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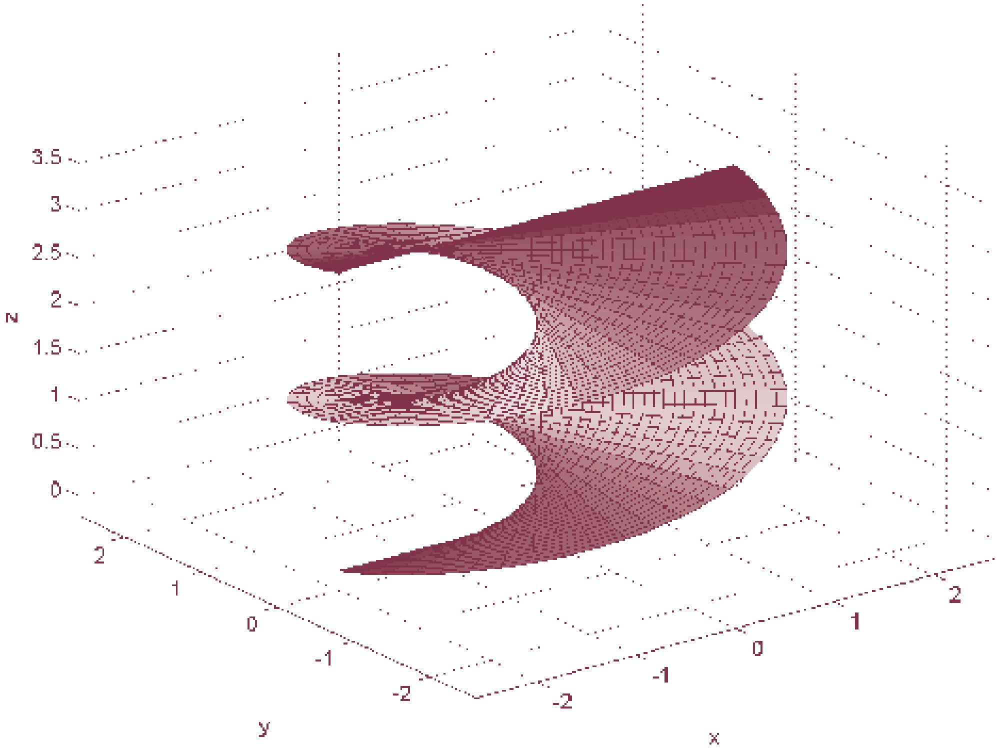

MATHEMATICA

3/2013

STUDIA
UNIVERSITATIS BABEŞ-BOLYAI
MATHEMATICA

3 / 2013

YEAR **(LVIII) 2013**
MONTH **SEPTEMBER**
ISSUE **3**

S T U D I A
UNIVERSITATIS BABEŞ-BOLYAI

MATHEMATICA
3

Redacţia: 400084 Cluj-Napoca, str. M. Kogălniceanu nr. 1
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Stud. Univ. Babeş-Bolyai Math. 58(2013), No. 3, 289–295

A proof of a covering correspondence

Tiberiu Coconeţ

Abstract. We show that the isomorphism between the Clifford extensions of two Brauer corresponding blocks of normal subgroups induces a defect group preserving bijection which coincides with the Harris-Knörr correspondence between their covering blocks.

Mathematics Subject Classification (2010): Primary 20C20. Secondary 16W50, 16S35.

Keywords: Group algebras, blocks, G-algebras, Brauer construction, group graded algebras, defect group.

1. Introduction

Clifford extensions for blocks were introduced by E.C. Dade in [5], where he proved that two Brauer correspondent blocks b and b_1 with defect group D of normal subgroups K and $N_K(D)$ of the finite groups H and $N_H(D)$ respectively, have isomorphic Clifford extensions.

Dade [5, Section 3] also gives a bijective correspondence between the blocks of a strongly graded algebra that cover a fixed block b of the identity component and the conjugacy classes of blocks of the twisted group algebra corresponding to the Clifford extension of b.

A generalization of Dade's main result is given in [3], where we prove an isomorphism of Clifford extensions for points of identity components of certain H/K-graded H-interior algebras, without assuming that the ground field is algebraically closed.

The aim of this paper is to establish a link between the above isomorphism of Clifford extensions and the result of M.E. Harris and R. Knörr [6] which states that the Brauer correspondence induces a bijection between the blocks of H covering b and the blocks of $N_H(D)$ covering b_1. Actually, there is some suggestion in [6] that such a connection is possible, but no details are given. Note also that a module-theoretic version of the Harris-Knörr correspondence was given by J. Alperin [1]. Here we prove that the isomorphism of Clifford extensions induces a defect group preserving bijective correspondence between the blocks of H covering b and the blocks of $N_H(D)$ covering b_1, which coincides with the Harris-Knörr correspondence.

We present our general setting in Section 2, while in Section 3 we review the required results on the defect groups of covering blocks. The details on the correspondence induced by the isomorphism of the Clifford extensions are presented in Section 4, following [3]. The last section is devoted to the proof of our main result, stated in Theorem 5.1. The reader is referred to [9] and [7] for general facts on block theory.

2. Preliminaries

2.1. Let $\mathcal{O}$ be a discrete valuation ring having residual field k of characteristic $p \geq 0$. Let K be a normal subgroup of the finite group H, denote $G = H/K$, and consider the group algebra $\mathcal{O}H$ regarded as a strongly G-graded algebra

$$A := \mathcal{O}H = \bigoplus_{\sigma \in G} \mathcal{O}\sigma,$$

which is also an H-algebra under the conjugation action of H. We fix a block b of the identity component $A_1 := \mathcal{O}K$ of A. We denote by D a defect group in K of the block b.

2.2. If H_b denotes the stabilizer of b in H, and G_b is the quotient H_b/K, as in [5] we consider the G_b-graded subalgebra

$$bC := bC_A(A_1) = (b\mathcal{O}H_b)^K = \bigoplus_{\sigma \in G_b} (b\mathcal{O}\sigma)^K = \bigoplus_{\sigma \in G_b} bC_\sigma^K$$

of A. We truncate bC by taking the components indexed by the normal subgroup

$$G[b] = \{\sigma \in G_b \mid bC_\sigma^K \cdot bC_{\sigma^{-1}}^K = bC_1^K\}$$

of G_b; this yields the strongly $G[b]$-graded G_b-algebra, and hence an H_b-algebra

$$C[b] := \bigoplus_{\sigma \in G[b]} bC_\sigma^K.$$

The identity component

$$bC_1^K = b(\mathcal{O}K)^K = bZ(\mathcal{O}K)$$

is a local ring such that the field

$$\hat{k}_1 = bZ(\mathcal{O}K)/J(bZ(\mathcal{O}K))$$

is a finite extension of k.

2.3. Consider also the quotient $C[b]/C[b]J(C[b]_1)$, which is the twisted group algebra of $G[b]$ over the field $\hat{k}_1$, corresponding to the *Clifford extension*

$$1 \to \hat{k}_1^* \to hU(C[b]/C[b]J(C[b]_1)) \to G[b] \to 1 \tag{2.1}$$

of the block b. Where by hU we denoted the homogeneous units of $C[b]/C[b]J(C[b]_1)$. Explicitly, the set of elements that satisfy

$$\bar{a} \in (C[b]/C[b]J(C[b]_1))^* \cap bC_g/bC_gJ(C[b]_1),$$

for some $g \in G[b]$. Since $bC_1 = C[b]_1$ is a H_b-algebra, the H_b-invariance of $J(C[b]_1)$ implies that the canonical map

$$C[b] \to C[b]/C[b]J(C[b]_1)$$

is a homomorphism of H_b-algebras.

Lemma 2.4. *The algebras bC^{H_b} and $C[b]^{H_b}$ have the same primitive idempotents.*

Proof. The proof of this statement is based on results of [5, Paragraph 3], which remain true even if the field k in not algebraically closed. One easily checks that in our setting [5, Lemma 3.3] is valid. So there is a two-sided ideal

$$I = \Big(\bigoplus_{\sigma \in G_b \backslash G[b]} bC_\sigma\Big) \oplus C[b]J(C[b]_1)$$

of bC that is both H_b-invariant and contained in $J(bC)$. This gives the equality

$$bC = C[b] \oplus \Big(\bigoplus_{\sigma \in G_b \backslash G[b]} bC_\sigma\Big) = C[b] + I = C[b] + J(bC);$$

showing that every primitive idempotent of bC belongs to $C[b]$. So, any block, that is a primitive idempotent of $Z(bC) = bC^{H_b}$, lies in $C[b]^{H_b}$. Conversely, any primitive idempotent of $C[b]^{H_b}$ remains primitive in bC^{H_b}, since I is contained in $J(bC)$. □

3. Remarks on defect groups

In this section we discuss the connections between the defect groups of blocks covering the block b of C_1 and the defect groups of primitive idempotents of $C[b]^{H_b}$. Some of the results have already been proven in [5, Paragraph 6 and 7], but for the sake of completeness we present them here. As a definition of a defect group of a block we will use [9, Paragraph 18] or [5, Paragraph 4]. Dade uses the maximal ideal corresponding to a block in order to define the defect group of that block. Nevertheless, one easily shows that both treatments lead to the same definition.

3.1. As it is well known, the blocks of H covering b are the primitive idempotents of $Z(s\mathcal{O}H)$, where

$$s = \mathrm{Tr}_{H_b}^{H}(b).$$

By [5, Proposition 4.9] we have the isomorphism

$$Z(s\mathcal{O}H) \simeq Z(b\mathcal{O}Hb) = Z(b\mathcal{O}H_b) = bC^{H_b}. \tag{3.1}$$

Using this and the results of Section 2 above, we see that the blocks of H that cover b are actually the primitive idempotents of $C[b]^{H_b}$.

3.2. We denote by B a block that covers b and by B' the correspondent of B through the isomorphism (3.1). Then $B = \mathrm{Tr}_{H_b}^{H}(B')$. Let Q denote a defect group in H_b of B'. This means that Q is with the properties $B' \in b\mathcal{O}Hb_Q^{H_b}$ and $B' \not\subseteq \mathrm{Ker}(\mathrm{Br}_Q)$, where Br_Q denotes the Brauer homomorphism with respect to Q. But then, since $B's = B'$ we get $B \in s\mathcal{O}H_Q^H$. For $x \in H \setminus H_b$ we also have $bb^x = 0$. Taking into account that $B' = bB' = bB'$, then obviously $BB' = B'$. This forces $B \not\subseteq \mathrm{Ker}(\mathrm{Br}_Q)$. We have shown that any block that covers b has a defect group in H that is contained in H_b.

3.3. By [8, Proposition 4.2], the block B' has a defect group Q (in H_b) satisfying $Q \cap K = D$. The ending of Paragraph 3.2 assures that Q is also a defect group of B. We can apply [8, Proposition 4.2] to obtain a defect group L of B in H that satisfies the same condition as Q, that is $L \cap K = D$. Thus, there is $y \in H$ with $L^y = Q$, and then $y \in N_H(D)$.

4. Clifford extensions of blocks

We keep the notations of the preceding sections. For the details on the following statements the reader is referred to [3].

4.1. The restriction to $bC = b(\mathcal{O}H_b)^K$ of the Brauer homomorphism

$$\mathrm{Br}_D^H : (\mathcal{O}H)^D \to kC_H(D)$$

gives the epimorphism

$$\mathrm{Br}_D^H : b(\mathcal{O}H_b)^K \to \bar{b}kC_H(D)_{\bar{b}}^{N_K(D)}, \tag{i}$$

where $\bar{b} = \mathrm{Br}_D^H(b)$.

Next let b_1 denote the Brauer correspondent of b, seen as a block of $\mathcal{O}N_K(D)$, also having defect group D. Repeating the construction of Section 2 for $N_H(D)$, $N_K(D)$ and b_1 in place of H, K and b respectively we easily obtain another Clifford extension

$$1 \to \hat{k}_2^* \to hU(C'[b_1]/C'[b_1]J(C'[b_1]_1)) \to G'[b_1] \to 1. \tag{4.1}$$

Here we used the $N_H(D)/N_K(D)$-graded centralizer

$$b_1C' := C_{\mathcal{O}N_H(D)}(\mathcal{O}N_K(D)) = \mathcal{O}N_H(D)^{N_K(D)}.$$

In extension (4.1) $C'[b_1]$ and $G'[b_1]$ stand for the analogous notation of $C[b]$ and of the group $G[b]$ respectively. Moreover, $\hat{k}_2$ is the field given by the quotient

$$C[b_1]_1/J(C[b_1]_1) = Z(b_1\mathcal{O}N_K(D))/J(b_1\mathcal{O}N_K(D)).$$

4.2. There is another epimorphism induced by the Brauer map

$$\mathrm{Br}_D^{N_H(D)} : b_1(\mathcal{O}N_H(D)_{b_1})^{N_K(D)} \to \bar{b}_1kC_H(D)_{\bar{b}_1}^{N_K(D)}, \tag{ii}$$

where $\bar{b}_1 = \mathrm{Br}_D^{N_H(D)}(b_1)$. As far as $\bar{b} = \bar{b}_1$ and

$$N_H(D)_b = N_H(D)_{b_1} = N_H(D)_{\bar{b}},$$

applied twice, [3, Theorem 4.1] gives the isomorphism

$$C[b]/C[b]J(C[b_1]_1) \simeq C'[b_1]/C'[b_1]J(C'[b_1]_1). \tag{4.2}$$

Note that the two quotients above are isomorphic as $N_H(D)_b/N_K(D) \simeq H_b/K$-algebras. In fact we have

$$\begin{aligned}(C[b]/C[b]J(C[b]_1))^{H_b} &= (C[b]/C[b]J(C[b]_1))^{H_b/K} \qquad (4.3)\\ \simeq (C'[b_1]/C'[b_1]J(C'[b_1]_1))^{N_H(D)_b/N_K(D)} &= (C'[b_1]/C'[b_1]J(C'[b_1]_1)^{N_H(D)_b}.\end{aligned}$$

Proposition 4.3. *There is a bijection between the primitive idempotents of $C[b]^{H_b}$ and the primitive idempotents of $C'[b_1]^{N_H(D)_b}$.*

Proof. The subalgebra of H_b-fixed elements of $C[b]$ lies in the center of $C[b]$, and the subalgebra of $N_H(D)_b$-fixed elements of $C'[b_1]$ lies in the center of $C'[b_1]$. Isomorphisms (4.2), (4.3) and [5, Lemma 3.1] give the desired bijection. □

Remark 4.4. Isomorphism (4.2), Proposition 4.3 and Lemma 2.4 give a bijection between the primitive idempotents of bC^{H_b} and the primitive idempotents of $b_1(C')^{N_H(D)_b}$. If $s' = \mathrm{Tr}_{N_H(D)_b}^{N_H(D)}(b_1)$, isomorphism (3.1) and its analogous isomorphism give a bijection between the blocks of $s\mathcal{O}H$ and the blocks of $s'\mathcal{O}N_H(D)$. Thus, we obtained a correspondence between the blocks of H that cover b and the blocks of $N_H(D)$ that cover b_1. We call this the *Clifford-Dade correspondence.*

5. The Harris - Knörr correspondence

With the above results and notations we have:

Theorem 5.1. *The isomorphic Clifford extensions of b and of b_1 define a defect group preserving bijective correspondence between blocks of $\mathcal{O}H$ covering b and blocks of $\mathcal{O}N_H(D)$ covering b_1. Moreover the Clifford-Dade correspondence between the blocks covering b and b_1 coincides with the Brauer correspondence.*

Proof. Remark 4.4 already gives a bijection between the blocks of H that cover b and the blocks of $N_H(D)$ that cover b_1. We prove that this bijection preserves the defect groups.

First of all let us emphasize that isomorphism (4.2) holds because the two Brauer homomorphisms introduced in (i) and (ii) verify

$$\mathrm{Br}_D^H(C[b]) = \mathrm{Br}_D^{N_H(D)}(C'[b_1]).$$

This last equality holds because both $C[b]$ and $C'[b_1]$ are crossed products. Taking a closer look at the proof of [3, Theorem 4.1] we observe that $C[b]/C[b]J(C[b]_1)$ as well as $C[b_1]/C[b_1]J(C'[b_1]_1)$ are both isomorphic to the twisted group algebra associated to the Clifford extension of $\bar{b} = \bar{b}_1$. It follows that the correspondence obtained in Proposition 4.3 connects the central idempotents B', which is primitive in $C[b]^{H_b}$, and B'_1, which is primitive in $C'[b_1]^{N_H(D)_b}$, that verify

$$\mathrm{Br}_D^H(B') = \mathrm{Br}_D^{N_H(D)}(B'_1). \tag{5.1}$$

Let B be the block covering b corresponding to B' through isomorphism (3.1). Note that it suffices to choose L, a defect group of B in H, such that $L \cap K = D$. Then, according to 3.3 there is $y \in H$ such that $Q := L^y$ is a defect group of B and of B' that is contained in H_b and satisfies $Q \cap K = D$; moreover $y \in N_H(D)$. Mackey decomposition and the equalities $Q \cap K = D$, $H_b = N_H(D)_b K$ prove

$$\mathrm{Br}_D^H((b\mathcal{O}H_b)_Q^{H_b}) = \mathrm{Br}_D^{N_H(D)}((b_1\mathcal{O}N_H(D)_b)_Q^{N_H(D)_b}) := \mathcal{I}'.$$

Indeed, since

$$\mathrm{Br}_D^H(b\mathcal{O}H_b^Q) = \mathrm{Br}_D^{N_H(D)}(b_1\mathcal{O}N_H(D)_b^Q),$$

if $\mathrm{Tr}_Q^{H_b}(a) \in (b\mathcal{O}H_b)_Q^{H_b}$ we have

$$\begin{aligned}\mathrm{Br}_D^H(\mathrm{Tr}_Q^{H_b}(a)) &= \mathrm{Br}_D^H(\sum_{x\in D\backslash H_b/Q} \mathrm{Tr}_{D\cap Q^x}^D(a^x)) \\ &= \mathrm{Br}_D^H(\sum_{x\in D\backslash N_H(D)_b/Q} a^x) = \mathrm{Tr}_Q^{N_H(D)_b}(\mathrm{Br}_D^H(a)) \\ &= \mathrm{Tr}_Q^{N_H(D)_b}(\mathrm{Br}_D^{N_H(D)}(a')) = \mathrm{Br}_D^{N_H(D)}(\mathrm{Tr}_Q^{N_H(D)_b}(a')).\end{aligned}$$

At this step [2, Proposition 1.5] gives the commutativity of the diagram

$$\begin{array}{ccccc} (b\mathcal{O}H_b)_Q^{H_b} & \xrightarrow{\mathrm{Br}_D^H} & \mathcal{I}' & \xleftarrow{\mathrm{Br}_D^{N_H(D)}} & (b_1\mathcal{O}N_H(D)_b)_Q^{N_H(D)_b} \\ & {\scriptstyle \mathrm{Br}_Q^H}\searrow & \downarrow{\scriptstyle \mathrm{Br}_{Q,D}} & \swarrow{\scriptstyle \mathrm{Br}_Q^{N_H(D)}} & \\ & & \mathrm{Br}_{Q,D}(\mathcal{I}'). & & \end{array}$$

This diagram and [9, Proposition 18.5 (d)] prove that there is a unique correspondent block $\tilde{B}_1' \in (b_1\mathcal{O}N_H(D)_b)^{N_H(D)_b}$ of B' with the same defect group Q in $N_H(D)_b$ as B'. Now we can clearly see, by the commutativity of the above diagram and equality (5.1), that

$$\mathrm{Br}_D^H(B') = \mathrm{Br}_D^{N_H(D)}(B_1') = \mathrm{Br}_D^{N_H(D)}(\tilde{B}_1') \neq 0.$$

This means $B_1' = \tilde{B}_1'$, and moreover that B_1 has defect group L. Hence, the Clifford-Dade correspondence preserves the defect groups. Furthermore, since the Clifford-Dade correspondence is given by the Brauer morphisms (i) and (ii) it is quite clear that it coincides with the Brauer correspondence. □

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Tiberiu Coconeţ
Babeş-Bolyai University
Faculty of Economics and Business Administration
Str. Teodor Mihali, nr. 58-60
400591 Cluj-Napoca, Romania
e-mail: tiberiu.coconet@math.ubbcluj.ro
tiberiu.coconet@econ.ubbcluj.ro

Stud. Univ. Babeş-Bolyai Math. 58(2013), No. 3, 297–323

Multivariate generalised fractional Pólya type integral inequalities

George A. Anastassiou

Abstract. Here we present a set of multivariate generalised fractional Pólya type integral inequalities on the ball and shell. We treat both the radial and non-radial cases in all possibilities. We give also estimates for the related averages.

Mathematics Subject Classification (2010): 26A33, 26D10, 26D15.

Keywords: Multivariate Pólya integral inequality, radial generalised fractional derivative, ball, shell.

1. Introduction

We mention the following famous Pólya's integral inequality, see [9], [10, p. 62], [11] and [12, p. 83].

Theorem 1.1. *Let $f(x)$ be differentiable and not identically a constant on $[a,b]$ with $f(a) = f(b) = 0$. Then the exists at least one point $\xi \in [a,b]$ such that*

$$|f'(\xi)| > \frac{4}{(b-a)^2} \int_a^b f(x)\,dx. \tag{1.1}$$

In [13], Feng Qi presents the following very interesting Pólya type integral inequality (1.2), which generalizes (1.1).

Theorem 1.2. *Let $f(x)$ be differentiable and not identically constant on $[a,b]$ with $f(a) = f(b) = 0$ and $M = \sup_{x\in[a,b]} |f'(x)|$. Then*

$$\left|\int_a^b f(x)\,dx\right| \leq \frac{(b-a)^2}{4} M, \tag{1.2}$$

where $\frac{(b-a)^2}{4}$ in (1.2) is the best constant.

The above motivate the current paper.

In this article we present multivariate fractional Pólya type integral inequalities in various cases, similar to (1.2).

For the last we need the following fractional calculus background.

Let $\alpha > 0$, $m = [\alpha]$ ($[\cdot]$ is the integral part), $\beta = \alpha - m$, $0 < \beta < 1$, $f \in C([a,b])$, $[a,b] \subset \mathbb{R}$, $x \in [a,b]$. The gamma function Γ is given by $\Gamma(\alpha) = \int_0^\infty e^{-t} t^{\alpha-1} dt$. We define the left Riemann-Liouville integral

$$\left(J_\alpha^{a+} f\right)(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)\, dt, \tag{1.3}$$

$a \le x \le b$. We define the subspace $C_{a+}^\alpha([a.b])$ of $C^m([a,b])$:

$$C_{a+}^\alpha([a,b]) = \left\{ f \in C^m([a,b]) : J_{1-\beta}^{a+} f^{(m)} \in C^1([a,b]) \right\}. \tag{1.4}$$

For $f \in C_{a+}^\alpha([a,b])$, we define the left generalized α-fractional derivative of f over $[a,b]$ as

$$D_{a+}^\alpha f := \left(J_{1-\beta}^{a+} f^{(m)} \right)', \tag{1.5}$$

see [1], p. 24. Canavati first in [5], introduced the above over $[0,1]$.

Notice that $D_{a+}^\alpha f \in C([a,b])$.

We need the following left fractional Taylor's formula, see [1], pp. 8-10, and in [5] the same over $[0,1]$ that appeared first.

Theorem 1.3. *Let $f \in C_{a+}^\alpha([a,b])$.*

(i) If $\alpha \ge 1$, then

$$f(x) = f(a) + f'(a)(x-a) + f''(a)\frac{(x-a)^2}{2} + ... + f^{(m-1)}(a)\frac{(x-a)^{m-1}}{(m-1)!} \tag{1.6}$$

$$+ \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} \left(D_{a+}^\alpha f\right)(t)\, dt, \quad \textit{all } x \in [a,b].$$

(ii) If $0 < \alpha < 1$, we have

$$f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} \left(D_{a+}^\alpha f\right)(t)\, dt, \quad \textit{all } x \in [a,b]. \tag{1.7}$$

Furthermore we need:

Let again $\alpha > 0$, $m = [\alpha]$, $\beta = \alpha - m$, $f \in C([a,b])$, call the right Riemann-Liouville fractional integral operator by

$$\left(J_{b-}^\alpha f\right)(x) := \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t)\, dt, \tag{1.8}$$

$x \in [a,b]$, see also [2], [6], [7], [8], [15]. Define the subspace of functions

$$C_{b-}^\alpha([a,b]) := \left\{ f \in C^m([a,b]) : J_{b-}^{1-\beta} f^{(m)} \in C^1([a,b]) \right\}. \tag{1.9}$$

Define the right generalized α-fractional derivative of f over $[a,b]$ as

$$D_{b-}^\alpha f = (-1)^{m-1} \left(J_{b-}^{1-\beta} f^{(m)} \right)', \tag{1.10}$$

see [2]. We set $D_{b-}^{0}f = f$. Notice that $D_{b-}^{\alpha}f \in C([a,b])$.

From [2], we need the following right Taylor fractional formula.

Theorem 1.4. *Let* $f \in C_{b-}^{\alpha}([a,b])$, $\alpha > 0$, $m := [\alpha]$. *Then*

(i) If $\alpha \geq 1$, *we get*

$$f(x) = \sum_{k=0}^{m-1} \frac{f^{(k)}(b)}{k!}(x-b)^k + \left(J_{b-}^{\alpha} D_{b-}^{\alpha} f\right)(x), \quad \text{all } x \in [a,b]. \tag{1.11}$$

(ii) If $0 < \alpha < 1$, *we get*

$$f(x) = J_{b-}^{\alpha} D_{b-}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} \left(D_{b-}^{\alpha} f\right)(t)\, dt, \quad \text{all } x \in [a,b]. \tag{1.12}$$

We need from [3]

Definition 1.5. *Let* $f \in C([a,b])$, $x \in [a,b]$, $\alpha > 0$, $m := [\alpha]$. *Assume that* $f \in C_{b-}^{\alpha}\left(\left[\frac{a+b}{2}, b\right]\right)$ *and* $f \in C_{a+}^{\alpha}\left(\left[a, \frac{a+b}{2}\right]\right)$. *We define the balanced Canavati type fractional derivative by*

$$D^{\alpha} f(x) := \begin{cases} D_{b-}^{\alpha} f(x), & \text{for } \frac{a+b}{2} \leq x \leq b, \\ D_{a+}^{\alpha} f(x), & \text{for } a \leq x < \frac{a+b}{2}. \end{cases} \tag{1.13}$$

In [4] *we proved the following fractional Pólya type integral inequality without any boundary conditions.*

Theorem 1.6. *Let* $0 < \alpha < 1$, $f \in C([a,b])$. *Assume* $f \in C_{a+}^{\alpha}\left(\left[a, \frac{a+b}{2}\right]\right)$ *and* $f \in C_{b-}^{\alpha}\left(\left[\frac{a+b}{2}, b\right]\right)$. *Set*

$$M_1(f) = \max\left\{ \left\|D_{a+}^{\alpha} f\right\|_{\infty,\left[a,\frac{a+b}{2}\right]}, \left\|D_{b-}^{\alpha} f\right\|_{\infty,\left[\frac{a+b}{2},b\right]} \right\}. \tag{1.14}$$

Then

$$\left|\int_a^b f(x)\,dx\right| \leq \int_a^b |f(x)|\,dx \leq \tag{1.15}$$

$$\frac{\left(\left\|D_{a+}^{\alpha} f\right\|_{\infty,\left[a,\frac{a+b}{2}\right]} + \left\|D_{b-}^{\alpha} f\right\|_{\infty,\left[\frac{a+b}{2},b\right]}\right)}{\Gamma(\alpha+2)} \left(\frac{b-a}{2}\right)^{\alpha+1} \leq M_1(f) \frac{(b-a)^{\alpha+1}}{\Gamma(\alpha+2)\, 2^{\alpha}}. \tag{1.16}$$

Inequalities (1.15), (1.16) *are sharp, namely they are attained by*

$$f_*(x) = \left\{ \begin{array}{ll} (x-a)^{\alpha}, & x \in \left[a, \frac{a+b}{2}\right] \\ (b-x)^{\alpha}, & x \in \left[\frac{a+b}{2}, b\right] \end{array} \right\}, \quad 0 < \alpha < 1. \tag{1.17}$$

Clearly here non zero constant functions f *are excluded.*

The last result also motivates this work.

Remark 1.7. (see [4]) When $\alpha \geq 1$, thus $m = [\alpha] \geq 1$, and by assuming that $f^{(k)}(a) = f^{(k)}(b) = 0$, $k = 0, 1, ..., m-1$, we can prove the same statements (1.15), (1.16), (1.17) as in Theorem 1.6. If we set there $\alpha = 1$ we derive exactly Theorem 1.2. So we have generalized Theorem 1.2. Again here $f^{(m)}$ cannot be a constant different than zero, equivalently, f cannot be a non-trivial polynomial of degree m.

We present Pólya type integral inequalities on the ball and shell.

2. Main results

We need

Remark 2.1. We define the ball $B(0,R)=\{x\in\mathbb{R}^N:|x|<R\}\subseteq\mathbb{R}^N$, $N\geq 2$, $R>0$, and the sphere

$$S^{N-1}:=\{x\in\mathbb{R}^N:|x|=1\},$$

where $|\cdot|$ is the Euclidean norm. Let $d\omega$ be the element of surface measure on S^{N-1} and let

$$\omega_N=\int_{S^{N-1}}d\omega=\frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}.$$

For $x\in\mathbb{R}^N-\{0\}$ we can write uniquely $x=r\omega$, where $r=|x|>0$ and $\omega=\frac{x}{r}\in S^{N-1}$, $|\omega|=1$. Note that $\int_{B(0,R)}dy=\frac{\omega_N R^N}{N}$ is the Lebesgue measure of the ball.

Following [14, pp. 149-150, exercise 6] and [16, pp. 87-88, Theorem 5.2.2] we can write $F:\overline{B(0,R)}\to\mathbb{R}$ a Lebesgue integrable function that

$$\int_{B(0,R)}F(x)\,dx=\int_{S^{N-1}}\left(\int_0^R F(r\omega)\,r^{N-1}dr\right)d\omega; \tag{2.1}$$

we use this formula a lot.

Initially the function $f:\overline{B(0,R)}\to\mathbb{R}$ is radial; that is, there exists a function g such that $f(x)=g(r)$, where $r=|x|$, $r\in[0,R]$, $\forall\, x\in\overline{B(0,R)}$, $\alpha>0$, $m=\lceil\alpha\rceil$. Here we assume that $g\in C([0,R])$ with $g\in C^{\alpha}_{0+}\left(\left[0,\frac{R}{2}\right]\right)$ and $g\in C^{\alpha}_{R-}\left(\left[\frac{R}{2},R\right]\right)$, such that $g^{(k)}(0)=g^{(k)}(R)=0$, $k=0,1,...,m-1$. In case of $0<\alpha<1$ then the last boundary conditions are void.

By assumption here and Theorem 1.3 we have

$$g(s)=\frac{1}{\Gamma(\alpha)}\int_0^s(s-t)^{\alpha-1}\left(D^{\alpha}_{0+}g\right)(t)\,dt, \tag{2.2}$$

all $s\in\left[0,\frac{R}{2}\right]$,

also it holds, by assumption and Theorem 1.4, that

$$g(s)=\frac{1}{\Gamma(\alpha)}\int_s^R(t-s)^{\alpha-1}\left(D^{\alpha}_{R-}g\right)(t)\,dt, \tag{2.3}$$

all $s\in\left[\frac{R}{2},R\right]$.

By (2.2) we get

$$|g(s)|\leq\frac{1}{\Gamma(\alpha)}\int_0^s(s-t)^{\alpha-1}\left|\left(D^{\alpha}_{0+}g\right)(t)\right|dt$$

$$\leq\left\|D^{\alpha}_{0+}g\right\|_{\infty,\left[0,\frac{R}{2}\right]}\frac{1}{\Gamma(\alpha)}\int_0^s(s-t)^{\alpha-1}dt=\frac{\left\|D^{\alpha}_{0+}g\right\|_{\infty,\left[0,\frac{R}{2}\right]}}{\Gamma(\alpha+1)}s^{\alpha}, \tag{2.4}$$

for any $s\in\left[0,\frac{R}{2}\right]$.

That is

$$|g(s)|\leq\frac{\left\|D^{\alpha}_{0+}g\right\|_{\infty,\left[0,\frac{R}{2}\right]}}{\Gamma(\alpha+1)}s^{\alpha}, \tag{2.5}$$

for any $s \in \left[0, \frac{R}{2}\right]$.

Similarly we obtain

$$|g(s)| \leq \frac{1}{\Gamma(\alpha)} \int_s^R (t-s)^{\alpha-1} \left| \left(D_{R-}^{\alpha} g \right)(t) \right| dt$$

$$\leq \frac{\left\| D_{R-}^{\alpha} g \right\|_{\infty, \left[\frac{R}{2}, R\right]}}{\Gamma(\alpha)} \int_s^R (t-s)^{\alpha-1} dt = \frac{\left\| D_{R-}^{\alpha} g \right\|_{\infty, \left[\frac{R}{2}, R\right]}}{\Gamma(\alpha+1)} (R-s)^{\alpha}, \tag{2.6}$$

for any $s \in \left[\frac{R}{2}, R\right]$.

I.e. it holds

$$|g(s)| \leq \frac{\left\| D_{R-}^{\alpha} g \right\|_{\infty, \left[\frac{R}{2}, R\right]}}{\Gamma(\alpha+1)} (R-s)^{\alpha}, \tag{2.7}$$

for any $s \in \left[\frac{R}{2}, R\right]$.

Next we observe that

$$\left| \int_{B(0,R)} f(y)\, dy \right| \leq \int_{B(0,R)} |f(y)|\, dy \overset{(2.1)}{=}$$

$$\int_{S^{N-1}} \left(\int_0^R |g(s)| s^{N-1} ds \right) d\omega = \left(\int_0^R |g(s)| s^{N-1} ds \right) \int_{S^{N-1}} d\omega =$$

$$\left(\int_0^R |g(s)| s^{N-1} ds \right) \frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} = \tag{2.8}$$

$$\frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \left\{ \int_0^{\frac{R}{2}} |g(s)| s^{N-1} ds + \int_{\frac{R}{2}}^R |g(s)| s^{N-1} ds \right\} \overset{\text{(by (2.5) and (2.7))}}{\leq}$$

$$\frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)\Gamma(\alpha+1)} \left\{ \left\| D_{0+}^{\alpha} g \right\|_{\infty, \left[0, \frac{R}{2}\right]} \int_0^{\frac{R}{2}} s^{\alpha+N-1} ds + \right.$$

$$\left. \left\| D_{R-}^{\alpha} g \right\|_{\infty, \left[\frac{R}{2}, R\right]} \int_{\frac{R}{2}}^R (R-s)^{\alpha} \left(\left(s - \frac{R}{2} \right) + \frac{R}{2} \right)^{N-1} ds \right\} = \tag{2.9}$$

$$\frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)\Gamma(\alpha+1)} \left\{ \frac{\left\| D_{0+}^{\alpha} g \right\|_{\infty, \left[0, \frac{R}{2}\right]}}{(\alpha+N)} \left(\frac{R}{2} \right)^{\alpha+N} + \left\| D_{R-}^{\alpha} g \right\|_{\infty, \left[\frac{R}{2}, R\right]} \cdot \right.$$

$$\left. \left[\sum_{k=0}^{N-1} \binom{N-1}{k} \left(\frac{R}{2} \right)^k \int_{\frac{R}{2}}^R (R-s)^{(\alpha+1)-1} \left(s - \frac{R}{2} \right)^{N-k-1} ds \right] \right\} =$$

$$\frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)\Gamma(\alpha+1)} \left\{ \frac{\left\| D_{0+}^{\alpha} g \right\|_{\infty, \left[0, \frac{R}{2}\right]}}{(\alpha+N)} \left(\frac{R}{2} \right)^{\alpha+N} + \right. \tag{2.10}$$

$$\left. \left\| D_{R-}^{\alpha} g \right\|_{\infty, \left[\frac{R}{2}, R\right]} \left[\sum_{k=0}^{N-1} \binom{N-1}{k} \left(\frac{R}{2} \right)^k \frac{\Gamma(\alpha+1)\Gamma(N-k)}{\Gamma(\alpha+N+1-k)} \left(\frac{R}{2} \right)^{\alpha+N-k} \right] \right\} =$$

$$\frac{\pi^{\frac{N}{2}} R^{\alpha+N}}{2^{\alpha+N-1}\Gamma\left(\frac{N}{2}\right)} \left\{ \frac{\left\|D_{0+}^{\alpha} g\right\|_{\infty,\left[0,\frac{R}{2}\right]}}{(\alpha+N)\,\Gamma(\alpha+1)} + \right.$$

$$\left. \left\|D_{R-}^{\alpha} g\right\|_{\infty,\left[\frac{R}{2},R\right]} (N-1)! \left[\sum_{k=0}^{N-1} \frac{1}{k!\Gamma(\alpha+N+1-k)}\right] \right\}. \quad (2.11)$$

We have proved that

$$\left|\int_{B(0,R)} f(y)\,dy\right| \leq \int_{B(0,R)} |f(y)|\,dy \leq$$

$$\frac{\pi^{\frac{N}{2}} R^{\alpha+N}}{2^{\alpha+N-1}\Gamma\left(\frac{N}{2}\right)} \left\{ \frac{\left\|D_{0+}^{\alpha} g\right\|_{\infty,\left[0,\frac{R}{2}\right]}}{(\alpha+N)\,\Gamma(\alpha+1)} + \right. \quad (2.12)$$

$$\left. (N-1)! \left\|D_{R-}^{\alpha} g\right\|_{\infty,\left[\frac{R}{2},R\right]} \left[\sum_{k=0}^{N-1} \frac{1}{k!\Gamma(\alpha+N+1-k)}\right] \right\}.$$

Consider now

$$g_*(s) = \begin{cases} s^{\alpha}, & s \in \left[0,\frac{R}{2}\right], \\ (R-s)^{\alpha}, & s \in \left[\frac{R}{2},R\right], \ \alpha > 0. \end{cases} \quad (2.13)$$

We have as in [4] that

$$D_{0+}^{\alpha} s^{\alpha} = \Gamma(\alpha+1), \ \text{ all } s \in \left[0,\frac{R}{2}\right], \quad (2.14)$$

and

$$\left\|D_{0+}^{\alpha} s^{\alpha}\right\|_{\infty,\left[0,\frac{R}{2}\right]} = \Gamma(\alpha+1).$$

Similarly as in [4] we get

$$D_{R-}^{\alpha} (R-s)^{\alpha} = \Gamma(\alpha+1), \ \text{ all } s \in \left[\frac{R}{2},R\right], \quad (2.15)$$

and

$$\left\|D_{R-}^{\alpha} (R-s)^{\alpha}\right\|_{\infty,\left[\frac{R}{2},R\right]} = \Gamma(\alpha+1). \quad (2.16)$$

That is

$$\left\|D_{0+}^{\alpha} g_*\right\|_{\infty,\left[0,\frac{R}{2}\right]} = \left\|D_{R-}^{\alpha} g_*\right\|_{\infty,\left[\frac{R}{2},R\right]} = \Gamma(\alpha+1). \quad (2.17)$$

Consequently we find that

$$\text{R.H.S. } (2.12) = \frac{\pi^{\frac{N}{2}} R^{\alpha+N}}{2^{\alpha+N-1}\Gamma\left(\frac{N}{2}\right)} \left\{ \frac{1}{(\alpha+N)} + \right.$$

$$\left. (N-1)!\Gamma(\alpha+1) \left[\sum_{k=0}^{N-1} \frac{1}{k!\Gamma(\alpha+N+1-k)}\right] \right\}. \quad (2.18)$$

Let $f_* : \overline{B(0,R)} \to \mathbb{R}$ be radial such that $f_*(x) = g_*(s)$, $s = |x|$, $s \in [0,R]$, $\forall$ $x \in \overline{B(0,R)}$.

Then we have

$$\text{L.H.S. } (2.12) = \int_{B(0,R)} f_*(y)\,dy \overset{(2.1)}{=}$$

$$\left(\int_0^R g_*(s)\,s^{N-1}ds\right)\frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} =$$

$$\frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}\left\{\int_0^{\frac{R}{2}} s^{\alpha+N-1}ds + \int_{\frac{R}{2}}^R (R-s)^\alpha\, s^{N-1}ds\right\} = \tag{2.19}$$

$$\frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}\left\{\left(\frac{R}{2}\right)^{\alpha+N}\frac{1}{(\alpha+N)} + \int_{\frac{R}{2}}^R (R-s)^\alpha\left(\left(s-\frac{R}{2}\right)+\frac{R}{2}\right)^{N-1}ds\right\} =$$

$$\frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}\left\{\frac{R^{\alpha+N}}{2^{\alpha+N}(\alpha+N)}+\right.$$

$$\left.\sum_{k=0}^{N-1}\binom{N-1}{k}\left(\frac{R}{2}\right)^k\int_{\frac{R}{2}}^R (R-s)^{(\alpha+1)-1}\left(s-\frac{R}{2}\right)^{N-k-1}ds\right\} = \tag{2.20}$$

$$\frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}\left\{\frac{R^{\alpha+N}}{2^{\alpha+N}(\alpha+N)}+\right.$$

$$\left.\sum_{k=0}^{N-1}\binom{N-1}{k}\left(\frac{R}{2}\right)^k\frac{\Gamma(\alpha+1)\Gamma(N-k)}{\Gamma(\alpha+N+1-k)}\left(\frac{R}{2}\right)^{\alpha+N-k}\right\} =$$

$$\frac{\pi^{\frac{N}{2}}R^{\alpha+N}}{\Gamma\left(\frac{N}{2}\right)2^{\alpha+N-1}}\left\{\frac{1}{(\alpha+N)}+\right.$$

$$\left.(N-1)!\Gamma(\alpha+1)\left[\sum_{k=0}^{N-1}\frac{1}{k!\Gamma(\alpha+N+1-k)}\right]\right\} \overset{(2.18)}{=} \text{R.H.S. } (2.12), \tag{2.21}$$

proving (2.12) sharp, in fact it is attained.

We have proved the following main result.

Theorem 2.2. *Let $f : \overline{B(0,R)} \to \mathbb{R}$ be radial; that is, there exists a function g such that $f(x) = g(s)$, $s = |x|$, $s \in [0,R]$, $\forall\, x \in \overline{B(0,R)}$, $\alpha > 0$. Assume that $g \in C([0,R])$, with $g \in C^\alpha_{0+}\left(\left[0,\frac{R}{2}\right]\right)$ and $g \in C^\alpha_{R-}\left(\left[\frac{R}{2},R\right]\right)$, such that $g^{(k)}(0) = g^{(k)}(R) = 0$, $k = 0,1,...,m-1$, $m = [\alpha]$. When $0 < \alpha < 1$ the last boundary conditions are void. Then*

$$\left|\int_{B(0,R)} f(y)\,dy\right| \le \int_{B(0,R)} |f(y)|\,dy \le$$

$$\frac{\pi^{\frac{N}{2}}R^{\alpha+N}}{2^{\alpha+N-1}\Gamma\left(\frac{N}{2}\right)}\left\{\frac{\left\|D^\alpha_{0+}g\right\|_{\infty,\left[0,\frac{R}{2}\right]}}{(\alpha+N)\Gamma(\alpha+1)}+\right. \tag{2.22}$$

$$\left.(N-1)!\left\|D^\alpha_{R-}g\right\|_{\infty,\left[\frac{R}{2},R\right]}\left[\sum_{k=0}^{N-1}\frac{1}{k!\Gamma(\alpha+N+1-k)}\right]\right\}.$$

Inequalities (2.22) are sharp, namely they are attained by a radial function f_* such that $f_*(x) = g_*(s)$, all $s \in [0, R]$, where

$$g_*(s) = \begin{cases} s^\alpha, & s \in \left[0, \frac{R}{2}\right], \\ (R-s)^\alpha, & s \in \left[\frac{R}{2}, R\right]. \end{cases} \tag{2.23}$$

We continue with

Remark 2.3. (Continuation of Remark 2.1) Here we assume that $\alpha \geq 1$. By (2.2) we get

$$|g(s)| \leq \frac{s^{\alpha-1}}{\Gamma(\alpha)} \left\| D_{0+}^\alpha g \right\|_{L_1\left(\left[0, \frac{R}{2}\right]\right)}, \tag{2.24}$$

all $s \in \left[0, \frac{R}{2}\right]$.

Also, by (2.3), we obtain

$$|g(s)| \leq \frac{(R-s)^{\alpha-1}}{\Gamma(\alpha)} \left\| D_{R-}^\alpha g \right\|_{L_1\left(\left[\frac{R}{2}, R\right]\right)}, \tag{2.25}$$

all $s \in \left[\frac{R}{2}, R\right]$.

Hence as in (2.8) we get

$$\int_{B(0,R)} |f(y)|\, dy \leq \frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \left(\int_0^R |g(s)|\, s^{N-1} ds \right) = \tag{2.26}$$

$$\frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \left\{ \int_0^{\frac{R}{2}} |g(s)|\, s^{N-1} ds + \int_{\frac{R}{2}}^R |g(s)|\, s^{N-1} ds \right\} \overset{\text{(by (2.24), (2.25))}}{\leq}$$

$$\frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)\Gamma(\alpha)} \left\{ \left(\int_0^{\frac{R}{2}} s^{N+\alpha-2} ds \right) \left\| D_{0+}^\alpha g \right\|_{L_1\left(\left[0, \frac{R}{2}\right]\right)} + \right. \tag{2.27}$$

$$\left. \left(\int_{\frac{R}{2}}^R (R-s)^{\alpha-1} s^{N-1} ds \right) \left\| D_{R-}^\alpha g \right\|_{L_1\left(\left[\frac{R}{2}, R\right]\right)} \right\} =$$

(acting the same as before, see (2.9)-(2.11))

$$\frac{\pi^{\frac{N}{2}} R^{\alpha+N-1}}{2^{\alpha+N-2}\Gamma\left(\frac{N}{2}\right)} \left\{ \frac{\left\| D_{0+}^\alpha g \right\|_{L_1\left(\left[0, \frac{R}{2}\right]\right)}}{(\alpha+N-1)\Gamma(\alpha)} + \right.$$

$$\left. (N-1)! \left\| D_{R-}^\alpha g \right\|_{L_1\left(\left[\frac{R}{2}, R\right]\right)} \left[\sum_{k=0}^{N-1} \frac{1}{k!\Gamma(\alpha+N-k)} \right] \right\} \overset{(1.13)}{=} \tag{2.28}$$

$$\frac{\pi^{\frac{N}{2}} R^{\alpha+N-1}}{2^{\alpha+N-2}\Gamma\left(\frac{N}{2}\right)} \left\{ \frac{\left\| D^\alpha g \right\|_{L_1\left(\left[0, \frac{R}{2}\right]\right)}}{(\alpha+N-1)\Gamma(\alpha)} + \right.$$

$$\left. (N-1)! \left\| D^\alpha g \right\|_{L_1\left(\left[\frac{R}{2}, R\right]\right)} \left[\sum_{k=0}^{N-1} \frac{1}{k!\Gamma(\alpha+N-k)} \right] \right\}. \tag{2.29}$$

We have proved

Theorem 2.4. *Here all terms and assumptions as in Theorem* 2.2, *however with* $\alpha \geq 1$. *Then*

$$\int_{B(0,R)} |f(y)|\, dy \leq \frac{\pi^{\frac{N}{2}} R^{\alpha+N-1}}{2^{\alpha+N-2}\Gamma\left(\frac{N}{2}\right)} \left\{ \frac{\left\|D_{0+}^{\alpha} g\right\|_{L_1\left(\left[0,\frac{R}{2}\right]\right)}}{(\alpha+N-1)\Gamma(\alpha)} + \right.$$

$$\left. (N-1)! \left\|D_{R-}^{\alpha} g\right\|_{L_1\left(\left[\frac{R}{2},R\right]\right)} \left[\sum_{k=0}^{N-1} \frac{1}{k!\Gamma(\alpha+N-k)}\right] \right\}. \tag{2.30}$$

We continue with

Remark 2.5. (Also a continuation of Remark 2.1) Let here $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$, with $\alpha > \frac{1}{q}$. By (2.2) we have

$$|g(s)| \leq \frac{1}{\Gamma(\alpha)} \int_0^s (s-t)^{\alpha-1} \left|\left(D_{0+}^{\alpha} g\right)(t)\right| dt \leq$$

$$\frac{1}{\Gamma(\alpha)} \left(\int_0^s (s-t)^{p(\alpha-1)} dt\right)^{\frac{1}{p}} \left(\int_0^s \left|\left(D_{0+}^{\alpha} g\right)(t)\right|^q dt\right)^{\frac{1}{q}} =$$

$$\frac{1}{\Gamma(\alpha)} \frac{s^{\alpha-1+\frac{1}{p}}}{(p(\alpha-1)+1)^{\frac{1}{p}}} \left\|D_{0+}^{\alpha} g\right\|_{L_q\left(\left[0,\frac{R}{2}\right]\right)}, \tag{2.31}$$

all $s \in \left[0, \frac{R}{2}\right]$.

Similarly by (2.3) we obtain

$$|g(s)| \leq \frac{1}{\Gamma(\alpha)} \int_s^R (t-s)^{\alpha-1} \left|\left(D_{R-}^{\alpha} g\right)(t)\right| dt \leq$$

$$\frac{1}{\Gamma(\alpha)} \left(\int_s^R (t-s)^{p(\alpha-1)} dt\right)^{\frac{1}{p}} \left(\int_s^R \left|\left(D_{R-}^{\alpha} g\right)(t)\right|^q dt\right)^{\frac{1}{q}} =$$

$$\frac{1}{\Gamma(\alpha)} \frac{(R-s)^{\alpha-1+\frac{1}{p}}}{(p(\alpha-1)+1)^{\frac{1}{p}}} \left\|D_{R-}^{\alpha} g\right\|_{L_q\left(\left[\frac{R}{2},R\right]\right)}, \tag{2.32}$$

all $s \in \left[\frac{R}{2}, R\right]$.

Hence it holds

$$\int_{B(0,R)} |f(y)|\, dy \overset{(2.8)}{=}$$

$$\frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \left\{ \int_0^{\frac{R}{2}} |g(s)|\, s^{N-1} ds + \int_{\frac{R}{2}}^R |g(s)|\, s^{N-1} ds \right\} \overset{\text{(by (2.31), (2.32))}}{\leq}$$

$$\frac{2\pi^{\frac{N}{2}}}{\Gamma(\alpha)\Gamma\left(\frac{N}{2}\right)(p(\alpha-1)+1)^{\frac{1}{p}}} \left\{ \left(\int_0^{\frac{R}{2}} s^{\alpha+N-2+\frac{1}{p}} ds\right) \left\|D_{0+}^{\alpha} g\right\|_{L_q\left(\left[0,\frac{R}{2}\right]\right)} + \right. \tag{2.33}$$

$$\left. \left(\int_{\frac{R}{2}}^R (R-s)^{\alpha-1+\frac{1}{p}} s^{N-1} ds\right) \left\|D_{R-}^{\alpha} g\right\|_{L_q\left(\left[\frac{R}{2},R\right]\right)} \right\} =$$

$$\frac{2\pi^{\frac{N}{2}}}{\Gamma(\alpha)\Gamma\left(\frac{N}{2}\right)(p(\alpha-1)+1)^{\frac{1}{p}}}\left\{\frac{\left(\frac{R}{2}\right)^{\left(\alpha+N-\frac{1}{q}\right)}}{\left(\alpha+N-\frac{1}{q}\right)}\left\|D_{0+}^{\alpha}g\right\|_{L_q\left(\left[0,\frac{R}{2}\right]\right)}+\right.$$

$$\left[\sum_{k=0}^{N-1}\binom{N-1}{k}\left(\frac{R}{2}\right)^{k}\left(\int_{\frac{R}{2}}^{R}(R-s)^{\left(\alpha+\frac{1}{p}-1\right)}\left(s-\frac{R}{2}\right)^{N-k-1}ds\right)\right]$$

$$\left.\left\|D_{R-}^{\alpha}g\right\|_{L_q\left(\left[\frac{R}{2},R\right]\right)}\right\}=$$

$$\frac{2\pi^{\frac{N}{2}}}{\Gamma(\alpha)\Gamma\left(\frac{N}{2}\right)(p(\alpha-1)+1)^{\frac{1}{p}}}\left\{\frac{R^{\left(\alpha+N-\frac{1}{q}\right)}}{\left(\alpha+N-\frac{1}{q}\right)2^{\left(\alpha+N-\frac{1}{q}\right)}}\left\|D_{0+}^{\alpha}g\right\|_{L_q\left(\left[0,\frac{R}{2}\right]\right)}+\right. \tag{2.34}$$

$$\left[\sum_{k=0}^{N-1}\frac{(N-1)!}{k!(N-k-1)!}\left(\frac{R}{2}\right)^{k}\frac{\Gamma\left(\alpha+\frac{1}{p}\right)\Gamma(N-k)}{\Gamma\left(\alpha+\frac{1}{p}+N-k\right)}\left(\frac{R}{2}\right)^{\alpha+\frac{1}{p}+N-k-1}\right]$$

$$\left.\left\|D_{R-}^{\alpha}g\right\|_{L_q\left(\left[\frac{R}{2},R\right]\right)}\right\}=$$

$$\frac{2\pi^{\frac{N}{2}}}{\Gamma(\alpha)\Gamma\left(\frac{N}{2}\right)(p(\alpha-1)+1)^{\frac{1}{p}}}\left\{\frac{R^{\left(\alpha+N-\frac{1}{q}\right)}}{\left(\alpha+N-\frac{1}{q}\right)2^{\left(\alpha+N-\frac{1}{q}\right)}}\left\|D_{0+}^{\alpha}g\right\|_{L_q\left(\left[0,\frac{R}{2}\right]\right)}+\right. \tag{2.35}$$

$$(N-1)!\Gamma\left(\alpha+\frac{1}{p}\right)\left(\frac{R^{\alpha+N-\frac{1}{q}}}{2^{\alpha+N-\frac{1}{q}}}\right)\cdot$$

$$\left.\left[\sum_{k=0}^{N-1}\frac{1}{k!\Gamma\left(\alpha+\frac{1}{p}+N-k\right)}\right]\left\|D_{R-}^{\alpha}g\right\|_{L_q\left(\left[\frac{R}{2},R\right]\right)}\right\}=$$

$$\frac{\pi^{\frac{N}{2}}R^{\alpha+N-\frac{1}{q}}}{\Gamma(\alpha)\Gamma\left(\frac{N}{2}\right)(p(\alpha-1)+1)^{\frac{1}{p}}2^{\alpha+N-\frac{1}{q}-1}}\left\{\frac{\left\|D_{0+}^{\alpha}g\right\|_{L_q\left(\left[0,\frac{R}{2}\right]\right)}}{\left(\alpha+N-\frac{1}{q}\right)}+\right.$$

$$\left.(N-1)!\Gamma\left(\alpha+\frac{1}{p}\right)\left[\sum_{k=0}^{N-1}\frac{1}{k!\Gamma\left(\alpha+\frac{1}{p}+N-k\right)}\right]\left\|D_{R-}^{\alpha}g\right\|_{L_q\left(\left[\frac{R}{2},R\right]\right)}\right\}. \tag{2.36}$$

We have proved the following

Theorem 2.6. *Let* $p,q>1:\frac{1}{p}+\frac{1}{q}=1$, $\alpha>\frac{1}{q}$. *All other terms and assumptions as in Theorem* 2.2. *Then*

$$\int_{B(0,R)}|f(y)|\,dy\leq$$

$$\frac{\pi^{\frac{N}{2}}R^{\alpha+N-\frac{1}{q}}}{\Gamma(\alpha)\Gamma\left(\frac{N}{2}\right)(p(\alpha-1)+1)^{\frac{1}{p}}2^{\alpha+N-\frac{1}{q}-1}}\left\{\frac{\left\|D_{0+}^{\alpha}g\right\|_{L_q\left(\left[0,\frac{R}{2}\right]\right)}}{\left(\alpha+N-\frac{1}{q}\right)}+\right.$$

$$(N-1)!\Gamma\left(\alpha+\frac{1}{p}\right)\left[\sum_{k=0}^{N-1}\frac{1}{k!\Gamma\left(\alpha+\frac{1}{p}+N-k\right)}\right]\left\|D_{R-}^{\alpha}g\right\|_{L_q\left(\left[\frac{R}{2},R\right]\right)}\Bigg\}. \quad (2.37)$$

Combining Theorems 2.2, 2.4, 2.6 we derive

Theorem 2.7. *Let any* $p,q>1:\frac{1}{p}+\frac{1}{q}=1$ *and* $\alpha\geq 1$*. And let* $f:\overline{B(0,R)}\to\mathbb{R}$ *be radial; that is, there exists a function* g *such that* $f(x)=g(s)$*,* $s=|x|$*,* $s\in[0,R]$*,* $\forall\ x\in\overline{B(0,R)}$*. Assume that* $g\in C([0,R])$*, with* $g\in C_{0+}^{\alpha}\left(\left[0,\frac{R}{2}\right]\right)$ *and* $g\in C_{R-}^{\alpha}\left(\left[\frac{R}{2},R\right]\right)$*, such that* $g^{(k)}(0)=g^{(k)}(R)=0$*,* $k=0,1,...,m-1$*,* $m=[\alpha]$*. When* $0<\alpha<1$ *the last boundary conditions are void. Then*

$$\left|\int_{B(0,R)}f(y)\,dy\right|\leq\int_{B(0,R)}|f(y)|\,dy\leq$$

$$\min\left\{\frac{\pi^{\frac{N}{2}}R^{\alpha+N}}{2^{\alpha+N-1}\Gamma\left(\frac{N}{2}\right)}\left\{\frac{\left\|D_{0+}^{\alpha}g\right\|_{\infty,\left[0,\frac{R}{2}\right]}}{(\alpha+N)\Gamma(\alpha+1)}+\right.\right.$$

$$\left.(N-1)!\left\|D_{R-}^{\alpha}g\right\|_{\infty,\left[\frac{R}{2},R\right]}\left[\sum_{k=0}^{N-1}\frac{1}{k!\Gamma(\alpha+N+1-k)}\right]\right\},$$

$$\frac{\pi^{\frac{N}{2}}R^{\alpha+N-1}}{2^{\alpha+N-2}\Gamma\left(\frac{N}{2}\right)}\left\{\frac{\left\|D_{0+}^{\alpha}g\right\|_{L_1\left(\left[0,\frac{R}{2}\right]\right)}}{(\alpha+N-1)\Gamma(\alpha)}+\right.$$

$$\left.(N-1)!\left\|D_{R-}^{\alpha}g\right\|_{L_1\left(\left[\frac{R}{2},R\right]\right)}\left[\sum_{k=0}^{N-1}\frac{1}{k!\Gamma(\alpha+N-k)}\right]\right\},$$

$$\frac{\pi^{\frac{N}{2}}R^{\alpha+N-\frac{1}{q}}}{\Gamma(\alpha)\Gamma\left(\frac{N}{2}\right)(p(\alpha-1)+1)^{\frac{1}{p}}2^{\alpha+N-\frac{1}{q}-1}}\left\{\frac{\left\|D_{0+}^{\alpha}g\right\|_{L_q\left(\left[0,\frac{R}{2}\right]\right)}}{\left(\alpha+N-\frac{1}{q}\right)}+\right.$$

$$\left.\left.(N-1)!\Gamma\left(\alpha+\frac{1}{p}\right)\left[\sum_{k=0}^{N-1}\frac{1}{k!\Gamma\left(\alpha+\frac{1}{p}+N-k\right)}\right]\left\|D_{R-}^{\alpha}g\right\|_{L_q\left(\left[\frac{R}{2},R\right]\right)}\right\}\right\}. \quad (2.38)$$

Note 2.8. It holds

$$Vol(B(0,R))=\frac{2\pi^{\frac{N}{2}}R^N}{\Gamma\left(\frac{N}{2}\right)N}. \quad (2.39)$$

The corresponding estimate on the average follows

Corollary 2.9. *Let all terms and assumptions as in Theorem 2.7. Then*

$$\left|\frac{1}{Vol(B(0,R))}\int_{B(0,R)}f(y)\,dy\right|\leq\frac{1}{Vol(B(0,R))}\int_{B(0,R)}|f(y)|\,dy\leq$$

$$\min\left\{\frac{NR^{\alpha}}{2^{\alpha+N}}\left\{\frac{\left\|D_{0+}^{\alpha}g\right\|_{\infty,\left[0,\frac{R}{2}\right]}}{(\alpha+N)\Gamma(\alpha+1)}+\right.\right.$$

$$(N-1)! \left\| D_{R-}^{\alpha} g \right\|_{\infty, \left[\frac{R}{2}, R\right]} \left[\sum_{k=0}^{N-1} \frac{1}{k! \Gamma(\alpha+N+1-k)} \right] \Bigg\},$$

$$\frac{NR^{\alpha-1}}{2^{\alpha+N-1}} \left\{ \frac{\left\| D_{0+}^{\alpha} g \right\|_{L_1\left(\left[0, \frac{R}{2}\right]\right)}}{(\alpha+N-1)\Gamma(\alpha)} + \right.$$

$$\left. (N-1)! \left\| D_{R-}^{\alpha} g \right\|_{L_1\left(\left[\frac{R}{2}, R\right]\right)} \left[\sum_{k=0}^{N-1} \frac{1}{k! \Gamma(\alpha+N-k)} \right] \right\},$$

$$\frac{NR^{\alpha-\frac{1}{q}}}{\Gamma(\alpha)\left(p(\alpha-1)+1\right)^{\frac{1}{p}} 2^{\alpha+N-\frac{1}{q}}} \left\{ \frac{\left\| D_{0+}^{\alpha} g \right\|_{L_q\left(\left[0, \frac{R}{2}\right]\right)}}{\left(\alpha+N-\frac{1}{q}\right)} + \right.$$

$$\left. (N-1)! \Gamma\left(\alpha+\frac{1}{p}\right) \left[\sum_{k=0}^{N-1} \frac{1}{k! \Gamma\left(\alpha+\frac{1}{p}+N-k\right)} \right] \left\| D_{R-}^{\alpha} g \right\|_{L_q\left(\left[\frac{R}{2}, R\right]\right)} \right\} \Bigg\}. \quad (2.40)$$

We continue with Pólya type inequalities on the ball for non-radial functions.

Theorem 2.10. *Let $f \in C\left(\overline{B(0,R)}\right)$ that is not necessarily radial, $0<\alpha<2$. Assume for any $\omega \in S^{N-1}$, that $f(\cdot\omega) \in C_{0+}^{\alpha}\left(\left[0, \frac{R}{2}\right]\right)$ and $f(\cdot\omega) \in C_{R-}^{\alpha}\left(\left[\frac{R}{2}, R\right]\right)$, such that $f(0) = f(R\omega) = 0$. When $0<\alpha<1$ the last boundary conditions are void. We further assume that*

$$\left\| \frac{\partial_{0+}^{\alpha} f(r\omega)}{\partial r^{\alpha}} \right\|_{\infty, \left(r \in \left[0, \frac{R}{2}\right]\right)}, \left\| \frac{\partial_{R-}^{\alpha} f(r\omega)}{\partial r^{\alpha}} \right\|_{\infty, \left(r \in \left[\frac{R}{2}, R\right]\right)} \le K, \quad (2.41)$$

for every $\omega \in S^{N-1}$, where $K>0$.

Then

(i)

$$\int_{B(0,R)} |f(y)| \, dy \le \frac{K \pi^{\frac{N}{2}} R^{\alpha+N}}{2^{\alpha+N-1} \Gamma\left(\frac{N}{2}\right)} \cdot \quad (2.42)$$

$$\left\{ \frac{1}{(\alpha+N)\Gamma(\alpha+1)} + (N-1)! \left[\sum_{k=0}^{N-1} \frac{1}{k! \Gamma(\alpha+N+1-k)} \right] \right\},$$

and

(ii)

$$\left| \frac{1}{Vol(B(0,R))} \int_{B(0,R)} f(y) \, dy \right| \le \frac{1}{Vol(B(0,R))} \int_{B(0,R)} |f(y)| \, dy \le \quad (2.43)$$

$$\frac{KNR^{\alpha}}{2^{\alpha+N}} \left\{ \frac{1}{(\alpha+N)\Gamma(\alpha+1)} + (N-1)! \left[\sum_{k=0}^{N-1} \frac{1}{k! \Gamma(\alpha+N+1-k)} \right] \right\}.$$

Proof. In Remark 2.1, see (2.8), (2.9), (2.10), (2.11), we proved that

$$\int_0^R |g(s)| s^{N-1} ds \leq \left(\frac{R}{2}\right)^{\alpha+N} \cdot$$

$$\left\{ \frac{\left\|D_{0+}^{\alpha} g\right\|_{\infty,\left[0,\frac{R}{2}\right]}}{(\alpha+N)\Gamma(\alpha+1)} + \left\|D_{R-}^{\alpha} g\right\|_{\infty,\left[\frac{R}{2},R\right]} (N-1)! \left[\sum_{k=0}^{N-1} \frac{1}{k!\Gamma(\alpha+N+1-k)}\right] \right\}. \tag{2.44}$$

In the above (2.44) we plug in $g(\cdot) = f(\cdot\omega)$, for $\omega \in S^{N-1}$ fixed, and we get

$$\int_0^R |f(s\omega)| s^{N-1} ds \overset{(2.41)}{\leq} K \left(\frac{R}{2}\right)^{\alpha+N} \cdot$$

$$\left\{ \frac{1}{(\alpha+N)\Gamma(\alpha+1)} + (N-1)! \left[\sum_{k=0}^{N-1} \frac{1}{k!\Gamma(\alpha+N+1-k)}\right] \right\} =: \lambda_1. \tag{2.45}$$

Consequently we obtain

$$\int_{B(0,R)} |f(y)| dy = \int_{S^{N-1}} \left(\int_0^R |f(s\omega)| s^{N-1} ds \right) d\omega \leq$$

$$\lambda_1 \int_{S^{N-1}} d\omega = \lambda_1 \frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}, \tag{2.46}$$

proving the claims. □

We continue with

Theorem 2.11. *Let $f \in C\left(\overline{B(0,R)}\right)$ that is not necessarily radial, $1 \leq \alpha < 2$. Assume for any $\omega \in S^{N-1}$, that $f(\cdot\omega) \in C_{0+}^{\alpha}\left(\left[0,\frac{R}{2}\right]\right)$ and $f(\cdot\omega) \in C_{R-}^{\alpha}\left(\left[\frac{R}{2},R\right]\right)$, such that $f(0) = f(R\omega) = 0$. We further assume*

$$\left\| \frac{\partial_{0+}^{\alpha} f(\cdot\omega)}{\partial r^{\alpha}} \right\|_{L_1\left(\left[0,\frac{R}{2}\right]\right)}, \left\| \frac{\partial_{R-}^{\alpha} f(\cdot\omega)}{\partial r^{\alpha}} \right\|_{L_1\left(\left[\frac{R}{2},R\right]\right)} \leq M, \tag{2.47}$$

for every $\omega \in S^{N-1}$, where $M > 0$.

Then

(i)

$$\int_{B(0,R)} |f(y)| dy \leq \frac{M\pi^{\frac{N}{2}} R^{\alpha+N-1}}{2^{\alpha+N-2}\Gamma\left(\frac{N}{2}\right)} \cdot \tag{2.48}$$

$$\left\{ \frac{1}{(\alpha+N-1)\Gamma(\alpha)} + (N-1)! \left[\sum_{k=0}^{N-1} \frac{1}{k!\Gamma(\alpha+N-k)}\right] \right\},$$

and

(ii)

$$\frac{1}{Vol(B(0,R))} \int_{B(0,R)} |f(y)| dy \leq \tag{2.49}$$

$$\frac{MNR^{\alpha-1}}{2^{\alpha+N-1}}\left\{\frac{1}{(\alpha+N-1)\,\Gamma(\alpha)}+(N-1)!\left[\sum_{k=0}^{N-1}\frac{1}{k!\Gamma(\alpha+N-k)}\right]\right\}.$$

Proof. In Remark 2.3, see (2.26), (2.27), (2.28), we proved that

$$\int_0^R |g(s)|\, s^{N-1}ds \leq \left(\frac{R}{2}\right)^{\alpha+N-1}\cdot$$

$$\left\{\frac{\left\|D_{0+}^{\alpha}g\right\|_{L_1\left(\left[0,\frac{R}{2}\right]\right)}}{(\alpha+N-1)\,\Gamma(\alpha)}+\left\|D_{R-}^{\alpha}g\right\|_{L_1\left(\left[\frac{R}{2},R\right]\right)}(N-1)!\left[\sum_{k=0}^{N-1}\frac{1}{k!\Gamma(\alpha+N-k)}\right]\right\}. \quad (2.50)$$

In the above (2.50) we plug in $g(\cdot)=f(\cdot\omega)$, for $\omega\in S^{N-1}$ fixed, and we derive

$$\int_0^R |f(s\omega)|\, s^{N-1}ds \overset{(2.47)}{\leq} M\left(\frac{R}{2}\right)^{\alpha+N-1}\cdot$$

$$\left\{\frac{1}{(\alpha+N-1)\,\Gamma(\alpha)}+(N-1)!\left[\sum_{k=0}^{N-1}\frac{1}{k!\Gamma(\alpha+N-k)}\right]\right\}=:\lambda_2. \quad (2.51)$$

Hence

$$\int_{B(0,R)}|f(y)|\,dy=\int_{S^{N-1}}\left(\int_0^R |f(s\omega)|\, s^{N-1}ds\right)d\omega\leq$$

$$\lambda_2\int_{S^{N-1}}d\omega=\lambda_2\frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}, \quad (2.52)$$

proving the claims. □

We further have

Theorem 2.12. *Let $p,q>1:\frac{1}{p}+\frac{1}{q}=1$, and $\frac{1}{q}<\alpha<2$. Let $f\in C\left(\overline{B(0,R)}\right)$ that is not necessarily radial. Assume for any $\omega\in S^{N-1}$, that $f(\cdot\omega)\in C_{0+}^{\alpha}\left(\left[0,\frac{R}{2}\right]\right)$ and $f(\cdot\omega)\in C_{R-}^{\alpha}\left(\left[\frac{R}{2},R\right]\right)$, such that $f(0)=f(R\omega)=0$. When $\frac{1}{q}<\alpha<1$ the last boundary condition is void. We further assume*

$$\left\|\frac{\partial_{0+}^{\alpha}f(\cdot\omega)}{\partial r^{\alpha}}\right\|_{L_q\left(\left[0,\frac{R}{2}\right]\right)},\ \left\|\frac{\partial_{R-}^{\alpha}f(\cdot\omega)}{\partial r^{\alpha}}\right\|_{L_q\left(\left[\frac{R}{2},R\right]\right)}\leq\Phi, \quad (2.53)$$

for every $\omega\in S^{N-1}$, where $\Phi>0$.

Then

(i)

$$\int_{B(0,R)}|f(y)|\,dy\leq\frac{\Phi\pi^{\frac{N}{2}}R^{\alpha+N-\frac{1}{q}}}{\Gamma(\alpha)\,\Gamma\left(\frac{N}{2}\right)(p(\alpha-1)+1)^{\frac{1}{p}}\,2^{\alpha+N-\frac{1}{q}-1}}\cdot \quad (2.54)$$

$$\left\{\frac{1}{\left(\alpha+N-\frac{1}{q}\right)}+(N-1)!\Gamma\left(\alpha+\frac{1}{p}\right)\left[\sum_{k=0}^{N-1}\frac{1}{k!\Gamma\left(\alpha+\frac{1}{p}+N-k\right)}\right]\right\},$$

and

(ii)

$$\frac{1}{Vol\left(B\left(0,R\right)\right)}\int_{B(0,R)}\left|f\left(y\right)\right|dy\leq\frac{\Phi NR^{\alpha-\frac{1}{q}}}{\Gamma\left(\alpha\right)\left(p\left(\alpha-1\right)+1\right)^{\frac{1}{p}}2^{\alpha+N-\frac{1}{q}}}\cdot \tag{2.55}$$

$$\left\{\frac{1}{\left(\alpha+N-\frac{1}{q}\right)}+\left(N-1\right)!\Gamma\left(\alpha+\frac{1}{p}\right)\left[\sum_{k=0}^{N-1}\frac{1}{k!\Gamma\left(\alpha+\frac{1}{p}+N-k\right)}\right]\right\}.$$

Proof. In Remark 2.5, see (2.33), (2.34), (2.35), (2.36), we proved that

$$\int_0^R\left|g\left(s\right)\right|s^{N-1}ds\leq$$

$$\left(\frac{R}{2}\right)^{\alpha+N-\frac{1}{q}}\frac{1}{\Gamma\left(\alpha\right)\left(p\left(\alpha-1\right)+1\right)^{\frac{1}{p}}}\cdot\left\{\frac{\left\|D_{0+}^{\alpha}g\right\|_{L_q\left(\left[0,\frac{R}{2}\right]\right)}}{\left(\alpha+N-\frac{1}{q}\right)}+\right. \tag{2.56}$$

$$\left.\left(N-1\right)!\Gamma\left(\alpha+\frac{1}{p}\right)\left[\sum_{k=0}^{N-1}\frac{1}{k!\Gamma\left(\alpha+\frac{1}{p}+N-k\right)}\right]\left\|D_{R-}^{\alpha}g\right\|_{L_q\left(\left[\frac{R}{2},R\right]\right)}\right\}.$$

In the above (2.56) we plug in $g\left(\cdot\right)=f\left(\cdot\omega\right)$, for $\omega\in S^{N-1}$ fixed, and we find

$$\int_0^R\left|f\left(s\omega\right)\right|s^{N-1}ds\overset{(2.53)}{\leq}\Phi\left(\frac{R}{2}\right)^{\alpha+N-\frac{1}{q}}\frac{1}{\Gamma\left(\alpha\right)\left(p\left(\alpha-1\right)+1\right)^{\frac{1}{p}}}\cdot$$

$$\left\{\frac{1}{\left(\alpha+N-\frac{1}{q}\right)}+\left(N-1\right)!\Gamma\left(\alpha+\frac{1}{p}\right)\left[\sum_{k=0}^{N-1}\frac{1}{k!\Gamma\left(\alpha+\frac{1}{p}+N-k\right)}\right]\right\}=:\lambda_3. \tag{2.57}$$

Thus

$$\int_{B(0,R)}\left|f\left(y\right)\right|dy=\int_{S^{N-1}}\left(\int_0^R\left|f\left(s\omega\right)\right|s^{N-1}ds\right)d\omega\leq$$

$$\lambda_3\int_{S^{N-1}}d\omega=\lambda_3\frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}, \tag{2.58}$$

proving the claims. □

We make

Remark 2.13. Let the spherical shell $A:=B\left(0,R_2\right)-\overline{B\left(0,R_1\right)}$, $0<R_1<R_2$, $A\subseteq\mathbb{R}^N$, $N\geq2$, $x\in\overline{A}$. Consider first that $f:\overline{A}\rightarrow\mathbb{R}$ is radial; that is, there exists g such that $f\left(x\right)=g\left(r\right)$, $r=|x|$, $r\in\left[R_1,R_2\right]$, $\forall\ x\in\overline{A}$. Here x can be written uniquely as $x=r\omega$, where $r=|x|>0$ and $\omega=\frac{x}{r}\in S^{N-1}$, $|\omega|=1$, see ([14], p. 149-150 and [1], p. 421), furthermore for general $F:\overline{A}\rightarrow\mathbb{R}$ Lebesgue integrable function we have that

$$\int_A F\left(x\right)dx=\int_{S^{N-1}}\left(\int_{R_1}^{R_2}F\left(r\omega\right)r^{N-1}dr\right)d\omega. \tag{2.59}$$

Let $d\omega$ be the element of surface measure on S^{N-1}, then

$$\omega_N := \int_{S^{N-1}} d\omega = \frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}. \tag{2.60}$$

Here

$$Vol\,(A) = \frac{\omega_N\left(R_2^N - R_1^N\right)}{N} = \frac{2\pi^{\frac{N}{2}}\left(R_2^N - R_1^N\right)}{N\Gamma\left(\frac{N}{2}\right)}. \tag{2.61}$$

We assume that $g \in C([R_1, R_2])$, and $\alpha > 0$, $m = [\alpha]$, such that $g \in C^{\alpha}_{R_1+}\left(\left[R_1, \frac{R_1+R_2}{2}\right]\right)$ and $g \in C^{\alpha}_{R_2-}\left(\left[\frac{R_1+R_2}{2}, R_2\right]\right)$, with $g^{(k)}(R_1) = g^{(k)}(R_2) = 0$, $k = 0, 1, ..., m-1$. When $0 < \alpha < 1$ the last boundary conditions are void.

By assumption here and Theorem 1.3 we have

$$g(s) = \frac{1}{\Gamma(\alpha)} \int_{R_1}^{s} (s-t)^{\alpha-1}\left(D^{\alpha}_{R_1+}g\right)(t)\,dt, \tag{2.62}$$

all $s \in \left[R_1, \frac{R_1+R_2}{2}\right]$,

also it holds, by assumption and Theorem 1.4, that

$$g(s) = \frac{1}{\Gamma(\alpha)} \int_{s}^{R_2} (t-s)^{\alpha-1}\left(D^{\alpha}_{R_2-}g\right)(t)\,dt, \tag{2.63}$$

all $s \in \left[\frac{R_1+R_2}{2}, R_2\right]$.

By (2.62) we get

$$|g(s)| \leq \frac{1}{\Gamma(\alpha)} \int_{R_1}^{s} (s-t)^{\alpha-1}\left|\left(D^{\alpha}_{R_1+}g\right)(t)\right| dt \tag{2.64}$$

$$\leq \left\|D^{\alpha}_{R_1+}g\right\|_{\infty,\left[R_1, \frac{R_1+R_2}{2}\right]} \frac{(s-R_1)^{\alpha}}{\Gamma(\alpha+1)}, \tag{2.65}$$

for any $s \in \left[R_1, \frac{R_1+R_2}{2}\right]$.

Similarly we obtain by (2.63) that

$$|g(s)| \leq \frac{1}{\Gamma(\alpha)} \int_{s}^{R_2} (t-s)^{\alpha-1}\left|\left(D^{\alpha}_{R_2-}g\right)(t)\right| dt \tag{2.66}$$

$$\leq \left\|D^{\alpha}_{R_2-}g\right\|_{\infty,\left[\frac{R_1+R_2}{2}, R_2\right]} \frac{(R_2-s)^{\alpha}}{\Gamma(\alpha+1)}, \tag{2.67}$$

for any $s \in \left[\frac{R_1+R_2}{2}, R_2\right]$.

Next we observe that

$$\left|\int_A f(y)\,dy\right| \leq \int_A |f(y)|\,dy \overset{(2.59)}{=} \tag{2.68}$$

$$\int_{S^{N-1}} \left(\int_{R_1}^{R_2} |g(s)|\, s^{N-1} ds\right) d\omega = \left(\int_{R_1}^{R_2} |g(s)|\, s^{N-1} ds\right) \frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} = \tag{2.69}$$

$$\frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \left\{\int_{R_1}^{\frac{R_1+R_2}{2}} |g(s)|\, s^{N-1} ds + \int_{\frac{R_1+R_2}{2}}^{R_2} |g(s)|\, s^{N-1} ds\right\} \overset{\text{(by (2.65) and (2.67))}}{\leq}$$

$$\frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)\Gamma\left(\alpha+1\right)}\left\{\left\|D_{R_1+}^{\alpha}g\right\|_{\infty,\left[R_1,\frac{R_1+R_2}{2}\right]}\int_{R_1}^{\frac{R_1+R_2}{2}}\left(s-R_1\right)^{\alpha}s^{N-1}ds\right.$$

$$\left.+\left\|D_{R_2-}^{\alpha}g\right\|_{\infty,\left[\frac{R_1+R_2}{2},R_2\right]}\int_{\frac{R_1+R_2}{2}}^{R_2}\left(R_2-s\right)^{\alpha}s^{N-1}ds\right\}= \tag{2.70}$$

$$\frac{\pi^{\frac{N}{2}}\left(N-1\right)!}{\Gamma\left(\frac{N}{2}\right)2^{\alpha+N-1}}\left\{\left\|D_{R_1+}^{\alpha}g\right\|_{\infty,\left[R_1,\frac{R_1+R_2}{2}\right]}\right.$$

$$\left(\sum_{k=0}^{N-1}\left(-1\right)^{N+k-1}\frac{\left(R_1+R_2\right)^k\left(R_2-R_1\right)^{N-k+\alpha}}{k!\Gamma\left(N-k+\alpha+1\right)}\right)+$$

$$\left.\left\|D_{R_2-}^{\alpha}g\right\|_{\infty,\left[\frac{R_1+R_2}{2},R_2\right]}\left[\sum_{k=0}^{N-1}\frac{\left(R_1+R_2\right)^k\left(R_2-R_1\right)^{\alpha+N-k}}{k!\Gamma\left(\alpha+1+N-k\right)}\right]\right\}. \tag{2.71}$$

We have proved that

$$\left|\int_A f\left(y\right)dy\right|\leq\int_A\left|f\left(y\right)\right|dy\leq\frac{\pi^{\frac{N}{2}}\left(N-1\right)!}{\Gamma\left(\frac{N}{2}\right)2^{\alpha+N-1}}\cdot$$

$$\left\{\left\|D_{R_1+}^{\alpha}g\right\|_{\infty,\left[R_1,\frac{R_1+R_2}{2}\right]}\left(\sum_{k=0}^{N-1}\frac{\left(-1\right)^{N+k-1}\left(R_1+R_2\right)^k\left(R_2-R_1\right)^{N-k+\alpha}}{k!\Gamma\left(N-k+\alpha+1\right)}\right)+\right.$$

$$\left.\left\|D_{R_2-}^{\alpha}g\right\|_{\infty,\left[\frac{R_1+R_2}{2},R_2\right]}\left[\sum_{k=0}^{N-1}\frac{\left(R_1+R_2\right)^k\left(R_2-R_1\right)^{\alpha+N-k}}{k!\Gamma\left(\alpha+1+N-k\right)}\right]\right\}. \tag{2.72}$$

Consider now $f_*:\overline{A}\to\mathbb{R}$ be radial such that $f_*\left(x\right)=g_*\left(s\right)$, $s=\left|x\right|$, $s\in\left[R_1,R_2\right]$, $\forall x\in\overline{A}$, where

$$g_*\left(s\right)=\begin{cases}\left(s-R_1\right)^{\alpha}, & s\in\left[R_1,\frac{R_1+R_2}{2}\right],\\ \left(R_2-s\right)^{\alpha}, & s\in\left[\frac{R_1+R_2}{2},R_2\right],\ \alpha>0.\end{cases} \tag{2.73}$$

We have, as in [4], that

$$\left\|D_{R_1+}^{\alpha}g_*\right\|_{\infty,\left[R_1,\frac{R_1+R_2}{2}\right]}=\Gamma\left(\alpha+1\right),\quad\text{and}\quad\left\|D_{R_2-}^{\alpha}g_*\right\|_{\infty,\left[\frac{R_1+R_2}{2},R_2\right]}=\Gamma\left(\alpha+1\right). \tag{2.74}$$

Hence

$$\text{R.H.S. (2.72) (applied on } g_*)=\frac{\Gamma\left(\alpha+1\right)\pi^{\frac{N}{2}}\left(N-1\right)!}{\Gamma\left(\frac{N}{2}\right)2^{\alpha+N-1}}\cdot$$

$$\left\{\sum_{k=0}^{N-1}\left(1+\left(-1\right)^{N+k-1}\right)\frac{\left(R_1+R_2\right)^k\left(R_2-R_1\right)^{\alpha+N-k}}{k!\Gamma\left(\alpha+1+N-k\right)}\right\}. \tag{2.75}$$

Furthermore we find

$$\text{L.H.S. (2.72) (applied on } f_*)=\int_A f_*\left(y\right)dy=$$

$$\left(\int_{R_1}^{R_2}g_*\left(s\right)s^{N-1}ds\right)\frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}=$$

$$\frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}\left\{\int_{R_1}^{\frac{R_1+R_2}{2}}(s-R_1)^{\alpha}s^{N-1}ds+\int_{\frac{R_1+R_2}{2}}^{R_2}(R_2-s)^{\alpha}s^{N-1}ds\right\}= \tag{2.76}$$

$$\frac{\pi^{\frac{N}{2}}(N-1)!\Gamma(\alpha+1)}{\Gamma\left(\frac{N}{2}\right)2^{N+\alpha-1}}\left\{\left(\sum_{k=0}^{N-1}\frac{(-1)^{N+k-1}(R_1+R_2)^k(R_2-R_1)^{N+\alpha-k}}{k!\Gamma(N+\alpha+1-k)}\right)+\right. \tag{2.77}$$

$$\left.\left(\sum_{k=0}^{N-1}\frac{(R_1+R_2)^k(R_2-R_1)^{\alpha+N-k}}{k!\Gamma(\alpha+1+N-k)}\right)\right\}=$$

$$\frac{\pi^{\frac{N}{2}}(N-1)!\Gamma(\alpha+1)}{\Gamma\left(\frac{N}{2}\right)2^{N+\alpha-1}}\left\{\sum_{k=0}^{N-1}\left((-1)^{N+k-1}+1\right)\frac{(R_1+R_2)^k(R_2-R_1)^{N+\alpha-k}}{k!\Gamma(N+\alpha+1-k)}\right\}. \tag{2.78}$$

So that we find

$$\text{R.H.S. (2.72) (applied on } g_*) = \text{L.H.S. (2.72) (applied on } f_*), \tag{2.79}$$

proving sharpness of (2.72).

We have proved the following

Theorem 2.14. *Let $f:\overline{A}\to\mathbb{R}$ be radial; that is, there exists a function g such that $f(x)=g(s)$, $s=|x|$, $s\in[R_1,R_2]$, $\forall\ x\in\overline{A}$, $\alpha>0$, $m=[\alpha]$. We assume that $g\in C([R_1,R_2])$, such that $g\in C^{\alpha}_{R_1+}\left(\left[R_1,\frac{R_1+R_2}{2}\right]\right)$ and $g\in C^{\alpha}_{R_2-}\left(\left[\frac{R_1+R_2}{2},R_2\right]\right)$,with $g^{(k)}(R_1)=g^{(k)}(R_2)=0$, $k=0,1,...,m-1$. When $0<\alpha<1$ the last boundary conditions are void. Then*

$$\left|\int_A f(y)\,dy\right|\le\int_A|f(y)|\,dy\le\frac{\pi^{\frac{N}{2}}(N-1)!}{\Gamma\left(\frac{N}{2}\right)2^{\alpha+N-1}}\cdot$$

$$\left\{\left\|D^{\alpha}_{R_1+}g\right\|_{\infty,\left[R_1,\frac{R_1+R_2}{2}\right]}\left(\sum_{k=0}^{N-1}\frac{(-1)^{N+k-1}(R_1+R_2)^k(R_2-R_1)^{N-k+\alpha}}{k!\Gamma(N-k+\alpha+1)}\right)+\right. \tag{2.80}$$

$$\left.\left\|D^{\alpha}_{R_2-}g\right\|_{\infty,\left[\frac{R_1+R_2}{2},R_2\right]}\left(\sum_{k=0}^{N-1}\frac{(R_1+R_2)^k(R_2-R_1)^{\alpha+N-k}}{k!\Gamma(\alpha+1+N-k)}\right)\right\}.$$

Inequalities (2.80) are sharp, namely they are attained by the radial function $f_*:\overline{A}\to\mathbb{R}$ such that $f_*(x)=g_*(s)$, $s=|x|$, $s\in[R_1,R_2]$, $\forall\ x\in\overline{A}$, where

$$g_*(s)=\begin{cases}(s-R_1)^{\alpha}, & s\in\left[R_1,\frac{R_1+R_2}{2}\right],\\ (R_2-s)^{\alpha}, & s\in\left[\frac{R_1+R_2}{2},R_2\right].\end{cases} \tag{2.81}$$

We continue with

Remark 2.15. Here $\alpha\ge1$. By (2.64) we get

$$|g(s)|\le\frac{(s-R_1)^{\alpha-1}}{\Gamma(\alpha)}\left\|D^{\alpha}_{R_1+}g\right\|_{L_1\left(\left[R_1,\frac{R_1+R_2}{2}\right]\right)}, \tag{2.82}$$

for any $s\in\left[R_1,\frac{R_1+R_2}{2}\right]$.

And by (2.66) we derive

$$|g(s)| \leq \frac{(R_2 - s)^{\alpha-1}}{\Gamma(\alpha)} \left\| D^{\alpha}_{R_2-} g \right\|_{L_1\left(\left[\frac{R_1+R_2}{2}, R_2\right]\right)}, \tag{2.83}$$

for any $s \in \left[\frac{R_1+R_2}{2}, R_2\right]$.

Hence

$$\int_A |f(y)| \, dy \overset{(2.69)}{=}$$

$$\frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \left\{ \int_{R_1}^{\frac{R_1+R_2}{2}} |g(s)| \, s^{N-1} ds + \int_{\frac{R_1+R_2}{2}}^{R_2} |g(s)| \, s^{N-1} ds \right\} \overset{\text{(by (2.82) and (2.83))}}{\leq} \tag{2.84}$$

$$\frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)\Gamma(\alpha)} \left\{ \left\| D^{\alpha}_{R_1+} g \right\|_{L_1\left(\left[R_1, \frac{R_1+R_2}{2}\right]\right)} \left(\int_{R_1}^{\frac{R_1+R_2}{2}} (s - R_1)^{\alpha-1} s^{N-1} ds \right) + \right. \tag{2.85}$$

$$\left. \left\| D^{\alpha}_{R_2-} g \right\|_{L_1\left(\left[\frac{R_1+R_2}{2}, R_2\right]\right)} \left(\int_{\frac{R_1+R_2}{2}}^{R_2} (R_2 - s)^{\alpha-1} s^{N-1} ds \right) \right\} =$$

$$\frac{\pi^{\frac{N}{2}} (N-1)!}{\Gamma\left(\frac{N}{2}\right) 2^{\alpha+N-2}} \left\{ \left\| D^{\alpha}_{R_1+} g \right\|_{L_1\left(\left[R_1, \frac{R_1+R_2}{2}\right]\right)} \right.$$

$$\left(\sum_{k=0}^{N-1} (-1)^{N+k-1} \frac{(R_1+R_2)^k (R_2 - R_1)^{N+\alpha-k-1}}{k! \Gamma(N+\alpha-k)} \right) + \tag{2.86}$$

$$\left. \left\| D^{\alpha}_{R_2-} g \right\|_{L_1\left(\left[\frac{R_1+R_2}{2}, R_2\right]\right)} \left(\sum_{k=0}^{N-1} \frac{(R_1+R_2)^k (R_2 - R_1)^{N+\alpha-k-1}}{k! \Gamma(N+\alpha-k)} \right) \right\}.$$

We have proved that

Theorem 2.16. *All terms and assumptions here as in Theorem* 2.14*, but with* $\alpha \geq 1$. *Then*

$$\int_A |f(y)| \, dy \leq \frac{\pi^{\frac{N}{2}} (N-1)!}{\Gamma\left(\frac{N}{2}\right) 2^{\alpha+N-2}} \cdot$$

$$\left\{ \left\| D^{\alpha}_{R_1+} g \right\|_{L_1\left(\left[R_1, \frac{R_1+R_2}{2}\right]\right)} \left(\sum_{k=0}^{N-1} (-1)^{N+k-1} \frac{(R_1+R_2)^k (R_2 - R_1)^{N+\alpha-k-1}}{k! \Gamma(N+\alpha-k)} \right) + \right.$$

$$\left. \left\| D^{\alpha}_{R_2-} g \right\|_{L_1\left(\left[\frac{R_1+R_2}{2}, R_2\right]\right)} \left(\sum_{k=0}^{N-1} \frac{(R_1+R_2)^k (R_2 - R_1)^{N+\alpha-k-1}}{k! \Gamma(N+\alpha-k)} \right) \right\}. \tag{2.87}$$

We continue with

Remark 2.17. Let $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$. Let $\alpha > \frac{1}{q}$. By (2.64) we get

$$|g(s)| \leq \frac{(s - R_1)^{\alpha-1+\frac{1}{p}}}{\Gamma(\alpha)(p(\alpha-1)+1)^{\frac{1}{p}}} \left\| D^{\alpha}_{R_1+} g \right\|_{L_q\left(\left[R_1, \frac{R_1+R_2}{2}\right]\right)}, \tag{2.88}$$

for any $s \in \left[R_1, \frac{R_1+R_2}{2}\right]$.

Similarly by (2.66) we derive

$$|g(s)| \leq \frac{(R_2 - s)^{\alpha - 1 + \frac{1}{p}}}{\Gamma(\alpha)(p(\alpha-1)+1)^{\frac{1}{p}}} \left\| D^{\alpha}_{R_2 -} g \right\|_{L_q\left(\left[\frac{R_1+R_2}{2}, R_2\right]\right)}, \tag{2.89}$$

for any $s \in \left[\frac{R_1+R_2}{2}, R_2\right]$.

Hence

$$\int_A |f(y)|\, dy =$$

$$\frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \left\{ \int_{R_1}^{\frac{R_1+R_2}{2}} |g(s)|\, s^{N-1} ds + \int_{\frac{R_1+R_2}{2}}^{R_2} |g(s)|\, s^{N-1} ds \right\} \leq$$

$$\frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)\Gamma(\alpha)(p(\alpha-1)+1)^{\frac{1}{p}}} \cdot$$

$$\left\{ \left\| D^{\alpha}_{R_1 +} g \right\|_{L_q\left(\left[R_1, \frac{R_1+R_2}{2}\right]\right)} \left(\int_{R_1}^{\frac{R_1+R_2}{2}} (s - R_1)^{\alpha - 1 + \frac{1}{p}} s^{N-1} ds \right) + \right.$$

$$\left. \left\| D^{\alpha}_{R_2 -} g \right\|_{L_q\left(\left[\frac{R_1+R_2}{2}, R_2\right]\right)} \left(\int_{\frac{R_1+R_2}{2}}^{R_2} (R_2 - s)^{\alpha - 1 + \frac{1}{p}} s^{N-1} ds \right) \right\} = \tag{2.90}$$

$$\frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)\Gamma(\alpha)(p(\alpha-1)+1)^{\frac{1}{p}}} \left\{ \left\| D^{\alpha}_{R_1 +} g \right\|_{L_q\left(\left[R_1, \frac{R_1+R_2}{2}\right]\right)} \left(\frac{(N-1)!\Gamma\left(\alpha + \frac{1}{p}\right)}{2^{\alpha + N - \frac{1}{q}}} \right) \cdot \right.$$

$$\left(\sum_{k=0}^{N-1} \frac{(-1)^{N+k-1} (R_1+R_2)^k (R_2-R_1)^{N-k+\alpha-\frac{1}{q}}}{k!\Gamma\left(N + \alpha + \frac{1}{p} - k\right)} \right) + \tag{2.91}$$

$$\left\| D^{\alpha}_{R_2 -} g \right\|_{L_q\left(\left[\frac{R_1+R_2}{2}, R_2\right]\right)} \left(\frac{(N-1)!\Gamma\left(\alpha + \frac{1}{p}\right)}{2^{\alpha + N - \frac{1}{q}}} \right) \cdot$$

$$\left. \left(\sum_{k=0}^{N-1} \frac{(R_1+R_2)^k (R_2-R_1)^{\alpha+N-k-\frac{1}{q}}}{k!\Gamma\left(\alpha + \frac{1}{p} + N - k\right)} \right) \right\} =$$

$$\frac{\pi^{\frac{N}{2}} (N-1)!\Gamma\left(\alpha + \frac{1}{p}\right)}{\Gamma\left(\frac{N}{2}\right)\Gamma(\alpha)(p(\alpha-1)+1)^{\frac{1}{p}} 2^{\alpha+N-\frac{1}{q}-1}} \cdot$$

$$\left\{ \left\| D^{\alpha}_{R_1 +} g \right\|_{L_q\left(\left[R_1, \frac{R_1+R_2}{2}\right]\right)} \left(\sum_{k=0}^{N-1} \frac{(-1)^{N+k-1} (R_1+R_2)^k (R_2-R_1)^{N+\alpha-k-\frac{1}{q}}}{k!\Gamma\left(N + \alpha + \frac{1}{p} - k\right)} \right) + \right.$$

$$\left. \left\| D^{\alpha}_{R_2 -} g \right\|_{L_q\left(\left[\frac{R_1+R_2}{2}, R_2\right]\right)} \left(\sum_{k=0}^{N-1} \frac{(R_1+R_2)^k (R_2-R_1)^{N+\alpha-k-\frac{1}{q}}}{k!\Gamma\left(\alpha + N + \frac{1}{p} - k\right)} \right) \right\}. \tag{2.92}$$

We have proved

Theorem 2.18. *Let* $p,q>1:\frac{1}{p}+\frac{1}{q}=1$, $\alpha>\frac{1}{q}$. *All terms and assumptions as in Theorem* 2.14. *Then*

$$\int_A |f(y)|\,dy \le \frac{\pi^{\frac{N}{2}}(N-1)!\Gamma\left(\alpha+\frac{1}{p}\right)}{\Gamma\left(\frac{N}{2}\right)\Gamma(\alpha)(p(\alpha-1)+1)^{\frac{1}{p}}2^{\alpha+N-\frac{1}{q}-1}}\cdot$$

$$\left\{\left\|D_{R_1+}^{\alpha}g\right\|_{L_q\left(\left[R_1,\frac{R_1+R_2}{2}\right]\right)}\left(\sum_{k=0}^{N-1}\frac{(-1)^{N+k-1}(R_1+R_2)^k(R_2-R_1)^{N+\alpha-k-\frac{1}{q}}}{k!\Gamma\left(N+\alpha+\frac{1}{p}-k\right)}\right)+\right.$$

$$\left.\left\|D_{R_2-}^{\alpha}g\right\|_{L_q\left(\left[\frac{R_1+R_2}{2},R_2\right]\right)}\left(\sum_{k=0}^{N-1}\frac{(R_1+R_2)^k(R_2-R_1)^{N+\alpha-k-\frac{1}{q}}}{k!\Gamma\left(\alpha+N+\frac{1}{p}-k\right)}\right)\right\}. \tag{2.93}$$

Combining Theorems 2.14, 2.16, 2.18 we derive

Theorem 2.19. *Let any* $p,q>1:\frac{1}{p}+\frac{1}{q}=1$. *And let* $f:\overline{A}\to\mathbb{R}$ *be radial; that is, there exists a function* g *such that* $f(x)=g(s)$, $s=|x|$, $s\in[R_1,R_2]$, $\forall\, x\in\overline{A}$; $\alpha\ge 1$, $m=[\alpha]$. *We assume that* $g\in C([R_1,R_2])$, *such that* $g\in C_{R_1+}^{\alpha}\left(\left[R_1,\frac{R_1+R_2}{2}\right]\right)$ *and* $g\in C_{R_2-}^{\alpha}\left(\left[\frac{R_1+R_2}{2},R_2\right]\right)$, *with* $g^{(k)}(R_1)=g^{(k)}(R_2)=0$, $k=0,1,...,m-1$. *Then*

$$\left|\int_A f(y)\,dy\right| \le \int_A |f(y)|\,dy \le \min\left\{\frac{\pi^{\frac{N}{2}}(N-1)!}{\Gamma\left(\frac{N}{2}\right)2^{\alpha+N-1}}\cdot\right.$$

$$\left\{\left\|D_{R_1+}^{\alpha}g\right\|_{\infty,\left[R_1,\frac{R_1+R_2}{2}\right]}\left(\sum_{k=0}^{N-1}\frac{(-1)^{N+k-1}(R_1+R_2)^k(R_2-R_1)^{N-k+\alpha}}{k!\Gamma(N-k+\alpha+1)}\right)+\right.$$

$$\left.\left\|D_{R_2-}^{\alpha}g\right\|_{\infty,\left[\frac{R_1+R_2}{2},R_2\right]}\left(\sum_{k=0}^{N-1}\frac{(R_1+R_2)^k(R_2-R_1)^{\alpha+N-k}}{k!\Gamma(\alpha+1+N-k)}\right)\right\},$$

$$\frac{\pi^{\frac{N}{2}}(N-1)!}{\Gamma\left(\frac{N}{2}\right)2^{\alpha+N-2}}\left\{\left\|D_{R_1+}^{\alpha}g\right\|_{L_1\left(\left[R_1,\frac{R_1+R_2}{2}\right]\right)}\cdot\right.$$

$$\left(\sum_{k=0}^{N-1}(-1)^{N+k-1}\frac{(R_1+R_2)^k(R_2-R_1)^{N+\alpha-k-1}}{k!\Gamma(N+\alpha-k)}\right)+$$

$$\left.\left\|D_{R_2-}^{\alpha}g\right\|_{L_1\left(\left[\frac{R_1+R_2}{2},R_2\right]\right)}\left(\sum_{k=0}^{N-1}\frac{(R_1+R_2)^k(R_2-R_1)^{N+\alpha-k-1}}{k!\Gamma(N+\alpha-k)}\right)\right\},$$

$$\frac{\pi^{\frac{N}{2}}(N-1)!\Gamma\left(\alpha+\frac{1}{p}\right)}{\Gamma\left(\frac{N}{2}\right)\Gamma(\alpha)(p(\alpha-1)+1)^{\frac{1}{p}}2^{\alpha+N-\frac{1}{q}-1}}\cdot$$

$$\left\{\left\|D_{R_1+}^{\alpha}g\right\|_{L_q\left(\left[R_1,\frac{R_1+R_2}{2}\right]\right)}\left(\sum_{k=0}^{N-1}\frac{(-1)^{N+k-1}(R_1+R_2)^k(R_2-R_1)^{N+\alpha-k-\frac{1}{q}}}{k!\Gamma\left(N+\alpha+\frac{1}{p}-k\right)}\right)+\right.$$

$$\left\| D^{\alpha}_{R_2 -} g \right\|_{L_q\left(\left[\frac{R_1+R_2}{2}, R_2\right]\right)} \left(\sum_{k=0}^{N-1} \frac{(R_1+R_2)^k (R_2-R_1)^{N+\alpha-k-\frac{1}{q}}}{k!\Gamma\left(\alpha+N+\frac{1}{p}-k\right)} \right) \Bigg\} \Bigg\}. \quad (2.94)$$

The corresponding estimate on the average follows

Corollary 2.20. *Let all terms and assumptions as in Theorem* 2.19. *Then*

$$\left| \frac{1}{Vol(A)} \int_A f(y)\, dy \right| \leq \frac{1}{Vol(A)} \int_A |f(y)|\, dy \leq \left(\frac{N!}{2^{\alpha+N}\left(R_2^N - R_1^N\right)} \right) \cdot$$

$$\min \Bigg\{ \Bigg\{ \left\| D^{\alpha}_{R_1 +} g \right\|_{\infty, \left[R_1, \frac{R_1+R_2}{2}\right]} \left(\sum_{k=0}^{N-1} \frac{(-1)^{N+k-1} (R_1+R_2)^k (R_2-R_1)^{N-k+\alpha}}{k!\Gamma(N-k+\alpha+1)} \right)$$

$$+ \left\| D^{\alpha}_{R_2 -} g \right\|_{\infty, \left[\frac{R_1+R_2}{2}, R_2\right]} \left(\sum_{k=0}^{N-1} \frac{(R_1+R_2)^k (R_2-R_1)^{\alpha+N-k}}{k!\Gamma(\alpha+1+N-k)} \right) \Bigg\},$$

$$2 \Bigg\{ \left\| D^{\alpha}_{R_1 +} g \right\|_{L_1\left(\left[R_1, \frac{R_1+R_2}{2}\right]\right)} \left(\sum_{k=0}^{N-1} (-1)^{N+k-1} \frac{(R_1+R_2)^k (R_2-R_1)^{N+\alpha-k-1}}{k!\Gamma(N+\alpha-k)} \right)$$

$$+ \left\| D^{\alpha}_{R_2 -} g \right\|_{L_1\left(\left[\frac{R_1+R_2}{2}, R_2\right]\right)} \left(\sum_{k=0}^{N-1} \frac{(R_1+R_2)^k (R_2-R_1)^{N+\alpha-k-1}}{k!\Gamma(N+\alpha-k)} \right) \Bigg\},$$

$$\frac{\Gamma\left(\alpha+\frac{1}{p}\right) 2^{\frac{1}{q}}}{\Gamma(\alpha)\left(p(\alpha-1)+1\right)^{\frac{1}{p}}} \cdot$$

$$\Bigg\{ \left\| D^{\alpha}_{R_1 +} g \right\|_{L_q\left(\left[R_1, \frac{R_1+R_2}{2}\right]\right)} \left(\sum_{k=0}^{N-1} \frac{(-1)^{N+k-1} (R_1+R_2)^k (R_2-R_1)^{N+\alpha-k-\frac{1}{q}}}{k!\Gamma\left(N+\alpha+\frac{1}{p}-k\right)} \right) +$$

$$\left\| D^{\alpha}_{R_2 -} g \right\|_{L_q\left(\left[\frac{R_1+R_2}{2}, R_2\right]\right)} \left(\sum_{k=0}^{N-1} \frac{(R_1+R_2)^k (R_2-R_1)^{N+\alpha-k-\frac{1}{q}}}{k!\Gamma\left(\alpha+N+\frac{1}{p}-k\right)} \right) \Bigg\} \Bigg\}. \quad (2.95)$$

We need

Definition 2.21. *(see* [1], *p. 287) Let* $\alpha > 0$, $m = [\alpha]$, $\beta := \alpha - m$, $f \in C^m\left(\overline{A}\right)$, *and* A *is a spherical shell. Assume that there exists* $\frac{\partial^{\alpha}_{R_1+} f(x)}{\partial r^{\alpha}} \in C\left(\overline{A}\right)$, *given by*

$$\frac{\partial^{\alpha}_{R_1+} f(x)}{\partial r^{\alpha}} := \frac{1}{\Gamma(1-\beta)} \frac{\partial}{\partial r} \left(\int_{R_1}^{r} (r-t)^{-\beta} \frac{\partial^m f(t\omega)}{\partial r^m} dt \right), \quad (2.96)$$

where $x \in \overline{A}$; *that is,* $x = r\omega$, $r \in [R_1, R_2]$, *and* $\omega \in S^{N-1}$.

We call $\frac{\partial^{\alpha}_{R_1+} f}{\partial r^{\alpha}}$ *the left radial generalised fractional derivative of* f *of order* α.

We also need to introduce

Definition 2.22. *Let $\alpha > 0$, $m = [\alpha]$, $\beta := \alpha - m$, $f \in C^m\left(\overline{A}\right)$, and A is a spherical shell. Assume that there exists $\frac{\partial^{\alpha}_{R_2-} f(x)}{\partial r^{\alpha}} \in C\left(\overline{A}\right)$, given by*

$$\frac{\partial^{\alpha}_{R_2-} f(x)}{\partial r^{\alpha}} := (-1)^{m-1} \frac{1}{\Gamma(1-\beta)} \frac{\partial}{\partial r} \left(\int_r^{R_2} (t-r)^{-\beta} \frac{\partial^m f(t\omega)}{\partial r^m} dt \right), \tag{2.97}$$

where $x \in \overline{A}$; that is, $x = r\omega$, $r \in [R_1, R_2]$, and $\omega \in S^{N-1}$.

We call $\frac{\partial^{\alpha}_{R_2-} f}{\partial r^{\alpha}}$ the right radial generalised fractional derivative of f of order α.

We present

Theorem 2.23. *Let the sperical shells $A := B(0, R_2) - \overline{B(0, R_1)}$, $0 < R_1 < R_2$, $A \subseteq \mathbb{R}^N$, $N \geq 2$; $A_1 := B\left(0, \frac{R_1+R_2}{2}\right) - \overline{B(0, R_1)}$, $A_2 := B(0, R_2) - \overline{B\left(0, \frac{R_1+R_2}{2}\right)}$. Let $f \in C\left(\overline{A}\right)$, not necessarily radial, $\alpha > 0$, $m = [\alpha]$. Assume that $\frac{\partial^{\alpha}_{R_1+} f}{\partial r^{\alpha}} \in C\left(\overline{A_1}\right)$, $\frac{\partial^{\alpha}_{R_2-} f}{\partial r^{\alpha}} \in C\left(\overline{A_2}\right)$. For each $\omega \in S^{N-1}$, we assume further that $f(\cdot\omega) \in C^{\alpha}_{R_1+}\left(\left[R_1, \frac{R_1+R_2}{2}\right]\right)$ and $f(\cdot\omega) \in C^{\alpha}_{R_2-}\left(\left[\frac{R_1+R_2}{2}, R_2\right]\right)$, with $\frac{\partial^k f(R_1\omega)}{\partial r^k} = \frac{\partial^k f(R_2\omega)}{\partial r^k} = 0$, $k = 0, 1, ..., m-1$. When $0 < \alpha < 1$ the last boundary conditions are void. Then*

(i)

$$\left| \int_A f(y)\, dy \right| \leq \int_A |f(y)|\, dy \leq \frac{\pi^{\frac{N}{2}} (N-1)!}{\Gamma\left(\frac{N}{2}\right) 2^{\alpha+N-1}} \cdot$$

$$\left\{ \left\| \frac{\partial^{\alpha}_{R_1+} f}{\partial r^{\alpha}} \right\|_{\infty, \overline{A_1}} \left(\sum_{k=0}^{N-1} \frac{(-1)^{N+k-1} (R_1+R_2)^k (R_2-R_1)^{N+\alpha-k}}{k!\Gamma(N+\alpha+1-k)} \right) + \right.$$

$$\left. \left\| \frac{\partial^{\alpha}_{R_2-} f}{\partial r^{\alpha}} \right\|_{\infty, \overline{A_2}} \left(\sum_{k=0}^{N-1} \frac{(R_1+R_2)^k (R_2-R_1)^{N+\alpha-k}}{k!\Gamma(N+\alpha+1-k)} \right) \right\}, \tag{2.98}$$

and

(ii)

$$\left| \frac{1}{Vol(A)} \int_A f(y)\, dy \right| \leq \frac{1}{Vol(A)} \int_A |f(y)|\, dy \leq \left(\frac{N!}{2^{\alpha+N}\left(R_2^N - R_1^N\right)} \right) \cdot \tag{2.99}$$

$$\left\{ \left\| \frac{\partial^{\alpha}_{R_1+} f}{\partial r^{\alpha}} \right\|_{\infty, \overline{A_1}} \left(\sum_{k=0}^{N-1} \frac{(-1)^{N+k-1} (R_1+R_2)^k (R_2-R_1)^{N+\alpha-k}}{k!\Gamma(N+\alpha+1-k)} \right) + \right.$$

$$\left. \left\| \frac{\partial^{\alpha}_{R_2-} f}{\partial r^{\alpha}} \right\|_{\infty, \overline{A_2}} \left(\sum_{k=0}^{N-1} \frac{(R_1+R_2)^k (R_2-R_1)^{N+\alpha-k}}{k!\Gamma(N+\alpha+1-k)} \right) \right\}.$$

Proof. By (2.69)-(2.71) we get

$$\int_{R_1}^{R_2} |g(s)|\, s^{N-1} ds \leq \left(\frac{\Gamma\left(\frac{N}{2}\right)}{2\pi^{\frac{N}{2}}} \right) \left(\frac{\pi^{\frac{N}{2}} (N-1)!}{\Gamma\left(\frac{N}{2}\right) 2^{\alpha+N-1}} \right) \cdot \tag{2.100}$$

$$\left\{ \left\| D^{\alpha}_{R_1+} g \right\|_{\infty, \left[R_1, \frac{R_1+R_2}{2}\right]} \left(\sum_{k=0}^{N-1} \frac{(-1)^{N+k-1} (R_1+R_2)^k (R_2-R_1)^{N+\alpha-k}}{k!\Gamma(N+\alpha+1-k)} \right) + \right.$$

$$\left\|D_{R_2-}^{\alpha}g\right\|_{\infty,\left[\frac{R_1+R_2}{2},R_2\right]}\left[\sum_{k=0}^{N-1}\frac{\left(R_1+R_2\right)^k\left(R_2-R_1\right)^{N+\alpha-k}}{k!\Gamma\left(N+\alpha+1-k\right)}\right]\right\}.$$

For fixed $\omega\in S^{N-1}$, $f(\cdot\omega)$ sets like a radial function on $\overline{A}$. Thus plugging $f(\cdot\omega)$ into (2.100), we get

$$\int_{R_1}^{R_2}\left|f(s\omega)\right|s^{N-1}ds\leq\left(\frac{\Gamma\left(\frac{N}{2}\right)}{2\pi^{\frac{N}{2}}}\right)\left(\frac{\pi^{\frac{N}{2}}(N-1)!}{\Gamma\left(\frac{N}{2}\right)2^{\alpha+N-1}}\right)\cdot \tag{2.101}$$

$$\left\{\left\|\frac{\partial_{R_1+}^{\alpha}f}{\partial r^{\alpha}}\right\|_{\infty,\overline{A_1}}\left(\sum_{k=0}^{N-1}\frac{(-1)^{N+k-1}\left(R_1+R_2\right)^k\left(R_2-R_1\right)^{N+\alpha-k}}{k!\Gamma\left(N+\alpha+1-k\right)}\right)+\right.$$

$$\left.\left\|\frac{\partial_{R_2-}^{\alpha}f}{\partial r^{\alpha}}\right\|_{\infty,\overline{A_2}}\left(\sum_{k=0}^{N-1}\frac{\left(R_1+R_2\right)^k\left(R_2-R_1\right)^{N+\alpha-k}}{k!\Gamma\left(N+\alpha+1-k\right)}\right)\right\}=:\gamma_1.$$

Therefore by (2.59) and (2.101) we derive

$$\int_A\left|f(y)\right|dy=\int_{S^{N-1}}\left(\int_{R_1}^{R_2}\left|f(s\omega)\right|s^{N-1}ds\right)d\omega\leq$$

$$\gamma_1\int_{S^{N-1}}d\omega=\gamma_1\frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}=\left(\frac{\pi^{\frac{N}{2}}(N-1)!}{\Gamma\left(\frac{N}{2}\right)2^{\alpha+N-1}}\right)\cdot \tag{2.102}$$

$$\left\{\left\|\frac{\partial_{R_1+}^{\alpha}f}{\partial r^{\alpha}}\right\|_{\infty,\overline{A_1}}\left(\sum_{k=0}^{N-1}\frac{(-1)^{N+k-1}\left(R_1+R_2\right)^k\left(R_2-R_1\right)^{N+\alpha-k}}{k!\Gamma\left(N+\alpha+1-k\right)}\right)+\right.$$

$$\left.\left\|\frac{\partial_{R_2-}^{\alpha}f}{\partial r^{\alpha}}\right\|_{\infty,\overline{A_2}}\left(\sum_{k=0}^{N-1}\frac{\left(R_1+R_2\right)^k\left(R_2-R_1\right)^{N+\alpha-k}}{k!\Gamma\left(N+\alpha+1-k\right)}\right)\right\},$$

proving the claims of the theorem. □

We give also

Theorem 2.24. *Let $f\in C\left(\overline{A}\right)$, not necessarily radial, $\alpha\geq 1$, $m=[\alpha]$. For each $\omega\in S^{N-1}$, we assume that $f(\cdot\omega)\in C_{R_1+}^{\alpha}\left(\left[R_1,\frac{R_1+R_2}{2}\right]\right)$ and $f(\cdot\omega)\in C_{R_2-}^{\alpha}\left(\left[\frac{R_1+R_2}{2},R_2\right]\right)$, with $\frac{\partial^k f(R_1\omega)}{\partial r^k}=\frac{\partial^k f(R_2\omega)}{\partial r^k}=0$, $k=0,1,...,m-1$. We further assume*

$$\left\|\frac{\partial_{R_1+}^{\alpha}f(\cdot\omega)}{\partial r^{\alpha}}\right\|_{L_1\left(\left[R_1,\frac{R_1+R_2}{2}\right]\right)},\ \left\|\frac{\partial_{R_2-}^{\alpha}f(\cdot\omega)}{\partial r^{\alpha}}\right\|_{L_1\left(\left[\frac{R_1+R_2}{2},R_2\right]\right)}\leq\Psi_1, \tag{2.103}$$

for every $\omega\in S^{N-1}$, where $\Psi_1>0$.

Then

(i)

$$\int_A\left|f(y)\right|dy\leq\frac{\Psi_1\pi^{\frac{N}{2}}(N-1)!}{\Gamma\left(\frac{N}{2}\right)2^{\alpha+N-2}}\cdot \tag{2.104}$$

$$\left\{\left(\sum_{k=0}^{N-1}\frac{(-1)^{N+k-1}\left(R_1+R_2\right)^k\left(R_2-R_1\right)^{N+\alpha-k-1}}{k!\Gamma\left(N+\alpha-k\right)}\right)+\right.$$

$$\left(\sum_{k=0}^{N-1}\frac{(R_1+R_2)^k(R_2-R_1)^{N+\alpha-k-1}}{k!\Gamma(N+\alpha-k)}\right)\Bigg\},$$

and

(ii)

$$\frac{1}{Vol(A)}\int_A|f(y)|\,dy\leq\frac{\Psi_1 N!}{2^{\alpha+N-1}\left(R_2^N-R_1^N\right)}\cdot \tag{2.105}$$

$$\Bigg\{\left(\sum_{k=0}^{N-1}\frac{(-1)^{N+k-1}(R_1+R_2)^k(R_2-R_1)^{N+\alpha-k-1}}{k!\Gamma(N+\alpha-k)}\right)+$$

$$\left(\sum_{k=0}^{N-1}\frac{(R_1+R_2)^k(R_2-R_1)^{N+\alpha-k-1}}{k!\Gamma(N+\alpha-k)}\right)\Bigg\}.$$

Proof. Similar to Theorem 2.23, using (2.84)-(2.86). □

We finish with

Theorem 2.25. *Let $f\in C\left(\overline{A}\right)$, not necessarily radial, $\alpha>\frac{1}{q}$, where $p,q>1:\frac{1}{p}+\frac{1}{q}=1$, $m=[\alpha]$. For each $\omega\in S^{N-1}$, we assume that $f(\cdot\omega)\in C^{\alpha}_{R_1+}\left(\left[R_1,\frac{R_1+R_2}{2}\right]\right)$ and $f(\cdot\omega)\in C^{\alpha}_{R_2-}\left(\left[\frac{R_1+R_2}{2},R_2\right]\right)$, with $\frac{\partial^k f(R_1\omega)}{\partial r^k}=\frac{\partial^k f(R_2\omega)}{\partial r^k}=0$, $k=0,1,...,m-1$. When $\frac{1}{q}<\alpha<1$ the last boundary conditions is void. We further assume*

$$\left\|\frac{\partial^{\alpha}_{R_1+}f(\cdot\omega)}{\partial r^{\alpha}}\right\|_{L_q\left(\left[R_1,\frac{R_1+R_2}{2}\right]\right)},\ \left\|\frac{\partial^{\alpha}_{R_2-}f(\cdot\omega)}{\partial r^{\alpha}}\right\|_{L_q\left(\left[\frac{R_1+R_2}{2},R_2\right]\right)}\leq\Psi_2, \tag{2.106}$$

for every $\omega\in S^{N-1}$, where $\Psi_2>0$.

Then

(i)

$$\int_A|f(y)|\,dy\leq\frac{\Psi_2\pi^{\frac{N}{2}}(N-1)!\Gamma\left(\alpha+\frac{1}{p}\right)}{\Gamma\left(\frac{N}{2}\right)\Gamma(\alpha)(p(\alpha-1)+1)^{\frac{1}{p}}2^{\alpha+N-\frac{1}{q}-1}}\cdot \tag{2.107}$$

$$\Bigg\{\left(\sum_{k=0}^{N-1}\frac{(-1)^{N+k-1}(R_1+R_2)^k(R_2-R_1)^{N+\alpha-k-\frac{1}{q}}}{k!\Gamma\left(N+\alpha+\frac{1}{p}-k\right)}\right)+$$

$$\left(\sum_{k=0}^{N-1}\frac{(R_1+R_2)^k(R_2-R_1)^{N+\alpha-k-\frac{1}{q}}}{k!\Gamma\left(\alpha+N+\frac{1}{p}-k\right)}\right)\Bigg\},$$

and

(ii)

$$\frac{1}{Vol(A)}\int_A|f(y)|\,dy\leq\frac{N!\Gamma\left(\alpha+\frac{1}{p}\right)\Psi_2}{2^{\alpha+N-\frac{1}{q}}\left(R_2^N-R_1^N\right)\Gamma(\alpha)(p(\alpha-1)+1)^{\frac{1}{p}}}\cdot \tag{2.108}$$

$$\left\{ \left(\sum_{k=0}^{N-1} \frac{(-1)^{N+k-1} (R_1+R_2)^k (R_2-R_1)^{N+\alpha-k-\frac{1}{q}}}{k!\Gamma\left(N+\alpha+\frac{1}{p}-k\right)} \right) + \left(\sum_{k=0}^{N-1} \frac{(R_1+R_2)^k (R_2-R_1)^{N+\alpha-k-\frac{1}{q}}}{k!\Gamma\left(\alpha+N+\frac{1}{p}-k\right)} \right) \right\}.$$

Proof. Similar to Theorem 2.23, using (2.90)-(2.92). □

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George A. Anastassiou
Department of Mathematical Sciences
University of Memphis
Memphis, TN 38152, U.S.A.
e-mail: `ganastss@memphis.edu`

Stud. Univ. Babeş-Bolyai Math. 58(2013), No. 3, 325–343

On Hermite-Hadamard type integral inequalities for n-times differentiable preinvex functions with applications

Muhammad Amer Latif

Abstract. In this paper some new Hermite-Hadamard type inequalities for n-times differentiable preinvex functions are established. Our established results generalize some of those results proved in recent papers for differentiable preinvex functions and n-times differentiable convex functions. Applications to some special means of our results are given as well.

Mathematics Subject Classification (2010): 26D07, 26D10, 26D99.

Keywords: Hermite-Hadamard's inequality, invex set, preinvex function, Hölder's inequality.

1. Introduction

The following definition for convex functions is well known in the mathematical literature: A function $f : I \to \mathbb{R}$, $\emptyset \neq I \subseteq \mathbb{R}$, is said to be convex on I if inequality

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y),$$

holds for all $x, y \in I$ and $t \in [0, 1]$.

Many inequalities have been established for convex functions but the most famous is the Hermite-Hadamard inequality, due to its rich geometrical significance and applications, which is stated as follows:

Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex mapping and $a, b \in I$ with $a < b$. Then

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}. \tag{1.1}$$

Both the inequalities hold in reversed direction if f is concave. Since its discovery in 1883, Hermite-Hadamard inequality (see [11]) has been considered the most useful inequality in mathematical analysis. Some of the classical inequalities for means can be derived from (1.1) for particular choices of the function f. A number of papers have been written on this inequality providing new proofs, noteworthy extensions, generalizations, refinements, counterparts and new Hermite-Hadamard-type inequalities and numerous applications, see [6]-[8], [10, 12], [14]-[16], [23]-[29] and the references therein.

By using the following result:

Lemma 1.1. [6]*Suppose $a,\ b \in I \subseteq \mathbb{R}$ with $a < b$ and $f : I^{\circ} \to \mathbb{R}$ is differentiable. If $f^{'} \in L(a, b)$, then*

$$\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx = \frac{b-a}{2}\int_0^1 (1-2t)\, f^{'}(ta + (1-t)\, b)\, dt.$$

Dragomir and Agarwal [6], established the following results connected with the right part of (1.1) and applied them for some elementary inequalities for real numbers and in numerical integration:

Theorem 1.2. [6]*Suppose $a,\ b \in I \subseteq \mathbb{R}$ with $a < b$ and $f : I^{\circ} \to \mathbb{R}$ is differentiable. If $f^{'} \in L(a, b)$ and $\left|f^{'}\right|$ is convex on $[a, b]$, then the following inequality holds:*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{b-a}{8}\left(\left|f^{'}(a)\right| + \left|f^{'}(b)\right|\right). \tag{1.2}$$

Theorem 1.3. [6]*Suppose $a,\ b \in I \subseteq \mathbb{R}$ with $a < b$ and $f : I^{\circ} \to \mathbb{R}$ is differentiable. If $f^{'} \in L(a, b)$ and $\left|f^{'}\right|^{\frac{p}{p-1}}$, $p > 1$, is convex on $[a, b]$, then the following inequality holds:*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{b-a}{2(1+p)^{\frac{1}{p}}}\left[\frac{\left|f^{'}(a)\right|^{\frac{p}{p-1}} + \left|f^{'}(b)\right|^{\frac{p}{p-1}}}{2}\right]^{\frac{p-1}{p}}. \tag{1.3}$$

Pearce and Pečarić [21], established the following result that gave an improvement and simplification of the constant in Theorem 1.3 and consolidate this result with Theorem 1.2 as Theorem 1.4 below. An analogous result, Theorem 1.5, is developed which relates in the same way to the first inequality in (1.1). Also, they develop analogous results base on concavity and apply them to special means and to estimates of the error term in the trapezoidal formula.

Theorem 1.4. [21]*Suppose $a,\ b \in I \subseteq \mathbb{R}$ with $a < b$ and $f : I^{\circ} \to \mathbb{R}$ is differentiable. If $f^{'} \in L(a, b)$ and $\left|f^{'}\right|^{q}$, $q \geq 1$, is convex on $[a, b]$, then the following inequality holds:*

$$\left|f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{b-a}{4}\left[\frac{\left|f^{'}(a)\right|^{q} + \left|f^{'}(b)\right|^{q}}{2}\right]^{\frac{1}{q}}. \tag{1.4}$$

Theorem 1.5. [21]*Suppose $a, b \in I \subseteq \mathbb{R}$ with $a < b$ and $f : I^\circ \to \mathbb{R}$ is differentiable. If $f' \in L(a,b)$ and $\left|f'\right|^q$, $q \geq 1$, is convex on $[a,b]$, then the following inequality holds:*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{b-a}{4}\left[\frac{\left|f'(a)\right|^q + \left|f'(b)\right|^q}{2}\right]^{\frac{1}{q}}. \tag{1.5}$$

In a recent paper, Dah-Yang Hwang [12], established new inequalities of Hermite-Hadamard type for n-times differentiable convex and concave functions and obtained better estimates of those results established in Theorem 1.4 and Theorem 1.5.

The main result from [12] is pointed out as follows:

Theorem 1.6. [12]*Suppose $f : I^\circ \subset \mathbb{R} \to \mathbb{R}$, $a, b \in I^\circ$ with $a < b$. If $f^{(n)}$ exists on I°, $f^{(n)} \in L(a,b)$ for $n \in \mathbb{N}$, $n \geq 1$ and $\left|f^{(n)}\right|^q$, $q \geq 1$, then we have the inequality:*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx - \sum_{k=2}^{n-1}\frac{(k-1)(b-a)^k}{2(k+1)!}f^{(k)}(a)\right|$$
$$= \frac{(n-1)^{1-\frac{1}{q}}(b-a)^n}{2(n+1)!}\left[\frac{(n^2-2)\left|f'(a)\right|^q + n\left|f'(b)\right|^q}{n+2}\right]^{\frac{1}{q}}. \tag{1.6}$$

The following lemma, which generalize Lemma 1.1, was used to establish the above result:

Lemma 1.7. [12]*Suppose $f : I^\circ \subset \mathbb{R} \to \mathbb{R}$, $a, b \in I^\circ$ with $a < b$. If $f^{(n)}$ exists on I° and $f^{(n)} \in L(a,b)$ for $n \in \mathbb{N}$, $n \geq 1$, then we have the identity:*

$$\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx - \sum_{k=2}^{n-1}\frac{(k-1)(b-a)^k}{2(k+1)!}f^{(k)}(a)$$
$$= \frac{(b-a)^n}{2n!}\int_0^1 t^{n-1}(n-2t)\,f^{(n)}(ta+(1-t)b)dt.$$

For more recent results for n-times differentiable functions we refer the readers to the latest research work of Wei-Dong Jiang et. al [15] and Shu-Hong et al. [21] concerning inequalities for n-times differentiable s-convex and m-convex functions respectively (see the references in these papers as well).

In recent years, lot of efforts have been made by many mathematicians to extend and to generalize the classical convexity. These studies include among others the work of Hanson in [9], Ben-Israel and Mond [5] and Pini [22]. Hanson in [9], introduced invex functions as a significant generalization of convex functions. Ben-Israel and Mond [5], gave the concept of preinvex functions . Pini [22], introduced the concept of prequasiinvex functions as a generalization of invex functions.

Let us recall some known results concerning invexity and preinvexity.

Let K be a subset in $\mathbb{R}^n$ and let $f : K \to \mathbb{R}$ and $\eta : K \times K \to \mathbb{R}^n$ be continuous functions. Let $x \in K$, then the set K is said to be invex at x with respect to $\eta(\cdot,\cdot)$, if

$$x + t\eta(y,x) \in K, \forall x, y \in K, t \in [0,1].$$

K is said to be an invex set with respect to η if K is invex at each $x \in K$. The invex set K is also called a η-connected set.

Definition 1.8. [30] *The function f on the invex set K is said to be preinvex with respect to η, if*

$$f(u + t\eta(v,u)) \le (1-t) f(u) + tf(v), \forall u, v \in K, t \in [0,1].$$

The function f is said to be preconcave if and only if $-f$ is preinvex.

It is to be noted that every preinvex function is convex with respect to the map $\eta(x,y) = x - y$ but the converse is not true see for instance [30].

Barani, Ghazanfari and Dragomir in [4], presented the following estimates of the right-side of a Hermite-Hadamard type inequality in which some preinvex functions are involved. These results generalize the results given above in Theorem 1.2 and Theorem 1.3.

Theorem 1.9. [4] *Let $K \subseteq \mathbb{R}$ be an open invex subset with respect to $\eta : K \times K \to \mathbb{R}$. Suppose that $f : K \to \mathbb{R}$ is a differentiable function .Assume $p \in \mathbb{R}$ with $p > 1$. If $\left|f'\right|^{\frac{p}{p-1}}$ is preinvex on K then, for every $a, b \in K$ with $\eta(b,a) \neq 0$, then the following inequality holds:*

$$\left|\frac{f(a) + f(a+\eta(b,a))}{2} - \frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(x)dx\right| \le \frac{|\eta(b,a)|}{2(1+p)^{\frac{1}{p}}} \left[\frac{\left|f'(a)\right|^{\frac{p}{p-1}} + \left|f'(b)\right|^{\frac{p}{p-1}}}{2}\right]^{\frac{p-1}{p}}. \quad (1.7)$$

Theorem 1.10. [4] *Let $K \subseteq \mathbb{R}$ be an open invex subset with respect to $\eta : K \times K \to \mathbb{R}$. Suppose that $f : K \to \mathbb{R}$ is a differentiable function. If $\left|f'\right|$ is preinvex on K then, for every $a, b \in K$ with $\eta(b,a) \neq 0$, then the following inequality holds:*

$$\left|\frac{f(a) + f(a+\eta(b,a))}{2} - \frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(x)dx\right| \le \frac{|\eta(b,a)|}{8} \left(\left|f'(a)\right| + \left|f'(b)\right|\right). \quad (1.8)$$

For more results on Hermite-Hadamard type inequalities for preinvex and log-preinvex functions, we refer the readers to the latest papers of M. Z. Sarikaya et. al , [27] and Noor [18]-[20].

The main purpose of the present paper is to establish new Hermite-Hadamard type inequalities in Section 2 that are connected with the right-side and left-side of Hermite-Hadamard inequality for preinvex functions but for n-times differentiable preinvex functions which generalize those results established for differentiable preinvex functions and n-times differentiable convex functions.

2. Main results

In order to prove our main results, we need the following lemmas:

Lemma 2.1. *Let $K \subseteq \mathbb{R}$ be an open invex subset with respect to $\eta : K \times K \to \mathbb{R}$ and $a, b \in K$ with $a < \eta(b,a)$. Suppose $f : K \to \mathbb{R}$ is a function such that $f^{(n)}$ exists on K for $n \in \mathbb{N}$, $n \geq 1$. If $f^{(n)}$ is integrable on $[a, a+\eta(b,a)]$, then for every $a, b \in K$ with $\eta(b,a) \neq> 0$, the following equality holds:*

$$-\frac{f(a)+f(a+\eta(b,a))}{2}+\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)\,dx$$
$$+\sum_{k=2}^{n-1}\frac{(-1)^k (k-1)(\eta(b,a))^k}{2(k+1)!}f^{(k)}(a+\eta(b,a))$$
$$=\frac{(-1)^{n-1}(\eta(b,a))^n}{2n!}\int_0^1 t^{n-1}(n-2t)\,f^{(n)}(a+t\eta(b,a))dt, \quad (2.1)$$

where the sum above takes 0 when $n=1$ and $n=2$.

Proof. The case $n=1$ is the Lemma 2.1 from [4]. Suppose (2.1) holds for $n-1$, i.e.

$$-\frac{f(a)+f(a+\eta(b,a))}{2}+\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)\,dx$$
$$+\sum_{k=2}^{n-2}\frac{(-1)^k (k-1)(\eta(b,a))^k}{2(k+1)!}f^{(k)}(a+\eta(b,a))$$
$$=\frac{(-1)^{n-2}(\eta(b,a))^{n-1}}{2(n-1)!}\int_0^1 t^{n-2}((n-1)-2t)\,f^{(n-1)}(a+t\eta(b,a))dt. \quad (2.2)$$

Now integrating by parts and using (2.2), we have

$$\frac{(-1)^{n-1}(\eta(b,a))^n}{2n!}\int_0^1 t^{n-1}(n-2t)\,f^{(n)}(a+t\eta(b,a))dt$$
$$\frac{(-1)^{n-1}(n-2)(\eta(b,a))^{n-1}}{2n!}f^{(n-1)}(a+\eta(b,a))$$
$$-\frac{f(a)+f(a+\eta(b,a))}{2}+\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)\,dx$$
$$+\sum_{k=2}^{n-2}\frac{(-1)^k (k-1)(\eta(b,a))^k}{2(k+1)!}f^{(k)}(a+\eta(b,a))$$
$$=-\frac{f(a)+f(a+\eta(b,a))}{2}+\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)\,dx$$

$$+\sum_{k=2}^{n-1}\frac{(-1)^k(k-1)(\eta(b,a))^k}{2(k+1)!}f^{(k)}(a+\eta(b,a)).$$

This completes the proof of the lemma. □

Lemma 2.2. *Let $K\subseteq\mathbb{R}$ be an open invex subset with respect to $\eta:K\times K\to\mathbb{R}$ and $a,\ b\in K$ with $a<\eta(b,a)$ Suppose $f:K\to\mathbb{R}$ is a function such that $f^{(n)}$ exists on K for $n\in\mathbb{N}$, $n\geq 1$. If $f^{(n)}$ is integrable on $[a,a+\eta(b,a)]$, then for every $a,b\in K$ with $\eta(b,a)\neq 0$, the following equality holds:*

$$\sum_{k=0}^{n-1}\frac{\left[(-1)^k+1\right](\eta(b,a))^k}{2^{k+1}(k+1)!}f^{(k)}\left(a+\frac{1}{2}\eta(b,a)\right)-\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)}f(x)\,dx$$
$$=\frac{(-1)^{n+1}(\eta(b,a))^n}{n!}\int_0^1 K_n(t)f^{(n)}(a+t\eta(b,a))dt, \quad (2.3)$$

where

$$K_n(t):=\begin{cases} t^n, & t\in\left[0,\frac{1}{2}\right] \\ (t-1)^n, & t\in\left(\frac{1}{2},1\right] \end{cases}.$$

Proof. For $n=1$, we have by integration by parts that

$$\eta(b,a)\int_0^1 K_1(t)f^{'}(a+t\eta(b,a))dt$$
$$=\eta(b,a)\left[\int_0^{\frac{1}{2}}tf^{'}(a+t\eta(b,a))dt+\int_{\frac{1}{2}}^1(t-1)f^{'}(a+t\eta(b,a))dt\right]$$
$$=f\left(a+\frac{1}{2}\eta(b,a)\right)-\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)}f(x)\,dx.$$

which is true. Suppose now that (2.3) is true for $n-1$, i.e.

$$\sum_{k=0}^{n-2}\frac{\left[(-1)^k+1\right](\eta(b,a))^k}{2^{k+1}(k+1)!}f^{(k)}\left(a+\frac{1}{2}\eta(b,a)\right)-\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)}f(x)\,dx$$
$$=\frac{(-1)^n(\eta(b,a))^{n-1}}{(n-1)!}\int_0^1 K_{n-1}(t)f^{(n-1)}(a+t\eta(b,a))dt. \quad (2.4)$$

Now by integration by parts and using (2.4), we have

$$\frac{(-1)^{n+1}(\eta(b,a))^n}{n!}\int_0^1 K_n(t)f^{(n)}(a+t\eta(b,a))$$
$$=\frac{(-1)^{n-1}(\eta(b,a))^{n-1}f^{(n-1)}(a+\frac{1}{2}\eta(b,a))}{2^n n!}+\frac{(\eta(b,a))^{n-1}f^{(n-1)}(a+\frac{1}{2}\eta(b,a))}{2^n n!}$$
$$+\frac{(-1)^n(\eta(b,a))^{n-1}}{(n-1)!}\int_0^1 K_{n-1}(t)f^{(n-1)}(a+t\eta(b,a))dt$$

$$= \frac{\left[(-1)^{n-1}+1\right](\eta(b,a))^{n-1}}{2^n n!} f^{(n-1)}(a+\frac{1}{2}\eta(b,a))$$
$$+\sum_{k=0}^{n-2} \frac{\left[(-1)^{k}+1\right](\eta(b,a))^{k}}{2^{k+1}(k+1)!} f^{(k)}\left(a+\frac{1}{2}\eta(b,a)\right) - \frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)\,dx$$
$$= \sum_{k=0}^{n-1} \frac{\left[(-1)^{k}+1\right](\eta(b,a))^{k}}{2^{k+1}(k+1)!} f^{(k)}\left(a+\frac{1}{2}\eta(b,a)\right) - \frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)\,dx.$$

This completes the proof of the lemma. □

We are now ready to give our first result.

Theorem 2.3. *Let $K \subseteq \mathbb{R}$ be an open invex subset with respect to $\eta : K \times K \to \mathbb{R}$. Suppose $f : K \to \mathbb{R}$ is a function such that $f^{(n)}$ exists on K and $f^{(n)}$ is integrable on $[a, a+\eta(b,a)]$ for $n \in \mathbb{N}$, $n \geq 2$. If $\left|f^{(n)}\right|$ is preinvex on K for $n \in \mathbb{N}$, $n \geq 2$, then for every $a, b \in K$ with $\eta(b,a) > 0$, we have the following inequality:*

$$\left|\frac{f(a)+f(a+\eta(b,a))}{2} - \frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)\,dx \right.$$
$$\left. -\sum_{k=2}^{n-1} \frac{(-1)^k (k-1)(\eta(b,a))^k}{2(k+1)!} f^{(k)}(a+\eta(b,a))\right|$$
$$\leq \frac{(\eta(b,a))^n}{2(n+1)!}\left(\frac{n\left|f^{(n)}(a)\right| + \left(n^2-2\right)\left|f^{(n)}(b)\right|}{n+2}\right). \quad (2.5)$$

Proof. Suppose $n \geq 2$. Let $a, b \in K$. Since K is an invex set with respect to η, for every $t \in [0,1]$ we have $a + t\eta(b,a) \in K$. By preinvexity of $\left|f^{(n)}\right|$ and Lemma 2.1, we get that

$$\left|-\frac{f(a)+f(a+\eta(b,a))}{2} + \frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)\,dx \right.$$
$$\left. +\sum_{k=2}^{n-1} \frac{(-1)^k (k-1)(\eta(b,a))^k}{2(k+1)!} f^{(k)}(a+\eta(b,a))\right|$$
$$\leq \frac{(\eta(b,a))^n}{2n!}\int_0^1 t^{n-1}(n-2t)\left|f^{(n)}(a+t\eta(b,a))\right| dt$$
$$\leq \frac{(\eta(b,a))^n}{2n!}\int_0^1 t^{n-1}(n-2t)\left((1-t)\left|f^{(n)}(a)\right| + t\left|f^{(n)}(b)\right|\right) dt$$
$$= \frac{(\eta(b,a))^n}{2n!}\left(\left|f^{(n)}(b)\right|\int_0^1 t^{n}(n-2t)\,dt + \left|f^{(n)}(a)\right|\int_0^1 t^{n-1}(n-2t)(1-t)\,dt\right). \quad (2.6)$$

Since

$$\int_0^1 t^n(n-2t)\,dt = \frac{n^2-2}{(n+1)(n+2)}$$

and

$$\int_0^1 t^{n-1}(n-2t)(1-t)\,dt = \frac{n}{(n+1)(n+2)},$$

we have from (2.6) the desired inequality (2.5).
This completes the proof of the theorem □

Theorem 2.4. *Let $K \subseteq \mathbb{R}$ be an open invex subset with respect to $\eta : K \times K \to \mathbb{R}$. Suppose $f : K \to \mathbb{R}$ is a function such that $f^{(n)}$ exists on K and $f^{(n)}$ is integrable on $[a, a+\eta(b,a)]$ for $n \in \mathbb{N}$, $n \geq 2$. If $\left|f^{(n)}\right|^q$, $q \geq 1$, is preinvex on K for $n \in \mathbb{N}$, $n \geq 2$, then for every $a, b \in K$ with $\eta(b,a) > 0$, we have the following inequality:*

$$\left|\frac{f(a)+f(a+\eta(b,a))}{2} - \frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)\,dx \right.$$
$$\left. - \sum_{k=2}^{n-1} \frac{(-1)^k (k-1)(\eta(b,a))^k}{2(k+1)!} f^{(k)}(a+\eta(b,a))\right|$$
$$\leq \frac{(\eta(b,a))^n (n-1)^{1-\frac{1}{q}}}{2(n+1)!}\left(\frac{(n^2-2)\left|f^{(n)}(a)\right|^q + n\left|f^{(n)}(b)\right|^q}{n+2}\right)^{\frac{1}{q}}. \quad (2.7)$$

Proof. Suppose that $n \geq 2$. For $q = 1$, we get the inequality (2.5). Assume now that $q > 1$, then by the preinvexity of $\left|f^{(n)}\right|^q$ on K, Lemma 2.1 and the Hölder's inequality, we have

$$\left|\frac{f(a)+f(a+\eta(b,a))}{2} - \frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)\,dx \right.$$
$$\left. - \sum_{k=2}^{n-1} \frac{(-1)^k (k-1)(\eta(b,a))^k}{2(k+1)!} f^{(k)}(a+\eta(b,a))\right|$$
$$\leq \frac{(\eta(b,a))^n}{2n!}\left(\int_0^1 t^{n-1}(n-2t)\,dt\right)^{1-\frac{1}{q}}\left(\int_0^1 t^{n-1}(n-2t)\left|f^{(n)}(a+t\eta(b,a))\right|^q dt\right)^{\frac{1}{q}}$$
$$\leq \frac{(\eta(b,a))^n}{2n!}\left(\int_0^1 t^{n-1}(n-2t)\,dt\right)^{1-\frac{1}{q}}$$
$$\times\left(\left|f^{(n)}(b)\right|^q \int_0^1 t^n(n-2t)\,dt + \left|f^{(n)}(a)\right|^q \int_0^1 t^{n-1}(n-2t)(1-t)\,dt\right)^{\frac{1}{q}}. \quad (2.8)$$

Since

$$\int_0^1 t^n(n-2t)\,dt = \frac{n^2-2}{(n+1)(n+2)},$$
$$\int_0^1 t^n(n-2t)\,dt = \frac{n-1}{n+1}$$

and

$$\int_0^1 t^{n-1}(n-2t)(1-t)\,dt = \frac{n}{(n+1)(n+2)},$$

we have from (2.8) that

$$\left|\frac{f(a)+f(a+\eta(b,a))}{2}-\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)}f(x)\,dx\right.$$
$$\left.-\sum_{k=2}^{n-1}\frac{(-1)^k(k-1)(\eta(b,a))^k}{2(k+1)!}f^{(k)}(a+\eta(b,a))\right|$$
$$\leq\frac{(\eta(b,a))^n}{2n!}\left(\int_0^1 t^{n-1}(n-2t)\,dt\right)^{1-\frac{1}{q}}\left(\int_0^1 t^{n-1}(n-2t)\left|f^{(n)}(a+t\eta(b,a))\right|^q dt\right)^{\frac{1}{q}}$$
$$\leq\frac{(\eta(b,a))^n(n-1)^{1-\frac{1}{q}}}{2(n+1)!}\left(\frac{n\left|f^{(n)}(a)\right|^q+\left(n^2-2\right)\left|f^{(n)}(b)\right|^q}{n+2}\right)^{\frac{1}{q}}. \tag{2.9}$$

Hence the proof of the theorem is complete. □

Corollary 2.5. *Suppose the assumptions of Theorem* 2.4 *are satisfied then for* $n=2$*, we have*

$$\left|\frac{f(a)+f(a+\eta(b,a))}{2}-\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)}f(x)\,dx\right|$$
$$\leq\frac{(\eta(b,a))^2}{12}\left(\frac{\left|f''(a)\right|^q+\left|f''(b)\right|^q}{2}\right)^{\frac{1}{q}}. \tag{2.10}$$

Corollary 2.6. *If we take* $q=1$ *in (*2.10*), we obtain*

$$\left|\frac{f(a)+f(a+\eta(b,a))}{2}-\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)}f(x)\,dx\right|$$
$$\leq\frac{(\eta(b,a))^2}{12}\left(\frac{\left|f''(a)\right|+\left|f''(b)\right|}{2}\right). \tag{2.11}$$

We note that the bound in (2.11) may be better than the bound in (1.8).

Corollary 2.7. *If* $\left|f^{(n)}\right|^q$ *is preinvex on* K *with respect to the function* $\eta(y,x)=y-x$*,* $q\geq 1$*,* $n\in\mathbb{N}$*,* $n\geq 2$*. Then* $\left|f^{(n)}\right|^q$ *is convex on* K*,* $q\geq 1$*,* $n\in\mathbb{N}$*,* $n\geq 2$ *and hence the inequality (*2.7*) becomes*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx-\sum_{k=2}^{n-1}\frac{(-1)^k(k-1)(b-a)^k}{2(k+1)!}f^{(k)}(b)\right|$$
$$\leq\frac{(b-a)^n(n-1)^{1-\frac{1}{q}}}{2(n+1)!}\left(\frac{n\left|f^{(n)}(a)\right|^q+\left(n^2-2\right)\left|f^{(n)}(b)\right|^q}{n+2}\right)^{\frac{1}{q}}. \tag{2.12}$$

Now we give some results related to left-side of Hermite-Hadamard's inequality for n-times differentiable preinvex functions.

Theorem 2.8. *Let $K \subseteq \mathbb{R}$ be an open invex subset with respect to $\eta : K \times K \to \mathbb{R}$. Suppose $f : K \to \mathbb{R}$ is a function such that $f^{(n)}$ exists on K and $f^{(n)}$ is integrable on $[a, a + \eta(b,a)]$ for $n \in \mathbb{N}$, $n \geq 1$. If $\left|f^{(n)}\right|$ is preinvex on K for $n \in \mathbb{N}$, $n \geq 1$, then for every $a, b \in K$ with $\eta(b,a) > 0$, we have the following inequality:*

$$\left|\sum_{k=0}^{n-1} \frac{\left[(-1)^k + 1\right](\eta(b,a))^k}{2^{k+1}(k+1)!} f^{(k)}\left(a + \frac{1}{2}\eta(b,a)\right) - \frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(x)\,dx\right| \leq \frac{(\eta(b,a))^n}{2^{n+1}(n+1)!}\left[\left|f^{(n)}(a)\right| + \left|f^{(n)}(b)\right|\right]. \quad (2.13)$$

Proof. Suppose $n \geq 1$. By using Lemma 2.2 and the preinvexity of $\left|f^{(n)}\right|$ on K for $n \in \mathbb{N}$, $n \geq 1$, we have

$$\begin{aligned}
&\left|\sum_{k=0}^{n-1} \frac{\left[(-1)^k + 1\right](\eta(b,a))^k}{2^{k+1}(k+1)!} f^{(k)}\left(a + \frac{1}{2}\eta(b,a)\right) - \frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(x)\,dx\right| \\
&\leq \frac{(\eta(b,a))^n}{n!}\left[\int_0^{\frac{1}{2}} t^n \left|f^{(n)}(a + t\eta(b,a))\right| dt + \int_{\frac{1}{2}}^1 (1-t)^n \left|f^{(n)}(a + t\eta(b,a))\right| dt\right] \\
&\leq \frac{(\eta(b,a))^n}{n!}\left[\int_0^{\frac{1}{2}} t^n \left((1-t)\left|f^{(n)}(a)\right| + t\left|f^{(n)}(b)\right|\right) dt\right. \\
&\quad \left. + \int_{\frac{1}{2}}^1 (1-t)^n \left((1-t)\left|f^{(n)}(a)\right| + t\left|f^{(n)}(b)\right|\right) dt\right] \\
&= \frac{(\eta(b,a))^n}{n!}\left[\left|f^{(n)}(a)\right|\left(\int_0^{\frac{1}{2}} t^n(1-t)\,dt + \int_{\frac{1}{2}}^1 (1-t)^{n+1}\,dt\right)\right. \\
&\quad \left. + \left|f^{(n)}(b)\right|\left(\int_0^{\frac{1}{2}} t^{n+1}dt + \int_{\frac{1}{2}}^1 t(1-t)^n\,dt\right)\right]. \quad (2.14)
\end{aligned}$$

Since

$$\int_0^{\frac{1}{2}} t^n(1-t)\,dt + \int_{\frac{1}{2}}^1 (1-t)^{n+1}\,dt = \frac{1}{2^{n+1}(n+1)}$$

and

$$\int_0^{\frac{1}{2}} t^{n+1}dt + \int_{\frac{1}{2}}^1 t(1-t)^n\,dt = \frac{1}{2^{n+1}(n+1)},$$

we get from (2.14) the desired inequality (2.13). This completes the proof of the theorem. □

The following results contains the powers of the absolute values of the n the derivative of the preinvex function.

Theorem 2.9. *Let $K \subseteq \mathbb{R}$ be an open invex subset with respect to $\eta : K \times K \to \mathbb{R}$. Suppose $f : K \to \mathbb{R}$ is a function such that $f^{(n)}$ exists on K and $f^{(n)}$ is integrable on*

$[a, a+\eta(b,a)]$ for $n \in \mathbb{N}$, $n \geq 1$. If $\left|f^{(n)}\right|^{\frac{p}{p-1}}$ is preinvex on K for $n \in \mathbb{N}$, $n \geq 1$, $p \in \mathbb{R}$, $p > 1$, then for every $a, b \in K$ with $\eta(b,a) > 0$, we have the following inequality:

$$\left|\sum_{k=0}^{n-1} \frac{\left[(-1)^k + 1\right](\eta(b,a))^k}{2^{k+1}(k+1)!} f^{(k)}\left(a + \frac{1}{2}\eta(b,a)\right) - \frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(x)\,dx\right|$$
$$\leq \frac{(\eta(b,a))^n}{2^{n+\frac{1}{p}}(np+1)^{\frac{1}{p}} n!} \left[\left(\frac{3\left|f^{(n)}(a)\right|^{\frac{p}{p-1}} + \left|f^{(n)}(b)\right|^{\frac{p}{p-1}}}{8}\right)^{\frac{p-1}{p}} + \left(\frac{\left|f^{(n)}(a)\right|^{\frac{p}{p-1}} + 3\left|f^{(n)}(b)\right|^{\frac{p}{p-1}}}{8}\right)^{\frac{p-1}{p}}\right]. \quad (2.15)$$

Proof. From Lemma 2.2 and the Hölder's integral inequality, we have

$$\left|\sum_{k=0}^{n-1} \frac{\left[(-1)^k + 1\right](\eta(b,a))^k}{2^{k+1}(k+1)!} f^{(k)}\left(a + \frac{1}{2}\eta(b,a)\right) - \frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(x)\,dx\right|$$
$$\leq \frac{(\eta(b,a))^n}{n!} \left[\left(\int_0^{\frac{1}{2}} t^{np} dt\right)^{\frac{1}{p}} \left(\int_0^{\frac{1}{2}} \left|f^{(n)}(a+t\eta(b,a))\right|^{\frac{p}{p-1}} dt\right)^{\frac{p-1}{p}} + \left(\int_{\frac{1}{2}}^1 (1-t)^{np}\,dt\right)^{\frac{1}{p}} \left(\int_{\frac{1}{2}}^1 \left|f^{(n)}(a+t\eta(b,a))\right|^{\frac{p}{p-1}} dt\right)^{\frac{p-1}{p}}\right]. \quad (2.16)$$

Since $\left|f^{(n)}\right|^{\frac{p}{p-1}}$ is preinvex on K for $n \in \mathbb{N}$, $n \geq 1$, $p \in \mathbb{R}$, $p > 1$, then for every $a, b \in K$ with $\eta(b,a) > 0$, we have

$$\int_0^{\frac{1}{2}} \left|f^{(n)}(a+t\eta(b,a))\right|^{\frac{p}{p-1}} dt \leq \left|f^{(n)}(a)\right|^{\frac{p}{p-1}} \int_0^{\frac{1}{2}} (1-t)\,dt + \left|f^{(n)}(b)\right|^{\frac{p}{p-1}} \int_0^{\frac{1}{2}} t dt = \frac{3}{8}\left|f^{(n)}(a)\right|^{\frac{p}{p-1}} + \frac{1}{8}\left|f^{(n)}(b)\right|^{\frac{p}{p-1}} \quad (2.17)$$

and

$$\int_0^{\frac{1}{2}} \left|f^{(n)}(a+t\eta(b,a))\right|^{\frac{p}{p-1}} dt \leq \left|f^{(n)}(a)\right|^{\frac{p}{p-1}} \int_{\frac{1}{2}}^1 (1-t)\,dt + \left|f^{(n)}(b)\right|^{\frac{p}{p-1}} \int_{\frac{1}{2}}^1 t dt = \frac{1}{8}\left|f^{(n)}(a)\right|^{\frac{p}{p-1}} + \frac{3}{8}\left|f^{(n)}(b)\right|^{\frac{p}{p-1}}. \quad (2.18)$$

Also

$$\int_0^{\frac{1}{2}} t^{np} dt = \int_{\frac{1}{2}}^{1} (1-t)^{np}\, dt = \frac{1}{2^{np+1}(np+1)}. \tag{2.19}$$

Using (2.17), (2.18) and (2.19) in (2.16), we get the required inequality (2.15). This completes the proof of the theorem. □

Theorem 2.10. *Let $K \subseteq \mathbb{R}$ be an open invex subset with respect to $\eta : K \times K \to \mathbb{R}$. Suppose $f : K \to \mathbb{R}$ is a function such that $f^{(n)}$ exists on K and $f^{(n)}$ is integrable on $[a, a+\eta(b,a)]$ for $n \in \mathbb{N}$, $n \geq 1$. If $\left|f^{(n)}\right|^q$ is preinvex on K for $n \in \mathbb{N}$, $n \geq 1$, $p \in \mathbb{R}$, $q \geq 1$, then for every $a, b \in K$ with $\eta(b,a) > 0$, we have the following inequality:*

$$\left|\sum_{k=0}^{n-1} \frac{\left[(-1)^k + 1\right](\eta(b,a))^k f^{(k)}\left(a + \frac{1}{2}\eta(b,a)\right)}{2^{k+1}(k+1)!} - \frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)\,dx\right| \leq \frac{(\eta(b,a))^n}{2^{n+1}(n+1)!}\left[\left(\frac{(n+3)\left|f^{(n)}(a)\right|^q + (n+1)\left|f^{(n)}(b)\right|^q}{2(n+2)}\right)^{\frac{1}{q}} + \left(\frac{(n+1)\left|f^{(n)}(a)\right|^q + (n+3)\left|f^{(n)}(b)\right|^q}{2(n+2)}\right)^{\frac{1}{q}}\right]. \tag{2.20}$$

Proof. The case $q = 1$ is the Theorem 2.8. Suppose $q > 1$, then from Lemma 2.2 and the power-mean integral inequality, we have

$$\left|\sum_{k=0}^{n-1} \frac{\left[(-1)^k + 1\right](\eta(b,a))^k f^{(k)}\left(a + \frac{1}{2}\eta(b,a)\right)}{2^{k+1}(k+1)!} - \frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)\,dx\right| \leq \frac{(\eta(b,a))^n}{n!}\left[\left(\int_0^{\frac{1}{2}} t^n dt\right)^{1-\frac{1}{q}}\left(\int_0^{\frac{1}{2}} t^n \left|f^{(n)}(a + t\eta(b,a))\right|^q dt\right)^{\frac{1}{q}} + \left(\int_{\frac{1}{2}}^{1} (1-t)^n\, dt\right)^{\frac{1}{q}}\left(\int_{\frac{1}{2}}^{1} \left|f^{(n)}(a + t\eta(b,a))\right|^q dt\right)^{\frac{1}{q}}\right]. \tag{2.21}$$

Since $\left|f^{(n)}\right|^q$ is preinvex on K for $n \in \mathbb{N}$, $n \geq 1$, $p \in \mathbb{R}$, $q > 1$, then for every $a, b \in K$ with $\eta(b,a) > 0$, we have

$$\begin{aligned}
&\int_0^{\frac{1}{2}} t^n \left|f^{(n)}(a + t\eta(b,a))\right|^q dt \\
&\leq \left|f^{(n)}(a)\right|^q \int_0^{\frac{1}{2}} t^n (1-t)\, dt + \left|f^{(n)}(b)\right|^q \int_0^{\frac{1}{2}} t^{n+1} dt \\
&= \frac{(n+3)\left|f^{(n)}(a)\right|^q + (n+1)\left|f^{(n)}(b)\right|^q}{2^{n+2}(n+1)(n+2)} \qquad (2.22)
\end{aligned}$$

and

$$\begin{aligned}
&\int_0^{\frac{1}{2}} (1-t)^n \left|f^{(n)}(a + t\eta(b,a))\right|^q dt \\
&\leq \left|f^{(n)}(a)\right|^q \int_{\frac{1}{2}}^1 (1-t)^{n+1} dt + \left|f^{(n)}(b)\right|^q \int_{\frac{1}{2}}^1 t(1-t)^n dt \\
&= \frac{(n+1)\left|f^{(n)}(a)\right|^q + (n+3)\left|f^{(n)}(b)\right|^q}{2^{n+2}(n+1)(n+2)}. \qquad (2.23)
\end{aligned}$$

Also

$$\int_0^{\frac{1}{2}} t^n dt = \int_{\frac{1}{2}}^1 (1-t)^n dt = \frac{1}{2^{n+1}(n+1)}. \qquad (2.24)$$

A usage of (2.22), (2.23) and (2.24) in (2.21) gives us the required inequality (2.20). This completes the proof of the theorem. □

Remark 2.11. If we take $n = 1$ in Theorem 2.8, Theorem 2.9 and Theorem 2.10, we obtain those results proved in [27] for differentiable preinvex functions.

Corollary 2.12. *Under the assumptions of Theorem* 2.9*, if f is preinvex on K with respect to the function $\eta(y,x) = y - x$, $q \geq 1$, $n \in \mathbb{N}$, $n \geq 1$. Then $\left|f^{(n)}\right|^{\frac{p-1}{p}}$ is convex on K, $p > 1$, $p \in \mathbb{R}$, $n \in \mathbb{N}$, $n \geq 1$ and hence we have the following inequality:*

$$\begin{aligned}
&\left|\sum_{k=0}^{n-1} \frac{\left[(-1)^k + 1\right](b-a)^k}{2^{k+1}(k+1)!} f^{(k)}\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)\,dx\right| \\
&\leq \frac{(b-a)^n}{2^{n+\frac{1}{p}}(np+1)^{\frac{1}{p}} n!} \left[\left(\frac{3\left|f^{(n)}(a)\right|^{\frac{p}{p-1}} + \left|f^{(n)}(b)\right|^{\frac{p}{p-1}}}{8}\right)^{\frac{p-1}{p}}\right. \\
&\qquad \left. + \left(\frac{\left|f^{(n)}(a)\right|^{\frac{p}{p-1}} + 3\left|f^{(n)}(b)\right|^{\frac{p}{p-1}}}{8}\right)^{\frac{p-1}{p}}\right]. \qquad (2.25)
\end{aligned}$$

Corollary 2.13. *Under the assumptions of Theorem* 2.10*, if f is preinvex on K with respect to the function $\eta(y,x) = y - x$, $q \geq 1$, $n \in \mathbb{N}$, $n \geq 1$. Then $\left|f^{(n)}\right|^q$ is convex on K, $q \geq 1$, $n \in \mathbb{N}$, $n \geq 1$ and hence we have the following inequality:*

$$\left|\sum_{k=0}^{n-1} \frac{\left[(-1)^k + 1\right](\eta(b,a))^k}{2^{k+1}(k+1)!} f^{(k)}\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)\,dx\right|$$
$$\leq \frac{(b-a)^n}{2^{n+1}(n+1)!}\left[\left(\frac{(n+3)\left|f^{(n)}(a)\right|^q + (n+1)\left|f^{(n)}(b)\right|^q}{2(n+2)}\right)^{\frac{1}{q}} + \left(\frac{(n+1)\left|f^{(n)}(a)\right|^q + (n+3)\left|f^{(n)}(b)\right|^q}{2(n+2)}\right)^{\frac{1}{q}}\right]. \quad (2.26)$$

Corollary 2.14. *If we take $q = 1$ in (*2.26*), we get the following inequality:*

$$\left|\sum_{k=0}^{n-1} \frac{\left[(-1)^k + 1\right](\eta(b,a))^k}{2^{k+1}(k+1)!} f^{(k)}\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)\,dx\right| \leq \frac{(b-a)^n}{2^n n!}\left[\frac{\left|f^{(n)}(a)\right| + \left|f^{(n)}(b)\right|}{2(n+1)}\right]. \quad (2.27)$$

Corollary 2.15. *If the conditions of Theorem* 2.8*, Theorem* 2.9 *and Theorem* 2.10 *are satisfied then for $n = 2$, we have the following inequalities respectively:*

$$\left|f\left(a + \frac{1}{2}\eta(b,a)\right) - \frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)\,dx\right| \leq \frac{(\eta(b,a))^2}{48}\left[\left|f''(a)\right| + \left|f''(b)\right|\right]. \quad (2.28)$$

$$\left|f\left(a + \frac{1}{2}\eta(b,a)\right) - \frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)\,dx\right|$$
$$\leq \frac{(\eta(b,a))^2}{8 \cdot 2^{\frac{1}{p}}(2p+1)^{\frac{1}{p}}}\left[\left(\frac{3\left|f''(a)\right|^{\frac{p}{p-1}} + \left|f''(b)\right|^{\frac{p}{p-1}}}{8}\right)^{\frac{p-1}{p}} + \left(\frac{\left|f''(a)\right|^{\frac{p}{p-1}} + 3\left|f''(b)\right|^{\frac{p}{p-1}}}{8}\right)^{\frac{p-1}{p}}\right], p > 1. \quad (2.29)$$

$$\left|f\left(a+\frac{1}{2}\eta(b,a)\right)-\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)}f(x)\,dx\right|$$
$$\leq\frac{(\eta(b,a))^2}{48}\left[\left(\frac{5|f''(a)|^q+3|f''(b)|^q}{8}\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\frac{3|f''(a)|^q+5|f''(b)|^q}{8}\right)^{\frac{1}{q}}\right],q\geq 1. \quad (2.30)$$

Remark 2.16. It may be noted that the inequalities (2.28), (2.29) and (2.30) may give better bounds than those proved in [27].

3. Applications to special means

In the following we give certain generalizations of some notions for a positive valued function of a positive variable.

Definition 3.1. [2]*A function $M:\mathbb{R}_+^2\to\mathbb{R}_+$, is called a Mean function if it has the following properties:*

1. *Homogeneity: $M(ax,ay)=aM(x,y)$, for all $a>0$,*
2. *Symmetry : $M(x,y)=M(y,x)$,*
3. *Reflexivity : $M(x,x)=x$,*
4. *Monotonicity: If $x\leq x'$ and $y\leq y'$, then $M(x,y)\leq M(x',y')$,*
5. *Internality:* $\min\{x,y\}\leq M(x,y)\leq\max\{x,y\}$.

We consider some means for arbitrary positive real numbers α,β (see for instance [2]).

1. The arithmetic mean:
$$A:=A(\alpha,\beta)=\frac{\alpha+\beta}{2}$$
2. The geometric mean:
$$G:=G(\alpha,\beta)=\sqrt{\alpha\beta}$$
3. The harmonic mean:
$$H:=H(\alpha,\beta)=\frac{2}{\frac{1}{\alpha}+\frac{1}{\beta}}$$
4. The power mean:
$$P_r:=P_r(\alpha,\beta)=\left(\frac{\alpha^r+\beta^r}{2}\right)^{\frac{1}{r}},\ r\geq 1$$
5. The identric mean:
$$I:=I(\alpha,\beta)=\begin{cases}\frac{1}{e}\left(\frac{\beta^\beta}{\alpha^\alpha}\right), & \alpha\neq\beta\\ \alpha, & \alpha=\beta\end{cases}$$

6. The logarithmic mean:

$$L := L(\alpha,\beta) = \frac{\alpha-\beta}{\ln|\alpha| - \ln|\beta|},\ \ |\alpha| \neq |\beta|$$

7. The generalized log-mean:

$$L_p := L_p(\alpha,\beta) = \left[\frac{\beta^{p+1}-\alpha^{p+1}}{(p+1)(\beta-\alpha)}\right],\ \alpha\neq\beta,\ p\in\mathbb{R}\backslash\{-1,0\}.$$

It is well known that L_p is monotonic nondecreasing over $p \in \mathbb{R}$, with $L_{-1} := L$ and $L_0 := I$. In particular, we have the following inequality $H \leq G \leq L \leq I \leq A$.

Now, let a and b be positive real numbers such that $a < b$. Consider the function $M := M(a,b) : [a, a+\eta(b,a)] \times [a, a+\eta(b,a)] \to \mathbb{R}^+$, which is one of the above mentioned means, therefore one can obtained variant inequalities for these means as follows:

Setting $\eta(b,a) = M(b,a)$ in (2.11), one can obtain the following interesting inequalities involving means:

$$\left|\frac{f(a)+f(a+M(b,a))}{2} - \frac{1}{M(b,a)}\int_a^{a+M(b,a)} f(x)\,dx\right| \leq \frac{(M(b,a))^2}{24}\left(\left|f''(a)\right| + \left|f''(b)\right|\right). \quad (3.1)$$

$$\left|f\left(a+\frac{1}{2}M(b,a)\right) - \frac{1}{M(b,a)}\int_a^{a+M(b,a)} f(x)\,dx\right| \leq \frac{(M(b,a))^2}{48}\left[\left|f''(a)\right| + \left|f''(b)\right|\right]. \quad (3.2)$$

$$\left|f\left(a+\frac{1}{2}M(b,a)\right) - \frac{1}{M(b,a)}\int_a^{a+M(b,a)} f(x)\,dx\right| \leq \frac{(M(b,a))^2}{8\cdot 2^{\frac{1}{p}}(2p+1)^{\frac{1}{p}}}\left[\left(\frac{3\left|f''(a)\right|^{\frac{p}{p-1}} + \left|f''(b)\right|^{\frac{p}{p-1}}}{8}\right)^{\frac{p-1}{p}} + \left(\frac{\left|f''(a)\right|^{\frac{p}{p-1}} + 3\left|f''(b)\right|^{\frac{p}{p-1}}}{8}\right)^{\frac{p-1}{p}}\right],\ p>1. \quad (3.3)$$

$$\left|f\left(a+\frac{1}{2}M(b,a)\right)-\frac{1}{M(b,a)}\int_a^{a+M(b,a)}f(x)\,dx\right|$$
$$\leq\frac{(M(b,a))^2}{48}\left[\left(\frac{5|f''(a)|^q+3|f''(b)|^q}{8}\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\frac{3|f''(a)|^q+5|f''(b)|^q}{8}\right)^{\frac{1}{q}}\right],\ q\geq 1. \quad (3.4)$$

Letting $M = A, G, H, P_r, I, L, L_p$ in (3.1), (3.2), (3.3) and (3.4), we get the inequalities involving means, and the details are left to the interested reader.

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Muhammad Amer Latif
University of Hail
Faculty of Science, Department of Mathematics
Hail 2440, Saudi Arabia
e-mail: m_amer_latif@hotmail.com

Stud. Univ. Babeş-Bolyai Math. 58(2013), No. 3, 345–354

Certain subclasses of analytic univalent functions generated by harmonic univalent functions

Santosh Joshi and Girish Shelake

Abstract. In this paper we define and investigate subclasses of analytic univalent functions generated by harmonic univalent and sense-preserving mappings.We obtain some inclusion theorems and convolution characterizations for above subclasses of univalent functions.

Mathematics Subject Classification (2010): 30C45, 30C55, 50E20.

Keywords: Harmonic mappings, univalent, sense-preserving, analytic, hypergeometric functions, subordination.

1. Introduction

A continuous complex-valued function $f = u + iv$ defined in a simply connected domain D is said to be harmonic in D if both u and v are real harmonic in D. In any simply connected domain we write

$$f = h + \bar{g} \tag{1.1}$$

where h and g are analytic in D. We call h the analytic part and g the co-analytic part of f. A necessary and sufficient condition for f to be locally univalent and sense-preserving in D is that $|h'(z)| > |g'(z)|$ for all z in D, see [6].

Every harmonic function $f = h + \bar{g}$ is uniquely determined by the coefficients of power series expansions in the unit disk $U = \{z : |z| < 1\}$ given by

$$h(z) = z + \sum_{n=2}^{\infty} A_n z^n, \quad g(z) = \sum_{n=1}^{\infty} B_n z^n, \quad z \in U, |B_1| < 1, \tag{1.2}$$

where $A_n \in \mathbb{C}$ for $n = 2, 3, 4, \ldots$ and $B_n \in \mathbb{C}$ for $n = 1, 2, 3, \ldots$. For further information about these mappings, one may refer to [4, 6, 7].

In 1984, Clunie and Sheil-Small [6] studied the family S_H of all univalent sense-preserving harmonic functions f of the form (1.1) in U, such that h and g are represented by (1.2). Note that S_H reduces to the well-known family S, the class of all normalized analytic univalent functions h given in (1.2), whenever the co-analytic part g of f is zero. Let K and K_H be the subclasses of S and S_H respectively such that images of $f(U)$ are convex.

In last two decades, several researchers have defined various subclasses of S using subordination. For the functions h and F analytic in U, we say h is subordinate to F $(h \prec F)$, if there exists an analytic function w in the unit disk U, with $w(0) = 0$ and $|w(z)| < 1$ such that $h(z) = F(w(z))$ for all z in U. Using subordination, we define two subclasses of S as follows:

$$S^*[A, B, \alpha, \gamma] = \left\{ f \in S : \frac{zf'(z)}{f(z)} \prec \frac{1 + \gamma[B + (A - B)(1 - \alpha)]z}{1 + \gamma Bz}, z \in U \right\},$$

$$K[A, B, \alpha, \gamma] = \left\{ f \in S : \frac{(zf'(z))'}{f'(z)} \prec \frac{1 + \gamma[B + (A - B)(1 - \alpha)]z}{1 + \gamma Bz}, z \in U \right\},$$

where $0 \le \alpha < 1, 0 < \gamma \le 1, -1 \le B < \gamma(B + (A - B)(1 - \alpha)) < A \le 1$. Note that the condition $|B| \le 1$ implies that the function $[1 + \gamma(B + (A - B)(1 - \alpha))z][1 + \gamma Bz]^{-1}$ is convex and univalent in U. For different values of parameters A, B, α and γ one can obtain several subclasses of S. For $\gamma = 1$ we get the subclasses defined by S. Joshi et.al[8].

Note that the convex domains are those domains that are convex in every direction. The following lemma will motivate us to construct certain analytic univalent function associated with $f \in S_H$.

Lemma 1.1 ([5, 6])**.** *A harmonic function $f = h + \bar{g}$ locally univalent in U is a univalent mapping of U and $f \in K_H$ if and only if $h - g$ is an analytic univalent mapping of U onto a domain convex in the direction of the real axis.*

For $f = h + \bar{g}$ in S_H, where h and g are given by (1.2), Lemma 1.1 led us to construct the function t with suitable normalization, given by

$$t(z) = \frac{h(z) - g(z)}{1 - B_1} = z + \sum_{n=2}^{\infty} \frac{A_n - B_n}{1 - B_1} z^n, \quad z \in U. \tag{1.3}$$

Since $f \in S_H$ is sense-preserving, it follows that $|B_1| < 1$. Hence the function t belongs to S. This observation has prompted us to define the following classes:

$$S_H[A, B, \alpha, \gamma] := \{f = h + \bar{g} \in S_H : t \in S^*[A, B, \alpha, \gamma]\},$$

$$K_H[A, B, \alpha, \gamma] := \{f = h + \bar{g} \in S_H : t \in K[A, B, \alpha, \gamma]\}.$$

In [2], Ahuja O.P connected hypergeometric functions with harmonic mappings $f = h + \bar{g}$ by defining the convolution operator Ω by

$$\Omega(f) := f \tilde{*} (\phi_1 + \bar{\phi}_2) = h * \phi_1 + \overline{g * \phi_2},$$

where $*$ denotes the convolution product of two power series and ϕ_1, ϕ_2 are defined by

$$\phi_1(z) = zF(a_1, b_1; c_1; z) = z + \sum_{n=2}^{\infty} \frac{(a_1)_{n-1}(b_1)_{n-1}}{(c_1)_{n-1}(1)_{n-1}} z^n,$$

$$\phi_2(z) = zF(a_2, b_2; c_2; z) = \sum_{n=1}^{\infty} \frac{(a_2)_{n-1}(b_2)_{n-1}}{(c_2)_{n-1}(1)_{n-1}} z^n.$$

Here $F(a, b; c; z)$ is a well-known hypergeometric function and a's, b's, c's are complex parameters with $c \neq 0, -1, -2, \ldots$. Corresponding to any function $f = h + \bar{g}$ given by (1.2), we have $\Omega(f) = H + \bar{G}$, where

$$H(z) = z + \sum_{n=2}^{\infty} \frac{(a_1)_{n-1}(b_1)_{n-1}}{(c_1)_{n-1}(1)_{n-1}} A_n z^n \quad \text{and}$$
$$G(z) = \sum_{n=1}^{\infty} \frac{(a_2)_{n-1}(b_2)_{n-1}}{(c_2)_{n-1}(1)_{n-1}} B_n z^n, |B_1| < 1. \tag{1.4}$$

We will frequently use the Gauss summation formula

$$F(a, b; c; 1) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n(1)_n} = \frac{\Gamma(c-a-b)\Gamma(c)}{\Gamma(c-a)\Gamma(c-b)}, \mathrm{Re}(c-a-b) > 0.$$

In the present paper, we study certain connections of the mappings $f = h + \bar{g}$ in S_H with the corresponding analytic functions in the classes $S^*[A, B, \alpha, \gamma]$ and $K[A, B, \alpha, \gamma]$. More precisely, we obtain some inclusion theorems and convolution characterization theorems for the classes $S_H[A, B, \alpha, \gamma]$ and $K_H[A, B, \alpha, \gamma]$.

2. Lemmas

Lemma 2.1. *A function h defined by the first equation in (1.2) is in $S^*[A, B, \alpha, \gamma]$ if*

$$\sum_{n=2}^{\infty} \{(n-1)(1+\gamma|B|) + \gamma(A-B)(1-\alpha)\}|A_n| \leq (A-B)(1-\alpha)\gamma.$$

Proof. In view of definition of $S^*[A, B, \alpha, \gamma]$, it follows that $h \in S^*[A, B, \alpha, \gamma]$ if and only if there exists an analytic function w such that

$$\frac{zh'(z)}{h(z)} = \frac{1 + \gamma[B + (A-B)(1-\alpha)]w(z)}{1 + \gamma B w(z)},$$

with $w(0) = 0$ and $|w(z)| < |z|$. Since $|w(z)| < 1$, the above equation is equivalent to

$$\left| \frac{\frac{zh'(z)}{h(z)} - 1}{\gamma[B + (A-B)(1-\alpha)] - \gamma B \frac{zh'(z)}{h(z)}} \right| < 1, \ z \in U.$$

On the other hand, on $|z| = 1$ we have

$$
\begin{aligned}
&|zh'(z) - h(z)| - |[B + (A - B)(1 - \alpha)]h(z) - Bzh'(z)|\gamma \\
&= \left|\sum_{n=2}^{\infty}(n-1)A_n z^n\right| \\
&\quad - \gamma\left|(A - B)(1-\alpha)z - \sum_{n=2}^{\infty}[(n-1)B - (A-B)(1-\alpha)]A_n z^n\right| \\
&\le \sum_{n=2}^{\infty}[(n-1)(1+\gamma|B|) + \gamma(A-B)(1-\alpha)]|A_n| - (A-B)(1-\alpha)\gamma \\
&\le 0,
\end{aligned}
$$

provided the given condition holds. Hence from the maximum modulus Theorem it follows that $h \in S^*[A, B, \alpha, \gamma]$. □

Lemma 2.2. *A function h defined by the first equation in (1.2) is in $K[A, B, \alpha, \gamma]$ if*

$$\sum_{n=2}^{\infty} n\{(n-1)(1+\gamma|B|) + \gamma(A-B)(1-\alpha)\}|A_n| \le (A-B)(1-\alpha)\gamma.$$

Proof. From the definition of $K[A, B, \alpha, \gamma]$, it follows that $h \in K[A, B, \alpha, \gamma]$ if and only if there exists an analytic function w such that

$$\frac{(zh'(z))'}{h'(z)} = \frac{1 + \gamma[B + (A-B)(1-\alpha)]w(z)}{1 + \gamma B w(z)},$$

with $w(0) = 0$ and $|w(z)| < |z| < 1$. This equality is equivalent to

$$\left|\frac{\frac{(zh'(z))'}{h'(z)} - 1}{\gamma[B + (A-B)(1-\alpha)] - \gamma B\frac{(zh'(z))'}{h'(z)}}\right| < 1,\ z \in U$$

□

The remaining steps of the proof are similar to the proof of Lemma 2.1.

Lemma 2.3 ([2]). *Let $f = h + \bar{g}$ where h and g are analytic functions of the form (1.2). If $a_j, b_j \in \mathbb{C}\backslash\{0\}, c_j \in \mathbb{R}$ are such that $c_j > |a_j| + |b_j| + 1$ for $j = 1, 2$ and the following inequalities*

(i) $\displaystyle\sum_{n=2}^{\infty}|A_n| + \sum_{n=1}^{\infty}|B_n| \le 1, |B_1| < 1,$

(ii) $\displaystyle\sum_{j=1}^{2}\left(\frac{|a_j b_j|}{c_j - |a_j| - |b_j| - 1} + 1\right)F(|a_j|, |b_j|; c_j; 1) \le 2$

are satisfied, then $\Omega(f)$ is sense-preserving harmonic and univalent in U; and so $\Omega(f) \in S_H$.

Lemma 2.4 ([2]). *If $a, b, c > 0$, then*

(i) $\displaystyle\sum_{n=1}^{\infty}(n-1)\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}} = \frac{ab}{c-a-b-1}F(a,b;c;1)$ *if* $c > a+b+1$,

(ii) $\displaystyle\sum_{n=2}^{\infty}(n-1)^2\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}} = \left(\frac{(a)_2(b)_2}{(c-a-b-2)_2} + \frac{ab}{c-a-b-1}\right)F(a,b;c;1)$
if $c > a+b+2$.

3. Main results

Theorem 3.1. *Let $f = h + \bar{g}$ be of the form (1.2), and for $j = 1, 2$, suppose $a_j, b_j \in \mathbb{C}\backslash\{0\}, c_j \in \mathbb{R}$ are such that $c_j > |a_j| + |b_j| + 1$ and $\Omega(f) \in S_H$. If the coefficient conditions*

(i) $\displaystyle\sum_{n=2}^{\infty}|A_n| + \sum_{n=1}^{\infty}|B_n| \leq 1$

(ii) $\displaystyle\sum_{j=1}^{2}\left(\frac{(1+\gamma|B|)}{\gamma(A-B)(1-\alpha)}\frac{|a_jb_j|}{c_j-|a_j|-|b_j|-1} + 1\right)F(|a_j|,|b_j|;c_j;1)$
$\leq (2 + |1-B_1|) < 4$

are satisfied, then $\Omega(f) \in S_H[A, B, \alpha, \gamma]$.

Proof. In order to prove that $\Omega(f) \in S_H[A, B, \alpha, \gamma]$, it suffices to prove that the function

$$\begin{aligned} T(z) &:= \frac{H(z) - G(z)}{1 - B_1} \\ &= z + \sum_{n=2}^{\infty}\left[\frac{(a_1)_{n-1}(b_1)_{n-1}}{(c_1)_{n-1}(1)_{n-1}}A_n - \frac{(a_2)_{n-1}(b_2)_{n-1}}{(c_2)_{n-1}(1)_{n-1}}B_n\right]\frac{1}{1-B_1}z^n \end{aligned} \tag{3.1}$$

is in $S^*[A, B, \alpha, \gamma]$. Note that $|A_n| \leq 1$ and $|B_n| \leq 1$, by the condition (i). As an application of the Lemma 2.1, the function $T \in S^*[A, B, \alpha, \gamma]$ provided that $Q_1 \leq 1$, where

$$\begin{aligned} Q_1 &:= \sum_{n=2}^{\infty}\left[\frac{(n-1)(1+\gamma|B|) + \gamma(A-B)(1-\alpha)}{(A-B)(1-\alpha)\gamma}\right] \\ &\quad\cdot\left|\frac{(a_1)_{n-1}(b_1)_{n-1}}{(c_1)_{n-1}(1)_{n-1}}\frac{A_n}{1-B_1} - \frac{(a_2)_{n-1}(b_2)_{n-1}}{(c_2)_{n-1}(1)_{n-1}}\frac{B_n}{1-B_1}\right| \\ &\leq \sum_{n=2}^{\infty}\left[\frac{(n-1)(1+\gamma|B|) + \gamma(A-B)(1-\alpha)}{\gamma(A-B)(1-\alpha)|1-B_1|}\right] \\ &\quad\cdot\left(\frac{(|a_1|)_{n-1}(|b_1|)_{n-1}}{(c_1)_{n-1}(1)_{n-1}} + \frac{(|a_2|)_{n-1}(|b_2|)_{n-1}}{(c_2)_{n-1}(1)_{n-1}}\right) \\ &= \frac{(1+\gamma|B|)}{|1-B_1|(A-B)(1-\alpha)\gamma} \end{aligned}$$

$$\cdot \sum_{n=2}^{\infty}(n-1)\left(\frac{(|a_1|)_{n-1}(|b_1|)_{n-1}}{(c_1)_{n-1}(1)_{n-1}}+\frac{(|a_2|)_{n-1}(|b_2|)_{n-1}}{(c_2)_{n-1}(1)_{n-1}}\right)$$
$$+\frac{1}{|1-B_1|}\sum_{n=2}^{\infty}\left(\frac{(|a_1|)_{n-1}(|b_1|)_{n-1}}{(c_1)_{n-1}(1)_{n-1}}+\frac{(|a_2|)_{n-1}(|b_2|)_{n-1}}{(c_2)_{n-1}(1)_{n-1}}\right)$$
$$=\frac{(1+\gamma|B|)}{|1-B_1|(A-B)(1-\alpha)\gamma}\left(\frac{|a_1b_1|}{c_1-|a_1|-|b_1|-1}F(|a_1|,|b_1|;c_1;1)\right.$$
$$\left.+\frac{|a_2b_2|}{c_2-|a_2|-|b_2|-1}F(|a_2|,|b_2|;c_2;1)\right)$$
$$+\frac{1}{|1-B_1|}(F(|a_1|,|b_1|;c_1;1)+F(|a_2|,|b_2|;c_2;1)-2)$$

by Lemma 2.3. Therefore, it follows that $T \in S^*[A,B,\alpha,\gamma]$ if the inequality

$$\frac{1}{|1-B_1|}\sum_{j=1}^{2}\left(\frac{(1+\gamma|B|)}{(A-B)(1-\alpha)\gamma}\frac{|a_jb_j|}{c_j-|a_j|-|b_j|-1}+1\right)$$
$$\cdot F(|a_j|,|b_j|;c_j;1)-\frac{2}{|1-B_1|}\leq 1$$

holds. But this inequality is true because of given condition (ii). □

Theorem 3.2. *Let $f=h+\bar{g}$ given by (1.2) be in S_H. If the inequality*

$$\sum_{n=2}^{\infty}\{(n-1)(1+\gamma|B|)+(A-B)(1-\alpha)\gamma\}|A_n|$$
$$+\sum_{n=1}^{\infty}\{(n-1)(1+\gamma|B|)+(A-B)(1-\alpha)\gamma\}|B_n|\leq(A-B)(1-\alpha)\gamma|1-B_1|$$

is satisfied, then $f\in S_H[A,B,\alpha,\gamma]$.

Proof. From the definition of $S_H[A,B,\alpha,\gamma]$, it suffices to prove that the function t given by (1.3) is in the class $S^*[A,B,\alpha,\gamma]$. As an application of Lemma 2.1, we only need to show that $Q_2\leq 1$, where

$$Q_2:=\sum_{n=2}^{\infty}\frac{(n-1)(1+\gamma|B|)+(A-B)(1-\alpha)\gamma}{(A-B)(1-\alpha)\gamma}\left|\frac{A_n-B_n}{1-B_1}\right|.$$

But

$$Q_2\leq\sum_{n=2}^{\infty}\frac{(n-1)(1+\gamma|B|)+(A-B)(1-\alpha)\gamma}{(A-B)(1-\alpha)\gamma}\left[\frac{|A_n|+|B_n|}{|1-B_1|}\right]$$

and thus $Q_2\leq 1$ holds because of the given condition. □

Theorem 3.3. *Let $f=h+\bar{g}$ be of the form (1.2) and for $j=1,2$, suppose $a_j,b_j\in\mathbb{C}\backslash\{0\}, c_j\in\mathbb{R}$ such that $c_j>|a_j|+|b_j|+2$ and $\Omega(f)\in S_H$. If the coefficient conditions*

(i) $\sum_{n=2}^{\infty}|A_n|+\sum_{n=1}^{\infty}|B_n|\leq 1,$

(ii) $$\sum_{j=1}^{2}\left\{\frac{(1+\gamma B)}{(A-B)(1-\alpha)\gamma}\frac{(|a_j|)_2(|b_j|)_2}{(c_j-|a_j|-|b_j|-2)_2}+\left(\frac{2(1+\gamma|B|)}{(A-B)(1-\alpha)\gamma}+1\right)\frac{|a_jb_j|}{c_j-|a_j|-|b_j|-1}+1\right\}F(|a_j|,|b_j|;c_j;1)\le 2+|1-B_1|<4$$

are satisfied, then $\Omega(f)\in K_H[A,B,\alpha,\gamma]$.

Proof. In view of the definition of $K_H[A,B,\alpha,\gamma]$ and the fact that $\Omega(f)\in S_H$, it suffices to prove that the function T given by (3.1) is in $K[A,B,\alpha,\gamma]$. Note that by the condition (i) we have $|A_n|\le 1$ and $|B_n|\le 1$. In the view of Lemma (2.2), the function $T\in K[A,B,\alpha,\gamma]$ provided that $Q_3\le 1$, where

$$\begin{aligned}
Q_3 &:= \sum_{n=2}^{\infty} n\left[\frac{(n-1)(1+\gamma|B|)+(A-B)(1-\alpha)\gamma}{(A-B)(1-\alpha)\gamma}\right]\\
&\quad\cdot\left|\frac{(a_1)_{n-1}(b_1)_{n-1}}{(c_1)_{n-1}(1)_{n-1}}\frac{A_n}{1-B_1}-\frac{(a_2)_{n-1}(b_2)_{n-1}}{(c_2)_{n-1}(1)_{n-1}}\frac{B_n}{1-B_1}\right|\\
&\le \sum_{n=2}^{\infty} n\left[\frac{(n-1)(1+\gamma|B|)+(A-B)(1-\alpha)\gamma}{(A-B)(1-\alpha)\gamma|1-B_1|}\right]\\
&\quad\cdot\left[\frac{(|a_1|)_{n-1}(|b_1|)_{n-1}}{(c_1)_{n-1}(1)_{n-1}}+\frac{(|a_2|)_{n-1}(|b_2|)_{n-1}}{(c_2)_{n-1}(1)_{n-1}}\right]\\
&= \frac{1+\gamma|B|}{|1-B_1|(A-B)(1-\alpha)\gamma}\sum_{n=2}^{\infty}[(n-1)^2+(n-1)](D_1+D_2)\\
&\quad+\frac{1}{|1-B_1|}\sum_{n=2}^{\infty}(n-1)(D_1+D_2)+\frac{1}{|1-B_1|}\sum_{n=2}^{\infty}(D_1+D_2)\\
&= \frac{1}{|1-B_1|}\left[\begin{array}{l}\dfrac{1+\gamma|B|}{(A-B)(1-\alpha)\gamma}\displaystyle\sum_{n=2}^{\infty}(n-1)^2(D_1+D_2)\\ +\left(\dfrac{1+\gamma|B|}{(A-B)(1-\alpha)\gamma}+1\right)\displaystyle\sum_{n=2}^{\infty}(n-1)(D_1+D_2)\\ +\displaystyle\sum_{n=2}^{\infty}(D_1+D_2)\end{array}\right],
\end{aligned}$$

where $D_j=\dfrac{(|a_j|)_{n-1}(|b_j|)_{n-1}}{(c_j)_{n-1}(1)_{n-1}}$ for $j=1,2$.

Using Lemma 2.4, we find that

$$Q_3\le\frac{1}{|1-B_1|}\sum_{j=1}^{2}\left\{\frac{(1+\gamma|B|)}{(A-B)(1-\alpha)\gamma}\frac{(|a_j|)_2(|b_j|)_2}{(c_j-|a_j|-|b_j|-2)_2}+\left(\frac{2(1+\gamma|B|)}{(A-B)(1-\alpha)\gamma}+1\right)\cdot\frac{|a_jb_j|}{c_j-|a_j|-|b_j|-1}+1\right\}F(|a_j|,|b_j|;c_j;1)-\frac{2}{|1-B_1|},$$

and this proves that $Q_3 \le 1$, if the condition (ii) holds. □

The proof of the next theorem is similar to the proof of Theorem 3.2 and hence it is omitted.

Theorem 3.4. *Let $f = h + \bar{g}$ given by (1.2) be in S_H. If the inequality*

$$\sum_{n=2}^{\infty} n\{(n-1)((1+\gamma|B|)+(A-B)(1-\alpha)\gamma\}|A_n|$$

$$+\sum_{n=1}^{\infty} n\{(n-1)(1+\gamma|B|)+(A-B)(1-\alpha)\gamma\}|B_n| \le (A-B)(1-\alpha)\gamma$$

is satisfied, then $f \in K_H[A,B,\alpha,\gamma]$.

The next two Theorems give characterizations of functions in $S_H[A,B,\alpha,\gamma]$ and $K_H[A,B,\alpha,\gamma]$.

Theorem 3.5. *If $f(z) = h(z) + \overline{g(z)} \in S_H$ then $f \in S_H[A,B,\alpha,\gamma]$ if and only if*

$$\frac{1}{z}[(h(z)-g(z)) * F_1(z)] \ne 0$$

for all z in U and all ξ, such that $|\xi| = 1$, where

$$F_1(z) := \frac{z + \left(\frac{\xi-(B+(A-B)(1-\alpha))\gamma}{(A-B)(1-\alpha)\gamma}\right) z^2}{(1-z)^2}.$$

Proof. By definition of $S_H[A,B,\alpha,\gamma]$, it is obvious that $f \in S_H[A,B,\alpha,\gamma]$ if and only if $t(z)$ given by (1.3) belongs to $S^*[A,B,\alpha,\gamma]$. But, $t \in S^*[A,B,\alpha,\gamma]$ if and only if

$$\frac{zt'(z)}{t(z)} \prec \frac{1+(B+(A-B)(1-\alpha))\gamma z}{1+\gamma B z},$$

that is

$$\frac{zt'(z)}{t(z)} \ne \frac{1+(B+(A-B)(1-\alpha))\gamma\varsigma}{1+\gamma B\varsigma}$$

for $z \in U$ and $|\varsigma| = 1$, which is equivalent to

$$\frac{1}{z}[(1+\gamma B\varsigma)zt' - (1+(B+(A-B)(1-\alpha))\gamma\varsigma)t] \ne 0.$$

Since

$$zt' = t * \frac{z}{(1-z)^2}, \quad t = t * \frac{z}{1-z},$$

the above inequality is equivalent to

$$\frac{1}{z}\left[t(z) * \left[\frac{-\gamma(A-B)(1-\alpha)\varsigma z + [1+(B+(A-B)(1-\alpha))\gamma\varsigma]z^2}{(1-z)^2}\right]\right]$$

$$= \frac{-\gamma(A-B)(1-\alpha)\varsigma}{(1-B_1)z}\left[(h(z)-g(z)) * \left(\frac{z + \left(\frac{\xi-\gamma(B+(A-B)(1-\alpha))}{(A-B)(1-\alpha)\gamma}\right) z^2}{(1-z)^2}\right)\right] \ne 0,$$

where $|-1/\varsigma| = |\xi| = 1$, and the result follows. □

Corollary 3.6. *If* $f(z) = h(z) + \overline{g(z)} \in S_H$*, then* $f \in K_H[A, B, \alpha, \gamma]$ *if and only if*

$$\frac{1}{z}[(h(z) - g(z)) * F_2(z)] \neq 0,$$

for all z *in* U *and all* ξ*, such that* $|\xi| = 1$*, where*

$$F_2(z) := \frac{z + \left(\frac{2\xi-(2B+(A-B)(1-\alpha))\gamma}{(A-B)(1-\alpha)\gamma}\right) z^2}{(1-z)^3}.$$

Proof. Note that $t \in K[A, B, \alpha, \gamma]$ if and only $zt'(z) \in S_H[A, B, \alpha, \gamma]$. If we let

$$p(z) = \frac{z + \left(\frac{\xi-(B+(A-B)(1-\alpha))\gamma}{(A-B)(1-\alpha)\gamma}\right) z^2}{(1-z)^2},$$

we note that

$$zp'(z) = \frac{z + \left(\frac{2\xi-2B\gamma-(A-B)(1-\alpha)\gamma}{(A-B)(1-\alpha)\gamma}\right) z^2}{(1-z)^3}.$$

Using the identity $zt' * p = t * zp'$, the result follows from Theorem 3.5. □

Acknowledgement. The authors express their sincerest thanks to the referee for useful suggestions.

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Santosh Joshi
Department of Mathematics
Walchand College, Sangli-416415
Maharashtra, India
e-mail: joshisb@hotmail.com

Girish Shelake
Department of Mathematics
Willingdon College, Sangli-416415
Maharashtra, India
e-mail: shelakegd@gmail.com

Stud. Univ. Babeş-Bolyai Math. 58(2013), No. 3, 355–360

Some sufficient conditions for starlike and convex functions

Sukhwinder Singh Billing

Abstract. Using the technique of differential subordination, in particular, a lemma due to Miller and Mocanu [1], we study a certain differential operator to obtain some sufficient conditions for starlike, convex, strongly starlike and strongly convex functions. In particular, we prove that if $f \in \mathcal{A}_n$, $\left(\frac{z\mathcal{F}'(z)}{\mathcal{F}(z)}\right)^\gamma \neq 0$, $z \in \mathbb{E}$ satisfies

$$\gamma\left(1+\frac{z\mathcal{F}''(z)}{\mathcal{F}'(z)}-\frac{z\mathcal{F}'(z)}{\mathcal{F}(z)}\right) \prec \frac{2n(1-\alpha)z}{(1-z)(1+(1-2\alpha)z)},\ 0 \leq \alpha < 1,\ z \in \mathbb{E},$$

then

$$\left(\frac{z\mathcal{F}'(z)}{\mathcal{F}(z)}\right)^\gamma \prec \frac{1+(1-2\alpha)z}{1-z},\ z \in \mathbb{E},$$

where $\mathcal{F}(z) = (1-\lambda)f(z) + \lambda z f'(z),\ 0 \leq \lambda \leq 1$ is univalent and γ is a non-zero complex number.

Mathematics Subject Classification (2010): 30C80, 30C45.

Keywords: Differential subordination, analytic function, starlike function, convex function.

1. Introduction

Let $\mathcal{H}$ be the class of functions analytic in the open unit disk $\mathbb{E} = \{z : |z| < 1\}$. For n a positive integer and $a \in \mathbb{C}$, let

$$\mathcal{H}[a,n] = \left\{f \in \mathcal{H} : f(z) = a + a_n z^n + a_{n+1}z^{n+1} + \cdots\right\}.$$

The class $\mathcal{A}_n$ of normalized analytic functions is defined as

$$\mathcal{A}_n = \left\{f \in \mathcal{H} : f(z) = z + a_{n+1}z^{n+1} + a_{n+2}z^{n+2} + \cdots\right\}.$$

Let $\phi : \mathbb{C}^2 \times \mathbb{E} \to \mathbb{C}$ and let h be univalent in $\mathbb{E}$. If p is analytic in $\mathbb{E}$ and satisfies the differential subordination

$$\phi(p(z), zp'(z); z) \prec h(z),\ \phi(p(0), 0; 0) = h(0), \tag{1.1}$$

then p is called a solution of the first order differential subordination (1.1). The univalent function q is called a dominant of the differential subordination (1.1) if $p \prec q$ for all p satisfying (1.1). A dominant $\tilde{q}$ that satisfies $\tilde{q} \prec q$ for all dominants q of (1.1), is said to be the best dominant of (1.1).

In what follows, only principal values of complex powers are considered.

Irmak and Şan [2] introduced two differential operators $\mathcal{V}[\gamma, \lambda; f](z)$ and $\mathcal{W}[\gamma, \lambda; f](z)$ defined as under:

$$\mathcal{V}[\gamma, \lambda; f](z) = \left(\frac{z\mathcal{F}'(z)}{\mathcal{F}(z)}\right)^{\gamma} \tag{1.2}$$

and

$$\mathcal{W}[\gamma, \lambda; f](z) = \gamma \frac{\mathcal{F}(z)}{\mathcal{F}'(z)} \left(\frac{z\mathcal{F}'(z)}{\mathcal{F}(z)}\right)' = \gamma \left(1 + \frac{z\mathcal{F}''(z)}{\mathcal{F}'(z)} - \frac{z\mathcal{F}'(z)}{\mathcal{F}(z)}\right) \tag{1.3}$$

where $\mathcal{F}(z) = (1-\lambda)f(z) + \lambda z f'(z),\ f \in \mathcal{A}_n,\ 0 \le \lambda \le 1$ is univalent and γ is a non-zero complex number and they proved the following results:

Theorem 1.1. *If $f \in \mathcal{A}_n$ satisfies the condition*

$$|\mathcal{W}[\gamma, \lambda; f](z)| < \beta,\ 0 < \beta \le 1,\ z \in \mathbb{E},$$

then

$$|\arg\{\mathcal{V}[\gamma, \lambda; f](z)\}| < \frac{\pi}{2}\beta,$$

where $\mathcal{V}[\gamma, \lambda; f](z)$ and $\mathcal{W}[\gamma, \lambda; f](z)$ are given by (1.2) and (1.3) respectively.

Theorem 1.2. *If $f \in \mathcal{A}_n$ satisfies*

$$\Re\{\mathcal{W}[\gamma, \lambda; f](z)\} < \frac{nM}{1+M}, M \ge 1,\ z \in \mathbb{E},$$

then

$$|\mathcal{V}[\gamma, \lambda; f](z) - 1| < M,$$

where $\mathcal{V}[\gamma, \lambda; f](z)$ and $\mathcal{W}[\gamma, \lambda; f](z)$ are given by (1.2) and (1.3) respectively.

The main objective of this paper is to determine some sufficient conditions for starlike, convex, strongly starlike and strongly convex functions in terms of the operator defined above in (1.3). We claim that our results improve the above stated results of Irmak and Şan [2] and their consequences.

To prove our main result, we shall use the following lemma of Miller and Mocanu [1, pp. 76].

Lemma 1.3. *Let h be starlike in $\mathbb{E}$, with $h(0) = 0$ and $a \ne 0$. If $p \in \mathcal{H}[a, n]$ satisfies*

$$\frac{zp'(z)}{p(z)} \prec h(z),$$

then

$$p(z) \prec q(z) = a \exp\left[\frac{1}{n}\int_0^z \frac{h(t)}{t}\, dt\right],$$

and q is the best (a, n) dominant.

2. Main result and applications

Theorem 2.1. *Let h be starlike in $\mathbb{E}$, with $h(0) = 0$. Let $f \in \mathcal{A}_n$ be such that $\mathcal{V}[\gamma, \lambda; f](z) \neq 0,\ z \in \mathbb{E}$ and satisfy*

$$\mathcal{W}[\gamma, \lambda; f](z) \prec h(z),\ z \in \mathbb{E},$$

then

$$\mathcal{V}[\gamma, \lambda; f](z) \prec q(z) = \exp\left[\frac{1}{n}\int_0^z \frac{h(t)}{t}\,dt\right],$$

and q is the best dominant. Here $0 \leq \lambda \leq 1$, γ is a non-zero complex number and $\mathcal{V}[\gamma, \lambda; f](z)$ and $\mathcal{W}[\gamma, \lambda; f](z)$ are given by (1.2) and (1.3) respectively.

Proof. Write $p(z) = \mathcal{V}[\gamma, \lambda; f](z)$, then a little calculation yields

$$\frac{zp'(z)}{p(z)} = \mathcal{W}[\gamma, \lambda; f](z).$$

Now the proof follows from Lemma 1.3. □

Select $h(z) = \dfrac{2n\beta z}{1 - z^2}$, $0 < \beta \leq 1$, $z \in \mathbb{E}$ in the above theorem, we obtain the following result.

Theorem 2.2. *Let γ be a non-zero complex number and $0 \leq \lambda \leq 1$. Let $f \in \mathcal{A}_n$ be such that $\mathcal{V}[\gamma, \lambda; f](z) \neq 0,\ z \in \mathbb{E}$ and satisfy*

$$\mathcal{W}[\gamma, \lambda; f](z) \prec \frac{2n\beta z}{1 - z^2} = h(z),\ 0 < \beta \leq 1,$$

where $h(\mathbb{E}) = \mathbb{C} \setminus \{w \in \mathbb{C} : \Re(w) = 0,\ |\Im(w)| \geq n\beta\}$, then

$$\mathcal{V}[\gamma, \lambda; f](z) \prec \left(\frac{1 + z}{1 - z}\right)^{\beta},\ z \in \mathbb{E},$$

here $\mathcal{V}[\gamma, \lambda; f](z)$ and $\mathcal{W}[\gamma, \lambda; f](z)$ are given by (1.2) and (1.3) respectively.

Remark 2.3. Note that the differential operator $\mathcal{W}[\gamma, \lambda; f](z)$ takes values in an extended region of the complex plane than the result of Irmak and Şan [2] stated in Theorem 1.1 to get the same conclusion.

Taking $\gamma - 1 = \lambda = 0$ and $\gamma = \lambda = 1$ in Theorem 2.2, we, respectively, obtain the following results.

Corollary 2.4. *(i) If $f \in \mathcal{A}_n$, $\dfrac{zf'(z)}{f(z)} \neq 0$, satisfies the inequality*

$$1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)} \prec h(z),$$

then

$$\frac{zf'(z)}{f(z)} \prec \left(\frac{1 + z}{1 - z}\right)^{\beta},\ 0 < \beta \leq 1,\ z \in \mathbb{E},\ i.e.$$

f is strongly starlike of order β where $h(\mathbb{E}) = \mathbb{C} \setminus \{w \in \mathbb{C} : \Re(w) = 0,\ |\Im(w)| \geq n\beta\}$.

(ii) If $f \in \mathcal{A}_n,\ 1 + \dfrac{zf''(z)}{f'(z)} \neq 0$, satisfies the inequality

$$1 + z\left(\frac{(zf'(z))''}{(zf'(z))'} - \frac{(zf'(z))'}{zf'(z)}\right) \prec h(z),$$

then

$$1 + \frac{zf''(z)}{f'(z)} \prec \left(\frac{1+z}{1-z}\right)^{\beta},\ 0 < \beta \leq 1,\ z \in \mathbb{E},\ i.e.$$

f is strongly convex of order β where $h(\mathbb{E}) = \mathbb{C} \setminus \{w \in \mathbb{C} : \Re(w) = 0,\ |\Im(w)| \geq n\beta\}$.

Remark 2.5. Note that the results in above corollary extend the corresponding results of Irmak and Şan [2] (Corollary 2.1) in the sense that the differential operators

$$1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)} \text{ and } 1 + z\left(\frac{(zf'(z))''}{(zf'(z))'} - \frac{(zf'(z))'}{zf'(z)}\right)$$

now take values in largely extended region of the complex plane as compared to their results.

When $h(z)$ in Theorem 2.1 is taken as $h(z) = \dfrac{n\alpha z}{1+\alpha z},\ 0 < \alpha \leq 1,\ z \in \mathbb{E}$, we have the following result.

Theorem 2.6. *For non-zero complex γ and $0 \leq \lambda \leq 1,\ 0 < \alpha \leq 1$, if $f \in \mathcal{A}_n$ with $\mathcal{V}[\gamma, \lambda; f](z) \neq 0,\ z \in \mathbb{E}$, satisfies*

$$\mathcal{W}[\gamma, \lambda; f](z) \prec \frac{n\alpha z}{1+\alpha z},\ z \in \mathbb{E},$$

then

$$|\mathcal{V}[\gamma, \lambda; f](z) - 1| < \alpha,\ z \in \mathbb{E},$$

where $\mathcal{V}[\gamma, \lambda; f](z)$ and $\mathcal{W}[\gamma, \lambda; f](z)$ are given by (1.2) and (1.3) respectively.

Selecting $\gamma - 1 = \lambda = 0$ and $\gamma = \lambda = 1$ in Theorem 2.6, we, respectively, obtain:

Corollary 2.7. *(i) If $f \in \mathcal{A}_n,\ \dfrac{zf'(z)}{f(z)} \neq 0$, satisfies the inequality*

$$\Re\left\{z\left(\frac{f''(z)}{f'(z)} - \frac{f'(z)}{f(z)}\right)\right\} < \frac{(n-1)\alpha - 1}{1+\alpha},\ 0 < \alpha \leq 1,\ z \in \mathbb{E},$$

then

$$\left|\frac{zf'(z)}{f(z)} - 1\right| < \alpha.$$

(ii) If $f \in \mathcal{A}_n,\ 1 + \dfrac{zf''(z)}{f'(z)} \neq 0$, satisfies the inequality

$$\Re\left\{z\left(\frac{(zf'(z))''}{(zf'(z))'} - \frac{(zf'(z))'}{zf'(z)}\right)\right\} < \frac{(n-1)\alpha - 1}{1+\alpha},\ 0 < \alpha \leq 1,\ z \in \mathbb{E},$$

then

$$\left|\frac{zf''(z)}{f'(z)}\right| < \alpha.$$

Write $h(z) = \dfrac{2n(1-\alpha)z}{(1-z)(1+(1-2\alpha)z)}$, $0 \le \alpha < 1$, $z \in \mathbb{E}$ in Theorem 2.1 to get the following result.

Theorem 2.8. *Let $f \in \mathcal{A}_n$ be such that $\mathcal{V}[\gamma,\lambda;f](z) \neq 0$, $z \in \mathbb{E}$ and satisfy*

$$\mathcal{W}[\gamma,\lambda;f](z) \prec \frac{2n(1-\alpha)z}{(1-z)(1+(1-2\alpha)z)},\ 0 \le \alpha < 1,\ z \in \mathbb{E},$$

then

$$\mathcal{V}[\gamma,\lambda;f](z) \prec \frac{1+(1-2\alpha)z}{1-z},\ z \in \mathbb{E},$$

for $0 \le \lambda \le 1$, γ a non-zero complex number and $\mathcal{V}[\gamma,\lambda;f](z)$ and $\mathcal{W}[\gamma,\lambda;f](z)$ are given by (1.2) and (1.3) respectively.

Note that above theorem, in turn, gives the following result of Irmak and Şan [2].

Corollary 2.9. *Let $0 \le \alpha < 1$, $0 \le \lambda \le 1$ and γ be a non-zero complex number. Let $\mathcal{V}[\gamma,\lambda;f](z)$ and $\mathcal{W}[\gamma,\lambda;f](z)$ be given by (1.2) and (1.3) respectively. If $f \in \mathcal{A}_n$ with $\mathcal{V}[\gamma,\lambda;f](z) \neq 0$, $z \in \mathbb{E}$, satisfies*

$$\Re\{\mathcal{W}[\gamma,\lambda;f](z)\} > \begin{cases} \frac{n\alpha}{2(\alpha-1)}, & 0 \le \alpha \le 1/2, \\ \frac{n(\alpha-1)}{2\alpha}, & 1/2 \le \alpha < 1 \end{cases},$$

then

$$\Re\{\mathcal{V}[\gamma,\lambda;f](z)\} > \alpha,\ z \in \mathbb{E}.$$

Setting $\gamma - 1 = \lambda = 0$ and $\gamma = \lambda = 1$ in above corollary, we obtain, respectively, the following results which offer a correct version of Corollary 2.3 of Irmak and Şan [2].

Corollary 2.10. *(i) Let $0 \le \alpha < 1$ and let $f \in \mathcal{A}_n$, $\dfrac{zf'(z)}{f(z)} \neq 0$, satisfy the inequality*

$$\Re\left\{z\left(\frac{f''(z)}{f'(z)} - \frac{f'(z)}{f(z)}\right)\right\} > \begin{cases} \frac{(n-2)\alpha+2}{2(\alpha-1)}, & 0 \le \alpha \le 1/2, \\ \frac{(n-2)\alpha-n}{2\alpha}, & 1/2 \le \alpha < 1 \end{cases},$$

then

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > \alpha,\ \textit{i.e. } f \textit{ is starlike of order } \alpha.$$

(ii) Let $0 \le \alpha < 1$ and let $f \in \mathcal{A}_n$, $1 + \dfrac{zf''(z)}{f'(z)} \neq 0$, satisfy the inequality

$$\Re\left\{z\left(\frac{(zf'(z))''}{(zf'(z))'} - \frac{(zf'(z))'}{zf'(z)}\right)\right\} > \begin{cases} \frac{(n-2)\alpha+2}{2(\alpha-1)}, & 0 \le \alpha \le 1/2, \\ \frac{(n-2)\alpha-n}{2\alpha}, & 1/2 \le \alpha < 1 \end{cases},$$

then

$$\Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha,\ \textit{i.e. } f \textit{ is convex of order } \alpha.$$

Setting $\gamma-1=\lambda=0,\ \gamma=\lambda=1$ and $\alpha=0$ in the Theorem 2.8, we, respectively, obtain the following results.

Corollary 2.11. *(i) Suppose* $f\in\mathcal{A}_n,\ \dfrac{zf'(z)}{f(z)}\neq 0$, *satisfies the condition*

$$z\left(\frac{f''(z)}{f'(z)}-\frac{f'(z)}{f(z)}\right)\prec G(z),\ \text{where } G(\mathbb{E})=\mathbb{C}\setminus\{w\in\mathbb{C}:\Re(w)=-1,|\Im(w)|\geq n\},$$

then

$$\frac{zf'(z)}{f(z)}\prec\frac{1+z}{1-z},\ z\in\mathbb{E},\ \text{i.e. } f \text{ is starlike in } \mathbb{E}.$$

(ii) Suppose $f\in\mathcal{A}_n,\ 1+\dfrac{zf''(z)}{f'(z)}\neq 0$, *satisfies the condition*

$$z\left(\frac{(zf'(z))''}{(zf'(z))'}-\frac{(zf'(z))'}{zf'(z)}\right)\prec G(z),$$

then

$$1+\frac{zf''(z)}{f'(z)}\prec\frac{1+z}{1-z},\ z\in\mathbb{E},\ \text{i.e. } f \text{ is convex in } \mathbb{E},$$

where $G(\mathbb{E})=\mathbb{C}\setminus\{w\in\mathbb{C}:\Re(w)=-1,|\Im(w)|\geq n\}$.

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Sukhwinder Singh Billing
Department of Mathematics
Sri Guru Granth Sahib World University
Fatehgarh Sahib-140 407, Punjab, India
e-mail: ssbilling@gmail.com

Stud. Univ. Babeş-Bolyai Math. 58(2013), No. 3, 361–369

On a generalization of Szász operators by multiple Appell polynomials

Serhan Varma

Abstract. In this paper, we define a form of positive linear operators by means of multiple Appell polynomials. Also, Kantorovich type generalization and simultaneous approximation of these operators are given. Convergence properties of our operators are verified with the help of the universal Korovkin-type property and the order of approximation is calculated by using classical modulus of continuity.

Mathematics Subject Classification (2010): 41A25, 41A36.

Keywords: Szász operator, modulus of continuity, rate of convergence, multiple Appell polynomials.

1. Introduction

Szász [8] introduced the well-known operators in 1950 by

$$S_n(f;x) = e^{-nx}\sum_{k=0}^{\infty}\frac{(nx)^k}{k!}f\left(\frac{k}{n}\right) \tag{1.1}$$

where $x \in [0,\infty)$ and $f \in C[0,\infty)$. It may be mentioned that these operators are also the examples of positive approximation processes discovered by Korovkin [6]. Many authors deal with the generalizations of Szász operators.

Jakimovski and Leviatan [5] gave a generalization of Szász operators involving the Appell polynomials. The Appell polynomials $p_k(x)$ have the following generating relation

$$g(u)\, e^{ux} = \sum_{k=0}^{\infty} p_k(x)\, u^k \ , \tag{1.2}$$

where $g(u) = \sum\limits_{k=0}^{\infty} a_k u^k$ is an analytic function in the disc $|u| < R$, $(R > 1)$ and $g(1) \neq 0$. Under the assumptions $p_k(x) \geq 0$ for $x \in [0,\infty)$, they proposed the linear positive operators as

$$P_n(f;x)=\frac{e^{-nx}}{g(1)}\sum_{k=0}^{\infty}p_k(nx)\,f\left(\frac{k}{n}\right)\quad . \tag{1.3}$$

For the special case $g(u)=1$, from the generating functions (1.2) one can easily find $p_k(x)=\frac{x^k}{k!}$ and from this fact (1.3) reduces Szász operators (1.1). The detailed approximation properties of the operators (1.3) were given by Wood ([9],[10]) and Ciupa ([2]-[4]).

In this paper, we construct an operator by the help of multiple Appell polynomials. First of all, we need the following results from the paper of Lee [7].

Definition 1.1. *Multiple polynomial system(multiple PS for short) means a set of polynomials* $\{p_{k_1,k_2}(x)\}_{k_1,k_2=0}^{\infty}$ *with* $\deg(p_{k_1,k_2})=k_1+k_2$ *for* $k_1,k_2\geq 0$.

Definition 1.2. *A multiple PS* $\{p_{k_1,k_2}(x)\}_{k_1,k_2=0}^{\infty}$ *is called multiple Appell if it has a generating function of the form*

$$A(t_1,t_2)\,e^{x(t_1+t_2)}=\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\frac{p_{k_1,k_2}(x)}{k_1!k_2!}t_1^{k_1}t_2^{k_2} \tag{1.4}$$

where A *has a series expansion*

$$A(t_1,t_2)=\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\frac{a_{k_1,k_2}}{k_1!k_2!}t_1^{k_1}t_2^{k_2} \tag{1.5}$$

with $A(0,0)=a_{0,0}\neq 0$.

Theorem 1.3. *Let* $\{p_{k_1,k_2}(x)\}_{k_1,k_2=0}^{\infty}$ *be a multiple PS. Then the following statements are all equivalent.*

(a) $\{p_{k_1,k_2}(x)\}_{k_1,k_2=0}^{\infty}$ *is a set of multiple Appell polynomials.*

(b) There exists a sequence $\{a_{k_1,k_2}\}_{k_1,k_2=0}^{\infty}$ *with* $a_{0,0}\neq 0$ *such that*

$$p_{k_1,k_2}(x)=\sum_{r_1=0}^{k_1}\sum_{r_2=0}^{k_2}\binom{k_1}{r_1}\binom{k_2}{r_2}a_{k_1-r_1,k_2-r_2}x^{r_1+r_2}.$$

(c) For every $k_1+k_2\geq 1$, *we have*

$$p'_{k_1,k_2}(x)=k_1p_{k_1-1,k_2}(x)+k_2p_{k_1,k_2-1}(x).$$

In view of these results, let us restrict the multiple Appell polynomials satisfying

(i) $A(1,1)\neq 0$ and $\frac{a_{k_1,k_2}}{A(1,1)}\geq 0$ for $k_1,k_2\in\mathbb{N}$,

(ii) (1.4) and (1.5) converge for $|t_1|<R_1,\ |t_2|<R_2\ (R_1,R_2>1)$.

Under the above restrictions, we introduce the following positive linear operators for $x\in[0,\infty)$

$$K_n(f;x)=\frac{e^{-nx}}{A(1,1)}\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\frac{p_{k_1,k_2}\left(\frac{nx}{2}\right)}{k_1!k_2!}f\left(\frac{k_1+k_2}{n}\right) \tag{1.6}$$

whenever the right-hand side of (1.6) exists.

Remark 1.4. We have to note the following special cases.

Case 1. For $t_2 = 0$, the generating functions given by (1.4) reduce to the generating functions for the Appell polynomials given by (1.2). By the help of Appell polynomials, Jakimovski and Leviatan constructed the operators (1.3) and gave the approximation properties in [5].

Case 2. For $t_2 = 0$ and $A(t_1, 0) = 1$, from the generating functions given by (1.4) we easily find $p_k(x) = x^k$. From this fact, one can obtain Szász operators (1.1).

The outline of the paper is as follows. In the following section, we obtain the uniform convergence of the operators (1.6) by using the universal Korovkin-type property and the order of approximation by virtue of classical modulus of continuity. Also, simultaneous approximation of the operators (1.6) is derived. In section 3, Kantorovich type generalization of our operators and the approximation properties are given.

2. Approximation properties of K_n operators

In this section, we state our main theorem with the help of the universal Korovkin-type property and calculate the order of approximation by modulus of continuity. We note that throughout the paper we use the following abbreviations for the partial derivatives

$$\frac{\partial A}{\partial t_i} = A_{t_i} \quad \text{and} \quad \frac{\partial^2 A}{\partial t_i \partial t_j} = A_{t_i t_j} \quad i, j = 1, 2 \ .$$

Lemma 2.1. *The operators given by* (1.6) *satisfy the following equalities*

$$K_n(1; x) = 1 \tag{2.1}$$

$$K_n(s; x) = x + \frac{A_{t_1}(1,1) + A_{t_2}(1,1)}{nA(1,1)} \tag{2.2}$$

$$\begin{aligned} K_n\left(s^2; x\right) &= x^2 + \frac{x}{n}\left(1 + \frac{2\left(A_{t_1}(1,1) + A_{t_2}(1,1)\right)}{A(1,1)}\right) \\ &+ \frac{1}{n^2 A(1,1)}\left\{A_{t_1}(1,1) + A_{t_2}(1,1)\right. \\ &\left. + A_{t_1 t_1}(1,1) + 2A_{t_1 t_2}(1,1) + A_{t_2 t_2}(1,1)\right\} \end{aligned} \tag{2.3}$$

where $x \geq 0$.

Proof. For $f(s) = 1$, we get from (1.6)

$$K_n(f; x) = \frac{e^{-nx}}{A(1,1)} \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} \frac{p_{k_1,k_2}\left(\frac{nx}{2}\right)}{k_1! k_2!}.$$

By taking $t_1 = 1$, $t_2 = 1$ and replacing x by $\frac{nx}{2}$ in the generating function (1.4), one can easily get (2.1).

For $f(s)=s$, we obtain from (1.6)

$$K_n(s;x)=\frac{e^{-nx}}{nA(1,1)}\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\frac{(k_1+k_2)\,p_{k_1,k_2}\left(\frac{nx}{2}\right)}{k_1!k_2!}.$$

Taking the partial derivatives of the generating function (1.4) with respect to t_1 and t_2 and later by taking $t_1=1$, $t_2=1$ and replacing x by $\frac{nx}{2}$, one can get (2.2).

Taking into account the second order partial derivatives of the generating function (1.4) with respect to t_1 and t_2, one can get (2.3) by using similar technique. □

Let us define the class of E as follows

$$E:=\left\{f:x\in[0,\infty)\,,\ \frac{f(x)}{1+x^2}\text{ is convergent as }x\to\infty\right\}.$$

Theorem 2.2. *Let $f\in C[0,\infty)\cap E$. Then*

$$\lim_{n\to\infty}K_n(f;x)=f(x)\,,$$

the convergence being uniform in each compact subset of $[0,\infty)$.

Proof. According to (2.1)-(2.3), we have

$$\lim_{n\to\infty}K_n\left(s^i;x\right)=x^i\ ,\quad i=0,1,2\ .$$

Since the above convergences are verified uniformly in each compact subset of $[0,\infty)$, we obtain the desired result by applying the universal Korovkin-type property (vi) of Theorem 4.1.4 in [1]. □

Let us recall the following definition.

Definition 2.3. *Let $f\in\tilde{C}[0,\infty)$ and $\delta>0$. The modulus of continuity $\omega(f;\delta)$ of the function f is defined by*

$$\omega(f;\delta):=\sup_{\substack{x,y\in[0,\infty)\\|x-y|\le\delta}}|f(x)-f(y)|$$

where $\tilde{C}[0,\infty)$ is the space of uniformly continuous functions on $[0,\infty)$.

Next, for the order of approximation we express the following.

Theorem 2.4. *Let $f\in\tilde{C}[0,\infty)\cap E$. We have the following inequality for the operators* (1.6)

$$\begin{aligned}&|K_n(f;x)-f(x)|\\&\le\left\{1+\sqrt{\begin{array}{c}x+\frac{1}{nA(1,1)}\left\{A_{t_1}(1,1)+A_{t_2}(1,1)+A_{t_1t_1}(1,1)\right.\\ \left.+2A_{t_1t_2}(1,1)+A_{t_2t_2}(1,1)\right\}\end{array}}\right\}\\&\quad\times\omega\left(f;\frac{1}{\sqrt{n}}\right).\end{aligned}\tag{2.4}$$

Proof. According to (2.1) and the property of modulus of continuity, the left-hand side of (2.4) leads to

$$\begin{aligned}
&\left|K_n(f;x)-f(x)\right| \\
&\leq \frac{e^{-nx}}{A(1,1)}\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\frac{p_{k_1,k_2}\left(\frac{nx}{2}\right)}{k_1!k_2!}\left|f\left(\frac{k_1+k_2}{n}\right)-f(x)\right| \\
&\leq \left\{1+\frac{1}{\delta}\frac{e^{-nx}}{A(1,1)}\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\frac{p_{k_1,k_2}\left(\frac{nx}{2}\right)}{k_1!k_2!}\left|\frac{k_1+k_2}{n}-x\right|\right\}\omega(f;\delta)\ .
\end{aligned} \tag{2.5}$$

By applying the Cauchy-Schwarz inequality for each sums, (2.5) becomes

$$\begin{aligned}
&\left|K_n(f;x)-f(x)\right| \\
&\leq \left\{1+\frac{1}{\delta}\left(\frac{e^{-nx}}{A(1,1)}\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\frac{p_{k_1,k_2}\left(\frac{nx}{2}\right)}{k_1!k_2!}\right)^{1/2}\right. \\
&\quad\left.\times\left(\frac{e^{-nx}}{A(1,1)}\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\frac{p_{k_1,k_2}\left(\frac{nx}{2}\right)}{k_1!k_2!}\left(\frac{k_1+k_2}{n}-x\right)^2\right)^{1/2}\right\}\omega(f;\delta) \\
&= \left\{1+\frac{1}{\delta}\left(K_n(1;x)\right)^{1/2}\left(K_n\left((s-x)^2;x\right)\right)^{1/2}\right\}\omega(f;\delta)
\end{aligned} \tag{2.6}$$

In view of Lemma 2.1, (2.6) gives (2.4) by taking $\delta=\delta_n=\frac{1}{\sqrt{n}}$. □

In the end of this section, we will give the following theorem for the simultaneous approximation. Let $\tilde{C}^r[0,\infty)$ be the space of r-times differentiable function such that $f^{(r)}$ is uniformly continuous on $[0,\infty)$.

Theorem 2.5. *Let $f\in\tilde{C}^r[0,\infty)\cap E$. We have the following inequality for the derivatives of the operators* (1.6)

$$\begin{aligned}
\left|K_n^{(r)}(f;x)-f^{(r)}(x)\right| \leq& \left\{1+\sqrt{\begin{array}{c} x+\frac{1}{nA(1,1)}\left\{A_{t_1}(1,1)+A_{t_2}(1,1)\right. \\ \left.+A_{t_1t_1}(1,1)+2A_{t_1t_2}(1,1)+A_{t_2t_2}(1,1)\right\}\end{array}}\right\} \\
&\times\omega\left(f^{(r)};\frac{1}{\sqrt{n}}+\frac{r}{n}\right)+\omega\left(f^{(r)};\frac{r}{n}\right).
\end{aligned} \tag{2.7}$$

Proof. By virtue of the result (c) from Theorem 1.3, we deduce that

$$K_n^{(r)}(f;x)=n^r\frac{e^{-nx}}{A(1,1)}\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\frac{p_{k_1,k_2}\left(\frac{nx}{2}\right)}{k_1!k_2!}\Delta_{1/n}^r f\left(\frac{k_1+k_2}{n}\right) \tag{2.8}$$

where $\Delta_{1/n}^{r} f\left(\frac{k_1+k_2}{n}\right)$ is the difference of order r of f with the step $\frac{1}{n}$. Taking into account the relation between finite difference and divided difference, (2.8) leads to

$$\begin{aligned} K_n^{(r)}(f;x) &= r!\frac{e^{-nx}}{A(1,1)}\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\frac{p_{k_1,k_2}\left(\frac{nx}{2}\right)}{k_1!k_2!}\frac{\Delta_{1/n}^{r} f\left(\frac{k_1+k_2}{n}\right)}{r!\left(\frac{1}{n}\right)^{r}} \\ &= r!\frac{e^{-nx}}{A(1,1)}\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\frac{p_{k_1,k_2}\left(\frac{nx}{2}\right)}{k_1!k_2!} \\ &\quad\times\left[\frac{k_1+k_2}{n},\frac{k_1+k_2+1}{n},...,\frac{k_1+k_2+r}{n};f\right] \\ &= r!\frac{e^{-nx}}{A(1,1)}\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\frac{p_{k_1,k_2}\left(\frac{nx}{2}\right)}{k_1!k_2!}h\left(\frac{k_1+k_2}{n}\right)=r!K_n(h;x) \end{aligned}$$

where $h(t)=\left[t,t+\frac{1}{n},...,t+\frac{r}{n};f\right]$. By using the inequality (2.4), we have

$$\begin{aligned} &\left|K_n^{(r)}(f;x)-f^{(r)}(x)\right| \\ &\leq r!\left|K_n(h;x)-h(x)\right|+\left|r!h(x)-f^{(r)}(x)\right| \\ &\leq r!\left\{1+\sqrt{\begin{array}{c} x+\frac{1}{nA(1,1)}\left\{A_{t_1}(1,1)+A_{t_2}(1,1)\right. \\ \left.+A_{t_1t_1}(1,1)+2A_{t_1t_2}(1,1)+A_{t_2t_2}(1,1)\right\}\end{array}}\right\} \\ &\quad\times\omega\left(h;\frac{1}{\sqrt{n}}\right)+\left|r!h(x)-f^{(r)}(x)\right| \end{aligned} \tag{2.9}$$

From the property of the modulus of continuity, we get

$$\begin{aligned} &\left|h(t+\delta)-h(t)\right| \\ &= \left|\left[t+\delta,t+\delta+\frac{1}{n},...,t+\delta+\frac{r}{n};f\right]-\left[t,t+\frac{1}{n},...,t+\frac{r}{n};f\right]\right| \\ &= \frac{1}{r!}\left|f^{(r)}\left(t+\delta+\frac{r}{n}\theta_1\right)-f^{(r)}\left(t+\frac{r}{n}\theta_2\right)\right| \\ &\leq \frac{1}{r!}\omega\left(f^{(r)};\delta+\frac{r}{n}\left|\theta_1-\theta_2\right|\right) \\ &\leq \frac{1}{r!}\omega\left(f^{(r)};\delta+\frac{r}{n}\right) \end{aligned}$$

where θ_1, $\theta_2\in(0,1)$. For $\delta=\frac{1}{\sqrt{n}}$, we obtain

$$\omega\left(h;\frac{1}{\sqrt{n}}\right)\leq\frac{1}{r!}\omega\left(f^{(r)};\frac{1}{\sqrt{n}}+\frac{r}{n}\right)\quad. \tag{2.10}$$

Otherwise,

$$\begin{aligned} \left|r!h(x)-f^{(r)}(x)\right| &= \left|r!\left[x,x+\frac{1}{n},...,x+\frac{r}{n};f\right]-f^{(r)}(x)\right| \\ &= \left|f^{(r)}\left(x+\frac{r}{n}\theta_3\right)-f^{(r)}(x)\right| \\ &\leq \omega\left(f^{(r)};\frac{r}{n}\theta_3\right)\leq\omega\left(f^{(r)};\frac{r}{n}\right) \end{aligned} \tag{2.11}$$

where $\theta_3 \in (0,1)$. Combining (2.10) and (2.11) with (2.9), we reach the desired result. □

3. Kantorovich type generalization of K_n operators

In this section, we give a Kantorovich type generalization of the operators K_n (1.6) with same restrictions

$$K_n^*(f;x)=\frac{ne^{-nx}}{A(1,1)}\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\frac{p_{k_1,k_2}\left(\frac{nx}{2}\right)}{k_1!k_2!}\int_{\frac{k_1+k_2}{n}}^{\frac{k_1+k_2+1}{n}}f(s)\,ds\ . \tag{3.1}$$

Now, let us give the following lemma which is used in the sequel.

Lemma 3.1. *The operators given by* (3.1) *satisfy the following equalities*

$$K_n^*(1;x) = 1 \tag{3.2}$$

$$K_n^*(s;x) = x+\frac{A_{t_1}(1,1)+A_{t_2}(1,1)}{nA(1,1)}+\frac{1}{2n} \tag{3.3}$$

$$\begin{aligned} K_n^*\left(s^2;x\right) &= x^2+\frac{2x}{n}\left(1+\frac{A_{t_1}(1,1)+A_{t_2}(1,1)}{A(1,1)}\right)+\frac{1}{n^2A(1,1)} \\ &\times\left\{\begin{array}{c} 2\left(A_{t_1}(1,1)+A_{t_2}(1,1)+A_{t_1t_2}(1,1)\right) \\ +A_{t_1t_1}(1,1)+A_{t_2t_2}(1,1) \end{array}\right\}+\frac{1}{3n^2} \end{aligned} \tag{3.4}$$

Proof. Above equalities (3.2)-(3.4) follow from Lemma 2.1 immediately. □

Next, we derive the following two theorems for the uniform convergence and the order of approximation.

Theorem 3.2. *Let* $f \in C[0,\infty)\cap E$. *Then*

$$\lim_{n\to\infty}K_n^*(f;x)=f(x),$$

the convergence being uniform in each compact subset of $[0,\infty)$.

Proof. From (3.2)-(3.4), we get

$$\lim_{n\to\infty}K_n^*\left(s^i;x\right)=x^i\ ,\quad i=0,1,2\ .$$

The proof is completed by virtue of the above uniform convergences in each compact subset of $[0,\infty)$ and the universal Korovkin-type property (vi) of Theorem 4.1.4 in [1]. □

Theorem 3.3. *Let $f \in \tilde{C}[0,\infty) \cap E$. We have the following inequality for the operators* (3.1)

$$\begin{aligned} &\left|K_n^*(f;x) - f(x)\right| \\ &\leq \left\{1 + \sqrt{x + \frac{1}{nA(1,1)}\left\{\begin{array}{c} 2\left(A_{t_1}(1,1) + A_{t_2}(1,1) + A_{t_1t_2}(1,1)\right) \\ +A_{t_1t_1}(1,1) + A_{t_2t_2}(1,1) \end{array}\right\} + \frac{1}{3n}}\right\} \\ &\times\omega\left(f;\frac{1}{\sqrt{n}}\right). \end{aligned} \tag{3.5}$$

Proof. From (3.2) and the property of modulus of continuity, the left-hand side of (3.5) becomes

$$\begin{aligned} &\left|K_n^*(f;x) - f(x)\right| \\ &\leq \frac{ne^{-nx}}{A(1,1)}\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\frac{p_{k_1,k_2}\left(\frac{nx}{2}\right)}{k_1!k_2!}\int_{\frac{k_1+k_2}{n}}^{\frac{k_1+k_2+1}{n}}\left|f(s) - f(x)\right|ds \\ &\leq \left\{1 + \frac{1}{\delta}\frac{ne^{-nx}}{A(1,1)}\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\frac{p_{k_1,k_2}\left(\frac{nx}{2}\right)}{k_1!k_2!}\int_{\frac{k_1+k_2}{n}}^{\frac{k_1+k_2+1}{n}}\left|s - x\right|ds\right\}\omega(f;\delta). \end{aligned}$$

By using the Cauchy-Schwarz inequality for the integral, we have

$$\begin{aligned} &\left|K_n^*(f;x) - f(x)\right| \\ &\leq \left\{1 + \frac{1}{\delta}\frac{e^{-nx}}{A(1,1)}\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\frac{p_{k_1,k_2}\left(\frac{nx}{2}\right)}{k_1!k_2!}\left(n\int_{\frac{k_1+k_2}{n}}^{\frac{k_1+k_2+1}{n}}(s-x)^2ds\right)^{1/2}\right\}\omega(f;\delta). \end{aligned} \tag{3.6}$$

By applying the Cauchy-Schwarz inequality for each sums, (3.6) leads to

$$\begin{aligned} &\left|K_n^*(f;x) - f(x)\right| \\ &\leq \left\{1 + \frac{1}{\delta}\left(\frac{e^{-nx}}{A(1,1)}\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\frac{p_{k_1,k_2}\left(\frac{nx}{2}\right)}{k_1!k_2!}\right)^{1/2}\right. \\ &\quad\left.\times\left(\frac{ne^{-nx}}{A(1,1)}\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\frac{p_{k_1,k_2}\left(\frac{nx}{2}\right)}{k_1!k_2!}\int_{\frac{k_1+k_2}{n}}^{\frac{k_1+k_2+1}{n}}(s-x)^2ds\right)^{1/2}\right\}\omega(f;\delta) \\ &= \left\{1 + \frac{1}{\delta}\left(K_n(1;x)\right)^{1/2}\left(K_n^*\left((s-x)^2;x\right)\right)^{1/2}\right\}\omega(f;\delta) \end{aligned} \tag{3.7}$$

Taking into account Lemma 2.1 and Lemma 3.1 in (3.7), we get the inequality (3.5) for $\delta = \delta_n = \frac{1}{\sqrt{n}}$. □

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Serhan Varma
e-mail: svarma@science.ankara.edu.tr
smartsero@hotmail.com
Ankara University Faculty of Science
Department of Mathematics
Tandoğan TR-06100
Ankara, Turkey

Stud. Univ. Babeş-Bolyai Math. 58(2013), No. 3, 371–380

On approximation of functions of one variable in spaces with a polynomial weight

Zbigniew Walczak

Abstract. In this paper we give some approximation theorems for a general class of discrete type operators. We discuss the linear and nonlinear cases.

Mathematics Subject Classification (2010): 41A36.

Keywords: Discrete operators, direct approximation theorems, polynomial weighted spaces.

1. Introduction

The creation of the basics of approximation theory can be attributed to the Russian mathematician Chebyshev, who formulated and examined the existence of polynomials furnishing best approximations from a particular function over 150 years ago. One of the first problems in the area was to find the polynomial that best approximated the function $f(x) = x^n$ in the interval $[-1, 1]$ in the class of algebraic polynomials having the degree $n - 1$. Solving that problem, Chebyshev defined polynomials $T_n(x) = cos(narccosx)$, which are now called Chebyshev polynomials and which have been widely used in uniform function approximation. The origin of function approximation theory are also connected with K. Weierstrass, S. N. Bernstein, L. Fejer and D. Jackson. It was at the turn of the 20th century that basic problems of continuous function approximation were formulated. The authors proved that, among other things, each continuous function on the closed and bounded interval could be approximated by an algebraic (trigonometric) polynomial with any predetermined order of accuracy. Another important issue was to efficiently obtain operators approximating a particular function with a predetermined accuracy. In addition, research was centred around estimating the rate of convergence of a series of polynomials to a particular function approximated by the polynomials.

Research on function approximation was justified by it being used in other mathematical fields (especially mathematical analysis, functional analysis and the theory of differential equations) and the progress in that field was influenced by other branches of science. I would like to mention that the Fourier series is used in physics and technology and examining many limit problems comes down to studying approximation issues. The merging of function approximation and other branches of science is now particularly visible thanks to numerical analysis and problems, e.g. Bernstein polynomials are widely used in e.g. computer graphics.

The development and directions of research on function approximation by linear operators have been defined in numerous publications and dissertations.

Our research has aimed at generalizing the aforementioned results concerning approximation of functions by positive linear operators. Such research usually needs to be carried out using more subtle proving methods, and results obtained in this way make it possible to come up with additional conclusions.

In the sections more important definitions and theorems are designated by consecutive figures. Definitions and certain properties of the polynomial weighted space and some other designations are denoted as in M. Becker [1].

Similarly to [1], let $p \in \mathbb{N}_0 := \{0, 1, 2, \dots\}$, and let

$$w_0(x) := 1, \quad w_p(x) := (1 + x^p)^{-1}, \qquad \text{if} \qquad p \geq 1. \tag{1.1}$$

Denote by C_p, $p \in \mathbb{N}_0$, the set of all real-valued functions f, continuous on $\mathbb{R}_0 := [0, \infty)$ and such that $w_p f$ is uniformly continuous and bounded on $\mathbb{R}_0$. The norm on C_p is defined by the formula

$$\|f\|_p \equiv \|f(\cdot)\|_p := \sup_{x \in \mathbb{R}_0} w_p(x) \, |f(x)|. \tag{1.2}$$

In the paper [12] it was constructed for any real function f on the interval $\mathbb{R}_0$ a sequence of positive linear operators S_n defined by

$$S_n(f; x) = \sum_{k=0}^{\infty} a_{nk}(x; q) f\left(\frac{k+q}{n}\right), \qquad n, q \in \mathbb{N} := \{1, 2, \cdots\}, \tag{1.3}$$

where $a_{nk}(x;q) := \frac{(nx)^k}{g(nx;q)(k+q)!}$ and $g(0;q) = \frac{1}{q!}$, $g(t;q) = \frac{1}{t^q}\left(e^t - \sum_{k=0}^{q-1} \frac{t^j}{j!}\right)$. These operators possess many remarkable properties. We present a few of them. It is known [12] that for $f \in C_p$, $p \in \mathbb{N}_0$

$$\lim_{n \to \infty} S_n(f; x) = f(x), \tag{1.4}$$

uniformly on every interval $[x_1, x_2]$, $x_2 > x_1 \geq 0$. In [12] it was proved that

$$\lim_{n \to \infty} n(S_n(f; x) - f(x)) = \frac{x}{2} f''(x) \tag{1.5}$$

for all $f \in C_p^2$.

The operators (1.3) are related to the well-known Szász-Mirakyan operators

$$B_n(f; x) := e^{-nx} \sum_{k=0}^{\infty} \frac{(nx)^k}{k!} f\left(\frac{k}{n}\right),$$

$x \in \mathbb{R}_0$, $n \in \mathbb{N}$.

In many papers various modification of B_n were introduced and examined. They have been studied intensively. We refer the reader to A. Ciupa [2]- [4], L. Rempulska, A. Thiel [9]- [10]. Many publications on the topic allude to the research of V.Gupta [5] and V. Gupta, P. Maheshwari [6], V. Gupta, R. Yadav [7] and V. Gupta, D. K. Verma [8]. Their results improve other related results in the literature.

The paper pays special attention to defining various classes of operators and examining their certain approximation properties. Because of properties of examined operators, classical (and widely used) methods of proving approximation theorems were employed, in which traditional mathematical problems were subject to subtle and sometimes difficult analytical techniques.

We shall use the modulus of continuity of $f \in C_p$,

$$\omega_1(f; C_p; t) := \sup_{0 \le h \le t} \|\Delta_h f(\cdot)\|_p, \qquad t \ge 0,$$

and the modulus of smoothness of $f \in C_p$

$$\omega_2(f; C_p; t) := \sup_{0 \le h \le t} \|\Delta_h^2 f(\cdot)\|_p, \qquad t \ge 0,$$

where

$$\Delta_h f(x) := f(x+h) - f(x), \ \Delta_h^2 f(x) := f(x) - 2f(x+h) + f(x+2h).$$

In this paper we shall denote by $M_k(\alpha, \beta), k = 1, 2, ...,$ suitable positive constants depending only on indicated parameters α, β.

Similarly as in the paper [14] we introduce the following class of operators in C_p.

Definition 1.1. *We define the class of operators S_n by the formula*

$$S_n(f; F_{n,r}; x) := \sum_{k=0}^{\infty} a_{nk}(x; q) F_{n,r}\left(f\left(\frac{k+q}{n}\right)\right), \ f \in C_p, \ p \in \mathbb{N}_0, \ q \in \mathbb{N}, \tag{1.6}$$

where $(F_{n,r})_1^\infty$, is a sequence of continuous functions on $\mathbb{R} := (-\infty, +\infty)$ such that $\sup_{x \in \mathbb{R}} w_r(|x|) \, |F_{n,r}(x) - x| \le \frac{M_1(r)}{b_n}$, $(b_n)_1^\infty$ is an increasing sequence of positive numbers with the property $\lim_{n\to\infty} b_n = \infty$.

2. Preliminary results

In this section we shall give some results, which we shall apply to the proofs of the main theorems.

First we give some properties of the operators S_n.

Lemma 2.1. ([12]) *Let $p \in \mathbb{N}_0$ and $q \in \mathbb{N}$ be fixed numbers. Then there exists $M_2(p, q)$ such that*

$$\|S_n(1/w_p)\|_p \le M_2(p, q). \tag{2.1}$$

Moreover for every $f \in C_p$ we have

$$\|S_n(f)\|_p \le M_2(p, q)\|f\|_p. \tag{2.2}$$

The formula (2.2) shows that $S_n(f)$ is a positive linear operators on C_p.

Now we shall give approximation theorems for S_n.

Theorem 2.2. ([12]) *Let $p \in \mathbb{N}_0$ be a fixed number. Then there exists $M_3(p,q)$ such that for every $f \in C_p$ and $n \in \mathbb{N}$ we have*

$$w_p(x)|S_n(f;x) - f(x)| \le M_3(p,q)\omega_1\left(f; C_p; \sqrt{\frac{x+1}{n}}\right), \ x \in \mathbb{R}_0. \qquad (2.3)$$

Now we shall give some properties of the operators (1.6).

Lemma 2.3. *Let $(F_{n,r})_1^\infty$, $n, r \in \mathbb{N}$, be a sequence of continuous functions on $\mathbb{R}$ such that $\sup_{x\in\mathbb{R}} w_r(|x|)\,|F_{n,r}(x) - x| \le \frac{M_1(r)}{b_n}$, where $(b_n)_1^\infty$ is an increasing sequence of positive numbers and $\lim_{n\to\infty} b_n = \infty$. For every $p \in \mathbb{N}_0$ we have*

$$\|S_n(f;F_{n,r})\|_{pr} \le M_4(p,q,r,b_1), \ f \in C_p.$$

The above inequality shows that $S_n(f;F_{n,r})$ is well-defined on the space C_{pr}.

Proof. For $f \in C_p$ and $p, q, r \in \mathbb{N}$ we have

$$w_{pr}(x)|S_n(f;F_{n,r};x)| \le w_{pr}(x)\sum_{k=0}^{\infty} a_{nk}(x;q)\left|F_{n,r}\left(f\left(\frac{k+q}{n}\right)\right)\right|.$$

$$\le w_{pr}(x)\sum_{k=0}^{\infty} a_{nk}(x;q)$$

$$\times\left\{\left|F_{n,r}\left(f\left(\frac{k+q}{n}\right)\right) - f\left(\frac{k+q}{n}\right)\right| + \left|f\left(\frac{k+q}{n}\right)\right|\right\}$$

From (1.3) by our assumption we get

$$w_{pr}(x)|S_n(f;F_{n,r};x)| \le w_{pr}(x)\sum_{k=0}^{\infty} a_{nk}(x;q)$$

$$\times\left\{\frac{M_1(r)}{b_n}\left(1 + \left|f\left(\frac{k+q}{n}\right)\right|^r\right) + \left|f\left(\frac{k+q}{n}\right)\right|\right\}$$

$$\le M_5(r,q,b_1)w_{pr}(x)\left\{1 + S_n\left(|f(t)|^r; x\right) + S_n\left(|f(t)|; x\right)\right\}$$

Observe that

$$w_{pr}(x)S_n\left(|f(t)|^r; x\right) \le M_6(p,q,r)\|f\|_{pr} w_{pr}(x)S_n\left(1/w_{pr}(t); x\right) \le M_7(p,q,r). \quad (2.4)$$

From this we immediately obtain

$$\|S_n(f;F_{n,r})\|_{pr} \le M_8(p,q,r,b_1), \ f \in C_p, \ p \in \mathbb{N}.$$

The proof is similar for $p = 0$. Thus the proof is completed. □

Theorem 2.4. *If the assumptions of Lemma 2.3 are satisfied then there exists $M_9(p,q,r)$ such that for every $f \in C_p$ and $p \in \mathbb{N}_0$ we have*

$$w_{pr}(x)|S_n(f;F_{n,r};x) - f(x)| \le M_9(p,q,r)\left\{b_n^{-1} + \omega_1\left(f; C_p; \sqrt{\frac{x+1}{n}}\right)\right\}. \quad (2.5)$$

Proof. By (1.6) and (1.3) we get

$$w_{pr}(x)(S_n(f;F_{n,r};x)-f(x))=w_{pr}(x)\sum_{k=0}^{\infty}a_{nk}(x;q)$$
$$\times\left\{\left(F_{n,r}\left(f\left(\frac{k+q}{n}\right)\right)-f\left(\frac{k+q}{n}\right)\right)+\left(f\left(\frac{k+q}{n}\right)-f\left(x\right)\right)\right\}$$
$$=w_{pr}(x)\sum_{k=0}^{\infty}a_{nk}(x;q)\left(F_{n,r}\left(f\left(\frac{k+q}{n}\right)\right)-f\left(\frac{k+q}{n}\right)\right)$$
$$+w_{pr}(x)(S_n(f(t);x)-f(x)).$$

From Theorem 2.2 we have

$$w_{pr}(x)|S_n(f(t);x)-f(x)|\le M_3(p,q,r)\omega_1\left(f;C_p;\sqrt{\frac{x+1}{n}}\right).$$

By our assumptions we get

$$w_{pr}(x)\sum_{k=0}^{\infty}a_{nk}(x;q)\left|F_{n,r}\left(f\left(\frac{k+q}{n}\right)\right)-f\left(\frac{k+q}{n}\right)\right|$$
$$\le\frac{M_1(r)}{b_n}w_{pr}(x)\sum_{k=0}^{\infty}a_{nk}(x;q)\left(1+\left|f\left(\frac{k+q}{n}\right)\right|^r\right)$$
$$=\frac{M_1(r)}{b_n}\left(1+w_{pr}(x)S_n\left(\left|f\left(t\right)\right|^r;x\right)\right).$$

Applying (2.4) we obtain (2.5). □

3. Main results

In this section we shall use the same method to obtain a general class of operators.

Similarly as in the paper [14] let Ω be the set of all infinite matrices $A=[a_{nk}(x)]_{n\in\mathbb{N},k\in\mathbb{N}_0}$, of functions $a_{nk}\in C_0$ having the following properties:

(a) $a_{nk}(x)\ge 0$ for $x\in\mathbb{R}_0$, $n\in\mathbb{N}$, $k\in\mathbb{N}_0$,

(b) $\sum_{k=0}^{\infty}a_{nk}(x)=1$ for $x\in\mathbb{R}_0$, $n\in\mathbb{N}$,

(c) $\sum_{k=0}^{\infty}k^p a_{nk}(x)$, $x\in\mathbb{R}_0$, $n\in\mathbb{N}$, $p\in\mathbb{N}$, is uniformly convergent on $\mathbb{R}_0$ and its sum is a function belonging to the space C_p

(d) for given $p\in\mathbb{N}$ there exists a positive constant $M_{10}(p,A)$ dependent on p and A such that the function

$$T_{n,p}(A;x):=\sum_{k=0}^{\infty}a_{nk}(x)\left(\frac{k}{n}-x\right)^p,\qquad x\in\mathbb{R}_0,\quad n\in\mathbb{N},\tag{3.1}$$

satisfies the conditions

$$\|T_{n,2p}(A;\cdot)\|_{2p}\le M_{10}(p,A)n^{-p},\qquad n\in\mathbb{N},\tag{3.2}$$

and

$$T_{n,1}(A;x)=0.\tag{3.3}$$

We introduce the following class of operators in C_p.

Definition 3.1. *Let $A \in \Omega$ and let $r \in \mathbb{N}$ be a fixed number. We define the class of operators S_n and $S_{n,p}$ by the formulas*

$$S_n(f;F_{n,r};A;x) := \sum_{k=0}^{\infty} a_{nk}(x) F_{n,r}\left(f\left(\frac{k}{n}\right)\right), \ f \in C_p, \ p \in \mathbb{N}_0, \tag{3.4}$$

$x \in \mathbb{R}_0$, where $(F_{n,r})_1^\infty$, is a sequence of continuous functions on $\mathbb{R} := (-\infty, +\infty)$ such that $\sup_{x\in\mathbb{R}} w_r(|x|)\,|F_{n,r}(x) - x| \le \frac{M_1(r)}{b_n}$, $(b_n)_1^\infty$ is an increasing sequence of positive numbers with the property $\lim_{n\to\infty} b_n = \infty$.

In this section we shall give some results, which we shall apply to the proofs of the main theorems.

Definition 3.2. *Let the matrix $A \in \Omega$ and let C_p for a given space with $p \in \mathbb{N}_0$. For $f \in C_p$ we define the operators*

$$K_n(f;A;x) := \sum_{k=0}^{\infty} a_{nk}(x) f\left(\frac{k}{n}\right), \qquad n \in \mathbb{N}, \quad x \in \mathbb{R}_0. \tag{3.5}$$

First we shall give some properties of the operators K_n.

Lemma 3.3. *Let $A \in \Omega$ and $p \in \mathbb{N}_0$. Then there exists $M_{11}(p,A)$ such that*

$$\|K_n(1/w_p;A)\|_p \le M_{11}(p,A). \tag{3.6}$$

Moreover for every $f \in C_p$ we have

$$\|K_n(f;A)\|_p \le M_{11}(p,A)\|f\|_p. \tag{3.7}$$

*The formulas (*3.6*) and (*3.7*) show that $K_n(f;A)$ is a positive linear operators on C_p.*

Proof. If $p = 0$, then by (1.1), (1.2) and the property (b) we have $\|K_n(1/w_0)\|_0 = 1$.

Let $p \in N$. By (3.5), 1.1), (3.1), the property (b) and the Hölder inequality we get

$$\begin{aligned} w_p(x)K_n(1/w_p(t);A;x) &= w_p(x)(1 + K_n(t^p;A;x)) \\ &= w_p(x)(1 + K_n(2^{p-1}(|t-x|^p + x^p);A;x)) \\ &= w_p(x)(1 + 2^{p-1}x^p + 2^{p-1}K_n(|t-x|^p;A;x)) \\ &\le M_{12}(p) + 2^{p-1}(w_p^2(x)T_{n,2p}(A;x))^{1/2} \\ &\le M_{12}(p) + 2^{p-1}(w_{2p}(x)T_{n,2p}(A;x))^{1/2} \end{aligned}$$

From this and by (3.2) we can write

$$\|K_n(1/w_p;A)\|_p \le M_{12}(p)(1 + \|T_{n,2p}(A)\|_{2p}^{1/2}) \le M_{11}(p,A).$$

The formulas (3.5) and (1.1) yield

$$\|K_n(f;A)\|_p \le \|f\|_p \|K_n(1/w_p;A)\|_p$$

for $f \in C_p$. Applying (3.6) we obtain (3.7). □

Now we shall give approximation theorems for K_n.

Theorem 3.4. *Let $p \in \mathbb{N}_0$ be a fixed number. Then there exists a positive constant $M_{13}(p, A)$ such that for every $f \in C_p^2$ we have*

$$w_p(x)|K_n(f; A; x) - f(x)| \leq M_{13}(p, A)\frac{\|f''\|_p(1+x)^2}{n}, \qquad n \in \mathbb{N},\ x \in \mathbb{R}_0. \tag{3.8}$$

Proof. For a fixed $x \in \mathbb{R}_0$ and $f \in C_p^2$ we have

$$f(t) = f(x) + f'(x)(t-x) + \int_x^t \int_x^s f''(u)duds, \quad t \in \mathbb{R}_0,$$

which yields

$$f(t) = f(x) + f'(x)(t-x) + \int_x^t (t-u)f''(u)du, \quad t \in \mathbb{R}_0.$$

From this and by (3.5) we deduce that

$$K_n(f(t); A; x) = f(x) + f'(x)K_n(t-x; A; x) + K_n\left(\int_x^t (t-u)f''(u)du; A; x\right) \tag{3.9}$$

for $n \in \mathbb{N}$. By (1.1) and (1.2) we can write

$$\left|\int_x^t (t-u)f''(u)du\right| \leq \|f''\|_p\left(\frac{1}{w_p(t)} + \frac{1}{w_p(x)}\right)(t-x)^2.$$

Applying the above inequality, the Hölder inequality (1.1), (3.1), (3.3) and (3.5), we derive from (3.9)

$$\begin{aligned} &w_p(x)\,|K_n(f; A; x) - f(x)| \\ &\leq \|f''\|_p\left\{w_p(x)K_n\left(\frac{(t-x)^2}{w_p(t)}; A; x\right) + T_{n,2}(A; x)\right\} \leq \\ &\leq M_{14}(p, A)\|f''\|_p(T_{n,4}(A; x))^{1/2}\{(w_p^2(x)K_n(1/w_p^2(t); A; x))^{1/2} + 1\} \\ &\leq M_{15}(p, A)\|f''\|_p(T_{n,4}(A; x))^{1/2}\{(w_{2p}(x)K_n(1/w_{2p}(t); A; x))^{1/2} + 1\} \end{aligned}$$

for $n \in \mathbb{N}$. Using (1.1), (1.2), (3.2) and (3.6), we obtain the desired estimate (3.8). □

Theorem 3.5. *Let $p \in \mathbb{N}_0$ be a fixed number. Then there exists $M_{16}(p, A)$ such that for every $f \in C_p$ and $n \in \mathbb{N}$ we have*

$$w_p(x)|K_n(f; A; x) - f(x)| \leq M_{16}(p, A)\omega_2\left(f; C_p; \frac{x+1}{n^{1/2}}\right), \ x \in \mathbb{R}_0. \tag{3.10}$$

Proof. Let $x \in \mathbb{R}_0$. Similarly as in [1] we apply the Stieklov function of $f \in C_p$

$$f_h(x) := \frac{4}{h^2}\int_0^{\frac{h}{2}}\int_0^{\frac{h}{2}} [f(x+s+t) - f(x+2(s+t))]dsdt \tag{3.11}$$

for $x \in \mathbb{R}_0$, $h > 0$. From (3.11) we get

$$f_h'(x) = \frac{1}{h^2}\int_0^{\frac{h}{2}} [8\Delta_{h/2}f(x+s) - 2\Delta_h f(x+2s)]ds,$$

$$f_h''(x) = \frac{1}{h^2}\left[8\Delta_{h/2}^2 f(x) - \Delta_h^2 f(x)\right].$$

Consequently

$$\|f_h - f\|_p \le \omega_2\left(f, C_p; h,\right), \tag{3.12}$$

$$\|f_h''\|_p \le 9h^{-2}\omega_2\left(f, C_p; h\right), \tag{3.13}$$

for $h > 0$. We see that $f_h \in C_p^2$ if $f \in C_p$. Hence, for $x \in \mathbb{R}_0$ and $n \in \mathbb{N}$, we can write

$$w_p(x)\left|K_n(f;A;x) - f(x)\right| \le w_p(x)\left\{\left|K_n\left(f - f_h;A;x\right)\right|\right.$$
$$\left. + \left|K_n\left(f_h;A;x\right) - f_h(x)\right| + \left|f_h(x) - f(x)\right|\right\} := Z_1 + Z_2 + Z_3.$$

By (3.7) and (3.12) we have

$$Z_1 \le M_{17}(p;A)\left\|f - f_h\right\|_p \le M_{17}(p;A)\omega_2\left(f, C_p; h\right), \quad Z_3 \le \omega_2\left(f, C_p; h\right).$$

Applying Theorem 3.4 and (3.13), we get

$$Z_2 \le M_{18}(p,A)\frac{\|f_h''\|_p(1+x)^2}{n} \le$$
$$\le M_{18}(p,A)\frac{9(1+x)^2}{h^2 n}\omega_2\left(f, C_p; h\right).$$

Combining these and setting $h = \frac{1+x}{n^{1/2}}$, for fixed $n \in \mathbb{N}$, we obtain the inequality (3.10). □

Now we shall give some properties of the operators (3.4).

Lemma 3.6. *Let $(F_{n,r})_1^\infty$, $n, r \in \mathbb{N}$, be a sequence of continuous functions on $\mathbb{R}$ such that $\sup_{x\in\mathbb{R}} w_r(|x|)\left|F_{n,r}(x) - x\right| \le \frac{M_{19}(r)}{b_n}$, where $(b_n)_1^\infty$ is an increasing sequence of positive numbers with the property $\lim_{n\to\infty} b_n = \infty$. For every $A \in \Omega$ and $p \in \mathbb{N}_0$ we have*

$$\|S_n(f;F_{n,r};A)\|_{pr} \le M_{20}(p,r,A,b_1), \;\; f \in C_p.$$

The above inequality show that $S_n(f;F_{n,r};A)$ is well-defined on the space C_{pr}.

Proof. For $f \in C_p$ and $p, r \in \mathbb{N}$ we have

$$w_{pr}(x)|S_n(f;F_{n,r};A;x)| \le w_{pr}(x)\sum_{k=0}^{\infty} a_{nk}(x)\left|F_{n,r}\left(f\left(\frac{k}{n}\right)\right)\right|.$$
$$\le w_{pr}(x)\sum_{k=0}^{\infty} a_{nk}(x)\left\{\left|F_{n,r}\left(f\left(\frac{k}{n}\right)\right) - f\left(\frac{k}{n}\right)\right| + \left|f\left(\frac{k}{n}\right)\right|\right\}$$

From (3.5) by our assumption we get

$$w_{pr}(x)|S_n(f;F_{n,r};A;x)|$$
$$\le w_{pr}(x)\sum_{k=0}^{\infty} a_{nk}(x)\left\{\frac{M_1(r)}{b_n}\left(1 + \left|f\left(\frac{k}{n}\right)\right|^r\right) + \left|f\left(\frac{k}{n}\right)\right|\right\}$$
$$\le M_{21}(r,A,b_1)w_{pr}(x)\left\{1 + K_n\left(\left|f\left(t\right)\right|^r;A;x\right) + K_n\left(\left|f\left(t\right)\right|;A;x\right)\right\}$$

Observe that

$$w_{pr}(x)K_n\left(\left|f\left(t\right)\right|^r;A;x\right)$$
$$\le M_{21}(p,r,A)\|f\|_{pr}w_{pr}(x)K_n\left(1/w_{pr}(t);A;x\right) \le M_{22}(p,r,A) \tag{3.14}$$

From this we immediately obtain

$$\|S_n(f;F_{n,r};A)\|_{pr} \leq M_{23}(p,r,A,b_1),\ f \in C_p,\ p \in \mathbb{N}.$$

The proof is similar for $p = 0$. Thus the proof is completed. □

Theorem 3.7. *If assumptions of Lemma 3.6 are satisfied then there exists* $M_{24}(p,r,A)$ *such that for every* $f \in C_p$ *and* $p \in \mathbb{N}_0$ *we have*

$$w_{pr}(x)|S_n(f;F_{n,r};A;x) - f(x)| \leq M_{24}(p,r,A)\left\{b_n^{-1} + \omega_2\left(f;C_p;\frac{x+1}{n^{1/2}}\right)\right\}. \quad (3.15)$$

Proof. By (3.4) and (3.5) we get

$$\begin{aligned} &w_{pr}(x)(S_n(f;F_{n,r};A;x) - f(x)) \\ &= w_{pr}(x)\sum_{k=0}^{\infty} a_{nk}(x)\left\{\left(F_{n,r}\left(f\left(\frac{k}{n}\right)\right) - f\left(\frac{k}{n}\right)\right) + \left(f\left(\frac{k}{n}\right) - f(x)\right)\right\} \\ &= w_{pr}(x)\sum_{k=0}^{\infty} a_{nk}(x)\left(F_{n,r}\left(f\left(\frac{k}{n}\right)\right) - f\left(\frac{k}{n}\right)\right) \\ &\quad + w_{pr}(x)(K_n(f(t);A;x) - f(x)). \end{aligned}$$

From Theorem 3.5 we have

$$w_{pr}(x)|K_n(f(t);A;x) - f(x)| \leq M_{25}(p,r,A)\omega_2\left(f;C_p;\frac{x+1}{n^{1/2}}\right).$$

By our assumptions we get

$$\begin{aligned} &w_{pr}(x)\sum_{k=0}^{\infty} a_{nk}(x)\left|F_{n,r}\left(f\left(\frac{k}{n}\right)\right) - f\left(\frac{k}{n}\right)\right| \\ &\leq \frac{M_1(r)}{b_n} w_{pr}(x)\sum_{k=0}^{\infty} a_{nk}(x)\left(1 + \left|f\left(\frac{k}{n}\right)\right|^r\right) \\ &= \frac{M_1(r)}{b_n}\left(1 + w_{pr}(x)K_n\left(|f(t)|^r;A;x\right)\right). \end{aligned}$$

Applying (3.14) we obtain (3.15). □

Now we shall give one example of operators of the $S_n(f;F_{n,r};A)$ type. The Baskakov operators

$$V_n(f;x) := \sum_{k=0}^{\infty}\binom{n-1+k}{k}x^k(1+x)^{-n-k}f\left(\frac{k}{n}\right), \qquad x \in \mathbb{R}_0,\ n \in \mathbb{N},$$

for $f \in C_p$, are generated by the matrix $A^* = [a_{nk}^*(x)]_{n\in\mathbb{N},\ k\in\mathbb{N}_0}$ with

$$a_{nk}^*(x) := \binom{n-1+k}{k}x^k(1+x)^{-n-k}, \qquad x \in \mathbb{R}_0,$$

i.e. $V_n(f;x) = K_n(f;A^*;x)$. If $F_{n,r}(x) = x$ for $n \in \mathbb{N}$ and $x \in \mathbb{R}$, then the operators $S_n(f;F_{n,r};A^*)$ and $V_n(f)$ are identical.

It is worth remarking that the introduced definitions also cover the case of nonlinear operators. To the best of the author's knowledge, there are not many publications on this topic. Another benefit from the definitions that we have proposed is the ability to use the research method to modify other positive linear operators known in literature. We would like to stress that the approximation theorems found in this paper covered results presented in many other papers.

Acknowledgment. The author is extremely thankful to the referee for making valuable suggestions, leading to the better presentation of the paper.

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Zbigniew Walczak
Poznań University of Technology
Institute of Mathematics
Piotrowo 3A, 60-965 Poznań, Poland
e-mail: `Zbigniew.Walczak@put.poznan.pl`

Stud. Univ. Babeş-Bolyai Math. 58(2013), No. 3, 381–392

An existence result for nonlinear hemivariational-like inequality systems

Andrea Éva Molnár and Orsolya Vas

Abstract. In this paper, we introduce and study a system of nonlinear hemivariational-like inequality system in reflexive Banach spaces. The novelty of our approach is that we prove the existence of at least one solution for our system without imposing any monotonicity assumptions or using nonsmooth critical point theory. The main tool is a fixed theorem due to T.C. Lin [Bull. Aust. Math. Soc. **34** (1986), 107-117.] Two applications are also given: the first one prove the existence of Nash generalized derivative points, while the second one the existence of at least one solution of a Schrödinger-type problem.

Mathematics Subject Classification (2010): 47J20, 54C60, 47H10.

Keywords: Fixed point theory, hemivariational inequality system, locally Lipschitz functional, set-valued operator.

1. Introduction

Hemivariational inequalities were introduced by Panagiotopoulos in the early eighties of the 20^{th} century (see Panagiotopoulos [20],[21]) as a generalization of variational inequalities. Actually, they are more than simple generalizations, since a hemivariational inequality is not equivalent to a minimum problem. The hemivariational inequality problem simplifies to a variational inequality problem, if we restrict the involved functionals to be convex (see for instance: Fichera [5]; Lions and Stampacchia [11]; Hartman and Stampacchia [6]).

A useful tool to study the hemivariational inequalities is the critical point theory for locally Lipschitz functions, which study was started by K. Chang in his pioneering work [3]. These types of inequalities have created a new field in nonsmooth analysis, because they are based on the concept of the Clarke's generalized gradient of locally Lipschitz functions. Since its appearance, the theory of hemivariational inequalities has given important results both in pure and applied mathematics and it is very useful to understand several problems of mechanics and engineering for nonconvex, nonsmooth energy functionals. For a better understanding of the theory of hemivariational inequalities and other types of inequalities of hemivariational type (variational-hemivariational, quasi-hemivariational, etc.) as well as for various applications to Nonsmooth Mechanics, Economics, Engineering and Geometry, the reader is reffered to the monographs of Naniewicz and Panagiotopoulos [17], Motreanu and Panagiotopoulos [15], Motreanu and Rădulescu [16], Kristály, Rădulescu and Varga [9] and their references.

In this paper we introduce a new type of inequality systems, which we call *nonlinear hemivariational–like inequality system.* In the aforementioned works, the solvability of the hemivariational inequality systems was studied via critical point theory for locally Lipschitz functions. The main goal of this paper is to study inequality problems in a general and unified framework (as nonlinear hemivariational-like inequalities can be reduced to variational-like inequalities of standard hemivariational inequalities) and establish an existence result without employing the nonsmooth critical point theory or imposing any monotonicity assumptions.

Concerning the applicability of our abstract result, in the last section of this paper, we present two possible applications: the first is related to the existence of Nash generalized derivative points, while the second deals with a Schrödinger-type problem. The notion of Nash generalized derivative point can be found in the paper of Kristály [8] who studied the existence of such points in the case of Riemannian manifolds. One year later, Repovš and Varga generalized this notion for systems of hemivariational inequalities (see Varga, Repovš [22]). We point out the fact that, in paper [22], the authors established two different proofs for the existence result, in the first proof using Ky Fan's version of Knaster-Kuratowsky-Mazurkiewicz fixed point theorem, while in the second one using Tarafdar's fixed point theorem. Our approach is slightly different, as we use a fixed point theorem due to Lin to prove our main result.

The paper is structured as follows. In Section 2, we recall some definitions and properties of locally Lipschitz functions and generalized gradients. Also we present Lin's fixed point theorem for set-valued mappings which will play an essential role in the proof of the main result. In Section 3, we describe the abstract framework in which we work, we formulate the system of nonlinear hemivariational-like inequalities and prove the main result. The last section is dedicated to some applications as we already mentioned above.

2. Preliminaries

Let $E_1, E_2, \dots E_n$ be Banach spaces for $i \in \{1, \dots, n\}$. Throughout the paper we denote by E_i^* the topological dual space of the Banach space E_i, while $\|\cdot\|_i$ and $\langle \cdot, \cdot \rangle_i$ denote the norm in E_i and the duality pairing between E_i^* and E_i, respectively for every $i \in \{1, \dots, n\}$.

In the following, we recall some basic definitions and properties from the theory developed by Clarke [4].

Definition 2.1. *Let E be a Banach space and $f : E \to \mathbb{R}$ be a functional. f is called locally Lipschitz if every point $v \in E$ possesses a neighborhood V such that*

$$|f(z) - f(w)| \leq K_v \|z - w\|, \quad \forall w, z \in V,$$

for a constant $K_v > 0$ which depends on V.

Definition 2.2. *Let E be a Banach space, E^* be its topological dual space and $f : E \to \mathbb{R}$ be a functional. The generalized derivative of a locally Lipschitz functional $f : E \to \mathbb{R}$ at the point $v \in E$ along the direction $w \in E$ is denoted by $f^0(v; w)$, i.e.*

$$f^0(v; w) = \limsup_{\substack{z \to v \\ t \searrow 0}} \frac{f(z + tw) - f(z)}{t}.$$

As follows, we recall here a result for locally Lipschitz functions (for the proof of this result see Clarke [4]).

Proposition 2.1. *If we consider a function $f : E \to \mathbb{R}$ on a Banach space E, which is locally Lipschitz of rank K_w near the point $w \in E$, then*

(a) *the function $w \rightsquigarrow f^0(v; w)$ is finite, subadditive, positively homogeneous and satisfies*

$$f^0(v; w)| \leq K_v \|w\|;$$

(b) *$f^0(v; w)$ is upper semicontinuous as a function of (v, w).*

Definition 2.3. *The generalized gradient of f at the point $v \in E$, which is a subset of E^*, is defined by*

$$\partial f(v) = \{y^* \in E^* : \ \langle y^*, w\rangle \leq f^0(v; w), \ \ \text{for each } w \in E\}.$$

Remark 2.1. *Using the Hahn-Banach theorem (see, for example Brezis* [2]*), it is easy to see that the set $\partial f(v)$ is nonempty for every $v \in E$.*

Similarly, we can define the partial generalized derivative and the partial generalized gradient of a locally Lipschitz functional in the i^{th} variable.

Definition 2.4. *Consider the function $f : E_1 \times \ldots \times E_i \times \ldots \times E_n \to \mathbb{R}$ which is locally Lipschitz in the i^{th} variable. The partial generalized derivative of $f(v_1, \ldots, v_i, \ldots, v_n)$ at the point $v_i \in E_i$ in the direction $w_i \in E_i$, denoted by $f^0_{,i}(v_1, \ldots, v_i, \ldots, v_n; w_i)$, is*

$$f^0_{,i}(v_1,\ldots,v_i,\ldots,v_n; w_i) = \limsup_{\substack{z_i \to v_i \\ t \searrow 0}} \frac{f(v_1,\ldots,z_i + tw_i,\ldots,v_n) - f(v_1,\ldots,z_i,\ldots,v_n)}{t},$$

while the partial generalized gradient of the mapping $v_i \rightsquigarrow f(v_1,\ldots,v_i,\ldots,v_n)$, denoted by $\partial_i f(v_1, \ldots, v_i, \ldots, v_n)$, is

$$\partial_i f(v_1,\ldots,v_i,\ldots,v_n) = \left\{y^*_i \in E^*_i : f^0_{,i}(v_1,\ldots,v_i,\ldots,v_n; w_i) \geq \langle y^*_i, w_i\rangle_{E_i}, \ \forall w_i \in E_i\right\}.$$

Definition 2.5. *Let E be a Banach space and $f : E \to \mathbb{R}$ be a locally Lipschitz functional. f is regular at the point $u \in E$, if for each $w \in E$ there exists the usual one-sided directional derivative $f'(v; w)$ and it coincides with the generalized directional derivative, i.e.*

$$f'(v; w) = f^0(v; w).$$

f is called regular if the above condition holds at every point $v \in E$.

We recall the following elementary lemma due to Clarke [4].

Lemma 2.1. *Let $f : Y_1 \times \ldots \times Y_n \to \mathbb{R}$ be a regular, locally Lipschitz functional. Then the following assertions are satisfied:*

(i) $\partial f(u_1,\ldots,u_i,\ldots,u_n) \subseteq \partial_1 f(u_1,\ldots,u_i,\ldots,u_n) \times \ldots \times \partial_n f(u_1,\ldots,u_i,\ldots,u_n)$;

(ii) $f^0(u_1,\ldots,u_i,\ldots,u_n; v_1,\ldots,v_i,\ldots,v_n) \leq \sum_{i=1}^{n} f^0_{,i}(u_1,\ldots,u_i,\ldots,u_n; v_i)$;

(iii) $f^0(u_1,\ldots,u_i,\ldots,u_n; 0,\ldots,v_i,\ldots,0) \leq f^0_{,i}(u_1,\ldots,u_i,\ldots,u_n; v_i)$.

At the end of this section we present Lin's fixed point theorem for set-valued mappings (see Lin [10]), which we shall use later to prove our main result.

Theorem 2.1. *Let E be a Hausdorff topological vector space and $D \subset E$ be a nonempty convex set. Let $\mathcal{U} \subseteq D \times D$ be a subset such that*

(a) *$(u,u) \in \mathcal{U}$ for each $u \in D$;*

(b) *for each $v \in D$ the set $\mathcal{V}(v) = \{w \in D :\ (v,w) \in \mathcal{U}\}$ is closed in D;*

(c) *for each $w \in D$ the set $\mathcal{W}(w) = \{v \in D :\ (v,w) \notin \mathcal{U}\}$ is either convex or empty;*

(d) *D has a nonempty compact convex subset D_0 such that the set M is compact, where M is defined as follows:*

$$M = \{w \in D :\ (v,w) \in \mathcal{U} \text{ for each } v \in D_0\}.$$

Then there exists a point $w_0 \in M$ such that $D \times \{w_0\}$ is a subset of $\mathcal{U}$.

3. Formulation of the problem

Let us consider the reflexive Banach spaces $X_1,\ldots,X_n$, the Banach spaces $Y_1,\ldots,Y_n$ and the bounded, closed, convex sets $D_i \subseteq X_i$ for $i \in \{1,\ldots,n\}$, where $n \in \mathbb{Z}_+$. We suppose that for $i \in \{1,\ldots,n\}$ there exist linear compact operators $A_i : X_i \to Y_i$, the single-valued functions $\eta_i : X_i \times X_i \to X_i$ and the nonlinear functionals $\phi_i : X_1 \times \ldots \times X_i \times \ldots \times X_n \times X_i \to \mathbb{R}$. We also assume that $J : Y_1 \times \ldots \times Y_n \to \mathbb{R}$ is a regular locally Lipschitz functional. We introduce the following notations:

- $X = X_1 \times \ldots \times X_n$, $Y = Y_1 \times \ldots \times Y_n$ and $D = D_1 \times \ldots \times D_n$;
- $\overline{u}_i = A_i(u_i)$, $\overline{\eta}_i(u_i,v_i) = A_i(\eta_i(u_i,v_i))$, for each $i = \overline{1,n}$;
- $u = (u_1,\ldots,u_n)$ and $\overline{u} = (\overline{u}_1,\ldots,\overline{u}_n)$;
- $\eta(u,v) = (\eta_1(u_1,v_1),\ldots,\eta_n(u_n,v_n))$ and $\overline{\eta}(u,v) = (\overline{\eta}_1(u_1,v_1),\ldots,\overline{\eta}_n(u_n,v_n))$;
- $\Phi : X \times X \to \mathbb{R}, \quad \Phi(u,v) = \sum_{i=1}^{n} \phi_i(u_1,\ldots,u_i,\ldots,u_n,\eta_i(u_i,v_i))$.

We are now in the position to formulate our problem. As we mentioned in the Introduction we will call this *nonlinear hemivariational-like inequality system*:

(NHLIS) Find $(u_1,\ldots,u_n) \in D_1 \times \ldots \times D_n$ such that

$$\begin{cases} \phi_1(u_1,\ldots,u_n,\eta_1(u_1,v_1)) + J^0_{,1}(\overline{u}_1,\ldots,\overline{u}_n;\overline{\eta}_1(u_1,v_1)) \geq 0 \\ \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \\ \phi_n(u_1,\ldots,u_n,\eta_n(u_n,v_n)) + J^0_{,n}(\overline{u}_1,\ldots,\overline{u}_n;\overline{\eta}_n(u_n,v_n)) \geq 0 \end{cases}$$

for all $(v_1,\ldots,v_n) \in D_1 \times \ldots \times D_n$.

In order to obtain our result, we need the following assertions:

- $(\mathcal{H})$ For each $i \in \{1, \ldots, n\}$ the mapping $\eta_i(\cdot, \cdot) : X_i \times X_i \to X_i$ satisfies the following conditions:
  - **(i)** $\eta_i(u_i, u_i) = 0$, for all $u_i \in X_i$;
  - **(ii)** $\eta_i(u_i, \cdot)$ is linear operator, for each $u_i \in X_i$;
  - **(iii)** for each $v_i \in X_i, \eta_i(u_i^m, v_i) \rightharpoonup \eta(u_i, v_i)$, whenever $u_i^m \rightharpoonup u_i$.
- (Φ) For every $i \in \{1, \ldots, n\}$, the functional $\phi_i : X_1 \times \ldots \times X_i \times \ldots \times X_n \times X_i \to \mathbb{R}$ satisfies
  - **(i)** $\phi_i(u_1, \ldots, u_i, \ldots, u_n, 0) = 0$, for all $u_i \in X_i$;
  - **(ii)** for all $v_i \in X_i$, the mapping $(u_1, \ldots, u_n) \rightsquigarrow \phi_i(u_1, \ldots, u_n; \eta_i(u_i, v_i))$ is weakly upper semicontinuous;
  - **(iii)** the mapping $v_i \rightsquigarrow \sum_{i=1}^{n} \phi_i(u_1, \ldots, u_n; \eta_i(u_i, v_i))$ is convex, for each $(u_1, \ldots, u_n) \in X_1 \times \ldots \times X_n$.

Remark 3.1. *Because $J^0_{,i}(u_1, \ldots, u_n; v_i)$ is convex and $\eta_i(u_i, \cdot)$ is linear for each $i \in \{1, \ldots, n\}$ and for each $(u_1, \ldots, u_n) \in X_1 \times \ldots \times X_n$, it follows that the mapping $v_i \rightsquigarrow J^0_{,i}(u_1, \ldots, u_n; \eta_i(u_i, v_i))$ is convex.*

Example 3.1. 1. *For $i \in \{1,\ldots,n\}$ let us choose the function $\eta_i : X_i \times X_i \to X_i$ as follows:*

$$\eta_i(u_i, v_i) = v_i - u_i, \text{ for all } u_i, v_i \in X_i.$$

In this case, $\eta_i(u_i, v_i)$ satisfies the assumptions **(i), (ii)** *from $(\mathcal{H})$. Notice that, with this choice we get back the problem formulated in Varga and Repovš's paper* [22].

2. *Let $B_i : X_i \to X_i$ be a linear compact operator, $\alpha_i > 0, \beta_i \in X_i$, $i \in \{1, \ldots, n\}$ and define a function $f : X_i \to X_i$ by $f_i(x) = \alpha_i B_i(x) + \beta_i$. If we take the function $\eta_i : X_i \times X_i \to X_i$ as follows:*

$$\eta_i(u_i, v_i) = f_i(v_i) - f_i(u_i), \text{ for all } u_i, v_i \in X_i, i \in \{1, \ldots, n\},$$

then it is clear that the conditions $(\mathcal{H})$ **(i)-(iii)** *hold for $\eta_i(u_i, v_i)$.*

Now we present the main result of this paper. We deal with the case when the sets D_i are nonempty, bounded, closed and convex.

Theorem 3.1. *Let us consider the nonempty, bounded, closed and convex sets $D_i \subset X_i$ for each $i \in \{1, \ldots, n\}$. If the conditions $(\mathcal{H})$ and (Φ) are fulfilled, then the system of nonlinear hemivariational-like inequalities* **(NHLIS)** *admits at least one solution.*

Next let us formulate the following hemivariational inequality:
(VHI) Find $u \in D$ such that for all $v \in D$ we have

$$\Phi(u, v) + J^0(\overline{u}; \overline{\eta}(u, v)) \geq 0.$$

Proposition 3.1. *If $u^0 = (u^0_1, \ldots, u^0_n) \in D_1 \times \ldots \times D_n$ is a solution of the inequality* **(VHI)** *and the assumptions $(\mathcal{H})$-(i) and (Φ)-(i) hold, then u^0 is also a solution of the system* **(NHLIS)**.

Proof. First of all, we fix a point $v_i \in D_i$, for each $i \in \{1,\dots,n\}$ and we assume that $v_j = u_j^0, j \neq i$. Let us suppose that u^0 solves **(VHI)**. Then, from Lemma 2.1 (ii), **($\mathcal{H}$)**-(i) and **(Φ)**-(i) we obtain

$$\begin{aligned} 0 &\leq \Phi(u^0,v) + J^0(\overline{u}^0;\overline{\eta}(u^0,v)) \\ &\leq \sum_{j=1}^{n} \phi_j(u_1^0,\dots,u_j^0,\dots,u_n^0,\eta_j(u_j^0,v_j)) + \sum_{j=1}^{n} J_{,j}^0\left(\overline{u}_1^0,\dots,\overline{u}_n^0;\overline{\eta}_j(u_j^0,v_j)\right) \\ &= \phi_i(u_1^0,\dots,u_i^0,\dots,u_n^0,\eta_i(u_i^0,v_i)) + J_{,i}^0(\overline{u}_1^0,\dots,\overline{u}_n^0,\overline{\eta}_i(u_i^0,v_i)), \forall i = \overline{1,n}. \end{aligned}$$

Hence, it follows that $(u_1^0,\dots,u_n^0) \in D_1 \times \dots \times D_n$ is a solution of our system **(NHLIS)** too. □

Proof of Theorem 3.1. If we take into consideration the Proposition 3.1, it is enough to prove that problem **(VHI)** has at least one solution. Toward this, let us introduce the set

$$\mathcal{U} = \{(v,u) \in D \times D : \Phi(u,v) + J^0(\overline{u};\overline{\eta}(u,v)) \geq 0\} \subset D \times D.$$

We will show that $\mathcal{U}$ satisfies all the conditions of Lin's theorem for the weak topology of the space X.

At first, we can observe that $(u,u) \in \mathcal{U}, \forall u \in D$. Indeed, if we fix a point $u \in D$, then from **($\mathcal{H}$)-(i)** and **(Φ)-(i)** we deduce that

$$\begin{aligned} \Phi(u,u) + J^0(\overline{u};\overline{\eta}(u,u)) &= \sum_{i=1}^{n} \phi_i(u_1,\dots,u_i,\dots,u_n,\eta_i(u_i,u_i)) + J^0(\overline{u};0) \\ &= \sum_{i=1}^{n} \phi_i(u_1,\dots,u_i,\dots,u_n,0) = 0, \end{aligned}$$

that is, $(u,u) \in \mathcal{U}$.

Let us consider now the set $\mathcal{V}(v) = \{u \in D : (v,u) \in \mathcal{U}\}$. The next step is to justify that this set is weakly closed in D.
It suffices to prove that for a fixed point $v \in D$ the function

$$F : D \to \mathbb{R}, F(u) = \Phi(u,v) + J^0(\overline{u};\overline{\eta}(u,v))$$

is weakly upper semicontinuous, which is equivalent with the fact that the mappings $\Phi(u,v)$ and $J^0(\overline{u};\overline{\eta}(u,v))$ are both weakly upper semicontinuous for the fixed $v \in D$. Let $\{u^m\} \subset D$ be a sequence which converges weakly to some $u \in D$. Then, for every $i \in \{1,\dots,n\}$, u_i^m converges weakly to u_i, when $m \to +\infty$. From **($\mathcal{H}$)-(iii)**, it follows that for all $i \in \{1,\dots,n\}$, $\eta_i(u_i^m,v_i)$ converges weakly to $\eta_i(u_i,v_i)$, when $m \to +\infty$.

Taking this fact into account and **(Φ)-(ii)**, we observe:

$$\begin{aligned}\limsup_{m\to\infty}\Phi(u^m,v) &= \limsup_{m\to\infty}\sum_{i=1}^n \phi_i(u_1^m,\ldots,u_n^m;\eta_i(u_i^m,v_i))\\ &\le \sum_{i=1}^n \limsup_{m\to\infty}\phi_i(u_1^m,\ldots,u_n^m;\eta_i(u_i^m,v_i))\\ &\le \sum_{i=1}^n \phi_i(u_1,\ldots,u_n;\eta_i(u_i,v_i))\\ &= \Phi(u,v),\end{aligned}$$

which means that the mapping $u\mapsto\Phi(u,v)$ is weakly upper semicontinuous. On the other hand, we assumed that A_i is a compact operator, for each $i\in\{1,\ldots,n\}$. Hence, we obtain that $\overline{u}^m$ converges strongly to some $\overline{u}\in D$ and for each $i\in\{1,\ldots,n\}$ and $v_i\in D_i$, $\overline{\eta_i}(u_i^m,v_i)$ converges strongly to $\overline{\eta_i}(u_i,v_i)$, so $\overline{\eta}(u^m,v)$ converges strongly to $\overline{\eta}(u,v)$, for each $v\in D$. Applying this fact, together with Proposition 2.1 (b), we derive that

$$\limsup_{m\to\infty} J^0(\overline{u}^m;\overline{\eta}(u^m,v))\le J^0(\overline{u};\overline{\eta}(u,v)).$$

Therefore, the function F is weakly upper semicontinuous. Hence, the set $\{u\in D: F(u)\ge\alpha\}$ is weakly closed, for every $\alpha\in\mathbb{R}$. If we choose $\alpha=0$, then we obtain that the set $\mathcal{V}(v)$ is weakly closed, which is exactly what we wanted to prove.

In the following, we consider the set $\mathcal{W}(u)=\{v\in D:(v,u)\notin\mathcal{U}\}$, for all $u\in D$, and we will establish that this set is either convex or empty.

We suppose that $\mathcal{W}(u)\neq\emptyset$ for a fixed $u\in D$ and we shall demonstrate that this set is convex. In order to obtain this fact, let us choose $v',v''\in\mathcal{W}(u)$, $\lambda\in]0,1[$ and let v^λ be their convex combination, i.e. $v^\lambda=(1-\lambda)v'+\lambda v''$. Applying **(Φ)-(iii)**, we get

$$\begin{aligned}\Phi(u,v^\lambda) &= \sum_{i=1}^n \phi_i\left(u_1,\ldots,u_n,\eta_i(u_i,v_i^\lambda)\right)\\ &= \sum_{i=1}^n \phi_i\left(u_1,\ldots,u_n,\eta_i(u_i,(1-\lambda)v_i'+\lambda v_i'')\right)\\ &\le (1-\lambda)\sum_{i=1}^n\phi_i\left(u_1,\ldots,u_n,\eta_i(u_i,v_i')\right)+\lambda\sum_{i=1}^n\phi_i\left(u_1,\ldots,u_n,\eta_i(u_i,v_i'')\right)\\ &= (1-\lambda)\Phi(u,v')+\lambda\Phi(u,v'')\},\quad \forall\lambda\in]0,1[.\end{aligned}$$

Hence, $v\rightsquigarrow\Phi(u,v)$ is convex. On the other hand, from Remark 3.1 we deduce that the mapping $v\rightsquigarrow J^0(\overline{u};\eta_i(\overline{u},\overline{v}))$ is convex. Therefore, the set $\mathcal{W}(u)$ is convex for the fixed $u\in D$.

It remains to justify that the set $M=\{u\in D:(v,u)\in\mathcal{U},\forall v\in D\}$ is weakly compact. Obviously, we can give the set M in the following form:

$$M=\bigcap_{v\in D}\mathcal{V}(v).$$

Since D is bounded, closed and convex subset of the reflexive space X, it follows that D also is a weakly compact subset of X. Taking into consideration that the set $\mathcal{V}$ is weakly closed in D, it follows that the set M is an intersection of weakly closed subsets of D. Therefore, the set M is weakly compact.

We proved that the set $\mathcal{U}$ satisfies the assumptions (1)-(4) of Lin's theorem. Using this theorem, we conclude that there exists $u^0 \in M \subseteq D$ such that $K \times \{u^0\} \subset \mathcal{U}$, which means that

$$\Phi(u^0, v) + J^0(\overline{u}^0; \overline{\eta}(u^0, v)) \geq 0, \forall v \in D.$$

Therefore u^0 is a solution of **(VHI)**, which – applying Proposition 3.1 – is also a solution of **(NHLIS)**. This completes the proof. □

Remark 3.2. *It is known that the solutions of hemivariational inequality systems on unbounded domains exist if we extend the assumptions for the bounded domains with a coercivity condition. Taking this into account, if we impose some coercivity conditions, it will ensure that Theorem* 3.1 *will also hold when the sets D_i are unbounded (for details, see Repovš, Varga* [22], *Remark 3.3).*

4. Applications

In this section we give some concrete applications of our main result. First, we give a generalization of Theorem 4.1 from Varga and Repovš's paper [22], then in particular case, we present an existence result for a Schrödinger-type problem.

4.1. Nash generalized derivative points

Let us consider the Banach spaces $E_1, \ldots, E_n$ for each $i \in \{1, \ldots, n\}$ and the nonempty set $D_i \subset E_i$. Let $g_i : D_1 \times \ldots \times D_i \times \ldots \times D_n \to \mathbb{R}$ be given functionals for $i \in \{1, \ldots, n\}$. In the following we define some useful notions reffered to Nash-type equilibrium problems which will be used throughout this subsection. Firstly, we recall the well-known notion introduced by Nash (see e.g. Nash [18, 19]):

Definition 4.1. *We say that an element $(u_1, \ldots, u_i, \ldots, u_n) \in D_1 \times \ldots \times D_i \times \ldots \times D_n$ is a Nash equilibrium point for the functionals $g_i, i \in \{1, \ldots, n\}$, if*

$$g_i(u_1, \ldots, u_i, \ldots, u_n) \leq g_i(u_1, \ldots, v_i, \ldots, u_n),$$

for all $i \in \{1, \ldots, n\}$ and for all $(v_1, \ldots, v_i, \ldots, v_n) \in D_1 \times \ldots \times D_i \times \ldots \times D_n$.

Let $D_i' \subset E_i$ be an open set such that for each $i \in \{1, \ldots, n\}$ we have that $D_i \subset D_i'$. We impose a condition on the functional $g_i : D_1 \times \ldots \times D_i' \times \times \ldots \times D_n \to \mathbb{R}$ for all $i \in \{1, \ldots, n\}$: the mapping $u_i \rightsquigarrow g_i(u_1, \ldots, u_i, \ldots, u_n)$ must be continuous and locally Lipschitz. Under these conditions we can give the definition of the Nash generalized derivative point. This notion was introduced by Kristály in [8] for functions defined in Riemannian manifolds. We give a little bit different form of this definition:

Definition 4.2. *A point $(u_1, \ldots, u_i, \ldots, u_n) \in D_1 \times \ldots \times D_i \times \ldots \times D_n$ is said to be a Nash generalized derivative point for the functionals $g_1, \ldots, g_i, \ldots, g_n$ if for all*

$i \in \{1,\dots,n\}$ and all $(v_1,\dots,v_i,\dots,v_n) \in D_1 \times \dots \times D_i \times \dots \times D_n$ the following inequality holds:

$$g^0_{i,i}(u_1,\dots,u_i,\dots,u_n; v_i - u_i) \geq 0.$$

Remark 4.1. 1. *We notice, that every Nash equilibrium point also is a Nash generalized derivative point.*

2. *In the paper* [7]*, Kassay, Kolumbán and Páles introduced the notion of Nash stationary point. We can show easily that if for every $i \in \{1,\dots,n\}$ the functionals g_i are differentiable with respect to the i^{th} variable, the notions of Nash generalized derivative point and Nash stationary point coincide.*

An application of the main theorem yields the existence result for a type of Nash generalized derivative points. Thus, if we take in Theorem 3.1 the following choices

$$\phi_i(u_1,\dots,u_i,\dots,u_n,\eta_i(u_i,v_i)) = g^0_{i,i}(u_1,\dots,u_i,\dots,u_n;\eta_i(u_i,v_i))$$

for $i \in \{1,\dots,n\}$ and set $J = 0$, and we suppose that the function

$$(u_1,\dots,u_i,\dots,u_n;v_i) \rightsquigarrow g_{i,i}(u_1,\dots,u_i,\dots,u_n;\eta_i(u_i,v_i))$$

is weakly upper semicontinuous for each $v_i \in D_i$, $i = \overline{1,n}$, then we obtain the theorem below:

Theorem 4.1. *For each $i \in \{1,\dots,n\}$ let $D_i \subset X_i$ be a nonempty, bounded, closed and convex set and let us assume that conditions $(\mathcal{H})$ and (Φ) hold true. In these conditions, there exists a point $(u^0_1,\dots,u^0_i,\dots,u^0_n) \in D_1 \times \dots \times D_n$ such that for all $(u_1,\dots,u_n) \in D_1 \times \dots \times D_n$ and $i \in \{1,\dots,n\}$ we have*

$$g^0_{i,i}(u^0_1,\dots,u^0_i,\dots,u^0_n;\eta_i(u^0_i,v_i)) \geq 0.$$

In order to highlight the importance of the previous result, we make the following remark:

Remark 4.2. *If we choose $\eta(u^0_i,v_i) = v_i - u^0_i$ for $i \in \{1,\dots,n\}$, then we give back the existence result for Nash generalized derivative points from Repovš and Varga's paper* [22].

As a second application, we state an existence result for a general hemivariational inequalities system. In order to establish this theorem, in Theorem 2.1 we need to require that the functionals $\phi_i : Y_1 \times \dots \times Y_i \times \dots \times Y_n \times Y_i \to \mathbb{R}$ be differentiable in the i^{th} variable for $i \in \{1,\dots,n\}$ and their derivatives $\phi'_i : Y_1 \times \dots \times Y_i \times \dots \times Y_n \times Y_i \to \mathbb{R}$ be continuous for $i \in \{1,\dots,n\}$.

Corollary 4.1. *Let us consider the regular, locally Lipschitz function $J : Y_1 \times Y_2 \times \dots \times Y_i \times \dots \times Y_n \to \mathbb{R}$ and the nonlinear functionals $\phi_i : Y_1 \times \dots \times Y_i \times \dots \times Y_n \times Y_i \to \mathbb{R}$ with their continuous derivatives $\phi'_i : Y_1 \times \dots \times Y_i \times \dots \times Y_n \times Y_i \to \mathbb{R}$ for $i \in \{1,\dots,n\}$. Let also $D_i \subset X_i$ be bounded, closed and convex sets for $i \in \{1,\dots,n\}$ and let us assume that conditions $(\mathcal{H})$ and (Φ) hold true. Then, there exists a point $u^0 = (u^0_1,\dots,u^0_n) \in D_1 \times \dots \times D_n$ such that*

$$\Phi'_i(\overline{u}^0,\overline{\eta_i}(u^0_i,u_i)) + J^0_i(\overline{u}^0;\overline{\eta_i}(u^0_i,u_i)) \geq 0,$$

for each $u = (u_1,\dots,u_n) \in D_1 \times \dots \times D_n$ and $i \in \{1,\dots,n\}$.

4.2. Schrödinger-type systems

Let $a_1, a_2 : \mathbb{R}^n \to \mathbb{R}$ $(n > 2)$ be two continuous functions such that the following assumptions hold:

- $\inf_{x \in \mathbb{R}^n} a_i(x) > 0$, $i = 1, 2$;
- $\text{meas}(\{x \in \mathbb{R}^n : a_i(x) \le M_i\}) < \infty$, for every $M_i > 0$, $i = 1, 2$.

For $i = 1, 2$, let us consider the Hilbert-spaces

$$X_i := \left\{ u \in W^{1,2}(\mathbb{R}^n) : \int_{\mathbb{R}^n} |\nabla u(x)|^2 + a_i(x)u^2(x)dx < \infty \right\},$$

which are equipped with the inner products

$$\langle u, v \rangle_{X_i} = \int_{\mathbb{R}^n} [\nabla u(x) \nabla v(x) + a_i(x)u(x)v(x)]\, dx.$$

It is known that for $q \in [2, 2^*]$, the space $W^{1,2}(\mathbb{R}^n) \hookrightarrow L^q(\mathbb{R}^n)$ is continuous, therefore $X_1 \times X_2 \hookrightarrow L^s(\mathbb{R}^n) \times L^r(\mathbb{R}^n)$ is continuous for $s, r \in [2, 2^*]$. If $s, r \in [2, 2^*)$, then $X_1 \times X_2 \hookrightarrow L^s(\mathbb{R}^n) \times L^r(\mathbb{R}^n)$ is compact (for the proof of the latter result see Bartsch, Wang [1]).

Let $G : \mathbb{R}^2 \to \mathbb{R}$ be a regular, locally Lipschitz-function, for which we assume the following:

(G1) There exist $c > 0$ and $r \in (2, 2^*), s \in (2, 2^*)$ such that

$$|w_u| \le c(|u| + |v| + |u|^{r-1}),$$

$$|w_v| \le c(|v| + |u| + |v|^{s-1}),$$

for all $(u, v) \in \mathbb{R}^2, w_u \in \partial_1 G(u, v), w_v \in \partial_2 G(u, v), i = \overline{1, 2}$, where $\partial_1 G(u, v)$ and $\partial_2 G(u, v)$ denote the (partial) generalized gradient of $G(\cdot, v)$ at the point u and v, respectively, while $2^* = \frac{2N}{N-2}, N > 2$ is the Sobolev critical exponent.

Let $K_1 \subset X_1, K_2 \subset X_2$ be two nonempty, convex, closed and bounded subsets. In the following let us denote by $G_1^0(u(x), v(x); w_1(x))$ and $G_2^0(u(x), v(x); w_2(x))$ the directional derivatives of G in the first and the second variable along the direction w_1 and, respectively w_2. Now, we are in the position to formulate the following Schrödinger-type problem:

(Sch-S) Find $(u_1, u_2) \in K_1 \times K_2$ such that for every $(v_1, v_2) \in K_1 \times K_2$

$$\langle \overline{u_1}, \overline{\eta}_1(u_1, v_1) \rangle + \int_{\mathbb{R}^n} G_1^0(\overline{u_1}(x), \overline{v_1}(x), \overline{\eta_1}(u_1(x), v_1(x)))dx \ge 0, \forall v_1 \in K_1;$$

$$\langle \overline{u_2}, \overline{\eta}_2(u_2, v_2) \rangle + \int_{\mathbb{R}^n} G_2^0(u_2(x), v_2(x), \overline{\eta_2}(\overline{u_2}(x), \overline{v_2}(x)))dx \ge 0, \forall v_2 \in K_2.$$

At the end of this section, as an application of the main result we show the existence of at least one solution of the previous problem.

Corollary 4.2. *If $K_1 \subset X_1$ and $K_2 \subset X_2$ are two nonempty, convex, closed and bounded subsets and η_1, η_2 satisfy the conditions* ($\mathcal{H}$) *and G satisfies* **(G1)***, then the problem* **(Sch-S)** *has at least one solution.*

Acknowledgements. The authors would like to thank to Professor Csaba Varga, Dr. Muzsnay Zoltán and Dr. Nicuşor Costea for useful remarks and helpful comments concerning this paper. The author Andrea Éva Molnár has been supported by Project POSDRU/CPP107/DMI 1.5/S/76841 "Modern Doctoral Studies: Internationalization and Interdisciplinarity"/"Studii doctorale moderne: internaţionalizare şi interdisciplinaritate" (Project co-financed by the Sectoral Operational Program For Human Resources Development 2007 - 2013, Babeş-Bolyai University, Cluj-Napoca, Romania).

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Andrea Éva Molnár and Orsolya Vas
Department of Mathematics, Babeş-Bolyai University
1, Kogălniceanu Str., 400084, Cluj-Napoca, Romania
E-mails: andrea.molnar@math.ubbcluj.ro vasleanka@yahoo.com

Stud. Univ. Babeş-Bolyai Math. 58(2013), No. 3, 393–400

The equiform differential geometry of curves in 4-dimensional galilean space $\mathbb{G}_4$

M. Evren Aydin and Mahmut Ergüt

Abstract. In this paper, we establish equiform differential geometry of curves in 4-dimensional Galilean space $\mathbb{G}_4$. We obtain the angle between the equiform Frenet vectors and their derivatives in $\mathbb{G}_4$. Also, we characterize generalized helices with respect to their equiform curvatures.

Mathematics Subject Classification (2010): 53A35.

Keywords: Equiform geometry, generalized helices.

1. Introduction

Differential geometry of the Galilean space $\mathbb{G}_3$ has been largely developed in O. Röschel's paper [12]. The Frenet formulas of a curve in 4-dimensional Galilean space $\mathbb{G}_4$ are given by [13]. The helices in $\mathbb{G}_3$ are characterized by [8]. The equiform differential geometry of isotropic spaces and Galilean-pseudo Galilean spaces are represented by [9, 4, 5]. In this paper, we construct equiform differential geometry of curves in $\mathbb{G}_4$.

The Galilean space is three dimensional complex projective space, $\mathbb{P}_3$, in which absolute figure $\{w, f, I_1, I_2\}$ consist of a real plane w (absolute plane), a real line $f \subset w$ (absolute line) and two complex conjugate points, I_1, $I_2 \in f$ (absolute points) [7].

The equiform geometry of Cayley - Klein space is defined by requesting that similarity group of the space preserves angles between planes and lines, respectively. Cayley-Klein geometries are studied for many years. However, they recently have become interesting again since their importance for other fields, like soliton theory [11], have been rediscovered. The positive aspect of this paper is the equiform Frenet formulas and equiform curvatures of $\mathbb{G}_3$ to generalize these of $\mathbb{G}_4$.

2. Preliminaries

Four-dimensional Galilean geometry can be described as the study of properties of four-dimensional space with coordinates that are invariant under general Galilean transformations

$$\begin{aligned}
x' &= (\cos\beta\cos\alpha - \cos\gamma\sin\beta\sin\alpha)\,x + (\sin\beta\cos\alpha - \cos\gamma\sin\beta\sin\alpha)\,y \\
&\quad + (\sin\gamma\sin\alpha)\,z + (v\cos\delta_1)\,t + a, \\
y' &= -(\cos\beta\sin\alpha + \cos\gamma\sin\beta\cos\alpha)\,x + (-\sin\beta\sin\alpha - \cos\gamma\cos\beta\cos\alpha)\,y \\
&\quad + (\sin\gamma\cos\alpha)\,z + (v\cos\delta_2)\,t + b, \\
z' &= (\sin\gamma\sin\beta)\,x - (\sin\gamma\cos\beta)\,y + (\cos\gamma)\,z + (v\cos\delta_3)\,t + c, \\
t' &= t + d,
\end{aligned}$$

where $\cos^2\delta_1 + \cos^2\delta_2 + \cos^2\delta_3 = 1$.
Given two vectors $\overrightarrow{\alpha}$ and $\overrightarrow{\beta}$ with the coordinates $(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ and $(\beta_1, \beta_2, \beta_3, \beta_4)$, respectively, then the Galilean scalar product g between the vectors is defined as follows

$$g\left(\overrightarrow{\alpha}, \overrightarrow{\beta}\right) = \begin{cases} \alpha_1\beta_1, & \text{if} \quad \alpha_1 \neq 0 \quad \text{or} \quad \beta_1 \neq 0, \\ \alpha_2\beta_2 + \alpha_3\beta_3 + \alpha_4\beta_4, & \text{if} \quad \alpha_1 = 0 \quad \text{and} \quad \beta_1 = 0. \end{cases} \tag{2.1}$$

For the vectors $\overrightarrow{\alpha}$, $\overrightarrow{\beta}$, $\overrightarrow{\gamma}$ with the coordinates $(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$, $(\beta_1, \beta_2, \beta_3, \beta_4)$, $(\gamma_1, \gamma_2, \gamma_3, \gamma_4)$, the cross product of $\mathbb{G}_4$ given by

$$\overrightarrow{\alpha} \times_{\mathbb{G}} \overrightarrow{\beta} \times_{\mathbb{G}} \overrightarrow{\gamma} = \begin{vmatrix} 0 & \overrightarrow{e}_2 & \overrightarrow{e}_3 & \overrightarrow{e}_4 \\ \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 \\ \beta_1 & \beta_2 & \beta_3 & \beta_4 \\ \gamma_1 & \gamma_2 & \gamma_3 & \gamma_4 \end{vmatrix}, \tag{2.2}$$

where $\overrightarrow{e}_i$ are the standard basis vectors.

Let $C : I \subset \mathbb{R} \longrightarrow \mathbb{G}_4$ be a curve, parametrized by the invariant parameter $s = x$, is given in the coordinate form

$$C(s) = (s, c_1(s), c_2(s), c_3(s)),$$

the Frenet vector fields of the curve C defined by

$$\begin{aligned}
V_1 &= (1, \dot{c}_1, \dot{c}_2, \dot{c}_3), \\
V_2 &= \frac{1}{k_1}(0, \ddot{c}_1, \ddot{c}_2, \ddot{c}_3), \\
V_3 &= \frac{1}{k_2}\left(0, \frac{d\left(\frac{1}{k_1}\ddot{c}_1\right)}{ds}, \frac{d\left(\frac{1}{k_1}\ddot{c}_2\right)}{ds}, \frac{d\left(\frac{1}{k_1}\ddot{c}_3\right)}{ds}\right), \\
V_4 &= V_1 \times_{\mathbb{G}} V_2 \times_{\mathbb{G}} V_3,
\end{aligned} \tag{2.3}$$

where k_1, k_2, k_3 are the first, second and third curvature functions, respectively, defined by

$$k_1 = \left((\ddot{c}_1)^2 + (\ddot{c}_2)^2 + (\ddot{c}_3)^2\right)^{\frac{1}{2}},$$
$$k_2 = \left[g\left(\dot{V}_2, \dot{V}_2\right)\right]^{\frac{1}{2}}, \tag{2.4}$$
$$k_3 = g\left(\dot{V}_3, V_4\right),$$

where the derivative with respect to s denote by a dot. Thus, the Frenet equations of $\mathbb{G}_4$ given by as follows ([13])

$$\begin{aligned} \dot{V}_1 &= k_1 V_2, \\ \dot{V}_2 &= k_2 V_3, \\ \dot{V}_3 &= -k_2 V_2 + k_3 V_4, \\ \dot{V}_4 &= -k_3 V_3. \end{aligned}$$

3. Frenet formulas in equiform geometry of $\mathbb{G}_4$

Let $C : I \subset \mathbb{R} \longrightarrow \mathbb{G}_4$ be a curve parametrized by arclength s. The equiform parameter of the curve $C(s)$ defined by

$$\sigma = \int \frac{ds}{\rho}, \tag{3.1}$$

where $\rho = \frac{1}{k_1}$ is radius of curvature of the curve. Considering the equation (3.1), it is written that

$$\frac{ds}{d\sigma} = \rho. \tag{3.2}$$

Suppose that h is a homothety with the center in the origin and the coefficient λ. If we take $\tilde{C} = h(C)$, then it can easily be seen that

$$\tilde{s} = \lambda s \quad \text{and} \quad \tilde{\rho} = \lambda \rho, \tag{3.3}$$

where $\tilde{s}$ is the arc-length parameter of $\tilde{C}$ and $\tilde{\rho}$ the radius of curvature of this curve. Hence σ is an equiform invariant parameter of C.

Remark 3.1. Denote by k_1, k_2, k_3 the curvature functions of the curve C. Then, the curvatures k_1, k_2, k_3 are not invariants of the homothety group, because from (2.4), it follows that

$$\tilde{k}_1 = \frac{1}{\lambda} k_1,\ \tilde{k}_2 = \frac{1}{\lambda} k_2,\ \tilde{k}_3 = \frac{1}{\lambda} k_3.$$

Now, if we get

$$\mathbb{V}_1 = \frac{dC}{d\sigma}, \tag{3.4}$$

then using (2.1), we have

$$\mathbb{V}_1 = \rho V_1. \tag{3.5}$$

Also, we define the vectors $\mathbb{V}_2, \mathbb{V}_3, \mathbb{V}_4$ by

$$\mathbb{V}_2 = \rho V_2, \quad \mathbb{V}_3 = \rho V_3, \quad \mathbb{V}_4 = \rho V_4 \tag{3.6}$$

Thus, $\{\mathbb{V}_1, \mathbb{V}_2, \mathbb{V}_3, \mathbb{V}_4\}$ is an equiform invariant tetrahedron of the curve C.

Now, we will find the derivatives of these vectors with respect to σ using by (3.2), (3.4) and (3.6). For this purpose, it can be written that

$$\mathbb{V}_1' = \frac{d}{d\sigma}(\mathbb{V}_1) = \dot{\rho}\mathbb{V}_1 + \mathbb{V}_2.$$

Similarly, we obtain

$$\begin{aligned}
\mathbb{V}_2' &= \frac{d\mathbb{V}_2}{d\sigma} = \dot{\rho}\mathbb{V}_2 + \frac{k_2}{k_1}\mathbb{V}_3, \\
\mathbb{V}_3' &= \frac{d\mathbb{V}_3}{d\sigma} = -\frac{k_2}{k_1}\mathbb{V}_2 + \dot{\rho}\mathbb{V}_3 + \frac{k_3}{k_1}\mathbb{V}_4, \\
\mathbb{V}_4' &= \frac{d\mathbb{V}_4}{d\sigma} = -\frac{k_3}{k_1}\mathbb{V}_3 + \dot{\rho}\mathbb{V}_4,
\end{aligned}$$

where the derivatives of the vectors $\mathbb{V}_1, \mathbb{V}_2, \mathbb{V}_3, \mathbb{V}_4$ with respect to σ are denoted by a dash $(')$.

Definition 3.2. The function $\mathbb{K}_i : I \longrightarrow \mathbb{R}$ $(i = 1, 2, 3)$ is defined by

$$\mathbb{K}_1 = \dot{\rho}, \ \mathbb{K}_2 = \frac{k_2}{k_1}, \ \mathbb{K}_3 = \frac{k_3}{k_1} \tag{3.7}$$

is called $i.th$ equiform curvature of the curve C. It is easy to prove that $\mathbb{K}_i$ is differential invariant of the group of equiform transformations.

Thus the formulas analogous to famous the Frenet formulas in the equiform geometry of the Galilean 4−space $\mathbb{G}_4$ have the following form:

$$\begin{aligned}
\mathbb{V}_1' &= \mathbb{K}_1\mathbb{V}_1 + \mathbb{V}_2, \\
\mathbb{V}_2' &= \mathbb{K}_1\mathbb{V}_2 + \mathbb{K}_2\mathbb{V}_3, \\
\mathbb{V}_3' &= -\mathbb{K}_2\mathbb{V}_2 + \mathbb{K}_1\mathbb{V}_3 + \mathbb{K}_3\mathbb{V}_4, \\
\mathbb{V}_4' &= -\mathbb{K}_3\mathbb{V}_3 + \mathbb{K}_1\mathbb{V}_4,
\end{aligned} \tag{3.8}$$

where the functions $\mathbb{K}_1$, $\mathbb{K}_2$, $\mathbb{K}_3$ is the equiform curvatures of this curve.

These formulas can be written in matrix form as follows:

$$\begin{bmatrix} \mathbb{V}_1' \\ \mathbb{V}_2' \\ \mathbb{V}_3' \\ \mathbb{V}_4' \end{bmatrix} = \begin{bmatrix} \mathbb{K}_1 & 1 & 0 & 0 \\ 0 & \mathbb{K}_1 & \mathbb{K}_2 & 0 \\ 0 & -\mathbb{K}_2 & \mathbb{K}_1 & \mathbb{K}_3 \\ 0 & 0 & -\mathbb{K}_3 & \mathbb{K}_1 \end{bmatrix} \begin{bmatrix} \mathbb{V}_1 \\ \mathbb{V}_2 \\ \mathbb{V}_3 \\ \mathbb{V}_4 \end{bmatrix}$$

Because of the equiform Frenet formulas (3.8), the below equalities regarding equiform curvatures can be given

$$\mathbb{K}_i = \begin{cases} \dfrac{1}{\rho^2} g\left(\mathbb{V}_j', \mathbb{V}_j\right), & (j = 1, 2, 3, 4), \quad \text{for } i = 1, \\ \dfrac{1}{\rho^2} g\left(\mathbb{V}_i', \mathbb{V}_{i+1}\right) = -\dfrac{1}{\rho^2} g\left(\mathbb{V}_i, \mathbb{V}_{i+1}'\right), & \text{for } i = 2, 3, \end{cases} \tag{3.9}$$

where $\rho = \frac{1}{k_1}$ is radius of curvature of C.

Theorem 3.3. *Let $C : I \subset \mathbb{R} \longrightarrow \mathbb{G}_4$ be a curve parametrized by arclength s, $\{\mathbb{V}_1, \mathbb{V}_2, \mathbb{V}_3, \mathbb{V}_4\}$ be the equiform invariant tetrahedron and the function $\mathbb{K}_i : I \longrightarrow \mathbb{R}$ $(i = 1, 2, 3)$ be i.th equiform curvature of the curve C. Then for $1 \leq i \leq 4$, the angle between the vectors $\mathbb{V}_i$ and $\mathbb{V}_i'$ is given as follows*

$$\measuredangle\left(\mathbb{V}_i, \mathbb{V}_i'\right) = \begin{cases} \rho\sqrt{\mathbb{K}_1^2 - 2\mathbb{K}_1 + 2} & \text{for } i = 1, \\ \arccos\left(\frac{\mathbb{K}_1}{\sqrt{\mathbb{K}_1^2 + \mathbb{K}_2^2}}\right), & \text{for } i = 2, \\ \arccos\left(\frac{\mathbb{K}_1}{\sqrt{\mathbb{K}_1^2 + \mathbb{K}_2^2 + \mathbb{K}_3^2}}\right), & \text{for } i = 3, \\ \arccos\left(\frac{\mathbb{K}_1}{\sqrt{\mathbb{K}_1^2 + \mathbb{K}_3^2}}\right), & \text{for } i = 4 \end{cases} \tag{3.10}$$

Proof. For $i = 1$, let θ_1 be the angle between the vectors $\mathbb{V}_1$ and $\mathbb{V}_1'$. Since these vectors are non-isotropic, it is obtained as follows

$$\begin{aligned} \theta_1 &= \left[g\left(\mathbb{V}_1 - \mathbb{V}_1', \mathbb{V}_1 - \mathbb{V}_1'\right)\right]^{\frac{1}{2}} \\ &= \rho\sqrt{\mathbb{K}_1^2 - 2\mathbb{K}_1 + 2}. \end{aligned}$$

For $i = 2$, denote by θ_2, the angle between the vectors $\mathbb{V}_2$ and $\mathbb{V}_2'$. The vectors $\mathbb{V}_2$ and $\mathbb{V}_2'$ are isotropic and we have

$$\begin{aligned} \cos\theta_2 &= \frac{g\left(\mathbb{V}_2, \mathbb{V}_2'\right)}{\left[g\left(\mathbb{V}_2, \mathbb{V}_2\right)\right]^{\frac{1}{2}}\left[g\left(\mathbb{V}_2', \mathbb{V}_2'\right)\right]^{\frac{1}{2}}} \\ &= \frac{\mathbb{K}_1}{\sqrt{\mathbb{K}_1^2 + \mathbb{K}_2^2}}. \end{aligned}$$

The others are obtained in a similar way.

4. The characterizations of the curves

The equiform curvatures $\mathbb{K}_i$ $(i = 1, 2, 3)$ in $\mathbb{G}_4$ have important geometric interpretation. For example,

(i) The equiform curvatures of a curve have following form

$$\mathbb{K}_2 = const., \ \mathbb{K}_3 = const., \tag{4.1}$$

if and only if the curve is generalized helix. Here, we do not set condition on $\mathbb{K}_1$.

(ii) If (4.1) holds and $\mathbb{K}_1$ is identically zero, then the curve is a $W-$curve.

Now, we present a few characterizations regarding a curve in $\mathbb{G}_4$ with respect to the its equiform curvatures.

Theorem 4.1. *Let C be a curve in $\mathbb{G}_4$ with the equiform invariant tetrahedron $\{\mathbb{V}_1, \mathbb{V}_2, \mathbb{V}_3, \mathbb{V}_4\}$ and with equiform curvatures $\mathbb{K}_1 \neq 0$. Then C has $\mathbb{K}_2 \equiv 0$ if and only if C lies fully in a 2-dimensional subspace of $\mathbb{G}_4$.*

Theorem 4.2. *Let C be a curve in $\mathbb{G}_4$ with the equiform invariant tetrahedron $\{\mathbb{V}_1, \mathbb{V}_2, \mathbb{V}_3, \mathbb{V}_4\}$ and with equiform curvatures $\mathbb{K}_1, \mathbb{K}_2 \neq 0$. Then C has $\mathbb{K}_3 \equiv 0$ if and only if C lies fully in a hyperplane of $\mathbb{G}_4$.*

Proof. If C has $\mathbb{K}_3 \equiv 0$, then from (3.8) , we have

$$\begin{aligned} C' &= \mathbb{V}_1, \\ C'' &= \mathbb{K}_1\mathbb{V}_1 + \mathbb{V}_2, \\ C''' &= \left(\rho\dot{\mathbb{K}}_1 + \mathbb{K}_1^2\right)\mathbb{V}_1 + 2\mathbb{K}_1\mathbb{V}_2 + \mathbb{K}_2\mathbb{V}_3, \\ C^{(4)} &= \left(\frac{d\left(\rho\dot{\mathbb{K}}_1 + \mathbb{K}_1^2\right)}{d\sigma} + \left(\rho\dot{\mathbb{K}}_1 + \mathbb{K}_1^2\right)\mathbb{K}_1\right)\mathbb{V}_1 \\ &\quad + \left(\rho\dot{\mathbb{K}}_1 + 3\mathbb{K}_1^2 + 2\rho\dot{\mathbb{K}}_1 - \mathbb{K}_2^2\right)\mathbb{V}_2 \\ &\quad + \left(3\mathbb{K}_1\mathbb{K}_2 + \rho\dot{\mathbb{K}}_2\right)\mathbb{V}_3. \end{aligned}$$

Hence, by using Mclauren expansion for C, given by

$$C(\sigma) = C(0) + C'(0)\sigma + C''(0)\frac{\sigma^2}{2!} + C'''(0)\frac{\sigma^3}{3!} +, ...,$$

we obtain that C lies fully in a hyperplane of $\mathbb{G}_4$ by spanned

$$\{C'(0), C''(0), C'''(0)\}.$$

Conversely, we suppose that C lies fully in a hyperplane Γ of $\mathbb{G}_4$. Then, there exist the points $p, q \in \mathbb{G}_4$ such that C satisfies the equation of Γ given by

$$g(C(\sigma) - p, q) = 0, \tag{4.2}$$

where $q \in \Gamma^{\perp}$. Differentiating (4.2) with respect to σ, we can write

$$g(C', q) = g(C'', q) = g(C''', q) = 0.$$

Since

$$C' = \mathbb{V}_1 \text{ and } C'' = \mathbb{K}_1\mathbb{V}_1 + \mathbb{V}_2,$$

it follows that

$$g(\mathbb{V}_1, q) = g(\mathbb{V}_2, q) = 0. \tag{4.3}$$

Similarly, we have

$$g(\mathbb{V}_3, q) = 0. \tag{4.4}$$

Again, differentiating (4.4)

$$\begin{aligned} 0 &= g(-\mathbb{K}_2\mathbb{V}_2 + \mathbb{K}_1\mathbb{V}_3 + \mathbb{K}_3\mathbb{V}_4, q) \\ 0 &= \mathbb{K}_3 g(\mathbb{V}_4, q), \end{aligned}$$

because $\mathbb{V}_4$ is the only vector perpendicular to $\{\mathbb{V}_1, \mathbb{V}_2, \mathbb{V}_3\}$, we obtain

$$\mathbb{K}_3 = 0,$$

this completes the proof. □

Last, we give a characterization for a generalized helix in $\mathbb{G}_4$ with respect to the curvatures in equiform geometry.

Theorem 4.3. *Let C be a curve with equiform invariant vector $\mathbb{V}_3$ in the equiform geometry of $\mathbb{G}_4$ is a generalized helix if and only if*

$$\mathbb{V}_3'' + \varphi_1\mathbb{V}_3 = \varphi_2\mathbb{V}_2 + \varphi_3\mathbb{V}_4, \tag{4.5}$$

where $\varphi_1 = \mathbb{K}_2^2 + \mathbb{K}_3^2 - \mathbb{K}_1^2 - \rho\dot{\mathbb{K}}_1$, $\varphi_2 = -2\mathbb{K}_1\mathbb{K}_2$ *and* $\varphi_3 = 2\mathbb{K}_1\mathbb{K}_3$.

Proof. Suppose that the curve C is a generalized helix. Thus, we have

$$\mathbb{K}_2 = const. \text{ and } \mathbb{K}_3 = const. \tag{4.6}$$

From (3.8) and (4.6), it is easy to prove that the equation (4.5) is satisfied.

Conversely, we assume that the equation (4.5) holds. Then from (3.8), it follows that

$$\mathbb{V}_3' = -\mathbb{K}_2\mathbb{V}_2 + \mathbb{K}_1\mathbb{V}_3 + \mathbb{K}_3\mathbb{V}_4, \tag{4.7}$$

and differentiating (4.7) with respect to σ

$$\begin{aligned} \mathbb{V}_3'' &= \left(-\rho\dot{\mathbb{K}}_2 - 2\mathbb{K}_1\mathbb{K}_2\right)\mathbb{V}_2 \\ &\quad + \left(\rho\dot{\mathbb{K}}_1 + \mathbb{K}_1^2 - \mathbb{K}_2^2 - \mathbb{K}_3^2\right)\mathbb{V}_3 \\ &\quad + \left(\rho\dot{\mathbb{K}}_3 + 2\mathbb{K}_1\mathbb{K}_3\right)\mathbb{V}_4, \end{aligned}$$

so, we obtain

$$\dot{\mathbb{K}}_2 = 0 \quad \text{and} \quad \dot{\mathbb{K}}_3 = 0$$

which completes the proof. □

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M. Evren Aydin and Mahmut Ergüt
Firat University, Department of Mathematics
23119, Elazig, Turkey
e-mail: aydnevren@gmail.com

Stud. Univ. Babeş-Bolyai Math. 58(2013), No. 3, 401–411

Semi-infinite optimization problems and their approximations

Augusta Raţiu and Dorel I. Duca

Abstract. In this paper, to the semi-infinite optimization problem (P), we attach the approximated semi-infinite optimization problems $(P_{1,0})$, $(P_{0,1})$ and $(P_{1,1})$ and some connections between the optimal solutions of the problems (P), $(P_{1,0})$, $(P_{0,1})$ and $(P_{1,1})$ are given.

Mathematics Subject Classification (2010): 90C90, 90C59, 90C34.

Keywords: Optimal solution, semi-infinite optimization, invex function, approximated optimization problem, connections between optimal solutions.

1. Introduction

We consider the optimization problem:

$$\begin{array}{ll} \min & f(x) \\ \text{such that:} & \\ & x \in X \\ & g_t(x) \leq 0,\ t \in T \\ & h_s(x) = 0,\ s \in S \end{array} \tag{P}$$

where X is a subset of $\mathbb{R}^n$, T and S are nonempty sets, and $f : X \to \mathbb{R}$, $g_t : X \to \mathbb{R}$, $t \in T$ and $h_s : X \to \mathbb{R}$, $s \in S$ are functions.

Let

$$\mathcal{F}(P) := \{x \in X : \ g_t(x) \leq 0, \ \ (t \in T), \ h_s(x) = 0, \ \ (s \in S)\},$$

denote the set of all feasible solutions of Problem (P).

Depending on the sets T and S, we can have the following problems: if the sets T and S are finite, then the Problem (P) is a classic optimization problem, otherwise, the Problem (P) is a semi-infinite optimization problem with infinite number of constraints.

The field of semi-infinite programming appeared in 1924, but the name was coined in 1962 by Kortanek, Cooper and Charnes in the papers [3, 4]. Optimization problems in this area are characterized with a finite number of variables and an infinite number of constraints, or an infinite number of variables and a finite number of constraints. This class of optimization problems contains both convex and nonconvex optimization problems.

In recent years, in this domain over 10 books and 1000 articles have been published, treating both theoretical and practical issues, e.g., Hettich and Kortanek in [9].

We can find, in the literature, many semi-infinite optimization models from mechanical stress of materials, robot trajectory planning, economics [13], optimal signal sets, production planning, digital filter design, time minimal heating or cooling of a ball [11], air pollution control, minimal norm problem in the space of polynomial, robust optimization, system and control [8], reverse Chebyshev approximation [10]. The stability analysis in semi-infinite optimization (SIO) became an important issue, e.g., [2, 6, 7]. Authors who have treated (SIO) problem would be: Rückmann and Shapiro [16], Dinh The Luc [14], Polak [15], Still [17], Krabs [12].

Among the assumptions of necessary, respectively sufficient conditions for the solutions of semi-infinite optimization problem, appears the compactness of the sets T and S. The results obtained in this paper do not require that the sets T and S to be compact. The idea is to replace the Problem (P) with another simple problem and to establish the implications between the optimal solutions of the two problems.

Let $\eta : X \times X \to X$ be a function, x^0 be an interior point of X. Assume that the functions $f : X \to \mathbb{R}$, $g_t : X \to \mathbb{R}$, $t \in T$ and $h_s : X \to \mathbb{R}$, $s \in S$ are differentiable at x^0.

In this paper, we propose to attach to Problem (P), the following three approximated problems:

The first problem is:

$$\begin{array}{ll} \min & f(x^0) + \left[\nabla f(x^0)\right]\left(\eta(x, x^0)\right) \\ \text{such that:} & \\ & x \in X \\ & g_t(x) \leq 0,\ t \in T \\ & h_s(x) = 0,\ s \in S \end{array} \tag{P$_{1,0}$}$$

called $(1, 0)$-η approximated optimization problem.

The second problem is:

$$\begin{array}{ll} \min & f(x) \\ \text{such that:} & \\ & x \in X \\ & g_t(x^0) + \left[\nabla g_t(x^0)\right]\left(\eta(x, x^0)\right) \leq 0,\ t \in T \\ & h_s(x^0) + \left[\nabla h_s(x^0)\right]\left(\eta(x, x^0)\right) = 0,\ s \in S \end{array} \tag{P$_{0,1}$}$$

called $(0, 1)$-η approximated optimization problem.

The third problem is

$$\begin{array}{ll} \min & f(x^0) + \left[\nabla f(x^0)\right]\left(\eta(x,x^0)\right) \\ \text{such that:} & \\ & x \in X \\ & g_t(x^0) + \left[\nabla g_t(x^0)\right]\left(\eta(x,x^0)\right) \le 0,\ t \in T \\ & h_s(x^0) + \left[\nabla h_s(x^0)\right]\left(\eta(x,x^0)\right) = 0,\ s \in S \end{array} \tag{P$_{1,1}$}$$

called $(1,1)$-η approximated optimization problem.

In the case where T and S are finite, the idea of approximating the Problem (P) appeared in several papers, e.g. [1, 5].

After presenting some definitions, in paragraph 3 some connections between the optimal solutions of the four problems: (P), $(P_{1,0})$, $(P_{0,1})$ and $(P_{1,1})$ are given.

2. Definitions and preliminary results

Definition 2.1. *Let X be a nonempty subset of $\mathbb{R}^n$, x^0 be an interior point of X, $f : X \to \mathbb{R}$ be a differentiable function at x^0 and $\eta : X \times X \to X$ be a function. We say that:*
(a) the function f is invex at x^0 with respect to (w.r.t.) η if

$$f(x) - f(x^0) \ge [\nabla f(x^0)](\eta(x,x^0)),\ \textit{for all } x \in X,$$

(b) the function f is incave at x^0 with respect to (w.r.t.) η if $(-f)$ is invex at x^0 w.r.t. η,
(c) the function f is avex at x^0 with respect to (w.r.t.) η if f is both invex and incave at x^0 w.r.t. η,
(d) the function f is pseudoinvex at x^0 with respect to (w.r.t.) η if

$$[\nabla f(x^0)](\eta(x,x^0)) \ge 0 \Rightarrow f(x) - f(x^0) \ge 0,\ \textit{for all } x \in X,$$

(e) the function f is quasi-incave at x^0 with respect to (w.r.t.) η if

$$f(x) - f(x^0) \ge 0 \Rightarrow [\nabla f(x^0)](\eta(x,x^0)) \ge 0,\ \textit{for all } x \in X.$$

In the following two theorems establish connections between the sets of feasible solutions of the problem (P) and the problems $(P_{0,1})$, $(P_{1,1})$.

Theorem 2.2. *Let X be a subset of $\mathbb{R}^n$, x^0 be an interior point of X, $\eta : X \times X \to X$ and f, g_t, $h_s : X \to \mathbb{R}$, $t \in T$, $s \in S$. Assume that:*
(a) for each $t \in T$, the function g_t is differentiable at x^0 and invex at x^0 w.r.t. η,
(b) for each $s \in S$, the function h_s is differentiable at x^0 and avex at x^0 w.r.t. η.

If

$$\begin{aligned} \mathcal{L} := \{x \in X : \ & g_t(x^0) + \left[\nabla g_t(x^0)\right]\left(\eta(x,x^0)\right) \le 0,\ t \in T \textit{ and} \\ & h_s(x^0) + \left[\nabla h_s(x^0)\right]\left(\eta(x,x^0)\right) = 0,\ s \in S\}, \end{aligned}$$

then

$$\mathcal{F}(P) \subseteq \mathcal{L}.$$

Proof. Let $x \in \mathcal{F}(P)$. This is equivalent with

$$g_t(x) \leq 0,\ t \in T,$$

and

$$h_s(x) = 0,\ s \in S.$$

From (a) and (b) we have

$$g_t(x) - g_t(x^0) \geq \left[\nabla g_t(x^0)\right]\left(\eta(x, x^0)\right),\ t \in T,$$

and

$$h_s(x) - h_s(x^0) = \left[\nabla h_s(x^0)\right]\left(\eta(x, x^0)\right),\ s \in S.$$

But

$$g_t(x) \leq 0,\ t \in T,$$

and

$$h_s(x) = 0,\ s \in S,$$

so

$$g_t(x^0) + \left[\nabla g_t(x^0)\right]\left(\eta(x, x^0)\right) \leq 0,\ t \in T,$$
$$h_s(x^0) + \left[\nabla h_s(x^0)\right]\left(\eta(x, x^0)\right) = 0,\ s \in S.$$

Consequently,

$$x \in \mathcal{L}.$$

□

Theorem 2.3. *Let X be a subset of $\mathbb{R}^n$, x^0 be an interior point of X, $\eta : X \times X \to X$ and f, g_t, $h_s : X \to \mathbb{R}$, $t \in T$, $s \in S$. Assume that:*
(a) for each $t \in T$, the function g_t is differentiable at x^0 and incave at x^0 w.r.t. η,
(b) for each $s \in S$, the function h_s is differentiable at x^0 and avex at x^0 w.r.t. η.

If

$$\mathcal{L} := \{x \in X : \quad g_t(x^0) + \left[\nabla g_t(x^0)\right]\left(\eta(x, x^0)\right) \leq 0,\ t \in T \text{ and } h_s(x^0) + \left[\nabla h_s(x^0)\right]\left(\eta(x, x^0)\right) = 0,\ s \in S\},$$

then

$$\mathcal{L} \subseteq \mathcal{F}(P).$$

Proof. Let $x \in \mathcal{L}$. This is equivalent with

$$\left[\nabla g_t(x^0)\right]\left(\eta(x, x^0)\right) + g_t(x^0) \leq 0,\ t \in T, \tag{2.1}$$

$$\left[\nabla h_s(x^0)\right]\left(\eta(x, x^0)\right) + h_s(x^0) = 0,\ s \in S. \tag{2.2}$$

From the hypotheses (a) and (b) we have

$$g_t(x) - g_t(x^0) \leq \left[\nabla g_t(x^0)\right]\left(\eta(x, x^0)\right),\ t \in T,$$

and

$$h_s(x) - h_s(x^0) = \left[\nabla h_s(x^0)\right]\left(\eta(x, x^0)\right), s \in S.$$

Now, from (2.1) and (2.2), we obtain

$$g_t(x) \leq 0,\ t \in T,$$

and

$$h_s(x) = 0,\ s \in S.$$

Hence,

$$x \in \mathcal{F}(P). \qquad \square$$

3. Main results

In this paragraph we present some connections between the optimal solutions of semi-infinite optimization problems (P) and $(P_{1,0})$, $(P_{0,1})$ and $(P_{1,1})$.

3.1. Approximate problem $(P_{1,0})$

For $(1,0)$-η approximated type we have the following results:

Theorem 3.1. *Let X be a subset of $\mathbb{R}^n$, x^0 be an interior point of X, $\eta : X \times X \to X$ and f, g_t, $h_s : X \to \mathbb{R}$, $t \in T$, $s \in S$. Assume that:*
(a) the function f is differentiable at x^0 and pseudoinvex at x^0 w.r.t. η,
(b) $\eta(x^0, x^0) = 0$.

If x^0 is an optimal solution for Problem $(P_{1,0})$, then x^0 is an optimal solution for Problem (P).

Proof. Obviously $\mathcal{F}(P) = \mathcal{F}(P_{1,0})$. On the other hand, the point x^0 is an optimal solution for $(P_{1,0})$, then $x^0 \in \mathcal{F}(P_{1,0})$ and

$$\begin{aligned} f(x^0) + \left[\nabla f(x^0)\right]\left(\eta(x^0, x^0)\right) \leq \\ \leq f(x^0) + \left[\nabla f(x^0)\right]\left(\eta(x, x^0)\right), \text{ for all } x \in \mathcal{F}(P_{1,0}). \end{aligned} \tag{3.1}$$

From (b) and (3.1) we obtain:

$$\left[\nabla f(x^0)\right]\left(\eta(x, x^0)\right) \geq 0, \text{ for all } x \in \mathcal{F}(P_{1,0}) = \mathcal{F}(P). \tag{3.2}$$

Now from (a) and (3.2) it follows:

$$f(x^0) \leq f(x), \text{ for all } x \in \mathcal{F}(P).$$

Hence x^0 is an optimal solution for Problem (P). $\square$

Theorem 3.2. *Let X be a subset of $\mathbb{R}^n$, x^0 be an interior point of X, $\eta : X \times X \to X$ and f, g_t, $h_s : X \to \mathbb{R}$, $t \in T$, $s \in S$. Assume that:*
(a) the function f is differentiable at x^0 and quasi-incave at x^0 w.r.t. η,
(b) $\eta(x^0, x^0) = 0$.

If x^0 is an optimal solution for Problem (P), then x^0 is an optimal solution for Problem $(P_{1,0})$.

Proof. Obviously $\mathcal{F}(P) = \mathcal{F}(P_{1,0})$. On the other hand, the point x^0 is an optimal solution for (P), then $x^0 \in \mathcal{F}(P)$ and

$$f(x^0) \leq f(x), \text{ for all } x \in \mathcal{F}(P). \tag{3.3}$$

Suppose that x^0 is not the optimal solution for Problem $(P_{1,0})$, which implies that there exists $x^1 \in \mathcal{F}(P_{1,0})$ such that

$$f(x^0) + \left[\nabla f(x^0)\right]\left(\eta(x^1, x^0)\right) < f(x^0) + \left[\nabla f(x^0)\right]\left(\eta(x^0, x^0)\right). \tag{3.4}$$

From (3.4) and (b) it follows:

$$\left[\nabla f(x^0)\right]\left(\eta(x^1, x^0)\right) < 0.$$

From (a) we obtain

$$f(x^1) < f(x^0),$$

which contradicts the optimality of x^0 for Problem (P).
Hence x^0 is an optimal solution for Problem $(P_{1,0})$. □

3.2. Approximate problem $(P_{0,1})$

For $(0,1)$-η approximated type we have the following results:

Theorem 3.3. *Let X be a subset of $\mathbb{R}^n$, x^0 be an interior point of X, $\eta : X \times X \to X$ and f, g_t, $h_s : X \to \mathbb{R}$, $t \in T$, $s \in S$. Assume that:*
(a) for each $t \in T$, the function g_t is differentiable at x^0 and invex at x^0 w.r.t. η,
(b) for each $s \in S$, the function h_s is differentiable at x^0 and avex at x^0 w.r.t. η,
(c) $x^1 \in \mathcal{F}(P)$.

If x^1 is an optimal solution for Problem $(P_{0,1})$, then x^1 is an optimal solution for Problem (P).

Proof. The point x^1 is an optimal solution for $(P_{0,1})$, we have

$$f(x^1) \leq f(x), \text{ for all } x \in \mathcal{F}(P_{0,1}). \tag{3.5}$$

By Theorem 2.2, we have

$$\mathcal{F}(P) \subseteq \mathcal{F}(P_{0,1}). \tag{3.6}$$

From (c), (3.5) and (3.6) we obtain

$$f(x^1) \leq f(x), \text{for all } x \in \mathcal{F}(P).$$

Hence, x^1 is an optimal solution for Problem (P). □

Theorem 3.4. *Let X be a subset of $\mathbb{R}^n$, x^0 be an interior point of X, $\eta : X \times X \to X$ and f, g_t, $h_s : X \to \mathbb{R}$, $t \in T$, $s \in S$. Assume that:*
(a) for each $t \in T$, the function g_t is differentiable at x^0 and incave at x^0 w.r.t. η,
(b) for each $s \in S$, the function h_s is differentiable at x^0 and avex at x^0 w.r.t. η,
(c) $x^1 \in \mathcal{F}(P_{0,1})$.

If x^1 is an optimal solution for Problem (P), then x^1 is an optimal solution for Problem $(P_{0,1})$.

Proof. The point x^1 is an optimal solution for (P), we have

$$f(x^1) \leq f(x), \text{ for all } x \in \mathcal{F}(P). \tag{3.7}$$

By Theorem 2.3, we have

$$\mathcal{F}(P_{0,1}) \subseteq \mathcal{F}(P). \tag{3.8}$$

From (c), (3.7) and (3.8) we obtain

$$f(x^1) \leq f(x), \text{for all } x \in \mathcal{F}(P_{0,1}).$$

Hence, x^1 is an optimal solution for Problem $(P_{0,1})$. □

3.3. Approximate problem $(P_{1,1})$

For $(1,1)$-η approximated type we have the following results:

Theorem 3.5. *Let X be a subset of $\mathbb{R}^n$, x^0 be an interior point of X, $\eta : X \times X \to X$ and f, g_t, $h_s : X \to \mathbb{R}$, $t \in T$, $s \in S$. Assume that:*
(a) the function f is differentiable at x^0 and pseudoinvex at x^0 w.r.t. η,
(b) for each $t \in T$, the function g_t is differentiable at x^0 and invex at x^0 w.r.t. η,
(c) for each $s \in S$, the function h_s is differentiable at x^0 and avex at x^0 w.r.t. η,
(d) $x^0 \in \mathcal{F}(P)$,
(e) $\eta(x^0, x^0) = 0$.

If x^0 is an optimal solution for Problem $(P_{1,1})$, then x^0 is an optimal solution for Problem (P).

Proof. The point x^0 is an optimal solution for $(P_{1,1})$, we have

$$\begin{aligned} f(x^0) + \left[\nabla f(x^0)\right] \left(\eta(x^0, x^0)\right) \leq \\ \leq f(x^0) + \left[\nabla f(x^0)\right] \left(\eta(x, x^0)\right), \text{ for all } x \in \mathcal{F}(P_{1,1}). \end{aligned} \tag{3.9}$$

By Theorem 2.2, we have

$$\mathcal{F}(P) \subseteq \mathcal{F}(P_{1,1}).$$

From (e) and (3.9) we obtain:

$$\left[\nabla f(x^0)\right] \left(\eta(x, x^0)\right) \geq 0, \text{ for all } x \in \mathcal{F}(P_{1,1}). \tag{3.10}$$

Now from (a) and (3.10) it follows:

$$f(x^0) \leq f(x), \text{ for all } x \in \mathcal{F}(P_{1,1}),$$

then, from (d),

$$f(x^0) \leq f(x), \text{ for all } x \in \mathcal{F}(P).$$

Hence x^0 is an optimal solution for Problem (P). □

Theorem 3.6. *Let X be a subset of $\mathbb{R}^n$, x^0 be an interior point of X, $\eta : X \times X \to X$ and f, g_t, $h_s : X \to \mathbb{R}$, $t \in T$, $s \in S$. Assume that:*
(a) the function f is differentiable at x^0 and quasi-incave at x^0 w.r.t. η,
(b) for each $t \in T$, the function g_t is differentiable at x^0 and incave at x^0 w.r.t. η,
(c) for each $s \in S$, the function h_s is differentiable at x^0 and avex at x^0 w.r.t. η,
(d) $x^0 \in \mathcal{F}(P_{1,1})$,
(e) $\eta(x^0, x^0) = 0$.

If x^0 is an optimal solution for Problem (P), then x^0 is an optimal solution for Problem $(P_{1,1})$.

Proof. The point x^0 is an optimal solution for (P), we have

$$f(x^0) \leq f(x), \text{ for all } x \in \mathcal{F}(P). \tag{3.11}$$

By Theorem 2.3, we have

$$\mathcal{F}(P_{1,1}) \subseteq \mathcal{F}(P).$$

From (3.11) and (a) it follows:

$$\left[\nabla f(x^0)\right] \left(\eta(x, x^0)\right) \geq 0, \text{ for all } x \in \mathcal{F}(P),$$

hence

$$\left[\nabla f(x^0)\right]\left(\eta(x,x^0)\right) \geq 0, \text{ for all } x \in \mathcal{F}(P_{1,1}).$$

Consequently,

$$f(x^0) + \left[\nabla f(x^0)\right]\left(\eta(x^0,x^0)\right) \leq$$
$$\leq f(x^0) + \left[\nabla f(x^0)\right]\left(\eta(x,x^0)\right), \text{for all } x \in \mathcal{F}(P_{1,1}).$$

Hence x^0 is an optimal solution for Problem $(P_{1,1})$. □

Theorem 3.7. *Let X be a subset of $\mathbb{R}^n$, x^0 be an interior point of X, $\eta : X \times X \to X$ and f, g_t, $h_s : X \to \mathbb{R}$, $t \in T$, $s \in S$. Assume that:*
(a) the function f is differentiable at x^0 and quasi-incave at x^0 w.r.t. η,
(b) for each $t \in T$, the function g_t is differentiable at x^0,
(c) for each $s \in S$, the function h_s is differentiable at x^0,
(d) $\eta(x^0,x^0) = 0$.

If x^0 is an optimal solution for Problem $(P_{0,1})$, then x^0 is an optimal solution for Problem $(P_{1,1})$.

Proof. Obviously $\mathcal{F}(P_{0,1}) = \mathcal{F}(P_{1,1})$. On the other hand, the point x^0 is an optimal solution for $(P_{0,1})$, then

$$f(x^0) \leq f(x), \text{ for all } x \in \mathcal{F}(P_{0,1}). \tag{3.12}$$

From (3.12) and (a) it follows:

$$\left[\nabla f(x^0)\right]\left(\eta(x,x^0)\right) \geq 0, \text{ for all } x \in \mathcal{F}(P_{0,1}). \tag{3.13}$$

Now from (d) and (3.13) it follows

$$f(x^0) + \left[\nabla f(x^0)\right]\left(\eta(x^0,x^0)\right) \leq f(x^0) + \left[\nabla f(x^0)\right]\left(\eta(x,x^0)\right),$$

for all $x \in \mathcal{F}(P_{0,1}) = \mathcal{F}(P_{1,1})$.
Hence x^0 is an optimal solution for Problem $(P_{1,1})$. □

Theorem 3.8. *Let X be a subset of $\mathbb{R}^n$, x^0 be an interior point of X, $\eta : X \times X \to X$ and f, g_t, $h_s : X \to \mathbb{R}$, $t \in T$, $s \in S$. Assume that:*
(a) the function f is differentiable at x^0 and pseudoinvex at x^0 w.r.t. η,
(b) for each $t \in T$, the function g_t is differentiable at x^0,
(c) for each $s \in S$, the function h_s is differentiable at x^0,
(d) $\eta(x^0,x^0) = 0$.

If x^0 is an optimal solution for Problem $(P_{1,1})$, then x^0 is an optimal solution for Problem $(P_{0,1})$.

Proof. Obviously $\mathcal{F}(P_{0,1}) = \mathcal{F}(P_{1,1})$. On the other hand, the point x^0 is an optimal solution for $(P_{1,1})$, then

$$f(x^0) + \left[\nabla f(x^0)\right]\left(\eta(x^0,x^0)\right) \leq f(x^0) + \left[\nabla f(x^0)\right]\left(\eta(x,x^0)\right), \quad \text{for all } x \in \mathcal{F}(P_{1,1}). \tag{3.14}$$

From (3.14) and (d) it follows:

$$\left[\nabla f(x^0)\right]\left(\eta(x,x^0)\right) \geq 0, \text{ for all } x \in \mathcal{F}(P_{1,1}).$$

Now from (a) and (3.14) it follows

$$f(x^0) \leq f(x), \text{ for all } x \in \mathcal{F}(P_{0,1}) = \mathcal{F}(P_{1,1}).$$

Hence x^0 is an optimal solution for Problem $(P_{0,1})$. □

Theorem 3.9. *Let X be a subset of $\mathbb{R}^n$, x^0 be an interior point of X, $\eta : X \times X \to X$ and f, g_t, $h_s : X \to \mathbb{R}$, $t \in T$, $s \in S$. Assume that:*
(a) the function f is differentiable at x^0,
(b) for each $t \in T$, the function g_t is differentiable at x^0, and invex at x^0 w.r.t. η,
(c) for each $s \in S$, the function h_s is differentiable at x^0, and avex at x^0 w.r.t. η,
(d) $x^1 \in \mathcal{F}(P_{1,0})$.

If x^1 is an optimal solution for Problem $(P_{1,1})$, then x^1 is an optimal solution for Problem $(P_{1,0})$.

Proof. By Theorem 2.2, we have $\mathcal{F}(P_{1,0}) \subseteq \mathcal{F}(P_{1,1})$. Now (d) implies that x^1 is an optimal solution for $(P_{1,0})$. □

Theorem 3.10. *Let X be a subset of $\mathbb{R}^n$, x^0 be an interior point of X, $\eta : X \times X \to X$ and f, g_t, $h_s : X \to \mathbb{R}$, $t \in T$, $s \in S$. Assume that:*
(a) the function f is differentiable at x^0,
(b) for each $t \in T$, the function g_t is differentiable at x^0, and incave at x^0 w.r.t. η,
(c) for each $s \in S$, the function h_s is differentiable at x^0, and avex at x^0 w.r.t. η.
(d) $x^1 \in \mathcal{F}(P_{1,1})$.

If x^1 is an optimal solution for Problem $(P_{1,0})$, then x^1 is an optimal solution for Problem $(P_{1,1})$

Proof. By Theorem 2.3, we have $\mathcal{F}(P_{1,1}) \subseteq \mathcal{F}(P_{1,0})$. Now (d) implies that x^1 is an optimal solution for $(P_{1,1})$. □

Theorem 3.11. *Let X be a subset of $\mathbb{R}^n$, x^0 be an interior point of X, $\eta : X \times X \to X$ and f, g_t, $h_s : X \to \mathbb{R}$, $t \in T$, $s \in S$. Assume that:*
(a) the function f is differentiable at x^0 and pseudoinvex at x^0 w.r.t. η,
(b) for each $t \in T$, the function g_t is differentiable at x^0, and incave at x^0 w.r.t. η,
(c) for each $s \in S$, the function h_s is differentiable at x^0, and avex at x^0 w.r.t. η,
(d) $x^0 \in \mathcal{F}(P_{0,1})$,
(e) $\eta(x^0, x^0) = 0$.

If x^0 is an optimal solution for Problem $(P_{1,0})$, then x^0 is an optimal solution for Problem $(P_{0,1})$.

Proof. By Theorem 2.3, we have $\mathcal{F}(P_{0,1}) \subseteq \mathcal{F}(P_{1,0})$. The point x^0 is an optimal solution for $(P_{1,0})$, then

$$f(x^0) + \left[\nabla f(x^0)\right]\left(\eta(x^0, x^0)\right) \leq f(x^0) + \left[\nabla f(x^0)\right]\left(\eta(x, x^0)\right), \text{ for all } x \in \mathcal{F}(P_{1,0}). \tag{3.15}$$

From (3.15) and (e) it follows:

$$\left[\nabla f(x^0)\right]\left(\eta(x, x^0)\right) \geq 0, \text{ for all } x \in \mathcal{F}(P_{1,0}). \tag{3.16}$$

Now from (a) and (3.16) it follows

$$f(x^0) \leq f(x), \text{ for all } x \in \mathcal{F}(P_{0,1}).$$

Hence x^0 is an optimal solution for Problem $(P_{0,1})$. □

Theorem 3.12. *Let X be a subset of $\mathbb{R}^n$, x^0 be an interior point of X, $\eta : X \times X \to X$ and f, g_t, $h_s : X \to \mathbb{R}$, $t \in T$, $s \in S$. Assume that:*
(a) the function f is differentiable at x^0 and quasi-incave at x^0 w.r.t. η,
(b) for each $t \in T$, the function g_t is differentiable at x^0 and invex at x^0 w.r.t. η,
(c) for each $s \in S$, the function h_s is differentiable at x^0 and avex at x^0 w.r.t. η,
(d) $x^0 \in \mathcal{F}(P_{1,0})$,
(e) $\eta(x^0, x^0) = 0$.

If x^0 is an optimal solution for Problem $(P_{0,1})$, then x^0 is an optimal solution for Problem $(P_{1,0})$

Proof. By Theorem 2.2, we have $\mathcal{F}(P_{1,0}) \subseteq \mathcal{F}(P_{0,1})$. The point x^0 is an optimal solution for $(P_{0,1})$, then

$$f(x^0) \le f(x), \text{ for all } x \in \mathcal{F}(P_{0,1}). \tag{3.17}$$

Assume that $x^0 \in \mathcal{F}(P_{1,0})$ is not the optimal solution for $(P_{1,0})$, then there exists $x^1 \in \mathcal{F}(P_{1,0})$ such that

$$f(x^0) + \left[\nabla f(x^0)\right]\left(\eta(x^0, x^0)\right) > f(x^0) + \left[\nabla f(x^0)\right]\left(\eta(x^1, x^0)\right) \tag{3.18}$$

From (3.18) and (e) it follows:

$$\left[\nabla f(x^0)\right]\left(\eta(x^1, x^0)\right) < 0. \tag{3.19}$$

Now from (a) it follows

$$f(x^1) < f(x^0).$$

which contradicts the optimality of x^0 for Problem $(P_{0,1})$. □

4. Conclusions

In this paper, three problems $(P_{1,0})$, $(P_{0,1})$ and $(P_{1,1})$ are presented, whose solutions give us information about the solutions of semi-infinite optimization problem (P).

Acknowledgements. This work was possible with the financial support of the Sectoral Operational Programme for Human Resources Development 2007-2013, co-financed by the European Social Fund, under the project number POSDRU/107/1.5/S/76841 with the title "Modern Doctoral Studies: Internationalization and Interdisciplinarity".

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Augusta Raţiu
Faculty of Mathematics and Computer Science
Babeş-Bolyai University, 1 Mihail Kogălniceanu
400084 Cluj-Napoca, Romania
e-mail: augu2003@yahoo.com

Dorel I. Duca
Faculty of Mathematics and Computer Science
Babeş-Bolyai University, 1 Mihail Kogălniceanu
400084 Cluj-Napoca, Romania
e-mail: dduca@math.ubbcluj.ro

Stud. Univ. Babeş-Bolyai Math. 58(2013), No. 3, 413–413

Retraction notice

Retraction notice to: ”The Riemann hypothesis” [Stud. Univ. Babeş-Bolyai Math. 57 (2012), No. 2, 195-208]

Dorin Ghisa
York University, Glendon College, 2275 Bayview Avenue, Toronto, Canada

Update of Dorin Ghisa
RETRACTED: The Riemann hypothesis
Stud. Univ. Babeş-Bolyai Math. 57 (2012), No. 2, 195-208.

This article has been retracted at the request of the Editors.
As a result of negative feedback from a number of readers, the editors asked the opinion of several specialists. The conclusion has been that at certain points, insufficient argumentation was provided in order to sustain the final major conclusion of the paper.
Apologies are offered to the readers of the journal that this was not noticed during the peer review process.

Stud. Univ. Babeş-Bolyai Math. 58(2013), No. 3, 415–417

Book reviews

Smaïl Djebali, Lech Górniewicz and Abdelghani Ouahab, Solution sets for differential equations and inclusions, Series in Nonlinear Analysis and Applications, Vol. 18, xix + 453 pp, Walter de Gruyter, Berlin - New York, 2013, ISBN: 978-3-11-029344-9, e-ISBN: 978-3-11-029356-2, ISSN: 0941-813X.

The book is concerned with the topological structure of the solution sets of differential equations and inclusions. with the aim to offer a comprehensive exposition of classical and recent results in this area. This direction of research was initiated by G. Peano in 1890 who proved that the set S of solutions to the Cauchy problem (1) $x'(t) = f(t, x(t))$, $t \in [t_0, a]$, $x(t_0) = x_0$, where $f : [0, a] \times \mathbb{R} \to \mathbb{R}$ is a continuous function, has the property that the sections $S(t) = \{x(t) : x \in S\}$ are nonempty, compact and connected for every t in a neighborhood of t_0. The result was successively extended, first by Kneser in 1923 to n dimensions, and then to various settings by other mathematicians – Hukuhara (1928), Aronszajn (1942). A turning point in this study was represented by a paper from 1969 by F. Browder and C. Gupta, a result that appears recurrently, in various hypostases, throughout the book.

The topological properties of the solution sets considered by the authors are acyclicity, the AR (absolute retract) property, being an R_δ-set (i.e., the intersection of a decreasing sequence of compact absolute retracts). The study is done in the first chapter, *Topological structure of fixed point sets*, in the more general context of fixed point sets for various kinds of mappings or of operator equations. This chapter contains also the proofs of some fundamental fixed point theorems for single-valued mappings (Banach, Brouwer, Schauder), as well as for set-valued mappings. The second chapter, *Existence theory for differential equations and inclusions,* contains the fundamental theorems of Picard-Lindelöf, Peano and Carathéodori, as well as Nagumo's results on the existence of solutions of differential equations on non-compact intervals and of differential inclusions.

The core of the book is formed by the chapters 3, *Solution sets for differential equations and inclusions,* and 4, *Impulsive differential inclusions and solution sets,* where the authors systematically examine the topological behavior of the solutions sets in various situations.

In order to make the book self-contained, the authors have included some auxiliary material from algebraic topology in Chapter 5, *Preliminary notions of topology*, and set-valued analysis in Chapter 6, *Background in multi-valued analysis,* completed in Appendix with other results on compactness and weak compactness in function spaces, the Bochner integral, C_0-semigroups.

The book ends with a rich bibliography, counting 506 items and including lots by the authors.

Written by experts and including many of their results and of their co-workers, and presenting in a unitary way a lot of results, both classical and recent, scattered to various publications, this research monograph will become an indispensable tool for the researchers in nonlinear ordinary and partial differential equations and inclusions, applied topology and topological fixed point theory. The self-contained style of exposition adopted by the authors, with a careful presentation of the needed background material, makes the book of great help for those desiring to start research in this field, too.

Radu Precup

Carlos Boss and Charles Schwartz, Functional Calculi, World Scientific, London - Singapore, 2013, x+215 pp, ISBN 978-981-4415-97-2.

The book is devoted to an exposition of functional calculi for various classes of linear operators, including the background material needed for the presentation. By a functional calculus one understands a construction which associates to an operator, or to a family of commuting operators, a homomorphism from a function space into a subspace of continuous linear operators. The simplest case is that of a polynomial $p(z) = \sum_{k=0}^{n} a_k z^k$ and an operator A on a Banach space X, to which one associates the operator $p(A) = \sum_{k=0}^{n} a_k A^k$. The so defined correspondence, $p \mapsto p(A)$, is a homomorphism from the algebra $\mathcal{P}$ of polynomials into the algebra $L(X)$ of continuous linear operators on X, and any functional calculus should be an extension of this homomorphism. The most familiar example is that of a continuous self adjoint linear operator A on a complex Hilbert space H, in which case there exists a projection valued measure E defined on the σ-algebra of Borel subsets of the spectrum $\sigma(A)$ of A, such that $A = \int_{\sigma(A)} z dE(z)$. This case is treated in Chapter 2, *Functions of a self adjoint operator*, with the extension to several commuting self adjoint operators given in Chapter 3. The fourth chapter is concerned with the spectral theorem for normal operators. The background material is developed in Chapter 1, *Vector and operator valued measures,* which presents the integration of scalar functions with respect to vector measures, with a stretch on operator valued measures and, in particular, on projection valued measures. The integration of vector functions with respect to scalar measures is treated in Chapter 5. The sixth chapter, *An abstract operational calculus*, is concerned with an axiomatic approach to operational calculus. The Riesz functional calculus, based on the theory of analytic vector functions, in particular on Cauchy's formula for such functions, is developed in the seventh chapter. The preamble of this chapter contains a brief but thorough presentation of basic results on vector analytic functions, including a proof, due to S. Grabiner (1976), of Runge's approximation theorem.

The last chapter of the book, Chapter 8, *Weyl's functional calculus*, is concerned with a functional calculus devised by H. Weyl and having its origins in quantum mechanics. The key tool is the so called Weyl transform (the authors use a slight modification of the original one), based on the Fourier transform of vector tempered distributions.

The basic text is completed by five appendices: A. *The Orlicz-Pettis theorem* (on unconditionally convergent series in Banach spaces), B. *The spectrum of an operator*, C. *Self adjoint, normal and unitary operators*, D. *Sesquilinear functionals,* and E. *Tempered distributions and the Fourier transform.* The most consistent of these is the last one (30 pages) which contains a quick presentation (with proofs) of the basic results in this area – the spaces $\mathcal{S}(\mathbb{R}^n)$ and $\mathcal{S}'(\mathbb{R}^n)$, the Fourier transform of functions and of tempered distributions, the Paley-Wiener theorem.

The bibliography contains 47 items, mainly textbooks, but some fundamental papers in spectral theory, or containing simpler proofs of known results, are included as well.

The book presents, in an accessible, self-contained way and in a relatively small number of pages, some basic results in the spectral theory of linear operators on Banach or on Hilbert space. Of great help is the auxiliary material which prevent the reader to lose time by looking through various treatises on functional analysis or measure theory. The book can be used for courses on functional analysis and operator theory, or for self-study, as an introduction to the subject.

Tiberiu Trif