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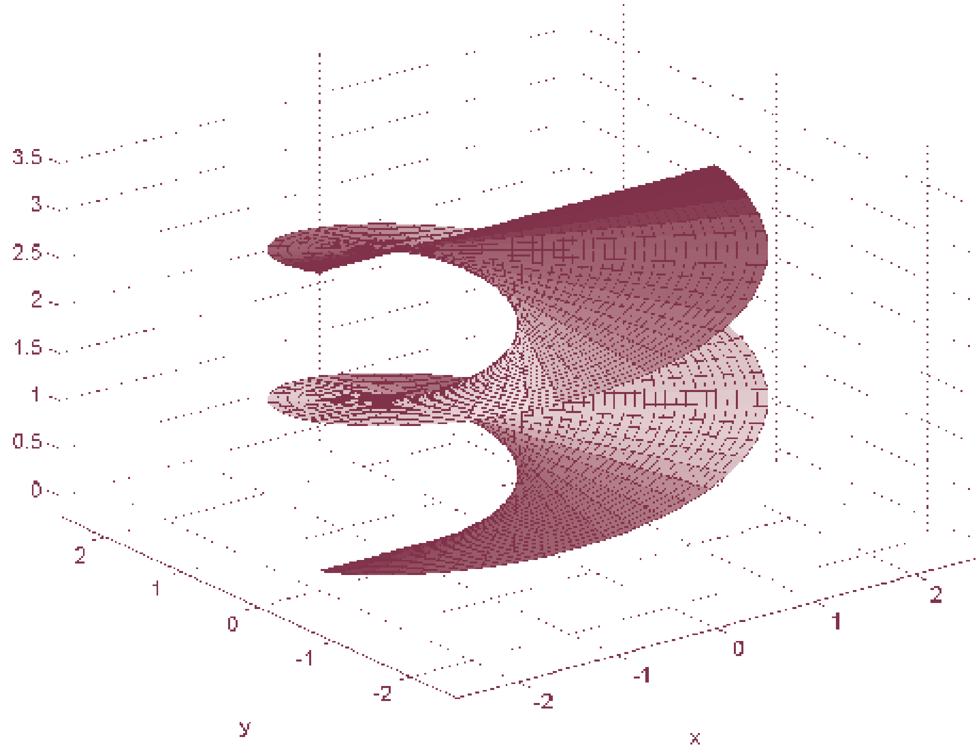

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CONTENTS

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Restricting the Clifford extensions of a pointed group

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Abstract. In this note we give a restriction of the isomorphism constructed in the main result of [1], which states that the Clifford extensions of two Brauer correspondent points are isomorphic, by using a defect pointed group instead of an ordinary defect group and by replacing the Brauer quotient with the multiplicity algebra of the mentioned pointed group.

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1. Introduction

Let p be a prime and $\mathcal{O}$ a discrete valuation ring such that k is the residue field of $\mathcal{O}$. We also consider a finite group G and a normal subgroup N of G.

An N-interior G-algebra is an $\mathcal{O}$-algebra A endowed with two group homomorphisms

$$N \to A^* \text{ and } G \to \mathrm{Aut}_{\mathcal{O}}(A).$$

We denote by A^* the group of invertible elements of A. Then

$$N \ni y \mapsto y \cdot 1 = 1 \cdot y \in A^*$$

and

$$G \ni x \mapsto \varphi(x) \in \mathrm{Aut}_{\mathcal{O}}(A);$$

i.e. any x determines an $\mathcal{O}$-algebra automorphism of A. We use standard notation impling conjugation on the right:

$$\varphi(x)(a) =: a^x \text{ and } a^y := y^{-1} \cdot a \cdot y,$$

for any $x \in G$, $y \in N$ and $a \in A$.

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Let H by any subgroup of G. A point α of H on A, denoted H_α, is a $(A^H)^*$-conjugacy class of a primitive idempotent $i \in A^H$. Throughout we use notations as: $H_{\{\alpha\}}$-denoting the stabilizer of α, provided that H acts on the subalgebra of A that contains the point α; H_i-denoting the stabilizer of some idempotent i and also $N_H(K_\alpha)$-the subgroup of H that normalizes the group K and stabilizes the point α.

We use [5, Theorem 8.20] to characterize fusions on an N-interior G-algebra A. Let α be a point of A^H and let $j \in \alpha$. Any element $x \in N_G(H)$ acts by conjugation on H and determines an automorphism. If this automorphism satisfies $y^{-1}y^{x^{-1}} \in N$ for any $y \in H$, then it determines an A-fusion from the pointed group H_α to itself if and only if there exists $a \in A^*$ such that for any $y \in H$ we have

$$aj = ja, \qquad a^y \cdot y^{-1}y^{x^{-1}} = a.$$

We have already introduced in [1] the so-called *Clifford extensions* of points. Constructing such an extension implies working with an N-interior G-algebra A, a point β of N on A, and with P, a defect group of β that is contained in N. Let $\bar{\beta}$ denote the correspondent point of β determined by the Brauer morphism. The main result of [1] states that the extension corresponding to β is isomorphic to the extension corresponding to $\bar{\beta}$. This result generalizes the main result of [2, Section 12].

In this paper we construct an analogous isomorphism of extensions by using a defect pointed group P_γ of β and by replacing the Brauer quotient of A^P with the multiplicity algebra of P_γ. The first replacement forces a new computation of the groups that appear in original construction of the Clifford extension of β. The second replacement, that is the replacement made with regard to the Brauer quotient, generates a slightly more complicated situation with respect to the gradings. It seems that the grading of the new quotient, that contains the multiplicity algebra as the identity component, depends on the units of a source algebra of the pointed group N_β.

We use standard notations and we refer the reader to [4] and [6] for details regarding the theory of G-algebras and pointed groups.

2. Existing constructions and results

2.1. Let us recall the notations and quote the existing results. For more details regarding the proofs of the following statements we refer to [1]. Let A be an N-interior G-algebra, P be a p-subgroup of N, and let $\beta \in \mathcal{P}(A^N|P_\gamma)$, that is a point of N on A with defect pointed group P_γ. We denote by $\bar{G}$ the quotient group G/N. Following [1, Proposition 3.5] we see that the action of G on A^N gives rise to the normalizer $N_G(N_\beta)$ which satisfies

$$N_G(N_\beta) = N_G(P)_{\{\beta\}}N.$$

2.2. Consider the algebra

$$\hat{A} = A \otimes_N G = \bigoplus_x A \otimes x,$$

where x runs through a set of representatives of the classes of $\bar{G}$. The product in $\hat{A}$ is given by

$$(a \otimes x)(b \otimes y) = ab^{x^{-1}} \otimes xy.$$

Then $\hat{A}$ is clearly a G-interior algebra via the following morphism of groups

$$G \to A^*, \text{ where } g \mapsto 1 \otimes g.$$

2.3. If $j \in \beta$, due to the action of $N_G(N_\beta)$ on A^N, for any $x \in N_G(N_\beta)$ we have $j^x = aja^{-1}$, for some $a \in (A^N)^*$. This allows the construction of the following two groups (since the group $A^* \otimes G < \hat{A}^*$ acts by conjugation on $\hat{A}$):

$$\widehat{N} := (A^N)_j^* \otimes N$$

and

$$\widehat{N_G(N_\beta)} := ((A^N)^* \otimes N_G(N_\beta))_j.$$

The exact sequence

$$1 \to \widehat{N} \to \widehat{N_G(N_\beta)} \to \bar{N}_G(N_\beta) \to 1 \qquad \text{(i)}$$

induces the isomorphism

$$\widehat{N_G(N_\beta)}/\widehat{N} \simeq \bar{N}_G(N_\beta) = N_G(N_\beta)/N.$$

Now $A_\beta := jAj$ is an $\widehat{N}$-interior $\widehat{N_G(N_\beta)}$-algebra, it is actually N-interior. Next we see that

$$\hat{A}_\beta := A_\beta \otimes_{\widehat{N}} \widehat{N_G(N_\beta)}$$

is an N-interior algebra, thus we consider the set

$$\bar{G}[\beta] = \{\bar{x} \in \bar{N}_G(N_\beta) \mid (A_\beta \otimes \hat{x})^N \cdot (A_\beta \otimes \hat{x}^{-1})^N = (A_\beta)^N\},$$

where $\hat{x}$ is a lifting of $\bar{x}$ via (i). This set turns out to be a normal subgroup of $\bar{N}_G(N_\beta)$, see [1, Proposition 2.7].

2.4. Let $N_G^A(N_\beta)$ denote the group consisting of elements $x \in N_G(N_\beta)$ such that the conjugation action of x on N induces an A-fusion from N_β to itself. We have

$$\bar{G}[\beta] \simeq \bar{N}_G^A(N_\beta).$$

Set $\widehat{N_G^A(N_\beta)}$, the inverse image of $\bar{G}[\beta]$ in $\widehat{N_G(N_\beta)}$, and set

$$\hat{A}[\beta] := A_\beta \otimes_{\widehat{N}} \widehat{N_G^A(N_\beta)}.$$

The algebra

$$\hat{A}[\beta]^N := \bigoplus_x (A_\beta \otimes \hat{x})^N,$$

where x runs through a set of representatives for the classes in $\bar{G}[\beta]$, is a crossed product and it is also an $\widehat{N_G(N_\beta)}$-algebra, since N and $\widehat{N_G^A(N_\beta)}$ are two normal subgroups of $\widehat{N_G(N_\beta)}$. The quotient

$$\hat{\bar{A}}[\beta]^N := \hat{A}[\beta]^N / J_{gr}(\hat{A}[\beta]^N)$$

is the twisted group algebra of $\hat{k} := A_\beta(N_\beta) = A^N_\beta/J(A^N_\beta)$ with $\bar{G}[\beta]$, actually a crossed product of $\hat{k}$ with $\bar{G}[\beta]$, and it corresponds uniquely to the Clifford extension of β

$$1 \to \hat{k}^* \to \widehat{\bar{N}^A_G(N_\beta)} \to \bar{G}[\beta] \to 1. \tag{1}$$

We denoted by J_{gr} the graded Jacobson radical. The qoutient $\hat{k}$ is a skew field whose center is an extension of k and $\widehat{\bar{N}^A_G(N_\beta)}$ denotes the group of homogeneous units of $\hat{\bar{A}}[\beta]^N$.

3. The Clifford extension of the multiplicity algebra

3.1. Consider $\beta \subseteq A^N$ as above, having defect pointed group P_γ. Since γ is a point of A^P, it corresponds uniquely to the $N_G(P_\gamma)$-invariant maximal ideal m_γ of A^P such that $\gamma \nsubseteq m_\gamma$. Consider the multiplicity algebra of P_γ :

$$A(P_\gamma) := A^P/m_\gamma.$$

The map

$$s_\gamma : A^P \to A(P_\gamma)$$

is an epimorphism of $N_G(P_\gamma)$-algebras. The restriction

$$s_\gamma : A^N_P \to A(P_\gamma)^{N_N(P_\gamma)}_P$$

is still an epimorphism of $N_G(P_\gamma)$-algebras and $\bar{\beta} := s_\gamma(\beta) \subseteq A(P_\gamma)^{N_N(P_\gamma)}$ is the unique correspondent point of β having P as a defect group, see [5, Theorem 6.14]. Set $\bar{j} := s_\gamma(j)$.

3.2. Denote by $N^A_G(P_\gamma)$ the subgroup of $N_G(P_\gamma)$ consisting of elements that determine A-fusions from P_γ to itself. Then

$$\bar{G}[\beta]_{\{\gamma\}} := \bar{G}[\beta] \cap \bar{N}^A_G(P_\gamma)$$

is the subgroup of $\bar{N}_G(N_\beta)$ such that any representative of any class determines A-fusions for both N_β and P_γ.

Denote by $\widehat{N^A_G(N_\beta)}_{\{\gamma\}}$ the inverse image of $\bar{G}[\beta]_{\{\gamma\}}$ via (i). In this case the algebra $\hat{A}^\gamma_\beta := A_\beta \otimes_{\hat{N}} \widehat{N^A_G(N_\beta)}_{\{\gamma\}}$ is N-invariant, hence

$$(\hat{A}^\gamma_\beta)^P := (A_\beta \otimes_{\hat{N}} \widehat{N^A_G(N_\beta)}_{\{\gamma\}})^P = \bigoplus_{\hat{x}} (A_\beta \otimes \hat{x})^P,$$

where $\hat{x}$ lifts a set of representatives of the classes in $\bar{G}[\beta]_{\{\gamma\}}$, is strongly $\bar{G}[\beta]_{\{\gamma\}}$-graded, it is actually a crossed-product.

Proposition 3.3. *With the above notations*

$$\widehat{m}_\gamma := m_\gamma \cdot (\hat{A}^\gamma_\beta)^P = (\hat{A}^\gamma_\beta)^P \cdot m_\gamma$$

is a two-sided ideal of $(\hat{A}^\gamma_\beta)^P$.

Proof. It suffices to prove the equality

$$m_\gamma \cdot (A_\beta \otimes \hat{x})^P = (A_\beta \otimes \hat{x})^P \cdot m_\gamma, \tag{*}$$

for any lifting $\hat{x}$.

The inclusion $P_\gamma \leq N_\beta$ provides an idempotent $i \in \gamma$ such that $ji = i = ji$. Thus the inclusion $A_\gamma \subseteq A_\beta$ gives the following homomorphism of groups

$$\phi : A_\gamma^* \to A_\beta^*, \text{ where } A_\gamma \ni a \mapsto a' := a + j - i \in A_\beta.$$

Then, if $x \in N_G(N_\beta)_{\{\gamma\}}$ determines an A-fusion from P_γ to itself we get

$$y^{x^{-1}} = y^a,$$

for some $a \in A_\gamma^*$ and for any $y \in P$. Clearly $a'a^{-1} = i \in A^P$, then we get

$$a'a^{-1} \cdot y = y \cdot a'a^{-1},$$

or equivalently $y^a = y^{a'}$, for any $y \in P$. We obtain $(a')^y \cdot y^{-1}xyx^{-1} = a'$ and for a lifting $\hat{x}$ of $\bar{x}$ this is equivalent to

$$(a' \otimes \hat{x})^y = a' \otimes \hat{x},$$

for any $y \in P$, as a homogeneous unit of $(\hat{A}_\beta^\gamma)^P$. Using all the above we can view

$$(\hat{A}_\beta^\gamma)^P = \bigoplus_{\hat{x}} (a' \otimes \hat{x}) \cdot A_\beta^P,$$

where $\hat{x}$ lifts via extension (i) a set of representatives of the classes of $\bar{G}[\beta]_{\{\gamma\}}$. All homogeneous units $a' \otimes \hat{x}$ of $(\hat{A}_\beta^\gamma)^P$ satisfy $a' \in N_{A^*}(P)$ and $a'i = ia'$. Hence for proving $(*)$ it suffices to prove

$$(a' \otimes \hat{x})^{-1} \cdot m_\gamma \cdot (a' \otimes \hat{x}) = m_\gamma.$$

Since $\hat{x}$ lifts an element of $\bar{N}_G(P_\gamma)$ the last equality is equivalent to

$$(a')^{-1} \cdot m_\gamma \cdot a' = m_\gamma.$$

The maximal ideal $(a')^{-1} \cdot m_\gamma \cdot a'$ of A^P can not contain γ, because otherwise

$$\gamma^{a'} = \gamma \subseteq m_\gamma,$$

which is a contradiction. □

The next proposition follows from the proof of the above proposition.

Proposition 3.4. *The group $\bar{G}[\beta]_{\{\gamma\}}$ is isomorphic to the subgroup of $\bar{N}_G^A(N_\beta)$ that consists of elements $\bar{x}$ such that for any lifting $\hat{x}$ the module $(A_\beta \otimes \hat{x})^P$ contains a homogeneous unit $a' \otimes \hat{x}$ satisfying $a' \in \phi(A_\gamma^*)$.*

Proof. Indeed, any element $x \in N_G(P_\gamma)_{\{\beta\}}$ that determines A-fusions for both P_γ and N_β lifts to $\hat{x}$ which gives, for any $y \in P$,

$$y^{\hat{x}^{-1}} = y^a = y^{a'},$$

for some $a \in A_\gamma^*$ and $a' = a + j - i$. □

3.5. In what follows it is more convenient to replace $\hat{N}$ with $(A^N)^*_j \otimes N_N(P_\gamma)$ and $\widehat{N_G(N_\beta)}$ with $((A^N)^* \otimes N_G(P_\gamma)_{\{\beta\}})_j$, since the two pairs give isomorphic quotients. We further denote

$$\widehat{N_G(N_\beta)}_{\{\gamma\}} := ((A^N)^* \otimes N_G(P_\gamma)_{\{\beta\}})_{j,i},$$

the subgroup of $((A^N)^* \otimes N_G(P_\gamma)_{\{\beta\}})_j$ whose elements also fix i. With this setting we have:

Lemma 3.6. *The group* $\widehat{N^A_G(N_\beta)}_{\{\gamma\}}$ *is* $\widehat{N_G(N_\beta)}_{\{\gamma\}}$*-invariant.*

Proof. Let $\hat{x} \in \widehat{N^A_G(N_\beta)}_{\{\gamma\}}$, $\hat{z} \in \widehat{N_G(N_\beta)}_{\{\gamma\}}$ and $a' \in \phi(A^*_\gamma)$ such that

$$(a' \otimes \hat{x})^y = a' \otimes \hat{x},$$

for any $y \in P$. Then $(a' \otimes \hat{x})^{\hat{z}} = (a')^{\hat{z}} \otimes \hat{x}^{\hat{z}}$ is also a homogeneous unit of $(A_\beta \otimes \hat{x}^{\hat{z}})^P$ verifying $\hat{x}^{\hat{z}} \in \widehat{N^A_G(N_\beta)}$ and $(a')^{\hat{z}} \in \phi(A^*_\gamma)$. □

3.7. Using the N-invariance of $\hat{A}^\gamma_\beta$ we obtain

$$\begin{aligned}(\bar{\hat{A}}^\gamma_\beta)^N :&= (\hat{A}^\gamma_\beta)^N / J_{gr}((\hat{A}^\gamma_\beta)^N)\\ &= \bigoplus_{\hat{x}} \left((A_\beta \otimes \hat{x})^N / J(A^N_\beta)(A_\beta \otimes \hat{x})^N\right),\end{aligned}$$

the crossed product of $\hat{k} = A_\beta(N_\beta)$ with $\bar{G}[\beta]_{\{\gamma\}}$, and simultaneously the twisted group algebra of $\hat{k}$ with $\bar{G}[\beta]_{\{\gamma\}}$ corresponding to the Clifford extension

$$1 \to \hat{k}^* \to \widehat{\bar{N}^A_G(N_\beta)}_{\{\gamma\}} \to \bar{G}[\beta]_{\{\gamma\}} \to 1. \tag{1'}$$

Clearly (1') is a subextension of (1).

3.8. Constructions similar to that of 2.3, making use of the action of $N_G(P_\gamma)$ on $A(P_\gamma)^{N_N(P_\gamma)}$, determine the exact sequence

$$\begin{aligned}1 &\to (A(P_\gamma)^{N_N(P_\gamma)})^*_{\bar{j}} \otimes N_N(P_\gamma) \to \\ &\to ((A(P_\gamma)^{N_N(P_\gamma)})^* \otimes N_G(P_\gamma)_{\{\bar{\beta}\}})_{\bar{j}} \to \bar{N}_G(P_\gamma)_{\{\bar{\beta}\}} \to 1.\end{aligned} \tag{ii}$$

We set

$$\widehat{N_N(P_\gamma)} := ((A(P_\gamma)^{N_N(P_\gamma)})^*_{\bar{j}} \otimes N_N(P_\gamma))_{\bar{i}} \text{ and}$$

$$\widehat{N_G(P_\gamma)}_{\{\bar{\beta}\}} := ((A(P_\gamma)^{N_N(P_\gamma)})^* \otimes N_G(P_\gamma)_{\{\bar{\beta}\}})_{\bar{j},\bar{i}},$$

where $\bar{i} = s_\gamma(i)$. One easily checks that $\widehat{N_N(P_\gamma)}$ is a normal subgroup in $\widehat{N_G(P_\gamma)}_{\{\bar{\beta}\}}$.

Lemma 3.9. *The following statements hold.*

a) *The groups* $N_G(P_\gamma)_{\{\beta\}}$ *and* $N_G(P_\gamma)_{\{\bar{\beta}\}}$ *coincide.*

b) *The groups* $N_G(P_\gamma)_i$ *and* $N_G(P_\gamma)_{\bar{i}}$ *coincide, hence* $N_N(P_\gamma)_i = N_N(P_\gamma)_{\bar{i}}$ *is a normal subgroup of* $\widehat{N_G(P_\gamma)}_{\{\bar{\beta}\}}$ *and of* $\widehat{N_G(N_\beta)}_{\{\gamma\}}$ *contained in* $\widehat{N^A_G(N_\beta)}_{\{\gamma\}}$.

Proof. The equality from assertion a) follows by using the epimorphism

$$s_\gamma : A_P^N \to A(P_\gamma)_P^{N_N(P_\gamma)}$$

of $N_G(P_\gamma)$-algebras and [5, Proposition 3.23], since $\bar\beta$ corresponds uniquely to β. For the proof of b), we note that the proof of the first equality of the statement is similar to that of a). The second equality is a consequence of the first equality. Next, for any $t \in N_N(P_\gamma)_i$ we have

$$A_\beta^P = (A_\beta \otimes t)^P.$$

This implies that any element $a' \in \phi((A_\gamma^*)^P) \subseteq (A_\beta^P)^*$ verifies

$$a' = a'_1 \otimes t \in (A_\beta^P)^*,$$

where a'_1 is a unit in A_β. Then, for a suitable $a \in A_\gamma^*$, we have

$$a'_1 = a't^{-1} = at^{-1} + j - i \in \phi(A_\gamma^*).$$

By our choices of a, a' and a'_1 we obtain $y^{t^{-1}} = y^{a'_1} = y^{at^{-1}}$, for all $y \in P$. Proposition 3.4 implies that $N_N(P_\gamma)_i$ is contained in $\widehat{N_G^A(N_\beta)}_{\{\gamma\}}$, since any element of N determines an A-fusion of N_β. We also have

$$N_N(P_\gamma)_i = N_N(P_\gamma) \cap \widehat{N_G(N_\beta)}_{\{\gamma\}} = N_N(P_\gamma) \cap \widehat{N_G(P_\gamma)}_{\{\bar\beta\}}.$$

Since $N_N(P_\gamma)$ is normal in $\widehat{N_G(N_\beta)}$ and in $((A(P_\gamma)^{N_N(P_\gamma)})^* \otimes N_G(P_\gamma)_{\{\bar\beta\}})_{\bar j}$, the statement follows. □

3.10. We denote by $\bar G[\bar\beta]_{\{\gamma\}}$ the normal subgroup of $\bar N_G(P_\gamma)_{\{\bar\beta\}}$ that is isomorphic to $\bar G[\beta]_{\{\gamma\}}$ and by $\widehat{N_G^{A(P_\gamma)}}(P_\gamma)_{\{\bar\beta\}}$ the inverse image of $\bar G[\bar\beta]_{\{\gamma\}}$ in the infinite group of (*ii*), i.e. $((A(P_\gamma)^{N_N(P_\gamma)})^* \otimes N_G(P_\gamma)_{\{\bar\beta\}})_{\bar j}$.

Remark 3.11. If $\hat x$ lifts an element of $\bar G[\bar\beta]_{\{\gamma\}}$ then, considering in the proof of Proposition 3.3 A in place of A_β , we obtain $(A \otimes \hat x)^P \cdot m_\gamma = m_\gamma \cdot (A \otimes \hat x)^P$. Then we set

$$\hat A(P_\gamma) := \bigoplus_{\hat x} \left((A \otimes \hat x)^P / (m_\gamma \cdot (A \otimes \hat x)^P)\right),$$

where $\hat x$ lifts a set of representatives of $\bar G[\bar\beta]_{\{\gamma\}}$. We denote

$$\overline{(A \otimes \hat x)^P} := (A \otimes \hat x)^P / (m_\gamma \cdot (A \otimes \hat x)^P),$$

for any lifting $\hat x$, and, using Lemma 3.9 and Proposition 3.3, we determine the strongly $\bar G[\bar\beta]_{\{\gamma\}}$-graded $N_N(P_\gamma)$-algebra

$$\hat A(P_\gamma)_{\bar\beta} := \bar j \hat A(P_\gamma) \bar j = (\hat A_\beta^\gamma)^P / \widehat m_\gamma = \bigoplus_{\hat x} \overline{(A_\beta \otimes \hat x)^P}.$$

Lemma 3.12. *The map*

$$\widehat s_\gamma : (\hat A_\beta^\gamma)^P \to \hat A(P_\gamma)_{\bar\beta},$$

sending $a \otimes \hat x$ to $\overline{a \otimes \hat x} := a \otimes \hat x + m_\gamma \cdot (A_\beta \otimes \hat x)^P$, is an epimorphism of $\bar G[\bar\beta]_{\{\gamma\}} \simeq \bar G[\beta]_{\{\gamma\}}$-strongly graded $N_N(P_\gamma)$-algebras.

Remark 3.13. We obtain the following characterization of $\bar{G}[\bar{\beta}]_{\{\gamma\}}$. Explicitly, it consists of elements $\bar{x} \in \bar{N}_G(P_\gamma)_{\bar{\beta}}$ such that x determines A-fusion of P_γ and for any lifting $\hat{x}$ we obtain

$$(\overline{(A_\beta \otimes \hat{x})^P})^{N_N(P_\gamma)} \cdot (\overline{(A_\beta \otimes \hat{x}^{-1})^P})^{N_N(P_\gamma)} = A(P_\gamma)_{\bar{\beta}}^{N_N(P_\gamma)}.$$

Further we construct a morphism of groups between $\widehat{N_G(N_\beta)}_{\{\gamma\}}$ and $\widehat{N_G(P_\gamma)}_{\{\bar{\beta}\}}$. Explicitly, if $\hat{n} := n \otimes z \in \widehat{N_G(N_\beta)}_{\{\gamma\}}$, where $z \in N_G(P_\gamma)_{\{\beta\}}$, then the image of $\hat{n}$ is $\widehat{s_\gamma(n)} := s_\gamma(n) \otimes z \in \widehat{N_G(P_\gamma)}_{\{\bar{\beta}\}}$. So that we obtain a well-defined morphism of groups:

$$\theta : \widehat{N_G(N_\beta)}_{\{\gamma\}}/N_N(P_\gamma)_i \to \widehat{N_G(P_\gamma)}_{\{\bar{\beta}\}}/N_N(P_\gamma)_{\bar{i}}.$$

3.14. By [5, Lemma 6.15], 3.13 and 3.12 and the above remark, the restriction

$$\widehat{s}_\gamma : (\hat{A}_\beta^\gamma)^N \to \hat{A}(P_\gamma)_{\bar{\beta}}^{N_N(P_\gamma)}$$

is an epimorphism of $\widehat{N_G(N_\beta)}_{\{\gamma\}}/N_N(P_\gamma)_i$-algebras, via the restriction determined by θ. By the definition of the action on $\overline{(A_\beta \otimes \hat{x})^P}$, this morphism verifies

$$\widehat{s}_\gamma((a \otimes \hat{x})^{\bar{\hat{n}}}) = (\widehat{s}_\gamma(a \otimes \hat{x}))^{\theta(\bar{\hat{n}})} := \overline{(n \otimes z)^{-1}(a \otimes \hat{x})(n \otimes z)},$$

for any $\bar{\hat{n}}$ in $\widehat{N_G(N_\beta)}_{\{\gamma\}}/N_N(P_\gamma)_i$.

Corollary 3.15. *The group $\bar{G}[\bar{\beta}]_{\{\gamma\}}$ is invariant under the conjugation action determined by the elements belonging to the image of θ.*

Proof. For any $\hat{x}$ that lifts an element of $\bar{G}[\bar{\beta}]_\gamma$ we have

$$\widehat{s}_\gamma((A_\beta \otimes \hat{x})^N) = (\overline{(A_\beta \otimes \hat{x})^P})^{N_N(P_\gamma)}.$$

Using Lemma 3.6, Remark 3.13 and 3.13 the result follows. □

3.16. Denote

$$\hat{k}_1 := A(P_\gamma)_{\bar{\beta}}(N_N(P_\gamma)_{\bar{\beta}}) = A(P_\gamma)_{\bar{\beta}}^{N_N(P_\gamma)}/J(A(P_\gamma)_{\bar{\beta}}^{N_N(P_\gamma)}).$$

The twisted group algebra

$$\hat{\bar{A}}(P_\gamma)_{\bar{\beta}}^{N_N(P_\gamma)} := \hat{A}(P_\gamma)_{\bar{\beta}}^{N_N(P_\gamma)}/J_{gr}(\hat{A}(P_\gamma)_{\bar{\beta}}^{N_N(P_\gamma)}),$$

of $\hat{k}_1$ with $\bar{G}[\bar{\beta}]_{\{\gamma\}}$, corresponds uniquely to the extension

$$1 \to \hat{k}_1^* \to \widehat{\bar{N}_G^{A(P_\gamma)}(P_\gamma)}_{\{\bar{\beta}\}} \to \bar{G}[\bar{\beta}]_{\{\gamma\}} \to 1. \tag{2}$$

We call this the Clifford extension of the multiplicity algebra of γ. Note that $\hat{k}_1$ is a skew field having the center a finite extension of k. Moreover, we observe that $\hat{\bar{A}}(P_\gamma)_{\bar{\beta}}^{N_N(P_\gamma)}$ is actually a crossed product of $\hat{k}_1$ with $\bar{G}[\bar{\beta}]_{\{\gamma\}}$, implying that it is a strongly $\bar{G}[\bar{\beta}]_{\{\gamma\}}$-graded algebra.

4. The restricted isomorphism of Clifford extensions

We summarize all of the above in the main result of the paper and we refer to [1, Theorem 4.1] for more details regarding the proof.

Theorem 4.1. *The following statements hold.*

(i) *The extensions* (1′) *and* (2) *are isomorphic.*

(ii) *The crossed products they correspond to are isomorphic as* $\widehat{N_G(N_\beta)}_{\{\gamma\}}/N_N(P_\gamma)_i$*-algebras.*

Proof. The epimorphism $\widehat{s}_\gamma$ of $\bar{G}[\beta]_{\{\gamma\}} \simeq \bar{G}[\bar{\beta}]_{\{\gamma\}}$-strongly graded algebras determines the epimorphism

$$\widehat{\bar{s}}_\gamma : (\hat{\bar{A}}^\gamma_\beta)^N \to \hat{\bar{A}}(P_\gamma)^{N_N(P_\gamma)}_{\bar{\beta}},$$

since $\widehat{s}_\gamma(J((A^\gamma_\beta)^N)) \subseteq J(A(P_\gamma)^{N_N(P_\gamma)}_{\bar{\beta}})$. The map $\widehat{\bar{s}}_\gamma$ is also a morphism of $\bar{G}[\beta]_{\{\gamma\}}$-strongly graded algebras. [5, Proposition 3.23] gives $\hat{k} \simeq \hat{k}_1$, and the first assertion follows from [3, Proposition 2.12]. As for the second assertion we use Lemma 3.12, Remark 3.13 and Corollary 3.15. □

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Asymptotic behavior of intermediate points in certain mean value theorems. III

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Abstract. The paper is devoted to the study of the asymptotic behavior of intermediate points in certain mean value theorems of integral and differential fractional calculus.

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1. Introduction

Let $a, b \in \mathbb{R}$ such that $a < b$, let $f : [a, b] \to \mathbb{R}$ be a continuous function, and let $g \in L^1[a, b]$ such that g does not change its sign in $[a, b]$. Then, according to the first mean value theorem of integral calculus (see, for instance, [11, Theorem 85.6], or [6]), for every $x \in (a, b]$ there exists $\xi_x \in (a, x)$ such that

$$\int_a^x f(t)g(t)dt = f(\xi_x) \int_a^x g(t)dt.$$

It was proved in [15, Theorem 2.2] that

$$\lim_{x \to a+} \frac{\xi_x - a}{x - a} = \sqrt[n]{\frac{k+1}{n+k+1}} \tag{1.1}$$

if, in addition, the functions f and g satisfy the following conditions:

(i) there exists a positive integer n such that f is n times differentiable at a, with $f^{(j)}(a) = 0$ for $1 \le j \le n-1$ and $f^{(n)}(a) \neq 0$;

(ii) $g \in C[a, b]$ and there exists a nonnegative integer k such that g is k times differentiable at a with $g^{(j)}(a) = 0$ for $0 \le j \le k-1$ and $g^{(k)}(a) \neq 0$.

Regarding (ii) we notice that the continuity of g is automatically assured if $k \ge 2$, at least on a small interval $[a, a+h] \subseteq [a, b]$ (this clearly suffices when dealing with the limit (1.1)).

Further, let n be a positive integer, and let $f : [a,b] \to \mathbb{R}$ be a function whose derivative $f^{(n)}$ exists on $[a,b]$. Then, according to the Lagrange-Taylor mean value theorem, for every $x \in (a,b]$ there exists some $\xi_x \in (a,x)$ such that

$$f(x) - T_{n-1}(f;a)(x) = \frac{f^{(n)}(\xi_x)}{n!}\,(x-a)^n,$$

where $T_m(h;a)$ denotes the mth Taylor polynomial associated with h and a

$$T_m(h;a)(x) := h(a) + h'(a)(x-a) + \cdots + \frac{h^{(m)}(a)}{m!}\,(x-a)^m,$$

provided that h is m times differentiable at a. It was proved by A. G. Azpeitia [4] that

$$\lim_{x\to a+} \frac{\xi_x - a}{x-a} = \binom{n+p}{p}^{-1/p} \tag{1.2}$$

if, in addition, f satisfies the following conditions:

(i) there exists a positive integer p such that $f \in C^{n+p}[a,b]$;
(ii) $f^{(n+j)}(a) = 0$ for $1 \le j < p$;
(iii) $f^{(n+p)}(a) \neq 0$.

This result was generalized by U. Abel [1], who derived for ξ_x a complete asymptotic expansion of the form

$$\xi_x = a + \sum_{k=1}^{\infty} \frac{c_k}{k!}\,(x-a)^k \quad (x \to a).$$

Azpeitia's result was generalized also by T. Trif [14], who obtained the asymptotic behavior of the intermediate point in the Cauchy-Taylor mean value theorem. For other results concerning the asymptotic behavior of the intermediate points in certain mean value theorems the reader is referred to [2, 7, 8, 9, 19].

The purpose of our paper is to establish asymptotic formulas, that are similar to (1.1) and (1.2), but in the framework of fractional calculus.

2. Fractional mean value theorems of integral calculus

K. Diethelm [6, Theorem 2.1] generalized the first mean value theorem of integral calculus to the framework of fractional calculus. Recall that given $\alpha > 0$, the Riemann-Liouville fractional primitive of order α of a function $f : [a,b] \to \mathbb{R}$ is defined by (see [12] or [5])

$$J_a^\alpha f(x) := \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)\,dt,$$

provided the right side is pointwise defined on $[a,b]$.

Theorem 2.1. ([6, Theorem 2.1]) *Let $\alpha > 0$, let $f : [a,b] \to \mathbb{R}$ be a continuous function, and let $g \in L^1[a,b]$ be a function which does not change its sign on $[a,b]$. Then for almost every $x \in (a,b]$ there exists some $\xi_x \in (a,x)$ such that*

$$J_a^\alpha(fg)(x) = f(\xi_x) J_a^\alpha g(x). \tag{2.1}$$

Moreover, if $\alpha \geq 1$ or $g \in C[a,b]$, then the existence of ξ_x is assured for every $x \in (a,b]$.

In the special case when $g(x) \equiv 1$, the above mean value theorem takes the following form.

Corollary 2.2. ([6, Corollary 2.2]) *If $\alpha > 0$ and $f : [a,b] \to \mathbb{R}$ is a continuous function, then for every $x \in (a,b]$ there exists some $\xi_x \in (a,x)$ such that*

$$J_a^\alpha f(x) = f(\xi_x)\,\frac{(x-a)^\alpha}{\Gamma(\alpha+1)}\,. \tag{2.2}$$

We point out that there is a misprint in the statement of [6, Corollary 2.2], where $\Gamma(\alpha)$ appears instead of $\Gamma(\alpha+1)$.

In what follows we intend to prove a fractional version of the second mean value theorem of integral calculus. For reader's convenience we recall first the second mean value theorem for Lebesgue integrals, which is usually stated for Riemann integrals (see, for instance, [3, Theorem 10.2.5] or [18, Theorem 1]).

Theorem 2.3. *Let $f : [a,b] \to [0,\infty)$ be a nondecreasing function, and let $g \in L^1[a,b]$. Then for every $x \in (a,b]$ there exists some $\xi_x \in [a,x]$ such that*

$$\int_a^x f(t)g(t)dt = f(x-0)\int_{\xi_x}^x g(t)dt.$$

The fractional version of Theorem 2.3 can be formulated as follows.

Theorem 2.4. *Let $\alpha > 0$, let $f : [a,b] \to [0,\infty)$ be a nondecreasing function, and let $g \in L^1[a,b]$. Then for almost every $x \in (a,b]$ there exists some $\xi_x \in [a,x]$ such that*

$$J_a^\alpha (fg)(x) = f(x-0)J_{\xi_x}^\alpha g(x). \tag{2.3}$$

Moreover, if $\alpha \geq 1$ or $g \in C[a,b]$, then the existence of ξ_x is assured for every $x \in (a,b]$.

Proof. Let $x \in (a,b]$. Under the assumptions of the theorem we have

$$J_a^\alpha (fg)(x) = \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1}f(t)g(t)dt = \int_a^x f(t)h(t)dt,$$

where $h : (a,x) \to \mathbb{R}$ is the function defined by $h(t) := (x-t)^{\alpha-1}g(t)/\Gamma(\alpha)$.

If $\alpha \geq 1$ or $g \in C[a,b]$, then $h \in L^1[a,x]$. By Theorem 2.3 it follows that there exists $\xi_x \in [a,x]$ such that

$$J_a^\alpha (fg)(x) = \int_a^x f(t)h(t)dt = f(x-0)\int_{\xi_x}^x h(t)dt = f(x-0)J_{\xi_x}^\alpha g(x).$$

If $0 < \alpha < 1$ and g is supposed only Lebesgue integrable, then the above argument still works, but the integrability of h holds only for almost all $x \in (a,b]$ (see [17, Theorem 4.2 (d)]). $\square$

3. Asymptotic behavior of intermediate points in fractional mean value theorems of integral calculus

The purpose of this section is to investigate the asymptotic behavior of the point ξ_x in (2.1) and (2.2) as the interval $[a, x]$ shrinks to zero. More precisely, we prove that under certain additional assumptions on f and g, the limit $\lim\limits_{x \to a+} \dfrac{\xi_x - a}{x - a}$ exists and we find its value. In the proof of the main result of this section we need the following

Lemma 3.1. *Let $\alpha > 0$, let p be a nonnegative integer, and let $\omega : [a, b] \to \mathbb{R}$ be a continuous function such that $\omega(x) \to 0$ as $x \to a+$. Then*

$$\int_a^x (x-t)^{\alpha-1}(t-a)^p \omega(t)dt = o\big((x-a)^{p+\alpha}\big) \quad (x \to a+).$$

Proof. Indeed, for every $x \in (a, b]$ we have

$$\int_a^x (x-t)^{\alpha-1}(t-a)^p \omega(t)dt = \Gamma(\alpha) J_a^\alpha(\omega g)(x),$$

where $g : [a, b] \to [0, \infty)$ is defined by $g(t) := (t-a)^p$. According to Theorem 2.1 there exists $\xi_x \in (a, x)$ such that

$$\begin{aligned} J_a^\alpha(\omega g)(x) &= \omega(\xi_x) J_a^\alpha g(x) = \frac{\omega(\xi_x)}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1}(t-a)^p dt \\ &= \omega(\xi_x)\, \frac{B(p+1, \alpha)}{\Gamma(\alpha)}\, (x-a)^{p+\alpha}, \end{aligned}$$

whence

$$\int_a^x (x-t)^{\alpha-1}(t-a)^p \omega(t)dt = \omega(\xi_x) B(p+1, \alpha)(x-a)^{p+\alpha}.$$

Since $\omega(x) \to 0$ as $x \to a+$, we obtain the conclusion. ☐

Theorem 3.2. *Let α be a positive real number and let $f, g : [a, b] \to \mathbb{R}$ be functions satisfying the following conditions:*

(i) *$f \in C[a, b]$ and there is a positive integer n such that f is n times differentiable at a with $f^{(j)}(a) = 0$ for $1 \le j \le n-1$ and $f^{(n)}(a) \ne 0$;*

(ii) *$g \in C[a, b]$, g does not change its sign in some interval $[a, a+h] \subseteq [a, b]$, and there is a nonnegative integer k such that g is k times differentiable at a with $g^{(j)}(a) = 0$ for $0 \le j \le k-1$ and $g^{(k)}(a) \ne 0$.*

Then the point ξ_x in (2.1) satisfies

$$\lim_{x \to a+} \frac{\xi_x - a}{x - a} = \sqrt[n]{\frac{(k+1)(k+2)\cdots(k+n)}{(\alpha+k+1)(\alpha+k+2)\cdots(\alpha+k+n)}}.$$

Proof. Without loosing the generality we may assume that $f(a) = 0$. Indeed, otherwise we replace f by the function $t \in [a, b] \mapsto f(t) - f(a)$. Note that if ξ_x satisfies (2.1), then ξ_x satisfies also

$$J_a^\alpha\Big(\big(f - f(a)\big)g\Big)(x) = \big(f(\xi_x) - f(a)\big) J_a^\alpha g(x).$$

We notice also that (2.1) is equivalent to

$$\int_a^x (x-t)^{\alpha-1} f(t)g(t)dt = f(\xi_x)\int_a^x (x-t)^{\alpha-1} g(t)dt. \tag{3.1}$$

By the Taylor expansions of f and g we have

$$\begin{aligned} f(t) &= \frac{f^{(n)}(a)}{n!}(t-a)^n + \omega(t)(t-a)^n, \\ g(t) &= \frac{g^{(k)}(a)}{k!}(t-a)^k + \varepsilon(t)(t-a)^k, \end{aligned}$$

where ω and ε are continuous functions on $[a,b]$ satisfying $\omega(t)\to 0$ and $\varepsilon(t)\to 0$ as $t\to a+$. Therefore we have

$$\begin{aligned} (x-t)^{\alpha-1} f(t)g(t) &= \frac{f^{(n)}(a)g^{(k)}(a)}{n!\,k!}(x-t)^{\alpha-1}(t-a)^{n+k} \\ &\quad +(x-t)^{\alpha-1}(t-a)^{n+k}\gamma(t), \end{aligned}$$

where γ is continuous on $[a,b]$ and $\gamma(t)\to 0$ as $t\to a+$. By applying Lemma 3.1 we deduce that

$$\begin{aligned} &\int_a^x (x-t)^{\alpha-1} f(t)g(t)dt \\ &\quad= \frac{f^{(n)}(a)g^{(k)}(a)}{n!\,k!}\,B(\alpha,n+k+1)\,(x-a)^{n+k+\alpha} + o\big((x-a)^{n+k+\alpha}\big) \end{aligned} \tag{3.2}$$

as $x\to a+$. On the other hand, since

$$(x-t)^{\alpha-1}g(t) = \frac{g^{(k)}(a)}{k!}(x-t)^{\alpha-1}(t-a)^k + (x-t)^{\alpha-1}(t-a)^k\varepsilon(t),$$

by Lemma 3.1 we get

$$\int_a^x (x-t)^{\alpha-1}g(t)dt = \frac{g^{(k)}(a)}{k!}\,B(\alpha,k+1)(x-a)^{k+\alpha} + o\big((x-a)^{k+\alpha}\big)$$

as $x\to a+$. Taking into account that

$$f(\xi_x) = \frac{f^{(n)}(a)}{n!}(\xi_x-a)^n + \omega(\xi_x)(\xi_x-a)^n$$

and that $0<\xi_x-a<x-a$, we obtain

$$\begin{aligned} &f(\xi_x)\int_a^x (x-t)^{\alpha-1}g(t)dt \\ &\quad= \frac{f^{(n)}(a)g^{(k)}(a)}{n!\,k!}\,B(\alpha,k+1)(\xi_x-a)^n(x-a)^{k+\alpha} \\ &\qquad +o\big((x-a)^{n+k+\alpha}\big) \end{aligned} \tag{3.3}$$

as $x \to a+$. By (3.1), (3.2) and (3.3) we conclude that

$$\frac{f^{(n)}(a)g^{(k)}(a)}{n!\,k!}\,B(\alpha,k+1)(\xi_x-a)^n(x-a)^{k+\alpha}$$

$$=\frac{f^{(n)}(a)g^{(k)}(a)}{n!\,k!}\,B(\alpha,n+k+1)\,(x-a)^{n+k+\alpha}+o\big((x-a)^{n+k+\alpha}\big)$$

as $x \to a+$. Multiplying both sides by $n!\,k!(x-a)^{-n-k-\alpha}/\big(f^{(n)}(a)g^{(k)}(a)\big)$ we get

$$\begin{aligned}\left(\frac{\xi_x-a}{x-a}\right)^n &= \frac{B(\alpha,n+k+1)}{B(\alpha,k+1)}+o(1)\\ &= \frac{(k+1)(k+2)\cdots(k+n)}{(\alpha+k+1)(\alpha+k+2)\cdots(\alpha+k+n)}+o(1)\end{aligned}$$

as $x \to a+$, whence the conclusion. □

Corollary 3.3. *Let $\alpha > 0$, and let $f : [a,b] \to \mathbb{R}$ be a function satisfying the condition* (i) *in Theorem* 3.2*. Then the point ξ_x in* (2.2) *satisfies*

$$\lim_{x\to a+}\frac{\xi_x-a}{x-a}=\sqrt[n]{\frac{n!}{(\alpha+1)(\alpha+2)\cdots(\alpha+n)}}\,.$$

In the special case when $\alpha = 1$, then Theorem 3.2 and Corollary 3.3 coincide with earlier results obtained by T. Trif [15, Theorem 2.2] and B. Zhang [20, Theorem 4], respectively.

Unfortunately, we were not able to prove a result similar to those stated in Theorem 3.2 and Corollary 3.3, but concerning the asymptotic behavior of the point ξ_x in formula (2.3).

4. Fractional mean value theorems of differential calculus

K. Diethelm [6] and P. Guo, C. P. Li, and G. R. Chen [10] extended recently also the classical Lagrange and Lagrange-Taylor mean value theorems to the framework of fractional calculus. Let $\alpha > 0$, and let $f : [a,b] \to \mathbb{R}$ be a given function. The Caputo fractional derivative of order α of f is defined by

$$D^{\alpha}_{*a}f := D^{\alpha}_{a}\Big(f - T_{\lceil\alpha\rceil-1}(f;a)\Big),$$

where $\lceil\cdot\rceil$ denotes the ceiling function that rounds up to the nearest integer, while D^{α}_{a} is the Riemann-Liouville differential operator, defined by

$$D^{\alpha}_{a}f := D^{\lceil\alpha\rceil}J_a^{\lceil\alpha\rceil-\alpha}f.$$

In the above formula D^m denotes the classical differential operator of order m, and $J^0_a f := f$.

Theorem 4.1. ([6, Theorem 2.3], [10, Theorem 3]) *Let $\alpha > 0$, and let $f \in C^{\lceil\alpha\rceil-1}[a,b]$ be a function such that $D^{\alpha}_{*a} f \in C[a,b]$. Then for every $x \in (a,b]$ there exists some $\xi_x \in (a,x)$ such that*

$$\frac{f(x) - T_{\lceil\alpha\rceil-1}(f;a)(x)}{(x-a)^{\alpha}} = \frac{1}{\Gamma(\alpha+1)}\, D^{\alpha}_{*a} f(\xi_x). \tag{4.1}$$

In the special case when $0 < \alpha \le 1$, then $T_{\lceil\alpha\rceil-1}(f;a)(x) = f(a)$, and Theorem 4.1 takes the following form.

Corollary 4.2. ([6, Corollary 2.4]) *Let $0 < \alpha \le 1$, and let $f \in C[a,b]$ be a function such that $D^{\alpha}_{*a} f \in C[a,b]$. Then for every $x \in (a,b]$ there exists some $\xi_x \in (a,x)$ such that*

$$\frac{f(x) - f(a)}{(x-a)^{\alpha}} = \frac{1}{\Gamma(\alpha+1)}\, D^{\alpha}_{*a} f(\xi_x). \tag{4.2}$$

Theorem 4.3. ([10, Theorem 4]) *Let $\alpha > 0$, and let $f, g \in C^{\lceil\alpha\rceil-1}[a,b]$ be functions such that $D^{\alpha}_{*a} f, D^{\alpha}_{*a} g \in C[a,b]$. Then for every $x \in (a,b]$ there exists some $\xi_x \in (a,x)$ such that*

$$D^{\alpha}_{*a} f(\xi_x)\Big(g(x) - T_{\lceil\alpha\rceil-1}(g;a)(x)\Big) = D^{\alpha}_{*a} g(\xi_x)\Big(f(x) - T_{\lceil\alpha\rceil-1}(f;a)(x)\Big). \tag{4.3}$$

Corollary 4.4. ([10, Corollary 3.6]) *Let $0 < \alpha \le 1$, and let $f, g \in C[a,b]$ be functions such that $D^{\alpha}_{*a} f, D^{\alpha}_{*a} g \in C[a,b]$. Then for every $x \in (a,b]$ there exists some $\xi_x \in (a,x)$ such that*

$$D^{\alpha}_{*a} f(\xi_x)\big(g(x) - g(a)\big) = D^{\alpha}_{*a} g(\xi_x)\big(f(x) - f(a)\big). \tag{4.4}$$

Remark 4.5. Let $\alpha > 0$, and let $f \in C^{\lceil\alpha\rceil-1}[a,b]$ such that $D^{\alpha}_{*a} f \in C[a,b]$. Further, let $g : [a,b] \to \mathbb{R}$ be the function defined by $g(t) := (t-a)^{\alpha}$. If $n := \lceil\alpha\rceil$, then $n-1 < \alpha \le n$, and $T_{\lceil\alpha\rceil-1}(g;a)(x) = T_{n-1}(g;a)(x) = 0$. On the other hand, for every $y \in (a,b)$ one has

$$\begin{aligned} D^{\alpha}_{*a} g(y) &= \frac{d^n}{dy^n}\, J^{n-\alpha}_a \Big(g - T_{\lceil\alpha\rceil-1}(g;a)\Big)(y) = \frac{d^n}{dy^n}\, J^{n-\alpha}_a g(y) \\ &= \Gamma(\alpha+1). \end{aligned}$$

Therefore, in this case (4.3) reduces to (4.1), while (4.4) reduces to (4.2). In other words, Theorem 4.3 coincides with Theorem 4.1, while Corollary 4.4 coincides with Corollary 4.2 in the special case when $g(t) := (t-a)^{\alpha}$.

It should be mentioned that similar fractional mean value theorems, but involving the Riemann-Liouville fractional derivative instead of the Caputo fractional derivative have been obtained by other authors (see, for instance, [10], [13], [16]).

5. Asymptotic behavior of intermediate points in fractional mean value theorems of differential calculus

Theorem 5.1. *Let $\alpha > 0$ be a non-integer number, let $n := \lceil\alpha\rceil$, and let $f : [a,b] \to \mathbb{R}$ be a function satisfying the following conditions:*

(i) *there exists a nonnegative integer p such that $f \in C^{n+p}[a,b]$;*

(ii) $f^{(n+j)}(a)=0$ *for* $0\le j<p$;
(iii) $f^{(n+p)}(a)\neq 0$.

Then the point ξ_x *in* (4.1) *satisfies*

$$\lim_{x\to a+}\frac{\xi_x-a}{x-a}=\big((n+p-\alpha)B(\alpha+1,n+p-\alpha)\big)^{\frac{1}{n+p-\alpha}}.$$

Proof. We note first that since $f\in C^{n+p}[a,b]$, the derivative $f^{(n-1)}$ must be absolutely continuous on $[a,b]$. By [5, Theorem 3.1] it follows that

$$D^{\alpha}_{*a}f=J^{n-\alpha}_{a}D^{n}f. \tag{5.1}$$

Due to (ii), by the Taylor expansion of f we have

$$f(x)-T_{n-1}(f;a)(x)=\frac{f^{(n+p)}(a)}{(n+p)!}\,(x-a)^{n+p}+\omega(x)(x-a)^{n+p}, \tag{5.2}$$

where ω is continuous on $[a,b]$ and satisfies $\omega(x)\to 0$ as $x\to a+$. On the other hand, by the Taylor expansion of $f^{(n)}$ we have

$$f^{(n)}(t)=\frac{f^{(n+p)}(a)}{p!}\,(t-a)^{p}+\varepsilon(t)(t-a)^{p}, \tag{5.3}$$

where ε is continuous on $[a,b]$ and satisfies $\varepsilon(t)\to 0$ as $t\to a+$.

Taking into account (5.1), equality (4.1) can be rewritten as

$$f(x)-T_{n-1}(f;a)(x)=\frac{(x-a)^{\alpha}}{\Gamma(\alpha+1)\Gamma(n-\alpha)}\int_a^{\xi_x}(\xi_x-t)^{n-\alpha-1}f^{(n)}(t)dt. \tag{5.4}$$

By (5.3) and Lemma 3.1 we find that

$$\begin{aligned}&\int_a^{\xi_x}(\xi_x-t)^{n-\alpha-1}f^{(n)}(t)dt\\&\quad=\frac{f^{(n+p)}(a)}{p!}\int_a^{\xi_x}(\xi_x-t)^{n-\alpha-1}(t-a)^{p}dt\\&\qquad+\int_a^{\xi_x}(\xi_x-t)^{n-\alpha-1}(t-a)^{p}\varepsilon(t)dt\\&\quad=\frac{\Gamma(n-\alpha)}{\Gamma(n+p+1-\alpha)}\,f^{(n+p)}(a)(\xi_x-a)^{n+p-\alpha}+o\big((\xi_x-a)^{n+p-\alpha}\big),\end{aligned}$$

whence

$$\begin{aligned}&\frac{(x-a)^{\alpha}}{\Gamma(\alpha+1)\Gamma(n-\alpha)}\int_a^{\xi_x}(\xi_x-t)^{n-\alpha-1}f^{(n)}(t)dt\\&\qquad=\frac{f^{(n+p)}(a)(x-a)^{\alpha}(\xi_x-a)^{n+p-\alpha}}{\Gamma(\alpha+1)\Gamma(n+p+1-\alpha)}+o\big((x-a)^{n+p}\big)\end{aligned} \tag{5.5}$$

as $x \to a+$, because $0 < \xi_x - a < x - a$. Taking now into account (5.2), (5.4), and (5.5), we find that

$$\frac{f^{(n+p)}(a)(x-a)^{n+p}}{\Gamma(n+p+1)} = \frac{f^{(n+p)}(a)(x-a)^{\alpha}(\xi_x-a)^{n+p-\alpha}}{\Gamma(\alpha+1)\Gamma(n+p+1-\alpha)} + o\big((x-a)^{n+p}\big)$$

as $x \to a+$. Multiplying both sides by

$$\Gamma(\alpha+1)\Gamma(n+p+1-\alpha)(x-a)^{-n-p}/f^{(n+p)}(a)$$

we get

$$\begin{aligned}\left(\frac{\xi_x-a}{x-a}\right)^{n+p-\alpha} &= \frac{\Gamma(\alpha+1)\Gamma(n+p+1-\alpha)}{\Gamma(n+p+1)} + o(1)\\ &= (n+p-\alpha)B(\alpha+1, n+p-\alpha) + o(1) \qquad (x \to a+),\end{aligned}$$

whence the conclusion. □

Corollary 5.2. *Let $0 < \alpha < 1$, and let $f : [a,b] \to \mathbb{R}$ be a function satisfying the following conditions:*

(i) *there exists a positive integer p such that $f \in C^p[a,b]$;*
(ii) *$f^{(j)}(a) = 0$ for $1 \le j < p$;*
(iii) *$f^{(p)}(a) \neq 0$.*

Then the point ξ_x in (4.2) *satisfies*

$$\lim_{x \to a+} \frac{\xi_x - a}{x - a} = \big((p-\alpha)B(\alpha+1, p-\alpha)\big)^{\frac{1}{p-\alpha}}.$$

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Subclass of meromorphic functions with positive coefficients defined by convolution

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Abstract. In this paper we introduce and study new class of meromorphic functions defined by convolution. We obtain coefficients inequalities, distortion theorems, extreme points, closure theorems and some other results for the modified Hadamard products. Finally, we obtain application involving an integral operator.

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1. Introduction

Let Σ denote the class of meromorphic functions of the form:

$$f(z) = \frac{1}{z} + \sum_{k=1}^{\infty} a_k z^k, \tag{1.1}$$

which are analytic in the punctured unit disc $U^* = \{z : z \in \mathbb{C} \text{ and } 0 < |z| < 1\} = U \backslash \{0\}$. Let $g \in \Sigma$, be given by

$$g(z) = \frac{1}{z} + \sum_{k=1}^{\infty} b_k z^k, \tag{1.2}$$

then the Hadamard product (or convolution) of f and g is given by

$$(f * g)(z) = \frac{1}{z} + \sum_{k=1}^{\infty} a_k b_k z^k = (g * f)(z). \tag{1.3}$$

A function $f \in \Sigma$ is meromorphic starlike of order β $(0 \leq \beta < 1)$ if

$$-\text{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \beta \ (z \in U), \tag{1.4}$$

and the class of all such functions is denoted by $\Sigma^*(\beta)$. A function $f \in \Sigma$ is meromorphic convex of order β $(0 \leq \beta < 1)$ if

$$-\mathrm{Re}\left\{1+\frac{zf''(z)}{f'(z)}\right\} > \beta \ (z \in U), \tag{1.5}$$

and the class of such functions is denoted by $\Sigma_k^*(\beta)$. The classes $\Sigma^*(\beta)$ and $\Sigma_k^*(\beta)$ are introduced and studied by Pommerenke [11], Miller [9], Mogra et al. [10], Cho [4], Cho et al. [5] and Aouf ([1] and [2]).

For $\alpha \geq 0,\ 0 \leq \beta < 1,\ 0 \leq \lambda < \frac{1}{2}$ and g is given by (1.2), with $b_k \geq 0$ $(k \geq 1)$, we denote by $M(f, g; \alpha, \beta, \lambda)$ the subclass of Σ consisting of functions of the form (1.1) and satisfying the analytic criterion:

$$-\mathrm{Re}\left\{\frac{z(f*g)'(z)+\lambda z^2(f*g)''(z)}{(1-\lambda)(f*g)(z)+\lambda z(f*g)'(z)}+\beta\right\}$$
$$\geq \alpha\left|\frac{z(f*g)'(z)+\lambda z^2(f*g)''(z)}{(1-\lambda)(f*g)(z)+\lambda z(f*g)'(z)}+1\right| (z \in U). \tag{1.6}$$

We note that for suitable choices of g , α and λ, we obtain the following subclasses:

$$M(f, \frac{1}{z(1-z)}; 0, \beta; 0) = \Sigma^*(\beta) \ (0 \leq \beta < 1)$$

and

$$M(f, \frac{1}{z(1-z)}; 0, \beta; 1) = \Sigma_k^*(\beta) \ (0 \leq \beta < 1)$$

(see Pommerenke [11]).

Also, we note that

(1) $M(f, g; \alpha, \beta, 0) = N(f, g; \alpha, \beta)$

$$= \left\{ f \in \Sigma : -\mathrm{Re}\left(\frac{z(f*g)'(z)}{(f*g)(z)}+\beta\right) \geq \alpha\left|\frac{z(f*g)'(z)}{(f*g)(z)}+1\right| \ (z \in U^*)\right\};$$

(2) Putting $g(z) = \frac{1}{z} + \sum_{k=1}^{\infty}\left(\frac{\ell+\delta k}{\ell}\right)^m z^k$ in (1.6), then the class

$$M(f, \frac{1}{z} + \sum_{k=1}^{\infty}\left(\frac{\ell+\delta k}{\ell}\right)^m z^k; \alpha, \beta, \lambda)$$

reduces to the class

$$M_{\delta,\ell}(m; \alpha, \beta, \lambda) = \left\{ f \in \Sigma : -\mathrm{Re}\left\{\frac{z(I^m(\delta,\ell)f(z))'+\lambda z^2(I^m(\delta,\ell)f(z))''}{(1-\lambda)(I^m(\delta,\ell)f(z))+\lambda z(I^m(\delta,\ell)f(z))'}+\beta\right\} \geq \alpha \right.$$
$$\left.\left|\frac{z(I^m(\delta,\ell)f(z))'+\lambda z^2(I^m(\delta,\ell)f(z))''}{(1-\lambda)(I^m(\delta,\ell)f(z))+\lambda z(I^m(\delta,\ell)f(z))'}+1\right| \ (\delta \geq 0;\ \ell > 0;\ m \in \mathbb{N}_0;\ z \in U)\right\},$$

where the operator

$$I^m(\delta, \ell)(z) = \frac{1}{z} + \sum_{k=1}^{\infty}\left(\frac{\ell+\delta k}{\ell}\right)^m z^k, \tag{1.7}$$

was introduced and studied by El-Ashwah [6, with $p = 1$] (see also Bulboacă et al. [3], El-Ashwah [7, with $p = 1$] and El-Ashwah et al. [8, with $p = 1$]).

2. Coefficient estimates

Unless otherwise mentioned, we assume throughout this paper that

$\alpha \geq 0,\ 0 \leq \beta < 1,\ 0 \leq \lambda < \frac{1}{2},\ g$ is given by (1.2) with $b_k > 0$ and $b_k \geq b_1$

$(k \geq 1)$.

Theorem 2.1. *Let the function f defined by (1.1). Then $f \in M(f, g; \alpha, \beta, \lambda)$ if and only if*

$$\sum_{k=1}^{\infty} [1 + \lambda(k-1)]\,[k(1+\alpha) + (\alpha+\beta)]\, a_k b_k \leq (1-\beta)(1-2\lambda). \tag{2.1}$$

Proof. Let the condition (2.1) holds true and using the fact that

$$-\mathrm{Re}(w) \geq \alpha\,|w+1| + \beta \text{ if and only if } -\mathrm{Re}\{(1+\alpha e^{i\theta})w + \alpha e^{i\theta}\} \geq \beta,$$

we have

$$-\mathrm{Re}\left\{\frac{z(f*g)'(z) + \lambda z^2 (f*g)''(z)}{(1-\lambda)(f*g)(z) + \lambda z (f*g)'(z)} + \beta\right\}$$

$$\geq \alpha \left|\frac{z(f*g)'(z) + \lambda z^2 (f*g)''(z)}{(1-\lambda)(f*g)(z) + \lambda z (f*g)'(z)} + 1\right|.$$

Hence

$$-\mathrm{Re}\left\{(1+\alpha e^{i\theta})\frac{z(f*g)'(z) + \lambda z^2 (f*g)''(z)}{(1-\lambda)(f*g)(z) + \lambda z (f*g)'(z)} + \alpha e^{i\theta}\right\} \geq \beta,$$

or, equivalently,

$$-\mathrm{Re}\left\{\frac{(1+\alpha e^{i\theta})\left[z(f*g)'(z) + \lambda z^2 (f*g)''(z)\right] + \alpha e^{i\theta}\left[(1-\lambda)(f*g)(z) + \lambda z (f*g)'(z)\right]}{(1-\lambda)(f*g)(z) + \lambda z (f*g)'(z)}\right\} \geq \beta,$$

where $-\pi \leq \theta < \pi$. Suppose that

$$G(z) = -(1+\alpha e^{i\theta})\left[z(f*g)'(z) + \lambda z^2 (f*g)''(z)\right]$$

$$-\alpha e^{i\theta}\left[(1-\lambda)(f*g)(z) + \lambda z (f*g)'(z)\right],$$

$$H(z) = (1-\lambda)(f*g)(z) + \lambda z (f*g)'(z),$$

and using the fact that

$$\mathrm{Re}(w) \geq \beta \text{ if and only if } |w - (1+\beta)| \leq |w + (1-\beta)| \text{ where } w = -(u+iv),$$

we need to prove that

$$|G(z) + (1-\beta)H(z)| \geq |G(z) - (1+\beta)H(z)| \text{ for } 0 \leq \beta < 1.$$

Then

$$|G(z) + (1-\beta)H(z)| - |G(z) - (1+\beta)H(z)|$$

$$\begin{aligned}
&= \left|(2-\beta)(1-2\lambda)\frac{1}{z} - \sum_{k=1}^{\infty}[k-(1-\beta)][1+\lambda(k-1)]a_k b_k z^k - \alpha e^{i\theta}.\right.\\
&\quad \left. .\sum_{k=1}^{\infty}(k+1)[1+\lambda(k-1)]a_k b_k z^k\right| - \left|\beta(1-2\lambda)\frac{1}{z} - \sum_{k=1}^{\infty}[k+(1+\beta)]\,.\right.\\
&\quad \left. .[1+\lambda(k-1)]a_k b_k z^k - \alpha e^{i\theta}\sum_{k=1}^{\infty}(k+1)[1+\lambda(k-1)]a_k b_k z^k\right|\\
&\geq (2-\beta)(1-2\lambda)\frac{1}{|z|} - \sum_{k=1}^{\infty}[k-(1-\beta)][1+\lambda(k-1)]a_k b_k |z|^k - \alpha.\\
&\quad .\sum_{k=1}^{\infty}(k+1)[1+\lambda(k-1)]a_k b_k |z|^k - \beta(1-2\lambda)\frac{1}{|z|} - \sum_{k=1}^{\infty}[k+(1+\beta)]\,.\\
&\quad .[1+\lambda(k-1)]a_k b_k |z|^k - \alpha\sum_{k=1}^{\infty}(k+1)[1+\lambda(k-1)]a_k b_k |z|^k\\
&= 2(1-\beta)(1-2\lambda)\frac{1}{|z|} - 2\sum_{k=1}^{\infty}(k+\beta)[1+\lambda(k-1)]a_k b_k |z|^k - 2\alpha.\\
&\quad .\sum_{k=1}^{\infty}(k+1)[1+\lambda(k-1)]a_k b_k |z|^k \geq 0.
\end{aligned}$$

On simplification we easily arrive at the inequality (2.1).
Conversely, suppose that f is in the class $M(f,g;\alpha,\beta,\lambda)$. Then

$$-\mathrm{Re}\left\{\frac{(1+\alpha e^{i\theta})\left[z(f*g)'(z)+\lambda z^2(f*g)''(z)\right]+\alpha e^{i\theta}\left[(1-\lambda)(f*g)(z)+\lambda z(f*g)'(z)\right]}{(1-\lambda)(f*g)(z)+\lambda z(f*g)'(z)}\right\} \geq \beta,$$

Hence

$$\mathrm{Re}\left\{\frac{(1-2\lambda)(1-\beta)\frac{1}{z} - \sum\limits_{k=1}^{\infty}\{k+\alpha e^{i\theta}(k+1)+\beta\}[1+\lambda(k-1)]a_k b_k z^k}{(1-2\lambda)\frac{1}{z} + \sum\limits_{k=1}^{\infty}[1+\lambda(k-1)]a_k b_k z^k}\right\} \geq 0,$$

If we now choose z to be real and $z \to 1^-$, we write

$$\sum_{k=1}^{\infty}[1+\lambda(k-1)]\,[k(1+\alpha)+(\alpha+\beta)]\,a_k b_k \leq (1-\beta)(1-2\lambda),$$

which completes the proof of Theorem 2.1.

Corollary 2.2. *Let the function f defined by (1.1) be in the class $M(f,g;\alpha,\beta,\lambda)$. Then*

$$a_k \leq \frac{(1-\beta)(1-2\lambda)}{[1+\lambda(k-1)]\,[k(1+\alpha)+(\alpha+\beta)]\,b_k}. \tag{2.2}$$

The result is sharp for the function

$$f(z) = \frac{1}{z} + \frac{(1-\beta)(1-2\lambda)}{[1+\lambda(k-1)]\,[k(1+\alpha)+(\alpha+\beta)]\,b_k}z^k. \tag{2.3}$$

3. Distortion theorems

Theorem 3.1. *Let the function f defined by (1.1) be in the class $M(f, g; \alpha, \beta, \lambda)$, then for $0 < |z| = r < 1$, we have*

$$\frac{1}{|z|} - \frac{(1-\beta)(1-2\lambda)}{(2\alpha+\beta+1)b_1}|z| \le |f(z)| \le \frac{1}{|z|} + \frac{(1-\beta)(1-2\lambda)}{(2\alpha+\beta+1)b_1}|z|. \tag{3.1}$$

The result is sharp for the function f given by

$$f(z) = \frac{1}{z} + \frac{(1-\beta)(1-2\lambda)}{(2\alpha+\beta+1)b_1}z. \tag{3.2}$$

Proof. It is easy to see from Theorem 2.1 that

$$\begin{aligned}(2\alpha+\beta+1)b_1\sum_{k=1}^{\infty} a_k &\le \sum_{k=1}^{\infty}[1+\lambda(k-1)]\,[k(1+\alpha)+(\alpha+\beta)]\,a_k b_k \\ &\le (1-\beta)(1-2\lambda).\end{aligned}$$

Then

$$\sum_{k=1}^{\infty} a_k \le \frac{(1-\beta)(1-2\lambda)}{(2\alpha+\beta+1)b_1}. \tag{3.3}$$

Making use of (3.3), we have

$$\begin{aligned}|f(z)| &\ge \frac{1}{|z|} - |z|\sum_{k=1}^{\infty} a_k \\ &\ge \frac{1}{|z|} - \frac{(1-\beta)(1-2\lambda)}{(2\alpha+\beta+1)b_1}|z|,\end{aligned} \tag{3.4}$$

and

$$\begin{aligned}|f(z)| &\le \frac{1}{|z|} + |z|\sum_{k=1}^{\infty} a_k \\ &\le \frac{1}{|z|} + \frac{(1-\beta)(1-2\lambda)}{(2\alpha+\beta+1)b_1}|z|,\end{aligned} \tag{3.5}$$

which proves the assertion (3.1), and this completes the proof of Theorem 3.1.

Theorem 3.2. *Let the function f defined by (1.1) be in the class $M(f, g; \alpha, \beta, \lambda)$, then for $0 < |z| = r < 1$, we have*

$$\frac{1}{|z|^2} - \frac{(1-\beta)(1-2\lambda)}{(2\alpha+\beta+1)b_1} \le |f'(z)| \le \frac{1}{|z|^2} + \frac{(1-\beta)(1-2\lambda)}{(2\alpha+\beta+1)b_1}. \tag{3.6}$$

The result is sharp for the function f given by (3.2).

Proof. From Theorem 2.1 and (3.3), we have

$$\sum_{k=1}^{\infty} k a_k \le \frac{(1-\beta)(1-2\lambda)}{(2\alpha+\beta+1)b_1}. \tag{3.7}$$

Since the remaining part of the proof is similar to the proof of Theorem 3.1, we omit the details.

4. Closure theorems

Let the functions f_j be defined, for $j = 1, 2, ..., m$, by

$$f_j(z) = \frac{1}{z} + \sum_{k=1}^{\infty} a_{k,j} z^k \ (a_{k,j} \geq 0). \tag{4.1}$$

Theorem 4.1. *Let the functions f_j $(j = 1, 2, ..., m)$ defined by (4.1) be in the class $M(f, g; \alpha, \beta, \lambda)$. Then the function h defined by*

$$h(z) = \frac{1}{z} + \sum_{k=1}^{\infty} \left(\frac{1}{m} \sum_{j=1}^{m} a_{k,j} \right) z^k, \tag{4.2}$$

also belongs to the class $M(f, g; \alpha, \beta, \lambda)$.

Proof. Since f_j $(j = 1, 2, ..., m)$ are in the class $M(f, g; \alpha, \beta, \lambda)$, it follows from Theorem 2.1, that

$$\sum_{k=1}^{\infty} [1 + \lambda(k-1)] [k(1+\alpha) + (\alpha+\beta)] a_{k,j} b_k \leq (1-\beta)(1-2\lambda),$$

for every $j = 1, 2, ..., m$. Hence

$$\begin{aligned} &\sum_{k=1}^{\infty} [1 + \lambda(k-1)] [k(1+\alpha) + (\alpha+\beta)] b_k \left(\frac{1}{m} \sum_{j=1}^{m} a_{k,j} \right) \\ &= \frac{1}{m} \sum_{j=1}^{m} \left(\sum_{k=1}^{\infty} [1 + \lambda(k-1)] [k(1+\alpha) + (\alpha+\beta)] a_{k,j} b_k \right) \\ &\leq (1-\beta)(1-2\lambda). \end{aligned}$$

From Theorem 2.1, it follows that $h \in M(f, g; \alpha, \beta, \lambda)$. This completes the proof of Theorem 4.1.

Theorem 4.2. *The class $M(f, g; \alpha, \beta, \lambda)$ is closed under convex linear combinations.*

Proof. Let the functions f_j $(j = 1, 2)$ defined by (4.1) be in the class $M(f, g; \alpha, \beta, \lambda)$. Then it is sufficient to show that the function

$$h(z) = \eta f_1(z) + (1-\eta) f_2(z) \ (0 \leq \eta \leq 1), \tag{4.3}$$

is in the class $M(f, g; \alpha, \beta, \lambda)$. Since for $0 \leq \eta \leq 1$,

$$h(z) = \frac{1}{z} + \sum_{k=1}^{\infty} [\eta a_{k,1} + (1-\eta) a_{k,2}] z^k, \tag{4.4}$$

with the aid of Theorem 2.1, we have

$$\begin{aligned} &\sum_{k=1}^{\infty} [1 + \lambda(k-1)] [k(1+\alpha) + (\alpha+\beta)] b_k [\eta a_{k,1} + (1-\eta) a_{k,2}] \\ &\leq \eta(1-\beta)(1-2\lambda) + (1-\eta)(1-\beta)(1-2\lambda) \\ &= (1-\beta)(1-2\lambda), \end{aligned}$$

which implies that $h \in M(f,g;\alpha,\beta,\lambda)$.

Theorem 4.3. *Let* $\sigma \geq 0$, *then*

$$M(f,g;\alpha,\beta,\lambda) \subseteq N(f,g;\alpha,\sigma),$$

where

$$\sigma = 1 - \frac{2(1-\beta)(1-2\lambda)(1+\alpha)}{(2\alpha+\beta+1)+(1-\beta)(1-2\lambda)}. \tag{4.5}$$

Proof. If $f \in M(f,g;\alpha,\beta,\lambda)$, then

$$\sum_{k=1}^{\infty} \frac{[1+\lambda(k-1)]\,[k(1+\alpha)+(\alpha+\beta)]\,b_k}{(1-\beta)(1-2\lambda)} a_k \leq 1. \tag{4.6}$$

We need to find the value of σ such that

$$\sum_{k=1}^{\infty} \frac{[k(1+\alpha)+(\alpha+\sigma)]\,b_k}{(1-\sigma)} a_k \leq 1. \tag{4.7}$$

Thus it is sufficient to show that

$$\frac{[k(1+\alpha)+(\alpha+\sigma)]}{(1-\sigma)} \leq \frac{[1+\lambda(k-1)]\,[k(1+\alpha)+(\alpha+\beta)]}{(1-\beta)(1-2\lambda)}.$$

Then

$$\sigma \leq 1 - \frac{(k+1)(1-\beta)(1-2\lambda)(1+\alpha)}{[1+\lambda(k-1)]\,[k(1+\alpha)+(\alpha+\beta)]+(1-\beta)(1-2\lambda)}.$$

Since

$$D(k) = 1 - \frac{(k+1)(1-\beta)(1-2\lambda)(1+\alpha)}{[1+\lambda(k-1)]\,[k(1+\alpha)+(\alpha+\beta)]+(1-\beta)(1-2\lambda)},$$

is an increasing function of k $(k \geq 1)$, we obtain

$$\sigma \leq D(1) = 1 - \frac{2(1-\beta)(1-2\lambda)(1+\alpha)}{(2\alpha+\beta+1)+(1-\beta)(1-2\lambda)}.$$

Theorem 4.4. *Let* $f_0(z) = \dfrac{1}{z}$ *and*

$$f_k(z) = \frac{1}{z} + \frac{(1-\beta)(1-2\lambda)}{[1+\lambda(k-1)]\,[k(1+\alpha)+(\alpha+\beta)]\,b_k} z^k \ (k \geq 1). \tag{4.8}$$

Then f *is in the class* $M(f,g;\alpha,\beta,\lambda)$ *if and only if can be expressed in the form*

$$f(z) = \sum_{k=0}^{\infty} \mu_k f_k(z), \tag{4.9}$$

where $\mu_k \geq 0$ *and* $\sum_{k=0}^{\infty} \mu_k = 1$.

Proof. Assume that

$$\begin{aligned} f(z) &= \sum_{k=0}^{\infty} \mu_k f_k(z) \\ &= \frac{1}{z} + \sum_{k=1}^{\infty} \frac{(1-\beta)(1-2\lambda)}{[1+\lambda(k-1)]\,[k(1+\alpha)+(\alpha+\beta)]\,b_k} \mu_k z^k. \end{aligned} \tag{4.10}$$

Then it follows that

$$\sum_{k=1}^{\infty} \frac{[1+\lambda(k-1)]\,[k(1+\alpha)+(\alpha+\beta)]\,b_k}{(1-\beta)(1-2\lambda)} . \frac{(1-\beta)(1-2\lambda)}{[1+\lambda(k-1)]\,[k(1+\alpha)+(\alpha+\beta)]\,b_k} \mu_k$$

$$= \sum_{k=1}^{\infty} \mu_k = 1 - \mu_0 \leq 1.$$

which implies that $f \in M(f, g; \alpha, \beta, \lambda)$.
Conversely, assume that the function f defined by (1.1) be in the class M $(f, g;$ $\alpha, \beta, \lambda)$. Then

$$a_k \leq \frac{(1-\beta)(1-2\lambda)}{[1+\lambda(k-1)]\,[k(1+\alpha)+(\alpha+\beta)]\,b_k}.$$

Setting

$$\mu_k = \frac{[1+\lambda(k-1)]\,[k(1+\alpha)+(\alpha+\beta)]\,b_k}{(1-\beta)(1-2\lambda)} a_k,$$

where

$$\mu_0 = 1 - \sum_{k=1}^{\infty} \mu_k \ ,$$

we can see that f can be expressed in the form (4.9).
Corollary 4.5. *The extreme points of the class $M(f, g; \alpha, \beta, \lambda)$ are the functions $f_0(z) = \frac{1}{z}$ and*

$$f_k(z) = \frac{1}{z} + \frac{(1-\beta)(1-2\lambda)}{[1+\lambda(k-1)]\,[k(1+\alpha)+(\alpha+\beta)]\,b_k} z^k \ (k \geq 1). \tag{4.11}$$

5. Modified Hadamard products

Let the functions f_j $(j = 1, 2)$ defined by (4.1). The modified Hadamard product of f_1 and f_2 is defined by

$$(f_1 * f_2)(z) = \frac{1}{z} + \sum_{k=1}^{\infty} a_{k,1} a_{k,2} z^k = (f_2 * f_1)(z). \tag{5.1}$$

Theorem 5.1. *Let the functions f_j $(j = 1, 2)$ defined by (4.1) be in the class $M(f, g; \alpha, \beta, \lambda)$. Then $f_1 * f_2 \in M(f, g; \alpha, \varphi, \lambda)$, where*

$$\varphi = 1 - \frac{2(1-\beta)^2(1-2\lambda)(1+\alpha)}{(2\alpha+\beta+1)^2 b_1 + (1-\beta)^2(1-2\lambda)}. \tag{5.2}$$

The result is sharp for the functions f_j $(j = 1, 2)$ *given by*

$$f_j(z) = \frac{1}{z} + \frac{(1-\beta)(1-2\lambda)}{(2\alpha+\beta+1)b_1} z \ (j = 1, 2). \tag{5.3}$$

Proof. Employing the technique used earlier by Schild and Silverman [12], we need to find the largest real parameter φ such that

$$\sum_{k=1}^{\infty} \frac{[1+\lambda(k-1)]\,[k(1+\alpha)+(\alpha+\varphi)]\,b_k}{(1-\varphi)(1-2\lambda)} a_{k,1}a_{k,2} \leq 1. \tag{5.4}$$

Since $f_j \in M(f, g; \alpha, \beta, \lambda)$ $(j = 1, 2)$, we readily see that

$$\sum_{k=1}^{\infty} \frac{[1+\lambda(k-1)]\,[k(1+\alpha)+(\alpha+\beta)]\,b_k}{(1-\beta)(1-2\lambda)} a_{k,1} \leq 1, \tag{5.5}$$

and

$$\sum_{k=1}^{\infty} \frac{[1+\lambda(k-1)]\,[k(1+\alpha)+(\alpha+\beta)]\,b_k}{(1-\beta)(1-2\lambda)} a_{k,2} \leq 1. \tag{5.6}$$

By the Cauchy-Schwarz inequality we have

$$\sum_{k=1}^{\infty} \frac{[1+\lambda(k-1)]\,[k(1+\alpha)+(\alpha+\beta)]\,b_k}{(1-\beta)(1-2\lambda)} \sqrt{a_{k,1}a_{k,2}} \leq 1. \tag{5.7}$$

Thus it is sufficient to show that

$$\frac{[1+\lambda(k-1)]\,[k(1+\alpha)+(\alpha+\varphi)]\,b_k}{(1-\varphi)(1-2\lambda)} a_{k,1}a_{k,2} \leq \frac{[1+\lambda(k-1)]\,[k(1+\alpha)+(\alpha+\beta)]\,b_k}{(1-\beta)(1-2\lambda)} \sqrt{a_{k,1}a_{k,2}}, \tag{5.8}$$

or equivalently, that

$$\sqrt{a_{k,1}a_{k,2}} \leq \frac{[k(1+\alpha)+(\alpha+\beta)]\,(1-\varphi)}{[k(1+\alpha)+(\alpha+\varphi)]\,(1-\beta)}. \tag{5.9}$$

Hence, in light of the inequality (5.7), it is sufficient to prove that

$$\frac{(1-\beta)(1-2\lambda)}{[1+\lambda(k-1)]\,[k(1+\alpha)+(\alpha+\beta)]\,b_k} \leq \frac{[k(1+\alpha)+(\alpha+\beta)]\,(1-\varphi)}{[k(1+\alpha)+(\alpha+\varphi)]\,(1-\beta)}. \tag{5.10}$$

It follows from (5.10) that

$$\varphi \leq 1 - \frac{(1-\beta)^2(1-2\lambda)(1+\alpha)(k+1)}{[1+\lambda(k-1)]\,[k(1+\alpha)+(\alpha+\beta)]^2\,b_k + (1-\beta)^2(1-2\lambda)}. \tag{5.11}$$

Now defining the function $E(k)$ by

$$E(k) = 1 - \frac{(1-\beta)^2(1-2\lambda)(1+\alpha)(k+1)}{[1+\lambda(k-1)]\,[k(1+\alpha)+(\alpha+\beta)]^2\,b_k + (1-\beta)^2(1-2\lambda)}. \tag{5.12}$$

We see that $E(k)$ is an increasing function of k $(k \geq 1)$. Therefore, we conclude that

$$\varphi \leq E(1) = 1 - \frac{2(1-\beta)^2(1-2\lambda)(1+\alpha)}{(2\alpha+\beta+1)^2 b_1 + (1-\beta)^2(1-2\lambda)}, \tag{5.13}$$

which evidently completes the proof of Theorem 5.1.
Using arguments similar to those in the proof of Theorem 5.1, we obtain the following theorem:

Theorem 5.2. *Let the function f_1 defined by (4.1) be in the class $M(f,g;\alpha,\beta,\lambda)$. Suppose also that the function f_2 defined by (4.1) be in the class $M(f,g;\alpha,\rho,\lambda)$. Then $f_1 * f_2 \in M(f,g;\alpha,\zeta,\lambda)$ where*

$$\zeta = 1 - \frac{2(1-\beta)(1-\rho)(1-2\lambda)(1+\alpha)}{(2\alpha+\beta+1)(2\alpha+\rho+1)b_1 + (1-\beta)(1-\rho)(1-2\lambda)}. \tag{5.14}$$

The result is sharp for the functions f_j $(j=1,2)$ given by

$$f_1(z) = \frac{1}{z} + \frac{(1-\beta)(1-2\lambda)}{(2\alpha+\beta+1)b_1} z\ , \tag{5.15}$$

and

$$f_2(z) = \frac{1}{z} + \frac{(1-\rho)(1-2\lambda)}{(2\alpha+\rho+1)b_1} z\ . \tag{5.16}$$

Theorem 5.3. *Let the functions f_j $(j=1,2)$ defined by (4.1) be in the class $M(f,g;\alpha,\beta,\lambda)$. Then the function*

$$h(z) = \frac{1}{z} + \sum_{k=1}^{\infty} (a_{k,1}^2 + a_{k,2}^2) z^k \tag{5.17}$$

belong to the class $M(f,g;\alpha,\varepsilon,\lambda)$, where

$$\varepsilon = 1 - \frac{4(1-\beta)^2(1-2\lambda)(1+\alpha)}{(2\alpha+\beta+1)^2 b_1 + 2(1-\beta)^2(1-2\lambda)}. \tag{5.18}$$

The result is sharp for the functions f_j $(j=1,2)$ defined by (5.3).

Proof. By using Theorem 2.1, we obtain

$$\sum_{k=1}^{\infty} \left\{ \frac{[1+\lambda(k-1)]\,[k(1+\alpha)+(\alpha+\beta)]\,b_k}{(1-\beta)(1-2\lambda)} \right\}^2 a_{k,1}^2$$

$$\leq \left\{ \sum_{k=1}^{\infty} \frac{[1+\lambda(k-1)]\,[k(1+\alpha)+(\alpha+\beta)]\,b_k}{(1-\beta)(1-2\lambda)} a_{k,1} \right\}^2 \leq 1, \tag{5.19}$$

and

$$\sum_{k=1}^{\infty} \left\{ \frac{[1+\lambda(k-1)]\,[k(1+\alpha)+(\alpha+\beta)]\,b_k}{(1-\beta)(1-2\lambda)} \right\}^2 a_{k,2}^2$$

$$\leq \left\{ \sum_{k=1}^{\infty} \frac{[1+\lambda(k-1)]\,[k(1+\alpha)+(\alpha+\beta)]\,b_k}{(1-\beta)(1-2\lambda)} a_{k,2} \right\}^2 \leq 1. \tag{5.20}$$

It follows from (5.19) and (5.20) that

$$\sum_{k=1}^{\infty} \frac{1}{2} \left\{ \frac{[1+\lambda(k-1)]\,[k(1+\alpha)+(\alpha+\beta)]\,b_k}{(1-\beta)(1-2\lambda)} \right\}^2 (a_{k,1}^2 + a_{k,2}^2) \leq 1. \tag{5.21}$$

Therefore, we need to find the largest ε such that

$$\frac{[1+\lambda(k-1)]\,[k(1+\alpha)+(\alpha+\varepsilon)]\,b_k}{(1-\varepsilon)(1-2\lambda)}$$

$$\leq \frac{1}{2}\left\{\frac{[1+\lambda(k-1)]\,[k(1+\alpha)+(\alpha+\beta)]\,b_k}{(1-\beta)(1-2\lambda)}\right\}^2, \tag{5.22}$$

that is

$$\varepsilon \leq 1-\frac{2(1-\beta)^2(1-2\lambda)(1+\alpha)(k+1)}{[1+\lambda(k-1)]\,[k(1+\alpha)+(\alpha+\beta)]^2\,b_k+2(1-\beta)^2(1-2\lambda)}. \tag{5.23}$$

Since

$$G(k)=1-\frac{2(1-\beta)^2(1-2\lambda)(1+\alpha)(k+1)}{[1+\lambda(k-1)]\,[k(1+\alpha)+(\alpha+\beta)]^2\,b_k+2(1-\beta)^2(1-2\lambda)}, \tag{5.24}$$

is an increasing function of k $(k \geq 1)$, we obtain

$$\varepsilon \leq G(1)=1-\frac{4(1-\beta)^2(1-2\lambda)(1+\alpha)}{(2\alpha+\beta+1)^2 b_1+2(1-\beta)^2(1-2\lambda)}, \tag{5.25}$$

and hence the proof of Theorem 5.3 is completed.

6. Integral operators

Theorem 6.1. *Let the functions f given by* (1.1) *be in the class $M(f,g;\alpha,\beta,\lambda)$. Then the integral operator*

$$F(z)=c\int_0^1 u^c\ f(uz)du\ \ (0<u\leq 1;\ \ c>0), \tag{6.1}$$

is in the class $M(f,g;\alpha,\xi,\lambda)$, where

$$\xi = 1-\frac{2c(1-\beta)(1+\alpha)}{(c+2)(2\alpha+\beta+1)+c(1-\beta)}. \tag{6.2}$$

The result is sharp for the function f given by (3.2).

Proof. Let $f \in M(f,g;\alpha,\beta,\lambda)$, then

$$\begin{aligned} F(z) &= c\int_0^1 u^c\ f(uz)du \\ &= \frac{1}{z}+\sum_{k=1}^{\infty}\frac{c}{k+c+1}a_k z^k. \end{aligned} \tag{6.3}$$

Thus it is sufficient to show that

$$\sum_{k=1}^{\infty}\frac{c[1+\lambda(k-1)]\,[k(1+\alpha)+(\alpha+\xi)]\,b_k}{(k+c+1)(1-\xi)(1-2\lambda)}a_k \leq 1. \tag{6.4}$$

Since $f \in M(f, g; \alpha, \beta, \lambda)$, then

$$\sum_{k=1}^{\infty} \frac{[1+\lambda(k-1)]\,[k(1+\alpha)+(\alpha+\beta)]\,b_k}{(1-\beta)(1-2\lambda)} a_k \leq 1. \tag{6.5}$$

From (6.4) and (6.5), we have

$$\frac{c\,[k(1+\alpha)+(\alpha+\xi)]}{(k+c+1)(1-\xi)} \leq \frac{[k(1+\alpha)+(\alpha+\beta)]}{(1-\beta)}.$$

Then

$$\xi \leq 1 - \frac{c(1-\beta)(k+1)(1+\alpha)}{(c+k+1)[k(1+\alpha)+(\alpha+\beta)]+c(1-\beta)}.$$

Since

$$Y(k) = 1 - \frac{c(1-\beta)(k+1)(1+\alpha)}{(c+k+1)[k(1+\alpha)+(\alpha+\beta)]+c(1-\beta)},$$

is an increasing function of k $(k \geq 1)$, we obtain

$$\xi \leq Y(1) = 1 - \frac{2c(1-\beta)(1+\alpha)}{(c+2)(2\alpha+\beta+1)+c(1-\beta)},$$

and hence the proof of Theorem 6.1 is completed.

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On a class of dichotomous evolution operators with strongly continuous families of projections

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Abstract. The aim of this paper is to present a concept of nonuniform exponential dichotomy through a certain class of strongly continuous evolution operators defined with the aid of a particular family of projections acting on the state space. This class easily emphasizes the fact that, in the case of uniform exponential dichotomy, the uniform exponential growth is essential in order to prove the boundedness of the dichotomic family of projections. The main result of the paper is the extension of the boundedness result in the nonuniform setting.

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1. Introduction

The exponential dichotomy property for linear dynamical systems has gained prominence since the appearance of two fundamental monographs of J. L. Massera and J. J. Schäffer [12], J. L. Daleckii and M. G. Krein [10]. These were followed by the important books of C. Chicone and Y. Latushkin [9] and L. Barreira and C. Valls [5].

Concerning the stability, unstability and dichotomy properties, it is worth to note that their study had an impressive development and several results were obtained, which characterizes these properties, connect them and study their preservations under small perturbations, which were successfully materialized in [2], [14], [17], [11], [16], [22], [19] and the references therein.

The study of concepts of nonuniform exponential dichotomies materialized in a large number of interesting research papers, from where we point out: [6], [7], [8], [13], [18], [21], [17].

In this paper we present a particular family of projections on the Banach space $l^\infty(\mathbb{N}^*, \mathbb{R})$, which satisfies a vast variety of properties, useful in constructing counterexamples (see for example [3] in discrete time). Attached to this family of projections, we give a particular type of evolution operator which will serve as an example to the importance of the growth property assumed in order to prove the existence of a constant upper-bound for the dichotomic family of projections.

In the final part of this paper we took a step forward in this direction: under the hypotheses of exponential growth and *nonuniform* exponential dichotomy, the family of projections is, by conclusion, exponentially bounded.

Several results in the uniform setting were obtained in this sense, and we point out the works [15], [20], [23], [1].

A first approach in the nonuniform case (under the assumption of nonuniform exponential growth and *uniform* asymptotic behavior) was successfully accomplished in [2] in discrete-time, from where the particular cases of exponential and polynomial upper-bounds of the projections were obtained. By using different methods and a stronger concept than the exponential dichotomy (the notion of admissibility), an exponential upper-bound - in terms of an auxiliary norm constructed on the state space - of the family of projections was obtained in [4].

2. Preliminaries

Let X be a real or complex Banach space, and $\mathcal{B}(X)$ the algebra of bounded linear operators acting on X. We denote by $\|\cdot\|$ the norm on X and on $\mathcal{B}(X)$, and let Δ be the set of all pairs of real nonnegative numbers (t,s) satisfying $t \geq s$.

Definition 2.1. *A map $U : \Delta \to \mathcal{B}(X)$ is called an* **evolution operator** *on X if the following conditions hold:*

(e_1) $U(t,t) = I$, for all $t \geq 0$ (I denoting the identity operator on X).
(e_2) $U(t,s)U(s,t_0) = U(t,t_0)$, for all $(t,s),(s,t_0) \in \Delta$.
Moreover, if
(e_3) for all $t \geq 0$ and for all $x \in X$ the maps $[0,t] \ni \tau \mapsto U(t,\tau)x \in X$ and $[t,\infty) \ni \tau \mapsto U(\tau,t)x \in X$ are continuous
then we say that $U : \Delta \to \mathcal{B}(X)$ is a **strongly continuous evolution operator**.

Definition 2.2. *Let $U : \Delta \to \mathcal{B}(X)$ be an evolution operator. We say that $U : \Delta \to \mathcal{B}(X)$ has an* **exponential growth** *if there exist $M, \omega > 0$, $\varepsilon \geq 0$ such that*

$$\|U(t,s)x\| \leq Me^{\varepsilon s}e^{\omega(t-s)}\|x\|, \quad \forall (t,s) \in \Delta, \forall x \in X.$$

In the particular case in which $\varepsilon = 0$, we say that U has a **uniform exponential growth**.

Remark 2.3. If an evolution operator $U : \Delta \to \mathcal{B}(X)$ has a uniform exponential growth then it obviously has a nonuniform exponential growth.

Example 2.4. Let $f : \mathbb{R}_+ \to (0,\infty)$ be a continuous function. For $(t,s) \in \Delta$ we define $U(t,s) : X \to X$ by

$$U(t,s)x = \frac{f(t)}{f(s)} \cdot x, \quad \forall x \in X.$$

We have that U is a strongly continuous evolution operator.

1. If $f(t)=e^t$, for al $t\geq 0$, it is easy to see that $U:\Delta\to\mathcal{B}(X)$ has a uniform exponential growth.
2. If $f(t)=t\cdot\cos t$, for all $t\geq 0$, one can see that $U:\Delta\to\mathcal{B}(X)$ has an exponential growth, which is not uniform.

Below, we will see that Definition 2.2 is not redundant.

Example 2.5. Let $X=\mathbb{R}$ and $A:\mathbb{R}_+\to\mathbb{R}_+$, $A(t)=e^t$. Consider the Cauchy problem

$$\begin{cases}\dot{x}(t) &= A(t)x(t), \quad t>0\\ x(0) &= e\end{cases}$$

The above stated problem has the solution $x(t)=e^{e^t}$, the corresponding evolution operator being $U:\Delta\to\mathcal{B}(\mathbb{R})$,

$$U(t,s)x=e^{e^t-e^s}\cdot x,\quad \forall(t,s,x)\in\Delta\times\mathbb{R}.$$

Assuming that there exist $M\geq 1$, $\varepsilon\geq 0$ and $\omega>0$ such that

$$\|U(t,s)\|\leq Me^{\varepsilon s}e^{\omega(t-s)},\quad \forall(t,s)\in\Delta,$$

choosing in the above inequality $s=0$, we obtain the contradiction

$$e^{e^t}\leq Me^{\omega t},\quad \forall t\geq 0.$$

The example from above shows us that even in the particular context of evolution operators arising from Cauchy problems, the exponential growth is not assured.

Definition 2.6. *A map $P:\mathbb{R}_+\to\mathcal{B}(X)$ is called a* **family of projections** *on X if*

$$P(t)^2=P(t),\qquad \text{for every } t\geq 0.$$

In addition,

(i) if there are $M\geq 1$ and $\gamma\geq 0$ such that

$$\|P(t)\|\leq Me^{\gamma t},\qquad \text{for all } t\geq 0$$

then we say that the family $P:\mathbb{R}_+\to\mathcal{B}(X)$ is **exponentially bounded**. *In the particular case when $\gamma=0$, P is called* **bounded***;*

(ii) if for all $t\geq 0$ and for all $x\in X$, the map

$$\mathbb{R}_+\ni t\mapsto P(t)x\in X$$

is continuous then we say that the family $P:\mathbb{R}_+\to\mathcal{B}(X)$ is **strongly continuous**.

Remark 2.7. If $P:\mathbb{R}_+\to\mathcal{B}(X)$ is a family of projections on X then

$$Q:\mathbb{R}_+\to\mathcal{B}(X) \text{ defined by } Q(t)=I-P(t)$$

is also a family of projections on X, which is called the **complementary family of projections** of P.

Definition 2.8. *Let $U:\Delta\to\mathcal{B}(X)$ be an evolution operator and $P:\mathbb{R}_+\to\mathcal{B}(X)$ a family of projections. We say that $P:\mathbb{R}_+\to\mathcal{B}(X)$ is:*

(i) **invariant** *for the evolution operator $U:\Delta\to\mathcal{B}(X)$ if for all $(t,s)\in\Delta$*

$$U(t,s)P(s)=P(t)U(t,s);$$

(ii) **strongly invariant** *for the evolution operator* $U : \Delta \to \mathcal{B}(X)$ *if it is invariant for* U *and for all* $(t, s) \in \Delta$ *the restriction*

$$U(t,s)_| : KerP(s) \to KerP(t)$$

is an isomorphism.

In what follows, if P is invariant for U, then we say that (U, P) is a **dichotomy pair**.

3. Nonuniform exponential dichotomies

Let (U, P) be a dichotomy pair.

Definition 3.1. *We say that* (U, P) *is* **exponentially dichotomic** *(e.d) if there exist constants* $N \geq 1, \beta > 0,\ \alpha \geq 0$ *such that for all* $(t, s, x) \in \Delta \times X$ *the following hold:*

(ed_1) $\|U(t,s)P(s)x\| \leq Ne^{\alpha s}e^{-\beta(t-s)}\|P(s)x\|$;
(ed_2) $Ne^{\alpha s}\|U(t,s)Q(s)x\| \geq e^{\beta(t-s)}\|Q(s)x\|$.

If $\alpha = 0$ *then we say that* (U, P) *is* **uniformly exponentially dichotomic** *(u.e.d).*

Remark 3.2. If a dichotomy pair (U, P) is (u.e.d) then it is also (e.d). The converse is not generally true, as we can see in the below example.

Example 3.3. Let $X = \mathbb{R}^2$, $f : \mathbb{R}_+ \to \mathbb{R}$,

$$f(t) = \frac{t}{1 + \{t\}}, \quad t \geq 0$$

where by $\{t\}$ we denoted the fractional part of the real number t. For the above defined function we have the following estimation:

$$f(t) - f(s) \geq \frac{1}{2}(t - s) - \frac{s}{2}, \quad \forall (t, s) \in \Delta.$$

We define $U : \Delta \to \mathcal{B}(\mathbb{R}^2)$ by

$$U(t,s)(x_1, x_2) = \left(e^{f(s)-f(t)}, e^{f(t)-f(s)}\right), \quad (t, s, x_1, x_2) \in \Delta \times \mathbb{R}^2.$$

Defining $P : \mathbb{R}_+ \to \mathcal{B}(X)$, by $P(t)(x_1, x_2) = (x_1, 0)$ for $t \geq 0$ and $(x_1, x_2) \in \mathbb{R}^2$, we have that (U, P) is a dichotomy pair and a straightforward estimation shows us that for all $(t, s) \in \Delta$, and for all $x = (x_1, x_2) \in \mathbb{R}^2$

$$\|U(t,s)P(s)x\| \leq e^{\frac{1}{2}s}e^{-\frac{1}{2}(t-s)}\|P(s)x\|$$
$$e^{\frac{1}{2}s}\|U(t,s)Q(s)x\| \geq e^{\frac{1}{2}(t-s)}\|Q(s)x\|.$$

Hence conditions (ed_1) and (ed_1) follow from above, from where (U, P) is e.d, but the dichotomy cannot be uniform since, by assuming the contrary, for $n \in \mathbb{N}$ setting $t_n = n + \frac{3}{2}$ and $s_n = n + 1$, with N, β given by Definition 3.1, we would obtain the contradiction

$$e^{\frac{n}{3}} \leq Ne^{-\frac{\beta}{2}}, \quad \forall n \in \mathbb{N}.$$

Remark 3.4. In [15] it is proven that if $U : \Delta \to \mathcal{B}(X)$ has uniform exponential growth then the uniform exponential dichotomy of $U : \Delta \to \mathcal{B}(X)$ implies that

$$\sup_{t\geq 0} \|P(t)\| < +\infty.$$

We will show that the uniform exponential growth of the dichotomic evolution operator $U : \Delta \to \mathcal{B}(X)$ is essential for the conclusion in the preceding remark to hold.

In what follows, we will present a family of projections which is strongly continuous and, by choosing an appropriate evolution operator, it will give a dichotomy pair with interesting properties.

Example 3.5. Let $X = l^\infty(\mathbb{N}^*, \mathbb{R})$ the Banach space of bounded real-valued sequences, endowed wit the sup-norm

$$\|x\|_\infty = \sup_{n\geq 1} |x_n|, \quad \text{for } x = (x_n)_{n\geq 1} \in l^\infty(\mathbb{N}^*, \mathbb{R}).$$

The norm on $\mathcal{B}(X)$ will be denoted as usual by $\|\cdot\|$.

For every $t \in \mathbb{R}_+$ we define $P(t) : l^\infty(\mathbb{N}^*, \mathbb{R}) \to l^\infty(\mathbb{N}^*, \mathbb{R})$, for $x = (x_1, x_2, \ldots, x_n, \ldots) \in l^\infty(\mathbb{N}^*, \mathbb{R})$, by

$$P(t)x = \big(x_1 + (e^t - 1)x_2\ ,\ 0\ ,\ x_3 + (e^t - 1)x_4\ ,\ 0\ ,\ \ldots\big)\,.$$

We denote by $Q(t) = I - P(t)$, for all $t \in \mathbb{R}_+$.

The properties of the family of operators $P : \mathbb{R}_+ \to \mathcal{B}(l^\infty(\mathbb{N}^*, \mathbb{R}))$ are pointed out by the following result.

Proposition 3.6. *For all $t, s \in \mathbb{R}_+$ and for all $x = (x_n)_{n\geq 1} \in l^\infty(\mathbb{N}^*, \mathbb{R})$, the following assertions hold:*

(i) $P(t)$ *is correctly defined,* $P(t) \in \mathcal{B}(l^\infty(\mathbb{N}^*, \mathbb{R}))$ *and* $\|P(t)\| = e^t$*;*
(ii) $P(t)$ *is a projection on* $l^\infty(\mathbb{N}^*, \mathbb{R})$*;*
(iii) $Q(t)x = ((1 - e^t)\,x_2\ ,\ x_2\ ,\ (1 - e^t)\,x_4\ ,\ x_4\ ,\ \ldots)$ *and* $\|Q(t)\| = \max\{1, e^t - 1\}$*;*
(iv)

$$\begin{aligned} \mathit{Range}P(t) &= \mathit{Range}P(s) \\ &= \{(y_n)_{n\geq 1} \in l^\infty(\mathbb{N}^*, \mathbb{R})\ :\ y_{2n} = 0, \forall n \in \mathbb{N}^*\} =: \mathcal{H}; \end{aligned}$$

(v)

$$\begin{aligned} \mathit{Range}Q(t) &= \big\{(x_n) \in l^\infty(\mathbb{N}^*, \mathbb{R})\ :\ x_{2n-1} + \big(e^t - 1\big)\, x_{2n} = 0, \forall n \in \mathbb{N}^*\big\} \\ &=: \mathcal{K}(t); \end{aligned}$$

(vi) the decomposition $l^\infty(\mathbb{N}^*, \mathbb{R}) = \mathcal{H} \oplus \mathcal{K}(t)$ *holds;*
(vii) $P(t)P(s) = P(s)$*;*
(viii) $Q(t)Q(s) = Q(t)$*;*
(ix) $Q(t)P(s) = 0$.

Proof. Let $t, s \in \mathbb{R}_+$ and $x = (x_n)_{n\geq 1} \in l^\infty(\mathbb{N}^*, \mathbb{R})$.

(i) Obviously $P(t)$ is a linear operator, and from

$$\|P(t)x\|_\infty = \sup_{n\geq 1} \left|x_{2n-1} + \left(e^t - 1\right) x_{2n}\right| \leq \left(1 + \left|e^t - 1\right|\right) \|x\|_\infty = e^t \|x\|_\infty,$$

we have that $P(t)$ is correctly defined, and $P(t) \in \mathcal{B}(X)$ with $\|P(t)\| \leq e^t$. Choosing $x = (1, 1, 1, \ldots) \in l^\infty(\mathbb{N}^*, \mathbb{R})$ we have that $\|x\|_\infty = 1$ and from

$$P(t)x = \left(1 + \left(e^t - 1\right) \cdot 1 \ , \ 0 \ , \ 1 + \left(e^t - 1\right) \cdot 1 \ , \ 0 \ , \ \ldots\right) = \left(e^t \ , \ 0 \ , \ e^t \ , \ 0 \ , \ \ldots\right),$$

we have that

$$\|P(t)x\|_\infty = e^t = e^t \|x\|_\infty,$$

from which it follows that $\|P(t)\| = e^t$.

(ii) Let $y = P(t)x$. Then we have that for all $n \in \mathbb{N}^*$, $y_{2n-1} = x_{2n-1} + (e^t - 1)\, x_{2n}$ and $y_{2n} = 0$. It follows from here that

$$\begin{aligned} P(t)^2 x = P(t)y &= \Big(y_1 + \left(e^t - 1\right) y_2 \ , \ 0 \ , \ y_3 + \left(e^t - 1\right) y_4 \ , \ 0 \ , \ \ldots\Big) \\ &= (y_1, 0, y_3, 0, \ldots) = P(t)x. \end{aligned}$$

(iii) The expression defining $Q(t)$ follows from a straightforward computation. Let $y = Q(t)x$. It follows that for $n \in \mathbb{N}^*$, $y_{2n-1} = (1 - e^t)\, x_{2n}$ and $y_{2n} = x_{2n}$. This implies that

$$\begin{aligned} \|Q(t)x\|_\infty = \sup_{n\geq 1} |y_n| &= \max\left\{\sup_{n\geq 1} |y_{2n-1}|, \sup_{n\geq 1} |y_{2n}|\right\} \\ &= \max\left\{\left(e^t - 1\right) \sup_{n\geq 1} |x_{2n}|, \sup_{n\geq 1} |x_{2n}|\right\} \\ &= \max\left\{1, e^t - 1\right\} \cdot \sup_{n\geq 1} |x_{2n}| \leq \max\left\{1, e^t - 1\right\} \cdot \|x\|_\infty. \end{aligned}$$

Choosing $x_0 = (0, 1, 0, 1, \ldots) \in l^\infty(\mathbb{N}^*, \mathbb{R})$ having $\|x_0\|_\infty = 1$, from

$$Q(t)x_0 = \left(1 - e^t \ , \ 1 \ , \ 1 - e^t \ , \ 1 \ , \ \ldots\right)$$

we obtain that $\|Q(t)x_0\|_\infty = \max\{1, e^t - 1\}\, \|x_0\|_\infty$, from which the validity of the assertion follows.

(iv) Let $y \in RangeP(t)$. Then there exists $z \in l^\infty(\mathbb{N}^*, \mathbb{R})$ with $y = P(t)z$, from which we deduce that $y_{2n} = 0$, for all $n \in \mathbb{N}^*$.
Conversely, let $y \in l^\infty(\mathbb{N}^*, \mathbb{R})$ be a bounded sequence having $y_{2n} = 0$, for all $n \in \mathbb{N}^*$. A straightforward calculation shows us that $P(t)y = y$, so $y \in RangeP(t)$.

(v) From the equivalence $P(t)x = 0 \Leftrightarrow x_{2n-1} + (e^t - 1)\, x_{2n} = 0$, for all $n \in \mathbb{N}^*$, it follows that the assertion is true.

(vi) The decomposition takes place, provided by the fact that $P(t) \in \mathcal{B}(l^\infty(\mathbb{N}^*, \mathbb{R}))$.

(vii) From $P(s)x \in \mathcal{H} = RangeP(t)$, $P(t)$ acting as the identity operator on its range, we deduce that $P(t)P(s)x = P(s)x$.

(viii) The desired relation follows from

$$\begin{aligned} Q(t)Q(s)x &= (I - P(t))(I - P(s))x = x - P(s)x - P(t)x + P(t)P(s)x \\ &= x - P(s)x - P(t)x + P(s)x = x - P(t)x = Q(t)x. \end{aligned}$$

(ix) From $P(s)x \in \mathcal{H} = RangeP(t)$, we have that $Q(t)P(s)x = 0$. □

Another property of the above defined family of projections is given by:

Proposition 3.7. *The family of projections* $P : \mathbb{R}_+ \to \mathcal{B}(l^\infty(\mathbb{N}^*, \mathbb{R}))$ *defined in Example* 3.5 *is strongly continuous.*

Proof. Let $x \in l^\infty(\mathbb{N}^*, \mathbb{R})$ and $\tau_0 \in \mathbb{R}_+$. A simple computation gives us that

$$P(\tau)x - P(\tau_0)x = ((e^\tau - e^{\tau_0})x_2\ ,\ 0\ ,\ (e^\tau - e^{\tau_0})x_4\ ,\ 0\ ,\ \ldots),$$

hence

$$\|P(\tau)x - P(\tau_0)x\|_\infty = |e^\tau - e^{\tau_0}| \sup_{n\geq 1} |x_{2n}| \leq |e^\tau - e^{\tau_0}|\, \|x\|_\infty \xrightarrow[\tau\to\tau_0]{} 0.$$

From this it easily follows that

$$\|Q(\tau)x - Q(\tau_0)x\|_\infty = \|x - P(\tau)x - x + P(\tau_0)x\|_\infty = \|P(\tau_0)x - P(\tau)x\|_\infty \xrightarrow[\tau\to\tau_0]{} 0$$

hence its complementary is also strongly continuous. □

For $(t, s) \in \Delta$, we define $U_P(t, s) : l^\infty(\mathbb{N}^*, \mathbb{R}) \to l^\infty(\mathbb{N}^*, \mathbb{R})$ by

$$U_P(t, s)x = e^{s-t}P(s)x + e^{t-s}Q(t)x, \quad \forall x \in l^\infty(\mathbb{N}^*, \mathbb{R}).$$

The following result will point out the basic properties that the above defined two-parameter family of bounded linear operators verifies.

Proposition 3.8. U_P *is a strongly continuous evolution operator on* $l^\infty(\mathbb{N}^*, \mathbb{R})$.

Proof. (e_1) We have that $U_P(t, t)x = e^0P(t)x + e^0Q(t)x = x$, for all $(t, s, x) \in \Delta \times l^\infty(\mathbb{N}^*, \mathbb{R})$.
(e_2) Let $(t, s), (s, t_0) \in \Delta$ and $x \in l^\infty(\mathbb{N}^*, \mathbb{R})$.

$$\begin{aligned}
&U_P(t, s)U_P(s, t_0)x = e^{s-t}P(s)U(s, t_0)x + e^{t-s}Q(t)U(s, t_0)x \\
&= e^{s-t}P(s)\left(e^{t_0-s}P(t_0)x + e^{s-t_0}Q(s)x\right) + e^{t-s}Q(t)\left(e^{t_0-s}P(t_0)x + e^{s-t_0}Q(s)x\right) \\
&= e^{t_0-t}P(s)P(t_0)x + e^{s-t}e^{s-t_0}P(s)Q(s)x + e^{t-s}e^{t_0-s}Q(t)P(t_0)x + e^{t-t_0}Q(t)Q(s)x \\
&= e^{t_0-t}P(t_0)x + e^{t-t_0}Q(t)x = U_P(t, t_0)x.
\end{aligned}$$

(e_3) Let $t \geq 0$ and $x \in l^\infty(\mathbb{N}^*, \mathbb{R})$. The continuity of the map $[0, t] \ni \tau \mapsto U(t, \tau)x \in l^\infty(\mathbb{N}^*, \mathbb{R})$ follows from the below estimations:

$$\begin{aligned}
&\|U_P(t, \tau)x - U_P(t, \tau_0)x\|_\infty \\
&\leq \left\|e^{\tau-t}P(\tau)x + e^{t-\tau}Q(t)x - e^{\tau_0-t}P(\tau_0)x - e^{t-\tau_0}Q(t)x\right\|_\infty \\
&\leq \left\|e^{\tau-t}P(\tau)x - e^{\tau_0-t}P(\tau_0)x\right\|_\infty + \left\|e^{t-\tau}Q(t)x - e^{t-\tau_0}Q(t)x\right\|_\infty \\
&\leq \left\|e^{\tau-t}P(\tau)x - e^{\tau-t}P(\tau_0)x\right\|_\infty + \left\|e^{\tau-t}P(\tau_0)x - e^{\tau_0-t}P(\tau_0)x\right\|_\infty + \\
&+ \left|e^{t-\tau} - e^{t-\tau_0}\right| \|Q(t)x\|_\infty \leq \\
&\leq \|P(\tau)x - P(\tau_0)x\|_\infty + \left|e^{\tau-t} - e^{\tau_0-t}\right| \|P(\tau_0)x\|_\infty + e^t\left|e^{-\tau} - e^{-\tau_0}\right| \|Q(t)x\|_\infty.
\end{aligned}$$

To prove that the map $[t,\infty) \ni \tau \mapsto U(\tau,t)x \in l^\infty(\mathbb{N}^*,\mathbb{R})$ is continuous, we will proceed as above:

$$\begin{aligned}
&\|U_P(\tau,t)x - U_P(\tau_0,t)x\|_\infty \\
&= \left\|e^{t-\tau}P(t)x + e^{\tau-t}Q(\tau)x - e^{t-\tau_0}P(t)x - e^{\tau_0-t}Q(\tau_0)x\right\|_\infty \\
&\leq \left|e^{t-\tau} - e^{t-\tau_0}\right| \|P(t)x\|_\infty + e^{\tau-t}\|Q(\tau)x - Q(\tau_0)x\|_\infty + \left|e^{\tau-t} - e^{\tau_0-t}\right| \|Q(\tau_0)x\|_\infty,
\end{aligned}$$

the right-hand side tending to zero as $\tau \to \tau_0$, provided by the fact that the map $\tau \mapsto e^{\tau-t}$ is bounded on the interval $[t,\tau_0+1]$. □

Regarding the growth of the above defined evolution operator, we state the following two results.

Proposition 3.9. *The evolution operator $U_P : \Delta \to \mathcal{B}(l^\infty(\mathbb{N}^*,\mathbb{R}))$ has an exponential growth.*

Proof. Let $(t,s) \in \Delta$ and $x \in l^\infty(\mathbb{N}^*,\mathbb{R})$. Setting $M = \omega = 2$ and $\varepsilon = 1$, and having in mind that

$$\begin{aligned}
\|U_P(t,s)x\|_\infty &= \left\|e^{s-t}P(s)x + e^{t-s}Q(t)x\right\|_\infty \leq e^{t-s}\left(\|P(s)\| + \|Q(t)\|\right)\|x\|_\infty = \\
&= e^{t-s}\left(e^s + \max\{1, e^t - 1\}\right)\|x\|_\infty \leq 2e^t e^{t-s}\|x\|_\infty = \\
&= 2e^s e^{2(t-s)}\|x\|_\infty,
\end{aligned}$$

we obtain the desired conclusion. □

Proposition 3.10. *The evolution operator U_P does not admit a uniform exponential growth.*

Proof. Assume by a contradiction that there exist $M,\omega > 0$ such that

$$\|U_P(t,s)x\|_\infty \leq Me^{\omega(t-s)}\|x\|_\infty, \quad \forall (t,s,x) \in \Delta \times l^\infty(\mathbb{N}^*,\mathbb{R}).$$

Let, in the above inequality, $t \geq 3$, $s = t-1$ and $x = (0,1,0,1,\dots) \in l^\infty(\mathbb{N}^*,\mathbb{R})$, having $\|x\|_\infty = 1$. This implies that

$$\|U_P(t,t-1)x\|_\infty \leq Me^\omega. \tag{3.1}$$

We have that

$$\frac{1}{e}P(t-1)x = \left(e^{t-2} - \frac{1}{e}, 0, e^{t-2} - \frac{1}{e}, 0, \dots\right) \tag{3.2}$$

$$eQ(t)x = \left(e - e^{t+1}, e, e - e^{t+1}, e, \dots\right). \tag{3.3}$$

From (3.2) and (3.3) it follows that

$$U_P(t,t-1)x = \left(e^{t-2} - e^{t+1} + e - \frac{1}{e}, e, e^{t-2} - e^{t+1} + e - \frac{1}{e}, e, \dots\right),$$

from which we deduce that

$$\|U_P(t,t-1)x\|_\infty = \max\left\{\left|e^{t-2} - e^{t+1} + e - \frac{1}{e}\right|, e\right\}. \tag{3.4}$$

By Lagrange's mean value theorem applied to the exponential function on the interval $[t-2, t+1] \subset [1, \infty)$, there exists $\xi_t \in (t-2, t+1)$ such that

$$e^{t+1} - e^{t-2} = 3e^{\xi_t} > 3e^{t-2} \geq 3e. \tag{3.5}$$

Hence

$$e^{t-2} - e^{t+1} + e - \frac{1}{e} = -3e^{\xi_t} + e - \frac{1}{e} < -2e - \frac{1}{e} < -2e < -e < 0. \tag{3.6}$$

By (3.4) and (3.6), we have that

$$\|U_P(t, t-1)x\|_\infty = e^{t+1} - e^{t-2} + \frac{1}{e} - e. \tag{3.7}$$

Finally, using (3.7), (3.5) and (3.1), we obtain the contradicting inequality

$$3e^{t-2} + \frac{1}{e} - e \leq e^{t+1} - e^{t-2} + \frac{1}{e} - e \leq Me^{\omega}, \quad \forall t \geq 3.$$

□

Proposition 3.11. *(U_P, P) is a dichotomy pair.*

Proof. Let $(t, s) \in \Delta$ and $x \in l^\infty(\mathbb{N}^*, \mathbb{R})$. The conclusion easily follows from

$$\begin{aligned} U_P(t,s)P(s)x &= e^{s-t}P(s)P(s)x + e^{t-s}Q(t)P(s)x \\ &= e^{s-t}P(s)x; \\ P(t)U_P(t,s)x &= P(t)\left(e^{s-t}P(s)x + e^{t-s}Q(t)x\right) \\ &= e^{s-t}P(t)P(s)x + e^{t-s}P(t)Q(t)x \\ &= e^{s-t}P(s)x. \end{aligned}$$

□

Corollary 3.12. *From the above proposition, we can state that for all $(t, s) \in \Delta$ we have:*

(i) $U_P(t, s)Q(s) = Q(t)U_P(t, s)$;
(ii) $U_P(t, s)\mathcal{H} \subset \mathcal{H}$;
(iii) $U_P(t, s)\mathcal{K}(s) \subset \mathcal{K}(t)$.

Proposition 3.13. *For all $(t, s) \in \Delta$ the restriction $U_P(t, s)_| : \mathcal{K}(s) \to \mathcal{K}(t)$ is an isomorphism.*

Proof. Let $(t, s) \in \Delta$. To prove the injectivity of $U_P(t, s)_|$, let $y \in \mathcal{K}(s)$ satisfying $U_P(t, s)_| y = 0$. Using the definition of $\mathcal{K}(s)$, we have that there exists $x \in l^\infty(\mathbb{N}^*, \mathbb{R})$ with $y = Q(s)x$. It follows that

$$U_P(t, s)_| y = U_P(t, s)Q(s)x = e^{t-s}Q(t)x$$

which implies $Q(t)x = 0$, so $P(t)x = x$. Hence $y = Q(s)x = Q(s)P(t)x = 0$.
To prove the surjectivity of the operator, let $z \in \mathcal{K}(t)$. It follows that there exists $y \in l^\infty(\mathbb{N}^*, \mathbb{R})$ with $z = Q(t)y$. Let $x = e^{s-t}Q(s)y \in \mathcal{K}(s)$. Then

$$U_P(t, s)_| x = e^{t-s}Q(t)x = e^{t-s}e^{s-t}Q(t)Q(s)y = Q(t)y = z.$$

□

For $t \geq 0$ we will refer to $\mathcal{H}$ and $\mathcal{K}(t)$ as to the *stable* and *unstable* subspaces at time t respectively.

Proposition 3.14. *There exist constants $N, \beta > 0$ such that*

$$\|U_P(t,s)P(s)x\|_\infty \leq Ne^{-\beta(t-s)}\|P(s)x\|_\infty, \quad \forall (t,s,x) \in \Delta \times l^\infty(\mathbb{N}^*, \mathbb{R}).$$

Proof. Choose $N = \beta = 1$. By Proposition 3.11 we have that for $(t,s,x) \in \Delta \times l^\infty(\mathbb{N}^*, \mathbb{R})$,

$$\|U_P(t,s)P(s)x\|_\infty = e^{s-t}\|P(s)x\|_\infty = Ne^{-\beta(t-s)}\|P(s)x\|_\infty. \qquad \square$$

Before stating the next result, we will need the following lemma, which gives us the sup-norm induced on $\mathcal{K}(t)$, $t \geq 0$.

Lemma 3.15. *For every $t \in \mathbb{R}_+$ and for every $x \in l^\infty(\mathbb{N}^*, \mathbb{R})$, we have that*

$$\|Q(t)x\|_\infty = \max\{1, e^t - 1\} \cdot \sup_{n\geq 1} |x_{2n}|.$$

Proof. It follows from Proposition 3.6, (iii). $\square$

Proposition 3.16. *There exist $N, \beta > 0$ such that*

$$\|U_P(t,s)Q(s)x\|_\infty \geq Ne^{\beta(t-s)}\|Q(s)x\|_\infty, \quad \forall (t,s,x) \in \Delta \times l^\infty(\mathbb{N}^*, \mathbb{R}).$$

Proof. Choose $N = \beta = 1$. Let $(t,s) \in \Delta$ and $x \in l^\infty(\mathbb{N}^*, \mathbb{R})$. We have that

$$\begin{aligned}\|U_P(t,s)Q(s)x\|_\infty &= e^{t-s}\|Q(t)x\|_\infty \\ &= e^{t-s}\max\{1, e^t - 1\}\sup_{n\geq 1}|x_{2n}| \\ &\geq e^{t-s}\max\{1, e^s - 1\}\sup_{n\geq 1}|x_{2n}| \\ &= e^{t-s}\|Q(s)x\|_\infty.\end{aligned}$$

$\square$

By synthesizing all of the above, we can state the following result which emphasizes the key properties of the evolution operator constructed in this section.

Theorem 3.17. *The following assertions hold:*

(i) $U_P : \Delta \to \mathcal{B}(l^\infty(\mathbb{N}^*, \mathbb{R}))$ *is a strongly continuous evolution operator on* $l^\infty(\mathbb{N}^*, \mathbb{R})$*;*

(ii) $U_P : \Delta \to \mathcal{B}(l^\infty(\mathbb{N}^*, \mathbb{R}))$ *has an exponential growth and* **does not have a uniform exponential growth***;*

(iii) (U_P, P) *is exponentially dichotomic.*

(iv) $\sup_{t\geq 0}\|P(t)\| = +\infty$.

Conclusion. In terms of Theorem 3.17, although the evolution operator U verifies all the conditions that makes it uniformly exponentially dichotomic, the property

$$\sup_{t\geq 0}\|P(t)\| < +\infty$$

fails, provided by the fact that the evolution operator does not admit a uniform exponential growth.

In the final part of this section, we will give a boundedness result of the dichotomic family of projections in the nonuniform case.

Theorem 3.18. *Let (U, P) be a dichotomy pair which is (e.d). If $U : \Delta \to \mathcal{B}(X)$ has an exponential growth then the family of projections $P : \mathbb{R}_+ \to \mathcal{B}(X)$ is exponentially bounded.*

Proof. Let $M, \omega > 0$ and $\varepsilon \geq 0$ given by the exponential growth and N, α, β given by the (e.d) property. Let $s \geq 0$, $x \in X$ and $t \geq s$. It follows that

$$(\|P(s)x\| - \|x\|) \frac{e^{\beta(t-s)}}{Ne^{\alpha s}} - Ne^{\alpha s}e^{-\beta(t-s)}\|P(s)x\|$$
$$\leq \frac{e^{\beta(t-s)}}{N}e^{-\alpha s}\|Q(s)x\| - Ne^{\alpha s}e^{-\beta(t-s)}\|P(s)x\|$$
$$\leq \|U(t,s)Q(s)x\| - \|U(t,s)P(s)x\| \leq \|U(t,s)x\| \leq Me^{\varepsilon s}e^{\omega(t-s)}\|x\|$$

from where

$$\left[\frac{e^{-\alpha s}e^{\beta(t-s)}}{N} - Ne^{\alpha s}e^{-\beta(t-s)}\right]\|P(s)x\| \leq Me^{\varepsilon s}e^{(\omega+\beta)(t-s)}\|x\|. \tag{3.8}$$

Consider

$$t = s + \frac{\alpha}{\beta}s + \frac{\ln N}{\beta} + 1 \geq s.$$

Then we have that

$$\frac{1}{N}e^{-\alpha s}e^{\beta(t-s)} = \frac{1}{N}e^{-\alpha s}Ne^{\beta}e^{\alpha s} = e^{\beta} \tag{3.9}$$

$$Ne^{\alpha s}e^{-\beta(t-s)} = Ne^{\alpha s}\frac{1}{N}e^{-\beta}e^{-\alpha s} = e^{-\beta} \tag{3.10}$$

$$e^{(\omega+\beta)(t-s)} = e^{\frac{\alpha(\omega+\beta)}{\beta}s} \cdot e^{(\omega+\beta)\left(\frac{\ln N}{\beta}+1\right)}. \tag{3.11}$$

By denoting

$$L = \frac{Me^{(\omega+\beta)\left(\frac{\ln N}{\beta}+1\right)}}{e^{\beta} - e^{-\beta}} \quad \text{şi} \quad \gamma = \frac{\alpha(\omega+\beta)}{\beta} + \varepsilon$$

from (3.9), (3.10), (3.11) and (3.8) we obtain that

$$\|P(s)x\| \leq Le^{\gamma s}\|x\|$$

which shows us that $P : \mathbb{R}_+ \to \mathcal{B}(X)$ is exponentially bounded. □

As it was expected, the result from [15] can be obtained using the above theorem, which is pointed out below.

Corollary 3.19. *Let (U, P) be a dichotomy pair which is (u.e.d). If $U : \Delta \to \mathcal{B}(X)$ has a uniform exponential growth then the family of projections $P : \mathbb{R}_+ \to \mathcal{B}(X)$ is bounded.*

Proof. It results from Theorem 3.18, by observing that if $\alpha = \varepsilon = 0$ then $\gamma = 0$. □

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Statistical convergence on probabilistic modular spaces

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Abstract. In this work, we introduce the concepts of statistical convergence and statistical Cauchy sequence on probabilistic modular spaces. After giving some useful characterizations for statistically convergent sequences, we display an example such that our method of convergence works but its classical case does not work. Also we define statistical limit points, statistical cluster points on probabilistic modular spaces. Finally, we give the relations between these notions and limit points of sequences on probabilistic modular spaces.

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Keywords: natural density, statistical convergence, modular space, probabilistic modular space.

1. Introduction

The theory of modular spaces was firstly presented by Nakano [13] and subsequently generalized by Musielak and Orlicz [10]. Later many researchers have investigated these spaces in [7, 8]. Also, Menger [9] has introduced the concept of probabilistic metric space which is an interesting and important generalization of the notion of a metric space. The probabilistic generalization of metric space appears when there is an uncertainty about the distance between the points and we know only the probabilities of possible values this distance can take. According to this work instead of associating a number−the distance $d(x,y)$ −to every pair (x,y), one should associate a distribution function $N_{x,y}$ and for any positive real number t, interpret $N_{x,y}(t)$ as the probability that the distance from x to y is less than t. An important family of probabilistic metric spaces are probabilistic normed spaces. For more details, the reader is referred to [1, 15]. After Menger's work, Fallahi and Nourouzi [3] have introduced probabilistic modular spaces in the probabilistic sense which are more general than probabilistic normed spaces and they investigated some basic properties of these spaces.

The concept of statistical convergence for sequences of real numbers was introduced by Fast [6]. Later on some generalizations and applications of this notion have been investigated by many authors [2, 4, 5, 11, 12]. Karakuş studied the concept of statistical convergence in probabilistic normed spaces [14]. In this paper we study the properties of the sequences which are statistically convergent in a probabilistic modular space. Also we define statistical limit points and statistical cluster points in a probabilistic modular space and prove some interesting results.

We recall some notations and basic definitions used in this paper:

A functional $\rho : X \to [0, +\infty]$ is said to be a *modular* on a real linear space X provided that the following conditions hold:

(i) $\rho(x) = 0$ iff x is the null vector θ,

(ii) $\rho(x) = \rho(-x)$,

(iii) $\rho(\alpha x + \beta y) \leq \rho(x) + \rho(y)$ for every $x, y \in X$ and for any $\alpha, \beta \geq 0$ with $\alpha + \beta = 1$.

Then the vector subspace $X_\rho = \{x \in X : \rho(ax) \to 0 \text{ as } a \to 0\}$ of X is called a *modular space.*

If A is a subset of $\mathbb{N}$, the set of natural numbers, then the *natural density* of A denoted by $\delta(A)$, is defined by

$$\delta(A) := \lim_n \frac{1}{n} |\{k \leq n : k \in A\}|$$

whenever the limit exists, where $|A|$ denotes the cardinality of the set A. The natural density may not exist for each set A. But the upper density $\bar{\delta}$ always exists for each set A identified as follows:

$$\bar{\delta}(A) := \limsup_n \frac{1}{n} |\{k \leq n : k \in A\}| .$$

A sequence $x = \{x_k\}$ of numbers is *statistically convergent* to L if

$$\delta(\{k \in \mathbb{N} : |x_k - L| \geq \varepsilon\}) = 0$$

for every $\varepsilon > 0$. In this case we write $st - \lim x = L$.

Note that every convergent sequence is statistically convergent to the same value. If x is statistically convergent, then x needs not to be convergent. It is also not necessarily bounded. For example, let $x = \{x_k\}$ be defined as

$$x_k := \begin{cases} \sqrt{k}, & \text{if } k \text{ is a square} \\ 1, & \text{otherwise.} \end{cases}$$

It is easy to see that $st - \lim x = 1$. But x is neither convergent nor bounded.

Definition 1.1. *A function $f : \mathbb{R} \to \mathbb{R}_0^+$ is called a distribution function if it is non-decreasing and left-continuous with* $\inf_{t \in \mathbb{R}} f(t) = 0$, *and* $\sup_{t \in \mathbb{R}} f(t) = 1$.

We will denote the set of all distribution functions by $\mathcal{D}$.

Definition 1.2. *A triangular norm, briefly called t-norm, is a binary operation on* $[0, 1]$ *which is continuous, commutative, associative, non-decreasing and has 1 as a neutral element, i.e., it is a continuous mapping* $\wedge : [0, 1] \times [0, 1] \to [0, 1]$ *such that for all* $a, b, c \in [0, 1]$:

1. $a \wedge 1 = a$,

2. $a \wedge b = b \wedge a$,
3. $c \wedge d \geq a \wedge b$ *if* $c \geq a$ *and* $d \geq b$,
4. $(a \wedge b) \wedge c = a \wedge (b \wedge c)$.

Definition 1.3. *A pair* (X, μ) *is called a probabilistic modular space* ($\mathcal{P}-$*modular space) if* X *is a real vector space,* μ *is a mapping from* X *into* $\mathcal{D}$ *(for* $x \in X$*, the function* $\mu(x)$ *is denoted by* μ_x*, and* $\mu_x(t)$ *is the value* μ_x *at* $t \in \mathbb{R}$*) satisfying the following conditions:*

1. $\mu_x(0) = 0$,
2. $\mu_x(t) = 1$ *for all* $t > 0$ *iff* $x = 0$,
3. $\mu_{-x}(t) = \mu_x(t)$,
4. $\mu_{\alpha x+\beta y}(s+t) \geq \mu_x(s) \wedge \mu_y(t)$ *for all* $x, y \in X$, *and* $\alpha, \beta, s, t \in \mathbb{R}_0^+$, $\alpha + \beta = 1$.

We say (X, μ) is $\beta-$*homogeneous*, where $\beta \in (0, 1]$ if,

$$\mu_{\alpha x}(t) = \mu_x\left(\frac{t}{|\alpha|^\beta}\right),$$

for every $x \in X$, $t > 0$, and $\alpha \in \mathbb{R} \setminus \{0\}$.

Example 1.4. Suppose that X is a real vector space and ρ is a modular on X. Define

$$\mu_x(t) = \begin{cases} 0, & t \leq 0, \\ \frac{t}{t+\rho(x)}, & t > 0, \end{cases}$$

for all $x \in X$. Then (X, μ) is a $\mathcal{P}-$modular space.

We recall that the concept of convergence and Cauchy sequence in a probabilistic modular space are studied in [3].

Definition 1.5. *Let* (X, μ) *be a* $\mathcal{P}-$*modular space.*

- *A sequence* $\{x_k\}$ *in* X *is said to be* $\mu-$*convergent to a point* $L \in X$ *and denoted by* $x_k \overset{\mu}{\to} L$ *or* $\mu - \lim x = L$, *if for every* $\varepsilon > 0$ *and* $\lambda \in (0, 1)$, *there exists a positive integer* k_0 *such that* $\mu_{x_k - L}(\varepsilon) > 1 - \lambda$, *for all* $k \geq k_0$.
- *A sequence* $\{x_k\}$ *in* X *is called a* $\mu-$*Cauchy sequence if for every* $\varepsilon > 0$ *and* $\lambda \in (0, 1)$, *there exists a positive integer* k_0 *such that* $\mu_{x_k - x_l}(\varepsilon) > 1 - \lambda$, *for all* $k, l \geq k_0$.
- *A subset* F *of* X *is said to be* $\mu-$*bounded if for every* $\lambda \in (0, 1)$, *there exists* $t > 0$ *such that* $\mu_x(t) > 1 - \lambda$ *for all* $x \in F$.
- *For* $x \in X$, $\varepsilon > 0$ *and* $0 < \lambda < 1$, *the ball centered at* x *with radius* λ *is defined by*

$$B(x, \lambda, \varepsilon) = \{y \in X : \mu_{x-y}(\varepsilon) > 1 - \lambda\}.$$

Remark 1.6. Let (X, μ) be a $\mathcal{P}-$modular space, and $\mu_x(t) = \frac{t}{t+\rho(x)}$, where $x \in X$ and $t \geq 0$. Then it can be easily seen that $x_n \overset{\rho}{\to} x$ if and only if $x_n \overset{\mu}{\to} x$.

2. Statistical convergence on $\mathcal{P}-$modular spaces

In this work we deal with the statistical convergence on probabilistic modular spaces. Now, we may obtain our main results.

Definition 2.1. *Let (X, μ) be a $\mathcal{P}-$modular space.*

- *We say that a sequence $x = \{x_k\}$ is statistically convergent to $L \in X$ with respect to the probabilistic modular μ (or briefly $st_\mu-$convergent) provided that for every $\varepsilon > 0$ and $\lambda \in (0,1)$*
$$\delta\left(\left\{k \in \mathbb{N} : \mu_{x_k - L}\left(\varepsilon\right) \leq 1 - \lambda\right\}\right) = 0, \tag{2.1}$$
or equivalently,
$$\lim_n \frac{1}{n}\left|\left\{k \leq n : \mu_{x_k - L}\left(\varepsilon\right) \leq 1 - \lambda\right\}\right| = 0$$
and denoted by $st_\mu - \lim x = L$.
- *We say that a sequence $x = \{x_k\}$ is a statistical Cauchy sequence with respect to the probabilistic modular μ (or briefly $st_\mu-$Cauchy) provided that for every $\varepsilon > 0$ and $\lambda \in (0,1)$, there is a positive integer $N = N(\varepsilon)$ such that*
$$\delta\left(\left\{k \in \mathbb{N} : \mu_{x_k - x_N}\left(\varepsilon\right) \leq 1 - \lambda\right\}\right) = 0,$$
or equivalently,
$$\lim_n \frac{1}{n}\left|\left\{k \leq n : \mu_{x_k - x_N}\left(\varepsilon\right) \leq 1 - \lambda\right\}\right| = 0.$$

By using (2.1) and well-known density properties, we have the following lemma.

Lemma 2.2. *Let (X, μ) be a $\mathcal{P}-$modular space.Then, for every $\varepsilon > 0$ and $\lambda \in (0,1)$, the following statements are equivalent:*

(i) $st_\mu - \lim x = L$,
(ii) $\delta\left(\left\{k \in \mathbb{N} : \mu_{x_k - L}\left(\varepsilon\right) \leq 1 - \lambda\right\}\right) = 0$,
(iii) $\delta\left(\left\{k \in \mathbb{N} : \mu_{x_k - L}\left(\varepsilon\right) > 1 - \lambda\right\}\right) = 1$,
(iv) $st - \lim \mu_{x_k - L}\left(\varepsilon\right) = 1$.

Theorem 2.3. *Let (X, μ) be a $\mathcal{P}-$modular space. If a sequence $x = \{x_k\}$ is $st_\mu-$convergent, then the $st_\mu-$limit is unique.*

Proof. Assume that $st_\mu - \lim x = L_1$, $st_\mu - \lim x = L_2$ and $\lambda \in (0,1)$. Choose $\eta \in (0,1)$ such that $(1-\eta) \wedge (1-\eta) > 1 - \lambda$. Then, for any $\varepsilon > 0$, define the following sets:

$$\begin{aligned} T_{\mu,1}\left(\eta, \varepsilon\right) &: = \left\{k \in \mathbb{N} : \mu_{x_k - L_1}\left(\varepsilon\right) \leq 1 - \eta\right\}, \\ T_{\mu,2}\left(\eta, \varepsilon\right) &: = \left\{k \in \mathbb{N} : \mu_{x_k - L_2}\left(\varepsilon\right) \leq 1 - \eta\right\}. \end{aligned}$$

Since $st_\mu - \lim x = L_1$, $\delta\left(T_{\mu,1}\left(\eta, \varepsilon\right)\right) = 0$ for all $\varepsilon > 0$. Also because of $st_\mu - \lim x = L_2$, we get $\delta\left(T_{\mu,2}\left(\eta, \varepsilon\right)\right) = 0$ for all $\varepsilon > 0$. Let $T_\mu\left(\eta, \varepsilon\right) = T_{\mu,1}\left(\eta, \varepsilon\right) \cap T_{\mu,2}\left(\eta, \varepsilon\right)$.

Then it can be easily seen that $\delta\left(T_{\mu}\left(\eta,\varepsilon\right)\right)=0$ which implies $\delta\left(\mathbb{N}/T_{\mu}\left(\eta,\varepsilon\right)\right)=1$. If $k\in\mathbb{N}/T_{\mu}\left(\eta,\varepsilon\right)$, then we get

$$\begin{aligned}\mu_{\frac{1}{2}(L_1-L_2)}\left(\varepsilon\right) &= \mu_{\frac{1}{2}(x_k-L_1)+\frac{1}{2}(L_2-x_k)}\left(\varepsilon\right)\\ &\geq \mu_{x_k-L_1}\left(\frac{\varepsilon}{2}\right)\wedge\mu_{x_k-L_2}\left(\frac{\varepsilon}{2}\right)\\ &> (1-\eta)\wedge(1-\eta).\end{aligned}$$

Because of $(1-\eta)\wedge(1-\eta)>1-\lambda$, it follows that

$$\mu_{L_1-L_2}\left(\varepsilon\right)>1-\lambda. \tag{2.2}$$

Since $\lambda>0$ is arbitrary, by (2.2) we have $\mu_{L_1-L_2}\left(\varepsilon\right)=1$ for all $\varepsilon>0$. This implies that $L_1=L_2$. □

Theorem 2.4. *Let (X,μ) be a $\beta-$homogeneous $\mathcal{P}-$modular space. If $\{x_k\}$ is a $st_\mu-$Cauchy sequence, possessing a subsequence which is $st_\mu-$convergent, then $\{x_k\}$ is $st_\mu-$convergent to the same limit.*

Proof. For a given $\lambda>0$ choose $\eta\in(0,1)$ such that $(1-\eta)\wedge(1-\eta)>1-\lambda$. Because of $\{x_k\}$ is a $st_\mu-$*Cauchy* sequence , there exists $t_1\in\mathbb{N}$, a positive integer $N=N(\varepsilon)$ and subset A_1 of density 1 such that $\mu_{x_k-x_N}\left(\frac{\varepsilon}{2^{\beta+1}}\right)>1-\eta$ holds for all $\varepsilon>0$, $k\in A_1$and $k\geq t_1$. Let $\{x_{k_i}\}$ be a subsequence of $\{x_k\}$ which is $st_\mu-$*convergent to* $L\in X$, then there exists $t_2\in\mathbb{N}$ and subset A_2 of density 1 such that $\mu_{x_{k_i}-L}\left(\frac{\varepsilon}{2^{\beta+1}}\right)>1-\eta$ holds for all $\varepsilon>0$, $k_i\in A_2$ and $k_i\geq t_2$. Take $t_0=\max\{t_1,t_2\}$ and $A=A_1\cap A_2$, then $\delta\left(A\right)=1$ and for all $\varepsilon>0$, $k\in A$ and $k\geq t_0$ we have

$$\begin{aligned}\mu_{x_k-L}\left(\varepsilon\right) &\geq \mu_{2\left(x_k-x_{k_i}\right)}\left(\frac{\varepsilon}{2}\right)\wedge\mu_{2\left(x_{k_i}-L\right)}\left(\frac{\varepsilon}{2}\right)\\ &= \mu_{x_k-x_{k_i}}\left(\frac{\varepsilon}{2^{\beta+1}}\right)\wedge\mu_{x_{k_i}-L}\left(\frac{\varepsilon}{2^{\beta+1}}\right)\\ &> (1-\eta)\wedge(1-\eta)>1-\lambda.\end{aligned}$$

That is $st_\mu-\lim x=L$. □

Theorem 2.5. *Let (X,μ) be a $\beta-$homogeneous $\mathcal{P}-$modular space. Then every $st_\mu-$convergent sequence is also a $st_\mu-$Cauchy sequence.*

Proof. Suppose that $\{x_k\}$ is a $st_\mu-$*convergent to* $L\in X$. Let $\lambda\in(0,1)$ and choose $\eta\in(0,1)$ such that $(1-\eta)\wedge(1-\eta)>1-\lambda$. There exists $k_0\in\mathbb{N}$ and subset A of density 1 such that $\mu_{x_k-L}\left(\frac{\varepsilon}{2^{\beta+1}}\right)>1-\eta$ holds for all $\varepsilon>0$, $k\in A$ and $k\geq k_0$. If $N=N(\varepsilon)$ is a positive integer,

$$\begin{aligned}\mu_{x_k-x_N}\left(\varepsilon\right) &\geq \mu_{2(x_N-L)}\left(\frac{\varepsilon}{2}\right)\wedge\mu_{2(x_k-L)}\left(\frac{\varepsilon}{2}\right)\\ &= \mu_{x_N-L}\left(\frac{\varepsilon}{2^{\beta+1}}\right)\wedge\mu_{x_k-L}\left(\frac{\varepsilon}{2^{\beta+1}}\right)\\ &> (1-\eta)\wedge(1-\eta)>1-\lambda,\end{aligned}$$

for every $\varepsilon>0$, $k\in A$ and $k\geq k_0$. That is $\{x_k\}$ is a $st_\mu-$*Cauchy* sequence. □

Theorem 2.6. *Let (X, μ) be a $\mathcal{P}-$modular space. If $x = \{x_k\}$ is a $\mu-$convergent sequence, then x is also a $st_\mu-$convergent sequence.*

Proof. Since x is $\mu-$*convergent*, for every $\varepsilon > 0$ and $\lambda \in (0,1)$, there is a number $k_0 \in \mathbb{N}$ such that $\mu_{x_k - L} > 1 - \lambda$ for all $k \geq k_0$. So the set $\{k \in \mathbb{N} : \mu_{x_k - L}(\varepsilon) \leq 1 - \lambda\}$ has at most finitely many terms. Because every finite subset of the natural numbers has density zero, we can easily see that $\delta(\{k \in \mathbb{N} : \mu_{x_k - L}(\varepsilon) \leq 1 - \lambda\}) = 0$, which completes the proof. □

Example 2.7. Define $\rho : \mathbb{R} \to \mathbb{R}$ by

$$\rho(x) = \begin{cases} 0, & x = 0 \\ 1, & x \neq 0 \end{cases}$$

for all $x \in \mathbb{R}$. Then $(\mathbb{R}, \rho)$ is a modular space. Let $a \wedge b = ab$ and $\mu_x(t) = \frac{t}{t + \rho(x)}$, where $x \in X$ and $t \geq 0$. Observe that $(\mathbb{R}, \mu)$ is a $\beta-$*homogeneous* $\mathcal{P}-$modular space. Now we define the sequence $x = \{x_k\}$ whose terms are given by

$$x_k := \begin{cases} 1, & \text{if } k = m^2 \ (m \in \mathbb{N}), \\ 0, & \text{otherwise.} \end{cases} \tag{2.3}$$

Then for *any* $\varepsilon > 0$ and for *every* $\lambda \in (0,1)$, let

$$T_\mu(\lambda, \varepsilon) := \{k \leq n : \mu_{x_k}(\varepsilon) \leq 1 - \lambda\}.$$

Since

$$\begin{aligned} T_\mu(\lambda, \varepsilon) &= \left\{k \leq n : \frac{\varepsilon}{\varepsilon + \rho(x_k)} \leq 1 - \lambda\right\} \\ &= \left\{k \leq n : \rho(x_k) \geq \frac{\lambda\varepsilon}{1 - \lambda} > 0\right\} \\ &= \{k \leq n : x_k = 1\} \\ &= \left\{k \leq n : k = m^2 \text{ and } m \in \mathbb{N}\right\}, \end{aligned}$$

we get

$$\frac{1}{n}|T_\mu(\lambda, \varepsilon)| \leq \frac{1}{n}\left|\left\{k \leq n : k = m^2 \text{ and } m \in \mathbb{N}\right\}\right| \leq \frac{\sqrt{n}}{n}$$

that is

$$\lim_n \frac{1}{n}|T_\mu(\lambda, \varepsilon)| = 0.$$

So, we have $st_\mu - \lim x = 0$. But, because $x = \{x_k\}$ given by (2.3) is not convergent in the space $(\mathbb{R}, \rho)$, by Remark 1.6, we also see that x is not convergent with respect to the probabilistic modular μ.

Theorem 2.8. *Let (X, μ) be a $\mathcal{P}-$modular space. Let $st_\mu - \lim x = L$ if and only if there exists an increasing index sequence $T = \{k_n\}_{n \in \mathbb{N}}$ of natural numbers such that $\delta(T) = 1$ and $\mu - \lim_{n \in T} x_n = L$, i.e., $\mu - \lim_n x_{k_n} = L$.*

Proof. Necessity: First suppose that $st_\mu - \lim x = L$. Then, for every $\varepsilon > 0$ and $j \in \mathbb{N}$, let

$$T(j,\varepsilon) := \left\{ n \in \mathbb{N} : \mu_{x_n - L}(\varepsilon) > 1 - \frac{1}{j} \right\}$$

Observe that, for $\varepsilon > 0$ and $j \in \mathbb{N}$,

$$T(j+1,\varepsilon) \subset T(j,\varepsilon). \tag{2.4}$$

Because $st_\mu - \lim x = L$, we can write

$$\delta(T(j,\varepsilon)) = 1, \quad (\varepsilon > 0 \text{ and } j \in \mathbb{N}). \tag{2.5}$$

Let r_1 be an arbitrary number of $T(1,\varepsilon)$. Then, by (2.5), there is a number $r_2 \in T(2,\varepsilon)$, $(r_2 > r_1)$, such that, for all $n \geq r_2$,

$$\frac{1}{n} \left| \left\{ k \leq n : \mu_{x_k - L}(\varepsilon) > 1 - \frac{1}{2} \right\} \right| > \frac{1}{2}.$$

Further, by (2.5), there is a number $r_3 \in T(3,\varepsilon)$, $(r_3 > r_2)$, such that, for all $n \geq r_3$,

$$\frac{1}{n} \left| \left\{ k \leq n : \mu_{x_k - L}(\varepsilon) > 1 - \frac{1}{3} \right\} \right| > \frac{2}{3},$$

and so on. Hence, by induction we can construct an increasing index sequence $\{r_j\}_{j\in\mathbb{N}}$ of natural numbers such that $r_j \in T(j,\varepsilon)$ and such that the following statement holds for all $n \geq r_j$ $(j \in \mathbb{N})$:

$$\frac{1}{n} \left| \left\{ k \leq n : \mu_{x_k - L}(\varepsilon) > 1 - \frac{1}{j} \right\} \right| > \frac{j-1}{j}. \tag{2.6}$$

Now we set the increasing index sequence T as follows:

$$T := \{n \in \mathbb{N} : 1 < n < r_1\} \cup \left\{ \bigcup_{j\in\mathbb{N}} \{n \in T(j,\varepsilon) : r_j \leq n < r_{j+1}\} \right\}. \tag{2.7}$$

Then by (2.4), (2.6) and (2.7) we conclude, for all n, $(r_j \leq n < r_{j+1})$, that

$$\frac{1}{n} |\{k \leq n : k \in T\}| \geq \frac{1}{n} \left| \left\{ k \leq n : \mu_{x_k - L}(\varepsilon) > 1 - \frac{1}{j} \right\} \right| > 1 - \frac{1}{j}.$$

Therefore it follows that $\delta(T) = 1$. Now choose a number $j \in \mathbb{N}$ and let $\varepsilon > 0$ such that $\frac{1}{j} < \varepsilon$. Suppose that $n \geq v_j$ and $n \in T$. Then, from the definition of T, there exists a number $m \geq j$ such that $v_m \leq n < v_{m+1}$ and $n \in T(j,\varepsilon)$. Hence, we get, for every $\varepsilon > 0$,

$$\mu_{x_n - L}(\varepsilon) > 1 - \frac{1}{j} > 1 - \varepsilon$$

for all $n \geq v_j$ and $n \in T$, which implies

$$\mu - \lim_{n\in T} x_n = L.$$

This completes the proof of necessity.

Sufficiency: Assume that there exists an increasing index sequence $T = \{k_n\}_{n\in\mathbb{N}}$ such that $\delta(T) = 1$ and $\mu - \lim_{n\in T} x_n = L$. Now, for any $\varepsilon > 0$ and $\lambda \in (0,1)$, there is a number n_0 such that for each $n \geq n_0$ the inequality $\mu_{x_n - L}(\varepsilon) > 1 - \lambda$ holds. Now define $S(\lambda, \varepsilon) := \{n \in \mathbb{N} : \mu_{x_n - L}(\varepsilon) \leq 1 - \lambda\}$. Then we have

$$S(\lambda, \varepsilon) \subset \mathbb{N} - \{k_{n_0}, k_{n_0+1}, k_{n_0+2}, ...\}.$$

Since $\delta(T) = 1$, we get $\delta(\mathbb{N} - \{k_{n_0}, k_{n_0+1}, k_{n_0+2}, ...\}) = 0$, which yields that $\delta(S(\lambda, \varepsilon)) = 0$.

Hence, we get $st_\mu - \lim x = L$. □

By a similar technique as in the above theorem one can get the following result at once.

Theorem 2.9. *Let (X, μ) be a $\mathcal{P}-$modular space and $x = \{x_k\}$ is a sequence in X. The following statements are equivalent:*

- (a) *x is a $st_\mu-$Cauchy sequence.*
- (b) *There exists an increasing index sequence $T = \{k_n\}$ of natural numbers such that $\delta(T) = 1$ and the subsequence $\{x_{k_n}\}_{n \in \mathbb{N}}$ is a $\mu-$Cauchy sequence.*

Remark 2.10. If $st_\mu - \lim_n x_n = L$, then there exists a sequence $y = \{y_n\}$ such that $\mu - \lim_n y_n = L$ and $\delta(\{n \in \mathbb{N} : x_n = y_n\}) = 1$.

Now, we show that statistical convergence on a $\mathcal{P}-$modular space has some properties similar to the properties of the usual convergence on $\mathbb{R}$.

Lemma 2.11. *Let (X, μ) be a $\beta-$homogeneous $\mathcal{P}-$modular space.*

1. *If $st_\mu - \lim x = L_1$ and $st_\mu - \lim y = L_2$,*
   then $st_\mu - \lim (x + y) = L_1 + L_2$.
2. *If $st_\mu - \lim x = L$ and $\alpha \in \mathbb{R}$, then $st_\mu - \lim \alpha x = \alpha L$.*
3. *If $st_\mu - \lim x = L_1$ and $st_\mu - \lim y = L_2$,*
   then $st_\mu - \lim (x - y) = L_1 - L_2$.

Proof. (1) Let $st_\mu - \lim x = L_1$, $st_\mu - \lim y = L_2$ and $\lambda \in (0, 1)$. There exists $\eta \in (0, 1)$ such that $(1 - \eta) \wedge (1 - \eta) > 1 - \lambda$. There exists $k_1 \in \mathbb{N}$ and subset A_1 of density 1 such that $\mu_{x_k - L}\left(\frac{\varepsilon}{2^{\beta+1}}\right) > 1 - \eta$ holds for all $\varepsilon > 0$, $k \in A_1$ and $k \geq k_1$. Also, there exists $k_2 \in \mathbb{N}$ and subset A_2 of density 1 such that $\mu_{y_k - L_2}\left(\frac{\varepsilon}{2^{\beta+1}}\right) > 1 - \eta$ holds for all $\varepsilon > 0$, $k \in A_2$ and $k \geq k_2$. Take $k_0 = \max\{k_1, k_2\}$ and $A = A_1 \cap A_2$, then $\delta(A) = 1$ and for every $\varepsilon > 0$, $k \in A$ and $k \geq k_0$ we have

$$\begin{aligned} \mu_{(x_k - L_1) + (y_k - L_2)}(\varepsilon) &\geq \mu_{2(x_k - L_1)}\left(\frac{\varepsilon}{2}\right) \wedge \mu_{2(y_k - L_2)}\left(\frac{\varepsilon}{2}\right) \\ &= \mu_{x_k - L_1}\left(\frac{\varepsilon}{2^{\beta+1}}\right) \wedge \mu_{y_k - L_2}\left(\frac{\varepsilon}{2^{\beta+1}}\right) \\ &> (1 - \eta) \wedge (1 - \eta) > 1 - \lambda, \end{aligned}$$

That is $st_\mu - \lim (x + y) = L_1 + L_2$.

(2) Let $st_\mu - \lim x = L$, $\varepsilon > 0$, $\lambda > 0$. We may assume that $\alpha = 0$. In this case

$$\mu_{0x_k - 0L}(\varepsilon) = \mu_0(\varepsilon) = 1 > 1 - \lambda.$$

So we get $\mu - \lim x_k = 0$. Then from Theorem 2.5 we have $st_\mu - \lim x_k = 0$.

Now we consider the case of $\alpha \in \mathbb{R}\,(\alpha \neq 0)$. Since $st_{\mu} - \lim x = L$, if we define the set

$$T_{\mu}(\lambda, \varepsilon) := \{k \in \mathbb{N} : \mu_{x_k - L}(\varepsilon) \leq 1 - \lambda\}$$

then we can say $\delta(T_{\mu}(\lambda, \varepsilon)) = 0$ for all $\varepsilon > 0$ which implies $\delta(\mathbb{N}/T_{\mu}(\lambda, \varepsilon)) = 1$. If $k \in \mathbb{N}/T_{\mu}(\lambda, \varepsilon)$, then we get

$$\begin{aligned}
\mu_{\alpha x_k - \alpha L}(\varepsilon) &= \mu_{x_k - L}\left(\frac{\varepsilon}{|\alpha|}\right) \\
&\geq \mu_{x_k - L}(\varepsilon) \wedge \mu_0\left(\frac{\varepsilon}{|\alpha|} - \varepsilon\right) \\
&= \mu_{x_k - L}(\varepsilon) \wedge 1 \\
&= \mu_{x_k - L}(\varepsilon) > 1 - \lambda
\end{aligned}$$

for $\alpha \in \mathbb{R}\,(\alpha \neq 0)$. That is

$$\delta(\{k \in \mathbb{N} : \mu_{\alpha x_k - \alpha L}(\varepsilon) \leq 1 - \lambda\}) = 0.$$

So $st_{\mu} - \lim \alpha x = \alpha L$.

(3) The proof is clear from (1) and (2). □

Theorem 2.12. *Let (X, μ) be a $\mathcal{P}-$modular space and $S_b^{\mu}(X)$ the space of bounded statistically convergent sequences on the $\mathcal{P}-$modular space. Then the set $S_b^{\mu}(X)$ is a closed linear subspace of the set $l_{\infty}^{\mu}(X)$.*

Proof. It is clear that $S_b^{\mu}(X) \subset \overline{S_b^{\mu}(X)}$. Now we show that $\overline{S_b^{\mu}(X)} \subset S_b^{\mu}(X)$. Let $y \in \overline{S_b^{\mu}(X)}$, then because of $B(y, \lambda, \varepsilon) \cap S_b^{\mu}(X) \neq \varnothing$, there is an $x \in B(y, \lambda, \varepsilon) \cap S_b^{\mu}(X)$.

Let $\varepsilon > 0$ and for a given $\eta > 0$ choose $\lambda \in (0, 1)$ such that $(1 - \lambda) \wedge (1 - \lambda) > 1 - \eta$. Since $x \in B(y, \lambda, \varepsilon) \cap S_b^{\mu}(X)$, there is a set $T \subseteq \mathbb{N}$ with $\delta(T) = 1$ such that

$$\mu_{y_n - x_n}\left(\frac{\varepsilon}{2}\right) > 1 - \lambda \text{ and } \mu_{x_n}\left(\frac{\varepsilon}{2}\right) > 1 - \lambda$$

for all $n \in T$. Then we have

$$\begin{aligned}
\mu_{y_n}(\varepsilon) &= \mu_{y_n - x_n + x_n}(\varepsilon) \\
&\geq \mu_{y_n - x_n}\left(\frac{\varepsilon}{2}\right) \wedge \mu_{x_n}\left(\frac{\varepsilon}{2}\right) \\
&> 1 - \eta,
\end{aligned}$$

for all $n \in T$. Therefore

$$\delta(\{n \in T : \mu_{y_n}(\varepsilon) > 1 - \eta\}) = 1$$

and so $y \in S_b^{\mu}(X)$. □

3. Statistical limit points and statistical cluster points on $\mathcal{P}-$modular spaces

The concepts of statistical limit points and statistical cluster points of real number sequences were given by Fridy in 1993 [5]. Also, he gives relations between them and the set of ordinary limit points. In this section we study the analogues of these notions on probabilistic modular spaces.

Definition 3.1. *Let (X,μ) be a $\mathcal{P}-$modular space. Then $l\in X$ is called a limit point of the sequence $x=\{x_k\}$ with respect to the probabilistic modular μ (or $\mu-$limit point) if there is a subsequence of x that converges to l with respect to the probabilistic modular μ. The set of all limit points of the sequence x is denoted by $L_\mu(x)$.*

Definition 3.2. *Let (X,μ) be a $\mathcal{P}-$modular space. If $\{x_{k(j)}\}$ is a subsequence of $x=\{x_k\}$ and $K:=\{k(j)\in\mathbb{N}:j\in\mathbb{N}\}$ then we abbreviate $\{x_{k(j)}\}$ by $\{x\}_K$ in this case $\delta(K)=0$. $\{x\}_K$ is called a thin subsequence or subsequence of density zero. Additionally, $\{x\}_K$ is a nonthin subsequence of x if K does not have density zero.*

Definition 3.3. *Let (X,μ) be a $\mathcal{P}-$modular space. $L\in X$ is called a statistical limit point of the sequence $x=\{x_k\}$ with respect to the probabilistic modular μ (or $st_\mu-$limit point) if there is a nonthin subsequence of x that converges to L with respect to the probabilistic modular μ and we say L is a $st_\mu-$limit point of the sequence $x=\{x_k\}$. The set of all $st_\mu-$limit points of the sequence x is denoted by $\Lambda_\mu(x)$.*

Definition 3.4. *Let (X,μ) be a $\mathcal{P}-$modular space. Then $\gamma\in X$ is called a statistical cluster point of the sequence $x=\{x_k\}$ with respect to the probabilistic modular μ (or $st_\mu-$cluster point) if for all $\varepsilon>0$ and $\lambda\in(0,1)$*

$$\overline{\delta}\left(\{k\in\mathbb{N}:\mu_{x_k-\gamma}(\varepsilon)>1-\lambda\}\right)>0.$$

In this case we say that γ is a $st_\mu-$cluster point of the sequence $x=\{x_k\}$. The set of all $st_\mu-$cluster points of the sequence x is denoted by $\Gamma_\mu(x)$.

Theorem 3.5. *Let (X,μ) be a $\mathcal{P}-$modular space. For any sequence $x\in X$ it holds $\Lambda_\mu(x)\subset\Gamma_\mu(x)$.*

Proof. Assume $L\in\Lambda_\mu(x)$, then there is a nonthin subsequence $\{x_{k(j)}\}$ of $x=\{x_k\}$ that is $st_\mu-$*convergent* to L, i.e. for all $\varepsilon>0$ and $\lambda\in(0,1)$

$$\delta\left(\left\{k(j)\in\mathbb{N}:\mu_{x_{k(j)}-L}(\varepsilon)>1-\lambda\right\}\right)=d>0.$$

So

$$\{k\in\mathbb{N}:\mu_{x_k-L}(\varepsilon)>1-\lambda\}\supset\left\{k(j)\in\mathbb{N}:\mu_{x_{k(j)}-L}(\varepsilon)>1-\lambda\right\},$$

we have

$$\begin{aligned}\{k\in\mathbb{N}:\mu_{x_k-L}(\varepsilon)>1-\lambda\} \quad\supset\quad & \{k(j)\in\mathbb{N}:j\in\mathbb{N}\}\\ & \setminus\left\{k(j)\in\mathbb{N}:\mu_{x_{k(j)}-L}(\varepsilon)\le 1-\lambda\right\}.\end{aligned}$$

Since $\{x_{k(j)}\}$ is $st_\mu-$*convergent* to L, the set

$$\left\{k(j)\in\mathbb{N}:\mu_{x_{k(j)}-L}(\varepsilon)\le 1-\lambda\right\}$$

is finite for every $\varepsilon > 0$. Hence,

$$\begin{aligned}\overline{\delta}\left(\{k \in \mathbb{N} : \mu_{x_k - L}(\varepsilon) > 1 - \lambda\}\right) &\geq \overline{\delta}\left(\{k(j) \in \mathbb{N} : j \in \mathbb{N}\}\right) \\ &\quad - \overline{\delta}\left(\left\{k(j) \in \mathbb{N} : \mu_{x_{k(j)} - L}(\varepsilon) \leq 1 - \lambda\right\}\right).\end{aligned}$$

Therefore

$$\overline{\delta}\left(\{k \in \mathbb{N} : \mu_{x_k - L}(\varepsilon) > 1 - \lambda\}\right) > 0$$

that is $L \in \Gamma_\mu(x)$. □

Theorem 3.6. *Let (X, μ) be a $\mathcal{P}-$modular space. For any sequence $x \in X$ it holds $\Gamma_\mu(x) \subseteq L_\mu(x)$.*

Proof. Suppose $\gamma \in \Gamma_\mu(x)$, then

$$\delta\left(\{k \in \mathbb{N} : \mu_{x_k - \gamma}(\varepsilon) > 1 - \lambda\}\right) > 0$$

for any $\varepsilon > 0$ and $\lambda \in (0, 1)$. Set $\{x\}_K$ a nonthin subsequence of x such that

$$K := \left\{k(j) \in \mathbb{N} : \mu_{x_{k(j)} - \gamma}(\varepsilon) > 1 - \lambda\right\}$$

for every $\varepsilon > 0$ and $\delta(K) \neq 0$. Since there are infinitely many elements in K, $\gamma \in L_\mu(x)$. □

Theorem 3.7. *Let (X, μ) be a $\mathcal{P}-$modular space. For a sequence $x = \{x_k\}$ with $st_\mu - \lim x = L$ it follows that $\Lambda_\mu(x) = \Gamma_\mu(x) = \{L\}$.*

Proof. First we show that $\Lambda_\mu(x) = \{L\}$. Assume that $\Lambda_\mu(x) = \{L, N\}$ $(L \neq N)$. Then we can write that there exist nonthin subsequences $\left\{x_{k(j)}\right\}$ and $\left\{x_{l(i)}\right\}$ of $x = \{x_k\}$ that are $st_\mu-$*convergent* to L and N, respectively. Because of $\left\{x_{l(j)}\right\}$ is $st_\mu-$*convergent* to N for every $\varepsilon > 0$ and $\lambda \in (0, 1)$

$$K := \left\{l(i) \in \mathbb{N} : \mu_{x_{l(i)} - N}(\varepsilon) \leq 1 - \lambda\right\}$$

is a finite set, so $\delta(K) = 0$. Then we observe that

$$\begin{aligned}\{l(i) \in \mathbb{N} : i \in \mathbb{N}\} &= \left\{l(i) \in \mathbb{N} : \mu_{x_{l(i)} - N}(\varepsilon) > 1 - \lambda\right\} \\ &\quad \cup \left\{l(i) \in \mathbb{N} : \mu_{x_{l(i)} - N}(\varepsilon) \leq 1 - \lambda\right\}\end{aligned}$$

which implies that

$$\delta\left(\left\{l(i) \in \mathbb{N} : \mu_{x_{l(i)} - N}(\varepsilon) > 1 - \lambda\right\}\right) \neq 0. \tag{3.1}$$

Since $st_\mu - \lim x = L$,

$$\delta\left(\{k \in \mathbb{N} : \mu_{x_k - L}(\varepsilon) \leq 1 - \lambda\}\right) = 0 \tag{3.2}$$

for every $\varepsilon > 0$. Hence, we get

$$\delta\left(\{k \in \mathbb{N} : \mu_{x_k - L}(\varepsilon) > 1 - \lambda\}\right) \neq 0.$$

For every $L \neq N$

$$\left\{l(i) \in \mathbb{N} : \mu_{x_{l(i)} - N}(\varepsilon) > 1 - \lambda\right\} \cap \{k \in \mathbb{N} : \mu_{x_k - L}(\varepsilon) > 1 - \lambda\} = \varnothing.$$

So

$$\left\{l(i) \in \mathbb{N} : \mu_{x_{l(i)} - N}(\varepsilon) > 1 - \lambda\right\} \subseteq \{k \in \mathbb{N} : \mu_{x_k - L}(\varepsilon) \leq 1 - \lambda\}.$$

Therefore

$$\overline{\delta}\left(\left\{l\left(i\right) \in \mathbb{N} : \mu_{x_{l(i)}-N}\left(\varepsilon\right) > 1-\lambda\right\}\right) \leq \overline{\delta}\left(\left\{k \in \mathbb{N} : \mu_{x_k-L}\left(\varepsilon\right) \leq 1-\lambda\right\}\right) = 0.$$

This contradicts (3.1). Hence $\Lambda_{\mu}\left(x\right) = \{L\}$.

Now suppose that $\Gamma_{\mu}\left(x\right) = \{L, M\}$ $(L \neq M)$. Then

$$\overline{\delta}\left(\left\{k \in \mathbb{N} : \mu_{x_k-M}\left(\varepsilon\right) > 1-\lambda\right\}\right) \neq 0. \tag{3.3}$$

Since

$$\left\{k \in \mathbb{N} : \mu_{x_k-L}\left(\varepsilon\right) > 1-\lambda\right\} \cap \left\{k \in \mathbb{N} : \mu_{x_k-M}\left(\varepsilon\right) > 1-\lambda\right\} = \varnothing$$

for every $L \neq M$, so

$$\left\{k \in \mathbb{N} : \mu_{x_k-L}\left(\varepsilon\right) \leq 1-\lambda\right\} \supseteq \left\{k \in \mathbb{N} : \mu_{x_k-M}\left(\varepsilon\right) > 1-\lambda\right\}.$$

Hence

$$\overline{\delta}\left(\left\{k \in \mathbb{N} : \mu_{x_k-L}\left(\varepsilon\right) \leq 1-\lambda\right\}\right) \geq \overline{\delta}\left(\left\{k \in \mathbb{N} : \mu_{x_k-M}\left(\varepsilon\right) > 1-\lambda\right\}\right). \tag{3.4}$$

From (3.3), the right hand-side of (3.4) is greater than zero and from (3.2), the left hand-side of (3.4) equals to zero. This is a contradiction. So $\Gamma_{\mu}\left(x\right) = \{L\}$. □

Theorem 3.8. *Let* (X, μ) *be a* $\mathcal{P}-$*modular space. Then the set* Γ_{μ} *is closed in* X *for each sequence* $x = \{x_k\}$ *of elements of* X.

Proof. Let $y \in \overline{\Gamma_{\mu}\left(x\right)}$. Take $\varepsilon > 0$ and $0 < \lambda < 1$. There exists $\gamma \in \Gamma_{\mu}\left(x\right) \cap B\left(y, \lambda, \varepsilon\right)$ such that

$$B\left(y, \lambda, \varepsilon\right) = \left\{x \in X : \mu_{y-x}\left(\varepsilon\right) > 1-\lambda\right\}.$$

Choose $\zeta > 0$ such that $B\left(\gamma, \zeta, \varepsilon\right) \subset B\left(y, \lambda, \varepsilon\right)$. We get

$$\left\{k \in \mathbb{N} : \mu_{y-x_k}\left(\varepsilon\right) > 1-\lambda\right\} \supset \left\{k \in \mathbb{N} : \mu_{\gamma-x_k}\left(\varepsilon\right) > 1-\zeta\right\}$$

so

$$\delta\left(\left\{k \in \mathbb{N} : \mu_{y-x_k}\left(\varepsilon\right) > 1-\lambda\right\}\right) \neq 0$$

and $y \in \Gamma_{\mu}$. □

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Inverse theorem for the iterates of modified Bernstein type polynomials

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Abstract. Gupta and Maheshwari [12] introduced a new sequence of Durrmeyer type linear positive operators P_n to approximate p^{th} Lebesgue integrable functions on $[0,1]$. It is observed that these operators are saturated with $O(n^{-1})$. In order to improve this slow rate of convergence, following Agrawal et al [2], we [3] applied the technique of an iterative combination to the above operators P_n and estimated the error in the L_p- approximation in terms of the higher order integral modulus of smoothness using some properties of the Steklov mean. The present paper is in continuation of this work. Here we have discussed the corresponding inverse result for the above iterative combination $T_{n,k}$ of the operators P_n.

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1. Introduction

Motivated by the definition of Phillips operators (cf. [1] and [15]), Gupta and Maheshwari [12] proposed modified Bernstein type polynomials P_n to approximate functions in $L_p[0,1]$ as follows:

For $f \in L_p[0,1], 1 \leq p < \infty$,

$$P_n(f;x) = \int_0^1 W_n(x,t)f(t)\,dt,\ x \in [0,1],$$

where $W_n(x,t) = n\sum_{\nu=1}^{n} p_{n,\nu}(x)p_{n-1,\nu-1}(t) + (1-x)^n\delta(t)$,

$$p_{n,\nu}(t) = \binom{n}{\nu} t^{\nu}(1-t)^{n-\nu},\ \ 0 \leq t \leq 1,$$

and $\delta(t)$ being the Dirac-delta function, is the kernel of the operators P_n.

Since the order of approximation by the operators P_n is, at best, $O(n^{-1})$, however smooth the function may be, following [3], the iterative combination $T_{n,k} : L_p[0,1] \to C^\infty[0,1]$ of these operators is defined as

$$T_{n,k}(f;x) = \left(I - (I - P_n)^k\right)(f;x) = \sum_{m=1}^{k} (-1)^{m+1} \binom{k}{m} P_n^m(f;x),\ k \in \mathbb{N},$$

where $P_n^0 \equiv I$ and $P_n^m \equiv P_n(P_n^{m-1})$ for $m \in \mathbb{N}$.

In order to improve the rate of convergence, Micchelli [16] introduced an iterative combination for Bernstein polynomials and obtained some direct and saturation results. Gonska and Zhou [11] showed that the iterative combinations can be regarded as iterated Boolean sums and obtained global direct and inverse results in the sup-norm. The iterated Boolean sums have also been studied by several other authors (e.g. [4],[8],[17],[18] and [21]) wherein they have obtained direct and saturation results. Ding and Cao [7] discussed direct and inverse theorems in the sup- norm for iterated Boolean sums of the multivariate Bernstein polynomials using the technique of K-functionals. Sinha et al [19] proved an inverse theorem in the L_p- norm for the Micchelli combination of Bernstein-Durrmeyer polynomials.

Gonska and Zhou [11] obtained the results in the sup- norm using the Ditzian Totik modulus of smoothness and $K-$ functional. Ding and Cao [7] also obtained the results in sup- norm using $K-$ functional. Sevy ([17] and [18]) considered the limits of the linear combinations of iterates of Bernstein and Durrmeyer polynomials in the sup- norm by keeping the degree n of the approximants as a constant while the order of iteration becomes infinite and showed that they converge to the Lagrange interpolation polynomial and the least square approximating polynomial on $[0,1]$ respectively. The more general results have been obtained in [21].

Motivated by the work of Sinha et al [19], Agrawal et al [3] considered the Micchelli combinations for the operator proposed by Gupta and Maheshwari [12] and obtained some direct results in L_p- norm. In the present paper, we continue the work done in [19] by proving a corresponding local inverse theorem in the L_p- norm.

The iterates are defined as

$$P_n^{m+1}(f;x) = \int_0^1 W_n(x,t) P_n^m(f;t)\,dt,\ x \in [0,1].$$

At every stage it uses the entire previous operator value. The analysis in L_p- case, therefore, differs from the study of operators in [10] and linear combinations of operators in [8]. The proof of the theorem is carried out by using the properties of Steklov means. Due to the presence of the Dirac- delta term in the kernel of these operators, the analysis of the proof is quite different. It uses the multinomial theorem, Hölder's inequality and the Fubini's theorem repeatedly.

Throughout the present paper, we assume that $I = [0,1]$, $I_j = [a_j, b_j], j = 1,2,$ where $0 < a_1 < a_2 < b_2 < b_1 < 1$ and by C we mean a positive constant not necessarily the same at each occurrence.

In [3], we obtained the following direct theorem:

Theorem 1.1. *Let $f \in L_p(I), p \geq 1$. Then, for sufficiently large values of n there holds*

$$\|T_{n,k}(f;x) - f(x)\|_{L_p(I_2)} \leq C\left(\omega_{2k}\Big(f, \frac{1}{\sqrt{n}}, p, I_1\Big) + n^{-k}\|f\|_{L_p(I)}\right),$$

where C is a constant independent of f and n.

Remark 1.2. From the above theorem, it follows that if $\omega_{2k}(f,\tau,p,I_1) = O(\tau^\alpha)$, as $\tau \to 0$ then $\|T_{n,k}(f;x) - f(x)\|_{L_p(I_2)} = O(n^{-\alpha/2})$, as $n \to \infty$, where $0 < \alpha < 2k$.

The aim of this paper is to characterize the class of functions for which

$$\|T_{n,k}(f;x) - f(x)\|_{L_p(I_2)} = O(n^{-\alpha/2}), \text{ as } n \to \infty, \text{ where } 0 < \alpha < 2k.$$

Thus, we prove the following theorem (*inverse theorem*):

Theorem 1.3. *Let $f \in L_p(I), p \geq 1$. Let $0 < \alpha < 2k$ and*

$$\|T_{n,k}(f;x) - f(x)\|_{L_p(I_1)} = O(n^{-\alpha/2}), \text{ as } n \to \infty.$$

Then, $\omega_{2k}(f,\tau,p,I_2) = O(\tau^\alpha)$, as $\tau \to 0$.

Remark 1.4. We observe that without any loss of generality we may assume that $f(0) = 0$. To prove it, let $f_1(t) = f(t) - f(0)$. By definition,

$$T_{n,k}(f_1;x) = \sum_{m=1}^{k} (-1)^{m+1} \binom{k}{m} P_n^m(f_1;x).$$

Further, using linearity,

$$P_n^m(f_1;x) = P_n^m(f;x) - f(0)P_n^m(1;x) = P_n^m(f;x) - f(0).$$

This implies that $T_{n,k}(f_1;x) = T_{n,k}(f;x) - f(0)$. This entails that

$$T_{n,k}(f_1;x) - f_1(x) = T_{n,k}(f;x) - f(x),$$

where $f_1(0) = 0$.

Since $f(0) = 0$ (in view of the above remark), it follows that $P_n f(0) = 0$. Consequently, $P_n^m f(0) = 0, \forall m \in \mathbb{N}$.

2. Preliminaries

In this section, we mention some definitions and prove auxiliary results which we need in establishing our main theorem.

Lemma 2.1. *Let $r > 0$ and ν be an integer such that $0 \leq \nu \leq n$. Then for every ν there holds*

$$\int_0^1 p_{n,\nu}(t)\left|\frac{\nu}{n} - t\right|^r dt = O\left(\frac{1}{n^{\frac{r}{2}+1}}\right), \text{ as } n \to \infty.$$

Proof. Let i be an integer such that $2i > r$. An application of Hölder's inequality in integral gives

$$\int_0^1 p_{n,\nu}(t)\left|\frac{\nu}{n}-t\right|^r dt$$
$$\leq \left(\int_0^1 p_{n,\nu}(t)\left|\frac{\nu}{n}-t\right|^{2i} dt\right)^{\frac{r}{2i}} \left(\int_0^1 p_{n,\nu}(t)\,dt\right)^{1-\frac{r}{2i}}. \tag{2.1}$$

It follows that

$$\int_0^1 t^j p_{n,\nu}(t)\,dt = \binom{n}{\nu} B(\nu+j+1, n-\nu+1) = \frac{(\nu+1)(\nu+2)...(\nu+j)}{(n+1)(n+2)...(n+j+1)}.$$

Hence, by binomial expansion

$$\int_0^1 p_{n,\nu}(t)\left(\frac{\nu}{n}-t\right)^{2i} dt$$
$$= \sum_{j=0}^{2i}\binom{2i}{j}(-1)^j\left(\frac{\nu}{n}\right)^{2i-j}\frac{(\nu+1)(\nu+2)...(\nu+j)}{(n+1)(n+2)...(n+j+1)}$$
$$= \frac{1}{(n+1)n^{2i}}\left\{\nu^{2i} - \binom{2i}{1}\nu^{2i-1}(\nu+1)\left(1+\frac{2}{n}\right)^{-1}\right. \tag{2.2}$$
$$+ \binom{2i}{2}\nu^{2i-2}(\nu+1)(\nu+2)\left(1+\frac{2}{n}\right)^{-1}\left(1+\frac{3}{n}\right)^{-1}$$
$$\left. + ... + (\nu+1)(\nu+2)...(\nu+2i)\prod_{s=2}^{2i+1}\left(1+\frac{s}{n}\right)^{-1}\right\}.$$

Now,

$$\prod_{s=2}^{j+1}\left(1+\frac{s}{n}\right)^{-1} = 1 + \frac{p_1(j)}{n} + \frac{p_2(j)}{n^2} + \frac{p_3(j)}{n^3} + ..., \tag{2.3}$$

where $p_1(j)$ is a second degree polynomial in j, $p_2(j)$ is a fourth degree polynomial in j and so on.

Similarly,

$$(\nu+1)(\nu+2)...(\nu+j) = \nu^j + q_1(j)\nu^{j-1} + q_2(j)\nu^{j-2} + ... + j!, \tag{2.4}$$

where $q_1(j)$ is a second degree polynomial in j, $q_2(j)$ is a fourth degree polynomial in j and so on.

Thus from (2.2)- (2.4), we have

$$\int_0^1 p_{n,\nu}(t)\left(\frac{\nu}{n}-t\right)^{2i}dt$$

$$= \frac{1}{(n+1)n^{2i}}\left\{\sum_{j=0}^{2i}\binom{2i}{j}(-1)^j\nu^{2i-j}(\nu^j+q_1(j)\nu^{j-1}+q_2(j)\nu^{j-2}+...)\times\right.$$
$$\left.\left(1+\frac{p_1(j)}{n}+\frac{p_2(j)}{n^2}+...\right)\right\}$$
$$= \frac{1}{(n+1)n^{2i}}\left\{\sum_{j=0}^{2i}\binom{2i}{j}(-1)^j(\nu^{2i}+q_1(j)\nu^{2i-1}+q_2(j)\nu^{2i-2}+...)\times\right.$$
$$\left.\left(1+\frac{p_1(j)}{n}+\frac{p_2(j)}{n^2}+...\right)\right\}$$
$$= O\left(\frac{1}{n^{i+1}}\right), \text{ as } n\to\infty. \tag{2.5}$$

This holds for every ν, where $0 \le \nu \le n$ and in view of the following identity:

$$\sum_{j=0}^{2i}(-1)^j\binom{2i}{j}j^m = \begin{cases} 0, & m=0,1,...,2i-1 \\ (2i)!, & m=2i. \end{cases}$$

Now, on combining (2.1), (2.5) and in view of $\int_0^1 p_{n,\nu}(t)dt = \frac{1}{n+1}$, we obtain

$$\int_0^1 p_{n,\nu}(t)\left|\frac{\nu}{n}-t\right|^r dt \le C\left(\frac{1}{n^{i+1}}\right)^{\frac{r}{2i}}\left(\frac{1}{n+1}\right)^{1-\frac{r}{2i}} = O\left(\frac{1}{n^{\frac{r}{2}+1}}\right). \qquad \square$$

For $m \in \mathbb{N}$, the m^{th} order moment for P_n is defined as

$$\mu_{n,m}(x) = P_n\left((t-x)^m; x\right).$$

Lemma 2.2. [2] *The elementary moments are* $\mu_{n,0}(x) = 1, \mu_{n,1}(x) = \frac{(-x)}{(n+1)}$ *and for* $m \ge 1$ *there holds the recurrence relation*

$$(n+m+1)\mu_{n,m+1}(x) = x(1-x)\left\{\mu'_{n,m}(x)+2m\,\mu_{n,m-1}(x)\right\}+(m(1-2x)-x)\mu_{n,m}(x).$$

Consequently,

(i) $\mu_{n,m}(x)$ *is a polynomial in* x *of degree* m;

(ii) $\mu_{n,m}(x) = O\left(n^{-[(m+1)/2]}\right)$, *as* $n\to\infty$, *uniformly in* $x \in I$, *where* $[\beta]$ *is the integer part of* β.

Corollary 2.3. *There holds for* $r > 0$

$$P_n(|t-x|^r; x) = O\left(n^{-r/2}\right), \text{ as } n\to\infty, \text{ uniformly in } x \in I.$$

Proof. Let s be an even integer $> r$. An application of Hölder's inequality in integral and Lemma 2.2 in the next step gives

$$P_n(|t-x|^r;x) = \int_0^1 W_n(x,t)|t-x|^r\,dt$$

$$\leq \left(\int_0^1 W_n(x,t)|t-x|^s\,dt\right)^{\frac{r}{s}} \left(\int_0^1 W_n(x,t)\,dt\right)^{1-\frac{r}{s}} \leq C(n^{-s/2})^{r/s} = Cn^{-r/2}. \quad \square$$

Lemma 2.4. [3] *There holds for $l \in \mathbb{N}$*

$$x^l(1-x)^l D^l\left(p_{n,\nu}(x)\right) = \sum_{\substack{2i+j\leq l\\ i,j\geq 0}} n^i(\nu-nx)^j q_{i,j,l}(x)p_{n,\nu}(x),$$

where $D \equiv \frac{d}{dx}$ and $q_{i,j,l}(x)$ are certain polynomials in x independent of n and ν.

Lemma 2.5. [3] *There holds for $k, l \in \mathbb{N}$*

$$T_{n,k}\left((t-x)^l;x\right) = O(n^{-k}),\ \textit{as } n\to\infty,\ \textit{uniformly in } x \in I.$$

Lemma 2.6. *Let $r > 0$ and $V_n(x,t) =: n\sum_{\nu=1}^{n} p_{n,\nu}(x)p_{n-1,\nu-1}(t)$, then*

$$\int_0^1 V_n(x,t)|x-t|^r\,dx = O(n^{-r/2}),\ \textit{as } n\to\infty,$$

uniformly for all t in $[0,1]$.

Proof. Let $J =: \int_0^1 V_n(x,t)|x-t|^r\,dx$ and s be an even integer $> r$. Then, proceeding along the lines of the proof of the Corollary 2.3 and in view of

$$\int_0^1 p_{n,\nu}(x)dx = \frac{1}{n+1}$$

we have

$$J \leq \left(\int_0^1 V_n(x,t)(x-t)^s dx\right)^{\frac{r}{s}} \left(\frac{n}{n+1}\right)^{1-\frac{r}{s}}.$$

We may write

$$\int_0^1 V_n(x,t)(x-t)^s\,dx = (-1)^s.n\sum_{i=0}^{s}\binom{s}{i}t^{s-i}(-1)^i\sum_{\nu=1}^{n}p_{n-1,\nu-1}(t)\int_0^1 p_{n,\nu}(x)x^i\,dx.$$

Since

$$\int_0^1 p_{n,\nu}(x)x^i\,dx = \frac{(\nu+1)...(\nu+i)}{(n+1)...(n+i+1)},$$

it follows that

$$\int_0^1 V_n(x,t)(x-t)^s\,dx$$
$$= (-1)^s.n\sum_{i=0}^{s}\binom{s}{i}t^{s-i}(-1)^i\sum_{\nu=0}^{n-1}p_{n-1,\nu}(t)\frac{(\nu+2)...(\nu+i+1)}{(n+1)...(n+i+1)}. \quad (2.6)$$

Now, $(\nu+2)...(\nu+i+1) = \nu^i + p_1(i)\nu^{i-1} + p_2(i)\nu^{i-2} + ...$, where $p_j(i)$ is a polynomial in i of degree $2j$. Moreover,

$$\nu^i = q_0(i)\nu^{(i)} + q_1(i)\nu^{(i-1)} + q_2(i)\nu^{(i-2)} + ... + q_{i-1}(i)\nu^{(1)}, \quad (2.7)$$

where $q_0(i) = q_{i-1}(i) = 1, \nu^{(j)} = \nu(\nu-1)(\nu-2)...(\nu-j+1), j = 0,1,2...,i$ and $q_j(i)$ is a polynomial in i of degree $2j$.

Utilizing (2.7) in (2.6) and using the properties of binomial coefficients, we get the required order. □

Definition 2.7. *Let $f \in L_p(I)$, $p \geq 1$. Then for sufficiently small $\eta > 0$, the Steklov mean $f_{\eta,m}$ of m^{th} order is defined as follows:*

$$f_{\eta,m}(t) = \eta^{-m}\int_{-\eta/2}^{\eta/2}\cdots\int_{-\eta/2}^{\eta/2}\Big(f(t) + (-1)^{m-1}\Delta^m_{\sum_{i=1}^m t_i}f(t)\Big)\,dt_1...dt_m,$$

where $t \in I_1$ and Δ_h^m is m^{th} order forward difference operator of step length h.

Lemma 2.8. *The function $f_{\eta,m}$ satisfies the following properties*

(a) $f_{\eta,m}$ has derivatives up to order m over I_1, $f_{\eta,m}^{(m-1)} \in AC(I_1)$ and $f_{\eta,m}^{(m)}$ exists a.e. and belongs to $L_p(I_1)$;
(b) $\|f_{\eta,m}^{(r)}\|_{L_p(I_2)} \leq C_r\,\eta^{-r}\omega_r(f,\eta,p,I_1), r = 1,2,...,m$;
(c) $\|f - f_{\eta,m}\|_{L_p(I_2)} \leq C_{m+1}\,\omega_m(f,\eta,p,I_1)$;
(d) $\|f_{\eta,m}\|_{L_p(I_2)} \leq C_{m+2}\|f\|_{L_p(I_1)}$;
(e) $\|f_{\eta,m}^{(m)}\|_{L_p(I_2)} \leq C_{m+3}\,\eta^{-m}\|f\|_{L_p(I_1)}$,

where $C_i's$ are certain constants that depend on i but are independent of f and η.

The proof follows from Theorem 18.17 ([13]) and ([20], Exercise 3.12, pp.165-166).

Lemma 2.9. *Let $f \in L_p(I)$, $p \geq 1$ and $r, m \in \mathbb{N}$. Then there holds*

$$\|P_n^m(f(t)(t-x)^r;x)\|_{L_p(I)} \leq C\,n^{-r/2}\|f\|_{L_p(I)}.$$

Proof. Using Remark 1.4

$$P_n^m(f(t)(t-x)^r;x) = \int_0^1\int_0^1...\int_0^1 V_n(x,t_1)V_n(t_1,t_2)...V_n(t_{m-1},t_m)(t_m-x)^r f(t_m)dt_m...dt_1.$$

A repeated use of Hölder's inequality and in view of $\int_0^1 V_n(x,t)dt = O(1)$ makes

$$|P_n^m(f(t)(t-x)^r;x)|^p \leq \int_0^1\int_0^1 ... \int_0^1 V_n(x,t_1)V_n(t_1,t_2)...V_n(t_{m-1},t_m)|t_m-x|^{rp}\,|f(t_m)|^p\, dt_m...dt_1.$$

We now consider integration on both sides. On the right side by virtue of Fubini's theorem, the integration is done with respect to x followed by $t_1, t_2, ..., t_m$ respectively. Thus

$$\int_0^1 |P_n^m(f(t)(t-x)^r;x)|^p\, dx \leq \int_0^1 ... \int_0^1 \left(\int_0^1 V_n(x,t_1)|t_m-x|^{rp}dx\right) \times V_n(t_1,t_2)...V_n(t_{m-1},t_m)\,|f(t_m)|^p\, dt_1...dt_m. \tag{2.8}$$

Let $s > rp$ be an integer. Then, using Hölder's inequality and in view of

$$\int_0^1 p_{n,\nu}(x)dx = \frac{1}{n+1}$$

we have

$$\int_0^1 V_n(x,t_1)|t_m-x|^{rp}\,dx \leq \left(\int_0^1 V_n(x,t_1)|t_m-x|^s dx\right)^{\frac{rp}{s}} \left(\frac{n}{n+1}\right)^{1-\frac{rp}{s}}. \tag{2.9}$$

By multinomial expansion

$$\begin{aligned}|t_m-x|^s &\leq (|t_m-t_{m-1}|+|t_{m-1}-t_{m-2}|+...+|t_1-x|)^s\\ &\leq \sum_{\substack{r_1+r_2+...+r_m=s,\\ r_k\geq 0,\,\forall 1\leq k\leq m}} \binom{s}{r_1,r_2,..,r_m}|t_m-t_{m-1}|^{r_m}...|t_1-x|^{r_1}.\end{aligned} \tag{2.10}$$

Now, we combine (2.8)-(2.10), resort Lemma 2.6 m times and Hölder's inequality $(m-1)$ times to reach

$$\begin{aligned}&\int_0^1 |P_n^m(f(t)(t-x)^r;x)|^p\, dx\\ &\leq C\left(\sum_{\substack{r_1+r_2+...+r_m=s,\\ r_k\geq 0,\,\forall 1\leq k\leq m}} \binom{s}{r_1,r_2,..,r_m} n^{-\frac{r_1+r_2+...+r_m}{2}}\right)^{\frac{rp}{s}} \int_0^1 |f(t_m)|^p\, dt_m\\ &\leq Cn^{-\frac{rp}{2}} m^{rp}\|f\|^p_{L_p(I)},\end{aligned}$$

using bound of multinomial coefficients. Taking p^{th} root on both sides we complete the proof of lemma. □

Lemma 2.10. *Let $m, s \in \mathbb{N}$ and $f \in L_p(I),\ p \geq 1$ have a compact support in $[a,b] \subset (0,1)$. Then there holds*

$$\left\| \frac{d^{2s}}{dx^{2s}} P_n^m(f;x) \right\|_{L_p[a,b]} \leq C\, n^s \|f\|_{L_p[a,b]}.$$

Proof. An application of Lemma 2.4 enables us to express

$$\begin{aligned} \frac{d^{2s}}{dx^{2s}} P_n^m(f;x) &= \frac{d^{2s}}{dx^{2s}} \int_0^1 W_n(x,v) P_n^{m-1}(f;v)\, dv \\ &= n \sum_{\nu=1}^{n} p_{n,\nu}(x) \sum_{\substack{2i+j\leq 2s \\ i,j\geq 0}} n^i \frac{(\nu - nx)^j q_{i,j,s}(x)}{(x(1-x))^{2s}} \times \\ &\quad \int_0^1 p_{n-1,\nu-1}(v) P_n^{m-1}(f;v)\, dv. \end{aligned} \tag{2.11}$$

When $p > 1$, applying Hölder's inequality twice, first for summation and then for integration, we obtain

$$\begin{aligned} &\left| \sum_{\nu=1}^{n} (\nu - nx)^j p_{n,\nu}(x)\, n \int_0^1 p_{n-1,\nu-1}(v) P_n^{m-1}(f;v)\, dv \right|^p \\ &\qquad \leq \sum_{\nu=1}^{n} |\nu - nx|^{jp} p_{n,\nu}(x)\, n \int_0^1 p_{n-1,\nu-1}(v) \left| P_n^{m-1}(f;v) \right|^p\, dv. \end{aligned} \tag{2.12}$$

The above inequality is true for $p = 1$, as well. Now, we integrate both sides of (2.12) with respect to x and take help of Lemma 2.1 in next step to obtain

$$\begin{aligned} &\int_a^b \left| \sum_{\nu=1}^{n} (\nu - nx)^j p_{n,\nu}(x)\, n \int_0^1 p_{n-1,\nu-1}(v) P_n^{m-1}(f;v)\, dv \right|^p dx \\ &\leq \sum_{\nu=1}^{n} \left(\int_a^b p_{n,\nu}(x) |\nu - nx|^{jp}\, dx \right) n \int_0^1 p_{n-1,\nu-1}(v) \left| P_n^{m-1}(f;v) \right|^p\, dv \\ &\leq \frac{C_1 n^{jp/2}}{n} . n \int_0^1 \left(\sum_{\nu=1}^{n} p_{n-1,\nu-1}(v) \right) \left| P_n^{m-1}(f;v) \right|^p\, dv \\ &\leq C_1 n^{jp/2} \| P_n^{m-1}(f;.) \|_{L_p(I)}^p. \end{aligned} \tag{2.13}$$

Let $C_2 =: \sup\limits_{x\in[a,b]} \sup\limits_{\substack{2i+j\leq 2s \\ i,j\geq 0}} \dfrac{|q_{i,j,s}(x)|}{(x(1-x))^{2s}}$.

We now combine (2.11) and (2.13) and conclude that

$$\begin{aligned}\left\|\frac{d^{2s}}{dx^{2s}}P_n^m(f;x)\right\|_{L_p[a,b]} &\leq C_1^{1/p}C_2\left(\sum_{\substack{2i+j\leq 2s\\ i,j\geq 0}} n^i n^{j/2}\right)\|P_n^{m-1}(f;.)\|_{L_p(I)}\\ &\leq C\,n^s\|f\|_{L_p(I)}=C\,n^s\|f\|_{L_p[a,b]}.\end{aligned}$$

Hence, the required result follows. □

Lemma 2.11. *Let $m,s\in\mathbb{N}$ and $f\in L_p[0,1],\ p\geq 1$ have a compact support in $[a,b]\subset(0,1)$. Moreover, let $f^{(2s-1)}\in AC[a,b]$ and $f^{(2s)}\in L_p[a,b]$, then*

$$\left\|\frac{d^{2s}}{dx^{2s}}P_n^m(f;x)\right\|_{L_p[a,b]}\leq C\|f^{(2s)}\|_{L_p[a,b]}.$$

Proof. Since P_n^m maps polynomials into polynomials of the same degree, using Lemma 2.4 we have

$$\begin{aligned}\frac{d^{2s}}{dx^{2s}}P_n^m(f;x) &= \frac{1}{(2s-1)!}\int_0^1\ldots\int_0^1\frac{d^{2s}}{dx^{2s}}(W_n(x,t_1))W_n(t_1,t_2)\ldots\times\\ &\qquad W_n(t_{m-1},t_m)\int_x^{t_m}(t_m-w)^{2s-1}f^{(2s)}(w)\,dw\,dt_m\ldots dt_1\\ &= \frac{1}{(2s-1)!}\sum_{\substack{2i+j\leq 2s\\ i,j\geq 0}} n^i\sum_{\nu=1}^{n}(\nu-nx)^j\frac{q_{i,j,s}(x)}{(x(1-x))^{2s}}\times\\ &\qquad p_{n,\nu}(x)\int_0^1\ldots\int_0^1 np_{n-1,\nu-1}(t_1)W_n(t_1,t_2)\ldots W_n(t_{m-1},t_m)\times\\ &\qquad\int_x^{t_m}(t_m-w)^{2s-1}f^{(2s)}(w)\,dw\,dt_m\ldots dt_1.\end{aligned}$$

Let us define $W_n(x,t)=0,t\notin[0,1]$. Now, we break the interval of integration in t_m in the following way:

There exists for each n an integer $r(n)$ such that

$$\frac{r}{\sqrt{n}}\leq\max\{1-a,b\}\leq\frac{r+1}{\sqrt{n}}.$$

Let $C = \sup_{x\in[a,b]} \sup_{\substack{2i+j\leq 2s\\ i,j\geq 0}} \frac{|q_{i,j,s}(x)|}{(x(1-x))^{2s}}$. Then

$$\left|\frac{d^{2s}}{dx^{2s}}P_n^m(f;x)\right| \leq C \sum_{\substack{2i+j\leq 2s\\ i,j\geq 0}} n^i \sum_{\nu=1}^{n} |\nu - nx|^j\, p_{n,\nu}(x)\times \tag{2.14}$$
$$\int_0^1 ... \int_0^1 np_{n-1,\nu-1}(t_1)W_n(t_1,t_2)...W_n(t_{m-2},t_{m-1})\times$$
$$\left\{\int_0^1 W_n(t_{m-1},t_m)|t_m - x|^{2s-1}\left|\int_x^{t_m} |f^{(2s)}(w)|dw\right| dt_m\right\} dt_{m-1}...dt_1.$$

The expression inside the curly bracket in (2.14), however is bounded by

$$\leq \sum_{l=0}^{r}\left\{\int_{x+\frac{l}{\sqrt{n}}}^{x+\frac{l+1}{\sqrt{n}}} W_n(t_{m-1},t_m)|t_m - x|^{2s-1}\int_x^{x+\frac{l+1}{\sqrt{n}}} |f^{(2s)}(w)|dw\, dt_m\right.$$
$$\left.+\int_{x-\frac{l+1}{\sqrt{n}}}^{x-\frac{l}{\sqrt{n}}} W_n(t_{m-1},t_m)|t_m - x|^{2s-1}\int_{x-\frac{l+1}{\sqrt{n}}}^{x} |f^{(2s)}(w)|dw\, dt_m\right\}$$
$$\leq \sum_{l=1}^{r}\left\{\frac{n^2}{l^4}\int_{x+\frac{l}{\sqrt{n}}}^{x+\frac{l+1}{\sqrt{n}}} W_n(t_{m-1},t_m)|t_m - x|^{2s+3}\int_x^{x+\frac{l+1}{\sqrt{n}}} |f^{(2s)}(w)|dw\, dt_m\right. \tag{2.15}$$
$$\left.+\frac{n^2}{l^4}\int_{x-\frac{l+1}{\sqrt{n}}}^{x-\frac{l}{\sqrt{n}}} W_n(t_{m-1},t_m)|t_m - x|^{2s+3}\int_{x-\frac{l+1}{\sqrt{n}}}^{x} |f^{(2s)}(w)|dw\, dt_m\right\}$$
$$+\int_{x-\frac{1}{\sqrt{n}}}^{x+\frac{1}{\sqrt{n}}} W_n(t_{m-1},t_m)|t_m - x|^{2s-1}\int_{x-\frac{1}{\sqrt{n}}}^{x+\frac{1}{\sqrt{n}}} |f^{(2s)}(w)|dw\, dt_m.$$

Using (2.15) in (2.14)

$$\left|\frac{d^{2s}}{dx^{2s}}P_n^m(f;x)\right| \leq C \sum_{\substack{2i+j\leq 2s\\ i,j\geq 0}} n^i \sum_{\nu=1}^{n} |\nu - nx|^j\, p_{n,\nu}(x)\times$$
$$\int_0^1 ... \int_0^1 np_{n-1,\nu-1}(t_1)W_n(t_1,t_2)...W_n(t_{m-2},t_{m-1})\times$$

$$\Big\{\sum_{l=1}^{r}\Big(\frac{n^2}{l^4}\int_{x+\frac{l}{\sqrt{n}}}^{x+\frac{l+1}{\sqrt{n}}} W_n(t_{m-1},t_m)|t_m-x|^{2s+3}\int_{x}^{x+\frac{l+1}{\sqrt{n}}}|f^{(2s)}(w)|dw\,dt_m$$

$$+\frac{n^2}{l^4}\int_{x-\frac{l+1}{\sqrt{n}}}^{x-\frac{l}{\sqrt{n}}} W_n(t_{m-1},t_m)|t_m-x|^{2s+3}\int_{x-\frac{l+1}{\sqrt{n}}}^{x}|f^{(2s)}(w)|dw\,dt_m\Big)$$

$$+\int_{x-\frac{1}{\sqrt{n}}}^{x+\frac{1}{\sqrt{n}}} W_n(t_{m-1},t_m)|t_m-x|^{2s-1}\int_{x-\frac{1}{\sqrt{n}}}^{x+\frac{1}{\sqrt{n}}}|f^{(2s)}(w)|dw\,dt_m\Big\}\,dt_{m-1}...dt_1,$$

$$= J_1+J_2+J_3,\ \text{say.}$$

In order to estimate J_1, J_2 and J_3, we use multinomial expansion

$$|t_m-x|^{2s+3} \le \sum_{\substack{r_1+r_2+\ldots+r_m=2s+3,\\ r_k\ge 0,\ \forall 1\le k\le m}} \binom{2s+3}{r_1,r_2,..,r_m}\times |t_m-t_{m-1}|^{r_m}|t_{m-1}-t_{m-2}|^{r_{m-1}}...|t_1-x|^{r_1}.$$

Thus

$$J_1 \le C\sum_{\substack{2i+j\le 2s\\ i,j\ge 0}} n^i\left(\sum_{l=1}^{r}\frac{n^2}{l^4}\left(\int_{x}^{x+\frac{l+1}{\sqrt{n}}}|f^{(2s)}(w)|dw\right)\right)\sum_{\nu=1}^{n}|\nu-nx|^j\,p_{n,\nu}(x)\times$$

$$\sum_{\substack{r_1+r_2+\ldots+r_m=2s+3,\\ r_i\ge 0,\ \forall 1\le i\le m}} \binom{2s+3}{r_1,r_2,..,r_m}\int_0^1...\int_0^1 np_{n-1,\nu-1}(t_1)W_n(t_1,t_2)...W_n(t_{m-1},t_m)\times$$

$$|t_m-t_{m-1}|^{r_m}|t_{m-1}-t_{m-2}|^{r_{m-1}}...|t_1-x|^{r_1}\,dt_m\,dt_{m-1}...dt_1.$$

A repeated application of Corollary 2.3 makes

$$J_1 \le C\sum_{\substack{2i+j\le 2s\\ i,j\ge 0}} n^i\left(\sum_{l=1}^{r}\frac{n^2}{l^4}\left(\int_{x}^{x+\frac{l+1}{\sqrt{n}}}|f^{(2s)}(w)|dw\right)\right)\times \tag{2.16}$$

$$\sum_{\nu=1}^{n}|\nu-nx|^j p_{n,\nu}(x)\Big\{\sum_{\substack{r_1+r_2+\ldots+r_m=2s+3,\\ r_k\ge 0,\ \forall 1\le k\le m}} \binom{2s+3}{r_1,r_2,..,r_m}\frac{1}{n^{(r_m+\ldots+r_2)/2}}\Big\}\times$$

$$\left(\int_0^1 np_{n-1,\nu-1}(t_1)|t_1-x|^{r_1}dt_1\right).$$

In order to obtain a bound for J_1 in (2.16) we require an estimate of

$$\int_0^1 np_{n-1,\nu-1}(t_1)|t_1-x|^{r_1}dt_1.$$

This is accomplished with the help of Lemma 2.1 and moments of Bernstein polynomials ([14], Theorem 1.5.1).

$$\int_0^1 np_{n-1,\nu-1}(t_1)|t_1-x|^{r_1}dt_1 \le \int_0^1 np_{n-1,\nu-1}(t_1)\left(\left|t_1-\frac{\nu-1}{n-1}\right|+\left|\frac{\nu-1}{n-1}-x\right|\right)^{r_1} dt_1$$

$$=\sum_{i_1=0}^{r_1}\binom{r_1}{i_1}\left|\frac{(\nu-nx)-(1-x)}{n-1}\right|^{i_1}\int_0^1 np_{n-1,\nu-1}(t_1)\left|t_1-\frac{\nu-1}{n-1}\right|^{r_1-i_1} dt_1$$

$$\le C\sum_{i_1=0}^{r_1}\binom{r_1}{i_1}n^{-(r_1-i_1)/2}\left|(\nu-nx)-(1-x)\right|^{i_1}.$$

Therefore,

$$J_1 \le C\sum_{\substack{2i+j\le 2s\\ i,j\ge 0}} n^i\left(\sum_{l=1}^{r}\frac{n^2}{l^4}\left(\int_x^{x+\frac{l+1}{\sqrt{n}}}|f^{(2s)}(w)|dw\right)\right)\sum_{\substack{r_1+r_2+\ldots+r_m=2s+3,\\ r_k\ge 0,\,\forall 1\le k\le m}}\binom{2s+3}{r_1,r_2,..,r_m}\times$$
$$\left\{\sum_{\nu=1}^{n}|\nu-nx|^j p_{n,\nu}(x)\left(\sum_{i_1=0}^{r_1}\binom{r_1}{i_1}n^{i_1/2}\left|\frac{(\nu-nx)-(1-x)}{n-1}\right|^{i_1}\right)\right\}\frac{1}{n^{(2s+3)/2}}$$
$$\le Cm^{2s+3}\frac{n^s}{n^{(2s+3)/2}}\left(\sum_{l=1}^{r}\frac{n^2}{l^4}\int_x^{x+\frac{l+1}{\sqrt{n}}}|f^{(2s)}(w)|dw\right). \quad (2.17)$$

We now take p norm in x in above. Let p,q be the conjugate exponents such that

$$\frac{1}{p}+\frac{1}{q}=1$$

and $\psi_l(x,.)$ denote the characteristic function of the interval $[x,x+\frac{l+1}{\sqrt{n}}]$. By using Hölder's inequality and Fubini's theorem

$$
\begin{aligned}
\int_a^b \left(\int_x^{x+\frac{l+1}{\sqrt{n}}} |f^{(2s)}(w)| dw \right)^p dx &\leq \left(\frac{l+1}{\sqrt{n}}\right)^{p/q} \int_a^b \int_x^{x+\frac{l+1}{\sqrt{n}}} |f^{(2s)}(w)|^p dw\, dx \\
&= \left(\frac{l+1}{\sqrt{n}}\right)^{p/q} \int_a^b \int_0^1 \psi_l(x,w) |f^{(2s)}(w)|^p dw\, dx \\
&= \left(\frac{l+1}{\sqrt{n}}\right)^{p/q} \int_0^1 \left(\int_a^b \psi_l(x,w)\, dx \right) |f^{(2s)}(w)|^p\, dw \\
&\leq \left(\frac{l+1}{\sqrt{n}}\right)^{p/q} \left(\frac{l+1}{\sqrt{n}}\right) \int_0^1 |f^{(2s)}(w)|^p\, dw \\
&= \left(\frac{l+1}{\sqrt{n}}\right)^{p} \|f^{(2s)}\|^p_{L_p[0,1]}.
\end{aligned}
$$

Hence,

$$
\left\| \int_x^{x+\frac{l+1}{\sqrt{n}}} |f^{(2s)}(w)| dw \right\|_{L_p[a,b]} \leq \left(\frac{l+1}{\sqrt{n}}\right) \|f^{(2s)}\|_{L_p(I)}.
$$

This implies by (2.17), that

$$
\|J_1\|_{L_p[a,b]} \leq Cm^{2s+3} \|f^{(2s)}\|_{L_p(I)} = Cm^{2s+3} \|f^{(2s)}\|_{L_p[a,b]}.
$$

In order to find estimates J_2 and J_3 we proceed in a similar manner and obtain the required order. Combining the estimates of J_1, J_2 and J_3, we complete the proof. □

3. Proof of Inverse Theorem

Proof. We choose numbers x_i and $y_i, i = 1, 2, 3$ that satisfy $0 < a_1 < x_1 < x_2 < x_3 < a_2 < b_2 < y_3 < y_2 < y_1 < b_1 < 1$.

We choose a function $g \in C_0^{2k}$ such that $\operatorname{supp} g \subset (x_2, y_2)$ with $g(x) = 1$ on $[x_3, y_3]$ and $\bar{f} = fg$.

Now, for all values of $\gamma \leq \tau$ we have

$$
\begin{aligned}
\left\|\Delta_\gamma^{2k} \bar{f}(x)\right\|_{L_p[x_2,y_2]} &\leqslant \left\|\Delta_\gamma^{2k} (\bar{f}(x) - T_{n,k}(\bar{f}; x))\right\|_{L_p[x_2,y_2]} + \left\|\Delta_\gamma^{2k} T_{n,k}(\bar{f}; x)\right\|_{L_p[x_2,y_2]} \\
&= \Sigma_1 + \Sigma_2, \text{ say.} \qquad (3.1)
\end{aligned}
$$

Let $\bar{f}_{\eta,2k}(x)$ denote the Steklov mean for the function $\bar{f}(x)$. Then, Lemmas 2.10 and 2.11 entail

$$\begin{aligned}\Sigma_2 &= \left\|\Delta_\gamma^{2k} T_{n,k}(\bar{f};x)\right\|_{L_p[x_2,y_2]} \\ &\le \gamma^{2k}\left\{ \left\|T_{n,k}^{(2k)}(\bar{f}-\bar{f}_{\eta,2k};x)\right\|_{L_p[x_1,y_1]} + \left\|T_{n,k}^{(2k)}(\bar{f}_{\eta,2k};x)\right\|_{L_p[x_1,y_1]} \right\} \\ &\le C\gamma^{2k}\left\{ n^k \|\bar{f}-\bar{f}_{\eta,2k}\|_{L_p[x_2,y_2]} + \|\bar{f}_{\eta,2k}^{(2k)}\|_{L_p[x_2,y_2]} \right\} \\ &\le C\gamma^{2k}\left(n^k + \frac{1}{\eta^{2k}}\right)\omega_{2k}(\bar{f},\eta,p,[x_2,y_2]), \qquad (3.2)\end{aligned}$$

for sufficiently small values of γ and η.

It follows from the hypothesis that a component of Σ_1 is bounded as

$$\begin{aligned}&\left\|\bar{f}(x) - T_{n,k}(\bar{f};x)\right\|_{L_p[x_2,y_2]} \\ &\quad\le \|g(x)(f(x)-T_{n,k}(f;x))\|_{L_p[x_2,y_2]} + \|T_{n,k}(f(t)(g(t)-g(x));x)\|_{L_p[x_2,y_2]} \\ &\quad\le \frac{C}{n^{\alpha/2}} + \|T_{n,k}(f(t)(g(t)-g(x));x)\|_{L_p[x_2,y_2]}\,. \qquad (3.3)\end{aligned}$$

We now establish that

$$\|T_{n,k}(f(t)(g(t)-g(x));x)\|_{L_p[x_2,y_2]} = O(n^{-\alpha/2}). \qquad (3.4)$$

This is a major point in the proof of our theorem. Assuming (3.4)to be true, it follows from (3.1)-(3.4) that

$$\left\|\Delta_\gamma^{2k}\bar{f}(x)\right\|_{L_p[x_2,y_2]} \le C_1\left\{\frac{1}{n^{\alpha/2}} + \gamma^{2k}\left(n^k + \frac{1}{\eta^{2k}}\right)\omega_{2k}(\bar{f},\eta,p,[x_2,y_2])\right\}. \qquad (3.5)$$

We choose $\eta = n^{-1/2}$ and take $\sup_{\gamma\le\tau}$ in (3.5) to obtain

$$\omega_{2k}(\bar{f},\tau,p,[x_2,y_2]) \le C\left\{\eta^\alpha + \left(\frac{\tau}{\eta}\right)^{2k}\omega_{2k}(\bar{f},\eta,p,[x_2,y_2])\right\}.$$

Now, making use of the Lemma ([6], p.696), we get

$$\omega_{2k}(\bar{f},\tau,p,[x_2,y_2]) \le C\,\tau^\alpha$$

and therefore

$$\omega_{2k}(f,\tau,p,I_2) = O(\tau^\alpha),\ \text{as}\,\tau\to 0.$$

The proof of (3.4) is accomplished by induction on α. When $\alpha\le 1$, by mean value theorem and Lemma 2.9

$$\begin{aligned}\|T_{n,k}(f(t)(g(t)-g(x));x)\|_{L_p[x_2,y_2]} &= \|T_{n,k}(f(t)(t-x)g'(\xi);x)\|_{L_p[x_2,y_2]} \\ &\le \frac{C}{n^{1/2}}\|f\|_{L_p(I)},\end{aligned}$$

where ξ lies between t and x. This proves (3.4) when $\alpha\le 1$.

We next assume that (3.4) is true when α lies in $[r-1,r)$ and prove that it is true for $\alpha\in[r,r+1)$. Let $f_{\eta,2k}(t)$ denote the Steklov mean. We express

$$
\begin{aligned}
f(t)&(g(t)-g(x))\\
&=\{(f(t)-f_{\eta,2k}(t))+(f_{\eta,2k}(t)-f_{\eta,2k}(x))+f_{\eta,2k}(x)\}(g(t)-g(x))\\
&=(f(t)-f_{\eta,2k}(t))(g(t)-g(x))+(f_{\eta,2k}(t)-f_{\eta,2k}(x))(g(t)-g(x))\\
&+f_{\eta,2k}(x)(g(t)-g(x))\\
&=\Sigma_3+\Sigma_4+\Sigma_5,\ \text{say.} \qquad (3.6)
\end{aligned}
$$

Let $\psi(t)$ denote the characteristic function of $[x_2-\delta_0, y_2+\delta_0]$. This entails that $\|P_n^m((f(t)-f_{\eta,2k}(t))(g(t)-g(x));x)\|_{L_p[x_2,y_2]}$

$$
\begin{aligned}
&\le \|P_n^m((f(t)-f_{\eta,2k}(t))(t-x)g'(\xi)\psi(t);x)\|_{L_p[x_2,y_2]}\\
&+\|P_n^m((f(t)-f_{\eta,2k}(t))(t-x)g'(\xi)(1-\psi(t));x)\|_{L_p[x_2,y_2]}\\
&\le \|g'\|_{C(I)}\,\|P_n^m(|f(t)-f_{\eta,2k}(t)|\,|t-x|\psi(t);x)\|_{L_p[x_2,y_2]}\\
&+\|g'\|_{C(I)}\delta_0^{-(2k-1)}\left\|P_n^m(|f(t)-f_{\eta,2k}(t)|\,|t-x|^{2k}(1-\psi(t));x)\right\|_{L_p[x_2,y_2]}\\
&\le Cn^{-1/2}\|(f-f_{\eta,2k})\psi(t)\|_{L_p[x_2,y_2]}+Cn^{-k}\|f-f_{\eta,2k}\|_{L_p(I)}\\
&\le Cn^{-1/2}\omega_{2k}(f,\eta,f,p,[x_1,y_1])\\
&+Cn^{-k}\|f\|_{L_p(I)}\ \text{(in view of Lemmas 2.9 and 2.8).} \qquad (3.7)
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\|T_{n,k}(\Sigma_3;x)\| &= \|\sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}P_n^m(\Sigma_3;x)\|\\
&\le Cn^{-1/2}\omega_{2k}(f,\eta,f,p,[x_1,y_1])+Cn^{-k}\|f\|_{L_p(I)}. \qquad (3.8)
\end{aligned}
$$

To obtain an estimate of Σ_5, we note that $g(t)$ is a very smooth function and hence

$$
\begin{aligned}
T_{n,k}(g(t)-g(x);x) &= \sum_{j=1}^{2k-1}\frac{g^{(j)}(x)}{j!}T_{n,k}((t-x)^j;x)\\
&+\frac{1}{(2k)!}T_{n,k}(g^{(2k)}(\xi)(t-x)^{2k};x), \qquad (3.9)
\end{aligned}
$$

where ξ lies between t and x.

Now, applying Lemmas 2.5 and 2.9 on the right hand side of (3.9) respectively, we have

$$
\|T_{n,k}(\Sigma_5;x)\|_{L_p[x_2,y_2]} \le Cn^{-k}\|f_{\eta,2k}\|_{L_p[x_2,y_2]} \le Cn^{-k}\|f\|_{L_p(I)}. \qquad (3.10)
$$

Since, by virtue of Lemma 2.8, $f_{\eta,2k}$ is $2k$ times differentiable, there follows

$$\Sigma_4 = \left\{\sum_{i=1}^{2k-1} \frac{(t-x)^i}{i!} f_{\eta,2k}^{(i)}(x) + \frac{1}{(2k-1)!}\int_x^t (t-w)^{2k-1} f_{\eta,2k}^{(2k)}(w)dw\right\}\times$$

$$\left\{\sum_{j=1}^{2k-2} \frac{(t-x)^j}{j!} g^{(j)}(x) + \frac{(t-x)^{2k-1}}{(2k-1)!} g^{(2k-1)}(\xi)\right\}$$

$$= \sum_{i=1}^{2k-1}\sum_{j=1}^{2k-2} \frac{(t-x)^{i+j}}{i!j!} f_{\eta,2k}^{(i)}(x) g^{(j)}(x)$$

$$+ \frac{g^{(2k-1)}(\xi)}{(2k-1)!} \sum_{i=1}^{2k-1} \frac{(t-x)^{2k+i-1}}{i!} f_{\eta,2k}^{(i)}(x)$$

$$+ \frac{1}{(2k-1)!}\int_x^t (t-w)^{2k-1} f_{\eta,2k}^{(2k)}(w)dw \times$$

$$\left\{\sum_{j=1}^{2k-2} \frac{(t-x)^j}{j!} g^{(j)}(x) + \frac{(t-x)^{2k-1}}{(2k-1)!} g^{(2k-1)}(\xi)\right\}$$

$$= \Sigma_6 + \Sigma_7 + \Sigma_8, \text{ say,} \tag{3.11}$$

where ξ lies between t and x.

By Lemma 2.5 and Theorem 3.1 ([9], p.5)

$$\|T_{n,k}(\Sigma_6; x)\|_{L_p[x_2,y_2]} \le Cn^{-k}\left(\sum_{i=1}^{2k-1} \|f_{\eta,2k}^{(i)}(x)\|_{L_p[x_2,y_2]}\right)$$

$$\le Cn^{-k}\left(\|f_{\eta,2k}\|_{L_p[x_2,y_2]} + \|f_{\eta,2k}^{(2k-1)}(x)\|_{L_p[x_2,y_2]}\right). \tag{3.12}$$

Similarly,

$$\|T_{n,k}(\Sigma_7; x)\|_{L_p[x_2,y_2]} \le Cn^{-k}\left(\|f_{\eta,2k}\|_{L_p[x_2,y_2]} + \|f_{\eta,2k}^{(2k-1)}(x)\|_{L_p[x_2,y_2]}\right). \tag{3.13}$$

We now examine a typical term of $T_{n,k}(\Sigma_8; x)$ expressed as

$$P_n^m\left((t-x)^i \int_x^t (t-w)^{2k-1} f_{\eta,2k}^{(2k)}(w)dw; x\right)$$

$$= P_n^m\left((t-x)^i \int_x^t (t-w)^{2k-1} f_{\eta,2k}^{(2k)}(w)\psi(w)dw; x\right)$$

$$+ P_n^m\left((t-x)^i \int_x^t (t-w)^{2k-1} f_{\eta,2k}^{(2k)}(w)(1-\psi(w))dw; x\right)$$

$$= \Sigma_9 + \Sigma_{10}, \text{ say.} \tag{3.14}$$

We may write

$$|\Sigma_9| \leq \int_0^1 ... \int_0^1 W_n(x,t_1)W_n(t_1,t_2)...W_n(t_{m-1},t_m)|t_m - x|^{2k+i-1} \times \tag{3.15}$$
$$\left| \int_x^{t_m} |f^{(2k)}_{\eta,2k}(w)|\psi(w)dw \right| dt_m dt_{m-1}...dt_2 dt_1.$$

Now, proceeding along the lines of the proof of Lemma 2.11, we obtain

$$\|\Sigma_9\|_{L_p[x_2,y_2]} \leq \frac{C}{n^{(2k+i)/2}} \|f^{(2k)}_{\eta,2k}\|_{L_p[x_2-\delta,y_2-\delta]}. \tag{3.16}$$

The presence of $(1-\psi(w))$ in Σ_{10} implies that $|w-x| > \delta_0$. Therefore

$$|\Sigma_{10}| \leq \int_0^1 ... \int_0^1 W_n(x,t_1)W_n(t_1,t_2)...W_n(t_{m-1},t_m) \times \tag{3.17}$$
$$|t_m - x|^{2k+i-1+2k} \left(\delta_0^{-2k} \left| \int_x^{t_m} |f^{(2k)}_{\eta,2k}(w)|(1-\psi(w))dw \right| dt_m dt_{m-1}...dt_2 dt_1 \right).$$

Proceeding along the lines of the proof of Lemma 2.11 again yields

$$\|\Sigma_{10}\|_{L_p[x_2,y_2]} \leq \frac{C}{n^{(4k+i)/2}} \|f^{(2k)}_{\eta,2k}\|_{L_p(I)}. \tag{3.18}$$

Combining (3.14), (3.16) and (3.18), we get

$$\|T_{n,k}(\Sigma_8; x)\|_{L_p[x_2,y_2]}$$
$$\leq C \left\{ \frac{1}{n^{(2k+i)/2}} \|f^{(2k)}_{\eta,2k}\|_{L_p[x_2-\delta_0,y_2+\delta_0]} + \frac{1}{n^{(4k+i)/2}} \|f^{(2k)}_{\eta,2k}\|_{L_p(I)} \right\}. \tag{3.19}$$

Utilizing (3.6)-(3.19), we are led to

$$\|T_{n,k}(f(t)(g(t)-g(x)); x)\|_{L_p[x_2,y_2]}$$
$$\leq C \Bigg\{ \frac{1}{n^{1/2}} \omega_{2k}(f,\eta,p,[x_1,y_1]) + \frac{1}{n^k}\|f\|_{L_p(I)} + \frac{1}{n^k}\Bigg(\|f_{\eta,2k}\|_{L_p[x_2,y_2]}$$
$$+ \|f^{(2k-1)}_{\eta,2k}\|_{L_p[x_2,y_2]} \Bigg) + \frac{1}{n^{(2k+1)/2}} \|f^{(2k)}_{\eta,2k}\|_{L_p[x_2-\delta_0,y_2-\delta_0]} + \frac{1}{n^{(4k+1)/2}} \|f^{(2k)}_{\eta,2k}\|_{L_p(I)} \Bigg\}.$$

This is further simplified by Lemma 2.8 by taking $\eta = n^{-1/2}$ for large values of n as

$$\|T_{n,k}(f(t)(g(t)-g(x)); x)\|_{L_p[x_2,y_2]}$$
$$\leq C \Bigg\{ \frac{1}{n^{1/2}} \omega_{2k}(f,n^{-1/2},p,[x_1,y_1]) + \frac{1}{n^k}\|f\|_{L_p(I)}$$
$$+ \frac{1}{n^{1/2}} \omega_{2k-1}(f,n^{-1/2},p,[x_1,y_1]) \Bigg\}. \tag{3.20}$$

The induction hypothesis implies that for $[c,d] \subset (a_1,b_1)$

$$\omega_{2k}(f,n^{-1/2},p,[c,d]) = O(n^{-\alpha/2}),\ n \to \infty. \tag{3.21}$$

This induces, by (Theorem 6.1.2, [20]),

$$\omega_{2k-1}(f, n^{-1/2}, p, [c,d]) = O(n^{-\alpha/2}),\ n \to \infty. \tag{3.22}$$

Incorporating (3.21) and (3.22) in (3.20), we obtain

$$\|T_{n,k}(f(t)(g(t)-g(x)); x)\|_{L_p[x_2,y_2]} = O(n^{-(\alpha+1)/2}),\ n \to \infty.$$

This proves (3.4) and hence the proof of the theorem follows. □

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A modification of generalized Baskakov-Kantorovich operators

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Abstract. In this paper, we give some direct results and weighted approximation properties for a modification of generalized Baskakov-Kantorovich operators.

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Keywords: Kantorovich operator, direct result, weighted approximation.

1. Introduction

The Baskakov operators, defined by V.A. Baskakov [7], and their Kantorovich type modification ([11], p.115) are given by

$$B_n(f;x) = \sum_{k=0}^{\infty} f\left(\frac{k}{n}\right)\binom{n+k-1}{k}\frac{x^k}{(1+x)^{n+k}}, \quad x \geq 0, n \in \mathbb{N}$$

and

$$V_n(f;x) = \sum_{k=0}^{\infty}\binom{n+k-1}{k}\frac{x^k}{(1+x)^{n+k}} n\int_{\frac{k}{n}}^{\frac{k+1}{n}} f(t)dt, \quad x \geq 0, n \in \mathbb{N},$$

respectively. In the literature there are many studies which include Baskakov operators, Baskakov-Kantorovich operators and their generalizations. Some of them are [1], [3]- [8], [10]- [13] and [16]- [27]. We now deal only with the works which are necessary for this paper. In the identity

$$(1-t)^{-x}e^{at} = \sum_{k=0}^{\infty} P_k(x,a)\frac{t^k}{k!},$$

where $a \geq 0$ is any constant and

$$P_k(x,a) = \sum_{i=0}^{k}\binom{k}{i}(x)_i a^{k-i}$$

with $(x)_0 = 1$, $(x)_i = x(x+1)\cdots(x+i-1)$ for $i \geq 1$ by setting $x = n$ and $t = \frac{x}{1+x}$, Miheşan [18] constructed the generalized Baskakov operators

$$B_n^a(f;x) = e^{-\frac{ax}{1+x}} \sum_{k=0}^{\infty} f\left(\frac{k}{n}\right) \frac{P_k(n,a)}{k!} \frac{x^k}{(1+x)^{n+k}}, \quad x \geq 0, n \in \mathbb{N}$$

for every $f \in C[0,\infty)$. He showed that these operators converge uniformly on $[0,b]$ for functions having exponential growth on positive x-axis and obtained the order of approximation with the help of the usual modulus of continuity. After that in [25], by proposing integral type modification of the operators B_n^a in the sense of Kantorovich as follows:

$$V_n^a(f;x) = ne^{-\frac{ax}{1+x}} \sum_{k=0}^{\infty} \frac{P_k(n,a)}{k!} \frac{x^k}{(1+x)^{n+k}} \int_{\frac{k}{n}}^{\frac{k+1}{n}} f(t)dt, \quad x \geq 0, n \in \mathbb{N}$$

Wafi and Khatoon proved a Voronovskaya type theorem in polynomial weight spaces for these operators. Note that for $a = 0$ the operators B_n^a and V_n^a reduce to the operators B_n and V_n, respectively. In 2010, Erençin and Başcanbaz-Tunca [12] presented the following generalization of the operators $B_n^a(f;x)$

$$L_n(f;x) = e^{-\frac{a_n x}{1+x}} \sum_{k=0}^{\infty} f\left(\frac{k}{b_n}\right) \frac{P_k(n,a_n)}{k!} \frac{x^k}{(1+x)^{n+k}}, \quad x \geq 0, n \in \mathbb{N}, \tag{1.1}$$

where (a_n) are (b_n) are two sequences of positive numbers such that

$$\lim_{n\to\infty} \frac{n}{b_n} = 1, \quad \lim_{n\to\infty} \frac{a_n}{b_n} = 0, \quad \lim_{n\to\infty} \frac{1}{b_n} = 0,$$

and investigated approximation properties of such operators by means of the weighted Korovkin type theorem given in [14, 15] and also introduced an application to functional differential equations which gives a recurrence relation for the monomials of that operators.

Very recently, Altomare, Montano and Leonessa [2] presented the modification of Szasz-Mirakyan-Kantorovich operators defined by

$$C_n(f;x) = e^{-nx} \sum_{k=0}^{\infty} \frac{(nx)^k}{k!} \left[\frac{n}{b_n - a_n} \int_{\frac{k+a_n}{n}}^{\frac{k+b_n}{n}} f(t)dt\right], \quad x \geq 0, n \in \mathbb{N},$$

where (a_n) and (b_n) are sequences of real numbers such that $0 \leq a_n < b_n \leq 1$. They introduced some approximation properties of these operators on continuous function spaces, weighted continuous function spaces and Lebesgue spaces and also obtained some estimates for the rate of convergence.

Inspired by that work, we consider the following Kantorovich type operators

$$K_n(f;x) = \sum_{k=0}^{\infty} S_{n,a_n}(k,x) \frac{b_n}{d_n - c_n} \int_{\frac{k+c_n}{b_n}}^{\frac{k+d_n}{b_n}} f(t)dt, \quad x \geq 0, n \in \mathbb{N}, \tag{1.2}$$

where

$$S_{n,a_n}(k,x) = e^{-\frac{a_n x}{1+x}} \frac{P_k(n,a_n)}{k!} \frac{x^k}{(1+x)^{n+k}}$$

and (a_n), (b_n), (c_n) and (d_n) are sequences of real numbers having the properties:
(i) $a_n \geq 0, \quad b_n \geq 1, \quad 0 \leq c_n < d_n \leq 1$
(ii) $\lim_{n\to\infty} \frac{n}{b_n} = 1, \quad \lim_{n\to\infty} \frac{a_n}{b_n} = 0.$
We remark that for $a_n = a$, $b_n = n$, $c_n = 0$ and $d_n = 1$ the operators $K_n(f;x)$ turn out to be the operators $V_n^a(f;x)$.

In the present paper, we first give some direct results. Next, we prove a weighted Korovkin type theorem and compute the order of approximation with the help of the weighted modulus of continuity for these operators.

2. Auxiliary results

By [12], we have

$$L_n(1;x) = 1 \tag{2.1}$$

$$L_n(t;x) = \frac{n}{b_n}x + \frac{a_n}{b_n}\frac{x}{1+x} \tag{2.2}$$

$$L_n(t^2;x) = \frac{n(n+1)}{b_n^2}x^2 + \frac{2a_n n}{b_n^2}\frac{x^2}{1+x} + \frac{a_n^2}{b_n^2}\frac{x^2}{(1+x)^2} + \frac{n}{b_n^2}x + \frac{a_n}{b_n^2}\frac{x}{1+x}, \tag{2.3}$$

where $L_n(f;x)$ is defined by (1.1).
In the sequel, we shall need to following lemmas.

Lemma 2.1. *The following equalities hold:*

$$\begin{aligned}L_n(t^3;x) =& \frac{n(n+1)(n+2)}{b_n^3}x^3 + \frac{3a_n n(n+1)}{b_n^3}\frac{x^3}{1+x} + \frac{3a_n^2 n}{b_n^3}\frac{x^3}{(1+x)^2} \\ &+ \frac{a_n^3}{b_n^3}\frac{x^3}{(1+x)^3} + \frac{3n(n+1)}{b_n^3}x^2 + \frac{6a_n n}{b_n^3}\frac{x^2}{1+x} + \frac{3a_n^2}{b_n^3}\frac{x^2}{(1+x)^2} \\ &+ \frac{n}{b_n^3}x + \frac{a_n}{b_n^3}\frac{x}{1+x}\end{aligned}$$

and

$$\begin{aligned}L_n(t^4;x) =& \frac{n(n+1)(n+2)(n+3)}{b_n^4}x^4 + \frac{4a_n n(n+1)(n+2)}{b_n^4}\frac{x^4}{1+x} \\ &+ \frac{6a_n^2 n(n+1)}{b_n^4}\frac{x^4}{(1+x)^2} + \frac{4a_n^3 n}{b_n^4}\frac{x^4}{(1+x)^3} + \frac{a_n^4}{b_n^4}\frac{x^4}{(1+x)^4} \\ &+ \frac{6n(n+1)(n+2)}{b_n^4}x^3 + \frac{18a_n n(n+1)}{b_n^4}\frac{x^3}{1+x} + \frac{18a_n^2 n}{b_n^4}\frac{x^3}{(1+x)^2} \\ &+ \frac{6a_n^3}{b_n^4}\frac{x^3}{(1+x)^3} + \frac{7n(n+1)}{b_n^4}x^2 + \frac{14a_n n}{b_n^4}\frac{x^2}{1+x} + \frac{7a_n^2}{b_n^4}\frac{x^2}{(1+x)^2} \\ &+ \frac{n}{b_n^4}x + \frac{a_n}{b_n^4}\frac{x}{1+x}.\end{aligned}$$

It can be proved in a similar way that of the proof of Lemma 2.1 in [18] or by using the recurrence relation given in [12].

Lemma 2.2. *For the operators $K_n(f;x)$ defined by (1.2), we have*

$$\begin{aligned} K_n(1;x) &= 1, \\ K_n(t;x) &= \frac{n}{b_n}x + \frac{a_n}{b_n}\frac{x}{1+x} + \frac{m_0(n)}{2b_n} \\ K_n(t^2;x) &= \frac{n(n+1)}{b_n^2}x^2 + \frac{2a_n n}{b_n^2}\frac{x^2}{1+x} + \frac{a_n^2}{b_n^2}\frac{x^2}{(1+x)^2} + \frac{nm_1(n)}{b_n^2}x \\ &+ \frac{a_n m_1(n)}{b_n^2}\frac{x}{1+x} + \frac{m_2(n)}{3b_n^2}, \end{aligned}$$

$$\begin{aligned} K_n(t^3;x) =& \frac{n(n+1)(n+2)}{b_n^3}x^3 + \frac{3a_n n(n+1)}{b_n^3}\frac{x^3}{1+x} + \frac{3a_n^2 n}{b_n^3}\frac{x^3}{(1+x)^2} \\ &+ \frac{a_n^3}{b_n^3}\frac{x^3}{(1+x)^3} + \frac{3n(n+1)m_3(n)}{2b_n^3}x^2 + \frac{3a_n n m_3(n)}{b_n^3}\frac{x^2}{1+x} \\ &+ \frac{3a_n^2 m_3(n)}{2b_n^3}\frac{x^2}{(1+x)^2} + \frac{nm_4(n)}{2b_n^3}x + \frac{a_n m_4(n)}{2b_n^3}\frac{x}{1+x} + \frac{m_5(n)}{4b_n^3}, \end{aligned}$$

and

$$\begin{aligned} K_n(t^4;x) =& \frac{n(n+1)(n+2)(n+3)}{b_n^4}x^4 + \frac{4a_n n(n+1)(n+2)}{b_n^4}\frac{x^4}{1+x} \\ &+ \frac{6a_n^2 n(n+1)}{b_n^4}\frac{x^4}{(1+x)^2} + \frac{4a_n^3 n}{b_n^4}\frac{x^4}{(1+x)^3} + \frac{a_n^4}{b_n^4}\frac{x^4}{(1+x)^4} \\ &+ \frac{2n(n+1)(n+2)m_6(n)}{b_n^4}x^3 + \frac{6a_n n(n+1)m_6(n)}{b_n^4}\frac{x^3}{1+x} \\ &+ \frac{6a_n^2 n m_6(n)}{b_n^4}\frac{x^3}{(1+x)^2} + \frac{2a_n^3 m_6(n)}{b_n^4}\frac{x^3}{(1+x)^3} + \frac{n(n+1)m_7(n)}{b_n^4}x^2 \\ &+ \frac{2a_n n m_7(n)}{b_n^4}\frac{x^2}{1+x} + \frac{a_n^2 m_7(n)}{b_n^4}\frac{x^2}{(1+x)^2} + \frac{nm_8(n)}{b_n^4}x + \frac{a_n m_8(n)}{b_n^4}\frac{x}{1+x} \\ &+ \frac{m_9(n)}{5b_n^4}, \end{aligned}$$

where
$m_0(n) = c_n + d_n$, $m_1(n) = c_n + d_n + 1$, $m_2(n) = c_n^2 + c_n d_n + d_n^2$,
$m_3(n) = c_n + d_n + 2$, $m_4(n) = 2(c_n^2 + c_n d_n + d_n^2) + 3(c_n + d_n) + 2$,
$m_5(n) = c_n^3 + c_n^2 d_n + c_n d_n^2 + d_n^3$, $m_6(n) = c_n + d_n + 3$,
$m_7(n) = 2(c_n^2 + c_n d_n + d_n^2) + 6(c_n + d_n) + 7$,
$m_8(n) = c_n^3 + c_n^2 d_n + c_n d_n^2 + d_n^3 + 2(c_n^2 + c_n d_n + d_n^2) + 2(c_n + d_n) + 1$ *and*
$m_9(n) = c_n^4 + c_n^3 d_n + c_n^2 d_n^2 + c_n d_n^3 + d_n^4$.

By using the definition of K_n, the equalities (2.1)- (2.3) and Lemma 2.1, it can be proved easily. So, we omit them.

Now in terms of the linearity of the operators K_n and Lemma 2.2 we can state the following lemma.

Lemma 2.3. *For the operators $K_n(f;x)$ defined by (1.2), we have*

$$\begin{aligned} K_n((t-x)^2;x) = &\left(\frac{n(n+1)}{b_n^2} - \frac{2n}{b_n} + 1\right)x^2 + \frac{2a_n}{b_n}\left(\frac{n}{b_n} - 1\right)\frac{x^2}{1+x} \\ &+ \frac{a_n^2}{b_n^2}\frac{x^2}{(1+x)^2} + \left(\frac{nm_1(n)}{b_n^2} - \frac{m_0(n)}{b_n}\right)x \\ &+ \frac{a_n m_1(n)}{b_n^2}\frac{x}{1+x} + \frac{m_2(n)}{3b_n^2} \end{aligned} \tag{2.4}$$

and

$$\begin{aligned} &K_n((t-x)^4;x) \\ &= \left(\frac{n(n+1)(n+2)(n+3)}{b_n^4} - \frac{4n(n+1)(n+2)}{b_n^3} + \frac{6n(n+1)}{b_n^2} - \frac{4n}{b_n} + 1\right)x^4 \\ &+ \frac{4a_n}{b_n}\left(\frac{n(n+1)(n+2)}{b_n^3} - \frac{3n(n+1)}{b_n^2} + \frac{3n}{b_n} - 1\right)\frac{x^4}{1+x} \\ &+ \frac{6a_n^2}{b_n^2}\left(\frac{n(n+1)}{b_n^2} - \frac{2n}{b_n} + 1\right)\frac{x^4}{(1+x)^2} + \frac{4a_n^3}{b_n^3}\left(\frac{n}{b_n} - 1\right)\frac{x^4}{(1+x)^3} \\ &+ \frac{a_n^4}{b_n^4}\frac{x^4}{(1+x)^4} + 2\left(\frac{n(n+1)(n+2)m_6(n)}{b_n^4} - \frac{3n(n+1)m_3(n)}{b_n^3} + \frac{3nm_1(n)}{b_n^2}\right. \\ &\left. - \frac{m_0(n)}{b_n}\right)x^3 + \frac{6a_n}{b_n}\left(\frac{n(n+1)m_6(n)}{b_n^3} - \frac{2nm_3(n)}{b_n^2} + \frac{m_1(n)}{b_n}\right)\frac{x^3}{1+x} \\ &+ \frac{6a_n^2}{b_n^2}\left(\frac{nm_6(n)}{b_n^2} - \frac{m_3(n)}{b_n}\right)\frac{x^3}{(1+x)^2} + \frac{2a_n^3 m_6(n)}{b_n^4}\frac{x^3}{(1+x)^3} \\ &+ \left(\frac{n(n+1)m_7(n)}{b_n^4} - \frac{2nm_4(n)}{b_n^3} + \frac{2m_2(n)}{b_n^2}\right)x^2 \\ &+ \frac{2a_n}{b_n}\left(\frac{nm_7(n)}{b_n^3} - \frac{m_4(n)}{b_n^2}\right)\frac{x^2}{1+x} + \frac{a_n^2 m_7(n)}{b_n^4}\frac{x^2}{(1+x)^2} \\ &+ \left(\frac{nm_8(n)}{b_n^4} - \frac{m_5(n)}{b_n^3}\right)x + \frac{a_n m_8(n)}{b_n^4}\frac{x}{1+x} + \frac{m_9(n)}{5b_n^4}, \end{aligned} \tag{2.5}$$

where $m_0(n)$,$m_1(n)$, $m_2(n)$, $m_3(n)$, $m_4(n)$,$m_5(n)$, $m_6(n)$, $m_7(n)$, $m_8(n)$ and $m_9(n)$ are given as in Lemma 2.2.

Lemma 2.4. *For the operators $K_n(f;x)$ defined by (1.2), we have*

$$K_n((t-x)^4;x) \leq 12m_7(n)A(n)(x^4+x^3+x^2+x+1),$$

where $m_7(n)$ given as in Lemma 2.2 and $A(n) = \max\{A_1(n), A_2(n)\}$ with

$$\begin{aligned} A_1(n) = &\left|\frac{n(n+1)(n+2)(n+3)}{b_n^4} - \frac{4n(n+1)(n+2)}{b_n^3} + \frac{6n(n+1)}{b_n^2} - \frac{4n}{b_n} + 1\right| \\ &+ \frac{a_n}{b_n}\left(\frac{n(n+1)(n+2)}{b_n^3} + \frac{n}{b_n}\right) + \frac{a_n^2}{b_n^2}\left(\frac{n(n+1)}{b_n^2} + 1\right) + \frac{a_n^3}{b_n^3}\left(\frac{n}{b_n} + \frac{a_n}{b_n}\right), \end{aligned}$$

$$A_2(n) = \frac{n(n+1)(n+2)}{b_n^4} + \frac{n}{b_n^2} + \frac{a_n}{b_n^2}\left(\frac{n(n+1)}{b_n^2}+1\right) + \frac{a_n^2}{b_n^3}\left(\frac{n}{b_n}+\frac{a_n}{b_n}\right).$$

Proof. From (2.5) we may write

$$\begin{aligned}
&K_n((t-x)^4;x)\\
\le&\left(\frac{n(n+1)(n+2)(n+3)}{b_n^4} - \frac{4n(n+1)(n+2)}{b_n^3} + \frac{6n(n+1)}{b_n^2} - \frac{4n}{b_n} + 1\right)x^4\\
&+\frac{4a_n}{b_n}\left(\frac{n(n+1)(n+2)}{b_n^3}+\frac{3n}{b_n}\right)\frac{x^4}{1+x} + \frac{6a_n^2}{b_n^2}\left(\frac{n(n+1)}{b_n^2}+1\right)\frac{x^4}{(1+x)^2}\\
&+\frac{4a_n^3 n}{b_n^4}\frac{x^4}{(1+x)^3} + \frac{a_n^4}{b_n^4}\frac{x^4}{(1+x)^4} + 2\left(\frac{n(n+1)(n+2)m_6(n)}{b_n^4} + \frac{3nm_1(n)}{b_n^2}\right)x^3\\
&+\frac{6a_n}{b_n}\left(\frac{n(n+1)m_6(n)}{b_n^3}+\frac{m_1(n)}{b_n}\right)\frac{x^3}{1+x} + \frac{a_n^2 n m_6(n)}{b_n^4}\frac{x^3}{(1+x)^2}\\
&+\frac{2a_n^3 m_6(n)}{b_n^4}\frac{x^3}{(1+x)^3} + \left(\frac{n(n+1)m_7(n)}{b_n^4}+\frac{2m_2(n)}{b_n^2}\right)x^2 + \frac{2a_n n m_7(n)}{b_n^4}\frac{x^2}{1+x}\\
&+\frac{a_n^2 m_7(n)}{b_n^4}\frac{x^2}{(1+x)^2} + \frac{nm_8(n)}{b_n^4}x + \frac{a_n m_8(n)}{b_n^4}\frac{x}{1+x} + \frac{m_9(n)}{5b_n^4}\\
\le&12\Bigg\{\left|\frac{n(n+1)(n+2)(n+3)}{b_n^4} - \frac{4n(n+1)(n+2)}{b_n^3} + \frac{6n(n+1)}{b_n^2} - \frac{4n}{b_n} + 1\right|x^4\\
&+\frac{a_n}{b_n}\left(\frac{n(n+1)(n+2)}{b_n^3}+\frac{n}{b_n}\right)\frac{x^4}{1+x} + \frac{a_n^2}{b_n^2}\left(\frac{n(n+1)}{b_n^2}+1\right)\frac{x^4}{(1+x)^2}\\
&+\frac{a_n^3 n}{b_n^4}\frac{x^4}{(1+x)^3} + \frac{a_n^4}{b_n^4}\frac{x^4}{(1+x)^4} + \left(\frac{n(n+1)(n+2)m_6(n)}{b_n^4} + \frac{nm_1(n)}{b_n^2}\right)x^3\\
&+\frac{a_n}{b_n}\left(\frac{n(n+1)m_6(n)}{b_n^3}+\frac{m_1(n)}{b_n}\right)\frac{x^3}{1+x} + \frac{a_n^2 n m_6(n)}{b_n^4}\frac{x^3}{(1+x)^2}\\
&+\frac{a_n^3 m_6(n)}{b_n^4}\frac{x^3}{(1+x)^3} + \left(\frac{n(n+1)m_7(n)}{b_n^4}+\frac{m_2(n)}{b_n^2}\right)x^2 + \frac{a_n n m_7(n)}{b_n^4}\frac{x^2}{1+x}\\
&+\frac{a_n^2 m_7(n)}{b_n^4}\frac{x^2}{(1+x)^2} + \frac{nm_8(n)}{b_n^4}x + \frac{a_n m_8(n)}{b_n^4}\frac{x}{1+x} + \frac{m_9(n)}{b_n^4}\Bigg\}.
\end{aligned}$$

Since $\frac{x^s}{(1+x)^l} \le x^s$ for all $x \ge 0$, $l \le s$ $(l, s = 1, 2, 3, 4)$ and $m_1(n)$, $m_2(n)$, $m_6(n)$, $m_8(n)$, $m_9(n) < m_7(n)$ for all $n \in \mathbb{N}$ one gets

$$K_n((t-x)^4;x) \le 12\Big\{A_1(n)x^4 + m_7(n)\left[A_2(n)x^3 + A_3(n)x^2 + A_4(n)x + A_5(n)\right]\Big\},$$

where

$$A_3(n) = \frac{n(n+1)}{b_n^4} + \frac{1}{b_n^2} + \frac{a_n}{b_n^3}\left(\frac{n}{b_n}+\frac{a_n}{b_n}\right), A_4(n) = \frac{1}{b_n^3}\left(\frac{n}{b_n}+\frac{a_n}{b_n}\right), A_5(n) = \frac{1}{b_n^4}.$$

Finally, since $A_5(n) \le A_4(n) < A_3(n) < A_2(n)$ for all $n \in \mathbb{N}$ we can write

$$K_n((t-x)^4;x) \le 12\Big\{A_1(n)x^4 + m_7(n)A_2(n)\left(x^3+x^2+x+1\right)\Big\}$$
$$\le 12m_7(n)\Big\{A_1(n)x^4 + A_2(n)\left(x^3+x^2+x+1\right)\Big\}$$

which gives the desired result. □

3. Direct results

Let $C_B[0,\infty)$ denote the space of real valued continuous and bounded functions f on the interval $[0,\infty)$, endowed with the norm

$$\|f\| = \sup_{0\le x<\infty} |f(x)|.$$

For any $\delta > 0$, Peetre's K-functional is defined by

$$K_2(f;\delta) = \inf_{g\in C_B^2[0,\infty)} \{\|f-g\| + \delta\, \|g''\|\},$$

where $C_B^2[0,\infty) = \{g \in C_B[0,\infty) : g', g'' \in C_B[0,\infty)\}$. By DeVore and Lorentz ([9], p.177, Theorem 2.4) there exists an absolute constant $C > 0$ such that

$$K_2(f;\delta) \le C\omega_2(f;\sqrt{\delta}), \tag{3.1}$$

where the second order modulus of smoothness of $f \in C_B[0,\infty)$ is defined as

$$\omega_2(f;\delta) = \sup_{0<h\le\delta}\ \sup_{0\le x<\infty} |f(x+2h) - 2f(x+h) + f(x)|.$$

Also usual modulus of continuity of $f \in C_B[0,\infty)$ is defined by

$$\omega(f;\delta) = \sup_{0<h\le\delta}\ \sup_{0\le x<\infty} |f(x+h) - f(x)|.$$

Now consider the following operator

$$\widehat{K}_n(f;x) = K_n(f;x) - f\left(\frac{n}{b_n}x + \frac{a_n}{b_n}\frac{x}{1+x} + \frac{m_0(n)}{2b_n}\right) + f(x),$$

where $m_0(n)$ given as in Lemma 2.2.

Lemma 3.1. *Let $g \in C_B^2[0,\infty)$. Then we have*

$$\left|\widehat{K}_n(g;x) - g(x)\right| \le \delta_n(x)\|g''\|,$$

where

$$\delta_n(x) = K_n((t-x)^2;x) + \left[\left(\frac{n}{b_n} - 1\right)x + \frac{a_n}{b_n}\frac{x}{1+x} + \frac{m_0(n)}{2b_n}\right]^2.$$

Proof. By the definition of the operators $\widehat{K}_n$ and Lemma 2.2 we get

$$\begin{aligned}\widehat{K}_n(t-x;x) &= K_n(t-x,x) - \left(\frac{n}{b_n}x + \frac{a_n}{b_n}\frac{x}{1+x} + \frac{m_0(n)}{2b_n} - x\right) \\ &= K_n(t,x) - xK_n(1,x) - \left(\frac{n}{b_n}x + \frac{a_n}{b_n}\frac{x}{1+x} + \frac{m_0(n)}{2b_n} - x\right) \\ &= 0.\end{aligned}$$

Let $g \in C_B^2[0,\infty)$ and $x \in [0,\infty)$. By Taylor's formula of g

$$g(t) - g(x) = (t-x)g'(x) + \int_x^t (t-u)g''(u)du, \quad t \in [0,\infty)$$

one may write

$$\begin{aligned}&\widehat{K}_n(g;x) - g(x) \\ =&g'(x)\widehat{K}_n(t-x,x) + \widehat{K}_n\left(\int_x^t (t-u)g''(u)du;x\right) \\ =&\widehat{K}_n\left(\int_x^t (t-u)g''(u)du;x\right) \\ =&K_n\left(\int_x^t (t-u)g''(u)du;x\right) \\ &- \int_x^{\frac{n}{b_n}x+\frac{a_n}{b_n}\frac{x}{1+x}+\frac{m_0(n)}{2b_n}} \left(\frac{n}{b_n}x + \frac{a_n}{b_n}\frac{x}{1+x} + \frac{m_0(n)}{2b_n} - u\right) g''(u)du.\end{aligned}$$

Now using the following inequalities

$$\left|\int_x^t (t-u)g''(u)du\right| \leq (t-x)^2 \|g''\|$$

and

$$\begin{aligned}&\left|\int_x^{\frac{n}{b_n}x+\frac{a_n}{b_n}\frac{x}{1+x}+\frac{m_0(n)}{2b_n}} \left(\frac{n}{b_n}x + \frac{a_n}{b_n}\frac{x}{1+x} + \frac{m_0(n)}{2b_n} - u\right) g''(u)du\right| \\ &\leq \left[\left(\frac{n}{b_n} - 1\right)x + \frac{a_n}{b_n}\frac{x}{1+x} + \frac{m_0(n)}{2b_n}\right]^2 \|g''\|\end{aligned}$$

we reach to

$$\begin{aligned}&\left|\widehat{K}_n(g;x) - g(x)\right| \\ &\leq \left\{K_n((t-x)^2;x) + \left[\left(\frac{n}{b_n} - 1\right)x + \frac{a_n}{b_n}\frac{x}{1+x} + \frac{m_0(n)}{2b_n}\right]^2\right\} \|g''\| \\ &= \delta_n(x)\|g''\|.\end{aligned}$$

□

Theorem 3.2. *Let $f \in C_B[0,\infty)$. Then for all $x \in [0,\infty)$ there exists a constant $A > 0$ such that*

$$|K_n(f;x) - f(x)| \le A\omega_2\left(f;\sqrt{\delta_n(x)}\right) + \omega\left(f;\left|\frac{n}{b_n} - 1\right|x + \frac{a_n}{b_n}\frac{x}{1+x} + \frac{m_0(n)}{2b_n}\right),$$

where $\delta_n(x)$ defined as in Lemma 3.1.

Proof. By means of the definitions of the operators $\widehat{K}_n$ and K_n we have

$$\begin{aligned}|K_n(f;x) - f(x)| \le & \left|\widehat{K}_n(f-g;x)\right| + |(f-g)(x)| + \left|\widehat{K}_n(g;x) - g(x)\right| \\ & + \left|f\left(\frac{n}{b_n}x + \frac{a_n}{b_n}\frac{x}{1+x} + \frac{m_0(n)}{2b_n}\right) - f(x)\right|\end{aligned}$$

and

$$\left|\widehat{K}_n(f;x)\right| \le |K_n(f;x)| + 2\|f\| \le \|f\|K_n(1;x) + 2\|f\| = 3\|f\|.$$

Thus we may conclude that

$$\begin{aligned}|K_n(f;x) - f(x)| \le & 4\|f-g\| + \left|\widehat{K}_n(g;x) - g(x)\right| \\ & + \left|f\left(\frac{n}{b_n}x + \frac{a_n}{b_n}\frac{x}{1+x} + \frac{m_0(n)}{2b_n}\right) - f(x)\right|.\end{aligned}$$

In the light of Lemma 3.1 one gets

$$\begin{aligned}|K_n(f;x) - f(x)| \le & 4\|f-g\| + \delta_n(x)\|g''\| \\ & + \omega\left(f;\left|\frac{n}{b_n} - 1\right|x + \frac{a_n}{b_n}\frac{x}{1+x} + \frac{m_0(n)}{2b_n}\right).\end{aligned}$$

Therefore taking the infimum over all $g \in C_B^2[0,\infty)$ on the right-hand side of the last inequality and considering (3.1), we find that

$$\begin{aligned}|K_n(f;x) - f(x)| &\le 4K_2(f;\delta_n(x)) + \omega\left(f;\left|\frac{n}{b_n} - 1\right|x + \frac{a_n}{b_n}\frac{x}{1+x} + \frac{m_0(n)}{2b_n}\right) \\ &\le 4C\omega_2\left(f;\sqrt{\delta_n(x)}\right) + \omega\left(f;\left|\frac{n}{b_n} - 1\right|x + \frac{a_n}{b_n}\frac{x}{1+x} + \frac{m_0(n)}{2b_n}\right) \\ &= A\omega_2\left(f;\sqrt{\delta_n(x)}\right) + \omega\left(f;\left|\frac{n}{b_n} - 1\right|x + \frac{a_n}{b_n}\frac{x}{1+x} + \frac{m_0(n)}{2b_n}\right)\end{aligned}$$

which completes the proof. □

Theorem 3.3. *Let $0 < \gamma \le 1$ and $f \in C_B[0,\infty)$. Then if $f \in Lip_M(\gamma)$, that is, the inequality*

$$|f(t) - f(x)| \le M\,|t-x|^\gamma, \quad x,t \in [0,\infty)$$

holds, then for each $x \in [0,\infty)$ we have

$$|K_n(f;x) - f(x)| \le \delta_n^{\frac{\gamma}{2}}(x),$$

where $\delta_n(x) = K_n((t-x)^2;x)$ and $M > 0$ is a constant.

Proof. Let $f \in C_B[0,\infty) \cap Lip_M(\gamma)$. By the linearity and monotonicity of the operators K_n we get

$$\begin{aligned}|K_n(f;x) - f(x)| &\leq K_n\left(|f(t) - f(x)|\,; x\right) \\ &\leq MK_n\left(|t-x|^\gamma\,; x\right) \\ &= M\sum_{k=0}^{\infty} S_{n,a_n}(k,x)\frac{b_n}{d_n - c_n}\int_{\frac{k+c_n}{b_n}}^{\frac{k+d_n}{b_n}} |t-x|^\gamma\, dt.\end{aligned}$$

Now applying the Hölder inequality two times successively with $p = \frac{2}{\gamma}$, $q = \frac{2}{2-\gamma}$, we obtain

$$\begin{aligned}|K_n(f;x) - f(x)| &\leq M\sum_{k=0}^{\infty} S_{n,a_n}(k,x)\left\{\frac{b_n}{d_n - c_n}\int_{\frac{k+c_n}{b_n}}^{\frac{k+d_n}{b_n}} (t-x)^2 dt\right\}^{\frac{\gamma}{2}} \\ &\leq MK_n\left((t-x)^2; x\right)^{\frac{\gamma}{2}} \\ &= M\delta_n^{\frac{\gamma}{2}}(x).\end{aligned}$$

This completes the proof. □

4. Weighted approximation properties

Now we introduce convergence properties of the operators K_n via the weighted Korovkin type theorem given by Gadjiev in [14, 15]. For this purpose we recall some definitions and notations.

Let $\rho(x) = 1 + x^2$ and $B_\rho[0,\infty)$ be the space of all functions having the property

$$|f(x)| \leq M_f\rho(x),$$

where $x \in [0,\infty)$ and M_f is a positive constant depending only on f. $B_\rho[0,\infty)$ is equipped with the norm

$$\|f\|_\rho = \sup_{0\leq x<\infty}\frac{|f(x)|}{1+x^2}.$$

$C_\rho[0,\infty)$ denotes the space of all continuous functions belonging to $B_\rho[0,\infty)$. By $C_\rho^0[0,\infty)$ we denote the subspace of all functions $f \in C_\rho[0,\infty)$ for which

$$\lim_{x\to\infty}\frac{|f(x)|}{\rho(x)} < \infty.$$

Theorem A [14, 15]**:** *Let $\{A_n\}$ be a sequence of positive linear operators acting from $C_\rho[0,\infty)$ to $B_\rho[0,\infty)$ and satisfying the conditions*

$$\lim_{n\to\infty} ||A_n(t^\nu; x) - x^\nu||_\rho = 0, \quad \nu = 0,1,2.$$

Then for any function $f \in C_\rho^0[0,\infty)$,

$$\lim_{n\to\infty} ||A_n(f;x) - f(x)||_\rho = 0.$$

Note that a sequence of linear positive operators A_n acts from $C_\rho[0,\infty)$ to $B_\rho[0,\infty)$ if and only if

$$||A_n(\rho;x)||_\rho \leq M\rho,$$

where $M\rho$ is positive constant. This fact is a simple result of the necessary and sufficient condition that

$$A_n(\rho;x) \leq M\rho(x)$$

given in [14, 15].

Theorem 4.1. *Let $\{K_n\}$ be the sequence of linear positive operators defined by (1.2). Then for each $f \in C_\rho^0[0,\infty)$, we have*

$$\lim_{n\to\infty} ||K_n(f;x) - f(x)||_\rho = 0.$$

Proof. Using Lemma 2.2, we may write

$$\sup_{0\leq x<\infty} \frac{|K_n(\rho;x)|}{1+x^2} \leq 1 + \frac{n(n+1)}{b_n^2} + \frac{2a_n n}{b_n^2} + \frac{a_n^2}{b_n^2} + \frac{nm_1(n)}{b_n^2} + \frac{a_n m_1(n)}{b_n^2} + \frac{m_2(n)}{3b_n^2}.$$

Since $\lim_{n\to\infty} \frac{n}{b_n} = 1$ we have $\lim_{n\to\infty} \frac{1}{b_n} = 0$. Thus under the conditions (i) and (ii), there exists a positive constant M^* such that

$$\frac{n(n+1)}{b_n^2} + \frac{2a_n n}{b_n^2} + \frac{a_n^2}{b_n^2} + \frac{nm_1(n)}{b_n^2} + \frac{a_n m_1(n)}{b_n^2} + \frac{m_2(n)}{3b_n^2} < M^*$$

for each n. Hence we get

$$||K_n(\rho;x)||_\rho \leq 1 + M^*$$

which shows that $\{K_n\}$ is a sequence of positive linear operators acting from $C_\rho[0,\infty)$ to $B_\rho[0,\infty)$.

In order to complete the proof, it is enough to prove that the conditions of Theorem A

$$\lim_{n\to\infty} ||K_n(t^\nu;x) - x^\nu||_\rho = 0, \quad \nu = 0,1,2$$

are satisfied. It is clear that

$$\lim_{n\to\infty} ||K_n(1;x) - 1||_\rho = 0.$$

By Lemma 2.2, we have

$$\begin{aligned} ||K_n(t;x) - x||_\rho &= \sup_{0\leq x<\infty} \left| \left(\frac{n}{b_n} - 1\right) \frac{x}{1+x^2} + \frac{a_n}{b_n} \frac{x}{(1+x)(1+x^2)} + \frac{m_0(n)}{2b_n} \frac{1}{1+x^2} \right| \\ &\leq \left| \frac{n}{b_n} - 1 \right| + \frac{a_n}{b_n} + \frac{m_0(n)}{b_n}. \end{aligned}$$

Thus taking into consideration the conditions (i) and (ii) we can conclude that

$$\lim_{n\to\infty} ||K_n(t;x) - x||_\rho = 0.$$

Similarly, one gets

$$
\begin{aligned}
&||K_n(t^2;x)-x^2||_\rho \\
&= \sup_{0\le x<\infty}\left|\left(\frac{n(n+1)}{b_n^2}-1\right)\frac{x^2}{1+x^2}+\frac{2a_n n}{b_n^2}\frac{x^2}{(1+x)(1+x^2)}+\frac{a_n^2}{b_n^2}\frac{x^2}{(1+x)^2(1+x^2)}\right. \\
&\quad \left.+\frac{nm_1(n)}{b_n^2}\frac{x}{1+x^2}+\frac{a_n m_1(n)}{b_n^2}\frac{x}{(1+x)(1+x^2)}+\frac{m_2(n)}{3b_n^2}\frac{1}{(1+x^2)}\right| \\
&\le \left|\frac{n(n+1)}{b_n^2}-1\right|+\frac{2a_n n}{b_n^2}+\frac{a_n^2}{b_n^2}+\frac{nm_1(n)}{b_n^2}+\frac{a_n m_1(n)}{b_n^2}+\frac{m_2(n)}{b_n^2}
\end{aligned}
$$

which leads to

$$\lim_{n\to\infty}||K_n(t^2;x)-x^2||_\rho = 0.$$

Thus the proof is completed. □

Now we compute the order of approximation of the operators K_n in terms of the weighted modulus of continuity $\Omega_2(f,\delta)$ (see[17]) defined by

$$\Omega_2(f,\delta) = \sup_{x\ge 0, 0<h\le\delta} = \frac{|f(x+h)-f(x)|}{1+(x+h)^2}, \quad f\in C_\rho^0[0,\infty)$$

and has the following properties:
(a) $\Omega_2(f,\delta)$ is monotone increasing function of δ,
(b) $\lim_{\delta\to 0^+}\Omega_2(f,\delta)=0$,
(c) for each $\lambda\in\mathbb{R}^+, \Omega_2(f,\lambda\delta)\le(\lambda+1)\Omega_2(f,\delta)$.

Theorem 4.2. *Let $\{K_n\}$ be the sequence of linear positive operators defined by (1.2). Then for each $f\in C_\rho^0[0,\infty)$, we have*

$$\sup_{0\le x<\infty}\frac{|K_n(f;x)-f(x)|}{(1+x^2)^3}\le C\Omega_2\left(f,[m_7(n)A(n)]^{\frac14}\right),$$

where C is positive constant and $m_7(n)$ and $A(n)$ defined as in Lemma 2.2 and Lemma 2.4, respectively.

Proof. For $x\ge 0$ and $t\ge 0$, by the definition of $\Omega_2(f,\delta)$ and the property (c), we may write

$$
\begin{aligned}
|f(t)-f(x)| &\le \left(1+(x+|t-x|)^2\right)\left(1+\frac{|t-x|}{\delta_n}\right)\Omega_2(f,\delta_n) \\
&\le 2(1+x^2)\left(1+(t-x)^2\right)\left(1+\frac{|t-x|}{\delta_n}\right)\Omega_2(f,\delta_n).
\end{aligned}
$$

By using the monotonicity of K_n and the following inequality (see [16])

$$\left(1+(t-x)^2\right)\left(1+\frac{|t-x|}{\delta_n}\right)\le 2(1+\delta_n^2)\left(1+\frac{(t-x)^4}{\delta_n^4}\right)$$

one gets

$$\begin{aligned}|K_n(f;x)-f(x)| &\le 2(1+x^2)K_n\left(\left(1+(t-x)^2\right)\left(1+\frac{|t-x|}{\delta_n}\right);x\right)\Omega_2(f,\delta_n)\\ &\le 4(1+\delta_n^2)(1+x^2)K_n\left(1+\frac{(t-x)^4}{\delta_n^4};x\right)\Omega_2(f,\delta_n)\\ &= 4(1+\delta_n^2)(1+x^2)\left[1+\frac{1}{\delta_n^4}K_n((t-x)^4;x)\right]\Omega_2(f,\delta_n)\\ &\le C_1(1+x^2)\left[1+\frac{1}{\delta_n^4}K_n((t-x)^4;x)\right]\Omega_2(f,\delta_n).\end{aligned}$$

With the help of the Lemma 2.4 this inequality leads to

$$|K_n(f;x)-f(x)| \le 12C_1(1+x^2)\left[1+\frac{m_7(n)A(n)}{\delta_n^4}(x^4+x^3+x^2+x+1)\right]\Omega_2(f,\delta_n)$$

which gives the required result. □

We observe that in Theorem 4.1 we have showed that K_n converges to f in the weighted space $C\rho[0,\infty)$. But in Theorem 4.2 we have computed the rate of convergence for these operators in the weighted space $C\rho^3[0,\infty)$.

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The weighted mean operator on ℓ^2 with weight sequence $w_n = (n+1)^p$ is hyponormal for $p = 2$

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Abstract. Posinormality is used to demonstrate that the weighted mean matrix whose weight sequence is the sequence of squares of positive integers is a hyponormal operator on ℓ^2.

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1. Introduction

In this paper, attention will be focused on an example of a weighted mean matrix that does not satisfy the sufficient conditions for hyponormality given in [**4**]. Nor does it satisfy the key lemma used in [5], so a somewhat different approach will be required here. The computations here are much more complex than those in [5], and, because of that, the computer software package SAGE [6] has been used as an aid.

If $B(H)$ denotes the set of all bounded linear operators on a Hilbert space H, then $A \in B(H)$ is said to be is *posinormal* (see [1], [2]) if $AA^* = A^*PA$ for some positive operator $P \in B(H)$, called the *interrupter*, and A is *hyponormal* if $A^*A - AA^* \geq 0$. Hyponormal operators are necessarily posinormal.

A lower triangular infinite matrix $M = [m_{ij}]$, acting through multiplication to give a bounded linear operator on ℓ^2, is *factorable* if its entries are of the form

$$m_{ij} = \begin{cases} a_i c_j & if \quad j \leq i \\ 0 & if \quad j > i \end{cases}$$

where a_i depends only on i and c_j depends only on j; the matrix M is *terraced* if $c_j = 1$ for all j. A *weighted mean matrix* is a lower triangular matrix with entries w_j/W_i, where $\{w_j\}$ is a nonnegative sequence with $w_0 > 0$, and $W_i = \sum_{j=0}^{i} w_j$. A weighted mean matrix is factorable, with $a_i = 1/W_i$ and $c_j = w_j$ for all i,j.

2. Main result

Under consideration here will be the weighted mean matrix M associated with the weight sequence $w_n = (n+1)^2$. As was also the case for $w_n = n+1$, this example is not easily seen to be hyponormal directly from the definition and fails to satisfy the sufficent conditions for hyponormality given in [4, Corollary 1], but it survives the necessary condition given in [4, Corollary 2]. Encouraged by the latter, we set out to prove that M is hyponormal. The next theorem will provide us our main tool – an expression for the interrupter P associated with the matrix M.

Theorem 2.1. *Suppose $M = [a_i c_j]$ is a lower triangular factorable matrix that acts as a bounded operator on ℓ^2 and that the following conditions are satisfied:*

(a) *both $\{a_n\}$ and $\{a_n/c_n\}$ are positive decreasing sequences that converge to 0, and*
(b) *the matrix B defined by $B = [b_{ij}]$ by*

$$b_{ij} = \begin{cases} c_i(\frac{1}{c_j} - \frac{1}{c_{j+1}}\frac{a_{j+1}}{a_j}) & if \quad i \le j; \\ -\frac{a_{j+1}}{a_j} & if \quad i = j+1; \\ 0 & if \quad i > j+1. \end{cases}$$

is a bounded operator on ℓ^2.
Then M is posinormal with interrupter $P = B^*B$. The entries of $P = [p_{ij}]$ are given by

$$p_{ij} = \begin{cases} \frac{c_j^2 c_{j+1}^2 a_{j+1}^2 + (\sum_{k=0}^{j} c_k^2)(c_{j+1}a_j - c_j a_{j+1})^2}{c_j^2 c_{j+1}^2 a_j^2} & if \quad i = j; \\ \frac{(c_i a_{i+1} - c_{i+1} a_i)[c_j(\sum_{k=0}^{j+1} c_k^2)a_{j+1} - c_{j+1}(\sum_{k=0}^{j} c_k^2)a_j]}{c_i c_{i+1} c_j c_{j+1} a_i a_j} & if \quad i > j; \\ \frac{(c_j a_{j+1} - c_{j+1} a_j)[c_i(\sum_{k=0}^{i+1} c_k^2)a_{i+1} - c_{i+1}(\sum_{k=0}^{i} c_k^2)a_i]}{c_i c_{i+1} c_j c_{j+1} a_i a_j} & if \quad i < j. \end{cases}$$

Proof. See [3]. □

We are now ready for the main result. The induction step in the proof below was aided by explicit computations using the computer software package SAGE [6].

Theorem 2.2. *The weighted mean matrix M associated with the weight sequence $w_n = (n+1)^2$ is hyponormal.*

Proof. One easily verifies that the weighed mean matrix M associated with $w_n = (n+1)^2$ satisfies the hypotheses of Theorem 2.1. For M to be hyponormal, we must have

$$\langle (M^*M - MM^*)f, f\rangle = \langle (M^*M - M^*PM)f, f\rangle = \langle (I-P)Mf, Mf\rangle \ge 0$$

for all f in ℓ^2. Consequently, we can conclude that M will be hyponormal when $Q :\equiv I - P \ge 0$; we note that the range of M contains all the e_n's from the standard orthonormal basis for ℓ^2.

Using the given weight sequence, we determine that the entries of $Q = [q_{mn}]$ are given by

$$q_{mn} = \begin{cases} \frac{60n^7+600n^6+2488n^5+5476n^4+6795n^3+4650n^2+1584n+207}{30(n+1)^3(n+2)^3(n+3)(2n+5)} & if \quad m=n; \\ -\frac{1}{30} \cdot \frac{10m^3+52m^2+93m+57}{(m+1)^2(m+2)^2(m+3)(2m+5)} \cdot \frac{(3n^2+7n+3)(2n+3)}{(n+1)(n+2)} & if \quad m>n; \\ -\frac{1}{30} \cdot \frac{10n^3+52n^2+93n+57}{(n+1)^2(n+2)^2(n+3)(2n+5)} \cdot \frac{(3m^2+7m+3)(2m+3)}{(m+1)(m+2)} & if \quad m<n. \end{cases}$$

In order to show that Q is positive, it suffices to show that Q_N, the N^{th} finite section of Q (involving rows $m = 0, 1, 2, ..., N$ and columns $n = 0, 1, 2, ..., N$), has positive determinant for each positive integer N. For columns $n = 0, 1, ..., N-1$, we multiply the $(n+1)^{st}$ column of Q_N by

$$z_n :\equiv \frac{(n+3)(2n+3)(3n^2+7n+3)}{(n+1)(2n+5)(3n^2+13n+13)}$$

and subtract from the n^{th} column. Call the new matrix Q'_N. Then we work with the rows of Q'_N. For $m = 0, 1, ..., N-1$, we multiply the $(m+1)^{st}$ row of Q'_N by z_m and subtract from the m^{th} row. This leads to the tridiagonal form

$$Y_N :\equiv \begin{pmatrix} d_0 & s_0 & 0 & \dots & 0 & 0 \\ s_0 & d_1 & s_1 & \dots & 0 & 0 \\ 0 & s_1 & d_2 & \dots & . & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & . & \dots & d_{N-1} & s_{N-1} \\ 0 & 0 & 0 & \dots & s_{N-1} & d_N \end{pmatrix},$$

where

$$d_n = q_{nn} - z_n q_{n,n+1} - z_n(q_{n+1,n} - z_n q_{n+1,n+1}) = q_{nn} - 2z_n q_{n,n+1} + z_n^2 q_{n+1,n+1} = \frac{144n^{11}+3192n^{10}+31216n^9+177540n^8+651210n^7+1613062n^6}{(n+1)^3(n+3)(n+4)(2n+5)^2(2n+7)(3n^2+13n+13)^2} + \frac{2743061n^5+3186210n^4+2460693n^3+1192988n^2+323673n+37086}{(n+1)^3(n+3)(n+4)(2n+5)^2(2n+7)(3n^2+13n+13)^2}$$

and $s_n = q_{n+1,n} - z_n q_{n+1,n+1} = -\frac{(n+3)(2n+3)(2n^2+10n+11)(3n^2+7n+3)}{(n+1)(n+2)(n+4)(2n+5)(2n+7)(3n^2+13n+13)}$ when $0 \le n \le N-1$; and

$$d_N = \frac{60N^7+600N^6+2488N^5+5476N^4+6795N^3+4650N^2+1584N+207}{30(N+1)^3(N+2)^3(N+3)(2N+5)}.$$

Note that $det\ Y_N = det\ Q'_N = det\ Q_N$. Next we transform Y_N into a triangular matrix with the same determinant, and we find that the new matrix has diagonal entries δ_n which are given by the recursion formula: $\delta_0 = d_0$, $\delta_n = d_n - s_{n-1}^2/\delta_{n-1}$ $(1 \le n \le N)$. An induction argument shows that

$$\delta_n \ge \frac{30(n+3)^6(2n+3)^2(2n^2+10n+11)^2(3n^2+7n+3)^2}{(n+1)^2(n+4)(2n+5)^2(2n+7)(3n^2+13n+13)^2 g(n)} > 0, \text{ where}$$

$$g(n) = 60n^7 + 1020n^6 + 7348n^5 + 29016n^4 + 67679n^3 + 93031n^2 + 69633n + 21860$$

for $0 \le n \le N-1$; note that $g(n)$ is the numerator obtained in d_N when N is replaced by $n+1$. Since d_N departs from the pattern set by the earlier d_n's, δ_N must be handled separately:

$$\delta_N \ge \frac{60N^7+600N^6+2488N^5+5476N^4+6795N^3+4650N^2+1584N+207}{30(N+1)^3(N+2)^4(N+3)(2N+5)} > 0.$$

Therefore $\det Q_N = \prod_{j=0}^{N} \delta_j > 0$, and the proof is complete. □

For the induction step in the proof above, the initial estimate for δ_n came from computing $\frac{s_{N-1}^2}{d_N}$ and then replacing N by $n+1$. From there, an adjustment was needed.

The verification of the induction step reduces to showing that a 19^{th} degree polynomial is positive for all $n \geq 1$. Below is the command that was given to SAGE to execute.

$n = \mathrm{var}\ ('n')$

expand((30 * (n + 2)^4 * (144 * n^11 + 3192 * n^10 + 31216 * n^9 + 177540 * n^8 + 651210 * n^7 + 1613062 * n^6 + 2743061 * n^5 + 3186210 * n^4 + 2460693 * n^3 + 1192988 * n^2 + 323673 * n + 37086) − (n + 1) * (n + 4) * (2 * n + 5) * (2 * n + 7) * (3 * n^2 + 13 * n + 13)^2 * (60 * (n − 1)^7 + 1020 * (n − 1)^6 + 7348 * (n − 1)^5 + 29016 * (n − 1)^4 + 67679 * (n − 1)^3 + 93031 * (n − 1)^2 + 69633 * (n − 1) + 21860)) * (60 * n^7 + 1020 * n^6 + 7348 * n^5 + 29016 * n^4 + 67679 * n^3 + 93031 * n^2 + 69633 * n + 21860) − 900 * (n + 1) * (n + 2)^4 * (n + 3)^7 * (2 * n + 3)^2 * (2 * n^2 + 10 * n + 11)^2 * (3 * n^2 + 7 * n + 3)^2)

And this is the resulting SAGE worksheet output, which we denote by $f(n)$.

f(n) = 220320 * n^19 + 8325216 * n^18 + 147344112 * n^17 + 1621610588 * n^16 + 12423804832 * n^15 + 70274637076 * n^14 + 303640886360 * n^13 + 1022365685883 * n^12 + 2710505167956 * n^11 + 5672704072899 * n^10 + 9319440019836 * n^9 + 11820506702133 * n^8 + 11159132582690 * n^7 + 7175130478741 * n^6 + 2225790478822 * n^5 − 894429232807 * n^4 − 1475079085458 * n^3 − 812545969449 * n^2 − 226952537400 * n − 26586925200

$f(0) = -26586925200$

$f(1) = 48058098267150$

$f(2) = 29447930357308764$

$f(3) = 2740303120043884194$

$f(4) = 100611201083636165760$

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Multiple symmetric solutions for some hemivariational inequalities

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Abstract. In the present paper we prove some multiplicity results for hemivariational inequalities defined on the unit ball or on the whole space. By variational methods, we demonstrate that the solutions of these inequalities are invariant by spherical cap symmetrization, the main tools being the symmetric version of Ekeland's variational principle proved by M. Squassina [11] and a nonsmooth version of the symmetric minimax principle due to J. Van Schaftingen [13].

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1. Introduction and main results

In this paper we are treating two different problems, which will be detailed below.

1.1. The first problem

Consider the following semi-linear elliptic differential inclusion problem, coupled with the homogeneous Dirichlet boundary condition:

$$\begin{cases} -\Delta_p u + |u|^{p-2}u \in \lambda \partial F(x, u(x)) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{$\mathcal{P}^1_\lambda$}$$

where λ is a positive parameter, $1 < p < N$, $\Omega \subset \mathbb{R}^N$ is the unit ball, $\Delta_p(u) = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is the p-Laplacian operator, and $\partial F(x, s)$ stands for the generalized gradient of the locally Lipschitz function $F : \Omega \times \mathbb{R} \to \mathbb{R}$ at the point $s \in \mathbb{R}$ with respect to the second variable (see for details Section 2). Here and in the sequel $|\cdot|$ denotes the Euclidean norm in $\mathbb{R}^N$.

Such problems arise mostly in mathematical physics, where solutions of elliptic problems correspond to certain equilibrium state of the physical system. This is the reason why problems of this type were intensively studied by several authors in the last years.

In the study of PDE-s are often used different symmetrization techniques. We can find many papers where the solutions are for e.g. radially symmetric functions (see Squassina [12]), axially symmetric functions (Kristály, Mezei in [7]) or has some symmetry properties with respect to certain group actions (Farkas, Mezei in [5]). Recently was applied the spherical cap and Schwarz symmetrization for such problems. Van Schaftingen in [13] and Squassina in [11] developed an abstract framework for the symmetrizations. Using their results, Filipucci, Pucci, Varga in [9] obtained existence results of some eigenvalue problems and Farkas, Varga in [6] proved multiplicity results for a model quasi-linear elliptic system in case of C^1 functionals.

The purpose of our paper is to extend these results for locally Lipschitz functions. We ensure the existence of multiple spherical cap symmetric solutions for the problem $(\mathcal{P}_\lambda^1)$, where the natural functional space is the Sobolev space $W_0^{1,p}(\Omega)$, endowed with its standard norm

$$\|u\| = \left(\int_\Omega |\nabla u(x)|^p + \int_\Omega |u(x)|^p\right)^{1/p}.$$

In order to obtain our result, we need the following assumptions on the function F:

$(\mathbf{F_1})$ $\displaystyle\lim_{|s|\to 0} \frac{\max\{|\xi| : \xi \in \partial F(x,s)\}}{|s|^{p-1}} = 0;$

$(\mathbf{F_2})$ $\displaystyle\lim_{|s|\to +\infty} \frac{\max\{|\xi| : \xi \in \partial F(x,s)\}}{|s|^{p-1}} = 0;$

$(\mathbf{F_3})$ There exists an $u_0 \in W_0^{1,p}(\Omega)$, $u_0 \neq 0$ such that

$$\int_\Omega F(x, u_0(x))dx > 0.$$

$(\mathbf{F_4})$ $F(x,s) = F(y,s)$ for a.e. $x, y \in \Omega$, with $|x| = |y|$ and all $s \in \mathbb{R}$;

$(\mathbf{F_5})$ $F(x,s) \leq F(x,-s)$ for a.e. $x \in \Omega$ and all $s \in \mathbb{R}^-$.

The first main result of the paper is the following:

Theorem 1.1. *Assume that $1 < p < N$. Let $\Omega \subset \mathbb{R}^N$ be the unit ball and $F : \Omega\times\mathbb{R} \to \mathbb{R}$ be a locally Lipschitz function with $F(x,0) = 0$, satisfying $(\mathbf{F_1})$-$(\mathbf{F_5})$. Then,*

- (a) *there exists a λ_F such that, for every $0 < \lambda \leq \lambda_F$ the problem $(\mathcal{P}_\lambda^1)$ has only the trivial solution;*
- (b) *there exists a λ_1 such that, for every $\lambda > \lambda_1$ the problem $(\mathcal{P}_\lambda^1)$ has at least two weak solutions in $W_0^{1,p}(\Omega)$, invariants by spherical cap symmetrization (for details, see Section 2).*

Remark 1.1. *Choosing $p = 3$, the function $F : \Omega \times \mathbb{R} \to \mathbb{R}$ defined by*

$$F(x,s) = \begin{cases} |x|(s^4 - s^2), & \text{if } |s| \leq 1 \\ |x| \ln s^2, & \text{if } |s| > 1. \end{cases} \tag{1.1}$$

fulfills the hypotheses $(\mathbf{F_1})$-$(\mathbf{F_5})$.

1.2. The second problem

Let $\Omega = \mathbb{R}^N$. Consider a real, separable, reflexive Banach space $(X, \|\cdot\|_X)$ and its topological dual $(X^\star, \|\cdot\|_{X^\star})$. Let $F : \Omega \times \mathbb{R} \to \mathbb{R}$ a locally Lipschitz function. In addition, let p be such that $2 \le p < N$, while $p^\star = \dfrac{Np}{N-p}$ denotes the Sobolev critical exponent.

Our second problem is formulated as follows:

Find $u \in X$ such that

$$\langle Au, v\rangle + \int_{\mathbb{R}^N} F_y^0(x; u(x); -v(x))dx \ge 0, \ \forall v \in X, \qquad (\mathcal{P}_\lambda^2)$$

where F_y^0 denotes the generalized directional derivative of F in the second variable.

In order to derive our second existence result, we need to impose the following hypotheses:

(CT) Suppose that for $r \in [p, p^\star]$, the inclusion $X \hookrightarrow L^r(\mathbb{R}^N)$ is continuous with the embedding constant C_r.

(**CP**) Assume that for $r \in (p, p^\star)$, the embedding $X \hookrightarrow L^r(\mathbb{R}^N)$ is compact.

Notice that $\langle\cdot,\cdot\rangle$ denotes the duality pairing between $X^\star$ and X and $||\cdot||_r$ is the norm of $L^r(\mathbb{R}^N)$.

Let $A : X \to X^\star$ be a potential operator with the potential $a : X \to \mathbb{R}$, that is, a is Gâteaux differentiable and for every $u, v \in X$ we have

$$\lim_{t\to 0} \frac{a(u+tv) - a(u)}{t} = \langle A(u), v\rangle.$$

For a potential we always assume that $a(0) = 0$. In addition, we suppose that $A : X \to X^\star$ satisfies the following properties:

($\mathbf{A_1}$) A is hemicontinuous, i.e. A is continuous on line segments in X and $X^\star$ equipped with the weak topology.

($\mathbf{A_2}$) A is homogeneous of degree $p-1$, i.e. for every $u \in X$ and $t > 0$ we have $A(tu) = t^{p-1}A(u)$.

($\mathbf{A_3}$) $A : X \to X^\star$ is a strongly monotone operator, i.e. there exists a continuous function $\tau : [0, \infty) \to [0, \infty)$ which is strictly positive on $(0, \infty)$, $\tau(0) = 0$, $\lim_{t\to\infty} \tau(t) = \infty$ and

$$\langle A(u) - A(v), u - v\rangle \ge \tau(||u - v||_X)||u - v||_X,$$

for all $u, v \in X$.

($\mathbf{A_4}$) $a(u) \ge c\|u\|_X^p$, for all $u \in X$, where c is a positive constant.

($\mathbf{A_5}$) $a(u^H) \le a(u)$, for all $u \in X$, where u^H denotes the polarization of u (for details, see Section 2.).

Remark 1.2. *By conditions* ($\mathbf{A_1}$) *and* ($\mathbf{A_2}$), *we have* $a(u) = \frac{1}{p}\langle A(u), u\rangle$.

Furthermore, we suppose that the following additional condition holds: there exists $c > 0$ and $r \in (p, p^\star)$ such that

($\mathbf{F_1'}$) $|\xi| \le c(|s|^{p-1} + |s|^{r-1}), \forall s \in \mathbb{R}, \xi \in F(x, s)$ and a.e. $x \in \mathbb{R}^N$.

Moreover, instead of ($\mathbf{F_2}$), we assume that:

($\mathbf{F_2'}$) there exists $q \in (0,p)$, $\nu \in (p,p^\star)$, $\alpha \in L^{\frac{\nu}{\nu-q}}(\mathbb{R}^N)$, $\beta \in L^1(\mathbb{R}^N)$ such that

$$F(z,s) \le \alpha(z)|s|^q + \beta(z)$$

for all $s \in \mathbb{R}$ and a.e. $z \in \mathbb{R}^N$.

Remark 1.3. *When Ω is the unit ball, by conditions* ($\mathbf{F_1}$) *and* ($\mathbf{F_2}$)*, we can deduce the assumption* ($\mathbf{F_1'}$)*. But in the case of this second problem when we assume that $\Omega = \mathbb{R}^N$, we really need the condition* ($\mathbf{F_1'}$)*.*

Now we can state our second main result:

Theorem 1.2. *Assume that $2 \le p < N$ and let $\Omega = \mathbb{R}^N$. Let $F : \Omega \times \mathbb{R} \to \mathbb{R}$ be a locally Lipschitz function, $A : X \to X^\star$ be a potential operator such that the conditions* ($\mathbf{A_1}$) $-$ ($\mathbf{A_5}$) *and* (**CT**), (**CP**), ($\mathbf{F_1}$), ($\mathbf{F_1'}$), ($\mathbf{F_2'}$), ($\mathbf{F_4}$), ($\mathbf{F_5}$) *are fulfilled. Then, there exists $\lambda_2 > 0$ such that for every $\lambda > \lambda_2$ the problem $(\mathcal{P}_\lambda^2)$ has two nontrivial solutions, which are invariants by spherical cap symmetrization.*

The energy functional related to the problem $(\mathcal{P}_\lambda^2)$ is defined as follows:

$$\mathscr{A}_\lambda(u) = a(u) - \lambda\tilde{\mathcal{F}}(u),$$

where $\tilde{\mathcal{F}} : X \to \mathbb{R}$ is a function defined by $\tilde{\mathcal{F}}(u) = \displaystyle\int_{\mathbb{R}^N} F(x,u(x))dx.$

Remark 1.4. *We observe that, using Proposition 5.1.2. from Cs. Varga and A. Kristály* [8]*, due to condition* ($\mathbf{F_1'}$)*, we have that*

$$\tilde{\mathcal{F}}^0(u;v) \le \int_{\mathbb{R}^N} F_y^0(x,u(x);v(x))dx. \tag{1.2}$$

Therefore, it follows that the critical points of the energy functional $\mathscr{A}_\lambda$ are the (weak) solutions of the problem $(\mathcal{P}_\lambda^2)$.

2. Preliminaries and abstract framework

In this section we give a brief overview on some preparatory results used in the sequel.

2.1. Locally Lipschitz functions

In the following, we recall some basic definitions and properties from the theory developed by F. Clarke [4].

Let E be a Banach space, E^* be its topological dual space, V be an open subset of E and $f : V \to \mathbb{R}$ be a functional.

Definition 2.1. *The functional $f : V \to \mathbb{R}$ is called locally Lipschitz if every point $v \in V$ possesses a neighborhood $\mathcal{V}$ such that*

$$|f(z) - f(w)| \le K_v\|z - w\|_E, \quad \forall w, z \in \mathcal{V},$$

for a constant $K_v > 0$ which depends on $\mathcal{V}$.

Definition 2.2. *The generalized derivative of a locally Lipschitz functional $f : V \to \mathbb{R}$ at the point $v \in V$ along the direction $w \in E$ is denoted by $f^0(v; w)$, i.e.*

$$f^0(v;w) = \limsup_{\substack{z \to v \\ t \searrow 0}} \frac{f(z+tw) - f(z)}{t}.$$

We recall here some useful properties of the generalized directional derivative for locally Lipschitz functions (see F. Clarke [4]).

Definition 2.3. *Let E be a Banach space. A locally Lipschitz functional $h : E \to \mathbb{R}$ is said to satisfy the* non-smooth Palais-Smale condition at level $c \in \mathbb{R}$ (*for brevity we shall use the notation $(PS)_c$-condition*) *if any sequence $\{u_n\} \subset E$ which satisfies*

(i) $h(u_n) \to c$;
(ii) *there exists $\{\varepsilon_n\} \subset \mathbb{R}$, $\varepsilon_n \downarrow 0$ such that $h^0(u_n; v - u_n) + \varepsilon_n \|v - u_n\|_e \geq 0$, for all $v \in E$ and all $n \in \mathbb{N}$*

admits a convergent subsequence. If this is true for every $c \in \mathbb{R}$, we say that h satisfies the non-smooth (PS)-condition.

Remark 2.1. *If we use the notation $\lambda_h(u) = \inf_{w \in \partial h(u)} \|w\|_{E^\star}$ (see K.-C. Chang* [3]*) and we replace the condition (ii) from the above definition with the following one:*
(ii)' $\lambda_h(u_n) \to 0$,
we obtain an equivalent definition with the Definition 2.3.

Definition 2.4. *The generalized gradient of $f : V :\to \mathbb{R}$ at the point $v \in V$ is a subset of E^*, defined by*

$$\partial f(v) = \{y^* \in E^* : \ \langle y^*, w\rangle \leq f^0(v; w), \quad \text{for each } w \in E\}. \tag{2.1}$$

Remark 2.2. *Using the Hahn-Banach theorem (see, for example H. Brezis* [2]*), it is easy to see that the set $\partial f(v)$ is nonempty for every $v \in E$.*

The next result will be crucial in the proofs of our main result.

Theorem 2.1. (Lebourg's Mean Value Theorem, F. Clarke [4]) *Let U be an open subset of a Banach space E, let x, y be two points of U such that the line segment $[x, y] = \{(1-t)x + ty : 0 \leq t \leq 1\}$ is contained in U and let $f : U \to \mathbb{R}$ be a locally Lipschitz function. Then there exists $u \in [x,\ y] \backslash \{x,\ y\}$ such that*

$$f(y) - f(x) = \langle z,\ y - x\rangle,$$

for some $z \in \partial f(u)$.

2.2. Abstract framework of symmetrization

Now we recall the definition of spherical cap symmetrization and polarization.

Definition 2.5 (Spherical cap symmetrization)**.** *Let $P \in \partial B(0,1) \cap \mathbb{R}^N$. The spherical cap symmetrization of the set A with respect to P is the unique set A^* such that $A^* \cap \{0\} = A \cap \{0\}$ and for any $r \geq 0$,*

$$A^* \cap \partial B(0,r) = B_g(rP, \rho) \cap \partial B(0,r) \text{ for some } \rho \geq 0,$$
$$\mathcal{H}^{N-1}(A^* \cap \partial B(0,r)) = \mathcal{H}^{N-1}(A \cap \partial B(0,r)),$$

where $\mathcal{H}^{N-1}$ is the outer Hausdorff $(N-1)$-dimensional measure and $B_g(rP,\rho)$ denotes the geodeisc ball on the sphere $\partial B(0,r)$ of center rP and radius ρ. By definition $B_g(rP,0)=\emptyset$.

Definition 2.6. *The spherical cap symmetrization of a function $f:\Omega\to\overline{\mathbb{R}}$ is the unique function $u^*:\Omega^*\to\overline{\mathbb{R}}$ such that, for all $c\in\mathbb{R}$,*

$$\{u^*>c\}=\{u>c\}^*.$$

Definition 2.7 (Polarization)**.** *A subset H of $\mathbb{R}^N$ is called a polarizer if it is a closed affine half-space of $\mathbb{R}^N$, namely the set of points x which satisfy $\alpha\cdot x\le\beta$ for some $\alpha\in\mathbb{R}^N$ and $\beta\in\mathbb{R}$ with $|\alpha|=1$. Given x in $\mathbb{R}^N$ and a polarizer H the reflection of x with respect to the boundary of H is denoted by x_H. The polarization of a function $u:\mathbb{R}^N\to\mathbb{R}^+$ by a polarizer H is the function $u^H:\mathbb{R}^N\to\mathbb{R}^+$ defined by*

$$u^H(x)=\begin{cases}\max\{u(x),u(x_H)\}, & \text{if } x\in H\\ \min\{u(x),u(x_H)\}, & \text{if } x\in\mathbb{R}^N\setminus H.\end{cases}\tag{2.2}$$

The polarization $C^H\subset\mathbb{R}^N$ of a set $C\subset\mathbb{R}^N$ is defined as the unique set which satisfies $\chi_{C^H}=(\chi_C)^H$, where χ denotes the characteristic function. The polarization u^H of a positive function u defined on $C\subset\mathbb{R}^N$ is the restriction to C^H of the polarization of the extension $\tilde{u}:\mathbb{R}^N\to\mathbb{R}^+$ of u by zero outside C. The polarization of a function which may change sign is defined by $u^H:=|u|^H$, for any given polarizer H.

Following J. Van Schaftingen [13], consider the abstract framework below:

Let X, V and W be three real Banach spaces, with $X\subset V\subset W$ and let $S\subset X$. For the clarity, we present some crucial abstract symmetrization and polarization results of J. Van Schaftingen [13] and of M. Squassina [11]. Let us first introduce the following main assumption.

Definition 2.8. *Let $\mathcal{H}_\star$ be a pathconnected topological space and denote by $h:S\times\mathcal{H}_\star\to S$, $(u,H)\mapsto u^H$, the polarization map. Let $\star:S\to V, u\mapsto u^\star$, be any symmetrization map. Assume that the following properties hold.*

1) *The embeddings $X\hookrightarrow V$ and $V\hookrightarrow W$ are continuous;*
2) *h is continuous;*
3) *$(u^\star)^H=(u^H)^\star=u^\star$ and $(u^H)^H=u^H$ for all $u\in S$ and $H\in\mathcal{H}_\star$;*
4) *for all $u\in S$ there exists a sequence $(H)_m\subset\mathcal{H}_\star$ such that $u^{H_1\dots H_m}\to u^\star$ in V;*
5) *$\|u^H-v^H\|_V\le\|u-v\|_V$ for all u, $v\in S$ and $H\in\mathcal{H}_\star$.*

Since there exists a map $\Theta:(X,\|\cdot\|_V)\to(S,\|\cdot\|_V)$ which is Lipschitz continuous, with Lipschitz constant $C_\Theta>0$, and such that $\Theta|_S=Id|_S$, both maps $h:S\times\mathcal{H}_\star\to S$ and $\star:S\to V$ can be extended to $h:X\times\mathcal{H}_\star\to S$ and $\star:X\to V$ by setting $u=(\Theta(u))^H$ and $u^\star=(\Theta(u))^\star$ for every $u\in X$ and $H\in\mathcal{H}_\star$.

The previous properties, in particular 4) and 5), and the definition of Θ easily yield that

$$\|u^H-v^H\|_V\le C_\Theta\|u-v\|_V,\qquad \|u^\star-v^\star\|_V\le C_\Theta\|u-v\|_V\tag{2.3}$$

for all u, $v\in X$ and for all $H\in\mathcal{H}_\star$.

Some known examples of spherical cap symmetrization with Dirichlet boundary and of Schwarz symmetrization are given by J. Van Schaftingen in [13].

2.3. Variational framework

We recall three results which will play an essential role in what follows.

Proposition 2.1. (*Proposition 3.3. of R. Filippucci et al.* [9]) *Let* $G : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ *be a Carathéodory function, satisfying property* $(\mathbf{F_4})$, *that is* $G(x, u) = G(y, u)$ *for a.e.* $x, y \in \mathbb{R}^N$, *with* $|x| = |y|$, *and all* $u \in \mathbb{R}$. *Then, for all* $H \in \mathcal{H}_\star$

$$\int_{\mathbb{R}^N} G(x, u(x))dx = \int_{\mathbb{R}^N} G(x, u^H(x))dx \tag{2.4}$$

along any $u : \mathbb{R}^N \to \mathbb{R}_0^+$, *with* $G(\cdot, u(\cdot)) \in L^1(\mathbb{R}^N)$.

Remark 2.3. *The statement of the above proposition remains valid if we choose* $\Omega = \Omega^H \subset \mathbb{R}$ *instead of the whole space* $\mathbb{R}^N$ *(see J. Van Schaftingen* [13, Proposition 2.19]*).*

In the paper of Cs. Varga and V. Varga [14] a quantitative deformation lemma is proved for locally Lipschitz functions. J. Van Schaftingen in [13], proves a symmetric version of this variational principle (see Theorem 3.5) for C^1 functionals. Using the mentioned results with slight modifications, we can prove the following symmetric variational principle for locally Lipschitz functionals.

Theorem 2.2. *Let* $(X, V, \star, \mathcal{H}_\star, S)$ *satisfy the assumptions of Definition* 2.8. *Denote by* $\kappa > 0$ *any constant with the property* $\|u\|_V \le \kappa\|u\|_X$ *for all* $u \in X$. *Let* $e \in X \setminus \{0\}$ *be fixed and*

$$\Gamma = \{\gamma : C([0, 1], X) : \gamma(0) = 0, \gamma(1) = e\}.$$

Consider also the locally Lipschitz functional $\Phi : X \to \mathbb{R}$, *which satisfies:*

1) $\infty > c := \inf_{\gamma \in \Gamma} \sup_{t \in [0,1]} \Phi(\gamma(t)) > a := \max\{\Phi(0), \Phi(e)\}$,
2) $\Phi(u^H) \le \Phi(u)$, *for all* $u \in S$ *and* $H \in \mathcal{H}_\star$.

Then for every $0 < \varepsilon < \frac{c-a}{2}$, $\delta > 0$ *and* $\gamma \in \Gamma$, *with the properties*

i) $\sup_{t \in [0,1]} \Phi(\gamma(t)) \le c + \varepsilon$;
ii) $\gamma([0, 1]) \subset S$;
iii) $\{\gamma(0), \gamma(1)\}^{H_0} = \{\gamma(0), \gamma(1)\}$ *for some* $H_0 \in \mathcal{H}_\star$,

there exists $u_\varepsilon \in X$ *such that*

a) $c - 2\varepsilon \le \Phi(u_\varepsilon) \le c + 2\varepsilon$;
b) $\|u_\varepsilon - u_\varepsilon^\star\|_V \le 2(2\kappa + 1)\delta$;
c) $\lambda_\Phi(u) \le 8\varepsilon/\delta$.

3. Proof of Theorem 1.1

Definition 3.1. *We say that* $u \in W_0^{1,p}(\Omega)$ *is a weak solution to problem* $(\mathcal{P}_\lambda^1)$ *if there exists* $\xi_F \in \partial F(x, u(x))$ *for a.e.* $x \in \Omega$ *such that for all* $v \in W_0^{1,p}(\Omega)$ *we have*

$$\int_\Omega (|\nabla u|^{p-2}\nabla u \nabla v + |u|^{p-2}uv)dx = \lambda \int_\Omega \xi_F v(x)dx. \tag{3.1}$$

We consider the functionals $I, \mathcal{F} : W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$I(u) = \frac{1}{p}\int_\Omega (|\nabla u|^p + |u|^p)dx, \qquad \mathcal{F}(u) = \int_\Omega F(x, u(x))dx.$$

Now, we can define the energy functional associated to the problem $(\mathcal{P}_\lambda^1)$ by

$$\mathscr{E}_\lambda(u) = I(u) - \lambda\mathcal{F}(u).$$

Remark 3.1. *If Ω is bounded, using* [10, Theorem 1.3]*, we have*

$$\partial\mathcal{F}(u) \subset \int_\Omega \partial F(x, u(x))dx.$$

Hence, the critical points of the energy functional $\mathscr{E}_\lambda$ are exactly the (weak) solutions of the problem $(\mathcal{P}_\lambda^1)$. So, instead of seeking for the solutions of the problem $(\mathcal{P}_\lambda^1)$, it is enough to look for the critical points of the energy functional $\mathscr{E}_\lambda$.

Before proving our main result, we prove that the functional $\mathscr{E}_\lambda$ is coercive and it satisfies the non-smooth Palais-Smale condition on $W_0^{1,p}(\Omega)$.

Lemma 3.1. *The functional $\mathscr{E}_\lambda : W_0^{1,p}(\Omega) \to \mathbb{R}$ is coercive for every $\lambda \geq 0$, that is, $\mathscr{E}_\lambda(u) \to \infty$ as $\|u\| \to \infty$, for all $u \in W_0^{1,p}(\Omega)$.*

Proof. Let us fix a $\lambda \geq 0$. In particular, from $(\mathbf{F_1})$, there exists a $\delta_1 > 0$ such that

$$|\xi| \leq \frac{1}{2}\cdot\frac{1}{p}c_p^{-p}\frac{1}{1+\lambda}|s|^{p-1}, |s| < \delta_1, \tag{3.2}$$

where c_p is the best Sobolev constant in the embedding $W_0^{1,p}(\Omega) \hookrightarrow L^q(\Omega)(q \in [1, p^*)])$.

Due to $(\mathbf{F_2})$, it follows that for every $\varepsilon > 0$ there exists $\delta_2 = \delta_2(\varepsilon) > 0$, such that

$$\max\{|\xi| : \xi \in \partial F(x, s)\} \leq \varepsilon|s|^{p-1}, |s| > \delta_2.$$

Moreover, if $\varepsilon = \frac{1}{2}\cdot\frac{1}{p}c_p^{-p}\cdot\frac{1}{1+\lambda}$, then for every $\xi \in \partial F(x, s)$ one, has

$$|\xi| \leq \frac{1}{2}\cdot\frac{1}{p}c_p^{-p}\frac{1}{1+\lambda}|s|^{p-1}, |s| > \delta_2. \tag{3.3}$$

Since the set-valued mapping ∂F is upper-semicontinuous, then there exists $C_F = \sup\{\partial F(x, [\delta_2, \delta_1])\}$, thus

$$|\xi| \leq \frac{1}{p}c_p^{-p}\frac{1}{1+\lambda}|s|^{p-1} + C_F, \quad \text{for all } s \in \mathbb{R}. \tag{3.4}$$

Now we can use Lebourg's mean value theorem (see Theorem 2.1), obtaining that:

$$|F(x, s)| = |F(x, s) - F(x, 0)| \leq |\xi_\theta s| \text{ for some } \xi_\theta \in \partial F(x, \theta s), \theta \in (0, 1).$$

Combining this inequality with the relation (3.4), we get

$$|F(x, s)| \leq \frac{1}{p}c_p^{-p}\frac{1}{1+\lambda}|s|^p + C_F|s|.$$

Moreover,

$$\mathscr{E}_\lambda(u) \geq \frac{1}{p}\|u\|^p - \frac{1}{p}\frac{\lambda}{1+\lambda}\left(\frac{\|u\|_p^p}{c_p^p}\right) - \lambda C_F\|u\|_1.$$

Therefore,

$$\begin{aligned}\mathscr{E}_\lambda(u) &\geq \frac{1}{p}\|u\|^p - \frac{1}{p}\frac{\lambda}{1+\lambda}\|u\|^p - \lambda\cdot C_1\|u\| \\ &= \frac{1}{p}\left(1-\frac{\lambda}{1+\lambda}\right)\|u\|^p - \lambda C_1\|u\| \to \infty\end{aligned}$$

as $\|u\| \to \infty$,where C_1 is a constant, which concludes our proof. □

Lemma 3.2. *For every $\lambda > 0$, $\mathscr{E}_\lambda$ satisfies the non-smooth Palais-Smale condition.*

Proof. Let $\lambda > 0$ be fixed. We consider a Palais-Smale sequence $\{u_n\} \subset W_0^{1,p}(\Omega)$ for $\mathscr{E}_\lambda$, i.e., for some $\varepsilon_n \to 0^+$, we have

$$\mathscr{E}_\lambda^o(u_n; u-u_n) \geq -\varepsilon_n\|u-u_n\| \tag{3.5}$$

and $\{\mathscr{E}_\lambda(u_n)\}$ is bounded in $W_0^{1,p}(\Omega)$. Since $\mathscr{E}_\lambda$ is coercive, the sequence $\{u_n\}$ is bounded. Therefore taking a subsequence if necessary, we may assume that $u_n \rightharpoonup u$ weakly in $W_0^{1,p}(\Omega)$ and $u_n \to u$ strongly in L^p (note that $W_0^{1,p}(\Omega) \hookrightarrow L^p(\Omega)$ is compact, see H. Brezis [2]). One clearly has,

$$\langle I'(u_n), u-u_n\rangle = \int_\Omega \left(|\nabla u_n|^{p-2}\nabla u_n\right)(\nabla u - \nabla u_n) + \int_\Omega |u_n|^{p-2}u_n(u-u_n),$$

and

$$\langle I'(u), u_n-u\rangle = \int_\Omega \left(|\nabla u|^{p-2}\nabla u\right)(\nabla u_n - \nabla u) + \int_\Omega |u|^{p-2}u(u_n-u).$$

Adding these two relations and from the fact that $|v-w|^p \leq (|v|^{p-2}v - |w|^{p-2}w)(v-w)$, one can conclude that

$$\begin{aligned}&\langle I'(u_n), u-u_n\rangle + \langle I'(u), u_n-u\rangle = \\ &\int_\Omega \left(|\nabla u_n|^{p-2}\nabla u_n - |\nabla u|^{p-2}\nabla u\right)(\nabla u - \nabla u_n) + \int_\Omega (|u_n|^{p-2}u_n - |u|^{p-2}u)(u-u_n) \\ &\leq \int_\Omega (-|\nabla u_n - \nabla u|^p - |u_n-u|^p) = -\|u_n-u\|^p.\end{aligned} \tag{3.6}$$

On the other hand, by the relations

$$\mathscr{E}_\lambda^o(u_n; u-u_n) = \langle I'(u_n); u-u_n\rangle + \lambda\mathcal{F}^o(u_n; u_n-u),$$

$$\mathscr{E}_\lambda^o(u; u_n-u) = \langle I'(u); u_n-u\rangle + \lambda\mathcal{F}^o(u; u-u_n),$$

and the inequalities (3.5) and (3.6), we have

$$\begin{aligned}\|u_n-u\|^p &\leq \varepsilon_n\|u-u_n\| - \mathscr{E}_\lambda^o(u; u_n-u) + \\ &+\lambda(\mathcal{F}^o(u_n; u_n-u) + \mathcal{F}^o(u; u-u_n))\end{aligned} \tag{3.7}$$

Since the sequence $\{u_n\}$ is bounded in $W_0^{1,p}(\Omega)$, we clearly have

$$\lim_{n\to\infty} \varepsilon_n \|u - u_n\| = 0. \tag{3.8}$$

Now fix $w^* \in \partial\mathscr{E}_\lambda(u)$. In particular, by the definition (2.1), we have $\langle w^*; u_n - u\rangle \leq \mathscr{E}_\lambda^o(u; u_n - u)$. Since $u_n \rightharpoonup u$ weakly in $W_0^{1,p}(\Omega)$, we obtain

$$\liminf_{n\to\infty} \mathscr{E}_\lambda^o(u; u_n - u) \geq 0. \tag{3.9}$$

Now, for the remaining two terms in the estimation (3.7), we use the fact that

$$\mathcal{F}^o(u; v) \leq \int_\Omega F^o(x, u(x); v(x))dx, \forall u, v \in W_0^{1,p}(\Omega).$$

Therefore,

$$\begin{aligned} \mathcal{F}^o(u_n; u_n - u) &\leq \int_\Omega F^o(x, u_n(x); u_n(x) - u(x))dx \\ &= \int_\Omega \max\{\xi(u_n(x) - u(x)) : \xi \in \partial F(x, u_n(x))\}dx \\ &\leq \int_\Omega |u_n(x) - u(x)| \cdot \max\{|\xi| : \xi \in \partial F(x, u_n(x))\}dx. \end{aligned}$$

From the upper semi-continuity property of ∂F, one has

$$\sup_{\substack{n\in\mathbb{N}\\ x\in\Omega}} \{|\xi| : \xi \in \partial F(x, u_n(x)\} < \infty.$$

Proceeding in the same way for $\mathcal{F}^o(u; u - u_n)$ and adding the outcomes, we obtain

$$\mathcal{F}^o(u_n; u_n - u) + \mathcal{F}^o(u; u - u_n) \leq K \cdot \int_\Omega |u_n(x) - u(x)| = K||u_n - u||_{L^1}, \tag{3.10}$$

where K is a constant. Since $u_n \to u$ strongly in $L^1(\Omega)$, we have that

$$\limsup_{n\to\infty} (\mathcal{F}^o(u_n; u_n - u) + \mathcal{F}^o(u; u - u_n)) \leq 0. \tag{3.11}$$

Now, combining the inequalities (3.8), (3.9) and (3.11), we obtain

$$\limsup_{n\to\infty} \|u - u_n\|^p \leq 0,$$

which means that $u_n \to u$ strongly in $W_0^{1,p}(\Omega)$. □

From the symmetric Ekeland's variational principle, given by M. Squassina in [11] (see Theorem 2.8), we can state the following corollary for locally Lipschitz functions.

Lemma 3.3. *Let $(X, V, \star, \mathcal{H}_\star, S)$ satisfy the assumptions given in Definition* 2.8, *with $V = L^p(\Omega)$, $X = W_0^{1,p}(\Omega)$ and with the further property that if $(u_n)_n \subset W_0^{1,p}(\Omega)$ such that $u_n \to u$ in $L^p(\Omega)$, then $u_n^\star \to u^\star$ in $L^p(\Omega)$. Assume that $\Phi : W_0^{1,p}(\Omega) \to \mathbb{R}$ is a locally Lipschitz functional bounded from below such that*

$$\Phi(u^H) \leq \Phi(u) \quad \text{for all } u \in S \text{ and } H \in \mathcal{H}_\star. \tag{3.12}$$

and for all $u \in W_0^{1,p}(\Omega)$ there exists $\xi \in S$, with $\Phi(\xi) \leq \Phi(u)$.

If Φ satisfies the $(PS)_{\inf \Phi}$ condition, then there exists $v \in W_0^{1,p}(\Omega)$, such that $\Phi(v) = \inf \Phi$ and $v = v^\star$ in $L^p(\Omega)$.

Proof. Put $\inf \Phi = d$. For the minimizing sequence $(u_n)_n$ we consider the following sequence:

$$\varepsilon_n = \begin{cases} \Phi(u_n) - d, & \text{if } \Phi(u_n) - d > 0 \\ \dfrac{1}{n}, & \text{if } \Phi(u_n) - d = 0. \end{cases}$$

Then $\Phi(u_n) \leq d + \varepsilon_n$ and $\varepsilon_n \to 0$ as $n \to \infty$. By [11, Theorem 2.8], there exists a sequence $(v_n)_n \subset W_0^{1,p}(\Omega)$ such that:

a) $\Phi(v_n) \leq \Phi(u_n)$;
b) $\lambda_\Phi(u_n) \to 0$;
c) $\|v_n - v_n^\star\|_p \to 0$;

Since Φ satisfies the $(PS)_d$ condition, there exists $v \in W_0^{1,p}(\Omega)$ such that $v_n \to v$ in $W_0^{1,p}(\Omega)$. Hence $v_n \to v$ in $L^p(\Omega)$ (because $W_0^{1,p}(\Omega)$ is compactly embedded in $L^p(\Omega)$) and so $v_n^\star \to v^\star$ in $L^p(\Omega)$ by assumption. In particular,

$$\|v - v^\star\|_p \leq \|v - v_n\|_p + \|v_n - v_n^\star\|_p + \|v_n^\star - v^\star\|_p \to 0.$$

Therefore $v = v^\star$ in $L^p(\Omega)$, as stated. □

Lemma 3.4. *One has,*

$$\mathscr{E}_\lambda(u^H) \leq \mathscr{E}_\lambda(u).$$

Proof. One has that $\|\nabla u^H\|_p = \|\nabla u\|_p$, and $\|u^H\|_p \leq \|u\|_p$ (see Van Schaftingen [13]). On the other hand, due to Proposition 2.1, one has

$$\int_\Omega F(x, u(x))dx = \int_\Omega F(x, u^H(x))dx,$$

therefore

$$\mathscr{E}_\lambda(u^H) \leq \mathscr{E}_\lambda(u).$$

□

Now we can prove our main result.

Proof of Theorem 1.1: (a) Suppose that $u \in W_0^{1,p}(\Omega)$ is a weak solution of $(\mathcal{P}_\lambda^1)$. Now, if we put $v = u$ as the test function in the relation (3.1), we obtain

$$||u||^p = \int_\Omega (|\nabla u|^p + |u|^p)dx = \lambda \int_\Omega \xi_F u dx \leq \lambda c_F \int_\Omega |u|^p dx \leq \lambda c_F c_p^p ||u||^p,$$

where $c_F = \max\limits_{s>0} \dfrac{\max\{|\xi| : \xi \in \partial F(x,s)\}}{s^{p-1}} > 0$. Therefore, if $\lambda < \frac{1}{c_F c_p^p}$, then $u = 0$.

(b) By Lemma 3.3 there exists the global minimum $v_\lambda = v_\lambda^*$ of the energy functional $\mathscr{E}_\lambda$.

We now turn to establish the existence of the second nontrivial solution of $(\mathcal{P}_\lambda^1)$.

From the assumption $(\mathbf{F_3})$, one has

$$\mathscr{E}_\lambda(u_0) = \frac{1}{p}\|u_0\|^p - \lambda \int_\Omega F(x, u_0(x))dx = A - \lambda B,$$

where $A = \frac{1}{p}\|u_0\|^p > 0$, and $B = \int_\Omega F(x, u_0(x))dx > 0$. Consequently, there exists $\lambda_0 > 0$ such that for every $\lambda > \lambda_0$, we have that $h(\lambda) = A - \lambda B < 0$, therefore

$$\mathscr{E}_\lambda(u_0) = \frac{1}{p}\|u_0\|^p - \lambda \int_\Omega F(x, u_0(x))dx < 0.$$

In fact, we may choose,

$$\lambda_0 = \frac{1}{p} \inf \left\{ \frac{\|u\|^p}{\mathcal{F}(u)} : u \in W_0^{1,p}(\Omega), \mathcal{F}(u) > 0 \right\}.$$

Now, fix $\lambda > \lambda_0$. From $(\mathbf{F_1})$ it follows that for fixed $\frac{1}{p\lambda c_p^p} > \varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$, such that

$$\max\{|\xi| : \xi \in \partial F(x, s)\} \leq \varepsilon |s|^{p-1}, |s| < \delta,$$

therefore for every $\xi \in \partial F(x, s)$, $|s| \leq \delta$ one has,

$$|\xi| \leq \varepsilon \cdot |s|^{p-1}. \tag{3.13}$$

Using the Lebourg's mean value theorem (see Theorem 2.1), we obtain:

$$|F(x, s)| = |F(x, s) - F(x, 0)| \leq |\xi_\theta s| \text{ for some } \xi_\theta \in \partial F(x, \theta s), \theta \in (0, 1),$$

which means that using the (3.13) iequality, we have

$$|F(x, s)| \leq \varepsilon \cdot |s|^p,$$

whenever $|s| \leq \delta$.

Thus, if $u \in W_0^{1,p}(\Omega)$ with $\|u\| = \rho < \min\left\{\frac{\delta}{c_p}, \|u_0\|\right\}$, then

$$\begin{aligned}
\mathscr{E}_\lambda(u) &= \frac{1}{p}||u||^p - \lambda\mathcal{F}(u) \\
&\geq \frac{1}{p}||u||^p - \varepsilon\lambda c_p^p \|u\|^p \\
&= ||u||^p \left(\frac{1}{p} - \varepsilon\lambda c_p^p\right) \\
&= \rho^p \left(\frac{1}{p} - \varepsilon\lambda c_p^p\right) > 0.
\end{aligned}$$

Since $\mathscr{E}_\lambda$ satisfies the Palais-Smale condition and

$$\inf_{\|u\|=\rho} \mathscr{E}_\lambda(u) > 0 = \mathscr{E}_\lambda(0) > \mathscr{E}_\lambda(u_0),$$

we are in the position to apply the Mountain Pass theorem, which means that $c = \inf_{\gamma\in\Gamma} \sup_{t\in[0,1]} \mathscr{E}_\lambda(\gamma(t))$ is a critical value of $\mathscr{E}_\lambda$, therefore there exists a critical point u such that $\mathscr{E}_\lambda(u) = c$.

From the definition of c, we have

$$\sup_{t\in[0,1]} \mathscr{E}_\lambda(\gamma(t)) \leq c + \frac{1}{n^2}.$$

From the above inequality and from the fact that we can choose $\gamma(0) = 0$ and $\gamma(1) = u = u^H$, we can apply Theorem 2.2 for $\varepsilon = \frac{1}{n^2}$, and $\delta = \frac{1}{n}$. Thus, there exists $u_n \in W_0^{1,p}(\Omega)$ such that

(a) $|\mathscr{E}_\lambda(u_n) - c| \le \frac{2}{n^2}$;
(b) $\|u_n - u_n^*\|_p \le 2(2\kappa + 1)\frac{1}{n}$;
(c) $\lambda_{\mathscr{E}_\lambda}(u_n) \le \frac{8}{n}$.

Since $\mathscr{E}_\lambda$ satisfies the Palais-Smale condition, up to a subsequence u_n converges to u in $W_0^{1,p}(\Omega)$, with $\mathscr{E}_\lambda(u) = c$, $\lambda_{\mathscr{E}_\lambda}(u) = 0$ and $u = u^*$. This means that u is a critical point for the energy functional $\mathcal{E}_\lambda$, different from the critical point obtained in (a) and it is invariant by spherical cap symmetrization as well. □

4. Proof of Theorem 1.2

Similarly to the previous section, we start this paragraph with the proofs of two properties of the energy functional $\mathscr{A}_\lambda$, namely that $\mathscr{A}_\lambda$ is coercive and it satisfies the Palais-Smale condition for every $\lambda > 0$.

Lemma 4.1. *Let the conditions* $(\mathbf{F_2'})$ *and* $(\mathbf{A_4})$ *be satisfied. Then the functional* $\mathscr{A}_\lambda : X \to \mathbb{R}$ *is coercive for each* $\lambda > 0$*, that is,* $\mathscr{A}_\lambda(u) \to \infty$ *as* $\|u\|_X \to \infty$*, for all* $u \in X$*.*

Proof. Due to $(\mathbf{F_2'})$, for all $u \in X$ we have:

$$F(x, u(x)) \le \alpha(x)|u(x)|^q + \beta(x).$$

Hence, by using Hölder's inequality, it follows that

$$\begin{aligned}\int_{\mathbb{R}^N} F(x,u(x))dx &\le \int_{\mathbb{R}^N} \alpha(x)|u(x)|^q dx + \int_{\mathbb{R}^N} \beta(x)dx \\ &\le \left[\int_{\mathbb{R}^N} \alpha(x)^{\frac{\nu}{\nu-q}}\right]^{\frac{\nu-q}{\nu}} \cdot \left[\int_{\mathbb{R}^N} [|u(x)|^q]^{\frac{\nu}{q}}\right]^{\frac{q}{\nu}} dx + \int_{\mathbb{R}^N} \beta(x)dx \\ &\le \|\alpha\|_{\frac{\nu}{\nu-q}} \cdot \|u\|_\nu^q + \|\beta\|_1. \end{aligned} \tag{4.1}$$

Since $X \hookrightarrow L^\nu(\mathbb{R}^N)$, when $\nu \in [p, p^*]$, one can find a number $C_\nu \ge 0$ such that

$$\|u\|_\nu^q \le C_\nu^q \|u\|_X^q. \tag{4.2}$$

Combining the relations (4.1) and (4.2), we obtain that for all $\lambda > 0$, we have

$$-\lambda \int_{\mathbb{R}^N} F(x,u(x))dx \ge -\lambda\|\alpha\|_{\frac{\nu}{\nu-q}} \cdot C_\nu^q \|u\|_X^q - \lambda\|\beta\|_1.$$

Therefore, from the definition of the energy functional $\mathscr{A}_\lambda$ and using the condition $(\mathbf{A_4})$, we get

$$\begin{aligned}\mathscr{A}_\lambda(u) &\ge a(u) - \lambda\|\alpha\|_{\frac{\nu}{\nu-q}} \cdot C_\nu^q\|u\|_X^q - \lambda\|\beta\|_1 \\ &\ge c\|u\|^p - \lambda\|\alpha\|_{\frac{\nu}{\nu-q}} \cdot C_\nu^q\|u\|_X^q - \lambda\|\beta\|_1.\end{aligned}$$

Taking into account $(\mathbf{F_2})'$ and the fact that $q \in (0, p)$, it follows that $\mathscr{A}_\lambda(u) \to +\infty$, whenever $\|u\|_X \to +\infty$. This completes the proof. □

Lemma 4.2. *If the conditions hold then for every $\lambda > 0$ the functional $\mathscr{A}_\lambda : X \to \mathbb{R}$ satisfies the Palais-Smale condition.*

Proof. The proof of this lemma is similar to the proofs of the Lemma 3.2 and of [8, Theorem 5.1.1]. □

Lemma 4.3. *Assume that* $(\mathbf{F_4}) - (\mathbf{F_5})$ *and* $(\mathbf{A_5})$ *are satisfied. Then, for all $H \in H_*$, we have*

$$\mathscr{A}_\lambda(u^H) \leq \mathscr{A}_\lambda(u), \forall u \in X.$$

Proof. From $(\mathbf{A_5})$, we have that $a(u^H) \leq a(u)$. Therefore, using $(\mathbf{F_4}) - (\mathbf{F_5})$ and taking inspiration from the proof of Lemma 4.6. in M. Squassina [11], we obtain

$$\int_{\mathbb{R}^N} F(x, u(x))dx \leq \int_{\mathbb{R}^N} F(x, u^H(x))dx.$$

Hence, by the definition of $\mathscr{A}_\lambda$, we have that for all $\lambda > 0$:

$$\mathscr{A}_\lambda(u) = \frac{1}{p}a(u) - \lambda \int_{\mathbb{R}^N} F(x, u(x))dx \geq \frac{1}{p}a(u^H) - \lambda \int_{\mathbb{R}^N} F(x, u^H(x))dx = \mathscr{A}_\lambda(u^H).$$

□

Proof of Theorem 1.2*:* The proof is similar to the proof of Theorem 1.1 so it is left to the reader.

4.1. Particular case

Let $V : \mathbb{R}^N \to \mathbb{R}$ a function such that:

(**V1**) $V_0 := \inf_{x \in \mathbb{R}^N} V(x) > 0$;

(**V2**) For every $M > 0$, we have $\text{meas}(\{x \in \mathbb{R}^N : b(x) \leq M\}) < \infty$;

(**V3**) For $x, y \in \mathbb{R}^N$, if $|x| \leq |y|$ then $V(x) \leq V(y)$.

The space $H = \{u \in H^1(\mathbb{R}^N) : \int_{\mathbb{R}^N} (|\nabla u|^2 + V(x)u^2 dx < \infty\}$, equipped with the inner product

$$\langle u, v \rangle_H = \int_{\mathbb{R}^N} (\nabla u \nabla v + V(x)uv)dx$$

is a Hilbert space. It is known that H is compactly embedded into $L^s(\mathbb{R}^n)$ for $s \in [2, 2^*)$ (see T. Bartsch, Z.-Q. Wang [1]).

A particular case of the problem (P_λ^2) can be formulated as follows: Find a positive $u \in H$ such that for each $v \in H$ we have

$$\int_{\mathbb{R}^N} (\nabla u \nabla v + V(x)uv)dx + \int_{\mathbb{R}^N} F_y^0(x, u(x) - v(x))dx \geq 0. \qquad (\mathbf{P}'_\lambda).$$

Similarly to the proof of Theorem 1.2, we can prove the next result:

Lemma 4.4. *If $f : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ satisfies the conditions* $(\mathbf{F1})$, $(\mathbf{F'_2})$, $(\mathbf{F_4})$, $(\mathbf{F_5})$ *and* $(\mathbf{V1}) - (\mathbf{V3})$, *then there exists two nontrivial solutions of the problem* $(\mathbf{P_\lambda^2})$, *which are invariants by the spherical cap symmetrization.*

Proof. Theorem 1.2 can be applied since the conditions (**A1**) − (**A5**) are fulfilled. Indeed, the assumptions (**A1**) − (**A5**) follow from the fact that $a(u) = \frac{1}{2}\langle u, u\rangle$. On the other hand, the condition (**V3**), implies (**A5**). Then, by Theorem 1.2, it follows that problem ($\mathbf{P}^{\mathbf{2}}_{\lambda}$) has two nontrivial solutions, which are invariants by the spherical cap symmetrization. □

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On the geometry of conformal Hamiltonian of the time-dependent coupled harmonic oscillators

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Abstract. In this paper we construct the distinguished (d-) geometry (in the sense of d-connections, d-torsions, d-curvatures, momentum geometrical gravitational and electromagnetic theories) for the conformal Hamiltonian of the time-dependent coupled oscillators on the dual 1-jet space $J^{1*}\left(\mathbb{R},\mathbb{R}^2\right)$.

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We dedicate this paper to the memory of Professor Gheorghe Atanasiu (1939-2014).

1. Introduction

The model of time-dependent coupled oscillators is used to investigate the dynamics of charged particle motion in the presence of time-varying magnetic fields. At the same time, the model of coupled harmonic oscillator has also been widely used to study the quantum effects in mesoscopic coupled electric circuits. For more details, please see [2].

If m_i $(i = 1, 2)$, ω_i $(i = 1, 2)$, and $k(t)$ are the time-dependent mass, frequency, and the coupling parameter, respectively, then the conformal Hamiltonian of the time-dependent coupled harmonic oscillators is given on the dual 1-jet space $J^{1*}\left(\mathbb{R},\mathbb{R}^2\right)$ by [2]

$$H(t,x,p) = h_{11}(t)e^{\sigma(x)}\left[\frac{\left(p_1^1\right)^2}{m_1(t)} + \frac{\left(p_2^1\right)^2}{m_2(t)}\right] + \mathcal{F}(t,x) = \tag{1.1}$$

$$= h_{11}(t)e^{\sigma(x)}\frac{\delta^{ij}}{m_i(t)}p_i^1 p_j^1 + \mathcal{F}(t,x) =$$

$$= h_{11}(t)g^{ij}(t,x)p_i^1 p_j^1 + \mathcal{F}(t,x),$$

where $\sigma : \mathbb{R}^2 \to \mathbb{R}$ is a smooth conformal function on $\mathbb{R}^2$, h_{11} is a Riemannian metric on $\mathbb{R}$, $(t, x, p) = (t, x^1, x^2, p_1^1, p_2^1)$ are the coordinates of the space $J^{1*}(\mathbb{R}, \mathbb{R}^2)$, and

$$\mathcal{F}(t,x) = \frac{m_1(t)\omega_1^2(t)x_1^2}{2} + \frac{m_2(t)\omega_2^2(t)x_2^2}{2} + \frac{k(t)\left(x_2 - x_1\right)^2}{2}.$$

The jet coordinates (t, x, p) transform by the rules:

$$\tilde{t} = \tilde{t}(t),\ \tilde{x}^i = \tilde{x}^i(x^j),\ \tilde{p}_i^1 = \frac{\partial x^j}{\partial \tilde{x}^i}\frac{d\tilde{t}}{dt}p_j^1, \tag{1.2}$$

where $i, j = 1, 2$, rank $(\partial\tilde{x}^i/\partial x^j) = 2$ and $d\tilde{t}/dt \neq 0$.

2. The canonical nonlinear connection

The *fundamental metrical d-tensor* induced by the conformal Hamiltonian metric (1.1) is defined by (see [1] and [3])

$$g^{ij}(t,x) = \frac{h^{11}}{2}\frac{\partial^2 H}{\partial p_i^1 p_j^1} = \frac{\delta^{ij}}{m_i(t)}e^{\sigma(x)} \Longrightarrow g_{jk}(t,x) = \delta_{jk}m_k(t)e^{-\sigma(x)}. \tag{2.1}$$

If we use the notations

$$\text{K}_{11}^1 = \frac{h^{11}}{2}\frac{dh^{11}}{dt},$$

$$\Gamma_{ij}^k \overset{def}{=} \frac{g^{kl}}{2}\left(\frac{\partial g_{li}}{\partial x^j} + \frac{\partial g_{lj}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^l}\right) = \frac{1}{2}\left(-\delta_i^k\sigma_j - \delta_j^k\sigma_i + \delta_{ij}\frac{m_j(t)}{m_k(t)}\sigma_k\right), \tag{2.2}$$

where $h^{11} = 1/h_{11} > 0$ and $\sigma_i = \partial\sigma/\partial x^i$, then we have the following geometrical result:

Proposition 2.1. *For the conformal Hamiltonian metric (*1.1*) the canonical nonlinear connection on the dual 1-jet space $J^{1*}(\mathbb{R}, \mathbb{R}^2)$ has the following components:*

$$N = \left(\underset{1}{N}{}_{(i)1}^{(1)} = \text{K}_{11}^1 p_i^1,\ \underset{2}{N}{}_{(i)j}^{(1)} = -\Gamma_{ij}^k p_k^1\right). \tag{2.3}$$

In other words, we have

$$\underset{2}{N}{}_{(i)j}^{(1)} = \frac{1}{2}\left(\sigma_i p_j^1 + \sigma_j p_i^1 - \delta_{ij}\frac{m_j(t)}{m_k(t)}\sigma_k p_k^1\right).$$

Proof. The canonical nonlinear connection produced by H on the dual 1-jet space $J^{1*}(\mathbb{R}, \mathbb{R}^2)$ has the following components:

$$\underset{1}{N}{}_{(i)1}^{(1)} = \text{K}_{11}^1 p_i^1$$

and

$$\underset{2}{N}{}_{(i)j}^{(1)} = \frac{h^{11}}{4}\left[\frac{\partial g_{ij}}{\partial x^k}\frac{\partial H}{\partial p_k^1} - \frac{\partial g_{ij}}{\partial p_k^1}\frac{\partial H}{\partial x^k} + g_{ik}\frac{\partial^2 H}{\partial x^j\partial p_k^1} + g_{jk}\frac{\partial^2 H}{\partial x^i\partial p_k^1}\right].$$

So, by direct computations, we obtain (2.3). □

3. N-linear Cartan canonical connection. d-Torsions and d-curvatures

The nonlinear connection N produces the following *dual adapted bases* of d-vector fields and d-covector fields:

$$\left\{\frac{\delta}{\delta t}, \frac{\delta}{\delta x^i}, \frac{\partial}{\partial p_i^1}\right\} \subset \mathcal{X}\left(J^{1*}\left(\mathbb{R}, \mathbb{R}^2\right)\right), \quad \left\{dt, dx^i, \delta p_i^1\right\} \subset \mathcal{X}^*\left(J^{1*}\left(\mathbb{R}, \mathbb{R}^2\right)\right), \tag{3.1}$$

where

$$\frac{\delta}{\delta t} = \frac{\partial}{\partial t} - \underset{1}{N}{}^{(1)}_{(r)1}\frac{\partial}{\partial p_r^1} = \frac{\partial}{\partial t} - \varkappa_{11}^1 p_r^1 \frac{\partial}{\partial p_r^1}, \tag{3.2}$$

$$\frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - \underset{2}{N}{}^{(1)}_{(r)i}\frac{\partial}{\partial p_r^1} = \frac{\partial}{\partial x^i} + \Gamma_{ri}^s p_s^1 \frac{\partial}{\partial p_r^1}, \tag{3.3}$$

$$\delta p_i^1 = dp_i^1 + \underset{1}{N}{}^{(1)}_{(i)1} dt + \underset{2}{N}{}^{(1)}_{(i)j} dx^j.$$

The naturalness of the geometrical adapted bases (3.1) is coming from the fact that, via a transformation of coordinates (1.2), their elements transform as the classical tensors. Therefore, the description of all subsequent geometrical objects on the dual 1-jet space $J^{1*}(\mathbb{R}, \mathbb{R}^2)$ (e.g., the Cartan canonical N-linear connection, its torsion and curvature, etc.) will be done in local adapted components. As a result, by direct computations, we obtain the following geometrical result:

Proposition 3.1. *The Cartan canonical N-linear connection produced by the conformal Hamiltonian metric (*1.1*) has the following adapted local components:*

$$C\Gamma(N) = \left(\varkappa_{11}^1,\ A_{j1}^i = \delta_j^i \frac{m_j'(t)}{2m_i(t)},\ H_{jk}^i = \Gamma_{jk}^i,\ C_{i(1)}^{j(k)} = 0\right). \tag{3.4}$$

Proof. The adapted components of Cartan canonical connection are given by the formulas

$$A_{j1}^i = \frac{g^{il}}{2}\frac{\delta g_{lj}}{\delta t} = \frac{g^{il}}{2}\frac{dg_{lj}}{dt} = \frac{\delta_j^i}{2m_i(t)}\frac{dm_j}{dt} = \delta_j^i \frac{m_j'(t)}{2m_i(t)},$$

$$H_{jk}^i = \frac{g^{ir}}{2}\left(\frac{\delta g_{jr}}{\delta x^k} + \frac{\delta g_{kr}}{\delta x^j} - \frac{\delta g_{jk}}{\delta x^r}\right) = \Gamma_{jk}^i,$$

$$C_{i(1)}^{j(k)} = -\frac{g_{ir}}{2}\left(\frac{\partial g^{jr}}{\partial p_k^c} + \frac{\partial g^{kr}}{\partial p_j^c} - \frac{\partial g^{jk}}{\partial p_r^c}\right) = 0.$$

Using the derivative operators (3.2) and (3.3), the direct calculations lead us to the desired results. □

Proposition 3.2. *The Cartan canonical connection of the conformal Hamiltonian metric (*1.1*) has four effective d-torsions:*

$$T_{1j}^r = -A_{j1}^r,\ P_{(r)1(1)}^{(1)\ (j)} = A_{r1}^j,$$

$$R_{(r)1j}^{(1)} = \frac{\partial \Gamma_{rj}^s}{\partial t} p_s^1,$$

$$R_{(r)ij}^{(1)} = -\mathfrak{R}_{rij}^k p_k^1,$$

where

$$\mathfrak{R}^l_{ijk} = \frac{\partial \Gamma^l_{ij}}{\partial x^k} - \frac{\partial \Gamma^l_{ik}}{\partial x^j} + \Gamma^s_{ij}\Gamma^l_{sk} - \Gamma^s_{ik}\Gamma^l_{sj}.$$

Proof. The Cartan canonical connection on the dual 1-jet space $J^{1*}(\mathbb{R}, \mathbb{R}^2)$ generally has six effective local d-tensors of torsion. For our particular Cartan canonical connection (3.4) these reduce only to four (the others are zero) [1]:

$$T^r_{1j} = -A^r_{j1}, \; P^{r(j)}_{i(1)} = C^{r(j)}_{i(1)} = 0,$$

$$P^{(1)\ (j)}_{(r)1(1)} = \frac{\partial \underset{1}{N}{}^{(1)}_{(r)1}}{\partial p^1_j} + A^j_{r1} - \delta^j_r \chi^1_{11} = A^j_{r1},$$

$$P^{(1)\ (j)}_{(r)i(1)} = \frac{\partial \underset{2}{N}{}^{(1)}_{(r)i}}{\partial p^1_j} + \Gamma^j_{ri} = 0,$$

$$R^{(1)}_{(r)1j} = \frac{\delta \underset{1}{N}{}^{(1)}_{(r)1}}{\delta x^j} - \frac{\delta \underset{2}{N}{}^{(1)}_{(r)j}}{\delta t} = -\frac{\partial \underset{2}{N}{}^{(1)}_{(r)j}}{\partial t} = \frac{\partial \Gamma^s_{rj}}{\partial t} p^1_s,$$

$$R^{(1)}_{(r)ij} = \frac{\delta \underset{2}{N}{}^{(1)}_{(r)i}}{\delta x^j} - \frac{\delta \underset{2}{N}{}^{(1)}_{(r)j}}{\delta x^i} = -\mathfrak{R}^k_{rij} p^1_k.$$

□

Proposition 3.3. *The Cartan canonical connection of the conformal Hamiltonian metric (*1.1*) has two effective d-curvatures:*

$$R^l_{i1k} = \frac{\partial A^l_{i1}}{\partial x^k} - \frac{\partial \Gamma^l_{ik}}{\partial t} + A^r_{i1}\Gamma^l_{rk} - \Gamma^r_{ik}A^l_{r1},$$

$$R^l_{ijk} = \mathfrak{R}^l_{ijk}.$$

Proof. A Cartan canonical connection on the dual 1-jet space $J^{1*}(\mathbb{R}, \mathbb{R})$ generally has five local d-tensors of curvature. For our particular Cartan canonical connection (3.4) these reduce only to two (the others are zero). So, we have [1]:

$$R^l_{i1k} = \frac{\delta A^l_{i1}}{\delta x^k} - \frac{\delta \Gamma^l_{ik}}{\delta t} + A^r_{i1}\Gamma^l_{rk} - \Gamma^r_{ik}A^l_{r1} = \frac{\partial A^l_{i1}}{\partial x^k} - \frac{\partial \Gamma^l_{ik}}{\partial t} + A^r_{i1}\Gamma^l_{rk} - \Gamma^r_{ik}A^l_{r1},$$

$$R^l_{ijk} = \frac{\delta \Gamma^l_{ij}}{\delta x^k} - \frac{\delta \Gamma^l_{ik}}{\delta x^j} + \Gamma^r_{ij}\Gamma^l_{rk} - \Gamma^r_{ik}\Gamma^l_{rj} = \frac{\partial \Gamma^l_{ij}}{\partial x^k} - \frac{\partial \Gamma^l_{ik}}{\partial x^j} + \Gamma^r_{ij}\Gamma^l_{rk} - \Gamma^r_{ik}\Gamma^l_{rj} = \mathfrak{R}^l_{ijk},$$

$$P^{l\ (k)}_{i1(1)} = \frac{\partial A^l_{i1}}{\partial p^1_k} = 0, \quad P^{l\ (k)}_{ij(1)} = \frac{\partial \Gamma^l_{ij}}{\partial p^1_k} = 0, \quad S^{l(j)(k)}_{i(1)(1)} = 0.$$

□

4. From the conformal Hamiltonian of the time-dependent coupled oscillators to field-like geometrical models

4.1. Momentum gravitational-like geometrical model

The conformal Hamiltonian metric (1.1) produces on the momentum phase space $J^{1*}(\mathbb{R},\mathbb{R}^2)$ the adapted metrical d-tensor (*momentum gravitational potential*)

$$\mathbb{G} = h_{11}dt \otimes dt + g_{ij}dx^i \otimes dx^j + h_{11}g^{ij}\delta p_i^1 \otimes \delta p_j^1,$$

where g_{jk} and g^{ij} are given by (2.1). We postulate that the momentum gravitational potential $\mathbb{G}$ is governed by the *geometrical Einstein equations*

$$\text{Ric}(C\Gamma(N)) - \frac{\text{Sc}(C\Gamma(N))}{2}\mathbb{G} = \mathcal{K}\mathbb{T}, \tag{4.1}$$

where:

• $\text{Ric}(C\Gamma(N))$ is the Ricci d-tensor associated with the Cartan canonical linear connection (3.4);

• $\text{Sc}(C\Gamma(N))$ is the *scalar curvature*;

• $\mathcal{K}$ is the Einstein constant and $\mathbb{T}$ in an intrinsic momentum *stress-energy* d-tensor of matter.

Therefore, using the adapted basis of vector fields, we can locally describe the global geometrical Einstein equations (4.1). Consequently, some direct computations lead to:

Proposition 4.1. *The Ricci tensor of the Cartan canonical connection of the conformal Hamiltonian metric (*1.1*) has the following two effective Ricci d-tensors:*

$$R_{11} := \text{K}_{11} = 0, \qquad R_{1i} = R^1_{1i1} = 0,$$
$$R_{i1} = R^r_{i1r}, \qquad R_{ij} = R^r_{ijr} = \mathfrak{R}^r_{ijr} := \mathfrak{R}_{ij},$$
$$R_{i(1)}^{\ \ (j)} := -P_{i(1)}^{\ \ (j)} = -P^{r\ \ (j)}_{ir(1)} = 0, \qquad R^{(i)}_{(1)1} := -P^{(i)}_{(1)1} = -P^{i\ \ (r)}_{r1(1)} = 0,$$

$$R_{1(1)}^{\ \ (j)} = -P^{1(j)}_{1(1)1} = 0,$$
$$R^{(i)(j)}_{(1)(1)} := -S^{(i)(j)}_{(1)(1)} = -S^{i(j)(r)}_{r(1)(1)} = 0,$$
$$R^{(i)}_{(1)j} := -P^{(i)}_{(1)j} = -P^{i\ \ (r)}_{rj(1)} = 0.$$

Corollary 4.2. *The scalar curvature of the Cartan canonical connection of the conformal Hamiltonian metric (*1.1*) is given by the following formula:*

$$\text{Sc}(C\Gamma(N)) = g^{ij}R_{ij} = g^{ij}\mathfrak{R}_{ij} := \mathfrak{R}.$$

Corollary 4.3. *The geometrical momentum Einstein-like equations produced by the conformal Hamiltonian metric (1.1) are locally described by*

$$\begin{cases} -\dfrac{\mathfrak{R}}{2}h_{11} = \mathcal{K}\mathbb{T}_{11} \\ \mathfrak{R}_{ij} - \dfrac{\mathfrak{R}}{2}g_{ij} = \mathcal{K}\mathbb{T}_{ij} \\ -\dfrac{\mathfrak{R}}{2}h_{11}g^{ij} = \mathcal{K}\mathbb{T}^{(i)(j)}_{(1)(1)} \\ 0 = \mathbb{T}_{1i}, \quad R_{i1} = \mathcal{K}\mathbb{T}_{i1} \\ 0 = \mathbb{T}^{\ \ (i)}_{1(1)}, \quad 0 = \mathbb{T}^{\ \ (j)}_{i(1)} \\ 0 = \mathbb{T}^{(i)}_{(1)1}, \quad 0 = \mathbb{T}^{(i)}_{(1)j}. \end{cases}$$

The *geometrical momentum conservation-like laws* produced by the conformal Hamiltonian metric (1.1) are postulated by the following formulas (for more details, please see [1]):

$$\begin{cases} \left[\dfrac{\mathfrak{R}}{2}\right]_{/1} = R^r_{1|r} \\ \left[\mathfrak{R}^r_j - \dfrac{\mathfrak{R}}{2}\delta^r_j\right]_{|r} = 0 \\ \left[\dfrac{\mathfrak{R}}{2}\delta^j_r\right]|^{(r)}_{(1)} = 0, \end{cases}$$

where "$_{/1}$", "$_{|r}$" and "$|^{(r)}_{(1)}$" are the local covariant derivatives induced by the Cartan canonical connection $C\Gamma(N)$, and we have

$$R^i_1 = g^{iq}R_{q1}, \quad \mathfrak{R}^i_j = g^{iq}\mathfrak{R}_{qj}.$$

4.2. Geometrical momentum electromagnetic-like 2-form

On the momentum phase space $J^{1*}(\mathbb{R}, \mathbb{R}^2)$, the distinguished geometrical electromagnetic 2-form is defined by

$$\mathbb{F} = F^{(i)}_{(1)j}\delta p^1_i \wedge dx^j,$$

where

$$F^{(i)}_{(1)j} = \frac{h^{11}}{2}\left[g^{jk}\underset{2}{N}{}^{(1)}_{(k)i} - g^{ik}\underset{2}{N}{}^{(1)}_{(k)i} + (g^{jk}\Gamma^r_{ki} - g^{ik}\Gamma^r_{kj})p^1_r\right].$$

By a direct calculation, the conformal Hamiltonian metric (1.1) produces the null momentum electromagnetic components

$$F^{(i)}_{(1)j} = 0.$$

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Book reviews

The Best Writing on Mathematics 2014, Edited by Mircea Pitici, Princeton University Press 2014, 376 pp., ISBN: 9780691164175, ISBN: 9781400865307 (eBook).

Mircea Pitici does, in the foreword of his book, a very good characterization of the relationship between mathematics and ”everything”. ”When we talk about mathematics, or when we teach it, or when we write about it, many of us feign detachment. It is almost a cultural universal to pretend that mathematics is ”out there,” independent of our whims and oddities. But doing mathematics and taking or writing about it are activities neither neutral nor innocent; we can only do them if we are engaged, and the engagement marks not only us (as thinkers and experimenters) but also those who watch us, listen to us, and think with us. Thus mathematics always requires full participation; without genuine involvement, there is no mathematics”. This may actually serve as the mission statement of any honest mathematician. Since Middle Ages, Mathematics has been thought as the fabric of this world, and the fundamental mathematical principles have been accepted as the principles all the living beings naturally obey. An accurate understanding of mathematics is essential for the educated person in order to be able to start asking questions and getting answers about the environment.

This volume, the fifth in a series edited by M. Pitici and published with PUP (2010, 2011, 2012, 2013), is a work of this fundamental nature. This is a collection of 20 previously published essays introducing to the general audience various topics of mathematics through applications and applicative and experimental fields of science.

A simple coverage of the titles show quite a nice variety of concerns:

The Prospects for Mathematics in a Multimedia Civilization (Philip J. Davies); *Fearful Symmetry* (Ian Stewart); *E pluribus unum: From Complexity, Universality* (Terence Tao); *Degrees of Separation* (Gregory Goth); *Randomness* (Charles Seife); *Randomness in Music* (Donald E. Knuth); *Playing the Odds* (Soren Johnson), *Machines of the Infinite* (John Pavlus); *Bridges, String Art, and Bezier Curves* (Renan Gross); *Slicing a Cone for Art and Science* (Daniel S. Silver); *High Fashion Meets Higher Mathematics* (Kelly Delp); *The Jordan Curve Theorem is Nontrivial* (Fionna Ross and William T. Ross); *Why Mathematics? What Mathematics*? (Anna Sfard); *Math Anxiety: Who Has It, Why It Helps, and How to Guard against It*; (Erin A. Maloney and Sian L. Beilock); *How Old Are the Platonic Solids*? (David R. Lloyd); *Early Modern Mathematical Instruments* (Jim Bennett); *A Revolution in Mathematics? What Really Happened a Century Ago and Why It Matters Today* (Frank Quinn);

Errors of Probability in Historical Context (Prakash Gorroochurn); *The End of Probability* (Ellie Ayache); *An abc Proof Too Tough Even for Mathematicians* (Kevin Hartnett).

This volume is compulsory reading for any person involved with, or interested in mathematics and its positioning in the nature as we know it.

Horia F. Pop

Hongyu Guo , Modern Mathematics and Applications in Computer Graphics and Vison, World Scientific, London-Singapore-Hong Kong 2014, xxiii + 408 pp, ISBN: 978-981-4449-32-8.

In contrast to many areas of Computer Science where discrete mathematics is mostly applied, computer graphics and vision utilize many domains of continuous mathematics. Unfortunately, many students in this area lack a sufficient knowledge of the needed results from this part of mathematics and the reading of books dedicated to specified topics is discouraging. The intention of the author of the present book is to supply them with an easy to handle and understand record of some results in these abstract areas of mathematics.

In order to make the book easy to read the author omitted all the proofs. This does not mean that, at the same time, the rigor is sacrificed – the book contains rigorous definitions (put in boxes with headings) and the formal enounces of theorems (highlighted with gray shades). Instead presenting the proofs, the author prefers to carefully explain and motivate the notions and the results with emphasis on the intuition, with many examples and historical and philosophical insights.

The book has four parts: I. Algebra; II. Geometry; III. Topology and more; IV. Applications. A preliminary chapter, Chapter 0, *Mathematical structures*, contains an essay on mathematics in historical perspective, its role in society, the relations between mathematics and reality (are the mathematical results reflecting some real things or a creation of the human mind).

The first part deals with linear algebra, tensor algebra, exterior algebra and geometric algebra. The part on geometry is concerned with projective geometry, differential geometry and elements of non-Euclidean geometry. The third part includes general topology, manifolds, Hilbert spaces and elements of measure theory and probability theory. In the last part of the book, based in part on some articles by the author, one shows how the elaborated machinery can be put to solve problems in computer vision and computer graphics and includes: color spaces, perspective analysis of images, quaternions and 3-D rotations, manifold learning in machine learning. Written in a pleasant and alive style, with suggestive quotations and witty comments of the author (also many photos illustrating the text are made by the author), the book will be of great help for students in computer science specializing in computer vision and computer graphics. Other students who use mathematics in their disciplines (physics, chemistry, biology, economics) will find the book as a good source of rapid and reliable information.

Dana Cobzaş

H. Scott Dumas, THE KAM STORY – A Friendly Introduction to the Content, History, and Significance of Classical Kolmogorov-Arnold-Moser Theory, World Scientific, London-Singapore-Hong Kong 2014, xv + 361 pp, ISBN: 978-981-4556-58-3.

The book tells the story of the discovery of the Kolmogorov-Arnold-Moser theory (KAM in short) as well as of the results and controversies preceding it. The story starts in 1954 at the International Congress of Mathematicians when A. N. Kolmogorvov presented an astonishing result with a sketch of the proof. The details, with completions and extensions, were supplied by Kolmogorov's student Vladimir Arnold and by the German-American-Suisse mathematician Jürgen Moser (Kolmogorov considered the analytic case while Moser treated the smooth one, with order of smoothness 333, currently reduced at 3). The theory has its origin in the paper published in 1890 by H. Poincaré in Acta Mathematica (a whole volume of 270 pages) who received for it the prize offered by Oscar II, King of Norway and Sweden, the results being then expanded in his three volume book *Les méthodes nouvelles de la mécanique céleste*, Paris, 1892, 1893 and 1899. This was turning point in the development of mathematics and mechanics that led also to a new domain - dynamical systems and chaos. As the jury heaps praise on the paper "it will change the course of astronomical dynamics for ever" - and it did indeed, even much more. Roughly speaking, Kolmogorov proved the existence of invariant tori of perturbed Hamiltonian systems of the form $H(\theta, I, \varepsilon) = h(I) + \varepsilon f(\theta, I, \varepsilon)$, where (θ, I) are the action-angles variables and $h(I)$ is a smooth (or even analytic) completely integrable Hamiltonian system. The problem is related to the n-body problem, the stability of the solar system and Boltzman ergodic hypothesis (invalidated by this theory), and in this extended form its origins can be traced back to Kepler, Newton, Lagrange, Hamilton, and others.

The book presents in an informal way the basics of classical KAM theory in a broader context, with emphasis on the evolution of ideas in historical perspective. Subsequent developments are discussed in Chapter 6, *Other results in Hamiltonian Perturbation Theory* (HPT) (the work of Chirikov and Nekhoroshev), and physical applications in Chapter 7. The author appeals as possible to original sources, correcting errors in attributing some results. Besides the mathematical problems the author discusses some general questions as the Russian, European and American ways of doing mathematics (a long time in mutual isolation, leading to some priority discussions) as well as some philosophical aspects - "the last laugh of Hegel". There are a lot of footnotes which make part of the text, completing it with comments of the author or from persons involved in these events. The main body of the book is completed by 6 very useful appendices: A. *Kolmogorov's 1954 paper*; B. *Overview of low-dimensional small divisors problem*; C. *East meets West–Russians, Europeans, Americans*; D. *Guide for further reading*; E. *Selected quotations*, and F. *Glossary* (80 pages, explaining the main notions used in the text–a welcome addition). Written in a live and accessible style, the book is addressed to a large audience, first of all to mathematicians and physicists of various specialties, as it does not require a background in dynamical systems. Some part of it can be read with benefit and enchantment by anybody interested in the evolution of scientific ideas.

Paul Blaga

Qamrul Hasan Ansari (Editor), Nonlinear Analysis–Approximation Theory, Optimization and Applications, Springer India, New Delhi, Heidelberg, 2014, xv + 352 pp, ISBN 978-81-322-1882-1; ISBN 978-81-322-1883-8 (eBook).

This is a collection of survey papers on some topics in nonlinear functional analysis–best approximation, optimization, fixed point theory, monotone operators and variational inequalities, equilibrium problems. The papers included in this volume emphasize the tight connections existing between these domains and show how results and methods from one area help to solve problems in another ones.

The papers on best approximation concern various continuity properties of the metric projection in Banach spaces (P. Veeramani and S. Rajesh), the use of various kinds of convergence of slices in the study of the geometry of Banach spaces in connection with applications to best approximation (P. Shunmugaraj). Two papers, *Best proximity points* (P. Veeramani and S. Rajesh) and *Best approximation in nonlinear functional analysis* (S. P. Singh and M. R. Singh), wheels around the famous result of Ky Fan on the existence of best proximity points and its relevance for various questions in fixed point theory, optimization and variational inequalities. The paper by J. Banaś, *Measures of noncompactness and well-posed minimization problems,* shows how some geometric properties of Banach spaces, as nearly strict convexity, nearly uniform convexity, nearly uniform smoothness, defined through various measures of noncompactness can be used to prove the well-posedness of some generalized minimization problems. Well posedness is also the subject of the paper by D. V. Pai, *Well-posedness, regularization, and viscosity solutions of minimization problems.* Iterative methods for finding fixed points and zeros of monotone operators, and for solving variational inequalities are presented in the papers *Hierarchical minimization problems and applications* (D. R. Sahu and Q. H. Ansari), *Triple hierarchical variational inequalities* (Q. H. Ansari, L.-C. Ceng and H. Gupta) and *Split feasibility and fixed point problems* (Q. H. Ansari and A. Rehan). Isotone projection cones in Hilbert spaces with applications to complementarity problems are discussed in the paper by M. Abbas and S. Z. Németh, *Isotone projection cones and nonlinear complementarity problems.*

Containing well written survey papers by renown experts, the volume will provide researchers in nonlinear analysis and related domains to a quick introduction and, at the same time, with a state-of-the-art in several very active area of current investigation.

S. Cobzaş