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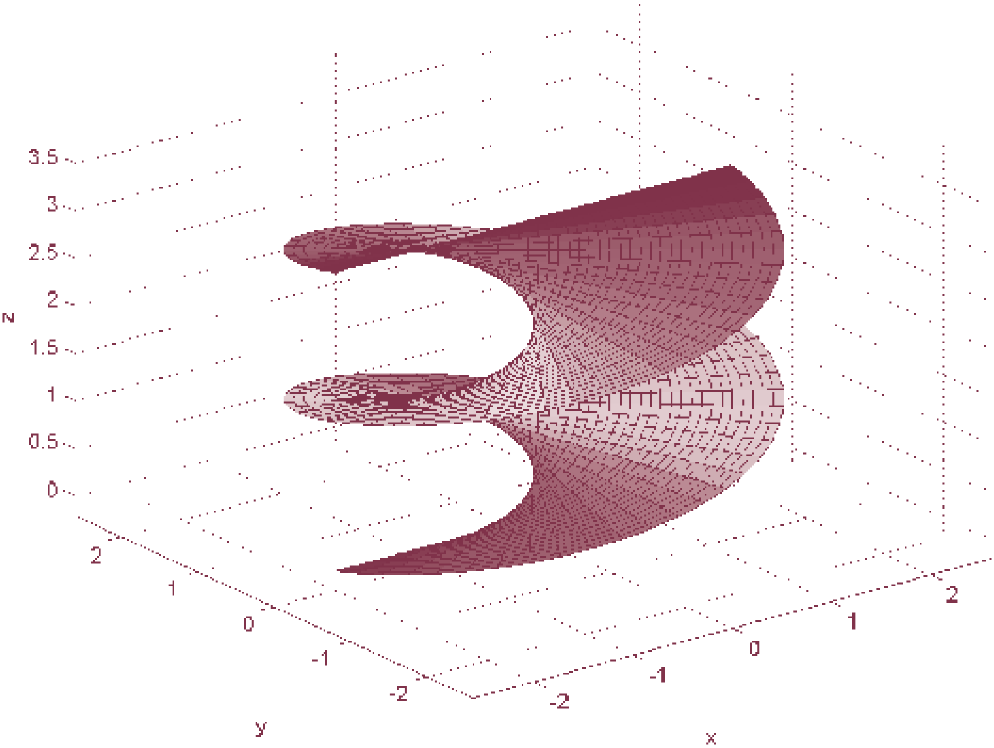

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New midpoint and trapezoidal-type inequalities for prequasiinvex functions via generalized fractional integrals

Seth Kermausuor and Eze R. Nwaeze

Abstract. In this work, we establish some new midpoint and trapezoidal type inequalities for prequasiinvex functions via the Katugampola fractional integrals. Some of the results obtained in this paper are generalizations of some earlier results in the literature.

Mathematics Subject Classification (2010): 26A33, 26A51, 26D10, 26D15.

Keywords: Hermite-Hadamard inequality, midpoint-type inequalities, trapezoidal-type inequalities, quasi-convex functions, prequasiinvex functions, Hölder's inequality, power mean inequality, Katugampola fractional integrals, Riemann-Liouville fractional integrals, Hadamard fractional integrals.

1. Introduction

A function $f : [a, b] \to \mathbb{R}$ is said to be convex on $[a, b]$ if

$$f(tx + (1 - t)y) \leq tf(x) + (1 - t)f(y)$$

for all $x, y \in [a, b]$ and $t \in [0, 1]$ (see [26, 28]). The following result which holds for convex functions is known in the literature as the Hermite-Hadamard inequality.

Theorem 1.1 ([10]). *If $f : [a, b] \to \mathbb{R}$ is convex on $[a, b]$ with $a < b$, then*

$$f\Big(\frac{a+b}{2}\Big) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}.$$

Many authors have studied and generalized the Hermite-Hadamard inequality in several ways via different classes of convex functions. For some recent results related to the Hermite-Hadamard inequality, we refer the interested reader to the papers [1, 22, 23, 13, 20, 21, 4, 9, 3, 2, 18, 19].

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The concept of quasi-convexity which generalizes the concept of convexity is defined as follows.

Definition 1.2 (See [26, 28]**).** A function $f : [a, b] \to \mathbb{R}$ is said to be quasi-convex on $[a, b]$ if

$$f(tx + (1-t)y) \le \max\{f(x), f(y)\}$$

for all $x, y \in [a, b]$ and $t \in [0, 1]$.

In [12], Ion introduced the following Hermite-Hadamard type inequalities also known as trapezoidal-type inequalities for quasi-convex functions as follows.

Theorem 1.3. *Let $f : [a, b] \to \mathbb{R}$ be a differentiable function on (a, b). If $|f'|$ is quasi-convex on $[a, b]$, then the following inequality holds:*

$$\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x)dx \right| \le \frac{b-a}{4} \max\{|f'(a)|, |f'(b)|\}.$$

Theorem 1.4. *Let $f : [a, b] \to \mathbb{R}$ be a differentiable function on (a, b). If $|f'|^{\frac{p}{p-1}}$, $p > 1$ is quasi-convex on $[a, b]$, then the following inequality holds:*

$$\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x)dx \right| \le \frac{b-a}{2(p+1)^{1/p}} \left(\max\left\{ |f'(a)|^{\frac{p}{p-1}}, |f'(b)|^{\frac{p}{p-1}} \right\} \right)^{\frac{p-1}{p}}.$$

For more results related to quasi-convex functions, we refer the interested reader to the papers [9, 3, 1, 2]. The concept of preinvexity was introduced in [5, 11, 32] as a generalization of convexity as follows.

Definition 1.5. Let $I \subseteq \mathbb{R}$ and $\eta : I \times I \to \mathbb{R}$ be a bifunction. I is said to be an invex set with respect to η, if

$$x + t\eta(y, x) \in I \text{ for all } x, y \in I \text{ and } t \in [0, 1].$$

If $I \subseteq \mathbb{R}$ is an invex set with respect to the bifunction η, then a function $f : I \to \mathbb{R}$ is said to be a preinvex function with respect to η, if

$$f(x + t\eta(y, x)) \le (1-t)f(x) + tf(y) \text{ for all } x, y \in I \text{ and } t \in [0, 1].$$

Remark 1.6. If $\eta(y, x) = y - x$ in Definition 1.5, then we have that f is a convex function. Thus, every convex function is a preinvex function with respect to the bifunction $\eta(y, x) = y - x$. However, not every preinvex function is a convex function (see [32] for more details).

In a similar way, the concept of quasi-convexity has been generalized in the following definition.

Definition 1.7 ([24]**).** If $I \subseteq \mathbb{R}$ is an invex set with respect to the bifunction η, then a function $f : I \to \mathbb{R}$ is said to be prequasiinvex with respect to η, if

$$f(x + t\eta(y, x)) \le \max\{f(x), f(y)\} \text{ for all } x, y \in I \text{ and } t \in [0, 1].$$

Remark 1.8. Every quasi-convex function is a prequasiinvex function with respect to the bifunction $\eta(y, x) = y - x$. However, not every prequasinvex function is a quasi-convex function (see [33] for more details).

Barani et al. [4] established the following trapezoidal-type inequalities for prequasiinvex functions which are generalizations of Theorem 1.3 and Theorem 1.4.

Theorem 1.9. *Let $A \subseteq \mathbb{R}$ be an open invex subset with respect to $\eta : A \times A \to \mathbb{R}$. Suppose that $f : A \to \mathbb{R}$ is a differentiable function. If $|f'|$ is prequasiinvex on A, then for every $a, b \in A$ the following inequality holds:*

$$\left| \frac{f(a) + f(a + \eta(b,a))}{2} - \frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(x)dx \right| \le \frac{|\eta(b,a)|}{4} \max\{|f'(a)|, |f'(b)|\}.$$

Theorem 1.10. *Let $A \subseteq \mathbb{R}$ be an open invex subset with respect to $\eta : A \times A \to \mathbb{R}$. Suppose that $f : A \to \mathbb{R}$ is a differentiable function. If $|f'|^{\frac{p}{p-1}}$ is prequasiinvex on A, then for every $a, b \in A$ the following inequality holds:*

$$\left| \frac{f(a) + f(a + \eta(b,a))}{2} - \frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(x)dx \right|$$
$$\le \frac{|\eta(b,a)|}{2(p+1)^{1/p}} \left(\max\left\{ |f'(a)|^{\frac{p}{p-1}}, |f'(b)|^{\frac{p}{p-1}} \right\} \right)^{\frac{p-1}{p}}.$$

For more information and results related to prequasiinvex functions, we refer the interested reader to the papers [24, 33, 13, 20, 21]. In [13], the author generalized Theorem 1.9 and Theorem 1.10 using the Riemann-Liouville fractional integrals.

Our goal in this paper is to provide some midpoint and trapizoidal type inequalities for functions whose derivative in absolute value to some exponents are prequasiinvex via the Katugampola fractional integrals. Some of our results generalize the results in [13]. We end this section with the definitions of the Riemann-Liouville, Hadamard and Katugampola fractional integrals and some preliminary results.

Definition 1.11 ([25]). The left- and right-sided Riemann-Liouville fractional integrals of order $\alpha > 0$ of f are defined by

$$J^{\alpha}_{a+} f(x) := \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)dt$$

and

$$J^{\alpha}_{b-} f(x) := \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t)dt$$

with $a < x < b$ and $\Gamma(\cdot)$ is the gamma function given by

$$\Gamma(x) := \int_0^{\infty} t^{x-1} e^{-t} dt, \ Re(x) > 0$$

with the property that $\Gamma(x+1) = x\Gamma(x)$.

Definition 1.12 ([29]). The left- and right-sided Hadamard fractional integrals of order $\alpha > 0$ of f are defined by

$$H^{\alpha}_{a+} f(x) := \frac{1}{\Gamma(\alpha)} \int_a^x \left(\ln \frac{x}{t} \right)^{\alpha-1} \frac{f(t)}{t} dt$$

and

$$H_{b-}^{\alpha}f(x) := \frac{1}{\Gamma(\alpha)}\int_x^b \left(\ln\frac{t}{x}\right)^{\alpha-1}\frac{f(t)}{t}dt.$$

Definition 1.13. $X_c^p(a,b)$ $(c \in \mathbb{R},\ 1 \le p \le \infty)$ denotes the space of all complex-valued Lebesgue measurable functions f for which $\|f\|_{X_c^p} < \infty$, where the norm $\|\cdot\|_{X_c^p}$ is defined by

$$\|f\|_{X_c^p} = \left(\int_a^b |t^c f(t)|^p \frac{dt}{t}\right)^{1/p} \quad (1 \le p < \infty)$$

and for $p = \infty$

$$\|f\|_{X_c^\infty} = \text{ess}\sup_{a \le t \le b} |t^c f(t)|.$$

In 2011, Katugampola [14] introduced a new fractional integral operator which generalizes the Riemann-Liouville and Hadamard fractional integrals as follows:

Definition 1.14. Let $[a,b] \subset \mathbb{R}$ be a finite interval. Then, the left- and right-sided Katugampola fractional integrals of order $\alpha > 0$ of $f \in X_c^p(a,b)$ are defined by

$${}^{\rho}I_{a+}^{\alpha}f(x) := \frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_a^x \frac{t^{\rho-1}}{(x^\rho - t^\rho)^{1-\alpha}}f(t)dt$$

and

$${}^{\rho}I_{b-}^{\alpha}f(x) := \frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_x^b \frac{t^{\rho-1}}{(t^\rho - x^\rho)^{1-\alpha}}f(t)dt$$

with $a < x < b$ and $\rho > 0$, if the integrals exist.

Remark 1.15. It is shown in [14] that the Katugampola fractional integral operators are well-defined on $X_c^p(a,b)$.

Theorem 1.16 ([14]). *Let $\alpha > 0$ and $\rho > 0$. Then for $x > a$*

1. $\lim_{\rho\to 1} {}^{\rho}I_{a+}^{\alpha}f(x) = J_{a+}^{\alpha}f(x)$,
2. $\lim_{\rho\to 0^+} {}^{\rho}I_{a+}^{\alpha}f(x) = H_{a+}^{\alpha}f(x)$.

Similar results also hold for the right-sided operators.

For more information about the Katugampola fractional integrals and related results, we refer the interested reader to the papers [6, 14, 15, 16, 17].

Lemma 1.17 (See [27, 31]). *For any $\alpha \in [0,1]$ and $x, y \in [0,1]$, we have*

$$|x^\alpha - y^\alpha| \le |x-y|^\alpha.$$

2. Main results

2.1. Midpoint-type inequalities

The following lemma is a generalization of [7, Lemma 16] via the Katugampola fractional integrals.

Lemma 2.1. *Let $\alpha, \rho > 0$, $I \subseteq \mathbb{R}$ be an open invex set with respect to the bifunction $\eta : I \times I \to \mathbb{R}$ and $f : I \to \mathbb{R}$ be a differentiable mapping on I. If $a, b > 0$ with $a < b$ such that $a^\rho, b^\rho \in I$, $\eta(b^\rho, a^\rho) > 0$ and $f' \in L_1\Big([a^\rho, a^\rho + \eta(b^\rho, a^\rho)]\Big)$, then the following equality via the fractional integrals holds:*

$$\begin{aligned}
&f\left(\frac{2a^\rho + \eta(b^\rho, a^\rho)}{2}\right) - \frac{\rho^\alpha \Gamma(\alpha+1)}{2\eta(b^\rho, a^\rho)^\alpha}\Big[{}^\rho I^\alpha_{a+} f(a^\rho + \eta(b^\rho, a^\rho)) \\
&+ {}^\rho I^\alpha_{\left(\sqrt[\rho]{a^\rho + \eta(b^\rho, a^\rho)}\right)-} f(a^\rho)\Big] \\
&= \frac{\eta(b^\rho, a^\rho)\rho}{2}\Big(I_1 + I_2 + I_3 + I_4\Big), \qquad (2.1)
\end{aligned}$$

where

$$\begin{aligned}
I_1 &= \int_0^{\sqrt[\rho]{1/2}} t^{(\alpha+1)\rho-1} f'(a^\rho + t^\rho \eta(b^\rho, a^\rho))dt, \\
I_2 &= -\int_0^{\sqrt[\rho]{1/2}} t^{(\alpha+1)\rho-1} f'(a^\rho + (1-t^\rho)\eta(b^\rho, a^\rho))dt, \\
I_3 &= \int_{\sqrt[\rho]{1/2}}^1 (t^{\alpha\rho} - 1)t^{\rho-1} f'(a^\rho + t^\rho \eta(b^\rho, a^\rho))dt
\end{aligned}$$

and

$$I_4 = \int_{\sqrt[\rho]{1/2}}^1 (1 - t^{\alpha\rho})t^{\rho-1} f'(a^\rho + (1-t^\rho)\eta(b^\rho, a^\rho))dt.$$

Proof. By integrating by parts, we have

$$\begin{aligned}
I_1 &= \int_0^{\sqrt[\rho]{1/2}} t^{(\alpha+1)\rho-1} f'(a^\rho + t^\rho \eta(b^\rho, a^\rho))dt \\
&= \frac{t^{\alpha\rho}}{(b^\rho - a^\rho)\rho} f(a^\rho + t^\rho \eta(b^\rho, a^\rho))\Big|_0^{\sqrt[\rho]{1/2}} \\
&\qquad - \frac{\alpha}{\eta(b^\rho, a^\rho)} \int_0^{\sqrt[\rho]{1/2}} t^{\alpha\rho-1} f(a^\rho + t^\rho \eta(b^\rho, a^\rho))dt \\
&= \frac{2^{-\alpha}}{\eta(b^\rho, a^\rho)\rho} f\left(\frac{2a^\rho + \eta(b^\rho, a^\rho)}{2}\right) \\
&\qquad - \frac{\alpha}{\eta(b^\rho, a^\rho)} \int_0^{\sqrt[\rho]{1/2}} t^{\alpha\rho-1} f(a^\rho + t^\rho \eta(b^\rho, a^\rho))dt. \qquad (2.2)
\end{aligned}$$

Similarly, we have

$$I_2 = \frac{2^{-\alpha}}{\eta(b^\rho, a^\rho)\rho} f\left(\frac{2a^\rho + \eta(b^\rho, a^\rho)}{2}\right) - \frac{\alpha}{\eta(b^\rho, a^\rho)} \int_0^{\sqrt[\rho]{1/2}} t^{\alpha\rho-1} f(a^\rho + (1-t^\rho)\eta(b^\rho, a^\rho))dt, \tag{2.3}$$

$$\begin{aligned} I_3 &= \int_{\sqrt[\rho]{1/2}}^1 (t^{\alpha\rho} - 1)t^{\rho-1} f'(a^\rho + t^\rho \eta(b^\rho, a^\rho))dt \\ &= \frac{t^{\alpha\rho} - 1}{\eta(b^\rho, a^\rho)\rho} f(a^\rho + t^\rho\eta(b^\rho, a^\rho))\Big|_{\sqrt[\rho]{1/2}}^1 \\ &\quad - \frac{\alpha}{\eta(b^\rho, a^\rho)} \int_{\sqrt[\rho]{1/2}}^1 t^{\alpha\rho-1} f(a^\rho + t^\rho\eta(b^\rho, a^\rho))dt \\ &= \frac{1-2^{-\alpha}}{\eta(b^\rho, a^\rho)\rho} f\left(\frac{2a^\rho + \eta(b^\rho, a^\rho)}{2}\right) \\ &\quad - \frac{\alpha}{\eta(b^\rho, a^\rho)} \int_{\sqrt[\rho]{1/2}}^1 t^{\alpha\rho-1} f(a^\rho + t^\rho\eta(b^\rho, a^\rho))dt \end{aligned} \tag{2.4}$$

and

$$I_4 = \frac{1-2^{-\alpha}}{\eta(b^\rho, a^\rho)\rho} f\left(\frac{2a^\rho + \eta(b^\rho, a^\rho)}{2}\right) - \frac{\alpha}{\eta(b^\rho, a^\rho)} \int_{\sqrt[\rho]{1/2}}^1 t^{\alpha\rho-1} f(a^\rho + (1-t^\rho)\eta(b^\rho, a^\rho))dt. \tag{2.5}$$

Now, by using (2.2), (2.3), (2.4) and (2.5), we have

$$\begin{aligned} \frac{2}{\eta(b^\rho, a^\rho)\rho} f\left(\frac{2a^\rho + \eta(b^\rho, a^\rho)}{2}\right) &- \frac{\alpha}{\eta(b^\rho, a^\rho)}\left[\int_0^1 t^{\alpha\rho-1} f(a^\rho + t^\rho\eta(b^\rho, a^\rho))dt \right. \\ &\left. + \int_0^1 t^{\alpha\rho-1} f(a^\rho + (1-t^\rho)\eta(b^\rho, a^\rho))dt\right] \\ &= I_1 + I_2 + I_3 + I_4. \end{aligned} \tag{2.6}$$

By using change of variables and Definition 1.14, we have

$$\int_0^1 t^{\alpha\rho-1} f(a^\rho + t^\rho\eta(b^\rho, a^\rho))dt = \frac{\rho^{\alpha-1}\Gamma(\alpha)}{\eta(b^\rho, a^\rho)^\alpha} {}^\rho I^\alpha_{\left(\sqrt[\rho]{a^\rho + \eta(b^\rho, a^\rho)}\right)-} f(a^\rho) \tag{2.7}$$

and

$$\int_0^1 t^{\alpha\rho-1} f(a^\rho + (1-t^\rho)\eta(b^\rho, a^\rho))dt = \frac{\rho^{\alpha-1}\Gamma(\alpha)}{\eta(b^\rho, a^\rho)^\alpha} {}^\rho I^\alpha_{a+} f(a^\rho + \eta(b^\rho, a^\rho)). \tag{2.8}$$

Substituting (2.7) and (2.8) in (2.6), we obtain

$$I_1 + I_2 + I_3 + I_4 = \frac{2}{\eta(b^\rho, a^\rho)\rho} f\left(\frac{2a^\rho + \eta(b^\rho, a^\rho)}{2}\right) - \frac{\rho^{\alpha-1}\Gamma(\alpha+1)}{\eta(b^\rho, a^\rho)^{\alpha+1}} \times \left[{}^\rho I^\alpha_{a+} f(a^\rho + \eta(b^\rho, a^\rho)) + {}^\rho I^\alpha_{\left(\sqrt[\rho]{a^\rho+\eta(b^\rho,a^\rho)}\right)-} f(a^\rho)\right]. \tag{2.9}$$

The desired identity in (2.1) follows from (2.9). Hence, the proof is complete. □

Remark 2.2. If we choose $\rho = 1$ in Lemma 2.1, then we obtain [7, Lemma 16]. Also, if $\rho \neq 1$ and $\eta(x, y) = x - y$ in Lemma 2.1, then we obtain [8, Lemma 2.1] with a minor mistake in the identities obtained in [8] where $\Gamma(\alpha + 1)$ should have been $\Gamma(\alpha)$ instead.

Theorem 2.3. *Under the conditions of Lemma* 2.1, *if* $|f'|^q, q \geq 1$ *is prequasiinvex on* I, *then the following inequality holds:*

$$\left| f\left(\frac{2a^\rho + \eta(b^\rho, a^\rho)}{2}\right) - \frac{\rho^\alpha\Gamma(\alpha+1)}{2\eta(b^\rho, a^\rho)^\alpha}\left[{}^\rho I^\alpha_{a+} f(a^\rho + \eta(b^\rho, a^\rho)) + {}^\rho I^\alpha_{\left(\sqrt[\rho]{a^\rho+\eta(b^\rho,a^\rho)}\right)-} f(a^\rho)\right]\right| \leq \eta(b^\rho, a^\rho)\left(\frac{1}{2} - \frac{1}{\alpha+1} + \frac{1}{2^\alpha(\alpha+1)}\right)\left(\max\left\{|f'(a^\rho)|^q, |f'(b^\rho)|^q\right\}\right)^{1/q}.$$

Proof. By using Lemma 2.1 and the properties of the absolute value, we have

$$\left| f\left(\frac{2a^\rho + \eta(b^\rho, a^\rho)}{2}\right) - \frac{\rho^\alpha\Gamma(\alpha+1)}{2\eta_1(b^\rho, a^\rho)^\alpha}\left[{}^\rho I^\alpha_{a+} f(a^\rho + \eta(b^\rho, a^\rho)) + {}^\rho I^\alpha_{\left(\sqrt[\rho]{a^\rho+\eta(b^\rho,a^\rho)}\right)-} f(a^\rho)\right]\right| \leq \frac{\eta(b^\rho, a^\rho)\rho}{2}\Big(|I_1| + |I_2| + |I_3| + |I_4|\Big). \tag{2.10}$$

By using the power mean inequality, we have

$$|I_1| \leq \left(\int_0^{\sqrt[\rho]{1/2}} t^{(\alpha+1)\rho-1}dt\right)^{1-1/q}\left(\int_0^{\sqrt[\rho]{1/2}} t^{(\alpha+1)\rho-1}|f'(a^\rho + t^\rho\eta(b^\rho, a^\rho))|^q dt\right)^{1/q}. \tag{2.11}$$

Using the prequasiinvexity of $|f'|^q$, we have

$$|f'(a^\rho + t^\rho\eta(b^\rho, a^\rho))|^q \leq \max\left\{|f'(a^\rho)|^q, |f'(b^\rho)|^q\right\}. \tag{2.12}$$

Substituting (2.12) in (2.11), we obtain

$$|I_1| \leq \frac{1}{2^{\alpha+1}(\alpha+1)\rho}\left(\max\left\{|f'(a^\rho)|^q, |f'(b^\rho)|^q\right\}\right)^{1/q}. \tag{2.13}$$

Using similar arguments, we deduce that

$$|I_2| \leq \frac{1}{2^{\alpha+1}(\alpha+1)\rho}\left(\max\left\{|f'(a^\rho)|^q, |f'(b^\rho)|^q\right\}\right)^{1/q}, \tag{2.14}$$

$$\begin{aligned}|I_3| &\leq \int_{\sqrt[\rho]{1/2}}^{1} |t^{\alpha\rho}-1| t^{\rho-1}dt \left(\max\left\{|f'(a^\rho)|^q, |f'(b^\rho)|^q\right\}\right)^{1/q} \\ &= \frac{1}{\rho}\int_{1/2}^{1}(1-u^\alpha)du\left(\max\left\{|f'(a^\rho)|^q, |f'(b^\rho)|^q\right\}\right)^{1/q} \\ &= \frac{1}{\rho}\Big(\frac{1}{2}-\frac{1}{\alpha+1}+\frac{1}{2^{\alpha+1}(\alpha+1)}\Big)\left(\max\left\{|f'(a^\rho)|^q, |f'(b^\rho)|^q\right\}\right)^{1/q} \end{aligned}\tag{2.15}$$

and

$$|I_4| \leq \frac{1}{\rho}\Big(\frac{1}{2}-\frac{1}{\alpha+1}+\frac{1}{2^{\alpha+1}(\alpha+1)}\Big)\left(\max\left\{|f'(a^\rho)|^q, |f'(b^\rho)|^q\right\}\right)^{1/q}. \tag{2.16}$$

The desired inequality follows from (2.10) by using (2.11)-(2.12). □

Corollary 2.4. *If in Theorem* 2.3 *we take* $\eta(x,y) = x-y$ *for all* $x, y \in I$, *i.e,* $|f'|^q, q \geq 1$, *is quasiconvex, then the following inequality holds:*

$$\begin{aligned}&\left|f\left(\frac{a^\rho+b^\rho}{2}\right)-\frac{\rho^\alpha\Gamma(\alpha+1)}{2(b^\rho-a^\rho)^\alpha}\left[{}^\rho I^\alpha_{a+}f(b^\rho)+{}^\rho I^\alpha_{b-}f(a^\rho)\right]\right| \\ &\leq (b^\rho-a^\rho)\Big(\frac{1}{2}-\frac{1}{\alpha+1}+\frac{1}{2^{\alpha}(\alpha+1)}\Big)\left(\max\left\{|f'(a^\rho)|^q, |f'(b^\rho)|^q\right\}\right)^{1/q}.\end{aligned}$$

Remark 2.5. It is worth noting that in [8, Theorem 2.8] the authors established another estimate for the left hand side of the inequality in Corollary 2.4 under the condition that $|f'|$ is convex. On the other hand, since every convex function is quasiconvex it follows that the inequality in Corollary 2.4 holds if $|f'|^q, q \geq 1$ is convex.

Theorem 2.6. *Under the conditions of Lemma* 2.1*, if* $|f'|^q, q > 1$ *is prequasiinvex on* I*, then the following inequality holds:*

$$\begin{aligned}&\left|f\left(\frac{2a^\rho+\eta(b^\rho,a^\rho)}{2}\right)-\frac{\rho^\alpha\Gamma(\alpha+1)}{2\eta(b^\rho,a^\rho)^\alpha}\left[{}^\rho I^\alpha_{a+}f(a^\rho+\eta(b^\rho,a^\rho)) \right.\right.\\ &\qquad\left.\left. + {}^\rho I^\alpha_{\left(\sqrt[\rho]{a^\rho+\eta(b^\rho,a^\rho)}\right)-}f(a^\rho)\right]\right| \\ &\leq \frac{\eta(b^\rho,a^\rho)}{2}\left[\left(\frac{1}{2^{\alpha r}(\alpha r+1)}\right)^{1/r}+\left(2\int_{1/2}^{1}|u^\alpha-1|^r du\right)^{1/r}\right] \\ &\qquad\times\left(\max\left\{|f'(a^\rho)|^q, |f'(b^\rho)|^q\right\}\right)^{1/q},\end{aligned}\tag{2.17}$$

where $\frac{1}{r}+\frac{1}{q}=1$. *In addition, if* $\alpha\in(0,1]$, *then we have the inequality*

$$\left|f\left(\frac{2a^\rho+\eta(b^\rho,a^\rho)}{2}\right)-\frac{\rho^\alpha\Gamma(\alpha+1)}{2\eta(b^\rho,a^\rho)^\alpha}\left[{}^\rho I^\alpha_{a+}f(a^\rho+\eta(b^\rho,a^\rho))\right.\right.$$
$$\left.\left.+{}^\rho I^\alpha_{\left(\sqrt[\rho]{a^\rho+\eta(b^\rho,a^\rho)}\right)-}f(a^\rho)\right]\right|$$
$$\leq \eta(b^\rho,a^\rho)\left(\frac{1}{2^{\alpha r}(\alpha r+1)}\right)^{1/r}\left(\max\left\{|f'(a^\rho)|^q,|f'(b^\rho)|^q\right\}\right)^{1/q}. \tag{2.18}$$

Proof. By using Lemma 2.1 and the properties of the absolute value, we have

$$\left|f\left(\frac{2a^\rho+\eta(b^\rho,a^\rho)}{2}\right)-\frac{\rho^\alpha\Gamma(\alpha+1)}{2\eta(b^\rho,a^\rho)^\alpha}\left[{}^\rho I^\alpha_{a+}f(a^\rho+\eta(b^\rho,a^\rho))\right.\right.$$
$$\left.\left.+{}^\rho I^\alpha_{\left(\sqrt[\rho]{a^\rho+\eta(b^\rho,a^\rho)}\right)-}f(a^\rho)\right]\right|$$
$$\leq \frac{\eta(b^\rho,a^\rho)\rho}{2}\Big(|I_1|+|I_2|+|I_3|+|I_4|\Big). \tag{2.19}$$

By using the Hölder's inequality, we have

$$|I_1|\leq\left(\int_0^{\sqrt[\rho]{1/2}}t^{\alpha\rho r}t^{\rho-1}dt\right)^{1/r}\left(\int_0^{\sqrt[\rho]{1/2}}t^{\rho-1}|f'(a^\rho+t^\rho\eta(b^\rho,a^\rho))|^q dt\right)^{1/q}. \tag{2.20}$$

Using the prequasiinvexity of $|f'|^q$, we have

$$|f'(a^\rho+t^\rho\eta(b^\rho,a^\rho))|^q\leq\max\left\{|f'(a^\rho)|^q,|f'(b^\rho)|^q\right\}. \tag{2.21}$$

Substituting (2.21) in (2.20), we obtain

$$|I_1|\leq\left(\frac{1}{2^{\alpha r+1}(\alpha r+1)\rho}\right)^{1/r}\left(\frac{1}{2\rho}\max\left\{|f'(a^\rho)|^q,|f'(b^\rho)|^q\right\}\right)^{1/q}$$
$$=\frac{1}{2\rho}\left(\frac{1}{2^{\alpha r}(\alpha r+1)}\right)^{1/r}\left(\max\left\{|f'(a^\rho)|^q,|f'(b^\rho)|^q\right\}\right)^{1/q}. \tag{2.22}$$

Using similar arguments, we deduce that

$$|I_2|\leq\frac{1}{2\rho}\left(\frac{1}{2^{\alpha r}(\alpha r+1)}\right)^{1/r}\left(\max\left\{|f'(a^\rho)|^q,|f'(b^\rho)|^q\right\}\right)^{1/q}, \tag{2.23}$$

$$|I_3|\leq\left(\int_{\sqrt[\rho]{1/2}}^1|t^{\alpha\rho}-1|^r t^{\rho-1}dt\right)^{1/r}\left(\max\left\{|f'(a^\rho)|^q,|f'(b^\rho)|^q\right\}\int_{\sqrt[\rho]{1/2}}^1 t^{\rho-1}dt\right)^{1/q}$$
$$=\left(\frac{1}{\rho}\int_{1/2}^1|u^\alpha-1|^r du\right)^{1/r}\left(\frac{1}{2\rho}\max\left\{|f'(a^\rho)|^q,|f'(b^\rho)|^q\right\}\right)^{1/q}$$
$$=\frac{1}{2\rho}\left(2\int_{1/2}^1|u^\alpha-1|^r du\right)^{1/r}\left(\max\left\{|f'(a^\rho)|^q,|f'(b^\rho)|^q\right\}\right)^{1/q} \tag{2.24}$$

and

$$|I_4| \leq \frac{1}{2\rho}\left(2\int_{1/2}^{1}|u^\alpha-1|^r du\right)^{1/r}\left(\max\left\{|f'(a^\rho)|^q,|f'(b^\rho)|^q\right\}\right)^{1/q}. \tag{2.25}$$

The inequality in (2.17) follows from (2.19) by using (2.20)-(2.21). Now, if $\alpha \in (0,1]$, then it follows from Lemma 1.17 that

$$\int_{1/2}^{1}|u^\alpha-1|^r du \leq \int_{1/2}^{1}(1-u)^{\alpha r}du = \frac{1}{2^{\alpha r+1}(\alpha r+1)}. \tag{2.26}$$

The inequality in (2.18) follows from (2.17) by using (2.26). Hence, the proof is complete. □

Corollary 2.7. *If in Theorem* 2.6 *we take* $\eta(x,y) = x-y$ *for all* $x,y \in I$, *i.e,* $|f'|^q, q>1$, *is quasiconvex, then the following inequality holds:*

$$\begin{aligned}&\left|f\left(\frac{a^\rho+b^\rho}{2}\right)-\frac{\rho^\alpha\Gamma(\alpha+1)}{2(b^\rho-a^\rho)^\alpha}\left[{}^\rho I^\alpha_{a+}f(b^\rho)+{}^\rho I^\alpha_{b-}f(a^\rho)\right]\right|\\ &\leq \frac{b^\rho-a^\rho}{2}\left[\left(\frac{1}{2^{\alpha r}(\alpha r+1)}\right)^{1/r}+\left(2\int_{1/2}^{1}|u^\alpha-1|^r du\right)^{1/r}\right]\\ &\quad\times\left(\max\left\{|f'(a^\rho)|^q,|f'(b^\rho)|^q\right\}\right)^{1/q},\end{aligned}$$

where $\frac{1}{r}+\frac{1}{q}=1$. *In addition, if* $\alpha \in (0,1]$, *then we have the inequality*

$$\begin{aligned}&\left|f\left(\frac{a^\rho+b^\rho}{2}\right)-\frac{\rho^\alpha\Gamma(\alpha+1)}{2(b^\rho-a^\rho)^\alpha}\left[{}^\rho I^\alpha_{a+}f(b^\rho)+{}^\rho I^\alpha_{b-}f(a^\rho)\right]\right|\\ &\leq (b^\rho-a^\rho)\left(\frac{1}{2^{\alpha r}(\alpha r+1)}\right)^{1/r}\left(\max\left\{|f'(a^\rho)|^q,|f'(b^\rho)|^q\right\}\right)^{1/q}.\end{aligned}$$

2.2. Trapezoidal-type inequalities

The following lemma is a generalization of Lemma 2.4 in [6] for the invex case.

Lemma 2.8. *Let* $\alpha,\rho>0$, $I\subseteq\mathbb{R}$ *be an open invex set with respect to the bifunction* $\eta: I\times I\to\mathbb{R}$ *and* $f: I\to\mathbb{R}$ *be a differentiable mapping on* I. *If* $a,b>0$ *with* $a<b$ *such that* $a^\rho,b^\rho\in I$, $\eta(b^\rho,a^\rho)>0$ *and* $f'\in L_1\Big([a^\rho,a^\rho+\eta(b^\rho,a^\rho)]\Big)$, *then the following equality via the fractional integrals holds:*

$$\begin{aligned}&\frac{f(a^\rho)+f(a^\rho+\eta(b^\rho,a^\rho))}{2}-\frac{\rho^\alpha\Gamma(\alpha+1)}{2\eta(b^\rho,a^\rho)^\alpha}\Big[{}^\rho I^\alpha_{\left(\sqrt[\rho]{a^\rho+\eta(b^\rho,a^\rho)}\right)-}f(a^\rho)\\ &\qquad\qquad+{}^\rho I^\alpha_{a+}f(a^\rho+\eta(b^\rho,a^\rho))\Big]\\ &=\frac{\eta(b^\rho,a^\rho)\rho}{2}\int_0^1[(1-t^\rho)^\alpha-t^{\rho\alpha}]t^{\rho-1}f'(a^\rho+(1-t^\rho)\eta(b^\rho,a^\rho))dt.\end{aligned} \tag{2.27}$$

Proof. We observe that

$$\int_0^1 [(1-t^\rho)^\alpha - t^{\rho\alpha}]t^{\rho-1}f'(a^\rho + (1-t^\rho)\eta(b^\rho, a^\rho))dt = I_1 - I_2,$$

where

$$I_1 = \int_0^1 (1-t^\rho)^\alpha t^{\rho-1} f'(a^\rho + (1-t^\rho)\eta(b^\rho, a^\rho))dt$$

and

$$I_2 = \int_0^1 t^{\alpha\rho} t^{\rho-1} f'(a^\rho + (1-t^\rho)\eta(b^\rho, a^\rho))dt.$$

By integrating by parts and change of variables, we have

$$\begin{aligned}
I_1 &= \int_0^1 (1-t^\rho)^\alpha t^{\rho-1} f'(a^\rho + (1-t^\rho)\eta(b^\rho, a^\rho))dt \\
&= -\frac{(1-t^\rho)^\alpha}{\eta(b^\rho, a^\rho)\rho} f(a^\rho + (1-t^\rho)\eta(b^\rho, a^\rho))\Big|_0^1 \\
&\quad - \frac{\alpha}{\eta(b^\rho, a^\rho)} \int_0^1 (1-t^\rho)^{\alpha-1} t^{\rho-1} f(a^\rho + (1-t^\rho)\eta(b^\rho, a^\rho))dt \\
&= \frac{1}{\eta(b^\rho, a^\rho)\rho} f(a^\rho + \eta(b^\rho, a^\rho)) \\
&\quad - \frac{\alpha}{\eta(b^\rho, a^\rho)} \int_0^1 (1-t^\rho)^{\alpha-1} t^{\rho-1} f(a^\rho + (1-t^\rho)\eta(b^\rho, a^\rho))dt \\
&= \frac{1}{\eta(b^\rho, a^\rho)\rho} f(a^\rho + \eta(b^\rho, a^\rho)) \\
&\quad - \frac{\alpha}{\eta(b^\rho, a^\rho)^{\alpha+1}} \int_a^{\sqrt[\rho]{a^\rho + \eta(b^\rho, a^\rho)}} (u^\rho - a^\rho)^{\alpha-1} u^{\rho-1} f(u^\rho)du. \qquad (2.28)
\end{aligned}$$

By using Definition 1.14 and (2.28), we have

$$I_1 = \frac{f(a^\rho + \eta(b^\rho, a^\rho))}{\eta(b^\rho, a^\rho)\rho} - \frac{\rho^{\alpha-1}\Gamma(\alpha+1)}{\eta(b^\rho, a^\rho)^{\alpha+1}} {}^\rho I^\alpha_{\left(\sqrt[\rho]{a^\rho + \eta(b^\rho, a^\rho)}\right)_-} f(a^\rho). \qquad (2.29)$$

By a similar argument, we have

$$I_2 = -\frac{f(a^\rho)}{\eta(b^\rho, a^\rho)\rho} + \frac{\rho^{\alpha-1}\Gamma(\alpha+1)}{\eta(b^\rho, a^\rho)^{\alpha+1}} {}^\rho I^\alpha_{a+} f(a^\rho + \eta(b^\rho, a^\rho)). \qquad (2.30)$$

By using (2.29) and (2.30), we have

$$\begin{aligned}
I_1 - I_2 = \frac{f(a^\rho) + f(a^\rho + \eta(b^\rho, a^\rho))}{\eta(b^\rho, a^\rho)\rho} - \frac{\rho^{\alpha-1}\Gamma(\alpha+1)}{\eta(b^\rho, a^\rho)^{\alpha+1}} \Big[&{}^\rho I^\alpha_{\left(\sqrt[\rho]{a^\rho + \eta(b^\rho, a^\rho)}\right)_-} f(a^\rho) \\
&+ {}^\rho I^\alpha_{a+} f(a^\rho + \eta(b^\rho, a^\rho)) \Big]. \qquad (2.31)
\end{aligned}$$

The desired identity in (2.27) follows from (2.31). □

Remark 2.9. If $\eta(x,y) = x - y$ in Lemma 2.8, then we obtain [6, Lemma 2.4] with minor mistakes in the identity obtained in [6] where $\Gamma(\alpha+1)$ should have been $\Gamma(\alpha)$ and $\frac{b^\rho - a^\rho}{2}$ should have been $\frac{(b^\rho - a^\rho)\rho}{2}$ instead.

Theorem 2.10. *Under the conditions of Lemma 2.8, if $|f'|^q, q \geq 1$ is prequasiinvex on I, then the following inequality holds:*

$$\begin{aligned}&\left|\frac{f(a^\rho)+f(a^\rho+\eta(b^\rho,a^\rho))}{2} - \frac{\rho^\alpha\Gamma(\alpha+1)}{2\eta(b^\rho,a^\rho)^\alpha}\left[{}^\rho I^\alpha_{\left(\sqrt[\rho]{a^\rho+\eta(b^\rho,a^\rho)}\right)-} f(a^\rho)\right.\right.\\ &\qquad\left.\left. + {}^\rho I^\alpha_{a+} f(a^\rho+\eta(b^\rho,a^\rho))\right]\right|\\ &\leq \frac{\eta(b^\rho,a^\rho)}{\alpha+1}\left(1-\frac{1}{2^\alpha}\right)\left(\max\left\{|f'(a^\rho)|^q, |f'(b^\rho)|^q\right\}\right)^{1/q}. \end{aligned} \tag{2.32}$$

Proof. Using Lemma 2.8, the power mean inequality and the prequasiinvexity of $|f'|^q$, we have

$$\begin{aligned}&\left|\frac{f(a^\rho)+f(a^\rho+\eta(b^\rho,a^\rho))}{2} - \frac{\rho^\alpha\Gamma(\alpha+1)}{2\eta(b^\rho,a^\rho)^\alpha}\left[{}^\rho I^\alpha_{\left(\sqrt[\rho]{a^\rho+\eta(b^\rho,a^\rho)}\right)-} f(a^\rho)\right.\right.\\ &\qquad\left.\left. + {}^\rho I^\alpha_{a+} f(a^\rho+\eta(b^\rho,a^\rho))\right]\right|\\ &\leq \frac{\eta(b^\rho,a^\rho)\rho}{2}\left(\int_0^1 \left|(1-t^\rho)^\alpha - t^{\rho\alpha}\right| t^{\rho-1}dt\right)^{1-1/q}\\ &\quad\times\left(\int_0^1 \left|(1-t^\rho)^\alpha - t^{\rho\alpha}\right| t^{\rho-1}\left|f'(a^\rho+(1-t^\rho)\eta(b^\rho,a^\rho))\right|^q dt\right)^{1/q}\\ &\leq \frac{\eta(b^\rho,a^\rho)\rho}{2}\left(\int_0^1 \left|(1-t^\rho)^\alpha - t^{\rho\alpha}\right| t^{\rho-1}dt\right)\left(\max\left\{|f'(a^\rho)|^q, |f'(b^\rho)|^q\right\}\right)^{1/q}\\ &= \frac{\eta(b^\rho,a^\rho)\rho}{2}\left(\frac{1}{\rho}\int_0^1 \left|(1-u)^\alpha - u^\alpha\right| du\right)\left(\max\left\{|f'(a^\rho)|^q, |f'(b^\rho)|^q\right\}\right)^{1/q}. \end{aligned} \tag{2.33}$$

Now, we observe that

$$\begin{aligned}\int_0^1 \left|(1-u)^\alpha - u^\alpha\right| du &= \int_0^{1/2}\left((1-u)^\alpha - u^\alpha\right)du + \int_{1/2}^1\left(u^\alpha - (1-u)^\alpha\right)du\\ &= \frac{1}{\alpha+1} - \frac{1}{2^\alpha(\alpha+1)} + \frac{1}{\alpha+1} - \frac{1}{2^\alpha(\alpha+1)}\\ &= \frac{2}{\alpha+1}\left(1-\frac{1}{2^\alpha}\right). \end{aligned} \tag{2.34}$$

The inequality in (2.32) follows from (2.33) and (2.34). □

Remark 2.11. If $\eta(x,y) = x - y$ in Theorem 2.10, then we recover the result in [30, Theorem 2.4]. Also, if $\rho = 1$ in Theorem 2.10, then we obtain the result in [13, Theorem 2.3].

Theorem 2.12. *Under the conditions of Lemma* 2.8*, if* $|f'|^q, q > 1$ *is prequasiinvex on* I*, then the following inequality holds:*

$$\left|\frac{f(a^\rho)+f(a^\rho+\eta(b^\rho,a^\rho))}{2}-\frac{\rho^\alpha\Gamma(\alpha+1)}{2\eta(b^\rho,a^\rho)^\alpha}\left[{}^\rho I^\alpha_{\left(\sqrt[\rho]{a^\rho+\eta(b^\rho,a^\rho)}\right)-}f(a^\rho)+{}^\rho I^\alpha_{a+}f(a^\rho+\eta(b^\rho,a^\rho))\right]\right|$$
$$\leq \frac{\eta(b^\rho,a^\rho)}{2}\left(\int_0^1\left|(1-u)^\alpha-u^\alpha\right|^r du\right)^{1/r}\left(\max\left\{|f'(a^\rho)|^q,|f'(b^\rho)|^q\right\}\right)^{1/q}, \quad (2.35)$$

where $\frac{1}{r}+\frac{1}{q}=1$*. In addition, if* $\alpha\in(0,1]$*, then we have the inequality*

$$\left|\frac{f(a^\rho)+f(a^\rho+\eta(b^\rho,a^\rho))}{2}-\frac{\rho^\alpha\Gamma(\alpha+1)}{2\eta(b^\rho,a^\rho)^\alpha}\left[{}^\rho I^\alpha_{\left(\sqrt[\rho]{a^\rho+\eta(b^\rho,a^\rho)}\right)-}f(a^\rho)+{}^\rho I^\alpha_{a+}f(a^\rho+\eta(b^\rho,a^\rho))\right]\right|$$
$$\leq \frac{\eta(b^\rho,a^\rho)}{2}\left(\frac{1}{\alpha r+1}\right)^{1/r}\left(\max\left\{|f'(a^\rho)|^q,|f'(b^\rho)|^q\right\}\right)^{1/q}. \quad (2.36)$$

Proof. Using Lemma 2.8, the Hölder's inequality and the prequasiinvexity of $|f'|^q$, we have

$$\left|\frac{f(a^\rho)+f(a^\rho+\eta(b^\rho,a^\rho))}{2}-\frac{\rho^\alpha\Gamma(\alpha+1)}{2\eta(b^\rho,a^\rho)^\alpha}\left[{}^\rho I^\alpha_{\left(\sqrt[\rho]{a^\rho+\eta(b^\rho,a^\rho)}\right)-}f(a^\rho)+{}^\rho I^\alpha_{a+}f(a^\rho+\eta(b^\rho,a^\rho))\right]\right|$$
$$\leq \frac{\eta(b^\rho,a^\rho)\rho}{2}\left(\int_0^1\left|(1-t^\rho)^\alpha-t^{\rho\alpha}\right|^r t^{\rho-1}dt\right)^{1/r}$$
$$\times\left(\int_0^1 t^{\rho-1}\left|f'(a^\rho+(1-t^\rho)\eta(b^\rho,a^\rho))\right|^q dt\right)^{1/q}$$
$$\leq \frac{\eta(b^\rho,a^\rho)\rho}{2}\left(\frac{1}{\rho}\int_0^1\left|(1-u)^\alpha-u^\alpha\right|^r du\right)^{1/r}\left(\frac{1}{\rho}\max\left\{|f'(a^\rho)|^q,|f'(b^\rho)|^q\right\}\right)^{1/q}$$
$$= \frac{\eta(b^\rho,a^\rho)}{2}\left(\int_0^1\left|(1-u)^\alpha-u^\alpha\right|^r du\right)^{1/r}\left(\max\left\{|f'(a^\rho)|^q,|f'(b^\rho)|^q\right\}\right)^{1/q}.$$

This proves the inequality in (2.35). By using Lemma 1.17 with $\alpha \in (0,1]$, we deduce that

$$\begin{aligned}\int_0^1 \Big|(1-u)^\alpha - u^\alpha\Big|^r du &\le \int_0^1 \Big|1-2u\Big|^{\alpha r} du \\ &= \int_0^{1/2} (1-2u)^{\alpha r} du + \int_{1/2}^1 (2u-1)^{\alpha r} du \\ &= \frac{1}{2(\alpha r+1)} + \frac{1}{2(\alpha r+1)} \\ &= \frac{1}{\alpha r+1}. \qquad (2.37)\end{aligned}$$

The inequality in (2.36) follows from (2.35) and (2.37). □

Remark 2.13. If $\rho = 1$ in the inequality (2.36) in Theorem 2.12, then we obtain the result in [13, Theorem 2.4].

Corollary 2.14. *If in Theorem 2.12 we take $\eta(x,y) = x - y$ for all $x, y \in I$, i.e, $|f'|^q, q > 1$, is quasiconvex, then the following inequality holds:*

$$\begin{aligned}&\left|\frac{f(a^\rho)+f(b^\rho)}{2} - \frac{\rho^\alpha \Gamma(\alpha+1)}{2(b^\rho - a^\rho)^\alpha}\left[{}^\rho I^\alpha_{b-} f(a^\rho) + {}^\rho I^\alpha_{a+} f(b^\rho)\right]\right| \\ &\le \frac{b^\rho - a^\rho}{2}\left(\int_0^1 \Big|(1-u)^\alpha - u^\alpha\Big|^r du\right)^{1/r} \left(\max\left\{|f'(a^\rho)|^q, |f'(b^\rho)|^q\right\}\right)^{1/q},\end{aligned}$$

where $\frac{1}{r} + \frac{1}{q} = 1$. In addition, if $\alpha \in (0,1]$, then we have the inequality

$$\begin{aligned}&\left|\frac{f(a^\rho)+f(b^\rho)}{2} - \frac{\rho^\alpha \Gamma(\alpha+1)}{2(b^\rho - a^\rho)^\alpha}\left[{}^\rho I^\alpha_{b-} f(a^\rho) + {}^\rho I^\alpha_{a+} f(a^\rho + \eta(b^\rho, a^\rho))\right]\right| \\ &\le \frac{b^\rho - a^\rho}{2}\left(\frac{1}{\alpha r+1}\right)^{1/r} \left(\max\left\{|f'(a^\rho)|^q, |f'(b^\rho)|^q\right\}\right)^{1/q}.\end{aligned}$$

3. Conclusion

We established two midpoint-type inequalities and two trapezoidal-type inequalities for functions whose derivatives in absolute value to some powers are prequasiinvex with respect to a bifunction η via the Katugampola fractional integral operators. By considering the bifunction $\eta(x,y) = x - y$, the results for quasiconvex functions has been obtained from our main results. Several other results can be obtained from our results by considering different bifunctions and/or different values of the parameters involved. In particular, if we take $\rho = 1$, then our results are in terms of the Riemann-Liouville fractional integrals. Also, we hope that under certain conditions on f and η, similar results via the Hadamard fractional integrals could be derived from our results by taking the limit as $\rho \to 0^+$. The details are left for the interested reader.

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Existence theory for implicit fractional q-difference equations in Banach spaces

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Abstract. This paper deals with some existence results for a class of implicit fractional q-difference equations. The results are based on the fixed point theory in Banach spaces and the concept of measure of noncompactness. An illustrative example is given in the last section.

Mathematics Subject Classification (2010): 26A33, 34G20.

Keywords: Fractional q-difference equation, implicit, measure of noncompactness, solution, fixed point.

1. Introduction

Fractional differential equations have recently been applied in various areas of engineering, mathematics, physics, and other applied sciences [27]. For some fundamental results in the theory of fractional calculus and fractional differential equations we refer the reader to the monographs [1, 2, 3, 20, 26, 30], the papers [21, 22, 29] and the references therein. Recently, considerable attention has been given to the existence of solutions of initial and boundary value problems for fractional differential equations and inclusions with Caputo fractional derivative; [2, 19]. Implicit fractional differential equations were analyzed by many authors; see, for instance [1, 2, 4, 12, 13, 14] and the references therein.

Fractional q-difference equations were initiated at the beginning of the 19th century [5, 15], and received significant attention in recent years. Some interesting details about initial and boundary value problems of q-difference and fractional q-difference equations can be found in [7, 8, 16, 17] and references therein.

Recently, in [3], the authors applied the measure of noncompactness to some classes of functional Riemann–Liouville or Caputo fractional differential equations in

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Banach spaces. Motivated by the above papers, we discuss the existence of solutions for the following implicit fractional q-difference equation

$$({}^C D_q^\alpha u)(t) = f(t, u(t), ({}^C D_q^\alpha u)(t)), \ t \in I := [0, T], \tag{1.1}$$

with the initial condition

$$u(0) = u_0, \tag{1.2}$$

where $q \in (0,1)$, $\alpha \in (0,1]$, $T > 0$, $f : I \times E \times E \to E$ is a given function, E is a real (or complex) Banach space with norm $\|\cdot\|$, and ${}^C D_q^\alpha$ is the Caputo fractional q-difference derivative of order α.

This paper initiates the study of implicite fractional q-difference equations on Banach spaces.

2. Preliminaries

Consider the Banach space $C(I) := C(I, E)$ of continuous functions from I into E equipped with the usual supremum (uniform) norm

$$\|u\|_\infty := \sup_{t \in I} \|u(t)\|.$$

As usual, $L^1(I)$ denotes the space of measurable functions $v : I \to E$ which are Bochner integrable with the norm

$$\|v\|_1 = \int_0^T \|v(t)\| dt.$$

Let us recall some definitions and properties of fractional q-calculus. For $a \in \mathbb{R}$, we set

$$[a]_q = \frac{1 - q^a}{1 - q}.$$

The q-analogue of the power $(a - b)^n$ is

$$(a - b)^{(0)} = 1, \ (a - b)^{(n)} = \Pi_{k=0}^{n-1}(a - bq^k); \ a, b \in \mathbb{R}, \ n \in \mathbb{N}.$$

In general,

$$(a - b)^{(\alpha)} = a^\alpha \Pi_{k=0}^{\infty} \left(\frac{a - bq^k}{a - bq^{k+\alpha}} \right); \ a, b, \alpha \in \mathbb{R}.$$

Definition 2.1. [18] The q-gamma function is defined by

$$\Gamma_q(\xi) = \frac{(1 - q)^{(\xi - 1)}}{(1 - q)^{\xi - 1}}; \ \xi \in \mathbb{R} - \{0, -1, -2, ...\}$$

Notice that the q-gamma function satisfies $\Gamma_q(1 + \xi) = [\xi]_q \Gamma_q(\xi)$.

Definition 2.2. [18] The q-derivative of order $n \in \mathbb{N}$ of a function $u : I \to E$ is defined by $(D_q^0 u)(t) = u(t)$,

$$(D_q u)(t) := (D_q^1 u)(t) = \frac{u(t) - u(qt)}{(1 - q)t}; \ t \neq 0, \quad (D_q u)(0) = \lim_{t \to 0} (D_q u)(t),$$

and

$$(D_q^n u)(t) = (D_q D_q^{n-1} u)(t); \ t \in I, \ n \in \{1, 2, \ldots\}.$$

Set $I_t := \{tq^n : n \in \mathbb{N}\} \cup \{0\}$.

Definition 2.3. [18] The q-integral of a function $u : I_t \to E$ is defined by

$$(I_q u)(t) = \int_0^t u(s) d_q s = \sum_{n=0}^{\infty} t(1-q)q^n f(tq^n),$$

provided that the series converges.

We note that $(D_q I_q u)(t) = u(t)$, while if u is continuous at 0, then

$$(I_q D_q u)(t) = u(t) - u(0).$$

Definition 2.4. [6] The Riemann–Liouville fractional q-integral of order $\alpha \in \mathbb{R}_+ := [0, \infty)$ of a function $u : I \to E$ is defined by $(I_q^0 u)(t) = u(t)$, and

$$(I_q^\alpha u)(t) = \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} u(s) d_q s; \ t \in I.$$

Lemma 2.5. [24] *For $\alpha \in \mathbb{R}_+ := [0, \infty)$ and $\lambda \in (-1, \infty)$ we have*

$$(I_q^\alpha (t-a)^{(\lambda)})(t) = \frac{\Gamma_q(1+\lambda)}{\Gamma(1+\lambda+\alpha)} (t-a)^{(\lambda+\alpha)}; \ 0 < a < t < T.$$

In particular,

$$(I_q^\alpha 1)(t) = \frac{1}{\Gamma_q(1+\alpha)} t^{(\alpha)}.$$

Definition 2.6. [25] The Riemann–Liouville fractional q-derivative of order $\alpha \in \mathbb{R}_+$ of a function $u : I \to E$ is defined by $(D_q^0 u)(t) = u(t)$, and

$$(D_q^\alpha u)(t) = (D_q^{[\alpha]} I_q^{[\alpha]-\alpha} u)(t); \ t \in I,$$

where $[\alpha]$ is the integer part of α.

Definition 2.7. [25] The Caputo fractional q-derivative of order $\alpha \in \mathbb{R}_+$ of a function $u : I \to E$ is defined by $({}^C D_q^0 u)(t) = u(t)$, and

$$({}^C D_q^\alpha u)(t) = (I_q^{[\alpha]-\alpha} D_q^{[\alpha]} u)(t); \ t \in I.$$

Lemma 2.8. [25] *Let $\alpha \in \mathbb{R}_+$. Then the following equality holds:*

$$(I_q^\alpha \, {}^C D_q^\alpha u)(t) = u(t) - \sum_{k=0}^{[\alpha]-1} \frac{t^k}{\Gamma_q(1+k)} (D_q^k u)(0).$$

In particular, if $\alpha \in (0, 1)$, then

$$(I_q^\alpha \, {}^C D_q^\alpha u)(t) = u(t) - u(0).$$

From the above lemma and in order to define a solution for the problem (1.1)-(1.2), we conclude with the following lemma.

Lemma 2.9. *Let $f : I \times E \times E \to E$ such that $f(\cdot, u, v) \in C(I)$, for each $u, v \in E$. Then the problem (1.1)-(1.2) is equivalent to the problem of obtaining solutions of the integral equation*

$$g(t) = f(t, u_0 + (I_q^\alpha g)(t), g(t)),$$

and if $g(\cdot) \in C(I)$ is the solution of this equation, then

$$u(t) = u_0 + (I_q^\alpha g)(t).$$

Definition 2.10. [9, 10, 11, 28] Let X be a Banach space and let Ω_X be the family of bounded subsets of X. The Kuratowski measure of noncompactness is the map $\mu : \Omega_X \to [0, \infty)$ defined by

$$\mu(M) = \inf\{\epsilon > 0 : M \subset \cup_{j=1}^{m} M_j, \text{diam}(M_j) \leq \epsilon\},$$

where $M \in \Omega_X$.

The measure of noncompactness satisfies the following properties

(1) $\mu(M) = 0 \Leftrightarrow \overline{M}$ is compact (M is relatively compact).
(2) $\mu(M) = \mu(\overline{M})$.
(3) $M_1 \subset M_2 \Rightarrow \mu(M_1) \leq \mu(M_2)$.
(4) $\mu(M_1 + M_2) \leq \mu(M_1) + \mu(M_2)$.
(5) $\mu(cM) = |c|\mu(M), \ c \in \mathbb{R}$.
(6) $\mu(\text{conv}\, M) = \mu(M)$.

For our purpose we will need the following fixed point theorem:

Theorem 2.11. *(Monch's fixed point theorem [23]). Let D be a bounded, closed and convex subset of a Banach space such that $0 \in D$, and let N be a continuous mapping of D into itself. If the implication*

$$V = \overline{conv} N(V) \text{ or } V = N(V) \cup \{0\} \Rightarrow \overline{V} \text{ is compact}, \tag{2.1}$$

holds for every subset V of D, then N has a fixed point.

3. Main results

In this section, we are concerned with existence results for the problem (1.1)-(1.2).

Definition 3.1. By a solution of problem (1.1)-(1.2), we mean a continuous function u that satisfies the equation (1.1) on I and the initial condition (1.2).

The following hypotheses will be used in the sequel.

(H_1) The function $f : I \times E \times E \to E$ is continuous.
(H_2) There exists a continuous function $p \in C(I, \mathbb{R}_+)$, such that

$$\|f(t, u, v)\| \leq p(t); \ \textit{for } t \in I, \ \textit{and } u, v \in E,$$

(H_3) For each bounded set $B \subset E$ and for each $t \in I$, we have

$$\mu(f(t, B, {}^C D_q^r B)) \leq p(t)\mu(B),$$

where ${}^C D_q^r B = \{{}^C D_q^r w : w \in B\}$, and μ is a measure of noncompactness on E.

Set

$$p^* = \sup_{t\in I} p(t), \ \textit{and}\ L := \sup_{t\in I} \int_0^T \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} d_q s.$$

Theorem 3.2. *Assume that the hypotheses* $(H_1)-(H_3)$ *hold. If*

$$\ell := Lp^* < 1, \tag{3.1}$$

then the problem (1.1)-(1.2) has at least one solution defined on I.

Proof. By using Lemma 2.9, we transform the problem (1.1)-(1.2) into a fixed point problem. Consider the operator $N : C(I) \to C(I)$ defined by

$$(Nu)(t) = u_0 + (I_q^\alpha g)(t);\ t \in I, \tag{3.2}$$

where $g \in C(I)$ such that

$$g(t) = f(t, u(t), g(t)),\ \textit{or}\ g(t) = f(t, u_0 + (I_q^\alpha g)(t), g(t)).$$

For any $u \in C(I)$ and each $t \in I$, we have

$$\begin{aligned}
\|(Nu)(t)\| &\le \|u_0\| + \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} |g(s)| d_q s \\
&\le \|u_0\| + \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} p(s) d_q s \\
&\le \|u_0\| + p^* \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} d_q s \\
&\le \|u_0\| + Lp^* \\
&:= R.
\end{aligned}$$

Thus

$$\|N(u)\|_\infty \le R. \tag{3.3}$$

This proves that N transforms the ball $B_R := B(0, R) = \{w \in C : \|w\|_\infty \le R\}$ into itself.
We shall show that the operator $N : B_R \to B_R$ satisfies all the assumptions of Theorem 2.11. The proof will be given in three steps.

Step 1. $N : B_R \to B_R$ *is continuous.*
Let $\{u_n\}_{n\in\mathbb{N}}$ be a sequence such that $u_n \to u$ in B_R. Then, for each $t \in I$, we have

$$\|(Nu_n)(t) - (Nu)(t)\| \ \le\ \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} \|(g_n(s) - g(s))\| d_q s,$$

where $g_n, g \in C(I)$ such that

$$g_n(t) = f(t, u_n(t), g_n(t)),$$

and

$$g(t) = f(t, u(t), g(t)).$$

Since $u_n \to u$ as $n \to \infty$ and f is continuous, we get

$$g_n(t) \to g(t)\ \text{as}\ n \to \infty,\ \text{for each}\ t \in I.$$

Hence

$$\|N(u_n) - N(u)\|_\infty \leq L\|g_n - g\|_\infty \to 0 \;\; as \;\; n \to \infty.$$

Step 2. *$N(B_R)$ is bounded and equicontinuous.*
Since $N(B_R) \subset B_R$ and B_R is bounded, then $N(B_R)$ is bounded.
Next, let $t_1, t_2 \in I$, $t_1 < t_2$ and let $u \in B_R$. Thus, we have

$$\|(Nu)(t_2) - (Nu)(t_1)\| \leq \left\| \int_0^{t_2} \frac{(t_2 qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} g(s) d_q s - \int_0^{t_1} \frac{(t_1 qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} g(s) d_q s \right\|.$$

where $g \in C(I)$ such that

$$g(t) = f(t, u(t), g(t)).$$

Hence, we get

$$\begin{aligned} \|(Nu)(t_2) - (Nu)(t_1)\| &\leq \int_{t_1}^{t_2} \frac{(t_2 qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} p(s) d_q s \\ &+ \int_0^{t_1} \left| \frac{(t_2 qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} - \frac{(t_1 qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} \right| d_q s \\ &\leq p^* \int_{t_1}^{t_2} \frac{(t_2 qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} p(s) d_q s \\ &+ p^* \int_0^{t_1} \left| \frac{(t_2 qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} - \frac{(t_1 qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} \right| d_q s. \end{aligned}$$

As $t_1 \longrightarrow t_2$, the right-hand side of the above inequality tends to zero.
Step 3. *The implication (2.1) holds.*
Now let V be a subset of B_R such that $V \subset \overline{N(V)} \cup \{0\}$. V is bounded and equicontinuous and therefore the function $t \to v(t) = \alpha(V(t))$ is continuous on I. By (H_3) and the properties of the measure μ, for each $t \in I$, we have

$$\begin{aligned} v(t) &\leq \mu((NV)(t) \cup \{0\}) \\ &\leq \mu((NV)(t)) \\ &\leq \int_0^t \frac{(tq - s)^{(\alpha-1)}}{\Gamma_q(\alpha)} p(s) \mu(V(s)) d_q s \\ &\leq \int_0^t \frac{(tq - s)^{(\alpha-1)}}{\Gamma_q(\alpha)} p(s) v(s) d_q s \\ &\leq Lp^* \|v\|_\infty. \end{aligned}$$

Thus

$$\|v\|_\infty \leq \ell \|v\|_\infty.$$

From (3.1), we get $\|v\|_\infty = 0$, that is, $v(t) = \mu(V(t)) = 0$, for each $t \in I$, and then $V(t)$ is relatively compact in E. In view of the Ascoli–Arzelà theorem, V is relatively compact in B_R. Applying now Theorem 2.11, we conclude that N has a fixed point which is a solution of the problem (1.1)-(1.2).

4. An example

Let

$$l^1 = \left\{ u = (u_1, u_2, \ldots, u_n, \ldots) : \sum_{n=1}^{\infty} |u_n| < \infty \right\}$$

be the Banach space with the norm

$$\|u\|_{l^1} = \sum_{n=1}^{\infty} |u_n|.$$

Consider the following problem of implicit fractional $\frac{1}{4}$−difference equations

$$\begin{cases} ({}^cD_{\frac{1}{4}}^{\frac{1}{2}} u_n)(t) = f_n(t, u(t), ({}^cD_{\frac{1}{4}}^{\frac{1}{2}} u)(t)); \ t \in [0,1], \\ u(0) = (0, 0, \ldots, 0, \ldots), \end{cases} \tag{4.1}$$

where

$$\begin{cases} f_n(t,u,v) = \dfrac{t^{\frac{-1}{4}}(2^{-n} + u_n(t)) \sin t}{64L(1 + \|u\|_{l^1} + \sqrt{t})(1 + \|u\|_{l^1} + \|v\|_{l^1})}, \ t \in (0,1], \\ \qquad f_n(0,u,v) = 0, \end{cases} .$$

with

$$f = (f_1, f_2, \ldots, f_n, \ldots), \ and \ u = (u_1, u_2, \ldots, u_n, \ldots).$$

For each $t \in (0,1]$, we have

$$\begin{aligned} \|f(t,u(t))\|_{l^1} &= \sum_{n=1}^{\infty} |f_n(s, u_n(s))| \\ &\leq \frac{t^{\frac{-1}{4}} |\sin t|}{64L(1 + \|u\|_{l^1} + \sqrt{t})(1 + \|u\|_{l^1} + \|v\|_{l^1})} (1 + \|u\|_{l^1}) \\ &\leq \frac{t^{\frac{-1}{4}} |\sin t|}{64L}. \end{aligned}$$

Thus, the hypothesis (H_2) is satisfied with

$$\begin{cases} p(t) = \dfrac{t^{\frac{-1}{4}} |\sin t|}{64L}; \ t \in (0,1], \\ \quad p(0) = 0. \end{cases}$$

So, we have $p^* \leq \frac{1}{64L}$, and then

$$Lp^* = \frac{1}{64} < 1.$$

Simple computations show that all conditions of Theorem 3.2 are satisfied. Hence, the problem (4.1) has at least one solution defined on $[0,1]$.

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Darboux problem for fractional partial hyperbolic differential inclusions on unbounded domains with delay

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Abstract. In this paper we investigate the existence of solutions of initial value problems (IVP for short), for partial hyperbolic functional and neutral differential inclusions of fractional order involving Caputo fractional derivative with finite delay by using the nonlinear alternative of Frigon type for multivalued admissible contraction in Fréchet spaces.

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Keywords: Partial functional differential inclusion, fractional order, solution, left-sided mixed Riemann-Liouville integral, Caputo fractional-order derivative, finite delay, Fréchet space, fixed point.

1. Introduction

In this paper we are concerned with the existence of solutions to fractional order initial value problem (IVP for short), for the system

$$({}^cD_0^r u)(t,x) \in F(t,x,u_{(t,x)}),\ \text{if } (t,x) \in J, \tag{1.1}$$

$$u(t,x) = \phi(t,x),\ \text{if } (t,x) \in \tilde{J}, \tag{1.2}$$

$$u(t,0) = \varphi(t),\ u(0,x) = \psi(x),\ (t,x) \in J, \tag{1.3}$$

where $\varphi(0) = \psi(0)$, $J := [0,\infty) \times [0,\infty)$, $\tilde{J} := [-\alpha,+\infty) \times [-\beta,+\infty) \backslash [0,\infty) \times [0,\infty)$, ${}^cD_0^r$ is the standard Caputo's fractional derivative of order $r = (r_1, r_2) \in (0,1] \times (0,1], F : J \times C([-\alpha,0] \times [-\beta,0],\mathbb{R}^n) \to \mathcal{P}(\mathbb{R}^n)$ is a multivalued map with compact valued, $\mathcal{P}(\mathbb{R}^n)$ is the family of all subsets of $\mathbb{R}^n$, $\phi \in C := C([-\alpha,0] \times [-\beta,0],\mathbb{R}^n)$ is a given continuous function with $\phi(t,0) = \varphi(t)$, $\phi(0,x) = \psi(x)$ for each $(t,x) \in J, \varphi : [0,\infty) \to \mathbb{R}^n, \psi : [0,\infty) \to \mathbb{R}^n$ are given absolutely continuous

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functions and C is the space of continuous functions on $[-\alpha, 0] \times [-\beta, 0]$.
We denote by $u_{(t,x)}$ the element of C defined by

$$u_{(t,x)}(s, \tau) = u(t + s, x + \tau); \ (s, \tau) \in [-\alpha, 0] \times [-\beta, 0],$$

here $u_{(t,x)}(\cdot, \cdot)$ represents the history of the state u.

Next we consider the following system of partial neutral hyperbolic differential inclusion of fractional order

$$^cD_0^r[u(t,x) - g(t, x, u_{(t,x)})] \in F(t, x, u_{(t,x)}), \text{ if } (t,x) \in J, \tag{1.4}$$

$$u(t,x) = \phi(t,x), \text{ if } (t,x) \in \tilde{J}, \tag{1.5}$$

$$u(t,0) = \varphi(t), \ u(0,x) = \psi(x), \ (t,x) \in J, \tag{1.6}$$

where F, ϕ, φ, ψ are as in problem (1.1)-(1.3) and $g : J \times C([-\alpha, 0] \times [-\beta, 0], \mathbb{R}^n) \to \mathbb{R}^n$ is a given continuous function.

It is well known that differential equations and inclusions of fractional order play a very important role in describing some real world problems. For example some problems in physics, mechanics, viscoelasticity, electrochemistry, control, porous media, electromagnetic, etc. (see [14, 20, 21, 22]). The theory of differential equations and inclusions of fractional order has recently received a lot of attention and now constitutes a significant branch of nonlinear analysis. Numerous research papers and monographs have appeared devoted to fractional differential equations and inclusions, for example see the monographs of Kilbas *et al.* [16], Lakshmikantham *et al.* [18], and the papers by Belarbi *et al.* [3], Benchohra *et al.* [4, 5, 6, 7] and the references therein.

Differential delay equations and inclusions, or functional differential equations and inclusions, have been used in modeling scientific phenomena for many years. Often, it has been assumed that the delay is either a fixed constant or is given as an integral in which case it is called a distributed delay; see for instance the books by Lakshmikantham *et al.* [19], Wu [25] and the papers [8, 13, 23].

In this paper, we present existence result for the problems (1.1)-(1.3) and (1.4)-(1.6). Our aim here is to give global existence results for the above problem. The fundamental tools applied here are essentially multi-valued version of nonlinear alternative of Frigon type [10].

2. Preliminaries

In this section, we introduce notations, definitions, and preliminary facts which are used throughout this paper. Let $n \in \mathbb{N}$ and $J_0 = [0, n] \times [0, n]$. By $C(J_0, \mathbb{R})$ we denote the Banach space of all continuous functions from J_0 into $\mathbb{R}^n$ with the norm

$$\|u\|_\infty = \sup_{(t,x) \in J_0} \|u(t,x)\|,$$

where $\| \cdot \|$ denotes a suitable complete norm on $\mathbb{R}^n$.
As usual, by $AC(J_0, \mathbb{R})$ we denote the space of absolutely continuous functions from J_0 into $\mathbb{R}^n$ and $L^1(J_0, \mathbb{R})$ is the space of Lebesgue-integrable functions $u : J_0 \to \mathbb{R}^n$ with the norm

$$\|u\|_{L^1} = \int_0^n \int_0^n \|u(t,x)\| dt dx.$$

Definition 2.1. [24] Let $r = (r_1, r_2) \in (0, \infty) \times (0, \infty), \theta = (0, 0)$ and $u \in L^1(J, \mathbb{R}^n)$. The left-sided mixed Riemann-Liouville integral of order r of u is defined by

$$(I_\theta^r u)(t, x) = \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^t \int_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1} u(s, \tau) d\tau ds.$$

In particular,

$$(I_\theta^\theta u)(t, x) = u(t, x), \ (I_\theta^\sigma u)(t, x) = \int_0^t \int_0^x u(s, \tau) d\tau ds; \text{ for almost all } (t, x) \in J,$$

where $\sigma = (1, 1)$.
For instance, $I_\theta^r u$ exists for all $r_1, r_2 \in (0, \infty) \times (0, \infty)$, when $u \in L^1(J, \mathbb{R}^n)$. Note also that when $u \in C(J, \mathbb{R}^n)$, then $(I_\theta^r u) \in C(J, \mathbb{R}^n)$, moreover

$$(I_\theta^r u)(t, 0) = (I_\theta^r u)(0, x) = 0; \ (t, x) \in J.$$

Example 2.2. Let $\lambda, \omega \in (-1, \infty)$ and $r = (r_1, r_2) \in (0, \infty) \times (0, \infty)$, then

$$I_\theta^r t^\lambda x^\omega = \frac{\Gamma(1+\lambda)\Gamma(1+\omega)}{\Gamma(1+\lambda+r_1)\Gamma(1+\omega+r_2)} t^{\lambda+r_1} x^{\omega+r_2}, \text{ for almost all } (t, x) \in J.$$

By $1 - r$ we mean $(1 - r_1, 1 - r_2) \in [0, 1) \times [0, 1)$. Denote by $D_{tx}^2 := \frac{\partial^2}{\partial t \partial x}$, the mixed second order partial derivative.

Definition 2.3. [24] Let $r \in (0, 1] \times (0, 1]$ and $u \in L^1(J, \mathbb{R}^n)$. The mixed fractional Riemann-Liouville derivative of order r of u is defined by the expression

$$D_\theta^r u(t, x) = (D_{tx}^2 I_\theta^{1-r} u)(t, x)$$

and the Caputo fractional-order derivative of order r of u is defined by the expression

$$({}^c D_0^r u)(t, x) = (I_\theta^{1-r} \frac{\partial^2}{\partial t \partial x} u)(t, x).$$

The case $\sigma = (1, 1)$ is included and we have

$$(D_\theta^\sigma u)(t, x) = ({}^c D_\theta^\sigma u)(t, x) = (D_{tx}^2 u)(t, x), \text{ for almost all } (t, x) \in J.$$

Example 2.4. Let $\lambda, \omega \in (-1, \infty)$ and $r = (r_1, r_2) \in (0, 1] \times (0, 1]$, then

$$D_\theta^r t^\lambda x^\omega = \frac{\Gamma(1+\lambda)\Gamma(1+\omega)}{\Gamma(1+\lambda-r_1)\Gamma(1+\omega-r_2)} t^{\lambda-r_1} x^{\omega-r_2}, \text{ for almost all } (t, x) \in J.$$

3. Some properties of set-valued maps

Let $(X, \|\cdot\|)$ be a Banach space. Denote

- $\mathcal{P}(X) = \{Y \subset X : Y \neq \emptyset\}$,
- $\mathcal{P}_{cl}(X) = \{Y \in \mathcal{P}(X) : Y \text{ closed}\}$,
- $\mathcal{P}_b(X) = \{Y \in \mathcal{P}(X) : Y \text{ bounded}\}$,
- $\mathcal{P}_{cp}(X) = \{Y \in \mathcal{P}(X) : Y \text{ compact}\}$,
- $\mathcal{P}_{cp,c}(X) = \{Y \in \mathcal{P}(X) : Y \text{ compact and convex}\}$.

For each $u \in C(J, \mathbb{R}^n)$, define the set of selections of F by

$$S_{F\circ u} = \{f \in L^1(J, \mathbb{R}^n) : f(t,x) \in F(t,x,u(t,x)) \ a.e. \ (t,x) \in J\}.$$

Let (X, d) be a metric space induced from the normed space $(X, \|\cdot\|)$. Consider $H_d : \mathcal{P}(X) \times \mathcal{P}(X) \longrightarrow \mathbb{R}_+ \cup \{\infty\}$ given by

$$H_d(A,B) = \max\left\{\sup_{a\in A} d(a,B), \sup_{b\in B} d(A,b)\right\},$$

where $d(A,b) = \inf_{a\in A} d(a,b)$, $d(a,B) = \inf_{b\in B} d(a,b)$. Then $(\mathcal{P}_{b,cl}(X), H_d)$ is a metric space and $(\mathcal{P}_{cl}(X), H_d)$ is a generalized metric space (see [17]).

Definition 3.1. A multivalued map $F : J \times \mathbb{R}^n \to \mathcal{P}(\mathbb{R}^n)$ is said to be Carathéodory if

(i) $(t,x) \longmapsto F(t,x,u)$ is measurable for each $u \in \mathbb{R}^n$;

(ii) $u \longmapsto F(t,x,u)$ is upper semicontinuous for almost all $(t,x) \in J$.

F is said to be L^1-Carathéodory if $(i), (ii)$ and the following condition holds;

(iii) for each $c > 0$, there exists $\sigma_c \in L^1(J, \mathbb{R}_+)$ such that

$$\begin{aligned}\|F(t,x,u)\|_{\mathcal{P}} &= \sup\{\|f\| : f \in F(t,x,u)\}\\ &\leq \sigma_c(t,x) \ \text{ for all } \|u\| \leq c \text{ and for } a.e. \ (t,x) \in J.\end{aligned}$$

For more details on multivalued maps see the books of Aubin and Cellina [1], Aubin and Frankowska [2], Deimling [9], Gorniewicz [12], Hu and Papageorgiou [15] and Kisielewiecz [17].

4. Some properties in Fréchet spaces

Let X be a Fréchet space with a family of semi-norms $\{\|\cdot\|_n\}_{n\in\mathbb{N}}$. We assume that the family of semi-norms $\{\|\cdot\|_n\}$ verifies :

$$\|u\|_1 \leq \|u\|_2 \leq \ \|u\|_3 \leq ... \quad for \ every \ u \in X.$$

Let $Y \subset X$, we say that Y is bounded if for every $n \in \mathbb{N}$, there exists $\overline{M}_n > 0$ such that

$$\|v\|_n \leq \overline{M}_n \quad for \ all \ v \in Y.$$

To X we associate a sequence of Banach spaces $\{(X^n, \|\cdot\|_n)\}$ as follows : For every $n \in \mathbb{N}$, we consider the equivalence relation $\sim_n$ defined by : $u \sim_n v$ if and only if $\|u - v\|_n = 0$ for $u, v \in X$. We denote $X^n = (X|_{\sim_n}, \|\cdot\|_n)$ the quotient space, the completion of X^n with respect to $\|\cdot\|_n$. To every $Y \subset X$, we associate a sequence $\{Y^n\}$ of subsets $Y^n \subset X^n$ as follows: For every $u \in X$, we denote $[u]_n$ the equivalence class of u of subset X^n and we defined $Y^n = \{[u]_n : u \in Y\}$. We denote $\overline{Y^n}$, $int_n(Y^n)$ and $\partial_n Y^n$, respectively, the closure, the interior and the boundary of Y^n with respect to $\|\cdot\|_n$ in X^n. For more information about this subject see [11].

Definition 4.1. A multivalued map $F : X \longrightarrow \mathcal{P}(X)$ is called an admissible contraction with constant $\{k_n\}_{n\in\mathbb{N}}$ if for each $n \in \mathbb{N}$ there exists $k_n \in (0,1)$ such that

(i) $H_d(F(u), F(v)) \leq k_n||u - v||_n$ for all $u, v \in X$.

(ii) For every $u \in X$ and every $\varepsilon \in (0,\infty)^n$, there exists $v \in F(u)$ such that

$$||u-v||_n \leq ||u-F(u)||_n + \varepsilon_n \quad \text{for every } n \in \mathbb{N}.$$

Theorem 4.2. *(Nonlinear alternative of Frigon type)* [10] *Let X be a Fréchet space and U an open neighborhood of the origin in X, and let $N : \overline{U} \to \mathcal{P}(X)$ be an admissible multivalued contraction. Assume that N is bounded. Then one of the following statements is holds:*

$(C1)$ *N has at least one fixed point;*
$(C2)$ *There exist $\lambda \in [0,1)$ and $u \in \partial U$ such that $u \in \lambda N(u)$.*

5. Existence of solutions

In this section, we give our main existence result for the problems (1.1)-(1.3) and (1.4)-(1.5). For each $n \in \mathbb{N}$ we set

$$C_n = C([-\alpha,n]\times[-\beta,n],\mathbb{R}^n)$$

and we define seminorms in $C_0 := C([-\alpha,\infty)\times[-\beta,\infty),\mathbb{R}^n)$ by:

$$\|u\|_n = \{\sup \|u(t,x)\| : -\alpha \leq t \leq n, -\beta \leq x \leq n\}.$$

Then C_0 is a Fréchet space with the family $\{\|\cdot\|_n\}$. of seminorms.

5.1. The functional case

Now we are able to state and prove our main theorem for the problem (1.1)-(1.3).

Before starting and proving this result, we give what we mean by a solution of the problem (1.1)-(1.3).

Definition 5.1. A function $u \in C_0$ is said to be a solution of (1.1)-(1.3) if there exists a function $f \in L^1(J,\mathbb{R}^n)$ with $f(t,x) \in F(t,x,u_{(t,x)})$ such that $({}^cD_0^r u)(t,x) = f(t,x)$ and u satisfies equations (1.3) on J and the condition (1.2) on $\tilde{J}$.

For the existence of solutions for the problem (1.1)-(1.3), we need the following lemma:

Lemma 5.2. *A function $u \in C_0$ is a solution of problem* (1.1)-(1.3) *if and only if u satisfies the equation*

$$u(t,x) = z(t,x) + \frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^t\int_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1}f(s,\tau)d\tau ds$$

for all $(t,x) \in J$ and the condition (1.2) *on $\tilde{J}$, where*

$$z(t,x) = \varphi(t) + \psi(x) - \varphi(0).$$

Our main existence result in this section is based on the nonlinear alternative of Frigon. We will need to introduce the following hypothesis:

$(H1)$ $F : J \times C([-\alpha,0]\times[-\beta,0],\mathbb{R}^n) \to \mathcal{P}_{cp,c}(\mathbb{R}^n)$ is a L^1-Carathéodory map.
$(H2)$ For each $n \in \mathbb{N}$, there exist $p_n \in L^1(J,\mathbb{R}_+)$ and $\Psi : [0,\infty) \to (0,\infty)$ continuous and nondecreasing such that

$$\|F(t,x,u)\|_{\mathcal{P}} \leq p_n(t,x)\Psi(||u||), \text{ for a.e. } (t,x) \in J_0 \text{ and each } u \in C,$$

$(H3)$ For each $n \in \mathbb{N}$, there exists $\ell_n \in L^1(J_0, \mathbb{R}^+)$ such that

$$H_d(F(t,x,u), F(t,x,v)) \leq \ell_n(t,x)|u-v|, \text{ for all } u, v \in C,$$

and

$$d(0, (F(t,x,0)) \leq \ell_n(t,x), \text{ a.e. } (t,x) \in J_0.$$

Where $C := C([-\alpha, 0] \times [-\beta, 0], \mathbb{R}^n)$.

$(H4)$ For each $n \in \mathbb{N}$, there exists a numbre $M_n > 0$ such that

$$\frac{M_n}{||z||_n + \frac{\Psi(M_n) p_n^* n^{r_1+r_2}}{\Gamma(r_1+1)\Gamma(r_2+1)}} > 1, \tag{5.1}$$

where $p_n^* = \sup_{(t,x) \in J_0} p_n(t,x)$.

Theorem 5.3. *Assume that hypotheses* $(H1)$-$(H4)$ *hold. If*

$$\frac{\ell_n^* n^{r_1+r_2}}{\Gamma(r_1+1)\Gamma(r_2+1)} < 1, \tag{5.2}$$

where

$$\ell_n^* = \sup_{(t,x) \in J_0} \ell_n(t,x),$$

then the IVP (1.1)-(1.3) *has at least one solution on* $[-\alpha, \infty] \times [-\beta, \infty]$.

Proof. Transform the problem (1.1)-(1.3) into a fixed point problem. Consider the operator $N : C_0 \to \mathcal{P}(C_0)$ defined by,

$$(Nu)(t,x) = h \in C_0$$

such that

$$h(t,x) = \begin{cases} \phi(t,x), & (t,x) \in \tilde{J}, \\ z(t,x) + \\ \dfrac{1}{\Gamma(r_1)\Gamma(r_2)} \displaystyle\int_0^t \int_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1} f(s,\tau) d\tau ds, & (t,x) \in J, \end{cases}$$

where $f \in S_{F,u}$.

Remark 5.4. For each $u \in C_0$, the set $S_{F,u}$ is nonempty since by $(H1)$, F has a mesurable selection.

Let u be a possible solution of the inclusion $u \in \lambda N(u)$ for some $0 < \lambda < 1$. Thus for each $(t,x) \in J_0$,

$$\begin{aligned} ||u(t,x)|| &= \lambda ||z(t,x)|| + \frac{\lambda}{\Gamma(r_1)\Gamma(r_2)} \int_0^t \int_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1} \\ &\quad ||f(s,\tau)|| d\tau ds \\ &\leq ||z(t,x)|| + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^t \int_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1} \\ &\quad p_n(s,\tau)\Psi(||u_{(s,\tau)}||) d\tau ds \\ &\leq ||z||_n + \frac{\Psi(||u||_n) p_n^* n^{r_1+r_2}}{\Gamma(r_1+1)\Gamma(r_2+1)}. \end{aligned}$$

This implies by $(H4)$ that, for each $(t,x)\in J_0$, we have

$$\frac{||u||_n}{||z||_n+\frac{\Psi(||u||_n)p_n^* n^{r_1+r_2}}{\Gamma(r_1+1)\Gamma(r_2+1)}}\le 1.$$

Then by condition (5.1) we have a contradiction, so there exists M_n such that $||u||_n\neq M_n$. Since for every $(t,x)\in J_0$, we have

$$\|u\|_n\le \max(\|\phi\|_C, M_n^*):=R_n.$$

Set

$$U=\{u\in C_0:\|u\|_n\le R_n+1\ for\ all\ n\in\mathbb{N}\}.$$

We shall show that $N:U\to\mathcal{P}(U)$ is a contraction and an admissible operator. First, we prove that N is a contraction; that is, there exists $\gamma<1$, such that

$$H_d(N(u)-N(u^*))\le\gamma||u-u^*||_n,\qquad for\ u,u^*\in U.$$

Let $u,u^*\in U$ and $h\in N(u)$. Then there exists $f(t,x)\in F(t,x,u_{(t,x)})$ such that for each $(t,x)\in J_0$,

$$h(t,x)=z(t,x)+\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^t\int_0^x(t-s)^{r_1-1}(x-\tau)^{r_2-1}f(s,\tau)d\tau ds.$$

From $(H3)$ it follows that

$$H_d(F(t,x,u_{(t,x)})-F(t,x,u^*_{(t,x)}))\le\ell_n(t,x)||u_{(s,\tau)}-u^*_{(s,\tau)}||.$$

Hence there is exists $f^*\in F(t,x,u^*_{(t,x)})$ such that

$$|f(t,x)-f^*(t,x)|\le\ell_n(t,x)||u_{(t,x)}-u^*_{(t,x)}||,\quad\forall(t,x)\in J_0.$$

Let us define for each $(t,x)\in J_0$,

$$h^*(t,x)=z(t,x)+\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^t\int_0^x(t-s)^{r_1-1}(x-\tau)^{r_2-1}f^*(s,\tau)d\tau ds.$$

Then we have

$$\begin{aligned}|h(t,x)\quad &-\quad h^*(t,x)|\le\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^t\int_0^x(t-s)^{r_1-1}(x-\tau)^{r_2-1}\\ &\times\quad |f(s,\tau)-f^*(s,\tau)|d\tau ds\\ &\le\quad \frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^t\int_0^x(t-s)^{r_1-1}(x-\tau)^{r_2-1}\ell_n(s,\tau)||u-u^*||\\ &\le\quad \frac{\ell_n^*||u-u^*||_n}{\Gamma(r_1)\Gamma(r_2)}\int_0^a\int_0^b(t-s)^{r_1-1}(x-\tau)^{r_2-1}d\tau ds,\end{aligned}$$

where $\ell_n^*=\sup_{(s,\tau)\in J_0}\ell_n(s,\tau)$. Therefore

$$\|h-h^*\|_n\le\frac{\ell_n^* n^{r_1+r_2}}{\Gamma(r_1+1)\Gamma(r_2+1)}\|u-u^*\|_n.$$

By an analogous relation, obtained by interchanging the roles of u and u^*, it follows that

$$H_d(N(u) - N(u^*)) \leq \frac{\ell_n^* n^{r_1+r_2}}{\Gamma(r_1+1)\Gamma(r_2+1)} ||u - u^*||_n.$$

Hence by (5.2), N is a contraction.
Now, $N : C_n \to \mathcal{P}_{cp}(C_n)$ is given by,

$$(Nu)(t,x) = h \in C_n$$

such that

$$h(t,x) = \begin{cases} \phi(t,x), & (t,x) \in \tilde{J}, \\ z(t,x)+ & \\ \dfrac{1}{\Gamma(r_1)\Gamma(r_2)} \displaystyle\int_0^t \int_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1} f(s,\tau) d\tau ds, & (t,x) \in J_0, \end{cases}$$

where $f \in S_{F,u}^n = \{f \in L^1(J_0, \mathbb{R}^n) : f(t,x) \in F(t,x,u_{(t,x)})\ a.e.\ (t,x) \in J_0\}$. From $(H2)$-$(H3)$ and since F is compact valued, we can prove that for every $u \in C_n, N(u) \in \mathcal{P}_{cp}(C_n)$, and there exists $u^* \in C_n$ such that $u^* \in N(u^*)$. (For the proof see *Benchohra et al.* [4]). Let $h \in C_n, u \in U$ and $\varepsilon > 0$. Now, if $\tilde{u} \in N(u^*)$, then we have

$$||u^* - \tilde{u}||_n \leq ||u^* - h||_n + ||\tilde{u} - h||_n.$$

Since h is arbitrary we may suppose that $h \in B(\tilde{u}, \varepsilon) = \{k \in C_n : ||k - \tilde{u}||_n \leq \varepsilon\}$. Therefore,

$$||u^* - \tilde{u}||_n \leq ||u^* - N(u^*)||_n + \varepsilon.$$

On the other hand, if $\tilde{u} \notin N(u^*)$, then $||\tilde{u} - N(u^*)||_n \neq 0$. Since $N(u^*)$ is compact, there exists $v \in N(u^*)$ such that $||\tilde{u} - N(u^*)||_n = ||\tilde{u} - v||_n$. Then we have

$$||u^* - v||_n \leq ||u^* - h||_n + ||v - h||_n.$$

Therefore,

$$||u^* - v||_n \leq ||u^* - N(u^*)||_n + \varepsilon.$$

So, N is an admissible operator contraction. By our choice of U, there is no $u \in \partial U$ such that $u \in \lambda N(u)$, for $\lambda \in (0,1)$. As a consequence of the nonlinear alternative of Frigon type, we deduce that N has a fixed point which is a solution to problem (1.1)-(1.3).

5.2. The neutral type case

Now, we present the existence of solutions to fractional order IVP (1.4)-(1.6).

Definition 5.5. A function $u \in C_0$ is said to be a solution of (1.4)-(1.6) if there exists a function $f \in L^1(J, \mathbb{R}^n)$ with $f(t,x) \in F(t,x,u_{(t,x)})$ such that

$${}^cD_0^r[u(t,x) - g(t,x,u_{(t,x)})] = f(t,x)$$

and u satisfies equations (1.6) on J and the condition (1.5) on $\tilde{J}$.

For the existence of solutions for the problem (1.4)-(1.6), we need the following lemma:

Lemma 5.6. *A function* $u \in C_0$ *is a solution of problem* (1.4)-(1.6) *if and only if* u *satisfies the equation*

$$\begin{aligned} u(t,x) &= z(t,x) + g(t,x,u_{(t,x)}) - g(t,0,u_{(t,0)}) \\ & -g(0,x,u_{(0,x)}) + g(0,0,u_{(0,0)}) \\ &+ \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^t \int_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1} f(s,\tau)d\tau ds \end{aligned}$$

for all $(t,x) \in J$ *and the condition* (1.5) *on* $\tilde{J}$, *where*

$$z(t,x) = \varphi(t) + \psi(x) - \varphi(0).$$

Theorem 5.7. *Assume* $(H1)$-$(H3)$ *and the following hypothesis holds.*

$(H5)$ *For each* $n \in \mathbb{N}$, *there exists* $d_n \in C(J_0, \mathbb{R}^n)$ *such that for each* $(t,x) \in J_0$ *we have*

$$\|g(t,x,u) - g(t,x,v)\| \leq d_n \|u - v\|, \ for\ each\ u \in C([-\alpha,0] \times [-\beta,0], \mathbb{R}^n).$$

$(H6)$ *For each* $n \in \mathbb{N}$, *there exists an numbre* $M_n > 0$ *such that*

$$\frac{M_n}{||z||_n + 4g^* + 4d_n M_n + \frac{\Psi(M_n)p_n^* n^{r_1+r_2}}{\Gamma(r_1+1)\Gamma(r_2+1)}} > 1, \tag{5.3}$$

where $p_n^* = \sup\limits_{(t,x)\in J} p_n(t,x)$ *and* $g^* = \sup\limits_{(s,\tau)\in J_0} ||g(t,x,0)||$.

If

$$4d_n + \frac{\ell_n^* n^{r_1+r_2}}{\Gamma(r_1+1)\Gamma(r_2+1)}, \tag{5.4}$$

where

$$\ell_n^* = \sup_{(t,x)\in J_0} \ell_n(t,x),$$

then there exists at least one solution for IVP (1.4)-(1.6) *on* $[-\alpha,\infty) \times [-\beta,\infty)$.

Proof. Transform the problem (1.4)-(1.6) into a fixed point problem. Consider the operator $N_1 : C_0 \to C_0$ defined by

$$(N_1 u)(t,x) = h_1 \in C_0$$

such that

$$h_1(t,x) = \begin{cases} \phi(t,x), & (t,x) \in \tilde{J}, \\ z(t,x) + g(t,x,u_{(t,x)}) - & \\ g(t,0,u_{(t,0)}) - g(0,x,u_{(0,x)}) + g(0,0,u) + & \\ \dfrac{1}{\Gamma(r_1)\Gamma(r_2)} \displaystyle\int_0^t \int_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1} f(s,\tau)d\tau ds, & (t,x) \in J, \end{cases}$$

where $f \in S_{F,u}$.

Remark 5.8. For each $u \in C_0$, the set $S_{F,u}$ is nonempty since by $(H1)$, F has a mesurable selection.

Let u be a possible solution of the inclusion $u \in \lambda N_1(u)$ for some $0 < \lambda < 1$. Thus for each $(t,x) \in J_0$,

$$\begin{aligned} u(t,x) &= \lambda[z(t,x) + g(t,x,u_{(t,x)}) + g(t,0,u_{(t,0)}) + g(0,x,u_{(0,x)}) + g(0,0,u)] \\ &+ \frac{\lambda}{\Gamma(r_1)\Gamma(r_2)} \int_0^t \int_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1}||f(s,\tau)||d\tau ds, \end{aligned}$$

then, we have

$$\begin{aligned} ||u(t,x)|| &\leq ||z(t,x)|| + ||g(t,x,u_{(t,x)})|| + ||g(t,0,u_{(t,0)})|| \\ & +||g(0,x,u_{(0,x)})|| + ||g(0,0,u)|| \\ &+ \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^t \int_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1}||f(s,\tau)||d\tau ds. \end{aligned}$$

This implies by $(H2)$ and $(H5)$ that, for each $(t,x) \in J_0$, we have

$$\begin{aligned} ||u(t,x)|| &\leq ||z(t,x)|| + 4d_n||u|| + ||g(t,x,0)|| \\ & +||g(t,0,0)|| + ||g(0,x,0)|| + ||g(0,0,0)|| \\ &+ \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^t \int_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1} \\ & p_n(s,\tau)\Psi(||u_{(s,\tau)}||)d\tau ds \\ &\leq ||z||_n + 4g^* + 4d_n||u||_n + \frac{\Psi(||u||_n)p_n^* n^{r_1+r_2}}{\Gamma(r_1+1)\Gamma(r_2+1)}. \end{aligned}$$

This implies by $(H6)$ that, for each $(t,x) \in J_0$, we have

$$\frac{||u||_n}{||z||_n + 4g^* + 4d_n||u||_n + \frac{\Psi(||u||_n)p_n^* n^{r_1+r_2}}{\Gamma(r_1+1)\Gamma(r_2+1)}} \leq 1.$$

Then by condition (5.3) we have a contradiction, so there exists M_n such that $||u||_n \neq M_n$. Since for every $(t,x) \in J_0$, we have

$$\|u\|_n \leq \max(\|\phi\|_C, M_n^*) := R_n'.$$

Set

$$U = \{u \in C_0 : \|u\|_n \leq R_n' + 1 \ for\ all\ n \in \mathbb{N}\}.$$

We shall show that $N_1 : U \to \mathcal{P}(U)$ is a contraction and an admissible operator. First, we prove that N_1 is a contraction; that is, there exists $\gamma < 1$, such that

$$H_d(N_1(u) - N_1(u^*)) \leq \gamma||u - u^*||_n, \quad for\ u, u^* \in U.$$

Let $u, u^* \in U$ and $h \in N_1(u)$. Then there exists $f(t,x) \in F(t,x,u_{(s,\tau)})$ such that for each $(t,x) \in J_0$,

$$\begin{aligned} h_1(t,x) &= z(t,x) + g(t,x,u_{(t,x)}) - g(t,0,u_{(t,0)}) \\ & -g(0,x,u_{(0,x)}) + g(0,0,u) \\ &+ \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^t \int_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1} f(s,\tau)d\tau ds. \end{aligned}$$

From $(H3)$ it follows that

$$H_d(F(t,x,u_{(t,x)}) - F(t,x,u^*_{(t,x)})) \leq \ell_n(t,x)||u_{(t,x)} - u^*_{(t,x)}||.$$

Hence there is exists $f^* \in F(t,x,u^*_{(t,x)})$ such that

$$|f(t,x) - f^*(t,x)| \leq \ell_n(t,x)||u_{(t,x)} - u^*_{(t,x)}||, \quad \forall (t,x) \in J_0.$$

Let us define $\forall (t,x) \in J_0$,

$$\begin{aligned} h_1^*(t,x) &= z(t,x) + g(t,x,u^*_{(t,x)}) - g(t,0,u^*_{(t,0)}) - g(0,x,u^*_{(0,x)}) + g(0,0,u^*) \\ &+ \frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^t\int_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1} f(s,\tau)d\tau ds. \end{aligned}$$

Then we have

$$||h_1(t,x) - h_1^*(t,x)|| \leq$$

$$\begin{aligned} &\leq \|g(t,x,u_{(t,x)}) - g(t,x,u^*_{(t,x)})\| + \|g(t,0,u_{(t,0)}) - g(t,0,u^*_{(t,0)})\| \\ &\quad +\|g(0,x,u_{(0,x)}) - g(0,x,u^*_{(0,x)})\| + \|g(0,0,u) - g(0,0,u^*)\| \\ &+ \frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^t\int_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1}||f(s,\tau) - f^*(s,\tau)||d\tau ds \\ &\leq d_n(\|u_{(t,x)} - u^*_{(t,x)}\|_n + \|u_{(t,0)} - u^*_{(t,0)}\|_n \\ &\quad +\|u_{(0,x)} - u^*_{(0,x)}\|_n + \|u - u^*\|_n) \\ &+ \frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^t\int_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1}\ell_n(s,\tau)||u - u^*||d\tau ds \\ &\leq 4d_n\|u - u^*\|_n \\ &+ \frac{\ell_n^*||u-u^*||_n}{\Gamma(r_1+1)\Gamma(r_2+1)}\int_0^t\int_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1}d\tau ds \\ &\leq \left(4d_n + \frac{\ell_n^* n^{r_1+r_2}}{\Gamma(r_1+1)\Gamma(r_2+1)}\right)\|u - u^*\|_n, \end{aligned}$$

where $\ell_n^* = \sup_{(s,\tau)\in J_0} \ell_n(s,\tau)$. Therefore

$$\|h_1 - h_1^*\|_n \leq \left(4d_n + \frac{\ell_n^* n^{r_1+r_2}}{\Gamma(r_1+1)\Gamma(r_2+1)}\right)\|u - u^*\|_n.$$

By an analogous relation, obtained by interchanging the roles of u and u^*, it follows that

$$H_d(N_1(u) - N_1(u^*)) \leq \left(4d_n + \frac{\ell_n^* n^{r_1+r_2}}{\Gamma(r_1+1)\Gamma(r_2+1)}\right)\|u - u^*\|_n.$$

Hence by (5.4), N_1 is a contraction.

Now, $N_1 : C_n \to \mathcal{P}_{cp}(C_n)$ is given by

$$(N_1 u)(t,x) = h_1 \in C_n$$

such that

$$h_1(t,x) = \begin{cases} \phi(t,x), & (t,x) \in \tilde{J}, \\ z(t,x) + g(t,x,u_{(t,x)}) \\ -g(t,0,u_{(t,0)}) - g(0,x,u_{(0,x)}) + g(0,0,u) + \\ \dfrac{1}{\Gamma(r_1)\Gamma(r_2)} \displaystyle\int_0^t \int_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1} f(s,\tau) d\tau ds, & (t,x) \in J, \end{cases}$$

where $f \in S^n_{F,u} = \{f \in L^1(J_0, \mathbb{R}^n) : f(t,x) \in F(t,x,u_{(t,x)})\ a.e.\ (t,x) \in J_0\}$. From $(H2)$-$(H3)$ and since F is compact valued, we can prove that for every $u \in C_n, N_1(u) \in \mathcal{P}_{cp}(C_n)$, and there exists $u^* \in C_n$ such that $u^* \in N_1(u^*)$. Let $h_1 \in C_n, u \in U$ and $\varepsilon > 0$. Now, if $\tilde{u} \in N_1(u^*)$, then we have

$$||u^* - \tilde{u}||_n \leq ||u^* - h_1||_n + ||\tilde{u} - h_1||_n.$$

Since h_1 is arbitrary we may supose that $h_1 \in B(\tilde{u}, \varepsilon) = \{k \in C_n : ||k - \tilde{u}||_n \leq \varepsilon\}$. Therefore,

$$||u^* - \tilde{u}||_n \leq ||u^* - N_1(u^*)||_n + \varepsilon.$$

On the other hand, if $\tilde{u} \notin N_1(u^*)$, then $||\tilde{u} - N_1(u^*)||_n \neq 0$. Since $N_1(u^*)$ is compact, there exists $v \in N_1(u^*)$ such that $||\tilde{u} - N_1(u^*)||_n = ||\tilde{u} - v||_n$. Then we have

$$||u^* - v||_n \leq ||u^* - h_1||_n + ||v - h_1||_n.$$

Therefore,

$$||u^* - v||_n \leq ||u^* - N_1(u^*)||_n + \varepsilon.$$

So, N_1 is an admissible operator contraction. By our choice of U, there is no $u \in \partial U$ such that $u \in \lambda N_1(u)$, for $\lambda \in (0,1)$. As a consequence of the nonlinear alternative of Frigon type, we deduce that N_1 has a fixed point which is a solution to problem (1.4)-(1.6).

6. Examples

As an application of our results we consider the following hyperbolic functional differential inclusions of the form

$$({}^cD^r_0 u)(t,x) \in F(t-1, x-2, u), \quad if\ (t,x) \in J := [0,\infty) \times [0,\infty), \tag{6.1}$$

$$u(t,0) = t,\ u(0,x) = x^2, \quad (t,x) \in J, \tag{6.2}$$

$$u(t,x) = t + x^2, \quad (t,x) \in \tilde{J} := [-1,\infty) \times [-2,\infty) \backslash [0,\infty) \times [0,\infty), \tag{6.3}$$

where $F : J \times C([-1,0] \times [-2,0], \mathbb{R}^n) \to \mathcal{P}(\mathbb{R}^n)$ is a multivalued map with compact valued, $\mathcal{P}(\mathbb{R}^n)$ is the family of all subsets of $\mathbb{R}^n$.

Thus under appropriate conditions on the function F as those mentioned in the hypotheses $(H1)$-$(H4)$ implies that problem (6.1)-(6.3) has at least one solution defined on $[-1,\infty) \times [-2,\infty)$.

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Blow-up results for damped wave equation with fractional Laplacian and non linear memory

Tayeb Hadj Kaddour and Ali Hakem

Abstract. The goal of this paper is to study the nonexistence of nontrivial solutions of the following Cauchy problem

$$\begin{cases} u_{tt} + (-\Delta)^{\beta/2} u + u_t = \int_0^t (t-\tau)^{-\gamma} |u(\tau,\cdot)|^p \, d\tau, \\ u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x), \quad x \in \mathbb{R}^n, \end{cases}$$

where $p > 1$, $0 < \gamma < 1$, $\beta \in (0,2)$ and $(-\Delta)^{\beta/2}$ is the fractional Laplacian operator of order $\frac{\beta}{2}$. Our approach is based on the test function method.

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1. Introduction

The main goal of this paper is to discuss the critical exponent to the following Cauchy problem

$$\begin{cases} u_{tt} + (-\Delta)^{\beta/2} u + u_t = \int_0^t (t-\tau)^{-\gamma} |u(\tau,\cdot)|^p \, d\tau, \\ u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x), \quad x \in \mathbb{R}^n, \end{cases} \tag{1.1}$$

where $(-\Delta)^s$, $s \in (0,1)$, is the fractional Laplacian operator defined by

$$(-\Delta)^s f(x) = C_{n,s} \; P.V \int_{\mathbb{R}^n} \frac{f(x)-f(y)}{|x-y|^{n+2s}} \, dy, \quad x \in \mathbb{R}^n, \tag{1.2}$$

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as long as the right-hand side exists, where $P.V$ stands for the Cauchy's principal value and

$$C_{n,s} = \frac{4^s \Gamma\left(\frac{n}{2}+s\right)}{\pi^{\frac{n}{2}} \Gamma(-s)}$$

is the normalization constant and Γ denotes the Gamma function. Indeed, the fractional Laplacian $(-\Delta)^s$, $s \in (0,1)$ is a pseudo-differential operator of symbol $p(x,\xi) = |\xi|^{2s}$, $\xi \in \mathbb{R}^n$, defined by

$$(-\Delta)^s v = \mathcal{F}^{-1}\big(|\xi|^{2s} \mathcal{F} v(\xi)\big), \quad \text{for all } v \in \mathcal{S}'(\mathbb{R}^n), \tag{1.3}$$

where $\mathcal{F}$ and $\mathcal{F}^{-1}$ are, respectively, the Fourier transform and its inverse. In fact $(-\Delta)^s$ is a particular case of Levy operator $\mathcal{L}$ defined by

$$\mathcal{L} v(x) = \mathcal{F}^{-1}\big(a(\xi) \mathcal{F} v(\xi)\big)(x), \quad \text{for all } v \in \mathcal{S}'(\mathbb{R}^n),\ x \in \mathbb{R}^n. \tag{1.4}$$

For more details about these notions, we refer to ([1], [8], [13], [9], [3], [14]) and the references therein.

Before we present our results, let us mention below some motivations for studying the problem of the type (1.1). In [2], Cazenave and al. considered the corresponding equation

$$\begin{cases} u_t - \Delta u = \displaystyle\int_0^t (t-\tau)^{-\gamma} |u(\tau,\cdot)|^{p-1} u(\tau,\cdot) d\tau, \\ 0 \le \gamma < 1, \quad u_0 \in C_0(\mathbb{R}^n). \end{cases} \tag{1.5}$$

It was shown that, if

$$p_\gamma = 1 + \frac{2(2-\gamma)}{(n-2+2\gamma)_+} \quad \text{and } p^* = \max\left(p_\gamma, \gamma^{-1}\right),$$

where

$$(n-2+2\gamma)_+ = \max(n-2+2\gamma,\, 0).$$

Then

1. If $\gamma \neq 0$, $p \le p^*$ and $u_0 > 0$, then the solution u of (1.5) blows up in finite time.
2. If $\gamma \neq 0, p > p^*$ and $u_0 \in L_{q^*}(\mathbb{R}^n)$ (where $q^* = \frac{(p-1)n}{4-2\gamma}$) with $\|u_0\|_{L_{q^*}}$ small enough, then u exists globally. In particular, They proved that the critical exponent in Fujita's sense p^* is not the one predicted by scaling. This is not a surprising result since it is well known that scaling is efficient only for parabolic equations and not for pseudo-parabolic ones. To show this, it is sufficient to note that, formally, equation (1.5) is equivalent to

$$D^\alpha_{0|t} u_t - D^\alpha_{0|t} \Delta u = \Gamma(\alpha)\, |u|^{p-1}\, u,$$

where $\alpha = 1-\gamma$ and $D^\alpha_{0|t}$ is the fractional derivative operator of order α ($\alpha \in (0,1)$) in Riemann-Liouville sense defined by

$$D^\alpha_{0|t} u = \frac{d}{dt} J^{1-\alpha}_{0|t} u, \tag{1.6}$$

and $J^{1-\alpha}_{0|t}$ is the fractional integral of order $1-\alpha$ defined by the formula (2.2) bellow.

In the special case $\gamma = 0$, Souplet [15] proved that the nonzero positive solution of (1.5) blows -up in finite time. Note that the classical damped wave equation with nonlinear memory, namely

$$u_{tt} - \Delta u + u_t = \int_0^t (t-\tau)^{-\gamma} |u(\tau,\cdot)|^p d\tau, \tag{1.7}$$

was investigated by Fino [4]. He studied the global existence and blow-up of solutions. He used as the main tool the weighted energy method with a weight similar to the one introduced by G. Todorova an B. Yardanov [16], while he employed the test function method to derive nonexistence results. In particular, he found the same p_γ and so the same critical exponent p^* founded by Cazenave and al in [2]. More recently, the Authors of [6] generalized the results of [2] and [4] by establishing nonexistence results for the following Cauchy problem:

$$\begin{cases} u_{tt} - \Delta u + D^\sigma_{0|t} u_t = \displaystyle\int_0^t (t-\tau)^{-\gamma} |u(\tau,\cdot)|^p d\tau, \quad t > 0. \\ u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x), \quad x \in \mathbb{R}^n. \end{cases} \tag{1.8}$$

Remark 1.1. Throughout, C denotes a positive constant, whose value may change from line to line.

2. Blow up solutions

This section is devoted to prove blow-up results of problem (1.1). The method which we will use for our task is the test function method considered by Mitidieri and Pohozaev ([10], [11]), Pohozaev and Tesei [12], Fino [4], Hadj-Kaddour and Hakem ([5], [6]); it was also used by Zhang [17].

Before that, one can show that the problem (1.1) can be written in the following form:

$$\begin{cases} u_{tt} + (-\Delta)^{\beta/2} u + u_t = \Gamma(\alpha) J^\alpha_{0|t}(|u|^p), \\ u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x), \quad \text{for all} \quad x \in \mathbb{R}^n, \end{cases} \tag{2.1}$$

where $\alpha = 1 - \gamma$ and $J^\alpha_{0|t}$ is the fractional integral of order α ($\alpha \in (0,1)$) defined for all $v \in L^1_{loc}(\mathbb{R})$, by

$$J^\alpha_{0|t} v(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{v(s)}{(t-s)^{1-\alpha}} ds, \tag{2.2}$$

where $(-\Delta)^{\beta/2}$ is the fractional Laplacian operator of order $\beta/2$, $\beta \in (0,2)$.

First, let us introduce what we mean by a weak solution for problem (2.1).

Definition 2.1. Let $T > 0$, $\gamma \in (0,1)$ and $\beta \in (0,2)$. A weak solution for the Cauchy problem (2.1) in $[0,T) \times \mathbb{R}^n$ with initial data $(u_0, u_1) \in L^1_{loc}(\mathbb{R}^n) \times L^1_{loc}(\mathbb{R}^n)$ is a locally integrable function $u \in L^p\big((0,T), L^p_{loc}(\mathbb{R}^n)\big)$ that satisfies

$$\Gamma(\alpha)\int_0^T\int_{\mathbb{R}^n} J^\alpha_{0|t}(|u|^p)\varphi(t,x)dtdx + \int_{\mathbb{R}^n}(u_0(x)+u_1(x))\varphi(0,x)dx$$
$$-\int_{\mathbb{R}^n} u_0(x)\varphi_t(0,x)dx = \int_0^T\int_{\mathbb{R}^n} u(t,x)\varphi_{tt}(t,x)dtdx$$
$$-\int_0^T\int_{\mathbb{R}^n} u(t,x)\varphi_t(t,x)dtdx - \int_0^T\int_{\mathbb{R}^n} u(t,x)(-\Delta)^{\beta/2}\varphi(t,x)dtdx, \qquad (2.3)$$

for all non-negative test function $\varphi \in \mathcal{C}^2([0,T]\times\mathbb{R}^n)$ such that $\varphi(T,\cdot) = \varphi_t(T,\cdot) = 0$ and $\alpha = 1-\gamma$. If $T = \infty$, we call u a global in time weak solution to (2.1).

Now, we are ready to state the main results of this paper. For all $\gamma \in (0,1)$, $\beta \in (0,2)$ and $n \in \mathbb{N}$, we put

$$p_\gamma(\beta) = 1 + \frac{\beta(2-\gamma)}{(n-\beta(1-\gamma))_+} \quad \text{and} \quad p^* = \max\{p_\gamma(\beta), \gamma^{-1}\}. \qquad (2.4)$$

Theorem 2.2. *Let $0<\gamma<1$, $p \in (1,\infty)$ for $n = 1,2$ and $1 < p < \frac{n}{n-2}$ for $n \geq 3$. We assume that $(u_0,u_1) \in H^1(\mathbb{R}^n)\times L^2(\mathbb{R}^n)$ satisfying the following relation:*

$$\int_{\mathbb{R}^n} u_i(x)dx > 0, \quad i = 0,1. \qquad (2.5)$$

Moreover, we suppose the condition

$$p \leq p^*.$$

Then, the problem (2.1) admits no global weak solution.

The proof of our main result is given in the next section.

3. Proofs

In this section, we give the proof of Theorem 2.2. For this task, we choose a test function for some $T>0$, as follows:

$$\varphi(t,x) = D^\alpha_{t|T}\psi(t,x) = \varphi_1^\ell(x)D^\alpha_{t|T}\varphi_2(t), \quad (t,x) \in \mathbb{R}_+\times\mathbb{R}^n, \qquad (3.1)$$

where $\ell > 1$ and $D^\alpha_{t|T}$ is the right fractional derivative operator of order α in the sense of Riemann-Liouville defined by

$$D^\alpha_{t|T}v(t) = -\frac{1}{\Gamma(1-\alpha)}\frac{\partial}{\partial t}\int_t^T \frac{v(s)}{(s-t)^\alpha}ds, \qquad (3.2)$$

and the functions φ_1 and φ_2 are given by

$$\varphi_1(x) = \phi\Big(\frac{x^2}{K}\Big), \quad \varphi_2(t) = \Big(1-\frac{t}{T}\Big)_+^\sigma, \qquad (3.3)$$

with $K>0$, $\sigma>1$ and ϕ is a smooth non-increasing function such that

$$\phi(s) = \begin{cases} 1 & \text{if } 0 \leq s \leq 1, \\ 0 & \text{if } s \geq 2, \end{cases} \quad 0 \leq \phi \leq 1 \text{ everywhere and } |\phi'(s)| \leq \frac{C}{s}. \qquad (3.4)$$

We also denote by Ω_K for the support of φ_1, that is

$$\Omega_K = supp\varphi_1 = \big\{x \in \mathbb{R}^n,\ |x|^2 \leq 2K\big\}, \tag{3.5}$$

and by Δ_K for the set containing the support of $\Delta\varphi_1$ which is defined as follows:

$$\Delta_K = \big\{x \in \mathbb{R}^n,\ K \leq |x|^2 \leq 2K\big\}. \tag{3.6}$$

Furthermore, for every $f, g \in \mathcal{C}([0,T])$ such that $D^{\alpha}_{0|t}f(t)$ and $D^{\alpha}_{t|T}g(t)$ exist and are continuous, for all $t \in [0,T],\ 0 < \alpha < 1$ we have the formula of integration by parts([14])

$$\int_0^t f(t)D^{\alpha}_{t|T}g(t)dt = \int_0^t \big(D^{\alpha}_{0|t}f(t)\big)g(t)dt, \tag{3.7}$$

Note also that, for all $u \in \mathcal{C}^n[0,T]$ and all integers $n \geq 0$, we have

$$(-1)^n\partial_t^n D^{\alpha}_{t|T}u(t) = D^{\alpha+n}_{t|T}u(t), \tag{3.8}$$

where ∂_t^n is the $n-$times ordinary derivative with respect to t. Moreover, for all $1 \leq q \leq \infty$, the following formula

$$\big(D^{\alpha}_{0|t} \circ I^{\alpha}_{0|t}\big)(u) = u \text{ for all } u \in L^q\left([0,T]\right), \tag{3.9}$$

holds almost everywhere on [0,T].

The following Lemmas are crucial in the proof of Theorem 2.2.

Lemma 3.1. *Let $\sigma > 1$ and φ_2 be the function defined by*

$$\varphi_2(t) = \Big(1 - \frac{t}{T}\Big)^{\beta}_{+}.$$

Then, for all $\alpha \in (0,1)$ we have

$$D^{\alpha}_{t|T}\varphi_2(t) = C_1T^{-\beta}(T-t)^{\beta-\alpha}_{+} = CT^{-\alpha}\Big(1 - \frac{t}{T}\Big)^{\beta-\alpha}_{+},$$

$$D^{\alpha+1}_{t|T}\varphi_2(t) = C_2T^{-\beta}(T-t)^{\beta-\alpha-1}_{+} = CT^{-\alpha-1}\Big(1 - \frac{t}{T}\Big)^{\beta-\alpha-1}_{+},$$

and

$$D^{\alpha+2}_{t|T}\varphi_2(t) = C_3T^{-\beta}(T-t)^{\beta-\alpha-2}_{+} = CT^{-\alpha-2}\Big(1 - \frac{t}{T}\Big)^{\beta-\alpha-2}_{+}.$$

In particular, for all $\alpha \in (0,1)$, one has

$$D^{\alpha+j}_{t|T}\varphi_2(0) = C_jT^{-\alpha-2}, \quad \textit{for all}\ \ j = 0,1,2, \tag{3.10}$$

and

$$C_j = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1-j)}, \quad j = 0,1,2. \tag{3.11}$$

Proof. The proof of Lemma 3.1 is straight-forward. For all $\alpha \in (0,1)$, we have by definition (3.2)

$$D^{\alpha}_{t|T}\varphi_2(t) = -\frac{1}{\Gamma(1-\alpha)}\frac{\partial}{\partial t}\int_t^T \frac{\varphi_2(s)}{(s-t)^{\alpha}}ds.$$

By using the Euler's change of variable

$$s \mapsto y = \frac{s-t}{T-t}, \tag{3.12}$$

we get,

$$\begin{aligned}
D^{\alpha}_{t|T}\varphi_2(t) &= \frac{1}{\Gamma(1-\alpha)}\frac{\partial}{\partial t}\int_t^T \frac{(1-\frac{s}{T})^{\beta}}{(s-t)^{\alpha}}ds \\
&= \frac{T^{-\beta}}{\Gamma(1-\alpha)}\frac{\partial}{\partial t}\left((T-t)^{\beta-\alpha+1}\int_0^1 y^{-\alpha}(1-y)^{\beta}dy\right) \\
&= \frac{(\beta-\alpha+1)\mathcal{B}(1-\alpha,\beta+1)}{\Gamma(1-\alpha)}T^{-\beta}(T-t)^{\beta-\alpha} \\
&= \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}T^{-\alpha}\left(1-\frac{t}{T}\right)^{\beta-\alpha},
\end{aligned}$$

where $\mathcal{B}$ is the *Beta function* defined by

$$\mathcal{B}(u,v) = \int_0^1 t^{u-1}(1-t)^{v-1}dt, \quad \mathcal{B}(u,v) = \frac{\Gamma(u)\Gamma(v)}{\Gamma(u+v)}. \tag{3.13}$$

For the second and the third, we apply directly formula (3.8) to show that

$$\forall t \in [0,T] : D^{\alpha+i}_{t|T}\varphi_2(t) = (-1)^i \partial_t D^{\alpha}_{t|T}\varphi_2(t), \quad \text{for all } i = 1,2.$$

Hence the result is conclude. □

Lemma 3.2 (Ju Cordoba). ([7]) *Let $0 \le \beta \le 2$, $\ell \ge 1$ and $(-\Delta)^{\beta/2}$ be the operator defined by (*1.3*). Then for all $\Psi \in D((-\Delta)^{\beta/2})$, the following inequality holds*

$$(-\Delta)^{\beta/2}\Psi^{\ell} \le \ell\Psi^{\ell-1}(-\Delta)^{\beta/2}\Psi.$$

Proof. (Theorem 2.2) The proof is by contradiction. Suppose that u is a global weak solution to (2.1). Introducing the test function defined by (3.1), using the formula of integration by parts (3.7) and the identity (3.9) we get easily

$$\begin{aligned}
\int_0^T\int_{\mathbb{R}^n} J^{\alpha}_{0|t}(|u|^p)\varphi(t,x)dtdx &= \int_0^T\int_{\mathbb{R}^n} I^{\alpha}_{0|t}(|u|^p)D^{\alpha}_{t|T}\psi(t,x)dtdx \\
&= \int_0^T\int_{\mathbb{R}^n} D^{\alpha}_{0|T}\big(J^{\alpha}_{0|T}(|u|^p)\big)\psi(t,x)dtdx \\
&= \int_0^T\int_{\mathbb{R}^n} |u|^p\psi(t,x)dtdx.
\end{aligned} \tag{3.14}$$

For the second term of the left-hand side of equality (2.3), thanks to Lemma 3.1, we have

$$\begin{aligned}
\int_{\mathbb{R}^n}\big(u_0(x)+u_1(x)\big)\varphi(0,x)dx &= \int_{\mathbb{R}^n}\big(u_0(x)+u_1(x)\big)\varphi_1^{\ell}(x)D^{\alpha}_{t|T}\varphi_2(t)_{|t=0}\,dx \\
&= CT^{-\alpha}\int_{\mathbb{R}^n}\big(u_0(x)+u_1(x)\big)\varphi_1^{\ell}(x)dx.
\end{aligned} \tag{3.15}$$

Analogously, we obtain for the third term of the left hand-side of the weak formulation (2.3)

$$\int_{\mathbb{R}^n} u_0(x)\varphi_t(0,x)dx = -CT^{-\alpha-1}\int_{\mathbb{R}^n} u_0(x)\varphi_1^{\ell}(x)dx. \tag{3.16}$$

Therefore, using formula (3.8) with $n=1$ and $n=2$, we get respectively

$$\int_0^T\int_{\mathbb{R}^n} u(t,x)\varphi_t(t,x)dtdx = -\int_0^T\int_{\mathbb{R}^n} u(t,x)\varphi_1^{\ell}(x)D_{t|T}^{\alpha+1}\varphi_2(t)dtdx, \tag{3.17}$$

and

$$\int_0^T\int_{\mathbb{R}^n} u(t,x)\varphi_{tt}(t,x)dtdx = \int_0^T\int_{\mathbb{R}^n} u(t,x)\varphi_1^{\ell}(x)D_{t|T}^{\alpha+2}\varphi_2(t)dtdx. \tag{3.18}$$

Finally for the third term of the right-hand side of the weak formulation (2.3), we obtain

$$\begin{aligned}&\int_0^T\int_{\mathbb{R}^n} u(t,x)(-\Delta)^{-\beta/2}\varphi(t,x)dtdx\\ &\leq \ell\times\int_0^T\int_{\mathbb{R}^n} u(t,x)\varphi_1^{\ell-1}(-\Delta)^{-\beta/2}\varphi_1(x)D_{t|T}^{\alpha}\varphi_2(t)dtdx,\end{aligned} \tag{3.19}$$

where we have used Lemma 3.2 with $\Psi=\varphi_1$.
Inserting all the formulas (3.14), (3.15), (3.16), (3.17), (3.18) and (3.19) in the weak formulation (2.3) we arrive at

$$\begin{aligned}&\Gamma(\alpha)\int_0^T\int_{\mathbb{R}^n}|u|^p\psi(t,x)dtdx + CT^{-\alpha}\int_{\mathbb{R}^n}\big(u_0(x)+u_1(x)\big)\varphi_1^{\ell}(x)dx\\ &+CT^{-\alpha-1}\int_{\mathbb{R}^n}u_0(x)\varphi_1^{\ell}(x)dx \leq C\Big(\int_0^T\int_{\mathbb{R}^n}|u(t,x)|\varphi_1^{\ell}(x)|D_{t|T}^{\alpha+2}\varphi_2(t)|dtdx\\ &+\int_0^T\int_{\mathbb{R}^n}|u(t,x)|\varphi_1^{\ell}(x)|D_{t|T}^{\alpha+1}\varphi_2(t)|dtdx\\ &\quad+\int_0^T\int_{\mathbb{R}^n}|u(t,x)|\varphi_1^{\ell-1}(-\Delta)^{-\beta/2}\varphi_1(x)|D_{t|T}^{\alpha}\varphi_2(t)|dtdx\Big),\end{aligned} \tag{3.20}$$

where $C>0$ independent of T. Next, using the fact that (2.5) imply

$$\int_{\mathbb{R}^n}\big(u_0(x)+u_1(x)\big)\varphi_1^{\ell}(x)dx > 0 \,\text{and}\, \int_{\mathbb{R}^n}u_0(x)\varphi_1^{\ell}(x)dx > 0, \tag{3.21}$$

we deduce easily from (3.20) the inequality

$$\int_0^T\int_{\mathbb{R}^n}|u|^p\psi(t,x)dtdx \leq C(J_1+J_2+J_3), \tag{3.22}$$

where

$$J_1=\int_0^T\int_{\mathbb{R}^n}|u(t,x)|\varphi_1^\ell(x)|D_{t|T}^{\alpha+2}\varphi_2(t)|dtdx, \tag{3.23}$$

$$J_2=\int_0^T\int_{\mathbb{R}^n}|u(t,x)|\varphi_1^\ell(x)|D_{t|T}^{\alpha+1}\varphi_2(t)|dtdx, \tag{3.24}$$

$$J_3=\int_0^T\int_{\mathbb{R}^n}|u(t,x)|\varphi_1^{\ell-1}(-\Delta)^{-\beta/2}\varphi_1(x)|D_{t|T}^{\alpha}\varphi_2(t)|dtdx. \tag{3.25}$$

Now, the main goal is to estimate the integrals $J_1,\ J_2$ and J_3. To do so, we apply the following $\varepsilon-$ *Young inequality*

$$AB\leq\varepsilon A^p+C(\varepsilon)B^q,\ pq=p+q,\quad C(\varepsilon)=(\varepsilon p)^{-q/p}q^{-1}.$$

It is quite easy to check that

$$\begin{aligned}J_1&=\int_0^T\int_{\mathbb{R}^n}|u(t,x)|\psi^{\frac{1}{p}}\psi^{-\frac{1}{p}}\varphi_1^\ell(x)|D_{t|T}^{\alpha+2}\varphi_2(t)|dtdx\\&\leq\varepsilon\int_0^T\int_{\mathbb{R}^n}|u|^p\psi dtdx+C(\varepsilon)\int_0^T\int_{\mathbb{R}^n}\varphi_1^\ell\varphi_2^{-\frac{1}{p-1}}|D_{t|T}^{\alpha+2}\varphi_2|^{\frac{p}{p-1}}dtdx.\end{aligned} \tag{3.26}$$

Similarly, for J_2 and J_3, we obtain

$$\begin{aligned}J_2&\leq\varepsilon\int_0^T\int_{\mathbb{R}^n}|u(t,x)|^p\psi(t,x)dtdx\\&\quad+C(\varepsilon)\int_0^T\int_{\mathbb{R}^n}\varphi_1^\ell(x)\varphi_2^{-\frac{1}{p-1}}|D_{t|T}^{\alpha+1}\varphi_2(t)|^{\frac{p}{p-1}}dtdx,\end{aligned} \tag{3.27}$$

$$\begin{aligned}J_3&\leq\varepsilon\int_0^T\int_{\mathbb{R}^n}|u(t,x)|^p\psi(t,x)dtdx\\&\quad+C(\varepsilon)\int_0^T\int_{\mathbb{R}^n}\varphi_1^{\ell-\frac{p}{p-1}}\big(-\Delta)^{\beta/2}\varphi_1\big)^{\frac{p}{p-1}}\varphi_2^{-\frac{1}{p-1}}|D_{t|T}^{\alpha}\varphi_2|^{\frac{p}{p-1}}dtdx.\end{aligned} \tag{3.28}$$

Plugging the estimates (3.26), (3.27), (3.28) into (3.22) we find, for ε small enough, the estimate

$$\begin{aligned}\int_0^T\int_{\mathbb{R}^n}|u|^p\psi(t,x)dtdx&\leq C\Big(\int_0^T\int_{\mathbb{R}^n}\varphi_1^\ell\varphi_2^{-\frac{1}{p-1}}|D_{t|T}^{\alpha+2}\varphi_2|^{\frac{p}{p-1}}dtdx\\&\quad+\int_0^T\int_{\mathbb{R}^n}\varphi_1^\ell(x)\varphi_2^{-\frac{1}{p-1}}|D_{t|T}^{\alpha+1}\varphi_2(t)|^{\frac{p}{p-1}}dtdx\\&\quad+\int_0^T\int_{\mathbb{R}^n}\varphi_1^{\ell-\frac{p}{p-1}}\big(-\Delta)^{\beta/2}\varphi_1\big)^{\frac{p}{p-1}}\varphi_2^{-\frac{1}{p-1}}|D_{t|T}^{\alpha}\varphi_2|^{\frac{p}{p-1}}dtdx\Big)\\&\leq C(I_1+I_2+I_3),\end{aligned} \tag{3.29}$$

where $C>0$ independent of T, and

$$I_1=\int_0^T\int_{\mathbb{R}^n}\varphi_1^\ell\varphi_2^{-\frac{1}{p-1}}|D_{t|T}^{\alpha+2}\varphi_2|^{\frac{p}{p-1}}dtdx, \tag{3.30}$$

$$I_2=\int_0^T\int_{\mathbb{R}^n}\varphi_1^\ell(x)\varphi_2^{-\frac{1}{p-1}}|D_{t|T}^{\alpha+1}\varphi_2(t)|^{\frac{p}{p-1}}dtdx, \tag{3.31}$$

$$I_3=\int_0^T\int_{\mathbb{R}^n}\varphi_1^{\ell-\frac{p}{p-1}}\big(-\Delta)^{\beta/2}\varphi_1\big)^{\frac{p}{p-1}}\varphi_2^{-\frac{1}{p-1}}|D_{t|T}^{\alpha}\varphi_2|^{\frac{p}{p-1}}dtdx. \tag{3.32}$$

The aim, now, is to estimate the integrals I_1,I_2 and I_3. We have to distinguish two cases:

Case of $p\le p_\gamma(\beta)$

At this stage, we introduce the scaled variables.

$$x=T^{\frac{1}{\beta}}y \quad \text{and} \quad t=T\tau. \tag{3.33}$$

Let $K=T^{1/\beta}$. Using Fubini's theorem, we get, for I_1

$$\begin{aligned}I_1&=\Big(\int_{\Omega_T}\varphi_1^\ell(x)dx\Big)\Big(\int_0^T\varphi_2(t)^{-\frac{1}{p-1}}|D_{t|T}^{\alpha+2}\varphi_2(t)|^{\frac{p}{p-1}}dt\Big)\\&=\Big(T^{\frac{n}{\beta}}\int_0^2\phi^\ell(y^2)dy\Big)\Big(T^{1-(\alpha+2)\frac{p}{p-1}}\int_0^1(1-\tau)^{-\frac{\beta}{p-1}+(\beta-\alpha-2)\frac{p}{p-1}}d\tau\Big)\\&=CT^{1-(\alpha+2)\frac{p}{p-1}+\frac{n}{\beta}},\end{aligned} \tag{3.34}$$

where we have used

$$\int_{\Omega_T}\varphi_1^\ell(x)dx=T^{\frac{n}{\beta}}\int_0^2\phi^\ell(y^2)dy=CT^{\frac{n}{\beta}}, \tag{3.35}$$

and

$$\int_0^1(1-\tau)^{-\frac{\beta}{p-1}+(\beta-\alpha-2)\frac{p}{p-1}}d\tau=C. \tag{3.36}$$

Similarly, for I_2 and I_3, we obtain

$$\begin{aligned}I_2&=\Big(\int_{\Omega_T}\varphi_1^\ell(x)dx\Big)\Big(\int_0^T\varphi_2(t)^{-\frac{1}{p-1}}|D_{t|T}^{\alpha+1}\varphi_2(t)|^{\frac{p}{p-1}}dt\Big)\\&=CT^{1-(\alpha+1)\frac{p}{p-1}+\frac{n}{\beta}},\end{aligned} \tag{3.37}$$

and

$$\begin{aligned}I_3&=\int_0^T\int_{\mathbb{R}^n}\varphi_1^{\ell-\frac{p}{p-1}}(x)\big(-\Delta)^{\beta/2}\varphi_1(x)\big)^{\frac{p}{p-1}}\varphi_2^{-\frac{1}{p-1}}(t)|D_{t|T}^{\alpha}\varphi_2(t)|^{\frac{p}{p-1}}dtdx\\&=\int_{\Omega_T}\varphi_1^{\ell-\frac{p}{p-1}}(x)\big(-\Delta)^{\beta/2}\varphi_1(x)\big)^{\frac{p}{p-1}}dx\int_0^T\varphi_2^{-\frac{1}{p-1}}(t)|D_{t|T}^{\alpha}\varphi_2(t)|^{\frac{p}{p-1}}dt\\&=CT^{1-(\alpha+\frac{2}{\beta})\frac{p}{p-1}+\frac{n}{\beta}}.\end{aligned} \tag{3.38}$$

Combining (3.38), (3.37) and (3.36), it holds from (3.29)

$$\int_0^T\int_{\Omega_T}|u(t,x)|^p\psi(t,x)dtdx\le CT^{-\delta}, \tag{3.39}$$

for some positive constant C independent of T and

$$\delta = 1 - (\alpha+1)\frac{p}{p-1} + \frac{n}{\beta}. \tag{3.40}$$

Now we distinguish between two other subcases as follows:
Sub-case: $p < p_\gamma(\beta)$
Noting that

$$p < p_\gamma(\beta) \Longleftrightarrow \delta > 0. \tag{3.41}$$

Then, by passing to the limit in (3.39) as T goes to ∞ and invoking the fact that

$$\lim_{T\longrightarrow\infty} \psi(t,x) = 1, \tag{3.42}$$

we get after applying the dominate convergence theorem of Lebesgue that

$$\int_0^{+\infty}\int_{\mathbb{R}^n} |u(t,x)|^p dtdx = 0. \tag{3.43}$$

This means that $u = 0$ and this is a contradiction.
The second case is:
Sub-case: $p = p_\gamma(\beta)$
First, we remark that the condition $p = p_\gamma(\beta)$ is equivalent to $\delta = 0$. Then, by taking the limit as $T \to \infty$ in (3.39) together with the consideration $\delta = 0$ we get

$$\int_0^{+\infty}\int_{\mathbb{R}^n} |u|^p\, dtdx < +\infty, \tag{3.44}$$

from which we can deduce that

$$\lim_{T\to\infty}\int_0^{+\infty}\int_{\Delta_T} |u|^p\, \psi dtdx = 0, \tag{3.45}$$

where Δ_T is defined by (3.6). Fixing arbitrarily R in $]0,T[$ for some $T > 0$ such that when $T \to \infty$ we don't have $R \to \infty$ at the same time and taking $K = R^{-\frac{1}{\beta}}T^{\frac{1}{\beta}}$. First, we apply the following Hölder's inequality

$$\int_X uv d\mu \le \Big(\int_X u^p d\mu\Big)^{\frac{1}{p}}\Big(\int_X v^q d\mu\Big)^{\frac{1}{q}}, \tag{3.46}$$

which happens for all $u \in L^p(X)$ and $v \in L^q(X)$ such that $p, q \in (1, +\infty)$ and $pq = p + q$ instead of $\varepsilon-$ *Young's* one to estimate the integral J_3 defined by (3.25) on the set

$$\Omega_{TR^{-1}} = \Big\{x \in \mathbb{R}^n : |x|^2 \le 2R^{-\frac{1}{\beta}}T^{\frac{1}{\beta}}\Big\} = supp\varphi_1. \tag{3.47}$$

Taking into account the fact that $supp\Delta\varphi_1 \subset \Delta_{TR^{-1}} \subset \Omega_{TR^{-1}}$ where $\Delta_{TR^{-1}}$ is defined by

$$\Delta_{TR^{-1}} = \Big\{x \in \mathbb{R}^n : R^{-\frac{1}{\beta}}T^{\frac{1}{\beta}} \le |x|^2 \le 2R^{-\frac{1}{\beta}}T^{\frac{1}{\beta}}\Big\}, \tag{3.48}$$

we obtain the estimate

$$\int_0^T\int_{\mathbb{R}^n}|u(t,x)|\varphi_1^{\ell-1}(-\Delta)^{-\beta/2}\varphi_1(x)|D^{\alpha}_{t|T}\varphi_2(t)|dtdx\le\Big(\int_0^T\int_{\Delta_{TR^{-1}}}|u|^p\psi dtdx\Big)^{\frac{1}{p}}$$
$$\times\Big(\int_0^T\int_{\Delta_{TR^{-1}}}\psi^{-\frac{q}{p}}\varphi_1^{(\ell-1)q}\big((-\Delta)^{\beta/2}\varphi_1\big)^q|D^{\alpha}_{t|T}\varphi_2|^q dtdx\Big)^{\frac{1}{q}},\tag{3.49}$$

while we estimate J_1 and J_2 by using $\varepsilon-$*Young* inequality as we did in the first case. Then we have to estimate the integrals $I_1,\ I_2$ and $\tilde{I}_3$ where I_1 and I_2 are given by (3.30) and (3.31) respectively and $\tilde{I}_3$ is defined by

$$\tilde{I}_3=\Big(\int_0^T\int_{\Delta_{TR^{-1}}}\psi^{-\frac{q}{p}}\varphi_1^{(\ell-1)q}\big((-\Delta)^{\beta/2}\varphi_1\big)^q|D^{\alpha}_{t|T}\varphi_2|^q dtdx\Big)^{\frac{1}{q}}.\tag{3.50}$$

For this task, we consider the scaled change of variables

$$x=R^{-\frac{1}{\beta}}T^{\frac{1}{\beta}}\quad\text{and } t=T^{\frac{1}{\beta}}\tau.\tag{3.51}$$

In this way, we find after using Fubini's theorem

$$I_1+I_2\le C\big(T^{-(\alpha+2)\frac{p}{p-1}+\frac{n}{\beta}+1}+T^{-(\alpha+1)\frac{p}{p-1}+\frac{n}{\beta}+1}\big)R^{-\frac{n}{\beta}}.\tag{3.52}$$

Moreover, taking into account the hypothesis $\delta=0$ we get from (3.52) the estimate

$$I_1+I_2\le CR^{-\frac{n}{\beta}},\tag{3.53}$$

for $C>0$ independent of R and T. In the other hand, we may estimate $\tilde{I}_3$ by using the same change of variables (3.51) as follows

$$\tilde{I}_3\le CR^{\frac{1}{\beta}-q\frac{n}{\beta}}.\tag{3.54}$$

Combining the estimates (3.54) and (3.53) together with (3.22), we obtain the inequality

$$\begin{aligned}\int_0^T\int_{\Omega_{TR^{-1}}}|u(t,x)|^p\psi(t,x)dtdx&\le CR^{-\frac{n}{\beta}}\\ +CR^{\frac{1}{\beta}-q\frac{n}{\beta}}\Big(\int_0^T\int_{\Delta_{TR^{-1}}}&|u(t,x)|^p\psi(t,x)dtdx\Big)^{\frac{1}{p}}.\end{aligned}\tag{3.55}$$

Using (3.45) and the fact that $\lim\limits_{T\to+\infty}\psi(t,x)=1$ we obtain from (3.55) as $T\to+\infty$.

$$\int_0^\infty\int_{\mathbb{R}^n}|u|^p dtdx\le CR^{-\frac{n}{\beta}},$$

which means that necessarily $R\to+\infty$ and this is a contradiction.
Now we deal with the second main result in Theorem 2.2.

Case of $p\le\frac{1}{\gamma}$

Even this case is divided into two subcases as follows:

2. i. Subcase of $p<\frac{1}{\gamma}$

In this case we take $K=R^{\frac{1}{\beta}}$, where R is a fixed positive number. Now let us turn to estimate the integrals $J_1,\ J_2$ and J_3 by using $\varepsilon-$*Young* inequality as we did in the first case, so we obtain the estimate (3.29). The aim, now, is to estimate the integrals I_1, I_2 and I_3 defined respectively by (3.30), (3.31) and (3.32), on the set

$$\Omega_R := \left\{x \in \mathbb{R}^n : |x| \leq 2R^{\frac{1}{\beta}}\right\} = supp\varphi_1, \tag{3.56}$$

since they are null outside Ω_R. For this reason, we consider the following scaled variables

$$x = R^{\frac{n}{\beta}} y \quad \text{and} \quad t = T\tau. \tag{3.57}$$

So, for I_1 we have

$$\begin{aligned} I_1 &= \Big(\int_{\Omega_R} \varphi_1^\ell(x)dx\Big)\Big(\int_0^T \varphi_2(t)^{-\frac{1}{p-1}} |D_{t|T}^{\alpha+2}\varphi_2(t)|^{\frac{p}{p-1}} dt\Big) \\ &= \Big(R^{\frac{n}{\beta}} \int_0^2 \phi^\ell(y^2)dy\Big)\Big(T^{1-(\alpha+2)\frac{p}{p-1}} \int_0^1 (1-\tau)^{-\frac{\beta}{p-1}+(\beta-\alpha-2)\frac{p}{p-1}} d\tau\Big) \\ &= CR^{\frac{n}{\beta}} T^{1-(\alpha+2)\frac{p}{p-1}}, \end{aligned} \tag{3.58}$$

for some constant $C > 0$ independent of R and T. In the same way, we obtain

$$I_2 = CR^{\frac{n}{\beta}} T^{1-(\alpha+1)\frac{p}{p-1}}, \tag{3.59}$$

where $C > 0$ is of R and T. Finally

$$I_3 = CR^{(\frac{n}{2}-\frac{p}{p-1})\frac{1}{\beta}} T^{1-\alpha\frac{p}{p-1}}. \tag{3.60}$$

Including the estimates (3.60), (3.59) and (3.58) into (3.29) we arrive at

$$\begin{aligned} \int_0^T \int_{\Omega_R} |u(t,x)|^p \psi(t,x)dtdx &= CR^{\frac{n}{\beta}} \left(T^{1-(\alpha+2)\frac{p}{p-1}} + T^{1-(\alpha+1)\frac{p}{p-1}}\right) \\ &\quad + CR^{\left(\frac{n}{2}-\frac{p}{p-1}\right)\frac{1}{\beta}} T^{1-\alpha\frac{p}{p-1}}. \end{aligned} \tag{3.61}$$

First, we note that $p < \frac{1}{\gamma}$ implies that

$$1 - \alpha\frac{p}{p-1} < 0.$$

Therefore, the fact that

$$\alpha\frac{p}{p-1} < (\alpha+1)\frac{p}{p-1} < (\alpha+2)\frac{p}{p-1}$$

together with

$$\lim_{T\to+\infty} \psi(t,x) = \varphi_1^\ell(x), \tag{3.62}$$

allow us after taking the limit as $T \to +\infty$ in (3.61) to obtain

$$\int_0^{+\infty} \int_{\Omega_R} |u(t,x)|^p \varphi_1^\ell(x)dtdx = 0. \tag{3.63}$$

Next, taking the limit as $R \to +\infty$ in (3.63). Using the fact that $\lim_{R\to+\infty} \varphi_1^\ell(x) = 1$, we get

$$\int_0^{+\infty} \int_{\mathbb{R}^n} |u(t,x)|^p dtdx = 0.$$

This implies that $u = 0$ which is contradiction.

2. ii. Subcase of $p = \frac{1}{\gamma}$

In this case, the assumption

$$p < \frac{n}{n-2} \qquad \text{if } n \geq 3, \tag{3.64}$$

is needed. First, we observe that (3.64) implies

$$\frac{n}{2} - \frac{p}{p-1} < 0. \tag{3.65}$$

Under these assumptions, remind our selves that $\alpha = 1-\gamma$, then we verify easily that

$$1 - \alpha\frac{p}{p-1} = 0, \;\; 1-(\alpha+1)\frac{p}{p-1} = -\frac{1}{1-\gamma} < 0, \tag{3.66}$$

and also

$$1-(\alpha+2)\frac{p}{p-1} = -\frac{2p}{p-1} = -\frac{2}{1-\gamma} < 0.$$

Hence, taking the limit as $T \to \infty$ in (3.61) with the considerations (3.66) and (3.62) we obtain

$$\int_0^\infty \int_{\Omega_R} |u(t,x)|^p \varphi_1^\ell(x) dt dx = CR^{\left(\frac{n}{2}-\frac{p}{p-1}\right)\frac{1}{\beta}}. \tag{3.67}$$

Finally, one can remark that if $n = 1, 2$ then $\frac{n}{2} - \frac{p}{p-1} < 0$ for all $p > 1$ and then by taking the limit as $R \to \infty$ in (3.67), using the facts that $\beta \in (0, 2)$ and

$$\lim_{R\to+\infty} \varphi_1^\ell(x) = 1,$$

one has

$$\int_0^\infty \int_{\mathbb{R}^n} |u(t,x)|^p dt dx = 0. \tag{3.68}$$

This implies that $u = 0$ and this is a contradiction.
If $n \geq 3$ then $\frac{n}{2} - \frac{p}{p-1}$ is negative then it is not hard to get (3.68) by letting $R \to \infty$ in (3.67), if we assume furthermore that (3.64) or equivalently (3.65) is satisfied. This achieved the proof of *Theorem* 2.2. □

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Majorization problems for certain starlike functions associated with the exponential function

Hesam Mahzoon

Abstract. Let $\mathcal{S}_e^*$ and $\mathcal{S}_B^*$ denote the class of analytic functions f in the open unit disc normalized by $f(0) = 0 = f'(0) - 1$ and satisfying, respectively, the following subordination relations:

$$\frac{zf'(z)}{f(z)} \prec e^z \quad \text{and} \quad \frac{zf'(z)}{f(z)} \prec e^{e^z - 1}.$$

In this article, we investigate majorization problems for the classes $\mathcal{S}_e^*$ and $\mathcal{S}_B^*$ without acting upon any linear or nonlinear operators.

Mathematics Subject Classification (2010): 30C45.

Keywords: Univalent, starlike, exponential function, majorization, subordination, Bell numbers.

1. Introduction

Let $\mathcal{H}$ be the set of analytic functions f on the open unit disc

$$\Delta = \{z \in \mathbb{C} : |z| < 1\}$$

where $\mathbb{C}$ denotes the complex plane. Also let $\mathcal{A}$ be a subclass of $\mathcal{H}$ that whose members are normalized by the condition $f(0) = 0 = f'(0) - 1$. Let the functions f and g belong to the class $\mathcal{H}$ and there exists a Schwarz function $\phi : \Delta \to \Delta$ with the conditions $\phi(0) = 0$ and $|\phi(z)| < 1$ such that $f(z) = g(\phi(z))$. Then we say that f is subordinate to g, written as $f(z) \prec g(z)$ or $f \prec g$. It is clear that if $f \prec g$, then

$$f(0) = g(0) \quad \text{and} \quad f(\Delta) \subset g(\Delta). \tag{1.1}$$

Also, if g is univalent (one-to-one) in Δ, then $f(z) \prec g(z)$ iff the conditions (1.1) hold true. The subclass of $\mathcal{A}$ consisting of all univalent functions $f(z)$ in Δ will be denoted by $\mathcal{U}$. A function $f \in \mathcal{A}$ is said to be starlike if f maps Δ onto a domain

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which is starlike with respect to origin. The class of starlike functions in $\mathcal{U}$ is denoted $\mathcal{S}^*$. Analytically, a function $f \in \mathcal{A}$ belongs to the class $\mathcal{S}^*$ iff

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > 0 \quad (z \in \Delta).$$

In 1992, Ma and Minda (see [15]) have introduced the class

$$\mathcal{S}^*(\varphi) := \left\{f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec \varphi(z)\right\}$$

where φ is analytic univalent function with $\operatorname{Re}\{\varphi(z)\} > 0$ $(z \in \Delta)$ and normalized by $\varphi(0) = 1$ and $\varphi'(0) > 0$. For special choices of φ, the class $\mathcal{S}^*(\varphi)$ becomes to the well-known subclasses of the starlike functions. For example, the class

$$\mathcal{S}^*((1 + Az)/(1 + Bz)) =: \mathcal{S}^*[A, B] \quad (-1 \leq B < A \leq 1)$$

was introduced by Janowski, see [8]. If we also let $\varphi(z) := (1 + (1 - 2\alpha)z)/(1 - z)$, then the class $\mathcal{S}^*(\varphi)$ $(0 \leq \alpha < 1)$ gives the well-known class of the starlike functions of order α. We recall that a function $f \in \mathcal{A}$ is starlike of order α iff

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha \quad (z \in \Delta).$$

The family of all such functions is denoted by $\mathcal{S}^*(\alpha)$. We put $\mathcal{S}^*(0) \equiv \mathcal{S}^*$. The family $\mathcal{S}^*(\alpha)$ for $\alpha \in [0, 1)$ is a subfamily of the univalent functions (e.g., see [7]) and the function

$$\mathrm{K}_\alpha(z) := \frac{z}{(1 - z)^{2(1-\alpha)}} = z + \sum_{n=2}^{\infty} c_n(\alpha) z^n \quad (z \in \Delta, 0 \leq \alpha < 1),$$

where

$$c_n(\alpha) := \frac{\prod_{k=2}^{n}(k - 2\alpha)}{(n - 1)!} \quad (n \geq 2),$$

is the well-known extremal function for the class $\mathcal{S}^*(\alpha)$.
In 2015, Mendiratta et al. [17] introduced the class $\mathcal{S}^*_e$ as follows:

$$\mathcal{S}^*_e := \left\{f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec e^z =: \varphi_0(z)\right\}.$$

An extremal function for the class $\mathcal{S}^*_e$ is

$$f_1(z) := z \exp\left(\int_0^z \frac{e^\zeta - 1}{\zeta} \mathrm{d}\zeta\right) = z + z^2 + \frac{3}{4}z^3 + \frac{17}{36}z^4 + \cdots.$$

This function f_1 also plays the role extremal for many extremal problems. We notice that the exponential function $\varphi_0(z) = e^z$ has positive real part in Δ and

$$\varphi_0(\Delta) = \{\zeta \in \mathbb{C} : |\log \zeta| < 1\} =: \Omega.$$

It is easy to see that Ω is symmetric with respect to the real axis, starlike with respect to 1 and $\varphi_0'(0) > 0$ (see Figure 1(a)). Thus we have

$$f \in \mathcal{S}^*_e \Leftrightarrow \left|\log\left\{\frac{zf'(z)}{f(z)}\right\}\right| < 1 \quad (z \in \Delta).$$

For more details about the class $\mathcal{S}_e^*$ one can refer to [17].
Motivated by the above defined classes, Kumar et al. [12] (see also [6]) defined the class $\mathcal{S}_B^*$ associated with the Bell numbers where

$$\mathcal{S}_B^* := \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec e^{e^z-1} =: Q(z) \right\} =: \mathcal{S}^*(Q).$$

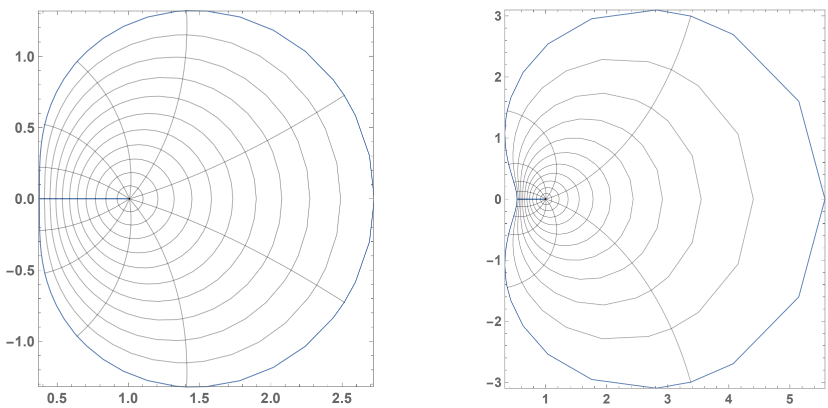
The function f_2 defined by

$$f_2(z) := z \exp\left(\int_0^z \frac{Q(\zeta)-1}{\zeta} \mathrm{d}\zeta \right) = z + z^2 + z^3 + \frac{17}{18}z^4 + \frac{245}{288}z^5 + \cdots,$$

belongs to the class $\mathcal{S}_B^*$ and serve as an extremal function in many problems. We also note that

$$Q(z) = e^{e^z-1} = \sum_{n=0}^{\infty} B_n \frac{z^n}{n!} = 1 + z + z^2 + \frac{5}{6}z^3 + \frac{5}{8}z^4 + \cdots \quad (z \in \Delta),$$

is starlike with respect to 1 (see Figure 1(b)) and its coefficients generate the Bell numbers. For a brief survey on these numbers, readers may refer to [4, 3].



(a) (b)

Figure 1. (a): The boundary curve of $\varphi_0(\Delta) = \exp(\Delta)$
(b): The boundary curve of $Q(\Delta) = \exp(\exp(\Delta) - 1)$

Also, for more details about some another subclasses of the starlike functions with various special cases of φ, see [10, 9, 11, 13, 14, 19, 20, 21].

The following theorem due to Carathéodory, see [5]:

Theorem A. *If the function $f \in \mathcal{H}$ satisfies the conditions*

$$|f(z)| \leq 1 \quad \text{and} \quad f(0) = 0,$$

then $|f'(z)| \leq 1$ for $|z| \leq \sqrt{2} - 1$.

Theorem B (below) is a generalization of the Theorem A which was proved by MacGregor, see [16]. Indeed, by letting $g(z) = z$, Theorem B reduces to the Theorem A.

Theorem B. *If $f(z)$ is majorized by $g(z)$ in Δ and $g(0)=0$, then*

$$\max_{|z|=r}|f'(z)| \leq \max_{|z|=r}|g'(z)|$$

for each number r in the interval $[0, \sqrt{2}-1]$.

We recall that a function $f \in \mathcal{H}$ is called to be majorized by $g \in \mathcal{H}$ written as

$$f(z) \ll g(z),$$

if there exists an analytic function ψ in Δ and satisfying the following conditions

$$|\psi(z)| \leq 1 \quad \text{and} \quad f(z) = \psi(z)g(z) \tag{1.2}$$

for all $z \in \Delta$. It should be noted that for the first time Mac-Gregor defined the concept of majorization. Indeed, he has been studied majorization problem for the class of starlike functions [16]. Recently, also many researchers have studied several majorization problems for certain subclasses of analytic functions which are defined by the concept of subordination, see for instance [1, 2, 25, 22, 23, 24].

The present paper aims to study majorization problems for the classes $\mathcal{S}_e^*$ and $\mathcal{S}_B^*$ without acting upon any linear or nonlinear operators to the above function classes.

2. Main Results

The following lemma (see [18]) will be needed in our investigation.

Lemma 2.1. *Let $\psi(z)$ be analytic in Δ and satisfying $|\psi(z)| \leq 1$ for all $z \in \Delta$. Then*

$$|\psi'(z)| \leq \frac{1-|\psi(z)|^2}{1-|z|^2}.$$

The first result of this section is continued in the following form.

Theorem 2.2. *Let the function f be in the class $\mathcal{A}$ and $g \in \mathcal{S}_e^*$. If $f(z)$ is majorized by $g(z)$ in Δ, then*

$$\max_{|z|=r}|f'(z)| \leq \max_{|z|=r}|g'(z)|$$

for each number r in the interval $[0, 0.323784]$ where $r_1 \approx 0.323784$ is the positive root of the equation

$$1 - r^2 - 2re^r = 0. \tag{2.1}$$

Proof. Let $f \in \mathcal{A}$ and the function g belongs to the class $\mathcal{S}_e^*$. Then by definition of the class $\mathcal{S}_e^*$ we have

$$\frac{zg'(z)}{g(z)} \prec e^z,$$

or equivalently

$$\frac{zg'(z)}{g(z)} = e^{\phi(z)} \quad (z \in \Delta), \tag{2.2}$$

where ϕ is a Schwarz function. With a simple calculation and since $|\phi(z)| \leq |z|$ (see [7]), (2.2) implies that

$$\left|\frac{g(z)}{g'(z)}\right| \leq re^r \quad (|z| = r < 1). \tag{2.3}$$

By the assumption since $f(z) \ll g(z)$ in Δ, thus there exists an analytic function ψ in Δ satisfying $|\psi(z)| \leq 1$ such that

$$f(z) = \psi(z)g(z) \quad (z \in \Delta). \tag{2.4}$$

Differentiating of both sides of (2.4) gives us

$$f'(z) = \psi'(z)g(z) + \psi(z)g'(z) = g'(z)\left(\psi'(z)\frac{g(z)}{g'(z)} + \psi(z)\right). \tag{2.5}$$

Now by (2.3), (2.5) and by Lemma 2.1 we get

$$\begin{aligned} |f'(z)| &\leq \left(|\psi(z)| + \frac{1-|\psi(z)|^2}{1-r^2} \times re^r\right)|g'(z)| \\ &= \left(\gamma + \frac{1-\gamma^2}{1-r^2} \times re^r\right)|g'(z)|, \end{aligned}$$

where $|\psi(z)| =: \gamma \in [0,1]$. We now define the function $\mu(\gamma, r)$ as follows

$$\mu(\gamma, r) := \gamma + \frac{1-\gamma^2}{1-r^2} \times re^r.$$

It is enough to consider r_1 as follows

$$r_1 = \max\{r \in [0,1) : \mu(\gamma, r) \leq 1, \forall \gamma \in [0,1]\}.$$

Therefore

$$\mu(\gamma, r) \leq 1 \Leftrightarrow \lambda(\gamma, r) \geq 0,$$

where $\lambda(\gamma, r) := 1 - r^2 - (1+\gamma)re^r$. We see that $\lambda(\gamma, r)$ is decreasing function with respect to γ and gets its minimum value in $\gamma = 1$, namely

$$\min\{\lambda(\gamma, r) : \gamma \in [0,1]\} = \lambda(1, r) = \lambda(r),$$

where $\lambda(r) := 1 - r^2 - 2re^r$. On the other hand, since $\lambda(0) = 1 > 0$ and $\lambda(1) = -2e < 0$, thus there exists a r_1 such that $\lambda(r) \geq 0$ for all $r \in [0, r_1]$ where r_1 is the smallest positive root of the Eq. (2.1). □

Since the identity function $g(z) = z$ belongs to the class $\mathcal{S}_e^*$, therefore we have the following result.

Corollary 2.3. *If a function $f \in \mathcal{A}$ satisfies the condition*

$$|f(z)| < 1 \quad (z \in \Delta),$$

then $|f'(z)| \leq 1$ for $|z| \leq 0.323784$.

The next result gives a same result for the class $\mathcal{S}_B^*$.

Theorem 2.4. *Let the function f be in the class $\mathcal{A}$ and $g \in \mathcal{S}_B^*$. If $f(z)$ is majorized by $g(z)$ in Δ, then*

$$\max_{|z|=r} |f'(z)| \leq \max_{|z|=r} |g'(z)| \quad (0 \leq r \leq r_2) \tag{2.6}$$

where r_2 is the smallest positive root of the equation

$$(1-r^2)e^{e^{-r}-1} - 2r = 0. \tag{2.7}$$

Proof. Let f belong to the class $\mathcal{A}$. If $g \in \mathcal{S}^*_B$ then the following subordination relation holds true:

$$\frac{zg'(z)}{g(z)} \prec e^{e^z - 1},$$

or equivalently

$$\frac{zg'(z)}{g(z)} = e^{e^{\phi(z)}-1} \quad (z \in \Delta), \tag{2.8}$$

where ϕ is a Schwarz function. With a simple calculation and since $|\phi(z)| \le |z|$, (2.8) yields that

$$\left|\frac{g(z)}{g'(z)}\right| \le \frac{r}{e^{e^{-r}-1}} \quad (|z| = r < 1). \tag{2.9}$$

On the other hand we have $f(z) \ll g(z)$ in Δ. Therefore by (2.4), (2.5), (2.9) and Lemma 2.1 we get

$$\begin{aligned} |f'(z)| &\le \left(|\psi(z)| + \frac{1 - |\psi(z)|^2}{1 - r^2} \times \frac{r}{e^{e^{-r}-1}}\right) |g'(z)| \\ &= \left(\gamma + \frac{1 - \gamma^2}{1 - r^2} \times \frac{r}{e^{e^{-r}-1}}\right) |g'(z)|, \end{aligned}$$

where $|\psi(z)| =: \gamma \in [0, 1]$. We define

$$\eta(\gamma, r) := \gamma + \frac{1 - \gamma^2}{1 - r^2} \times \frac{r}{e^{e^{-r}-1}}.$$

Therefore we are looking for r_2 such that (2.6) holds. It is sufficient to consider r_2 as follows:

$$r_2 = \max\{r \in [0, 1) : \eta(\gamma, r) \le 1, \forall \gamma \in [0, 1]\}.$$

Thus

$$\eta(\gamma, r) \le 1 \Leftrightarrow \theta(\gamma, r) \ge 0,$$

where $\theta(\gamma, r) := (1 - r^2)(e^{e^{-r}-1}) - r(1 + \gamma)$. We see that $\frac{\partial \theta}{\partial \gamma} = -r < 0$. In conclusion, $\theta(\gamma, r)$ gets its minimum value in $\gamma = 1$, namely

$$\min\{\theta(\gamma, r) : \gamma \in [0, 1]\} = \theta(1, r) = \theta(r),$$

where $\theta(r) := (1 - r^2)(e^{e^{-r}-1}) - 2r$. We have $\theta(0) = 1 > 0$ and $\theta(1) = -2 < 0$. So there exists a r_2 such that $\theta(r) \ge 0$ for all $r \in [0, r_2]$ where r_2 is the smallest positive root of the Eq. (2.7). This completes the proof. □

If we let $g(z) = z$ in the above Theorem 2.4, then we get the following.

Corollary 2.5. *If a function $f \in \mathcal{A}$ satisfies the condition*

$$|f(z)| < 1 \quad (z \in \Delta),$$

then $|f'(z)| \le 1$ for all z which $|z| \le r_2$, where r_2 is the smallest positive root of the Eq. (2.7).

Remark 2.6. Figure 2 shows the roots r_1 and r_2 in Theorem 2.2 and Theorem 2.4, respectively, are approximately equal.

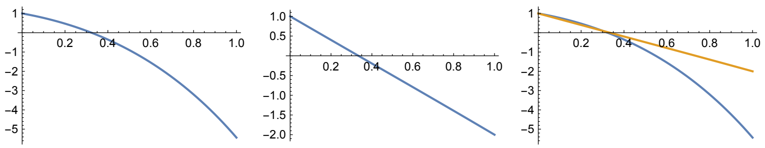


FIGURE 2. graph of Eq. (2.1) (left), graph of Eq. (2.7) (centre), graph of both Eqs. (2.1) and (2.7) (right)

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Coefficient estimates for a subclass of analytic functions by Srivastava-Attiya operator

Mostafa Jafari, Ahmad Motamednezad and Ebrahim Analouei Adegani

Abstract. In this paper, we investigate bounds of the coefficients for subclass of analytic and bi-univalent functions. The results presented in this paper would generalize and improve some recent works and other authors.

Mathematics Subject Classification (2010): 30C45, 30C50.

Keywords: Analytic functions, bi-univalent functions, coefficient estimates, Srivastava-Attiya operator, subordination.

1. Introduction

Let $\mathcal{A}$ be a class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

which are analytic in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$. Further, let $\mathcal{S}$ denote the class of functions $f \in \mathcal{A}$ which are univalent in $\mathbb{U}$.

For $f(z)$ defined by (1.1) and $h(z)$ defined by

$$h(z) = z + \sum_{n=2}^{\infty} b_n z^n,$$

the Hadamard product $(f * h)(z)$ of the functions $f(z)$ and $h(z)$ defined by

$$(f * h)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n,$$

In 2007, Srivastava and Attiya [21] (see also Răducanu and Srivastava [18] and Prajapat and Goyal [17]) for the class $\mathcal{A}$ introduced and investigated linear operator

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$\mathcal{J}_\mu^b : \mathcal{A} \to \mathcal{A}$ that defined in terms of the Hadamard product by

$$\mathcal{J}_\mu^b f(z) = z + \sum_{k=2}^{\infty} \Theta_k a_k z^k,$$

where

$$\Theta_k = \left| \left(\frac{1+b}{k+b} \right)^\mu \right|,$$

and (throughout this paper unless otherwise mentioned) the parameters μ, b are considered as $\mu \in \mathbb{C}$ and $b \in \mathbb{C} \setminus \{0, -1, -2, \cdots\}$, (see for more details [20]).

Remark 1.1. (1) For $\mu = 1$ and $b = \upsilon$ ($\upsilon > -1$), we get generalized Libera-Bernardi integral operator [19];
(2) For $\mu = \sigma$ ($\sigma > 0$) and $b = 1$, we get Jung-Kim-Srivastava integral operator [12].

For each $f \in \mathcal{S}$, the Koebe one-quarter theorem [9] ensures that the image of $\mathbb{U}$ under f contains a disk of radius $\frac{1}{4}$. Hence every function $f \in \mathcal{S}$ has an inverse f^{-1}, which is defined by

$$f^{-1}(f(z)) = z \qquad (z \in \mathbb{U}),$$

and

$$f(f^{-1}(w)) = w \qquad \left(|w| < r_0(f); r_0(f) \geq \frac{1}{4} \right),$$

where

$$g(w) = f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3) w^3 + \cdots . \tag{1.2}$$

A function $f \in \mathcal{A}$ is said to be bi-univalent in $\mathbb{U}$ if both f and f^{-1} are univalent in $\mathbb{U}$. Let Σ denote the class of bi-univalent functions in $\mathbb{U}$ given by (1.1).

Recently many researchers have introduced and investigated several interesting subclasses of the bi-univalent function class Σ and they have found non-sharp estimates on the first two Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$ and other problems, see for example, [3, 2, 4, 5, 6, 7, 8, 10, 11, 13, 14, 15, 22, 23, 24].

For two functions f and g that are analytic in $\mathbb{U}$, we say that the function f is *subordinate* to g and write $f(z) \prec g(z)$, if there exists a Schwarz function ω, that is analytic in $\mathbb{U}$ with $\omega(0) = 0$ and $|\omega(z)| < 1$ such that $f(z) = g(\omega(z))$ for all $z \in \mathbb{U}$.

In particular, if the function g is univalent in $\mathbb{U}$, then $f(z) \prec g(z)$ if and only if $f(0) = g(0)$ and $f(\mathbb{U}) \subseteq g(\mathbb{U})$.

In this work, we obtain estimates of coefficients for a subclass of bi-univalent functions considered by Selvaraj et al. [20]. The results presented in this paper would generalize and improve some recent works and other authors.

2. The subclass $\mathcal{S}_{\Sigma,t}^{\mu,b}(\gamma,\lambda,\phi)$

Throughout this paper, we assume that ϕ is an analytic function with positive real part in the unit disk $\mathbb{U}$, satisfying $\phi(0)=1,\ \phi'(0)>0$ and symmetric with respect to the real axis. Such a function has series expansion of the form

$$\phi(z)=1+B_1z+B_2z^2+B_3z^3+\cdots \quad (B_1>0), \tag{2.1}$$

Let that $u(z)$ and $v(z)$ are Schwarz function in $\mathbb{U}$ with

$$u(0)=v(0)=0,\ |u(z)|<1,\ |v(z)|<1$$

and suppose that

$$u(z)=\sum_{n=1}^{\infty}p_nz^n \quad \text{and} \quad v(z)=\sum_{n=1}^{\infty}q_nz^n \quad (z\in\mathbb{U}). \tag{2.2}$$

Then [16, p. 172]

$$|p_1|\le 1, \quad |p_2|\le 1-|p_1|^2, \quad |q_1|\le 1, \quad |q_2|\le 1-|q_1|^2. \tag{2.3}$$

By (2.1), we get

$$\phi(u(z))=1+B_1p_1z+(B_1p_2+B_2p_1^2)z^2+\cdots \quad (z\in\mathbb{U}) \tag{2.4}$$

and

$$\phi(v(w))=1+B_1q_1w+(B_1q_2+B_2q_1^2)w^2+\cdots \quad (w\in\mathbb{U}). \tag{2.5}$$

In 2014, Selvaraj et al. [20] introduced subclass of Σ and obtained estimates on the coefficients $|a_2|$ and $|a_3|$ for functions in this subclass as follows:

Definition 2.1. [20] A function $f\in\Sigma$ given by (1.1) is said to be in the class $\mathcal{S}_{\Sigma,t}^{\mu,b}(\gamma,\lambda,\phi)$ if the following conditions are satisfied:

$$1+\frac{1}{\gamma}\left(\frac{[(1-t)z]^{1-\lambda}(\mathcal{J}_\mu^bf(z))'}{[\mathcal{J}_\mu^bf(z)-\mathcal{J}_\mu^bf(tz)]^{1-\lambda}}-1\right)\prec\phi(z),$$

and

$$1+\frac{1}{\gamma}\left(\frac{[(1-t)w]^{1-\lambda}(\mathcal{J}_\mu^bg(w))'}{[\mathcal{J}_\mu^bg(w)-\mathcal{J}_\mu^bg(tw)]^{1-\lambda}}-1\right)\prec\phi(w),$$

where $|t|\le 1\ (t\ne 1);\ \gamma\in\mathbb{C}\backslash\{0\};\ \lambda\ge 0;\ z,w\in\mathbb{U}$ and g is given by (1.2).

Theorem 2.2. [20] *Let the function $f(z)$ given by (1.1) be in the class $\mathcal{S}_{\Sigma,t}^{\mu,b}(\gamma,\lambda,\phi)$. Then*

$$|a_2|\le\frac{|\gamma|B_1\sqrt{2B_1}}{\sqrt{|\gamma B_1^2\Lambda(\lambda,t)\Xi(\lambda,t)-2(B_2-B_1)[\Lambda(\lambda,t)+2]^2\Theta_2^2+2\gamma B_1^2\Upsilon(\lambda,t)\Theta_3|}} \tag{2.6}$$

and

$$|a_3|\le\frac{B_1|\gamma|}{\Upsilon(\lambda,t)\Theta_3}+\left(\frac{B_1|\tau|}{[\Lambda(\lambda,t)+2]\Theta_2}\right)^2, \tag{2.7}$$

where

$$\Lambda(\lambda,t)=(\lambda-1)(1+t),\ \Upsilon(\lambda,t)=[(\lambda-1)(1+t+t^2)+3]$$

and

$$\Xi(\lambda,t)=[(\lambda-2)(1+t)+4].$$

3. Coefficient estimates

In the section, we get that the following theorem which is an refinement of inequalities (2.6) and (2.7).

Theorem 3.1. *Let the function* $f(z)$ *given by* (1.1) *be in the class* $\mathcal{S}^{\mu,b}_{\Sigma,t}(\gamma,\lambda,\phi)$, $|t|\le 1$ $(t\neq 1)$, $\gamma\in\mathbb{C}\backslash\{0\}$ *and* $\lambda\ge 0$. *Then*

$$|a_2|\le \frac{|\gamma|B_1\sqrt{2B_1}}{\sqrt{2B_1[\Lambda(\lambda,t)+2]^2\Theta_2^2+|\gamma B_1^2\Lambda(\lambda,t)\Xi(\lambda,t)-2B_2[\Lambda(\lambda,t)+2]^2\Theta_2^2+2\gamma B_1^2\Upsilon(\lambda,t)\Theta_3|}}$$

and

$$|a_3|\le\begin{cases}\dfrac{|\tau|B_1}{\Upsilon(\lambda,t)\Theta_3} & B_1\le\dfrac{[(\lambda-1)(1+t)+2]^2\Theta_2^2}{|\gamma|\Theta_3[(\lambda-1)(1+t+t^2)+3]}\\[2ex] \dfrac{\Phi(\Theta_1,\Theta_2,\lambda,t)}{\Psi(\Theta_1,\Theta_2,\lambda,t)\Upsilon(\lambda,t)\Theta_3} & B_1>\dfrac{[(\lambda-1)(1+t)+2]^2\Theta_2^2}{|\gamma|\Theta_3[(\lambda-1)(1+t+t^2)+3]}.\end{cases}$$

where

$$\begin{aligned}\Phi(\Theta_1,\Theta_2,\lambda,t)=&|\tau|B_1\left|\gamma B_1^2\Lambda(\lambda,t)\Xi(\lambda,t)-2B_2[\Lambda(\lambda,t)+2]^2\Theta_2^2+2\gamma B_1^2\Upsilon(\lambda,t)\Theta_3\right|\\&+2|\gamma|^2\Theta_3\Upsilon(\lambda,t)B_1^3,\end{aligned}$$

and

$$\begin{aligned}\Psi(\Theta_1,\Theta_2,\lambda,t)=&2B_1[\Lambda(\lambda,t)+2]^2\Theta_2^2\\&+|\gamma B_1^2\Lambda(\lambda,t)\Xi(\lambda,t)-2B_2[\Lambda(\lambda,t)+2]^2\Theta_2^2+2\gamma B_1^2\Upsilon(\lambda,t)\Theta_3|.\end{aligned}$$

Proof. Let $f\in\mathcal{S}^{\mu,b}_{\Sigma,t}(\gamma,\lambda,\phi)$ and $g=f^{-1}$. Then there are analytic functions $u,v:\mathbb{U}\to\mathbb{U}$, with $u(0)=v(0)=0$, given by (2.2) such that

$$1+\frac{1}{\gamma}\left(\frac{[(1-t)z]^{1-\lambda}(\mathcal{J}^b_\mu f(z))'}{[\mathcal{J}^b_\mu f(z)-\mathcal{J}^b_\mu f(tz)]^{1-\lambda}}-1\right)=\phi(u(z)),\tag{3.1}$$

and

$$1+\frac{1}{\gamma}\left(\frac{[(1-t)w]^{1-\lambda}(\mathcal{J}^b_\mu g(w))'}{[\mathcal{J}^b_\mu g(w)-\mathcal{J}^b_\mu g(tw)]^{1-\lambda}}-1\right)=\phi(v(w)).\tag{3.2}$$

From (2.4), (2.5), (3.1) and (3.2), we obtain

$$[(\lambda-1)(1+t)+2]\Theta_2a_2=\gamma B_1p_1,\tag{3.3}$$

$$\begin{aligned}&[(\lambda-1)(1+t+t^2)+3]\Theta_3a_3+\frac{1}{2}(\lambda-1)(1+t)[(\lambda-2)(1+t)+4]\Theta_2^2a_2^2\\=&\gamma[B_1p_2+B_2p_1^2],\end{aligned}\tag{3.4}$$

$$-[(\lambda-1)(1+t)+2]\Theta_2a_2=\gamma B_1q_1,\tag{3.5}$$

and

$$[(\lambda-1)(1+t+t^2)+3]\Theta_3(2a_2^2-a_3) + \frac{1}{2}(\lambda-1)(1+t)[(\lambda-2)(1+t)+4]\Theta_2^2a_2^2 = \gamma[B_1q_2+B_2q_1^2]. \tag{3.6}$$

From (3.3) and (3.5), we get

$$p_1 = -q_1. \tag{3.7}$$

Adding (3.4) and (3.6), and using (3.7), we have

$$\big((\lambda-1)(1+t)[(\lambda-2)(1+t)+4]\Theta_2^2 + 2\Theta_3[(\lambda-1)(1+t+t^2)+3]\big)\, a_2^2 - 2\gamma B_2 p_1^2 = \gamma B_1(p_2+q_2). \tag{3.8}$$

From (3.3), we have

$$\big(\gamma B_1^2\{(\lambda-1)(1+t)[(\lambda-2)(1+t)+4]\Theta_2^2 + 2\Theta_3[(\lambda-1)(1+t+t^2)+3]\} -2B_2[(\lambda-1)(1+t)+2]^2\Theta_2^2)a_2^2 = \gamma^2B_1^3(p_2+q_2).$$

By (2.3) and (3.3), we get

$$\begin{aligned}&|\big(\gamma B_1^2\{(\lambda-1)(1+t)[(\lambda-2)(1+t)+4]\Theta_2^2 + 2\Theta_3[(\lambda-1)(1+t+t^2)+3]\} \\ &- 2B_2[(\lambda-1)(1+t)+2]^2\Theta_2^2)a_2^2| \le |\tau|^2B_1^3(|p_2|+|q_2|) \\ &\le 2|\gamma|^2B_1^3(1-|p_1|^2) \\ &= 2|\gamma|^2B_1^3 - 2B_1[(\lambda-1)(1+t)+2]^2\Theta_2^2|a_2|^2.\end{aligned}$$

Therefore,

$$|a_2| \le \tag{3.9}$$

$$\le \frac{|\gamma|B_1\sqrt{2B_1}}{\sqrt{2B_1[\Lambda(\lambda,t)+2]^2\Theta_2^2 + |\gamma B_1^2\Lambda(\lambda,t)\Xi(\lambda,t) - 2B_2[\Lambda(\lambda,t)+2]^2\Theta_2^2 + 2\gamma B_1^2\Upsilon(\lambda,t)\Theta_3|}},$$

where

$$\Lambda(\lambda,t) = (\lambda-1)(1+t),\ \ \Upsilon(\lambda,t) = [(\lambda-1)(1+t+t^2)+3]$$

and

$$\Xi(\lambda,t) = [(\lambda-2)(1+t)+4].$$

Next, in order to find the bound on the coefficient $|a_3|$, by subtracting (3.6) from (3.4), and using (3.7), we get

$$2[(\lambda-1)(1+t+t^2)+3]\Theta_3a_3 = 2\Theta_3[(\lambda-1)(1+t+t^2)+3]a_2^2 + \tau B_1(p_2-q_2). \tag{3.10}$$

Using (2.3) and (3.7), we have

$$\begin{aligned}&2[(\lambda-1)(1+t+t^2)+3]\Theta_3|a_3| \\ &\le |\gamma|B_1(|p_2|+|q_2|) + 2\Theta_3[(\lambda-1)(1+t+t^2)+3]|a_2|^2 \\ &\le 2|\gamma|B_1(1-|p_1|^2) + 2\Theta_3[(\lambda-1)(1+t+t^2)+3]|a_2|^2.\end{aligned}$$

From (3.3), we get

$$\begin{aligned}&|\gamma|B_1[(\lambda-1)(1+t+t^2)+3]\Theta_3|a_3|\\ &\leq \left[|\gamma|\Theta_3[(\lambda-1)(1+t+t^2)+3]B_1-[(\lambda-1)(1+t)+2]^2\Theta_2^2\right]|a_2|^2+|\gamma|^2B_1^2.\end{aligned}$$

From (3.9), for $\left[|\gamma|\Theta_3[(\lambda-1)(1+t+t^2)+3]B_1-[(\lambda-1)(1+t)+2]^2\Theta_2^2\right] > 0$ we have

$$\begin{aligned}&|\gamma|B_1[(\lambda-1)(1+t+t^2)+3]\Theta_3|a_3|\\ &\leq \left[|\gamma|\Theta_3[(\lambda-1)(1+t+t^2)+3]B_1-[(\lambda-1)(1+t)+2]^2\Theta_2^2\right]\\ &\times\frac{2|\gamma|^2B_1^3}{2B_1[\Lambda(\lambda,t)+2]^2\Theta_2^2+|\gamma B_1^2\Lambda(\lambda,t)\Xi(\lambda,t)-2B_2[\Lambda(\lambda,t)+2]^2\Theta_2^2+2\gamma B_1^2\Upsilon(\lambda,t)\Theta_3|}\\ &+|\gamma|^2B_1^2.\end{aligned}$$

Therefore,

$$\begin{aligned}|a_3| &\leq \left[|\gamma|\Theta_3[(\lambda-1)(1+t+t^2)+3]B_1-[(\lambda-1)(1+t)+2]^2\Theta_2^2\right]\\ &\quad\times\frac{2|\gamma|B_1^2}{\Psi(\Theta_1,\Theta_2,\lambda,t)[(\lambda-1)(1+t+t^2)+3]\Theta_3}+\frac{|\gamma|B_1}{[(\lambda-1)(1+t+t^2)+3]\Theta_3},\end{aligned}$$

where

$$\begin{aligned}\Psi(\Theta_1,\Theta_2,\lambda,t) =&2B_1[\Lambda(\lambda,t)+2]^2\Theta_2^2\\ &+|\gamma B_1^2\Lambda(\lambda,t)\Xi(\lambda,t)-2B_2[\Lambda(\lambda,t)+2]^2\Theta_2^2+2\gamma B_1^2\Upsilon(\lambda,t)\Theta_3|.\end{aligned}$$

Consequently,

$$|a_3|\leq\begin{cases}\dfrac{|\gamma|B_1}{[(\lambda-1)(1+t+t^2)+3]\Theta_3} & B_1\leq\dfrac{[(\lambda-1)(1+t)+2]^2\Theta_2^2}{|\gamma|\Theta_3[(\lambda-1)(1+t+t^2)+3]}\\[2ex] \dfrac{\Phi(\Theta_1,\Theta_2,\lambda,t)}{\Psi(\Theta_1,\Theta_2,\lambda,t)[(\lambda-1)(1+t+t^2)+3]\Theta_3} & B_1>\dfrac{[(\lambda-1)(1+t)+2]^2\Theta_2^2}{|\gamma|\Theta_3[(\lambda-1)(1+t+t^2)+3]},\end{cases}$$

where

$$\begin{aligned}\Phi(\Theta_1,\Theta_2,\lambda,t) =& |\tau|B_1\left|\gamma B_1^2\Lambda(\lambda,t)\Xi(\lambda,t)-2B_2[\Lambda(\lambda,t)+2]^2\Theta_2^2+2\gamma B_1^2\Upsilon(\lambda,t)\Theta_3\right|\\ &+2|\gamma|^2\Theta_3[(\lambda-1)(1+t+t^2)+3]B_1^3.\end{aligned}$$

This completes the proof. □

Remark 3.2. Theorem 3.1 is an improvement of the estimates obtained by Selvaraj et al. [20] in Theorem 2.2. For the coefficient $|a_2|$, it is clear that

$$\begin{aligned}&\frac{|\gamma|B_1\sqrt{2B_1}}{\sqrt{2B_1[\Lambda(\lambda,t)+2]^2\Theta_2^2+|\gamma B_1^2\Lambda(\lambda,t)\Xi(\lambda,t)-2B_2[\Lambda(\lambda,t)+2]^2\Theta_2^2+2\gamma B_1^2\Upsilon(\lambda,t)\Theta_3|}}\\ &\leq\frac{|\gamma|B_1\sqrt{2B_1}}{\sqrt{|\gamma B_1^2\Lambda(\lambda,t)\Xi(\lambda,t)-2(B_2-B_1)[\Lambda(\lambda,t)+2]^2\Theta_2^2+2\gamma B_1^2\Upsilon(\lambda,t)\Theta_3|}}.\end{aligned}$$

On the other hand, for the coefficient $|a_3|$, we make the following cases:

(i) For $B_1 \leq \dfrac{[(\lambda-1)(1+t)+2]^2\Theta_2^2}{|\gamma|\Theta_3[(\lambda-1)(1+t+t^2)+3]}$, it is clear that

$$\frac{|\gamma|B_1}{\Upsilon(\lambda,t)\Theta_3} \leq \frac{B_1|\gamma|}{\Upsilon(\lambda,t)\Theta_3} + \left(\frac{B_1|\tau|}{[\Lambda(\lambda,t)+2]\Theta_2}\right)^2.$$

(ii) For $B_1 > \dfrac{[(\lambda-1)(1+t)+2]^2\Theta_2^2}{|\gamma|\Theta_3[(\lambda-1)(1+t+t^2)+3]}$, it is clear that

$$\frac{\Phi(\Theta_1,\Theta_2,\lambda,t)}{\Psi(\Theta_1,\Theta_2,\lambda,t)\Upsilon(\lambda,t)\Theta_3} \leq \frac{B_1|\gamma|}{\Upsilon(\lambda,t)\Theta_3} + \left(\frac{B_1|\tau|}{[\Lambda(\lambda,t)+2]\Theta_2}\right)^2.$$

Remark 3.3. If we set $\lambda = 0$ in Theorem 3.1, then we get an improvement of the estimates obtained by Selvaraj et al. [20, Corollary 2.1].

Remark 3.4. If we set $\lambda = 1$ in Theorem 3.1, then we get an improvement of the estimates obtained by Selvaraj et al. [20, Corollary 2.2].

Remark 3.5. If $\mathcal{J}_\mu^b f(z)$ be the identity map and $\lambda = 0$ in Theorem 3.1, then we get an improvement of the estimates obtained by Selvaraj et al. [20, Corollary 2.3].

Remark 3.6. If $\mathcal{J}_\mu^b f(z)$ be the identity map and $\lambda = 1$ in Theorem 3.1, then we get an improvement of the estimates obtained by Selvaraj et al. [20, Corollary 2.4].

Remark 3.7. If $\mathcal{J}_\mu^b f(z)$ be the identity map and $\gamma = 1,\ t = 0$ in Theorem 3.1, then we get an improvement of the estimates obtained by Deniz [8, Theorem 2.8].

Remark 3.8. If $\mathcal{J}_\mu^b f(z)$ be the identity map and $\gamma = 1,\ \lambda = 1$ in Theorem 3.1 is an improvement of the estimates obtained by Ali et al. in [3, Theorem 2.1].

Remark 3.9. If we take

$$\phi(z) = \frac{1+Az}{1+Bz} = 1 + (A-B)z + (B-A)Bz^2 + \cdots (-1 \leq B < A \leq 1, z \in \mathbb{U})$$

and

$$\varphi(z) = \left(\frac{1+z}{1-z}\right)^\alpha = 1 + 2\alpha z + 2\alpha^2 z^2 + \cdots \ (0 < \alpha \leq 1,\ z \in \mathbb{U}),$$

which gives $B_1 = A - B,\ B_2 = (B-A)B$ and $B_1 = 2\alpha,\ B_2 = 2\alpha^2$, in Theorem 3.1, then we can deduce interesting results analogous, respectively. Also, for suitable choices the parameter μ and b in Theorems 3.1 and some Remarks above we have an improvement of results involving Libera-Bernardi integral operator [19] and Jung-Kim-Srivastava integral operator [12].

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Existence for stochastic sweeping process with fractional Brownian motion

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Abstract. This paper is devoted to the study of a convex stochastic sweeping process with fractional Brownian by time delay. The approach is based on discretizing stochastic functional differential inclusions.

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Keywords: Sweeping process, evolution inclusion, perturbation, normal cone, fixed point.

1. Introduction

The so-called sweeping process is a particular differential inclusion of the general form

$$-x'(t) \in N_{C(t)}(x(t))\, a, e.\, t \in [0, T] \tag{1.1}$$

$$x(0) \in C(0) \tag{1.2}$$

where $C(t)$ is a convex time dependance set,and $N_C(t)(x(t))$ is the normal cone to $C(t)$ at $x(t)$.The sweeping process, introduced by Moreau in the early 1970s, and extensively studied by himself and other authors (see, e.g., [2, 7, 8, 5]).These models prove to be quite useful in elastoplasticity, non smooth mechanics, convex optimization, mathematical economics, queuing theory, etc. In this paper, we propose a simple extension of the sweeping process. More precisely, We consider the problem formally

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expressed by

$$\begin{cases} -dx(t) & \in N_{C_1(t)}(x(t))dt + G^1(t, x_t, y_t)dB^{H_1} \ a,e.\, t \in J := [0,T] \\ -dy(t) & \in N_{C_2(t)}(y(t))dt + G^2(t, x_t, y_t)dB^{H_2} \ a,e.\, t \in J := [0,T] \\ x(t) & = \phi(t), t \in [-r, 0],\, x(0) \in C_1(0) \\ y(t) & = \overline{\phi}(t), t \in [-r, 0],\, y(0) \in C_2(0) \end{cases} \tag{1.3}$$

where $C_1(t), C_2(t)$ is convex for all t, X is a real separable Hilbert space with inner product $\langle \cdot, \cdot \rangle$ induced by norm $\|\cdot\|$, $G^j : M_2([-r, 0], X) \times M_2([-r, 0], X) \to L^0_{Q_{H_j}}(Y, X)$ are given functions. Here, $L^0_{Q_{H_j}}(Y, X)$ denotes the space of all Q_{H_j}-Hilbert-Schmidt operators from Y into X,B^{H_j} is sequence of mutually independent fractional Brownian motions with $H_1 \neq H_2$ i.e ($B^{H_1} \neq B^{H_2}$) for each $j = 1, 2$, with Hurst parameter $H_j > \frac{1}{2}$. Here $y(\cdot, \cdot) : [-r, T] \times \Omega \to X$, then for any $t \geq 0$, $y_t(\cdot, \cdot) : [-r, 0] \times \Omega \to X$ is given by:

$$y_t(\theta, \omega) = y(t + \theta, \omega), \text{ for } \theta \in [-r, 0], \ \omega \in \Omega.$$

Here $y_t(\cdot)$ represents the history of the state from time $t - r$, up to the present time t. Let $M^2([-r, 0], X)$ be the following space defined by

$$M^2([-r, 0], X) = \{\phi, \overline{\phi} \colon [-r, 0] \times \Omega \to X, \quad \phi, \overline{\phi} \in C([-r, 0], L^2(\Omega, X))\},$$

endowed with the norm

$$||\phi(t)||_{M^2_{\mathcal{F}_0}} = \int_{-r}^0 |\phi(t)|^2 dt$$

Now, for a given $T > 0$, we define

$$\begin{cases} M^2([-r, T], X) = y\colon [-r, T] \times \Omega \to X, \quad \phi, \overline{\phi} \in C([-r, T], L^2(\Omega, X)) \text{ and} \\ \qquad \sup_{t \in [0,T]} E(|y(t)|^2) < \infty, \ \int_{-r}^0 |\phi(t)|^2 dt < \infty. \end{cases}$$

Endowed with the norm

$$\|y\|_{M^2_{\mathcal{F}_b}} = \sup_{-r \leq s \leq T} (\mathbb{E}\|y(s)\|^2)^{\frac{1}{2}}.$$

Random differential and integral equations play an important role in characterizing many social, physical, biological and engineering problems; see for instance the monographs by Da Prato and Zabczyk [3], Gard [4],Sobzyk [10] and Tsokos and Padgett [11]. For example, a stochastic model for drug distribution in a biological system was described by Tsokos and Padgett [11] to a closed system with a simplified heat, one organ or capillary bed, and re-circulation of a blood with a constant rate of flow, where the heart is considered as a mixing chamber of constant volume. For the basic theory concerning stochastic differential equations see the monographs by Bharucha-Reid [1], Mao[6], Øksendal[9], Tsokos and Padgett [11].

This paper is organized as follows. In Section 2 and 3, we recall some definitions and results that will be used in all the sequel. Section 4 is devoted to the study of the existence problem of (1.3).In Section 5, we restrict our attention to the case when the perturbation with F.

2. Basic definitions of stochastic calculus

In this section, we introduce notations, definitions, and preliminary facts which are used throughout this paper.Actually we will borrow them from [?].Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space with a filtration $(\mathcal{F} = \mathcal{F}_t)_{t\geq 0}$ satisfying the usual conditions (i.e. right continuous and $\mathcal{F}_0$ containing all $\mathbb{P}$-null sets).

For a stochastic process $x(\cdot,\cdot) : [0,T]\times\Omega \to X$ we will write $x(t)$ (or simply x when no confusion is possible) instead of $x(t,\omega)$.

Definition 2.1. Given $H_1, H_2 \in (0,1)$,$H_1 \neq H_2$ a continuous centered Gaussian process B^H is said to be a two-sided one-dimensional fractional Brownian motion (fBm) with Hurst parameter H_j,$j = 1,2$ if its covariance function $R_{H_j}(t,s) = \mathbb{E}[B^{H_j}(t))B^{H_j}(s)]$ satisfies

$$R_{H_j}(t,s) = \frac{1}{2}(|t|^{2H_j} + |s|^{2H_j} - |t-s|^{2H_j}) \quad t,s \in [0,T].$$

It is known that $B^H(t)$ with $H_j > \frac{1}{2}$ admits the following Volterra representation

$$B^{H_j}(t) = \int_0^t K_{H_j}(t,s)dW(s) \tag{2.1}$$

where W is a standard Brownian motion given by

$$W(t) = B^{H_j}((K^*_{H_j})^{-1}\xi_{[0,t]}),$$

and the Volterra kernel the kernel $K(t,s)$ is given by

$$K_{H_j}(t,s) = c_{H_j} s^{1/2-H_j} \int_s^t (u-s)^{H_j-\frac{3}{2}} \left(\frac{u}{s}\right)^{H_j-\frac{1}{2}} du, \quad t \geq s,$$

where $c_{H_j} = \sqrt{\frac{H_j(2H_j-1)}{\beta(2H_j-2,H_j-\frac{1}{2})}}$ and $\beta(\cdot,\cdot)$ denotes the Beta function, $K(t,s) = 0$ if $t \leq s$, and it holds

$$\frac{\partial K_{H_j}}{\partial t}(t,s) = c_H \left(\frac{t}{s}\right)^{H_j-\frac{1}{2}} (t-s)^{H_j-\frac{3}{2}},$$

and the kernel $K^*_{H_j}$ is defined as follows. Denote by $\mathcal{E}$ the set of step functions on $[0,T]$. Let $\mathcal{H}$ be the Hilbert space defined as the closure of $\mathcal{E}$ with respect to the scalar product

$$\langle \chi_{[0,t]}, \chi_{[0,s]} \rangle_{\mathcal{H}} = R_{H_j}(t,s),$$

and consider the linear operator $K^*_{H_j}$ from $\mathcal{E}$ to $L^2([0,T])$ defined by,

$$(K^*_{H_j}\phi^j)(t) = \int_s^T \phi^j(t)\frac{\partial K_{H_j}}{\partial t}(t,s)dt.$$

Notice that,

$$(K^*_{H_j}\chi_{[0,t]})(s) = K_{H_j}(t,s)\chi_{[0,t]}(s).$$

The operator $K^*_{H_j}$ is an isometry between $\mathcal{E}$ and $L^2([0,T])$ which can be extended to the Hilbert space $\mathcal{H}$. In fact, for any $s,t \in [0,T]$ we have

$$\langle K^*_{H_j}\chi_{[0,t]}, K^*_{H_j}\chi_{[0,t]}\rangle_{L^2([0,T])} = \langle \chi_{[0,t]}, \chi_{[0,s]}\rangle_{\mathcal{H}} = R_{H_j}(t,s).$$

In addition, for any $\phi^j \in \mathcal{H}$,

$$\int_0^T \phi^j(s)dB^{H_j}(s) = \int_0^T (K^*_{H_j}\phi^j)(s)dW(s),$$

if and only if $K^*_{H_j}\phi \in L^2([0,T])$. Next we are interested in considering an fBm with values in a Hilbert space and giving the definition of the corresponding stochastic integral.

Definition 2.2. An $\mathcal{F}_t$-adapted process ϕ^j on $[0,T]\times\Omega \to X$ is an elementary or simple process if for a partition $\psi = \{\bar{t}_0 = 0 < \bar{t}_1 < \ldots < \bar{t}_n = T\}$ and $(\mathcal{F}_{\bar{t}_i})$-measurable X-valued random variables $(\phi^j_{\bar{t}_i})_{1\leq i\leq n}$, ϕ_t satisfies

$$\phi^j_t(\omega) = \sum_{i=1}^n \phi^j_i(\omega)\chi_{(\bar{t}_{i-1},\bar{t}_i]}(t), \quad \text{for } \ 0\leq t\leq T, \quad \omega\in\Omega.$$

The Itô integral of the simple process ϕ^j is defined as

$$I_{H_j}(\phi^j) = \int_0^T \phi^j(s)dB^{H_j}(s) = \sum_{i=1}^n \phi^j(\bar{t}_i)(B_l^{H_j}(\bar{t}_i) - B_l^{H_j}(\bar{t}_{i-1})), \tag{2.2}$$

whenever $\phi^j_{\bar{t}_i} \in L^2(\Omega, \mathcal{F}_{\bar{t}_i}, \mathbb{P}, X)$ for all $i \leq n$.

Let $(X, \langle\cdot,\cdot\rangle, |\cdot|_X)$, $(Y, \langle\cdot,\cdot\rangle, |\cdot|_Y)$ be separable Hilbert spaces. Let $\mathcal{L}(Y,X)$ denote the space of all linear bounded operators from Y into X. Let $e_n, n = 1,2,\ldots$ be a complete orthonormal basis in Y and $Q_{H_j} \in \mathcal{L}(Y,X)$ be an operator defined by $Q_{H_j}e_n = \lambda^j_n e_n$ with finite trace $trQ_{H_j} = \sum_{n=1}^\infty \lambda^j_n < \infty$ where λ^j_n, $n = 1,2,\ldots$, are non-negative real numbers. Let $(\beta_n^{H_j})_{n\in N}$ be a sequence of two-sided one-dimensional standard fractional Brownian motions mutually independent on $(\Omega,\mathcal{F},\mathbb{P})$. If we define the infinite dimensional fBm on Y with covariance Q_{H_j} as

$$B^{H_j}(t) = \sum_{n=1}^\infty \sqrt{\lambda_n}\beta_n^{H_j}(t)e_n, \tag{2.3}$$

then it is well defined as an Y-valued Q_{H_j}-cylindrical fractional Brownian motion (see [?]) and we have

$$\mathbb{E}\langle\beta_l^{H_j}(t), x\rangle\langle\beta_k^H(s), y\rangle = R_{H_{lk}}(t,s)\langle Q_{H_j}(x), y\rangle, \quad x,y\in Y \quad and\, s,t\in[0,T]$$

such that

$$R_{H^j_{lk}} = \frac{1}{2}\{|\, t\,|^{2H_j} + |\, s\,|^{2H_j} + |\, t-s\,|^{2H_j}\}\delta_{lk} \quad t,s\in[0,T],$$

where

$$\delta_{lj} = \begin{cases} 1 & k = l, \\ 0, & k \neq l. \end{cases}$$

In order to define Wiener integrals with respect to a $Q_{H_j} - fBm$, we introduce the space $L^0_{Q_{Hj}} := L^0_{Q_{Hj}}(Y, X)$ of all Q_{H_j}−Hilbert-Schmidt operators $\varphi^j : Y \longrightarrow X$. We recall that $\varphi^j \in L(Y, X)$ is called a Q_{H^j}−Hilbert-Schmidt operator, if

$$\|\varphi^j\|^2_{L^0_{Q_{H_j}}} = \|\varphi Q^{1/2}_{H_j}\|^2_{HS} = tr(\varphi_j Q \varphi_j^*) < \infty.$$

Definition 2.3. Let $\phi^j(s), s \in [0, T]$, be a function with values in $L^0_{Q_{Hj}}(Y, X)$. The Wiener integral of ϕ^j with respect to fBm given by (2.3) is defined by

$$\int_0^T \phi^j(s) dB^{H_j}(s) = \sum_{n=1}^{\infty} \int_0^t \sqrt{\lambda_n} \phi^j(s) e_n d\beta_n^{H_j}$$
$$= \sum_{n=1}^{\infty} \int_0^T \sqrt{\lambda_n} K^*_{H_j}(\phi^j e_n)(s) d\beta_n. \tag{2.4}$$

Notice that if

$$\sum_{n=1}^{\infty} \|\phi Q^{1/2} e_n\|_{L^{1/H_j}([0,T];X)} < \infty, \tag{2.5}$$

the next result ensures the convergence of the series in the previous definition. It can be proved by similar arguments to those used to prove Lemma 2.4 in Caraballo *et al.* [?].

Lemma 2.4. *For any* $\phi^j : [0, T] \to L^0_{Q_{Hj}}(Y, X)$ *such that* (2.5) *holds, and for any* $\alpha, \beta \in [0, T]$ *with* $\alpha > \beta$, *for each* $j = 1, 2$

$$\mathbb{E}\left|\int_\alpha^\beta \phi^j(s) dB^{H_j}(s)\right|_X^2 \leq c_2(H_j) H_j (2H_j - 1)(\alpha - \beta)^{2H_j - 1} \sum_{n=1}^{\infty} \int_\alpha^\beta \left|\phi^j(s) Q^{1/2} e_n\right|_X^2 ds. \tag{2.6}$$

where $c_2(H_j)$ *is a constant depending on* H_j. *If, in addition,*

$$\sum_{n=1}^{\infty} |\phi^j Q^{1/2} e_n|_X \text{ is uniformly convergent for } t \in [0, T],$$

then,

$$\mathbb{E}\left|\int_\alpha^\beta \phi^j(s) dB^{H_j}(s)\right|_X^2 \leq c_2(H_j) H_j (2H_j - 1)(\alpha - \beta)^{2H_j - 1} \int_\alpha^\beta \left\|\phi^j(s)\right\|^2_{L^0_{Q_{Hj}}} ds. \tag{2.7}$$

3. Nonsmooth analysis

Let $x, y \in X$; the projection of x,y into $C_j \subset X$ is the set

$$Proj(y, C_j) = \{z \in C_j \ : d(z, C_j) = ||z - y||\}.$$

This set is nonempty if, for example, C_j is weakly closed.Let C_j be a closed subset of space X;and let $x, y \in C_i$: We say that a vector $v \in X$ is a proximal normal to C_j at z if $v = y - z$ for some $y \in X$ with $z \in Proj(y, C_j)$. We denote by $N^p(z, C_j)$.

the normal cone. One can show that $\eta \in N^p(y, C_j)$ if and only if there exists M such that the following proximal normal inequality holds,

$$\langle \eta, z - y \rangle \leq M||z - y||,$$

for all $z \in C_j$. (In general, M will depend on x). On the other hand

$$N^p(z, C_j) = \bigcup_{n=1}^{\infty} \left\{ v \in X \ : \ d(y + \frac{v}{n}) = \frac{||v||}{n} \right\}.$$

This cone is convex, but in general not closed . An useful characterization of the proximal normal cone is the following (see,e.g., [?], Proposition 1.1.5(a)):

$$N^p(z, C_j) = \cup_{\mu > 0}\{v \in X \ : \ \langle v, a - z\rangle \leq \mu||z - y||^2, \, a \in C_j\}.$$

If C_j is closed and convex then we have

$$z \in N^p(z, C_j) \Longleftrightarrow y \in C_j \text{ and } \langle z, y\rangle = \sigma(z, C_i) \Longleftrightarrow y \in C_j, x \in \partial\varphi_{C_j}(y)$$

where σ is the support function of a subset C_j of X, $\partial\varphi_{C_j}$ is the subdifferential in the sense of convex analysis and C_i is the indicator function of a subset C_j of X

$$\partial\varphi_{C_j}(y) = \begin{cases} 0, & \text{if} \quad y \in C_j, \\ \emptyset, & \text{if} \, y \in C_j. \end{cases}$$

We define the Bouligand cone by

$$T_{C_j}(x) = \left\{ v \in X \ : \ \lim_{h \to 0} \inf \frac{d(z + hv, C_j)}{h} \right\} = \bigcap_{\epsilon > 0} \bigcap_{\delta > 0} \bigcup_{0 < h < \delta} \Big(\frac{C_j - z}{h} + \epsilon \overline{B}(0, 1) \Big).$$

For more informations about nonsmooth analysis we see the monographs of Clarke and Ledyaev et al [?] and Clarke [?].

3.1. Multi-valued analysis

$$\mathcal{P}_{cl}(X) = \{y \in \mathcal{P}(X) : y \text{ closed } \},$$
$$\mathcal{P}_{b}(X) = \{y \in \mathcal{P}(X) : y \text{ bounded } \},$$
$$\mathcal{P}_{c}(X) = \{y \in \mathcal{P}(X) : y \text{ convex } \},$$
$$\mathcal{P}_{cp}(X) = \{y \in \mathcal{P}(X) : y \text{ compact } \}.$$

Consider $H_d : \mathcal{P}(X) \times \mathcal{P}(X) \longrightarrow \mathbb{R}^n_+ \cup \{\infty\}$ defined by

$$H_d(A, B) \quad := \quad \begin{pmatrix} H_{d_1}(A, B) \\ \dots \\ H_{d_n}(A, B) \end{pmatrix}$$

Let (X, d) be a generalized metric space with

$$d(x, y) \quad := \quad \begin{pmatrix} d_1(x, y) \\ \dots \\ d_n(x, y) \end{pmatrix}$$

Notice that d is a generalized metric space on X if and only if d_i, $i = 1,..,n$ are metrics on X,

$$H_d(A,B) = \max\left\{\sup_{a\in A} d(a,B), \sup_{b\in B} d(A,b)\right\},$$

where $d(A,b) = \inf_{a\in A} d(a,b), d(a,B) = \inf_{b\in B} d(a,b)$. Then, $(\mathcal{P}_{b,cl}(X), H_d)$ is a metric space and $(\mathcal{P}_{cl}(X), H_d)$ is a generalized metric space.

A multivalued map $F : X \longrightarrow \mathcal{P}(X)$ is convex (closed) valued if $F(y)$ is convex (closed) for all $y \in X$, F is bounded on bounded sets if $F(B) = \bigcup_{y\in B} F(y)$ is bounded in X for all $B \in \mathcal{P}_b(X)$. F is called upper semi-continuous (u.s.c. for short) on X if for each $y_0 \in X$ the set $F(y_0)$ is a nonempty, closed subset of X, and for each open set $\mathcal{U}$ of X containing $F(y_0)$, there exists an open neighborhood $\mathcal{V}$ of y_0 such that $F(\mathcal{V}) \in \mathcal{U}$. F is said to be completely continuous if $F(B)$ is relatively compact for every $B \in \mathcal{P}_b(X)$.

If the multivalued map F is completely continuous with nonempty compact valued, then F is u.s.c. if and only if F has a closed graph, i.e., $x_n \longrightarrow x_*$, $y_n \longrightarrow y_*$, $y_n \in F(x_n)$ imply $y_* \in F(x_*)$.

A multi-valued map $F : J \longrightarrow \mathcal{P}_{cp,c}$ is said to be measurable if for each $y \in X$, the mean-square distance between y and $F(t)$ is measurable.

Definition 3.1. The set-valued map $F : J \times X \times X \to \mathcal{P}(X \times X)$ is said to be L^2-Carathéodory if

(i). $t \mapsto F(t,v)$ is measurable for each $v \in X \times X$;
(ii). $v \mapsto F(t,v)$ is u.s.c. for almost all $t \in J$;
(iii). for each $q > 0$, there exists $h_q \in L^1(J,\mathbb{R}^+)$ such that

$$\|F(t,v)\|^2 := \sup_{f\in F(t,v)} \|f\|^2 \le h_q(t), \text{for all } \|v\|^2 \le q \text{ and for a.e. } t \in J.$$

We denote the graph of G to be the set $gr(G) = \{(x,y) \in X \times Y, \quad y \in G(x)\}$.

Lemma 3.2. [?] *If $G : X \to P_{cl}(Y)$ is u.s.c., then $gr(G)$ is a closed subset of $X \times Y$. Conversely, if G is locally compact and has nonempty compact values and a closed graph, then it is u.s.c.*

Lemma 3.3. [?] *If $G : X \to P_{cp}(Y)$ is quasicompact and has a closed graph, then G is u.s.c.*

Definition 3.4. A set-valued operator $G : J \longrightarrow \mathcal{P}_{cl}(X)$ is said to be a contraction if there exists $0 \le \gamma < 1$ such that

$$H_d(G(x), G(y)) \le \gamma d(x,y), \text{ for all } x,y \in X,$$

The following two results are easily deduced from the limit properties.

Lemma 3.5. (See e.g. [?], Theorem 1.4.13) *If $G : X \to \mathcal{P}_{cp}(X)$ is u.s.c., then for any $x_0 \in X$,*

$$\limsup_{x\to x_0} G(x) = G(x_0).$$

Lemma 3.6. (See e.g. [?], Lemma 1.1.9) *If Let $(K_n)_{n\in N} \subset K \subset X$ be a sequence of subsets where K is compact in the separable Banach space X. Then*

$$\overline{co}(\limsup_{n\to\infty} K_n) = \cap_{N>0}\overline{co}(\cup_{n\geq N} K_n)$$

where $\overline{co}A$ refers to the closure of the convex hull of A.

The second one is due to Mazur, 1933:

Lemma 3.7. (Mazur's Lemma, ([?] [Theorem 21.4])) *Let X be a normed space and $\{x_k\}_{k\in N} \subset X$ be a sequence weakly converging to a limit $x \in X$. Then there exists a sequence of convex combinations $y_m = \sum_{k=1}^{m} \alpha_{mk} x_k$ with $\alpha_{mk} > 0$ for $k = 1, 2, .., m$ and $\sum_{k=1}^{m} \alpha_{mk} = 1$, which converges strongly to x.*

Lemma 3.8. [?] $C : [0, T] \to \mathcal{P}_{cl}(X)$ *such that*

(i). *C is Hausdorff lower semicontinuous at $t = 0$;*
(ii). *∂C is Hausdorff upper semicontinuous at $t = 0$;*
(iii). *there exist $x \in X$ and $r_0 > 0$ such that $B(x, r_0) \subseteq C(0)$*

Then for every $r \in (0, r_0)$ there exists $\delta > 0$ such that $B(x, r) \subset C(r)$ for all $t \in [0, \delta]$.

4. Statement of the main results

Definition 4.1. A function $x, y \in M^2([-r, T], X)$, is said to be a solution of (1.3) if x, y satisfies the equation

$$\begin{cases} dx(t) & \in N^p(x(t), C_1(t))dt + G^1(t, x_t, y_t)dB^{H_1} \;\; a, e.\, t \in [0, T] \\ dy(t) & \in N^p(y(t), C_2(t))dt + G^2(t, x_t, y_t)dB^{H_2} \;\; a, e.\, t \in [0, T] \end{cases}$$

and the conditions $(x(t), y(t)) \in (C_1(t), C_2(t))$, for all $t \in [0, T]$.

First, we will list the following hypotheses which will be imposed in our main theorem. In this section,

(H_1) $C_j(t)$ is convex for every $t \in [0, T]$ and there exists $\lambda > 0$ such that

$$H_{d_j}(C_j(t), C_j(s)) \leq \lambda|t - s|,$$

for all $t, s \in [0, T]$,

(H_2) there exists a positive constant α_j, β_j for each $j = 1, 2$ such that

$$\mathbb{E}|G^j(t, x, y) - G^j(t, \overline{x}, \overline{y})| \leq \alpha_j||x - \overline{x}||_{M^2_{\mathcal{F}_0}} + \beta_j||y - \overline{y}||_{M^2_{\mathcal{F}_0}},$$

for all $t \in [0, T]$ and $x, y, \overline{x}, \overline{y} \in M^2([-r, 0], X)$

Theorem 4.2. *Assume that (H_1) and (H_2) hold. Then, problem (1.3) possesses a unique solution on $[0, T]$.*

Proof. The existence part. Therefore, we pass immediately to uniqueness. We shall obtain the solution by a well-establish discretization procedure.
The following discretization scheme lies at the heart of many proofs for sweeping processes. Consider for every $n \in \mathbb{N}$, the following partition of $[0,T]$,

$$t_{n,i} := \frac{iT}{2^n},\ 0 \le i \le 2^n \text{ and } I_{n,i} = (t_{n,i}, t_{n,i+1}],\ \text{if } 0 \le i \le 2^n - 1,\ n \ge 0.$$

$$x_{n,0} = \begin{cases} \phi(t), & t \in [-r, 0], \\ \phi(0), & t \in [0, t_{n,0}], \end{cases}$$

for any $I_{n,0} = (t_{n,0}, t_{n,1}]$, we have

$$x_{n,1} = \begin{cases} x_{n,0}(t),\ t \in [-r, t_{n,0}], \\ proj\Big(\phi(0) + G^1(t_{n,0}, x(_{n,0})_{t_{n,0}}, y(_{n,0})_{t_{n,0}})(B^{H_1}(t_{n,1}) - B^{H_1}(t_{n,0}), C_1(t_{n,1})\Big), \\ t \in [t_{n,0}, t_{n,1}] \end{cases}$$

for any $I_{n,1} = (t_{n,1}, t_{n,2}]$, we have

$$x_{n,2} = \begin{cases} x_{n,1}(t), t \in [-r, t_{n,1}], \\ proj\Big(x_{n,1}(t_{n,1}) + G^1(t_{n,1}, x(_{n,1})_{t_{n,1}}, y(_{n,1})_{t_{n,1}})(B^{H_1}(t_{n,2}) \\ -B^{H_1}(t_{n,1}), C_1(t_{n,2})\Big), \\ t \in [t_{n,1}, t_{n,2}]. \end{cases}$$

With the same argument we can define recursively

$$x_{n,i+1} = \begin{cases} x_{n,i}(t), t \in [-r, t_{n,i}], \\ proj\Big(x_{n,i}(t_{n,i}) \\ +G^1(t_{n,i}, x(_{n,i})_{t_{n,1}}, y(_{n,i})_{t_{n,1}})(B^{H_1}(t_{n,i+1}) \\ -B^{H_1}(t_{n,i}), C_1(t_{n,i+1})\Big),\ t \in [t_{n,i}, t_{n,i+1}]. \end{cases}$$

Estimate (x_n, y_n) by norm $M^2([-r,T], X) \times M^2([-r,T], X)$, since (x_n, y_n) is piecewise affine, by direct calculations,

$$\sup\{\sqrt{E|x_{n,i+1}(t) - x_{n,i}(t)|^2} \quad : \ t \in [-r, T]\} \le \lambda \frac{T}{2^n}. \tag{4.1}$$

Observe that $(x_{n,i}(t), y_{n,i}(t)) \in (C_1(t_{n,i}), C_2(t_{n,i}))$,and

$$\mathbb{E}|x_{n,i+1}(t) - x_{n,i}(t)| \le \mathbb{E}H_{d_1}(C_1(t_{n,i}), C_1(t_{n,i+1})) \le \lambda \frac{T}{2^n} \tag{4.2}$$

and

$$\mathbb{E}|y_{n,i+1}(t) - y_{n,i}(t)| \le \mathbb{E}H_{d_2}(C_2(t_{n,i}), C_2(t_{n,i+1})) \le \lambda \frac{T}{2^n}, \tag{4.3}$$

for all $t \in (t_{n,i-1}, t_{n,i}]$, for every $0 \le i \le 2^n$.

By affine interpolation we define a corresponding sequence of approximate solutions $x_n, y_n \in M^2([-r,T],X)$; for $t \in I_{n,i}$ the explicit formula is

$$x_n(t) = \begin{cases} x_{n,i}(t), & t \in [-r, t_{n,i}] \\ x_{n,i}(t_{n,i}) + \frac{t-t_{n,i}}{\epsilon_n}(x_{n,i+1}(t) - x_{n,i}(t)) & \\ +G^1(t_{n,i}, x(_{n,i})_{t_{n,i}})(B^{H_1}(t) - B^{H_1}(t_{n,1})), & t \in [t_{n,i}, t_{n,i+1}] \end{cases}$$

and

$$y_n(t) = \begin{cases} y_{n,i}(t), & t \in [-r, t_{n,i}] \\ y_{n,i}(t_{n,i}) + \frac{t-t_{n,i}}{\epsilon_n}(y_{n,i+1}(t) - y_{n,i}(t)) & \\ +G^2(t_{n,i}, x(_{n,i})_{t_{n,i}}, y(_{n,i})_{t_{n,i}})(B^{H_2}(t) - B^{H_2}(t_{n,1})), & t \in [t_{n,i}, t_{n,i+1}] \end{cases}$$

where $\epsilon_n = \frac{T}{2^n}$ and for every $0 \leq i \leq 2^n - 1$.

From the definition of normal proximal cone, we have

$$\begin{aligned} dx_n(t) \in &-N(x_{n,i+1}, C_1(t_{n,i+1}))dt \\ &+ G^1(t_{n,i}, x(_{n,i})_{t_{n,i}}, y(_{n,i})_{t_{n,i}})(B^{H_1}(t) - B^{H_1}(t_{n,1})). \end{aligned} \tag{4.4}$$

and

$$\begin{aligned} dy_n(t) \in &-N(y_{n,i+1}, C_2(t_{n,i+1}))dt \\ &+ G^2(t_{n,i}, x(_{n,i})_{t_{n,i}}, y(_{n,i})_{t_{n,i}})(B^{H_2}(t) - B^{H_2}(t_{n,1})). \end{aligned} \tag{4.5}$$

Now we prove that $\{x_n, y_n,\ n \in \mathbb{N}\}$ is compact in $M^2([-r,T],X)$, for each $z_n = (x_n, y_n)$ in $M^2([-r,T],X) \times M^2([-r,T],X)$.

Step 1. $\{(x_n, y_n)\ n \in \mathbb{N}\}$ are bounded sets in $M^2([-r,T],X) \times M^2([-r,T],X)$.

We obtain

$$\begin{aligned} |x_n(t)| &\leq |x_{n,i}(t)| + |x_{n,i+1}(t) - x_{n,i}(t)| \\ &+b|G^1(t_{n,i}, x(_{n,i})_{t_{n,i}}, y(_{n,i})_{t_{n,i}})||(B^{H_1}(t) - B^{H_1}(t_{n,1}))| \\ &\leq |x_{n,0}(t)| + \sum_{k=1}^{i+1} |x_{n,k-1}(t) - x_{n,k}(t)| \\ &+T|G^1(t_{n,i}, x(_{n,i}, y(_{n,i})_{t_{n,i}}, y(_{n,i})_{t_{n,i}})||(B^{H_1}(t) - B^{H_1}(t_{n,1}))| \\ &\leq ||\phi|| + 2T + T\Big(|G^1(t_{n,i}, x(_{n,i})_{t_{n,i}}, y(_{n,i})_{t_{n,i}}) \\ &-G^1(t_{n,i}, 0, 0)| + |G^1(t_{n,i}, 0, 0)|\Big)|(B^{H_1}(t) - B^{H_1}(t_{n,1}))| \\ &\leq ||\phi|| + 2T + T\Big(\alpha_1||(x_{n,i})_{t_{n,i}}||_{M^2_{\mathcal{F}_0}} \\ &+\beta_1||(y_{n,i})_{t_{n,i}}||_{M^2_{\mathcal{F}_0}} + |G^1(t_{n,i}, 0, 0)|\Big)|(B^{H_1}(t) - B^{H_1}(t_{n,1}))|. \end{aligned}$$

By definition $(x_{n,i}, y_{n,i})$ we can prove that there exist $M, \overline{M} > 0$ such that

$$\sup\{\mathbb{E}|x_{n,i}(t)|\ :\ t \in [-r,T]\} \leq M$$

and

$$\sup\{\mathbb{E}|y_{n,i}(t)| \ : \ t \in [-r,T]\} \leq \overline{M}.$$

Hence, by using (4.2) and (4.3), we have

$$\begin{aligned}
\mathbb{E}|x_n(t)|^2 \leq & 2\mathbb{E}||\phi||^2 + 4T^2 + 2T^2\Big(\alpha_1 E||(x_{n,i})_{t_{n,i}}||^2 + \beta_1 E||(y_{n,i})_{t_{n,i}}||^2 \\
& + \sup_{t\in[0,b]} |G^1(t,0,0)|^2\Big)\mathbb{E}|(B^{H_1}(t) - B^{H_1}(t_{n,1}))|^2 \\
\leq & 2\mathbb{E}||\phi||^2 + 4T^2 + 2T^2\Big(\alpha_1 \mathbb{E}||(x_{n,i})_{t_{n,i}}||^2 + \beta_1 E||(y_{n,i})_{t_{n,i}}||^2 \\
& + \sup_{t\in[0,T]} |G^1(t,0,0)|^2\Big)|t - t_{n,1}|^{2H_1} \\
\leq & 2\mathbb{E}||\phi||^2 + 4T^2 + 2T^2\Big(\alpha_1 M + \beta_1 \overline{M} + \sup_{t\in[0,T]} |G^1(t,0,0)|^2\Big)|t - t_{n,1}|^{2H_1} \\
\leq & 2\mathbb{E}||\phi||^2 + 4T^2 + 2T^2\Big(\alpha_1 M + \beta_1 \overline{M} + \sup_{t\in[0,T]} |G^1(t,0,0)|^2\Big)T^{2H_1} = l_1.
\end{aligned}$$

Similarly, we have

$$\mathbb{E}|y_n(t)|^2 \leq 2\mathbb{E}||\overline{\phi}||^2 + 4T^2 + 2T^2\Big(\alpha_2 \overline{M} + \beta_2 \overline{M} + \sup_{t\in[0,T]} |G^2(t,0,0)|^2\Big)T^{2H_2} = l_2.$$

which implies that

$$\begin{pmatrix} \mathbb{E}|x_n(t)|^2 \\ \mathbb{E}|y_n(t)|^2 \end{pmatrix} \leq \begin{pmatrix} l_1 \\ l_2 \end{pmatrix}$$

Step 2. $\{(x_n, y_n)\ n \in \mathbb{N}\}$ are equicontinuous sets in $M^2([-r,T],X) \times M^2([-r,T],X)$. Let $\tau_1, \tau_2 \in [t_{n,i}, t_{n,i+1}], \tau_1 < \tau_2$. Thus

$$\begin{aligned}
& \mathbb{E}|x_n(\tau_2) - x_n(\tau_1)|^2 \\
= \ & \mathbb{E}\Big|\frac{\tau_2 - \tau_1}{\epsilon_n}(x_{n,i+1} - x_{n,i}) + G^1(t_{n,i}, x(_{n,i})_{t_{n,i}}, y(_{n,i})_{t_{n,i}})(B^{H_1}(\tau_2) - B^{H_1}(\tau_1))\Big|^2 \\
\leq \ & 2|\tau_2 - \tau_1|^2 + 2\Big(\alpha_1 M + \beta_1 \overline{M} + \sup_{t\in[0,T]} |G^1(t,0,0)|^2\Big)|\tau_2 - \tau_1|^{2H_1}.
\end{aligned}$$

Similarly

$$\begin{aligned}
\mathbb{E}|y_n(\tau_2) - y_n(\tau_1)|^2 \leq & 2|\tau_2 - \tau_1|^2 \\
& + 2\Big(\alpha_2 M + \beta_2 \overline{M} + \sup_{t\in[0,T]} |G^2(t,0,0)|^2\Big)|\tau_2 - \tau_1|^{2H_2}.
\end{aligned}$$

The right-hand side tends to zero as $\tau_2 - \tau_1 \to 0$, and ϵ sufficiently small. From Steps 1, 2. By the Arzela-Ascoli theorem, we conclude that there is a subsequence of (x_n, y_n), again denoted (x_n, y_n) which converges to $(x, y) \in M^2([-r,T],X)$.
Now, we prove that $(x(t), y(t)) \in (C_1(t), C_2(t))$. Let $\rho_n(t)$,$\mu_n(t)$ be two functions from $[0,T]$ into $[0,T]$ defined by

$$\begin{aligned}
\rho_n(t) = t_{n,i}, \qquad & \text{if} \quad t \in [t_{n,i}, t_{n,i+1}), \qquad \rho_n(0) = 0 \\
\mu_n(t) = t_{n,i+1} \qquad & \text{if} \quad t \in [t_{n,i}, t_{n,i+1}), \qquad \mu_n(0) = 0,
\end{aligned}$$

for all $t \in [0,T]$. From (4.4) and (4.5) we have

$$dx_n(t) \in -N(x_n(\mu_n(t)), C_1(\mu_n(t)))dt \\ +G^1(t_{\rho_n(t)}, x_{\rho_n(t)}, y_{\rho_n(t)})dB^{H_1}(\rho_n(t)), \text{ a.e. } t \in [0,T] \tag{4.6}$$

and

$$dy_n(t) \in -N(x_n(\mu_n(t)), C_2(\mu_n(t)))dt \\ +G^2(t_{\rho_n(t)}, x_{\rho_n(t)}, y_{\rho_n(t)})dB^{H_2}(\rho_n(t)), \text{ a.e. } t \in [0,T]. \tag{4.7}$$

Moreover, for all n large enough,we have

$$\rho_n(t) \to t, \qquad \mu_n(t) \to t \qquad \text{uniformly on} \quad [0,b]$$

Since $|\rho_n(t) - t| \leq \frac{T}{2^n}$ and $|\mu_n(t) - t| \leq \frac{T}{2^n}$. Thus

$$|y_n(\rho_n(t)) - y_n(t)| \leq H_{d_1}(C_1(\rho_n(t)), C_1(t)) \leq \lambda|\rho_n(t) - t|,$$

which immediately yields

$$\sup\{\sqrt{\mathbb{E}|y_n(\rho_n(t)) - y_n(t)|^2} \ : \ t \in [0,T]\} \leq \lambda\sqrt{E|\rho_n(t) - t|^2} \to 0 \,\text{as}\, n \to \infty.$$

Let $t \in [0,T]$.From (4.1) for each $n \in \mathbb{N}$,$t_{n,i} \in I_{n,i}$ for some i,

$$\begin{aligned} |x_n(t) - C_1(t)| &\leq |x_n(t) - x_n(t_{n,i})| + d(x_n(t_{n,i}), C_1(t)) \\ &\leq \lambda\frac{T}{2^n} + H_{d_1}(C_1(t_{n,i}), C_1(t)). \end{aligned}$$

Thus

$$|x_n(t) - C_1(t)| \leq \lambda\frac{T}{2^{n-1}}. \tag{4.8}$$

Since (x_n, y_n) is defined by linear interpolation, we obtain

$$|x_n'(t)| \leq \frac{1}{\epsilon_n} \sup_i |x_{n,i+1}(t) - x_{n,i}(t)|,$$

and

$$|y_n'(t)| \leq \frac{1}{\epsilon_n} \sup_i |y_{n,i+1}(t) - y_{n,i}(t)|.$$

By letting $n \to \infty$ in(4.8) for all $t \in [0,T]$,we obtain that

$$(x(t), y(t)) \in (C_1, C_2).$$

Now, we prove that the sequences of composition mappings $(x_n \circ \mu_n, y \circ \mu_n)$ and $(x_n \circ \rho_n, y \circ \rho_n)$ converge uniforms to (x_t, y_t) in $M^2([-r,0], X)$

$$\begin{aligned} \mathbb{E}|x_n(\rho_n(t) + \tau) - x(t+\tau)|^2 &\leq 3\mathbb{E}|x_n(\rho_n(t) + \tau) - x_n(t+\tau)|^2 \\ &+ 3\mathbb{E}|x_n(\rho_n(t) + \tau) - x_n(\mu_n(t) + \tau)|^2 \\ &+ 3\mathbb{E}|x_n(\mu_n(t) + \tau) - x_n(t+\tau)|^2. \end{aligned}$$

Thus

$$\begin{aligned} \sup_{\tau \in [-r,0]} \mathbb{E}|(x_n)_{\rho_n(t)} - x_t|^2 &\leq 3\lambda^2\mathbb{E}|\rho_n(t) - t|^2 + 3\mathbb{E}|\rho_n(t) - \mu_n(t)|^2 \\ &+ 3 \sup_{\tau \in [-r,T]} \mathbb{E}|x_n(\mu_n(t)) - x(t)|^2 \to 0 \quad \text{as} \quad n \to \infty. \end{aligned}$$

Since $|(\rho_n(t)-\tau)-(t-\tau)| \leq \frac{T}{2^n}$ and $|\mu_n(t)-\rho_n(t)| \leq \frac{T}{2^{n-1}}$. We can pass to the limit when $n \to \infty$, we deduce from

$$(x_{\rho_n(t)}, y_{\rho_n(t)}) \to (x_t, y_t) \in M^2([-r,0], X)$$

and,the fact that $G^i(.,.,.)$ is a continuous function then we have

$$G^i(\rho_n(t), x_{\rho_n(t)}, y_{\rho_n(t)}) \to G^i(t, x_t, y_t).$$

Now, we show that

$$dx(t) \in -N(x(t), C_1(t))dt + G^1(t, x_t, y_t)dB^{H_1}(t), \text{ a.e. } t \in [0,T]. \tag{4.9}$$

and

$$dy(t) \in -N(y(t), C_2(t))dt + G^2(t, x_t, y_t)dB^{H_2}(t), \text{ a.e. } t \in [0,T]. \tag{4.10}$$

Since (x_n, y_n) is bounded in $X \times X$,there exists a subsequence of (x_n, y_n)converge to (x, y). Then

$$\begin{aligned}\int_0^T &\sigma\Big(-x_n'(t) + G^1(t,(x_n)_t,(y_n)_t)dB^{H_1}(t), C_1(\mu_n(t))\Big)dt \\ &\leq \int_0^T \Big(-x_n'(t) + G^1(t,(x_n)_t,(y_n)_t)dB^{H_1}(t), x(\mu_n(t))\Big)dt. \end{aligned}\tag{4.11}$$

Using the fact that $\sigma(., C_j(t))$ is lower semicontinuous [**?**],then

$$\begin{aligned}\liminf_{n\to\infty} \int_0^T &\sigma\Big(-x_n'(t) + G^1(t,(x_n)_t,(y_n)_t)dB^{H_1}(t), C_1(\mu_n(t))\Big)dt \\ &\geq \int_0^T \Big(-x'(t) + G^1(t,x_t,y_t)dB^{H_1}(t), C_1(t)\Big)dt. \end{aligned}\tag{4.12}$$

By (5.16) and (5.18),we obtain

$$\begin{aligned}\int_0^T &\Big(-x'(t) + G^1(t,x_t,y_t)dB^{H_1}(t), C_1(t)\Big)dt \\ &\geq \int_0^T \sigma\Big(-x'(t) + G^1(t,x_t,y_t)dB^{H_1}(t), C_1(t)\Big)dt. \end{aligned}\tag{4.13}$$

Thus,

$$dx(t) \in -N(x(t), C_1(t))dt + G^1(t, x_t, y_t)dB^{H_1}(t), \text{ a.e. } t \in [0,T].$$

and

$$dy(t) \in -N(y(t), C_2(t))dt + G^2(t, x_t, y_t)dB^{H_2}(t), \text{ a.e. } t \in [0,T].$$

Finally, we prove the uniqueness of solutions of the problem (1.3).Let us assume that (x, y) and $(\overline{x}, \overline{y})$ are two solutions of (1.3).

$$d\overline{x}(t) \in -N(\overline{x}(t), C_1(t))dt + G^1(t, \overline{x}_t, \overline{y}_t)dB^{H_1}(t), \text{ a.e. } t \in [0,T],$$

and

$$d\overline{y}(t) \in -N(\overline{y}(t), C_2(t))dt + G^2(t, \overline{x}_t, \overline{y}_t)dB^{H_2}(t), \text{ a.e. } t \in [0,T].$$

Since $C(t) = (C_1(t), C_2(t))$ is a convex set, then

$$T_{C_j}(z) = \cup_{h>0}\overline{\frac{C_j(t)-z}{h}},$$

for all $t \in [0,T]$,

$$T_{C_j}(z) \subset \{v \in X : \langle v, \xi \rangle \leq 0 \text{ for all } \quad \xi \in N^p(z,\xi)\},$$

which immediately yields

$$\Big\langle x'(t) - \overline{x}'(t) + \Big(G^1(t,x_t,y_t) - G^1(t,\overline{x}_t,\overline{y}_t)\Big)dB^{H_1}(t), x(t) - \overline{x}(t)\Big\rangle \leq 0.$$

Thus, we deduce

$$\Big\langle x'(t) - \overline{x}'(t), x(t) - \overline{x}(t)\Big\rangle + \Big\langle \Big(G^1(t,x_t,y_t) - G^1(t,\overline{x}_t,\overline{y}_t)\Big)dB^{H_1}(t), x(t) - \overline{x}(t)\Big\rangle \leq 0.$$

By assumptions (H_1), (H_2) imply

$$\begin{aligned} \frac{1}{2}.\frac{d}{dt}\Big|x(t) - \overline{x}(t)\Big|^2 &\leq \alpha_1 ||x_t - \overline{x}_t||_{M^2_{\mathcal{F}_0}}\Big|x(t) - \overline{x}(t)\Big|dB^{H_1}(t) \\ &+\beta_1 ||y_t - \overline{y}_t||_{M^2_{\mathcal{F}_0}}\Big|x(t) - \overline{x}(t)\Big|dB^{H_1}(t) \end{aligned} \tag{4.14}$$

and

$$\begin{aligned} \frac{1}{2}.\frac{d}{dt}\Big|y(t) - \overline{y}(t)\Big|^2 &\leq \alpha_2 ||x_t - \overline{x}_t||_{M^2_{\mathcal{F}_0}}\Big|y(t) - \overline{y}(t)\Big|dB^{H_1}(t) \\ &+\beta_2 ||y_t - \overline{y}_t||_{M^2_{\mathcal{F}_0}}\Big|y(t) - \overline{y}(t)\Big|dB^{H_1}(t). \end{aligned} \tag{4.15}$$

Integrating (4.14)and (4.15) over $(0,t)$ we arrive at

$$\begin{aligned} \Big|x(t) - \overline{x}(t)\Big|^2 &\leq \alpha_1 \int_0^t ||x_s - \overline{x}_s||_{M^2_{\mathcal{F}_0}}\Big|x(s) - \overline{x}(s)\Big|dB^{H_1}(s) \\ &+ \beta_1 \int_0^t ||y_s - \overline{y}_s||_{M^2_{\mathcal{F}_0}}\Big|x(s) - \overline{x}(s)\Big|dB^{H_1}(s) \\ &\leq \alpha_1 \int_0^t \sup_{s\in[0,t]} \sqrt{E|x(s) - \overline{x}(s)|^2}\Big|x(s) - \overline{x}(s)\Big|dB^{H_1}(s) \\ &+ \beta_1 \int_0^t \sup_{s\in[0,t]} \sqrt{E|y(s) - \overline{y}(s)|^2}\Big|x(s) - \overline{x}(s)\Big|dB^{H_1}(s). \end{aligned}$$

Then, for each $t \in [0,T]$ and thanks to Lemma 2.4,

$$\begin{aligned} \mathbb{E}\Big|x(t) - \overline{x}(t)\Big|^4 &\leq 2\alpha_1 \mathbb{E}\Big|\int_0^t \sup_{s\in[0,t]} \sqrt{\mathbb{E}|x(s) - \overline{x}(s)|^2}\Big|x(s) - \overline{x}(s)\Big|dB^{H_1}(s)\Big|^2 \\ &+ 2\beta_1 \mathbb{E}\Big|\int_0^t \sup_{s\in[0,t]} \sqrt{\mathbb{E}|y(s) - \overline{y}(s)|^2}\Big|x(s) - \overline{x}(s)\Big|dB^{H_1}(s)\Big|^2 \\ &\leq 2c_2(H_1)H_1(2H_1-1)T^{2H_1-1}\alpha_1 \int_0^t \sup_{s\in[0,t]} E|x(s) - \overline{x}(s)|^4 ds \\ &+ 2c_2(H_1)H_1(2H_1-1)T^{2H_1-1}\beta_1 \\ &\quad \int_0^t \sup_{s\in[0,t]} \mathbb{E}|x(s) - \overline{x}(s)|^2 E|y(s) - \overline{y}(s)|^2 ds. \end{aligned}$$

Thus

$$\mathbb{E}\Big|x(t)-\overline{x}(t)\Big|^4 \leq A_1\int_0^t \sup_{s\in[0,t]}\mathbb{E}|x(s)-\overline{x}(s)|^4ds + B_1\int_0^t \sup_{s\in[0,t]}\mathbb{E}|y(s)-\overline{y}(s)|^4ds,$$

where

$$A_1 = 2c_2(H_1)H_1(2H_1-1)T^{2H_1-1}(2\alpha_1+\beta_1)$$

and

$$B_1 = c_2(H_1)H_1(2H_1-1)T^{2H_1-1}\beta_1.$$

In the same way, we also have

$$\begin{aligned}\mathbb{E}\Big|y(t)-\overline{y}(t)\Big|^4 &\leq 2c_2(H_2)H_2(2H_2-1)T^{2H_2-1}\alpha_2\int_0^t \sup_{s\in[0,t]}\mathbb{E}|y(s)-\overline{y}(s)|^4ds\\ &+ 2c_2(H_2)H_2(2H_2-1)T^{2H_2-1}\beta_2\\ &\int_0^t \sup_{s\in[0,t]} E|x(s)-\overline{x}(s)|^2\mathbb{E}|y(s)-\overline{y}(s)|^2ds,\end{aligned}$$

and, consequently,

$$\mathbb{E}\Big|y(t)-\overline{y}(t)\Big|^4 \leq A_2\int_0^t \sup_{s\in[0,t]}\mathbb{E}|y(s)-\overline{y}(s)|^4ds + B_2\int_0^t \sup_{s\in[0,t]}\mathbb{E}|x(s)-\overline{x}(s)|^4ds,$$

where

$$A_3 = c_2(H_2)H_2(2H_2-1)T^{2H_2-1}(2\alpha_2+\beta_2),$$

and

$$A_4 = c_2(H_2)H_2(2H_2-1)T^{2H_2-1}\beta_2.$$

Adding these we obtain

$$\begin{aligned}\mathbb{E}\Big|x(t)-\overline{x}(t)\Big|^4+\mathbb{E}\Big|y(t)-\overline{y}(t)\Big|^4 &\leq A_*\int_0^t \sup_{s\in[0,t]}\mathbb{E}|x(s)-\overline{x}(s)|^4ds\\ &+ B_*\int_0^t \sup_{s\in[0,t]}\mathbb{E}|y(s)-\overline{y}(s)|^4ds,\end{aligned}$$

where $A_* = A_1+B_2,\ \ B_* = A_2+B_1$. Then

$$\begin{aligned}\sup_{s\in[0,t]}\mathbb{E}\Big|x(t)-\overline{x}(t)\Big|^4+\mathbb{E}\Big|y(t)-\overline{y}(t)\Big|^4 &\leq A_{**}\int_0^t \sup_{s\in[0,t]}\Big(\mathbb{E}|x(s)-\overline{x}(s)|^4\\ &+\mathbb{E}|y(s)-\overline{y}(s)|^4\Big)ds,\end{aligned}$$

where $A_{**} = \max\{A_*, B_*\}$.
By a generalization of Gronwall inequality, we have

$$\sup_{s\in[0,t]}\mathbb{E}\Big|x(t)-\overline{x}(t)\Big|^4+\mathbb{E}\Big|y(t)-\overline{y}(t)\Big|^4 = 0 \Longrightarrow (x(t),y(t)) = (\overline{x}(t),\overline{y}(t)),\ \text{a.e. } t\in[0,T].$$

The proof is therefore complete. □

5. Perturbation Problem (1.3)

To prove the main result we will need the following auxiliary inclusion:

$$\begin{cases} -dx(t) & \in N_{C_1(t)}(x(t))dt + F^1(t, x_t, y_t)dt \\ & +G^1(t, x_t, y_t)dB^{H_1}, \text{ a.e. } t \in [0, T] \\ -dy(t) & \in N_{C_2(t)}(y(t))dt + F^2(t, x_t, y_t)dt \\ & +G^2(t, x_t, y_t)dB^{H_2}, \text{ a.e. } t \in [0, T] \\ x(t) & = \phi(t), t \in [-r, 0], \; x(0) \in C_1(0) \\ y(t) & = \overline{\phi}(t), t \in [-r, 0], \; y(0) \in C_2(0) \end{cases} \tag{5.1}$$

Very recently in the case where $G^i = 0$ the perturbation problem was studied by Castaing et al . [**?**]. The aim in those works, is to study the existence of a solution of the problem (5.1) and investigated the topological structure of the solution set. The goal of this section is to study the existence result of the problem (5.1).

Theorem 5.1. *Assume that* (H_1) *and* (H_2) *and the conditions .*

(H_3) $F^j : [0, T] \times M^2([-r, 0], X) \times M^2([-r, 0], X) \to \mathcal{P}_{cp,cv}(X)$ *be a u.s.c. Carathedory multimap, and for each* $t \in [0, T]$, *scalarly* $\mathcal{L}([0, T]) \otimes \mathcal{B}(M^2([-r, 0], X), X)$ *measurable, where* $\mathcal{L}([0, T])$ *is the* $\sigma-$ *algebra of Lebesgue measurable sets of* $[0, T]$ *and* $\mathcal{B}(M^2)$ *is the Borel tribe of* M^2 *and* $|F^j(t, x, y)| \leq k_j$ *for all* $(t, x, y) \in [0, T] \times M^2([-r, 0], X) \times M^2([-r, 0], X)$ *or some constant* $k_j > 0$.

Then, problem (5.1) *has at least one solution on* $[0, T]$.

Proof. Consider for every $n \in \mathbb{N}$, the following partition of $[0, T]$,

$$t_{n,i} := \frac{iT}{2^n}, \; 0 \leq i \leq 2^n \text{ and } I_{n,i} = (t_{n,i}, t_{n,i+1}], \text{ if } 0 \leq i \leq 2^n - 1, \; n \geq 0.$$

$$x_{n,0} = \begin{cases} \phi(t), & t \in [-r, 0], \\ \phi(0), & t \in [0, t_{n,0}], \end{cases}$$

for any $I_{n,0} = (t_{n,0}, t_{n,1}]$, we have

$$x_{n,1} = \begin{cases} x_{n,0}(t), \; t \in [-r, t_{n,0}], \\ proj\Big(\phi(0) + g_0^1(t_{n,0}) \\ +G^1(t_{n,0}, x(_{n,0})_{t_{n,0}}, y(_{n,0})_{t_{n,0}})(B^{H_1}(t_{n,1}) \\ -B^{H_1}(t_{n,0}), C(t_{n,1})\Big), \; t \in [t_{n,0}, t_{n,1}]. \end{cases}$$

Similarly,for any $I_{n,1} = (t_{n,1}, t_{n,2}]$, we have

$$x_{n,2} = \begin{cases} x_{n,1}(t), \ t \in [-r, t_{n,1}], \\ proj\Big(x_{n,1}(t_{n,1}) + g_0^1(t_{n,1}) \\ +G^1(t_{n,1}, x(_{n,1})_{t_{n,1}}, y(_{n,1})_{t_{n,1}})(B^{H_1}(t_{n,2}) \\ -B^{H_1}(t_{n,1}), C(t_{n,2})\Big), t \in [t_{n,1}, t_{n,2}]. \end{cases}$$

With the same argument we can define recursively, for any $I_{n,i} = (t_{n,i}, t_{n,i+1}]$,

$$x_{n,i+1} = \begin{cases} x_{n,i}(t), t \in [-r, t_{n,i}], \\ proj\Big(x_{n,i}(t_{n,i}) + g_0^1(t_{n,i}) \\ +G^1(t_{n,i}, x(_{n,i})_{t_{n,1}}, y(_{n,i})_{t_{n,1}})(B^{H_1}(t_{n,i+1}) \\ -B^{H_1}(t_{n,i}), C(t_{n,i+1})\Big), t \in [t_{n,i}, t_{n,i+1}] \end{cases}$$

where

$$g_0^j(t,u) = \min\{|x| \ : x \in F^j(t,u)\}.$$

By construction, we have $(x_{n,i}, y_{n,i}) \in (C_1, C_2)$, for all $t \in [t_{n,i-1}, t_{n,i}]$.
Then for every $0 \le i \le 2^n$,

$$|x_{n,i+1}(t) - x_{n,i}(t)| \le H_{d_1}(C_1(t_{n,i}), C_1(t_{n,i+1})) \le \lambda \frac{T}{2^n}$$

and

$$|y_{n,i+1}(t) - y_{n,i}(t)| \le H_{d_2}(C_1(t_{n,i}), C_1(t_{n,i+1})) \le \lambda \frac{T}{2^n}$$

and, consequently,

$$\sup\Big\{\sqrt{\mathbb{E}|x_{n,i+1}(t) - x_{n,i}(t)|^2} \ : t \in [-r, T]\Big\} \le \lambda \frac{T}{2^n} \tag{5.2}$$

and

$$\sup\Big\{\sqrt{\mathbb{E}|y_{n,i+1}(t) - y_{n,i}(t)|^2} \ : t \in [-r, T]\Big\} \le \lambda \frac{T}{2^n} \tag{5.3}$$

Put

$$x_n(t) = \begin{cases} x_{n,i}(t), & t \in [-r, t_{n,i}] \\ x_{n,i}(t_{n,i}) + \frac{t - t_{n,i}}{\epsilon_n}(x_{n,i+1}(t) - x_{n,i}(t)) + (t - t_{n,i}) g_0^1(t_{n,i}) & \\ +G^1(t_{n,i}, x_{t_{n,i}}, y_{t_{n,i}})(B^{H_1}(t) - B^{H_1}(t_{n,1})), & t \in [t_{n,i}, t_{n,i+1}]. \end{cases}$$

and

$$y_n(t) = \begin{cases} y_{n,i}(t), & t \in [-r, t_{n,i}] \\ y_{n,i}(t_{n,i}) + \frac{t - t_{n,i}}{\epsilon_n}(y_{n,i+1}(t) - y_{n,i}(t)) + (t - t_{n,i}) g_0^2(t_{n,i}) & \\ +G^2(t_{n,i}, x_{t_{n,i}}, y_{t_{n,i}})(B^{H_2}(t) - B^{H_2}(t_{n,1})), & t \in [t_{n,i}, t_{n,i+1}]. \end{cases}$$

Since (x_n, y_n) is defined by linear interpolation, we have

$$|x_n'(t)| \le \frac{1}{\epsilon_n} \sup_i |x_{n,i+1}(t) - x_{n,i}(t)|$$

and

$$|y_n'(t)| \leq \frac{1}{\epsilon_n} \sup_i |y_{n,i+1}(t) - y_{n,i}(t)|.$$

Using the fast that the projections are non-expansive, thus

$$|x_{n,i+1}(t) - proj(x_{n,i}(t), C_1(t_{n,i+1}))| \leq \epsilon_n |g_0^1(t_{n,i})| \leq \epsilon_n k_1.$$

and

$$|y_{n,i+1}(t) - proj(y_{n,i}(t), C_2(t_{n,i+1}))| \leq \epsilon_n |g_0^2(t_{n,i})| \leq \epsilon_n k_2.$$

Hence

$$|x_{n,i+1}(t) - x_{n,i}(t)| \leq \epsilon_n (k_1 + \lambda). \tag{5.4}$$

Thus

$$|x_n'(t)| \leq k_1 + \lambda \quad \text{ and } \sup_{t\in J} |x_n'(t)|^2 \leq (k_1 + \lambda)^2. \tag{5.5}$$

From the definition of normal proximal cone, we have

$$dx_n(t) \in -N(x_{n,i+1}, C_1(t_{n,i+1}))dt + g_0^1(t_{n,i})dt$$

$$+G^1(t_{n,i}, x(_{n,i})_{t_{n,i}}, y(_{n,i})_{t_{n,i}})(B^{H_1}(t) - B^{H_1}(t_{n,1})), \text{ a.e. } t \in [0,T] \tag{5.6}$$

and

$$dy_n(t) \in -N(y_{n,i+1}, C_2(t_{n,i+1}))dt + g_0^2(t_{n,i})dt$$

$$+G^2(t_{n,i}, x(_{n,i})_{t_{n,i}}, y(_{n,i})_{t_{n,i}})(B^{H_2}(t) - B^{H_2}(t_{n,1})), \text{ a.e. } t \in [0,T]. \tag{5.7}$$

Now we prove that $\{(x_n, y_n) \quad , n \in \mathbb{N}\}$ is compact in $M^2([-r,T],X) \times M^2([-r,T],X)$.
Step 1. $\{(x_n, y_n)\ n \in \mathbb{N}\}$ are bounded sets in $M^2([-r,T],X) \times M^2([-r,T],X)$.
We have

$$\begin{aligned}
|x_n(t)| \leq & |x_{n,i}(t)| + |x_{n,i+1}(t) - x_{n,i}(t)| + T|g_0^1(t_{n,i}, x(_{n,i})_{t_{n,i}}, y(_{n,i})_{t_{n,i}})| \\
& + |G^1(t_{n,i}, x(_{n,i})_{t_{n,i}}, y(_{n,i})_{t_{n,i}})||(B^{H_1}(t) - B^{H_1}(t_{n,1}))| \\
\leq & |x_{n,0}(t)| + 2\sum_{k=1}^{i+1} |x_{n,k-1}(t) - x_{n,k}(t)| + Tk_1 \\
& + |G^1(t_{n,i}, x(_{n,i}, y(_{n,i})_{t_{n,i}}, y(_{n,i})_{t_{n,i}})||(B^{H_1}(t) - B^{H_1}(t_{n,1}))| \\
\leq & ||\phi|| + 2T + \Big(|G^1(t_{n,i}, x(_{n,i})_{t_{n,i}}, y(_{n,i})_{t_{n,i}}) - G^1(t_{n,i}, 0, 0)| \\
& + |G^1(t_{n,i}, 0, 0)|\Big)|(B^{H_1}(t) - B^{H_1}(t_{n,1}))| \\
\leq & ||\phi|| + 2T + Tk_1 \\
& + T\Big(\alpha_1 ||(x_{n,i})_{t_{n,i}}||_{M^2_{\mathcal{F}_0}} + \beta_1 ||(y_{n,i})_{t_{n,i}}||_{M^2_{\mathcal{F}_0}} \\
& + |G^1(t_{n,i}, 0, 0)|\Big)|(B^{H_1}(t) - B^{H_1}(t_{n,1}))|.
\end{aligned}$$

Then,

$$\begin{aligned}\mathbb{E}|x_n(t)|^2 \leq & 2(||\phi||^2 + 2T + Tk_1)^2 + 2T^2\Big(\alpha_1 M + \beta_1 \overline{M} \\ & + \sup_{t\in[0,T]} |G^1(t,0,0)|^2\Big)\mathbb{E}|(B^{H_1}(t) - B^{H_1}(t_{n,1}))|^2 \\ \leq & 2(||\phi||^2 + 2T + Tk_1)^2 \\ & + 2T^2\Big(\alpha_1 M + \beta_1 \overline{M} + \sup_{t\in[0,T]} |G^1(t,0,0)|^2\Big)T^{2H_1} := \bar{l}_1.\end{aligned}$$

Hence

$$\sup\{\sqrt{\mathbb{E}|x_n(t)|^2} \ : \ t \in [-r,T]\} \leq \bar{l}_1.$$

and

$$\sup\{\sqrt{\mathbb{E}|y_n(t)|^2} \ : \ t \in [-r,T]\} \leq \bar{l}_2.$$

Which implies that

$$\begin{pmatrix} \mathbb{E}|x_n(t)|^2 \\ \mathbb{E}|y_n(t)|^2 \end{pmatrix} \leq \begin{pmatrix} \bar{l}_1 \\ \bar{l}_2 \end{pmatrix}$$

Step 2. $\{(x_n, y_n), \ n \in \mathbb{N}\}$ are equicontinuous sets in $M^2([-r,T],X)$.
Let $\tau_1, \tau_2 \in [t_{n,i}, t_{n,i+1}], \tau_1 < \tau_2$. Thus

$$\begin{aligned} & \mathbb{E}|x_n(\tau_2) - x_n(\tau_1)|^2 \\ = \ & \mathbb{E}\Big|\frac{\tau_2 - \tau_1}{\epsilon_n}(x_{n,i+1} - x_{n,i}) + (\tau_2 - \tau_1)g_0^1(t_{n,i}, x(_{n,i})_{t_{n,i}}, y(_{n,i})_{t_{n,i}}) \\ + \ & G^1(t_{n,i}, x(_{n,i})_{t_{n,i}}, y(_{n,i})_{t_{n,i}})(B^{H_1}(\tau_2) - B^{H_1}(\tau_1))\Big|^2 \\ \leq \ & 3|\tau_2 - \tau_1|^2 + 3\Big(\alpha_1 M + \beta_1 \overline{M} + \sup_{t\in[0,T]} |G^1(t,0,0)|^2\Big)|\tau_2 - \tau_1|^{2H_1} \\ + \ & 3k_1^2|\tau_2 - \tau_1|^2. \end{aligned}$$

Similarly,

$$\begin{aligned}\mathbb{E}|y_n(\tau_2) - y_n(\tau_1)|^2 \leq & 3|\tau_2 - \tau_1|^2 + 3\Big(\alpha_2 M + \beta_2 \overline{M} + \sup_{t\in[0,T]} |G^2(t,0,0)|^2\Big)|\tau_2 - \tau_1|^{2H_2} \\ & + 3k_2^2|\tau_2 - \tau_1|^2.\end{aligned}$$

The right-hand side tends to zero as $\tau_2 - \tau_1 \to 0$, and ϵ sufficiently small. From Steps 1, 2, by the Arzela-Ascoli theorem, we conclude that there is a subsequence of (x_n, y_n), again denoted (x_n, y_n) which converges to (x, y) in $M^2([-r,T],X) \times M^2([-r,T],X)$. It remains to prove that $(x(t), y(t)) \in (C_1(t), C_2(t))$. Let $t \in [0,T]$,from (5.5) ,we obtain

$$\begin{aligned} 0 \le |x_n(t) - C_1(t)| &= d(x_n(t), C_1(t)) \\ &\le \Big|x_n(t) - x_n(t_{n,i})\Big| + d(x_n(t_{n,i}), C_1(t)) \\ &\le (k_1+\lambda)|t - t_{n,i}| + H_{d_1}(C_1(t_{n,i}), C_1(t)) \\ &\le \frac{(k_1+\lambda)b}{2^{n-1}}. \end{aligned}$$

Then

$$|x_n(t) - C_1(t)| \le \frac{(k_1+\lambda)T}{2^{n-1}}. \tag{5.8}$$

and

$$|y_n(t) - C_2(t)| \le \frac{(k_2+\lambda)T}{2^{n-1}}. \tag{5.9}$$

By letting $n \to \infty$ in (5.8) and (5.9) ,we obtain that

$$(x(t), y(t)) \in (C_1, C_2) \tag{5.10}$$

Now, we define, for $t \in [0, T]$

$$\rho_n(t) = t_{n,i}, \qquad \mu_n(t) = t_{n,i+1} \quad \text{if} \quad t \in [t_{n,i}, t_{n,i+1}).$$

Hence, by using (4.4) and (4.5) we have

$$\begin{aligned} dx_n(t) \in &-N(x_n(\mu_n(t)), C_1(\mu_n(t)))dt + g_0^1(t_{\rho_n(t)}, x_{\rho_n(t)}, y_{\rho_n(t)}) \\ &+G^1(t_{\rho_n(t)}, x_{\rho_n(t)}, y_{\rho_n(t)})dB^{H_1}(\rho_n(t)) \text{ a,e. } t \in [0,T]. \end{aligned} \tag{5.11}$$

and

$$\begin{aligned} dy_n(t) \in &-N(x_n(\mu_n(t)), C_2(\mu_n(t)))dt + g_0^2(t_{\rho_n(t)}, x_{\rho_n(t)}, y_{\rho_n(t)}) \\ &+G^2(t_{\rho_n(t)}, x_{\rho_n(t)}, y_{\rho_n(t)})dB^{H_2}(\rho_n(t)) \; t \in \text{ a,e. } t \in [0,T]. \end{aligned} \tag{5.12}$$

Hence

$$\rho_n(t) \to t, \qquad \mu_n(t) \to t \qquad \text{uniformly on} \quad [0, b]$$

Since $|\rho_n(t) - t| \le \frac{T}{2^n}$ and $|\mu_n(t) - t| \le \frac{T}{2^n}$. Moreover,

$$|x_n(\rho_n(t)) - x_n(t)| \le H_{d_1}(C_1(\rho_n(t)), C_1(t)) \le \lambda|\rho_n(t) - t|.$$

Similarly,

$$|y_n(\rho_n(t)) - y_n(t)| \le H_{d_2}(C_2(\rho_n(t)), C_2(t)) \le \lambda|\rho_n(t) - t|.$$

Therefore,

$$\sup\{\sqrt{\mathbb{E}|x_n(\rho_n(t)) - x_n(t)|^2} \;:\; t \in [0,T]\} \le \lambda\sqrt{\mathbb{E}|\rho_n(t) - t|^2} \to 0 \text{ as } n \to \infty.$$

and

$$\sup\{\sqrt{\mathbb{E}|y_n(\rho_n(t)) - y_n(t)|^2} \;:\; t \in [0,T]\} \le \lambda\sqrt{\mathbb{E}|\rho_n(t) - t|^2} \to 0 \text{ as } n \to \infty.$$

In Theorem (4.2) was proved that $(x_{\rho_n(t)}, y_{\rho_n(t)})$ converge to (x_t, y_t) in $M^2([-r, T], X)$.
Let $v_n^j(t) = g_0^j(\rho_n(t), (x_n)_{\rho_n(t)}), (y_n)_{\rho_n(t)}))$.From H_3 we have $|v_n^j(t)| \le k_j$ for $n \in \mathbb{N}$ implies that $v_n^j(t) \in lB(0,1)$, hence $(v_n^j)_{n\in\mathbb{N}}$ which converges weakly to some limit $v^j \in L^2(J, X)$. Since $F(., x, y)$ is u.s.c. with closed and convex values and $F^j(., ., .)$

is bounded for each $j = 1, 2$, then exists a sequence $\{F_m\}_{m\in\mathbb{N}}$ of globally u.s.c. set-valued mappings on $J \times M^2([-r, 0], X) \times M^2([-r, 0], X)$ with convex compact values in $X \times X$ satisfying the following conditions:

$$||F^j_m(t, x, y)|| \leq k_j,$$

for all $(t, x, y) \in J \times M^2([-r, 0], X) \times M^2([-r, 0], X)$ and $j = 1, 2$,

$$F^j_{m+1}(t, x, y) \subset F^j_m(t, x, y), \quad F(t, x, y) = \cap_{m\geq 1} F^j_m(t, x, y).$$

Now we need to prove that $\upsilon^j(t) \in F^j(t, x_t, y_t)$, for a.e. $t \in J$. Lemma 3.7 yields the existence of constants $\alpha^n_i \geq 0$, $l = 1, 2.., k(n)$ and $j = 1, 2$ such that $\sum_{l=1}^{k(n)} \alpha^n_l = 1$ and the sequence of convex combinations $\psi^j_n(.) = \sum_{l=1}^{k(n)} \alpha^n_l \upsilon^j_l(.)$ converges strongly to some limit $\upsilon^j \in L^2(J, X)$. Since F^j takes convex values, using Lemma 3.6, we obtain that

$$\begin{aligned}
\upsilon^j(t) &\in \bigcap_{n\geq 1} \overline{\{\psi^j_n(t)\}}, \quad a.e \quad t \in J \\
&\subset \bigcap_{n\geq 1} \overline{co}\{\upsilon^j_k(t), \quad k \geq n\} \\
&\subset \bigcap_{n\geq 1} \overline{co}\{\bigcup_{k\geq n} F^j_m(\rho_k(t), (x_k)_{\rho_k(t)}, (y_k)_{\mu_k(t)})\} \\
&= \overline{co}\{\limsup_{k\to\infty} F^j_m(\mu_k(t), (x_k)_{\mu_k(t)}, (y_k)_{\mu_k(t)})\}. \qquad (5.13)
\end{aligned}$$

Since F^j_m is u.s.c. and has compact values, then by Lemma 3.5, we have

$$\limsup_{n\to\infty} F^j_m(\rho_n(t), (x_n)_{\rho_n(t)}, (y_n)_{\rho_n(t)}) = F^j_m(t, x_t, y_t) \quad \text{for a.e} \quad t \in J.$$

This and (5.13) imply that $\upsilon^j(t) \in \overline{co}(F^j(t, x_t, y_t)$. Since, for each $j = 1, 2$, $F^j_m(., ., .)$ has closed, convex values, we deduce that $\upsilon^j(t) \in F^j_m(t, x_t, y_t)$ for a.e. $t \in J$,then $\upsilon^j(t) \in F^j(t, x_t, y_t)$.

We can pass to the limit when $n \to \infty$, we deduce from

$$(x_{\rho_n(t)}, y_{\rho_n(t)}) \to (x_t, y_t) \in M^2([-r, 0], X) \text{ as } n \to \infty.$$

Using the fact that $G^j(., ., .)$ is a continuous function then we have

$$G^j(\rho_n(t), x_{\rho_n(t)}, y_{\rho_n(t)}) \to G^j(t, x_t, y_t) \text{ as } n \to \infty.$$

Now, we show that

$$dx(t) \in -N(x(t), C_1(t))dt + \upsilon^1(t)dt + G^1(t, x_t, y_t)dB^{H_1}(t) \text{ a.e. } t \in [0, T]. \qquad (5.14)$$

and

$$dy(t) \in -N(y(t), C_2(t))dt + \upsilon^2(t)dt + G^2(t, x_t, y_t)dB^{H_2}(t) \text{ a.e. } t \in [0, T]. \qquad (5.15)$$

Since (x_n, y_n) is bounded in $X \times X$,there exists a subsequence of (x_n, y_n)converge to (x, y).Then

$$\int_0^T \sigma\Big(- x'_n(t) + v_n^1(t) + G^1(t, (x_n)_t, (y_n)_t)dB^{H_1}(t), C_1(\mu_n(t))\Big)dt$$
$$\leq \int_0^T \Big(- x'_n(t) + v_n^1(t) + G^1(t, (x_n)_t, (y_n)_t)dB^{H_1}(t), x(\mu_n(t))\Big)dt. \quad (5.16)$$

Using the fact that $\sigma(., C_1(t))$ is lower semicontinuous ,then

$$\liminf_{n\to\infty} \int_0^T \sigma\Big(- x'_n(t) + v_n^1(t) + G^1(t, (x_n)_t, (y_n)_t)dB^{H_1}(t), C_1(\mu_n(t))\Big)dt$$
$$\geq \int_0^T \Big(- x'(t) + v^1(t) + G^1(t, x_t, y_t)dB^{H_1}(t), C_1(t)\Big)dt. \quad (5.17)$$

By (5.16) and (5.18),we obtain

$$\int_0^T \Big(- x'(t) + v^1(t) + G^1(t, x_t, y_t)dB^{H_1}(t), C_1(t)\Big)dt$$
$$\geq \int_0^T \sigma\Big(- x'(t) + v^1(t) + G^1(t, x_t, y_t)dB^{H_1}(t), C_1(t)\Big)dt. \quad (5.18)$$

Thus,

$$dx(t) \in -N(x(t), C_1(t))dt + F^1(t, x_t, y_t)dt + G^1(t, x_t, y_t)dB^{H_1}(t), \text{ a.e. } t \in [0, T].$$

and

$$dy(t) \in -N(y(t), C_2(t))dt + F^1(t, x_t, y_t)dt + G^2(t, x_t, y_t)dB^{H_2}(t), \text{ a.e. } t \in [0, T].$$

and the proof is finished. □

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Unsteady flow of Bingham fluid in a thin layer with mixed boundary conditions

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Abstract. In this paper we consider the dynamic system for Bingham fluid in a three-dimensional thin domain with Fourier and Tresca boundary condition. We study the existence and uniqueness results for the weak solution, then we establish its asymptotic behavior, when the depth of the thin domain tends to zero. This study yields a mechanical laws that give a new description of the behavior this system.

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1. Introduction

This work gives an extension to describe the flow of fluids in a dynamic system to some of the results obtained in a series of papers [1, 2, 4, 5, 9], in which the authors considered a stationary case only of the general equations describing the motion of some fluid flows in bounded thin domain, with slip and mixed boundary conditions. The aim of this paper is to study the asymptotic analysis of an incompressible Bingham fluid in a dynamic regime in a three dimensional thin domain mixed boundary and subject to slip phenomenon on a part of the boundary. We are interested here in the existence and uniqueness for this problem and also its behavior when the thickness of the thin domain tends to zero.

This fluid enters the category of non-Newtonian fluids, and there are many milieus in nature and industry exhibiting the behavior of the Bingham fluid. For example, heavy crude oils, colloid solutions...See also historical ref [3]. More specifically, the model under study is mainly related for lubrication problems in a lot of mechanical papers [10, 11, 13] when the gap between the solid surfaces is very weak. In this dynamic system, the non-slip condition is caused by the chemical structure between the

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lubricants and the surrounding surfaces. On the contrary, tangential stresses, when they reach a certain threshold, destroy the chemical structure and induce a slip phenomena. This phenomenon is implicitly expressed by the Reynolds equation, which was mathematically posed during 1985 in [12].

Thus, following the same ideas as in [5]. The departure point is the laws of conservation, which includes here the effect of the acceleration-dependent inertia forces. A friction law of Tresca and the Fourier boundary condition are assumed on the boundary, so fall into the scope of the work of [8]. Then we will compare our results to stationary problem in [1, 2, 4, 5].

This work is also devoted to prove our results, with suitable conditions on the initial data, contrary to what was assumed in [7, p. 289-290] where the initial conditions for the data were null. The main difficulty here is to estimate the solutions of the problem, due to the fractional term for the Bingham constitutive law and the assumption coming from the initial velocity. The proofs presented in this paper are based on regularization methods and classical results for elliptic variational derived from [6, 7]. The plan of this paper is as follow, we present in section 2, some notation and the weak formulation of problem. In section 3, we give the main results on existence results by the regularization methods. In section 4, we introduce a scaling as in [5, 8], we give some needed estimates on the velocity and pressure, also the convergence results. In sections 5 we present the limit problem and we give the mechanical interpretation of the results.

2. Preliminaries and variational formulation

Let ω be fixed region in the surface $x' = (x_1, x_2) \in \mathbb{R}^2$, and let $h \in C^2(\omega)$ be a smooth positive function such that $0 < \underline{h} \leq h(x') \leq \overline{h}$ for all $(x', 0) \in \omega$. Consider an incompressible Bingham fluid occupying the domain

$$\begin{aligned} \Omega^\varepsilon &= \left\{x = (x', x_3) \in \mathbb{R}^3 : \ (x', 0) \in \omega, \quad 0 < x_3 < \varepsilon h(x')\right\}, \\ Q^\varepsilon &= \Omega^\varepsilon \times \left]0, T\right[. \end{aligned}$$

where $\varepsilon \in \left]0, 1\right[$ and $T > 0$. Noting Γ^ε the boundary of Ω^ε, we have $\Gamma^\varepsilon = \bar{\omega} \cup \bar{\Gamma}_1^\varepsilon \cup \bar{\Gamma}_L^\varepsilon$, and Γ_1^ε the upper boundary of equation $x_3 = \varepsilon h(x')$, Γ_L^ε is the lateral boundary. We denote by $\mathbb{S}_n$ $(n = 2, 3)$ the space of symmetric tensors, while '.' and $|.|$ will represent the inner product and the Euclidean norm on $\mathbb{S}_n$ or $\mathbb{R}^n$. We consider the rate of deformation operator defined for every $u^\varepsilon \in H^1(\Omega^\varepsilon)^3$ by $D(u^\varepsilon) = \dfrac{1}{2}(\nabla u^\varepsilon + (\nabla u^\varepsilon)^T)$. Let ν denote the unit outer normal on Γ^ε, and we write u^ε for its trace on Γ^ε, also

$$u_\nu^\varepsilon = u^\varepsilon.\nu, \quad u_\tau^\varepsilon = u^\varepsilon - u_\nu^\varepsilon.\nu, \quad \sigma_\nu^\varepsilon = (\sigma^\varepsilon.\nu).\nu \text{ and } \sigma_\tau^\varepsilon = \sigma^\varepsilon.\nu - (\sigma_\nu^\varepsilon).\nu$$

be, respectively, the components of the normal, the tangential of u^ε on Γ^ε, the normal and the tangential of σ^ε on Γ^ε.

The unstable flow of Bingham fluid that will be studied in this paper is given by the following mechanical problem.

Problem P. Find the velocity fields $u^\varepsilon = (u_1^\varepsilon, u_2^\varepsilon, u_3^\varepsilon)$ and the scalar pressure p^ε such that

$$\frac{\partial u^\varepsilon}{\partial t} - \operatorname{div}(\sigma^\varepsilon) = -\nabla p^\varepsilon + f^\varepsilon \quad \text{in } \Omega^\varepsilon \times [0,T]\,, \tag{2.1}$$

$$\operatorname{div}(u^\varepsilon) = 0 \qquad \text{in } \Omega^\varepsilon \times [0,T]\,, \tag{2.2}$$

$$\left\{\begin{array}{ll} \sigma_{ij}^\varepsilon = \varepsilon^{-1}\alpha \dfrac{D_{ij}(u^\varepsilon)}{|D(u^\varepsilon)|} + 2\mu D_{ij}(u^\varepsilon) & \text{if } |D(u^\varepsilon)| \neq 0 \\ |\sigma^\varepsilon| \leq \varepsilon^{-1}\alpha & \text{if } |D(u^\varepsilon)| = 0 \end{array}\right. \quad \text{in } \Omega^\varepsilon \times [0,T]\,, \tag{2.3}$$

$$u^\varepsilon = 0 \text{ on } \Gamma_L^\varepsilon \times]0,T[\,, \tag{2.4}$$

$$u^\varepsilon \cdot \nu = 0 \quad \text{on } (\omega \cup \Gamma_1^\varepsilon) \times]0,T[\,, \tag{2.5}$$

$$\sigma_\tau(u^\varepsilon) = -l^\varepsilon u^\varepsilon \text{ on } \Gamma_1^\varepsilon \times]0,T[\,, \tag{2.6}$$

$$\left\{\begin{array}{l} |\sigma_\tau^\varepsilon| < \varepsilon^{-1}k \Rightarrow u_\tau^\varepsilon(t) = 0 \\ |\sigma_\tau^\varepsilon| = \varepsilon^{-1}k \Rightarrow \exists\lambda \geq 0 \; u_\tau^\varepsilon(t) = -\lambda\sigma_\tau^\varepsilon \end{array}\right. \quad \text{on } \omega \times [0,T]\,, \tag{2.7}$$

$$u^\varepsilon(x,0) = u_0^\varepsilon(x) \;\; \forall x \in \Omega^\varepsilon. \tag{2.8}$$

Here, the flow is given by equation (2.1), where $f^\varepsilon = (f_1^\varepsilon, f_2^\varepsilon, f_3^\varepsilon)$ denote the volume force of density. The equation (2.2) represent the incompressibility condition. Relation (2.3) represents the constitutive law of Bingham fluid of viscosity μ and plasticity threshold α, where μ, $\alpha > 0$ are constants independent of ε. The condition (2.4) is the Dirichlet boundary. (2.5) give the non-slip condition of velocity on Γ_1^ε and ω. (2.4) represent the Fourier condition on Γ_1^ε, where $l^\varepsilon > 0$ is a given constant. Condition (2.7) represents a Tresca's friction law on ω, where k is a coefficient independent of ε, finally, the initial velocity is a given by (2.8), with $u_0^\varepsilon \neq 0$ is a given function.

Now, we us consider the following function spaces

$$\begin{aligned} K^\varepsilon &= \left\{\phi \in H^1(\Omega^\varepsilon)^3 : \phi = 0 \text{ on } \Gamma_L^\varepsilon,\ \phi.\nu = 0 \text{ on } \omega \cup\ \Gamma_1^\varepsilon\right\}, \\ K_{\text{div}}^\varepsilon &= \{\phi \in K^\varepsilon : \operatorname{div}(\phi) = 0 \text{ in } \Omega^\varepsilon\ \}, \\ L_0^2(\Omega^\varepsilon) &= \left\{q \in L^2(\Omega^\varepsilon) : \int_{\Omega^\varepsilon} q\, dx = 0\right\}. \end{aligned}$$

Let us introduce the bilinear forms a, $\breve{a}$ and functional J^ε defined by

$$a(u^\varepsilon, \phi - u^\varepsilon) = 2\mu \int_{\Omega^\varepsilon} D_{ij}(u^\varepsilon)\, D_{ij}(\phi - u^\varepsilon)\, dx,$$

$$\breve{a}(u^\varepsilon, \phi - u^\varepsilon) = a(u^\varepsilon, \phi - u^\varepsilon) + l^\varepsilon \int_{\Gamma_1^\varepsilon} u^\varepsilon.(\phi - u^\varepsilon)\, d\tau,$$

$$J^\varepsilon(\phi) = \varepsilon^{-1} \int_\omega k\, |\phi_\tau|\ dx' + \sqrt{2}\alpha\varepsilon^{-1} \int_{\Omega^\varepsilon} |D(\phi)|\, dx.$$

J^ε is convex and continuous but non differentiable in K^ε.

Following [5, 8], the variational inequality of the problem (2.1)-(2.8) is given by

Problem Pv. Find $\{u^\varepsilon, p^\varepsilon\}$ where $u^\varepsilon(t) \in K^\varepsilon_{\rm div}$, $\dfrac{\partial u^\varepsilon}{\partial t}(t) \in K^\varepsilon$ and $p^\varepsilon(t) \in L^2_0(\Omega^\varepsilon)$ such that

$$\int\limits_{\Omega^\varepsilon} \frac{\partial u^\varepsilon}{\partial t}(t).\left(\phi - u^\varepsilon(t)\right) dx + \check{a}(u^\varepsilon(t), \phi - u^\varepsilon(t)) - \int\limits_{\Omega^\varepsilon} p^\varepsilon \, div\ (\phi)\, dx + J^\varepsilon(\phi) - J^\varepsilon(u^\varepsilon(t)) \geq \int\limits_{\Omega^\varepsilon} f^\varepsilon.\left(\phi - u^\varepsilon(t)\right) dx \ \forall t\ \in]0, T[\ \ \forall \phi \in K^\varepsilon \tag{2.9}$$

with

$$u^\varepsilon(0) = u^\varepsilon_0\,(\neq 0)\,. \tag{2.10}$$

Notation. To simplify the writing, we will denote the norm in $L^2\left(\Omega^\varepsilon\right)^3$ by$\|.\|_{0,\Omega^\varepsilon}$ and the norm in $H^s\left(\Omega^\varepsilon\right)^3$ by $\|.\|_{s,\Omega^\varepsilon}$, the inner products on the space $L^2\left(\Omega^\varepsilon\right)^3$ designed by $(.,.)$ and le $\langle .,. \rangle$ denote the duality pairing between $(K^\varepsilon_{\rm div})'$ and $K^\varepsilon_{\rm div}$.

3. Existence and uniqueness results

We establish here a theorem of existence of weak solutions for Pv.

Theorem 3.1. *We make the following assumptions :*

$$f^\varepsilon,\ \frac{\partial f^\varepsilon}{\partial t} \in L^2\left(0, T; L^2\left(\Omega^\varepsilon\right)^3\right),\ f^\varepsilon\left(0\right) \in L^2\left(\Omega^\varepsilon\right)^3 \tag{3.1}$$

$$k \in C^\infty_0(\omega),\ k > 0 \ \textit{does not depend on } t, \tag{3.2}$$

$$u^\varepsilon_0 \in H^2(\Omega^\varepsilon)^3 \cap H^1_0(\Omega^\varepsilon)^3,\ \left(D\left(u^\varepsilon_0\right)\right)_\tau = 0 \ \textit{on } \omega \cup \Gamma^\varepsilon_1, \tag{3.3}$$

$$\exists \eta > 0 \quad \left|D\left(u^\varepsilon_0\right)\right| \geq \varepsilon^{-1}\eta \ \ \textit{a.e. in } \Omega^\varepsilon. \tag{3.4}$$

Under these assumptions, there exist a function u^ε unique solution of (2.9)-(2.10) with

$$u^\varepsilon,\ \frac{\partial u^\varepsilon}{\partial t} \in L^\infty\left(0, T; L^2\left(\Omega^\varepsilon\right)^3\right) \cap L^2\left(0, T; H^1\left(\Omega^\varepsilon\right)^3\right). \tag{3.5}$$

Remark 3.1. The hypothesis $\langle u^\varepsilon_0 \neq 0\rangle$ leads us to make additional techniques in the resolution of (2.9)-(2.10). First, we introduce two technical lemmas in the following paragraph, which will be used to obtain the needed estimates, then we will give the demonstration of theorem 3.1.

3.1. Regularization

For $\zeta > 0$, we consider the operator ψ_ζ and Ψ_ζ defined by

$$\psi_\zeta \ : L^2\left(\omega\right)^2 \rightarrow L^2\left(\omega\right)^2,\quad v \rightarrow \psi_\zeta(v) = |v|^{\zeta-1}\, v$$

$$\Psi_\zeta : H^1\left(\Omega^\varepsilon\right)^{3\times3} \rightarrow H^1\left(\Omega^\varepsilon\right)^{3\times3},\quad \sigma \rightarrow \Psi_\zeta(\sigma) = |\sigma|^{\zeta-1}\,\sigma$$

From [7], we approach J^ε by differentiable family;

$$J^\varepsilon_\zeta(v) = \varepsilon^{-1}\int_\omega k(x')\frac{|v_\tau|^{(1+\zeta)}}{1+\zeta}dx' + \sqrt{2}\alpha\varepsilon^{-1}\int_{\Omega^\varepsilon}\frac{|D\left(v\right)|^{(1+\zeta)}}{1+\zeta}dx,$$

we have

$$\left\langle \left(\ J^\varepsilon_\zeta\right)'(v)\,, \phi\right\rangle = \varepsilon^{-1}\int_\omega k\psi_\zeta(v_\tau).\phi_\tau dx' + \sqrt{2}\alpha\varepsilon^{-1}\int_{\Omega^\varepsilon}\Psi_\zeta(D\left(v\right)).D\left(\phi\right)dx. \tag{3.6}$$

Then, we can approach the inequality (2.9) by the following equation, for all $\phi \in K^{\varepsilon}_{\text{div}}$:

$$\left(\frac{\partial u^{\varepsilon}_{\zeta}}{\partial t}(t), \phi\right) + \breve{a}\left(u^{\varepsilon}_{\zeta}(t), \phi\right) + \left\langle \left(J^{\varepsilon}_{\zeta}\right)'(v), \phi\right\rangle = \left(f^{\varepsilon}(t), \phi\right) \tag{3.7}$$

with

$$u^{\varepsilon}_{\zeta}(0) = u^{\varepsilon}_{0} \tag{3.8}$$

Lemma 3.1. Let $G : \mathbb{S}^{\star}_{3} \to \mathbb{S}_{3}$ be defined by $G(\tau) = |\tau|^{\zeta-1}\tau$ such that $\zeta \in]0,1[$. Let $\sigma \in H^{1}(\Omega^{\varepsilon})^{3\times 3}$, we suppose that there exist a strictly positive constant β such that $|\sigma| \geq \beta$ a. e. in $\overline{\Omega}^{\varepsilon}$, then

$$Go\sigma \in H^{1}(\Omega^{\varepsilon})^{3\times 3} \text{ and } \frac{\partial}{\partial x_k}(Go\sigma) = \left(\frac{\partial G}{\partial \tau_{ij}} o\sigma\right)\frac{\partial \sigma_{ij}}{\partial x_k} \quad \forall i,j,k \in \{1,2,3\}.$$

Proof. We have $|G(\tau)| = |\tau|^{\zeta}$ $\forall \tau \in \mathbb{S}^{\star}_{3}$. Since $|\sigma| \geq \beta$, and therefore

$$|Go\sigma| = |\sigma|^{\zeta} = |\sigma|\,|\sigma|^{\zeta-1} \leq \beta^{\zeta-1}|\sigma|,$$

as a consequence $Go\sigma \in L^{2}(\Omega^{\varepsilon})^{3\times 3}$.
Similarly,by a standard calculation of differentiation of a composition, we have

$$\begin{aligned}\left|\left(\frac{\partial G}{\partial \tau_{ij}} o\sigma\right)\frac{\partial \sigma_{ij}}{\partial x_k}\right| &= \left||\sigma|^{\zeta-1}\left(\left(\zeta-1\right)\sigma^{2}_{ij}|\sigma|^{-2}+1\right)\frac{\partial \sigma_{ij}}{\partial x_k}\right| \\ &\leq |\sigma|^{\zeta-1}\left|\frac{\partial \sigma_{ij}}{\partial x_k}\right| \leq \beta^{\zeta-1}\left|\frac{\partial \sigma_{ij}}{\partial x_k}\right| \end{aligned} \tag{3.9}$$

and thus $\left(\frac{\partial G}{\partial \tau_{ij}} o\sigma\right)\frac{\partial \sigma_{ij}}{\partial x_k} \in L^{2}(\Omega^{\varepsilon})^{3\times 3}$. It remains to verify that

$$\int_{\Omega^{\varepsilon}} (Go\sigma)\,.\frac{\partial \Phi}{\partial x_k} dx = \int_{\Omega^{\varepsilon}}\left(\frac{\partial G}{\partial \tau_{ij}} o\sigma\right)\frac{\partial \sigma_{ij}}{\partial x_k}.\Phi dx \quad \forall \Phi \in \mathcal{C}^{1}_{0}(\Omega^{\varepsilon})^{3\times 3}.$$

By Friedrich Theorem (see [6, p. 265]), there exists a sequence σ_n in $\mathcal{C}^{\infty}_{0}\left(\mathbb{R}^{3}\right)^{3\times 3}$ such that $\sigma_n \to \sigma$ in $L^{2}(\Omega^{\varepsilon})^{3\times 3}$ and $\nabla \sigma_n \to \nabla \sigma$ in $L^{2}(W^{\varepsilon})^{3\times 3\times 3}$ for all open W^{ε} with $\overline{W^{\varepsilon}} \subset \Omega^{\varepsilon}$. Then, we can follow the proof with an argument similar to that used in proof of [6, Proposition 9.5]. □

Lemma 3.2. Let $\varepsilon, \zeta \in]0,1[$. If u^{ε}_{0} verifies the assumptions (3.3), (3.4). Then $\left(J^{\varepsilon}_{\zeta}\right)'(u^{\varepsilon}_{0})$ belong to $L^{2}(\Omega^{\varepsilon})^{3}$, moreover, there exist a constant $\gamma > 0$ does not depend on Ω^{ε}, such that

$$\left\|\left(J^{\varepsilon}_{\zeta}\right)'(u^{\varepsilon}_{0})\right\|_{0,\Omega^{\varepsilon}} \leq \varepsilon^{-1}\gamma\left\|u^{\varepsilon}_{0}\right\|_{2,\Omega^{\varepsilon}}. \tag{3.11}$$

Proof. Using Green's formula in (3.6) and using the assumption (3.3), we get

$$\left\langle \left(J^{\varepsilon}_{\zeta}\right)'(u^{\varepsilon}_{0}), \phi\right\rangle = -\sqrt{2}\alpha\varepsilon^{-1}\int_{\Omega^{\varepsilon}}\left\{\mathrm{Div}\left(\Psi_{\zeta}(D(u^{\varepsilon}_{0})\right)\right\}\phi\, dx \tag{3.12}$$

Applying lemma 3.1 for $\sigma = D(u^{\varepsilon}_{0})$ and $\beta = \varepsilon^{-1}\eta$, clearly $\Psi_{\zeta}(D(u^{\varepsilon}_{0})) \in H^{1}(\Omega^{\varepsilon})^{3\times 3}$. By [7] we can write the Gelfand triple

$$K^{\varepsilon}_{\text{div}} \subset L^{2}(\Omega^{\varepsilon})^{3} \subset \left(K^{\varepsilon}_{\text{div}}\right)',$$

and it follows the following relation :

$$\left(\left(J_\zeta^\varepsilon\right)'\left(u_0^\varepsilon\right),\phi\right)=\left\langle\left(J_\zeta^\varepsilon\right)'\left(u_0^\varepsilon\right),\phi\right\rangle \quad \forall\phi\in\ L^2(\Omega^\varepsilon)^3.$$

By comparison with (3.12), we find

$$\left(J_\zeta^\varepsilon\right)'\left(u_0^\varepsilon\right)=-\sqrt{2}\alpha\varepsilon^{-1}\mathrm{Div}\left(\Psi_\zeta(D\left(u_0^\varepsilon\right)\right).$$

But, due to fact that (3.9) we have$\|\Psi_\zeta(D\left(u_0^\varepsilon\right))\|_{1,\Omega^\varepsilon}\leq\eta^{\zeta-1}\left\|D\left(u_0^\varepsilon\right)\right\|_{1,\Omega^\varepsilon}$. Then, using Sobolev injection related to Div and D, the relation (3.11) can be easily deduced with $\gamma=\sqrt{6}\alpha\eta^{\zeta-1}$. □

3.2. Demonstration of Theorem 3.1

First, we seek to estimate the solution independently of ζ. Let $t\in[0,T]$. As

$$\left\langle\left(J_\zeta^\varepsilon\right)'\left(u_\zeta^\varepsilon\right),u_\zeta^\varepsilon\right\rangle\geq 0,$$

the equation (3.7) for $\phi=u_\zeta^\varepsilon(t)$ becomes

$$\frac{1}{2}\frac{d}{dt}\left\|u_\zeta^\varepsilon(t)\right\|_{0,\Omega^\varepsilon}^2+a\left(u_\zeta^\varepsilon(t),u_\zeta^\varepsilon(t)\right)+l^\varepsilon\left\|u_\zeta^\varepsilon(t)\right\|_{0,\Gamma_1^\varepsilon}^2\leq(f^\varepsilon(t),u_\zeta^\varepsilon(t)). \tag{3.13}$$

By [5] there exist a constant $C_k>0$ such that

$$a\left(u_\zeta^\varepsilon(t),u_\zeta^\varepsilon(t)\right)+l^\varepsilon\left\|v\left(t\right)\right\|_{0,\Gamma_1^\varepsilon}^2\geq 2\mu C_K\left\|v\left(t\right)\right\|_{1,\Omega^\varepsilon}^2\quad\forall v\left(t\right)\in K_{\mathrm{div}}^\varepsilon.$$

Then, by the integral of (3.13) relative to t, and using a Gronwall-type argument we obtain

$$\left\|u_\zeta^\varepsilon(t)\right\|_{0,\Omega^\varepsilon}^2+\int_0^t\left\|u_\zeta^\varepsilon(\sigma)\right\|_{1,\Omega^\varepsilon}^2d\sigma\leq c \tag{3.14}$$

Now, we derive (3.7) in t and taking $\phi=\dfrac{\partial u_\zeta^\varepsilon}{\partial t}(t)$,

$$\begin{aligned}&\left(\frac{\partial^2u_\zeta^\varepsilon}{\partial t^2}(t),\frac{\partial u_\zeta^\varepsilon}{\partial t}(t)\right)+a\left(\frac{\partial u_\zeta^\varepsilon}{\partial t}(t),\frac{\partial u_\zeta^\varepsilon}{\partial t}(t)\right)+l^\varepsilon\left\|\frac{\partial u_\zeta^\varepsilon}{\partial t}(t)\right\|_{0,\Gamma_1^\varepsilon}^2\\&+\left\langle\frac{d}{dt}\left(\ J_\zeta^\varepsilon\right)'\left(u_\zeta^\varepsilon(t)\right),\frac{\partial u_\zeta^\varepsilon}{\partial t}(t)\right\rangle=\left(\frac{\partial f^\varepsilon}{\partial t}(t),\frac{\partial u_\zeta^\varepsilon}{\partial t}(t)\right).\end{aligned} \tag{3.15}$$

Taking into account $K_{\mathrm{div}}^\varepsilon\subset L^2(\Omega^\varepsilon)^3\subset\left(K_{\mathrm{div}}^\varepsilon\right)'$ and by [12], the following inequality holds: there exists a positives constants ρ and λ, such that

$$a(v,v)+\rho\left\|v\right\|_{0,\Omega^\varepsilon}^2\geq\lambda\left\|v\right\|_{1,\Omega^\varepsilon}^2\ \ \forall v\in K^\varepsilon.$$

We know that the operator $\left(J_\zeta^\varepsilon \right)'$ is monotonous, we have

$$\begin{aligned} & \left\langle \frac{d}{dt} \left(J_\zeta^\varepsilon \right)' (\phi(t)) , \phi'(t) \right\rangle \\ = & \int_\omega k^\varepsilon \lim_{s \to 0} \frac{\psi_\zeta(\phi_\tau(t+s)) - \psi_\zeta(\phi_\tau(t))}{s} . \frac{\phi_\tau(t+s) - \phi_\tau(t)}{s} dx' \\ & + \sqrt{2} \alpha \varepsilon^{-1} \int_{\Omega^\varepsilon} \lim_{s \to 0} \frac{\Psi_\zeta(\phi(t+s)) - \Psi_\zeta(\phi(t))}{s} . \frac{\phi(t+s) - \phi(t)}{s} dx' \\ \geq & \ 0. \end{aligned}$$

So, the formula (3.15) becomes

$$\begin{aligned} & \left\| \frac{\partial u_\zeta^\varepsilon}{\partial t}(t) \right\|_{0,\Omega^\varepsilon}^2 + \lambda \int_0^t \left\| \frac{\partial u_\zeta^\varepsilon}{\partial t}(s) \right\|_{1,\Omega^\varepsilon}^2 ds + 2 l^\varepsilon \int_0^t \left\| \frac{\partial u_\zeta^\varepsilon}{\partial t}(s) \right\|_{0,\Gamma_1^\varepsilon}^2 ds \\ \leq & \left\| \frac{\partial u_\zeta^\varepsilon}{\partial t}(0) \right\|_{0,\Omega^\varepsilon}^2 + (\rho + 1) \int_0^t \left\| \frac{\partial u_\zeta^\varepsilon}{\partial t}(s) \right\|_{0,\Omega^\varepsilon}^2 ds + \int_0^t \left\| \frac{\partial f^\varepsilon}{\partial t}(s) \right\|_{0,\Omega^\varepsilon}^2 ds \end{aligned} \tag{3.16}$$

But, $\dfrac{\partial u_\zeta^\varepsilon}{\partial t}(0)$ is defined by, for all $\phi \in K_{\text{div}}^\varepsilon$,

$$\left(\frac{\partial u_\zeta^\varepsilon}{\partial t}(0), \phi \right) = (f^\varepsilon(0), \phi) - a(u_0^\varepsilon, \phi) - \left\langle \left(J_\zeta^\varepsilon \right)' (u_0^\varepsilon) , \phi \right\rangle$$

Consequently, we deduce that

$$\frac{\partial u_\zeta^\varepsilon}{\partial t}(0) = f^\varepsilon(0) - A(u_0^\varepsilon) - \left(J_\zeta^\varepsilon \right)' (u_0^\varepsilon) \text{ in } L^2 \left(\Omega^\varepsilon \right)^3 \tag{3.17}$$

where $A(u_0^\varepsilon) \in \mathcal{L} \left(K_{\text{div}}^\varepsilon ; K_{\text{div}}^{\varepsilon'} \right)$ is given by Riesz's representation theorem,

$$\langle A(u_0^\varepsilon), \phi \rangle = a(u_0^\varepsilon, \phi).$$

According to lemma 3.2 and the assumptions (3.1), we have

$$\left\| \frac{\partial u_\zeta^\varepsilon}{\partial t}(0) \right\|_{0,\Omega^\varepsilon} \leq \text{cte (independent of } \zeta).$$

This, joined to (3.16) and using a Gronwall lemma, shows that

$$\left\| \frac{\partial u_\zeta^\varepsilon}{\partial t}(t) \right\|_{0,\Omega^\varepsilon}^2 + \int_0^t \left\| \frac{\partial u_\zeta^\varepsilon}{\partial t}(s) \right\|_{1,\Omega^\varepsilon}^2 ds + l^\varepsilon \int_0^t \left\| \frac{\partial u_\zeta^\varepsilon}{\partial t}(s) \right\|_{0,\Gamma_1^\varepsilon}^2 ds \leq c. \tag{3.18}$$

By (3.14) and (3.18), we can extract from u_ζ^ε a sequence denoted u_δ^ε such that the following convergences in $L^\infty \left(0, T; L^2 \left(\Omega^\varepsilon \right)^3 \right) \cap L^2 \left(0, T; H^1 \left(\Omega^\varepsilon \right)^3 \right)$:

$$u_\delta^\varepsilon \longrightarrow u^\varepsilon, \ \frac{\partial u_\delta^\varepsilon}{\partial t} \longrightarrow \frac{\partial u^\varepsilon}{\partial t}.$$

We deduce from equation (3.7) that

$$\left(\frac{\partial u_\delta^\varepsilon}{\partial t}, \phi - u_\delta^\varepsilon\right) + a\left(u_\delta^\varepsilon, \phi - u_\delta^\varepsilon\right) + l^\varepsilon \int_{\Gamma_1^\varepsilon} u_\delta^\varepsilon\left(\phi - u_\delta^\varepsilon\right) d\tau + J_\delta^\varepsilon\left(\phi\right)$$
$$+J_\delta^\varepsilon\left(u_\delta^\varepsilon\right) - \left(f^\varepsilon, \phi - u_\delta^\varepsilon\right) = J_\delta^\varepsilon\left(\phi\right) - J_\delta^\varepsilon\left(u_\delta^\varepsilon\right) - \left\langle \left(J_\delta^\varepsilon\right)'\left(u_\delta^\varepsilon\right), \phi - u_\delta^\varepsilon\right\rangle \geq 0$$

Finally, passing to the limit in δ as in [12], and using the semi-continuous inferior of the function $u \to \int_0^T \breve{a}(u,u)dt$ and $v \to \int_0^T J^\varepsilon\left(v\right)dt$ for $L^2\left(0,T;K_{\text{div}}^\varepsilon\right)$ with the weak topology, to obtain (2.9)-(2.10).
The proof of uniqueness is analogous to [8], and this concludes the proof of theorem 3.1. □

4. Some estimates and convergence

4.1. The rescaled problem

To estimate the solutions $\{u^\varepsilon, p^\varepsilon\}$ we use the scaling $z = x_3/\varepsilon$ and the following fixed domains

$$\begin{aligned} \Omega &= \left\{(x', z) \in \mathbb{R}^3 : (x', 0) \in \omega, \quad 0 < z < h(x')\right\}, \\ Q &= \Omega \times]0, T[\,. \end{aligned}$$

We denote by Γ_1 is the upper boundary of the equation $z = h\left(x\right)$ and Γ_L is the lateral boundary. This rescaling maps the spaces K^ε, $K_{\text{div}}^\varepsilon$ and $L_0^2(\Omega^\varepsilon)$ onto the spaces K, K_{div} and $L_0^2(\Omega)$ respectively, are defined by:

$$\begin{aligned} K &= \left\{\phi \in H^1(\Omega)^3 : \phi = 0 \text{ on } \Gamma_L,\ \phi.\nu = 0 \text{ on } \omega \cup\ \Gamma_1\right\}, \\ K_{\text{div}} &= \left\{\phi \in K : \text{div}(\phi) = 0 \text{ in } \Omega\ \right\}, \\ L_0^2(\Omega) &= \left\{q \in L^2(\Omega) : \int_\Omega q\ dx = 0\right\}. \end{aligned}$$

We denote by $\widehat{u}^\varepsilon = (\widehat{u}_1^\varepsilon, \widehat{u}_2^\varepsilon, \widehat{u}_3^\varepsilon)$ and $\widehat{p}^\varepsilon$ the rescaling of the solution by $\{u^\varepsilon, p^\varepsilon\}$ of problem (2.9)-(2.10). For any $(x', z, t) \in Q$, we set

$$\begin{aligned} \widehat{u}_i^\varepsilon\left(x', z, t\right) &= u_i^\varepsilon\left(x', x_3, t\right)\ i = 1, 2,\ \widehat{u}_3^\varepsilon\left(x', z, t\right) = \varepsilon^{-1} u_3^\varepsilon\left(x', x_3, t\right), \\ \left(\widehat{u}_0^\varepsilon\right)_i\left(x', z\right) &= \left(u_0^\varepsilon\right)_i\left(x', x_3\right)\ i = 1, 2,\ \left(\widehat{u}_0^\varepsilon\right)_3\left(x', z\right) = \varepsilon^{-1}\left(u_0^\varepsilon\right)_3\left(x', x_3\right), \\ \widehat{p}^\varepsilon\left(x', z, t\right) &= \varepsilon^2 p^\varepsilon\left(x', x_3, t\right), \end{aligned}$$

and defining the rescaled force by

$$f^\varepsilon\left(x', x_3, t\right) = \varepsilon^{-2} \widehat{f}\left(x', z, t\right).$$

To meet our needs in paragraph 4.2, according to [5] we must assume

$$\begin{cases} \mu C(\Gamma_1^\varepsilon) \leq l^\varepsilon \\ \text{where } C(\Gamma_1^\varepsilon) = 2\|\frac{\partial}{\partial x_2} h^\varepsilon\|_{\mathcal{C}(\bar{\omega})}(1 + \|\frac{\partial}{\partial x_1} h^\varepsilon\|_{\mathcal{C}(\bar{\omega})}^2 \\ l^\varepsilon = \varepsilon^{-1}\ l \ \text{ and } l \text{ be not dependent on } \varepsilon \end{cases} \tag{4.1}$$

One can check that $\{\widehat{u}^\varepsilon, \widehat{p}^\varepsilon\}$ solves the rescaled problem

$$\left.\begin{array}{c}
\sum_{i=1,2} \varepsilon^2 \left(\frac{\partial \widehat{u}_i^\varepsilon}{\partial t}, \widehat{\phi}_i - \widehat{u}_i^\varepsilon \right) + \varepsilon^4 \left(\frac{\partial \widehat{u}_3^\varepsilon}{\partial t}, \widehat{\phi}_3 - \widehat{u}_3^\varepsilon \right) + \widehat{a}\left(\widehat{u}^\varepsilon, \widehat{\phi} - \widehat{u}^\varepsilon \right) \\
- \sum_{i=1,2} \int_\Omega \widehat{p}^\varepsilon \frac{\partial(\widehat{\phi}_i - \widehat{u}_i^\varepsilon)}{\partial x_i} dx' dz - \int_\Omega \frac{1}{\varepsilon} \widehat{p}^\varepsilon \frac{\partial(\widehat{\phi}_3 - \widehat{u}_3^\varepsilon)}{\partial z} dx' dz \\
+ \sum_{i=1,2} l \int_{\Gamma_1} \widehat{u}_i^\varepsilon \left(\widehat{\phi}_i - \widehat{u}_i^\varepsilon \right) d\tau + l \int_{\Gamma_1} \varepsilon^2 \widehat{u}_3^\varepsilon \left(\widehat{\phi}_3 - \widehat{u}_3^\varepsilon \right) d\tau \\
+\sqrt{2}\alpha\varepsilon^{-1} \int_\Omega \left(\left| \widetilde{D}\left(\widehat{\phi}\right) \right| - \left| \widetilde{D}\left(\widehat{u}_\tau^\varepsilon\right) \right| \right) dx + \int_\omega k \left(\left| \widehat{\phi}_\tau \right| - \left| (\widehat{u}^\varepsilon)_\tau \right| \right) dx' \\
\geq \sum_{i=1,2} \int_\Omega \widehat{f}_i \left(\widehat{\phi}_i - \widehat{u}_i^\varepsilon \right) dx' dz + \varepsilon \int_\Omega \widehat{f}_3 \left(\widehat{\phi}_3 - \widehat{u}_3^\varepsilon \right) dx' dz \\
\forall \widehat{\phi} \in K, \ \forall t \in]0, T[, \\
\widehat{u}^\varepsilon(0) = \widehat{u}_0^\varepsilon,
\end{array}\right\} \quad (4.2)$$

where

$$\begin{aligned}
\widehat{a}\left(\widehat{u}^\varepsilon(t), \widehat{\phi} - \widehat{u}^\varepsilon(t) \right) = & \sum_{i,j=1,2} \int_\Omega \varepsilon^2 \mu \left(\frac{\partial \widehat{u}_i^\varepsilon}{\partial x_j} + \frac{\partial \widehat{u}_j^\varepsilon}{\partial x_i} \right) \frac{\partial}{\partial x_j} (\widehat{\phi}_i - \widehat{u}_i^\varepsilon) dx' dz \\
& + \sum_{i=1,2} \int_\Omega \mu \left(\frac{\partial \widehat{u}_i^\varepsilon}{\partial z} + \varepsilon^2 \frac{\partial \widehat{u}_3^\varepsilon}{\partial x_i} \right) \frac{\partial}{\partial z} (\widehat{\phi}_i - \widehat{u}_i^\varepsilon) dx' dz \\
& + \int_\Omega 2\mu\varepsilon^2 \frac{\partial \widehat{u}_3^\varepsilon}{\partial z} \frac{\partial(\widehat{\phi}_3 - \widehat{u}_3^\varepsilon)}{\partial z} dx' dz \\
& + \sum_{j=1,2} \int_\Omega \mu\varepsilon^2 \left(\varepsilon^2 \frac{\partial \widehat{u}_3^\varepsilon}{\partial x_j} + \frac{\partial \widehat{u}_j^\varepsilon}{\partial z} \right) \frac{\partial}{\partial x_j} (\widehat{\phi}_3 - \widehat{u}_3^\varepsilon) dx' dz,
\end{aligned}$$

and

$$\left| \widetilde{D}(v) \right| = \left[\frac{1}{4} \sum_{i,j=1}^{2} \varepsilon^2 \left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right)^2 + \frac{1}{2} \sum_{i=1}^{2} \left(\frac{\partial v_i}{\partial z} + \varepsilon^2 \frac{\partial v_3}{\partial x_i} \right)^2 + \varepsilon^2 \left(\frac{\partial v_3}{\partial z} \right)^2 \right]^{\frac{1}{2}}.$$

4.2. Estimates of solutions

We have the following estimate theorem

Theorem 4.1. *Assume that* (4.1) *hold, and let* $\{u^\varepsilon, p^\varepsilon\}$ *be a solution of problem (2.9)-(2.10). Then, there exist three constants* C, $\tilde{C}$ and $\tilde{C}'$ *independents of* ε *such that*

$$\left.\begin{array}{l}
\sum_{i=1}^{2} \left(\|\varepsilon \widehat{u}_i^\varepsilon(t)\|_{0,\Omega}^2 + \int_0^t \left\| \frac{\partial \widehat{u}_i^\varepsilon}{\partial z}(s) \right\|_{0,\Omega}^2 ds + \int_0^t \left\| \varepsilon^2 \frac{\partial \widehat{u}_3^\varepsilon}{\partial x_i}(s) \right\|_{0,\Omega}^2 ds \right) + \\
\left\| \varepsilon^2 \widehat{u}_3^\varepsilon(t) \right\|_{0,\Omega}^2 + \int_0^t \left\| \varepsilon \frac{\partial \widehat{u}_3^\varepsilon}{\partial z}(s) \right\|_{0,\Omega}^2 ds + \sum_{i,j=1}^{2} \int_0^t \left\| \varepsilon \frac{\partial \widehat{u}_i^\varepsilon}{\partial x_j}(s) \right\|_{0,\Omega}^2 ds \leq C,
\end{array}\right. \quad (4.3)$$

$$\sum_{i=1,2} \|\widehat{u}_i^\varepsilon\|_{L^2(Q)}^2 + \|\varepsilon \widehat{u}_3^\varepsilon\|_{L^2(Q)}^2 \leq \tilde{C}, \quad (4.4)$$

$$\sum_{i=1,2} \|\varepsilon \frac{\partial \widehat{u}_i^\varepsilon}{\partial t}\|^2_{L^2(Q)} + \|\varepsilon^2 \frac{\partial \widehat{u}_3^\varepsilon}{\partial t}\|^2_{L^2(Q)} \leq \tilde{C}'. \tag{4.5}$$

Proof. From [8], we recall the following inequalities (Poincaré, Korn and Young respectively)

$$\|u^\varepsilon(t)\|^2_{0,\Omega^\varepsilon} \leq 2\overline{h}^2 \varepsilon^2 \|\nabla u^\varepsilon(t)\|^2_{0,\Omega^\varepsilon} + 2\overline{h}\varepsilon \int_{\Gamma_1^\varepsilon} \|u^\varepsilon(t)\|^2_{0,\Gamma_1^\varepsilon} d\tau, \tag{4.6}$$

$$\mu \|\nabla u^\varepsilon(t)\|^2_{0,\Omega^\varepsilon} \leq a(u^\varepsilon(t), u^\varepsilon(t)) + \mu C(\Gamma_1^\varepsilon) \int_{\Gamma_1^\varepsilon} \|u^\varepsilon(t)\|^2_{0,\Gamma_1^\varepsilon} d\tau, \tag{4.7}$$

$$ab \leq \theta^2 \frac{a^2}{2} + \theta^{-2}\frac{b^2}{2}, \ \forall (a,b) \in \mathbb{R}^2, \ \forall \theta \in \mathbb{R}^*.$$

Integrating (2.9) over $[0,t]$ and choosing $\phi = 0$, we have

$$\begin{aligned} &\frac{1}{2}\|u^\varepsilon(t)\|^2_{0,\Omega^\varepsilon} + \int_0^t a(u^\varepsilon(s), u^\varepsilon(s))ds + l^\varepsilon \int_0^t \|u^\varepsilon(s)\|^2_{0,\Gamma_1^\varepsilon} ds \\ &\leq \frac{1}{2}\|u_0^\varepsilon\|^2_{0,\Omega} + \int_0^t (f^\varepsilon(s), u^\varepsilon(s))\, ds. \end{aligned} \tag{4.8}$$

Hence, by using Hölder, Poincaré and Young inequalities for $\theta = \sqrt{\mu/2}$,

$$a = \|\nabla u^\varepsilon(s)\|_{0,\Omega^\varepsilon} \text{ and } b = \varepsilon\overline{h}\, \|f^\varepsilon(s)\|_{0,\Omega^\varepsilon},$$

then $\theta = \sqrt{l^\varepsilon/2}$, $a = \|u^\varepsilon(s)\|^2_{0,\Gamma_1^\varepsilon}$ and $b = \sqrt{\overline{h}\varepsilon}\, \|f^\varepsilon(s)\|^2_{0,\Omega^\varepsilon}$, respectively, we get

$$\begin{aligned} \left|\int_0^t (f^\varepsilon(s), u^\varepsilon(s))ds\right| &\leq \frac{\mu}{4}\int_0^t \|\nabla u^\varepsilon(s)\|^2_{0,\Omega^\varepsilon} ds + \frac{2\varepsilon^2\overline{h}^2}{\mu}\int_0^t \|f^\varepsilon(s)\|^2_{0,\Omega^\varepsilon} ds \\ &+ \frac{l^\varepsilon}{4}\int_0^t \|u^\varepsilon(s)\|^2_{0,\Gamma_1^\varepsilon} ds + \frac{2\overline{h}\varepsilon}{l^\varepsilon}\int_0^t \|f^\varepsilon(s)\|^2_{0,\Omega^\varepsilon} ds. \end{aligned} \tag{4.9}$$

Ignoring the first term of (4.8) and combining (4.1), (4.7) and (4.9) we infer

$$\begin{aligned} &\frac{\mu}{4}\int_0^t \|\nabla u^\varepsilon(s)\|^2_{0,\Omega^\varepsilon} ds + \frac{l^\varepsilon}{4}\int_0^t \|u^\varepsilon(s)\|^2_{0,\Gamma_1^\varepsilon} ds \\ &\leq \frac{1}{2}\|u_0^\varepsilon\|^2_{0,\Omega} + \left(\frac{2\varepsilon^2\overline{h}^2}{\mu} + \frac{2\overline{h}\varepsilon}{l^\varepsilon}\right)\int_0^t \|f^\varepsilon(s)\|^2_{0,\Omega^\varepsilon} ds, \end{aligned}$$

multiplying the last inequality by $4\varepsilon^2$ and passing to the fixed domain in the right hand, we get

$$\varepsilon^2 \int_0^t \|\nabla u^\varepsilon(s)\|^2_{0,\Omega^\varepsilon} ds + \varepsilon \int_0^t \|u^\varepsilon(s)\|^2_{0,\Gamma_1^\varepsilon} ds \leq C \tag{4.10}$$

where C does not depend on ε.

We change again to the fixed domain in the first term of inequality (4.10), we find (4.3). From (4.6) and (4.10), it is easy to obtain a constant $\tilde{C} = \max(2\overline{h}^2, 2\overline{h})C$, such that

$$\varepsilon^{-1}\int_0^t \|u^\varepsilon(s)\|^2_{0,\Omega^\varepsilon}\, ds \le \tilde{C}$$

In fact, the last estimate is equivalent to (4.4).
Now, from (3.15) and as

$$\left\langle \left(J^\varepsilon_\zeta\right)''\left(u^\varepsilon_\zeta\right), \frac{\partial}{\partial t}u^\varepsilon_\zeta\right\rangle \ge 0,$$

we have

$$\begin{aligned}\frac{1}{2}\left\|\frac{\partial u^\varepsilon_\zeta}{\partial t}(t)\right\|^2_{0,\Omega^\varepsilon} + \int_0^t a\left(\frac{\partial u^\varepsilon_\zeta}{\partial t}(s), \frac{\partial u^\varepsilon_\zeta}{\partial t}(s)\right) ds + l^\varepsilon \int_0^t \left\|\frac{\partial u^\varepsilon_\zeta}{\partial t}(s)\right\|^2_{0,\Gamma^\varepsilon_1} ds \\ \le \tfrac{1}{2}\left\|\frac{\partial u^\varepsilon_\zeta}{\partial t}(0)\right\|^2_{0,\Omega^\varepsilon} + \int_0^t \left\|\frac{\partial f^\varepsilon}{\partial t}(s)\right\|_{0,\Omega^\varepsilon}\left\|\frac{\partial u^\varepsilon_\zeta}{\partial t}(s)\right\|_{0,\Omega^\varepsilon} ds\end{aligned} \tag{4.11}$$

By applying the inequality (4.6) for $\frac{\partial}{\partial t}u^\varepsilon_\zeta$ and the Young successively, we get

$$\begin{aligned}\left|\int_0^t \left(\frac{\partial f^\varepsilon}{\partial t}(s), \frac{\partial u^\varepsilon_\zeta}{\partial t}(s)\right) ds\right| \le{}& \frac{\mu}{8}\int_0^t \left\|\nabla\frac{\partial u^\varepsilon_\zeta}{\partial t}(s)\right\|^2_{0,\Omega^\varepsilon} ds + \frac{4\varepsilon^2\overline{h}^2}{\mu}\int_0^t \left\|\frac{\partial f^\varepsilon}{\partial t}(s)\right\|^2_{0,\Omega^\varepsilon} ds \\ &+ \frac{3l^\varepsilon}{4}\int_0^t \left\|\frac{\partial u^\varepsilon_\zeta}{\partial t}(s)\right\|^2_{0,\Gamma^\varepsilon_1} ds + \frac{2\overline{h}\varepsilon}{3l^\varepsilon}\int_0^t \left\|\frac{\partial f^\varepsilon}{\partial t}(s)\right\|^2_{0,\Omega^\varepsilon} ds.\end{aligned} \tag{4.12}$$

From (4.11), (4.12) and using (4.7) we obtain

$$\begin{aligned}\frac{1}{2}\left\|\frac{\partial u^\varepsilon_\zeta}{\partial t}(t)\right\|^2_{0,\Omega^\varepsilon} + \frac{\mu}{16}\int_0^t \left\|\nabla\frac{\partial u^\varepsilon_\zeta}{\partial t}(s)\right\|^2_{0,\Omega^\varepsilon} ds + \frac{l^\varepsilon}{16}\int_0^t \left\|\frac{\partial u^\varepsilon_\zeta}{\partial t}(s)\right\|^2_{0,\Gamma^\varepsilon_1} ds \le \\ \frac{1}{2}\left\|\frac{\partial u^\varepsilon_\zeta}{\partial t}(0)\right\|^2_{0,\Omega^\varepsilon} + \frac{4\varepsilon^2\overline{h}^2}{\mu}\int_0^t \left\|\frac{\partial f^\varepsilon}{\partial t}(s)\right\|^2_{0,\Omega^\varepsilon} ds + \frac{2\overline{h}\varepsilon}{3l^\varepsilon}\int_0^t \left\|\frac{\partial f^\varepsilon}{\partial t}(s)\right\|^2_{0,\Omega^\varepsilon} ds\end{aligned} \tag{4.13}$$

We must estimate $\frac{\partial u^\varepsilon_\zeta}{\partial t}(0)$. Starting from the equation (3.17) and taking into account the assumptions (3.1), (3.3), then applying lemma 3.2, we conclude

$$\begin{aligned}\left\|\frac{\partial u^\varepsilon_\zeta}{\partial t}(0)\right\|_{0,\Omega^\varepsilon} &\le \|f^\varepsilon(0)\|_{0,\Omega^\varepsilon} + \|A(u^\varepsilon_0)\|_{0,\Omega^\varepsilon} + \left\|\left(J^\varepsilon_\zeta\right)'(u^\varepsilon_0)\right\|_{0,\Omega^\varepsilon} \\ &\le \|f^\varepsilon(0)\|_{0,\Omega^\varepsilon} + 2\sqrt{3}\mu\,\|u^\varepsilon_0\|_{2,\Omega^\varepsilon} + \varepsilon^{-1}\gamma\,\|u^\varepsilon_0\|_{2,\Omega^\varepsilon}.\end{aligned}$$

We recall that $\varepsilon \in]0,1[$, by multiplying the last inequality by $\varepsilon^{\frac{5}{2}}$ and the fact that $\varepsilon^3 \left\| u_0^\varepsilon \right\|_{2,\Omega^\varepsilon}^2 \leq \left\| \widehat{u}_0 \right\|_{2,\Omega}^2$, we deduce

$$\varepsilon^{\frac{5}{2}} \left\| \frac{\partial u_\zeta^\varepsilon}{\partial t}(0) \right\|_{0,\Omega^\varepsilon} \leq c_0 \tag{4.14}$$

with

$$c_0 = \left\| \widehat{f}(0) \right\|_{0,\Omega} + 2\sqrt{3}\mu \left\| \widehat{u}_0 \right\|_{2,\Omega} + \gamma \left\| \widehat{u}_0 \right\|_{2,\Omega} .$$

Consequently, it follows from (4.13)-(4.14) and passing to the limit when $\zeta \to 0$, we find (after multiplying by $2\varepsilon^5$)

$$\frac{\mu}{8}\varepsilon^5 \int_0^t \left\| \nabla \frac{\partial u^\varepsilon}{\partial t}(s) \right\|_{0,\Omega^\varepsilon}^2 ds + \frac{l}{8}\varepsilon^4 \int_0^t \left\| \frac{\partial u^\varepsilon}{\partial t}(s) \right\|_{0,\Gamma_1^\varepsilon}^2 ds \leq C' \tag{4.15}$$

with

$$C' = (c_0)^2 + \frac{8\overline{h}^2}{\mu} \left\| \frac{\partial}{\partial t}\widehat{f} \right\|_{L^2(Q)}^2 + \frac{4\overline{h}}{3l} \left\| \frac{\partial}{\partial t}\widehat{f} \right\|_{L^2(Q)}^2$$

is a constant independent of ε.
We apply the inequality (4.6) for $\dfrac{\partial u^\varepsilon}{\partial t}$ in the estimate (4.15), that implies that there exists a constant $\tilde{C}'$ independent of ε such that

$$\varepsilon \int_0^t \left\| \frac{\partial u^\varepsilon}{\partial t}(s) \right\|_{0,\Omega^\varepsilon}^2 ds \leq \tilde{C}'.$$

Finally, passing this estimate to the fixed domain Ω to get (4.5). □

Theorem 4.2. *Under the hypotheses of theorem 4.1 there exists a constant C independent of ε such that*

$$\left\| \frac{\partial \hat{p}^\varepsilon}{\partial x_i} \right\|_{L^2(0,T,H^{-1}(\Omega))} \leq C, \; i = 1,2 \text{ and } \left\| \frac{\partial \hat{p}^\varepsilon}{\partial z} \right\|_{L^2(0,T,H^{-1}(\Omega))} \leq C\varepsilon. \tag{4.16}$$

Proof. Let ξ in $L^2(0,T,H_0^1(\Omega))$, putting in (4.2) $\phi = \widehat{u}^\varepsilon + \tilde{\xi}$, where $\tilde{\xi} = (\xi,0,0)$ or $\tilde{\xi} = (0,\xi,0)$ and integrating over $[0,t]$ we find for $i = 1,2$,

$$\int_0^t \left(\frac{\partial \widehat{p}^\varepsilon}{\partial x_i}(s), \xi(s) \right) ds \leq \int_0^t \varepsilon^2 \left(\frac{\partial \widehat{u}_i^\varepsilon}{\partial t}(s), \xi(s) \right) ds$$

$$+\mu \sum_{i,j=1,2} \int_0^t \int_\Omega \varepsilon^2 \left(\frac{\partial \widehat{u}_i^\varepsilon}{\partial x_j} + \frac{\partial \widehat{u}_j^\varepsilon}{\partial x_i} \right)(s) \frac{\partial \xi}{\partial x_j}(s)\, dx'dzds$$

$$+\sqrt{2}\alpha \int_0^t \int_\Omega \left(\left| \widetilde{D}\left(\widehat{u}^\varepsilon + \tilde{\xi} \right) \right| - \left| \widetilde{D}\left(\widehat{u}^\varepsilon \right) \right| \right) dx'dzds$$

$$+\mu \sum_{i=1,2} \int_0^t \int_\Omega \left(\frac{\partial \widehat{u}_i^\varepsilon}{\partial z} + \varepsilon^2 \frac{\partial \widehat{u}_3^\varepsilon}{\partial x_i} \right)(s) \frac{\partial \xi}{\partial z}(s)\, dx' dz ds - \int_0^t \left(\widehat{f}_i(s), \xi(s) \right) ds.$$

The Hölder inequality and estimates (4.3)-(4.5) show the continuity of the linear functional

$$\xi \to \int_0^t \left(\frac{\partial \widehat{p}^\varepsilon}{\partial x_i}(s), \xi(s) \right) ds,$$

which proves (4.16) for $i = 1, 2$. In addition, case $i = 3$ follows from the choice $\phi = \widehat{u}^\varepsilon(t) \pm \xi$ with $\xi \equiv (0, 0, \xi)$. □

4.3. Convergence u^ε and p^ε

To establish a limit solution of the problem, we introduce the following space,

$$V_z = \left\{ v = (v_1, v_2) \in L^2(\Omega)^2 : \frac{\partial v}{\partial z} \in L^2(\Omega)^2;\ v = 0 \text{ on } \Gamma_L \right\}.$$

From [6], $L^2(0, T, V_z)$ is a Banach space. We show the following result:

Theorem 4.3. *Under the hypotheses of theorem 4.1, for any solution* $\{u^\varepsilon, p^\varepsilon\}$*, there exist* $u^\star = (u_1^\star, u_2^\star) \in L^2(0, T, V_z)$ *and* $p^\star \in L^2(0, T, L_0^2(\Omega))$ *such that when* ε *tends to 0 we have the following convergences in* $L^2(0, T, V_z)$:

$$(\widehat{u}_1^\varepsilon, \widehat{u}_2^\varepsilon) \rightharpoonup (u_1^\star, u_2^\star), \quad \varepsilon^2 \left(\frac{\partial}{\partial t} \widehat{u}_1^\varepsilon, \frac{\partial}{\partial t} \widehat{u}_2^\varepsilon \right) \rightharpoonup 0 \tag{4.17}$$

the following convergences in $L^2(Q)$:

$$\varepsilon \widehat{u}_3^\varepsilon \rightharpoonup 0, \quad \varepsilon^3 \frac{\partial \widehat{u}_3^\varepsilon}{\partial t} \rightharpoonup 0, \quad \varepsilon \frac{\partial \widehat{u}_i^\varepsilon}{\partial x_j} \rightharpoonup 0, \quad \varepsilon^2 \frac{\partial \widehat{u}_3^\varepsilon}{\partial x_j} \rightharpoonup 0, \quad \varepsilon \frac{\partial \widehat{u}_3^\varepsilon}{\partial z} \rightharpoonup 0 \tag{4.18}$$

$(1 \le i, j \le 2)$, *and the convergence* $\widehat{p}^\varepsilon \rightharpoonup p^\star$ *in* $L^2(0, T, L_0^2(\Omega))$.
Moreover, $p^\star$ *depends only on* x'.

Proof. In particular (4.3), (4.4) we have

$$\left\| \frac{\partial \widehat{u}_i^\varepsilon}{\partial z} \right\|_{L^2(Q)}^2 \le C \text{ and } \|\widehat{u}_i^\varepsilon\|_{L^2(Q)}^2 \le \tilde{C},$$

for $i = 1, 2$, we deduce the first convergence of (4.17). Similarly, from (4.15) and (4.5) we find the second. For the rest of the proof, we use the same steps in the stationary case as in [1, 5]. □

5. On the limit model

By a classical semi continuity argument and using the convergence results of the theorem 4.3, we deduce that (4.2) leads to the system

$$\left.\begin{array}{c}
\sum_{i=1}^{2}\mu\int_{\Omega}\frac{\partial u_i^{\star}}{\partial z}(t)\frac{\partial}{\partial z}(\widehat{\phi}_i-u_i^{\star}(t))dx'dz \\
-\int_{\Omega}p^{\star}(x',t)\left(\frac{\partial\widehat{\phi}_1}{\partial x_1}+\frac{\partial\widehat{\phi}_2}{\partial x_2}\right)dx'dz \\
-\int_{\omega}p^{\star}(x',t)\left(\widehat{\phi}_1(x',h(x'))\frac{\partial h}{\partial x_1}+\widehat{\phi}_2(x',h(x'))\frac{\partial h}{\partial x_2}\right)dx' \\
+\sum_{i=1}^{2}l\int_{\Gamma_1}u_i^{\star}(t)\left(\widehat{\phi}_i-u_i^{\star}(t)\right)d\tau \\
+\alpha\int_{\Omega}\left(\left|\frac{\partial\widehat{\phi}}{\partial z}\right|-\left|\frac{\partial u^{\star}}{\partial z}(t)\right|\right)dx'dz+\int_{\omega}k(|\widehat{\phi}|-|u^{\star}(t)|)dx' \\
\geq\sum_{i=1}^{2}(\widehat{f}_i(t),\widehat{\phi}_i-u_i^{\star}(t))\quad\forall\widehat{\phi}\in\Pi(K),\forall t\in\,]0,T[, \\
\\
u_i^{\star}(x',z,0)=\widehat{u}_{0,i},\quad i=1,2
\end{array}\right\}\tag{5.1}$$

where

$$\Pi(K)=\left\{\left(\widehat{\phi}_1,\widehat{\phi}_2\right)\in H^1(\Omega)^2:\widehat{\phi}=\left(\widehat{\phi}_1,\widehat{\phi}_2,\widehat{\phi}_3\right)\in K\right\}.$$

Theorem 5.1. *Under the assumptions of theorem 4.1, the limit solution $\{u^{\star},\ p^{\star}\}$ satisfies:*

$$-\frac{\partial}{\partial z}\sigma_i^{\star}(t)=\widehat{f}_i(t)-\frac{\partial}{\partial x_i}p^{\star}(t),\ i=1,2,\ \text{in } L^2(\Omega),\tag{5.2}$$

$$u_i^{\star}(0)=\widehat{u}_{0,i},\ i=1,2\tag{5.3}$$

for a.e. $t\in\,]0,T[$, where $\sigma^{\star}=(\sigma_i^{\star})_{i=1,2}$ checks the constitutive law of Bingham fluid, as follows

$$\left\{\begin{array}{ll}\sigma^{\star}=\mu\frac{\partial u^{\star}}{\partial z}+\alpha\frac{\partial u^{\star}/\partial z}{|\partial u^{\star}/\partial z|},\ \text{if}\ \left|\frac{\partial u^{\star}}{\partial z}\right|\neq 0, \\ |\sigma^{\star}|\leq\alpha,\qquad\text{if}\ \left|\frac{\partial u^{\star}}{\partial z}\right|=0\end{array}\right.\quad\text{in } Q\tag{5.4}$$

Proof. Let $\psi=(\psi_1,\psi_2)\in H_0^1(\Omega)^2$, putting in (5.1) $\widehat{\phi}=u^{\star}(t)\pm\lambda\psi$ ($\lambda>0$) and dividing the inequality obtained by λ, as λ tends to zero, for any t it follows that

$$\sum_{i=1}^{2}\mu\int_{\Omega}\frac{\partial u_i^{\star}}{\partial z}(t)\frac{\partial}{\partial z}\psi dx'dz-\int_{\Omega}p^{\star}(x',t)\left(\frac{\partial\psi_1}{\partial x_1}+\frac{\partial\psi_2}{\partial x_2}\right)dx'dz$$
$$+\sum_{i=1}^{2}\alpha\int_{\Omega}\left\{\left|\frac{\partial u^{\star}}{\partial z}(t)\right|^{-1}\frac{\partial u_i^{\star}}{\partial z}(t)\right\}\frac{\partial}{\partial z}\psi_i dx'dz=\sum_{i=1}^{2}\int_{\Omega}\widehat{f}_i(t)\psi_i dx'dz$$

when

$$\left|\frac{\partial u^\star}{\partial z}(t)\right| \neq 0.$$

By Green's formula, we obtain

$$-\sum_{i=1}^{2}\int_\Omega \mu\frac{\partial^2 u_i^\star}{\partial z^2}(t)\,\psi_i dx' + \sum_{i=1}^{2}\int_\Omega \frac{\partial p^\star}{\partial x_i}(x',t)\psi_i dx'dz$$
$$-\sum_{i=1}^{2}\alpha\int_\Omega \frac{\partial}{\partial z}\left\{\left|\frac{\partial u^\star}{\partial z}(t)\right|^{-1}\frac{\partial u_i^\star}{\partial z}(t)\right\}\psi_i dx'dz = \sum_{i=1}^{2}\int_\Omega \widehat{f}_i\,(t)\,\psi_i dx'dz.$$

Therefore, from this equality and fact that $\widehat{f} \in L^2(Q)$ we get (5.2). Similarly, the second case of (5.4) can be recovered by [7]. The condition (5.3) is a consequence directly of (4.17), (4.18) and the condition $\widehat{u}^\varepsilon(0) = \widehat{u}_0^\varepsilon$. □

Now we are in a position to deduce the equations corresponding for problem (5.1)-(5.4).

Remark 5.1. Note that the term related to inertia effects does not exist in the limit equation in (5.2), means that the limit problem (5.2) - (5.4) is in equilibrium at each time instant. Therefore, the Reynolds equation is obtained in a manner similar to the stationary case as in [1], and from [2] the Tresca boundary condition can be recovered. Indeed, the case $\alpha = 0$ corresponds to the Stokes flow, and has been studied in [8].

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Finite time blow-up for quasilinear wave equations with nonlinear dissipation

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Abstract. In this paper we consider a class of quasilinear wave equations

$$u_{tt} - \Delta_\alpha u - \omega_1 \Delta u_t - \omega_2 \Delta_\beta u_t + \mu |u_t|^{m-2} u_t = |u|^{p-2} u,$$

associated with initial and Dirichlet boundary conditions. Under certain conditions on α, β, m, p, we show that any solution with positive initial energy, blows up in finite time. Furthermore, a lower bound for the blow-up time will be given.

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1. Introduction

In this paper, we would like to study the blow-up of solutions of the following initial boundary value problem of a quasilinear wave equation

$$\begin{cases} u_{tt} - \Delta_\alpha u - \omega_1 \Delta u_t - \omega_2 \Delta_\beta u_t + \mu |u_t|^{m-2} u_t = |u|^{p-2} u, & x \in \Omega, \quad t > 0, \\ u(x,t) = 0, & x \in \partial\Omega, \quad t > 0, \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), & x \in \Omega. \end{cases} \tag{1.1}$$

Here, Ω is a bounded domain of $\mathbb{R}^n$ with a smooth boundary $\partial\Omega$. Additionally, we assume that

$$u_0 \in W_0^{1,\alpha}(\Omega), \quad u_1 \in L^2(\Omega), \tag{1.2}$$

and $\alpha, \beta, \omega_1, \omega_2, \mu, m, p$ are positive constants, with

$$\begin{cases} 2 < p \le \alpha^* = \frac{\alpha n}{n-\alpha}, & \text{for } n > \alpha, \\ 2 < p < \infty, & \text{for } n = \alpha. \end{cases} \tag{1.3}$$

The operator Δ_α is the classical α-Laplacian given by:

$$\Delta_\alpha u = \operatorname{div}\left(|\nabla u|^{\alpha-2} \nabla u\right).$$

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Notice that $\Delta_\beta u_t$ is a quasilinear strong damping term, and it is degenerate when $\beta > 2$.

Nonlinear hyperbolic equations of the type (1.1) have been investigated in the papers [2, 5, 7, 9, 15], and the references therein. Several examples of this type arise in physics, for example, the problem (1.1) represents a longitudinal motion of a viscoelastic rod obeying the nonlinear Voight model.

Zhijiang [14] proved a blow up result for the problem (1.1) when the initial energy is sufficiently negative. This result was extended by Messaoudi and Houari [8] to a situation when the solution has negative initial energy. Liu and Wang [6] studied a more general model including (1.1), and by improving the arguments in [14] and [8] they established a blow-up result in the subcritical initial energy case, i.e. $E(0) < d$, where $E(0)$ is the initial energy and d is the depth of the potential well.

For $\alpha = \beta = m = 2$, equation in (1.1) reduces to the linearly damped wave equation

$$u_{tt} - \Delta u + \omega \Delta u_t + \mu u_t = |u|^{p-2}u. \tag{1.4}$$

Gazzola and Squassina [3] studied (1.4) and gave a necessary and sufficient conditions for blow-up if $E(0) < d$. Recently, Yang and Xu [13] gave a sufficient condition for blow-up if $E(0) > d$. Sun et al. [12] obtained, for (1.4), an estimate of the lower bound for the blow-up time when $2 < p \leq \frac{2(n-1)}{n-2}$. This work was extended by Guo and Liu [4] to the case when the exponent $p \in \left(\frac{2(n-1)}{n-2}, \frac{2(n^2-2)}{n-2}\right]$. Later, in the case of $\omega > 0$, Baghaei [1] improved the results in [12] and [4] by enlarging the upper bound for p to 2^*.

In related work, Song and Xue [11] studied the following nonlinear wave equation with strong damping

$$u_{tt} - \Delta u + \int_0^t g((t-\tau)\Delta u(\tau)d\tau - \Delta u_t = |u|^{p-2}u. \tag{1.5}$$

They introduced a new technique to obtain a finite time blow-up result with arbitrary high initial energy in the case of linear strong damping. By applying the technique similar to that in [11], Song [10] extended the result in [11] to the case of nonlinear weak damping $\mu|u_t|^{m-2}u_t$ in place of $-\Delta u_t$ in (1.5).

In this paper, by using the technique in [10], we give sufficient conditions for finite time blow-up of solutions of (1.1), in the case $E(0) \geq d$. Furthermore, by using the techniques in [4], we obtain a lower bound for the blow-up time.

2. Preliminaries

We denote by $\|.\|_p$ the $L^p(\Omega)$ norm $(2 \leq p < \infty)$, and by $(.,.)$ the L^2 inner product. We introduce the following functional space

$$\mathcal{H} := L^\infty([0,T), W_0^{1,\alpha}(\Omega)) \cap W^{1,\infty}([0,T), L^2(\Omega)) \\ \cap W^{1,\beta}([0,T), W^{1,\beta}(\Omega)) \cap W^{1,m}([0,T), L^m(\Omega)),$$

for $T > 0$, and the energy functional

$$E(t) := \frac{1}{2}\|\nabla u\|_\alpha^\alpha + \frac{1}{2}\|u_t\|_2^2 - \frac{1}{p}\|u\|_p^p.$$

We define also the following constant

$$\lambda = B_*^{-\frac{p}{p-\alpha}},$$

where B_* is the best constant of the Sobolev embedding $W_0^{1,\alpha}(\Omega) \hookrightarrow L^p(\Omega)$. Finally, we characterize the depth of the potential well d as follows:

$$d = \left(\frac{1}{\alpha} - \frac{1}{p}\right)\lambda^2.$$

Lemma 2.1. *Let u be a global solution to problem (1.1). Then we have*

$$E'(t) = -\omega_1\|\nabla u_t\|_2^2 - \omega_2\|\nabla u_t\|_\beta^\beta - \mu\|u_t\|_m^m, \quad \forall t \geq 0.$$

As a consequence, we have the following inequalities:

$$E(t) \leq E(0), \quad \forall t \geq 0, \tag{2.1}$$

and

$$-E'(t) \geq \omega_1\|\nabla u_t\|_2^2, \quad -E'(t) \geq \omega_2\|\nabla u_t\|_\beta^\beta, \quad -E'(t) \geq \mu\|u_t\|_m^m. \tag{2.2}$$

Subsequently, we state the following theorems (see [6]).

Theorem 2.2 (Local existence). *Assume that conditions (1.2) and (1.3) hold. Then problem (1.1) has a unique local solution $u \in \mathcal{H}$.*

Theorem 2.3 (Blow-up for $E(0) < d$). *Assume (1.2) and (1.3) hold. Assume further that $\alpha, \beta, m \geq 2$ and $p > \alpha > \max\{m, \beta\}$. Suppose $E(0) < d$ and*

$$\|\nabla u_0\|_\alpha > \lambda. \tag{2.3}$$

Then u blows up in finite time.

3. Finite time blow-up

In this section we extend the blow-up result in [8] to the case $E(0) \geq d$. Here is our main result:

Theorem 3.1 (Blow-up for $E(0) \geq d$). *Assume (1.2), (2.3) and (1.3) hold. Assume further that $\alpha, \beta, m > 2$, $\alpha > \beta$ and $p > \max\{m, \alpha\}$. Suppose $E(0) \geq d$ and*

$$(u_t(0), u(0)) > ME(0), \tag{3.1}$$

where $M > 0$ is defined in (3.7), then the solution $u \in \mathcal{H}$ of (1.1) blows up in finite time.

Proof. Assume by contradiction that $u(t)$ is a global solution of (1.1). Setting

$$F(t) := \frac{1}{2}\|u(t)\|_2^2,$$

it follows from (1.1) that

$$\begin{aligned} F''(t) &= \|u_t\|_2^2 + \|u\|_p^p - \|\nabla u\|_\alpha^\alpha \\ &\quad - \omega_1(\nabla u_t, \nabla u) - \omega_2(|\nabla u_t|^{\beta-2}\nabla u_t, u) - \mu(|u_t|^{m-2}u_t, u). \end{aligned} \tag{3.2}$$

By using Hölder's inequality and Young's inequality, we estimate the two last terms in the right-hand side of the previous equation, as follows

$$(\nabla u_t, \nabla u) \le \eta\|\nabla u\|_2^2 + \tfrac{1}{4\eta}\|\nabla u_t\|_2^2, \quad \eta > 0,$$

$$(|\nabla u_t|^{\beta-2}\nabla u_t, u) \le \tfrac{1}{\beta}\sigma^\beta\|\nabla u\|_\beta^\beta + \tfrac{\beta-1}{\beta}\sigma^{\beta/(1-\beta)}\|\nabla u_t\|_\beta^\beta, \quad \sigma > 0,$$

$$(|u_t|^{m-2}u_t, u) \le \tfrac{1}{m}\delta^m\|u\|_m^m + \tfrac{m-1}{m}\delta^{m/(1-m)}\|u_t\|_m^m, \quad \delta > 0.$$

So, thatnks to the convexity of the function y^x/x for $y \ge 0$ and $x > 0$, we have

$$\begin{aligned} \frac{\delta^m}{m}\|u\|_m^m &\le \frac{s}{2}\delta^m\|u\|_2^2 + \frac{1-s}{p}\delta^m\|u\|_p^p, \qquad s = \frac{p-m}{p-2}, \\ \frac{1}{\beta}\sigma^\beta\|\nabla u\|_\beta^\beta &\le \frac{\theta}{2}\sigma^\beta\|\nabla u\|_2^2 + \frac{1-\theta}{\alpha}\sigma^\beta\|\nabla u\|_\alpha^\alpha, \qquad \theta = \frac{\alpha-\beta}{\alpha-2}. \end{aligned}$$

Hence, (3.2) becomes

$$\begin{aligned} F''(t) &\ge \|u_t\|_2^2 - \left[1 + \frac{\omega_2(1-\theta)}{\alpha}\sigma^\beta\right]\|\nabla u\|_\alpha^\alpha - \frac{\mu s}{2}\delta^m\|u\|_2^2 \\ &\quad - \left(\omega_1\eta + \frac{\omega_2\theta}{2}\sigma^\beta\right)\|\nabla u\|_2^2 + \left[1 - \frac{\mu(1-s)}{p}\delta^m\right]\|u\|_p^p \\ &\quad - \frac{\omega_1}{4\eta}\|\nabla u_t\|_2^2 - \omega_2\frac{\beta-1}{\beta}\sigma^{\beta/(1-\beta)}\|\nabla u_t\|_\beta^\beta - \mu\frac{m-1}{m}\delta^{-\frac{m}{m-1}}\|u_t\|_m^m. \end{aligned} \tag{3.3}$$

Next, since $u(t)$ is global and $E(0) \ge d$, then by Theorem 2.3, $E(t) \ge d$, $\forall t \ge 0$. Thus, using the embedding $L^\alpha(\Omega) \hookrightarrow L^2(\Omega)$ and the inequality

$$z^b \le (z+a)\left(z + \frac{1}{a}\right), \quad z \ge 0,\ 0 < b \le 1,\ a > 0,$$

we obtain

$$\begin{aligned} \|\nabla u\|_2^2 &\le c\|\nabla u\|_\alpha^2 \\ &= c\left[\|\nabla u\|_\alpha^\alpha\right]^{2/\alpha} \\ &\le c\left(1 + \frac{1}{d}\right)\left[\|\nabla u\|_\alpha^\alpha + d\right] \\ &\le C\left[\|\nabla u\|_\alpha^\alpha + E(t)\right], \quad \forall t \ge 0. \end{aligned} \tag{3.4}$$

By using Lemma 2.1 and (2.2), we get

$$\begin{aligned}
&\frac{d}{dt}\left\{F'(t)-\left[\frac{1}{4\eta}+\frac{\beta-1}{\beta}\sigma^{\frac{-\beta}{\beta-1}}+\frac{m-1}{m}\delta^{-\frac{m}{m-1}}\right]E(t)\right\}\\
&\quad\geq F''(t)+\frac{\omega_1}{4\eta}\|\nabla u_t\|_2^2+\omega_2\frac{\beta-1}{\beta}\sigma^{-\frac{\beta}{\beta-1}}\|\nabla u_t\|_\beta^\beta+\mu\frac{m-1}{m}\delta^{-\frac{m}{m-1}}\|u_t\|_m^m.
\end{aligned}$$

Adding and subtracting $p(1-\varepsilon)E(t)$, for $\varepsilon\in(0,1)$, in the right-hand side of the last inequality, and using (3.4) and the Poincaré inequality we obtain

$$\begin{aligned}
&\frac{d}{dt}\left\{F'(t)-\left[\frac{1}{4\eta}+\frac{\beta-1}{\beta}\sigma^{-\frac{\beta}{\beta-1}}+\frac{m-1}{m}\delta^{-\frac{m}{m-1}}\right]E(t)\right\}\\
&\geq \|u_t\|_2^2-\frac{\mu s}{2}\delta^m\|u\|_2^2-\left[1+\frac{\omega_2(1-\theta)}{\alpha}\sigma^\beta\right]\|\nabla u\|_\alpha^\alpha\\
&-\left(\omega_1\eta+\frac{\omega_2\theta}{2}\sigma^\beta\right)\|\nabla u\|_2^2+\left[1-\frac{\mu(1-s)}{p}\delta^m\right]\|u\|_p^p\\
&\geq\left[1+\frac{p}{2}(1-\varepsilon)\right]\|u_t\|_2^2-\frac{\mu s}{2}\delta^m\|u\|_2^2+k(\varepsilon)\|\nabla u\|_\alpha^\alpha\\
&-\left(\omega_1\eta+\frac{\omega_2\theta}{2}\sigma^\beta\right)\|\nabla u\|_2^2+\left[\varepsilon-\frac{\mu(1-s)}{p}\delta^m\right]\|u\|_p^p-p(1-\varepsilon)E(t)\\
&\geq\left[1+\frac{p}{2}(1-\varepsilon)\right]\|u_t\|_2^2-\frac{\mu s}{2}\delta^m\|u\|_2^2+\gamma(\varepsilon)\|\nabla u\|_2^2\\
&+\left[\varepsilon-\frac{\mu(1-s)}{p}\delta^m\right]\|u\|_p^p-\left[k(\varepsilon)+p(1-\varepsilon)\right]E(t)\\
&\geq\left[1+\frac{p}{2}(1-\varepsilon)\right]\|u_t\|_2^2+\left\{\gamma(\varepsilon)B-\frac{\mu s}{2}\delta^m\right\}\|u\|_2^2\\
&+\left[\varepsilon-\frac{\mu(1-s)}{p}\delta^m\right]\|u\|_p^p-\left[k(\varepsilon)+p(1-\varepsilon)\right]E(t),
\end{aligned}\tag{3.5}$$

where

$$\begin{aligned}
k(\varepsilon)&=\frac{1}{\alpha}\left[p(1-\varepsilon)-\alpha-\omega_2(1-\theta)\sigma^\beta\right],\\
\gamma(\varepsilon)&=\frac{k(\varepsilon)}{C}-\omega_1\eta-\frac{\omega_2\theta}{2}\sigma^\beta,
\end{aligned}$$

and B is the best constant of Poincaré inequality

$$\|\nabla u\|_2^2\geq B\|u\|_2^2.$$

Therefore, taking $\eta=\varepsilon$, $\sigma=\varepsilon$,

$$\delta=\left[\frac{p\varepsilon}{\mu(1-s)}\right]^{1/m},$$

setting

$$\gamma_1(\varepsilon)=\frac{1}{4\varepsilon}+\frac{\beta-1}{\beta}\varepsilon^{-\frac{\beta}{\beta-1}}+\frac{m-1}{m}\left(\frac{1-s}{p\varepsilon}\right)^{-\frac{1}{m-1}},$$

and substituting in (3.5), we arrive at

$$\frac{d}{dt}\left[F'(t)-\gamma_1(\varepsilon)E(t)\right] \geq \left[1+\frac{p}{2}(1-\varepsilon)\right]\|u_t\|_2^2 + \left[\gamma(\varepsilon)B - \frac{ps}{2(1-s)}\varepsilon\right]\|u\|_2^2 - \left[k(\varepsilon)+p(1-\varepsilon)\right]E(t).$$

By using the Schwarz inequality, we have

$$2\left[1+\tfrac{p}{2}(1-\varepsilon)\right]^{1/2}\left[\gamma(\varepsilon)B-\tfrac{ps}{2(1-s)}\varepsilon\right]^{1/2}(u_t,u) \leq \left[1+\tfrac{p}{2}(1-\varepsilon)\right]\|u_t\|_2^2 + \left[\gamma(\varepsilon)B-\tfrac{ps}{2(1-s)}\varepsilon\right]\|u\|_2^2.$$

Consequently, we obtain

$$\begin{aligned}\frac{d}{dt}\left[F'(t)-\gamma_1(\varepsilon)E(t)\right] &\geq a(\varepsilon)(u_t,u) - \left[k(\varepsilon)+p(1-\varepsilon)\right]E(t)\\ &= a(\varepsilon)\left[F'(t)-\gamma_2(\varepsilon)E(t)\right],\end{aligned} \tag{3.6}$$

where

$$\begin{aligned} a(\varepsilon) &= 2\left[1+\tfrac{p}{2}(1-\varepsilon)\right]^{1/2}\left[\gamma(\varepsilon)B-\tfrac{ps}{2(1-s)}\varepsilon\right]^{1/2},\\ \gamma_2(\varepsilon) &= \tfrac{k(\varepsilon)+p(1-\varepsilon)}{a(\varepsilon)}.\end{aligned}$$

Since

$$\gamma(\varepsilon)B - \frac{ps}{2(1-s)}\varepsilon \to \begin{cases} \frac{B(p-\alpha)}{\alpha C} > 0 & \text{as } \varepsilon \to 0^+ \\ -\left[\frac{\alpha+\omega_2(1-\theta)}{\alpha C}+\omega_1+\frac{\omega_2\theta}{2}\right]B - \frac{ps}{2(1-s)} < 0 & \text{as } \varepsilon \to 1^-, \end{cases}$$

then, there exists $\varepsilon_* \in (0,1)$, such that

$$a(\varepsilon_*) = 0 \text{ and } a(\varepsilon) > 0, \quad \forall \varepsilon \in (0,\varepsilon_*).$$

Hence, we have

$$\gamma_1(\varepsilon)-\gamma_2(\varepsilon) \to \begin{cases} +\infty & \text{as } \varepsilon \to 0^+ \\ -\infty & \text{as } \varepsilon \to \varepsilon_*^-. \end{cases}$$

Therefore, there exists $\varepsilon_0 \in (0,\varepsilon_*)$, such that $\gamma_1(\varepsilon_0) = \gamma_2(\varepsilon_0) > 0$. So, by setting

$$\begin{aligned} L(t) &= F'(t) - \gamma_1(\varepsilon_0)E(t),\\ M &= \gamma_1(\varepsilon_0),\end{aligned} \tag{3.7}$$

and by using (2.3), we obtain

$$\begin{aligned} L(0) &= (u_t(0),u(0)) - \gamma_1(\varepsilon_0)E(0)\\ &> (u_t(0),u(0)) - ME(0) > 0.\end{aligned}$$

Moreover, with this choice of ε_0, (3.6) becomes

$$\frac{d}{dt}L(t) \geq a(\varepsilon_0)L(t),$$

which gives

$$L(t) \geq L(0)e^{a(\varepsilon_0)t}, \qquad \forall t \geq 0,$$

and hence

$$F'(t) \geq L(0)e^{a(\varepsilon_0)t}, \qquad \forall t \geq 0.$$

By integrating this last inequality over $(0, t)$, we get

$$\|u(t)\|_2^2 = 2F(t) \geq 2F(0) + 2\frac{L(0)}{a(\varepsilon_0)}\left[e^{a(\varepsilon_0)t} - 1\right], \quad \forall t \geq 0. \tag{3.8}$$

On the other hand, by using Hölder's inequality and (2.2), we have

$$\begin{aligned}
\|u(t)\|_2 &\leq \|u(0)\|_2 + \int_0^t \|u_\tau(\tau)\|_2 d\tau \\
&\leq \|u(0)\|_2 + C\int_0^t \|u_\tau(\tau)\|_m d\tau \\
&\leq \|u(0)\|_2 + Ct^{\frac{m-1}{m}} \int_0^t \|u_\tau(\tau)\|_m^m d\tau \\
&\leq \|u(0)\|_2 + Ct^{\frac{m-1}{m}} \int_0^t \frac{-1}{\mu}\frac{dE(\tau)}{d\tau} d\tau \\
&\leq \|u(0)\|_2 + Ct^{\frac{m-1}{m}} \left[\frac{E(0) - E(t)}{\mu}\right]^{1/m} \\
&\leq \|u(0)\|_2 + C\left[\frac{E(0)}{\mu}\right]^{1/m} t^{\frac{m-1}{m}},
\end{aligned}$$

which clearly contradicts (3.8). □

4. Lower bound for the blow-up time

In this section, we give a lower bound for the blow-up time $T_{\max}$. To this end, we define

$$G(t) := \frac{1}{p}\|u(t)\|_p^p.$$

Theorem 4.1. *Let u be the solution of (1.1), and assume that*

$$\begin{cases} 2 < p \leq \frac{\alpha(n-2)+2n}{2(n-\alpha)}, & \text{for } n > \alpha, \\ 2 < p < \infty, & \text{for } n = \alpha. \end{cases}$$

Then

$$T_{\max} \geq \int_{G(0)}^{+\infty} \left\{\tau + A_1 \tau^{\frac{2}{\alpha}(p-1)} + A_2\right\}^{-1} d\tau,$$

where A_1 and A_2 are positive constants to be determined later in the proof.

Proof. By using inequality (2.1), we have

$$\frac{1}{2}\|u_t\|_2^2 + \frac{1}{\alpha}\|\nabla u\|_\alpha^\alpha = E(t) + \frac{1}{p}\|u(t)\|_p^p \leq E(0) + G(t). \tag{4.1}$$

Next, using the Schwarz inequality, the Sobolev-type inequality

$$\|u\|_q \leq C_q \|\nabla u\|_\alpha, \quad \forall q \in [1, \alpha^*], \quad \forall u \in W_0^{1,\alpha}(\Omega), \tag{4.2}$$

inequality (4.1) yields

$$\begin{aligned} G'(t) &= (|u|^{p-2}u, u_t) \\ &\leq \frac{1}{2}\|u_t\|_2^2 + \frac{1}{2}\|u\|_{2(p-1)}^{2(p-1)} \\ &\leq \frac{1}{2}\|u_t\|_2^2 + \frac{C_{2(p-1)}^{2(p-1)}}{2}\|\nabla u\|_\alpha^{2(p-1)} \\ &\leq E(0) + G(t) + \frac{C_{2(p-1)}^{2(p-1)}}{2}\left[\alpha E(0) + \alpha G(t)\right]^{\frac{2}{\alpha}(p-1)}. \end{aligned} \tag{4.3}$$

From (4.3) and Jensen's inequality, we obtain the differential inequality

$$G'(t) \leq G(t) + A_1 \left[G(t)\right]^{\frac{2}{\alpha}(p-1)} + A_2, \tag{4.4}$$

with

$$A_1 = C_*^{2(p-1)} 2^{\frac{2}{\alpha}(p-1)-2} \alpha^{\frac{2}{\alpha}(p-1)} \quad \text{and} \quad A_2 = E(0) + A_1 \left[E(0)\right]^{\frac{2}{\alpha}(p-1)}.$$

Hence, we get

$$T_{\max} \geq \int_0^{T_{\max}} \left\{G(s) + A_1 \left[G(s)\right]^{\frac{2}{\alpha}(p-1)} + A_2\right\}^{-1} G'(s) ds.$$

Since $\lim_{t \to T_{\max}^-} G(t) = +\infty$, so the previous inequality implies

$$T_{\max} \geq \int_{G(0)}^{+\infty} \left\{\tau + A_1 \tau^{\frac{2}{\alpha}(p-1)} + A_2\right\}^{-1} d\tau.$$

□

In the next theorem, when $n > \alpha$, the upper bound for p is enlarged. We define

$$H(t) := \frac{1}{\sigma}\|u(t)\|_\sigma^\sigma,$$

where $\sigma = \frac{\alpha(n-2)+2n}{2(n-\alpha)}$. Then, we have

Theorem 4.2. *Let u be the solution of (1.1), and assume that*

$$\frac{\alpha(n-2)+2n}{2(n-\alpha)} < p \leq \frac{\alpha n(2n-\alpha+2)-2\alpha^2}{2n(n-\alpha)}. \tag{4.5}$$

Then

$$T_{\max} \geq \int_{H(0)}^{+\infty} \left\{B_1 \tau^{b_1} + B_2 \tau^{b_2} + B_3\right\}^{-1} d\tau,$$

where B_1, B_2, B_3, b_1 and b_2 are positive constants to be determined later in the proof.

Proof. By using inequality (2.1), we have

$$\frac{1}{2}\|u_t\|_2^2 + \frac{1}{\alpha}\|\nabla u\|_\alpha^\alpha = E(t) + \frac{1}{p}\|u(t)\|_p^p \le E(0) + \frac{1}{p}\|u(t)\|_p^p. \tag{4.6}$$

Using the Schwarz inequality, the Sobolev-type inequality (4.2), with $q = \alpha^*$, and inequality (4.6) we get

$$\begin{aligned} H'(t) &= (|u|^{\sigma-2}u, u_t) \\ &\le \frac{1}{2}\|u_t\|_2^2 + \frac{1}{2}\|u\|_{2(\sigma-1)}^{2(\sigma-1)} \\ &\le \frac{1}{2}\|u_t\|_2^2 + \frac{C_*^{\alpha^*}}{2}\|\nabla u\|_\alpha^{\alpha^*} \\ &\le E(0) + \frac{1}{p}\|u\|_p^p + \frac{C_*^{\alpha^*}}{2}\left[\alpha E(0) + \frac{\alpha}{p}\|u\|_p^p\right]^{\frac{n}{n-\alpha}}. \end{aligned} \tag{4.7}$$

Next, the interpolation inequality, the Sobolev inequality and Young's inequality give

$$\begin{aligned} \|u\|_p^p &\le \|u\|_{\alpha^*}^{\theta p}.\|u\|_\sigma^{(1-\theta)p}, \qquad \theta = \frac{\alpha^*(p-\sigma)}{p(\alpha^*-\sigma)}, \\ &\le C_*^{\theta p}\|\nabla u\|_\alpha^{\theta p}.\|u\|_\sigma^{(1-\theta)p}, \\ &\le \frac{1}{\alpha}\|\nabla u\|_\alpha^\alpha + B\|u\|_\sigma^r, \end{aligned} \tag{4.8}$$

where

$$B = C_*\left(1 - \frac{\theta p}{\alpha}\right)(p\theta C_*)^{\frac{p\theta}{\alpha - p\theta}} \quad \text{and} \quad r = \frac{\alpha p(1-\theta)}{\alpha - \theta p}.$$

Note that in virtue of (4.5), we have $\alpha > \theta p$. Hence, by (2.1) we have

$$\|u\|_p^p \le E(0) + \frac{1}{p}\|u\|_p^p + B\|u\|_\sigma^r, \tag{4.9}$$

which gives

$$\frac{1}{p}\|u\|_p^p \le \frac{1}{p-1}\left(E(0) + B\|u\|_\sigma^r\right).$$

Inserting this last inequality in (4.7), and using Jensen's inequality, we obtain

$$\begin{aligned} H'(t) &\le \frac{pE(0)}{p-1} + \frac{B}{p-1}\|u\|_\sigma^r + \frac{C_*^{\alpha^*}}{2}\left[\frac{\alpha pE(0)}{p-1} + \frac{\alpha B}{p-1}\|u\|_\sigma^r\right]^{\frac{n}{n-\alpha}} \\ &= B_1\left(H(t)\right)^{b_1} + B_2\left(H(t)\right)^{b_2} + B_3, \end{aligned} \tag{4.10}$$

where

$$\begin{aligned} B_1 &= \frac{B\sigma^r}{p-1}, \quad B_2 = \frac{C_*^{\alpha^*}}{2}2^{\frac{\alpha}{n-\alpha}}\left[\frac{\alpha B\sigma^r}{p-1}\right]^{\frac{n}{n-\alpha}}, \\ B_3 &= \frac{pE(0)}{p-1} + \frac{C_*^{\alpha^*}}{2}2^{\frac{\alpha}{n-\alpha}}\left[\frac{\alpha pE(0)}{p-1}\right]^{\frac{n}{n-\alpha}}, \\ b_1 &= \frac{r}{\sigma}, \quad b_2 = \frac{rn}{\sigma(n-\alpha)}. \end{aligned}$$

Finally, integrating inequality (4.10) over $(0, T_{\max})$ we get

$$T_{\max} \geq \int_0^{T_{\max}} \left\{B_1 (H(s))^{b_1} + B_2 (H(s))^{b_2} + B_3\right\}^{-1} H'(s)ds,$$

and so

$$T_{\max} \geq \int_{H(0)}^{+\infty} \left\{B_1 \tau^{b_1} + B_2 \tau^{b_2} + B_3\right\}^{-1} d\tau.$$

□

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Global nonexistence of solution for coupled nonlinear Klein-Gordon with degenerate damping and source terms

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Abstract. In this article we consider a coupled system of nonlinear Klein-Gordon equations with degenerate damping and source terms. We prove, with positive initial energy, the global nonexistence of solutions by concavity method.

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1. Introduction

We consider the following system

$$\begin{cases} u_{tt} - \Delta u_t - div\left(|\nabla u|^{\alpha-2}\nabla u\right) - div\left(|\nabla u_t|^{\beta_1-2}\nabla u_t\right) \\ \qquad +a_1 |u_t|^{m-2} u_t + m_1^2 u = f_1(u,v), \\ v_{tt} - \Delta v_t - div\left(|\nabla v|^{\alpha-2}\nabla v\right) - div\left(|\nabla v_t|^{\beta_2-2}\nabla v_t\right) \\ \qquad +a_2 |v_t|^{r-2} v_t + m_2^2 v = f_2(u,v), \end{cases} \tag{1.1}$$

where $u = u(t,x)$, $v = v(t,x)$, $x \in \Omega$, a bounded domain of $\mathbb{R}^N$ $(N \geq 1)$ with a smooth boundary $\partial\Omega$, $t > 0$ and a_1, a_2, b_1, b_2, m_1, $m_2 > 0$ and β_1, β_2, m, $r \geq 2$, $\alpha > 2$, and the two functions $f_1(u,v)$ and $f_2(u,v)$ given by

$$\begin{aligned} f_1(u,v) &= b_1|u+v|^{2(\rho+1)}(u+v) + b_2|u|^{\rho}u|v|^{(\rho+2)} \\ f_2(u,v) &= b_1|u+v|^{2(\rho+1)}(u+v) + b_2|u|^{(\rho+2)}|v|^{\rho}v. \end{aligned} \tag{1.2}$$

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The system (1.1) is supplemented by the following initial and boundary conditions

$$\begin{cases} (u(0), v(0)) = (u_0, v_0), (u_t(0), v_t(0)) = (u_1, v_1),\ x \in \Omega \\ u(x) = v(x) = 0 \qquad x \in \partial\Omega. \end{cases} \tag{1.3}$$

Originally the interaction between the source term and the damping term in the wave equation is given by

$$u_{tt} - \Delta u + a\,|u_t|^{m-2}\,u_t = b\,|u|^{p-2}\,u, \text{ in } \ \Omega \times (0, T)\,, \tag{1.4}$$

where Ω is a bounded domain of $\mathbb{R}^N$, $N \geq 1$ with a smooth boundary $\partial\Omega$, has an exciting history. It has been shown that the existence and the asymptotic behavior of solutions depend on a crucial way on the parameters m, p and on the nature of the initial data. More precisely, it is well known that in the absence of the source term $|u|^{p-2}\,u$ then a uniform estimate of the form

$$\|u_t\,(t)\|_2 + \|\nabla u\,(t)\|_2 \leq C, \tag{1.5}$$

holds for any initial data $(u_0, u_1) = (u(0), u_t(0))$ in the energy space $H_0^1\,(\Omega) \times L^2\,(\Omega)\,,$ where C is a positive constant independent of t. The estimate (1.5) shows that any local solution u of problem (1.4) can be continued in time as long as (1.5) is verified. This result has been proved by several authors. See for example [2, 5, 7, 15, 20, 3]. On the other hand in the absence of the damping term $|u_t|^{m-2}\,u_t$, the solution of (1.4) ceases to exist and there exists a finite value T^* such that

$$\lim_{t \to T^*} \|u\,(t)\|_p = +\infty, \tag{1.6}$$

the reader is refereed to Ball [1] and Kalantarov & Ladyzhenskaya [6] for more details. When both terms are present in equation (1.4), the situation is more delicate. This case has been considered by Levine in [8, 9], where he investigated problem (1.4) in the linear damping case ($m = 2$) and showed that any local solution u of (1.4) cannot be continued in $(0, \infty) \times \Omega$ whenever the initial data are large enough (negative initial energy). The main tool used in [8] and [9] is the "concavity method". This method has been a widely applicable tool to prove the blow up of solutions in finite time of some evolution equations. The basic idea of this method is to construct a positive functional $\theta\,(t)$ depending on certain norms of the solution and show that for some $\gamma > 0$, the function $\theta^{-\gamma}\,(t)$ is a positive concave function of t. Thus there exists T^* such that $\lim\limits_{t \to T^*} \theta^{-\gamma}\,(t) = 0$. Since then, the concavity method became a powerful and simple tool to prove blow up in finite time for other related problems. Unfortunately, this method is limited to the case of a linear damping. Georgiev and Todorova [4] extended Levine's result to the nonlinear damping case ($m > 2$). In their work, the authors considered the problem (1.4) and introduced a method different from the one known as the concavity method. They showed that solutions with negative energy continue to exist globally 'in time' if the damping term dominates the source term (i.e.$m \geq p$) and blow up in finite time in the other case (i.e.$p > m$) if the initial energy is sufficiently negative. Their method is based on the construction of an auxiliary function L which is a perturbation of the total energy of the system and satisfies the differential inequality

$$\frac{dL(t)}{dt} \geq \xi L^{1+\nu}(t) \tag{1.7}$$

In $[0,\infty)$, where $\nu > 0$. Inequality (1.7) leads to a blow up of the solutions in finite tim $t \geq L(0)^{-\nu}\xi^{-1}\nu^{-1}$, provided that $L(0) > 0$. However the blow up result in [4] was not optimal in terms of the initial data causing the finite time blow up of solutions. Thus several improvement have been made to the result in [4] (see for example [10, 11, 12, 18]. In particular, Vitillaro in [18] combined the arguments in [4] and [11] to extend the result in [4] to situations where the damping is nonlinear and the solution has positive initial energy.

In [19], Yang, studied the problem

$$\begin{aligned} u_{tt} - \Delta u_t - div\left(|\nabla u|^{\alpha-2}\nabla u\right) - div\left(|\nabla u_t|^{\beta-2}\nabla u_t\right) \\ +a|u_t|^{m-2}u_t = b|u|^{p-2}u, \end{aligned} \tag{1.8}$$

in $(0,T)\times\Omega$ with initial conditions and boundary condition of Dirichlet type. He showed that solutions blow up in finite time T^* under the condition $p > \max\{\alpha, m\}$, $\alpha > \beta$, and the initial energy is sufficiently negative (see condition (ii) in [19][Theorem 2.1]). In fact this condition made it clear that there exists a certain relation between the blow-up time and $|\Omega|$. ([19], [Remark 2]).

Messaoudi and Said-Houari [13] improved the result in [19] and showed that the blow up of solutions of problem (1.8) takes place for negative initial data only regardless of the size of Ω.

The absence of the terms m_1u^2 and m_2v^2, equations (1.1) take the form:

$$\begin{cases} u_{tt} - \Delta u_t - div\left(|\nabla u|^{\alpha-2}\nabla u\right) - div\left(|\nabla u_t|^{\beta_1-2}\nabla u_t\right) \\ \qquad +a_1|u_t|^{m-2}u_t = f_1(u,v), \\ v_{tt} - \Delta v_t - div\left(|\nabla v|^{\alpha-2}\nabla v\right) - div\left(|\nabla v_t|^{\beta_2-2}\nabla v_t\right) \\ \qquad +a_2|v_t|^{r-2}v_t = f_2(u,v), \end{cases}$$

In [16] Rahmoun. A and Ouchenane. D proved the global nonexistence result, Under an appropriate assumptions on the initial data and under some restrictions on the parameter ; β_1;β_2; m; r and on the nonlinear functions f_1 and f_2.

2. Preliminaries

In this section, we introduce some notations and some technical lemmas to be used throughout this paper. By $\|.\|_q$, we denote the usual $L^q(\Omega)$-norm. The constants $C, c, c_1, c_2, \ldots$, used throughout this paper are positive generic constants, which may be different in various occurrences. We define

$$F(u,v) = \frac{1}{2(\rho+2)}\left[b_1|u+v|^{2(\rho+2)} + 2b_2|uv|^{\rho+2}\right].$$

Then, it is clear that, from (1.2), we have

$$uf_1(u,v) + vf_2(u,v) = 2(\rho+2)F(u,v). \tag{2.1}$$

The following lemma was introduced and proved in [14]

Lemma 2.1. *There exist two positive constants* c_0 *and* c_1 *such that*

$$\frac{c_0}{2(\rho+2)}\left(|u|^{2(\rho+2)}+|v|^{2(\rho+2)}\right)\leq F(u,v)\leq \frac{c_1}{2(\rho+2)}\left(|u|^{2(\rho+2)}+|v|^{2(\rho+2)}\right). \quad (2.2)$$

The energy functional is given by

$$\begin{aligned} E(t) &= \frac{1}{2}\left(\|u_t\|_2^2+\|v_t\|_2^2\right)+\frac{1}{\alpha}\left(\|\nabla u\|_\alpha^\alpha+\|\nabla v\|_\alpha^\alpha\right) \\ &\quad +m_1^2\|u\|_2^2+m_2^2\|v\|_2^2-\int_\Omega F(u,v)\,dx. \end{aligned} \quad (2.3)$$

Let us define the constant r_α as follows

$$r_\alpha=\frac{N\alpha}{N-\alpha},\quad \textit{if } N>\alpha,\ r_\alpha>\alpha \textit{ if } N=\alpha,\ \textit{and } r_\alpha=\infty \textit{ if } N<\alpha. \quad (2.4)$$

The inequality below is the key to prove the global nonexistence of solution. A similar version of this lemma was first introduced in [17]

Lemma 2.2. *Suppose that* $\alpha>2$, *and* $2<2(\rho+2)<r_\alpha$. *Then there exists* $\eta>0$ *such that the inequality*

$$\|u+v\|_{2(\rho+2)}^{2(\rho+2)}+2\|uv\|_{\rho+2}^{\rho+2}\leq \eta\left(\|\nabla u\|_\alpha^\alpha+\|\nabla v\|_\alpha^\alpha\right)^{\frac{2(\rho+2)}{\alpha}}, \quad (2.5)$$

holds.

Proof. It is clear that by using the Minkowski's inequality, we get

$$\|u+v\|_{2(\rho+2)}^2\leq 2(\|u\|_{2(\rho+2)}^2+\|v\|_{2(\rho+2)}^2),$$

the embedding $W_0^{1,\alpha}\hookrightarrow L^{2(\rho+2)}(\Omega)$ gives

$$\|u\|_{2(\rho+2)}^2\leq C\|\nabla u\|_\alpha^2\leq C(\|\nabla u\|_\alpha^\alpha)^{\frac{2}{\alpha}}\leq C(\|\nabla u\|_\alpha^\alpha+\|\nabla v\|_\alpha^\alpha)^{\frac{2}{\alpha}},$$

and similary, we have

$$\|v\|_{2(\rho+2)}^2\leq C\|\nabla u\|_\alpha^\alpha+\|\nabla v\|_\alpha^\alpha)^{\frac{2}{\alpha}}.$$

Thus, we deduce from the above estimates that

$$\|u+v\|_{2(\rho+2)}^2\leq C(\|\nabla u\|_\alpha^\alpha+\|\nabla v\|_\alpha^\alpha)^{\frac{2}{\alpha}}, \quad (2.6)$$

also, Hölder and Young's inequalities give

$$\begin{aligned} \|uv\|_{(\rho+2)} &\leq \|u\|_{2(\rho+2)}\|v\|_{2(\rho+2)} \\ &\leq C(\|\nabla u\|_{2(\rho+2)}^2+\|\nabla v\|_{2(\rho+2)}^2) \\ &\leq C(\|\nabla u\|_\alpha^\alpha+\|\nabla v\|_\alpha^\alpha)^{\frac{2}{\alpha}}. \end{aligned} \quad (2.7)$$

Collecting the estimates (2.6) and (2.7), then (2.5) holds. This completes the proof of Lemma 2.2 □

Lemma 2.3. *Let $v > 0$ be a real positive number and L be a solution of the ordinary differential inequality*

$$\frac{dL(t)}{dt} \geq \xi L^{1+v}(t), \tag{2.8}$$

defined in $[0, \infty)$.
If $L(0) > 0$, *then the solution ceacesto exist for* $t \geq L(0)^{-v} \xi^{-1} v^{-1}$.

Proof. Direct integration of (2.8) gives

$$L^{-v}(0) - L^{-v}(t) \geq \xi v t.$$

Thus we obtain the following estimate

$$L^{v}(t) \geq \left[L^{-v}(0) - \xi v t\right]^{-1}. \tag{2.9}$$

It is clear that the right-hand side of (2.9) is unbounded when

$$\xi v t = L^{-v}(0).$$

This completes the proof. □

In the following lemma, we show that the total energy of our system is a nonincreasing function of t.

Lemma 2.4. *Let* (u, v) *be the solution of system* (1.1)-(1.3), *then the energy functional is a non-increasing function for all* $t \geq 0$

$$\begin{aligned}\frac{dE(t)}{dt} &= -\|\nabla u_t\|_2^2 - \|\nabla v_t\|_2^2 - \|\nabla u_t\|_{\beta_1}^{\beta_1} - \|\nabla v_t\|_{\beta_2}^{\beta_2} \\ &\quad -a_1\|u_t\|_m^m - a_2\|v_t\|_r^r - m_1^2\|u\|_2^2 - m_2^2\|v\|_2^2. \end{aligned} \tag{2.10}$$

Proof. We multiply the first equation in (1.1) by u_t and second equation by v_t and integrate over Ω, using integration by parts, we obtain (2.10). □

3. Global nonexistence result

In this section, we prove that, under some restrictions on the initial data and under som restrictions on the parameter $\alpha, \beta_1, \beta_2, m, r$, then the lifespan of solution of problem (1.1)- (1.3) is finite

Theorem 3.1. *Suppose that* β_1, β_2, m, $r \geq 2$, $\alpha > 2$, $\rho > -1$ *such that* β_1, $\beta_2 < \alpha$, *and* $\max\{m, r\} < 2(\rho + 2) < r_\alpha$, *where* r_α *is the Sobolev critical exponent of* $W_0^{1,\alpha}(\Omega)$. *defined in* (2.4). *Assume further that*

$$E(0) < E_1, \qquad (\|\nabla u_0\|_\alpha^\alpha + \|\nabla v_0\|_\alpha^\alpha)^{\frac{1}{\alpha}} + m_1^2\|u_0\|_2^2 + m_2^2\|v_0\|_2^2 > \zeta_1.$$

Then, any weak solutions of (1.1)-(1.3) *cannot exist for all time. Here the constants* E_1 *and* ζ_1 *are defined in* (3.1).

In order to prove our result and for the sake of simplicity, we take $b_1 = b_2 = 1$ and introduce the following

$$B = \eta^{\frac{1}{2(\rho+2)}}, \qquad \zeta_1 = B^{\frac{-2(\rho+2)}{2(\rho+2)-\alpha}}, \qquad E_1 = \left(\frac{1}{\alpha} - \frac{1}{2(\rho+2)}\right)\zeta_1^{\alpha}, \tag{3.1}$$

where η is the optimal constant in (2.5).
The following lemma allows us to prove a blow up result for a large class of initial data. This lemma is similar to the one in [17] and has its origin in [18]

Lemma 3.2. *Let (u, v) be a solution of (1.1)-(1.3). Assume that $\alpha > 2$, $\rho > -1$. Assume further that $E(0) < E_1$ and*

$$\left(\|\nabla u_0\|_\alpha^\alpha + \|\nabla v_0\|_\alpha^\alpha\right)^{\frac{1}{\alpha}} + m_1^2 \|u_0\|_2^2 + m_2^2 \|v_0\|_2^2 > \zeta_1. \tag{3.2}$$

Then there exists a constant $\zeta_2 > \zeta_1$ such that

$$\left(\|\nabla u\|_\alpha^\alpha + \|\nabla v\|_\alpha^\alpha\right)^{\frac{1}{\alpha}} + m_1^2 \|u\|_2^2 + m_2^2 \|v\|_2^2 > \zeta_2, \tag{3.3}$$

and

$$\left[\|u+v\|_{2(\rho+2)}^{2(\rho+2)} + 2\|uv\|_{\rho+2}^{\rho+2}\right]^{\frac{1}{2(\rho+2)}} \geq B\zeta_2, \ \forall t \geq 0. \tag{3.4}$$

Proof. We first note, by (2.3) and the definition of B, that

$$\begin{aligned} E(t) &\geq \frac{1}{\alpha}\left(\|\nabla u\|_\alpha^\alpha + \|\nabla v\|_\alpha^\alpha\right) + m_1^2\|u\|_2^2 + m_2^2\|v\|_2^2 \\ &\quad - \frac{1}{2(\rho+2)}\left[|u+v|^{2(\rho+2)} + 2|uv|^{\rho+2}\right] \\ &\geq \frac{1}{\alpha}\left(\|\nabla u\|_\alpha^\alpha + \|\nabla v\|_\alpha^\alpha\right) + m_1^2\|u\|_2^2 + m_2^2\|v\|_2^2 \\ &\quad - \frac{\eta}{2(\rho+2)}\left(\|\nabla u\|_\alpha^\alpha + \|\nabla v\|_\alpha^\alpha\right)^{\frac{2(\rho+2)}{\alpha}} \\ &\geq \frac{1}{\alpha}\zeta^\alpha - \frac{\eta}{2(\rho+2)}\zeta^{2(\rho+2)}, \end{aligned} \tag{3.5}$$

where $\zeta = \left[\|\nabla u\|_\alpha^\alpha + \|\nabla v\|_\alpha^\alpha + m_1^2\|u\|_\alpha^\alpha + m_2^2\|v\|_\alpha^\alpha\right]^{\frac{1}{\alpha}}$. It is not hard to verify that g is increasing for $0 < \zeta < \zeta_1$, decreasing for $\zeta > \zeta_1$, $g(\zeta) \to -\infty$ as $\zeta \to +\infty$, and

$$g(\zeta_1) = \frac{1}{\alpha}\zeta_1^\alpha - \frac{B^{2(\rho+2)}}{2(\rho+2)}\zeta_1^{2(\rho+2)} = E_1,$$

where ζ_1 is given in (3.1). Therefore, since $E(0) < E_1$, there exists $\zeta_2 > \zeta_1$ such that $g(\zeta_2) = E(0)$.
If we set $\zeta_0 = \left[\|\nabla u(0)\|_\alpha^\alpha + \|\nabla v(0)\|_\alpha^\alpha\right]^{\frac{1}{\alpha}} + m_1^2\|u(0)\|_2^2 + m_2^2\|v(0)\|_2^2$, then by (3.5) we have $g(\zeta_0) \leq E(0) = g(\zeta_2)$, which implies that $\zeta_0 \geq \zeta_2$.
Now, establish (3.3), we suppose by contradiction that

$$\left(\|\nabla u_0\|_\alpha^\alpha + \|\nabla v_0\|_\alpha^\alpha\right)^{\frac{1}{\alpha}} + m_1^2\|u_0\|_2^2 + m_2^2\|v_0\|_2^2 < \zeta_2,$$

for some $t_0 > 0$; by the continuity of $\|\nabla u(.)\|_\alpha^\alpha + \|\nabla v(.)\|_\alpha^\alpha + m_1^2 \|u(.)\|_2^2 + m_2^2 \|v(.)\|_2^2$ we can choose t_0 such that

$$(\|\nabla u(t_0)\|_\alpha^\alpha + \|\nabla v(t_0)\|_\alpha^\alpha)^{\frac{1}{\alpha}} + m_1^2 \|u(t_0)\|_2^2 + m_2^2 \|v(t_0)\|_2^2 > \zeta_1.$$

Again, the use of (3.5) leads to

$$E(t_0) \geq g(\|\nabla u(t_0)\|_\alpha^\alpha + \|\nabla v(t_0)\|_\alpha^\alpha) + m_1^2 \|u(t_0)\|_2^2 + m_2^2 \|v(t_0)\|_2^2 > g(\zeta_2) = E(0).$$

This is impossible since $E(t) \leq E(0)$, for all $t \in [0,T)$. Hence, (3.3) is established. To prove (3.4), we make use of (2.3) to get

$$\begin{aligned} &\frac{1}{\alpha}(\|\nabla u_0\|_\alpha^\alpha + \|\nabla v_0\|_\alpha^\alpha) + m_1^2 \|u_0\|_2^2 + m_2^2 \|v_0\|_2^2 \\ &\leq E(0) + \frac{1}{2(\rho+2)}\left[\|u+v\|_{2(\rho+2)}^{2(\rho+2)} + 2\|uv\|_{\rho+2}^{\rho+2}\right]. \end{aligned}$$

Consequently, (3.3) yields

$$\begin{aligned} \frac{1}{2(\rho+2)}\left[\|u+v\|_{2(\rho+2)}^{2(\rho+2)} + 2\|uv\|_{\rho+2}^{\rho+2}\right] &\geq \frac{1}{\alpha}(\|\nabla u\|_\alpha^\alpha + \|\nabla v\|_\alpha^\alpha) - E(0) \\ &\geq \frac{1}{\alpha}\zeta_2^\alpha - E(0) \\ &\geq \frac{1}{\alpha}\zeta_2^\alpha - g(\zeta_2) \qquad (3.6) \\ &= \frac{B^{2(\rho+2)}}{2(\rho+2)}\zeta_2^{2(\rho+2)}. \end{aligned}$$

Therefore, (3.6) and (3.1) yield the desired result. □

Proof. (of Theorem 3.1). We suppose that the solution exists for all time and set

$$H(t) = E_1 - E(t). \qquad (3.7)$$

By using (2.3) and (3.7) we get

$$\begin{aligned} H'(t) &= \|\nabla u_t\|_2^2 + \|\nabla v_t\|_2^2 + \|\nabla u_t\|_{\beta_1}^{\beta_1} + \|\nabla v_t\|_{\beta_2}^{\beta_2} \\ &\quad + a_1\|u_t\|_m^m + a_2\|v_t\|_r^r + m_1^2 \|u\|_2^2 + m_2^2 \|v\|_2^2. \end{aligned}$$

From (2.10), It is clear that for all $t \geq 0$, $H'(t) > 0$. Therefore, we have

$$\begin{aligned} 0 < H(0) \leq H(t) &= E_1 - \frac{1}{2}\left(\|u_t\|_2^2 + \|v_t\|_2^2 + m_1^2 \|u\|_2^2 + m_2^2 \|v\|_2^2\right) \\ &\quad -\frac{1}{\alpha}(\|\nabla u\|_\alpha^\alpha + \|\nabla v\|_\alpha^\alpha) \\ &\quad + \frac{1}{2(\rho+2)}\left[\|u+v\|_{2(\rho+2)}^{2(\rho+2)} + 2\|uv\|_{\rho+2}^{\rho+2}\right]. \qquad (3.8) \end{aligned}$$

From (2.3) and (3.3), we obtain, for all $t \geq 0$,

$$\begin{aligned} &E_1 - \frac{1}{2}\left(\|u_t\|_2^2 + \|v_t\|_2^2 + m_1^2 \|u\|_2^2 + m_2^2 \|v\|_2^2\right) - \frac{1}{\alpha}(\|\nabla u\|_\alpha^\alpha + \|\nabla v\|_\alpha^\alpha) \\ &< E_1 - \frac{1}{\alpha}\zeta_1^\alpha = -\frac{1}{2(\rho+2)}\zeta_1^\alpha < 0. \end{aligned}$$

Hence,

$$0 < H(0) \leq H(t) \leq \frac{1}{2(\rho+2)}\left[\|u+v\|_{2(\rho+2)}^{2(\rho+2)} + 2\|uv\|_{\rho+2}^{\rho+2}\right], \ \forall t \geq 0.$$

Then by (2.2), we have

$$0 < H(0) \leq H(t) \leq \frac{c_1}{2(\rho+2)}\left[\|u\|_{2(\rho+2)}^{2(\rho+2)} + \|v\|_{2(\rho+2)}^{2(\rho+2)}\right], \ \forall t \geq 0. \tag{3.9}$$

We then define

$$L(t) = H^{1-\sigma}(t) + \varepsilon \int_\Omega (uu_t + vv_t))dx, \tag{3.10}$$

for ε small to be chosen later and

$$\begin{aligned} 0 \quad < \quad \sigma \leq \min\left\{\frac{1}{2}, \frac{\alpha - m}{2(\rho+2)(m-1)}, \frac{\alpha - r}{2(\rho+2)(r-1)}, \right. \\ \left. \frac{(\alpha-2)}{2(\rho+2)}, \frac{\alpha-\beta_1}{2(\rho+2)(\beta_1-1)}, \frac{\alpha-\beta_2}{2(\rho+2)(\beta_2-1)}\right\}. \end{aligned} \tag{3.11}$$

Our goal is to show that $L(t)$ satisfies the differential inequality (1.7). Indeed, taking the derivative of (3.10), using (1.1) and adding subtracting $\varepsilon kH(t)$, we obtain

$$\begin{aligned} L'(t) &= (1-\sigma) H^{-\sigma}(t) H'(t) + \varepsilon k H(t) \\ &+\varepsilon\left(1+\frac{k}{2}\right)\left(\|u_t\|_2^2 + \|v_t\|_2^2 + m_1^2\|u\|_2^2 + m_2^2\|v\|_2^2\right) \\ &+\varepsilon(1-k)\int_\Omega F(u,v) - \varepsilon k E_1 \\ &-\varepsilon\int_\Omega \nabla u \nabla u_t dx - \varepsilon \int_\Omega \nabla v \nabla v_t dx \\ &+\varepsilon\left(\frac{k}{\alpha}-1\right)(\|\nabla u\|_\alpha^\alpha + \|\nabla v\|_\alpha^\alpha) \\ &-\varepsilon\int_\Omega |\nabla u_t|^{\beta_1-2}\nabla u_t \nabla u dx - \varepsilon\int_\Omega |\nabla v_t|^{\beta_2-2}\nabla v_t\nabla v dx \\ &-\varepsilon a_1\int_\Omega |u_t|^{m-2} u_t u dx - \varepsilon a_2 \int_\Omega |v_t|^{r-2} v_t v dx. \end{aligned} \tag{3.12}$$

We then exploit Young's inequality to get for $\mu_i, \lambda_i, \delta_i > 0$ $i = 1, 2$

$$\begin{aligned} \int_\Omega \nabla u \nabla u_t dx &\leq \frac{1}{4\mu_1}\|\nabla u\|_2^2 + \mu_1\|\nabla u_t\|_2^2, \\ \int_\Omega \nabla v \nabla v_t dx &\leq \frac{1}{4\mu_2}\|\nabla v\|_2^2 + \mu_2\|\nabla v_t\|_2^2, \end{aligned} \tag{3.13}$$

and

$$\begin{aligned} \int_\Omega |\nabla u_t|^{\beta_1-1}\nabla u dx &\leq \frac{\lambda_1^{\beta_1}}{\beta_1}\|\nabla u\|_{\beta_1}^{\beta_1} + \frac{\beta_1-1}{\beta_1}\lambda_1^{-\beta_1/(\beta_1-1)}\|\nabla u_t\|_{\beta_1}^{\beta_1}, \\ \int_\Omega |\nabla v_t|^{\beta_2-1}\nabla v dx &\leq \frac{\lambda_2^{\beta_2}}{\beta_2}\|\nabla v\|_{\beta_2}^{\beta_2} + \frac{\beta_2-1}{\beta_2}\lambda_2^{-\beta_2/(\beta_2-1)}\|\nabla v_t\|_{\beta_1}^{\beta_1}, \end{aligned} \tag{3.14}$$

and also

$$\int_\Omega |u_t|^{m-2} u_t u dx \le \frac{\delta_1^m}{m} \|u\|_m^m + \frac{m-1}{m}\delta_1^{-m/(m-1)} \|u_t\|_m^m,$$

$$\int_\Omega |v_t|^{r-2} v_t v dx \le \frac{\delta_2^r}{r} \|v\|_r^r + \frac{r-1}{r}\delta_2^{-r/(r-1)} \|v_t\|_r^r. \tag{3.15}$$

A substitution of (3.13)-(3.15)) in (3.12) and using (2.2) yields

$$\begin{aligned} L'(t) \ \ge\ & (1-\sigma) H^{-\sigma}(t) H'(t) + \varepsilon k H(t) \\ & +\varepsilon\left(1+\frac{k}{2}\right)\left(\|u_t\|_2^2 + \|v_t\|_2^2 + m_1^2\|u\|_2^2 + m_2^2\|v\|_2^2\right) \\ & +\varepsilon\left(\frac{c_0}{2(\rho+2)} - \frac{kc_1}{2(\rho+2)}\right)\left(\|u\|_{2(\rho+2)}^{2(\rho+2)} + \|v\|_{2(\rho+2)}^{2(\rho+2)}\right) - \varepsilon k E_1 \\ & -\frac{\varepsilon}{4\mu_1}\|\nabla u\|_2^2 - \mu_1\varepsilon\|\nabla u_t\|_2^2 - \frac{\varepsilon}{4\mu_2}\|\nabla v\|_2^2 - \varepsilon\mu_2\|\nabla v_t\|_2^2 \\ & +\varepsilon\left(\frac{k}{\alpha}-1\right)\left(\|\nabla u\|_\alpha^\alpha + \|\nabla v\|_\alpha^\alpha\right) \\ & -\varepsilon\frac{\lambda_1^{\beta_1}}{\beta_1}\|\nabla u\|_{\beta_1}^{\beta_1} - \varepsilon\frac{\beta_1-1}{\beta_1}\lambda_1^{-\beta_1/(\beta_1-1)}\|\nabla u_t\|_{\beta_1}^{\beta_1} \\ & -\varepsilon\frac{\lambda_2^{\beta_2}}{\beta_2}\|\nabla v\|_{\beta_2}^{\beta_2} - \varepsilon\frac{\beta_2-1}{\beta_2}\lambda_2^{-\beta_2/(\beta_2-1)}\|\nabla v_t\|_{\beta_1}^{\beta_1} \\ & -a_1\varepsilon\frac{\delta_1^m}{m}\|u\|_m^m - a_1\varepsilon\frac{m-1}{m}\delta_1^{-m/(m-1)}\|u_t\|_m^m \\ & -a_2\varepsilon\frac{\delta_2^r}{r}\|v\|_r^r - a_2\varepsilon\frac{r-1}{r}\delta_2^{-r/(r-1)}\|v_t\|_m^m. \end{aligned} \tag{3.16}$$

Let us choose δ_1, δ_2, μ_1, μ_2, λ_1, and λ_2 such that

$$\begin{cases} \delta_1^{-m/(m-1)} = M_1 H^{-\sigma}(t) \\ \delta_2^{-r/(r-1)} = M_2 H^{-\sigma}(t) \\ \mu_1 = M_3 H^{-\sigma}(t) \\ \mu_2 = M_4 H^{-\sigma}(t) \\ \lambda_1^{-\beta_1/(\beta_1-1)} = M_5 H^{-\sigma}(t) \\ \lambda_2^{-\beta_2/(\beta_2-1)} = M_6 H^{-\sigma}(t), \end{cases} \tag{3.17}$$

for M_1, M_2, M_3, M_4, M_5 and M_6 large constants to be fixed later. Thus, by using (3.17), and for

$$M = M_3 + M_4 + (\beta_1-1)M_5/\beta_1 + (\beta_2-1)M_6/\beta_2 + (m-1)M_1/m + (r-1)M_2/r,$$

then, inequality (3.16) takes the form

$$\begin{aligned}
L^{'}(t) \geq\ & ((1-\sigma)-\varepsilon M) H^{-\sigma}(t) H^{'}(t)+\varepsilon k H(t) \\
& +\varepsilon\left(1+\frac{k}{2}\right)\left(\|u_t\|_2^2+\|v_t\|_2^2+m_1^2\|u\|_2^2+m_2^2\|v\|_2^2\right) \\
& +\varepsilon\left(\frac{c_0}{2(\rho+2)}-\frac{kc_1}{2(\rho+2)}\right)\left(\|u\|_{2(\rho+2)}^{2(\rho+2)}+\|v\|_{2(\rho+2)}^{2(\rho+2)}\right) \\
& -\varepsilon k E_1+\varepsilon\left(\frac{k}{\alpha}-1\right)\left(\|\nabla u\|_\alpha^\alpha+\|\nabla v\|_\alpha^\alpha\right) \\
& -\frac{\varepsilon}{4M_3} H^\sigma(t)\|\nabla u\|_2^2-\frac{\varepsilon}{4M_4} H^\sigma(t)\|\nabla v\|_2^2 \\
& -\frac{a_1\varepsilon}{m} M_1^{-(m-1)} H^{\sigma(m-1)}(t)\|u\|_m^m \\
& -\frac{a_2\varepsilon}{r} M_2^{-(r-1)} H^{\sigma(r-1)}(t)\|v\|_r^r \\
& -\varepsilon\frac{M_5^{-(\beta_1-1)}}{\beta_1} H^{\sigma(\beta_1-1)}(t)\|\nabla u\|_{\beta_1}^{\beta_1} \\
& -\varepsilon\frac{M_6^{-(\beta_2-1)}}{\beta_2} H^{\sigma(\beta_2-1)}(t)\|\nabla u\|_{\beta_2}^{\beta_2}. \qquad (3.18)
\end{aligned}$$

We then use the two embedding

$$L^{2(\rho+2)}(\Omega) \hookrightarrow L^m(\Omega), W_0^{1,\alpha} \hookrightarrow L^{2(\rho+2)}(\Omega),$$

and (3.9) to get

$$\begin{aligned}
H^{\sigma(m-1)}(t)\|u\|_m^m \leq\ & c_2(\|u\|_{2(\rho+2)}^{2\sigma(m-1)(\rho+2)+m} \\
& +\|v\|_{2(\rho+2)}^{2\sigma(m-1)(\rho+2)}\|u\|_{2(\rho+2)}^m) \\
\leq\ & c_2(\|\nabla u\|_\alpha^{2\sigma(m-1)(\rho+2)+m} \\
& +\|\nabla v\|_\alpha^{2\sigma(m-1)(\rho+2)}\|\nabla u\|_\alpha^m). \qquad (3.19)
\end{aligned}$$

Similarly, the embedding $L^{2(\rho+2)}(\Omega) \hookrightarrow L^r(\Omega)$, $W_0^{1,\alpha} \hookrightarrow L^{2(\rho+2)}(\Omega)$ and (3.9) give

$$\begin{aligned}
H^{\sigma(r-1)}(t)\|v\|_r^r \leq\ & c_3(\|v\|_{2(\rho+2)}^{2\sigma(r-1)(\rho+2)+r} \\
& +\|u\|_{2(\rho+2)}^{2\sigma(r-1)(\rho+2)}\|v\|_{2(\rho+2)}^r) \\
\leq\ & c_3(\|\nabla v\|_\alpha^{2\sigma(r-1)(\rho+2)+r} \\
& +\|\nabla u\|_\alpha^{2\sigma(r-1)(\rho+2)}\|\nabla v\|_\alpha^r). \qquad (3.20)
\end{aligned}$$

Furthermore, the two embedding $W_0^{1,\alpha} \hookrightarrow L^{2(\rho+2)}(\Omega)$, $L^\alpha(\Omega) \hookrightarrow L^2(\Omega)$, yields

$$\begin{aligned}
H^\sigma(t)\|\nabla u\|_2^2 \leq\ & c_4\left(\|u\|_{2(\rho+2)}^{2\sigma(\rho+2)}\|\nabla u\|_2^2+\|v\|_{2(\rho+2)}^{2\sigma(\rho+2)}\|\nabla u\|_2^2\right) \\
\leq\ & c_4\left(\|\nabla u\|_\alpha^{2\sigma(\rho+2)+2}+\|\nabla v\|_\alpha^{2\sigma(\rho+2)}\|\nabla u\|_\alpha^2\right), \qquad (3.21)
\end{aligned}$$

and

$$\begin{aligned} H^{\sigma}(t)\|\nabla v\|_2^2 &\leq c_5\left(\|\nabla u\|_\alpha^{2\sigma(\rho+2)}\|\nabla v\|_\alpha^2+\|\nabla v\|_\alpha^{2\sigma(\rho+2)}\|\nabla v\|_\alpha^2\right) \\ &= c_5\left(\|\nabla u\|_\alpha^{2\sigma(\rho+2)}\|\nabla v\|_\alpha^2+\|\nabla v\|_\alpha^{2\sigma(\rho+2)+2}\right). \end{aligned} \tag{3.22}$$

Since $max(\beta_1,\beta_2)<\alpha$ then we have

$$\begin{aligned} H^{\sigma(\beta_1-1)}(t)\|\nabla u\|_{\beta_1}^{\beta_1} &\leq c_6(\|\nabla u\|_\alpha^{2\sigma(\beta_1-1)(\rho+2)}\|\nabla u\|_\alpha^{\beta_1} \\ &\quad +\|\nabla v\|_\alpha^{2\sigma(\beta_1-1)(\rho+2)}\|\nabla u\|_\alpha^{\beta_1}) \\ &= c_6(\|\nabla u\|_\alpha^{2\sigma(\beta_1-1)(\rho+2)+\beta_1} \\ &\quad +\|\nabla v\|_\alpha^{2\sigma(\beta_1-1)(\rho+2)}\|\nabla u\|_\alpha^{\beta_1}), \end{aligned} \tag{3.23}$$

and

$$\begin{aligned} H^{\sigma(\beta_2-1)}(t)\|\nabla v\|_{\beta_2}^{\beta_2} &\leq c_7(\|\nabla u\|_\alpha^{2\sigma(\beta_2-1)(\rho+2)}\|\nabla v\|_\alpha^{\beta_2} \\ &\quad +\|\nabla v\|_\alpha^{2\sigma(\beta_2-1)(\rho+2)}\|\nabla v\|_\alpha^{\beta_2}) \\ &= c_7(\|\nabla u\|_\alpha^{2\sigma(\beta_2-1)(\rho+2)}\|\nabla v\|_\alpha^{\beta_2} \\ &\quad +\|\nabla v\|_\alpha^{2\sigma(\beta_2-1)(\rho+2)+\beta_2}), \end{aligned} \tag{3.24}$$

for some positive constants c_2, c_3, c_4, c_5, c_6 and c_7. By using (3.11) and the algebraic inequality

$$z^{\nu}\leq(z+1)\leq\left(1+\tfrac{1}{a}\right)(z+a),\quad \forall z\geq 0,\ 0<\nu\leq 1,\ a\geq 0. \tag{3.25}$$

We have, for all $t\geq 0$,

$$\begin{cases} \|\nabla u\|_\alpha^{2\sigma(m-1)(\rho+2)+m}\leq d\left(\|\nabla u\|_\alpha^\alpha+H(0)\right)\leq d\left(\|\nabla u\|_\alpha^\alpha+H(t)\right) \\ \|\nabla v\|_\alpha^{2\sigma(r-1)(\rho+2)+r}\leq d\left(\|\nabla v\|_\alpha^\alpha+H(t)\right) \\ \|\nabla u\|_\alpha^{2\sigma(\rho+2)+2}\leq d\left(\|\nabla u\|_\alpha^\alpha+H(t)\right) \\ \|\nabla v\|_\alpha^{2\sigma(\rho+2)+2}\leq d\left(\|\nabla v\|_\alpha^\alpha+H(t)\right) \\ \|\nabla u\|_\alpha^{2\sigma(\beta_1-1)(\rho+2)+\beta_1}\leq d\left(\|\nabla u\|_\alpha^\alpha+H(t)\right) \\ \|\nabla v\|_\alpha^{2\sigma(\beta_2-1)(\rho+2)+\beta_2}\leq d\left(\|\nabla v\|_\alpha^\alpha+H(t)\right), \end{cases} \tag{3.26}$$

where $d=1+1/H(0)$.
Also keeping in mind the fact that $max(m,r)<\alpha$, using Yong's inequality, the

inequality (3.25) togrther withe (3.11), we conclude

$$\begin{cases} \|\nabla v\|_{\alpha}^{2\sigma(m-1)(\rho+2)}\|\nabla u\|_{\alpha}^{m}\leq C\left(\|\nabla v\|_{\alpha}^{\alpha}+\|\nabla u\|_{\alpha}^{\alpha}\right) \\ \|\nabla u\|_{\alpha}^{2\sigma(r-1)(\rho+2)}\|\nabla v\|_{\alpha}^{r}\leq C\left(\|\nabla u\|_{\alpha}^{\alpha}+\|\nabla v\|_{\alpha}^{\alpha}\right) \\ \|\nabla v\|_{\alpha}^{2\sigma(\rho+2)}\|\nabla u\|_{\alpha}^{2}\leq C\left(\|\nabla v\|_{\alpha}^{\alpha}+\|\nabla u\|_{\alpha}^{\alpha}\right) \\ \|\nabla u\|_{\alpha}^{2\sigma(\rho+2)}\|\nabla v\|_{\alpha}^{2}\leq C\left(\|\nabla u\|_{\alpha}^{\alpha}+\|\nabla v\|_{\alpha}^{\alpha}\right) \\ \|\nabla v\|_{\alpha}^{2\sigma(\beta_1-1)(\rho+2)}\|\nabla u\|_{\alpha}^{\beta_1}\leq C\left(\|\nabla v\|_{\alpha}^{\alpha}+\|\nabla u\|_{\alpha}^{\alpha}\right) \\ \|\nabla u\|_{\alpha}^{2\sigma(\beta_2-1)(\rho+2)}\|\nabla v\|_{\alpha}^{\beta_2}\leq C\left(\|\nabla u\|_{\alpha}^{\alpha}+\|\nabla v\|_{\alpha}^{\alpha}\right), \end{cases} \tag{3.27}$$

where C is a generic positive constant. Taking into account (3.19)- (3.27), then (3.18) takes the form

$$\begin{aligned} L^{'}(t) \geq\ & ((1-\sigma)-\varepsilon M)\,H^{-\sigma}(t)\,H^{'}(t) \\ & +\varepsilon\left(1+\frac{k}{2}\right)\left(\|u_t\|_2^2+\|v_t\|_2^2+m_1^2\|u\|_2^2+m_2^2\|v\|_2^2\right) \\ & +\varepsilon(\left[k/\alpha-1-kE_1\zeta_2^{-a}\right]-CM_1^{-(m-1)}-CM_2^{-(r-1)} \\ & -\frac{C}{4}M_3^{-1}-\frac{C}{4}M_4^{-1}-CM_5^{-(\beta_1-1)} \\ & -CM_6^{-(\beta_2-1)}-1)\left(\|\nabla u\|_{\alpha}^{\alpha}+\|\nabla v\|_{\alpha}^{\alpha}\right) \\ & +\varepsilon\left(k-CM_1^{-(m-1)}-CM_2^{-(r-1)}-\frac{C}{4}M_3^{-1}-\frac{C}{4}M_4^{-1}\right. \\ & \left.-CM_5^{-(\beta_1-1)}-CM_6^{-(\beta_2-1)}\right)H(t) \\ & +\varepsilon\left(\frac{c_0}{2(\rho+2)}-\frac{kc_1}{2(\rho+2)}\right)\left(\|u\|_{2(\rho+2)}^{2(\rho+2)}+\|v\|_{2(\rho+2)}^{2(\rho+2)}\right), \end{aligned} \tag{3.28}$$

for some constant k. Using $k=c_0/c_1$, we arrive at

$$\begin{aligned} L^{'}(t) \geq\ & ((1-\sigma)-\varepsilon M)\,H^{-\sigma}(t)\,H^{'}(t) \\ & +\varepsilon\left(1+\frac{c_0}{2c_1}\right)\left(\|u_t\|_2^2+\|v_t\|_2^2+m_1^2\|u\|_2^2+m_2^2\|v\|_2^2\right) \\ & +\varepsilon\left(\overline{c}-CM_1^{-(m-1)}-CM_2^{-(r-1)}-\frac{C}{4}M_3^{-1}-\frac{C}{4}M_4^{-1}\right. \\ & \left.-CM_5^{-(\beta_1-1)}-CM_6^{-(\beta_2-1)}-1\right)\left(\|\nabla u\|_{\alpha}^{\alpha}+\|\nabla v\|_{\alpha}^{\alpha}\right) \\ & +\varepsilon\left(c_0/c_1-CM_1^{-(m-1)}-CM_2^{-(r-1)}-\frac{C}{4}M_3^{-1}-\frac{C}{4}M_4^{-1}\right. \\ & \left.-CM_5^{-(\beta_1-1)}-CM_6^{-(\beta_2-1)}\right)H(t), \end{aligned} \tag{3.29}$$

where $\overline{c} = k/\alpha - 1 - kE_1\zeta_2^{-2} = c_0/(c_1\alpha) - 1 - (c_0/c_1) E_1\zeta_2^{-2} > 0$ since $\zeta_2 > \zeta_1$.
At this point, and for large values of M_1, M_2, M_3, M_4, M_5 and M_6, we can find positive constants Λ_1 and Λ_2 such that (3.29) becomes

$$\begin{aligned} L^{'}(t) &\geq ((1-\sigma) - M\varepsilon) H^{-\sigma}(t) H^{'}(t) \\ &+\varepsilon\left(1+\frac{c_0}{2c_1}\right)\left(\|u_t\|_2^2 + \|v_t\|_2^2 + m_1^2\|u\|_2^2 + m_2^2\|v\|_2^2\right) \\ &+\varepsilon\Lambda_1\left(\|\nabla u\|_\alpha^\alpha + \|\nabla v\|_\alpha^\alpha\right) + \varepsilon\Lambda_2 H(t). \end{aligned} \tag{3.30}$$

Once M_1, M_2, M_3, M_4, M_5 and M_6 are fixed (hence, Λ_1 and Λ_2), we pick ε small enough so that $((1-\sigma) - M\varepsilon) \geq 0$ and

$$L(0) = H^{1-\sigma}(0) + \int_\Omega [u_0.u_t + v_0.v_t]\,dx > 0.$$

From these and (3.30) becomes

$$\begin{aligned} L^{'}(t) &\geq \varepsilon\Gamma(H(t) + \|u_t\|_2^2 + \|v_t\|_2^2 + m_1^2\|u\|_2^2 + m_2^2\|v\|_2^2 \\ &+ \|\nabla u\|_\alpha^\alpha + \|\nabla v\|_\alpha^\alpha). \end{aligned} \tag{3.31}$$

Thus, we have $L(t) \geq L(0) > 0$, for all $t \geq 0$. Next, by Holder's and Young's inequalities, we estimate

$$\begin{aligned} &\left(\int_\Omega u.u_t(x,t)\,dx + \int_\Omega v.v_t(x,t)\,dx\right)^{\frac{1}{1-\sigma}} \\ \leq\ & C\left(\|u\|_{2(\rho+2)}^{\frac{\tau}{1-\sigma}} + \|u_t\|_2^{\frac{s}{1-\sigma}} + \|v\|_{2(\rho+2)}^{\frac{\tau}{1-\sigma}} + \|v_t\|_2^{\frac{s}{1-\sigma}}\right) \\ \leq\ & C\left(\|\nabla u\|_\alpha^{\frac{\tau}{1-\sigma}} + \|u_t\|_2^{\frac{s}{1-\sigma}} + \|\nabla v\|_\alpha^{\frac{\tau}{1-\sigma}} + \|v_t\|_2^{\frac{s}{1-\sigma}}\right), \end{aligned} \tag{3.32}$$

for $\frac{1}{\tau} + \frac{1}{s} = 1$. We take $s = 2(1-\sigma)$, to get $\frac{\tau}{1-\sigma} = \frac{2}{1-2\sigma}$.
By using (3.11) and (3.25) we get

$$\|\nabla u\|_\alpha^{\frac{2}{(1-2\sigma)}} \leq d\left(\|\nabla u\|_\alpha^\alpha + H(t)\right),$$

and

$$\|\nabla v\|_\alpha^{\frac{2}{(1-2\sigma)}} \leq d\left(\|\nabla v\|_\alpha^\alpha + H(t)\right),\ \forall t \geq 0.$$

Therefore, (3.32) becomes

$$\begin{aligned} &\left(\int_\Omega u.u_t(x,t)\,dx + \int_\Omega v.v_t(x,t)\,dx\right)^{\frac{1}{1-\sigma}} \\ &\leq C(\|\nabla u\|_\alpha^\alpha + \|\nabla v\|_\alpha^\alpha + \|u_t\|_2^2 + \|v_t\|_2^2 \\ &\quad + m_1^2\|u\|_2^2 + m_2^2\|v\|_2^2 + H(t)), \forall t \geq 0. \end{aligned} \tag{3.33}$$

Also, since

$$\begin{aligned} L^{\frac{1}{1-\sigma}}(t) &= \left(H^{1-\sigma}(t)+\varepsilon\int_{\Omega}(u.u_t+v.v_t)(x,t)\,dx\right)^{\frac{1}{(1-\sigma)}} \\ &\leq C\left(H(t)+\left|\int_{\Omega}(u.u_t(x,t)+v.v_t(x,t))\,dx\right|^{\frac{1}{(1-\sigma)}}\right) \\ &\leq C[H(t)+\|\nabla u\|_{\alpha}^{\alpha}+\|\nabla v\|_{\alpha}^{\alpha}+\|u_t\|_2^2+\|v_t\|_2^2 \\ &\quad +m_1^2\|u\|_2^2+m_2^2\|v\|_2^2],\ \forall t\geq 0. \end{aligned} \tag{3.34}$$

Combining withe (3.34) and (3.31), we arrive at

$$L^{'}(t)\geq a_0 L^{\frac{1}{1-\sigma}}(t),\ \forall t\geq 0. \tag{3.35}$$

Finally, a simple integration of (3.35) gives the desired result.This completes the proof of Theorem (3.1) □

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Global existence and stability of solution for a $p-$Kirchhoff type hyperbolic equation with damping and source terms

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Abstract. In this paper, we consider a nonlinear $p-$Kirchhoff type hyperbolic equation with damping and source terms

$$u_{tt} - M\left(\int_{\Omega} |\nabla u|^p \, dx\right) \Delta_p u + |u_t|^{m-2} u_t = |u|^{r-2} u.$$

Under suitable assumptions and positive initial energy, we prove the global existence of solution by using the potential energy and Nehari's functionals. Finally, the stability of equation is established based on Komornik's integral inequality.

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Keywords: $p-$Kirchhoff type hyperbolic equation, global existence, source term, Komornik's integral inequality.

1. Introduction

In this article, we consider the following value problem

$$\begin{cases} u_{tt} - M\left(\int_{\Omega} |\nabla u|^p \, dx\right) \Delta_p u + |u_t|^{m-2} u_t = |u|^{r-2} u, & (x,t) \in \Omega \times (0,T), \\ u(x,t) = 0, & (x,t) \in \partial\Omega \times (0,), \\ u(x,0) = u_0(x), \ u_t(x,0) = u_1(x), & x \in \Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^n, n \geq 1$ with smooth boundary $\partial\Omega$ and

$$M(s) = a + bs$$

with positive parameters a, b, $\Delta_p u = div(|\nabla u|^{p-2} \nabla u)$, $p \geq 2$.

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In the past few years, much effort has been devoted to nonlocal problems because of their wide applications in both physics and biology. For exemple the following hyperbolic equation with a nonlocal coefficient are as follows:

$$\varepsilon u_{tt}^{\varepsilon} + u_t^{\varepsilon} - M\left(\int_{\Omega} |\nabla u^{\varepsilon}|^p \, dx\right) \Delta_p u^{\varepsilon} = f\left(x,\ t,\ u^{\epsilon}\right), \tag{1.2}$$

where $M(s) = a + bs,\ a, b > 0$ and $p > 1$, in a bounded domain $\Omega \subset \mathbb{R}^n$ is a potential model for damped small transversal vibrations of an elastic string with uniform density ε (see [6]). For $p = 2$, such nonlocal equations were first proposed by Kirchhoff [7] in 1883 and therefore were usually referred to as Kirchhoff equations.

Equation (1.1) can be viewed as a generalization of a model introduced by Kirchhoff [15]. The following Kirchhoff type equation

$$u_{tt} - M\left(\|\nabla u\|_2^2\right) \Delta u + g\left(u_t\right) = f\left(u\right), \tag{1.3}$$

have been discussed by many authors. For $g(u_t) = u_t$, the global existence and blow up results can by found in ([13], [15]), for $g(u_t) = |u_t|^{m-2} u_t$, $p > 2$, the main results of existence and blow up are in ([5], [11]). The absence of the damping term $|u_t|^{m-2} u_t$ in equation (1.1), when $M(s) = a + bs^{\gamma}$ $(\gamma > 0)$ and $p = 2$, the existence of the global solution was investigated by many authors (see [1]-[4], [9], [10], [15], [16]). The works of K. Ono [12]-[14] deal with equation (1.3) in two cases with $f(u) = |u|^{r-2} u$, $p > 2$. In the first case, for $g(u_t) = -u_t$ or u_t, he considered $M(s) = a + bs^{\gamma}$, where $a \geq 0$, $b \geq 0$, $a + b > 0$, $\gamma > 0$. He showed that the local solutions blow up at finite time with $E(0) > 0$ by applying the concavity method. Moreover, he combined the so-called potential well method and concavity method to show blow-up properties with $E(0) > 0$. While in the second case, for $g(u_t) = |u_t|^{m-2} u_t$, $m > 2$, he treated $M(s) = a + bs^{\gamma}$, where $b > 0$, $a = 0$ and $\gamma \geq 1$. He proved that the local solution is not global when $p > max(2\gamma + 2, m)$ and $E(0) < 0$.

The paper is organized as follows. In section 2, we introduce some notations and Lemma needed in the next sections to prove the main result. In section 3, we use the energy and Nihari functionals to prove the global existence of the solutions. In section 4, we use the energy method to prove the result based on Komornik's integral inequality.

2. Preliminaries

We begin this section with some notations and definitions. Denote by $\|.\|_p$, the $L^p(\Omega)$ norm of a Lebesgue function $u \in L^p(\Omega)$ for $p \geq 1$. We use $W_0^{1,p}(\Omega)$ to denote the well-known Sobolev space such that both u and $|\nabla u|$ are in $W_0^{1,p}(\Omega)$ equipped with the norm $\|u\|_{W_0^{1,p}(\Omega)} = \|\nabla u\|_p$.

Lemma 2.1. *Let s be a number with $2 \leq s \leq +\infty$ if $n \leq p$ and $2 \leq s \leq \frac{pn}{n-p}$ if $n > p$. Then there is a constant c_* depending on Ω and s such that*

$$\|u\|_s \leq c_* \|\nabla u\|_p, \quad \forall u \in W_0^{1,p}(\Omega).$$

Theorem 2.2. *Suppose that* $(u_0, u_1) \in W_0^{1,p}(\Omega) \times L^2(\Omega)$ *and*

$$2p < r \leq p^*,$$

where

$$p^* = \begin{cases} \frac{np}{n-p}, & \text{if } n > p, \\ +\infty & \text{if } n \leq p. \end{cases}$$

Then problem (1.1) *has a unique weak solution such that*

$$\begin{aligned} u &\in L^\infty\left((0,T),\ W_0^{1,p}(\Omega)\right), \\ u_t &\in L^\infty\left((0,T),\ L^2(\Omega)\right) \cap L^m(\Omega \times (0,\ T)), \\ u_{tt} &\in L^2\left((0,T),\ W^{-1,p'}(\Omega)\right). \end{aligned}$$

3. Global existence

In this section, we state and prove our result, we define the potential energy functional and the Nehari's functional, by the following

$$E(t) = E(u(t)) = \frac{1}{2}\|u_t(t)\|_2^2 + \frac{a}{p}\|\nabla u(t)\|_p^p + \frac{b}{2p}\|\nabla u(t)\|_p^{2p} - \frac{1}{r}\|u(t)\|_r^r. \quad (3.1)$$

$$J(t) = J(u(t)) = \frac{a}{p}\|\nabla u(t)\|_p^p + \frac{b}{2p}\|\nabla u(t)\|_p^{2p} - \frac{1}{r}\|u(t)\|_r^r. \quad (3.2)$$

$$I(t) = I(u(t)) = a\|\nabla u(t)\|_p^p + b\|\nabla u(t)\|_p^{2p} - \|u(t)\|_r^r. \quad (3.3)$$

We can considering $a = b = 1$, and this does not change the general result of (1.1).

Lemma 3.1. *Under the assumptions of theorem* 2.2, *we have*

$$E'(t) = -\|u_t(t)\|_m^m \leq 0, \quad t \in [0,\ T]. \quad (3.4)$$

and

$$E(t) \leq E(0).$$

Proof. We multiply the first equation of (1.1) by u_t and integrating over the domain Ω, we get

$$\frac{d}{dt}\left(\frac{1}{2}\|u_t\|_2^2 + \frac{1}{p}\int_\Omega |\nabla u(t)|^p\,dx + \frac{1}{2p}\left(\int_\Omega |\nabla u(t)|^p\,dx\right)^2 - \frac{1}{r}\|u(t)\|_r^r\right) = -\|u_t(t)\|_m^m,$$

then

$$E'(t) = -\|u_t(t)\|_m^m \leq 0.$$

Integrating (3.4) over $(0,\ t)$, we obtain $E(t) \leq E(0)$. □

Lemma 3.2. *Assume that the assumptions of theorem 2.2 hold,*

$$I(0) > 0,$$

and

$$\beta_1 + \beta_2 < 1, \tag{3.5}$$

where

$$\beta_1 := \alpha c_*^r \left(\frac{pr}{r-p} E(0)\right)^{\frac{r-p}{p}}, \ \beta_2 := (1-\alpha) c_*^r \left(\frac{2pr}{r-2p} E(0)\right)^{\frac{r-2p}{2p}}$$

with $0 < \alpha < 1$, c_* *is the best embedding constant of* $W_0^{1,\,p}(\Omega) \hookrightarrow L^r(\Omega)$, *then* $I(t) > 0$, *for all* $t \in [0,\ T]$.

Proof. By continuity, there exists T_*, such that

$$I(t) \geq 0, \quad \text{for all } t \in [0,\ T_*]. \tag{3.6}$$

Now, we have for all $t \in [0,\ T_*]$:

$$\begin{aligned} J(t) &= J(u(t)) = \frac{1}{p}\|\nabla u(t)\|_p^p + \frac{1}{2p}\|\nabla u(t)\|_p^{2p} - \frac{1}{r}\|u(t)\|_r^r \\ &\geq \frac{1}{p}\|\nabla u(t)\|_p^p + \frac{1}{2p}\|\nabla u(t)\|_p^{2p} - \frac{1}{r}\left(\|\nabla u(t)\|_p^p + \|\nabla u(t)\|_p^{2p} - I(t)\right) \\ &\geq \frac{r-p}{pr}\|\nabla u(t)\|_p^p + \frac{r-2p}{2pr}\|\nabla u(t)\|_p^{2p} + \frac{1}{r} I(t) \end{aligned}$$

using (3.6), we obtain

$$\frac{r-p}{pr}\|\nabla u(t)\|_p^p + \frac{r-2p}{2pr}\|\nabla u(t)\|_p^{2p} \leq J(t), \qquad \text{for all } t \in [0,\ T_*]. \tag{3.7}$$

By the definition of E, we get

$$\|\nabla u(t)\|_p^p \leq \frac{pr}{r-p} E(t) \leq \frac{pr}{r-p} E(0) \tag{3.8}$$

and

$$\|\nabla u(t)\|_p^{2p} \leq \frac{2pr}{r-2p} E(t) \leq \frac{2pr}{r-2p} E(0) \tag{3.9}$$

On the other hand, we have

$$\|u(t)\|_r^r = \alpha \ \|u(t)\|_r^r + (1-\alpha)\|u(t)\|_r^r$$

By the embedding of $W_0^{1,\,p}(\Omega) \hookrightarrow L^r(\Omega)$, we obtain

$$\begin{aligned} \|u(t)\|_r^r &\leq \alpha\ c_*^r \|\nabla u(t)\|_p^r + (1-\alpha) c_*^r \|\nabla u(t)\|_p^r \\ &\leq \alpha\ c_*^r \|\nabla u(t)\|_p^{r-p} \times \|\nabla u(t)\|_p^p + (1-\alpha) c_*^r \|\nabla u(t)\|_p^{r-2p} \times \|\nabla u(t)\|_p^{2p} \end{aligned}$$

By (3.8) and (3.9), we get

$$\|u(t)\|_r^r \leq \beta_1 \|\nabla u(t)\|_p^p + \beta_2 \|\nabla u(t)\|_p^{2p}, \qquad \text{for all } t \in [0,\ T_*]. \tag{3.10}$$

Since $\beta_1 + \beta_2 < 1$, then

$$\|u(t)\|_r^r < \|\nabla u(t)\|_p^p + \|\nabla u(t)\|_p^{2p}, \qquad \text{for all } t \in [0,\ T_*].$$

This implies that

$$I(t) > 0, \qquad \text{for all } t \in [0,\ T_*].$$

By repeating the above procedure, we can extend T_* to T. □

Theorem 3.3. *Under the assumptions of lemma* 3.2, *the local solution of* (1.1) *is global.*

Proof. We have

$$\begin{aligned} E(u(t)) &= \frac{1}{2}\|u_t(t)\|_2^2 + \frac{1}{p}\|\nabla u(t)\|_p^p + \frac{1}{2p}\|\nabla u(t)\|_p^{2p} - \frac{1}{r}\|u(t)\|_r^r \\ &\geq \frac{1}{2}\|u_t(t)\|_2^2 + \frac{r-p}{pr}\|\nabla u(t)\|_p^p + \frac{r-2p}{2pr}\|\nabla u(t)\|_p^{2p}. \end{aligned}$$

So that

$$\|u_t(t)\|_2^2 + \|\nabla u(t)\|_p^p \leq C\ E(t). \tag{3.11}$$

By Lemma 3.1, we obtain

$$\|u_t(t)\|_2^2 + \|\nabla u(t)\|_p^p \leq C\ E(0). \tag{3.12}$$

This implies that the local solution is global in time. □

4. Stability of solution

In this section our main result is established based in Komornik's integral inequality [8]. For this, we need the following Lemma:

Lemma 4.1. *Suppose that the assumptions of Lemma* 3.2 *and* $m > p$, *hold, then there exists a positive constant* c *such that*

$$\int_\Omega |u(t)|^m\,dx \leq cE(t). \tag{4.1}$$

Proof. By using (3.8), we obtain

$$\begin{aligned} \int_\Omega |u(t)|^m\,dx = \|u(t)\|_m^m &\leq c_*^m \|\nabla u(t)\|_p^m \\ &\leq c_*^m \|\nabla u(t)\|_p^{m-p} \times \|\nabla u(t)\|_p^p \\ &\leq c_*^m \|\nabla u(t)\|_p^{m-p} \times \frac{rp}{r-p}E(t) \leq cE(t). \end{aligned}$$

□

Now, we state our main result:

Theorem 4.2. *Let the assumptions of Lemma* 3.2, *then, there exists constants* $C, \zeta > 0$, *such that*

$$\begin{aligned} E(t) &\leq \frac{C}{(1+t)^{\frac{2}{m-2}}}, \quad \text{for all } t \geq 0 \quad \text{if } m > 2. \\ E(t) &\leq Ce^{-\zeta t}, \qquad \text{for all } t \geq 0 \quad \text{if } m = 2. \end{aligned}$$

Proof. Multiplying first equation of (1.1) by $u(t)E^q(t)$ $(q>0)$, and integrating over $\Omega\times(S,\ T)$, we obtain

$$\int_S^T\int_\Omega E^q(t)\left[u(t)u_{tt}(t)-u(t)\left(M\left(\int_\Omega|\nabla u|^p\,dx\right)\Delta_p u+|u_t|^{m-2}u_t\right)\right]dxdt$$

$$=\int_S^T E^q(t)\int_\Omega|u(t)|^r\,dxdt$$

So that

$$\int_S^T\int_\Omega E^q(t)\Big[(u(t)u_t(t))_t-|u_t(t)|^2+|\nabla u(t)|^p+\|\nabla u(t)\|_p^p|\nabla u(t)|^p$$

$$+u(t)|u_t|^{m-2}u_t\Big]dxdt=\int_S^T E^q(t)\int_\Omega|u(t)|^r\,dxdt$$

We add and subtract the term

$$\int_S^T E^q(t)\int_\Omega\Big[\beta_1|\nabla u(t)|^p+\beta_2\|\nabla u(t)\|_p^p|\nabla u(t)|^p+(2+\beta_1+\beta_2)|u_t(t)|^2\Big]dxdt,$$

and use (3.10), to get

$$(1-\beta_1)\int_S^T E^q(t)\int_\Omega\Big[|\nabla u(t)|^p+|u_t(t)|^2\Big]dxdt$$

$$+(1-\beta_2)\int_S^T E^q(t)\int_\Omega\Big[\|\nabla u(t)\|_p^p|\nabla u(t)|^p+|u_t(t)|^2\Big]dxdt$$

$$+\int_S^T E^q(t)\int_\Omega\Big[(u(t)u_t(t))_t-(3-\beta_1-\beta_2)|u_t(t)|^2\Big]dxdt$$

$$+\int_S^T E^q(t)\int_\Omega u(t)u_t(t)|u_t(t)|^{m-2}\,dxdt$$

$$=-\int_S^T E^q(t)\int_\Omega\Big[\beta_1|\nabla u(t)|^p+\beta_2\|\nabla u(t)\|_p^p|\nabla u(t)|^p-|u(t)|^r\Big]dxdt\le 0. \qquad (4.2)$$

It is clear that

$$\gamma\int_S^T E^q(t)\int_\Omega\left[\frac{1}{p}|\nabla u(t)|^p+\frac{1}{2p}\|\nabla u(t)\|_p^p|\nabla u(t)|^p+\frac{|u_t(t)|^2}{2}-\frac{|u(t)|^r}{r}\right]dxdt$$

$$\le (1-\beta_1)\int_S^T E^q(t)\int_\Omega\left[\frac{1}{p}|\nabla u(t)|^p+\frac{|u_t(t)|^2}{2}\right]dxdt$$

$$+(1-\beta_2)\int_S^T E^q(t)\int_\Omega\left[\frac{1}{2p}\|\nabla u(t)\|_p^p|\nabla u(t)|^p+\frac{|u_t(t)|^2}{2}\right]dxdt \tag{4.3}$$

where $\gamma=Min\left((1-\beta_1),\ (1-\beta_2)\right)$. By (4.2) , (4.3) and definition of $E(t)$, we get

$$\begin{aligned}\gamma\int_S^T E^{q+1}(t)\,dt \le & -\int_S^T E^q(t)\int_\Omega (u(t)\,u_t(t))_t\,dxdt\\ &+(3-\beta_1-\beta_2)\int_S^T E^q(t)\int_\Omega |u_t(t)|^2\,dxdt\\ &-\int_S^T E^q(t)\int_\Omega u(t)\,u_t(t)\,|u_t(t)|^{m-2}\,dxdt.\end{aligned} \tag{4.4}$$

Using the definition of $E(t)$ and the following expression

$$\begin{aligned}\frac{d}{dt}\left(E^q(t)\int_\Omega u(t)\,u_t(t)\,dx\right) = & \; qE^{q-1}(t)\frac{d}{dt}E(t)\int_\Omega u(t)\,u_t(t)\,dx\\ &+E^q(t)\int_\Omega (u(t)\,u_t(t))_t\,dx.\end{aligned}$$

Inequality (4.4), becomes

$$\gamma\int_S^T E^{q+1}(t)\,dt\le q\int_S^T E^{q-1}(t)\frac{d}{dt}E(t)\int_\Omega u(t)\,u_t(t)\,dx$$

$$-\int_S^T\frac{d}{dt}\left(E^q(t)\int_\Omega u(t)\,u_t(t)\,dx\right)dt-\int_S^T E^q(t)\int_\Omega u(t)\,u_t(t)\,|u_t(t)|^{m-2}\,dxdt$$

$$+(3-\beta_1-\beta_2)\int_S^T E^q(t)\int_\Omega |u_t(t)|^2\,dxdt. \tag{4.5}$$

In the sequel, we denote by c the various constants.
We estimate the terms in the right-hand side of (4.5) as follow:

By (3.4) and Young's inequality, we obtain

$$\begin{aligned} & q\int_S^T E^{q-1}(t)\frac{d}{dt}E(t)\int_\Omega u(t)\,u_t(t)\,dx \\ \leq\ & q\int_S^T E^{q-1}(t)\left(-E^{'}(t)\right)\int_\Omega \left[\frac{1}{p}\left|u(t)\right|^p+\frac{p-1}{p}\left|u_t(t)\right|^{\frac{p}{p-1}}\right]dxdt \end{aligned} \tag{4.6}$$

Since, $1\leq\frac{p}{p-1}<2$, by the embedding of $L^2(\Omega)\hookrightarrow L^{\frac{p}{p-1}}(\Omega)$, we have

$$\begin{aligned} & q\int_S^T E^{q-1}(t)\frac{d}{dt}E(t)\int_\Omega u(t)\,u_t(t)\,dx \\ \leq\ & q\int_S^T E^{q-1}(t)\left(-E^{'}(t)\right)\int_\Omega \left[\frac{1}{p}\left|u(t)\right|^p+c\frac{p-1}{p}\left|u_t(t)\right|^{2}\right]dxdt \end{aligned}$$

Thus, by (3.11), we find

$$\begin{aligned} & q\int_S^T E^{q-1}(t)\frac{d}{dt}E(t)\int_\Omega u(t)\,u_t(t)\,dx \\ \leq\ & c\int_S^T E^{q}(t)\left(-E^{'}(t)\right)dt \\ \leq\ & cE^{q+1}(S)-cE^{q+1}(T) \\ \leq\ & cE^{q}(0)\,E(S)\leq cE(S). \end{aligned} \tag{4.7}$$

For the second term, we have

$$\begin{aligned} & -\int_S^T \frac{d}{dt}\left(E^q(t)\int_\Omega u(t)\,u_t(t)\,dx\right)dxdt \\ \leq\ & \left|E^q(t)\int_\Omega u(S)\,u_t(S)\,dx-E^q(t)\int_\Omega u(T)\,u_t(T)\,dx\right| \\ \leq\ & E^q(t)\left|\int_\Omega u(x,S)\,u_t(x,S)\,dx\right|+E^q(t)\left|\int_\Omega u(x,T)\,u_t(x,T)\,dx\right| \\ \leq\ & cE^{q+1}(S)+cE^{q+1}(T) \\ \leq\ & cE^{q}(0)\,E(S)\leq cE(S). \end{aligned} \tag{4.8}$$

For the third term, we use the following Young inequality:

$$XY \leq \frac{\varepsilon}{\lambda_1} X^{\lambda_1} + \frac{1}{\lambda_2 \varepsilon^{\frac{\lambda_2}{\lambda_1}}} Y^{\lambda_2},\ X,\ Y \geq 0,\ \varepsilon > 0 \text{ and } \frac{1}{\lambda_1} + \frac{1}{\lambda_2} = 1,$$

with $\lambda_1 = m$, $\lambda_2 = \frac{m}{m-1}$.
By (3.4) and Lemma 4.1, we have

$$\begin{aligned}
& -\int_S^T E^q(t) \int_\Omega u(t) u_t(t) |u_t(t)|^{m-2} \, dxdt \\
\leq & \int_S^T E^q(t) \left(\varepsilon c \int_\Omega |u(t)|^m \, dx + c_\varepsilon \int_\Omega |u_t(t)|^m \, dx \right) dt \\
\leq & \ \varepsilon c \int_S^T E^q(t) \int_\Omega |u(t)|^m \, dxdt + c_\varepsilon \int_S^T E^q(t) \left(-E'(t) \right) dt \\
\leq & \ \varepsilon c \int_S^T E^{q+1}(t) \, dt + c_\varepsilon E(S). \qquad (4.9)
\end{aligned}$$

For the last term of (4.5), we have

$$\begin{aligned}
& (3 - \beta_1 - \beta_2) \int_S^T E^q(t) \int_\Omega |u_t(t)|^2 \, dxdt \\
\leq & \ c \int_S^T E^q(t) \left(\int_\Omega |u_t(t)|^m \, dx \right)^{\frac{2}{m}} dt \\
\leq & \ c \int_S^T E^q(t) \left(-E'(t) \right)^{\frac{2}{m}} dt. \qquad (4.10)
\end{aligned}$$

By Young's inequality with $\lambda_1 = (q+1)/q$ and $\lambda_2 = q+1$, we have

$$\int_S^T E^q(t) \left(-E'(t) \right)^{\frac{2}{m}} dt \leq \varepsilon c \int_S^T E^{q+1}(t) \, dt + c_\varepsilon \int_S^T \left(-E'(t) \right)^{\frac{2(q+1)}{m}} dt.$$

We take $q = \frac{m}{2} - 1$, to find

$$\int_S^T E^q(t) \left(-E'(t) \right)^{\frac{2}{m}} dt \leq \varepsilon c \int_S^T E^{q+1}(t) \, dt + c_\varepsilon \int_S^T \left(-E'(t) \right) dt.$$

This implies

$$\int_S^T E^q(t)\left(-E'(t)\right)^{\frac{2}{m}} dt \le \varepsilon c \int_S^T E^{q+1}(t)\,dt + c_\varepsilon E(S). \tag{4.11}$$

Substituting (4.11) into (4.10), we obtain

$$(3-\beta_1-\beta_2)\int_S^T E^q(t)\int_\Omega |u_t(t)|^2\,dxdt \le \varepsilon c\int_S^T E^{q+1}(t)\,dt + c_\varepsilon E(S). \tag{4.12}$$

By insert (4.7), (4.8), (4.9) and (4.12) in (4.5), we arrive at

$$\gamma\int_S^T E^{\frac{m}{2}}(t)\,dt \le \varepsilon c\int_S^T E^{\frac{m}{2}}(t)\,dt + c_\varepsilon E(S).$$

Choosing ε small enough for that

$$\int_S^T E^{\frac{m}{2}}(t)\,dt \le cE(S).$$

By taking T goes to ∞, we get

$$\int_S^\infty E^{\frac{m}{2}}(t)\,dt \le cE(S).$$

By Komornik's integral inequality yields the result. □

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Applications of the deferred generalized de la Vallée Poussin means in approximation of continuous functions

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Abstract. In this paper we have proved a theorem which show the degree of approximation of periodic functions by some generalized means of their Fourier series. In addition, our result is extended to two-dimensional setting as well.

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1. Introduction

Let f be a 2π-periodic function, $f \in L[0, 2\pi]$, and

$$\frac{a_0}{2} + \sum_{k=1}^{\infty}(a_k \cos kx + b_k \sin kx), \tag{1.1}$$

its Fourier series at the point x, where

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos kx dx,\ (k = 0, 1, \ldots);\ \ b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin kx dx,\ (k = 1, 2, \ldots).$$

By

$$\|f\| = \sup_{0\le x\le 2\pi} |f(x)|$$

we denote the sup-norm of f over $[0, 2\pi]$, and by $C[0, 2\pi]$ the class of all 2π-periodic continuous functions defined in $[0, 2\pi]$.

In 1928, was G. Alexits [4] who studied the degree of approximation of function a $f \in \text{Lip}\alpha$ by Cesàro means (C, δ) of its Fourier series. This study may be considered as a starting point for other studies of this nature, and another type of similar studies

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can be found in [6]-[9]. Recent studies of other researchers can be found in [1], [5], and [7].

For our purpose, we are going to recall a result proved in [6]. To do this we need first to present the generalized Vallée Poussin mean given in [10].

Let $\sum_{n=1}^{\infty} w_n$ be a given infinite series and let s_n be its n-th partial sum. Let $\lambda := (\lambda_n)$ be a monotone non-decreasing sequence of integers such that $\lambda_1 = 1$ and $\lambda_{n+1} - \lambda_n \leq 1$.

The mean

$$V_n(\lambda) = \frac{1}{\lambda_n} \sum_{m=n-\lambda_n}^{n-1} s_m, \quad (n \geq 1), \tag{1.2}$$

is called the n-th generalized de la Vallée Poussin mean of the sequence (s_n) generated by sequence (λ_n).

For n-th partial sum

$$s_n(f;x) = \frac{a_0}{2} + \sum_{k=1}^{n} (a_k \cos kx + b_k \sin kx)$$

of the series (1.1), its n-th generalized de la Vallée Poussin mean is defined by

$$V_n(\lambda; f; x) = \frac{1}{\lambda_n} \sum_{m=n-\lambda_n}^{n-1} s_m(f;x), \quad (n \geq 1), \tag{1.3}$$

and the modulus of continuity of $f(x)$, for a given real number $\delta > 0$, is defined as follows

$$\omega(f;\delta) := \sup_{|x-y| \leq \delta} |f(x) - f(y)|,$$

where $x, y \in [0, 2\pi]$.

Throughout this paper we write $u = \mathcal{O}(v)$ if there exists a positive constant K, such that $u \leq Kv$. Now, we are ready to recall the result mentioned above.

Theorem 1.1 ([6]). *Let $f \in C[0, 2\pi]$ and $\omega(f;t)$ be its modulus of continuity satisfying the following conditions as $t \to +0$:*

$$\int_t^{\frac{\pi}{2}} u^{-2} \omega(f;u) du = \mathcal{O}(F(t)), \tag{1.4}$$

where $F(t) \geq 0$, and

$$\int_0^t F(u) du = \mathcal{O}(tF(t)). \tag{1.5}$$

Then

$$\|f - V_n(\lambda; f)\| = \mathcal{O}\left(\frac{1}{\lambda_n} F\left(\frac{\pi}{2\lambda_n}\right)\right). \tag{1.6}$$

For our further investigation let $a := (a_n)$ and $b := (b_n)$ be sequences of non-negative integers with condition

$$1 \leq b_n - a_n + \lambda_n, \quad (n = 1, 2, \dots). \tag{1.7}$$

Whence, we are in able to generalize the mean $V_n(\lambda)$ defined in (1.2) as follows.

The mean

$$V_n(\lambda, a, b) = \frac{1}{b_n - a_n + \lambda_n} \sum_{m=a_n-\lambda_n}^{b_n-1} s_m, \quad (n \geq 1), \tag{1.8}$$

is called the n-th deferred generalized de la Vallée Poussin mean of the sequence (s_n) generated by sequences λ, a, and b.

It is the purpose of this paper to estimate the deviation $f - V_n(\lambda, a, b)$ in the sup-norm, which in fact generalize Theorem 1.1 (as well as we extend it in the two-dimensional setting, see subsec. 3.2). To do this we need some helpful lemmas given in next section.

2. Auxiliary lemma

Next lemma has been proved implicitly in [6].

Lemma 2.1. *Let* (1.4) *hold. Then,* $\omega(f; t) = \mathcal{O}(tF(t))$.

Now, we prove next helpful lemma.

Lemma 2.2. *Denote by*

$$\mathbb{K}_n^{a,b}(t) := \sum_{m=a_n-\lambda_n}^{b_n-1} D_m(t) = \sum_{m=a_n-\lambda_n}^{b_n-1} \frac{\sin{(2m+1)\, t}}{\sin t}$$

the deferred de la Vallée Poussin kernel, where $D_m(t) := \frac{\sin(2m+1)t}{\sin t}$.
Then,

(i) $\mathbb{K}_n^{a,b}(t) = \dfrac{\sin(b_n - a_n + \lambda_n)t \sin(b_n + a_n - \lambda_n)t}{\sin^2 t}$,

(ii) $|\mathbb{K}_n^{a,b}(t)| = \mathcal{O}\left(\dfrac{b_n - a_n + \lambda_n}{t}\right), \quad 0 < t \leq \dfrac{\pi}{2(b_n - a_n + \lambda_n)}$,

(iii) $|\mathbb{K}_n^{a,b}(t)| = \mathcal{O}\left(\dfrac{1}{t^2}\right), \quad \dfrac{\pi}{2(b_n - a_n + \lambda_n)} < t \leq \dfrac{\pi}{2}$.

Proof. (i) We have

$$\begin{aligned}
\mathbb{K}_n^{a,b}(t) &= \sum_{m=a_n-\lambda_n}^{b_n-1} \frac{\sin{(2m+1)\, t}}{\sin t} \\
&= \sum_{m=0}^{b_n-1} \frac{2\sin{(2m+1)\, t}\sin t}{2\sin^2 t} - \sum_{m=0}^{a_n-\lambda_n-1} \frac{2\sin{(2m+1)\, t}\sin t}{2\sin^2 t} \\
&= \frac{1-\cos(2b_n t)}{2\sin^2 t} - \frac{1-\cos(a_n-\lambda_n)t}{2\sin^2 t} \\
&= \frac{\sin^2(b_n t) - \sin^2(a_n-\lambda_n)t}{\sin^2 t} \\
&= \frac{\sin(b_n - a_n + \lambda_n)t \sin(b_n + a_n - \lambda_n)t}{\sin^2 t}.
\end{aligned}$$

(ii) Using the inequalities $|\sin\beta| \leq 1$, $|\sin\beta| \leq \beta$, and $\sin\beta \geq \frac{2}{\pi}\beta$ for $0 < \beta \leq \frac{\pi}{2}$, we have:

$$|\mathbb{K}_n^{a,b}(t)| \leq \frac{\pi^2(b_n - a_n + \lambda_n)t}{4t^2} = \mathcal{O}\left(\frac{b_n - a_n + \lambda_n}{t}\right).$$

(iii) Similarly, using the inequalities $|\sin\beta| \leq 1$ and $\sin\beta \geq \frac{2}{\pi}\beta$ for $0 < \beta \leq \frac{\pi}{2}$, we also have:

$$|\mathbb{K}_n^{a,b}(t)| \leq \frac{\pi^2}{4t^2} = \mathcal{O}\left(\frac{1}{t^2}\right).$$

The proof is completed. □

In the sequel we pass to the main result.

3. Main result

3.1. Approximation by deferred generalized de la Vallée Poussin mean of single Fourier series

Here, we prove the following.

Theorem 3.1. *Let $f \in C[0, 2\pi]$ and $\omega(f;t)$ be its modulus of continuity satisfying conditions (1.4) and (1.5) as $t \to +0$, where $F(t) \geq 0$.*
Then

$$\|f - V_n(\lambda, a, b; f)\| = \mathcal{O}\left(\frac{1}{b_n - a_n + \lambda_n} F\left(\frac{\pi}{2(b_n - a_n + \lambda_n)}\right)\right). \tag{3.1}$$

Proof. After some calculation we have:

$$s_m(f;x) = \frac{1}{\pi}\int_0^{\frac{\pi}{2}} [f(x+2t) + f(x-2t)]D_m(t)dt,$$

where $D_m(t) = \frac{\sin(2m+1)t}{\sin t}$.
Denoting by $V_n(\lambda, a, b; f; x)$ the deferred generalized de la Vallée Poussin mean of $s_m(f;x)$, i.e.,

$$V_n(\lambda, a, b; f; x) := \frac{1}{b_n - a_n + \lambda_n}\sum_{m=a_n-\lambda_n}^{b_n-1} s_m(f;x),$$

we get:

$$V_n(\lambda, a, b; f; x) - f(x) = \frac{1}{(b_n - a_n + \lambda_n)\pi}\int_0^{\frac{\pi}{2}} \psi_x(t)\mathbb{K}_n^{a,b}(t)dt,$$

where

$$\psi_x(t) := f(x+2t) + f(x-2t) - f(x).$$

Whence,

$$
\begin{aligned}
\|V_n(\lambda,a,b;f)-f\| &\le \frac{1}{(b_n-a_n+\lambda_n)\pi}\int_0^{\frac{\pi}{2}}|\psi_x(t)||\mathbb{K}_n^{a,b}(t)|dt\\
&\le \frac{4}{(b_n-a_n+\lambda_n)\pi}\left(\int_0^{\frac{\pi}{2(b_n-a_n+\lambda_n)}}+\int_{\frac{\pi}{2(b_n-a_n+\lambda_n)}}^{\frac{\pi}{2}}\right)\omega(f;t)|\mathbb{K}_n^{a,b}(t)|dt\\
&:= \mathbb{P}_1+\mathbb{P}_2. \qquad (3.2)
\end{aligned}
$$

Using Lemma 2.2, part (ii), we obtain:

$$|\mathbb{P}_1| = \mathcal{O}(1)\int_0^{\frac{\pi}{2(b_n-a_n+\lambda_n)}} t^{-1}\omega(f;t)dt,$$

and applying Lemma 2.1, (1.4) and (1.5), we get:

$$
\begin{aligned}
|\mathbb{P}_1| &= \mathcal{O}(1)\int_0^{\frac{\pi}{2(b_n-a_n+\lambda_n)}}\int_t^{\frac{\pi}{2}} u^{-2}\omega(f;u)dudt\\
&= \mathcal{O}(1)\int_0^{\frac{\pi}{2(b_n-a_n+\lambda_n)}} F(t)dt\\
&= \mathcal{O}\left(\frac{1}{b_n-a_n+\lambda_n}F\left(\frac{\pi}{2(b_n-a_n+\lambda_n)}\right)\right). \qquad (3.3)
\end{aligned}
$$

To estimate $\mathbb{P}_2$, we use Lemma 2.2, part (iii). Namely, based on (1.4), we have

$$
\begin{aligned}
|\mathbb{P}_2| &= \mathcal{O}\left(\frac{1}{(b_n-a_n+\lambda_n)\pi}\right)\int_{\frac{\pi}{2(b_n-a_n+\lambda_n)}}^{\frac{\pi}{2}} t^{-2}\omega(f;t)dt\\
&= \mathcal{O}\left(\frac{1}{b_n-a_n+\lambda_n}F\left(\frac{\pi}{2(b_n-a_n+\lambda_n)}\right)\right). \qquad (3.4)
\end{aligned}
$$

Finally, inserting (3.2) and (3.3) into (3.4), we immediately obtain (3.1) as required. The proof is completed. □

Remark 3.2. Since, in general, $\lambda_n \le b_n - a_n + \lambda_n$, then we observe that the degree of approximation obtained in Theorem 3.1 is not worse than that appears in Theorem 1.1.

Remark 3.3. For $b_n = a_n = n$, we immediately obtain the result given in [6].

Further, let the sequences $a := (a_n)$ and $b := (b_n)$ be of non-negative integers with conditions

$$a_n < b_n, \quad n = 1, 2, \ldots, \qquad (3.5)$$

and

$$\lim_{n\to\infty} b_n = +\infty. \qquad (3.6)$$

If $\lambda_n = 1$ for all $n \ge 1$, then the deferred de la Vallée Poussin mean

$$V_n(1, a+2, b+1; f; x)$$

reduces to

$$D_a^b(f;x) := \frac{1}{b_n - a_n} \sum_{m=a_n+1}^{b_n} s_m(f;x),$$

which is the deferred Cesàro mean of the sum $s_n(f;x)$ introduced in [2]. In the same paper, it was shown that (3.5) and (3.6) are conditions of regularity for D_a^b. Consequently, if conditions (3.5) and (3.6) are satisfied, then from Theorem 3.1 we deduce the following.

Corollary 3.4. *Let $f \in C[0,2\pi]$ and $\omega(f;t)$ be its modulus of continuity satisfying conditions (1.4) and (1.5) as $t \to +0$, where $F(t) \geq 0$.*
Then

$$\|f - D_a^b(f)\| = \mathcal{O}\left(\frac{1}{b_n - a_n} F\left(\frac{\pi}{2(b_n - a_n)}\right)\right).$$

Also, if we take $\lambda_n = n$, $a_n = n$, $b_n = n+1$, $\forall n \geq 1$, then the deferred generalized de la Vallée Poussin mean reduces to ordinary Cesàro mean of the sum $s_n(f;x)$,

$$\sigma_n(f;x) := \frac{1}{n+1} \sum_{m=0}^{n} s_m(f;x).$$

Therefore, Theorem 3.1 also implies:

Corollary 3.5. *Let $f \in C[0,2\pi]$ and $\omega(f;t)$ be its modulus of continuity satisfying conditions (1.4) and (1.5) as $t \to +0$, where $F(t) \geq 0$.*
Then

$$\|f - \sigma_n(f)\| = \mathcal{O}\left(\frac{1}{n+1} F\left(\frac{\pi}{2(n+1)}\right)\right).$$

Let us specify the function $F(t)$ as follows:

$$F(t) = \begin{cases} t^{\gamma-1}, & 0 < \gamma < 1; \\ \log\left(\frac{\pi}{t}\right), & \gamma = 1. \end{cases}$$

Using this function the following estimations from Theorem 3.1, Corollary 3.4, and Corollary 3.5 can be deduced (of course all other conditions are maintaining):

(a) From Theorem 3.1:

$$\|f - V_n(\lambda, a, b; f)\| = \begin{cases} \mathcal{O}_\gamma\left(\frac{1}{(b_n - a_n + \lambda_n)^\gamma}\right), & 0 < \gamma < 1; \\ \frac{\log(2(b_n - a_n + \lambda_n))}{b_n - a_n + \lambda_n}, & \gamma = 1. \end{cases}$$

(b) From Corollary 3.4:

$$\|f - D_a^b(f)\| = \begin{cases} \mathcal{O}_\gamma\left(\frac{1}{(b_n - a_n)^\gamma}\right), & 0 < \gamma < 1; \\ \frac{\log(2(b_n - a_n))}{b_n - a_n}, & \gamma = 1. \end{cases}$$

(c) From Corollary 3.5 (this is a particular case of a result given in [4]):

$$\|f - \sigma_n(f)\| = \begin{cases} \mathcal{O}_\gamma\left(\frac{1}{(n+1)^\gamma}\right), & 0 < \gamma < 1; \\ \frac{\log(2(n+1))}{n+1}, & \gamma = 1. \end{cases}$$

3.2. Approximation by deferred generalized de la Vallée Poussin mean of double Fourier series

Let $C([-\pi,\pi]^2)$ be the class of real-valued functions of two variables that are continuous on $[-\pi,\pi]\times[-\pi,\pi] := [-\pi,\pi]^2$ and 2π periodic with respect to x and y. We recall that the double Fourier series of the function $f(x,y)\in C([-\pi,\pi]^2)$ is defined by

$$f(x,y)\sim\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\lambda_{mn}\Big[a_{mn}\cos mx\cos ny+b_{mn}\sin mx\cos ny \\ +c_{mn}\cos mx\sin ny+d_{mn}\sin mx\sin ny\Big],$$

where

$$\lambda_{mn}=\begin{cases}1/4, & \text{if } m=n=0,\\ 1/2, & \text{if } m>0, n=0\vee m=0, n>0,\\ 1, & \text{if } m>0, n>0,\end{cases}$$

and

$$a_{mn}=\frac{1}{\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}f(u,v)\cos mu\cos nv du dv,$$
$$b_{mn}=\frac{1}{\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}f(u,v)\sin mu\cos nv du dv,$$
$$c_{mn}=\frac{1}{\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}f(u,v)\cos mu\sin nv du dv,$$
$$d_{mn}=\frac{1}{\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}f(u,v)\sin mu\sin nv du dv,$$

are the Fourier coefficients of the function $f(x,y)$.

The sequence $\{s_{m,n}(f;x,y)\}$ represents the sequence of partial sums of the double Fourier series which can be rewritten in integral form by

$$s_{m,n}(x,y):=s_{m,n}(f;x,y):=\frac{1}{\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}f(x+u,y+v)D_m(u)D_n(v)du dv.$$

To my best knowledge the double de la Vallée Poussin mean of $s_{m,n}(x,y)$ is defined by (see [3])

$$V_{m,n}^{(p,q)}(f;x,y):=\frac{1}{(p+1)(q+1)}\sum_{k=n}^{n+p}\sum_{\ell=m}^{m+q}s_{k,\ell}(x,y),\quad p\geq 0,\ q\geq 0. \tag{3.7}$$

The mean $V_{m,n}^{(p,q)}(f;x,y)$ is generalized in [11] as follows (for our purposes we modify it "a little bit"). Let $\lambda:=(\lambda_m)$ and $\mu:=(\mu_n)$ be two monotone non-decreasing sequences of integers such that $\lambda_1=\mu_1=1$, $\lambda_{m+1}-\lambda_m\leq 1$, and $\mu_{n+1}-\mu_n\leq 1$.

The mean

$$V_{m,n}^{\lambda,\mu}(f;x,y)=\frac{1}{\lambda_m\mu_n}\sum_{k=m-\lambda_m}^{m-1}\sum_{k=n-\mu_n}^{n-1}s_{k,\ell}(x,y),\quad (m,n\geq 1), \tag{3.8}$$

is called the (mn)-th deferred generalized de la Vallée-Poussin mean of the sequence $(s_{k,\ell}(x,y))$ generated by sequences (λ_m) and (μ_n).

The (total) modulus of continuity of a continuous function $f(x,y)$, 2π-periodic in each variable, in symbols $f\in C([-\pi,\pi]^2)$, is defined by (see [12], page 283)

$$\omega_1(f,\delta_1,\delta_2)=\sup_{x,y}\sup_{|u|\le\delta_1,|v|\le\delta_2}|f(x+u,y+v)-f(x,y)|,\quad \delta_1,\delta_2\ge 0.$$

To estimate the deviation

$$\max_{(x,y)\in Q}\left|V_{m,n}^{\lambda,\mu}(f;x,y)-f(x,y)\right|,$$

which is the main result of this subsection, first we denote

$$\begin{aligned}\phi_{xy}(s,t):=f(x+s,y+t)&+f(x-s,y+t)\\&+f(x+s,y-t)+f(x-s,y-t)-4f(x,y).\end{aligned}$$

Now, we are in able to prove the following.

Theorem 3.6. *Let $f\in C([-\pi,\pi]^2)$, $\omega_1(f,s,t)=\mathcal{O}\left(\omega^{(1)}(s)\omega^{(2)}(t)\right)$, where $\omega^{(1)}(s)$ and $\omega^{(2)}(t)$ are two non-negative functions of modulus type satisfying conditions (1.4) and (1.5) as $s,t\to+0$, and $F_1(s),F_2(t)\ge 0$ two mediate functions. Then*

$$\max_{(x,y)\in Q}\left|V_{m,n}^{\lambda,\mu}(f;x,y)-f(x,y)\right|=\mathcal{O}\left(\frac{1}{\lambda_m\lambda_n}F_1\left(\frac{\pi}{2\lambda_m}\right)F_2\left(\frac{\pi}{2\lambda_n}\right)\right).$$

Proof. After some transforms we get:

$$V_{m,n}^{\lambda,\mu}(f;x,y)-f(x,y)=\frac{1}{\pi^2}\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\phi_{xy}(2s,2t)K_{mn}^{\lambda,\mu}(s,t)dsdt, \tag{3.9}$$

where

$$K_{mn}^{\lambda,\mu}(s,t):=\frac{1}{\lambda_m\mu_n}\sum_{k=m-\lambda_m}^{m-1}\sum_{\ell=n-\mu_n}^{n-1}\frac{\sin(2k+1)s}{\sin s}\frac{\sin(2\ell+1)t}{\sin t}.$$

Without difficulty the quantity $K_{mn}^{\lambda,\mu}(s,t)$ can be written as

$$K_{mn}^{\lambda,\mu}(s,t)=\frac{\sin(\lambda_m s)\sin[(2m-\lambda_m)s]\sin(\mu_n t)\sin[(2n-\mu_n)t]}{\lambda_m\mu_n\sin^2 s\sin^2 t}.$$

Therefore, we have:

$$\begin{aligned}&|V_{m,n}^{\lambda,\mu}(f;x,y)-f(x,y)|\le\left(\frac{4}{\pi}\right)^2\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\omega_1(f,s,t)|K_{mn}^{\lambda,\mu}(s,t)|dsdt\\&=\mathcal{O}\left(\int_0^{\frac{\pi}{2\lambda_m}}\int_0^{\frac{\pi}{2\mu_n}}+\int_{\frac{\pi}{2\lambda_m}}^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2\mu_n}}+\int_0^{\frac{\pi}{2\lambda_m}}\int_{\frac{\pi}{2\mu_n}}^{\frac{\pi}{2}}+\int_{\frac{\pi}{2\lambda_m}}^{\frac{\pi}{2}}\int_{\frac{\pi}{2\mu_n}}^{\frac{\pi}{2}}\right)\\&:=\mathcal{O}\left(\mathbb{S}_1+\mathbb{S}_2+\mathbb{S}_3+\mathbb{S}_4\right).\end{aligned}\tag{3.10}$$

Using Jordan's inequality $\sin\nu \geq \frac{2}{\pi}\nu$ for $0 < \nu \leq \frac{\pi}{2}$, given assumptions, and Lemma 2.1, we obtain:

$$\begin{aligned}\mathbb{S}_1 &= \mathcal{O}(1)\int_0^{\frac{\pi}{2\lambda_m}}\int_0^{\frac{\pi}{2\mu_n}} s^{-1}t^{-1}\omega_1(f,s,t)dsdt \\ &= \mathcal{O}\left(\frac{1}{\lambda_m\mu_n}F_1\left(\frac{\pi}{2\lambda_m}\right)F_2\left(\frac{\pi}{2\mu_n}\right)\right).\end{aligned} \tag{3.11}$$

Using the same arguments and Lemma 2.2, we also obtain:

$$\begin{aligned}\mathbb{S}_2 &= \mathcal{O}(1)\int_{\frac{\pi}{2\lambda_m}}^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2\mu_n}} s^{-2}t^{-1}\omega_1(f,s,t)dsdt \\ &= \mathcal{O}\left(\frac{1}{\lambda_m\mu_n}F_1\left(\frac{\pi}{2\lambda_m}\right)F_2\left(\frac{\pi}{2\mu_n}\right)\right).\end{aligned} \tag{3.12}$$

With very similar reasoning, we get:

$$\begin{aligned}\mathbb{S}_3 &= \mathcal{O}(1)\int_0^{\frac{\pi}{2\lambda_m}}\int_{\frac{\pi}{2\mu_n}}^{\frac{\pi}{2}} s^{-1}t^{-2}\omega_1(f,s,t)dsdt \\ &= \mathcal{O}\left(\frac{1}{\lambda_m\mu_n}F_1\left(\frac{\pi}{2\lambda_m}\right)F_2\left(\frac{\pi}{2\mu_n}\right)\right).\end{aligned} \tag{3.13}$$

Finally, based on given assumptions, and Lemma 2.2 twice, we have:

$$\begin{aligned}\mathbb{S}_4 &= \mathcal{O}(1)\int_{\frac{\pi}{2\lambda_m}}^{\frac{\pi}{2}}\int_{\frac{\pi}{2\mu_n}}^{\frac{\pi}{2}} s^{-2}t^{-2}\omega_1(f,s,t)dsdt \\ &= \mathcal{O}\left(\frac{1}{\lambda_m\mu_n}F_1\left(\frac{\pi}{2\lambda_m}\right)F_2\left(\frac{\pi}{2\mu_n}\right)\right).\end{aligned} \tag{3.14}$$

Subsequently, inserting (3.11), (3.12),(3.13), and (3.14) into (3.9), the requested estimation follows.
The proof is completed. □

Specifying functions $F_i(z)$, $(i = 1, 2)$, by:

$$F_i(z) = \begin{cases} z^{\gamma_i - 1}, & 0 < \gamma_i < 1; \\ \log\left(\frac{\pi}{z}\right), & \gamma_i = 1 \end{cases}$$

then Theorem 3.6 implies:

Corollary 3.7. *Let $f \in C([-\pi,\pi]^2)$, $\omega_1(f,s,t) = \mathcal{O}\left(\omega^{(1)}(s)\omega^{(2)}(t)\right)$, where $\omega^{(1)}(s)$ and $\omega^{(2)}(t)$ are two non-negative functions of modulus type satisfying conditions (1.4) and (1.5) as $s,t \to +0$. Then*

$$\max_{(x,y)\in Q}\left|V_{m,n}^{\lambda,\mu}(f;x,y) - f(x,y)\right| = \begin{cases} \mathcal{O}\left(\frac{1}{\lambda_m^{\gamma_1}\mu_n^{\gamma_2}}\right), & 0 < \gamma_1, \gamma_2 < 1; \\ \mathcal{O}\left(\frac{\log(2\mu_n)}{\lambda_m^{\gamma_1}\mu_n}\right), & 0 < \gamma_1 < 1, \gamma_2 = 1; \\ \mathcal{O}\left(\frac{\log(2\lambda_m)}{\lambda_m\mu_n^{\gamma_2}}\right), & \gamma_1 = 1, 0 < \gamma_2 < 1; \\ \mathcal{O}\left(\frac{\log(2\lambda_m)\log(2\mu_n)}{\lambda_m\mu_n}\right), & \gamma_1 = \gamma_2 = 1 \end{cases}$$

In particular case, it is clear that $V^{m,n}_{m+1,n+1}(f;x,y) \equiv \sigma_{m,n}(f;x,y)$, which is the double Fejèr mean of the sequence $(s_{k,\ell}(x,y))$. Thus, Theorem 3.6 also implies:

Corollary 3.8. *Let $f \in C([-\pi,\pi]^2)$, $\omega_1(f,s,t) = \mathcal{O}\left(\omega^{(1)}(s)\omega^{(2)}(t)\right)$, where $\omega^{(1)}(s)$ and $\omega^{(2)}(t)$ are two non-negative functions of modulus type satisfying conditions (1.4) and (1.5) as $s,t \to +0$. Then*

$$\max_{(x,y)\in Q} |\sigma_{m,n}(f;x,y) - f(x,y)| = \begin{cases} \mathcal{O}\left(\frac{1}{(m+1)^{\gamma_1}(n+1)^{\gamma_2}}\right), & 0<\gamma_1,\gamma_2<1; \\ \mathcal{O}\left(\frac{\log(2(n+1))}{(m+1)^{\gamma_1}(n+1)}\right), & 0<\gamma_1<1, \gamma_2=1; \\ \mathcal{O}\left(\frac{\log(2(m+1))}{(m+1)(n+1)^{\gamma_2}}\right), & \gamma_1=1, 0<\gamma_2<1; \\ \mathcal{O}\left(\frac{\log(2(m+1))\log(2(n+1))}{(m+1)(n+1)}\right), & \gamma_1=\gamma_2=1 \end{cases}$$

Let $a := (a_n)$, $b := (b_n)$, $c := (c_n)$, and $d := (d_n)$ be sequences of non-negative integers with conditions

$$1 \le b_m - a_m + \lambda_m, \quad 1 \le d_n - c_n + \mu_n, \quad (m,n = 1,2,\dots). \tag{3.15}$$

The mean $V^{\lambda,\mu}_{m,n}(f;x,y)$ can be generalized further by

$$V^{\lambda,\mu}_{m,n}(a,b,c,d;f;x,y) = \frac{1}{\lambda_m \mu_n} \sum_{k=a_m-\lambda_m}^{b_m-1} \sum_{k=c_n-\mu_n}^{d_n-1} s_{k,\ell}(x,y), \quad (m,n \ge 1), \tag{3.16}$$

is called the (mn)-th double deferred generalized de la Vallée Poussin mean of the sequence $(s_{k,\ell}(x,y))$ generated by sequences (λ_m) and (μ_n).

Remark 3.9. Note that for $a_m = b_m = m$ and $c_n = d_n = n$, for all $m,n \ge 1$, we obtain

$$V^{\lambda,\mu}_{m,n}(a,b,c,d;f;x,y) \equiv V^{\lambda,\mu}_{m,n}(f;x,y),$$

and

$$V^{m,n}_{m+1,n+1}(a,b,c,d;f;x,y) \equiv \sigma_{m,n}(f;x,y).$$

The mean $V^{\lambda,\mu}_{m,n}(a,b,c,d;f;x,y)$ given by (3.16) can be used to prove the following general theorem.

Theorem 3.10. *Let $f \in C([-\pi,\pi]^2)$, $\omega_1(f,s,t) = \mathcal{O}\left(\omega^{(1)}(s)\omega^{(2)}(t)\right)$, where $\omega^{(1)}(s)$ and $\omega^{(2)}(t)$ are two non-negative functions of modulus type satisfying conditions (1.4) and (1.5) as $s,t \to +0$, and $F_1(s), F_2(t) \ge 0$ two mediate functions. Then*

$$\begin{aligned} &\max_{(x,y)\in Q} \left|V^{\lambda,\mu}_{m,n}(a,b,c,d;f;x,y) - f(x,y)\right| \\ &= \mathcal{O}\Bigg(\frac{1}{(b_m-a_m+\lambda_m)(d_n-c_n+\mu_n)} \\ &\quad \times F_1\left(\frac{\pi}{2(b_m-a_m+\lambda_m)}\right) F_2\left(\frac{\pi}{2(d_n-c_n+\mu_n)}\right)\Bigg). \end{aligned}$$

Proof. Because of the similarity with the proof of Theorem 3.6 we omit the proof of this theorem. □

Remark 3.11. One should note that Theorem 3.6 is a particular case of Theorem 3.10 (when $a_m = b_m$ and $c_n = d_n$; $\forall m, n \geq 1$). Moreover, it covers Corollary 3.7 and Corollary 3.8 as well (when $a_m = b_m$, $c_n = d_n$, $\lambda_m = m$, and $\mu_n = n$; $\forall m, n \geq 1$).

Further, let $a := (a_m)$, $b := (b_m)$, $c := (c_n)$, and $d := (d_n)$ be sequences of non-negative integers with conditions

$$a_m < b_m, \quad c_n < d_n, \quad (m, n = 1, 2, \dots), \tag{3.17}$$

and

$$\lim_{m\to\infty} b_m = +\infty, \quad \lim_{n\to\infty} d_n = +\infty. \tag{3.18}$$

If $\lambda_m = 1$ and $\mu_n = 1$ for all $m, n \geq 1$, then the double deferred de la Vallée Poussin mean $V_{m,n}^{\lambda,\mu}(a+2, b+1, c+2, d+1; f; x, y)$ reduces to

$$D_{a,c}^{b,d}(f; x, y) := \frac{1}{(b_m - a_m)(d_n - c_n)} \sum_{k=a_m+1}^{b_m} \sum_{\ell=c_n+1}^{d_n} s_{k,\ell}(f; x, y),$$

which is the double deferred Cesàro mean of the sum $s_{k,\ell}(f; x, y)$ introduced implicitly in [13]. It was shown there, that (3.17) and (3.18) are conditions of regularity for $D_{a,c}^{b,d}$. Therefore, if conditions (3.17) and (3.18)) are satisfied, then Theorem 3.10 implies the following.

Corollary 3.12. *Let* $f \in C([-\pi, \pi]^2)$, $\omega_1(f, s, t) = \mathcal{O}\left(\omega^{(1)}(s)\omega^{(2)}(t)\right)$, *where* $\omega^{(1)}(s)$ *and* $\omega^{(2)}(t)$ *are two non-negative functions of modulus type satisfying conditions (*1.4*) and (*1.5*) as* $s, t \to +0$, *and* $F_1(s), F_2(t) \geq 0$ *two mediate functions. Then*

$$\max_{(x,y)\in Q} \left|D_{a,c}^{b,d}(f; x, y) - f(x, y)\right|$$
$$= \mathcal{O}\left(\frac{1}{(b_m - a_m)(d_n - c_n)} F_1\left(\frac{\pi}{2(b_m - a_m)}\right) F_2\left(\frac{\pi}{2(d_n - c_n)}\right)\right).$$

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Positive definite kernels on the set of integers, stability, some properties and applications

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Abstract. We define and investigate a class of positive definite kernel so called equivalent-kernel. We formulate and prove an analogous of Paley-Wiener theorem in the context of positive definite kernel. The main ingredient in the proof is Kolmogorov decomposition. Finally, some applications to stochastic processes are given.

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Introduction

Positive definite kernels play a prominent role in some applications such as numerical solution of partial differential equations, machine learning, computer graphics, problem moment and probability theory. In the present work we explore some properties of positive definite kernels. For this kernels one obtains some similar results to equivalents bases in Banach spaces and Riesz bases in Hilbert spaces. An important tool to be used is a version of a classic result due to Kolmogorov, which will be called a Kolmogorov decomposition of the positive definite kernel K (see [3]). We will use Kolmogorov decomposition of a positive definite kernel to obtain a characterization results of equivalents kernels (see Theorem 3.3). This result is similar to a known result for equivalents bases, Riesz bases and stochastic processes. Using the above, one obtains an analogue Paley-Wiener Theorem (see [8]) in the context of positive definite kernels (see Theorem 3.4). Finally, some applications to stochastic processes are given.

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1. Paley-Wiener theorem

Orthonormal bases are very important in Hilbert space theory. There is another less known but also very useful type of bases: the Riesz bases. This section will be devoted to them. More about these bases can be found in Young's book [8].

Definition 1.1. A basis in a Hilbert space is a *Riesz basis* if it is equivalent to an orthonormal basis.

The fundamental criterium of stability, and historically the first one, is due to Paley and Wiener [7]. It is based on the known fact that a linear bounded operator T on a Banach space is invertible if

$$\|I - T\| < 1.$$

Theorem 1.2. *(Paley -Wiener) Let $\{x_n\}_{n\in\mathbb{N}}$ be a basis in the Banach space X, and suppose that $\{y_n\}_{n\in\mathbb{N}}$ is a sequence of elements of X such that*

$$\left\|\sum_{n=1}^{N} c_n(x_n - y_n)\right\| \leq \lambda \left\|\sum_{n=1}^{N} c_n x_n\right\|,$$

for all $N \in \mathbb{N}$, some constant λ, with $0 \leq \lambda < 1$ and for any sequence of scalars $\{c_n\}_{n\in\mathbb{N}}$. Then $\{y_n\}_{n\in\mathbb{N}}$ is a basis for X equivalent to $\{x_n\}_{n\in\mathbb{N}}$.

See [8, Theorem 10] for a proof.

2. Kolmogorov decomposition theorem

2.1. The Hilbert space associated to a positive definite operator valued kernel

Let $\{\mathcal{H}_n\}_{n\in\mathbb{Z}}$ be a family of Hilbert spaces. An *operator valued kernel* on $\mathbb{Z}$ to $\{\mathcal{H}_n\}_{n\in\mathbb{Z}}$ is an application $K : \mathbb{Z} \times \mathbb{Z} \to \bigcup_{m,n\in\mathbb{Z}} \mathcal{L}(\mathcal{H}_m, \mathcal{H}_n)$ such that

$$K(n, m) \in \mathcal{L}(\mathcal{H}_m, \mathcal{H}_n) \quad \text{for } n, m \in \mathbb{Z}.$$

In this section and the following one, unless it is otherwise stated, all the kernels will be operator valued ones.

A sequence $\{h_n\}$ in $\oplus_{n\in\mathbb{Z}}\mathcal{H}_n$ is said to have *finite support* if $h_n = 0$ except for a finite number of integers n.

A kernel K on $\mathbb{Z}$ to $\{\mathcal{H}_n\}_{n\in\mathbb{Z}}$ is a *positive definite kernel* if

$$\sum_{n,m\in\mathbb{Z}} \langle K(n, m)h_m, h_n\rangle_{\mathcal{H}_n} \geq 0,$$

for every sequence $\{h_n\}$ in $\oplus_{n\in\mathbb{Z}}\mathcal{H}_n$ with finite support.

Let K be a positive definite kernel. Let $\mathcal{F}$ be the linear space of elements $\bigoplus_{n\in\mathbb{Z}} \mathcal{H}_n$ and $\mathcal{F}_o$ be the space of elements in $\mathcal{F}$ with finite support.

Define $B_K : \mathcal{F}_o \times \mathcal{F}_o \to \mathbb{C}$ with

$$B_K(f, g) = \sum_{m,n\in\mathbb{Z}} \langle K(n, m)f_m, g_n\rangle_{\mathcal{H}_n}, \tag{2.1}$$

for $f, g \in \mathcal{F}_o$, $f = \{f_n\}$, $g = \{g_n\}$, $f_n, g_n \in \mathcal{H}_n$.

Note that B_K satisfies all the properties of an inner product, except for the fact that the set

$$\mathcal{N}_K = \{h \in \mathcal{F}_o : B_K(h,h) = 0\},$$

could be non-trivial.

According to the Cauchy-Schwarz inequality

$$\mathcal{N}_K = \{h \in \mathcal{F}_o : B_K(h,g) = 0, \text{ for all } g \in \mathcal{F}_o\},$$

hence $\mathcal{N}_K$ is a linear subspace of $\mathcal{F}_o$.

The quotient space $\mathcal{F}_o/\mathcal{N}_K$ is also a linear subspace. If $[h]$ stands for the class of the element h in $\mathcal{F}_o/\mathcal{N}_K$, then the application

$$\langle [h], [g] \rangle = B_K(h,g), \quad h, g \in \mathcal{F}_o,$$

is well defined. To prove that $\langle \cdot, \cdot \rangle$ is an inner product on $\mathcal{F}_o/\mathcal{N}_K$ is straightforward.

The completion of $\mathcal{F}_o/\mathcal{N}_K$ with respect to the norm induced by this inner product is a Hilbert space. It is known as *the Hilbert space associated to the positive definite kernel K* and it is denoted by $\mathcal{H}_K$. The inner product and the norm of $\mathcal{H}_K$ will be represented as $\langle \cdot, \cdot \rangle_{\mathcal{H}_K}$ and $\| \cdot \|_{\mathcal{H}_K}$ respectively. This norm will be named as *the norm induced by K*.

2.2. Kolmogorov Decomposition Theorem

The following theorem is a version of the classic result of Kolmogorov (see [5] for a historical review).

Theorem 2.1 (Kolmogorov). *Let K be a positive definite kernel. Then there exists a Hilbert space $\mathcal{H}_K$ and a map V defined on $\mathbb{Z}$ such that $V(n)$ belongs to $\mathcal{L}(\mathcal{H}_n, \mathcal{H}_K)$ for each $n \in \mathbb{Z}$ and*

(a) $K(n,m) = V^*(n)V(m)$ *if* $n, m \in \mathbb{Z}$.

(b) $\mathcal{H}_K = \bigvee_{n \in \mathbb{Z}} V(n)\mathcal{H}_n$.

(c) *The decomposition is unique in the following sense: if $\mathcal{H}'$ is another Hilbert space and V' defined on $\mathbb{Z}$ is an application such that $V'(n) \in \mathcal{L}(\mathcal{H}_n, \mathcal{H}_K)$ for each $n \in \mathbb{Z}$ that satisfies* (a) *and* (b)*, then there exists a unitary operator $\Phi : \mathcal{H}_K \to \mathcal{H}'$ such that $\Phi V(n) = V'(n)$ for all $n \in \mathbb{Z}$.*

A proof of this theorem can be found in [3, Theorem 3.1].

An application V that satisfies the property (a) in Theorem 2.1 will be called *The Kolmogorov Decomposition of the Kernel K* or simply, a *Decomposition of the kernel K* (see [3]). The property (b) is referred to as the *minimality property* of Kolmogorov Decomposition. The meaning of property (c) is that, under the minimality condition (b), the Kolmogorov decomposition is essentially unique.

3. Some results for positive definite kernels

3.1. Equivalent definite positive kernels

Suppose the family of Hilbert spaces $\{\mathcal{H}_n\}_{n \in \mathbb{Z}}$ reduces to a single space, i.e. $\mathcal{H}_n = \mathcal{H}$ for all $n \in \mathbb{Z}$.

In this section some results given in [1] are extended to the case of kernel to operator valued.

Definition 3.1. Let $K_1, K_2 : \mathbb{Z}\times\mathbb{Z} \to \mathcal{L}(\mathcal{H})$ be two positive definite kernels.

It is said that K_1 and K_2 are *equivalent* if there exist two constants A, B with $0 < A \leq B$ such that

$$A\|[h]_{K_1}\|^2_{\mathcal{H}_{K_1}} \leq \|[h]_{K_2}\|^2_{\mathcal{H}_{K_2}} \leq B\|[h]_{K_1}\|^2_{\mathcal{H}_{K_1}},$$

for $h \in \mathcal{F}_o$.

Remark 3.2. Let $K : \mathbb{Z}\times\mathbb{Z} \to \mathcal{L}(\mathcal{H})$ be a positive definite kernel. Let $h \in \mathcal{F}_o$ and $\{h_n\}_{n\in\mathbb{Z}}$ a sequence in $\mathcal{H}$ with finite support.

By virtue of the definition of norm induced by the kernel K and Kolmogorov decomposition theorem it is obtained

$$\begin{aligned}\|[h]\|^2_{\mathcal{H}_K} &= \langle [h],[h]\rangle_{\mathcal{H}_K} = \sum_{n,m\in\mathbb{Z}} \langle K(n,m)h_m, h_n\rangle_{\mathcal{H}} \\ &= \sum_{m,n\in\mathbb{Z}} \langle V_K(n)^* V_K(m)h_m, h_n\rangle_{\mathcal{H}} = \left\| \sum_{n\in\mathbb{Z}} V_K(n)h_n \right\|^2_{\mathcal{H}}.\end{aligned}$$

The following is one of our results.

Theorem 3.3. *Let $K_1, K_2 : \mathbb{Z}\times\mathbb{Z} \to \mathcal{L}(\mathcal{H})$ be two positive definite kernels. Then the following conditions are equivalent:*

(i) *The kernels K_1 y K_2 are equivalents.*

(ii) *There exists a linear bounded bijective application, with bounded inverse*

$$\Phi : \mathcal{H}_{K_1} \to \mathcal{H}_{K_2},$$

such that

$$\Phi V_{K_1}(n) = V_{K_2}(n) \quad \text{for all } n \in \mathbb{Z}.$$

(iii) *There exist two constants A, B with $0 < A \leq B$ such that*

$$\begin{aligned} A \sum_{n,m\in\mathbb{Z}} \langle K_1(n,m)h_m, h_n\rangle_{\mathcal{H}} &\leq \sum_{n,m\in\mathbb{Z}} \langle K_2(n,m)h_m, h_n\rangle_{\mathcal{H}} \\ &\leq B \sum_{n,m\in\mathbb{Z}} \langle K_1(n,m)h_m, h_n\rangle_{\mathcal{H}}, \end{aligned}$$

for all sequence with finite support $\{h_n\}_{n\in\mathbb{Z}} \subset \mathcal{H}$.

Proof. Let V_{K_1} and V_{K_2} be the Kolmogorov decomposition of the kernels K_1, K_2 and Let $\mathcal{H}_{K_1}$ and $\mathcal{H}_{K_2}$ the associated Hilbert spaces.
Remark 3.2 allows us to write condition (iii) in the following way: there exist two constants A and B with $0 < A \leq B$ such that

$$A\|[h]_{K_1}\|^2_{\mathcal{H}_{K_1}} \leq \|[h]_{K_2}\|^2_{\mathcal{H}_{K_2}} \leq B\|[h]_{K_1}\|^2_{\mathcal{H}_{K_1}},$$

for $h \in \mathcal{F}_o$.
Consequently the conditions (i) and (iii) are equivalents.

Next, suppose that condition (ii) is true. Since Φ is a linear bounded and invertible operator, then there exist two constants a_o, b_o with $0 < a_o \leq b_o$ such that

$$a_o\|f\|_{\mathcal{H}_{K_1}} \leq \|\Phi(f)\|_{\mathcal{H}_{K_2}} \leq b_o\|f\|_{\mathcal{H}_{K_1}},$$

for all $f \in \mathcal{H}_{K_1}$.
Let $f \in \mathcal{H}_{K_1}$ given by

$$f = \sum_{n\in\mathbb{Z}} V_{K_1}(n)h_n,$$

where $\{h_n\}_{n\in\mathbb{Z}}$ is a sequence in $\mathcal{H}$ with finite support.
Then

$$a_o^2\left\|\sum_{n\in\mathbb{Z}} V_{K_1}(n)h_n\right\|^2_{\mathcal{H}_{K_1}} \leq \left\|\sum_{n\in\mathbb{Z}} V_{K_2}(n)h_n\right\|^2_{\mathcal{H}_{K_2}} \leq b_o^2\left\|\sum_{n\in\mathbb{Z}} V_{K_1}(n)h_n\right\|^2_{\mathcal{H}_{K_1}}.$$

On the other hand, since K_1 and K_2 are positive definite kernels, by the Kolmogorov decomposition theorem we have

$$K_1(n,m) = V^*_{K_1}(n)V_{K_1}(m), \quad m,n \in \mathbb{Z}$$

and

$$K_2(n,m) = V^*_{K_2}(n)V_{K_2}(m), \quad m,n \in \mathbb{Z}.$$

Taking in to account the above expression we have that

$$\begin{aligned}\left\|\sum_{n\in\mathbb{Z}} V_{K_1}(n)h_n\right\|^2_{\mathcal{H}_{K_1}} &= \left\langle \sum_{m\in\mathbb{Z}} V_{K_1}(m)h_m, \sum_{n\in\mathbb{Z}} V_{K_1}(n)h_n \right\rangle_{\mathcal{H}_{K_1}} \\ &= \sum_{m,n\in\mathbb{Z}} \langle V_{K_1}(n)^* V_{K_1}(m)h_m, h_n\rangle_{\mathcal{H}} \\ &= \sum_{m,n\in\mathbb{Z}} \langle K_1(n,m)h_m, h_n\rangle_{\mathcal{H}},\end{aligned}$$

similarly,

$$\left\|\sum_{n\in\mathbb{Z}} V_{K_2}(n)h_n\right\|^2_{\mathcal{H}_{K_2}} = \sum_{m,n\in\mathbb{Z}} \langle K_2(n,m)h_m, h_n\rangle_{\mathcal{H}}.$$

Thus, choosing $A = a_o^2$ and $B = b_o^2$ we have

$$\begin{aligned}A \sum_{m,n\in\mathbb{Z}} \langle K_1(n,m)h_m, h_n\rangle_{\mathcal{H}} &\leq \sum_{m,n\in\mathbb{Z}} \langle K_2(n,m)h_m, h_n\rangle_{\mathcal{H}} \\ &\leq B \sum_{m,n\in\mathbb{Z}} \langle K_1(n,m)h_m, h_n\rangle_{\mathcal{H}},\end{aligned}$$

where $\{h_n\}_{n\in\mathbb{Z}}$ is a sequence in $\mathcal{H}$ with finite support.
Now, let us suppose that condition (iii) is valid.
The application $\Phi_o : \mathcal{F}_{o,K_1} \to \mathcal{F}_{o,K_2}$ is defined as follows

$$\Phi_o\left(\sum_{n\in\mathbb{Z}} V_{K_1}(n)h_n\right) = \sum_{n\in\mathbb{Z}} V_{K_2}(n)h_n,$$

where $\{h_n\}_{n\in\mathbb{Z}}$ is a sequence in $\mathcal{H}$ with finite support. It is not hard to prove that Φ_o is a linear operator.
In what follows we will proof that Φ_o is a bounded above and bounded below operator. By the Kolmogorov decomposition theorem we obtain

$$\sum_{m,n\in\mathbb{Z}} \langle K_2(n,m)h_m, h_n\rangle_{\mathcal{H}} = \sum_{m,n\in\mathbb{Z}} \langle V_{K_2}(n)^* V_{K_2}(m)h_m, h_n\rangle_{\mathcal{H}}.$$

Taking into account the above result and the way that the operator Φ_o was defined we arrive to the next result

$$\begin{aligned}\sum_{m,n\in\mathbb{Z}} \langle K_2(n,m)h_m, h_n\rangle_{\mathcal{H}} &= \left\langle \sum_{m\in\mathbb{Z}} V_{K_2}(m)h_m, \sum_{n\in\mathbb{Z}} V_{K_2}(n)h_n \right\rangle_{\mathcal{H}_{K_2}} \\ &= \left\| \sum_{n\in\mathbb{Z}} V_{K_2}(n)h_n \right\|^2_{\mathcal{H}_{K_2}} = \left\| \Phi_o\left(\sum_{n\in\mathbb{Z}} V_{K_1}(n)h_n \right) \right\|^2_{\mathcal{H}_{K_2}}.\end{aligned}$$

In a similar way we have

$$\sum_{m,n\in\mathbb{Z}} \langle K_1(n,m)h_m, h_n\rangle_{\mathcal{H}} = \left\| \sum_{n\in\mathbb{Z}} V_{K_1}h_n \right\|^2_{\mathcal{H}_{K_1}}.$$

By (iii),

$$A\left\| \sum_{n\in\mathbb{Z}} V_{K_1}(n)h_n \right\|^2_{\mathcal{H}_{K_1}} \le \left\| \Phi_o\left(\sum_{n\in\mathbb{Z}} V_{K_1}(n)h_n \right) \right\|^2_{\mathcal{H}_{K_2}} \le B\left\| \sum_{n\in\mathbb{Z}} V_{K_1}(n)h_n \right\|^2_{\mathcal{H}_{K_1}}.$$

The last chain of inequalities shows us that Φ_o is a bounded above and bounded below operator. Even more the domain and the range of Φ_o are dense in the spaces $\mathcal{H}_{K_1}$ and $\mathcal{H}_{K_2}$ respectively. Then this operator can be extended to a bounded operator with bounded inverse say $\Phi : \mathcal{H}_{K_1} \to \mathcal{H}_{K_2}$. By construction

$$\Phi V_{K_1}(n) = V_{K_2}(n) \quad \text{for all} \quad n \in \mathbb{Z}.$$ □

Theorem 3.3 has similarities with results referring to equivalent basic sequences in Banach spaces, for more details on the topic (see [6, 2]).

Our next stability result for positive definite kernels is similar to a stability theorem for equivalent bases due to Paley-Wiener (see [8, Theorem 10]).

In first place we will fix the notation. Given two positive definite kernels $K : \mathbb{Z}\times\mathbb{Z} \to \mathcal{L}(\mathcal{H})$ and $K_1 : \mathbb{Z}\times\mathbb{Z} \to \mathcal{L}(\mathcal{H})$, let V_K and V_{K_1} the Kolmogorov decompositions of K and K_1 respectively and let $\mathcal{H}_K$ and $\mathcal{H}_{K_1}$ the induced Hilbert spaces.

Theorem 3.4. *Let $K : \mathbb{Z}\times\mathbb{Z} \to \mathcal{L}(\mathcal{H})$ and $K_1 : \mathbb{Z}\times\mathbb{Z} \to \mathcal{L}(\mathcal{H})$ be two positive definite kernels. If $V_{K_1}(n) \in \mathcal{L}(\mathcal{H}, \mathcal{H}_K)$ for all $n \in \mathbb{Z}$ and satisfies*

$$\left\| \sum_{n\in\mathbb{Z}} (V_K(n) - V_{K_1}(n))h_n \right\|_{\mathcal{H}_K} \le \lambda \left\| \sum_{n\in\mathbb{Z}} V_K(n)h_n \right\|_{\mathcal{H}_K},$$

for any sequence with finite support $\{h_n\}_{n\in\mathbb{Z}} \subset \mathcal{H}$, where $\lambda \in (0,1)$, then K_1 is equivalent to K.

Proof. Let us define the operator $T : \mathcal{H}_K \to \mathcal{H}_K$ as follows

$$T\left(\sum_{n\in\mathbb{Z}} V_K(n)h_n\right) = \sum_{n\in\mathbb{Z}} (V_K(n) - V_{K_1}(n))h_n,$$

where $\{h_n\}_{n\in\mathbb{Z}}$ is a sequence in $\mathcal{H}$ with finite support.
By hypothesis T is well defined and it is a linear operator. From the definition of T and by hypothesis we have.

$$\left\| T\left(\sum_{n\in\mathbb{Z}} V_K(n)h_n\right)\right\|^2_{\mathcal{H}_K} \leq \lambda^2 \left\|\sum_{n\in\mathbb{Z}} V_K(n)h_n\right\|^2_{\mathcal{H}_K}.$$

Hence, T is a bounded operator and moreover

$$\|T\| \leq |\lambda| < 1.$$

Next, let us consider the operator $I - T : \mathcal{H}_K \to \mathcal{H}_K$, as usual $I : \mathcal{H}_K \to \mathcal{H}_K$ is the identity operator.
Since $\|T\| < 1$, by a well known functional analysis Theorem, $I - T$ is an invertible bounded linear operator. Moreover,

$$\begin{aligned}(I-T)\left(\sum_{n\in\mathbb{Z}} V_K(n)h_n\right) &= \sum_{n\in\mathbb{Z}} V_K(n)h_n - T\left(\sum_{n\in\mathbb{Z}} V_K(n)h_n\right)\\ &= \sum_{n\in\mathbb{Z}} V_K(n)h_n - \left(\sum_{n\in\mathbb{Z}} (V_K(n) - V_{K_1}(n))h_n\right)\\ &= \sum_{n\in\mathbb{Z}} V_{K_1}(n)h_n.\end{aligned}$$

From the above, it follows that there are positive constants m and M with $m \leq M$ such that

$$\begin{aligned}m\left\|\sum_{n\in\mathbb{Z}} V_K(n)h_n\right\|_{\mathcal{H}_K} &\leq \left\|(I-T)\left(\sum_{n\in\mathbb{Z}} V_K(n)h_n\right)\right\|_{\mathcal{H}_K}\\ &= \left\|\sum_{n\in\mathbb{Z}} V_{K_1}(n)h_n\right\|_{\mathcal{H}_K}\\ &\leq M\left\|\sum_{n\in\mathbb{Z}} V_K(n)h_n\right\|_{\mathcal{H}_K}.\end{aligned}$$

By Remark 3.2

$$\left\|\sum_{n\in\mathbb{Z}} V_K(n)h_n\right\|^2_{\mathcal{H}_K} = \sum_{m,n\in\mathbb{Z}} \langle K(n,m)h_m, h_n\rangle_{\mathcal{H}}.$$

By hypothesis $V_{K_1}(n) \in \mathcal{L}(\mathcal{H}, \mathcal{H}_K)$ for all $n \in \mathbb{Z}$, thus $V_{K_1}(n)h_n \in \mathcal{H}_K$. Then

$$\begin{aligned}\sum_{n,m\in\mathbb{Z}} \langle K_1(n,m)h_m, h_n\rangle_{\mathcal{H}} &= \sum_{m,n\in\mathbb{Z}} \langle V_{K_1}(n)^* V_{K_1}(m)h_m, h_n\rangle_{\mathcal{H}} \\ &= \sum_{m,n\in\mathbb{Z}} \langle V_{K_1}(m)h_m, V_{K_1}(n)h_n\rangle_{\mathcal{H}_K} \\ &= \left\| \sum_{n\in\mathbb{Z}} V_{K_1}(n)h_n \right\|^2_{\mathcal{H}_K}.\end{aligned}$$

Replacing these expressions in the above inequalities, we derive the existence of positive constants A and B with $A \leq B$ such that

$$\begin{aligned}A \sum_{m,n\in\mathbb{Z}} \langle K(n,m)h_m, h_n\rangle_{\mathcal{H}} &\leq \sum_{m,n\in\mathbb{Z}} \langle K_1(n,m)h_m, h_n\rangle_{\mathcal{H}} \\ &\leq B \sum_{m,n\in\mathbb{Z}} \langle K(n,m)h_m, h_n\rangle_{\mathcal{H}},\end{aligned}$$

for all sequences $\{h_n\}_{n\in\mathbb{Z}}$ in $\mathcal{H}$ with finite support.

Applying Theorem 3.3, it follows that K_1 is equivalent to K. □

4. Applications to stochastic processes

4.1. Multivariate stochastic processes

In this section it will be used the decomposition of the covariance Kernels of the stochastic processes (see [3], Section 1, Chapter 6).

Let (Ω, F, P) be a probability space, where F is a σ-algebra of subsets of Ω and P is a probability measure on F. A *stochastic variable* is a function $x : \Omega \to \mathbb{C}$, which is measurable with respect to the σ-algebra F. *A stochastic process* is a family $\{x_n\}_{n\in\mathbb{Z}}$ of stochastic variables. Let $L^2(P)$ be the Hilbert space of the measurable functions from F to Ω with integrable square, this is,

$$L^2(P) = \left\{ x : \Omega \to \mathbb{C} : x \text{ is a measurable function and } \int_\Omega |x(\omega)|^2 dP(\omega) < +\infty \right\}$$

equipped with the inner product

$$\langle x, y\rangle_{L^2(P)} = \int_\Omega x(\omega)\overline{y(\omega)} dP(\omega).$$

From here on, only stochastic processes with variables in $L^2(P)$ will be considered.

The mean-value variable is defined by

$$m_n = E(x_n) = \int_\Omega x_n(\omega) dP(\omega)$$

and it is convenient to assume that $m_n = 0$ for all $n \in \mathbb{Z}$. The correlation of the stochastic process $\{x_n\}_{n\in\mathbb{Z}}$ is given by

$$K(m,n) = K_{mn} = \int_\Omega x_n(\omega)\overline{x_m(\omega)} dP(\omega) = \langle x_n, x_m\rangle_{L^2(P)}.$$

for all $m, n \in \mathbb{Z}$.

It is straightforward that the correlation kernel of this process is a positive definite kernel. In fact

$$\begin{aligned}\sum_{i,j=m}^{n} K_{ij}\lambda_j\overline{\lambda}_i &= \sum_{i,j=m}^{n} \langle x_j, x_i\rangle_{L^2(P)}\lambda_j\overline{\lambda}_i \\ &= \sum_{i,j=m}^{n} \langle \lambda_j x_j, \lambda_i x_i\rangle_{L^2(P)} \\ &= \left\| \sum_{j=m}^{n} \lambda_j x_j \right\|_{L^2(P)}^2 \geq 0,\end{aligned}$$

for all $m, n \in \mathbb{Z}, m \leq n$, and $\lambda_k \in \mathbb{C}$, where $k = m, m+1, ..., n$.

A stochastic process $\{x_n\}_{n\in\mathbb{Z}}$ is said to be *stationary (in a wide sense)* if its correlation kernel is a Toeplitz kernel, that is

$$K(m,n) = K_{n-m} \quad \text{for all} \quad m, n \in \mathbb{Z}.$$

In this case it can be used the Naimark Decomposition Theorem in order to associate the stationary stochastic process $\{x_n\}_{n\in\mathbb{Z}}$ with the Hilbert space $\mathcal{H}_K$, the unitary operator $S \in L(\mathcal{H}_K)$ and the operator $Q \in L(\mathbb{C}, \mathcal{H}_K)$ such that

$$K_n = Q^* S^n Q, \quad n \in \mathbb{Z}.$$

The geometric settings for the prediction problem can be extended in order to deal with the multivariate case too. Let notice that a random variable $x_n : \Omega \to \mathbb{C}$, of a stochastic process $\{x_n\}_{n\in\mathbb{Z}} \subset L^2(P)$, can be interpreted as an operator from $\mathbb{C}$ to $L^2(P)$ defining $\widetilde{x}_n : \mathbb{C} \to L^2(P)$ as

$$\widetilde{x}_n(\lambda) = \lambda x_n,$$

and the elements of the correlation kernel of the process can be calculated according to the rule

$$K(m,n) = (\widetilde{x}_m)^* \widetilde{x}_n.$$

Also, it must be noticed that many stochastic processes have the same correlation kernel. Having this in mind it is convenient to adopt the following terminology. The main object used to describe a *multivariate process* will be its correlation kernel K which is supposed to be positive definite and $K(m,n) \in \mathcal{L}(\mathcal{H}_n, \mathcal{H}_m)$ for all $m, n \in \mathbb{Z}$, where $\mathbf{H} = \{\mathcal{H}_n\}_{n\in\mathbb{Z}}$ is a family of Hilbert spaces.

Definition 4.1. A pair $[\mathcal{K}, X]$, where $\mathcal{K}$ is a Hilbert space and $X = \{X_n\}_{n\in\mathbb{Z}}$ is a family of operators X_n in $\mathcal{L}(\mathcal{H}_n, \mathcal{K})$, is called a *geometric model of the multivariate process* with correlation kernel K, if

$$K(m,n) = X_m^* X_n.$$

The Kolmogorov Decomposition Theorem shows that given a positive definite kernel K, there exists a geometric model of the multivariate process with correlation kernel K. If $[\mathcal{K}, X]$ is the geometric model of the multivariate process with covariance kernel K then $\mathcal{H}_X$ will be the subspace of $\mathcal{K}$ generated for this model, that is,

$$\mathcal{H}_X = \bigvee_{n\in\mathbb{Z}} X_n \mathcal{H}_n. \tag{4.1}$$

If $[\mathcal{K}', X']$ is another geometric model of the same process, then the Kolmogorov Decomposition Theorem guarantees the existence of an unitary operator $\Phi : \mathcal{H}_X \to \mathcal{H}_{X'}$ such that $\Phi X_n = X'_n$ for all $n \in \mathbb{Z}$. This means that the geometry of the process is essentially determined by the choise of a geometric model such that

$$\mathcal{K} = \bigvee_{n\in\mathbb{Z}} X_n \mathcal{H}_n. \tag{4.2}$$

4.2. Equivalent multivariate stochastic processes

From here on, $\mathcal{H}_n = \mathcal{H}$ for all $n \in \mathbb{Z}$ and the covariance kernels of the processes will be positive definite.

Theorem 4.2 (Isomorphism). *Let $[\mathcal{W}, X]$ be the geometric model of a multivariate process and let $K : \mathbb{Z} \times \mathbb{Z} \to \mathcal{L}(\mathcal{H})$ be the kernel of covariance associated with the process. Then there exists an unit operator $\Phi : \mathcal{H}_K \to \mathcal{H}_X$ such that*

$$\Phi V_K(n) = X_n \quad \textit{for all} \quad n \in \mathbb{Z}.$$

Proof. Let $[\mathcal{W}, X]$, $X = \{X_n\}_{n\in\mathbb{Z}}$ be a geometric model of a multivariate process and $K : \mathbb{Z} \times \mathbb{Z} \to \mathcal{L}(\mathcal{H})$ be the kernel of covariance associated with the process.
It follows that the covariance kernel and the space generated by the process is given by

$$K(n, m) = X_n^* X_m \quad \text{and} \quad \mathcal{H}_X = \bigvee_{n\in\mathbb{Z}} X_n \mathcal{H}.$$

On the other hand, since K is a positive definite kernel one more time by the Kolmogorov decomposition theorem there exists a Hilbert space $\mathcal{H}_K$ and an application $V_K(n) \in \mathcal{L}(\mathcal{H}, \mathcal{H}_K)$ for all $n \in \mathbb{Z}$ such that

$$K(n, m) = V_K^*(n) V_K(m) \quad \text{and} \quad \mathcal{H}_K = \bigvee_{n\in\mathbb{Z}} V_K(n) \mathcal{H}.$$

Let us define the application $\Phi : \mathcal{H}_K \to \mathcal{H}_X$ in the following way

$$\Phi \left(\sum_{n\in\mathbb{Z}} V_K(n) h_n \right) = \sum_{n\in\mathbb{Z}} X_n h_n,$$

where $\{h_n\}_{n\in\mathbb{Z}}$ is a sequence with finite support in $\mathcal{H}$.

Then we have

$$\left\|\Phi\left(\sum_{n\in\mathbb{Z}} V_K(n)h_n\right)\right\|^2_{\mathcal{H}_X} = \left\|\sum_{n\in\mathbb{Z}} X_n h_n\right\|^2_{\mathcal{H}_X} = \sum_{m,n\in\mathbb{Z}} \langle X_m h_m, X_n h_n\rangle_{\mathcal{H}_X}$$
$$= \sum_{m,n\in\mathbb{Z}} \langle K(n,m)h_m, h_n\rangle_{\mathcal{H}} = \sum_{m,n\in\mathbb{Z}} \langle V_K^*(n)V_K(m)h_m, h_n\rangle_{\mathcal{H}}$$
$$= \sum_{m,n\in\mathbb{Z}} \langle V_K(m)h_m, V_K(n)h_n\rangle_K = \left\|\sum_{n\in\mathbb{Z}} V_K(n)h_n\right\|^2_{\mathcal{H}_K},$$

all of this show us that the application Φ can be extended by continuity to an unit operator from $\mathcal{H}_K$ over $\mathcal{H}_X$ and moreover $\Phi V_K(n) = X_n$ for all $n \in \mathbb{Z}$. □

Definition 4.3. Two geometric models of multivariate processes $[\mathcal{K}, X]$ and $[\mathcal{L}, Y]$ are said to be *equivalent* , if $\dim(\mathcal{H}_X) = \dim(\mathcal{H}_Y)$ and there are two constants A, B with $0 < A \le B$ such that

$$A\left\|\sum_{n\in\mathbb{Z}} X_n h_n\right\|^2_{\mathcal{H}_X} \le \left\|\sum_{n\in\mathbb{Z}} Y_n h_n\right\|^2_{\mathcal{H}_Y} \le B\left\|\sum_{n\in\mathbb{Z}} X_n h_n\right\|^2_{\mathcal{H}_X},$$

where $\{h_n\}_{n\in\mathbb{Z}}$ is a sequence in $\mathcal{H}$ with finite support.

By Theorem 4.2 and definitions we have the following.

Proposition 4.4. *Let $[\mathcal{W}, X]$ and $[\mathcal{W}_1, Y]$ be two geometric model of multivariate process and let K_1 and K_2 be two kernels of covariance associated with the processes. Then K_1 and K_2 are equivalent kernels if and only if $X = \{X_n\}_{n\in\mathbb{Z}}$ and $Y = \{Y_n\}_{n\in\mathbb{Z}}$ are equivalent processes.*

As an application we give the proof of the results obtained in [4].

Theorem 4.5. *Let $[\mathcal{K}, X]$ and $[\mathcal{L}, Y]$ be two geometric models of multivariate processes. The following conditions are equivalent:*

(i) *The models of the multivariate processes $[\mathcal{K}, X]$ and $[\mathcal{L}, Y]$ are equivalent.*

(ii) *There is a bijective bounded linear application with bounded inverse $\psi : \mathcal{H}_X \to \mathcal{H}_Y$ such that*
$$\psi X_n = Y_n \quad \textit{for all } n \in \mathbb{Z}.$$

(iii) *There exist two constants A, B with $0 < A \le B$ such that*
$$A\left\|\sum_{n\in\mathbb{Z}} X_n h_n\right\|^2_{\mathcal{H}_X} \le \left\|\sum_{n\in\mathbb{Z}} Y_n h_n\right\|^2_{\mathcal{H}_Y} \le B\left\|\sum_{n\in\mathbb{Z}} X_n h_n\right\|^2_{\mathcal{H}_X},$$
for each sequence with finite support $\{h_n\}_{n\in\mathbb{Z}} \subset \mathcal{H}$.

Proof. The equivalence between (i) and (iii) follows by definition. Next, we are going to show that (i) implies (ii) to this end let us assume that $X = \{X_n\}_{n\in\mathbb{Z}}$ and $Y = \{Y_n\}_{n\in\mathbb{Z}}$ are equivalent processes let K_1 and K_2 be the kernels of covariance associated with the processes $X = \{X_n\}_{n\in\mathbb{Z}}$ and $Y = \{Y_n\}_{n\in\mathbb{Z}}$ respectively. Since $X = \{X_n\}_{n\in\mathbb{Z}}$ and $Y = \{Y_n\}_{n\in\mathbb{Z}}$ are equivalent, then by proposition 4.4 we concluded that K_1 and

K_2 are equivalent kernels. By Theorem 3.3, there exists a biyective bounded linear application linear with bounded inverse $\Phi : \mathcal{H}_{K_1} \to \mathcal{H}_{K_2}$ such that

$$\Phi V_{K_1}(n) = V_{K_2}(n) \quad \text{for all } n \in \mathbb{Z}.$$

Let us consider the operators $\phi_1 : \mathcal{H}_{K_1} \to \mathcal{H}_X$ such that

$$\phi_1 V_{K_1}(n) = X_n \quad \text{for all} \quad n \in \mathbb{Z}$$

and $\phi_2 : \mathcal{H}_{K_2} \to \mathcal{H}_Y$ such that

$$\phi_2 V_{K_2}(n) = Y_n \quad \text{for all} \quad n \in \mathbb{Z}.$$

From the above it follows that

$$\phi_2^{-1} \Phi \phi_1^{-1} X_n = Y_n \quad \text{for all} \quad n \in \mathbb{Z}.$$

Now suppose that (ii) holds then there is a bijective bounded linear application with bounded inverse $\psi : \mathcal{H}_X \to \mathcal{H}_Y$ such that

$$\psi X_n = Y_n \quad \text{for all } n \in \mathbb{Z}.$$

Let K_1 and K_2 be two kernels of covariance associated with the processes $X = \{X_n\}_{n\in\mathbb{Z}}$ and $Y = \{Y_n\}_{n\in\mathbb{Z}}$, respectively.
Let us consider the operators $\phi_1 : \mathcal{H}_{K_1} \to \mathcal{H}_X$ such that

$$\phi_1 V_{K_1}(n) = X_n \quad \text{for all} \quad n \in \mathbb{Z}$$

and $\phi_2 : \mathcal{H}_{K_2} \to \mathcal{H}_Y$ such that

$$\phi_2 V_{K_2}(n) = Y_n \quad \text{for all} \quad n \in \mathbb{Z}.$$

From the above it follows that

$$\phi_2^{-1} \psi \phi_1 V_{K_1}(n) = V_{K_2}(n) \quad \text{for all} \quad n \in \mathbb{Z}.$$

By Theorem 3.3, we obtain $\dim(\mathcal{H}_{K_1}) = \dim(\mathcal{H}_{K_2})$ and there exist two positive constants A, B, $A \leq B$ such that

$$\begin{aligned} A \sum_{n,m\in\mathbb{Z}} \langle K_1(n,m) h_m, h_n \rangle_{\mathcal{H}} &\leq \sum_{n,m\in\mathbb{Z}} \langle K_2(n,m) h_m, h_n \rangle_{\mathcal{H}} \\ &\leq B \sum_{n,m\in\mathbb{Z}} \langle K_1(n,m) h_m, h_n \rangle_{\mathcal{H}}, \end{aligned}$$

where $\{h_n\}_{n\in\mathbb{Z}}$ is a sequence in $\mathcal{H}$ with finite support.
The result comes up from the fact that $K_1(m,n) = X_m^* X_n$ and $K_2(m,n) = Y_m^* Y_n$. □

In the multivariate stochastic processes setting it is possible to obtain a result similar to that of the theorem on stability (see Theorem 1.2).

The following is our result about stability of multivariate stochastic processes.

Theorem 4.6. *Let $[\mathcal{W}, Y]$ be a geometrical model of a multivariate stochastic process, $\mathcal{H}_Y$ the subspace generated by the process, and suppose $X_n \in \mathcal{L}(\mathcal{H}, \mathcal{H}_Y)$ for all $n \in \mathbb{Z}$ such that*

$$\left\| \sum_{n\in\mathbb{Z}} (Y_n - X_n) h_n \right\|_{\mathcal{H}_Y} \leq \delta \left\| \sum_{n\in\mathbb{Z}} Y_n h_n \right\|_{\mathcal{H}_Y}, \tag{4.3}$$

for some constant δ, $0 < \delta < 1$, and any sequence $\{h_n\}_{n\in\mathbb{Z}}$ in $\mathcal{H}$ with finite support. Then the geometric model of the multivariate process $[\mathcal{K}, X]$ is equivalent to $[\mathcal{W}, Y]$.

Proof. Let K and K_1 be two kernels of covariance associated with the processes $Y = \{Y_n\}_{n\in\mathbb{Z}}$ and $X = \{X_n\}_{n\in\mathbb{Z}}$, respectively.

Let us consider the operators $\Phi_1 : \mathcal{H}_K \to \mathcal{H}_Y$ such that

$$\Phi_1 V_K(n) = Y_n \quad \text{for all} \quad n \in \mathbb{Z}$$

and $\Phi_2 : \mathcal{H}_{K_1} \to \mathcal{H}_X$ such that

$$\Phi_2 V_{K_1}(n) = X_n \quad \text{for all} \quad n \in \mathbb{Z}.$$

From the above and hypothesis we have

$$\mathcal{H}_{K_1} \subset \mathcal{H}_K \quad \text{and} \quad \Phi_2\left(\sum_{n\in\mathbb{Z}} V_{K_1}(n)h_n\right) = \sum_{n\in\mathbb{Z}} X_n h_n = \Phi_1\left(\sum_{n\in\mathbb{Z}} V_{K_1}(n)h_n\right).$$

Then

$$\begin{aligned}
\left\|\sum_{n\in\mathbb{Z}}(V_K(n) - V_{K_1}(n))h_n\right\|_{\mathcal{H}_K} &= \left\|\Phi_1 \sum_{n\in\mathbb{Z}}(V_K(n) - V_{K_1}(n))h_n\right\|_{\mathcal{H}_Y} \\
&= \left\|\sum_{n\in\mathbb{Z}}(\Phi_1 V_K(n) - \Phi_2 V_{K_1}(n))h_n\right\|_{\mathcal{H}_Y} \\
&= \left\|\sum_{n\in\mathbb{Z}}(Y_n - X_n)h_n\right\|_{\mathcal{H}_Y} \\
&\leq \delta \left\|\sum_{n\in\mathbb{Z}} Y_n h_n\right\|_{\mathcal{H}_Y} = \delta \left\|\sum_{n\in\mathbb{Z}} \Phi_1 V_K(n)h_n\right\|_{\mathcal{H}_Y} \\
&= \delta \left\|\sum_{n\in\mathbb{Z}} V_K(n)h_n\right\|_{\mathcal{H}_K},
\end{aligned}$$

for any sequence $\{h_n\}_{n\in\mathbb{Z}}$ in $\mathcal{H}$ with finite support.

Finally, by Theorem 3.4 it follows that K_1 and K are equivalent kernels. Therefore $X = \{X_n\}_{n\in\mathbb{Z}}$ is equivalent to $Y = \{Y_n\}_{n\in\mathbb{Z}}$. □

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Hybrid conjugate gradient-BFGS methods based on Wolfe line search

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Abstract. In this paper, we present some hybrid methods for solving unconstrained optimization problems. These methods are defined using proper combinations of the search directions and included parameters in conjugate gradient and quasi-Newton method of Broyden–Fletcher–Goldfarb–Shanno (CG-BFGS). Their global convergence under the Wolfe line search is analyzed for general objective functions. Numerical experiments show the superiority of the modified hybrid (CG-BFGS) method with respect to some existing methods.

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1. Introduction

Conjugate gradient methods are very important ones for solving unconstrained optimization problems, especially for large scale problems. It is well known that Fletcher-Reeves (FR) [7], Conjugate Descent (CD) [6] and Dai-Yuan (DY) [4] conjugate gradient methods have strong convergence properties, but they may not perform well in practice. On the other hand, Hestnes-Stiefel (HS) [9], Polak-Ribiere-Polyak (PRP) [13, 14] and Liu-Storey (LS) [12] conjugate gradient methods may not converge in general, but they often perform better than FR, CD and DY. To combine the best numerical performances of the LS method and the global convergence properties of the CD method, Yang et al. [17] proposed a hybrid LS-CD method. Dai and Liao [3] proposed an efficient conjugate gradient method (Dai-Liao type method). Later, some more efficient Dai-Liao type conjugate gradient method, known as DHSDL and DLSDL were proposed in [21].

The rest of this paper is organized as follows. In Section 2, we give various possibilities to determine the step size and the search direction. A hybridization of

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the conjugate gradient method (CG) and the BFGS method will also be presented. In Section 3, we consider the modification of LSCD method, termed as MLSCD and the modification of (DHSDL and DLSDL) termed as MMDL [15] and we prove the global convergence using the Wolfe line search instead of backtracking line search used by the authors in [15]. In Section 4, we consider the hybrid method BFGS-CG termed as H-BFGS-CG1 in [15] and we prove the global convergence with the Wolfe line search termed WH-BFGS-CG. In section 5, we report some numerical results and compare the performance of the different considered methods. Finally, we give some conclusions to end this paper.

2. Preliminaries

Consider the following unconstrained optimization problem

$$\min f(x), \quad x \in \mathbb{R}^n, \tag{2.1}$$

where $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ is a continuously differentiable function. Let g_k be the gradient of $f(x)$ at the current iterative point x_k, then the classical conjugate gradient method for (2.1) is given by

$$x_{k+1} = x_k + \alpha_k d_k, \tag{2.2}$$

in which $\alpha_k > 0$ is the step size found by one of the line search methods, and d_k is the search direction defined by

$$d_k = \begin{cases} -g_0, & k = 0, \\ -g_k + \beta_k d_{k-1}, & k \geq 1, \end{cases} \tag{2.3}$$

where β_k is an appropriately defined real scalar, known as the conjugate gradient parameter.

Since Fletcher and Reeves introduced the nonlinear conjugate gradient method in 1964, many formulae have been proposed using various modifications of the conjugate gradient direction d_k and the parameter β_k. The most popular parameters β_k are:

$$\begin{aligned}
\beta_k^{FR} &= \frac{\| g_k \|^2}{\| g_{k-1} \|^2}, \quad \beta_k^{CD} = -\frac{\| g_k \|^2}{g_{k-1}^T d_{k-1}}, \quad \beta_k^{DY} = \frac{\| g_k \|^2}{y_{k-1}^T d_{k-1}}, \\
\beta_k^{HS} &= \frac{g_k^T y_{k-1}}{y_{k-1}^T d_{k-1}}, \quad \beta_k^{PRP} = \frac{g_k^T y_{k-1}}{\| g_{k-1} \|^2}, \quad \beta_k^{LS} = -\frac{g_k^T y_{k-1}}{g_{k-1}^T d_{k-1}}, \\
\beta_k^{DHSDL} &= \frac{\| g_k \|^2 - \frac{\|g_k\|}{\|g_{k-1}\|} \mid g_k^T g_{k-1} \mid}{\mu \mid g_k^T d_{k-1} \mid + d_{k-1}^T y_{k-1}} - t\frac{g_k^T s_{k-1}}{d_{k-1}^T y_{k-1}}, \quad \mu > 1, \ t > 0, \\
\beta_k^{DLSDL} &= \frac{\| g_k \|^2 - \frac{\|g_k\|}{\|g_{k-1}\|} \mid g_k^T g_{k-1} \mid}{\mu \mid g_k^T d_{k-1} \mid - d_{k-1}^T g_{k-1}} - t\frac{g_k^T s_{k-1}}{d_{k-1}^T y_{k-1}}, \quad \mu > 1, \ t > 0,
\end{aligned}$$

where

$$y_{k-1} = g_k - g_{k-1}, \ s_{k-1} = x_k - x_{k-1}$$

and $\| \cdot \|$ denotes the Euclidean vector norm.

In this paper, the step size α_k is determined using the following Wolfe line search conditions

$$\begin{aligned} f(x_k+\alpha_k d_k) &\le f(x_k)+\rho\alpha_k g_k^T d_k, \\ g_{k+1}^T d_k &\ge \sigma g_k^T d_k,\ 0<\rho<\sigma<1. \end{aligned} \tag{2.4}$$

To combine the best numerical performances of the PRP method and the global convergence properties of the FR method, Touati-Ahmed and Storey [16] proposed a hybrid PRP-FR method which is called the H1 method in [19], with the gradient parameter is defined as

$$\beta_k^{H1}=\max\{0,\min\{\beta_k^{PRP},\beta_k^{FR}\}\}. \tag{2.5}$$

Gilbert and Nocedal in [8] modified (2.5) to

$$\beta_k=\max\{-\beta_k^{FR},\min\{\beta_k^{PRP},\beta_k^{FR}\}\}.$$

A hybrid HS-DY conjugate gradient method was proposed by Dai and Yuan in [5], termed as the H2 method in [19] where the gradient parameter is defined as

$$\beta_k^{H2}=\max\{0,\min\{\beta_k^{HS},\beta_k^{DY}\}\}. \tag{2.6}$$

We consider hybrid CG methods where the search direction d_k, $k\ge 1$, from (2.3) is modified using one of the following tow rules [15]

$$d_k=\mathcal{D}(\beta_k,g_k,d_{k-1})=-\left(1+\beta_k\frac{g_k^T d_{k-1}}{\|g_k\|^2}\right)g_k+\beta_k d_{k-1} \tag{2.7}$$

$$d_k=\mathcal{D}_1(\beta_k,g_k,d_{k-1})=-B_k g_k+\mathcal{D}(\beta_k,g_k,d_{k-1}) \tag{2.8}$$

and the conjugate gradient parameter β_k is defined using some proper combinations of the parameters β_k given above and already defined hybridizations of these parameters.

Zhang et al. in [20, 18] proposed a modification to the FR method, termed as the MFR method, using the search direction

$$d_k=\mathcal{D}(\beta_k^{FR},g_k,d_{k-1}) \tag{2.9}$$

Zhang in [18] also proposed a modified DY method, which is known as the MDY method, using the search direction

$$d_k=\mathcal{D}(\beta_k^{DY},g_k,d_{k-1}) \tag{2.10}$$

The MFR and MDY methods posses very useful property

$$g_k^T d_k=-\|g_k\|^2 \tag{2.11}$$

If the exact line search is used, then MFR and the MDY methods reduce to the FR and the DY methods, respectively.

The MFR method has proven to be globally convergent for non convex functions with the Wolfe line search or the Armijo line search, and it is very efficient in real computations [20].

However, it is not known whether the MDY method converges globally. So, in [19], the authors replaced β_k^{FR} in (2.9) and β_k^{DY} in (2.10) by β_k^{H1} and β_k^{H2}, respectively. Then, they defined new hybrid PRP-FR and HS-DY methods, which they call the NH1 method and the NH2 method, respectively. These methods are based on the search directions

$$\text{NH1} : d_k = \mathcal{D}(\beta_k^{H1}, g_k, d_{k-1}) \tag{2.12}$$

$$\text{NH2} : d_k = \mathcal{D}(\beta_k^{H2}, g_k, d_{k-1}). \tag{2.13}$$

It is clear that NH1 and NH2 are descent methods, they satisfy (2.11).
On the other hand, the search direction d_k in quasi-Newton methods is obtained as a solution of the linear algebraic system

$$B_k d_k = -g_k, \tag{2.14}$$

where B_k is an approximation of the Hessian. The initial approximation is the identity matrix ($B_0 = I$) and the subsequent updates B_k are defined by an appropriate formula.
Here, we are interested in the BFGS update formula, defined by

$$B_{k+1} = B_k + \frac{y_k y_k^T}{s_k^T y_k} - \frac{B_k s_k s_k^T B_k}{s_k^T B_k s_k}, \tag{2.15}$$

where $s_k = x_{k+1} - x_k, \quad y_k = g_{k+1} - g_k$. The next secant equation must hold

$$B_{k+1} s_k = y_k, \tag{2.16}$$

which is possible only if the curvature condition

$$y_k^T s_k > 0 \tag{2.17}$$

is satisfied.

The three-term hybrid BFGS conjugate gradient method was proposed in [10]. That method uses best properties of both BFGS and CG methods and defines a hybrid BFGS-CG method for solving some selected unconstrained optimization problems, resulting in improvement in the total number of iterations and the CPU time.

3. Modification of LSCD, DHSDL and DLSDL methods

3.1. A modified LSCD conjugate gradient method

We consider the modification of LSCD method, defined in [17] by

$$\beta_k^{LSCD} = \max\left\{0, \min\left\{\beta_k^{LS}, \beta_k^{CD}\right\}\right\}, \tag{3.1}$$

$$d_k = \begin{cases} -g_0 & k = 0 \\ d_k = -g_k + \beta_k^{LSCD} d_{k-1} & k \geq 1, \end{cases}$$

and define the MLSCD method [15] with the search direction

$$d_k = \mathcal{D}(\beta_k^{LSCD}, g_k, d_{k-1}). \tag{3.2}$$

Now, we give the algorithm of this method using the Wolfe line search.

3.1.1. Algorithm WMLSCD.

- Step0: Given a starting point x_0 and a parameter $0 < \varepsilon < 1$.
- Step1: Set $k = 0$ and compute $d_0 = -g_0$.
- Step2: If $\|g_k\| \leq \varepsilon$, STOP; else go to Step3.
- Step3: Find the step size $\alpha_k \in]0, 1]$ using the Wolfe line search.
- Step4: Compute $x_{k+1} = x_k + \alpha_k d_k$.
- Step5: Compute $y_k = g_{k+1} - g_k$ and go to Step6.
- Step6: Compute

$$\begin{aligned} \beta_{k+1}^{LS} &= -\frac{y_k^T g_{k+1}}{g_k^T d_k}, \ \beta_{k+1}^{CD} = -\frac{\|g_{k+1}\|^2}{g_k^T d_k}, \\ \beta_{k+1}^{LSCD} &= \max\left\{0, \min\left\{\beta_{k+1}^{LS}, \beta_{k+1}^{CD}\right\}\right\}. \end{aligned}$$

- Step7: Compute the search direction $d_{k+1} = \mathcal{D}(\beta_{k+1}^{LSCD}, g_{k+1}, d_k)$.
- Step8: Let $k := k + 1$ and go to Step2.

3.1.2. Convergence of the WMLSCD conjugate gradient method. It is easy to prove the next theorem.

Theorem 3.1. *Let β_k be any CG parameter. Then, the search direction*

$$d_k = \mathcal{D}(\beta_k, g_k, d_{k-1})$$

satisfies

$$g_k^T d_k = -\|g_k\|^2. \tag{3.3}$$

To prove the global convergence of the WMLSCD method, we need the following assumptions.

Assumption 3.1 The level set $\mathcal{L} = \{x \in \mathbb{R}^n / f(x) \leq f(x_0)\}$ is bounded.

Assumption 3.2 The function f is continuously differentiable in some neighbourhood $\mathcal{N}$ of $\mathcal{L}$ and its gradient is Lipschitz continuous. Namely, there exists a constant $L > 0$ such that

$$\|g(x) - g(y)\| \leq L\|x - y\|, \text{ for all } x, y \in \mathcal{N}. \tag{3.4}$$

It is well known that if Assumption 3.2 holds, then there exists a positive constant γ, such that

$$\|g_k\| \leq \gamma, \forall k \tag{3.5}$$

The next lemma, often called the Zoutendijk condition [22], is used to prove the global convergence of nonlinear CG method.

Lemma 3.2. [15] *Let the Assumption 3.1 and Assumption 3.2 be satisfied. Let the sequence $\{x_k\}$ be generated by the MLSCD method with the Wolfe line search. Then it holds that*

$$\sum_{k=1}^{\infty} \frac{\|g_k\|^4}{\|d_k\|^2} < +\infty \tag{3.6}$$

Theorem 3.3. *Let the Assumption 3.1 and Assumption 3.2 hold. Then, the sequence $\{x_k\}$ generated by the WMLSCD method with the Wolfe line search satisfies*

$$\liminf_{k\to\infty} \|g_k\| = 0 \tag{3.7}$$

Proof. In order to gain the contradiction, let us suppose that (3.7) does not hold. Then, there exists a constant $c > 0$ such that

$$\|g_k\| \geq c, \text{ for all } k \tag{3.8}$$

Clearly, (3.2) can be rewritten into the form

$$d_k = -l_k g_k + \beta_k^{LSCD} d_{k-1}, \; l_k = 1 + \beta_k^{LSCD} \frac{g_k^T d_{k-1}}{\|g_k\|^2}. \tag{3.9}$$

Now from (3.9), it follows that

$$d_k + l_k g_k = \beta_k^{LSCD} d_{k-1}$$

which further implies

$$(d_k + l_k g_k)^2 = \left(\beta_k^{LSCD} d_{k-1}\right)^2$$

$$\Longleftrightarrow \|d_k\|^2 + 2l_k d_k^T g_k + l_k^2 \|g_k\|^2 = \left(\beta_k^{LSCD}\right)^2 \|d_{k-1}\|^2,$$

and subsequently

$$\|d_k\|^2 = \left(\beta_k^{LSCD}\right)^2 \|d_{k-1}\|^2 - 2l_k d_k^T g_k - l_k^2 \|g_k\|^2. \tag{3.10}$$

Notice that

$$\beta_k^{LSCD} = \max\left\{0, \min\left\{\beta_k^{LS}, \beta_k^{CD}\right\}\right\} \leq \left|\beta_k^{CD}\right| \tag{3.11}$$

Dividing both sides of (3.10) by $(g_k^T d_k)^2$, we get from (3.11), (3.3), (3.8) and the definition of β_k^{CD} that

$$\begin{aligned} \frac{\|d_k\|^2}{\|g_k\|^4} &= \frac{\|d_k\|^2}{(g_k^T d_k)^2} = \left(\beta_k^{LSCD}\right)^2 \frac{\|d_{k-1}\|^2}{(g_k^T d_k)^2} - \frac{2l_k d_k^T g_k}{(g_k^T d_k)^2} - l_k^2 \frac{\|g_k\|^2}{(g_k^T d_k)^2} \\ &\leq \left(\beta_k^{CD}\right)^2 \frac{\|d_{k-1}\|^2}{(g_k^T d_k)^2} - \frac{2l_k}{g_k^T d_k} - l_k^2 \frac{\|g_k\|^2}{(g_k^T d_k)^2} \\ &= \left(\frac{\| g_k \|^2}{-g_{k-1}^T d_{k-1}}\right)^2 \frac{\|d_{k-1}\|^2}{(g_k^T d_k)^2} - \frac{2l_k}{g_k^T d_k} - l_k^2 \frac{\|g_k\|^2}{(g_k^T d_k)^2} \end{aligned}$$

Finally

$$\frac{\|d_k\|^2}{\|g_k\|^4} \leq \left(\frac{\| g_k \|^2}{-g_{k-1}^T d_{k-1}}\right)^2 \frac{\|d_{k-1}\|^2}{(g_k^T d_k)^2} - \frac{2l_k}{g_k^T d_k} - l_k^2 \frac{\|g_k\|^2}{(g_k^T d_k)^2} \tag{3.12}$$

Now, applying (3.3), (3.12) becomes

$$\begin{aligned}
\frac{\|d_k\|^2}{\|g_k\|^4} &\leq \frac{\| g_k \|^4}{\| g_{k-1} \|^4}{}^2 \frac{\|d_{k-1}\|^2}{\| g_k \|^4} - \frac{2l_k}{\| g_k \|^2} - l_k^2 \frac{\|g_k\|^2}{\| g_k \|^4} \\
&= \frac{\|d_{k-1}\|^2}{\| g_{k-1} \|^4} + \frac{2l_k}{\| g_k \|^2} - l_k^2 \frac{1}{\| g_k \|^2} \\
&= \frac{\|d_{k-1}\|^2}{\| g_{k-1} \|^4} - \frac{(l_k - 1)^2}{\| g_k \|^2} + \frac{1}{\| g_k \|^2} \\
&\leq \frac{\|d_{k-1}\|^2}{\| g_{k-1} \|^4} + \frac{1}{\| g_k \|^2} \\
&\leq \sum_{j=0}^{k} \frac{1}{\|g_j\|^2} \\
&\leq \frac{k+1}{c^2}.
\end{aligned}$$

The last inequalities imply

$$\sum_{k\geq 1} \frac{\|g_k\|^4}{\|d_k\|^2} \geq c^2 \sum_{k\geq 1} \frac{1}{k+1} = \infty$$

which contradicts to (3.6). This completes the proof. □

3.2. A modified DHSDL and DLSDL conjugate gradient method

In this part, we have the hybrid MMDL method, proposed in [15], which is defined by the search direction d_k as follows

$$\beta_k^{MMDL} = \max\left\{0, \min\left\{\beta_k^{DHSDL}, \beta_k^{DLSDL}\right\}\right\}$$

$$d_k = \mathcal{D}(\beta_k^{MMDL}, g_k, d_{k-1}).$$

We give the algorithm of this method where we have changed the backtracking line search by the Wolfe line search.

3.2.1. Algorithm WMMDL.

- Step0: Given a starting point x_0, a parameter $0 < \varepsilon < 1$ and $\mu > 1$.
- Step1: Set $k = 0$ and compute $d_0 = -g_0$.
- Step2: If $\|g_k\| \leq \varepsilon$, STOP; else go to Step3.
- Step3: Find the step size $\alpha_k \in]0, 1]$ using the Wolfe line search.
- Step4: Compute $x_{k+1} = x_k + \alpha_k d_k$.
- Step5: Compute $y_k = g_{k+1} - g_k$, $s_k = x_{k+1} - x_k$ and go to Step6.
- Step6: Compute

$$\beta_{k+1}^{DHSDL} = \frac{\| g_{k+1} \|^2 - \frac{\|g_{k+1}\|}{\|g_k\|} \mid g_{k+1}^T g_k \mid}{\mu \mid g_{k+1}^T d_k \mid + d_k^T y_k} - \alpha_k \frac{g_{k+1}^T s_k}{d_k^T y_k}$$

$$\beta_{k+1}^{DLSDL} = \frac{\| g_{k+1} \|^2 - \frac{\|g_{k+1}\|}{\|g_k\|} \mid g_{k+1}^T g_k \mid}{\mu \mid g_{k+1}^T d_k \mid - d_k^T g_k} - \alpha_k \frac{g_{k+1}^T s_k}{d_k^T y_k}$$

$$\beta_{k+1}^{MMDL} = \max\left\{0, \min\left\{\beta_{k+1}^{DHSDL}, \beta_{k+1}^{DLSDL}\right\}\right\}.$$

- Step7: Compute the search direction $d_{k+1} = \mathcal{D}(\beta_{k+1}^{MMDL}, g_{k+1}, d_k)$.
- Step8: Let $k := k+1$ and go to Step2.

3.2.2. Convergence of the WMMDL conjugate gradient method. The following theorem prove the global convergence of the WMMDL method.

Theorem 3.4. *Let the Assumption 3.1 and Assumption 3.2 be satisfied. Then the sequence $\{x_k\}$ generated by the WMMDL method with the Wolfe line search satisfies*

$$\liminf_{k\to\infty} \|g_k\| = 0 \tag{3.13}$$

Proof. Assume, on the contrary, that (3.13) does not hold. Then, there exists a constant $c > 0$ such that

$$\|g_k\| \geq c, \text{ for all } k \tag{3.14}$$

Denote

$$l_k = 1 + \beta_k^{MMDL} \frac{g_k^T d_{k-1}}{\|g_k\|^2}$$

Then we can write

$$d_k + l_k g_k = \beta_k^{MMDL} d_{k-1}$$

and further

$$(d_k + l_k g_k)^2 = \left(\beta_k^{MMDL} d_{k-1}\right)^2$$

$$\Longleftrightarrow \|d_k\|^2 + 2l_k d_k^T g_k + l_k^2 \|g_k\|^2 = \left(\beta_k^{MMDL}\right)^2 \|d_{k-1}\|^2.$$

Thus,

$$\|d_k\|^2 = \left(\beta_k^{MMDL}\right)^2 \|d_{k-1}\|^2 - 2l_k d_k^T g_k - l_k^2 \|g_k\|^2. \tag{3.15}$$

Having in view, $\mu > 1$ as well as $d_k^T g_k < 0$ and applying the extended conjugacy condition $d_k^T y_{k-1} = -\alpha g_k^T s_{k-1}$, $\alpha > 0$, which was exploited in [3, 21], we get

$$\begin{aligned}
\beta_{k+1}^{DHSDL} &= \frac{\| g_{k+1} \|^2 - \frac{\|g_{k+1}\|}{\|g_k\|} \mid g_{k+1}^T g_k \mid}{\mu \mid g_{k+1}^T d_k \mid + d_k^T y_k} - \alpha_k \frac{g_{k+1}^T s_k}{d_k^T y_k} \\
&\leq \frac{\| g_{k+1} \|^2 - \frac{\|g_{k+1}\|}{\|g_k\|} \mid g_{k+1}^T g_k \mid}{\mu \mid g_{k+1}^T d_k \mid + d_k^T y_k} \\
&= \frac{\| g_{k+1} \|^2 - \frac{\|g_{k+1}\|}{\|g_k\|} \mid g_{k+1}^T g_k \mid}{\mu \mid g_{k+1}^T d_k \mid + d_k^T (g_{k+1} - g_k)} \\
&= \frac{\| g_{k+1} \|^2 - \frac{\|g_{k+1}\|}{\|g_k\|} \mid g_{k+1}^T g_k \mid}{\mu \mid g_{k+1}^T d_k \mid + d_k^T g_{k+1} - d_k^T g_k} \\
&\leq \frac{\| g_{k+1} \|^2}{\mu \mid g_{k+1}^T d_k \mid + d_k^T g_{k+1} - d_k^T g_k} \\
&\leq \frac{\| g_{k+1} \|^2}{-d_k^T g_k}.
\end{aligned}$$

Further

$$\begin{aligned}\beta_{k+1}^{DLSDL} &= \frac{\| g_{k+1} \|^2 - \frac{\|g_{k+1}\|}{\|g_k\|} \mid g_{k+1}^T g_k \mid}{\mu \mid g_{k+1}^T d_k \mid - d_k^T g_k} - \alpha_k \frac{g_{k+1}^T s_k}{d_k^T y_k} \\ &\leq \frac{\| g_{k+1} \|^2 - \frac{\|g_{k+1}\|}{\|g_k\|} \mid g_{k+1}^T g_k \mid}{\mu \mid g_{k+1}^T d_k \mid - d_k^T g_k} \\ &\leq \frac{\| g_{k+1} \|^2}{\mu \mid g_{k+1}^T d_k \mid - d_k^T g_k} \\ &\leq \frac{\| g_{k+1} \|^2}{-d_k^T g_k}.\end{aligned}$$

Now, we conclude

$$\beta_k^{MMDL} = \max\left\{0, \min\left\{\beta_k^{DHSDL}, \beta_k^{DLSDL}\right\}\right\} \leq \frac{\| g_k \|^2}{-d_{k-1}^T g_{k-1}} \tag{3.16}$$

Next, dividing both sides of (3.15) by $(g_k^T d_k)^2$, we get from (3.3), (3.16) and (3.14) that

$$\begin{aligned}\frac{\|d_k\|^2}{\|g_k\|^4} &= \frac{\|d_k\|^2}{(g_k^T d_k)^2} = \left(\beta_k^{MMDL}\right)^2 \frac{\|d_{k-1}\|^2}{(g_k^T d_k)^2} - \frac{2 l_k d_k^T g_k}{(g_k^T d_k)^2} - l_k^2 \frac{\|g_k\|^2}{(g_k^T d_k)^2} \\ &= \left(\beta_k^{MMDL}\right)^2 \frac{\|d_{k-1}\|^2}{(g_k^T d_k)^2} - \frac{2 l_k}{g_k^T d_k} - l_k^2 \frac{\|g_k\|^2}{(g_k^T d_k)^2} \\ &\leq \left(\frac{\| g_k \|^2}{-g_{k-1}^T d_{k-1}}\right)^2 \frac{\|d_{k-1}\|^2}{(g_k^T d_k)^2} - \frac{2 l_k}{g_k^T d_k} - l_k^2 \frac{\|g_k\|^2}{(g_k^T d_k)^2} \\ &= \frac{\| g_k \|^4}{\| g_{k-1} \|^4}^2 \frac{\|d_{k-1}\|^2}{\| g_k \|^4} - \frac{2 l_k}{\| g_k \|^2} - l_k^2 \frac{\|g_k\|^2}{\| g_k \|^4} \\ &= \frac{\|d_{k-1}\|^2}{\| g_{k-1} \|^4} + \frac{2 l_k}{\| g_k \|^2} - l_k^2 \frac{1}{\| g_k \|^2} \\ &= \frac{\|d_{k-1}\|^2}{\| g_{k-1} \|^4} - \frac{(l_k - 1)^2}{\| g_k \|^2} + \frac{1}{\| g_k \|^2} \\ &\leq \frac{\|d_{k-1}\|^2}{\| g_{k-1} \|^4} + \frac{1}{\| g_k \|^2} \\ &\leq \sum_{j=0}^{k} \frac{1}{\|g_j\|^2} \\ &\leq \frac{k+1}{c^2}.\end{aligned}$$

These inequalities imply

$$\sum_{k \geq 1} \frac{\|g_k\|^4}{\|d_k\|^2} \geq c^2 \sum_{k \geq 1} \frac{1}{k+1} = \infty$$

Therefore, $\|g_k\| \geq c$ causes a contradiction to (3.6). Consequently, (3.13) is verified. This completes the proof. □

4. Hybrid BFGS-CG methods

It is known that conjugate gradient method are better compared to the quasi-Newton method in terms of the CPU time. In addition, BFGS is more costly in terms of the memory storage requirements than CG. On the other hand, the quasi-Newton methods are better in terms of the number of iterations and the number of function evaluations. For this purpose, various hybridizations of quasi-Newton methods and CG methods have been proposed by various researchers.

In [10], the authors proposed a hybrid search direction that combines the quasi-Newton and CG methods, where d_k is defined by

$$d_k = \begin{cases} -B_k g_k & k = 0 \\ -B_k g_k + \eta(-g_k + \beta_k d_{k-1}) & k \geq 1, \end{cases}$$

$$\text{where } \eta > 0 \text{ and } \beta_k = \frac{g_k^T g_{k-1}}{g_k^T d_{k-1}}.$$

A hybrid direction search between BFGS update of the Hessian matrix and the conjugate parameter β_k was proposed in [1, 11].

4.1. WH-BFGS-CG method

P. S. Stanimirovic et al. proposed in [15] a three-term hybrid BFGS-CG method, called H-BFGS-CG, defined by the search direction

$$d_k = \begin{cases} -B_k g_k, & k = 0 \\ \mathcal{D}_1(\beta_{k+1}^{LSCD}, g_k, d_{k-1}), & k \geq 1 \end{cases} \tag{4.1}$$

The following algorithm correspond to this method, where we have changed the backtracking line search by the Wolfe line search.

4.1.1. Algorithm WH-BFGS-CG.

- Step0: Given a starting point x_0 and a parameter $0 < \varepsilon < 1$.
- Step1: Set $k = 0$ and compute g_0, $B_0 = I$, $d_0 = -B_0 g_0$.
- Step2: If $\|g_k\| \leq \varepsilon$, STOP; else go to Step3.
- Step3: Find the step size $\alpha_k \in]0, 1]$ using the Wolfe line search.
- Step4: Compute $x_{k+1} = x_k + \alpha_k d_k$.
- Step5: Compute $y_k = g_{k+1} - g_k$, $s_k = x_{k+1} - x_k$ and go to Step6.
- Step6: Compute

$$\beta_{k+1}^{LS} = -\frac{y_k^T g_{k+1}}{g_k^T d_k}, \ \beta_{k+1}^{CD} = -\frac{\|g_{k+1}\|^2}{g_k^T d_k},$$

$$\beta_{k+1}^{LSCD} = \max\left\{0, \min\left\{\beta_{k+1}^{LS}, \beta_{k+1}^{CD}\right\}\right\}.$$

- Step7: Compute B_{k+1} using (2.15).
- Step8: Compute the search direction $d_{k+1} = \mathcal{D}_1(\beta_{k+1}^{LSCD}, g_{k+1}, d_k)$.
- Step9: Let $k := k + 1$ and go to Step2.

4.2. Convergence analysis of WH-BFGS-CG method

Assumption 4.1:

$H1$: The objective function f is twice continuously differentiable.

$H2$: The level set $\mathcal{L}$ is convex. Moreover, there exist positive constants c_1 and c_2 such that

$$c_1 \|z\|^2 \le z^T H(x) z \le c_2 \|z\|^2, \text{ for all } z \in \mathbb{R}^n \text{ and } x \in \mathcal{L},$$

where $H(x)$ is the Hessian of f.

$H3$: The gradient g is Lipschitz continuous at the point x^*, that is, there exists a positive constant c_3 satisfying

$$\|g(x) - g(x^*)\| \le c_3 \|x - x^*\|,$$

for all x in a neighbourhood of x^*.

Theorem 4.1. [2] *Let $\{B_k\}$ be generated by the BFGS update formula* (2.15)*, where $s_k = x_{k+1} - x_k$, $y_k = g_{k+1} - g_k$. Assume that the matrix B_k is symmetric positive definite and satisfies* (2.16) *and* (2.17) *for all k. Furthermore, assume that $\{s_k\}$ and $\{y_k\}$ satisfy the inequality*

$$\frac{\|y_k - G_* s_k\|}{\|s_k\|} \le \epsilon_k,$$

for some symmetric positive definite matrix G_* and for some sequence $\{\epsilon_k\}$ possessing the property

$$\sum_{k=1}^{\infty} \epsilon_k < \infty,$$

then

$$\lim_{k \longrightarrow \infty} \frac{\|(B_k - G_*) s_k\|}{\|s_k\|} = 0,$$

and the sequences $\{\|B_k\|\}, \{\|B_k^{-1}\|\}$ are bounded.

Theorem 4.2. *(Sufficient descent and global convergence) Consider Algorithm WH-BFGS-CG. Assume that the conditions H1, H2 and H3 in Assumption 4.1 are satisfied as well as conditions of Theorem* 4.1*. Then*

$$\lim_{k \to \infty} \|g_k\|^2 = 0.$$

Proof. From (4.1), we have

$$\begin{aligned} g_k^T d_k &= -g_k^T B_k g_k - g_k^T g_k - \beta_k^{LSCD} g_k^T d_{k-1} + \beta_k^{LSCD} g_k^T d_{k-1} \\ &\le -c_1 \|g_k\|^2 - \|g_k\|^2 = -(c_1 + 1) \|g_k\|^2 \\ &\le -\|g_k\|^2, \quad 0 < c_1 + 1 \le 1, \end{aligned}$$

then

$$g_k^T d_k \le -\|g_k\|^2. \tag{4.2}$$

We conclude that the sufficient descent holds.

Further, from Wolfe line search conditions and (4.2), it holds

$$f(x_k) - f(x_k + \alpha_k d_k) \ge -\rho \alpha_k g_k^T d_k \ge \rho \alpha_k \|g_k\|^2. \tag{4.3}$$

Since $f(x_k)$ is decreasing and the sequence $f(x_k)$ is bounded below and by the condition $H2$, we have

$$\lim_{k\to\infty} f(x_k) - f(x_k + \alpha_k d_k) = 0. \tag{4.4}$$

Hence (4.3) and (4.4) imply

$$\lim_{k\to\infty} \rho\alpha_k \|g_k\|^2 = 0.$$

Now, since $\rho > 0$ and $\alpha_k > 0$, we have

$$\lim_{k\to\infty} \|g_k\|^2 = 0.$$

This completes the proof. □

5. Numerical results

In this section, some numerical results are reported to illustrate the behaviours of WMLSCD, WMMDL and WH-BFGS-CG methods. The step size α_k is determined using the Wolfe line search.

We use the Matlab Langage with a precision $\varepsilon = 10^{-6}$.

We designate by:

- k: The number of iterations required to obtain the solution.
- Time: The execution time in second.

Example 5.1. We take the function

$$f(x) = \sum_{i=1}^{n}(\exp(x_i) - x_i).$$

We take as starting point $x_0 = (1, 1, \ldots, 1)^T$.
The minimum of this function is reached at the point

$$x^* = (0, 0, \ldots, 0)^T \text{ and } f(x^*) = n.$$

The results obtained are summarised in the following tables:
For $n = 3$, we have

Methods	k	Time	$\| g_k \|$
WMLSCD	19	0.149532	$8.0732e-07$
WMMDL	19	0.161138	$8.0732e-07$
WH-BFGS-GC	5	0.073673	$1.4372e-08$

For $n = 100$, we have

Methods	k	Time	$\| g_k \|$
WMLSCD	22	3.883876	$5.8263e-07$
WMMDL	22	3.803220	$5.8263e-07$
WH-BFGS-GC	5	1.622640	$8.2976e-08$

For $n = 500$, we have

Methods	k	Time	$\| g_k \|$
WMLSCD	24	74.325460	$6.4631e-07$
WMMDL	24	70.101070	$6.4631e-07$
WH-BFGS-GC	5	21.087659	$1.8554e-07$

Example 5.2. We take the function

$$f(x) = \sum_{i=1}^{n} \ln(\exp(x_i) + \exp(-x_i)).$$

We take as starting point $x_0 = (1.1, 1.1, \ldots, 1.1)^T$
The minimum of this function is reached at the point

$$x^* = (0, 0, \ldots, 0)^T \text{ and } f(x^*) = n \ln(2).$$

The results obtained are summarised in the following tables:
For $n = 3$, we have

Methods	k	Time	$\| g_k \|$
WMLSCD	96	0.348543	$9.6801e-07$
WMMDL	95	0.443647	$9.5309e-07$
WH-BFGS-GC	47	0.375461	$8.4400e-08$

For $n = 100$, we have

Methods	k	Time	$\| g_k \|$
WMLSCD	104	40.083872	$9.9132e-07$
WMMDL	104	83.918822	$9.9369e-07$
WH-BFGS-GC	66	20.465962	$8.4827e-07$

For $n = 200$, we have

Methods	k	Time	$\| g_k \|$
WMLSCD	107	83.209667	$9.1391e-07$
WMMDL	108	80.273199	$9.2334e-07$
WH-BFGS-GC	69	52.410529	$8.2027e-07$

For $n = 300$, we have

Methods	k	Time	$\| g_k \|$
WMLSCD	109	171.535865	$9.8675e-07$
WMMDL	111	205.430203	$9.5399e-07$
WH-BFGS-GC	70	110.807414	$7.9846e-07$

Commentaries: The numerical tests show clearly that the proposed hybrid algorithm WH-BFGS-GC Wolfe based on line search is more efficient in terms of number of iterations and computation time than WMLSCD and WMMDL methods.

6. Conclusion

We have considered the hybrid conjugate gradient methods, MLSCD, MMDL and H-BFGS-CG, for solving unconstrained optimization problems where we have changed the backtracking line search given in [15] by the Wolfe line search. Firstly, we have shown that the obtained WMLSCD, WMMDL and WH-BFGS-CG algorithms are globally convergent for general functions.

Secondly, the numerical simulations confirm the effectiveness of the approach WH-BFGS-CG. In fact, the WH-BFGS-CG method is the most efficient in terms of number of iterations and computation time compared to WMLSCD and WMMDL methods which was not the case with backtracking line search, where the computation time of H-BFGS-GC was greater than MLSCD and MMDL [15].

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Analysis of quasistatic viscoelastic viscoplastic piezoelectric contact problem with friction and adhesion

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Abstract. In this paper we study the process of bilateral contact with adhesion and friction between a piezoelectric body and an insulator obstacle, the so-called foundation. The material's behavior is assumed to be electro-viscoelastic-viscoplastic; the process is quasistatic, the contact is modeled by a general non-local friction law with adhesion. The adhesion process is modeled by a bonding field on the contact surface. We derive a variational formulation for the problem and then, under a smallness assumption on the coefficient of friction, we prove the existence of a unique weak solution to the model.The proofs are based on a general results on elliptic variational inequalities and fixed point arguments.

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Keywords: Viscoelastic, viscoplastic, piezoelectric, bilateral contact, non local Coulomb friction, adhesion, quasi-variational inequality, weak solution, fixed point.

1. Introduction

A piezoelectric body is one that produces an electric charge when a mechanical stress is applied (the body is squeezed or stretched). Conversely, a mechanical deformation (the body shrinks or expands) is produced when an electric field is applied. This kind of materials appears usually in the industry as switches in radiotronics, electroacoustics or measuring equipments. Piezoelectric materials for which the mechanical properties are elastic are also called electro-elastic materials, those for which the mechanical properties are viscoelastic are also called electro-viscoelastic materials and those for which the mechanical properties are viscoplastic are also called electro-viscoplastic materials. Therfore, a viscoelastic-viscoplastic piezoelectric contact problems are considered. Different models have been developed to describe the

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interaction between the electrical and mechanical fields (see, e.g. [2, 14, 18] and the references therein). A static frictional contact problem for electric-elastic material was considered in [3], under the assumption that the foundation is insulated. Electro-elastic-visco-plastic and elastic-visco-plastic contact problems were recently studied in [13, 15].

Adhesion may take place between parts of the contacting surfaces. It may be intentional, when surfaces are bonded with glue, or unintentional, as a seizure between very clean surfaces. The adhesive contact is modeled by a bonding field on the contact surface, denoted in this paper by β; it describes the pointwise fractional density of active bonds on the contact surface, and sometimes referred to as the intensity of adhesion. Following [11], [12], the bonding field satisfies the restrictions $0 \leq \beta \leq 1$; when $\beta = 1$ at a point of the contact surface, the adhesion is complete and all the bonds are active; when $\beta = 0$ all the bonds are inactive, severed, and there is no adhesion; when $0 < \beta < 1$ the adhesion is partial and only a fraction β of the bonds is active. Basic modelling can be found in [11, 12]. Analysis of models for adhesive contact can be found in [7, 4, 6].

In this work we continue in this line of research, where we extend the result established in [8]. The novelty here lies in the fact that we consider a viscoelastic-viscoplastic piezoelectric body, the contact is bilateral and the friction is described by a nonlocal version of Coulomb's law of dry friction with adhesion. A similar boundary conditions are used in [20], where the constitutive law of the material is viscoelastic.

This paper is structured as follows. In Section 2 we present the viscoelastic-viscoplaastic piezoelectric contact model with friction and adhesion and provide comments on the contact boundary conditions. In Section 3 we list the assumptions on the data and derive the variational formulation. In Section 4, we present our main existence and uniqueness result, Theorem (4.1), which states the unique weak solvability of the contact problem under a smallness assumption on the coefficient of friction.

2. The model

We consider a body made of a piezoelectric material which occupies the domain $\Omega \subset \mathbb{R}^d (d = 2, 3)$ with a smooth boundary $\partial\Omega = \Gamma$ and a unit outward normal ν. The body is acted upon by body forces of density f_0 and has volume free electric charges of density q_0. It is also constrained mechanically and electrically on the boundary. To describe these constraints we assume a partition of Γ into three open disjoint parts Γ_1, Γ_2 and Γ_3, on the one hand, and a partition of $\Gamma_1 \cup \Gamma_2$ into two open parts Γ_a and Γ_b, on the other hand. We assume that $meas\ \Gamma_1 > 0$ and $meas\ \Gamma_a > 0$. The body is clamped on Γ_1 and, therefore, the displacement field vanishes there. Surface tractions of density f_2 act on Γ_2. We also assume that the electrical potential vanishes on Γ_a and a surface electrical charge of density q_2 is prescribed on Γ_b. On Γ_3 the body is in adhesive and frictional contact with an insulator obstacle, the so-called foundation.

We are interested in the deformation of the body on the time interval $[0, T]$. The process is assumed to be quasistatic, i.e. the inertial effects in the equation of motion are neglected. We denote by $x \in \Omega \cup \Gamma$ and $t \in [0, T]$ the spatial and the time variable, respectively, and, to simplify the notation, we do not indicate in what follows the dependence of various functions on x or t. Here and everywhere in this paper, i, j, k, $l = 1, ..., d$, summation over two repeated indices is implied, and the index that follows a comma represents the partial derivative with respect to the corresponding component of x. The dot above variable represents the time derivatives.

We denote by $\mathbb{S}^d$ the space of second-order symmetric tensors on $\mathbb{R}^d$ ($d = 2, 3$) and by ".", $\|.\|$ the inner product and the norm on $\mathbb{S}^d$ and $\mathbb{R}^d$, respectively, that is $u.v = u_i v_i$, $\|v\| = (v.v)^{1/2}$ for $u = (u_i)$, $v = (v_i) \in \mathbb{R}^d$, and $\sigma.\tau = \sigma_{ij}\tau_{ij}$, $\|\sigma\| = (\sigma.\sigma)^{1/2}$ for $\sigma = (\sigma_{ij})$, $\tau = (\tau_{ij}) \in \mathbb{S}^d$. We also use the usual notation for the normal components and the tangential parts of vectors and tensors, respectively, given by $v_\nu = v \cdot \nu$, $v_\tau = v - v_\nu \nu$, $\sigma_\nu = \sigma_{ij}\nu_i\nu_j$, and $\sigma_\tau = \sigma\nu - \sigma_\nu \nu$. With these assumptions, the classical model for the process is the following.

Problem ($\mathcal{P}$). Find a displacement field $u : \Omega \times [0,T] \to \mathbb{R}^d$, a stress field $\sigma : \Omega \times [0,T] \to \mathbb{S}^d$, an electric potential $\varphi : \Omega \times [0,T] \to \mathbb{R}$, an electric displacement field $D : \Omega \times [0,T] \to \mathbb{R}^d$ *and a bonding field* $\beta : \Omega \times [0,T] \to \mathbb{R}$ *such that*

$$\begin{aligned}\sigma(x,t) &= \mathcal{A}\varepsilon(\dot{u}(x,t)) + \mathcal{F}\varepsilon(u(x,t)) \\ &\quad + \int_0^t \mathcal{G}(\sigma(x,s), \varepsilon(u(x,s))ds - \mathcal{E}^*\mathbf{E}(\varphi(x,t)) \qquad \text{in } \Omega \times (0,T),\end{aligned} \tag{2.1}$$

$$D = \mathcal{B}\mathbf{E}(\varphi) + \mathcal{E}\varepsilon(u) \qquad \text{in } \Omega \times (0,T), \tag{2.2}$$

$$Div\sigma + f_0 = 0 \qquad \text{in } \Omega \times (0,T), \tag{2.3}$$

$$divD = q_0 \qquad \text{in } \Omega \times (0,T), \tag{2.4}$$

$$u = 0 \qquad \text{on } \Gamma_1 \times (0,T), \tag{2.5}$$

$$\sigma\nu = f_2 \qquad \text{on } \Gamma_2 \times (0,T), \tag{2.6}$$

$$u_\nu = 0, \qquad \text{on } \Gamma_3 \times (0,T), \tag{2.7}$$

$$\begin{cases} \bullet\ \|\sigma_\tau + \gamma_\tau \beta^2 R_\tau(u_\tau)\| \le \mu p(|R\sigma_\nu|), \\ \bullet\ \|\sigma_\tau + \gamma_\tau \beta^2 R_\tau(u_\tau)\| < \mu p(|R\sigma_\nu|) \\ \quad \Rightarrow \dot{u}_\tau = 0, \\ \bullet\ \|\sigma_\tau + \gamma_\tau \beta^2 R_\tau(u_\tau)\| = \mu p(|R\sigma_\nu|) \\ \quad \Rightarrow \exists\ \lambda > 0, \text{such that:} \\ \quad \sigma_\tau + \gamma_\tau \beta^2 R_\tau(u_\tau) = -\lambda\ \dot{u}_\tau, \end{cases} \qquad \text{on } \Gamma_3 \times (0,T), \tag{2.8}$$

$$\dot{\beta}(t) = -(\beta(t)\gamma_\tau \|R_\tau(u_\tau(t))\|^2 - \varepsilon_a)_+ \qquad \text{on } \Gamma_3 \times (0,T), \tag{2.9}$$

$$\varphi = 0 \qquad \text{on } \Gamma_a \times (0,T), \tag{2.10}$$

$$D.\nu = q_2 \qquad \text{on } \Gamma_b \times (0,T), \tag{2.11}$$

$$D.\nu = 0 \qquad \text{on } \Gamma_3 \times (0,T), \tag{2.12}$$

$$u(0) = u_0 \qquad \text{in } \Omega, \tag{2.13}$$

$$\beta(0) = \beta_0 \qquad \text{on } \Gamma_3. \tag{2.14}$$

Equations (2.1) and (2.2) represent the electro-viscoelastic-viscoplastic constitutive law of the material in which $\sigma = (\sigma_{ij})$ is the stress tensor, $\varepsilon(u) = (\varepsilon_{ij}(u))$ denotes the linearized strain tensor, $\mathcal{A}$ and $\mathcal{F}$ are the elasticity and viscosity tensors, respectivelly, $\mathcal{G}$ denotes a viscoplastic function, $\mathbf{E}(\varphi) = -\nabla\varphi$ is the electric field, $\mathcal{E} = (e_{ijk})$ represents the third-order piezoelectric tensor, $\mathcal{E}^* = (e^*_{ijk})$ where $e^*_{ijk} = e_{kij}$ is its transpose such that:

$$\mathcal{E}\sigma.\upsilon = \sigma.\mathcal{E}^*\upsilon \quad \forall\sigma \in \mathbb{S}^d,\ \upsilon \in \mathbb{R}^d, \tag{2.15}$$

$D = (D_1, ..., D_d)$ is the electric displacement vector and $\mathcal{B} =(\mathcal{B}_{ij})$ denotes the electric permittivity tensor. Equations (2.3) and (2.4) are the equilibrium equations for the stress and electric-displacement fields, respectively, in which "*Div*" and "*div*" denote the divergence operators for tensor and vector valued functions, respectively. Conditions (2.5) and (2.6) are the displacement and traction boundary conditions in which $\sigma\nu$ represents the Cauchy stress vector, whereas (2.10) and (2.11) represent the electric boundary conditions. Note that we need to impose assumption (2.12) for physical reasons. Indeed, this condition models the case when the obstacle is a perfect insulator and was used in [3, 9]. Condition (2.7) represents the bilateral contact, where u_ν represents the normal displacement. Conditions (2.8) is a non local *Coulomb's* law of friction coupled with adhesion in which μ denotes the coefficient of friction and γ_τ is a given adhesion coefficients, u_τ and σ_τ are tangential components of vector u and tensor σ, respectively, σ_ν represents the normal stress, $\dot{u}_\tau$ is the tangential velocity on the bondary, the operator $R : H^{-\frac{1}{2}} \to L^2(\Gamma)$ (see e.eg. [10]) is a linear continuous operator used to regularize the normal trace of stress which is too rough on Γ, p is a non-negative function, the so-called friction bound, and R_τ is the truncation operator defined by

$$R_\tau(\upsilon) = \begin{cases} \upsilon & \text{if } \|\upsilon\| \le L, \\ L\dfrac{\upsilon}{\|\upsilon\|} & \text{if } \|\upsilon\| > L. \end{cases}$$

Here $L > 0$ is the characteristic length of the bond, beyond which it does not offer any additional traction (see e.eg. [19]). The evolution of the bonding field is governed by the differential equation (2.9) with given positive adhesion coefficients γ_τ and ε_a where $r_+ = \max\{0, r\}$. Finally, (2.13) and (2.14) represent the initial conditions in which u_0 and β_0 are the prescribed initial displacement and bonding fields, respectively.

3. Preliminaries and variational formulation

In this section, we list the assumptions on the data and derive a variational formulation for the contact problem. To this end we need to introduce some notation and preliminaries. We use the notation H, H_1, $\mathcal{H}$ and $\mathcal{H}_1$ for the following spaces

$$H = \{\upsilon = (\upsilon_i) \ |\upsilon_i \in L^2(\Omega), i = \overline{1,d}\}, \quad H_1 = \{\upsilon = (\upsilon_i) \mid \varepsilon(\upsilon) \in \mathcal{H}\},$$
$$\mathcal{H} = \{\tau = (\tau_{ij}) \ |\tau_{ij} = \tau_{ji} \in L^2(\Omega), i,j = \overline{1,d}\},\ \mathcal{H}_1 = \{\tau \in \mathcal{H} \ |Div\tau \in H\}.$$

The spaces H, H_1, $\mathcal{H}$ and $\mathcal{H}_1$ are real *Hilbert* spaces endowed with the canonical inner products given by

$$(u,v)_H = \int_\Omega u_i v_i \, dx, \quad (u,v)_{H_1} = (u,v)_H + (\varepsilon(u), \varepsilon(v)_{\mathcal{H}}, .$$
$$(\sigma,\tau)_{\mathcal{H}} = \int_\Omega \sigma_{ij}\tau_{ij} \; dx, \quad (\sigma,\tau)_{\mathcal{H}_1} = (\sigma,\tau)_{\mathcal{H}} + (Div\sigma, Div\tau)_H,$$

such that $\varepsilon : H_1 \longrightarrow \mathcal{H}$ and $Div : \mathcal{H}_1 \longrightarrow H$ are the deformation and divergence operators, respectively defined by

$$\begin{array}{l} \varepsilon(v) = (\varepsilon_{ij}(v)), \quad \varepsilon_{ij}(v) = \frac{1}{2}(v_{i,j} + v_{j,i}) \quad \forall v \in H_1, \\ Div\,\tau = (\tau_{ij,j}) \quad \forall \tau \in \mathcal{H}_1. \end{array}$$

and the associated norms are denoted by $\|\cdot\|_H$, $\|\cdot\|_{H_1}$, $\|\cdot\|_{\mathcal{H}}$ and $\|\cdot\|_{\mathcal{H}_1}$, respectively. We recall that for every element $v \in H_1$ we denote by v the trace γv of v on Γ. If $\sigma \in C^1(\overline{\Omega})^{\mathbb{N}\times\mathbb{N}}$ then, the following *Green's* formula holds

$$(\sigma, \varepsilon(v))_{\mathcal{H}} + (Div\sigma, v)_H = \int_\Gamma \sigma\nu \cdot v \; da, \quad \forall v \in H_1. \tag{3.1}$$

For every real *Hilbert* space X we employ the usual notation for the spaces $L^p(0,T;X)$ and $W^{k,p}(0,T;X)$, $p \in [0,\infty]$, $k = 1,2,...$

We now list the assumptions on the problem's data.

$$\left\{ \begin{array}{ll} (a) & \mathcal{A} = (a_{ijkl}) : \Omega \times \mathbb{S}^d \longrightarrow \mathbb{S}^d \text{ such that} \\ & \mathcal{A}(x,\tau) = (a_{ijkl}(x)\tau_{kl}) \; \forall \; \tau = (\tau_{ij}) \in \mathbb{S}^d, \text{ a.e. } x \in \Omega. \\ (b) & a_{ijkl} = a_{jikl} = a_{klij} \in L^\infty(\Omega), 1 \le i,j,k,l \le d. \\ (c) & \text{there exists } m_{\mathcal{A}} > 0 \text{ such that:} \\ & a_{ijkl}\tau_{ij}\tau_{kl} \ge m_{\mathcal{A}}||\tau||^2 \;\; \forall \; \tau \in \mathbb{S}^d, \text{ a.e. } x \in \Omega. \end{array} \right. \tag{3.2}$$

$$\left\{ \begin{array}{ll} (a) & \mathcal{F} = (f_{ijkl}) : \Omega \times \mathbb{S}^d \longrightarrow \mathbb{S}^d \text{ such that:} \\ & \mathcal{F}(x,\tau) = (f_{ijkl}(x)\tau_{kl}) \; \forall \; \tau = (\tau_{ij}) \in \mathbb{S}^d, \text{ a.e. } x \in \Omega. \\ (b) & f_{ijkl} = f_{jikl} = f_{klij} \in L^\infty(\Omega), 1 \le i,j,k,l \le d. \\ (c) & \text{there exists } m_{\mathcal{A}} > 0 \text{ such that} \\ & f_{ijkl}\tau_{ij}\tau_{kl} \ge m_{\mathcal{F}}||\tau||^2 \;\; \forall \; \tau \in \mathbb{S}^d, \text{ a.e. } x \in \Omega. \end{array} \right. \tag{3.3}$$

$$\left\{ \begin{array}{ll} (a) & \mathcal{E} : \Omega \times \mathbb{S}^d \longrightarrow \mathbb{R}^d \text{ such that:} \\ & \mathcal{E}(x,\varepsilon) = (e_{ijk}(x)\varepsilon_{jk}) \quad \forall \varepsilon = (\varepsilon_{ij}) \in \mathbb{S}^d, \text{ a.e. } x \in \Omega, \\ (b) & e_{ijk} = e_{ikj} \in L^\infty(\Omega). \end{array} \right. \tag{3.4}$$

$$\left\{ \begin{array}{ll} (a) & \mathcal{B} : \Omega \times \mathbb{R}^d \to \mathbb{R}^d \text{ such that:} \\ & \mathcal{B}(x,\mathbf{E}) = (\mathcal{B}_{ij}(x)E_j) \quad \forall \mathbf{E} = (E_i) \in \mathbb{R}^d, \quad \text{a.e. } \mathbf{x} \in \Omega, \\ (c) & \mathcal{B}_{ij} = \mathcal{B}_{ji} \in L^\infty(\Omega), \\ (d) & \text{there exists } m_{\mathcal{B}} > 0 \text{ such that } \mathcal{B}_{ij}(x)E_iE_j \ge m_{\mathcal{B}}\|\mathbf{E}\|^2 \\ & \forall \mathbf{E} = (E_i) \in \mathbb{R}^d, \text{ a.e. } x \in \Omega. \end{array} \right. \tag{3.5}$$

$$\begin{cases} \text{(a)} & p : \Gamma_3 \times \mathbb{R} \longrightarrow \mathbb{R}_+. \\ \text{(b)} & \text{there exists } L_p > 0 \text{ such that} \\ & |p(x, r_1) - p(x, r_2)| \leq L_p |r_1 - r_2|, \\ & \forall r_1,\ r_2 \in \mathbb{R},\ \text{a.e. } x \in \Gamma_3, \\ \text{(c)} & x \longmapsto p(x, r) \text{ is Lebesgue measurable on } \Gamma_3, \\ \text{(d)} & \text{the mapping } x \longmapsto p(x, 0) \in L^2(\Gamma_3). \end{cases} \tag{3.6}$$

$$\begin{cases} \text{(a)} & \mathcal{G} : \Omega \times \mathbb{S}^d \times \mathbb{S}^d \longrightarrow \mathbb{S}^d \\ \text{(b)} & \text{there exists } L_{\mathcal{G}} > 0 \text{ such that} \\ & \|\mathcal{G}(x, \sigma_1, \varepsilon_1) - \mathcal{G}(x, \sigma_2, \varepsilon_2)\| \leq L_{\mathcal{G}} \|\sigma_1 - \sigma_2\| \\ & \forall\ \sigma_1, \sigma_2, \varepsilon_1, \varepsilon_2 \in \mathbb{S}^d,\ \text{a.e. } x \in \Omega, \\ \text{(b)} & \text{for any } \sigma, \varepsilon \in \mathbb{S}^d,\ x \longmapsto \mathcal{G}(x, \sigma, \varepsilon) \text{ is measurable,} \\ \text{(c)} & \text{the mapping } x \longmapsto \mathcal{G}(x, 0, 0) \text{ belongs to } \mathcal{H}. \end{cases} \tag{3.7}$$

The forces, tractions, volume and surface free charge densities satisfy

$$f_0 \in W^{1,2}(0, T; H), \quad f_2 \in W^{1,2}(0, T; L^2(\Gamma_2)^d), \tag{3.8}$$

$$q_0 \in W^{1,2}(0, T; L^2(\Omega)), \quad q_2 \in W^{1,2}(0, T; L^2(\Gamma_b)). \tag{3.9}$$

The adhesion coefficient γ_τ and the limit bound ε_a satisfy the conditions

$$\gamma_\tau \in L^\infty(\Gamma_3), \quad \epsilon_a \in L^2(\Gamma_3), \quad \gamma_\tau,\ \epsilon_a \geq 0 \quad \text{a.e. on } \Gamma_3. \tag{3.10}$$

Also, we assume that the initial bonding field satisfies the condition

$$\beta_0 \in L^2(\Gamma_3),\ 0 \leq \beta_0 \leq 1 \ \text{ a.e. on } \Gamma_3, \tag{3.11}$$

Finally, the coefficient of friction μ is assumed to satisfy

$$\mu \in L^\infty(\Gamma_3), \quad \mu(x) \geq 0 \quad \text{a.e. on } \Gamma_3. \tag{3.12}$$

Let now consider the closed subspace of H_1 defined by

$$V = \{\ v \in H_1 \mid v = 0 \text{ on } \Gamma_1,\ v_\nu = 0 \text{ on } \Gamma_3\}. \tag{3.13}$$

Since $meas\,(\Gamma_1) > 0$, the following Korn's inequality holds

$$\|\varepsilon(v)\|_{\mathcal{H}} \geq C_K \|v\|_{H_1} \quad \forall v \in V, \tag{3.14}$$

where the proof my be found in [16] (p. 79). Equiping V with the inner product

$$(u, v)_V = (\varepsilon(u), \varepsilon(v))_{\mathcal{H}}, \tag{3.15}$$

and let $\|\cdot\|_V$ be the associated norm. We deduce from *Korn's* inequality that $\|.\|_{H_1}$ and $\|.\|_V$ are eauivalente norme on V. Then $(V, \|.\|_V)$ is a real *Hilbert* space. Next, we assume that the initial displacement satisfies the condition

$$u_0 \in\ V. \tag{3.16}$$

We also introduce the following spaces

$$W = \{\ \psi \in H^1(\Omega) \mid \psi = 0 \text{ on } \Gamma_a\ \}, \tag{3.17}$$

$$\mathcal{W} = \{\ D = (D_i) \mid D_i \in L^2(\Omega),\ div\ D \in L^2(\Omega)\}. \tag{3.18}$$

Since $meas\,(\Gamma_a) > 0$ it is well known that W is a real *Hilbert* space endowed with the inner product

$$(\varphi, \psi)_W = (\nabla\varphi, \nabla\psi)_{L^2(\Omega)^d}, \tag{3.19}$$

and the associated norm is $\|\cdot\|_W$. Also we have the following *Friedrichs-Poincaré* inequality

$$\|\nabla\psi\|_{L^2(\Omega)^d} \geq C_F\,\|\psi\|_{H^1(\Omega)} \quad \forall\,\psi \in W, \tag{3.20}$$

where $C_F > 0$ is a constant which depends only on Ω and Γ_a. The space $\mathcal{W}$ is a real *Hilbert* space endowed with the inner product

$$(D, \mathbf{E})_{\mathcal{W}} = \int_\Omega D \cdot \mathbf{E}\, dx + \int_\Omega divD \cdot div\mathbf{E}\, dx,$$

and the associated norm is $\|\cdot\|_{\mathcal{W}}$. Moreover, by the *Sobolev* trace theorem, there exist two positive constants C_0 and $\tilde{C}_0$ depending only on Ω, Γ_1 and Γ_3 such that

$$\|v\|_{L^2(\Gamma_3)^d} \leq C_0\|v\|_V \quad \forall v \in V, \quad \|\psi\|_{L^2(\Gamma_3)} \leq \tilde{C}_0\|\psi\|_W \quad \forall\,\psi \in W. \tag{3.21}$$

It follows from proprieties of R that there existe a constant C_R depending only on Ω, Γ_3 and R such that

$$\|R\sigma_\nu\|_{L^2(\Gamma_3)} \leq C_R\|\sigma_\nu\|_{\mathcal{H}_1} \quad \forall \sigma \in \mathcal{H}_1. \tag{3.22}$$

Next, we define the two mappings $f : [0, T] \longrightarrow V$ and $q : [0, T] \longrightarrow W$, respectively, by

$$(f(t), v)_V = \int_\Omega f_0(t) \cdot v\, dx + \int_{\Gamma_2} f_2(t) \cdot v\, da, \tag{3.23}$$

$$(q(t), \psi)_W = \int_\Omega q_0(t)\psi\, dx - \int_{\Gamma_b} q_2(t)\psi\, da, \tag{3.24}$$

for all $v \in V$, $\psi \in W$ and $t \in [0, T]$. We note that the definitions of f and q are based on the *Riesz* representation theorem. Moreover, it follows from assumptions (3.8) and (3.9) that

$$f \in W^{1,2}(0, T; V), \tag{3.25}$$

$$q \in W^{1,2}(0, T; W). \tag{3.26}$$

Also, we introduce the set

$$\mathcal{Q} = \{\beta \in L^\infty(0, T; L^2(\Gamma_3)) \;/\; 0 \leq \beta(t) \leq 1 \;\forall\, t \in [0, T],\; a.e.\ on\ \Gamma_3\}. \tag{3.27}$$

Now, let us define the adhesion functional $j_{ad} : L^2(\Gamma_3) \times V \times V \longrightarrow \mathbb{R}$ and the friction functional $j_{fr} : \mathcal{H}_1 \times V \longrightarrow \mathbb{R}$, respectivelly, by

$$j_{ad}(\beta, u, v) = \int_{\Gamma_3} \gamma_\tau \beta^2 R_\tau(u_\tau) \cdot v_\tau da, \tag{3.28}$$

$$j_{fr}(\sigma, v) = \int_{\Gamma_3} \mu p(|R\sigma_\nu|) \cdot \|v_\tau\| da. \tag{3.29}$$

Using a standard procedure based on *Green's formulas* (see (3.1)) we can derive the following variational formulation of the problem (2.1)–(2.14).

Problem ($\mathcal{P}^V$). Find a displacement field $u : [0,T] \to V$, a stress field $\sigma : \Omega \times [0,T] \to \mathcal{H}$, an electric potential $\varphi : [0,T] \to W$, *and a bonding field* $\beta : [0,T] \to L^2(\Gamma_3)$ *such that*

$$\sigma(t) = \mathcal{A}\varepsilon(\dot{u}(t)) + \mathcal{F}\varepsilon(u(t)) + \int_0^t \mathcal{G}(\sigma(x,s), \varepsilon(u(x,s))ds - \mathcal{E}^* \boldsymbol{E}(\varphi(t)) \tag{3.30}$$

$$\begin{gathered}(\sigma(t), \varepsilon(\omega) - \varepsilon(\dot{u}(t))_{\mathcal{H}} + j_{ad}(\beta(t), u(t), \omega - \dot{u}(t)) \\ + j_{fr}(\sigma(t), \omega) - j_{fr}(\sigma(t), \dot{u}(t)) \geq (f(t), \omega - \dot{u}(t))_V, \\ \forall\, v \in V, \quad \forall t \in [0,T],\end{gathered} \tag{3.31}$$

$$\begin{gathered}(\mathcal{B}\nabla\varphi(t), \nabla\psi)_{L^2(\Omega)^d} - (\mathcal{E}\varepsilon(u(t), \nabla\psi)_H = (q(t), \psi)_W, \\ \forall \psi \in W,\ \forall t \in [0,T],\end{gathered} \tag{3.32}$$

$$\dot{\beta}(t) = -(\gamma_\tau \beta(t) \left\| R_\tau(u_\tau(t)) \right\|^2) - \epsilon_a)_+ \ , \text{ a.e. } t \in (0,T), \tag{3.33}$$

$$u(0) = u_0 \tag{3.34}$$

$$\beta(0) = \beta_0. \tag{3.35}$$

4. Existence and uniqueness result

Theorem 4.1. *Assume that* (3.2)*–*(3.12) *and* (3.16) *hold. Then, there exists a constant* $\mu_0 > 0$ *such that Problem* $\mathcal{P}^V$ *has a unique solution* $(u, \sigma, \varphi,\, \beta)$ *if* $\|\mu\|_{L^\infty(\Gamma_3)} < \mu_0$*. Moreover, the solution satisfies*

$$\boldsymbol{u} \in W^{2,2}(0,T;V), \tag{4.1}$$

$$\sigma \in W^{1,2}(0,T;\mathcal{H}_1), \tag{4.2}$$

$$\varphi \in W^{1,2}(0,T;W). \tag{4.3}$$

$$\beta \in W^{1,\infty}(0,T;L^2(\Gamma_3)) \cap \mathcal{Q}. \tag{4.4}$$

A quintuple of functions $(u,\, \sigma,\, \varphi,\, D,\, \beta)$ which satisfies (2.1), (2.2) and (3.30), (3.35) is called a *weak solution* of the contact Problem ($\mathcal{P}$). We conclude by Theorem (4.1) that, under the assumptions (3.2)–(3.12) and (3.16), there exists a unique weak solution of Problem ($\mathcal{P}$). To precise the regularity of the weak solution we note that the constitutive relations (2.2), the assumptions (3.4)–(3.5) and the regularity (4.3) implies that $\boldsymbol{D} \in W^{1,2}(0,T;L^2(\Omega)^d)$. Moreover, using again (2.2) combined with (3.32) and the notation (3.24) and choosing $\psi \in C_0^\infty(\Omega)$ we find that *div* $D(t) = q_0(t)$ for all $t \in [0,T]$. It follows now from the regularities (3.9) that $divD \in W^{1,2}(0,T;L^2(\Omega))$, which shows that

$$D \in W^{1,2}(0,T;\mathcal{W}). \tag{4.5}$$

We conclude that the weak solution $(u, \sigma, \varphi, D, \beta)$ of the piezoelectric contact problem ($\mathcal{P}$) has the regularity (4.1)–(4.5).

The proof of Theorem(4.1) will be carried out in several steps. We assume in the following that the conditions, (3.2)–(3.12) and (3.16), of Theorem(4.1) hold and below we denote by "c" a generic positive constant which is independent of time and whose value may change from place to place. In the first step, let $\eta \in W^{1,2}(0,T;V)$,

$\kappa \in L^2(0,T;\mathcal{H})$ and $\lambda \in W^{1,2}(0,T;\mathcal{H}_1)$ be a given functions. We introduce the function $z_\kappa \in W^{1,2}(0,T;\mathcal{H})$ defined by

$$z_\kappa(t) = \int_0^t \kappa(s)ds \quad \forall t \in [0,T], \tag{4.6}$$

and we consider the following intermediate problem.

Problem ($\mathcal{P}_1^V$). Find $u_{\kappa\eta\lambda} : [0,T] \to V$ and $\sigma_{\kappa\eta\lambda} : [0,T] \to \mathcal{H}_1$ *such that*

$$\sigma_{\kappa\eta\lambda}(t) = \mathcal{A}\varepsilon(\dot{u}_{\kappa\eta\lambda}(t)) + \mathcal{F}\varepsilon(u_{\kappa\eta\lambda}(t)) + z_\kappa(t) + \varepsilon(\eta(t)). \tag{4.7}$$

$$\begin{gathered}(\mathcal{A}\varepsilon(\dot{u}_{\kappa\eta\lambda}(t)), \varepsilon(\omega) - \varepsilon(\dot{u}_{\kappa\eta\lambda}(t))_{\mathcal{H}} + (\mathcal{F}\varepsilon(u_{\kappa\eta\lambda}(t)), \varepsilon(\omega) - \varepsilon(\dot{u}_{\kappa\eta\lambda}(t))_{\mathcal{H}} \\ +(z_\kappa(t), \varepsilon(\omega) - \varepsilon(\dot{u}_{\kappa\eta\lambda}(t))_{\mathcal{H}} + (\varepsilon(\eta(t)), \varepsilon(\omega) - \varepsilon(\dot{u}_{\kappa\eta\lambda}(t))_{\mathcal{H}} \\ +j_{fr}(\lambda(t), \omega) - j_{fr}(\lambda(t), \dot{u}_{\kappa\eta\lambda}(t)) \geq (f(t), \omega - \dot{u}_{\kappa\eta\lambda}(t))_V \\ \forall\, \omega \in V, \quad \forall t \in [0,T],\end{gathered} \tag{4.8}$$

$$u_{\kappa\eta\lambda}(0) = u_0. \tag{4.9}$$

Lemma 4.1. *Problem $\mathcal{P}_1^V$ has a unique solution $(u_{\kappa\eta\lambda}, \sigma_{\kappa\eta\lambda})$. Moreover, the solution satisfies*

$$\begin{aligned}&\mathbf{a})u_{\kappa\eta\lambda} \in W^{2,2}(0,T;V), \\ &\mathbf{b})\sigma_{\kappa\eta\lambda} \in W^{1,2}(0,T;\mathcal{H}_1), \\ &\mathbf{c})Div\sigma_{\kappa\eta\lambda} + f_0 = 0.\end{aligned} \tag{4.10}$$

Proof. We denote by $\tilde{\sigma}_{\kappa\eta\lambda}$ and j_λ the elements given by

$$\tilde{\sigma}_{\kappa\eta\lambda}(t) \quad = \sigma_{\kappa\eta\lambda}(t) - z_\kappa(t) - \varepsilon(\eta(t)). \tag{4.11}$$

$$j_\lambda(\omega) \quad = j_{fr}(\lambda, \omega) \quad \forall \omega \in V. \tag{4.12}$$

By (3.15) and Riesz's representation theorem we deduce that there exists an element $f_{\kappa\eta} \in W^{1,2}(0,T;V)$ such that

$$(f_{\kappa\eta}(t), \upsilon)_V = (f(t) - \eta(t), \upsilon)_V + (z_\kappa(t), \varepsilon(\upsilon))_{\mathcal{H}}. \tag{4.13}$$

Since $f, \eta \in W^{1,2}(0,T;V)$ and $z_\kappa \in W^{1,2}(0,T;\mathcal{H})$ we deduce that $f_{\kappa\eta} \in W^{1,2}(0,T;V)$. Moreover, using (4.7), (4.8), (4.9), (4.11) and (4.12) leads us to consider the following variational problem.

Problem ($\mathcal{P}_2^V$). Find $u_{\kappa\eta\lambda} : [0,T] \to V$ and $\tilde{\sigma}_{\kappa\eta\lambda} : [0,T] \to \mathcal{H}_1$ *such that*

$$\tilde{\sigma}_{\kappa\eta\lambda}(t) = \mathcal{A}\varepsilon(\dot{u}_{\kappa\eta\lambda}(t)) + \mathcal{F}\varepsilon(u_{\kappa\eta\lambda}(t)). \tag{4.14}$$

$$\begin{gathered}(\tilde{\sigma}_{\kappa\eta\lambda}(t), \varepsilon(\omega) - \varepsilon(\dot{u}_{\kappa\eta\lambda}(t))_{\mathcal{H}} + j_\lambda(\omega) - j_\lambda(\dot{u}_{\kappa\eta\lambda}(t)) \\ \geq (f_{\kappa\eta}(t), \omega - \dot{u}_{\kappa\eta\lambda}(t))_V \quad \forall\, \omega \in V, \quad \forall t \in [0,T],\end{gathered}$$

$$u_{\kappa\eta\lambda}(0) = u_0, \tag{4.15}$$

Note that V is a closed subspace of H_1 and the fonctional j_λ is convex lower semicontinuous on V such that $j \neq +\infty$. By a classical results for elliptic variational inequalities (see e.g. [5], Theorem (4.1) page 348) there exists a unique solution $(u_{\kappa\eta\lambda}, \tilde{\sigma}_{\kappa\eta\lambda})$ for the variational problem $\mathcal{P}_2^V$ stisfying the regularity condition

$$u_{\kappa\eta\lambda} \in W^{2,2}(0,T;V), \; \tilde{\sigma}_{\kappa\eta\lambda} \in W^{1,2}(0,T;\mathcal{H}_1). \tag{4.16}$$

Next, kepping in mind (4.7) we put $\omega = \dot{u}_{\kappa\eta\lambda}(t) \pm \upsilon$ where $\upsilon \in \mathcal{D}(\Omega)^d$ in (4.8) to obtain $Div\sigma_{\kappa\eta\lambda} + f_0 = 0$.

Finally, we deduce that $(u_{\kappa\eta\lambda}, \sigma_{\kappa\eta\lambda})$ is the unique solution of the variational problem $\mathcal{P}_1^V$ stisfying condition (4.10), which concludes the proof of Lemma (4.1). □

In the second step we use the displacement field $u_{\kappa\eta\lambda}$ obtained in Lemma(4.1) to obtain the following existence and uniqueness result for the electric potential field.

Lemma 4.2. *There exists a unique function* $\varphi_{\kappa\eta\lambda} \in W^{1,2}(0,T;W)$ *such that*

$$(\mathcal{B}\nabla\varphi_{\kappa\eta\lambda}(t), \nabla\psi)_{L^2(\Omega)^d} - (\mathcal{E}\varepsilon(u_{\kappa\eta\lambda}(t)), \nabla\psi)_{L^2(\Omega)^d} = (q(t), \psi)_W \quad \forall\psi \in W, \quad \forall\, t \in [0\ T], \tag{4.17}$$

Moreover, if φ_1 *and* φ_2 *are the solution of (*4.17*) for* $u_1, u_2 \in W^{2,2}(0,T;V)$*, respectivelly, then we have*

$$\|\varphi_1(t) - \varphi_2(t)\|_W \le c\|u_1(t) - u_2(t)\|_V ds, \quad \forall\, t \in [0,T], \quad a.e.\ on\ \Gamma_3 \tag{4.18}$$

Proof. Let $u_{\kappa\eta\lambda} \in W^{2,2}(0,T;V)(0,T;V)$ be the function defined in Lemma (4.1). As in [1], using *Riesz's* representation theorem we may define the operator $\mathcal{L}_{\kappa\eta\lambda} : W \longrightarrow W$ by

$$(\mathcal{L}_{\kappa\eta\lambda}(\varphi(t)), \psi)_W = (\mathcal{B}\nabla\varphi(t), \nabla\psi)_{L^2(\Omega)^d} - (\mathcal{E}\varepsilon(u_{\eta\lambda}(t)), \nabla\psi)_{L^2(\Omega)^d} \quad \forall\psi \in W,\ \forall\, t \in [0,T]. \tag{4.19}$$

It follows from assumptions (3.4) and (3.5) that the operator $\mathcal{L}_{\kappa\eta\lambda}$ is stongly monotone Lipschitz continuous on W. Then, we deduce that there exists a unique element $\varphi_{\kappa\eta\lambda}(t) \in W$ satisfies,

$$\mathcal{L}_{\kappa\eta\lambda}(\varphi_{\kappa\eta\lambda}(t)) = q(t) \quad \forall t\ \in [0,T]. \tag{4.20}$$

Thus, it follows from (4.19) and (4.20) that $\varphi_{\kappa\eta\lambda}(t) \in W$ is the unique solution of equation (4.17). Let now $t_1, t_2 \in [0,T]$ and for the sake of simplicity we use the notations $\varphi_i = \varphi_{\kappa\eta\lambda}(t_i)$, $u_i = u_{\kappa\eta\lambda}(t_i)$, $q_i = q(t_i)$ for $i = 1,2$. Using (4.17), (3.4) and (3.5) we find that

$$\|\varphi_1 - \varphi_2\|_W \le c(\|u_1 - u_2\|_V + \|q_1 - q_2\|_W),$$

the previous inequality yields

$$\|\varphi_{\kappa\eta\lambda}(t_1) - \varphi_{\kappa\eta\lambda}(t_2)\|_W \le c(\|u_{\kappa\eta\lambda}(t_1) - u_{\kappa\eta\lambda}(t_2)\|_V + \|q(t_1) - q(t_2)\|_W). \tag{4.21}$$

Since $u_{\kappa\eta\lambda} \in W^{2,2}(0,T;V)$ and $q \in W^{1,2}(0,T;W)$, it follows that

$$\varphi_{\kappa\eta\lambda} \in W^{1,2}(0,T;W).$$

Assume now that φ_1 and φ_2 are the solution of (4.17) for $u_1, u_2 \in W^{2,2}(0,T;V)$, respectively. Arguments similar to those used in proof of (4.21) leads to (4.18), which concludes the proof of Lemma (4.2). □

In the third step, for $u_{\kappa\eta\lambda}$ obtained in Lemma (4.1), we solve equation (3.33) for the adhesion field.

Problem ($\mathcal{P}^{\beta_{\kappa\eta\lambda}}$). Find a bonding field $\beta_{\kappa\eta\lambda} : [0,T] \to L^2(\Gamma_3)$ such that

$$\dot{\beta}_{\kappa\eta\lambda}(t) = -(\gamma_\tau \beta_{\kappa\eta\lambda}(t) \|R_\tau(u_{\kappa\eta\lambda\tau}(t))\|^2 - \varepsilon_a)_+ \text{ a.e. } t \in (0,T), \tag{4.22}$$

$$\beta_{\kappa\eta\lambda}(0) = \beta_0. \tag{4.23}$$

Lemma 4.3. *There exists a unique solution $\beta_{\kappa\eta\lambda}$ to Problem $\mathcal{P}^{\beta_{\kappa\eta\lambda}}$ satisfing $\beta_{\kappa\eta\lambda} \in W^{1,\infty}(0,T, L^2(\Gamma_3)) \cap \mathcal{Q}$. Moreover, if β_1 and β_2 are the solution of (4.22)-(4.23) for $u_1, u_2 \in W^{2,2}(0,T;V)$, respectivelly, then we have*

$$\begin{aligned} \|\beta_1(t) - \beta_2(t)\|_{L^2(\Gamma_3)} \leq c \int_0^t \|u_1(s) - u_2(s)\|_V ds, \\ \forall\ t \in [0,T], \quad a.e. \text{ on } \Gamma_3 \end{aligned} \tag{4.24}$$

Proof. The proof of Lemma 4.3 is based on a version of *Cauchy-Lipschitz* theorem (see, e.g., [17], page 48), by arguments similar to those used in [7]. □

In the fourth step, for $\eta \in W^{1,2}(0,T;V)$, $\kappa \in L^2(0,T;\mathcal{H})$ and $\lambda \in W^{1,2}(0,T;\mathcal{H}_1)$ we denote by $u_{\kappa\eta\lambda}$, $\varphi_{\kappa\eta\lambda}$ and $\beta_{\kappa\eta\lambda}$ the functions obtained in Lemmas (4.1), (4.2) and (4.3), respectively. We now define the operator $\Lambda_{\kappa\eta} : L^2(0,T;\mathcal{H}_1) \longrightarrow L^2(0,T;\mathcal{H}_1)$ by

$$\Lambda_{\kappa\eta}\lambda = \sigma_{\kappa\eta\lambda}. \tag{4.25}$$

Lemma 4.4. *For all $\lambda \in L^2(0,T;\mathcal{H}_1)$ the function $\Lambda_{\kappa\eta}\lambda$ belongs to $W^{1,2}(0,T;\mathcal{H}_1)$. Moreover, The operator $\Lambda_{\kappa\eta}$ has a unique fixed point $\lambda_{\kappa\eta} \in W^{1,2}(0,T;\mathcal{H}_1)$.*

Proof. Let $t_1, t_2 \in [0,T]$. Keeping in mind (3.2), (3.3), (3.15) and using (4.7) written for $t = t_1$ and $t = t_2$ we find that

$$\begin{aligned} \|\sigma_{\kappa\eta\lambda}(t_1) - \sigma_{\kappa\eta\lambda}(t_2)\|_{\mathcal{H}} \leq{}& c(\|\dot{u}_{\kappa\eta\lambda}(t_1) - \dot{u}_{\kappa\eta\lambda}(t_2)\|_V + \|u_{\kappa\eta\lambda}(t_1) - u_{\kappa\eta\lambda}(t_2)\|_V \\ &+ \|z_\kappa(t_1) - z_\kappa(t_2)\|_{\mathcal{H}} + \|\eta(t_1) - \eta(t_2)\|_V). \end{aligned} \tag{4.26}$$

On the other hand, we have

$$\begin{aligned} \|\sigma_{\kappa\eta\lambda}(t_1) - \sigma_{\kappa\eta\lambda}(t_2)\|_{\mathcal{H}_1} \leq{}& \|\sigma_{\kappa\eta\lambda}(t_1) - \sigma_{\kappa\eta\lambda}(t_2)\|_{\mathcal{H}} \\ &+ \|Div\sigma_{\kappa\eta\lambda}(t_1) - Div\sigma_{\kappa\eta\lambda}(t_2)\|_H, \end{aligned}$$

using (4.10)(c), (4.26) and the previous inequality we obtain

$$\begin{aligned} \|\sigma_{\kappa\eta\lambda}(t_1) - \sigma_{\kappa\eta\lambda}(t_2)\|_{\mathcal{H}_1} \leq{}& c(\|\dot{u}_{\kappa\eta\lambda}(t_1) - \dot{u}_{\kappa\eta\lambda}(t_2)\|_V + \|u_{\kappa\eta\lambda}(t_1) - u_{\kappa\eta\lambda}(t_2)\|_V \\ &+ \|z_\kappa(t_1) - z_\kappa(t_2)\|_{\mathcal{H}} + \|\eta(t_1) - \eta(t_2)\|_V) \\ &+ \|f_0(t_1) - f_0(t_2)\|_H. \end{aligned} \tag{4.27}$$

Now, we get from (4.25) that

$$\begin{aligned} \|\Lambda_{\kappa\eta}\lambda(t_1) - \Lambda_{\kappa\eta}\lambda(t_2)\|_{\mathcal{H}_1} \leq{}& c(\|\dot{u}_{\kappa\eta\lambda}(t_1) - \dot{u}_{\kappa\eta\lambda}(t_2)\|_V + \|u_{\kappa\eta\lambda}(t_1) - u_{\kappa\eta\lambda}(t_2)\|_V \\ &+ \|z_\kappa(t_1) - z_\kappa(t_2)\|_{\mathcal{H}} + \|\eta(t_1) - \eta(t_2)\|_V) \\ &+ \|f_0(t_1) - f_0(t_2)\|_H. \end{aligned} \tag{4.28}$$

Since

$$\dot{u}_{\kappa\eta\lambda} \in W^{1,2}(0,T;V),\ u_{\kappa\eta\lambda} \in W^{2,2}(0,T;V),\ z_\kappa \in W^{1,2}(0,T;\mathcal{H}),\ \eta \in W^{1,2}(0,T;V)$$

and $f_0 \in W^{1,2}(0,T;H)$, it follows that

$$\Lambda_{\kappa\eta}\lambda \in W^{1,2}(0,T;\mathcal{H}_1) \tag{4.29}$$

Let now $\lambda_1, \lambda_2 \in L^2(0,T;\mathcal{H}_1)$ and let $t \in [0,T]$. We use the notation $u_i = u_{\kappa\eta\lambda_i}$, $\sigma_i = \sigma_{\kappa\eta\lambda_i}$ $\dot{u}_i = \dot{u}_{\kappa\eta\lambda_i}$ for $i = 1,2$. In (4.8) written for $\lambda = \lambda_1$, we take $\omega = \dot{u}_2$, and also written for $\lambda = \lambda_2$, we take $\omega = \dot{u}_1$. After adding the resulting inequalities and using (3.2), (3.3), (3.6), (3.12), (3.15), (3.21), (3.22), (3.29) with some elementary calculus we find that

$$\|\dot{u}_1(t) - \dot{u}_2(t)\|_V \leq \frac{L_p C_0 C_R \|\mu\|_{L^\infty(\Gamma_3)}}{m_{\mathcal{A}}}\|\lambda_1(t) - \lambda_2(t)\|_{\mathcal{H}_1} + \frac{C_{\mathcal{F}}}{m_{\mathcal{A}}}\int_0^t \|\dot{u}_1(s) - \dot{u}_2(s)\|_V ds, \tag{4.30}$$

and, after a Gronwall argument, we obtain

$$\|\dot{u}_1(t) - \dot{u}_2(t)\|_V \leq \frac{L_p C_0 C_R \|\mu\|_{L^\infty(\Gamma_3)}}{m_{\mathcal{A}}}\|\lambda_1(t) - \lambda_2(t)\|_{\mathcal{H}_1}. \tag{4.31}$$

Next, from (4.10)(c) we have $Div\sigma_1(t) = Div\sigma_2(t)$. Moreover, using (4.7), (3.2), (3.3), (3.15) and (3.21) we obtain

$$\|\sigma_1(t) - \sigma_2(t)\|_{\mathcal{H}_1} = \|\sigma_1(t) - \sigma_2(t)\|_{\mathcal{H}} \leq c(\|\dot{u}_1(t) - \dot{u}_2(t\|_V + \|u_1(t) - u_2(t)\|_V). \tag{4.32}$$

Now, using using (4.32) and Young's inequality we obtain

$$\|\sigma_1(t) - \sigma_2(t)\|^2_{\mathcal{H}_1} \leq c(\|\dot{u}_1(t) - \dot{u}_2(t)\|^2_V + \|u_1(t) - u_2(t)\|^2_V),$$

where, we deduce by using (4.25) that

$$\|\Lambda_{\kappa\eta}\lambda_1(t) - \Lambda_{\kappa\eta}\lambda_2(t)\|^2_{\mathcal{H}_1} \leq c(\|\dot{u}_1(t) - \dot{u}_2(t)\|^2_V + \int_0^t \|\dot{u}_1(s) - \dot{u}_2(s)\|^2_V ds). \tag{4.33}$$

We combine now (4.31) and (4.33) to obtain

$$\|\Lambda_{\kappa\eta}\lambda_1(t) - \Lambda_{\kappa\eta}\lambda_2(t)\|^2_{\mathcal{H}_1} \leq c(\|\lambda_1(t) - \lambda_2(t)\|^2_{\mathcal{H}_1} + \int_0^t \|\lambda_1(s) - \lambda_2(s)\|^2_{\mathcal{H}_1} ds),$$

and, reiterating this inequality m times, yields

$$\|\Lambda^m_{\kappa\eta}\lambda_1 - \Lambda^m_{\kappa\eta}\lambda_2\|^2_{L^2(0,T;\mathcal{H}_1)} \leq \frac{c^m(m+T)^m}{m!}\|\lambda_1 - \lambda_2\|^2_{L^2(0,T;\mathcal{H}_1)},$$

which implies that for m sufficiently large, $\Lambda^m_{\kappa\eta}$ is contraction on the Banach space $L^2(0,T;\mathcal{H}_1)$. Therefore, there exists a unique $\lambda_{\kappa\eta} \in L^2(0,T;\mathcal{H}_1)$ such that $\Lambda^m_{\kappa\eta}\lambda_{\kappa\eta} = \lambda_{\kappa\eta}$ where we deduce that $\lambda_{\kappa\eta}$ is the unique fixed point of $\Lambda_{\kappa\eta}$. Moreover, equality (4.25) implies that $\lambda_{\kappa\eta} \in W^{1,2}(0,T;\mathcal{H}_1)$, which concludes the proof of Lemma (4.4). □

Now, let $\lambda_{\kappa\eta}$ the fixed point of the operator $\Lambda_{\kappa\eta}$. We use *Riesz*'s representation theorem to define the operator $\mathit{\Lambda}_\kappa : L^2(0,T;V) \longrightarrow L^2(0,T;V)$ by

$$(\Lambda_\kappa\eta(t), \upsilon)_V = j(\beta_{\kappa\eta\lambda_{\kappa\eta}}(t), u_{\kappa\eta\lambda_{\kappa\eta}}(t), \upsilon) + (\mathcal{E}^*\boldsymbol{E}(\varphi_{\kappa\eta\lambda_{\kappa\eta}}(t)), \varepsilon(\upsilon)), \tag{4.34}$$

for all $\upsilon \in V$ and $t \in [0,T]$. We have the following result.

Lemma 4.5. *For all $\eta \in L^2(0,T;V)$ the function $\Lambda_\kappa\eta$ belongs to $W^{1,2}(0,T;V)$. Moreover, there exists a constant $\mu_0 > 0$ such that the operator Λ_κ has a unique fixed point $\eta_\kappa \in W^{1,2}(0,T;V)$ if $\|\mu\|_{L^\infty(\ \Gamma_3)} \leq \mu_0$.*

Proof. Let $\eta \in L^2(0,T;V)$ and let $t_1, t_2 \in [0,T]$. Using (4.34), (3.28), (3.21) and keeping in mind the inequality $0 \leq \beta_{\kappa\eta\lambda_{\kappa\eta}}(t) \leq 1$ and the properies of the operators R_ν, R_τ and $\mathcal{E}^*$ we find that

$$\begin{aligned} \|\Lambda_\kappa\eta(t_1) - \Lambda_\kappa\eta(t_2)\|_V \leq c(&\|u_{\kappa\eta\lambda_{\kappa\eta}}(t_1) - u_{\kappa\eta\lambda_{\kappa\eta}}(t_2)\|_V \\ &+ \|\beta_{\kappa\eta\lambda_{\kappa\eta}}(t_1) - \beta_{\kappa\eta\lambda_{\kappa\eta}}(t_2)\|_{L^2(\Gamma_3)} \\ &+ \|\varphi_{\kappa\eta\lambda_{\kappa\eta}}(t_1) - \varphi_{\kappa\eta\lambda_{\kappa\eta}}(t_2)\|_W). \end{aligned} \tag{4.35}$$

Since

$$u_{\kappa\eta\lambda_{\kappa\eta}} \in W^{2,2}(0,T;V),\ \beta_{\kappa\eta\lambda_{\kappa\eta}} \in W^{1,\infty}(0,T,L^2(\Gamma_3)) \cap \mathcal{Q}$$

and $\varphi_{\kappa\eta\lambda_{\kappa\eta}} \in W^{1,2}(0,T;W)$ we deduce that $\Lambda_\kappa\eta \in W^{1,2}(0,T;V)$.
Let now $\eta_1, \eta_2 \in L^2(0,T;V)$ and let $u_i = u_{\kappa\eta_i\lambda_{\kappa\eta_i}}$, $\dot{u}_i = \dot{u}_{\kappa\eta_i\lambda_{\kappa\eta_i}}$, $\beta_i = \beta_{\kappa\eta_i\lambda_{\kappa\eta_i}}$, $\varphi_i = \varphi_{\kappa\eta_i\lambda_{\kappa\eta_i}}$, $\sigma_i = \sigma_{\kappa\eta_i\lambda_{\kappa\eta_i}}$ for $i = 1,2$. Arguments similar to those used in the proof of (4.35) lead to

$$\begin{aligned} \|\Lambda_\kappa\eta_1(t) - \Lambda_\kappa\eta_2(t)\|_V \leq c(&\|u_1(t) - u_2(t)\|_V \\ &+ \|\beta_1(t - \beta_2(t)\|_{L^2(\Gamma_3)} + \|\varphi_1(t) - \varphi_2(t)\|_W). \end{aligned} \tag{4.36}$$

We combine now (4.18), (4.24) and (4.36) to obtain

$$\begin{aligned} \|\Lambda_\kappa\eta_1(t) - \Lambda_\kappa\eta_2(t)\|_V \leq c(&\|u_1(t) - u_2(t)\|_V \\ &+ \int_0^t \|u_1(s) - u_2(s)\|_V ds). \end{aligned} \tag{4.37}$$

Moreover, since $u_1(0) = u_2(0) = u_0$ we have

$$\|u_1(t) - u_2(t)\|_V \leq c\int_0^t \|\dot{u}_1(s) - \dot{u}_2(s)\|_V ds. \tag{4.38}$$

From (4.37) and (4.38) we find

$$\|\Lambda_\kappa\eta_1(t) - \Lambda_\kappa\eta_2(t)\|_V \leq c\int_0^t \|\dot{u}_1(s) - \dot{u}_2(s)\|_V ds \qquad \forall\ t \in [0\ T]\,. \tag{4.39}$$

On the other hand, keeping in mind that $\lambda_{\kappa\eta_i} = \sigma_i$, using (4.8) and by arguments similar to those used in (4.30) we find that

$$m_{\mathcal{A}}\|\dot{u}_1(t) - \dot{u}_2(t)\|_V \leq L_p C_0 C_R \|\mu\|_{L^\infty(\Gamma_3)} \|\sigma_1(t) - \sigma_2(t)\|_{\mathcal{H}} + \|\eta_1(t) - \eta_2(t)\|_V + C_{\mathcal{F}} \int_0^t \|\dot{u}_1(s) - \dot{u}_2(s)\|_V ds,$$

and, after a Gronwall argument, we obtain

$$m_{\mathcal{A}}\|\dot{u}_1(t) - \dot{u}_2(t)\|_V \leq L_p C_0 C_R \|\mu\|_{L^\infty(\Gamma_3)} \|\sigma_1(t) - \sigma_2(t)\|_{\mathcal{H}}, + \|\eta_1(t) - \eta_2(t)\|_V. \quad (4.40)$$

Now, by (4.10)(c) it follows that $Div\sigma_1(t) = Div\sigma_2(t)$. Then, from (4.7), (3.2), (3.3) and (3.15) we find that

$$\|\sigma_1(t) - \sigma_2(t)\|_{\mathcal{H}_1} = \|\sigma_1(t) - \sigma_2(t)\|_{\mathcal{H}} \leq C_{\mathcal{A}} \|\dot{u}_1(t) - \dot{u}_2(t)\|_V + C_{\mathcal{F}} \|u_1(t) - u_2(t)\|_V + \|\eta_1(t) - \eta_2(t)\|_V, \quad (4.41)$$

where we deduce that

$$\|\sigma_1(t) - \sigma_2(t)\|_{\mathcal{H}_1} = \|\sigma_1(t) - \sigma_2(t)\|_{\mathcal{H}} \leq C_{\mathcal{A}} \|\dot{u}_1(t) - \dot{u}_2(t)\|_V + \|\eta_1(t) - \eta_2(t)\|_V + C_{\mathcal{F}} \int_0^t \|\dot{u}_1(s) - \dot{u}_2(s)\|_V ds, \quad (4.42)$$

We combine now (4.40) and (4.42) to obtain

$$m_{\mathcal{A}}\|\dot{u}_1(t) - \dot{u}_2(t)\|_V \leq C_{\mathcal{A}} L_p C_0 C_R \|\mu\|_{L^\infty(\Gamma_3)} \|\dot{u}_1(t) - \dot{u}_2(t)\|_V + (L_p C_0 C_R \|\mu\|_{L^\infty(\Gamma_3)} + 1) \|\eta_1(t) - \eta_2(t)\|_V + L_p C_0 C_R \|\mu\|_{L^\infty(\Gamma_3)} C_{\mathcal{F}} \int_0^t \|\dot{u}_1(s) - \dot{u}_2(s)\|_V ds. \quad (4.43)$$

Now, we take $\|\mu\|_{L^\infty(\Gamma_3)} \leq \mu_0$ such that

$$\mu_0 = \frac{m_{\mathcal{A}}}{C_{\mathcal{A}} L_p C_0 C_R}. \quad (4.44)$$

Using (4.43) and after a Gronwall argument we find that

$$(m_{\mathcal{A}} - C_{\mathcal{A}} L_p C_0 C_R \|\mu\|_{L^\infty(\Gamma_3)}) \|\dot{u}_1(t) - \dot{u}_2(t)\|_V \leq (L_p C_0 C_R \|\mu\|_{L^\infty(\Gamma_3)} + 1) \|\eta_1(t) - \eta_2(t)\|_V,$$

where, we deduce that for $\|\mu\|_{L^\infty(\Gamma_3)} \leq \mu_0$ we have

$$\|\dot{u}_1(t) - \dot{u}_2(t)\|_V \leq c \|\eta_1(t) - \eta_2(t)\|_V. \quad (4.45)$$

We combine now (4.45) and (4.39) to see that

$$\|\Lambda_\kappa \eta_1(t) - \Lambda_\kappa \eta_2(t)\|_V \leq c \int_0^t \|\eta_1(s) - \eta_2(s)\|_V ds \qquad \forall\, t \in [0, T] \quad (4.46)$$

and by Cauchy-Schwartz inequality we deduce that

$$\|\Lambda_\kappa\eta_1(t)-\Lambda_\kappa\eta_2(t)\|_V^2\leq c\int_0^t\|\eta_1(s)-\eta_2(s)\|_V^2ds\qquad\forall\, t\in[0,T]\tag{4.47}$$

Reiterating this inequality m times yields

$$\|\Lambda_\kappa^m\eta_1-\Lambda_\kappa^m\eta_2\|^2_{L^2(0,T;V)}\leq\frac{c^mT^m}{m!}\|\eta_1-\eta_2\|^2_{L^2(0,T;V)}.$$

which implies that, for $\|\mu\|_{L^\infty(\Gamma_3)}\leq\mu_0$ and m sufficiently large, a power Λ_κ^m of Λ_κ is a contraction in the *Banach* space $L^2(0,T;V)$. Thus, there exists a unique element $\eta_\kappa\in L^2(0,T;V)$ such that $\Lambda_\kappa^m\eta_\kappa=\eta_\kappa$ and η_κ is also the unique fixed point of Λ_κ, i.e $\Lambda_\kappa\eta_\kappa=\eta_\kappa$. The regularity $\eta_\kappa\in W^{1,2}(0,T;V)$ follows from the regularity $\Lambda_\kappa\eta_\kappa\in W^{1,2}(0,T;V)$, which concludes the proof of Lemma (4.5). □

Next, let $\|\mu\|_{L^\infty(\Gamma_3)}\leq\mu_0$ and $\lambda_{\kappa\eta}$, η_κ the fixed points of operators $\Lambda_{\kappa\eta}$, Λ_κ respectivelly. We put $u_k=u_{\kappa\eta_\kappa\lambda_{\kappa\eta}}$, $\sigma_k=\sigma_{\kappa\eta_\kappa\lambda_{\kappa\eta}}$, $\varphi_k=\varphi_{\kappa\eta_\kappa\lambda_{\kappa\eta}}$ and $\beta_k=\beta_{\kappa\eta_\kappa\lambda_{\kappa\eta}}$ for the solutions obtened in lemmas (4.1), (4.2), (4.3). Moreover, we define the operator $\mathit{\Lambda}:L^2(0,T;\mathcal{H})\longrightarrow L^2(0,T;\mathcal{H})$ by

$$\mathit{\Lambda}\kappa=\mathcal{G}(\sigma_\kappa,\varepsilon(u_\kappa)),\tag{4.48}$$

such that

$$\sigma_\kappa(t)=\mathcal{A}\varepsilon(\dot u_\kappa(t))+\mathcal{F}\varepsilon(u_\kappa(t))+z_\kappa(t)+\mathcal{E}^*\boldsymbol{E}(\varphi_\kappa(t)).\tag{4.49}$$

$$\begin{gathered}(\sigma_\kappa(t),\varepsilon(\omega)-\varepsilon(\dot u_\kappa(t))_{\mathcal{H}}+j_{ad}(\beta_\kappa(t),u_\kappa(t),\omega-\dot u_\kappa(t))\\ +j_{fr}(\sigma_\kappa(t),\omega)-j_{fr}(\sigma_\kappa(t),\dot u_\kappa(t))\geq(f_\kappa(t),\omega-\dot u_\kappa(t))_V\\ \forall\,\omega\in V,\quad\forall t\in[0,T].\end{gathered}\tag{4.50}$$

$$(f_\kappa(t),\upsilon)_V=(f(t),\upsilon)_V+(z_\kappa(t),\varepsilon(\upsilon))_{\mathcal{H}}.\tag{4.51}$$

Lemma 4.6. *The function $\mathit{\Lambda}\kappa$ belongs to $W^{1,2}(0,T;\mathcal{H})$ and the operator $\mathit{\Lambda}$ has a unique fixed point $\kappa^*\in L^2(0,T;\mathcal{H})$.*

Proof. Let $\kappa\in L^2(0,T;\mathcal{H})$ and let $t_1,t_2\in[0,T]$. Using (4.48), (3.7) and (3.15) we find that

$$\|\mathit{\Lambda}\kappa(t_1)-\mathit{\Lambda}\kappa(t_2)\|_{\mathcal{H}}\leq L_{\mathcal{G}}(\|\sigma_\kappa(t_1)-\sigma_\kappa(t_2)\|_{\mathcal{H}}+\|u_\kappa(t_1)-u_\kappa(t_2)\|_V).$$

Since $u_\kappa\in W^{2,2}(0,T;V)$, $\sigma_\kappa\in W^{1,2}(0,T;\mathcal{H}_1)$ we deduce that $\Lambda\kappa\in W^{1,2}(0,T;\mathcal{H})$.

Next, let $\kappa_1,\kappa_2\in L^2(0,T;\mathcal{H})$. For the sake of simplicity, we put $u_i=u_{\kappa_i}$, $\sigma_i=\sigma_{\kappa_i}$, $\beta_i=\beta_{\kappa_i}$, $\varphi_i=\varphi_{\kappa_i}$ and $z_i=z_{\kappa_i}$. Usin again (4.48), (3.7) and (3.15) we obtain

$$\|\mathit{\Lambda}\kappa_1(t)-\mathit{\Lambda}\kappa_2(t)\|_{\mathcal{H}}\leq L_{\mathcal{G}}(\|\sigma_1(t)-\sigma_2(t)\|_{\mathcal{H}}+\|u_1(t)-u_2(t)\|_V).\tag{4.52}$$

On the other hand, by arguments similar to those used in (4.30) the inequality (4.50) leads to

$$\begin{aligned}&(\sigma_1(t)-\sigma_2(t),\varepsilon(\dot u_1)-\varepsilon(\dot u_2)_{\mathcal H}\le\\&+j_{ad}(\beta_1,u_1,\dot u_2-\dot u_1)+j_{ad}(\beta_1,u_1,\dot u_2-\dot u_1)\\&(\mathcal E^*\boldsymbol E(\varphi_1(t))-\mathcal E^*\boldsymbol E(\varphi_2(t)),\varepsilon(\dot u_1)-\varepsilon(\dot u_2))_{\mathcal H}\\&+j_{fr}(\sigma_1(t),\dot u_2(t))-j_{fr}(\sigma_1(t),\dot u_1(t))\\&+j_{fr}(\sigma_2(t),\dot u_1(t))-j_{fr}(\sigma_2(t),\dot u_2(t)).\end{aligned}\tag{4.53}$$

Using (4.49), (3.2), (3.3), (3.6), (3.21), (3.22), (3.12), (3.28), (3.29) and the previous inequality and after some algebric manipulation we find that

$$\begin{aligned}m_{\mathcal A}\|\dot u_1(t)-\dot u_2(t)\|_V^2\le{}&(c\|u_1(t)-u_2(t)\|_V+\|z_1(t)-z_2(t)\|_V\\&+c\|\varphi_1(t)-\varphi_2(t)\|_W+c\|\beta_1(t)-\beta_2(t)\|_{L^2(\Gamma_3)}\\&+L_pC_0C_R\|\mu\|_{L^\infty(\Gamma_3)}\|\sigma_1(t)-\sigma_2(t)\|_{\mathcal H})\|\dot u_1(t)-\dot u_2(t)\|_V,\end{aligned}$$

where we deduce that

$$\begin{aligned}m_{\mathcal A}\|\dot u_1(t)-\dot u_2(t)\|_V\le{}&c\|u_1(t)-u_2(t)\|_V+\|z_1(t)-z_2(t)\|_{\mathcal H}\\&+c\|\varphi_1(t)-\varphi_2(t)\|_W+c\|\beta_1(t)-\beta_2(t)\|_{L^2(\Gamma_3)}\\&+L_pC_0C_R\|\mu\|_{L^\infty(\Gamma_3)}\|\sigma_1(t)-\sigma_2(t)\|_{\mathcal H}.\end{aligned}$$

We combine now (4.18), (4.24) and the previous inequality to obtain

$$\begin{aligned}m_{\mathcal A}\|\dot u_1(t)-\dot u_2(t)\|_V\le{}&L_pC_0C_R\|\mu\|_{L^\infty(\Gamma_3)}\|\sigma_1(t)-\sigma_2(t)\|_{\mathcal H}\\&+\|z_1(t)-z_2(t)\|_{\mathcal H}+c\|u_1(t)-u_2(t)\|_V\\&+\int_0^t\|u_1(s)-u_2(s)\|_V ds.\end{aligned}\tag{4.54}$$

Moreover, since $u_1(0)=u_2(0)=u_0$ we have

$$\|u_1(t)-u_2(t)\|_V\le\int_0^t\|\dot u_1(s)-\dot u_2(s)\|_V ds.\tag{4.55}$$

From (4.54) and (4.55) we find

$$\begin{aligned}m_{\mathcal A}\|\dot u_1(t)-\dot u_2(t)\|_V\le{}&L_pC_0C_R\|\mu\|_{L^\infty(\Gamma_3)}\|\sigma_1(t)-\sigma_2(t)\|_{\mathcal H}\\&+\|z_1(t)-z_2(t)\|_{\mathcal H}+c\int_0^t\|\dot u_1(s)-\dot u_2(s)\|_V ds,\end{aligned}$$

and after a Gronwall argument we find that

$$\begin{aligned}m_{\mathcal A}\|\dot u_1(t)-\dot u_2(t)\|_V\le{}&\|z_1(t)-z_2(t)\|_{\mathcal H}\\&+L_pC_0C_R\|\mu\|_{L^\infty(\Gamma_3)}\|\sigma_1(t)-\sigma_2(t)\|_{\mathcal H}.\end{aligned}\tag{4.56}$$

On the other hand, using (4.49), (3.2), (3.3) we find that

$$\|\sigma_1(t)-\sigma_2(t)\|_{\mathcal{H}} \le C_{\mathcal{A}}\|\dot u_1(t)-\dot u_2(t)\|_V + C_{\mathcal{F}}\|u_1(t)-u_2(t)\|_V + \|z_1(t)-z_2(t)\|_V + c\|\varphi_1(t)-\varphi_2(t)\|_W, \tag{4.57}$$

where, we deduce from (4.18) that

$$\|\sigma_1(t)-\sigma_2(t)\|_{\mathcal{H}} \le \|z_1(t)-z_2(t)\|_{\mathcal{H}} + C_{\mathcal{A}}\|\dot u_1(t)-\dot u_2(t)\|_V + C_{\mathcal{F}}\|u_1(t)-u_2(t)\|_V \tag{4.58}$$

Combining (4.56) and (4.58) we obtain

$$\begin{aligned} m_{\mathcal{A}}\|\dot u_1(t)-\dot u_2(t)\|_V &\le \|z_1(t)-z_2(t)\|_{\mathcal{H}} \\ &+L_pC_0C_R\|\mu\|_{L^\infty(\Gamma_3)}C_{\mathcal{A}}\|\dot u_1(t)-\dot u_2(t)\|_V \\ &+L_pC_0C_R\|\mu\|_{L^\infty(\Gamma_3)}\|z_1(t)-z_2(t)\|_{\mathcal{H}} \\ &+L_pC_0C_R\|\mu\|_{L^\infty(\Gamma_3)}C_{\mathcal{F}}\|u_1(t)-u_2(t)\|_V. \end{aligned}$$

It follows now from the previous inequality that

$$\begin{aligned} m_{\mathcal{A}}\|\dot u_1(t)-\dot u_2(t)\|_V &\le (1+L_pC_0C_R\|\mu\|_{L^\infty(\Gamma_3)})\|z_1(t)-z_2(t)\|_{\mathcal{H}} \\ &+ C_{\mathcal{A}}L_pC_0C_R\|\mu\|_{L^\infty(\Gamma_3)}\|\dot u_1(t)-\dot u_2(t)\|_V \\ &+ L_pC_0C_R\|\mu\|_{L^\infty(\Gamma_3)}C_{\mathcal{F}}\int_0^t \|\dot u_1(s)-\dot u_2(s)\|_V ds, \end{aligned}$$

and after a Gronwall argument we find that

$$\begin{aligned} m_{\mathcal{A}}\|\dot u_1(t)-\dot u_2(t)\|_V &\le (1+L_pC_0C_R\|\mu\|_{L^\infty(\Gamma_3)})\|z_1(t)-z_2(t)\|_{\mathcal{H}} \\ &+ C_{\mathcal{A}}L_pC_0C_R\|\mu\|_{L^\infty(\Gamma_3)}\|\dot u_1(t)-\dot u_2(t)\|_V. \end{aligned}$$

Since, $\|\mu\|_{L^\infty(\ \Gamma_3)} \le \mu_0$ the previous inequality leads to

$$\|\dot u_1(t)-\dot u_2(t)\|_V \le c\|z_1(t)-z_2(t)\|_{\mathcal{H}} \tag{4.59}$$

Moreover, since $u_1(0)=u_2(0)=u_0$ we have

$$\|u_1(t)-u_2(t)\|_V \le \int_0^t \|\dot u_1(s)-\dot u_2(s)\|_V ds \le c\int_0^t \|z_1(s)-z_2(s)\|_{\mathcal{H}} ds. \tag{4.60}$$

Combining, (4.58), (4.59) and (4.60) we find

$$\|\Lambda\kappa_1(t)-\Lambda\kappa_2(t)\|_{\mathcal{H}} \le c\|z_1(t)-z_2(t)\|_{\mathcal{H}} + c\int_0^t \|z_1(s)-z_2(s)\|_{\mathcal{H}} ds. \tag{4.61}$$

Now, from (4.6) we have $z_1(0)=z_2(0)=0$. Then,

$$\|z_1(t)-z_2(t)\|_V \le \int_0^t \|\dot z_1(s)-\dot z_2(s)\|_{\mathcal{H}} ds. \tag{4.62}$$

Therefore, combining (4.61) and (4.62) we obtain

$$\|\Lambda\kappa_1(t) - \Lambda\kappa_2(t)\|_{\mathcal{H}} \leq c \int_0^t \|\dot{z}_1(s) - \dot{z}_2(s)\|_{\mathcal{H}} ds. \tag{4.63}$$

Finally, using (4.6) and Cauchy-Schwartz inequality we find

$$\|\Lambda\kappa_1(t) - \Lambda\kappa_2(t)\|_{\mathcal{H}}^2 \leq c \int_0^t \|\kappa_1(s) - \kappa_2(s)\|_{\mathcal{H}}^2 ds.$$

Reiterating this inequality m times yields

$$\|\Lambda^m \kappa_1 - \Lambda^m \kappa_2\|_{L^2(0,T;\mathcal{H})}^2 \leq \frac{c^m T^m}{m!} \|\kappa_1 - \kappa_2\|_{L^2(0,T;\mathcal{H})}^2.$$

which implies that, for m sufficiently large, a power Λ^m of Λ is a contraction in the *Banach* space $L^2(0,T;\mathcal{H})$. Thus, there exists a unique element $\kappa^* \in L^2(0,T;\mathcal{H})$ such that $\Lambda^m \kappa^* = \kappa^*$ and κ^* is also the unique fixed point of Λ, i.e $\Lambda\kappa^* = \kappa^*$, which concludes the proof of Lemma (4.6). □

Now, we have all the ingredients necessary to prove Theorem 4.1.

Existence: Let κ^*, η_κ, $\lambda_{\kappa\eta}$ be the fixed points of operators Λ, Λ_κ, $\Lambda_{\kappa\eta}$, respectively, and $(u,\sigma) = (u_{\kappa\eta\lambda}, \sigma_{\kappa\eta\lambda})$ the solution of the variational problem $\mathcal{P}_1^V$ with $\kappa = \kappa^*$, $\eta = \eta_\kappa$, $\lambda = \lambda_{\kappa\eta}$. We also denote by $\varphi = \varphi_{\kappa\eta\lambda}$ and $\beta = \beta_{\kappa\eta\lambda}$ the solution of problems (4.17) and $\mathcal{P}^{\beta_{\kappa\eta\lambda}}$, respectively, with $\kappa = \kappa^*$, $\eta = \eta_\kappa$, $\lambda = \lambda_{\kappa\eta}$. Clearly, it follows from (4.6), (4.25), (4.34) and (4.48) that (3.30)-(3.35) holds. We conclude that (u, σ, φ, D, β) is a solution of Problem $\mathcal{P}^V$ and it satisfies (4.1)-(4.5).

Uniqueness: The uniqueness of the solution follows from the uniqueness of the fixed points of Λ, Λ_κ, $\Lambda_{\kappa\eta}$ and from the uniqueness part of Lemmas (4.1), (4.2) and (4.3).

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Mathematical modelling of free convection in a square cavity filled with a bidisperse porous medium for large values of Rayleigh number

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Abstract. A free convection problem for bidisperse porous media is considered. The numerical solutions are obtained using an algorithm based on an nonuniform grid. Results for some values of the governing parameters when Rayleigh number is equal to 10^4 are provided.

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Keywords: Numerical results, free convection, square cavity, bidisperse porous medium, nonuniform grid.

1. Introduction

Fluid flow and heat transfer in porous media represented the subject of intensive research in the last decades. A comprehensive presentation of the volume of work in this domain can be found in [14]. In the past several years, there has been much interest in double porosity materials, the so called bidisperse porous media (BDPM). The literature in the field is extensive, see for instance [20] and references therein. A very good description of the mathematical models concerning heat transfer and fluid flow in BDPM can be found in the excellent chapter [15] in the book [10].

A new mathematical model which describes the flow and heat transfer in a square cavity filled with BDPM was considered in [18]. It represents an extension of the classical problem of steady Darcy free convection for a monodisperse (regular) porous medium by following the model proposed in [16] and [17]. The basic equations were transformed in terms of dimensionless stream functions and temperatures and an algorithm based on finite difference method was provided to obtain the numerical solutions.

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The current paper represents a continuation of [18] and we use a different approach to obtain the numerical results. More exactly, we consider, as in [19], a nonuniform grid to a better capture of the phenomena near the boundaries. The novelty consists in the fact that, in contrast with [18], we provide numerical solutions for large values of Rayleigh number.

The rest of the paper is structured as follows. The basic equations and preliminary materials are presented in Section 2. In Section 3 we describe the numerical algorithm and give some test results. Finally, in Section 4 we provide and discuss our principal results in the form of tables and figures.

2. Basic equations

A porous medium is a material consisting of a solid matrix with a interconnected void saturated by a fluid. A bidisperse porous medium, as it is mentioned in [16] or [17], is composed of clusters of large particles that are agglomerations of small particles. Examples of BDPM are beds of porous and fractured rocks, coal deposits or bidisperse catalysts. There exists a wide range of applications in geophysics, medicine or food industry, see [10], [14], [22] or [20]. Fluid flow and heat transfer in BDPM were studied for various configurations as vertical and wavy plates, channels or cylindrical geometries. The problem of steady Darcy free convection in an enclosure was analyzed in [18]. More exactly, the geometry of the model consists in a square cavity with a given size filled with BDPM, see Figure 1a. The horizontal walls are adiabatic whereas the vertical walls are kept at constant but different temperatures. The physical problem is represented mathematically by the following set of partial differential equations introduced in [18] along with the corresponding boundary conditions illustrated in Figure 1a

$$\frac{\partial u_f}{\partial x} + \frac{\partial v_f}{\partial y} = 0, \tag{2.1}$$

$$\frac{\partial u_p}{\partial x} + \frac{\partial v_p}{\partial y} = 0, \tag{2.2}$$

$$\frac{\partial p}{\partial x} = -\frac{\mu}{K_f} u_f - \xi(u_f - u_p), \tag{2.3}$$

$$\frac{\partial p}{\partial x} = -\frac{\mu}{K_p} u_p - \xi(u_p - u_f), \tag{2.4}$$

$$\frac{\partial p}{\partial y} = -\frac{\mu}{K_f} v_f - \xi(v_f - v_p) + \rho g \hat{\beta}(T_F - T_0), \tag{2.5}$$

$$\frac{\partial p}{\partial y} = -\frac{\mu}{K_p} v_p - \xi(v_p - v_f) + \rho g \hat{\beta}(T_F - T_0), \tag{2.6}$$

$$\phi(\rho c)_f \left(u_f \frac{\partial T_f}{\partial x} + v_f \frac{\partial T_f}{\partial y} \right) = \phi k_f \nabla^2 T_f + h(T_p - T_f), \tag{2.7}$$

$$(1-\phi)(\rho c)_p \left(u_p \frac{\partial T_p}{\partial x} + v_p \frac{\partial T_p}{\partial y} \right) = (1-\phi) k_p \nabla^2 T_p + h(T_f - T_p), \tag{2.8}$$

where

$$T_F = \frac{\phi T_f + (1-\phi)\varepsilon T_p}{\phi + (1-\varepsilon)\phi}, \quad T_0 = \frac{T_h + T_c}{2}.$$

Here the subscripts f and p are related to the macrophase and to the microphase, respectively. Moreover, (x, y) represent the Cartesian coordinates, (u, v) are the filtration velocity components, T is the temperature, p is the pressure, K is the permeability, g is the magnitude of the acceleration due to gravity, c is the specific heat at constant pressure, h is the inter-phase heat transfer coefficient, ϕ is the volume fraction of the f-phase, μ is the dynamic viscosity, ρ is the fluid density, ξ is the coefficient for momentum transfer between the two phases, ε is the porosity within the p-phase and $\hat{\beta}$ is the volumetric thermal expansion. In order to obtain a dimensionless form of (2.1)-(2.8) the following variables are considered

$$p = \frac{\mu k_f}{(\rho c)_f K_f} P, (u_f, v_f) = \frac{\phi k_f}{(\rho c)_f L}(U_f, V_f), (u_p, v_p) = \frac{(1-\phi)k_p}{(\rho c)_p L}(U_p, V_p)$$

$$(x, y) = L(X, Y), T_f = (T_h - T_c)\theta_f + T_0, T_p = (T_h - T_c)\theta_p + T_0.$$

The previous dimensionless variables are substituted in (2.1)-(2.8). Proceeding as in [18], we introduce the stream functions ψ_f and ψ_p given by

$$(U_f, U_p) = \frac{\partial}{\partial Y}(\psi_f, \psi_p), \ (V_f, V_p) = -\frac{\partial}{\partial X}(\psi_f, \psi_p)$$

and eliminate the pressure P. The governing equations for continuity, momentum and energy are transformed in the following dimensionless nonlinear system

$$-(1+\sigma_f)\nabla^2 \psi_f + \beta \sigma_f \nabla^2 \psi_p = Ra \left(\tau \frac{\partial \theta_f}{\partial X} + (1-\tau) \frac{\partial \theta_p}{\partial X} \right) \tag{2.9}$$

$$\sigma_f \nabla^2 \psi_f - \beta \left(\sigma_f + \frac{1}{K_r} \right) \nabla^2 \psi_p = Ra \left(\tau \frac{\partial \theta_f}{\partial X} + (1-\tau) \frac{\partial \theta_p}{\partial X} \right) \tag{2.10}$$

$$\nabla^2 \theta_f = \phi \left(\frac{\partial \psi_f}{\partial Y} \frac{\partial \theta_f}{\partial X} - \frac{\partial \psi_f}{\partial X} \frac{\partial \theta_f}{\partial Y} \right) + H(\theta_f - \theta_p) \tag{2.11}$$

$$\nabla^2 \theta_p = (1-\phi) \left(\frac{\partial \psi_p}{\partial Y} \frac{\partial \theta_p}{\partial X} - \frac{\partial \psi_p}{\partial X} \frac{\partial \theta_p}{\partial Y} \right) + H\gamma(\theta_p - \theta_f), \tag{2.12}$$

where Ra denotes the Rayleigh number, σ_f represents the inter-phase momentum transfer parameter, K_r is the permeability ratio, H is the inter-phase heat transfer parameter, γ is the modified thermal conductivity ratio, β denotes the modified thermal diffusivity ratio and τ incorporates the porosity of micropores and are defined as follows

$$Ra = \frac{\rho g\hat{\beta}(T_h - T_c)K_f L(\rho c)_f}{\phi\mu k_f},\ \sigma_f = \frac{\xi K_f}{\mu},\ \beta = \frac{(1-\phi)k_p(\rho c)_f}{\phi k_f(\rho c)_p}$$

$$K_r = \frac{K_p}{K_f},\ H = \frac{hL^2}{\phi k_f},\ \gamma = \frac{\phi k_f}{(1-\phi)k_p},\ \tau = \frac{\phi}{\phi + (1-\phi)\varepsilon}.$$

More details about their significance and their values can be found in [8]. The independent variables (X, Y) belong to $[0,1] \times [0,1]$ and the corresponding boundary conditions are given by

$$\begin{cases} \psi_f = \psi_p = 0,\ \theta_f = \theta_p = \frac{1}{2} \text{ at } X = 0 \\ \psi_f = \psi_p = 0,\ \theta_f = \theta_p = -\frac{1}{2} \text{ at } X = 1 \\ \psi_f = \psi_p = \frac{\partial\theta_f}{\partial Y} = \frac{\partial\theta_p}{\partial Y} = 0 \text{ at } Y \in \{0,1\}. \end{cases} \tag{2.13}$$

In the rest of the paper we use the following values, considered in [16] or [17], $\phi = 0.5$, $\tau = 0.625$, $H \in \{10^{-2}, 10^2\}$, $K_r \in \{10^{-3}, 10^{-1}\}$, $\sigma_f \in \{10^{-1}, 1\}$, $Ra \in \{10^2, 10^3, 10^4\}$, $\beta \in \{1, 10\}$ and $\gamma \in \{10^{-2}, 1, 10^2\}$.

In addition, physical quantities of interest are the mean Nusselt numbers at the heated wall, given in the following dimensionless form

$$Nu_f = -\int_0^1 \left(\frac{\partial\theta_f}{\partial X}\right)_{X=0} dY, \quad Nu_p = -\int_0^1 \left(\frac{\partial\theta_p}{\partial X}\right)_{X=0} dY. \tag{2.14}$$

Moreover, an overall Nusselt number can be obtained

$$Nu_{all} = \frac{\gamma}{1+\gamma} Nu_f + \frac{\gamma}{1+\gamma} Nu_p. \tag{2.15}$$

3. Numerical algorithm

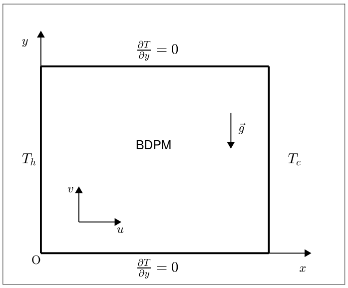
A central-finite difference scheme was used in [18] to obtain the numerical solutions of equations (2.9)-(2.12) subject to boundary conditions (2.13). Moreover, the nonlinear system of discretized equations was solved using a Gauss-Seidel iteration technique. The following convergence criterion was used to check the convergence of the method

$$\|\lambda_{new} - \lambda_{old}\| / \|\lambda_{new}\| \le \delta, \tag{3.1}$$

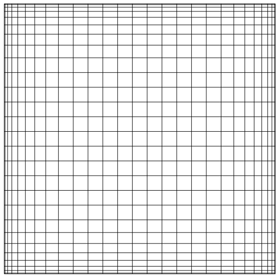
where δ is a prescribed error, λ represents the unknowns ψ or θ and $\|\cdot\|$ is a given norm.

As we mentioned in Section 1, we change the algorithm proposed in [18]. More exactly, we consider a variable grid near the walls to determine the numerical solutions. The step size varies as a quadratic function. In order to illustrate the grid structure, we represent in Figure 1b the mesh of size 28×28. The smallest step size is near the boundaries, while the largest step size is in the middle of the domain. To define this grid we consider the variable grid layer thickness (v.g.l.t.), b, and the number of nodes in v.g.l.t., n_b. This allows us to compute the first step in v.g.l.t., h_b, and the total number of nodes in one direction, n. For all results presented in this paper, choosing

$\delta = 10^{-9}$ in (3.1) proves to be sufficiently small such that any smaller value produces similar results. The numerical experiments were performed on the computer cluster Kotys (see [4]).



(A) The physical model



(B) The mesh with the size 28×28

In the rest of this section we provide some results related to the grid pattern and the validation of the algorithm. To determine the grid structure, we performed numerical simulations for various values of v.g.l.t., b, and different number of nodes in v.g.l.t., n_b. However, in order to save the space we restrict to present only the results from Table 1. This analysis help us to conclude that the suitable grid for the cases $Ra = 10^2$ or $Ra = 10^3$ can be based on 102×102 points, *i.e.* $b = 0.2$ and $n_b = 30$. Moreover, for the case $Ra = 10^4$ all the results are obtained using 119×119 points, *i.e.* $b = 0.2$ and $n_b = 35$.

TABLE 1. Results for different grids at $Ra = 10^4$ when $\sigma_f = 1, K_r = 10^{-1}, \beta = 1, H = 10^{-2}, \gamma = 10^{-2}$

b	n_b	h_b	n	Nu_f	Nu_p	$\max\lvert\psi_f\rvert$	$\max\lvert\psi_p\rvert$
0.2	35	0.00017	119	24.946	9.073	91.586	22.896
0.2	40	0.00013	137	24.956	9.078	91.618	22.904
0.3	40	0.00019	104	24.930	9.069	91.571	22.892
0.3	45	0.00015	117	24.947	9.074	91.603	22.900
0.4	45	0.00020	99	24.926	9.068	91.567	22.891
0.4	50	0.00016	110	24.942	9.073	91.597	22.899

Finally, Table 2 contains a comparison between the computed values of Nusselt number with the results from the open literature for different values of Rayleigh number. As it can be seen, the obtained results show a good agreement with the results reported by the mentioned authors. Therefore, we are confident that the results reported in the present paper are accurate.

At the end of this section, we mention that more details about numerical methods for partial differential equations can be found in [7] or [21]. Moreover, numerical results

TABLE 2. Comparison of Nusselt number for $\phi = \tau = \beta = 1$, $K_r = 10^{-4}$ and $\sigma_f = H = \gamma = 0$

Authors	Ra			
	10	10^2	10^3	10^4
[1]	1.079	3.160	14.060	48.330
[2]	–	3.113	–	48.900
[3]	–	4.200	15.800	50.800
[9]	–	3.141	13.448	42.583
[11]	–	3.118	13.637	48.117
[13]	1.065	2.801	–	–
[18]	–	–	13.664	–
[19]	1.078	3.108	13.613	48.208
[23]	–	3.097	12.960	51.000
Present	1.079	3.108	13.603	48.370

based on spline functions for the problem of natural convection in a square cavity filled with a fluid-saturated porous medium are provided in [12].

4. Results and discussion

In this section we present numerical results for the streamlines, isotherms and mean Nusselt numbers for the values of the parameters introduced in Section 2. More exactly, we consider constant some parameters and check the effect of the other ones. Tacking into account the fact that numerical results for $Ra \in \{10^2, 10^3\}$ were analyzed in [18], we restrict our attention to the case $Ra = 10^4$. Concerning the parameters which describe the porosities all results are given for the following values $\phi = 0.5$ and $\tau = 0.625$. Table 3 contains the values of the mean Nusselt numbers Nu_f and Nu_p defined in (2.14) and Table 4 provides the maximum absolute value of stream functions. Figs. 2–6 show the streamlines and their maximum absolute value (up) and isotherms (bottom) for $Ra = 10^4$, whereas Figs. 7–8 depict results for $Ra \in \{10^2, 10^3, 10^4\}$.

We analyze these results in the following. First of all, we can observe that for all values of governing parameters when $Ra = 10^4$ the flow is unicellular. Moreover, the results given in Table 4 show that the flow in p-phase is much slower than the flow in f-phase. From the position of isotherms in f-phase, which are not parallel with the vertical walls, we conclude that there exists a predominant convective heat transfer in macropahse.

For small values of H and γ, *i.e.* an intense thermal non-equilibrium effect is considered, the isotherms in p-phase are almost parallel with the vertical walls of the cavity, see Figure 2 or Figure 8. We deduce that in this case the heat transfer is mainly conductive in microphase. The difference between the streamlines for the two phases seems to be negligible, see Figure 2 or Figure 7. However, there exists an important difference between maximum absolute values of stream functions. For large values of H and γ (thermal equilibrium) we observe an increasing in p-phase of convection effect, see the isotherms in Figs. 3–4. Moreover, for $H = \gamma = 10^2$ the isotherms have a very similar form, see Figure 6, the two phases being in thermal equilibrium. In addition, we observe that a thermal boundary layer near the vertical boundaries is presented. The flow in both phases is stratified and the dimension of central cells increases with the increase of H and γ. Also, the position of streamlines in Figs. 3–6 shows the existence of a boundary layer type flow.

Using the results in Table 3 we deduce that the values of Nusselt numbers increase by increasing K_r from 10^{-3} to 10^{-1}. Moreover, the results provided in Table 4 show that the maximum absolute value of stream function decreases in f-phase and increases in p-phase. The same behavior can be observed comparing Figure 2 and Figure 5 and it is in agreement with the physical situation. More exactly, K_r represents the ratio of micropermeability to macropermeability and small values of it suggest that the flow and convective heat transfer are reduced in microphase.

The increase of inter-phase momentum transfer parameter σ_f from 10^{-1} to 1 implies that the maximum absolute value of stream function decreases in macrophase and increases in microphase, see Table 4 or Figs. 3–4. This behavior is not surprising since σ_f is a measure of the way in which momentum is transferred between the two phases. An analogue situation is encountered for the heat flux, see Table 3, excepting the case $H = \gamma = 10^2$ when a strong thermal equilibrium exists and the values of both Nusselt numbers decrease.

Finally, we analyze the influence of Rayleigh number, Ra, on the flow and Nusselt numbers. We observe that the Nusslet numbers and the maximum of stream functions increase with the increasing of Ra, see Table 5 or Figure 7. An identic behavior is observed for the regular case, see Table 2. As we mentioned above, for large values of Rayleigh number we have convective heat transfer in macrophase, see Figs. 2–6 and Figure 8. Moreover, the conduction dominates the heat transfer in p-phase for $Ra = 10^2$ or $Ra = 10^3$, see Figure 8. The convection effect influences the heat transfer in microphase for $Ra = 10^4$ and it is more important when H and γ increase, *i.e.* the heat transfer between the phases occurs more rapidly.

Finally, we point out that the subject of a further paper can be represented by the study of this problem in triangular cavities with curved sides. To this end, we can use interpolation procedures introduced in [5] or [6].

TABLE 3. Nusselt numbers

γ	H	K_r	σ_f	$\beta = 1$		$\beta = 10$	
				Nu_f	Nu_p	Nu_f	Nu_p
10^{-2}	10^{-2}	10^{-3}	0.1	29.374	1.002	29.397	1.000
			1	21.586	1.005	21.604	1.000
		10^{-1}	0.1	31.673	6.468	29.610	1.255
			1	24.940	9.073	23.297	1.444
	10^{2}	10^{-3}	0.1	26.639	1.082	26.493	1.077
			1	18.691	1.080	18.456	1.069
		10^{-1}	0.1	32.852	6.468	28.253	1.355
			1	25.971	9.143	22.206	1.583
10^{2}	10^{-2}	10^{-3}	0.1	29.476	1.088	29.497	1.085
			1	21.659	1.088	21.675	1.081
		10^{-1}	0.1	31.673	6.437	29.694	1.320
			1	24.940	9.046	23.351	1.498
	10^{2}	10^{-3}	0.1	32.184	18.654	32.184	18.645
			1	23.149	15.264	23.149	15.250
		10^{-1}	0.1	32.359	19.876	32.371	18.812
			1	24.885	17.754	24.918	16.127

TABLE 4. Maximum absolute value of streamlines

γ	H	K_r	σ_f	$\beta = 1$		$\beta = 10$	
				$\max\lvert\psi_f\rvert$	$\max\lvert\psi_p\rvert$	$\max\lvert\psi_f\rvert$	$\max\lvert\psi_p\rvert$
10^{-2}	10^{-2}	10^{-3}	0.1	225.31	0.27	224.38	0.02
			1	123.20	0.36	122.25	0.03
		10^{-1}	0.1	140.39	16.51	220.36	2.59
			1	91.58	22.89	134.56	3.36
	10^{2}	10^{-3}	0.1	249.03	0.29	250.48	0.03
			1	151.01	0.45	152.76	0.04
		10^{-1}	0.1	142.66	16.78	225.99	2.65
			1	96.61	24.15	143.93	3.59
10^{2}	10^{-2}	10^{-3}	0.1	219.46	0.26	218.66	0.02
			1	119.92	0.35	119.07	0.03
		10^{-1}	0.1	139.89	16.45	215.70	2.53
			1	91.43	22.85	131.99	3.29
	10^{2}	10^{-3}	0.1	97.47	0.11	97.47	0.01
			1	70.23	0.21	70.24	0.02
		10^{-1}	0.1	97.84	11.51	98.02	1.15
			1	75.23	18.80	75.55	1.88

TABLE 5. Variation of results with Ra when $\sigma_f = 1$, $K_r = 10^{-1}$, $\beta = 1$, $H = 10^{-2}$, $\gamma = 10^2$

Ra	Nu_f	Nu_p	$\max\lvert\psi_f\rvert$	$\max\lvert\psi_p\rvert$
10^2	1.485	1.049	3.80	0.95
10^3	6.464	2.085	21.45	5.36
10^4	24.940	9.046	91.43	22.85

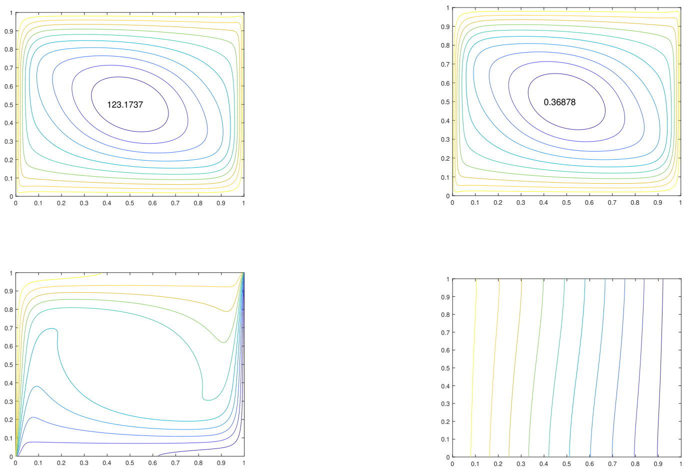


FIGURE 2. Streamlines and isotherms for $K_r = 10^{-3}$, $\sigma_f = 1$, $\beta = 1$, $H = 10^{-2}$, $\gamma = 1$: f-phase (left), p-phase (right)

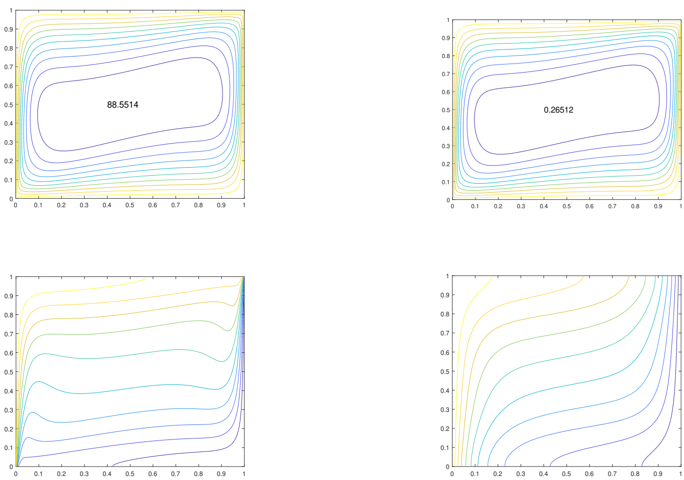


FIGURE 3. Streamlines and isotherms for $K_r = 10^{-3}$, $\sigma_f = 1$, $\beta = 1$, $H = 10^{2}$, $\gamma = 1$: f-phase (left), p-phase (right)

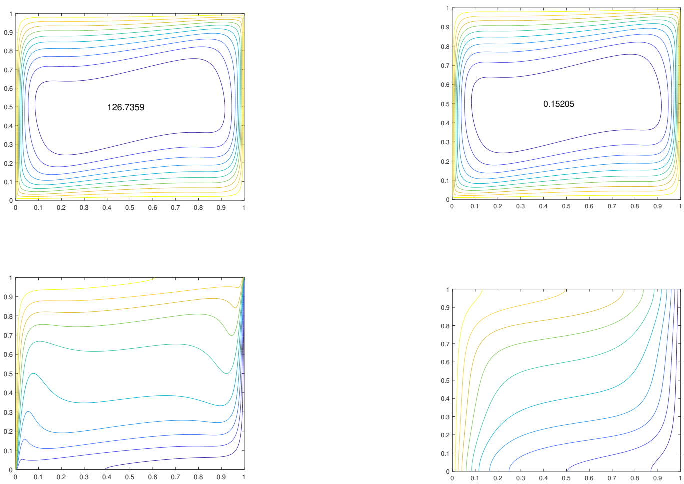


FIGURE 4. Streamlines and isotherms for $K_r = 10^{-3}$, $\sigma_f = 10^{-1}$, $\beta = 1$, $H = 10^2$, $\gamma = 1$: f-phase (left), p-phase (right)

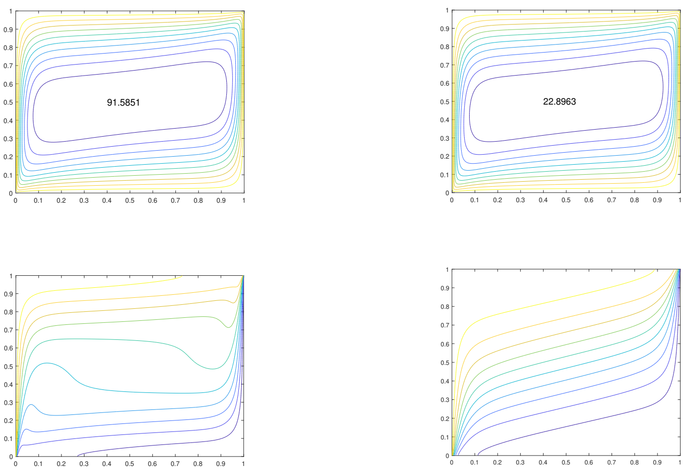


FIGURE 5. Streamlines and isotherms for $K_r = 10^{-1}$, $\sigma_f = 1$, $\beta = 1$, $H = 10^{-2}$, $\gamma = 1$: f-phase (left), p-phase (right)

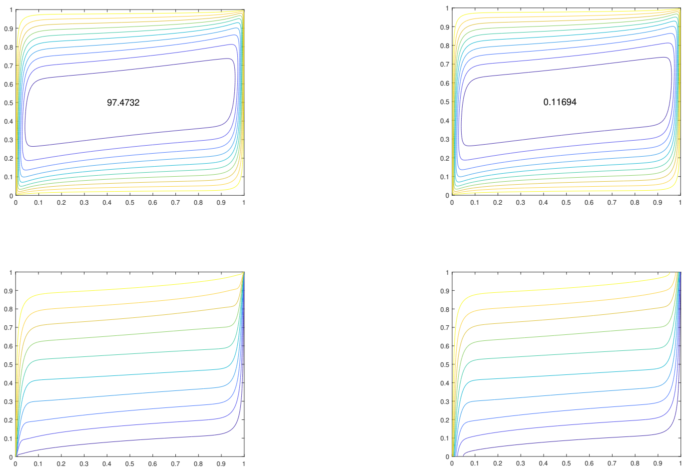
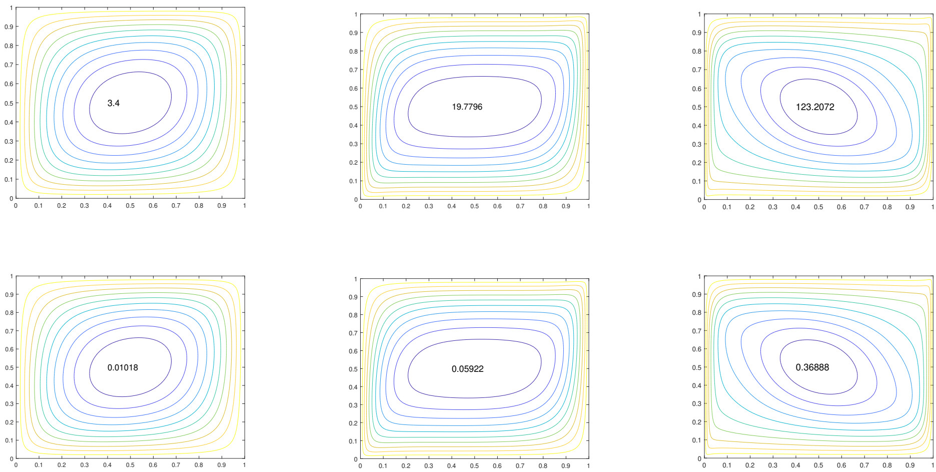
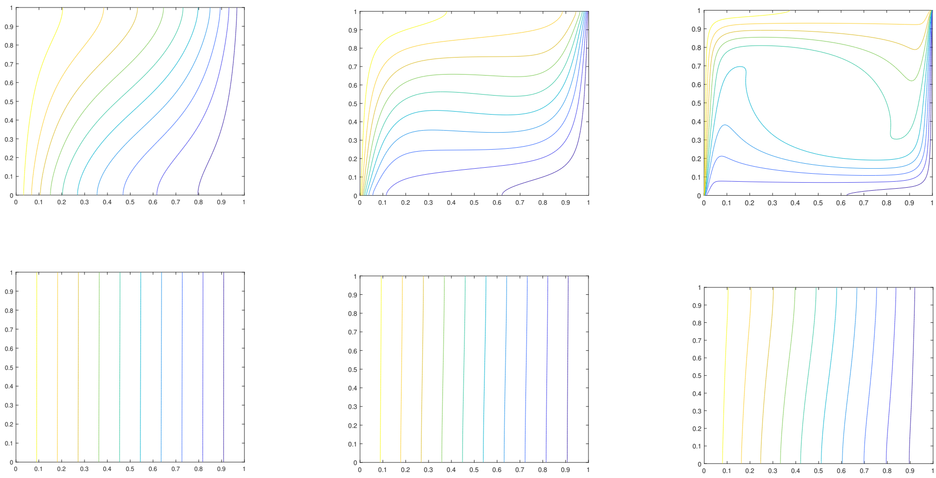


FIGURE 6. Streamlines and isotherms for $K_r = 10^{-3}$, $\sigma_f = 10^{-1}$, $\beta = 1$, $H = 10^2$, $\gamma = 10^2$: f-phase (left), p-phase (right)



(A) $Ra = 10^3$ (B) $Ra = 10^3$ (C) $Ra = 10^4$

FIGURE 7. Streamlines for $K_r = 10^{-3}$, $\sigma_f = 1$, $\beta = 1$, $H = 10^{-2}$, $\gamma = 10^{-2}$: f-phase (up), p-phase (bottom)



(A) $Ra = 10^2$ (B) $Ra = 10^3$ (C) $Ra = 10^4$

FIGURE 8. Isotherms for $K_r = 10^{-3}$, $\sigma_f = 1$, $\beta = 1$, $H = 10^{-2}$, $\gamma = 10^{-2}$: f-phase (up), p-phase (bottom)

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Book reviews

Alexey R. Alimov and Igor' G. Tsar'kov, Geometric approximation theory,
Springer Monographs in Mathematics. Cham: Springer 2022, xxi+508 p.
ISBN: 978-3-030-90950-5/hbk; 978-3-030-90953-6/pbk; 978-3-030-90951-2/ebook).

The origins of abstract approximation theory can be traced back to the years 50s of the 19th century when P.L. Chebyshev considered the problem of uniform approximation of continuous functions by polynomials in connection with some technical problems (the construction of some mechanisms as "parallelograms" which transform a circular motion into a rectilinear one, devices used for steam engines). This proves that approximation theory had, and still have, important applications in various scientific and technical domains. Since then the domain developed in many directions by the contributions of many mathematicians and applied scientists.

The present book contains an encyclopedic presentations of a lot of topics in approximation theory in concrete as well as in general Banach spaces, starting with some classical and ending with some very recent results. The first chapter contains some preliminaries. Some classical results on best approximation in the space $C[a, b]$ are presented in the second chapter, including Chebyshev alternation theorem, de la Vallée Poussin and Haar theorems and Mairhuber theorem (the space $C(Q)$ contains a Chebyshev subspace of dimension $n \geq 2$ only if Q is homeomorphic to a subset of the unit circle). Applications are given to Remez's algorithm. Best approximation by rational functions in $C[a, b]$ and in L^p is treated in the 11th chapter.

Chapter 3, *Best approximation in Euclidean spaces* (meaning inner product spaces) contains Kolmogorov criterion on the characterization of best approximation elements and Phelps theorem on the convexity of sets with Lipschitz metric projection. The 4th chapter is dedicated to some notions (approximative compactness, bounded compactness as well as their generalizations, done by Blatter, to a regular mode of convergence) that are very efficient tools in proving existence results in best approximation.

The fifths chapter is concerned with solarity properties of sets and their role in the characterization of best approximation elements, continuity and differentiability properties of the metric projection. Notice that solarity is a recurrent topic of the book. Various types of suns and the relations between them are considered in Chapter 10, *Solarity of Chebyshev sets*, including recent important contributions of the authors.

An old and still unsolved problem in best approximation is that of the convexity of Chebyshev sets – *is any Chebyshev subset of a Hilbert space convex*? In Chapter 5, *Convexity of Chebyshev sets and suns*, the authors present five proofs (of Berdyshev-Klee-Vlasov, Asplund, Konyagin, Vlasov and Brosowski) on the convexity of Chebyshev sets in $\mathbb{R}^n$. Johnson's counterexample of a nonconvex Chebyshev set in an incomplete inner product space and a presentation of Klee caverns are included as well. Other counterexamples (Dunham's example of a Chebyshev set with an isolated point, Klee's example of a discrete Chebyshev set and Koshcheev example of a disconnected sun) are given in Chapter 7, *Connectedness and approximative properties of sets*.

Chapter 8 is concerned with the existence of Chebyshev subspaces in finite and infinite dimensional spaces, with emphasis on the space $L^1(\mu)$. The influence of some geometric properties of Banach spaces (Efimov-Stechkin property, uniform convexity and uniform smoothness) on the approximative properties of their subset is discussed in the 9th chapter.

Chapter 13, *Approximation of vector-valued functions*, contains some results of Zukhovickii, Stechkin, Tsar'kov, Garkavi, Koshcheev, a.o., on the extension of the results on best approximation in spaces of real-valued functions (characterization, Haar condition, Chebyshev systems, etc) to the case of the space $C(Q, X)$, where Q is a compact Hausdorff topological space and X a Banach space.

Chapter 14 is devoted to a detailed study of Jung constant defined as the radius of the smallest ball covering any set of diameter 1. This is a very important tool in the geometry of Banach spaces with applications to best approximation and to fixed point theory for nonexpansive mappings (the inverse of Jung constant is called the coefficient of normality of the corresponding Banach space) and for condensing mappings. Chapter 15 contains a detailed study of Chebyshev centers, a notion related to best approximation (simultaneous approximation) and having important practical applications as, for instance, to optimal location problem. One studies the existence and uniqueness of Chebyshev centers, continuity, stability and selections for the Chebyshev center map, algorithms for finding Chebyshev centers and applications.

Chapter 16 is concerned with several kinds of widths (Kolmogorov, Alexandrov, Fourier, Bernstein) which are strongly related to approximation theory, allowing to compare the efficiency of the approximation by various classes of approximating sets (algebraic or trigonometric polynomials, rational functions, etc).

The last chapter, Chapter 17, *Approximation properties of arbitrary sets in linear normed spaces. Almost Chebyshev sets and sets of almost uniqueness,* is concerned with genericity properties (in the sense of Baire category) and porosity results in best approximation problems and in the study of farthest points (existence and uniqueness), a direction of research initiated by S. B. Stechkin in 1963.

The book contains also three appendices: A. *Chebyshev systems of functions in the spaces* C, C^n *and* L^p, B. *Radon, Helly and Carathódory theorems. Decomposition theorem*, and C. *Some open problems.* Some open problems are also formulated throughout the main text.

The bibliography counts 632 items.

Written by two experts with substantial contributions to the domain, this book incorporates a lot of results, both classical but also new ones situated in the focus of current research (including authors' results). It can be warmly recommended to a large community of mathematicians interested in best approximation and its relations to Banach space geometry, but it can also be used for graduate courses in approximation theory.

Notice that a two volume preliminary version of the book was published in Russian (Ontoprint, Moskva, 2017 and 2018), but the present one is entirely rewritten, updated and enlarged. (A review of the Russian edition was published in Stud. Univ. Babeş-Bolyai, Mathematica **63** (2018), no. 4.).

S. Cobzaş

Saeed Zakeri, A Course in Complex Analysis, Princeton University Press, 2021, xii+428 pages, hardback, ISBN: 9780691207582, ebook, ISBN: 9780691218502.

The book under review is an excellent introduction to Complex Analysis.

The author managed to put together in a harmonious way a large variety of classical results of the theory. Here is a list with the most important topics and results with complete self-contained proofs in the book: the Cauchy-Riemann equations, Cauchy's theorems and their homology versions, Liouville's theorem and its hyperbolic version, the identity theorem, the open mapping theorem, the maximum principle for holomorphic and harmonic functions, the residue theorem, the argument principle, Möbius maps and their dynamics, conformal metrics, the Schwarz-Pick lemma and Ahlfor's generalization, Montel's theorem and its generalization, the convergence results of Weierstrass, Hurwitz and Vitali, Marty's theorem, the Riemann mapping theorem, Koebe's distorsion bounds for the class of schlicht functions, the Carathéodory extension theorem, the solution of the Dirichlet problem on the disk with the Poisson kernel, the Fatou theorem, harmonic measures and Blaschke products, Weierstrass' factorization theorem, Jensen's formula, Mittag-Leffler's theorem, elliptic functions, Runge's theorem, Schönflies' theorem, conformal models of finitely connected domains, natural boundaries, Ostrowski's theorem, the monodromy theorem, the Schwarz reflection principle for analytic arcs, the Hausdorff measure and holomorphic removability, the Schwarz-Christoffel formula, Bloch's theorem, Schottky's theorem, Picard's theorems, Zalcman's rescaling theorem, branched coverings, the Riemann-Hurwitz formula, the modular group, the uniformization theorem for spherical domains, the characterization of hyperbolic domains, holomorphic covering maps of topological annuli.

Each chapter ends with a generous list of problems. Even though the book doesn't include the solutions, the problems have short solutions and are not too hard, but sufficiently challenging to motivate the reader to go again through the theory, and thus to understand better the key ideas of each chapter.

All the arguments are very rigorous and presented in depth, without burdening the reader with unnecessary details. The exposition is clear and intuitive with lots of suggestive examples. Moreover, the coloring of the definitions and the beautiful pictures make the study of the book a pleasant experience. Some pictures are so well designed that they represent proofs without words (a nice example is the picture that illustrates the jumping principle for the winding number). Furthermore, the historical marginal notes and the pictures of the mathematicians that obtained the results are very welcome.

As a minor drawback, we believe that the section dedicated to the covering properties of the exponential map is superfluous, taking into account the section about covering spaces, because the ideas in the particular case are pretty much the same as in the general setting.

The book is dedicated to graduate students and advanced undergraduate students. The main prerequisite is a basic background knowledge of Real Analysis, Topology and Measure Theory. In order to truly appreciate the geometric viewpoint and to enjoy the intuition behind some analytic results, we believe the reader should have some knowledge of Differential Geometry of curves and surfaces (in particular, tangent vectors, curvature of curves/surfaces, conformal maps and geodesics).

We encourage the reader to take a look also at the website of the book, where the author provides, for each chapter, additional comments, explanations, problems and an errata: http://qcpages.qc.cuny.edu/ zakeri/CAbook/ACCA.html

Mihai Iancu

Shahriar Shahriari, An Invitation to Combinatorics, Cambridge Mathematical Textbooks, xv + 613 p. 2022. ISBN 978-1-108-47654-6/hbk; 978-1-108-56870-8/ebook.

Combinatorics is a branch of mathematics that deals with counting problems and some other related concepts. Knowledge of the basic principles of combinatorics could greatly simplify the task of counting. The present book attempts at an accessible, amicable and conversational exposition of the art and the science of counting.

The first three chapters, 1. *Induction and recurrence relations*, 2. *The Pigeonhole Principle and Ramsey Theory,* and 3. *Counting, probability, balls and boxes*, are concerned with the foundational or fundamental concepts of combinatorics. These include induction, recurrence relations, the pigeonhole principle, multisets, graphs, Ramsey theory, Schur, Van der Waerden and graph Ramsey numbers, besides the fundamental principles of counting, such as the addition principle and the multiplication principle.

The next four chapters, 4. *Permutations and combinations*, 5. *Binomial and multinomial coefficients*, 6. *Stirling numbers,* and 7. *Integer partitions,* capitalize on the foundational concepts and introduce various techniques and special kinds of numbers that simplify the task of counting. These include permutations, falling factorials, combinations, binomial coefficients, lattice paths, Ming-Catalan numbers, Stirling numbers (both of the first and of the second kind), partitions of integers and pentagonal number theorem.

The last four chapters, 8. *The Inclusion-Exclusion Principle*, 9. *Generating functions*, 10. *Graph theory,* and 11. *Posets, matchings, and Boolean lattices*, are concerned with some advanced combinatorics concepts such as the inclusion-exclusion principle, combinations of multisets, restricted permutations, generating functions, basics of graph theory, posets (partially ordered sets), total orders and the matching problem.

The book also contains ten collaborative mini-projects meant for groups of three or four students to work and explore things collaboratively. There is a great emphasis on problem-solving and guided discovery.

The book has been written in a conversational style making it both accessible and engaging for the readers. The book is an excellent invitation to the world of combinatorial thinking.

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