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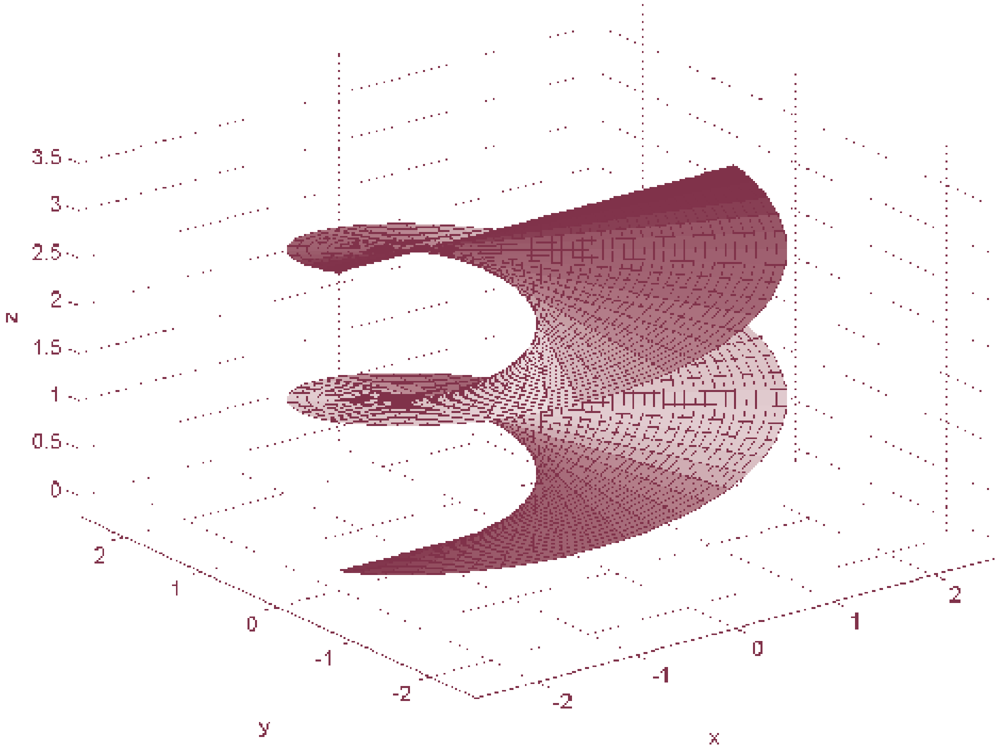

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Primes of the form $\pm a^2 \pm qb^2$

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To the memory of Professor Mircea-Eugen Craioveanu (1942-2012)

Abstract. Representations of primes by simple quadratic forms, such as $\pm a^2 \pm qb^2$, is a subject that goes back to Fermat, Lagrange, Legendre, Euler, Gauss and many others. We are interested in a comprehensive list of such results, for $q \leq 20$. Some of the results can be established with elementary methods and we exemplify that in some instances. We are introducing new relationships between various representations.

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1. Introduction

Let us consider the following three types of representations for a natural number:

$$\mathcal{E}(q) := \{n \in \mathbb{N} | n = a^2 + qb^2, \text{ with } a, b \in \mathbb{Z}\}, \tag{1.1}$$

$$\mathcal{H}_1(q) := \{n \in \mathbb{N} | n = qb^2 - a^2, \text{ with } a, b \in \mathbb{Z}\}, \text{ and} \tag{1.2}$$

$$\mathcal{H}_2(q) := \{n \in \mathbb{N} | n = a^2 - qb^2, \text{ with } a, b \in \mathbb{Z}\}. \tag{1.3}$$

We are going to denote by $\mathcal{P}$ the set of prime numbers. In this paper we want to exemplify how standard elementary methods can be used to obtain the representations stated in the next three theorems:

Theorem 1.1. *For a prime p we have $p \in \mathcal{E}(q)$ if and only if*

(i) *(Fermat) $(q = 1)$ $p = 2$ or $p \equiv 1 \pmod 4$*
(ii) *(Fermat) $(q = 2)$ $p = 2$ or $p \equiv 1$ or $3 \pmod 8$*
(iii) *(Fermat-Euler) $(q = 3)$ $p = 3$ or $p \equiv 1 \pmod 6$*
(iv) *$(q = 4)$ $p \equiv 1 \pmod 4$*
(v) *(Lagrange) $(q = 5)$ $p = 5$ or $p \equiv j^2 \pmod{20}$ for some $j \in \{1, 3\}$*
(vi) *$(q = 6)$ $p \equiv 1$ or $7 \pmod{24}$*
(vii) *$(q = 7)$ $p = 7$ or $p \equiv j^2 \pmod{14}$ for some $j \in \{1, 3, 5\}$*
(viii) *$(q = 8)$ $p \equiv 1 \pmod 8$*
(ix) *$(q = 9)$ $p \equiv j^2 \pmod{36}$ for some $j \in \{1, 5, 7\}$*

(X) *($q = 10$) $p \equiv j$ (mod 40) for some $j \in \{1, 9, 11, 19\}$*
(XI) *($q = 12$) $p \equiv j$ (mod 48) for some $j \in \{1, 13, 25, 37\}$*
(XII) *($q = 13$) $p \equiv j^2$ (mod 52) for some $j \in \{1, 3, 5, 7, 9, 11\}$*
(XIII) *($q = 15$) $p \equiv j$ (mod 60) for some $j \in \{1, 19, 31, 49\}$*
(XIV) *($q = 16$) $p \equiv 1$ (mod 8)*

We are going to prove (VII), in order introduce the method that will be employed several times. One may wonder what is the corresponding characterization for $q = 11$ or $q = 14$. It turns out that an answer cannot be formulated only in terms of residue classes as shown in ([19]). We give in Theorem 1.4 possible characterizations whose proofs are based on non-elementary techniques which are described in [6].

Theorem 1.2. *For a prime p we have $p \in \mathcal{H}_1(q)$ if and only if*
(I) *($q = 1$) $p \neq 2$*
(II) *($q = 2$) $p = 2$ or $p \equiv \pm 1$ (mod 8) (i.e. $p \equiv 1$ or $p \equiv 1$ (mod 8))*
(III) *($q = 3$) $p \in \{2, 3\}$ or $p \equiv 11$ (mod 12)*
(IV) *($q = 4$) $p \equiv 3$ (mod 4)*
(V) *($q = 5$) $p = 5$ or $p \equiv \pm j^2$ (mod 20) for some $j \in \{1, 3\}$*
(VI) *($q = 6$) $p = 2$ or $p \equiv j$ (mod 24) for some $j \in \{5, 23\}$*
(VII) *($q = 7$) $p = 7$ or $p \equiv j$ (mod 14) for some $j \in \{3, 5, 13\}$*
(VIII) *($q = 8$) $p = 7$ or $p \equiv -j^2$ (mod 32) for some $j \in \{1, 3, 5, 7\}$*
(IX) *($q = 9$) $p \equiv -1$ (mod 6)*
(X) *($q = 10$) $p \equiv j$ (mod 40) for some $j \in \{1, 9, 31, 39\}$*
(XI) *($q = 11$) $p \in \{2, 11\}$ or $p \equiv -j^2$ (mod 44) for some $j \in \{1, 3, 5, 7, 9\}$*

In this case, for exemplification, we show (V).

Theorem 1.3. *For a prime p we have $p \in \mathcal{H}_2(q)$ if and only if*
(I) *($q = 1$) $p \neq 2$*
(II) *($q = 2$) $p = 2$ or $p \equiv \pm 1$ (mod 8)*
(III) *($q = 3$) $p \equiv 1$ (mod 12)*
(IV) *($q = 4$) $p \equiv 1$ (mod 4)*
(V) *($q = 5$) $p = 5$ or $p \equiv \pm j^2$ (mod 20) for some $j \in \{1, 3\}$*
(VI) *($q = 6$) $p = 3$ or $p \equiv j$ (mod 24) for some $j \in \{1, 19\}$*
(VII) *($q = 7$) $p = 2$ or $p \equiv j$ (mod 14) for some $j \in \{1, 9, 11\}$*
(VIII) *($q = 8$) $p = 7$ or $p \equiv j^2$ (mod 32) for some $j \in \{1, 3, 5, 7\}$*
(IX) *($q = 9$) $p \equiv 1$ (mod 6)*
(X) *($q = 10$) $p \equiv j$ (mod 40) for some $j \in \{1, 9, 31, 39\}$*
(XI) *($q = 11$) $p \equiv j^2$ (mod 44) for some $j \in \{1, 3, 5, 7, 9\}$*

We observe that for $q = 2$, $q = 5$, $q = 10$ the same primes appear for both characterizations in Theorem 1.2 and Theorem 1.3. There are several questions that can be raised in relation to this observation:

Problem 1. Determine all values of q, for which we have

$$\mathcal{H}_1(q) \cap \mathcal{P} = \mathcal{H}_2(q) \cap \mathcal{P}. \tag{1.4}$$

Problem 2. If the equality (1.4) holds true for relatively prime numbers q_1 and q_2, does is it hold true for $q_1 q_2$?

In [6], David Cox begins his classical book on the study of (1.1), with a detailed and well documented historical introduction of the main ideas used and the difficulties encountered in the search of new representations along time. The following abstract characterization in [6] brings more light into this subject:

(Theorem 12.23 in [6]**)** *Given a positive integer q, there exists an irreducible polynomial with integer coefficients f_q of degree $h(-4q)$, such that for every odd prime p not dividing q,*

$$p = a^2 + qb^2 \Leftrightarrow \text{the equations} \begin{cases} x^2 \equiv -q \ (\text{mod } p) \\ \\ f_q(x) \equiv 0 \ (\text{mod } p) \end{cases}$$

have integer solutions. An algorithm for computing f_q exists. ($h(D)$ is the number of classes of primitive positive definite quadratic forms of discriminant D).

While some of the representations included here are classical, others may be more or less known. We found some of the polynomials included here by computational experimentations. For more details in this direction see [1], [2], [5], [6], [7], [15] and [19].

Theorem 1.4. *For an odd prime p we have $p = a^2 + qb^2$ for some integers a, b if and only if*

(I) *($q = 11$) $p > 2$ and the equation*

$$(X^3 + 2X)^2 + 44 \equiv 0 \ (\text{mod } p) \quad \text{has a solution,}$$

(II) *(Euler's conjecture) ($q = 14$) the equations*

$$X^2 + 14 \equiv 0 \text{ and } (X^2 + 1)^2 - 8 \equiv 0 \ (\text{mod } p) \text{ have solutions}$$

(III) *($q = 17$) the equations $X^2 + 17 \equiv 0$ and $(X^2 - 1)^2 + 16 \equiv 0$ (mod p) have solutions*

(IV) *($q = 18$) the equation $(X^2 - 3)^2 + 18(2^2) \equiv 0$ (mod p) has a solution*

(V) *($q = 19$) the equation $(X^3 - 4x)^2 + 19(4^2) \equiv 0$ (mod p) has a solution*

(VI) *($q = 20$) the equation $(X^4 - 4)^2 + 20X^4 \equiv 0$ (mod p) has a solution*

(XXI) *($q = 21$) the equation $(X^4 + 4)^2 + 84X^4 \equiv 0$ (mod p) has a solution*

(XXII) *($q = 22$) $p > 22$ and the equation $(x^2 + 3)^2 + 22(4^2) \equiv 0$ (mod p) has a solution*

(XXIII) *($q = 23$) the equation $(X^3 + 15X)^2 + 23(19^2) \equiv 0$ (mod p) has a solution*

(XXIV) *($q = 24$) the equation $(X^4 + 4)^2 + 24(2X)^4 \equiv 0$ (mod p) has a solution*

(XXV) *($q = 25$) $p > 25$ the equation $X^4 + 100 \equiv 0$ (mod p) has a solution*

(XXVI) *($q = 26$)*

(XXVII) *(Gauss) ($q = 27$) $p \equiv 1$ (mod 3) and the equation $X^3 \equiv 2$ (mod p) has a solution*

(XXVIII) *($q = 28$)*

(XXVIV) *($q = 29$) $p \equiv 1$ (mod 4) and the equation $(X^3 - X)^2 + 116 = 0$ (mod p)* *has a solution*

(xxx) *($q = 30$)*

(xxxi) *($q = 31$) (L. Kronecker, pp. 88* [6]*) the equation*

$$(X^3 - 10X)^2 + 31(X^2 - 1)^2 \equiv 0 \ (mod\ p) \ \ \textit{has a solution}$$

(xxxii) *($q = 32$) $p \equiv 1$ (mod 8) and the equation*

$$(X^2 - 1)^2 \equiv -1 \ (mod\ p) \ \textit{has a solution.}$$

(xxxvii) *($q = 37$) the equation $X^4 + 31X^2 + 9 = 0$ (mod p) has a solution*

(lxiv) *(Euler's conjecture) ($q = 64$) $p \equiv 1$ (mod 4) and the equation $X^4 \equiv 2$ (mod p) has a solution.*

Our interest in this subject came from studding the problem of finding all equilateral triangles, in the three dimensional space, having integer coordinates for their vertices (see [3], [8], [9], and [12]). It turns out that such equilateral triangles exist only in planes $\mathcal{P}_{a,b,c,f} := \{(x, y, z) \in \mathbb{R}^3 : ax + by + cz = f, f \in \mathbb{Z}\}$ where a, b, and c are in such way

$$a^2 + b^2 + c^2 = 3d^2 \tag{1.5}$$

for some integer d and side-lengths of the triangles are of the form

$$\ell = d\sqrt{2(m^2 - mn + n^2)}$$

for some integers m and n. Let us include here a curious fact that we ran into at that time.

Proposition 1.5. [8] *An integer t which can be written as $t = 3x^2 - y^2$ with $x, y \in \mathbb{Z}$ is the sum of two squares if and only if t is of the form $t = 2(m^2 - mn + n^2)$ for some integers m and n.*

If we introduce the sets

$$A := \{t \in \mathbb{Z} | t = 3x^2 - y^2, x, y \in \mathbb{Z}\},$$

$$B := \{t \in \mathbb{Z} | t = x^2 + y^2, x, y \in \mathbb{Z}\}$$

and

$$C := \{t \in \mathbb{Z} | t = 2(x^2 - xy + y^2), x, y \in \mathbb{Z}\}$$

then we actually have an interesting relationship between these sets.

Theorem 1.6. *For the sets defined above, one has the inclusions*

$$A \cap B \subsetneqq C, \quad B \cap C \subsetneqq A, \quad \textit{and} \quad C \cap A \subsetneqq B. \tag{1.6}$$

We include a proof of this theorem in the Section 4. The inclusions in (1.6) are strict as one can see from Figure 1.

Let us observe that there are primes p with the property that $2p$ is in all three sets A, B and C. We will show that these primes are the primes of the form $12k + 1$

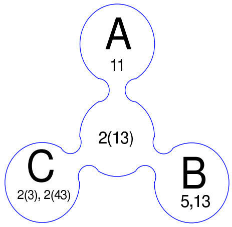

FIGURE 1. *"God created the integers, all else is the work of man." Leopold Kronecker*

for some integer k. Some representations for such primes are included next:

$$13 = (1^2 + 5^2)/2 = 3^2 - 3(4) + 4^2 = [3(3^2) - 1]/2$$

$$37 = (5^2 + 7^2)/2 = 3^2 - 3(7) + 7^2 = [3(5)^2 - 1]/2 \tag{1.7}$$

$$61 = (1^2 + 11^2)/2 = 4^2 - 4(9) + 9^2 = [3(9^2) - 11^2]/2.$$

It is natural to ask whether or not the next forms in the Theorem 1.1 aren't related to similar parameterizations for regular or semi-regular simplices in $\mathbb{Z}^n$ for bigger values of n. In [20], Isaac Schoenberg gives a characterization of those n's for which a regular simplex exists in $\mathbb{Z}^n$. Let us give the restatement of Schoenberg's result which appeared in [16]: *all n such that $n+1$ is a sum of 1, 2, 4 or 8 odd squares.*

As interesting corollaries of these statements we see that if one prime p has some representation it must have some other type of representation(s). Let us introduce a notation for these classes of primes:

$$\mathcal{P}_q := \{p \ \text{ odd prime} | p = a^2 + qb^2 \text{ for some } a, b \in \mathbb{N}\}.$$

So we have $\mathcal{P}_1 = \mathcal{P}_4$, $\mathcal{P}_8 = \mathcal{P}_{16}$ (Gauss, see [21]), $\mathcal{P}_5 \subset \mathcal{P}_1$, $\mathcal{P}_{10} \subset \mathcal{P}_2$, ...In the same spirit, we must bring to reader's attention, that in the case $q = 32$ there exists a characterization due to Barrucand and Cohn [1], which can be written with our notation as

$$\mathcal{P}_{32} = \{p \mid p \equiv 1 \ (mod\ 8), \text{there exists } x \text{ such that } x^8 \equiv -4 \ (mod\ p)\}.$$

We observe that (**xxxii**) in Theorem 1.4 implies this characterization because $x^8+4 = (x^4-2x^2+2)(x^4+2x^2+2)$ and clearly $(x^2-1)^2+1 = x^4-2x^2+2$. In fact, the two statements are equivalent. Indeed, if a is a solution of $x^8 + 4 \equiv 0$ (mod p) then we either have $x^4 - 2x^2 + 2 \equiv 0$ (mod p) or $x^4 + 2x^2 + 2 \equiv 0$ (mod p). We know that there exists a solution b of $x^2+1 \equiv 0$ (mod p). Hence if $a^4+2a^2+2 \equiv 0$ (mod p) then $(ab)^4 - 2(ab)^2 + 2 \equiv 0$ (mod p) which shows that the equation $x^4 - 2x^2 + 2 \equiv 0$ (mod p) always has a solution.

Also, another classical result along these lines is Kaplansky's Theorem ([14]):

Theorem 1.7. *A prime of the form $16n+9$ is in $\mathcal{P}_{32} \setminus \mathcal{P}_{64}$ or in $\mathcal{P}_{64} \setminus \mathcal{P}_{32}$. For a prime p of the form $16n+1$ we have $p \in \mathcal{P}_{32} \cap \mathcal{P}_{64}$ or $p \notin \mathcal{P}_{64} \cup \mathcal{P}_{32}$.*

For further developments similar to Kaplansky's result we refer to [2]. One can show that the representations in Theorem 1.1 are unique (see Problem 3.23 in [6]).

2. Case (vii)

We are going to use elementary methods in the next three sections and the well known Law of Reciprocity.

Theorem 2.1. **[Gauss]** *For every p and q odd prime numbers we have*

$$\left(\frac{p}{q}\right)\left(\frac{q}{p}\right) = (-1)^{\frac{p-1}{2}\frac{q-1}{2}}. \tag{2.1}$$

with notation $\left(\frac{\cdot}{p}\right)$, defined for every odd prime p and every a coprime with p known as the Legendre symbol:

$$\left(\frac{a}{p}\right) = \begin{cases} 1 \; if \; the \; equation \; x^2 \equiv a \; (mod \; p) \; has \; a \; solution, \\ \\ -1 \; if \; the \; equation \; x^2 \equiv a \; (mod \; p) \; has \; no \; solution \end{cases} \tag{2.2}$$

We think that this method can be used to prove all the statements in Theorem 1.1, Theorem 1.3 and Theorem 1.2. We learned about this next technique from [17] and [18].

Necessity. If $p = x^2 + 7y^2$ then $p \equiv x^2 \pmod 7$. Clearly we may assume $p > 7$. Therefore, x may be assumed to be different of zero. Then the residues of $p \pmod 7$ are 1, 2, or 4. Let us suppose that $p \equiv r \pmod{14}$ with $r \in \{0, 1, 2, ..., 13\}$. Because p is prime, r must be an odd number, not a multiple of 7 and which equals 1, 2 or 4 (mod 7). This leads to only three such residues, i.e. $r \in \{1, 9, 11\}$, which are covered by the odd squares j^2, $j \in \{1, 3, 5\}$.

Sufficiency. We may assume that $p > 2$. Let us use the hypothesis to show that the equation $x^2 = -7$ has a solution. Let p be a prime of the form $14k + r$, $r \in \{1, 9, 11\}$, $k \in \mathbb{N} \cup \{0\}$. By the Quadratic Reciprocity, we have $\left(\frac{7}{p}\right)\left(\frac{p}{7}\right) = (-1)^{\frac{3(p-1)}{2}}$. Since $\left(\frac{-1}{p}\right) = (-1)^{\frac{p-1}{2}}$, then

$$\left(\frac{-7}{p}\right) = (-1)^{\frac{p-1}{2}}\left(\frac{7}{p}\right) = (-1)^{\frac{p-1}{2}+\frac{3(p-1)}{2}}\left(\frac{p}{7}\right) = \left(\frac{r'}{7}\right),$$

where $p = 7(2k') + r', r' \in \{0, 1, .., 6\}$. This shows that if $r' \in \{1, 2, 4\}$ we have a solution x_0 for the equation $x^2 \equiv -7$ (mod p).

Let us now apply the Pigeonhole Principle: we let $m \in \mathbb{N}$ be in such a way that $m^2 < p < (m + 1)^2$. We consider the function $g : \{0, 1, 2, ..., m\} \times \{0, 1, 2, ..., m\} \to \{0, 1, 2, ..., p - 1\}$ defined by $g(u, v) \equiv u + vx_0$ (mod p). Since $(m + 1)^2 > p$, we must have two distinct pairs (a'', b'') and (a', b') such that $g(a'', b'') = g(a', b')$. Then $a'' - a' \equiv (b' - b'')x_0$ (mod p). Then, if we let $a = a'' - a'$, and $b = b' - b''$ we get that $0 < q := a^2 + 7b^2 \equiv b^2(x_0^2 + 7) \equiv 0$ (mod p). But, $q = a^2 + 7b^2 \le m^2 + 7m^2 = 8m^2 < 8p$. It follows that $q \in \{p, 2p, 3p, 4p, 5p, 6p, 7p\}$. We need to eliminate the cases

$q \in \{2p, 3p, 4p, 5p, 6p, 7p\}$. If $q = 7p$ then $7p = a^2 + 7b^2$ which implies that a is a multiple of 7, or $a = 7a'$, which gives $p = b^2 + 7a'^2$ as wanted.

If $q = 3p$, then $q = 3(14k' + r) = 7\ell + s$ where $s \in \{3, 5, 6\}$. But this is impossible because $q \equiv a^2 \pmod 7$. The same argument works if $q = 6p$, because $r' \in \{1, 2, 4\}$ if and only if $6r' \in \{3, 5, 6\} \pmod 7$. Similarly, the case $p = 5p$ is no difference.

If $q = 2p$ or $a^2 + 7b^2 = 2p$ implies that a and b cannot be both odd, since in this case $a^2 + 7b^2$ is a multiple of 8 and $2p$ is not. Therefore a and b must be both even, but that shows that $2p$ is a multiple of 4. Again this is not the case.

Finally, if $q = 4p$ then the argument above works the same way but in the end we just simplify by a 4.

3. Cases $q \in \{11, 17, 19\}$

Given a big prime p, the characterizations in Theorem 1.4 cannot be easily checked. Instead, one can show a similar result that is slightly less but in the same spirit of Theorem 1.1.

Theorem 3.1. *(i) A prime $p > 17$ is of the form $a^2 + 17b^2$ or $2p = a^2 + 17b^2$, for some $a, b \in \mathbb{N}$ if and only if $p \equiv (2j + 1)^2$ (mod 68) for some $j = 0, ..., 7$.*

(ii) The representation of a prime as in part (a) is exclusive, i.e. a prime p cannot be of the form $a^2 + 17b^2$ and at the same time $2p = x^2 + 17y^2$, for some $x, y \in \mathbb{N}$.

Proof. (i)
" $\Rightarrow$ " If the prime p can be written $p = a^2 + 17b^2$ then $p \equiv a^2 \pmod{17}$ with a not divisible by 17. We observe that a and b cannot be both odd or both even. Then $p \equiv 1 \pmod 4$. If $p = 68k + r$ with $r \in \{0, 1, 2, ..., 67\}$ then $r \equiv 1 \pmod 4$, not a multiple of 17 and a quadratic residue modulo 17, i.e. $r = 17\ell + r'$ with $r' \in \{1, 2, 4, 8, 9, 13, 15, 16\}$. This gives $r \in \{1, 9, 13, 21, 25, 33, 49, 53\}$. One can check that these residues are covered in a one-to-one way by the odd squares j^2, $j \in \{1, 3, 5, 7, 9, 11, 13, 15\}$.

If $2p = a^2 + 17b^2$ then $2p \equiv a^2 \pmod{17}$ with a not divisible by 17. In this case a and b must be both odd and then $2p = a^2 + 17b^2 \equiv 2 \pmod 8$. This implies, as before, that $p \equiv 1 \pmod 4$. If $p = 68k + r$ with $r \in \{0, 1, 2, ..., 67\}$ then $r \equiv 1$ (mod 4), not divisible by 17 and $2r$ is a quadratic residue modulo 17. Interestingly enough, we still have $r \in \{1, 9, 13, 21, 25, 33, 49, 53\}$.
" $\Leftarrow$ " We have $p \equiv j^2 \pmod{17}$ and so $(\frac{p}{17}) = 1$. By the Theorem 2.1, we have $(\frac{17}{p})(\frac{p}{17}) = (-1)^{8\frac{(p-1)}{2}} = 1$ which implies $(\frac{17}{p}) = 1$.

Since $(\frac{-1}{p}) = (-1)^{\frac{p-1}{2}}$, we get that $(\frac{-17}{p}) = (-1)^{\frac{p-1}{2}}$. If $p = 68k + j^2$ with $j \in \{1, 3, 5, 7, 9, 11, 13, 15\}$, we see that $(\frac{-17}{p}) = 1$. Therefore $x^2 \equiv -17$ (mod p) has a solution x_0. As in the case $q = 7$, if we use the same idea of the Pi Pigeonhole Principle we obtain that $q = a^2 + 17b^2 < 18p$ for some $a, b \in \mathbb{Z}$ and $q \equiv 0$ (mod p). Hence $q = \ell p$ with $\ell \in \{1, 2, ..., 17\}$. We may assume that $gcd(a, b) = 1$, otherwise we can simplify the equality $q = \ell p$ by $gcd(a, b)$ which cannot be p. Clearly if $\ell = 1$, $\ell = 2$ or $\ell = 17$ we are done. Since $q \equiv 0$, 1 or 2 (mod 4) and $p \equiv 1 \pmod 4$ we cannot have $\ell \in \{3, 7, 11, 15\}$. If $\ell \in \{4, 8, 12, 16\}$, $\ell = 4\ell'$, we can simplify the equality by a

4 and reduce this case to $\ell' \in \{1, 2, 3, 4\}$. Each one of these situations leads to either the conclusion of our claim or it can be excluded as before or reduced again by a 4.

(Case $\ell = 5$ or $\ell = 10$) Hence $q = \ell p = a^2 + 17b^2 \equiv a^2 + 2b^2 \equiv 0 \pmod 5$. If b is not a multiple of 5 then this implies $x^2 \equiv -2 \pmod 5$ which is not true. Hence b must be a multiple of 5 and then so must be a. Then the equality $\ell p = a^2 + 17b^2$ implies that ℓp is a multiple of 25 which is not possible.

(Case $\ell = 6$ or $\ell = 14$) In this case we must have a and b odd and then $q = 2(4s + 1) = \ell p$ which is not possible.

(Case $\ell = 13$) In this case $4q = (2a)^2 + 17(2b)^2 = 2p(3^2 + 17(1)^2)$. We will use Euler's argument ([6], Lemma 1.4, p. 10) here. If we calculate $M = (2b)^2[3^2 + 17(1)^2] - 4q = [3(2b) - 2a][3(2b) + 2a]$, we see that $2(13)$ divides M and so it divides either $3(2b) - 2a$ or $3(2b) + 2a$. Without loss of generality we may assume that $2(13)$ divides $3(2b) - 2a$. Hence, we can write $3(2b) - 2a = 2(13)d$ for some $d \in \mathbb{Z}$. Next, we calculate

$$2a + 17d = 3(2b) - 2(13)d + 17d = 3(2b) - 9d,$$

which implies that $2a + 17d = 3e$ for some $e \in \mathbb{Z}$. Also, from the above equality we get that $2b = e + 3d$. Then

$$2p(26) = 4q = (2a)^2 + 17(2b)^2 = (3e - 17d)^2 + 17(e + 3d)^2 = 26(e^2 + 17d^2) \Rightarrow$$
$$2p = e^2 + 17d^2.$$

(Case $\ell = 9$) We have $4q = (2a)^2 + 17(2b)^2 = 2p(1^2 + 17(1)^2)$. We calculate $M = (2b)^2[1^2 + 17(1)^2] - 4q = (2b - 2a)(2b + 2a)$, we see that $2(9)$ divides M and so it divides either $2b - 2a$ or $2b + 2a$. We need to look into two possibilities now. First $2(9)$ divides one of the factors $2b - 2a$ or $2b + 2a$, or $2(3)$ divides each one of them. In the second situation we can see that 3 divides $4a = 2b + 2a - (2b - 2a)$ and so 3 must divide b too. This last possibility is excluded by the assumption that $gcd(a, b) = 1$. Without loss of generality we may assume that $2(9)$ divides $2b - 2a$. Hence, we can write $2b - 2a = 2(9)d$ for some $d \in \mathbb{Z}$. We set, $2a = e - 17d$ and observe that $2b = 2a + 18d = e - 17d + 18d = e + d$. Then

$$2p(18) = 4q = (2a)^2 + 17(2b)^2 = (e - 17d)^2 + 17(e + d)^2 = 18(e^2 + 17d^2) \Rightarrow$$
$$2p = e^2 + 17d^2.$$

(ii) To show this claim, we may use Euler's argument as above. □

For primes q which are multiples of four minus one, the patterns suggest that we have to change the modulo but also there are more trickier changes. Let us look at the cases $q = 11$ and $q = 19$. In case $q = 11$, we have seen that the quadratic form $a^2 + 11b^2$ in Theorem 1.1 can be separated by a polynomial from the other possible forms of representing primes which are quadratic residues of odd numbers modulo 22.

Theorem 3.2. *(i) A prime $p > 11$ is of the form $a^2 + 11b^2$ or $3p = a^2 + 11b^2$, for some $a, b \in \mathbb{N}$ if and only if $p \equiv (2j + 1)^2$ (mod 22) for some $j = 0, ..., 4$.*

(ii) A prime $p > 19$ satisfies $4p = a^2 + 19b^2$, for some $a, b \in \mathbb{N}$ if and only if $p \equiv (2j + 1)^2$ (mod 38) for some $j = 0, ..., 8$.

(iii) The representations of a prime as in part (i) are exclusive, i.e. a prime p cannot be in both representations.

We leave these proofs for the interested reader.

4. Proof of Theorem 1.6

Clearly the inclusions $A \cap B \subset C$ and $C \cap A \subset B$ are covered by Proposition 1.5. To show $B \cap C \subset A$ we will first prove it for $t = 2p$ with p a prime. Since $2p = a^2 + b^2$ we have $a^2 \equiv -b^2 \pmod p$. Because $p > 2$, a cannot be divisible by p and so it has an inverse (mod p) say a^{-1}. This shows that $x_0 = ba^{-1}$ is a solution of the equation $x^2 \equiv -1 \pmod p$. Similarly since $2p = 2(x^2 - xy + y^2)$ we get that $4(x^2 - xy + y^2) = (2x - y)^2 + 3y^2 \equiv 0 \pmod p$. This gives a solution y_0 of the equation $x^2 \equiv -3$ (mod p). So, we have $z_0 = x_0 y_0$ satisfying $z_0^2 \equiv 3 \pmod p$. Let us now apply the Pigeonhole Principle as before: we let $m \in \mathbb{N}$ be in such a way that $m^2 < p < (m+1)^2$. We consider the function $g : \{0, 1, 2, ..., m\} \times \{0, 1, 2, ..., m\} \to \{0, 1, 2, ..., p-1\}$ defined by $g(u, v) \equiv u + vz_0 \pmod p$. Since $(m+1)^2 > p$, we must have two distinct pairs (a'', b'') and (a', b') such that $g(a'', b'') = g(a', b')$. Then $a'' - a' \equiv (b' - b'')z_0 \pmod p$. Then, if we let $r = a'' - a'$, and $s = b' - b''$ we get that $q := r^2 - 3s^2 \equiv s^2(z_0^2 - 3) \equiv 0 \pmod p$. So, q needs to be a multiple of p. If $q = 0$ then $r = \pm s\sqrt{3}$ which is not possible because r and s are integers not both equal to zero. If $q > 0$ then $0 < q \le r^2 < p$, which is again impossible. It remains that $q < 0$, and so $0 < -q = 3s^2 - r^2 \le 3s^2 < 3p$. This leaves only two possibilities for q: either $q = -p$ or $q = -2p$. Hence, we need to exclude the case $3s^2 - r^2 = p$. This implies $4p = 12s^2 - 4r^2 = (2x - y)^2 + 3y^2$. Then $4r^2 + (2x - y)^2 \equiv 0 \pmod 3$. Since -1 is not a quadratic residue modulo 3 we must have r divisible by 3 which is gives $p = 3$ but we cannot have $6 = a^2 + b^2$. It remains that $2p = 3s^2 - r^2$. Let us observe that in this case s and r cannot be both even or of different parities since p must be of the form $4k + 1$. Hence, we have the representation $p = (\frac{3s+r}{2})^2 - 3(\frac{s+r}{2})^2$.

To prove the inclusion in general we just need to observe that for any number $t \in B \cap C$ and a prime $p > 2$ dividing t, then if p is of the form $4k + 3$ then it divides a and b and so p^2 divides t. The same is true if p is of the form $6k - 1$. Clearly all the primes that appear in the decomposition of t to an even power they can be factored out and reduce the problem to factors of the form $12k + 1$ but for these factors we can apply the above argument and use the identities:

$$(y^2 - 3x^2)(v^2 - 3u^2) = (3ux + vy)^2 - 3(xv + uy)^2,$$
$$2(x^2 - 3y^2) = 3(x + y)^2 - (x + 3y)^2.$$

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The hyperbolic Desargues theorem in the Poincaré model of hyperbolic geometry

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To the memory of Professor Mircea-Eugen Craioveanu (1942-2012)

Abstract. In this note, we present the hyperbolic Desargues theorem in the Poincaré disc of hyperbolic geometry.

Mathematics Subject Classification (2010): 30F45, 20N99, 51B10, 51M10.

Keywords: Hyperbolic geometry, hyperbolic triangle, gyrovector.

1. Introduction

Hyperbolic Geometry appeared in the first half of the 19^{th} century as an attempt to understand Euclid's axiomatic basis of Geometry. It is also known as a type of non-Euclidean Geometry, being in many respects similar to Euclidean Geometry. Hyperbolic Geometry includes similar concepts as distance and angle. Both these geometries have many results in common but many are different.

There are known many models for Hyperbolic Geometry, such as: Poincaré disc model, Poincaré half-plane, Klein model, Einstein relativistic velocity model, etc. The hyperbolic geometry is a non-Euclidean geometry. Here, in this study, we present a proof of Desargues theorem in the Poincaré disc model of hyperbolic geometry. The well-known Desargues theorem states that if the three straight lines joining the corresponding vertices of two triangles and all meet in a point, then the three intersections of pairs of corresponding sides lie on a straight line [1]. This result has a simple statement but it is of great interest. We just mention here few different proofs given by N. A. Court [2], H. Coxeter [3], C. Durell [4], H. Eves [5], C.Ogilvy [6], W. Graustein [7].

We begin with the recall of some basic geometric notions and properties in the Poincaré disc. Let D denote the unit disc in the complex z - plane, i.e.

$$D = \{z \in \mathbb{C} : |z| < 1\}.$$

The most general Möbius transformation of D is

$$z \to e^{i\theta} \frac{z_0 + z}{1 + \overline{z_0} z} = e^{i\theta}(z_0 \oplus z),$$

which induces the Möbius addition $\oplus$ in D, allowing the Möbius transformation of the disc to be viewed as a Möbius left gyro-translation

$$z \to z_0 \oplus z = \frac{z_0 + z}{1 + \overline{z_0} z}$$

followed by a rotation. Here $\theta \in \mathbb{R}$ is a real number, $z, z_0 \in D$, and $\overline{z_0}$ is the complex conjugate of z_0. Let $Aut(D, \oplus)$ be the automorphism group of the grupoid $(D, \oplus)$. If we define

$$gyr : D \times D \to Aut(D, \oplus), gyr[a, b] = \frac{a \oplus b}{b \oplus a} = \frac{1 + a\overline{b}}{1 + \overline{a}b},$$

then is true gyro-commutative law

$$a \oplus b = gyr[a, b](b \oplus a).$$

A gyro-vector space $(G, \oplus, \otimes)$ is a gyro-commutative gyro-group $(G, \oplus)$ that obeys the following axioms:

(1) $gyr[\mathbf{u}, \mathbf{v}]\mathbf{a}\cdot\, gyr[\mathbf{u}, \mathbf{v}]\mathbf{b} = \mathbf{a} \cdot \mathbf{b}$ for all points $\mathbf{a}, \mathbf{b}, \mathbf{u}, \mathbf{v} \in G$.

(2) G admits a scalar multiplication, $\otimes$, possessing the following properties. For all real numbers $r, r_1, r_2 \in \mathbb{R}$ and all points $\mathbf{a} \in G$:

(G1) $1 \otimes \mathbf{a} = \mathbf{a}$

(G2) $(r_1 + r_2) \otimes \mathbf{a} = r_1 \otimes \mathbf{a} \oplus r_2 \otimes \mathbf{a}$

(G3) $(r_1 r_2) \otimes \mathbf{a} = r_1 \otimes (r_2 \otimes \mathbf{a})$

(G4) $\frac{|r| \otimes \mathbf{a}}{\|r \otimes \mathbf{a}\|} = \frac{\mathbf{a}}{\|\mathbf{a}\|}$

(G5) $gyr[\mathbf{u}, \mathbf{v}](r \otimes \mathbf{a}) = r \otimes gyr[\mathbf{u}, \mathbf{v}]\mathbf{a}$

(G6) $gyr[r_1 \otimes \mathbf{v}, r_1 \otimes \mathbf{v}] = 1$

(3) Real vector space structure $(\|G\|, \oplus, \otimes)$ for the set $\|G\|$ of one-dimensional "vectors"

$$\|G\| = \{\pm \|\mathbf{a}\| : \mathbf{a} \in G\} \subset \mathbb{R}$$

with vector addition $\oplus$ and scalar multiplication $\otimes$, such that for all $r \in \mathbb{R}$ and $\mathbf{a}, \mathbf{b} \in G$,

(G7) $\|r \otimes \mathbf{a}\| = |r| \otimes \|\mathbf{a}\|$

(G8) $\|\mathbf{a} \oplus \mathbf{b}\| \leq \|\mathbf{a}\| \oplus \|\mathbf{b}\|$

Definition 1.1. *The hyperbolic distance function in D is defined by the equation*

$$d(a, b) = |a \ominus b| = \left| \frac{a - b}{1 - \overline{a}b} \right|.$$

Here, $a \ominus b = a \oplus (-b)$, for $a, b \in D$.

For further details we refer to the recent book of A. Ungar [8].

Theorem 1.2. (The Menelaus's Theorem for Hyperbolic Gyrotriangle) *Let ABC be a gyrotriangle in a Möbius gyrovector space $(V_s, \oplus, \otimes)$ with vertices $A, B, C \in V_s$, sides $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbf{V}_s$, and side gyrolengths $a, b, c \in (-s, s)$, $\mathbf{a} = \ominus B \oplus C$, $\mathbf{b} = \ominus C \oplus A$, $\mathbf{c} = \ominus A \oplus B$, $a = \|\mathbf{a}\|$, $b = \|\mathbf{b}\|$, $c = \|\mathbf{c}\|$. If l is a gyroline not through any vertex of a gyrotriangle ABC such that l meets BC in D, CA in E, and AB in F, then*

$$\frac{(AF)_\gamma}{(BF)_\gamma} \cdot \frac{(BD)_\gamma}{(CD)_\gamma} \cdot \frac{(CE)_\gamma}{(AE)_\gamma} = 1,$$

where $v_\gamma = \frac{v}{1-\frac{v^2}{s^2}}$.

(See [9])

Theorem 1.3. (Converse of Menelaus's Theorem for Hyperbolic Gyrotriangle) *If D lies on the gyroline BC, E on CA, and F on AB such that*

$$\frac{(AF)_\gamma}{(BF)_\gamma} \cdot \frac{(BD)_\gamma}{(CD)_\gamma} \cdot \frac{(CE)_\gamma}{(AE)_\gamma} = 1,$$

then D, E, and F are collinear.

(See [9])

2. Main results

In this section we prove the Desargues theorem in the Poincaré disc model of hyperbolic geometry.

Theorem 2.1. The Desargues Theorem for Hyperbolic Gyrotriangle *If ABC, $A'B'C'$ are two gyrotriangles such that the gyrolines AA', BB', CC' meet in O, and BC and $B'C'$ meet at L, CA and $C'A'$ at M, AB and $A'B'$ at N, then L, M, and N are collinear.*

Proof. If we use Menelaus's theorem in the gyrotriangle OBC, cut by the gyroline $B'C'$ (See Theorem 1.2, Figure 1), we get

$$\frac{(LC)_\gamma}{(LB)_\gamma} \cdot \frac{(B'B)_\gamma}{(B'O)_\gamma} \cdot \frac{(C'O)_\gamma}{(C'C)_\gamma} = 1. \tag{2.1}$$

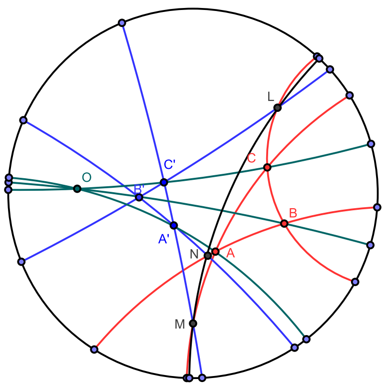

Figure 1

If we use Menelaus's theorem in the gyrotriangle OCA, cut by the gyroline $C'A'$, we get

$$\frac{(MA)_\gamma}{(MC)_\gamma} \cdot \frac{(C'C)_\gamma}{(C'O)_\gamma} \cdot \frac{(A'O)_\gamma}{(A'A)_\gamma} = 1. \tag{2.2}$$

If we use Menelaus's theorem in the gyrotriangle OAB, cut by the gyroline $A'B'$, we get

$$\frac{(NB)_\gamma}{(NA)_\gamma} \cdot \frac{(A'A)_\gamma}{(A'O)_\gamma} \cdot \frac{(B'O)_\gamma}{(B'B)_\gamma} = 1. \tag{2.3}$$

Multiplying the relations (2.1), (2.2) and (2.3), we obtain

$$\frac{(LC)_\gamma}{(LB)_\gamma} \cdot \frac{(MA)_\gamma}{(MC)_\gamma} \cdot \frac{(NB)_\gamma}{(NA)_\gamma} = 1, \tag{2.4}$$

and by Theorem 1.3 we get that the gyropoints L, M, and N are collinear. □

Naturally, one may wonder whether the converse of the Desargues theorem exists. Indeed, a partially converse theorem does exist as we show in the following theorem.

Theorem 2.2. (Converse of Desargues Theorem for Hyperbolic Gyrotriangle) *Let ABC, $A'B'C'$ are two gyrotriangles such that the gyrolines BC and $B'C'$ meet at L, CA and $C'A'$ at M, AB and $A'B'$ at N, and the gyropoints L, M, and N are collinear. If two of the gyrolines AA', BB', CC' meet, then all three are concurrent.*

Proof. Let O be a point of intersection of gyrolines AA' and BB'. Then N is the point of intersection of gyrolines $AB, A'B'$, and MN. If we use Desargues theorem for gyrotriangles $LB'B$ and MAA' we obtain that the points of intersection of the gyrolines AA' and BB', LB and MA, MA' and LB' respectively, are collinear. So, the gyropoints O, C, and C' are collinear, the conclusion follows. □

Many of the theorems of Euclidean geometry are relatively similar form in the Poincaré model of hyperbolic geometry, Desargues theorem is an example in this respect. In the Euclidean limit of large s, $s \to \infty$, v_γ reduces to v, so Desargues theorem for hyperbolic triangle reduces to the Desargues theorem of Euclidean geometry.

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New aspects in the use of canonoid transformations

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To the memory of Professor Mircea-Eugen Craioveanu (1942-2012)

Abstract. Canonoid transformations with respect to a locally Hamiltonian vector field are studied through the concept of generating function and the Helmholtz theory of the inverse problem. The case of dimension two is connected with the Liouville equation. The use of such transformations for determining first integrals is illustrated with two examples: the Whittaker system (in dimension four) and the damped harmonic oscillator (in the dimension two).

Mathematics Subject Classification (2010): 37C80, 37C05, 37C10, 70H15.

Keywords: Locally Hamiltonian vector field, canonoid transformation, the inverse problem, Helmholtz conditions, integrating factor, generating function, Liouville equation.

1. Introduction

The theory of *canonoid transformations* is, by now, a well-known approach in geometrical dynamics. Introduced by Saletan in his famous book [20] as a generalization of classical notion of *canonical transformation*, the concept of canonoid diffeomorphism has its roots in the work of Sophus Lie [16] as it is pointed out by P. Havas in the MR review of [7]: "the most general canonoid transformation for a particular Hamiltonian is given in Lie Theorem III." Important contributions to this theory are given by Cariñena and co-workers [2]-[5], Negri and co-workers [18], [21] as well as in [9] and [15]. A careful analysis of this concept was performed recently in [11] and for Nambu mechanics in [8].

The aim of the present paper is to point out new features of canonoid transformations, for example in order to obtain conservation laws (first integrals) of a given dynamical system. The framework consists in a pair (M, X) with M a smooth manifold of even dimension, $\dim M = 2n$, and $X \in \mathcal{X}(M)$ a vector field on M generating the ODE system:

$$\dot{x}^i = \frac{dx^i}{dt} = X^i\left(x^1, ..., x^{2n}\right) \tag{1.1}$$

where $\left(x^i\right)_{i=\overline{1,2n}}$ are local coordinates on M and X has the local expression

$$X = X^i \frac{\partial}{\partial x^i}.$$

We call X a *locally Hamiltonian vector field* if there exists a symplectic form $\omega \in \Omega^2(M)$ such that:

$$\mathcal{L}_X \omega = 0, \tag{1.2}$$

in other words, ω is a symplectic structure *associated to* X. Then, $\phi \in Diff(M)$ is called *canonoid* with respect to the pair (X, ω) (conform [1, p. 155]) if the new vector field $Y = \phi_*(X)$ is locally Hamiltonian with the same associated symplectic structure i.e. $\mathcal{L}_Y \omega = 0$. It follows n first integrals $\alpha_0, ..., \alpha_{n-1} \in C^\infty(M)$ for X, or for the system (1), given by [3]:

$$(\phi^* \omega)^{n-k} \wedge \omega^k = \alpha_k \omega^n. \tag{1.3}$$

An important remark here is that α's can be independent or not, trivial or not.

A canonoid transformation may be locally found in the classical way [4] by solving the system of partial differential equations which results from projecting both sides of the equality:

$$\mathcal{L}_X (\theta - \phi^* \theta) = dF \tag{1.4}$$

on the canonical-Darboux base $(dq^a, dp_a)_{a=\overline{1,n}}$; here θ is a local *potential* of ω, i.e. $\omega = d\theta$, and $F \in C^\infty(M)$ is called *the generating function* of the diffeomorphism ϕ.

Let us recall that in [5] a coordinate-free description of canonoid transformations is included, but we prefer here local computations in order to handle concrete examples. More precisely, in the following section we set a pair (M, X) and build, using the Helmholtz method of integrating factor in solving *the inverse problem*, a local symplectic form associated to X. In the next section, using the obtained geometrical framework, we study the existence of a canonoid transformation and corresponding first integrals. In the last section, the theory is applied to a four dimensional differential system, considered by Whittacker, and to a two dimensional system, namely the damped harmonic oscillator.

Another important remark here is that for $n = 1$ the unique (non-null) coefficient of the associated symplectic structure, which appears as integrating factor in the Helmholtz conditions, is solution of the celebrated *Liouville equation* [17], [10]. This equation is a main tool in statistical mechanics where a solution is called *probability density function* [22], while in mathematics is called *last multiplier* [12], [19]. A feature of this equation is that it does not always admits solutions [13].

2. The inverse problem

Let M be a real, smooth and orientable, $2n$-dimensional manifold, $C^\infty(M)$ the real algebra of smooth real functions on M, $\mathcal{X}(M)$ the Lie algebra of vector fields and $\Omega^k(M)$ the $C^\infty(M)$-module of k-differential forms, $0 \le k \le n$. Fix $X \in \mathcal{X}(M)$ which we suppose that it is not locally Hamiltonian with respect to the 2-form

$$\omega_0 = \sum_{a=1}^{n} dx^a \wedge dx^{n+a}.$$

In order to build a symplectic structure associated to X we follows the approach of Helmholtz based on the notion of *integrating factor* namely a set $c_{ij} = c_{ij}(t,x) \in C^\infty(\mathbb{R}\times M)$ such that the equivalent to (1.1) system $c_{ij}\left(\dot{x}^j - X^j\right) = 0$ is *self-adjoint.* The self-adjoint conditions are given by:

$$\left\{\begin{array}{c} c_{ij} + c_{ji} = 0 \\ \frac{\partial c_{ij}}{\partial x^h} + \frac{\partial c_{jh}}{\partial x^i} + \frac{\partial c_{hi}}{\partial x^j} = 0 \\ \frac{\partial c_{ij}}{\partial t} = \frac{\partial D_i}{\partial x^j} - \frac{\partial D_j}{\partial x^i}, \quad D_i = -c_{ij}X^j. \end{array}\right. \tag{2.1}$$

A global formulation of the Helmholtz conditions can be derived in terms of differential forms; namely if we consider the time-dependent 2-forms:

$$\left\{\begin{array}{c} \omega = \frac{1}{2}c_{ij}dx^i \wedge dx^j \\ \Omega = \omega + i_X\omega \wedge dt = \frac{1}{2}c_{ij}dx^i \wedge dx^j + D_i dx^i \wedge dt \end{array}\right. \tag{2.2}$$

then the Helmholtz conditions reduce to the closedeness of Ω: $d\Omega = 0$.

The following result is straightforward: *The vector field X admits a locally Hamiltonian description if and only if the system* (2.1) *admits an autonomous, i.e. time-independent, and non-degenerate, i.e.* $\det(c_{ij}) \neq 0$, *solution. In this case, the associated symplectic form is ω given by* (2.2).

Actually, the determination of the form ω comes from the integration of the system formed by the first two equations of (2.1) and by the equation:

$$X^h \frac{\partial c_{ij}}{\partial x^h} + c_{ih}\frac{\partial X^h}{\partial x^j} + c_{hj}\frac{\partial X^h}{\partial x^i} = 0 \tag{2.3}$$

which may be obtained from self-adjointness conditions (2.1).

3. Canonoid transformations and associated first integrals

Suppose that we found a symplectic 2-form ω such that X is locally Hamiltonian with respect to ω. If $\theta = A_j dx^j \in \Omega^1(M)$ is a local potential of ω, i.e. $d\theta = \omega$, and $F \in C^\infty(M)$ is a given function then the canonoid transformation ϕ having F as generating function is determined by the relation:

$$\mathcal{L}_X \phi^*\theta = dF. \tag{3.1}$$

Searching ϕ with local expression $\overline{x}^i = \varphi^i\left(x^1,...,x^{2n}\right)$ the previous equation becomes:

$$\overline{c}_{ij}\frac{\partial \varphi^i}{\partial x^h}\frac{\partial \varphi^j}{\partial x^r}X^h + \frac{\partial}{\partial x^r}\left(\overline{A}_j X^s \frac{\partial \varphi^j}{\partial x^s}\right) = \frac{\partial F}{\partial x^r}, \tag{3.2}$$

with $\overline{c}_{ij}(x) = c_{ij}(\phi(x))$ and $\overline{A}_j(x) = A_j(\phi(x))$. Considering the vector field $Y = \phi_*(X)$, the equation (3.2) may be rewritten as:

$$\overline{c}_{ij}\frac{\partial \varphi^j}{\partial x^r}\overline{Y}^r + \frac{\partial}{\partial x^r}\left(\overline{A}_j \overline{X}^j\right) = \frac{\partial F}{\partial x^r} \tag{3.3}$$

if Y has the expression $Y = Y^i \frac{\partial}{\partial x^i}$ and $\overline{Y}^i(x) = Y^i(\phi(x))$.

Now, let us consider the Hamiltonian function H [6] defined by $i_X\omega = dH$ and the Poisson structure [6] defined by the symplectic form ω. The associated Poisson bracket is expressed as $\{f, g\} = c^{jk}\frac{\partial f}{\partial x^j}\frac{\partial f}{\partial x^k}$ and the local components of the vector field $X = X_H$ are $X^k = c^{kj}\frac{\partial H}{\partial x^j}$ where $\left(c^{jk}\right)$ is the inverse matrix of the matrix (c_{jk}).

Taking into account that $\overline{Y}^i = \{\varphi^i, H\}$ the last equation reads:

$$\overline{c}_{ij}\frac{\partial \varphi^j}{\partial x^r}\{\varphi^i, H\} + \frac{\partial}{\partial x^r}\left(\overline{A}_j\left\{\varphi^j, H\right\}\right) = \frac{\partial F}{\partial x^r}. \tag{3.4}$$

A straightforward computation give us the first integral:

$$\alpha_{n-1} = \overline{c}_{ij}\left\{\varphi^i, \varphi^j\right\}. \tag{3.5}$$

There is also another way to find a canonoid transformation. If X is locally Hamiltonian with respect to ω, a vector field $Y = Y^i\frac{\partial}{\partial x^i}$ is locally Hamiltonian with respect to the same 2-form ω if and only if $\left(Y^1, ..., Y^{2n}\right)$ is a solution of the system (2.3). If we find such a solution we have to look for a transformation ψ such that $\psi_* Y = X$. This equality becomes:

$$\frac{\partial \psi^i}{\partial x^j}Y^j = X^i(\psi(x)) \tag{3.6}$$

where $\overline{x}^i = \psi^i\left(x^1, ..., x^{2n}\right)$ is the local expression of the diffeomorphism ψ. Obviously, if ψ is canonoid with respect to Y then $\phi = \psi^{-1}$ is canonoid with respect to X.

In the particular case $n = 2$ let α_0, α_1 be determined from the pair (X, ϕ) and β_0, β_1 similarly found and related to the pair (Y, ψ). A direct computation shows that:

$$\alpha_0 = \frac{1}{\phi^*\beta_0}, \alpha_1 = \frac{\phi^*\beta_1}{\phi^*\beta_0}. \tag{3.7}$$

Another particular case is $n = 1$. The equation (2.3) is reduced to:

$$X(\xi) + \xi divX = 0 \tag{3.8}$$

where ξ is the integrating factor and $divX$ is the divergence of the vector field X which is locally Hamiltonian with respect to $\omega = \xi dx^1 \wedge dx^2$. But (3.8) is exactly the Liouville equation discussed at the end of Introduction. The unique first integral associated to the pair (X, ϕ) in this case is $\alpha_0 = \overline{\xi}\left\{\varphi^1, \varphi^2\right\}$ with $\overline{\xi}(x) = \xi(\phi(x))$.

4. Examples

Let us consider the system of second-order ODE of Whittaker, [23]:

$$\begin{cases} \ddot{q}^1 - q^1 = 0 \\ \ddot{q}^2 - \dot{q}^1 = 0 \end{cases} \tag{4.1}$$

which does not admit a classical Lagrangian formulation [14]. If we use the notation $q^1 = x^1, q^2 = x^2, \dot{q}^1 = x^3, \dot{q}^2 = x^4$, we get the first-order equivalent system:

$$\begin{cases} \dot{x}^1 = x^3 \\ \dot{x}^2 = x^4 \\ \dot{x}^3 = x^1 \\ \dot{x}^4 = x^3 \end{cases} \tag{4.2}$$

An admissible symplectic structure for the vector field

$$X = x^3 \frac{\partial}{\partial x^1} + x^4 \frac{\partial}{\partial x^2} + x^1 \frac{\partial}{\partial x^3} + x^3 \frac{\partial}{\partial x^4}$$

is:

$$\omega = dx^1 \wedge dx^2 + dx^1 \wedge dx^3 + dx^2 \wedge dx^4 - dx^3 \wedge dx^4. \tag{4.3}$$

The Hamiltonian function H and the potential θ are respectively given by:

$$\begin{cases} H = \left(x^3\right)^2 + \frac{1}{2}\left[\left(x^4\right)^2 - \left(x^1\right)^2\right] - x^1 x^4 \\ \theta = -\left(x^2 + x^3\right) dx^1 + \left(x^2 - x^3\right) dx^4. \end{cases} \tag{4.4}$$

Another vector field which is locally Hamiltonian with respect to the same symplectic form is, for example, $Y = \frac{\partial}{\partial x^1}$. The canonoid transformation ϕ with $Y = \phi_*(X)$ and its generating function F are given respectively by:

$$\begin{cases} \varphi^1 = \ln\left(x^1 + x^3\right) \\ \varphi^2 = \sin\left[\left(x^3\right)^2 - \left(x^1\right)^2\right] \\ \varphi^3 = x^4 - x^1 \\ \varphi^4 = \cos\left[x^2 - x^3 + \left(x^1 - x^4\right)\ln\left(x^1 + x^3\right)\right] \end{cases} \tag{4.5}$$

and:

$$F = \sin\left[\left(x^3\right)^2 - \left(x^1\right)^2\right] - x^1 - x^2 - x^3 + x^4. \tag{4.6}$$

The first integral (3.5) is:

$$\alpha_1 = \cos\left[\left(x^3\right)^2 - \left(x^1\right)^2\right] + \sin\left[x^2 - x^3 + \left(x^1 - x^4\right)\ln\left(x^1 + x^3\right)\right] \times$$
$$\times \left[\left(x^1 - x^4\right)\cos\left(\left(x^3\right)^2 - \left(x^1\right)^2\right) - 1\right]. \tag{4.7}$$

Now, let us consider the equation of the damped harmonic oscillator:

$$\ddot{x} + c\dot{x} + kx = 0 \tag{4.8}$$

for which we assume that $c > 0, k > 0$ and $c^2 - 4k > 0$. The vector field

$$X = x^2 \frac{\partial}{\partial x^1} - \left(kx^1 + cx^2\right)\frac{\partial}{\partial x^2}$$

where we have used the notations $x = x^1$ and $\dot{x} = x^2$ has the integrating factor:

$$\xi = \left(a_1 x^1 + x^2\right)^{-\frac{c}{a_2}} \tag{4.9}$$

where:

$$a_1 = \frac{c + \sqrt{c^2 - 4k}}{2}, \quad a_2 = \frac{c - \sqrt{c^2 - 4k}}{2}. \tag{4.10}$$

The Hamiltonian function has the form:

$$H = \frac{\left(a_1 x^1 + x^2\right)^{a_1 \sigma}}{\sigma\left(a_1 \sigma + 1\right)} \left(\sigma x^2 - x^1\right), \sigma = -\frac{1}{a_2}. \tag{4.11}$$

A canonoid transformation for the vector field X is :

$$\begin{cases} \varphi^1 = \frac{\sigma}{a_1} \ln\left(a_1 x^1 + x^2\right) \\ \varphi^2 = -\sigma \ln\left(a_1 x^1 + x^2\right) + \left(a_1 x^1 + x^2\right)^A \left(a_2 x^1 + x^2\right)^B \end{cases} \tag{4.12}$$

with:

$$A = \frac{2a_1}{\sqrt{c^2 - 4k}}, B = \frac{-2a_2}{\sqrt{c^2 - 4k}}. \tag{4.13}$$

The vector fields X and $Y = \phi_* X = \frac{1}{a_1}\frac{\partial}{\partial x^1} - \frac{\partial}{\partial x^2}$ are locally Hamiltonian with respect to the same symplectic form $\omega = \xi dx^1 \wedge dx^2$. Finally, the first integral is $\alpha_0 = H^\beta$ with $\beta = \frac{c}{\sqrt{c^2 - 4k}}$.

5. Conclusions

0) The canonoid transformations provides useful information about the geometrical (symplectic structures and therefore volume forms) and dynamical (first integrals and bi-Hamiltonian description) objects which can be naturally associated to a given dynamical system.
1) The theory of these transformations has deep connections with other fundamental theoretical and applied constructions namely the theory of inverse problem and the Liouville equation.
2) The important structures generated by a canonoid transformation can be essential steps toward two remarkable approaches: the complete integrability of Liouville-Arnold type and the numerical integrators.
3) From the previous remarks it seems that this type of transformations to be more adapted than the canonical maps to some "in present" complicated or strange dynamical systems.
4) Due to the connection with the Liouville equation we can call the Helmholtz conditions of self-adjointness as *generalized Liouville equations* or *Liouville equations of higher even dimension* and maybe this fact opens a new way to connect the classical (Newtonian) mechanics to statistical physics.

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The group of isometries of the French rail ways metric

Vasile Bulgărean

To the memory of Professor Mircea-Eugen Craioveanu (1942-2012)

Abstract. In this paper we study the isometry groups $\mathrm{Iso}_{d_{F,p}}(\mathbb{R}^n)$, where $d_{F,p}$ is the French rail ways metric given by (2.1). We derive some general properties of $\mathrm{Iso}_{d_{F,p}}(\mathbb{R}^n)$ and we discuss the particular situation when $X = \mathbb{R}^n$ and $d = d_2$ is the Euclidean metric.

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1. Introduction

Let (X, d) be a metric space. The map $f : X \to X$ is called an *isometry* with respect to the metric d (or a *d-isometry*), if it is surjective and it preserves the distances. That is for any points $x, y \in X$ the relation $d(f(x), f(y)) = d(x, y)$ holds. From this relation it follows that f is injective, hence it is bijective. Denote by $\mathrm{Iso}_d(X)$ the set of all isometries of the metric space (X, d). It is clear that $(\mathrm{Iso}_d(X), \circ)$ is a subgroup of $(S(X), \circ)$, where $S(X)$ denotes the group of all bijective transformations $f : X \to X$. We will call $(\mathrm{Iso}_d(X), \circ)$ the *group of isometries* of the metric space (X, d). A general, important and complicated problem is to described the group $(\mathrm{Iso}_d(X), \circ)$. This problem was formulated in the paper [2] for metric spaces with a metric that is not given by a norm defined by an inner product.

Some results in direction to solve this problem for some particular metrics are the following. D.J. Schattschneider [17] found an elementary proof for the property that group $\mathrm{Iso}_{d_1}(\mathbb{R}^2)$ is the semi-direct product of D_4 and $T(2)$, where d_1 is the "taxicab metric" defined by (1.2) (for $p = 1$ and $n = 2$) and D_4 and $T(2)$ are the symmetry group of the square and the translations group of $\mathbb{R}^2$, respectively. A similar result holds for the group $\mathrm{Iso}_{d_1}(\mathbb{R}^3)$, i.e. this group is isomorphic to the semi-direct product of groups D_h and $T(3)$, where D_h is the symmetry group of the Euclidean octahedron and $T(3)$ is the translations group of $\mathbb{R}^3$. This was recently proved by O. Gelisgen,

R. Kaya [7]. In fact the "taxicab metric" generates many interesting non-Euclidean geometric properties (see the book of E.F. Krause [10] or the thesis of G. Chen [4]). Another result concerning the isometry group of the plane $\mathbb{R}^2$ with respect to the "Chinese checker metric" d_C, where

$$d_C(x,y) = \max\{|x^1 - x^2|, |y^1 - y^2|\} + (\sqrt{2}-1)\min\{|x^1 - x^2|, |y^1 - y^2|\}, \quad (1.1)$$

was recently obtained by R. Kaya, O. Gelisgen, S. Ekmekci, A. Bayar [9].They have showed that this group is isomorphic to the semi-direct product the Dihedral group D_8, the Euclidean symmetry group of the regular octagon and $T(2)$.

In the paper [1] we have considered $X = \mathbb{R}^n$, and for any real number $p \geq 1$ the metric d_p defined by

$$d_p(x,y) = \left(\sum_{i=1}^{n} |x^i - y^i|^p \right)^{1/p}, \quad (1.2)$$

where $x = (x^1, ..., x^n), y = (y^1, ..., y^n) \in \mathbb{R}^n$. If $p = \infty$, then the metric d_∞ is defined by

$$d_\infty(x,y) = \max\{|x^1 - y^1|, ..., |x^n - y^n|\}. \quad (1.3)$$

In the case $p = 2$, we get the well-known Euclidean metric on $\mathbb{R}^n$. In this case we have the Ulam's Theorem which states that $\mathrm{Iso}_{d_2}(\mathbb{R}^n)$ is isomorphic to the semi-direct product of orthogonal group $O(n)$ and $T(n)$, where $T(n)$ is the group of translations of $\mathbb{R}^n$ (see for instance the references [5], [16]). The situation $p \neq 2$ is very interesting. The main result of [1] consists in a complete description of the groups $\mathrm{Iso}_{d_p}(\mathbb{R}^n)$ for $p \geq 1, p \neq 2$, and $p = \infty$. We have proved that in case $p \neq 2$ all these groups are isomorphic and consequently, they are not depending on number p. The main ingredient in the proof of the above result is the Mazur-Ulam Theorem about the isometries between normed linear spaces (see the original reference [12], the monograph [6], the references [14], [15] for some extensions, and [18], [19] for new proofs).

In this paper we study some properties of the isometry group of the so called French railroad metric.

2. The main results

Let (X, d) be a metric space and let $f : X \to X$ be a map. The minimal displacement of f, denoted by $\lambda(f)$, is the greatest lower bound of the displacement function of f, that is

$$\lambda(f) = \inf_{x \in X} d(x, f(x)).$$

The minimal set of f, denoted by $\mathrm{Min}(f)$, is the subset of X defined as

$$\mathrm{Min}(f) = \{x \in X : d(x, f(x)) = \lambda(f)\}.$$

According to the monograph of A. Papadopoulos [13], we have the following general classification of the isometries of a metric space X in terms of the invariants $\lambda(f)$ and $\mathrm{Min}(f)$. Consider $f : X \to X$ to be an isometry. Then f is said to be

1. *Parabolic* if $\mathrm{Min}(f) = \Phi$.
2. *Elliptic* if $\mathrm{Min}(f) \neq \Phi$ and $\lambda(f) = 0$. Thus, f is elliptic if and only if $\mathrm{Fix}(f) \neq \Phi$.

3. *Hyperbolic* if $\mathrm{Min}(f) \neq \Phi$ and $\lambda(f) > 0$.

France is a centralized country: every train that goes from one French city to another has to pass through Paris. This is slightly exaggerated, but not too much. This motivates the name French railroad metric for the following construction. Let (X, d) be a metric space, and fix $p \in X$. Define a new metric $d_{F,p}$ on X by letting

$$d_{F,p}(x,y) = \begin{cases} 0, & \text{if and only if } x = y \\ d(x,p) + d(p,y), & \text{if } x \neq y \end{cases} \tag{2.1}$$

The French railroad metric generates a very poor geometry. In fact, the geometric properties are concentrated at the point p. For instance, consider X to be the plane $\mathbb{R}^2$, $p = O$, the origin, and d is the standard Euclidean metric d_2. Then for two distinct points A and B the perpendicular bisector of the segment $[AB]$ exists if and only if $AO = BO$. Similarly, the Menger segment $[AB]$ (see the monograph of W. Benz [3, p.43]) has a midpoint, if and only if $AO = BO$. In the case $AO \neq BO$, the segment $[AB]$ consists only in the points A and B.

The main purpose of this paper is to derive some general properties of the group $\mathrm{Iso}_{d_{F,p}}(X)$. We will give an equivalent property for $\mathrm{Iso}_{d_{F,0}}(\mathbb{R}^n)$, the isometry group of the space $\mathbb{R}^n$ with the fixed point the origin $p = 0$, and the Euclidean metric d_2, and we will discuss the complexity of an effective description in the cases $n = 1$ and $n = 2$.

Let $\mathrm{Iso}_d^{(p)}(X)$ be the subgroup of $\mathrm{Iso}_d(X)$ consisting in all isometries fixing the point p, i.e.

$$\mathrm{Iso}_d^{(p)}(X) = \{h \in \mathrm{Iso}_d(X) : h(p) = p\}.$$

Theorem 2.1. *The following relation* $\mathrm{Iso}_d^{(p)}(X) \subseteq \mathrm{Iso}_{d_{F,p}}(X)$ *holds, that is* $\mathrm{Iso}_d^{(p)}(X)$ *is a subgroup of* $\mathrm{Iso}_{d_{F,p}}(X)$.

Proof. Let $f \in \mathrm{Iso}_d^{(p)}(X)$. For every $x, y \in X, x \neq y$, we have

$$\begin{aligned} d_{F,p}(f(x), f(y)) &= d(f(x), p) + d(p, f(y)) = d(f(x), f(p)) + d(f(p), f(y)) \\ &= d(x,p) + d(p,y) = d_{F,p}(x,y), \end{aligned}$$

hence, f belongs to $\mathrm{Iso}_{d_{F,p}}(X)$. □

For $x \neq y$, the relation $d_{F,p}(f(x), f(y)) = d_{F,p}(x,y)$ is equivalent to

$$d(f(x), p) + d(p, f(y)) = d(x,p) + d(p,y). \tag{2.2}$$

Theorem 2.2. *For every isometry* $f \in \mathrm{Iso}_{F,p}(X)$*, the point* p *is fixed, that is* $f(p) = p$.

Proof. Let start to investigate the topology generated by metric $d_{F,p}$. Consider the ball of radius r and centered in the point C, that is the set

$$B_{F,p}(C; r) = \{x \in X : d_{F,p}(C, x) \leq r\}.$$

The inequality $d_{F,p}(C,x) \leq r$ is equivalent to $d(C,p) + d(p,x) \leq r$. If $C \neq p$, then we obtain $d(p,x) \leq r - d(C,p)$. This means that $B_{F,p}(C; r) = B_d(p; r - d(C,p)$ if $d(C,p) < r$ and $B_{F,p}(C; r) = \{C\}$ if $d(C,p) \geq r$. Therefore, the neighborhoods basis at the point p generated by $d_{F,p}$ coincides with the the neighborhoods basis at the point p generated by the metric d.

For every $x, y \in X, x \neq y$, we have $d_{F,p}(x,y) = d(x,p) + d(p,y) \geq d(x,y)$, hence

$$d_{F,p}(x,y) \geq d(x,y). \tag{2.3}$$

Because f is a $d_{F,p}$-isometry, from (2.3) it follows

$$d(f(x), f(y)) \leq d_{F,p}(f(x), f(y)) = d_{F,p}(x,y).$$

Take $y = p$ in relation (2.2) and obtain

$$d(f(x), p) + d(p, f(p)) = d(x,p). \tag{2.4}$$

Clearly, f is continuous in the topology generated by the metric $d_{F,p}$. According to the result from the beginning of the proof, we have $x \to p$ in the topology of $d_{F,p}$ if and only if $x \to p$ in the topology of d. That is $d_{F,p}(x,p) \to 0$ if and only if $d(x,p) \to 0$, hence the relation (2.4) implies $d(f(p), p) = 0$, hence $f(p) = p$. □

From the result in Theorem 2.2, it follows :

Corollary 2.3. *For every metric space (X, d) and for every point $p \in X$, the metric space $(X, d_{F,p})$ is of elliptic type, that is every its isometry is elliptic.*

3. Comments on the case $X = \mathbb{R}^n$ and $d = d_2$

Let consider $X = \mathbb{R}^n$ with the Euclidean metric $d = d_2$, and the point p to be the origin of $\mathbb{R}^n$. If f is an isometry with respect to the induced French railroad metric, then f satisfies relation (2.4) hence, for every $x, y \in \mathbb{R}^n$, we have

$$||f(x)|| + ||f(y)|| = ||x|| + ||y||. \tag{3.1}$$

According to Theorem 2.2, the origin of $\mathbb{R}^n$ is a fixed point of f, i.e. we have $f(0) = 0$. Therefore, the relation (3.1) is equivalent to

$$||f(x)|| = ||x||, \quad x \in \mathbb{R}^n. \tag{3.2}$$

Therefore, $f \in \mathrm{Iso}_{d_{F,0}}(\mathbb{R}^n)$ if and only if it is bijective and it satisfies the functional equation (3.2). The equation (3.2) ensures the continuity of f only at the point 0. For the sake of simplicity, let us denote by G_n the isometry group $\mathrm{Iso}_{d_{F,0}}(\mathbb{R}^n)$. If f is linear, then the equation (3.2) means that it is an orthogonal transformation of $\mathbb{R}^n$. It follows that the real orthogonal group $O(n, \mathbb{R})$ is a subgroup of G_n. The group $\{-1_{\mathbb{R}^n}, 1_{\mathbb{R}^n}\}$ is a normal subgroup of G_n.

Also, if we impose some smoothness conditions we obtain interesting situations. Denote by $G_n^k, k = 0, 1, \cdots, \infty$, the subgroup of G_n consisting in all $d_{F,0}$-isometries of class C^k. Clearly, we get

$$G_n^\infty \subset \cdots \subset G_n^k \subset \ldots \subset G_n^1 \subset G_n^0 \subset G_n.$$

On the other hand, if $f \in G_n^k$, then the restriction $f\,|_{S^{n-1}}$ is an element of the group $\mathrm{Diff}^k(S^{n-1})$ of the C^k- diffeomorphisms of the unity $(n-1)$-dimensional sphere of the space $\mathbb{R}^n$. As we expect, a bijective extension of a C^k- diffeomorphism $h : S^{n-1} \to S^{n-1}$ to the space $\mathbb{R}^n$, preserving the property to satisfy (3.2), is not unique. This means that it is very possible that the group G_n^k is larger than $\mathrm{Diff}^k(S^{n-1})$.

In the case $n = 1$, the group G_1 is defined by all bijective maps $f : \mathbb{R} \to \mathbb{R}$, continuous at 0, and satisfying the functional equation $|f(x)| = |x|, x \in \mathbb{R}$. The relation $|f(x)| = |x|$ is equivalent to $f^2(x) - x^2 = 0$, i.e. $(f(x) - x)(f(x) + x) = 0$. Therefore, the general possible form of the maps in G_1 is

$$f_A(x) = \begin{cases} x & \text{if} \quad x \in \mathbb{R} \setminus A \\ -x & \text{if} \quad x \in A \end{cases}$$

where A is an arbitrary subset of $\mathbb{R}$. For all real number $a \neq 0$, the map

$$f_a(x) = \begin{cases} x & \text{if} \quad x \neq -a, a \\ -a & \text{if} \quad x = a \\ a & \text{if} \quad x = -a \end{cases}$$

belongs to G_1. Moreover, we have $f_a \circ f_a = 1_{\mathbb{R}}$ hence $\{1_{\mathbb{R}}, f_a\}$ is a subgroup of G_1. This means that the group G_1 has infinitely many subgroups isomorphic to $\mathbb{Z}_2$. Another subgroup isomorphic to $\mathbb{Z}_2$ is $G_1^{\infty} = \{-1_{\mathbb{R}}, 1_{\mathbb{R}}\}$, and it contains all smooth diffeomorphisms of the real line satisfying the functional equation $|f(x)| = |x|$, $x \in \mathbb{R}$. It is a normal subgroup of G_1. Clearly, $1_{\mathbb{R}}$ preserves the orientation of the 0-dimensional sphere $S^0 = \{-1, 1\}$, and $-1_{\mathbb{R}}$ reverses the orientation of $S^0 = \{-1, 1\}$.

In the case $n = 2$, we can identify $\mathbb{R}^2$ with the complex plane $\mathbb{C}$, and then the group G_2 is defined by all bijective maps $f : \mathbb{C} \to \mathbb{C}$, continuous at 0, and satisfying the functional equation $|f(z)| = |z|, z \in \mathbb{C}$. The orthogonal group $O(2, \mathbb{R})$ is the symmetry group of the circle $S^1 = \{z \in \mathbb{C} : |z| = 1\}$. It is isomorphic (as a real Lie group) to the circle group $(S^1, \cdot)$, also known as $U(1)$. In this case, the circle group $(S^1, \cdot)$ is a subgroup of G_2. Consequently, all subgroups of $(S^1, \cdot)$ are subgroups of G_2.

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Barycentric and trilinear coordinates of some remarkable points of a hyperbolic triangle

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To the memory of Professor Mircea-Eugen Craioveanu (1942-2012)

Abstract. In this paper we establish the barycentric and trilinear equations of the altitudes and perpendicular bisectors of a hyperbolic triangle and we compute the barycentric and trilinear coordinates of the orthocenter and circumcenter. We, also, indicate necessary and sufficient conditions for these two points to be ordinary points.

Mathematics Subject Classification (2010): 51M09, 51M10.

Keywords: Trilinear coordinates, barycentric coordinates, hyperbolic plane.

1. Introduction

The purpose of this paper is to give some methods to compute the barycentric and trilinear coordinates for some important points in the hyperbolic triangle. For this, we will use the *Cayley-Klein model*, also called the *projective model*.

In the following, we shall consider the hyperbolic triangle ABC in which the *trilinear coordinates* are defined in a natural way, as the hyperbolic distances from an arbitrary point M in the plane of the triangle to the sides of the triangle. The *barycentric coordinates* are obtained from trilinear coordinates, multiplying the values by the hyperbolic sines of the hyperbolic lengths of the sides of the triangle.

The definition of these coordinates can be given, also, by specifying a particular choice of the polarity that defines the Absolute. This is reflected in the definition of the polarity matrices $[c_{\mu\nu}]$ and $[C_{\mu\nu}]$. We remind these matrices for both of the coordinates systems:

For the trilinear coordinates system:

$$[c_{\mu\nu}] = \frac{1}{\Gamma}\begin{pmatrix} \sin^2 a & \sin A \sin B \cosh c & \sin A \sin C \cosh b \\ \sin A \sin B \cosh c & \sin^2 B & \sin B \sin C \cosh A \\ \sin A \sin C \cosh b & \sin B \sin C \cos a & \sin^2 C \end{pmatrix}$$

and

$$[C_{\mu\nu}] = \begin{pmatrix} -1 & \cos C & \cos B \\ \cos C & -1 & \cos A \\ \cos B & \cos A & -1 \end{pmatrix}$$

For barycentric coordinates system:

$$[c_{\mu\nu}] = \begin{pmatrix} 1 & \cosh c & \cosh b \\ \cosh c & 1 & \cosh a \\ \cosh b & \cosh a & 1 \end{pmatrix}$$

and

$$[C_{\mu\nu}] = \frac{1}{\Gamma} \begin{pmatrix} -\sinh^2 a & \sinh a \sinh b \cos C & \sinh a \sinh c \cos B \\ \sinh a \sinh b \cos C & -\sinh^2 b & \sinh b \sinh c \cos A \\ \sinh a \sinh c \cos B & \sinh b \sinh c \cos A & -\sinh^2 c \end{pmatrix}.$$

We can pass from point coordinates to line coordinates (and the other way around) by using the relations:

$$\begin{cases} x_\mu = c_{\mu\nu} \cdot \xi_\nu \\ \xi_\mu = C_{\mu\nu} x_\mu. \end{cases}$$

where x_μ are the point coordinates and ξ_ν are the line coordinates.

See [1, 2] for details.

Remark 1.1. A different approach to barycentric coordinates, using the Poincaré disk model, was taken by A. Ungar (see [4]).

2. Barycentric and trilinear equation of the altitudes. Coordinates of the Orthocenter

In the following we will present the coordinates in barycentric coordinates. Having the result in barycentric coordinates system, the reader can easily obtain at any time the coordinates in trilinear coordinate system by dividing each component with the hyperbolic sinus of corresponding side length of the triangle.

Having ABC a hyperbolic triangle, we denote by A' the orthogonal projection of the vertex A on the side BC, B' the orthogonal projection of the vertex B on the side AC and C' the orthogonal projection of the vertex C on the side AB.

For start, we want to obtain the equation of the line AA'. We know that in barycentic coordinates, A is defined by $(1, 0, 0)$.

A general equation of a line, both in barycentric and trilinear coordinates, is of the form:

$$\alpha_0 X_0 + \alpha_1 X_1 + \alpha_2 X_2 = 0. \tag{2.1}$$

Because AA' passes through A, this means that AA' is of the form: $\alpha_1 x_1 + \alpha_2 x_2 = 0$, or, to put it another way, the *line* coordinates of AA', denoted by ξ are:

$$\xi = [0, \alpha_1, \alpha_2]. \tag{2.2}$$

We also know that the side BC, denoted by η has the line coordinates

$$\eta = [1, 0, 0] \tag{2.3}$$

As $AA' \perp BC$, we have the relation

$$[\xi, \eta] = 0. \tag{2.4}$$

By definition,

$$\begin{aligned}[\xi, \eta] = C_{00} \cdot \xi_0 \nu_0 + C_{01} \cdot \xi_0 \nu_1 + C_{02} \cdot \xi_0 \nu_2 + \\ C_{10} \cdot \xi_1 \nu_0 + C_{11} \cdot \xi_1 \nu_1 + C_{12} \cdot \xi_1 \nu_2 + \\ C_{20} \cdot \xi_2 \nu_0 + C_{21} \cdot \xi_2 \nu_1 + C_{22} \cdot \xi_2 \nu_2.\end{aligned}$$

From 2.2 and 2.3 we have:

$$\begin{aligned}[\xi, \eta] = C_{00} \cdot 0 \cdot 1 + C_{01} \cdot 0 \cdot 1 + C_{02} \cdot 0 \cdot 0 + \\ C_{10} \cdot \alpha_1 \cdot 1 + C_{11} \cdot \alpha_1 \cdot 0 + C_{12} \cdot \alpha_1 \cdot 1 + \\ C_{20} \cdot \alpha_2 \cdot 1 + C_{21} \cdot \alpha_2 \cdot 0 + C_{22} \cdot \alpha_2 \cdot 0 = \\ = C_{10} \cdot \alpha_1 + C_{20} \cdot \alpha_2.\end{aligned}$$

If we use the matrix $[C_{\mu\nu}]$ for barycentric coordinate, we obtain:

$$[\xi, \eta] = \sinh a \cdot \sinh b \cdot \cos C \cdot \alpha_1 + \sinh a \cdot \sinh c \cdot \cos B \cdot \alpha_2.$$

By using the condition (2.4), we have:

$$\sinh a \cdot \sinh b \cdot \cos C \cdot \alpha_1 + \sinh a \cdot \sinh c \cdot \cos B \cdot \alpha_2 = 0$$

thus, the relation between α_1 and α_2 is:

$$\alpha_2 = -\alpha_1 \frac{\sinh b \cdot \cos C}{\sinh c \cdot \cos B}.$$

In conclusion we have the following coordinates for ξ

$$\xi = \left[0, 1, -\frac{\sinh b \cdot \cos C}{\sinh c \cdot \cos B}\right] \tag{2.5}$$

or, more simplified:

$$\xi = [0, \sinh c \cdot \cos B, -\sinh b \cdot \cos C].$$

Thus the equation of the altitude AA' is:

$$AA' : X_1 \sinh c \cos B - X_2 \sinh b \cdot \cos C = 0.$$

By performing the same computation for the other two altitudes, we obtain the following

Theorem 2.1. *The equations of the altitudes of the hyperbolic triangle ABC, written in the barycentric coordinates determined by the triangle, are:*

$$\begin{aligned} AA' &: X_1 \sinh c \cos B - X_2 \sinh b \cdot \cos C = 0, \\ BB' &: X_0 \sinh c \cdot \cos A - X_2 \sinh a \cdot \cos C = 0, \\ CC' &: X_0 \sinh b \cdot \cos A - X_1 \sinh a \cdot \cos B = 0. \end{aligned} \tag{2.6}$$

By solving the system (2.6), we obtain:

Consequence 2.2. *The barycentric coordinates of the orthocenter of the hyperbolic triangle ABC are given by:*

$$H\left(\frac{1}{\sinh b\sinh c\cos A},\frac{1}{\sinh c\sinh a\cos B},\frac{1}{\sinh a\sinh b\cos C}\right). \tag{2.7}$$

The orthocenter is a real point (i.e. the altitudes do *intersect), iff we have $(H,H)>0$, i.e. iff*

$$\begin{aligned}&\left(\sinh^2 a\cos^2 B+2\sinh a\sinh b\cosh c\cos A\cos B+\sinh^2 b\cos^2 A\right)\cos^2 C+\\&+\left(2\sinh a\cosh b\sinh c\cos A\cos^2 B+2\cosh a\sinh b\sinh c\cos^2 A\cos B\right)\cos C+\\&+\sinh^2 c\cos^2 A\cos^2 B>0\end{aligned} \tag{2.8}$$

If we want to use trilinear coordinates, instead, we simple apply a coordinate change to the equations from the Theorem 2.1 and the Consequence 2.2 and we get:

Theorem 2.3. *The equations of the altitudes of the hyperbolic triangle ABC, written in the trilinear coordinates determined by the triangle, are:*

$$\begin{aligned}AA'&: x_1\cos B-x_2\cos C=0,\\BB'&: x_0\cos A-x_2\cos C=0,\\CC'&: x_0\cos A-x_1\cos B=0\end{aligned} \tag{2.9}$$

Corollary 2.4. *The trilinear coordinates of the orthocenter of the hyperbolic triangle ABC are given by:*

$$H\left(\frac{1}{\cos A},\frac{1}{\cos B},\frac{1}{\cos C}\right). \tag{2.10}$$

The orthocenter is an ordinary point iff $(H,H)>0$, i.e. iff

$$\begin{aligned}&\cos^2 A\cos^2 B\sin^2 C+\\&+\left(2\cosh a\cos^2 A\cos B\sin B+2\cosh b\cos A\sin A\cos^2 B\right)\cos C\sin C+\\&\left(\cos^2 A\sin^2 B+2\cosh c\cos A\sin A\cos B\sin B+\sin^2 A\cos^2 B\right)\cos^2 C>0.\end{aligned} \tag{2.11}$$

It can be proved that the equations (2.8) and (2.11) are equivalent.

3. Barycentric and trilinear equation of the perpendicular bisectors. Coordinates of the Circumcenter

In order to obtain the line coordinates of the perpendicular bisectors, we use the already known coordinates of the midpoints of the sides of the triangle (see [1]).

If we consider A'' to be the midpoint of BC, B'' – the midpoint of AC and C'' – the midpoint of AB, we have their coordinates in trilinear coordinates:

$$A''\left(0, \sinh\frac{a}{2}\sin C, \sinh\frac{a}{2}\sin B\right);$$
$$B''\left(\sinh\frac{a}{2}\sin C, 0, \sinh\frac{b}{2}\sin A\right);$$
$$C''\left(\sinh\frac{c}{2}\sin B, \sinh\frac{c}{2}\sin A, 0\right);$$

or

$$\begin{aligned} &A''\left(0, \sin C, \sin B\right); \\ &B''\left(\sin C, 0, \sin A\right); \\ &C''\left(\sin B, \sin A, 0\right). \end{aligned} \tag{3.1}$$

From these, we can easily obtain the barycentric coordinates (see [1]).

We denote by ξ the line perpendicular to BC at the point A''. Then the general equation of ξ (in trilinear coordinates) is:

$$\xi : \alpha_0 x_0 + \alpha_1 x_1 + \alpha_2 x_2.$$

We also know that the equation of BC is $\eta : x_0 = 0$. So we have:

$$\xi = [\alpha_0, \alpha_1.\alpha_2],$$
$$\eta = [1, 0, 0].$$

Because we know that A'' is on ξ we have:

$$\alpha_1 \sinh\frac{a}{2}\sin C + \alpha_2 \sinh\frac{a}{2}\sin B = 0$$
$$\alpha_2 = -\alpha_1 \cdot \frac{\sin C}{\sin B}$$

If we replace in the quation of ξ we get:

$$\xi : \alpha_0 \sin B x_0 + \alpha_1 \sin B x_1 - \alpha_1 \sin C x_2 = 0;$$

We know that $\xi \perp \eta$, which implies that $[\xi, \eta] = 0$.

Using the relation (2.4), we obtain:

$$\begin{aligned} [\xi, \eta] &= C_{00}\xi_0\eta_0 + C_{10}\xi_1\eta_1 + C_{20}\xi_2\eta_0 \\ &= C_{00}\alpha_0 \sin B + C_{10}\alpha_1 \sin B - C_{20}\alpha_1 \sin C \\ &= -\alpha_0 \sin B + \cos C \sin B \alpha_1 - \cos B \sin C \alpha_1 \\ &= 0 \end{aligned}$$

Thus

$$\alpha_0 = \alpha_1 \frac{\sin C \cos C - \cos B \sin C}{\sin B} = \alpha_1 \frac{\sin(B - C)}{\sin B}$$

If we replace in the general equation for AA'', we have the form for the perpendicular bisector AA'':

$$\xi_{AA''} = [\sin B \cos C - \cos B \sin C, \sin B, -\sin C]. \tag{3.2}$$

After similar computations, we obtain the theorem:

Theorem 3.1. *The trilinear line coordinates of the perpendicular bisectors of the sides of the hyperbolic triangle ABC are*

$$\begin{aligned}\xi_{AA''} &= [\sin B\cos C-\cos B\sin C, \sin B, -\sin C],\\ \xi_{BB''} &= [\sin A, \sin C\cos A-\cos C\sin A, -\sin C],\\ \xi_{CC''} &= [\sin A, -\sin B, \cos B\sin A-\cos A\sin B].\end{aligned}\tag{3.3}$$

or

$$\begin{aligned}\xi_{AA''} &= [\sin(B-C), \sin B, -\sin C],\\ \xi_{BB''} &= [\sin A, \sin(C-A), -\sin C],\\ \xi_{CC''} &= [\sin A, -\sin B, \sin(A-B)].\end{aligned}\tag{3.4}$$

After solving the system of equations of the perpendicular bisectors, we get

Corollary 3.2. *The trilinear coordinates of the circumcenter of the hyperbolic triangle ABC are*

$$\begin{aligned}&O\left(\sin B-\sin(C-A), \sin(C-B)+\sin A,\right.\\ &\left.\sin(C-A)\sin(C-B)+\sin A\sin B\right).\end{aligned}\tag{3.5}$$

O is an ordinary point iff $(O,O)>0$, i.e. iff

$$\begin{aligned}&\big(\sin^2 C\sin^2(C-A)+2\,\cosh a\sin B\sin C\sin(C-A)+\sin^2 B\big)\cdot\\ &\cdot\sin^2(C-B)+\big(-2\cosh b\sin A\sin C\sin^2(C-A)+\\ &\big(2\sin A\sin B\sin^2 C+(2\cosh b+\cosh a)\sin A\sin B\sin C-\\ &-2\cosh c\sin A\sin B\big)\sin(C-A)+2\cosh a\sin A\sin^2 B\sin C+\\ &+(2\cosh c+2)\sin A\sin^2 B\big)\sin(C-B)+\sin^2 A\sin^2(C-A)+\\ &+\big((-2\cosh c-2)\sin^2 A\sin B-2\cosh b\sin^2 A\sin B\sin C\big)\sin(C-A)+\\ &+\sin^2 A\sin^2 B\sin^2 C+(2\cosh b+2\cosh a)\sin^2 A\sin^2 B\sin C+\\ &+(2\cosh c+2)\sin^2 A\sin^2 B>0.\end{aligned}\tag{3.6}$$

If we pass to the barycentric coordinates, we get immediately, from the Theorem (3.1):

Theorem 3.3. *The barycentric line coordinates of the perpendicular bisectors of the sides of the hyperbolic triangle ABC are*

$$\begin{aligned}\xi_{AA''} &= [\sinh b\sinh c\sin(B-C), \sinh a\sinh c\sin B, -\sinh a\sinh b\sin C],\\ \xi_{BB''} &= [\sinh b\sinh c\sin A, \sinh a\sinh c\sin(C-A), -\sinh a\sinh b\sin C],\\ \xi_{CC''} &= [\sinh b\sinh c\sin A, -\sinh a\sinh c\sin B, \sinh a\sinh b\sin(A-B)].\end{aligned}\tag{3.7}$$

Also, the consequence (3.2) gives rise to the consequence

Consequence 3.4. *The barycentric coordinates of the circumcenter of the hyperbolic triangle ABC are*

$$\begin{aligned}&O\left(\sinh a(\sin B-\sin(C-A)), \sinh b(\sin(C-B)+\sin A),\right.\\ &\left.\sinh c(\sin(C-A)\sin(C-B)+\sin A\sin B)\right).\end{aligned}\tag{3.8}$$

The point O is ordinary iff $(O,O)>0$, i.e. iff

$$\begin{aligned}
&\big(\sinh^2 c \sin^2(C-A) + 2\cosh a \sinh b \sinh c \sin(C-A) + \sinh^2 b\big)\cdot \\
&\cdot \sin^2(C-B) + \big(-2\sinh a \cosh b \sinh c \sin^2(C-A) + \\
&+ \big(\big(2\sinh^2 c \sin A + 2\sinh a \cosh b \sinh c\big)\sin B + \\
&+2\cosh a \sinh b \sinh c \sin A - 2\sinh a \sinh b \cosh c)\sin(C-A) + \\
&+ \ (2\cosh a \sinh b \sinh c \sin A + 2\sinh a \sinh b \cosh c)\sin B + \\
&+2\sinh^2 b \sin A\big)\sin(C-B) + \sinh^2 a \sin^2(C-A) + \\
&+ \big(\big(-2\sinh a \cosh b \sinh c \sin A - 2\ \sinh^2 a\big)\sin B - \\
&-2\sinh a \sinh b \cosh c \sin A)\sin(C-A) + \big(\sinh^2 c \sin^2 A + \\
&+2\sinh a \cosh b \sinh c \sin A + \sinh^2 a\big)\sin^2 B + \\
&+ \big(2\cosh a \sinh b \sinh c \sin^2 A + 2\sinh a \sinh b \cosh c \sin A\big)\sin B + \\
&+ \sinh^2 b \sin^2 A > 0.
\end{aligned} \tag{3.9}$$

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Oeljeklaus-Toma manifolds and locally conformally Kähler metrics. A state of the art

Liviu Ornea and Victor Vuletescu

To the memory of Professor Mircea-Eugen Craioveanu (1942-2012)

Abstract. We review several properties about Oeljeklaus-Toma manifolds, especially the locally conformally Kähler ones, with focus on the non-existence of certain complex submanifolds.

Mathematics Subject Classification (2010): 53C55, 32J18.

Keywords: Compact complex manifolds, algebraic number fields, algebraic units, locally conformally Kähler metrics, complex submanifold.

1. Introduction

The idea of associating compact complex manifolds to number fields is present since the very beginnings of complex geometry. If one was to write a history of this ideas, he would probably start from elliptic curves, which subtle links to number theory were felt by L. Kronecker and K. Weierstrass, would then include H. Weyl, whose research on complex tori have roots in the study of number fields units, and would then arrive to A. Weil who extended this line of research to Kähler manifolds.

The goal of the present paper is to give an account on the recent progress in a highly interesting class of compact complex manifolds associated to certain number fields introduced by K. Oeljeklaus & M. Toma in 2005. Despite being a relatively new topic, this kind of manifolds already provided a number of surprising results in the non-Kähler geometry, as we shall see below.

2. Basic facts from algebraic number theory

We recall (cf. e.g. [7]) that an (abstract) number field is a finite extension K of $\mathbb{Q}$; it follows that K is isomorphic (as $\mathbb{Q}-$algebras) to $\mathbb{Q}[X]/(f)$ where $f \in \mathbb{Z}[X]$ is a (monic) irreducibile polynomial. An abstract number field K can be embedded into $\mathbb{C}$

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by mapping $X(\mathrm{mod} f) \in K$ to α, where α is a root of f. It follows that K has exactly n embeddings into $\mathbb{C}$, where $n = \deg(f)$. Usually, one divides the roots of f into two subsets: the real ones, and call the corresponding embeddings *real embeddings* of K, and the complex, non-real ones, that come in pairs of conjugate numbers (and call the resulting embeddings accordingly, *complex embeddings*). We shall denote by s the number of real embeddings and by $2t$ the number of complex ones; hence $n = s + 2t$.

An *algebraic integer* of K is an element $a \in K$ satisfying a monic equation with integer coefficients. The set of all algebraic integers of K forms a ring, usually denoted by $\mathcal{O}_K$. For instance, if $p > 2$ is some prime number and $K = \mathbb{Q}(\zeta_p)$ (where ζ_p is a primitive root of unity of order p) then $\mathcal{O}_K = \mathbb{Z}[\zeta_p]$. But in the general case, such nice descriptions of the ring of integers are no longer available. Eventually, let us recall that seen as $\mathbb{Z}-$module, $\mathcal{O}_K$ is free of rank n.

The invertible elements of $\mathcal{O}_K$ are called *units*, and the (multiplicative) group of units is denoted $\mathcal{O}_K^*$. By the celebrated Dirichlet's units theorem, $\mathcal{O}_K^*$ is a group of rank $s+t-1$. For instance, if $K = \mathbb{Q}(\sqrt{3})$ then any solution $(a, b) \in \mathbb{Z}^2$ of the Pell equation

$$x^2 - 3y^2 = 1$$

will define a unit $a + b\sqrt{3} \in \mathcal{O}_K^*$. By contrast, in $K = \mathbb{Q}(i\sqrt{3})$ the only units are ± 1 and $\pm\epsilon$, where ϵ is a non-real root of unity of order 3. Again, in the general case, there are no immediate descriptions of the group of units.

3. Oeljeklaus-Toma manifolds

3.1. The construction

The following construction was done in [8].

Fix a number field K with s real embeddings and $2t > 0$ complex embeddings. Suppose the embeddings $\sigma_1, \ldots, \sigma_n$ of K are labelled in such a way that the first s ones are real, and $\sigma_{s+k} = \overline{\sigma}_{s+t+k}$ for all $k,\ 1 \le k \le t$.

We say that a unit $u \in \mathcal{O}_K^*$ is *totally positive* if $\sigma_i(u) > 0$ for all real embeddings $\sigma_i, 1 \le i \le s$. The set $\mathcal{O}_K^{*,+}$, of totally positive units form a subgroup of $\mathcal{O}_K^*$, obviously of finite index - since for any unit u, its square u^2 is totally positive.

Let $\mathbb{H} = \{z \in \mathbb{C}\,;\, \mathrm{Im}\, z > 0\}$ be the upper half-plane. For any $a \in \mathcal{O}_K$ denote by T_a the automorphism of $\mathbb{H}^s \times \mathbb{C}^t$ given by

$$T_a(z_1, \ldots, z_{t+s}) = (z_1 + \sigma_1(a), \ldots, z_{s+t} + \sigma_{s+t}(a))\,.$$

Similary, for any totally positive unit u, let R_u be the automorphism of $\mathbb{H}^s \times \mathbb{C}^t$ defined by

$$R_u(z_1, \ldots, z_{t+s}) = (\sigma_1(u)z_1, \ldots, \sigma_{s+t}(u)z_{t+s})\,.$$

Note that the totally positivity of u is needed for R_u to act on $\mathbb{H}^s \times \mathbb{C}^t$.

The above maps define for any subgroup $U \subset \mathcal{O}_K^{*,+}$ a fixed-point-free action of the semidirect product $U \ltimes \mathcal{O}_K$ on $\mathbb{H}^s \times \mathbb{C}^t$. The main point is that one can always find subgroups U such that the above action is also discrete and cocompact; such subgroups are called *admissible subgroups*. Note that if U is an admissible subgroup then necessarily one has $\mathrm{rank}_{\mathbb{Z}}(U) + \mathrm{rank}_{\mathbb{Z}}(\mathcal{O}_K) = 2(s+t)$, hence $\mathrm{rank}_{\mathbb{Z}}(U) = s$. This explains why the condition $t > 0$ is needed: otherwise we would have that the rank of $\mathcal{O}_K^*$ is $s-1$, strictly less than s, and hence admissible subgroups could not exist.

By definition, if U is an admissible subgroup, the compact quotient

$$\mathbb{H}^s \times \mathbb{C}^t / U \ltimes \mathcal{O}_K$$

is called *an Oeljeklaus-Toma manifold* and is usually denoted by $X(K, U)$.

Remark 3.1. For $s = t = 1$, one recovers the familiar Inoue surface S_M, [5]. This is known to be (real) homogeneous, indeed a solvmanifold. Accordingly, H. Kasuya proved the following:

Proposition 3.2. [6, §6] *Oeljeklaus-Toma manifolds are solvmanifolds.*

Indeed, Kasuya proved that

$$X(K, U) = G/U \ltimes \mathcal{O}_K, \quad \text{with} \quad G = \mathbb{R}^s \ltimes_\phi (\mathbb{R}^s \times \mathbb{C}^t).$$

Here ϕ acts as follows:

$$\phi(t_1, \ldots, t_s) = \operatorname{diag}\left(e^{t_1}, \ldots, e^{t_s}, e^{\psi_1 + \sqrt{-1}\phi_1}, \ldots, e^{\psi_t + \sqrt{-1}\phi_t}\right),$$

where $\psi_k = \frac{1}{2}\sum_1^s b_{ik} t_i$, $\varphi_k = \sum_1^s c_{ik} t_i$, with the coefficients b_{ik}, c_{ik} given by expressing $|\sigma_{s+k}(a)| = e^{\frac{1}{2}\sum_1^s b_{ik} t_i}$, and hence $\sigma_{s+k}(a) = e^{\frac{1}{2}\sum_1^s b_{ik} t_i + \sum_1^s c_{ik} t_i}$.

The natural complex structure on $\mathbb{R}^s \ltimes_\phi (\mathbb{R}^s \times \mathbb{C}^t)$ is seen to descend to the quotient and to be integrable, but the induced complex structure is G left-invariant and *not* G right-invariant, and hence $X(K, U)$ is not a complex Lie group. This is in accordance with the result proven in [8] (that we also recall below, see 3.5) that the biholomorphism group of $X(K, U)$ is discrete.

3.2. Basic invariants

We next investigate the basic invariants of Oeljeklaus-Toma manifolds. We start by looking at their Betti numbers.

Theorem 3.3. ([8]) *If K is a number field with s real embeddings and t complex embeddings, and if U is an admissible subgroup of $\mathcal{O}_K^*$, then:*

a) $b_1(X(K, U)) = s$;

b) if, in addition, there is no proper subfield $L \subset K$ such that $U \subset \mathcal{O}_L^*$, then

$$b_2(X(K, U)) = \binom{s}{2}.$$

Sketch of proof. The basic idea to compute $H^i(X(K, U), \mathbb{Q})$ is as follows. Since the universal cover of $X(K, U)$ is contractible, one is reduced to compute the group cohomology $H^i(U \ltimes \mathcal{O}_K, \mathbb{Q})$. Next, as $U \ltimes \mathcal{O}_K$ is a semidirect product of two abelian groups, and since the cohomology of abelian groups is well-known, one can simply use the Lyndon-Hochschild-Serre spectral sequence

$$E_2^{p,q} = H^p(U, H^q(\mathcal{O}_K, \mathbb{Q})) \Rightarrow H^{p+q}(U \ltimes \mathcal{O}_K, \mathbb{Q}).$$

Now the claims of a) and b) follow by a careful inspection of the differentials in the above spectral sequence.

Alternatively, one can prove a) directly, along the following lines. An immediate computation shows that

$$T_a R_u T_a^{-1} R_u^{-1} = T_{(1-u)a}$$

for any $u \in U$ and any $a \in \mathcal{O}_K$. Now one can check that the subgroup of $\mathcal{O}_K$ generated by elements of the form $(1-u)a$ is of finite index, so U is a quotient of $H_1(X(K,U),\mathbb{Z})$ by a finite subgroup; consequently, the first Betti number will equal the rank of U.

Remark 3.4. In fact, one can explicitely exhibit s linearly independent closed 1-forms on $X(K,U)$. Indeed, if we let $z_k = x_k + iy_k$ for all k, then the differential forms

$$\omega_k = \frac{1}{y_k} dy_k, k = 1, \ldots, s \tag{3.1}$$

defined on $\mathbb{H}^s \times \mathbb{C}^t$ are $U \ltimes \mathcal{O}_K-$invariant, hence descending to forms on $X(K,U)$.

Next, we look at some analytical invariants.

Theorem 3.5. ([8]) *On any Oeljelkaus-Toma manifold $X = X(K,U)$ the holomorphic vector bundles: Ω^1_X, the holomorphic tangent bundle $\mathcal{T}_X$ and any positive power $\mathcal{K}^n_X$ of the canonical bundle have no global holomorphic sections. Consequently, $H^{1,0}(X) = H^0(X, \Omega^1_X) = 0$, X has finitely many automorphisms and its Kodaira dimension is $-\infty$.*

By contrast, $h^{0,1}(X) = \dim_{\mathbb{C}} H^1(X, \mathcal{O}_X) \geq s$. In particular, since $h^{0,1} \neq h^{1,0}$, it follows that for $s > 0$ the manifold X cannot carry Kähler metrics.

Sketch of proof. The assertions on the absence of global sections of all the vector bundles in the statement follow in a rather direct way: one shows that the corresponding bundles on the universal cover have no non-zero global sections which are invariant under $U \ltimes \mathcal{O}_K$. The key ingredient is the following fact: if one factors $\mathbb{H}^s \times \mathbb{C}^t$ by $\mathcal{O}_K$ only (hence getting a non-compact manifold, which covers X), the quotient has no global non-constant holomorphic function (exactly as in the compact case).

4. Oeljeklaus-Toma manifolds and locally conformally Kähler geometry

4.1. LCK geometry

At this point, we recall the notion of *locally conformally Kähler manifold, LCK* for short, see [3]. By definition, a hermitian metric g on a complex manifold X is LCK if X can be covered by open subsets

$$X = \bigcup_{\alpha \in A} U_\alpha$$

with the property that on each U_α there exists a Kähler metric g_α wich is conformal to the restriction of g to U_α,

$$g_\alpha = e^{-f_\alpha} g_{|U_\alpha}$$

for some smooth function f_α defined on U_α. If one of the U_α equals the whole X, we say that g is *globally conformally Kähler*, GCK for short.

There are at least two different, equivalent ways, of saying that a hermitian metric g is LCK. One of them is as follows. Let g be a hermitian metric on the complex manifold X and let ω be its associated Kähler form,

$$\omega(X, Y) = g(XJY)$$

where J is the almost-complex structure of X. Then g is LCK if and only if there exists a closed 1-form θ (called *the Lee form* of g) such that

$$d\omega = \theta \wedge \omega.$$

Notice that g is GCK if and only if θ is exact.

An equivalent definition is as follows. Let $\widetilde{X}$ be the universal cover of X. Then X has an LCK metric iff $\widetilde{X}$ has a Kähler metric Ω upon which the fundamental group of X (seen as the group of deck transforms of $\widetilde{X}$) acts by homotheties,

$$\gamma^*(\Omega) = \chi(\gamma)\Omega, \forall \gamma \in \pi_1(X) \tag{4.1}$$

for some $\chi(\gamma) \in \mathbb{R}_{>0}$. Notice that in order to obtain non-GCK metrics on X, at least one $\chi(\gamma)$ above should be different from 1.

This last way of characterizing LCK manifolds is particularly useful in exhibiting examples. For instance, we can see that the so-called *diagonal Hopf manifolds* are LCK. Recall that such a manifold is by definition the quotient of $\mathbb{C}^n \setminus \{0\}$ under the action of $\mathbb{Z}$ generated by the map

$$(z_1, \ldots, z_n) \mapsto (\alpha z_1, \ldots, \alpha z_n)$$

where $\alpha \in \mathbb{C}, |\alpha| \neq 1$. Clearly, in this way, the action of $\mathbb{Z}$ is by homotheties with respect to the standard flat metric on $\mathbb{C}^n \setminus \{0\}$,

$$\omega_{flat} = dz_1 \wedge d\overline{z}_1 + \cdots + dz_n \wedge d\overline{z}_n.$$

In fact, all Hopf manifolds $\mathbb{C}^n \setminus \{0\}/\langle A\rangle$, with A being a linear operator with eigenvalues of strictly smaller than 1 absolute values are LCK, see [11].

Locally conformally Kähler metrics were introduced for the first time by I. Vaisman in the mid 80's. Since then, by the effort of many people, it was shown that almost all non-Kähler compact complex surfaces have LCK metrics, see [1], [2]. Still, in higher dimensions, until the paper of Oeljeklaus-Toma appeared, the only known LCK structures known were basically Hopf manifolds (and their complex submanifolds).

Theorem 4.1. ([8]) *Let K be a number field with $t = 1$ complex embeddings. Then, for any admissible group of totally positive units U, the manifold $X(K, U)$ has an LCK metric.*

Proof. Let $H : \mathbb{H}^s \times \mathbb{C} \to \mathbb{C}$ be the map

$$H(z_1, \ldots, z_s, z_{s+1}) = \prod_{i=1}^{s} \frac{1}{Im(z_i)} + |z_{s+1}|^2.$$

By direct computation, one checks that H is a *Kähler potential,* that is, its associated $(1, 1)-$ form

$$\omega = \sqrt{-1}\partial\overline{\partial}H$$

is a Kähler metric. Clearly, any translation T_a $(a \in \mathcal{O}_K)$ leaves ω invariant, while for any $u \in U$ we have

$$R_u^*(\omega) = |\sigma(u)|^2\omega$$

where σ is the only (up to complex conjugation) complex embedding of K.

We see that $U \ltimes \mathcal{O}_K$ acts by homotheties upon ω, hence $X(K, U)$ has a LCK metric. On the other hand, this metric will not be GCK, as $X(K, U)$ cannot carry Kähler metrics.

Remark 4.2. 1. For $s = t = 1$, the above metric coincides with the one found by F. Tricerri in [13] on the Inoue surfaces of type S_M.

2. We stress that, unlike $\sqrt{-1}\partial\overline{\partial}H$, the above potential H *is not* acted on by homotheties. Moreover, no potential with this automorphy property can exist on Oeljeklaus-Toma manifolds, as this would impose the deck group to be isomorphic to $\mathbb{Z}$, [10].

Remark 4.3. A very important subclass of LCK manifolds is defined in terms of the Lee form. Namely, if (X, g) is an LCK manifold with Lee form θ, then (X, g) is called a *Vaisman manifold* if

$$\nabla\theta = 0$$

where ∇ is the Levi-Civita connection of the metric g. Typical examples are the diagonal Hopf manifolds (see [4] for Hopf surfaces or [11] for higher dimensions); other examples appear on surfaces, [1]. Compact Vaisman manifolds have very good geometric properties and are intimately related to Sasakian manifolds.

It is easily seen that the LCK metric in [8] is not Vaisman. Moreover:

Proposition 4.4. ([6]) *Oeljeklaus-Toma manifolds cannot carry any Vaisman metric.*

This is again consistent with the result in [1] that no Inoue surface can carry Vaisman metrics. Kasuya's proof uses the homogeneous presentation of the Oeljeklaus-Toma manifold and a characterization of the existence of Vaisman metrics on certain types of solvmanifolds in terms of cohomology of Lie algebras.

4.2. The Vaisman conjecture

The Vaisman conjecture. In [14], it was asserted that any compact LCK manifold X should have at least one odd Betti number of odd degree:

$$b_{2k+1}(X) = 1 (\mathrm{mod}\, 2)$$

for some k. The conjecture was a long-standing one, until the paper of [8] appeared. The counter-example given there is as follows. Take any number field with $s = 2, t = 1$ and any admissible subgroup $U \subset \mathcal{O}_K^{*,+}$. Then the manifold $X(K, U)$ will carry an LCK metric, by the above 4.1. On the other hand, $X(K, U)$ is of (complex) dimension $s+t = 3$ and its first Betti number is $b_1(X) = s = 2$ by 3.3, a). Consequently, one also has $b_5(X) = b_1(X) = 2$ from Poincaré duality. As $X(K, U)$ carries a global, nowhere vanishing 1-form (recall the forms defined in (3.1)), its Euler-Poincaré characteristic vanishes, so $b_3(X)$ is also even. We see $X(K, U)$ is indeed a counter-example to Vaisman's conjecture.

4.3. Submanifolds of Oeljelkaus-Toma manifolds

As noticed already, in the case $s = t = 1$, the Oeljeklaus-Toma manifolds are Inoue surfaces of type S_M. This particular kind of surfaces are remarkable, as they carry no closed analytic curve. It is thus a natural question to ask about submanifolds, or more general, closed analytic subspaces of Oeljeklaus-Toma manifolds. Of course, for convenient choices of the number field K and for the admissible subgroup U, the corresponding manifold $X(K, U)$ will contain proper submanifolds. For instance, if K is a proper extension of another number field L and if $U \subset \mathcal{O}_L^*$, then $X(L, U) \subset X(K, U)$, see [8] for details. It is thus reasonable to restrict our attention to the cases with "nice geometry", more exactly to the case $t = 1$, where the existence of LCK metrics holds. In this case, one has:

Theorem 4.5. ([9]) *Let K be a number field with $t = 1$ and let $X = X(K, U)$ be an associated Oeljeklaus-Toma manifold. If $Y \subset X$ is a closed connected reduced analytic subspace, then either $Y = X$ or Y is a point. In other words, X carries no proper closed analytic subspaces, i.e. it is a* simple manifold*, in the sense of Campana. In particular, LCK Oeljeklaus-Toma manifolds do not admit non-constant meromorphic functions.*

The proof relies on two deep facts. One is of purely geometrico-differential nature: the LCK metric leads to a "highly-positive" $(1, 1)-$form, derived from the Lee form of the metric. The positivity of this form implies that a certain, very naturally defined foliation Σ on X has a very intriguing property: if a closed connected analytical subspace Y of X contains a point z sitting on the leaf Σ_x, then the whole Σ_z is contained in Y. Now, if $Y \subset X$ is a proper analytic subspace (i.e. $\dim(Y) > 0$) one shows that the closure of the leaves is the whole X; but to achieve this, one has to use a deep result in algebraic number theory, namely the so-called "strong adelic approximation theorem".

In the very general case (hence without restricting to $t = 1$, i.e. to LCK geometry), one can show

Theorem 4.6. ([15]) *Let X be an Oeljeklaus-Toma manifold. Then X carries no closed 1-dimensional analytic subspaces.*

Recently the same author obtained an extension of this theorem, to

Theorem 4.7. ([16]) *Let X be an Oeljeklaus-Toma manifold. If $S \subset X$ is a smooth compact surface, then S is a Inoue surface.*

An interesting (and apparently rather difficult) question imposes by itself:

Question 4.8. Is it true that if $X = X(K, U)$ is an Oeljeklaus-Toma manifold and if $X' \subset X$ is a connected, closed, reduced, analytical space, then X' is of the form $X' = X(K', U)$ with $K' \subset K$ and $U \subset U$ (i.e X' is obtained by the procedure described at the beginning of the section)?

Note that an affirmative answer would imply all theorems above, as fields with $t_K = 1$ complex embeddings have no proper subfields K' with $t_{K'} > 0$, thus we would get Theorem 4.5, and also Theorem 4.6, since in quadratic imaginary fields the rank of the group of units is zero and Theorem 4.7, as OT-surfaces are Inoue surfaces, according to Remark 3.1.

4.4. LCK rank of Oeljeklaus-Toma manifolds

The deep interplay between geometry and number theory, emphasized in the skecth of proof of 4.5 is actually much more extended. We illustrate this in the following.

Recall that one of the possible definitions of an LCK metric on a manifold X involves the homothety factors described by relation (4.1). Note that if $\gamma \in \pi_1(X)$ is a deck-transformation with $\chi(\gamma) = 1$, then actually γ is an isometry of the Kähler metric Ω. Hence, a natural question occurs: "how many" of the elements $\gamma \in \pi_1(X)$ are "honest homotheties", i.e. with $\chi(\gamma) \neq 1$? Put in a more rigourous setup:

Question 4.9. Determine how large can be the rank of the group

$$\{\chi(\gamma)\,;\, \gamma \in \pi_1(X)\}. \tag{4.2}$$

Of course, for LCK, non-GCK manifolds, this LCK rank is bounded from below by 1 (as at least one of the γ's must not be an isometry) and from above by the first Betti number of X. Until the Oeljeklaus-Toma manifolds appeared, in all examples known, the rank above actually had only these two extremal values: either 1, or $b_1(X)$. Some Oeljeklaus-Toma manifolds are -so far- the only known examples when this rank is non-trivial; more precisely, we have:

Theorem 4.10. ([12]) *Let K be a number field with $t = 1$ and $X = X(K, U)$ be an Oeljeklaus-Toma manifold. Then, the rank of the above group (defined in (4.2)) is different form 1 and $b_1(X)$ if and only if K is a quadratic extension of a (totally real) number field. In this last case, the rank equals $\frac{b_1(X)}{2}$, and this possibility occurs for Oeljeklaus-Toma manifolds of arbitrary high dimensions.*

The basic idea behind the proof is as follows. For any $u \in U$ (seen as an element in $\pi_1(X)$), the automorphy factor $\chi(u)$ is actually $|\sigma(u)|$, where σ is the only (up to complex conjugation) complex embedding of K. Now, if the rank is different from $b_1(X)$, then at least one $u \in U$ must have $|\sigma(u)| = 1$. This forces u to be a *reciprocal unit,* i.e. its minimal polynomial over $\mathbb{Q}$ to be a reciprocal one. But if u is a reciprocal unit, then the field $K' = \mathbb{Q}(u + \frac{1}{u})$ is a subfield of K, of relative degree 2, and it can be easily shown that K' must actually be totally real. Eventually, to produce infinitely many examples of Oeljeklaus-Toma manifolds with non-trivial rank, one reverses the process. One starts with a totally real number field K' (for instance with cyclotomic fields) and extend it to a field $K \supset K'$ with $[K : K'] = 2$, taking care to ramify precisely one real embedding of K'.

4.5. Oeljeklaus-Toma manifolds with $t > 1$

As already noticed, the main ingredient (apart from the number-theoretical ones) in most of the results above is the existence of LCK metrics. So far, existence of such metrics is known to hold on Oeljeklaus-Toma manifolds $X(K, U)$ for which the number field K has precisely $t = 1$ complex embeddings. It is thus natural to ask whether this condition can be dropped.

Actually, as already noticed in [8], in the other "extreme case", i.e. if K is a number field with $s = 1$ real embedings (and $t > 1$ complex ones) then for any choice of the admissible group of units U, the resulting Oeljeklaus-Toma manifold $X(K, U)$ has no LCK metric.

There are (so far) at least two results showing that probably, if K is a number field with $t > 1$ complex embedings, then no Oeljeklaus-Toma manifold $X(K, U)$ carries an LCK metric.

The first one was already recalled (4.4): an Oeljeklaus-Toma manifold cannot carry Vaisman metrics. But this does not rule out the possibility of existence of non-Vaisman, LCK metrics. However, when there are "too many" complex embeddings, this is not true. More exactly, we have:

Theorem 4.11. ([17]) *let K be a number field with $t > 2s$. then for any admissible group of units U, the Oeljeklaus-Toma manifold $X(K, U)$ carries no LCK metric.*

The proof relies again on the interplay between differential geometry and number theory. Namely, first one shows that if an LCK metric exists on $X(K, U)$ then, by looking at the automorphy factors $\chi(u)$ of any unit $u \in U$ one gets that $|\sigma(u)|$ is the same for *any* complex embedding σ of K. But then, one exploits a nice fact about algebraic integers with "many" Galois conjugates of the same absolute value: their minimal polynomial f must actually be of the form $f(X) = g(X^t)$, and from here one easily derives a contradiction.

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Exact discrete Morse functions on surfaces

Vasile Revnic

To the memory of Professor Mircea-Eugen Craioveanu (1942-2012)

Abstract. In this paper, we review some basic facts about discrete Morse theory, we introduce the Morse-Smale characteristic for a finite simplicial complex, and we construct $\mathbb{Z}_2$-exact discrete Morse functions on the torus with two holes $\mathbf{T}_2$ and on the dunce hat $\mathbf{DH}$.

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Let K be a finite simplicial complex. A function $f : K \to \mathbb{R}$ is a *discrete Morse function* if for every simplex $\alpha^{(p)} \in K$ we have simultaneously :

$$\#\{\beta^{(p+1)} > \alpha^{(p)} \mid f(\beta) \leq f(\alpha)\} \leq 1, \quad \#\{\gamma^{(p+1)} < \alpha^{(p)} \mid f(\gamma) \geq f(\alpha)\} \leq 1,$$

where $\#A$ denotes the cardinality of the set A.

Note that a discrete Morse function is not a continuous function on the complex K since we did not considered any topology on K. Rather, it is an assignment of a single number to each simplex.

The other main ingredient in discrete Morse theory is the notion of a critical simplex of discrete function. A p-dimensional simplex $\alpha^{(p)}$ is *critical* if the following relations hold simultaneously :

$$\#\{\beta^{(p+1)} > \alpha^{(p)} \mid f(\beta) \leq f(\alpha)\} = 0, \quad \#\{\gamma^{(p+1)} < \alpha^{(p)} \mid f(\gamma) \geq f(\alpha)\} = 0.$$

The study of the discrete version of the Morse theory was initiated by R.Forman [9], [10].

If K is a m-dimensional simplicial complex with a discrete Morse function, then let $\mu_j = \mu_j(f)$ denote the number of critical simplices of dimension j of the function f. For any field F, let $\beta_j = \dim H_j(K, F)$ be the j-th Betti number with respect to F, $j = 0, 1, \ldots, m$.

Let K be a m-dimensional simplicial complex with a discrete Morse function f. Then the following relations also hold in the discrete context.

(1) The weak discrete Morse's inequalities.

(i) $\mu_j \geq \beta_j, j = 0, 1, \ldots, m$;

(ii) $\mu_0 - \mu_1 + \mu_2 - \cdots + (-1)^m \mu_m = \beta_0 - \beta_1 + \beta_2 - \cdots + (-1)^m \beta_m = \chi(K)$.

The last relation is called Euler' s relation.

(2) Also, the strong discrete Morse's inequalities are valid in this context, that is for each $j = 0, 1, \ldots, m-1$, we have

$$\mu_j - \mu_{j-1} + \cdots + (-1)^j \mu_0 \geq \beta_j - \beta_{j-i} + \cdots + (-1)^j \beta_0.$$

Let K be a m-dimensional simplicial complex containing exactly c_j simplices of dimension j, for each $j = 0, 1, \ldots, m$. Let $C_j(K, \mathbb{Z})$ denote the space $\mathbb{Z}^{c_j}$. More precisely, $C_j(K, \mathbb{Z})$ denotes the free Abelian group generated by the j-simplices of K, each endowed with an orientation. Then for each j, there are boundary maps $\partial_j : C_j(K, \mathbb{Z}) \to C_{j-1}(K, \mathbb{Z})$, such that $\partial_{j-1} \circ \partial_j = 0$.

The resulting differential complex

$$0 \longrightarrow C_m(K, \mathbb{Z}) \xrightarrow{\partial_m} C_{m-1}(K, \mathbb{Z}) \xrightarrow{\partial_{m-1}} \ldots \xrightarrow{\partial_1} C_0(K, \mathbb{Z}) \longrightarrow 0$$

calculates the singular homology of K. That is, if we define the quotient space

$$H_j(C, \partial) = \operatorname{Ker}(\partial_j)/\operatorname{Im}(\partial_{j+1}),$$

then for each j we have the isomorphism

$$H_j(C, \partial) \cong H_j(K, \mathbb{Z}),$$

where $H_j(K, \mathbb{Z})$ denotes the singular homology of K.

The discrete Morse theory is the main tool in studying some geometric properties of finite simplicial complexes. In this respect we refer to the papers of D.Andrica and I.C.Lazăr [3]-[6], K.Crowley [8], and I.C. Lazăr [11], [12].

1. The discrete Morse-Smale characteristic

Consider K^m to be a m-dimensional finite simplicial complex.

The discrete Morse-Smale characteristic of K was considered in paper [13], and it is a natural extension of the well-known Morse-Smale characteristic of a manifold (see the monograph [2]).

Let $\Omega(K)$ be the set of all discrete Morse functions defined on K. It is clear that $\Omega(K)$ is nonempty, because, for instance, the trivial example of discrete Morse function defined by $f(\sigma) = \dim \sigma$, $\sigma \in K$.

For $f \in \Omega(K)$, let $\mu_j(f)$ be the number of j-dimensional critical simplices of f, $j = 0, 1, \ldots, m$.

Let $\mu(f)$ be the number defined as follows:

$$\mu(f) = \sum_{j=0}^{m} \mu_j(f),$$

i.e. $\mu(f)$ is the total number of critical simplices of f. The number

$$\gamma(K) = \min\{\mu(f) : \ f \in \Omega(K)\}$$

is called the *discrete Morse Smale characteristic* of the simplicial complex K. So, the discrete Morse-Smale characteristic represents the minimal number of critical simplices for all discrete Morse functions defined on K.

In analogous way, one can define the numbers $\gamma_j(K), j = 0, 1, \ldots, m$, by

$$\gamma_j(K) = \min\{\mu_j(f) : \ f \in \Omega(K)\},$$

that is the minimal numbers of critical of j-dimensional simplices, for all discrete Morse functions defined on K.

The effective computation of these numbers associated to a finite simplicial complex is an extremely complicated problem in combinatorial topology. A finite algorithm for the determination of these numbers for any simplicial complex is not yet known.

2. Exact discrete Morse functions and F-perfect Morse functions

Consider the finite m-dimensional simplicial complex K. For $j = 0, 1, \ldots, m$, let $H_j(K, F)$ be the singular homology groups with the coefficients in the field F, and let $\beta_j(K, F) = \mathrm{rank} H_j(K, F) = \dim_F H_j(K, F)$ be the Betti numbers with respect to F.

For every $f \in \Omega(K)$, we have the discrete weak Morse inequalities :

$$\mu_j(f) \geq \beta_j(K, F), \quad j = 0, 1, \ldots, m.$$

The discrete Morse function $f \in \Omega(K)$ is called *exact* (or *minimal*) if $\mu_j(f) = \gamma_j(K)$, for all $j = 0, 1, \ldots, m$. So, an exact discrete Morse function has a minimal number of critical simplices in each dimension.

The discrete Morse function $f \in \Omega(K)$ is called *F-perfect* if $\mu_j(f) = \beta_j(K, F), \quad j = 0, 1, \ldots, m$.

Using the discrete weak Morse inequalities and the definition of the discrete Morse-Smale characteristic, we obtain the inequalities:

$$\mu_j(f) \geq \min\{\mu_j(f) : \ f \in \Omega(K)\} = \gamma_j(K) \geq \beta_j(K, F).$$

Theorem. *The simplicial complex K has F-perfect discrete Morse functions if and only if $\gamma(K) = \beta(K, F)$, where*

$$\beta(K, F) = \sum_{j=0}^{m} \beta_j(K, F)$$

is the total Betti number of K with respect to the field F.

Proof. To prove the direct implication, let $f \in \Omega(K)$ be a fixed F-perfect discrete Morse function. Using the weak Morse inequalities, it follows:

$$\mu(f) = \sum_{j=0}^{m} \mu_j(f) \geq \sum_{j=0}^{m} \beta_j(K, F) = \beta(K, F),$$

hence $\mu(f) \geq \beta(K, F)$. Using the definition of the discrete Morse-Smale characteristic of K, we get

$$\gamma(K) = \min\{\mu(f) : \ f \in \Omega(K)\} \geq \beta(K, F).$$

Because f is a discrete F-perfect Morse function on K, we have $\mu(f) = \beta(K, F)$. On the other hand, clearly we have the inequality

$$\gamma(K) = \min\{\mu(g) : \ g \in \Omega(K)\} \leq \beta(K, F).$$

Therefore, we get $\gamma(K) \leq \beta(K, F)$ and the desired relation follows.

For the converse implication, let $f \in \Omega(K)$ be a discrete Morse function. From the relations

$$\mu(f) = \sum_{j=0}^{m} \mu_j(f) \text{ and } \beta(K, F) = \sum_{j=0}^{m} \beta_j(K, F),$$

using the hypothesis $\gamma(K) = \beta(K, F)$, it follows

$$\sum_{j=0}^{m} [\mu_j(f) - \beta_j(K, F)] = 0.$$

From the discrete weak Morse inequalities, we get $\mu_j(f) - \beta_j(K, F) \geq 0, \quad j = 0, 1, \ldots, m$. All in all, the following relations hold $\mu_j(f) = \beta_j(K, F), \quad j = 0, 1, \ldots, m$. Therefore, f is a discrete F-perfect Morse function. □

If K is a simplicial complex of dimension m, one knows that $C_j(K, \mathbb{Z})$, $j = 0, 1, \ldots, m$, is a finitely generated free Abelian group generated by the j-simplices in K. Since subgroups and quotient groups of finitely generated groups are again finitely generated, it follows that the homology group $H_j(K, \mathbb{Z})$ is finitely generated. Therefore, by the fundamental theorem about such groups, we can write $H_j(K, \mathbb{Z}) \simeq A_j \oplus B_j$, where A_j is a free group and B_j is the torsion subgroup of $H_j(K, \mathbb{Z})$.

Therefore, the singular homology groups $H_j(K, \mathbb{Z})$, $j = 0, 1, \ldots, m$, are finitely generated. For every $j = 0, 1, \ldots, m$, we can write

$$H_j(K, \mathbb{Z}) \simeq (\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}) \oplus (\mathbb{Z}_{n_{j_1}} \oplus \cdots \oplus \mathbb{Z}_{n_{j_{\beta(j)}}}),$$

where $\mathbb{Z}$ is taken β_j times in the free group, $j = 0, 1, \ldots, m$. Here β_j represents the Betti numbers of K with respect to the group $(\mathbb{Z}, +)$, that is we have $\beta_j(K, \mathbb{Z}) = \text{rank} H_j(K, \mathbb{Z})$, $j = 0, 1, \ldots, m$.

3. A $\mathbb{Z}_2$-exact Morse function on the two holes torus $\mathbf{T}_2$

The torus with two holes $\mathbf{T}_2$ is the connected sum of two copies of torus $\mathbb{T}^2$, that is $\mathbf{T}_2 = \mathbb{T}^2 \# \mathbb{T}^2$. In this section we consider the triangulation of $\mathbb{T}^2$ represented in Figure 1. The singular homology of the torus with two holes $\mathbf{T}_2$ is given by

$$H_0(\mathbf{T}_2) = \mathbb{Z}, \quad H_1(\mathbf{T}_2) = \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}, \quad H_2(\mathbf{T}_2) = \mathbb{Z}.$$

Then, using the universal coefficients formula, we easily obtain

$$H_0(\mathbf{T}_2, \mathbb{Z}_2) \simeq Z_2, \quad H_1(\mathbf{T}_2, \mathbb{Z}_2) \simeq \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2, \quad H_2(\mathbf{T}_2, \mathbb{Z}_2) \simeq \mathbb{Z}_2.$$

This implies that the $\mathbb{Z}_2$-Betti numbers of $\mathbf{T}_2$ are given by

$$\beta_0(\mathbf{T}_2, \mathbb{Z}_2) = 1, \quad \beta_1(\mathbf{T}_2, \mathbb{Z}_2) = 4, \quad \beta_2(\mathbf{T}_2, \mathbb{Z}_2) = 1,$$

hence the $\mathbb{Z}_2$- total Betti number of $\mathbf{T}_2$ is

$$\beta(\mathbf{T}_2, \mathbb{Z}_2) = \sum_{j=0}^{2} \beta_j(\mathbf{T}_2, \mathbb{Z}_2) = 1 + 4 + 1 = 6.$$

We obtain $\gamma(\mathbf{T}_2) = \beta(\mathbf{T}_2, \mathbb{Z}_2) = 6$, and this relation implies that we can define on the simplicial complex given by the triangulation of torus with two holes $\mathbf{T}_2$ in Figure 1, a discrete Morse function with exactly six critical simplices. This function is $\mathbb{Z}_2$-exact and is defined in Figure 1, where the critical simplices are encircled.

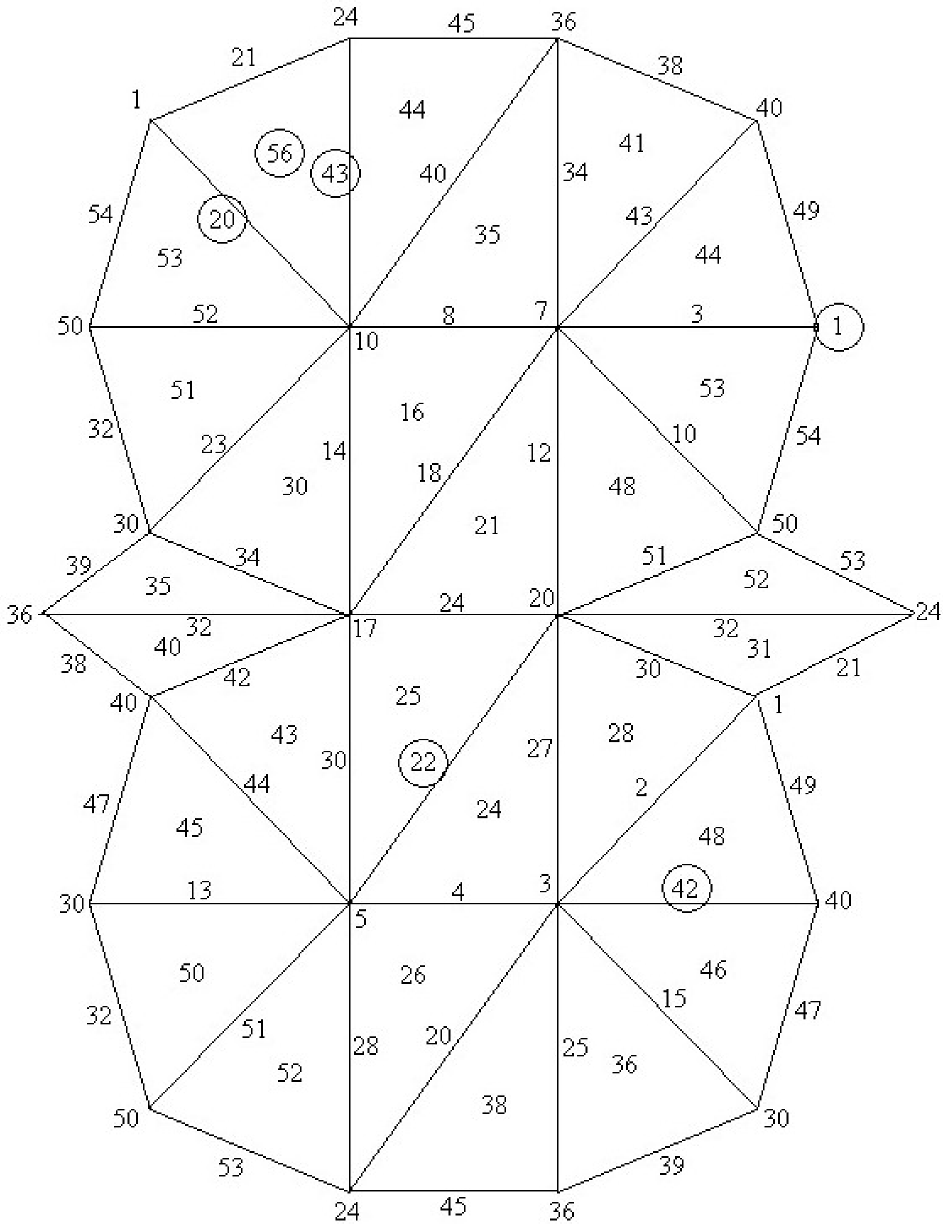

Figure 1.

4. A $\mathbb{Z}_2$-exact Morse function on the dunce hat DH

In topology, the *dunce hat* **DH** is a compact topological space formed by taking a solid triangle and gluing all three sides together, with the orientation of one side reversed. Simply gluing two sides oriented in the same direction would yield a cone much like the layman's dunce cap, but the gluing of the third side results in identifying the base of the cap with a line joining the base to the point.

The space **DH** is contractible, but not collapsible. Contractibility can be easily seen by noting that the dunce hat embeds in the 3-ball and the 3-ball deformation retracts onto the dunce hat. Alternatively, note that the dunce hat is the CW-complex obtained by gluing the boundary of a 2-cell onto the circle. The gluing map is homotopic to the identity map on the circle and so the complex is homotopy equivalent to the disc. By contrast, it is not collapsible because it does not have a free face.

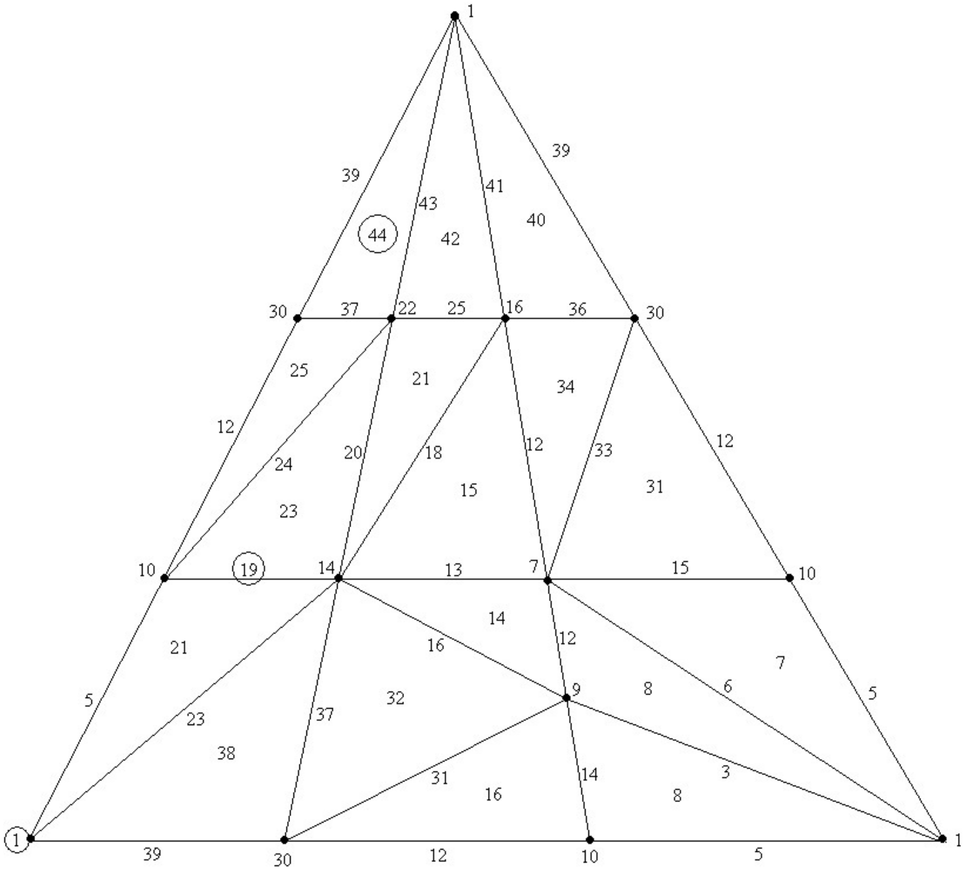

Figure 2.

The name is due to E. C. Zeeman [15], who observed that any contractible 2-complex (such as the dunce hat) after taking the Cartesian product with the closed unit interval seemed to be collapsible. This observation became known as the Zeeman conjecture and was shown by Zeeman to imply the Poincaré conjecture.

We consider the triangulation of the dunce hat **DH** which is shown in Figure 2.

The singular homology of the dunce hat **DH** is

$$H_0(\mathbf{DH}) = \mathbb{Z}, \quad H_1(\mathbf{DH}) = \mathbb{Z}, \quad H_2(\mathbf{DH}) = \mathbb{Z}.$$

Then, using again the universal coefficients formula, we obtain

$$H_0(\mathbf{DH}, \mathbb{Z}_2) \simeq \mathbb{Z}_2, \quad H_1(\mathbf{DH}, \mathbb{Z}_2) \simeq \mathbb{Z}_2, \quad H_2(\mathbf{DH}, \mathbb{Z}_2) \simeq \mathbb{Z}_2.$$

This implies that the $\mathbb{Z}_2$-Betti numbers of **DH** are given by

$$\beta_0(\mathbf{DH}, \mathbb{Z}_2) = 1, \quad \beta_1(\mathbf{DH}, \mathbb{Z}_2) = 1, \quad \beta_2(\mathbf{DH}, \mathbb{Z}_2) = 1,$$

and the total Betti number is

$$\beta(\mathbf{DH}, \mathbb{Z}_2) = \sum_{j=0}^{2} \beta_j(\mathbf{DH}, \mathbb{Z}_2) = 1 + 1 + 1 = 3.$$

We obtain $\gamma(\mathbf{DH}) = \beta(\mathbf{DH}, \mathbb{Z}_2) = 3$, and this property implies that we can define on the simplicial complex given by the triangulation of **DH** in Figure 2 a discrete Morse function with exactly three critical simplices. This function is $\mathbb{Z}_2$-exact and is defined in Figure 2, the critical simplices are encircled.

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Symplectic connections on supermanifolds: Existence and non-uniqueness

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To the memory of Professor Mircea-Eugen Craioveanu (1942-2012)

Abstract. We show, in this note, that on any symplectic supermanifold, even or odd, there exist an infinite dimensional affine space of symmetric connections, compatible to the symplectic form.

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Keywords: Symplectic supermanifolds, symplectic connections.

1. Introduction

The connections compatible with a symplectic form have been studied for several decades, by now. They were introduced by Ph. Tondeur, in 1961 (see [12]), for the more general situation of an almost-symplectic manifold. Nevertheless, they became really important lately, in the early ninetieth, when Fedosov ([7]) discovered that they may be useful in the deformation quantization. Therefore, a symplectic manifold endowed with a symmetric connection, compatible with the symplectic form, has been baptized with the name of *Fedosov manifold*. A recent review of the theory of symplectic connections can be found in [5]. A few years later, the notion of symplectic connection has been extended to symplectic supermanifolds and the corresponding objects (namely symplectic supermanifolds, even or odd, endowed with a symplectic connection) have been named *Fedosov supermanifolds* (see [9]). It is the aim of this note to show that, as in the case of symplectic manifolds, on a symplectic supermanifold (odd or even, it doesn't matter), symplectic connections exist in abundance. The language we use is slightly different from that used in the original papers, because we use a coordinate-free approach (see [2], [3], [4]).

As it is well-known, there are several approaches to supermanifolds, not entirely equivalent. The differences are not very important for this paper. Nevertheless, to avoid ambiguities, we state from the very beginning that for us "supermanifold" means "supermanifold in the sense of Berezin and Leites"[1]. For details, see [1], [6], [10], [11].

2. Symplectic connections on supermanifolds

Definition 2.1. *Let $\mathcal{M}$ be an arbitrary, finite dimensional, supermanifold. A connection (a covariant derivative) on this supermanifold is a mapping $\nabla : \mathcal{X}(\mathcal{M}) \times \mathcal{X}(\mathcal{M}) \to \mathcal{X}(\mathcal{M})$ for which the following conditions are fulfilled:*

(i) ∇ is additive in both arguments:

$$\nabla_{X_1+Y_2} Y = \nabla_{X_1} Y + \nabla_{X_2} Y, \quad \nabla_X (Y_1 + Y_2) = \nabla_X Y_1 + \nabla_X Y_2;$$

(ii) $\nabla_{fX} Y = f \nabla_X Y$;

(iii) $\nabla_X (fY) = X(f) \cdot Y + (-1)^{|X|\cdot|f|} \nabla_X Y$,

where in the first two relations X, Y, X_1, X_2, Y_1, Y_2 are arbitrary vector fields and f an arbitrary superfunction, while in the last equality all the entries are assumed to be homogeneous.

The torsion tensor can be defined here in a similar manner to the corresponding tensor for connections on ordinary (ungraded) manifolds:

Definition 2.2. *Let ∇ be a connection on a supermanifold. The* torsion *of the connection is the tensor field (twice covariant and once contravariant) defined by*

$$T(X, Y) = \nabla_X Y - (-1)^{|X|\cdot|Y|} \nabla_Y X - [X, Y],$$

for any homogeneous vector fields X and Y. Also by analogy with the classical case, a connection on a supermanifold is called symmetric *if its torsion vanishes. Thus, the connection is symmetric iff for any homogeneous vector fields X and Y we have*

$$\nabla_X Y - (-1)^{|X|\cdot|Y|} \nabla_Y X = [X, Y].$$

It can be shown easily that, using the same methods from the classical differential geometry, the covariant derivative on supermanifolds can be extended to arbitrary tensor fields, not just vector fields. The interesting case for us is the one of twice covariant tensor fields. Thus, if g is a twice covariant *homogeneous* tensor field on a supermanifold $\mathcal{M}$, then we have

$$(\nabla_X g)(Y, Z) \equiv \nabla_X g(Y, Z) = X(g(Y, Z)) - (-1)^{|X|\cdot|g|} g(\nabla_X Y, Z) - \\ - (-1)^{|X|\cdot(|Y|+|g|)} g(Y, \nabla_X Z).$$

We are interested, in this paper, in the particular case of a *homogeneous* symplectic supermanifold, i.e. a supermanifold endowed with a homogeneous 2-form ω, which is both closed and non-degenerate.

Definition 2.3. *Let $(\mathcal{M}, \omega)$ be a homogeneous symplectic supermanifold (hereafter, it will be called, simply,* symplectic supermanifold*). A connection ∇ on $\mathcal{M}$ is called* symplectic *it is both symmetric and compatible to the symplectic form. Thus, a symplectic connection on a symplectic supermanifold is a connection ∇ for which:*

[1] These supermanifolds are also called "graded manifolds", especially in the Western literature.

(i) the torsion tensor vanishes, i.e.

$$\nabla_X Y - (-1)^{|X|\cdot|Y|}\nabla_Y X = [X,Y]$$

and

(ii) it is compatible to the symplectic form, i.e.

$$\nabla_X\omega(Y,Z) = X(\omega(Y,Z)) - (-1)^{|X|\cdot|\omega|}\omega(\nabla_X Y, Z) - \\ - (-1)^{|X|\cdot(|Y|+|\omega|)}\omega(Y,\nabla_X Z) = 0,$$

for any homogeneous vector fields X, Y, Z.

3. Existence and uniqueness results for symplectic connections

Theorem 3.1 (Existence). *Let $(\mathcal{M},\omega)$ be a symplectic supermanifold. Then on $\mathcal{M}$ there is at least a symplectic connection.*

Proof. The proof we are going to give is an adaptation of the proof from the classical symplectic geometry of manifolds. Namely, we notice, first of all, that on M there is at least a symmetric connection, ∇^0. To proof this, it is enough to consider a Riemannian metric on $\mathcal{M}$ (which we know we can find) and take ∇^0 to be the Levi-Civita connection associated to this metric, which, we also know, exists (and it is even unique). Of course, ∇^0 is not a symplectic connection, in most situations, and what we shall do is to "correct" this connection to get a symplectic one.

We define now a twice covariant and once contravariant tensor field N through the relation

$$\nabla^0_X\omega(Y,Z) = (-1)^{|\omega|\cdot|X|}\omega(N(X,Y),Z). \tag{3.1}$$

We shall proof some properties of N, for later use. First, we claim that

$$\omega(N(X,Y),Z) = -(-1)^{|Y|\cdot|Z|}\omega(N(X,Z),Y). \tag{3.2}$$

Indeed, we have

$$\omega(N(X,Y),Z) = (-1)^{|\omega|\cdot|X|}\nabla^0_X\omega(Y,Z) = \\ = -(-1)^{|\omega|\cdot|X|}(-1)^{|Y|\cdot|Z|}\nabla^0_X(Z,Y) = -(-1)^{|Y|\cdot|Z|}\omega(N(X,Z),Y).$$

Another important property of N, which follows, this time, from the closeness of the symplectic form, is the following:

$$\omega(N(X,Y),Z) + (-1)^{|X|(|Y|+|Z|)}\omega(N(Y,Z),X) + \\ + (-1)^{|Z|(|X|+|Y|)}\omega(N(Z,X),Y) = 0 \tag{3.3}$$

As mentioned before, to prove (3.3), we shall start from the closeness of the symplectic form and we shall use the symmetry of the connection ∇^0, as well as the definition of the tensor N. Thus, we have

$$
\begin{aligned}
0 &= d\omega(X,Y,Z) = (-1)^{|\omega|\cdot|X|}X(\omega(Y,Z))- \\
&-(-1)^{|Y|(|\omega|+|X|)}Y(\omega(X,Z)) + (-1)^{|Z|(|\omega|+|X|+|Y|)}Z(\omega(X,Y))- \\
&-\omega([X,Y],Z) + (-1)^{|Y|\cdot|Z|}\omega([X,Z],Y) - (-1)^{|X|(|Y|+|Z|)}\omega([Y,Z],X) = \\
&= (-1)^{|\omega|\cdot|X|}X(\omega(Y,Z)) - (-1)^{|Y|(|\omega|+|X|)}Y(\omega(X,Z))+ \\
&+(-1)^{|Z|(|\omega|+|X|+|Y|)}Z(\omega(X,Y)) - \omega\left(\nabla^0_X Y - (-1)^{|X|\cdot|Y|}\nabla^0_Y X, Z\right) + \\
&+(-1)^{|Y|\cdot|Z|}\omega\left(\nabla^0_X Z - (-1)^{|X|\cdot|Z|}\nabla^0_Z X, Y\right) - \\
&-(-1)^{|X|(|Y|+|Z|)}\omega\left(\nabla^0_Y Z - (-1)^{|Y|\cdot|Z|}\nabla^0_Z Y, X\right) = \\
&= (-1)^{|\omega|\cdot|X|}X(\omega(Y,Z)) - (-1)^{|Y|(|\omega|+|X|)}Y(\omega(X,Z))+ \\
&+(-1)^{|Z|(|\omega|+|X|+|Y|)}Z(\omega(X,Y)) - \omega\left(\nabla^0_X Y, Z\right) + \\
&+(-1)^{|X|\cdot|Y|}\omega\left(\nabla^0_Y X, Z\right) + (-1)^{|Y|\cdot|Z|}\omega\left(\nabla^0_X Z, Y\right) - \\
&-(-1)^{(|X|+|Y|)|Z|}\omega\left(\nabla^0_Z X, Y\right) - (-1)^{|X|(|Y|+|Z|)}\omega\left(\nabla^0_Y Z, X\right)\big) + \\
&+(-1)^{|X|(|Y|+|Z|)+|Y|\cdot|Z|}\omega\left(\nabla^0_Z Y, X\right) = (-1)^{|\omega|\cdot|X|}\left[X(\omega(Y,Z))-\right. \\
&-(-1)^{|\omega|\cdot|X|}\omega\left(\nabla^0_X Y, Z\right) - (-1)^{|X|(|\omega|+|Y|}\omega\left(Y, \nabla^0_X Z\right)\Big] - \\
&-(-1)^{|Y|(|\omega|+|X|}\Big[Y(\omega(X,Z)) - (-1)^{|\omega|\cdot|Y|}\omega\left(\nabla^0_Y X, Z\right) - \\
&-(-1)^{|Y|(|\omega|+|X|}\omega\left(X, \nabla^0_Y Z\right)\Big] + (-1)^{|Z|(|\omega|+|X|+|Y|}\Big[Z(\omega(X,Y))- \\
&-(-1)^{|\omega|\cdot|Z|}\omega\left(\nabla^0_Z X, Y\right) - (-1)^{|Z|(|\omega|+|X|}\omega\left(X, \nabla^0_Z Y\right)\Big] = \\
&= (-1)^{|\omega|\cdot|X|}\nabla^0_X\omega(Y,Z) - (-1)^{|Y|(|\omega|+|X|}\nabla^0_Y\omega(X,Z)+ \\
&+(-1)^{|Z|(|\omega|+|X|+|Y|)}\nabla^0_Z\omega(X,Y)
\end{aligned}
$$

We define now a new connection, ∇, by letting

$$\nabla_X Y = \nabla^0_X Y + \frac{1}{3}N(X,Y) + \frac{(-1)^{|X|\cdot|Y|}}{3}N(Y,X). \tag{3.4}$$

We start by proving that this is, indeed, a connection. ∇ is, obviously, bi-additive and homogeneous in the first variable. Moreover, we have

$$
\begin{aligned}
\nabla_X(fY) &= \nabla^0_X(fY) + \frac{1}{3}N(X,fY) + \frac{(-1)^{|X|\cdot|Y|}}{3}N(fY,X) = \\
&= f\nabla^0_X Y + (-1)^{|f|\cdot|X|}X(f)\cdot Y + f\Bigg(\frac{1}{3}N(X,Y)+ \\
&+\frac{(-1)^{|X|\cdot|Y|}}{3}N(Y,X)\Bigg) = f\nabla_X Y + (-1)^{|f|\cdot|X|}X(f)\cdot Y,
\end{aligned}
$$

hence ∇ is a connection.

We claim that ∇ is a symplectic connection. Let's check first that ∇ is symmetric. Indeed, we have

$$\begin{aligned}
\nabla_X Y - (-1)^{|X|\cdot|Y|}\nabla_Y X = \nabla^0_X Y + \frac{1}{3}N(X,Y) + \frac{(-1)^{|X|\cdot|Y|}}{3}N(Y,X) - \\
- (-1)^{|X|\cdot|Y|}\left(\nabla^0_Y X + \frac{1}{3}N(Y,X) + \frac{(-1)^{|Y|\cdot|X|}}{3}N(X,Y)\right) = \\
= \nabla^0_X Y - (-1)^{|X|\cdot|Y|}\nabla^0_Y X = [X,Y],
\end{aligned}$$

where we used the fact that the connection ∇^0 is symmetric. Finally, we show that the connection is compatible with the symplectic form. We have

$$\begin{aligned}
&\nabla_X \omega(Y,Z) = X(\omega(Y,Z)) - (-1)^{|\omega|\cdot|X|}\omega\left(\nabla_X Y, Z\right) - \\
&- (-1)^{|X|(|\omega|+|Y|}\omega\left(Y, \nabla_X Z\right) = X(\omega(Y,Z)) - (-1)^{|\omega|\cdot|X|}\omega\Big(\nabla^0_X Y + \\
&+ \frac{1}{3}N(X,Y) + \frac{(-1)^{|X|\cdot|Y|}}{3}N(Y,X), Z\Big) - (-1)^{|X|(|\omega|+|Y|}\omega\Big(Y, \nabla^0_X Z + \\
&+ \frac{1}{3}N(X,Z) + \frac{(-1)^{|X|\cdot|Z|}}{3}N(Z,X)\Big) = X(\omega(Y,Z)) - \\
&- (-1)^{|\omega|\cdot|X|}\omega\left(\nabla^0_X Y, Z\right) - (-1)^{|X|(|\omega|+|Y|)}\omega\left(Y, \nabla^0_X Z\right) - \\
&- \frac{1}{3}(-1)^{|\omega|\cdot|X|}\omega(N(X,Y),Z) - \frac{1}{3}(-1)^{|X|(|\omega|+|Y|)}\omega(N(Y,X),Z) - \\
&- \frac{1}{3}(-1)^{|X|(|\omega|+|Y|)}\omega(Y,N(X,Z)) - \frac{1}{3}(-1)^{|X|(|\omega|+|Y|+|Z|)}\omega(Y,N(Z,X)) = \\
&= \nabla^0_X \omega(Y,Z) - \frac{1}{3}(-1)^{|\omega|\cdot|X|}\omega(N(X,Y),Z) + \\
&+ \frac{1}{3}(-1)^{|X|(|\omega|+|Y|+|Z|)}\omega(N(Y,Z),X) - \frac{1}{3}(-1)^{|\omega|\cdot|X|}\omega(N(X,Y),Z) + \\
&+ \frac{1}{3}(-1)^{|\omega|\cdot|X|+|Z|(|X|+|Y|}\omega(N(Z,X),Y) = (-1)^{|\omega|\cdot|X|}\omega(N(X,Y),Z) - \\
&- \frac{2}{3}(-1)^{|\omega|\cdot|X|}\omega(N(X,Y),Z) + \\
&+ \frac{1}{3}(-1)^{|\omega|\cdot|X|}\Big((-1)^{|X|(|Y|+|Z|)}\omega(N(Y,Z),X) + \\
&+ (-1)^{|Z|(|X|+|Y|)}\omega(N(Z,X),Y)\Big) = \frac{1}{3}(-1)^{|\omega|\cdot|X|}\Big(\omega(N(X,Y),Z) + \\
&+ (-1)^{|X|(|Y|+|Z|)}\omega(N(Y,Z),X) + (-1)^{|Z|(|X|+|Y|)}\omega(N(Z,X),Y)\Big) = 0,
\end{aligned}$$

which proves that, indeed, ∇ is a symplectic connection.

Thus, on any symplectic supermanifold there is *at least* a symplectic connection. As we shall prove next, there are, actually, infinitely many.

We notice, first of all, that the difference of two symplectic connections is allways a symplectic connection. Let now ∇ be a symplectic connection. Any other connection on $\mathcal{M}$ should be of the form

$$\nabla'_X Y = \nabla_X Y + S(X,Y),$$

where S is a $(2,1)$ tensor field on $\mathcal{M}$. If we want ∇' to be symplectic, first of all it should be symmetric, which means:

$$\nabla'_X Y - (-1)^{|X|\cdot|Y|}\nabla'_Y X = [X,Y],$$

i.e.

$$\nabla_X Y + S(X,Y) - (-1)^{|X|\cdot|Y|}\nabla_Y X - (-1)^{|X|\cdot|Y|}S(Y,X) = [X,Y].$$

As ∇ is symmetric, it follows that S should verify the relation

$$S(X,Y) = (-1)^{|X|\cdot|Y|}S(Y,X),$$

meaning that S is supersymmetric. Now we should ask that ∇' should, also, be compatible to the symplectic form. We have:

$$\begin{aligned}
&\nabla'_X\omega(Y,Z) = X\left(\omega(Y,Z)\right) - (-1)^{|\omega|\cdot|X|}\omega\left(\nabla'_X Y, Z\right) - \\
&- (-1)^{|X|(|\omega|+|Y|}\omega\left(Y, \nabla'_X Z\right) = \\
&= \underbrace{X\left(\omega(Y,Z)\right) - (-1)^{|\omega|\cdot|X|}\omega\left(\nabla_X Y, Z\right) - (-1)^{|X|(|\omega|+|Y|}\omega\left(Y,\nabla_X Z\right)}_{=0} - \\
&- (-1)^{|\omega|\cdot|X|}\omega\left(S(X,Y),Z\right) - (-1)^{|X|(|\omega|+|Y|}\omega\left(Y,S(X,Z)\right) = \\
&= (-1)^{|\omega|\cdot|X|}\left[\omega\left(S(X,Y),Z\right) + (-1)^{|X|\cdot|Y|}\omega\left(Y,S(X,Z)\right)\right] = \\
&= (-1)^{|\omega|\cdot|X|}\left[\omega\left(S(X,Y),Z\right) - (-1)^{|Y|\cdot|Z|}\omega\left(S(X,Z),Y\right)\right].
\end{aligned}$$

Thus, ∇' is a symplectic connection if and only if

$$\omega\left(S(X,Y),Z\right) = (-1)^{|Y|\cdot|Z|}\omega\left(S(X,Z),Y\right),$$

i.e. the 3-covariant tensor field $\omega\left(S(X,Y),Z\right)$ is *totally graded symmetric*. The conclusion is, as in the classical, ungraded, case, that the set of all symplectic connections on a given symplectic supermanifold is an infinite dimensional affine space. □

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The minimum number of critical points of circular Morse functions

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To the memory of Professor Mircea-Eugen Craioveanu (1942-2012)

Abstract. The minimum number of critical points for circular Morse functions on closed connected surfaces has been computed by the authors in [4]. Some bounds for the minimum characteristic number of closed connected orientable surfaces embedded in the first Heisenberg group with respect to its horizontal distribution are also given by [4]. In this paper we provide a more elementary proof for the minimum number of critical points of circular Morse functions and the details for the bounds on the mentioned minimum characteristic number.

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1. Introduction

In this paper we show that the circular Morse-Smale characteristic of a closed connected surface Σ is, except for the projective plane, the absolute value $|\chi(\Sigma)|$ of its Euler-Poincaré characteristic.

Definition 1.1. If M is a differential manifold, then the *circular Morse-Smale characteristic* of M is defined by

$$\gamma_{_{S^1}}(M) := \min\{\mathrm{card}(C(f)) : f \in \mathcal{F}(M, S^1)\}, \tag{1.1}$$

where $\mathcal{F}(M, S^1)$ stands for the set of all circular Morse functions $f : M \to S^1$.

Note that the *Morse-Smale characteristic* of a manifold M is defined by

$$\gamma(M) = \min\{\mathrm{card}(C(f)) : f \in \mathfrak{F}(M)\},$$

where $\mathfrak{F}(M)$ denotes the set of all real-valued Morse functions defined on M, and it was studied by Andrica in [1, pp.106-129]. The *circular Morse-Smale characteristic* was defined by Andrica and Mangra [2, 3].

Proposition 1.1. ([4]) *If $\widetilde{M}$ is a k-fold cover of M, then $\gamma_{_{S^1}}(\widetilde{M}) \leq k \cdot \gamma_{_{S^1}}(M)$.*

Constructing a circular Morse function on the closed connected orientable surface Σ_g, of genus g, with exactly $2(g-1)$ critical points is part of the strategy to compute the circular Morse-Smale characteristic of the surface Σ_g. We achieve this goal by producing a suitable embedding of Σ_g in $\mathbb{R}^3 \setminus Oz$, where Oz stands for the z-axis $\{(x,0,0) : x \in \mathbb{R}\}$, alongside a submersion $f : \mathbb{R}^3 \setminus Oz \longrightarrow S^1$, whose restriction $f\big|_{\Sigma_g}$ is a circular Morse function with exactly $2(g-1)$ critical points. In fact the suitable submersion is

$$f(x,y,z) = \frac{1}{\sqrt{x^2+y^2}}(x,y,0). \tag{1.2}$$

In this respect we need to characterize somehow the critical points of such a restriction.

Proposition 1.2. *Let $\Sigma \subseteq \mathbb{R}^3$ be a regular surface and $f : \mathbb{R}^3 \longrightarrow N$ be a submersion, where N is either the real line or the circle S^1. The point $p = (x_0, y_0, z_0) \in \Sigma$ is critical for the restriction $f|_\Sigma$ if and only if the tangent plane of Σ at p is the tangent plane at p to the fiber $\mathcal{F}_p := f^{-1}(f(p))$ of the submersion (*1.2*) through p.*

Proposition 1.2 follows from the following more general statement.

Proposition 1.3. *Let M^m, N^n, P^p, $m \geq n > p$ be differential manifolds, let $f : M \to N$ be a differential map and $g : N \to P$ be a submersion. Then $x \in M$ is a regular point of $g \circ f$ if and only if $f \pitchfork_x \mathcal{F}_x$, where $\mathcal{F}_x$ stands for the fiber $g^{-1}(g(x))$ of g through x.*

Proof. Recall that we have the transversality property $f \pitchfork_x \mathcal{F}_x$ if and only if $\mathrm{Im}(df)_x + \ker(dg)_{f(x)} = T_{f(x)}(N)$, i.e. $\mathrm{Im}(df)_x + \ker(dg)_{f(x)} = T_{f(x)}(N)$, as $T_{f(x)}(\mathcal{F}_x) = \ker(dg)_{f(x)}$.

Assume that $x \in R(g \circ f)$, i.e. $\mathrm{Im} d(g\circ f)_x = T_{(g\circ f)(x)}(N)$. We only need to show that $T_{f(x)}(N) \subseteq \mathrm{Im}(df)_x + \ker(dg)_{f(x)}$, as the opposite inclusion is obvious. Consider $v \in T_{f(x)}(N)$ and observe that there exists $u \in T_x(M)$ such that $(dg)_{f(x)}(v) = d(g\circ f)_x(u)$, since $\mathrm{Im}[d(g\circ f)_x] = T_{(g\circ f)(x)}(N)$. Consequently we obtain successively:

$$\begin{aligned}
(dg)_{f(x)}(v) = d(g\circ f)_x(u) &\Leftrightarrow (dg)_{f(x)}(v) = (dg)_{f(x)}\left((df_x)(u)\right)\\
&\Leftrightarrow (dg)_{f(x)}(v) - (dg)_{f(x)}\left((df_x)(u)\right) = 0\\
&\Leftrightarrow (dg)_{f(x)}\left(v - (df_x)(u)\right) = 0\\
&\Leftrightarrow v - (df_x)(u) \in \ker(dg)_{f(x)}\\
&\Leftrightarrow v \in (df_x)(u) + \ker(dg)_{f(x)} \subseteq \mathrm{Im}(df)_x + \ker(dg)_{f(x)}.
\end{aligned}$$

In order to prove the opposite inclusion, we use the property of g to be a submersion and observe that we have successively:

$$\begin{aligned}
\mathrm{Im}(df)_x + \ker(dg)_{f(x)} = T_{f(x)}(N) &\quad \Rightarrow\\
(dg)_{f(x)}\left[\mathrm{Im}(df)_x + \ker(dg)_{f(x)}\right] = (dg)_{f(x)}\left[T_{f(x)}(N)\right] &\quad \Leftrightarrow\\
(dg)_{f(x)}\left[\mathrm{Im}(df)_x\right] + (dg)_{f(x)}\left[\ker(dg)_{f(x)}\right] = T_{g(f(x))}(N) &\quad \Leftrightarrow\\
(dg)_{f(x)}\left[\mathrm{Im}(df)_x\right] = T_{(g\circ f)(x)}(N) &\quad \Leftrightarrow\\
Im\left((dg)_{f(x)} \circ df)_x\right) = T_{(g\circ f)(x)}(N) &\quad \Leftrightarrow\\
Im\left(d(g\circ f)_x\right) = T_{(g\circ f)(x)}(N) &\quad \Leftrightarrow \quad x \in R(g\circ f).
\end{aligned}$$

$\square$

2. The circular Morse-Smale characteristic of closed surfaces

According to [4, Corollary 1.3], $\gamma_{S^1}(S^2) = \gamma(S^2) = 2$ and $\gamma_{S^1}(\mathbb{RP}^2) = \gamma(\mathbb{RP}^2) = 3$. Also $\gamma_{S^1}(\Sigma_1) = \gamma_{S^1}(T^2) = 0$, as the projection $T^2 = S^1 \times S^1 \to S^1$ is a submersion and it has no critical points. More generally, we shall prove the following:

Theorem 2.1. *The circular Morse-Smale characteristic of a closed surface $\Sigma \neq \mathbb{RP}^2$ is*

$$\gamma_{S^1}(\Sigma) = |\chi(\Sigma)| \tag{2.1}$$

2.1. The case of the closed orientable surfaces

In this case we only need to prove Theorem 2.1 for the compact orientable surface Σ_g of genus $g \geq 1$, as it is obvious for $\Sigma = S^2$ (see [4, Corollary 1.3]). In this respect we need:

1. to show that $\mu(F) := \mu_0(F) + \mu_1(F) + \mu_2(F) \geq 2(g-1)$ for every circular Morse function $F : \Sigma_g \longrightarrow S^1$, where $\mu_j(F)$ stands for the number of critical of index j of F and $\mu(F)$ for the total number card$(C(F))$ of critical points of F;
2. to produce a circular Morse function on Σ_g with exactly $2(g-1)$ critical points.

In order to do so, we first observe that

$$2 - 2g = \mu_0(F) - \mu_1(F) + \mu_2(F). \tag{2.2}$$

Indeed, by using the Poincaré-Hopf Theorem one obtains

$$2 - 2g = \chi(\Sigma_g) = \sum_{p \in C(F)} \mathrm{ind}_p(\nabla F),$$

where ∇F is the gradient vector field of F with respect to some Riemann metric on Σ_g. To finish the proof of relation 2.2, we just need to observe that the index of the gradient vector field ∇F at a critical point of index one is -1 and the index of ∇F at the critical points of index zero and two is 1. Indeed the local behavior of F around the critical points of index one is $F = x^2 - y^2$ and its gradient behaves locally around such a point like the vector field $(x, -y)$. The degree of its normalized restriction to the circle S^1 is -1 as the normalized restriction is a diffeomorphism which reverses the orientation. Similarly, the index of ∇F at a critical point of index zero or two is one as the local behavior of F around such a critical point is $F = x^2 + y^2$ or $F = -x^2 - y^2$ and its gradient behaves locally around such a point like the vector field (x, y) or $(-x, -y)$ respectively. The normalized restrictions of these vector fields to the circle S^1 are diffeomorphisms preserving the orientation and their degree is therefore one. Thus, the relation (2.2) is now completely proved via the Poincaré-Hopf Theorem.

For the second item of the above observation we prove the following

Lemma 2.2. *The surface Σ_g can be suitably embedded into the three dimensional space $\mathbb{R}^3 \setminus Oz$ such that the restriction $f\big|_{\Sigma_g} : \Sigma_g \longrightarrow S^1$ is a circular Morse function with exactly $2(g-1)$ critical points, where $f : \mathbb{R}^3 \setminus Oz \longrightarrow S^1$ is the submersion given by*

$$f(x, y, z) = \frac{1}{\sqrt{x^2 + y^2}}(x, y, 0).$$

2.1.1. The embedding of Σ_g into $\mathbb{R}^3 \setminus Oz$. Recall that $\Sigma_1 = T^2 = S^1 \times S^1$ is being usually identified with the surface of revolution in $\mathbb{R}^3$ obtained by rotating a circle in the plane xOz centered at a point on the x-axis around the z-axis. The radius of the circle is supposed to be strictly smaller than the distance from the origin to its center. A certain embedding of the surface Σ_g in $\mathbb{R}^3$, obtained from the one of Σ_1 on which we perform some surgery, will be useful in our approach. However the above mentioned embedding of Σ_1 in $\mathbb{R}^3$ has one circle on '*its top*' and one circle on '*its bottom*', where the Gauss curvature vanishes. The two circles form the critical set of the height function $f_{\vec{k}}$ in the direction of the z-axis, on the embedded copy of T^2 in $\mathbb{R}^3$. Thus, this height function is not a Morse function.

In order to construct our suitable embedding of Σ_g we need to rotate around the z-axis a closed convex curve of nonconstant curvature with a unique center of symmetry, on the x-axis, which lies in the plane xOz and has no overlaps with the z-axis, rather than a circle with the same properties except the requirement on the curvature. This curve is also required to contain two segments mutually symmetric with respect to the x-axis, one on '*its top*' and the other on '*its bottom*'. These two segments form the critical set of the height function $f_{\vec{k}}$ restricted to the curve itself.

Instead of rotating a circle within the plane xOz, we consider the embedding of Σ_1 obtained by rotating, around the z-axis a closed convex curve described above. The

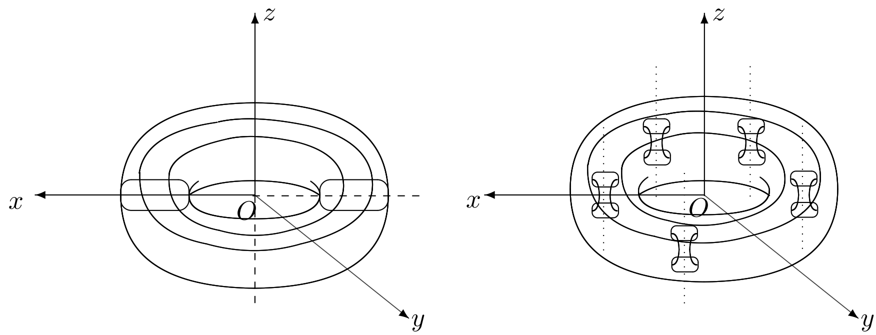

FIGURE 1. An embedded copy of Σ_6 constructed out of an embedded copy of Σ_1

obtained copy of Σ_1 is flat on the two annuli $\mathcal{A}$ and $\mathcal{A}'$ generated by the two symmetric segments of the generating curve, which lie in two horizontal parallel planes. Consider the points $p_1, \dots p_{g-1} \in \mathcal{A}$ and $q_1, \dots, q_{g-1} \in \mathcal{A}'$ such that the lines p_iq_i, $i = 1, \dots, g-1$ are vertical, i.e. parallel to the z-axis. In order to obtain a topological copy of the surface Σ_g we next remove some small open discs $D_1, \dots, D_{g-1} \subseteq \mathcal{A}$ centered at $p_1, \dots p_{g-1}$ and $D'_1, \dots, D'_{g-1} \subseteq \mathcal{A}'$ centered at $q_1, \dots, q_{g-1}$ respectively. The radii of the disks D_i and D'_i are supposed to be the same. We next consider suitable planar curves

$$\gamma_i : [0,1] \longrightarrow \mathrm{cl}\,(\mathcal{B}) \cap \pi_i, \; i = 1, \dots, g-1$$

such that $\gamma_i(0) \in \partial D_i$ and $\gamma_i(1) \in \partial D'_i$, where$p_i q_i \cap xOy = \{(x_i, y_i, 0)\}$, π_i is the plane parallel to xOz through the point $(x_i, y_i, 0)$ (i.e. $\pi_i : y = y_i$) and $\mathcal{B}$ is the bounded component of the complement of the embedded copy of Σ_1. The curves γ_i are chosen in such a way to complete, by their rotation around the axes $p_i q_i$, the embedded copy of $\Sigma_1 \setminus [D_1 \cup \ldots \cup D_{g-1} \cup D'_1 \cup \ldots \cup D'_{g-1}]$ up to a smooth embedded copy of Σ_g.

2.1.2. The cardinality of the set $C\left(f\big|_{\Sigma_g}\right)$ and the nondegeneracy of its points. Since our embedded copy of Σ_g is constructed out of several surfaces of revolutions, we are going to investigate the critical set of the restriction of the submersion (1.2) to such a surface, by using the geometric interpretation coming from Proposition 1.2.

Proposition 2.3. card $\big(C(f|_{\Sigma_g})\big) = 2(g-1)$.

Proof. Every surface of revolution Σ around a vertical line of equations $x = x_0$, $y = y_0$ can be parametrized as follows:

$$\begin{cases} x = x_0 + \alpha(v)\cos u \\ y = y_0 + \alpha(v)\sin u \quad & u \in (0, 2\pi), v \in [0,1]. \\ z = \beta(v) \end{cases}$$

In our considerations the function α is supposed to be strictly positive. Recall that a point $p(u,v) = (x(u,v), y(u,v), z(u,v))$ is, according to Proposition 1.3, critical for the restriction $f|_\Sigma$ if and only if the tangent plane of Σ at $p(u,v)$ contains the fiber of f through $p(u,v)$, i.e. its equation is $y(u,v)x = x(u,v)y$. On the other hand the equation of the tangent plane of Σ at $p(u,v)$ is

$$\begin{aligned} (x - x(u,v))\alpha(v)\beta'(v)\cos u + (y - y(u,v))\alpha(v)\beta'(v)\sin u \\ - \alpha(v)\alpha'(v)(z - z(u,v)) = 0 \end{aligned} \tag{2.3}$$

The two planes are equal, i.e. $p(u,v) \in C\left(f|_\Sigma\right)$, if and only if

$$\begin{cases} \alpha(v)\beta'(v)\cos u \cdot x(u,v) + \alpha(v)\beta'(v)\sin u \cdot y(u,v) = 0 \\ \alpha(v)\alpha'(v) = 0, \end{cases}$$

or equivalently

$$\begin{cases} x_0 \cos u + y_0 \sin u + \alpha(v) = 0 \\ \alpha'(v) = 0. \end{cases} \tag{2.4}$$

The equation $x_0 \cos u + y_0 \sin u + \alpha(v) = 0$ is equivalent to $\cos(u - \alpha) = -\frac{\alpha(v)}{x_0}\cos x$, and has two solutions on the interval $(-x, 2\pi - x)$, where $\tan x = \frac{y_0}{x_0}$ and x_0 is assumed to be nonzero. Since $\cos x = \frac{x_0}{\sqrt{x_0^2 + y_0^2}}$, the condition $|\frac{\alpha(v)}{x_0}\cos x| < 1$ is equivalent to $\alpha(v) < \sqrt{x_0^2 + y_0^2}$. Up to now we use the fact that $x_0^2 + y_0^2 > 0$ several times. Note that for $x_0 = y_0 = 0$ the restriction $f|_\Sigma$ has no critical points at all, as the first equation of the system (2.4) has no solutions in such a case. In particular the restriction $f|_{\Sigma_1}$ has no critical points at all as the embedded copy of Σ_1 is a surface of revolution around the z-axis, i.e. $x_0 = y_0 = 0$.

We now recall that $p_i q_i \cap xOy = (x_i, y_i, 0)$ and choose

$$\gamma_i : [0,1] \longrightarrow \mathrm{cl}\,(\mathcal{B}) \cap \pi_i, \ \gamma_i(t) = (\alpha_i(t), y_i, \beta_i(t)).$$

such that $\alpha_i(0) = \alpha_i(1)$, the equations $\alpha_i'(v) = 0$ has one solution in $(0,1)$, $\alpha_i'' > 0$ and $\lim\limits_{v\to 0} \beta_i'(v) = -\infty$, $\lim\limits_{v\to 1} g_i'(v) = +\infty$. With such choices of the functions α_i and β_i, the revolution surfaces of the curves γ_i around the axes p_iq_i completes the surface $\Sigma_1 \setminus [D_1 \cup \ldots \cup D_{g-1} \cup D_1' \cup \ldots \cup D_{g-1}']$ up to a smooth embedded copy of Σ_g. Moreover the restriction of f to each of these revolution surfaces has exactly two critical points. Thus, the restriction $f|_{\Sigma_g}$ has precisely $2(g-1)$ critical points. □

Proposition 2.4. *The restriction $f|_{\Sigma_g}$ is a circular Morse function, i.e. its critical points are nondegenerated. Moreover the critical points of $f|_{\Sigma_g}$ have all index* 1.

Proof. The local representations of the restriction $f|_{\Sigma_g}$ have one of the following form:

$$\varphi(u,v) = x_0 + \alpha(v)\cos u \text{ or } \psi(u,v) = y_0 + \alpha(v)\sin u.$$

The nodegeneracy of a critical point (u_0, v_0), via the local representations φ or ψ, is quite obvious as det $\left(\mathrm{Hess}_{(u_0,v_0)}\varphi\right)$ or det $\left(\mathrm{Hess}_{(u_0,v_0)}\psi\right)$ is either

$$-\alpha(v_0)\alpha''(v_0)\cos^2 u_0 - (\alpha'(v_0))^2 \sin^2 u_0 < 0$$

or

$$-\alpha(v_0)\alpha''(v_0)\sin^2 u_0 - (\alpha'(v_0))^2 \cos^2 u_0 < 0$$

respectively.

Thus the critical point (u_0, v_0) of the local representation φ or ψ of the restriction $f|_{\Sigma_g}$ is, indeed, non-degenerate of index one. □

Proof of Theorem 2.1 *in the orientable case.* We only need to treat the case $g \geq 2$ as for $g \in \{0,1\}$ we obviously have $\gamma_{_{S^1}}(\Sigma_0) = \gamma_{_{S^1}}(S^2) = \gamma(S^2) = 2$ and $\gamma_{_{S^1}}(\Sigma_1) = \gamma_{_{S^1}}(T^2) = 0$. For the inequality $\gamma_{_{S^1}}(\Sigma_g) \geq 2(g-1)$ we just need to use relation (2.2) that is $2 - 2g = \mu_0(F) - \mu_1(F) + \mu_2(F) \geq -\mu_1(F)$, for every circular Morse function $F : \Sigma_g \longrightarrow S^1$. This shows that $2(g-1) \leq \mu_1(F) \leq \mu_0(F) + \mu_1(F) + \mu_2(F) = \mu(F)$, for every circular Morse function $F : \Sigma_g \longrightarrow S^1$, and the inequality $2(g-1) \leq \gamma_{_{S^1}}(\Sigma_g)$ therefore. The opposite inequality is proved by the existence of the circular Morse function $f\big|_{\Sigma_g}$ which has exactly $2(g-1)$ critical points.

Remark 2.5. No real valued Morse function defined on a compact manifold M^m $(m \geq 2)$ can merely have critical points of index one, as the global minimum of such a function has index zero and its global maximum has index m. Thus the restriction $f|_{\Sigma_g}$ cannot be lifted to any map $\widetilde{f} : \Sigma_g \longrightarrow \mathbb{R}$, i.e. $\exp \circ \widetilde{f} = f$ and the induced group homomorphism $f_* : \pi(\Sigma_g) \longrightarrow \mathbb{Z} = \pi(S^1)$ is nontrivial therefore.

Proof of Theorem 2.1 *in the non-orientable case.* In this case we rely on Proposition 1.1 in order to prove the inequality

$$\gamma_{_{S^1}}\left(g\mathbb{RP}^2\right) \geq |\chi\left(g\mathbb{RP}^2\right)|,$$

for $g \geq 2$, where $k\mathbb{RP}^2$ stands for the connected sum $\mathbb{RP}^2 \# \mathbb{RP}^2 \# \cdots \# \mathbb{RP}^2$ of k copies of the projective plane. Indeed, by applying Proposition 1.1 to the orientable double cover

$$\Sigma_{g-1} \to g\mathbb{RP}^2$$

we obtain successively:

$$\begin{aligned}\gamma_{S^1}\big(g\mathbb{RP}^2\big) &\geq \tfrac{1}{2}\gamma_{S^1}\big(\Sigma_{g-1}\big) = \tfrac{1}{2}|\chi\big(\Sigma_{g-1}\big)| \\ &= \tfrac{1}{2}|2-2(g-1)| = |2-g| = |\chi\big(g\mathbb{RP}^2\big)|.\end{aligned}$$

For the opposite inequality we first recall that

$$f : \mathbb{RP}^2 \longrightarrow \mathbb{R},\ f([x_1, x_2, x_3]) = \frac{x_1^2 + 2x_2^2 + 3x_3^2}{x_1^2 + x_2^2 + x_3^2},$$

is a perfect Morse function with exactly three critical points of indices $0, 1, 2$, i.e a minimum point p, a maximum point q and a saddle point s. If $\varepsilon > 0$ is small enough, then the inverse images $D := f^{-1}(-\infty, f_2(p)+\varepsilon)$ and $D' := f^{-1}(f(q)-\varepsilon, \infty)$ are open disks and the inverse image $f^{-1}[f(p) + \varepsilon, f(q) - \varepsilon] = \mathbb{RP}^2 \setminus (D_1 \cup D_2)$ is a compact surface with two circular boundary components $f^{-1}(f(p) + \varepsilon)$ and $f^{-1}(f(q) - \varepsilon)$. Observe that the restriction

$$f\big|_{\mathbb{RP}^2\setminus(D\cup D')} : \mathbb{RP}^2 \setminus (D_1 \cup D_2) \longrightarrow [f(p) + \varepsilon, f(q) - \varepsilon]$$

has one critical point of index one, i.e the saddle point s. We next glue successively g copies of $\mathbb{RP}^2 \setminus (D \cup D')$, say

$$M_1 := \mathbb{RP}^2\setminus(D_1\cup D_1'), \ldots, M_g := \mathbb{RP}^2\setminus(D_g \cup D_g'),$$

along the circular boundaries

$$\partial D_i' := f_i^{-1}(f_i(q) - \varepsilon) \subset M_i \text{ and } \partial D_{i+1} := f_{i+1}^{-1}(f_{i+1}(p) + \varepsilon) \subset M_{i+1}$$

of

$$D_i' := f_i^{-1}(f_i(q) - \varepsilon, \infty) \text{ and } D_{i+1} := f_{i+1}^{-1}(-\infty, f_{i+1}(p) + \varepsilon),$$

where

$$f_i := f + iL : \mathbb{RP}^2 \longrightarrow \mathbb{R}, \quad (i = 1, \ldots, g-1)$$

and

$$L := \text{length}([f(p) + \varepsilon, f(q) - \varepsilon]) = f(q) - f(p) - 2\varepsilon.$$

The obtained surface is $g\mathbb{RP}^2 \setminus (D_1 \cup D_g')$. Note that f_i is a Morse function with one saddle point which is constant on each of the circular boundaries $\partial D_i = f_i^{-1}(f_i(p)+\varepsilon)$ and $\partial D_i' = f_i^{-1}(f_i(q) - \varepsilon)$ of M_i. Moreover, the equalities $f_i\big|_{\partial D_i'} = f_{i+1}\big|_{\partial D_{i+1}}$ hold for every $i = 1, \ldots, g-1$, which shows that the function

$$F : g\mathbb{RP}^2 \setminus (D_1 \cup D_g') \longrightarrow \mathbb{R},\ F\big|_{M_i} := f_i$$

is well defined. In fact, F is a Morse function with g saddle points which is constant on the circle boundaries

$$\partial D_1 = f_1^{-1}(f_1(p) + \varepsilon) \subset M_1 \text{ and } \partial D_g' = f_g^{-1}(f_g(q) - \varepsilon) \subset M_g.$$

Identifying the circle boundaries ∂D_1 and $\partial D_g'$ of $g\mathbb{RP}^2 \setminus (D_1 \cup D_g')$, via a suitable diffeomorphism $\varphi : \partial D_1 \longrightarrow \partial D_g'$, we get the non-orientable surface $(g + 2)\mathbb{RP}^2$. Identifying $\min F$ with $\max F$ in $\operatorname{Im}(F)$ we obtain the circle S^1. Also, the Morse function

$$g\mathbb{RP}^2 \setminus (D_1 \cup D_g') \longrightarrow \operatorname{Im}(F),\ x \mapsto F(x)$$

descends to a circular Morse function

$$f_0 : (g+2)\mathbb{RP}^2 = g\mathbb{RP}^2 \setminus (D_1 \cup D'_g)\Big/\{x = \varphi(x)\} \to S^1 = \mathrm{Im}(F)\Big/\{\min F = \max F\}$$

with g saddle points. This shows that the inequality $\gamma_{S^1}\left((g+2)\mathbb{RP}^2\right) \leq g$ holds for all $g \geq 1$.

Therefore, we provided the second proof of Theorem 2.1 in the non-orientable cases $g\mathbb{RP}^2$ with $g \geq 3$. On the other hand the Klein Bottle $2\mathbb{RP}^2$ is a fibration over S^1 with fiber S^1, which shows that $\gamma_{S^1}\left(2\mathbb{RP}^2\right) = 0 = |\chi(2\mathbb{RP}^2)|$. □

3. On the number of characteristic points

The horizontal distribution of the first Heisenberg group $\mathbb{H}^1 = (\mathbb{R}^3, *)$ is $\mathcal{H} = \mathrm{span}(X, Y) = \{\mathcal{H}_p := \mathrm{span}(X_p, Y_p)\}_{p\in\mathbb{H}^1}$, where $X = \partial_x + 2y_i\partial_t$ and $Y = \partial_y - 2x\partial_t$. Let us consider a surface $S \subseteq \mathbb{R}^3$ which is C^1 smooth. The *characteristic set* [5, 6] of S with respect to $\mathcal{H}$ is defined as

$$C(S, \mathcal{H}) := \{p \in S \,:\, T_pS = \mathcal{H}_p\}.$$

Definition 3.1. *If S is a C^1 smooth surface which can be embedded into $\mathbb{R}^3$, then the* minimum characteristic number *of S relative to $\mathcal{H}$ on $\mathbb{R}^3$ is defined as*

$$mcn(S, \mathcal{H}) := \min\{\mathrm{card}\,(C(f(S), \mathcal{H})) : f \in \mathrm{Embed}(S, \mathbb{R}^3)\},$$

where $\mathrm{Embed}(S, \mathbb{R}^3)$ *stands for the set of all embeddings of S into $\mathbb{R}^3$.*

Theorem 3.2. *If $g \geq 2$, then $2g - 2 \leq mcn(\Sigma_g, \mathcal{H}) \leq 4g - 4$.*

For the lower bound $2g - 2$ of $mcn(\Sigma_g, \mathcal{H})$ we refer the reader to [4] and for the upper bound $4g - 4$ we need to construct an embedding of Σ_g in $\mathbb{R}^3$ with $4g - 4$ characteristic points with respect to the horizontal distribution of the first Heisenberg group $\mathbb{H}^1 = (\mathbb{R}^3, *)$. In this respect we shall use the possibility to embed Σ_1 in $\mathbb{R}^3$ as a revolution surface and construct a suitable embedding of Σ_g out of Σ_1 by performing some surgery on Σ_1. The handles we plan to glue are surfaces of revolution as well. In fact, we shall use the embedding of Σ_g described in the previous section. Therefore we need to investigate the size of the characteristic sets of revolution surfaces $S \subset \mathbb{R}^3$ with respect to the horizontal distribution of the first Heisenberg group $\mathbb{H}^1 = (\mathbb{R}^3, *)$.

3.1. Revolution surfaces in $\mathbb{H}^1$ with low number of horizontal points

Every revolution surface S obtained by rotating a plane curve $x = \alpha(v), z = v$ $(\alpha > 0)$ around the vertical line $x = x_0, y = y_0$ admits a local parametrization of type

$$\begin{aligned} x &= x_0 + \alpha(v)\cos u \\ y &= y_0 + \alpha(v)\sin u \quad,\ u \in I, v \in J, \\ z &= v \end{aligned}$$

where I is an open interval of length 2π and J will be symmetric with respect to the origin, i.e. $J = (-a, a)$. The function f is subject to the following requirements:

$$\alpha \text{ is bounded },\ \alpha'' > 0 \text{ and } \lim_{v\to\pm a} \alpha'(v) = \pm\infty. \tag{3.1}$$

The vector equation of our revolution surface is

$$\vec{r} = (x_0 + \alpha(v)\cos u)\partial_x + (x_0 + \alpha(v)\sin u)\partial_y + v\partial_t$$

and

$$\begin{array}{rcl} \vec{r}_u & = & -(\alpha(v)\sin u)\partial_x + (\alpha(v)\cos u)\partial_y \\ \vec{r}_v & = & (\alpha'(v)\cos u)\partial_x + (\alpha'(v)\sin u)\partial_y + \partial_t \\ \vec{r}_u \wedge \vec{r}_v & = & (\alpha(v)\cos u)\partial_x + (\alpha(v)\sin u)\partial_y - \alpha(v)\alpha'(v)\partial_t. \end{array}$$

On the other hand the horizontal vector fields of the distribution $\mathcal{H}$ are $X = \partial_x + 2y\partial_t$, $Y = \partial_y - 2x\partial_t$ and their vector product is

$$X \wedge Y = -2y\partial_x + 2x\partial_y + \partial_t.$$

Thus, the point $r(u,v) := (x(u,v), y(u,v), z(u,v)) \in S$ is a horizontal point if and only if the vectors $\vec{r}_u \wedge \vec{r}_v, X \wedge Y$ are linearly dependent at $r(u,v)$, i.e.

$$\begin{array}{rcrcl} \sin u & + & 2\alpha(v)\alpha'(v)\cos u & = & -2x_0\alpha'(v) \\ 2\alpha(v)\alpha'(v)\sin u & - & \cos u & = & -2y_0\alpha'(v). \end{array}$$

Thus

$$\begin{array}{rcl} \sin u & = & -2\alpha'(v)\dfrac{x_0 + 2y_0\alpha(v)\alpha'(v)}{1 + 4\alpha^2(v)(\alpha'(v))^2} \\ \cos u & = & -2\alpha'(v)\dfrac{2x_0\alpha(v)\alpha'(v) - y_0}{1 + 4\alpha^2(v)(\alpha'(v))^2}. \end{array} \tag{3.2}$$

Remark 3.3. *No revolution surface around the z-axis has $\mathcal{H}$-tangency points, as the equations (3.2) have no solutions at all for $x_0 = y_0 = 0$.*

The identity $\sin^2 u + \cos^2 u = 1$ leads us to the equation

$$(\alpha'(v))^2 = \frac{1}{4\left(||(x_0, y_0)||^2 - \alpha^2(v)\right)}, \tag{3.3}$$

which has at least two solutions on the interval $J = (-a, a)$, as the right hand side of (3.3) is bounded and $(\alpha')^2$ covers the positive real half line $[0, \infty)$ twice, once on the interval $(-a, o]$ and once on the interval $[0, a)$. For suitable choices of the function α, the equation (3.3) has precisely two solutions. Such a choice is

$$\alpha(v) = 2 - \sqrt{\frac{2 - v^2}{2}} \tag{3.4}$$

for $a = \sqrt{2}$ and $||(x_0, y_0)|| = 3$. Indeed, for the choice (3.4) of the function α the equation (3.3) becomes:

$$4v^2\sqrt{2(2 - v^2)} = -v^4 - 9v^2 + 2. \tag{3.5}$$

Note that the equation (3.5) has precisely two solutions, as can be easily checked.

Proof of Theorem 3.2. The closed convex curve in the plane xOz described at the beginning of the section (2.1.1) is supposed to have its unique center at the point $(3, 0, 0)$. The coordinates of the points p_i and q_i have the forms (x_i, y_i, z_i) and $(x_i, y_i, -z_i)$ respectively, for $i = 1, \ldots, g - 1$. Moreover $||(x_i, y_i)||^2 := x_i^2 + y_i^2 = 3$

for all $i = 1, \dots, g-1$. The handles we use within our surgery process are revolution surfaces around the vertical lines $x = x_i, y = y_i$ of parametrized equations

$$\begin{array}{l} x = x_i + \alpha(v)\cos u \\ y = y_i + \alpha(v)\sin u \\ z = v \end{array}, \quad u \in I, v \in J,$$

We denote by v_i and v_i' the roots of the equations

$$(\alpha'(v))^2 = \frac{1}{4(||(x_i, y_i)||^2 - \alpha^2(v))}, \tag{3.6}$$

with the choice (3.4) for the function f. The equations which corresponds to (3.2)

$$\begin{cases} \sin u & = & -2\alpha'(v_i)\dfrac{x_i + 2y_i\alpha(v_i)\alpha'(v_i)}{1 + 4\alpha^2(v_i)(\alpha'(v_i))^2} \\ \\ \cos u & = & -2\alpha'(v_i)\dfrac{2x_i\alpha(v_i)\alpha'(v_i) - y_i}{1 + 4\alpha^2(v_i)(\alpha'(v_i))^2}, \end{cases} \tag{3.7}$$

$$\begin{cases} \sin u & = & -2\alpha'(v_i')\dfrac{x_i + 2y_i\alpha(v_i')\alpha'(v_i')}{1 + 4\alpha^2(v_i')(\alpha'(v_i'))^2} \\ \\ \cos u & = & -2\alpha'(v_i')\dfrac{2x_i\alpha(v_i')\alpha'(v_i') - y_i}{1 + 4\alpha^2(v_i')(\alpha'(v_i'))^2}. \end{cases} \tag{3.8}$$

Since the graphs of the sine and cosine functions on each interval of length 2π are intersected at most twice by any straight line parallel to the u-axis, it follows that the equations (3.7) as well as (3.8) have at most two roots for each $i = 1, \dots, g-1$. On the other hand the surface Σ_g embedded in $\mathbb{H}^1$ the way described right after Theorem 3.2 has no other $\mathcal{H}$-characteristic points. Indeed, on the two annuli $\mathcal{A}$ and $\mathcal{A}'$ the tangent planes to Σ_g are parallel to the xOy plane, a parallelism relation which happens for the planes of the distribution $\mathcal{H}$ just along the z-axis. This shows that Σ_g, embedded in $\mathbb{R}^3 \setminus Oz$ as described before, has no extra characteristic points as two annuli have no common points with the z-axis. The remaining part of our embedded Σ_g is completely contained in Σ_1 which is, in its turn, a revolution surface around the z-axis and has no $\mathcal{H}$-tangency points, as we saw in Remark 3.3. Thus, our embedded surface Σ_g has at most $4(g-1)$ $\mathcal{H}$-tangency points. □

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About a class of rational TC-Bézier curves with two shape parameters

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To the memory of Professor Mircea-Eugen Craioveanu (1942-2012)

Abstract. In this paper, we will study some properties concerning the cubic rational trigonometric Bézier curve attached at a class of cubic trigonometric Bézier curves with two shape parameters (for short TC-Bézier curves) introduced in paper [6].

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1. Introduction

In the following lines, we will present some well known results about Bézier curves.

A Bézier curve is defined using the classical Bernstein polynomials, in the following way:

$$P(t) = \sum_{i=0}^{n} B_{i,n}(t)p_i \tag{1.1}$$

where p_i with $i = \overline{0,n}$, represent the control points attached to Bézier curve and

$$B_{i,n}(t) = \binom{n}{i} t^i (1-t)^{n-i}$$

with $t \in [0,1]$ represent the Bernstein polynomials.

A cubic Bézier curve can be obtained for $n = 3$ and have the following form:

$$P(t) = \binom{3}{0}(1-t)^3 p_0 + \binom{3}{1} t(1-t)^2 p_1 + \binom{3}{2} t^2(1-t)p_2 + \binom{3}{3} t^3 p_3$$

or, for short:

$$P(t) = (1-t)^3 p_0 + 3t(1-t)^2 p_1 + 3t^2(1-t)p_2 + t^3 p_3$$

A rational Bézier curve is given by:

$$x(t) = \frac{w_0 p_0 B_{0,n}(t) + ... + w_n p_n B_{n,n}(t)}{w_0 B_{0,n}(t) + ... + w_n B_{n,n}(t)} \tag{1.2}$$

Here, w_i with $i = \overline{0,n}$ represent the weights of the control points p_i. We can rewrite (1.2) in the following way:

$$x(t) = \frac{\sum_{i=0}^{n} w_i p_i B_{i,n}(t)}{\sum_{i=0}^{n} w_i B_{i,n}(t)} \tag{1.3}$$

The authors of paper [6], H. Liu, L. Li and Z. Daming, have replaced the classical Bernstein base of the cubic Bézier curve with a new one which has 2 parameters λ and μ.

The trigonometric base choosed by the authors of paper [6] for the cubic TC Bézier curve, is:

$$\begin{cases} B_{0,3}(t) = 1 - (1+\lambda)\sin t + \lambda \sin^2 t \\ B_{1,3}(t) = (1+\lambda)\sin t - (1+\lambda)\sin^2 t \\ B_{2,3}(t) = (1+\mu)\cos t - (1+\mu)\cos^2 t \\ B_{3,3}(t) = 1 - (1+\mu)\cos t + \mu \cos^2 t \end{cases} \tag{1.4}$$

where $t \in \left[0, \frac{\pi}{2}\right]$ and $\lambda, \mu \in [-1, 1]$.

Other results concerning classical and trigonometric Bézier curves are obtained in the following papers:[1], [2], [3], [4], [5], [7] and [8].
Next, we will present some important results obtained in paper [6].

Theorem 1.1. ([6]) *The basis functions (1.4) have the following properties:*
(1) Nonnegativity and partition of unity: $B_{i,3}(t) \geq 0$, $i \in \{0,1,2,3\}$.
(2) Monotonicity: For a given parameter t, $B_{0,3}(t)$ and $B_{3,3}(t)$ are monotonically decreasing for the shape parameters λ and μ; respectively; $B_{1,3}(t)$ and $B_{2,3}(t)$ are monotonically increasing for the shape parameters λ and μ; respectively;
(3) Symmetry: $B_{i,3}(t;\lambda,\mu) = B_{3-i,3}(\frac{\pi}{2} - t; \lambda, \mu)$ *for* $i = \overline{0,3}$.

Definition 1.2. ([6]) *Given points p_i, ($i = \overline{0,3}$) in $\mathbb{R}^2$, $\mathbb{R}^3$, then*

$$r(t) = \sum_{i=0}^{3} p_i B_{i,3}(t) \tag{1.5}$$

$t \in \left[0, \frac{\pi}{2}\right]$; $\lambda, \mu \in [0, 1]$, *is called a cubic trigonometric Bézier curve with two shape parameters, i.e. the TC-Bézier curve for short.*

Theorem 1.3. *([6]) (partial enounce) The cubic TC-Bézier curves (1.5) have the following properties:*
(1) Terminal properties:

$$\begin{cases} r(0) = p_0 \\ r\left(\frac{\pi}{2}\right) = p_3; \end{cases} \qquad \begin{cases} r'(0) = (1+\lambda)(p_1 - p_0) \\ r'\left(\frac{\pi}{2}\right) = (1+\mu)(p_3 - p_2); \end{cases}$$

$$\begin{cases} r''(0) = 2\lambda p_0 - 2(1+\lambda)p_1 + (1+\mu)p_2 + (1-\mu)p_3 \\ r''\left(\frac{\pi}{2}\right) = (1-\lambda)p_0 + (1+\lambda)p_1 - 2(1+\mu)p_2 + 2\mu p_3; \end{cases}$$

(2) Symmetry: p_0, p_1, p_2, p_3 *and* p_3, p_2, p_1, p_0 *define the same TC-Bézier curve in different parametrizations.*
(3) Convex hull property: The entire TC-Bézier segment must lie inside its control polygon spanned by p_0, p_1, p_2, p_3.

For more details on TC-Bézier curves, please see [6].

2. Main results

Using the TC-Bézier curve presented before in this paper, we can introduce the cubic rational TC-Bézier curves, as follows:

$$r(t) = \frac{\sum_{i=0}^{3} w_i p_i B_{i,3}(t)}{\sum_{i=0}^{3} w_i B_{i,3}(t)} \tag{2.1}$$

with $\lambda, \mu \in [-1, 1]$ and w_i are the weights of the control points p_i with $i = \overline{0,3}$ and $B_{i,3}(t)$ represent the trigonometric basis introduced in (1.4).

We can rewrite (2.1) in the following way:

$$r(t) = \frac{(1-(1+\lambda)\sin t+\lambda\sin^2 t)w_0p_0+(1+\lambda)(\sin t-\sin^2 t)w_1p_1+(1+\mu)(\cos t-\cos^2 t)w_2p_2+(1-(1+\mu)\cos t+\mu\cos^2 t)w_3p_3}{(1-(1+\lambda)\sin t+\lambda\sin^2 t)w_0+(1+\lambda)(\sin t-\sin^2 t)w_1+(1+\mu)(\cos t-\cos^2 t)w_2+(1-(1+\mu)\cos t+\mu\cos^2 t)w_3}$$

where $\lambda, \mu \in [-1, 1]$, $t \in \left[0, \frac{\pi}{2}\right]$.

Theorem 2.1. *The curvature in* $t = 0$ *for the rational TC-Bézier curve (2.1), is:*

$$K(0) = \left(\frac{1+\mu}{1+\lambda}\right)\frac{w_0}{w_1^2}\left(w_3\frac{\|\overline{p_0p_1} \times \overline{p_0p_3}\|}{\|\overline{p_0p_1}\|^3} - w_2\frac{\|\overline{p_0p_1} \times \overline{p_0p_2}\|}{\|\overline{p_0p_1}\|^3}\right)$$

Proof. We start with r(t) defined in (2.1). After tedious computations for $\overline{r'(t)}$ and $\overline{r''(t)}$, one obtains for $t = 0$, the following result:

$$\overline{r'(0)} = -\frac{w_1}{w_0}(p_0\lambda - p_1\lambda + p_0 - p_1) = -(\lambda+1)\frac{w_1}{w_0}\overline{p_0p_1}$$

Then, we obtain:

$$\overline{r''(0)} = -\frac{1}{w_0^2}[w_0w_2(1+\mu)\overline{p_0p_2} -$$
$$-2\left(w_1^2 + 2w_1^2\lambda + w_1^2\lambda^2 - w_0w_1\lambda - w_0w_1\lambda^2\right)\overline{p_0p_1} - w_0w_3(1+\mu)\overline{p_0p_3}]$$

From the curvature definition, for $t = 0$, we know that:

$$K(0) = \frac{\left\|\overline{r'(0)} \times \overline{r''(0)}\right\|}{\left\|\overline{r'(0)}\right\|^3}$$

Now, we compute:

$$\overline{r'(0)} \times \overline{r''(0)} = -\lambda^2 \frac{w_1}{w_0^3} \left[w_0 w_2 (1+\mu)(\overline{p_0 p_1} \times \overline{p_0 p_2} - w_0 w_3 (1+\mu)(\overline{p_0 p_1} \times \overline{p_0 p_3}) \right] =$$

$$\lambda^2 \frac{w_1}{w_0^2} \left[w_3 (\overline{p_0 p_1} \times \overline{p_0 p_3}) - w_2 (\overline{p_0 p_1} \times \overline{p_0 p_2}) \right]$$

Also, one obtains:

$$\left\| \overline{r'(0)} \right\|^3 = \lambda^3 \frac{w_1^3}{w_0^3} \left\| \overline{p_0 p_1} \right\|^3$$

Finally, we get:

$$K(0) = \frac{\lambda^2 \frac{w_0 w_1}{w_0^3} \left[w_3 (1+\mu) \left\| \overline{p_0 p_1} \times \overline{p_0 p_3} \right\| - w_2 (1+\mu) \left\| \overline{p_0 p_1} \times \overline{p_0 p_2} \right\| \right]}{\lambda^3 \frac{w_1^3}{w_0^3} \left\| \overline{p_0 p_1} \right\|^3}$$

$$= \left(\frac{1+\mu}{1+\lambda} \right) \frac{w_0}{w_1^2} \left(w_3 \frac{\left\| \overline{p_0 p_1} \times \overline{p_0 p_3} \right\|}{\left\| \overline{p_0 p_1} \right\|^3} - w_2 \frac{\left\| \overline{p_0 p_1} \times \overline{p_0 p_2} \right\|}{\left\| \overline{p_0 p_1} \right\|^3} \right)$$

and this complete the proof. □

Remark 2.2. For the particular case, when we have the same weights $w_0 = w_1$, one obtains one of the well known results from Theorem 1.3, which was:

$$r'(0) = (1+\lambda)(p_1 - p_0).$$

Next, we will reparametrizate the TC-Bézier rational curve and we take $t = \arcsin(u)$ with $t \in [0,1] \subset \left[0, \frac{\pi}{2}\right]$.
After reparametrization, we get:

$$r(t) = \frac{(1-(1+\lambda)u+\lambda u^2)w_0 p_0+(1+\lambda)(u-u^2)w_1 p_1+(1+\mu)(\sqrt{1-u^2}-1+u^2)w_2 p_2+(1-(1+\mu)\sqrt{1-u^2}+\mu(1-u^2))w_3 p_3}{(1-(1+\lambda)u+\lambda u^2)w_0+(1+\lambda)(u-u^2)w_1+(1+\mu)(\sqrt{1-u^2}-1+u^2)w_2+(1-(1+\mu)\sqrt{1-u^2}+\mu(1-u^2))w_3} \tag{2.2}$$

Remark 2.3. For $\lambda = \mu = 1$, in the above expression (2.2), one obtains the following TC-Bézier rational curve:

$$r(t) = \frac{(1-u)^2 w_0 p_0+2(u-u^2)w_1 p_1+2(\sqrt{1-u^2}-1+u^2)w_2 p_2+(1-\sqrt{1-u^2})^2 w_3 p_3}{(1-u)^2 w_0+2(u-u^2))w_1+2(\sqrt{1-u^2}-1+u^2)w_2+(1-\sqrt{1-u^2})^2 w_3} \tag{2.3}$$

Theorem 2.4. *The hodograph of the TC-Bézier rational curve (2.3), for $u = 0$, is*

$$2\frac{w_1}{w_0}(p_1 - p_0).$$

Proof. We start with the above expression of the TC-Bézier rational curve (2.3), and we compute:

$$\overline{r'(u)} = \frac{-2(1-u)w_0 p_0+(2-4u)w_1 p_1+\left(-\frac{2u}{\sqrt{(1-u^2)}}+4u\right)w_2 p_2+\frac{2(1-\sqrt{1-u^2})w_3 p_3 u}{\sqrt{1-u^2}}}{(1-u)^2 w_0+(2u-2u^2)w_1+(2\sqrt{1-u^2}+2u^2-2)w_2+(1-\sqrt{1-u^2})^2 w_3}$$

$$-\frac{(1-u)^2 w_0 p_0+(2u-2u^2)w_1 p_1+(2\sqrt{1-u^2}+2u^2-2)w_2 p_2+(1-\sqrt{1-u^2})^2 w_3 p_3}{(1-u)^2 w_0+(2u-2u^2)w_1+(2\sqrt{1-u^2}+2u^2-2)w_2+(1-\sqrt{1-u^2})^2 w_3} \cdot$$

$$\cdot \left(-2(1-u)w_0 p_0 + (2-4u)w_1 p_1 + \left(-\frac{2u}{\sqrt{(1-u^2)}} + 4u \right) w_2 p_2 + \frac{2(1-\sqrt{1-u^2})w_3 p_3 u}{\sqrt{1-u^2}} \right)$$

Replacing in the above expression $u = 0$, we get:

$$2\frac{w_1}{w_0}(p_1 - p_0). \tag{2.4}$$

and this end the proof of the theorem. $\square$

Remark 2.5. The hodograph of the classical rational Bézier curve for $u = 0$ is

$$3\frac{w_1}{w_0}(p_1 - p_0),$$

and this is a closed result obtained by us in (2.4).

Conclusion. In this paper we proved that the two shape parameters of one TC-Bézier rational curve have a key role when we compute the curvature of the curve. The computation of the torsion for this class of TC-Bézier rational curve is not an easy task. In a future paper we will try to continue our investigations on TC-Bézier rational curves.

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On some asymptotic properties of the Rössler dynamical system

Răzvan M. Tudoran and Anania Gîrban

To the memory of Professor Mircea-Eugen Craioveanu (1942-2012)

Abstract. In this paper we present a method to stabilize asymptotically the Lyapunov stable equilibrium states of the Rössler dynamical system.

Mathematics Subject Classification (2010): 70H05, 70H09, 70H14.

Keywords: Hamiltonian systems, Rössler system, stability theory, dissipation.

1. Introduction

The Rössler dynamical system [12] has been widely investigated over the last years, mainly from the chaotic dynamics perspective. In this work we are concerned with the analysis of the conservative properties of this system. Among the studied topics related to the conservative properties of the Rössler dynamical system, one can mention various types of integrability, namely Darboux integrability ([8], [13]), formal and analytic integrability [7], the description of the global dynamics in the Poincaré sphere [6] and a dynamical analysis from the Hamiltonian point of view [14].

The aim of this work is to analyze further the Rössler dynamical system from the stability theory point of view. More exactly, we present a method to associate to each Lyapunov stable equilibrium state of the Rössler system, a special type of dissipative system in such a way that each Lyapunov stable equilibrium state of the Rössler system generates a one dimensional attracting neighborhood for the dissipative system.

The structure of the paper is as follows. In the second section of this work, we recall from [14] the geometric framework adopted in our study, namely a Hamiltonian realization of the Rössler system. In the third section of the paper we recall from [14] the main results regarding the Lyapunov stability analysis of the equilibrium states of the Rössler system. In the fourth section, we recall the definition of a metriplectic system and construct explicitly a metriplectic perturbation associated to the Rössler system. The metriplectic perturbation of the Rössler system, prove to have all the equilibrium states of the Rössler system. The last part of the paper contains the main results, namely it describes explicitly the method of associating, to each Lyapunov stable equilibrium state of the Rössler system, a special type of metriplectic system, in such a way that each Lyapunov stable equilibrium of the unperturbed system generates a one dimensional attracting neighborhood for the dissipative system.

For details on Hamiltonian dynamics, and respectively metriplectic dissipative systems, see, e.g. [1], [2], [11], [3], [4], [9].

2. Setting of the problem from the Poisson geometry point of view

As the purpose of this paper is to study a special type of perturbations of a Hamiltonian system from the Poisson dynamics and geometry point of view, the first step in this approach is to prepare the geometric framework of the problem. The results from this chapter are from [14].

The Rössler system we consider for our study, is governed by the equations:

$$\begin{cases} \dot{x} = -y - z \\ \dot{y} = x \\ \dot{z} = xz. \end{cases} \tag{2.1}$$

Note that in the article [13] it is proved that the above system it is the only case when the Rössler system it is completely integrable.

Let us recall now some results from [14] concerning the geometric framework of the problem. The following proposition from [14] provides a Hamiltonian formulation of the Rössler system on an appropriate Poisson manifold.

Theorem 2.1. *The dynamics* (2.1) *admit the following Hamilton-Poisson realization:*

$$(\mathbb{R}^3, \nu\Pi_C, H) \tag{2.2}$$

where

$$\Pi_C(x,y,z) = \begin{bmatrix} 0 & e^{-y} & ze^{-y} \\ -e^{-y} & 0 & 0 \\ -ze^{-y} & 0 & 0 \end{bmatrix}$$

is the Poisson structure generated by the smooth function $C(x,y,z) := ze^{-y}$, the rescaling ν is given by $\nu(x,y,z) = -e^{y}$, and the Hamiltonian $H \in C^{\infty}(\mathbb{R}^3,\mathbb{R})$ is given by $H(x,y,z) := \dfrac{1}{2}(x^2+y^2)+z$.

Note that, by Poisson structure generated by the smooth function C, we mean the Poisson structure generated by the Poisson bracket

$$\{f,g\}_C := \nabla C \cdot (\nabla f \times \nabla g),$$

for any smooth functions $f, g \in C^{\infty}(\mathbb{R}^3,\mathbb{R})$.

Next remark from [14] provides a class of first integrals for all the Hamiltonian dynamical systems modeled on the Poisson manifold $(\mathbb{R}^3, \nu\Pi_C)$.

Remark 2.2. By definition we have that the center of the Poisson algebra $(C^{\infty}(\mathbb{R}^3,\mathbb{R}), \{\cdot,\cdot\}_C)$ is generated by the Casimir invariant $C(x,y,z) = ze^{-y}$.

3. Equilibrium states and Lyapunov stability

In this short section, we recall some results from [14] regarding the Lyapunov stability of the equilibrium states of the Rössler system (2.2). As our main purpose is to perturb the Rössler system in such a way that, each Lyapunov stable equilibrium of the unperturbed system, turnes to an asymptotically stable equilibrium for the perturbed system, we do not consider here the unstable equilibrium states of the unperturbed system.

Note that the set of equilibrium states of the Rössler system is given by

$$\mathcal{E} := \{(0, -M, M) : M \in \mathbb{R}\}.$$

Let us recall from [14], the following theorem describing the stability properties of the equilibrium states of the Rössler system.

Theorem 3.1. *Let $e_M = (0, -M, M) \in \mathcal{E}$ be an arbitrary equilibrium state of the Rössler system* (2.1)*. The equilibrium $e_M \in \mathcal{E}$ is Lyapunov stable for $M > -1$ and unstable for $M \leq -1$.*

Proof. See [14]. □

4. Metriplectic perturbations of the system (2.2)

The purpose of this section is to associate to the Rössler system (2.2), a class of metriplectic systems (parameterized by a smooth real function $\varphi \in C^{\infty}(\mathbb{R}, \mathbb{R})$) in such a way that the equilibrium states of the Hamilton-Poisson system (2.2) are also equilibrium states for all the associated metriplectic systems. By metriplectic system we mean a dynamical system consisting of a compatible pair consisting of a conservative system (modeled by a Hamiltonian system), together with a dissipative (nonconservative) system (modeled by a gradient system with respect to a symmetric tensor G). For details regarding the properties of metriplectic systems, see e.g. [10], [3].

Let us give first the definition of a general metriplectic perturbation of a Hamiltonian system on the Poisson manifold $(\mathbb{R}^3, \nu\Pi_C)$.

Definition 4.1. *A metriplectic perturbation of a Hamiltonian system on $(\mathbb{R}^3, \nu\Pi_C)$ is a dynamical system of the type:*

$$\dot{u} = \nu(u)\Pi_C(u) \cdot \nabla H(u) + G(u) \cdot \nabla(\varphi \circ C)(u), \; u^T = (x, y, z) \in \mathbb{R}^3,$$

where $\nu, H, C \in C^{\infty}(\mathbb{R}^3, \mathbb{R})$, $C(x, y, z) = ze^{-y}$, $\nu(x, y, z) = -e^y$, G is a symmetric covariant tensor, and $\varphi \in C^{\infty}(\mathbb{R}, \mathbb{R})$, such that the following compatibility conditions hold:

(i) $G \cdot \nabla H = \bar{0}$,
(ii) $(\nabla(\varphi \circ C))^T \cdot G \cdot \nabla(\varphi \circ C) \leq 0$.

Let us now construct a metriplectic perturbation of the Rössler system (2.2). In order to do that, we associate to the Hamiltonian $H \in C^{\infty}(\mathbb{R}^3, \mathbb{R})$, $H(x, y, z) = \frac{1}{2}(x^2+y^2)+z$, of the system (2.2), a second order covariant symmetric tensor, given by

$G = \nabla H \otimes \nabla H - \|\nabla H\|^2 \, \mathrm{Id}$, in order to get a candidate for a metriplectic perturbation of the system (2.2).

Note that, in coordinates:

$$G(x, y, z) = \begin{bmatrix} -y^2 - 1 & xy & x \\ xy & -x^2 - 1 & y \\ x & y & -x^2 - y^2 \end{bmatrix}.$$

Next proposition gives a family of metriplectic perturbations of the Rössler system, parameterized by a smooth real function $\varphi \in C^\infty(\mathbb{R}, \mathbb{R})$.

Proposition 4.2. *The system:*

$$\dot{u} = \nu(u)\Pi_C(u) \cdot \nabla H(u) + G \cdot \nabla(\varphi \circ C)(u), \; u^T = (x, y, z), \tag{4.1}$$

is a metriplectic perturbation of the Rössler system, where $\nu, H, C \in C^\infty(\mathbb{R}^3, \mathbb{R})$ *are given by* $\nu(x, y, z) = -e^y$, $H(x, y, z) = \dfrac{1}{2}(x^2 + y^2) + z$, *and respectively* $C(x, y, z) = ze^{-y}$.

Proof. In order to obtain the conclusion, we need to check the condition (i) and respectively (ii) from the above definition. The condition (i) follows by straightforward computations. To verify the condition (ii), note that:

$$\begin{aligned} (\nabla(\varphi \circ C)(x, y, z))^T \cdot G(x, y, z) \cdot \nabla(\varphi \circ C)(x, y, z) = \\ = -\left[\varphi'\left(ze^{-y}\right)\right]^2 \cdot e^{-2y} \left[x^2 + x^2 z^2 + (y + z)^2\right] \\ \leq 0. \end{aligned}$$

□

Before analyzing the equilibrium states of the metriplectic system, let us write the system in coordinates.

Remark 4.3. The metriplectic system (4.1) is given in coordinates by:

$$\begin{cases} \dot{x} = -y - z + \varphi'\left(ze^{-y}\right) \cdot xe^{-y}(1 - yz), \\ \dot{y} = x + \varphi'\left(ze^{-y}\right) \cdot e^{-y}(y + z + x^2 z), \\ \dot{z} = xz - \varphi'\left(ze^{-y}\right) \cdot e^{-y}(x^2 + y^2 + yz). \end{cases} \tag{4.2}$$

Next remark gives a relation between the equilibrium states of the Hamilton-Poisson system (2.2) and the associated metriplectic perturbation (4.1).

Remark 4.4. All of the equilibrium states of the Rössler system (2.2) are also equilibrium states for the perturbed metriplectic system (4.1), for any smooth real function $\varphi \in C^\infty(\mathbb{R}, \mathbb{R})$.

5. Asymptotically stabilizing the metriplectically perturbed system

The aim of this section is to discuss the asymptotic stability of some special equilibrium states of the metriplectic system (4.1). In the previous section we obtained that for any smooth real function $\varphi \in C^\infty(\mathbb{R}, \mathbb{R})$, all the equilibrium states of the Rössler system (2.1) are also equilibrium states of his metriplectic perturbation (4.1). The aim of this section is to make use of this important property in order to metriplectically perturb the Rössler system in such a way that each Lyapunov stable equilibrium of the unperturbed system, generates a one-dimensional attracting neighborhood for the associated metriplectically perturbed system.

Before stating the main results of this paper, let us recall the principle of LaSalle [5].

Theorem 5.1. *Let $x_0 \in \mathbb{R}^n$ be an equilibrium state of the dynamical system $\dot{x} = f(x)$, where $f \in C^\infty(\mathbb{R}^n, \mathbb{R}^n)$, and let U be a compact neighborhood around x_0. Suppose there exists $L : U \to \mathbb{R}$ a C^1 function with $L(x) > 0$ for $x \neq x_0$, $L(x_0) = 0$ and $\dot{L}(x) \leq 0$. Let $E := \{x \in U : \dot{L}(x) = 0\}$ and $M \subset E$ be the largest dynamically invariant subset of E. Then there exists $V \subset U$ a neighborhood of x_0 such that the ω-limit set $\omega(x) \subset M$ for all $x \in V$.*

Let us now state the main result of this article.

Theorem 5.2. *Let $e_M \in \mathcal{E}$ be a Lyapunov stable equilibrium state of the Rössler systems* (2.1)*, and respectively* (4.1)*. Then there exists a smooth function $\varphi_{e_M} \in C^\infty(\mathbb{R}, \mathbb{R})$, a compact neighborhood K around e_M and a neighborhood $U \subset K$ such that any solution of the metriplectic system* (4.1) *(corresponding to φ_{e_M}) starting in U approaches $K \cap \mathcal{E}$.*

Proof. Let $e_M = (0, -M, M) \in \mathcal{E}$ be a Lyapunov stable equilibrium state of the Rössler system (2.1). Recall from Theorem (3.1) that e_M is a Lyapunov stable equilibrium state for the system (2.1) if and only if $M > -1$. Recall that e_M it is also an equilibrium state for the system (4.1) for any smooth real function φ. In order to prove the theorem, we construct a Lyapunov type function that verifies the hypothesis of LaSalle's principle. Let

$$(x, y, z) \in \mathbb{R}^3 \mapsto L_{\varphi_{e_M}}(x, y, z) = \frac{1}{2}(x^2 + y^2) + z + \varphi_{e_M}\left(ze^{-y}\right) \in \mathbb{R}$$

be a smooth real function, where $\varphi_{e_M} \in C^\infty(\mathbb{R}, \mathbb{R})$ is given by

$$\varphi_{e_M}(t) = e^{-2M} \cdot \frac{M+2}{M+1} \cdot \frac{t^2}{2} - e^{-M} \cdot \left[\frac{M(M+2)}{M+1} + 1\right] \cdot t.$$

Using these functions, we construct a candidate for a Lyapunov type function that verifies LaSalle's principle.

Let $L_{e_M} \in C^\infty(\mathbb{R}^3, \mathbb{R})$ be the smooth function given by

$$L_{e_M}(x, y, z) = L_{\varphi_{e_M}}(x, y, z) - L_{\varphi_{e_M}}(0, -M, M).$$

Note that the condition $L_{e_M}(e_M) = 0$ is automatically satisfied, and also we have that $\mathbf{d}L_{e_M}(e_M) = 0$. Hence, to check the first condition of LaSalle's principle, i.e., $L_{e_M}(x, y, z) > L_{e_M}(e_M) = 0$, locally for $(x, y, z) \neq e_M = (0, -M, M)$, it is enough to prove that $\mathbf{d}^2 L_{e_M}(e_M)$ is positive definite.

This is true indeed, because

$$\mathbf{d}^2 L_{e_M}(e_M) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & M^2 + (M+1)^{-1} & 1 - M(M+2)(M+1)^{-1} \\ 0 & 1 - M(M+2)(M+1)^{-1} & (M+2)(M+1)^{-1} \end{bmatrix}$$

is positive definite, since $M > -1$.

To check the last condition of LaSalle's principle we compute first $\dot{L}_{e_M}$.

$$\begin{aligned} \dot{L}_{e_M}(x,y,z) &= [\nabla H(x,y,z) + \nabla(\varphi_{e_M} \circ C)(x,y,z)]^T (\dot{x}, \dot{y}, \dot{z})^T \\ &= [\nabla H(x,y,z) + \nabla(\varphi_{e_M} \circ C)(x,y,z)]^T [\nu(x,y,z)\Pi_C(x,y,z)\nabla H(x,y,z) \\ &+ G(x,y,z)\nabla(\varphi_{e_M} \circ C)(x,y,z)] \\ &= -\left[\varphi'_{e_M}\left(ze^{-y}\right)\right]^2 \cdot e^{-2y} \cdot \left[x^2 + x^2 z^2 + (y+z)^2\right] \\ &\leq 0. \end{aligned}$$

Using the above relation and the analytic expression of φ'_{e_M}, we get that

$$E_{e_M} := \{(x,y,z) \in \mathbb{R}^3 \mid \dot{L}_{e_M}(x,y,z) = 0\} = \mathcal{E} \cup \Sigma_M,$$

where $\Sigma_M := \left\{(x,y,z) \in \mathbb{R}^3 \mid ze^{-y} = e^M[M + (M+1)(M+2)^{-1}]\right\}$ is a symplectic leaf of the Poisson manifold $(\mathbb{R}^3, \nu\Pi_C)$, and consequently a dynamically invariant set. Hence, the largest dynamically invariant subset $\mathcal{M}_{e_M} \subseteq E_{e_M}$ coincides with E_{e_M}. Now the conclusion follows from LaSalle's principle together with the remark that $e_M = (0, -M, M) \in \mathcal{E}$. □

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On operads in terms of finite pointed sets

Andor Lukács

Abstract. We prove that the definition of operads in terms of finite pointed sets is equivalent to the classical definition.

Mathematics Subject Classification (2010): 18D50, 55P48.

Keywords: Operads.

1. Introduction

Operads are algebraic structures that model various kinds of algebras such as commutative, associative, Lie, Poisson, etc. They were introduced by J. P. May in [9] as a tool to study the algebraic structures inherent in iterated loop spaces. May's work was continued by J. M. Boardman and R. M. Vogt on homotopy invariant algebraic structures in topological spaces, where operads played a central role. Starting from the nineties, operads had their renaissance, due to the works of M. Kontsevich on graph homology, of Ginzburg an Kapranov on generalized Koszul duality, and of P. Deligne on the structure of Hochschild cohomology among others (see [6, 4]).

In the literature there are mainly two equivalent definitions of operads that are used: the first one is the classical definition of May ([9]), and the second is the "$\circ_i$-definition", that also appears in the reference book of M. Markl, S. Shnider and J. Stasheff ([8]). It is folklore that these two definitions are equivalent (and an outline of the proof can be found in [8]).

If one examines any of these two definitions of operads, one can see that the role of the natural numbers is to keep track of the arity of the abstract operations as well as to label the inputs of these operations. This approach has certain disadvantages which become apparent when we compose two abstract operations. For example, to get the labels of the resulting operation right, one has to adjust the labels of the composed operations accordingly. This adjustment gives rise to not wanted technicalities in many cases, for instance when proving that something is an operad: one will need to use block permutations to prove equivariance and associativity for example.

A possible remedy to this problem can be given by labeling our operations in P with finite sets, and when a composition occurs just take the disjoint union of the reoccurring labels for the new operation. This approach has been used in the past for example by V. Hinich and A. Vaintrob in [5]. They credit P. Deligne and J. S. Milne for the formalism (see [2]). The "finite pointed sets" approach to operads was used also by P. van der Laan in his thesis [7]. None of these sources prove that the finite pointed set approach to operads is equivalent to the classical one.

The aim of this paper is to prove that the definition of operads in terms of finite pointed sets is equivalent to the classical definitions.

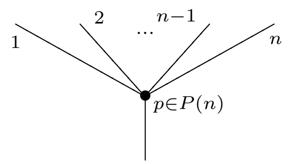
The paper is organised as follows. In Section 2 we describe operads intuitively. The goal here is to have a picture about operads in general, hence the technical details are omited (although the $\circ_i$-definition of operads in symmetrical monodal categories appears in all detail as a consequence of our constructions in Section 4). The reader interested in the technical details can find these in the work of J. P. May ([9]) and in the reference book of M. Markl, S. Shnider and J. Stasheff ([8]). In Section 3 we define operads in terms of finite pointed sets. In Section 4 we prove that this definition is equivalent to the definition in terms of the $\circ_i$ operations, hence to any of the two classical definitions, in the categorical sense.

2. An intuitive description of operads

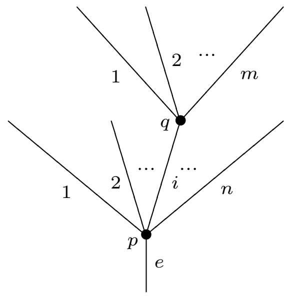
Intuitively, an operad in the classical sense consists of a "space"(vector space over a field $\Bbbk$, topological space, or more generally, an object in a symmetric monoidal category) $P(n)$ together with a right action of the symmetric group Σ_n on $P(n)$ for every $n \in \mathbb{N}$, an identity element $\mathrm{id} \in P(1)$ and composition maps

$$\circ_i \colon P(n) \otimes P(m) \longrightarrow P(n+m-1), \quad i = 1, 2, \ldots, n$$

for all n and m. The nature of the axioms this data has to satisfy comes from the intuition that the space $P(n)$ is thought of as a space of operations with n inputs and one output:



The action of the groups Σ_n permutes the inputs and the composition $p \circ_i q$ of two operations gives a new operation by using the output of q as the i-th input of p. This operation can be visualised as grafting the tree for q on the i-th leaf of the tree for p:



The unit $\mathrm{id} \in P(1)$ can be thought of as an operation which takes one input and gives it back as output.

The axioms that the operad P has to satisfy are the formal consequences of the above intuition. In fact, the intuition can be made to a rigourous example of an operad: if the underlying category is the category of vector spaces over a field $\Bbbk$ and if V is such a vector space, define

$$\mathrm{End}_V(n) := \mathcal{V}ect_{\Bbbk}(\underbrace{V \otimes \cdots \otimes V}_{n \text{ times}}, V)$$

and follow the description above to define the rest of the structure. This operad is called the *endomorphism operad on* V. It has a prominent role in the theory of operads not only because it models the abstract definition of operads, but also because it can "realize" on the space V the algebraic structure encoded in an operad P. To be more precise, note that any map of operads $\alpha\colon P \longrightarrow \mathrm{End}_V$ takes an "abstract" n-ary operation of $P(n)$ to a "concrete" n-ary operation $V \otimes \cdots \otimes V \longrightarrow V$ and the various compatibility conditions for α impose algebraic relations between these concrete operations on the End_V side. For particular operads in $\mathcal{V}ect_{\Bbbk}$ one can describe in this way various kinds of $\Bbbk$-algebras (e.g. associative, commutative, Lie, Poisson, Leibnitz, etc). This provides a justification for the following terminology: in the literature a vector space V together with an operad map $\alpha\colon P \longrightarrow \mathrm{End}_V$ is called a *P-algebra*.

A rigourous definition of an operad that follows the intuition given above can be found in [8], although the original definition (the one by J.P. May in [9]) differs from this approach. May's definition collects the $\circ_i$ composition maps for a given $P(n)$, $i = 1, 2, \ldots, n$ under one big composition map

$$P(n) \otimes \big(P(m_1) \otimes P(m_2) \otimes \cdots \otimes P(m_n)\big) \xrightarrow{\ \gamma\ } P(m_1 + m_2 + \cdots + m_n)\ .$$

An $n+1$-tuple of operations $(p, q_1, q_2, \ldots, q_n)$ is sent by γ to a new operation which we usually write as $p(q_1, q_2, \ldots, q_n)$ and visualise as n trees corresponding to the operations q_i, grafted upon the leaves of the tree corresponding to the operation p. The equivalence of the two definitions follows from the existence of the unit-operation $\mathrm{id} \in P(1)$, and a proof of this can be found in [8]. For example, the operation

$$\circ_i\colon P(n) \otimes P(m) \longrightarrow P(n + m - 1)$$

can be obtained from

$$\gamma\colon P(n)\otimes\big(P(1)\otimes\cdots P(1)\otimes \underset{i\text{-th}}{P(m)}\otimes P(1)\otimes\cdots\otimes P(1)\big)\longrightarrow P(m+n-1).$$

3. Operads in terms of finite pointed sets

Denote by $\mathcal{F}in_*$ the category of finite pointed sets (X,x_0) and basepoint-preserving bijections. To any ordered pair $((X,x_0),(Y,y_0))\in\mathcal{F}in_*\times\mathcal{F}in_*$ and $x\in X$, $x\neq x_0$ we render $(X\sqcup_x Y,x_0)\in\mathcal{F}in_*$, defined as

$$X\sqcup_x Y = X\sqcup Y\setminus\{x,y_0\}.$$

The following properties of the $\sqcup_x$ operations are going to be important for the definition of operads:

Associativity. If $(X,x_0),(Y,y_0),(Z,z_0)\in\mathcal{F}in_*$ and $x,x'\in X$, $y\in Y$ such that $x_0\neq x\neq x'\neq x_0$ and $y\neq y_0$ then

$$(X\sqcup_x Y)\sqcup_y Z = X\sqcup_x(Y\sqcup_y Z),$$
$$(X\sqcup_x Y)\sqcup_{x'} Z = (X\sqcup_{x'} Z)\sqcup_x Y.$$

Equivariance. If $\sigma\colon (X,x_0)\longrightarrow(X',x_0')$ and $\tau\colon(Y,y_0)\longrightarrow(Y',y_0')$ are maps in $\mathcal{F}in_*$ and $x\in X$, $x\neq x_0$ then σ and τ induce a map

$$\sigma\circ_x\tau\colon (X\sqcup_x Y,x_0)\longrightarrow(X'\sqcup_{\sigma(x)}Y',x_0')$$

in $\mathcal{F}in_*$, defined as

$$\sigma\circ_x\tau|_{X\setminus\{x\}}=\sigma|_{X\setminus\{x\}},$$
$$\sigma\circ_x\tau|_{Y\setminus\{y_0\}}=\tau|_{Y\setminus\{y_0\}}.$$

Unit. For any pointed set with two elements $(U,u_0)=(\{u,u_0\},u_0)$ and any other pointed set (X,x_0) together with an element $x\in X\setminus\{x_0\}$ there are maps

$$e_{ux_0}\colon (X,x_0)\longrightarrow(U\sqcup_u X,u_0)\quad\text{and}\quad e_{ux}\colon(X,x_0)\longrightarrow(X\sqcup_x U,x_0),$$

where e_{ux_0} sends x_0 to u_0 and is the identity elsewhere, and e_{ux} sends x to u and is the identity elsewhere.

Let $(\mathcal{E},\otimes,I,a,l,r,s)$ be a symmetric monoidal category.

Definition 3.1. *A contravariant functor $P\colon\mathcal{F}in_*^{\mathrm{op}}\longrightarrow\mathcal{E}$ is called a collection or a $\mathcal{F}in_*$-module in $\mathcal{E}$.*

If P is a collection in $\mathcal{E}$ then for any map $\sigma\colon(X,x_0)\longrightarrow(X',x_0')$ in $\mathcal{F}in_*$ the induced map $P(\sigma)\colon P(X',x_0')\longrightarrow P(X,x_0)$ can be considered as acting on the right on $P(X',x_0')$. We will write instead of $P(\sigma)$ just σ.

Definition 3.2. *An operad in $\mathcal{E}$ is a collection $P\colon\mathcal{F}in_*^{\mathrm{op}}\longrightarrow\mathcal{E}$ with structure maps*

$$\circ_x\colon P(X,x_0)\otimes P(Y,y_0)\longrightarrow P(X\sqcup_x Y,x_0)$$

for any (X,x_0), $(Y,y_0)\in\mathcal{F}in_*$ and $x\in X$, $x\neq x_0$, which satisfy the following three conditions:

Associativity. *For any (X,x_0), (Y,y_0) and $(Z,z_0)\in\mathcal{F}in_*$, and any $x,x'\in X$, $y\in Y$ such that $x_0\neq x\neq x'\neq x_0$ and $y\neq y_0$ the following diagrams commute:*

$$\begin{tikzcd}
P(X,x_0)\otimes P(Y,y_0)\otimes P(Z,z_0) \arrow[r, "\circ_x\otimes\mathrm{id}"] \arrow[d, "\mathrm{id}\otimes\circ_y"'] & P(X\sqcup_x Y,x_0)\otimes P(Z,z_0) \arrow[d, "\circ_y"] \\
P(X,x_0)\otimes P(Y\sqcup_y Z,y_0) \arrow[r, "\circ_x"] & P(X\sqcup_x Y\sqcup_y Z,x_0)
\end{tikzcd}$$

$$\begin{tikzcd}
P(X,x_0)\otimes P(Y,y_0)\otimes P(Z,z_0) \arrow[r, "\circ_x\otimes\mathrm{id}"] \arrow[d, "\mathrm{id}\otimes s"'] & P(X\sqcup_x Y,x_0)\otimes P(Z,z_0) \arrow[dd, "\circ_{x'}"] \\
P(X,x_0)\otimes P(Z,z_0)\otimes P(Y,y_0) \arrow[d, "\circ_{x'}\otimes\mathrm{id}"'] & \\
P(X\sqcup_{x'} Z,x_0)\otimes P(Y,y_0) \arrow[r, "\circ_x"] & P(X\sqcup_x Y\sqcup_{x'} Z,x_0),
\end{tikzcd}$$

where $s\colon P(Y,y_0)\otimes P(Z,z_0)\longrightarrow P(Z,z_0)\otimes P(Y,y_0)$ is the symmetry of $\mathcal{E}$.

Equivariance. *For any $\sigma\colon(X,x_0)\longrightarrow(X',x_0')$, $\tau\colon(Y,y_0)\longrightarrow(Y',y_0')$ maps in $\mathcal{F}in_*$ and $x\in X$, $x\neq x_0$ the following diagram commutes:*

$$\begin{tikzcd}
P(X',x_0')\otimes P(Y',y_0') \arrow[r, "\circ_{\sigma(x)}"] \arrow[d, "\sigma\otimes\tau"'] & P(X'\sqcup_{\sigma(x)} Y',x_0') \arrow[d, "\sigma\circ_x\tau"] \\
P(X,x_0)\otimes P(Y,y_0) \arrow[r, "\circ_x"] & P(X\sqcup_x Y,x_0).
\end{tikzcd}$$

Unit. *For any set with two elements $(U,u_0)=(\{u,u_0\},u_0)\in\mathcal{F}in_*$ there is a map $\eta_{(U,u_0)}\colon I\longrightarrow P(U,u_0)$, for which the compositions*

$$I\otimes P(X,x_0)\xrightarrow{\eta_U\otimes\mathrm{id}}P(U,u_0)\otimes P(X,x_0)\xrightarrow{\circ_u}P(U\sqcup_u X,u_0)\xrightarrow{e_{ux_0}}P(X,x_0),$$

$$P(X,x_0)\otimes I\xrightarrow{\mathrm{id}\otimes\eta_U}P(X,x_0)\otimes P(U,u_0)\xrightarrow{\circ_x}P(X\sqcup_x U,x_0)\xrightarrow{e_{ux}}P(X,x_0)$$

are the left and right identities in the monoidal category $\mathcal{E}$ for any $(X,x_0)\in\mathcal{F}in_*$.

The following diagram commutes for any two-point sets (X,x_0) and (X',x_0'):

$$\begin{tikzcd}
I \arrow[r, "\eta_X"] \arrow[d, equal] & P(X,x_0) \arrow[d, "\alpha"] \\
I \arrow[r, "\eta_{X'}"] & P(X',x_0')
\end{tikzcd}$$

where $\alpha\colon(X',x_0')\longrightarrow(X,x_0)$ is the obvious (unique) map.

Definition 3.3. *Let P and Q be operads in $\mathcal{E}$. A morphism of operads $\mu\colon P\longrightarrow Q$ is an equivariant natural transformation from P to Q which is compatible with all the operations $\circ_x$ and unit maps η_U. Explicitly, such a μ is a family of maps*

$\mu_{(X,x_0)}\colon P(X,x_0) \longrightarrow Q(X,x_0)$, *such that the following diagrams commute for any possible choice of* x, σ *and* η*:*

$$\begin{tikzcd} P(X',x_0') \arrow[r, "\mu_{X'}"] \arrow[d, "\sigma"'] & Q(X',x_0') \arrow[d, "\sigma"] \\ P(X,x_0) \arrow[r, "\mu_X"] & Q(X,x_0) \end{tikzcd} \qquad \begin{tikzcd} P(X,x_0)\otimes P(Y,y_0) \arrow[r, "\circ_x"] \arrow[d, "\mu_X\otimes\mu_Y"'] & P(X\sqcup_x Y,x_0) \arrow[d, "\mu_{X\sqcup_x Y}"] \\ Q(X,x_0)\otimes Q(Y,y_0) \arrow[r, "\circ_x"] & Q(X\sqcup_x Y,x_0) \end{tikzcd}$$

$$\begin{tikzcd} I \arrow[r, "\eta_U"] \arrow[d, equal] & P(U,u_0) \arrow[d, "\mu_U"] \\ I \arrow[r, "\eta_U"] & Q(U,u_0) \end{tikzcd}$$

With these maps, operads in $\mathcal{E}$ form the category of $\mathcal{F}in_*$-operads $\mathcal{O}p_{\mathcal{F}in_*}$.

4. Equivalence with the classical definition

In this Section we are going to prove that the category of $\mathcal{F}in_*$-operads arising from our definition is equivalent to the classical one, given in terms of the $\circ_i$ operations (see [8], pp. 46) which in turn is equivalent to the original definition of May [9].

In the following we are going to denote the pointed set $(\{0,1,\dots,n\},0) \in \mathcal{F}in_*$ by $\langle n\rangle$. Instead of $P(\langle n\rangle)$ let us write $P(n)$. If P is an operad in $\mathcal{E}$ then any composition map $\circ_x\colon P(X,x_0)\otimes P(Y,y_0) \longrightarrow P(X\sqcup_x Y,x_0)$ gives rise to a new one $\circ_i\colon P(m)\otimes P(n) \longrightarrow P(\langle m\rangle \sqcup_i \langle n\rangle)$ via the actions of some pointed bijections $\sigma\colon \langle m\rangle \longrightarrow (X,x_0)$ with $\sigma(i)=x$ and $\tau\colon \langle n\rangle \longrightarrow (Y,y_0)$, because of the equivariance condition:

$$\circ_x = (\sigma \circ_i \tau)^{-1}(\circ_i)(\sigma\otimes\tau)).$$

This property suggests to study more the structures induced by the operad axioms on the objects $P(m)$. Define the *renumbering map* $\varphi_i\colon \langle m+n-1\rangle \longrightarrow \langle m\rangle \sqcup_i \langle n\rangle$,

$$\varphi_i(k) := \begin{cases} k\in\langle m\rangle & \text{if } k<i,\\ (k-n+1)\in\langle m\rangle & \text{if } k>i+n-1,\\ (k-i+1)\in\langle n\rangle & \text{if } i\le k\le i+n-1.\end{cases} \tag{4.1}$$

We can infer that the composition of φ_i with $\circ_i$ defines a new operation, denoted by $\bullet_i$ which is written only in terms of the sets $\langle m\rangle$:

$$\bullet_i := \varphi_i\circ_i\colon P(m)\otimes P(n) \longrightarrow P(m+n-1).$$

In the following we look at the axioms – induced by the associativity, equivariance and unit axioms for P – that this new operations satisfy.

Associativity. Let $\circ_i\colon P(m)\otimes P(n) \longrightarrow P(\langle m\rangle \sqcup_u \langle n\rangle)$ and $\circ_j\colon P(n)\otimes P(p) \longrightarrow P(\langle n\rangle \sqcup_j \langle p\rangle)$ be two operations. To avoid confusion we write $j_{\langle n\rangle}$ instead of j, when

it is necessary to indicate the set from which j is taken.
The squares of the following diagram commute:

$$\begin{tikzcd}
P(m)\otimes P(n)\otimes P(p) \arrow[r, "\circ_i\otimes\mathrm{id}"] \arrow[d, "\mathrm{id}\otimes\circ_j"'] & P(\langle m\rangle\sqcup_i\langle n\rangle)\otimes P(p) \arrow[r, "\varphi_i\otimes\mathrm{id}"] \arrow[d, "\circ_{j_{\langle n\rangle}}"'] & P(m+n-1)\otimes P(p) \arrow[d, "k"'] \\
P(m)\otimes P(\langle n\rangle\sqcup_j\langle p\rangle) \arrow[r, "\circ_i"] \arrow[d, "\mathrm{id}\otimes\varphi_j"'] & P(\langle m\rangle\sqcup_i\langle n\rangle\sqcup_{j_{\langle n\rangle}}\langle p\rangle) \arrow[r, "\varphi_i\circ_k\mathrm{id}"] \arrow[d, "\mathrm{id}\circ_i\varphi_j"'] & P(\langle m+n-1\rangle\sqcup_k\langle p\rangle) \arrow[d, "\varphi_k"'] \\
P(m)\otimes P(n+p-1) \arrow[r, "\circ_i"] & P(\langle m\rangle\sqcup_i\langle n+p-1\rangle) \arrow[r, "\varphi_i"] & P(m+n+p-2),
\end{tikzcd}$$

where $k=\varphi_i^{-1}(j_{\langle n\rangle})$. Indeed, the commutativity of the first square is just an associativity condition of the operad P, the second and third squares are equivariance conditions of P. The commutativity of the last square follows from a straightforward computation. With the use of the $\bullet_i$ operations the bordering square of the diagram above can be written as

$$\begin{tikzcd}
P(m)\otimes P(n)\otimes P(p) \arrow[r, "\bullet_i\otimes\mathrm{id}"] \arrow[d, "\mathrm{id}\otimes\bullet_j"'] & P(m+n-1)\otimes P(p) \arrow[d, "\bullet_{j+i-1}"] \\
P(m)\otimes P(n+p-1) \arrow[r, "\bullet_i"] & P(m+n+p-2).
\end{tikzcd}$$

We proceed similarly for operations $\circ_i\colon P(m)\otimes P(n)\longrightarrow P(\langle m\rangle\sqcup_i\langle n\rangle)$ and $\circ_j\colon P(m)\otimes P(p)\longrightarrow P(\langle m\rangle\sqcup_j\langle p\rangle)$, where $i\neq j$. In this case we use the second axiom for associativity of the operad P. We obtain the diagram

$$\begin{tikzcd}
P(m)\otimes P(n)\otimes P(n) \arrow[r, "\circ_i\otimes\mathrm{id}"] \arrow[d, "\mathrm{id}\otimes s"'] & P(\langle m\rangle\sqcup_i\langle n\rangle)\otimes P(p) \arrow[r, "\varphi_i\otimes\mathrm{id}"] \arrow[dd, "\circ_{j_{\langle m\rangle}}"'] & P(m+n-1)\otimes P(p) \arrow[dd, "\circ_k"'] \\
P(m)\otimes P(p)\otimes P(n) \arrow[d, "\circ_j\otimes\mathrm{id}"'] & & \\
P(\langle m\rangle\sqcup_j\langle p\rangle)\otimes P(n) \arrow[r, "\circ_{i_{\langle m\rangle}}"] \arrow[d, "\varphi_j\otimes\mathrm{id}"'] & P(\langle m\rangle\sqcup_{i_{\langle m\rangle}}\langle n\rangle\sqcup_{j_{\langle m\rangle}}\langle p\rangle) \arrow[r, "\varphi_i\circ_k\mathrm{id}"] \arrow[d, "\varphi_j\circ_l\mathrm{id}"'] & P(\langle m+n-1\rangle\sqcup_k\langle p\rangle) \arrow[d, "\varphi_k"'] \\
P(m+p-1)\otimes P(n) \arrow[r, "\circ_l"] & P(\langle m+p-1\rangle\sqcup_l\langle n\rangle) \arrow[r, "\varphi_l"] & P(m+n+p-2),
\end{tikzcd}$$

where $l=\varphi_j^{-1}(i_{\langle m\rangle})$ and $k=\varphi_i^{-1}(j_{\langle m\rangle})$. Again, only the commutativity of the last square must be checked, because the other squares are commutative from the associativity and equivariance properties of P. If we write the bordering square with the operations $\bullet_i$, then we have

$$\begin{tikzcd}
P(m) \otimes P(n) \otimes P(p) \arrow[r, "\bullet_i \otimes \mathrm{id}"] \arrow[d, "\mathrm{id} \otimes s"'] & P(m+n-1) \otimes P(p) \arrow[dd, "\bullet_{j+n-1}"] \\
P(m) \otimes P(p) \otimes P(n) \arrow[d, "\bullet_j \otimes \mathrm{id}"'] & \\
P(m+p-1) \otimes P(n) \arrow[r, "\bullet_i"] & P(m+n+p-2)
\end{tikzcd}$$

if $i < j$: in this case $l = i$ and $k = j + n - 1$;

$$\begin{tikzcd}
P(m) \otimes P(n) \otimes P(p) \arrow[r, "\bullet_i \otimes \mathrm{id}"] \arrow[d, "\mathrm{id} \otimes s"'] & P(m+n-1) \otimes P(p) \arrow[dd, "\bullet_j"] \\
P(m) \otimes P(p) \otimes (n) \arrow[d, "\bullet_j \otimes \mathrm{id}"'] & \\
P(m+p-1) \otimes P(n) \arrow[r, "\bullet_{i+p-1}"] & P(m+n+p-2)
\end{tikzcd}$$

if $i > j$: in this case $l = i + p - 1$ and $k = j$.
The obtained three commutative diagrams are the associativity axioms for the $\bullet_i$ operations. After a suitable renumbering, they can be expressed in the following equations:

$$\bullet_j(\bullet_i \otimes \mathrm{id}) = \begin{cases} \bullet_i(\mathrm{id} \otimes \bullet_{j-i+1}), & \text{if } 1 \leq i \leq j \leq n \leq m+n-1; \\ \bullet_i(\bullet_{j+n-1} \otimes \mathrm{id})(\mathrm{id} \otimes s), & \text{if } n \leq i+n-1 < j \leq m+n-1; \\ \bullet_{i+p-1}(\bullet_j \otimes \mathrm{id})(\mathrm{id} \otimes s), & \text{if } 1 \leq j < i \leq m. \end{cases} \tag{4.2}$$

Equivariance. Let $\sigma\colon \langle m\rangle \longrightarrow \langle m\rangle$, $\tau\colon \langle n\rangle \longrightarrow \langle n\rangle$ be two maps in $\mathcal{F}in_*$. The equivariance property of P induces the commutative diagram

$$\begin{tikzcd}
P(m) \otimes P(n) \arrow[r, "\circ_i"] \arrow[d, "\sigma \otimes \tau"'] & P(\langle m\rangle \sqcup_i \langle n\rangle) \arrow[r, "\varphi_i"] \arrow[d, "\sigma \circ_k \tau"'] & P(m+n-1) \arrow[d, dotted, "\sigma \bullet_k \tau"] \\
P(m) \otimes P(n) \arrow[r, "\circ_k"] & P(\langle m\rangle \sqcup_k \langle n\rangle) \arrow[r, "\varphi_k"] & P(m+n-1),
\end{tikzcd}$$

where $\sigma(k) = i$ and $\sigma \bullet_k \tau\colon \langle m+n-1\rangle \longrightarrow \langle m+n-1\rangle$,

$$\sigma \bullet_k \tau = (\varphi_i)^{-1}(\sigma \circ_k \tau)(\varphi_k).$$

A straightforward computation shows that

$$\sigma \bullet_k \tau = \sigma_{(1,\ldots,1,n,1,\ldots,1)} \circ (\mathrm{id} \times \cdots \times \mathrm{id} \times \tau \times \mathrm{id} \times \cdots \times \mathrm{id}), \tag{4.3}$$

where on the right hand side of the equation, n and τ occur at the k^{th} position. We infer that the equivariance property induces the commutativity of the diagram

$$\begin{tikzcd}
P(m)\otimes P(n) \arrow[r, "\bullet_{\sigma(k)}"] \arrow[d, "\sigma\otimes\tau"'] & P(m+n-1) \arrow[d, "\sigma\bullet_k\tau"] \\
P(m)\otimes P(n) \arrow[r, "\bullet_k"] & P(m+n-1)
\end{tikzcd}$$

or

$$(\sigma \bullet_k \tau)\bullet_{\sigma(k)} = (\bullet_k)(\sigma\otimes\tau). \tag{4.4}$$

Unit. Let us take in the unit condition for an operad P the two-element pointed set $(X, x_0) = \langle 1\rangle$. It follows that for any $n \in \mathbb{N}^*$ we have

$$e_{xy_0} = \varphi_1 = \mathrm{id}\colon \langle n\rangle \longrightarrow \langle 1\rangle \sqcup_1 \langle n\rangle,$$

$$e_{xy} = \varphi_i\colon \langle n\rangle \longrightarrow \langle n\rangle \sqcup_i \langle 1\rangle,$$

hence the unit conditions for the $\bullet_i$ operations say that the following compositions must be the corresponding left and right identities in $\mathcal{E}$:

$$I\otimes P(n) \xrightarrow{\eta\otimes\mathrm{id}} P(1)\otimes P(n) \xrightarrow{\bullet_1} P(n); \tag{4.5}$$

$$P(n)\otimes I \xrightarrow{\mathrm{id}\otimes\eta} P(n)\otimes P(1) \xrightarrow{\bullet_i} P(n). \tag{4.6}$$

These properties imply the following definition:

Definition 4.1. *Let Σ denote the symmetric groupoid (i.e. the category whose objects are the finite sets $[n] = \{1, 2, \ldots, n\}$ for every $n \in \mathbb{N}^*$ and the maps are permutations $\sigma\colon [n] \longrightarrow [n]$). A Σ-operad in a symmetric monoidal category $\mathcal{E}$ is a contravariant functor $P\colon \Sigma^{\mathrm{op}} \longrightarrow \mathcal{E}$ with operations*

$$\bullet_i\colon P(m)\otimes P(n) \longrightarrow P(m+n-1)$$

for every $1 \le i \le m$ (here we denote $P([m])$ by $P(m)$), which satisfy the conditions given in equations (4.2), (4.4), (4.5) and (4.6).

This definition agrees with Markl, Shinder and Stasheff's definition of an operad in [8], and it is equivalent to the definition given by May [9]. *Morphisms of* Σ*-operads* are defined as that of operads: they are collections of maps $\mu_m\colon P(m) \longrightarrow Q(m)$, for which the following diagrams commute:

$$\begin{tikzcd}
P(m) \arrow[r, "\mu_m"] \arrow[d, "\sigma"'] & Q(m) \arrow[d, "\sigma"] \\
P(m) \arrow[r, "\mu_m"] & Q(m)
\end{tikzcd}
\qquad
\begin{tikzcd}
P(m)\otimes P(n) \arrow[r, "\bullet_i"] \arrow[d, "\mu_m\otimes\mu_n"'] & P(m+n-1) \arrow[d, "\mu_{m+n-1}"] \\
Q(m)\otimes Q(n) \arrow[r, "\bullet_i"] & Q(m+n-1)
\end{tikzcd}
\qquad
\begin{tikzcd}
I \arrow[r, "\eta"] \arrow[d, equal] & P(1) \arrow[d, "\mu_1"] \\
I \arrow[r, "\eta"] & Q(1)
\end{tikzcd}$$

It follows that we have a category of Σ-operads in $\mathcal{E}$, which we denote by $\mathcal{O}p_\Sigma$.

We turn to prove that $\mathcal{O}p_\Sigma$ and $\mathcal{O}p_{\mathcal{F}in_*}$ are equivalent categories. For this, first observe that the usual restriction and extension functors $R\colon \mathcal{F}in_* \longrightarrow \Sigma$ and $E\colon \Sigma \longrightarrow \mathcal{F}in_*$ are equivalences and even $RE = \mathrm{id}$ is satisfied. Denote the induced functors on the categories of $\mathcal{E}$-collections by $R^\#\colon \mathcal{E}^{\Sigma^{\mathrm{op}}} \longrightarrow \mathcal{E}^{\mathcal{F}in_*^{\mathrm{op}}}$ and

$E^{\#}\colon \mathcal{E}^{\mathcal{F}in_*^{\mathrm{op}}} \longrightarrow \mathcal{E}^{\Sigma^{\mathrm{op}}}$. By a slight abuse of notation, we will not distinguish between the finite set $[n]$ and the finite pointed set $(\langle n\rangle, 0)$ in what follows.

Lemma 4.2. *$P\colon \mathcal{F}in_*^{op} \longrightarrow \mathcal{E}$ defines a $\mathcal{F}in_*$-operad if and only if $E^{\#}(P)\colon \Sigma^{op} \longrightarrow \mathcal{E}$ defines a Σ-operad.*

Proof. If P is a $\mathcal{F}in_*$-operad, then (by the abuse of notation mentioned above) $E^{\#}(P)(n) = P(n)$ and $E^{\#}(P)(\sigma) = P(\sigma)$ for any $n \in \mathbb{N}^*$ and $\sigma \in \Sigma_n$. The construction of the $\bullet_i$ operations as above gives a Σ-operad structure to $E^{\#}(P)$.
Conversely, suppose that $E^{\#}(P)$ is a Σ-operad. Then we have operations

$$\bullet_i\colon P(m) \otimes P(n) \longrightarrow P(m+n-1)$$

which satisfy the respective associativity, equivariance and unit conditions.
First, define the operations

$$\circ_i\colon P(m) \otimes P(n) \longrightarrow P(\langle m\rangle \sqcup_i \langle n\rangle)$$

with the composition: $\circ_i := \varphi_i^{-1}\bullet_i$ where the maps φ_i are defined by (4.1). It follows from the Σ-equivariance condition that the diagram

$$\begin{tikzcd} P(m)\otimes P(n) \arrow[r, "\circ_{\sigma(i)}"] \arrow[d, "\sigma\otimes\tau"'] & P(\langle m\rangle \sqcup_i \langle n\rangle) \arrow[d, "\sigma\circ_i\tau"] \\ P(m)\otimes P(n) \arrow[r, "\circ_i"] & P(\langle m\rangle \sqcup_i \langle n\rangle) \end{tikzcd} \tag{4.7}$$

also commutes.
Second, define the operations $\circ_x\colon P(X) \otimes P(Y) \longrightarrow P(X \sqcup_x Y)$ by requiring the diagram

$$\begin{tikzcd} P(X)\otimes P(Y) \arrow[r, "\circ_x"] \arrow[d, "\sigma\otimes\tau"'] & P(X \sqcup_x Y) \arrow[d, "\sigma\circ_i\tau"] \\ P(m)\otimes P(n) \arrow[r, "\circ_i"] & P(\langle m\rangle \sqcup_i \langle n\rangle) \end{tikzcd}$$

to be commutative. Here $\sigma\colon \langle m\rangle \longrightarrow X$, $\tau\colon \langle n\rangle \longrightarrow Y$ are chosen maps in $\mathcal{F}in_*$ with the property that $\sigma(i) = x$. The operations $\circ_x$ do not depend on the choice of σ and τ, because of the commutative square (4.7). Indeed, if σ' and τ' define an operation $(\circ_x)' \neq \circ_x$ by

$$\begin{tikzcd} P(X)\otimes P(Y) \arrow[r, "(\circ_x)'"] \arrow[d, "\sigma'\otimes\tau'"'] & P(X \sqcup_x Y) \arrow[d, "\sigma'\circ_i\tau'"] \\ P(m)\otimes P(n) \arrow[r, "\circ_i"] & P(\langle m\rangle \sqcup_i \langle n\rangle) \end{tikzcd}$$

then patching together the last two diagrams follows that the diagram (4.7) is not commutative with the maps $\sigma'\sigma^{-1}$ and $\tau'\tau^{-1}$, which is contradiction.
Thus the operations $\circ_x$ are well defined. The axioms for the $\mathcal{F}in_*$-operad are easily verified: we just have to do the diagram-chasing with $\bullet_i$ and $\circ_x$ backwards. $\square$

Lemma 4.3. *$\mu\colon P \longrightarrow Q$ is a map of $\mathcal{F}in_*$-operads if and only if $E^{\#}(\mu)$ is a map of Σ-operads.*

Proof. A straightforward check, using the maps φ_i and that $E^{\#}(\mu)_n = \mu_{\langle n\rangle}$. $\square$

Theorem 4.4. *The categories $\mathcal{O}p_{\mathcal{F}in_*}$ and $\mathcal{O}p_{\Sigma}$ are equivalent.*

Proof. For any Σ-operad Q we have $E^{\#}R^{\#}(Q) = Q$. We infer by Lemma 4.2 that $R^{\#}(Q)$ is an operad. This and Lemma 4.2 again show that $E^{\#}$ is an essentially surjective functor when viewed between the operad-categories.
On the other hand, because $E^{\#}$ is fully faithful, Lemma 4.3 implies that $E^{\#}$ is also fully faithful between the operad-categories. Hence $\mathcal{O}p_{\mathcal{F}in_*}$ and $\mathcal{O}p_{\Sigma}$ are equivalent. $\square$

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Some remarks on restriction maps between cohomology of fusion systems

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Abstract. We define a restriction map between two cohomology algebras of some saturated fusion systems which are chosen in a particular situation. We find conditions for this map to induce an injective map between the varieties which can be associated to these finitely generated graded commutative cohomology algebras. Some minimal examples for which we can apply our results are also given.

Mathematics Subject Classification (2010): 20C20, 16EXX.

Keywords: fusion system, finite group, cohomology, variety.

1. Preliminaries

Saturated fusion systems on finite p-groups are intensively studied in the last years by mathematicians from different areas such as: modular representation theory, algebraic topology and finite groups. A saturated fusion system $\mathcal{F}$ on a finite p-group P is a category whose objects are the subgroups of P and whose morphisms satisfy certain axioms mimicking the behavior of a finite group G having P as a Sylow subgroup. The axioms of saturated fusion systems were invented by Puig in early 1990's. See [1] for a detailed exposition of results and definitions involving fusion systems.

The cohomology algebra of a p-local finite group with coefficients in $\mathbb{F}_p$ is introduced in [3, §5] and is equal with cohomology algebra of a saturated fusion system. Let k be an algebraically closed field of characteristic p. We denote by $\mathrm{H}^*(G,k)$ the cohomology algebra of the group G with trivial coefficients. As in [6] we will use the language of homotopy classes of chain maps (see [6, 2.8]). We denote by $\mathrm{H}^*(\mathcal{F})$ the algebra of stable elements of $\mathcal{F}$, i.e. the cohomology algebra of the saturated fusion system $\mathcal{F}$, which is the subalgebra of $\mathrm{H}^*(P,k)$ consisting of elements $[\zeta] \in \mathrm{H}^*(P,k)$ such that

$$\mathrm{res}^P_Q([\zeta]) = \mathrm{res}_\varphi([\zeta]),$$

for any $\varphi \in \mathrm{Hom}_{\mathcal{F}}(Q, P)$ and any subgroup Q of P. This is the main object of study in this paper. Moreover Broto, Levi and Oliver showed that any saturated fusion system $\mathcal{F}$ has a non-unique $P-P$-biset X with certain properties formulated by Linckelmann and Webb (see [3, Proposition 5.5]). Such a $P-P$-biset X is called a *characteristic biset*. Using this biset, S. Park noticed in [8] a result which says that a saturated fusion system can be realized by a finite group. This finite group is $G = \mathrm{Aut}(X_P)$, that is the group of bijections of the characteristic biset X, preserving the right P-action. So, by [8, Theorem 3], we identify $\mathcal{F}$ with $\mathcal{F}_P(G)$ which is the fusion system on P such that for every $Q, R \leq P$ we have

$$\mathrm{Hom}_{\mathcal{F}_P(G)}(Q, R) = \{\varphi : Q \to R \mid \exists x \in G \ s.t. \ \varphi(u) = xux^{-1}, \forall u \in Q\}.$$

Using this identification we will define a restriction map from the cohomology algebra of the group G with coefficients in the field k to the cohomology algebra of the fusion system, $\mathrm{H}^*(\mathcal{F})$. We denote this map by $\rho_{\mathcal{F},G}$, and we have the following proposition.

Proposition 1.1. *Let $\mathcal{F}$ be a saturated fusion system on P and let X be a characteristic $P-P$-biset. Let $G = \mathrm{Aut}(X_P)$ and then we identify $\mathcal{F}$ with $\mathcal{F}_P(G)$. We have* $\mathrm{res}^G_P(\mathrm{H}^*(G, k)) \subseteq \mathrm{H}^*(\mathcal{F})$*, hence there is a homomorphism of algebras*

$$\rho_{\mathcal{F},G} : \mathrm{H}^*(G, k) \to \mathrm{H}^*(\mathcal{F}),$$

given by $\rho_{\mathcal{F},G}([\zeta]) = \mathrm{res}^G_P([\zeta])$, for any $[\zeta] \in \mathrm{H}^*(G, k)$.

Next we will define the main restriction map of this article, between the cohomology algebras of two saturated fusion systems. This is done by considering the following situation:

Situation $(*)$. *Let Q be a finite p-subgroup of a finite p-group P. Let G be a finite group which realizes a saturated fusion system $\mathcal{G}$ on P (i.e. $\mathcal{G} = \mathcal{F}_P(G)$) and $\mathcal{F}$ a fusion subsystem (i.e subcategory and fusion system) of $\mathcal{G}$ on Q. We assume that there is H which realizes $\mathcal{F}$ and $Q \leq H \leq G$.*

The next example assure us that there are cases of saturated fusion systems in Situation $(*)$.

Example 1.2. Let H be a finite subgroup of a finite group G with P a Sylow p-subgroup of G such that $P \cap H \neq \{1\}$. Then $\mathcal{F} = \mathcal{F}_{P\cap H}(H)$ and $\mathcal{G} = \mathcal{F}_P(G)$ are in Situation $(*)$.

It is easy to verify that in Situation $(*)$ the restriction map res^P_Q induces a well-defined homomorphism of algebras

$$\mathrm{res}_{\mathcal{G},\mathcal{F}} : \mathrm{H}^*(\mathcal{G}) \to \mathrm{H}^*(\mathcal{F}),$$

given by $\mathrm{res}_{\mathcal{G},\mathcal{F}}([\zeta]) = \mathrm{res}^P_Q([\zeta])$ for any $[\zeta] \in \mathrm{H}^*(\mathcal{G})$.

Now we set some notations, which are known to appear in the Quillen stratification of V_G ([4, Definition 8.4.4, Theorem 8.5.2]) of group cohomology ring. Let E be a p-subgroup of G. The restriction map $\mathrm{res}^G_E : \mathrm{H}^*(G, k) \to \mathrm{H}^*(E, k)$ induces a map on varieties, which we denote

$$r^*_{G,E} : V_E \to V_G.$$

As usual we define the subvariety of V_E

$$V_E^+ = V_E \setminus \bigcup_{F<E} (\mathrm{res}_F^E)^*(V_F),$$

and denote the subvarieties of V_E

$$V_{G,E} = r_{G,E}^*(V_E), \; V_{G,E}^+ = r_{G,E}^*(V_E^+).$$

Finally we set $W_G(E) = N_G(E)/C_G(E)$, the Weyl group. Similarly to the group cohomology ring case, since $\mathrm{H}^*(\mathcal{F})$ is a graded commutative finitely generated k-algebra we can associate the spectrum of maximal ideals, i.e. the variety denoted $V_{\mathcal{F}}$. Varieties for cohomology algebras of particular cases of saturated fusion systems were studied in [7], for fusion systems associated to block algebras of finite groups. See also [2, Chapter 5] for more results regarding varieties.

Theorem 1.3. *We assume that we are in Situation* $(*)$*.*

(i) *The following diagram is commutative*

$$\begin{tikzcd} \mathrm{H}^*(G,k) \arrow[r, "\rho_{\mathcal{G},G}"] \arrow[d, "\mathrm{res}_H^G"] & \mathrm{H}^*(\mathcal{G}) \arrow[d, "\mathrm{res}_{\mathcal{G},\mathcal{F}}"] \\ \mathrm{H}^*(H,k) \arrow[r, "\rho_{\mathcal{F},H}"] & \mathrm{H}^*(\mathcal{F}) \end{tikzcd}$$

(ii) *If* $\mathrm{Ker}(\mathrm{res}_{\mathcal{G},\mathcal{F}})$ *has a nilpotent ideal then* $\mathrm{res}_{\mathcal{G},\mathcal{F}}$ *induces a finite surjective map*

$$\mathrm{res}_{\mathcal{G},\mathcal{F}}^* : V_{\mathcal{F}} \to V_{\mathcal{G}}.$$

In Situation $(*)$ if $Q = P$ then the restriction $\mathrm{res}_{\mathcal{G},\mathcal{F}}$ becomes the inclusion map, hence $\mathrm{Ker}(\mathrm{res}_{\mathcal{F},\mathcal{G}})$ is a nilpotent ideal. Therefore exist cases for which Theorem 1.3, (ii) is true. The next definitions allow us to find conditions for which $\mathrm{res}_{\mathcal{G},\mathcal{F}}^*$ is injective.

Definition 1.4. *Let* $\mathcal{G}, \mathcal{F}$ *be two saturated fusion systems in Situation* $(*)$*. We say that the pair* $(\mathcal{F}, H)$ *is* **weakly elementary embedded** *in* $(\mathcal{G}, G)$ *if:*

(1) *Whenever* E *is an elementary abelian* p*-subgroup of* H *then* $W_G(E) \cong W_H(E)$*;*
(2) *If two elementary abelian* p*-subgroups of* H *are* G*-conjugate then they are also* H*-conjugate.*

The main result of this article is the following theorem.

Theorem 1.5. *In Situation* $(*)$ *we assume that* $\rho_{\mathcal{F},H}^*$ *is injective. If* $(\mathcal{F}, H)$ *is weakly elementary embedded in* $(\mathcal{G}, G)$ *then* $\mathrm{res}_{\mathcal{G},\mathcal{F}}^*$ *is injective.*

Using Theorem 1.3, (ii) and Theorem 1.5 it is easy to check the following corollary. The proof is left for the reader.

Corollary 1.6. *We assume that we are under the hypothesis of Theorem* 1.5 *such that* $Q = P$*. Then* $\mathrm{res}_{\mathcal{G},\mathcal{F}}^*$ *is a bijective map.*

We notice from Example 1.2 that there are some minimal examples for which the above theorem and corollary can be applied.

2. Proofs of the results

Proof of Proposition 1.1. Let Q be a subgroup of P, $\varphi \in \mathrm{Hom}_{\mathcal{F}}(Q,P)$ and let $[\zeta] \in \mathrm{H}^*(G,k)$. We have to prove that

$$\mathrm{res}^P_Q(\mathrm{res}^G_P([\zeta])) = \mathrm{res}_{\varphi}(\mathrm{res}^G_P([\zeta])).$$

We denote by $\overline{\varphi} = i_1 \circ \varphi$, where $i_1 : P \to G$ is the inclusion. Then we will prove that

$$\mathrm{res}^G_Q([\zeta]) = \mathrm{res}_{\overline{\varphi}}([\zeta]).$$

We consider S a Sylow p-subgroup of G such that $P \leq S$. Then $\mathcal{F}_P(G)$ is a full subcategory of $\mathcal{F}_S(G)$, hence $\varphi \in \mathrm{Hom}_{\mathcal{F}_S(G)}(Q,P)$. If we take $\varphi' = i_2 \circ \varphi$, where $i_2 : P \to S$ is the inclusion, then $\varphi' \in \mathrm{Hom}_{\mathcal{F}_S(G)}(Q,S)$. By Cartan-Eilenberg stable elements theorem ([5, XII, Theorem 10.1]) we have that

$$\mathrm{res}^S_Q(\mathrm{res}^G_S([\zeta])) = \mathrm{res}_{\varphi'}(\mathrm{res}^G_S([\zeta])).$$

Since $\overline{\varphi} = i_3 \circ \varphi'$, where $i_3 : S \to G$ is the inclusion, we get the above, desired condition.

Proof of Theorem 1.3. (i) is easy to check since we have compositions of restrictions. For (ii) we have that $\mathrm{H}^*(\mathcal{F})$ is a $\rho_{\mathcal{F},H}(\mathrm{H}^*(H,k))$-submodule of $\mathrm{H}^*(Q,k)$. Since $\mathrm{H}^*(Q,k)$ is noetherian as $\mathrm{res}^H_Q(\mathrm{H}^*(H,k))$-module it follows that $\mathrm{H}^*(\mathcal{F})$ is a finitely generated $\rho_{\mathcal{F},H}(\mathrm{H}^*(H,k))$-module. Now $\mathrm{H}^*(H,k)$ is a finitely generated $\mathrm{res}^G_H(\mathrm{H}^*(G,k))$-module. Then we obtain that $\mathrm{H}^*(\mathcal{F})$ is finitely generated as $(\rho_{\mathcal{F},H} \circ \mathrm{res}^G_H)(\mathrm{H}^*(G,k))$-module, hence by (i) we get that $\mathrm{H}^*(\mathcal{F})$ is finitely generated as $(\mathrm{res}_{\mathcal{G},\mathcal{F}} \circ \rho_{\mathcal{G},G})(\mathrm{H}^*(G,k))$-module. Since $(\mathrm{res}_{\mathcal{G},\mathcal{F}} \circ \rho_{\mathcal{G},G})(\mathrm{H}^*(G,k))$ is a subalgebra of $\mathrm{res}_{\mathcal{G},\mathcal{F}}(\mathrm{H}^*(\mathcal{G}))$ we obtain that $\mathrm{H}^*(\mathcal{F})$ is finitely generated as $\mathrm{res}_{\mathcal{G},\mathcal{F}}(\mathrm{H}^*(\mathcal{G}))$-module, thus $\mathrm{res}_{\mathcal{G},\mathcal{F}}$ is a finite map. Now $\mathrm{res}_{\mathcal{G},\mathcal{F}}$ is also a dominant map (see [2, Section 5.4]), because $\mathrm{Ker}(\mathrm{res}_{\mathcal{G},\mathcal{F}})$ is a nilpotent ideal. We conclude that it is surjective, see [2, Theorem 5.4.7].

Proof of Theorem 1.5. Let $m_1, m_2 \in V_{\mathcal{F}}$ such that $\mathrm{res}^*_{\mathcal{G},\mathcal{F}}(m_1) = \mathrm{res}^*_{\mathcal{G},\mathcal{F}}(m_2)$. By Theorem 1.3, (i) we have that

$$\rho^*_{\mathcal{G},G} \circ \mathrm{res}^*_{\mathcal{G},\mathcal{F}} = (\mathrm{res}^G_H)^* \circ \rho^*_{\mathcal{F},H};$$

From [4, Theorem 8.5.2] (Quillen stratification) applied to V_H there is $E_1 \leq H$ an elementary abelian p-subgroup and $\gamma_1 \in V^+_{E_1}$ such that $\rho^*_{\mathcal{F},H}(m_1) = r^*_{H,E_1}(\gamma_1)$. Similarly there is $E_2 \leq H$ an elementary abelian p-subgroup and $\gamma_2 \in V^+_{E_2}$ such that $\rho^*_{\mathcal{F},H}(m_2) = r^*_{H,E_2}(\gamma_2)$, hence

$$((\mathrm{res}^G_H)^* \circ \rho^*_{\mathcal{F},H})(m_1) = ((\mathrm{res}^G_H)^* \circ r^*_{H,E_1})(\gamma_1),$$
$$((\mathrm{res}^G_H)^* \circ \rho^*_{\mathcal{F},H})(m_2) = ((\mathrm{res}^G_H)^* \circ r^*_{H,E_2})(\gamma_2).$$

From the above relations it follows that

$$((\mathrm{res}^G_H)^* \circ r^*_{H,E_1})(\gamma_1) = ((\mathrm{res}^G_H)^* \circ r^*_{H,E_2})(\gamma_2),$$

that is

$$r^*_{G,E_1}(\gamma_1) = r^*_{G,E_2}(\gamma_2) \in V^+_{G,E_1} \cap V^+_{G,E_2},$$

thus E_1, E_2 are G-conjugate, and by Definition 1.4, (2) we get that they are H-conjugate. From this we can choose now $E_1 = E_2 = E$ and $r^*_{G,E}(\gamma_1) = r^*_{G,E}(\gamma_2) \in$

$V_{G,E}^{+}$. By the Quillen stratification for $\mathrm{H}^*(G,k)$ we have $V_{G,E}^{+} \cong V_E^{+}/W_G(E)$ and this inseparable isogeny is given by $r_{G,E}$. We obtain that γ_1, γ_2 are in the same orbit of the action of $W_G(E)$ on V_E^{+}. By Definition 1.4, (1) it follows that γ_1, γ_2 are in the same orbit of the action of $W_H(E)$ on V_E^{+}, then $r_{H,E}^*(\gamma_1) = r_{H,E}^*(\gamma_2)$. We conclude that $\rho_{\mathcal{F},H}^*(m_1) = \rho_{\mathcal{F},H}^*(m_2)$, hence $m_1 = m_2$ since $\rho_{\mathcal{F},H}^*$ is injective.

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Integral operator defined by q-analogue of Liu-Srivastava operator

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Abstract. In this paper, we shall give an application of q-analogues theory in geometric function theory. We introduce an integral operator for meromorphic functions involving the q-analogue of differential operator. We also investigate several properties for this operator.

Mathematics Subject Classification (2010): 30C45.

Keywords: q-analogue, meromorphic function, Liu-Srivastava operator, integral operator.

1. Introduction

The theory of q-analogues or q-extensions of classical formulas and functions based on the observation that

$$\lim_{q\to 1}\frac{1-q^{\alpha}}{1-q}=\alpha, |q|<1,$$

therefore the number $(1-q^{\alpha})/(1-q)$ is sometimes called the basic number $[\alpha]_q$. In this work we derive q-analogue of Liu-Srivastava operator and employ this new differential operator to define an integral operator for meromorphic functions.

Let Σ denote the class of functions of the form

$$f(z)=\frac{1}{z}+\sum_{k=1}^{\infty}a_k z^k, \tag{1.1}$$

which are analytic in the punctured open unit disk

$$\mathbb{U}^*=\{z: z\in\mathbb{C}, 0<|z|<1\}=\mathbb{U}\backslash\{0\}.$$

For complex parameters α_i,β_j $(i=1,\dots,r,\ j=1,\dots,s,\ \alpha_i\in\mathbb{C},\ \beta_j\in\mathbb{C}\backslash\{0,-1,-2,\dots\})$ the basic hypergeometric function (or q- hypergeometric function) is the q-analogue of the familiar hypergeometric function and it is defined as follows:

$$\psi(\alpha_1,\ldots,\alpha_r;\beta_1,\ldots,\beta_s,q,z)=\sum_{k=0}^{\infty}\frac{(\alpha_1,q)_k\ldots(\alpha_r,q)_k}{(q,q)_k(\beta_1,q)_k\ldots(\beta_s,q)_k}\left[(-1)^k q^{\binom{k}{2}}\right]^{1+s-r} z^k, \tag{1.2}$$

with $\binom{k}{2}=k(k-1)/2$, where $q\neq 0$ when $r>s+1$, $(r,s\in\mathbb{N}_0=\mathbb{N}\cup\{0\})$, and $(\alpha,q)_k$ is the q-analogue of the Pochhammer symbol $(\alpha)_k$ defined by

$$(\alpha,q)_k=\begin{cases}1, & k=0;\\ (1-\alpha)(1-\alpha q)(1-\alpha q^2)\ldots(1-\alpha q^{k-1}), & k\in\mathbb{N}.\end{cases}$$

It is clear that

$$\lim_{q\to 1}\frac{(q^\alpha;q)_k}{(1-q)^k}=(\alpha)_k.$$

The radius of convergence ρ of the basic hypergeometric series (1.2) for $|q|<1$ is given by

$$\rho=\begin{cases}\infty, & \text{if} \quad r<s+1;\\ 1, & \text{if} \quad r=s+1;\\ 0, & \text{if} \quad r>s+1.\end{cases}$$

The basic hypergeometric series defined by (1.2) was first introduced by Heine in 1846. Therefore it is sometimes called Heine's series. For more details concerning the q-theory the reader may refer to (see [1],[2]).

Now for $z\in\mathbb{U}$, $|q|<1$, and $r=s+1$, the basic hypergeometric function defined in (1.2) takes the form

$${}_r\Phi_s(\alpha_1,\ldots,\alpha_r;\beta_1,\ldots,\beta_s,q,z)=\sum_{k=0}^{\infty}\frac{(\alpha_1;q)_k\ldots(\alpha_r;q)_k}{(q;q)_k(\beta_1;q)_k\ldots(\beta_s;q)_k}z^k$$

which converges absolutely in the open unit disk $\mathbb{U}$.

Corresponding to the function ${}_r\Phi_s(\alpha_1,\ldots,\alpha_r;\beta_1,\ldots,\beta_s,q,z)$, consider

$$\begin{aligned}{}_r\mathcal{G}_s(\alpha_1,\ldots,\alpha_r;\beta_1,\ldots,\beta_s,q,z)&=\frac{1}{z}\quad {}_r\Phi_s(\alpha_1,\ldots,\alpha_r;\beta_1,\ldots,\beta_s,q,z)\\ &=\frac{1}{z}+\sum_{k=1}^{\infty}\frac{(\alpha_1,q)_{k+1}\ldots(\alpha_r,q)_{k+1}}{(q,q)_{k+1}(\beta_1,q)_{k+1}\ldots(\beta_s,q)_{k+1}}z^k.\end{aligned}$$

Next, we define the linear operator $\mathcal{L}_s^r(\alpha_1,\ldots,\alpha_r;\beta_1,\ldots,\beta_s;q):\Sigma\to\Sigma$ by

$$\begin{aligned}\mathcal{L}_s^r(\alpha_1,\ldots,\alpha_r;\beta_1,\ldots,\beta_s;q)f(z)&={}_r\,\mathcal{G}_s(\alpha_1,\ldots,\alpha_r;\beta_1,\ldots,\beta_s,q,z)*f(z)\\ &=\frac{1}{z}+\sum_{k=1}^{\infty}\nabla_s^r(\alpha_1,q,k)a_k z^k\end{aligned} \tag{1.3}$$

where

$$\nabla_s^r(\alpha_1,q,k)=\frac{(\alpha_1,q)_{k+1}\ldots(\alpha_r,q)_{k+1}}{(q,q)_{k+1}(\beta_1,q)_{k+1}\ldots(\beta_s,q)_{k+1}}.$$

For the sake of simplicity we write

$$\mathcal{L}_s^r(\alpha_1, \ldots, \alpha_r; \beta_1, \ldots, \beta_s; q)f(z) = \mathcal{L}_s^r[\alpha_1, q]f(z).$$

Remark 1.1. **i.** For $\alpha_i = q^{\alpha_i}, \beta_j = q^{\beta_j}, \alpha_i > 0, \beta_j > 0, (i = 1, \ldots, r; j = 1, \ldots, s, r = s+1), q \to 1$ the operator $\mathcal{L}_s^r[\alpha_1, q]f(z) = \mathcal{H}_s^r[\alpha_1]f(z)$ which was investigated by Liu and Srivastava [3].

ii. For $r = 2, s = 1, \alpha_2 = q, q \to 1$, the operator $\mathcal{L}_1^2[\alpha_1, q, \beta_1, q]f(z) = \mathcal{L}[\alpha_1; \beta_1]f(z)$ was introduced and studied by Liu and srivastava [4]. Further, we note in passing that this operator $\mathcal{L}[\alpha_1; \beta_1]f(z)$ is closely related to the Carlson-Shaffer operator $\mathcal{L}[\alpha_1; \beta_1]f(z)$ defined on the space of analytic univalent functions in $\mathbb{U}$.

iii. For $r = 1, s = 0, \alpha_1 = \lambda + 1, q \to 1$, the operator $\mathcal{L}_0^1[\lambda + 1, q]f(z) = \mathcal{D}^\lambda f(z) = \frac{1}{z((1-z)^{\lambda+1}} * f(z)(\lambda > -1)$, where $\mathcal{D}^\lambda$ is the differential operator which was introduced by Ganigi and Uralegadi [5], and then it was generalized by Yang [6].

Analogue to the integral operator defined in [7] which involving q-hypergeometric functions on the normalized analytic functions, we now define the following integral operator on the space of meromorphic functions in the class Σ using the differential operator $\mathcal{L}_s^r[\alpha_1, q]$ defined in (1.3).

Definition 1.2. *Let $n \in \mathbb{N}, i \in \{1, 2, \ldots, n\}, \gamma_i > 0$. We define the integral operator $\mathcal{H}(f_1, f_2, \ldots, f_n)(z) : \Sigma^n \to \Sigma$ by*

$$\mathcal{H}(f_1, f_2, \ldots, f_n)(z) = \frac{1}{z^2} \int_0^z (u\, \mathcal{L}_s^r[\alpha_1, q]f_1(u))^{\gamma_1} \ldots (u\, \mathcal{L}_s^r[\alpha_1, q]f_n(u))^{\gamma_n} du. \quad (1.4)$$

For the sake of simplicity, we write $\mathcal{H}(z)$ instead of $\mathcal{H}(f_1, f_2, \ldots, f_n)(z)$.

We observe that in (1.4) for $r = 1, s = 0, a_1 = q$, we obtain the integral operator introduced and studied by Mohammed and Darus [8], see also ([9],[10],[11]).

The following definitions introduce subclasses of Σ which are of meromorphic starlike functions.

Definition 1.3. *Let a function $f \in \Sigma$ be analytic in $\mathbb{U}^*$. Then f is in the class $\Sigma_{r,s}^*(\alpha_1, q, \delta, b)$ if and only if, f satisfies*

$$\Re\left\{1 - \frac{1}{b}\left(\frac{z(\mathcal{L}_s^r[\alpha_1, q]f)'(z)}{\mathcal{L}_s^r[\alpha_1, q]f(z)} + 1\right)\right\} > \delta,$$

where $\mathcal{L}_s^r[\alpha_1, q]f$ defined in (1.3) and $b \in \mathbb{C}\backslash\{0\}, 0 \le \delta < 1$.

Definition 1.4. *Let a function $f \in \Sigma$ be analytic in $\mathbb{U}^*$. Then f is in the class $\Sigma_{r,s}^*\mathcal{U}(\alpha_1, q, \alpha, \delta, b)$ if and only if, f satisfies*

$$\Re\left\{1 - \frac{1}{b}\left(\frac{z(\mathcal{L}_s^r[\alpha_1, q]f)'(z)}{\mathcal{L}_s^r[\alpha_1, q]f(z)} + 1\right)\right\} > \alpha\left|\frac{1}{b}\left(\frac{z(\mathcal{L}_s^r[\alpha_1, q]f)'(z)}{\mathcal{L}_s^r[\alpha_1, q]f(z)} + 1\right)\right| + \delta,$$

where $\mathcal{L}_s^r[\alpha_1, q]f$ defined in (1.3) and $\alpha \ge 0, -1 \le \delta < 1, \alpha + \delta \ge 0, b \in \mathbb{C}\backslash\{0\}$.

Definition 1.5. *Let a function $f \in \Sigma$ be analytic in $\mathbb{U}^*$. Then f is in the class $\Sigma^*_{r,s}\mathcal{UH}(\alpha_1, q, \alpha, b)$ if and only if, f satisfies*

$$\left|1 - \frac{1}{b}\left(\frac{z(\mathcal{L}^r_s[\alpha_1, q]f)'(z)}{\mathcal{L}^r_s[\alpha_1, q]f(z)} + 1\right) - 2\alpha(\sqrt{2} - 1)\right| < \Re\left\{\sqrt{2}\left(1 - \frac{1}{b}\left(\frac{z(\mathcal{L}^r_s[\alpha_1, q]f)'(z)}{\mathcal{L}^r_s[\alpha_1, q]f(z)} + 1\right)\right)\right\} + 2\alpha(\sqrt{2} - 1),$$

where $\mathcal{L}^r_s[\alpha_1, q]f$ defined in (1.3) and $\alpha > 0, b \in \mathbb{C}\backslash\{0\}$.

For $r = 1, s = 0$ and $\alpha_1 = q$ in Definitions 1.3, 1.4 and 1.5, we obtain $\Sigma^*_b(\delta), \Sigma^*\mathcal{U}(\alpha, \delta, b)$ and $\Sigma^*\mathcal{UH}(\alpha, b)$ the classes of meromorphic functions, introduced and studied by Mohammed and Darus [12].

Now, let us introduce the following families of subclasses of meromorphic functions $\Sigma\mathcal{F}_1(\delta, b), \Sigma\mathcal{F}_2(\alpha, \delta, b)$ and $\Sigma\mathcal{F}_3(\alpha, b)$ as follows.

Definition 1.6. *Let a function $f \in \Sigma$ be analytic in $\mathbb{U}^*$. Then f is in the class $\Sigma\mathcal{F}_1(\delta, b)$ if and only if, f satisfies*

$$\Re\left\{1 - \frac{1}{b}\left(\frac{z(zf''(z) + 3f'(z))}{zf'(z) + 2f(z)} + 1\right)\right\} > \delta, \tag{1.5}$$

where $b \in \mathbb{C}\backslash\{0\}, 0 \le \delta < 1$.

Definition 1.7. *Let a function $f \in \Sigma$ be analytic in $\mathbb{U}^*$. Then f is in the class $\Sigma\mathcal{F}_2(\alpha, \delta, b)$ if and only if, f satisfies*

$$\Re\left\{1 - \frac{1}{b}\left(\frac{z(zf''(z) + 3f'(z))}{zf'(z) + 2f(z)} + 1\right)\right\} > \alpha\left|\frac{1}{b}\left(\frac{z(zf''(z) + 3f'(z))}{zf'(z) + 2f(z)} + 1\right)\right| + \delta, \tag{1.6}$$

where $\alpha \ge 0, -1 \le \delta < 1, \alpha + \delta \ge 0, b \in \mathbb{C}\backslash\{0\}$.

Definition 1.8. *Let a function $f \in \Sigma$ be analytic in $\mathbb{U}^*$. Then f is in the class $\Sigma\mathcal{F}_3(\alpha, b)$ if and only if, f satisfies*

$$\left|1 - \frac{1}{b}\left(\frac{z(zf''(z) + 3f'(z))}{zf'(z) + 2f(z)} + 1\right) - 2\alpha(\sqrt{2} - 1)\right| < \Re\left\{\sqrt{2}\left(1 - \frac{1}{b}\left(\frac{z(zf''(z) + 3f'(z))}{zf'(z) + 2f(z)} + 1\right)\right)\right\} + 2\alpha(\sqrt{2} - 1), \tag{1.7}$$

where $\alpha > 0, b \in \mathbb{C}\backslash\{0\}$.

2. Main results

In this section, we investigate some properties for the integral operator $\mathcal{H}(z)$ defined by (1.4)of the subclasses given by Definitions 1.3, 1.4 and 1.5

Theorem 2.1. *For $i \in \{1, 2, \dots, n\}$, let $\gamma_i > 0$ and $f_i \in \Sigma^*_{r,s}(\alpha_1, q, \delta_i, b)(0 \le \delta < 1)$ and $b \in \mathbb{C}\backslash\{0\}$. If*

$$0 < \sum_{i=1}^{n} \gamma_i(1 - \delta_i) \le 1,$$

then $\mathcal{H}(z)$ *is in the class* $\Sigma\mathcal{F}_1(\mu, b)$, $\mu = 1 - \sum_{i=1}^{n} \gamma_i(1-\delta_i)$

Proof. A differentiation of $\mathcal{H}(z)$ which is defined by (1.4), we obtain

$$z^2\mathcal{H}'(z) + 2z\mathcal{H}(z) = (z\ \mathcal{L}_s^r[\alpha_1, q]f_1(z))^{\gamma_1} \dots (z\ \mathcal{L}_s^r[\alpha_1, q]f_n(z))^{\gamma_n}, \tag{2.1}$$

$$\begin{aligned} z^2\mathcal{H}''(z) &+ 4z\mathcal{H}'(z) + 2\mathcal{H}(z) \\ &= \sum_{i=1}^{n} \gamma_i \Big(\frac{z(\mathcal{L}_s^r[\alpha_1, q]f)_i'(z) + \mathcal{L}_s^r[\alpha_1, q]f_i(z)}{z\mathcal{L}_s^r[\alpha_1, q]f_i(z)} \Big) \\ &\qquad [(z\ \mathcal{L}_s^r[\alpha_1, q]f_1(z))^{\gamma_1} \dots (z\ \mathcal{L}_s^r[\alpha_1, q]f_n(z))^{\gamma_n}] \end{aligned} \tag{2.2}$$

Then from (2.1) and (2.2),we obtain

$$\frac{z^2\mathcal{H}''(z) + 4z\mathcal{H}'(z) + 2\mathcal{H}(z)}{z^2\mathcal{H}'(z) + 2z\mathcal{F}(z)} = \sum_{i=1}^{n} \gamma_i \left(\frac{(\mathcal{L}_s^r[\alpha_1, q]f_i)'(z)}{\mathcal{L}_s^r[\alpha_1, q]f_i(z)} + \frac{1}{z} \right). \tag{2.3}$$

By multiplying (2.3) with z we have

$$\frac{z^2\mathcal{H}''(z) + 4z\mathcal{H}'(z) + 2\mathcal{H}_{\gamma_i}(z)}{z\mathcal{H}'_{\gamma_i}(z) + 2\mathcal{H}_{\gamma_i}(z)} = \sum_{i=1}^{n} \gamma_i \left(\frac{z(\mathcal{L}_s^r[\alpha_1, q]f_i)'(z)}{\mathcal{L}_s^r[\alpha_1, q]f_i(z)} + 1 \right).$$

That is equivalent to

$$\frac{z\,(z\mathcal{H}''(z) + 3\mathcal{H}'(z))}{z\mathcal{H}'(z) + 2\mathcal{H}(z)} + 1 = \sum_{i=1}^{n} \gamma_i \left(\frac{z(\mathcal{L}_s^r[\alpha_1, q]f_i)'(z)}{\mathcal{L}_s^r[\alpha_1, q]f_i(z)} + 1 \right). \tag{2.4}$$

Equivalently, (2.4) can be written as

$$1 - \frac{1}{b}\left\{ \frac{z(z\mathcal{H}''(z) + 3\mathcal{H}'(z))}{z\mathcal{H}'(z) + 2\mathcal{H}(z)} + 1 \right\} = \sum_{i=1}^{n} \gamma_i \left\{ 1 - \frac{1}{b}\left(\frac{z(\mathcal{L}_s^r[\alpha_1, q]f_i)'(z)}{\mathcal{L}_s^r[\alpha_1, q]f_i(z)} + 1 \right) \right\} + 1 - \sum_{i=1}^{n} \gamma_i.$$

Taking the real part of both sides of the last expression, we have

$$\begin{aligned} \Re &\left\{ 1 - \frac{1}{b}\left(\frac{z(z\mathcal{H}''(z) + 3\mathcal{H}'(z))}{z\mathcal{H}'(z) + 2\mathcal{H}(z)} + 1 \right) \right\} \\ &= \sum_{i=1}^{n} \gamma_i \Re \left\{ 1 - \frac{1}{b}\left(\frac{z(\mathcal{L}_s^r[\alpha_1, q]f_i)'(z)}{\mathcal{L}_s^r[\alpha_1, q]f_i(z)} + 1 \right) \right\} + 1 - \sum_{i=1}^{n} \gamma_i. \end{aligned}$$

Since $f_i \in \Sigma_{r,s}^*(\alpha_1, q, \delta_i, b)$, hence

$$\Re\left\{ 1 - \frac{1}{b}\left(\frac{z(z\mathcal{H}''(z) + 3\mathcal{H}'(z))}{z\mathcal{H}'(z) + 2\mathcal{H}(z)} + 1 \right) \right\} > \sum_{i=1}^{n} \gamma_i \delta_i + 1 - \sum_{i=1}^{n} \gamma_i.$$

Therefore

$$\Re\left\{ 1 - \frac{1}{b}\left(\frac{z(z\mathcal{H}''(z) + 3\mathcal{H}'(z))}{z\mathcal{H}'(z) + 2\mathcal{H}(z)} + 1 \right) \right\} > 1 - \sum_{i=1}^{n} \gamma_i(1-\delta_i).$$

Then $\mathcal{H}(z) \in \Sigma\mathcal{F}_1(\mu, b)$, $\quad \mu = 1 - \sum_{i=1}^{n} \gamma_i(1-\delta_i)$ □

Theorem 2.2. *For $i \in \{1, 2, \ldots, n\}$, let $\gamma_i > 0$ and $f_i \in \Sigma^*_{r,s}\mathcal{U}(\alpha, \delta, b)(\alpha \geq 0, -1 \leq \delta < 1, \alpha + \delta \geq 0)$ and $b \in \mathbb{C}\backslash\{0\}$. If*

$$\sum_{i=1}^{n} \gamma_i \leq 1,$$

then $\mathcal{H}(z)$ is in the class $\Sigma\mathcal{F}_2(\alpha, \delta, b)$.

Proof. Since $f_i \in \Sigma^*_{r,s}\mathcal{U}(\alpha_1, q, \alpha, \delta, b)$, it follows from Definition 1.3 that

$$\Re\left\{1 - \frac{1}{b}\left(\frac{z(\mathcal{L}^r_s[\alpha_1, q]f_i)'(z)}{\mathcal{L}^r_s[\alpha_1, q]f_i(z)} + 1\right)\right\} > \alpha\left|\frac{1}{b}\left(\frac{z(\mathcal{L}^r_s[\alpha_1, q]f_i)'(z)}{\mathcal{L}^r_s[\alpha_1, q]f_i(z)} + 1\right)\right| + \delta. \quad (2.5)$$

Considering (2.2) and (2.5) we obtain

$$\Re\left\{1 - \frac{1}{b}\left(\frac{z(z\mathcal{H}''(z) + 3\mathcal{H}'(z))}{z\mathcal{H}'(z) + 2\mathcal{H}(z)} + 1\right)\right\} - \alpha\left|\frac{1}{b}\left(\frac{z(z\mathcal{H}''(z) + 3\mathcal{H}'(z))}{z\mathcal{H}'(z) + 2\mathcal{H}(z)} + 1\right)\right| - \delta.$$

$$= 1 - \sum_{i=1}^{n}\gamma_i + \sum_{i=1}^{n}\gamma_i\Re\left\{1 - \frac{1}{b}\left(\frac{z(\mathcal{L}^r_s[\alpha_1, q]f_i)'(z)}{\mathcal{L}^r_s[\alpha_1, q]f_i(z)} + 1\right)\right\}$$
$$- \alpha\left|\sum_{i=1}^{n}\gamma_i\frac{1}{b}\left(\frac{z(\mathcal{L}^r_s[\alpha_1, q]f_i)'(z)}{\mathcal{L}^r_s[\alpha_1, q]f_i(z)} + 1\right)\right| - \delta$$

$$> 1 - \sum_{i=1}^{n}\gamma_i + \sum_{i=1}^{n}\gamma_i\left\{\alpha\left|\frac{1}{b}\left(\frac{z(\mathcal{L}^r_s[\alpha_1, q]f_i)'(z)}{\mathcal{L}^r_s[\alpha_1, q]f_i(z)} + 1\right)\right| + \delta\right\}$$
$$- \alpha\sum_{i=1}^{n}\gamma_i\left|\frac{1}{b}\left(\frac{z(\mathcal{L}^r_s[\alpha_1, q]f_i)'(z)}{\mathcal{L}^r_s[\alpha_1, q]f_i(z)} + 1\right)\right| - \delta$$

$$= (1 - \delta)(1 - \sum_{i=1}^{n}\gamma_i) \geq 0.$$

This completes the proof. □

Theorem 2.3. *For $i \in \{1, 2, \ldots, n\}$, let $\gamma_i > 0$ and $f_i \in \Sigma^*\mathcal{UH}(\alpha, b)$ $(\alpha > 0$ and $b \in \mathbb{C}\backslash\{0\})$. If*

$$\sum_{i=1}^{n} \gamma_i \leq 1,$$

then $\mathcal{H}(z)$ is in the class $\Sigma\mathcal{F}_3(\alpha, b)$.

Proof. Since $f_i \in \Sigma^*_{r,s}\mathcal{UH}(\alpha_1, q, \alpha, b)$, it follows from Definition 1.4 that

$$\Re\left\{\sqrt{2}\left(1 - \frac{1}{b}\left(\frac{z(\mathcal{L}^r_s[\alpha_1, q]f_i)'(z)}{\mathcal{L}^r_s[\alpha_1, q]f_i(z)} + 1\right)\right)\right\} + 2\alpha(\sqrt{2} - 1)$$
$$- \left|1 - \frac{1}{b}\left(\frac{z(\mathcal{L}^r_s[\alpha_1, q]f_i)'(z)}{\mathcal{L}^r_s[\alpha_1, q]f_i(z)} + 1\right) - 2\alpha(\sqrt{2} - 1)\right| > 0. \quad (2.6)$$

Considering (2.2) and (2.6), we obtain

$$\begin{aligned}
&\Re\left\{\sqrt{2}\left(1-\frac{1}{b}\left(\frac{z(z\mathcal{H}''(z)+3\mathcal{H}'(z))}{z\mathcal{H}'(z)+2\mathcal{H}(z)}+1\right)\right)\right\}+2\alpha(\sqrt{2}-1)\\
&\qquad -\left|1-\frac{1}{b}\left(\frac{z(z\mathcal{H}''(z)+3\mathcal{H}'(z))}{z\mathcal{H}'(z)+2\mathcal{H}(z)}+1\right)-2\alpha(\sqrt{2}-1)\right| \qquad (2.7)\\
&=\Re\left\{\sqrt{2}\left[1-\sum_{i=1}^{n}\gamma_i\frac{1}{b}\left(\frac{z(\mathcal{L}_s^r[\alpha_1,q]f_i)'(z)}{\mathcal{L}_s^r[\alpha_1,q]f_i(z)}+1\right)\right]\right\}+2\alpha(\sqrt{2}-1)\\
&\qquad -\left|1-\sum_{i=1}^{n}\gamma_i\frac{1}{b}\left(\frac{z(\mathcal{L}_s^r[\alpha_1,q]f_i)'(z)}{\mathcal{L}_s^r[\alpha_1,q]f_i(z)}+1\right)-2\alpha(\sqrt{2}-1)\right|\\
&=\sqrt{2}-\sqrt{2}\sum_{i=1}^{n}\gamma_i\Re\frac{1}{b}\left(\frac{z(\mathcal{L}_s^r[\alpha_1,q]f_i)'(z)}{\mathcal{L}_s^r[\alpha_1,q]f_i(z)}+1\right)+2\alpha(\sqrt{2}-1)\\
&\qquad -\left|1-\sum_{i=1}^{n}\gamma_i\frac{1}{b}\left(\frac{z(\mathcal{L}_s^r[\alpha_1,q]f_i)'(z)}{\mathcal{L}_s^r[\alpha_1,q]f_i(z)}+1\right)-2\alpha(\sqrt{2}-1)\right|\\
&=\sqrt{2}+\sqrt{2}\sum_{i=1}^{n}\gamma_i\Re\left\{1-\frac{1}{b}\left(\frac{z(\mathcal{L}_s^r[\alpha_1,q]f_i)'(z)}{\mathcal{L}_s^r[\alpha_1,q]f_i(z)}+1\right)\right\}-\sqrt{2}\sum_{i=1}^{n}\gamma_i+2\alpha(\sqrt{2}-1)\\
&\qquad -\left|1+\sum_{i=1}^{n}\gamma_i\left[1-\frac{1}{b}\left(\frac{z(\mathcal{L}_s^r[\alpha_1,q]f_i)'(z)}{\mathcal{L}_s^r[\alpha_1,q]f_i(z)}+1\right)-2\alpha(\sqrt{2}-1)\right]-\sum_{i=1}^{n}\gamma_i\right.\\
&\qquad\qquad \left.+2\alpha(\sqrt{2}-1)\sum_{i=1}^{n}\gamma_i-2\alpha(\sqrt{2}-1)\right|\\
&=\sqrt{2}\left(1-\sum_{i=1}^{n}\gamma_i\right)+2\alpha(\sqrt{2}-1)+\sqrt{2}\sum_{i=1}^{n}\gamma_i\Re\left\{1-\frac{1}{b}\left(\frac{z(\mathcal{L}_s^r[\alpha_1,q]f_i)'(z)}{\mathcal{L}_s^r[\alpha_1,q]f_i(z)}+1\right)\right\}\\
&\qquad -\left|[1-2\alpha(\sqrt{2}-1)]\left(1-\sum_{i=1}^{n}\gamma_i\right)+\sum_{i=1}^{n}\gamma_i\left[1-\frac{1}{b}\left(\frac{z(\mathcal{L}_s^r[\alpha_1,q]f_i)'(z)}{\mathcal{L}_s^r[\alpha_1,q]f_i(z)}+1\right)\right.\right.\\
&\qquad\qquad \left.\left.-2\alpha(\sqrt{2}-1)\right]\right|\\
&\geq\sqrt{2}\left(1-\sum_{i=1}^{n}\gamma_i\right)+2\alpha(\sqrt{2}-1)+\sqrt{2}\sum_{i=1}^{n}\gamma_i\Re\left\{1-\frac{1}{b}\left(\frac{z(\mathcal{L}_s^r[\alpha_1,q]f_i)'(z)}{\mathcal{L}_s^r[\alpha_1,q]f_i(z)}+1\right)\right\}\\
&-\sum_{i=1}^{n}\gamma_i\left|1-\frac{1}{b}\left(\frac{z(\mathcal{L}_s^r[\alpha_1,q]f_i)'(z)}{\mathcal{L}_s^r[\alpha_1,q]f_i(z)}+1\right)-2\alpha(\sqrt{2}-1)\right|-|1-2\alpha(\sqrt{2}-1)|\left(1-\sum_{i=1}^{n}\gamma_i\right)
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{i=1}^{n} \gamma_i \Bigg\{ \Re\sqrt{2}\left[1-\frac{1}{b}\left(\frac{z(\mathcal{L}_s^r[\alpha_1,q]f_i)'(z)}{\mathcal{L}_s^r[\alpha_1,q]f_i(z)}+1\right)\right] + 2\alpha(\sqrt{2}-1) \\
&\qquad - \left|1-\frac{1}{b}\left(\frac{z(\mathcal{L}_s^r[\alpha_1,q]f_i)'(z)}{\mathcal{L}_s^r[\alpha_1,q]f_i(z)}+1\right) - 2\alpha(\sqrt{2}-1)\right| \Bigg\} + \sqrt{2}\left(1-\sum_{i=1}^{n}\gamma_i\right) \\
&\qquad\qquad + 2\alpha(\sqrt{2}-1) - 2\alpha(\sqrt{2}-1)\sum_{i=1}^{n}\gamma_i - |1-2\alpha(\sqrt{2}-1)|\left(1-\sum_{i=1}^{n}\gamma_i\right) \\
&> \left[\sqrt{2}+2\alpha(\sqrt{2}-1) - |1-2\alpha(\sqrt{2}-1)|\right]\left(1-\sum_{i=1}^{n}\gamma_i\right) \\
&> \left(1-\sum_{i=1}^{n}\gamma_i\right)\min\left\{(\sqrt{2}-1)(1+4\alpha), \sqrt{2}+1\right\} \geq 0.
\end{aligned}
$$

This completes the proof. □

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Book reviews

Will H H. Moore & David A. Siegel, A Mathematical Course for Political & and Social Research, Princeton University Press, Princeton and Oxford, 2013, xix+430 pp, ISBN 978-0-691-15995-9 (hardback), ISBN 978-0-691-15917-1 (paperback).

The aim of the present book is to introduce the political scientists to some mathematical tools used in their discipline. In spite of its abstract character, mathematics helps them to develop rigorous theories based on the observed data and phenomena and, at the same time, gives them rigorous tests on the implications of developed theories. As the book is addressed to an audience with little prior knowledge of mathematics (usually at the high school level), the formal mathematics rigor is sacrificed in the favor of intuition, compensated by some footnotes and comments providing formal formalism. The book is divided into five main parts: I. *Building blocks*, II. *Calculus in one dimension*, III. *Probability,* IV. *Linear algebra*, and V. *Multivariate calculus and optimization.*

The first part is concerned with basic tools used in mathematics: notation, basic results on computation in arithmetic and algebra, functions, relations and utility (closely related to the subject matter of the book). This part ends with some notions from calculus – sequences and series, continuous functions – completed in the second part with differentiation and integration of functions of one variable, with applications to extrema. The third part of the book is concerned with an important topic for political sciences, namely probability theory. Although statistics is an essential tool in political sciences, the stretch is here on probability, including only a brief discussion on statistical inference. The fourth part discusses vector, matrices, vector spaces, and ends with a brief discussion of some more advanced topics – eigenvalues and Markov chains. The last part of the book contains some results on calculus in several variables with applications to optimization – unconstrained, and constrained both with equality and inequality constraints.

The text is completed with many worked examples, exercises, and applications to various topics in political and social sciences. Written in an intuitive and accessible way, this book can be used as a primer for math novices in the social sciences as well as a handy reference for the researchers in this area.

Nicolae Popovici

Igor Kriz and Aleš Pultr, Introduction to Mathematical Analysis, Birkhäuser-Springer, Basel, 2013, ISBN 978-3-0348-0635-0; ISBN 978-3-0348-0636-7 (eBook); DOI 10.1007/978-3-0348-0636-7, xx+510 pp.

As the authors mention in the Preface, their aim is "to write a book which the students may want to keep after the course is over, and which could serve them as a bridge to higher mathematics". With this aim in mind the authors included in their book some topics from topology, calculus of real functions of one and several real variables, elements of complex analysis, some differential and Riemannian geometry, elements of functional analysis, as well as some applications.

Some basic tools from topology, viewed as a background of the whole analysis (understood in a large sense), are treated in Chapters 2 and 7, *Metric and topological spaces*, I and II, respectively. Calculus of real functions of one or several real variables is treated in chapters 1. *Preliminaries*, 3. *Multivariable differential calculus,* 4. *Integration I: Multivariable Riemann integral and basic ideas toward the Lebesgue integral,* 5. *Integration II: Measurable functions, measure and the techniques of Lebesgue integration,* 8. *Line integrals and Green's theorem.* The authors treat first Riemann's integral and then the Lebesgue integral is introduced via Daniel's method. The basics of complex analysis are developed in Chapters 10. *Complex analysis I: Basic concepts,* and 13. *Complex analysis II: Further topics.* The chapters concerned with differential geometry are: 12. *Smooth manifolds, differential forms and Stokes' theorem,* and 14. *Tensor calculus and Riemannian geometry.* Two final chapters are devoted to some results from functional analysis: 16. *Banach and Hilbert spaces: Elements of functional analysis,* and 17. *A few applications of Hilbert spaces,* (including a Hilbert space proof of the Radon-Nikodym theorem). Other applications included in the book are to differential equations, in Chapters 6. *Systems of ordinary differential equations,* and 7. *Systems of linear differential equations,* and to calculus of variations in Chapter 14. *Calculus of variations and the geodesic equation.*

Some supplementary material is included in Chapter 11. *Multilinear algebra* (tensor products and the exterior Grassmann algebra are presented by the means of homological algebra), and in two appendices, A. *Linear algebra I: Vector spaces*, and B. *Linear algebra II: More about matrices.*

Each chapter ends with a set of well chosen exercises completing the main text. Treating in a unified and coherent way several topics from mathematical analysis, both real and complex, differential and Riemannian geometry, functional analysis, and their applications, the present well written book is a valuable addition to the existing ones on similar topics. It can be used by graduate students in mathematics and researchers in mathematics and other areas (physics, chemistry, economics) to find a rigorous foundations and details on several topics in analysis. The instructors can recommend the book as a supplementary material for their courses.

S. Cobzaş

Niels Lauritzen, Undergraduate Convexity – From Fourier and Motzkin to Kuhn and Tucker, World Scientific, London - Singapore - Beijing, 2013, xiv + 283 pages, ISBN: 978-981-4412-51-3 and 978-981-4452-76-2 (pbk).

As the author says in the Preface – "Convexity is a key concept in modern mathematics with rich applications in economics and optimization". The aim of this book is to present at an elementary level (the prerequisites are some familiarity with calculus and linear algebra) the basic results on convexity in the finite dimensional setting, i.e. in the space $\mathbb{R}^n$. The book can be divided into three main parts – Chapters 1–6 are devoted to convex sets, Chapters 7–9 to convex functions, and the last chapter, Chapter 10, *Convex optimization*, to applications. In spite of its elementary level some consistent applications are included as well. A special attention is paid to the algorithmic questions as, e.g., to find whether a point belongs to the convex hull of a finite set of points.

The first two chapters of the book present some results on Fourier-Motzkin elimination method (a generalization of Gauss' method) to solve systems of linear inequalities and some results on affine spaces and subspaces, affine independence, affine hulls.

The study of convexity starts in the third chapter, 3. *Convex subsets*, with some elemenatray properties, convex hulls (Carathéodori's theorem), faces, extreme points, and a presentation of an algorithm, based on Carathéodori's theorem and on Bland's rule from the simplex method, to decide if a point is in the convex hull of a finite subset of $\mathbb{R}^d$.

Chapter 4, *Polyhedra*, is devoted to this important class of convex sets. Applications are given to Farkas' lemma and Gordan's theorem, Markov chains, doubly stochastic matrices, and to Hall's marriage problem. This study is continued in Chapter 5, *Computation with polyhedra*, where two important algorithms – the double description method for polyhedra and the simplex algorithm – are presented. Other properties of the convex subsets of $\mathbb{R}^d$, as the existence and characterization of projections onto closed convex sets, supporting hyperplanes, separation of convex sets, are treated in the sixth chapter, *Closed convex subsets and separating hyperplanes*, the last of the first part.

The study of convex functions begins in Chapter 7, *Convex functions*, with convex functions of one variable (continuity and differentiability properties, local minima), the case of functions of several variables being postponed to Chapter 9, *Convex functions of several variables*. For reader's convenience a chapter, 8. *Differentiable functions of several variables,* contains a presentation (with full proofs) of basic results from the differential calculus for vector functions. Note that the corresponding results for functions of one variable were proved in the seventh chapter as well. Chapter 9 contains characterizations of the convexity of differentiable functions of several variables in terms of the monotony of the first differential and positivity of the second differential. For this last result nice and simple proofs of Sylvester's criterium of the positive definiteness of matrices, as well as of the more delicate question of positive semidefiniteness, are included . This chapter contains also a discussion on the spectral properties of symmetric matrices and a study of quadratic forms (Sylvester's law of inertia).

As we yet mentioned the last chapter of the book is dedicated to applications – Karush-Kuhn-Tuccker optimality conditions, Lagrangians and saddle points, duality.

Two appendices, A. *Analysis,* and B. *Linear (in)dependence and the rank of a matrix,* complete the main text with some useful notions and results.

The book, based on one quarter courses, *Konvekse Mœngder* and *Konvekse Funktioner*, taught for several years to undergraduate students in mathematics, economics and computer science at Aarhus University, is didactically written in a pleasant and alive style, with careful motivation of the considered notions, illuminating examples and pictures, and relevant historical remarks. The front cover contains a picture of Johan Ludvig William Valdemar Jensen, the creator of convex functions, some quotations from his papers being included in the book. As a matter of fact, Jensen worked as a telephone engineer in København and never acquired a formal degree in mathematics or held an academic position, and has done mathematics for its beauty and his own enjoyment. The author dedicates this book to him "as a tribute to the joyful and genuine pursuit of mathematics". All in all, this is a remarkable book, a readable and attractive introduction to the multi-faced domain of convexity and its applications.

Nicolae Popovici

Ioannis Farmakis and Martin Moskowitz, Fixed Point Theorems and Their Applications, World Scientific, London - Singapore - Beijing, 2013, xi + 234 pages, ISBN: 978-981-4458-91-7.

The book presents some classical fixed point theorems, with emphasis on those with an algebraic or geometric flavor and their applications. For instance, Brouwer's fixed point theorem, proved in the first chapter via Milnor's approach, is applied to the existence of positive eigenvalues and positive proper vectors of positive matrices (the Perron-Frobenius theorem) and to a glimpse of Google research engine. Similarly, in Chapter 2, *Fixed point theorems in analysis,* Schauder-Tychonoff's fixed point theorem is applied to Peano's existence theorem for differential equations. This chapter contains also a proof of the Markov-Kakutani fixed point theorem for families of affine mappings, with applications to Lie and amenable groups.

Chapter 3, *The Lefcshetz fixed point theorem,* presents the algebraic-topological fixed point theorem of S. Lefschetz, including a brief discussion on manifolds, Lie groups, transversality and a proof of the Atiyah and Singer fixed point theorem that led them to the Fields Medal index theorem for elliptic operators.

Chapter 4, *Fixed point theorems in geometry,* is devoted to the fixed point theorem of E. Cartan on compact groups of isomorphisms on Hadamard manifolds. and to the theorems of Preissmann and Weinstein on fixed points on manifolds with negative, respectively positive, curvature.

Chapter 5, *Fixed point theorems of volume preserving maps,* starts with a proof of Poincaré's recurrence theorem for volume preserving maps and includes also fixed point theorem in symplectic geometry, a discussion on Arnold's conjecture on the number of fixed points of maps on such manifolds, Poincaré's last geometric theorem on fixed points on tori, and a study of Anosov diffeomorphisms. The chapter concludes with a presentation of Lefschetz zeta function and its applications. Chapter 6, *Borel's fixed point theorem in algebraic geometry*, treats Borel's fixed point theorem for solvable algebraic groups acting on a complex projective variety.

The seventh chapter, *Miscellaneous fixed point theorems*, is concerned with applications to number theory (little Fermat's theorem and Fermat's two square theorem), Jordan's theorem on fixed points of finite groups of transformations and a fixed point for the holomorphic mappings in the unit disc, due to the second named author. The last chapter of the book, Chapter 8, *A fixed point theorem in set theory*, contains a proof of Knaster-Tarski theorem on fixed points of order preserving functions on Banach lattices with application to Schröder-Cantor-Bernstein theorem from set theory.

The book presents interest mainly by some more special fixed point theorems in algebraic topology, algebraic geometry, and differential and symplectic geometry, as well as by the interesting applications of fixed point results to various areas of mathematics. Written in a way that the chapters can be used independently, it appeals to a large audience.

Adrian Petruşel