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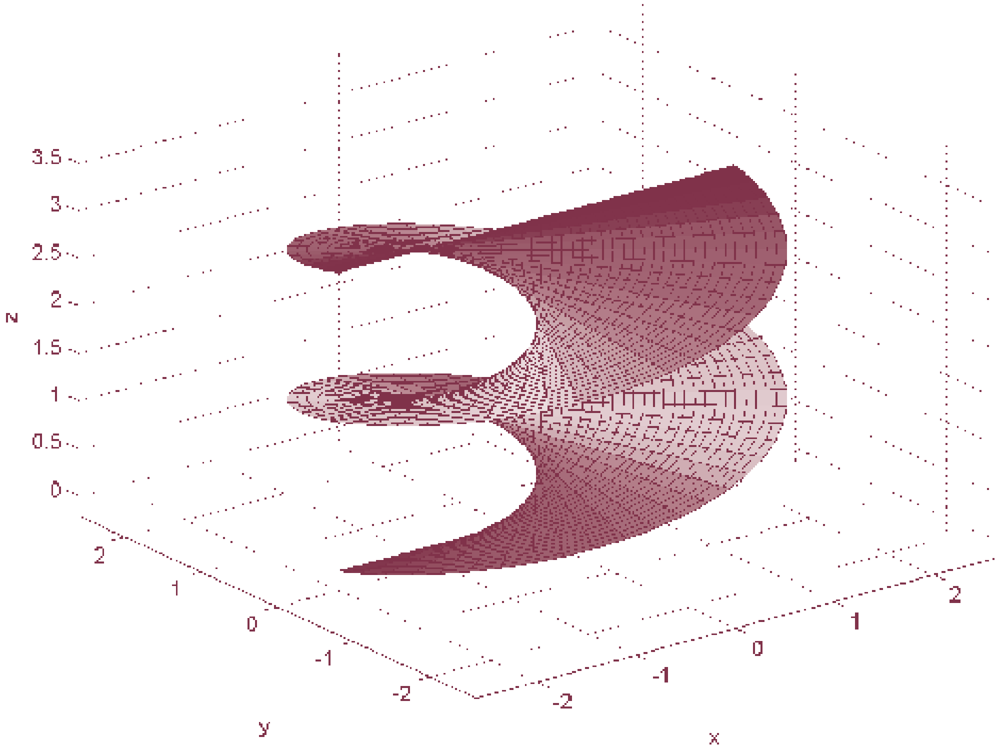

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On the hyper-Wiener index of unicyclic graphs with given matching number

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Abstract. We determine the minimum hyper-Wiener index of unicyclic graphs with given number of vertices and matching number, and characterize the extremal graphs.

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Keywords: Hyper-Wiener index, Wiener index, distance, matching number, unicyclic graphs.

1. Introduction

Let G be a simple connected graph with vertex set $V(G)$. For $u, v \in V(G)$, the distance $d_G(u, v)$ or d_{uv} between u and v in G is the length of a shortest path connecting them. The Wiener index of G is defined as [7, 13]

$$W(G) = \sum_{\{u,v\} \subseteq V(G)} d_{uv}.$$

The Wiener index has found various applications in chemical research [11] and has been studied extensively in mathematics [3, 4].

As a variant of the Wiener index, the hyper-Wiener index of the graph G is defined as [8]

$$WW(G) = \sum_{\{u,v\} \subseteq V(G)} \binom{d_{uv}+1}{2} = \frac{1}{2} \sum_{\{u,v\} \subseteq V(G)} \left(d_{uv}^2 + d_{uv}\right).$$

This graph invariant was proposed by Randić [12] for a tree and extended by Klein et al. [8] to a connected graph. It is used to predict physicochemical properties of organic compounds [1], and has also been extensively studied, see, e.g., [2, 5, 9, 10, 14].

Du and Zhou [4] determined the minimum Wiener indices of trees and unicyclic graphs with given number of vertices and matching number, respectively, and characterize the extremal graphs. Recently, Yu et al. [15] gave the minimum hyper-Wiener index of trees with given number of vertices and matching number, and characterized the unique extremal graph. We now determine the minimum hyper-Wiener index of unicyclic graphs with given number of vertices and matching number, and characterize the extremal graphs.

2. Preliminaries

For a connected graph G with $u \in V(G)$, let $W_G(u) = \sum_{v \in V(G)} d_{uv}$, and

$$WW_G(u) = \sum_{v \in V(G)} \binom{d_{uv}+1}{2}.$$

For $u \in V(G)$, let $d_G(u)$ be the degree of u in G, and the eccentricity of u, denoted by $ecc(u)$, is the maximum distance from u to all other vertices in G. Let S_n be the n-vertex star.

Lemma 2.1. *Let G be an n-vertex connected graph with a pendent vertex x being adjacent to vertex y, and let z be a neighbor of y different from x, where $n \geq 4$. Then*

$$WW(G) - WW(G-x) \geq 6n - 8 - 3d_G(y)$$

with equality if and only if $ecc(y) = 2$. Moreover, if $d_G(y) = 2$, then

$$WW(G) - WW(G-x-y) \geq 16n - 36 - 7d_G(z)$$

with equality if and only if $ecc(z) = 2$.

Proof. Note that

$$\begin{aligned} WW_G(x) &= \sum_{u \in V(G)\setminus\{x\}} \binom{1+d_{uy}+1}{2} \\ &= \sum_{u \in V(G)\setminus\{x\}} \binom{d_{uy}+1}{2} + \sum_{u \in V(G)\setminus\{x\}} (d_{uy}+1) \\ &= WW_G(y) - 1 + W_G(y) - 1 + n - 1 \\ &= WW_G(y) + W_G(y) + n - 3. \end{aligned}$$

Then

$$\begin{aligned} WW(G) - WW(G-x) &= WW_G(x) = WW_G(y) + W_G(y) + n - 3 \\ &\geq \binom{1+1}{2} d_G(y) + \binom{2+1}{2}(n-1-d_G(y)) \\ &\quad + d_G(y) + 2(n-1-d_G(y)) + n - 3 \\ &= 6n - 8 - 3d_G(y) \end{aligned}$$

with equality if and only if $ecc(y) = 2$.

If $d_G(y) = 2$, then $W_G(y) = W_G(z) + n - 4$,

$$\begin{aligned} WW_G(y) &= 1 + \sum_{u \in V(G)\setminus\{x,y\}} \binom{1 + d_{uz} + 1}{2} \\ &= 1 + \sum_{u \in V(G)\setminus\{x,y\}} \binom{d_{uz} + 1}{2} + \sum_{u \in V(G)\setminus\{x,y\}} (d_{uz} + 1) \\ &= 1 + WW_G(z) - 1 - 3 + W_G(z) - 1 - 2 + n - 2 \\ &= WW_G(z) + W_G(z) + n - 8, \end{aligned}$$

and thus

$$\begin{aligned} & WW(G) - WW(G - x - y) \\ = \;& WW_G(x) + WW_G(y) - 1 = 2WW_G(y) + W_G(y) + n - 4 \\ = \;& 2(WW_G(z) + W_G(z) + n - 8) + (W_G(z) + n - 4) + n - 4 \\ = \;& 2WW_G(z) + 3W_G(z) + 4n - 24 \\ \geq \;& 2\left(\binom{1+1}{2} d_G(z) + \binom{2+1}{2}(n - 1 - d_G(z))\right) \\ & +3\left[d_G(z) + 2(n - 1 - d_G(z))\right] + 4n - 24 \\ = \;& 16n - 36 - 7d_G(z) \end{aligned}$$

with equality if and only if $ecc(z) = 2$. □

Let C_n be a cycle with n vertices.

Lemma 2.2. [6, 8] *Let u be a vertex on the cycle C_r with $r \geq 3$. Then*

$$W_{C_r}(u) = \begin{cases} \frac{r^2-1}{4} & \text{if } r \text{ is odd,} \\ \frac{r^2}{4} & \text{if } r \text{ is even,} \end{cases}$$

$$WW_{C_r}(u) = \begin{cases} \frac{(r-1)(r+1)(r+3)}{24} & \text{if } r \text{ is odd,} \\ \frac{r(r+1)(r+2)}{24} & \text{if } r \text{ is even.} \end{cases}$$

For integers n and r with $3 \leq r \leq n$, let $S_{n,r}$ be the graph formed by attaching $n - r$ pendent vertices to a vertex of the cycle C_r.

Lemma 2.3. [14] *Let G be an n-vertex unicyclic graph with cycle length r, where $3 \leq r \leq n$. Then*

$$WW(G) \geq \begin{cases} \frac{72n^2+(2r^3+18r^2-98r-90)n-r^4-15r^3+25r^2+87r}{48} & \text{if } r \text{ is odd} \\ \frac{72n^2+(2r^3+18r^2-92r-72)n-r^4-15r^3+22r^2+72r}{48} & \text{if } r \text{ is even} \end{cases}$$

with equality if and only if $G = S_{n,r}$.

3. Results

For integers n and m with $2 \leq m \leq \lfloor \frac{n}{2} \rfloor$, let $\mathbb{U}(n,m)$ be the set of unicyclic graphs with n vertices and matching number m, and let $U_{n,m}$ be the unicyclic graph obtained by attaching a pendent vertex to $m-2$ noncentral vertices and adding an edge between two other noncentral vertices of the star S_{n-m+2}. Obviously, $U_{n,m} \in \mathbb{U}(n,m)$. By direct calculation, $WW(U_{n,m}) = \frac{1}{2}(3n^2 + m^2 + 6nm - 19n - 23m + 42)$.

For integer $m \geq 3$, let $\mathbb{U}_1(m)$ be the set of graphs in $\mathbb{U}(2m,m)$ containing a pendent vertex whose neighbor is of degree two. Let $\mathbb{U}_2(m) = \mathbb{U}(2m,m) \setminus \mathbb{U}_1(m)$. Let $H_{8,5}$ be the graph obtained by attaching three pendent vertices to three consecutive vertices of C_5. Let $H_{8,6}$ be the graph obtained by attaching two pendent vertices to two adjacent vertices of C_6. Let $H'_{8,6}$ be the graph obtained by attaching two pendent vertices to two vertices of distance two of C_6. Let $H''_{8,6}$ be the graph obtained by attaching two pendent vertices to two vertices of distance three of C_6.

Lemma 3.1. *Let $G \in \mathbb{U}_2(m)$ with $m \geq 4$. Then $WW(G) \geq \frac{1}{2}(25m^2 - 61m + 42)$ with equality if and only if $G = H_{8,5}$.*

Proof. Since $G \in \mathbb{U}_2(m)$, it is easily seen that $G = C_{2m}$ or G is a graph of maximum degree three obtained by attaching some pendent vertices to a cycle. If $G = C_{2m}$, then by Lemma 2.2,

$$\begin{aligned} WW(C_{2m}) &= \frac{(2m)^2(2m+1)(2m+2)}{48} = \frac{1}{6}(2m^4 + 3m^3 + m^2) \\ &> \frac{1}{2}(25m^2 - 61m + 42). \end{aligned}$$

Suppose that $G \neq C_{2m}$. Then G is a graph of maximum degree three obtained by attaching some pendent vertices to a cycle C_r, where $m \leq r \leq 2m-1$.

Case 1. $r = m$. Then every vertex on the cycle has degree three, and for any pendent vertex x and its neighbor y, by Lemmas 2.1 and 2.2, we have

$$\begin{aligned} WW(G) &= \frac{1}{2}m(WW_G(x) + WW_G(y)) \\ &= \frac{1}{2}m(2WW_G(y) + W_G(y) + 2m - 3) \\ &= \frac{1}{2}m\left(2\sum_{u \in V(C_m)} \binom{d_{uy}+1}{2} + 2\sum_{u \in V(G)\setminus V(C_m)} \binom{d_{uy}+1}{2}\right. \\ &\quad \left. + \sum_{u \in V(C_m)} d_{uy} + \sum_{u \in V(G)\setminus V(C_m)} d_{uy} + 2m - 3\right) \\ &= \frac{1}{2}m\left(2WW_{C_m}(y) + 2\sum_{u \in V(C_m)} \binom{1 + d_{uy} + 1}{2}\right. \\ &\quad \left. + W_{C_m}(y) + \sum_{u \in V(C_m)} (d_{uy} + 1) + 2m - 3\right) \end{aligned}$$

$$= \frac{1}{2}m\left(2WW_{C_m}(y) + 2\sum_{u\in V(C_m)}\binom{d_{uy}+1}{2} + 2\sum_{u\in V(C_m)}(d_{uy}+1)\right.$$
$$\left. + W_{C_m}(y) + \sum_{u\in V(C_m)}(d_{uy}+1) + 2m - 3\right)$$
$$= \frac{1}{2}m\left(4WW_{C_m}(y) + 4W_{C_m}(y) + 5m - 3\right)$$
$$= \begin{cases} \frac{1}{12}(m^4 + 9m^3 + 29m^2 - 27m) & \text{if } m \text{ is odd} \\ \frac{1}{12}(m^4 + 9m^3 + 32m^2 - 18m) & \text{if } m \text{ is even} \end{cases}$$
$$> \frac{1}{2}(25m^2 - 61m + 42).$$

Case 2. $r = m + 1$. Then there are precisely two adjacent vertices on the cycle of degree two in G. Let G' be the graph obtained from G by attaching two pendent vertices to the two adjacent vertices of degree two in G. For any pendent vertex x and its neighbor y in G', by the above conclusion and Lemma 2.2, we have

$$\begin{aligned} WW(G) &= WW(G') - 2WW_{G'}(x) + \binom{3+1}{2} \\ &= \frac{1}{2}(m+1)\left(4WW_{C_{m+1}}(y) + 4W_{C_{m+1}}(y) + 5(m+1) - 3\right) \\ &\quad -2(2WW_{C_{m+1}}(y) + 3W_{C_{m+1}}(y) + 4m + 1) + 6 \\ &= \frac{1}{2}\left((4m-4)WW_{C_{m+1}}(y) + (4m-8)W_{C_{m+1}}(y) + 5m^2 - 9m + 10\right) \\ &= \begin{cases} \frac{1}{12}(m^4 + 11m^3 + 35m^2 - 77m + 42) & \text{if } m \text{ is odd} \\ \frac{1}{12}(m^4 + 11m^3 + 32m^2 - 86m + 60) & \text{if } m \text{ is even} \end{cases} \\ &\geq \frac{1}{2}(25m^2 - 61m + 42) \end{aligned}$$

with equality if and only if $m = 4$, i.e., $G = H_{8,5}$.

Case 3. $m + 2 \leq r \leq 2m - 1$. First we consider the subcase $m \geq 5$. By Lemma 2.3,

$$\begin{aligned} WW(G) &\geq WW(S_{2m,r}) \\ &= \begin{cases} \frac{1}{48}\left(-r^4 + (4m-15)r^3 + (36m+25)r^2 \right. \\ \left. +(87 - 196m)r + 288m^2 - 180m\right) & \text{if } r \text{ is odd,} \\ \frac{1}{48}\left(-r^4 + (4m-15)r^3 + (36m+22)r^2 \right. \\ \left. +(72 - 184m)r + 288m^2 - 144m\right) & \text{if } r \text{ is even.} \end{cases} \end{aligned}$$

Let $f(r) = 48WW(S_{2m,r})$. For odd r, we have

$$f'(r) = -4r^3 + (12m - 45)r^2 + (72m + 50)r + 87 - 196m,$$

from which it is easy to check that $f'(r) > 0$, and thus $f(r)$ is increasing with respect to r, implying that

$$\begin{aligned} WW(G) &\geq \frac{1}{48} f(r) \geq \frac{1}{48} f(m+2) \\ &= \frac{1}{48}(3m^4 + 37m^3 + 195m^2 - 421m + 138) \\ &> \frac{1}{2}(25m^2 - 61m + 42). \end{aligned}$$

For even r, by similar arguments as above,

$$\begin{aligned} WW(G) &\geq \frac{1}{48} f(r) \geq \frac{1}{48} f(m+2) \\ &= \frac{1}{48}(3m^4 + 37m^3 + 204m^2 - 388m + 96) \\ &> \frac{1}{2}(25m^2 - 61m + 42). \end{aligned}$$

Now we consider the subcase $m = 4$. Then $r = 6, 7$, $G = H_{8,6}, H'_{8,6}, H''_{8,6}$ or $H_{8,7}$, and the hyper-Wiener indices of these four graph are respectively equal to 106, 110, 115, and 109, all larger than $99 = \frac{1}{2}(25 \times 4^2 - 61 \times 4 + 42)$.

The result follows by combining Cases 1-3. □

Let $H_{6,3}$ be the graph obtained by attaching a vertex to every vertex of a triangle. Let $H_{6,4}$ be the graph obtained by attaching two pendent vertices to two adjacent vertices of a quadrangle. Let $H_{6,5}$ be the graph obtained by attaching a pendent vertex to C_5. Then the following Lemma may be checked easily.

Lemma 3.2. *Among the graphs in* $\mathbb{U}(6,3)$, $H_{6,5}$ *is the unique graph with minimum hyper-Wiener index* 39, *and* $U_{6,3}$, $H_{6,3}$, $H_{6,4}$ *and* C_6 *are the unique graphs with the second minimum hyper-Wiener index* 42.

For $G \in \mathbb{U}_1(m)$, a vertex triple of G, denoted by (x, y, z), consist of three vertices x, y and z, where x is a pendent vertex of G whose neighbor y is of degree two, and z is the neighbor of y different from x. For the vertex triple (x, y, z) and a perfect matching M with $|M| = m$, we have $xy \in M$ and $d_G(z) \leq m + 1$.

Lemma 3.3. *Let* $G \in \mathbb{U}(8,4)$. *Then* $WW(G) \geq 99$ *with equality if and only if* $G = U_{8,4}$ *or* $H_{8,5}$.

Proof. If $G \in \mathbb{U}_2(4)$, then by Lemma 3.1, $WW(G) \geq \frac{1}{2}(25 \times 4^2 - 61 \times 4 + 42) = 99$ with equality if and only if $G = H_{8,5}$. Suppose that $G \in \mathbb{U}_1(4)$. Let (x, y, z) be a vertex triple of G. Then $G - x - y \in \mathbb{U}(6,3)$. If $G - x - y \neq H_{6,5}$, then by Lemma 2.1,

$$WW(G) \geq WW(G - x - y) + 16 \times 8 - 36 - 7d_G(z) \geq 42 + 92 - 7 \times 5 = 99$$

with equalities if and only if $G - x - y = U_{6,3}$, $H_{6,3}$, $H_{6,4}$ or C_6, $d_G(z) = 5$ and $ecc(z) = 2$, i.e., $G = U_{8,4}$. If $G - x - y = H_{6,5}$, then $d_G(z) \leq 4$, and by Lemma 2.1,

$$WW(G) \geq WW(H_{6,5}) + 16 \times 8 - 36 - 7d_G(z) \geq 39 + 92 - 7 \times 4 = 103 > 99.$$

The result follows. □

Lemma 3.4. *Let* $G \in \mathbb{U}(10,5)$. *Then* $WW(G) \geq 181$ *with equality if and only if* $G = U_{10,5}$.

Proof. If $G \in \mathbb{U}_2(5)$, then by Lemma 3.1, $WW(G) > \frac{1}{2}(25 \times 5^2 - 61 \times 5 + 42) = 181$. Suppose that $G \in \mathbb{U}_1(5)$. Let (x, y, z) be a vertex triple of G. Then $G-x-y \in \mathbb{U}(8,4)$, and by Lemmas 2.1 and 3.3,

$$WW(G) \geq WW(G-x-y) + 16 \times 10 - 36 - 7d_G(z) \geq 99 + 124 - 7 \times 6 = 181$$

with equalities if and only if $G - x - y = U_{8,4}$ or $H_{8,5}$, $d_G(z) = 6$ and $ecc(z) = 2$, i.e., $G = U_{10,5}$. □

Proposition 3.5. *Let* $G \in \mathbb{U}(2m, m)$, *where* $m \geq 2$.
(*i*) *If* $m = 3$, *then* $WW(G) \geq 39$ *with equality if and only if* $G = H_{6,5}$;
(*ii*) *If* $m \neq 3$, *then*

$$WW(G) \geq \frac{1}{2}(25m^2 - 61m + 42)$$

with equality if and only if $G = U_{4,2}$, C_4 *for* $m = 2$, $G = U_{8,4}$, $H_{8,5}$ *for* $m = 4$, *and* $G = U_{2m,m}$ *for* $m \geq 5$.

Proof. The case $m = 2$ is obvious since $\mathbb{U}(4,2) = \{U_{4,2}, C_4\}$ and $WW(U_{4,2}) = WW(C_4) = 10$. The cases $m = 3$ and $m = 4$ follow from Lemmas 3.2 and 3.3, respectively.

Suppose that $m \geq 5$. Let $g(m) = \frac{1}{2}(25m^2 - 61m + 42)$. We prove the result by induction on m. If $m = 5$, then the result follows from Lemma 3.4. Suppose that $m \geq 6$ and the result holds for graphs in $\mathbb{U}(2m-2, m-1)$. Let $G \in \mathbb{U}(2m, m)$. If $G \in \mathbb{U}_2(m)$, then by Lemma 3.1, $WW(G) > g(m)$. If $G \in \mathbb{U}_1(m)$, then for a vertex triple (x, y, z) of G, $G - x - y \in \mathbb{U}(2m-2, m-1)$, and thus by Lemma 2.1 and the induction hypothesis,

$$\begin{aligned} WW(G) &\geq WW(G-x-y) + 32m - 36 - 7d_G(z) \\ &\geq g(m-1) + 32m - 36 - 7(m+1) \\ &= \frac{1}{2}(25m^2 - 61m + 42) = g(m) \end{aligned}$$

with equality if and only if $G - x - y = U_{2m-2,m-1}$, $d_G(z) = m + 1$ and $ecc(z) = 2$, i.e., $G = U_{2m,m}$. □

Let $H_{7,5}$ be the graph obtained by attaching two pendent vertices to a vertex of C_5.

Theorem 3.6. *Let* $G \in \mathbb{U}(n, m)$, *where* $2 \leq m \leq \lfloor \frac{n}{2} \rfloor$.
(*i*) *If* $(n, m) = (6, 3)$, *then* $WW(G) \geq 39$ *with equality if and only if* $G = H_{6,5}$;
(*ii*) *If* $(n, m) \neq (6, 3)$, *then*

$$WW(G) \geq \frac{1}{2}(3n^2 + m^2 + 6nm - 19n - 23m + 42)$$

with equality if and only if $G = U_{4,2}$, C_4 *for* $(n, m) = (4, 2)$, $G = U_{5,2}$, C_5 *for* $(n, m) = (5, 2)$, $G = U_{7,3}$, $H_{7,5}$ *for* $(n, m) = (7, 3)$, $G = U_{8,4}$, $H_{8,5}$ *for* $(n, m) = (8, 4)$ *and* $G = U_{n,m}$ *otherwise.*

Proof. The case $(n,m)=(6,3)$ follows from Lemma 3.2. Suppose that $(n,m)\neq(6,3)$. Let $g(n,m)=\frac{1}{2}(3n^2+m^2+6nm-19n-23m+42)$.

If $G=C_n$ with $n\geq 7$, then by Lemma 2.2, $WW(G)>g(n,m)$.

If $G\neq C_n$ with $n>2m$, then there exist a pendent vertex x and a maximum matching M such that x is not M-saturated in G [16], and thus $G-x\in\mathbb{U}(n-1,m)$. Let y be the unique neighbor of x. Since M contains one edge incident with y, and there are $n-m$ edges of G outside M, we have $d_G(y)\leq n-m+1$.

Case 1. $m=2$. The result for $n=4$ follows from Proposition 3.5. If $n=5$, then by Lemma 2.3, the minimum hyper-Wiener index is achieved only by $S_{5,3}$, $S_{5,4}$, or C_5, and thus the result follows by noting that $WW(S_{5,3})=WW(C_5)=20<23=WW(S_{5,4})$ and $S_{5,3}=U_{5,2}$. If $n\geq 6$, then by Lemma 2.3, the minimum hyper-Wiener index is achieved only by $S_{n,3}$ or $S_{n,4}$, and thus the result follows by noting that $WW(S_{n,3})=\frac{1}{2}(3n^2-7n)<\frac{1}{2}(3n^2-n-24)=WW(S_{n,4})$ and $S_{n,3}=U_{n,2}$.

Case 2. $m=3$. Suppose first that $n=7$. Then $G-x\in\mathbb{U}(6,3)$. If $G-x=H_{6,5}$, then $d_G(y)\leq 4$, and by Lemma 2.1,

$$WW(G)\geq WW(G-x)+6\times 7-8-3d_G(y)\geq 39+34-12=61$$

with equalities if and only if $d_G(y)=4$ and $ecc(y)=2$, i.e., $G=H_{7,5}$, while if $G-x\neq H_{6,5}$, then by Lemmas 2.1 and 3.2,

$$WW(G)\geq WW(G-x)+6\times 7-8-3d_G(y)\geq 42+34-15=61$$

with equalities if and only if $G-x=U_{6,3}$, $H_{6,3}$, $H_{6,4}$ or C_6, $d_G(y)=5$ and $ecc(y)=2$, i.e., $G=U_{7,3}$. It follows that $WW(G)\geq 61$ with equality if and only if $G=H_{7,5}$ or $U_{7,3}$. For $n\geq 8$, we prove the result by induction on n. If $n=8$, then $G-x\in\mathbb{U}(7,3)$, and by Lemma 2.1,

$$WW(G)\geq WW(G-x)+6\times 8-8-3d_G(y)\geq 61+40-3\times 6=83$$

with equalities if and only if $G=H_{7,5}$ or $U_{7,3}$, $d_G(y)=6$ and $ecc(y)=2$, i.e., $G=U_{8,4}$. Suppose that $n\geq 9$ and the result holds for graphs in $\mathbb{U}(n-1,3)$. By Lemma 2.1 and the induction hypothesis,

$$\begin{aligned} WW(G) &\geq WW(G-x)+6n-8-3d_G(y)\\ &\geq g(n-1,3)+6n-8-3(n-2)\\ &= \frac{1}{2}(3n^2-n-18)=g(n,3)\end{aligned}$$

with equalities if and only if $G-x=U_{n-1,3}$, $d_G(y)=n-2$ and $ecc(y)=2$, i.e., $G=U_{n,3}$.

Case 3. $m=4$. The case $n=8$ follows from Lemma 3.3. For $n\geq 9$, we prove the result by induction on n. If $n=9$, then $G-x\in\mathbb{U}(8,4)$, and by Lemmas 2.1 and 3.3,

$$WW(G)\geq WW(G-x)+6\times 9-8-3d_G(y)\geq 99+46-3\times 6=127$$

with equalities if and only if $G=U_{8,4}$ or $H_{8,5}$, $d_G(y)=6$ and $ecc(y)=2$, i.e., $G=U_{9,4}$. Suppose that $n\geq 10$ and the result holds for graphs in $\mathbb{U}(n-1,4)$.

By Lemma 2.1 and the induction hypothesis,

$$\begin{aligned} WW(G) &\geq WW(G-x)+6n-8-3d_G(y) \\ &\geq g(n-1,4)+6n-8-3(n-3) \\ &= \frac{1}{2}(3n^2+5n-34)=g(n,4) \end{aligned}$$

with equalities if and only if $G-x=U_{n-1,4}$, $d_G(y)=n-3$ and $ecc(y)=2$, i.e., $G=U_{n,4}$.

Case 4. $m \geq 5$. We prove the result by induction on n (for fixed m). If $n=2m$, then the result follows from Proposition 3.5. Suppose that $n>2m$ and the result holds for graphs in $\mathbb{U}(n-1,m)$. Let $G \in \mathbb{U}(n,m)$. By Lemma 2.1 and the induction hypothesis,

$$\begin{aligned} WW(G) &\geq WW(G-x)+6n-8-3d_G(y) \\ &\geq g(n-1,m)+6n-8-3(n-m+1) \\ &= \frac{1}{2}(3n^2+m^2+6nm-19n-23m+42)=g(n,m) \end{aligned}$$

with equalities if and only if $G-x=U_{n-1,m}$, $d_G(y)=n-m+1$ and $ecc(y)=2$, i.e., $G=U_{n,m}$. □

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Fractional order partial hyperbolic differential equations involving Caputo's derivative

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Abstract. In the present paper we investigate the existence and uniqueness of solutions of the Darboux problem for the initial value problems (IVP for short), for some classes of hyperbolic fractional order differential equations by using some fixed point theorems.

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Keywords: Partial hyperbolic differential equation, fractional order, left-sided mixed Riemann-Liouville integral, Caputo fractional-order derivative, solution, fixed point.

1. Introduction

The idea of fractional calculus and fractional order differential equations and inclusions has been a subject of interest not only among mathematicians, but also among physicists and engineers. Indeed, we can find numerous applications in rheology, control, porous media, viscoelasticity, electrochemistry, electromagnetism, etc. [14, 15, 19, 20, 22, 27]. There has been a significant development in ordinary and partial fractional differential equations in recent years; see the monographs of Abbas *et al.* [3], Kilbas *et al.* [17], Miller and Ross [21], Samko *et al.* [26], the papers of Abbas and Benchohra [1, 2], Abbas *et al.* [4, 5], Belarbi *et al.* [8], Benchohra *et al.* [9, 10, 11], Diethelm [13], Kaufmann and Mboumi [16], Kilbas and Marzan [18], Mainardi [19], Podlubny *et al* [25], Vityuk [28], Vityuk and Golushkov [29], Vityuk and Mykhailenko [30, 31], Zhang [32] and the references therein.

Applied problems require definitions of fractional derivative allowing the utilization of physically interpretable initial conditions. Caputo's fractional derivative, originally introduced by Caputo [12] and afterwards adopted in the theory of linear viscoelasticity, satisfies this demand. For a consistent bibliography on this topic, historical remarks and examples we refer to [6, 7, 23, 24].

In [33], Zhang considered the existence and uniqueness of positive solutions for the following fractional order system

$$\begin{cases} D_\theta^r u(x,y) = f(x,y,u(x,y), D_\theta^{\rho_1} u(x,y),\ldots, \\ D_\theta^{\rho_n} u(x,y)); \text{ if } (x,y) \in (0,a] \times (0,b], \\ u(x,0) = u(0,y) = 0, \end{cases} \tag{1.1}$$

where $r = (\alpha, \beta) \in (0,1] \times (0,1],\ \rho_i = (\delta_i, \gamma_i);\ i = 1,\ldots,n$, and $0 \le \gamma_i < \alpha,\ 0 \le \delta_i < \beta$, and D_θ^r is Riemann-Liouville fractional derivative.

In the present paper we investigate the existence and uniqueness of solutions to fractional order system

$${}^cD_\theta^r u(x,y) = f(x,y,u(x,y),{}^c D_\theta^\rho u(x,y)); \text{ if } (x,y) \in J := [0,a] \times [0,b], \tag{1.2}$$

$$\begin{cases} u(x,0) = \varphi(x);\ x \in [0,a], \\ u(0,y) = \psi(y);\ y \in [0,b], \\ \varphi(0) = \psi(0), \end{cases} \tag{1.3}$$

where $a,b > 0,\ \theta = (0,0),\ r = (r_1,r_2),\ \rho = (\rho_1,\rho_2),\ 0 < \rho_i < r_i \le 1;\ i = 1,2,\ {}^cD_\theta^r$ is the standard Caputo's fractional derivative of order r, $f : J \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is a given function, $\varphi : [0,a] \to \mathbb{R}^n$, and $\psi : [0,b] \to \mathbb{R}^n$ are given absolutely continuous functions. We present three results for the problem (1.2)-(1.3), the two first results are based on Schauder's Fixed Point Theorem (Theorems 3.3 and 3.4) and the third one on Banach's contraction principle (Theorem 3.5). As an extension to the problem (4.1)-(4.2), we present two similar results (Theorems 4.1 and 4.2).

2. Preliminaries

In this section, we introduce notations, definitions, and preliminary facts which are used throughout this paper. By $C(J)$ we denote the Banach space of all continuous functions from J into $\mathbb{R}^n$ with the norm

$$\|w\|_\infty = \sup_{(x,y)\in J} \|w(x,y)\|,$$

where $\|.\|$ denotes a suitable complete norm on $\mathbb{R}^n$.

As usual, by $AC(J)$ we denote the space of absolutely continuous functions from J into $\mathbb{R}^n$ and $L^1(J)$ is the space of Lebesgue-integrable functions $w : J \to \mathbb{R}^n$ with the norm

$$\|w\|_{L^1} = \int_0^a \int_0^b \|w(x,y)\| dy dx.$$

Definition 2.1. [29] *Let $r = (r_1,r_2) \in (0,\infty) \times (0,\infty),\ \theta = (0,0)$ and $u \in L^1(J)$. The left-sided mixed Riemann-Liouville integral of order r of u is defined by*

$$(I_\theta^r u)(x,y) = \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^x \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} u(s,t) dt ds.$$

In particular,

$$(I_\theta^\theta u)(x,y) = u(x,y),\ (I_\theta^\sigma u)(x,y) = \int_0^x \int_0^y u(s,t)dtds;\ \text{for a.a } (x,y) \in J,$$

where $\sigma = (1,1)$.

For instance, $I_\theta^r u$ exists for all $r_1, r_2 \in (0,\infty)$, when $u \in L^1(J)$. Note also that when $u \in C(J)$, then $(I_\theta^r u) \in C(J)$, moreover

$$(I_\theta^r u)(x,0) = (I_\theta^r u)(0,y) = 0;\ x \in [0,a],\ y \in [0,b].$$

Example 2.2. Let $\lambda, \omega \in (-1,\infty)$ and $r = (r_1, r_2) \in (0,\infty) \times (0,\infty)$, then

$$I_\theta^r x^\lambda y^\omega = \frac{\Gamma(1+\lambda)\Gamma(1+\omega)}{\Gamma(1+\lambda+r_1)\Gamma(1+\omega+r_2)} x^{\lambda+r_1} y^{\omega+r_2},\ \text{for almost all } (x,y) \in J.$$

By $1-r$ we mean $(1-r_1, 1-r_2) \in [0,1) \times [0,1)$. Denote by $D_{xy}^2 := \frac{\partial^2}{\partial x \partial y}$, the mixed second order partial derivative.

Definition 2.3. [29] *Let $r \in (0,1] \times (0,1]$ and $u \in L^1(J)$. The Caputo fractional-order derivative of order r of u is defined by the expression ${}^cD_\theta^r u(x,y) = (I_\theta^{1-r} D_{xy}^2 u)(x,y)$.*

The case $\sigma = (1,1)$ is included and we have

$$({}^cD_\theta^\sigma u)(x,y) = (D_{xy}^2 u)(x,y),\ \text{for almost all } (x,y) \in J.$$

Example 2.4. Let $\lambda, \omega \in (-1,\infty)$ and $r = (r_1, r_2) \in (0,1] \times (0,1]$, then

$$D_\theta^r x^\lambda y^\omega = \frac{\Gamma(1+\lambda)\Gamma(1+\omega)}{\Gamma(1+\lambda-r_1)\Gamma(1+\omega-r_2)} x^{\lambda-r_1} y^{\omega-r_2},\ \text{for almost all } (x,y) \in J.$$

For $w, {}^c D_\theta^\rho w \in C(J)$, denote

$$\|w(x,y)\|_1 = \|w(x,y)\| + \|{}^cD_\theta^\rho w(x,y)\|.$$

We define the space X as the following

$$X = \{w \in C(J) \text{ having the Caputo fractional derivative of order } \rho,$$
$$\text{and } {}^cD_\theta^\rho w \in C(J)\}.$$

In the space X we define the norm

$$\|w\|_X = \sup_{(x,y)\in J} \|w(x,y)\|_1.$$

It is easy to see that $(X, \|.\|_X)$ is a Banach space.

3. Existence of solutions

Let us start by defining what we mean by a solution of the problem (1.2)-(1.3).

Definition 3.1. *A function $u \in X$ is said to be a solution of (1.2)-(1.3) if u satisfies equation (1.2) and conditions (1.3) on J.*

For the existence of solutions for the problem (1.2)-(1.3) we need the following lemma. Its proof is easily and left to the reader.

Lemma 3.2. *A function $u \in X$ is a solution of problem (1.2)-(1.3) if and only if u satisfies*

$$u(x,y) = \mu(x,y) + I_\theta^r f(x,y,u(x,y),{}^c D_\theta^\rho u(x,y)); \ (x,y) \in J,$$

where

$$\mu(x,y) = \varphi(x) + \psi(y) - \varphi(0).$$

Further, we present conditions for the existence of a solution of problem (1.2)-(1.3) by using Schauder's Fixed Point Theorem. In the following result we assume a sublinear growth condition on the right hand side, namely the function f.

Theorem 3.3. *Assume*

(H_1) *The function $f : J \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is continuous,*
(H_2) *There exist constants $c, c_i > 0;\ i = 0,1$ and $0 < \tau_j < 1;\ j = 0,1$ such that*

$$\|f(x,y,u(x,y),{}^c D_\theta^\rho u)\| \leq c + c_0\|u\|^{\tau_0} + c_1\|{}^c D_\theta^\rho u\|^{\tau_1},$$

for any $u \in \mathbb{R}^n$ and all $(x,y) \in J$.

Then there exists at least a solution for IVP (1.2)-(1.3) on J.

Proof. Transform the problem (1.2)-(1.3) into a fixed point problem. Consider the operator $N : X \to X$ defined by,

$$N(u)(x,y) = \mu(x,y) + I_\theta^r f(x,y,u(x,y),{}^c D_\theta^\rho u(x,y)). \tag{3.1}$$

By Lemma 3.2, the problem of finding the solutions of the IVP (1.2)-(1.3) is reduced to finding the solutions of the operator equation $N(u) = u$. Differentiating both sides of (3.1) by applying the Caputo fractional derivative, we get

$${}^c D_\theta^\rho (Nu)(x,y) = {}^c D_\theta^\rho \mu(x,y) + I_\theta^{r-\rho} f(x,y,u(x,y),{}^c D_\theta^\rho u(x,y)). \tag{3.2}$$

Since $N(u)$ and ${}^c D_\theta^\rho (Nu)$ are continuous on J, then N maps X into itself.

From (H_1) and the Arzela-Ascoli Theorem, the operator N is completely continuous.

Let $\tau = \max\{\tau_0, \tau_1\}$ and $B_R = \{u \in X : \|u\|_X \leq R\}$ be a closed bounded and convex subset of X, where

$$R > \max\{1, A, B, C, D\},$$

where

$$A = 4\|\mu\|_\infty + \frac{4ca^{r_1}b^{r_2}}{\Gamma(1+r_1)\Gamma(1+r_2)},$$

$$B = 4\|{}^c D_\theta^\rho \mu\|_\infty + \frac{4ca^{r_1-\rho_1}b^{r_2-\rho_2}}{\Gamma(1+r_1-\rho_1)\Gamma(1+r_2-\rho_2)},$$

$$C = \left(\frac{\Gamma(1+r_1)\Gamma(1+r_2)}{4(c_0+c_1+2)a^{r_1}b^{r_2}}\right)^{\frac{1}{1-\tau}},$$

$$D = \left(\frac{\Gamma(1+r_1-\rho_1)\Gamma(1+r_2-\rho_2)}{4(c_0+c_1+2)a^{r_1-\rho_1}b^{r_2-\rho_2}}\right)^{\frac{1}{1-\tau}}.$$

By (H_2), for every $u \in B_R$ and $(x, y) \in J$ we have

$$\begin{aligned}
& \|N(u)(x,y)\| \\
\leq\ & \|\mu(x,y)\| + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^x \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} \\
\times\ & \|f(s,t,u(s,t),^c D_\theta^\rho u(s,t)))\| dtds \\
\leq\ & \|\mu(x,y)\| + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^x \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} \\
\times\ & (c + c_0\|u(s,t)\|^{\tau_0} + c_1\|^c D_\theta^\rho u(s,t)\|^{\tau_1}) dtds \\
\leq\ & \|\mu\|_\infty + \frac{a^{r_1}b^{r_2}}{\Gamma(1+r_1)\Gamma(1+r_2)}(c + c_0\|u\|_X^{\tau_0} + c_1\|^c D_\theta^\rho u\|_X^{\tau_1}) \\
\leq\ & \|\mu\|_\infty + \frac{a^{r_1}b^{r_2}}{\Gamma(1+r_1)\Gamma(1+r_2)}\Big(c + (c_0+1)R^{\tau_0} + (c_1+1)R^{\tau_1}\Big) \\
\leq\ & \|\mu\|_\infty + \frac{a^{r_1}b^{r_2}}{\Gamma(1+r_1)\Gamma(1+r_2)}\Big(c + (c_0+c_1+2)R^{\tau}\Big) \\
=\ & \|\mu\|_\infty + \frac{a^{r_1}b^{r_2}}{\Gamma(1+r_1)\Gamma(1+r_2)}\Big(c + R^{\tau-1}R(c_0+c_1+2)\Big) \\
\leq\ & \frac{R}{4} + \frac{R}{4} = \frac{R}{2},
\end{aligned}$$

and

$$\begin{aligned}
& \|^c D_\theta^\rho N(u)(x,y)\| \\
\leq\ & \|^c D_\theta^\rho \mu(x,y)\| + \frac{1}{\Gamma(r_1-\rho_1)\Gamma(r_2-\rho_2)} \int_0^x \int_0^y (x-s)^{r_1-\rho_1-1} \\
\times\ & (y-t)^{r_2-\rho_2-1} f(s,t,u(s,t),^c D_\theta^\rho u(s,t)))\| dtds \\
\leq\ & \|^c D_\theta^\rho \mu(x,y)\| + \frac{1}{\Gamma(r_1-\rho_1)\Gamma(r_2-\rho_2)} \int_0^x \int_0^y (x-s)^{r_1-\rho_1-1} \\
\times\ & (y-t)^{r_2-\rho_2-1}(c + c_0\|u(s,t)\|^{\tau_0} + c_1\|^c D_\theta^\rho u(s,t)\|^{\tau_1}) dtds \\
\leq\ & \|^c D_\theta^\rho \mu\|_\infty + \frac{a^{r_1-\rho_1}b^{r_2-\rho_2}}{\Gamma(1+r_1-\rho_1)\Gamma(1+r_2-\rho_2)}(c + c_0\|u\|_X^{\tau_0} + c_1\|^c D_\theta^\rho u\|_X^{\tau_1}) \\
\leq\ & \|^c D_\theta^\rho \mu\|_\infty + \frac{a^{r_1-\rho_1}b^{r_2-\rho_2}}{\Gamma(1+r_1-\rho_1)\Gamma(1+r_2-\rho_2)} \\
\times\ & \Big(c + (c_0+1)R^{\tau_0} + (c_1+1)R^{\tau_1}\Big) \\
\leq\ & \|^c D_\theta^\rho \mu\|_\infty + \frac{a^{r_1-\rho_1}b^{r_2-\rho_2}}{\Gamma(1+r_1-\rho_1)\Gamma(1+r_2-\rho_2)}\Big(c + R^{\tau-1}R(c_0+c_1+2)\Big) \\
\leq\ & \frac{R}{4} + \frac{R}{4} = \frac{R}{2}.
\end{aligned}$$

Thus, for every $u \in B_R$ and $(x,y) \in J$ we have

$$\begin{aligned} \|N(u)(x,y)\|_1 &= \|N(u)(x,y)\| + \|^c D_\theta^\rho N(u)(x,y)\| \\ &\leq \frac{R}{2} + \frac{R}{2} = R. \end{aligned}$$

Hence $\|N(u)\|_X \leq R$ for $u \in B_R$, that is, $N(B_R) \subseteq B_R$. Schauder's fixed point theorem implies that the operator N has at least a fixed point $u^* \in B_R$. By Lemma 3.2, the problem (1.2)-(1.3) has a solution $u^* \in B_R$.
In the following result we assume a superlinear growth condition on the function f.

Theorem 3.4. *Assume* (H_1) *and the following hypothesis holds*

(H_2') *There exist constants* $d_i > 0;\ i = 0,1$ *and* $\nu_j > 1;\ j = 0,1$ *such that*

$$\|f(x,y,u(x,y),^c D_\theta^\rho u)\| \leq d_0\|u\|^{\nu_0} + d_1\|^c D_\theta^\rho u\|^{\nu_1},$$

for any $u \in \mathbb{R}^n$ *and all* $(x,y) \in J$.

Then the IVP (1.2)*-*(1.3) *has at least a solution on* J.

Proof. Consider the operator N defined by (3.1). In a similar way as in Theorem 3.3, we can complete this proof, provided if we take the closed, bounded and convex subset $B_R = \{u \in X : \|u\|_X \leq R\}$ of the space X, where

$$R < \min\left\{1, \frac{\|\mu\|_\infty}{3}, \mathcal{A}, \mathcal{B}\right\},$$

where

$$\mathcal{A} = \left(\frac{\Gamma(1+r_1)\Gamma(1+r_2)}{3(c_0+c_1+2)a^{r_1}b^{r_2}}\right)^{\frac{1}{1-\nu}},$$

$$\mathcal{B} = \left(\frac{\Gamma(1+r_1-\rho_1)\Gamma(1+r_2-\rho_2)}{3(c_0+c_1+2)a^{r_1-\rho_1}b^{r_2-\rho_2}}\right)^{\frac{1}{1-\nu}},$$

and

$$\nu = \min\{\nu_0, \nu_1\}.$$

Now, we present a uniqueness result for the problem (1.2)-(1.3) based on Banach's contraction principle.

Theorem 3.5. *Assume* (H_1) *and the following hypothesis holds*

(H_3) *There exist positive functions* $g, h \in C(J)$ *satisfying*

$$(I_\theta^r g)(x,y) + (I_\theta^{r-\rho})g(x,y) < \frac{1}{2},\ (I_\theta^r h)(x,y) + (I_\theta^{r-\rho}h)(x,y) < \frac{1}{2},$$

such that

$$\|f(x,y,u,^c D_\theta^\rho u) - f(x,y,v,^c D_\theta^\rho v)\| \leq g(x,y)\|u-v\| + h(x,y)\|^c D_\theta^\rho u -^c D_\theta^\rho v\|,$$

for all $(x,y) \in J$ *and* $u, v \in \mathbb{R}^n$.

Then the IVP (1.2)*-*(1.3) *has a unique solution on* J.

Proof. Consider the operator N defined in (3.1). Let $u, v \in X$. By assumption (H_3), for $(x, y) \in J$, we have

$$\begin{aligned}
& \|N(u)(x,y) - N(v)(x,y)\|_1 \\
= \; & \|I_\theta^r \Big(f(x,y,u(x,y),{}^c D_\theta^\rho u(x,y)) - f(x,y,v(x,y),{}^c D_\theta v^\rho(x,y)) \Big)\| \\
+ \; & \|{}^c D_\theta^r I_\theta^\rho \Big(f(x,y,u(x,y),{}^c D_\theta^\rho u(x,y)) - f(x,y,v(x,y),{}^c D_\theta v^\rho(x,y)) \Big)\| \\
\leq \; & I_\theta^r \Big(g(x,y)\|u(x,y) - v(x,y)\| + h(x,y)\|{}^c D_\theta^\rho u(x,y) -{}^c D_\theta^\rho v(x,y)\| \Big) \\
+ \; & I_\theta^{r-\rho} \Big(g(x,y)\|u(x,y) - v(x,y)\| + h(x,y)\|{}^c D_\theta^\rho u(x,y) -{}^c D_\theta^\rho v(x,y)\| \Big) \\
\leq \; & \Big(I_\theta^r g(x,y) + I_\theta^{r-\rho} g(x,y) \Big) \|u(x,y) - v(x,y)\| \\
+ \; & \Big(I_\theta^r h(x,y) + I_\theta^{r-\rho} h(x,y) \Big) \|{}^c D_\theta^\rho u(x,y) -{}^c D_\theta^\rho v(x,y)\| \\
\leq \; & \frac{1}{2}\|u(x,y) - v(x,y)\| + \frac{1}{2}\|{}^c D_\theta^\rho u(x,y) -{}^c D_\theta^\rho v(x,y)\| \\
\leq \; & \frac{1}{2}\|u(x,y) - v(x,y)\|_1.
\end{aligned}$$

Hence

$$\|N(u) - N(v)\|_X \leq \frac{1}{2}\|u - v\|_X,$$

which implies that N is a contraction operator. Then Banach's Contraction Principle assures that the operator N has a unique fixed point $u^* \in X$.

4. More general existence results

In this section, we present (without proof) two existence results to the more general class of fractional order IVP for the system

$$\begin{aligned}
{}^c D_\theta^r u(x,y) = f(x,y,u(x,y),{}^c D_\theta^{\rho_1} u(x,y),{}^c D_\theta^{\rho_2} u(x,y), ..., \\
{}^c D_\theta^{\rho_m} u(x,y)); \text{ if } (x,y) \in J,
\end{aligned} \tag{4.1}$$

$$\begin{cases} u(x,0) = \varphi(x); \; x \in [0,a], \\ u(0,y) = \psi(y); \; y \in [0,b], \\ \varphi(0) = \psi(0), \end{cases} \tag{4.2}$$

where $J := [0,a] \times [0,b], \; a, b > 0, \; \theta = (0,0), \; r = (r_1, r_2), \; \rho_i = (\rho_{i,1}, \rho_{i,2}), \; 0 < \rho_{i,j} < r_j \leq 1; \; i = 1, \ldots, m, \; j = 1, 2$ and f is a given continuous function.

For $w, {}^c D_\theta^{\rho_i} w \in C(J); \; i = 1, \ldots, m$, denote

$$\|w(x,y)\|_1 = \|w(x,y)\| + \sum_{i=1}^m \|{}^c D_\theta^{\rho_i} w(x,y)\|.$$

We define the following space

$$\begin{aligned}
\widetilde{X} = \{ & w \in C(J) \text{ having the Caputo fractional derivative of order } \rho_i, \\
& \text{and } {}^c D_\theta^{\rho_i} w \in C(J); \; i = 1, \ldots, m\}.
\end{aligned}$$

The space $\widetilde{X}$ is a Banach space with the norm

$$\|w\|_{\widetilde{X}} = \sup_{(x,y)\in J} \|w(x,y)\|_1.$$

The following result for the problem (4.1)-(4.2) is based on Schauder's fixed point theorem.

Theorem 4.1. *Assume that the function f satisfying one of the following conditions:*

(H_4) *There exist constants $c, c_i > 0;\ i = 0,\dots,m$ and $0 < \tau_j < 1;\ j = 1,\dots,m$ such that*

$$\|f(x,y,u(x,y),{}^cD_\theta^{\rho_1}u,{}^cD_\theta^{\rho_2}u,...,{}^cD_\theta^{\rho_m}u)\| \le c + c_0\|u\|^{\tau_0} + \sum_{i=1}^{m} c_i\|{}^cD_\theta^{\rho_i}u\|^{\tau_i},$$

for any $u \in \mathbb{R}^n$ and all $(x,y) \in J$.

(H_4') *There exist constants $d_i > 0;\ i = 0,1,\dots,m$ and $\nu_j > 1;\ j = 0,1,\dots,m$ such that*

$$\|f(x,y,u(x,y),{}^cD_\theta^{\rho_1}u,{}^cD_\theta^{\rho_2}u,...,{}^cD_\theta^{\rho_m}u)\| \le d_0\|u\|^{\nu_0} + \sum_{i=1}^{m} d_i\|{}^cD_\theta^{\rho_i}u\|^{\nu_i},$$

for any $u \in \mathbb{R}^n$ and all $(x,y) \in J$.

Then there exists at least a solution for IVP (4.1)-(4.2) on J.

By means of the Banach contraction principle, we have the following result for problem (4.1)-(4.2).

Theorem 4.2. *Assume*

(H_5) *There exist positive functions $g, h_i \in C(J);\ i = 1,\dots,m$ satisfying*

$$(I_\theta^r g)(x,y) + \sum_{i=1}^{m}(I_\theta^{r-\rho_i}g)(x,y) < \frac{1}{2},$$

$$\sum_{i=1}^{m}(I_\theta^r h_i)(x,y) + \sum_{j=1}^{m}\sum_{i=1}^{m}(I_\theta^{r-\rho_j}h_i)(x,y) < \frac{1}{2},$$

such that

$$\|f(x,y,u,{}^cD_\theta^{\rho_1}u,\dots,{}^cD_\theta^{\rho_m}u) - f(x,y,v,{}^cD_\theta^{\rho_1}v,\dots,{}^cD_\theta^{\rho_m}v)\| \le g(x,y)\|u-v\|$$

$$+ \sum_{i=1}^{m} h_i(x,y)\|{}^cD_\theta^{\rho_i}u - {}^cD_\theta^{\rho_i}v\|,$$

for all $(x,y) \in J$ and $u, v \in \mathbb{R}^n$.

Then the IVP (4.1)-(4.2) has a unique solution on J.

5. An example

As an application of our results we consider the following partial hyperbolic differential equations of the form

$$^{c}D_{\theta}^{r}u(x,y)=\frac{72}{72+9xy^2|u(x,y)|+8x^2y|^{c}D_{\theta}^{\rho}u(x,y)|};\ \text{ if } (x,y)\in[0,1]\times[0,1], \quad (5.1)$$

$$u(x,0)=x,\ u(0,y)=y^2;\ x,y\in[0,1]. \quad (5.2)$$

Set for $(x,y)\in[0,1]\times[0,1]$

$$f(x,y,u(x,y),^{c}D_{\theta}^{r}u(x,y))=\frac{72}{72+9xy^2|u(x,y)|+8x^2y|^{c}D_{\theta}^{\rho}u(x,y)|}.$$

Clearly, the function f is continuous. For each $u,\overline{u},\ v,\overline{v}\in\mathbb{R}$ and $(x,y)\in[0,1]\times[0,1]$ we have

$$\begin{aligned}&|f(x,y,u(x,y),v(x,y))-f(x,y,\overline{u}(x,y),\overline{v}(x,y))|\\ \leq\ &\frac{1}{8}xy^2\|u-\overline{u}\|+\frac{1}{9}x^2y\|v-\overline{v}\|.\end{aligned}$$

Hence condition (H_3) is satisfied with

$$g(x,y)=\frac{1}{8}xy^2 \text{ and } h(x,y)=\frac{1}{9}x^2y.$$

For each $(x,y)\in[0,1]\times[0,1]$ and $0<r_i<\rho_i\leq 1;\ i=1,2$ we have

$$(I_{\theta}^{r}g)(x,y)+(I_{\theta}^{r-\rho}g)(x,y)\leq\frac{2\Gamma(2)\Gamma(3)}{8\Gamma(2)\Gamma(3)}=\frac{1}{4}<\frac{1}{2},$$

and

$$(I_{\theta}^{r}h)(x,y)+(I_{\theta}^{r-\rho}h)(x,y)\leq\frac{2\Gamma(2)\Gamma(3)}{9\Gamma(2)\Gamma(3)}=\frac{2}{9}<\frac{1}{2}.$$

Consequently, Theorem 3.5 implies that problem (5.1)-(5.2) has a unique solution on $[0,1]\times[0,1]$.

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Stability in neutral nonlinear dynamic equations on time scale with unbounded delay

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Abstract. Let $\mathbb{T}$ be a time scale which is unbounded above and below and such that $0 \in \mathbb{T}$. Let $id - r : \mathbb{T} \to \mathbb{T}$ be such that $(id - r)(\mathbb{T})$ is a time scale. We use the contraction mapping theorem to obtain stability results about the zero solution for the following neutral nonlinear dynamic equations with unbounded delay

$$x^{\triangle}(t) = -a(t) x^{\sigma}(t) + b(t) G\left(x^2(t), x^2(t - r(t))\right) + c(t) \left(x^2\right)^{\widetilde{\triangle}}(t - r(t)), \ t \in \mathbb{T},$$

and

$$x^{\triangle}(t) = -a(t) x^{\sigma}(t) + b(t) G(x(t), x(t - r(t))) + c(t) x^{\widetilde{\triangle}}(t - r(t)), \ t \in \mathbb{T},$$

where $f^{\triangle}$ is the $\triangle$-derivative on $\mathbb{T}$ and $f^{\widetilde{\triangle}}$ is the $\triangle$-derivative on $(id - r)(\mathbb{T})$. We provide interesting examples to illustrate our claims.

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1. Introduction

The concept of time scales analysis is a fairly new idea. In 1988, it was introduced by the German mathematician Stefan Hilger in his Ph.D. thesis [11]. It combines the traditional areas of continuous and discrete analysis into one theory. After the publication of two textbooks in this area (by Bohner and Peterson, 2001, 2003, [5]-[6]), more and more researchers were getting involved in this fast-growing field of mathematics.

The study of dynamic equations brings together the traditional research areas of (ordinary and partial) differential and difference equations. It allows one to handle these two research areas at the same time, hence shedding light on the reasons for their seeming discrepancies. In fact, many new results for the continuous and discrete cases have been obtained by studying the more general time scales case (see [1]-[4], [8]-[13] and the references therein).

The reader can find more details on the materials and basic properties used in our section 2 in the first chapter of Bohner and Peterson book [5] pages 1-50 and can find good examples of dynamic equations in Chapter 2 [6] pages 17-46.

We have studied dynamic nonlinear equations with functional delay on a time scale and have obtained some interesting results concerning the existence of periodic solutions (see [1]-[3]) and this work is a continuation. Here, we focus on two neutral nonlinear dynamic equations which, for our delight, have not been yet studied by mean of fixed point technic on time scales.

There is no doubt that the Liapunov method have been used successfully to investigate stability properties of wide variety of ordinary, functional and partial equations. Nevertheless, the application of this method to problem of stability in differential equations with delay has encountered serious difficulties if the delay is unbounded or if the equation has unbounded term (see [7]-[10] and references therein). It has been noticed (see [8]-[10]) that some of theses difficulties vanish by using the fixed point technic. Other advantages of fixed point theory over Liapunov's method is that the conditions of the former are average while those of the latter are pointwise.

Below, we consider the following neutral nonlinear dynamic equations with unbounded delay given by

$$\begin{aligned} x^{\triangle}(t) &= -a(t)\,x^{\sigma}(t) + b(t)\,G\left(x^{2}(t), x^{2}(t-r(t))\right) \\ &\quad + c(t)\left(x^{2}\right)^{\widetilde{\triangle}}(t-r(t)),\ t \in \mathbb{T}, \end{aligned} \tag{1.1}$$

and

$$x^{\triangle}(t) = -a(t)\,x^{\sigma}(t) + b(t)\,G\left(x(t), x(t-r(t))\right) + c(t)\,x^{\widetilde{\triangle}}(t-r(t)),\ t \in \mathbb{T}, \tag{1.2}$$

where $\mathbb{T}$ is an unbounded above and below time scale. Throughout this paper we assume that $0 \in \mathbb{T}$ for convenience. We also assume that $a, b : \mathbb{T} \to \mathbb{R}$ are continuous and that $c : \mathbb{T} \to \mathbb{R}$ is continuously delta-differentiable. In order for the function $x(t-r(t))$ to be well-defined and differentiable over $\mathbb{T}$, we assume that $r : \mathbb{T} \to \mathbb{R}$ is positive and twice continuously delta-differentiable, and that $id - r : \mathbb{T} \to \mathbb{T}$ is an increasing mapping such that $(id - r)(\mathbb{T})$ is closed where id is the identity function on the time scale $\mathbb{T}$. Throughout this paper, intervals subscripted with a $\mathbb{T}$ represent real intervals intersected with $\mathbb{T}$. For example, for $a, b \in \mathbb{T}$, $[a,b]_{\mathbb{T}} = [a,b] \cap \mathbb{T} = \{t \in \mathbb{T} : a \leq b\}$.

In recent years, when $\mathbb{T} = \mathbb{R}$, a number of investigators had studied stability of differential equations by mean of various fixed point techniques (see [7]-[10] and papers therein and we refer to [14] for fixed point theorems). In this work we use the fixed point technique based on the contraction mapping theorem to prove that the zero solution solution of equation 1.1 (respectively 1.2) is stable and illustrate our theory by giving examples.

In Section 2, we present some preliminary material that we will need through the remainder of the paper. We present our main results on stability by using the contraction mapping principle in Section 3 and we provide two examples to illustrate our work.

2. Preliminaries

In this section, we consider some advanced topics in the theory of dynamic equations on a time scales. Again, we remind that for a review of this topic we direct the reader to the monographs of Bohner and Peterson [5], Chapter 1 and Chapter 2, pages 1-78 and [6] pages 1-16.

A time scale $\mathbb{T}$ is a closed nonempty subset of $\mathbb{R}$. For $t \in \mathbb{T}$ the forward jump operator σ, and the backward jump operator ρ, respectively, are defined as $\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}$ and $\rho(t) = \sup\{s \in \mathbb{T} : s < t\}$. These operators allow elements in the time scale to be classified as follows. We say t is right scattered if $\sigma(t) > t$ and right dense if $\sigma(t) = t$. We say t is left scattered if $\rho(t) < t$ and left dense if $\rho(t) < t$. The graininess function $\mu : \mathbb{T} \to [0, \infty)$, is defined by $\mu(t) = \sigma(t) - t$ and gives the distance between an element and its successor. We set $\inf \emptyset = \sup \mathbb{T}$ and $\sup \emptyset = \inf \mathbb{T}$. If $\mathbb{T}$ has a left scattered maximum M, we define $\mathbb{T}^k = \mathbb{T} \setminus \{M\}$. Otherwise, we define $\mathbb{T}^k = \mathbb{T}$. If $\mathbb{T}$ has a right scattered minimum m, we define $\mathbb{T}_k = \mathbb{T} \setminus \{m\}$. Otherwise, we define $\mathbb{T}_k = \mathbb{T}$.

Let $t \in \mathbb{T}^k$ and let $f : \mathbb{T} \to \mathbb{R}$. The delta derivative of $f(t)$, denoted $f^{\triangle}(t)$, is defined to be the number (when it exists), with the property that, for each $\epsilon > 0$, there is a neighborhood U of t such that

$$\left| f(\sigma(t)) - f(s) - f^{\triangle}(t)[\sigma(t) - s] \right| \leq \epsilon |\sigma(t) - s|,$$

for all $s \in U$. If $\mathbb{T} = \mathbb{R}$ then $f^{\triangle}(t) = f'(t)$ is the usual derivative. If $\mathbb{T} = \mathbb{Z}$ then $f^{\triangle}(t) = \triangle f(t) = f(t+1) - f(t)$ is the forward difference of f at t.

A function f is right dense continuous (rd-continuous), $f \in C_{rd} = C_{rd}(\mathbb{T}, \mathbb{R})$, if it is continuous at every right dense point $t \in \mathbb{T}$ and its left-hand limits exist at each left dense point $t \in \mathbb{T}$. The function $f : \mathbb{T} \to \mathbb{R}$ is differentiable on $\mathbb{T}^k$ provided $f^{\triangle}(t)$ exists for all $t \in \mathbb{T}^k$.

We are now ready to state some properties of the delta-derivative of f. Note $f^{\sigma}(t) = f(\sigma(t))$.

Theorem 2.1. [5, Theorem 1.20] *Assume $f, g : \mathbb{T} \to \mathbb{R}$ are differentiable at $t \in \mathbb{T}^k$ and let α be a scalar.*

(i) $(f+g)^{\triangle}(t) = f^{\triangle}(t) + g^{\triangle}(t)$.

(ii) $(\alpha f)^{\triangle}(t) = \alpha f^{\triangle}(t)$.

(ii) The product rules

$$\begin{aligned} (fg)^{\triangle}(t) &= f^{\triangle}(t) g(t) + f^{\sigma}(t) g^{\triangle}(t), \\ (fg)^{\triangle}(t) &= f(t) g^{\triangle}(t) + f^{\triangle}(t) g^{\sigma}(t). \end{aligned}$$

(iv) If $g(t) g^{\sigma}(t) \neq 0$ then

$$\left(\frac{f}{g}\right)^{\triangle}(t) = \frac{f^{\triangle}(t) g(t) - f(t) g^{\triangle}(t)}{g(t) g^{\sigma}(t)}.$$

The next theorem is the chain rule on time scales.

Theorem 2.2 (Chain Rule). [5, Theorem 1.93] *Assume* $\nu : \mathbb{T} \to \mathbb{R}$ *is strictly increasing and* $\widetilde{\mathbb{T}} := \nu(\mathbb{T})$ *is a time scale. Let* $\omega : \widetilde{\mathbb{T}} \to \mathbb{R}$. *If* $\nu^{\triangle}(t)$ *and* $\omega^{\widetilde{\triangle}}(\nu(t))$ *exist for* $t \in \mathbb{T}^k$, *then* $(\omega \circ \nu)^{\triangle} = \left(\omega^{\widetilde{\triangle}} \circ \nu\right) \nu^{\triangle}$.

In the sequel we will need to differentiate and integrate functions of the form $f(t - r(t)) = f(\nu(t))$ where, $\nu(t) := t - r(t)$. Our next theorem is the substitution rule.

Theorem 2.3 (Substitution). [5, Theorem 1.98]*Assume* $\nu : \mathbb{T} \to \mathbb{R}$ *is strictly increasing and* $\widetilde{\mathbb{T}} := \nu(\mathbb{T})$ *is a time scale. If* $f : \mathbb{T} \to \mathbb{R}$ *is rd-continuous function and* ν *is differentiable with rd-continuous derivative, then for* $a, b \in \mathbb{T}$,

$$\int_a^b f(t)\, \nu^{\triangle}(t) \triangle t = \int_{\nu(a)}^{\nu(b)} \left(f \circ \nu^{-1}\right)(s)\, \widetilde{\triangle} s.$$

A function $p : \mathbb{T} \to \mathbb{R}$ is said to be regressive provided $1 + \mu(t) p(t) \neq 0$ for all $t \in \mathbb{T}^k$. The set of all regressive rd-continuous function $f : \mathbb{T} \to \mathbb{R}$ is denoted by $\mathcal{R}$. The set of all positively regressive functions $\mathcal{R}^+$, is given by $\mathcal{R}^+ = \{f \in \mathcal{R} : 1 + \mu(t) f(t) > 0 \text{ for all } t \in \mathbb{T}\}$.

Let $p \in \mathcal{R}$ and $\mu(t) \neq 0$ for all $t \in \mathbb{T}$. The exponential function on $\mathbb{T}$ is defined by

$$e_p(t, s) = \exp\left(\int_s^t \frac{1}{\mu(z)} Log\left(1 + \mu(z) p(z)\right) \triangle z\right). \tag{2.1}$$

It is well known that if $p \in \mathcal{R}^+$, then $e_p(t, s) > 0$ for all $t \in \mathbb{T}$. Also, the exponential function $y(t) = e_p(t, s)$ is the solution to the initial value problem $y^{\triangle} = p(t) y$, $y(s) = 1$. Other properties of the exponential function are given by the following lemma.

Lemma 2.4. [5, Theorem 2.36]*Let* $p, q \in \mathcal{R}$. *Then*

(*i*) $e_0(t, s) = 1$ *and* $e_p(t, t) = 1$;

(*ii*) $e_p(\sigma(t), s) = (1 + \mu(t) p(t))\, e_p(t, s)$;

(*iii*) $\dfrac{1}{e_p(t, s)} = e_{\ominus p}(t, s)$, *where* $\ominus p(t) = -\dfrac{p(t)}{1 + \mu(t) p(t)}$;

(*iv*) $e_p(t, s) = \dfrac{1}{e_p(s, t)} = e_{\ominus p}(s, t)$;

(*v*) $e_p(t, s)\, e_p(s, r) = e_p(t, r)$;

(*vi*) $e_p^{\triangle}(., s) = p e_p(., s)$ *and* $\left(\dfrac{1}{e_p(., s)}\right)^{\triangle} = -\dfrac{p(t)}{e_p^{\sigma}(., s)}$.

3. Stability by fixed point theory

We begin our work by considering the neutral nonlinear dynamic equation with an unbounded delay

$$x^{\triangle}(t) = -a(t) x^{\sigma}(t) + b(t) G\left(x(t), x(t - r(t))\right) + c(t) x^{\widetilde{\triangle}}(t - r(t)), \ t \in \mathbb{T}, \tag{3.1}$$

where a, b, c and r are defined as before. Here, we assume $G(x,y)$ is locally Lipschitz continuous in x and y. That is, there is a $L > 0$ so that if $|x|, |y|, |z|$ and $|w| \leq L$, then

$$|G(x,y) - G(z,w)| \leq k_1 |x-z| + k_2 |y-w|, \tag{3.2}$$

for some positive constants $k_1$and k_2.
Also, we assume

$$G(0,0) = 0. \tag{3.3}$$

In addition to the conditions on r mentioned in Section 1, we need that

$$r^{\triangle}(t) \neq 1, \ \forall t \in \mathbb{T}. \tag{3.4}$$

Furthermore, the exponential function $e_{\ominus a}(t,0)$ must satisfy

$$e_{\ominus a}(t,0) \to 0 \text{ as } t \to \infty, \tag{3.5}$$

as well as the initial value problem $y^{\triangle}(t) = -a(t) y^{\sigma}(t)$, $y(0) = 1$. As such, we require that $a(t) \geq 0$ for all $t \in \mathbb{T}$. Since $a(t) \geq 0$ for all $t \in \mathbb{T}$, then $1 + \mu(t) a(t) \geq 1 > 0$ for all t and so $a \in \mathcal{R}^{+}$.

We begin by inverting equation (3.1) to obtain an equivalent equation . To do this, we use the variation of parameter formula to rewrite the equation as an integral mapping equation suitable for the contraction mapping theorem. So, in this step we need only to know what does a solution of (3.1) looks like. From now on, $\psi(t)$ denotes a real valued function with domain $(-\infty, 0]_{\mathbb{T}}$.

Lemma 3.1. *Suppose (3.4) holds. If $x(t)$ is a solution of equation (3.1) on an interval $[0,T)_{\mathbb{T}}$, $(T > 0)$ satisfying the initial condition $x(t) = \psi(t)$ for $t \in (-\infty, 0]_{\mathbb{T}}$, then $x(t)$ is a a solution of the integral equation*

$$\begin{aligned} x(t) = {} & \left(\psi(0) - \frac{c(0)}{1 - r^{\triangle}(0)} x(-r(0)) \right) e_{\ominus a}(t,0) + \frac{c(t)}{1 - r^{\triangle}(t)} x(t - r(t)) \\ & - \int_0^t \left[h(s) x^{\sigma}(s - r(s)) - b(s) G(x(s), x(s - r(s))) \right] e_{\ominus a}(t,s) \triangle s, \end{aligned} \tag{3.6}$$

where

$$h(s) = \frac{\left(c^{\triangle}(s) + c^{\sigma}(s) a(s)\right)\left(1 - r^{\triangle}(s)\right) + r^{\triangle\triangle}(s) c(s)}{\left(1 - r^{\triangle}(s)\right)\left(1 - r^{\triangle}(\sigma(s))\right)}. \tag{3.7}$$

Conversely, if a rd-continuous function $x(t)$ satisfies $x(t) = \psi(t)$ for $t \in (-\infty, 0]_{\mathbb{T}}$ and is a solution of (3.6) on some interval $[0,T)_{\mathbb{T}}$, $(T > 0)$, then $x(t)$ is a solution of equation (3.1) on $[0,T)_{\mathbb{T}}$.

Proof. We begin by rewriting (3.1) as

$$x^{\triangle}(t) + a(t) x^{\sigma}(t) = b(t) G(x(t), x(t - r(t))) + c(t) x^{\tilde{\triangle}}(t - r(t)).$$

Multiply both sides of the above equation by $e_a(t,0)$ and then we integrate from 0 to t to obtain

$$\begin{aligned}&\int_0^t \left(e_a(s,0)\,x(s)\right)^{\triangle}\triangle s\\&=\int_0^t \left[b(s)\,G(x(s),x(s-r(s)))+c(s)\,x^{\widetilde{\triangle}}(s-r(s))\right]e_a(s,0)\triangle s.\end{aligned}$$

As a consequence, we arrive at

$$\begin{aligned}&e_a(t,0)\,x(t)-x(0)\\&=\int_0^t \left[b(s)\,G(x(s),x(s-r(s)))+c(s)\,x^{\widetilde{\triangle}}(s-r(s))\right]e_a(s,0)\triangle s.\end{aligned}$$

Add $x(0)$ to both sides and multiply them by $e_{\ominus a}(t,0)$ to obtain

$$\begin{aligned}x(t)&=x(0)\,e_{\ominus a}(t,0)\\&+\int_0^t \left[b(s)\,G(x(s),x(s-r(s)))+c(s)\,x^{\widetilde{\triangle}}(s-r(s))\right]e_{\ominus a}(t,s)\triangle s.\end{aligned}\tag{3.8}$$

Here we have used Lemma 2.4 to simplify the exponential. We want to pull the factor $x^{\widetilde{\triangle}}(s-r(s))$ from under the integral in (3.8). Clearly

$$\begin{aligned}&\int_0^t c(s)\,x^{\widetilde{\triangle}}(s-r(s))\,e_{\ominus a}(t,s)\triangle s\\&=\int_0^t x^{\widetilde{\triangle}}(s-r(s))\left(1-r^{\triangle}(s)\right)\frac{c(s)}{(1-r^{\triangle}(s))}e_{\ominus a}(t,s)\triangle s.\end{aligned}$$

Using the integration by parts formula we get

$$\int_0^t f^{\triangle}(s)\,g(s)\triangle s=(fg)(t)-(fg)(0)-\int_0^t f^{\sigma}(s)\,g^{\triangle}(s)\triangle s,$$

and Theorems 2.2 and 2.3 implie

$$\begin{aligned}&\int_0^t c(s)\,x^{\widetilde{\triangle}}(s-r(s))\,e_{\ominus a}(t,s)\triangle s\\&=\frac{c(t)}{1-r^{\triangle}(t)}x(t-r(t))-\frac{c(0)}{1-r^{\triangle}(0)}x(-r(0))\,e_{\ominus a}(t,0)\\&-\int_0^t h(s)\,x^{\sigma}(s-r(s))\,e_{\ominus a}(t,s)\triangle s,\end{aligned}\tag{3.9}$$

where h is given by (3.7). Finally, by substituting the right hand side of (3.9) into (3.8) we obtain (3.6). Conversely, suppose that a rd-continuous function $x(t)$ satisfying $x(t)=\psi(t)$ for $t\in(-\infty,0]_{\mathbb{T}}$ and is a solution of (3.6) on an interval $[0,T)_{\mathbb{T}}$. Then it is $\triangle$-differentiable on $[0,T)_{\mathbb{T}}$. By $\triangle$-differentiating (3.6) we obtain (3.1). □

Now, let $\psi:(-\infty,0]_{\mathbb{T}}\to\mathbb{R}$ be a given bounded $\triangle$-differentiable initial function. We say that $x:=x(.,0,\psi)$ is a solution of (3.1) if $x(t)=\psi(t)$ for $t\leq 0$ and satisfies (3.1) for $t\geq 0$.

We say the zero solution of (3.1) is stable at t_0 if for each $\epsilon > 0$, there is a $\delta = \delta(\epsilon) > 0$ such that [$\psi : (-\infty, 0]_{\mathbb{T}} \to \mathbb{R}$ with $\|\psi\| < \delta$] implies $|x(t, t_0, \psi)| < \epsilon$.

Let $C_{rd} = C_{rd}(\mathbb{T}, \mathbb{R})$ be the space of all rd-continuous functions from $\mathbb{T} \to \mathbb{R}$ and define the set S_ψ by

$$S_\psi = \{\varphi \in C_{rd} : \|\varphi\| \leq L,\ \varphi(t) = \psi(t) \text{ if } t \leq 0 \text{ and } \varphi(t) \to 0 \text{ as } t \to \infty\}.$$

Then $(S_\psi, \|.\|)$ is a Banach space where $\|.\|$ is the supremum norm (we refer to [7, Example 1.2.2, page 18] for the proof that S_ψ is a Banach space).

For the next theorem we assume there is an $\alpha > 0$ such that

$$\left|\frac{c(t)}{1 - r^{\triangle}(t)}\right| + \int_0^t (|h(s)| + (k_1 + k_2)|b(s)|)\, e_{\ominus a}(t, s) \triangle s \leq \alpha < 1,\ t \geq 0, \tag{3.10}$$

and

$$t - r(t) \to \infty \text{ as } t \to \infty. \tag{3.11}$$

Theorem 3.2. *If (*3.2*)-(*3.5*), (*3.10*) and (*3.11*) hold, then every solution $x(., 0, \psi)$ in C_{rd} of (*3.1*) with a small continuous initial function ψ, is bounded and tends to zero as $t \to \infty$. Moreover, the zero solution is stable at $t_0 = 0$.*

Proof. For α and L, find an appropriate $\delta > 0$ such that

$$\left|1 - \frac{c(0)}{1 - r^{\triangle}(0)}\right| \delta + \alpha L \leq L.$$

Let $\psi : (-\infty, 0]_{\mathbb{T}} \to \mathbb{R}$ be a given small bounded initial function with $\|\psi\| < \delta$. Define the mapping $P : S_\psi \to S_\psi$ by

$$(P\varphi)(t) = \psi(t),\ \text{if } t \leq 0,$$

and

$$(P\varphi)(t) = \left(\varphi(0) - \frac{c(0)}{1 - r^{\triangle}(0)}\varphi(-r(0))\right) e_{\ominus a}(t, 0) + \frac{c(t)}{1 - r^{\triangle}(t)}\varphi(t - r(t))$$
$$- \int_0^t [h(s)\varphi^{\sigma}(s - r(s)) - b(s)G(\varphi(s), \varphi(s - r(s)))]\, e_{\ominus a}(t, s) \triangle s,\ t \geq 0.$$

Clearly, $P\varphi$ is continuous when φ is such. Let $\varphi \in S_\psi$, then using (3.10) in the definition of $P\varphi$ and applying (3.2) and (3.3), we obtain

$$|(P\varphi)(t)| \leq \left|1 - \frac{c(0)}{1 - r^{\triangle}(0)}\right| \delta + \left|\frac{c(t)}{1 - r^{\triangle}(t)}\right| L$$
$$+ \int_0^t [|h(s)||\varphi^{\sigma}(s - r(s))| + |b(s)||G(\varphi(s), \varphi(s - r(s)))|]\, e_{\ominus a}(t, s) \triangle s$$
$$\leq \left|1 - \frac{c(0)}{1 - r^{\triangle}(0)}\right| \delta + L\left\{\left|\frac{c(t)}{1 - r^{\triangle}(t)}\right| + \int_0^t (|h(s)| + (k_1 + k_2)|b(s)|)\, e_{\ominus a}(t, s) \triangle s\right\}$$
$$\leq \left|1 - \frac{c(0)}{1 - r^{\triangle}(0)}\right| \delta + L\alpha,$$

which implies that $|(P\varphi)(t)| \leq L$ for the chosen δ. Thus we have $\|P\varphi\| \leq L$.

Next we show that $(P\varphi)(t) \to 0$ as $t \to \infty$. By (3.5) and (3.11), the first term in the definition of $(P\varphi)(t)$ tends to zero. Also, the second term on the right-hand side tends to zero because of (3.11) and the fact that $\varphi \in S_\psi$. It remains to show that the integral term tends to zero as $t \to \infty$.

Let $\epsilon > 0$ be arbitrary and $\varphi \in S_\psi$. Then $\|\varphi\| \leq L$ and there exists $t_1 > 0$ such that $|\varphi(t)|$, $|\varphi(t - r(t))|$ and $|\varphi^\sigma(t - r(t))| < \epsilon$ for $t \geq t_1$. By condition (3.5), there exists $t_2 > t_1$ such that for $t > t_2$

$$e_{\ominus a}(t, t_1) < \frac{\epsilon}{\alpha L}.$$

For $t > t_2$, we have

$$\begin{aligned}
&\left| \int_0^t [h(s)\varphi^\sigma(s - r(s)) - b(s) G(\varphi(s), \varphi(s - r(s)))] e_{\ominus a}(t, s) \triangle s \right| \\
&\leq \int_0^t [|h(s)| |\varphi^\sigma(s - r(s))| + |b(s)| |G(\varphi(s), \varphi(s - r(s)))|] e_{\ominus a}(t, s) \triangle s \\
&\leq L \int_0^{t_1} (|h(s)| + (k_1 + k_2)|b(s)|) e_{\ominus a}(t, s) \triangle s \\
&+ \epsilon \int_{t_1}^{t_2} (|h(s)| + (k_1 + k_2)|b(s)|) e_{\ominus a}(t, s) \triangle s \\
&\leq L e_{\ominus a}(t, t_1) \int_0^{t_1} (|h(s)| + (k_1 + k_2)|b(s)|) e_{\ominus a}(t_1, s) \triangle s + \alpha\epsilon \\
&\leq \alpha L e_{\ominus a}(t, t_1) + \alpha\epsilon \leq \epsilon + \alpha\epsilon.
\end{aligned}$$

Hence $(P\varphi)(t) \to 0$ as $t \to \infty$.

It remains to show that P is a contraction under the supremum norm. For this, let $\varphi, \phi \in S_\psi$ then

$$\begin{aligned}
|(P\varphi)(t) - (P\phi)(t)| &\leq \left| \frac{c(t)}{1 - r^\triangle(t)} \right| \|\varphi - \phi\| \\
&+ \int_0^t |h(s)(\varphi^\sigma(s - r(s)) - \phi^\sigma(s - r(s)))| e_{\ominus a}(t, s) \triangle s \\
&+ \int_0^t |b(s)(G(\varphi(s), \varphi(s - r(s))) - G(\phi(s), \phi(s - r(s))))| e_{\ominus a}(t, s) \triangle s
\end{aligned}$$

$$\leq \left\{ \left| \frac{c(t)}{1 - r^\triangle(t)} \right| + \int_0^t (|h(s)| + (k_1 + k_2)|b(s)|) e_{\ominus a}(t, s) \triangle s \right\} \|\varphi - \phi\| \leq \alpha \|\varphi - \phi\|.$$

Thus, by the contraction mapping principle, P has a unique fixed point in S_ψ which solves (3.1), bounded and tends to zero as $t \to \infty$. The stability of the zero solution at $t_0 = 0$ follows from the above work by simply replacing L by ϵ. $\square$

Some stability result obtained on $\mathbb{R}$ for similar linear equations with delay via fixed point technic can be found in [8] (see also [7]). The authors in [13] have obtained results of stability for a nonlinear dynamic delay equation but with no neutral term.

Example 3.3. Let

$$\begin{aligned} \mathbb{T} &= (-\infty, -1] \cup \left\{ (1/2)^{\mathbb{Z}} - 1 \right\} \\ &= (-\infty, -1] \cup \left\{ ..., \left(1 - 2^n\right)/2^n, ..., -3/4, -1/2, 0, 1, 3, ..., 2^n - 1, ... \right\}. \end{aligned}$$

Then for any small continuous initial function $\psi : (-\infty, 0]_{\mathbb{T}} \to \mathbb{R}$, every solution $x\left(., 0, \psi\right)$ of the nonlinear neutral dynamic equation

$$\begin{aligned} x^{\triangle}(t) &= -3x^{\sigma}(t) + (3/2)\, c_0 \left(\sin\left(x(t)\right) + \sin\left(x\left(t/2 - 1/2\right)\right)\right) \\ &\quad + c_0 x^{\widetilde{\triangle}}\left(t/2 - 1/2\right), \end{aligned} \tag{3.12}$$

where c_0 is a positive constant, is bounded and goes to 0 as $t \to \infty$.

Indeed, in (3.12) we have $r(t) = t/2 + 1/2$. Let $t \in (1/2)^{\mathbb{Z}} - 1$. Then there exists an $n \in \mathbb{Z}$ such that $t = (1/2)^n - 1$. Hence

$$\begin{aligned} t - r(t) &= \frac{1}{2}\left(\left(\frac{1}{2}\right)^n - 1\right) - \frac{1}{2} \\ &= \left(\frac{1}{2}\right)^{n+1} - 1 \in \mathbb{T}. \end{aligned}$$

So, $id - r : \mathbb{T} \to \mathbb{T}$. Furthermore $(id - r)(\mathbb{T})$ is a time scale. Also, $t - r(t) = t/2 - 1/2 \to \infty$ as $t \to \infty$ and $(t - r(t))^{\triangle} = (t/2 - 1/2)^{\triangle} = 1/2$. Consequently, conditions (3.4) and (3.11) are satisfied. Since $1 + 3\mu(t) > 0$ for all $t \in \mathbb{T}$, then $3 \in \mathcal{R}^+$ and condition (3.5) is satisfied as well.
Also, in (3.12), we have

$$G\left(x(t), x\left(t/2 - 1/2\right)\right) = \sin\left(x(t)\right) + \sin\left(x\left(t/2 - 1/2\right)\right).$$

Clearly $G(0, 0) = 0$ and $G(x, y)$ is locally Lipschitz continuous in x and y. That is, there is a $L > 0$ so that if $|x|, |y|, |z|$ and $|w| \leq L$, then

$$\begin{aligned} \left|G(x, y) - G(z, w)\right| &= \left|\sin(x) + \sin(y) - (\sin(z) + \sin(w))\right| \\ &\leq \left|\sin(x) - \sin(z)\right| + \left|\sin(y) - \sin(w)\right| \\ &\leq |x - z| + |y - w|. \end{aligned}$$

One may easily check that $h(s) = 6c_0$. Also

$$\begin{aligned}
&\left|\frac{c(t)}{1 - r^{\triangle}(t)}\right| + \int_0^t \left(|h(s)| + (k_1 + k_2)|b(s)|\right) e_{\ominus a}(t,s) \triangle s \\
&= 2c_0 + 9c_0 \int_0^t e_{\ominus 3}(t,s) \triangle s \\
&= 2c_0 + 3c_0 - 3c_0 e_{\ominus 3}(t,0) \\
&\leq 5c_0.
\end{aligned}$$

Hence, (3.10) is satisfied for $c_0 \leq \frac{\alpha}{5}$, $\alpha \in (0,1)$. Let ψ be a given initial function which is continuous with $|\psi(t)| \leq \delta$ for all $t \in \mathbb{T}$ and define

$$S_\psi = \{\varphi \in C_{rd} : \|\varphi\| \leq L,\ \varphi(t) = \psi(t) \text{ if } t \leq 0 \text{ and } \varphi(t) \to 0 \text{ as } t \to \infty\}.$$

Define

$$(P\varphi)(t) = \psi(t) \text{ if } t \leq 0,$$

and

$$\begin{aligned}
(P\varphi)(t) &= (\psi(0) - 2c_0\psi(-1/2)) e_{\ominus 3}(t,0) + 2c_0\varphi(t/2 - 1/2) \\
&- \int_0^t [6c_0\varphi^\sigma(s/2 - 1/2) - (3/2)c_0(\sin(\varphi(s)) + \sin(\varphi(s/2 - 1/2)))] \\
&\times e_{\ominus 3}(t,s) \triangle s,\ t \geq 0.
\end{aligned}$$

Then, for $\varphi \in S_\psi$ with $\|\varphi\| \leq L$, we have

$$\|P\varphi\| \leq (1 - 2c_0)\delta + 5c_0 L \leq (1 - 2c_0)\delta + \alpha L.$$

This implies that $\|P\varphi\| \leq L$, for $L \geq \frac{(1-2c_0)\delta}{1-\alpha}$. To see that P defines a contraction mapping, we let $\varphi, \phi \in S_\psi$. Then

$$\begin{aligned}
|(P\varphi)(t) - (P\phi)(t)| &\leq 2c_0\|\varphi - \phi\| + 3c_0\|\varphi - \phi\| \\
&\leq \alpha\|\varphi - \phi\|.
\end{aligned}$$

Hence, by Theorem 3.2, every solution $x(.,0,\psi)$ of (3.12) with small continuous initial function $\psi : (-\infty, 0]_{\mathbb{T}} \to \mathbb{R}$, is in S_ψ, bounded and goes to zero as $t \to \infty$.

Next we turn our attention to the nonlinear neutral dynamic equation with unbounded delay

$$\begin{aligned}
x^{\triangle}(t) &= -a(t)x^\sigma(t) + b(t)G\left(x^2(t), x^2(t - r(t))\right) \\
&+ c(t)\left(x^2\right)^{\widetilde{\triangle}}(t - r(t)),\ t \in \mathbb{T}, \qquad (3.13)
\end{aligned}$$

where a, b, c, r and G are defined as before.

We use the variation of parameter to get the solution

$$x(t) = \left(x(0) - \frac{c(0)}{1 - r^{\triangle}(0)} x^2(-r(0))\right) e_{\ominus a}(t,0) + \frac{c(t)}{1 - r^{\triangle}(t)} x^2(t - r(t)) - \int_0^t \left[h(s)\left(x^2\right)^{\sigma}(s - r(s)) - b(s) G\left(x^2(s), x^2(s - r(s))\right)\right] e_{\ominus a}(t,s) \triangle s,$$

where h is given by (3.7).
Let

$$S_{\psi} = \{\varphi \in C_{rd} : \|\varphi\| \leq L,\ \varphi(t) = \psi(t) \text{ if } t \leq 0 \text{ and } \varphi(t) \to 0 \text{ as } t \to \infty\}.$$

For the next theorem we assume there is an $\alpha > 0$ such that

$$L\left\{\left|\frac{c(t)}{1 - r^{\triangle}(t)}\right| + \int_0^t \left(|h(s)| + (k_1 + k_2)|b(s)|\right) e_{\ominus a}(t,s) \triangle s\right\} \leq \alpha < 1/2,\ t \geq 0. \tag{3.14}$$

Theorem 3.4. *If (3.2)-(3.5), (3.11) and (3.14) hold, then every solution $x(.,0,\psi)$ of (3.13) with a small continuous initial function ψ, is bounded and tends to zero as $t \to \infty$. Moreover, the zero solution is stable at $t_0 = 0$.*

Proof. For α and L, find an appropriate $\delta > 0$ such that

$$\delta + \left|\frac{c(0)}{1 - r^{\triangle}(0)}\right| \delta^2 + \alpha L \leq L.$$

Let $\psi : (-\infty, 0]_{\mathbb{T}} \to \mathbb{R}$ be a given small bounded initial function with $\|\psi\| < \delta$. Define the mapping $P : S_{\psi} \to S_{\psi}$ by

$$(P\varphi)(t) = \psi(t), \text{ if } t \leq 0,$$

and

$$(P\varphi)(t) = \left(\varphi(0) - \frac{c(0)}{1 - r^{\triangle}(0)} \varphi^2(-r(0))\right) e_{\ominus a}(t,0) + \frac{c(t)}{1 - r^{\triangle}(t)} \varphi^2(t - r(t)) - \int_0^t \left[h(s)\left(\varphi^2\right)^{\sigma}(s - r(s)) - b(s) G\left(\varphi^2(s), \varphi^2(s - r(s))\right)\right] e_{\ominus a}(t,s) \triangle s,\ t \geq 0.$$

Clearly, $P\varphi$ is continuous when φ is such. Let $\varphi \in S_\psi$, then using (3.14) in the definition of $P\varphi$ and applying (3.2) and (3.3), we have

$$\begin{aligned}
&|(P\varphi)(t)| \\
&\le \delta + \left|\frac{c(0)}{1-r^\triangle(0)}\right|\delta^2 + \left|\frac{c(t)}{1-r^\triangle(t)}\right| L \\
&+ \int_0^t \left[|h(s)|\left|\left(\varphi^2\right)^\sigma (s-r(s))\right| + |b(s)|\left|G\left(\varphi^2(s), \varphi^2(s-r(s))\right)\right|\right] e_{\ominus a}(t,s)\triangle s \\
&\le \delta + \left|\frac{c(0)}{1-r^\triangle(0)}\right|\delta^2 \\
&+ L^2\left\{\left|\frac{c(t)}{1-r^\triangle(t)}\right| + \int_0^t (|h(s)| + (k_1+k_2)|b(s)|) e_{\ominus a}(t,s)\triangle s\right\} \\
&\le \delta + \left|\frac{c(0)}{1-r^\triangle(0)}\right|\delta^2 + L\alpha,
\end{aligned}$$

which implies that $|(P\varphi)(t)| \le L$ for the chosen δ. Thus we have $\|P\varphi\| \le L$.

Next we show that $(P\varphi)(t) \to 0$ as $t \to \infty$. By (3.5) and (3.11), the first term in the definition of $(P\varphi)(t)$ tends to zero. Also, the second term on the right-hand side tends to zero because of (3.11) and the fact that $\varphi \in S_\psi$. It remains to show that the integral term tends to zero as $t \to \infty$.
Let $\epsilon > 0$ be arbitrary and $\varphi \in S_\psi$. Then $\|\varphi\| \le L$ and there exists $t_1 > 0$ such that $|\varphi(t)|$, $|\varphi(t-r(t))|$ and $|\varphi^\sigma(t-r(t))| < \epsilon$ for $t \ge t_1$. By condition (3.5), there exists $t_2 > t_1$ such that for $t > t_2$

$$e_{\ominus a}(t,t_1) < \frac{\epsilon}{\alpha L^2}.$$

For $t > t_2$, we have

$$\begin{aligned}
&\left|\int_0^t \left[h(s)\left(\varphi^2\right)^\sigma(s-r(s)) - b(s)G\left(\varphi^2(s), \varphi^2(s-r(s))\right)\right] e_{\ominus a}(t,s)\triangle s\right| \\
&\le \int_0^t [\,|h(s)|\left|\left(\varphi^2\right)^\sigma(s-r(s))\right| \\
&+ |b(s)|\left|G\left(\varphi^2(s), \varphi^2(s-r(s))\right)\right|\,] e_{\ominus a}(t,s)\triangle s \\
&\le L^2\int_0^{t_1} (|h(s)| + (k_1+k_2)|b(s)|) e_{\ominus a}(t,s)\triangle s \\
&+ \epsilon^2\int_{t_1}^{t_2} (|h(s)| + (k_1+k_2)|b(s)|) e_{\ominus a}(t,s)\triangle s \\
&\le L^2 e_{\ominus a}(t,t_1)\int_0^{t_1} (|h(s)| + (k_1+k_2)|b(s)|) e_{\ominus a}(t_1,s)\triangle s + \alpha\epsilon^2 \\
&\le \alpha L^2 e_{\ominus a}(t,t_1) + \alpha\epsilon^2 \le \epsilon + \alpha\epsilon^2.
\end{aligned}$$

Hence $(P\varphi)(t) \to 0$ as $t \to \infty$.

It remains to show that P is a contraction under the supremum norm. For this, let $\varphi, \phi \in S_\psi$ then

$$\begin{aligned}
&\left|(P\varphi)(t) - (P\phi)(t)\right| \\
&\leq \left\{ \left| \frac{c(t)}{1 - r^{\triangle}(t)} \right| + \int_0^t (|h(s)| + (k_1 + k_2)|b(s)|) e_{\ominus a}(t,s) \triangle s \right\} \left\| \varphi^2 - \phi^2 \right\| \\
&\leq (2L) \{ \left| \frac{c(t)}{1 - r^{\triangle}(t)} \right| \\
&+ \int_0^t (|h(s)| + (k_1 + k_2)|b(s)|) e_{\ominus a}(t,s) \triangle s \} \left\| \varphi - \phi \right\| \\
&\leq (2\alpha) \left\| \varphi - \phi \right\|.
\end{aligned}$$

Thus, by the contraction mapping principle, P has a unique fixed point in S_ψ which solves (3.13), is bounded and tends to zero as $t \to \infty$. The stability of the zero solution at $t_0 = 0$ follows from the above work by simply replacing L by ϵ. □

Example 3.5. Let

$$\begin{aligned}
\mathbb{T} &= (-\infty, -1] \cup \left\{ (1/3)^{\mathbb{Z}} - 1 \right\} \\
&= (-\infty, -1] \cup \{..., (1 - 3^n)/3^n, ..., -8/9, -2/3, 0, 2, 8, ..., 3^n - 1, ...\}.
\end{aligned}$$

Then for any small continuous initial function $\psi : (-\infty, 0]_{\mathbb{T}} \to \mathbb{R}$, every solution $x(., 0, \psi)$ of the nonlinear neutral dynamic equation

$$\begin{aligned}
x^{\triangle}(t) &= -2x^{\sigma}(t) + c_0 \left(\sin\left(x^2(t)\right) + \cos\left(x^2(t/3 - 2/3)\right) - 1\right) \\
&\quad + 2c_0 \left(x^2\right)^{\widetilde{\triangle}} (t/3 - 2/3),
\end{aligned} \tag{3.15}$$

where c_0 is a positive constant, bounded and goes to 0 as $t \to \infty$.

Indeed, in (3.15) we have $r(t) = 2t/3 + 2/3$. Let $t \in (1/3)^{\mathbb{Z}} - 1$. Then there exists an $n \in \mathbb{Z}$ such that $t = (1/3)^n - 1$. Hence

$$\begin{aligned}
t - r(t) &= \frac{1}{3}\left(\left(\frac{1}{3}\right)^n - 1\right) - \frac{2}{3} \\
&= \left(\frac{1}{3}\right)^{n+1} - 1 \in \mathbb{T}.
\end{aligned}$$

So, $id - r : \mathbb{T} \to \mathbb{T}$. Furthermore $(id - r)(\mathbb{T})$ is a time scale. Also, $t - r(t) = t/3 - 2/3 \to \infty$ as $t \to \infty$ and $(t - r(t))^{\triangle} = (t/3 - 2/3)^{\triangle} = 1/3$. Consequently, conditions (3.4) and (3.11) are satisfied. Since $1 + 2\mu(t) > 0$ for all $t \in \mathbb{T}$, then $2 \in \mathcal{R}^+$ and condition (3.5) is satisfied as well.
Also, in (3.15), we have

$$G\left(x^2(t), x^2(t/3 - 2/3)\right) = \sin\left(x^2(t)\right) + \cos\left(x^2(t/3 - 2/3)\right) - 1.$$

Clearly $G(0,0)=0$ and $G(x,y)$ is locally Lipschitz continuous in x and y. That is, there is a $L>0$ so that if $|x|$, $|y|$, $|z|$ and $|w|\leq L$, then

$$\begin{aligned} |G(x,y)-G(z,w)| &= |\sin(x)+\cos(y)-(\sin(z)+\cos(w))| \\ &\leq |\sin(x)-\sin(z)|+|\cos(y)-\cos(w)| \\ &\leq |x-z|+|y-w|. \end{aligned}$$

One may easily arrive at $h(s)=6c_0$. Also

$$\begin{aligned} &L\left\{\left|\frac{c(t)}{1-r^{\triangle}(t)}\right|+\int_0^t\left(|h(s)|+(k_1+k_2)|b(s)|\right)e_{\ominus a}(t,s)\triangle s\right\} \\ &=L\left(3c_0+8c_0\int_0^t e_{\ominus 2}(t,s)\triangle s\right) \\ &=L\left\{3c_0+4c_0-4c_0e_{\ominus 2}(t,0)\right\} \\ &\leq 7Lc_0. \end{aligned}$$

Hence, (3.14) is satisfied for $c_0\leq\dfrac{\alpha}{7L}$, $\alpha\in(0,1/2)$. Let ψ be a given initial function that is continuous with $|\psi(t)|\leq\delta$ for all $t\in\mathbb{T}$ and define

$$S_\psi=\{\varphi\in C_{rd}:\|\varphi\|\leq L,\ \varphi(t)=\psi(t)\ \text{ if } t\leq 0 \text{ and } \varphi(t)\to 0 \text{ as } t\to\infty\}.$$

Define

$$(P\varphi)(t)=\psi(t)\ \text{ if } t\leq 0,$$

and

$$\begin{aligned} (P\varphi)(t) &= \left(\psi(0)-3c_0\psi^2(-2/3)\right)e_{\ominus 2}(t,0)+3c_0\varphi^2(t/3-2/3) \\ &\quad -\int_0^t\Big[6c_0\left(\varphi^2\right)^\sigma(s/3-2/3)-c_0\left(\sin\left(\varphi^2(s)\right)+\cos\left(\varphi^2(s/3-2/3)\right)-1\right)\Big] \\ &\qquad\qquad\times e_{\ominus 2}(t,s)\triangle s,\ \ t\geq 0. \end{aligned}$$

Then, for $\varphi\in S_\psi$ with $\|\varphi\|\leq L$, we have

$$\|P\varphi\|\leq\delta+3c_0\delta^2+7c_0L^2\leq\delta+3c_0\delta^2+\alpha L.$$

This implies that $\|P\varphi\|\leq L$, for $L\geq\dfrac{\delta+3c_0\delta^2}{1-\alpha}$. To see that P defines a contraction mapping, we let $\varphi,\phi\in S_\psi$. Then

$$\begin{aligned} |(P\varphi)(t)-(P\phi)(t)| &\leq 3c_0\left\|\varphi^2-\phi^2\right\|+4c_0\left\|\varphi^2-\phi^2\right\| \\ &\leq 14c_0L\|\varphi-\phi\| \\ &\leq 2\alpha\|\varphi-\phi\|. \end{aligned}$$

Hence, by Theorem 3.4, every solution $x(.,0,\psi)$ of (3.15) with small continuous initial function $\psi:(-\infty,0]_{\mathbb{T}}\to\mathbb{R}$, is in S_ψ, bounded and goes to zero as $t\to\infty$.

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The pairs of linear positive operators according to a general method of construction

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Abstract. We provide an estimate between discrete operators and their associated integral operators. A lot of examples are given.

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1. Introduction

Using a well-known method of construction for the pairs of linear positive operators, we give an estimate of the difference between the terms of these pairs. Let $(L_n)_{n\geq 1}$ be a sequence of linear positive operators, $L_n : \mathcal{L} \longrightarrow \mathcal{F}(I)$ having the form

$$L_n(f;x) = \sum_{k\geq 0} h_{n,k}(x)\nu_{n,k}(f),\ x \in I,\ f \in \mathcal{L}, \tag{1.1}$$

where $\mathcal{L}$ is the space of all real measurable bounded functions on I, for which $L_n f$ is well defined and $\mathcal{F}(I)$ is the space of all real valued functions defined on I.

Also the functions $f \in \mathcal{L}$ are $\mu_{n,k}$- integrable on I, $\mu_{n,k}$ being probability Borel measures on I and so that, the linear positive functional

$$\nu_{n,k}(f) = \int_I f(u)d\mu_{n,k}(u),\ f \in \mathcal{L} \tag{1.2}$$

is well defined for each $n \geq 1$ and $k \geq 0$. We assume that, $x_{n,k} \in I$ is the barycenter of the probability Borel measure $\mu_{n,k}$, i.e.

$$x_{n,k} = \nu_{n,k}(e_1) = \int_I u d\mu_{n,k}(u). \tag{1.3}$$

As usual, $e_i(x) = x^i$, $x \in I$, $i = 0, 1, 2, \ldots$ denote the test functions. In (1.1) we consider the positive functions $h_{n,k} \in \mathbf{C}_B(I)$ so that, $\sum_{k\geq 0} h_{n,k}(x) = 1$. With these remarks, the linear positive operators (1.1) preserve the constant functions. It is well known (see [3], [4], [8], [9], [22], [25]) that the sequence of positive linear operators (1.1) is associated with the next sequence of linear positive operators

$$P_n(f;x) = \sum_{k\geq 0} h_{n,k}(x) f\left(x_{n,k}\right), \ n \geq 1, \ k \geq 0, \ x \in I, \ f \in \mathcal{L} \tag{1.4}$$

where we consider that $\mathcal{L}$ is the common set of all real functions f on I for which $L_n f$, $\nu_{n,k}(f)$, $P_n f$ are well defined. We remark that

$$L_n\left(e_1;x\right) = P_n\left(e_1;x\right) = \sum_{k\geq 0} h_{n,k}(x) x_{n,k}.$$

In the next section, we present an estimate on the difference between the terms of the pair of operators (L_n, P_n).

2. An estimate on the difference $|L_n f - P_n f|$

The basic result for the next theorem is the barycenter inequality of ν a probability Radon measure on I

$$\nu(h) \geq h(b), \ h \in \mathbf{C}_B(I) \text{ convex},$$

with $b = \nu(e_1)$ the barycenter of probability Radon measure ν.

Indeed, if $h = \dfrac{\|f''\|}{2} e_2 \pm f$, $f \in \mathbf{C}_B^2(I)$, then the barycenter inequality becomes

$$|\nu(f) - f(b)| \leq \frac{\|f''\|}{2}\left[\nu(e_2) - b^2\right],$$

where $\|\cdot\|$ is the uniform norm.

Theorem 2.1. *If $(L_n)_{n\geq 1}$, $(P_n)_{n\geq 1}$, are two sequences of linear positive operators defined as (1.1) respectively (1.4) for $f \in \mathbf{C}_B^2(I) \subset \mathcal{L}$, then for $x \in I$ we shall have the estimation*

$$|L_n(f;x) - P_n(f;x)| \leq \frac{\|f''\|}{2} \sum_{k\geq 0} h_{n,k}(x) \left[\nu_{n,k}(e_2) - \left(\nu_{n,k}(e_1)\right)^2\right]. \tag{2.1}$$

3. Some examples

The main purpose of the present paper is to establish results of type (2.1) for a number of well-known pairs operators (L_n, P_n) used in approximation theory. In our examples, the functions $h_{n,k}(x)$ are the next discrete probability functions:

(i). $p_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}$, $x \in [0,1]$, $0 \leq k \leq n$, $n \geq 1$ (the binomial probability)

(ii). $s_{n,k}(x) = e^{-nx}\dfrac{(nx)^k}{k!}$, $x \geq 0$, $k \geq 0$, $n \geq 1$ (the Poisson probability)

(iii). $\pi_{n,k}(x) = \binom{n+k-1}{k}\dfrac{x^k}{(1+x)^{n+k}}$, $x \geq 0$, $k \geq 0$, $n \geq 1$ (the negative binomial probability or the Pascal probability).

Also, we consider the next probability density functions:

(iv). $\gamma_n(x) = \begin{cases} 0 & , x < 0 \\ \dfrac{1}{\Gamma(n)} x^{n-1} e^{-x} & , x \geq 0,\ n > 0 \end{cases}$ (the Gamma probability density function),

$\Gamma(a) = \int\limits_0^\infty e^{-x} x^{a-1} dx,\ a > 0$ (the Gamma function)

(v). $\beta_{k,n}(x) = \begin{cases} 0 & , x \notin [0,1] \\ \dfrac{1}{B(k,n)} x^{k-1}(1-x)^{n-1} & , x \in [0,1],\ k > 0,\ n > 0 \end{cases}$ (the Beta probability density function),

$B(k,n) = \int\limits_0^1 x^{k-1}(1-x)^{n-1} dx,\ k > 0,\ n > 0$ (the Beta function of the first kind)

(vi). $b_{k,n}(x) = \begin{cases} 0 & , x < 0 \\ \dfrac{1}{B(k,n)} \cdot \dfrac{x^{k-1}}{(1+x)^{n+k}} & , x \geq 0,\ k > 0,\ n > 0 \end{cases}$ (the Inverse-Beta probability density function),

$B(k,n) = \int\limits_0^\infty \dfrac{x^{k-1}}{(1+x)^{n+k}} dx,\ k > 0,\ n > 0$ (the Beta function of the second kind)

(vii). $\omega(x) = \begin{cases} 0 & , x \notin [a,b] \\ \dfrac{1}{b-a} & , x \in [a,b],\ a < b \end{cases}$ (the uniform continuous probability density function on $[a,b]$).

It is easy to see that, between the discrete probability functions and the probability density functions there are a link in the next sense:

$$\begin{cases} p_{n,k}(x) &= \dfrac{1}{n+1} \beta_{k+1,n-k+1}(x) \\ &= \beta_{k+1,n-k+1}(x) \int\limits_0^1 p_{n,k}(x) dx,\ 0 \leq k \leq n,\ n \geq 1,\ x \in [0,1], \\ s_{n,k}(x) &= \gamma_{k+1}(nx),\ k \geq 0,\ n \geq 1,\ x \geq 0, \\ \pi_{n,k}(x) &= \dfrac{1}{n-1} b_{k+1,n-1}(x),\ k \geq 0,\ n > 1,\ x \geq 0. \end{cases}$$

Using a general method of construction for the pairs of linear positive operators, we give the next examples.

A. Let $h_{n,k}(x) = p_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}$, $n \geq 1$, $k = \overline{0,n}$, $I = [0,1]$ be, the binomial probability.

A1. Taking

$$\nu_{n,k}(f) = \begin{cases} \int_0^1 f(u)\beta_{k,n-k}(u)du & , 1 \le k \le n-1,\, n \ge 2 \\ f(0) & , k=0 \\ f(1) & , k=n \end{cases}$$
$$= \begin{cases} (n-1)\int_0^1 f(u)p_{n-2,k-1}(u)du & , 1 \le k \le n-1,\, n \ge 2 \\ f(0) & , k=0 \\ f(1) & , k=n \end{cases}$$

and

$$x_{n,k} = \nu_{n,k}(e_1) = \begin{cases} \int_0^1 u\beta_{k,n-k}(u)du = \frac{k}{n} & , 1 \le k \le n-1,\, n \ge 2 \\ 0 & , k=0 \\ 1 & , k=n \end{cases}$$

we obtain the pair of operators $(DB_n(f;x), B_n(f;x))$ where

$$DB_n(f;x) = p_{n,0}(x)f(0) + (n-1)\sum_{k=1}^{n-1} p_{n,k}(x)\int_0^1 f(u)p_{n-2,k-1}(u)du$$
$$+p_{n,n}(x)f(1)$$

is the genuine Bernstein - Durrmeyer operator, defined and investigated by Goodman, T.N.T., Sharma, A. [15], [16] and $B_n(f;x) = \sum_{k=0}^{n} p_{n,k}(x)f\left(\frac{k}{n}\right)$ is the classical Bernstein operator.

We have with (2.1), for $f \in \mathbf{C}^2[0,1]$ the next estimation

$$|DB_n(f;x) - B_n(f;x)| \le \frac{\|f''\|}{2} \cdot \frac{n-1}{n+1} \cdot \frac{x(1-x)}{n}.$$

A2. If we have

$$\nu^*_{n,k}(f) = \int_0^1 f(u)\beta_{k+1,n-k+1}(u)du = (n+1)\int_0^1 f(u)p_{n,k}(u)du$$

and

$$x_{n,k} := \nu^*_{n,k}(e_1) = \frac{B(k+2,n-k+1)}{B(k+1,n-k+1)} = \frac{k+1}{n+2},\ 0 \le k \le n,$$

then, we get the pair of operators $(DB_n^*(f,x), B_n^*(f,x))$ with

$$\begin{aligned} DB_n^*(f;x) &= \sum_{k=0}^{n} p_{n,k}(x) \int_0^1 f(u)\beta_{k+1,n-k+1}(u)du \\ &= (n+1)\sum_{k=0}^{n} p_{n,k}(x) \int_0^1 f(u)p_{n,k}(u)du \end{aligned}$$

the classical Bernstein-Durrmeyer operator defined by Durrmeyer J.L.[12] and extensively studied by Derriennic M. M. [10], Ditzian Z., Ivanov K. [11], Gonska H.H., Zhou X. [14] and $B_n^*(f;x) = \sum_{k=0}^{n} p_{n,k}(x) f\left(\frac{k+1}{n+2}\right)$, the Bernstein-Stancu operator [27]. For they, if $f \in \mathbf{C}^2[0,1]$ then

$$|DB_n^*(f;x) - B_n^*(f;x)| \leq \frac{\|f''\|}{2} \cdot \frac{n(n-1)x(1-x)+n+1}{(n+2)^2(n+3)}.$$

A3. For the functional $\lambda_{n,k}(f) = (n+1)\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} f(u)du$ associated with the uniform continuous probability density function

$$\omega_{n,k}(u) = \begin{cases} 0 & , u \notin \left[\frac{k}{n+1}, \frac{k+1}{n+1}\right] \\ n+1 & , u \in \left[\frac{k}{n+1}, \frac{k+1}{n+1}\right] \end{cases}$$

we have

$$x_{n,k} := \lambda_{n,k}(e_1) = (n+1)\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} udu = \frac{2k+1}{2(n+1)},\ 0 \leq k \leq n,\ n \geq 1.$$

So, the pair of operators becomes $(KB_n(f;x), B_n^{**}(f;x))$ with

$$KB_n(f;x) = (n+1)\sum_{k=0}^{n} p_{n,k}(x) \int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} f(u)du \tag{3.1}$$

the Bernstein-Kantorovich [18] operator and

$$B_n^{**}(f;x) = \sum_{k=0}^{n} p_{n,k}(x) f\left(\frac{2k+1}{2(n+1)}\right)$$

the Bernstein-Stancu [27] operator.

Using the Theorem 2.1 we have for $f \in \mathbf{C}^2[0,1]$ the estimate

$$|KB_n^*(f;x) - B_n^{**}(f;x)| \leq \frac{\|f''\|}{24(n+1)^2}.$$

A4. Consider now, the linear positive functional

$$\lambda^*_{n,k}(f;a_n,b_n) = \frac{n+1}{b_n-a_n}\int\limits_{\frac{k+a_n}{n+1}}^{\frac{k+b_n}{n+1}} f(u)du,\ 0 \le a_n < b_n \le 1,$$

which can be associated with the uniform continuous probability density function

$$\omega^*_{n,k}(u;a_n,b_n) = \begin{cases} 0 & , u \notin \left[\frac{k+a_n}{n+1}, \frac{k+b_n}{n+1}\right] \\ \frac{n+1}{b_n-a_n} & , u \in \left[\frac{k+a_n}{n+1}, \frac{k+b_n}{n+1}\right] \end{cases}, 0 \le a_n < b_n \le 1$$

and

$$x_{n,k} := \lambda^*_{n,k}(e_1;a_n,b_n) = \frac{2k+a_n+b_n}{2(n+1)},\ 0 \le k \le n,\ n \ge 1.$$

We obtain the pair of operators $(AL_n(f;x), BS_n(f;x))$ with

$$AL_n(f;x) = \sum_{k=0}^{n} p_{n,k}(x)\left(\frac{n+1}{b_n-a_n}\int\limits_{\frac{k+a_n}{n+1}}^{\frac{k+b_n}{n+1}} f(u)du\right)$$

$0 \le a_n < b_n \le 1$, $f \in \mathbf{C}[0,1]$, $x \in [0,1]$, a generalization of Bernstein-Kantorovich operators (3.1) which was given by Altomare F., Leonessa V. [2] and its associated operators is the Bernstein-Stancu type operator [27]

$$BS_n(f;x) = \sum_{k=0}^{n} p_{n,k}(x) f\left(\frac{2k+a_n+b_n}{2(n+1)}\right),\ (a_n \to 0,\ b_n \to 1,\ n \to \infty).$$

If $f \in \mathbf{C}^2[0,1]$ we obtain with (2.1) the next estimate

$$|AL_n(f;x) - BS_n(f;x)| \le \|f''\| \frac{(b_n-a_n)^2}{24(n+1)^2}.$$

A5. Taking for $a, b > -1$, $\alpha \ge 0$, $c := c_n = [n^\alpha]$ the positive linear beta functional

$$T^{a,b,c}_{k,n}(f) = \frac{1}{B\left(ck+a+1, c(n-k)+b+1\right)}\int\limits_0^1 f(u)u^{ck+a}(1-u)^{c(n-k)+b}du$$

and its associated linear positive beta operator

$$T^{a,b,c}_{n}(f;x)$$

$$= \frac{1}{B\left(cnx+a+1, cn(1-x)+b+1\right)}\int\limits_0^1 f(u)u^{cnx+a}(1-u)^{cn(1-x)+b}du$$

$x \in [0,1]$, we have with (1.1) a linear positive operator defined and investigated by Mache D. H. [19], [20], which represents a link between the Durrmeyer operator with Jacobi weights (for $\alpha = 0$) and the classical Bernstein operator

$$DM_n(f;x) = \sum_{k=0}^{n} p_{n,k}(x) T_{n,k}^{a,b,c}(f) = B_n\left(T_n^{a,b,c}\right)(f;x).$$

Because $T_{n,k}^{a,b,c}(e_1) = \dfrac{ck+a+1}{cn+a+b+2}$, Rasa I. [25] using (1.4) defined and investigated a new linear positive operator associated with $DM_n(f)$,

$$R_n(f;x) = \sum_{k=0}^{n} p_{n,k}(x) f\left(\frac{ck+a+1}{cn+a+b+2}\right).$$

If $f \in \mathbf{C}^2[0,1]$ then for the pair of operators $(DM_n f, R_n f)$ with Theorem 2.1 we have the estimate

$$\begin{aligned}&|DM_n(f;x) - R_n(f;x)| \\ &\le \frac{\|f''\|}{2} \cdot \frac{c^2 n(n-1)x(1-x) + cn(b-a)x + cn(a+1) + (a+1)(b+1)}{2(cn+a+b+2)^2(cn+a+b+3)}.\end{aligned}$$

B. Let $h_{n,k} := s_{n,k}(x) = e^{-nx}\dfrac{(nx)^k}{k!}$, $x \ge 0$, $k \ge 0$, $n \ge 1$ be, the Poisson probability function.

B1. If

$$\tau_{n,k}(f) = \begin{cases} n\int_0^\infty f(u)\gamma_k(nu)du = n\int_0^\infty f(u)s_{n,k-1}(u)du \\ \qquad = \dfrac{\int_0^\infty f(u)s_{n,k-1}(u)du}{\int_0^\infty s_{n,k-1}(u)du} & , k \ge 1 \\ 0 & , k = 0 \end{cases}$$

is a linear positive functional and

$$x_{n,k} := \tau_{n,k}(e_1) = \begin{cases} n\int_0^\infty u\gamma_k(nu)du = n\int_0^\infty s_{n,k-1}(u)du = \dfrac{k}{n} & , k \ge 1 \\ 0 & , k = 0 \end{cases}$$

then we have with (1.4) the classical Szasz-Mirakjan operator

$$S_n(f;x) = \sum_{k=0}^{\infty} s_{n,k}(x) f\left(\frac{k}{n}\right). \tag{3.2}$$

and with (1.1) the Phillips operator [23] or the genuine Szasz-Durrmeyer operator

$$\begin{aligned} DS_n(f,x) &= s_{n,0}(x)f(0) + \sum_{k=1}^{\infty} s_{n,k}(x) \int_0^{\infty} f(u)s_{n,k-1}(u)du \\ &= s_{n,0}(x)f(0) + \sum_{k=1}^{\infty} s_{n,k}(x) \int_0^{\infty} f(u)\gamma_k(nu)du \end{aligned}$$

For these two operators, with the Theorem 2.1, we obtain the estimation

$$\begin{aligned} |DS_n(f;x) - S_n(f;x)| &\leq \frac{\|f''\|}{2} \sum_{k\geq 0} s_{n,k}(x) \left[\tau_{n,k}(e_2) - (\tau_{n,k}(e_1))^2\right] \\ &= \frac{\|f''\|}{2} \cdot \frac{x}{n}, \end{aligned}$$

$x \geq 0$, $f \in \mathbf{C}_B^2[0,\infty)$.

B2. We consider the linear positive functional

$$\tau^*_{n,k}(f) = n \int_0^{\infty} f(u)s_{n,k}(u)du = n \int_0^{\infty} f(u)\gamma_{k+1}(nu)du$$

and

$$x_{n,k} := \tau^*_{n,k}(e_1) = n \int_0^{\infty} u\gamma_{k+1}(nu)du = \frac{k+1}{n},\ k \geq 0.$$

So, using (1.4) we get a modification of Szasz-Mirakjan operator

$$S^*_n(f;x) = \sum_{k=0}^{\infty} s_{n,k}(x) f\left(\frac{k+1}{n}\right)$$

and with (1.1) the Szasz-Durrmeyer type operator, which was defined and studied by Mazhar, Totik [21]

$$\begin{aligned} DS^*_n(f,x) &= n\sum_{k=0}^{\infty} s_{n,k}(x) \int_0^{\infty} f(u)s_{n,k}(u)du \\ &= n\sum_{k=0}^{\infty} s_{n,k}(x) \int_0^{\infty} f(u)\gamma_{k+1}(nu)du. \end{aligned}$$

If $f \in \mathbf{C}_B^2[0,\infty)$ then

$$\begin{aligned} |DS^*_n(f;x) - S^*_n(f;x)| &\leq \frac{\|f''\|}{2} \sum_{k\geq 0} s_{n,k}(x) \left[\tau^*_{n,k}(e_2) - \left(\tau^*_{n,k}(e_1)\right)^2\right] \\ &\leq \frac{\|f''\|}{2} \left(\frac{x}{n} + \frac{1}{n^2}\right). \end{aligned}$$

B3. Taking the linear positive functional

$$\varphi_{n,k}(f)=\begin{cases} f(0) & ,\ k=0\\ \int\limits_0^\infty f(u)b_{k,n+1}(u)du & ,\ k>0,\end{cases}$$

with $b_{k,n+1}(u) = \begin{cases} 0 & ,\ u\le 0 \text{ or } k=0\\ \dfrac{1}{B(k,n+1)}\cdot\dfrac{u^k}{(1+u)^{n+k+1}} & ,\ u>0,\ k>0\end{cases}$ the Inverse-Beta probability density function we obtain the knots

$$x_{n,k}:=\varphi_{n,k}(e_1)=\begin{cases} 0 & ,\ k=0\\ \int\limits_0^\infty u b_{k,n+1}(u)du=\dfrac{B(k+1,n)}{B(k,n+1)}=\dfrac{k}{n} & ,\ k>0.\end{cases}$$

According to (1.1) we have the Szasz-Inverse Beta operator, defined by Govil N.K., Gupta, V., Noor M. A., [17] and studied by Finta Z., Govil N. K., Gupta V., [13], Cismaiu C., [5], [6], [7]

$$SA_n(f,x) \quad = \quad f(0)s_{n,0}(x)+\sum_{k=1}^{\infty}s_{n,k}(x)\int_0^\infty f(u)b_{k,n+1}(u)du$$

and according to (1.4) we have the Szasz-Mirakjan S_nf operator (3.2). For the pair of operators (SA_nf, S_nf), if $f\in\mathbf{C}_B^2[0,\infty)$ we get the next estimate

$$\begin{aligned}|SA_n(f;x)-S_n(f;x)| &\le \frac{\|f''\|}{2}\sum_{k=1}^{\infty}s_{n,k}(x)\left[\varphi_{n,k}(e_2)-(\varphi_{n,k}(e_1))^2\right]\\ &\le \frac{\|f''\|}{2}\cdot\frac{1}{n-1}\left(x(x+1)+\frac{x}{n}\right),\ n>1.\end{aligned}$$

B4. For the linear functional

$$\phi_{n,k}(f)=\int_{\frac{k}{n}}^{\frac{k+1}{n}} f(u)\rho_{n,k}(u)du=n\int_{\frac{k}{n}}^{\frac{k+1}{n}} f(u)du,\ n\ge 1,\ k\ge 0$$

with

$$\rho_{n,k}(u)=\begin{cases} 0 & ,\ u\notin\left[\dfrac{k}{n+1},\dfrac{k+1}{n+1}\right]\\ n & ,\ u\in\left[\dfrac{k}{n+1},\dfrac{k+1}{n+1}\right],\ n\ge 1,\ k\ge 0\end{cases}$$

the uniform continuous probability density function, we have

$$x_{n,k}:=\phi_{n,k}(e_1)=\int_{\frac{k}{n}}^{\frac{k+1}{n}} u\rho_{n,k}(u)du=\frac{2k+1}{2n},\ n\ge 1,\ k\ge 0$$

and with (1.4) we obtain a modification of Szasz-Mirakjan operator

$$S_n^{**}(f;x) = \sum_{k=0}^{\infty} s_{n,k}(x) f\left(\frac{2k+1}{2n}\right).$$

The associated operator with (1.1) is the classical Szasz-Kantorovich operator [18] $K_n S_n^{**} : L_1([0,\infty)) \longrightarrow B([0,\infty))$,

$$L_1([0,\infty)) = \left\{ f : [0,\infty) \longrightarrow \mathbb{R}, \text{ measurable on } [0,\infty), \int_0^{\infty} |f(x)|\, dx < \infty \right\}$$

defined

$$K_n S_n^{**}(f;x) = n \sum_{k=0}^{\infty} s_{n,k}(x) \int_{\frac{k}{n}}^{\frac{k+1}{n}} f(u)du$$

and so, using the Theorem 2.1 we obtain

$$|K_n S_n^{**}(f;x) - S_n^{**}(f;x)| \quad \leq \quad \frac{\|f''\|}{2} \cdot \frac{1}{12n^2},\ f \in \mathbf{C}_B^2[0,\infty).$$

C. Now, we consider the negative binomial probability or the Pascal probability

$$h_{n,k}(x) := \pi_{n,k}(x) = \frac{1}{n-1} b_{k+1,n-1}(x) = \binom{n+k-1}{k} \frac{x^k}{(1+x)^{n+k}}$$

$x \geq 0,\ k \geq 0,\ n > 1$.

C1. Let $\sigma_{n,k}(f) = (n-1)\int_0^{\infty} f(u)\pi_{n,k}(u)du = \int_0^{\infty} f(u) b_{k+1,n-1}(u)du$,be the linear functional which is defined whenever f is either a real-valued bounded measurable function on $[0,\infty)$ or a continuous function on $[0,\infty)$ such that $f(x) = O\left(x^r\right),\ 0 < r < n-1$. We obtain

$$x_{n,k} := \sigma_{n,k}(e_1) = \int_0^{\infty} u b_{k+1,n-1}(u)du = \frac{B(k+2,n-2)}{B(k+1,n-1)} = \frac{k+2}{n-2},\ n > 2.$$

We get with (1.1) the Baskakov-Durrmeyer operator, defined and investigated by Sahai A., Prasad G. [26]:

$$\begin{aligned} DM_n(f,x) &= (n-1)\sum_{k=0}^{\infty} \pi_{n,k}(x) \int_0^{\infty} f(u)\pi_{n,k}(u)du \\ &= \sum_{k=0}^{\infty} \pi_{n,k}(x) \int_0^{\infty} f(u) b_{k+1,n-1}(u)du,\ n > 1,\ x \geq 0 \end{aligned}$$

and with (1.4) a modification of Baskakov operator

$$M_n(f;x) = \sum_{k=0}^{\infty} \pi_{n,k}(x) f\left(\frac{k+1}{n-2}\right),\ n > 2.$$

So, for $n > 3$, $x \geq 0$, $f \in \mathbf{C}_B^2[0,\infty)$ with (2.1) we obtain the estimate

$$|DM_n(f;x) - M_n(f;x)| \quad \leq \quad \frac{\|f''\|}{2} \cdot \frac{n(n+1)x^2 + (n-2)(nx+1)}{(n-2)^2(n-3)}.$$

C2. If the functional (1.2) is

$$\nu_{n,k}(f) := (n+1)\int_0^\infty f(u)\pi_{n+2,k-1}(u)du = \int_0^\infty f(u)b_{k,n+1}(u)du,\ k \geq 1$$

then we get

$$x_{n,k} := \nu_{n,k}(e_1) = \begin{cases} \int_0^\infty u b_{k,n+1}(u)du = \dfrac{B(k+1,n)}{B(k,n+1)} = \dfrac{k}{n} & ,\ k > 0 \\ 0 & ,\ k = 0 \end{cases}$$

So, we have with (1.1) the genuine Baskakov-Durrmeyer operator

$$\begin{aligned} DM_n^*(f,x) &= f(0)\pi_{n,0}(x) + (n+1)\sum_{k=1}^\infty \pi_{n,k}(x)\int_0^\infty f(u)\pi_{n+2,k-1}(u)du \\ &= f(0)\pi_{n,0}(x) + (n+1)\sum_{k=1}^\infty \pi_{n,k}(x)\int_0^\infty f(u)b_{k,n+1}(u)du \end{aligned}$$

and with (1.4) the classical Baskakov operator

$$M_n^*(f;x) = \sum_{k=0}^\infty \pi_{n,k}(x) f\left(\frac{k}{n}\right).$$

Using the Inverse-Beta operator or the Stancu operator of the second kind [28]:

$$W_n(f;x) = \begin{cases} f(0) & ,\ x = 0 \\ \dfrac{1}{B(nx,n+1)} \int_0^\infty f(u)\dfrac{u^{nx-1}}{(1+u)^{nx+n+1}}du & ,\ x > 0, \end{cases}$$

for which

$$\begin{cases} W_n(e_0;x) = 1 \\ W_n(e_1;x) = x \\ W_n(e_2;x) = x^2 + \dfrac{x(x+1)}{n-1} & ,\ n > 1 \end{cases}$$

we have $DM_n^*(f) = M_n^*(W_n)(f;x)$. If $f \in \mathbf{C}_B^2[0,\infty)$, $n > 1$, $x \geq 0$, then we obtain the next estimation

$$\begin{aligned} &|DM_n^*(f;x) - M_n^*(f;x)| \\ &\leq \frac{\|f''\|}{2}\sum_{k=1}^\infty \pi_{n,k}(x)\left[W_n\left(e_2;\frac{k}{n}\right) - \left(W_n\left(e_1,\frac{k}{n}\right)\right)^2\right] \\ &\leq \frac{\|f''\|}{2} \cdot \frac{(n+1)x^2 + x}{n(n-1)} + \frac{x}{n-1}. \end{aligned}$$

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An asymptotic formula for Jain's operators

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Abstract. We investigate a class of linear positive operators of discrete type depending on a real parameter. By additional conditions imposed on this parameter, the considered sequence turns into an approximation process.

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Keywords: Approximation process, Voronovskaja theorem, Jain's operator.

1. Introduction

In 1970 G.C. Jain has introduced in [1] a new class of positive linear operators based on a Poisson-type distribution. In 1984 starting from Jain's operator, S.Umar and Q. Razi introduced in [6] a class of modified Szász-Mirakjan operators and studied their approximation properties. Later on, in 1995 L. Rempulska approached in [5] a Voronovskaja type result for some operators of Szász-Mirakjan type.

Present paper aims to prove a Voronovskaja type result for a class of linear positive operators of discrete type depending on a real parameter. In first section of this paper, we collect some basic results concerning Jain's operator, $P_n^{[\beta]}$, and we also compute other similar relations starting from those who are already proved by Jain.

In Section 2 will be highlighted the main results obtained and Section 3 will host the proofs of the stated results.

First of all, we recall the form of a Poisson-type distribution.

Lemma 1.1. ([1]) *For* $0 < \alpha < \infty, \quad |\beta| < 1,$ *let*

$$\omega_\beta(k,\alpha) = \alpha(\alpha + k\beta)^{k-1} e^{-(\alpha+k\beta)}/k! \ ; \ k \in \mathbb{N}_0. \tag{1.1}$$

then

$$\sum_{k=0}^{\infty} \omega_\beta(k,\alpha) = 1. \tag{1.2}$$

Lemma 1.2. ([1]) *Let*

$$S(r,\alpha,\beta) = \sum_{k=0}^{\infty} (\alpha + \beta k)^{k+r-1} e^{-(\alpha+\beta k)}/k!, \ r = 0, 1, 2, ... \tag{1.3}$$

and

$$\alpha S(0,\alpha,\beta)=1. \tag{1.4}$$

Then

$$S(r,\alpha,\beta)=\sum_{k=0}^{\infty}\beta^k(\alpha+k\beta)S(r-1,\alpha+k\beta,\beta). \tag{1.5}$$

The functions $S(r,\alpha,\beta)$ satisfy the recurrence formula

$$S(r,\alpha,\beta)=\alpha S(r-1,\alpha,\beta)+\beta S(r,\alpha+\beta,\beta). \tag{1.6}$$

The above formula implies

$$S(1,\alpha,\beta)=\sum_{k=0}^{\infty}\beta^k=\frac{1}{1-\beta} \tag{1.7}$$

and

$$S(2,\alpha,\beta)=\sum_{k=0}^{\infty}\frac{\beta^k(\alpha+k\beta)}{1-\beta}=\frac{\alpha}{(1-\beta)^2}+\frac{\beta^2}{(1-\beta)^3}. \tag{1.8}$$

We easily get

Lemma 1.3. *Let S be the function defined in Lemma* 1.2. *Then, one has*

(i) $S(3,\alpha,\beta)=\dfrac{\alpha^3}{(1-\beta)^3}+\dfrac{3\alpha\beta^2}{(1-\beta)^4}+\dfrac{\beta^3+2\beta^4}{(1-\beta)^5}$,

(ii) $S(4,\alpha,\beta)=\dfrac{\alpha^3}{(1-\beta)^4}+\dfrac{6\alpha^2\beta^2}{(1-\beta)^5}+\dfrac{\alpha\beta^3(11\beta+4)}{(1-\beta)^6}+\dfrac{6\beta^6+8\beta^5+\beta^4}{(1-\beta)^7}$.

The operator defined by Jain is given by

$$(P_n^{[\beta]}f)(x)=\sum_{k=0}^{\infty}\omega_\beta(k,nx)\cdot f\left(\frac{k}{n}\right),\quad f\in C[0,\infty), \tag{1.9}$$

where $0\le\beta<1$ and $\omega_\beta(k,\alpha)$ has been defined in (1.1).

Remark 1.4. If we take $\beta=0$ in (1.9) we obtain Szász -Mirakjan operator [3], [4].

$$(P_n^{[0]}f)(x)\equiv(S_nf)(x)=e^{-nx}\sum_{k=0}^{\infty}\frac{(nx)^k}{k!}\cdot\frac{k}{n},\quad x\ge 0. \tag{1.10}$$

We denote by $e_j(t)$ the monomial of degree j, $e_j(t)=t^j$.
Taking in view Lemma 1.2, in [1] has been established the following identities.

$$(P_n^{[\beta]}e_0)(x)=1. \tag{1.11}$$

$$(P_n^{[\beta]}e_1)(x)=xS(1,nx+\beta,\beta)=\frac{x}{1-\beta}. \tag{1.12}$$

$$\begin{aligned}(P_n^{[\beta]}e_2)(x)&=\frac{x}{n}\Big[S(2,nx+2\beta,\beta)+S(1,nx+\beta,\beta)\Big]\\&=\frac{x^2}{(1-\beta)^2}+\frac{x}{n(1-\beta)^3}.\end{aligned} \tag{1.13}$$

2. Main results

In what follows, $C_2[0,\infty)$ represents the space of all continuous functions having the second derivative continuous.

In this section we first define the function

$$\varphi_x \in C_2[0,\infty),\ \ \varphi_x(t) = t - x. \tag{2.1}$$

We also compute the values of $P_n^{[\beta]}$ on φ_x^3 and φ_x^4.

In order to present our main theorem, we need the following lemmas.

Lemma 2.1. *The operators defined by* (1.9) *verify the following identities.*

(i) $$(P_n^{[\beta]}e_3)(x) = \frac{x^3}{(1-\beta)^3} + \frac{3x^2}{n(1-\beta)^4} - \frac{x(6\beta^4 - 6\beta^3 - 2\beta - 1)}{n^2(1-\beta)^5}.$$

(ii) $$(P_n^{[\beta]}e_4)(x) = \frac{x^4}{(1-\beta)^4} + \frac{6x^3}{n(1-\beta)^5} - \frac{x^2(36\beta^4 - 72\beta^3 + 36\beta^2 - 8\beta - 7)}{n^2(1-\beta)^6} + \frac{x(105\beta^5 - 14\beta^4 - 2\beta^3 + 12\beta^2 + 8\beta + 1)}{n^3(1-\beta)^7}.$$

Remark 2.2. Examining the relations (i) and (ii) in Lemma 2.1, based on Korovkin theorem [2] and Theorem 2.1 in [1], we may observe that $(P_n^{[\beta]})_{n\ge 1}$ does not form an approximation process. In order to transform it into an approximation process, we replace the constant β by a number $\beta_n \in [0,1)$.

If

$$\lim_{n\to\infty} \beta_n = 0, \tag{2.2}$$

then Lemma 2.1 ensures us that $\lim_{n\to\infty} (P_n^{[\beta_n]}e_j)(x) = x^j$, $j = \overline{0,2}$ uniformly in $C([0,\infty))$.

On the basis of relations (1.12), (1.13) and Lemma 1.2 we deduce the following identities.

$$\begin{aligned}(P_n^{[\beta_n]}\varphi_x)(x) &= \sum_{k=0}^{\infty}(nx + k\beta_n)^{k-1}\cdot e^{-(nx+k\beta_n)}\frac{1}{k!}\cdot\frac{k-x}{n}\\ &= (P_n^{[\beta_n]}e_1)(x) - x(P_n^{[\beta_n]}e_0)(x)\\ &= \frac{x}{1-\beta_n} - x.\end{aligned} \tag{2.3}$$

$$\begin{aligned}(P_n^{[\beta_n]}\varphi_x^2)(x) &= \sum_{k=0}^{\infty}(nx + k\beta_n)^{k-1}\cdot e^{-(nx+k\beta_n)}\frac{1}{k!}\cdot\frac{(k-x)^2}{n^2}\\ &= (P_n^{[\beta_n]}e_2)(x) - 2x(P_n^{[\beta_n]}e_1)(x) + x^2(P_n^{[\beta_n]}e_0)(x)\\ &= \frac{x^2}{(1-\beta_n)^2} - \frac{2x^2}{1-\beta_n} + x^2 + \frac{x}{n(1-\beta_n)^3}.\end{aligned} \tag{2.4}$$

where φ_x is defined by (2.1).

Lemma 2.3. *Let the operator* $P_n^{[\beta_n]}$ *be defined by relation* (1.9) *and let* φ_x *be given by* (2.1). *Then*

(i) $$(P_n^{[\beta_n]}\varphi_x^3)(x) = \frac{x^3}{(1-\beta_n)^3} - \frac{3x^3}{(1-\beta_n)^2} + \frac{3x^3}{1-\beta_n} - x^3 + \frac{3x^2}{n(1-\beta_n)^4} - \frac{3x^2}{n(1-\beta_n)^3} - \frac{x(6\beta_n^4 - 6\beta_n^3 - 2\beta_n - 1)}{n^2(1-\beta_n)^5}.$$

(ii) $$(P_n^{[\beta_n]}\varphi_x^4)(x) = \frac{x^4}{(1-\beta_n)^4} - \frac{4x^4}{(1-\beta_n)^3} + \frac{6x^4}{(1-\beta_n)^2} - \frac{4x^4}{1-\beta_n} + x^4 + \frac{6x^3}{n(1-\beta_n)^5} - \frac{12x^3}{n(1-\beta_n)^4} + \frac{6x^3}{n(1-\beta_n)^3} - \frac{x^2(36\beta_n^4 - 72\beta_n^3 + 36\beta_n^2 - 8\beta_n - 7)}{n^2(1-\beta_n)^6} + \frac{4x^2(6\beta_n^4 - 6\beta_n^3 - 2\beta_n - 1)}{n^2(1-\beta_n)^5} + \frac{x(105\beta_n^5 - 14\beta_n^4 - 2\beta_n^3 + 12\beta_n^2 + 8\beta_n + 1)}{n^3(1-\beta_n)^7}.$$

Lemma 2.4. *Let* $P_n^{[\beta_n]}$ *be the Jain operator and let* φ_x *be defined in* (2.1). *In addition, if (2.2) holds, then*

$$P_n^{[\beta_n]}\varphi_x^4 \le \frac{12x^3}{n(1-\beta_n)^5} + \frac{24x^2}{n^2(1-\beta_n)^5} + \frac{106x}{n^3(1-\beta_n)^7}.$$

We may now present the main result.

Theorem 2.5. *Let* $f \in C_2([0,\infty))$ *and let the operator* $P_n^{[\beta_n]}$ *be defined as in* (1.9). *If* (2.2) *holds, then*

$$\lim_{n\to\infty} n\left(P_n^{[\beta_n]}(f;x) - f(x)\right) = \frac{x}{2}f''(x), \quad \forall x > 0.$$

3. Proofs

Proof of Lemma 1.3.

(i) $$S(3,\alpha,\beta) = \sum_{k=0}^{\infty}\beta^k(\alpha+k\beta)S(2,\alpha+k\beta,\beta)$$
$$= \sum_{k=0}^{\infty}\beta^k(\alpha+k\beta)\left(\frac{\alpha+k\beta}{(1-\beta)^2} + \frac{\beta^2}{(1-\beta)^3}\right)$$
$$= \frac{1}{(1-\beta)^2}\sum_{k=0}^{\infty}\beta^k(\alpha+k\beta)(\alpha+k\beta) + \frac{\beta^2}{(1-\beta)^3}\sum_{k=0}^{\infty}\beta^k(\alpha+k\beta)$$
$$= \frac{1}{(1-\beta)^2}\left(\frac{\alpha^2}{1-\beta} + \frac{2\alpha\beta^2}{(1-\beta)^2} + \frac{\beta^3(1+\beta)}{(1-\beta)^3}\right) + \frac{\beta^2}{(1-\beta)^3}\left(\frac{\alpha}{1-\beta} + \frac{\beta^2}{(1-\beta)^2}\right) = \frac{\alpha^3}{(1-\beta)^3} + \frac{3\alpha\beta^2}{(1-\beta)^4} + \frac{\beta^3+2\beta^4}{(1-\beta)^5}.$$

(ii)
$$S(4,\alpha,\beta)=\sum_{k=0}^{\infty}\beta^k(\alpha+k\beta)S(3,\alpha+k\beta,\beta)$$
$$=\sum_{k=0}^{\infty}\beta^k(\alpha+k\beta)\left[\frac{(\alpha+k\beta)^2}{(1-\beta)^3}+\frac{3(\alpha+k\beta)\beta^2}{(1-\beta)^4}+\frac{\beta^3+2\beta^4}{(1-\beta)^5}\right]$$
$$=\frac{1}{(1-\beta)^3}\sum_{k=0}^{\infty}\beta^k(\alpha+k\beta)^3+\frac{3\beta^2}{(1-\beta)^4}\sum_{k=0}^{\infty}\beta^k(\alpha+k\beta)^2$$
$$+\frac{\beta^3+2\beta^4}{(1-\beta)^5}\sum_{k=0}^{\infty}\beta^k(\alpha+k\beta)$$
$$=\frac{\alpha^3}{(1-\beta)^4}+\frac{6\alpha^2\beta^2}{(1-\beta)^5}+\frac{\alpha\beta^3(11\beta+4)}{(1-\beta)^6}+\frac{6\beta^6+8\beta^5+\beta^4}{(1-\beta)^7}.$$ □

Proof of Lemma 2.1.

(i)
$$P_n^{[\beta]}(e_3;x)=xn\sum_{k=0}^{\infty}(nx+k\beta)^{k-1}\cdot e^{-(nx+k\beta)}\frac{1}{k!}\cdot\frac{k^3}{n^3}$$
$$=\frac{x}{n^2}\Big[S(3,nx+3\beta,\beta)+3S(2,nx+2\beta,\beta)+S(1,nx+\beta,\beta)\Big]$$
$$=\frac{x^3}{(1-\beta)^3}+\frac{3x^2}{n(1-\beta)^4}-\frac{x(6\beta^4-6\beta^3-2\beta-1)}{n^2(1-\beta)^5}.$$

(ii)
$$(P_n^{[\beta]}e_4)(x)=xn\sum_{k=0}^{\infty}(nx+k\beta)^{k-1}\cdot e^{-(nx+k\beta)}\frac{1}{k!}\cdot\frac{k^4}{n^4}$$
$$=\frac{x}{n^3}\Big[S(4,nx+4\beta,\beta)+6S(3,nx+3\beta,\beta)$$
$$+7S(2,nx+2\beta,\beta)+S(1,nx+\beta,\beta)\Big]$$
$$=\frac{x^4}{(1-\beta)^4}+\frac{6x^3}{n(1-\beta)^5}-\frac{x^2(36\beta^4-72\beta^3+36\beta^2-8\beta-7)}{n^2(1-\beta)^6}$$
$$+\frac{x(105\beta^5-14\beta^4-2\beta^3+12\beta^2+8\beta+1)}{n^3(1-\beta)^7}.$$ □

Proof of Lemma 2.3.

(i)
$$(P_n^{[\beta_n]}\varphi_x^3)(x)=(P_n^{[\beta_n]}e_3)(x)-3x(P_n^{[\beta_n]}e_2)(x)+3x^2(P_n^{[\beta_n]}e_1)(x)-x^3(P_n^{[\beta_n]}e_0)(x)$$
$$=\frac{x^3}{(1-\beta_n)^3}-\frac{3x^3}{(1-\beta_n)^2}+\frac{3x^3}{1-\beta_n}-x^3+\frac{3x^2}{n(1-\beta_n)^4}$$
$$-\frac{3x^2}{n(1-\beta_n)^3}-\frac{x(6\beta_n^4-6\beta_n^3-2\beta_n-1)}{n^2(1-\beta_n)^5}.$$

(ii)
$$(P_n^{[\beta_n]}\varphi_x^4)(x) = (P_n^{[\beta_n]}e_4)(x) - 4x(P_n^{[\beta_n]}e_3)(x) + 6x^2(P_n^{[\beta_n]}e_2)(x)$$
$$-4x^3(P_n^{[\beta_n]}e_1)(x) + x^4(P_n^{[\beta_n]}e_0)(x)$$
$$= \frac{x^4}{(1-\beta_n)^4} - \frac{4x^4}{(1-\beta_n)^3} + \frac{6x^4}{(1-\beta_n)^2} - \frac{4x^4}{1-\beta_n} + x^4$$
$$+\frac{6x^3}{n(1-\beta_n)^5} - \frac{12x^3}{n(1-\beta_n)^4} + \frac{6x^3}{n(1-\beta_n)^3}$$
$$-\frac{x^2(36\beta_n^4 - 72\beta_n^3 + 36\beta_n^2 - 8\beta_n - 7)}{n^2(1-\beta_n)^6}$$
$$+\frac{4x^2(6\beta_n^4 - 6\beta_n^3 - 2\beta_n - 1)}{n^2(1-\beta_n)^5}$$
$$+\frac{x(105\beta_n^5 - 14\beta_n^4 - 2\beta_n^3 + 12\beta_n^2 + 8\beta_n + 1)}{n^3(1-\beta_n)^7}. \qquad \square$$

Proof of Lemma 2.4. Starting from relation (*ii*) in Lemma 2.3, the entire proof of Lemma 2.4 is based on the following simple increases:

$$\frac{6x^3}{n(1-\beta_n)^3} \le \frac{6x^3}{n(1-\beta_n)^5},$$

$$6\beta_n^4 - 6\beta_n^3 - 2\beta_n - 1 \le 6, \quad 105\beta_n^5 - 14\beta_n^4 - 2\beta_n^3 + 12\beta_n^2 + 8\beta_n + 1 \le 106. \qquad \square$$

Proof of Theorem 2.5. Let $f, f', f'' \in C_2([0,\infty))$ and $x \in [0,\infty)$ be fixed. By the Taylor formula we have

$$f(t) = f(x) + f'(x)(t-x) + \frac{1}{2}f''(x)(t-x)^2 + r(t;x)(t-x)^2, \tag{3.1}$$

where $r(t;x)$ is the Peano form of the remainder, $r(\cdot;x) \in C_2([0,\infty))$ and

$$\lim_{t\to x} r(t;x) = 0.$$

Let φ_x be given by (2.1). We apply $P_n^{[\beta_n]}$ to (3.1) and we get

$$(P_n^{[\beta_n]}f)(x) - f(x) = (P_n^{[\beta_n]}\varphi_x)(x)\cdot f'(x) + \frac{1}{2}(P_n^{[\beta_n]}\varphi_x^2)(x)\cdot f''(x)$$
$$+ (P_n^{[\beta_n]}\varphi_x^2 \cdot r(\cdot;x))(x). \tag{3.2}$$

Using the relations (2.3) and (2.4) one obtains

$$(P_n^{[\beta_n]}f)(x) - f(x) = \left(\frac{x}{1-\beta_n} - x\right)f'(x)$$
$$+\frac{1}{2}\left[\frac{x^2}{(1-\beta_n)^2} - \frac{2x^2}{1-\beta_n} + x^2 + \frac{x}{n(1-\beta_n)^3}\right]f''(x)$$
$$+ (P_n^{[\beta_n]}\varphi_x^2 \cdot r(\cdot;x))(x) \tag{3.3}$$

For the last term, by applying the Cauchy-Schwartz inequality, we get

$$0 \le |(P_n^{[\beta_n]}\varphi_x^2 \cdot r(\cdot;x))(x)| \le \sqrt{(P_n^{[\beta_n]}\varphi_x^4)(x)} \cdot \sqrt{(P_n^{[\beta_n]}r^2(\cdot;x))(x)} \tag{3.4}$$

We have marked that $\lim\limits_{t\to x} r(t,x) = 0$. In harmony with Remark 2.2 we have

$$\lim_{n\to\infty} P_n^{[\beta_n]}(r^2(x,x);x) = 0. \tag{3.5}$$

On the basis of (2.2) , (3.4), (3.5) and Lemma 2.4 , we get that

$$\lim_{n\to\infty} n\left(P_n^{[\beta_n]}(f;x) - f(x)\right) = \frac{x}{2}f''(x). \qquad \square$$

Remark 3.1. Considering Jain's operator P_n^β and taking $\beta = \beta_n$, with β_n satisfying (2.2) we have rediscovered the genuine Voronovskaja result for Szász operators (1.10). The same genuine Voronovskaja result was found once again in in [5, *Eq. (20)*] while studying some operators of Szász-Mirakjan type.

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Two remarks on harmonic Bergman spaces in $\mathbb{B}^n$ and $\mathbb{R}^{n+1}_+$

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Abstract. Sharp estimates on distances in spaces of harmonic functions in the unit ball and the upper half space are obtained.

Mathematics Subject Classification (2010): 42B15, 42B30.

Keywords: Distance estimates, harmonic function, unit ball, Bergman spaces.

1. Introduction and preliminaries

In this note we obtain distance estimates in spaces of harmonic functions on the unit ball and on the upper half space. This line of investigation can be considered as a continuation of papers [1], [4] and [5]. These results are contained in the second section of the paper. The first section is devoted to preliminaries and main definitions which are needed for formulations of main results. Almost all objects we define and definitions can be found in [2] and in [6].

Let $\mathbb{B}$ be the open unit ball in $\mathbb{R}^n$, $\mathbb{S} = \partial\mathbb{B}$ is the unit sphere in $\mathbb{R}^n$, for $x \in \mathbb{R}^n$ we have $x = rx'$, where $r = |x| = \sqrt{\sum_{j=1}^n x_j^2}$ and $x' \in \mathbb{S}$. Normalized Lebesgue measure on $\mathbb{B}$ is denoted by $dx = dx_1 \dots dx_n = r^{n-1} dr dx'$ so that $\int_{\mathbb{B}} dx = 1$. We denote the space of all harmonic functions in an open set Ω by $h(\Omega)$. In this paper letter C designates a positive constant which can change its value even in the same chain of inequalities.

For $0 < p < \infty$, $0 \le r < 1$ and $f \in h(\mathbb{B})$ we set

$$M_p(f, r) = \left(\int_{\mathbb{S}} |f(rx')|^p dx' \right)^{1/p},$$

with the usual modification to cover the case $p = \infty$.

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For $0 < p < \infty$ and $\alpha > -1$ we consider weighted harmonic Bergman spaces $A^p_\alpha = A^p_\alpha(\mathbb{B})$ defined by

$$A^p_\alpha = \left\{ f \in h(\mathbb{B}) : \|f\|^p_{A^p_\alpha} = \int_{\mathbb{B}} |f(x)|^p (1-|x|^2)^\alpha dx < \infty \right\}.$$

For $p = \infty$ this definition is modified in a standard manner:

$$A^\infty_\alpha = A^\infty_\alpha(\mathbb{B}) = \left\{ f \in h(\mathbb{B}) : \|f\|_{A^\infty_\alpha} = \sup_{x \in \mathbb{B}} |f(x)|(1-|x|^2)^\alpha < \infty \right\}, \quad \alpha > -1.$$

These spaces are complete metric spaces for $0 < p \le \infty$, they are Banach spaces for $p \ge 1$.

Next we need certain facts on spherical harmonics and the Poisson kernel, see [2] for a detailed exposition. Let $Y^{(k)}_j$ be the spherical harmonics of order k, $1 \le j \le d_k$, on $\mathbb{S}$. Next,

$$Z^{(k)}_{x'}(y') = \sum_{j=1}^{d_k} Y^{(k)}_j(x') \overline{Y^{(k)}_j(y')}$$

are zonal harmonics of order k. Note that the spherical harmonics $Y^{(k)}_j$, ($k \ge 0$, $1 \le j \le d_k$) form an orthonormal basis of $L^2(\mathbb{S}, dx')$. Every $f \in h(\mathbb{B})$ has an expansion

$$f(x) = f(rx') = \sum_{k=0}^\infty r^k b_k \cdot Y^k(x'),$$

where $b_k = (b^1_k, \dots, b^{d_k}_k)$, $Y^k = (Y^{(k)}_1, \dots, Y^{(k)}_{d_k})$ and $b_k \cdot Y^k$ is interpreted in the scalar product sense: $b_k \cdot Y^k = \sum_{j=1}^{d_k} b^j_k Y^{(k)}_j$.

We denote the Poisson kernel for the unit ball by $P(x, y')$, it is given by

$$\begin{aligned} P(x,y') = P_{y'}(x) &= \sum_{k=0}^\infty r^k \sum_{j=1}^{d_k} Y^{(k)}_j(y') Y^{(k)}_j(x') \\ &= \frac{1}{n\omega_n} \frac{1-|x|^2}{|x-y'|^n}, \; x = rx' \in \mathbb{B}, \quad y' \in \mathbb{S}, \end{aligned}$$

where ω_n is the volume of the unit ball in $\mathbb{R}^n$. We are going to use also a Bergman kernel for A^p_β spaces, this is the following function

$$Q_\beta(x,y) = 2 \sum_{k=0}^\infty \frac{\Gamma(\beta+1+k+n/2)}{\Gamma(\beta+1)\Gamma(k+n/2)} r^k \rho^k Z^{(k)}_{x'}(y'), \; x = rx', \; y = \rho y' \in \mathbb{B}. \tag{1.1}$$

For details on this kernel we refer to [2], where the following theorem can be found.

Theorem 1.1. [2] *Let $p \ge 1$ and $\beta \ge 0$. Then for every $f \in A^p_\beta$ and $x \in \mathbb{B}$ we have*

$$f(x) = \int_0^1 \int_{\mathbb{S}^{n-1}} Q_\beta(x,y) f(\rho y')(1-\rho^2)^\beta \rho^{n-1} d\rho dy', \; y = \rho y'.$$

This theorem is a cornerstone for our approach to distance problems in the case of the unit ball. The following lemma gives estimates for this kernel, see [2], [3].

Lemma 1.2. *1. Let $\beta > 0$. Then, for $x = rx', y = \rho y' \in \mathbb{B}$ we have*

$$|Q_\beta(x,y)| \le \frac{C}{|\rho x - y'|^{n+\beta}}.$$

2. Let $\beta > -1$. Then

$$\int_{\mathbb{S}^{n-1}} |Q_\beta(rx',y)|dx' \le \frac{C}{(1-r\rho)^{1+\beta}},\ |y| = \rho, \quad 0 \le r < 1.$$

3. Let $\beta > n-1$, , $0 \le r < 1$ and $y' \in \mathbb{S}^{n-1}$. Then

$$\int_{\mathbb{S}^{n-1}} \frac{dx'}{|rx'-y'|^\beta} \le \frac{C}{(1-r)^{\beta-n+1}}.$$

Lemma 1.3. [2] *Let $\alpha > -1$ and $\lambda > \alpha + 1$. Then*

$$\int_0^1 \frac{(1-r)^\alpha}{(1-r\rho)^\lambda} dr \le C(1-\rho)^{\alpha+1-\lambda},\ 0 \le \rho < 1.$$

Lemma 1.4. *For $\delta > -1$, $\gamma > n + \delta$ and $\beta > 0$ we have*

$$\int_{\mathbb{B}} |Q_\beta(x,y)|^{\frac{\gamma}{n+\beta}} (1-|y|)^\delta dy \le C(1-|x|)^{\delta-\gamma+n},\ x \in \mathbb{B}.$$

Proof. Using Lemma 1.2 and Lemma 1.3 we obtain:

$$\int_{\mathbb{B}} |Q_\beta(x,y)|^{\frac{\gamma}{n+\beta}} (1-|y|)^\delta dy \le C \int_{\mathbb{B}} \frac{(1-|y|)^\delta}{|\rho r x' - y'|^\gamma} dy$$

$$\le C \int_0^1 (1-\rho)^\delta \int_{\mathbb{S}} \frac{dy'}{|\rho r x' - y'|^\gamma} dy' d\rho$$

$$\le C \int_0^1 (1-\rho)^\delta (1-r\rho)^{n-\gamma-1} d\rho \le C(1-r)^{n+\delta-\gamma}.$$ □

We set $\mathbb{R}^{n+1}_+ = \{(x,t) : x \in \mathbb{R}^n, t > 0\} \subset \mathbb{R}^{n+1}$. We usually denote points in $\mathbb{R}^{n+1}_+$ by $z = (x,t)$ or $w = (y,s)$ where $x, y \in \mathbb{R}^n$ and $s, t > 0$.

For $0 < p < \infty$ and $\alpha > -1$ we consider spaces

$$\tilde{A}^p_\alpha(\mathbb{R}^{n+1}_+) = \tilde{A}^p_\alpha = \left\{ f \in h(\mathbb{R}^{n+1}_+) : \int_{\mathbb{R}^{n+1}_+} |f(x,t)|^p t^\alpha dx dt < \infty \right\}.$$

Also, for $p = \infty$ and $\alpha > 0$, we set

$$\tilde{A}^\infty_\alpha(\mathbb{R}^{n+1}_+) = \tilde{A}^\infty_\alpha = \left\{ f \in h(\mathbb{R}^{n+1}_+) : \sup_{(x,t)\in\mathbb{R}^{n+1}_+} |f(x,t)| t^\alpha < \infty \right\}.$$

These spaces have natural (quasi)-norms, for $1 \le p \le \infty$ they are Banach spaces and for $0 < p \le 1$ they are complete metric spaces.

We denote the Poisson kernel for $\mathbb{R}^{n+1}_+$ by $P(x,t)$, i.e.

$$P(x,t) = c_n \frac{t}{(|x|^2+t^2)^{\frac{n+1}{2}}},\ x \in \mathbb{R}^n, t > 0.$$

For an integer $m \geq 0$ we introduce a Bergman kernel $Q_m(z,w)$, where $z = (x,t) \in \mathbb{R}^{n+1}_+$ and $w = (y,s) \in \mathbb{R}^{n+1}_+$, by

$$Q_m(z,w) = \frac{(-2)^{m+1}}{m!} \frac{\partial^{m+1}}{\partial t^{m+1}} P(x-y, t+s).$$

The terminology is justified by the following result from [2].

Theorem 1.5. *Let $0 < p < \infty$ and $\alpha > -1$. If $0 < p \leq 1$ and $m \geq \frac{\alpha+n+1}{p} - (n+1)$ or $1 \leq p < \infty$ and $m > \frac{\alpha+1}{p} - 1$, then*

$$f(z) = \int_{\mathbb{R}^{n+1}_+} f(w) Q_m(z,w) s^m dyds, \ f \in \tilde{A}^p_\alpha, \quad z \in \mathbb{R}^{n+1}_+. \tag{1.2}$$

The following elementary estimate of this kernel is contained in [2]:

$$|Q_m(z,w)| \leq C \left[|x-y|^2 + (s+t)^2 \right]^{-\frac{n+m+1}{2}}, \quad z = (x,t), w = (y,s) \in \mathbb{R}^{n+1}_+. \tag{1.3}$$

2. Estimates for distances in harmonic function spaces in the unit ball and related problems in $\mathbb{R}^{n+1}_+$

In this section we investigate distance problems both in the case of the unit ball and in the case of the upper half space. The method we use here originated in [7], see [1], [4], [5] for various modification of this method.

Lemma 2.1. *Let $0 < p < \infty$ and $\alpha > -1$. Then there is a $C = C_{p,\alpha,n}$ such that for every $f \in A^p_\alpha(\mathbb{B})$ we have*

$$|f(x)| \leq C(1-|x|)^{-\frac{\alpha+n}{p}} \|f\|_{A^p_\alpha}, \ x \in \mathbb{B}.$$

Proof. We use subharmonic behavior of $|f|^p$ to obtain

$$|f(x)|^p \leq \frac{C}{(1-|x|)^n} \int_{B(x, \frac{1-|x|}{2})} |f(y)|^p dy$$

$$\leq C \frac{(1-|x|)^{-\alpha}}{(1-|x|)^n} \int_{B(x, \frac{1-|x|}{2})} |f(y)|^p (1-|y|)^\alpha dy \leq C(1-|x|)^{-\alpha-n} \|f\|^p_{A^p_\alpha}. \qquad \square$$

This lemma shows that A^p_α is continuously embedded in $A^\infty_{\frac{\alpha+n}{p}}$ and motivates the distance problem that is investigated in Theorem 2.3.

Lemma 2.2. *Let $0 < p < \infty$ and $\alpha > -1$. Then there is $C = C_{p,\alpha,n}$ such that for every $f \in \tilde{A}^p_\alpha$ and every $(x,t) \in \mathbb{R}^{n+1}_+$ we have*

$$|f(x,t)| \leq C y^{-\frac{\alpha+n+1}{p}} \|f\|_{\tilde{A}^p_\alpha}. \tag{2.1}$$

The above lemma states that $\tilde{A}^p_\alpha$ is continuously embedded in $\tilde{A}^\infty_{\frac{\alpha+n+1}{p}}$, its proof is analogous to that of Lemma 2.1.

For $\epsilon > 0$, $t > 0$ and $f \in h(\mathbb{B})$ we set

$$U_{\epsilon,t}(f) = U_{\epsilon,t} = \{x \in \mathbb{B} : |f(x)|(1-|x|)^t \geq \epsilon\}.$$

Theorem 2.3. *Let* $p > 1$, $\alpha > -1$, $t = \frac{\alpha+n}{p}$ *and* $\beta > \max(\frac{\alpha+n}{p} - 1, \frac{\alpha}{p})$. *Set, for* $f \in A^\infty_{\frac{\alpha+n}{p}}(\mathbb{B})$:

$$t_1(f) = \operatorname{dist}_{A^\infty_{\frac{\alpha+n}{p}}}(f, A^p_\alpha),$$

$$t_2(f) = \inf\left\{\epsilon > 0 : \int_{\mathbb{B}} \left(\int_{U_{\epsilon,t}} |Q_\beta(x,y)|(1-|y|)^{\beta-t}dy\right)^p (1-|x|)^\alpha dx < \infty\right\}.$$

Then $t_1(f) \asymp t_2(f)$.

Proof. We begin with inequality $t_1(f) \geq t_2(f)$. Assume $t_1(f) < t_2(f)$. Then there are $0 < \epsilon_1 < \epsilon$ and $f_1 \in A^p_\alpha$ such that $\|f - f_1\|_{A^\infty_t} \leq \epsilon_1$ and

$$\int_{\mathbb{B}} \left(\int_{U_{\epsilon,t}(f)} |Q_\beta(x,y)|(1-|y|)^{\beta-t}dy\right)^p (1-|x|)^\alpha dx = +\infty.$$

Since $(1-|x|)^t|f_1(x)| \geq (1-|x|)^t|f(x)| - (1-|x|)^t|f(x) - f_1(x)|$ for every $x \in \mathbb{B}$ we conclude that $(1-|x|)^t|f_1(x)| \geq (1-|x|)^t|f(x)| - \epsilon_1$ and therefore

$$(\epsilon - \epsilon_1)\chi_{U_{\epsilon,t}(f)}(x)(1-|x|)^{-t} \leq |f_1(x)|, \ x \in \mathbb{B}.$$

Hence

$$\begin{aligned} +\infty &= \int_{\mathbb{B}} \left(\int_{U_{\epsilon,t}(f)} |Q_\beta(x,y)|(1-|y|)^{\beta-t}dy\right)^p (1-|x|)^\alpha dx \\ &= \int_{\mathbb{B}} \left(\int_{\mathbb{B}} \frac{\chi_{U_{\epsilon,t}(f)}(y)}{(1-|y|)^t} |Q_\beta(x,y)|(1-|y|)^\beta dy\right)^p (1-|x|)^\alpha dx \\ &\leq C_{\epsilon,\epsilon_1} \int_{\mathbb{B}} \left(\int_{\mathbb{B}} |f_1(y)||Q_\beta(x,y)|(1-|y|)^\beta dy\right)^p (1-|x|)^\alpha dx = M, \end{aligned}$$

and we are going to prove that M is finite, arriving at a contradiction. Let q be the exponent conjugate to p. We have, using Lemma 1.4,

$$\begin{aligned} I(x) &= \left(\int_{\mathbb{B}} |f_1(y)|(1-|y|)^\beta |Q_\beta(x,y)|dy\right)^p \\ &= \left(\int_{\mathbb{B}} |f_1(y)|(1-|y|)^\beta |Q_\beta(x,y)|^{\frac{1}{n+\beta}(\frac{n}{p}+\beta-\epsilon)} |Q_\beta(x,y)|^{\frac{1}{n+\beta}(\frac{n}{q}+\epsilon)}dy\right)^p \\ &\leq \int_{\mathbb{B}} |f_1(y)|^p(1-|y|)^{p\beta}|Q_\beta(x,y)|^{\frac{n+p\beta-p\epsilon}{n+\beta}}dy \left(\int_{\mathbb{B}} |Q_\beta(x,y)|^{\frac{n+q\epsilon}{n+\beta}}dy\right)^{p/q} \\ &\leq C(1-|x|)^{-p\epsilon} \int_{\mathbb{B}} |f_1(y)|^p(1-|y|)^{p\beta}|Q_\beta(x,y)|^{\frac{n+p\beta-p\epsilon}{n+\beta}}dy \end{aligned}$$

for every $\epsilon > 0$. Choosing $\epsilon > 0$ such that $\alpha - p\epsilon > -1$ we have, by Fubini's theorem and Lemma 1.4:

$$\begin{aligned} M &\leq C \int_{\mathbb{B}} |f_1(y)|^p(1-|y|)^{p\beta} \int_{\mathbb{B}} (1-|x|)^{\alpha-p\epsilon}|Q_\beta(x,y)|^{\frac{n+p\beta-p\epsilon}{n+\beta}}dxdy \\ &\leq C \int_{\mathbb{B}} |f_1(y)|^p(1-|y|)^\alpha dy < \infty. \end{aligned}$$

In order to prove the remaining estimate $t_1(f) \leq Ct_2(f)$ we fix $\epsilon > 0$ such that the integral appearing in the definition of $t_2(f)$ is finite and use Theorem 1.1, with $\beta > \max(t-1, 0)$:

$$f(x) = \int_{\mathbb{B}\setminus U_{\epsilon,t}(f)} Q_\beta(x,y)f(y)(1-|y|^2)^\beta dy + \int_{U_{\epsilon,t}(f)} Q_\beta(x,y)f(y)(1-|y|^2)^\beta dy$$
$$= f_1(x) + f_2(x).$$

Since, by Lemma 1.4, $|f_1(x)| \leq 2^\beta \int_{\mathbb{B}} |Q_\beta(x,y)|(1-|w|)^{\beta-t}dy \leq C(1-|x|)^{-t}$ we have $\|f_1\|_{A_t^\infty} \leq C\epsilon$. Thus it remains to show that $f_2 \in A_\alpha^p$ and this follows from

$$\|f_2\|_{A_\alpha^p}^p \leq \|f\|_{A_t^\infty}^p \int_{\mathbb{B}} \left(\int_{U_{\epsilon,t}(f)} |Q_\beta(x,y)|(1-|y|^2)^{\beta-t}dy \right)^p (1-|x|)^\alpha dx < \infty. \qquad \square$$

The above theorem has a counterpart in the $\mathbb{R}_+^{n+1}$ setting. As a preparation for this result we need the following analogue of Lemma 1.4.

Lemma 2.4. *For $\delta > -1$, $\gamma > n+1+\delta$ and $m \in \mathbb{N}_0$ we have*

$$\int_{\mathbb{R}_+^{n+1}} |Q_m(z,w)|^{\frac{\gamma}{n+m+1}} s^\delta dy ds \leq Ct^{\delta-\gamma+n+1}, \ t > 0.$$

Proof. Using Fubini's theorem and estimate (1.3) we obtain

$$I(t) = \int_{\mathbb{R}_+^{n+1}} |Q_m(z,w)|^{\frac{\gamma}{n+m+1}} s^\delta dy ds \leq C \int_0^\infty s^\delta \left(\int_{\mathbb{R}^n} \frac{dy}{[|y|^2 + (s+t)^2]^\gamma} \right) ds$$
$$= C \int_0^\infty s^\delta (s+t)^{n-\gamma} ds = Ct^{\delta-\gamma+n+1}. \qquad \square$$

For $\epsilon > 0$, $\lambda > 0$ and $f \in h(\mathbb{R}_+^{n+1})$ we set:

$$V_{\epsilon,\lambda}(f) = \{(x,t) \in \mathbb{R}_+^{n+1} : |f(x,t)|t^\lambda \geq \epsilon\}.$$

Theorem 2.5. *Let $p > 1$, $\alpha > -1$, $\lambda = \frac{\alpha+n+1}{p}$, $m \in \mathbb{N}_0$ and $m > \max(\frac{\alpha+n+1}{p} - 1, \frac{\alpha}{p})$. Set, for $f \in \tilde{A}^\infty_{\frac{\alpha+n+1}{p}}(\mathbb{R}_+^{n+1})$:*

$$s_1(f) = \operatorname{dist}_{\tilde{A}^\infty_{\frac{\alpha+n+1}{p}}}(f, \tilde{A}^p_\alpha),$$

$$s_2(f) = \inf \left\{ \epsilon > 0 : \int_{\mathbb{R}_+^{n+1}} \left(\int_{V_{\epsilon,\lambda}} Q_m(z,w)s^{m-\lambda} dy ds \right)^p t^\alpha dx dt < \infty \right\}.$$

Then $s_1(f) \asymp s_2(f)$.

The proof of this theorem closely parallels the proof of the previous one, in fact, the role of Lemma 1.4 is taken by Lemma 2.4 and the role of Theorem 1.1 is taken by Theorem 1.5. We leave details to the reader.

Remark. Results of this note very recently were extended by the authors to all values of positive p. Proofs of these assertions are heavily based on the well-known so-called Whitney decomposition of the upper halfspace of $\mathbb{R}^{n+1}$ and the unit ball B and some nice properties and estimates of the related Whitney cubes and harmonic functions on them, which partially can be found in [6].

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On a subalgebra of $L^1_w(G)$

İsmail Aydın

Abstract. Let G be a locally compact abelian group with Haar measure. We define the spaces $B_{1,w}(p,q) = L^1_w(G) \cap (L^p, \ell^q)(G)$ and discuss some properties of these spaces. We show that $B_{1,w}(p,q)$ is an $S_w(G)$ space. Furthermore we investigate compact embeddings and the multipliers of $B_{1,w}(p,q)$.

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Keywords: Multipliers, compact embedding, amalgam spaces.

1. Introduction

Let G be a locally compact abelian group with Haar measure μ. An amalgam space $(L^p, \ell^q)(G)$ $(1 \le p, q \le \infty)$ is a Banach space of measurable (equivalence classes of) functions on G which belong locally to L^p and globally to ℓ^q. Several authors have introduced special cases of amalgams. Among others N. Wiener [28], [29], P. Szeptycki [25], T. S. Liu, A. Van Rooij and J. K. Wang [19], H. E. Krogstad [17] and H. G. Feichtinger [8]. For a historical background of amalgams see [11]. The first systematic study of amalgams on the real line was undertaken by F. Holland [16]. In 1979 J. Stewart [24] extended the definition of Holland to locally compact abelian groups using the Structure Theorem for locally compact groups.

For $1 \le p < \infty$, the spaces $B^p(G) = L^1(G) \cap L^p(G)$ is a Banach algebra with respect to the norm $\|.\|_{B^p(G)}$ defined by $\|f\|_{B^p(G)} = \|f\|_1 + \|f\|_p$ and usual convolution product. The Banach algebras $B^p(G)$ have been studied by C. R. Warner [27], L. Y. H. Yap [30], and others. L. Y. H. Yap [31] extended some of the results on $B^p(G)$ to the Segal algebras

$$B(p,q)(G) = L^1(G) \cap L(p,q)(G),$$

where $L(p,q)(G)$ is Lorentz spaces. The purpose of this paper is to discuss some properties of the spaces $B_{1,w}(p,q) = L^1_w(G) \cap (L^p, \ell^q)(G)$. Also we investigate the spaces of all multipliers from $L^1_w(G)$ into $B_{1,w}(p,q)$ and $(B_{1,w}(p,q))^*$ over $L^1_w(G)$.

2. Preliminaries

The translation operator T_y is given by $T_y f(x) = f(x-y)$ for $x \in G$. $(B, \|.\|_B)$ is called (strongly) translation invariant if one has $T_y f \in B$ $\big($ and $\|T_y f\|_B = \|f\|_B\big)$ for all $f \in B$ and $y \in G$. A space $(B, \|.\|_B)$ is called strongly character invariant if one has $M_t f(x) = \langle x, t\rangle f(x) \in B$ and $\|M_t f\|_B = \|f\|_B$ for all $f \in B$, $x \in G$ and $t \in \widehat{G}$, where $\widehat{G}$ is the dual group of G. A Banach function space (shortly BF-space) on G is a Banach space $(B, \|.\|_B)$ of measurable functions which is continuously embedded into $L^1_{loc}(G)$, i.e. for any compact subset $K \subset G$ there exists some constant $C_K > 0$ such that $\|f\chi_K\|_1 \le C_K \|f\|_B$ for all $f \in B$. A BF-space is called solid if $g \in B$, $f \in L^1_{loc}(G)$ and $|f(x)| \le |g(x)|$ locally almost every where (shortly l.a.e) implies $f \in B$ and $\|f\|_B \le \|g\|_B$. It is easy to see that $(B, \|.\|_B)$ is solid iff it is a $L^\infty-$module. $C_c(G)$ will denote the linear space of continuous functions on G, which have compact support.

Definition 2.1. A strictly positive, continous function w satisfying $w(x) \ge 1$ and $w(x+y) \le w(x)w(y)$ for all $x, y \in G$ will be called a weight function. Let $1 \le p < \infty$. Then the weighted Lebesgue space $L^p_w(G) = \{f : fw \in L^p(G)\}$ is a Banach space with norm $\|f\|_{p,w} = \|fw\|_p$ and its dual space $L^{p'}_{w^{-1}}(G)$, where $\frac{1}{p} + \frac{1}{p'} = 1$. Moreover, if $1 < p < \infty$, then $L^p_w(G)$ is a reflexive Banach space. Particularly, for $p = 1$, $L^1_w(G)$ is a Banach algebra under convolution, called a Beurling algebra. It is obvious that $\|.\|_1 \le \|.\|_{1,w}$ and $L^1_w(G) \subset L^1(G)$. We say that $w_1 \prec w_2$ if and only if there exists a $C > 0$ such that $w_1(x) \le Cw_2(x)$ for all $x \in G$. Two weight functions are called equivalent and written $w_1 \approx w_2$, if $w_1 \prec w_2$ and $w_2 \prec w_1$. It is known that $L^p_{w_2}(G) \subset L^p_{w_1}(G)$ iff $w_1 \prec w_2$. A weight function w is said to satisfy the Beurling-Domar (shortly BD) condition, if

$$\sum_{n\ge 1} n^{-2} \log w(nx) < \infty$$

for all $x \in G$ [6].

Definition 2.2. Let V and W be two Banach modules over a Banach algebra A. Then a multiplier from V into W is a bounded linear operator T from V into W, which commutes with module multiplication, i.e. $T(av) = aT(v)$ for $a \in A$ and $v \in V$. We denote by $Hom_A(V, W)$ the space of all multipliers from V into W. Also we write $Hom_A(V, V) = Hom_A(V)$. It is known that

$$Hom_A(V, W^*) \cong (V \otimes_A W)^*,$$

where W^* is dual of W and $V \otimes_A W$ is the $A-$module tensor product of V and W [Corollary 2.13, 21].

We will denote by $M(G)$ the space of bounded regular Borel measures on G. We let

$$M(w) = \left\{ \mu \in M(G) : \int_G w d|\mu| < \infty \right\}.$$

It is known that the space of multipliers from $L^1_w(G)$ to from $L^1_w(G)$ is homeomorphic to $M(w)$ [12].

A kind of generalization of Segal algebra was defined in [3], as follows:

Definition 2.3. Let $S_w(G) = S_w$ be a subalgebra of $L^1_w(G)$ satisfying the following conditions:

S1) S_w is dense in $L^1_w(G)$.

S2) S_w is a Banach algebra under some norm $\|.\|_{S_w}$ and invariant under translations.

S3) $\|T_a f\|_{S_w} \leq w(a) \|f\|_{S_w}$ for all $a \in G$ and for each $f \in S_w$.

S4) If $f \in S_w$, then for every $\varepsilon > 0$ there exists a neighborhood U of the identity element of G such that $\|T_y f - f\|_{S_w} < \varepsilon$ for all $y \in U$.

S5) $\|f\|_{1,w} \leq \|f\|_{S_w}$ for all $f \in S_w$.

Definition 2.4. We denote by $L^p_{loc}(G)$ $(1 \leq p \leq \infty)$ the space of (equivalence classes of) functions on G such that f restricted to any compact subset E of G belongs to $L^p(G)$. Let $1 \leq p, q \leq \infty$. The amalgam of L^p and ℓ^q on the real line is the normed space

$$(L^p, \ell^q) = \left\{ f \in L^p_{loc}(\mathbb{R}) : \|f\|_{pq} < \infty \right\},$$

where

$$\|f\|_{pq} = \left[\sum_{n=-\infty}^{\infty} \left[\int_n^{n+1} |f(x)|^p \, dx \right]^{q/p} \right]^{1/q}. \tag{2.1}$$

We make the appropriate changes for p, q infinite. The norm $\|.\|_{pq}$ makes (L^p, ℓ^q) into a Banach space [16].

The following definition of $(L^p, \ell^q)(G)$ is due to J. Stewart [24]. By the Structure Theorem [Theorem 24.30, 15], $G = \mathbb{R}^a \times G_1$, where a is a nonnegative integer and G_1 is a locally compact abelian group which contains an open compact subgroup H. Let $I = [0,1)^a \times H$ and $J = \mathbb{Z}^a \times T$, where T is a transversal of H in G_1, i.e. $G_1 = \bigcup_{t \in T} (t + H)$ is a coset decomposition of G_1. For $\alpha \in J$ we define $I_\alpha = \alpha + I$, and therefore G is equal to the disjoint union of relatively compact sets I_α. We normalize μ so that $\mu(I) = \mu(I_\alpha) = 1$ for all α. Let $1 \leq p, q \leq \infty$. The amalgam space $(L^p, \ell^q)(G) = (L^p, \ell^q)$ is a Banach space

$$\left\{ f \in L^p_{loc}(G) : \|f\|_{pq} < \infty \right\},$$

where

$$\begin{aligned}
\|f\|_{pq} &= \left[\sum_{\alpha \in J} \|f\|^q_{L^p(I_\alpha)} \right]^{1/q} \quad \text{if } 1 \leq p, q < \infty, \qquad (2.2)\\
\|f\|_{\infty q} &= \left[\sum_{\alpha \in J} \sup_{x \in I_\alpha} |f(x)|^q \right]^{1/q} \quad \text{if } p = \infty, \ 1 \leq q < \infty,\\
\|f\|_{p\infty} &= \sup_{\alpha \in J} \|f\|_{L^p(I_\alpha)} \quad \text{if } 1 \leq p < \infty, \ q = \infty.
\end{aligned}$$

If $G = \mathbb{R}$, then we have $J = \mathbb{Z}$, $I_\alpha = [\alpha, \alpha + 1)$ and (2.2) becomes (2.1).

The amalgam spaces (L^p, ℓ^q) satisfy the following relations and inequalities [24]:

$$(L^p, \ell^{q_1}) \subset (L^p, \ell^{q_2}) \quad q_1 \leq q_2 \tag{2.3}$$

$$(L^{p_1},\ell^q)\subset(L^{p_2},\ell^q)\quad p_1\geq p_2 \tag{2.4}$$
$$(L^p,\ell^p)=L^p \tag{2.5}$$
$$(L^p,\ell^q)\subset L^p\cap L^q,\quad p\geq q \tag{2.6}$$
$$L^p\cup L^q\subset(L^p,\ell^q),\quad p\leq q \tag{2.7}$$
$$\|f\|_{pq_2}\leq\|f\|_{pq_1},\quad q_1\leq q_2 \tag{2.8}$$
$$\|f\|_{p_2q}\leq\|f\|_{p_1q},\quad p_1\geq p_2. \tag{2.9}$$

Note that $C_c(G)$ is included in all amalgam spaces. If $1\leq p,q<\infty$, then the dual space of (L^p,ℓ^q) is isometrically isomorphic to $\left(L^{p'},\ell^{q'}\right)$, where $1/p+1/p'=1/q+1/q'=1$.

Definition 2.5. Let A be a Banach algebra. A Banach space B is said to be a Banach $A-$module if there exists a bilinear operation $\cdot:A\times B\to B$ such that

(*i*) $(f\cdot g)\cdot h=f\cdot(g\cdot h)$ for all $f,g\in A$, $h\in B$.

(*ii*) For some constant $C\geq1$, $\|f\cdot h\|_B\leq C\|f\|_A\|h\|_B$ for all $f\in A$, $h\in B$ [7].

Theorem 2.6. If p,q,r,s are exponents such that $1/p+1/r-1=1/m\leq1$ and $1/q+1/s-1=1/n\leq1$, then

$$(L^p,\ell^q)*(L^r,\ell^s)\subset(L^m,\ell^n).$$

Moreover, if $f\in(L^p,\ell^q)$ and $g\in(L^r,\ell^s)$, then

$$\begin{aligned}\|f*g\|_{mn}&\leq 2^a\|f\|_{pq}\|g\|_{rs}\ \text{if } m\neq1\\ \|f*g\|_{1n}&\leq 2^{2a}\|f\|_{1q}\|g\|_{1s}\end{aligned} \tag{2.10}$$

$([1],[2],[23])$.

Theorem 2.7. Let $1\leq p,q\leq\infty$. If for each $a\in G$ and $f\in(L^p,\ell^q)$, then

$$\|T_af\|_{pq}\leq2^a\|f\|_{pq},$$

i.e. the amalgam space (L^p,ℓ^q) is translation invariant $([23])$.

Theorem 2.8. Let $1\leq p,q<\infty$. Then the mapping $y\to T_y$ is continuous from G into (L^p,ℓ^q) $([23])$.

Now we use the fact that (L^p,ℓ^q) has an equivalent translation-invariant norm $\|.\|_{pq}^{\sharp}$. The following theorem was first introduced in [1].

Theorem 2.9. A function f belongs to (L^p,ℓ^q), $1\leq p,q\leq\infty$, iff the function $f^\sharp$ on G defined by

$$f^\sharp(x)=\|f\|_{L^p(x+E)}$$

belongs to $L^q(G)$. If $\|f\|_{pq}^{\sharp}=\left\|f^\sharp\right\|_q$, then

$$2^{-a}\|f\|_{pq}\leq\|f\|_{pq}^{\sharp}\leq2^a\|f\|_{pq},$$

where E is open precompact neighborhood of 0 and

$$\|f\|_{pq}^{\sharp}=\left[\int_G\|f\|_{L^p(x+E)}^q\,dx\right]^{1/q}$$

$([1],[23],[11])$.

Definition 2.10. A net $\{e_\alpha\}$ in a commutative, normed algebra A is an approximate identity, abbreviated a.i., if for all $a \in A$, $\lim_\alpha e_\alpha a = a$ in A.

Proposition 2.11. Let $1 \le p, q < \infty$. If $\{e_\alpha\}$ is an a.i. in $L^1(G)$, then $\{e_\alpha\}$ is also an a.i. in (L^p, ℓ^q), i.e.

$$\lim_\alpha \|e_\alpha * f - f\|_{pq} = 0$$

for all $f \in (L^p, \ell^q)$ ([23]).

The proof the following Lemma is easy.

Lemma 2.12. Let $1 \le p, q < \infty$. Let $\{f_n\}$ be a sequence in (L^p, ℓ^q) and $\|f_n - f\|_{pq} \to 0$, where $f \in (L^p, \ell^q)$. Then $\{f_n\}$ has a subsequence which converges pointwise almost everywhere to f.

3. The space $B_{1,w}(p, q)$

Let $1 \le p, q < \infty$. We define the vector space $B_{1,w}(p, q) = L^1_w(G) \cap (L^p, \ell^q)(G)$ and equip this space with the sum norm

$$\|f\|^{1,w}_{pq} = \|f\|_{1,w} + \|f\|_{pq}$$

where $f \in B_{1,w}(p, q)$. In this section we will discuss some properties of this space.

Theorem 3.1. The space $\left(B_{1,w}(p, q), \|.\|^{1,w}_{pq}\right)$ is a Banach algebra with respect to convolution.

Proof. Let $\{f_n\}$ be a Cauchy sequence in $B_{1,w}(p, q)$. Clearly $\{f_n\}$ is a Cauchy sequence in $L^1_w(G)$ and (L^p, ℓ^q). Since $L^1_w(G)$ and (L^p, ℓ^q) are Banach spaces, then there exist $f \in L^1_w(G)$ and $g \in (L^p, \ell^q)$ such that $\|f_n - f\|_{1,w} \to 0$, $\|f_n - g\|_{pq} \to 0$. Hence there exists a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ which convergence pointwise to f almost everywhere. Also we obtain $\|f_{n_k} - g\|_{pq} \to 0$ and there exists a subsequence $\left\{f_{n_{k_l}}\right\}$ of $\{f_{n_k}\}$ which convergence pointwise to g almost everywhere by Lemma 2.12. Therefore $f = g$ almost everywhere, $\|f_n - f\|^{1,w}_{pq} \to 0$ and $f \in B_{1,w}(p, q)$. That means $B_{1,w}(p, q)$ is a Banach space.

Let $f, g \in B_{1,w}(p, q)$ be given. Since $L^1_w(G)$ is a Banach algebra under convolution, then $f * g \in L^1_w(G)$ and

$$\|f * g\|_{1,w} \le \|f\|_{1,w} \|g\|_{1,w}. \tag{3.1}$$

Since the amalgam space (L^p, ℓ^q) is a Banach $L^1(G)-$module by [23], then we write

$$\|f * g\|_{pq} \le C \|f\|_1 \|g\|_{pq}, \tag{3.2}$$

where $C \ge 1$. By using (3.1), (3.2) and the definition of $\|.\|^{1,w}_{pq}$ we have

$$\begin{aligned}\|f * g\|^{1,w}_{pq} &= \|f * g\|_{1,w} + \|f * g\|_{pq} \\ &\le \|f\|_{1,w} \|g\|_{1,w} + C \|f\|_1 \|g\|_{pq} \\ &= C \|f\|_{1,w} \left(\|g\|_{1,w} + \|g\|_{pq}\right) \\ &\le C \|f\|^{1,w}_{pq} \|g\|^{1,w}_{pq}.\end{aligned}$$

□

Proposition 3.2. The space $\left(B_{1,w}(p,q), \|.\|_{pq}^{1,w}\right)$ is a solid BF-space on G.

Proof. Let $K \subset G$ be given a compact subset and $f \in B_{1,w}(p,q)$. Then we have

$$\int_K |f(x)|\, dx \le \|f\|_1 \le \|f\|_{pq}^{1,w}.$$

Let $f \in B_{1,w}(p,q)$ and $g \in L^\infty(G)$. Since $L_w^1(G)$ and (L^p, ℓ^q) are solid BF-space [9], then

$$\begin{aligned} \|fg\|_{pq}^{1,w} &= \|fg\|_{1,w} + \|fg\|_{pq} \\ &\le \|f\|_{1,w} \|g\|_\infty + \|f\|_{pq} \|g\|_\infty = \|f\|_{pq}^{1,w} \|g\|_\infty . \end{aligned}$$

This completes the proof. □

Proposition 3.3. (**i**) The space $B_{1,w}(p,q)$ is translation invariant and for every $f \in B_{1,w}(p,q)$ the inequality $\|T_a f\|_{pq}^{1,w} \le w(a)\|f\|_{pq}^{1,w}$ holds.

(**ii**) The mapping $y \to T_y f$ is continuous from G into $B_{1,w}(p,q)$ for every $f \in B_{1,w}(p,q)$.

Proof. (**i**) Let $f \in B_{1,w}(p,q)$. Then it is easy to show that $T_a f \in L_w^1(G)$ and $\|T_a f\|_{1,w} \le w(a)\|f\|_{1,w}$ for all $a \in G$. By Theorem 2.9, we write

$$(T_y f)^\sharp (x) = \|T_y f\|_{L^p(x+E)} = \|f\|_{L^p(x+y+E)} = f^\sharp(x+y) = T_{-y} f^\sharp(x).$$

This implies that

$$\|T_y f\|_{pq}^\sharp = \left\|(T_y f)^\sharp\right\|_q = \left\|T_{-y} f^\sharp\right\|_q = \left\|f^\sharp\right\|_q = \|f\|_{pq}^\sharp .$$

Hence we have

$$\|T_a f\|_{pq}^{1,w} \le w(a)\|f\|_{pq}^{1,w} + \|f\|_{pq}^\sharp \le w(a)\|f\|_{pq}^{1,w} .$$

(**ii**) Let $f \in B_{1,w}(p,q)$. Then $f \in L_w^1(G)$ and $f \in (L^p, \ell^q)$. It is well known that the translation operator is continuous from G into $L_w^1(G)$ ([10], [20]). Thus for any $\varepsilon > 0$, there exists a neighbourhood U_1 of unit element of G such that

$$\|T_y f - f\|_{1,w} < \frac{\varepsilon}{2} \tag{3.3}$$

for all $y \in U_1$. Also by using Theorem 2.8, there exists a neighbourhood U_2 of unit element of G such that

$$\|T_y f - f\|_{pq} < \frac{\varepsilon}{2} \tag{3.4}$$

for all $y \in U_2$. Let $U = U_1 \cap U_2$. By using (3.3) and (3.4), then we obtain

$$\begin{aligned} \|T_y f - f\|_{pq}^{1,w} &= \|T_y f - f\|_{1,w} + \|T_y f - f\|_{pq} \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \end{aligned}$$

for all $y \in U$. This completes the proof. □

Theorem 3.4. The space $B_{1,w}(p,q)$ is a S_w algebra.

Proof. We have already proved the some conditions in Theorem 3.1 and Proposition 3.3 for S_w algebra. We now prove that $B_{1,w}(p,q)$ is dense in $L^1_w(G)$. Since $C_c(G) \subset B_{1,w}(p,q)$ and $C_c(G)$ is dense in $L^1_w(G)$, then $B_{1,w}(p,q)$ is dense in $L^1_w(G)$. □

Proposition 3.5. The space $\left(B_{1,w}(p,q), \|.\|^{1,w}_{pq}\right)$ is strongly character invariant and the map $t \to M_t f$ is continuous from $\widehat{G}$ into $B_{1,w}(p,q)$ for all $f \in B_{1,w}(p,q)$.

Proof. The spaces $L^1_w(G)$ and (L^p, ℓ^q) are strongly character invariant and the map $t \to M_t f$ is continuous from $\widehat{G}$ into this spaces ([10], [22]). Hence the proof is completed. □

Proposition 3.6. $B_{1,w}(p,q)$ is a essential Banach $L^1_w(G)-$module.

Proof. Let $f \in B_{1,w}(p,q)$ and $g \in L^1_w(G)$. Since (L^p, ℓ^q) is an essential Banach $L^1(G)-$module, then we have

$$\begin{aligned}
\|f*g\|^{1,w}_{pq} &= \|f*g\|_{1,w} + \|f*g\|_{pq} \\
&\leq \|f\|_{1,w}\|g\|_{1,w} + \|f\|_{pq}\|g\|_1 \\
&= \|f\|^{1,w}_{pq}\|g\|_{1,w}.
\end{aligned}$$

Also, by using Proposition 2.11, then $\|e_\alpha * f - f\|^{1,w}_{pq} \to 0$. Hence $L^1_w(G)*B_{1,w}(p,q) = B_{1,w}(p,q)$ by Module Factorization Theorem [26]. This completes the proof. □

Consider the mapping Φ from $B_{1,w}(p,q)$ into $L^1_w(G)\times(L^p,\ell^q)$ defined by $\Phi(f) = (f,f)$. This is a linear isometry of $B_{1,w}(p,q)$ into $L^1_w(G)\times(L^p,\ell^q)$ with the norm

$$|||(f,f)||| = \|f\|_{1,w} + \|f\|_{pq}, \ (f \in B_{1,w}(p,q)).$$

Hence it is easy to see that $B_{1,w}(p,q)$ is a closed subspace of the Banach space $L^1_w(G)\times(L^p,\ell^q)$. Let

$$H = \{(f,f) : f \in B_{1,w}(p,q)\}$$

and

$$K = \left\{ \begin{array}{c} (\varphi,\psi) : (\varphi,\psi) \in L^\infty_{w^{-1}}(G) \times \left(L^{p'}, \ell^{q'}\right), \\ \int\limits_G f(x)\varphi(x)dx + \int\limits_G f(y)\psi(y)dy = 0, \text{ for all } (f,f) \in H \end{array} \right\},$$

where $1/p + 1/p' = 1$ and $1/q + 1/q' = 1$.

The following Proposition is easily proved by Duality Theorem 1.7 in [18].

Proposition 3.7. The dual space $(B_{1,w}(p,q))^*$ of $B_{1,w}(p,q)$ is isomorphic to

$$L^\infty_{w^{-1}}(G) \times \left(L^{p'}, \ell^{q'}\right)/K.$$

Proposition 3.8. If p,q,r,s are exponents such that $1/p + 1/r - 1 = 1/m \leq 1$ and $1/q + 1/s - 1 = 1/n \leq 1$, then

$$B_{1,w}(p,q) * B_{1,w}(r,s) \subset B_{1,w}(m,n).$$

Moreover, if $f \in B_{1,w}(p,q)$ and $g \in B_{1,w}(r,s)$, then there exists a $C \geq 1$ such that

$$\|f*g\|^{1,w}_{mn} \leq C\|f\|^{1,w}_{pq}\|g\|^{1,w}_{rs}.$$

Proof. Let $f \in B_{1,w}(p,q)$ and $g \in B_{1,w}(r,s)$. By Theorem 2.6 we have

$$\begin{aligned} \|f*g\|_{mn}^{1,w} &= \|f*g\|_{1,w} + \|f*g\|_{mn} \\ &\le \|f\|_{1,w}\|g\|_{1,w} + C\|f\|_{pq}\|g\|_{rs} \\ &\le C\|f\|_{1,w}\|g\|_{rs}^{1,w} + C\|f\|_{pq}\|g\|_{rs}^{1,w} \\ &= C\|f\|_{pq}^{1,w}\|g\|_{rs}^{1,w}. \end{aligned}$$

Hence $B_{p,q}^1(G) * B_{r,s}^1(G) \subset B_{m,n}^1(G)$. □

4. Inclusions of the spaces $B_{1,w}(p,q)$

Proposition 4.1. (**i**) If $q_1 \le q_2$ and $w_2 \prec w_1$, then $B_{1,w_1}(p,q_1) \subset B_{1,w_2}(p,q_2)$.
(**ii**) If $p_1 \ge p_2$ and $w_2 \prec w_1$, then $B_{1,w_1}(p_1,q) \subset B_{1,w_2}(p_2,q)$.

Proof. By using (2.8) and (2.9), then the proof is completed. □

Lemma 4.2. For any $f \in B_{1,w}(p,q)$ and $z \in G$ there exist constants $C_1(f), C_2(f) > 0$ such that

$$C_1(f)w(z) \le \|T_z f\|_{pq}^{1,w} \le C_2(f)w(z).$$

Proof. Let $f \in B_{1,w}(p,q)$. Then by Lemma 2.2 in [10], there exists a constant $C_1(f) > 0$ such that

$$C_1(f)w(z) \le \|T_z f\|_{1,w}. \tag{4.1}$$

By using (4.1), we have

$$C_1(f)w(z) \le \|T_z f\|_{1,w} + \|T_z f\|_{pq} = \|T_z f\|_{pq}^{1,w} \le w(z)\|f\|_{pq}^{1,w}. \tag{4.2}$$

If we combine (4.1) and (4.2), we obtain the inequality

$$C_1(f)w(z) \le \|T_z f\|_{pq}^{1,w} \le C_2(f)w(z),$$

with $C_2(f) = \|f\|_{pq}^{1,w}$. □

The following lemma is easily proved by using the closed graph theorem.

Lemma 4.3. Let w_1 and w_2 be two weights. Then $B_{1,w_1}(p,q) \subset B_{1,w_2}(p,q)$ if and only if there exists a constant $C > 0$ such that $\|f\|_{pq}^{1,w_2} \le C\|f\|_{pq}^{1,w_1}$ for all $f \in B_{1,w_1}(p,q)$.

Proposition 4.4. Let w_1 and w_2 be two weights. Then $B_{1,w_1}(p,q) \subset B_{1,w_2}(p,q)$ if and only if $w_2 \prec w_1$.

Proof. The sufficiency of condition is obvious. Suppose that $B_{1,w_1}(p,q) \subset B_{1,w_2}(p,q)$. By Lemma 4.2, there exist C_1, C_2, C_3 and $C_4 > 0$ such that

$$C_1 w_1(z) \le \|T_z f\|_{pq}^{1,w_1} \le C_2 w_1(z) \tag{4.3}$$

and

$$C_3 w_2(z) \le \|T_z f\|_{pq}^{1,w_2} \le C_4 w_2(z) \tag{4.4}$$

for $z \in G$. Since $T_z f \in B_{1,w_1}(p,q)$ for all $f \in B_{1,w_1}(p,q)$, then there exists a constant $C > 0$ such that

$$\|T_z f\|_{pq}^{1,w_2} \le C\|T_z f\|_{pq}^{1,w_1} \tag{4.5}$$

by Lemma 4.3. If one using (4.3), (4.4) and (4.5),we obtain

$$C_3 w_2(z) \leq \|T_z f\|_{pq}^{1,w_2} \leq C \|T_z f\|_{pq}^{1,w_1} \leq C C_2 w_1(z).$$

That means $w_2 \prec w_1$. □

Corollary 4.5. Let w_1 and w_2 be two weights. Then $B_{1,w_1}(p,q) = B_{1,w_2}(p,q)$ if and only if $w_1 \approx w_2$.

Now by using the techniques in [14], we investigate compact embeddings of the spaces $B_{1,w}(p,q)$. Also we will take $G = \mathbb{R}^d$ with Lebesgue measure dx for compact embedding.

Lemma 4.6. Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence in $B_{1,w}(p,q)$. If $\{f_n\}$ converges to zero in $B_{1,w}(p,q)$, then $\{f_n\}$ converges to zero in the vague topology (which means that

$$\int_{\mathbb{R}^d} f_n(x)k(x)dx \to 0$$

for $n \to \infty$ for all $k \in C_c(\mathbb{R}^d)$, see [4]).

Proof. Let $k \in C_c(\mathbb{R}^d)$. We write

$$\left| \int_{\mathbb{R}^d} f_n(x)k(x)dx \right| \leq \|k\|_\infty \|f_n\|_1 \leq \|k\|_\infty \|f_n\|_{pq}^{1,w}. \tag{4.6}$$

Hence by (4.6) the sequence $\{f_n\}_{n\in\mathbb{N}}$ converges to zero in vague topology. □

Theorem 4.7. Let w, ν be two weights on $\mathbb{R}^d$. If $\nu \prec w$ and $\frac{\nu(x)}{w(x)}$ doesn't tend to zero in $\mathbb{R}^d$ as $x \to \infty$, then the embedding of the space $B_{1,w}(p,q)$ into $L_\nu^1(\mathbb{R}^d)$ is never compact.

Proof. Firstly we assume that $w(x) \to \infty$ as $x \to \infty$. Since $\nu \prec w$, there exists $C_1 > 0$ such that $\nu(x) \leq C_1 w(x)$. This implies $B_{1,w}(p,q) \subset L_\nu^1(\mathbb{R}^d)$. Let $(t_n)_{n\in\mathbb{N}}$ be a sequence with $t_n \to \infty$ in $\mathbb{R}^d$. Also since $\frac{\nu(x)}{w(x)}$ doesn't tend to zero as $x \to \infty$ then there exists $\delta > 0$ such that $\frac{\nu(x)}{w(x)} \geq \delta > 0$ for $x \to \infty$. For the proof the embedding of the space $B_{1,w}(p,q)$ into $L_\nu^1(\mathbb{R}^d)$ is never compact, take any fixed $f \in B_{1,w}(p,q)$ and define a sequence of functions $\{f_n\}_{n\in\mathbb{N}}$, where $f_n = w(t_n)^{-1} T_{t_n} f$. This sequence is bounded in $B_{1,w}(p,q)$. Indeed we write

$$\|f_n\|_{pq}^{1,w} = \left\|w(t_n)^{-1} T_{t_n} f\right\|_{pq}^{1,w} = w(t_n)^{-1} \|T_{t_n} f\|_{pq}^{1,w}. \tag{4.7}$$

By Lemma 4.2, we know $\|T_y f\|_{pq}^{1,w} \approx w(y)$. Hence there exists $M > 0$ such that $\|T_y f\|_{pq}^{1,w} \leq M w(y)$. By using (4.7), we write

$$\|f_n\|_{pq}^{1,w} = w(t_n)^{-1} \|T_{t_n} f\|_{pq}^{1,w} \leq M w(t_n)^{-1} w(t_n) = M.$$

Now we will prove that there wouldn't exists norm convergence of subsequence of $\{f_n\}_{n\in\mathbb{N}}$ in $L_\nu^1(\mathbb{R}^d)$. The sequence obtained above certainly converges to zero in the vague topology. Indeed for all $k \in C_c(\mathbb{R}^d)$ we write

$$\left| \int\limits_{\mathbb{R}^d} f_n(x)k(x)dx \right| \leq \frac{1}{w(t_n)} \int\limits_{\mathbb{R}^d} |T_{t_n} f(x)| \, |k(x)| \, dx \tag{4.8}$$
$$= \frac{1}{w(t_n)} \|k\|_\infty \|T_{t_n} f\|_1 = \frac{1}{w(t_n)} \|k\|_\infty \|f\|_1 .$$

Since right hand side of (4.8) tends zero for $n \to \infty$, then we have

$$\int\limits_{\mathbb{R}^d} f_n(x)k(x)dx \to 0.$$

Finally by Lemma 4.6, the only possible limit of $\{f_n\}_{n\in\mathbb{N}}$ in $L^1_\nu(\mathbb{R}^d)$ is zero. It is known by Lemma 2.2 in [10] that $\|T_y f\|_{1,\nu} \approx \nu(y)$. Hence there exists $C_2 > 0$ and $C_3 > 0$ such that

$$C_2\nu(y) \leq \|T_y f\|_{1,\nu} \leq C_3\nu(y). \tag{4.9}$$

From (4.9) and the equality

$$\|f_n\|_{1,\nu} = \left\|w(t_n)^{-1} T_{t_n} f\right\|_{1,\nu} = w(t_n)^{-1} \|T_{t_n} f\|_{1,\nu}$$

we obtain

$$\|f_n\|_{1,\nu} = w(t_n)^{-1} \|T_{t_n} f\|_{1,\nu} \geq C_2 w(t_n)^{-1}\nu(t_n). \tag{4.10}$$

Since $\frac{\nu(t_n)}{w(t_n)} \geq \delta > 0$ for all t_n, by using (4.10) we write

$$\|f_n\|_{1,\nu} \geq C_2 w(t_n)^{-1}\nu(t_n) \geq C_2\delta.$$

It means that there would not be possible to find norm convergent subsequence of $\{f_n\}_{n\in\mathbb{N}}$ in $L^1_\nu(\mathbb{R}^d)$.

Now we assume that w is a constant or bounded weight function. Since $\nu \prec w$, then $\frac{\nu(x)}{w(x)}$ is also constant or bounded and doesn't tend to zero as $x \to \infty$. We take a function $f \in B_{1,w}(p,q)$ with compactly support and define the sequence $\{f_n\}_{n\in\mathbb{N}}$ as in (4.7). Thus $\{f_n\}_{n\in\mathbb{N}} \subset B_{1,w}(p,q)$. It is easy to show that $\{f_n\}_{n\in\mathbb{N}}$ is bounded in $B_{1,w}(p,q)$ and converges to zero in the vague topology. Then there would not possible to find norm convergent subsequence of $\{f_n\}_{n\in\mathbb{N}}$ in $L^1_\nu(\mathbb{R}^d)$. This completes the proof. □

Proposition 4.8. Let w_1, w_2 be Beurling weight functions on $\mathbb{R}^d$. If $w_2 \prec w_1$ and $\frac{w_2(x)}{w_1(x)}$ doesn't tend to zero in $\mathbb{R}^d$ then the embedding $i : B_{1,w_1}(p,q) \hookrightarrow B_{1,w_2}(p,q)$ is never compact.

Proof. The proof can be obtained by means of Proposition 4.4, Proposition 4.3 and Theorem 4.7. □

5. Multipliers of $B_{1,w}(p,q)$

Now we discuss multipliers of the spaces $B_{1,w}(p,q)$. We define the space

$$M_{B_{1,w}(p,q)} = \{\mu \in M(w) : \|\mu\|_M \le C(\mu)\}$$

where

$$\|\mu\|_M = \sup\left\{ \frac{\|\mu * f\|^{1,w}_{pq}}{\|f\|_{1,w}} : f \in L^1_w(G),\ f \neq 0,\ \widehat{f} \in C_c(\widehat{G}) \right\}.$$

By the Proposition 2.1 in [13], we have $M_{B_{1,w}(p,q)} \neq \{0\}$.

Proposition 5.1. If w satisfies (BD), then for a linear operator $T : L^1_w(G) \to B_{1,w}(p,q)$ the following are equivalent:

(**i**) $T \in Hom_{L^1_w(G)}\left(L^1_w(G), B_{1,w}(p,q)\right)$.

(**ii**) There exists a unique $\mu \in M_{B_{1,w}(p,q)}$ such that $Tf = \mu * f$ for every $f \in L^1_w(G)$. Moreover the correspondence between T and μ defines an isomorphism between $Hom_{L^1_w(G)}\left(L^1_w(G), B_{1,w}(p,q)\right)$ and $M_{B_{1,w}(p,q)}$.

Proof. It is known that $B_{1,w}(p,q)$ is a S_w space by Theorem 3.4. Thus, the proof is completed by Proposition 2.4 in [13]. □

Theorem 5.2. If w satisfies (BD) and $T \in Hom_{L^1_w(G)}(B_{1,w}(p,q))$, then there exists a unique pseudo measure $\sigma \in \left(A(\widehat{G})\right)^*$ (see [20]), such that $Tf = \sigma * f$ for all $f \in B_{1,w}(p,q)$.

Proof. It is known that $B_{1,w}(p,q)$ is a S_w space by Theorem 3.4 and an essential Banach module over $L^1_w(G)$ by Proposition 3.6. Thus, the proof is completed by Theorem 5 in [5]. □

Proposition 5.3. The multiplier space $Hom_{L^1_w(G)}\left(L^1_w(G), (B_{1,w}(p,q))^*\right)$ is isomorphic to $L^\infty_{w^{-1}}(G) \times \left(L^{p'}, \ell^{q'}\right)/K$.

Proof. By Proposition 3.6, we write $L^1_w(G) * B_{1,w}(p,q) = B_{1,w}(p,q)$. Hence by Corollary 2.13 in [21] and Proposition 3.7, we have

$$\begin{aligned} Hom_{L^1_w(G)}\left(L^1_w(G), (B_{1,w}(p,q))^*\right) &= \left(L^1_w(G) * B_{1,w}(p,q)\right)^* = (B_{1,w}(p,q))^* \\ &= L^\infty_{w^{-1}}(G) \times \left(L^{p'}, \ell^{q'}\right)/K. \end{aligned}$$

□

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About some links between the Dini-Hadamard-like normal cone and the contingent one

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Abstract. The primary goal of this paper is to furnish an alternative description for the contingent normal cone, similar to the one that exists for the Fréchet one, but by using a directional convergence in place of the usual one. In fact, we actually prove that the same description is available not only for the contingent normal cone, but also for the Dini-Hadamard normal cone and the Dini-Hadamard-like one. Furthermore, we show that although in the case of the Dini-Hadamard subdifferential the geometric construction agrees with the analytical one, in the case of the Dini-Hadamard-like one the analytical construction is only greater than the geometrical one.

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1. Introduction

It is well known that nonsmooth functions, sets with nonsmooth boundaries and set-valued mappings appear naturally and frequently in various areas of mathematics and applications, especially in those related to optimization, stability, variational systems and control systems. Actually, the study of the local behavior of nondifferentiable objects is accomplished in the framework of *nonsmooth analysis* whose origin goes back in the early 1960's, when control theorists and nonlinear programmers attempted to deal with necessary optimality conditions for problems with nonsmooth data or with nonsmooth functions (such as the pointwise maximum of several smooth functions) that arise even in many problems with smooth data. Since then, nonsmooth analysis has come to play an important role in functional analysis, optimization, mechanics and plasticity, differential equations (as in the theory of viscosity solutions), control theory etc, becoming an active and fruitful area of mathematics.

One of the most important topics in nonsmooth analysis is the study of different kinds of tangent cones and normal cones to arbitrary sets. It is also worth mentioning here that the most successfully construction used in this framework turns out to be the so-called *contingent cone*, independently introduced in 1930 by Bouligand [5] and by Severi [18] in the context of contingent equations and differential geometry. Later, under the name of *cone of variations admissible by equality constrints*, the same cone was rediscovered and used in optimization theory by Dubovitskii and Milyutin [7, 8] (for more details about related tangential constructions we refer the reader to Aubin and Frankowska [2]). On the other hand, it is interesting to observe that there is a close relationship between the Dini-Hadamard directional derivative and the contingent cone (see [1, 15]). Thus, exploring such well known results but also a variational description available for the Dini-Hadamard subdifferential of calm functions (see [12]), we are able to provide the main result of the paper, a nice characterization of the contingent normal cone (the polar cone to the contingent one) via a sort of directional limes superior. Moreover, we even show that this kind of description holds also true for the Dini-Hadamard-like normal cone and for the Dini-Hadamard one. Finally, we study the relationship between various kinds of geometrical and analytical subdifferential constructions, pointing out the key role of a decoupled construction in characterizing the Dini-Hadamard-like subdifferential.

2. Preliminary notions and results

Consider a Banach space X and its topological dual space X^*. We denote the *open ball* with center $\overline{x} \in X$ and radius $\delta > 0$ in X by $B(\overline{x}, \delta)$, while $\overline{B}_X$ and S_X stand for the *closed unit ball* and the *unit sphere* of X, respectively. Having a set $C \subseteq X$, $\delta_C : X \to \overline{\mathbb{R}} \cup \{+\infty\}$, defined by $\delta_C(x) = 0$ for $x \in C$ and $\delta_C(x) = +\infty$, otherwise, denotes its *indicator function*. Given a function $f : X \to \overline{\mathbb{R}}$ which is finite at $\overline{x}$, we usually denote by $\text{epi} f = \{(x, \alpha) \in X \times \mathbb{R} : \ f(x) \leq \alpha\}$ the *epigraph* of f.

One of the most attractive constructions,

$$d^{DH} f(\overline{x}; h) := \liminf_{\substack{u \to h \\ t \downarrow 0}} \frac{f(\overline{x} + tu) - f(\overline{x})}{t} \tag{2.1}$$

which appeared in the 1970's was called *lower semiderivative* by Penot [15], *contingent derivative/epiderivative* by Aubin [1], *lower Dini* (or *Dini-Hadamard*) *directional derivative* by Ioffe [9, 10] and *subderivative* by Rockafellar and Wets [17].

Since in the case of real functions, the Dini-Hadamard directional derivative goes back to the classical *derivative numbers* by Dini [6], in general it can be described in a geometrical way via the contingent cone as follows

$$d^{DH} f(\overline{x}; h) = \inf\{\alpha \in \mathbb{R} : (h, \alpha) \in T((\overline{x}, f(\overline{x})); \text{epi} f)\}, \tag{2.2}$$

where for a given set $C \subset X$ with $\overline{x} \in C$,

$$T(\overline{x}; C) := \text{Limsup}_{t \downarrow 0} \frac{C - \overline{x}}{t}$$

denotes the *contingent* (or the *Bouligand*) *cone* to C at $\overline{x}$, equivalently described as the collection of those $v \in X$ such that there are sequences $(x_n) \subset C$ and $(\alpha_n) \subset \mathbb{R}_+$

with the property that $(x_n) \to \overline{x}$ and $(\alpha_n(x_n - \overline{x})) \to v$ as $n \to \infty$. Note also here that the contingent cone can be viewed (see [1]) in the following way

$$T(\overline{x}; C) = \bigcap_{\substack{\varepsilon>0 \\ \delta>0}} \bigcup_{t\in(0,\delta)} (t^{-1}(C - \overline{x}) + \varepsilon \overline{B}_X), \tag{2.3}$$

i.e. the set of all vectors v so that one can find sequences $t_n \downarrow 0, u_n \to v$ with the property that $\overline{x} + t_n u_n \in C$ for all $n \in \mathbb{N}$.

Consequently, the Dini-Hadamard subdifferential of f at $\overline{x}$, that is

$$\partial^{DH} f(\overline{x}) := \{x^* \in X^* : \langle x^*, h\rangle \le d^{DH} f(\overline{x}; h) \text{ for all } h \in X\}, \tag{2.4}$$

can also be expressed by means of the polar cone to the contingent one, i.e.

$$\partial^{DH} f(x) = \{x^* \in X^* : (x^*, -1) \in T^\circ((x, f(x)); \mathrm{epi} f)\}, \tag{2.5}$$

where given a subcone $K \subseteq X$, its polar cone K° is defined by

$$K^\circ := \{x^* \in X^* : \sup_{x\in K} \langle x^*, x\rangle \le 0\}.$$

It is worth emphasizing here that, accordingly to [1, 15], if f is the indicator function of $C \subseteq X$ and $\overline{x} \in C$, then $d^{DH} f(\overline{x}; h)$ (viewed as a function in the second variable) is the indicator function of the contingent cone $T(\overline{x}; C)$ and hence

$$\partial^{DH} \delta_C(x) = T^\circ(x; C). \tag{2.6}$$

Let us also remark that $d^{DH} f(\overline{x}; \cdot)$ is in general not convex (as it is actually *typical* concave in some particular instances, see Bessis and Clarke [3]), but lower semicontinuous. However, the Dini-Hadamard subdifferential of f at $\overline{x}$ is always a convex set.

Similarly, following the two steps procedure of constructing the Dini-Hadamard subdifferential, but employing a directional convergence in place of the usual one, we can define (see [12]) the *Dini-Hadamard-like subdifferential* of f at $\overline{x}$, i.e. the following set

$$\widetilde{\partial} f(\overline{x}) := \{x^* \in X^* : \langle x^*, h\rangle \le \widetilde{D}_d f(\overline{x}; h) \ \ \forall h \in X \ \forall d \in X \setminus \{0\}\}, \tag{2.7}$$

where

$$\widetilde{D}_d f(\overline{x}; h) := \sup_{\delta>0} \inf_{\substack{u\in B(h,\delta)\cap(h+[0,\delta]\cdot B(d,\delta)) \\ t\in(0,\delta)}} \frac{f(\overline{x} + tu) - f(\overline{x})}{t}, \tag{2.8}$$

labeled as the *Dini-Hadamard-like directional derivative* of f at $\overline{x}$ in the direction $h \in X$ through $d \in X \setminus \{0\}$, extend somehow the Dini-Hadamard directional derivative, while the essential idea was inspired by the fruitful relationship between sponges and directionally convergent sequences (we refer the reader to [16, Lemma 2.1]). As usual, in case $|f(\overline{x})| = \infty$, we set $\partial^{DH} f(\overline{x}) = \widetilde{\partial} f(\overline{x}) = \emptyset$.

Actually, introduced by Treiman [19], the sponge turns out to be very useful for characterizing the Dini-Hadamard subdifferential. Actually, the idea behind this concept was the fact that a neighborhood is in general not broad enough to characterize this kind of subdifferential constructions.

Definition 2.1. *A set $S \subseteq X$ is said to be a* sponge *around $\overline{x} \in X$ if for all $h \in X \setminus \{0\}$ there exist $\lambda > 0$ and $\delta > 0$ such that $\overline{x} + [0, \lambda] \cdot B(h, \delta) \subseteq S$.*

In fact, as one can easily observe from the definition above, the singular point 0 is ignored. Furthermore, every neighborhood of a point $\overline{x} \in X$ is also a sponge around $\overline{x}$, but the converse is not true (see for instance [4, Example 2.2]). However, in case S is a convex set or X is a finite dimensional space (here one can make use of the fact that the unit sphere is compact), then S is also a neighborhood of $\overline{x}$.

As regards the links between the two subdifferentials above, one can easily observe that the following inclusion

$$\partial^{DH} f(\overline{x}) \subseteq \widetilde{\partial} f(\overline{x}) \tag{2.9}$$

holds always true, but it can be even strict (see the discussion after [12, Theorem 3.1]). However, in case X is finite dimensional or the function f is calm at $\overline{x}$, i.e. there exists $c \geq 0$ and $\delta > 0$ such that $f(x) - f(\overline{x}) \geq -c\|x - \overline{x}\|$ for all $x \in B(\overline{x}, \delta)$, in particular if f is locally Lipschitz at $\overline{x}$, then the Dini-Hadamard subdifferential agrees with the Dini-Hadamard-like one.

Although the Dini-Hadamard-like subdifferential as well as the Dini-hadamard one (if additionally a calmness assumption is fulfilled, too) of a given function $f : X \to \overline{\mathbb{R}}$ at a point $\overline{x}$ with $|f(\overline{x})| < +\infty$ can be described via the following variational description (see [12, Theorem 3.1])

$$\begin{aligned} \widetilde{\partial} f(\overline{x}) := \{x^* \in X^* : \forall \varepsilon > 0\ \exists S \text{ a sponge around } \overline{x} \text{ such that } \forall x \in S \\ f(x) - f(\overline{x}) \geq \langle x^*, x - \overline{x}\rangle - \varepsilon\|x - \overline{x}\|\}, \end{aligned} \tag{2.10}$$

or, in other words,

$$x^* \in \widetilde{\partial} f(\overline{x}) \Leftrightarrow \begin{array}{c} \forall \varepsilon > 0\ \forall u \in S_X\ \exists \delta > 0 \text{ such that} \\ \forall s \in (0, \delta)\ \forall v \in B(u, \delta) \text{ for } x := \overline{x} + sv \text{ one has} \\ f(x) - f(\overline{x}) \geq \langle x^*, x - \overline{x}\rangle - \varepsilon\|x - \overline{x}\|. \end{array} \tag{2.11}$$

it seems that the following decoupled construction, introduced in [13], was designed not only to derive exact subdifferential formulae for Dini-Hadamard and Dini-Hadamard-like subgradients (see for instance [14]), as it was especially introduced from the necessity to deal with appropriate derivative-like constructions on product spaces. The reason is that, due to the very special structure of the spongious sets, the cartesian product of two sponges is, in general, not a sponge.

Thus, given a function $f : X \times Y \to \overline{\mathbb{R}}$ defined on a product of two Banach spaces X and Y, the following subdifferential construction

$$\begin{aligned} \widetilde{\partial}_{\!r} f(\overline{x}, \overline{y}) := \{(x^*, y^*) \in X^* \times Y^* : \forall \varepsilon > 0\ \exists S_1 \text{ a sponge around } \overline{x}, \\ \exists S_2 \text{ a sponge around } \overline{y} \text{ such that } \forall (x, y) \in S_1 \times S_2 \\ f(x, y) - f(\overline{x}, \overline{y}) \geq \langle (x^*, y^*), (x - \overline{x}, y - \overline{y})\rangle - \varepsilon\|(x - \overline{x}, y - \overline{y})\|\}, \end{aligned} \tag{2.12}$$

denotes the *decoupled Dini-Hadamard-like (lower) subdifferential* of f at $(\overline{x}, \overline{y})$, where $X \times Y$ is a Banach space with respect to the *sum norm*

$$\|(x, y)\| := \|x\| + \|y\|$$

imposed on $X \times Y$ unless otherwise stated. It is interesting to observe that the last notion is actually quite different than the Dini-Hadamard-like one, since, at first sight, neither $\widetilde{\partial} f(\overline{x}, \overline{y}) \nsubseteq \widetilde{\partial}_{\mathfrak{r}} f(\overline{x}, \overline{y})$ nor the opposite inclusion $\widetilde{\partial}_{\mathfrak{r}} f(\overline{x}, \overline{y}) \nsubseteq \widetilde{\partial} f(\overline{x}, \overline{y})$ is valid.

3. Relationships between subgradients and normal cones

Let us begin our exposure with a few remarks. First of all, following relation (2.6) above, one can easily observe that the Dini-Hadamard normal cone to a set $C \subseteq X$ at $\overline{x} \in C$, naturally introduced via the Dini-Hadamard subdifferential to the indicator function, can also be expressed via the polar cone to the contingent one, in fact the contingent normal cone, i.e.

$$N^{DH}(\overline{x}; C) := \partial^{DH} \delta_C(\overline{x}) = T^{\circ}(\overline{x}; C) := N(\overline{x}; C). \tag{3.1}$$

On the other hand, relations (2.5) and (3.1) clearly yield

$$\partial^{DH} f(\overline{x}) = \{x^* \in X^* : (x^*, -1) \in N^{DH}((\overline{x}, f(\overline{x})); \mathrm{epi} f))\}. \tag{3.2}$$

In fact, the latter actually says that the analytic Dini-Hadamard subdifferential ∂_a^{DH}, as introduced in (2.4), always agrees with the geometrical one, ∂_g^{DH}, as defined in (3.2). However, this is no longer the case for the Dini-Hadamard-like subdifferential. The reason is that, for this particular construction, one can state a similar result like in (3.2) only by making use of the corresponding decoupled one.

Proposition 3.1. *Let $f : X \to \overline{\mathbb{R}}$ be a given function finite at $\overline{x}$. Then*

$$\widetilde{\partial} f(\overline{x}) = \{x^* \in X^* : (x^*, -1) \in \widetilde{N}_{\mathfrak{r}}((\overline{x}, f(\overline{x})); \mathit{epi} f)\}, \tag{3.3}$$

where $\widetilde{N}_{\mathfrak{r}}((\overline{x}, \overline{y}); C) := \widetilde{\partial}_{\mathfrak{r}} \delta((\overline{x}, \overline{y}); C)$ stands for the decoupled Dini-Hadamard-like normal cone *to $C \subset X \times Y$ at $(\overline{x}, \overline{y})$.*

Proof. To justify the inclusion " $\subseteq$ ", we have to show that

$$(x^*, -1) \in \widetilde{\partial}_{\mathfrak{r}} \delta((\overline{x}, f(\overline{x})); \mathrm{epi} f),$$

whenever $x^* \in \widetilde{\partial} f(\overline{x})$. To proceed, pick any $\varepsilon > 0$ and observe that there exists a sponge S_1 around $\overline{x}$ such that for all $x \in S_1$

$$f(x) - f(\overline{x}) \geq \langle x^*, x - \overline{x} \rangle - \varepsilon \|x - \overline{x}\|. \tag{3.4}$$

Further, for any $(x, y) \in (S_1 \times \overline{\mathbb{R}}) \setminus \mathrm{epi} f$ the following estimate

$$\delta_{\mathrm{epi} f}(x, y) \geq \langle (x^*, -1), (x - \overline{x}, y - f(\overline{x})) \rangle - \varepsilon(\|x - \overline{x}\| + \|y - f(\overline{x})\|)$$

holds true. Thus, it remains us to show the latter inequality for an arbitrary $(x, y) \in \mathrm{epi} f$.

Indeed, relation (3.4) above leads to

$$\begin{aligned} \langle x^*, x - \overline{x} \rangle - \varepsilon(\|x - \overline{x}\| + \|y - f(\overline{x})\|) &\leq \langle x^*, x - \overline{x} \rangle - \varepsilon \|x - \overline{x}\| \\ &\leq f(x) - f(\overline{x}) \leq y - f(\overline{x}), \end{aligned}$$

and consequently $(x^*, -1) \in \widetilde{N}_{\mathfrak{r}}((\overline{x}, f(\overline{x})); \mathrm{epi} f)\}$, which ends the proof of the first inclusion.

For the reverse one, take an arbitrary element on the right-hand side of (3.3) and assume on the contrary that $x^* \notin \widetilde{\partial} f(\overline{x})$. Then there exists $\gamma > 0$ such that for any natural number k one can find $x_k \in S_1 \cap B(\overline{x}, \frac{1}{k})$ with the following property

$$f(x_k) - f(\overline{x}) - \langle x^*, x_k - \overline{x}\rangle + \gamma\|x - \overline{x}\| < 0,$$

where (obviously) $x_k \neq \overline{x}$. Hence, taking $a_k := f(\overline{x}) + \langle x^*, x_k - \overline{x}\rangle - \gamma\|x_k - \overline{x}\|$ one clearly gets $(x_k, a_k) \in \text{epi}f$ for all $k \in \mathbb{N}$ and additionally $(a_k)_k \to f(\overline{x})$.

Further

$$\begin{aligned}\frac{\langle x^*, x_k - \overline{x}\rangle - (a_k - f(\overline{x}))}{\|(x_k, a_k) - (\overline{x}, f(\overline{x}))\|} &= \frac{\gamma\|x_k - \overline{x}\|}{\|(x_k - \overline{x}, \langle x^*, x_k - \overline{x}\rangle - \gamma\|x_k - \overline{x}\|)\|} \\ &\geq \frac{\gamma}{1 + \|x^*\| + \gamma} > 0.\end{aligned}$$

On the other hand, taking into account the fact that

$$(x^*, -1) \in \widetilde{N}_r((\overline{x}, f(\overline{x})); \text{epi}f),$$

for a fixed $\gamma' \in (0, \frac{\gamma}{1+\|x^*\|+\gamma})$ there exist two sponges S_1 around $\overline{x}$ and S_2 around $f(\overline{x})$ (which is in fact a neighborhood) such that for any $(x, y) \in S_1 \times S_2$ one has

$$\delta_{\text{epi}f}(x, y) \geq \langle (x^*, -1), (x - \overline{x}, y - f(\overline{x}))\rangle - \gamma'\|(x - \overline{x}, y - f(\overline{x}))\|.$$

Consequently, for all large enough $k \in \mathbb{N}$, $(x_k, a_k) \in S_1 \times S_2 \cap \text{epi}f$, and hence

$$\gamma' \geq \frac{\langle x^*, x_k - \overline{x}\rangle - (a_k - f(\overline{x}))}{\|(x_k, a_k) - (\overline{x}, f(\overline{x}))\|},$$

which is a contradiction. Finally, $x^* \in \widetilde{\partial} f(\overline{x})$ and the proof is complete. □

A valuable characterization of the contingent normal cone, similar to the one that exist for the Fréchet normal cone (see for instance [11, Definition 1.1]), but by replacing the usual convergence with a directional one, will be provided in the sequel. In fact, one can say more.

Theorem 3.2. *Let C be a nonempty subset of X and $\overline{x} \in C$. Then*

$$\begin{aligned} N(\overline{x}; C) &= N^{DH}(\overline{x}; C) = \widetilde{N}(\overline{x}; C) \\ &= \{x^* \in X^* : \inf_{\delta\in(0,1)} \sup_{x\in(\overline{x}+(0,\delta)\cdot B(u,\delta))\cap C} \frac{\langle x^*, x - \overline{x}\rangle}{\|x - \overline{x}\|} \leq 0 \ \forall u \in S_X\}, \end{aligned} \quad (3.5)$$

where $\widetilde{N}(\overline{x}; C) := \widetilde{\partial}\delta_C(\overline{x})$.

Proof. First, it is sufficient to take into account relation (3.1) above and also to observe that the indicator function $\delta_C(\overline{x})$ is calm at $\overline{x}$, in order to justify the equalities

$$N(\overline{x}; C) = N^{DH}(\overline{x}; C) = \widetilde{N}(\overline{x}; C).$$

Thus, it remains us to show only that

$$\widetilde{N}(\overline{x}; C) = \{x^* \in X^* : \inf_{\delta\in(0,1)} \sup_{x\in(\overline{x}+(0,\delta)\cdot B(u,\delta))\cap C} \frac{\langle x^*, x - \overline{x}\rangle}{\|x - \overline{x}\|} \leq 0 \ \forall u \in S_X\}.$$

So, let us justify only the inclusion " ⊆ ", since the reverse one can be done similarly, but by reversing the steps ordering. Take $x^* \in \widetilde{N}(\overline{x}; C)$ and consider arbitrary $\varepsilon > 0$

and $u \in S_X$. Then, if we take also into account the variational description (2.11), one gets $\delta \in (0,1)$ such that for all $x \in (\overline{x} + (0,\delta) \cdot B(u,\delta)) \cap C$

$$0 \geq \langle x^*, x - \overline{x} \rangle - \varepsilon \| x - \overline{x} \|.$$

Thus,

$$\sup_{x \in (\overline{x} + (0,\delta) \cdot B(u,\delta)) \cap C} \frac{\langle x^*, x - \overline{x} \rangle}{\| x - \overline{x} \|} \leq \varepsilon$$

and consequently

$$\inf_{\delta \in (0,1)} \sup_{x \in (\overline{x} + (0,\delta) \cdot B(u,\delta)) \cap C} \frac{\langle x^*, x - \overline{x} \rangle}{\| x - \overline{x} \|} \leq \varepsilon.$$

Finally, this gives, by passing to the limit as $\varepsilon \downarrow 0$, that x^* satisfies the inequality in the right-hand side of (3.6) and the proof of the theorem is complete. □

Finally, we illustrate the relationship between the analytic Dini-Hadamard-like subdifferential and the geometrical one, concluding that this kind of construction doesn't follows at all the behavior of the Fréchet subdifferential (see, for instance, the results in [11, Section 1.3]).

Corollary 3.3. *Let $f : X \to \overline{\mathbb{R}}$ be an arbitrary function and $\overline{x} \in X$. Then*

$$\widetilde{\partial}_g f(\overline{x}) \subsetneq \widetilde{\partial}_a f(\overline{x}), \tag{3.6}$$

where $\widetilde{\partial}_g f(\overline{x}) := \{x^* \in X^* : (x^*, -1) \in \widetilde{N}((\overline{x}, f(\overline{x})); epi f)\}$ stands for the geometric Dini-Hadamard-like subdifferential of f at $\overline{x}$, while $\widetilde{\partial}_a f(\overline{x}) := \widetilde{\partial} f(\overline{x})$ denotes the analytical one.

Proof. It is easy to check that

$$\partial^{DH} f(\overline{x}) = \{x^* \in X^* : (x^*, -1) \in \widetilde{N}((\overline{x}, f(\overline{x})); \text{epi} f)\}$$

if we take into account relation (3.2) and Theorem 3.2 above. Hence, in view of Proposition 3.1 the following inclusion

$$\widetilde{N}((\overline{x}, f(\overline{x})); \text{epi} f)\} \subseteq \widetilde{N}_t((\overline{x}, f(\overline{x})); \text{epi} f)\},$$

holds true, but it can be even strict, since $\partial^{DH} f(\overline{x}) \subsetneq \widetilde{\partial} f(\overline{x})$. Consequently, $\widetilde{\partial}_g f(\overline{x}) = \partial^{DH} f(\overline{x}) \subsetneq \widetilde{\partial}_a f(\overline{x})$ and the proof of the corollary is complete. □

Finally, let us illustrate the relationships between various subgradients studied above, which are in fact direct consequences of the discussions made in this subsection.

Corollary 3.4. *Let $f : X \to \overline{\mathbb{R}}$ be an arbitrary function and $\overline{x} \in X$. Then*

$$\partial_a^{DH} f(\overline{x}) = \partial_g^{DH} f(\overline{x}) = \widetilde{\partial}_g f(\overline{x}) \subsetneq \widetilde{\partial}_a f(\overline{x}), \tag{3.7}$$

while the equalities hold true in case f is calm at $\overline{x}$.

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Approximate fixed point theorems for generalized T-contractions in metric spaces

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Abstract. In this paper, we introduce the concept of T-asymptotically regular mapping and establish a lemma for ϵ-fixed points of two commuting mappings in metric spaces. This lemma is used for proving approximate fixed point theorems for various types of contraction mappings in the framework of metric spaces. Our results in this paper extends and improves upon, among others, the corresponding results of Berinde given in [2].

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1. Introduction

Fixed point property of various types of mappings has been used to solve many problems in applied mathematics. In several situations of practical utility, the mapping under consideration may not have an exact fixed point due to some restriction on the space or the map. Besides that, there may arise many situations in real life where the existence of fixed points is not strictly required, but that of 'nearly fixed points' is more than enough. In such cases, we can make use of the concept of ϵ-fixed points (approximate fixed points) which is one type of 'nearly fixed points'. Let us consider the metric space (X, d) and T a self map of this metric space. Suppose that we would like to find an approximate solution of $Tx = x$. If there exists a point $x_0 \in X$ such that $d(Tx_0, x_0) < \epsilon$, where ϵ is a positive number, then x_0 is called an approximate solution of the equation $Tx = x$ or we can say that $x_0 \in X$ is an approximate fixed point (or ϵ-fixed point) of T.

Approximate fixed point property for various types of mappings have been a prominent area of research of many mathematicians for the last few years. In 2006, Berinde [2] proved quantitative and qualitative approximate fixed point theorems for various types of well known contractions on metric spaces. It was proved that even by weakening the conditions by giving up the completeness of the space, the existence of ϵ-fixed points is still guaranteed for operators satisfying Kannan, Chatterjea and Zamfirescu type of conditions on metric spaces.

In 2009, Beiranvand et al. [1], introduced the notions of T-Banach contraction and T-contractive mapping and extended the Banach contraction principle (see [4]) and Edelstein's fixed point theorem [7]. In the same year, Moradi [9] introduced the T-Kannan contractive type mappings, extending in this way the well-known Kannan's fixed point theorem given in [8]. The corresponding versions of T-contractive, T-Kannan mappings and T-Chatterjea contractions on cone metric spaces were studied in [10]. The same authors [12], then studied the existence of fixed points of T-Zamfirescu and T-weak contraction mappings defined on a complete cone metric space. Later, in [11] they studied the existence of fixed points for T-Zamfirescu operators in complete metric spaces and proved a convergence theorem of T-Picard iteration for the class of T-Zamfirescu operators.

Inspired and motivated by the above facts, we prove approximate fixed point theorems for the classes of T-Banach contraction, T-Kannan contraction, T-Chatterjea contraction, T-Zamfirescu operators and T-almost contraction. Here we mention that we consider operators in metric spaces, not in complete metric spaces which is the usual framework for fixed point theorems.

Let (X, d) be a metric space and $T, S : X \to X$ be two commuting mappings. Here we introduce the concept of T-asymptotically regular mapping in a metric space and then establish a lemma regarding approximate fixed points of the commuting mappings in metric spaces. We use this lemma to prove qualitative theorems for various types of contractions on metric spaces.

We need the following definitions to prove our main results:

Definition 1.1. *Let (X, d) be a metric space. Let $f : X \to X, \epsilon > 0$ and $x \in X$. Then x_0 is an* **ϵ-fixed point(approximate fixed point) of** *f if*

$$d(f(x_0), x_0) < \epsilon.$$

Note. The set of all ϵ-fixed points of f, for a given ϵ can be denoted by

$$F_\epsilon(f) = \{x \in X : x \text{ is an } \epsilon - \text{fixed point of } f\}.$$

Definition 1.2. *Let (X, d) be a metric space and $f : X \to X$. Then f has the* **approximate fixed point property (a.f.p.p)** *if for every $\epsilon > 0$,*

$$F_\epsilon(f) \neq \phi.$$

Definition 1.3. *Let (X, d) be a metric space, $f : X \to X$ is said to be asymptotically regular if*

$$d(f^n(x), f^{n+1}(x)) \to 0 \ \text{ as } \ n \to \infty, \quad \text{ for all } \ x \in X.$$

Definition 1.4. *Let (X, d) be a metric space, $T, S : X \to X$ be two functions. S is called T-asymptotically regular if*

$$d(TS^n(x), TS^{n+1}(x)) \to 0 \ \text{ as } \ n \to \infty, \quad \text{ for all } \ x \in X.$$

Definition 1.5. *Let (X,d) be a metric space and $T,S : X \to X$ be two functions. S is said to be T-***Banach contraction** *(TB contraction) if there exists $a \in [0,1)$ such that*

$$d(TSx,TSy) \leq ad(Tx,Ty), \qquad \text{for all } x,y \in X.$$

If we take $T = I$, the identity map, then we obtain the definition of *Banach's contraction* (see [4]).

Definition 1.6. *Let (X,d) be a metric space and $T,S : X \to X$ be two functions. S is said to be T-***Kannan contraction** *(TK contraction) if there exists $b \in [0,\frac{1}{2})$ such that*

$$d(TSx,TSy) \leq b(d(Tx,TSx)+d(Ty,TSy]), \qquad \text{for all } x,y \in X.$$

Here when $T = I$, the identity map, we get *Kannan operator* [8].

Definition 1.7. *Let (X,d) be a metric space and $T,S : X \to X$ be two functions. S is said to be T-***Chatterjea contraction** *(TC contraction) if there exists $c \in [0,\frac{1}{2})$ such that*

$$d(TSx,TSy) \leq c[d(Tx,TSy)+d(Ty,TSx]), \qquad \text{for all } x,y \in X.$$

When $T = I$, the identity map, in the above definition, it becomes *Chatterjea operator* [6].

Definition 1.8. *Let (X,d) be a metric space and $T,S : X \to X$ be two functions. S is said to be T-***Zamfirescu operator** *(TZ operator) if there are real numbers $0 \leq a < 1, 0 \leq b < \frac{1}{2}, 0 \leq c < \frac{1}{2}$ such that for all $x,y \in X$ at least one of the conditions is true:*

(TZ_1) : $d(TSx,TSy) \leq ad(Tx,Ty)$,
(TZ_2) : $d(TSx,TSy) \leq b[d(Tx,TSx)+d(Ty,TSy]),$
(TZ_3) : $d(TSx,TSy) \leq c[d(Tx,TSy)+d(Ty,TSx]).$

When the function T is equated to I, the identity map, we obtain the definition of *Zamfirescu operator* introduced in [13].

Definition 1.9. *Let (X,d) be a metric space and $T,S : X \to X$ be two functions. S is said to be T-***almost contraction** *if there exists $\delta \in [0,1)$ and $L \geq 0$ such that*

$$d(TSx,TSy) \leq \delta d(Tx,Ty)+Ld(Ty,TSx), \qquad \text{for all } x,y \in X.$$

When $T = I$, the identity map, in the above definition, we obtain the definition of almost contraction, the concept introduced by Berinde (see [3], [5]).

In 2004, this concept was introduced by Berinde as *weak contraction* [3] and later in 2008, it was renamed by himself as *almost contraction* [5].

In order to prove our main results we need the following lemma:

Lemma 1.10. *Let (X,d) be a metric space and $T,S : X \to X$ be two commuting mappings. If S is T-asymptotically regular, then S has approximate fixed point property.*

Proof. Let $x_0 \in X$. Since $S : X \to X$ is T-asymptotically regular, we have

$$d(TS^n(x_0), TS^{n+1}(x_0)) \to 0 \text{ as } n \to \infty, \quad \text{for all } x \in X,$$

which gives that for every $\epsilon > 0$, there exists $n_0(\epsilon) \in N$ such that

$$d(TS^n(x_0), TS^{n+1}(x_0)) < \epsilon, \text{ for all } n \geq n_0(\epsilon)$$

$$i.e., d(TS^n(x_0), TS(S^n(x_0))) < \epsilon, \text{ for all } n \geq n_0(\epsilon).$$

Since T and S are commuting mappings, this implies that for every $\epsilon > 0$, there exists $n_0(\epsilon) \in N$ such that

$$d(TS^n(x_0), ST(S^n(x_0))) < \epsilon, \text{ for all } n \geq n_0(\epsilon).$$

Denote $y_0 = TS^n(x_0)$. Then we get that for every $\epsilon > 0$, there exists $y_0 \in X$ such that

$$d(y_0, Sy_0) < \epsilon, \text{ for all } n \geq n_0(\epsilon).$$

So for each $\epsilon > 0$, there exists an ϵ-fixed point of S in X, namely y_0 which means that S has approximate fixed point property.

2. Main results

Theorem 2.1. *Let (X, d) be a metric space and $T, S : X \to X$ be two commuting mappings where S is a TB-contraction. Then for every $\epsilon > 0$,*

$$F_\epsilon(S) \neq \phi,$$

i.e., S has approximate fixed point property.

Proof. Let $\epsilon > 0, x \in X$. Then

$$\begin{aligned} d(TS^n(x), TS^{n+1}(x)) &= d(TS(S^{n-1}(x)), TS(S^n(x))) \\ &\leq ad(TS^{n-1}(x), TS^n(x)) \\ &\leq a^2 d(TS^{n-2}(x), TS^{n-1}(x)) \\ &\leq \ldots\ldots\ldots\ldots \\ &\leq a^n d(Tx, TSx). \end{aligned}$$

Since $a \in [0, 1)$, from the above inequality we get that

$$d(TS^n(x), TS^{n+1}(x)) \to 0 \text{ as } n \to \infty, \quad \text{for all } x \in X,$$

which implies that S is T-asymptotically regular. Now by applying Lemma 1.10 we obtain that for every $\epsilon > 0$,

$$F_\epsilon(S) \neq \phi,$$

which means that S has approximate fixed point property.

Corollary 2.2. [2, *Theorem* 2.1] *Let (X, d) be a metric space and $f : X \to X$ an a-contraction. Then for every $\epsilon > 0$,*

$$F_\epsilon(f) \neq \phi.$$

Theorem 2.3. *Let (X, d) be a metric space and $T, S : X \to X$ be two commuting mappings where S is a TK-contraction. Then for every $\epsilon > 0$,*

$$F_\epsilon(S) \neq \phi,$$

i.e., S has approximate fixed point property.

Proof. Let $\epsilon > 0, x \in X$. Then

$$\begin{aligned} d(TS^n(x), TS^{n+1}(x)) &= d(TS(S^{n-1}(x)), TS(S^n(x))) \\ &\leq b[d(TS^{n-1}(x), TS(S^{n-1}(x))) + d(TS^n(x), TS(S^n(x)))] \\ &= b[d(TS^{n-1}(x), TS^n(x)) + d(TS^n(x), TS^{n+1}(x))]. \end{aligned}$$

Thus we have the inequality

$$(1 - b)d(TS^n(x), TS^{n+1}(x)) \leq b[d(TS^{n-1}(x), TS^n(x))],$$

which implies that

$$\begin{aligned} d(TS^n(x), TS^{n+1}(x)) &\leq \frac{b}{1-b} d(TS^{n-1}(x), TS^n(x)) \\ &\leq \left(\frac{b}{1-b}\right)^2 d(TS^{n-2}(x), TS^{n-1}(x)) \\ &\leq \cdots\cdots\cdots\cdots \\ &\leq \left(\frac{b}{1-b}\right)^n d(Tx, TSx). \end{aligned}$$

Since $b \in [0, \frac{1}{2})$, from the above inequality we get that

$$d(TS^n(x), TS^{n+1}(x)) \to 0 \text{ as } n \to \infty, \quad \text{for all } x \in X,$$

which implies that S is T-asymptotically regular. Now by applying Lemma 1.10 we obtain that for every $\epsilon > 0$,

$$F_\epsilon(S) \neq \phi,$$

which means that S has approximate fixed point property.

Corollary 2.4. [2, *Theorem* 2.2] *Let (X, d) be a metric space and $f : X \to X$ a Kannan operator. Then for every $\epsilon > 0$,*

$$F_\epsilon(f) \neq \phi.$$

Theorem 2.5. *Let (X, d) be a metric space and $T, S : X \to X$ be two commuting mappings where S is a TC-contraction. Then for every $\epsilon > 0$,*

$$F_\epsilon(S) \neq \phi,$$

i.e., S has approximate fixed point property.

Proof. Let $\epsilon > 0, x \in X$. Then

$$\begin{aligned} d(TS^n(x), TS^{n+1}(x)) &= d(TS(S^{n-1}(x)), TS(S^n(x))) \\ &\leq c[d(TS^{n-1}(x), TS(S^n(x))) + d(TS^n(x), TS(S^{n-1}(x)))] \\ &= c[d(TS^{n-1}(x), TS^{n+1}(x) + d(TS^n(x), TS^n(x))] \\ &= c[d(TS^{n-1}(x), TS^{n+1}(x))]. \end{aligned}$$

Now we have,

$$d(TS^{n-1}(x),TS^{n+1}(x)) \leq d(TS^{n-1}(x),TS^{n}(x)) + d(TS^{n}(x),TS^{n+1}(x)),$$

which implies that

$$d(TS^{n}(x),TS^{n+1}(x)) \leq c[d(TS^{n-1}(x),TS^{n}(x)) + d(TS^{n}(x),TS^{n+1}(x))],$$

which gives

$$(1-c)d(TS^{n}(x),TS^{n+1}(x)) \leq c[d(TS^{n-1}(x),TS^{n}(x)].$$

Thus we have the inequality,

$$\begin{aligned} d(TS^{n}(x),TS^{n+1}(x)) &\leq \frac{c}{1-c} d(TS^{n-1}(x),TS^{n}(x)) \\ &\leq \left(\frac{c}{1-c}\right)^2 d(TS^{n-2}(x),TS^{n-1}(x)) \\ &\leq \ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ &\leq \left(\frac{c}{1-c}\right)^n d(Tx,TSx). \end{aligned}$$

Since $c \in [0, \frac{1}{2})$, from the above inequality we get that

$$d(TS^{n}(x),TS^{n+1}(x)) \to 0 \text{ as } n \to \infty, \quad \text{for all } x \in X,$$

which implies that S is T-asymptotically regular. Now by applying Lemma 1.10 we obtain that for every $\epsilon > 0$,

$$F_{\epsilon}(S) \neq \phi,$$

which means that S has approximate fixed point property.

Corollary 2.6. [2, *Theorem* 2.3] *Let* (X,d) *be a metric space and* $f : X \to X$ *a Chatterjea operator. Then for every* $\epsilon > 0$,

$$F_{\epsilon}(f) \neq \phi.$$

Theorem 2.7. *Let* (X,d) *be a metric space and* $T,S : X \to X$ *be two commuting mappings where* S *is a* TZ*-operator. Then for every* $\epsilon > 0$,

$$F_{\epsilon}(S) \neq \phi,$$

i.e., S *has approximate fixed point property.*

Proof. Let $\epsilon > 0, x \in X$. If (TZ_2) holds, then

$$\begin{aligned} d(TSx,TSy) &\leq b[d(Tx,TSx) + d(Ty,TSy)] \\ &\leq b[d(Tx,TSx)] + b[d(Ty,Tx) + d(Tx,TSx) + d(TSx,TSy)] \\ &= 2b[d(Tx,TSx)] + b[d(Tx,Ty)] + b[d(TSx,TSy)]. \end{aligned}$$

The above inequality gives

$$(1-b)d(TSx,TSy) \leq 2b[d(Tx,TSx)] + b[d(Tx,Ty)],$$

which implies

$$d(TSx,TSy) \leq \frac{2b}{1-b}[d(Tx,TSx)] + \frac{b}{1-b}[d(Tx,Ty)]. \tag{2.1}$$

If (TZ_3) holds, then

$$\begin{aligned} d(TSx,TSy) &\le c[d(Tx,TSy)+d(Ty,TSx)] \\ &\le c[d(Tx,TSx)+d(TSx,TSy)+d(Ty,Tx)+d(Tx,TSx)] \\ &= c[d(TSx,TSy)]+2c[d(Tx,TSx)]+c[d(Tx,Ty)]. \end{aligned}$$

The above inequality gives

$$(1-c)d(TSx,TSy) \le 2c[d(Tx,TSx)]+c[d(Tx,Ty)],$$

which implies

$$d(TSx,TSy) \le \frac{2c}{1-c}[d(Tx,TSx)]+\frac{c}{1-c}[d(Tx,Ty)]. \tag{2.2}$$

Denote

$$\delta = \max\left\{a, \frac{b}{1-b}, \frac{c}{1-c}\right\}.$$

Then we have $0 \le \delta < 1$ and in view of (TZ_1), (2.1) and (2.2), it results that the inequality

$$d(TSx,TSy) \le 2\delta[d(Tx,TSx)]+\delta[d(Tx,Ty)] \tag{2.3}$$

holds for all $x, y \in X$.
Using (2.3), we get

$$\begin{aligned} d(TS^n(x),TS^{n+1}(x)) &= d(TS(S^{n-1}(x)),TS(S^n(x))) \\ &\le 2\delta[d(TS^{n-1}(x),TS(S^{n-1}(x))]+\delta[d(TS^{n-1}(x),TS^n(x))] \\ &= 3\delta[d(TS^{n-1}(x),TS^n(x))]. \end{aligned}$$

Thus we obtain that

$$\begin{aligned} d(TS^n(x),TS^{n+1}(x)) &\le 3\delta[d(TS^{n-1}(x),TS^n(x))] \\ &\le (3\delta)^2\,[d(TS^{n-2}(x),TS^{n-1}(x))] \\ &\le \ldots\ldots\ldots\ldots\ldots\ldots \\ &\le (3\delta)^n\,[d(Tx,TSx)]. \end{aligned}$$

Since $\delta \in [0,1)$, the above inequality gives

$$d(TS^n(x),TS^{n+1}(x)) \to 0 \text{ as } n \to \infty, \quad \text{for all } x \in X,$$

which implies that S is T-asymptotically regular. Now by applying Lemma 1.10 we obtain that for every $\epsilon > 0$,

$$F_\epsilon(S) \neq \phi,$$

which means that S has approximate fixed point property.

Corollary 2.8. [2, *Theorem* 2.4] *Let (X,d) be a metric space and $f : X \to X$ a Zamfirescu operator. Then for every $\epsilon > 0$,*

$$F_\epsilon(f) \neq \phi.$$

Theorem 2.9. *Let (X, d) be a metric space and $T, S : X \to X$ be two commuting mappings where S is a T-almost contraction. Then for every $\epsilon > 0$,*

$$F_\epsilon(S) \neq \phi,$$

i.e., S has approximate fixed point property.

Proof. Let $\epsilon > 0, x \in X$. Then

$$\begin{aligned} d(TS^n(x), TS^{n+1}(x)) &= d(TS(S^{n-1}(x)), TS(S^n(x))) \\ &\leq \delta d(TS^{n-1}(x), TS^n(x)) + Ld(TS^n(x), TS^n(x)) \\ &= \delta d(TS^{n-1}(x), TS^n(x)) \\ &\leq \delta^2 d(TS^{n-2}(x), TS^{n-1}(x)) \\ &\leq \ldots\ldots\ldots\ldots \\ &\leq \delta^n d(Tx, TSx). \end{aligned}$$

Since $\delta \in [0, 1)$, from the above inequality we get that

$$d(TS^n(x), TS^{n+1}(x)) \to 0 \text{ as } n \to \infty, \quad \text{for all } x \in X,$$

which implies that S is T-asymptotically regular. Now by applying Lemma 1.10 we obtain that for every $\epsilon > 0$,

$$F_\epsilon(S) \neq \phi,$$

which means that S has approximate fixed point property.

Corollary 2.10. [2, *Theorem* 2.5] *Let (X, d) be a metric space and $f : X \to X$ an almost contraction. Then for every $\epsilon > 0$,*

$$F_\epsilon(f) \neq \phi.$$

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Some corrected optimal quadrature formulas

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Abstract. The optimal 3-point quadrature formulae of closed type are derived and the estimations of error in terms of a variety on norms involving the second derivative are given. The corrected quadrature rules of the optimal quadrature formulae are considered. These results are obtained from an inequalities point of view.

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1. Introduction

The problem to construct the optimal quadratures formulas was studied by many authors. The first results were obtained by A. Sard, L.S. Meyers and S.M. Nikolski. In the last years a number of authors have obtained in many different ways the optimal quadrature formulas ([1], [5], [6], [10], [14], [15]). In this section we present the classical methods to construct this kind of quadrature formulas.

Let $\mathcal{H}$ be the class of sufficiently smooth functions $f : [a, b] \to \mathbb{R}$ and we consider the following quadrature formula with degree of exactness equal $n - 1$

$$\int_a^b f(x)dx = \sum_{i=0}^{m} \sum_{k=0}^{z_i-1} A_{ki} f^{(k)}(x_i) + \mathcal{R}_n[f], \tag{1.1}$$

where the nodes $a \le x_0 < x_1 < \cdots < x_m \le b$ have the multiplicities z_i, $1 \le z_i \le n$.

The quadrature formula (1.1) is called **optimal in the sense of Sard** in the space $\mathcal{H}$, if

$$\mathcal{E}_{m,n}(\mathcal{H}, A) = \sup_{f\in\mathcal{H}} |\mathcal{R}_n[f]|$$

attains the minimum value with regard to A, where $A = \{A_{ki}\}_{i=0}^{m}\,{}_{k=0}^{z_i-1}$ are the coefficients of quadrature formula.

The quadrature formula (1.1) is called **optimal in sense Nikolski** in the space $\mathcal{H}$, if

$$\mathcal{E}_{m,n}(\mathcal{H}, A, X) = \sup_{f\in\mathcal{H}} |\mathcal{R}_n[f]|$$

attains the minimum value with regard to A and X, where $A = \{A_{ki}\}_{i=0}^{m}\,{}_{k=0}^{z_i-1}$ are the coefficients and $X = (x_0, x_1, \ldots, x_m)$ are the nodes of quadrature formula.

We denote

$$W_p^n[a,b] := \left\{ f \in C^{n-1}[a,b], f^{(n-1)} \text{ absolutely continuous }, \left\|f^{(n)}\right\|_p < \infty \right\}$$

with

$$\|f\|_p := \left\{ \int_a^b |f(x)|^p \, dx \right\}^{\frac{1}{p}}, \quad \text{for } 1 \le p < \infty,$$
$$\|f\|_\infty := \sup_{x\in[a,b]} |f(x)| .$$

If $f \in W_p^n[a,b]$, by using Peano$'$s theorem, the remainder term can be written

$$\mathcal{R}_n[f] = \int_a^b K_n(t) f^{(n)}(t) dt,$$

where $K_n(t) = \mathcal{R}_n\left[\dfrac{(x-t)_+^{n-1}}{(n-1)!}\right]$.

For the remainder term we have the evaluation

$$|\mathcal{R}_n[f]| \le \left[\int_a^b \left|f^{(n)}(t)\right|^p dt\right]^{\frac{1}{p}} \left[\int_a^b |K_n(t)|^q \, dt\right]^{\frac{1}{q}}, \quad \frac{1}{p} + \frac{1}{q} = 1, \tag{1.2}$$

with remark that in the cases $p = 1$ and $p = \infty$ this evaluation is

$$|\mathcal{R}_n[f]| \le \int_a^b \left|f^{(n)}(t)\right| dt \sup_{t\in[a,b]} |K_n(t)| , \tag{1.3}$$

$$|\mathcal{R}_n[f]| \le \sup_{t\in[a,b]} \left|f^{(n)}(t)\right| \int_a^b |K_n(t)| \, dt. \tag{1.4}$$

The φ-function method is a model of constructing the quadrature formulas and was given by D.V. Ionescu ([9]). Suppose that $f \in C^r[a,b]$ and for some given $n \in \mathbb{N}$ consider the nodes $a = x_0 < ... < x_n = b$. On each interval $[x_{k-1}, x_k]$, $k = 1, ..., n$, it is considered a function φ_k, $k = 1, ..., n$, with the property that

$$\varphi_k^{(r)} = 1, \; k = 1, ..., n. \tag{1.5}$$

One defines the function φ as follows

$$\varphi|_{[x_{k-1},x_k]} = \varphi_k, \; k = 1, ..., n, \tag{1.6}$$

i.e., the restriction of the function φ to the interval $[x_{k-1}, x_k]$ is φ_k.

Using the integration by parts of the integral

$$S(f):=\int_a^b f(x)dx=\sum_{k=1}^n\int_{x_{k-1}}^{x_k}\varphi_k^{(r)}(x)f(x)dx,$$

one obtains the following quadrature formula

$$\int_a^b f(x)dx=\sum_{k=0}^n\sum_{j=0}^{r-1}A_{kj}f^{(j)}(x_k)+\mathcal{R}_n(f), \tag{1.7}$$

with

$$\mathcal{R}_n(f)=(-1)^r\int_a^b\varphi(x)f^{(r)}(x)dx \tag{1.8}$$

and

$$\begin{aligned}A_{0j}&=(-1)^{j+1}\varphi_1^{(r-j-1)}(x_0),\\ A_{kj}&=(-1)^j(\varphi_k-\varphi_{k+1})^{(r-j-1)}(x_k),\ k=1,\ldots,n-1,\\ A_{nj}&=(-1)^j\varphi_n^{(r-j-1)}(x_n),\ j=0,1,\ldots,r-1.\end{aligned} \tag{1.9}$$

In [1], T. Cătinaş and G. Coman studied the optimality in sense of Nikolski for a quadrature formula, using the method of φ-function.

In [16], N. Ujević and L. Mijić constructed a class of quadrature formulas of close type with 3 nodes. Let

$$K_2(\alpha,\beta,\gamma,\delta;t)=\begin{cases}\frac{1}{2}(t-\alpha)(t-\beta),\ t\in\left[a,\frac{a+b}{2}\right],\\ \\ \frac{1}{2}(t-\gamma)(t-\delta),\ t\in\left(\frac{a+b}{2},b\right],\end{cases}$$

be a function which depends on the parameters $\alpha,\beta,\gamma,\delta\in\mathbb{R}$. Integrating by parts the integral $\int_a^b K_2(\alpha,\beta,\gamma,\delta;t)f''(t)dt$, and putting conditions that the coefficients of the first derivatives to be zero, N. Ujević and L. Mijić were constructed the following class of quadrature formulas of close type

$$\begin{aligned}\int_a^b f(t)dt&=A_0(\alpha,\beta,\gamma,\delta)f(a)+A_1(\alpha,\beta,\gamma,\delta)f\left(\frac{a+b}{2}\right)\\ &+A_2(\alpha,\beta,\gamma,\delta)f(b)+\mathcal{R}[f],\end{aligned}$$

where

$$\mathcal{R}[f]=\int_a^b K_2(\alpha,\beta,\gamma,\delta)f''(t)dt.$$

The parameters $\alpha,\beta,\gamma,\delta$ are obtained putting conditions that the remainder term which is evaluated in sense of (1.4) to be minimal, namely $\int_a^b|K_2(\alpha,\beta,\gamma,\delta)|\,dt$ to attains the minimum value.

The main result obtained by N. Ujević and L. Mijić in the above described procedure is formulated bellow.

Theorem 1.1. [16] *Let $I \subset \mathbb{R}$ be an open interval such that $[0,1] \subset I$ and let $f : I \to \mathbb{R}$ be a twice differentiable function such that f'' is bounded and integrable. Then we have*

$$\left| \int_0^1 f(t)dt - \frac{\sqrt{2}}{8} f(0) - \left(1 - \frac{\sqrt{2}}{4}\right) f\left(\frac{1}{2}\right) - \frac{\sqrt{2}}{8} f(1) \right| \leq \frac{2-\sqrt{2}}{48} \|f''\|_\infty. \quad (1.10)$$

The main purpose of the section 2 is to derive a quadrature formula of close type with 3-points which is optimal in sense Nikolski, namely we calculate the coefficients A_i, $i = \overline{0,2}$ and the node $a_1 \in (a,b)$ such that the quadrature formula

$$\int_a^b f(t)dt = A_0 f(a) + A_1 f(a_1) + A_2 f(b) + \mathcal{R}_2[f],$$

to be optimal, considering that the remainder term is evaluated in sense of (1.2) in the cases $p = 1$, $p = 2$ and $p = \infty$.

For the simplicity, in this paper we choose $[a,b] = [0,1]$. The corresponding results in the arbitrary interval $[a,b]$ can be obtained using the following lemma.

Lemma 1.2. [11] *If $-\infty < \alpha < \beta < +\infty$ and w is a weight function on (α,β) and $\int_\alpha^\beta f(t)w(t)dt = \sum_{i=0}^m A_i f(x_i) + r_m[f]$, $f \in L_w^1(\alpha,\beta)$, then*

$$W(x) = w\left(\alpha + (\beta-\alpha)\frac{x-a}{b-a}\right), \ x \in (a,b), \ -\infty < a < b < +\infty,$$

is a weight function on (a,b) and

$$\int_a^b F(x)W(x)dx = \frac{b-a}{\beta-\alpha}\sum_{i=0}^m A_i F\left(a + (b-a)\frac{x_i-\alpha}{\beta-\alpha}\right) + \mathcal{R}_m[F],$$

where $F \in L_w^1(a,b)$ and $\mathcal{R}_m[F] = \frac{b-a}{\beta-\alpha} r_m[\tilde{F}]$, $\tilde{F}(t) = F\left(a + (b-a)\frac{t-\alpha}{\beta-\alpha}\right)$.

2. The optimal 3-point quadrature formula of closed type

Let

$$\int_0^1 f(x)dx = A_0 f(0) + A_1 f(a_1) + A_2 f(1) + \mathcal{R}_2[f] \quad (2.1)$$

be a quadrature formula with degree of exactness equal 1.

Since the quadrature formula has degree of exactness 1, the remainder term verifies the conditions $\mathcal{R}_2[e_i] = 0$, $e_i(x) = x^i$, $i = 0, 1$, namely

$$\begin{cases} A_0 + A_1 + A_2 = 1 \\ A_1 a_1 + A_2 = \dfrac{1}{2} \end{cases} \quad (2.2)$$

and using Peano's theorem the remainder term has the following integral representation

$$\mathcal{R}_2[f] = \int_0^1 K_2(t) f''(t)dt, \text{ where} \tag{2.3}$$

$$K_2(t) = \mathcal{R}_2\left[(x-t)_+\right] = \begin{cases} \frac{1}{2}t^2 - A_0 t, & 0 \le t \le a_1, \\ \frac{1}{2}(1-t)^2 - A_2(1-t), & a_1 < t \le 1. \end{cases} \tag{2.4}$$

Theorem 2.1. *For $f \in W_\infty^2[0,1]$, the quadrature formula of the form (2.1), optimal with regard to the error, is*

$$\int_0^1 f(x)dx = \frac{\sqrt{2}}{8} f(0) + \frac{4-\sqrt{2}}{4} f\left(\frac{1}{2}\right) + \frac{\sqrt{2}}{8} f(1) + \mathcal{R}_2^{[1]}[f], \tag{2.5}$$

with

$$\mathcal{R}_2^{[1]}[f] = \int_0^1 K_2^{[1]}(t) f''(t)dt,\ K_2^{[1]}(t) = \begin{cases} \frac{1}{2}t^2 - \frac{\sqrt{2}}{8}t, & 0 \le t \le \frac{1}{2}, \\ \frac{1}{2}(1-t)^2 - \frac{\sqrt{2}}{8}(1-t), & \frac{1}{2} < t \le 1, \end{cases} \tag{2.6}$$

$$\left|\mathcal{R}_2^{[1]}[f]\right| \le \frac{2-\sqrt{2}}{48} \|f''\|_\infty \approx 0.0122 \|f''\|_\infty.$$

Proof. The remainder term (2.3) can be evaluate in the following way

$$\left|\mathcal{R}_2^{[1]}[f]\right| \le \|f''\|_\infty \int_0^1 \left|K_2^{[1]}(t)\right| dt.$$

The quadrature formula is optimal with regard to the error if

$$\int_0^1 \left|K_2^{[1]}(t)\right| dt \to \text{minimum}.$$

We have

$$\begin{aligned} \mathcal{I}(A_0, A_2, a_1) &= \int_0^1 |K_2^{[1]}(t)|dt = \int_0^{a_1} |K_2^{[1]}(t)|dt + \int_{a_1}^1 |K_2^{[1]}(t)|dt \\ &= \int_0^{2A_0} \left(A_0 t - \frac{1}{2}t^2\right) dt + \int_{2A_0}^{a_1} \left(\frac{1}{2}t^2 - A_0 t\right) dt \\ &+ \int_{a_1}^{1-2A_2} \left[\frac{1}{2}(1-t)^2 - A_2(1-t)\right] dt + \int_{1-2A_2}^1 \left[A_2(1-t) - \frac{1}{2}(1-t)^2\right] dt \\ &= \frac{4}{3}A_0^3 - \frac{1}{2}a_1^2 A_0 + \frac{1}{6}a_1^3 + \frac{4}{3}A_2^3 - \frac{1}{2}(1-a_1)^2 A_2 + \frac{1}{6}(1-a_1)^3. \end{aligned}$$

Putting condition that the partial derivatives to be zero, namely

$$\begin{cases} \dfrac{\partial \mathcal{I}(A_0, A_2, a_1)}{\partial A_0} = 4A_0^2 - \dfrac{1}{2}a_1^2 = 0, \\ \dfrac{\partial \mathcal{I}(A_0, A_2, a_1)}{\partial A_2} = 4A_2^2 - \dfrac{1}{2}(1-a_1)^2 = 0, \\ \dfrac{\partial \mathcal{I}(A_0, A_2, a_1)}{\partial a_1} = -a_1 A_0 + \dfrac{a_1^2}{2} - \dfrac{1}{2}(1-a_1)^2 + A_2(1-a_1) = 0, \end{cases}$$

we find the following values for the coefficients and the node of optimal quadrature formula

$$A_0 = A_2 = \frac{\sqrt{2}}{8},\ A_1 = 1 - \frac{\sqrt{2}}{4},\ a_1 = \frac{1}{2},\ \text{and} \int_0^1 |K_2^{[1]}(t)|dt = \frac{2-\sqrt{2}}{48}. \qquad \square$$

Remark 2.2. The optimal quadrature (2.5) coincides with the quadrature formula (1.10) obtained by N. Ujević and L. Mijić in [16], but this quadrature formula was obtained in different way than in [16]. This result motivated us to seek the quadrature formulas of type (2.1) such that the estimation of its error to be best possible in p-norm for $p = 2$ and $p = 1$.

Remark 2.3. For the remainder term of quadrature formula (2.5) can be established the following two estimations

$$\begin{aligned} \left|\mathcal{R}_2^{[1]}[f]\right| &\le \left[\int_0^1 \left(K_2^{[1]}(t)\right)^2 dt\right]^{\frac{1}{2}} \|f''\|_2 = \frac{1}{16}\sqrt{\frac{22-15\sqrt{2}}{15}}\|f''\|_2 \\ &\approx 0.0143\|f''\|_2,\ f \in W_2^2[0,1], \\ \left|\mathcal{R}_2^{[1]}[f]\right| &\le \sup_{t\in[0,1]} |K_2^{[1]}(t)| \cdot \|f''\|_1 = \frac{2-\sqrt{2}}{16}\|f''\|_1 \approx 0.0366\|f''\|_1,\ f \in W_1^2[0,1]. \end{aligned}$$

Theorem 2.4. *For $f \in W_2^2[0,1]$, the quadrature formula of the form (2.1), optimal with regard to the error, is*

$$\int_0^1 f(x)dx = \frac{3}{16}f(0) + \frac{5}{8}f\left(\frac{1}{2}\right) + \frac{3}{16}f(1) + \mathcal{R}_2^{[2]}[f], \tag{2.7}$$

with

$$\mathcal{R}_2^{[2]}[f] = \int_0^1 K_2^{[2]}(t)f''(t)dt,\quad K_2^{[2]}(t) = \begin{cases} \dfrac{1}{2}t^2 - \dfrac{3}{16}t,\ 0 \le t \le \dfrac{1}{2}, \\ \dfrac{1}{2}(1-t)^2 - \dfrac{3}{16}(1-t),\ \dfrac{1}{2} < t \le 1, \end{cases} \tag{2.8}$$

and

$$\left|\mathcal{R}_2^{[2]}[f]\right| \le \frac{\sqrt{5}}{160}\|f''\|_2 \approx 0.0140\|f''\|_2.$$

Proof. The remainder term (2.3) can be evaluated in the following way

$$\left|\mathcal{R}_2^{[2]}[f]\right| \leq \left[\int_0^1 \left(K_2^{[2]}(t)\right)^2 dt\right]^{\frac{1}{2}} \|f''\|_2.$$

The quadrature formula is optimal with regard to the error if

$$\int_0^1 \left(K_2^{[2]}(t)\right)^2 dt \to \text{minimum}.$$

We have

$$\begin{aligned}
\mathcal{I}(A_0, A_2, a_1) &= \int_0^1 \left(K_2^{[2]}(t)\right)^2 dt = \int_0^{a_1} \left(\frac{1}{2}t^2 - A_0 t\right)^2 dt \\
&+ \int_{a_1}^1 \left[\frac{1}{2}(1-t)^2 - A_2(1-t)\right]^2 dt \\
&= \frac{1}{20}a_1^5 - \frac{A_0}{4}a_1^4 + \frac{A_0^2}{3}a_1^3 + \frac{(1-a_1)^5}{20} - A_2\frac{(1-a_1)^4}{4} + A_2^2\frac{(1-a_1)^3}{3}.
\end{aligned}$$

Putting condition that the partial derivatives to be zero, namely

$$\begin{cases}
\dfrac{\partial \mathcal{I}(A_0, A_2, a_1)}{\partial A_0} = -\dfrac{a_1^4}{4} + \dfrac{2}{3}A_0 a_1^3 = 0, \\[2ex]
\dfrac{\partial \mathcal{I}(A_0, A_2, a_1)}{\partial A_2} = -\dfrac{(1-a_1)^4}{4} + \dfrac{2}{3}A_2(1-a_1)^3 = 0, \\[2ex]
\dfrac{\partial \mathcal{I}(A_0, A_2, a_1)}{\partial a_1} = \dfrac{a_1^4}{4} - A_0 a_1^3 + A_0^2 a_1^2 - \dfrac{(1-a_1)^4}{4} + A_2(1-a_1)^3 - A_2^2(1-a_1)^2 = 0,
\end{cases}$$

we find the following values for the coefficients and the node of optimal quadrature formula

$$A_0 = A_2 = \frac{3}{16},\ A_1 = \frac{5}{8},\ a_1 = \frac{1}{2},\ \text{and}\ \int_0^1 \left(K_2^{[2]}(t)\right)^2 dt = \frac{1}{2^{10}\cdot 5}. \qquad \square$$

Remark 2.5. For the remainder term of quadrature formula (2.7) can be established the following two estimations

$$\left|\mathcal{R}_2^{[2]}[f]\right| \leq \int_0^1 |K_2^{[2]}(t)| dt \cdot \|f''\|_\infty = \frac{19}{1536}\|f''\|_\infty \approx 0.0124\|f''\|_\infty, \text{ for } f \in W_\infty^2[0,1],$$

$$\left|\mathcal{R}_2^{[2]}[f]\right| \leq \sup_{t\in[0,1]} |K_2^{[2]}(t)| \cdot \|f''\|_1 = \frac{1}{32}\|f''\|_1 \approx 0.0313\|f''\|_1, \text{ for } f \in W_1^2[0,1].$$

Theorem 2.6. *For $f \in W_1^2[0,1]$, the quadrature formula of the form (2.1), optimal with regard to the error, is*

$$\int_0^1 f(x)dx = \frac{\sqrt{2}-1}{2}f(0) + (2-\sqrt{2})f\left(\frac{1}{2}\right) + \frac{\sqrt{2}-1}{2}f(1) + \mathcal{R}_2^{[3]}[f], \qquad (2.9)$$

with

$$\mathcal{R}_2^{[3]}[f]=\int_0^1 K_2^{[3]}(t)f''(t)dt,\ K_2^{[3]}(t)=\begin{cases}\frac{1}{2}t^2-\frac{\sqrt{2}-1}{2}t,\ 0\le t\le\frac{1}{2},\\ \\ \frac{1}{2}(1-t)^2-\frac{\sqrt{2}-1}{2}(1-t),\ \frac{1}{2}<t\le 1,\end{cases}\tag{2.10}$$

and

$$\left|\mathcal{R}_2^{[3]}[f]\right|\le\frac{3-2\sqrt{2}}{8}\|f''\|_1\approx 0.0214\|f''\|_1.$$

Proof. The remainder term (2.3) can be evaluated in the following way

$$\left|\mathcal{R}_2^{[3]}[f]\right|\le\|f''\|_1\cdot\sup_{0\le t\le 1}|K_2^{[3]}(t)|.$$

The quadrature formula is optimal with regard to the error if

$$\sup_{0\le t\le 1}|K_2^{[3]}(t)|\to\text{minimum}.$$

We have

$$\sup_{t\in[0,a_1]}|K_2^{[3]}(t)|=\max\left\{|K_2^{[3]}(A_0)|,|K_2^{[3]}(a_1)|\right\}=\max\left\{\frac{1}{2}A_0^2,\left|\frac{1}{2}a_1^2-A_0a_1\right|\right\},$$

$$\sup_{t\in[a_1,1]}|K_2^{[3]}(t)|=\max\left\{|K_2^{[3]}(a_1)|,|K_2^{[3]}(1-A_2)|\right\}=\max\left\{\left|\frac{1}{2}a_1^2-A_0a_1\right|,\frac{1}{2}A_2^2\right\},$$

therefore

$$\sup_{t\in[0,1]}|K_2^{[3]}(t)|=\max\left\{\frac{1}{2}A_0^2,\frac{1}{2}A_2^2,\frac{1}{2}a_1^2-A_0a_1\right\}.\tag{2.11}$$

Putting condition that $\sup_{t\in[0,1]}|K_2^{[3]}(t)|$ to attains the minimum value, which in our case is equivalent with $K_2^{[3]}(a_1)=-K_2^{[3]}(t_{\min})$, where $t_{\min}\in\{A_0,1-A_2\}$, we obtain $A_0=(\sqrt{2}-1)a_1$, respectively $A_2=(\sqrt{2}-1)(1-a_1)$. Since $K\in C[0,1]$, namely $K_2^{[3]}(a_1-0)=K_2^{[3]}(a_1+0)$, we can find the following relation

$$\frac{1}{2}a_1^2-(\sqrt{2}-1)a_1^2=\frac{1}{2}(1-a_1)^2-(\sqrt{2}-1)(1-a_1)^2.$$

From the above equality we obtain $a_1=\frac{1}{2}$ and the values for the coefficients of the optimal quadrature formula are $A_0=A_2=\frac{\sqrt{2}-1}{2}$, $A_1=2-\sqrt{2}$. From (2.11) it follows $\sup_{t\in[0,1]}|K_2^{[3]}(t)|=\frac{1}{2}A_0^2=\frac{3-2\sqrt{2}}{8}$. □

Remark 2.7. For the remainder term of quadrature formula (2.9) can be established the following two estimations

$$\begin{aligned}\left|\mathcal{R}_2^{[3]}[f]\right| \leq \int_0^1 |K_2^{[3]}(t)|dt \cdot \|f''\|_\infty &= \frac{37\sqrt{2}-52}{24}\|f''\|_\infty \\ &\approx 0.0136\|f''\|_\infty,\ f \in W_\infty^2[0,1], \\ \left|\mathcal{R}_2^{[3]}[f]\right| \leq \left[\int_0^1 \left(K_2^{[3]}(t)\right)^2 dt\right]^{\frac{1}{2}} \|f''\|_2 &= \frac{1}{8}\sqrt{\frac{78-55\sqrt{2}}{15}}\|f''\|_2 \\ &\approx 0.0151\|f''\|_2,\ f \in W_2^2[0,1].\end{aligned}$$

Remark 2.8. If we denote by $C_p^{[i]}$ the constants which appear in estimations of the following type

$$\left|\mathcal{R}_2^{[i]}[f]\right| \leq C_p^{[i]} \left\|f''\right\|_p,$$

where $i = 1,2,3$, $p = \infty, 2$, respectively 1, and $f \in W_p^2[0,1]$, from the above results the inequalities $C_\infty^{[1]} \leq C_\infty^{[2]} \leq C_\infty^{[3]}$, $C_2^{[2]} \leq C_2^{[1]} \leq C_2^{[3]}$ and $C_1^{[3]} \leq C_1^{[2]} \leq C_1^{[1]}$ are true. Therefore, we can assert that our results are better than Ujević and Mijić's result, if we consider 2-norm, respectively 1-norm.

3. The corrected quadrature formulae

In recent years some authors have considered so called perturbed (corrected) quadrature rules (see [2], [3], [4], [7], [8], [17]). By corrected quadrature rule we mean the formula which involves the values of the first derivative in end points of the interval not only the values of the function in certain points. These formulae have a higher degree of exactness than the original rule. The estimate of the error in corrected rule is better then in the original rule, in generally.

The main purpose of this section is to derive corrected rule of the optimal quadrature formulae obtained in previous section. Here we will show that the corrected formula improves the original formula. We mention that the corrected formula of (2.5) was considered by N. Ujević and L. Mijić in [16].

Let

$$\int_0^1 f(x)dx = A_0 f(0) + A_1 f\left(\frac{1}{2}\right) + A_2 f(1) + A\left[f'(1) - f'(0)\right] + \tilde{\mathcal{R}}_2[f], \tag{3.1}$$

where

$$\tilde{\mathcal{R}}_2[e_i] = 0,\ i = 0,1, \text{ and } A = \int_0^1 K_2(t)dt$$

be the corrected quadrature formula of the rule (2.1).

Since the remainder term has degree of exactness 1 we can write

$$\tilde{\mathcal{R}}_2[f] = \int_0^1 \tilde{K}_2(t) f''(t)dt, \text{ where} \tag{3.2}$$

$$\tilde{K}_2(t) = \tilde{\mathcal{R}}_2\left[(x-t)_+\right] = K_2(t) - A. \tag{3.3}$$

From the relation (3.3) we remark that $\int_0^1 \tilde{K}_2(t)dt = 0$. Now we will calculate the coefficient A from corrected optimal quadrature formulas obtained in previous section. If we consider $f(x) = \frac{x^2}{2}$ in (3.1) we find

$$A = \frac{1}{6} - \frac{1}{2}A_1 - \frac{1}{2}A_2. \tag{3.4}$$

Using relations (3.3) and (3.4) we construct the following corrected quadrature formula of (2.5), (2.7), respectively (2.9):

$$\begin{aligned}\int_0^1 f(x)dx &= \frac{\sqrt{2}}{8}f(0) + \frac{4-\sqrt{2}}{4}f\left(\frac{1}{2}\right) + \frac{\sqrt{2}}{8}f(1) \\ &+ \frac{4-3\sqrt{2}}{96}\left[f'(1) - f'(0)\right] + \tilde{\mathcal{R}}_2^{[1]}[f],\end{aligned} \tag{3.5}$$

where

$$\tilde{\mathcal{R}}_2^{[1]}[f] = \int_0^1 \tilde{K}_2^{[1]}(t)f''(t)dt, \tag{3.6}$$

$$\tilde{K}_2^{[1]}(t) = \begin{cases} \frac{1}{2}t^2 - \frac{\sqrt{2}}{8}t - \frac{4-3\sqrt{2}}{96}, & 0 \le t \le \frac{1}{2} \\ \frac{1}{2}(1-t)^2 - \frac{\sqrt{2}}{8}(1-t) - \frac{4-3\sqrt{2}}{96}, & \frac{1}{2} < t \le 1, \end{cases}$$

$$\int_0^1 f(x)dx = \frac{3}{16}f(0) + \frac{5}{8}f\left(\frac{1}{2}\right) + \frac{3}{16}f(1) - \frac{1}{192}\left[f'(1) - f'(0)\right] + \tilde{\mathcal{R}}_2^{[2]}[f], \tag{3.7}$$

where

$$\tilde{\mathcal{R}}_2^{[2]}[f] = \int_0^1 \tilde{K}_2^{[2]}(t)f''(t)dt, \tag{3.8}$$

$$\tilde{K}_2^{[2]}(t) = \begin{cases} \frac{1}{2}t^2 - \frac{3}{16}t + \frac{1}{192}, & 0 \le t \le \frac{1}{2} \\ \frac{1}{2}(1-t)^2 - \frac{3}{16}(1-t) + \frac{1}{192}, & \frac{1}{2} < t \le 1, \end{cases}$$

respectively

$$\begin{aligned}\int_0^1 f(x)dx &= \frac{\sqrt{2}-1}{2}f(0) + (2-\sqrt{2})f\left(\frac{1}{2}\right) + \frac{\sqrt{2}-1}{2}f(1) \\ &+ \frac{4-3\sqrt{2}}{24}\left[f'(1) - f'(0)\right] + \tilde{\mathcal{R}}_2^{[3]}[f],\end{aligned} \tag{3.9}$$

where

$$\tilde{\mathcal{R}}_2^{[3]}[f] = \int_0^1 \tilde{K}_2^{[3]}(t)f''(t)dt, \tag{3.10}$$

$$\tilde{K}_2^{[3]}(t) = \begin{cases} \frac{1}{2}t^2 - \frac{\sqrt{2}-1}{2}t - \frac{4-3\sqrt{2}}{24}, & 0 \le t \le \frac{1}{2} \\ \frac{1}{2}(1-t)^2 - \frac{\sqrt{2}-1}{2}(1-t) - \frac{4-3\sqrt{2}}{24}, & \frac{1}{2} < t \le 1, \end{cases}$$

Denote by $\tilde{C}_p^{[i]}$ the constant which appear in estimations of the remainder term of corrected quadrature formulas, namely

$$\left|\tilde{\mathcal{R}}_2^{[i]}[f]\right| \le \tilde{C}_p^{[i]} \left\|f''\right\|_p ,$$

where $i = 1,2,3$, $p = \infty, 2$, respectively 1, and $f \in W_p^2[0,1]$. The constants $\tilde{C}_p^{[i]}$ can be calculated in a similar way with the constants $C_p^{[i]}$ defined in Remark 2.8. From the bellow table follows that for $p = \infty$ and $p = 2$ the corrected formula improves the original formula.

$\diagdown i$	1	2	3
$C_\infty^{[i]}$	$\frac{2-\sqrt{2}}{48} \approx 0.0122$	$\frac{19}{1536} \approx 0.0124$	$\frac{37\sqrt{2}-52}{24} \approx 0.0136$
$\tilde{C}_\infty^{[i]}$	$\frac{5}{96}\sqrt{6} - \frac{29}{432}\sqrt{3} \approx 0.0113$	$\frac{19}{13824}\sqrt{57} \approx 0.0104$	$\frac{\sqrt{3(13-9\sqrt{2})^3}}{27} \approx 0.0091$
$C_2^{[i]}$	$\frac{1}{16}\sqrt{\frac{22-15\sqrt{2}}{15}} \approx 0.0143$	$\frac{\sqrt{5}}{160} \approx 0.0140$	$\frac{1}{8}\sqrt{\frac{78-55\sqrt{2}}{15}} \approx 0.0151$
$\tilde{C}_2^{[i]}$	$\frac{\sqrt{470-300\sqrt{2}}}{480} \approx 0.0141$	$\frac{\sqrt{155}}{960} \approx 0.0130$	$\left(\frac{1}{90} - \frac{1}{128}\sqrt{2}\right) \approx 0.0112$
$C_1^{[i]}$	$\frac{2-\sqrt{2}}{16} \approx 0.0366$	$\frac{1}{32} \approx 0.0313$	$\frac{3-2\sqrt{2}}{8} \approx 0.0214$
$\tilde{C}_1^{[i]}$	$\left(\frac{1}{12} - \frac{1}{32}\sqrt{2}\right) \approx 0.0391$	$\frac{7}{192} \approx 0.0365$	$\left(\frac{5}{24} - \frac{1}{8}\sqrt{2}\right) \approx 0.0316$

Theorem 3.1. *Let $f : [0,1] \to \mathbb{R}$ be an absolutely continuous function such that $f'' \in L[0,1]$ and there exist real number $m[f], M[f]$ such that $m[f] \le f''(t) \le M[f]$, $t \in [0,1]$. Then*

$$\left|\tilde{\mathcal{R}}_2^{[1]}[f]\right| \le \frac{M[f]-m[f]}{2}\left(\frac{5\sqrt{6}}{96} - \frac{29\sqrt{3}}{432}\right) \approx 11306 \times 10^{-6} \cdot \frac{M[f]-m[f]}{2}, \tag{3.11}$$

$$\left|\tilde{\mathcal{R}}_2^{[2]}[f]\right| \le \frac{M[f]-m[f]}{2} \cdot \frac{19\sqrt{57}}{13824} \approx 10377 \times 10^{-6} \cdot \frac{M[f]-m[f]}{2}, \tag{3.12}$$

$$\left|\tilde{\mathcal{R}}_2^{[3]}[f]\right| \le \frac{M[f]-m[f]}{2} \cdot \frac{\sqrt{3(13-9\sqrt{2})^3}}{27} \approx 9104 \times 10^{-6} \cdot \frac{M[f]-m[f]}{2}. \tag{3.13}$$

If there exist a real number $m[f]$ such that $m[f] \le f''(t)$, $t \in [0,1]$, then

$$\begin{aligned} \left|\tilde{\mathcal{R}}_2^{[1]}[f]\right| &\le \frac{1}{4}\left(\frac{1}{3} - \frac{\sqrt{2}}{8}\right)(f'(1) - f'(0) - m[f]) \\ &\approx 39139 \times 10^{-6}\,(f'(1) - f'(0) - m[f]), \end{aligned} \tag{3.14}$$

$$\left|\tilde{\mathcal{R}}_2^{[2]}[f]\right| \le \frac{7}{192}\,(f'(1)-f'(0)-m[f]) \approx 36458\times 10^{-6}\,(f'(1)-f'(0)-m[f]), \tag{3.15}$$

$$\left|\tilde{\mathcal{R}}_2^{[3]}[f]\right| \le \frac{5-3\sqrt{2}}{24}\cdot(f'(1)-f'(0)-m[f]) \approx 31557\times 10^{-6}\,(f'(1)-f'(0)-m[f]). \tag{3.16}$$

If there exist a real number $M[f]$ such that $f''(t)\le M[f]$, $t\in[0,1]$, then

$$\left|\tilde{\mathcal{R}}_2^{[1]}[f]\right| \le \frac{1}{4}\left(\frac{1}{3}-\frac{\sqrt{2}}{8}\right)[M[f]-(f'(1)-f'(0))] \approx 39139\times 10^{-6}\,[M[f]-(f'(1)-f'(0))],$$

$$\left|\tilde{\mathcal{R}}_2^{[2]}[f]\right| \le \frac{7}{192}\,[M[f]-(f'(1)-f'(0))] \approx 36458\times 10^{-6}\,[M[f]-(f'(1)-f'(0))],$$

$$\left|\tilde{\mathcal{R}}_2^{[3]}[f]\right| \le \frac{5-3\sqrt{2}}{24}\,[M[f]-(f'(1)-f'(0))] \approx 31557\times 10^{-6}\,[M[f]-(f'(1)-f'(0))].$$

Proof. Since $\int_0^1 \tilde{K}_2(t)dt=0$, the remainder term (3.2) can be written in the following way

$$\tilde{\mathcal{R}}_2[f]=\int_0^1 \tilde{K}_2(t)\left(f''(t)-\frac{M[f]+m[f]}{2}\right)dt.$$

Therefore

$$\left|\tilde{\mathcal{R}}_2[f]\right| \le \left\|f''-\frac{M[f]+m[f]}{2}\right\|_\infty\cdot\|\tilde{K}_2\|_1 \le \frac{M[f]-m[f]}{2}\cdot\|\tilde{K}_2\|_1.$$

Calculating the norm of the kernel $\tilde{K}_2$ from the integral representation of the remainder term (3.6), (3.8) and (3.10), respectively, the first part of the theorem is proved.

To prove the relations (3.14), (3.15) and (3.16) respectively, we consider the following estimation of remainder term

$$\left|\tilde{\mathcal{R}}_2[f]\right| = \left|\int_0^1 \tilde{K}_2(t)\,(f''(t)-m)\,dt\right| \le \sup_{t\in[0,1]}|\tilde{K}_2(t)|\cdot\int_0^1 (f''(t)-m)\,dt = \left\|\tilde{K}_2\right\|_\infty\cdot(f'(1)-f'(0)-m).$$

The last part of the theorem can be proved using the following estimation of remainder term

$$\left|\tilde{\mathcal{R}}_2[f]\right| = \left|\int_0^1 \tilde{K}_2(t)\,(f''(t)-M)\,dt\right| \le \sup_{t\in[0,1]}|\tilde{K}_2(t)|\cdot\int_0^1 (M-f''(t))\,dt = \left\|\tilde{K}_2\right\|_\infty\cdot[M-(f'(1)-f'(0))].$$

□

Let $f, g : [a, b] \to \mathbb{R}$ be integrable functions on $[a, b]$. The functional

$$T(f,g) := \frac{1}{b-a}\int_a^b f(t)g(t)dt - \frac{1}{b-a}\int_a^b f(t)dt \cdot \frac{1}{b-a}\int_a^b g(t)dt, \tag{3.17}$$

is well known in the literature as the Čebyšev functional. It was proved that $T(f,f) \geq 0$ and the inequality $|T(f,g)| \leq \sqrt{T(f,f)} \cdot \sqrt{T(g,g)}$ holds. Denote by $\sigma(f,a,b) = \sqrt{T(f,f)}$.

Theorem 3.2. *Let $f : [0,1] \to \mathbb{R}$ be an absolutely continuous function such that $f'' \in L_2[0,1]$. Then*

$$\left|\tilde{\mathcal{R}}_2^{[1]}[f]\right| \leq \sqrt{\frac{47}{23040} - \frac{\sqrt{2}}{768}} \cdot \sigma(f'';0,1) \approx 14089 \times 10^{-6}\sigma(f'';0,1), \tag{3.18}$$

$$\left|\tilde{\mathcal{R}}_2^{[2]}[f]\right| \leq \frac{\sqrt{155}}{960} \cdot \sigma(f'';0,1) \approx 12969 \times 10^{-6}\sigma(f'';0,1), \tag{3.19}$$

$$\left|\tilde{\mathcal{R}}_2^{[3]}[f]\right| \leq \frac{1}{120}\sqrt{320-225\sqrt{2}} \cdot \sigma(f'';0,1) \approx 11186 \times 10^{-6}\sigma(f'';0,1). \tag{3.20}$$

Proof. The remainder term of the corrected quadrature formula (3.2) can be written in such way

$$\tilde{\mathcal{R}}_2[f] = \int_0^1 \tilde{K}_2(t)f''(t)dt = \int_0^1 \left[K_2(t) - \int_0^1 K_2(t)dt\right] f''(t)dt$$
$$= \int_0^1 K_2(t)f''(t)dt - \int_0^1 K_2(t)dt \cdot \int_0^1 f''(t)dt = T(K_2, f'').$$

From the above relation we obtain

$$\left|\tilde{\mathcal{R}}_2[f]\right| = |T(K_2, f'')| \leq \sqrt{T(K_2,K_2)}\sqrt{T(f'',f'')} = \sigma(K_2;0,1) \cdot \sigma(f'';0,1).$$

Calculating $\sigma(K_2;0,1)$ for the kernel defined in (2.6), (2.8) and (2.10), respectively, the theorem is proved. □

Remark 3.3. The inequalities (3.18), (3.19) and (3.20), respectively, are sharp in the sense that the constants $\sqrt{\frac{47}{23040} - \frac{\sqrt{2}}{768}}$, $\frac{\sqrt{155}}{960}$ and $\frac{1}{120}\sqrt{320 - 225\sqrt{2}}$ respectively, cannot be replaced by a smaller ones. To prove that we define the functions

$$F^{[i]}(x) = \int_0^x \left(\int_0^t K_2^{[i]}(u)du\right) dt, \quad i = 1,2,3. \tag{3.21}$$

For the function (3.21) the right-hand side of the inequalities (3.18), (3.19) and (3.20), respectively are equal with $T(K_2^{[i]}, K_2^{[i]})$, $i = 1,2,3$, respectively, and the left-hand side becomes

$$\left|\tilde{\mathcal{R}}_2^{[i]}[f]\right| = \left|\int_0^1 \tilde{K}_2^{[i]}(t) \cdot K_2^{[i]}(t)dt\right| =$$
$$\left|\int_0^1 \left[K_2^{[i]}(t) - \int_0^1 K_2^{[i]}(t)dt\right] K_2^{[i]}(t)dt\right| = T(K_2^{[i]}, K_2^{[i]}), \; i = 1,2,3.$$

Remark 3.4. Denote by Z_i, $i = 1,2,3$, the constants which appear in one of the following types of estimations obtained in Theorem 3.1 and Theorem 3.2, namely

$$\left|\tilde{\mathcal{R}}_2^{[i]}[f]\right| \leq Z_i \cdot \frac{M[f]-m[f]}{2},$$

$$\left|\tilde{\mathcal{R}}_2^{[i]}[f]\right| \leq Z_i \cdot (f'(1)-f'(0)-m[f]),$$

$$\left|\tilde{\mathcal{R}}_2^{[i]}[f]\right| \leq Z_i \cdot (M[f]-[f'(1)-f'(0)])$$

or

$$\left|\tilde{\mathcal{R}}_2^{[i]}[f]\right| \leq Z_i \cdot \sigma(f'';0,1).$$

Since for every $i = 1,2,3$ we have $Z_3 \leq Z_2 \leq Z_1$, for the corrected quadrature formulas, our results are better than Ujević and Mijić's results obtained in [16].

The corrected quadrature formulas (3.5), (3.7), and (3.9), respectively have degree of exactness 3, which is higher than the original rule, namely for $j = \overline{1,3}$, $\tilde{R}_2^{[j]}[e_i] = 0$ and $\tilde{R}_2^{[j]}[e_4] \neq 0$, where $e_i(x) = x^i$, $i = \overline{0,4}$. Using Peano's Theorem, the remainder term can be written

$$\mathcal{R}_4[f] = \int_0^1 K_4(t) f^{(4)}(t),\ \ K_4(t) = \mathcal{R}_4\left[\frac{(x-t)_+^3}{3!}\right], \tag{3.22}$$

where by $\mathcal{R}_4$ we denote the new integral representation of the remainder term of these quadrature formulas.

In the next part of this paper, using relation (3.22), we will give new estimations of the remainder term in quadrature formulas (3.5), (3.7), and (3.9), respectively.

Theorem 3.5. *If $f \in C^4[0,1]$, then the remainder term of quadrature formula (3.5)has the integral representation*

$$\mathcal{R}_4^{[1]}[f] = \int_0^1 K_4^{[1]}(t) f^{(4)}(t)dt,\ \ \textit{where}$$

$$K_4^{[1]}(t) = \begin{cases} \dfrac{1}{24}t^2\left(t^2 - \dfrac{\sqrt{2}}{2}t - \dfrac{4-3\sqrt{2}}{8}\right), & 0 \leq t \leq \dfrac{1}{2}, \\[2ex] \dfrac{1}{24}(1-t)^2\left((1-t)^2 - \dfrac{\sqrt{2}}{2}(1-t) - \dfrac{4-3\sqrt{2}}{8}\right), & \dfrac{1}{2} < t \leq 1, \end{cases}$$

and the following estimations hold

$$\begin{aligned}\left|\mathcal{R}_4^{[1]}[f]\right| &\leq \sqrt{\int_0^1 \left(K_4^{[1]}(t)\right)^2 dt}\sqrt{\int_0^1 \left(f^{(4)}(t)\right)^2 dt} \\ &= \frac{\sqrt{23170-15645\sqrt{2}}}{80640}\|f^{(4)}\|_2 \approx 4.008\times 10^{-4}\|f^{(4)}\|_2,\end{aligned}$$

$$\begin{aligned}\left|\mathcal{R}_4^{[1]}[f]\right| &\leq \int_0^1 \left|K_4^{[1]}(t)\right| dt\cdot \sup_{t\in[0,1]}|f^{(4)}(t)| \\ &= \frac{200-171\sqrt{2}-(90-68\sqrt{2})\sqrt{5-3\sqrt{2}}+2(15-8\sqrt{2})\sqrt{43-30\sqrt{2}}}{11520}\|f^{(4)}\|_\infty \\ &\approx 2.946\times 10^{-4}\cdot\|f^{(4)}\|_\infty,\end{aligned}$$

$$\begin{aligned}\left|\mathcal{R}_4^{[1]}[f]\right| &\leq \sup_{t\in[0,1]}|K_4^{[1]}(t)|\cdot\int_0^1 |f^{(4)}(t)|dt \\ &= \frac{2-\sqrt{2}}{768}\cdot\|f^{(4)}\|_1 \approx 7.627\times 10^{-4}\cdot\|f^{(4)}\|_1.\end{aligned}$$

Theorem 3.6. *If $f\in C^4[0,1]$, then the remainder term of quadrature formula (3.7)has the integral representation*

$$\mathcal{R}_4^{[2]}[f] = \int_0^1 K_4^{[2]}(t)f^{(4)}(t)dt, \text{ where}$$

$$K_4^{[2]}(t) = \begin{cases} \frac{1}{24}t^2\left(t^2-\frac{3}{4}t+\frac{1}{16}\right), & 0\leq t\leq \frac{1}{2}, \\ \frac{1}{24}(1-t)^2\left((1-t)^2-\frac{3}{4}(1-t)+\frac{1}{16}\right), & \frac{1}{2}<t\leq 1,\end{cases}$$

and the following estimations hold

$$\begin{aligned}\left|\mathcal{R}_4^{[2]}[f]\right| &\leq \sqrt{\int_0^1 \left(K_4^{[2]}(t)\right)^2 dt}\sqrt{\int_0^1 \left(f^{(4)}(t)\right)^2 dt} \\ &= \frac{\sqrt{2905}}{161280}\|f^{(4)}\|_2 \approx 3.342\times 10^{-4}\|f^{(4)}\|_2,\end{aligned}$$

$$\begin{aligned}\left|\mathcal{R}_4^{[2]}[f]\right| &\leq \int_0^1 \left|K_4^{[2]}(t)\right| dt\cdot \sup_{t\in[0,1]}|f^{(4)}(t)| \\ &= \frac{125\sqrt{5}-103}{737280}\cdot\|f^{(4)}\|_\infty \approx 2.394\times 10^{-4}\cdot\|f^{(4)}\|_\infty,\end{aligned}$$

$$
\begin{aligned}
\left|\mathcal{R}_4^{[2]}[f]\right| &\leq \sup_{t\in[0,1]} |K_4^{[2]}(t)| \cdot \int_0^1 |f^{(4)}(t)|dt \\
&= \frac{1}{1536} \cdot \|f^{(4)}\|_1 \approx 6.51 \times 10^{-4} \cdot \|f^{(4)}\|_1.
\end{aligned}
$$

Theorem 3.7. *If $f \in C^4[0,1]$, then the remainder term of quadrature formula (3.9)has the integral representation*

$$
\mathcal{R}_4^{[3]}[f] = \int_0^1 K_4^{[3]}(t) f^{(4)}(t)dt, \text{ where}
$$

$$
K_4^{[3]}(t) = \begin{cases} \dfrac{1}{24}t^2\left(t^2 - 2(\sqrt{2}-1)t - \dfrac{4-3\sqrt{2}}{2}\right), \; 0 \leq t \leq \dfrac{1}{2}, \\[2ex] \dfrac{1}{24}(1-t)^2\left((1-t)^2 - 2(\sqrt{2}-1)(1-t) - \dfrac{4-3\sqrt{2}}{2}\right), \; \dfrac{1}{2} < t \leq 1, \end{cases}
$$

and the following estimations hold

$$
\begin{aligned}
\left|\mathcal{R}_4^{[3]}[f]\right| &\leq \sqrt{\int_0^1 \left(K_4^{[3]}(t)\right)^2 dt}\sqrt{\int_0^1 \left(f^{(4)}(t)\right)^2 dt} \\
&= \frac{\sqrt{68530 - 48405\sqrt{2}}}{40320}\|f^{(4)}\|_2 \approx 2.148 \times 10^{-4}\|f^{(4)}\|_2,
\end{aligned}
$$

$$
\begin{aligned}
\left|\mathcal{R}_4^{[3]}[f]\right| &\leq \int_0^1 \left|K_4^{[3]}(t)\right| dt \cdot \sup_{t\in[0,1]} |f^{(4)}(t)| \\
&= \frac{78470 - 55487\sqrt{2} - 32(550 - 389\sqrt{2})\sqrt{10-7\sqrt{2}}}{5760} \cdot \|f^{(4)}\|_\infty \\
&\approx 1.461 \times 10^{-4} \cdot \|f^{(4)}\|_\infty,
\end{aligned}
$$

$$
\begin{aligned}
\left|\mathcal{R}_4^{[3]}[f]\right| &\leq \sup_{t\in[0,1]} |K_4^{[3]}(t)| \cdot \int_0^1 |f^{(4)}(t)|dt \\
&= \frac{3-2\sqrt{2}}{384} \cdot \|f^{(4)}\|_1 \approx 4.468 \times 10^{-4} \cdot \|f^{(4)}\|_1.
\end{aligned}
$$

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Solving nonlinear oscillators using a modified homotopy analysis method

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Abstract. In this paper, a new algorithm called the modified homotopy analysis method (MHAM) is presented to solve a nonlinear oscillators. The proposed scheme is based on the homotopy analysis method (HAM), Laplace transform and Padé approximants. Several illustrative examples are given to demonstrate the effectiveness of the present method. Results obtained using the scheme presented here agree well with those derived from the modified homotopy perturbation method (MHPM). The results reveal that the MHAM is an effective and convenient for solving nonlinear differential equations.

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Keywords: Nonlinear oscillators, homotopy analysis method, Laplace transform; Padé approximants.

1. Introduction

The study of nonlinear oscillators is of crucial importance in all areas of physics and engineering, as well as in other disciplines. It is very difficult to solve nonlinear problems and, in general, it is often more difficult to get an analytic approximation than a numerical one to a given nonlinear problem. Several methods have been used to find approximate solutions to strongly nonlinear oscillators. Such methods include variational iteration method [1, 2, 3, 4, 5, 6, 7], Adomian decomposition method [8, 9], differential transform method [10], and harmonic balance based methods [11, 12, 13, 14, 15]. Surveys of the literature with multitudinous references and useful bibliographies have been given in Refs. [16, 17]. Recently, Momani et al [18], proposed a powerful analytic method, namely modified homotopy perturbation method. This method is based on the homotopy perturbation method, the Laplace transformation and Padé approximants. The approximate solution of the MHPM displays the periodic behavior which is characteristic of the oscillatory equations. The homotopy analysis method (HAM) [19] yields rapidly convergent series solutions by using few iterations for both linear and nonlinear differential equations. The HAM was successfully applied to solve many nonlinear problems such as Riccati differential equation of fractional order [20], fractional KdV-Burgers-Kuramoto equation [21], systems of fractional algebraic-differential equations [22], and so on. In this paper, we developed a symbolic algorithm to find the solution of nonlinear oscillators by a modified homotopy analysis method (MHAM). The MHAM is based on the homotopy analysis method (HAM), Laplace transform and Padé approximants. Finally, we make a numerical comparison between our method and the MHPM. The structure of this paper is as follows. In section 2 we describe the homotopy analysis method and briefly discuss Padé approximants. In Section 3 we present three examples to show the efficiency and simplicity of the method.

2. Homotopy analysis method

The HAM has been extended by many authors to solve linear and nonlinear fractional differential equations [19, 20, 21, 22]. In this section the basic ideas of the homotopy analysis method are introduced. To show the basic idea, let us consider the following nonlinear oscillator equation

$$y^{''}(t) + c_1 y(t) + c_2 y^2(t) + c_3 y^3(t) = \epsilon F(t, y(t), y^{'}(t)),\ t \geq 0, \tag{2.1}$$

subject to the initial conditions

$$y(0) = a,\ y^{'}(0) = b, \tag{2.2}$$

where $c_i,\ i = 1, 2, 3,$ are positive real numbers and ϵ is a parameter (not necessarily small). We assume that the function $F(t, y(t), y^{'}(t))$ is an arbitrary nonlinear function of its arguments. Now, we can construct the so-called zero-order deformation equations of the equation (2.1) by

$$\begin{aligned}(1-q)L[\phi(t;q) - y_0(t)] &= q\ \hbar[\frac{d^2}{dt^2}\phi(t;q) + c_1\phi(t;q) + c_2\phi^2(t;q) \\ &\quad + c_3\phi^3(t;q) - \epsilon F(t, \phi(t;q), \frac{d}{dt}\phi(t;q))],\end{aligned} \tag{2.3}$$

where $q \in [0, 1]$ is an embedding parameter, L is an auxiliary linear operator satisfy $L(0) = 0$, $y_0(t)$ is an initial guess satisfies the initial condition (2.2), $\hbar \neq 0$ is an auxiliary parameter and $\phi(t; q)$ is an unknown function. Obviously, when $q = 0$ and when $q = 1$, we have $\phi(t; 0) = y_0(t)$ and $\phi(t; 1) = y(t)$.Thus as q increasing from 0 to 1, $\phi(t; q)$ varies from $y_0(t)$ to $y(t)$. Expanding $\phi(t; q)$ in Taylor series with respect to q, one has

$$\phi(t;q) = y_0(t) + \sum_{m=1}^{\infty} y_m(t) q^m, \tag{2.4}$$

where

$$y_m(t) = \frac{1}{m!}\frac{\partial^m \phi(t;q)}{\partial q^m}|_{q=0}. \tag{2.5}$$

If the auxiliary parameter h and the initial guess $y_0(t)$ are so properly chosen, then the series (2.4) converges at $q = 1$, one has

$$y(t) = y_0(t) + \sum_{m=1}^{\infty} y_m(t). \tag{2.6}$$

Define the vector

$$\overrightarrow{y}_m(t) = \{y_0(t), y_1(t), \ldots, y_m(t)\}. \tag{2.7}$$

Differentiating the zero-order deformation equation (2.3) m times with respective to q, then setting $q = 0$ and finally dividing them by $m!$, we have the so-called mth-order deformation equations

$$L[y_m(t) - \chi_m y_{(m-1)}(t)] = \hbar \Re_m(\overrightarrow{y}_{m-1}(t)), \tag{2.8}$$

where

$$\begin{aligned} \Re_m(\overrightarrow{y}_{m-1}(t)) &= y''_{m-1}(t) + c_1 y_{m-1}(t) + c_2 \sum_{i=0}^{m-1} y_i(t) y_{m-i-1}(t) \\ &\quad + c_3 \sum_{i=0}^{m-1} y_{m-i-1}(t) \sum_{j=0}^{i} y_j(t) y_{i-j}(t) \\ &\quad - \frac{\epsilon}{(m-1)!} \frac{\partial^{m-1}}{\partial q^{m-1}} [F(t, \phi(t;q), \frac{d}{dt}\phi(t;q))]|_{q=0}, \end{aligned} \tag{2.9}$$

and

$$\chi_m = \begin{cases} 0, & m \le 1 \\ 1, & m > 1. \end{cases} . \tag{2.10}$$

The mth-order approximation of $y(t)$ is given by $y(t) = \sum_{i=0}^{m} y_i(t)$. This power series can be transformed into Padé series easily. Padé series is defined in the following

$$a_0 + a_1 x + a_2 x^2 + \ldots = \frac{p_0 + p_1 x + \ldots + p_M x^M}{1 + q_1 x + \ldots + q_L x^L}. \tag{2.11}$$

Multiply both sides of (2.11) by the denominator of right-hand side in (2.11). We have

$$\begin{aligned} a_l + \sum_{k=l}^{M} a_{l-k} q_k &= p_l, \ (l = 0, 1, \ldots, M), \\ a_l + \sum_{k=l}^{L} a_{l-k} q_k &= 0, \ (l = M+1, \ldots, M+L). \end{aligned} \tag{2.12}$$

Solving the linear equation in (2.12), we have q_k $(k = 1, \ldots, L)$, and substituting into (2.11), we have p_k $(l = 1, \ldots, L)$ [23]. We use Mathematica to obtain diagonal Padé approximants of various orders, such as [2/2] or [4/4].

3. Numerical results

To demonstrate the effectiveness of the method we consider the following three examples of nonlinear oscillator equation.

3.1. Example 1

Consider the following Helmholtz equation

$$y^{''}(t) + 2y(t) + y^2(t) = 0, \quad t > 0, \tag{3.1}$$

subject to the initial conditions

$$y(0) = 0.1, \ y^{'}(0) = 0. \tag{3.2}$$

We start with initial approximation

$$y_0(t) = 0.1. \tag{3.3}$$

In view of the algorithm presented in the previous section, if we select the auxiliary linear operator as

$$L = \frac{d^2}{dt^2} \tag{3.4}$$

we can construct the homotopy as

$$y_m(t) = \chi_m y_{(m-1)}(t) + \hbar \int_0^t (t-\tau)\Re_m(\overrightarrow{y}_{m-1}(t))d\tau, \tag{3.5}$$

where

$$\Re_m(\overrightarrow{y}_{m-1}(t)) = y^{''}_{m-1}(t) + 2y_{m-1}(t) + \sum_{i=0}^{m-1} y_i(t)y_{m-i-1}(t). \tag{3.6}$$

Using formula (3.5), the fifth-term approximate solution for equation (3.1) is given by

$$\begin{aligned} y(t) \quad = \quad & 0.1 + 0.42\hbar t^2 + 0.63\hbar^2 t^2 + 0.42\hbar^3 t^2 + 0.105\hbar^4 t^2 + 0.1155\hbar^2 t^4 \\ & +0.154\hbar^3 t^4 + 0.05775\hbar^4 t^4 + 0.00711667\hbar^3 t^6 + 0.0053375\hbar^4 t^6 \\ & +0.000142083\hbar^4 t^8. \end{aligned} \tag{3.7}$$

Setting $\hbar = -1$ in Eq (3.7), then we have

$$y(t) = 0.1 - 0.105t^2 + 0.0925t^4 - 0.00177917t^6 + 0.000142083t^8. \tag{3.8}$$

In order to improve the accuracy of the homotopy analysis solution of the Helmholtz equation we need to implement the following technique. First applying the Laplace transformation to the previous series solution, then we get

$$\widetilde{y}_m(s) = \frac{0.1}{s} - \frac{0.21}{s^3} + \frac{0.462}{s^5} - \frac{1.281}{s^7} + \frac{5.7288}{s^9}. \tag{3.9}$$

Now, let $s = \frac{1}{t}$ in (3.9), then we have

$$\widetilde{y}_m(t) = 0.1t - 0.21t^3 + 0.462t^5 - 1.281t^7 + 5.7288t^9.$$

The [4/4] Padé approximation gives

$$\left[\frac{4}{4}\right] = \frac{0.1t + 1.27t^3}{1 + 14.8t^2 + 26.46t^4}.$$

Recalling $t = 1/s$, we obtain [4/4] in terms of s

$$\left[\frac{4}{4}\right] = \frac{1.27s + 0.1s^3}{26.46 + 14.8s^2 + s^4}.$$

By using the inverse Laplace transformation to the [4/4] Padé approximation, we obtain the same solution obtained in Momani et al. [18] using the modified homotopy perturbation method

$$y(t) = 0.0998141\cos(1.4423t) + 0.000185858\cos(3.56648t). \qquad (3.10)$$

Figure 1 shows the series solution (3.10) exhibit the periodic behavior which is characteristic of the oscillatory Helmholtz equation (3.1) and (3.2).

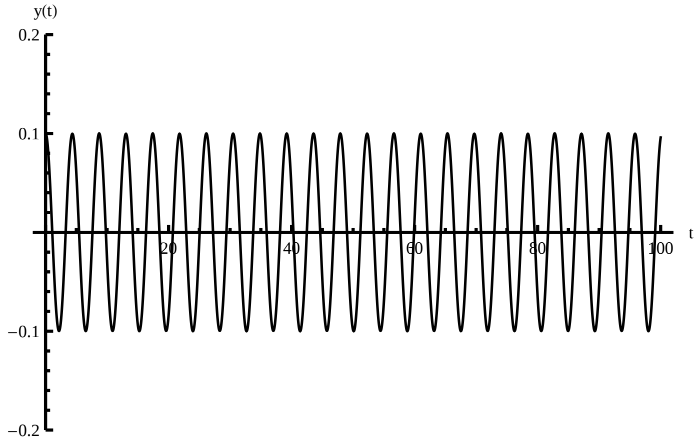

Figure 1. Plots of Eq. (3.10)

3.2. Example 2

Consider the following nonlinear equation

$$y''(t) + y(t) = -0.1y^2(t)y'(t), \quad t > 0, \qquad (3.11)$$

subject to the initial conditions

$$y(0) = 1, \; y'(0) = 0. \qquad (3.12)$$

Select the initial guess as

$$y_0(t) = 1, \qquad (3.13)$$

and the auxiliary linear operator (3.4), then we have the homotopy (3.5) where

$$\Re_m(\overrightarrow{y}_{m-1}(t)) = y''_{m-1}(t) + y_{m-1}(t) + 0.1\sum_{i=0}^{m-1} y'_{m-i-1}(t)\sum_{j=0}^{i} y_j(t)y_{i-j}(t). \qquad (3.14)$$

Using formula (3.5), the fifth-term approximate solution for equation (3.11) is given by

$$\begin{aligned} y(t) &= 1+2\hbar t^2+3\hbar^2t^2+2\hbar^3t^2+0.5\hbar^4t^2+0.1\hbar^2t^3+0.133\hbar^3t^3 \\ &\quad +0.05\hbar^4t^3+0.25\hbar^2t^4+0.34\hbar^3t^4+0.126\hbar^4t^4+0.027\hbar^3t^5 \\ &\quad +0.020008\hbar^4t^5+0.00555556\hbar^3t^6+0.0045694\hbar^4t^6 \\ &\quad +0.001369\hbar^4t^7+0.0000248\hbar^4t^8. \end{aligned} \tag{3.15}$$

Setting $\hbar=-1$ in Eq (3.15), then we have

$$\begin{aligned} y(t) &= 1-0.5t^2+0.01667t^3+0.0413t^4-0.00666t^5 \\ &\quad -0.00098611t^6+0.0013691t^7+0.0000248t^8. \end{aligned} \tag{3.16}$$

Applying the Laplace transformation to the previous series solution, then we get

$$\widetilde{y}_m(s)=\frac{1}{s}-\frac{1}{s^3}+\frac{0.1}{s^4}+\frac{0.99}{s^5}-\frac{0.799}{s^6}-\frac{0.71}{s^7}+\frac{6.9}{s^8}+\frac{1}{s^9}. \tag{3.17}$$

Let $s=\frac{1}{t}$ in (3.17), then we have

$$\widetilde{y}_m(t)=t-t^3+0.1t^4+0.99t^5-0.799t^6-0.71t^7+6.9t^8+t^9.$$

The [4/4] Padé approximation gives

$$\left[\frac{4}{4}\right]=\frac{t+0.3335t^2+9.16122t^3+0.313787t^4}{1+0.3335t+10.1612t^2+0.547287t^3+9.13787t^4}.$$

Recalling $t=1/s$, we obtain [4/4] in terms of s

$$\left[\frac{4}{4}\right]=\frac{0.313787+9.16122s+0.3335s^2+s^3}{9.13787+0.547287s+10.1612s^2+0.3335s^3+s^4}.$$

By using the inverse Laplace transformation to the [4/4] Padé approximation, we obtain the same solution obtained in Momani et al. [18] using the modified homotopy perturbation method

$$\begin{aligned} y(t) &= e^{(-0.013-0.999i)t}((0.5+0.0111i)+(0.5-0.0111i)e^{-1.99it}) \\ &\quad +e^{(-0.15-3.02i)t}((0.0003-0.002i)+(0.0003+0.002i)e^{-6it}). \end{aligned} \tag{3.18}$$

The equation (3.11) called the "unplugged" van der Pol, and all its solutions are expected to oscillate with decreasing to zero. Figure 2 shows the series solution (3.18) of the oscillatory nonlinear equation (3.11) and (3.12).

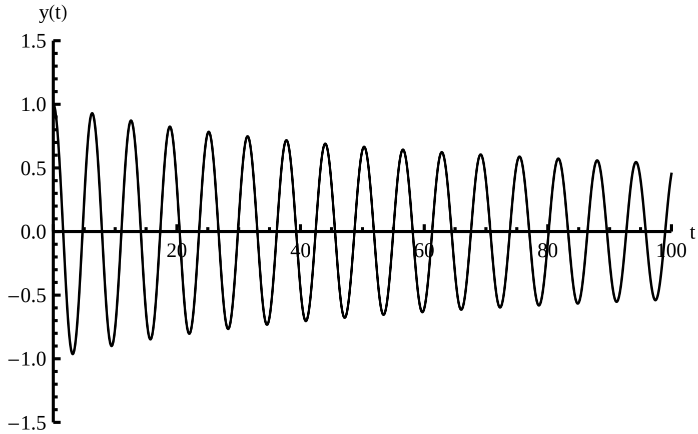

Figure 2. Plots of Eq. (3.18)

3.3. Example 3

Consider the following nonlinear equation

$$y''(t) + y(t) + 0.45y^2(t) = y(t)y'(t), \quad t > 0, \tag{3.19}$$

subject to the initial conditions

$$y(0) = 0.1,\ y'(0) = 0. \tag{3.20}$$

Take

$$y_0(t) = 0.1, \tag{3.21}$$

and the auxiliary linear operator (3.4), then we have the homotopy (3.5) where

$$\begin{aligned} \Re_m(\overrightarrow{y}_{m-1}(t)) &= y''_{m-1}(t) + y_{m-1}(t) + 0.45 \sum_{i=0}^{m-1} y_i(t) y_{m-i-1}(t) \\ &\quad - \sum_{i=0}^{m-1} y'_i(t) y_{m-i-1}(t). \end{aligned} \tag{3.22}$$

The fifth-term approximate solution for equation (3.19) is given by

$$\begin{aligned} y(t) &= 0.1 + 0.21\hbar t^2 + 0.31\hbar^2 t^2 + 0.21\hbar^3 t^2 + 0.05\hbar^4 t^2 \\ &\quad -0.01\hbar^2 t^3 - 0.0139\hbar^3 t^3 - 0.005\hbar^4 t^3 + 0.0285\hbar^2 t^4 \\ &\quad +0.038\hbar^3 t^4 + 0.014\hbar^4 t^4 - 0.0019\hbar^3 t^5 - 0.0014\hbar^4 t^5 \\ &\quad +0.0008536\hbar^3 t^6 + 0.000665\hbar^4 t^6 - 0.0000524\hbar^4 t^7 \\ &\quad +8.1389 \times 10^{-6}\hbar^4 t^8. \end{aligned} \tag{3.23}$$

Setting $\hbar = -1$ in Eq (3.23), then we have

$$\begin{aligned} y(t) &= 0.1 - 0.05t^2 - 0.00174t^3 + 0.0047t^4 + 0.00046t^5 \\ &\quad -0.0001889t^6 - 0.0000524t^7 + 8.1 \times 10^{-6} t^8. \end{aligned} \tag{3.24}$$

Applying the Laplace transformation to the previous series solution, then we get

$$\begin{aligned}\widetilde{y}_m(s) &= \frac{0.1}{s} - \frac{0.1045}{s^3} - \frac{0.01045}{s^4} + \frac{0.11286}{s^5} + \frac{0.0554373}{s^6} \\ &\quad - \frac{0.136028}{s^7} - \frac{0.264279}{s^8} + \frac{0.32816}{s^9}. \qquad (3.25)\end{aligned}$$

Let $s = \frac{1}{t}$ in (3.25), then we have

$$\begin{aligned}\widetilde{y}_m(t) &= 0.1t - 0.104t^3 - 0.01045t^4 + 0.11286t^5 + 0.0554372t^6 \\ &\quad - 0.136028t^7 - 0.264279t^8 + 0.32816t^9.\end{aligned}$$

The [4/4] Padé approximation gives

$$\left[\frac{4}{4}\right] = \frac{0.1t + 0.0643181t^2 + 90.570534t^3 - 0.0226522t^4}{1 + 0.643181t + 6.75034t^2 + 0.550102t^3 + 5.99272t^4}.$$

Recalling $t = 1/s$, we obtain [4/4] in terms of s

$$\left[\frac{4}{4}\right] = \frac{-0.0226522 + 0.570534s + 0.0643181s^2 + 0.1s^3}{5.99272 + 0.550102s + 6.75034s^2 + 0.643181s^3 + s^4}.$$

By using the inverse Laplace transformation to the [4/4] Padé approximation, we obtain

$$\begin{aligned}y(t) &= e^{(-0.3-2.4i)t}((0.0001 + 0.0003i) - (0.0001 - 0.0003i)e^{-4.7it}) \\ &\quad + e^{(-1.02i)t}((0.05 - 0.001i)e^{0.01t} + (0.05 + 0.001i)e^{(0.01+2.1i)t}). \qquad (3.26)\end{aligned}$$

The above results are in excellent agreement with the results obtained by Momani et al. [18] using the modified homotopy perturbation method. Figure 3 shows the series solution (3.26) of the oscillatory nonlinear equation (3.19) and (3.20).

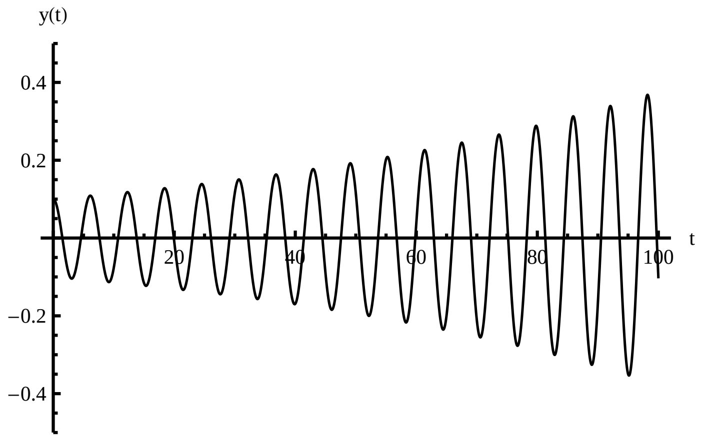

Figure 3. Plots of Eq. (3.26)

4. Conclusions

In this work, we proposed an efficient modification of the HAM which introduces an efficient tool for solving nonlinear oscillatory equations. The modified algorithm has been successfully implemented to find approximate solutions for many problems. The comparison of the result obtained by MHAM with that obtained by MHPM confirms our belief of the efficiency of our techniques. The basic idea described in this paper is expected to be further employed to find periodic solutions to nonlinear fractional oscillatory equations.

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A kind of bilevel traveling salesman problem

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Abstract. The present paper highlights a type of a bilevel optimization problem on a graph. It models a real practical problem. Let N be a finite set, $G = (N, E)$ be a weighted graph and let $I \subset N$. Let $\mathcal{C}_1$, respectively $\mathcal{C}_2$, be the set of those subgraphs $G_1 = (N_1, E_1)$, respectively $G_2 = (N_2, E_2)$, of G which fulfill some given conditions in each case. Let a and b be positive numbers and let g be a natural value function defined on the set of subgraphs of G. We study the following bilevel programming problem:

$$\begin{cases} ah(G_1) + bh(G_2) \;\rightarrow\; \min \\ \text{such that} \\ G_1 \in \mathcal{C}_1, \\ G_2 \in S^*(G_1), \end{cases}$$

where $h(G_i)$ represents the value of a Hamiltonian circuit of minimum value corresponding to the subgraph $G_i, i = 1, 2$, and

$$S^*(G_1) \;=\; \operatorname{argmin}\{g(G_2) \,|\, G_2 \in \mathcal{C}_2 \text{ and } N_1 \cap N_2 \cap I = \emptyset\}.$$

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1. Introduction

Multilevel programming and, subsequently bilevel programming, have lately become important areas in optimization. The investigations of such types of problems are strongly motivated by their actual real-life applications in areas such as economics, medicine, engineering etc. The increasing number of these applications have led mathematicians to develop new theories and mathematical models.

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In mathematical terms, the bilevel programming problem is an optimization problem where a subset of the variables is constrained to be an optimal solution of a given optimization problem parameterized by the remaining variables. References [2], [3] and [6] are useful papers for studies concerning the bilevel and multilevel programming.

Also, as it is well known, the traveling salesman problem (TSP) is one of the oldest and most studied combinatorial problem. From the mathematical point of view, researches concerning the TSP had an important role in development of the graph theory. Reference [1] presents various aspects of the TSP, especially the ones related to the methods and algorithms of solving it. References [4] and [5] provide some of the problems known under the generic name of *The Vehicle Routing Problem*, which represent a generalization of the TSP.

In the present paper, a problem of generating new types of routes is studied, using the bilevel optimization problem as a mathematical tool.

2. The practical problem

The problem studied in the present paper is based on an actual practical problem, named by us *The Milk Collection Problem*:[1] A dairy products manufacturing company collects twice a day the milk from a certain area. Collection points are located only on roads linking villages in the area. The milk is brought to the collection points by the owners. The quantity of milk delivered depends on the time when the collection is scheduled. Some providers can bring the milk to the collection points only in the morning. Others only in the evening, and some of them both in the morning and in the evening. There exists the possibility for some providers, who deliver milk in the morning, to store it (in conditions that do not impair the milk quality) and to offer it for delivery only in the evening. The others do not have this possibility. The providers impose that, either, the entire quantity of milk offered will be collected by the dairy products manufacturing company, or nothing. The milk is collected by the dairy products manufacturing company, in the morning and in the evening, using a collector tank, which has a capacity denoted by $\overline{Q}$.

The problem that arises is that of planning the providers:

- those who bring milk to the collection points in the morning, and the milk is collected by the collector tank in the morning;
- those who bring milk to the collection points in the morning, but it is necessary to store it until evening, when it will be collected by the collector tank;
- those who bring milk to the collection points in the evening, and the milk is collected by the collector tank in the evening,

such that the total cost required for milk collection in a day to be minimum and, a collection point to be visited by the collector tank at most once in the morning and at most once in the evening.

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The providers planning must satisfy the following requirements:
a) the quantity of milk collected in the morning not to exceed the capacity $\overline{Q}$ of the collector tank;
b) the quantity of milk collected in the evening (which may be from the evening milk or from the stored one) not to exceed the capacity $\overline{Q}$ of the collector tank;
c) the quantity of milk collected in the morning, and the quantity collected in the evening, must be greater than a specified quantity, denoted by $\underline{Q}$, in order to ensure the continuity in the production process;
d) the quantity of stored milk to be minimum and in the same time fulfilling the conditions a)-c).

3. The mathematical model of the milk collection problem

Let n be the number of the providers. Let us denote by $N = \{1, ..., n\}$. As well, let us denote by L_i, $i \in \{0, 1, ..., n+1\}$, the collection point where the provider i brings the milk. Let L_0 be the location where the collector tank starts and L_{n+1} be the location where the collector tank must return. We agree that they coincide, so we have $L_0 = L_{n+1}$. The collector tank transportation cost between each two locations $i, j \in \{0, 1, \ldots, n+1\}$ is known, and it is denoted by c_{ij}. By q_i^1, respectively q_i^2, we denote the quantity of milk that can be delivered by the provider $i \in N$ in the morning, respectively in the evening.

Let I_1 be the set of indices corresponding to the collection points where providers can deliver milk only in the morning, but can not store it. Let I_2 be the set of indices corresponding to the collection points where providers can deliver milk both in the morning and in the evening, but in case they deliver milk in the morning, do not accept to store it. Let I_3 be the set of indices corresponding to the collection points where providers can deliver milk both in the morning and in the evening, and accept to store the morning milk in case it is required. Let I_4 be the set of indices corresponding to the collection points where providers can deliver milk only in the evening.

It is obvious that

$$I_1 \cap I_2 = \emptyset,\ I_1 \cap I_3 = \emptyset,\ I_1 \cap I_4 = \emptyset,\ I_2 \cap I_3 = \emptyset,\ I_2 \cap I_4 = \emptyset,\ I_3 \cap I_4 = \emptyset,$$

$$I_1 \cup I_2 \cup I_3 \cup I_4 = N.$$

Now, let us consider the complete undirected graph $G = (\tilde{N}; E)$, where

$$N = \{1, \ldots, n\},\ \ \tilde{N} = N \cup \{0\} \cup \{n+1\} = \{0, 1, ..., n, n+1\}$$

and

$$E = \{\{ij\} \mid i \in \tilde{N},\ j \in \tilde{N},\ j \neq i\}.$$

The graph vertices correspond to the locations L_i, $i \in \{0, 1, \ldots, n+1\}$. Let us denote by Λ the set of subgraphs $\Gamma = (N_\Gamma, E_\Gamma)$ of $G = (\tilde{N}; E)$. We weight the graph G using the cost matrix $C = [c_{ij}]_{i,j \in \tilde{N}}$, where $c_{i,j}$, for $i \neq j$, is the minimum transport cost (of the collector tank) from the location i to location j and $c_{ii} = +\infty$, for each $i \in \tilde{N}$. As well, we attach to each node $i \in \{1, \ldots, n\}$ two positive weights, q_i^1 and q_i^2.

In order to elaborate the mathematical model for this problem, we consider two subgraphs $G_1 = (N_1, E_1)$ and $G_2 = (N_2, E_2)$ of G as variables. The set of nodes of the first subgraph, N_1, corresponds to the indices of the collection points where the milk is collected in the morning. The set of nodes of the second subgraph, N_2, corresponds to the indices of the collection points where the milk is collected in the evening.

The main objective is to determine the sets N_1 and N_2, such that the total transport cost in one day to be minimum and the problem restrictions to be satisfied.

Let us note that, for a fixed set N_1, it is obtained a minimum cost of the morning collection if it is followed a Hamiltonian circuit of minimum value in G_1. Analogous, for a fixed set N_2, it is obtained a minimum cost of the evening collection (stored milk or delivered to the collection points only in the evening) if it is followed a Hamiltonian circuit of minimum value in G_2. Therefore, if for a subgraph Γ of G, we denote by $h(\Gamma)$ the value of a Hamiltonian circuit of minimum value corresponding to the subgraph Γ, then the minimum cost of milk collection in one day it is equal to $h(G_1) + h(G_2)$.

The graphs G_1 and G_2 can not be chosen randomly; they must fulfill the problem restrictions. Thus, in the morning the collector tank can collect only from the nodes in which the providers deliver milk only in the morning. Therefore, $N_1 \subseteq I_1 \cup I_2 \cup I_3$. In the evening the collector tank can collect milk only from the nodes in which the providers deliver milk only in the evening or in which the morning milk was stored until evening. Therefore, $N_2 \subseteq I_2 \cup I_3 \cup I_4$.

In each collection point, where the providers can deliver milk both in the morning and in the evening, and where there exists the possibility to store the morning milk until evening, the quantity of milk delivered in the morning is collected just once: in the morning or in the evening. Therefore, the following condition occurs:

$$N_1 \cap N_2 \cap I_3 = \emptyset.$$

The quantity of milk collected in the morning, equal to $\sum_{i \in N_1} q_i^1$, must be greater than, or equal to, $\underline{Q}$, and can not exceed the collector tank capacity $\overline{Q}$. Also, the quantity of milk collected in the evening, equal to $\sum\limits_{i \in N_2 \cap I_3} q_i^1 + \sum\limits_{i \in N_2} q_i^2$, can not exceed the collector tank capacity $\overline{Q}$ and must be greater than, or equal to, $\underline{Q}$.

Let S be the set of all pairs (G_1, G_2) of subgraphs of the weighted graph G, $G_1 = (N_1, E_1)$ and $G_2 = (N_2, E_2)$, satisfying the conditions (3.1)-(3.5):

$$N_1 \subseteq I_1 \cup I_2 \cup I_3, \tag{3.1}$$

$$N_2 \subseteq I_2 \cup I_3 \cup I_4, \tag{3.2}$$

$$N_1 \cap N_2 \cap I_3 = \emptyset, \tag{3.3}$$

$$\underline{Q} \leq \sum_{i \in N_1} q_i^1 \leq \overline{Q}, \tag{3.4}$$

$$\underline{Q} \leq \sum_{i \in N_2 \cap I_3} q_i^1 + \sum_{i \in N_2} q_i^2 \leq \overline{Q}. \tag{3.5}$$

In order to plan the evening collection, it is necessary that the planning of the morning collection and of the stored milk, to be done with respect to the capacity type restrictions (3.4) and (3.5). For a given N_2, the quantity of stored milk is equal to $\sum_{i\in N_2\cap I_3} q_i^1$.

Let g be the real function defined on the set Λ of subgraphs of G, given by

$$g(\Gamma) \;=\; \sum_{i\in N_\Gamma\cap I_3} q_i^1,\; \forall\, \Gamma=(N_\Gamma,E_\Gamma)\in\Lambda. \tag{3.6}$$

For each $G_1\in\Lambda$, let us denote by

$$S(G_1)=\{G_2\in\Lambda\,|\,(G_1,G_2)\in S\}.$$

If $S(G_1)\neq\emptyset$, then the minimum quantity of stored milk (which can be determined taking into account the morning planning, i.e. knowing N_1) is obtained solving the following problem:

$$\begin{cases} g(G_2)\;\rightarrow\;\min \\ G_2\in S(G_1). \end{cases}$$

Let us denote by $S^*(G_1)$ the set of the optimal solutions of this problem; so, $S^*(G_1)\;=\;\text{argmin}\{g(G_2)\,|\,G_2\in S(G_1)\}$.

Under these circumstances, the milk collection problem is reduced to solve the following bilevel programming problem:

$$(BP)\quad \begin{cases} h(G_1)\,+\,h(G_2)\;\rightarrow\;\min \\ \text{such that} \\ (G_1,G_2)\in S, \\ G_2\in S^*(G_1). \end{cases}$$

Furthermore, a method for solving the problem (BP) is given, in a little more general context.

4. Generalization of the mathematical model for the milk collection problem

Let N be a finite set, $G=(N,E)$ be a weighted graph and let $I\subset N$. Let Λ be the set of subgraphs $\Gamma=(N_\Gamma,E_\Gamma)$ of G with $N_\Gamma\neq\emptyset$ and $E_\Gamma\neq\emptyset$.

Let $\mathcal{C}_1$ be the set of those elements $G_1=(N_1,E_1)$ of Λ which fulfill some given conditions. Also, let $\mathcal{C}_2$ be the set of those elements $G_2=(N_2,E_2)$ of Λ which fulfill other given conditions. In both cases, the conditions are some restrictions imposed to be fulfilled by the set of nodes N_1, respectively N_2. It can be defined, for example, by inequalities or equalities, which are generated by some given functions, or by some inclusions. For example, regarding the milk collection problem, $\mathcal{C}_1$ it is the set of those subgraphs which verify the conditions (3.1) and (3.4), while $\mathcal{C}_2$ it is the set of those subgraphs which verify the conditions (3.2) and (3.5).

Furthermore, for a subgraph $\Gamma\in\Lambda$, we denote by $h(\Gamma)$ the value of a Hamiltonian circuit of minimum value corresponding to it.

Now, let a and b be positive numbers and let $F:\Lambda\times\Lambda\rightarrow\mathbb{R}_+$ be the function given by

$$F(G_1,G_2)\;=\;a\cdot h(G_1)+b\cdot h(G_2),\;\forall\,(G_1,G_2)\in\Lambda\times\Lambda. \tag{4.1}$$

Also, let $g : \Lambda \rightarrow \mathbb{N}$ be a given function. For the milk collection problem, the function g returns the quantity of stored morning milk which it is collected in the evening.

The bilevel problem proposed to be solved is

$$(PBG) \quad \begin{cases} F(G_1, G_2) \rightarrow \min \\ G_1 \in \mathcal{C}_1, \\ G_2 \in S^*(G_1), \end{cases}$$

where $S^*(G_1)$ it is the set of the optimal solutions of the problem

$$(P(G_1)) \quad \begin{cases} g(G_2) \rightarrow \min \\ G_2 \in \mathcal{C}_2, \\ N_1 \cap N_2 \cap I = \emptyset. \end{cases}$$

Let us denote by

$$S = \{(G_1, G_2) \in \Lambda \times \Lambda \,|\, G_1 \in \mathcal{C}_1,\ G_2 \in \mathcal{C}_2,\ N_1 \cap N_2 \cap I = \emptyset\},$$

and

$$S_1 = \{G_1 \in \Lambda \mid \exists\, G_2 \in \Lambda \text{ s. t. } (G_1, G_2) \in S\},$$

i.e.

$$S_1 = \{G_1 \in \mathcal{C}_1 \mid \exists\, G_2 \in \mathcal{C}_2 \text{ s. t. } N_1 \cap N_2 \cap I = \emptyset\}.$$

For each $G_1 \in S_1$ we consider the set

$$S(G_1) = \{G_2 \in \mathcal{C}_2 \mid (G_1, G_2) \in S\} = \{G_2 \in \mathcal{C}_2 \mid N_1 \cap N_2 \cap I = \emptyset\}.$$

It is easy to see that $S(G_1)$ it is the set of feasible solutions of the problem $(P(G_1))$.

In what follows, we will use the lexicographic ordering relation in $\mathbb{R}^2$, denoted by $<_{\text{lex}}$, namely:

If we consider the points $x = (x_1, x_2)^T$ and $y = (y_1, y_2)^T$, then we have $x <_{\text{lex}} y$ if: (i) $x_1 < y_1$; or (ii) $x_1 = y_1$ and $x_2 < y_2$.

We denote $x \leq_{\text{lex}} y$ if $x <_{\text{lex}} y$ or $x = y$. The relation $\leq_{\text{lex}}$ is a total order relation on $\mathbb{R}^2$.

If A it is a subset of $\mathbb{R}^2$, $f = (f_1, f_2) : A \rightarrow \mathbb{R}^2$ it is a given function and $B \subseteq A$, then the requirement to determine the minimum point of the set $f(B)$ in relation to the lexicographic ordering relation and to determine the points of B for which this minimum is reached, is denoted by

$$(P) \quad \begin{cases} f(x) \rightarrow \text{lex} - \min \\ x \in B. \end{cases}$$

The minimum point $(f_1^*, f_2^*) \in \mathbb{R}^2$ of the set $f(B)$ is called optimum of f on B or optimal value of the problem (P); the set of points $b \in B$, such that $f(b) = (f_1^*, f_2^*)$, is called optimal solution of the problem (P).

Recalling our problem, let $H \in 2^I$. We consider the problems

$$(P_1(H)) \quad \begin{cases} h(G_1) \rightarrow \min \\ G_1 \in \mathcal{C}_1, \\ N_1 \cap I = H, \end{cases}$$

and

$$(P_2(H)) \quad \begin{cases} \begin{pmatrix} g(G_2) \\ h(G_2) \end{pmatrix} \to \text{lex} - \min \\ G_2 \in \mathcal{C}_2, \\ N_2 \cap H = \emptyset. \end{cases}$$

Let us denote by h_1^H the optimal value of the problem $(P_1(H))$ and by (g_2^H, h_2^H) the optimal value of the problem $(P_2(H))$.

Also, we define the function $\tilde{F} : 2^I \to \mathbb{R}$,

$$\tilde{F}(H) = a \cdot h_1^H + b \cdot h_2^H, \ \forall\, H \in 2^I, \tag{4.2}$$

and we consider the problem

$$(PP) \quad \begin{cases} \tilde{F}(H) \to \min \\ H \in 2^I. \end{cases}$$

Furthermore, we establish relations between the feasible solutions, and then between the optimal solutions, of the problems (PBG) and (PP).

Lemma 4.1. If $G_1^0 \in \mathcal{C}_1$, G_2^0 it is a feasible solution of the problem $(P(G_1^0))$ and $H^0 = N_1^0 \cap I$, then G_2^0 it is a feasible solution of the problem $(P_2(H^0))$.

Proof. As $G_2^0 \in S(G_1^0)$ we deduce that $G_2^0 \in \mathcal{C}_2$ and $N_1^0 \cap N_2^0 \cap I = \emptyset$. From the last equality, considering that $H^0 = N_1^0 \cap I$, we get that

$$N_2^0 \cap H^0 = N_2^0 \cap N_1^0 \cap I = \emptyset.$$

So, G_2^0 is a feasible solution of the problem $(P_2(H^0))$.$\diamond$

Lemma 4.2. If (G_1^0, G_2^0) it is a feasible solution of the problem (PBG) and $H^0 = N_1^0 \cap I$, then

i) G_1^0 it is a feasible solution of the problem $(P_1(H^0))$ and

ii) G_2^0 it is a feasible solution of the problem $(P_2(H^0))$.

Proof. Since (G_1^0, G_2^0) it is a feasible solution of the problem (PBG) and $H^0 = N_1^0 \cap I$, immediately results that: i) G_1^0 it is a feasible solution of the problem $(P_1(H^0))$; and ii) G_2^0 it is an optimal solution of the problem $(P(G_1^0))$. Because any optimal solution is a feasible one, applying Lemma 4.1, we deduce that G_2^0 it is a feasible solution of the problem $(P_2(H^0))$.$\diamond$.

Lemma 4.3. If $H^0 \in 2^I$, G_1^0 it is a feasible solution of the problem $(P_1(H^0))$ and G_2^0 it is an optimal solution of the problem $(P_2(H^0))$, then (G_1^0, G_2^0) it is a feasible solution of the problem (PBG).

Proof. Since G_1^0 it is a feasible solution of the problem $(P_1(H^0))$, we have

$$G_1^0 \in \mathcal{C}_1, \tag{4.3}$$

$$N_1^0 \cap I = H^0. \tag{4.4}$$

Because any optimal solution it is a feasible one, we have

$$G_2^0 \in \mathcal{C}_2, \tag{4.5}$$

$$N_2^0 \cap H^0 = \emptyset. \tag{4.6}$$

From (4.4) and (4.6), it results that

$$N_1^0 \cap N_2^0 \cap I \ = \ N_2^0 \cap H^0 = \emptyset. \tag{4.7}$$

In view of (4.5) and (4.7), we deduce that G_2^0 it is a feasible solution of the problem $(P(G_1^0))$, i.e. $G_2^0 \in S(G_1^0)$. We must prove that G_2^0 it is an optimal solution of this problem. Let us suppose the opposite. Then, there exists $G_2 \in S(G_1^0)$ such that $g(G_2) < g(G_2^0)$, which implies

$$\begin{pmatrix} g(G_2) \\ h(G_2) \end{pmatrix} <_{\text{lex}} \begin{pmatrix} g(G_2^0) \\ h(G_2^0) \end{pmatrix}. \tag{4.8}$$

Recalling Lemma 4.1, since $G_2 \in S(G_1^0)$, we deduce that G_2 it is a feasible solution of the problem $(P_2(H^0))$; the inequality (4.8) contradicts the fact that G_2^0 it is an optimal solution of the problem $(P_2(H^0))$. So, $G_2^0 \in S^*(G_1^0)$. Considering now the fact that $G_1^0 \in \mathcal{C}_1$, it results that (G_1^0, G_2^0) it is a feasible solution of the problem (PBG).$\diamond$

Theorem 4.4. If (G_1^0, G_2^0) it is an optimal solution of the problem (PBG), then taking $H^0 = N_1^0 \cap I$, the following sentences are true:
i) G_1^0 it is an optimal solution of the problem $(P_1(H^0))$;
ii) G_2^0 it is an optimal solution of the problem $(P_2(H^0))$;
iii) H^0 it is an optimal solution of the problem (PP).
Proof. i) Based on Lemma 4.2, we get that G_1^0 it is a feasible solution of the problem $(P_1(H^0))$. Let us suppose that G_1^0 it is not an optimal solution of the problem $(P_1(H^0))$. Then, there exists a feasible solution G_1 of the problem $(P_1(H^0))$, such that

$$h(G_1) < h(G_1^0). \tag{4.9}$$

As G_1 it is a feasible solution of the problem $(P_1(H^0))$, we have $G_1 \in \mathcal{C}_1$ and $N_1 \cap I = H^0$. Then

$$N_1 \cap N_2^0 \cap I = H^0 \cap N_2^0 = N_1^0 \cap I \cap N_2^0 = \emptyset.$$

Therefore, $G_2^0 \in S(G_1)$. Two cases can occur:
1) $G_2^0 \in S^*(G_1)$; or 2) $G_2^0 \notin S^*(G_1)$.

If $G_2^0 \in S^*(G_1)$, then (G_1, G_2^0) it is a feasible solution of the problem (PBG). Since $a > 0$, from (4.9), we deduce that

$$ah(G_1) + bh(G_2^0) \; < \; ah(G_1^0) + bh(G_2^0),$$

which contradicts the optimality of (G_1^0, G_2^0).

Now, let us suppose that $G_2^0 \notin S^*(G_1)$. Then, there exists $G_2 \in S(G_1)$ such that

$$g(G_2) < g(G_2^0). \tag{4.10}$$

Because $G_2 \in S(G_1)$, we have $G_2 \in \mathcal{C}_2$ and $N_2 \cap N_1 \cap I = \emptyset$. On the other hand, G_1 being a feasible solution of $(P_1(H^0))$, we have $N_1 \cap I = H^0$. It results that

$$N_1^0 \cap N_2 \cap I \; = \; H^0 \cap N_2 \; = \; N_1 \cap I \cap N_2 \; = \; \emptyset.$$

So, $G_2 \in S(G_1^0)$. Therefore, (4.10) contradicts the fact that $G_2^0 \in S^*(G_1^0)$.

Hence, G_1^0 it is an optimal solution of the problem $(P_1(H^0))$.

ii) From Lemma 4.2, ii), G_2^0 it is a feasible solution of $(P_2(H^0))$. Let us suppose that G_2^0 it is not an optimal solution of $(P_2(H^0))$. Since the set of solutions of the problem $(P_2(H^0))$ is a finite and nonempty set, the problem will have optimal solutions. Let $\tilde{G}_2 = (\tilde{N}_2, \tilde{E}_2)$ be an optimal solution of this problem. Based on Lemma 4.3,

$(G_1^0, \tilde{G}_2)$ it is a feasible solution of the problem (PBG). As we supposed the contrary, we have

$$\begin{pmatrix} g(\tilde{G}_2) \\ h(\tilde{G}_2) \end{pmatrix} <_{\text{lex}} \begin{pmatrix} g(G_2^0) \\ h(G_2^0) \end{pmatrix}. \tag{4.11}$$

Two cases can occur:
1) $g(\tilde{G}_2) < g(G_2^0)$; or 2) $g(\tilde{G}_2) = g(G_2^0)$ and $h(\tilde{G}_2) < h(G_2^0)$.

In the first case, we deduce that $G_2^0 \notin S^*(G_1^0)$, which contradicts the fact that (G_1^0, G_2^0) it is a feasible solution of (PBG).

In the second case, we have

$$F(G_1^0, \tilde{G}_2) = ah(G_1^0) + bh(\tilde{G}_2) \, < \, ah(G_1^0) + bh(G_2^0) = F(G_1^0, G_2^0),$$

which contradicts the optimality of (G_1^0, G_2^0).

iii) Since the set 2^I is a nonempty and finite set, the problem (PP) has an optimal solution. Let us suppose that H^0 it is not an optimal solution of the problem (PP), and let H^* be the optimal solution of the problem (PP). Under these circumstances, we have

$$\tilde{F}(H^*) < \tilde{F}(H^0). \tag{4.12}$$

Let us notice that, from i) and ii), taking into account the way in which the functions F and $\tilde{F}$ are defined (see (4.1), (4.2)), we have

$$\tilde{F}(H^0) \, = \, ah_1^{H^0} + bh_2^{H^0} \, = \, ah(G_1^0) + bh(G_2^0) = F(G_1^0, G_2^0). \tag{4.13}$$

Now, let G_1^* be an optimal solution of the problem $(P_1(H^*))$ and G_2^* be an optimal solution of the problem $(P_2(H^*))$. Then,

$$\tilde{F}(H^*) \, = \, ah_1^{H^*} + bh_2^{H^*} \, = \, ah(G_1^*) + bh(G_2^*) \, = \, F(G_1^*, G_2^*). \tag{4.14}$$

From (4.12)-(4.14), it results that

$$F(G_1^*, G_2^*) \, < \, F(G_1^0, G_2^0). \tag{4.15}$$

On the other hand, applying Lemma 4.3, we deduce that (G_1^*, G_2^*) it is a feasible solution of the problem (PBG). Hence, (4.15) contradicts the optimality of (G_1^0, G_2^0).⋄

Theorem 4.5. If H^0 it is an optimal solution of the problem (PP) and G_1^0, respectively G_2^0, it is an optimal solution of the problem $(P_1(H^0))$, respectively $(P_2(H^0))$, then (G_1^0, G_2^0) it is an optimal solution of the problem (PBG).

Proof. As G_1^0, respectively G_2^0, it is an optimal solution of $(P_1(H^0))$, respectively $(P_2(H^0))$, we get that $h_1^{H^0} = h(G_1^0)$ and $h_2^{H^0} = h(G_2^0)$. Therefore,

$$\tilde{F}(H^0) = ah(G_1^0) + bh(G_2^0) = F(G_1^0, G_2^0). \tag{4.16}$$

On the other hand, applying Lemma 4.3, we get that (G_1^0, G_2^0) it is a feasible solution of (PBG). If (G_1^0, G_2^0) it is not an optimal solution of the problem (PBG), then there exists $(\tilde{G}_1 = (\tilde{N}_1, \tilde{E}_1), \tilde{G}_2 = (\tilde{N}_2, \tilde{E}_2))$ feasible solution of (PBG), such that

$$ah(\tilde{G}_1) + bh(\tilde{G}_2) \, = \, F(\tilde{G}_1, \tilde{G}_2) < F(G_1^0, G_2^0) \, = \, ah(G_1^0) + bh(G_2^0). \tag{4.17}$$

From (4.17) we deduce that: $h(\tilde{G}_1) < h(G_1^0)$ or $h(\tilde{G}_2) < h(G_2^0)$.

Let $\tilde{H} = \tilde{N}_1 \cap I$. As $(\tilde{G}_1, \tilde{G}_2)$ it is a feasible solution of (PBG), we have $\tilde{G}_2 \in S^*(\tilde{G}_1)$; this implies that $h_2^{\tilde{H}} = F(\tilde{G}_1, \tilde{G}_2)$. Hence, (4.17) implies $\tilde{F}(\tilde{H}) < \tilde{F}(H^0)$. This contradicts the optimality of H^0.$\diamond$

Let

$$\lambda \geq 1 + \max\{F(G_1, G_2),\ \forall\, (G_1, G_2) \in \Lambda\}. \tag{4.18}$$

Let $G_1 \in S_1$ and $H \in 2^I$, fulfilling the following condition

$$N_1 \cap I = H. \tag{4.19}$$

Let us consider the problem

$$(PL_2(H)) \quad \begin{cases} \lambda \cdot g(G_2) + F(G_1, G_2) \ \rightarrow \ \min \\ G_2 \in \mathcal{C}_2, \\ H \cap N_2 = \emptyset. \end{cases}$$

Theorem 4.6. If $G_1 \in S_1$ and $H \in 2^I$ such that the condition (4.19) is fulfilled, then an element G_2 it is an optimal solution of the problem $(PL_2(H))$ if and only if it is an optimal solution of the problem $(P_2(H))$.

Proof. First, let us remark that both problems have the same set of feasible solutions.

Necessity. Let G_2 be an optimal solution of the problem $(PL_2(H))$. Let us suppose that G_2 it is not an optimal solution of the problem $(P_2(H))$. Two cases can occur:

1) there exists G_2^* a feasible solution of the problem $(P_2(H))$, such that $g(G_2^*) < g(G_2)$; or

2) there exists G_2^* a feasible solution of the problem $(P_2(H))$, such that $g(G_2) = g(G_2^*)$ and $F(G_1, G_2^*) < F(G_1, G_2)$.

As $\lambda > 0$, in the first case we obtain that

$$\lambda \cdot g(G_2^*) \ < \ \lambda \cdot g(G_2). \tag{4.20}$$

If $F(G_1, G_2^*) \leq F(G_1, G_2)$, then $\lambda \cdot g(G_2^*) + F(G_1, G_2^*) < \lambda \cdot g(G_2) + F(G_1, G_2)$. This contradicts the hypothesis that (G_1, G_2) it is an optimal solution of the problem $(PL_2(H))$.

Let us now suppose that $F(G_1, G_2^*) > F(G_1, G_2)$. As $g(G_2) \in \mathbb{N}$ and $g(G_2^*) \in \mathbb{N}$, based on $g(G_2^*) < g(G_2)$, we have $g(G_2) - g(G_2^*) \geq 1$. Therefore,

$$\frac{F(G_1, G_2) - F(G_1, G_2^*)}{g(G_2) - g(G_2^*)} \ \leq \ \frac{F(G_1, G_2) - F(G_1, G_2^*)}{1} \ \leq \ F(G_1, G_2) \ < \ \lambda. \tag{4.21}$$

It follows that $\lambda \cdot g(G_2^*) + F(G_1, G_2^*) < \lambda \cdot g(G_2) + F(G_1, G_2)$, which contradicts the hypothesis that (G_1, G_2) it is an optimal solution of the problem $(PL_2(H))$.

If we consider case 2), then we immediately get that $\lambda \cdot g(G_2^*) + F(G_1, G_2^*) < \lambda \cdot g(G_2) + F(G_1, G_2)$, which contradicts the optimality of (G_1, G_2) for the problem $(PL_2(H))$.

Sufficiency. Let G_2 be an optimal solution of the problem $(P_2(H))$ and let G_2^* be a feasible solution of the problem $(PL(H))$. Since G_2^* it is a feasible solution of the problem $(P_2(H))$, we get that

$$\begin{pmatrix} g(G_2) \\ F(G_1, G_2) \end{pmatrix} <_{\text{lex}} \begin{pmatrix} g(G_2^*) \\ F(G_1, G_2^*) \end{pmatrix}.$$

Three cases are possible:
1) $g(G_2) < g(G_2^*)$;
2) $g(G_2) = g(G_2^*)$ and $F(G_1, G_2^*) < F(G_1, G_2)$;
3) $g(G_2) = g(G_2^*)$ and $F(G_1, G_2^*) = F(G_1, G_2)$.

Since $g(G_2) \in \mathbb{N}$ and $g(G_2^*) \in \mathbb{N}$, in the first case we have

$$g(G_2) - g(G_2^*) \leq -1.$$

Therefore,

$$\begin{aligned} &\lambda \cdot g(G_2) + F(G_1, G_2) - (\lambda g(G_2^*) + F(G_1, G_2^*)) \\ &= \lambda \cdot (g(G_2) - g(G_2^*)) + F(G_1, G_2) - F(G_1, G_2^*) \\ &\leq -\lambda + F(G_1, G_2) - F(G_1, G_2^*) \\ &\leq -1 - F(G_1, G_2^*) < 0. \end{aligned}$$

In the cases 2) and 3) it results that

$$\lambda \cdot g(G_2) + F(G_1, G_2) \leq \lambda g(G_2^*) + F(G_1, G_2^*).$$

Since G_2^* it is a feasible solution chosen arbitrary, it results that G_2 it is an optimal solution of the problem $(PL_2(H))$.$\diamond$

Based on Theorem 4.6, we can reduce the solving of the problem (PBG) to solving $2^{|I|}$ couples of problems $(P_1(H), P_2(H))$, where the parameter H belongs to the set 2^I. Based on Theorem 4.6, the solving of the problem $(P_2(H))$ can be replaced by solving the problem $(PL_2(H))$.

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Book reviews

Peter Duren, Invitation to Classical Analysis, Pure and Applied Undergraduate Texts, Volume 17, American Mathematical Society, Providence, Rhode Island, 2012, xiii+392 pp; ISBN: 978-0-8218-6932-1.

In modern calculus textbooks, as, for instance, those by Walter Rudin or Jean Dieudonné, the focus is on the theoretical foundations of the subject, with less attention paid to concrete applications. The classical treatises on calculus, as well as those directed to physicists or engineers, contain such topics but treated in a less rigorous manner. The aim of the present book is to fill in this gap by presenting some substantial topics in classical analysis with full rigorous proofs and in historical perspective. As a thread running through the book one can mention the calculation of the sum of the series $\sum_{n=1}^{\infty} n^{-2}$. This problem posed by Johann Bernoulli, known as the *Basel problem*, was solved in 1735 by Leonard Euler who proved the remarkable equality $\sum_{n=1}^{\infty} n^{-2} = \pi^2/6$ (known as Euler's sum). Since then many proofs of this equality have been found, some of them being recorded in this book.

Although the reader is assumed to have acquired a good command of basic principles of mathematical analysis, the first chapter of the book contains a review of them, for convenience and easy reference. Chapters 2, *Special sequences*, and 3, *Power series*, contain also some standard material, along with some more special topics – the product formulae of Vieta and Wallis, Stirling's formula, a first treatment of Euler's sum, nowhere differentiable continuous functions.

In Ch. 4, *Inequalities*, beside some standard material, one can mention an elementary straightforward proof of Hilbert's inequality recently found by David Ullrich (to appear in American Mathematical Monthly). Ch. 5, *Infinite products*, is devoted to a topic less treated, or neglected at all, in modern courses of calculus. Ch. 6, *Approximation by polynomials*, contains the proofs given by Lebesgue, Landau, and Bernstein, to uniform approximation of continuous functions by polynomials, along with some refinements due to Pál, Fekete, and Müntz-Szasz. A proof of the Stone-Weierstrass approximation theorem is also included. Abel's Theorem, proved in Chapter 3, asserts that if the power series $\sum_{n=0}^{\infty} a_n x^n$ converges in $(-1, 1)$ and $\sum_{n=0}^{\infty} a_n = A$, then $\lim_{x\to 1-} \sum_{n=0}^{\infty} a_n x^n = A$. Tauber's classical theorem asserts that the converse result holds under the additional hypothesis that $na_n \to 0$. The result was extended by Hardy and Littlewood to the case of boundedness of the sequence $\{na_n\}$ and for Cesàro summability, which is more general than Abel's summability. The proofs of these results, given by Karamata, are presented in the seventh chapter of the book.

Fourier series are treated in Chapter 8 at an elementary level, meaning the exclusion of Lebesgue integration and complex analysis. In spite of these restrictions some spectacular results can be derived, proving the power of Fourier analysis.

Chapters 9, *The Gamma function*, and 14, *Elliptic functions*, are dealing with these special classes of functions, while Bessel functions and hypergeometric functions are treated in Ch. 13, *Differential equations.* Ch. 11, *Bernoulli numbers*, is concerned mainly with the properties of these numbers and applications to the calculation of the sum of the series $\zeta(2k) = \sum_{n=1}^{\infty} n^{-2k}$. Applications of Riemann zeta function to number theory are considered in Ch. 10, *Topics in number theory*, along with other results in this area. Ch. 12, *The Cantor set*, is concerned with cardinal numbers, Cantor set, Cantor-Scheeffer function, space-filling curves.

The book is very well organized – detailed name and subject indexes, historical notes on the evolution of mathematical ideas and their relevance for physical applications, capsule scientific biographies of the major contributors and a gallery of portraits. It is devoted to undergraduates who learned the basic principles of analysis, and are prepared to explore substantial topics in classical analysis. It is designed for self-study, but can also serve as a text for advanced courses in calculus.

The book reflects the delight the author experienced when writing it, a delight that will be surely shared by its readers as well.

Tiberiu Trif

Peter D. Lax and Lawrence Zalcman, Complex Proofs of Real Analysis, University Lecture Notes, Volume 58, American Mathematical Society, Providence, Rhode Island, 2012, xi+90 pp; ISBN- 978-0-8218-7599-9.

A famous saying of Poncelet asserts that "*Between two truths of the real domain, the easiest and shortest path quite often passes through the complex domain*". Since this dictum was endorsed and popularized by Hadamard, it is usually attributed to him. The present book is a brilliant illustration of this claim - the authors show how the method of complex analysis can be used to provide quick and elegant proofs of a wide variety of results in various areas of analysis. Beside analysis, a proof of the Prime Number Theorem ($\lim_{x\to 0} \pi(x)/\left(\frac{x}{\log x}\right) = 1$) based on contour integration of Riemann zeta function is included. Other earlier results obtained by the methods of complex analysis concern evaluations of the values of $\zeta(2k) = \sum_{n=1}^{\infty} n^{-2k}$, a proof of the fundamental theorem of algebra, and applications to approximation theory – uniform weighted approximation in $C_0(\mathbb{R})$ and Müntz's theorem.

The core of the book is formed by the applications to operator theory, in Chapter 3, and to harmonic analysis, in Chapter 4. Among the applications to operator theory we mention Rosenblum's elegant proof of Fuglede-Putnam theorem, Toeplitz operators and their inversion, Beurling's characterization of invariant subspaces of the unilateral shift on the Hardy space H^2, Szegö's theorem in prediction theory, Riesz-Thorin convexity theorem with applications to the boundedness of the Hilbert transform on $L^p(\mathbb{R}^n)$, $1 < p < \infty$.

The applications to harmonic analysis include D. J. Newman's proof of the uniqueness of the Fourier transform in $L^1(\mathbb{R})$, uniqueness and nonuniqueness results for the Radon transform, Paley-Wiener theorem and Titchmarsh convolution theorem, Hardy's theorem on the Fourier transform.

The fifth chapter of the book contains a proof of the famous Kahane-Gleason-Želazko theorem giving conditions for a linear functional on a commutative Banach algebra with unit to be multiplicative. The Fatou-Julia theorem in complex dynamics is treated in the sixth chapter. A coda deals briefly with two unusual applications of complex analysis – to fluid dynamics and to statistical mechanics (the stochastic Loewner evolution equation).

The book is fairly self-contained, the prerequisites being a standard course in complex analysis and familiarity with some results in functional analysis and Fourier transform. Some more specialized topics from complex analysis, as Liouville's theorem in Banach spaces, the Borel-Crathéodory inequality, Phragmen-Lindlöf theorem, and some results on normal families, are presented in Appendices. Relevant historical remarks and bibliographical references accompany each topic included in the book.

Written in a lively and entertaining style, but mathematically rigorous at the same time, the book is addressed to all mathematicians interested in elegant proofs of some fundamental results in mathematics. The instructors of complex analysis can use it as a source to enrich their lectures with nice examples.

Gabriela Kohr

Martin Väth, Topological Analysis – From the basics to the triple degree for nonlinear Fredholm inclusions, Series in Nonlinear Analysis and Applications, Vol. 16, ix+490 pp, Walter de Gruyter, Berlin - New York, 2012, ISBN: 978-3-11-027722-7, e-ISBN: 978-3-11-027733-3, ISSN: 0941-813X.

The degree theory, originally developed by Leray and Schauder in the thirties of the last century for a rather restricted class of equations, turned to be one of the most powerful tools of nonlinear analysis. Since then it has been successively extended to encompass much larger classes of equations, including even those involving noncompact or multivalued maps. The aim of the present monograph is to give a self-contained introduction to the whole area, culminating with a general degree theory for function triples, the first monograph treatment of this notion in such generality.

The book is divided into three parts: I. *Topology and multivalued maps*, II. *Coincidence degree for Fredholm maps*, and III. *Degree for function triples.*

The presentation is given in logical order, meaning from general results to particular ones, rather that in a didactic order. For instance, many results concerning single-valued continuous functions, developed in the second chapter devoted to topology, are obtained as particular cases of those referring to multivalued maps - mean-value results for continuous functions, some results on proper maps. This chapter contains also a a detailed treatment of separation axioms, including two less known – T_5 (every subset is T_4) and T_6 (perfectly normal spaces). Some specific results to metric spaces, as measures of noncompactness and condensing maps, embedding and extension results, are treated in the third chapter. The fourth chapter deals with homotopies, retracts, ANR and AR spaces, while the last one of this part (Ch. 5) is concerned with some advanced topological tools – covering space theory, dimension theory, Vietoris map.

The second part of the book contains some results from linear functional analysis – linear bounded operators on Banach spaces, Fredholm operator and an orientation theory for families of Fredholm operators based on determinants associated with them – and basic nonlinear functional analysis – Gâteaux and Fréchet differentiation, inverse and implicit function theorems, orientation for families of nonlinear Fredholm maps on Banach manifolds, a brief but fairly complete treatment of Brouwer degree with applications. This part ends with an introduction to Benevieri-Furi degrees, based on a definition of orientation by which the degree theory in infinite dimensional setting reduces to the finite dimensional case. The Leray-Schauder degree (the infinite dimensional version of Brouwer degree) is postponed to Chapter 13, where it is obtained as a particular case of a more general notion.

The highlight of the book is the third part where a very general degree theory, which unifies a lot of known degrees theories, is developed. It is concerned with problems of the form: (1) $F(x) \in \varphi(\Phi(x))$, where (a) F is a nonlinear Fredholm operator of index 0; (b) Φ is a multivalued mapping with acyclic values $\Phi(x)$; (c) φ is continuous , and (d) the composition $\varphi \circ \Phi$ is, roughly speaking, "more compact that F is proper". The key notion throughout the book is that of orientation with is gradually extended from linear Fredholm operators on Banach spaces to nonlinear Fredholm maps on Banach manifolds and on Banach bundles. The book contains also an account on Banach manifolds, which are the basic tools of the degree theory.

By the choice of material this excellent book can be used for several purposes. First as supplementary material for various courses in topology or functional analysis. Even in the standard part of the book, concerning topology and functional analysis, some shorter and elegant proofs to known results are given. Also the author pays a special attention to foundation – the necessity of the Axiom of Choice for various results is carefully checked.

Second, as a self-contained introduction to the degree theory in various of its hypostases, starting with Brouwer degree in Eucildean spaces and on manifolds and culminating with the degree theory for function triples, a very active and important domain of research with many applications in various areas of mathematics.

Radu Precup

Anne Greenbaum and Timothy P. Chartier, Numerical Methods. Design, Analysis, and Computer Implementation of Algorithms, Princeton University Press, 2012, 470 pp.; ISBN: 978-0-691-15122-9.

This book is a concise and modern exposition of both standard (Solution of nonlinear equations – Chapter 3, Numerical Linear Algebra – Chapters 7 and 12, Floating-point arithmetic – Chapter 5, Condition and stability of algorithms – Chapter 6, Interpolation – Chapter 8, Numerical Differentiation – Chapter 9, Numerical

Integration – Chapter 10, Ordinary Differential Equations – Chapters 11 and 13, Partial Differential Equations) and nontraditional (Mathematical modeling – Chapter 1, Monte Carlo Methods – Chapter 3, fractals – section 4.6, Markov chains – section 12.1.5) topics on numerical analysis.

The good balance between mathematical rigor, practical and computational aspect of numerical methods, and computer programs offers the instructor the flexibility to focuss on different aspects of numerical methods, depending on the aim of the course, the background and the interests of students.

Biographical information about mathematicians and short discussions on the history of numerical methods humanize the text.

The book contains extensive examples, presented in an intuitive way with high quality figure (some of them quite spectacular) and useful MATLAB codes. MATLAB exercises and routines are well integrated within the text, and a concise introduction into MATLAB is given in Chapter 2. The emphasis is on program's numerical and graphical capabilities and its applications, not on its syntax. The usage of MATLAB Toolbox `chebfun` facilitates presentation of recent results on interpolation at Chebyshev points and provides symbolic capabilities at speed of numeric procedures. A large variety of problems graded by the difficulty point of view are included. Applications are modern and up to date (e.g. information retrieval and animation, classical applications from physics and engineering).

Appendices on linear algebra and multivariate Taylor's theorem help the understanding of theoretical results in the text. A reach and comprehensive list of references is attached at the end of the book. Supplementary materials are available online.

I am sure this text will become a great title for the subject.

Intended audience: especially graduate students in mathematics and computer science, but also useful to applied mathematicians, computer scientists, engineers and physicists interested in applications of numerical analysis.

Radu Trîmbiţaş

Darryl D Holm, Geometric Mechanics, 2nd Edition, Imperial College Press, London, 2011.
Part I: **Dynamics and Symmetry**, xxiv + 441 pp, ISBN: 13 978-1-84816-774-2.
Part II: **Rotating, Translating and Rolling**, xx + 390 pp, ISBN: 13 978-1-84816-777-3.

The two volumes of Darryl Holm's *Geometric Mechanics* offer an attractive introduction to the tools and language of modern geometric mechanics. These volumes are designed for advanced undergraduates and beginning graduate students in mathematics, physics and engineering. The minimal prerequisite for reading *Geometric Mechanics* is a working knowledge of linear algebra, multivaluable calculus and some familiarity with Hamilton's principle, Euler-Lagrange variational principles and canonical Poisson brackets in classical mechanics, at the beginning undergraduate level.

In this second edition the author preserves the organization of the first edition (2008). However, the substance of the text has been rewritten throughout to improve the flow and to enrich the development of the material. In particular, the role of

Noether's theorem about the implications of Lie group symmetries for conservation laws of dynamical systems has been emphasized throughout, with many applications. Many worked examples on adjoint and coadjoint actions of Lie groups on smooth manifolds have been added. The enhanced coursework examples have been expanded.

The first volume contains six chapters: Fermat's ray optics; Newton, Lagrange, Hamilton and the rigid body; Lie, Poincaré, Cartan: Differential forms; Resonances and S^1 reduction; Elastic spherical pendulum; Maxwell–Bloch laser-matter equations, and two appendixes: Enhanced coursework and Exercises for review and further study.

The chapters of the second volume are: Galileo; Newton, Lagrange, Hamilton and the rigid body; Quaternions; Adjoint and coadjoint actions; The special orthogonal group $SO(3)$; Adjoint and coadjoint semidirect-product group actions; Euler–Poincaré and Lie–Poisson equations on $SE(3)$; Heavy-top equations; The Euler–Poincaré theorem; Lie–Poisson Hamiltonian form of a continuum spin chain; Momentum maps; Round, rolling rigid bodies. This volume ends with four appendices: Geometrical structure of classical mechanics; Lie groups and Lie algebras; Enhanced coursework; Poincaré's 1901 paper.

The two volumes of the second edition of Holm's *Geometric Mechanics* are ideal for classroom use, student projects and self-study.

Ferenc Szenkovits

Jaroslaw Kurzweil, Generalized Ordinary Differential equations – Non Absolutely Continuous Solutions, Series in Real Analysis, Vol. 11, World Scientific, London - Singapore - Beijing, 2012, ix + 197 pages, ISBN: 13 978-981-4324-02-1 and 10 981-4324-02-7.

Using a Riemann type approach, the author of the present book discovered in the fifties a new kind of integral, called non-absolutely convergent integral, which is more general than Lebesgue's integral and agrees with it in case of absolute integrability. Since R. Henstock independently arrived at the same conclusions, approximatively at the same time, the integral is known as the Henstock-Kurzweil (HK) integral. A good presentation of this integral, along with other types of integral, is given in the book by D. S. Kurtz and Ch. W. Swartz *Theories of Integration - The integrals of Riemann, Lebesgue, Henstock-Kurzweil and Mc Shane*, World Scientific, London-Singapore-Beijing, 2012, as well as in two previous books by the author – Teubner, Leipzig, 1980 (in German) and World Scientific 2000. The main point of the HK integral is the validity of a very general form of the fundamental theorem of calculus $\int_a^b f' = f(b) - f(a)$ for any differentiable function f. One of the main application of this new integral, given in a paper published by the author in 1957, was to the existence of generalized solutions and continuous dependence on the parameter for generalized ordinary differential equations (GODE), where solutions of infinite variation can occur. The aim of the present book is to give a systematic development of this subject. Since the main motivation came from Kapitza's pendulum equation studied by P. Kapitza in 1951, the first part of the book (Chapters 2-4) is devoted to this equation.

The second part of the book (Chapters 5-13) is concerned with strong Riemann solutions of the differential equation $\frac{d}{dt}x = D_tG(x, \tau, t)$.

The existence and properties of strong Henstock-Kurzweil (SKH) solutions of the above equations are studied in the third part (Chapters 14-18), based on some averaging techniques. The fourth part (Chapters 19-24) deals with SKH solutions of the equation $\frac{d}{dt}x = F(x, \tau, t)$, where $F : X \times [a, b]^2 \to X$, X a Banach space, is a mapping satisfying some natural conditions. The fifth (and the last, Chapters 25-27) part of the book is concerned with GODEs of the form $\frac{d}{dt}x = D_tG(x, \tau, t)$, $\frac{d}{dt}x = D_tG^\circ(x, \tau, t)$, where $G^\circ(x, t) = (SR)\int_a^t D_tG(x, \sigma, s)$.

The book is an important contribution to the area of differential equations, proving the power and versatility of the generalized integral discovered by Henstock and Kurzweil.

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