r}D_{\frac{\partial}{\partial y^{(4)s}}}\frac{\delta}{\delta y^{(k-1)i}} + ... + \underset{(2)}{A}^{l}{}_{j}\underset{(1)}{N}^{r}{}_{l}\underset{(k-4)}{N}^{s}{}_{r}\underset{(k-1)}{C}^{m}{}_{is}\frac{\delta}{\delta y^{(k-1)m}}\Big) + ...$$

So, we have obtained (2.1_1).

$$D_{\frac{\bar\delta}{\delta y^{(1)j}}}\frac{\bar\delta}{\delta y^{(k-1)i}} = \underset{(1)}{\bar C}^{s}{}_{ij}\frac{\bar\delta}{\delta y^{(k-1)s}} = \underset{(1)}{\bar C}^{s}{}_{ij}\frac{\delta}{\delta y^{(k-1)s}}.$$

$$D_{\frac{\bar\delta}{\delta y^{(1)j}}}\frac{\bar\delta}{\delta y^{(k-1)i}} = D_{\left(\frac{\delta}{\delta y^{(1)j}}+\underset{(1)}{A}^{l}{}_{j}\frac{\partial}{\partial y^{(2)l}}+\underset{(2)}{A}^{l}{}_{j}\frac{\partial}{\partial y^{(3)l}}+...+\underset{(k-2)}{A}^{l}{}_{j}\frac{\delta}{\delta y^{(k-1)l}}\right)}\frac{\delta}{\delta y^{(k-1)i}} =$$

$$= \Big(\underset{(1)}{C}^{s}{}_{ij} + \underset{(1)}{A}^{l}{}_{j}\underset{(2)}{C}^{s}{}_{il} + \underset{(2)}{A}^{l}{}_{j}\underset{(3)}{C}^{s}{}_{il} + ... + \underset{(k-2)}{A}^{l}{}_{j}\underset{(k-1)}{C}^{s}{}_{il}\Big)\frac{\delta}{\delta y^{(k-1)s}} +$$

$$+\underset{(1)}{A}^{l}{}_{j}\underset{(1)}{N}^{s}{}_{l}D_{\frac{\partial}{\partial y^{(3)s}}}\frac{\delta}{\delta y^{(k-1)i}} + ... + \underset{(1)}{A}^{l}{}_{j}\underset{(k-3)}{N}^{r}{}_{l}\underset{(k-1)}{C}^{s}{}_{ir}\frac{\delta}{\delta y^{(k-1)s}} +$$

$$+\underset{(2)}{A}^{l}{}_{j}\underset{(1)}{N}^{s}{}_{l}D_{\frac{\partial}{\partial y^{(4)s}}}\frac{\delta}{\delta y^{(k-1)i}} + ... + \underset{(2)}{A}^{l}{}_{j}\underset{(k-4)}{N}^{r}{}_{l}\underset{(k-1)}{C}^{s}{}_{ir}\frac{\delta}{\delta y^{(k-1)s}} + ...$$

So, we have obtained (2.2_2).

$$D_{\frac{\bar\delta}{\delta y^{(k-2)j}}}\frac{\bar\delta}{\delta y^{(k-1)i}} = \underset{(k-2)}{\bar C}^{s}{}_{ij}\frac{\bar\delta}{\delta y^{(k-1)s}} = \underset{(k-2)}{\bar C}^{s}{}_{ij}\frac{\delta}{\delta y^{(k-1)s}}.$$

$$D_{\frac{\bar\delta}{\delta y^{(k-2)j}}}\frac{\bar\delta}{\delta y^{(k-1)i}} = \underset{(k-2)}{\bar C}^{s}{}_{ij}\frac{\bar\delta}{\delta y^{(k-1)s}} + \underset{(1)}{A}^{l}{}_{j}\underset{(k-1)}{\bar C}^{s}{}_{il}\frac{\delta}{\delta y^{(k-1)s}}.$$

$$D_{\frac{\bar\delta}{\delta p_j}}\frac{\bar\delta}{\delta y^{(k-1)i}} = \bar C_{i}^{js}\frac{\bar\delta}{\delta y^{(k-1)s}} = \bar C_{i}^{js}\frac{\delta}{\delta y^{(k-1)s}};$$

$$D_{\frac{\bar\delta}{\delta p_j}}\frac{\bar\delta}{\delta y^{(k-1)i}} = D_{\frac{\bar\delta}{\delta p_j}}\frac{\delta}{\delta y^{(k-1)i}} = C_{i}^{js}\frac{\delta}{\delta y^{(k-1)s}};$$

So, we have obtained (2.2_{k-1}).

$$D_{\frac{\bar\delta}{\delta p_j}}\frac{\bar\delta}{\delta y^{(k-2)i}} = \bar C_{i}^{js}\frac{\bar\delta}{\delta y^{(k-2)s}} = \bar C_{i}^{js}\left(\frac{\delta}{\delta y^{(k-2)s}} + \underset{(1)}{A}^{l}{}_{s}\frac{\partial}{\partial y^{(k-1)l}}\right) =$$

$$= \bar C_{i}^{js}\frac{\delta}{\delta y^{(k-2)s}} + \bar C_{i}^{js}\underset{(1)}{A}^{l}{}_{s}\frac{\delta}{\delta y^{(k-1)l}}.$$

$$D_{\frac{\bar\delta}{\delta p_j}}\frac{\bar\delta}{\delta y^{(k-2)i}} = D_{\frac{\bar\delta}{\delta p_j}}\left(\frac{\delta}{\delta y^{(k-2)i}} + \underset{(1)}{A}^{l}{}_{i}\frac{\partial}{\partial y^{(k-1)l}}\right) =$$

$$= C_{i}^{js}\frac{\bar\delta}{\delta y^{(k-2)s}} + \left(\frac{\delta\underset{(1)}{A}^{s}{}_{i}}{\delta p_j} + \underset{(1)}{A}^{l}{}_{i}C_{l}^{js}\right)\frac{\bar\delta}{\delta y^{(k-1)s}}.$$

So, we have:

$$\bar C_{i}^{js} = C_{i}^{js} \tag{2.4}$$

$$\bar{C}_i^{\ jl} \underset{(1)}{A}{}^s{}_l = \frac{\delta \underset{(1)}{A}{}^s{}_i}{\delta p_j} + \underset{(1)}{A}{}^l{}_i C_l^{\ js}. \tag{2.5}$$

Analogous if we calculate $D_{\frac{\bar{\delta}}{\delta y^{(k-1)j}}} \frac{\bar{\delta}}{\delta y^{(k-2)i}}$ in two manner we obtain:

$$\underset{(k-1)}{\overline{C}}{}^s{}_{ij} = \underset{(k-1)}{C}{}^s{}_{ij}, \tag{2.6}$$

$$\underset{(k-1)}{\overline{C}}{}^l{}_{ij} \underset{(1)}{A}{}^s{}_l = \frac{\delta \underset{(1)}{A}{}^s{}_i}{\delta y^{(k-1)j}} + \underset{(1)}{A}{}^l{}_i \underset{(k-1)}{C}{}^s{}_{lj}. \tag{2.7}$$

We have:

$$\underset{(\alpha)}{A}{}^i{}_{j|k} = \frac{\delta \underset{(\alpha)}{A}{}^i{}_j}{\delta x_k} + \underset{(\alpha)}{A}{}^m{}_j H^i{}_{mk} - \underset{(\alpha)}{A}{}^i{}_m H^m{}_{jk},\ (\alpha = 1, 2, ..., k-1). \tag{2.8}$$

Using $(2.8), (2.7), (2.6), (2.5), (2.4), (2.2_{k-1}), (2.2_2)$ in the relation obtained analogous from $D_{\frac{\bar{\delta}}{\delta x^j}} \frac{\bar{\delta}}{\delta y^i}$, we obtain: $\underset{(1)}{A}{}^h{}_{i|j} = 0$. In the same manner we get $A_{ih|j} = 0$. □

Theorem 2.2. *Let N and $\bar{N}$ be two nonlinear connections on $T^{*k}M$, $(k \geq 2,\ k \in \mathbb{N})$, with local coefficients*

$$\left(\underset{(1)}{N}{}^j{}_i \left(x, y^{(1)}, ..., y^{(k-1)}, p\right), ..., \underset{(k-1)}{N}{}^j{}_i \left(x, y^{(1)}, ..., y^{(k-1)}, p\right), N_{ij} \left(x, y^{(1)}, ..., y^{(k-1)}, p\right)\right),$$

$$\left(\underset{(1)}{\bar{N}}{}^j{}_i \left(x, y^{(1)}, ..., y^{(k-1)}, p\right), ..., \underset{(k-1)}{\bar{N}}{}^j{}_i \left(x, y^{(1)}, ..., y^{(k-1)}, p\right), \overline{N}_{ij} \left(x, y^{(1)}, ..., y^{(k-1)}, p\right)\right),$$

$(i, j = 1, 2, ..., n)$, *respectively.*

If

$$D\Gamma(N) = \left(H^i{}_{jh}, \underset{(\alpha)}{C}{}^i{}_{jh}, C_i^{\ jh}\right)$$

and

$$D\bar{\Gamma}(\bar{N}) = \left(\bar{H}^i{}_{jh}, \underset{(\alpha)}{\bar{C}}{}^i{}_{jh}, \bar{C}_i^{\ jh}\right),$$

$(\alpha = 1, ..., k-1)$ *are the local coefficients of two $N-$, respectively $\bar{N}$-linear connections, D, respectively $\bar{D}$ on the differentiable manifold $T^{*k}M, (k \geq 2, k \in \mathbb{N})$, then there exists only one system of tensor fields*

$$\left(\underset{(1)}{A}{}^i{}_j, ..., \underset{(k-1)}{A}{}^i{}_j, A_{ij}, B^i{}_{jh}, \underset{(1)}{D}{}^i{}_{jh}, ..., \underset{(k-1)}{D}{}^i{}_{jh}, D_i^{\ jh}\right)$$

such that:

$$\begin{cases} \underset{(\alpha)}{\bar{N}}{}^i{}_j = \underset{(\alpha)}{N}{}^i{}_j - \underset{(\alpha)}{A}{}^i{}_j, (\alpha = 1, ..., k-1), \\ \bar{N}_{ij} = N_{ij} - A_{ij}, \\ \bar{H}^i{}_{sj} = H^i{}_{sj} + \underset{(1)}{A}{}^m{}_j \Big[\underset{(1)}{C}{}^i{}_{sm} + \underset{(1)}{N}{}^l{}_m \underset{(2)}{C}{}^i{}_{sl} + ... + \underset{(k-2)}{N}{}^l{}_m \underset{(k-1)}{C}{}^i{}_{sl} + \underset{(1)}{N}{}^l{}_m \underset{(1)}{N}{}^t{}_l \underset{(3)}{C}{}^i{}_{st} + \\ +... + \left(\underset{(1)}{N}{}^l{}_m \underset{(k-3)}{N}{}^t{}_l + ... + \underset{(k-3)}{N}{}^l{}_m \underset{(1)}{N}{}^t{}_l \right) \underset{(k-1)}{C}{}^i{}_{st} + ... + \underbrace{\underset{(1)}{N} ... \underset{(1)}{N}}_{(k-2)} \underset{(k-1)}{C} \Big] + \\ + \underset{(2)}{A}{}^m{}_j \Big[\underset{(2)}{C}{}^i{}_{sm} + \underset{(1)}{N}{}^l{}_m \underset{(3)}{C}{}^i{}_{sl} + ... + \underset{(k-3)}{N}{}^l{}_m \underset{(k-1)}{C}{}^i{}_{sl} + ... + \underbrace{\underset{(1)}{N} ... \underset{(1)}{N}}_{(k-3)} \underset{(k-1)}{C} \Big] + \\ +... + \underset{(k-2)}{A}{}^m{}_j \left(\underset{(k-2)}{C}{}^i{}_{sm} + \underset{(1)}{N}{}^l{}_m \underset{(k-1)}{C}{}^i{}_{sl} \right) + \underset{(k-1)}{A}{}^m{}_j \underset{(k-1)}{C}{}^i{}_{sm} - A_{jm} C_s{}^{im} - B^i{}_{sj}, \\ \underset{(1)}{\overline{C}}{}^i{}_{sj} = \underset{(1)}{C}{}^i{}_{sj} + \underset{(1)}{A}{}^m{}_j \Big[\underset{(2)}{C}{}^i{}_{sm} + \underset{(1)}{N}{}^r{}_m \underset{(3)}{C}{}^i{}_{sr} + ... + \underset{(k-3)}{N}{}^r{}_m \underset{(k-1)}{C}{}^i{}_{sr} + ...+ \\ + \underbrace{\underset{(1)}{N} \cdot ... \cdot \underset{(1)}{N}}_{(k-3)} \underset{(k-1)}{C} \Big] + ... + \underset{(k-3)}{A}{}^m{}_j \Big[\underset{(k-2)}{C}{}^i{}_{sm} + \underset{(1)}{N}{}^r{}_m \underset{(k-1)}{C}{}^i{}_{sr} \Big] + \\ + \underset{(k-2)}{A}{}^m{}_j \underset{(k-1)}{C}{}^i{}_{sm} - \underset{(1)}{D}{}^i{}_{sj}, \\ \underset{(k-2)}{\overline{C}}{}^i{}_{sj} = \underset{(k-2)}{C}{}^i{}_{sj} + \underset{(1)}{A}{}^l{}_j \underset{(k-1)}{C}{}^i{}_{sl} - \underset{(k-2)}{D}{}^i{}_{sj}, \\ \underset{(k-1)}{\bar{C}}{}^i{}_{sj} = \underset{(k-1)}{C}{}^i{}_{sj} - \underset{(k-1)}{D}{}^i{}_{sj}, \\ \bar{C}_s{}^{ij} = C_s{}^{ij} - D_s{}^{ij}, \end{cases} \tag{2.9}$$

with:

$$\begin{cases} \underset{(1)}{A}{}^h{}_{i|j} = 0, \\ A_{ih|j} = 0, (i, j, h = 1, 2..., n), \end{cases} \tag{2.10}$$

where $"_{|}"$ *denotes the* $h-$*covariant derivative with respect to* $D\Gamma(N)$.

Proof. The first equality (2.9) determines uniquely the tensor fields: $\underset{(\alpha)}{A}{}^i{}_j, (\alpha = 1, ..., k-1)$.The second equality (2.9) determines uniquely the tensor field A_{ij}. Since $\underset{(\alpha)}{C}{}^i{}_{jh}, (\alpha = 1, ..., k-1)$ and C_i^{jh} are $d-$tensor fields, the third equation (2.9) determines uniquely the tensor field $B^i{}_{jh}$. Similarly the fourth,... and the last equation (2.9) determines the tensor field $D_i{}^{jh}$ respectively. □

We have immediately:

Theorem 2.3. *If* $D\Gamma(N) = \left(H^i{}_{jh}, \underset{(\alpha)}{C}{}^i{}_{jh}, C_i{}^{jh}\right)(\alpha = 1, ..., k-1)$, *are the local coefficients of an* $N-$*linear connection* D *on* $T^{*k}M$ *and*

$$\left(\underset{(1)}{A}{}^i{}_j, ..., \underset{(k-1)}{A}{}^i{}_j, A_{ij}, B^i{}_{jh}, \underset{(1)}{D}{}^i{}_{jh}, ..., \underset{(k-1)}{D}{}^i{}_{jh}, D_i{}^{jh}\right),$$

is a system of tensor fields on $T^{*k}M$, *then* $D\bar{\Gamma}\left(\bar{N}\right) = \left(\bar{H}^i{}_{jh}, \underset{(\alpha)}{\bar{C}}{}^i{}_{jh}, \bar{C}_i{}^{jh}\right)$, $(\alpha = 1, ..., k-1)$, *given by* $(2.9)-(2.10)$ *are the local coefficients of an* $\bar{N}-$*linear connection,* $\bar{D}$, *on* $T^{*k}M, (k \geq 2, k \in \mathbb{N})$.

Following the definition given by M. Matsumoto [4, 5] in the case of Finsler spaces, we have:

Definition 2.1. i) The system of tensor fields:

$$\left(\underset{(1)}{A}{}^i{}_j, ..., \underset{(k-1)}{A}{}^i{}_j, A_{ij}, B^i{}_{jh}, \underset{(1)}{D}{}^i{}_{jh}, ..., \underset{(k-1)}{D}{}^i{}_{jh}, D_i{}^{jh}\right), \ (k \geq 2, k \in \mathbb{N})$$

is called the difference tensor fields of $D\Gamma(N)$ to $D\bar{\Gamma}\left(\bar{N}\right)$.

ii) The mapping: $D\Gamma(N) \longrightarrow D\bar{\Gamma}\left(\bar{N}\right)$ given by $(2.9)-(2.10)$ is called a transformation of $N-$linear connection to $\bar{N}-$linear connection on $T^{*k}M$, and it is noted by:

$$t\left(\underset{(1)}{A}{}^i{}_j, ..., \underset{(k-1)}{A}{}^i{}_j, A_{ij}, B^i{}_{jh}, \underset{(1)}{D}{}^i{}_{jh}, ..., \underset{(k-1)}{D}{}^i{}_{jh}, D_i{}^{jh}\right).$$

Theorem 2.4. *The set* $\mathcal{T}$ *of the transformations of* $N-$*linear connections to* $\bar{N}-$*linear connections on* $T^{*k}M, (k \geq 2, k \in \mathbb{N})$ *together with the composition of mappings isn't a group.*

Proof. Let

$$t\left(\underset{(1)}{A}{}^i{}_j, ..., \underset{(k-1)}{A}{}^i{}_j, A_{ij}, B^i{}_{jh}, \underset{(1)}{D}{}^i{}_{jh}, ..., \underset{(k-1)}{D}{}^i{}_{jh}, D_i{}^{jh}\right) : D\Gamma(N) \longrightarrow D\bar{\Gamma}\left(\bar{N}\right)$$

and

$$t\left(\underset{(1)}{\bar{A}}{}^i{}_j, ..., \underset{(k-1)}{\bar{A}}{}^i{}_j, \bar{A}_{ij}, \bar{B}^i{}_{jh}, \underset{(1)}{\bar{D}}{}^i{}_{jh}, ..., \underset{(k-1)}{\bar{D}}{}^i{}_{jh}, \bar{D}_i{}^{jh}\right) : D\bar{\Gamma}\left(\bar{N}\right) \longrightarrow D\bar{\bar{\Gamma}}\left(\bar{\bar{N}}\right),$$

be two transformations from $\mathcal{T}$, given by $(2.9)-(2.10)$.

From (2.9) we have:

$$\underset{(\alpha)}{\bar{\bar{N}}}{}^i{}_j = \underset{(\alpha)}{N}{}^i{}_j - \left(\underset{(\alpha)}{A}{}^i{}_j + \underset{(\alpha)}{\bar{A}}{}^i{}_j\right), \ (\alpha = 1, ..., k-1), \ \bar{\bar{N}}_{ij} = N_{ij} - \left(A_{ij} + \bar{\bar{A}}_{ij}\right).$$

We obtain for example:

$$\underset{(k-2)}{\bar{\bar{C}}}{}^i{}_{jh} = \underset{(k-2)}{C}{}^i{}_{jh} + \left(\underset{(1)}{A}{}^l{}_h + \underset{(1)}{\bar{A}}{}^l{}_h\right) \cdot \underset{(k-1)}{C}{}^i{}_{jl} - \left(\underset{(k-2)}{D}{}^i{}_{jh} + \underset{(k-2)}{\bar{D}}{}^i{}_{jh} + \underset{(k-1)}{D}{}^i{}_{jl}\underset{(1)}{\bar{A}}{}^l{}_h\right).$$

So $\underset{(k-2)}{\bar{\bar{C}}}{}^i{}_{jh}$ hasn't the form (2.9). It follows that the composition of two transformations from $\mathcal{T}$ isn't a transformation from $\mathcal{T}$, that is $\mathcal{T}$, together with the composition of mappings isn't a group. □

Remark 2.1. If we consider $\underset{(\alpha)}{A}{}^{i}{}_{j} = 0, (\alpha = 1, ..., k-1)$ and $A_{ij} = 0$ in (2.10) we obtain the set $\mathcal{T}_N$ of transformations of $N-$linear connections corresponding to the same nonlinear connection N :

$$\mathcal{T}_N = \left\{ t\left(\underbrace{0,...,0}_{(k)}, B^i{}_{jh}, \underset{(1)}{D}{}^i{}_{jh}, ..., \underset{(k-1)}{D}{}^i{}_{jh}, D_i{}^{jh}\right) \in \mathcal{T} \right\}.$$

We have:

Theorem 2.5. *The set $\mathcal{T}_N$ of the transformations of $N-$linear connections to $N-$linear connections on $T^{*k}M, (k \geq 2, k \in \mathbb{N})$, together with the composition of mappings is a group. This group, acts effectively and transitively on the set of $N-$linear connections.*

Proof. Let $t\left(\underbrace{0,...,0}_{(k)}, B^i{}_{jh}, \underset{(1)}{D}{}^i{}_{jh}, ..., \underset{(k-1)}{D}{}^i{}_{jh}, D_i{}^{jh}\right) : D\Gamma(N) \longrightarrow D\bar{\Gamma}(N)$ be a transformation from $\mathcal{T}_N$,given by (2.11) :

$$\begin{cases} \underset{(\alpha)}{\bar{N}}{}^i{}_j = \underset{(\alpha)}{N}{}^i{}_j, (\alpha = 1, ..., k-1), \\ \bar{N}_{ij} = N_{ij}, \\ \bar{H}^i{}_{jh} = H^i{}_{jh} - B^i{}_{jh}, \\ \underset{(\alpha)}{\bar{C}}{}^i{}_{jh} = \underset{(\alpha)}{C}{}^i{}_{jh} - \underset{(\alpha)}{D}{}^i{}_{jh}, (\alpha = 1, ..., k-1), \\ \bar{C}_i{}^{jh} = C_i{}^{jh} - D_i{}^{jh}, (i, j, h = 1, 2, ..., n). \end{cases} \tag{2.11}$$

The composition of two transformations from $\mathcal{T}_N$ is a transformation from $\mathcal{T}_N$, given by:

$$t\left(\underbrace{0,...,0}_{(k)}, \bar{B}^i{}_{jh}, \underset{(1)}{\bar{D}}{}^i{}_{jh}, ..., \underset{(k-1)}{\bar{D}}{}^i{}_{jh}, \bar{D}_i{}^{jh}\right) \circ t\left(\underbrace{0,...,0}_{(k)}, B^i{}_{jh}, \underset{(1)}{D}{}^i{}_{jh}, ..., \underset{(k-1)}{D}{}^i{}_{jh}, D_i{}^{jh}\right)$$

$$= t\left(\underbrace{0,...,0}_{(k)}, B^i{}_{jh}, +\bar{B}^i{}_{jh}, \underset{(1)}{D}{}^i{}_{jh} + \underset{(1)}{\bar{D}}{}^i{}_{jh}, ..., \underset{(k-1)}{D}{}^i{}_{jh} + \underset{(k-1)}{\bar{D}}{}^i{}_{jh}, D_i{}^{jh} + \bar{D}_i{}^{jh}\right).$$

The inverse of a transformation from $\mathcal{T}_N$ is the following transformation from $\mathcal{T}_N$:

$$t\left(0, 0, 0, -B^i{}_{jh}, -\underset{(1)}{D}{}^i{}_{jh}, ..., -\underset{(k-1)}{D}{}^i{}_{jh}, -D_i{}^{jh}\right) : D\Gamma(N) \longrightarrow D\bar{\Gamma}(N).$$

The transformations (2.11) preserve all $N-$linear connections D if:

$$B^i{}_{jh} = \underset{(1)}{D}{}^i{}_{jh} = ... = \underset{(k-1)}{D}{}^i{}_{jh} = D_i{}^{jh} = 0, (i, j, h = 1, 2, ..., n).$$

Therefore $\mathcal{T}_N$ acts effectively on the set of $N-$linear connections. From the Theorem 2.2 results that $\mathcal{T}_N$ acts transitively on this set. □

Let us consider:

$$\mathcal{T}_{NH}=\left\{t\left(\underbrace{0,...,0}_{(k+1)},\underset{(1)}{D}{}^{i}{}_{jh},...,\underset{(k-1)}{D}{}^{i}{}_{jh},D_{i}{}^{jh}\right)\in\mathcal{T}_{N}\right\},$$

$$\mathcal{T}_{N\underset{(1)}{C}}=\left\{t\left(\underbrace{0,...,0}_{(k)},B^{i}{}_{jh},0,\underset{(2)}{D}{}^{i}{}_{jh},...,\underset{(k-1)}{D}{}^{i}{}_{jh},D_{i}{}^{jh}\right)\in\mathcal{T}_{N}\right\},$$

....

$$\mathcal{T}_{N\underset{(k-1)}{C}}=\left\{t\left(\underbrace{0,...,0}_{(k)},B^{i}{}_{jh},\underset{(1)}{D}{}^{i}{}_{jh},...,\underset{(k-2)}{D}{}^{i}{}_{jh},0,D_{i}{}^{jh}\right)\in\mathcal{T}_{N}\right\},$$

$$\mathcal{T}_{NC}=\left\{t\left(\underbrace{0,...,0}_{(k)},B^{i}{}_{jh},\underset{(1)}{D}{}^{i}{}_{jh},...,\underset{(k-1)}{D}{}^{i}{}_{jh},0\right)\in\mathcal{T}_{N}\right\},$$

$$\mathcal{T}_{N\underset{(1)}{C}...\underset{(k-1)}{C}C}=\left\{t\left(\underbrace{0,...,0}_{(k)},B^{i}{}_{jh},\underbrace{0,...,0}_{(k)}\right)\in\mathcal{T}_{N}\right\},(k\geq 2,k\in\mathbb{N}).$$

Proposition 2.1. *The sets:*$\mathcal{T}_{NH},\mathcal{T}_{N\underset{(1)}{C}},...,\mathcal{T}_{N\underset{(k-1)}{C}},...,\mathcal{T}_{NC},\mathcal{T}_{N\underset{(1)}{C}...\underset{(k-1)}{C}C}$ *are Abelian subgroups of* $\mathcal{T}_{N}$.

Proposition 2.2. *The group* $\mathcal{T}_{N}$ *preserves the nonlinear connection* N, $\mathcal{T}_{NH}$ *preserves the nonlinear connection* N *and the component* $H^{i}{}_{jh}$ *of the local coefficients* $D\Gamma(N)$; $\mathcal{T}_{N\underset{(1)}{C}}$ *preserves the nonlinear connection* N *and the component* $\underset{(1)}{C}{}^{i}{}_{jh}$ *of the local coefficients* $D\Gamma(N)$, ..., $\mathcal{T}_{N\underset{(k-1)}{C}}$ *preserves the nonlinear connection* N *and the component* $\underset{(k-1)}{C}{}^{i}{}_{jh}$ *of the local coefficients* $D\Gamma(N)$, $\mathcal{T}_{NC}$ *preserves the nonlinear connection* N *and the component* $C_{i}{}^{jh}$ *of the local coefficients* $D\Gamma(N)$ *and* $\mathcal{T}_{N\underset{(1)}{C}...\underset{(k-1)}{C}C}$ *preserves the nonlinear connection* N *and the components* $\underset{(1)}{C}{}^{i}{}_{jh},,...,\underset{(k-1)}{C}{}^{i}{}_{jh},C_{i}{}^{jh}$ *of the local coefficients* $D\Gamma(N)$.

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Totally supra $b-$continuous and slightly supra $b-$continuous functions

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Abstract. In this paper, totally supra b-continuity and slightly supra b-continuity are introduced and studied. Furthermore, basic properties and preservation theorems of totally supra b-continuous and slightly supra b-continuous functions are investigated and the relationships between these functions and their relationships with some other functions are investigated.

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1. Introduction and preliminaries

In 1983, A. S. Mashhour et al. [11] introduced the supra topological spaces. In 1996, D. Andrijevic [1] introduced and studied a class of generalized open sets in a topological space called b-open sets. This type of sets discussed by El-Atike [10] under the name of $\gamma-$open sets. Also, in recent years, Ekici has studied some relationships of $\gamma-$open sets [5, 6, 8, 9]. In 2010, O. R. Sayed et al. [12] introduced and studied a class of sets and a class of maps between topological spaces called supra b-open sets and supra b-continuous functions, respectively. Now we introduce the concepts of totally supra b-continuous and slightly supra b-continuous functions and investigate several properties for these concepts.

Throughout this paper (X,τ), (Y,ρ) and (Z,σ) (or simply X, Y and Z) denote topological spaces on which no separation axioms are assumed unless explicitly stated. For a subset A of (X,τ), the closure and the interior of A in X are denoted by $Cl(A)$ and $Int(A)$, respectively. The complement of A is denoted by $X-A$. In the space (X,τ), a subset A is said to be b-open [1] if $A \subseteq Cl(Int(A)) \cup Int(Cl(A))$. A subcollection $\mu \subseteq 2^X$ is called a supra topology [11] on X if X, $\phi \in \mu$ and μ is closed under arbitrary union. (X,μ) is called a supra topological space. The elements of μ

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are said to be supra open in (X, μ) and the complement of a supra open set is called a supra closed set. The supra closure of a set A, denoted by $Cl^{\mu}(A)$, is the intersection of supra closed sets including A. The supra interior of a set A, denoted by $Int^{\mu}(A)$, is the union of supra open sets included in A. The supra topology μ on X is associated with the topology τ if $\tau \subseteq \mu$.

Definition 1.1. [12] *Let (X, μ) be a supra topological space. A set A is called a supra b-open set if $A \subseteq Cl^{\mu}(Int^{\mu}(A)) \cup Int^{\mu}(Cl^{\mu}(A))$. The complement of a supra b-open set is called a supra b-closed set.*

Definition 1.2. [2] *Let (X, μ) be a supra topological space. A set A is called a supra α-open set if $A \subseteq Int^{\mu}(Cl^{\mu}(Int^{\mu}(A)))$. The complement of a supra α-open set is called a supra α-closed set.*

Theorem 1.3. [12]. *(i) Arbitrary union of supra b-open sets is always supra b-open.*

(ii) Finite intersection of supra b-open sets may fail to be supra b-open.

Lemma 1.4. [12] *The intersection of a supra α-open set and a supra b-open set is a supra b-open set.*

Definition 1.5. [12] *The supra b-closure of a set A, denoted by $Cl^{\mu}_{b}(A)$, is the intersection of supra b-closed sets including A. The supra b-interior of a set A, denoted by $Int^{\mu}_{b}(A)$, is the union of supra b-open sets included in A.*

Definition 1.6. [7] *A function $f : X \longrightarrow Y$ is called:*

(1) slightly γ-continuous at a point $x \in X$ if for each clopen subset V in Y containing $f(x)$, there exists a γ-open subset U of X containing x such that $f(U) \subset V$.

(2) slightly γ-continuous if it has this property at each point of X.

Definition 1.7. [3, 7] *A function $f : X \longrightarrow Y$ is called:*

(i) γ-irresolute if for each γ-open subset G of Y, $f^{-1}(G)$ is γ-open in X.

(ii) γ-open if for every γ-open subset A of X, $f(A)$ is γ-open in Y.

Definition 1.8. [7] *A space X is called γ-connected provided that X is not the union of two disjoint nonempty γ-open sets.*

Definition 1.9. [3] *A space X is said to be:*

(i) $\gamma - T_1$ if for each pair of distinct points x and y of X, there exist γ-open sets U and V containing x and y, respectively such that $y \notin U$ and $x \notin V$.

(ii) $\gamma - T_2$ (γ-Hausdorff) if for each pair of distinct points x and y of X, there exist disjoint γ-open sets U and V in X such that $x \in U$ and $y \in V$.

Definition 1.10. [3, 7] *A space X is said to be:*

(i) $\gamma-$Lindelöf if every γ-open cover of X has a countable subcover.

(ii) $\gamma-$closed-compact if every γ-closed cover of X has a finite subcover.

(iii) $\gamma-$closed-Lindelöf if every cover of X by γ-closed sets has a countable subcover.

2. Totally supra $b-$continuous functions

In this section, the notion of totally supra b-continuous functions is introduced. If A is both supra b-open and supra b-closed, then it is said to be supra b-clopen.

Definition 2.1. [12] *Let (X, τ) and (Y, ρ) be two topological spaces and μ be an associated supra topology with τ. A function $f : (X, \tau) \longrightarrow (Y, \rho)$ is called a supra b-continuous function if the inverse image of each open set in Y is supra b-open in X.*

Definition 2.2. *Let (X, τ) and (Y, ρ) be two topological spaces and μ be an associated supra topology with τ. A function $f : (X, \tau) \longrightarrow (Y, \rho)$ is called a totally supra b-continuous function if the inverse image of each open set in Y is supra b-clopen in X.*

Remark 2.3. Every totally supra b-continuous function is supra b-continuous but the converse need not be true as it can be seen from the following example.

Example 2.4. Let $X = \{a, b, c\}$ and $\tau = \{X, \phi, \{a, b\}\}$ be a topology on X. The supra topology μ is defined as follows: $\mu = \{X, \phi, \{a\}, \{a, b\}\}$. Let $f : (X, \tau) \longrightarrow (X, \tau)$ be a function defined as follows: $f(a) = a$, $f(b) = c$, $f(c) = b$. The inverse image of the open set $\{a, b\}$ is $\{a, c\}$ which is supra b-open but it is not supra b-clopen. Then f is supra b-continuous but it is not totally supra b-continuous.

Definition 2.5. *A supra topological space (X, μ) is called supra $b-$connected if it is not the union of two nonempty disjoint supra b-open sets.*

Theorem 2.6. *A supra topological space (X, μ) is supra $b-$connected if and only if X and ϕ are the only supra b-clopen subsets of X.*

Proof. Obvious. □

Theorem 2.7. *Let (X, τ) be a topological spaces and μ be an associated supra topology with τ. If $f : (X, \tau) \longrightarrow (Y, \rho)$ is a totally supra b-continuous surjection and (X, μ) is supra $b-$connected, then (Y, ρ) is an indiscrete space.*

Proof. Suppose that (Y, ρ) is not an indiscrete space and let V be a proper nonempty open subset of (Y, ρ). Since f is a totally supra b-continuous function, then $f^{-1}(V)$ is a proper nonempty supra b-clopen subset of X. Therefore $X = f^{-1}(V) \cup (X - f^{-1}(V))$ and X is a union of two nonempty disjoint supra b-open sets, which is a contradiction. Therefore X must be an indiscrete space. □

Theorem 2.8. *Let (X, τ) be a topological space and μ be an associated supra topology with τ. The supra topological space (X, μ) is supra $b-$connected if and only if every totally supra b-continuous function from (X, τ) into any T_0-space (Y, ρ) is a constant map.*

Proof. $\Rightarrow$) Suppose that $f : (X, \tau) \longrightarrow (Y, \rho)$ is a totally supra b-continuous function, where (Y, ρ) is a T_0-space. Assume that f is not constant and $x, y \in X$ such that $f(x) \neq f(y)$. Since (Y, ρ) is T_0, and $f(x)$ and $f(y)$ are distinct points in Y, then there is an open set V in (Y, ρ) containing only one of the points $f(x)$, $f(y)$. We take the case $f(x) \in V$ and $f(y) \notin V$. The proof of the other case is similar. Since f is a totally supra b-continuous function, $f^{-1}(V)$ is a supra b-clopen subset of X and $x \in f^{-1}(V)$, but $y \notin f^{-1}(V)$. Since $X = f^{-1}(V) \cup (X - f^{-1}(V))$, X is a union of two nonempty disjoint supra b-open subsets of X. Thus (X, μ) is not supra $b-connected$, which is a contradiction.

$\Leftarrow$) Suppose that (X, μ) is not a supra $b-connected$ space, then there is a proper nonempty supra b-clopen subset A of X. Let $Y = \{a, b\}$ and $\rho = \{Y, \phi, \{a\}, \{b\}\}$, define $f : (X, \tau) \longrightarrow (Y, \rho)$ by $f(x) = a$ for each $x \in A$ and $f(x) = b$ for $x \in X - A$. Clearly f is not constant and totally supra b-continuous where Y is T_0, and thus we have a contradiction. □

Definition 2.9. *A supra topological space X is said to be:*

(i) supra $b - T_1$ if for each pair of distinct points x and y of X, there exist supra b-open sets U and V containing x and y, respectively such that $y \notin U$ and $x \notin V$.

(ii) supra $b - T_2$ if for each pair of distinct points x and y in X, there exist disjoint supra b-open sets U and V in X such that $x \in U$ and $y \in V$.

Theorem 2.10. *Let (X, τ) and (Y, ρ) be two topological spaces and μ be an associated supra topology with τ. Let $f : (X, \tau) \longrightarrow (Y, \rho)$ be a totally supra b-continuous injection. If Y is T_0 then (X, μ) is supra $b - T_2$.*

Proof. Let $x, y \in X$ with $x \neq y$. Since f is injection, $f(x) \neq f(y)$. Since Y is T_0, there exists an open subset V of Y containing $f(x)$ but not $f(y)$, or containing $f(y)$ but not $f(x)$. Thus for the first case we have, $x \in f^{-1}(V)$ and $y \notin f^{-1}(V)$. Since f is totally supra b-continuous and V is an open subset of Y, $f^{-1}(V)$ and $X - f^{-1}(V)$ are disjoint supra b-clopen subsets of X containing x and y, respectively. The second case is proved in the same way. Thus X is supra $b - T_2$. □

Definition 2.11. *Let (X, τ) be a topological space and μ be an associated supra topology with τ. A function $f : (X, \tau) \longrightarrow Y$ is called a strongly supra b-continuous function if the inverse image of every subset of Y is a supra b-clopen subset of X.*

Remark 2.12. Every strongly supra b-continuous function is totally supra b-continuous, but the converse need not be true as the following example shows.

Example 2.13. Let $X = \{a, b, c\}$ and $\tau = \{X, \phi\}$ be a topology on X. The supra topology μ is defined as follows: $\mu = \{X, \phi, \{a, c\}\}$. Let $f : (X, \tau) \longrightarrow (X, \tau)$ be the identity function, then f is totally supra b-continuous but it is not strongly supra b-continuous.

3. Slightly supra $b-$continuous functions

In this section, the notion of slightly supra b-continuous functions is introduced and characterizations and some relationships of slightly supra b-continuous functions and basic properties of slightly supra b-continuous functions are investigated and obtained.

Definition 3.1. *Let (X,τ) and (Y,ρ) be two topological spaces and μ be an associated supra topology with τ. A function $f:(X,\tau)\longrightarrow(Y,\rho)$ is called a slightly supra b-continuous function at a point $x\in X$ if for each clopen subset V in Y containing $f(x)$, there exists a supra b-open subset U in X containing x such that $f(U)\subseteq V$. The function f is said to be slightly supra b-continuous if it has this property at each point of X.*

Remark 3.2. Every supra b-continuous function is slightly supra b-continuous but the converse need not be true as it can be seen from the following example.

Example 3.3. Let R and N be the real numbers and natural numbers, respectively. Take two topologies on R as $\tau=\{R,\phi\}$ and $\rho=\{R,\phi,R-N\}$ and μ be the associated supra topology with τ defined as $\mu=\{R,\phi,N\}$. Let $f:(R,\tau)\longrightarrow(R,\rho)$ be an identity function. Then, f is slightly supra b-continuous, but it is not supra b-continuous.

Remark 3.4. Since every totally supra b-continuous function is supra b-continuous then every totally supra b-continuous function is slightly supra b-continuous but the converse need not be true. The function f in Example 3.3 is slightly supra b-continuous but it is not totally supra b-continuous.

Remark 3.5. Since every strongly supra b-continuous function is totally supra b-continuous then every strongly supra b-continuous function is slightly supra b-continuous but the converse need not be true. The function f in Example 2.13 is slightly supra b-continuous but it is not strongly supra b-continuous.

Theorem 3.6. *Let (X,τ) and (Y,ρ) be two topological spaces and μ be an associated supra topology with τ. The following statements are equivalent for a function $f:(X,\tau)\longrightarrow(Y,\rho)$:*

(1) f is slightly supra b-continuous;
(2) for every clopen set $V\subseteq Y$, $f^{-1}(V)$ is supra b-open;
(3) for every clopen set $V\subseteq Y$, $f^{-1}(V)$ is supra b-closed;
(4) for every clopen set $V\subseteq Y$, $f^{-1}(V)$ is supra b-clopen.

Proof. (1) $\Rightarrow$ (2): Let V be a clopen subset of Y and let $x\in f^{-1}(V)$. Since f is slightly supra b-continuous, by (1) there exists a supra b-open set U_x in X containing x such that $f(U_x)\subseteq V$; hence $U_x\subseteq f^{-1}(V)$. We obtain that $f^{-1}(V)=\cup\{U_x : x\in f^{-1}(V)\}$. Thus, $f^{-1}(V)$ is supra b-open.

(2) $\Rightarrow$ (3): Let V be a clopen subset of Y. Then $Y-V$ is clopen. By (2) $f^{-1}(Y-V)=X-f^{-1}(V)$ is supra b-open. Thus $f^{-1}(V)$ is supra b-closed.

(3) $\Rightarrow$ (4): It can be shown easily.

(4) $\Rightarrow$ (1): Let $x\in X$ and V be a clopen subset in Y with $f(x)\in V$. Let $U=f^{-1}(V)$. By assumption U is supra b-clopen and so supra b-open. Also $x\in U$ and $f(U)\subseteq V$. □

Corollary 3.7. [7] *Let (X,τ) and (Y,ρ) be topological spaces. The following statements are equivalent for a function $f:X\longrightarrow Y$:*

(1) f is slightly γ-continuous;
(2) for every clopen set $V\subset Y, f^{-1}(V)$ is γ-open;

(3) for every clopen set $V \subset Y, f^{-1}(V)$ is γ-closed;
(4) for every clopen set $V \subset Y, f^{-1}(V)$ is γ-clopen.

Theorem 3.8. *Every slightly supra b-continuous function into a discrete space is strongly supra b-continuous.*

Proof. Let $f : X \longrightarrow Y$ be a slightly supra b-continuous function and Y be a discrete space. Let A be any subset of Y. Then A is a clopen subset of Y. Hence $f^{-1}(A)$ is supra b-clopen in X. Thus f is strongly supra b-continuous. □

Definition 3.9. *Let (X, τ) and (Y, ρ) be two topological spaces and μ, η be associated supra topologies with τ and ρ, respectively. A function $f : (X, \tau) \to (Y, \rho)$ is called a supra b-irresolute function if the inverse image of each supra b-open set in Y is a supra b-open set in X.*

Theorem 3.10. *Let (X, τ), (Y, ρ) and (Z, σ) be topological spaces and μ, η be associated supra topologies with τ and ρ, respectively. Let $f : (X, \tau) \to (Y, \rho)$ and $g : (Y, \rho) \to (Z, \sigma)$ be functions. Then, the following properties hold:*

(1) If f is supra b-irresolute and g is slightly supra b-continuous, then gof is slightly supra b-continuous.

(2) If f is slightly supra b-continuous and g is continuous, then gof is slightly supra b-continuous.

Proof. (1) Let V be any clopen set in Z. Since g is slightly supra b-continuous, $g^{-1}(V)$ is supra b-open. Since f is supra b-irresolute, $f^{-1}(g^{-1}(V)) = (gof)^{-1}(V)$ is supra b-open. Therefore, gof is slightly supra b-continuous.

(2) Let V be any clopen set in Z. By the continuity of g, $g^{-1}(V)$ is clopen. Since f is slightly supra b-continuous, $f^{-1}(g^{-1}(V)) = (gof)^{-1}(V)$ is supra b-open. Therefore, gof is slightly supra b-continuous. □

Corollary 3.11. *Let (X, τ), (Y, ρ) and (Z, σ) be topological spaces and μ, η be associated supra topologies with τ and ρ, respectively. If $f : (X, \tau) \to (Y, \rho)$ is a supra b-irresolute function and $g : (Y, \rho) \to (Z, \sigma)$ is a supra b-continuous function, then gof is slightly supra b-continuous.*

Corollary 3.12. [7] *Let $f : X \to Y$ and $g : Y \to Z$ be functions. Then, the following properties hold:*

(1) If f is γ-irresolute and g is slightly γ-continuous, then $gof : X \to Z$ is slightly γ-continuous.

(2) If f is γ-irresolute and g is γ-continuous, then $gof : X \to Z$ is slightly γ-continuous.

Definition 3.13. *A function $f : (X, \tau) \to (Y, \rho)$ is called a supra b-open function if the image of each supra b-open set in X is a supra b-open set in Y.*

Theorem 3.14. *Let (X, τ), (Y, ρ) and (Z, σ) be topological spaces and μ, η be associated supra topologies with τ and ρ, respectively. Let $f : (X, \tau) \to (Y, \rho)$ be a supra b-irresolute, supra b-open surjection and $g : (Y, \rho) \to (Z, \sigma)$ be a function. Then g is slightly supra b-continuous if and only if gof is slightly supra b-continuous.*

Proof. $\Rightarrow$) Let g be slightly supra b-continuous. Then by Theorem 3.10, gof is slightly supra b-continuous.

$\Leftarrow$) Let gof be slightly supra b-continuous and V be clopen set in Z. Then $(gof)^{-1}(V)$ is supra b-open. Since f is a supra b-open surjection, then $f((gof)^{-1}(V)) = g^{-1}(V)$ is supra b-open in Y. This shows that g is slightly supra b-continuous. □

Corollary 3.15. [7] *$f : X \to Y$ be surjective, γ-irresolute and γ-open and $g : Y \to Z$ be a function. Then $gof : X \to Z$ is slightly γ-continuous if and only if g is slightly γ-continuous.*

Theorem 3.16. *Let (X, τ) be a topological space and μ be an associated supra topology with τ. If $f : (X, \tau) \longrightarrow (Y, \rho)$ is a slightly supra b-continuous function and (X, μ) is supra $b-connected$, then Y is connected.*

Proof. Suppose that Y is a disconnected space. Then there exist nonempty disjoint open sets U and V such that $Y = U \cup V$. Therefore, U and V are clopen sets in Y. Since f is slightly supra b-continuous, $f^{-1}(U)$ and $f^{-1}(V)$ are supra b-open in X. Moreover, $f^{-1}(U)$ and $f^{-1}(V)$ are disjoint and $X = f^{-1}(U) \cup f^{-1}(V)$. Since f is surjective, $f^{-1}(U)$ and $f^{-1}(V)$ are nonempty. Therefore, X is not supra $b-connected$. This is a contradiction and hence Y is connected. □

Corollary 3.17. [7] *If $f : X \to Y$ is slightly γ-continuous surjective function and X is γ-connected space, then Y is a connected space.*

Corollary 3.18. *The inverse image of a disconnected space under a slightly supra b-continuous surjection is supra $b - disconnected$.*

Recall that a space X is said to be (1) locally indiscrete if every open set of X is closed in X, (2) 0-dimensional if its topology has a base consisting of clopen sets.

Theorem 3.19. *Let (X, τ) be a topological space and μ be an associated supra topology with τ. If $f : (X, \tau) \longrightarrow (Y, \rho)$ is a slightly supra b-continuous function and Y is locally indiscrete, then f is supra b-continuous.*

Proof. Let V be any open set of Y. Since Y is locally indiscrete, V is clopen and hence $f^{-1}(V)$ are supra b-open in X. Therefore, f is supra b-continuous. □

Theorem 3.20. *Let (X, τ) be a topological space and μ be an associated supra topology with τ. If $f : (X, \tau) \longrightarrow (Y, \rho)$ is a slightly supra b-continuous function and Y is 0-dimensional, then f is supra b-continuous.*

Proof. Let $x \in X$ and $V \subseteq Y$ be any open set containing $f(x)$. Since Y is 0-dimensional, there exists a clopen set U containing $f(x)$ such that $U \subseteq V$. But f is slightly supra b-continuous then there exists a supra b-open set G containing x such that $f(x) \in f(G) \subseteq U \subseteq V$. Hence f is supra b-continuous. □

Corollary 3.21. [7] *If $f : X \to Y$ is slightly γ-continuous and Y is 0-dimensional, then f is γ-continuous.*

Theorem 3.22. *Let (X,τ) be a topological space and μ be an associated supra topology with τ. Let $f:(X,\tau)\longrightarrow(Y,\rho)$ be a slightly supra b-continuous injection and Y is 0-dimensional. If Y is T_1 (resp. T_2), then X is supra $b-T_1$ (resp. supra $b-T_2$).*

Proof. We prove only the second statement, the prove of the first being analogous. Let Y be T_2. Since f is injective, for any pair of distinct points $x,y\in X$, $f(x)\neq f(y)$. Since Y is T_2, there exist open sets V_1, V_2 in Y such that $f(x)\in V_1$, $f(y)\in V_2$ and $V_1\cap V_2=\phi$. Since Y is 0-dimensional, there exist clopen sets U_1, U_2 in Y such that $f(x)\in U_1\subseteq V_1$ and $f(y)\in U_2\subseteq V_2$. Consequently $x\in f^{-1}(U_1)\subseteq f^{-1}(V_1)$, $y\in f^{-1}(U_2)\subseteq f^{-1}(V_2)$ and $f^{-1}(U_1)\cap f^{-1}(U_2)=\phi$. Since f is slightly supra b-continuous, $f^{-1}(U_1)$ and $f^{-1}(U_2)$ are supra b-open sets and this implies that X is supra $b-T_2$. □

Definition 3.23. *A space X is said to be:*

(i) clopen T_1 [4,7] if for each pair of distinct points x and y of X, there exist clopen sets U and V containing x and y, respectively such that $y\notin U$ and $x\notin V$.

(ii) clopen T_2 (clopen Hausdorff or ultra-Hausdorff) [13] if for each pair of distinct points x and y in X, there exist disjoint clopen sets U and V in X such that $x\in U$ and $y\in V$.

Theorem 3.24. *Let (X,τ) be a topological space and μ be an associated supra topology with τ. Let $f:(X,\tau)\longrightarrow(Y,\rho)$ be a slightly supra b-continuous injection and Y is clopen T_1, then X is supra $b-T_1$.*

Proof. Suppose that Y is clopen T_1. For any distinct points x and y in X, there exist clopen sets V and W such that $f(x)\in V$, $f(y)\notin V$ and $f(y)\in W$, $f(x)\notin W$. Since f is slightly supra b-continuous, $f^{-1}(V)$ and $f^{-1}(W)$ are supra b-open subsets of X such that $x\in f^{-1}(V)$, $y\notin f^{-1}(V)$ and $y\in f^{-1}(W)$, $x\notin f^{-1}(W)$. This shows that X is supra $b-T_1$. □

Corollary 3.25. [7] *If $f:X\to Y$ is slightly γ-continuous injection and Y is clopen T_1, then X is $\gamma-T_1$.*

Theorem 3.26. *Let (X,τ) be a topological space and μ be an associated supra topology with τ. Let $f:(X,\tau)\longrightarrow(Y,\rho)$ be a slightly supra b-continuous injection and Y is clopen T_2, then X is supra $b-T_2$.*

Proof. For any pair of distinct points x and y in X, there exist disjoint clopen sets U and V in Y such that $f(x)\in U$ and $f(y)\in V$. Since f is slightly supra b-continuous, $f^{-1}(U)$ and $f^{-1}(V)$ are supra b-open subsets of X containing x and y, respectively. Therefore $f^{-1}(U)\cap f^{-1}(V)=\phi$ because $U\cap V=\phi$. This shows that X is supra $b-T_2$. □

Definition 3.27. [13] *A space X is said to be mildly compact (resp. mildly Lindelöf) if every clopen cover of X has a finite (resp. countable) subcover.*

Definition 3.28. *A supra topological space (X,μ) is called supra b-compact (resp. supra b-Lindelöf) if every supra b-open cover of X has a finite (resp. countable) subcover.*

Theorem 3.29. *Let (X, τ) be a topological space and μ be an associated supra topology with τ. Let $f : (X, \tau) \longrightarrow (Y, \rho)$ be a slightly supra b-continuous surjection, then the following statements hold:*

(1) if (X, μ) is supra b-compact, then Y is mildly compact.

(2) if (X, μ) is supra b-Lindelöf, then Y is mildly Lindelöf.

Proof. We prove (1), the proof of (2) being entirely analogous.

Let $\{V_\alpha : \alpha \in \Delta\}$ be a clopen cover of Y. Since f is slightly supra b-continuous, $\{f^{-1}(V_\alpha) : \alpha \in \Delta\}$ is a supra b-open cover of X . Since X is supra b-compact, there exists a finite subset Δ_0 of Δ such that $X = \cup\{f^{-1}(V_\alpha) : \alpha \in \Delta_0\}$. Thus we have $Y = \cup\{V_\alpha : \alpha \in \Delta_0\}$ which means that Y is mildly compact. □

Definition 3.30. *A supra topological space (X, μ) is called supra b-closed compact (resp. supra b-closed Lindelöf) if every cover of X by supra b-closed sets has a finite (resp. countable) subcover.*

Theorem 3.31. *Let (X, τ) be a topological space and μ be an associated supra topology with τ. Let $f : (X, \tau) \longrightarrow (Y, \rho)$ be a slightly supra b-continuous surjection, then the following statements hold:*

(1) if (X, μ) is supra b-closed compact, then Y is mildly compact.

(2) if (X, μ) is supra b-closed Lindelöf, then Y is mildly Lindelöf.

Proof. It can be obtained similarly as Theorem 3.29. □

Corollary 3.32. [7] *Let $f : X \to Y$ be a slightly γ-continuous surjection. Then the following statements hold:*

(1) if X is γ-Lindelöf, then Y is mildly Lindelöf.

(2) if X is γ-compact, then Y is mildly compact.

(3) if X is γ–closed-compact, then Y is mildly compact.

(4) if X is γ-closed-Lindelöf, then Y is mildly Lindelöf.

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