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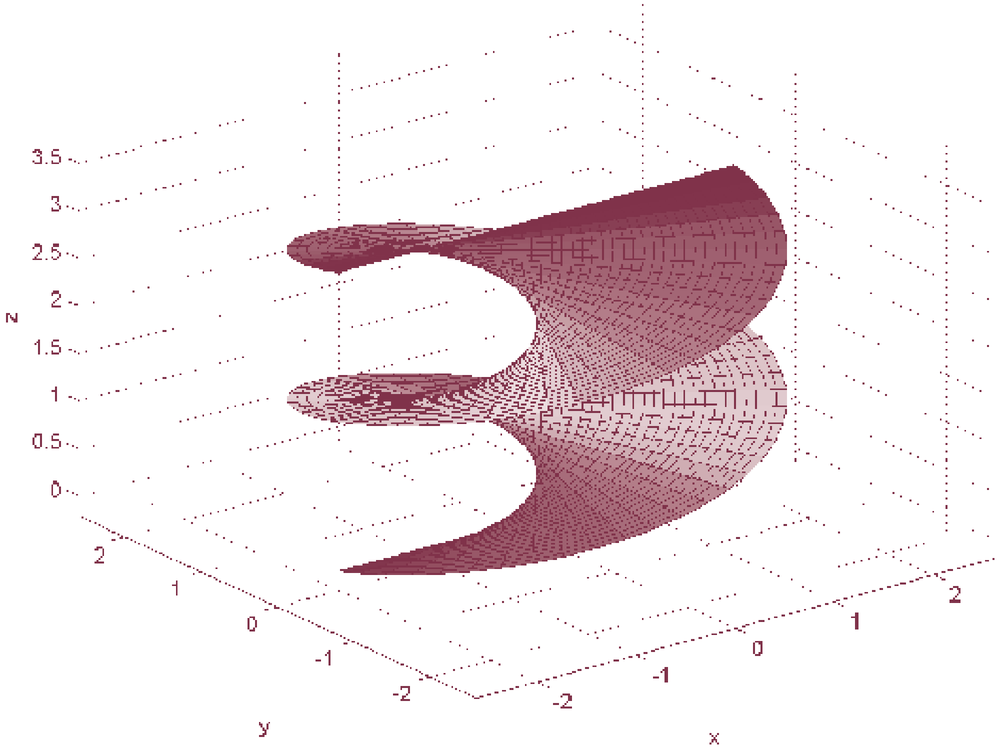

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Preface

The conference ”Geometric Function Theory and Applications - GFTA 2011”, Cluj-Napoca, Romania, September 04-08, 2011, was organized by the Babeş-Bolyai University Cluj-Napoca, Faculty of Mathematics and Computer Science, in collaboration with the University ”1 Decembrie 1918” Alba Iulia, Kinki University - Department of Mathematics - Osaka, Japan, TC Istanbul Kultur University, Istanbul, Turkey and University Kebangsaan Malaysia, Bangi, Selangor, Malaysia. In addition to purely mathematical conference objectives, its seventh edition, has proposed celebration of Professor Dr. Petru T. MOCANU, member of the Romanian Academy, founder of the Romanian school of Geometric Function Theory with the occasion of 80 years old anniversary. A bibliographical note is in this volume.

13 plenary lectures and 57 conferences by section were attended by 88 mathematicians from 16 countries: Saudi Arabia, Canada, Finland, Germany, Iran, Israel, Italy, Japan, Malaysia, Poland, Macedonia, Serbia, USA, Turkey, United Kingdom and Romania.

Plenary conferences were held by the following Professors: Ian GRAHAM (University of Toronto), Filippo BRACCI (Universita di Toma ”Tor Vergata”), Oliver ROTH (Universty of Wuerzburg), Gabriela KOHR (Babeş-Bolyai University of Cluj-Napoca), Mihai PASCU (”Transilvania” University Braşov), Shigeyoshi OWA (Kinki University Ossaka), Fatima AL OBOUDI (Riyadh University for Girls), Matti VUORINEN (Turku University), Rosihan ALI (University Sains Malaysia, Penang), Milutin OBRADOVIC, Wolfgang WENDLAND (University of Stuttgart), Maslina DARUS (Kebangsaan University Malaysia), Mihai CRISTEA (University of Bucharest).

A number of conferences held at GFTA 2011, to which we add a conference in the near future and dedicated to professor P.T. Mocanu, are included in this volume.

Grigore Ştefan Sălăgean,
President of the Organizing Committee of GFTA 2011

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Professor Petru T. Mocanu at his 80th anniversary

Grigore Ştefan Sălăgean

Professor **Petru T. Mocanu** was born in June 1, 1931 in Brăila, Romania. He attended primary and secondary school in Brăila, then university studies (1950-1953) and higher studies (1953-1957) at the Faculty of Mathematics, University of Cluj (now "Babeş-Bolyai" University of Cluj-Napoca). In 1959 he defended his doctoral dissertation under the guidance of the great Romanian mathematician G. Călugăreanu. The name of the doctoral thesis is *Variational methods in the theory of univalent functions.*

He worked at the "Babeş-Bolyai" University of Cluj-Napoca as Assistant Professor (1953-1957), Lecturer (1957-1962), Associate Professor (1962-1970) and Full Professor (since 1970).

He was Visiting Professor at Conakry, Guinea (1966-1967) and at Bowling Green State University, Ohio, USA (1992, fall semester), and invited talks at: Lodz (Poland) - 1966, Lublin (Poland) - 1970, Jyvaskyla (Finland) - 1973, 6 universities in USA - 1973, Lodz (Poland) - 1976, Rouen (France) - 1990, University of Michigan and Iowa University (USA) - 1992, Hagen (Germany) - 1991-1993, 1996, 2000, 2001, 2003-2008, Debrecen (Hungary) - 1996, Univ. of Chisinau (Moldavia), 1998, Wuerzburg (Germany) - 1998, 2000 - 2002, 2003, Dortmund (Germany) - 2000, 2002, 2006, Nicosia (Cyprus) - 2004, Istanbul (Turky) - 2007, 2009.

Professor P. T. Mocanu was appointed Dean of the Faculty of Mathematics (1968-1976 and 1984-1987), Head of the Chair of Function Theory (1976-1984 and 1990-2000), Head of Department of Mathematics and Vice-Rector of "Babeş-Bolyai" University (1990-1992) and the president of the Romanian Mathematical Society.

He is also the Chief Editor of *Mathematica (Cluj)* and member of Editorial Board of *Studia Universitatis Babeş-Bolyai. Mathematica*, *Bulletin de Mathématiques S.S.M.R*, *Proceedings of the Romanian Academy* and *Analele Univ. Oradea.*

This text was partially published in [1].

Since 1972 Professor P. T. Mocanu has been a Ph. D. supervisor; 37 students had taken Ph. D. At least 13 of his former Ph. D. students are now full university professors.

During his fertile teaching activity he teaches the basic course on Complex Analysis and many other special courses (Univalent Functions, Differential Subordinations, Geometric Function Theory, Measure Theory, Hardy Spaces etc.). He is also the author of two handbooks on Complex Analysis, which are very useful for Romanian students in mathematics.

Professor P. T. Mocanu was many years the Chairman of the Seminar of Geometric Function Theory, Department of Mathematics, "Babeş-Bolyai" University and he is the Head of Romanian School on Univalent Functions.

Professor P. T. Mocanu is: Member of Romanian Academy, Member of the American Mathematical Society, Doctor Honoris Causa of University "Lucian Blaga", Sibiu (Romania) - 1998 and of University of Oradea (Romania) - 2000.

The scientific activity of Professor P. T. Mocanu is very rich; he has more than 180 papers in the field of Geometric Function Theory (Univalent Functions) and two important monographs: *Geometric Theory of Univalent Functions* (Romanian), Ed. Casa Cartii de Stiinta, Cluj-Napoca, 1999, 410 pages and 2006, 460 pages (with T. Bulboacă and Gr. St. Sălăgean) and *Differential Subordinations: Theory and Applications*, Marcel Dekker, Inc., New York, Basel, 2000, 459 pages (with S. S. Miller) (this monograph is cited more then 520 times [2]). About 300 mathematicians cite a lot of his results. The top ten most cited papers are cited over 1530 times ([2]). The method of differential subordinations (admissible function method) due to Professor P. T. Mocanu and Professor S. S. Miller is well known and very useful in the research activity in the field of Geometric Function Theory.

Professor P. T. Mocanu obtained important results in the following domains: extremal problems in the theory of univalent functions, new classes of univalent functions (well known is the class of alpha-convex functions, known also as Mocanu functions), integral operators on classes of univalent functions, differential subordinations and superordinations (together professor S. S. Miller they introduced and developed this theory), conditions of diffeomorphism in the complex plane, sufficient conditions for injectivity, starlikeness or convexity, application of complex analysis in geometrical optics.

In Mathematical Reviews (MathSciNet) (2011) the name Mocanu is mentioned in about 400 reviews ("Review Text"), 50 in "Title" and more than 770 in "Anywhere". The concept of "Differential Subordination", is mentioned in about 300 reviews (Review Text), more than 250 in "Title" and more than 450 in "Anywhere". The concept of "alpha-convex function", is mentioned in 95 reviews (Review Text), 83 in "Title" and 146 in "Anywhere". There are 356 citations by 128 authors in MR Citation Database.

Collaborators: Hassoon S. Al-Amiri, Valeriu Anisiu, Kit C. Chan, Dan Coman, Paul J. Eenigenburg, Richard Fournier, Sorin Gh, Gal, John T. Gresser, Hidetaka Hamada, Miodrag Iovanov, Gabriela Kohr, Mirela Kohr, Sanford S. Miller, Grigor Moldovan, Gheorghe Oros, Mihai Popovici, Maxwell O. Reade, Dumitru Ripeanu, Stephan Ruscheweyh, Grigore Ştefan Sălăgean, Ioan Şerb, Steven M. Seubert, Ion D. Teodorescu, Gheorghe Toder, Xanthopoulos I. Xanthopoulos, Eligius J. Zlotkiewicz.

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A note on strong differential superordinations using a generalized Sălăgean operator and Ruscheweyh operator

Alina Alb Lupaş

Abstract. In the present paper we establish several strong differential superordinations regardind the new operator DR_λ^m defined by convolution product of the extended Sălăgean operator and Ruscheweyh derivative, $DR_\lambda^m : \mathcal{A}_{n\zeta}^* \to \mathcal{A}_{n\zeta}^*$, $DR_\lambda^m f(z,\zeta) = (D_\lambda^m * R^m) f(z,\zeta)$, $z \in U$, $\zeta \in \overline{U}$, where $R^m f(z,\zeta)$ denote the extended Ruscheweyh derivative, $D_\lambda^m f(z,\zeta)$ is the extended generalized Sălăgean operator and $\mathcal{A}_{n\zeta}^* = \{f \in \mathcal{H}(U \times \overline{U}),\ f(z,\zeta) = z + a_{n+1}(\zeta) z^{n+1} + \ldots,\ z \in U,\ \zeta \in \overline{U}\}$ is the class of normalized analytic functions.

Mathematics Subject Classification (2010): 30C45, 30A20, 34A40.

Keywords: Strong differential superordination, convex function, best subordinant, extended differential operator, convolution product.

1. Introduction

Denote by U the unit disc of the complex plane $U = \{z \in \mathbb{C} : |z| < 1\}$, $\overline{U} = \{z \in \mathbb{C} : |z| \le 1\}$ the closed unit disc of the complex plane and $\mathcal{H}(U \times \overline{U})$ the class of analytic functions in $U \times \overline{U}$.

Let

$$\mathcal{A}_{n\zeta}^* = \{f \in \mathcal{H}(U \times \overline{U}),\ f(z,\zeta) = z + a_{n+1}(\zeta) z^{n+1} + \ldots,\ z \in U,\ \zeta \in \overline{U}\},$$

where $a_k(\zeta)$ are holomorphic functions in $\overline{U}$ for $k \ge 2$, and

$$\mathcal{H}^*[a,n,\zeta] = \{f \in \mathcal{H}(U \times \overline{U}),\ f(z,\zeta) = a + a_n(\zeta) z^n + a_{n+1}(\zeta) z^{n+1} + \ldots,\ z \in U,\ \zeta \in \overline{U}\},$$

for $a \in \mathbb{C}$, $n \in \mathbb{N}$, $a_k(\zeta)$ are holomorphic functions in $\overline{U}$ for $k \ge n$.

Denote by

$$K_{n\zeta} = \left\{ f \in \mathcal{H}(U \times \overline{U}) : \operatorname{Re} \frac{z f_z''(z,\zeta)}{f_z'(z,\zeta)} + 1 > 0 \right\}$$

the class of convex function in $U \times \overline{U}$.

We also extend the differential operators presented above to the new class of analytic functions $\mathcal{A}^*_{n\zeta}$ introduced in [10].

Definition 1.1. [5] *For $f \in \mathcal{A}^*_{n\zeta}$, $\lambda \geq 0$ and $n, m \in \mathbb{N}$, the operator D^m_λ is defined by $D^m_\lambda : \mathcal{A}^*_{n\zeta} \to \mathcal{A}^*_{n\zeta}$,*

$$\begin{aligned}
D^0_\lambda f(z,\zeta) &= f(z,\zeta) \\
D^1_\lambda f(z,\zeta) &= (1-\lambda) f(z,\zeta) + \lambda z f'_z(z,\zeta) = D_\lambda f(z,\zeta), ..., \\
D^{m+1}_\lambda f(z,\zeta) &= (1-\lambda) D^m_\lambda f(z,\zeta) + \lambda z \left(D^m_\lambda f(z,\zeta)\right)'_z \\
&= D_\lambda \left(D^m_\lambda f(z,\zeta)\right), \quad z \in U, \zeta \in \overline{U}.
\end{aligned}$$

Remark 1.2. [5] If $f \in \mathcal{A}^*_{n\zeta}$ and $f(z) = z + \sum_{j=n+1}^{\infty} a_j(\zeta) z^j$, then

$$D^m_\lambda f(z,\zeta) = z + \sum_{j=n+1}^{\infty} \left[1 + (j-1)\lambda\right]^m a_j(\zeta) z^j, \text{ for } z \in U, \ \zeta \in \overline{U}.$$

Definition 1.3. [4] *For $f \in \mathcal{A}^*_{n\zeta}$, $n, m \in \mathbb{N}$, the operator R^m is defined by*

$$\begin{aligned}
R^m &: \mathcal{A}^*_{n\zeta} \to \mathcal{A}^*_{n\zeta}, \\
R^0 f(z,\zeta) &= f(z,\zeta), \\
R^1 f(z,\zeta) &= z f'_z(z,\zeta), ..., \\
(m+1) R^{m+1} f(z,\zeta) &= z \left(R^m f(z,\zeta)\right)'_z + m R^m f(z,\zeta), \quad z \in U, \ \zeta \in \overline{U}.
\end{aligned}$$

Remark 1.4. [4] If $f \in \mathcal{A}^*_{n\zeta}$, $f(z,\zeta) = z + \sum_{j=n+1}^{\infty} a_j(\zeta) z^j$, then

$$R^m f(z,\zeta) = z + \sum_{j=n+1}^{\infty} C^m_{m+j-1} a_j(\zeta) z^j, \ z \in U, \ \zeta \in \overline{U}.$$

As a dual notion of strong differential subordination G.I. Oros has introduced and developed the notion of strong differential superordinations in [9].

Definition 1.5. [9] *Let $f(z,\zeta)$, $H(z,\zeta)$ analytic in $U \times \overline{U}$. The function $f(z,\zeta)$ is said to be strongly superordinate to $H(z,\zeta)$ if there exists a function w analytic in U, with $w(0) = 0$ and $|w(z)| < 1$, such that $H(z,\zeta) = f(w(z),\zeta)$, for all $\zeta \in \overline{U}$. In such a case we write $H(z,\zeta) \prec\prec f(z,\zeta)$, $z \in U$, $\zeta \in \overline{U}$.*

Remark 1.6. [9] (i) Since $f(z,\zeta)$ is analytic in $U \times \overline{U}$, for all $\zeta \in \overline{U}$, and univalent in U, for all $\zeta \in \overline{U}$, Definition 1.5 is equivalent to $H(0,\zeta) = f(0,\zeta)$, for all $\zeta \in \overline{U}$, and $H\left(U \times \overline{U}\right) \subset f\left(U \times \overline{U}\right)$.

(ii) If $H(z,\zeta) \equiv H(z)$ and $f(z,\zeta) \equiv f(z)$, the strong superordination becomes the usual notion of superordination.

Definition 1.7. [9] *We denote by* Q^* *the set of functions that are analytic and injective on* $\overline{U}\times\overline{U}\backslash E(f,\zeta)$*, where* $E(f,\zeta)=\{y\in\partial U:\lim\limits_{z\to y}f(z,\zeta)=\infty\}$*, and are such that* $f_z'(y,\zeta)\neq 0$ *for* $y\in\partial U\times\overline{U}\backslash E(f,\zeta)$*. The subclass of* Q^* *for which* $f(0,\zeta)=a$ *is denoted by* $Q^*(a)$.

We have need the following lemmas to study the strong differential superordinations.

Lemma 1.8. [9] *Let* $h(z,\zeta)$ *be a convex function with* $h(0,\zeta)=a$ *and let* $\gamma\in\mathbb{C}^*$ *be a complex number with* $\mathrm{Re}\gamma\geq 0$*. If* $p\in\mathcal{H}^*[a,n,\zeta]\cap Q^*$, $p(z,\zeta)+\frac{1}{\gamma}zp_z'(z,\zeta)$ *is univalent in* $U\times\overline{U}$ *and*

$$h(z,\zeta)\prec\prec p(z,\zeta)+\frac{1}{\gamma}zp_z'(z,\zeta),\qquad z\in U,\ \zeta\in\overline{U},$$

then

$$q(z,\zeta)\prec\prec p(z,\zeta),\qquad z\in U,\ \zeta\in\overline{U},$$

where $q(z,\zeta)=\frac{\gamma}{nz^{\frac{\gamma}{n}}}\int_0^z h(t,\zeta)\,t^{\frac{\gamma}{n}-1}dt$, $z\in U$, $\zeta\in\overline{U}$*. The function* q *is convex and is the best subordinant.*

Lemma 1.9. [9] *Let* $q(z,\zeta)$ *be a convex function in* $U\times\overline{U}$ *and let*

$$h(z,\zeta)=q(z,\zeta)+\frac{1}{\gamma}zq_z'(z,\zeta),\ z\in U,\ \zeta\in\overline{U},$$

where $\mathrm{Re}\gamma\geq 0$.

If $p\in\mathcal{H}^*[a,n,\zeta]\cap Q^*$, $p(z,\zeta)+\frac{1}{\gamma}zp_z'(z,\zeta)$ *is univalent in* $U\times\overline{U}$ *and*

$$q(z,\zeta)+\frac{1}{\gamma}zq_z'(z,\zeta)\prec\prec p(z,\zeta)+\frac{1}{\gamma}zp_z'(z,\zeta),\qquad z\in U,\ \zeta\in\overline{U},$$

then

$$q(z,\zeta)\prec\prec p(z,\zeta),\qquad z\in U,\ \zeta\in\overline{U},$$

where $q(z,\zeta)=\frac{\gamma}{nz^{\frac{\gamma}{n}}}\int_0^z h(t,\zeta)\,t^{\frac{\gamma}{n}-1}dt$, $z\in U$, $\zeta\in\overline{U}$*. The function* q *is the best subordinant.*

2. Main results

Definition 2.1. [2] *Let* $\lambda\geq 0$ *and* $m\in\mathbb{N}\cup\{0\}$*. Denote by* DR_λ^m *the operator given by the Hadamard product (the convolution product) of the extended generalized Sălăgean operator* D_λ^m *and the extended Ruscheweyh operator* R^m, $DR_\lambda^m:\mathcal{A}_{n\zeta}^*\to\mathcal{A}_{n\zeta}^*$,

$$DR_\lambda^m f(z,\zeta)=\left(D_\lambda^m * R^m\right)f(z,\zeta).$$

Remark 2.2. [2] If $f\in\mathcal{A}_{n\zeta}^*$, $f(z,\zeta)=z+\sum\limits_{j=n+1}^{\infty}a_j(\zeta)z^j$, then

$$DR_\lambda^m f(z,\zeta)=z+\sum_{j=n+1}^{\infty}C_{m+j-1}^m\left[1+(j-1)\lambda\right]^m a_j^2(\zeta)z^j,\ z\in U,\ \zeta\in\overline{U}.$$

Remark 2.3. For $\lambda = 1$ we obtain the Hadamard product SR^m ([1], [3], [7], [8]) of the extended Sălăgean operator S^m and the extended Ruscheweyh operator R^m.

Theorem 2.4. *Let $h(z,\zeta)$ be a convex function in $U \times \overline{U}$ with $h(0,\zeta) = 1$. Let $m \in \mathbb{N}$, $\lambda \geq 0$, $f(z,\zeta) \in \mathcal{A}^*_{n\zeta}$, $F(z,\zeta) = I_c(f)(z,\zeta) = \frac{c+2}{z^{c+1}} \int_0^z t^c f(t,\zeta)\,dt$, $z \in U$, $\zeta \in \overline{U}$, $\mathrm{Re}\, c > -2$, and suppose that $(DR^m_\lambda f(z,\zeta))'_z$ is univalent in $U \times \overline{U}$, $(DR^m_\lambda F(z,\zeta))'_z \in \mathcal{H}^*[1,n,\zeta] \cap Q^*$ and*

$$h(z,\zeta) \prec\prec (DR^m_\lambda f(z,\zeta))'_z, \quad z \in U,\ \zeta \in \overline{U}, \tag{2.1}$$

then

$$q(z,\zeta) \prec\prec (DR^m_\lambda F(z,\zeta))'_z, \quad z \in U,\ \zeta \in \overline{U},$$

where $q(z,\zeta) = \frac{c+2}{nz^{\frac{c+2}{n}}} \int_0^z h(t,\zeta) t^{\frac{c+2}{n}-1} dt$. The function q is convex and it is the best subordinant.

Proof. We have

$$z^{c+1} F(z,\zeta) = (c+2) \int_0^z t^c f(t,\zeta)\,dt$$

and differentiating it, with respect to z, we obtain

$$(c+1) F(z,\zeta) + z F'_z(z,\zeta) = (c+2) f(z,\zeta)$$

and

$$(c+1) DR^m_\lambda F(z,\zeta) + z (DR^m_\lambda F(z,\zeta))'_z = (c+2) DR^m_\lambda f(z,\zeta), \quad z \in U,\ \zeta \in \overline{U}.$$

Differentiating the last relation with respect to z we have

$$(DR^m_\lambda F(z,\zeta))'_z + \frac{1}{c+2} z (DR^m_\lambda F(z,\zeta))''_{z^2} = (DR^m_\lambda f(z,\zeta))'_z, \quad z \in U,\ \zeta \in \overline{U}. \tag{2.2}$$

Using (2.2), the strong differential superordination (2.1) becomes

$$h(z,\zeta) \prec\prec (DR^m_\lambda F(z,\zeta))'_z + \frac{1}{c+2} z (DR^m_\lambda F(z,\zeta))''_{z^2}. \tag{2.3}$$

Denote

$$p(z,\zeta) = (DR^m_\lambda F(z,\zeta))'_z, \quad z \in U,\ \zeta \in \overline{U}. \tag{2.4}$$

Replacing (2.4) in (2.3) we obtain

$$h(z,\zeta) \prec\prec p(z,\zeta) + \frac{1}{c+2} z p'_z(z,\zeta), \quad z \in U,\ \zeta \in \overline{U}.$$

Using Lemma 1.8 for $\gamma = c+2$, we have

$$q(z,\zeta) \prec\prec p(z,\zeta),\ z \in U,\ \zeta \in \overline{U},\ \text{i.e. } q(z,\zeta) \prec\prec (DR^m_\lambda F(z,\zeta))'_z,\ z \in U,\ \zeta \in \overline{U},$$

where $q(z,\zeta) = \frac{c+2}{nz^{\frac{c+2}{n}}} \int_0^z h(t,\zeta) t^{\frac{c+2}{n}-1} dt$. The function q is convex and it is the best subordinant. □

Corollary 2.5. *Let* $h(z,\zeta)=\frac{1+(2\beta-\zeta)z}{1+z}$, *where* $\beta\in[0,1)$. *Let* $m\in\mathbb{N}$, $\lambda\geq 0$, $f(z,\zeta)\in\mathcal{A}^*_{n\zeta}$, $F(z,\zeta)=I_c(f)(z,\zeta)=\frac{c+2}{z^{c+1}}\int_0^z t^c f(t,\zeta)\,dt$, $z\in U$, $\zeta\in\overline{U}$, $\operatorname{Re}c>-2$, *and suppose that* $(DR^m_\lambda f(z,\zeta))'_z$ *is univalent in* $U\times\overline{U}$, $(DR^m_\lambda F(z,\zeta))'_z\in\mathcal{H}^*[1,n,\zeta]\cap Q^*$ *and*

$$h(z,\zeta)\prec\prec(DR^m_\lambda f(z,\zeta))'_z,\quad z\in U,\ \zeta\in\overline{U},\tag{2.5}$$

then

$$q(z,\zeta)\prec\prec(DR^m_\lambda F(z,\zeta))'_z,\qquad z\in U,\ \zeta\in\overline{U},$$

where q *is given by* $q(z,\zeta)=2\beta-\zeta+\frac{(c+2)(1+\zeta-2\beta)}{nz^{\frac{c+2}{n}}}\int_0^z\frac{t^{\frac{c+2}{n}-1}}{t+1}dt$, $z\in U$, $\zeta\in\overline{U}$. *The function* q *is convex and it is the best subordinant.*

Proof. Following the same steps as in the proof of Theorem 2.4 and considering $p(z,\zeta)=(DR^m_\lambda F(z,\zeta))'_z$, the strong differential superordination (2.5) becomes

$$h(z,\zeta)=\frac{1+(2\beta-\zeta)z}{1+z}\prec\prec p(z,\zeta)+\frac{1}{c+2}zp'_z(z,\zeta),\qquad z\in U,\ \zeta\in\overline{U}.$$

By using Lemma 1.8 for $\gamma=c+2$, we have $q(z,\zeta)\prec\prec p(z,\zeta)$, i.e.

$$q(z,\zeta)=\frac{c+2}{nz^{\frac{c+2}{n}}}\int_0^z h(t,\zeta)t^{\frac{c+2}{n}-1}dt=\frac{c+2}{nz^{\frac{c+2}{n}}}\int_0^z\frac{1+(2\beta-\zeta)t}{1+t}t^{\frac{c+2}{n}-1}dt$$

$$=2\beta-\zeta+\frac{(c+2)(1+\zeta-2\beta)}{nz^{\frac{c+2}{n}}}\int_0^z\frac{t^{\frac{c+2}{n}-1}}{t+1}dt\prec\prec(DR^m_\lambda F(z,\zeta))'_z,\qquad z\in U,\ \zeta\in\overline{U}.$$

The function q is convex and it is the best subordinant. □

Theorem 2.6. *Let* $q(z,\zeta)$ *be a convex function in* $U\times\overline{U}$ *and let*

$$h(z,\zeta)=q(z,\zeta)+\frac{1}{c+2}zq'_z(z,\zeta),$$

where $z\in U$, $\zeta\in\overline{U}$, $\operatorname{Re}c>-2$.

Let $m\in\mathbb{N}$, $\lambda\geq 0$, $f(z,\zeta)\in\mathcal{A}^*_{n\zeta}$, $F(z,\zeta)=I_c(f)(z,\zeta)=\frac{c+2}{z^{c+1}}\int_0^z t^c f(t,\zeta)\,dt$, $z\in U$, $\zeta\in\overline{U}$, *and suppose that* $(DR^m_\lambda f(z,\zeta))'_z$ *is univalent in* $U\times\overline{U}$, $(DR^m_\lambda F(z,\zeta))'_z\in\mathcal{H}^*[1,n,\zeta]\cap Q^*$ *and*

$$h(z,\zeta)\prec\prec(DR^m_\lambda f(z,\zeta))'_z,\quad z\in U,\ \zeta\in\overline{U},\tag{2.6}$$

then

$$q(z,\zeta)\prec\prec(DR^m_\lambda F(z,\zeta))'_z,\qquad z\in U,\ \zeta\in\overline{U},$$

where $q(z,\zeta)=\frac{c+2}{nz^{\frac{c+2}{n}}}\int_0^z h(t,\zeta)t^{\frac{c+2}{n}-1}dt$. *The function* q *is the best subordinant.*

Proof. Following the same steps as in the proof of Theorem 2.4 and considering $p(z,\zeta)=(DR^m_\lambda F(z,\zeta))'_z$, $z\in U$, $\zeta\in\overline{U}$, the strong differential superordination (2.6) becomes

$$h(z,\zeta)=q(z,\zeta)+\frac{1}{c+2}zq'_z(z,\zeta)\prec\prec p(z,\zeta)+\frac{1}{c+2}zp'_z(z,\zeta),\qquad z\in U,\ \zeta\in\overline{U}.$$

Using Lemma 1.9 for $\gamma = c + 2$, we have

$$q(z,\zeta) \prec\prec p(z,\zeta), \ z \in U, \ \zeta \in \overline{U}, \text{ i.e. } q(z,\zeta) \prec\prec (DR_\lambda^m F(z,\zeta))'_z, \ z \in U, \ \zeta \in \overline{U},$$

where $q(z,\zeta) = \frac{c+2}{nz^{\frac{c+2}{n}}} \int_0^z h(t,\zeta) t^{\frac{c+2}{n}-1} dt$. The function q is the best subordinant. □

Theorem 2.7. *Let $h(z,\zeta)$ be a convex function, $h(0,\zeta) = 1$. Let $\lambda \geq 0$, $m, n \in \mathbb{N}$, $f(z,\zeta) \in \mathcal{A}^*_{n\zeta}$ and suppose that $(DR_\lambda^m f(z,\zeta))'_z$ is univalent and $\frac{DR_\lambda^m f(z,\zeta)}{z} \in \mathcal{H}^*[1,n,\zeta] \cap Q^*$. If*

$$h(z,\zeta) \prec\prec (DR_\lambda^m f(z,\zeta))'_z, \qquad z \in U, \ \zeta \in \overline{U}, \tag{2.7}$$

then

$$q(z,\zeta) \prec\prec \frac{DR_\lambda^m f(z,\zeta)}{z}, \qquad z \in U, \ \zeta \in \overline{U},$$

where $q(z,\zeta) = \frac{1}{nz^{\frac{1}{n}}} \int_0^z h(t,\zeta) t^{\frac{1}{n}-1} dt$. The function q is convex and it is the best subordinant.

Proof. Consider

$$p(z,\zeta) = \frac{DR_\lambda^m f(z,\zeta)}{z} = \frac{z + \sum_{j=n+1}^{\infty} C_{m+j-1}^m [1 + (j-1)\lambda]^m a_j^2(\zeta) z^j}{z}$$
$$= 1 + \sum_{j=n+1}^{\infty} C_{m+j-1}^m [1 + (j-1)\lambda]^m a_j^2(\zeta) z^{j-1}.$$

Evidently $p \in \mathcal{H}^*[1,n,\zeta]$.

We have $p(z,\zeta) + zp'_z(z,\zeta) = (DR_\lambda^m f(z,\zeta))'_z$, $z \in U$, $\zeta \in \overline{U}$.

Then (2.7) becomes

$$h(z,\zeta) \prec\prec p(z,\zeta) + zp'_z(z,\zeta), \qquad z \in U, \ \zeta \in \overline{U}.$$

By using Lemma 1.8 for $\gamma = 1$, we have

$$q(z,\zeta) \prec\prec p(z,\zeta), \quad z \in U, \ \zeta \in \overline{U}, \quad \text{i.e.} \quad q(z,\zeta) \prec\prec \frac{DR_\lambda^m f(z,\zeta)}{z}, \quad z \in U, \ \zeta \in \overline{U},$$

where $q(z,\zeta) = \frac{1}{nz^{\frac{1}{n}}} \int_0^z h(t,\zeta) t^{\frac{1}{n}-1} dt$. The function q is convex and it is the best subordinant. □

Corollary 2.8. *Let $h(z,\zeta) = \frac{1+(2\beta-\zeta)z}{1+z}$ be a convex function in $U \times \overline{U}$, where $0 \leq \beta < 1$. Let $\lambda \geq 0$, $m, n \in \mathbb{N}$, $f(z,\zeta) \in \mathcal{A}^*_{n\zeta}$ and suppose that $(DR_\lambda^m f(z,\zeta))'_z$ is univalent and $\frac{DR_\lambda^m f(z,\zeta)}{z} \in \mathcal{H}^*[1,n,\zeta] \cap Q^*$. If*

$$h(z,\zeta) \prec\prec (DR_\lambda^m f(z,\zeta))'_z, \qquad z \in U, \ \zeta \in \overline{U}, \tag{2.8}$$

then

$$q(z,\zeta) \prec\prec \frac{DR_\lambda^m f(z,\zeta)}{z}, \qquad z \in U, \ \zeta \in \overline{U},$$

where q is given by $q(z,\zeta) = 2\beta - \zeta + \frac{1+\zeta-2\beta}{nz^{\frac{1}{n}}} \int_0^z \frac{t^{\frac{1}{n}-1}}{1+t} dt$, $z \in U$, $\zeta \in \overline{U}$. *The function q is convex and it is the best subordinant.*

Proof. Following the same steps as in the proof of Theorem 2.7 and considering

$$p(z,\zeta) = \frac{DR_\lambda^m f(z,\zeta)}{z},$$

the strong differential superordination (2.8) becomes

$$h(z,\zeta) = \frac{1+(2\beta-\zeta)z}{1+z} \prec\prec p(z,\zeta) + zp'_z(z,\zeta), \quad z \in U,\ \zeta \in \overline{U}.$$

By using Lemma 1.8 for $\gamma = 1$, we have $q(z,\zeta) \prec\prec p(z,\zeta)$, i.e.

$$q(z,\zeta) = \frac{1}{nz^{\frac{1}{n}}} \int_0^z h(t,\zeta)\, t^{\frac{1}{n}-1} dt = \frac{1}{nz^{\frac{1}{n}}} \int_0^z t^{\frac{1}{n}-1} \frac{1+(2\beta-\zeta)t}{1+t} dt$$

$$= 2\beta - \zeta + \frac{1+\zeta-2\beta}{nz^{\frac{1}{n}}} \int_0^z \frac{t^{\frac{1}{n}-1}}{1+t} dt \prec\prec \frac{DR_\lambda^m f(z,\zeta)}{z}, \quad z \in U,\ \zeta \in \overline{U}.$$

The function q is convex and it is the best subordinant. □

Theorem 2.9. *Let $q(z,\zeta)$ be convex in $U \times \overline{U}$ and let h be defined by*

$$h(z,\zeta) = q(z,\zeta) + zq'_z(z,\zeta).$$

If $\lambda \geq 0$, $m,n \in \mathbb{N}$, $f(z,\zeta) \in \mathcal{A}^*_{n\zeta}$, suppose that $\left(DR_\lambda^m f(z,\zeta)\right)'_z$ is univalent, $\frac{DR_\lambda^m f(z,\zeta)}{z} \in \mathcal{H}^*[1,n,\zeta] \cap Q^*$ and satisfies the strong differential superordination

$$h(z,\zeta) = q(z,\zeta) + zq'_z(z,\zeta) \prec\prec \left(DR_\lambda^m f(z,\zeta)\right)'_z, \quad z \in U,\ \zeta \in \overline{U}, \tag{2.9}$$

then

$$q(z,\zeta) \prec\prec \frac{DR_\lambda^m f(z,\zeta)}{z}, \quad z \in U,\ \zeta \in \overline{U},$$

where $q(z,\zeta) = \frac{1}{nz^{\frac{1}{n}}} \int_0^z h(t,\zeta) t^{\frac{1}{n}-1} dt$. *The function q is the best subordinant.*

Proof. Let

$$p(z,\zeta) = \frac{DR_\lambda^m f(z,\zeta)}{z} = \frac{z + \sum_{j=n+1}^{\infty} C_{m+j-1}^m [1+(j-1)\lambda]^m a_j^2(\zeta) z^j}{z}$$

$$= 1 + \sum_{j=n+1}^{\infty} C_{m+j-1}^m [1+(j-1)\lambda]^m a_j^2(\zeta) z^{j-1}.$$

Evidently $p \in \mathcal{H}^*[1,n,\zeta]$.

Differentiating, we obtain $p(z,\zeta) + zp'_z(z,\zeta) = \left(DR_\lambda^m f(z,\zeta)\right)'_z$, $z \in U$, $\zeta \in \overline{U}$, and (2.9) becomes

$$q(z,\zeta) + zq'_z(z,\zeta) \prec\prec p(z,\zeta) + zp'_z(z,\zeta), \quad z \in U,\ \zeta \in \overline{U}.$$

Using Lemma 1.9 for $\gamma = 1$, we have

$$q(z,\zeta) \prec\prec p(z,\zeta), \quad z \in U,\ \zeta \in \overline{U},\ \text{i.e.}$$

$$q(z,\zeta) = \frac{1}{nz^{\frac{1}{n}}} \int_0^z h(t,\zeta) t^{\frac{1}{n}-1} dt \prec\prec \frac{DR_\lambda^m f(z,\zeta)}{z}, \quad z \in U, \zeta \in \overline{U},$$

and q is the best subordinant. □

Theorem 2.10. *Let $h(z,\zeta)$ be a convex function, $h(0,\zeta) = 1$. Let $\lambda \geq 0$, $m, n \in \mathbb{N}$, $f(z,\zeta) \in \mathcal{A}_{n\zeta}^*$ and suppose that $\left(\frac{zDR_\lambda^{m+1} f(z,\zeta)}{DR_\lambda^m f(z,\zeta)}\right)_z'$ is univalent and $\frac{DR_\lambda^{m+1} f(z,\zeta)}{DR_\lambda^m f(z,\zeta)} \in \mathcal{H}^*[1,n,\zeta] \cap Q^*$. If*

$$h(z,\zeta) \prec\prec \left(\frac{zDR_\lambda^{m+1} f(z,\zeta)}{DR_\lambda^m f(z,\zeta)}\right)_z', \qquad z \in U,\ \zeta \in \overline{U}, \tag{2.10}$$

then

$$q(z,\zeta) \prec\prec \frac{DR_\lambda^{m+1} f(z,\zeta)}{DR_\lambda^m f(z,\zeta)}, \qquad z \in U,\ \zeta \in \overline{U},$$

where $q(z,\zeta) = \frac{1}{nz^{\frac{1}{n}}} \int_0^z h(t,\zeta) t^{\frac{1}{n}-1} dt$. The function q is convex and it is the best subordinant.

Proof. Consider

$$p(z,\zeta) = \frac{DR_\lambda^{m+1} f(z,\zeta)}{DR_\lambda^m f(z,\zeta)} = \frac{z + \sum_{j=n+1}^{\infty} C_{m+j}^{m+1} [1 + (j-1)\lambda]^{m+1} a_j^2(\zeta) z^j}{z + \sum_{j=n+1}^{\infty} C_{m+j-1}^{m} [1 + (j-1)\lambda]^{m} a_j^2(\zeta) z^j}$$

$$= \frac{1 + \sum_{j=n+1}^{\infty} C_{m+j}^{m+1} [1 + (j-1)\lambda]^{m+1} a_j^2(\zeta) z^{j-1}}{1 + \sum_{j=n+1}^{\infty} C_{m+j-1}^{m} [1 + (j-1)\lambda]^{m} a_j^2(\zeta) z^{j-1}}.$$

Evidently $p \in \mathcal{H}^*[1,n,\zeta]$.
We have

$$p_z'(z,\zeta) = \frac{\left(DR_\lambda^{m+1} f(z,\zeta)\right)_z'}{DR_\lambda^m f(z,\zeta)} - p(z,\zeta) \cdot \frac{\left(DR_\lambda^m f(z,\zeta)\right)_z'}{DR_\lambda^m f(z,\zeta)}.$$

Then

$$p(z,\zeta) + zp_z'(z,\zeta) = \left(\frac{zDR_\lambda^{m+1} f(z,\zeta)}{DR_\lambda^m f(z,\zeta)}\right)_z'.$$

Then (2.10) becomes

$$h(z,\zeta) \prec\prec p(z,\zeta) + zp_z'(z,\zeta), \qquad z \in U,\ \zeta \in \overline{U}.$$

By using Lemma 1.8 for $\gamma = 1$, we have

$$q(z,\zeta) \prec\prec p(z,\zeta),\ z \in U,\ \zeta \in \overline{U}, \quad \text{i.e.} \quad q(z,\zeta) \prec \frac{DR_{\lambda}^{m+1} f(z,\zeta)}{DR_{\lambda}^{m} f(z,\zeta)},\ z \in U,\ \zeta \in \overline{U},$$

where $q(z,\zeta) = \frac{1}{nz^{\frac{1}{n}}} \int_0^z h(t,\zeta)\, t^{\frac{1}{n}-1} dt$. The function q is convex and it is the best subordinant. □

Corollary 2.11. *Let $h(z,\zeta) = \frac{1+(2\beta-\zeta)z}{1+z}$ be a convex function in $U \times \overline{U}$, where $0 \le \beta < 1$. Let $\lambda \ge 0$, $m,n \in \mathbb{N}$, $f(z,\zeta) \in \mathcal{A}^*_{n\zeta}$ and suppose that $\left(\frac{zDR_{\lambda}^{m+1} f(z,\zeta)}{DR_{\lambda}^{m} f(z,\zeta)}\right)'_z$ is univalent, $\frac{DR_{\lambda}^{m+1} f(z,\zeta)}{DR_{\lambda}^{m} f(z,\zeta)} \in \mathcal{H}^*[1,n,\zeta] \cap Q^*$. If*

$$h(z,\zeta) \prec\prec \left(\frac{zDR_{\lambda}^{m+1} f(z,\zeta)}{DR_{\lambda}^{m} f(z,\zeta)}\right)'_z, \quad z \in U,\ \zeta \in \overline{U}, \tag{2.11}$$

then

$$q(z,\zeta) \prec\prec \frac{DR_{\lambda}^{m+1} f(z,\zeta)}{DR_{\lambda}^{m} f(z,\zeta)}, \quad z \in U,\ \zeta \in \overline{U},$$

where q is given by $q(z,\zeta) = 2\beta - \zeta + \frac{1+\zeta-2\beta}{nz^{\frac{1}{n}}} \int_0^z \frac{t^{\frac{1}{n}-1}}{1+t} dt$, $z \in U$, $\zeta \in \overline{U}$. The function q is convex and it is the best subordinant.

Proof. Following the same steps as in the proof of Theorem 2.10 and considering

$$p(z,\zeta) = \frac{DR_{\lambda}^{m+1} f(z,\zeta)}{DR_{\lambda}^{m} f(z,\zeta)},$$

the strong differential superordination (2.11) becomes

$$h(z,\zeta) = \frac{1+(2\beta-\zeta)z}{1+z} \prec\prec p(z,\zeta) + zp'_z(z,\zeta), \quad z \in U,\ \zeta \in \overline{U}.$$

By using Lemma 1.8 for $\gamma = 1$, we have $q(z,\zeta) \prec\prec p(z,\zeta)$, i.e.

$$q(z,\zeta) = \frac{1}{nz^{\frac{1}{n}}} \int_0^z h(t,\zeta)\, t^{\frac{1}{n}-1} dt = \frac{1}{nz^{\frac{1}{n}}} \int_0^z t^{\frac{1}{n}-1} \frac{1+(2\beta-\zeta)t}{1+t} dt$$

$$= 2\beta - \zeta + \frac{1+\zeta-2\beta}{nz^{\frac{1}{n}}} \int_0^z \frac{t^{\frac{1}{n}-1}}{1+t} dt \prec\prec \frac{DR_{\lambda}^{m+1} f(z,\zeta)}{DR_{\lambda}^{m} f(z,\zeta)}, \quad z \in U,\ \zeta \in \overline{U}.$$

The function q is convex and it is the best subordinant. □

Theorem 2.12. *Let $q(z,\zeta)$ be convex in $U \times \overline{U}$ and let h be defined by*

$$h(z,\zeta) = q(z,\zeta) + zq'_z(z,\zeta).$$

If $\lambda \ge 0$, $m,n \in \mathbb{N}$, $f(z,\zeta) \in \mathcal{A}^*_{n\zeta}$, suppose that $\left(\frac{zDR_{\lambda}^{m+1} f(z,\zeta)}{DR_{\lambda}^{m} f(z,\zeta)}\right)'_z$ is univalent, $\frac{DR_{\lambda}^{m+1} f(z,\zeta)}{DR_{\lambda}^{m} f(z,\zeta)} \in \mathcal{H}^*[1,n,\zeta] \cap Q^*$ and satisfies the strong differential superordination

$$h(z,\zeta) = q(z,\zeta) + zq'_z(z,\zeta) \prec\prec \left(\frac{zDR_{\lambda}^{m+1} f(z,\zeta)}{DR_{\lambda}^{m} f(z,\zeta)}\right)'_z, \quad z \in U,\ \zeta \in \overline{U}, \tag{2.12}$$

then

$$q(z,\zeta) \prec\prec \frac{DR_{\lambda}^{m+1} f(z,\zeta)}{DR_{\lambda}^{m} f(z,\zeta)}, \qquad z \in U,\ \zeta \in \overline{U},$$

where $q(z,\zeta) = \frac{1}{nz^{\frac{1}{n}}} \int_0^z h(t,\zeta)\, t^{\frac{1}{n}-1} dt$*. The function* q *is the best subordinant.*

Proof. Let

$$p(z,\zeta) = \frac{DR_{\lambda}^{m+1} f(z,\zeta)}{DR_{\lambda}^{m} f(z,\zeta)} = \frac{z + \sum_{j=n+1}^{\infty} C_{m+j}^{m+1} [1+(j-1)\lambda]^{m+1} a_j^2(\zeta) z^j}{z + \sum_{j=n+1}^{\infty} C_{m+j-1}^{m} [1+(j-1)\lambda]^{m} a_j^2(\zeta) z^j}$$

$$= \frac{1 + \sum_{j=n+1}^{\infty} C_{m+j}^{m+1} [1+(j-1)\lambda]^{m+1} a_j^2(\zeta) z^{j-1}}{1 + \sum_{j=n+1}^{\infty} C_{m+j-1}^{m} [1+(j-1)\lambda]^{m} a_j^2(\zeta) z^{j-1}}.$$

Evidently $p \in \mathcal{H}^*[1,n,\zeta]$.

Differentiating with respect to z, we obtain

$$p(z,\zeta) + zp_z'(z,\zeta) = \left(\frac{zDR_{\lambda}^{m+1} f(z,\zeta)}{DR_{\lambda}^{m} f(z,\zeta)} \right)_z', \ z \in U,\ \zeta \in \overline{U},$$

and (2.12) becomes

$$q(z,\zeta) + zq_z'(z,\zeta) \prec\prec p(z,\zeta) + zp_z'(z,\zeta), \qquad z \in U,\ \zeta \in \overline{U}.$$

Using Lemma 1.9 for $\gamma = 1$, we have

$$q(z,\zeta) \prec\prec p(z,\zeta), \quad z \in U,\ \zeta \in \overline{U}, \quad \text{i.e.}$$

$$q(z,\zeta) = \frac{1}{nz^{\frac{1}{n}}} \int_0^z h(t,\zeta)\, t^{\frac{1}{n}-1} dt \prec\prec \frac{DR_{\lambda}^{m+1} f(z,\zeta)}{DR_{\lambda}^{m} f(z,\zeta)}, \ z \in U, \zeta \in \overline{U},$$

and q is the best subordinant. □

Theorem 2.13. *Let* $h(z,\zeta)$ *be a convex function,* $h(0,\zeta) = 1$*. Let* $\lambda \geq 0$*,* $m,n \in \mathbb{N}$*,* $f(z,\zeta) \in \mathcal{A}_{n\zeta}^*$ *and suppose that* $\frac{m+1}{(m\lambda+1)z} DR_{\lambda}^{m+1} f(z,\zeta) - \frac{m(1-\lambda)}{(m\lambda+1)z} DR_{\lambda}^{m} f(z,\zeta)$ *is univalent and* $\left(DR_{\lambda}^{m} f(z,\zeta)\right)_z' \in \mathcal{H}^*[1,n,\zeta] \cap Q^*$*. If*

$$h(z,\zeta) \prec\prec \frac{m+1}{(m\lambda+1)z} DR_{\lambda}^{m+1} f(z,\zeta) - \frac{m(1-\lambda)}{(m\lambda+1)z} DR_{\lambda}^{m} f(z,\zeta), \qquad z \in U,\ \zeta \in \overline{U}, \tag{2.13}$$

then

$$q(z,\zeta) \prec\prec \left(DR_{\lambda}^{m} f(z,\zeta)\right)_z', \qquad z \in U,\ \zeta \in \overline{U},$$

where $q(z,\zeta) = \frac{m\lambda+1}{n\lambda z^{\frac{m\lambda+1}{n\lambda}}} \int_0^z h(t,\zeta)\, t^{\frac{(m-n)\lambda+1}{n\lambda}} dt$*. The function* q *is convex and it is the best subordinant.*

Proof. With notation

$$p(z,\zeta) = \left(DR_\lambda^m f(z,\zeta)\right)'_z = 1 + \sum_{j=n+1}^{\infty} C_{m+j-1}^m \left[1 + (j-1)\lambda\right]^m j a_j^2(\zeta) z^{j-1}$$

and $p(0,\zeta) = 1$, we obtain for $f(z,\zeta) = z + \sum_{j=n+1}^{\infty} a_j(\zeta) z^j$,

$$p(z,\zeta) + z p'_z(z,\zeta)$$

$$= \frac{m+1}{\lambda z} DR_\lambda^{m+1} f(z,\zeta) - \left(m - 1 + \frac{1}{\lambda}\right)\left(DR_\lambda^m f(z,\zeta)\right)'_z - \frac{m(1-\lambda)}{\lambda z} DR_\lambda^m f(z,\zeta)$$

and

$$p(z,\zeta) + \frac{\lambda}{m\lambda+1} z p'_z(z,\zeta) = \frac{m+1}{(m\lambda+1)z} DR_\lambda^{m+1} f(z,\zeta) - \frac{m(1-\lambda)}{(m\lambda+1)z} DR_\lambda^m f(z,\zeta).$$

Evidently $p \in \mathcal{H}^*[1,n,\zeta]$.

Then (2.13) becomes

$$h(z,\zeta) \prec\prec p(z,\zeta) + \frac{\lambda}{m\lambda+1} z p'_z(z,\zeta), \quad z \in U,\ \zeta \in \overline{U}.$$

By using Lemma 1.8 for $\gamma = m + \frac{1}{\lambda}$, we have

$$q(z,\zeta) \prec\prec p(z,\zeta),\ z \in U,\ \zeta \in \overline{U},\ \text{i.e. } q(z,\zeta) \prec\prec \left(DR_\lambda^m f(z,\zeta)\right)'_z,\ z \in U,\ \zeta \in \overline{U},$$

where $q(z,\zeta) = \frac{m\lambda+1}{n\lambda z^{\frac{m\lambda+1}{n\lambda}}} \int_0^z h(t,\zeta)\, t^{\frac{(m-n)\lambda+1}{n\lambda}} dt$. The function q is convex and it is the best subordinant. □

Corollary 2.14. *Let* $h(z,\zeta) = \frac{1+(2\beta-\zeta)z}{1+z}$ *be a convex function in* $U \times \overline{U}$*, where* $0 \le \beta < 1$*. Let* $\lambda \ge 0$*,* $m,n \in \mathbb{N}$*,* $f(z,\zeta) \in \mathcal{A}^*_{n\zeta}$ *and suppose that* $\frac{m+1}{(m\lambda+1)z} DR_\lambda^{m+1} f(z,\zeta) - \frac{m(1-\lambda)}{(m\lambda+1)z} DR_\lambda^m f(z,\zeta)$ *is univalent,* $\left(DR_\lambda^m f(z,\zeta)\right)'_z \in \mathcal{H}^*[1,n,\zeta] \cap Q^*$*. If*

$$h(z,\zeta) \prec\prec \frac{m+1}{(m\lambda+1)z} DR_\lambda^{m+1} f(z,\zeta) - \frac{m(1-\lambda)}{(m\lambda+1)z} DR_\lambda^m f(z,\zeta), \quad z \in U,\ \zeta \in \overline{U}, \tag{2.14}$$

then

$$q(z,\zeta) \prec\prec \left(DR_\lambda^m f(z,\zeta)\right)'_z, \quad z \in U,\ \zeta \in \overline{U},$$

where q *is given by* $q(z,\zeta) = 2\beta - \zeta + \frac{(1+\zeta-2\beta)(m\lambda+1)}{\lambda n z^{\frac{m\lambda+1}{\lambda n}}} \int_0^z \frac{t^{\frac{m\lambda+1}{\lambda n}-1}}{1+t} dt$*,* $z \in U$*,* $\zeta \in \overline{U}$*. The function* q *is convex and it is the best subordinant.*

Proof. Following the same steps as in the proof of Theorem 2.13 and considering $p(z,\zeta) = \left(DR_\lambda^m f(z,\zeta)\right)'_z$, the strong differential superordination (2.14) becomes

$$h(z,\zeta) = \frac{1+(2\beta-\zeta)z}{1+z} \prec\prec p(z,\zeta) + z p'_z(z,\zeta), \quad z \in U,\ \zeta \in \overline{U}.$$

By using Lemma 1.8 for $\gamma = \frac{m\lambda+1}{\lambda}$, we have $q(z,\zeta) \prec\prec p(z,\zeta)$, i.e.

$$q(z,\zeta)=\frac{m\lambda+1}{n\lambda z^{\frac{m\lambda+1}{n\lambda}}}\int_0^z h(t,\zeta)\, t^{\frac{(m-n)\lambda+1}{n\lambda}}dt = \frac{m\lambda+1}{n\lambda z^{\frac{m\lambda+1}{n\lambda}}}\int_0^z t^{\frac{(m-n)\lambda+1}{n\lambda}}dt\frac{1+(2\beta-\zeta)\,t}{1+t}dt$$

$$= 2\beta-\zeta+\frac{(1+\zeta-2\beta)(m\lambda+1)}{\lambda n z^{\frac{m\lambda+1}{\lambda n}}}\int_0^z \frac{t^{\frac{m\lambda+1}{\lambda n}-1}}{1+t}dt \prec\prec \left(DR_\lambda^m f(z,\zeta)\right)'_z,\ z\in U,\ \zeta\in\overline{U}.$$

The function q is convex and it is the best subordinant. □

Theorem 2.15. *Let $q(z,\zeta)$ be convex in $U\times\overline{U}$ and let h be defined by*

$$h(z,\zeta) = q(z,\zeta) + \frac{\lambda}{m\lambda+1} z q'_z(z,\zeta),\ \lambda\geq 0,\ m,n\in\mathbb{N}.$$

If $f(z,\zeta)\in\mathcal{A}^*_{n\zeta}$, suppose that $\frac{m+1}{(m\lambda+1)z}DR_\lambda^{m+1}f(z,\zeta) - \frac{m(1-\lambda)}{(m\lambda+1)z}DR_\lambda^m f(z,\zeta)$ is univalent and $\left(DR_\lambda^m f(z,\zeta)\right)'_z\in\mathcal{H}^*[1,n,\zeta]\cap Q^*$ and satisfies the strong differential superordination

$$h(z,\zeta) = q(z,\zeta) + \frac{\lambda}{m\lambda+1} z q'_z(z,\zeta) \prec\prec \tag{2.15}$$

$$\frac{m+1}{(m\lambda+1)z}DR_\lambda^{m+1}f(z,\zeta) - \frac{m(1-\lambda)}{(m\lambda+1)z}DR_\lambda^m f(z,\zeta),\ z\in U,\zeta\in\overline{U},$$

then

$$q(z,\zeta)\prec\prec \left(DR_\lambda^m f(z,\zeta)\right)'_z,\quad z\in U,\ \zeta\in\overline{U},$$

where $q(z,\zeta) = \frac{m\lambda+1}{n\lambda z^{\frac{m\lambda+1}{n\lambda}}}\int_0^z h(t,\zeta)\, t^{\frac{(m-n)\lambda+1}{n\lambda}}dt$. The function q is the best subordinant.

Proof. Let $p(z,\zeta) = \left(DR_\lambda^m f(z,\zeta)\right)' = 1+\sum_{j=n+1}^{\infty} C^m_{m+j-1}\left[1+(j-1)\lambda\right]^m j a_j^2(\zeta)\, z^{j-1}$.

Differentiating, we obtain

$$p(z,\zeta) + z p'_z(z,\zeta)$$

$$= \frac{m+1}{\lambda z}DR_\lambda^{m+1}f(z,\zeta) - \left(m-1+\frac{1}{\lambda}\right)\left(DR_\lambda^m f(z,\zeta)\right)'_z - \frac{m(1-\lambda)}{\lambda z}DR_\lambda^n f(z,\zeta)$$

and

$$p(z,\zeta) + \frac{\lambda}{m\lambda+1} z p'_z(z,\zeta)$$

$$= \frac{m+1}{(m\lambda+1)z}DR_\lambda^{m+1}f(z,\zeta) - \frac{m(1-\lambda)}{(m\lambda+1)z}DR_\lambda^m f(z,\zeta),\ z\in U,\ \zeta\in\overline{U},$$

and (2.15) becomes

$$q(z,\zeta) + \frac{\lambda}{m\lambda+1} z q'_z(z,\zeta) \prec\prec p(z,\zeta) + \frac{\lambda}{m\lambda+1} z p'_z(z,\zeta),\quad z\in U,\ \zeta\in\overline{U}.$$

Using Lemma 1.9 for $\gamma = m+\frac{1}{\lambda}$, we have $q(z,\zeta)\prec\prec p(z,\zeta)$, $z\in U$, $\zeta\in\overline{U}$, i.e.

$$q(z,\zeta) = \frac{m\lambda+1}{n\lambda z^{\frac{m\lambda+1}{n\lambda}}}\int_0^z h(t,\zeta)\, t^{\frac{(m-n)\lambda+1}{n\lambda}}dt \prec\prec \left(DR_\lambda^m f(z,\zeta)\right)'_z,\quad z\in U,\ \zeta\in\overline{U},$$

and q is the best subordinant. □

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Transformation of the traveller wave shape in propagation on a straight and inclined bed

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Abstract. By continuing the papers of the second author based on the relative movements method, we consider Gerstner's potential to study the traveller wave relative movement with respect to a moving dihedron with the translation velocity $c = \text{ct.}$, driven on a horizontal bed. In the second case we present the traveller wave relation motion with respect to a moving dihedron, driven on the inclined bed with the angle $\alpha = \text{ct.}$

Mathematics Subject Classification (2010): 76B15, 76B07, 76M25.

Keywords: Relative motion of traveller wave, mobile dihedron, inclined plane, straight or inclined bottom.

In this aim we shall apply the relative movements method, used with special success at the liquid viscous flow through rotor channels of radial turbo machines, through pipelines in vibration state, as well as at aquaplaning phenomenon of a car or airplane tyre.

We shall consider the Ferdinand von Gerstner's potential [1], whose constant C we determined as function of the wave height h, its wave-length λ and water deep in the horizontal channel H (Figure 5) [2], [3]

$$\Phi_1(X_1, Y_1, T_1) = C(h, \lambda, H)\text{ch}k(H - Y_1) \cdot \cos(kX_1 - \omega T_1),$$

using in this particular case the relative movement of the traveller wave with respect to the moving dihedron driven with his propagation velocity c, but taking then into consideration the bed slope, supposed to be constant $\alpha = \text{ct.}$

1. Traveller wave relative movement with respect to the moving dihedron with velocity c

The relations between the absolute, relative and transport variables being (Figure 5)

$$X_1(X, Y, T) = X + X_0(T_1) = X + cT_1, \tag{1.1}$$

$$Y_1(X,Y,T) = Y, \tag{1.2}$$

the time running in the same manner in both fixed and mobile dihedrons, from which by the total differentiation with respect to the time we shall obtain the relations between the absolute, relative and transport velocity components:

$$\begin{gathered} U_1(X_1,Y_1,T_1) = \frac{\partial \Phi_1}{\partial X_1} \\ = -kC\mathrm{ch}k(H-Y_1)\cdot\sin(kX_1-\omega T_1) = U(X,Y,T)+U_0 = U+c, \end{gathered} \tag{1.3}$$

$$V_1(X_1,Y_1,T_1) = \frac{\partial \Phi_1}{\partial Y_1} = -kC\mathrm{sh}k(H-Y_1)\cdot\cos(kX_1-\omega T_1) = V(X,Y,T), \tag{1.4}$$

from which one can deduce the expressions of the relative velocity components:

$$\begin{gathered} U = -kC\mathrm{ch}k(H-Y)\cdot\sin kX - c = \frac{dX}{dT}, \\ X - X_0 = [-kC\mathrm{ch}k(H-Y)\cdot\sin kX - c](T-T_0) \end{gathered} \tag{1.5}$$

$$\begin{gathered} V = -kC\mathrm{sh}k(H-Y)\cos kX = \frac{dY}{dT} \to \\ Y - Y_0 = -kC\mathrm{sh}k(H-Y)\cos kX(T-T_0), \end{gathered} \tag{1.6}$$

from which, eliminating the time and considering that at $X_0 = 0$ we have $Y_0 = H = 1m$, one shall obtain the two relations of iterative calculus, considering in the first relation that $Y_0 = H$

$$Y(Y_0 = H) = H + \frac{kC\mathrm{sh}kH\cdot\cos kX}{kC\mathrm{ch}kH\cdot\sin kX + c}(X-0), \tag{1.7}$$

and then considering that $Y_0 = Y$, calculated from the anterior relation (1.6)

$$Y(Y_0 = Y) = Y + \frac{kC\mathrm{sh}k(H-Y)\cdot\cos kX}{kC\mathrm{ch}k(H-Y)\cdot\sin kX + c}X, \tag{1.8}$$

the traveller wave trajectory being represented by the two values sequences for

$$H = 1m,\ \lambda = 4m,\ T_0 = 1,671s,\ k = 2\pi/\lambda = 1,57rad/m,\ C = 0,05222$$

and $c = \lambda/T_0 = 2,393776m/s$, presented in the Table 1, as in the Figures 1 and 2.

Table 1

$X(m)$	$Y(Y_0 = 1)(m)$	$Y(Y)(m)$	$X/Lambda$	$Y(Y_0 = 1)(m)$	$Y(Y)(m)$
0	1	1	0	1	1
0, 5	0, 9704	0, 9423	0, 125	0, 9979	0, 9958
1	0, 9212	0, 8525	0, 25	0, 9918	0, 9838
1, 5	0, 9212	0, 8520	0, 375	0, 9823	0, 9652
2	0, 9998	0, 9995	0, 5	0, 9404	0, 9423
2, 5	1, 1310	1, 2923	0, 625	0, 9570	0, 9173
3	1, 2362	1, 5865	0, 75	0, 9436	0, 8925
3, 5	1, 2080	1, 5097	0, 875	0, 9313	0, 8703
4	1, 0011	1, 0022	1	0, 9212	0, 8525

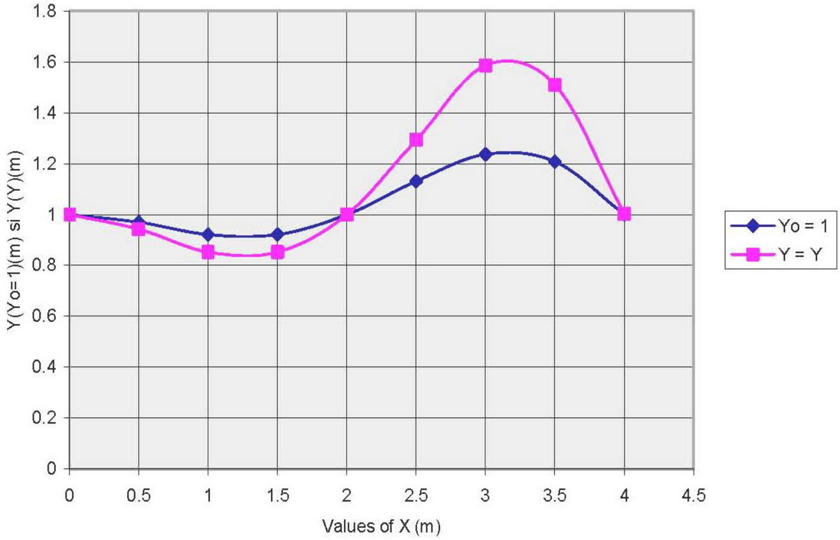

Fig. 1. The variations with abscissa X (m) of the two traveller wave trajectories

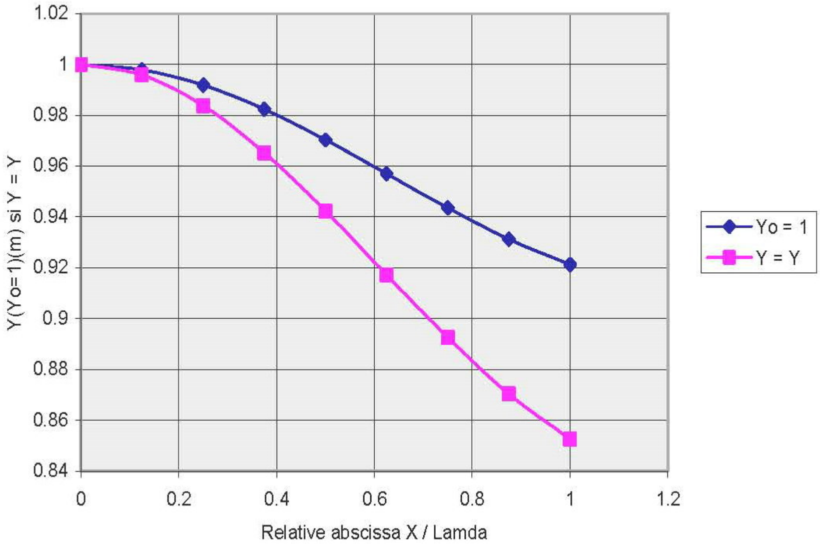

Fig. 2. The variation with relative abscissa X/λ of the two traveller wave trajectories

Concerning the variations (1.5) and (1.8), these are given in the Table 2 and are represented in the Figure 3 in the case when $H = 1 =$ constant and in the Figure 4 in the case when one considers Y variable and one takes anterior value calculated for the anterior X.

Table 2

$X(m)$	$Y(X)\ H = 1$	$X(m)$	$Y(X)\ H - Y$
0	0	0	0
$0,5$	0.026259	$0,5$	0.026259
1	$5.78E-05$	1	$5.54E-05$
$1,5$	-0.07865	$1,5$	-0.07864
2	-0.15748	2	-0.17993
$2,5$	-0.14849	$2,5$	-0.2048
3	-0.00062	3	-0.0009
$3,5$	0.206953	$3,5$	0.20729
4	0.315083	4	0.218097
$4,5$	0.237129	$4,5$	0.163315
5	0.001444	5	0.001101
$5,5$	-0.28739	$5,5$	-0.28688
6	-0.4723	6	-0.76075
$6,5$	-0.38721	$6,5$	-1.54749
7	-0.00336	7	-0.55962
$7,5$	0.44214	$7,5$	1.208477
8	0.63033	8	-0.09131
$8,5$	0.449407		
9	0.004678		
$9,5$	-0.49472		
10	-0.78693		

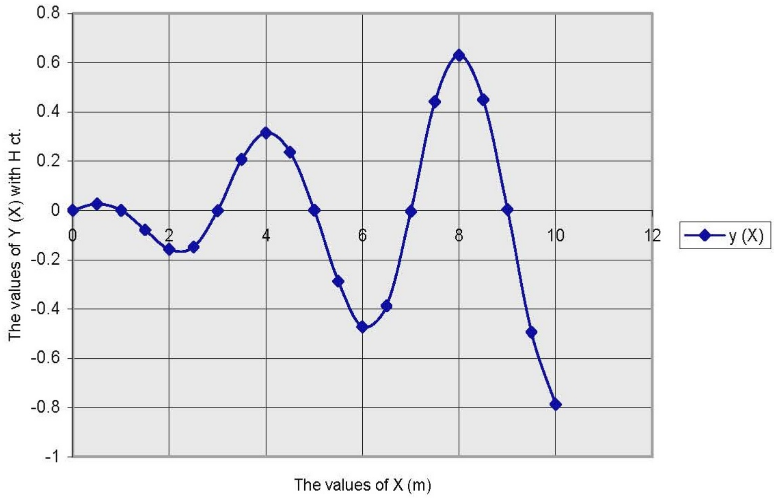

Fig. 3. The wave amplitude variation values $Y(X)$ as function of abscissa X in the consideration case of the position from the proximity of the wave free surface considered at the deep $H = 1$, supposed constant

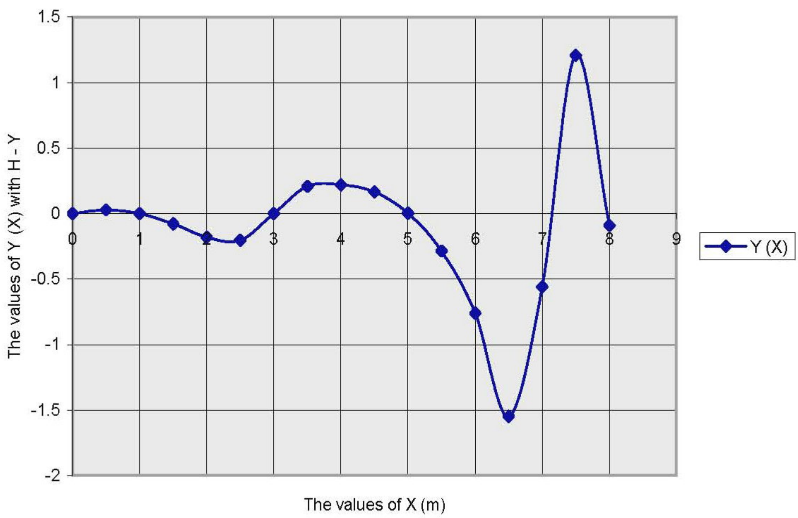

Fig. 4. The wave amplitude variation values $Y(X)$ as function of abscissa X in the consideration case of the position from the proximity of the wave free surface, considering that the deep is variable at the distance $H - Y$

2. The traveller wave relative motion with respect to the moving dihedron driven on the inclined plane

To study the wave free surface modification, by its propagation on a bed having a constant inclination $\alpha = $ ct., hypothesis sure in the case of ideal liquid, one shall utilize the relative motion method [3], by that the wave tries to immobilize with respect to the Cartesian dihedron $X0Y$, movable on these two coordinate axes through a translation motion with respect to the fixed absolute dihedron $X_1 0_1 Y_1$ (Figure 5).

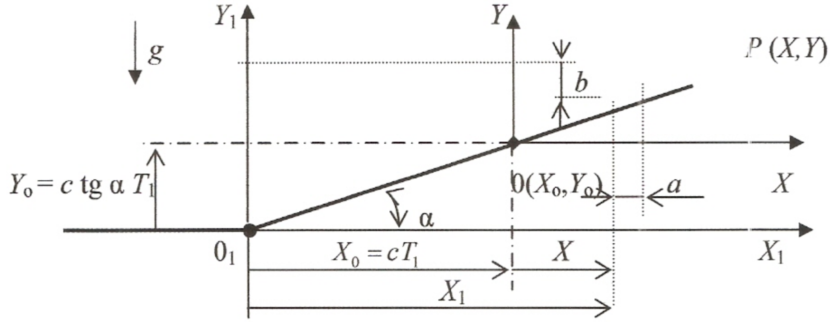

Fig. 5. Coordinate axes of the absolute and relative dihedron

In the hypothesis of wave propagation constant velocity $c = $ ct., the translation velocity of the mobile dihedron, solidary with the bed bottom, will be:

- velocity c for the axis origin $0X$ on the axis direction 0_1X_1 and consequently,
- velocity c tgα for the axis origin $0Y$ on the axis direction 0_1Y_1, the ratio of the two coordinates of the origin 0 of the moving dihedron being in each moment $Y_0/X_0 = $ tgα.

In this case the relations between absolute and relative variables will be:

$$X_1(X,Y,T) = X + X_0(T_1) = X + cT_1, \tag{2.1}$$

$$Y_1(X,Y,T) = Y + Y_0(T_1) = Y + c\ \mathrm{tg}\alpha T_1. \tag{2.2}$$

The relations between the absolute, relative and transport velocities are obtained by total derivation in time of the relations (2.1) and (2.2), partial derivative having not physical sense

$$U_1 = \frac{\partial \Phi_1}{\partial X_1} = -kC\mathrm{ch}k(Y_1)\sin(kX_1 - \omega T_1) = U(X,Y,T) + c, \tag{2.3}$$

$$V_1 = \frac{\partial \Phi_1}{\partial Y_1} = kC\mathrm{sh}k(Y_1)\cos(kX_1 - \omega T_1) = V(X,Y,T) + c\ \mathrm{tg}\alpha, \tag{2.4}$$

the time running in the same manner in any point of the absolute or relative dihedron in which we introduced the two components of the translation velocity of the moving dihedron:

$$\frac{dX_0}{dT} = U_0 = c = \mathrm{ct.} \quad \text{and} \quad \frac{dY_0}{dT} = V_0 = c\ \mathrm{tg}\alpha = \mathrm{ct.} \tag{2.5}$$

and integrating with respect to the time, we shall obtain

$$X - X_0 = \left[-\frac{C}{c\ \mathrm{tg}\alpha}\mathrm{sh}k(Y + c\ \mathrm{tg}\alpha T) \cdot \sin kX - cT \right](T - T_0) \tag{2.6}$$

$$Y - Y_0 = \left[\frac{C}{c\ \mathrm{tg}\alpha}\mathrm{ch}k(Y + c\ \mathrm{tg}\alpha T) \cdot \cos kX - c\ \mathrm{tg}\alpha T \right](T - T_0) \tag{2.7}$$

and eliminating the time one shall have

$$Y - Y_0 = \frac{\dfrac{C}{c\ \text{tg}\alpha}\text{ch}k(Y + c\ \text{tg}\alpha T)\cdot\cos kX - c\ \text{tg}\alpha T}{-\dfrac{C}{c\ \text{tg}\alpha}\text{sh}k(Y + c\ \text{tg}\alpha T)\cdot\sin kX - cT}(X - X_0), \tag{2.8}$$

which one shall use for the diverse time moments, for instance $T = 0$, in that case the relation becomes with $\text{tg}\alpha = \text{tg}20^\circ = 0,364$ being represented in the Figure 6

$$Y = H - \frac{\dfrac{C}{c\ \text{tg}\alpha}\text{ch}kY\cdot\cos kX}{\dfrac{C}{c\ \text{tg}\alpha}\text{sh}kY\cdot\sin kX}X \cong H - \frac{\dfrac{C}{c\ \text{tg}\alpha}\text{ch}kY\cdot\cos kX}{\dfrac{C}{c\ \text{tg}\alpha}\text{sh}kY\cdot k}. \tag{2.9}$$

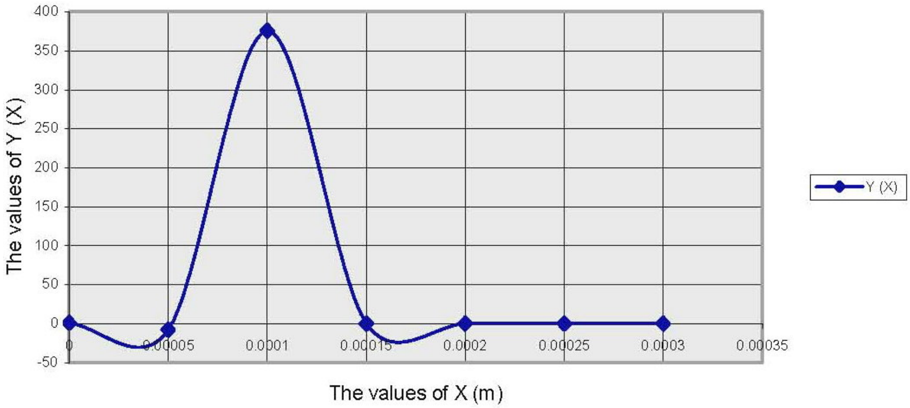

Fig. 6. The values variation $Y(X)$ for the initial moment $T = 0$

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On conditions for univalence of two integral operators

Daniel Breaz and Virgil Pescar

Abstract. In this paper we consider two integral operators. These operators was made based on the fact that the number of functions from their composition is entire part of the complex number modulus. The complex number is equal with the sum of the powers related to the functions from the composition of the integral operator.

Mathematics Subject Classification (2010): 30C45.

Keywords: Integral operator, univalence, integer part, modulus.

1. Introduction

Consider $\mathcal{U}$ the open unit disk. Let consider $\mathcal{A}$ be the class of analytic functions defined by $f(z) = z + a_2z^2 + ...$ We denote $\mathcal{S}$ be the class of univalent functions.

Theorem 1.1. [4] *If the function f belongs to the class $\mathcal{S}$, then for any complex number γ, $|\gamma| \leq \frac{1}{4}$, the function*

$$F_\gamma(z) = \int_0^z \left(\frac{f(t)}{t}\right)^\gamma dt$$

is in the class $\mathcal{S}$.

Theorem 1.2. [2] *If the function f is regular in unit disk $\mathcal{U}$, $f(z) = z + a_2z^2 + ...$ and*

$$\left(1 - |z|^2\right)\left|\frac{zf''(z)}{f'(z)}\right| \leq 1,$$

for all $z \in \mathcal{U}$, then the function f is univalent in $\mathcal{U}$.

Theorem 1.3. [5] *Let α be a complex number,* $\mathrm{Re}\alpha > 0$ *and $f(z) = z + a_2z^2 + ...$ be a regular function in $\mathcal{U}$. If*

$$\frac{1 - |z|^{2\mathrm{Re}\alpha}}{\mathrm{Re}\alpha}\left|\frac{zf''(z)}{f'(z)}\right| \leq 1,$$

for all $z \in \mathcal{U}$*, then for any complex number* β, $\mathrm{Re}\beta \geq \mathrm{Re}\alpha$*, the function*

$$F_\beta(z) = \left[\beta \int_0^z t^{\beta-1} f'(t)dt\right]^{\frac{1}{\beta}}$$

is in the class $\mathcal{S}$.

Theorem 1.4. [3] *If the function* g *is regular in* $\mathcal{U}$ *and* $|g(z)| < 1$ *in* $\mathcal{U}$*, then for all* $\xi \in \mathcal{U}$*, the following inequalities hold*

$$\left|\frac{g(\xi) - g(z)}{1 - \overline{g(z)}g(\xi)}\right| \leq \left|\frac{\xi - z}{1 - \overline{z}\xi}\right| \tag{1.1}$$

and

$$|g'(z)| \leq \frac{1 - |g(z)|^2}{1 - |z|^2},$$

the equalities hold in the case $g(z) = \epsilon\frac{z+u}{1+\overline{u}z}$*, where* $|\epsilon| = 1$ *and* $|u| < 1$.

Remark 1.5. [3] For $z = 0$, from inequality (1.1) we obtain for every $\xi \in \mathcal{U}$,

$$\left|\frac{g(\xi) - g(0)}{1 - \overline{g(0)}g(\xi)}\right| \leq |\xi|$$

and hence,

$$|g(\xi)| \leq \frac{|\xi| + |g(0)|}{1 + \overline{g(0)}g(\xi)}.$$

Considering $g(0) = a$ and $\xi = z$, then

$$|g(z)| \leq \frac{|z| + |a|}{1 + |a|\,|z|},$$

for all $z \in \mathcal{U}$.

Theorem 1.6. [7] *Let* $\gamma \in \mathbb{C}$, $f \in \mathcal{S}$, $f(z) = z + a_2z^2 + \ldots$.

If

$$\left|\frac{zf'(z) - f(z)}{zf(z)}\right| \leq 1,\ \forall\, z \in \mathcal{U}$$

and

$$|\gamma| \leq \frac{1}{\max_{|z|\leq 1}\left[\left(1 - |z|^2\right) \cdot |z| \cdot \frac{|z|+|a_2|}{1+|a_2||z|}\right]},$$

then

$$F_\gamma(z) = \int_0^z \left(\frac{f(t)}{t}\right)^\gamma dt$$

is in the class $\mathcal{S}$.

Theorem 1.7. [7] *Let* $\alpha, \beta, \gamma \in \mathbb{C}$, $f \in \mathcal{S}$, $f(z) = z + a_2z^2 + \ldots$.

If

$$\left|\frac{zf'(z) - f(z)}{zf(z)}\right| \leq 1,\ \forall\, z \in \mathcal{U},$$

$$\mathrm{Re}\beta \geq \mathrm{Re}\alpha > 0$$

and

$$|\gamma| \leq \frac{1}{\max_{|z|\leq 1}\left[\frac{1-|z|^{2\mathrm{Re}\alpha}}{\mathrm{Re}\alpha}\cdot|z|\cdot\frac{|z|+|a_2|}{1+|a_2||z|}\right]},$$

then

$$G_{\beta,\gamma}(z) = \left[\beta\int_0^z t^{\beta-1}\left(\frac{f(t)}{t}\right)^{\gamma}dt\right]^{\frac{1}{\beta}}$$

is in the class $\mathcal{S}$.

We define the next two integral operators

$$F_{[|\delta|]}(z) = \int_0^z \left(\frac{f_1(t)}{t}\right)^{\alpha_1}\cdot\ldots\cdot\left(\frac{f_{[|\delta|]}(t)}{t}\right)^{\alpha_{[|\delta|]}}dt,$$

where $\delta\in\mathbb{C}$, $|\delta|\notin[0,1)$, $\alpha_i\in\mathbb{C}$, $f_i\in\mathcal{A}$, $i=\overline{1,[|\delta|]}$, $\alpha_1+\ldots+\alpha_{[|\delta|]}=\delta$ and

$$G_{[|\gamma|]}(z) = \left[\gamma\int_0^z t^{\gamma-1}\left(\frac{f_1(t)}{t}\right)^{\alpha_1}\cdot\ldots\cdot\left(\frac{f_{[|\gamma|]}(t)}{t}\right)^{\alpha_{[|\gamma|]}}dt\right]^{\frac{1}{\gamma}},$$

$\gamma\in\mathbb{C}$, $|\gamma|\notin[0,1)$, $\alpha_i\in\mathbb{C}$, $f_i\in\mathcal{A}$, $i=\overline{1,[|\gamma|]}$, $\alpha_1+\ldots+\alpha_{[|\gamma|]}=\gamma$.

2. Main results

Theorem 2.1. *Let $\delta\in\mathbb{C}$, $|\delta|\notin[0,1)$, $\alpha_i\in\mathbb{C}$, for $i=\overline{1,[|\delta|]}$ and $\alpha_1+\ldots+\alpha_{[|\delta|]}=\delta$. If $f_i\in\mathcal{A}$, $f_i(z)=z+a_2^iz^2+\ldots$, for $i=\overline{1,[|\delta|]}$ and*

$$\left|\frac{zf_i'(z)-f_i(z)}{zf_i(z)}\right|\leq 1,\ \forall i=\overline{1,[|\delta|]},\ z\in\mathcal{U}, \tag{2.1}$$

$$\frac{|\alpha_1|+\ldots+\left|\alpha_{[|\delta|]}\right|}{\left|\alpha_1\cdot\ldots\cdot\alpha_{[|\delta|]}\right|}\leq 1, \tag{2.2}$$

$$\left|\alpha_1\cdot\ldots\cdot\alpha_{[|\delta|]}\right|\leq\frac{1}{\max_{|z|\leq 1}\left[\left(1-|z|^2\right)\cdot|z|\cdot\frac{|z|+|c|}{1+|c||z|}\right]}, \tag{2.3}$$

where

$$|c| = \frac{\left|\alpha_1a_2^1+\ldots+\alpha_{[|\delta|]}a_2^{[|\delta|]}\right|}{\left|\alpha_1\cdot\ldots\cdot\alpha_{[|\delta|]}\right|},$$

then

$$F_{[|\delta|]}(z) = \int_0^z\left(\frac{f_1(t)}{t}\right)^{\alpha_1}\cdot\ldots\cdot\left(\frac{f_{[|\delta|]}(t)}{t}\right)^{\alpha_{[|\delta|]}}dt$$

is in the class $\mathcal{S}$.

Proof. We have $f_i\in\mathcal{A}$, for all $i=\overline{1,[|\delta|]}$ and $\frac{f_i(z)}{z}\neq 0$, for all $i=\overline{1,[|\delta|]}$.

Let g be the function $g(z)=\left(\frac{f_1(z)}{z}\right)^{\alpha_1}\cdot\ldots\cdot\left(\frac{f_{[|\delta|]}(z)}{z}\right)^{\alpha_{[|\delta|]}}$, $z\in\mathcal{U}$. We have $g(0)=1$.

Consider the function

$$h(z) = \frac{1}{\left|\alpha_1 \cdot ... \cdot \alpha_{[|\delta|]}\right|} \cdot \frac{F''_{[|\delta|]}(z)}{F'_{[|\delta|]}(z)}, z \in \mathcal{U}.$$

The function $h(z)$ has the form:

$$h(z) = \frac{1}{\left|\alpha_1 \cdot ... \cdot \alpha_{[|\delta|]}\right|} \sum_{i=1}^{[|\delta|]} \alpha_i \frac{zf_i'(z) - f_i(z)}{zf_i(z)}.$$

Also,

$$h(0) = \frac{1}{\left|\alpha_1 \cdot ... \cdot \alpha_{[|\delta|]}\right|} \sum_{i=1}^{[|\delta|]} \alpha_i a_2^i.$$

By using the relations (2.1) and (2.2) we obtain that $|h(z)| < 1$ and

$$|h(0)| = \frac{\left|\alpha_1 a_2^1 + ... + \alpha_{[|\delta|]} a_2^{[|\delta|]}\right|}{\left|\alpha_1 \cdot ... \cdot \alpha_{[|\delta|]}\right|} = |c|.$$

Applying Remark 1.5 for the function h we obtain

$$\frac{1}{\left|\alpha_1 \cdot ... \cdot \alpha_{[|\delta|]}\right|} \cdot \left|\frac{F''_{[|\delta|]}(z)}{F'_{[|\delta|]}(z)}\right| \leq \frac{|z| + |c|}{1 + |c|\,|z|}, \forall z \in \mathcal{U}$$

and

$$\left|\left(1 - |z|^2\right) \cdot z \cdot \frac{F''_{[|\delta|]}(z)}{F'_{[|\delta|]}(z)}\right| \leq \left|\alpha_1 \cdot ... \cdot \alpha_{[|\delta|]}\right| \left(1 - |z|^2\right) \cdot |z| \cdot \frac{|z| + |c|}{1 + |c|\,|z|}, \forall z \in \mathcal{U}. \quad (2.4)$$

Consider the function $H : [0,\ 1] \to \mathbb{R}$ defined by

$$H(x) = (1 - x^2)x \frac{x + |c|}{1 + |c|\,x};\ x = |z|.$$

We have

$$H\left(\frac{1}{2}\right) = \frac{3}{8} \cdot \frac{1 + 2|c|}{2 + |c|} > 0 \Rightarrow \max_{x \in [0,1]} H(x) > 0.$$

Using this result and from (2.4) we have:

$$\left|\left(1 - |z|^2\right) \cdot z \cdot \frac{F''_{[|\delta|]}(z)}{F'_{[|\delta|]}(z)}\right| \leq \left|\alpha_1 \cdot ... \cdot \alpha_{[|\delta|]}\right| \cdot \max_{|z|<1} \left[\left(1 - |z|^2\right) \cdot |z| \cdot \frac{|z| + |c|}{1 + |c|\,|z|}\right], \forall z \in \mathcal{U}. \quad (2.5)$$

Applying the condition (2.3) in the form (2.5) we obtain that

$$\left(1 - |z|^2\right) \cdot \left|z \cdot \frac{F''_{[|\delta|]}(z)}{F'_{[|\delta|]}(z)}\right| \leq 1, \forall z \in \mathcal{U},$$

and from Theorem 1.2 we obtain that $F_{[|\delta|]} \in \mathcal{S}$.

Theorem 2.2. *Let* $\gamma,\ \delta \in \mathbb{C},\ |\gamma| \notin [0,1),\ \alpha_i \in \mathbb{C}$, *for* $i = \overline{1, [|\gamma|]}$, $\alpha_1 + ... + \alpha_{[|\gamma|]} = \gamma$. *If* $f_i \in \mathcal{A},\ f_i(z) = z + a_2^i z^2 + ...$, *for* $i = \overline{1, [|\gamma|]}$ *and*

$$\left|\frac{zf_i'(z) - f_i(z)}{zf_i(z)}\right| \leq 1,\ \forall i = \overline{1, [|\gamma|]},\ z \in \mathcal{U}, \tag{2.6}$$

$$\frac{|\alpha_1| + ... + \left|\alpha_{[|\gamma|]}\right|}{\left|\alpha_1 \cdot ... \cdot \alpha_{[|\gamma|]}\right|} \leq 1, \tag{2.7}$$

$$\mathrm{Re}\gamma \geq \mathrm{Re}\delta > 0,$$

$$\left|\alpha_1 \cdot ... \cdot \alpha_{[|\gamma|]}\right| \leq \frac{1}{\max_{|z|\leq 1}\left[\left(1 - |z|^2\right) \cdot |z| \cdot \frac{|z|+|c|}{1+|c||z|}\right]}, \tag{2.8}$$

where

$$|c| = \frac{\left|\alpha_1 a_2^1 + ... + \alpha_{[|\gamma|]} a_2^{[|\gamma|]}\right|}{\left|\alpha_1 \cdot ... \cdot \alpha_{[|\gamma|]}\right|},$$

then

$$G_{[|\gamma|]}(z) = \left[\gamma \int_0^z t^{\gamma-1} \left(\frac{f_1(t)}{t}\right)^{\alpha_1} \cdot ... \cdot \left(\frac{f_{[|\gamma|]}(t)}{t}\right)^{\alpha_{[|\gamma|]}} dt\right]^{\frac{1}{\gamma}}$$

is in the class $\mathcal{S}$.

Proof. We consider the function

$$h(z) = \int_0^z \left(\frac{f_1(t)}{t}\right)^{\alpha_1} \cdot ... \cdot \left(\frac{f_{[|\gamma|]}(t)}{t}\right)^{\alpha_{[|\gamma|]}} dt.$$

Let be the function

$$p(z) = \frac{1}{\left|\alpha_1 \cdot ... \cdot \alpha_{[|\gamma|]}\right|} \cdot \frac{h''(z)}{h'(z)},\ z \in \mathcal{U}.$$

The function $p(z)$ has the form:

$$p(z) = \frac{1}{\left|\alpha_1 \cdot ... \cdot \alpha_{[|\gamma|]}\right|} \sum_{i=1}^{[|\gamma|]} \alpha_i \frac{zf_i'(z) - f_i(z)}{zf_i(z)}.$$

By using the relations (2.6) and (2.7) we obtain $|p(z)| < 1$ and

$$|p(0)| = \frac{\left|\alpha_1 a_2^1 + ... + \alpha_{[|\gamma|]} a_2^{[|\gamma|]}\right|}{\left|\alpha_1 \cdot ... \cdot \alpha_{[|\gamma|]}\right|} = |c|.$$

Applying Remark 1.5 for the function h we obtain

$$\frac{1}{\left|\alpha_1 \cdot ... \cdot \alpha_{[|\gamma|]}\right|} \cdot \left|\frac{h''(z)}{h'(z)}\right| \leq \frac{|z| + |c|}{1 + |c|\,|z|}, \forall z \in \mathcal{U}$$

and

$$\left|\frac{1 - |z|^{2\mathrm{Re}\delta}}{\mathrm{Re}\delta} \cdot z \cdot \frac{h''(z)}{h'(z)}\right| \leq \left|\alpha_1 \cdot ... \cdot \alpha_{[|\gamma|]}\right| \frac{1 - |z|^{2\mathrm{Re}\delta}}{\mathrm{Re}\delta} \cdot |z| \cdot \frac{|z| + |c|}{1 + |c|\,|z|}, \forall z \in \mathcal{U}. \tag{2.9}$$

Consider the function $Q : [0,\ 1] \to \mathbb{R}$ defined by

$$Q(x) = \frac{1 - x^{2\mathrm{Re}\delta}}{\mathrm{Re}\delta} \cdot x \cdot \frac{x + |c|}{1 + |c|\, x};\ x = |z|\,.$$

We have $Q\left(\frac{1}{2}\right) > 0 \Rightarrow \max_{x \in [0,1]} Q(x) > 0$.

Using this result in (2.9), we have:

$$\frac{1 - |z|^{2\mathrm{Re}\delta}}{\mathrm{Re}\delta} \left|\frac{zh''(z)}{h'(z)}\right| \le \left|\alpha_1 \cdot \ldots \cdot \alpha_{[|\gamma|]}\right| \cdot \max_{|z|<1} \left[\frac{1 - |z|^{2\mathrm{Re}\delta}}{\mathrm{Re}\delta} \cdot |z| \cdot \frac{|z| + |c|}{1 + |c|\,|z|}\right], \forall z \in \mathcal{U}. \tag{2.10}$$

Applying the condition (2.8) in the relation (2.10), we obtain that

$$\frac{1 - |z|^{2\mathrm{Re}\delta}}{\mathrm{Re}\delta} \left|\frac{zh''(z)}{h'(z)}\right| \le 1, \forall z \in \mathcal{U}$$

and from Theorem 1.3, we obtain that $G_{[|\gamma|]} \in \mathcal{S}$.

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Subordination of certain subclass of convex function

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Abstract. In this paper we study the subordination of a certain subclass of convex functions with negative coefficients.

Mathematics Subject Classification (2010): 30C45.

Keywords: Analytic functions, positive coefficients, negative coefficients, subordination, convolution product, Pochammer symbol.

1. Introduction

Let $\mathcal{H}(U)$ be the set of functions which are regular in the unit disc U,

$$\mathcal{A} = \{f \in \mathcal{H}(U) : f(0) = f'(0) - 1 = 0\}$$

and $S = \{f \in \mathcal{A} : f \text{ is univalent in } U\}$.

In [11], the subfamily T of S consisting of functions f,

$$f(z) = z - \sum_{j=2}^{\infty} a_j z^j, \ a_j \geq 0, j = 2, 3, ..., \ z \in U, \tag{1.1}$$

was introduced.

Thus, we have the subfamily $S - T$ consisting of functions f of the form

$$f(z) = z + \sum_{j=2}^{\infty} a_j z^j, \ a_j \geq 0, j = 2, 3, ..., \ z \in U \tag{1.2}$$

Let consider N to be the class of all functions Φ which are analytic, convex, univalent in U and normalized by $\Phi(0) = 1$, $Re(\Phi(z)) > 0$ $(z \in U)$. Making use of the subordination principle of the analytic functions, many authors investigated the subclasses $S^\star(\Phi)$, $K(\Phi)$ and $C(\Phi, \psi)$ of the class $\mathcal{A}$, $\Phi, \psi \in N$ (see [4]), as follows:

$$S^\star(\Phi) := \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec \Phi(z) \in U \right\}$$

$$K(\Phi) := \left\{ f \in \mathcal{A} : 1 + \frac{zf''(z)}{f'(z)} \prec \Phi(z) \in U \right\} \tag{1.3}$$

$$C(\Phi, \psi) := \left\{ f \in \mathcal{A} : \exists g \in S^{\star}(\Phi)\ s.t.\ \frac{zf'(z)}{g(z)} \prec \psi(z) \in U \right\}.$$

Let $g(z) \in \mathcal{A}$, $g(z) = z + \sum_{j\geq 2} b_j z^j$. Then, the Hadamard product (or convolution) $f * g$ is defined by

$$f(z) * g(z) = (f * g)(z) = z + \sum_{j\geq 2} a_j b_j z^j.$$

If $g(z) \in \mathcal{A}$, $g(z) = z - \sum_{j\geq 2} b_j z^j$, the Hadamard product (or convolution) $f * g$ is defined by

$$f(z) * g(z) = (f * g)(z) = z - \sum_{j\geq 2} a_j b_j z^j.$$

Next, we have the basic idea of subordination as following: if f and g are analytic in U, then the function f is said to be subordinate to g, such as

$$f \prec g \text{ or } f(z) \prec g(z)\ (z \in U),$$

iff there exist the Schwarz function w, analytic in U, with $w(0) = 0$ and $|w(z)| < 1$, such that $f(z) = g(w(z))\ (z \in U)$.

Let $\psi : \mathbb{C}^2 \times U \to \mathbb{C}$ and h analytic in U. If p and $\psi(p(z), zp'(z); z)$ are univalent in U and satisfy the first-order differential superordination

$$h(z) \prec \psi(p(z), zp'(z); z), \quad for\ z \in U, \tag{1.4}$$

then p is considered to be a function of differential superordination. The analytic function q is a subordination of the differential superordination solutions, or more simple a subordination, if $q \prec p$ for all p that satisfy (1.4).

An univalent subordination $\tilde{q}$ that satisfies $q \prec \tilde{g}$ for all subordinations (1.4) is said to be the best subordination for (1.4). The best subordination is unique up to a rotation of U.

We continue our paper with already studied operators and known theories concerning the subordination principle that have to help us in our research.

2. Preliminary results

Let D^n be the Sălăgean differential operator (see [10]) $D^n : \mathcal{A} \to \mathcal{A}$, $n \in \mathbb{N}$, defined as:

$$D^0 f(z) = f(z),\ D^1 f(z) = Df(z) = zf'(z),\ D^n f(z) = D(D^{n-1} f(z)) \tag{2.1}$$

and $D^k,\ D^k : \mathcal{A} \to \mathcal{A},\ k \in \mathbb{N} \cup \{0\}$, of form:

$$D^0 f(z) = f(z),\ \ldots,\ D^k f(z) = D(D^{k-1} f(z)) = z + \sum_{n=2}^{\infty} n^k a_n z^n. \tag{2.2}$$

Definition 2.1. [5] *Let* $\beta, \lambda \in \mathbb{R}$, $\beta \geq 0$, $\lambda \geq 0$ *and* $f(z) = z + \sum_{j=2}^{\infty} a_j z^j$. *We denote by* D_λ^β *the linear operator defined by*

$$D_\lambda^\beta : A \to A, \quad D_\lambda^\beta f(z) = z + \sum_{j=n+1}^{\infty} [1 + (j-1)\lambda]^\beta a_j z^j . \tag{2.3}$$

Remark 2.2. In [1], we have introduced the following operator concerning the functions of form (1.1):

$$D_\lambda^\beta : A \to A, \quad D_\lambda^\beta f(z) = z - \sum_{j=n+1}^{\infty} [1 + (j-1)\lambda]^\beta a_j z^j . \tag{2.4}$$

The neighborhoods concerning the class of functions defined using the operator (2.4) is studied in [3].

Definition 2.3. [13] *We denote by* Q *the set of functions that are analytic and injective on* $\overline{U} - E(f)$, *where* $E(f) = \{\zeta \in \partial U : \lim_{z \to \zeta} f(z) = \infty\}$, *and* $f'(\zeta) \neq 0$ *for* $\zeta \in \partial U - E(f)$. *The subclass of* Q *for which* $f(0) = a$ *is denoted by* $Q(a)$.

Lemma 2.4. [13] *Let* h *be a convex function with* $h(0) = a$, *and let* $\gamma \in \mathbb{C} - \{0\}$ *be a complex number with* $Re\gamma \geq 0$. *If* $p \in \mathcal{H}[a, n] \cap Q$, $p(z) + \frac{1}{\gamma} zp'(z)$ *is univalent in* U *and*

$$h(z) \prec p(z) + \frac{1}{\gamma} zp'(z), \ for \ z \in U,$$

then

$$q(z) \prec p(z), \ for \ z \in U,$$

where $q(z) = \frac{\gamma}{nz^{\gamma/n}} \int_0^z h(t) t^{\gamma/n-1} dt$, *for* $z \in U$. *The function* q *is convex and it is the best subordination.*

Lemma 2.5. [13] *Let* q *be a convex function and let* $h(z) = q(z) + \frac{1}{\gamma} zq'(z)$, *for* $z \in U$, *where* $Re\gamma \geq 0$. *If* $p \in \mathcal{H}[a, n] \cap Q$, $p(z) + \frac{1}{\gamma} zp'(z)$ *is univalent in* U *and*

$$q(z) + \frac{1}{\gamma} zp'(z) \prec p(z) + \frac{1}{\gamma} zp'(z), \ for \ z \in U,$$

then

$$q(z) \prec p(z), \ for \ z \in U,$$

where $q(z) = \frac{\gamma}{nz^{\gamma/n}} \int_0^z h(t) t^{\gamma/n-1} dt$, *for* $z \in U$. *The function* q *is the best subordination.*

Definition 2.6. *For* $f \in \mathcal{A}$, *the generalized derivative operator* $\mu_{\lambda_1,\lambda_2}^{n,m}$ *is defined by*

$$\mu_{\lambda_1,\lambda_2}^{n,m} : \mathcal{A} \to \mathcal{A}$$

$$\mu_{\lambda_1,\lambda_2}^{n,m} f(z) = z + \sum_{k\geq 2} \frac{[1+\lambda_1(k-1)]^{m-1}}{[1+\lambda_2(k-1)]^m} c(n,k) a_k z^k, \ (z \in U), \tag{2.5}$$

where $n, m \in \mathbb{N}$, $\lambda_2 \geq \lambda_1 \geq 0$ *and* $c(n,k) = \dfrac{(n+1)_{k-1}}{(1)_{k-1}}$, $(x)_k$ *is the Pochammer symbol (or the shifted factorial).*

Remark 2.7. If we denote by $(x)_k$ the Pochammer symbol, we define it as follows:

$$(x)_k = \begin{cases} 1 & \text{for } \ k = 0, \ x \in \mathbb{C} - \{0\} \\ x(x+1)(x+2)\cdot \ldots \cdot (x+k-1) & \text{for } \ k \in \mathbb{N}^\star \ \textit{and} \ x \in \mathbb{C}. \end{cases}$$

Lemma 2.8. [14] *Let* $0 < a \leq c$. *If* $c \geq 2$ *or* $a + c \geq 3$, *then the function*

$$h(a,c;z) = z + \sum_{k\geq 2} \frac{(a)_{k-1}}{(c)_{k-1}} z^k \ (z \in U),$$

belongs to the class K *of convex functions (defined in (*1.3*)).*

Lemma 2.9. [12] *Let* $\Phi \in \mathcal{A}$, *convex in* U, *with* $\Phi(0) = 1$ *and*

$$Re(\beta\Phi(z) + \gamma) > 0 \ (\beta, \gamma \in \mathbb{C}\,; \ z \in U).$$

If $p(z)$ *is analytic in* U, $p(0) = \Phi(0)$, *then*

$$p(z) + \frac{zp'(z)}{\beta p(z) + \gamma} \prec \Phi(z) \Rightarrow p(z) \prec \Phi(z).$$

Next we study the subordination of a certain subclass of convex functions defined by using the Hadamard (convolution) product.

3. Main results

We consider the following operator (see [8], [9]):

$$\psi_1(z) = \sum_{k\geq 1} \frac{1+c}{k+c} z^k \ (Re\{c\} \geq 0\,; \ z \in U). \tag{3.1}$$

Let the operator $D_{\lambda_1,\lambda_2}^{n,\beta} f(z)$, $n \in \mathbb{N}, \beta \geq 0$, $\lambda_1, \lambda_2 \geq 0$ to be the following:

$$\begin{aligned} D_{\lambda_1,\lambda_{2\ominus}}^{n,\beta} f(z) &= \mu_{\lambda_1,\lambda_2}^{n,\beta} f(z) * \psi_1(z) \\ &= z - \sum_{k\geq 2} \frac{[1-\lambda_1(k-1))]^{\beta-1}}{[1-\lambda_2(k-1))]^{\beta}} \cdot \frac{1+c}{k+c} \cdot c(n,k) \cdot a_k z^k, \end{aligned} \tag{3.2}$$

where $f(z)$ is of form (1.1) and

$$\begin{aligned} D_{\lambda_1,\lambda_{2\oplus}}^{n,\beta} f(z) &= \mu_{\lambda_1,\lambda_2}^{n,\beta} f(z) * \psi_1(z) \\ &= z + \sum_{k\geq 2} \frac{[1-\lambda_1(k-1))]^{\beta-1}}{[1-\lambda_2(k-1))]^{\beta}} \cdot \frac{1+c}{k+c} \cdot c(n,k) \cdot a_k z^k, \end{aligned} \tag{3.3}$$

where $f(z)$ is of form (1.2).

Furthermore, we consider $D_{\lambda_1,\lambda_2}^{n,\beta} f(z)$ to be of form (3.2) or (3.3).

Definition 3.1. *Let $f(z)$ of form* (1.2)*, $z \in U$. We say that f is in the class $K_\lambda^\beta(\Phi(z))$ if:*

$$1+\frac{z(D^{n,\beta}_{\lambda_1,\lambda_2}f(z))''}{(D^{n,\beta}_{\lambda_1,\lambda_2}f(z))'}\prec\Phi(z),\quad n\in\mathbb{N},\beta\geq 0,\ \lambda_1,\lambda_2\geq 0,\quad z\in U,$$

where the function Φ is analytic, convex and univalent in U, normalized by

$$\Phi(0)=1,\ Re(\Phi(z))>0\ (z\in U).$$

Remark 3.2. From Definition 3.1, we have the class $K_\lambda^\beta(\Phi(z))$ as follows:

$$K_\lambda^\beta(\Phi(z))=\left\{f(z)\in S:1+\frac{z(D^{n,\beta}_{\lambda_1,\lambda_2}f(z))''}{(D^{n,\beta}_{\lambda_1,\lambda_2}f(z))'}\prec\Phi(z),\ \Phi(z)\in S,\ \Phi\text{ is convex}, z\in U\right\},$$

where $n\in\mathbb{N},\beta\geq 0,\ \lambda_1,\lambda_2\geq 0$.

Theorem 3.3. *Let the function $\Phi(z)$ to be analytic, convex and univalent in U, normalized by $\Phi(0)=1$, $Re(\Phi(z))>0\ (z\in U)$. Let $\lambda\geq 0$, $\gamma,\chi\in\mathbb{C}$, with $Re(\chi\Phi(z)+\gamma)>0$, $n\in\mathbb{N}$, $f\in\mathcal{A}_n$ and suppose that*

$$[D^{n,\beta}_{\lambda_1,\lambda_2}f(z)]'+\frac{(n+1)[(D^{n+1,\beta}_{\lambda_1,\lambda_2}f(z)-D^{n,\beta}_{\lambda_1,\lambda_2}f(z)]'}{\chi[D^{n,\beta}_{\lambda_1,\lambda_2}f(z)]'+\gamma},\ \beta\geq 0,\ \lambda_1,\lambda_2\geq\ (z\in U),$$

is univalent and the operator $D^{n,\beta}_{\lambda_1,\lambda_2}f(z)$ is in $\mathcal{H}[1,n]\cap Q$. If

$$\Phi(z)\prec[D^{n,\beta}_{\lambda_1,\lambda_2}f(z)]'+\frac{(n+1)[(D^{n+1,\beta}_{\lambda_1,\lambda_2}f(z)-D^{n,\beta}_{\lambda_1,\lambda_2}f(z)]'}{\chi[D^{n,\beta}_{\lambda_1,\lambda_2}f(z)]'+\gamma},\ z\in U,\tag{3.4}$$

then

$$q(z)\prec[D^{n,\beta}_{\lambda_1,\lambda_2}f(z)]'\ for\ z\in U,$$

where $q(z)=\dfrac{1}{nz^{\frac{u(z)}{n}}}\cdot\displaystyle\int_0^z\Phi(t)\cdot t^{\frac{u(t)}{n-1}}\,dt$, $u(z)=\chi\Phi(z)+\gamma,\ z\in U$.

Proof. We are going to prove the Theorem 3.3 by taking into account the operator $D^{n,\beta}_{\lambda_1,\lambda_{2\oplus}}f(z)$. We use the notation $D^{n,\beta}_{\lambda_1,\lambda_2}f(z)=D^{n,\beta}_{\lambda_1,\lambda_{2\oplus}}f(z)$ for simplification.

Let

$$\Phi(z)=\frac{z(D^{n,\beta}_{\lambda_1,\lambda_2}f(z))'}{(D^{n,\beta}_{\lambda_1,\lambda_2}f(z))}=\frac{[h^{n,\beta}_{\lambda_1,\lambda_2}*\psi_1*zf'](z)}{[h^{n,\beta}_{\lambda_1,\lambda_2}*\psi_1*f](z)},$$

where $h^{n,\beta}_{\lambda_1,\lambda_2}(z)=z+\displaystyle\sum_{k\geq 2}\frac{[1-\lambda_1(k-1))]^{\beta-1}}{[1-\lambda_2(k-1))]^{\beta}}\cdot c(n,k)\cdot a_k z^k$, $\beta\geq 0,\ \lambda_1,\lambda_2\geq\ (z\in U)$.

Thus, we obtain from $Re(\chi\Phi(z)+\gamma)>0$ that $|\chi|+|\gamma|\leq 2$.

We consider $p(z)=[D^{n,\beta}_{\lambda_1,\lambda_2}f(z)]'$ and we obtain the following:

$$p(z)+zp'(z)=[h^{n,\beta}_{\lambda_1,\lambda_2}*\psi_1*zf'](z)=[D^{n,\beta}_{\lambda_1,\lambda_2}f(z)]'+(n+1)[[D^{n+1,\beta}_{\lambda_1,\lambda_2}f(z)]'-[D^{n,\beta}_{\lambda_1,\lambda_2}f(z)]']$$

and

$$p(z)+\frac{zp'(z)}{\chi p(z)+\gamma}=[D^{n,\beta}_{\lambda_1,\lambda_2}f(z)]'\cdot\left[1-\frac{n+1}{\chi[D^{n,\beta}_{\lambda_1,\lambda_2}f(z)]'+\gamma}\right]+\frac{(n+1)[D^{n+1,\beta}_{\lambda_1,\lambda_2}f(z)]'}{\chi[D^{n,\beta}_{\lambda_1,\lambda_2}f(z)]'+\gamma}$$

$$=[D^{n,\beta}_{\lambda_1,\lambda_2}f(z)]'+\frac{(n+1)[(D^{n+1,\beta}_{\lambda_1,\lambda_2}f(z)-D^{n,\beta}_{\lambda_1,\lambda_2}f(z)]'}{\chi[D^{n,\beta}_{\lambda_1,\lambda_2}f(z)]'+\gamma}.$$

It is obviously that $p\in\mathcal{H}[1,n]$.

Further, we see that (3.4) can be written as follows

$$\Phi(z)\prec p(z)+\frac{zp'(z)}{\chi p(z)+\gamma},\ z\in U.$$

Making use of Lemma 2.9, we obtain

$$q(z)\prec p(z),\ z\in U\ i.e.\ q(z)\prec[D^{n,\beta}_{\lambda_1,\lambda_2}f(z)]'\ for\ z\in U,$$

where $q(z)=\frac{1}{nz^{\frac{u(z)}{n}}}\cdot\int\limits_0^z\Phi(t)\cdot t^{\frac{u(t)}{n-1}}\,dt$. The function q is convex and it is the best subordinant.

Remark 3.4. The proof is similar for $D^{n,\beta}_{\lambda_1,\lambda_2}f(z)$ of form (3.2).

Example 3.5. If we consider $\beta\in\mathbb{N}$ and $\psi_1(z)=1_z$ we obtain the operator $D^{n,\beta}_{\lambda_1,\lambda_{2\oplus}}f(z)$. Therefore we have

$$D^{n,\beta}_{\lambda_1,\lambda_{2\oplus}}f(z)=z+\sum_{k\geq2}\frac{[1-\lambda_1(k-1))]^{\beta-1}}{[1-\lambda_2(k-1))]^{\beta}}\cdot c(n,k)\cdot a_kz^k=\mu^{n,\beta}_{\lambda_1,\lambda_2}f(z),$$

which is a particular case of the Theorem 3.3. Thus, the open problem from [5] concerning the subordination of the class of convex functions is solved.

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An application of generalized integral operator

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Abstract. In this paper the authors introduced a new certain integral operator for analytic univalent functions defined in the open unit disc U. The object of this paper is to give an application of this operator to the differential inequalities.

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Keywords: Analytic functions, integral operator, multiplier transformations.

1. Introduction

Let A denote the class of functions of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

which are analytic in the open unit disc $\mathrm{U} = \{z \in \mathbb{C} : |z| < 1\}$.

In [3], Cătaş extended the multiplier transformations and defined the operator $I^m(\lambda, l)$ on A by the following series

$$I^m(\lambda, l)f(z) = z + \sum_{n=2}^{\infty} \left[\frac{1 + l + \lambda(n-1)}{1 + l}\right]^m a_n z^n,\ z \in \mathrm{U},$$

where $\lambda \geq 0$, $l \geq 0$, and $m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$. We note that $I^0(1,0)f(z) = f(z)$ and $I^1(1,0)f(z) = zf'(z)$.

Now, we define the integral operator $J^m(\lambda, l) : A \to A$, with $\lambda > 0$, $l \geq 0$, and $m \in \mathbb{N}_0$ as follows:

$$\begin{aligned}
J^0(\lambda, l)f(z) &= f(z), \\
J^1(\lambda, l)f(z) &= \frac{1+l}{\lambda} z^{1-\frac{1+l}{\lambda}} \int_0^z t^{\frac{1+l}{\lambda}-2} f(t)\,\mathrm{d}\,t, \\
J^2(\lambda, l)f(z) &= \frac{1+l}{\lambda} z^{1-\frac{1+l}{\lambda}} \int_0^z t^{\frac{1+l}{\lambda}-2} J^1(\lambda, l)f(t)\,\mathrm{d}\,t,
\end{aligned}$$

and, in general,

$$J^m(\lambda,l)f(z)=\frac{1+l}{\lambda}z^{1-\frac{1+l}{\lambda}}\int_0^z t^{\frac{1+l}{\lambda}-2}J^{m-1}(\lambda,l)f(t)\,\mathrm{d}\,t$$

$$=\underbrace{J^1(\lambda,l)\left(\frac{z}{1-z}\right)*J^1(\lambda,l)\left(\frac{z}{1-z}\right)*\cdots*J^1(\lambda,l)\left(\frac{z}{1-z}\right)*f(z)}_{m\text{ times}}. \tag{1.2}$$

We note that if $f\in A$, then from (1.1) and (1.2), we have

$$J^m(\lambda,l)f(z)=z+\sum_{n=2}^{\infty}\left[\frac{1+l}{1+l+\lambda(n-1)}\right]^m a_nz^n,\ z\in\mathrm{U}, \tag{1.3}$$

for $\lambda>0$, $l\geq 0$, and $m\in\mathbb{N}_0$. From (1.3), it is easy to verify that

$$\lambda z(J^{m+1}(\lambda,l)f(z))'=(1+l)J^m(\lambda,l)f(z)-(1+l-\lambda)J^{m+1}(\lambda,l)f(z), \tag{1.4}$$

whenever $\lambda>0$.

We note that:

(i) $J^m(1,1)f(z)=I^mf(z)$ (see Flett [4], and Uralegaddi and Somanatha [9]);
(ii) $J^m(1,0)f(z)=I^mf(z)$, $m\in\mathbb{N}_0$ (see Sălăgean [8]);
(iii) $J^{\alpha}(1,1)f(z)=I^{\alpha}f(z)$, $\alpha>0$ (see Jung et al. [5]);
(iv) $J^m(\lambda,0)f(z)=J_{\lambda}^{-m}f(z)$, $m\in\mathbb{N}_0$ (see Patel [7]).

For our purpose, we introduce the next definition:

Definition 1.1. *Let H be the set of complex-valued function $h(r,s,t):\mathbb{C}^3\to\mathbb{C}$ such that:*

(i) $h(r,s,t)$ is continuous in a domain $D\subset\mathbb{C}^3$;
(ii) $(1,1,1)\in D$ and $|h(1,1,1)|<1$;
(iii) $\left|h\left(e^{i\theta},\left(1-\frac{\lambda}{l+1}\right)e^{i\theta}+\frac{\lambda}{l+1}\zeta e^{i\theta},\right.\right.$

$$\left.\left.\left(1-\frac{\lambda}{l+1}\right)^2e^{i\theta}+\left(2\frac{\lambda}{l+1}-\left(\frac{\lambda}{l+1}\right)^2\right)\zeta e^{i\theta}+\left(\frac{\lambda}{l+1}\right)^2Le^{i\theta}\right)\right|\geq 1$$

whenever

$$\left(e^{i\theta},\left(1-\frac{\lambda}{l+1}\right)e^{i\theta}+\frac{\lambda}{l+1}\zeta e^{i\theta},\right.$$

$$\left.\left(1-\frac{\lambda}{l+1}\right)^2e^{i\theta}+\left(2\frac{\lambda}{l+1}-\left(\frac{\lambda}{l+1}\right)^2\right)\zeta e^{i\theta}+\left(\frac{\lambda}{l+1}\right)^2Le^{i\theta}\right)\in D,$$

with $\mathrm{Re}\left(e^{-i\theta}L\right)>\zeta(\zeta-1)$ *for all real θ, and for $\zeta\geq 1$.*

2. Main result

To prove our main result we shall need the following lemma due to Miller and Mocanu:

Lemma 2.1. [6] *Let* $w(z) = a + w_n z^n + \dots$ *be analytic in* U, *with* $w(z) \not\equiv a$. *If* $z_0 = r_0 e^{i\theta}$ $(0 < r_0 < 1)$, *and* $|w(z_0)| = \max_{|z| \le r_0} |w(z)|$. *Then,*

$$z_0 w'(z_0) = \zeta w(z_0),$$

and

$$\operatorname{Re}\left[1 + \frac{z_0 w''(z_0)}{w'(z_0)}\right] \ge \zeta, \tag{2.1}$$

where ζ *is a real number, and* $\zeta \ge 1$.

Theorem 2.2. *Let* $h(r,s,t) \in H$, *and let* $f \in A$ *satisfying*

$$\left(J^m(\lambda,l)f(z), J^{m-1}(\lambda,l)f(z), J^{m-2}(\lambda,l)f(z)\right) \in D \subset \mathbb{C}^3 \tag{2.2}$$

and

$$\left|h\left(J^m(\lambda,l)f(z), J^{m-1}(\lambda,l)f(z), J^{m-2}(\lambda,l)f(z)\right)\right| < 1 \tag{2.3}$$

for all $z \in$ U, *and for some* $\lambda > 0$, $l \ge 0$, *and* $m \ge 2$. *Then, we have*

$$|J^m(\lambda,l)f(z)| < 1, \ z \in \mathrm{U}.$$

Proof. If we define the function w by

$$J^m(\lambda,l)f(z) = w(z), \ m \in \mathbb{N}_0,$$

with $f \in A$, then we have $w \in A$, and $w(z) \neq 0$ at least for one $z \in$ U. With the aid of the identity (1.4), we obtain

$$J^{m-1}(\lambda,l)f(z) = \left(1 - \frac{\lambda}{l+1}\right) w(z) + \frac{\lambda}{l+1} z w'(z)$$

and

$$J^{m-2}(\lambda,l)f(z) = \left(1 - \frac{\lambda}{l+1}\right)^2 w(z) + \left(2\frac{\lambda}{l+1} - \left(\frac{\lambda}{l+1}\right)^2\right) z w'(z) + \left(\frac{\lambda}{l+1}\right)^2 z^2 w''(z).$$

We claim that $|w(z)| < 1$ for all $z \in$ U. Otherwise, there exists a point $z_0 \in$ U such that $\max_{|z|<|z_0|} |w(z)| = |w(z)| = 1$. Letting $w(z_0) = e^{i\theta}$ and using Lemma 2.1 we deduce that

$$J^m(\lambda,l)f(z_0) = w(z_0) = e^{i\theta},$$

$$J^{m-1}(\lambda,l)f(z_0) = \left(1 - \frac{\lambda}{l+1}\right) e^{i\theta} + \left(\frac{\lambda}{l+1}\right) \zeta e^{i\theta},$$

and

$$J^{m-2}(\lambda,l)f(z_0) = \left(1 - \frac{\lambda}{l+1}\right)^2 e^{i\theta} + \left(2\frac{\lambda}{l+1} - \left(\frac{\lambda}{l+1}\right)^2\right) \zeta e^{i\theta} + \left(\frac{\lambda}{l+1}\right)^2 L e^{i\theta},$$

where $L = z_0^2 w''(z_0)$, and $\zeta \geq 1$.

Further, an application of (2.1) from Lemma 2.1 gives that

$$\operatorname{Re} \frac{z_0 w''(z_0)}{w'(z_0)} = \operatorname{Re} \frac{z_0^2 w''(z_0)}{\zeta e^{i\theta}} \geq \zeta - 1,$$

or

$$\operatorname{Re}\left(e^{-i\theta} L\right) \geq \zeta(\zeta - 1).$$

Since $h(r, s, t) \in H$, we have

$$\left|h\left(J^m(\lambda, l)f(z_0), J^{m-1}(\lambda, l)f(z_0), J^{m-2}(\lambda, l)f(z_0)\right)\right|$$

$$= \left|h\left(e^{i\theta}, \left(1 - \frac{\lambda}{l+1}\right) e^{i\theta} + \frac{\lambda}{l+1}\zeta e^{i\theta},\right.\right.$$

$$\left.\left.\left(1 - \frac{\lambda}{l+1}\right)^2 e^{i\theta} + \left(2\frac{\lambda}{l+1} - \left(\frac{\lambda}{l+1}\right)^2\right)\zeta e^{i\theta} + \left(\frac{\lambda}{l+1}\right)^2 L e^{i\theta}\right)\right| \geq 1,$$

which contradicts the condition (2.3) of the theorem, and therefore we conclude that

$$|J^m(\lambda, l)f(z)| < 1, \ z \in \mathrm{U}.$$

□

Corollary 2.3. *Let $h(r, s, t) = s$ and $f \in A$ satisfying the conditions* (2.2) *and* (2.3) *for $m \geq 2$. Then,*

$$\left|J^{m+j}(\lambda, l)f(z)\right| < 1, \ z \in \mathrm{U},$$

for $j \geq 0$, $\lambda > 0$, $l \geq 0$, $m \geq 2$.

Proof. Since $h(r, s, t) = s \in H$, with the aid of the above theorem we have that

$$\left|J^{m-1}(\lambda, l)f(z)\right| < 1, \ z \in \mathrm{U},$$

implies

$$|J^m(\lambda, l)f(z)| < 1, \ z \in \mathrm{U}, \ (m \geq 2),$$

and from here it follows

$$\left|J^{m+j}(\lambda, l)f(z)\right| < 1, \ z \in \mathrm{U}, \ (j \geq 0).$$

□

Remark 2.4. (i) Putting $l = 0$ and $\lambda = 1$ in the above results we obtain the results obtained by Aouf et al. [1];

(ii) Putting $\lambda = l = 1$ in the above results we obtain the results obtained by Aouf et al. [2, Theorem 1 and Corollary 1] respectively;

(iii) Putting $l = 0$ in the above results we obtain the corresponding results for the operator $J_\lambda^{-m} f(z)$.

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The Riemann Hypothesis

Dorin Ghisa

This paper is dedicated to the 80th birthday of Professor Petru T. Mocanu

Abstract. The domain of the Riemann Zeta function and that of its derivative appear as branched covering Riemann surfaces $(\overline{\mathbb{C}}, f)$. The fundamental domains, which are the leafs of those surfaces are revealed. For this purpose, pre-images of the real axis by the two functions are taken and a thorough study of their geometry is performed. The study of intertwined curves generated in this way, allowed us to prove that the Riemann Zeta function has only simple zeros and finally that the Riemann Hypothesis is true. A version of this paper containing color visualization of the conformal mappings of the fundamental domains can be found in [10].

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1. Introduction

The Riemann Zeta function is one of the most studied transcendental functions in view of its many applications in number theory, algebra, complex analysis, statistics, as well as in physics. Another reason why this function has drawn so much attention is the celebrated Riemann Hypothesis (RH) regarding its non trivial zeros, which has resisted proof or disproof until now.

Hopefully, starting from now, people can write RP instead of RH, meaning the Riemann Property (of non trivial zeros).

The RP proof will be derived from the global mapping properties of Zeta function. The Riemann conjecture prompted the study of at lest local mapping properties in the neighborhood of non trivial zeros. There are known color visualizations of the module, the real part and the imaginary part of Zeta function at some of those points (see Wolfram MathWorld), however they do not offer an easy way to visualize the global behavior of this function. We perfected an idea found in [8], page 213.

The Riemann Zeta function has been obtained by analytic continuation [1], page 178 of the series

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s}, \qquad s = \sigma + it \tag{1.1}$$

which converges uniformly on the half plane $\sigma \geq \sigma_0$, where $\sigma_0 > 1$ is arbitrarily chosen. It is known [1], page 215, that Riemann function $\zeta(s)$ is a meromorphic function in the complex plane having a single simple pole at $s = 1$ with the residue 1. The representation formula

$$\zeta(s) = -\frac{\Gamma(1-s)}{2\pi i} \int_C [(-z)^{s-1}/(e^z - 1)]dz \tag{1.2}$$

where Γ is the Euler function and C is an infinite curve turning around the origin, which does not enclose any multiple of $2\pi i$, allows one to see that $\zeta(-2m) = 0$ for every positive integer m and that there are no other zeros of ζ on the real axis. However, the function ζ has infinitely many other zeros (so called, *non trivial* ones), which are all situated in the *(critical) strip* $\{s = \sigma + it : 0 < \sigma < 1\}$. The famous RH says that these zeros are actually on the *(critical) line* $\sigma = 1/2$.

We will make reference to the Laurent expansion of $\zeta(s)$ for $|s - 1| > 0$:

$$\zeta(s) = \frac{1}{s-1} + \sum_{n=0}^{\infty} [(-1)^n/n!]\gamma_n (s-1)^n, \tag{1.3}$$

where γ_n are the Stieltjes constants:

$$\gamma_n = \lim_{m\to\infty} [\sum_{k=1}^{m} (\log k)^n/k - (\log m)^{n+1}/(m+1)] \tag{1.4}$$

as well as to the functional equation [1], page 216:

$$\zeta(s) = 2^s \pi^{s-1} \sin\frac{\pi s}{2} \Gamma(1-s)\zeta(1-s). \tag{1.5}$$

As predicted, the proof of RH will have a lot of consequences in number theory, although RP is a property of conformal mappings. It is probably safe to say that the conformal mappings made their royal entrance into the realm of number theory through the RP. By the global mapping properties of the Riemann Zeta function we understand the conformal mapping properties of its fundamental domains.

The concept of a fundamental domain, as it appears here, has been formulated by Ahlfors, who also emphasized its importance as a tool in the study of analytic functions. He said: *Whatever the advantage of such a representation may be, the clearest picture of the Riemann surface is obtained by direct consideration of the fundamental regions in the z-plane* ([1], page 99).

Starting with the study of Blaschke products ([4], [5], [7]), we followed a true program of revealing fundamental domains for different classes of analytic functions ([2], [3], [6]), drawing in the end the conclusion that this can be done for any function f which is locally conformal throughout the Riemann sphere, except for an at most countable set of points in which $f'(z) = 0$, or which are multiple poles of f, or which are isolated essential singularities.

In the case of functions having essential singularities, the Big Picard Theorem has been instrumental in finding fundamental domains [2]. Once this achieved, by using the technique of simultaneous continuations [7], we obtained an essentially enriched type of Picard Theorem, saying that *every neighborhood V of an essential singularity of an analytic function f contains infinitely many fundamental domains*, i.e., after Ahlfors definition, *domains which are conformally mapped by f onto the whole complex plane with a slit.*

Obviously, f takes in V every value, except possibly those at the ends of the slit (lacunary values) infinitely many times. Moreover, we have infinitely many disjoint domains in V which are mapped conformally by f onto a neighborhood of every such value.

The Riemann Zeta function has a unique essential singularity at the point ∞ of the Riemann sphere. Thus, its fundamental domains should accumulate to infinity and only there, in the sense that every compact set from the complex plane intersects only a finite number of fundamental domains of this function, while in the exterior of a compact set there are infinitely many such domains.

2. The pre-image by ζ of the real axis

In order to make the paper self contained, let us repeat some of the results obtained in [3]. The pre-image by ζ of the real axis can be viewed as simultaneous continuation over the real axis from a real value starting from all the points in which that value is assumed.

By the Big Picard Theorem, every value z_0 from the z-plane ($z = \zeta(s)$), if it is not a lacunary value, is taken by the function ζ in infinitely many points s_n accumulating to ∞ and only there. This is true, in particular, for $z_0 = 0$. By [9], the Zeta function has only simple zeros, hence a small interval I of the real axis containing 0 will have as pre-image by ζ the union of infinitely many Jordan arcs γ_n passing each one through a zero s_n of ζ, and vice-versa, every zero s_n belongs to some arcs γ_n. Since $\zeta(\sigma) \in \mathbb{R}$, for $\sigma \in \mathbb{R}$, and by the formula (1.5), the *trivial zeros* of ζ are simple zeros and the arcs γ_n corresponding to these zeros are intervals of the real axis, if I is small enough.

Due to the fact that ζ is analytic (except at $s = 1$), between two consecutive trivial zeros of ζ there is at least one zero of the derivative ζ', i.e. at least one branch point of ζ. Since we have also $\zeta'(\sigma) \in \mathbb{R}$ for $\sigma \in \mathbb{R}$, if we perform simultaneous continuations over the real axis of the components included in $\mathbb{R}$ of the pre-image by ζ of I, we will encounter at some moments those branch points and the continuations follow on unbounded curves crossing the real axis at the respective points. As shown in [3], the respective curves are unbounded and they do not intersect each other.

Only the continuation of the interval containing the zero $s = -2$ stops at the unique pole $s = 1$, since $\lim_{\sigma \nearrow 1} \zeta(\sigma) = \infty$. Similarly, if instead of $z_0 = 0$ we take another real z_0 greater than 1 and perform the same operations, since $\lim_{\sigma \searrow 1} \zeta(\sigma) = \infty$, the continuation over the interval $(1, \infty)$ stops again at $s = 1$. In particular, the pre-image by ζ of this interval can contain no zero of ζ.

Thus, if we color for example red, the pre-image by ζ of the negative real half axis and let black the pre-image of the positive real half axis, then all the components of the pre-image of the interval $(1, +\infty)$ will be black, while those of the interval $(-\infty, 1)$ will have a part red and another part black, the junction of the two colors corresponding to a zero of ζ (trivial or not).

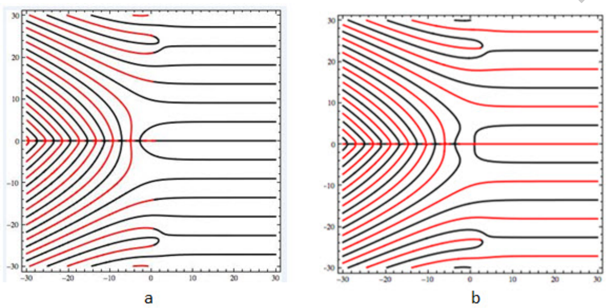

FIGURE 1

Fig. 1 represents the pre-images by ζ (Fig. 1a) and by ζ' (Fig. 1b) of the real axis in boxes of [-30,30]x[-30,30].

We notice the existence of branch points on the real axis and their color alternation, as well as the trivial zeros between them. If we superpose the two pictures, the branch points of the first, will coincide with the zeros of the second.

The components passing through non trivial zeros of the pre-image of the real axis form a more complex configuration, which has a lot to do with the special status of the value $z = 1$: on one hand this value is taken in infinitely many points of the argument plane, and on the other hand it behaves like a lacunary value, since it is obtained as a limit as s tends to infinity on some unbounded curves. We called it *quasi lacunary value.*

For the function ζ' the value $z = 0$ is quasi lacunary.

Due to the symmetry with respect to the real axis ($\zeta(\overline{s}) = \overline{\zeta(s)}$), it is sufficient to deal only with the upper half plane. Let $x_0 \in (1, +\infty)$ and let $s_k \in \zeta^{-1}(\{x_0\}) \setminus \mathbb{R}$. Continuation over $(1, +\infty)$ from s_k is either an unbounded curve Γ'_k such that $\lim_{\sigma \to +\infty} \zeta(\sigma + it) = 1$, by (1.1), and $\lim_{\sigma \to -\infty} \zeta(\sigma + it) = +\infty$, where $s = \sigma + it \in \Gamma'_k$, or there are points u such that $\zeta(u) = 1$, thus the continuation can take place over the whole real axis.

Theorem 2.1. *Consecutive curves Γ'_k and Γ'_{k+1} form strips S_k which are infinite in both directions. The function ζ maps these strips (not necessarily bijectively) onto the complex plane with a slit alongside the interval $[1, +\infty)$ of the real axis.*

Proof. Here $k \in \mathbb{N}$ is chosen in such a way that if $it_k \in \Gamma'_k$, then the sequence (t_k) is increasing. If two consecutive curves Γ'_k and Γ'_{k+1} met at a point s, one of the domains bounded by them would be mapped by ζ onto the complex plane with a slit alongside the real axis from 1 to $\zeta(s)$. Such a domain must contain a pole of ζ, but this cannot happen, since the only pole of ζ is $s = 1$. When a point s travels on Γ'_k and then on Γ'_{k+1} leaving the strip at left, $\zeta(s)$ moves on the real axis from 1 to ∞ and back. $\square$

When the continuation can take place over the whole real axis, we obtain unbounded curves each containing a non trivial zero of ζ and a point u with $\zeta(u) = 1$. Such a point u is necessarily interior to a strip S_k since the border of every S_k, which is included in $\zeta^{-1}((1, +\infty))$, and the set $\zeta^{-1}(\{1\})$ are disjoint.

Let us denote by $u_{k,j}$ the points of S_k for which $\zeta(u_{k,j}) = 1$, by $\Gamma_{k,j}$, $j \neq 0$ the components of $\zeta^{-1}(\mathbb{R})$ containing $u_{k,j}$ and by $s_{k,j}$ the non trivial zero of ζ situated on $\Gamma_{k,j}$. As shown in [3] Theorem 7, every S_k contains a unique component $\Gamma_{k,0}$ with the property that $\lim_{\sigma->+\infty} \zeta(\sigma + it) = 1$ and $\lim_{\sigma->-\infty} \zeta(\sigma + it) = -\infty$. where $\sigma + it \in \Gamma_{k,0}$ (i.e. which is projected bijectively by ζ onto the interval $(-\infty, 1)$) and a finite number $j_k - 1$ of components such that each one is projected bijectively by ζ onto the whole real axis. Therefore S_k contains j_k curves $\Gamma_{k,j}$, j_k non trivial zeros and $j_k - 1$ points $u_{k,j}$ with $\zeta(u_{k,j}) = 1$. We call S_k a j_k*-strip*. Here $j \in \mathbb{Z}$ is chosen in such a way that $\Gamma_{k,j}$ and $\Gamma_{k,j+1}$ are consecutive in the same sense as Γ'_k.

Theorem 2.2. *When the continuation takes place over the whole real axis, the components $\Gamma_{k,j}$ are such that the branches corresponding to both the positive and the negative real half axis contain only points $\sigma + it$ with $\sigma < 0$ for $|\sigma|$ big enough.*

Proof. A point traveling in the same direction on a circle γ centered at the origin of the z-plane meets consecutively the positive and the negative real half axis. Thus the pre-image of γ should meet consecutively the branches corresponding to the pre-image of the positive and the negative real half axis, which is possible only if the condition of the theorem is fulfilled. $\square$

In what follows, this *color alternation condition* will appear repeatedly. One of its immediate applications is that the real zeros of ζ' are simple and they alternate with the trivial zeros of ζ. Indeed, since the color change can happen only at a zero of ζ and at $s = 1$ and by the formula (1.5) the trivial zeros of ζ are simple, between two of them there must be one and only one zero of ζ', otherwise the color alternation condition would be violated.

Another application is that those components of pre-images of circles centered at the origin of radius $\rho > 1$ which cross a Γ'_k, will continue to cross alternatively red and black components of the pre-image of the real axis indefinitely, i.e. they are unbounded components.

Theorem 2.3. *Every strip S_k contains a unique unbounded component of the pre-image of the unit disc.*

Proof. To see this, it is enough to take the pre-image of a ray making an angle α with the real half axis and let $\alpha \to 0$. A point $s \in \zeta^{-1}(\{z\})$ with $|z| = 1$, $\arg z = \alpha$, tends to ∞ as $\alpha \to 0$ if and only if the corresponding component of the pre-image of that ray tends to $\Gamma_{k,0}$ as $\alpha \to 0$, which happens if and only if the component of the pre-image of the closed unit disc containing the point s is unbounded. The uniqueness of $\Gamma_{k,0}$ implies the uniqueness of such a component. $\square$

An even better way to visualize the geometry of conformal mappings realized by ζ is to take the pre-image of an orthogonal net formed with rays passing through the origin and circles centered at the origin. The annuli so formed are colored as in Fig. 2, where the points in the annuli and their pre-images by ζ have the same color, saturation and brightness. The color saturation in the annuli is decreasing clockwise. We show in the Fig. 2 (a,b,c) annuli of radii up to 10^4, although we investigated strips around the critical line with values of t up to 10^9 (see pictures at the end in [10]). We notice that to the quadrilaterals of the net from the z-plane correspond quadrilaterals in the argument plane of ζ which not only have the same colors but also the same conformal module. By using a convenient scale on the rays, we can arrange that all these quadrilaterals have the same module. The components of the pre-images of the four quadrilaterals having a vertex in $z = 1$ are unbounded if they don't have an $u_{k,j}$ as a vertex. It makes sense to say that the module of such an unbounded quadrilateral is that of its (bounded) image by ζ.

The theorems above guarantee that the landscape shown in Fig. 2(d,e,f) repeats itself indefinitely with variations regarding only the numbers of zeros in a strip S_k.

On the other hand, this landscape gives one an idea how difficult can it be to tackle the Riemann Hypothesis by number theoretical methods and suggests to rather use conformal mappings tools.

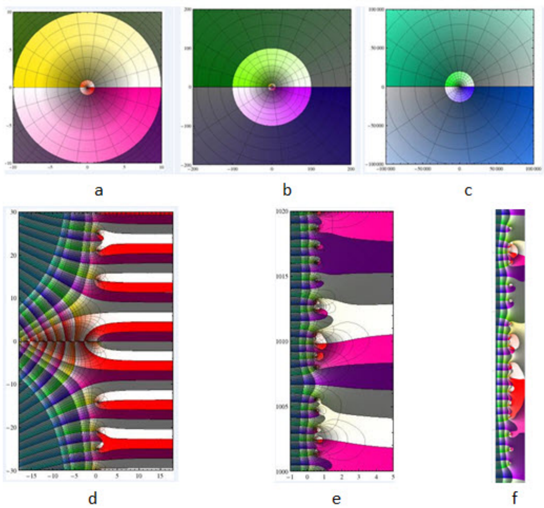

Figure 2

3. Fundamental domains of the Riemann Zeta function

If S_k is a j_k-strip, then the pre-image of a small circle γ of radius ρ centered at the origin will have j_k components situated in S_k which are closed Jordan curves each containing a unique zero of ζ. If ρ is small enough, then these curves are disjoint. As ρ increases, the curves expand. When two of them touch each other at a points $v_{k,j}$, they fuse into a unique closed curve containing the two respective zeros. Continuing to increase ρ, some other points $v_{k,j}$ are reached. These are branch points of ζ. It has been shown in [3] and [9] that there are exactly $j_k - 1$ points $v_{k,j}$ in every j_k-strip. The intersection of S_k with the pre-image of the segment between 1 and $\zeta(v_{k,j})$ is a set of unbounded curves starting at $u_{k,j}$ or arcs connecting two such points. These curves and arcs together with the pre-image of the interval $[1, +\infty)$ of the real axis bound sub-strips included in S_k which are fundamental domains $\Omega_{k,j}$ of ζ. Every fundamental domain $\Omega_{k,j}$ contains a unique simple zero $s_{k,j}$ of ζ.

The simplicity of the zeros of ζ has been proved in [9] by using the so-called *intertwined curves.* Namely, it has been shown that the components Υ'_k of the pre-image by ζ' of the interval $(-\infty, 0)$ of the real axis such that $\lim_{\sigma->-\infty} \zeta'(\sigma + it) = -\infty$ and $\lim_{\sigma->+\infty} \zeta'(\sigma + it) = 0$, $\sigma + it \in \Upsilon'_k$ form infinite strips Σ_k containing $j_k - 1$ simple zeros of ζ'.These zeros belong to the components $\Upsilon_{k,j}$, $j \neq 0$ of the pre-image by ζ' of the real axis. There is also in Σ_k a unique component $\Upsilon_{k,0}$ of the pre-image of the interval $(0, +\infty)$. These components follow also the color alternation rule. It has been proved in [9] that there is a one to one correspondence between those Γ'_k and Υ'_k, respectively $\Gamma_{k,j}$ and $\Upsilon_{k,j}$ which intersect each other (intertwined curves). Also, by using different colors for the components of the pre-image by ζ and by ζ' of the positive and negative real half axes, a *color matching rule* has been proved in [9]. The simplicity of the zeros of Zeta function, as well as of those of any derivative of ζ is a consequence of the two rules: the color alternation rule for ζ and ζ' and the color matching rule.

Fig. 3 illustrates the concept of intertwined curves, parts of which are shown in a box $[-10, 10] \times [30, 90]$.

The color matching rule means that *blue matches red and yellow matches black* for all the components except for $\Gamma_{k,0}$ and $\Upsilon_{k,0}$.

For the seek of space economy, the height of the box has been divided in three.

An inspection of the curves $\Gamma_{k,j}$ from the Fig. 3 shows that they have all the turning point (the point in which the tangent to $\Gamma_{k,j}$ is vertical) on the blue part, i.e. on the part corresponding to the positive real half axis. We will prove next that this happens for any curve $\Gamma_{k,j}$.

Theorem 3.1. *Let* $x -> s_{k,j}(x) = \sigma_{k,j}(x) + it_{k,j}(x)$ *be the parametric equation of* $\Gamma_{k,j}$ *such that*

$\zeta(s_{k,j}(x)) = x$, $x \in \mathbb{R}$, $s_{k,j} = \sigma_{k,j} + it_{k,j} = s_{k,j}(0)$,

If $\sigma'_{k,j}(x_0) = 0$ *, then* $x_0 > 0$.

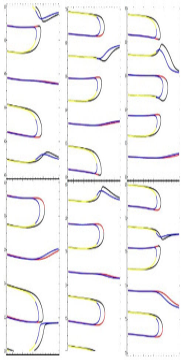

FIGURE 3

Proof. In terms of the zeros $s_{k,j}$, the *Hadamard Product Formula* (see Wikipedia/ Riemann Zeta Function) can be written as:

$$\zeta(s) = [1/2(s-1)\Gamma(1+s/2)]e^{As}\prod_{k=1}^{\infty}\prod_{j\in J_k}(1-s/\overline{s}_{k,j})(1-s/s_{k,j})\exp\{s(1/\overline{s}_{k,j}+1/s_{k,j})\},$$

where $A = 0.549269234...$

Let

$$\varphi_{k,j}(x) = [1 - s_{k,j}(x)/\overline{s}_{k,j}][1 - s_{k,j}(x)/s_{k,j}].$$

Since all the zeros of ζ are simple, in the neighborhood of $s_{k,j}$, the dominating factor of $\zeta(s_{k,j}(x))$ is $\varphi_{k,j}(x)$. We have

$$\varphi_{k,j}(x) = 1 - 2s_{k,j}(x)\sigma_{k,j}/|s_{k,j}|^2 + s_{k,j}^2(x)/|s_{k,j}|^2$$

$$= [1/|s_{k,j}|^2][(\sigma_{k,j}(x) - \sigma_{k,j})^2 + t_{k,j}^2 - t_{k,j}^2(x) + 2t_{k,j}(x)(\sigma_{k,j}(x) - \sigma_{k,j})i].$$

Thus, if $t_{k,j}(x) < t_{k,j}$, then $\operatorname{Re}\varphi_{k,j}(x) > 0$ and for $|\sigma_{k,j}(x) - \sigma_{k,j}|$ small enough, if $t_{k,j}(x) > t_{k,j}$ then $\operatorname{Re}\varphi_{k,j}(x) < 0$. Also $\operatorname{Im}\varphi_{k,j}(x) > 0$ if and only if $\sigma_{k,j}(x) > \sigma_{k,j}$.

Suppose that for $x < x_0 < 0$, we have $\sigma_{k,j}(x) > \sigma_{k,j}$, $t_{k,j}(x) < t_{k,j}$, i.e. the curve $\Gamma_{k,j}$ turns back at $s(x_0)$, where $x_0 < 0$, i.e. on the branch corresponding to the negative half axis, which is situated below the other branch. Then

$$\pi/2 < \arg s'_{k,j}(0) < \pi,\ 0 < \arg\varphi_{k,j}(x) < \pi/2 \tag{3.1}$$

Since

$$\varphi'_{k,j}(x) = \frac{2}{|s_{k,j}|^2}[s_{k,j}(x) - \sigma_{k,j}]s'_{k,j}(x) \text{ and } s_{k,j}(0) = s_{k,j},$$

we have

$$\varphi'_{k,j}(0) = \frac{2}{|s_{k,j}|^2}[s_{k,j} - \sigma_{k,j}]s'_{k,j}(0) = \frac{2it_{k,j}}{|s_{k,j}|^2}s'_{k,j}(0),$$

hence

$$\arg \varphi'_{k,j}(0) = \frac{\pi}{2} + \arg s'_{k,j}(0) \tag{3.2}$$

Since $\varphi'_{k,j}(0) = \lim\limits_{x->0}[\varphi_{k,j}(0) - \varphi_{k,j}(x)]/x = \lim\limits_{x->0}[-\varphi_{k,j}(x)]/x$ and $x < 0$, we have that

$$0 \leq \arg \varphi'_{k,j}(0) \leq \pi/2$$

which does not agree with (3.1) and (3.2).

If the curve turns back on the part corresponding to the positive half axis, and the relative position of the two branches is the same, then for $x > 0$ small enough, we have $\sigma(x) > \sigma_{k,j}$, $t(x) > t_{k,j}$, hence $\pi/2 < \arg \varphi_{k,j}(x) < \pi$ and $0 < \arg s'(0) < \pi/2$.

Then $\pi/2 < \arg \varphi'_{k,j}(0) < \pi$, which agrees with the formula (3.2). Similar arguments are valid when the relative position of two branches is reversed. The conclusion is that all the curves $\Gamma_{k,j}$ turn back at points corresponding to $x > 0$, as it appears in the pictures illustrating the theorems of this article. This result is used in the next section.

4. Proof of the Riemann Hypothesis

From the relation (1.5) and the fact that $\zeta(\overline{s}) = \overline{\zeta(s)}$, it can be easily inferred that the non trivial zeros of ζ appear in quadruplets: $s, \overline{s}, 1-s$ and $1-\overline{s}$. Proving the RH is equivalent to showing that ζ cannot have two zeros of the form $s_1 = \sigma_1 + it$, $s_2 = \sigma_2 + it$ in the critical strip. In particular, if we suppose that $s_1 = \sigma + it$ and $s_2 = 1 - \overline{s}_1 = 1 - \sigma + it$, $0 < \sigma < 1/2$ are zeros of ζ, then $s_1 = s_2$, i.e. $\sigma = 1 - \sigma$, hence $\sigma = 1/2$. In other words, all the non trivial zeros of ζ are of the form $s = 1/2 + it$.

There are two situations to be examined, namely a), when s_1 and s_2 are consecutive zeros in the same strip S_k and b), when s_1 is the last zero of S_k and s_2 is the first zero of S_{k+1}.

Once assured that $\sigma = 1/2$ in the two cases, a simple induction argument guarantees that this happens for all non trivial zeros in the upper half plane, and then, due to the symmetry $\zeta(\overline{s}) = \overline{\zeta(s)}$, the truth of the Riemann Hypothesis follows.

For the seek of completeness, let us present that induction argument.

Since S_1 contains only one zero $s_{1,0}$, the case a) is void for it.

Then, applying the case b) to S_1 and S_2 one concludes that $s_{1,0} = 1/2 + it_{1,0}$, for some positive number $t_{1,0}$.

Suppose that for an arbitrary k all the zeros in S_k are of the form $s_{k,j} = 1/2 + it_{k,j}$ and apply the case b) to the last zero $s_{k,j'}$ from S_k and the first zero $s_{k+1,j''}$ from S_{k+1}.

The conclusion is that $s_{k+1,j''} = 1/2 + it_{k+1,j''}$ for some positive number $t_{k+1,j''}$.

Then, applying the case a) to every couple of consecutive zeros from S_{k+1} one can draw the conclusion that all the zeros in S_{k+1} are of the form $1/2 + it_{k+1,j}$ for some positive numbers $t_{k+1,j}$, which completes the induction.

Case a). Suppose that the respective zeros are $s_{k,j-1} = \sigma_{k,j-1} + it$ and $s_{k,j} = \sigma_{k,j} + it$. Let $\Gamma_{k,j}$ and $\Upsilon_{k,j}$, respectively $\Gamma_{k,j-1}$ and $\Upsilon_{k,j-1}$ be the corresponding intertwined curves with $s_{k,j} \in \Gamma_{k,j}$ and $v_{k,j} \in \Upsilon_{k,j}$, respectively $s_{k,j-1} \in \Gamma_{k,j-1}$ and $v_{k,j-1} \in \Upsilon_{k,j-1}$.

Fig. 4 represents such a hypothetical situation.

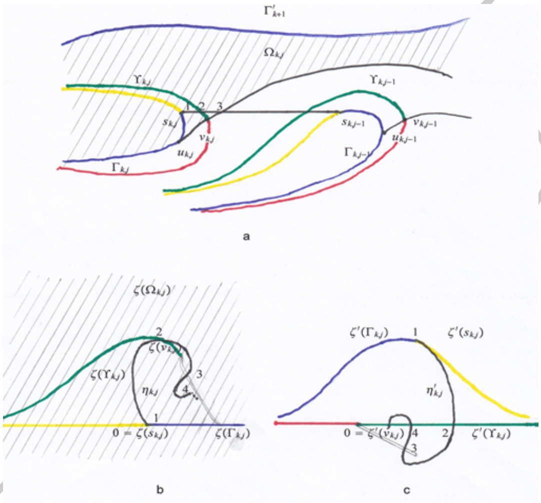

FIGURE 4

If $\lambda -> (1-\lambda)s_{k,j-1} + \lambda s_{k,j}$, $0 \leq \lambda \leq 1$ is the parametric equation of the segment I between $s_{k,j}$ and $s_{k,j-1}$, then

$$z(\lambda) = \zeta((1-\lambda)s_{k,j} + \lambda s_{k,j-1}),\ 0 \leq \lambda \leq 1 \tag{4.1}$$

is a parametric equation of a closed curve $\eta_{k,j}$ passing through the origin. The curve is obviously smooth except possibly at the origin and at $v_{k,j}$, if $v_{k,j} \in I$. Since $v_{k,j}$ is a simple zero of ζ' we have in $v_{k,j}$ a *star configuration,* (as in [1], page 133) of two orthogonal curves, the image by ζ of one of which passes through the origin. The tangent to this curve $\eta_{k,j}$ at $v_{k,j}$ still exists. In any case we can differentiate in (4.1) with respect to λ, $0 < \lambda < 1$ and we get:

$$z'(\lambda) = \zeta'((1-\lambda)s_{k,j-1} + \lambda s_{k,j})(s_{k,j-1} - s_{k,j}). \tag{4.2}$$

Since $s_{k,j-1} - s_{k,j} = \sigma_{k,j-1} - \sigma_{k,j} > 0$, we have

$$\arg z'(\lambda) = \arg \zeta'((1-\lambda)s_{k,j} + \lambda s_{k,j-1}) \tag{4.3}$$

which shows that the tangent to $\eta_{k,j}$ at $z(\lambda)$ makes an angle with the positive real half axis which is equal to the argument of $\zeta'((1-\lambda)s_{k,j} + \lambda s_{k,j-1})$. This last point describes an arc $\eta'_{k,j}$ starting at $\zeta'(s_{k,j})$ and ending at $\zeta'(s_{k,j-1})$, when λ varies from 0 to 1. Due to the color alternation rule and the color matching rule, both ends of $\eta'_{k,j}$ must be situated in the upper half plane. The right hand side in (4.3) exists also at the ends of this arc, which means that the limits as $\lambda \searrow 0$ and as $\lambda \nearrow 1$ of the left hand side also exist. These limits are two vectors starting at the origin with the first pointing to the upper half plane and the second pointing to the lower half plane. Indeed, the oriented interval I exits from the *parabola-like* curve $\Gamma_{k,j}$ and enters the *parabola-like* curve $\Gamma_{k,j-1}$ whose interiors are conformally mapped by ζ onto the lower half plane. This contradicts the equality (4.3) and the fact that both ends of $\eta'_{k,j}$ are situated in the upper half plane.

The conclusion is that a configuration as postulated in Fig. 4a is impossible.

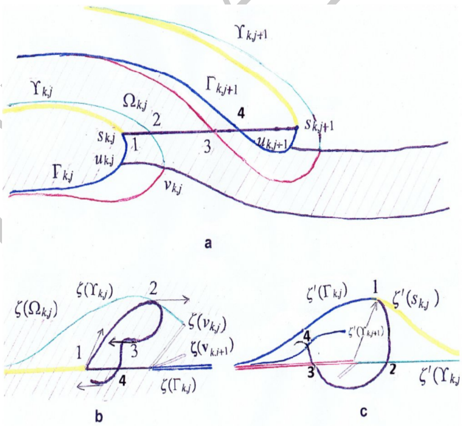

FIGURE 5

Let us deal with a hypothetical situation as represented in Fig. 5a. The points 1,2,3,... are mapped in tandem by ζ and ζ' into the points represented by the same numbers in Fig. 5b, respectively 5c. The number 1 is the origin in Fig. 5b and it is $\zeta'(s_{k,j})$ in Fig. 5c. Point 2 is the intersection with the positive half axis of $\eta'_{k,j}$, hence $\arg \zeta'((1+\lambda)s_{k,j} + \lambda s_{k,j-1}) = 0$ at the point 2, thus $\arg z'(\lambda) = 0$, which means that the tangent to $\eta_{k,j}$ at that point is horizontal and positively oriented with respect to the real axis. Point 3 corresponds in Fig. 5a to the intersection of I with $\Upsilon_{k,j+1}$ and is situated in Fig. 5c on the negative real half axis. Then $\eta_{k,j}$ has a horizontal tangent at the corresponding point 3 oriented in the negative direction of the real axis. At point 4 the interval I crosses $\Gamma_{k,j+1}$, hence $\eta'_{k,j}$ will cross the curve $\zeta'(\Gamma_{k,j+1})$, while $\eta_{k,j}$ will cross the real axis. In order for $\eta_{k,j}$ to reach the origin, it must have another point with horizontal tangent. This forces $\eta'_{k,j}$ to turn back to the real axis, crossing in its way again $\zeta'(\Gamma_{k,j+1})$ and so on. This means that I intersects again $\Gamma_{k,j+1}$ and $\Upsilon_{k,j+1}$, before reaching $s_{k,j+1}$, which is absurd. Thus a configuration like that postulated in Fig. 5a is impossible.

The final conclusion is that ζ cannot have two zeros of the form $s_{k,j} = \sigma + it_{k,j}$ and $s_{k,j+1} = 1 - \sigma + it_{k,j+1}$.

Case b). Suppose now that we deal with the zeros $s_{k,j'}$ and $s_{k+1,j}$ belonging to the adjacent strips S_k, respectively S_{k+1}.We make similar notations as in the case a) and we seek to find a contradiction by inspecting the images by ζ and by ζ' of the interval I between the two zeros whose parametric equation is

$$\lambda - > (1-\lambda)s_{k+1,j} + \lambda s_{k,j'}.$$

Fig. 6 illustrates the new hypothetical situation.

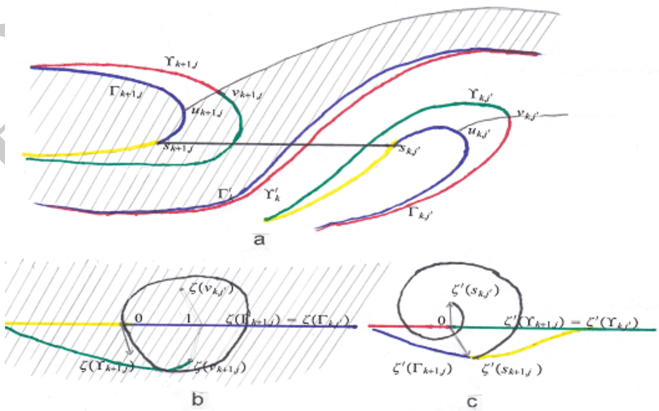

FIGURE 6

This time $\zeta'(s_{k,j'})$ belongs to the upper half plane, while $\zeta'(s_{k+1,j})$ belongs to the lower half plane. We notice again that $\lim_{\lambda \searrow 0} z'(\lambda)$ and $\lim_{\lambda \nearrow 1} z'(\lambda)$ exist and represent the half-tangents to $\eta_{k,j}$ at the origin. Both of these vectors point to the lower half plane, which contradicts the fact that $\zeta'(s_{k,j'})$ belongs to the upper half plane.

Thus, a configuration like that in Fig. 6a is impossible.

The argument corresponding to the configuration shown in Fig. 7 goes as follows. The position vectors of the points on $\eta'_{k,j}$ point to the lower half plane after point 5.

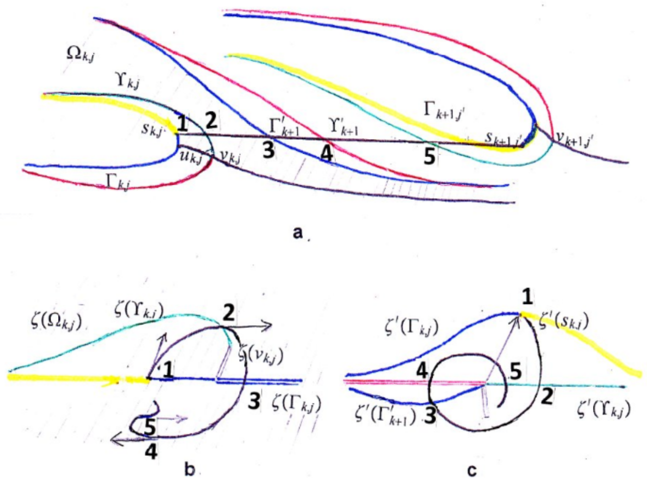

FIGURE 7

However, $\eta_{k,j}$ must turn to the origin after the corresponding position 5, which forces the tangent to it to point towards the upper half plane, contradicting the equality (4.3).

Thus, a configuration like that shown in Fig. 7a is not possible.

Consequently, all the imaginable positions of two consecutive non trivial zeros having the same imaginary part bring us to contradictions.

This completes the proof of the Riemann Hypothesis. □

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On a complex multidimensional approximation theorem and its applications to a complex operator-valued moment problems

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Abstract. The present note gives a proof to an approximation theorem of some holomorphic function, positively defined on a compact set in the unit polydisc in C^{2m}, with complex polynomials of a special type. This theorem is only enunciated in [3], without proof. We also underline in Theorem 5.1, section 5, some useful applications of the approximation theorem in solving a complex, operator-valued moment problem on the closed unit polydisc in C^m. The presented Theorem 5.1 gives another proof of Theorem 3.4. and Theorem 1.2.17. in [27], [28].

Mathematics Subject Classification (2010): 42A70, 15B05, 15B48.

Keywords: Holomorphic function, subnormal operator, joint spectral measure, linear functional, scalar measure associated to a spectral measure, complex and real moment problem.

1. Introduction

The present note, in Section 3 and Section 4, presents the proof of a theorem of uniformly approximation on compacta of a holomorphic, multivariable, complex function, positively defined on some compact set in the multidimensional unit polydisc $\mathrm{D}^m \times \mathrm{D}^m \subset \mathrm{C}^{2m}$, with polynomials in $2m$ complex variables, of a special type. This theorem was only enunciated in [3], without proof. The proof of this theorem in the present note is based on establishing an equivalence of the theorem with an intrinsic characterization theorem of subnormal tuples of commuting operators. The complex multidimensional approximation theorem is a useful tool in the approach of the complex, operator-valued moment problem on the closed unit polydisc $\overline{\mathrm{D}}^m \subset \mathrm{C}^m$. Based on this theorem, in Theorem 5.1, Section 5 of this note, it is given another proof of Theorem 3.4 in [26] and Theorem 1.2.17 in [27]. The complex operator-valued moment problem on the unit polydisc or on semialgebraic, nonvoid, compact sets in R^{2m} was simplified by testing the positivity of the moment functional associated with the moment sequence not on the whole space of the positively defined polynomials on the semialgebraic compact sets, but on some "smaller test subsets" e.g. [13], [21], [26], [27]. The proofs of Theorem 3.4 in [26] and Theorem 1.2.17 in [27] used Cassier's method for solving multidimensional real moment problems on semialgebraic nonvoid compact sets [5]. The operator-valued complex moment problem in theorem 2 is solved by applying the mentioned approximation Theorem 4.1 to obtain the same "test subset" as in Theorem 3.4, Theorem 1.2.27 [27], [28].

2. Preliminaries

Let $m \in \mathrm{N}^*$ be arbitrary, $I = (i_1, ..., i_m) \in \mathrm{N}^m$, $z = (z_1, ..., z_m) \in \mathrm{C}^m$, $t = (t_1, ..., t_m) \in \mathrm{R}^m$ denote the complex, respectively the real Euclidean space, $z^I = z_1^{i_1} ... z_m^{i_m}$, $|I| = i_1 + ... + i_m$ the length of $I = (i_1, ..., i_m) \in \mathrm{Z}_+^m$ and

$$\overline{D}^m = \{z = (z_1, ..., z_m) \in \mathrm{C}^m,\ |z_i| \le 1,\ 1 \le i \le m\}$$

the closed multidimensional unit polydisc. Let also H be a complex, separable Hilbert space and $B(H)$ denote the algebra of bounded, linear operators on H. For the commuting operators $S_1, ..., S_m$ in $B(H)$, the operator tuple $(S_1, ..., S_m)$ is called subnormal if there exists a Hilbert space $K \supset H$ and m commuting normals $(N_1, ..., N_m)$ such that $N_i H \subset H$ and $N_i|_H = S_i$ for all $1 \le i \le m$. The operator tuple $N = (N_1, ..., N_m)$ is referred to as a commuting normal extension of S. We denote with $S^I = S^{i_1} \circ ... \circ S^{i_m}$, with $C_K^P = C_{k_1}^{p_1} \times ... \times C_{k_m}^{p_m}$ for $K = (k_1, ..., k_m), P = (p_1, ..., p_m)$ multiindices in Z_+^m with $p_i \le k_i$, $\forall 1 \le i \le m$. For characterizing a subnormal tuple of commuting operators $S = (S_1, ..., S_m)$ in $B(H)$, we used in this note Ito's and Lubin's criterions of subnormality (Theorem 1 in [11], respectively Theorem 3.2 in [12]). This criterions are:

"A commuting tuple of operators $S = (S_1, ...S_m)$ in $B(H)$ is subnormal if and only if for any finite number of multiindices $I, J \in \mathrm{N}^m$ and any finite number of elements $x_I,\ x_J \in H$ we have the positivity condition:

$$\sum_{I,J} < S^I x_J, S^J x_I >_H \ge 0 (th.1[11],"$$

"$T_1, ..., T_m$ have commuting normal extension if and only if there exists a positive operator valued measure ρ defined on some m-dimensional rectangle R such that $T^{\bullet J} T^J = \int_R t^{2J} \mathrm{d}\rho t$ for all J (Theorem 3.2 [12])."

We also denote with $P(\mathrm{C}^m)$ the algebra of all complex polynomial functions in $z_1, ..., z_m, \overline{z_1}, ..., \overline{z_m}$.

3. A subnormality characterization of a commuting tuple of operators

Proposition 3.1. *Let $S_1, ..., S_m$ be m commuting operators in $B(H)$. Statements (i) and (ii) below are equivalent:*

(i) $S = (S_1, ...S_m)$ is a subnormal tuple where each S_i is a contraction.

The following inequality occurs:

(ii) $\sum_{0\le p_i\le k_i;1\le i\le m} (-1)^{|P|} C_K^P S^{*P} \circ S^P \ge 0,$ for $\forall\ K = (k_1, ..., k_m) \in Z_+^m$ *with* $P = (p_1, ..., p_m)$.

Proof. We shall prove that $(i) \Rightarrow (ii)$. Let $S = (S_1, ..., S_m)$ be a commuting tuple of subnormal contractions on an arbitrary, complex, separable Hilbert space H. In this case, there exists a Hilbert space $K \supset H$, commuting normals $N_1, ..., N_m$ such that $N_i H \subseteq H$ and $N_i|_H = S_i$, $\forall 1 \le i \le m$. According to Halmos's paper [9], if B is a normal operator on $K \supseteq H$, $A = B|_H$ and P is the projection of K on H, we have $A \circ P = B \circ P$ and $A^* \circ P = P \circ B^*$. Using these equalities for $B = N_i$ and $A = S_i$, for an arbitrary $u \in H$, statement (ii) becomes:

$$\sum_{0\le p_i\le k_i;1\le i\le m} (-1)^{p_1+p_2+...+p_m} C_{k_1}^{p_1}...C_{k_m}^{p_m} S_1^{*p_1} \circ ... \circ S_m^{*p_m} \circ S_1^{p_1} \circ ... \circ S_m^{p_m} u$$

$$= \sum_{0\le p_i\le k_i;1\le i\le m} (-1)^{p_1+p_2+...+p_m} C_{k_1}^{p_1}...C_{k_m}^{p_m} S_1^{*p_1} \circ ... \circ S_m^{*p_m} \circ S_1^{p_1} \circ ... \circ S_m^{p_m} Pu$$

$$= P(\sum_{0\le p_i\le k_i;1\le i\le m} (-1)^{p_1} C_{k_1}^{p_1} N_1^{*p_1} N_1^{P_1} ...(-1)^{p_m} C_{k_m}^{p_m} N_m^{*p_m} N_m^{p_m})u$$

$$= P[\prod_{1\le i\le m} (1 - z_i w_i)^{k_i} (N_i N_i^*)]u.$$

Because all S_i are contractions and $(N_1, ..., N_m)$ is the normal minimal extension of $(S_1, ..., S_m)$, we have $||N_i|| = ||S_i|| \le 1$; this imply

$$\prod_{0\le i\le m} (1 - z_i w_i)^{k_i} (N_i, N_i^*) \ge 0$$

e.g. $(< (1 - N_i^* \circ N_i)^2)u, u >=< (1 - N_i^* N_i)u, (1 - N_i^* N_i)u >\ge 0.)$ Applying again Halmos's equalities in [4] and returning to S_i operators, we obtain

$$\sum_{0\le p_i\le k_i;1\le i\le m} (-1)^{|P|} C_K^P S^{*P} \circ S^P \ge 0, \text{ for } K = (k_1, ..., km), P = (p_1, ..., p_m) \in Z_+^m$$

the required inequality.

Conversely. $(ii) \Rightarrow (i)$. If

$$\sum_{0\le p_i\le k_i;1\le i\le m} (-1)^{|P|} C_K^P S^{*P} \circ S^P \ge 0,$$

is true for any $K = (k_1, ..., k_m)$, $P = (p_1, ..., p_m) \in Z_+^m$, $p_i \le k_i$, $\forall 1 \le i \le m$ taking account that the $\{S_i\}_{i=1}^m$ is a family of commuting one, by multiplying the above inequality with $S^{*T} \circ S^T$, $T \in N^m$, we obtain

$$\sum_{0\le p_i\le k_i;1\le i\le m} (-1)^{p_1+p_2+...+p_m} C_{k_1}^{p_1}...C_{k_m}^{p_m} < S^{T+P}u, S^{T+P}u >\ge 0$$

for any vector $u \in H$ and any multiindicees $K,\ T,\ P \in N^m$. For every $u \in H$, we define the linear functional $\varphi_u : N^m \to R$, by $\varphi_u(T) =< S^{T+P}u, S^{T+P}u >$. Let Δ_i the operators $\Delta_i : L(N^m, R) \to L(N^m, R)$,

$$\Delta_i \varphi_u(T) = \varphi_u(T_1, ..., T_i + 1, ..., T_m) - \varphi_u(T_1, ..., T_m).$$

For $\Delta_i - s$ operators, condition (ii) in hypothesis is $\Delta_1^{k_1} \circ ... \circ \Delta_1^{k_m} \varphi_u(T) \geq 0$; that is the functional φ_u is completely monotonic for every $u \in H$. From [10], in these conditions, for a completely monotonic functional there exists positive, scalar, Borel measures μ_u such that $\varphi_u(P) = \int_{[0,1]^m} x^P \mathrm{d}\mu_u(x)$. With the help of the scalar measures μ_u, we define the semispectral measure: $< \rho(A)u, u >_H = \mu_u(A)$ for any Borel set $A \in Bor([0,1]^m)$. In these conditions,

$$\varphi_u(P) = \int_{[0,1]^m} x^P \mathrm{d} < \rho(x)u, u > = < S^P u, S^P u >_H = < S^{*P} S^P u, u > .$$

By the uniqueness of the representation theorem of a normal operator with respect to a spectral measure, we obtain $[S^*S]^P = \int_{[0,1]^m} x^P \mathrm{d}\rho(x)$. We prove in the sequel that $(S_1, ..., S_m)$ verify Ito's necessary and sufficient condition for subnormality by using the above identity; that is we prove that $\sum_{I,J} < S^{I+J} x_I, S^{I+J} x_J >_H \geq 0$ for any finite family $\{x_I\} \in H$. With the obtained representation, we have:

$$\sum_{I,J} < S^{I+J} x_I, S^{I+J} x_J >_H = \sum_{I,J} \int_{[0,1]^m} z^{I+J} \mathrm{d} < \rho(z) x_I, x_J >_H$$

$$= \sum_{I,J} \int_{[0,1]^m} \mathrm{d} < \rho^{\frac{1}{2}}(z) \sum_{I,J} z^I x_I, \rho^{\frac{1}{2}} \sum_{I,J} z^J x_J >_H \geq 0.$$

From this calculus, via Theorem 3.2 [12], it results that $S = (S_1, ..., S_m)$ is a subnormal tuple.

4. The main approximation theorem

In this section we prove the following:

Theorem 4.1. *Let* D^m, $m \in \mathrm{N}^*$ *denote the open unit polydisc in the complex space* C^m *and* $A^2(D^m \times D^m)$ *denote the space of continuous functions on the closed polydisc* $\overline{D}^m \times \overline{D}^m$, *analytic on* $D^m \times D^m$, *equipped with the topology of uniform convergence on compacta. Let* $M = \{f \in A^2(D^m \times D^m),\ f(z, \overline{z}) \geq 0\ , \forall\ z \in D^m\}$. *If* C *denotes the convex hull of the set* $\{f \in A^2(D^m \times D^m),\ f(z, w) = p(z) \prod_{i=1}^m (1 - z_i w_i)^{k_i} \overline{p(\overline{w})}\}$ *where* k_i *are non-negative integers and* p *is an* m*-variable complex polynomial, then* $\overline{\mathrm{C}} = M$, *where* $\overline{\mathrm{C}}$ *denotes the closure of* C *in the topology of uniform convergence on compacta.*

This theorem appears in [3], only enounced, without proof. The proof of this approximation theorem in the present note is based on establishing an equivalence of Theorem 4.1 in this section with Proposition 3.1 in section 3 (a criterion of subnormality of a m-tuple $(A_1, ..., A_m)$ of m commuting bounded operators, acting on a prescribed Hilbert space). In the sequel, we give some applications of this theorem to an operator-valued complex moment problem.

Proof. First of all we shall prove that Proposition 3.1 $\Rightarrow$ Theorem 4.1. Let $f \in C$, that is $f(z, w) = p(z) \prod_{i=1}^m (1 - z_i w_i)^{k_i} \overline{p(\overline{w})}$ and $|z_i| \leq 1$, $|w_i| \leq 1$, $1 \leq i \leq m$; in these conditions, $f \in A^2(D^m \times D^m)$ and $f(z, \overline{z}) \geq 0$ that is $f \in M$, consequently $\overline{C} \subseteq M$.

Conversely, we prove that, also $M \subseteq \overline{C}$; otherwise, if $M - \overline{C} \neq \Phi$, let $h_0 \in M - \overline{C}$. Because $\overline{C}$ is a close set, from Hahn Banach theorem, there exists a linear functional $\Lambda : A^2(D^m \times D^m) \to \mathrm{C}$ such that

$$\mathrm{Re}\Lambda\,(h_0) < 0 \leq \mathrm{Re}\Lambda\,(f),\ \forall f \in \overline{C}. \tag{4.1}$$

We extend Λ to the space of continuous functionals on $C(\overline{D}^m \times \overline{D}^m)$. By Riesz representation theorem, there exists $\nu_\Lambda \in Bor(\overline{D}^m \times \overline{D}^m)$ such that $\Lambda(f) = \int_{\overline{D}^m \times \overline{D}^m} f \mathrm{d}\nu_\Lambda$. With this identification, we have

$$\mathrm{Re}\,[\int_{\overline{D^m} \times \overline{D^m}} h_0 \mathrm{d}\nu_\Lambda] < 0 \leq \mathrm{Re} \int_{\overline{D^m} \times \overline{D^m}} f \mathrm{d}\nu_\Lambda,\ \forall f \in \overline{C}. \tag{4.2}$$

Let $\varphi : \overline{D}^m \times \overline{D}^m \to \overline{D}^m \times \overline{D}^m$, $\varphi(z,w) = (\overline{w}, \overline{z})$, the measure $\mu = \frac{1}{2}(\nu_\Lambda + \overline{\nu_\Lambda \circ \varphi})$ and $h \in M$ an arbitrary element. With the help of h, we define

$$g(z,w) = h(z,w) - \overline{h \circ \varphi}(z,w) = h(z,w) - \overline{h(\overline{w}, \overline{z})}.$$

From this construction, g is holomorphic on $D^m \times D^m$, continuous on $\overline{D}^m \times \overline{D}^m$ and $g(z, \overline{z}) = 0$ for all $(z, \overline{z}) \in D^m \times D^m$. If we consider the set $B = \{(z, \overline{z}), z \in D^m\}$, because $g|_B = 0$ and $g \in A^2(D^m \times D^m)$, $2m \geq 2$, it results that $g(z,w) = 0$ for all $(z,w) \in D^m \times D^m$. From this remark, $h = \overline{h \circ \varphi}$ on $\overline{D}^m \times \overline{D}^m$. Inequality (4.2) becomes

$$\begin{aligned}
\mathrm{Re}\,[\int_{\overline{D}^m \times \overline{D}^m} h \mathrm{d}\nu_\Lambda] &= \frac{1}{2}[\int_{\overline{D}^m \times \overline{D}^m} h \mathrm{d}\nu_\Lambda + \overline{h} \mathrm{d}\overline{\nu}_\Lambda] \\
&= \frac{1}{2}[\int_{\overline{D}^m \times \overline{D}^m} h \mathrm{d}\nu_\Lambda + \int_{\overline{D}^m \times \overline{D}^m} \overline{h \circ \varphi} \mathrm{d}\overline{\nu_\Lambda \circ \varphi} \\
&= \frac{1}{2}[\int_{\overline{D}^m \times \overline{D}^m} h \mathrm{d}\nu_\Lambda + \int_{\overline{D}^m \times \overline{D}^m} h \mathrm{d}\overline{\nu_\Lambda \circ \varphi}] \\
&= \int_{\overline{D}^m \times \overline{D}^m} h \frac{1}{2} \mathrm{d}[\nu_\Lambda + \overline{\nu \circ \varphi}] = \int_{\overline{D}^m \times \overline{D}^m} h \mathrm{d}\mu.
\end{aligned} \tag{4.3}$$

Using the new constructed measure μ, inequality (4.2) becomes:

$$\int_{\overline{D}^m \times \overline{D}^m} h_0 \mathrm{d}\mu < 0 \leq \int_{\overline{D}^m \times \overline{D}^m} f \mathrm{d}\mu \ \ \forall f \in \overline{C}. \tag{4.4}$$

We shall construct, with the help of Proposition 3.1, a scalar measure σ defined on $\overline{D}^m$, generated by the spectral measure associated with a commuting tuple of subnormals defined on $\overline{D}^m$, and prove, that on some dense set in $A^2(D^m \times D^m)$, the two measures are equal.

Let P the C- vector space of polynomials with complex coefficients in $z-$variable and $N = \{p \in P$ with $\int_{\overline{D}^m \times \overline{D}^m} p(z)\overline{p(\overline{w})} \mathrm{d}\mu(z,w) = 0\}$. The set N is a C-vector subspace in P; on the coset P/N, with the help of the definition of the subspace N, we define the hermitian product $< p + N, q + N >= \int_{\overline{D}^m \times \overline{D}^m} p(z)\overline{q(\overline{w})} \mathrm{d}\mu(z,w)$ and consider $H^2(\mu)$ the Hilbert space obtained as the separate completion of P/N with respect to this hermitian product. Let the operators $S_i : P \to P$, $[S_i p](z) = z_i p(z)$ for all $1 \leq i \leq m$. Because $||p||^2 - ||S_i||^2 = (\int_{\overline{D}^m \times \overline{D}^m} (1 - z_i w_i) p(z)\overline{p(\overline{w})} \mathrm{d}\mu(z,w) \geq 0$ on the subspace P, (|| || represents the norm generated by the introduced hermitian product), the operators S_i are contractions; we extend S_i to $H^2(\mu)$ with preserving the norm and the commutation relations; the extensions will be denoted also $S_i,\ 1 \leq i \leq m$. For the tuple $(S_1, ..., S_m)$ we shall verify condition (ii) in Proposition 3.1, for testing if it is a subnormal one. That is:

$$< \sum_{0 \leq p_i \leq k_i; 1 \leq i \leq m} (-1)^{|P|} C_K^P S_1^{*p_1} \circ ... \circ S_m^{*p_m} \circ S_1^{p_1} \circ ... \circ S_m^{p_m} p, p >_{H^2(\mu)}$$

$$= \int_{\overline{D}^m \times \overline{D}^m} \sum_{0 \leq p_i \leq k_i; 1 \leq i \leq m} (-1)^{|P|} C_K^P z_1^{p_1} ... z_m^{p_m} w_1^{p_1} ... w_m^{p_m} p(z) \overline{p(\overline{w})} \mathrm{d}\mu(z, w)$$

$$= \int_{\overline{D}^m \times \overline{D}^m} p(z) \prod_{i=1}^m (1 - z_i w_i)^{k_i} \overline{p(\overline{w})} \mathrm{d}\mu(z, w) \geq 0,$$

with $P = (p_1, ..., p_m),\ K = (k_1, ..., k_m)$, inequality that is exactly condition (ii) in Proposition 3.1. Because this proposition is true, there exists the joint spectral measure associated to the subnormal tuple $(S_1, ..., S_m)$ and, consequently, the scalar measures generated by it. Let $\mathrm{d}\sigma(A) =< E_S(A)1, 1 >_{H^2(\mu)}$ defined for all Borel sets A in $Bor(D^m)$ where E_S is the joint spectral measure associated with S. We have then,

$$[S_1^{p_1} \circ ... \circ S_m^{p_m} 1, S_1^{n_1} \circ ... \circ S_m^{n_m} 1] = [N_1^{p_1} \circ ... \circ N_m^{p_m} 1, N_1^{n_1} \circ ... \circ N_m^{n_m} 1] = \int_{\overline{D}^m} z^P \overline{z}^N \mathrm{d}\sigma(z)$$

for all multiindices $P = (p_1, ..., p_m), N = (n_1, ..., n_m)$.

From the definition of the scalar product on $H^2(\mu)$, we have also

$$[S^P 1, S^N 1] = \int_{\overline{D}^m \times \overline{D}^m} z^P w^N \mathrm{d}\mu(z, w)$$

for all multiindices $P = (p_1, ..., p_m), N = (n_1, ..., n_m))$. Because the polynomials in (z, w) are uniformly dense on compacta in $A^2(D^m \times D^m)$, we have also

$$\int_{\overline{D}^m \times \overline{D}^m} h(z, w) \mathrm{d}\mu(z, w) = \int_{\overline{D}^m} h(z, \overline{z}) \mathrm{d}\sigma(z),\ \forall h \in A^2(D^m \times D^m).$$

The above inequality happens in particular for $h_0 \in M - \overline{C}$. From the definition of M, we have $h_0(z, \overline{z}) \geq 0$, the scalar measure $\mathrm{d}\sigma$ is a positive one, we have than

$$0 > \int_{\overline{D}^m \times \overline{D}^m} h_0(z, w) \mathrm{d}\mu(z, w) = \int_{\overline{D}^m} h(z, \overline{z}) \mathrm{d}\sigma(z) \geq 0$$

which represents a contradiction; that is $M = \overline{C}$.

Conversely. Theorem 4.1 $\Rightarrow$ Proposition 3.1. Let $S = (S_1, ..., S_m)$ a commuting tuple of operators like in hypothesis of Proposition 3.1; the implication $(i) \Rightarrow (ii)$ is always true; we will prove that in conditions of Theorem 4.1, also $(ii) \Rightarrow (i)$ happens. In this case,

$$\prod_{i=1}^m (1 - z_i w_i)^{k_i} (S, S^*) \geq 0;$$

from this inequalities it results that all S_i operators are contractions and consequently for each $0 \leq r < 1$, we have also

$$\prod_{i=1}^{m}(1 - z_i wi)^{k_i}(rS, rS^*) \geq 0,$$

inequality that imply

$$p(rS)^* \prod_{i=1}^{m}(1 - z_i wi)^{k_i}(rS, rS^*)p(rS) \geq 0.$$

Let the holomorphic function $f(z, w) = \sum_{I,J} a_{IJ} z^I w^J$, $f \in A^2(D^m \times D^m)$ and the subnormal tuple of contractions $S = (S_1, ..., S_m)$ with the spectrum in $\overline{D}^m$. We define using the analytic calculus $f(rS, rS^*) = \sum_{I,J} a_{IJ}(rS)^I(rS^*)^J$. If $f \in C$ with

$$f(z, w) = p(z) \prod_{i=1}^{m}(1 - z_i w_i)^{k_i} \overline{p(\overline{w})},$$

we will also have $f(rS, rS^*) \geq 0$ for every such an element $f \in C$ and for every $0 \leq r < 1$. Because Theorem 4.1 is true, we also have $f(rS, rS^*) \geq 0$ for every $f \in M$. In particular, for any polynomial $p(z, w)$ with $p(z, \overline{z}) \geq 0$, also $p(rS, rS^*) \geq 0$ happens; subsequently we have:

$$\sum_{I,J} < r^{I+J} S^{I+J} x_I, r^{I+J} S^{I+J} x_J >_{H^2(\mu)} \geq 0.$$

Passing to the limit for $r \to 1$, we obtain

$$\sum_{I,J} < S^{I+J} x_I, S^{I+J} x_J >_{H^2(\mu)} \geq 0,$$

that is Lubin's condition for subnormality [12].

We prove that $S = (S_1, ...S_m)$ is a subnormal tuple (exactly condition (i) in Proposition 3.1) that is, Theorem 4.1 $\Leftrightarrow$ Proposition 3.1. From above, we have Theorem 4.1 $\Leftrightarrow$ Proposition 3.1. Because Proposition 3.1 is true, as it was shown in Section 3, Theorem 4.1 is also true.

5. Applications of Theorem 4.1 in solving an operator-valued complex moment problem

In this section, we give a useful application of Theorem 4.1, in the present paper, in solving the operator-valued complex moment problem on the complex unit polydisc $\overline{D}^m \subset \mathrm{C}^m$, $m > 1$.

We recall, that given a sequence of bounded operators $\Gamma = (\Gamma_{\alpha,\beta})_{\alpha,\beta \in \mathrm{Z}^m}$, $\Gamma_{\alpha,\beta} = \Gamma^*_{\beta,\alpha}$ acting on an arbitrary Hilbert space H, the operator-valued moment problem asks for necessary and sufficient conditions on Γ such that there exists a positive operator-valued measure F_Γ on $\overline{D}^m$ such that

$$\int_{\overline{D}^m} z^\alpha \overline{z}^\beta \mathrm{d}F_\Gamma(z), \ \forall \ \alpha, \beta \in \mathrm{Z}^m_+. \tag{5.1}$$

In [27], this multidimensional operator-valued moment problem is solved in Theorem 3.7 of the paper by passing, from the complex multidimensional moment problem on $\overline{D}^m \subset \mathrm{C}^m$, to a real multidimensional operator-valued problem on the set $\tau(\overline{D}^m) \subset \mathrm{C}^m$ with τ the mapping $\tau : \mathrm{C}^m \to \mathrm{R}^m$, $\tau(z_1, ..., z_m) = (|z_1|^2, ..., |z_m|^2) \in \mathrm{R}^m$. The multidimensional real operator-valued moment problem is solved in [27] by applying Cassier's results in [5] for solving the scalar real moment problem, naturally generated by the operator-valued one on the compact $\tau(\overline{D}^m)$. In the sequel, by a standard polarization argument for the obtained scalar representing measures, the operator-valued representing measure, solution for the real multidimensional operator-valued moment problem on $\tau(\overline{D}^m)$ is obtained. Via the inverse mapping τ^{-1}, the solution of the real multidimensional operator-valued moment problem furnished a solution of (5.1).

The enounce of the complex operator-valued moment problem from Theorem 5.1 in this note is quite the same with that of Theorem 3.7 in [27]. It adds, only, an additional condition on the sequence of operators $\Gamma = (\Gamma_{\alpha,\beta})$ in hypothesis, condition that ensure a direct proof of the operator-valued complex moment problem by using the Hahn-Banach theorem, Riesz representation theorem and Theorem 4.1 in this note.

Theorem 5.1. *Let $\Gamma = (\Gamma_{\alpha,\beta})_{\alpha,\beta\in \mathrm{Z}_+^m}$ be a sequence of bounded linear operators acting on an arbitrary, complex Hilbert space H, such that $\Gamma_{\alpha,\beta} = (\Gamma^*_{\alpha,\beta})$ for all $\alpha, \beta \in \mathrm{Z}_+^m$, with $\Gamma_{0,0} = Id_H$ and $\{\Gamma_{\alpha,\beta}(x)\}_{\alpha,\beta\in \mathrm{Z}_+^m}$ a bounded sequence in H for all $x \in H$. The following assertions are equivalent:*

(i) The following inequalities occur:

$$\sum_{\alpha,\beta\in \mathrm{Z}_+^m} \sum_{0\leq\theta\leq K} (-1)^{|\theta|} C_K^\theta c_\alpha \overline{c}_\beta \Gamma_{\alpha+\theta,\beta+\theta} \geq 0, \ \forall\ K \in \mathrm{Z}_+^m \tag{5.2}$$

and all sequences of complex numbers $\{c_\alpha\}_{\alpha\in \mathrm{Z}_+^m}$ with finite support.

(ii) There exists a positive operator-valued measure F_Γ on $\overline{D}^m$ such that

$$\Gamma_{\alpha,\beta} = \int_{\overline{D}^m} z^\alpha \overline{z}^\beta \mathrm{d}F_\Gamma(z) \ \text{ for all } \alpha, \beta \in \mathrm{Z}_+^m. \tag{5.3}$$

Proof. $(i) \Rightarrow (ii)$ For the data $\Gamma = (\Gamma_{\alpha,\beta})$ we set $L_\Gamma(z^\alpha \overline{z}^\beta) = \Gamma_{\alpha,\beta}$ for all $\alpha, \beta \in \mathrm{Z}_+^m$ and extend L_Γ on $P(\mathrm{C}^m)$ by linearity. For all $x \in H$, we denote with $L_\Gamma^x : P(\mathrm{C}^m) \to H$ the obtained linear functional $L_\Gamma^x(z^\alpha \overline{z}^\beta) =< \Gamma_{\alpha,\beta}x, x >_H$ from L_Γ, for all α, β in Z_+^m. From the hypothesis, the sequences $\{\Gamma_{\alpha,\beta}(x)\}_{\alpha,\beta\in \mathrm{Z}_+^n} \subset H$ are bounded $\forall\ x \in H$, it results that L_Γ^x are linear, continuous functionals on $P(\mathrm{C}^m)$. Because of (i), we have also

$$< \sum_{\alpha,\beta\in \mathrm{I}\subset \mathrm{Z}_+^m} \sum_{0\leq\theta\leq k} (-1)^{|\theta|} C_K^\theta c_\alpha \overline{c}_\beta \Gamma_{\alpha+\theta,\beta+\theta} x, x >_H \geq 0$$

with I a finite set; that is we have

$$L_\Gamma^x(|p(z)|^2 \prod_{i=1}^m (1 - z_i \overline{z}_i)^{k_i}) \geq 0, \ \text{for all } K = (k_1, ..., k_m) \in \mathrm{Z}_+^m \text{ and all } x \in H. \tag{5.4}$$

Let $f \in P(\mathrm{C}^m \times \mathrm{C}^m)$ be the function

$$f(z,w) = \sum_{\alpha,\beta \in \mathrm{I} \subset \mathrm{Z}_+^m} a_{\alpha,\beta} z^\alpha w^\beta,$$

I finite, such that $f(z,\overline{z}) \geq 0$,$\forall\ z \in D^m$; it results that $a_{\alpha,\beta} = \overline{a}_{\beta,\alpha}$. Because also $\Gamma_{\alpha,\beta} = \Gamma^*_{\beta,\alpha}$, it follows that $L_x^\Gamma(f(z,\overline{z})) \in \mathrm{R}$. From Theorem 4.1 in section 4, such an analytical function is uniformly approximate on compacta with polynomials of type $p(z)\prod_{i=1}^m (1 - z_i w_i)^{k_i} \overline{p(\overline{w})}$ for arbitrary $k_i \in \mathrm{N}$ and analytic polynomials $p \in P(\mathrm{C}^m)$. In this case, from (5.4), we have also $L_x^\Gamma(f(z,\overline{z})) \geq 0$; that is, also from hypothesis and from Weierstrass approximation theorem, $L_x^\Gamma : P(\mathrm{C}^m) \to \mathrm{C}$ is a positive, continuous, linear functional. With the Hahn-Banach theorem, we extend L_x^Γ on $C(\overline{\mathrm{D}}^m)$ preserving the linearity, continuity and positivity. In this case, from Riesz representation theorem, there exists a positive Radon measure $\mu(x,.)$ on $\overline{\mathrm{D}}^m$ such that $L_x^\Gamma(z^\alpha \overline{z}^\beta) = \int_{\overline{\mathrm{D}}^m} z^\alpha \overline{z}^\beta \mathrm{d}\mu(x,z) =< \Gamma_{\alpha,\beta} x, x >_H$ for all multiindices $\alpha, \beta \in \mathrm{Z}_+^m$. Because $\mu(x,B)$ is positive for all $B \in Bor(\overline{\mathrm{D}}^m)$, for any couple of vectors $u, v \in$H, we associate by a standard polarization argument, the scalar measure

$$\mu(u,v,B) = \frac{1}{4}[\mu(u+v,B) - \mu(u-v,B) + i\mu(u+iv,B) - i\mu(u-iv,B)].$$

For this measure, we have also

$$\int_{\overline{\mathrm{D}}^m} z^\alpha \overline{z}^\beta \mathrm{d}\mu(u,v,z) = \frac{1}{4}[\int_{\overline{\mathrm{D}}^m} z^\alpha \overline{z}^\beta \mathrm{d}\mu(u+v,z)$$

$$-\int_{\overline{\mathrm{D}}^m} z^\alpha \overline{z}^\beta \mathrm{d}\mu(u-v,z) + i\int_{\overline{\mathrm{D}}^m} z^\alpha \overline{z}^\beta \mathrm{d}\mu(u+iv,z) - i\int_{\overline{\mathrm{D}}^m} z^\alpha \overline{z}^\beta \mathrm{d}\mu(u-iv,z)]$$

$$=< \Gamma_{\alpha,\beta} u, v >_H \text{ for all } \alpha, \beta \in \mathrm{Z}_+^m. \tag{5.5}$$

From (5.5) and from the definition of L_u^Γ, we obtain

$$\int_{\overline{\mathrm{D}}^m} z^\alpha \overline{z}^\beta \mathrm{d}\mu(u,z) = \int_{\overline{\mathrm{D}}^m} z^\alpha \overline{z}^\beta \mathrm{d}\mu(u,u,z) =< \Gamma_{\alpha,\beta} u, u >_H; \tag{5.6}$$

that is, because the scalar moment problem on compacta is determinate, $\mu(z,u,u) = \mu(z,u)$. The mapping $< \Gamma_{\alpha,\beta} u, v >_H$ is linear in the first argument, antilinear in the second one; the same is true for the measure $\mu(.,.,z) : H \times H \to \mathrm{C}$. In the same time, we have also

$$0 \leq \mu(B,u) = \int_B d\mu(u,u,z) \leq \int_{\overline{\mathrm{D}}^m} d\mu(u,v,z) =< \Gamma_{0,0} u, u >_H \leq ||u||^2 = 1$$

for u with $||u|| = 1$; it follows that the bilinear form $\mu(u,v,z)$ has on the unit sphere 0 and 1 bounds. In these conditions, there exists for $\forall\ B \in Bor(\overline{\mathrm{D}}^m)$ a bounded selfadjoint operator $F_\Gamma(B)$ such that $\mu(u,v,B) =< F(B)u, v >_H$. Because $0 \leq \mu(u,u,B) =< F_\Gamma(B)u, u >_H \leq 1$, it follows that $F_\Gamma(B) \geq 0$ (is a positive operator). In the given conditions, (5.6) can be written symbolically $\Gamma_{\alpha,\beta} = \int_{\overline{\mathrm{D}}^m} z^\alpha \overline{z}^\beta \mathrm{d}F_\Gamma(z)$ for all $\alpha, \beta \in\ \mathrm{Z}_+^m$, that is condition (ii).

Conversely. $(ii) \Rightarrow (i)$. If we have a positive operator-valued measure F_Γ defined on $Bor(\overline{\mathrm{D}}^m)$, such that $\Gamma_{\alpha,\beta} = \int_{\overline{\mathrm{D}}^m} z^\alpha \overline{z}^\beta \mathrm{d}F_\Gamma(z)$, it follows that

$$\sum_{\alpha,\beta\in \mathrm{Z}_+^m} \sum_{0\leq\theta\leq k} (-1)^{|\theta|} C_k^\theta c_\alpha \overline{c}_\beta \Gamma_{\alpha+\theta,\beta+\theta}$$

$$= \sum_{\alpha,\beta\in \mathrm{Z}_+^m} \sum_{0\leq\theta\leq k} (-1)^{|\theta|} C_k^\theta c_\alpha \overline{c}_\beta \int_{\overline{\mathrm{D}}^m} z^{\alpha+\theta}\overline{z}^{\beta+\theta} \mathrm{d}F_\Gamma(z)$$

$$= \int_{\mathrm{D}^m} (\sum_{\alpha,\beta\in \mathrm{Z}_+^m} \sum_{0\leq\theta\leq k} (-1)^{|\theta|} C_k^\theta c_\alpha \overline{c}_\beta z^{\alpha+\theta}\overline{z}^{\beta+\theta} \mathrm{d}F_\Gamma(z)$$

$$= \int_{\mathrm{D}^m} |p(z)|^2 \prod_{i=1}^m (1-|z_i|^2)^{k_i} \mathrm{d}F_\Gamma(z) \geq 0$$

for any multiindices $k = (k_1, ..., k_m) \in \mathrm{Z}_+^m$ and any arbitrary analytic polynomial

$$p(z) = \sum_{\alpha\in \mathrm{I}\subset \mathrm{Z}_+^m} c_\alpha z^\alpha,$$

I finite; that is condition (i).

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Univalence criterion for a certain general integral operator

Vasile Marius Macarie

Abstract. In this paper we consider a general integral operator, the class of analytic functions defined in the open unit disk and two classes of univalent functions. By imposing supplementary conditions for these functions we determine sufficient univalence conditions for the considered general operator. Some particular results are also presented.

Mathematics Subject Classification (2010): 30C45.

Keywords: Open unit disk, analytic function, univalent function, integral operator.

1. Introduction

Let $\mathcal{A}$ be the class of analytic functions f defined in the open unit disk of the complex plane $U = \{z \in \mathbb{C} : |z| < 1\}$ and normalized by the conditions

$$f(0) = f'(0) - 1 = 0.$$

We consider S the class of all functions in $\mathcal{A}$ which are univalent in U and denote by P the class of the functions h which are analytic in U, $h(0) = 1$ and Re $h(z) > 0$ for all $z \in U$.

We define the class $S(\alpha)$ with $0 < \alpha \leq 2$ consisting of all functions $f \in \mathcal{A}$ that satisfy the conditions $f(z) \neq 0$ and $\left|\left(\frac{z}{f(z)}\right)''\right| \leq \alpha,\ z \in U$. Singh [4] proved that if $f \in S(\alpha)$ then the following relation is true:

$$\left|\frac{z^2 f'(z)}{f^2(z)} - 1\right| \leq \alpha |z|^2,\ z \in U.$$

In this paper we introduce a general integral operator

$$H_{n,p}(z) = \left\{\beta \int_0^z t^{\beta-1} \prod_{i=1}^{n} \left(\frac{f_i(t)}{t}\right)^{\frac{1}{\gamma}} \prod_{j=1}^{p} (h_j(t))^{\delta} \mathrm{d}t\right\}^{\frac{1}{\beta}}, \tag{1.1}$$

with $f_i \in S(\alpha_i)$ for all $i = 1, 2, ..., n$ and $h_j \in P$ for $j = 1, 2, ..., p$ and we obtain sufficient conditions for its univalency.

For proving our main results we need the following theorems:

Theorem 1.1. [3]. *Let α be a complex number, Re $\alpha > 0$ and $f \in \mathcal{A}$. If*

$$\frac{1 - |z|^{2Re\ \alpha}}{Re\ \alpha} \left| \frac{zf''(z)}{f'(z)} \right| \leq 1 \tag{1.2}$$

for all $z \in U$, then for any complex number β, Re $\beta \geq$ Re α, the function

$$F_\beta(z) = \left\{ \beta \int_0^z u^{\beta-1} f'(u) du \right\}^{\frac{1}{\beta}} \tag{1.3}$$

is in the class S.

Lemma 1.2. [2]. *(General Schwarz Lemma). Let the function f be regular in the disk $U_R = \{z \in \mathbb{C} : |z| < R\}$, with $|f(z)| < M$ for fixed M. If f has one zero with multiplicity order bigger than m for $z = 0$, then*

$$|f(z)| \leq \frac{M}{R^m} \cdot |z|^m \quad (z \in U_R).$$

The equality can hold only if

$$f(z) = e^{i\theta} \cdot \frac{M}{R^m} \cdot z^m,$$

where θ is constant.

2. Main results

Theorem 2.1. *Let $f_i \in S(\alpha_i)$, $0 < \alpha_i \leq 2$, $f_i(z) = z + a_3^i z^3 + a_4^i z^4 + ...$, $M_i \geq 1$, for all $i = 1, 2, ...n$, $h_j \in P$, $N_j > 0$, for all $j = 1, 2, ...p$ and $\delta, \gamma \in \mathbb{C}$ with*

$$\mathrm{Re}\ \gamma \geq \frac{1}{|\gamma|} \left(n + \sum_{i=1}^{n} (\alpha_i + 1) M_i \right) + |\delta| \sum_{j=1}^{p} N_j. \tag{2.1}$$

If

$$|f_i(z)| \leq M_i \text{ for all } i = 1, 2, ..., n, \ (z \in U) \tag{2.2}$$

and

$$\left| \frac{zh_j'(z)}{h_j(z)} \right| \leq N_j \text{ for all } j = 1, 2, ..., p, \ (z \in U) \tag{2.3}$$

then for every complex number β, Re $\beta \geq$ Re γ *the integral operator $H_{n,p}(z)$ defined by (1.1) is in the class S.*

Proof. Let us define the function

$$g(z) = \int_0^z \prod_{i=1}^{n} \left(\frac{f_i(t)}{t} \right)^{\frac{1}{\gamma}} \prod_{j=1}^{p} (h_j(t))^\delta \mathrm{d}t.$$

We have

$$g'(z)=\prod_{i=1}^{n}\left(\frac{f_i(z)}{z}\right)^{\frac{1}{\gamma}}\prod_{j=1}^{p}(h_j(z))^{\delta}$$

and, hence

$$\frac{zg''(z)}{g'(z)}=\frac{1}{\gamma}\sum_{i=1}^{n}\left(\frac{zf_i'(z)}{f_i(z)}-1\right)+\delta\sum_{j=1}^{p}\frac{zh_j'(z)}{h_j(z)}. \tag{2.4}$$

From (2.4) we obtain

$$\frac{1-|z|^{2\mathrm{Re}\gamma}}{\mathrm{Re}\gamma}\left|\frac{zg''(z)}{g'(z)}\right|\le\frac{1-|z|^{2\mathrm{Re}\gamma}}{\mathrm{Re}\gamma}\left[\frac{1}{|\gamma|}\sum_{i=1}^{n}\left(\left|\frac{zf_i'(z)}{f_i(z)}\right|+1\right)+|\delta|\sum_{j=1}^{p}\left|\frac{zh_j'(z)}{h_j(z)}\right|\right]$$

$$\le\frac{1-|z|^{2\mathrm{Re}\gamma}}{\mathrm{Re}\gamma}\left[\frac{1}{|\gamma|}\sum_{i=1}^{n}\left(\left|\frac{z^2f_i'(z)}{f_i^2(z)}\right|\cdot\left|\frac{f_i(z)}{z}\right|+1\right)+|\delta|\sum_{j=1}^{p}\left|\frac{zh_j'(z)}{h_j(z)}\right|\right] \tag{2.5}$$

From (2.2) applying general Schwarz Lemma we have $\left|\frac{f_i(z)}{z}\right|\le M_i$ for all $i=1,2,...,n$. Using the last relation, (2.3) and (2.5) we obtain

$$\frac{1-|z|^{2\mathrm{Re}\gamma}}{\mathrm{Re}\gamma}\left|\frac{zg''(z)}{g'(z)}\right|\le\frac{1-|z|^{2\mathrm{Re}\gamma}}{\mathrm{Re}\gamma}\left\{\frac{1}{|\gamma|}\sum_{i=1}^{n}\left[\left(\left|\frac{z^2f_i'(z)}{f_i^2(z)}-1\right|+1\right)M_i+1\right]+\right.$$
$$\left.+|\delta|\sum_{j=1}^{p}N_j\right\} \tag{2.6}$$

Because $f_i\in S(\alpha_i)$ for all $i=1,2,...,n$ we have

$$\left|\frac{z^2f_i'(z)}{f_i^2(z)}-1\right|\le\alpha_i|z|^2\ for\ all\ i=1,2,...,n,\ (z\in U). \tag{2.7}$$

From (2.6) and (2.7) it results

$$\frac{1-|z|^{2\mathrm{Re}\gamma}}{\mathrm{Re}\gamma}\left|\frac{zg''(z)}{g'(z)}\right|\le\frac{1-|z|^{2\mathrm{Re}\gamma}}{\mathrm{Re}\gamma}\left\{\frac{1}{|\gamma|}\sum_{i=1}^{n}\left[\left(\alpha_i|z|^2+1\right)M_i+1\right]+\right.$$
$$\left.+|\delta|\sum_{j=1}^{p}N_j\right\}\le\frac{1}{\mathrm{Re}\gamma}\left[\frac{1}{|\gamma|}\left(n+\sum_{i=1}^{n}(\alpha_i+1)M_i\right)+|\delta|\sum_{j=1}^{p}N_j\right] \tag{2.8}$$

From (2.1) and (2.8) we have

$$\frac{1-|z|^{2\mathrm{Re}\gamma}}{\mathrm{Re}\gamma}\left|\frac{zg''(z)}{g'(z)}\right|\le1,\ (z\in U)$$

and applying Theorem 1.1 we obtain that the integral operator $H_{n,p}(z)$ defined in (1.1) is in the class S. □

Letting $\alpha_i=\alpha$ for all $i=1,2,...,n$ in Theorem 2.1 we have

Corollary 2.2. *Let* $f_i \in S(\alpha)$, $0 < \alpha \le 2$, $f_i(z) = z + a_3^i z^3 + a_4^i z^4 + \ldots$, $M_i \ge 1$, *for all* $i = 1, 2, \ldots n$, $h_j \in P$, $N_j > 0$, *for all* $j = 1, 2, \ldots p$ *and* $\delta, \gamma \in \mathbb{C}$ *with*

$$\mathrm{Re}\ \gamma \ge \frac{n + (\alpha+1)\sum\limits_{i=1}^{n} M_i}{|\gamma|} + |\delta| \sum_{j=1}^{p} N_j.$$

If

$$|f_i(z)| \le M_i \text{ for all } i = 1, 2, \ldots, n,\ (z \in U)$$

and

$$\left| \frac{z h_j'(z)}{h_j(z)} \right| \le N_j \text{ for all } j = 1, 2, \ldots, p,\ (z \in U)$$

then for every complex number β, $\mathrm{Re}\ \beta \ge \mathrm{Re}\ \gamma$ *the integral operator* $H_{n,p}(z)$ *defined by* (1.1) *is in the class* S.

Letting $M_i = M$ for all $i = 1, 2, \ldots, n$ and $N_j = N$ for all $j = 1, 2, \ldots, p$ in Corollary 2.2 we have

Corollary 2.3. *Let* $f_i \in S(\alpha)$, $0 < \alpha \le 2$, $f_i(z) = z + a_3^i z^3 + a_4^i z^4 + \ldots$ *for all* $i = 1, 2, \ldots n$, $h_j \in P$ *for all* $j = 1, 2, \ldots p$, $M \ge 1$, $N > 0$, *and* $\delta, \gamma \in \mathbb{C}$ *with*

$$\mathrm{Re}\ \gamma \ge \frac{n[1 + (\alpha+1)M]}{|\gamma|} + p|\delta|N.$$

If

$$|f_i(z)| \le M \text{ for all } i = 1, 2, \ldots, n,\ (z \in U)$$

and

$$\left| \frac{z h_j'(z)}{h_j(z)} \right| \le N \text{ for all } j = 1, 2, \ldots, p,\ (z \in U)$$

then for every complex number β, $\mathrm{Re}\ \beta \ge \mathrm{Re}\ \gamma$ *the integral operator* $H_{n,p}(z)$ *defined by* (1.1) *is in the class* S.

Letting $n = 1$ and $p = 1$ in Corollary 2.3 we have

Corollary 2.4. *Let* $f \in S(\alpha)$, $0 < \alpha \le 2$, $f(z) = z + a_3 z^3 + a_4 z^4 + \ldots$, $h \in P$, $M \ge 1$, $N > 0$, *and* $\delta, \gamma \in \mathbb{C}$ *with*

$$\mathrm{Re}\ \gamma \ge \frac{1 + (\alpha+1)M}{|\gamma|} + |\delta|N.$$

If

$$|f(z)| \le M\ (z \in U)$$

and

$$\left| \frac{z h'(z)}{h(z)} \right| \le N\ (z \in U)$$

then for every complex number β, $\mathrm{Re}\ \beta \ge \mathrm{Re}\ \gamma$ *the integral operator*

$$H(z) = \left\{ \beta \int_0^z t^{\beta-1} \left(\frac{f(t)}{t} \right)^{\frac{1}{\gamma}} (h(t))^{\delta}\, \mathrm{d}t \right\}^{\frac{1}{\beta}}$$

is in the class S.

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Differential sandwich theorems involving certain convolution operator

Shahram Najafzadeh and Ali Ebadian

Abstract. In the present paper a certain convolution operator of analytic functions is defined. Moreover, subordination- and superordination- preserving properties for a class of analytic operators defined on the space of normalized analytic functions in the open unit disk is obtained. We also apply this to obtain sandwich results and generalizations of some known results.

Mathematics Subject Classification (2010): 30C45.

Keywords: Analytic function, convolution operator, differential subordination, differential superordination, sandwich theorem, best dominant and best subordinant.

1. Introduction

Let $H = H(\Delta)$ denote the class of analytic functions in the open unit disk

$$\Delta = \{z : |z| < 1\}$$

and

$$A := \{f \in H : f(0) = f'(0) - 1 = 0\}.$$

For a positive integer number n and $a \in \mathbb{C}$, let

$$\mathcal{H}[a, n] = \{f \in H(\Delta) : f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \dots\}.$$

Let f and F be members of the analytic function class H. The function f is said to be subordinate to F or F is said to be superordinate of f, if there exists a function w analytic in Δ, with $w(0) = 0$, and $|w(z)| < 1$ $(z \in \Delta)$ such that $f(z) = F(w(z))$ and we write $f \prec F$ or $f(z) \prec F(z)$ $(z \in \Delta)$. If the function F is univalent in $(z \in \Delta)$, then we have

$$f \prec F \iff f(0) = F(0) \text{ and } f(\Delta) \subset F(\Delta).$$

Let $\varphi : \mathbb{C}^2 \times \Delta \longrightarrow \mathbb{C}$ and h be analytic in Δ. If p is analytic in Δ and satisfies the(first-order) differential subordination

$$\varphi(p(z), zp'(z); z) \prec h(z) \tag{1.1}$$

then p is called a solution of the differential subordination. The univalent function q is called a dominant of the solutions of the differential subordination, or dominant if $p \prec q$ for all p satisfying (1.1). A dominant $\tilde{q}$ that satisfies $q \prec \tilde{q}$ for all dominants of (1.1) is said to be the best dominant.

Let $\varphi : \mathcal{C}^2 \times \Delta \longrightarrow \mathcal{C}$ and h be analytic in Δ. If p and $\varphi(p(z), zp'(z); z)$ are univalent in Δ and satisfies the(first-order) differential superordination

$$h(z) \prec \varphi(p(z), zp'(z); z) \tag{1.2}$$

then p is called a solution of the differential superordination. An analytic function q is called a subordinant of the solution of the differential superordination, or more simply a subordinant if $q \prec p$ for all q satisfying (1.2). A univalent subordinant $\tilde{q}$ that satisfies $\tilde{q} \prec q$ for all subordinant of (1.2) is said to be the best subordinant.

Ali et al [1] have obtained sufficient conditions for certain normalized analytic functions $f(z)$ to satisfy $q_1 \prec \frac{zf'(z)}{f(z)} \prec q_2(z)$, where q_1 and q_2 are given univalent functions in Δ with $q_1(0) = 1$ and $q_2(0) = 1$.
For two functions $f_j(z)(j = 1, 2)$, given by

$$f_j(z) = z + \sum_{k=2}^{\infty} a_{k,j} z^k \qquad (j = 1, 2)$$

we define the Hadamard product (or convolution) of $f_1(z)$ and $f_2(z)$ by

$$(f_1 * f_2)(z) := z + \sum_{k=2}^{\infty} a_{k,1} a_{k,2} z^k = (f_2 * f_1)(z) \qquad (z \in \Delta)$$

In terms of the Pochhammer symbol(or the shifted factorial), define $(\kappa)_n$ by

$$(\kappa)_0 = 1, \ \textit{and} \ (\kappa)_n = \kappa(\kappa+1)(\kappa+2)\ldots(\kappa+n-1) \qquad (n \in \mathbb{N} := \{1, 2, \ldots\})$$

also, define a function $\phi_a^\lambda(b, c; z)$ by

$$\phi_a^\lambda(b, c; z) := 1 + \sum_{n=1}^{\infty} (\frac{a}{a+n})^\lambda \frac{(b)_n}{(a)_n (c)_n} z^n, \qquad (z \in \Delta) \tag{1.3}$$

where

$$b \in \mathbb{C}, c \in \mathbb{R} \backslash Z_0^-, a \in \mathbb{C} \backslash Z_0^- (Z_0^- = \{0, -1, -2, \ldots\}); \lambda \geq 0$$

Corresponding to the function $\phi_a^\lambda(b, c; z)$, given by (1.3), we introduce the following convolution operator

$$L_a^\lambda(b, c; \beta) f(z) := \phi_a^\lambda(b, c; z) * (\frac{f(z)}{z})^\beta \qquad (f \in A, \beta \in \mathbb{C} \backslash 0, z \in \Delta) \tag{1.4}$$

It is easy to see that

$$z(\phi_a^\lambda(b, c; z))' = a\phi_a^\lambda(b, c; z) - a\phi_a^{\lambda+1}(b, c; z) \tag{1.5}$$

and

$$z(L_a^{\lambda+1}(b, c; \beta) f(z))' = aL_a^\lambda(b, c; \beta) f(z) - aL_a^{\lambda+1}(b, c; \beta) f(z) \tag{1.6}$$

The operator $L_a^\lambda(b, c; \beta) f(z)$ includes, as its special cases, Komatu integral operator(see [4], [6], [11]), some fractional calculus operators(see [4], [13], [14]) and Carlson-Shaffer operator(see [2]).

Making use of the principle of subordinantion between analytic functions Miller et all [9] obtained some interesting subordination theorems involving certain operators. Also Miller and Mocanu [8] considered subordination-preserving properties of certain integral operator investigations as the dual concept of differential subordination. In the present investigation, we obtain the subordination and superordination-preserving properties of the convolution operator L_a^λ defined by (1.4) with the Sandwich-type theorems.

2. Definitions and preliminaries

The following definitions and Lemmas will be required in our present investigation.

Definition 2.1. *If* $0 \leq \alpha < 1$ *and* $\lambda \geq 0, a \in \mathbb{C}\backslash Z_0^-(Z_0^- = \{0,-1,-2,\ldots\})$*, let* $\mathcal{L}_a^\lambda(\alpha)$ *denote the class of functions* $f \in A$ *wich satisfies the inequality*

$$Re[L_a^\lambda(b,c;\beta)f(z)] > \alpha$$

For $a = 1$, we set $\mathcal{L}_1^\lambda(\alpha) = \mathcal{L}^\lambda(\alpha)$.

Definition 2.2. [7] *We denote by* Q *the set of function* q *that are analytic and injective on* $\overline{\Delta}\backslash E(q)$ *where*

$$E(q) = \{\xi \in \Delta : \lim_{z\to\xi} q(z) = \infty\}$$

and $h'(\xi) \neq 0$ *for* $\xi \in \partial\Delta\backslash E(q)$*.*

Lemma 2.3. [7] *Let* $h(z)$ *be analytic and convex univalent in* Δ *and* $h(0) = a$*. Also let* $p(z)$ *be analytic in* Δ *with* $p(0) = a$*. If* $p(z) + \frac{zp'(z)}{\gamma} \prec h(z)$*, where* $\gamma \neq 0$ *and* $Re\gamma \geq 0$*, then* $p(z) \prec q(z) \prec h(z)$*, where*

$$q(z) = \frac{\gamma}{z^\gamma}\int_0^z h(t)t^{\gamma-1}dt$$

Furthermore $q(z)$ *is a convex function and is the best dominant.*

Lemma 2.4. [8] *Let* $h(z)$ *be convex in* Δ*,* $h(0) = a, \gamma \neq 0$ *and* $Re\gamma \geq 0$*. Also* $p \in \mathcal{H}[a,n] \cap Q$*. If* $p(z) + \frac{zp'(z)}{\gamma}$ *is univalent in* Δ*,* $h(z) \prec p(z) + \frac{zp'(z)}{\gamma}$ *and*

$$q(z) = \frac{\gamma}{z^\gamma}\int_0^z h(t)t^{\gamma-1}dt$$

then $q(z) \prec p(z)$*, and* $q(z)$ *is a convex function and is the best subordinant.*

Lemma 2.5. [12] *Let* $q(z)$ *be a convex univalent function in* Δ *and* $\psi, \gamma \in \mathbb{C}$ *with* $Re(1 + \frac{zq''(z)}{q'(z)}) > max\{0, -Re\frac{\psi}{\gamma}\}$*,* $h(0) = a, \gamma \neq 0$ *and* $Re\gamma \geq 0$*. If* $p(z)$ *is analytic in* Δ *and* $\psi p(z) + \gamma zp'(z) \prec \psi q(z) + \gamma zq'(z)$ *then* $p(z) \prec q(z)$*, and* $q(z)$ *is the best dominant.*

Lemma 2.6. [10] *Let* $q(z)$ *be a convex univalent function in* Δ *and* $\eta \in \mathbb{C}$*, assume that* $Re\eta > 0$*. If* $p(z) \in \mathcal{H}[a,n] \cap Q$*and* $p(z) + \eta zp'(z) \prec q(z) + \eta zq'(z)$ *which implies that* $q(z) \prec p(z)$*, and* $q(z)$ *is the best subordinant.*

3. Differential subordination defined by convolution operator

In this section some differential subordinations are set using the convolution operator and concrete example of convex functions.

Theorem 3.1. *If* $0 \leq \alpha < 1$ *and* $\lambda \geq 0, a \in \mathbb{C}\backslash Z_0^- (Z_0^- = \{0, -1, -2, \dots\})$, *then we have*

$$\mathcal{L}_a^\lambda(\alpha) \subset \mathcal{L}_a^{\lambda+1}(\delta)$$

where

$$\delta(\alpha, a) = a\beta(a) + a(2\alpha - 1)\beta(a + 1)$$

and

$$\beta(x) = \int_0^1 \frac{t^{x-1}}{1+t} dt$$

the result is sharp.

Proof. First not that $f \in \mathcal{L}_a^\lambda(\alpha)$ and

$$z(L_a^{\lambda+1}(b, c; \beta)f(z))' = aL_a^\lambda(b, c; \beta)f(z) - aL_a^{\lambda+1}(b, c; \beta)f(z) \tag{3.1}$$

we define $p(z) = L_a^{\lambda+1}(b, c; \beta)f(z)$ from the relation(1.1) we have

$$L_a^\lambda(b, c; \beta)f(z) = p(z) + \frac{zp'(z)}{a}$$

now from Lemma 2.3, for $\gamma = a$ it follows that

$$p(z) = L_a^{\lambda+1}(b, c; \beta)f(z) \prec q(z) = \frac{a}{z^a} \int_0^z \frac{1 + (2\alpha - 1)t}{1 + t} t^{a-1} dt$$

therefore we have

$$\mathcal{L}_a^\lambda(\alpha) \subset \mathcal{L}_a^{\lambda+1}(\delta)$$

where

$$\delta = MinReq(z)_{|z|\leq 1} = q(1) = a\beta(a) + a(2\alpha - 1)\beta(a + 1)$$

Furthermore $q(z)$ is a convex function and is the best dominant. □

For the class $\mathcal{L}^\lambda$ we obtain the next corollary.

Corollary 3.2. *If* $0 \leq \alpha < 1$ *and* $\lambda \geq 0$, *then we have*

$$\mathcal{L}^\lambda(\alpha) \subset \mathcal{L}^{\lambda+1}(\delta)$$

where

$$\delta = \delta(\alpha) = 2\alpha - 1 + 2(1 - \alpha) \ln 2$$

and the result is sharp.

Theorem 3.3. *Let* $h \in H(\Delta)$, *with* $h(0) = 1, h'(0) \neq 0$, *which verifies the inequality*

$$Re[1 + \frac{zh''(z)}{h'(z)}] > -\frac{1}{2}(z \in \Delta).$$

If $f \in A$ *and satisfies the differential subordination*

$$L_a^\lambda(b, c; \beta)f(z) \prec h(z) \tag{3.2}$$

then

$$L_a^{\lambda+1}(b,c;\beta)f(z) \prec q(z)(z \in \Delta) \tag{3.3}$$

where

$$q(z) = \frac{a}{z^a}\int_0^z h(t)t^{a-1}dt$$

The function $q(z)$ is convex and is the best dominant.

Proof. Let

$$p(z) = L_a^{\lambda+1}(b,c;\beta)f(z) \tag{3.4}$$

Differentiating (3.4) with respect to z, we have $p'(z) = (L_a^{\lambda+1}(b,c;\beta)f(z))'$. From the relation (3.1) we have

$$\frac{zp'(z)}{a} + p(z) = L_a^{\lambda}(b,c;\beta)f(z)$$

now, in view of (3.4), we obtain the following subordination

$$\frac{zp'(z)}{a} + p(z) \prec h(z)$$

then from Lemma 2.3 for $\gamma = a$ we conclude that

$$p(z) = L_a^{\lambda+1}(b,c;\beta)f(z) \prec q(z)$$

where

$$q(z) = \frac{a}{z^a}\int_0^z h(t)t^{a-1}dt$$

and $q(z)$ is the best dominant. □

Taking $\lambda = 0$ in the Theorem 3.3 we arrive the following corollary.

Corollary 3.4. *Let $h \in H(\Delta)$, with $h(0) = 1, h'(0) \neq 0$, and $Re(1 + \frac{zh''(z)}{h'(z)}) > -\frac{1}{2}$ $(z \in \Delta)$. If $f \in A$ and satisfies $(\frac{f(z)}{z})^\beta \prec h(z)$, then $K_a(b,c;\beta) \prec q(z)$ where*

$$q(z) = \frac{a}{z^a}\int_0^z h(t)t^{a-1}dt.$$

The function $q(z)$ is the best dominant.

Putting $\gamma \in \mathbb{C}$. By setting $a = \gamma + \beta, \lambda = 0$, and $b = c = 1$ in the Theorem 3.3, we get the following corollary.

Corollary 3.5. *Let $h \in H(\Delta)$, with $h(0) = 1, h'(0) \neq 0$, which satisfies the inequality*

$$Re(1 + \frac{zh''(z)}{h'(z)}) > -\frac{1}{2}\ (z \in \Delta).$$

If $f \in A$ and satisfies the differential subordination $(\frac{f(z)}{z})^\beta \prec h(z)$, then

$$\frac{\gamma+\beta}{z^{\gamma+\beta}}\int_0^z u^{\gamma-1}(f(u))^\beta du \prec \frac{1}{z}\int_0^z h(u)du.$$

The function $\frac{1}{z}\int_0^z h(u)du$ is the best dominant.

Corollary 3.6. *Let $0 < R \leq 1$ and let $h(z)$ be convex in Δ, defined by*

$$h(z) = 1 + Rz + \frac{Rz}{2 + Rz},$$

with $h(0) = 1$. If $f \in A$ satisfies the following differential subordination

$$L_a^{\lambda}(b, c; \beta)f(z) \prec h(z)$$

then

$$L_a^{\lambda+1}(b, c; \beta)f(z) \prec q(z)(z \in \Delta)$$

where

$$q(z) = \frac{a}{z^a} \int_0^z 1 + Rt + \frac{Rt}{2 + Rt} t^{a-1} dt,$$

$$q(z) = z^{a-1} + Ra(\frac{z^a}{a+1} + \frac{M(z)}{z})$$

where

$$M(z) = \int_0^z \frac{t^a}{2 + Rt} dt$$

The function $q(z)$ is convex and is the best dominant.

If $a = 1$, the Corollary 3.6 becomes:

Corollary 3.7. *Let $0 < r \leq 1$ and let $h(z)$ be convex in Δ, defined by*

$$h(z) = 1 + Rz + \frac{Rz}{2 + Rz},$$

with $h(0) = 1$. If $f \in A$ and suppose that

$$L^{\lambda}(b, c; \beta)f(z) \prec h(z)$$

then

$$L^{\lambda+1}(b, c; \beta)f(z) \prec q(z)(z \in \Delta)$$

where

$$q(z) = \frac{1}{z} \int_0^z 1 + Rt + \frac{Rt}{2 + Rt} dt,$$

$$q(z) = 2 + \frac{Rz}{2} - \frac{2}{Rz} \log(2 + Rz)$$

The function $q(z)$ is convex and is the best dominant.

By taking $R = 1$ in the Corollary 3.7 we have the following corollary.

Corollary 3.8. *Let $h(z)$ be convex in Δ, defined by $h(z) = 1 + z + \frac{z}{2+z}$, with $h(0) = 1$. If $f \in A$, satisfies the differential subordination*

$$L^{\lambda}(b, c; \beta)f(z) \prec h(z)$$

then

$$L^{\lambda+1}(b, c; \beta)f(z) \prec q(z)(z \in \Delta)$$

where

$$q(z) = 2 + \frac{z}{2} - \frac{2}{z} \log(2 + z)$$

The function $q(z)$ is convex and is the best dominant.

Corollary 3.9. *Let $h(z)$ be convex in Δ, defined by $h(z) = 1 + z + \frac{z}{2+z}$, with $h(0) = 1$, and suppose that $\gamma \in \mathbb{C}, a = \gamma + \beta, \lambda = 0, b = c = 1$. If $f \in A$ and satisfies the differential subordination $(\frac{f(z)}{z})^\beta \prec h(z)$, then*

$$\frac{\gamma+\beta}{z^{\gamma+\beta}} \int_0^z u^{\gamma-1}(f(u))^\beta du \prec q(z) = 2 + \frac{z}{2} - \frac{2}{z}\log(2+z)$$

The function $q(z)$ is convex and is the best dominant.

Corollary 3.10. *Let $h(z) = \frac{1+(2\alpha-1)z}{1+z}$ be convex function in Δ, with $h(0) = 1$. If $f \in \mathcal{L}^\lambda(\alpha)$ and $L^\lambda(b,c;\beta)f(z) \prec h(z)$ then*

$$L^{\lambda+1}(b,c;\beta)f(z) \prec q(z)(z \in \Delta)$$

where

$$q(z) = 2\alpha - 1 + 2(1-\alpha)\frac{\log(1+z)}{z}$$

The function $q(z)$ is convex and is the best dominant.

Theorem 3.11. *Let $q(z)$ be a convex function $q(0) = 1$, and let h be a function such that*

$$h(z) = q(z) + \frac{zq'(z)}{q(z)} \ (z \in \Delta).$$

If $f \in H(\Delta)$ and verifies the differential subordination

$$L_a^\lambda(b,c;\beta)f(z) \prec h(z) \tag{3.5}$$

then

$$L_a^{\lambda+1}(b,c;\beta)f(z) \prec q(z)(z \in \Delta)$$

and this result is sharp.

Proof. We have

$$z(L_a^{\lambda+1}(b,c;\beta)f(z))' = aL_a^\lambda(b,c;\beta)f(z) - aL_a^{\lambda+1}(b,c;\beta)f(z) \tag{3.6}$$

Let $p(z) = L_a^{\lambda+1}(b,c;\beta)f(z)$, then from (3.5) and (3.6), we have

$$p(z) + \frac{zp'(z)}{a} \prec q(z) + \frac{zq'(z)}{a}$$

An application of Lemma 2.6, we conclude that $p(z) \prec q(z)$ or $L_a^{\lambda+1}(b,c;\beta)f(z) \prec q(z)(z \in \Delta)$ and this result is sharp. □

Theorem 3.12. *Let $h \in H(\Delta)$, with $h(0) = 1, h'(0) \neq 0$, which satisfies the inequality*

$$Re(1 + \frac{zh''(z)}{h'(z)}) > -\frac{1}{2} \ (z \in \Delta).$$

If $f \in A$ and verifies the differential subordination

$$(L_a^{\lambda+1}(b,c;\beta)f(z))' \prec h(z)$$

then

$$\frac{L_a^{\lambda+1}(b,c;\beta)f(z)}{z} \prec q(z)$$

where

$$q(z) = \frac{1}{z}\int_0^z h(t)t^{a-1}dt$$

the function $q(z)$ is the best dominant.

Proof. Let us define the function f by

$$f(z) = \frac{L_a^{\lambda+1}(b,c;\beta)f(z)}{z} \tag{3.7}$$

Differentiating logarithmically with respect to z, we have

$$\frac{zp'(z)}{p(z)} = \frac{z(L_a^{\lambda+1}(b,c;\beta)f(z))'}{L_a^{\lambda+1}(b,c;\beta)f(z)} - 1$$

and

$$p(z) + zp'(z) = (L_a^{\lambda+1}(b,c;\beta)f(z))'$$

Now, from (3.7) we obtain

$$p(z) + zp'(z) \prec h(z)$$

Then, by Lemma 2.3 , for $\gamma = 1$ we have $p(z) \prec q(z)$ or

$$\frac{L_a^{\lambda+1}(b,c;\beta)f(z)}{z} \prec \frac{1}{z}\int_0^z h(t)dt$$

and the function $q(z)$ is the best dominant. Therefore, we complete the proof of theorem 3.12. □

Suppose that$\lambda = 0$ and in the Theorem 3.12 we have the following result.

Corollary 3.13. *Let $h \in H(\Delta)$, with $h(0) = 1, h'(0) \neq 0$, which satisfies the inequality*

$$Re(1 + \frac{zh''(z)}{h'(z)}) > -\frac{1}{2}\ (z \in \Delta).$$

If $f \in A$ and $(K_a(b,c;\beta)f(z))' \prec h(z)$ then

$$\frac{K_a(b,c;\beta)f(z)}{z} \prec \frac{1}{z}\int_0^z h(t)dt,$$

and the function $\frac{1}{z}\int_0^z h(t)dt$ is the best dominant.

By taking $\gamma \in \mathbb{C}, a = \gamma + \beta, \lambda = 0$, and $b = c = 1$ in Theorem 3.12 we get the following result.

Corollary 3.14. *Let $h \in H(\Delta), h(0) = 1, h'(0) \neq 0$. If*

$$Re(1 + \frac{zh''(z)}{h'(z)}) > -\frac{1}{2}\ (z \in \Delta)$$

and if $f \in A$

$$\frac{-(\gamma+\beta)}{z^{\gamma+\beta+1}}\int_0^z u^{\gamma-1}(f(u))^\beta du + \frac{\gamma+\beta}{z^{\beta+1}} \prec h(z)$$

then

$$\frac{\gamma+\beta}{z^{\gamma+\beta-1}}\int_0^z u^{\gamma-1}(f(u))^\beta du \prec \frac{1}{z}\int_0^z h(u)du$$

The function $\frac{1}{z}\int_0^z h(u)du$ is the best dominant.

Corollary 3.15. *Let $0 < R \leq 1$ and let $h(z)$ be convex in Δ, defined by*

$$h(z) = 1 + Rz + \frac{Rz}{2 + Rz},$$

with $h(0) = 1$. If $f \in A$ satisfies the following differential subordination

$$(L^{\lambda+1}(b, c; \beta) f(z))' \prec h(z)$$

then

$$\frac{L^{\lambda+1}(b, c; \beta) f(z)}{z} \prec q(z)$$

where

$$q(z) = \frac{1}{z} \int_0^z 1 + Rt + \frac{Rt}{2 + Rt} dt,$$

$$q(z) = 1 + \frac{Rz}{2} + \frac{RM(z)}{z}$$

where

$$M(z) = \frac{z}{R} - \frac{2}{R^2}(\ln(2 + Rz)) - \frac{2}{R} \ln 2, (z \in \Delta)$$

The function $q(z)$ is convex and is the best dominant.

Suppose that $\gamma \in \mathbb{C}, a = \gamma + \beta, \lambda = 0$, and $b = c = 1$ in the Corollary 3.15 we have the following corollary.

Corollary 3.16. *Let $h(z)$ be convex in Δ, defined by $h(z) = 1 + z + \frac{z}{2+z}$, with $h(0) = 1$. If $f \in A$, satisfies the differential subordination*

$$\frac{-(\gamma + \beta)}{z^{\gamma+\beta+1}} \int_0^z u^{\gamma-1}(f(u))^{\beta} du + \frac{\gamma + \beta}{z^{\beta+1}} \prec h(z)$$

then

$$\frac{\gamma + \beta}{z^{\gamma+\beta-1}} \int_0^z u^{\gamma-1}(f(u))^{\beta} du \prec \frac{1}{z} \int_0^z h(u) du$$

where

$$q(z) = 2 + \frac{z}{2} - \frac{2}{z} \log(2 + z)$$

The function $q(z)$ is convex and is the best dominant.

Corollary 3.17. *Let $h(z) = \frac{1+(2\alpha-1)z}{1+z}$ be convex function in Δ, with $h(0) = 1$. If $f \in \mathcal{L}^{\lambda}(\alpha)$ and*

$$(L^{\lambda+1}(b, c; \beta) f(z))' \prec h(z)$$

then

$$\frac{L^{\lambda+1}(b, c; \beta) f(z)}{z} \prec q(z)$$

where

$$q(z) = 2\alpha - 1 + 2(1 - \alpha) \frac{\log(1 + z)}{z}$$

The function $q(z)$ is convex and is the best dominant.

Theorem 3.18. *Let $q(z)$ be a convex function, $q(0) = 1$, and*

$$h(z) = q(z) + \frac{zq'(z)}{q(z)} \ (z \in \Delta).$$

If $f \in H(\Delta)$ and satisfies the differential subordination

$$(L_a^{\lambda+1}(b, c; \beta)f(z))' \prec h(z) \tag{3.8}$$

then

$$\frac{L_a^{\lambda+1}(b, c; \beta)f(z)}{z} \prec q(z)(z \in \Delta)$$

and this result is sharp.

Proof. Let

$$p(z) = \frac{L_a^{\lambda+1}(b, c; \beta)f(z)}{z} \tag{3.9}$$

Logarithmic differentiation of (3.9) and through a little simplification we obtain

$$p(z) + zp'(z) = (L_a^{\lambda+1}(b, c; \beta)f(z))'$$

now by using Lemma 2.6, we conclude that

$$\frac{L_a^{\lambda+1}(b, c; \beta)f(z)}{z} \prec q(z)$$

and this result is sharp. □

4. Differential superordination defined by convolution operator

The results of this section are obtained with differential superordination method.

Theorem 4.1. *Let $h \in H(\Delta)$ be convex function in Δ, with $h(0) = 1$, and $f \in A$. Assume that $L_a^{\lambda}(b, c; \beta)f(z)$ is univalent with $L_a^{\lambda+1}(b, c; \beta)f(z) \in \mathcal{H}[1, n] \cap Q$. If $h(z) \prec L_a^{\lambda}(b, c; \beta)f(z)$ then*

$$q(z) \prec L_a^{\lambda+1}(b, c; \beta)f(z) \tag{4.1}$$

where

$$q(z) = \frac{a}{z^a} \int_0^z h(t)t^{a-1}dt$$

The function $q(z)$ is the best subordinant.

Proof. If we let $p(z) = L_a^{\lambda+1}(b, c; \beta)f(z)$ then from the relation (1.6) we have $p(z) + \frac{zp'(z)}{a} = L_a^{\lambda}(b, c; \beta)f(z)$. Now according to Lemma 2.4 we get the desired result (4.1). □

Corollary 4.2. *Suppose that $\gamma \in \mathbb{C}, a = \gamma + \beta, \lambda = 0$, and $b = c = 1$. Let $h \in H(\Delta)$ be convex function in Δ, with $h(0) = 1$, and $f \in A$. Assume that $(\frac{f(z)}{z})^{\beta}$ is univalent with $\frac{\gamma+\beta}{z^{\gamma+\beta}} \int_0^z u^{\gamma-1}(f(u))^{\beta}du \in \mathcal{H}[1, n] \cap Q$. If $h(z) \prec (\frac{f(z)}{z})^{\beta}$ then*

$$\frac{1}{z} \int_0^z h(u)du \prec \frac{\gamma + \beta}{z^{\gamma+\beta}} \int_0^z u^{\gamma-1}(f(u))^{\beta}du$$

and $\frac{1}{z} \int_0^z h(u)du$ is the best subordinant.

Corollary 4.3. *Let $h(z)$ be convex in Δ, defined by $h(z) = 1 + z + \frac{z}{2+z}$, with $h(0) = 1$. Suppose that $\gamma \in \mathbb{C}, a = \gamma + \beta, \lambda = 0, b = c = 1$, and $f \in A$ and $(\frac{f(z)}{z})^{\beta}$ is univalent with $\frac{\gamma+\beta}{z^{\gamma+\beta}} \int_0^z u^{\gamma-1}(f(u))^{\beta} du \in \mathcal{H}[1,n] \cap Q$.If $h(z) \prec (\frac{f(z)}{z})^{\beta}$ then $q(z) \prec \frac{\gamma+\beta}{z^{\gamma+\beta}} \int_0^z u^{\gamma-1}(f(u))^{\beta} du$ where $q(z) = 2 + \frac{z}{2} - \frac{2}{z}\log(2+z)$. The function $q(z)$ is the best subordinant.*

Corollary 4.4. *Let $h(z) = \frac{1+(2\alpha-1)z}{1+z}$ be convex function in Δ, with $h(0) = 1$. Assume that $f \in \mathcal{L}^{\lambda+1}(\alpha)$ and $L^{\lambda}(b,c;\beta)f(z)$ is univalent with $L^{\lambda+1}(b,c;\beta)f(z) \in \mathcal{H}[1,n] \cap Q$. If $h(z) \prec L^{\lambda}(b,c;\beta)f(z)$ then*

$$q(z) \prec L_a^{\lambda+1}(b,c;\beta)f(z)$$

where

$$q(z) = 2\alpha - 1 + 2(1-\alpha)\frac{\log(1+z)}{z}$$

The function $q(z)$ is the best subordinant.

Theorem 4.5. *Let $h \in H(\Delta)$ be convex function in Δ, with $h(0) = 1$, and $f \in A$. Assume that $(L_a^{\lambda+1}(b,c;\beta)f(z))'$ is univalent with $\frac{L_a^{\lambda+1}(b,c;\beta)f(z)}{z} \in \mathcal{H}[1,n] \cap Q$. If $h(z) \prec (L_a^{\lambda+1}(b,c;\beta)f(z))'$ then*

$$q(z) \prec \frac{L_a^{\lambda+1}(b,c;\beta)f(z)}{z}$$

where

$$q(z) = \frac{1}{z}\int_0^z h(t)dt$$

The function $q(z)$ is the best subordinant.

5. Sandwich results

Combining results of differential subordinations and superordinations, we arrive at the following "Sandwich results".

Theorem 5.1. *Let $q_1(z)$ be convex univalent in the open unit disk, and $q_2(z)$ univalent in the open unite disk Δ and $f \in A$. Also let $L_a^{\lambda}(b,c;\beta)f(z)$ be univalent with $L_a^{\lambda+1}(b,c;\beta)f(z) \in \mathcal{H}[1,n] \cap Q$. The following subordinate relationship $q_1(z) \prec L_a^{\lambda}(b,c;\beta)f(z) \prec q_1(z)$ implies $q_1(z) \prec L_a^{\lambda+1}(b,c;\beta)f(z) \prec q_2(z)$. Moreover the functions $q_1(z), q_2(z)$, are, respectively the best subordinant and the best dominant.*

Theorem 5.2. *Suppose that $q_1(z)$ is convex univalent, and let $q_2(z)$ be univalent Δ and $f \in A$. If $(L_a^{\lambda+1}(b,c;\beta)f(z))'$ is univalent with $\frac{L_a^{\lambda+1}(b,c;\beta)f(z)}{z} \in \mathcal{H}[1,n] \cap Q$. If $q_1(z) \prec (L_a^{\lambda+1}(b,c;\beta)f(z))' \prec q_2(z)$ then $q_1(z) \prec \frac{L_a^{\lambda+1}(b,c;\beta)f(z)}{z} \prec q_2(z)$ and $q_1(z), q_2(z)$, are, respectively the best subordinant and the best dominant.*

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Dominants and best dominants in fuzzy differential subordinations

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Abstract. The theory of differential subordination was introduced by S.S. Miller and P.T. Mocanu in [1] and [2] then developed in many other papers. Using the notion of differential subordination, in [5] the authors define the notion of fuzzy subordination and in [6] they define the notion of fuzzy differential subordination. In this paper, we determine conditions for a function to be a dominant of the fuzzy differential subordination and we also give the best dominant.

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1. Introduction and preliminaries

Let U denote the unit disc of the complex plane:

$$U = \{z \in \mathbb{C} : \ |z| < 1\}, \ \overline{U} = \{z \in \mathbb{C} : \ |z| \le 1\}, \ \partial U = \{z \in \mathbb{C} : \ |z| = 1\}$$

and $\mathcal{H}(U)$ denote the class of analytic functions in U.

For $a \in \mathbb{C}$ and $n \in \mathbb{N}$, we denote by

$$\mathcal{H}[a, n] = \{f \in \mathcal{H}(U); \ f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \dots, \ z \in U\}$$

and

$$A_n = \{f \in \mathcal{H}(U) : \ f(z) = z + a_{n+1} z^{n+1} + a_{n+2} z^{n+2} + \dots, \ z \in U\},$$

with $A_1 = A$.

Let

$$S = \{f \in A; \ f \text{ univalent in } U\}$$

be the class of holomorphic and univalent functions in the open unit disc U, with conditions $f(0) = 0$, $f'(0) = 1$, that is the holomorphic and univalent functions with the following power series development

$$f(z) = z + a_2 z^2 + \dots, \quad z \in U.$$

Denote by

$$S^* = \left\{ f \in A;\ \operatorname{Re} \frac{zf'(z)}{f(z)} > 0,\ z \in U \right\},$$

the class of normalized starlike functions in U,

$$K = \left\{ f \in A;\ \operatorname{Re} \frac{zf''(z)}{f'(z)} + 1 > 0,\ z \in U \right\}$$

the class of normalized convex functions in U and by

$$C = \left\{ f \in A :\ \exists\, \varphi \in K;\ \operatorname{Re} \frac{f'(z)}{\varphi'(z)} > 0,\ z \in U \right\}$$

the class of normalized close-to-convex functions in U.

An equivalent formulation for close-to-convexity would involve the existence of a starlike function h (not necessarily normalized) such that

$$\operatorname{Re} \frac{zf'(z)}{h(z)} > 0, \quad z \in U.$$

Kaplan [1] and Sakaguchi [9] showed that $f \in S$ if

$$\operatorname{Re} \left[\frac{zf''(z)}{f'(z)} + 1 \right] > -\frac{1}{2}.$$

In order to prove our original results, we use the following definitions and lemmas:

Definition 1.1. [3, p. 21, Definition 2.26] *We denote by Q the set of functions q that are analytic and injective on $\overline{U} \setminus E(q)$, where*

$$E(q) = \left\{ \zeta \in \partial U :\ \lim_{z \to \zeta} q(z) = \infty \right\},$$

and are such that $q'(\zeta) \neq 0$, for $\zeta \in \partial U \setminus E(q)$. The set $E(q)$ is called exception set.

Definition 1.2. [5] *Let X be a non-empty set. An application $F : X \to [0,1]$ is called fuzzy subset.*

An alternate definition, more precise, would be the following:

A pair (A, F_A), where $F_A : X \to [0,1]$ and

$$A = \{x \in X :\ 0 < F_A(x) \le 1\} = \operatorname{supp}(A, F_A),$$

is called fuzzy subset.

The function F_A is called membership function of the fuzzy subset (A, F_A).

Definition 1.3. [6] *Let two fuzzy subsets of X, (M, F_M) and (N, F_N). We say that the fuzzy subsets M and N are equal if and only if $F_M(x) = F_N(x)$, $x \in X$ and we denote this by $(M, X_M) = (N, F_N)$. The fuzzy subset (M, F_M) is contained in the fuzzy subset (N, F_N) if and only if $F_M(x) \le F_N(x)$, $x \in X$ and we denote the inclusion relation by $(M, F_M) \subseteq (N, F_N)$.*

Definition 1.4. [5] *Let $D \subset \mathbb{C}$, $z_0 \in D$ be a fixed point, and let the functions $f, g \in \mathcal{H}(D)$. The function f is said to be fuzzy subordinate to g and write $f \prec_F g$ or $f(z) \prec_F g(z)$ if the following conditions are satisfied:*

(i) $f(z_0) = g(z_0)$

(ii) $F_{f(D)}f(z) \leq F_{g(D)}g(z)$, $z \in U$,

where

$$f(D) = \operatorname{supp}(D, F_{f(D)}) = \{z \in \mathbb{C} \mid 0 < F_{f(D)}(z) \leq 1\},$$

$$g(D) = \operatorname{supp}(D, F_{g(D)}) = \{z \in \mathbb{C} \mid 0 < F_{g(D)}(z) \leq 1\}.$$

Definition 1.5. [6] *Let Ω be a set in $\mathbb{C}$, $q \in Q$ and n be a positive integer. The class of admissible functions $\Psi_n[\Omega, q]$ consists of those functions $\psi : \mathbb{C}^3 \times U \to \mathbb{C}$ that satisfy the admissibility condition:*

$$F_\Omega \psi(r, s, t; z) = 0, \quad z \in U, \tag{1.1}$$

whenever $r = q(\zeta)$, $s = m\zeta q'(\zeta)$,

$$\operatorname{Re}\frac{t}{s} + 1 \geq m\operatorname{Re}\left[\frac{\zeta q''(\zeta)}{q'(\zeta)} + 1\right], \quad z \in U,$$

$\zeta \in \partial U \setminus E(q)$ and $m \geq n$. We write $\Psi_1[\Omega, q]$ as $\Psi[\Omega, q]$. In the special case when Ω is a simply connected domain, $\Omega \neq \mathbb{C}$, and h is conformal mapping of U into Ω we denote this class by $\Psi_n[h(U), q]$ or $\Psi_n[h, q]$.

If $\mathbb{C}^2 \times U \to \mathbb{C}$, then the admissibility condition (1.1) reduces to

$$F_\Omega \psi(q(\zeta), m\zeta q'(\zeta); z) = 0 \tag{1.2}$$

when $z \in U$, $\zeta \in \partial U \setminus E(q)$ and $m \geq n$.

Definition 1.6. [6] *Let $\psi : \mathbb{C}^3 \times U \to \mathbb{C}$ and let h be univalent in U with $h(0) = \psi(a, 0, 0; 0)$. If p is analytic in U with $p(0) = a$ and satisfies the (second-order) fuzzy differential subordination*

$$F_{\psi(\mathbb{C}^3 \times U)}\psi(p(z), zp'(z), z^2p''(z)) \leq F_{h(U)}h(z), \quad z \in U, \tag{1.3}$$

then p is called a fuzzy solution of the fuzzy differential subordination. The univalent function q is called a fuzzy dominant of the fuzzy solutions of the fuzzy differential subordination, or more simple a fuzzy dominant, if $p(0) = q(0)$ and $F_{p(U)}f(z) \leq F_{q(U)}q(z)$, $z \in U$, for all p satisfying (1.3). A fuzzy dominant $\widetilde{q}$ that satisfies $\widetilde{q}(0) = q(0)$ and $F_{\widetilde{q}(U)}\widetilde{q}(z) \leq F_{q(U)}q(z)$, $z \in U$, for all fuzzy dominants q of (1.3) is said to be the fuzzy best dominant of (1.3). Note that the fuzzy best dominant is unique up to a rotation of U. If we require the more restrictive condition $p \in \mathcal{H}[a, n]$, then p will be called an (a, n)-fuzzy solution, q an (a, n)-fuzzy dominant, and $\widetilde{q}$ the best (a, n)-fuzzy dominant.

Definition 1.7. [8] *A function $L(z, t)$, $z \in U$, $t \geq 0$, is a fuzzy subordination chain if $L(\cdot, t)$ is analytic and univalent in U for all $t \geq 0$, $L(z, t)$ is continuously differentiable on $[0, \infty)$ for all $z \in U$, and $F_{L[U\times(0,\infty)]}L(z, t_1) \leq F_{L[U\times(0,\infty)]}L(z, t_2)$, when $t_1 \leq t_2$.*

Lemma 1.8. [6, Th. 2.4] *Let h and q be univalent in U, with $q(0) = a$, and let $h_\rho(z) = h(\rho z)$ and $q_\rho(z) = q(\rho z)$. Let $\psi : \mathbb{C}^3 \times U \to \mathbb{C}$ satisfy one of the following conditions:*

(i) $\psi \in \Psi_n[h, q_\rho]$ for some $\rho \in (0, 1)$, or

(ii) there exists $\rho_0 \in (0, 1)$ such that $\psi \in \Psi_n[h_\rho, q_\rho]$, for all $\rho \in (\rho_0, 1)$.

If $p \in \mathcal{H}[a, 1]$ and $\psi(p(z), zp'(z), z^2p''(z))$ is analytic in U, $\psi(a, 0, 0; 0) = h(0)$ and

$$F_{\psi(\mathbb{C}^3\times U)}\psi(p(z), zp'(z), z^2p''(z)) \leq F_{h(U)}h(z), \quad z \in U,$$

then

$$F_{p(U)}p(z) \leq F_{q(U)}q(z), \quad z \in U.$$

Lemma 1.9. [6, Th. 2.6] *Let h be univalent in U, and let $\psi : \mathbb{C}^3 \times U \to \mathbb{C}$. Suppose that the differential equation*

$$\psi(q(z), zq'(z), n(n-1)zq'(z) + n^2z^{2n}q''(z); z) = h(z),$$

has a solution q, with $q(0) = a$, and one of the following conditions is satisfied:

(i) $q \in Q$ and $\psi \in \Psi_n[h, q]$;

(ii) q is univalent in U and $\psi \in \Psi_n[h, q_\rho]$ for some $\rho \in (0, 1)$, or

(iii) q is univalent in U and there exists $\rho_0 \in (0, 1)$ such that $\psi \in \Psi_n[h_\rho, q_\rho]$ for all $\rho \in (\rho_0, 1)$.

If $p \in \mathcal{H}[a, n]$, $\psi(p(z), zp'(z), z^2p''(z))$ is analytic in U, and p satisfies

$$F_{\psi(\mathbb{C}^3\times U)}\psi(p(z), zp'(z), z^2p''(z)) \leq F_{h(U)}h(z),$$

then

$$F_{p(U)}p(z) \leq F_{q(U)}q(z), \quad z \in U.$$

Lemma 1.10. [7] *If $L_\gamma : A \to A$ is the integral operator defined by $L_\gamma[f] = F$, given by*

$$L_\gamma[f](z) = F(z) = \frac{\gamma + 1}{z^\gamma} \int_0^z f(t)t^{\gamma-1}dt,$$

and $\operatorname{Re}\gamma > 0$ then

(i) $L_\gamma[S^*] \subset S^*$;

(ii) $L_\gamma[K] \subset K$;

(iii) $L_\gamma[\mathbb{C}] \subset \mathbb{C}$.

Lemma 1.11. [3, Lemma 2.2.d, p. 24] *Let $q \in Q$ with $q(0) = a$, and let*

$$p(z) = a + a_nz^n + a_{n+1}z^{n+1} + \dots$$

be analytic in U with $p(z) \not\equiv a$ and $n \geq 1$. If p is not subordinate to q, there exist points $z_0 = r_0e^{i\theta_0} \in U$ and $\zeta_0 \in \partial U \setminus E(q)$ and an $m \geq n \geq 1$ for which $p(U_{r_0}) \subset q(U)$,

(i) $p(z_0) = q(\zeta_0)$;

(ii) $z_0p'(z_0) = m\zeta_0q'(\zeta_0)$, and

(iii) $\operatorname{Re}\dfrac{z_0p''(z_0)}{p'(z_0)} + 1 \geq m\operatorname{Re}\left[\dfrac{\zeta_0q''(\zeta_0)}{q'(\zeta_0)} + 1\right].$

Lemma 1.12. [8, p. 159] *The function*

$$L(z,t) = a_1(t)z + a_2(t)z^2 + \dots$$

with $a_1(t) \neq 0$ *for* $t \geq 0$ *and* $\lim_{t\to\infty} |a_1(t)| = \infty$ *is a subordination chain if and only if*

$$\operatorname{Re}\left[\frac{z\partial L(z,t)/\partial z}{\partial L(z,t)/\partial t}\right] > 0, \quad z \in U,\ t \geq 0.$$

2. Main results

Theorem 2.1. *Let* h *be analytic in* U, *let* ϕ *be analytic in a domain* D *containing* $h(U)$ *and suppose*

(a) $\operatorname{Re}\phi[h(z)] > 0,\ z \in U$ *and*

(b) $h(z)$ *is convex.*

If p *is analytic in* U, *with* $p(0) = h(0)$, $p(U) \subset D$ *and* $\psi : \mathbb{C}^2 \times U \to \mathbb{C}$,

$$\psi(p(z), zp'(z)) = p(z) + zp'(z) \cdot \phi[p(z)]$$

is analytic in U, *then*

$$F_{\psi(\mathbb{C}^2\times U)}\psi(p(z), zp'(z)) \leq F_{h(U)}h(z), \tag{2.1}$$

implies

$$F_{p(U)}p(z) \leq F_{h(U)}h(z), \quad z \in U,$$

where

$$\begin{aligned}\psi(\mathbb{C}^2 \times U) &= \operatorname{supp}(\mathbb{C}^2 \times U, F_{\psi(\mathbb{C}^2\times U)}\psi(p(z), zp'(z))) \\ &= \{z \in \mathbb{C};\ 0 < F_{\psi(\mathbb{C}^2\times U)}\psi(p(z), zp'(z)) \leq 1\}, \\ h(U) = \operatorname{supp}\{U, F_{h(U)}h(z)) &= \{z \in \mathbb{C} :\ 0 < F_{h(U)}h(z) \leq 1\}.\end{aligned}$$

Proof. Without loss of generality we can assume that p and h satisfy the conditions of the theorem on the closed disc $\overline{U}$. If not, then we can replace $p(z)$ by $p_\rho(z) = p(\rho z)$, and $h(z)$ by $h_\rho(z) = h(\rho z)$, where $0 < \rho < 1$. These new functions satisfy the conditions of the theorem on $\overline{U}$. We would then prove that

$$F_{p_\rho(U)}p_\rho(z) \leq F_{p(U)}p(z), \text{ for all } 0 < \rho < 1.$$

By letting $\rho \to 1$, we obtain

$$F_{p(U)}p(z) \leq F_{h(U)}h(z), \quad z \in U.$$

In order to prove the theorem, we apply Lemma 1.8, and we show that $\psi \in \Psi_1[h_\rho, h_\rho]$, for all $\rho \in (0,1)$.

Suppose (a) and (b) are satisfied, but p is not fuzzy subordinate to h.

According to Lemma 1.11, there are points $z_0 \in U$ and $\zeta_0 \in \partial U$, and $m \geq 1$, with $p(z_0) = h(\zeta_0)$, $z_0p'(z_0) = m\zeta_0 q'(\zeta_0)$ such that

$$\begin{aligned}\psi_0 &= \psi(p(z_0), zp'(z_0)) \\ &= p(z_0) + z_0p'(z_0) \cdot \phi[p(z_0)] = h(\zeta_0) + m\zeta_0 h'(\zeta_0) \cdot \phi[h(\zeta_0)].\end{aligned} \tag{2.2}$$

From (2.2), we have

$$\psi_0 = h(\zeta_0) + m\zeta_0 h'(\zeta_0)\phi[h(\zeta_0)], \quad \zeta_0 \in \partial U,\ |\zeta_0| = 1,\ m \geq 1 \tag{2.3}$$

which gives

$$\frac{\rho_0 - h(\zeta_0)}{\zeta_0 h'(\zeta_0)} = m\phi[h(\zeta_0)]. \tag{2.4}$$

Using the conditions from the hypothesis of the theorem, we have:

$$\operatorname{Re} \frac{\psi_0 - h(\zeta_0)}{\zeta_0 h'(\zeta_0)} = \operatorname{Re} m\phi[h(\zeta_0)] > 0, \tag{2.5}$$

which implies

$$\left| \arg \frac{\psi_0 - h(\zeta_0)}{\zeta_0 h(\zeta_0)} \right| < \frac{\pi}{2}$$

which is equivalent to

$$|\arg[\psi_0 - h(\zeta_0)] - \arg[\zeta_0 h'(\zeta_0)]| < \frac{\pi}{2}. \tag{2.6}$$

Since $\zeta_0 h'_\rho(\zeta_0)$ is the outer normal at the border of the convex domain $h_\rho(U)$ at $h_\rho(\zeta_0)$, from (2.6) we get that $\psi_0 \notin h_\rho(U)$ which means

$$F_{h(U)}\psi(p(z_0), z_0 p'(z_0), z_0) = F_{h(U)}\psi(h(\zeta_0), m\zeta_0 h'(\zeta_0), z_0) = 0. \tag{2.7}$$

Using Definition 1.5, from (2.7) we have $\psi \in \Psi[h(U), h]$.

Using condition (i) from Lemma 1.8, we have

$$F_{p(U)}p(z) \leq F_{q(U)}q(z), \quad z \in U.$$

By carefully selecting the function ϕ we obtain the following corollaries.

If we let $\phi(w) = \beta w + r$ is Theorem 2.1 we obtain the following corollary:

Corollary 2.2. *Let β and γ be complex numbers with $\beta \neq 0$ and let β and h be analytic in U with $h(0) = p(0)$. If*

$$Q(z) = \beta h(z) + \gamma$$

satisfies

(a) $\operatorname{Re} Q(z) = \operatorname{Re}[\beta h(z) + \gamma] > 0,$

and

(b) p is convex

then

$$F_{\psi(\mathbb{C}^2 \times U)}[p(z) + zp'(z)(\beta p(z) + \gamma)] \leq F_{h(U)}h(z)$$

implies that

$$F_{p(U)}p(z) \leq F_{h(U)}h(z), \quad z \in U.$$

If we let $\phi(w) = \dfrac{1}{\beta w + \gamma}$ in Theorem 2.1, we obtain the following corollary:

Corollary 2.3. *Let β and γ be complex numbers with $\beta \neq 0$, and let p and h be analytic in U with $h(0) = p(0)$.*

If $Q(z) = \beta h(z) + \gamma$ satisfies

a) $\operatorname{Re} Q(z) > 0,\ z \in U$

and

b) Q is convex
then

$$F_{\psi(\mathbb{C}^2\times U)}\left[p(z)+\frac{zp'(z)}{\beta p(z)+\gamma}\right]\leq F_{h(U)}h(z),$$

implies

$$F_{p(U)}p(z)\leq F_{h(U)}h(z),\quad z\in U.$$

If we let $\phi(w)=\dfrac{1}{(\beta w+\gamma)^2}$ in Theorem 2.1, we obtain the following corollary:

Corollary 2.4. *Let β and γ be complex numbers with $\beta\neq 0$ and let p and h be analytic in U with $h(0)=p(0)$.*

If $Q(z)=\beta h(z)+\gamma$ satisfies
(i) $\operatorname{Re} Q^2(z)>0,\ z\in U$ and
(ii) Q is convex, then

$$F_{\psi(\mathbb{C}^2\times U)}\left[p(z)+\frac{zp'(z)}{(\beta p(z)+\gamma)^2}\right]\leq F_{h(U)}h(z),$$

implies

$$F_{p(U)}p(z)\leq F_{q(U)}q(z),\quad z\in U.$$

Theorem 2.5. *Let h be convex in U and let $P:U\to\mathbb{C}$, with $\operatorname{Re}P(z)>0$. If p is analytic in U and $\psi:\mathbb{C}^2\times U\to\mathbb{C}$,*

$$\psi(p(z),zp'(z))=p(z)+P(z)zp'(z),\tag{2.8}$$

is analytic in U, then

$$F_{\psi(\mathbb{C}^2\times U)}[p(z)+P(z)zp'(z)]\leq F_{h(U)}h(z),\tag{2.9}$$

implies

$$F_{p(U)}p(z)\leq F_{h(U)}h(z),\quad z\in U.$$

Proof. We next show that ψ is a proper admissible function. It seems like we can use Lemma 1.8 with $q=h$, and show that $\psi\in\Psi[h,h]$.

Unfortunately, we do not know the specific boundary behavior of h and thus cannot use this result. Instead we require the use of the limiting form of the theorem as given in part (ii) of Lemma 1.8. We only need to show that $\psi\in\Psi[h_\rho,h_\rho]$ for $0<\rho<1$, where $h_\rho(z)=h(\rho z)$. In this case the admissibility condition (1.1) reduces to showing

$$\psi_0=\psi[h_\rho(\zeta),m\zeta_0h'_\rho(\zeta_0);z]=h_\rho(\zeta_0)+mP(z)\zeta_0h'_\rho(\zeta)\notin h_\rho(U),\tag{2.10}$$

where $|\zeta_0|=1$, $z\in U$, and $m\geq 1$.

From (2.10), we have

$$\lambda=\frac{\psi_0-h_\rho(\zeta_0)}{\zeta_0h_\rho(\zeta_0)}=mP(z),\quad z\in U.\tag{2.11}$$

From $\operatorname{Re}P(z)>0$, $m\geq n$, we obtain

$$\operatorname{Re}\lambda=\operatorname{Re}mP(z)>0,$$

which gives

$$\left|\arg \frac{\psi_0 - h_\rho(\zeta_0)}{\zeta h'(\zeta_0)}\right| = |\arg mP(z)| < \frac{\pi}{2}. \tag{2.12}$$

Since $h_\rho(U)$ is convex, $h_\rho(\zeta_0) \in h_\rho(\partial U)$, and $\zeta_0 h'_\rho(\zeta_0)$ is the outer normal to $h_\rho(\partial U)$ at $h_\rho(\zeta_0)$, from (2.12) we conclude that $\psi_0 \notin h_\rho(U)$, which gives

$$F_{h_\rho(U)}[h_\rho(\zeta_0) + P(z)m\zeta_0 h'_\rho(\zeta_0); z] = 0, \tag{2.13}$$

and using Definition 1.5, we have

$$\psi \in \Psi_n[h_\rho(U), h_\rho].$$

Using Lemma 1.8, we deduce

$$F_{p(U)}p(z) \leq F_{q(U)}q(z), \quad z \in U.$$

Theorem 2.6. (Hallenbeck and Ruscheweyh) *Let h be a convex function with $h(0) = a$, and let $\gamma \in \mathbb{C}^*$ be a complex number with $\operatorname{Re}\gamma \geq 0$. If $p \in \mathcal{H}[a, n]$ with $p(0) = a$ and*

$$\psi : \mathbb{C}^2 \times U \to \mathbb{C}, \quad \psi(p(z), zp'(z)) = p(z) + \frac{1}{\gamma}zp'(z)$$

is analytic in U, then

$$F_{\psi(\mathbb{C}^2\times U)}\left[p(z) + \frac{1}{\gamma}zp'(z)\right] \leq F_{h(U)}h(z), \tag{2.14}$$

implies

$$F_{p(U)}p(z) \leq F_{q(U)}q(z) \leq F_{h(U)}h(z), \quad z \in U,$$

where

$$q(z) = \frac{\gamma}{nz^{\gamma/n}} \int_0^z h(t)t^{\frac{\gamma}{n}-1}dt. \tag{2.15}$$

The function q is convex and is the fuzzy best (a, n)-dominant.

Proof. We can apply Theorem 2.5. From (2.14) we obtain

$$F_{p(U)}p(z) \leq F_{h(U)}h(z), \quad z \in U. \tag{2.16}$$

The integral given by (2.15), with the exception of a different normations $q(0) = a$ has the form

$$q(z) = \frac{\gamma}{nz^{\gamma/n}} \int_0^z h(t)t^{\frac{\gamma}{n}-1}dt = \frac{\gamma}{nz^{\gamma/n}} \int_0^z (a + a_n t^n + \ldots)t^{\frac{\gamma}{n}-1}dt$$

$$= a + \frac{a_n}{\frac{\gamma}{n} + n}z^n + \ldots, \quad z \in U,$$

which gives $q \in \mathcal{H}[a, n]$.

Since h is convex and $\operatorname{Re}\frac{\gamma}{n} \geq 0$, we deduce from part (ii) of Lemma 1.10 that q is convex and univalent.

A simple calculation shows that q also satisfies the differential equation

$$q(z) + \frac{nz}{\gamma}zq'(z) = h(z) = \psi[q(z), zq'(z)], \quad z \in U. \tag{2.17}$$

Since q is the univalent solution of the differential equation (2.17) associated with (2.14), we can prove that it is the best dominant by applying Lemma 1.9. Without loss of generality, we can assume that h and q are analytic and univalent on $\overline{U}$, and $q'(\zeta) \neq 0$ for $|\zeta| = 1$. If not, then we could replace h with $h_\rho(z) = h(\rho z)$, and q with $q_\rho(z) = q(\rho z)$.

These new functions would then have the desired properties and we would prove the theorem using part (iii) of Lemma 1.9.

With our assumption, we will use part (i) of Lemma 1.9 and so we only need to show that $\psi \in \Psi_n[h, q]$. This is equivalent to showing that

$$\psi_0 = \psi(q(\zeta), m\zeta q'(\zeta)) = q(\zeta) + \frac{m\zeta q'(\zeta)}{\gamma} \notin h(U) \tag{2.18}$$

when $|\zeta| = 1$, $z \in U$ and $m \geq n$.

From (2.17) we obtain

$$\psi_0 = q(\zeta) + \frac{m}{n}[h(\zeta) - q(\zeta)].$$

Since $h(U)$ is a convex domain, and

$$H_{q(U)}q(z) \leq F_{h(U)}h(z), \quad z \in U,$$

and $\dfrac{m}{n} \geq 1$, we conclude that $\psi_0 \notin h(U)$, which implies

$$F_{h(U)}\psi(q(\zeta), m\zeta q'(\zeta); z) = 0.$$

Using Definition 1.5, from condition (1.1) we get

$$\psi \in \Psi_n[h(U), q].$$

Using Lemma 1.9, from condition (i) we obtain

$$F_{p(U)}p(z) \leq F_{q(U)}q(z), \quad z \in U.$$

Therefore, q is the fuzzy best (a, n)-dominant.

Theorem 2.7. *Let q be a convex function in U and let the function*

$$h(z) = q(z) + n\alpha z q'(z), \tag{2.19}$$

where $\alpha > 0$ and $n \in \mathbb{N}^*$.

If the function $p \in H[q(0), n]$, and $\psi : \mathbb{C}^2 \times U \to \mathbb{C}$,

$$\psi(p(z), zp'(z)) = p(z) + \alpha z p'(z)$$

is analytic in U, then

$$F_{\psi(\mathbb{C}^2 \times U)}[p(z) + \alpha n z p'(z)] \leq F_{h(U)}h(z), \tag{2.20}$$

implies

$$F_{p(U)}p(z) \leq F_{q(U)}q(z), \quad z \in U$$

and q is fuzzy best $(q(0), n)$-dominant.

Proof. Step I. We prove that function h is univalent.

Differentiating (2.19), we have

$$h'(z) = q'(z) + n\alpha[q'(z) + zq''(z)] \tag{2.21}$$

which gives

$$\frac{h'(z)}{q'(z)} = 1 + n\alpha\left[1 + \frac{zq''(z)}{q'(z)}\right], \quad z \in U. \tag{2.22}$$

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Some properties on generalized close-to-star functions

Hatice Esra Ozkan

Abstract. Let $f(z) = a_1 z + a_1 z^2 + \cdots, a_1 \neq 0$, be regular in $|z| < 1$ and have there no zeros except at the origin. Reade ([3]) and the Sakaguchi ([2]) showed that a necessary and sufficient condition for $f(z)$ to be a member of the class $C(k)$ is that $f(z)$ has a representation of the form

$$f(z) = s(z)(p(z))^k$$

where $s(z)$ is a regular function starlike with respect to the origin for $|z| < 1$, k is a positive constant, and $p(z)$ is a regular function with positive real part in $|z| < 1$. The class of close-to-star functions introduced by Reade ([4]) is equivalent to $C(1)$. In this paper we define the class $C(k, A, B)$ $(-1 \leq B < A \leq 1, k$ is positive constant) which contains the functions of the form

$$f(z) = s(z)(p(z))^k$$

where $s(z)$ is a regular Janowski starlike function, and $p(z)$ is a regular function with positive real part in $|z| < 1$. The aim of this paper is to give some properties and distortion theorems for this class.

Mathematics Subject Classification (2010): 30C45.

Keywords: Distortion theorem, radius of starlikeness.

1. Introduction

Let Ω be the family of functions $\phi(z)$ regular in $\mathbb{D} = \{z \quad | \quad |z| < 1\}$ and satisfying the conditions $\phi(0) = 0$, $|\phi(z)| < 1$ for all $z \in \mathbb{D}$.

The set $\mathcal{P}$ is the set of all functions of the form

$$f(z) = 1 + p_1 z + p_2 z^2 + \cdots = 1 + \sum_{n=1}^{\infty} p_n z^n$$

that are regular in $\mathbb{D}$, and such that for $z \in \mathbb{D}$,

$$Re(f(z)) > 0.$$

Any function in $\mathcal{P}$ is called a function with positive real part in $\mathbb{D}$ ([1]).

Next, for arbitrary fixed numbers A, B, given by $-1 \leq B < A \leq 1$, we denote by $\mathcal{P}(A,B)$ the family of functions $p(z) = 1 + p_1 z + p_2 z^2 + \cdots$ regular in $\mathbb{D}$ and such that $p(z)$ is in $\mathcal{P}(A,B)$ if and only if

$$p(z) = \frac{1 + A\phi(z)}{1 + B\phi(z)}$$

for some function $\phi(z) \in \Omega$ and every $z \in \mathbb{D}$ ([5]).

Let $\mathcal{S}^*(A,B)$ denote the family of functions $s(z) = z + c_2 z^2 + c_3 z^3 + \cdots$ regular in $\mathbb{D}$, and such that $s(z)$ is in $\mathcal{S}^*(A,B)$ if and only if

$$z\frac{s'(z)}{s(z)} = p(z)$$

for some $p(z)$ is in $\mathcal{P}(A,B)$ and all $z \in \mathbb{D}$ ([5]).

Moreover, $F(z) = z + \alpha_2 z^2 + \alpha_3 z^3 + \cdots$ and $G(z) = z + \beta_2 z^2 + \beta_3 z^3 + \cdots$ are analytic functions in $\mathbb{D}$, if there exist a function $\phi(z) \in \Omega$ such that $F(z) = G(\phi(z))$ for every $z \in \mathbb{D}$, then we say that $F(z)$ is subordinate to $G(z)$ and we write $F(z) \prec G(z)$ ([1]).

Finally, let $f(z) = z + a_2 z^2 + \cdots$ be analytic function in $\mathbb{D}$, if there exists a function $s(z) \in \mathcal{S}^*(A,B)$, such that

$$\frac{f(z)}{s(z)} = (p(z))^k$$

where $p(z) \in \mathcal{P}$, k is a positive constant, then the function is called the generalized close-to-star. The class of these functions is denoted by $C(k,A,B)$.

Lemma 1.1. [1] *Let $p(z)$ be an element of $\mathcal{P}$, then*

$$\frac{-2r}{1-r^2} \leq \left| z\frac{p'(z)}{p(z)} \right| \leq \frac{2r}{1-r^2} \tag{1.1}$$

$$\frac{-2r}{1-r^2} \leq Re\left(z\frac{p'(z)}{p(z)} \right) \leq \frac{2r}{1-r^2}. \tag{1.2}$$

Lemma 1.2. [5] *Let $s(z)$ be an element of $\mathcal{S}^*(A,B), (-1 \leq B < A \leq 1)$, then*

$$\frac{1-Ar}{1-Br} \leq \left| z\frac{s'(z)}{s(z)} \right| \leq \frac{1+Ar}{1+Br} \tag{1.3}$$

$$\frac{1-Ar}{1-Br} \leq Re\left(z\frac{s'(z)}{s(z)} \right) \leq \frac{1+Ar}{1+Br}. \tag{1.4}$$

2. Main Results

Theorem 2.1. *Let $f(z)$ be an element of $C(k,A,B)$, $(-1 \leq B < A \leq 1$, k is positive constant$)$, then*

$$\left| \frac{-2kr}{1-r^2} + \frac{1-Ar}{1-Br} \right| \leq \left| z\frac{f'(z)}{f(z)} \right| \leq \frac{2kr}{1-r^2} + \frac{1+Ar}{1+Br}. \tag{2.1}$$

Proof. If $f(z)$ be an element of $C(k,A,B)$, then we write

$$f(z) = s(z)(p(z))^k.$$

If we take the logarithmic derivative of the last equality, then we have

$$z\frac{f'(z)}{f(z)} = z\frac{s'(z)}{s(z)} + kz\frac{p'(z)}{p(z)}, \tag{2.2}$$

by applying triangle inequality for the equality (2.2), we obtain

$$\left|z\frac{s'(z)}{s(z)}\right| - k\left|z\frac{p'(z)}{p(z)}\right| \le \left|z\frac{f'(z)}{f(z)}\right| \le \left|z\frac{s'(z)}{s(z)}\right| + k\left|z\frac{p'(z)}{p(z)}\right|, \tag{2.3}$$

using lemma 1.1 in the inequality (2.3), then we obtain (2.1). □

Corollary 2.2. *For $A = 1$, $B = -1$, then*

$$\left|\frac{1-2(k+1)r+r^2}{1-r^2}\right| \le \left|z\frac{f'(z)}{f(z)}\right| \le \frac{1+2(k+1)r+r^2}{1-r^2}. \tag{2.4}$$

This result was obtained by Sakaguchi ([3]).

Corollary 2.3. *If $f(z) \in C(k,A,B)$ $(-1 \le B < A \le 1, k$ is positive constant), then*

$$\frac{1-(2k+A)r+(2kB-1)r^2+Ar^3}{(1-r^2)(1-Br)} \le Re\left(z\frac{f'(z)}{f(z)}\right) \le \frac{1+(2k+A)r+(2kB-1)r^2-Ar^3}{(1-r^2)(1+Br)}.$$

This inequality is simple consequence of inequality (2.1).

Corollary 2.4. [6] *The radius of starlikeness of the class $C(k,A,B)$ is the smallest positive root of the equations*

$$\psi(r) = 1-(2k+A)r+(2kB-1)r^2+Ar^3 = 0.$$

Proof. If $f(z) \in C(k,A,B)$, then we have

$$Re\left(z\frac{f'(z)}{f(z)}\right) \ge \frac{1-(2k+A)r+(2kB-1)r^2+Ar^3}{(1-r^2)(1-Br)} = \frac{\psi(r)}{(1-r^2)(1-Br)}. \tag{2.5}$$

The denominator of the expression on the right-hand side of the inequality (2.5) is positive for $0 \le r < 1$,

$$\psi(0) = 1,$$
$$\psi(1) = 1-(2k+A)+(2kB-1)+A = -2k+2kB = -2k(1-B) \le 0.$$

Thus the smallest positive root r_0 of the equation $\psi(r) = 0$ lies between 0 and 1.

Therefore the inequality $Re\left(z\frac{f'(z)}{f(z)}\right) > 0$ is valid $r = |z| = r_0$. Hence the radius of starlikeness for $C(k,A,B)$ is not less than r_0. Thus the corollary is proved. □

Theorem 2.5. *Let $f(z)$ be an element of $C(k,A,B)$ $(-1 \le B < A \le 1$, k is positive constant), then*

$$\frac{r(1-r)^k}{(1+r)^k(1-Br)^{\frac{B-A}{B}}} \le |f(z)| \le \frac{r(1+r)^k}{(1-r)^k(1+Br)^{\frac{B-A}{B}}}, \quad B \neq 0;$$

$$e^{-Ar} r \left(\frac{1-r}{1+r}\right)^k \le |f(z)| \le e^{Ar} r \left(\frac{1+r}{1-r}\right)^k, \quad B = 0.$$

Proof. Using corollary 2.3 and the equality

$$Re\left(z\frac{f'(z)}{f(z)}\right) = r\frac{\partial}{\partial r}\log|f(z)|$$

then we have

$$\frac{1-(2k+A)r+(2kB-1)r^2+Ar^3}{r(1-r^2)(1-Br)} \le \frac{\partial}{\partial r}\log|f(z)| \le \frac{1+(2k+A)r+(2kB-1)r^2-Ar^3}{r(1-r^2)(1+Br)}, \quad B \neq 0; \tag{2.6}$$

$$\frac{1-(2k+A)r-r^2+Ar^3}{r(1-r^2)} \le \frac{\partial}{\partial r}\log|f(z)| \le \frac{1+(2k+A)r-r^2-Ar^3}{r(1-r^2)}, \quad B = 0. \tag{2.7}$$

Integrating both sides of the inequalities (2.6) and (2.7) we get the results. □

Corollary 2.6. *Let $f(z)$ be an element of $C(k,A,B)$ $(-1 \le B < A \le 1$, k is positive constant), then*

$$\left(\frac{1-r}{1+r}\right)^k \frac{1}{(1-Br)^{\frac{B-A}{B}}} \left|-\frac{2kr}{1-r^2}+\frac{1-Ar}{1-Br}\right| \le |f'(z)| \le \left(\frac{1+r}{1-r}\right)^k \frac{1}{(1+Br)^{\frac{B-A}{B}}}\left[\frac{2kr}{1-r^2}+\frac{1+Ar}{1+Br}\right], B \neq 0;$$

$$\left(\frac{1-r}{1+r}\right)^k e^{-Ar}\left|-\frac{2kr}{1-r^2}+1-Ar\right| \le |f'(z)| \le \left(\frac{1+r}{1-r}\right)^k e^{Ar}\left[\frac{2kr}{1-r^2}+1+Ar\right], B = 0.$$

This corollary is simple consequence of theorem 2.1 *and theorem* 2.5.

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Growth and distortion theorem for the Janowski alpha-spirallike functions in the unit disc

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Abstract. Let A be the class of all analytic functions in the open unit disc $\mathbb{D} = \{z| \, |z| < 1\}$ of the form $f(z) = z + a_2z^2 + a_3z^3 + \cdots$. Let $g(z)$ be an element of A satisfying the condition

$$e^{i\alpha} z\frac{g'(z)}{g(z)} = \frac{1 + A\phi(z)}{1 + B\phi(z)}$$

where $|\alpha| < \frac{\pi}{2}$, $-1 \leq B < A \leq 1$ and $\phi(z)$ is analytic in $\mathbb{D}$ and satisfies the conditions $\phi(0) = 0$, $|\phi(z)| < 1$ for every $z \in \mathbb{D}$. Then $g(z)$ is called Janowski α-spirallike functions in the unit disc. The class of such functions is denoted by $S^*_\alpha(A, B)$. The aim of this paper is to give growth and distortion theorems for the class $S^*_\alpha(A, B)$.

Mathematics Subject Classification (2010): 30C45.

Keywords: Growth theorem, distortion theorem, radius of starlikeness.

1. Introduction

Let Ω be the family of functions $\phi(z)$ which are regular in the open unit disc $\mathbb{D}$ and satisfying the conditions $\phi(0) = 0$, $|\phi(z)| < 1$ for all $z \in \mathbb{D}$.

Next, for arbitrary fixed numbers A, B, $-1 \leq B < A \leq 1$, denote by $\mathcal{P}(A, B)$, the family of functions $p(z) = 1 + p_1z + p_2z^2 + \cdots$ regular in $\mathbb{D}$, such that $p(z)$ in $\mathcal{P}(A, B)$ if and only if

$$p(z) = \frac{1 + A\phi(z)}{1 - B\phi(z)} \tag{1.1}$$

for some function $\phi(z) \in \Omega$, and for all $z \in \mathbb{D}$. At the same time, this class can be represented by $Rep(z) > \frac{1-A}{1-B} > 0$.

Let $F(z) = z + \alpha_2z^2 + \alpha_3z^3 + \cdots$ and $G(z) = z + \beta_2z^2 + \beta_3z^3 + \cdots$ be analytic functions in $\mathbb{D}$. If there exists a function $\phi(z) \in \Omega$ such that $F(z) = G(\phi(z))$ for every $z \in \mathbb{D}$, then we say that $F(z)$ is subordinate to $G(z)$, and we write $F(z) \prec G(z)$. We also note that if $F(z) \prec G(z)$, then $F(\mathbb{D}) \subset G(\mathbb{D})$ ([1]).

Moreover, let $S^*_\alpha(A,B)$ denote the family of functions $f(z) = z + a_2z^2 + \cdots$ regular in $\mathbb{D}$, such that $f(z)$ is in $S^*_\alpha(A,B)$ if and only if there is a real number α for which,

$$e^{i\alpha} z\frac{f'(z)}{f(z)} = \cos\alpha p(z) + i\sin\alpha, |\alpha| < \frac{\pi}{2}, p(z) \in \mathcal{P}(A,B) \tag{1.2}$$

is true for every $z \in \mathbb{D}$. Then the class $S^*_\alpha(A,B)$ is called the Janowski α-spirallike functions.

The following lemma is due to I. S. Jack and plays very important role for our proof of Theorem 2.1 ([2]).

Lemma 1.1. *Let $\phi(z)$ be regular in the unit disc $\mathbb{D}$ with $\phi(0) = 0$. Then if $|\phi(z)|$ obtains its maximum value on the circle $|z| = r$ at the point z_1, one has $z_1\phi'(z_1) = k\phi(z_1)$, for some $k \geq 1$.*

2. Main results

Theorem 2.1.

$$f(z) \in S^*_\alpha(A,B) \Leftrightarrow (z\frac{f'(z)}{f(z)} - 1) \prec \begin{cases} \frac{e^{-i\alpha}(A-B)\cos\alpha.z}{1+Bz}; & B \neq 0, \\ e^{-i\alpha}(A\cos\alpha)z; & B = 0, \end{cases} \tag{2.1}$$

Proof. Let $f(z)$ be an element of $S^*_\alpha(A,B)$. We define the functions $\phi(z)$ by;

$$\frac{f(z)}{z} = \begin{cases} (1 + B\phi(z))^{\frac{(A-B)\cos\alpha e^{-i\alpha}}{B}}; & B \neq 0, \\ e^{A\cos\alpha e^{-i\alpha}\phi(z)}; & B = 0, \end{cases} \tag{2.2}$$

where $(1 + B\phi(z))^{\frac{(A-B)\cos\alpha e^{-i\alpha}}{B}}$ and $e^{A\cos\alpha e^{-i\alpha}\phi(z)}$ have the value 1 at $z = 0$. Then $\phi(z)$ is analytic and $\phi(0) = 0$. If we take the logarithmic derivative from (2.2) and after simple calculations, we get

$$(z\frac{f'(z)}{f(z)} - 1) = \begin{cases} \frac{(A-B)\cos\alpha e^{-i\alpha} z\phi'(z)}{1+B\phi(z)}; & B \neq 0, \\ A\cos\alpha.e^{-i\alpha}z\phi'(z); & B = 0, \end{cases} \tag{2.3}$$

We can easily conclude that this subordination is equivalent to $|\phi(z)| < 1$ for all $z \in \mathbb{D}$. On the contrary let's assume that there exists $z_1 \in \mathbb{D}$, such that $|\phi(z)|$ attains its maximum value on the circle $|z| = r$, that is $|\phi(z_1)| = 1$. Then when the conditions $z_1\phi'(z_1) = k\phi(z_1)$, $k \geq 1$ are satisfied for such $z_1 \in \mathbb{D}$ (Using I.S.Jack's Lemma), we obtain;

$$(z_1\frac{f'(z_1)}{f(z_1)} - 1) = \begin{cases} \frac{(A-B)\cos\alpha e^{-i\alpha}k\phi(z_1)}{1+B\phi(z_1)} = F_1(\phi(z_1)) \notin F_1(\mathbb{D}); & B \neq 0, \\ A\cos\alpha e^{-i\alpha}k\phi(z_1) = F_2(\phi(z_1)) \notin F_2(\mathbb{D}); & B = 0, \end{cases} \tag{2.4}$$

which contradicts (2.1) implying that the assumption is wrong , i.e.,
$|\phi(z)| < 1$ for all $z \in \mathbb{D}$. This shows that,

$$f(z) \in S^*_\alpha(A,B) \Rightarrow (z\frac{f'(z)}{f(z)} - 1) \prec \begin{cases} \frac{(A-B)\cos\alpha e^{-i\alpha}z}{1+Bz}; & B \neq 0, \\ A\cos\alpha.e^{-i\alpha}z; & B = 0, \end{cases} \tag{2.5}$$

Conversely,

$$(z\frac{f'(z)}{f(z)}-1)\prec\begin{cases}\frac{(A-B)\cos\alpha e^{-i\alpha}}{1+Bz}; & B\neq 0,\\ A\cos\alpha.e^{-i\alpha}z; & B=0,\end{cases}\Rightarrow$$

$$e^{i\alpha}z\frac{f'(z)}{f(z)}=\begin{cases}cos\alpha\frac{1+A\phi(z)}{1+B\phi(z)}+i\sin\alpha; & B\neq 0,\\ \cos\alpha(1+A\phi(z))+i\sin\alpha; & B=0,\end{cases}$$

This shows that $f(z)\in S^*_\alpha(A,B)$. □

Corollary 2.2. *Marx-Strohacker inequality for the class $S^*_\alpha(A,B)$ is;*

$$\begin{cases}\left|(\frac{f(z)}{z})^{\frac{B.e^{i\alpha}}{(A-B)\cos\alpha}}-1\right|<1; & B\neq 0,\\ \\ \left|\log(\frac{f(z)}{z})^{\frac{e^{i\alpha}}{A\cos\alpha}}\right|<1; & B=0,\end{cases} \tag{2.6}$$

Proof. The proof of this corollary is a simple consequence of Theorem 2.1. Indeed,

$$\frac{f(z)}{z}=(1+B\phi(z))^{\frac{A-B}{B}\cos\alpha e^{-i\alpha}}\Rightarrow\left|(\frac{f(z)}{z})^{\frac{B.e^{i\alpha}}{(A-B)\cos\alpha}}-1\right|<1$$

$$\frac{f(z)}{z}=e^{A\cos\alpha e^{-i\alpha}\phi(z)}\Rightarrow\left|\log(\frac{f(z)}{z})^{\frac{e^{i\alpha}}{A\cos\alpha}}\right|<1$$

□

Theorem 2.3. *The radius of starlikeness of the class $S^*_\alpha(A,B)$ is,*

$$r=\begin{cases}\frac{2}{(A-B)\cos\alpha+\sqrt{((A-B)^2\cos^2\alpha+4[AB\cos^2\alpha+B^2\sin^2\alpha])}}; & B\neq 0,\\ \frac{1}{A\cos\alpha}; & B=0,\end{cases} \tag{2.7}$$

This radius is sharp because the extremal function is;

$$f(z)=\begin{cases}z(1+Bz)^{\frac{A-B}{B}\cos\alpha e^{-i\alpha}}; & B\neq 0,\\ ze^{A\cos\alpha e^{-i\alpha}z}; & B=0,\end{cases} \tag{2.8}$$

with $\zeta=\frac{r(r-e^{i\alpha})}{1-re^{i\alpha}}$ and we obtain,

$$\zeta\frac{f'(\zeta)}{f(\zeta)}=\begin{cases}\frac{1-(A-B)\cos\alpha r-(AB\cos^2\alpha+B^2\sin^2\alpha)r^2}{1-B^2r^2}; & B\neq 0,\\ 1-Ar\cos\alpha; & B=0,\end{cases} \tag{2.9}$$

Proof. Using (1.2) we get;

$$p(z)=\frac{1}{\cos\alpha}(e^{i\alpha}z\frac{f'(z)}{f(z)}-i\sin\alpha) \tag{2.10}$$

On the other hand, since $p(z)\in\mathcal{P}(A,B)$, then we have,

$$\left|p(z)-\frac{1-ABr^2}{1-B^2r^2}\right|\leq\frac{(A-B)r}{1-B^2r^2} \tag{2.11}$$

The inequality (2.11) was obtained by W. Janowski [5]. Using (2.10) in (2.11) and after straightforward calculations we get:

$$\begin{cases} \frac{1-(A-B)\cos\alpha r-(AB\cos^2\alpha+B^2\sin^2\alpha)r^2}{1-B^2r^2} \leq Re z\frac{f'(z)}{f(z)} \\ \quad \leq \frac{1+(A-B)\cos\alpha r-(AB\cos^2\alpha+B^2\sin^2\alpha)r^2}{1-B^2r^2}; & B \neq 0, \\ 1-A\cos\alpha r \leq Re z\frac{f'(z)}{f(z)} \leq 1+A\cos\alpha r; & B=0, \end{cases} \tag{2.12}$$

The inequalities (2.12) shows that this theorem is true. □

Corollary 2.4. *If we take $A=1$, $B=-1$ we obtain,*

$$r = \frac{1}{\cos\alpha + |\sin\alpha|} \tag{2.13}$$

This is the radius of starlikeness of class of α-spirallike functions. This result was obtained independently and using different methods by both Robertson [4] and Libera [3]. We also note that if we give another special values to A and B, we obtain the radius of starlikeness of the subclass of α-spirallike functions.

Corollary 2.5. *Let $f(z)$ be an element of $S^*_\alpha(A,B)$, then*

$$r(1-Br)^{\frac{(A-B)\cos\alpha(\cos\alpha+1)}{2B}}(1+Br)^{\frac{(A-B)\cos\alpha(\cos\alpha-1)}{2B}} \leq |f(z)| \leq$$
$$r(1-Br)^{\frac{(A-B)\cos\alpha(\cos\alpha-1)}{2B}}(1+Br)^{\frac{(A-B)\cos\alpha(\cos\alpha+1)}{2B}}; B \neq 0,$$
$$re^{-(A\cos\alpha)r} \leq |f(z)| \leq re^{(A\cos\alpha)r}; B=0$$

Proof. Using (2.12),

$$Re(z\frac{f'(z)}{f(z)}) = r\frac{\partial}{\partial r}\log|f(z)|$$

and after the straightforward calculations we get the result. Also we note that these inequalities are sharp. Because the extremal function was given in Theorem 2.3. □

Corollary 2.6. *If $f(z) \in S^*_\alpha(A,B)$, then*

$$[(1-Ar)\cos\alpha-(1-Br)\sin\alpha]F(A,B,\cos\alpha,-r) \leq |f'(z)| \leq$$
$$[(1+Ar)\cos\alpha+(1+Br)\sin\alpha]F(A,B,\cos\alpha,r)$$

where

$$F(A,B,\cos\alpha,r) = (1+Br)^{\frac{(A-B)\cos\alpha(\cos\alpha+1)}{2B}-1}(1-Br)^{\frac{(A-B)\cos\alpha(\cos\alpha-1)}{2B}}$$

This inequality is sharp.

Proof. The proof of this corollary is based on the following observations

i. $p(z) \in P(A,B) \Rightarrow \frac{1-Ar}{1-Br} \leq |p(z)| \leq \frac{1+Ar}{1+Br}$

ii. $p(z) = \frac{1}{\cos\alpha}(e^{i\alpha}z\frac{f'(z)}{f(z)} - i\sin\alpha)$, $f(z) \in S^*_\alpha(A,B), p(z) \in P(A,B)$

iii. Corollary 2.5. using (i) and (ii) and after simple calculations we get:

$$\frac{(1-Ar)\cos\alpha-(1-Br)\sin\alpha}{1-Br} \leq \left|z\frac{f'(z)}{f(z)}\right| \leq \frac{(1+Ar)\cos\alpha+(1+Br)\sin\alpha}{1+Br} \tag{2.14}$$

Considering (2.14) and Corollary 2.5 we get the result. □

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A generalization of Goluzin's univalence criterion

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Abstract. In this paper we obtain a sufficient condition for univalence and quasiconformal extension of an analytic function, which generalizes the well known condition for univalency established by G. M. Goluzin.

Mathematics Subject Classification (2010): 30C45.

Keywords: Loewner chain, analytic functions, univalence criteria.

1. Introduction

Let $\mathcal{U}_r = \{z \in \mathbb{C} : |z| < r\}$ $0 < r \leq 1$ be the disk of radius r centered at 0 and let $\mathcal{U} = \mathcal{U}_1$ be the unit disk.

Denote by $\mathcal{A}$ the class of analytic functions in $\mathcal{U}$ which satisfy the usual normalization $f(0) = f'(0) - 1 = 0$.

During the time many criteria which guarantee the univalence of a function in $\mathcal{A}$ have been obtained. Some of these univalence criteria (see [11], [12], [16], [17]) involve the expression $z^2 f'(z)/f^2(z) - 1$.

In this paper we obtain a generalization of a univalence criterion due to Goluzin (see [7]) in which the logarithmic derivative of $z^2 f'(z)/f^2(z)$ is contained.

2. Loewner chains and quasiconformal extensions

Before proving our main result we need a brief summary of theory of Loewner chains.

A function $L(z,t) : \mathcal{U} \times [0,\infty) \to \mathbb{C}$ is said to be a *Loewner chain* or a *subordination chain* if:

(i) $L(z,t)$ is analytic and univalent in $\mathcal{U}$ for all $t \geq 0$.

(ii) $L(z,t) \prec L(z,s)$ for all $0 \leq t \leq s < \infty$, where the symbol " $\prec$ " stands for subordination.

The following result due to Pommerenke is often used to obtain univalence criteria.

Theorem 2.1. *([14], [15]) Let $L(z,t) = a_1(t)z + \dots$ be an analytic function in $\mathcal{U}_r$ $(0 < r \le 1)$ for all $t \ge 0$. Suppose that:*

(i) *$L(z,t)$ is a locally absolutely continuous function of $t \in [0,\infty)$, locally uniform with respect to $z \in \mathcal{U}_r$.*

(ii) *$a_1(t)$ is a complex valued continuous function on $[0,\infty)$ such that $a_1(t) \neq 0$, $\lim_{t\to\infty} |a_1(t)| = \infty$ and*

$$\left\{ \frac{L(z,t)}{a_1(t)} \right\}_{t\ge 0}$$

is a normal family of functions in $\mathcal{U}_r$.

(iii) *There exists an analytic function $p : \mathcal{U}_r \times [0,\infty) \to \mathbb{C}$ satisfying $\Re p(z,t) > 0$ for all $(z,t) \in \mathcal{U} \times [0,\infty)$ and*

$$z\frac{\partial L(z,t)}{\partial z} = p(z,t)\frac{\partial L(z,t)}{\partial t} \ , z \in \mathcal{U}_r \ , a.e \ \ t \ge 0. \tag{2.1}$$

Then, for each $t \ge 0$, the function $L(z,t)$ has an analytic and univalent extension to the whole disk $\mathcal{U}$, i.e $L(z,t)$ is a Loewner chain.

Let k be a constant in $[0,1)$. Recall that a homeomorphism f of $G \subset \mathbb{C}$ is said to be k-*quasiconformal* if $\partial_z f$ and $\partial_{\bar z} f$ are locally integrable on G and satisfy $|\partial_{\bar z} f| \le k|\partial_z f|$ almost everywhere in G.

The method of constructing quasiconformal extension criteria is based on the following result due to Becker (see [3], [4] and also [5]).

Theorem 2.2. *Supose that $L(z,t)$ is a subordination chain. Consider*

$$w(z,t) = \frac{p(z,t) - 1}{p(z,t) + 1} \ , \ z \in \mathcal{U} \ , \ t \ge 0$$

where $p(z,t)$ is defined by (2.1). If

$$|w(z,t)| \le k, \ 0 \le k < 1$$

for all $z \in \mathcal{U}$ and $t \ge 0$, then $L(z,t)$ admits a continuous extension to $\bar{\mathcal{U}}$ for each $t \ge 0$ and the function $F(z,\bar z)$ defined by

$$F(z,\bar z) = \begin{cases} L(z,0) & , \text{if} \quad |z| < 1 \\ L\left(\dfrac{z}{|z|}, \log|z|\right) & , \text{if} \quad |z| \ge 1. \end{cases}$$

is a k-quasiconformal extension of $L(z,0)$ to $\mathbb{C}$.

Examples of quasiconformal extension criteria can be found in [1], [2], [13] and more recently in [8], [9], [10].

3. Univalence criterion

In this section making use of Theorem 2.1 we obtain a univalence criterion for analytic functions in $\mathcal{U}$ which involves the logarithmic derivative of $z^2 f'(z)/f^2(z)$.

Theorem 3.1. *Let $f \in \mathcal{A}$ and let m be a positive real number. If*

$$\left|(1-|z|^{m+1})z\frac{d}{dz}\left(\log\frac{z^2f'(z)}{f^2(z)}\right)-\frac{m-1}{2}|z|^{m+1}\right| \leq \frac{m+1}{2}|z|^{m+1} \tag{3.1}$$

for all $z \in \mathcal{U}$, then the function f is univalent in the unit disk.

Proof. Let a be a positive real number and let the function $h(z,t)$ defined by

$$h(z,t) = 1-(e^{mat}-e^{-at})z\left[\frac{f'(e^{-at}z)}{f(e^{-at}z)}-\frac{e^{at}}{z}\right].$$

For all $t \geq 0$ and $z \in \mathcal{U}$ we have $e^{-at}z \in \mathcal{U}$ and from the analyticity of f in $\mathcal{U}$ it follows that $h(z,t)$ is also analytic in $\mathcal{U}$. Since $h(0,t)=1$, there exists a disk $\mathcal{U}_{r_1}$, $0<r_1<1$ in which $h(z,t)\neq 0$ for all $t\geq 0$. Then the function $L(z,t)$ defined by

$$L(z,t) = f(e^{-at}z)+\frac{(e^{mat}-e^{-at})zf'(e^{-at}z)}{1-(e^{mat}-e^{-at})z\left[\dfrac{f'(e^{-at}z)}{f(e^{-at}z)}-\dfrac{e^{at}}{z}\right]} \tag{3.2}$$

is analytic in $\mathcal{U}_{r_1}$, for all $t \geq 0$. If $L(z,t) = a_1(t)z + a_2(t)z^2 + \dots$ is Taylor expansion of $L(z,t)$ in $\mathcal{U}_{r_1}$, then it can be checked that we have $a_1(t) = e^{mat}$ and therefore $a_1(t) \neq 0$ for all $t \geq 0$ and $\lim_{t\to\infty}|a_1(t)| = \infty$.

From the analyticity of $L(z,t)$ in $\mathcal{U}_{r_1}$, it follows that there exists a number r_2, $0<r_2<r_1$, and a constant $K = K(r_2)$ such that

$$|\, L(z,t)/a_1(t)\,| < K, \qquad \forall z \in \mathcal{U}_{r_2}, \quad t \geq 0,$$

and thus $\{L(z,t)/a_1(t)\}$ is a normal family in $\mathcal{U}_{r_2}$. From the analyticity of $\partial L(z,t)/\partial t$, for all fixed numbers $T>0$ and r_3, $0<r_3<r_2$, there exists a constant $K_1>0$ (that depends on T and r_3) such that

$$\left|\frac{\partial L(z,t)}{\partial t}\right| < K_1, \qquad \forall z \in \mathcal{U}_{r_3}, \quad t\in[0,T].$$

It follows that the function $L(z,t)$ is locally absolutely continuous in $[0,\infty)$, locally uniform with respect to $z \in \mathcal{U}_{r_3}$. The function $p(z,t)$ defined by

$$p(z,t) = z\frac{\partial L(z,t)}{\partial z}\bigg/\frac{\partial L(z,t)}{\partial t}$$

is analytic in a disk $\mathcal{U}_r$, $0<r<r_3$, for all $t\geq 0$.

In order to prove that the function $p(z,t)$ is analytic and has positive real part in $\mathcal{U}$, we will show that the function

$$w(z,t) = \frac{p(z,t)-1}{p(z,t)+1}\ ,\ z\in\mathcal{U}_r\ ,\ t\geq 0$$

is analytic in $\mathcal{U}$ and

$$|w(z,t)|<1 \text{ for all } z\in\mathcal{U} \text{ and } t\geq 0. \tag{3.3}$$

Elementary calculation gives

$$w(z,t) = \frac{(1+a)\mathcal{G}(z,t)+1-ma}{(1-a)\mathcal{G}(z,t)+1+ma}, \tag{3.4}$$

where $\mathcal{G}(z,t)$ is given by

$$\mathcal{G}(z,t) = \left(e^{(m+1)at} - 1\right)\left[2 + \frac{e^{-at}zf''(e^{-at}z)}{f'(e^{-at}z)} - 2\frac{e^{-at}zf'(e^{-at}z)}{f(e^{-at}z)}\right]. \tag{3.5}$$

It is easy to prove that the condition (3.3) is equivalent to

$$\left|\mathcal{G}(z,t) - \frac{m-1}{2}\right| < \frac{m+1}{2} \quad \text{for all} \quad z \in \mathcal{U} \text{ and } t \geq 0. \tag{3.6}$$

For $t = 0$ and $z = 0$ the inequality (3.6) becomes

$$\left|\mathcal{G}(z,0) - \frac{m-1}{2}\right| = \left|\mathcal{G}(0,t) - \frac{m-1}{2}\right| = \left|\frac{m-1}{2}\right| < \frac{m+1}{2}. \tag{3.7}$$

Since m is a positive number, the last inequality holds true.

Let t be a fixed number, $t > 0$ and let $z \in \mathcal{U}$, $z \neq 0$. Since $|e^{-at}z| \leq e^{-at} < 1$ for all $z \in \overline{\mathcal{U}} = \{z \in \mathbb{C} : |z| \leq 1\}$, from (3.1) we conclude that the function $\mathcal{G}(z,t)$ is analytic in $\overline{\mathcal{U}}$. Using the maximum modulus principle it follows that for each $t > 0$, arbitrary fixed, there exists $\theta = \theta(t) \in \mathbb{R}$ such that

$$|\mathcal{G}(z,t)| < \max_{|\xi|=1} |\mathcal{G}(\xi,t)| = |\mathcal{G}(e^{i\theta},t)|. \tag{3.8}$$

Denote $u = e^{-at} \cdot e^{i\theta}$. Then $|u| = e^{-at} < 1$, $e^{(m+1)at} = 1/|u|^{m+1}$ and therefore

$$\mathcal{G}(e^{i\theta},t) = \left(\frac{1}{|u|^{m+1}} - 1\right) \cdot u \cdot \left[\frac{2}{u} + \frac{f''(u)}{f'(u)} - 2\frac{f'(u)}{f(u)}\right]$$

$$= \left(\frac{1}{|u|^{m+1}} - 1\right) \cdot u \frac{d}{du}\left(\log \frac{u^2 f'(u)}{f^2(u)}\right)$$

Since $u \in \mathcal{U}$, the inequality (3.1) implies

$$\left| \mathcal{G}(e^{i\theta},t) - \frac{m-1}{2} \right| \leq \frac{m+1}{2} \tag{3.9}$$

From (3.7), (3.8) and (3.9) we conclude that the inequality (3.6) holds true for all $z \in \mathcal{U}$ and $t \geq 0$. It follows that $L(z,t)$ is a Loewner chain and hence the function $L(z,0) = f(z)$ is univalent in $\mathcal{U}$. □

Remark 3.2. For $m = 1$ Theorem 3.1 specializes to a univalent criterion due to Goluzin [7].

The function $f(z) = \dfrac{z}{1+cz}$, $z \in \mathcal{U}$ satisfies the condition (3.1) of the Theorem 3.1 for all positive real numbers m and all complex numbers c.

4. Quasiconformal extension

In this section we obtain a quasiconformal extension of the univalence condition given in Theorem 3.1.

Theorem 4.1. *Let $f \in \mathcal{A}$. Let also $m \in \mathbb{R}_+$ and $k \in [0,1)$. If the inequality*

$$\left|(1-|z|^{m+1})z\frac{d}{dz}\left(\log\frac{z^2f'(z)}{f^2(z)}\right)-\frac{m-1}{2}|z|^{m+1}\right| \le k\frac{m+1}{2}|z|^{m+1} \tag{4.1}$$

is true for all $z \in \mathcal{U}$ then, the function f has an l-quasiconformal extension to $\mathbb{C}$, where

$$l = \frac{(1-a)^2+k|1-a^2|}{|1-a^2|+k(1-a)^2} \quad \textit{and } a > 0.$$

Proof. In the proof of Theorem 3.1 has been shown that the function $L(z,t)$ given by (3.2) is a subordination chain in $\mathcal{U}$. Applying Theorem 2.2 to the function $w(z,t)$ given by (3.4), we obtain that the condition

$$\left|\frac{(1+a)\mathcal{G}(z,t)+1-ma}{(1-a)\mathcal{G}(z,t)+1+ma}\right| < l,\ z \in \mathcal{U},\ t \ge 0 \text{ and } l \in [0,1) \tag{4.2}$$

where $\mathcal{G}(z,t)$ is defined by (3.5), implies l-quasiconformal extensibility of f.

Lenghty but elementary calculation shows that the last inequality (4.2) is equaivalent to

$$\left|\mathcal{G}(z,t)-\frac{a(1+l^2)(m-1)+(1-l^2)(ma^2-1)}{2a(1+l^2)+(1-l^2)(1+a^2)}\right| \le \frac{2al(1+m)}{2a(1+l^2)+(1-l^2)(1+a^2)}. \tag{4.3}$$

It is easy to check that, under the assumption (4.1) we have

$$\left|\mathcal{G}(z,t)-\frac{m-1}{2}\right| \le k\frac{m+1}{2}. \tag{4.4}$$

Consider the two disks Δ and Δ' defined by (4.3) and (4.4) respectively, where $\mathcal{G}(z,t)$ is replaced by a complex variable ζ. Our theorem will be proved if we find the smalest $l \in [0,1)$ for which Δ' is contained in Δ. This will be so if and only if the distance apart of the centers plus the smalest radius is equal, at most, to the largest radius. So, we are required to prove that

$$\left|\frac{a(1+l^2)(m-1)+(1-l^2)(ma^2-1)}{2a(1+l^2)+(1-l^2)(1+a^2)}-\frac{m-1}{2}\right|+k\frac{m+1}{2}$$

$$\le \frac{2al(1+m)}{2a(1+l^2)+(1-l^2)(1+a^2)}$$

or equivalently

$$\frac{(1-l^2)|1-a^2|}{2[2a(1+l^2)+(1-l^2)(1+a^2)]} \le \frac{2al}{2a(1+l^2)+(1-l^2)(1+a^2)}-\frac{k}{2} \tag{4.5}$$

with the condition

$$\frac{2al}{2a(1+l^2)+(1-l^2)(1+a^2)}-\frac{k}{2} \ge 0. \tag{4.6}$$

We will solve inequalities (4.5) and (4.6) for $1-a^2>0$. In a similar way they can be solved for $1-a^2<0$.

If in (4.5) the inequality sign is replaced by equal we obtain the following two solutions:

$$L_1 = \frac{(1-a)^2 + k(1-a^2)}{1-a^2+k(1-a)^2}\ ,\ L_2 = -\frac{(1+a)^2+k(1-a^2)}{1-a^2+k(1-a)^2}.$$

Therefore, the solution of inequality (4.5) is $l \leq L_2$ and $L_1 \leq l$. Since $L_2 < 0$ it remains $L_1 \leq l$.

After similar calculations, from inequality (4.6), we have $l \leq \mathcal{L}_2$ and $\mathcal{L}_1 \leq l$, where

$$\mathcal{L}_1 = \frac{-2a + \sqrt{4a^2 + (1-a^2)^2k^2}}{k(1-a)^2}\ ,\ \mathcal{L}_2 = \frac{-2a - \sqrt{4a^2+(1-a^2)^2k^2}}{k(1-a)^2}.$$

Since $\mathcal{L}_2 < 0$, we get $\mathcal{L}_1 \leq l$.

It can be checked that $\mathcal{L}_1 \leq L_1$. It follows $L_1 \leq l < 1$ and thus the proof is complete. □

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Relations between two kinds of derivatives on analytic functions II

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Abstract. We consider Ruscheweyh derivative $D^n f(z)$ and Sălăgean derivative $d^n f(z)$, for $n \in \{0, 1, 2, \ldots\}$, on a class

$$\mathcal{T} = \{f : f(z) = z - \sum_{n=2}^{\infty} a_n z^n \ (a_n \geqq 0,\ n = 2, 3, \ldots) \text{ is analytic in } |z| < 1\}.$$

On this paper we study relations between two subclasses $\mathcal{TR}(n, m; \alpha)$ and $\mathcal{TS}(n, m; \beta)$ of $\mathcal{T}$, where $\mathcal{TR}(n, m; \alpha) = \left\{ f \in \mathcal{T} : \mathrm{Re}\dfrac{D^{n+m} f(z)}{D^n f(z)} > \alpha, |z| < 1 \right\}$ and $\mathcal{TS}(n, m; \beta) = \left\{ f \in \mathcal{T} : \mathrm{Re}\dfrac{d^{n+m} f(z)}{d^n f(z)} > \beta, |z| < 1 \right\}$ for $\alpha \in [0, 1)$, $\beta \in [0, 1)$, $n = 0, 1, 2, \ldots$ and $m = 1, 2, \ldots$.

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1. Introduction

We consider a class of functions $f(z)$ defined by

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

analytic in the unit open disk $|z| < 1$, and we denote by $\mathcal{A}$ the class of such functions. We also denote by $\mathcal{T}$ a subclass of the class $\mathcal{A}$ satisfying

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n \qquad (a_n \geqq 0,\ n \in \mathcal{N}_2 = \mathcal{N} - \{1\}), \tag{1.2}$$

where $\mathcal{N}$ is the set of positive integers. For any $\beta \in [0, 1) = \{x : 0 \leqq x < 1\}$ a function $f(z)$, which is in the class $\mathcal{A}$ and satisfies $\mathrm{Re}\left\{\dfrac{zf'(z)}{f(z)}\right\} > \beta$ in $|z| < 1$, is called starlike of order β and we denote by $\mathcal{S}^*(\beta)$ the class of such functions. We denote by $\mathcal{S}^*$ the class $\mathcal{S}^*(0)$.

We also consider two kinds of derivatives, namely Ruscheweyh derivative ([1]) D^n and Sălăgean derivative ([2]) d^n for $n \in \mathcal{N}_0 = \mathcal{N} \cup \{0\}$ by

$$D^0 f(z) = f(z), \quad D^1 f(z) = Df(z) = zf'(z), \quad D^n f(z) = \frac{z(z^{n-1}f(z))^{(n)}}{n!} \quad (n \in \mathcal{N}_2)$$

and

$$d^0 f(z) = f(z), \quad d^1 f(z) = df(z) = zf'(z), \quad d^n f(z) = d(d^{n-1}f(z)) \quad (n \in \mathcal{N}_2),$$

respectively.

For $n \in \mathcal{N}_0, m \in \mathcal{N}$ and $\beta \in [0,1)$, Sekine([4]) introduced the following class

$$\mathcal{S}(n,m;\beta) = \left\{ f \in \mathcal{A} : \mathrm{Re}\frac{d^{n+m}f(z)}{d^n f(z)} > \beta, |z| < 1 \right\} \tag{1.3}$$

as a subclass of the class $\mathcal{A}$. For $n \in \mathcal{N}_0$, $m \in \mathcal{N}$ and $\alpha \in [0,1)$, we([3]) introduced the following class

$$\mathcal{R}(n,m;\alpha) = \left\{ f \in \mathcal{A} : \mathrm{Re}\frac{D^{n+m}f(z)}{D^n f(z)} = \mathrm{Re}\frac{n!(z^{n+m-1}f(z))^{(n+m)}}{(n+m)!(z^{n-1}f(z))^{(n)}} > \alpha, |z| < 1 \right\} \tag{1.4}$$

as another subclass of the class $\mathcal{A}$. We express $\mathcal{R}(n,1;\frac{1}{2})$, $\mathcal{S}(n,1;0)$ and $\mathcal{S}(n,m;0)$ as $\mathcal{R}(n)$ ([1]), $\mathcal{S}(n)$ ([2]) and $\mathcal{S}(n,m)$, respectively. Next let $\mathcal{TR}(n,m;\alpha)$, $\mathcal{TS}(n,m;\beta)$, $\mathcal{TR}(n,m)$ and $\mathcal{TS}(n,m)$ denote the classes $\mathcal{T}\cap\mathcal{R}(n,m;\alpha)$, $\mathcal{T}\cap\mathcal{S}(n,m;\beta)$, $\mathcal{T}\cap\mathcal{R}(n,m)$ and $\mathcal{T}\cap\mathcal{S}(n,m)$, respectively.

In the papers ([6], [5]), we researched a relation among subclasses $\mathcal{TR}(n,m;\alpha)$ and $\mathcal{TS}(n,m;\beta)$, respectively. In this paper, we will discuss a relation among subclasses mixed with $\mathcal{TR}(n,m;\alpha)$ and $\mathcal{TS}(n,m;\beta)$.

2. Preliminaries

2.1. Fundamental results

In this subsection, we show some useful fundamental results to prove our main theorem.

Theorem 2.1. ([1]) *The relation $\mathcal{R}(n+1) \subset \mathcal{R}(n) \subset \mathcal{S}^*$ holds for all $n \in \mathcal{N}_0$.*

Theorem 2.2. ([2]) *The relation $\mathcal{S}(n+1) \subset \mathcal{S}(n) \subset \mathcal{S}^*$ holds for all $n \in \mathcal{N}_0$.*

Theorem 2.3. ([4]) *If $\displaystyle\sum_{k=2}^{\infty} \frac{k^n(k^m-\beta)}{1-\beta}|a_k| \leqq 1$ for $n \in \mathcal{N}_0, m \in \mathcal{N}, \beta \in [0,1)$ and $f \in \mathcal{A}$, then $f \in \mathcal{S}(n,m;\beta)$.*

Theorem 2.4. ([4]) *For $n \in \mathcal{N}_0, m \in \mathcal{N}, \beta \in [0,1)$ and $f \in \mathcal{T}$, we have that*

$$f \in \mathcal{TS}(n,m;\beta) \iff \sum_{k=2}^{\infty} \frac{k^n(k^m-\beta)}{1-\beta} a_k \leqq 1. \tag{2.1}$$

The following theorem is a result to indicate a sufficient condition for $f \in \mathcal{R}(n,m;\alpha)$.

Theorem 2.5. ([5]) *If for $n \in \mathcal{N}_0$, $m \in \mathcal{N}$, $\alpha \in [0,1)$ and $f \in \mathcal{A}$,*

$$\sum_{k=2}^{\infty} \frac{\begin{pmatrix} n+m+k-1 \\ k-1 \end{pmatrix} - \alpha \begin{pmatrix} n+k-1 \\ k-1 \end{pmatrix}}{1-\alpha} |a_k| \leqq 1,$$

where $\begin{pmatrix} a \\ 0 \end{pmatrix} = 1$ for $a \in \mathcal{N}$ and $\begin{pmatrix} a \\ b \end{pmatrix} = \dfrac{a(a-1) \times \ldots \times (a-b+1)}{b!}$ *for $a, b \in \mathcal{N}$ and $a \geqq b$, then $f \in \mathcal{R}(n,m;\alpha)$.*

The following theorem is a useful result to indicate a necessary and sufficient condition for $f \in \mathcal{TR}(n,m;\alpha)$.

Theorem 2.6. ([5]) *For $n \in \mathcal{N}_0$, $m \in \mathcal{N}$, $\alpha \in [0,1)$ and $f \in \mathcal{T}$, we have that*

$$f \in \mathcal{TR}(n,m;\alpha) \iff \sum_{k=2}^{\infty} \frac{\begin{pmatrix} n+m+k-1 \\ k-1 \end{pmatrix} - \alpha \begin{pmatrix} n+k-1 \\ k-1 \end{pmatrix}}{1-\alpha} a_k \leqq 1. \qquad (2.2)$$

We obtain the following corollary of Theorem (2.5) replacing m by 1.

Corollary 2.7. ([5]) *If* $\displaystyle\sum_{k=2}^{\infty} \frac{\begin{pmatrix} n+k \\ k-1 \end{pmatrix} - \alpha \begin{pmatrix} n+k-1 \\ k-1 \end{pmatrix}}{1-\alpha} |a_k| \leqq 1$ *for $n \in \mathcal{N}_0$ and $f \in \mathcal{A}$, then $f \in \mathcal{R}(n,1;\alpha)$.*

We obtain the following corollary of Theorem (2.6) replacing m by 1.

Corollary 2.8. ([5]) For $n \in \mathcal{N}_0$ and $f \in \mathcal{T}$, we have that

$$f \in \mathcal{TR}(n,1;\alpha) \iff \sum_{k=2}^{\infty} \frac{\begin{pmatrix} n+k \\ k-1 \end{pmatrix} - \alpha \begin{pmatrix} n+k-1 \\ k-1 \end{pmatrix}}{1-\alpha} a_k \leqq 1.$$

2.2. Examples

Before proving our theorem, we present two examples. Their proof can be found in [3].

Example 2.9. The following relations hold true for $m \in \mathcal{N}_2$ and $0 \leqq \beta < 1$:
(a) $\mathcal{TR}(0,m;\alpha) \subsetneqq \mathcal{TS}(0,m;\beta)$ for $1 - \frac{1-\beta}{m!} \leqq \alpha < 1$,
(b) $\mathcal{TS}(0,m;\beta) \subsetneqq \mathcal{TR}(0,m;\alpha)$ for $0 \leqq \alpha < 1 - \frac{m}{2^m-1}(1-\beta)$,
(c) $\mathcal{TR}(0,m;\alpha) \not\subset \mathcal{TS}(0,m;\beta)$ for $1 - \frac{m}{2^m-1}(1-\beta) < \alpha < 1 - \frac{1-\beta}{m!}$
and
(d) $\mathcal{TS}(0,m;\beta) \not\subset \mathcal{TR}(0,m;\alpha)$ for $1 - \frac{m}{2^m-1}(1-\beta) < \alpha < 1 - \frac{1-\beta}{m!}$.

Example 2.10. The following relations hold true for $m \in \mathcal{N}_2$ and $0 \leqq \beta < 1$:
(a) $\mathcal{TR}(1,m;\alpha) \subsetneqq \mathcal{TS}(1,m;\beta)$ for $1 - \frac{1-\beta}{(m+1)!} \leqq \alpha < 1$,
(b) $\mathcal{TS}(1,m;\beta) \not\subset \mathcal{TR}(1,m;\alpha)$ and $\mathcal{TR}(1,m;\alpha) \not\subset \mathcal{TS}(1,m;\beta)$
for $1 - \frac{m}{2(2^m-1)}(1-\beta) < \alpha < 1 - \frac{(1-\beta)}{(m+1)!}$

and
(c) $\mathcal{TS}(1,m;\beta) \subsetneqq \mathcal{TR}(1,m;\alpha)$ for $0 \leqq \alpha \leqq 1 - \frac{m}{2(2^m-1)}(1-\beta)$.

3. Main result

Theorem 3.1. *The following relation holds true for* $n \in \mathcal{N}_0$, $m \in \mathcal{N}_2$ and $0 \leqq \beta < 1$:

$$\mathcal{TR}(n,m;\alpha) \subsetneqq \mathcal{TS}(n,m;\beta) \text{ for } 1 - \frac{1-\beta}{(m+n)!} \leqq \alpha < 1.$$

Proof. In order to prove that $\mathcal{TR}(n,m;\alpha) \subsetneqq \mathcal{TS}(n,m;\beta)$ for $1 - \frac{1-\beta}{(m+n)!} \leqq \alpha < 1$, $n \in \mathcal{N}_0$, $m \in \mathcal{N}_2$, $\beta \in [0,1)$, we have to prove that $G(\alpha,\beta;k,n,m) \leq 0$, where

$$G(\alpha,\beta;k,n,m) =$$

$$(1-\alpha)k^{n+m} - \frac{1-\beta}{(n+m)!}\prod_{l=0}^{n+m-1}(k+l) - \beta(1-\alpha)k^n + \frac{\alpha(1-\beta)}{n!}\prod_{l=0}^{n-1}(k+l),$$

for $k,m \in \mathcal{N}_2$, $n \in \mathcal{N}_0$, $1 - \frac{1-\beta}{(m+n)!} \leqq \alpha < 1$ $\beta \in [0,1)$. We show that $G(\alpha,\beta;k,n,m)$ is a decreasing function of α for all β,k,n,m (with the conditions in the Theorem) and that

$$G(1 - \frac{1-\beta}{(n+m)!}, \beta; k,n,m) \leq 0. \tag{3.1}$$

We have

$$\frac{\partial}{\partial\alpha}G(\alpha,\beta;k,n,m) = -k^{n+m} + \beta k^n + \frac{1-\beta}{n!}\prod_{l=0}^{n-1}(k+l) = H(n,m;\beta);$$

and we will prove that

$$\frac{\partial}{\partial\beta}H(n,m;\beta) = k^n - \frac{1}{n!}\prod_{l=0}^{n-1}(k+l) \geq 0, \quad n \in \mathcal{N},$$

or equivalently, that

$$n!k^n - \prod_{l=0}^{n-1}(k+l) \geq 0, \tag{3.2}$$

by the mathematical induction.

Case $n = 1$. $\frac{\partial}{\partial\beta}H(1,m;\beta) = k - k = 0$;

Suppose that (3.2)holds true.

Case $(n+1)$. We have

$$(n+1)!k^{n+1} = (n+1)k\{n!k^n\} > (n+1)k\prod_{l=0}^{n-1}(k+l) > \prod_{l=0}^{n}(k+l)$$

because $(n+1)k > k+n$.

From $\dfrac{\partial}{\partial\beta}H(n,m;\beta)\geq 0$ we have that

$$H(n,m;\beta)\leq H(n,m;1)=-k^{n+m}+k^{n}<0$$

for all $n\in\mathcal{N}_0$, $\beta\in[0,1)$. From $\dfrac{\partial}{\partial\alpha}G(\alpha,\beta;k,n,m)=H(n,m;\beta)<0$ we deduce that $G(\alpha,\beta;k,n,m)$ is a decreasing function of α; then

$$G(\alpha,\beta;k,n,m)\leq G\left(1-\frac{1-\beta}{(n+m)!},\beta;k,n,m\right) \tag{3.3}$$

Now we have to show that (3.1) holds. We can write

$$G\left(1-\frac{1-\beta}{(n+m)!},\beta;k,n,m\right)=\frac{1-\beta}{(n+m)!}L(\beta;k;n,m), \tag{3.4}$$

where

$$L(\beta;k;n,m)=k^{n+m}-\prod_{l=0}^{n+m-1}(k+l)+\beta k^{n}+\frac{(n+m)!}{n!}\prod_{l=0}^{n-1}(k+l)-\frac{1-\beta}{n!}\prod_{l=0}^{n-1}(k+l).$$

But

$$\frac{\partial}{\partial\beta}L(\beta;k;n,m)=k^{n}+\frac{1}{n!}\prod_{l=0}^{n-1}(k+l)>0,$$

hence

$$L(\beta;k;n,m)\leq L(1;k;n,m). \tag{3.5}$$

Now we prove that $L(1;k;n,m)\leq 0$ by mathematical induction with respect to n. We have

$$L(1;k;1,m)=k^{1+m}-k(k+1)\times\ldots\times(k+m)+k+(1+m)!k=k\phi(k),$$

where the function

$$\phi(x)=x^{m}-(x+1)\times\ldots\times(x+m)+1+(1+m)!\text{ for }x\geq 2$$

is increasing, because

$$\phi'(x)=mx^{m-1}-(x+1)\times\ldots\times(x+m)\sum_{s=1}^{m}\frac{1}{x+s}<0.$$

Since

$$\phi(x)\leq\phi(2)=2^{m}-3\times\ldots\times(m+2)+(m+1)!+1=2^{m}-\frac{(m+1)!m}{2}+1<0,$$

for $m\geq 2$, we obtain that $L(1;k;1,m)<0$ for $m,\ k\geq 2$.

Now we suppose that $L(1;k;n,m)<0$, that is

$$k^{n+m}+k^{n}<\prod_{l=0}^{n+m-1}(k+l)-\frac{(n+m)!}{n!}\prod_{l=0}^{n-1}(k+l) \tag{3.6}$$

We have (the case $(n+1)$)

$$k^{n+1+m}+k^{n+1}=k(k^{n+m}+k^{n})<k\left[\prod_{l=0}^{n+m-1}(k+l)-\frac{(n+m)!}{n!}\prod_{l=0}^{n-1}(k+l)\right], \tag{3.7}$$

where we used (3.6). The following inequalities

$$k\left[\prod_{l=0}^{n+m-1}(k+l)-\frac{(n+m)!}{n!}\prod_{l=0}^{n-1}(k+l)\right]<\prod_{l=0}^{n+m}(k+l)-\frac{(n+m+1)!}{(n+1)!}\prod_{l=0}^{n}(k+l),$$

$$k\prod_{l=0}^{n-1}(k+l)\left[\prod_{l=n}^{n+m-1}(k+l)-\frac{(n+m)!}{n!}\right]<$$

$$(k+n)\prod_{l=0}^{n-1}(k+l)\left[\prod_{l=n}^{n+m}(k+l)-\frac{(n+m+1)!}{n!}\right],$$

$$k\left[\prod_{l=n}^{n+m-1}(k+l)-\frac{(n+m)!}{n!}\right]<(k+n)\left[\prod_{l=n}^{n+m}(k+l)-\frac{(n+m+1)!}{n!}\right],$$

$$k(k+n)\times\ldots\times(k+n+m-1)-k(n+1)\times\ldots\times(n+m)<$$
$$(k+n)(k+n+1)\times\ldots\times(k+n+m)-(k+n)(n+2)\times\ldots\times(n+m+1)$$

and

$$\begin{aligned}&(k+n)\times\ldots\times(k+n+m-1)[k-(k+n+m)]-\\&(n+2)\times\ldots\times(n+m)[k(n+1)-(k+n)(n+m+1)]<0\end{aligned}\tag{3.8}$$

are equivalent to (3.7). We denote

$$\begin{aligned}M(k,m,n)=&-(k+n)\times\ldots\times(k+n+m-1)(n+m)-\\&(n+2)\times\ldots\times(n+m)[k(n+1)-(k+n)(n+m+1)].\end{aligned}\tag{3.9}$$

Since $k\geq 2$ we deduce
$M(k,m,n)\leq -(2+n)\times\ldots\times(2+n+m-2)(k+n+m-1)(n+m)+(n+2)\times\ldots\times(n+m)[n^2+n+nm+km]$
$=(n+2)\times\ldots\times(n+m)[-(k+n+m-1)(n+m)+(n^2+n+nm+km)]$
$=(n+2)\times\ldots\times(n+m)[(2-k)n+m(1-n)-m^2]<0.$

From $M(k,m,n)<0$, notation (3.9) and the equivalence of the inequalities between (3.7) and (3.8), we obtain that (3.6) holds and this means that

$$L(1;k;n,m)\leq 0.\tag{3.10}$$

Combining (3.10), (3.5), (3.4) and (3.3) we complete the proof. □

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A subclass of analytic functions

Andreea-Elena Tudor

Abstract. In the present paper, by means of Carlson-Shaffer operator and a multiplier transformation, we consider a new class of analytic functions $\mathcal{BL}(m, l, a, c, \mu, \alpha, \lambda)$. A sufficient condition for functions to be in this class and the angular estimates are provided.

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Keywords: Analytic functions, Carlson-Shaffer operator, Linear operator.

1. Introduction

Let $\mathcal{A}_n$ denote the class of functions f of the form

$$f(z) = z + \sum_{k=n+1}^{\infty} a_k z^k, \;\; z \in U \tag{1.1}$$

which are analytic in the open unit disc $U = \{z \in \mathbb{C} : |z| < 1\}$.

Let $\mathcal{H}(U)$ be the space of holomorphic functions in U.

For $a \in \mathbb{C}$ and $n \in \mathbb{N}$, we denote by

$$\mathcal{H}[a, n] = \left\{f \in \mathcal{H}(U), \quad f(z) = a + a_n z^n + a_{n+1} z^{n+1} + ..., z \in U\right\}$$

Let the function $\phi(a, c; z)$ be given by

$$\phi(a, c; z) := \sum_{k=0}^{\infty} \frac{(a)_k}{(c)_k} z^{k+1} \quad (c \neq 0, -1, -2, ...; z \in U)$$

where $(x)_k$ is the *Pochhammer symbol* defined by

$$(x)_k := \begin{cases} 1, & k = 0 \\ x(x+1)(x+2)...(x+k-1), & k \in \mathbb{N}^* \end{cases}$$

Carlson and Shaffer[1] introduced a linear operator $L(a, c)$, corresponding to the function $\phi(a, c; z)$, defined by the following Hadamard product:

$$L(a, c) := \phi(a, c; z) * f(z) = z + \sum_{k=1}^{\infty} \frac{(a)_k}{(c)_k} a_{k+1} z^{k+1} \tag{1.2}$$

We note that

$$z\left(L(a,c)f(z)\right)' = aL(a+1,c)f(z) - (a-1)L(a,c)f(z)$$

For $a = m+1$ and $c = 1$ we obtain the *Ruscheweyh derivative of* f, (see [9]).

In [2], N.E. Cho and H. M. Srivastava introduced a linear operator of the form:

$$\mathcal{I}(m,l)f(z) = z + \sum_{k=2}^{\infty}\left(\frac{l+k}{l+1}\right)^m a_k z^k, \quad m \in \mathbb{Z},\ l \geq 0 \tag{1.3}$$

We note that

$$z\left(\mathcal{I}(m,l)f(z)\right)' = (l+1)\mathcal{I}(m+1,l)f(z) - l\mathcal{I}(m,l)f(z)$$

For $l = 0$ we obtain Sălăgean operator introduced in [10].

Let now $\mathcal{L}(m,l,a,c,\lambda)$ be the operator defined by:

$$\mathcal{L}(m,l,a,c,\lambda)f(z) = \lambda\mathcal{I}(m,l)f(z) + (1-\lambda)L(a,c)f(z)$$

For $\lambda = 0$ we get *Carlson-Shaffer* operator introduced in [1], for $\lambda = 1$ we get linear operator in [2] and for $a = m+1, c = 1, l = 0$ we get generalized Sălăgean and Ruscheweyh operator introduced by A. Alb Lupaş in [4]

By means of operator $\mathcal{L}(m,l,a,c,\lambda)$ we introduce the following subclass of analytic functions:

Definition 1.1. *We say that a function* $f \in \mathcal{A}_n$ *is in the class* $\mathcal{BL}(m,l,a,c,\mu,\alpha,\lambda)$, $n,m \in \mathbb{N}$, $l,\mu,\lambda \geq 0$, $\alpha \in [0,1)$ *if*

$$\left|\frac{\mathcal{L}(m+1,l,a+1,c,\lambda)f(z)}{z}\left(\frac{z}{\mathcal{L}(m,l,a,c,\lambda)f(z)}\right)^{\mu} - 1\right| < 1-\alpha,\ z \in U. \tag{1.4}$$

Remark 1.2. The class $\mathcal{BL}(m,l,a,c,\mu,\alpha,\lambda)$ includes various classes of analytic univalent functions, such as:

- $\mathcal{BL}(0,0,a,c,1,\alpha,1) \equiv \mathcal{S}^*(\alpha)$
- $\mathcal{BL}(1,0,a,c,1,\alpha,1) \equiv \mathcal{K}(\alpha)$
- $\mathcal{BL}(0,0,a,c,0,\alpha,1) \equiv \mathcal{R}(\alpha)$
- $\mathcal{BL}(0,0,a,c,2,\alpha,1) \equiv \mathcal{B}(\alpha)$ introduced by Frasin and Darus in [7]
- $\mathcal{BL}(0,0,a,c,\mu,\alpha,1) \equiv \mathcal{B}(\mu,\alpha)$ introduced by Frasin and Jahangiri in [6]
- $\mathcal{BL}(m,0,a,c,\mu,\alpha,1) \equiv \mathcal{BS}(m,\mu,\alpha)$ introduced by A. Alb Lupaş and A. Cătaş in [5]
- $\mathcal{BL}(m,0,m+1,1,\mu,\alpha,0) \equiv \mathcal{BR}(m,\mu,\alpha)$ introduced by A. Alb Lupaş and A. Cătaş in [4]
- $\mathcal{BL}(m,0,m+1,1,\mu,\alpha,\lambda) \equiv \mathcal{BL}(m,\mu,\alpha,\lambda)$ introduced by A. Alb Lupaş and A. Cătaş in [3].

To prove our main result we shall need the following lemmas:

Lemma 1.3. [6] *Let* p *be analytic in* U, *with* $p(0) = 1$, *and suppose that*

$$Re\left(1 + \frac{zp'(z)}{p(z)}\right) > \frac{3\alpha-1}{2\alpha},\ z \in U \tag{1.5}$$

Then $Re\ p(z) > \alpha$ *for* $z \in U$ *and* $\alpha \in [-1/2, 1)$.

Lemma 1.4. [8] *Let $p(z)$ be an analytic function in U, $p(0)=1$ and $p(z)\neq 0,\ z\in U$. If there exist a point $z_0\in U$ such that*

$$|\arg(p(z))|<\frac{\pi}{2}\alpha,\quad |z|<|z_0|,\quad |\arg(p(z_0))|=\frac{\pi}{2}\alpha$$

with $0<\alpha\leq 1$, then we have

$$\frac{z_0p'(z_0)}{p(z_0}=ik\alpha$$

where

$$k\geq\frac{1}{2}\left(a+\frac{1}{a}\right)\geq 1\quad \textit{when}\ \arg(p(z_0))=\frac{\pi}{2}\alpha,$$

$$k\leq\frac{1}{2}\left(a+\frac{1}{a}\right)\leq -1\quad \textit{when}\ \arg(p(z_0))=-\frac{\pi}{2}\alpha,$$

$$p(z_0)^{1/\alpha}=\pm ai,\quad a>0.$$

2. Main results

In the first theorem we provide sufficient condition for functions to be in the class $\mathcal{BL}(m,l,a,c,\mu,\alpha,\lambda)$

Theorem 2.1. *Let $f\in\mathcal{A}_n,\ n,m\in\mathbb{N},l,\mu,\lambda\geq 0,\alpha\in[1/2,1)$. If*

$$\frac{\lambda(l+1)\mathcal{I}(m+2,l)f(z)-\lambda l\mathcal{I}(m+1,l)f(z)+(1-\lambda)(a+1)L(a+2,c)f(z)-(1-\lambda)aL(a+1,c)f(z)}{\mathcal{L}(m+1,l,a+1,c,\lambda)}$$

$$-\mu\frac{\lambda(l+1)\mathcal{I}(m+1,l)f(z)-\lambda l\mathcal{I}(m,l)f(z)+(1-\lambda)aL(a+1,c)f(z)-(1-\lambda)(a-1)L(a,c)f(z)}{\mathcal{L}(m,l,a,c,\lambda)}$$

$$+\mu\prec 1+\frac{3\alpha-1}{2\alpha}z,\quad z\in U \tag{2.1}$$

then $f\in\mathcal{BL}(m,l,a,c,\mu,\alpha,\lambda)$

Proof. If we denote by

$$p(z)=\frac{\mathcal{L}(m+1,l,a+1,c,\lambda)f(z)}{z}\left(\frac{z}{\mathcal{L}(m,l,a,c,\lambda)f(z)}\right)^{\mu} \tag{2.2}$$

where

$$p(z)=1+p_1z+p_2z^2+...,\ p(z)\in\mathcal{H}[1,1],$$

then, after a simple differentiation, we obtain

$$\frac{zp'(z)}{p(z)}=\frac{\lambda(l+1)\mathcal{I}(m+2,l)f(z)-\lambda l\mathcal{I}(m+1,l)f(z)+(1-\lambda)(a+1)L(a+2,c)f(z)}{\mathcal{L}(m+1,l,a+1,c,\lambda)}$$

$$-\frac{(1-\lambda)aL(a+1,c)f(z)}{\mathcal{L}(m+1,l,a+1,c,\lambda)}-\mu\frac{(1-\lambda)aL(a+1,c)f(z)-(1-\lambda)(a-1)L(a,c)f(z)}{\mathcal{L}(m,l,a,c,\lambda)}$$

$$-\mu\frac{\lambda(l+1)\mathcal{I}(m+1,l)f(z)-\lambda l\mathcal{I}(m,l)f(z)}{\mathcal{L}(m,l,a,c,\lambda)}-1+\mu$$

Using (2.1), we get

$$Re\left(1+\frac{zp'(z)}{p(z)}\right)>\frac{3\alpha-1}{2\alpha}.$$

Thus, from Lemma 1.3, we get

$$Re\left\{\frac{\mathcal{L}(m+1,l,a+1,c,\lambda)f(z)}{z}\left(\frac{z}{\mathcal{L}(m,l,a,c,\lambda)f(z)}\right)^{\mu}\right\}>\alpha.$$

Therefore, $f\in\mathcal{BL}(m,l,a,c,\mu,\alpha,\lambda)$, by Definition 1.1. □

If we take $a=l$ in Theorem 2.1, we obtain the following corollary:

Corollary 2.2. *Let $f\in\mathcal{A}_n,\ n,m\in\mathbb{N},l,\mu,\lambda\geq 0,\alpha\in[1/2,1)$. If*

$$\frac{(l+1)\mathcal{L}(m+2,l,a+2,c,\lambda)}{\mathcal{L}(m+1,l,a+1,c,\lambda)}-\mu\frac{(l+1)\mathcal{L}(m+1,l,a+1,c,\lambda)}{\mathcal{L}(m,l,a,c,\lambda)}-l+\mu(l+1)\prec$$

$$\prec 1+\frac{3\alpha-1}{2\alpha}z,\ \ z\in U$$

then $f\in\mathcal{BL}(m,l,a,c,\mu,\alpha,\lambda)$

Next, we prove the following theorem:

Theorem 2.3. *Let $f(z)\in\mathcal{A}$. If $f(z)\in\mathcal{BL}(m,l,l+1,c,\mu,\alpha,\lambda)$, then*

$$\left|\arg\frac{\mathcal{L}(m,l,l+1,c,\lambda)}{z}\right|<\frac{\pi}{2}\alpha$$

for $0<\alpha\leq 1$ and $2/\pi\arctan(\alpha/(l+1))-\alpha(\mu-1)=1$

Proof. If we denote by

$$p(z)=\frac{\mathcal{L}(m,l,l+1,c,\lambda)}{z}$$

where

$$p(z)=1+p_1z+p_2z^2+...,\ \ p(z)\in\mathcal{H}[1,1],$$

then, after a simple differentiation, we obtain

$$\frac{\mathcal{L}(m+1,l,l+2,c,\lambda)f(z)}{z}\left(\frac{z}{\mathcal{L}(m,l,l+1,c,\lambda)f(z)}\right)^{\mu}=\left(\frac{1}{p(z)}\right)^{\mu-1}\left(1+\frac{1}{l+1}\frac{zp'(z)}{p(z)}\right)$$

Suppose there exists a point $z_0\in U$ such that

$$|\arg(p(z))|<\frac{\pi}{2}\alpha,\ \ |z|<|z_0|,\ \ |\arg(p(z_0))|=\frac{\pi}{2}\alpha$$

Then, from Lemma 1.4, we obtain:

- If $\arg(p(z_0))=\pi/2\,\alpha$, then

$$\arg\left(\frac{\mathcal{L}(m+1,l,l+2,c,\lambda)f(z_0)}{z_0}\left(\frac{z_0}{\mathcal{L}(m,l,l+1,c,\lambda)f(z_0)}\right)^{\mu}\right)$$

$$=\arg\left(\frac{1}{p(z_0)}\right)^{\mu-1}\left(1+\frac{1}{l+1}\frac{z_0p'(z_0)}{p(z_0)}\right)=-(\mu-1)\frac{\pi}{2}\alpha+\arg\left(1+\frac{1}{l+1}ik\alpha\right)$$

$$\geq -(\mu-1)\frac{\pi}{2}\alpha+\arctan\frac{\alpha}{l+1}=\frac{\pi}{2}\left(\frac{2}{\pi}\arctan\frac{\alpha}{l+1}-\alpha(\mu-1)\right)=\frac{\pi}{2}$$

- If $\arg(p(z_0)) = -\pi/2\,\alpha$, then
$$\arg\left(\frac{\mathcal{L}(m+1,l,l+2,c,\lambda)f(z_0)}{z_0}\left(\frac{z_0}{\mathcal{L}(m,l,l+1,c,\lambda)f(z_0)}\right)^{\mu}\right) \leq -\frac{\pi}{2}$$
These contradict the assumption of the theorem.
Thus, the function $p(z)$ satisfy the inequality
$$|\arg(p(z))| < \frac{\pi}{2}\alpha, \quad z \in U. \qquad \square$$

If we get, in Theorem 2.2, $m = l = 0, \mu = 2$ and $\lambda = 1$, we obtain the following corollary, proved by B. A. Frasin and M. Darus in [7]:

Corollary 2.4. *Let $f(z) \in \mathcal{A}$. If $f(z) \in \mathcal{B}(\alpha)$, then*
$$\left|\arg\left(\frac{f(z)}{z}\right)\right| < \frac{\pi}{2}\alpha, \ z \in U$$
for some $\alpha(0 < \alpha < 1)$ and $(2/\pi)\tan^{-1}\alpha - \alpha = 1$.

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Convolution properties of Sălăgean-type harmonic univalent functions

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Abstract. Jahangiri et al. [5] defined "modified Salagean operator" for harmonic univalent functions in the unit disk. In that study, they also obtained necessary and sufficient coefficient conditions for the Salagean-type class of harmonic univalent functions. By using those coefficient conditions, we investigate some convolution properties for the Salagean-type class of harmonic univalent functions.

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Keywords: Harmonic, univalent, starlike, convex, convolution.

1. Introduction

Let H denote the family of continuous complex valued harmonic functions which are harmonic in the open unit disk $U = \{z : |z| < 1\}$ and let A be the subclass of H consisting of functions which are analytic in U. A function harmonic in U may be written as $f = h + \overline{g}$, where h and g are members of A. In this case, f is sense-preserving if $|h'(z)| > |g'(z)|$ in U. See Clunie and Sheil-Small [2]. To this end, without loss of generality, we may write

$$h(z) = z + \sum_{k=2}^{\infty} a_k z^k, \qquad g(z) = \sum_{k=1}^{\infty} b_k z^k. \tag{1.1}$$

One shows easily that the sense-preserving property implies that $|b_1| < 1$.

Let SH denote the family of functions $f = h + \overline{g}$ which are harmonic, univalent, and sense-preserving in U for which $f(0) = f_z(0) - 1 = 0$.

For the harmonic function $f = h + \overline{g}$, we call h the analytic part and g the co-analytic part of f. Note that SH reduces to the class S of normalized analytic univalent functions in U if the co-analytic part of f is identically zero.

In 1984 Clunie and Sheil-Small [2] investigated the class SH as well as its geometric subclasses and obtained some coefficient bounds. Since then, there has been several related papers on SH and its subclasses such as Avcı and Zlotkiewicz [1], Silverman [9], Silverman and Silvia [10], Jahangiri [4] studied the harmonic univalent functions.

The differential operator D^n $(n \in \mathbb{N}_0)$ was introduced by Salagean [7]. For $f = h + \overline{g}$ given by (1.1), Jahangiri et al. [5] defined the modified Salagean operator of f as

$$D^n f(z) = D^n h(z) + (-1)^n \overline{D^n g(z)}, \tag{1.2}$$

where

$$D^n h(z) = z + \sum_{k=2}^{\infty} k^n a_k z^k \quad \text{and} \quad D^n g(z) = \sum_{k=1}^{\infty} k^n b_k z^k.$$

For $0 \leq \alpha < 1$, Jahangiri et al. [5] defined the class $SH(n, \alpha)$ which consist of functions f of the form (1.1) such that

$$\operatorname{Re}\left(\frac{D^{n+1} f(z)}{D^n f(z)}\right) \geq \alpha, \quad 0 \leq \alpha < 1 \tag{1.3}$$

where $D^n f(z)$ defined by (1.2).

If the co-analytic part of $f = h + \overline{g}$ is identically zero, then the family $SH(n, \alpha)$ turns out to be the class $S(n, \alpha)$ introduced by Salagean [7] for the analytic case. The class $SH(n, \alpha)$ includes a variety of well-known subclasses of SH. Such as,

(i) $SH(0, 0) = SH^*$, is the class of harmonic starlike functions ([1], [9], [10]),

(ii) $SH(0, \alpha) = SH^*(\alpha)$, is the class of harmonic starlike functions of order α ([3], [4]),

(iii) $SH(1, 0) = KH$, is the class of harmonic convex functions ([1], [9], [10]),

(iv) $SH(1, \alpha) = KH(\alpha)$, is the class of harmonic convex functions of order α ([3], [4]) in U.

We let the subclass $\overline{SH}(n, \alpha)$ consist of harmonic functions $F = H + \bar{G}$ in SH so that H and G are of the form

$$H(z) = z - \sum_{k=2}^{\infty} a_k z^k, \; G(z) = (-1)^n \sum_{k=1}^{\infty} b_k z^k, \quad a_k, \; b_k \geq 0. \tag{1.4}$$

Let $f_j(z) \in SH$ $(j = 1, 2, \ldots, m)$ be given by

$$f_j(z) = z + \sum_{k=2}^{\infty} a_{k,j} \; z^k + \sum_{k=1}^{\infty} \overline{b_{k,j}} \; \overline{z}^k. \tag{1.5}$$

The convolution is defined by

$$(f_1 * \ldots * f_m)(z) = z + \sum_{k=2}^{\infty} \left(\prod_{j=1}^{m} a_{k,j}\right) z^k + \sum_{k=1}^{\infty} \left(\prod_{j=1}^{m} \overline{b_{k,j}}\right) \overline{z}^k. \tag{1.6}$$

Let $F_j(z) \in \overline{SH}(n, \alpha)$ $(j = 1, 2, \ldots, m)$ be given by

$$F_j(z) = z - \sum_{k=2}^{\infty} a_{k,j} z^k + (-1)^n \sum_{k=1}^{\infty} b_{k,j} \overline{z}^k, \quad a_{k,j}, \; b_{k,j} \geq 0. \tag{1.7}$$

The convolution is defined by

$$(F_1 * \ldots * F_m)(z) = z - \sum_{k=2}^{\infty} \left(\prod_{j=1}^{m} a_{k,j} \right) z^k + (-1)^n \sum_{k=1}^{\infty} \left(\prod_{j=1}^{m} b_{k,j} \right) \overline{z}^k. \tag{1.8}$$

Owa and Srivastava [6] studied convolution and generalized convolution properties of the classes $M_n(\alpha)$ and $N_n(\alpha)$, Al-Shaqsi and Darus [8] investigated such properties for the classes $SH^*(\alpha)$ and $KH(\alpha)$, by especially using Cauchy-Schwarz and Hölder inequalities. In this paper, we investigate convolution properties for the Salagean-type class of harmonic univalent functions by using above mentioned techniques.

2. Main Results

Lemma 2.1. [5] *Let $F = H + \overline{G}$ given by (1.4). Then $F \in \overline{SH}(n, \alpha)$ if and only if*

$$\sum_{k=2}^{\infty} k^n (k - \alpha) a_k + \sum_{k=1}^{\infty} k^n (k + \alpha) b_k \leq 1 - \alpha. \tag{2.1}$$

Theorem 2.2. *If $F_j(z) \in \overline{SH}(n, \alpha_j)$ $(j = 1, 2, \ldots, m)$ then $(F_1 * \ldots * F_m)(z) \in \overline{SH}(n, \beta)$, where*

$$\beta = 1 - \frac{\prod_{j=1}^{m} (1 - \alpha_j)}{2^n \prod_{j=1}^{m} (2 - \alpha_j) - \prod_{j=1}^{m} (1 - \alpha_j)}.$$

Proof. We use the principle of mathematical induction in our proof of Theorem 2.2. Let $F_1(z) \in \overline{SH}(n, \alpha_1)$ and $F_2(z) \in \overline{SH}(n, \alpha_2)$.
By using Lemma 2.1, we have

$$\sum_{k=2}^{\infty} k^n \left(\frac{k - \alpha_j}{1 - \alpha_j} \right) a_{k,j} + \sum_{k=1}^{\infty} k^n \left(\frac{k + \alpha_j}{1 - \alpha_j} \right) b_{k,j} \leq 1, \quad (j = 1, 2). \tag{2.2}$$

Then, we have

$$\left[\sum_{k=2}^{\infty} \left(\sqrt{k^n \left(\frac{k - \alpha_1}{1 - \alpha_1} \right) a_{k,1}} \right)^2 \times \sum_{k=2}^{\infty} \left(\sqrt{k^n \left(\frac{k - \alpha_2}{1 - \alpha_2} \right) a_{k,2}} \right)^2 \right]^{\frac{1}{2}}$$
$$+ \left[\sum_{k=1}^{\infty} \left(\sqrt{k^n \left(\frac{k + \alpha_1}{1 - \alpha_1} \right) b_{k,1}} \right)^2 \times \sum_{k=1}^{\infty} \left(\sqrt{k^n \left(\frac{k + \alpha_2}{1 - \alpha_2} \right) b_{k,2}} \right)^2 \right]^{\frac{1}{2}} \leq 1.$$

Thus, by applying the Cauchy-Schwarz inequality, we have

$$\sum_{k=2}^{\infty}\sqrt{k^{2n}\frac{(k-\alpha_1)(k-\alpha_2)}{(1-\alpha_1)(1-\alpha_2)}a_{k,1}a_{k,2}}$$

$$\leq \left[\sum_{k=2}^{\infty}\left(\sqrt{k^n\left(\frac{k-\alpha_1}{1-\alpha_1}\right)a_{k,1}}\right)^2\right]^{\frac{1}{2}}\times\left[\sum_{k=2}^{\infty}\left(\sqrt{k^n\left(\frac{k-\alpha_2}{1-\alpha_2}\right)a_{k,2}}\right)^2\right]^{\frac{1}{2}}$$

and

$$\sum_{k=1}^{\infty}\sqrt{k^{2n}\frac{(k+\alpha_1)(k+\alpha_2)}{(1-\alpha_1)(1-\alpha_2)}b_{k,1}b_{k,2}}$$

$$\leq \left[\sum_{k=1}^{\infty}\left(\sqrt{k^n\left(\frac{k+\alpha_1}{1-\alpha_1}\right)b_{k,1}}\right)^2\right]^{\frac{1}{2}}\times\left[\sum_{k=1}^{\infty}\left(\sqrt{k^n\left(\frac{k+\alpha_2}{1-\alpha_2}\right)b_{k,2}}\right)^2\right]^{\frac{1}{2}}.$$

Then, we get

$$\sum_{k=2}^{\infty}\sqrt{k^{2n}\frac{(k-\alpha_1)(k-\alpha_2)}{(1-\alpha_1)(1-\alpha_2)}a_{k,1}a_{k,2}}+\sum_{k=1}^{\infty}\sqrt{k^{2n}\frac{(k+\alpha_1)(k+\alpha_2)}{(1-\alpha_1)(1-\alpha_2)}b_{k,1}b_{k,2}}\leq 1.$$

Therefore, if

$$\sum_{k=2}^{\infty}k^n\left(\frac{k-\gamma}{1-\gamma}\right)a_{k,1}a_{k,2}\leq \sum_{k=2}^{\infty}\sqrt{k^{2n}\frac{(k-\alpha_1)(k-\alpha_2)}{(1-\alpha_1)(1-\alpha_2)}a_{k,1}a_{k,2}}\ ,$$

and

$$\sum_{k=1}^{\infty}k^n\left(\frac{k+\gamma}{1-\gamma}\right)b_{k,1}b_{k,2}\leq \sum_{k=1}^{\infty}\sqrt{k^{2n}\frac{(k+\alpha_1)(k+\alpha_2)}{(1-\alpha_1)(1-\alpha_2)}b_{k,1}b_{k,2}}$$

that is, if

$$\sqrt{a_{k,1}a_{k,2}}\leq \left(\frac{1-\gamma}{k-\gamma}\right)\sqrt{\frac{(k-\alpha_1)(k-\alpha_2)}{(1-\alpha_1)(1-\alpha_2)}}\quad (k=2,3,\ldots),$$

and

$$\sqrt{b_{k,1}b_{k,2}}\leq \left(\frac{1-\gamma}{k+\gamma}\right)\sqrt{\frac{(k+\alpha_1)(k+\alpha_2)}{(1-\alpha_1)(1-\alpha_2)}}\quad (k=1,2,\ldots)$$

then $(F_1 * F_2)(z)\in\overline{SH}(n,\gamma)$.

We also note that the inequality (2.2) yields

$$\sum_{k=2}^{\infty}\sqrt{k^n\left(\frac{k-\alpha_j}{1-\alpha_j}\right)a_{k,j}}\leq 1$$

and

$$\sum_{k=1}^{\infty}\sqrt{k^n\left(\frac{k+\alpha_j}{1-\alpha_j}\right)b_{k,j}}\leq 1$$

and so we get,

$$\sqrt{a_{k,j}} \leq \sqrt{\frac{1-\alpha_j}{k^n\left(k-\alpha_j\right)}} \quad (j=1,2;\ k=2,3,\ldots),$$

and

$$\sqrt{b_{k,j}} \leq \sqrt{\frac{1-\alpha_j}{k^n\left(k+\alpha_j\right)}} \quad (j=1,2;\ k=1,2,\ldots).$$

Consequently, if

$$\sqrt{\frac{\left(1-\alpha_1\right)\left(1-\alpha_2\right)}{k^{2n}\left(k-\alpha_1\right)\left(k-\alpha_2\right)}} \leq \frac{1-\gamma}{k-\gamma}\sqrt{\frac{\left(k-\alpha_1\right)\left(k-\alpha_2\right)}{\left(1-\alpha_1\right)\left(1-\alpha_2\right)}},\ (k=2,3,\ldots),$$

and

$$\sqrt{\frac{\left(1-\alpha_1\right)\left(1-\alpha_2\right)}{k^{2n}\left(k+\alpha_1\right)\left(k+\alpha_2\right)}} \leq \frac{1-\gamma}{k+\gamma}\sqrt{\frac{\left(k+\alpha_1\right)\left(k+\alpha_2\right)}{\left(1-\alpha_1\right)\left(1-\alpha_2\right)}},\ (k=1,2,\ldots)$$

that is, if

$$\frac{k-\gamma}{1-\gamma} \leq \frac{k^n\left(k-\alpha_1\right)\left(k-\alpha_2\right)}{\left(1-\alpha_1\right)\left(1-\alpha_2\right)}, \quad (k=2,3,\ldots)$$

and

$$\frac{k+\gamma}{1-\gamma} \leq \frac{k^n\left(k+\alpha_1\right)\left(k+\alpha_2\right)}{\left(1-\alpha_1\right)\left(1-\alpha_2\right)}, \quad (k=1,2,\ldots)$$

then we have $\left(F_1 * F_2\right)(z) \in \overline{SH}\left(n,\gamma\right).$
Then, we see that

$$\gamma \leq 1-\frac{(k-1)\left(1-\alpha_1\right)\left(1-\alpha_2\right)}{k^n\left(k-\alpha_1\right)\left(k-\alpha_2\right)-\left(1-\alpha_1\right)\left(1-\alpha_2\right)} = \phi(k) \quad (k=2,3,\ldots),$$

and

$$\gamma \leq 1-\frac{(k+1)\left(1-\alpha_1\right)\left(1-\alpha_2\right)}{k^n\left(k+\alpha_1\right)\left(k+\alpha_2\right)+\left(1-\alpha_1\right)\left(1-\alpha_2\right)} = \varphi(k) \quad (k=1,2,\ldots).$$

Since $\phi(k)$ for $k \geq 2$ and $\varphi(k)$ for $k \geq 1$ increasing,

$$\gamma \leq 1-\frac{\left(1-\alpha_1\right)\left(1-\alpha_2\right)}{2^n\left(2-\alpha_1\right)\left(2-\alpha_2\right)-\left(1-\alpha_1\right)\left(1-\alpha_2\right)},$$

and

$$\gamma \leq 1-\frac{2\left(1-\alpha_1\right)\left(1-\alpha_2\right)}{\left(1+\alpha_1\right)\left(1+\alpha_2\right)+\left(1-\alpha_1\right)\left(1-\alpha_2\right)},$$

and also,

$$\frac{\left(1-\alpha_1\right)\left(1-\alpha_2\right)}{2^n\left(2-\alpha_1\right)\left(2-\alpha_2\right)-\left(1-\alpha_1\right)\left(1-\alpha_2\right)} \leq \frac{2\left(1-\alpha_1\right)\left(1-\alpha_2\right)}{\left(1+\alpha_1\right)\left(1+\alpha_2\right)+\left(1-\alpha_1\right)\left(1-\alpha_2\right)},$$

then $\left(F_1 * F_2\right)(z) \in \overline{SH}\left(n,\gamma\right)$, where

$$\gamma = 1-\frac{\left(1-\alpha_1\right)\left(1-\alpha_2\right)}{2^n\left(2-\alpha_1\right)\left(2-\alpha_2\right)-\left(1-\alpha_1\right)\left(1-\alpha_2\right)}.$$

Next, we suppose that $(F_1 * \ldots * F_m)(z) \in \overline{SH}(n, \beta)$, where

$$\beta = 1 - \frac{\prod_{j=1}^{m}(1-\alpha_j)}{2^n \prod_{j=1}^{m}(2-\alpha_j) - \prod_{j=1}^{m}(1-\alpha_j)}.$$

We can show that, $(F_1 * \ldots * F_{m+1})(z) \in \overline{SH}(n, \delta)$, where

$$\delta = 1 - \frac{(1-\beta)(1-\alpha_{m+1})}{2^n (2-\beta)(2-\alpha_{m+1}) - (1-\beta)(1-\alpha_{m+1})}.$$

Since

$$(1-\beta)(1-\alpha_{m+1}) = \frac{\prod_{j=1}^{m+1}(1-\alpha_j)}{2^n \prod_{j=1}^{m}(2-\alpha_j) - \prod_{j=1}^{m}(1-\alpha_j)},$$

and

$$(2-\beta)(2-\alpha_{m+1}) = \frac{\prod_{j=1}^{m+1}(1-\alpha_j)}{2^n \prod_{j=1}^{m}(2-\alpha_j) - \prod_{j=1}^{m}(1-\alpha_j)},$$

we have

$$\delta = 1 - \frac{\prod_{j=1}^{m+1}(1-\alpha_j)}{2^n \prod_{j=1}^{m+1}(2-\alpha_j) - \prod_{j=1}^{m+1}(1-\alpha_j)}.$$ □

Corollary 2.3. *If* $F_j(z) \in \overline{SH}(n, \alpha)$ $(j = 1, 2, \ldots, m)$, *then* $(F_1 * \ldots * F_m)(z) \in \overline{SH}(n, \beta)$, *where*

$$\beta = 1 - \frac{(1-\alpha)^m}{2^n(2-\alpha)^m - (1-\alpha)^m}.$$

Corollary 2.4. *If* $F_j(z) \in SH^*(\alpha_j)$ $(j = 1, 2, \ldots, m)$, *then* $(F_1 * \ldots * F_m)(z) \in SH^*(\beta)$, *where*

$$\beta = 1 - \frac{\prod_{j=1}^{m}(1-\alpha_j)}{\prod_{j=1}^{m}(2-\alpha_j) - \prod_{j=1}^{m}(1-\alpha_j)}.$$

Corollary 2.5. *If* $F_j(z) \in KH(\alpha_j)$ $(j = 1, 2, \ldots, m)$, *then* $(F_1 * \ldots * F_m)(z) \in KH(\beta)$, *where*

$$\beta = 1 - \frac{\prod_{j=1}^{m}(1-\alpha_j)}{2\prod_{j=1}^{m}(2-\alpha_j) - \prod_{j=1}^{m}(1-\alpha_j)}.$$

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Distortion theorem and the radius of convexity for Janowski-Robertson functions

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Abstract. In this note, we consider another family of functions that includes the class of convex functions as a proper subfamily. For $-\frac{\pi}{2} < \alpha < \frac{\pi}{2}$, we say that $f(z) \in C_\alpha(A,B)$ if
i) $f(z) \in A$
ii) $f'(z) \neq 0 \in \mathbb{D}$, $e^{i\alpha}(1 + z\frac{f''(z)}{f'(z)}) = \cos\alpha p(z) + i\sin\alpha$, where $p(z)$ is analytic in $\mathbb{D}$ and satisfies the conditions $p(0) = 1$, $p(z) = \frac{1+A\phi(z)}{1+B\phi(z)}$, $-1 \leq B < A \leq 1$, $\phi(z)$ analytic in $\mathbb{D}$, and $\phi(0) = 0$, $|\phi(z)| < 1$ for every $z \in \mathbb{D}$. The class of $C_\alpha(A,B)$ is called Janowski-Robertson class. The aim of this paper is to give a distortion theorem and the radius of convexity for the class $C_\alpha(A,B)$.

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1. Introduction

Let Ω be the family of functions $\phi(z)$ analytic in the open unit disc

$$\mathbb{D} = \{z| \quad |z| < 1\}$$

and satisfy the conditions $\phi(0) = 0$, $|\phi(z)| < 1$ for every $z \in \mathbb{D}$.

Next, for arbitrary fixed numbers A, B, $-1 \leq B < A \leq 1$, denote by $P(A,B)$ the family of functions $p(z) = 1 + p_1 z + p_2 z^2 + \cdots$ analytic in $\mathbb{D}$, and such that $p(z)$ is in $P(A,B)$ if and only if

$$p(z) = \frac{1 + A\phi(z)}{1 + B\phi(z)} \tag{1.1}$$

for some $\phi(z) \in \Omega$, and every $z \in \mathbb{D}$.

Moreover, let $C_\alpha(A,B)$ denote the family of functions $f(z) = z + a_2 z^2 + \cdots$ analytic in $\mathbb{D}$ and such that $f(z)$ is in $C_\alpha(A,B)$ if and only if

$$e^{i\alpha}(1 + z\frac{f''(z)}{f'(z)}) = \cos\alpha p(z) + i\sin\alpha \tag{1.2}$$

for some functions $p(z) \in P(A,B)$, all $z \in \mathbb{D}$ and some real constant α ($|\alpha| < \frac{\pi}{2}$).

Finally, let $F(z) = z + \alpha_2 z^2 + \alpha_3 z^3 + \cdots$ and $G(z) = z + \beta_2 z^2 + \beta_3 z^3 + \cdots$ be analytic functions in $\mathbb{D}$, if there exists a function $\phi(z) \in \Omega$ such that $F(z) = G(\phi(z))$ for every $z \in \mathbb{D}$, then we say that $F(z)$ is subordinate to $G(z)$, and we write $F(z) \prec G(z)$. We also note that if $F(z) \prec G(z)$, then $F(\mathbb{D}) \subset G(\mathbb{D})$.

The radius of convexity for the family of analytic functions is defined in the following manner.

$$R(f) = \sup[r|Re(1 + z\frac{f''(z)}{f'(z)}) > 0, |z| < r]$$

2. Main Results

Theorem 2.1. *Let $f(z)$ be an element of $C_\alpha(A,B)$ then,*

$$\begin{cases} \frac{(1-Br)^{\frac{(A-B)}{2B}(1+\cos\alpha)\cos\alpha}}{(1+Br)^{\frac{(A-B)}{2B}(1-\cos\alpha)\cos\alpha}} \le |f'(z)| \le \frac{(1+Br)^{\frac{(A-B)}{2B}(1+\cos\alpha)\cos\alpha}}{(1-Br)^{\frac{(A-B)}{2B}(1-\cos\alpha)\cos\alpha}}; \ B \neq 0, \\ e^{-A\cos\alpha r} \le |f'(z)| \le e^{A\cos\alpha r}; \qquad B = 0. \end{cases} \tag{2.1}$$

Proof. Since

$$f(z) \in C_\alpha(A,B) \Leftrightarrow e^{i\alpha}(1 + z\frac{f''(z)}{f'(z)}) = \cos\alpha p(z) + i\sin\alpha \tag{2.2}$$

and

$$B \neq 0 \Rightarrow \left|p(z) - \frac{1-ABr^2}{1-B^2r^2}\right| \le \frac{(A-B)r}{1-B^2r^2} \tag{2.3}$$

Then we have,

$$\left|\frac{1}{\cos\alpha}[e^{i\alpha}(1 + z\frac{f''(z)}{f'(z)}) - i\sin\alpha] - \frac{1-ABr^2}{1-B^2r^2}\right| \le \frac{(A-B)r}{1-B^2r^2} \tag{2.4}$$

After simple calculations from (2.4) we obtain,

$$\frac{-(A-B)\cos\alpha(1+Br\cos\alpha)r}{(1-Br)(1+Br)} \le Re(z\frac{f''(z)}{f'(z)}) \le \frac{(A-B)\cos\alpha(1-Br\cos\alpha)r}{(1-Br)(1+Br)} \tag{2.5}$$

On the other hand we have

$$Re(z\frac{f''(z)}{f'(z)}) = r\frac{\partial}{\partial r}\log|f'(z)| \tag{2.6}$$

Considering (2.5) and (2.6) together we get;

$$\frac{-(A-B)\cos\alpha(1+Br\cos\alpha)}{(1-Br)(1+Br)} \le \frac{\partial}{\partial r}\log|f'(z)| \le \frac{(A-B)\cos\alpha(1-Br\cos\alpha)}{(1-Br)(1+Br)} \tag{2.7}$$

then after integration, we obtain (2.1). On the other hand,

$$B = 0 \Rightarrow |p(z) - 1| \le Ar \Rightarrow \tag{2.8}$$

$$\left|\frac{1}{\cos\alpha}[e^{i\alpha}(1 + z\frac{f''(z)}{f'(z)}) - i\sin\alpha] - 1\right| \le Ar \Rightarrow \tag{2.9}$$

$$\left|e^{i\alpha}(1+z\frac{f''(z)}{f'(z)})-i\sin\alpha-\cos\alpha\right|\leq A\cos\alpha.r\Rightarrow \tag{2.10}$$

$$\left|e^{i\alpha}(1+z\frac{f''(z)}{f'(z)})-e^{i\alpha}\right|\leq A\cos\alpha.r \tag{2.11}$$

$$\left|z\frac{f''(z)}{f'(z)}\right|\leq A\cos\alpha.r \tag{2.12}$$

$$-\left|z\frac{f''(z)}{f'(z)}\right|\geq -A\cos\alpha.r \tag{2.13}$$

Therefore,

$$-A\cos\alpha.r\leq Re(z\frac{f''(z)}{f'(z)})=r\frac{\partial}{\partial r}\log|f'(z)|\leq\left|z\frac{f''(z)}{f'(z)}\right|\leq A\cos\alpha.r \tag{2.14}$$

$$-A\cos\alpha\leq\frac{\partial}{\partial r}\log|f'(z)|\leq A\cos\alpha \tag{2.15}$$

If we integrate the last inequality, then we obtain the result. □

Corollary 2.2. *If* $f(z)\in C_\alpha(1,-1)$ *then;*

$$\frac{(1+r)^{-\cos^2\alpha-\cos\alpha}}{(1-r)^{\cos^2-\cos\alpha}}\leq|f'(z)|\leq\frac{(1+r)^{-\cos^2\alpha+\cos\alpha}}{(1-r)^{\cos^2+\cos\alpha}} \tag{2.16}$$

This is the distortion theorem for Robertson functions.

Corollary 2.3. *If we give another special values to* A *and* B *we obtain another distortion inequalities.*

Remark 2.4. If we give special values to A and B, we obtain that new inequalities for the Janowski-Robertson functions. The special values of A and B can be ordered in the following manner.

i. $A=1-2\alpha$, $B=-1$, $0\leq\alpha<1$
ii. $A=1$, $B=0$
iii. $A=\alpha$, $B=0$, $0<\alpha<1$
iv. $A=\alpha$, $B=-\alpha$, $0<\alpha<1$
v. $A=1$, $B=-1+\frac{1}{M}$, $M>\frac{1}{2}$.

Corollary 2.5. *The radius of convexity of the class* $C_\alpha(A,B)$ *is;*

$$\begin{cases} r=\frac{2}{(A-B)\cos\alpha+\sqrt{(A+B)^2\cos^2\alpha+4B^2\sin^2\alpha}}; & B\neq 0,\\ r=\frac{1}{A\cos\alpha}; & B=0.\end{cases} \tag{2.17}$$

Proof. The inequality (2.4) can be written in the following form,

$$Re(1+z\frac{f''(z)}{f'(z)})\geq\frac{1-(A-B)\cos\alpha r-B(A\cos^2\alpha+B\sin^2\alpha)r^2}{1-B^2r^2}, B\neq 0 \tag{2.18}$$

$$Re(1+z\frac{f''(z)}{f'(z)})\geq 1-A\cos\alpha.r, B=0 \tag{2.19}$$

The inequality (2.18) and (2.19) show that this corollary is true. □

Corollary 2.6. *The radius of convexity of the class $C_\alpha(1,-1)$ is*

$$r = \frac{1}{\cos\alpha + |\sin\alpha|}$$

We also note that all these results are sharp because the extremal function is

$$f(z) = \begin{cases} \displaystyle\int_0^z (1+B\zeta)^{\frac{A-B}{B}} d\zeta; \ B \neq 0, \\ \displaystyle\int_0^z e^{A\zeta} d\zeta; \qquad B = 0. \end{cases} \tag{2.20}$$

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Families of positive operators with reverse order

Şebnem Yildiz Pestil

Abstract. The objective of this paper is to present several invariant subspace results for collections of positive operators on Banach lattices.For any family $\mathcal{C}$ of positive operators in $L(E)$,I will reverse order.

Mathematics Subject Classification (2010): Banach lattices, invariant.

Keywords: 47B60.

1. Introduction

The objective of this paper is to present several invariant subspace results for collections of positive operators on Banach lattices. For any vector in a Banach lattice define $x^+ = x \vee 0$, $x^- = (-x) \vee 0$, $|x| = x \vee (-x)$. Throughout this paper X will denote a real or a complex Banach space and E will denote a real Banach lattice.

A collection S of bounded operators on a Banach space is said to be a *multiplicative* (*additive*) *semigroup* if for each pair $S, T \in S$ the operator ST (resp. $S + T$) also belongs to S.

We assume that all collections or families of operators under consideration in this section are non-empty. $\mathcal{C}$ denotes a non-empty collection of positive operators on a Banach lattice E. For all $x \in X$, we let $\mathcal{C}x = \{Cx : C \in \mathcal{C}\}$, and therefore $\|\mathcal{C}x\| = \sup\{\|Cx\| : C \in \mathcal{C}\}$.

For a subset D of a Banach space, we let

$$||D|| = \sup_{x \in D} ||x|| .$$

Accordingly for a set $\mathcal{C} \subset \mathcal{L}(E)$,

$$||\mathcal{C}|| = \sup_{C \in \mathcal{C}} ||C|| .$$

For any $A \in \mathcal{L}(X)$ we define

$$A\mathcal{C} = \{AC : C \in \mathcal{C}\}$$

and

$$\mathcal{C}A = \{CA : C \in \mathcal{C}\} .$$

For each $n \in N$, we shall also use the notation

$$\mathcal{C}^n = \{C_1C_2...C_n : C_1, ..., C_n \in \mathcal{C}\}$$

and hope that whenever this notation is used it will no cause any confusion with the standard Cartesian product notation. The commutant of $\mathcal{C}$ is the unital algebra of operators defined by

$$\mathcal{C}^{'} = \{A \in \mathcal{L}(X) : AC = CA \ for \ all \ C \in \mathcal{C}\}.$$

Definition 1.1. *A subspace $V \subset X$ is said to be $\mathcal{C} - invariant$ if V is C-invariant for each $C \in \mathcal{C}$.*

A collection $\mathcal{C}$ of operators is said to be $non - transitive$ if there exists a non-trivial closed $\mathcal{C}$-invariant subspace. Otherwise, the family $\mathcal{C}$ is called transitive.

Definition 1.2. *A family $\mathcal{C}$ of operators in $\mathcal{L}(X)$ is said to be:*
(1) *(locally) quasinilpotent at a point $x \in X$ if $\lim_{n\to\infty} \sqrt[n]{\|\mathcal{C}^n x\|} = 0$ and*
(2) *finitely quasinilpotent at a point $x \in X$ if every finite subcollection of $\mathcal{C}$ is (locally) quasinilpotent at x. Finitely quasinilpotent algebras of operators were considered by V.S. Shulman.*

If $T : X \to X$ is a bounded operator on a Banach space, then $\mathcal{Q}_T$ denotes the subset of X consisting of all vectors at which T is locally quasinilpotent,

$$\mathcal{Q}_T = \left\{x \in X : \lim_{n\to\infty} \|T^n x\|^{\frac{1}{n}} = 0\right\}.$$

Definition 1.3. *Let $T : X \to X$ is a bounded operator on a Banach space and V is a subspace of X. Then V is non-trivial if $V \neq \{0\}$ $V \neq X$. We say that V is invariant under T if $T(V) \subseteq V$. Also, V is said to be hyperinvariant for T or $T-$hyperinvariant whenever V is invariant under every bounded operator on X that commutes with T, i.e., if $S \in \mathcal{L}(X)$ and $ST = TS$ imply that $S(V) \subseteq V$.*

Definition 1.4. *Let $T : X \to X$ be a bounded operator on a Banach space and denoted by $\mathcal{Q}_T$, the set of all points where T is locally qusinilpotent, i.e.*

$$\mathcal{Q}_T = \left\{x \in X : \lim_{n\to\infty} \|T^n x\|^{\frac{1}{n}} = 0\right\}.$$

$\mathcal{Q}_T$ is a T-hyperinvariant vector subspace.

Let $x, y \in \mathcal{Q}_T$ and fix $\epsilon > 0$. Pick some n_0 such that $\|T^n x\| < \epsilon^n$ and $\|T^n y\| < \epsilon^n$ hold for all $n \geq n_0$. It follows that $\|T^n(x + y)\|^{\frac{1}{n}} \leq (\|T^n x\| + \|T^n y\|)^{\frac{1}{n}} < (2\epsilon^n)^{\frac{1}{n}} < 2\epsilon$ for all $n \geq n_0$.

Therefore, $\lim_{n\to\infty} \|T^n(x + y)\|^{\frac{1}{n}} = 0$, and so $x + y \in \mathcal{Q}_T$. Also note that if λ is scalar, then

$$\lim_{n\to\infty} \|T^n(\lambda x)\|^{\frac{1}{n}} = \lim_{n\to\infty} \|\lambda T^n(x)\|^{\frac{1}{n}} = \lim_{n\to\infty} |\lambda|^{\frac{1}{n}} \lim_{n\to\infty} \|T^n x\|^{\frac{1}{n}} = 0$$

and so $\lambda x \in \mathcal{Q}_T$. Consequently $\mathcal{Q}_T$ is a vector subspace of X.

Finally, let us show that $\mathcal{Q}_T$ is a T-hyperinvariant subspace. To see this assume that an operator $S \in \mathcal{L}(X)$ satisfies $TS = ST$ and let $x_0 \in \mathcal{Q}_T$. Then we have,

$$\|T^n(Sx_0)\|^{\frac{1}{n}} = \|S(T^n x_0)\|^{\frac{1}{n}} \leq \|S\|^{\frac{1}{n}} \|T^n x_0\|^{\frac{1}{n}} \to 0.$$

This implies $Sx_0 \in \mathcal{Q}_T$, and so $\mathcal{Q}_T$ is a T-hyperinvariant subspace.

To generalize this to a collection of operators $\mathcal{C}$, we let

$$\mathcal{Q}_{\mathcal{C}}^f = \{x \in X : \mathcal{C} \text{ is finitely quasinilpotent at } x\}.$$

$\mathcal{C}$ denotes a non empty collection of positive operators on a Banach lattice E.

The presence of the order structure on E leads naturally to a modification of the set $\mathcal{Q}_c^f$.

$$\mathcal{Q}_{\mathcal{C}}^f = \left\{x \in E : |x| \in \mathcal{Q}_c^f\right\}.$$

For a positive operator $C : E \to E$ on a Banach lattice, we denote by $[C\rangle$ the collection of all positive operators $A : E \to E$ such that $[A, C] \geq 0$

$$[C\rangle = \{A \in \mathcal{L}(E)_+ : AC - CA \geq 0\}.$$

In accordance with this notation we also let

$$[\mathcal{C}\rangle = \{A \in \mathcal{L}(E)_+ : C \in \mathcal{C}, AC - CA \geq 0\}.$$

Similarly,

$$\langle\mathcal{C}] = \{A \in \mathcal{L}(E)_+ : C \in \mathcal{C}, AC - CA \leq 0\}.$$

We need to introduce two additional collections associated with an arbitrary collection $\mathcal{C}$ of positive operators on E.

The first of these collections is *multiplicative semigroup generated by* $\mathcal{C}$ in $\mathcal{L}(E)$. It is the smallest semigroup of operators that contains $\mathcal{C}$ and it will be denoted by $S_{\mathcal{C}}$. $S_{\mathcal{C}}$ consists of all finite products of operators in $\mathcal{C}$.

$$S_{\mathcal{C}} = \bigcup_{n=1}^{\infty} \mathcal{C}^n.$$

The second collection denoted by $\mathcal{D}_{\mathcal{C}}$, is also a large collection of positive operators that is defined,

$$\mathcal{D}_{\mathcal{C}} = \left\{ D \in \mathcal{L}(E)_+ : \exists\{T_1, ..., T_k\} \subseteq \langle\mathcal{C}] \text{ and } \{S_1, ..., S_k\} \subseteq S_{\mathcal{C}} \text{ such that } D \leq \sum_{i=1}^{k} S_i T_i \right\}.$$

Proposition 1.5. *For any family $\mathcal{C}$ of positive operators in $L(E)$ the set $< \mathcal{C}]$ is a norm closed additive and multiplicative semigroup in $L(E)$ and contains the zero and the identity operators.*

Proof. $\mathcal{C}$ is norm closed and the operators 0 and I belong to $< \mathcal{C}]$. Now take two arbitrary operators S, T in $< \mathcal{C}]$. Then for each operator $C \in \mathcal{C}$ we have $SC \leq CS$ and $TC \leq CT$. Adding up the two inequalities, we get $(S + T)\,C \leq C\,(S + T)$ then $S + T \in < \mathcal{C}]$. Consequently,

$$STC = S\,(TC) \leq SCT = (SC)\,T \leq CST.$$

Therefore, $ST \in < \mathcal{C}]$. □

Proposition 1.6. *If $\mathcal{C}$ is a family of positive operators, then the collection $\mathcal{D}_{\mathcal{C}}$ is an additive and multiplicative semigroup in $\mathcal{L}(E)$.*

Proof. Pick any two operators D_1 and D_2 in $D_{\mathcal{C}}$. Hence

$$D_j \leq \sum_{i=1}^{n_j} S_{j,i} T_{j,i}$$

for some $T_{j,i} \in \langle \mathcal{C}]$, where $j = 1, 2$. $D_1 + D_2$ belongs to $\mathcal{D}_{\mathcal{C}}$. Let us verify that $D_1 D_2 \in \mathcal{D}_{\mathcal{C}}$. Indeed,

$$D_1 D_2 \leq \left[\sum_{k=1}^{n_1} S_{1,k} T_{1,k}\right] \left[\sum_{i=1}^{n_2} S_{2,i} T_{2,i}\right] = \sum_{k=1}^{n_1} \sum_{i=1}^{n_2} S_{1,k} T_{1,k} S_{2,i} T_{2,i}.$$

Since $T_{j,i} \in \langle \mathcal{C}]$, it follows that $T_{j,i} \in \langle S_{\mathcal{C}}]$ and hence $T_{1,k} S_{2,i} \leq S_{2,i} T_{1,k}$. Therefore,

$$D_1 D_2 \leq \sum_{i=1}^{n_1} \sum_{k=1}^{n_2} S_{2,k} S_{2,i} T_{1,k} T_{2,i}.$$

Since $\langle \mathcal{C}]$ and $S_{\mathcal{C}}$ are semigroups, we have that $T_{1,k} T_{2,i} \in \langle \mathcal{C}]$, $S_{1,k} S_{2,i} \in S_{\mathcal{C}}$. □

Proposition 1.7. *Each ideal $[D_c x]$ is both $\mathcal{C} -$ invariant and $\langle \mathcal{C}]$-invariant.*

Proof. Take any $y \in [D_c x]$. Since D_c is an additive semigroup, it follows that $|y| \leq \lambda D x$ for some scalar λ and $D \in D_c$. By the definition of D_c there exist operators $T \in \langle \mathcal{C}]$ and $S_i \in S_c$ $(i = 1, 2, 3, .., n)$ such that $D \leq \sum_{i=1}^{n} S_i T_i$, and so

$$|y| \leq \lambda \sum_{i=1}^{n} S_i T_i x.$$

Fix $C \in \mathcal{C}$ and consider the vector Cy. From $CT_i \geq T_i C$ for each i, we see that

$$|Cy| \leq C\,|y| \leq \lambda \sum_{i=1}^{n} C S_i T_i x.$$

Since $CS_i \in S_{\mathcal{C}}$ for each i we see that

$$K = \sum_{i=1}^{n} (CS_i) T_i \in \mathcal{D}_{\mathcal{C}}.$$

Therefore,

$$|Cy| \leq \lambda \sum_{i=1}^{n} (CS_i) T_i x = \lambda K x$$

and $Cy \in [D_c x]$. $[D_c x]$ is $\mathcal{C}]$-invariant.

Let $T \in \langle \mathcal{C}]$. Since $\langle \mathcal{C}]$ is a multiplicative semigroup, $TT_i \in \langle \mathcal{C}]$ for each i, and hence the operator $L = \sum_{i=1}^{n} S_i (TT_i)$ belongs to $\mathcal{D}_{\mathcal{C}}$.

$$|Ty| \leq T\,|y| \leq \lambda \sum_{i=1}^{n} S_i T T_i x = \lambda L x.$$

Consequently, $Ty \in [\mathcal{D}_{\mathcal{C}} x]$. □

Proposition 1.8. *The ideal $\hat{\mathcal{Q}}_c^f$ is $\langle \mathcal{C}]$-invariant.*

Proof. Fix $x \in \hat{\mathcal{Q}}_c^f$ that is $\|\mathcal{G}^n |x|\|^{\frac{1}{n}} \to 0$ for each finite subset $\mathcal{G}$ of $\mathcal{C}$. We must prove that Tx belong to $\hat{\mathcal{Q}}_c^f$ for each $C \in \mathcal{C}$ and each $T \in \langle \mathcal{C}]$. Fix $C \in \mathcal{C}$, $T \in \langle \mathcal{C}]$ and let $\mathcal{F} = \{C_1, ..., C_k\}$ be a finite subset of $\mathcal{C}$.

$C_i T \geq T C_i$ for each $1 \leq i \leq k$. For each operator $F \in \mathcal{F}^n$ we have $FT \geq TF$, and therefore,

$$\|T\mathcal{F}^n |x|\|^{\frac{1}{n}} \leq \|\mathcal{F}^n T |x|\|^{\frac{1}{n}} \leq \|T\|^{\frac{1}{n}} \|\mathcal{F}^n |x|\|^{\frac{1}{n}} \to 0$$

Consequently, $\|\mathcal{F}^n |Tx|\|^{\frac{1}{n}} \to 0$, and so $Tx \in \hat{\mathcal{Q}}_c^f$. The ideal $\hat{\mathcal{Q}}_c^f$ is also $\langle \mathcal{C}]$-invariant. □

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Distortion theorems for certain subclasses of typically real functions

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Abstract. In this paper we discuss the class $\mathcal{T}(\frac{1}{2})$ of typically real functions in the unit disk $\Delta = \{z \in \mathbb{C} : |z| < 1\}$ which are given by the formula

$$f(z) = \int_{-1}^{1} \frac{z}{\sqrt{1-2zt+z^2}} d\mu(t).$$

Three other classes: $\mathcal{K}_R$, $\mathcal{S}_R^*(\frac{1}{2})$ and $\mathcal{K}_R(i)$, consisting of convex functions, starlike functions of order 1/2 and convex in the direction of the imaginary axis, all with real coefficients, are contained in $\mathcal{T}(\frac{1}{2})$. The main idea of the paper is to obtain some distorsion results concerning $\mathcal{T}(\frac{1}{2})$ and apply them in solving analogous problems in $\mathcal{K}_R$, $\mathcal{S}_R^*(\frac{1}{2})$, $\mathcal{K}_R(i)$.

Mathematics Subject Classification (2010): 30C45, 30C25.

Keywords: Typically real functions, univalent functions, distortion theorems.

1. Preliminaries

Let $\mathcal{T}(\frac{1}{2})$ denote a subclass of typically real functions $\mathcal{T}$ consisting of functions given by the formula

$$f(z) = \int_{-1}^{1} f_t(z) d\mu(t) \ , z \in \Delta \ , \tag{1.1}$$

where $\Delta = \{z \in \mathbb{C} : |z| < 1\}$,

$$f_t(z) = \frac{z}{\sqrt{1-2zt+z^2}} \ , \tag{1.2}$$

μ is a probability measure on $[-1, 1]$ and the square root is chosen that $\sqrt{1} = 1$. This class was introduced and discussed by Szynal in [6], [3].

Observe that the kernel functions $f_t \in \mathcal{T}(\frac{1}{2})$ and analogous functions

$$k_t(z) = \frac{z}{1-2zt+z^2}$$

in the class $\mathcal{T}$ are connected by a simple relation

$$(f_t(z))^2 = zk_t(z) \ . \tag{1.3}$$

All functions of the class $\mathcal{T}(\frac{1}{2})$ have real coefficients given by

$$a_n = \int_{-1}^{1} P_{n-1}(t)d\mu(t) \ , \tag{1.4}$$

where P_n is the n-th Legendre polynomial. From the properties of the Legendre polynomials we know that $|P_n| \leq 1$ for every $n \in \mathbb{N}$. Hence all coefficients of every function $f \in \mathcal{T}(\frac{1}{2})$ are bounded by 1.

Similar property holds also for the following subclasses of univalent functions: $\mathcal{K}$ - convex functions, $\mathcal{S}^*(\frac{1}{2})$ - starlike functions of order 1/2, $\mathcal{K}(i)$ - functions that are convex in the direction of the imaginary axis and $\mathcal{K}_R$, $\mathcal{S}^*_R(\frac{1}{2})$, $\mathcal{K}_R(i)$ consisting of functions with real coefficients.

For the classes with real coefficients the integral formulae are known. They are a consequence of known relations between $\mathcal{K}_R$, $\mathcal{K}_R(i)$, $\mathcal{S}^*_R(\frac{1}{2})$ and $\mathcal{T}$, $\mathcal{P}_R$, where $\mathcal{P}_R$ is the class of functions having real coefficients and a positive real part.

Namely, for functions normalized by the equalities $f(0) = f'(0) - 1 = 0$ and $z \in \Delta$ we have

$$f \in \mathcal{K}_R \quad \text{iff} \quad 1 + z\tfrac{f''(z)}{f'(z)} \in \mathcal{P}_R \ , \tag{1.5}$$

$$f \in \mathcal{K}_R(i) \quad \text{iff} \quad zf'(z) \in \mathcal{T} \ , \tag{1.6}$$

$$f \in \mathcal{S}^*_R(\tfrac{1}{2}) \quad \text{iff} \quad \operatorname{Re} \tfrac{zf'(z)}{f(z)} > \tfrac{1}{2} \ . \tag{1.7}$$

The relation (1.7) can be written as follows

$$f \in \mathcal{S}^*_R(\tfrac{1}{2}) \quad \text{iff} \quad 2\tfrac{zf'(z)}{f(z)} - 1 \in \mathcal{P}_R \ . \tag{1.8}$$

From (1.5), (1.6), (1.8) we obtain

$$f \in \mathcal{K}_R, \quad \text{iff} \quad f'(z) = \exp\left(-\int_0^{\pi} \ln(1 - 2z\cos\varphi + z^2)\, d\mu(\varphi)\right),$$

$$f \in \mathcal{K}_R(i) \quad \text{iff} \quad f(z) = \int_0^{\pi} h_\varphi(z) d\mu(\varphi) \ ,$$

$$\text{where } h_\varphi(z) = \begin{cases} \frac{z}{1-z} & \text{for} \quad \varphi = 0\,, \\ \frac{1}{2i\sin\varphi} \ln\frac{1-ze^{-i\varphi}}{1-ze^{i\varphi}} & \text{for} \quad \varphi \in (0,\pi)\,, \\ \frac{z}{1+z} & \text{for} \quad \varphi = \pi\,. \end{cases}$$

$$f \in \mathcal{S}^*_R(\tfrac{1}{2}) \quad \text{iff} \quad f(z) = z\exp\int_0^{\pi} \ln\frac{1}{\sqrt{1-2z\cos\varphi + z^2}}\, d\mu(\varphi) \ ,$$

where $\mu \in P_{[0,\pi]}$.

Investigating classes $\mathcal{K}_R$, $\mathcal{K}_R(i)$, $\mathcal{S}^*_R(\frac{1}{2})$ with the use of their integral formulae is rather difficult. It is easier, in some cases, to obtain results in $\mathcal{T}(\frac{1}{2})$ and than to transfer them onto the classes mentioned above.

Remark 1.1. It is easy to check that

a) $f_t(z) = \frac{z}{\sqrt{1-2zt+z^2}} \in \mathcal{S}_R^*\left(\frac{1}{2}\right)$ for all $t \in [-1,1]$,
b) $f_1(z) = \frac{z}{1-z}$ and $f_{-1}(z) = \frac{z}{1+z}$ belong to $\mathcal{K}_R$,
c) $g_\alpha(z) = \alpha\frac{z}{1-z} + (1-\alpha)\frac{z}{1+z} \in \mathcal{K}_R(i)$ for all $\alpha \in [0,1]$,
d) $g_{1/2}(z) = \frac{z}{1-z^2} \in \mathcal{S}_R^*$.

In [4] it was proved that

$$\mathcal{K}_R \subset \mathcal{S}_R^*\left(\tfrac{1}{2}\right) \subset \mathcal{T}\left(\tfrac{1}{2}\right) \tag{1.9}$$

and

$$\mathcal{K}_R \subset \mathcal{K}_R(i) \subset \mathcal{T}(\tfrac{1}{2}) \ . \tag{1.10}$$

Moreover, there exist functions in $\mathcal{T}(\frac{1}{2})$ that are not univalent. For example, for every fixed $t \in (0,1)$ the function

$$f(z) = \frac{1}{2}\left[\frac{z}{\sqrt{1-2tz+z^2}} + \frac{z}{\sqrt{1+2tz+z^2}}\right] \quad , \quad z \in \Delta$$

is locally univalent in the disk Δ_{r_t}, where $r_t \in (0,1)$, and is not univalent in any disk Δ_r with $r \geq r_t$. Discussing these functions, one can prove that the radius of univalence for $\mathcal{T}\left(\frac{1}{2}\right)$ is not greater then $\sqrt{7}/3 = 0.881\ldots$. From this reason, we conclude that $\mathcal{K}_R$, $\mathcal{K}_R(i)$ and $\mathcal{S}_R^*(\frac{1}{2})$ are proper subclasses of $\mathcal{T}(\frac{1}{2})$.

2. Main results for $\mathcal{T}\left(\frac{1}{2}\right)$

At the beginning observe that for any function f with real coefficients, if D is symmetric with respect to the real axis then $f(D)$ also has the same property. It is a reason why in problems involving sets $f(D)$ one can discuss only a set $f(D^+)$ and than apply the reflection of this set with respect to the real axis. Throughout the paper we write $D^+ = D \cap \{z : \operatorname{Im} z > 0\}$.

Let $D_z(\mathcal{T}(\frac{1}{2}))$ denote the region of values $f(z)$ for a fixed $z \in \Delta$ while f varies the class $\mathcal{T}(\frac{1}{2})$. The important property of functions of the class $\mathcal{T}\left(\frac{1}{2}\right)$ is established in the following lemma.

Lemma 2.1. [4] *For a fixed $x \in (-1,1)$ the set $D_x(\mathcal{T}(\frac{1}{2}))$ coincides with the segment $[\frac{x}{1+x}, \frac{x}{1-x}]$. For a fixed $z \in \Delta^+$ the set $D_z(\mathcal{T}(\frac{1}{2}))$ is a convex set whose boundary consists of a curve $\{f_t(z) :\ t \in [-1,1]\}$ and a line segment $\{\alpha\frac{z}{1-z} + (1-\alpha)\frac{z}{1+z} :\ \alpha \in [0,1]\}$.*

We also need the distortion theorem for $\mathcal{T}$ obtained by Goluzin.

Theorem 2.2. [1] *For $f \in \mathcal{T}$ and $z \in \Delta \setminus \{0\}$ we have*

a)

$$|f(z)| \leq \begin{cases} \left|\frac{z}{(1-z)^2}\right| & \operatorname{Re}\frac{1+z^2}{z} \geq 2 \\ \frac{1}{\left|\operatorname{Im}\frac{1+z^2}{z}\right|} & \left|\operatorname{Re}\frac{1+z^2}{z}\right| < 2 \\ \left|\frac{z}{(1+z)^2}\right| & \operatorname{Re}\frac{1+z^2}{z} \leq -2 \ , \end{cases} \tag{2.1}$$

b)

$$\arg\frac{z}{(1+z)^2} \le \arg f(z) \le \arg\frac{z}{(1-z)^2} \quad \textit{for} \quad z\in\Delta^+\ , \tag{2.2}$$

$$\arg\frac{z}{(1-z)^2} \le \arg f(z) \le \arg\frac{z}{(1+z)^2} \quad \textit{for} \quad z\in\Delta^-\ . \tag{2.3}$$

Basing on the above facts we obtain some distortion results for $\mathcal{T}\left(\frac{1}{2}\right)$.

Theorem 2.3. *For* $f\in\mathcal{T}\left(\frac{1}{2}\right)$ *and* $z\in\Delta\setminus\{0\}$ *we have*

$$|f(z)| \le \begin{cases} \left|\frac{z}{1-z}\right| & \operatorname{Re}\frac{1+z^2}{z}\ge 2 \\ |z|\sqrt{\frac{|z|}{(1-|z|^2)|\operatorname{Im} z|}} & \left|\operatorname{Re}\frac{1+z^2}{z}\right| < 2 \\ \left|\frac{z}{1+z}\right| & \operatorname{Re}\frac{1+z^2}{z}\le -2\ . \end{cases} \tag{2.4}$$

The extremal functions in this theorem are: $f(z)=\frac{z}{1-z}$, $f(z)=\frac{z}{\sqrt{1-2t_0z+z^2}}$, where $t_0=\frac{1}{2}\operatorname{Re}\frac{1+z^2}{z}$, and $f(z)=\frac{z}{1+z}$.
The sets which appear in Theorem 2.2 and in Theorem 2.3 are presented in Fig. 1.
Proof. Assume that $f\in\mathcal{T}\left(\frac{1}{2}\right)$ and $z\in\Delta^+$. The relation (1.3) leads to

$$\begin{aligned} \max\left\{|f(z)|^2 : f\in\mathcal{T}(1/2)\right\} &= \max\left\{|w|^2 : w\in D_z(\mathcal{T}(1/2))\right\} = \\ \max\left\{|f_t(z)|^2 : t\in[-1,1]\right\} &= \max\left\{|zk_t(z)| : t\in[-1,1]\right\} = \\ |z|\max\left\{|w| : w\in D_z(\mathcal{T})\right\} &= |z|\max\left\{|h(z)| : h\in\mathcal{T}\right\}\ . \end{aligned} \tag{2.5}$$

The inequality (2.4) is a simple consequence of (2.5) and Theorem 2.2. □

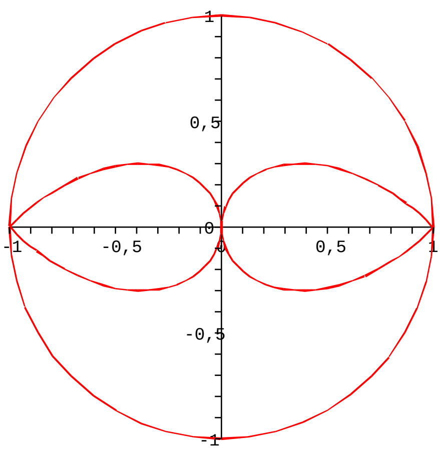

FIGURE 1. Subsets of Δ described in Theorems 2.2 and 2.3.

Theorem 2.4. *For $f \in \mathcal{T}\left(\frac{1}{2}\right)$ and $z \in \Delta$ we have*

$$|f(z)| \geq \begin{cases} \left|\frac{z}{1+z}\right| & \operatorname{Re}\frac{1}{z} \geq 1 \\ \left|\frac{\operatorname{Im} z}{1-z^2}\right| & \left|\operatorname{Re}\frac{1}{z}\right| < 1 \\ \left|\frac{z}{1-z}\right| & \operatorname{Re}\frac{1}{z} \leq -1 \ . \end{cases} \tag{2.6}$$

The extremal functions in this theorem are: $f(z) = \frac{z}{1+z}$, $f(z) = \frac{z(1+(2\alpha-1)z)}{1-z^2}$, where $2\alpha - 1 = -\operatorname{Re}\frac{1}{z}$, and $f(z) = \frac{z}{1-z}$.

It can be easily observed that both sets given by the inequalities: $\operatorname{Re}\frac{1}{z} \geq 1$ and $\operatorname{Re}\frac{1}{z} \leq -1$ are disks with the same radius 1 and centered in 1 and -1 respectively.

To prove Theorem 2.4 it is enough to write

$$\min\{|f(z)| : f \in \mathcal{T}(1/2)\} = \min\{|g_\alpha(z)| : \alpha \in [-1,1]\} = \left|\frac{z^2}{1-z^2}\right| \min\left\{\left|\frac{1}{z} + (2\alpha - 1)\right| : \alpha \in [-1,1]\right\} \tag{2.7}$$

and observe that for $z \neq 0$

$$\min\left\{\left|\frac{1}{z} + p\right| : p \in [-1,1]\right\} = \begin{cases} \left|\frac{1}{z} - 1\right| & \text{if} \quad \operatorname{Re}\left(\frac{1}{z} - 1\right) \geq 0 \\ \left|\operatorname{Im}\frac{1}{z}\right| & \text{if} \quad \left|\operatorname{Re}\frac{1}{z}\right| < 1 \\ \left|\frac{1}{z} + 1\right| & \text{if} \quad \operatorname{Re}\left(\frac{1}{z} + 1\right) \leq 0 \ . \end{cases}$$

From Theorem 2.3 and Theorem 2.4 we get

Corollary 2.5. *For $f \in \mathcal{T}\left(\frac{1}{2}\right)$ and $|z| = r$ we have*

$$\frac{r}{1+r} \leq |f(z)| \leq \frac{r}{1-r}. \tag{2.8}$$

Equalities hold for $f(z) = \frac{z}{1-z}$ at points $z = \pm r$ and for $f(z) = \frac{z}{1+z}$ at points $z = \mp r$.

Proof. We shall prove only the upper estimate. The proof of the lower one is similar and will be omitted.

Assume that $z = re^{i\varphi}$, where r is a fixed number from $(0,1)$, and φ varies in $[0,\pi]$. According to Theorem 2.3 we shall estimate $|f(z)|$ in three sets separately.

I. If $\operatorname{Re}\frac{1+z^2}{z} \geq 2$ (and $|z| = r$) then $\cos\varphi \geq \frac{2r}{1+r^2}$ and

$$\left|\frac{z}{1-z}\right| = \frac{r}{\sqrt{1 - 2r\cos\varphi + r^2}} \leq \frac{r}{1-r} \ .$$

II. If $|\operatorname{Re}\frac{1+z^2}{z}| < 2$ (and $|z| = r$) then $\varphi \in (\varphi_0, \pi - \varphi_0)$, where $\varphi_0 = \arccos\frac{2r}{1+r^2}$ and

$$|z|\sqrt{\frac{|z|}{(1-|z|^2)\,|\operatorname{Im} z|}} = \frac{r}{\sqrt{(1-r^2)\sin\varphi}} \leq \frac{r\sqrt{1+r^2}}{1-r^2} \ .$$

III. If $\operatorname{Re}\frac{1+z^2}{z} \le -2$ (and $|z| = r$) then $\cos\varphi \le -\frac{2r}{1+r^2}$ and

$$\left|\frac{z}{1+z}\right| = \frac{r}{\sqrt{1+2r\cos\varphi+r^2}} \le \frac{r}{1-r} \ .$$

Since $\frac{r\sqrt{1+r^2}}{1-r^2} \le \frac{r}{1-r}$ the proof of the right hand side inequality in (2.8) is complete. □

Similarly as in Theorems 2.3 and 2.4 we can estimate the argument of $f(z)$. Taking into account Lemma 2.1, the relation (1.3) and Theorem 2.2 we obtain for $z \in \Delta^+$

$$\begin{aligned} &\max\{\arg f(z) : f \in \mathcal{T}(1/2)\} = \max\{\arg f_t(z) : t \in [-1,1]\} = \\ &\frac{1}{2}\max\left\{\arg\left(f_t(z)\right)^2 : t \in [-1,1]\right\} = \frac{1}{2}\max\{\arg zk_t(z) : t \in [-1,1]\} = \\ &\frac{1}{2}\left(\arg z + \max\{\arg k_t(z) : t \in [-1,1]\}\right) = \\ &\frac{1}{2}\left(\arg z + \arg k_1(z)\right) = \frac{1}{2}\arg\left(f_1(z)\right)^2 = \arg f_1(z) \ . \end{aligned} \tag{2.9}$$

The above equalities hold if minimum is taken instead of maximum and the functions k_{-1} and f_{-1} instead of k_1 and f_1. In the same way we can estimate $\arg f(z)$ for $z \in \Delta^-$. This argument leads us to the following theorem.

Theorem 2.6. *For $f \in \mathcal{T}\left(\frac{1}{2}\right)$ we have*

$$\arg\frac{z}{1+z} \le \arg f(z) \le \arg\frac{z}{1-z} \quad \textit{for} \quad z \in \Delta^+ \tag{2.10}$$

$$\arg\frac{z}{1-z} \le \arg f(z) \le \arg\frac{z}{1+z} \quad \textit{for} \quad z \in \Delta^- \ . \tag{2.11}$$

The extremal functions are: $f(z) = \frac{z}{1+z}$ and $f(z) = \frac{z}{1-z}$, respectively.
In the paper [4] the following theorem was proved.

Theorem 2.7. *For every $r \in (0,1)$*

$$\bigcup_{f\in\mathcal{T}\left(\frac{1}{2}\right)} f(\Delta_r) = f_{-1}(\Delta_r) \cup f_1(\Delta_r) \ .$$

In other words it means that each set $f(\Delta_r)$ for $f \in \mathcal{T}\left(\frac{1}{2}\right)$ is included in $f_{-1}(\Delta_r) \cup f_1(\Delta_r)$. Both sets $f_{-1}(\Delta_r)$, $f_1(\Delta_r)$ are disks, centered in $-\frac{r^2}{1-r^2}$ and $\frac{r^2}{1-r^2}$ respectively and having the same radius $\frac{r}{1-r^2}$. Therefore

Corollary 2.8. *If $f \in \mathcal{T}\left(\frac{1}{2}\right)$ then for $|z| = r$*

$$a)\ |\operatorname{Re} f(z)| \le \frac{r}{1-r} \ ,$$

$$b)\ |\operatorname{Im} f(z)| \le \frac{r}{1-r^2} \ .$$

Remark 2.9. The result of Corollary 2.5 can be obtained considering the class $\mathcal{R}$ consisting of functions satisfying in Δ the condition

$$\operatorname{Re}\frac{f(z)}{z} > \frac{1}{2} .$$

A function $f \in \mathcal{R}$ can be associated with a function $p \in \mathcal{P}_R$ as follows

$$f(z) = \frac{1}{2}\, z\left(p(z)+1\right) .$$

From estimates valid for $\mathcal{P}_R$ we get for $|z| = r$ that

$$|f(z)| \le \frac{1}{2}\, r\left(\frac{1+r}{1-r}+1\right) = \frac{r}{1-r} . \tag{2.12}$$

Analogously,

$$f'(z) = \frac{1}{2}\ \left(zp'(z)+p(z)+1\right)$$

results in

$$|f'(z)| \le \frac{1}{2}\ \left(\frac{2r}{1-r^2}+\frac{1+r}{1-r}+1\right) = \frac{1}{(1-r)^2} . \tag{2.13}$$

Equalities in the above estimates hold if $p(z) = \frac{1+z}{1-z}$ or $p(z) = \frac{1-z}{1+z}$, and consequently, if $f(z) = \frac{z}{1-z}$ or $f(z) = \frac{z}{1+z}$.

On the other hand, it was proved (see,[5]) that $\mathcal{T}\left(\frac{1}{2}\right) \subset \mathcal{R}$. Moreover, the extremal functions in (2.12) and (2.13) belong to $\mathcal{T}\left(\frac{1}{2}\right)$. Therefore,

Corollary 2.10. *For $f \in \mathcal{T}\left(\frac{1}{2}\right)$ and $|z| = r$ we have*

$$|f'(z)| \le \frac{1}{(1-r)^2} . \tag{2.14}$$

3. Conclusions for other classes

Taking into account extremal functions in the results stated above we obtain the conclusions concerning subclasses of $\mathcal{T}\left(\frac{1}{2}\right)$ mentioned in Section 1.
From Theorems 2.3 and 2.4 we get

Corollary 3.1. *For $f \in \mathcal{S}_R^*\left(\frac{1}{2}\right)$ and $z \in \Delta \setminus \{0\}$ we have*

$$|f(z)| \le \begin{cases} \left|\frac{z}{1-z}\right| & \operatorname{Re}\frac{1+z^2}{z} \ge 2 \\ |z|\sqrt{\frac{|z|}{(1-|z|^2)|\operatorname{Im} z|}} & \left|\operatorname{Re}\frac{1+z^2}{z}\right| < 2 \\ \left|\frac{z}{1+z}\right| & \operatorname{Re}\frac{1+z^2}{z} \le -2 . \end{cases} \tag{3.1}$$

Corollary 3.2. *For $f \in \mathcal{K}_R(i)$ and $z \in \Delta \setminus \{0\}$ we have*

$$|f(z)| \geq \begin{cases} \left|\frac{z}{1+z}\right| & \mathrm{Re}\,\frac{1}{z} \geq 1 \\ \left|\frac{\mathrm{Im}\,z}{1-z^2}\right| & \left|\mathrm{Re}\,\frac{1}{z}\right| < 1 \\ \left|\frac{z}{1-z}\right| & \mathrm{Re}\,\frac{1}{z} \leq -1\ . \end{cases} \tag{3.2}$$

From the results stated above and from Corollaries 2.8 and 2.10 we conclude

Corollary 3.3. *If A is one of the following classes $\mathcal{K}_R$, $\mathcal{K}_R(i)$, $\mathcal{S}^*_R(\frac{1}{2})$, then for $|z| = r$ and $f \in A$*

$$a)\ |\,\mathrm{Re}\, f(z)| \leq \frac{r}{1-r}\ ,$$
$$b)\ |\,\mathrm{Im}\, f(z)| \leq \frac{r}{1-r^2}\ ,$$
$$c)\ \frac{r}{1+r} \leq |f(z)| \leq \frac{r}{1-r}\ ,$$
$$d)\ |f'(z)| \leq \frac{1}{(1-r)^2}\ .$$

The estimates in c) and d) are well-known. They were obtained by Gronwall and Loewner (for $\mathcal{K}$) and by Robertson (for $\mathcal{K}(i)$, $\mathcal{S}^*(\frac{1}{2})$), see for example [2]. Of course, they are true also for functions with real coefficients.

Corollary 3.4. *If A is one of the following classes $\mathcal{K}_R$, $\mathcal{K}_R(i)$, $\mathcal{S}^*_R(\frac{1}{2})$ then for $f \in A$*

$$\arg\frac{z}{1+z} \leq \arg f(z) \leq \arg\frac{z}{1-z} \quad \textit{for} \quad z \in \Delta^+$$
$$\arg\frac{z}{1-z} \leq \arg f(z) \leq \arg\frac{z}{1+z} \quad \textit{for} \quad z \in \Delta^-\ .$$

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Book reviews

Daniel Liberzon, Calculus of Variations and Optimal Control. A Concise Introduction, Princeton University Press, Princeton and Oxford, 2012, xv + 235 pp, ISBN: 978-0-691-15187-8.

It is tautology to say that there are many books on calculus of variations and optimal control. Besides this D. Liberzon decided to write one more and the result is in front of us. The author explicitly motivates his decision on writing a new book in its Preface. He wanted a book satisfying all the following features:

- appropriate presentation level – the author wanted a friendly introductory text accessible to graduate students;
- logical and notational consistency among topics – the author wanted a unification of notations and presentation between calculus of variations, optimal control, and the Hamilton-Jacobi-Bellman theory;
- proof of the maximum principle – the author emphasizes that a complete proof is rather long, but is indispensable in a consistent book;
- historical perspective – the topic allows some very instructive views on the development of the calculus of variations, optimal control, and dynamic programming, based on the contributions of many experts and schools of research. At the same time at each step there are fascinating and challenging examples in technological and in economic fields;
- manageable size – the present book is designed for a one semester lecture and therefore its length is rather limited.

We have to mention clearly and undoubtedly that the author succeeded to achieve his goals.

Chapter 1, *Introduction*, presents some results related to finite-dimensional and infinite-dimensional optimization.

Chapter 2, *Calculus of variations*, introduces some examples, then basic calculus of variations problem, first-order conditions for weak extrema, Hamiltonian formalism and mechanics, variational problems with constraints, and second-order conditions.

Chapter 3, *From calculus of variations to optimal control*, deals with necessary conditions for strong extrema, calculus of variations versus optimal control, optimal control problem formulation and assumptions, and variational approach to the fixed-time, free-endpoint problem.

Chapter 4, *The maximum principle*, introduces statement of the maximum principle, proof of the maximum principle (when the Lagrangian does not depend on time), discussion of the maximum principle, time-optimal control problems, and existence of optimal controls.

Chapter 5, *The Hamilton-Jacobi-Bellman theory*, deals with dynamic programming and the HJB equation, HJB equation versus the maximum principle, and viscosity solutions of the HJB equations.

Chapter 6, *The linear quadratic regulator*, contains the topics: finite-horizon LQR problem and infinite-horizon LQR problem.

Chapter 7, *Advanced topics*, is dedicated to maximum principle on manifolds, HJB equation, canonical equations, and characteristics, Riccati equations and inequalities in robust control, and maximum principle for hybrid control systems.

Each chapter ends with a rich and useful section of notes and references. The exercises are merely problems or even theorems. The author of the book presents a large list of references and a detailed index of notions, names, and symbols. The graphical presentation of the book is pleasant.

As final remarks we have to emphasize that this book is well written, it fully deserves all its goals mentioned at the beginning of the review, and is a pleasure to read it.

Marian Mureşan

Advanced Courses of Mathematical Analysis, IV, *Proceedings of the Fourth International School – In the Memory of Professor Antonio Aizpuru Tomás*, F. Javier-Pérez-Fernández and Fernando Rambla-Barreno, (Editors), World Scientific Publishers, London - Singapore 2012, xii+247 pp, ISBN:13-978-981-4335-80-5 and 10-981-4335-80-0.

This is the fourth International Course of Mathematical Analysis in Andalusia held in Jerez (Cádiz), 8–12 September 2009. The first one took place in Cádiz (2002), the second in Granada and the third in Rábida (Huelva) (2007), all their Proceedings being published with World Scientific. This course was initially planned to be held in Alméria, but after the sad and premature death of Professor Antonio Aizpuru at the age of 53, one of the pioneers of these events, the organizers decided to change the place to Cádiz and to dedicate it to the memory of Professor Aizpuru. A biographical sketch presented at the opening ceremony is included in the present volume: F. Javier Pérez-Fernández, *In memoriam: Professor Antonio Aizpuru.* The lectures consisted of seminars, called mini-courses, taking three hours along three days, and plenary talks of one hour each. Presented by renown experts in their fields, the lectures survey some specific domains, contain the state-of-the-art and emphasize open problems deserving further attention.

The first part of the book, Part A, contains the written versions of three mini-courses, Pietro Aiena, *Weyl theorems for bounded linear operators on Banach spaces*, Joe Diestel and Angela Spalsbury, *Finitely additive measures in action*, and Thomas Schlumprecht, *Sampling and recovery of bandlimited functions and applications to signal processing.*

Part B contains some plenary lectures, two of them, F. J. García-Pacheco, *Some results on the local theory of normed spaces since 2002*, and J. B. Seoane-Sepúlveda, *Summability and lineability in the work of Antonio Aizpuru Tomás*, are concerned with areas related to the work Professor Aizpuru, surveying some of his essential contributions.

Other lectures (there are 6 besides the two mentioned above) deal with topics as isometric shifts between spaces of continuous (J. Araujo), uniform algebras of holomorphic functions on the unit ball of some complex Banach spaces (R. M. Aron and P. Galindo), linear preservers on Banach algebras and Kaplansky problem (M. Mbekhta), bounded approximation property and Banach operator ideals (E. Oja), generalizations of Banach-Stone theorem to linear and bilinear mappings on spaces of continuous or Lipschitz functions (F. Rambla-Barreno), and Hardy-minus-Identity operator on some function spaces (J. Soria).

As for the previous ones, the aim of the present course was to bring together prominent specialists in real, complex and functional analysis to expose new results and to pose open questions and techniques which could be effective in their solution. By collecting results in various domains of Complex Analysis, Functional Analysis and Measure Theory, in the focus of current research but scattered in the literature, the present volume is a valuable reference for researchers and graduate students and a valuable source inspiration for future research.

S. Cobzaş

Douglas S. Kurtz and Charles W. Swartz Theories of Integration - The integrals of Riemann, Lebesgue, Henstock-Kurzweil and Mc Shane, World Scientific, London - Singapore - Beijing, 2012, xv + 294 pages, ISBN: 13 978-981-4368-99-5 and 10 981-4368-99-7.

The book contains a clear and thorough presentation of four types of integrals – Riemann, Lebesgue, Henstock-Kurzweil and McShane – first on intervals in $\mathbb{R}$ and then on subsets of $\mathbb{R}^n$. The integrals of Denjoy and Perron are only briefly mentioned. The main idea of the authors is to show how successive generalizations of the notion of integral fix some deficiencies of the previous ones – the advantage of Lebesgue integral over the Riemann integral consists in countable additivity and better convergence theorems, while the Henstock-Kurzweil offers a very general form of the fundamental theorem of calculus.

The book starts with a short presentation of the notion of areas of plane figures, including the exhaustion method of Archimedes for the calculation of the area of a disc.

Riemann's integration theory is developed in the second chapter and includes Lebesgue's criterion of integrability, the change of variable formula, and a treatment of improper integrals.

The third chapter, *Convergence theorems and the Lebesgue integral*, starts with Lebesgue's axioms for a general integral and their consequences. Lebesgue measurable sets in $\mathbb{R}^n$ are introduced via outer measures (Carathéodori's definition) and then one proves some fundamental results for measurable functions – Egorov and Luzin theorems, the approximation by step functions. The chapter ends with the presentation of some fundamental results on the Lebesgue integral, including Mikusinski's remarkable characterization of Lebesgue integrability through series of step functions, which is then used in the proofs of the theorems of Fubini and Tonelli.

The definition and the basic properties of Henstock-Kurzweil integral are presented in the fourth chapter, *Fundamental theorem of calculus and the Henstock-Kurzweil integral*. Here it is also proved the a.e. differentiability of monotone and absolutely continuous functions and variants of the fundamental theorem of calculus (the validity of $\int_a^b f' = f(b) - f(a)$) for various types of integrals. One shows also that a function f is Lebesgue integrable iff $|f|$ is Henstock-Kurzweil integrable.

The last chapter of the book, Ch. 5, *Absolute integrability and the McShane integral*, is concerned with the properties of the McShane integral which is, in fact, equivalent to the Lebesgue integral, allowing a Riemann type treatment of it. A key notion used in Chapters 4 and 5 in the proofs of convergence results is that of uniform integrability.

The book is clearly written with cleaver proofs of some fundamental results in real analysis. All the notions and results are accompanied by comments and examples and each chapter ends with a set of exercises completing the main text (some of them ask to fill in details of some proofs).

This is the second edition of a successful book. With respect to the first one, beside corrections and new proofs to some results, some new material has been added as, for instance, the convolution product and approximate identities with applications to Weierstrass type approximation results. The chapters are relatively independent, so that each one can be used for an introduction to a specific topic. The book can be recommended as a base text for introductory courses in real analysis or for self-study.

Valer Anisiu

Martin Moskowitz and Fortios Paliogiannis Functions of several variables, World Scientific, London - Singapore - Beijing, 2011, xv + 716 pages, ISBN: 13 978-981-4299-27-5 (pbk) and 10 981-4299-27-8 (pbk).

This is a course on Mathematical Analysis dedicated to functions of several variables. The Calculus of real-valued functions of one real variable is assumed to be known by the potential reader, so that the authors pass directly in the first two chapters, 1. *Basic features of Euclidean space* $\mathbb{R}^n$, and 2. *Functions on Euclidean spaces*, to several variables by an introduction to the Euclidean space $\mathbb{R}^n$, its topology and continuous functions on subsets of $\mathbb{R}^n$. The differentiability theory is studied in the third chapter, *Differential calculus in several variables*, which includes mean value theorems, Taylor's formula, extrema and conditioned extrema, studied through the implicit function theorem. Some pleasant surprises here are a proof of Sylvester's criterium on the positivity of a quadratic form and the inclusion of Morse lemma on critical values of differentiable vector functions. Integral calculus, meaning Riemann and Darboux integrals over Jordan measurable subsets of $\mathbb{R}^n$ and including Fubini's theorem, is treated in Ch. 4, *Integral calculus in several variables*. As more advanced topics we mention Sard's theorem on singular values of differentiable mappings and Urysohn's theorem on the partition of unity. The change of variables formula, one of the most challenging theorems in Calculus is treated in the fifth chapter, containing also a fairly complete treatment of improper multiple integrals with applications to some remarkable improper integrals as the classical $\int_{-\infty}^{+\infty} \exp(-x^2)dx$ and Euler's Gamma and Beta functions. This chapter includes also a brief introduction to the Fourier transform and to the Schwartz space.

For the sake of simplicity and accessibility, the presentation in Chapter 6, *Line and surface integrals*, is restricted to $\mathbb{R}^3$. This allows to the authors to treat in details, and at the same time in an intuitive form, some deep results as Green's and Stokes' theorems, and Poincaré's lemma on closed forms. A brief discussion on differential forms in $\mathbb{R}^3$ and exterior differentiation as well as Milnor's analytic proof of the Brouwer's fixed point theorem are also included.

In the last two chapters, 7, *Elements of ordinary and partial differential equations*, and 8, *An introduction to the calculus of variations*, the so far developed machinery is applied to these two important areas of mathematics.

To make the book more attractive the authors have include a lot of examples and applications from physics, mechanics and economy.

Each chapter ends with a set of solved problems and exercises, some of these being accompanied with solutions, inviting the reader to check his/her comprehension of the theoretical matter.

Written at an intermediate level, between undergraduate and upper undergraduate, the present book could be a good companion for both undergraduate students desiring to know more on Mathematical Analysis, as well as for upper level undergraduate (or even graduate) who want to clarify some notions from their undergraduate course.

Tiberiu Trif