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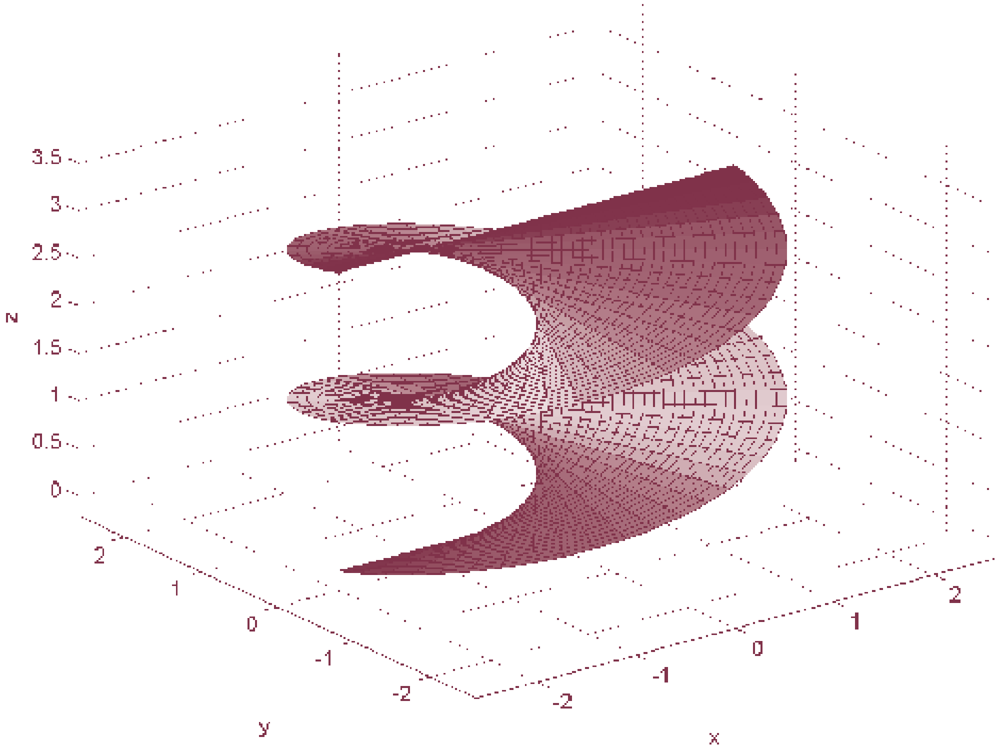

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Multivariate weighted fractional representation formulae and Ostrowski type inequalities

George A. Anastassiou

Abstract. Here we derive multivariate weighted fractional representation formulae involving ordinary partial derivatives of first order. Then we present related multivariate weighted fractional Ostrowski type inequalities with respect to uniform norm.

Mathematics Subject Classification (2010): 26A33, 26D10, 26D15.

Keywords: Multivariate fractional integral, weighted representation formula, multivariate weighted Ostrowski inequality.

1. Introduction

Let $f : [a, b] \to \mathbb{R}$ be differentiable on $[a, b]$, and $f' : [a, b] \to \mathbb{R}$ be integrable on $[a, b]$. Suppose now that $w : [a, b] \to [0, \infty)$ is some probability density function, i.e. it is a nonnegative integrable function satisfying $\int_a^b w(t)\,dt = 1$, and $W(t) = \int_a^t w(x)\,dx$ for $t \in [a, b]$, $W(t) = 0$ for $t \le a$ and $W(t) = 1$ for $t \ge b$. Then, the following identity (Pecarić, [5]) is the weighted generalization of the Montgomery identity ([4])

$$f(x) = \int_a^b w(t) f(t)\,dt + \int_a^b P_w(x, t) f'(t)\,dt, \tag{1.1}$$

where the weighted Peano Kernel is

$$P_w(x, t) := \begin{cases} W(t), & a \le t \le x, \\ W(t) - 1, & x < t \le b. \end{cases} \tag{1.2}$$

In [1] we proved

Theorem 1.1. *Let $w : [a, b] \to [0, \infty)$ be a probability density function, i.e. $\int_a^b w(t)\,dt = 1$, and set $W(t) = \int_a^t w(x)\,dx$ for $a \le t \le b$, $W(t) = 0$ for $t \le a$ and $W(t) = 1$ for $t \ge b$, $\alpha \ge 1$, and f is as in (1.1). Then the generalization of the weighted Montgomery* *identity for fractional integrals is the following*

$$f(x) = (b-x)^{1-\alpha}\Gamma(\alpha) J_a^\alpha (w(b) f(b)) \\ - J_a^{\alpha-1}(Q_w(x,b) f(b)) + J_a^\alpha (Q_w(x,b) f'(b)), \tag{1.3}$$

where the weighted fractional Peano Kernel is

$$Q_w(x,t) := \begin{cases} (b-x)^{1-\alpha}\Gamma(\alpha) W(t), & a \le t \le x, \\ (b-x)^{1-\alpha}\Gamma(\alpha)(W(t)-1), & x < t \le b, \end{cases} \tag{1.4}$$

i.e. $Q_w(x,t) = (b-x)^{1-\alpha}\Gamma(\alpha) P_w(x,t)$.

When $\alpha = 1$ then the weighted generalization of the Montgomery identity for fractional integrals in (1.3) reduces to the weighted generalization of the Montgomery identity for integrals in (1.1).

So for $\alpha \ge 1$ and $x \in [a,b)$ we can rewrite (1.3) as follows

$$f(x) = (b-x)^{1-\alpha}\int_a^b (b-t)^{\alpha-1} w(t) f(t)\,dt \\ - (b-x)^{1-\alpha}(\alpha-1)\int_a^b (b-t)^{\alpha-2} P_w(x,t) f(t)\,dt \\ + (b-x)^{1-\alpha}\int_a^b (b-t)^{\alpha-1} P_w(x,t) f'(t)\,dt. \tag{1.5}$$

In this article based on (1.5), we establish a multivariate weighted general fractional representation formula for $f(x)$, $x \in \prod_{i=1}^m [a_i,b_i] \subset \mathbb{R}^m$, and from there we derive an interesting multivariate weighted fractional Ostrowski type inequality. We finish with an application.

2. Main Results

We make

Assumption 2.1. Let $f \in C^1\left(\prod_{i=1}^m [a_i,b_i]\right)$.

Assumption 2.2. Let $f : \prod_{i=1}^m [a_i,b_i] \to \mathbb{R}$ be measurable and bounded, such that there exist $\frac{\partial f}{\partial x_j} : \prod_{i=1}^m [a_i,b_i] \to \mathbb{R}$, and it is x_j-integrable for all $j = 1,...,m$. Furthermore $\frac{\partial f}{\partial x_i}(t_1,...,t_i,x_{i+1},...,x_m)$ it is integrable on $\prod_{j=1}^i [a_j,b_j]$, for all $i = 1,...,m$, for any $(x_{i+1},...,x_m) \in \prod_{j=i+1}^m [a_j,b_j]$.

Convention 2.3. We set

$$\prod_{j=1}^0 \cdot = 1. \tag{2.1}$$

Notation 2.4. Here $x = \overrightarrow{x} = (x_1,...,x_m) \in \mathbb{R}^m$, $m \in \mathbb{N} - \{1\}$. Likewise $t = \overrightarrow{t} = (t_1,...,t_m)$, and $d\overrightarrow{t} = dt_1 dt_2 ... dt_m$. Here w_i, W_i correspond to $[a_i,b_i]$, $i = 1,...,m$, and are as w, W of Theorem 1.1.

We need

Definition 2.5. *(see [2] and [3]) Let $\prod_{i=1}^{m}[a_i,b_i]\subset\mathbb{R}^m$, $m\in\mathbb{N}-\{1\}$, $a_i<b_i$, $a_i,b_i\in\mathbb{R}$. Let $\alpha>0$, $f\in L_1\left(\prod_{i=1}^{m}[a_i,b_i]\right)$. We define the left mixed Riemann-Liouville fractional multiple integral of order α:*

$$\left(I_{a+}^{\alpha}f\right)(x):=\frac{1}{(\Gamma(\alpha))^m}\int_{a_1}^{x_1}...\int_{a_m}^{x_m}\left(\prod_{i=1}^{m}(x_i-t_i)\right)^{\alpha-1}f(t_1,...,t_m)\,dt_1...dt_m, \quad (2.2)$$

where $x_i\in[a_i,b_i]$, $i=1,...,m$, and $x=(x_1,...,x_m)$, $a=(a_1,...,a_m)$, $b=(b_1,...,b_m)$.

We present the following multivariate weighted fractional representation formula

Theorem 2.6. *Let f as in Assumption 2.1 or Assumption 2.2, $\alpha\geq1$, $x_i\in[a_i,b_i)$, $i=1,...,m$. Here P_{w_i} corresponds to $[a_i,b_i]$, $i=1,...,m$, and it is as in (1.2). The probability density function w_j is assumed to be bounded for all $j=1,...,m$. Then*

$$f(x_1,...,x_m)=\left(\prod_{j=1}^{m}(b_j-x_j)\right)^{1-\alpha}(\Gamma(\alpha))^m\left(I_{a+}^{\alpha}\left(\prod_{j=1}^{m}w_j\right)f\right)(b)$$
$$+\sum_{i=1}^{m}A_i(x_1,...,x_m)+\sum_{i=1}^{m}B_i(x_1,...,x_m), \quad (2.3)$$

where for $i=1,...,m$:

$$A_i(x_1,...,x_m):=-(\alpha-1)\left(\prod_{j=1}^{i}(b_j-x_j)\right)^{1-\alpha}\int_{\prod_{j=1}^{i}[a_j,b_j]}\left(\prod_{j=1}^{i-1}(b_j-t_j)\right)^{\alpha-1} \quad (2.4)$$
$$\cdot(b_i-t_i)^{\alpha-2}\left(\prod_{j=1}^{i-1}w_j(t_j)\right)P_{w_i}(x_i,t_i)f(t_1,...,t_i,x_{i+1},...,x_m)\,dt_1...dt_i,$$

and

$$B_i(x_1,...,x_m):=\left(\prod_{j=1}^{i}(b_j-x_j)\right)^{1-\alpha}\int_{\prod_{j=1}^{i}[a_j,b_j]}\left(\prod_{j=1}^{i}(b_j-t_j)\right)^{\alpha-1} \quad (2.5)$$
$$\cdot\left(\prod_{j=1}^{i-1}w_j(t_j)\right)P_{w_i}(x_i,t_i)\frac{\partial f}{\partial x_i}(t_1,...,t_i,x_{i+1},...,x_m)\,dt_1...dt_i.$$

Proof. We have that

$$f(x_1,x_2,...,x_m)\overset{(1.5)}{=}(b_1-x_1)^{1-\alpha}\int_{a_1}^{b_1}(b_1-t_1)^{\alpha-1}w_1(t_1)f(t_1,x_2,...,x_m)\,dt_1$$
$$+A_1(x_1,...,x_m)+B_1(x_1,...,x_m). \quad (2.6)$$

Similarly it holds

$$f(t_1,x_2,...,x_m)\overset{(1.5)}{=}(b_2-x_2)^{1-\alpha}\int_{a2}^{b_2}(b_2-t_2)^{\alpha-1}w_2(t_2)f(t_1,t_2,x_3,...,x_m)\,dt_2$$

$$-(\alpha-1)(b_2-x_2)^{1-\alpha}\int_{a_2}^{b_2}(b_2-t_2)^{\alpha-2}P_{w_2}(x_2,t_2)f(t_1,t_2,x_3,...,x_m)\,dt_2$$

$$+(b_2-x_2)^{1-\alpha}\int_{a_2}^{b_2}(b_2-t_2)^{\alpha-1}P_{w_2}(x_2,t_2)\frac{\partial f}{\partial x_2}(t_1,t_2,x_3,...,x_m)\,dt_2. \quad (2.7)$$

Next we plug (2.7) into (2.6).

We get

$$f(x_1,...,x_m)=((b_1-x_1)(b_2-x_2))^{1-\alpha}$$

$$\cdot\int_{a_1}^{b_1}\int_{a_2}^{b_2}((b_1-t_1)(b_2-t_2))^{\alpha-1}w_1(t_1)w_2(t_2)f(t_1,t_2,x_3,...,x_m)\,dt_1dt_2 \quad (2.8)$$

$$+A_2(x_1,...,x_m)+B_2(x_1,...,x_m)+A_1(x_1,...,x_m)+B_1(x_1,...,x_m).$$

We continue as above.

We also have

$$f(t_1,t_2,x_3,...,x_m)\overset{(1.5)}{=}(b_3-x_3)^{1-\alpha}$$

$$\cdot\int_{a_3}^{b_3}(b_3-t_3)^{\alpha-1}w_3(t_3)f(t_1,t_2,t_3,x_4,...,x_m)\,dt_3$$

$$-(\alpha-1)(b_3-x_3)^{1-\alpha}\int_{a_3}^{b_3}(b_3-t_3)^{\alpha-2}P_{w_3}(x_3,t_3)f(t_1,t_2,t_3,x_4,...,x_m)\,dt_3 \quad (2.9)$$

$$+(b_3-x_3)^{1-\alpha}\int_{a_3}^{b_3}(b_3-t_3)^{\alpha-1}P_{w_3}(x_3,t_3)\frac{\partial f}{\partial x_3}(t_1,t_2,t_3,x_4,...,x_m)\,dt_3.$$

We plug (2.9) into (2.8). Therefore it holds

$$f(x_1,...,x_m)=\left(\prod_{j=1}^{3}(b_j-x_j)\right)^{1-\alpha}\int_{\prod_{j=1}^{3}[a_j,b_j]}\left(\prod_{j=1}^{3}(b_j-t_j)\right)^{\alpha-1}$$

$$\cdot\left(\prod_{j=1}^{3}w_j(t_j)\right)f(t_1,t_2,t_3,x_4,...,x_m)\,dt_1dt_2dt_3$$

$$+\sum_{j=1}^{3}A_j(x_1,...,x_m)+\sum_{j=1}^{3}B_j(x_1,...,x_m). \quad (2.10)$$

Continuing similarly we finally obtain

$$f(x_1,...,x_m)=\left(\prod_{j=1}^{m}(b_j-x_j)\right)^{1-\alpha}$$

$$\cdot\int_{\prod_{j=1}^{m}[a_j,b_j]}\left(\prod_{j=1}^{m}(b_j-t_j)\right)^{\alpha-1}\left(\prod_{j=1}^{m}w_j(t_j)\right)f\left(\overrightarrow{t}\right)d\overrightarrow{t}$$

$$+\sum_{i=1}^{m}A_i(x_1,...,x_m)+\sum_{i=1}^{m}B_i(x_1,...,x_m), \quad (2.11)$$

that is proving the claim. $\square$

We make

Remark 2.7. Let $f \in C^1\left(\prod_{i=1}^m [a_i, b_i]\right)$, $\alpha \geq 1$, $x_i \in [a_i, b_i)$, $i = 1, ..., m$. Denote by

$$\|f\|_\infty^{\sup} := \sup_{x \in \prod_{i=1}^m [a_i, b_i]} |f(x)|. \tag{2.12}$$

From (1.2) we get that

$$|P_w(x,t)| \leq \left\{ \begin{array}{ll} W(x), & a \leq t \leq x, \\ 1 - W(x), & x < t \leq b \end{array} \right\}$$

$$\leq \max\{W(x), 1 - W(x)\} = \frac{1 + |2W(x) - 1|}{2}. \tag{2.13}$$

That is

$$|P_w(x,t)| \leq \frac{1 + |2W(x) - 1|}{2}, \tag{2.14}$$

for all $t \in [a,b]$, where $x \in [a,b]$ is fixed.

Consequently it holds

$$|P_{w_i}(x_i, t_i)| \leq \frac{1 + |2W_i(x_i) - 1|}{2}, \quad i = 1, ..., m. \tag{2.15}$$

Assume here that

$$w_j(t_j) \leq K_j, \tag{2.16}$$

for all $t_j \in [a_j, b_j]$, where $K_j > 0$, $j = 1, ..., m$.

Therefore we derive

$$|B_i(x_1, ..., x_m)| \leq \left(\prod_{j=1}^i (b_j - x_j)\right)^{1-\alpha} \left(\prod_{j=1}^{i-1} K_j\right)$$

$$\cdot \left(\frac{1 + |2W_i(x_i) - 1|}{2}\right) \left\|\frac{\partial f}{\partial x_i}\right\|_\infty^{\sup} \prod_{j=1}^i \left(\int_{a_j}^{b_j} (b_j - t_j)^{\alpha-1} dt_j\right). \tag{2.17}$$

That is

$$|B_i(x_1, ..., x_m)| \leq \left(\prod_{j=1}^i (b_j - x_j)\right)^{1-\alpha} \left(\frac{\prod_{j=1}^i (b_j - a_j)^\alpha}{\alpha^i}\right) \left(\prod_{j=1}^{i-1} K_j\right)$$

$$\cdot \left(\frac{1 + |2W_i(x_i) - 1|}{2}\right) \left\|\frac{\partial f}{\partial x_i}\right\|_\infty^{\sup}, \tag{2.18}$$

for all $i = 1, ..., m$.

Based on the above and Theorem 2.6 we have established the following multivariate weighted fractional Ostrowski type inequality.

Theorem 2.8. *Let $f \in C^1\left(\prod_{i=1}^m [a_i, b_i]\right)$, $\alpha \geq 1$, $x_i \in [a_i, b_i)$, $i = 1, ..., m$. Here P_{w_i} corresponds to $[a_i, b_i]$, $i = 1, ..., m$, and it is as in (1.2). Assume that $w_j(t_j) \leq K_j$, for all $t_j \in [a_j, b_j]$, where $K_j > 0$, $j = 1, ..., m$. And $A_i(x_1, ..., x_m)$ is as in (2.4), $i = 1, ..., m$. Then*

$$\left| f(x_1, ..., x_m) - \left(\prod_{j=1}^m (b_j - x_j)\right)^{1-\alpha} (\Gamma(\alpha))^m \left(I_{a+}^{\alpha}\left(\prod_{j=1}^m w_j\right) f\right)(b) \right.$$

$$\left. - \sum_{i=1}^m A_i(x_1, ..., x_m) \right| \leq \sum_{i=1}^m \left\{ \left(\prod_{j=1}^i (b_j - x_j)\right)^{1-\alpha} \left(\frac{\prod_{j=1}^i (b_j - a_j)^{\alpha}}{\alpha^i}\right) \right.$$

$$\left. \cdot \left(\prod_{j=1}^{i-1} K_j\right) \left(\frac{1 + |2W_i(x_i) - 1|}{2}\right) \left\| \frac{\partial f}{\partial x_i} \right\|_{\infty}^{\sup} \right\}. \tag{2.19}$$

3. Application

Here we operate on $[0,1]^m$, $m \in \mathbb{N} - \{1\}$. We notice that

$$\int_0^1 \left(\frac{e^{-x}}{1 - e^{-1}}\right) dx = 1, \tag{3.1}$$

and

$$\frac{e^{-x}}{1 - e^{-1}} \leq \frac{1}{1 - e^{-1}}, \text{ for all } x \in [0,1]. \tag{3.2}$$

So here we choose as w_j the probability density function

$$w_j^*(t_j) := \frac{e^{-t_j}}{1 - e^{-1}}, \tag{3.3}$$

$j = 1, ..., m$, $t_j \in [0,1]$.

So we have the corrsponding W_j as

$$W_j^*(t_j) = \frac{1 - e^{-t_j}}{1 - e^{-1}}, \ \ t_j \in [0,1], \tag{3.4}$$

and the corresponding P_{w_j} as

$$P_{w_j}^*(x_j, t_j) = \begin{cases} \frac{1 - e^{-t_j}}{1 - e^{-1}}, & 0 \leq t_j \leq x_j, \\ \frac{e^{-1} - e^{-t_j}}{1 - e^{-1}}, & x_j < t_j \leq 1, \end{cases} \tag{3.5}$$

$j = 1, ..., m$.

Set $\overrightarrow{0} = (0, ..., 0)$ and $\overrightarrow{1} = (1, ..., 1)$.

First we apply Theorem 2.6.

We have

Theorem 3.1. *Let $f \in C^1([0,1]^m)$, $\alpha \geq 1$, $x_i \in [0,1)$, $i = 1, ..., m$. Then*

$$f(x_1, ..., x_m) = \left(\prod_{j=1}^{m}(1-x_j)\right)^{1-\alpha}\left(\frac{\Gamma(\alpha)}{1-e^{-1}}\right)^m \left(I^{\alpha}_{\overrightarrow{0}+}\left(e^{-\sum_{j=1}^{m} t_j} f(\cdot)\right)\right)\left(\overrightarrow{1}\right)$$

$$+\sum_{i=1}^{m} A_i^*(x_1, ..., x_m) + \sum_{i=1}^{m} B_i^*(x_1, ..., x_m), \tag{3.6}$$

where for $i = 1, ..., m$:

$$A_i^*(x_1, ..., x_m) := \frac{-(\alpha-1)}{(1-e^{-1})^{i-1}}\left(\prod_{j=1}^{i}(1-x_j)\right)^{1-\alpha}\int_{[0,1]^i}\left(\prod_{j=1}^{i-1}(1-t_j)\right)^{\alpha-1} \tag{3.7}$$

$$\cdot (1-t_i)^{\alpha-2} e^{-\sum_{j=1}^{i-1} t_j} P^*_{w_i}(x_i, t_i) f(t_1, ..., t_i, x_{i+1}, ..., x_m)\, dt_1...dt_i,$$

and

$$B_i^*(x_1, ..., x_m) := \frac{\left(\prod_{j=1}^{i}(1-x_j)\right)^{1-\alpha}}{(1-e^{-1})^{i-1}}\int_{[0,1]^i}\left(\prod_{j=1}^{i}(1-t_j)\right)^{\alpha-1} \tag{3.8}$$

$$\cdot e^{-\sum_{j=1}^{i-1} t_j} P^*_{w_i}(x_i, t_i)\frac{\partial f}{\partial x_i}(t_1, ..., t_i, x_{i+1}, ..., x_m)\, dt_1...dt_i.$$

Above we set $\sum_{i=1}^{0} \cdot = 0$.

Finally we apply Theorem 2.8.

Theorem 3.2. *Let $f \in C^1([0,1]^m)$, $\alpha \geq 1$, $x_i \in [0,1)$, $i = 1, ..., m$. Here $P^*_{w_i}$ is as in (3.5) and $A_i^*(x_1, ..., x_m)$ as in (3.7), $i = 1, ..., m$. Then*

$$\left| f(x_1, ..., x_m) - \left(\prod_{j=1}^{m}(1-x_j)\right)^{1-\alpha}\left(\frac{\Gamma(\alpha)}{1-e^{-1}}\right)^m \left(I^{\alpha}_{\overrightarrow{0}+}\left(e^{-\sum_{j=1}^{m} t_j} f(\cdot)\right)\right)\left(\overrightarrow{1}\right)\right.$$

$$\left. -\sum_{i=1}^{m} A_i^*(x_1, ..., x_m)\right| \leq \sum_{i=1}^{m}\left\{\frac{\left(\prod_{j=1}^{i}(1-x_j)\right)^{1-\alpha}}{\alpha^i (1-e^{-1})^{i-1}}\right.$$

$$\left.\cdot\left(\frac{1+|2W_i^*(x_i)-1|}{2}\right)\left\|\frac{\partial f}{\partial x_i}\right\|_{\infty}^{\sup}\right\}. \tag{3.9}$$

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On some integral inequalities for twice differentiable mappings

Mehmet Zeki Sarıkaya, Erhan Set and M. Emin Ozdemir

Abstract. In this paper, we establish several new inequalities for some twice differantiable mappings. Then, we apply these inequalities to obtain new midpoint, trapezoid and perturbed trapezoid rules. Finally, some applications for special means of real numbers are provided.

Mathematics Subject Classification (2010): 26D15, 41A55, 26D10.

Keywords: Convex function, Ostrowski inequality and special means.

1. Introduction

In 1938 Ostrowski obtained a bound for the absolute value of the difference of a function to its average over a finite interval. The theorem is well known in the literature as Ostrowski's integral inequality [14]:

Theorem 1.1. *Let $f : [a,b] \to \mathbb{R}$ be a differentiable mapping on (a,b) whose derivative $f' : (a,b) \to \mathbb{R}$ is bounded on (a,b), i.e., $\|f'\|_\infty = \sup_{t\in(a,b)} |f'(t)| < \infty$. Then, the inequality holds:*

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)dt \right| \le \left[\frac{1}{4} + \frac{(x-\frac{a+b}{2})^2}{(b-a)^2} \right](b-a)\|f'\|_\infty \tag{1.1}$$

for all $x \in [a,b]$. The constant $\frac{1}{4}$ is the best possible.

In 1976, Milovanović and Pečarić proved a generalization of the Ostrowski inequality for n-times differentiable mappings (see for example [13, p.468]). Dragomir and Wang ([10], [11]) extended the result (1.1) and applied the extended result to numerical quadrature rules and to the estimation of error bounds for some special means. Also, Sofo and Dragomir [18] extended the result (1.1) in the L_p norm. Dragomir ([6]-[8]) further extended the (1.1) to incorporate mappings of bounded variation, Lipschitzian and monotonic mappings. For recent results and generalizations conserning Ostrowski's integral inequality see [1]-[13], [18], [19], and the references therein.

In [4], Cerone and Dragomir find the following perturbed trapezoid inequalities:

Theorem 1.2. *Let $f : [a,b] \to \mathbb{R}$ be such that the derivative f' is absolutely continuous on $[a,b]$. Then, the inequality holds:*

$$\left| \int_a^b f(t)dt - \frac{b-a}{2}\left[f(b)+f(a)\right] + \frac{(b-a)^2}{8}\left[f'(b)-f'(a)\right] \right| \tag{1.2}$$

$$\leq \begin{cases} \frac{(b-a)^3}{24}\left\|f''\right\|_\infty & \text{if } \quad f'' \in L_\infty[a,b] \\ \frac{(b-a)^{2+\frac{1}{q}}}{8(2q+1)^{\frac{1}{q}}}\left\|f''\right\|_p & \text{if } \quad f'' \in L_p[a,b],\ p>1,\ \frac{1}{p}+\frac{1}{q}=1 \\ \frac{(b-a)^2}{8}\left\|f''\right\|_1 & \text{if } \quad f'' \in L_1[a,b] \end{cases}$$

for all $t \in [a,b]$.

In recent years a number of authors have considered an error analysis for some known and some new quadrature formulas. They used an approach from the inequalities point of view. For example, the midpoint quadrature rule is considered in [4],[15],[17], the trapezoid rule is considered in [4],[16],[20]. In most cases estimations of errors for these quadrature rules are obtained by means of derivatives and integrands.

In this article, we first derive a general integral identity for twice derivatives functions. Then, we apply this identity to obtain our results and using functions whose twice derivatives in absolute value at certain powers are convex, we obtained new inequalities related to the Ostrowski's type inequality. Finally, we gave some applications for special means of real numbers.

2. Main results

In order to prove our main results, we need the following Lemma:

Lemma 2.1. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a twice differentiable function on I° with $f'' \in L_1[a,b]$, then*

$$\frac{1}{b-a}\int_a^b f(u)du - \frac{1}{2}\left[f(x)+f(a+b-x)\right]$$

$$+\frac{1}{2}(x-\frac{a+3b}{4})\left[f'(x)-f'(a+b-x)\right]$$

$$= \frac{(b-a)^2}{2}\int_0^1 k(t)\, f''(ta+(1-t)b)dt \tag{2.1}$$

where

$$k(t) := \begin{cases} t^2, & 0 \leq t < \frac{b-x}{b-a} \\ \left(t-\frac{1}{2}\right)^2, & \frac{b-x}{b-a} \leq t < \frac{x-a}{b-a} \\ (t-1)^2, & \frac{x-a}{b-a} \leq t \leq 1 \end{cases}$$

for any $x \in \left[\frac{a+b}{2}, b\right]$.

Proof. It suffices to note that

$$\begin{aligned} I &= \int_0^1 k(t) f''(ta+(1-t)b)dt \\ &= \int_0^{\frac{b-x}{b-a}} t^2 f''(ta+(1-t)b)dt + \int_{\frac{b-x}{b-a}}^{\frac{x-a}{b-a}} \left(t-\frac{1}{2}\right)^2 f''(ta+(1-t)b)dt \\ &\quad + \int_{\frac{x-a}{b-a}}^1 (t-1)^2 f''(ta+(1-t)b)dt \\ &= I_1 + I_2 + I_3. \end{aligned}$$

By inegration by parts, we have the following identity

$$\begin{aligned} I_1 &= \int_0^{\frac{b-x}{b-a}} t^2 f''(ta+(1-t)b)dt \\ &= \frac{t^2}{(a-b)} f'(ta+(1-t)b) \Big|_0^{\frac{b-x}{b-a}} - \frac{2}{a-b}\int_0^{\frac{b-x}{b-a}} t f'(ta+(1-t)b)dt \\ &= \frac{1}{(a-b)}\left(\frac{b-x}{b-a}\right)^2 f'(x) \\ &\quad - \frac{2}{a-b}\left[\frac{t}{(a-b)} f(ta+(1-t)b)\Big|_0^{\frac{b-x}{b-a}} - \frac{1}{a-b}\int_0^{\frac{b-x}{b-a}} f(ta+(1-t)b)dt\right] \\ &= -\frac{(b-x)^2}{(b-a)^3} f'(x) - \frac{2(b-x)}{(b-a)^3} f(x) + \frac{2}{(b-a)^2}\int_0^{\frac{b-x}{b-a}} f(ta+(1-t)b)dt. \end{aligned}$$

Similarly, we observe that

$$I_2 = \int_{\frac{b-x}{b-a}}^{\frac{x-a}{b-a}} \left(t-\frac{1}{2}\right)^2 f''(ta+(1-t)b)dt$$

$$= \frac{(a+b-2x)^2}{4(b-a)^3}\left[f'(x) - f'(a+b-x)\right] + \frac{(a+b-2x)}{(b-a)^3}\left[f(x)+f(a+b-x)\right]$$

$$+\frac{2}{(b-a)^2}\int_{\frac{b-x}{b-a}}^{\frac{x-a}{b-a}} f(ta+(1-t)b)dt$$

and

$$I_3 = \int_{\frac{x-a}{b-a}}^{1} (t-1)^2 f''(ta+(1-t)b)dt$$

$$= \frac{(b-x)^2}{(b-a)^3} f'(a+b-x) - \frac{2(b-x)}{(b-a)^3} f(a+b-x) + \frac{2}{(b-a)^2}\int_{\frac{x-a}{b-a}}^{1} f(ta+(1-t)b)dt.$$

Thus, we can write

$$I = I_1 + I_2 + I_3 = \frac{1}{(b-a)^2}\left(x - \frac{a+3b}{4}\right)[f'(x) - f'(a+b-x)]$$

$$-\frac{1}{(b-a)^2}[f(x) + f(a+b-x)] + \frac{2}{(b-a)^2}\int_0^1 f(ta+(1-t)b)dt.$$

Using the change of the variable $u = ta + (1-t)b$ for $t \in [0,1]$ and by multiplying the both sides by $(b-a)^2/2$ which gives the required identity (2.1). □

Now, by using the above lemma, we prove our main theorems:

Theorem 2.2. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a twice differentiable function on I° such that $f'' \in L_1[a,b]$ where $a, b \in I$, $a < b$. If $|f''|$ is convex on $[a,b]$, then the following inequality holds:*

$$\left|\frac{1}{b-a}\int_a^b f(u)du - \frac{1}{2}[f(x) + f(a+b-x)] + \frac{1}{2}(x - \frac{a+3b}{4})[f'(x) - f'(a+b-x)]\right|$$

$$\leq \frac{1}{(b-a)}\left[(b-x)^3 + \left(x - \frac{a+b}{2}\right)^3\right]\left(\frac{|f''(a)| + |f''(b)|}{6}\right) \tag{2.2}$$

for any $x \in \left[\frac{a+b}{2}, b\right]$.

Proof. From Lemma 2.1 and by the definition $k(t)$, we get

$$\left|\frac{1}{b-a}\int_a^b f(u)du - \frac{1}{2}[f(x) + f(a+b-x)] + \frac{1}{2}\left(x - \frac{a+3b}{4}\right)[f'(x) - f'(a+b-x)]\right|$$

$$\leq \frac{(b-a)^2}{2}\int_0^1 |k(t)||f''(ta+(1-t)b)|dt$$

$$= \frac{(b-a)^2}{2}\left\{\int_0^{\frac{b-x}{b-a}} t^2|f''(ta+(1-t)b)|dt + \int_{\frac{b-x}{b-a}}^{\frac{x-a}{b-a}}\left(t - \frac{1}{2}\right)^2|f''(ta+(1-t)b)|dt\right.$$

$$\left.+\int_{\frac{x-a}{b-a}}^{1}(t-1)^2|f''(ta+(1-t)b)|dt\right\}$$

$$= \frac{(b-a)^2}{2}\{J_1 + J_2 + J_3\} \tag{2.3}$$

Investigating the three separate integrals, we may evaluate as follows:

By the convexity of $|f''|$, we arrive at

$$\begin{aligned} J_1 &\leq \int_0^{\frac{b-x}{b-a}} \left(t^3 \left|f''(a)\right| + (t^2 - t^3) \left|f''(b)\right|\right) dt \\ &= \frac{(b-x)^4}{4(b-a)^4} \left|f''(a)\right| + \left(\frac{(b-x)^3}{3(b-a)^3} - \frac{(b-x)^4}{4(b-a)^4}\right) \left|f''(b)\right|, \end{aligned}$$

$$\begin{aligned} J_2 &\leq \int_{\frac{b-x}{b-a}}^{\frac{x-a}{b-a}} \left[\left(t-\frac{1}{2}\right)^2 t \left|f''(a)\right| + \left(t-\frac{1}{2}\right)^2 (1-t) \left|f''(b)\right|\right] dt \\ &= \frac{1}{3(b-a)^3}\left(x-\frac{a+b}{2}\right)^3 \left|f''(a)\right| + \frac{1}{3(b-a)^3}\left(x-\frac{a+b}{2}\right)^3 \left|f''(b)\right|, \end{aligned}$$

$$\begin{aligned} J_3 &\leq \int_{\frac{x-a}{b-a}}^{1} \left[(t-1)^2 t \left|f''(a)\right| + (1-t)^3 \left|f''(b)\right|\right] dt \\ &= \left(\frac{(b-x)^3}{3(b-a)^3} - \frac{(b-x)^4}{4(b-a)^4}\right) \left|f''(a)\right| + \frac{(b-x)^4}{4(b-a)^4} \left|f''(b)\right|. \end{aligned}$$

By rewrite J_1, J_2, J_3 in (2.3), we obtain (2.2) which completes the proof. □

Corollary 2.3 (Perturbed Trapezoid inequality). *Under the assumptions Theorem* 2.2 *with* $x = b$, *we have*

$$\left|\frac{1}{b-a}\int_a^b f(u)du - \frac{f(b)+f(a)}{2} + \frac{(b-a)}{8}[f'(b) - f'(a)]\right| \leq \frac{(b-a)^2}{48}(|f''(a)| + |f''(b)).$$

Remark 2.4. We choose $|f''(x)| \leq M, \quad M > 0$ in Corollary 2.3, then we recapture the first part of the inequality (1.2).

Corollary 2.5 (Trapezoid inequality). *Under the assumptions Theorem* 2.2 *with* $x = b$ *and* $f'(a) = f'(b)$ *in Theorem* 2.2, *we have*

$$\left|\frac{1}{b-a}\int_a^b f(u)du - \frac{f\left(a\right)+f\left(b\right)}{2}\right| \leq \frac{(b-a)^2}{48}(\left|f''\left(a\right)\right| + \left|f''\left(b\right)\right|). \tag{2.4}$$

Corollary 2.6 (Midpoint inequality). *Under the assumptions Theorem* 2.2 *with* $x = \frac{a+b}{2}$ *in Theorem* 2.2, *we have*

$$\left|\frac{1}{b-a}\int_a^b f(u)du - f\left(\frac{a+b}{2}\right)\right| \leq \frac{(b-a)^2}{48}(\left|f''\left(a\right)\right| + \left|f''\left(b\right)\right|). \tag{2.5}$$

Another similar result may be extended in the following theorem

Theorem 2.7. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a twice differentiable function on I° such that $f'' \in L_1[a,b]$ where $a, b \in I$, $a < b$. If $|f''|^q$ is convex on $[a,b]$, $q > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$, then*

$$\begin{aligned}
&\left| \frac{1}{b-a} \int_a^b f(u)du - \frac{1}{2}[f(x) + f(a+b-x)] \right. \\
&\left. + \frac{1}{2}\left(x - \frac{a+3b}{4}\right)[f'(x) - f'(a+b-x)] \right| \qquad (2.6) \\
&\leq \frac{2^{\frac{1}{p}-1}}{(2p+1)^{\frac{1}{p}} (b-a)^{\frac{1}{p}}} \left[(b-x)^{2p+1} \right. \\
&\left. + \left(x - \frac{a+b}{2}\right)^{2p+1} \right]^{\frac{1}{p}} \left(\frac{|f''(a)|^q + |f''(b)|^q}{2} \right)^{\frac{1}{q}}
\end{aligned}$$

for any $x \in \left[\frac{a+b}{2}, b\right]$.

Proof. From Lemma 2.1, by the definition $k(t)$ and using Hölder's inequality, it follows that

$$\begin{aligned}
&\left| \frac{1}{b-a} \int_a^b f(u)du - \frac{1}{2}[f(x) + f(a+b-x)] \right. \\
&\left. + \frac{1}{2}\left(x - \frac{a+3b}{4}\right)[f'(x) - f'(a+b-x)] \right| \qquad (2.7) \\
&\leq \frac{(b-a)^2}{2} \int_0^1 |k(t)||f''(ta + (1-t)b)|dt \\
&\leq \frac{(b-a)^2}{2} \left(\int_0^1 |k(t)|^p dt \right)^{\frac{1}{p}} \left(\int_0^1 |f''(ta + (1-t)b)|^q dt \right)^{\frac{1}{q}}.
\end{aligned}$$

Since $|f''|^q$ is convex on $[a,b]$, we know that for $t \in [0,1]$

$$|f''(ta + (1-t)b)|^q \leq t\,|f''(a)|^q + (1-t)\,|f''(b)|^q,$$

hence, a simple computation shows that

$$\int_0^1 |f''(ta + (1-t)b)|^q \, dt \leq \frac{|f''(a)|^q + |f''(b)|^q}{2} \qquad (2.8)$$

also,

$$\begin{aligned}
\int_0^1 |k(t)|^p \, dt &= \int_0^{\frac{b-x}{b-a}} t^{2p} dt + \int_{\frac{b-x}{b-a}}^{\frac{x-a}{b-a}} \left| t - \frac{1}{2} \right|^{2p} dt + \int_{\frac{x-a}{b-a}}^1 (1-t)^{2p} \, dt \\
& \qquad (2.9) \\
&= \frac{2}{(2p+1)(b-a)^{2p+1}} \left[(b-x)^{2p+1} + \left(x - \frac{a+b}{2}\right)^{2p+1} \right].
\end{aligned}$$

Using (2.8) and (2.9) in (2.7), we obtain (2.6). □

Corollary 2.8 (Perturbed Trapezoid inequality). *Under the assumptions Theorem* 2.7 *with* $x = b$*, we have*

$$\left|\frac{1}{b-a}\int_a^b f(u)du - \frac{f(b)+f(a)}{2} + \frac{(b-a)}{8}[f'(b)-f'(a)]\right| \leq \frac{(b-a)^2}{8(2p+1)^{\frac{1}{p}}}\left(\frac{|f''(a)|^q+|f''(b)|^q}{2}\right)^{\frac{1}{q}}.$$

Corollary 2.9 (Trapezoid inequality). *Under the assumptions Theorem* 2.7 *with* $x = b$ *and* $f'(a) = f'(b)$ *in Theorem* 2.7*, we have*

$$\left|\frac{1}{b-a}\int_a^b f(u)du - \frac{f(a)+f(b)}{2}\right| \leq \frac{(b-a)^2}{8(2p+1)^{\frac{1}{p}}}\left(\frac{|f''(a)|^q+|f''(b)|^q}{2}\right)^{\frac{1}{q}}. \tag{2.10}$$

Corollary 2.10 (Midpoint inequality). *Under the assumptions Theorem* 2.7 *with* $x = \frac{a+b}{2}$ *in Theorem* 2.7*, we have*

$$\left|\frac{1}{b-a}\int_a^b f(u)du - f\left(\frac{a+b}{2}\right)\right| \leq \frac{(b-a)^2}{8\,(2p+1)^{\frac{1}{p}}}\left(\frac{|f''(a)|^q+|f''(b)|^q}{2}\right)^{\frac{1}{q}}. \tag{2.11}$$

Theorem 2.11. *Let* $f : I \subset \mathbb{R} \to \mathbb{R}$ *be a twice differentiable function on* I° *such that* $f'' \in L_1[a,b]$ *where* $a, b \in I$*,* $a < b$*. If* $|f''|^q$ *is convex on* $[a,b]$ *and* $q \geq 1$*, then*

$$\begin{aligned}&\left|\frac{1}{b-a}\int_a^b f(u)du - \frac{1}{2}[f(x)+f(a+b-x)]\right.\\&\left.+\frac{1}{2}(x-\frac{a+3b}{4})[f'(x)-f'(a+b-x)]\right|\\&\leq \frac{1}{3(b-a)}\left[(b-x)^3+\left(x-\frac{a+b}{2}\right)^3\right]\left(\frac{|f''(a)|^q+|f''(b)|^q}{2}\right)^{\frac{1}{q}}\end{aligned} \tag{2.12}$$

for any $x \in \left[\frac{a+b}{2}, b\right]$.

Proof. From Lemma 2.1, by the definition $k(t)$ and using power mean inequality, it follows that

$$\begin{aligned}&\left|\frac{1}{b-a}\int_a^b f(u)du - \frac{1}{2}[f(x)+f(a+b-x)]\right.\\&\left.+\frac{1}{2}(x-\frac{a+3b}{4})[f'(x)-f'(a+b-x)]\right|\\&\leq \frac{(b-a)^2}{2}\int_0^1 |k(t)||f''(ta+(1-t)b)|dt\\&\leq \frac{(b-a)^2}{2}(\int_0^1 |k(t)|dt)^{1-\frac{1}{q}}(\int_0^1 |k(t)||f''(ta+(1-t)b)|^q dt)^{\frac{1}{q}}.\end{aligned} \tag{2.13}$$

Since $|f''|^q$ is convex on $[a,b]$, we know that for $t\in[0,1]$

$$|f''(ta+(1-t)b)|^q \leq t\,|f''(a)|^q+(1-t)\,|f''(b)|^q,$$

hence, by simple computation

$$\begin{aligned}\int_0^1 |k(t)|\,dt &= \int_0^{\frac{b-x}{b-a}} t^2dt+\int_{\frac{b-x}{b-a}}^{\frac{x-a}{b-a}}\left|t-\frac{1}{2}\right|^2dt+\int_{\frac{x-a}{b-a}}^1(1-t)^2\,dt \\ &= \frac{2}{3\,(b-a)^3}\left[(b-x)^3+\left(x-\frac{a+b}{2}\right)^3\right],\end{aligned} \tag{2.14}$$

and

$$\begin{aligned}&\int_0^1 |k(t)|\,|f''(ta+(1-t)b)|^q\,dt \\ \leq\; &\int_0^1 |k(t)|\left(t\,|f''(a)|^q+(1-t)\,|f''(b)|^q\right)dt \\ =\; &\int_0^{\frac{b-x}{b-a}}\left(t^3\,|f''(a)|^q+(t^2-t^3)\,|f''(b)|^q\right)dt \\ &+\int_{\frac{b-x}{b-a}}^{\frac{x-a}{b-a}}\left[\left(t-\frac{1}{2}\right)^2t\,|f''(a)|^q+\left(t-\frac{1}{2}\right)^2(1-t)\,|f''(b)|^q\right]dt \\ &+\int_{\frac{x-a}{b-a}}^1\left[(t-1)^2\,t\,|f''(a)|^q+(1-t)^3\,|f''(b)|^q\right]dt \\ =\; &\frac{(b-x)^4}{4(b-a)^4}\,|f''(a)|^q+\left(\frac{(b-x)^3}{3(b-a)^3}-\frac{(b-x)^4}{4(b-a)^4}\right)|f''(b)|^q \\ &+\frac{1}{3(b-a)^3}\left(x-\frac{a+b}{2}\right)^3|f''(a)|^q+\frac{1}{3(b-a)^3}\left(x-\frac{a+b}{2}\right)^3|f''(b)|^q \\ &+\left(\frac{(b-x)^3}{3(b-a)^3}-\frac{(b-x)^4}{4(b-a)^4}\right)|f''(a)|^q+\frac{(b-x)^4}{4(b-a)^4}\,|f''(b)|^q \\ =\; &\frac{1}{3\,(b-a)^3}\left[(b-x)^3+\left(x-\frac{a+b}{2}\right)^3\right]\left(|f''(a)|^q+|f''(b)|^q\right).\end{aligned} \tag{2.15}$$

Using (2.14) and (2.15) in (2.13), we obtain (2.12). □

Corollary 2.12. *Under the assumptions Theorem* 2.11 *with* $x = b$, *we have*

$$\left|\frac{1}{b-a}\int_a^b f(u)du - \frac{f(b)+f(a)}{2} + \frac{(b-a)}{8}[f'(b)-f'(a)]\right|$$
$$\le \frac{(b-a)^2}{24}\left(\frac{|f''(a)|^q + |f''(b)|^q}{2}\right)^{\frac{1}{q}}.$$

Corollary 2.13. *Under the assumptions Theorem* 2.11 *with* $x = b$ *and* $f'(a) = f'(b)$ *in Theorem* 2.11, *we have*

$$\left|\frac{1}{b-a}\int_a^b f(u)du - \frac{f(a)+f(b)}{2}\right| \le \frac{(b-a)^2}{24}\left(\frac{|f''(a)|^q + |f''(b)|^q}{2}\right)^{\frac{1}{q}}. \tag{2.16}$$

Corollary 2.14. *Under the assumptions Theorem* 2.11 *with* $x = \frac{a+b}{2}$ *in Theorem* 2.11, *we have*

$$\left|\frac{1}{b-a}\int_a^b f(u)du - f\left(\frac{a+b}{2}\right)\right| \le \frac{(b-a)^2}{24}\left(\frac{|f''(a)|^q + |f''(b)|^q}{2}\right)^{\frac{1}{q}}. \tag{2.17}$$

Remark 2.15. When considering that $24 > 8(2p+1)^{\frac{1}{p}}$, $p > 1$, the bounded of Corrolaries 2.12-2.14 are better than Corrolaries 2.8-2.10 respectively.

Open Problem. Under what conditions that the result of Theorem 2.7 and Theorem 2.11 is comparable.

3. Applications for special means

Recall the following means:
(a) The arithmetic mean

$$A = A(a,b) := \frac{a+b}{2}, \ a,b \ge 0;$$

(b) The geometric mean

$$G = G(a,b) := \sqrt{ab}, \ a,b \ge 0;$$

(c) The harmonic mean

$$H = H(a,b) := \frac{2ab}{a+b}, \ a,b > 0;$$

(d) The logarithmic mean

$$L = L(a,b) := \begin{cases} a & if \quad a = b \\ \frac{b-a}{\ln b - \ln a} & if \quad a \ne b \end{cases}, \qquad a,b > 0;$$

(e) The identric mean

$$I = I(a,b) := \begin{cases} a & if \quad a = b \\ \frac{1}{e}\left(\frac{b^b}{a^a}\right)^{\frac{1}{b-a}} & if \quad a \neq b \end{cases}, \quad a,b > 0;$$

(f) The $p-$logarithmic mean:

$$L_p = L_p(a,b) := \begin{cases} \left[\frac{b^{p+1}-a^{p+1}}{(p+1)(b-a)}\right]^{\frac{1}{p}} & \text{if} \quad a \neq b \\ a & \text{if} \quad a = b \end{cases}, \quad p \in \mathbb{R}\diagdown\{-1,0\}\,; \ a,b > 0.$$

It is also known that L_p is monotonically nondecreasing in $p \in \mathbb{R}$ with $L_{-1} := L$ and $L_0 := I$. The following simple relationships are known in the literature

$$H \leq G \leq L \leq I \leq A.$$

Now, using the results of Section 2, some new inequalities are derived for the above means.

Proposition 3.1. *Let $p \geq 2$ and $0 < a < b$. Then we have the inequality:*

$$\left|L_p^p(a,b) - A\left(a^p, b^p\right)\right| \leq p\,(p-1)\,\frac{(b-a)^2}{24}A\left(a^{p-2}, b^{p-2}\right).$$

Proof. The assertion follows from (2.4) applied for $f(x) = x^p$, $x \in [a,b]$ .We omitted the details. ☐

Proposition 3.2. *Let $0 < a < b$. Then we have the inequality:*

$$\left|L^{-1}(a,b) - A^{-1}(a,b)\right| \leq \frac{(b-a)^2}{12}A\left(a^{-3}, b^{-3}\right).$$

Proof. The assertion follows from (2.5) applied for $f(x) = \frac{1}{x}$, $x \in [a,b]$. We omitted the details. ☐

Proposition 3.3. *Let $q > 1$ and $0 < a < b$. Then we have the inequality:*

$$\left|\ln I(a,b) - \ln G(a,b)\right| \leq \frac{(b-a)^2}{8\,(2p+1)^{\frac{1}{p}}}\left[A\left(a^{-2q}, b^{-2q}\right)\right]^{1/q}.$$

Proof. The assertion follows from (2.10) applied for $f(x) = -\ln x$, $x \in [a,b]$. ☐

Proposition 3.4. *Let $p \geq 2$ and $0 < a < b$. Then we have the inequality:*

$$\left|L_p^p(a,b) - A^p(a,b)\right| \leq p\,(p-1)\,\frac{(b-a)^2}{8\,(2p+1)^{1/p}}\left[A\left(a^{q(p-2)}, b^{q(p-2)}\right)\right]^{1/q}.$$

Proof. The assertion follows from (2.11) applied for $f(x) = x^p$, $x \in [a,b]$. ☐

Proposition 3.5. *Let $p > 1$ and $0 < a < b$. Then we have the inequality:*

$$\left|L^{-1}(a,b) - H^{-1}(a,b)\right| \leq \frac{(b-a)^2}{12}\left[A\left(a^{-3q}, b^{-3q}\right)\right]^{1/q}.$$

Proof. The assertion follows from (2.16) applied for $f(x) = \frac{1}{x}$, $x \in [a, b]$. $\square$

Proposition 3.6. *Let $q > 1$, and $0 < a < b$. Then we have the inequality:*

$$|\ln I(a,b) - \ln A(a,b)| \leq \frac{(b-a)^2}{24} \left[A\left(a^{-2q}, b^{-2q}\right)\right]^{1/q}.$$

Proof. The assertion follows from (2.17) applied for $f(x) = -\ln x$, $x \in [a, b]$. $\square$

4. Applications for composite quadrature formula

Let d be a division $a = x_0 < x_1 < ... < x_{n-1} < x_n = b$ of the interval $[a, b]$ and $\xi = (\xi_0, ..., \xi_{n-1})$ a sequence of intermediate points, $\xi_i \in [x_i, x_{i+1}]$, $i = \overline{0, n-1}$. Then the following result holds:

Theorem 4.1. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a twice differentiable function on I° such that $f'' \in L_1[a, b]$ where $a, b \in I$, $a < b$. If $|f''|$ is convex on $[a, b]$ then we have*

$$\int_a^b f(u)du = A(f, f', d, \xi) + R(f, f', d, \xi)$$

where

$$\begin{aligned} A(f, f', d, \xi) \quad : &= \sum_{i=0}^{n-1} \frac{h_i}{2} \left[f(\xi_i) + f(x_i + x_{i+1} - \xi_i)\right] \\ &- \sum_{i=0}^{n-1} \frac{h_i}{2} \left(\xi_i - \frac{x_i + 3x_{i+1}}{4}\right) \left[f'(\xi_i) - f'(x_i + x_{i+1} - \xi_i)\right]. \end{aligned}$$

The remainder $R(f, f', d, \xi)$ satisfies the estimation:

$$|R(f, f', d, \xi)| \leq \sum_{i=0}^{n-1} \left[(x_{i+1} - \xi_i)^3 + \left(\xi_i - \frac{x_i + x_{i+1}}{2}\right)^3\right] \left(\frac{|f''(x_i)| + |f''(x_{i+1})|}{6}\right) \tag{4.1}$$

for any choice ξ of the intermediate points.

Proof. Apply Theorem 2.2 on the interval $[x_i, x_{i+1}]$, $i = \overline{0, n-1}$ to get

$$\left| \frac{h_i}{2} \left[f(\xi_i) + f(x_i + x_{i+1} - \xi_i)\right] - \frac{h_i}{2} \left(\xi_i - \frac{x_i + 3x_{i+1}}{4}\right) \right.$$

$$\left. \times \left[f'(\xi_i) - f'(x_i + x_{i+1} - \xi_i)\right] - \int_a^b f(u)du \right|$$

$$\leq \left[(x_{i+1} - \xi_i)^3 + \left(\xi_i - \frac{x_i + x_{i+1}}{2}\right)^3\right] \left(\frac{|f''(x_i)| + |f''(x_{i+1})|}{6}\right).$$

$\square$

Summing the above inequalities over i from 0 to $n-1$ and using the generalized triangle inequality, we get the desired estimation (4.1).

Corollary 4.2. *The following perturbed trapezoid rule holds:*

$$\int_a^b f(u)du = T(f,f',d) + R_T(f,f',d)$$

where

$$T(f,f',d) := \sum_{i=0}^{n-1} \frac{h_i}{2}\left[f(x_i) + f(x_{i+1})\right] - \sum_{i=0}^{n-1} \frac{(h_i)^2}{8}\left[f'(x_{i+1}) - f'(x_i)\right]$$

and the remainder term $R_T(f,f',d)$ satisfies the estimation,

$$R_T(f,f',d) \leq \sum_{i=0}^{n-1} \frac{(h_i)^3}{48}(|f''(x_i)| + |f''(x_{i+1})|).$$

Corollary 4.3. *The following midpoint rule holds:*

$$\int_a^b f(u)du = M(f,d) + R_M(f,d)$$

where

$$M(f,d) := \sum_{i=0}^{n-1} h_i \left[f(\frac{x_i + x_{i+1}}{2})\right]$$

and the remainder term $R_M(f,d)$ satisfies the estimation,

$$R_M(f,d) \leq \sum_{i=0}^{n-1} \frac{(h_i)^3}{48}(|f''(x_i)| + |f''(x_{i+1})|).$$

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Fekete-Szegő problem for a new class of analytic functions with complex order defined by certain differential operator

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Abstract. In this paper, we obtain Fekete-Szegő inequalities for a new class of analytic functions $f \in \mathcal{A}$ for which $1 + \frac{1}{b}[(1-\gamma)\frac{D_\lambda^n(f*g)(z)}{z} + \gamma(D_\lambda^n(f*g)(z))' - 1]$ $(\gamma, \lambda \geq 0; b \in \mathbb{C}^* = \mathbb{C}\setminus\{0\}; n \in \mathbb{N}_0; z \in U)$ lies in a region starlike with respect to 1 and is symmetric with respect to the real axis.

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1. Introduction

Let $\mathcal{A}$ denote the class of functions f of the form:

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \tag{1.1}$$

which are analytic in the open unit disc $U = \{z \in \mathbb{C} \text{ and } |z| < 1\}$. Further let S denote the family of functions of the form (1.1) which are univalent in U, and $g \in \mathcal{A}$ be given by

$$g(z) = z + \sum_{k=2}^{\infty} g_k z^k. \tag{1.2}$$

A classical theorem of Fekete-Szegő [8] states that, for $f \in S$ given by (1.1), that

$$\left|a_3 - \mu a_2^2\right| \leq \begin{cases} 3 - 4\mu, & if\ \mu \leq 0, \\ 1 + 2\exp\left(\dfrac{-2\mu}{1-\mu}\right), & if\ 0 \leq \mu \leq 1, \\ 4\mu - 3, & if\ \mu \geq 1. \end{cases} \tag{1.3}$$

The result is sharp.

Given two functions f and g, which are analytic in U with $f(0) = g(0)$, the function f is said to be subordinate to g if there exists a function w, analytic in U, such that $w(0) = 0$ and $|w(z)| < 1\,(z \in U)$ and $f(z) = g(w(z))$ $(z \in U)$. We denote this subordination by $f(z) \prec g(z)$ ([10]).
Let φ be an analytic function with positive real part on U, which satisfies $\varphi(0) = 1$ and $\varphi'(0) > 0$, and which maps the unit disc U onto a region starlike with respect to 1 and symmetric with respect to the real axis. Let $S^*(\varphi)$ be the class of functions $f \in S$ for which

$$\frac{zf'(z)}{f(z)} \prec \varphi(z), \tag{1.4}$$

and $C(\varphi)$ be the class of functions $f \in S$ for which

$$1 + \frac{zf''(z)}{f'(z)} \prec \varphi(z). \tag{1.5}$$

The classes of $S^*(\varphi)$ and $C(\varphi)$ were introduced and studied by Ma and Minda [9]. The familier class $S^*(\alpha)$ of starlike functions of order α and the class $C(\alpha)$ of convex functions of order α $(0 \le \alpha < 1)$ are the special cases of $S^*(\varphi)$ and $C(\varphi)$, respectively, when $\varphi(z) = \frac{1+(1-2\alpha)z}{1-z}$ $(0 \le \alpha < 1)$.
Ma and Minda [9] have obtained the Fekete-Szegő problem for the functions in the class $C(\varphi)$.

Definition 1.1. (Hadamard Product or Convolution) *Given two functions f and g in the class $\mathcal{A}$, where f is given by (1.1) and g is given by (1.2) the Hadamard product (or convolution) of f and g is defined (as usual) by*

$$(f * g)(z) = z + \sum_{k=2}^{\infty} a_k g_k z^k = (g * f)(z). \tag{1.6}$$

For the functions f and g defined by (1.1) and (1.2) respectively, the linear operator $D_\lambda^n : \mathcal{A} \longrightarrow \mathcal{A}$ $(\lambda \ge 0; n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}, \mathbb{N} = \{1, 2, 3, ...\})$ is defined by(see [4], see also [7, with $p = 1$]):

$$\begin{aligned} D_\lambda^0(f * g)(z) &= (f * g)(z), \\ D_\lambda^n(f * g)(z) &= D_\lambda(D_\lambda^{n-1}(f * g)(z)) \\ &= z + \sum_{k=2}^{\infty} [1 + \lambda(k-1)]^n a_k g_k z^k \ (\lambda \ge 0; n \in \mathbb{N}_0). \end{aligned} \tag{1.7}$$

Remark 1.2. (i) Taking $g(z) = \dfrac{z}{1-z}$, then operator $D_\lambda^n(f * \dfrac{z}{1-z})(z) = D_\lambda^n f(z)$, was introduced and studied by Al-Oboudi [2];
(ii) Taking $g(z) = \dfrac{z}{1-z}$ and $\lambda = 1$, then operator $D_1^n(f * \dfrac{z}{1-z})(z) = D^n f(z)$, was introduced by Sălăgean [12].

Using the operator D_λ^n we introduce a new class of analytic functions with complex order as following:

Definition 1.3. *For $b \in \mathbb{C}^* = \mathbb{C} \setminus \{0\}$ let the class $M_\lambda^n(f, g; \gamma, b; \varphi)$ denote the subclass of $\mathcal{A}$ consisting of functions f of the form (1.1) and g of the form (1.2) and satisfying the following subordination:*

$$1 + \frac{1}{b}\left[(1-\gamma)\frac{D_\lambda^n(f*g)(z)}{z} + \gamma\left(D_\lambda^n(f*g)(z)\right)' - 1\right] \prec \varphi(z), \tag{1.8}$$

$$(\gamma, \lambda \geq 0;\ n \in \mathbb{N}_0).$$

Specializing the parameters γ, λ, b, n, g and φ, we obtain the following subclasses studied by various authors:

(i) $M_\lambda^0\left(f, z + \sum\limits_{k=2}^{\infty} k^n z^k; \gamma, b; \frac{1+Az}{1+Bz}\right) = M_1^n\left(f, \frac{z}{1-z}; \gamma, b; \frac{1+Az}{1+Bz}\right)$ $= G_n(\gamma, b, A, B)$ $(\gamma, \lambda \geq 0, -1 \leq B < A \leq 1, b \in \mathbb{C}^*, n \in \mathbb{N}_0)$ (Sivasubramanian et al. [14]);

(ii) $M_\lambda^0\left(f, g; \gamma, b; \frac{1+(1-2\alpha)z}{1-z}\right) = S(f, g; \gamma, \alpha, b)$ $(0 \leq \alpha < 1, \gamma \geq 0, b \in \mathbb{C}^*)$ (Aouf et al. [5]);

(iii) $M_\lambda^0\left(f, z + \sum\limits_{k=2}^{\infty} k^n z^k; \gamma, b; \frac{1+z}{1-z}\right) = M_1^n\left(f, \frac{z}{1-z}; \gamma, b; \frac{1+z}{1-z}\right) = G_n(\gamma, b)$ $(\gamma \geq 0, b \in \mathbb{C}^*, n \in \mathbb{N}_0)$ (Aouf [3]);

(iv) $M_\lambda^0\left(f, \frac{z}{1-z}; 1, b; (1-\ell)\frac{1+Az}{1+Bz} + \ell\right) = R_\ell^b(A, B)$ $(b \in \mathbb{C}^*, 0 \leq \ell < 1, -1 \leq B < A \leq 1)$ (Reddy and Reddy [11]);

(v) $M_\lambda^0\left(f, \frac{z}{1-z}; 1, b; \varphi\right) = R_b(\varphi)$ $(b \in \mathbb{C}^*)$ (Ali et al. [1]).

Also we note that:

(i) If $g(z) = z + \sum\limits_{k=2}^{\infty} \Psi_k(\alpha_1) z^k$ (or $g_k = \Psi_k(\alpha_1)$), where

$$\Psi_k(\alpha_1) = \frac{(\alpha_1)_{k-1} \cdots\cdots (\alpha_q)_{k-1}}{(\beta_1)_{k-1} \cdots\cdots (\beta_s)_{k-1}(k-1)!} \tag{1.9}$$

$(\alpha_i > 0, i = 1, ..., q; \beta_j > 0, j = 1, ..., s; q \leq s+1; q, s \in \mathbb{N} = \{1, 2, ...\})$, where $(\nu)_k$ is the Pochhammer symbol defined in terms to the Gamma function Γ, by

$$(\nu)_k = \frac{\Gamma(\nu+k)}{\Gamma(\nu)} = \begin{cases} 1, & \text{if } k = 0, \\ \nu(\nu+1)(\nu+2)...(\nu+k-1), & \text{if } k \in \mathbb{N}, \end{cases}$$

then the class $M_\lambda^n(f, z + \sum\limits_{k=2}^{\infty} \Psi_k(\alpha_1) z^k; \gamma, b; \varphi)$ reduces to the class

$$\begin{aligned} &M_{\lambda,q,s}^n([\alpha_1]; \gamma, b; \varphi) \\ = &\left\{ f \in \mathcal{A} : 1 + \frac{1}{b}\left[(1-\gamma)\frac{D_\lambda^n(\alpha_1, \beta_1) f(z)}{z} + \gamma(D_\lambda^n(\alpha_1, \beta_1) f(z))' - 1\right] \prec \varphi(z), \right. \\ &\left. \gamma, \lambda \geq 0; b \in \mathbb{C}^*; n \in \mathbb{N}_0 \right\}, \end{aligned}$$

where, the operator $D_\lambda^n(\alpha_1,\beta_1)$ was defined as (see Selvaraj and Karthikeyan [13], see also El-Ashwah and Aouf [6]):

$$D_\lambda^n(\alpha_1,\beta_1)f(z)=z+\sum_{k=2}^{\infty}[1+\lambda(k-1)]^n\frac{(\alpha_1)_{k-1}\cdots(\alpha_q)_{k-1}}{(\beta_1)_{k-1}\cdots(\beta_s)_{k-1}(1)_{k-1}}a_kz^k$$

(ii) $M_\lambda^n(f,g;1,b;\varphi)=G_\lambda^n(f,g;b;\varphi)=\{f(z)\in\mathcal{A}:1+\frac{1}{b}[(D_\lambda^n(f*g)(z))'-1]\prec\varphi(z)$ $(\lambda\geq 0;b\in\mathbb{C}^*;n\in\mathbb{N}_0)\}$;

(iii) $M_\lambda^n(f,g;0,b;\varphi)=R_\lambda^n(f,g;b;\varphi)=\{f(z)\in\mathcal{A}:1+\frac{1}{b}[\frac{D_\lambda^n(f*g)(z)}{z}-1]\prec\varphi(z)$ $(\lambda\geq 0;b\in\mathbb{C}^*;n\in\mathbb{N}_0)\}$;

(iv) $M_\lambda^n\left(f,g;\gamma,(1-\rho)\cos\eta e^{-i\eta};\varphi\right)=E_{\lambda,\rho}^{n,\eta}(f,g;\gamma;\varphi)=\{f(z\in\mathcal{A}:e^{i\eta}[(1-\gamma)$ $\cdot\frac{D_\lambda^n(f*g)(z)}{z}+\gamma\left(D_\lambda^n(f*g)(z)\right)']\prec(1-\rho)\cos\eta\varphi(z)+i\sin\eta+\rho\cos\eta$ $(|\eta|\leq\frac{\pi}{2};$ $\gamma,\lambda\geq 0;\ 0\leq\rho<1;b\in\mathbb{C}^*;n\in\mathbb{N}_0)\}$.

In this paper, we obtain the Fekete-Szegő inequalities for functions in the class $M_\lambda^n(f,g;\gamma,b;\varphi)$.

2. Fekete-Szegő problem

Unless otherwise mentioned, we assume in the reminder of this paper that $\lambda\geq 0$, $b\in\mathbb{C}^*$ and $z\in U$.

To prove our results, we shall need the following lemmas:

Lemma 2.1. [9] *If* $p(z)=1+c_1z+c_2z^2+.....(z\in U)$ *is a function with positive real part in* U *and* μ *is a complex number, then*

$$\left|c_2-\mu c_1^2\right|\leq 2\max\{1;|2\mu-1|\}. \tag{2.1}$$

The result is sharp for the functions given by

$$p(z)=\frac{1+z^2}{1-z^2}\ \text{and}\ p(z)=\frac{1+z}{1-z}\ \ (z\in U). \tag{2.2}$$

Lemma 2.2. [9] *If* $p_1(z)=1+c_1z+c_2z^2+.....$ *is a function with positive real part in* U, *then*

$$\left|c_2-\nu c_1^2\right|\leq\begin{cases}-4\nu+2, & if\quad \nu\leq 0,\\ 2, & if\quad 0\leq\nu\leq 1,\\ 4\nu-2, & if\quad \nu\geq 1.\end{cases}$$

When $\nu<0$ *or* $\nu>1$, *the equality holds if and only if* $p_1(z)=\frac{1+z}{1-z}$ *or one of its rotations. If* $0<\nu<1$, *then the equality holds if and only if* $p_1(z)=\frac{1+z^2}{1-z^2}$ *or one of its rotations. If* $\nu=0$, *the equality holds if and only if*

$$p_1(z)=\left(\frac{1}{2}+\frac{1}{2}\gamma\right)\frac{1+z}{1-z}+\left(\frac{1}{2}-\frac{1}{2}\gamma\right)\frac{1-z}{1+z}\quad(0\leq\gamma\leq 1),$$

or one of its rotations. If $\nu = 1$, *the equality holds if and only if*

$$\frac{1}{p_1(z)} = \left(\frac{1}{2} + \frac{1}{2}\gamma\right)\frac{1+z}{1-z} + \left(\frac{1}{2} - \frac{1}{2}\gamma\right)\frac{1-z}{1+z} \quad (0 \leq \gamma \leq 1).$$

Also the above upper bound is sharp and it can be improved as follows when $0 < \nu < 1$:

$$\left|c_2 - \nu c_1^2\right| + \nu \left|c_1\right|^2 \leq 2 \ (0 < \nu < \frac{1}{2}),$$

and

$$\left|c_2 - \nu c_1^2\right| + (1-\nu)\left|c_1\right|^2 \leq 2 \ (\frac{1}{2} < \nu < 1).$$

Using Lemma 2.1, we have the following theorem:

Theorem 2.3. *Let* $\varphi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + ...$, *where* $\varphi(z) \in \mathcal{A}$ *and* $\varphi'(0) > 0$. *If* $f(z)$ *given by (1.1) belongs to the class* $M_\lambda^n(f, g; \gamma, b; \varphi)$ *and if* μ *is a complex order, then*

$$\left|a_3 - \mu a_2^2\right| \leq \frac{B_1 |b|}{(1+2\lambda)^n (1+2\gamma) g_3} \max\left\{1, \left|\frac{B_2}{B_1} - \frac{(1+2\lambda)^n (1+2\gamma) g_3}{(1+\lambda)^{2n} (1+\gamma)^2 g_2^2}\mu b B_1\right|\right\}. \tag{2.3}$$

The result is sharp.

Proof. If $f \in M_\lambda^n(f, g; \gamma, b; \varphi)$, then there exists a Schwarz function w analytic in U with $w(0) = 0$ and $|w(z)| < 1$ in U and such that

$$1 + \frac{1}{b}\left[(1-\gamma)\frac{D_\lambda^n(f*g)(z)}{z} + \gamma\left(D_\lambda^n(f*g)(z)\right)' - 1\right] = \varphi(w(z)). \tag{2.4}$$

Define the function p_1 by

$$p_1(z) = \frac{1+w(z)}{1-w(z)} = 1 + c_1 z + c_2 z^2 + ... \ . \tag{2.5}$$

Since w is a Schwarz function, we see that $\mathrm{Re} p_1(z) > 0$ and $p_1(0) = 1$.
Let define the function p by:

$$p(z) = 1 + \frac{1}{b}\left[(1-\gamma)\frac{D_\lambda^n(f*g)(z)}{z} + \gamma\left(D_\lambda^n(f*g)(z)\right)' - 1\right] = 1 + b_1 z + b_2 z^2 + ... \tag{2.6}$$

In view of the equations (2.4), (2.5) and (2.6), we have

$$\begin{aligned} p(z) &= \varphi\left(\frac{p_1(z)-1}{p_1(z)+1}\right) = \varphi\left(\frac{c_1 z + c_2 z^2 + ...}{2 + c_1 z + c_2 z^2 + ...}\right) \\ &= \varphi\left(\frac{1}{2}c_1 z + \frac{1}{2}\left(c_2 - \frac{c_1^2}{2}\right) z^2 + ...\right) \\ &= 1 + \frac{1}{2}B_1 c_1 z + \left[\frac{1}{2}B_1\left(c_2 - \frac{c_1^2}{2}\right) + \frac{1}{4}B_2 c_1^2\right] z^2 + ... \end{aligned} \tag{2.7}$$

Thus

$$b_1 = \frac{1}{2}B_1 c_1 \quad \text{and} \quad b_2 = \frac{1}{2}B_1\left(c_2 - \frac{c_1^2}{2}\right) + \frac{1}{4}B_2 c_1^2. \tag{2.8}$$

Since

$$1+\frac{1}{b}\left[(1-\gamma)\frac{D_{\lambda}^{n}(f*g)(z)}{z}+\gamma\left(D_{\lambda}^{n}(f*g)(z)\right)'-1\right]$$

$$=1+\left(\frac{1}{b}(1+\lambda)^{n}(1+\gamma)a_2g_2\right)z+\left(\frac{1}{b}(1+2\lambda)^{n}(1+2\gamma)a_3g_3\right)z^2+...,$$

from (2.6) and (2.8), we obtain

$$a_2=\frac{B_1c_1b}{2(1+\lambda)^{n}(1+\gamma)g_2}, \tag{2.9}$$

and

$$a_3=\frac{B_1c_2b}{2(1+2\lambda)^{n}(1+2\gamma)g_3}+\frac{c_1^2}{4(1+2\lambda)^{n}(1+2\gamma)g_3}\left[(B_2-B_1)b\right]. \tag{2.10}$$

Therefore, we have

$$a_3-\mu a_2^2=\frac{B_1b}{2(1+2\lambda)^{n}(1+2\gamma)g_3}\left[c_2-\nu c_1^2\right], \tag{2.11}$$

where

$$\nu=\frac{1}{2}\left[1-\frac{B_2}{B_1}+\frac{(1+2\lambda)^{n}(1+2\gamma)g_3\mu}{(1+\lambda)^{2n}(1+\gamma)^2g_2^2}B_1b\right]. \tag{2.12}$$

Our result now follows by an application of Lemma 2.1. The result is sharp for the functions f satisfying

$$1+\frac{1}{b}\left[(1-\gamma)\frac{D_{\lambda}^{n}(f*g)(z)}{z}+\gamma\left(D_{\lambda}^{n}(f*g)(z)\right)'-1\right]=\varphi(z^2), \tag{2.13}$$

and

$$1+\frac{1}{b}\left[(1-\gamma)\frac{D_{\lambda}^{n}(f*g)(z)}{z}+\gamma\left(D_{\lambda}^{n}(f*g)(z)\right)'-1\right]=\varphi(z). \tag{2.14}$$

This completes the proof of Theorem 2.3.

Remark 2.4. (i) Taking $\gamma=1$, $n=0$ and $g(z)=\frac{z}{1-z}$ in Theorem 2.3, we obtain the result obtained by Ali et al. [1, Theorem 2.3, with $k=1$];
(ii) Taking $\gamma=1$, $n=0$, $g(z)=\frac{z}{1-z}$ and $\varphi(z)=(1-\ell)\frac{1+Az}{1+Bz}+\ell$ $(0\leq\ell<1,$ $-1\leq B<A\leq1)$ in Theorem 2.3, we obtain the result obtained by Reddy and Reddy [11, Theorem 4].

Also by specializing the parameters in Theorem 2.3, we obtain the following new sharp results.

Putting $n=0$, $g(z)=z+\sum_{k=2}^{\infty}k^{n}z^{k}$ $(n\in\mathbb{N}_0)$ and $\varphi(z)=\frac{1+Az}{1-Bz}$ $(-1\leq B<A\leq1)$ (or equivalently, $B_1=A-B$ and $B_2=-B(A-B)$) in Theorem 2.3, we obtain the corollary:

Corollary 2.5. *If f given by (1.1) belongs to the class $G_n(\gamma, b; A, B)$, then for any complex number μ, we have*

$$\left|a_3 - \mu a_2^2\right| \leq \frac{(A-B)\,|b|}{(1+2\gamma)\,3^n} \max\left\{1, \left|\frac{(1+2\gamma)\,3^n}{(1+\gamma)^2\,2^{2n}}\mu\,(A-B)\,b + B\right|\right\}. \qquad (2.15)$$

The result is sharp.

Putting $n = 0$ and $\varphi(z) = \frac{1+(1-2\alpha)z}{1-z}$ $(0 \leq \alpha < 1)$ in Theorem 2.3, we obtain the following corollary:

Corollary 2.6. *If f given by (1.1) belongs to the class $S(f, g; \gamma, \alpha, b)$, then for any complex number μ, we have*

$$\left|a_3 - \mu a_2^2\right| \leq \frac{2\,(1-\alpha)\,|b|}{(1+2\gamma)\,g_3} \max\left\{1, \left|1 - \frac{2\,(1+2\gamma)\,g_3}{(1+\gamma)^2\,g_2^2}\mu\,(1-\alpha)\,b\right|\right\}. \qquad (2.16)$$

The result is sharp.

Putting $n = 0$, $g(z) = z + \sum_{k=2}^{\infty} k^n z^k$ $(n \in \mathbb{N}_0)$ and $\varphi(z) = \frac{1+z}{1-z}$ in Theorem 2.3, we obtain the following corollary:

Corollary 2.7. *If f given by (1.1) belongs to the class $G_n(\gamma, b)$, then for any complex number μ, we have*

$$\left|a_3 - \mu a_2^2\right| \leq \frac{2\,|b|}{(1+2\gamma)\,3^n} \max\left\{1, \left|1 - \frac{(1+2\gamma)\,3^n}{(1+\gamma)^2\,2^{2n-1}}\mu b\right|\right\}. \qquad (2.17)$$

The result is sharp.

Putting $\gamma = 1$ in Theorem 2.3, we obtain the following corollary:

Corollary 2.8. *If f given by (1.1) belongs to the class $G_\lambda^n(f, g; b; \varphi)$, then for any complex number μ, we have*

$$\left|a_3 - \mu a_2^2\right| \leq \frac{B_1\,|b|}{3\,(1+2\lambda)^n\,g_3} \max\left\{1, \left|\frac{B_2}{B_1} - \frac{3\,(1+2\lambda)^n\,g_3}{4\,(1+\lambda)^{2n}\,g_2^2}\mu B_1 b\right|\right\}. \qquad (2.18)$$

The result is sharp.

Putting $\gamma = 0$ in Theorem 2.3, we obtain the following corollary:

Corollary 2.9. *If f given by (1.1) belongs to the class $R_\lambda^n(f, g; b; \varphi)$, then for any complex number μ, we have*

$$\left|a_3 - \mu a_2^2\right| \leq \frac{B_1\,|b|}{(1+2\lambda)^n\,g_3} \max\left\{1, \left|\frac{B_2}{B_1} - \frac{(1+2\lambda)^n\,g_3}{(1+\lambda)^{2n}\,g_2^2}\mu B_1 b\right|\right\}. \qquad (2.19)$$

The result is sharp.

Putting $(1-\rho)\cos\eta e^{-i\eta}$ $\left(0 \leq \rho < 1; |\eta| \leq \frac{\pi}{2}\right)$ in Theorem 2.3, we obtain the following corollary:

Corollary 2.10. *If f given by (1.1) belongs to the class $E_{\lambda,\rho}^{n,\eta}(f,g;\gamma;\varphi)$, then for any complex number μ, we have*

$$\left|a_3-\mu a_2^2\right| \le \tfrac{(1-\rho)B_1\cos\eta}{(1+2\lambda)^n(1+2\gamma)g_3}\max\left\{1,\left|\frac{B_2}{B_1}e^{i\eta}-\tfrac{(1+2\lambda)^n(1+2\gamma)(1-\rho)\cos\eta}{(1+\lambda)^{2n}(1+\gamma)^2g_2^2}g_3\mu B_1\right|\right\}. \tag{2.20}$$

The result is sharp.

Using Lemma 2.2, we have the following theorem:

Theorem 2.11. *Let $\varphi(z)=1+B_1z+B_2z^2+B_3z^3+\ldots$, $(b>0; B_i>0; i\in\mathbb{N})$. Also let*

$$\sigma_1=\frac{(1+\lambda)^{2n}(1+\gamma)^2g_2^2(B_2-B_1)}{(1+2\lambda)^n(1+2\gamma)g_3bB_1^2},$$

and

$$\sigma_2=\frac{(1+\lambda)^{2n}(1+\gamma)^2g_2^2(B_2+B_1)}{(1+2\lambda)^n(1+2\gamma)g_3bB_1^2}.$$

If f is given by (1.1) belongs to the class $M_\lambda^n(f,g;\gamma,b;\varphi)$, then we have the following sharp results:
(i) If $\mu\le\sigma_1$, then

$$\left|a_3-\mu a_2^2\right|\le\frac{b}{(1+2\lambda)^n(1+2\gamma)g_3}\left[B_2-\frac{(1+2\lambda)^n(1+2\gamma)g_3b}{(1+\lambda)^{2n}(1+\gamma)^2g_2^2}\mu B_1^2\right]; \tag{2.21}$$

(ii) If $\sigma_1\le\mu\le\sigma_2$, then

$$\left|a_3-\mu a_2^2\right|\le\frac{bB_1}{(1+2\lambda)^n(1+2\gamma)g_3}; \tag{2.22}$$

(iii) If $\mu\ge\sigma_2$, then

$$\left|a_3-\mu a_2^2\right|\le\frac{b}{(1+2\lambda)^n(1+2\gamma)g_3}\left[-B_2+\frac{(1+2\lambda)^n(1+2\gamma)g_3b}{(1+\lambda)^{2n}(1+\gamma)^2g_2^2}\mu B_1^2\right]. \tag{2.23}$$

Proof. For $f\in M_\lambda^n(f,g;\gamma,b;\varphi)$, $p(z)$ given by (2.6) and p_1 given by (2.5), then a_2 and a_3 are given as in Theorem 2.3. Also

$$a_3-\mu a_2^2=\frac{B_1b}{2(1+2\lambda)^n(1+2\gamma)g_3}\left[c_2-\nu c_1^2\right], \tag{2.24}$$

where

$$\nu=\frac{1}{2}\left[1-\frac{B_2}{B_1}+\frac{(1+2\lambda)^n(1+2\gamma)g_3\mu}{(1+\lambda)^{2n}(1+\gamma)^2g_2^2}B_1b\right]. \tag{2.25}$$

First, if $\mu\le\sigma_1$, then we have $\nu\le0$, and by applying Lemma 2.2 to equality (2.24), we have

$$\left|a_3-\mu a_2^2\right|\le\frac{b}{(1+2\lambda)^n(1+2\gamma)g_3}\left[B_2-\frac{(1+2\lambda)^n(1+2\gamma)g_3b}{(1+\lambda)^{2n}(1+\gamma)^2g_2^2}\mu B_1^2\right],$$

which is evidently inequality (2.21) of Theorem 2.11.

If $\mu = \sigma_1$, then we have $\nu = 0$, therefore equality holds if and only if

$$p_1(z) = (\tfrac{1+\gamma}{2})\frac{1+z}{1-z} + (\tfrac{1-\gamma}{2})\frac{1-z}{1+z} \ (0 \le \gamma \le 1; z \in U).$$

Next, if $\sigma_1 \le \mu \le \sigma_2$, we note that

$$\max\left\{\frac{1}{2}\left[1 - \frac{B_2}{B_1} + \frac{(1+2\lambda)^n(1+2\gamma)\,g_3\mu}{(1+\lambda)^{2n}(1+\gamma)^2\,g_2^2}B_1 b\right]\right\} \le 1, \tag{2.26}$$

then applying Lemma 2.2 to equality (2.24), we have

$$\left|a_3 - \mu a_2^2\right| \le \frac{b}{(1+2\lambda)^n(1+2\gamma)\,g_3},$$

which is evidently inequality (2.22) of Theorem 2.11.
If $\sigma_1 < \mu < \sigma_2$, then we have

$$p_1(z) = \frac{1+z^2}{1-z^2}.$$

Finally, If $\mu \ge \sigma_2$, then we have $\nu \ge 1$, therefore by applying Lemma 2.2 to (2.24), we have

$$\left|a_3 - \mu a_2^2\right| \le \frac{b}{(1+2\lambda)^n(1+2\gamma)\,g_3}\left[\frac{(1+2\lambda)^n(1+2\gamma)\,g_3 b}{(1+\lambda)^{2n}(1+\gamma)^2\,g_2^2}\mu B_1^2 - B_2\right],$$

which is evidently inequality (2.23) of Theorem 2.11.
If $\mu = \sigma_2$, then we have $\nu = 1$, therefore equality holds if and only if

$$\frac{1}{p_1(z)} = \frac{1+\gamma}{2}\frac{1+z}{1-z} + \frac{1-\gamma}{2}\frac{1-z}{1+z} \ (0 \le \gamma \le 1; z \in U).$$

To show that the bounds are sharp, we define the functions $K_\varphi^s (s \ge 2)$ by

$$1 + \frac{1}{b}\left[(1-\gamma)\tfrac{D_\lambda^n(K_\varphi^s * g)(z)}{z} + \gamma\left(D_\lambda^n(K_\varphi^s * g)(z)\right)' - 1\right] = \varphi(z^{s-1}), \tag{2.27}$$

$$K_\varphi^s(0) = 0 = K_\varphi^{'s}(0) - 1,$$

and the functions F_t and G_t $(0 \le t \le 1)$ by

$$1 + \frac{1}{b}\left[(1-\gamma)\tfrac{D_\lambda^n(F_t * g)(z)}{z} + \gamma\left(D_\lambda^n(F_t * g)(z)\right)' - 1\right] = \varphi\left(\tfrac{z(z+t)}{1+tz}\right), \tag{2.28}$$

$$F_t(0) = 0 = F_t'(0) - 1,$$

and

$$1 + \frac{1}{b}\left[(1-\gamma)\tfrac{D_\lambda^n(G_t * g)(z)}{z} + \gamma\left(D_\lambda^n(G_t * g)(z)\right)' - 1\right] = \varphi\left(-\tfrac{z(z+t)}{1+tz}\right), \tag{2.29}$$

$$G_t(0) = 0 = G_t'(0) - 1.$$

Cleary the functions K_φ^s, F_t and $G_t \in M_\lambda^n(f, g; \gamma, b; \varphi)$. Also we write $K_\varphi = K_\varphi^2$.
If $\mu < \sigma_1$ or $\mu > \sigma_2$, then the equality holds if and only if f is K_φ or one of its rotations. When $\sigma_1 < \mu < \sigma_2$, then the equality holds if f is K_φ^3 or one of its rotations. If $\mu = \sigma_1$, then the equality holds if and only if f is F_t or one of its rotations. If $\mu = \sigma_2$, then the equality holds if and only if f is G_t or one of its rotations.

Remark 2.12. Taking $\gamma = 1$, $b = 1$, $n = 0$ and $g(z) = \dfrac{z}{1-z}$ in Theorem 2.11, we obtain the result obtained by Ali et al. [1, Corollary 2.5, with $k = 1$].

Also, using Lemma 2.2 we have the following theorem:

Theorem 2.13. *For $\varphi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + ...,$ $(b > 0; B_i > 0; i \in \mathbb{N})$ and $f(z)$ given by (1.1) belongs to the class $M_\lambda^n(f, g; \gamma, b; \varphi)$ and $\sigma_1 \le \mu \le \sigma_2$, then in view of Lemma 2.2, Theorem 2.11 can be improved. Let*

$$\sigma_3 = \frac{(1+\lambda)^{2n}(1+\gamma)^2 g_2^2 B_2}{(1+2\lambda)^n (1+2\gamma) g_3 b B_1^2},$$

(i) If $\sigma_1 \le \mu \le \sigma_3$, then

$$\left|a_3 - \mu a_2^2\right| + \tfrac{(1+\lambda)^{2n}(1+\gamma)^2 g_2^2}{(1+2\lambda)^n(1+2\gamma)g_3 b B_1}\left[1 - \frac{B_2}{B_1} + \tfrac{(1+2\lambda)^n(1+2\gamma)g_3}{(1+\lambda)^{2n}(1+\gamma)^2 g_2^2}\mu b B_1\right]|a_2|^2 \le \frac{B_1 b}{(1+2\lambda)^n(1+2\gamma)g_3}; \tag{2.30}$$

(ii) If $\sigma_3 \le \mu \le \sigma_2$, then

$$\left|a_3 - \mu a_2^2\right| + \tfrac{(1+\lambda)^{2n}(1+\gamma)^2 g_2^2}{(1+2\lambda)^n(1+2\gamma)g_3 b B_1}\left[1 + \frac{B_2}{B_1} - \tfrac{(1+2\lambda)^n(1+2\gamma)g_3}{(1+\lambda)^{2n}(1+\gamma)^2 g_2^2}\mu b B_1\right]|a_2|^2 \le \frac{B_1 b}{(1+2\lambda)^n(1+2\gamma)g_3}. \tag{2.31}$$

Proof. For the values of $\sigma_1 \le \mu \le \sigma_3$, we have

$$\begin{aligned}
&\left|a_3 - \mu a_2^2\right| + (\mu - \sigma_1)|a_2|^2 \\
&= \tfrac{B_1 b}{2(1+2\lambda)^n(1+2\gamma)g_3}\left|c_2 - \nu c_1^2\right| + \left(\mu - \tfrac{(1+\lambda)^{2n}(1+\gamma)^2 g_2^2 (B_2 - B_1)}{(1+2\lambda)^n(1+2\gamma)g_3 b B_1^2}\right)\tfrac{B_1^2 b^2}{4(1+2\lambda)^{2n}(1+\gamma)^2 g_2^2}|c_1|^2 \\
&= \frac{B_1 b}{(1+2\lambda)^n(1+2\gamma)g_3}\left\{\frac{1}{2}\left(\left|c_2 - \nu c_1^2\right| + \nu|c_1|^2\right)\right\}.
\end{aligned} \tag{2.32}$$

Now applying Lemma 2.2 to equality (2.32), we have

$$\left|a_3 - \mu a_2^2\right| + (\mu - \sigma_1)|a_2|^2 \le \frac{B_1 b}{(1+2\lambda)^n(1+2\gamma)g_3},$$

which is the inequality (2.30) of Theorem 2.13.

Next, for the values of $\sigma_3 \le \mu \le \sigma_2$, we have

$$\begin{aligned}
&\left|a_3 - \mu a_2^2\right| + (\sigma_2 - \mu)|a_2|^2 \\
&= \tfrac{bB_1}{2(1+2\lambda)^n(1+2\gamma)g_3}\left|c_2 - \nu c_1^2\right| + \left(\tfrac{(1+\lambda)^{2n}(1+\gamma)^2 g_2^2 (B_2 + B_1)}{(1+2\lambda)^n(1+2\gamma)g_3 b B_1^2} - \mu\right) \\
&\quad \cdot \tfrac{B_1^2 b^2}{4(1+2\lambda)^{2n}(1+\gamma)^2 g_2^2}|c_1|^2 \\
&= \frac{B_1 b}{(1+2\lambda)^n(1+2\gamma)g_3}\left\{\frac{1}{2}\left(\left|c_2 - \nu c_1^2\right| + (1-\nu)|c_1|^2\right)\right\}.
\end{aligned} \tag{2.33}$$

Now applying Lemma 2.2 to equality (2.33), we have

$$\left|a_3 - \mu a_2^2\right| + (\sigma_2 - \mu)|a_2|^2 \le \frac{B_1 b}{(1+2\lambda)^n(1+2\gamma)g_3},$$

which is the inequality (2.31). This completes the proof of Theorem 2.13.

Remark 2.14. (i) Specializing the parameters γ, λ, b, n, g and φ in Theorem 2.11 and Theorem 2.13, we obtain the corresponding results of the classes $G_n(\gamma, b, A, B)$, $S(f, g; \gamma, \alpha, b)$, $G_n(\gamma, b)$, $R_\ell^b(A, B)$, $M_{\lambda,q,s}^n([\alpha_1]; \gamma, b; \varphi)$, $G_\lambda^n(f, g; b; \varphi)$, $R_\lambda^n(f, g; b; \varphi)$ and $E_{\lambda,\rho}^{n,\eta}(f, g; \gamma; \varphi)$, as special cases as defined before.

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Improvement of a result due to P.T. Mocanu

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Abstract. A result concerning the starlikeness of the image of the Alexander operator is improved in this paper. The techniques of differential subordinations are used.

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1. Introduction

Let $U = \{z \in \mathbb{C} : |z| < 1\}$ be the open unit disk in in the complex plane. Let $\mathcal{A}$ be the class of analytic functions f, which are defined on the unit disk U and have the properties $f(0) = f'(0) - 1 = 0$. The subclass of $\mathcal{A}$, consisting of functions for which the domain $f(U)$ is starlike with respect to 0 is denoted by S^*. An analytic characterization of S^* is given by

$$S^* = \left\{ f \in \mathcal{A} : \operatorname{Re} \frac{zf'(z)}{f(z)} > 0,\ z \in U \right\}.$$

Another subclass of $\mathcal{A}$ we deal with is the class of close-to-convex functions denoted by C. A function $f \in \mathcal{A}$ belongs to the class C if and only if there is a starlike function $g \in S^*$, so that $\operatorname{Re} \frac{zf'(z)}{g(z)} > 0, \quad z \in U$. We note that C and S^* contain univalent functions. The Alexander integral operator is defined by the equality:

$$A(f)(z) = \int_0^z \frac{f(t)}{t}\, dt.$$

The authors of [1] pp. $310 - 311$ proved the following result:

Theorem 1.1. *Let A be the Alexander operator and let $g \in \mathcal{A}$ satisfy*

$$\operatorname{Re} \frac{zg'(z)}{g(z)} \geq \left| \operatorname{Im} \frac{z(zg'(z))'}{g(z)} \right|,\ z \in U. \tag{1.1}$$

If $f \in \mathcal{A}$ and

$$\operatorname{Re} \frac{zf'(z)}{g(z)} > 0,\ z \in U,$$

then $F = A(f) \in S^*$.

Improvements of this result can be found in [3], [4] and [6]. In this paper we put the problem to determine the smallest c_1 such that the following theorems hold.

Theorem 1.2. *Let* A *be the Alexander operator and let* $g \in \mathcal{A}$ *satisfy*

$$\operatorname{Re} \frac{zg'(z)}{g(z)} \geq c_1 \left| \operatorname{Im} \frac{z(zg'(z))'}{g(z)} \right|, \ z \in U. \tag{1.2}$$

If $f \in \mathcal{A}$ *and*

$$\operatorname{Re} \frac{zf'(z)}{g(z)} > 0, \ z \in U,$$

then $F = A(f) \in S^*$.

In [5] it has been proved that $A(C) \not\subseteq S^*$, and this result shows that $c_1 > 0$. We are not able to determine the the best value of c_1, but we will give a new improvement for Theorem 1.1 in the present paper. In order to do this, we need some lemmas, which are exposed in the next section.

2. Preliminaries

Let f and g be analytic functions in U. The function f is said to be subordinate to g, written $f \prec g$, if there is a function w analytic in U, with $w(0) = 0$, $|w(z)| < 1$, $z \in U$ and $f(z) = g(w(z))$, $z \in U$. Recall that if g is univalent, then $f \prec g$ if and only if $f(0) = g(0)$ and $f(U) \subset g(U)$.

Lemma 2.1. [1] (Miller-Mocanu) *Let* $p(z) = a + \sum_{k=n}^{\infty} a_k z^k$ *be analytic in* U *with* $p(z) \not\equiv a$, $n \geq 1$ *and let* $q : U \to \mathbb{C}$ *be an analytic and univalent function with* $q(0) = a$. *If* p *is not subordinate to* q, *then there are two points* $z_0 \in U$, $|z_0| = r_0$ *and* $\zeta_0 \in \partial U$ *and a real number* $m \in [n, \infty)$, *so that* q *is defined in* ζ_0, $p(U(0, r_0)) \subset q(U)$, *and:*

(i) $p(z_0) = q(\zeta_0)$

(ii) $z_0 p'(z_0) = m \zeta_0 q'(\zeta_0)$

and

(iii) $\operatorname{Re}\left(1 + \frac{z_0 p''(z_0)}{p'(z_0)}\right) \geq m \operatorname{Re}\left(1 + \frac{\zeta_0 q''(\zeta_0)}{q'(\zeta_0)}\right)$.

We note that $z_0 p'(z_0)$ *is the outward normal to the curve* $p(\partial U(0, r_0))$ *at the point* $p(z_0)$, *while* $\partial U(0, r_0)$ *denotes the border of the disc* $U(0, r_0)$.

In [6] the following result is proved:

Lemma 2.2. [6] *Let* $g \in \mathcal{A}$ *be a function, which satisfies the condition*

$$\left| \operatorname{Im} \frac{zg'(z)}{g(z)} \right| \leq 1, \ z \in U. \tag{2.1}$$

If $f \in \mathcal{A}$ *and*

$$\operatorname{Re} \frac{zf'(z)}{g(z)} > 0, \ z \in U,$$

then $F = A(f) \in S^*$.

3. The main result

Theorem 3.1. *Let $g \in \mathcal{A}$ be a function such that*

$$\operatorname{Re}\frac{zg'(z)}{g(z)} \geq \frac{2}{5}\left|\operatorname{Im}\frac{z(zg'(z))'}{g(z)}\right|, \ z \in U. \tag{3.1}$$

If $f \in \mathcal{A}$ and

$$\operatorname{Re}\frac{zf'(z)}{g(z)} > 0, \ z \in U,$$

then $F = A(f) \in S^*$.

Proof. If we denote $p(z) = \frac{zg'(z)}{g(z)}$, then (3.1) becomes

$$\operatorname{Re} p(z) > \frac{2}{5}\big|\operatorname{Im}[zp'(z) + p^2(z)]\big|, \quad z \in U. \tag{3.2}$$

We will prove that

$$p \prec q \ \text{ where } \ q(z) = 1 + \frac{2}{\pi}\log\frac{1+z}{1-z}, \ z \in U.$$

If the subordination $p \prec q$ does not hold, then according to Lemma 2.1, there are two points $z_2 \in U$, $\zeta_2 = e^{i\theta_2}$ and a real number $m_2 \in [1, \infty)$ such that

$$p(z_2) = q(\zeta_2) = 1 + \frac{2}{\pi}\log\frac{1+\zeta_2}{1-\zeta_2} = 1 + \frac{2}{\pi}(\ln|\cot\frac{\theta_2}{2}| \pm i\frac{\pi}{2})$$

and

$$z_2p'(z_2) = m_2\zeta_2q'(\zeta_2) = \frac{2m_2i}{\pi\sin\theta_2}.$$

We discuss the case $\theta_2 \in (0, \pi)$, the other case $\theta_2 \in [-\pi, 0)$ is similar. If $\theta_2 \in [0, \pi]$ and $x = \cot\frac{\theta_2}{2}$, then

$$p(z_2) = 1 + \frac{2}{\pi}\left(\ln|\cot\frac{\theta_2}{2}| + i\frac{\pi}{2}\right)$$

and we get

$$\big|\operatorname{Im}[z_2p'(z_2) + p^2(z_2)]\big| - \frac{5}{2}\operatorname{Re} p(z_2)$$

$$= \frac{2m_2}{\pi\sin\theta_2} + 2\left[1 + \frac{2}{\pi}\ln\left(\cot\frac{\theta_2}{2}\right)\right] - \frac{5}{2}\left[1 + \frac{2}{\pi}\ln\left(\cot\frac{\theta_2}{2}\right)\right]$$

$$\geq \frac{1+x^2}{\pi x} - \frac{1}{2}\big[1 + \frac{2}{\pi}\ln(x)\big] = \frac{1+x^2}{\pi x} - \frac{1}{2} - \frac{1}{\pi}\ln(x) \geq 0, \ x \in (0, \infty).$$

This contradicts (3.2), and consequently the subordination

$$\frac{zg'(z)}{g(z)} = p(z) \prec q(z) = 1 + \frac{2}{\pi}\log\frac{1+z}{1-z}$$

holds. This subordination implies $\left|\operatorname{Im}\frac{zg'(z)}{g(z)}\right| \leq 1, \ z \in U$ and so according to Lemma 2.2 we have $F = A(f) \in S^*$. □

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A comprehensive class of harmonic functions defined by convolution and its connection with integral transforms and hypergeometric functions

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Abstract. For given two harmonic functions Φ and Ψ with real coefficients in the open unit disk $\mathbb{D}$, we study a class of harmonic functions

$$f(z) = z - \sum_{n=2}^{\infty} A_n z^n + \sum_{n=1}^{\infty} B_n \bar{z}^n \ (A_n, B_n \geq 0)$$

satisfying $\operatorname{Re} \dfrac{(f * \Phi)(z)}{(f * \Psi)(z)} > \alpha$ $(0 \leq \alpha < 1, z \in \mathbb{D})$; $*$ being the harmonic convolution. Coefficient inequalities, growth and covering theorems, as well as closure theorems are determined. The results obtained extend several known results as special cases. In addition, we study the class of harmonic functions f that satisfy $\operatorname{Re} f(z)/z > \alpha$ $(0 \leq \alpha < 1, z \in \mathbb{D})$. As an application, their connection with certain integral transforms and hypergeometric functions is established.

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1. Introduction

Let $\mathcal{H}$ denote the class of all complex-valued harmonic functions f in the open unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ normalized by $f(0) = 0 = f_z(0) - 1$. Such functions can be written in the form $f = h + \bar{g}$, where

$$h(z) = z + \sum_{n=2}^{\infty} A_n z^n \quad \text{and} \quad g(z) = \sum_{n=1}^{\infty} B_n z^n \tag{1.1}$$

are analytic in $\mathbb{D}$. We call h the analytic part and g the co-analytic part of f. Let $\mathcal{S}_H$ be the subclass of $\mathcal{H}$ consisting of univalent and sense-preserving functions. In 1984, Clunie and Sheil-Small [5] initiated the study of the class $\mathcal{S}_H$ and its subclasses.

For analytic functions $\phi(z) = z + \sum_{n=2}^{\infty} A_n z^n$ and $\psi(z) = z + \sum_{n=2}^{\infty} A'_n z^n$, their convolution (or Hadamard product) is defined as $(\phi * \psi)(z) = z + \sum_{n=2}^{\infty} A_n A'_n z^n$, $z \in \mathbb{D}$. In the harmonic case, with $f = h + \bar{g}$ and $F = H + \bar{G}$, their harmonic convolution is defined as $f * F = h * H + \overline{g * G}$. Harmonic convolutions are investigated in [7, 8, 10, 15, 16].

Let $\mathcal{TH}$ be the subclass of $\mathcal{H}$ consisting of functions $f = h + \bar{g}$ so that h and g take the form

$$h(z) = z - \sum_{n=2}^{\infty} A_n z^n \quad , \quad g(z) = \sum_{n=1}^{\infty} B_n z^n \quad (A_n, B_n \geq 0). \tag{1.2}$$

Making use of the convolution structure for harmonic mappings, we study a new subclass $\mathcal{TH}(\Phi_i, \Psi_j; \alpha)$ introduced in the following:

Definition 1.1. *Suppose that $i, j \in \{0, 1\}$. Let the functions Φ_i, Ψ_j given by*

$$\Phi_i(z) = z + \sum_{n=2}^{\infty} p_n z^n + (-1)^i \sum_{n=1}^{\infty} q_n \bar{z}^n$$

and

$$\Psi_j(z) = z + \sum_{n=2}^{\infty} u_n z^n + (-1)^j \sum_{n=1}^{\infty} v_n \bar{z}^n$$

are harmonic in $\mathbb{D}$ with $p_n > u_n \geq 0$ $(n = 2, 3, \ldots)$ and $q_n > v_n \geq 0$ $(n = 1, 2, \ldots)$. Then a function $f \in \mathcal{H}$ is said to be in the class $\mathcal{H}(\Phi_i, \Psi_j; \alpha)$ if and only if

$$\operatorname{Re} \frac{(f * \Phi_i)(z)}{(f * \Psi_j)(z)} > \alpha \quad (z \in \mathbb{D}), \tag{1.3}$$

where $*$ denotes the harmonic convolution as defined above and $0 \leq \alpha < 1$. Further we define the class $\mathcal{TH}(\Phi_i, \Psi_j; \alpha)$ by

$$\mathcal{TH}(\Phi_i, \Psi_j; \alpha) = \mathcal{H}(\Phi_i, \Psi_j; \alpha) \cap \mathcal{TH}.$$

The family $\mathcal{TH}(\Phi_i, \Psi_j; \alpha)$ includes a variety of well-known subclasses of harmonic functions as well as many new ones. For example

$$\mathcal{TH}\left(\frac{z}{(1-z)^2} - \frac{\bar{z}}{(1-\bar{z})^2}, \frac{z}{1-z} + \frac{\bar{z}}{1-\bar{z}}; \alpha\right) \equiv \mathcal{TS}_H^*(\alpha);$$

and

$$\mathcal{TH}\left(\frac{z+z^2}{(1-z)^3} + \frac{\bar{z}+\bar{z}^2}{(1-\bar{z})^3}, \frac{z}{(1-z)^2} - \frac{\bar{z}}{(1-\bar{z})^2}; \alpha\right) \equiv \mathcal{TK}_H(\alpha)$$

are the classes of sense-preserving harmonic univalent functions f which are fully starlike of order α $(0 \leq \alpha < 1)$ and fully convex of order α $(0 \leq \alpha < 1)$ respectively (see [11, 12, 16]). These classes were studied by Silverman and Silvia [19] for the case $\alpha = 0$. Recall that fully starlike functions of order α and fully convex functions of order α are characterized by the conditions

$$\frac{\partial}{\partial \theta} \arg f(re^{i\theta}) > \alpha \quad (0 \leq \theta < 2\pi, 0 < r < 1)$$

and

$$\frac{\partial}{\partial\theta}\left(\arg\left\{\frac{\partial}{\partial\theta}f(re^{i\theta})\right\}\right) > \alpha, \quad (0 \le \theta < 2\pi, 0 < r < 1)$$

respectively. In the similar fashion, it is easy to see that the subclasses of $\mathcal{TH}$ introduced by Ahuja *et al.* [2, 3], Dixit *et al.* [6], Frasin [9], Murugusundaramoorthy *et al.* [14] and Yalçin *et al.* [20, 21] are special cases of our class $\mathcal{TH}(\Phi_i, \Psi_j; \alpha)$ for suitable choice of the functions Φ_i and Ψ_j.

In Section 2, we obtain the coefficient inequalities, growth and covering theorems, as well as closure theorems for functions in the class $\mathcal{TH}(\Phi_i, \Psi_j; \alpha)$. In particular, the invariance of the class $\mathcal{TH}(\Phi_i, \Psi_j; \alpha)$ under certain integral transforms and connection with hypergeometric functions is also established.

The study of harmonic mappings defined by using hypergeometric functions is a recent area of interest [1, 4, 17]. Let $F(a, b, c; z)$ be the Gaussian hypergeometric function defined by

$$F(a, b, c; z) := \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n (1)_n} z^n, \quad z \in \mathbb{D} \tag{1.4}$$

which is the solution of the second order homogeneous differential equation

$$z(1-z)w''(z) + [c - (a+b+1)z]w'(z) - abw(z) = 0,$$

where a, b, c are complex numbers with $c \neq 0, -1, -2, \ldots$, and $(\theta)_n$ is the Pochhammer symbol: $(\theta)_0 = 1$ and $(\theta)_n = \theta(\theta+1)\ldots(\theta+n-1)$ for $n = 1, 2, \ldots$. Since the hypergeometric series in (1.4) converges absolutely in $\mathbb{D}$, it follows that $F(a, b, c; z)$ defines an analytic function in $\mathbb{D}$ and plays an important role in the theory of univalent functions. We have obtained necessary and sufficient conditions for a harmonic function associated with hypergeometric functions to be in the class $\mathcal{TH}(\Phi_i, \Psi_j; \alpha)$. The well-known Gauss's summation theorem: if $\mathrm{Re}(c - a - b) > 0$ then

$$F(a, b, c; 1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}, \quad c \neq 0, -1, -2, \ldots$$

will be frequently used in this paper.

For $\gamma > -1$ and $-1 \le \delta < 1$, let $L_\gamma : \mathcal{H} \to \mathcal{H}$ and $G_\delta : \mathcal{H} \to \mathcal{H}$ be the integral transforms defined by

$$L_\gamma[f](z) := \frac{\gamma+1}{z^\gamma}\int_0^z t^{\gamma-1}h(t)\,dt + \overline{\frac{\gamma+1}{z^\gamma}\int_0^z t^{\gamma-1}g(t)\,dt}, \tag{1.5}$$

and

$$G_\delta[f](z) := \int_0^z \frac{h(t)-h(\delta t)}{(1-\delta)t}\,dt + \overline{\int_0^z \frac{g(t)-g(\delta t)}{(1-\delta)t}\,dt}. \tag{1.6}$$

where $f = h + \overline{g} \in \mathcal{H}$ and $z \in \mathbb{D}$. It has been shown that the class $\mathcal{TH}(\Phi_i, \Psi_j; \alpha)$ is preserved under these integral transforms.

For $0 \le \alpha < 1$, let

$$\mathcal{TU}_H(\alpha) := \mathcal{TH}\left(\frac{z}{1-z} + \frac{\bar{z}}{1-\bar{z}}, z; \alpha\right)$$

The last section of the paper investigates the properties of functions in the class $\mathcal{TU}_H(\alpha)$. Moreover, inclusion relations are obtained between the classes $\mathcal{TU}_H(\alpha)$, $\mathcal{TS}^*_H(\alpha)$ and $\mathcal{TK}_H(\alpha)$ under certain milder conditions.

2. The Class $\mathcal{TH}(\Phi_i, \Psi_j; \alpha)$

The first theorem of this section provides a sufficient coefficient condition for a function to be in the class $\mathcal{H}(\Phi_i, \Psi_j; \alpha)$.

Theorem 2.1. *Let the function $f = h + \bar{g}$ be such that h and g are given by* (1.1). *Furthermore, let*

$$\sum_{n=2}^{\infty} \frac{p_n - \alpha u_n}{1-\alpha}|A_n| + \sum_{n=1}^{\infty} \frac{q_n - (-1)^{j-i}\alpha v_n}{1-\alpha}|B_n| \leq 1 \tag{2.1}$$

where $0 \leq \alpha < 1$, $i, j \in \{0,1\}$, $p_n > u_n \geq 0$ $(n = 2,3,\ldots)$ and $q_n > v_n \geq 0$ $(n = 1,2,\ldots)$. Then $f \in \mathcal{H}(\Phi_i, \Psi_j; \alpha)$.

Proof. Using the fact that $\operatorname{Re} w > \alpha$ if and only if $|w-1| < |w+1-2\alpha|$, it suffices to show that

$$|C(z) + (1-2\alpha)D(z)| - |C(z) - D(z)| \geq 0, \tag{2.2}$$

where

$$C(z) = (f * \Phi_i)(z) = z + \sum_{n=2}^{\infty} A_n p_n z^n + (-1)^i \sum_{n=1}^{\infty} B_n q_n \bar{z}^n$$

and

$$D(z) = (f * \Psi_j)(z) = z + \sum_{n=2}^{\infty} A_n u_n z^n + (-1)^j \sum_{n=1}^{\infty} B_n v_n \bar{z}^n.$$

Substituting for $C(z)$ and $D(z)$ in (2.2) and making use of (2.1) we obtain

$$\begin{aligned}
&|C(z) + (1-2\alpha)D(z)| - |C(z) - D(z)| \\
&= \left| 2(1-\alpha)z + \sum_{n=2}^{\infty}(p_n + (1-2\alpha)u_n)A_n z^n + (-1)^i \sum_{n=1}^{\infty}(q_n + (-1)^{j-i}(1-2\alpha)v_n)B_n \bar{z}^n \right| \\
&\quad - \left| \sum_{n=2}^{\infty}(p_n - u_n)A_n z^n + (-1)^i \sum_{n=1}^{\infty}(q_n - (-1)^{j-i}v_n)B_n \bar{z}^n \right| \\
&\geq 2(1-\alpha)|z| - \sum_{n=2}^{\infty}(p_n + (1-2\alpha)u_n)|A_n||z|^n - \sum_{n=1}^{\infty}(q_n + (-1)^{j-i}(1-2\alpha)v_n)|B_n||z|^n \\
&\quad - \sum_{n=2}^{\infty}(p_n - u_n)|A_n||z|^n - \sum_{n=1}^{\infty}(q_n - (-1)^{j-i}v_n)|B_n||z|^n \\
&= 2(1-\alpha)|z| \left[1 - \sum_{n=2}^{\infty} \frac{p_n - \alpha u_n}{1-\alpha}|A_n||z|^{n-1} - \sum_{n=1}^{\infty} \frac{q_n - (-1)^{j-i}\alpha v_n}{1-\alpha}|B_n||z|^{n-1} \right] \\
&> 2(1-\alpha)|z| \left[1 - \sum_{n=2}^{\infty} \frac{p_n - \alpha u_n}{1-\alpha}|A_n| - \sum_{n=1}^{\infty} \frac{q_n - (-1)^{j-i}\alpha v_n}{1-\alpha}|B_n| \right] \geq 0.
\end{aligned}$$

The harmonic mappings

$$f(z) = z + \sum_{n=2}^{\infty} \frac{1-\alpha}{p_n - \alpha u_n} x_n z^n + \sum_{n=1}^{\infty} \frac{1-\alpha}{q_n - (-1)^{j-i}\alpha v_n} \bar{y}_n \bar{z}^n,$$

where $\sum_{n=2}^{\infty} |x_n| + \sum_{n=1}^{\infty} |y_n| = 1$, show that the coefficient bound given by (2.1) is sharp. □

Remark 2.2. In addition to the hypothesis of Theorem 2.1, if we assume that $p_n \geq n$ $(n = 2, 3, \ldots)$ and $q_n \geq n$ $(n = 1, 2, \ldots)$ then it is easy to deduce that $n(1-\alpha) \leq p_n - \alpha u_n$ $(n = 2, 3, \ldots)$ and $n(1-\alpha) \leq q_n - (-1)^{j-i}\alpha v_n$ $(n = 1, 2, \ldots)$ so that

$$\sum_{n=2}^{\infty} n|A_n| + \sum_{n=1}^{\infty} n|B_n| \leq \sum_{n=2}^{\infty} \frac{p_n - \alpha u_n}{1-\alpha}|A_n| + \sum_{n=1}^{\infty} \frac{q_n - (-1)^{j-i}\alpha v_n}{1-\alpha}|B_n| \leq 1.$$

By [12, Theorem 1, p. 472], $f \in \mathcal{S}_H$ and maps $\mathbb{D}$ onto a starlike domain.

Theorem 2.1 gives a sufficient condition for the harmonic function $\phi_1 + \overline{\phi_2}$ to be in the class $\mathcal{H}(\Phi_i, \Psi_j; \alpha)$ where $\phi_1(z) \equiv \phi_1(a_1, b_1, c_1; z)$ and $\phi_2(z) \equiv \phi_2(a_2, b_2, c_2; z)$ are the hypergeometric functions defined by

$$\phi_1(z) := zF(a_1, b_1, c_1; z) \quad \text{and} \quad \phi_2(z) := F(a_2, b_2, c_2; z) - 1. \tag{2.3}$$

Corollary 2.3. *Let $a_k, b_k, c_k > 0$ for $k = 1, 2$. Furthermore, let*

$$\sum_{n=2}^{\infty} \frac{p_n - \alpha u_n}{1-\alpha} \frac{(a_1)_{n-1}(b_1)_{n-1}}{(c_1)_{n-1}(1)_{n-1}} + \sum_{n=1}^{\infty} \frac{q_n - (-1)^{j-i}\alpha v_n}{1-\alpha} \frac{(a_2)_n(b_2)_n}{(c_2)_n(1)_n} \leq 1 \tag{2.4}$$

where $0 \leq \alpha < 1$, $i, j \in \{0, 1\}$, $p_n > u_n \geq 0$ $(n = 2, 3, \ldots)$ and $q_n > v_n \geq 0$ $(n = 1, 2, \ldots)$. Then $\phi_1 + \overline{\phi_2} \in \mathcal{H}(\Phi_i, \Psi_j; \alpha)$, ϕ_1 and ϕ_2 being given by (2.3).

The next corollary provides a sufficient condition for $\psi_1 + \overline{\psi_2}$ to belong to the class $\mathcal{H}(\Phi_i, \Psi_j; \alpha)$ where $\psi_1(z) \equiv \psi_1(a_1, b_1, c_1; z)$ and $\psi_2(z) \equiv \psi_2(a_2, b_2, c_2; z)$ are analytic functions defined by

$$\psi_1(z) := \int_0^z F(a_1, b_1, c_1; t)\,dt \quad \text{and} \quad \psi_2(z) := \int_0^z (F(a_2, b_2, c_2; t) - 1)\,dt. \tag{2.5}$$

Corollary 2.4. *Suppose that $a_k, b_k, c_k > 0$ for $k = 1, 2$ and*

$$\sum_{n=2}^{\infty} \frac{p_n - \alpha u_n}{1-\alpha} \frac{(a_1)_{n-1}(b_1)_{n-1}}{(c_1)_{n-1}(1)_n} + \sum_{n=2}^{\infty} \frac{q_n - (-1)^{j-i}\alpha v_n}{1-\alpha} \frac{(a_2)_{n-1}(b_2)_{n-1}}{(c_2)_{n-1}(1)_n} \leq 1 \tag{2.6}$$

where $0 \leq \alpha < 1$, $i, j \in \{0, 1\}$, $p_n > u_n \geq 0$ $(n = 2, 3, \ldots)$ and $q_n > v_n \geq 0$ $(n = 2, 3, \ldots)$. Then $\psi_1 + \overline{\psi_2} \in \mathcal{H}(\Phi_i, \Psi_j; \alpha)$, ψ_1 and ψ_2 being given by (2.5).

It is worth to remark that Theorems 2.2, 2.4 and 2.11 of [4] are particular cases of Corollary 2.3, while [4, Theorem 2.8] follows as a special case of Corollary 2.4. We next show that the coefficient condition (2.1) is also necessary for functions in $\mathcal{TH}(\Phi_i, \Psi_j; \alpha)$.

Theorem 2.5. *Let the function $f = h + \bar{g}$ be such that h and g are given by* (1.2). *Then $f \in \mathcal{TH}(\Phi_i, \Psi_j; \alpha)$ if and only if*

$$\sum_{n=2}^{\infty} \frac{p_n - \alpha u_n}{1-\alpha} A_n + \sum_{n=1}^{\infty} \frac{q_n - (-1)^{j-i}\alpha v_n}{1-\alpha} B_n \leq 1 \tag{2.7}$$

where $0 \leq \alpha < 1$, $i, j \in \{0, 1\}$, $p_n > u_n \geq 0$ $(n = 2, 3, \dots)$ and $q_n > v_n \geq 0$ $(n = 1, 2, \dots)$.

Proof. The sufficient part follows by Theorem 2.1 upon noting that $\mathcal{TH}(\Phi_i, \Psi_j; \alpha) \subset \mathcal{H}(\Phi_i, \Psi_j; \alpha)$. For the necessary part, let $f \in \mathcal{TH}(\Phi_i, \Psi_j; \alpha)$. Then (1.3) yields

$$\begin{aligned} &\operatorname{Re}\left(\frac{(f * \Phi_i)(z)}{(f * \Psi_j)(z)} - \alpha\right) \\ &= \operatorname{Re}\left(\frac{(1-\alpha)z + \sum_{n=2}^{\infty}(p_n - \alpha u_n)A_n z^n + (-1)^i \sum_{n=1}^{\infty}(q_n - (-1)^{j-i}\alpha v_n)B_n \bar{z}^n}{z + \sum_{n=2}^{\infty} A_n u_n z^n + (-1)^j \sum_{n=1}^{\infty} B_n v_n \bar{z}^n}\right) \\ &\geq \frac{(1-\alpha) - \sum_{n=2}^{\infty}(p_n - \alpha u_n)A_n |z|^{n-1} - \sum_{n=1}^{\infty}(q_n - (-1)^{j-i}\alpha v_n)B_n |z|^{n-1}}{1 + \sum_{n=2}^{\infty} A_n u_n |z|^{n-1} + \sum_{n=1}^{\infty} B_n v_n |z|^{n-1}} > 0. \end{aligned}$$

The above inequality must hold for all $z \in \mathbb{D}$. In particular, choosing the values of z on the positive real axis and letting $z \to 1^-$ we obtain the required condition (2.7). □

Theorem 2.5 immediately yields the following three corollaries.

Corollary 2.6. *For $f = h + \bar{g} \in \mathcal{TH}(\Phi_i, \Psi_j; \alpha)$ where h and g are given by* (1.2), *we have*

$$A_n \leq \frac{1-\alpha}{p_n - \alpha u_n} \ (n = 2, 3, \dots) \quad \text{and} \quad B_n \leq \frac{1-\alpha}{q_n - (-1)^{j-i}\alpha v_n} \ (n = 1, 2, \dots);$$

the result being sharp, for each n.

Corollary 2.7. *Let $a_k, b_k, c_k > 0$ for $k = 1, 2$ and ϕ_1, ϕ_2 be given by* (2.3). *Then a necessary and sufficient condition for the harmonic function $\Phi(z) = 2z - \phi_1(z) + \overline{\phi_2(z)}$ to be in the class $\mathcal{TH}(\Phi_i, \Psi_j; \alpha)$ is that* (2.4) *is satisfied.*

Corollary 2.8. *If $a_k, b_k, c_k > 0$ for $k = 1, 2$, then $\Psi(z) = 2z - \psi_1(z) + \overline{\psi_2(z)} \in \mathcal{TH}(\Phi_i, \Psi_j; \alpha)$ if and only if condition* (2.6) *holds, where ψ_1, ψ_2 are given by* (2.5).

Note that [4, Theorem 2.6] is a particular case of Corollary 2.7. By making use of Theorem 2.5, we obtain the following growth estimate for functions in the class $\mathcal{TH}(\Phi_i, \Psi_j; \alpha)$.

Theorem 2.9. *Let $f \in \mathcal{TH}(\Phi_i, \Psi_j; \alpha)$, $\sigma_n = p_n - \alpha u_n$ $(n = 2, 3, \dots)$ and $\Gamma_n = q_n - (-1)^{j-i}\alpha v_n$ $(n = 1, 2, \dots)$. If $\{\sigma_n\}$ and $\{\Gamma_n\}$ are non-decreasing sequences, then*

$$|f(z)| \leq (1 + B_1)|z| + \frac{1-\alpha}{\eta}\left(1 - \frac{q_1 - (-1)^{j-i}\alpha v_1}{1-\alpha} B_1\right)|z|^2,$$

and

$$|f(z)| \geq (1 - B_1)|z| - \frac{1-\alpha}{\eta}\left(1 - \frac{q_1 - (-1)^{j-i}\alpha v_1}{1-\alpha} B_1\right)|z|^2$$

for all $z \in \mathbb{D}$, where $\eta = \min\{\sigma_2, \Gamma_2\}$ and $B_1 = f_{\bar{z}}(0)$.

Proof. Writing $f = h + \bar{g}$ where h and g are given by (1.2), we have

$$\begin{aligned}
|f(z)| &\le (1+B_1)|z| + \sum_{n=2}^{\infty}(A_n+B_n)|z|^n \\
&\le (1+B_1)|z| + \frac{1-\alpha}{\eta}\sum_{n=2}^{\infty}\left(\frac{\eta}{1-\alpha}A_n + \frac{\eta}{1-\alpha}B_n\right)|z|^2 \\
&\le (1+B_1)|z| + \frac{1-\alpha}{\eta}\sum_{n=2}^{\infty}\left(\frac{p_n-\alpha u_n}{1-\alpha}A_n + \frac{q_n-(-1)^{j-i}\alpha v_n}{1-\alpha}B_n\right)|z|^2 \\
&\le (1+B_1)|z| + \frac{1-\alpha}{\eta}\left(1 - \frac{q_1-(-1)^{j-i}\alpha v_1}{1-\alpha}B_1\right)|z|^2
\end{aligned}$$

using the hypothesis and applying Theorem 2.5.

The proof of the left hand inequality follows on lines similar to that of the right hand side inequality. □

The covering result for the class $\mathcal{TH}(\Phi_i, \Psi_j; \alpha)$ follows from the left hand inequality of Theorem 2.9.

Corollary 2.10. *Under the hypothesis of Theorem* 2.9*, we have*

$$\left\{w \in \mathbb{C} : |w| < \frac{1}{\eta}(\eta - 1 + \alpha + (q_1 - (-1)^{j-i}\alpha v_1 - \eta)B_1)\right\} \subset f(\mathbb{D}).$$

Using Theorem 2.5 it is easily seen that the class $\mathcal{TH}(\Phi_i, \Psi_j; \alpha)$ is convex and closed with respect to the topology of locally uniform convergence so that the closed convex hull of $\mathcal{TH}(\Phi_i, \Psi_j; \alpha)$ equals itself. The next theorem determines the extreme points of $\mathcal{TH}(\Phi_i, \Psi_j; \alpha)$.

Theorem 2.11. *Suppose that* $0 \le \alpha < 1$, $i, j \in \{0, 1\}$, $p_n > u_n \ge 0$ $(n = 2, 3, \dots)$ *and* $q_n > v_n \ge 0$ $(n = 1, 2, \dots)$*. Set*

$$h_1(z) = z, \quad h_n(z) = z - \frac{1-\alpha}{p_n - \alpha u_n}z^n \; (n = 2, 3, \dots) \quad \textit{and}$$

$$g_n(z) = z + \frac{1-\alpha}{q_n - (-1)^{j-i}\alpha v_n}\bar{z}^n \; (n = 1, 2, \dots).$$

Then $f \in \mathcal{TH}(\Phi_i, \Psi_j; \alpha)$ *if and only if it can be expressed in the form*

$$f(z) = \sum_{n=1}^{\infty}(X_n h_n + Y_n g_n)(z), \quad X_n \ge 0, \quad Y_n \ge 0 \quad \textit{and} \quad \sum_{n=1}^{\infty}(X_n + Y_n) = 1. \quad (2.8)$$

In particular, the extreme points of $\mathcal{TH}(\Phi_i, \Psi_j; \alpha)$ *are* $\{h_n\}$ *and* $\{g_n\}$.

Proof. Let $f = h + \bar{g} \in \mathcal{TH}(\Phi_i, \Psi_j; \alpha)$ where h and g are given by (1.2). Setting

$$X_n = \frac{p_n - \alpha u_n}{1-\alpha}A_n \; (n = 2, 3, \dots) \quad \text{and} \quad Y_n = \frac{q_n - (-1)^{j-i}\alpha v_n}{1-\alpha}B_n \; (n = 1, 2, \dots),$$

we note that $0 \le X_n \le 1$ $(n = 2, 3, \ldots)$ and $0 \le Y_n \le 1$ $(n = 1, 2, \ldots)$ by Corollary 2.6. We define

$$X_1 = 1 - \sum_{n=2}^{\infty} X_n - \sum_{n=1}^{\infty} Y_n.$$

By Theorem 2.5, $X_1 \ge 0$ and f can be expressed in the form (2.8).

Conversely, for functions of the form (2.8) we obtain

$$f(z) = z - \sum_{n=2}^{\infty} \mu_n z^n + \sum_{n=1}^{\infty} \nu_n \bar{z}^n,$$

where $\mu_n = (1-\alpha)X_n/(p_n - \alpha u_n)$ $(n = 2, 3, \ldots)$ and $\nu_n = (1-\alpha)Y_n/(q_n - (-1)^{j-i}\alpha v_n)$ $(n = 1, 2, \ldots)$. Since

$$\sum_{n=2}^{\infty} \frac{p_n - \alpha u_n}{1-\alpha} \mu_n + \sum_{n=1}^{\infty} \frac{q_n - (-1)^{j-i}\alpha v_n}{1-\alpha} \nu_n = \sum_{n=2}^{\infty} X_n + \sum_{n=1}^{\infty} Y_n = 1 - X_1 \le 1,$$

it follows that $f \in \mathcal{TH}(\Phi_i, \Psi_j; \alpha)$ by Theorem 2.5. □

For harmonic functions of the form

$$f(z) = z - \sum_{n=2}^{\infty} A_n z^n + \sum_{n=1}^{\infty} B_n \bar{z}^n \quad \text{and} \quad F(z) = z - \sum_{n=2}^{\infty} A'_n z^n + \sum_{n=1}^{\infty} B'_n \bar{z}^n, \tag{2.9}$$

where $A_n, B_n, A'_n, B'_n \ge 0$, we define the product $\hat{*}$ of f and F as

$$(f \hat{*} F)(z) = z - \sum_{n=2}^{\infty} A_n A'_n z^n + \sum_{n=1}^{\infty} B_n B'_n \bar{z}^n = (F \hat{*} f)(z), \quad z \in \mathbb{D}.$$

Suppose that $\mathcal{I}$ and $\mathcal{J}$ are subclasses of $\mathcal{TH}$. We say that a class $\mathcal{I}$ is closed under $\hat{*}$ if $f \hat{*} F \in \mathcal{I}$ for all f, $F \in \mathcal{I}$. Similarly, the class $\mathcal{I}$ is closed under $\hat{*}$ with members of $\mathcal{J}$ if $f \hat{*} F \in \mathcal{I}$ for all $f \in \mathcal{I}$ and $F \in \mathcal{J}$. In general, the class $\mathcal{TH}(\Phi_i, \Psi_j; \alpha)$ is not closed under the product $\hat{*}$. The analytic function $f(z) = z - 2z^2$ $(z \in \mathbb{D})$ belongs to $\mathcal{TH}(z + z^2/2, z; 0)$, but $(f \hat{*} f)(z) = z - 4z^2 \notin \mathcal{TH}(z + z^2/2, z; 0)$. However, we shall show that the class $\mathcal{TH}(\Phi_i, \Psi_j; \alpha)$ is closed under $\hat{*}$ with certain members of $\mathcal{TH}$.

Theorem 2.12. *Suppose that $f, F \in \mathcal{TH}$ are given by* (2.9) *with $A'_n \le 1$ and $B'_n \le 1$. If $f \in \mathcal{TH}(\Phi_i, \Psi_j; \alpha)$ then $f \hat{*} F \in \mathcal{TH}(\Phi_i, \Psi_j; \alpha)$.*

Proof. In view of Theorem 2.5, it suffices to show that the coefficients of $f \hat{*} F$ satisfy condition (2.7). Since

$$\begin{aligned} &\sum_{n=2}^{\infty} \frac{p_n - \alpha u_n}{1-\alpha} A_n A'_n + \sum_{n=1}^{\infty} \frac{q_n - (-1)^{j-i}\alpha v_n}{1-\alpha} B_n B'_n \\ &\qquad \le \sum_{n=2}^{\infty} \frac{p_n - \alpha u_n}{1-\alpha} A_n + \sum_{n=1}^{\infty} \frac{q_n - (-1)^{j-i}\alpha v_n}{1-\alpha} B_n \le 1, \end{aligned}$$

the result follows immediately. □

By imposing some restrictions on the coefficients of Φ_i, it is possible for the class $\mathcal{TH}(\Phi_i, \Psi_j; \alpha)$ to be closed under the product $\hat{*}$, as seen by the following corollary.

Corollary 2.13. *If* $f, F \in \mathcal{TH}(\Phi_i, \Psi_j; \alpha)$ *with* $p_n \geq 1$ $(n = 2, 3, \ldots)$ *and* $q_n \geq 1$ $(n = 1, 2, \ldots)$ *then* $f \hat{*} F \in \mathcal{TH}(\Phi_i, \Psi_j; \alpha)$.

Proof. Suppose that f and F are given by (2.9). Using the hypothesis and Corollary 2.6 it is easy to see that $A'_n \leq 1$ $(n = 2, 3, \ldots)$ and $B'_n \leq 1$ $(n = 1, 2, \ldots)$. The result now follows by Theorem 2.12. □

In view of Corollary 2.13, it follows that the classes $\mathcal{TS}^*_H(\alpha)$ and $\mathcal{TK}_H(\alpha)$ are closed under the product $\hat{*}$. The next corollary shows that the class $\mathcal{TH}(\Phi_i, \Psi_j; \alpha)$ is preserved under certain integral transforms.

Corollary 2.14. *If* $f \in \mathcal{TH}(\Phi_i, \Psi_j; \alpha)$ *then* $L_\gamma[f]$ *and* $G_\delta[f]$ *belong to* $\mathcal{TH}(\Phi_i, \Psi_j; \alpha)$, *where* L_γ *and* G_δ *are integral transforms defined by* (1.5) *and* (1.6) *respectively.*

Proof. From the representations of $L_\gamma[f]$ and $G_\delta[f]$, it is easy to deduce that

$$L_\gamma[f](z) = f(z) \hat{*} \left(z - \sum_{n=2}^{\infty} \frac{\gamma + 1}{\gamma + n} z^n + \sum_{n=1}^{\infty} \frac{\gamma + 1}{\gamma + n} \overline{z}^n \right),$$

and

$$G_\delta[f](z) = f(z) \hat{*} \left(z - \sum_{n=2}^{\infty} \frac{1 - \delta^n}{1 - \delta} \frac{z^n}{n} + \sum_{n=1}^{\infty} \frac{1 - \delta^n}{1 - \delta} \frac{\overline{z}^n}{n} \right),$$

where $z \in \mathbb{D}$. The proof of the corollary now follows by invoking Theorem 2.12. □

The next two theorems provide sufficient conditions for the product $\hat{*}$ of $f \in \mathcal{TH}(\Phi_i, \Psi_j; \alpha)$ with certain members of $\mathcal{TH}$ associated with hypergeometric functions to be in the class $\mathcal{TH}(\Phi_i, \Psi_j; \alpha)$.

Theorem 2.15. *Let* $f \in \mathcal{TH}(\Phi_i, \Psi_j; \alpha)$ *and* $\Phi(z) = 2z - \phi_1(z) + \overline{\phi_2(z)}$; ϕ_1 *and* ϕ_2 *being given by* (2.3). *If* $a_k, b_k > 0$, $c_k > a_k + b_k$ *for* $k = 1, 2$ *and if*

$$F(a_1, b_1, c_1; 1) + F(a_2, b_2, c_2; 1) \leq 3,$$

then $f \hat{*} \Phi \in \mathcal{TH}(\Phi_i, \Psi_j; \alpha)$.

Proof. Writing $f = h + \overline{g}$ where h and g are given by (1.2), note that

$$(f \hat{*} \Phi)(z) = z - \sum_{n=2}^{\infty} \frac{(a_1)_{n-1}(b_1)_{n-1}}{(c_1)_{n-1}(1)_{n-1}} A_n z^n + \sum_{n=1}^{\infty} \frac{(a_2)_n (b_2)_n}{(c_2)_n (1)_n} B_n \overline{z}^n, \quad z \in \mathbb{D}.$$

Applying Corollary 2.6, we deduce that

$$\begin{aligned}
\sum_{n=2}^{\infty} \frac{p_n - \alpha u_n}{1 - \alpha} \frac{(a_1)_{n-1}(b_1)_{n-1}}{(c_1)_{n-1}(1)_{n-1}} A_n + \sum_{n=1}^{\infty} \frac{q_n - (-1)^{j-i} \alpha v_n}{1 - \alpha} \frac{(a_2)_n (b_2)_n}{(c_2)_n (1)_n} B_n \\
\leq \sum_{n=2}^{\infty} \frac{(a_1)_{n-1}(b_1)_{n-1}}{(c_1)_{n-1}(1)_{n-1}} + \sum_{n=1}^{\infty} \frac{(a_2)_n (b_2)_n}{(c_2)_n (1)_n} \\
= F(a_1, b_1, c_1; 1) + F(a_2, b_2, c_2; 1) - 2 \leq 1.
\end{aligned}$$

Theorem 2.5 now gives the desired result. □

Theorem 2.16. *Let $f \in \mathcal{TH}(\Phi_i, \Psi_j; \alpha)$, $a_k, b_k > 0$ and $c_k > a_k + b_k$ for $k = 1, 2$. Furthermore, if*

$$F(a_1, b_1, c_1; 1) + F(a_2, b_2, c_2; 1) \leq 4,$$

then $f \hat{*} \Psi \in \mathcal{TH}(\Phi_i, \Psi_j; \alpha)$ where $\Psi(z) = 2z - \psi_1(z) + \overline{\psi_2(z)}$; ψ_1 and ψ_2 being given by (2.5).

Proof. For $f = h + \overline{g}$ where h and g are given by (1.2), we have

$$(f \hat{*} \Psi)(z) = z - \sum_{n=2}^{\infty} \frac{(a_1)_{n-1}(b_1)_{n-1}}{(c_1)_{n-1}(1)_n} A_n z^n + \sum_{n=2}^{\infty} \frac{(a_2)_{n-1}(b_2)_{n-1}}{(c_2)_{n-1}(1)_n} B_n \overline{z}^n, \quad z \in \mathbb{D}.$$

A simple calculation shows that

$$\begin{aligned}
\sum_{n=2}^{\infty} \frac{p_n - \alpha u_n}{1-\alpha} \frac{(a_1)_{n-1}(b_1)_{n-1}}{(c_1)_{n-1}(1)_n} A_n + \sum_{n=2}^{\infty} \frac{q_n - (-1)^{j-i}\alpha v_n}{1-\alpha} \frac{(a_2)_{n-1}(b_2)_{n-1}}{(c_2)_{n-1}(1)_n} B_n \\
\leq \sum_{n=2}^{\infty} \frac{(a_1)_{n-1}(b_1)_{n-1}}{(c_1)_{n-1}(1)_n} + \sum_{n=2}^{\infty} \frac{(a_2)_{n-1}(b_2)_{n-1}}{(c_2)_{n-1}(1)_n} \\
\leq \frac{1}{2} \sum_{n=2}^{\infty} n \frac{(a_1)_{n-1}(b_1)_{n-1}}{(c_1)_{n-1}(1)_n} + \frac{1}{2} \sum_{n=2}^{\infty} n \frac{(a_2)_{n-1}(b_2)_{n-1}}{(c_2)_{n-1}(1)_n} \\
= \frac{1}{2}(F(a_1, b_1, c_1; 1) + F(a_2, b_2, c_2; 1)) - 1 \leq 1,
\end{aligned}$$

using the hypothesis and Corollary 2.6. Hence $f \hat{*} \Psi \in \mathcal{TH}(\Phi_i, \Psi_j; \alpha)$ by Theorem 2.5. □

We close this section by determining the convex combination properties of the members of the class $\mathcal{TH}(\Phi_i, \Psi_j; \alpha)$.

Theorem 2.17. *The class $\mathcal{TH}(\Phi_i, \Psi_j; \alpha)$ is closed under convex combinations.*

Proof. For $k = 1, 2, \ldots$ suppose that $f_k \in \mathcal{TH}(\Phi_i, \Psi_j; \alpha)$ where

$$f_k(z) = z - \sum_{n=2}^{\infty} a_{n,k} z^n + \sum_{n=1}^{\infty} b_{n,k} \overline{z}^n, \quad z \in \mathbb{D}.$$

For $\sum_{k=1}^{\infty} t_k = 1$, $0 \leq t_k \leq 1$, the convex combination of f_k's may be written as

$$f(z) = \sum_{k=1}^{\infty} t_k f_k(z) = z - \sum_{n=2}^{\infty} \gamma_n z^n + \sum_{n=1}^{\infty} \delta_n \overline{z}^n, \quad z \in \mathbb{D}$$

where $\gamma_n = \sum_{k=1}^{\infty} t_k a_{n,k}$ $(n = 2, 3, \ldots)$ and $\delta_n = \sum_{k=1}^{\infty} t_k b_{n,k}$ $(n = 1, 2, \ldots)$. Since

$$\begin{aligned}\sum_{n=2}^{\infty} \frac{p_n - \alpha u_n}{1-\alpha}\gamma_n + \sum_{n=1}^{\infty} \frac{q_n - (-1)^{j-i}\alpha v_n}{1-\alpha}\delta_n \\ &= \sum_{k=1}^{\infty} t_k \left(\sum_{n=2}^{\infty} \frac{p_n - \alpha u_n}{1-\alpha} a_{n,k} + \sum_{n=1}^{\infty} \frac{q_n - (-1)^{j-i}\alpha v_n}{1-\alpha} b_{n,k} \right) \\ &\leq \sum_{k=1}^{\infty} t_k = 1,\end{aligned}$$

it follows that $f \in \mathcal{TH}(\Phi_i, \Psi_j; \alpha)$ by Theorem 2.5. □

3. A particular case

For $0 \leq \alpha < 1$, set $\mathcal{U}_H(\alpha) := \mathcal{H}(z/(1-z) + \bar{z}/(1-\bar{z}), z; \alpha)$. Then $\mathcal{U}_H(\alpha)$ denote the set of all harmonic functions $f \in \mathcal{H}$ that satisfy $\operatorname{Re} f(z)/z > \alpha$ for $z \in \mathbb{D}$ and let $\mathcal{TU}_H(\alpha) := \mathcal{U}_H(\alpha) \cap \mathcal{TH}$. Applying Theorem 2.5 with $p_n - 1 = 0 = u_n$ $(n = 2, 3, \ldots)$, $q_n - 1 = 0 = v_n$ $(n = 1, 2, \ldots)$ and using the results of Section 2, we obtain

Theorem 3.1. *Let the function $f = h + \bar{g}$ be such that h and g are given by* (1.2) *and $0 \leq \alpha < 1$. Then $f \in \mathcal{TU}_H(\alpha)$ if and only if*

$$\sum_{n=2}^{\infty} \frac{A_n}{1-\alpha} + \sum_{n=1}^{\infty} \frac{B_n}{1-\alpha} \leq 1.$$

Furthermore, if $f \in \mathcal{TU}_H(\alpha)$ then $A_n \leq 1-\alpha$ $(n = 2, 3, \ldots)$, $B_n \leq 1-\alpha$ $(n = 1, 2, \ldots)$ and

$$(1 - B_1)|z| - (1 - \alpha - B_1)|z|^2 \leq |f(z)| \leq (1 + B_1)|z| + (1 - \alpha - B_1)|z|^2 \tag{3.1}$$

for all $z \in \mathbb{D}$. In particular, the range $f(\mathbb{D})$ contains the disk $|w| < \alpha$.

Moreover, the extreme points of the class $\mathcal{TU}_H(\alpha)$ are $\{h_n\}$ and $\{g_n\}$ where $h_1(z) = z$, $h_n(z) = z - (1-\alpha)z^n$ $(n = 2, 3, \ldots)$ and $g_n(z) = z + (1-\alpha)\bar{z}^n$ $(n = 1, 2, \ldots)$.

Theorem 3.2. *Let $a_k, b_k > 0$, $c_k > a_k + b_k$ for $k = 1, 2$. Then a necessary and sufficient condition for the harmonic function $\Phi(z) = 2z - \phi_1(z) + \overline{\phi_2(z)}$ to be in the class $\mathcal{TU}_H(\alpha)$ is that*

$$F(a_1, b_1, c_1; 1) + F(a_2, b_2, c_2; 1) \leq 3 - \alpha,$$

where ϕ_1 and ϕ_2 are given by (2.3).

The upper bound given in (3.1) for $f \in \mathcal{TU}_H(\alpha)$ is sharp and equality occurs for the function $f(z) = z + B_1\bar{z} + (1 - \alpha - B_1)\bar{z}^2$ for $B_1 \leq 1 - \alpha$. In a similar fashion, comparable results to Corollary 2.8 and Theorems 2.15, 2.16 for the class $\mathcal{TU}_H(\alpha)$ may also be obtained. For further investigation of results regarding $\mathcal{TU}_H(\alpha)$, we need to prove the following simple lemma.

Lemma 3.3. *Let $f = h + \bar{g} \in \mathcal{H}$ where h and g are given by* (1.1) *with $B_1 = g'(0) = 0$. Suppose that $\lambda \in (0, 1]$.*

(i) *If* $\sum_{n=2}^{\infty}(|A_n|+|B_n|)\leq\lambda$ *then* $f\in\mathcal{U}_H(1-\lambda)$;
(ii) *If* $\sum_{n=2}^{\infty}n(|A_n|+|B_n|)\leq\lambda$ *then* $f\in\mathcal{U}_H(1-\lambda/2)$ *and is starlike of order* $2(1-\lambda)/(2+\lambda)$.

The results are sharp.

Proof. Part (i) follows by Theorem 3.1. For the proof of (ii), note that

$$\sum_{n=2}^{\infty}(|A_n|+|B_n|)\leq\frac{1}{2}\sum_{n=2}^{\infty}n(|A_n|+B_n|)\leq\frac{\lambda}{2}.$$

By part (i) of the lemma, $f\in\mathcal{U}_H(1-\lambda/2)$. The order of starlikeness of f follows by [15, Theorem 3.6]. The functions $z+\lambda\bar{z}^2$ and $z+\lambda\bar{z}^2/2$ show that the results in (i) and (ii) respectively are best possible. □

Using Corollary 2.13, Theorem 3.1 and Lemma 3.3(i), we obtain the following corollary.

Corollary 3.4. *The class* $\mathcal{TU}_H(\alpha)$ *is closed under the product* $\hat{*}$. *In fact*

$$\mathcal{TU}_H(\alpha)\hat{*}\mathcal{TU}_H(\beta)\subset\mathcal{TU}_H(1-(1-\alpha)(1-\beta))$$

for $\alpha,\beta\in[0,1)$.

A well-known classical result involving differential inequalities in univalent function theory is Marx Strohhäcker theorem [13, Theorem 2.6(a), p. 57] which states that if f is an analytic function in $\mathbb{D}$ with $f(0)=0=f'(0)-1$ then

$$\operatorname{Re}\left(1+\frac{zf''(z)}{f'(z)}\right)>0 \quad\Rightarrow\quad \operatorname{Re}\frac{zf'(z)}{f(z)}>\frac{1}{2} \quad\Rightarrow\quad \operatorname{Re}\frac{f(z)}{z}>\frac{1}{2} \quad (z\in\mathbb{D}).$$

The function $f(z)=z/(1-z)$ shows that all these implications are sharp. However, this theorem does not extend to univalent harmonic mappings, that is, if $f\in\mathcal{S}_H$ maps $\mathbb{D}$ onto a convex domain then it is not true in general that $\operatorname{Re} f(z)/z>0$ for all $z\in\mathbb{D}$.. To see this, consider the harmonic half-plane mapping

$$L(z)=\frac{z-\frac{1}{2}z^2}{(1-z)^2}+\overline{\frac{-\frac{1}{2}z^2}{(1-z)^2}},\quad z\in\mathbb{D}$$

which maps the unit disk $\mathbb{D}$ univalently onto the half-plane $\operatorname{Re} w>-1/2$. Figure 1 shows that the function $L(z)/z$ does not have a positive real part in $\mathbb{D}$.

Denote by $\mathcal{TS}_H^{*0}(\alpha)$, $\mathcal{TK}_H^0(\alpha)$ and $\mathcal{TU}_H^0(\alpha)$, the classes consisting of functions f in $\mathcal{TS}_H^*(\alpha)$, $\mathcal{TK}_H(\alpha)$ and $\mathcal{TU}_H(\alpha)$ respectively, for which $f_{\bar{z}}(0)=0$. The next theorem connects the relation between these three classes.

Theorem 3.5. *For* $0\leq\alpha<1$, *the following sharp inclusions hold:*

$$\mathcal{TK}_H^0(\alpha)\subset\mathcal{TU}_H^0\left(\frac{3-\alpha}{2(2-\alpha)}\right); \tag{3.2}$$

and

$$\mathcal{TS}_H^{*0}(\alpha)\subset\mathcal{TU}_H^0\left(\frac{1}{2-\alpha}\right). \tag{3.3}$$

Proof. Let $f = h + \bar{g} \in \mathcal{TH}$ where h and g are given by (1.2). If $f \in \mathcal{TK}_H^0(\alpha)$ then

$$\sum_{n=2}^{\infty} n(A_n + B_n) \leq \frac{1}{2-\alpha} \sum_{n=2}^{\infty} n(n-\alpha)(A_n + B_n) \leq \frac{1-\alpha}{2-\alpha}$$

using [11]. By Lemma 3.3(ii) $f \in \mathcal{TU}_H^0((3-\alpha)/(2(2-\alpha)))$. Regarding the other inclusion, note that if $f \in \mathcal{TS}_H^{*0}(\alpha)$ then

$$\sum_{n=2}^{\infty} (A_n + B_n) \leq \frac{1}{2-\alpha} \sum_{n=2}^{\infty} (n-\alpha)(A_n + B_n) \leq \frac{1-\alpha}{2-\alpha}$$

by [12, Theorem 2, p. 474]. This shows that $f \in \mathcal{TU}_H^0(1/(2-\alpha))$ by Lemma 3.3(i) as desired. The analytic functions $z - (1-\alpha)z^2/(2(2-\alpha))$ and $z - (1-\alpha)z^2/(2-\alpha)$ show that inclusions in (3.2) and (3.3) respectively are sharp. $\square$

Remark 3.6. The proof of Theorem 3.5 shows that if $f \in \mathcal{TK}_H^0(\alpha)$ then f is starlike of order $2/(5-3\alpha)$ by applying Lemma 3.3(ii). This gives the inclusion

$$\mathcal{TK}_H^0(\alpha) \subset \mathcal{TS}_H^{*0}\left(\frac{2}{5-3\alpha}\right).$$

It is not known whether this inclusion is sharp for $\alpha \in (0,1)$. However, if $\alpha = 0$ then the inclusion $\mathcal{TK}_H^0(0) \subset \mathcal{TS}_H^{*0}(2/5)$ is sharp with the extremal function as $f(z) = z + \bar{z}^2/4$.

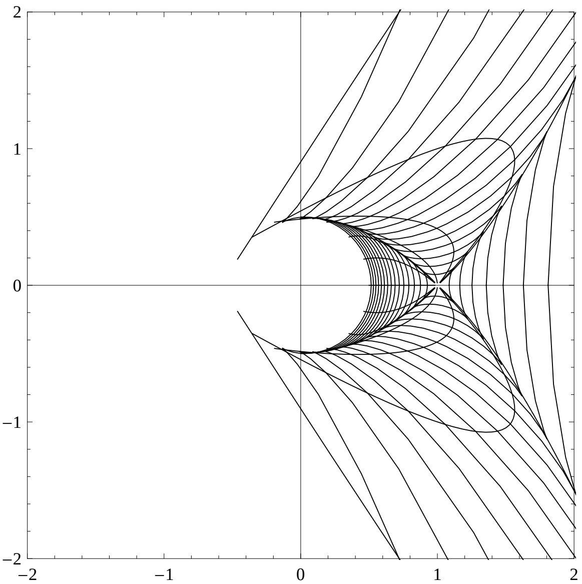

FIGURE 1. Graph of the function $L(z)/z$.

The functions in the class $\mathcal{TU}_H(\alpha)$ need not be univalent in $\mathbb{D}$. The last theorem of this section determines the radius of univalence, starlikeness and convexity of the class $\mathcal{TU}_H^0(\alpha)$.

Theorem 3.7. *The radius of univalence of the class $\mathcal{TU}_H^0(\alpha)$ is $1/(2(1-\alpha))$. This bound is also the radius of starlikeness of $\mathcal{TU}_H^0(\alpha)$. The radius of convexity of the class $\mathcal{TU}_H^0(\alpha)$ is $1/(4(1-\alpha))$.*

Proof. Let $f = h + \bar{g} \in \mathcal{TU}_H^0(\alpha)$ where h and g are given by (1.2) and let $r \in (0,1)$ be fixed. Then $r^{-1}f(rz) \in \mathcal{TU}_H^0(\alpha)$ and we have

$$\sum_{n=2}^{\infty} n(A_n + B_n)r^{n-1} \leq \sum_{n=2}^{\infty} \left(\frac{A_n}{1-\alpha} + \frac{B_n}{1-\alpha}\right) \leq 1$$

provided $nr^{n-1} \leq 1/(1-\alpha)$ which is true if $r \leq 1/(2(1-\alpha))$. In view of [18, Theorem 1, p. 284], f is univalent and starlike in $|z| < 1/(2(1-\alpha))$. Regarding the radius of convexity, note that

$$\sum_{n=2}^{\infty} n^2(A_n + B_n)r^{n-1} \leq \sum_{n=2}^{\infty} \left(\frac{A_n}{1-\alpha} + \frac{B_n}{1-\alpha}\right) \leq 1$$

provided $n^2r^{n-1} \leq 1/(1-\alpha)$ which is true if $r \leq 1/(4(1-\alpha))$. The function $f(z) = z + (1-\alpha)\bar{z}^2$ shows that these bounds are sharp. This completes the proof of the theorem. □

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On the global uniform asymptotic stability of time-varying dynamical systems

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Abstract. The objective of this work is twofold. In the first part, we present sufficient conditions for global uniform asymptotic stability and/or practical stability in terms of Lyapunov-like functions for nonlinear time varying systems. Furthermore, an illustrative numerical example is presented.

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1. Introduction

The problem of stability analysis of nonlinear time-varying systems has attracted the attention of several researchers and has produced a vast body of important results [1, 2, 13, 11] and the references therein.

This fact motivated to study systems whose desired behavior is asymptotic stability about the origin of the state space or a close approximation to this, e.g., all state trajectories are bounded and approach a sufficiently small neighborhood of the origin [5] and references therein. Quite often, one also desires that the state approaches the origin (or some sufficiently small neighborhood of it) in a sufficiently fast manner. To this end, the authors of [6] introduce a concept of exponential rate of convergence and for a specific class of uncertain systems they present controllers which guarantee this behavior. This property is referred to us as practical stability (see [4] for the more explanation and [3]).

On the other hand, the asymptotic stability is more important than stability, also the desired system may be unstable and yet the system may oscillate sufficiently near this state that its performance is acceptable, thus the notion of practical stability is more suitable in several situations than Lyapunov stability (see [9], [10]).

This paper aims, as first objective, to provide sufficient conditions that ensure the global uniform practical stability of system (2.1). A new quick proof for the results of [4] is also presented. Next, sufficient conditions for the GUPAS are presented.

Moreover, an example in dimensional two is given to illustrate the applicability of the result.

2. Definitions and tools

Consider the time varying system described by the following:

$$\dot{x} \quad = \quad f(t,x) + g(t,x) \tag{2.1}$$

where $f : \mathbb{R}^+ \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ and $g : \mathbb{R}^+ \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ are piecewise continuous in t and locally Lipschitz in x on $\mathbb{R}^+ \times \mathbb{R}^n$. We consider also the associated nominal system

$$\dot{x} \quad = \quad f(t,x) \tag{2.2}$$

For all $x_0 \in \mathbb{R}^n$ and $t_0 \in \mathbb{R}$, we will denote by $x(t;t_0,x_0)$, or simply by $x(t)$, the unique solution of (2.1) at time t_0 starting from the point x_0.

Unless otherwise stated, we assume throughout the paper that the functions encountered are sufficiently smooth. We often omit arguments of functions to simplify notation, $\|.\|$ stands for the Euclidean norm vectors. We recall now some standard concepts from stability and practical stability theory; any book on Lyapunov stability can be consulted for these; particularly good references are [7, 8]: $\mathcal{K}$ is the class of functions $\mathbb{R}^+ \to \mathbb{R}^+$ which are zero at the origin, strictly increasing and continuous. $\mathcal{K}_\infty$ is the subset of $\mathcal{K}$ functions that are unbounded. $\mathcal{L}$ is the set of functions $\mathbb{R}^+ \to \mathbb{R}^+$ which are continuous, decreasing and converging to zero as their argument tends to $+\infty$. $\mathcal{KL}$ is the class of functions $\mathbb{R}^+ \times \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ which are class $\mathcal{K}$ on the first argument and class $\mathcal{L}$ on the second argument. A positive definite function $\mathbb{R}^+ \to \mathbb{R}^+$ is one that is zero at the origin and positive otherwise. We define the closed ball $B_r := \Big\{ x \in \mathbb{R}^n : \ \|x\| \leq r \Big\}$.

In order to simplify the notation, we use the following notation.

$$\nabla_t V + \nabla_x^\top V \cdot f(t,x) \quad = \quad \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} \cdot f(t,x) \tag{2.3}$$

where $V : \mathbb{R} \times \mathbb{R}^n \longrightarrow \mathbb{R}$.

Next, we need the definitions given below.

Definition 2.1 (uniform stability of B_r).

1. *The solutions of system* (2.1) *are said to be uniformly bounded if for all α and any $t_0 \geq 0$ there exists a $\beta(\alpha) > 0$ so that $\|x(t)\| < \beta$ for all $t \geq t_0$ whenever $\|x_0\| < \alpha$.*
2. *The ball B_r is said to be uniformly stable if for all $\varepsilon > r$, there exists $\delta := \delta(\varepsilon)$ such that for all $t_0 \geq 0$*

$$\|x_0\| < \delta \implies \|x(t)\| < \varepsilon, \ \forall t \geq t_0. \tag{2.4}$$

3. *B_r is globally uniformly stable if it is uniformly stable and the solutions of system* (2.1) *exist and are globally uniformly bounded.*

Definition 2.2 (uniform attractivity of B_r). *B_r is globally uniformly attractive if for all $\varepsilon > r$ and $c > 0$, there exists $T =: T(\varepsilon, c) > 0$ such that for all $t_0 \geq 0$,*

$$\|x(t)\| < \varepsilon \ \forall t \geq t_0 + T, \ \|x_0\| < c.$$

System (2.2) is globally uniformly practically asymptotically stable (GUPAS) if there exists $r \geq 0$ such that B_r is globally uniformly stable and globally uniformly attractive.

Sufficient condition for GUPAS is the existence of a class $\mathcal{KL}$ function β and a constant $r > 0$ such that, given any initial state x_0, ensuing trajectory $x(t)$ satisfies:

$$\|x(t)\| \quad \leq \quad r + \beta(\|x_0\|, t), \quad \forall t \geq t_0. \tag{2.5}$$

If the class $\mathcal{KL}$ function β on the above relation (2.5) is of the form $\beta(r, t) = kr^{-\lambda t}$, with $\lambda, k > 0$ we say that the system (2.2) is globally uniformly practically exponentially stable (GUPES). It is also, worth to notice that if in the above definitions, we take $r = 0$, then one deals with the standard concept of GUAS and GUES. Moreover, in the rest of this paper, we study the asymptotic behavior of a small ball centered at the origin for $0 \leq\| x(t) \| -r$, so that if $r = 0$ in the above definitions we find the classical definition of the uniform asymptotic stability of the origin viewed as an equilibrium point (see [8] for more details).

In the sequel and in the order to solve the problem of GUAS and uniform exponential convergence to the ball B_r of the perturbed system (2.1), we introduce two technical lemmas, where the proof of the second one is given in appendixes, that will be crucial in establishing the main result of this work.

Lemma 2.3. [12] *Let $\varphi : [0, +\infty) \longrightarrow [0, +\infty)$ be a continuous function, ε is a positive real number and λ is a strictly positive real number. Assume that for all $t \in [0, +\infty)$ and $0 \leq u \leq t$, we have*

$$\varphi(t) - \varphi(u) \quad \leq \quad \int_u^t (-\lambda\varphi(s) + \varepsilon) \, \mathrm{d}s. \tag{2.6}$$

Then

$$\varphi(t) \quad \leq \quad \frac{\varepsilon}{\lambda} + \varphi(0) \exp(-\lambda t). \tag{2.7}$$

Lemma 2.4. *Let $y : [0, +\infty[\longrightarrow [0, +\infty[$ be a differentiable function, α be a class $\mathcal{K}_\infty$ function and c be a positive real number. Assume that for all $t \in [0, +\infty[$ we have,*

$$\dot{y}(t) \quad \leq \quad -\alpha\Big(y(t)\Big) + c. \tag{2.8}$$

Then, there exists a class $\mathcal{KL}$ function β_α such that

$$y(t) \quad \leq \quad \alpha^{-1}(2c) + \beta_\alpha\Big(y(0), t\Big). \tag{2.9}$$

Proof. Let α of class $\mathcal{K}_\infty$ and c be such that (2.8) holds.
Since α is a class $\mathcal{K}_\infty$ function, then there exists some constant d such that $\alpha(d) = 2c$. Using this, one sees that

$$\dot{y}(t) \leq -\frac{\alpha(y(t))}{2}, \quad \text{whenever } y(t) \geq d. \tag{$*$}$$

Let us introduce the set

$$S = \{y : y(t) \leq d\}.$$

We claim that the set S is forward invariant. That is to say that if $x(t_0) \in S$ for some $t_0 \geq 0$, then $x(t) \in S$ for all $t \geq t_0$. Indeed, suppose to the contrary that there exists $a > t_0$ such that $y(a) > d$.
Consider the set

$$\Delta = \{x < a \in \mathbb{R}_+ \text{ such that } y(t) > d,\ \forall t \in]x, a[\}$$

Since $t_1 \in \Delta$ and the continuity of y on $\mathbb{R}_+$, we have $\Delta \neq \emptyset$. Also, since Δ is lower bounded by t_0 then, $m = \inf \Delta$ is finite.
We know that $m \in \bar{\Delta}$, where $\bar{\Delta}$ denotes the closure of Δ. So that, we get by continuity $y(m) \geq d$, $m \geq t_0$ moreover, we have $y(t) > d$ for $t \in]m, a[$. It follows that the inequality in the left hand side of $(*)$ holds for each $t \in]m, a[$, and therefore that the absolutely continuous function $y(t)$ has a negative derivative almost everywhere on the interval $]m, a[$. Thus $y(a) \leq y(t_0) \leq d$. This contradicts the fact that $y(a) > d$.
So S must indeed be forward invariant, as claimed.
We continue now the proof of the Lemma. We distinguish the two possible cases:

- If $y(0) \leq d$, then $y(t) \leq d$ for all $t \geq 0$ as claimed above.
- If $y(0) > d$, then there exists $t_1 > 0$ (possibly $t_1 = +\infty$) such that $y(t) > d$, for all $t \in [0, t_1[$ and $y(t) \leq d$, for all $t \geq t_1$, with the understanding that the second case does not happen if $t_1 = +\infty$.
  Now we apply Lemma 2.2 in [14], to obtain a class $\mathcal{KL}$ function β_α such that $y(t) \leq \beta_\alpha(y(0), t), \forall t \in [0, t_1[$.

So every where we have,

$$y(t) \leq d + \beta_\alpha\Big(y(0), t\Big) = \alpha^{-1}(2c) + \beta_\alpha\Big(y(0), t\Big).$$

which completes the proof.

3. Main results

In what follows, with the aid of the previous Lemmas we give some new results on GUAS and practical stability.

3.1. Global uniform asymptotic stability

In this section we suppose that the origin $x = 0$ is equilibrium point for system (2.2) and the perturbation g vanishes, that is $g(t, 0) = 0, \forall t \geq 0$.
Consider the nonlinear system (2.2) and introduce a set of assumptions.
Assume that:

(A1). There exists continuously differentiable $V : [0, +\infty[\times\mathbb{R}^n \longrightarrow [0, +\infty[$, such that

$$c_1\|x\|^c \leq V(t, x) \leq c_2\|x\|^c \quad (3.1a)$$

$$\nabla_t V + \nabla_x^\top V f(t, x) \leq -c_3\|x\|^c \quad (3.1b)$$

$$\left|\nabla_x^\top V g(t, x)\right| \leq a(\|x\|)b(t) \quad (3.1c)$$

where $c_1, c_2,\ c_3$ and c are strictly positive constants, $a\ :\ [0,+\infty[\longrightarrow\mathbb{R}$ and $b:[0,+\infty[\longrightarrow\mathbb{R}$ are continuous functions satisfying:

$$\int_0^\infty b(s)\,\mathrm{d}s \quad\leq\quad b_\infty \qquad \lim_{t\to\infty} b(t)=0 \tag{3.2}$$

for some constant b_∞.

(A2). There exist some constants $k>0$ and $r\geq 0$, such that

$$a(\|x\|) \quad\leq\quad k\|x\|^c, \quad \text{for all} \quad \|x\|\geq r. \tag{3.3}$$

We state the following result.

Proposition 3.1. *Under assumptions (A1) and (A2), the equilibrium point $x=0$ of* (2.1) *is globally uniformly asymptotically stable.*

Proof. We first prove the forward completeness of system (2.1) by proving that the finite escape time phenomenon does not occur, secondly we prove global uniform asymptotic stability in the sense of Lyapunov.

No finite escape time:

The time derivative of $V(t,x)$ along the trajectories of system (2.1) is given by:

$$\dot{V}(t,x) \;=\; \nabla_t V + \nabla_x^\top V.f(t,x) + \nabla_x^\top .g(t,x) \tag{3.4}$$

From inequalities (3.1a), (3.1b) and (3.1c), we have

$$\begin{aligned}\|x(t)\|^c \quad&\leq\quad \frac{1}{c_1}\,V(0,x(0)) + \frac{1}{c_1}\int_0^t \dot{V}(s,x(s))\,\mathrm{d}s\\ &\leq\quad \frac{c_2}{c_1}\,\|x(0)\|^c + \frac{1}{c_1}\int_0^t b(s)a(\|x(s)\|)\,\mathrm{d}s.\end{aligned}$$

Let

$$\alpha_r = \sup_{\|x\|\leq r} a(\|x\|)$$

and

$$\alpha_b = \sup_{t\geq 0} b(t).$$

From (3.2) and the continuity of a and b we have α_r and α_b are finite.

It yields, by taking into account assumption $(A2)$,

$$\|x(t)\|^c \quad\leq\quad \frac{c_2}{c_1}\,\|x(0)\|^c + \frac{\alpha_b}{c_1}\int_0^t \left(\alpha_r + k\,\|x(s)\|^c\right)\,\mathrm{d}s.$$

An application of Gronwall's Lemma shows that the solutions exist for all $t\geq 0$.

Global uniform asymptotic stability:

Since $\lim\limits_{t\to\infty} b(t)=0$, there exists a time $t_1\geq 0$, such that $b(t)\leq\frac{c_3}{2k}$ for all $t\geq t_1$. Hence, by (3.3), we obtain for all $\|x\|\geq r$ and for all $t\geq t_1$,

$$\dot{V} \quad\leq\quad -c_3\,\|x\|^c + \frac{c_3}{2k}\,a(\|x\|) \;\leq\; -\frac{c_3}{2}\,\|x\|^c.$$

So,

$$\dot{V}(t,x) \quad \leq \quad -\frac{c_3}{2}V(t,x).$$

This implies that the sphere $S = \{x : \|x\| \leq r\}$ is globally attractive, that is, $\lim_{t\to\infty} d(x(t),S) = 0$ (d is the distance[1] between x and S). Thus, boundedness of solutions follows.
We invoke now Lemma 2.3 to show that the solution $x(t)$ goes to zero. Put $x_0 = x(t_0)$. Since the solutions are bounded, given any $c' > 0$, there exists $\beta_1 > 0$, such that for all $|x_0| < c'$ we have $|x(t)| < \beta_1$ for all $t \geq t_0$. From now on, we fix an arbitrary constant $c' > 0$ and x_0 such that $|x_0| < c'$.
Define:

$$M_a = \sup_{|x|<\beta_1} a(\|x\|).$$

By continuity of $a(.)$ on $\mathbb{R}^+$, M_a is finite. Observe also that M_a is independent of x_0 for $|x_0| < c'$.
Using (3.1) and (3.4), we obtain

$$\dot{V}(t,x) \quad \leq \quad -\frac{c_3}{c_2}V(t,x) + M_a b(t). \tag{3.5}$$

Let $\epsilon > 0$ be arbitrary. We shall prove that there exists a time $T > 0$ such that $\|x(t)\| < \epsilon$, for all $t \geq T + t_0$.
First notice that, since $\lim_{t\to\infty} b(t) = 0$, there exists a time T_1, such that

$$M_a b(t) \quad \leq \quad \frac{1}{2}\frac{c_3 c_1}{c_2}\epsilon^c, \quad \forall t \geq T_1. \tag{3.6}$$

Integrating inequality (3.5) from $u \in [T_1, t]$ to $t \geq T_1$, on both sides of the inequality, we get

$$V(t,x(t)) - V(u,x(u)) \quad \leq \quad \int_u^t \frac{c_3}{c_2}\left(-V(s,x(s)) + \frac{\epsilon^c c_1}{2}\right) \mathrm{d}s \tag{3.7}$$

Using Lemma 2.3, the inequality above implies that,

$$V(t,x(t)) \quad \leq \quad \frac{\epsilon^c c_1}{2} + V(T_1, x(T_1)) \exp\left(-\frac{c_3}{c_2}(t - T_1)\right) \tag{3.8}$$

$$\leq \quad \frac{\epsilon^c c_1}{2} + c_2\|x(T_1)\|^c \exp\left(-\frac{c_3}{c_2}(t - T_1)\right) \tag{3.9}$$

$$\leq \quad \frac{\epsilon^c c_1}{2} + c_2\beta_1^c \exp\left(-\frac{c_3}{c_2}(t - T_1)\right). \tag{3.10}$$

with $\beta_1 = \|x(T_1)\|$.

This is because $x(t)$ is bounded by β_1 for all $t \geq t_0 \geq 0$.
On the other hand, there exists a time T_2 (which is independent of x_0), such that

$$c_2\beta_1^c \exp\left(-\frac{c_3}{c_2}(t - T_1)\right) \quad \leq \quad \frac{1}{2}\epsilon^c c_1, \quad \forall t \geq T_2. \tag{3.11}$$

[1] $d(x,S) = \inf_{s\in S} \|x - s\|$

Using inequalities (3.8) and (3.11), we obtain

$$V(t,x(t)) \quad \leq \quad \epsilon^c c_1, \ \forall t \geq T = \max\left(T_1, T_2\right).$$

Using the fact that,

$$c_1\|x(t)\|^c \leq V(t,x(t)), \ \forall t \geq T + t_0.$$

We have,

$$\|x(t)\| \leq \epsilon, \ \forall t \geq T + t_0.$$

Therefore, the present proof is complete.
Following the same analysis as above one can prove the following proposition in the case when we replace $c_i\|x\|^c$, $i = 1,2,3$ by some $\mathcal{K}_\infty$ functions.

Proposition 3.2. *Under assumptions (A'1) below and (A2), the origin $x = 0$ of* (2.1) *is globally uniformly asymptotically stable equilibrium point,*

where

(A'1). There exists continuously differentiable $V : \mathbb{R}^+ \times \mathbb{R}^n \longrightarrow \mathbb{R}^+$ such that

$$\alpha_1(\|x\|) \ \leq \ V(t,x) \ \leq \ \alpha_2(\|x\|) \tag{3.12a}$$

$$\nabla_t V + \nabla_x^\top V f(t,x) \ \leq \ -\alpha_3(\|x\|) \tag{3.12b}$$

$$\left|\nabla_x^\top V g(t,x)\right| \ \leq \ a(\|x\|)b(t) \tag{3.12c}$$

for some class $\mathcal{K}_\infty$ functions $\alpha_i, i = 1,2,3$ and a, b satisfying (3.2).

3.2. Global uniform practical stability

In this section, it is worth to notice that the origin is not required to be an equilibrium point for the system (2.2). This may be in many situations meaningful from a practical point of view specially, when stability for uncertain systems is investigated. As pointed out in [3], necessary conditions for system (2.2) to be globally uniformly practically exponentially stable have been derived in [4] as follows.

Theorem 3.3. [4] *Consider system* (2.2). *Let $V : \mathbb{R}_+ \times \mathbb{R}^n \longrightarrow \mathbb{R}$ be a continuously differentiable function, such that*

$$c_1\|x\|^2 \ \leq \ V(t,x) \quad \leq \quad c_2\|x\|^2 \tag{3.13a}$$

$$\nabla_t V + \nabla_x^\top V f(t,x) \quad \leq \quad -c_3 V(t,x) + r \tag{3.13b}$$

for all $t \geq 0$, $x \in \mathbb{R}^n$, with c_1, c_2, c_3 are positive constants and r non negative constant. Then the ball B_α is globally uniformly exponentially stable, with $\alpha = \frac{r}{c_1 c_3}$.

The proof proposed in [4] is very clever but here, using lemma 2.3, we can give a shorter proof.
Proof. Indeed, the time derivative of V along the trajectories of system (2.1) is

$$\dot{V}(t,x) \ = \ \nabla_t V + \nabla_x^\top V f(t,x). \tag{3.14}$$

Using equation (3.13b) we get

$$\dot{V}(t,x) \quad \leq \quad -c_3 V(t,x) + r. \tag{3.15}$$

Now, integrating both sides the above inequality from $t \geq 0$ to $u \in [0, t]$, we obtain

$$V(t, x(t)) - V(u, x(u)) \leq \int_u^t (-c_3 V(s, x(s)) + r) \, \mathrm{d}s. \tag{3.16}$$

By applying lemma 2.3 with $\lambda = c_3$ and $\varepsilon = r$, it yields

$$V(t, x(t)) \leq \frac{r}{c_3} + V(0, x(0)) \exp(-\lambda t) \tag{3.17}$$

Which implies that,

$$\begin{aligned} \|x(t)\| &\leq \left(\tfrac{r}{c_1 c_3} + \frac{1}{c_1} V(0, x(0)) \exp(-\lambda t) \right)^{\frac{1}{2}} \\ &\leq \sqrt{\tfrac{r}{c_1 c_3}} + \sqrt{\frac{1}{c_1} V(0, x(0))} \ \exp\left(-\tfrac{\lambda}{2} t\right), \\ &\leq \sqrt{\tfrac{r}{c_1 c_3}} + \sqrt{\frac{1}{c_1} c_2} \ \|x(0))\| \ \exp\left(-\tfrac{\lambda}{2} t\right). \end{aligned}$$

This completes the proof.
The previous theorem can be generalized as follows.

Theorem 3.4. *Consider system* (2.2)*. Let* $V : \mathbb{R}_+ \times \mathbb{R}^n \longrightarrow \mathbb{R}$ *be a continuously differentiable function, such that*

$$c_1 \|x\|^c \leq V(t, x) \leq c_2 \|x\|^c \tag{3.18a}$$

$$\nabla_t V + \nabla_x^\top V f(t, x) \leq -c_3 V(t, x) + r \tag{3.18b}$$

for all $t \geq 0$, $x \in \mathbb{R}^n$, *with* c, c_1, c_2, c_3 *are positive constants and* r *a non negative constant. Then,*

1. *if* $c \geq 1$, *the ball* B_{α_1}, *where* $\alpha_1 = \left(\frac{r}{c_1 c_3}\right)^{\frac{1}{c}}$, *is globally uniformly exponentially stable.*
2. *if* $c \leq 1$, *the ball* B_{α_2}, *where* $\alpha_2 = 2^{\frac{1}{c} - 1} \left(\frac{r}{c_1 c_3}\right)^{\frac{1}{c}}$, *is globally uniformly exponentially stable.*

Proof. Following a similar reasoning as above, one can prove easily that

$$\|x(t)\| \leq \left(\frac{r}{c_1 c_3} + c_2 c_1^{-1} \ \|x(0)\|^c \ \exp(-\lambda t) \right)^{\frac{1}{c}} \tag{3.19}$$

1. If $c \geq 1$, by using the fact that $(a + b)^\varepsilon \leq a^\varepsilon + b^\varepsilon$, for all $a, b \geq 0$ and $\varepsilon \in]0, 1]$, one obtains

$$\|x(t)\| \leq \left(\frac{r}{c_1 c_3} \right)^{\frac{1}{c}} + (c_2 c_1^{-1})^{\frac{1}{c}} \ \|x(0)\| \exp\left(-\frac{\lambda}{c} t\right) \tag{3.20}$$

2. If $c \leq 1$. Since $(a + b)^p \leq 2^{p-1} (a^p + b^p)$, for all $a, b \geq 0$ and $p \geq 1$, one can get the conclusion by using a similar reasoning as above. This completes the proof.

Another interesting result relying upon the notion of global uniform asymptotic practical stability is the following theorem.

Theorem 3.5. *Consider the nonlinear system* (2.2). *Assume that there exists a continuously differentiable function* $V : \mathbb{R}_+ \times \mathbb{R}^n \longrightarrow \mathbb{R}$ *satisfying*

$$\alpha_1(\|x\|) \leq V(t,x) \leq \alpha_2(\|x\|), \tag{3.21a}$$

$$\nabla_t V + \nabla_x^\top V f(t,x) \leq -\alpha_3(V(t,x)) + c \tag{3.21b}$$

for some class $\mathcal{K}_\infty$ *functions* α_1, α_2, α_3, α_4 *and* $c \geq 0$. *Then the ball* B_r *is globally uniformly asymptotically practically stable, where* $r = \alpha_1^{-1}\left(\alpha_3^{-1}(2c)\right)$.

Proof. Let $x_0 \in \mathbb{R}^n$ and consider the trajectory $x(t)$ with the initial condition $x(0) = x_0$ and define $y(t) = V(t, x(t))$.
Equation (3.21b) implies that

$$\dot{y}(t) \leq -\alpha_3(y(t)) + c \tag{3.22}$$

Hence, from lemma 2.4 one one can deduce the existence of a class $\mathcal{KL}$ function β such that

$$V(t,x(t)) \leq \alpha_3^{-1}(2c) + \beta\Big(V(0,x_0),t\Big). \tag{3.23}$$

Put $\beta_d(r,s) = \beta(\alpha_2(r),s)$ which is a class $\mathcal{K}_\infty$ function. Thus, using (3.21a), the inequality above implies that

$$\|x(t)\| \leq \alpha_1^{-1}\Big(\alpha_3^{-1}(2c) + \beta_d\Big(\|x_0\|,t\Big)\Big) \tag{3.24}$$

$$\leq \alpha_1^{-1}\left(\alpha_3^{-1}(2c)\right) + \alpha_1^{-1}\Big(2\beta_d\Big(\|x_0\|,t\Big)\Big). \tag{3.25}$$

Here, we use the following general fact, a weak form of the triangle inequality which holds for any function γ of class $\mathcal{K}$ and any $a, b \geq 0$ (see Lemma 3):

$$\gamma(a+b) \leq \gamma(2a) + \gamma(2b). \tag{3.26}$$

It then holds that the GUPAS is fulfilled with $r = \alpha_1^{-1}\left(\alpha_3^{-1}(2c)\right)$ and

$$\beta\Big(s,t\Big) = \alpha_1^{-1}\Big(\beta_d\Big(s,t\Big)\Big).$$

Lemma 3.6. *let* $\gamma : [0,+\infty[\to [0,+\infty[$ *a function of class* $\mathcal{K}$. *Then, for all* $x, y \geq 0$, *we have:*

$$\gamma(x+y) \leq \gamma(2x) + \gamma(2y).$$

Proof. Clearly, if $x = 0$ or $y = 0$, then (3.26) holds. Let $x > 0$ and $y > 0$ and define the sets $I_x = [0,x]$, $I_y = [0,y]$ which are compact for every fixed x and y (since they are closed and bounded subsets of $[0,+\infty[\subset \mathbb{R}$). If $y \leq x$, the point $y \in I_x$. As a consequence, we have

$$\gamma(x+y) \leq \max_{s \in I_x} \gamma(x+s) := \gamma_1(x).$$

γ_1 is well defined due to the continuity of γ and the compactness of I_x. But, if $s \in I_x$, we have $s+x \leq 2x$. Since γ is a non decreasing function, then $\gamma_1(x) = \gamma(2x)$. If $x \leq y$, the point $x \in I_y$. By using the same argument as above, we conclude that

$$\gamma(x+y) \leq \max_{t \in I_y} \gamma(y+t) := \gamma_2(y) = \gamma(2y).$$

So, for all $x \geq 0$ and $y \geq 0$,

$$\gamma(x+y) \leq \gamma(2x) + \gamma(2y).$$

Example 3.7. We present now an example that implement the previous theorem. Consider the following planar system:

$$\dot{x} = \begin{pmatrix} -x_1 + \frac{\epsilon\, x_1}{1+x_1^2}\exp(-x_1^2) \\ -x_2 + \frac{\epsilon\, x_2}{1+x_2^2}\exp(-t^2) \end{pmatrix} + \begin{pmatrix} -x_2 \\ x_1 \end{pmatrix} a(t)$$

where $x = (x_1, x_2)^\top \in \mathbb{R}^2$, $a(t)$ is a continuous bounded function and $\epsilon > 0$. Choosing the quadratic Lyapunov function $V(t,x) = \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2$. The time derivative of V along the trajectories of the system is bounded by

$$\dot{V}(t,x) \quad \leq \quad -2V(t,x) + \epsilon.$$

Let $\epsilon = \frac{1}{2}$, by application of theorem 3.3, this planar system is globally practically exponentially stable. Moreover, the ball $B_{\frac{1}{2}}$ is globally uniformly practically exponentially stable. Now, for ϵ small enough, the state approaches the origin (or some sufficiently small neighborhood of it) in a sufficiently fast manner.

Conclusion. In this paper new sufficient conditions are established to prove the global uniform practical stability of a certain class of time-varying dynamical system. Moreover, a new proof for the result of [4] is presented. The effectiveness of the conditions obtained in this paper is verified in a numerical example.

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Higher order iterates of Szasz-Mirakyan-Baskakov operators

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Abstract. In this paper, we discuss the generalization of Szasz-Mirakyan-Baskakov type operators defined in [7], using the iterative combinations in ordinary and simultaneous approximations. We have better estimates in higher order modulus of continuity for these operators in simultaneous approximation.

Mathematics Subject Classification (2010): 41A25, 41A28.

Keywords: Iterative combinations, Szasz-Mirakyan operators, simultaneous approximation, Steklov mean, modulus of continuity.

1. Introduction

Lebesgue integrable functions f on $[0,\infty)$ are defined by

$$H[0,\infty)=\left\{f:\int_0^\infty \frac{|f(t)|}{(1+t)^n}dt<\infty, n\in N\right\}$$

A new sequence of linear positive operators was introduced by Gupta-Srivastava [4] in 1995. They combined Szasz-Mirakyan and Baskakov operators as

$$S_n(f;x)=(n-1)\sum_{v=0}^{\infty} q_{n,v}(x)\int_0^\infty p_{n,v}(t)f(t)dt, \quad \forall x\in[0,\infty), \tag{1.1}$$

where

$$\begin{aligned} p_{n,v}(t) &= \frac{(n+v-1)!}{v!(n-1)!}\frac{t^v}{(1+t)^{n+v}}, \\ q_{n,v}(x) &= \frac{e^{-nx}(nx)^v}{v!}, \quad 0\le x<\infty. \end{aligned}$$

We define the norm $\|.\|$ on $C_\gamma[0,\infty)$ by

$$\|f\|_\gamma=\sup_{0\le t<\infty}|f(t)|t^{-\gamma},$$

where $C_\gamma[0,\infty) = \{f \in C[0,\infty) : |f(t)| \leq Mt^\gamma, \gamma > 0\}$. It can be noticed that the order of approximation by these operators (1.1) is at best of $O(n^{-1})$, howsoever smooth the function may be. So in order to improve the rate of convergence, we consider the iterative combinations $R_{n,v} : H[0,\infty) \to C^\infty[0,\infty)$ of the operators $S_n(f,x)$ described as below

$$R_{n,v}(f(t),x) = (I-(I-S_n)^v)(f;x) = \sum_{r=1}^{v}(-1)^{r+1}\begin{pmatrix} v \\ r \end{pmatrix} S_n^r(f(t);x), \tag{1.2}$$

where $S_n^0 = I$ and $S_n^r = S_n(S_n^{r-1})$ for $r \in N$.

The purpose of this paper is to obtain the corresponding general results in terms of $(2k+2)^{th}$ order modulus of continuity by using properties of linear approximating method, namely Steklov Mean. In the present paper, we use the notations

$$\begin{aligned} I &\equiv [a,b], \quad 0<a<b<\infty, \\ I_i &\equiv [a_i,b_i], \quad 0<a_1<a_2<...<b_2<b_1<\infty;\ i=1,2,\ldots \end{aligned}$$

Also $\|.\|_{C(I)}$ is sup-norm on the interval I and having not same value in different cases by constant C. Some approximation properties for similar type operators were discussed in [3] and [7]. Very recently D. Sharma et al [8] obtained some results on similar type of operators.

2. Auxiliary results

In this section, we obtain some important lemmas which will be useful for the proof of our main theorem.

Lemma 2.1. [6] *For $m \in N^0$, we define*

$$U_{n,m}(x) = \sum_{v=0}^{\infty} p_{n,v}(x)\left(\frac{v}{n}-x\right)^m,$$

then $U_{n,0}=1$, $U_{n,1}=0$. Further, there holds the recurrence formula

$$nU_{n,m+1}(x) = x\left[U'_{n,m}(x) + mU_{n,m-1}(x)\right], \quad m \geq 1.$$

Consequently

1. *$U_{n,m}(x)$ is a polynomial in x of degree $\leq m$.*
2. *$U_{n,m}(x) = O(n^{-[m+1]/2})$, where $[\zeta]$ is integral part of ζ.*

Lemma 2.2. [4] *There exists the polynomials $\phi_{i,j,r}(x)$ independent of n and v such that*

$$x^r(1+x)^r\frac{d^r}{dx^r}p_{n,v}(x) = \sum_{\substack{2i+j\leq r;\\ i,j\geq 0}} n^i(v-nx)^j\phi_{i,j,r}(x)p_{n,v}(x);$$

$$x^r\frac{d^r}{dx^r}q_{n,v}(x) = \sum_{\substack{2i+j\leq r;\\ i,j\geq 0}} n^i(v-nx)^j\phi_{i,j,r}(x)q_{n,v}(x).$$

Lemma 2.3. [3] *We assume that* $0 < a_1 < a_2 < b_2 < b_1 < \infty$, *for sufficiently small* $\delta > 0$, *then* $(2k+2)^{th}$ *ordered Steklov mean* $g_{2k+2,\delta}(t)$ *which corresponds to* $g(t) \in C_\gamma[0,\infty)$, *is defined as*

$$g_{2k+2,\delta}(t) = \delta^{-(2k+2)} \int_{-\delta/2}^{\delta/2} \int_{-\delta/2}^{\delta/2} \dots \int_{-\delta/2}^{\delta/2} [g(t) - \Delta_\eta^{2k+2} g(t)] \prod_{i=1}^{2k+2} dt_i,$$

where

$$\eta = \frac{1}{2k+2} \sum_{i=1}^{2k+2} t_i, \quad \forall t \in [a,b].$$

It is easily checked in [1], [2] *and* [5] *that*

1. $g_{2k+2,\delta}$ *has continuous derivatives upto order* $(2k+2)$ *on* $[a,b]$;
2. $\|g_{2k+2,\delta}^{(r)}\|_{C[a_1,b_1]} \leq K\delta^{-r}\omega_r(g,\delta,a,b), \quad r = 1,2,...(2k+2)$;
3. $\|g - g_{2k+2,\delta}\|_{C[a_1,b_1]} \leq K\omega_{2k+2}(g,\delta,a,b)$;
4. $\|g_{2k+2,\delta}\|_{C[a_1,b_1]} \leq K\|g\|_\gamma$.

Here K *is a constant not necessarily same at different places.*

Lemma 2.4. *For the* m^{th} *order moment* $T_{n,m}(x), m \in N^0$ *defined by*

$$T_{n,m}(x) = (n-1)\sum_{v=0}^{\infty} q_{n,v}(x) \int_0^\infty p_{n,v}(t)(t-x)^m dt, \qquad \forall x \in [0,\infty)$$

we obtain

$$T_{n,0}(x) = 1 \tag{2.1}$$

$$T_{n,1}(x) = \frac{1+2x}{n-2}, \qquad n > 2 \tag{2.2}$$

$$T_{n,2}(x) = \frac{(n+6)x^2 + 2x(n+3) + 2}{(n-2)(n-3)}, \qquad n > 3 \tag{2.3}$$

and the recurrence relation for $n > (m+2)$

$$(n-m-2)T_{n,m+1}(x) = x[T'_{n,m}(x) + m(2+x)T_{n,m-1}(x)] + (m+1)(1+2x)T_{n,m}(x) \tag{2.4}$$

Further, for all $x \in [0,\infty)$, *we have* $T_{n,m}(x) = O(n^{-[m+1]/2})$.

Proof. Obviously (2.1)-(2.3) can be easily proved by using the definition of $T_{n,m}(x)$. To prove the recurrence relation (2.4), we proceed by taking

$$T'_{n,m}(x) = (n-1)\sum_{v=0}^{\infty} q'_{n,v}(x) \int_0^\infty p_{n,v}(t)(t-x)^m dt - mT_{n,m-1}(x)$$

Multiplying by x on both sides and then using identity $xq'_{n,v}(x) = (v - nx)q_{n,v}(x)$, we have

$$\begin{aligned} x[T'_{n,m}(x) + mT_{n,m-1}(x)] &= (n-1)\sum_{v=0}^{\infty}(v-nx)q_{n,v}(x)\int_0^{\infty} p_{n,v}(t)(t-x)^m dt \\ &= (n-1)\sum_{v=0}^{\infty}(v-nx)q_{n,v}(x)\int_0^{\infty} p_{n,v}(t)(t-x)^m dt \\ &= (n-1)\sum_{v=0}^{\infty} q_{n,v}(x)\int_0^{\infty} (v-nx)p_{n,v}(t)(t-x)^m dt \\ &= (n-1)\sum_{v=0}^{\infty} q_{n,v}(x)\int_0^{\infty} (v-nt)p_{n,v}(t)(t-x)^m dt \\ &\quad +(n-1)\sum_{v=0}^{\infty} q_{n,v}(x)\int_0^{\infty} np_{n,v}(t)(t-x)^{m+1} dt \\ &= (n-1)\sum_{v=0}^{\infty} q_{n,v}(x)\int_0^{\infty} (v-nt)p_{n,v}(t)(t-x)^m dt \\ &\quad +nT_{n,m+1}(x). \end{aligned}$$

Again, using identity $t(1+t)p'_{n,v}(t) = (v - nt)p_{n,v}(t)$ in RHS, we get

$$\begin{aligned} & x[T'_{n,m}(x) + mT_{n,m-1}(x)] \\ = \ & (n-1)\sum_{v=0}^{\infty} q_{n,v}(x)\int_0^{\infty} t(1+t)p'_{n,v}(t)(t-x)^m dt + nT_{n,m+1}(x) \\ = \ & (n-1)\sum_{v=0}^{\infty} q_{n,v}(x)\int_0^{\infty} [(1+2x)(t-x) + (t-x)^2 + x(1+x)]p'_{n,v}(t) \\ & \times (t-x)^m dt + nT_{n,m+1}(x) \\ = \ & -(m+1)(1+2x)T_{n,m}(x) - (m+2)T_{n,m+1}(x) - mx(1+x)T_{n,m-1}(x) \\ & +nT_{n,m+1}(x). \end{aligned}$$

This leads to our required result (2.4).

Further, for every $m \in N^0$, the m^{th} order moment $T^{(p)}_{n,m}$ for the operator S^p_n is defined by

$$T^{(p)}_{n,m}(x) = S^p_n((t-x)^m, x).$$

If we adopt the convention $T^{(1)}_{n,m}(x) = T_{n,m}(x)$, obviously $T^{(p)}_{n,m}(x)$ is of degree m.

Theorem 2.5. *Let $f \in C_\gamma[0,\infty)$, if $f^{(2v+p+2)}$ exists at a point $x \in [0,\infty)$, then*

$$\lim_{n\to\infty} n^{v+1}[R^{(p)}_{n,v}(f;x) - f^{(p)}(x)] = \sum_{k=p}^{2v+p+2} Q(k,v,p,x)f^{(k)}(x), \tag{2.5}$$

where $Q(k,v,p,x)$ are certain polynomials in x. Further if $f^{(2v+p+2)}$ is continuous on $(a-\eta, b+\eta) \subset [0,\infty)$ and $\eta > 0$, then this theorem holds uniformly in $[a,b]$.

Proof will be along the similar lines [4].

3. Main results

In this section, we establish the direct theorem.

Theorem 3.1. *Let $f \in H[0,\infty)$ be bounded on every finite subinterval of $[0,\infty)$ and $f(t) = O(t^\alpha)$ as $t \to \infty$ for some $\alpha > 0$.*
If $f^{(p)}$ exists and is continuous on $(a-\eta, b+\eta) \subset [0,\infty)$, for some $\eta > 0$. Then

$$\|R^{(p)}_{n,v}(f(t);x) - f^{(p)}(x)\|_{C(I_2)} \leq K\{n^{-v}\|f\|_{C(I_1)} + \omega_{2v+2}(f^{(p)}, n^{-1/2}, I_1)\}$$

where constant K is independent of f and n.

Proof. We can write

$$\begin{aligned} & \left\|R^{p}_{n,v}(f(t);x) - f^{(p)}(x)\right\|_{C(I_2)} \\ \leq\ & \|R^{(p)}_{n,v}(f - f_{\eta,2v+2};x)\|_{C(I_2)} + \left\|R^{(p)}_{n,v}(f_{\eta,2v+2};x) - f^{(p)}_{\eta,2v+2}(x)\right\|_{C(I_2)} \\ & + \left\|f^{(p)}(x) - f^{(p)}_{\eta,2v+2}(x)\right\|_{C(I_2)} \\ =:\ & P_1 + P_2 + P_3. \end{aligned}$$

By the property of Steklov Mean and $f^{(p)}_{\eta,2v+2}(x) = (f^{(p)})_{\eta,2v+2}(x)$, we get

$$P_3 \leq K\omega_{2v+2}(f^{(p)}, \eta, I_1).$$

To estimate P_2, applying Theorem 2.5 and interpolation property from [2], we have

$$\begin{aligned} P_2 &\leq Kn^{-(v+1)} \sum_{i=p}^{2v+p+2} \|f^{(i)}_{\eta,2v+2}(x)\|_{C(I_2)} \\ &\leq Kn^{-(v+1)} \left(\|f_{\eta,2v+2}\|_{C(I_2)} + \|(f^{(p)}_{\eta,2v+2})^{(2v+2)}\|_{C(I_2)}\right). \end{aligned}$$

Hence by using properties (2) and (4) of Steklov Mean, we get

$$P_2 \leq Kn^{-(v+1)}[\|f\|_{C(I_1)} + (\eta)^{-2v-2}\omega_{2v+2}(f^{(p)}, \eta, I_1)].$$

Suppose a^* and b^* be such that

$$0 < a_1 < a^* < a_2 < b_2 < b^* < b_1 < \infty.$$

In order to estimate P_1, let $F = f - f_{\eta,2v+2}$. Then, by hypothesis, we have

$$F(t) = \sum_{i=0}^{p} \frac{F^{(i)}(x)}{i!}(t-x)^i + \frac{F^{(p)}(\xi) - F^{(p)}(x)}{p!}(t-x)^p\psi(t) + h(t,x)(1-\psi(t)), \quad (3.1)$$

where ξ lies between t and x, and ψ is the characteristic function of the interval $[a^*, b^*]$. For $t \in [a^*, b^*]$ and $x \in [a_2, b_2]$, we get

$$F(t) = \sum_{i=0}^{p} \frac{F^{(i)}(x)}{i!}(t-x)^i + \frac{F^{(p)}(\xi) - F^{(p)}(x)}{p!}(t-x)^p,$$

and for $t \in [0,\infty) \setminus [a^*, b^*]$, $x \in [a_2, b_2]$, we define

$$h(t,x) = F(t) - \sum_{i=0}^{p} \frac{F^{(i)}(x)}{i!}(t-x)^i.$$

Now operating $R_{n,v}^p$ on both the sides of (3.1), we have the three terms on right side namely E_1, E_2 and E_3 respectively. By using (1.2) and Lemma 2.4, we get

$$\begin{aligned} E_1 &= \sum_{i=0}^{p} \frac{F^{(i)}(x)}{i!} \sum_{r=1}^{v} (-1)^{r+1} \begin{pmatrix} v \\ r \end{pmatrix} D^p \left(S_n^r((t-x)^i; x \right), \quad D \equiv \frac{d}{dx} \\ &= \frac{F^{(p)}(x)}{p!} \sum_{r=1}^{v} (-1)^{r+1} \begin{pmatrix} v \\ r \end{pmatrix} D^p \left(S_n^r(t^p; x) \right) \\ &\to F^{(p)}(x), \end{aligned}$$

when $n \to \infty$ uniformly in I_2. Therefore

$$\|E_1\|_{C(I_2)} \leq K \left\| f^{(p)} - f_{\eta,2v+2}^{(p)} \right\|_{C(I_2)}.$$

To obtain E_2, we have

$$\begin{aligned} \|E_2\|_{C(I_2)} &\leq \frac{2}{p!} \left\| f^{(p)} - f_{\eta,2v+2}^{(p)} \right\|_{C[a^*,b^*]} \sum_{r=1}^{v} (n-1) \begin{pmatrix} v \\ r \end{pmatrix} \sum_{v=0}^{\infty} |q_{n,v}^{(p)}(x)| \\ &\quad \times \int_0^\infty p_{n,v}(t) S_n^{r-1} \left(|t-x|^p, x \right) dt. \end{aligned}$$

Using Lemma 2.2, Cauchy Schwartz Inequality and Lemma 2.1

$$\begin{aligned} &(n-1) \sum_{v=0}^{\infty} |q_{n,v}^{(p)}(x)| \int_0^\infty p_{n,v}(t) S_n^{r-1}(|t-x|^p, x) dt \\ \leq\; & K \sum_{\substack{2r+j \leq p; \\ r,j \geq 0}} n^r \phi_{r,j,p}(x) x^{-p} (n-1) \sum_{v=0}^{\infty} q_{n,v}(x)(v-nx)^j \\ & \times \int_0^\infty p_{n,v}(t) S_n^{r-1}(|t-x|^p, x) dt \\ \leq\; & K \sum_{\substack{2r+j \leq p; \\ r,j \geq 0}} n^r \left(\sum_{v=0}^{\infty} q_{n,v}(x)(v-nx)^{2j} \right)^{1/2} \\ & \times \left((n-1) \sum_{v=0}^{\infty} q_{n,v}(x) \int_0^\infty p_{n,v}(t) S_n^{r-1}(|t-x|^{2p}, x) dt \right)^{1/2} \\ =\; & K \sum_{\substack{2r+j \leq p; \\ r,j \geq 0}} n^r O(n^{j/2}) O(n^{-p/2}) \\ =\; & O(1) \end{aligned}$$

as $n \to \infty$ uniformly in I_2. Therefore,

$$\|E_2\| \leq K\|f^{(p)} - f^{(p)}_{\eta,2v+2}\|_{C(a^*,b^*)}.$$

Since $t \in [0,\infty) \setminus [a^*, b^*]$ and $x \in [a_2, b_2]$, we can choose a $\delta > 0$ in such a way that $|t-x| \geq \delta$. If $\beta \geq \max\{\alpha, p\}$ be an integer, we can find a positive constant Q such that $|h(t,x)| \leq Q|t-x|^\beta$ whenever $|t-x| \geq \delta$. Again applying Lemma 2.2, Cauchy Schwartz Inequality three times, Lemma 2.1 and Lemma 2.4, we get

$$\begin{aligned}
|E_3| &\leq K\sum_{r=0}^{v}\binom{v}{r}\sum_{\substack{2r+j\leq p;\\ r,j\geq 0}} n^r\left(\sum_{v=0}^{\infty} q_{n,v}(x)(v-nx)^{2j}\right)^{1/2} \\
&\quad\times\left((n-1)\sum_{v=0}^{\infty} q_{n,v}(x)\int_0^\infty p_{n,v}(t)S_n^{r-1}((1-\psi(t))(t-x)^{2\beta};t)dt\right)^{1/2} \\
&\leq K\sum_{r=0}^{v}\binom{v}{r}\sum_{\substack{2r+j\leq p;\\ r,j\geq 0}} n^r\left(\sum_{v=0}^{\infty} q_{n,v}(x)(v-nx)^{2j}\right)^{1/2} \\
&\quad\times\left((n-1)\sum_{v=0}^{\infty} q_{n,v}(x)\int_0^\infty p_{n,v}(t)S_n^{r-1}(\frac{(t-x)^{2m}}{\delta^{2m-2\beta}};t)dt\right)^{1/2} \\
&\leq K\sum_{\substack{2r+j\leq p;\\ r,j\geq 0}} n^r O(n^{j/2})O(n^{-m/2}), \qquad m > \beta, \forall m \in I.
\end{aligned}$$

Hence $\|E_3\| = O(1)$, as $n \to \infty$, uniformly in I_2. Combining the estimates of E_1, E_2 and E_3, we get

$$P_1 \leq K\|f^{(p)} - f^{(p)}_{\eta,2v+2}\|_{C(a*,b*)} \leq K\omega_{2v+2}(f^{(p)},\eta,I_1).$$

Substituting $\eta = n^{-1/2}$, we get the required theorem.

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Approximation with an arbitrary order by generalized Szász-Mirakjan operators

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Abstract. By using two given arbitrary sequences $\alpha_n > 0$, $\beta_n > 0$, $n \in \mathbb{N}$ with the property that $\lim_{n\to\infty} \beta_n/\alpha_n = 0$, in this very short note we modify the generalized Szász-Mirakjan operator based on the Sheffer polynomials in such a way that on each compact subinterval in $[0, +\infty)$ the order of uniform approximation is $\omega_1(f; \sqrt{\beta_n/\alpha_n})$. These modified generalized operators can uniformly approximate a Lipschitz 1 function, on each compact subinterval of $[0, \infty)$ with an arbitrary good order of approximation $\sqrt{\beta_n/\alpha_n}$.

Mathematics Subject Classification (2010): 41A36, 41A25.

Keywords: Generalized Szász-Mirakjan operator, Sheffer polynomials, order of approximation.

1. Introduction

In [8], Szász introduced and investigated the approximation properties of the linear and positive operators attached to continuous functions $f : [0, \infty) \to \mathbb{R}$,

$$S_n(f)(x) = e^{-nx} \sum_{k=0}^{\infty} \frac{(nx)^k}{k!} f(k/n).$$

Generalizing the above operators, in [5] Jakimovski and Leviatan introduced and studied the qualitative approximation properties of the operators given by

$$P_n(f)(x) = \frac{e^{-nx}}{A(1)} \sum_{k=0}^{\infty} p_k(nx) f(k/n),$$

where p_k are the Appell polynomials defined by the generating function

$$A(t)e^{tx} = \sum_{k=0}^{\infty} p_k(x)t^k, \; A(z) = \sum_{k=0}^{\infty} c_k z^k, \; c_0 \neq 0,$$

is an analytic function in a disc $|z| < R$, $(R > 1)$ and $A(1) \neq 0$. For $A(z) = 1$, one recapture the Szász-Mirakjan operators.

In [4], Ismail introduced and studied the qualitative approximation properties of a generalization of the Jakimovski-Leviatan operators, given by

$$T_n(f)(x) = \frac{e^{-nxH(1)}}{A(1)} \sum_{k=0}^{\infty} p_k(nx) f(k/n),$$

where p_k are the Sheffer polynomials (more general than the Appell polynomials) defined by

$$A(t)e^{xH(t)} = \sum_{k=0}^{\infty} p_k(x)t^k, x \geq 0, |t| < R, \tag{1.1}$$

with $A(z) = \sum_{k=0}^{\infty} c_k z^k$, $H(z) = \sum_{k=1}^{\infty} h_k z^k$, analytic functions in a disk $|z| < R$, $(R > 1)$, $A(1) \neq 0$, $H'(1) = 1$, $c_k, h_k \in \mathbb{R}$, for all $k \geq 1$, $c_0 \in \mathbb{R}$, $c_0 \neq 0$, $h_1 \neq 0$, and supposing that $p_k(x) \geq 0$ for all $x \in [0, \infty)$, $k \geq 0$.

Quantitative estimate of the order $\omega_1(f; 1/\sqrt{n})$ in approximation by the $T_n(f)(x)$ operators were obtained by Sucu-Ibikli in [7].

By using two sequences of real numbers, $(\alpha_n)_n$, $(\beta_n)_n$ with the properties that $\lim_{n\to\infty} \frac{\beta_n}{\alpha_n} = 0$, in [1] Cetin and Ispir introduced a remarkable generalization of the Szász-Mirakjan operators attached to analytic functions f of exponential growth in a compact disk of the complex plane, $|z| < R$,

$$S_n(f; \alpha_n, \beta_n)(z) = e^{-\alpha_n z/\beta_n} \sum_{k=0}^{\infty} \frac{1}{k!} \left(\frac{\alpha_n z}{\beta_n}\right)^k \cdot f\left(\frac{k\beta_n}{\alpha_n}\right), z \in \mathbb{C}, |z| < R,$$

which approximate f in any compact disk $|z| \leq r$, $r < R$, with the approximation order $\frac{\beta_n}{\alpha_n}$.

The main aim of this short note is to consider the Ismail's kind generalization of the above operator, but attached to a real function of real variable defined on $[0, +\infty)$,

$$T_n(f; \alpha_n, \beta_n)(x) = \frac{e^{-\alpha_n x H(1)/\beta_n}}{A(1)} \sum_{k=0}^{\infty} p_k\left(\frac{\alpha_n x}{\beta_n}\right) \cdot f\left(\frac{k\beta_n}{\alpha_n}\right), x \in [0, \infty),$$

under the above hypothesis on A, H and p_k, obtaining the order of approximation $\omega_1(f; \sqrt{\beta_n/\alpha_n})$ which, for example, in the case of Lipschitz 1 functions on $[0, \infty)$ gives the order of uniform approximation $O(\sqrt{\beta_n/\alpha_n})$ on each compact subinterval of $[0, \infty)$.

Notice that for $\alpha_n = n$, $\beta_n = 1$ for all $n \in \mathbb{N}$, we recapture the above Ismail's generalization of the Szász-Mirakjan operators. Also, evidently that $T_n(f; \alpha_n, \beta_n)(z)$ generalize the operators introduced in [1].

Since the sequence β_n/α_n, $n \in \mathbb{N}$, can evidently be chosen to converge to zero with an arbitrary small order, it seems that in the class of Szász-Mirakjan type operators, the generalization $T_n(f; \alpha_n, \beta_n)$, $n \in \mathbb{N}$, represents the best possible construction and the most general.

2. Main results

Since $T_n(f;\alpha_n,\beta_n)$, $n\in\mathbb{N}$ are positive linear operators, we will follow the standard line of study. Firstly, we need the following lemma.

Lemma 2.1. *Denoting $e_k(x)=x^k$, for all $x\in[0,\infty)$ and $n\in\mathbb{N}$ we have:*
(i) $T_n(e_0;\alpha_n,\beta_n)(x)=1$;
(ii) $T_n(e_1;\alpha_n,\beta_n)(x)=x+\frac{\beta_n}{\alpha_n}\cdot\frac{A'(1)}{A(1)}$;
(iii) $T_n(e_2;\alpha_n,\beta_n)(x)=x^2+x\frac{\beta_n}{\alpha_n}\left(\frac{2A'(1)}{A(1)}+H''(1)+1\right)+\frac{\beta_n^2}{\alpha_n^2}\left(\frac{A''(1)+A'(1)}{A(1)}\right)$;
(iv) $T_n((\cdot-x)^2;\alpha_n,\beta_n)(x)=x\frac{\beta_n}{\alpha_n}(H''(1)+1)+\frac{\beta_n^2}{\alpha_n^2}\cdot\frac{A'(1)+A''(1)}{A(1)}$.

Proof. (i) If in (1.1) we take $t=1$ and replace x by $\frac{x\alpha_n}{\beta_n}$ then we obtain

$$A(1)\cdot e^{xH(1)\alpha_n/\beta_n}=\sum_{k=0}^{\infty}p_k(x\alpha_n/\beta_n),$$

which evidently is equivalent to $T_n(e_0;\alpha_n,\beta_n)(x)=1$.

(ii) Differentiating with respect to t the generation formula (1.1), we get

$$A'(t)\cdot e^{xH(t)}+A(t)\cdot x\cdot H'(t)\cdot e^{xH(t)}=\sum_{k=1}^{\infty}p_k(x)\cdot k\cdot t^{k-1}.$$

Taking above $t=1$ and replacing x by $x\frac{\alpha_n}{\beta_n}$, it follows

$$A'(1)\cdot e^{xH(1)\alpha_n/\beta_n}+A(1)\cdot\frac{x\alpha_n}{\beta_n}\cdot e^{xH(1)\alpha_n/\beta_n}=\sum_{k=1}^{\infty}p_k(x\alpha_n/\beta_n)\cdot k.$$

Multiplying both sides by $\frac{e^{-xH(1)\alpha_n/\beta_n}}{A(1)}\cdot\frac{\beta_n}{\alpha_n}$, it follows

$$\frac{A'(1)}{A(1)}\cdot\frac{\beta_n}{\alpha_n}+x=\frac{e^{-x(1)\alpha_n/\beta_n}}{A(1)}\cdot\sum_{k=1}^{\infty}p_k(x\alpha_n/\beta_n)\cdot\frac{k\beta_n}{\alpha_n}=T_n(e_1;\alpha_n,\beta_n)(x).$$

(iii) Differentiating (1.1) twice with respect to t, we get

$$A''(t)e^{xH(t)}+x[2A'(t)\cdot H'(t)+A(t)H''(t)]e^{xH(t)}+x^2A(t)[H'(t)]^2\cdot e^{xH(t)}$$
$$=\sum_{k=0}^{\infty}p_k(x)k(k-1)t^{k-2}.$$

Taking here $t=1$, replacing x by $x\cdot\frac{\alpha_n}{\beta_n}$ and then multiplying both sides by $\frac{e^{-xH(1)\alpha_n/\beta_n}}{A(1)}\cdot\frac{\beta_n^2}{\alpha_n^2}$, it follows

$$\frac{A''(1)}{A(1)}\cdot\frac{\beta_n^2}{\alpha_n^2}+x\cdot\frac{\beta_n}{\alpha_n}\left(\frac{2A'(1)}{A(1)}+H''(1)\right)+x^2$$
$$=\frac{e^{-xH(1)\alpha_n/\beta_n}}{A(1)}\cdot\sum_{k=0}^{\infty}p_k(x\alpha_n/\beta_n)\frac{k^2\beta_n^2}{\alpha_n^2}-\frac{e^{-xH(1)\alpha_n/\beta_n}}{A(1)}\cdot\frac{\beta n}{\alpha_n}\cdot\sum_{k=0}^{\infty}p_k(x\alpha_n/\beta_n)\frac{k\beta_n}{\alpha_n}$$
$$=T_n(e_2;\alpha_n,\beta_n)(x)-\frac{\beta_n}{\alpha_n}\cdot T_n(e_1;\alpha_n,\beta_n)(x),$$

which by using (ii) too implies

$$T_n(e_2;\alpha_n,\beta_n)(x)$$
$$= \frac{A''(1)}{A(1)}\cdot\frac{\beta_n^2}{\alpha_n^2} + x\cdot\frac{\beta_n}{\alpha_n}\left(\frac{2A'(1)}{A(1)} + H''(1)\right) + x^2 + \frac{\beta_n}{\alpha_n}\left(x + \frac{\beta_n}{\alpha_n}\cdot\frac{A'(1)}{A(1)}\right),$$

leading to the required formula.

(iv) It is an immediate consequence of (i)-(iii) and of the linearity of $T_n(\cdot;\alpha_n,\beta_n)$. □

The main result of this section is the following.

Theorem 2.2. *Let $f:[0,\infty)\to\mathbb{R}$ be uniformly continuous on $[0,\infty)$. Denote $\omega_1(f;\delta)=\sup\{|f(x)-f(y)|;|x-y|\le\delta,\ x,y\in[0,\infty)\}$. For all $x\in[0,\infty)$, $n\in\mathbb{N}$ we have*

$$|T_n(f;\alpha_n,\beta_n)(x) - f(x)|$$
$$\le \left(1+\sqrt{(H''(1)+1)x + \frac{\beta_n}{\alpha_n}\cdot\frac{A'(1)+A''(1)}{A(1)}}\right)\cdot\omega_1(f;\sqrt{\beta_n/\alpha_n}).$$

Proof. By the standard theory (see e.g. Shisha-Mond [6] where although the results are obtained for continuous functions on compact intervals, the reasonings remain the same if the functions are (uniformly) continuous on $[0,+\infty)$), we have

$$|T_n(f;\alpha_n,\beta_n)(x) - f(x)| \le (1+\delta^{-1}\sqrt{T_n((\cdot - x)^2;\alpha_n,\beta_n)(x)})\omega_1(f;\delta).$$

Replacing $\delta=\sqrt{\frac{\beta_n}{\alpha_n}}$ and using Lemma 2.1, (iv), we arrive at the desired estimate. □

As an immediate consequence of Theorem 2.2 we get the following.

Corollary 2.3. *Suppose that there exists $L>0$ such that $|f(x)-f(y)|\le L|x-y|$, for all $x,y\in[0,\infty)$. We have*

$$|T_n(f;\alpha_n,\beta_n)(x) - f(x)|$$
$$\le L\left(1+\sqrt{(H''(1)+1)x + \frac{\beta_n}{\alpha_n}\cdot\frac{A'(1)+A''(1)}{A(1)}}\right)\cdot\sqrt{\beta_n/\alpha_n}.$$

Remark 2.4. In order to get uniform convergence in the above results, the expression under the square root in the above estimations must be bounded, fact which holds when x belong to a compact subinterval of $[0,+\infty)$.

Remark 2.5. The optimality of the estimates in Theorem 2.2 and Corollary 2.3 consists in the fact that given an arbitrary sequence of strictly positive numbers $(\gamma_n)_n$, with $\lim_{n\to\infty}\gamma_n=0$, we always can find the sequences α_n, β_n satisfying $\omega_1(f;\sqrt{\beta_n/\alpha_n})\le\gamma_n$ for all $n\in\mathbb{N}$ in the case of Theorem 2.2 and $\sqrt{\frac{\beta_n}{\alpha_n}}\le\gamma_n$ for all $n\in\mathbb{N}$, in the case of Corollary 2.3.

Remark 2.6. For $\alpha_n=n$ and $\beta_n=1$ we recapture the results in [7], but the estimates there are essentially weaker than those in the present results.

Remark 2.7. If f is uniformly continuous on $[0,+\infty)$ then it is well known that its growth on $[0,+\infty)$ is linear, i.e. there exist $\alpha,\beta>0$ such that $|f(x)|\le \alpha x+\beta$, for all $x\in[0,+\infty)$ (see e.g. [2], p. 48, Problème 4, or [3]).

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Some properties of Sobolev algebras modelled on Lorentz spaces

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Abstract. In this paper, firstly Lorentz-Sobolev spaces $W^k_{L(p,q)}(\mathbb{R}^n)$ of integer order are introduced and some of their important properties are emphasized. Also, the Banach spaces $A^k_{L(p,q)}(\mathbb{R}^n) = L^1(\mathbb{R}^n) \cap W^k_{L(p,q)}(\mathbb{R}^n)$ (Lorentz-Sobolev algebras in the sense of H.Reiter) are studied. Then, using a result due to H.C.Wang, it is showed that Banach convolution algebras $A^k_{L(p,q)}(\mathbb{R}^n)$ do not have weak factorization. Lastly, it is found that the multiplier algebra of $A^k_{L(p,q)}(\mathbb{R}^n)$ coincides with the measure algebra $M(\mathbb{R}^n)$ for $1 < p < \infty$ and $1 \leq q < \infty$.

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1. Introduction

Let $\mathbb{R}^n$ denote the n-dimensional real Euclidean space. If $\alpha = (\alpha_1, ..., \alpha_n)$ is an n-tuple of nonnegative integers α_j, then we call α a *multi-index* and denote by x^α the monomial $x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}$, which has degree $|\alpha| = \sum_{j=1}^n \alpha_j$. Similarly, if $D_j = \frac{\partial}{\partial x_j}$ for $1 \leq j \leq n$, then

$$D^\alpha = D_1^{\alpha_1} D_2^{\alpha_2} ... D_n^{\alpha_n}$$

denotes a differential operator of order $|\alpha|$. For given two locally integrable functions f and g on $\mathbb{R}^n$, we say that $\frac{\partial^{|\alpha|} f}{\partial x^\alpha} = g$ (weak derivative of f) if

$$\int_{\mathbb{R}^n} f(x) \frac{\partial^{|\alpha|}\varphi}{\partial x^\alpha}(x)\, dx = (-1)^{|\alpha|} \int_{\mathbb{R}^n} g(x)\, \varphi(x)\, dx$$

for all $\varphi \in C_0^\infty(\mathbb{R}^n)$, where $C_0^\infty(\mathbb{R}^n)$ is the space of all smooth functions with compact support.

If we define a functional $\|\cdot\|_{k,p}$, where k is a nonnegative integer and $1 \leq p \leq \infty$, as follows:

$$\|f\|_{k,p} = \sum_{0 \leq |\alpha| \leq k} \|D^{\alpha} f\|_p \qquad \text{if} \quad 1 \leq p \leq \infty, \tag{1.1}$$

for any function $f \in L^p(\mathbb{R}^n)$, then we can consider two vector spaces on which $\|\cdot\|_{k,p}$ is a norm:

(i) $W^{k,p}(\mathbb{R}^n) := \{f \in L^p(\mathbb{R}^n) : D^{\alpha} f \in L^p(\mathbb{R}^n) \text{ for } 0 \leq |\alpha| \leq k\}$, where $D^{\alpha} f$ is the weak partial derivative of f,

(ii) $W_0^{k,p}(\mathbb{R}^n) :=$ the closure of $C_0^{\infty}(\mathbb{R}^n)$ in the space $W^{k,p}(\mathbb{R}^n)$.

Equipped with the appropriate norm (1.1), these are called *Sobolev spaces* over $\mathbb{R}^n$. Clearly, $W^{0,p}(\mathbb{R}^n) = L^p(\mathbb{R}^n)$, and if $1 \leq p < \infty$, $W_0^{0,p}(\mathbb{R}^n) = L^p(\mathbb{R}^n)$ since $C_0^{\infty}(\mathbb{R}^n)$ is dense in $L^p(\mathbb{R}^n)$. Also, $W^{k,p}(\mathbb{R}^n)$ is a Banach space for $1 \leq p \leq \infty$ and a reflexive space with its associate space $W^{-k,p'}(\mathbb{R}^n)$ if $1 < p < \infty$ and $\frac{1}{p} + \frac{1}{p'} = 1$. For any k, one can see the obvious chain of imbeddings

$$W_0^{k,p}(\mathbb{R}^n) \hookrightarrow W^{k,p}(\mathbb{R}^n) \hookrightarrow L^p(\mathbb{R}^n).$$

Sobolev spaces of integer order were introduced by S.L. Sobolev in [15,16]. These spaces are defined over an arbitrary domain $\Omega \subset \mathbb{R}^n$ by using subspaces of Lebesgue spaces. Many generalizations and specializations of these spaces have been constructed and studied in years. In particular, there are extensions that allow arbitrary real values of k, weighted spaces that introduce weight functions into the L^p-norms and other generalizations involve different orders of differentiation and different L^p-norms in different coordinate directions and Orlicz-Sobolev spaces. Finally, there has been much work on Sobolev spaces and its related areas. To an interested reader, we can suggest our main reference book [1] and the references therein.

2. Preliminaries

Definition 2.1. *Let G be a locally compact abelian group, and $(B(G), \|\cdot\|_B)$ be a Banach space of complex-valued measurable functions on G. $B(G)$ is called a homogeneous Banach space if the following are satisfied:*

H1. *$L_s f \in B(G)$ and $\|L_s f\|_B = \|f\|_B$ for all $f \in B(G)$ and $s \in G$, where $L_s f(x) = f(x - s)$.*

H2. *$s \to L_s f$ is a continuous map from G into $(B(G), \|\cdot\|_B)$.*

Definition 2.2. *A homogeneous Banach algebra on G is a subalgebra $B(G)$ of $L^1(G)$ such that $B(G)$ is itself a Banach algebra with respect to a norm $\|\cdot\|_B \geq \|\cdot\|_1$ and satisfies $H1$ and $H2$.*

Definition 2.3. *A homogeneous Banach algebra $B(G)$ is called a Segal algebra if it is dense in $L^1(G)$.*

Definition 2.4. *Let G be a locally compact abelian group with character group Γ. A Segal algebra $B(G)$ is called isometrically character-invariant if for every character $\varkappa$ and every $f \in B(G)$ one has $\varkappa f \in B(G)$ and $\|\varkappa f\|_B = \|f\|_B$. In other words, if $f \to \varkappa f$ is an isometry of $B(G)$, for all $\varkappa \in \Gamma$.*

Definition 2.5. *Let G be a locally compact abelian group with character group Γ, and μ be a positive Radon measure on Γ. A Banach algebra $(B(G), \|\cdot\|_B)$ in $L^1(G)$ is an $F^\mu-$algebra if $\widehat{B(G)} \subset L^p(G)$ for some $p \in (0,\infty)$ where "$\widehat{\ }$" denotes the Fourier transform.*

Definition 2.6. *Let G be a locally compact abelian group with character group Γ, and μ be a positive Radon measure on Γ. A Banach algebra $(B(G), \|\cdot\|_B)$ in $L^1(G)$ is a $P^\mu-$algebra if there exist two sequences (Δ_n) and (θ_n) of subsets of Γ, a sequence (f_n) in $B(G)$ and a sequence $c_n \geq 1$ satisfying*
p1. $\Delta_i \cap \Delta_j = \varnothing$ if $i \neq j$, $\theta_n \subset Int(\Delta_n)$, $\mu(\theta_n) = \alpha > 0, \mu(\Delta_n) = \beta < \infty$ for n=1,2,$\cdots$.(Int:=Interior)
p2. $0 \leq \widehat{f_n} \leq 1$, $Supp\widehat{f_n} \subset \Delta_n$, $\widehat{f_n}(\theta_n) = 1$ for each n=1,2,$\cdots$.
p3. $\|f_n\|_B \leq c_n$, $\sum_{n=1}^{\infty}\left(\frac{1}{c_n^a}\right) < \infty$, $\sum_{n=1}^{\infty}\left(\frac{1}{c_n^b}\right) = \infty$ for some $a, b \in (0,\infty)$.

An algebra is an $F^\mu P^\mu-$algebra if it is both F^μ and $P^\mu-$algebra. It is simply called $FP-$algebra if μ is the Haar measure on Γ.

Definition 2.7. *Let B be a Banach algebra. B is said to have weak factorization if, given $f \in B$, there are $f_1, \cdots, f_n, g_1, \cdots, g_n \in B$ such that $f = \sum_{i=1}^{n} f_i g_i$.*

Theorem 2.8. ([18, p.42]) *A homogeneous Banach space $(B(G), \|\cdot\|_B)$ is a homogeneous Banach algebra if and only if $B(G)$ is a linear subspace of $L^1(G)$ with $\|\cdot\|_B \geq \|\cdot\|_1$.*

Definition 2.9. *Let G be a (noncompact) locally compact abelian group. The translation coefficient K_E of a homogeneous Banach space E on G is the infimum of the constants K such that*

$$\limsup_{s\to\infty} \|f + L_s f\|_E \leq K \|f\|_E, \ \ \forall f \in E.$$

For the convenience of the reader, we now review briefly what we need from the theory of Lorentz spaces. Let (G, Σ, μ) be a measure space and let f be a measurable function on G. For each $y > 0$, the rearrangement of f is defined by

$$f^*(t) = \inf\{y > 0 : \mu\{x \in G : \ |f(x)| > y\} \leq t\}, \ t > 0,$$

where $\inf \emptyset = \infty$. Also the average function of f is defined by

$$f^{**}(t) = \frac{1}{t}\int_0^t f^*(s)\,ds \ , \ t > 0.$$

Note that $f^*(\cdot)$ and $f^{**}(\cdot)$ are non-increasing and right continuous functions on $(0,\infty)$ [3, 10] .For $p, q \in (0,\infty)$, we define

$$\|f\|_{p,q}^* = \left(\frac{q}{p}\int_0^\infty [f^*(t)]^q\, t^{\frac{q}{p}-1} dt\right)^{\frac{1}{q}}, \ \ \|f\|_{p,q} = \left(\frac{q}{p}\int_0^\infty [f^{**}(t)]^q\, t^{\frac{q}{p}-1} dt\right)^{\frac{1}{q}}. \tag{2.1}$$

Also, if $0 < p \leq \infty$ and $q = \infty$, we define

$$\|f\|_{p,\infty}^* = \sup_{t>0} t^{\frac{1}{p}} f^*(t) \ \text{ and } \ \|f\|_{p,\infty} = \sup_{t>0} t^{\frac{1}{p}} f^{**}(t).$$

For $0 < p < \infty$ and $0 < q \leq \infty$, Lorentz spaces are denoted by $L(p,q)(G)$ and defined to be the vector spaces of all measurable functions f on G such that $\|f\|_{p,q}^{*} < \infty$. We know that $\|f\|_{p,p}^{*} = \|f\|_{p}$ and so $L^{p}(G) = L(p,p)(G)$. It is also known that if $1 < p < \infty$ and $1 \leq q \leq \infty$, then

$$\|f\|_{p,q}^{*} \leq \|f\|_{p,q} \leq \frac{p}{p-1}\|f\|_{p,q}^{*} \tag{2.2}$$

for each $f \in L(p,q)(G)$ and $\left(L(p,q)(G), \|\cdot\|_{p,q}\right)$ is a Banach space [3,10].

In [19], it is found that $B(p,q)(G) := L^{1}(G) \cap L(p,q)(G)$ is a normed space with the norm $\|\cdot\|_{B} = \|\cdot\|_{1} + \|\cdot\|_{p,q}$ and a Segal algebra for $1 < p < \infty$, $1 \leq q < \infty$. Nevertheless, some other properties of $B(p,q)(G)$ spaces are showed in [7].

3. The $W_{L(p,q)}^{k}(\mathbb{R}^{n})$ and $A_{L(p,q)}^{k}(\mathbb{R}^{n})$ spaces

If one looks for "*Sobolev algebras*" in literature, one sees that there are a lot of published papers about Sobolev algebras obtained by using different function spaces that are defined over different groups or sets. These spaces have been investigated under several respects, and mostly applied to the study of strongly nonlinear variational problems and partial differential equations.

In the sense of our study, we attach importance to [4-6,17]. In [5], Orlicz-Sobolev spaces that are multiplicative Banach algebras are characterized. In [6], it is showed that the space $L_{\alpha}^{p}(G) \cap L^{\infty}(G)$ is an algebra with respect to pointwise multiplication, where G is a connected unimodular Lie group. Also, sufficient conditions for the Sobolev spaces to form an algebra under pointwise multiplication have been given in [17].

In [4], Chu defined $A_{k}^{p}(\mathbb{R}^{n}) = L^{1}(\mathbb{R}^{n}) \cap W^{k,p}(\mathbb{R}^{n})$ spaces and showed some algebraic properties of these spaces (Segal algebras). In this section, we will generalize his results to Lorentz-Sobolev spaces and Lorentz-Sobolev algebras.

Definition 3.1. *Lorentz-Sobolev spaces are defined by*

$$W_{L(p,q)}^{k}(\mathbb{R}^{n}) = \{f \in L(p,q)(\mathbb{R}^{n}) : D^{\alpha}f \in L(p,q)(\mathbb{R}^{n})\} \tag{3.1}$$

for all $\alpha \in \mathbb{N}_{0}^{n}$ with $|\alpha| \leq k$ where k is a nonnegative integer, $p \in (1,\infty)$ and $q \in [1,\infty)$. Also they are equipped with the norm

$$\|f\|_{W_{L(p,q)}^{k}(\mathbb{R}^{n})} = \sum_{|\alpha| \leq k} \|D^{\alpha}f\|_{p,q}. \tag{3.2}$$

Clearly, if $k = 0$, then $W_{L(p,q)}^{k}(\mathbb{R}^{n}) = L(p,q)(\mathbb{R}^{n})$. Besides this, if we define $W_{L(p,q)}^{k,0}(\mathbb{R}^{n})$ as the space of the closure of $C_{0}^{\infty}(\mathbb{R}^{n})$ in the space $W_{L(p,q)}^{k}(\mathbb{R}^{n})$, then it is easy to see that $W_{L(p,q)}^{0,0}(\mathbb{R}^{n}) = L(p,q)(\mathbb{R}^{n})$ where $p \in (1,\infty)$ and $q \in [1,\infty)$. For any k, the chain of imbeddings

$$W_{L(p,q)}^{k,0}(\mathbb{R}^{n}) \hookrightarrow W_{L(p,q)}^{k}(\mathbb{R}^{n}) \hookrightarrow L(p,q)(\mathbb{R}^{n}) \tag{3.3}$$

is also clear. Instead of dealing with Lorentz-Sobolev spaces $W^k_{L(p,q)}(\mathbb{R}^n)$, we can pay attention to the completion of the set

$$\left\{ f \in C^k(\mathbb{R}^n) : \|f\|_{W^k_{L(p,q)}(\mathbb{R}^n)} < \infty \right\}$$

with respect to the norm in (3.2). Because, it is easy to show that these spaces are equal.

Now, we are going to give two propositions without their (easy) proofs . One can prove them by using the same methods as those used for abstract Sobolev spaces.

Proposition 3.2. *$W^k_{L(p,q)}(\mathbb{R}^n)$ is a (homogeneous) Banach space with $\|\cdot\|_{W^k_{L(p,q)}(\mathbb{R}^n)}$.*

Proposition 3.3. *If $p, q \in (1, \infty)$, then $W^k_{L(p,q)}(\mathbb{R}^n)$ spaces are reflexive. In other words, the associate space of $W^k_{L(p,q)}(\mathbb{R}^n)$ is $W^{-k}_{L(p',q')}(\mathbb{R}^n)$ where $\frac{1}{p} + \frac{1}{p'} = 1$ and $\frac{1}{q} + \frac{1}{q'} = 1$.*

After this point, we are going to deal with the algebraic structures of $L^1(\mathbb{R}^n) \cap W^k_{L(p,q)}(\mathbb{R}^n)$ spaces. For this reason, we will call this intersection space as $A^k_{L(p,q)}(\mathbb{R}^n)$ and endow it with the sum norm

$$\|f\|_A := \|f\|_1 + \|f\|_{W^k_{L(p,q)}(\mathbb{R}^n)} \tag{3.4}$$

for all $f \in A^k_{L(p,q)}(\mathbb{R}^n)$.

Proposition 3.4. *$A^k_{L(p,q)}(\mathbb{R}^n)$ is a Segal algebra on $\mathbb{R}^n$ if $p \in (1, \infty)$ and $q \in [1, \infty)$.*

Proof. Let $p \in (1, \infty)$ and $q \in [1, \infty)$. Since $W^k_{L(p,q)}(\mathbb{R}^n)$ and $L^1(\mathbb{R}^n)$ are homogeneous Banach spaces, it is easy to see that $A^k_{L(p,q)}(\mathbb{R}^n)$ is also a homogeneous Banach space under the sum norm $\|\cdot\|_A \geq \|\cdot\|_1$ by [11]. By a result of Theorem 2.8, we get $A^k_{L(p,q)}(\mathbb{R}^n)$ is a homogeneous Banach algebra. By [1, 2.19.Theorem], we know that $C_0^\infty(\mathbb{R}^n)$ is dense in $L^1(\mathbb{R}^n)$ and is contained in $W^k_{L(p,q)}(\mathbb{R}^n)$. Therefore, $A^k_{L(p,q)}(\mathbb{R}^n)$ is a Segal algebra on $\mathbb{R}^n$. □

Theorem 3.5. *$A^k_{L(p,q)}(\mathbb{R}^n)$ is an FP-algebra for $p \in (1, \infty)$ and $q \in [1, \infty)$.*

Proof. Firstly, we are going to show the $P-$algebra property of $A^k_{L(p,q)}(\mathbb{R}^n)$ spaces.

(i) Let

$$\begin{aligned} \Delta_m &= \left[m - \frac{1}{4}, m + \frac{1}{4}\right] \times \cdots \times \left[m - \frac{1}{4}, m + \frac{1}{4}\right] \quad (n - \text{times}) \\ \Omega_m &= \left[m - \frac{1}{8}, m + \frac{1}{8}\right] \times \cdots \times \left[m - \frac{1}{8}, m + \frac{1}{8}\right] \quad (n - \text{times}) \end{aligned}$$

and

$$\Delta'_m = \left[m - \frac{1}{4}, m + \frac{1}{4}\right], \ \Omega'_m = \left[m - \frac{1}{8}, m + \frac{1}{8}\right]$$

for $m \geq 1$. By [18, 1.8.Theorem], there exists a generalized trapezium function $f_1 \in L^1(\mathbb{R})$ such that $0 \leq \widehat{f_1} \leq 1$, supp$\widehat{f_1} \subset \Delta'_1$ and $\widehat{f_1}(\Omega'_1) = 1$.

If we let $f_m(t) = e^{i(m-1)t} f_1(t)$, then it is easy to see that $0 \le \widehat{f_m} \le 1$, $\mathrm{supp}\widehat{f_m} \subset \Delta'_m$ and $\widehat{f_m}(\Omega'_m) = 1$ for $m \ge 2$. If we define F_m by $F_m(x_1,...,x_n) = f_m(x_1)\cdots f_m(x_n)$ for $m = 1,2,...,$ then $F_m \in L^1(\mathbb{R}^n)$, $\widehat{F_m}(t_1,...,t_n) = \widehat{f_m}(t_1)\cdots\widehat{f_m}(t_n)$ and $0 \le \widehat{F_m} \le 1$, $\mathrm{supp}\widehat{F_m} \subset \Delta_m$, $\widehat{F_m}(\Omega_m) = 1$. If $P\left(L^1(\mathbb{R}^n)\right)$ is the set of all f in $L^1(\mathbb{R}^n)$ whose Fourier transform $\widehat{f}$ has compact support, then it is seen that $F_m \in P\left(L^1(\mathbb{R}^n)\right)$. Since $P\left(L^1(\mathbb{R}^n)\right)$ is dense in every homogeneous Banach algebra [18, 3.7.Theorem], we have $F_m \in A^k_{L(p,q)}(\mathbb{R}^n)$. For $1 \le j \le k$ and $m \ge 2$, the equality

$$\begin{aligned} f_m^{(j)}(t) &= \left(e^{i(m-1)t}\right)^{(j)} f_1(t) + \binom{j}{1}\left(e^{i(m-1)t}\right)^{(j-1)} f_1'(t) + \\ &\quad + \binom{j}{2}\left(e^{i(m-1)t}\right)^{(j-2)} f_1''(t) + ... + \binom{j}{j}\left(e^{i(m-1)t}\right) f_1^{(j)}(t) \\ &= i^j (m-1)^j e^{i(m-1)t} f_1(t) + \binom{j}{1} i^{j-1}(m-1)^{j-1} e^{i(m-1)t} f_1'(t) \\ &\quad + \binom{j}{2} i^{j-2}(m-1)^{j-2} e^{i(m-1)t} f_1''(t) + ... + \binom{j}{j}\left(e^{i(m-1)t}\right) f_1^{(j)}(t) \end{aligned}$$

is written. Since $f_m \in P\left(L^1(\mathbb{R})\right) \subset A^k_{L(p,q)}(\mathbb{R})$, if

$$M = \max\left\{\|f_1\|_{p,q},\ \|f_1'\|_{p,q},...,\left\|f_1^{(j)}\right\|_{p,q}\right\}$$

then, we get

$$\begin{aligned} \left\|f_m^{(j)}\right\|_{p,q} &= \left\|i^j (m-1)^j e^{i(m-1)t} f_1(t) + ... + \left(e^{i(m-1)t}\right) f_1^{(j)}(t)\right\|_{p,q} \\ &\le (m-1)^j \|f_1\|_{p,q} + (m-1)^{j-1}\binom{j}{1}\|f_1'(t)\|_{p,q} + ... + \left\|f_1^{(j)}(t)\right\|_{p,q} \\ &\le 2^j (m-1)^j M. \end{aligned} \tag{3.5}$$

Again, for $1 \le |\alpha| = j \le k$ and $0 \le j_i \le j$, $j_1 + \cdots + j_n = j$, it can be written by (3.5) that

$$\begin{aligned} \|D^\alpha F_m(x_1,\cdots,x_n)\|_{p,q} &= \left\|f_m^{(j_1)}(x_1) f_m^{(j_2)}(x_2)\cdots f_m^{(j_n)}(x_n)\right\|_{p,q} \\ &\le \left(2^j (m-1)^j M\right)^n \le \left(2^k (m-1)^k M\right)^n \end{aligned}$$

and so

$$\|F_m\|_A = \|F_m\|_1 + \|F_m\|_{W^k_{L(p,q)}(\mathbb{R}^n)} = \|F_m\|_1 + \sum_{|\alpha|\le k}\|D^\alpha F_m\|_{p,q}$$

$$\begin{aligned} &= \|F_m\|_1 + \|F_m\|_{p,q} + \sum_{|\alpha|=1}\|D^\alpha F_m\|_{p,q} + \sum_{|\alpha|=2}\|D^\alpha F_m\|_{p,q} + \ldots + \sum_{|\alpha|=k}\|D^\alpha F_m\|_{p,q} \\ &\le \|F_m\|_1 + \|F_m\|_{p,q} + \left[\binom{k}{1} + \binom{k}{2} + ... + \binom{k}{k}\right]\left(2^k (m-1)^k M\right)^n \\ &\le B(m-1)^{kn} \end{aligned} \tag{3.6}$$

for $m \geq 2$ and some constant $B > 0$. Since we can take B and C_1 large enough such that $C_m = B(m-1)^{kn} \geq 1$ for $m = 2, 3, ...,$ $C_1 > \|F_1\|_A$ and $C_1 > 1$, we have

$$\sum_{m=1}^{\infty} \frac{1}{C_m^{k+1}} < \infty \qquad \text{but} \qquad \sum_{m=1}^{\infty} \frac{1}{C_m^{1/kn}} = \infty, \quad \text{for } k \geq 1.$$

Thus we get the result.

Now let $k = 0$. Then $A^0_{L(p,q)}(\mathbb{R}^n) = L^1(\mathbb{R}^n) \cap W^0_{L(p,q)}(\mathbb{R}^n) = L^1(\mathbb{R}^n) \cap L(p,q)(\mathbb{R}^n) = B(p,q)(\mathbb{R}^n)$. Since $B(p,q)(\mathbb{R}^n)$ is a character invariant Segal algebra and every character Segal algebra is a $P-$algebra by [7] and [18, 4.9.Theorem], we get that $A^0_{L(p,q)}(\mathbb{R}^n)$ is a $P-$algebra.

(ii) It is obvious from (3.3) that $A^k_{L(p,q)}(\mathbb{R}^n) \subset B(p,q)(\mathbb{R}^n)$. Since $B(p,q)(\mathbb{R}^n)$ is a Segal algebra with $\widehat{B(p,q)(\mathbb{R}^n)} \subset L(p,q)(\mathbb{R}^n)$ for $p \in (1,\infty)$ and $q \in [1,\infty)$ by [3, Lemma 3.8], we get $B(p,q)(\mathbb{R}^n)$ is an $F-$algebra for $p \in (1,\infty)$ and $q \in [1,\infty)$ by [18, 4.5.Definition]. It is known from [18, Theorem 4.6] that $F-$algebra property is a going-down property. In other words, if B is an $F-$algebra and A is a subalgebra of B, then A is also an $F-$algebra. Therefore, $A^k_{L(p,q)}(\mathbb{R}^n)$ is an $F-$algebra due to $A^k_{L(p,q)}(\mathbb{R}^n) \subset B(p,q)(\mathbb{R}^n)$.

(i) and (ii) give the result. □

In [18, Theorem 8.8], it is proved that an $FP-$algebra does not admit the weak factorization property. So, we can write the following theorem.

Theorem 3.6. *$A^k_{L(p,q)}(\mathbb{R}^n)$ does not admit the weak factorization property.*

Remark 3.7. We know that a character invariant Segal algebra on the locally compact abelian group G has weak factorization if and only if it is equal to $L^1(G)$, by [8, Theorem 2.2]. For $p \in (1,\infty)$ and $q \in [1,\infty)$, we have $A^k_{L(p,q)}(\mathbb{R}^n) \subsetneqq L^1(\mathbb{R}^n)$. Therefore, an alternative proof for the preceding theorem may be done by showing character invariance of $A^k_{L(p,q)}(\mathbb{R}^n)$.

Theorem 3.8. [2] *Suppose S is a Segal algebra in $L^1(G)$ of the form $L^1(G) \cap E$, where G is a noncompact locally compact abelian group, E is a homogeneous Banach space on G. If the translation coefficient K_E of E is less than 2, then the multipliers space of S is isometrically isomorphic to the space $M(G)$ of all bounded regular Borel measures on G.*

Theorem 3.9. *The multipliers space of $A^k_{L(p,q)}(\mathbb{R}^n)$ is isometrically isomorphic to $M(\mathbb{R}^n)$ for $p \in (1,\infty)$ and $q \in [1,\infty)$.*

Proof. Let $f \in A^k_{L(p,q)}(\mathbb{R}^n)$. Then,

$$\begin{aligned} \|f + L_s f\|_{W^k_{L(p,q)}(\mathbb{R}^n)} &= \sum_{|\alpha| \leq k} \|D^\alpha (f + L_s f)\|_{p,q} \\ &\leq \|f + L_s f\|_{p,q} + \sum_{1 \leq |\alpha| \leq k} \|D^\alpha f\|_{p,q} + \sum_{1 \leq |\alpha| \leq k} \|L_s D^\alpha f\|_{p,q} \\ &= \|f + L_s f\|_{p,q} + 2 \sum_{1 \leq |\alpha| \leq k} \|D^\alpha f\|_{p,q} \end{aligned}$$

can be written. If $f=0$ (a.e.), then it is trivial that

$$\limsup_{|s|\to\infty}\|f+L_sf\|_{W^k_{L(p,q)}(\mathbb{R}^n)}=0.$$

Now let $f\neq 0$. Gürkanlı showed in [9, Lemma 4.1] that $K_{L(p,q)(G)}=2^{\frac{1}{p}}$ for $p\in(1,\infty)$ and $q\in[1,\infty)$. Then, we get

$$\begin{aligned}\limsup_{|s|\to\infty}\|f+L_sf\|_{W^k_{L(p,q)}(\mathbb{R}^n)} &\leq \limsup_{|s|\to\infty}\|f+L_sf\|_{p,q}+2\sum_{1\leq|\alpha|\leq k}\|D^\alpha f\|_{p,q}\\ &=2^{\frac{1}{p}}\|f\|_{p,q}+2\sum_{1\leq|\alpha|\leq k}\|D^\alpha f\|_{p,q}\\ &=2^{\frac{1}{p}}\|f\|_{p,q}+2\|f\|_{p,q}-2\|f\|_{p,q}+2\sum_{1\leq|\alpha|\leq k}\|D^\alpha f\|_{p,q}\\ &=\left(2^{\frac{1}{p}}-2\right)\|f\|_{p,q}+2\sum_{|\alpha|\leq k}\|D^\alpha f\|_{p,q}\\ &=\|f\|_{W^k_{L(p,q)}(\mathbb{R}^n)}\left(2-\frac{\left(2-2^{\frac{1}{p}}\right)\|f\|_{p,q}}{\|f\|_{W^k_{L(p,q)}(\mathbb{R}^n)}}\right).\end{aligned}$$

Since $0<\|f\|_{p,q}\leq\|f\|_{W^k_{L(p,q)}(\mathbb{R}^n)}$, $0<2-2^{\frac{1}{p}}<1$ and $0<2-\frac{\left(2-2^{\frac{1}{p}}\right)\|f\|_{p,q}}{\|f\|_{W^k_{L(p,q)}(\mathbb{R}^n)}}<2$ for all $p\in(1,\infty)$, we see that $K_{W^k_{L(p,q)}(\mathbb{R}^n)}<2$. Therefore, the multipliers space of $A^k_{L(p,q)}(\mathbb{R}^n)$ is isometrically isomorphic to $M(\mathbb{R}^n)$ by the preceding theorem for $p\in(1,\infty)$ and $q\in[1,\infty)$. □

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A note on Zamfirescu's operators in Kasahara spaces

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Abstract. The aim of this paper is to give local and global fixed point results for Zamfirescu's operators in Kasahara spaces. Since the domain invariance for Zamfirescu's operators is not always satisfied, we use in our proofs the successive approximations method. Our local results extend and generalize Krasnoselskii's local fixed point theorem by replacing the context of metric space with a more general one: the Kasahara space. On the other hand, instead of contractions we use Zamfirescu's operators. As application, a homotopy result on large Kasahara spaces is given.

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1. Introduction and preliminaries

In 1972, T. Zamfirescu gives in [10] several fixed point theorems for single-valued mappings of contractive type in metric spaces, obtaining generalizations for Banach-Caccioppoli's contraction principle, Kannan's, Edelstein's and Singh's theorems. Four years later, S. Kasahara gives in [4] some generalizations of the Banach-Caccioppoli's contraction principle showing that this principle holds even if the functional d of the metric space (X, d) does not necessarily satisfy all of the axioms of the metric. In this sense S. Kasahara replaces the context of metric spaces and proves his theorems in d-complete L-spaces. In 2010, I.A. Rus introduces in [8] the notion of Kasahara space and gives similar fixed point results to those given by S. Kasahara.

In order to give the notion of Kasahara space, we recall first the notion of L-space which was given by M. Fréchet in [2].

Definition 1.1. *Let X be a nonempty set. Let*

$$s(X) := \big\{(x_n)_{n\in\mathbb{N}} \mid x_n \in X,\ n \in \mathbb{N}\big\}.$$

Let $c(X)$ be a subset of $s(x)$ and $Lim : c(X) \to X$ be an operator. By definition the triple $(X, c(X), Lim)$ is called an L-space (denoted by $(X, \to)$) if the following conditions are satisfied:

(i) *if $x_n = x$, for all $n \in \mathbb{N}$, then $(x_n)_{n\in\mathbb{N}} \in c(X)$ and $Lim(x_n)_{n\in\mathbb{N}} = x$.*

(ii) *if $(x_n)_{n\in\mathbb{N}} \in c(X)$ and $Lim(x_n)_{n\in\mathbb{N}} = x$, then for all subsequences $(x_{n_i})_{i\in\mathbb{N}}$ of $(x_n)_{n\in\mathbb{N}}$ we have that $(x_{n_i})_{i\in\mathbb{N}} \in c(X)$ and*

$$Lim(x_{n_i})_{i\in\mathbb{N}} = x.$$

Remark 1.2. For examples and more considerations on L-spaces, see [9].

We recall now the notion of Kasahara space, introduced by I.A. Rus in [8].

Definition 1.3. *Let $(X, \to)$ be an L-space and $d : X \times X \to \mathbb{R}_+$ be a functional. The triple $(X, \to, d)$ is a Kasahara space if and only if the following compatibility condition between $\to$ and d holds: for all $(x_n)_{n\in\mathbb{N}} \subset X$ with*

$$\sum_{n\in\mathbb{N}} d(x_n, x_{n+1}) < +\infty \;\Rightarrow\; (x_n)_{n\in\mathbb{N}} \text{ converges in } (X, \to).$$

Some examples of Kasahara spaces are presented bellow.

Example 1.4 (The trivial Kasahara space). Let (X, d) be a complete metric space. Let $\overset{d}{\to}$ be the convergence structure induced by the metric d on X. Then $(X, \overset{d}{\to}, d)$ is a Kasahara space.

Example 1.5 (I.A. Rus [8]**).** Let (X, ρ) be a complete semimetric space (see [5], [7]) with $\rho : X \times X \to \mathbb{R}_+$ continuous. Let $d : X \times X \to \mathbb{R}_+$ be a functional such that there exists $c > 0$ with $\rho(x, y) \le cd(x, y)$, for all $x, y \in X$. Then $(X, \overset{\rho}{\to}, d)$ is a Kasahara space.

Example 1.6 (I.A. Rus [8]**).** Let (X, ρ) be a complete quasimetric space (see [7]) with $\rho : X \times X \to \mathbb{R}_+$. Let $d : X \times X \to \mathbb{R}_+$ be a functional such that there exists $c > 0$ with $\rho(x, y) \le cd(x, y)$, for all $x, y \in X$. Then $(X, \overset{\rho}{\to}, d)$ is a Kasahara space.

Example 1.7 (S. Kasahara [4]**).** Let X denote the closed interval $[0, 1]$ and $\to$ be the usual convergence structure on $\mathbb{R}$. Let $d : X \times X \to \mathbb{R}_+$ be defined by

$$d(x, y) = \begin{cases} |x - y|, & \text{if } x \neq 0 \text{ and } y \neq 0 \\ 1, & \text{otherwise .} \end{cases}$$

Then $(X, \to, d)$ is a Kasahara space.

We recall also a very useful tool which helps us to prove the uniqueness of the fixed point for operators defined on Kasahara spaces.

Lemma 1.8 (Kasahara's lemma [4]**).** *Let $(X, \to, d)$ be a Kasahara space. Then*

$$d(x, y) = d(y, x) = 0 \;\Rightarrow\; x = y, \text{ for all } x, y \in X.$$

Remark 1.9. For more considerations on Kasahara spaces, see [4], [8] and the references therein.

In [10], T. Zamfirescu gives several fixed point theorems in metric spaces (X, d) for a specific contractive type operator $f : X \to X$ which satisfies at least one of the following conditions:

(i) there exists $\alpha \in [0, 1[$ such that

$$d(f(x), f(y)) \le \alpha d(x, y), \text{ for all } x, y \in X;$$

(ii) there exists $\beta \in [0, \frac{1}{2}[$ such that

$$d(f(x), f(y)) \le \beta[d(x, f(x)) + d(y, f(y))], \text{ for all } x, y \in X;$$

(iii) there exists $\gamma \in [0, \frac{1}{2}[$ such that

$$d(f(x), f(y)) \le \gamma[d(x, f(y)) + d(y, f(x))], \text{ for all } x, y \in X.$$

Remark 1.10. If $f : X \to X$ is a Zamfirescu operator defined on a metric space (X, d) then there exists a number $\delta \in [0, 1[$, $\delta := \max\{\alpha, \frac{\beta}{1-\beta}, \frac{\gamma}{1-\gamma}\}$ such that at least one of the above conditions (i), (ii) and (iii) holds, where α, β and γ are replaced with δ.

In our results, the considered Zamfirescu's operators, are defined as follows.

Definition 1.11. *Let $(X, \to, d)$ be a Kasahara space. The mapping $f : X \to X$ is called Zamfirescu operator if there exists $\delta \in [0, \frac{1}{2}[$ such that for each $x, y \in X$ at least one of the following conditions is true:*

(1_z) $d(f(x), f(y)) \le \delta d(x, y)$;
(2_z) $d(f(x), f(y)) \le \delta[d(x, f(x)) + d(y, f(y))]$;
(3_z) $d(f(x), f(y)) \le \delta[d(x, f(y)) + d(y, f(x))]$.

Throughout this paper we give some fixed point results for Zamfirescu's operators in the Kasahara space $(X, \to, d)$, where $d : X \times X \to \mathbb{R}_+$ is a premetric. We recall the notion of premetric in the following definition.

Definition 1.12. *Let X be a nonempty set. A distance functional $d : X \times X \to \mathbb{R}_+$ is called premetric if and only if the following conditions hold:*

(1) $d(x, x) = 0$, *for all* $x \in X$;
(2) $d(x, z) \le d(x, y) + d(y, z)$, *for all* $x, y, z \in X$.

Lemma 1.13. *Let $(X, \to, d)$ be a Kasahara space, where $d : X \times X \to \mathbb{R}_+$ is a premetric. If $f : X \to X$ is a Zamfirescu operator, then f has at most one fixed point.*

Proof. Let $x^*, y^* \in X$ be two fixed points for the Zamfirescu operator f.

Then $x^* = f(x^*)$ and $y^* = f(y^*)$.

Suppose that f satisfies the condition (1_z). Then we have

$$d(x^*, y^*) = d(f(x^*), f(y^*)) \le \delta d(x^*, y^*) \Rightarrow d(x^*, y^*) = 0.$$

Similarly, we get $d(y^*, x^*) = 0$. By Kasahara's lemma 1.8, it follows that $x^* = y^*$.

Assume that f satisfies the condition (2_z). We get that

$$d(x^*, y^*) = d(f(x^*), f(y^*)) \le \delta[d(x^*, f(x^*)) + d(y^*, f(y^*))] \Rightarrow d(x^*, y^*) = 0.$$

Similarly, we get $d(y^*, x^*) = 0$ and by Kasahara's lemma 1.8, it follows that $x^* = y^*$.

Finally, if f satisfies the condition (3_z), we have

$$d(x^*,y^*) = d(f(x^*),f(y^*)) \leq \delta[d(x^*,y^*) + d(y^*,x^*)].$$

Similarly, we have $d(y^*,x^*) \leq \delta[d(y^*,x^*) + d(x^*,y^*)]$.

Hence, we obtain $d(x^*,y^*) + d(y^*,x^*) \leq 2\delta[d(x^*,y^*) + d(y^*,x^*)]$ and we have further that $(1-2\delta)[d(x^*,y^*)+d(y^*,x^*)] \leq 0$. It follows that $d(x^*,y^*) = d(y^*,x^*) = 0$ and by Kasahara's lemma 1.8, we get $x^* = y^*$. □

Let $(X,\to)$ be an L-space and $f : X \to X$ be an operator. The following notations and notions will be needed in the sequel of this paper:

- $F_f := \{x \in X \mid x = f(x)\}$ the set of all fixed points for f.
- $Graph(f) := \{(x,y) \in X \times X \mid y = f(x)\}$ the graph of f. We say that f has closed graph with respect to $\to$ or $Graph(f)$ is closed in $X \times X$ with respect to $\to$ if and only if for any sequences $(x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}} \subset X$ with $y_n = f(x_n)$ for all $n \in \mathbb{N}$ and $x_n \to x \in X$, $y_n \to y \in X$, as $n \to \infty$, we have that $y = f(x)$.
- $\tilde{B}(x_0,r) := \big\{x \in X \mid d(x_0,x) \leq r\big\}$ the right closed ball centered in $x_0 \in X$ with radius $r \in \mathbb{R}_+$.
- A sequence $(x_n)_{n\in\mathbb{N}} \subset X$ is called sequence of successive approximations for f starting from a given point $x_0 \in X$ if $x_{n+1} = f(x_n)$, for all $n \in \mathbb{N}$. Notice that $x_n = f^n(x_0)$, for all $n \in \mathbb{N}$.

The aim of this paper is to give local and global fixed point results for Zamfirescu's operators in Kasahara spaces. Since the domain invariance for Zamfirescu's operators is not always satisfied, we use in our proofs the successive approximations method. Our local results extend and generalize Krasnoselskii's local fixed point theorem by replacing the context of metric space with a more general one: the Kasahara space. On the other hand, instead of contractions we use Zamfirescu's operators. As application, a homotopy result on large Kasahara spaces is given.

2. Fixed point results in Kasahara spaces

We begin this section by presenting our main local fixed point result which extends and generalizes Krasnoselskii's theorem.

Theorem 2.1 (Krasnoselskii (see e.g. [3]**)).** *Let (X,d) be a complete metric space. Let $x_0 \in X$, $r \in \mathbb{R}_+$ and $f : \tilde{B}(x_0,r) \to X$ be an operator.*

If there exists $\alpha \in [0,1[$ such that $d(f(x),f(y)) \leq \alpha d(x,y)$, for all $x,y \in X$ and $d(x_0,f(x_0)) < (1-\alpha)r$ then f has at least one fixed point in $\tilde{B}(x_0,r)$.

Remark 2.2. Let $(X,\to,d)$ be a Kasahara space, where $d : X\times X \to \mathbb{R}_+$ is a premetric. Let $x_0 \in X$ and $r \in \mathbb{R}_+$. If d is continuous on X with respect to the second argument, then

(*i*) the right closed ball $\tilde{B}(x_0,r)$ is a closed set in X with respect to $\to$, i.e., for any sequence $(z_n)_{n\in\mathbb{N}} \subset \tilde{B}(x_0,r)$, with $z_n \to z \in X$, as $n \to \infty$, we get that $z \in \tilde{B}(x_0,r)$;

(*ii*) $(\tilde{B}(x_0,r),\to,d)$ is a Kasahara space.

Our first main result is the following.

Theorem 2.3. *Let $(X, \to, d)$ be a Kasahara space, where $d : X \times X \to \mathbb{R}_+$ is a premetric. Let $x_0 \in X$, $r \in \mathbb{R}_+$ and $f : \tilde{B}(x_0, r) \to X$ be a Zamfirescu operator. We assume that:*

(*i*) *Graph(f) is closed in $X \times X$ with respect to $\to$;*
(*ii*) *$d(x_0, f(x_0)) \leq (1-\delta)r$;*
(*iii*) *d is continuous with respect to the second argument.*

Then

(1°) *f has at most one fixed point $x^* \in \tilde{B}(x_0, r)$ and $f^n(x_0) \to x^*$, as $n \to \infty$.*
(2°) *at least one of the following estimations holds:*

$$d(x_n, x^*) \leq \delta^n r, \text{ for all } n \in \mathbb{N}, \tag{2.1}$$

$$d(x_n, x^*) \leq \frac{\delta^n r}{(1-2\delta)(1-\delta)^{n-2}}, \text{ for all } n \in \mathbb{N}, \tag{2.2}$$

where $x^* \in F_f$ and $(x_n)_{n\in\mathbb{N}}$ is the sequence of successive approximations for f starting from x_0.

Proof. (1°). Let us consider the sequence of successive approximations $(x_n)_{n\in\mathbb{N}}$, $x_n = f^n(x_0)$, for all $n \in \mathbb{N}$, starting from $x_0 \in X$. By the assumption (*ii*) it follows that the Zamfirescu operator f is a graphic contraction on $\tilde{B}(x_0, r)$.

Indeed, if f satisfies (1_z) in Definition 1.11, then by choosing $y = f(x_0)$ we have

$$d(f(x_0), f^2(x_0)) \leq \delta d(x_0, f(x_0)).$$

If condition (2_z) is satisfied, then

$$d(f(x_0), f^2(x_0)) \leq \delta[d(x_0, f(x_0)) + d(f(x_0), f^2(x_0))]$$

which implies that $d(f(x_0), f^2(x_0)) \leq \frac{\delta}{1-\delta} d(x_0, f(x_0))$.

If condition (3_z) is satisfied, then

$$\begin{aligned} d(f(x_0), f^2(x_0)) &\leq \delta[d(x_0, f^2(x_0)) + d(f(x_0), f(x_0))] \\ &\leq \delta[d(x_0, f(x_0)) + d(f(x_0), f^2(x_0))]. \end{aligned}$$

So we obtain again that $d(f(x_0), f^2(x_0)) \leq \frac{\delta}{1-\delta} d(x_0, f(x_0))$.

By the same assumption (*ii*) we have that $f(x_0) \in \tilde{B}(x_0, r)$.

On the other hand, $d(x_0, f^2(x_0)) \leq d(x_0, f(x_0)) + d(f(x_0), f^2(x_0))$ which implies further that $d(x_0, f^2(x_0)) \leq (1-\delta^2)r$ or $d(x_0, f^2(x_0)) \leq r$, i.e., $f^2(x_0) \in \tilde{B}(x_0, r)$.

By mathematical induction, we get that for all $n \in \mathbb{N}$, $f^n(x_0) \in \tilde{B}(x_0, r)$ and that at least one of the following chains of estimations holds:

$$d(f^n(x_0), f^{n+1}(x_0)) \leq \delta d(f^{n-1}(x_0), f^n(x_0)) \leq \ldots \leq \delta^n d(x_0, f(x_0)),$$

or

$$d(f^n(x_0), f^{n+1}(x_0)) \leq \frac{\delta}{1-\delta} d(f^{n-1}(x_0), f^n(x_0)) \leq \ldots \leq \left(\frac{\delta}{1-\delta}\right)^n d(x_0, f(x_0)).$$

Knowing that $\delta \in [0, \frac{1}{2}[$, the series $\sum_{n\in\mathbb{N}} \delta^n$ and $\sum_{n\in\mathbb{N}} \big(\frac{\delta}{1-\delta}\big)^n$ are convergent. It follows that

$$\sum_{n\in\mathbb{N}} d(x_n, x_{n+1}) = \sum_{n\in\mathbb{N}} d(f^n(x_0), f^{n+1}(x_0)) < +\infty.$$

Since $(X, \to, d)$ is a Kasahara space, by (iii) we get that $(\tilde{B}(x_0, r), \to, d)$ is also a Kasahara space. Hence, the sequence $(x_n)_{n\in\mathbb{N}}$ is convergent in $\tilde{B}(x_0, r)$, so there exists an element $x^* \in \tilde{B}(x_0, r)$ such that $x_n \to x^*$, as $n \to \infty$.

Knowing that $Graph(f)$ is closed in $X \times X$ with respect to $\to$, we get that $x^* \in F_f$. The uniqueness of the fixed point is assured by Lemma 1.13.

(2°). Let $p \in \mathbb{N}$, $p \geq 1$. Since

$$d(f^n(x_0), f^{n+p}(x_0)) \leq \sum_{k=n}^{n+p-1} d(f^k(x_0), f^{k+1}(x_0))$$

we have at least one of the following two estimations:

$$d(f^n(x_0), f^{n+p}(x_0)) \leq \delta^n \left(\sum_{k=0}^{\infty} \delta^k \right) d(x_0, f(x_0)) \leq \frac{\delta^n}{1-\delta} d(x_0, f(x_0)),$$

or

$$d(f^n(x_0), f^{n+p}(x_0)) \leq \big(\frac{\delta}{1-\delta}\big)^n \left[\sum_{k=0}^{\infty} \big(\frac{\delta}{1-\delta}\big)^k \right] d(x_0, f(x_0)),$$

that is

$$d(f^n(x_0), f^{n+p}(x_0)) \leq \big(\frac{\delta}{1-\delta}\big)^n \frac{1-\delta}{1-2\delta}\, d(x_0, f(x_0)).$$

By letting $p \to \infty$ and by the assumption (ii), we get the estimations (2.1) and (2.2). □

We have also a global variant for Theorem 2.3.

Corollary 2.4. *Let $(X, \to, d)$ be a Kasahara space where $d : X \times X \to \mathbb{R}_+$ is a premetric, continuous with respect to the second argument. Let $f : X \to X$ be a Zamfirescu operator, having closed graph with respect to $\to$. Then*

(1°) *f has at least one fixed point $x^* \in X$ and $f^n(x_0) \to x^*$, as $n \to \infty$;*

(2°) *for all $n \in \mathbb{N}$, at least one of the following estimations holds:*

$$d(x_n, x^*) \leq \frac{\delta^n}{1-\delta} d(x_0, x_1) \quad or \quad d(x_n, x^*) \leq \left(\frac{\delta}{1-\delta}\right)^n \frac{1-\delta}{1-2\delta}\, d(x_0, x_1)$$

where $x^* \in F_f$ and $(x_n)_{n\in\mathbb{N}}$ is the sequence of successive approximations for f starting from x_0.

Proof. Fix $x_0 \in X$ and choose $r \in \mathbb{R}_+$ such that $d(x_0, f(x_0)) \leq (1-\delta)r$. The conclusions follow from Theorem 2.3. □

Remark 2.5. Regarding the Corollary 2.4, notice that the functional d must not necessarily be a premetric in order to prove the existence of fixed points for an operator $f : X \to X$ satisfying one of the conditions (1_z) or (2_z) from the Definition 1.11. However, the functional d must be at least a premetric in the case when f satisfies condition (3_z).

Remark 2.6. The global fixed point result given in Corollary 2.4 extends and generalizes Maia's fixed point theorem (see Theorem 1 in M.G. Maia [6]) in the sense that the set X endowed with two metrics is replaced by a Kasahara space. On the other hand, Zamfirescu's operators are used instead of contractions.

The following result is a generalization of Theorem 2.3.

Corollary 2.7. *Let $(X, \to, d)$ be a Kasahara space, where $d : X \times X \to \mathbb{R}_+$ is a premetric. Let $x_0 \in X$, $r \in \mathbb{R}_+$ and $f : \tilde{B}(x_0, r) \to X$ be an operator. We consider the function $\delta : \mathbb{R}_+^2 \to [0, \frac{1}{2}[$ with $\limsup\limits_{s \to t^+} \delta(s) < \frac{1}{2}$, for all $t \in \mathbb{R}_+^2$.*

Assume that:

(i) $Graph(f)$ is closed in $X \times X$ with respect to $\to$;

(ii) for all $x, y \in \tilde{B}(x_0, r)$, f satisfies at least one of the following conditions:

$(1'_z)$ $d(f(x), f(y)) \leq \delta\big(d(x,y), d(y,x)\big) \cdot d(x,y)$;

$(2'_z)$ $d(f(x), f(y)) \leq \delta\big(d(x, f(x)), d(y, f(y))\big) \cdot [d(x, f(x)) + d(y, f(y))]$;

$(3'_z)$ $d(f(x), f(y)) \leq \delta\big(d(x, f(y)), d(y, f(x))\big) \cdot [d(x, f(y)) + d(y, f(x))]$;

(iii) $d(x_0, f(x_0)) \leq (1 - \delta(\cdot, \cdot))r$;

(iv) d is continuous on X with respect to the second argument.

Then the following statements hold:

(1°) f has at least one fixed point $x^* \in \tilde{B}(x_0, r)$ and $f^n(x_0) \to x^*$, as $n \to \infty$.

(2°) at least one of the relations (2.1) and (2.2) holds.

Proof. We follow the proof of Theorem 2.3. □

3. An extension to large Kasahara spaces

The aim of this section is to present an extension of our fixed point results to large Kasahara spaces. As application, a homotopy result is given. To reach our purpose, we recall first the notion of large Kasahara space.

Definition 3.1 (I.A. Rus, [8]). *Let $(X, \to)$ be an L-space, $(G, +, \leq, \stackrel{G}{\to})$ be an L-space ordered semigroup with unity, 0 be the least element in $(G, \leq)$ and $d_G : X \times X \to G$ be an operator. The triple $(X, \to, d_G)$ is a large Kasahara space if and only if the following compatibility condition between $\to$ and d_G holds:*

- *if $(x_n)_{n \in \mathbb{N}} \subset X$ is a Cauchy sequence (in a certain sense) with respect to d_G then $(x_n)_{n \in \mathbb{N}}$ converges in $(X, \to)$.*

As in the previous section, we will consider the Kasahara space $(X, \to, d)$ where $d : X \times X \to \mathbb{R}_+$ is a premetric.

In order to obtain a large Kasahara space, we need to define a certain notion of Cauchy sequence with respect to the premetric d. We must take also into account the fact that d is not symmetric.

Definition 3.2. *Let (X, d) be a premetric space with $d : X \times X \to \mathbb{R}_+$ and let $(x_n)_{n\in\mathbb{N}}$ be a sequence in X. Then $(x_n)_{n\in\mathbb{N}}$ is a right-Cauchy sequence with respect to d if and only if*

$$\lim_{\substack{n\to\infty \\ m\to\infty}} d(x_n, x_m) = 0,$$

i.e., for any $\varepsilon > 0$, there exists $k \in \mathbb{N}$ such that $d(x_n, x_m) < \varepsilon$, for every $m, n \in \mathbb{N}$ with $m \geq n \geq k$.

The following notion of large Kasahara space arises.

Definition 3.3. *Let $(X, \to)$ be an L-space. Let $d : X \times X \to \mathbb{R}_+$ be a premetric on X. The triple $(X, \to, d)$ is a large Kasahara space if and only if the following compatibility condition between $\to$ and d holds:*

$$\text{if } (x_n)_{n\in\mathbb{N}} \subset X \text{ with } \lim_{\substack{n\to\infty \\ m\to\infty}} d(x_n, x_m) = 0 \text{ then } (x_n)_{n\in\mathbb{N}} \text{ converges in } (X, \to).$$

Remark 3.4. Let $(X, \to, d)$ be a large Kasahara space in the sense of Definition 3.3. Then $(X, \to, d)$ is a Kasahara space.

Indeed, let $(x_n)_{n\in\mathbb{N}}$ be a sequence in X with $\sum_{n\in\mathbb{N}} d(x_n, x_{n+1}) < +\infty$. By following S. Kasahara (see [4]), for all $\varepsilon > 0$, there exists $k \in \mathbb{N}$ such that for all $n, m \in \mathbb{N}$, with $m > n \geq k$, we have $d(x_n, x_m) \leq \sum_{i=n}^{m-1} d(x_i, x_{i+1}) < \varepsilon$.

Hence $\lim_{\substack{n\to\infty \\ m\to\infty}} d(x_n, x_m) = 0$ and since $(X, \to, d)$ is a large Kasahara space, we get that $(x_n)_{n\in\mathbb{N}}$ is convergent in $(X, \to)$. The conclusion follows from Definition 1.3.

Remark 3.5. Let $(X, \to, d)$ be a large Kasahara space in the sense of Definition 3.3. Then Theorem 2.3 and Corollaries 2.4 and 2.7 hold.

As application of Theorem 2.3 in large Kasahara spaces in the sense of Definition 3.3, we present a homotopy result which extends some similar homotopy results given on a set endowed with two metrics by A. Chiş in [1].

In our application, the following notion need to be defined.

Definition 3.6. *Let $(X, \to, d)$ be a large Kasahara space in the sense of Definition 3.3. A subset U of X is an open set with respect to d if there exists a right ball $B(x_0, r) := \{x \in X \mid d(x_0, x) < r\}$, $r > 0$, $x_0 \in U$ such that $B(x_0, r) \subset U$.*

Theorem 3.7. *Let $(X, \xrightarrow{\rho}, d)$ be a large Kasahara space in the sense of Definition 3.3, where $\rho : X \times X \to \mathbb{R}_+$ is a complete metric on X, $\xrightarrow{\rho}$ is the convergence structure induced by ρ on X and $d : X \times X \to \mathbb{R}_+$ is a continuous premetric on X.*

Let $Q \subset X$ be a closed set with respect to ρ. Let $U \subset X$ be an open set with respect to d and assume that $U \subset Q$.

Suppose $H : Q \times [0, 1] \to X$ satisfies the following properties:

(*i*) $x \neq H(x,\lambda)$ *for all* $x \in Q \setminus U$ *and all* $\lambda \in [0,1]$*;*

(*ii*) *for all* $\lambda \in [0,1]$ *and* $x, y \in Q$*, there exist* $\alpha \in [0,1[$ *and* $\beta \in [0,\frac{1}{2}[$ *such that one of the following conditions holds:*

(*ii*$_1$) $d(H(x,\lambda), H(y,\lambda)) \leq \alpha d(x,y)$*;*

(*ii*$_2$) $d(H(x,\lambda), H(y,\lambda)) \leq \beta[d(x, H(x,\lambda)) + d(y, H(y,\lambda))]$*;*

(*iii*) $H(x,\lambda)$ *is continuous in* λ *with respect to* d*, uniformly for* $x \in Q$*;*

(*iv*) H *is uniformly continuous from* $U \times [0,1]$ *endowed with the metric* d *on* U *into* (X,ρ)*;*

(*v*) H *is continuous from* $Q \times [0,1]$ *endowed with the metric* ρ *on* Q *into* (X,ρ)*.*

In addition, assume that H_0 *has a fixed point. Then for each* $\lambda \in [0,1]$ *we have that* H_λ *has a fixed point* $x_\lambda \in U$*. (here* $H_\lambda(\cdot) = H(\cdot,\lambda)$*)*

Proof. Let $A := \{\lambda \in [0,1] \mid \text{there exists } x \in U \text{ such that } x = H(x,\lambda)\}$.

Since H_0 has a fixed point and (*i*) holds, we have that $0 \in A$ so the set A is nonempty. We will show that A is open and closed in $[0,1]$ and so, by the connectedness of $[0,1]$, we will have $A = [0,1]$ and the proof will be complete.

First we show that A is closed in $[0,1]$.

Let $(\lambda_k)_{k\in\mathbb{N}}$ be a sequence in A with $\lambda_k \to \lambda \in [0,1[$ as $k \to \infty$. By the definition of A, for each $k \in \mathbb{N}$, there exists $x_k \in U$ such that $x_k = H(x_k,\lambda_k)$. Now we have

$$\begin{aligned} d(x_k,x_j) &= d(H(x_k,\lambda_k), H(x_j,\lambda_j)) \\ &\leq d(H(x_k,\lambda_k), H(x_k,\lambda)) + d(H(x_k,\lambda), H(x_j,\lambda)) \\ &\quad + d(H(x_j,\lambda), H(x_j,\lambda_j)) \end{aligned} \tag{3.1}$$

- If H satisfies (*ii*$_1$) then by (3.1) we get

$$\begin{aligned} & d(x_k,x_j) \leq d(H(x_k,\lambda_k), H(x_k,\lambda)) + \alpha d(x_k,x_j) + d(H(x_j,\lambda), H(x_j,\lambda_j)) \\ & \Leftrightarrow (1-\alpha) d(x_k,x_j) \leq d(H(x_k,\lambda_k), H(x_k,\lambda)) + d(H(x_j,\lambda), H(x_j,\lambda_j)) \end{aligned}$$

- If H satisfies (*ii*$_2$) then by (3.1) we have

$$\begin{aligned} d(x_k,x_j) &\leq d(H(x_k,\lambda_k), H(x_k,\lambda)) + d(H(x_j,\lambda), H(x_j,\lambda_j)) \\ &\quad + \beta[d(x_k, H(x_k,\lambda)) + d(x_j, H(x_j,\lambda))] \\ &= (d(H(x_k,\lambda_k), H(x_k,\lambda)) + d(H(x_j,\lambda), H(x_j,\lambda_j)) \\ &\quad + \beta[d(H(x_k,\lambda_k), H(x_k,\lambda)) + d(H(x_j,\lambda_j), H(x_j,\lambda))]. \end{aligned}$$

By (*iii*), letting $k, j \to \infty$ we get that the sequence $(x_k)_{k\in\mathbb{N}}$ is a Cauchy sequence with respect to d. Since $(X, \xrightarrow{\rho}, d)$ is a large Kasahara space, we get that $(x_k)_{k\in\mathbb{N}}$ is convergent in $(X, \xrightarrow{\rho})$. Moreover, since $Q \subset X$ is a closed set with respect to the complete metric ρ, there exists $x \in Q$ such that $\lim_{k\to\infty} \rho(x_k, x) = 0$.

We show next that $x = H(x,\lambda)$. Indeed, we have

$$\begin{aligned} \rho(x, H(x,\lambda)) &\leq \rho(x,x_k) + \rho(x_k, H(x,\lambda)) \\ &= \rho(x,x_k) + \rho(H(x_k,\lambda_k), H(x,\lambda)). \end{aligned}$$

By (*v*) and letting $k \to \infty$, we have $\rho(x, H(x,\lambda)) = 0$, so $x = H(x,\lambda)$ and by (*i*) we get that $x \in U$. Hence $\lambda \in A$ and so A is closed in $[0,1]$.

We show next that A is open in $[0,1]$.

Let $\lambda_0 \in A$ and $x_0 \in U$ such that $x_0 = H(x_0, \lambda_0)$. Since U is open with respect to d, by Definition 3.6 there exists a right ball $B(x_0, r) := \{x \in X \mid d(x_0, x) < r\}$, $r > 0$ such that $B(x_0, r) \subset U$. By (iii), H is uniformly continuous on $B(x_0, r)$.

Let $\varepsilon = \left(1 - \max\left\{\alpha, \frac{\beta}{1-\beta}\right\}\right)r > 0$. By the uniform continuity of H, there exists $\eta = \eta(r) > 0$ such that for each $\lambda \in [0,1]$ with $|\lambda - \lambda_0| \leq \eta$ we have $d(H(x, \lambda_0), H(x, \lambda)) < \varepsilon$ for any $x \in B(x_0, r)$. Since this property holds for $x = x_0$, we get $d(x_0, H(x_0, \lambda)) = d(H(x_0, \lambda_0), H(x_0, \lambda)) < \left(1 - \max\left\{\alpha, \frac{\beta}{1-\beta}\right\}\right)r$ for any $\lambda \in [0,1]$ with $|\lambda - \lambda_0| \leq \eta$.

By (ii), (iv) and (v) together with Theorem 2.3 in the context of large Kasahara spaces defined as in Definition 3.3, (in this case $\delta := \max\left\{\alpha, \frac{\beta}{1-\beta}\right\}$ and $f = H_\lambda$) we obtain the existence of $x_\lambda \in B(x_0, r)$ such that $x_\lambda = H_\lambda(x_\lambda)$ for any $\lambda \in [0,1]$ with $|\lambda - \lambda_0| \leq \eta$. Consequently A is open in $[0,1]$. □

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Some fixed point theorems on cartesian product in terms of vectorial measures of noncompactness

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Abstract. In this paper we study a system of operatorial equations in terms of some vectorial measures of noncompactness. The basic tools are the cartesian hull of a subset of a cartesian product and some classical fixed point principle.

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1. Introduction

Let X_i, $i = \overline{1,m}$, be some nonempty sets, $X := \prod_{i=1}^{m} X_i$ and $f : X \to X$ be an operator. In this case the fixed point equation

$$x = f(x),$$

where $x = (x_1, \ldots, x_m)$ and $f = (f_1, \ldots, f_m)$ takes the following form

$$\begin{cases} x_1 = f_1(x_1, \ldots, x_m) \\ \vdots \\ x_m = f_m(x_1, \ldots, x_m) \end{cases}$$

In this paper we shall study the above system of operatorial equation in the case when X_i, $i = \overline{1,m}$, are metric spaces. In order to do this, we introduce the cartesian hull and vectorial measure of noncompactness.

2. Preliminaries

Let (X, d) be a metric space. In this paper we shall use the following notations:
$\mathcal{P}(X) = \{Y |\ Y \subset X\}$
$P(X) = \{Y \subset X |\ Y \text{ is nonempty}\}$, $P_b(X) := \{Y \in P(X) |\ Y \text{ is bounded}\}$,
$P_{cl}(X) := \{Y \in P(X) |\ Y \text{ is closed}\}$, $P_{b,cl}(X) := P_b(X) \cap P_{cl}(X)$,

$P_{cp}(X) := \{Y \in P(X) |\ Y \text{ is compact}\}$.
If X is a Banach space then $P_{cv}(X) := \{Y \in P(X) |\ Y \text{ is convex}\}$
Let $f : X \to X$ is an operator. Then, we denote by $F_f := \{x \in X |\ x = f(x)\}$ the fixed point set of the operator f.

Definition 2.1. *A matrix $S \in \mathbb{R}_+^{m\times m}$ is called a matrix convergent to zero iff $S^k \to 0$ as $k \to +\infty$.*

Theorem 2.2. *(see* [2], [16], [18], [20], [23]*) Let $S \in \mathbb{R}_+^{m\times m}$. The following statements are equivalent:*

(i) *S is a matrix convergent to zero;*
(ii) *$S^k x \to 0$ as $k \to +\infty$, $\forall x \in \mathbb{R}^m$;*
(iii) *$I_m - S$ is non-singular and*

$$(I_m - S)^{-1} = I_m + S + S^2 + \ldots$$

(iv) *$I_m - S$ is non-singular and $(I_m - S)^{-1}$ has nonnegative elements;*
(v) *$\lambda \in \mathbb{C}$, $\det(S - \lambda I_m) = 0$ imply $|\lambda| < 1$;*
(vi) *there exists at least one subordinate matrix norm such that $\|S\| < 1$.*

The matrices convergent to zero were used by A. I. Perov [15] to generalize the contraction principle in the case of generalized metric spaces with the metric taking values in the positive cone of $\mathbb{R}^m$. For fixed point principle in such spaces see [16], [20], [22], [23].

3. Closure operators. Cartesian hull of a subset of a cartesian product

Let X be a nonempty set. By definition an operator $\eta : \mathcal{P}(X) \to \mathcal{P}(X)$ is a closure operator if:

(i) $Y \subset \eta(Y)$, $\forall Y \in \mathcal{P}(X)$;
(ii) $Y, Z \in \mathcal{P}(X)$, $Y \subset Z \Longrightarrow \eta(Y) \subset \eta(Z)$;
(iii) $\eta \circ \eta = \eta$.

In a real linear space X, the following operators are closure operators:

$$\eta : \mathcal{P}(X) \to \mathcal{P}(X),\ \eta(Y) := \text{linear hull of } Y;$$
$$\eta : \mathcal{P}(X) \to \mathcal{P}(X),\ \eta(Y) := \text{affine hull of } Y;$$
$$\eta : \mathcal{P}(X) \to \mathcal{P}(X),\ \eta(Y) := coY := \text{convex hull of } Y;$$

In a topological space X, the operator $\eta : \mathcal{P}(X) \to \mathcal{P}(X)$ defined by $\eta(Y) := \overline{Y}$ is a closure operator. In a linear topological space X, the operator $\eta : \mathcal{P}(X) \to \mathcal{P}(X)$ defined by $\eta(Y) := \overline{co}Y := \overline{coY}$ is a closure operator.

The main property of a closure operator is given by:

Lemma 3.1. *Let X be a nonempty set and $\eta : \mathcal{P}(X) \to \mathcal{P}(X)$ a closure operator. Let $(Y_i)_{i\in I}$ be a family of subsets of X such that $\eta(Y_i) = Y_i$ for all $i \in I$. Then*

$$\eta\left(\bigcap_{i\in I} Y_i\right) = \bigcap_{i\in I} Y_i.$$

In our considerations, in this paper, we need the following example of closure operator.

Let X_i, $i = \overline{1,m}$, be some nonempty sets and $X := \prod_{i=1}^{m} X_i$ their cartesian product. Let us denote by π_i, $i = \overline{1,m}$, the canonical projection on X_i, i.e.,

$$\pi_i : X \to X_i, \quad (x_1, \ldots, x_m) \mapsto x_i, \; i = \overline{1,m}.$$

Definition 3.2. *Let $Y \subset X$ be a subset of X. By the cartesian hull of Y we understand the subset*

$$caY := \pi_1(Y) \times \ldots \times \pi_m(Y).$$

Remark 3.3. In the paper [11] the set caY is denoted by $[Y]$.

Lemma 3.4. *The operator*

$$ca : \mathcal{P}(X) \to \mathcal{P}(X), \; Y \mapsto caY$$

is a closure operator.

Proof. We remark that:

1) $Y \subset caY$, for all $Y \in \mathcal{P}(X)$;
2) $Y, Z \in \mathcal{P}(X)$, $Y \subset Z$ then $caY \subset caZ$;
3) $ca(caY) = caY$, for all $Y \in \mathcal{P}(X)$.

So, $ca : \mathcal{P}(X) \to \mathcal{P}(X)$ is a closure operator. □

Remark 3.5. $caY = Y$ if and only if Y is a cartesian product, i.e., there exists $Y_i \subset X_i$, $i = \overline{1,m}$, such that $Y = \prod_{i=1}^{m} Y_i$.

We denote by $P_{ca}(X) := \{Y \in P(X) | \; Y \text{ is cartesian set } \}$.

Remark 3.6. From Lemma 3.1 and 3.4 it follows that the intersection of an arbitrary family of cartesian sets is a cartesian set.

Lemma 3.7. *Let $Y \subset X$ be a nonempty cartesian product subset of X and $f : Y \to Y$ an operator. Then $f(caf(Y)) \subset caf(Y)$.*

Proof. We remark that $f(Y) \subset ca\,f(Y) \subset Y$. □

The above lemmas will be basic for our proofs.

4. Measures of noncompactness. Examples

Let (X, d) be a complete metric space and $\delta : P_b(X) \to \mathbb{R}_+$

$$\delta(Y) := \sup\{d(a, b) | \; a, b \in Y\}.$$

be the diameter functional on X. The Kuratowski measure of noncompactness on X is defined by $\alpha_K : P_b(X) \to \mathbb{R}_+$

$$\alpha_K(Y) := \inf\left\{\varepsilon > 0 | \; Y = \bigcup_{i=1}^{m} Y_i, \; \delta(Y_i) \leq \varepsilon, \; m \in \mathbb{N}^*\right\}.$$

The Hausdorff measure of noncompactness on X is defined by $\alpha_H : P_b(X) \to \mathbb{R}_+$

$$\alpha_H(Y) := \inf\{\varepsilon > 0 |\ Y \text{ can be covered by a finitely many balls of radius} \le \varepsilon\}.$$

If we denote by α one of the functionals α_K and α_H then we have (see [1], [3], [5], [8], [19], [22], [4], ...):

Theorem 4.1. *The functional α has the following properties:*

(i) $\alpha(A) = 0 \Longrightarrow \overline{A}$ *is compact;*
(ii) $\alpha(A) = \alpha(\overline{A})$, $\forall A \in P_b(X)$*;*
(iii) $A \subset B,\ A, B \in P_b(X) \Longrightarrow \alpha(A) \le \alpha(B)$*;*
(iv) *If* $A_n \in P_{b,cl}(X)$, $A_{n+1} \subset A_n$ *and* $\alpha(A_n) \to 0$ *as* $n \to +\infty$ *then* $A_\infty := \bigcap_{n\in\mathbb{N}} A_n \neq \emptyset$ *and* $\alpha(A_\infty) = 0$.

In the case of a Banach space we have that

(v) $\alpha(coA) = \alpha(A)$, $\forall A \in P_b(X)$.

Let (X, d) be a complete metric space. By definition (see [19]), a functional

$$\alpha : P_b(X) \to \mathbb{R}_+$$

is called an abstract measure of noncompactness on X iff:

(i) $\alpha(A) = 0 \Longrightarrow \bar{A}$ is compact;
(ii) $\alpha(A) = \alpha(\bar{A})$, for all $A \in P_b(X)$;
(iii) $A \subset B,\ A, B \in P_b(X) \Longrightarrow \alpha(A) \le \alpha(B)$;
(iv) If $A_n \in P_{b,cl}(X)$, $A_{n+1} \subset A_n$ and $\alpha(A_n) \to 0$ as $n \to +\infty$ then $A_\infty := \bigcap_{n\in\mathbb{N}} A_n \neq \emptyset$ and $\alpha(A_\infty) = 0$.

In the case of a Banach space we add to these axioms the following:

(v) $\alpha(coA) = \alpha(A)$, for all $A \in P_b(X)$.

We remark that the Kuratowski's measure of noncompactness, α_K, the Hausdorff's measure of noncompactness, α_H and the diameter functional, δ, are examples of measure of noncompactness in the sense of the above definition (see [3], [7], [8], [9], [12], [19], ...). For other notions of abstract measures of noncompactness see [5], [14], [19] ...

5. Vectorial measures of noncompactness on a cartesian product of some metric spaces

Let (X_i, d_i), $i = \overline{1,m}$, be some complete metric spaces and let $X := \prod_{i=1}^{m} X_i$ their cartesian product. We consider on X the cartesian product topology. By definition a subset Y of X is a bounded subset if $\pi_i(Y) \in P_b(X_i)$, $i = \overline{1,m}$. Let α^i be a measure of noncompactness on X_i, $i = \overline{1,m}$. We consider on $P_b(X)$ the following vectorial functional

$$\alpha : P_b(X) \to \mathbb{R}^m_+,\ \alpha(Y) := \left(\alpha^1(\pi_1(Y)), \ldots, \alpha^m(\pi_m(Y))\right)^T.$$

We have:

Lemma 5.1. *The functional α has the following properties:*

(i) $Y \in P_b(X)$, $\alpha(Y) = 0 \Longrightarrow \overline{caY}$ *is compact;*

(i') $\alpha(caY) = \alpha(Y)$, *for all* $Y \in P_b(X)$*;*

(ii) $\alpha(\bar{Y}) = \alpha(Y)$, *for all* $Y \in P_b(X)$*;*

(iii) $Y \subset Z$, $Y, Z \in P_b(X) \Longrightarrow \alpha(Y) \leq \alpha(Z)$*;*

(iv) $Y_n \in P_{b,cl,ca}(X)$, $Y_{n+1} \subset Y_n$, $\alpha(Y_n) \to 0$ *as* $n \to +\infty$ *then* $Y_\infty := \bigcap_{n \in \mathbb{N}} Y_n \neq \emptyset$, $Y_\infty \in P_{b,cl,ca}(X)$ *and* $\alpha(Y_\infty) = 0$.

If $(X_i, |\cdot|_i)$, $i = \overline{1,m}$, *are Banach spaces then we have*

(v) $\alpha(coY) = \alpha(Y)$, *for all* $Y \in P_b(X)$.

Proof. The proof follows from the definition of α and from the definition of α^i. □

If we take $\alpha^i := \alpha^i_K$, $i = \overline{1,m}$, we have, by definition, the Kuratowski vectorial measure of noncompactness and if we take $\alpha^i := \alpha^i_H$, $i = \overline{1,m}$, we have the Hausdorff vectorial measure of noncompactness.

6. Fixed point theorems in terms of vectorial measures of noncompactness

Definition 6.1. *Let* $S \in \mathbb{R}^{m \times m}_+$ *be a matrix convergent to zero and* (X_i, d_i), $i = \overline{1,m}$, *complete metric spaces. Let* α^i *be a measure of noncompactness on* X_i, $i = \overline{1,m}$, *and* α *the corresponding vectorial measure of noncompactness on* $X := \prod_{i=1}^m X_i$. *An operator* $f : X \to X$ *is by definition an* (α, S)*-contraction iff:*

(i) $A \in P_b(X) \Longrightarrow f(A) \in P_b(X)$*;*

(ii) $\alpha(f(A)) \leq S\alpha(A)$, *for all* $A \in P_{b,ca}(X)$ *such that* $f(A) \subset A$.

If the condition (ii) is satisfied for all $A \in P_{b,ca}(X)$ then f is called a strict (α, S)-contraction.

Lemma 6.2. *Let* $Y \in P_{b,cl,ca}(X)$. *Let* $f : Y \to Y$ *be an operator such that:*

(i) f *is continuous;*

(ii) f *is an* (α, S)*-contraction.*

Then, there exists $A^* \in P_{b,cl,ca}(Y)$ *such that* $f(A^*) \subset A^*$ *and* $\alpha(A^*) = 0$.

Proof. Let $Y_1 := \overline{caf(Y)}$, $Y_2 := \overline{caf(Y_1)}$, ..., $Y_{n+1} := \overline{caf(Y_n)}$, ... It is clear that $Y_n \in P_{b,cl,ca}(Y)$, $Y_{n+1} \subset Y_n$ and $f(Y_n) \subset Y_n$. Moreover, from Lemma 5.1 and (ii) we have

$$\alpha(Y_n) = \alpha\left(\overline{caf(Y_{n-1})}\right) = \alpha(f(Y_{n-1})) \leq S\alpha(Y_{n-1}) \leq \ldots \leq S^n\alpha(Y),$$

therefore, $\alpha(Y_n) \to 0$ as $n \to +\infty$. From these we have that

$$Y_\infty := \bigcap_{n \in \mathbb{N}} Y_n \neq \emptyset,\ Y_\infty \in P_{b,cl,ca}(Y),\ f(Y_\infty) \subset Y_\infty \text{ and } \alpha(Y_\infty) = 0.$$

So, $A^* := Y_\infty$. □

In the case of Banach spaces, if $Y \in P_{b,cl,ca,co}(Y)$ then we have in addition that $coY_\infty = Y_\infty$. In the construction of the sequence set $(Y_n)_{n\in\mathbb{N}^*}$ we take $Y_{n+1} := \overline{co\,(caf\,(Y_n))}$.

From Lemma 6.2 we have the following basic fixed point principle in the case of metric spaces:

Theorem 6.3. *Let (X_i, d_i), $i = \overline{1,m}$, be some complete metric spaces and let $X := \prod_{i=1}^{m} X_i$ their cartesian product. Let $Y \in P_{b,cl,ca}(X)$ and $f : Y \to Y$ such that:*

(i) *f is continuous;*
(ii) *f is an (α, S)-contraction;*
(iii) *$A \in P_{b,cl,ca}(Y)$, $\alpha(A) = 0$ and $f(A) \subset A$ implies that $F_f \cap A \neq \emptyset$.*

Then

(a) *$F_f \neq \emptyset$;*
(b) *$\alpha(F_f) = 0$.*

Proof. (a) From Lemma 6.2, there exists $A^* \in P_{b,cl,ca}(Y)$ such that $f(A^*) \subset A^*$ and $\alpha(A^*) = 0$ and from condition (iii) it follows that $F_f \cap A^* \neq \emptyset$, i.e., $F_f \neq \emptyset$.

(b) We remark that $F_f \subset A^* = Y_\infty$ (see the proof of Lemma 6.2) and

$$0 \leq \alpha(F_f) \leq \alpha(Y_\infty) = 0.$$

□

If we take $\alpha := \delta$, the vectorial diameter functional, then from Theorem 6.3 we have:

Theorem 6.4. *Let (X_i, d_i), $i = \overline{1,m}$, be some complete metric spaces and $X := \prod_{i=1}^{m} X_i$. Let $Y \in P_{b,cl,ca}(X)$ and $f : Y \to Y$ such that:*

(i) *f is continuous;*
(ii) *f is an (δ, S)-contraction.*

Then $F_f = \{x^*\}$.

Proof. From Lemma 6.2, there exists $A^* \in P_{b,cl,ca}(Y)$ such that $f(A^*) \subset A^*$ and $\delta(A^*) = 0$. From $\delta(A^*) = 0$ we have that $A^* = \{x^*\}$ and $f(A^*) \subset A^*$ implies that $x^* \in F_f$. Also, from Theorem 6.3 we have that $\delta(F_f) = 0$, so $F_f = \{x^*\}$. □

In the case of Banach spaces we have:

Theorem 6.5. *Let $(X_i, |\cdot|_i)$, $i = \overline{1,m}$, be Banach spaces, $X := \prod_{i=1}^{m} X_i$ and $Y \in P_{b,cl,cv,ca}(X)$. Let $f : Y \to Y$ be such that:*

(i) *f is continuous;*
(ii) *f is an (α, S)-contraction.*

Then

(a) *$F_f \neq \emptyset$;*
(b) *$\alpha(F_f) = 0$.*

Proof. Let $Y_1 := \overline{co\,(caf\,(Y))}$, $Y_2 := \overline{co\,(caf\,(Y_1))}$, ..., $Y_{n+1} := \overline{co\,(caf\,(Y_n))}$, $n \in \mathbb{N}^*$. We remark that $Y_n \in P_{b,cl,cv,ca}(Y)$, $f(Y_n) \subset Y_n$, $Y_{n+1} \subset Y_n$ and

$$\alpha(Y_n) = \alpha\left(\overline{co\,(caf\,(Y_{n-1}))}\right) = \alpha(f(Y_{n-1})) \leq S\alpha(Y_{n-1}) \leq \ldots \leq S^n\alpha(Y),$$

therefore, $\alpha(Y_n) \to 0$ as $n \to +\infty$. These imply that

$$Y_\infty := \bigcap_{n\in\mathbb{N}} Y_n \neq \emptyset,\ Y_\infty \in P_{b,cl,cv,ca}(Y),\ f(Y_\infty) \subset Y_\infty \text{ and } \alpha(Y_\infty) = 0.$$

Since Y_∞ is a compact convex subset in the Banach space $X = \prod_{i=1}^{m} X_i$ (we take, for example, on X the norm $|x|_\infty = \max\{|x_1|, \ldots |x_m|\}$, which generates the cartesian product topology on X), from Schauder's fixed point theorem we have that $F_f \neq \emptyset$. But $F_f \subset Y_\infty$ is a closed subset of the compact subset Y_∞, so, F_f is a nonempty compact subset. □

For the operator $f : \prod_{i=1}^{m} X_i \to \prod_{i=1}^{m} X_i$, in the terms of vectorial norm, we have:

Theorem 6.6. *Let* $(X_i, |\cdot|_i)$, $i = \overline{1,m}$, *be Banach spaces,* $X := \prod_{i=1}^{m} X_i$, $\|x\| := (|x_1|_1, \ldots, |x_m|_m)^T$, *and* $f : X \to X$ *such that:*

(i) *f is continuous;*
(ii) *f is an (α, S)-contraction;*
(iii) *there exists $T \in \mathbb{R}_+^{m\times m}$ and a vector $M \in \mathbb{R}_+^m$ such that:*
  (1) *T is a matrix convergent to zero;*
  (2) *$\|f(x)\| \leq T\|x\| + M$, for all $x \in X$.*

Then

(a) *$F_f \neq \emptyset$;*
(b) *$\alpha(F_f) = 0$.*

Proof. Let $R = (R_1, \ldots, R_m)^T \in \mathbb{R}_+^m$, with $R_i > 0$, $i = \overline{1,m}$. We denote by

$$D_R := \{x \in X \mid \|x\| \leq R\}.$$

It is clear that $D_R \in P_{b,cl,ca,co}(X)$.

First we shall prove that there exists $R^0 \in \mathbb{R}_+^m$ such that

$$f(D_R) \subset D_R,\ \forall R \in \mathbb{R}_+^m,\ R \geq R^0.$$

Let $R \in \mathbb{R}_+^m$ and $x \in D_R$, from $(iii)_{(2)}$ we have

$$\|f(x)\| \leq TR + M.$$

To prove that $f(D_R) \subset D_R$ it is sufficient to have an R such that

$$TR + M \leq R \Leftrightarrow M \leq (I_m - T)R \Leftrightarrow (I_m - T)^{-1} M \leq R.$$

So, we can take $R^0 := (I_m - T)^{-1} M$. We remark that

$$f|_{D_R} : D_R \to D_R,\ \forall R \geq R^0,$$

satisfies the conditions from the Theorem 6.5 with $Y = D_R$. □

Remark 6.7. The above results generalize some results given in [7], [16], [17], [21], [24].

Remark 6.8. For the vector-valued norm versus scalar norms see [16], [17], [20].

Remark 6.9. For the condition *(iii)* in the scalar case see [3], [8], [9], [10], [12], [13].

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Coupled fixed point theorems for mixed monotone operators and applications

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Abstract. In this paper we present some existence and uniqueness results for a coupled fixed point problem associated to a pair of singlevalued satisfying the mixed monotone property. We also provide an application to a first-order differential system with periodic boundary value conditions.

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1. Introduction

In the study of the fixed points for an operator, it is sometimes useful to consider a more general concept, namely coupled fixed point. The concept of coupled fixed point for nonlinear operators was introduced and studied by Opoitsev (see V.I. Opoitsev [7]-[9]) and then, in 1987, by D. Guo and V. Lakshmikantham (see [4]) in connection with coupled quasisolutions of an initial value problem for ordinary differential equations. Later, a new research direction for the theory of coupled fixed points in ordered metric space was initiated by T. Gnana Bhaskar and V. Lakshmikantham in [3] and by V. Lakshmikantham and L. Ćirić in [5]. T. Gnana Bhaskar and V. Lakshmikantham [3] introduced the notion of the mixed monotone property of a given operator. Furthermore, they proved some coupled fixed point theorems for operators which satify the mixed monotone property and presented as an application, the existence and uniqueness of a solution for a periodic boundary value problem. Their approach is based on some contractive type conditions on the operator.

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For more applications see for example A. C. M. Ran, M. C. B. Reurings [12], J. J. Nieto and R. R. López [6], W. Sintunavarat, P. Kumam, and Y. J. Cho [15], V. Lakshmikantham and L. Ćirić [5], C. Urs [17].

Let X be a nonempty set. A mapping $d : X \times X \to \mathbb{R}^m$ is called a vector-valued metric on X if the following properties are satisfied:

(a) $d(x, y) \geq 0$ for all $x, y \in X$; if $d(x, y) = 0$, then $x = y$;
(b) $d(x, y) = d(y, x)$ for all $x, y \in X$;
(c) $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y \in X$.

A set endowed with a vector-valued metric d is called generalized metric space. The notions of convergent sequence, Cauchy sequence, completeness, open subset and closed subset are similar to those for usual metric spaces.

We denote by $M_{mm}(\mathbb{R}_+)$ the set of all $m \times m$ matrices with positive elements and by I the identity $m \times m$ matrix. If $x, y \in \mathbb{R}^m$, $x = (x_1, ..., x_m)$ and $y = (y_1, ..., y_m)$, then, by definition:

$$x \leq y \text{ if and only if } x_i \leq y_i \text{ for } i \in \{1, 2, ..., m\}.$$

Notice that we will make an identification between row and column vectors in $\mathbb{R}^m$.

For the proof of the main results we need the following theorems. A classical result in matrix analysis is the following theorem (see G. Allaire and S. M. Kaber [1], I. A. Rus [13], R. S. Varga [18]).

Theorem 1.1. *Let $A \in M_{mm}(\mathbb{R}_+)$. The following assertions are equivalents:*
(i) A is convergent towards zero;
(ii) $A^n \to 0$ as $n \to \infty$;
(iii) The eigenvalues of A are in the open unit disc, i.e $|\lambda| < 1$, for every $\lambda \in \mathbb{C}$ with $\det(A - \lambda I) = 0$;
(iv) The matrix $(I - A)$ is nonsingular and

$$(I - A)^{-1} = I + A + ... + A^n + ...; \tag{1.1}$$

(v) The matrix $(I - A)$ is nonsingular and $(I - A)^{-1}$ has nonnegative elements;
(vi) $A^n q \to 0$ and $qA^n \to 0$ as $n \to \infty$, for each $q \in \mathbb{R}^m$.

In the second section of this paper, the aim is to present in the setting of an ordered metric space, a Gnana Bhaskar-Lakshmikantham type theorem for the coupled fixed point problem of a pair of singlevalued operators satisfying a generalized mixed monotone assumption. In the third section we study the existence and uniqueness of the solution to a periodic boundary value system, as an application to the coupled fixed point theorem.

2. Existence and uniqueness results for coupled fixed point

Let X be a nonempty set endowed with a partial order relation denoted by $\leq$. Then we denote

$$X_{\leq} := \{(x_1, x_2) \in X \times X \ : \ x_1 \leq x_2 \text{ or } x_2 \leq x_1\}.$$

If $f : X \to X$ is an operator then we denote the cartesian product of f with itself as follows:

$$f \times f : X \times X \to X \times X, \text{ given by } (f \times f)(x_1, x_2) := (f(x_1), f(x_2)).$$

Definition 2.1. *Let X be a nonempty set. Then $(X, d, \leq)$ is called an ordered generalized metric space if:*
(i) (X, d) is a generalized metric space in the sense of Perov;
(ii) $(X, \leq)$ is a partially ordered set;

The following result will be an important tool in our approach.

Theorem 2.2. *Let $(X, d, \leq)$ be an ordered generalized metric space and let $f : X \to X$ be an operator. We suppose that:*
(1) for each $(x, y) \notin X_{\leq}$ there exists $z(x, y) := z \in X$ such that $(x, z), (y, z) \in X_{\leq}$;
(2) $X_{\leq} \in I(f \times f)$;
(3) $f : (X, d) \to (X, d)$ is continuous;
(4) the metric d is complete;
(5) there exists $x_0 \in X$ such that $(x_0, f(x_0)) \in X_{\leq}$;
(6) there exists a matrix $A \in M_{mm}(\mathbb{R}_+)$ which converges to zero, such that

$$d(f(x), f(y)) \leq Ad(x, y), \text{ for each } (x, y) \in X_{\leq}.$$

Then $f : (X, d) \to (X, d)$ is a Picard operator.

Proof. Let $x \in X$ be arbitrary. Since $(x_0, f(x_0)) \in X_{\leq}$, by (6) and (4), we get that there exists $x^* \in X$ such that $(f^n(x_0))_{n\in\mathbb{N}} \to x^*$ as $n \to +\infty$. By (3) we get that $x^* \in Fix(f)$.

If $(x, x_0) \in X_{\leq}$, then by (2), we have that $(f^n(x), f^n(x_0)) \in X_{\leq}$, for each $n \in \mathbb{N}$. Thus, by (6), we get that $(f^n(x))_{n\in\mathbb{N}} \to x^*$ as $n \to +\infty$.

If $(x, x_0) \notin X_{\leq}$, then by (1), it follows that there exists $z(x, x_0) := z \in X_{\leq}$ such that $(x, z), (x_0, z) \in X_{\leq}$. By the fact that $(x_0, z) \in X_{\leq}$, as before, we get that $(f^n(z))_{n\in\mathbb{N}} \to x^*$ as $n \to +\infty$. This together with the fact that $(x, z) \in X_{\leq}$ implies that $(f^n(x))_{n\in\mathbb{N}} \to x^*$ as $n \to +\infty$.

Finally, the uniqueness of the fixed point follows by the contraction condition (6) using again the assumption (1). □

Remark 2.3. The conclusion of the above theorem holds if instead the hypothesis (2) we put:
(2') $f : (X, \leq) \to (X, \leq)$ is monotone increasing
or
(2") $f : (X, \leq) \to (X, \leq)$ is monotone decreasing.

Of course, it is easy to remark that assertion (2) in Theorem 2.2 is more general. For example, if we consider the ordered metric space $(\mathbb{R}^2, d, \leq)$, then $f : \mathbb{R}^2 \to \mathbb{R}^2, f(x_1, x_2) := (g(x_1, x_2), g(x_1, x_2))$ satisfies (2), for any $g : \mathbb{R}^2 \to \mathbb{R}$.

Remark 2.4. Condition (5) from the above theorem is equivalent with:

$$(5') \quad f \text{ has a lower or an upper fixed point in } X.$$

Remark 2.5. For some similar results see Theorem 4.2 and Theorem 4.7 in A. Petruşel, I.A. Rus [10].

We will apply the above result for the coupled fixed point problem generated by two operators.

Let X be a nonempty set endowed with a partial order relation denoted by $\leq$. If we consider $z := (x, y), w := (u, v)$ two arbitrary elements of $Z := X \times X$, then, by definition

$$z \preceq w \text{ if and only if } (x \geq u \text{ and } y \leq v).$$

Notice that $\preceq$ is a partial order relation on Z.

We denote

$$Z_{\preceq} = \{(z, w) := ((x, y), (u, v)) \in Z \times Z \ : \ z \preceq w \text{ or } w \preceq z\}.$$

Let $T : Z \to Z$ be an operator defined by

$$T(x, y) := \begin{pmatrix} T_1(x, y) \\ T_2(x, y) \end{pmatrix} = (T_1(x, y), T_2(x, y)) \tag{2.1}$$

The cartesian product of T and T will be denoted by $T \times T$ and it is defined in the following way

$$T \times T : Z \times Z \to Z \times Z, \quad (T \times T)(z, w) := (T(z), T(w)).$$

We recall the following existence and uniqueness theorem for the coupled fixed point of a pair of singlevalued operators which satisfy the mixed monotone property (see A. Petrusel, G. Petrusel, and C. Urs [11]).

Theorem 2.6. *Let $(X, d, \leq)$ be an ordered and complete metric space and let $T_1, T_2 : X \times X \to X$ be two operators. We suppose:*

(i) for each $z = (x, y), w = (u, v) \in X \times X$ which are not comparable with respect to the partial ordering $\preceq$ on $X \times X$, there exists $t := (t_1, t_2) \in X \times X$ (which may depend on (x, y) and (u, v)) such that t is comparable (with respect to the partial ordering $\preceq$) with both z and w, i.e.,

$$((x \geq t_1 \text{ and } y \leq t_2) \text{ or } (x \leq t_1 \text{ and } y \geq t_2) \text{ and}$$
$$((u \geq t_1 \text{ and } v \leq t_2) \text{ or } (u \leq t_1 \text{ and } v \geq t_2));$$

(ii) for all $(x \geq u$ and $y \leq v)$ or $(u \geq x$ and $v \leq y)$ we have

$$\begin{cases} T_1(x, y) \geq T_1(u, v) \\ T_2(x, y) \leq T_2(u, v) \end{cases} \text{or} \begin{cases} T_1(u, v) \geq T_1(x, y) \\ T_2(u, v) \leq T_2(x, y) \end{cases}$$

(iii) $T_1, T_2 : X \times X \to X$ are continuous;

(iv) there exists $z_0 := (z_0^1, z_0^2) \in X \times X$ such that

$$\begin{cases} z_0^1 \geq T_1(z_0^1, z_0^2) \\ z_0^2 \leq T_2(z_0^1, z_0^2) \end{cases} \text{or} \begin{cases} T_1(z_0^1, z_0^2) \geq z_0^1 \\ T_2(z_0^1, z_0^2) \leq z_0^2 \end{cases}$$

(v) there exists a matrix $A = \begin{pmatrix} k_1 & k_2 \\ k_3 & k_4 \end{pmatrix} \in M_2(\mathbb{R}_+)$ *convergent toward zero such that*

$$\begin{aligned} d(T_1(x,y), T_1(u,v)) &\leq k_1 d(x,u) + k_2 d(y,v) \\ d(T_2(x,y), T_2(u,v)) &\leq k_3 d(x,u) + k_4 d(y,v) \end{aligned}$$

for all $(x \geq u$ *and* $y \leq v)$ *or* $(u \geq x$ *and* $v \leq y)$.

Then there exists a unique element $(x^*, y^*) \in X \times X$ *such that*

$$x^* = T_1(x^*, y^*) \text{ and } y^* = T_2(x^*, y^*)$$

and the sequence of the succesive aproximations $(T_1^n(w_0^1, w_0^2), T_2^n(w_0^1, w_0^2))$ *converges to* (x^*, y^*) *as* $n \to \infty$, *for all* $w_0 = (w_0^1, w_0^2) \in X \times X$.

3. An application to periodic boundary value system

We study now the existence and uniqueness of the solution to a periodic boundary value system, as an application to coupled fixed point Theorem 2.6 for mixed monotone type singlevalued operators in the framework of partially ordered metric space.

In a similar context M. D. Rus [14] investigated the fixed point problem for a system of multivariate operators which are coordinate-wise uniformly monotone, in the setting of quasi-ordered sets. As an application, a new abstract multidimensional fixed point problem was studied. Additionally an application to a first-order differential system with periodic boundary value conditions was presented (see M. D. Rus [14]). V. Berinde and M. Borcut studied in [2] also the existence and uniqueness of solutions to a tripled fixed point problem.

We denote the partial order relation by $\preceq$ on $C(I) \times C(I)$. If we consider $z := (x, y)$ and $w := (u, w)$ two arbitrary elements of $C(I) \times C(I)$, then by definition

$$z \preceq w \text{ if and only if } (x \geq u \text{ and } y \leq v),$$

where $x \geq u$ means that $x(t) \geq u(t)$, for all $t \in I$.

We consider the first-order periodic boundary value system:

$$\begin{cases} x'(t) = f_1(t, x(t), y(t)) \\ y'(t) = f_2(t, x(t), y(t)) \\ x(0) = x(T) \\ y(0) = y(T) \end{cases} \text{for all } t \in I := [0, T] \tag{3.1}$$

where $T > 0$ and $f_1, f_2 : I \times \mathbb{R}^2 \to \mathbb{R}$ under the assumptions:

(a1) f_1, f_2 are continuous;

(a2) there exist $\lambda > 0$ and $\mu_1, \mu_2, \mu_3, \mu_4 > 0$ such that

$$0 \leq f_1(t, x, y) - f_1(t, u, v) + \lambda(x - u) \leq \lambda[\mu_1(x - u) + \mu_2(y - v)]$$

$$-\lambda\left[\mu_3(x - u) + \mu_4(y - v)\right] \leq f_2(t, x, y) - f_2(t, u, v) + \lambda(y - v) \leq 0$$

for all $t \in I$ and $x, y, u, v \in \mathbb{R}$.

(a3) for each $z := (x, y), w := (u, w) \in C(I) \times C(I)$ which are not comparable with respect to the partial ordering $\preceq$ on $C(I) \times C(I)$ there exists $p := (p_1, p_2) \in$

$C(I)\times C(I)$ such that p is comparable (with respect to the partial ordering $\preceq$) with both z and w, i.e.

$$((x \geq p_1 \text{ and } y \leq p_2) \text{ or } (x \leq p_1 \text{ and } y \leq p_2) \text{ and}$$
$$(u \geq p_1 \text{ and } v \leq p_2) \text{ or } (u \leq p_1 \text{ and } v \leq p_2)).$$

(a4) for all ($x \geq u$ and $y \leq v$) or ($u \geq x$ and $v \leq y$) we have

$$\begin{cases} f_1(t,x,y) \geq f_1(t,u,v) \\ f_2(t,x,y) \leq f_1(t,u,v) \end{cases} \text{or} \begin{cases} f_1(t,u,v) \geq f_1(t,x,y) \\ f_2(t,u,v) \leq f_2(t,x,y) \end{cases}.$$

(a5) there exists $z_0 := (z_0^1, z_0^2) \in C(I)\times C(I)$ such that the following relations hold:

$$\text{(a5')} \begin{cases} z_0^1(t) \geq f_1(t, z_0^1(t), z_0^2(t)) \\ z_0^2(t) \geq f_2(t, z_0^1(t), z_0^2(t)) \end{cases} \text{or} \begin{cases} f_1(t, z_0^1(t), z_0^2(t)) \geq z_0^1(t) \\ f_2(t, z_0^1(t), z_0^2(t)) \leq z_0^2(t) \end{cases}$$

(a5") $(1+\lambda)\int_0^T G_\lambda(t,s) z_0^1(s)ds \geq z_0^1(t)$

$$(1+\lambda)\int_0^T G_\lambda(t,s) z_0^2(s)ds \leq z_0^2(t)$$

for all $t \in I$.

(a6) the matrix $S := \begin{pmatrix} \mu_1 & \mu_2 \\ \mu_3 & \mu_4 \end{pmatrix}$ is convergent to zero.

Lemma 3.1. *Let $x \in C^1(I)$ be such that it satisfies the periodic boundary value problem*

$$\begin{cases} x'(t) = h(t) \\ x(0) = x(T) \end{cases} t \in I,$$

with $h \in C(I)$. Then for some $\lambda \neq 0$ the above problem is equivalent to

$$x(t) = \int_0^T G_\lambda(t,s)(h(s) + \lambda x(s))ds, \text{ for all } t \in I,$$

where

$$G_\lambda(t,s) = \begin{cases} \frac{e^{\lambda(T+s-t)}}{e^{\lambda T}-1}, & \text{if } 0 \leq s < t \leq T \\ \frac{e^{\lambda(s-t)}}{e^{\lambda T}-1}, & \text{if } 0 \leq t < s \leq T \end{cases}.$$

The problem (3.1) is equivalent to the coupled fixed point problem

$$\begin{cases} x = F_1(x,y) \\ y = F_2(x,y) \end{cases},$$

with $X = C(I)$ and $F_1, F_2 : X^2 \to X$ defined by

$$F_1(x,y)(t) = \int_0^T G_\lambda(t,s)\left[f_1(s,x(s),y(s)) + \lambda x(s)\right] ds$$
$$F_2(x,y)(t) = \int_0^T G_\lambda(t,s)\left[f_2(s,x(s),y(s)) + \lambda y(s)\right] ds$$

We consider the complete metric d induced by the sup-norm on X,

$$d(x,y)=\sup_{t\in I}|x(t)-y(t)|\,,\ \text{for } x,y\in C(I)$$

For $x,y,u,v\in X$, we also denote $\widetilde{d}((x,y),(u,v)):=\begin{pmatrix} d(x,u)\\ d(y,v)\end{pmatrix}$.

Note that if $(x,y)\in X\times X$ is a coupled fixed point of F, then we have

$$x(t)=F_1(x,y)(t) \text{ and } y(t)=F_2(x,y)(t) \text{ for all } t\in I,$$

where $F(x,y)(t):=(F_1(x,y)(t),F_2(x,y)(t))$.

Our main result in this section is the following theorem regarding the existence and uniqueness of the solution to a periodic boundary value system.

Theorem 3.2. *Consider the problem (*3.1*) under the assumptions (a1)-(a6). Then there exists a unique solution (x^*,y^*) of the first-order boundary value problem* 3.1.

Proof. We verify the conditions of Theorem 2.6.

$$F_1(x,y)(t)-F_1(u,v)(t)$$
$$=\int_0^T G_\lambda(t,s)\left[f_1(s,x(s),y(s))+\lambda x(s)\right]ds-\int_0^T G_\lambda(t,s)\left[f_1(s,u(s),v(s))+\lambda u(s)\right]ds$$
$$=\int_0^T G_\lambda(t,s)[f_1(s,x(s),y(s))-f_1(s,u(s),v(s))+\lambda(x(s)-u(s))]ds$$

for all $t\in I$.

From the first condition in (a2) and the positivity of G_λ (for $\lambda>0$) it follows that

$$F_1(x,y)(t)-F_1(u,v)(t)\geq 0.$$

We get that

$$F_1(x,y)(t)\geq F_1(u,v)(t).$$

In a similar way we obtain

$$F_2(x,y)(t)-F_2(u,v)(t)$$
$$=\int_0^T G_\lambda(t,s)\left[f_2(s,x(s),y(s))+\lambda y(s)\right]ds-\int_0^T G_\lambda(t,s)\left[f_2(s,u(s),v(s))+\lambda v(s)\right]ds$$
$$=\int_0^T G_\lambda(t,s)[f_2(s,x(s),y(s))-f_2(s,u(s),v(s))+\lambda(y(s)-v(s))]ds$$

From the second inequality in (a2) and the positivity of G_λ (for $\lambda>0$) we have

$$F_2(x,y)(t)-F_2(u,v)(t)\leq 0.$$

Hence it follows that

$$F_2(x,y)(t)\leq F_2(u,v)(t).$$

So we get that for all ($x\geq u$ and $y\leq v$) or ($u\geq x$ and $v\leq y$) we have

$$\begin{cases} F_1(x,y)(t)\geq F_1(u,v)(t)\\ F_2(x,y)(t)\leq F_2(u,v)(t)\end{cases} \text{or} \begin{cases} F_1(u,v)(t)\geq F_1(x,y)(t)\\ F_2(u,v)(t)\leq F_2(x,y)(t)\end{cases}$$

Hence the second hypothesis of Theorem 2.6 is satisfied.

We know that

$$F_1(z_0^1, z_0^2)(t) = \int_0^T G_\lambda(t,s)[f_1(s, z_0^1(s), z_0^2(s)) + \lambda z_0^1(s)]ds,$$

and we have from condition (a5') that

$$f_1(t, z_0^1(t), z_0^2(t)) \geq z_0^1(t).$$

So we get that

$$\begin{aligned} F_1(z_0^1, z_0^2)(t) &\geq \int_0^T G_\lambda(t,s)[z_0^1(s) + \lambda z_0^1(s)]ds \\ &= (1+\lambda)\int_0^T G_\lambda(t,s) z_0^1(s)ds \geq z_0^1(t) \end{aligned}$$

for all $t \in I$.

Finally we obtain that

$$F_1(z_0^1, z_0^2)(t) \geq z_0^1(t), \text{ for all } t \in I.$$

$$F_2(z_0^1, z_0^2)(t) = \int_0^T G_\lambda(t,s)[f_2(s, z_0^1(s), z_0^2(s)) + \lambda z_0^2(s)]ds$$

From condition (a5') we know that

$$f_2(t, z_0^1(t), z_0^2(t)) \leq z_0^2(t).$$

It follows that

$$\begin{aligned} F_2(z_0^1, z_0^2)(t) &\leq \int_0^T G_\lambda(t,s)[z_0^2(s) + \lambda z_0^2(s)]ds \\ &= (1+\lambda)\int_0^T G_\lambda(t,s) z_0^2(s)ds \leq z_0^2(t). \end{aligned}$$

Hence we have that

$$F_2(z_0^1, z_0^2)(t) \leq z_0^2(t).$$

We conclude that the fourth hypothesis of Theorem 2.6 is satisfied.

$$|F_1(x,y)(t) - F_1(u,v)(t)|$$

$$= \left| \int_0^T G_\lambda(t,s)[f_1(s, x(s), y(s)) + \lambda x(s)]ds - \int_0^T G_\lambda(t, s[f_1(s, u(s), v(s)) + \lambda u(s)]ds \right|$$

$$= \left| \int_0^T G_\lambda(t,s)[f_1(s, x(s), y(s)) - f_1(s, u(s), v(s)) + \lambda x(s) - \lambda u(s)]ds \right|$$

$$\leq \lambda \int_0^T G_\lambda(t,s)(|\mu_1(x(s) - u(s))| + |\mu_2(y(s) - v(s))|)ds$$

$$\leq \mu_1 d(x,u) + \mu_2 d(y,v).$$

Taking the $\sup_{t \in I}$ in the above relation we have

$$d(F_1(x,y), F_1(u,v)) \leq \mu_1 d(x,u) + \mu_2 d(y,v).$$

In a similar way we get that

$$d(F_2(x,y),F_2(u,v)) \leq \mu_3 d(x,u) + \mu_4 d(y,v).$$

Then we have

$$\begin{pmatrix} d(F_1(x,y),F_1(u,v)) \\ d(F_2(x,y),F_2(u,v)) \end{pmatrix} \leq \begin{pmatrix} \mu_1 d(x,u) + \mu_2 d(y,v) \\ \mu_3 d(x,u) + \mu_4 d(y,v) \end{pmatrix} = \begin{pmatrix} \mu_1 & \mu_2 \\ \mu_3 & \mu_4 \end{pmatrix} \begin{pmatrix} d(x,u) \\ d(y,v) \end{pmatrix} = S \cdot \widetilde{d}((x,y),(u,v)),$$

where S is a matrix convergent to zero.

Since all the hypotheses of Theorem 2.6 are satisfied we get that the periodic boundary value problem (3.1) has a unique solution on $C(I) \times C(I)$. $\square$

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Book reviews

Simeon Reich and Alexander J. Zaslavski, Genericity in Nonlinear Analysis, Developments in Mathematics, Vol. 34, Springer New York Heidelberg Dordrecht London, 2014, ISBN 978-1-4614-9532-1; ISBN 978-1-4614-9533-8 (eBook), xiii + 520 pp.

The book is concerned with generic (in the sense of Baire category) results for various problems in Nonlinear Analysis. As the authors best explain in the Preface, one considers a class of problems in some functional space equipped with a complete metric. It is known that for some elements in this functional space the corresponding problem possesses a solution (a solution with desirable properties) and for some elements such solutions do not exist. Under these circumstances it is natural to ask if a solution (a solution with desirable properties) exists for most (in the sense of Baire category) elements, meaning that this holds for a dense G_δ subset of the considered function space. In some cases, "most" is taken in the stronger sense of σ-porousity (a σ-porous set is of first Baire category and, in finite dimension, of Lebesgue measure zero too).

The classes of problems to which this general procedure is applied are: fixed point problems for both single- and set-valued mappings, infinite products of operators, best approximation, discrete and continuous descent methods for minimization in Banach spaces, and the structure of minimal energy configurations with rational numbers in the Aubry-Mather theory.

The first chapter of the book, *Introduction*, contains an overview of the principal results on fixed points and infinite products, which are then treated in detail in chapters 2. *Fixed Point Results and Convergence of Powers of Operators*, 3. *Contractive Mappings*, 6. *Infinite Products*, and 9. *Set-Valued Mappings*. Some results are presented in the general framework of a hyperbolic space, meaning a metric space (X, ρ) endowed with a family M of metric lines (isometric images of $\mathbb{R}^1$) such that every pair of points in X is joined by a unique metric line in M and the metric satisfies an inequality, expressing, intuitively, the fact that the length of a median in a triangle is less or equal than half the length of the base. The study of genericity in fixed point problems was initiated by G. Vidossich (1974) and F. De Blasi and J. Myjak (1976).

For a nonempty, bounded and closed subset K of a Banach space X one denotes by $\mathcal{A}$ the family of all nonexpansive mappings $A : K \to K$ (meaning that $\|Ax-Ay\| \le \|x - y\|$, for all $x, y \in K$). Equipped with the metric $d(A, B) = \sup\{\|Ax - Bx\| : x \in K\}$ $\mathcal{A}$ is a complete metric space. A strict contraction is a γ-Lipschitz mapping with $0 \leq \gamma < 1$. A contractive mapping (in Rakotch's sense) is a mapping $A \in \mathcal{A}$ such that there exists a decreasing function $\phi^A : [0, \mathrm{diam}(K)] \to [0, 1]$ such that $\|Ax - Ay\| \leq \phi^A(\|x - y\|)\,\|x - y\|$, for all $x, y \in K$. E. Rakotch (1962) proved that, under the above hypotheses, every contractive mapping in $\mathcal{A}$ has a unique fixed point x_A and that the sequence of iterates $(A^n x)$ converges to x_A uniformly on K. The authors show that there exists a G_δ dense subset $\mathcal{F}$ of $\mathcal{A}$ such that every $A \in \mathcal{F}$ is contractive. It is worth to notice that the set of all strict contractions is of first Baire category in $\mathcal{A}$, even if X is a Hilbert space. Also there exists a subset $\mathcal{F}$ of $\mathcal{A}$ such that $\mathcal{A} \setminus \mathcal{F}$ is σ-porous and the conclusions of Rakotch's theorem hold for every $A \in \mathcal{F}$. Similar results, with respect to some Hausdorff-type metrics, are obtained for contractive set-valued mappings. The more delicate problem of non-expansive set-valued mappings is considered as well. Other generic results concern Mann-Ishikava iteration, stability of fixed points and the well-posedness of fixed point problems. The case of mappings which are non-expansive with respect to the Bergman metric (a topic of intense study in the recent years) is treated in Chapter 5, *Relatively Nonexpansive Operators with Respect to Bregman Distances.*

In Chapter 7, *Best Approximation*, one obtains generic and porousity results for generalized problems of best approximation in Banach spaces, or in the more general context of hyperbolic spaces. These results extend those obtained By S. B. Stechkin (1963), M. Edelstein (1968), Ka Sing Lau (1978), F. De Blasi and J. Myjak (1991,1998), Chong Li (2000), and others. The generalizations consist both in replacing the norm by a function f having some appropriate properties (convexity will do) and admitting that the set A, where the optimal points are checked, can vary. As a sample I do mention the following result: There exists a set $\mathcal{F}$ with σ-porous (with respect to some Hausdorf-type metric) complement in the space $S(X)$ of nonempty closed subsets of a complete hyperbolic space X such that the minimization problem is well posed for all points in X, excepting a subset of first Baire category (Theorem 7.5).

Generic and porousity results are obtained also for other classes of problems in chapters 4. *Dynamical Systems with Convex Lyapunov Functions*, 8. *Descent Methods*, and 10. *Minimal Configurations in the Aubry-Mather Theory.*

The book, based almost exclusively on the original results of the authors published in the last years, contains a lot of interesting and deep generic existence results for some classes of problems in nonlinear analysis. By bringing together results spread through various journals, it will be of great help for researchers in fixed point theory, optimization, best approximation and dynamical systems. Being carefully written, with complete proofs and illuminating examples, it can serve also as an introductory book to this areas of current research.

S. Cobzaş

Jürgen Appell, Józef Banaś and Nelson Merentes, Bounded Variation and Around, Series in Nonlinear Analysis and Applications, Vol. 17, xiii + 319 pp, Walter de Gruyter, Berlin - New York, 2014, ISBN: 978-3-11-026507-1, e-ISBN: 978-3-11-026511-8, ISSN: 0941-813X.

Functions of bounded variation were defined and studied first by Camille Jordan (1881) in order to extend the Dirichlet convergence criterium for Fourier series, who proved that these functions can be represented as differences of nondecreasing functions. Charles De la Vallée Poussin (1915) introduced the functions of bounded second variation and proved that these functions can be represented as differences of convex functions. T. Popoviciu (1934) generalized both these results by considering functions of bounded k-variation and proving that they can be written as differences of functions convex of order k, another class of functions defined and studied by T. Popoviciu (in *Introduction* his name is wrongly mentioned as Mihael T. Popoviciu – in item [253] the author is M. T. Popoviciu, but M. comes rather from Monsieur than from Mihael). Later on many extensions of this notion were considered, motivated mainly by their applications to Fourier series. We shall mention some of them which are treated in detail in this book.

In what follows f will be a function $f : [a, b] \to \mathbb{R}$ and P a partition $a = t_0 < t_1 < \cdots < t_m = b$ of the interval $[a, b]$.

F. Riesz (1910) considered the following variation $\mathrm{Var}_p^R(f, P) = \sum_{j=1}^m |f(t_j) - f(t_{j-1})|^p/(t_j - t_{j-1})^{p-1}$, and the total variation $\mathrm{Var}_p^R(f) = \sup_P \mathrm{Var}_p^R(f, P)$ where $p \geq 1$. He proved that a function f is of bounded Riesz variation iff it is absolutely continuous and $f' \in L^p([a, b])$. In this case $\mathrm{Var}_p^R(f) = \|f'\|_p^p$. He also used this class of functions to give representations for the duals of the spaces $L^p([a, b])$.

N. Wiener (1924) considered the total variation $\mathrm{Var}_p^W(f)$ as the supremum with respect to P of the variations $\mathrm{Var}_p^W(f, P) = \sum_{j=1}^m |f(t_j) - f(t_{j-1})|^p$. This was extended by L. Young (1934) to variations of the form $\mathrm{Var}_\phi^W(f, P) = \sum_{j=1}^m \phi(|f(t_j) - f(t_{j-1})|)$, where ϕ is a Young function. A similar extension to Riesz definition was given by Yu. T. Medvedev (1953). Daniel Waterman (1976) considered a more general notion of bounded variation using infinite sequences $\Lambda = (\lambda_n)$ of positive numbers with $\lambda_n \to 0$ and $\sum_n \lambda_n = \infty$, and infinite systems of nonoverlapping intervals $([a_n, b_n])$ in [a,b]: $\mathrm{Var}_\Lambda(f) = \sum_{j=1}^\infty \lambda_j |f(t_j) - f(t_{j-1})|$. Other generalizations were considered by M. Schramm (1982) (containing all the above mentioned notions of bounded variation, but difficult to handle, due to its technicality), B. Korenblum (1975), and others.

The aim of this book is to present a detailed and systematic account of all these notions of bounded variations, the properties of the corresponding spaces, and relations between various classes of functions with bounded variation. In general, the authors restrict the treatment to functions of one real variable, excepting the second chapter, *Classical BV-spaces*, where functions of several variables of bounded variations are considered as well. Applications are given to nonlinear composition operator (in Chapter 5) and to nonlinear superposition operator (in Chapter 6), as well as to convergence of Fourier series and to integral representations, via Riemann-Stieltjes-type integrals, of continuous linear functionals on various Banach function spaces.

Integrals of Riemann-Stieltjes-type, corresponding to various notions of bounded variation, are treated in detail in Chapter 4, *Riemann-Stieltjes integrals.*

The book is the first one that presents in a systematic and exhaustive way the many-faceted aspects of the notion of bounded variation and its applications. Some open problems are stated throughout the book, and each chapter ends with a set of exercises, completing the main text (the more difficult ones are marked by *). The book is very well organized with an index of symbols, a notion index, and a rich bibliography (329 items). The prerequisites are modest – some familiarity with real analysis, functional analysis and elements of operator theory – and so it can be used both for an introduction to the subject or as a reference text as well.

Tiberiu Trif

Lucio Boccardo and Gisella Croce, Elliptic Partial Differential Equations - Existence and Regularity of Distributional Solutions, Studies in Mathematics, Vol. 55, x + 192 pp, Walter de Gruyter, Berlin - New York, 2014, ISBN: 978-3-11-031540-0, e-ISBN: 978-3-11-031542-4, ISSN: 0179-0986.

The book is concerned with the existence and regularity of weak solutions to elliptic problems in divergence form (1) $-\mathrm{div}(a(x,u,\nabla u) = f$ in Ω and $u = 0$ on $\partial\Omega$, where $\Omega \subset \mathbb{R}^N$ is open and bounded, $f \in H^{-1}(\Omega)$ and $a : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ is elliptic, that is $a(x,s,\xi) \cdot \xi \geq \alpha|\xi|^2$, $\xi \in \mathbb{R}^N$. The approach is based on methods from real and functional analysis. In order to make the book self-contained, many auxiliary results are proved with full details. Also, for reader's convenience, other results from real analysis, functional analysis, and Sobolev spaces, are collected in appendices at the end of some chapters, with reference to the recent book by H. Brezis, *Functional Analysis, Sobolev Spaces and Partial Differential Equations*, Springer 2011.

The second chapter of the book (actually the first, because the Introduction is numbered as Chapter 1), *Some fixed point theorems*, contains Banach-Caccioppoli (called also Banach-Picard, or simply Banach, depending on the nationality of the author) contraction principle, Brouwer's fixed point theorem (Milnor's proof), and Schauder fixed point theorem. The third chapter, *Preliminaries of real analysis*, is concerned with some convergence results in L^p-spaces, other results on L^p-spaces being collected in an Appendix at the end of the chapter. The second part of this chapter contains a brief introduction to Marcinkiewicz spaces.

The study of elliptic equations starts in Chapter 4, *Linear and semilinear elliptic equations.* The existence results for this kind of equations are proved via Lax-Milgram and Stampacchia's theorems, whose full proofs are included. The Appendix to this chapter reviews some results in functional analysis (projections in Hilbert space and Riesz' theorem) and on Sobolev spaces.

The problems treated in Chapter 5, *Nonlinear elliptic equations*, are more difficult, due to the nonlinearity F in the equation $-\mathrm{div}(a(x,u,\nabla u) = F(x,u,\nabla u)$. The approach is based on some surjectivity results for pseudomonotone coercive operators on reflexive Banach spaces. The basic result proved here is the Leray-Lions existence theorem.

Chapter 6, *Summability of the solutions*, is concerned with some regularity results. One shows that the regularity of the solution depends on the regularity of the source f – if f belongs to a Lebesgue or Marcinkiewicz space, then the solution belongs to the same type of space (with modified exponents). This study continues in Chapter 7, *H^2 regularity for linear problems.* In Chapter 8, *Spectral analysis of linear operators,* one studies the eigenvalues of elliptic operators with applications to semilinear equations.

The last chapter of the first part of the book, 9, *Calculus of variations and Euler's equation*, is concerned with the existence of minimizers of weakly lsc integral functionals (De Giorgi's results), Euler's equation, and Ekeland's variational principle with applications.

The second part of the book is devoted to more specialized topics treated in chapters 10, *Natural growth problems*, 11, *Problems with low summable sources*, 12, *Uniqueness* (for both monotone and non-monotone elliptic operators), 13, *A problem with polynomial growth*, and 14, *A problem with degenerate coercivity.*

Based on undergraduate and Ph.D. courses, taught by the first author at La Sapienza University of Rome, this elegant book (dedicated to Bernardo Dacorogna for his 60th anniversary) presents, in an accessible but rigorous way, some basic results on elliptic partial differential equations. It can be used for undergraduate or graduate courses, or for introduction to this active area of investigation.

Radu Precup

Boris S. Mordukhovich and Nguyen Mau Nam, An Easy Path to Convex Analysis and Applications, Synthesis Lectures on Mathematics and Statistics, Vol. 6, No. 2, 218 pp., Morgan & Claypool, 2014, ISBN-10: 1627052372 ; ISBN-13: 978-1627052375.

The aim of the present book is to prepare the reader for the study of more advanced topics in nonsmooth and variational analysis. The authors have adopted a geometric approach, emphasizing the connections of normal and tangent cones with the subdifferentials of convex functions as well as their relevance for optimization problems. In this way they offer intuitive and more digestible models for the variety of cones considered in nonsmooth analysis. The accessibility is further stressed by the restriction to the framework of finite dimensional Euclidean space $\mathbb{R}^n$.

The first chapter of the book, *Convex sets and functions*, presents the basic notions and results in this area – convex hull, operations with convex sets, topological properties of convex sets, the algebraic interior, convex functions. Applications are given to the distance functions and to the optimal (or marginal) value function μ, defined by $\mu(x) = \inf\{\varphi(x, y) : y \in F(x)\}$, where $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^p$ is a set-valued mapping and $\varphi : \mathbb{R}^n \to \overline{\mathbb{R}} := \mathbb{R} \cup \{+\infty\}$ is an extended real-valued function.

The development of the subdifferential calculus for convex functions, given in the second chapter, is based on a general separation theorem for convex subsets of $\mathbb{R}^n$ and on the normal cone $N(\bar{x}; \Omega)$ to a convex set Ω. Calculus rules for normal cones are established and the continuity and differentiability properties of convex functions are studied. The authors consider two kinds of subdifferentials for a convex function $f : \Omega \to \overline{\mathbb{R}}$ – the singular subdifferential $\partial^\infty f(\bar{x}) = \{v \in \mathbb{R}^n : (v, 0) \in N((\bar{x}, f(\bar{x})); \text{epi} f)\}$

and the usual subdifferential $\partial f(\bar{x})$, defined as the set of subgradients of f at $\bar{x}$. As applications, the subdifferentials of the distance function and of the optimal value function are calculated. Some connections of the subdifferential of the optimal value function with the coderivative of the set-valued mapping F, a key tool in variational analysis, are established.

Chapter 3, *Remarkable consequences of convexity*, starts with the proof of the equivalence between the Fréchet and Gâteaux differentiability of convex functions and continues with Carathéodory's theorem on the convex hulls of subsets of $\mathbb{R}^n$, Radon's theorem, Helly's intersection theorem, and Farkas lemma on systems of linear inequalities. One introduces the tangent cone $T(\bar{x};\Omega)$ and one proves the duality relations $N(\bar{x};\Omega) = [T(\bar{x};\Omega)]^\circ$ and $T(\bar{x};\Omega) = [N(\bar{x};\Omega)]^\circ$, where $K^\circ = \{v \in \mathbb{R}^n : \langle v, x\rangle \le 0, \forall x \in K\}$ denotes the polar cone to a cone $K \subset \mathbb{R}^n$.

The last chapter, Ch. 4, *Applications to optimization and location problems*, contains some recent results of the authors on the Fermat-Torricelli and Sylvester problems – two problems with geometric flavor, where the methods of convex analysis proved to be very efficient for their solution.

The book is clearly and carefully written, with elegant and full proofs to almost all results. Some notions are accompanied by nicely drawn illustrative pictures, and the exercises at the end of each chapter help the reader to a broader and deeper understanding of the results from the main text.

The book contains an accessible, with a strong intuitive support but reasonably complete, introduction to some basic results in convex analysis in $\mathbb{R}^n$. It is recommended as a preliminary (or a companion) lecture to more advanced texts as, for instance, the monumental treatise of the first named author, B. S. Mordukhovich, *Variational Analaysis and Generalized Differentiation*, Vols. I and II, Springer, 2006 (up to now, Google Scholar counts 2015 citations of these volumes). It can be used also as a textbook for graduate or advanced undergraduate courses.

S. Cobzaş