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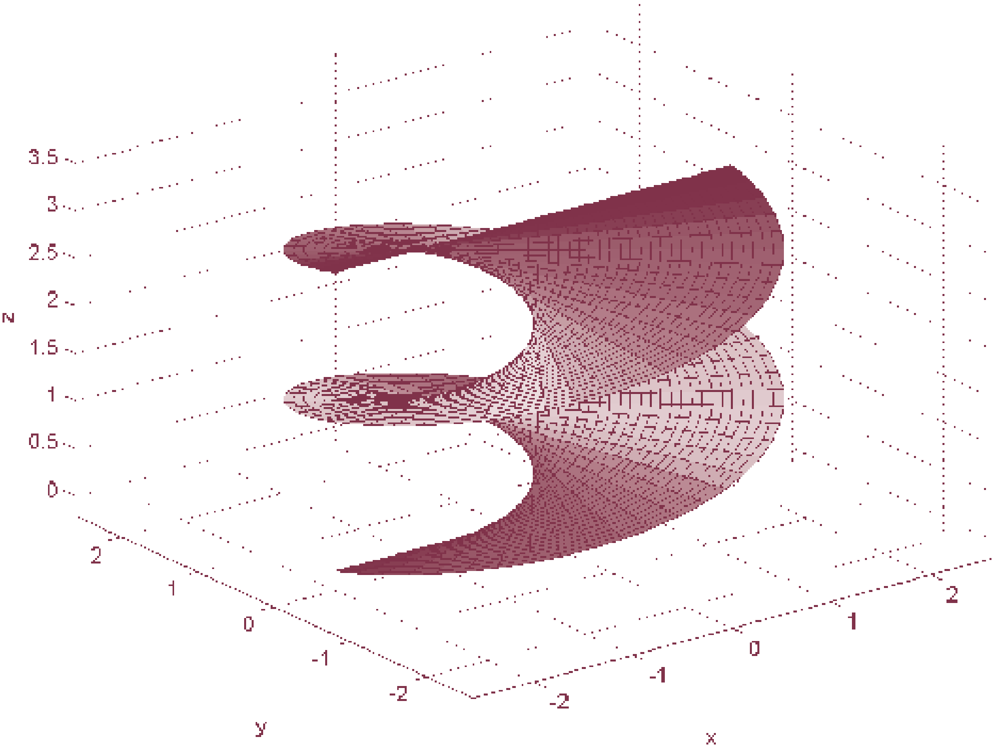

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CONTENTS

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Differential sandwich theorem for certain class of analytic functions associated with an integral operator

Luminiţa-Ioana Cotîrlă and Adriana Cătaş

Abstract. In this paper we obtain some applications of first order differential subordination and superordination result involving an integral operator for certain normalized analytic function.

Mathematics Subject Classification (2010): 30C45.

Keywords: Integral operator, subordination and superordination, analytic functions, sandwich theorem.

1. Introduction and preliminaries

Let $\mathcal{A}$ denote the class of functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k,\ a_k \geq 0, \tag{1.1}$$

which are analytic and univalent in the open unit disk $U = \{z : |z| < 1\}$.

If f and g are analytic functions in U, we say that f is subordinate to g in U, written symbolically as $f \prec g$ or $f(z) \prec g(z)$ if there exists a Schwarz function $w(z)$ analytic in U, with $w(0) = 0$ and $|w(z)| < 1$, such that $f(z) = g(w(z))$, $z \in U$. In particular, if the function g is univalent in U, the subordination $f \prec g$ is equivalent to $f(0) = g(0)$ and $f(U) \subset g(U)$ (see [2], [3]).

For the function f given by (1.1) and $g \in \mathcal{A}$ given by $g(z) = z + \sum_{k=2}^{\infty} b_k z^k$, the Hadamard product (or convolution) of f and g is defined by

$$(f * g)(z) = z + \sum_{k=2}^{\infty} a_k b_k z^k = (g * f)(z).$$

The set of all functions f that are analytic and injective on $\overline{U} - E(f)$, denote by Q where

$$E(f) = \{\zeta \in \partial U : \lim_{z \to \zeta} f(z) = \infty\}$$

and are such that $f'(\zeta) \neq 0$ for $\zeta \in \partial U \backslash E(f)$, (see [4]).

If $\psi : \mathbb{C}^3 \times U \to \mathbb{C}$ and h is univalent in U with $q \in Q$. In [3] Miller and Mocanu consider the problem of determining conditions on admissible functions ψ such that

$$\psi(p(z), zp'(z), z^2p''(z); z) \prec h(z) \tag{1.2}$$

implies that $p(z) \prec q(z)$ for all functions $p \in \mathcal{H}[a, n]$ that satisfy the differential subordination (1.2).

Let $\phi : \mathbb{C}^3 \times U \to \mathbb{C}$ and $h \in \mathcal{H}$ with $q \in \mathcal{H}[a, n]$. In [4] and [5] is studied the dual problem and determined conditions on ϕ such that

$$h(z) \prec \phi(p(z), zp'(z), z^2p''(z); z) \tag{1.3}$$

implies $q(z) \prec p(z)$ for all functions $p \in Q$ that satisfy the above subordination. They also found conditions so that the functions q is the largest function with this property, called the best subordinant of the subordination (1.3).

Let $\mathcal{H}(U)$ be the class of analytic functions in the open unit disc.

For n a positive integer and $a \in \mathbb{C}$ let

$$\mathcal{H}[a, n] = \{f \in \mathcal{H} : f(z) = a + a_n z^n + \ldots\}.$$

The integral operator I^m of a function f is defined in [6] by

$$I^0 f(z) = f(z),$$

$$I^1 f(z) = If(z) = \int_0^z f(t)t^{-1}dt,$$

$$\ldots$$

$$I^m f(z) = I\left(I^{m-1} f(z)\right), \ z \in U.$$

Lemma 1.1. [3] *Let q be univalent in U, $\zeta \in \mathbb{C}^*$ and suppose that*

$$\operatorname{Re}\left\{1 + \frac{zq''(z)}{q'(z)}\right\} > \max\left\{0, -\operatorname{Re}\left(\frac{1}{\zeta}\right)\right\}. \tag{1.4}$$

If p is analytic in U with $p(0) = q(0)$ and

$$p(z) + \zeta zp'(z) \prec q(z) + \zeta zq'(z) \tag{1.5}$$

then $p(z) \prec q(z)$ and q is the best dominant.

Lemma 1.2. [3] *Let the function q be univalent in the unit disk and let θ, φ be analytic in domain D containing $q(U)$ with $\varphi(w) \neq 0$, where $w \in q(U)$. Set*

$$Q(z) = zq'(z)\varphi(q(z)) \quad \text{and} \quad h(z) = \theta(q(z)) + Q(z).$$

Suppose that

$$Q \text{ is starlike univalent in } U;$$

$$\operatorname{Re}\left\{\frac{zh'(z)}{Q(z)}\right\} > 0, \ \text{for } z \in U.$$

If p is analytic with $p(0) = q(0)$, $p(U) \subseteq D$ and

$$\theta(p(z)) + zp'(z)\varphi(p(z)) \prec \theta(q(z)) + zq'(z)\varphi(q(z)) \tag{1.6}$$

then $p(z) \prec q(z)$ and q is the best dominant.

Lemma 1.3. [1] *Let q be convex in the unit disc U, $q(0) = a$ and $\zeta \in \mathbb{C}$, $\mathrm{Re}\,(\zeta) > 0$. If $p \in \mathcal{H}[a, 1] \cap Q$ and $p(z) + \zeta zp'(z)$ is univalent in U then*

$$q(z) + \zeta zq'(z) \prec p(z) + \zeta zp'(z) \tag{1.7}$$

implies $q(z) \prec p(z)$ and q is the best subordinant.

Lemma 1.4. [2] *Let the function q be convex and univalent in the unit disc U and θ and φ be analytic in a domain D containing $q(U)$. Suppose that*
1. $\mathrm{Re}\left\{\dfrac{\theta'(q(z))}{\varphi(q(z))}\right\} > 0$ *for $z \in U$ and*
2. $Q(z) = zq'(z)\varphi(q(z))$ *is starlike univalent in U.*
If $p \in \mathcal{H}[q(0), 1] \cap Q$ with $p(U) \subseteq D$ and $\theta(p(z)) + zp'(z)\varphi(p(z))$ is univalent in U and

$$\theta(q(z)) + zq'(z)\varphi(q(z)) \prec \theta(p(z)) + zp'(z)\varphi(p(z)) \tag{1.8}$$

then $q(z) \prec p(z)$ and q is the best subordinant.

2. Main results

Theorem 2.1. *Let q be univalent in U, with $q(0) = 1$ and $q(z) \neq 0$ for all $z \in U$, and let $\sigma \in \mathbb{C}^*$, $f \in \mathcal{A}$ and suppose that f and g satisfy the next conditions:*

$$\frac{I^{m+1}\left(f(z)\right)}{z} \neq 0, z \in U \tag{2.1}$$

and

$$\mathrm{Re}\,\left\{1 + \frac{zq''(z)}{q'(z)} - \frac{zq'(z)}{q(z)}\right\} > 0 \text{ for } z \in U. \tag{2.2}$$

If

$$\frac{I^{m}\left(f(z)\right)}{I^{m+1}\left(f(z)\right)} \prec 1 + \frac{zq'(z)}{\sigma q(z)}, \tag{2.3}$$

then

$$\left(\frac{I^{m+1}\left(f(z)\right)}{z}\right)^{\sigma} \prec q(z)$$

and q is the best dominant of (2.3).

Proof. Let

$$p(z) = \left(\frac{I^{m+1}\left(f(z)\right)}{z}\right)^{\sigma}, \ z \in U. \tag{2.4}$$

Because the integral operator I^m satisfies the identity $z\left[I^{m+1}\left(f(z)\right)\right]' = I^m\left(f(z)\right)$ and the function $p(z)$ is analytic in U, by differentiating (2.4) logarithmically with respect to z, we obtain

$$\frac{zp'(z)}{p(z)} = \sigma\left(\frac{I^{m}\left(f(z)\right)}{I^{m+1}\left(f(z)\right)} - 1\right). \tag{2.5}$$

In order to prove our result we will use Lemma 1.2. In this lemma we consider

$$\theta(w) = 1 \text{ and } \varphi(w) = \frac{1}{\sigma w},$$

then θ is analytic in $\mathbb{C}$ and $\varphi(w) \neq 0$ is analytic in $\mathbb{C}^*$. Also, if we let

$$Q(z) = zq'(z)\varphi(q(z)) = \frac{zq'(z)}{\sigma q(z)}$$

and

$$h(z) = \theta(q(z)) + Q(z) = 1 + \frac{zq'(z)}{\gamma\sigma q(z)}$$

from (2.2) we see that $Q(z)$ is a starlike function in U. We also have

$$\operatorname{Re}\left\{\frac{zh'(z)}{Q(z)}\right\} = \operatorname{Re}\left\{1 + \frac{zq''(z)}{q'(z)} - \frac{zq'(z)}{q(z)}\right\} > 0 \text{ for } z \in U$$

and then, by using Lemma 1.2 we deduce that subordination (2.3) implies $p(z) \prec q(z)$ and the function q is the best dominant of (2.3). □

Taking $q(z) = \frac{1+Az}{1+Bz}$ $(-1 \leq B < A \leq 1)$ in Theorem 2.1, it easy to check that the assumption

$$p(z) + \frac{1}{\sigma}zp'(z) \prec q(z) + \frac{\alpha}{\sigma}zq'(z)$$

holds, hence we obtain the next result.

Corollary 2.2. *Let $\sigma \in \mathbb{C}^*$ and $f \in \mathcal{A}$. Suppose*

$$\frac{I^{m+1}(f(z))}{z} \neq 0, z \in U.$$

If

$$\frac{I^m(f(z))}{I^{m+1}(f(z))} \prec 1 + \frac{z(A-B)}{\sigma(1+Az)(1+Bz)},$$

then

$$\left(\frac{I^{m+1}(f(z))}{z}\right)^{\sigma} \prec \frac{1+Az}{1+Bz}$$

and $q(z) = \frac{1+Az}{1+Bz}$ is the best dominant.

Taking $q(z) = \frac{1+z}{1-z}$ in Theorem 2.1, it easy to check that the assumption

$$p(z) + \frac{1}{\sigma}zp'(z) \prec q(z) + \frac{\alpha}{\sigma}zq'(z)$$

holds, hence we obtain the next result.

Corollary 2.3. *Let $\sigma \in \mathbb{C}^*$ and $f \in \mathcal{A}$. Suppose*

$$\frac{I^{m+1}(f(z))}{z} \neq 0, z \in U.$$

If

$$\frac{I^m(f(z))}{I^{m+1}(f(z))} \prec 1 + \frac{2z}{\sigma(1-z)(1+z)},$$

then

$$\left(\frac{I^{m+1}(f(z))}{z}\right)^{\sigma} \prec \frac{1+z}{1-z}$$

and $q(z) = \frac{1+z}{1-z}$ *is the best dominant.*

Theorem 2.4. *Let q be univalent in U, with $q(0) = 1$. Let $\sigma \in \mathbb{C}^*$ and $t, \nu, \eta \in \mathbb{C}$ with $\nu + \eta \neq 0$. Let $f \in \mathcal{A}$ and suppose that f and g satisfy the next conditions*

$$\frac{\upsilon I^{m+1}(f(z)) + \eta I^{m+2}(f(z))}{(\upsilon + \eta) z} \neq 0, z \in U \tag{2.6}$$

and

$$\operatorname{Re}\left\{1 + \frac{zq''(z)}{q'(z)}\right\} > \max\{0, -\operatorname{Re}t\}, \ z \ \in U. \tag{2.7}$$

If

$$\begin{aligned} \psi(z) = {} & t\left[\frac{\upsilon I^{m+1}(f(z)) + \eta I^{m+2}(f(z))}{(\upsilon + \eta) z}\right]^{\sigma} \\ & + \sigma\left[\frac{\upsilon z\left(I^{m+1}(f(z))\right)' + z\eta\left(I^{m+2}(f(z))\right)'}{\upsilon I^{m+1}(f(z)) + \eta I^{m+2}(f(z))} - 1\right] \end{aligned} \tag{2.8}$$

and

$$\psi(z) \prec tq(z) + \frac{zq'(z)}{q(z)} \tag{2.9}$$

then

$$\left[\frac{\upsilon I^{m+1}(f(z)) + \eta I^{m+2}(f(z))}{(\upsilon + \eta) z}\right]^{\sigma} \prec q(z)$$

and q is the best dominant.

Proof. Let

$$p(z) = \left[\frac{\upsilon I^{m+1}(f(z)) + \eta I^{m+2}(f(z))}{(\upsilon + \eta) z}\right]^{\sigma}, z \in U. \tag{2.10}$$

According to (2.3) the function $p(z)$ is analytic in U and differentiating (2.10) logarithmically with respect to z, we obtain

$$\frac{zp'(z)}{p(z)} = \sigma\left[\frac{\upsilon z\left(I^{m+1}(f(z))\right)' + z\eta\left(I^{m+2}(f(z))\right)'}{\upsilon I^{m+1}(f(z)) + \eta I^{m+2}(f(z))} - 1\right] \tag{2.11}$$

and hence

$$zp'(z) = \sigma\left[\frac{\upsilon I^{m+1}(f(z)) + \eta I^{m+2}(f(z))}{(\upsilon + \eta) z}\right]^{\sigma} \cdot \left[\frac{\upsilon z\left(I^{m+1}(f(z))\right)' + z\eta\left(I^{m+2}(f(z))\right)'}{\upsilon I^{m+1}(f(z)) + \eta I^{m+2}(f(z))} - 1\right].$$

In order to prove our result we will use Lemma 1.2. In this lemma we consider

$$\theta(w) = tw \quad \text{and} \quad \varphi(w) = \frac{1}{w}$$

then θ is analytic in $\mathbb{C}$ and $\varphi(w) \neq 0$ is analytic in $\mathbb{C}^*$. Also if we let

$$Q(z) = zq'(z)\varphi(q(z)) = \sigma\left[\frac{\upsilon z\left(I^{m+1}(f(z))\right)' + z\eta\left(I^{m+2}(f(z))\right)'}{\upsilon I^{m+1}(f(z)) + \eta I^{m+2}(f(z))} - 1\right]$$

and

$$h(z) = \theta(q(z)) + Q(z)$$

$$= t\left[\frac{\upsilon I^{m+1}(f(z)) + \eta I^{m+2}(f(z))}{(\upsilon+\eta)z}\right]^{\sigma} + \sigma\left[\frac{\upsilon z\left(I^{m+1}(f(z))\right)' + z\eta\left(I^{m+2}(f(z))\right)'}{\upsilon I^{m+1}(f(z)) + \eta I^{m+2}(f(z))} - 1\right]$$

from (2.6) we see that $Q(z)$ is a starlike function in U. We also have

$$\text{Re}\left\{\frac{zh'(z)}{Q(z)}\right\} = \text{Re}\left\{t + 1 + \frac{zq''(z)}{q'(z)}\right\} > 0 \text{ for } z \in U$$

and then, by using Lemma 1.2 we deduce that the subordination (2.9) implies $p(z) \prec q(z)$. □

Taking $q(z) = \frac{1+Az}{1+Bz}$ $(-1 \leq B < A \leq 1)$ in Theorem 2.4 and according to

$$\frac{zp'(z)}{p(z)} = \sigma\left(\frac{I^{m+1}(f(z))}{I^{m+2}(f(z))} - 1\right)$$

the condition (2.7) becomes $\max\{0, -\text{Re}(t)\} \leq \frac{1-|B|}{1+|B|}$. Hence, for the special case $\upsilon = 1$ and $\eta = 0$ we obtain the following result.

Corollary 2.5. *Let* $t \in \mathbb{C}$ *with* $\max\{0, -\text{Re}(t)\} \leq \frac{1-|B|}{1+|B|}$. *Let* $f \in \mathcal{A}$ *and suppose that*

$$\frac{I^{m+1}(f(z))}{z} \neq 0, z \in U.$$

If

$$t\left[\frac{I^{m+1}(f(z))}{z}\right]^{\sigma} + \sigma\left[\frac{z\left(I^{m+1}(f(z))\right)'}{I^{m+1}(f(z))} - 1\right] \prec t\frac{1+Az}{1+Bz} + \frac{(1-B)z}{(1+Az)(1+Bz)}$$

then

$$\left(\frac{I^{m+1}(f(z))}{z}\right)^{\sigma} \prec \frac{1+Az}{1+Bz}$$

and $q(z) = \frac{1+Az}{1+Bz}$ *is the best dominant.*

Taking $\upsilon = m = 1,\ \eta = 0$ and $q(z) = \frac{1+z}{1-z}$ in Theorem 2.1, we obtain the next result.

Corollary 2.6. *Let* $f \in \mathcal{A}$ *and suppose that* $\frac{I^2(f(z))}{z} \neq 0, z \in U$, $\sigma \in \mathbb{C}^*$. *If*

$$t\left[\frac{I^2(f(z))}{z}\right]^{\sigma} + \sigma\left[\frac{z\left(I^2(f(z))\right)'}{I^2(f(z))} - 1\right] \prec t\frac{1+z}{1-z} + \frac{2z}{(1+z)(1-z)}$$

then

$$\left[\frac{I^2(f(z))}{z}\right]^{\sigma} \prec \frac{1+z}{1-z}$$

and $q(z) = \dfrac{1+z}{1-z}$ *is the best dominant.*

Theorem 2.7. *Let q be convex in U, with $q(0)=1$. Let $\sigma\in\mathbb{C}^*$ and $t,\nu,\eta\in\mathbb{C}$ with $\nu+\eta\neq 0$ and* $\operatorname{Re} t>0$. *Let $f\in\mathcal{A}$ and suppose that f satisfies the next conditions:*

$$\frac{\upsilon I^{m+1}(f(z))+\eta I^{m+2}(f(z))}{(\upsilon+\eta)z}\neq 0, z\in U \tag{2.12}$$

and

$$\left[\frac{\upsilon I^{m+1}(f(z))+\eta I^{m+2}(f(z))}{(\upsilon+\eta)z}\right]^{\sigma}\in\mathcal{H}[q(0),1]\cap Q. \tag{2.13}$$

If the function ψ given by (2.8) *is univalent in U and*

$$tq(z)+\frac{zq'(z)}{q(z)}\prec\psi(z), \tag{2.14}$$

then

$$q(z)\prec\left[\frac{\upsilon I^{m+1}(f(z))+\eta I^{m+2}(f(z))}{(\upsilon+\eta)z}\right]^{\sigma}$$

and $q(z)$ is the best subordinant of (2.14).

Proof. Let

$$p(z)=\left[\frac{\upsilon I^{m+1}(f(z))+\eta I^{m+2}(f(z))}{(\upsilon+\eta)z}\right]^{\sigma},\ z\in U. \tag{2.15}$$

According to (2.12) the function $p(z)$ is analytic in U and differentiating (2.15) logarithmically with respect to z, we obtain

$$\frac{zp'(z)}{p(z)}=\sigma\left[\frac{\upsilon z\left(I^{m+1}(f(z))\right)'+z\eta\left(I^{m+2}(f(z))\right)'}{\upsilon I^{m+1}(f(z))+\eta I^{m+2}(f(z))}-1\right]. \tag{2.16}$$

In order to prove our result we will use Lemma 1.4. In this lemma we consider

$$Q(z)=zq'(z)\varphi(q(z))=\sigma\left[\frac{\upsilon z\left(I^{m+1}(f(z))\right)'+z\eta\left(I^{m+2}(f(z))\right)'}{\upsilon I^{m+1}(f(z))+\eta I^{m+2}(f(z))}-1\right]$$

and

$$h(z)=\theta(q(z))+Q(z)$$

$$=t\left[\frac{\upsilon I^{m+1}(f(z))+\eta I^{m+2}(f(z))}{(\upsilon+\eta)z}\right]^{\sigma}+\sigma\left[\frac{\upsilon z\left(I^{m+1}(f(z))\right)'+z\eta\left(I^{m+2}(f(z))\right)'}{\upsilon I^{m+1}(f(z))+\eta I^{m+2}(f(z))}-1\right]$$

from (2.12) we see that $Q(z)$ is a starlike function in U. We also have

$$\operatorname{Re}\left\{\frac{zh'(z)}{Q(z)}\right\}=\operatorname{Re}\left\{t+1+\frac{zq''(z)}{q'(z)}\right\}>0 \text{ for } z\in U$$

and then, by using Lemma 1.4 we deduce that the subordination (2.14) implies $q(z)\prec p(z)$ and the proof is completed. □

Corollary 2.8. *Let q_1, q_2 are two convex functions in U, with $q_1(0) = q_2(0) = 1$, $\sigma \in \mathbb{C}^*$, $t, \nu, \eta \in \mathbb{C}$ with $\nu + \eta \neq 0$ and* $\mathrm{Re}\, t > 0$. *Let $f \in \mathcal{A}$ and suppose that f satisfies the next conditions:*

$$\frac{\upsilon I^{m+1}(f(z)) + \eta I^{m+2}(f(z))}{(\upsilon + \eta) z} \neq 0, z \in U$$

and

$$\left(\frac{\upsilon I^{m+1}(f(z)) + \eta I^{m+2}(f(z))}{(\upsilon + \eta) z} \right)^{\sigma} \in \mathcal{H}[q(0), 1] \cap Q.$$

If the function $\psi(z)$ given by (2.8) *is univalent in U and*

$$tq_1(z) + \frac{zq_1'(z)}{q_1(z)} \prec \psi(z) \prec tq_2(z) + \frac{zq_2'(z)}{q_2(z)}$$

then

$$q_1(z) \prec \left(\frac{\upsilon I^{m+1}(f(z)) + \eta I^{m+2}(f(z))}{(\upsilon + \eta) z} \right)^{\sigma} \prec q_2(z) \tag{2.17}$$

and q_1, q_2 are respectively, the best subordinant and the best dominant of (2.17).

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On the stability of solutions of fractional non conformable differential equations

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Abstract. In this note we obtain sufficient conditions under which we can guarantee the stability of solutions of a fractional differential equations of non conformable type and we obtain some fractional analogous theorems of the direct Lyapunov method for a given class of equations of motion.

Mathematics Subject Classification (2010): 34A08.

Keywords: Fractional non conformable system of equations, Lyapunov second method, stability, asymptotic stability, instability.

1. Introduction

Fractional calculus concerns the generalization of differentiation and integration to non-integer (fractional) orders. The subject has a long mathematical history being discussed for the first time already in the correspondence of Leibniz with L'Hopital when this replied "What does $\frac{d^n}{dx^n}f(x)$ mean if n=$\frac{1}{2}$?" in September 30 of 1695. Over the centuries many mathematicians have built up a large body of mathematical knowledge on fractional integrals and derivatives. Although fractional calculus is a natural generalization of calculus, and although its mathematical history is equally long, it has, until recently, played a negligible role in physics. One reason could be that, until recently, the basic facts were not readily accessible even in the mathematical literature (see [13]). The nature of many systems makes that they can be more precisely modeled using fractional differential equations. The differentiation and integration of arbitrary orders have found applications in diverse fields of science and engineering like viscoelasticity, electrochemistry, diffusion processes, control theory, heat conduction, electricity, mechanics, chaos, and fractals (see [5], [6] and [13]). Lyapunov's Second or Direct Method is unique in that it does not require a characterization of the solutions to determine stability. This method often allows us to determine whether a differential equation is stable without knowing anything about what the solutions look like, so it is ideal for dealing with nonlinear systems. The method uses a supplementary function called a Lyapunov function to determine properties of the asymptotic behavior of solutions of a differential equation. It is known that the method of Lyapunov functions is a tool used in the analysis of stability, in many classes of differential equations of disturbed movement, so it is interesting to investigate an extension of the method for non-integer order systems (see [9] and [10] and bibliography there). Such extension is based on the concept of a local fractional derivative non conformable, defined by the authors in a previous paper (see [2]) which is presented below. In this paper the application of a fractional-like derivative of the Lyapunov function for the stability analysis of solutions of the equations of perturbed motion with a fractional-like derivative of the state vector is discussed. Some fractional analogous theorems of the direct Lyapunov method for a given class of equations of motion are presented.

2. Preliminary results

It is necessary to present some necessary definitions for our work. Be $\alpha \in (0,1]$ and define a continuous function $f : [t_0, +\infty) \to \mathbb{R}$.

First, let's remember the definition of $N_1^{\alpha} f(t)$, a non conformable fractional derivative of a function in a point t defined in [9] and that is the basis of our results, that are close resemblance of those found in classical qualitative theory.

Definition 2.1. Given a function $f : [t_0, +\infty) \to \mathbb{R}$, $t_0 > 0$. Then the N-derivative of f of order α is defined by

$$N_1^{\alpha} f(t) = \lim_{\varepsilon \to 0} \frac{f(t + \varepsilon e^{t^{-\alpha}}) - f(t)}{\varepsilon}$$

for all $t > 0$, $\alpha \in (0,1)$. If f is $\alpha-$differentiable in some $(0,a)$, and $\lim_{t\to 0^+} N_1^{(\alpha)} f(t)$ exists, then define $N_1^{(\alpha)} f(0) = \lim_{t \to 0^+} N_1^{(\alpha)} f(t)$.

If the N-derivative of the function $x(t)$ of order α exists and is finite in (t_0, ∞), we will say that $x(t)$ is N-differentiable in $I = (t_0, \infty)$.

Remark 2.2. The use in definition 2.1 of the limit of a certain incremental quotient, instead of the integral used in the classical definitions of fractional derivatives, allows us to give the following interpretation of the N-derivative. Suppose that the point moves in a straight line in $\mathbb{R}_+$. For the moments $t_1 = t$ and $t_2 = t + he^{t^{-\alpha}}$ where $h > 0$ and $\alpha \in (0,1]$ and we denote $S(t_1)$ and $S(t_2)$ the path traveled by point P at time t_1 and t_2 so we have

$$\frac{S(t_2) - S(t_1)}{t_2 - t_1} = \frac{S(t + he^{t^{-\alpha}}) - S(t)}{he^{t^{-\alpha}}}$$

this is the average N-speed of point P over time $he^{t^{-\alpha}}$. Let's consider

$$\operatorname*{Lim}_{h\to 0} \frac{S(t + he^{t^{-\alpha}}) - S(t)}{he^{t^{-\alpha}}}.$$

When $\alpha = 1$, this is the usual instantaneous velocity of a point P at any time $t > 0$. If $\alpha \in (0, 1)$ this is the instantaneous q-speed of the point P for any $t > 0$. Therefore, the physical meaning of the N-derivative is the instantaneous q-change rate of the state vector of the considered mechanics or another nature of the system.

Remark 2.3. The N-derivative solves almost all the insufficiencies that are indicated to the classical fractional derivatives. In particular we have the following result.

Theorem 2.4. *(See [2]) Let f and g be N-differentiable at a point $t > 0$ and $\alpha \in (0, 1]$. Then*

a) $N_1^\alpha(af + bg)(t) = aN_1^\alpha(f)(t) + bN_1^\alpha(g)(t)$.
b) $N_1^\alpha(t^p) = e^{t^{-\alpha}} pt^{p-1},\ p \in \mathbb{R}$.
c) $N_1^\alpha(\lambda) = 0,\ \lambda \in \mathbb{R}$.
d) $N_1^\alpha(fg)(t) = fN_1^\alpha(g)(t) + gN_1^\alpha(f)(t)$.
e) $N_1^\alpha\left(\dfrac{f}{g}\right)(t) = \dfrac{gN_1^\alpha(f)(t) - fN_1^\alpha(g)(t)}{g^2(t)}$.
f) *If, in addition, f is differentiable then $N_1^\alpha(f) = e^{t^{-\alpha}} f'(t)$.*
g) *Being f differentiable and $\alpha = n$ integer, we have $N_1^n(f)(t) = e^{t^{-n}} f'(t)$.*

Remark 2.5. The relations a), c), d) and e) are similar to the classical results mathematical analysis, these relationships are not established (or do not occur) for fractional derivatives of global character (see [5] and [13] and bibliography there). The relation c) is maintained for the fractional derivative of Caputo. Cases c), f) and g) are typical of this non conformable local fractional derivative.

Now we will present the equivalent result, for N_1^α, of the well-known chain rule of classic calculus and that is basic in the Second Method of Lyapunov, for the study of stability of perturbed motion.

Theorem 2.6. (See [2]) *Let $\alpha \in (0, 1]$, g N-differentiable at $t > 0$ and f differentiable at $g(t)$ then*

$$N_1^\alpha(f \circ g)(t) = f'(g(t))N_1^\alpha g(t).$$

Definition 2.7. The non conformable fractional integral of order α is defined by the expression

$${}_NJ_{t_0}^\alpha f(t) = \int_{t_0}^t \frac{f(s)}{e^{s^{-\alpha}}} ds.$$

The following statement is analogous to the one known from the Ordinary Calculus.

Theorem 2.8. *Let f be N-differentiable function in (t_0, ∞) with $\alpha \in (0, 1]$. Then for all $t > t_0$ we have*

a) *If f is differentiable ${}_NJ_{t_0}^\alpha\left(N_1^\alpha f(t)\right) = f(t) - f(t_0)$.*
b) $N_1^\alpha\left({}_NJ_{t_0}^\alpha f(t)\right) = f(t)$.

Proof. a) From definition we have

$${}_NJ_{t_0}^\alpha\left(N_1^\alpha f(t)\right) = \int_{t_0}^t \frac{N_1^\alpha f(s)}{e^{s^{-\alpha}}} ds = \int_{t_0}^t \frac{f'(s)e^{s^{-\alpha}}}{e^{s^{-\alpha}}} ds = f(t) - f(t_0).$$

b) Analogously we have

$$N_1^{\alpha}\left({}_N J_{t_0}^{\alpha} f(t)\right) = e^{t^{-\alpha}} \frac{d}{dt}\left[\int_{t_0}^{t} \frac{f(s)}{e^{s^{-\alpha}}} ds\right] = f(t). \qquad \square$$

3. N-derivative of the Lyapunov function and conditions of stability and instability of movement

Consider the following system of fractional N-differential equations

$$N_1^{\alpha} x(t) = f(t, x(t)), \tag{3.1}$$

$$x(t_0) = x_0. \tag{3.2}$$

where $x \in \mathbb{R}^n$, $f \in C(\mathbb{R}_+ \times \mathbb{R}^n, \mathbb{R}^n)$, $t_0 > 0$. It is further assumed that for $(t_0, x_0) \in \operatorname{int}(\mathbb{R}_+ \times \mathbb{R}^n)$ the initial value problem (3.1-3.2) has a solution $x(t) \in C^{\alpha}(I)$ for all $t > t_0 > 0$. In addition, it is assumed that $f(t, 0) = 0$ for all $t > t_0 > 0$.

Let for equation (3.1) a Lyapunov-type function $V(t, x) \in C^{\alpha}(I \times \mathbb{R}^n)$ be constructed in some way such that $V(t, 0) = 0$ for all $t > 0$. Introduce the notation $S_r = \{x \in \mathbb{R}^n : \|x\| < r, r > 0\}$.

Definition 3.1. Let V be a continuous and α-differentiable function (scalar or vector), $V : I \times S_r \to \mathbb{R}^p (p = 1$ or $p = m$, respectively), and $x(t)$ be the solution of the IVP (3.1-3.2), which exists and is defined on $I \times S_r$. Corresponding to V(t,x) we define for $(t, x) \in I \times S_r$ the function

$${}_+N_{(3.1)}^{\alpha} V(t, x) = \lim_{h \to 0} \sup \frac{[V(t + h, x + h^{\alpha} f(t, x)) - V(t, x)]}{h} \tag{3.3}$$

is the N-derivative of $V(t, x)$ with respect to the system (3.1) (or along the solutions of system (3.1)).

We will now present the results analogous to those known from the Second Method of Lyapunov, for the study of the stability of systems (3.1).

With $C(\mathbb{R})$ and $CI(\mathbb{R})$ we respectively denote the families of continuous functions and increasing continuous functions defined on $\mathbb{R}$.

Definition 3.2. (see [11]). $CS(\mathbb{R}) = \{h \in C(\mathbb{R}) : \ xh(x) > 0, \ x \neq 0\}$

Definition 3.3. (see [11]). $CC(\mathbb{R}) := CI(\mathbb{R}) \cap CS(\mathbb{R})$.

Definition 3.4. A continuous function $\beta : [0, t) \to [0, +\infty)$ is said to belong to class-K if it is strictly increasing and $\beta(0) = 0$.

Theorem 3.5. *Suppose that for the system (3.1) there is a function N-differentiable $V(t, x)$ and the functions $a, b \in K$, such that*

i) $V(t, x) \geq a(\|x\|)$,
ii) $V(t, x) \leq b(\|x\|)$, *and*

$${}_+N_{(3.1)}^{\alpha} V(t, x) \leq 0, \tag{3.4}$$

for all $(t, x) \in I \times S_r$. Then the solution $x = 0$ of the system (3.1) is uniformly stable.

Proof. Let $x(t)$ the solution of system (3.1) which satisfies the initial condition $(t_0, x_0) \in I \times S_r$ and that exists for all $t \geq t_0$. Let $t_0 \in I$ and $0 < \varepsilon < r$. Under conditions i), ii) of the theorem let's choose $\delta = \delta(\varepsilon) > 0$ such that

$$b(\delta) < a(\varepsilon) \tag{3.5}$$

Let's prove that if $\|x_0\| < \delta$ then $\|x(t)\| < \varepsilon$ for all $t \geq t_0$. If this were not true, then there is a solution $x = x(t)$ such that for $\|x_0\| < \delta$ there exists $t_1 > t_0$ what satisfies $\|x(t_1)\| = \varepsilon$, and $\|x(t)\| < \varepsilon$ for all $t \in [t_0, t_1)$.

Under Theorem 8 and condition (3.2), we have

$$V(t, x(t)) - V(t_0, x_0) =_N J^{\alpha}_{t_0} \left(N^{\alpha}_1 V(t, x(t))\right)$$

and so

$$V(t, x(t)) - V(t_0, x_0) \leq 0 \tag{3.6}$$

of this last inequality for $t = t_1$ we get

$$a(\varepsilon) \leq V(t_1, x(t_1)) \leq V(t_0, x_0) \leq b(\|x\|) < a(\varepsilon) \tag{3.7}$$

The resulting inequality is evidently false. This proves Theorem (3.5). □

Next, we present the conditions that guarantee the asymptotic stability of the null solution of the fractional system (3.1).

Theorem 3.6. *In addition to the conditions i)-ii) of the previous theorem, suppose that instead of condition (*3.4*), we have*

$$_+N^{\alpha}_{(3.1)} V(t, x) \leq -c(\|x\|), \tag{3.8}$$

*for all $(t, x) \in I \times S_r$ and c is a function of class K. Then the solution $x = 0$ of the system (*3.1*) is uniform asymptotically stable.*

Proof. Under the conditions of the theorem, the solution $x = 0$ of the system (3.1) is uniformly stable since the conditions of the previous theorem are satisfied. We show that this solution is uniformly asymptotically stable.

Let $0 < \varepsilon < r$ and $\delta = \delta(\varepsilon) > 0$ as before. For $\varepsilon_0 \leq r$ let's choose $\delta_0 = \delta_0(\varepsilon_0) > 0$ and we consider the solution $x(t)$ with initial conditions $t_0 \in I$ and $\|x_0\| < \delta_0$. For $t_0 < t \leq t_0 + T(\varepsilon)$, where $T(\varepsilon)$ will be defined by an implicit expression that will be specified later, such a solution satisfies $\|x(t)\| \geq \delta(\varepsilon)$. Let's prove that under the conditions of the theorem this is impossible. From (3.8) and Theorem 2.8 we obtain

$$\begin{aligned} V(t, x(t)) - V(t_0, x_0) &= {}_N J^{\alpha}_{t_0} \left(N^{\alpha}_1 V(t, x(t))\right) \leq -{}_N J^{\alpha}_{t_0} \left(c(\|x(t)\|)\right) \\ V(t, x(t)) - V(t_0, x_0) &\leq -\int_{t_0}^{t} \frac{c(\|x(t)\|)}{e^{(s-t_0)^{-\alpha}}} ds. \end{aligned} \tag{3.9}$$

We denote by

$${}_N J^{\alpha}_{t_0}(e) = \int_{t_0}^{t} \frac{ds}{e^{(s-t_0)^{-\alpha}}} = E(t) - E(t_0),$$

so we have from (3.9)

$$V(t, x(t)) \leq b(\delta_0) - c(\delta(\epsilon))_N J^{\alpha}_{t_0}(e) \tag{3.10}$$

For $t = t_0 + T(\varepsilon)$ the inequality (3.10) we can write it as

$$0 < a(\delta(\varepsilon)) \leq V(t_0 + T(\varepsilon), x(t_0 + T(\varepsilon))) \leq b(\delta_0) - c(\delta(\varepsilon))[E(T(\varepsilon))] \leq 0.$$

This contradiction shows that there is $t_1 \in [t_0, t_0 + T(\varepsilon)]$ for which $\|x(t_1)\| < \delta(\varepsilon)$. Therefore, the estimate $\|x(t)\| < \varepsilon$ is true for all $t \geq t_0 + T(\varepsilon)$ as $\|x_0\| < \delta_0$ and $\operatorname*{Lim}_{t\to\infty} \|x(t)\| = 0$ uniformly in $t_0 \in I$. This proves Theorem. □

Next, we will establish the conditions for the instability of the solution $x = 0$ of the system (3.1).

Theorem 3.7. *Suppose that for the system (*3.1*) there is an N-differentiable Lyapunov function V(t,x) such that on $I \times B_q$ with $B_q \subset B_\varepsilon$ satisfies the assumptions*

i) $0 < V(t, x) \leq b(\|x\|)$,

ii) ${}_{+}N^{\alpha}_{(3.1)} V(t, x) \leq \lambda V(t, x) + W(t, x)$, *with* $\lambda > 0$ *and* $V : I \times B_q \to \mathbb{R}_+$, $W(t, x) \geq 0$;

iii) *the solution* $x = 0$ *belongs to* δB_q*;*

iv) $V(t, x) = 0$ *on* $I \times (\delta B_q \cap B_\varepsilon)$.

Then the solution $x = 0$ is unstable.

Proof. From assumptions ii), and Theorem 2.8 we have

$$V(t, x(t)) \geq V(t_0, x(t_0)) \exp\left[\lambda_N J^{\alpha}_{t_0}(e)\right], t \geq t_0, \tag{3.11}$$

Let the solution with initial condition $x_0 \in N$ be a neighborhood of $x = 0$. So that for any $t \geq t_0$ satisfying the estimate (3.11) along the solution $x(t)$, then it is clear that for $t \to \infty$, the function $V(t, x(t))$ grows indefinitely, whereas under the conditions of Theorem 3.5 is bounded. Therefore, for the solution $x(t)$ there exists t' such that $x(t')$ will leave the region B_ε. This shows the instability of the solution $x = 0$ of the system (3.1), which proves the theorem. □

Example 3.8. We consider the Lienard N-fractional system

$$\begin{cases} N^{\alpha}_1 x(t) = y - F(t) \\ N^{\alpha}_1 y(t) = -g(x(t)) \end{cases} \tag{3.12}$$

with

$$F(x) = \int_0^x f(r)dr$$

and we take the Lyapunov Function

$$V(t, x, y) = \frac{y^2}{2} + G(x), \tag{3.13}$$

with

$$G(x) = \int_0^x g(s)ds.$$

Under assumptions on the continuous functions f and g:

1. $f(x) > 0$ for all $x \in \mathbb{R}$,
2. $xg(x) > 0$ for all $x \neq 0$,

we have the stability of solution $x = y = 0$ of system (3.12). From (3.13) we have

$$_{+}N^{\alpha}_{(3.1)}V(t,x,y) = -g(x)F(x) \leq 0.$$

By virtue of Theorem 3.5 the desired result is obtained.

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On oscillatory second order nonlinear impulsive systems of neutral type

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Abstract. In this work, the necessary and sufficient conditions for oscillation of a class of second order neutral impulsive systems are established and our impulse satisfies a discrete neutral nonlinear equation of similar type. Further, one illustrative example showing applicability of the new result is included.

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1. Introduction

Impulsive differential equations are now recognized as an excellent source of models to simulate processes and phenomena observed in control theory, physics, chemistry, population dynamics, industrial robotics, biotechnologies, economics and to mention a few. Due to the wide range application of this theory to the real world problem, a good number of interests has been given to study impulsive differential equations, since it is much richer than the corresponding theory of differential equations without impulse effect. We refer the readers to the monographs [1, 2, 10, 13, 14] and [18], where a number of properties of their solutions are discussed and the references cited there in.

In [28], Tripathy has considered the impulsive system

$$(E_1)\begin{cases}\big(y(t)+p(t)y(t-\tau)\big)'+q(t)G\big(y(t-\sigma)\big)=0,\ t\neq\tau_k,\ k\in\mathbb{N},\\ \Delta\big(y(\tau_k)+p(\tau_k)y(\tau_k-\tau)\big)+q(\tau_k)G\big(y(\tau_k-\sigma)\big)=0,\ k\in\mathbb{N},\end{cases}$$

and studied the oscillatory character of solutions of the system. For all ranges of $p(t)$, he has established the oscillation criteria for the impulsive system (E_1) which is highly nonlinear and G could be linear, sublinear or superlinear. In [29], Tripathy

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and Santra have made an attempt to establish the necessary and sufficient condition for oscillation of a class of forced impulsive differential equations of the form

$$\begin{cases} \big(y(t)+p(t)y(t-\tau)\big)' + q(t)G\big(y(t-\sigma)\big) = f(t), & t \neq \tau_k,\ k \in \mathbb{N}, \\ \Delta\big(y(\tau_k)+p(\tau_k)y(\tau_k-\tau)\big) + r(\tau_k)G\big(y(\tau_k-\sigma)\big) = g(\tau_k), & k \in \mathbb{N}. \end{cases}$$

In an another paper [30], Tripathy and Santra have studied the characterization of the impulsive system

$$(E_2)\begin{cases} \big(y(t)-ry(t-\tau)\big)' + qy(t-\sigma) = 0,\ t \neq \tau_k,\ k \in \mathbb{N}, \\ \Delta\big(y(\tau_k)-ry(\tau_k-\tau)\big) + py(\tau_k-\sigma) = 0,\ k \in \mathbb{N}, \end{cases}$$

and linearized oscillation of the system

$$(E_3)\begin{cases} \big(y(t)-r(t)g(y(t-\tau))\big)' + q(t)f\big(y(t-\sigma)\big) = 0,\ t \neq \tau_k,\ k \in \mathbb{N}, \\ \Delta\big(y(\tau_k)-r(\tau_k)g(y(\tau_k-\tau))\big) + p(\tau_k)f\big(y(\tau_k-\sigma)\big) = 0, \quad k \in \mathbb{N}. \end{cases}$$

They have established the conditions pertaining the oscillation of the system (E_2) using the pulsatile constant and hence the linearized oscillation results carried out for (E_3) by using its limiting equation (E_2).

Motivated by the works [28, 29, 30], an attempt is made here to discuss the oscillation properties of a class of second order neutral impulsive system of the form:

$$(E)\begin{cases} \big(r(t)(y(t)+p(t)y(t-\tau))'\big)' + q(t)G\big(y(t-\sigma)\big) = 0, & t \neq \tau_k, k \in \mathbb{N}, \\ \Delta\big(r(\tau_k)(y(\tau_k)+p(\tau_k)y(\tau_k-\tau))'\big) + q(\tau_k)G\big(y(\tau_k-\sigma)\big) = 0, & k \in \mathbb{N}, \end{cases}$$

where $\tau, \sigma \in \mathbb{R}_+ = (0, +\infty)$; $\tau_1, \tau_2, \cdots, \tau_k, \cdots$ are the fixed moments of impulse effect; $p(\tau_k)$, $r(\tau_k)$ and $q(\tau_k)$ are real sequences for $k \in \mathbb{N}$; $G \in C(\mathbb{R}, \mathbb{R})$ is nondecreasing such that $xG(x) > 0$ for $x \neq 0$; $q, r \in C(\mathbb{R}_+, \mathbb{R}_+)$; $p \in PC(\mathbb{R}_+, \mathbb{R})$, and

$$\Delta\big(r(\tau_k)z'(\tau_k)\big) = r(\tau_k+0)z'(\tau_k+0) - r(\tau_k-0)z'(\tau_k-0);$$
$$y(\tau_k-0) = y(\tau_k) \quad \text{and} \quad y(\tau_k-\tau-0) = y(\tau_k-\tau), \quad k \in \mathbb{N}.$$

The objective of this work is to establish the necessary and sufficient conditions for oscillation of the impulsive system (E). Here, we are concerned with oscillating systems which remain oscillating after being perturbed by instantaneous change of state. We may note that this type of work is very rare in the literature signifying that the impulse of the differential equation follows a difference equation of same type. In this direction, we refer the reader to some of the related works [3, 4, 5, 6, 7, 8, 9, 11, 12, 15, 16, 17, 19, 26, 27, 32, 33, 34] and the references cited there in.

Definition 1.1. A function $y : [-\rho, +\infty) \to \mathbb{R}$ is said to be a solution of (E) with initial function $\phi \in C([-\rho, 0], \mathbb{R})$, if $y(t) = \phi(t)$ for $t \in [-\rho, 0]$, $y \in PC(\mathbb{R}_+, \mathbb{R})$, $z(t) = y(t) + p(t)y(t-\tau)$ and $r(t)z'(t)$ are continuously differentiable for $t \in \mathbb{R}_+$, and $y(t)$ satisfies (E) for all sufficiently large $t \geq 0$, where $\rho = \max\{\tau, \sigma\}$, $PC(\mathbb{R}_+, \mathbb{R})$ is the set of all functions $U : \mathbb{R}_+ \to \mathbb{R}$ which are continuous for $t \in \mathbb{R}_+, t \neq \tau_k, k \in \mathbb{N}$, continuous from the left- side for $t \in \mathbb{R}_+$, and have discontinuity of the first kind at the points $\tau_k \in \mathbb{R}_+, k \in \mathbb{N}$.

Definition 1.2. A nontrivial solution $y(t)$ of (E) is said to be nonoscillatory, if there exists a point $t_0 \geq 0$ such that $y(t)$ has a constant sign for $t \geq t_0$. Otherwise, the solution $y(t)$ is said to be oscillatory.

Definition 1.3. A solution $y(t)$ of (E) is said to be regular, if it is defined on some interval $[T_y, +\infty) \subset [t_0, +\infty)$ and

$$\sup\{|y(t)| : t \geq T_y\} > 0$$

for every $T_y \geq T$. A regular solution $y(t)$ of (E_1) is said to be eventually positive (eventually negative), if there exists $t_1 > 0$ such that $y(t) > 0$ $(y(t) < 0)$ for $t \geq t_1$.

2. Main results

This section deals with the necessary and sufficient conditions for oscillation of all solutions of the impulsive system (E). We introduce the following assumptions for our use in the sequel:

(A_0) $\int_0^\infty \frac{dt}{r(t)} < \infty$ if and only if $\sum_{k=1}^\infty \frac{1}{r(\tau_k)} < \infty$;
(A_1) $0 < \tau_1 < \tau_2 < \cdots$ and $\lim_{k\to\infty} \tau_k = +\infty$;
(A_2) $p \in PC(\mathbb{R}_+, \mathbb{R})$, $p_k = p(\tau_k - 0) = p(\tau_k)$, $r_k = r(\tau_k - 0) = r(\tau_k)$ and $q_k = q(\tau_k - 0) = q(\tau_k)$, $k \in \mathbb{N}$.

Theorem 2.1. *Let $-1 < -a \leq p(t) \leq 0$, $a > 0$ and $t \in \mathbb{R}_+$. Assume that (A_0), (A_1) and (A_2) hold. Furthermore, assume that*

(A_3) $G(-u) = -G(u)$, $u \in \mathbb{R}$

and

(A_4) $\int_\sigma^\infty q(t)G\big(CR(t-\sigma)\big)dt + \sum_{k=1}^\infty q(\tau_k)G\big(CR(\tau_k - \sigma)\big) < +\infty$ *for every constant* $C > 0$

hold, where $R(t) = \int_0^t \frac{ds}{r(s)}$. Then every unbounded solution of the system (E) oscillates if and only if

(A_5) $\int_0^\infty \frac{ds}{r(s)} < +\infty.$

Proof. Let $y(t)$ be a regular solution of (E) which is unbounded. So, there exists $t_0 > 0$ such that $y(t) > 0$ or < 0, for $t \geq t_0$. Without loss of generality and because of (A_3), we may assume that $y(t) > 0$, $\quad y(t-\tau) > 0$ and $y(t-\sigma) > 0$, for $t \geq t_1 > t_0 + \rho$. Setting

$$z(t) = y(t) + p(t)y(t-\tau) \tag{2.1}$$

in the system (E), it follows that

$$\begin{aligned} \big(r(t)z'(t)\big)' &= -q(t)G\big(y(t-\sigma)\big) < 0, \quad t \neq \tau_k \\ \Delta\big(r(\tau_k)z'(\tau_k)\big) &= -q_k\, G\big(y(\tau_k - \sigma)\big) < 0, \quad k \in \mathbb{N} \end{aligned} \tag{2.2}$$

for $t \geq t_1$. Hence, there exists $t_2 > t_1$ such that $r(t)z'(t)$ is nonincreasing on $[t_2, \infty)$. Since $z(t)$ is monotonic, then there exists $t_3 > t_2$ such that $z(t) > 0$ or < 0, for

$t \geq t_3$. Indeed, $z(t) < 0$ for $t \geq t_3$ implies that $y(t) < y(t-\tau)$, $y(\tau_k) < y(\tau_k - \tau)$, $y(\tau_k + 0) < y(\tau_k + 0 - \tau)$ and hence

$$\begin{aligned} &y(t) < y(t-\tau) < y(t-2\tau) < \cdots < y(t_3), \quad t \neq \tau_k, \\ &y(\tau_k) < y(\tau_k - \tau) < y(\tau_k - 2\tau) < \cdots < y(t_3), \quad k \in \mathbb{N}, \\ &y(\tau_k + 0) < y(\tau_k + 0 - \tau) < y(\tau_k + 0 - 2\tau) < \cdots < y(t_3), \quad k \in \mathbb{N}, \end{aligned}$$

that is, $y(t)$ is bounded, which is absurd. Hence, $z(t) > 0$ for $t \geq t_3$. If $r(t)z'(t) > 0$ for $t \geq t_3$, then $r(t)z'(t)$ is nonincreasing on $[t_3, \infty)$ and hence there exist a constant $C > 0$ and $t_4 > t_3$ such that $r(t)z'(t) \leq C$ for $t \geq t_4$. Consequently,

$$z(t) \leq z(t_4) + \sum_{t_4 \leq \tau_k < t} z'(\tau_k) + C \int_{t_4}^{t} \frac{ds}{r(s)},$$

since $r(\tau_k)z'(\tau_k) \leq C$. Therefore, the last inequality becomes

$$z(t) \leq z(t_4) + C \left[\int_{t_4}^{t} \frac{ds}{r(s)} + \sum_{t_4 \leq \tau_k < t} \frac{1}{r(\tau_k)} \right] < \infty,$$

as $t \to \infty$ due to (A_0). On the other hand, $y(t)$ is unbounded, and thus there exists $\{\eta_n\}$ such that $\eta_n \to \infty$ as $n \to \infty$, $y(\eta_n) \to \infty$ as $n \to \infty$ and

$$y(\eta_n) = \max\{y(s) : t_3 \leq s \leq \eta_n\}.$$

Therefore,

$$\begin{aligned} z(\eta_n) &= y(\eta_n) + p(\eta_n)y(\eta_n - \tau) \\ &\geq (1-a)y(\eta_n) \to +\infty, \quad \text{as} \quad t \to \infty \end{aligned}$$

implies that $z(t)$ (ultimately $z(\tau_k)$ for $k \in \mathbb{N}$) is unbounded, a contradiction.
Obviously, the case $r(t)z'(t) < 0,\ z(t) > 0$ for $t \geq t_3$ is not possible.
Hence, every unbounded solution of the system (E) oscillates.
Next, we suppose that (A_5) doesn't hold. Assume that

$$\int_0^{\infty} \frac{ds}{r(s)} = +\infty$$

and due to our assumption (A_4), let

$$\int_T^{\infty} q(t)G\big(CR(t-\sigma)\big)dt + \sum_{k=1}^{\infty} q_k\, G\big(CR(\tau_k - \sigma)\big) \leq \frac{C}{4},\ C > 0.$$

Let's consider

$$\begin{aligned} M = \{y : y \in C([T-\rho, +\infty), \mathbb{R}), y(t) = 0 \qquad for \quad t \in [T-\rho, T] \quad \text{and} \\ \frac{C}{4}\big[R(t) - R(T)\big] \leq y(t) \leq C\big[R(t) - R(T)\big]\ \} \end{aligned}$$

and define $\Phi : M \to C([T-\rho, +\infty), \mathbb{R})$ such that

$$(\Phi y)(t) = \begin{cases} 0, & t \in [T-\rho, T) \\ -p(t)y(t-\tau) + \int_T^t \frac{1}{r(u)} \big[\frac{C}{4} + \int_u^\infty q(s)G\big(y(s-\sigma)\big)ds \\ + \sum_{k=1}^\infty q_k G\big(y(\tau_k - \sigma)\big)\big]du, & t \geq T. \end{cases}$$

For every $y \in M$,

$$\begin{aligned} (\Phi y)(t) &\geq \int_T^t \frac{1}{r(u)} \left[\frac{C}{4} + \int_u^\infty q(s)G\big(y(s-\sigma)\big)ds + \sum_{k=1}^\infty q_k G\big(y(\tau_k - \sigma)\big) \right] du \\ &\geq \frac{C}{4} \int_T^t \frac{du}{r(u)} = \frac{C}{4}\big[R(t) - R(T)\big] \end{aligned}$$

and $y(t) \leq CR(t)$ implies that

$$\begin{aligned} (\Phi y)(t) &\leq -p(t)y(t-\tau) + \frac{C}{2} \int_T^t \frac{du}{r(u)} \\ &\leq aC\big[R(t-\tau) - R(T)\big] + \frac{C}{2}\big[R(t) - R(T)\big] \\ &\leq aC\big[R(t) - R(T)\big] + \frac{C}{2}\big[R(t) - R(T)\big] \\ &= \left(a + \frac{1}{2}\right) C\big[R(t) - R(T)\big] \\ &\leq C\big[R(t) - R(T)\big] \end{aligned}$$

implies that $(\Phi y)(t) \in M$. Define $u_n : [T-\rho, +\infty) \to \mathbb{R}$ by the recursive formula

$$u_n(t) = \big(\Phi u_{n-1}\big)(t), \quad n \geq 1,$$

with the initial condition

$$u_0(t) = \begin{cases} 0, & t \in [T-\rho, T) \\ \frac{C}{4}\big[R(t) - R(T)\big], & t \geq T. \end{cases}$$

Inductively it is easy to verify that

$$\frac{C}{4}\big[R(t) - R(T)\big] \leq u_{n-1}(t) \leq u_n(t) \leq C\big[R(t) - R(T)\big].$$

for $t \geq T$. Therefore for $t \geq T - \rho$, $\lim_{n\to\infty} u_n(t)$ exists. Let

$$\lim_{n\to\infty} u_n(t) = u(t) \text{ for } t \geq T - \rho.$$

By the Lebesgue's dominated convergence theorem $u \in M$ and $\big(\Phi u\big)(t) = u(t)$, where $u(t)$ is a solution of the impulsive system (E) on $[T-\rho, \infty)$ such that $u(t) > 0$. Hence, (A_5) is necessary. This completes the proof of the theorem. □

Remark 2.1. In Theorem 2.1, G could be linear, sublinear or superlinear.

Theorem 2.2. *Let $-1 < -a \leq p(t) \leq 0$, $a > 0$ for $t \in \mathbb{R}_+$. Assume that $(A_1) - (A_3)$ and (A_5) hold. Furthermore, assume that*

(A_6) $\int_T^\infty \frac{1}{r(t)} \left[\int_T^t q(s)G\big(CR_1(s-\sigma)\big)ds + \sum_{k=1}^\infty q(\tau_k)G\big(CR_1(\tau_k - \sigma)\big) \right] dt = +\infty$

and

(A_7) $\int_T^\infty q(s)ds + \sum_{k=1}^\infty q(\tau_k) = +\infty$

hold for every constants $C, T > 0$, where $R_1(t) = \int_t^\infty \frac{ds}{r(s)}$. Then every solution of the system (E) either oscillates or converges to zero.

Proof. Let $y(t)$ be a regular solution of (E). Proceeding as in Theorem 2.1, we have (2.2) for $t \geq t_1$. Hence, there exists $t_2 > t_1$ such that $r(t)z'(t)$ and $z(t)$ are of constant sign on $[t_2, \infty)$. If $z(t) < 0$ for $t \geq t_2$, then $y(t)$ is bounded. Consequently, $\lim_{t\to\infty} z(t)$ exists. As a result,

$$\begin{aligned} 0 &\geq \lim_{t\to\infty} z(t) = \limsup_{t\to\infty} z(t) \\ &\geq \limsup_{t\to\infty}\big(y(t) - a\, y(t-\tau)\big) \\ &\geq \limsup_{t\to\infty} y(t) + \liminf_{t\to\infty}\big(-a\, y(t-\tau)\big) \\ &= (1-a)\limsup_{t\to\infty} y(t) \end{aligned}$$

implies that $\limsup_{t\to\infty} y(t) = 0$ $[\because 1 - a > 0]$ and thus $\lim_{t\to\infty} y(t) = 0$ for $t \neq \tau_k$, $k \in \mathbb{N}$. We may note that $\{y(\tau_k - 0)\}_{k\in\mathbb{N}}$ and $\{y(\tau_k + 0)\}_{k\in\mathbb{N}}$ are sequences of reals, and because of continuity of y

$$\lim_{k\to\infty} y(\tau_k - 0) = 0 = \lim_{k\to\infty} y(\tau_k + 0)$$

due to

$$\liminf_{t\to\infty} y(t) = 0 = \limsup_{t\to\infty} y(t).$$

Hence for all t and τ_k, $k \in \mathbb{N}$, $\lim_{t\to\infty} y(t) = 0$. Let $z(t) > 0$ for $t \geq t_2$. If $r(t)z'(t) < 0$ for $t \geq t_2$, then $z(t)$ is bounded and hence $\lim_{t\to\infty} z(t)$ exists. Therefore, for $s \geq t > t_2$, $r(s)z'(s) \leq r(t)z'(t)$ implies that

$$z'(s) \leq \frac{r(t)z'(t)}{r(s)},$$

that is,

$$z(s) \leq z(t) + r(t)z'(t)\int_t^s \frac{d\theta}{r(\theta)}.$$

Because $r(t)z'(t)$ is nonincreasing, we can find a constant $C > 0$ such that $r(t)z'(t) \leq -C$ for $t \geq t_2$. As a result,

$$z(s) \leq z(t) - C\int_t^s \frac{d\theta}{r(\theta)}$$

and hence $0 \leq z(t) - CR_1(t)$ for $t \geq t_2$. Ultimately, $z(\tau_k) \geq CR_1(\tau_k)$, $k \in \mathbb{N}$. From the system (2.2) it is easy to see that

$$\begin{gathered} \big(r(t)z'(t)\big)' + q(t)G\big(CR_1(t-\sigma)\big) \leq 0, \quad t \neq \tau_k \\ \Delta\big(r(\tau_k)z'(\tau_k)\big) + q(\tau_k)\, G\big(CR_1(\tau_k - \sigma)\big) \leq 0,\ k \in \mathbb{N}. \end{gathered}$$

Integrating the last inequality from t_2 to $t(> t_2)$, we obtain

$$\big[r(s)z'(s)\big]_{t_2}^{t} + \int_{t_2}^{t} q(s)G\big(CR_1(s-\sigma)\big)ds - \sum_{t_3 \le \tau_k < t} \Delta\big(r(\tau_k)z'(\tau_k)\big) \le 0,$$

that is,

$$\begin{aligned}\int_{t_2}^{t} q(s)G\big(CR_1(s-\sigma)\big)ds + \sum_{t_2 \le \tau_k < t} q_k G\big(CR_1(\tau_k-\sigma)\big) &\le -\big[r(s)z'(s)\big]_{t_2}^{t}\\ &\le -r(t)z'(t)\end{aligned}$$

implies that

$$\frac{1}{r(t)}\left[\int_{t_2}^{t} q(s)G\big(CR_1(s-\sigma)\big)ds + \sum_{t_2 \le \tau_k < t} q_k G\big(CR_1(\tau_k-\sigma)\big)\right] \le -z'(t)$$

and further integration of the preceding inequality, we have

$$\begin{aligned}\int_{t_3}^{u} \frac{1}{r(t)} & \left[\int_{t_3}^{t} q(s)G\big(CR_1(s-\sigma)\big)ds + \sum_{t_3 \le \tau_k < t} q_k G\big(CR_1(\tau_k-\sigma)\big)\right] dt\\ &\le -\big[z(t)\big]_{t_3}^{u} + \sum_{t_3 \le \tau_k < u} \Delta z(\tau_k)\\ &= -\big[z(t)\big]_{t_3}^{u} + \sum_{t_3 \le \tau_k < u} \big[z(\tau_k+0) - z(\tau_k-0)\big]\\ &\le z(t_3) + \sum_{t_3 \le \tau_k < u} z(\tau_k+0)\\ &< +\infty.\end{aligned}$$

Ultimately,

$$\int_{t_3}^{\infty} \frac{1}{r(t)}\left[\int_{t_3}^{t} q(s)G\big(CR_1(s-\sigma)\big)ds + \sum_{k=1}^{\infty} q_k G\big(CR_1(\tau_k-\sigma)\big)\right] dt < \infty,$$

gives a contradiction to (A_6). Hence, $r(t)z'(t) > 0$ for $t \ge t_2$. As $z(t)$ is nondecreasing on $[t_2,\infty)$, there exist a constant $C > 0$ and $t_3 > t_2$ such that $z(t) \ge C$ for $t \ge t_3$. Therefore, the system (2.2) becomes

$$\begin{aligned}\big(r(t)z'(t)\big)' + q(t)G(C) \le 0, &\quad t \ne \tau_k\\ \Delta\big(r(\tau_k)z'(\tau_k)\big) + q(\tau_k)\, G(C) \le 0, &\quad k \in \mathbb{N}.\end{aligned}$$

We integrate the preceding inequality from t_3 to $+\infty$ and obtain

$$\int_{t_3}^{\infty} q(s)ds + \sum_{t_3 \le \tau_k < \infty} q(\tau_k) < +\infty,$$

which is a contradiction to (A_7). Thus the proof of the theorem is complete. □

Theorem 2.3. *Let* $-1 < -a \le p(t) \le 0$, $a > 0$ *for* $t \in \mathbb{R}_+$. *Assume that* (A_5) *and*

(A_8) $\int_0^\infty q(s)ds + \sum_{k=1}^\infty q(\tau_k)dt < \infty$
hold. Then the impulsive system (E) *admits a positive bounded solution.*

Proof. Due to (A_5), it is easy to verify that

$$\int_0^\infty \frac{1}{r(s)}\Big[\int_s^\infty q(t)dt + \sum_{k=1}^\infty q(\tau_k)\Big]ds < +\infty. \tag{2.3}$$

Let there exist $T > \rho$ such that

$$G\big(R(t)\big)\int_T^t \frac{1}{r(s)}\Big[\int_s^\infty q(\theta)d\theta + \sum_{k=1}^\infty q(\tau_k)\Big]ds \le \frac{R(t)}{4},\ T \ge \rho.$$

Consider

$$M = \{y \in C([T-\sigma, +\infty), \mathbb{R}) : y(t) = \frac{R(t)}{4},\ t \in [T-\rho, T];$$
$$\frac{R(t)}{4} \le y(t) \le R(t)\ for\ t \ge T\}$$

and let $\Phi : M \to M$ be defined by

$$(\Phi y)(t) = \begin{cases} (\Phi y)(T), & t-\rho \le t \le T, \\ -p(t)y(t-\tau) + \frac{R(t)}{4} + \int_T^t \frac{1}{r(s)}\big[\int_s^\infty q(\theta)G\big(y(\theta-\sigma)\big)d\theta \\ + \sum_{k=1}^\infty q(\tau_k)G\big(y(\tau_k-\sigma)\big)\big]ds, & t \ge T. \end{cases}$$

For every $y \in M$, $(\Phi y)(t) \ge \frac{R(t)}{4}$ and

$$(\Phi y)(t) \le aR(t) + \frac{R(t)}{4} + G(R(t))\int_T^t \frac{1}{r(s)}\Big[\int_s^\infty q(\theta)d\theta + \sum_{k=1}^\infty q(\tau_k)\Big]ds$$
$$\le aR(t) + \frac{R(t)}{4} + \frac{R(t)}{4} = \left(a + \frac{1}{2}\right)R(t) \le R(t)$$

implies that $(\Phi y) \in M$. Proceeding as in the proof of Theorem 2.1, we conclude that the operator T has a fixed point $u \in M$, that is, $u(t) = (Tu)(t)$, $t \ge T - \rho$. Therefore, $u(t)$ is a solution of the impulsive system (E) with $\frac{R(t)}{4} \le u(t) \le R(t)$ for $t \ge T$ which is regular and does not tend to zero as $t \to \infty$ when the limit exists. Thus the theorem is proved. □

Theorem 2.4. *Let* $0 \le p(t) \le a < \infty$ *for* $t \in \mathbb{R}_+$. *Assume that* (A_1) − (A_3) *and* (A_5) *hold. Furthermore, assume that*

(A_9) *there exists* $\lambda > 0$ *such that* $G(u) + G(v) \ge \lambda G(u+v)$ *for* $u, v \in \mathbb{R}_+$,
(A_{10}) $G(uv) \le G(u)G(v)$, $u, v \in \mathbb{R}_+$,
(A_{11}) $\int_T^\infty \frac{1}{r(t)}\Big[\int_{T_1}^t Q(s)G\big(CR_1(s-\sigma)\big)ds + \sum_{k=1}^\infty Q(\tau_k)G\big(CR_1(\tau_k-\sigma)\big)\Big]dt$ $= +\infty$, $T, T_1 > 0$

and

(A_{12}) $\int_T^\infty Q(t)dt + \sum_{k=1}^\infty Q(\tau_k) = +\infty$, $T > \rho$
hold, where $Q(t) = \min\{q(t), q(t-\tau)\}$, $t \ge \tau$. *Then every solution of the impulsive system* (E) *oscillates.*

Proof. On the contrary, let $y(t)$ be a regular nonoscillatory solution of (E). Proceeding as in Theorem 2.1, we have two cases namely $z(t) > 0, r(t)z'(t) < 0$ and $z(t) > 0$, $r(t)z'(t) > 0$ for $t \in [t_2, \infty)$. Consider the former one. Ultimately, $y(t)$ is bounded. Using the same type of argument as in the proof of the Theorem 2.2, we obtain that $z(t) \geq CR_1(t)$ for $t \geq t_2$. From the system (E) it is easy to see that

$$\begin{aligned}&\big(r(t)z'(t)\big)' + q(t)G\big(y(t-\sigma)\big)\\ &+ G(a)\Big[\big(r(t-\tau)z'(t-\tau)\big)' + q(t-\tau)G\big(y(t-\tau-\sigma)\big)\Big] = 0,\ t \neq \tau_k,\end{aligned}$$

$$\begin{aligned}&\Delta\big(r(\tau_k)z'(\tau_k)\big) + q(\tau_k)G\big(y(\tau_k-\sigma)\big)\\ &+ G(a)\big[\Delta\big(r(\tau_k-\tau)z'(\tau_k-\tau)\big) + q(\tau_k-\tau)G\big(y(\tau_k-\tau-\sigma)\big)\big] = 0,\ k \in \mathbb{N}.\end{aligned}$$

Using (A_9) and (A_{10}) in the above system, it follows that

$$\begin{aligned}\big(r(t)z'(t)\big)' + G(a)\big(r(t-\tau)z'(t-\tau)\big)' + \lambda Q(t)G\big(z(t-\sigma)\big) &\leq 0\\ \Delta\big(r(\tau_k)z'(\tau_k)\big) + G(a)\Delta\big(r(\tau_k-\tau)z'(\tau_k-\tau)\big) + \lambda Q(\tau_k)G\big(z(\tau_k-\sigma)\big) &\leq 0,\end{aligned} \tag{2.4}$$

where $z(t) \leq y(t) + ay(t-\tau)$. Ultimately, (2.4) reduces to

$$\begin{aligned}\big(r(t)z'(t)\big)' + G(a)\big(r(t-\tau)z'(t-\tau)\big)' + \lambda Q(t)G\big(CR_1(t-\sigma)\big) &\leq 0\\ \Delta\big(r(\tau_k)z'(\tau_k)\big) + G(a)\Delta\big(r(\tau_k-\tau)z'(\tau_k-\tau)\big) + \lambda Q(\tau_k)G\big(CR_1(\tau_k-\sigma)\big) &\leq 0\end{aligned}$$

for $t \geq t_3 > t_2$, $t \neq \tau_k$, $k \in \mathbb{N}$. Integrating the last system from t_3 to t $(> t_3)$, we get

$$\begin{aligned}&\big[r(s)z'(s)\big]_{t_3}^{t} + G(a)\big[r(s-\tau)z'(s-\tau)\big]_{t_3}^{t} - \sum_{t_3 \leq \tau_k < t} \Delta\big(r(\tau_k)z'(\tau_k)\big)\\ &-G(a)\sum_{t_3 \leq \tau_k < t} \Delta\big(r(\tau_k-\tau)z'(\tau_k-\tau)\big) + \lambda\int_{t_3}^{t} Q(s)G\big(CR_1(s-\sigma)\big)ds \leq 0,\end{aligned}$$

that is,

$$\begin{aligned}\lambda\Big[\int_{t_3}^{t} Q(s)G\big(CR_1(s-\sigma)\big)ds \quad &+ \sum_{t_3 \leq \tau_k < t} Q(\tau_k)G\big(CR_1(\tau_k-\sigma)\big)\Big]\\ &\leq -\big[r(s)z'(s) + G(a)\big(r(s-\tau)z'(s-\tau)\big)\big]_{t_3}^{t}\\ &\leq -\big[r(t)z'(t) + G(a)\big(r(t-\tau)z'(t-\tau)\big)\big]\\ &\leq -\big(1+G(a)\big)r(t)z'(t).\end{aligned}$$

Therefore,

$$\frac{\lambda}{1+G(a)}\frac{1}{r(t)}\bigg[\int_{t_3}^{t} Q(s)G\big(CR_1(s-\sigma)\big)ds + \sum_{t_3 \leq \tau_k < t} Q(\tau_k)G\big(CR_1(\tau_k-\sigma)\big)\bigg] \leq -z'(t).$$

Integrating the above inequality, we obtain

$$\frac{\lambda}{1+G(a)}\int_{t_3}^{\infty}\frac{1}{r(t)}\bigg[\int_{t_3}^{t} Q(s)G\big(CR_1(s-\sigma)\big)ds + \sum_{t_3 \leq \tau_k < t} Q_k G\big(CR_1(\tau_k-\sigma)\big)\bigg]dt < \infty$$

which is a contradiction to (A_{11}). If the latter case holds, then there exist a constant $C > 0$ and $t_3 > t_2$ such that $z(t) \geq C$ for $t \geq t_3$. From (2.4), it follows that

$$\begin{aligned}\big(r(t)z'(t)\big)' + G(a)\big(r(t-\tau)z'(t-\tau)\big)' + \lambda Q(t)G(C) \leq 0\\ \Delta\big(r(\tau_k)z'(\tau_k)\big) + G(a)\Delta\big(r(\tau_k-\tau)z'(\tau_k-\tau)\big) + \lambda Q(\tau_k)G(C) \leq 0.\end{aligned}$$

Integrating the last inequality from t_3 to $+\infty$, we get a contradiction to (A_{12}). This completes the proof of the theorem. □

Theorem 2.5. *Let $0 \leq p(t) \leq R(t) < 1$ for $t \in \mathbb{R}_+$. Assume that (A_5) and (A_8) hold. Furthermore, assume that G is Lipschitzian on the intervals of the form $[a, b]$, $0 < a < b < \infty$. Then the impulsive system (E) admits a positive bounded solution.*

Proof. Proceeding as in the proof of Theorem 2.3, we get (2.3). So, there exists $T > \rho$ such that

$$\int_T^\infty \frac{1}{r(s)}\left[\int_s^\infty q(t)dt + \sum_{k=1}^\infty q(\tau_k)\right]ds < \frac{1-R(t)}{3L}.$$

where $L = \max\{L_1, G(1)\}$, L_1 is the Lipschitz constant of G on $\left[\frac{1-R(t)}{2}, 1\right]$ for $t \geq T$. Let $X = BC([T,\infty),\mathbb{R})$ be the space of real valued continuous functions on $[T,\infty)$. Indeed, X is a Banach space with respect to the sup norm defined by

$$\|y\| = \sup\{|y(t)| : t \geq T\}.$$

Define

$$S = \{v \in X : \frac{1-R(t)}{2} \leq v(t) \leq 1,\ t \geq T\}.$$

We notice that S is a closed and convex subspace of X. Let $\Phi : S \to S$ be such that

$$(\Phi y)(t) = \begin{cases}(\Phi y)(T+\rho), & t \in [T, T+\rho],\\ -p(t)y(t-\tau) + \frac{5+R(t)}{6} - \int_t^\infty \frac{1}{r(s)}\big[\int_s^\infty q(u)G\big(y(u-\sigma)\big)du \\ +\sum_{k=1}^\infty q(\tau_k)G\big(y(\tau_k-\sigma)\big)\big]ds, t \geq T+\rho.\end{cases}$$

For every $y \in X$, $(\Phi y)(t) \leq \frac{5+R(t)}{6} < 1$ and

$$(\Phi y)(t) \geq -R(t) + \frac{5+R(t)}{6} - \frac{1-R(t)}{3} = \frac{1}{2}(1-R(t))$$

implies that $\Phi y \in S$. For $y_1, y_2 \in S$,

$$\begin{aligned}|(\Phi y_1)(t) - (\Phi y_2)(t)| \leq R(t)&|y_1(t-\tau) - y_2(t-\tau)|\\ &+ \int_t^\infty \frac{1}{r(s)}\left[\int_s^\infty q(u)|G\big(y_1(u-\sigma)\big) - G\big(y_2(u-\sigma)\big)|du\right.\\ &\left.+ \sum_{k=1}^\infty q_k|G\big(y_1(\tau_k-\sigma)\big) - G\big(y_2(\tau_k-\sigma)\big)|\right]ds,\end{aligned}$$

that is,

$$|(\Phi y_1)(t) - (\Phi y_2)(t)| \leq R(t)\|y_1 - y_2\| + \|y_1 - y_2\| L_1 \times \int_t^\infty \frac{1}{r(s)} \left[\int_s^\infty q(u)du + \sum_{k=1}^\infty q_k \right] ds \leq \left(R(t) + \frac{1 - R(t)}{3} \right) \|y_1 - y_2\|$$

implies that

$$|(\Phi y_1)(t) - (\Phi y_2)(t)| \leq \mu \|y_1 - y_2\|,$$

where

$$\left(R(t) + \frac{1 - R(t)}{3} \right) \leq \frac{1 + 2\alpha}{3} = \mu < 1$$

and $\alpha = \limsup_{t\to\infty} R(t)$ ($\because R(t) < \infty, R'(t) > 0$). Therefore, Φ is a contraction. Using Banach's fixed point theorem, it follows that Φ has a unique fixed point $y(t)$ in $\left[\frac{1-R(t)}{2}, 1\right]$. This completes the proof of the theorem. □

Theorem 2.6. *Let $1 < a_1 \leq p(t) \leq a_2 < \infty$, $a_1^2 \geq a_2$ for $t \in \mathbb{R}_+$. Assume that (A_5) and (A_8) hold. Let G be Lipschitzian on intervals of the form $[a, b]$, $0 < a < b < \infty$. Then the impulsive system (E) admits a positive bounded solution.*

Proof. Proceeding as in the proof of Theorem 2.3, we have obtained (2.3). Let

$$\int_T^\infty \frac{1}{r(t)} \left[\int_t^\infty q(s)ds + \sum_{k=1}^\infty q(\tau_k) \right] dt < \frac{a_1 - 1}{4L},$$

where $L = \max\{L_1, L_2\}$, L_1 is the Lipschitz constant of G on $[a, b]$, $L_2 = G(b)$ with

$$a = \frac{4\mu(a_1^2 - a_2) - a_2(a_1 - 1)}{4{a_1}^2 a_2}$$

$$b = \frac{a_1 - 1 + 4\mu}{4a_1}, \quad \mu > \frac{a_2(a_1 - 1)}{4({a_1}^2 - a_2)} > 0.$$

Let $X = BC([T, \infty), \mathbb{R})$ be the space of real valued functions defined on $[T, \infty)$. Indeed, X is a Banach space with respect to sup norm defined by

$$||y|| = \sup\{|y(t)| : t \geq T\}.$$

Define

$$S = \{u \in X : \ a \leq u(t) \leq b, \ t \geq T\}.$$

Let $\Phi : S \to S$ be such that

$$(\Phi y)(t) = \begin{cases} \Phi y(T + \rho), & t \in [T, T + \rho] \\ -\frac{y(t+\tau)}{p(t+\tau)} + \frac{\mu}{p(t+\tau)} + \frac{1}{p(t+\tau)} \int_T^{t+\tau} \frac{1}{r(s)} \Big[\int_s^\infty q(v) G\big(y(v - \sigma)\big) dv \\ + \sum_{k=1}^\infty q(\tau_k) G\big(y(\tau_k - \sigma)\big) \Big] ds, & t \geq T + \rho. \end{cases}$$

For every $y \in S$,

$$\begin{aligned}(\Phi y)(t) &\le \frac{G(b)}{p(t+\tau)} \int_T^{t+\tau} \frac{1}{r(s)} \left[\int_s^{\infty} q(v)dv + \sum_{k=1}^{\infty} q(\tau_k)\right] ds + \frac{\mu}{p(t+\tau)} \\ &\le \frac{G(b)}{p(t+\tau)} \int_T^{\infty} \frac{1}{r(s)} \left[\int_s^{\infty} q(v)dv + \sum_{k=1}^{\infty} q(\tau_k)\right] ds + \frac{\mu}{p(t+\tau)} \\ &\le \frac{1}{a_1}\left[\frac{a_1-1}{4} + \mu\right] = b\end{aligned}$$

and

$$(\Phi y)(t) \ge -\frac{y(t+\tau)}{p(t+\tau)} + \frac{\mu}{p(t+\tau)} > -\frac{b}{a_1} + \frac{\mu}{a_2} = a$$

implies that $\Phi y \in S$. For $y_1, y_2 \in S$

$$\begin{aligned}|(\Phi y_1)(t) - (\Phi y_2)(t)| \le& \frac{1}{|p(t+\tau)|}|y_1(t+\tau) - y_2(t+\tau)| \\ &+ \frac{G(b)}{|p(t+\tau)|} \int_T^{t+\tau} \frac{1}{r(s)} \left[\int_s^{\infty} q(v)|y_1(v-\sigma) - y_2(v-\sigma)|dv \right. \\ &\left. + \sum_{k=1}^{\infty} q(\tau_k)|y_1(\tau_k-\sigma) - y_2(\tau_k-\sigma)|\right] ds,\end{aligned}$$

that is,

$$\begin{aligned}|(\Phi y_1)(t) - (\Phi y_2)(t)| &\le \frac{1}{a_1}||y_1 - y_2|| + \frac{G(b)}{a_1}||y_1 - y_2|| \\ &\quad \times \int_T^{t+\tau} \frac{1}{r(s)} \left[\int_s^{\infty} q(v)dv + \sum_{k=1}^{\infty} q(\tau_k)\right] ds \\ &< \frac{1}{a_1}||y_1 - y_2||\left(1 + \frac{a_1-1}{4}\right).\end{aligned}$$

Therefore,

$$||(\Phi y_1) - (\Phi y_2)|| \le \left(\frac{1}{a_1} + \frac{a_1-1}{4a_1}\right) ||y_1 - y_2||.$$

As $\left(\frac{1}{a_1} + \frac{a_1-1}{4a_1}\right) < 1$, Φ is a contraction mapping. We note that S is a closed convex subset of X and hence by the Banach's fixed point theorem Φ has a unique fixed point, that is, $\Phi y(t) = y(t)$ on $[a, b]$. Thus the proof of the theorem is complete. □

Theorem 2.7. *Let $-\infty < -a_1 \le p(t) \le -a_2 < -1$ for $t \in \mathbb{R}_+$, where $a_1, a_2 > 0$. Assume that $(A_1) - (A_3)$ and $(A_5) - (A_7)$ hold. If*

(A_{13}) $\int_T^{\infty} \frac{1}{r(t)} \left[\int_T^t q(s)ds + \sum_{k=1}^{\infty} q(\tau_k)\right] dt = +\infty,$

then every bounded solution of the system (E) either oscillates or converges to zero.

Proof. Let $y(t)$ be a bounded regular solution of (E). Proceeding as in Theorem 2.1, it follows that $z(t)$ and $r(t)z'(t)$ are monotonic functions on $[t_2,\infty)$. Since $y(t)$ is bounded, then $z(t)$ is bounded and hence $\lim_{t\to\infty} z(t)$ exists. Using the arguments as in the proof of Theorem 2.2, we get contradictions to (A_6) and (A_7) for the cases $z(t) > 0, r(t)z'(t) < 0$ and $z(t) > 0, r(t)z'(t) > 0$ respectively. Consider the case $z(t) < 0,\ r(t)z'(t) > 0$ for $t \geq t_2$. We claim that $\lim_{t\to\infty} z(t) = 0$. If not, there exist $\beta < 0$ and $t_3 > t_2$ such that $z(t+\tau-\sigma) < \beta$ for $t \geq t_3$. Hence, $z(t) \geq -a_1 y(t-\tau)$ implies that $y(t-\sigma) \geq -a_1^{-1}\beta$ for $t \geq t_3$. Consequently, the impulsive system (2.2) reduces to

$$\begin{aligned} \big(r(t)z'(t)\big)' + G\big(-a_1^{-1}\beta\big)q(t) \leq 0, &\quad t \neq \tau_k \\ \Delta\big(r(\tau_k)z'(\tau_k)\big) + G\big(-a_1^{-1}\beta\big)q(\tau_k) \leq 0, &\quad k \in \mathbb{N} \end{aligned} \tag{2.5}$$

for $t \geq t_3$. Integrating (2.5) from t_3 to $+\infty$, we get

$$\left[\int_{t_3}^{\infty} q(s)ds + \sum_{t_3 \leq \tau_k \leq \infty} q(\tau_k)\right] < \infty$$

which is a contradiction to (A_7). So, our claim holds and

$$\begin{aligned} 0 &= \lim_{t\to\infty} z(t) = \liminf_{t\to\infty}\big(y(t) + p(t)y(t-\tau)\big) \\ &\leq \liminf_{t\to\infty}(y(t) - a_2\, y(t-\tau)) \\ &\leq \limsup_{t\to\infty} y(t) + \liminf_{t\to\infty}\big(-a_2\, y(t-\tau)\big) \\ &= (1-a_2)\limsup_{t\to\infty} y(t) \end{aligned}$$

implies that $\limsup_{t\to\infty} y(t) = 0 \quad [\because 1 - a_2 < 0]$. Thus, $\lim_{t\to\infty} y(t) = 0$.
Let $z(t) < 0,\ r(t)z'(t) < 0$ for $t \geq t_2$. Proceeding as in the previous case, we get (2.5). Integrating (2.5) from t_3 to t, we obtain

$$\int_{t_3}^{t} q(s)G\big(-a_1^{-1}\beta\big)ds + \sum_{t_3 \leq \tau_k \leq t} q(\tau_k)G\big(-a_1^{-1}\beta\big) \leq -r(t)z'(t),$$

that is,

$$\frac{1}{r(t)}\left[\int_{t_3}^{t} q(s)G\big(-a_1^{-1}\beta\big)ds + \sum_{t_3 \leq \tau_k \leq t} q(\tau_k)G\big(-a_1^{-1}\beta\big)\right] \leq -z'(t)$$

for $t \geq t_3$. Further integration of the above inequality from t_3 to $+\infty$, we get

$$\int_{t_3}^{\infty} \frac{1}{r(t)}\left[\int_{t_3}^{t} q(s)ds + \sum_{t_3 \leq \tau_k \leq t} q(\tau_k)\right]dt < \infty$$

which contradicts (A_{13}). Thus $\lim_{t\to\infty} z(t) = 0$. Rest of this case follows from the previous case. This completes the proof of the theorem. □

Theorem 2.8. *Let $-\infty < -a_1 \leq p(t) \leq -a_2 < -1$ for $t \in \mathbb{R}_+$, where $a_1, a_2 > 0$ such that $4a_2 > a_1$. Assume that (A_5) and (A_8) hold. Furthermore, assume that G is Lipschitzian on the intervals of the form $[a,b]$, $0 < a < b < \infty$. Then the system (E) admit a positive bounded solution.*

Proof. Proceeding as in the proof of Theorem 2.3, we get (2.3). So, it is possible to find $T > \rho$ such that

$$\int_T^\infty \frac{1}{r(s)}\Big[\int_s^\infty q(t)dt + \sum_{k=1}^\infty q(\tau_k)\Big]ds < \frac{a_2-1}{4L},$$

where $L = \max\{L_1, G(1)\}$, L_1 is the Lipschitz constant of G on $(a,1)$,

$$a = \frac{(a_2-1)(4a_2-a_1)}{4a_1a_2}.$$

Let $X = BC([T,\infty),\mathbb{R})$ be the space of real valued continuous functions defined on $[T,\infty)$. Indeed, X is a Banach space with the supremum norm defined by

$$||y|| = \sup\{|y(t)| : t \geq T\}.$$

Define

$$S = \{v \in X : \ a \leq v(t) \leq 1, \ t \geq T\}.$$

We may note that S is a closed and convex subspace of X. Let $\Psi : S \to S$ be such that

$$(\Psi y)(t) = \begin{cases} \Psi y(T+\rho), \qquad t \in [T, T+\rho] \\ -\frac{y(t+\tau)}{p(t+\tau)} - \frac{a_2-1}{p(t+\tau)} + \frac{1}{p(t+\tau)}\int_T^{t+\tau}\frac{1}{r(s)}\Big[\int_s^\infty q(u)G\big(y(u-\sigma)\big)du \\ +\sum_{k=1}^\infty q(\tau_k)G\big(y(\tau_k-\sigma)\big)\Big]ds, \qquad t \geq T+\rho. \end{cases}$$

For every $y \in S$,

$$\begin{aligned}(\Psi y)(t) &\leq -\frac{y(t+\tau)}{p(t+\tau)} - \frac{a_2-1}{p(t+\tau)} \\ &\leq \frac{1}{a_2} + \frac{a_2-1}{a_2} = 1\end{aligned}$$

and

$$\begin{aligned}(\Psi y)(t) \geq &-\frac{a_2-1}{p(t+\tau)} + \frac{1}{p(t+\tau)} \\ &\times \int_T^{t+\tau}\frac{1}{r(s)}\Big[\int_s^\infty q(u)G\big(y(u-\sigma)\big)du + \sum_{k=1}^\infty q(\tau_k)G\big(y(\tau_k-\sigma)\big)\Big]ds \\ \geq &\ \frac{a_2-1}{a_1} + \frac{G(1)}{p(t+\tau)}\int_T^{t+\tau}\frac{1}{r(s)}\Big[\int_s^\infty q(u)du + \sum_{k=1}^\infty q(\tau_k)\Big]ds \\ \geq &\ \frac{a_2-1}{a_1} - \frac{G(1)}{a_2}\int_T^{\infty}\frac{1}{r(s)}\Big[\int_s^\infty q(u)du + \sum_{k=1}^\infty q(\tau_k)\Big]ds \\ \geq &\ \frac{a_2-1}{a_1} - \frac{a_2-1}{4a_2} = a\end{aligned}$$

implies that $(\Psi y) \in S$. For $y_1, y_2 \in S$, we have that

$$
\begin{aligned}
|(\Psi y_1)(t) - (\Psi y_2)(t)| \leq & \frac{1}{|p(t+\tau)|}|y_1(t+\tau) - y_2(t+\tau)| \\
& + \frac{L_1}{|p(t+\tau)|}\int_T^{t+\tau} \frac{1}{r(s)}[\int_s^\infty q(u)|y_1(u-\sigma) - y_2(u-\sigma)|du \\
& + \sum_{k=1}^{\infty} q(\tau_k)|y_1(\tau_k - \sigma) - y_2(\tau_k - \sigma)|]ds,
\end{aligned}
$$

that is,

$$|(\Psi y_1)(t) - (\Phi y_2)(t)| \leq \frac{1}{a_2}||y_1 - y_2|| + \frac{a_2 - 1}{4a_2}||y_1 - y_2||$$

implies that

$$||(\Psi y_1) - (\Psi y_2)|| \leq \mu ||y_1 - y_2||,$$

where $\mu = \frac{1}{a_2}\left(1 + \frac{a_2 - 1}{4}\right) < 1$. Therefore, Ψ is a contraction. By the Banach's fixed point theorem, Ψ has a unique fixed point $y \in S$. It is easy to see that $\lim_{t\to\infty} y(t) \neq 0$. This completes the proof of the theorem. □

3. Discussion and example

It is worth observation that we could succeed to establish the necessary and sufficient conditions for oscillation of all solutions of the impulsive system (E_1) when $-1 < p(t) \leq 0$ only. However, we failed to obtain the necessary and sufficient conditions for the other ranges of $p(t)$ and hence the undertaken problem is open for other ranges of $p(t)$. May be some other method is required to overcome the problem.

We conclude this section with the following example:

Example 3.1. Consider the impulsive system

$$
(E_4) \begin{cases} \left(r(t)(y(t) + p(t)y(t-1))'\right)' + q(t)y(t-1) = 0, \ t \neq \tau_k \\ \Delta\left(r(\tau_k)(y(\tau_k) + p(\tau_k)\, y(\tau_k - 1))'\right) + q(\tau_k)\, y(\tau_k - 1) = 0, \quad k \in \mathbb{N}, \end{cases}
$$

where $-1 < p(t) = e^{-t} - 1 \leq 0$, $q(t) = e^{-t}$, $r(t) = e^t$, $R(t) = 1 - e^{-t}$, $G(x) = x$, $\rho = 1$ and $\tau_k = 2^k$, $k \in \mathbb{N}$. Clearly, all conditions of Theorem 2.1 are satisfied. Thus by Theorem 2.1, every unbounded solution of the system (E_4) oscillates.

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Existence of solutions for a p-Laplacian Kirchhoff type problem with nonlinear term of superlinear and subcritical growth

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Abstract. This paper is concerned by the study of the existence of nonnegative and nonpositive solutions for a nonlocal quasilinear Kirchhoff problem by using the Mountain Pass lemma technique.

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Keywords: Kirchhoff problem, nonlocal quasilinear problem, variational methods, nonnegative and nonpositive solutions.

1. Introduction

Many research are interested to study the existence of nontrivial solutions of Kirchhoff type equations for its huge importance. The Kirchhoff equation was introduced for the first time in 1876, which describe the free transverse vibrations of a tight rope of length L and a constant density (assumed to be equal to 1). The rope is described by a variable x taking its values in the interval $[0, L]$. The system of equations describing this phenomena and which was given by Kirchhoff is

$$u_{tt} - \Big(g(\lambda)u_x\Big)_x = 0,\ 0 < x < L,\ t > 0, \tag{1.1}$$

$$v_{tt} - \Big(g(\lambda)(1 + v_x)\Big)_x = 0,\ 0 < x < L,\ t > 0, \tag{1.2}$$

$$u(0,t) = u(L,t) = v(0,t) = v(L,t),\ \ t \geq 0, \tag{1.3}$$

where λ is the deformation of the cord given by $\lambda(x,t) = \Big(|1+v_x|^2 + |u_x|^2\Big)^{\frac{1}{2}} - 1$, and $g(\lambda) = \frac{\sigma(\lambda)}{1+\lambda}$ with $\sigma(\lambda)$ represents the rope (cord) constraint corresponding to λ; finally and most important, the unknowns $u(x,t)$ and $v(x,t)$ represent the transversal and longitudinal displacements of the material point x at the time t. In order to separate the unknowns u and v and under some hypotheses, one can obtain

$$\begin{aligned} &u_{tt} - \Big(T_0 + \frac{E}{2L}\int_0^L |u_x|^2\,dx\Big)u_{xx} = 0, \quad 0 < x < L, t > 0,\\ &u(0,t) = u(L,t) = 0, \qquad\qquad\qquad\qquad t \geq 0, \end{aligned}$$

which is named the Kirchhoff equation. T_0 and $\frac{E}{2L}$ are two physical constants.
Since then, many researchers are interested in the Kirchhoff equation for its importance and it has been the subject of many studies; we cite here, in particular [4], which treats the following Kirchhoff type problem

$$\begin{cases} -\Big(a + b\displaystyle\int_\Omega |\nabla u|^2\,dx\Big)\Delta u = f(x,u), & \text{in } \Omega,\\ u = 0, & \text{on } \partial\Omega, \end{cases}$$

with 4-superlinear growth as $|u| \to +\infty$, and using minimax methods, it gives two interesting results, the existence of nontrivial solutions, and the existence of sign-changing solutions and multiple solutions. We cite also [3] which treat the existence and multiplicity of solutions for the semilinear elliptic problem given by

$$\begin{cases} -\Delta u + \ell(x)u = f(x,u), & \text{in } \Omega,\\ u = 0, & \text{on } \partial\Omega, \end{cases}$$

by using the Mountain Pass technique. Note that, our work is practically based on the papers [3] and [4].
Let us consider the following nonlocal[1] Kirchhoff problem

$$\begin{cases} -\Big(a + b\displaystyle\int_\Omega |\nabla u|^p\,dx\Big)^{p-1}\Delta_p u + \ell(x)|u|^{p-2}u = f(x,u), & \text{in } \Omega,\\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.4}$$

where Δ_p is the p Laplacian operator: $-\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$, Ω is a smooth bounded domain in $\mathbb{R}^N$ with $N \geq 3$, a, b two strictly positive real numbers, $\ell \in L^{\frac{N}{p}}(\Omega) \cap L^\infty(\Omega)$ and f is a real continuous function defined on $\overline{\Omega} \times \mathbb{R}$. The induced norm in $W_0^{1,p}(\Omega)$ is given by

$$||u|| := \Big(\int_\Omega |\nabla u|^p\,dx\Big)^{\frac{1}{p}}, \quad \forall u \in W_0^{1,p}(\Omega).$$

2. Statement of the main result

The operator L defined by $Lu = -(a + b||u||^p)^{p-1}\Delta_p u + \ell|u|^{p-2}u$ possesses an unbounded eigenvalues sequence

$$\lambda_1 < \lambda_2 \leq \cdots \leq \lambda_n \to +\infty \text{ as } n \to +\infty,$$

[1]It is called nonlocal because of the term $M(||u||^p) = a + b||u||^p$ which implies that the equation is no more a punctual identity [1].

where λ_1 is simple and is characterized by

$$\lambda_1 = \inf_{u \in W_0^{1,p}(\Omega), u \neq 0} \frac{(a+b||u||^p)^{p-1}||u||^p + \displaystyle\int_\Omega \ell(x)|u|^p\,dx}{\displaystyle\int_\Omega |u|^p\,dx}.$$

Remark 2.1. Our purpose in this remark is to study the following eigenvalue problem

$$L\phi = -(a+b||\phi||^p)^{p-1}\Delta_p\phi + \ell(x)|\phi|^{p-2}\phi = \lambda|\phi|^{p-2}\phi. \tag{2.1}$$

Let λ_k and $\widetilde{\phi}_k$ respectively eigenvalues and eigenfunctions of the operator

$$-\Delta_p + g|\phi|^{p-2} : W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega)$$

with $g \in L^\infty(\Omega)$ (see [5], [6]), which means that

$$-\Delta_p\widetilde{\phi}_k + g(x)|\widetilde{\phi}_k|^{p-2}\widetilde{\phi}_k = \lambda_k|\widetilde{\phi}_k|^{p-2}\widetilde{\phi}_k; \tag{2.2}$$

especially for $g(x) = \dfrac{\ell(x)}{(a+b||\widetilde{\phi}_k||^p)^{p-1}}$, i.e.,

$$-\Delta_p\widetilde{\phi}_k + \frac{\ell(x)}{(a+b||\widetilde{\phi}_k||^p)^{p-1}}|\widetilde{\phi}_k|^{p-2}\widetilde{\phi}_k = \lambda_k|\widetilde{\phi}_k|^{p-2}\widetilde{\phi}_k, \tag{2.3}$$

multiplying by $(a+b||\widetilde{\phi}_k||^p)^{p-1}$, we obtain

$$-(a+b||\widetilde{\phi}_k||^p)^{p-1}\Delta_p\widetilde{\phi}_k + \ell(x)|\widetilde{\phi}_k|^{p-2}\widetilde{\phi}_k = \lambda_k(a+b||\widetilde{\phi}_k||^p)^{p-1}|\widetilde{\phi}_k|^{p-2}\widetilde{\phi}_k, \tag{2.4}$$

so the sequence $(\widehat{\lambda}_k)$ defined by

$$\widehat{\lambda}_k = \lambda_k(a+b||\widetilde{\phi}_k||^p)^{p-1}$$

consist of eigenvalues of the operator L associated to the eigenfunctions $\widetilde{\phi}_k$. Since λ_1 is simple and strictly positive (see[5]) , it follows that $\widehat{\lambda}_1$, the first eigenvalue of (2.1), is also simple and strictly positive.

Proposition 2.1. *If λ is an eigenvalue of the operator L, then, there exist λ_k and $\widetilde{\phi}_k$ such that*

$$\lambda = \lambda_k(a+b||\widetilde{\phi}_k||^p)^{p-1}.$$

Proof. As λ is an eigenvalue of the operator L, one has that there exists $\phi \in W_0^{1,p}(\Omega)$ with $\phi \not\equiv 0$ which satisfies

$$-(a+b||\phi||^p)^{p-1}\Delta_p\phi + \ell(x)|\phi|^{p-2}\phi = \lambda|\phi|^{p-2}\phi \text{ in } \Omega, \quad \phi = 0 \text{ on } \partial\Omega,$$

and this implies that

$$(a+b||\phi||^p)^{p-1}\int_\Omega |\nabla\phi|^p\,dx + \int_\Omega \ell(x)|\phi|^p\,dx = \lambda\int_\Omega |\phi|^p\,dx,$$

as a result

$$\lambda = \frac{(a+b||\phi||^p)^{p-1}||\phi||^p + \int_\Omega \ell(x)|\phi|^p\,dx}{\int_\Omega |\phi|^p\,dx},$$

and that

$$-\Delta_p\phi + \frac{\ell(x)}{(a+b||\phi||^p)^{p-1}}|\phi|^{p-2}\phi = \frac{\lambda}{(a+b||\phi||^p)^{p-1}}|\phi|^{p-2}\phi,$$

consequently, there exists $k \in \mathbb{N}^*$ such that $\lambda_k = \frac{\lambda}{(a+b||\widetilde{\phi}_k||^p)^{p-1}}$ and $\phi = \widetilde{\phi}_k$ for some eigenfunction associated to λ_k. $\lambda_k = \frac{\lambda}{(a+b||\widetilde{\phi}_k||^p)^{p-1}}$ implies that $\lambda = \lambda_k(a+b||\widetilde{\phi}_k||^p)^{p-1}$ and this conclude the proof of the proposition. □

For $p < N$ and concerning the embedding mapping $W_0^{1,p}(\Omega) \hookrightarrow L^r(\Omega)$, it is continuous for $r \in [1, p^*]$ and compact for $r \in [1, p^*)$ with $p^* = \frac{pN}{N-p}$, so we have that $S_r|u|_r \leq ||u||$ for all $u \in W_0^{1,p}(\Omega)$, where $|\cdot|_r$ denotes the norm in $L^r(\Omega)$ and S_r is the best constant corresponding to the embedding mapping (see [2]).

In this paper, we assume that f is a continuous function on $\overline{\Omega} \times \mathbb{R}$ and satisfies:

(H1) for every $M > 0$, there exists a constant $L_M > 0$ such that $|f(x,s)| \leq L_M$ for $|s| \leq M$ and a.e. $x \in \Omega$,

(H2) $\lim\limits_{|s|\to\infty} \dfrac{f(x,s)}{|s|^{p^*-2}s} = 0$, uniformly in a.e. $x \in \Omega$,

(H3) there exist a function $m \in L^{\frac{N}{p}}(\Omega)$, and a subset $\Omega' \subset \Omega$ with $|\Omega'| > 0$ such that

$$\limsup_{s\to 0} \frac{pF(x,s)}{|s|^p} \leq m(x) \leq \lambda_1,$$

uniformly in a.e. $x \in \Omega$, and $m < \lambda_1$ in Ω', where $F(x,s) = \int_0^s f(x,t)\,dt$ and $|\cdot|$ is the Lebesgue measure,

(H4) $\lim\limits_{|s|\to\infty} \dfrac{F(x,s)}{s^{p^2}} = +\infty$ uniformly in a.e. $x \in \Omega$,

(H5) let $\overline{F}(x,u) = \dfrac{1}{p^2} f(x,u)u - F(x,u)$, then $\overline{F}(x,u) \to +\infty$ as $|u| \to +\infty$ uniformly in $x \in \Omega$, and there exists $\sigma > \max\{1, \frac{N}{p}\}$ such that $|f(x,u)|^\sigma \leq C\overline{F}(x,u)(|u|^{p-1})^\sigma$ for $|u|$ large.

Furthermore, we suppose that one of the two conditions is satisfied $\Big(\ell(x) - m(x) \geq 0\Big)$ or $\Big(\ell(x) \geq 0$ and $a^{p-1} \geq \frac{|m|_{L^\infty}}{S_p^p}$ when $p \geq 2\Big)$.

Example: consider the function

$$f(x,s) = \begin{cases} s^3\ln(1+s) + \frac{s^4}{4(1+s)} - \frac{1}{4}[s^3+s^2+s],\ s \geq 0, \\ s^3\ln(1-s) - \frac{s^4}{4(1-s)} - \frac{1}{4}[s^3-s^2+s],\ s < 0, \end{cases}$$

then f satisfies all the above hypotheses for $p = 2$ and $N = 3$.

Our main result is the following theorem

Theorem 2.1. *Assume that hypotheses (H1)-(H5) hold, and that $sf(x,s) \geq 0$ for $s \in \mathbb{R}$ and a.e. $x \in \Omega$. Then problem (1.4) has at least a nonnegative solution and a nonpositive solution.*

3. Preliminaries

Let $E = W_0^{1,p}(\Omega)$ and define the functional

$$\Phi(u) = \frac{1}{p}\widehat{M}(||u||^p) + \frac{1}{p}\int_\Omega \ell(x)|u|^p\,dx - \int_\Omega F(x,u)\,dx, \quad u \in E,$$

where $\widehat{M}(t) = \int_0^t [M(s)]^{p-1}\,ds$ and $M(s) = a + bs$, in other words,

$$\Phi(u) = \frac{1}{bp^2}\Big[(a + b||u||^p)^p - a^p\Big] + \frac{1}{p}\int_\Omega \ell(x)|u|^p\,dx - \int_\Omega F(x,u)\,dx, \quad u \in E.$$

The variational formulation associated to the problem is

$$\Big[M(||u||^p)\Big]^{p-1}\int_\Omega |\nabla u|^{p-2}\nabla u\nabla v\,dx + \int_\Omega \ell(x)|u|^{p-2}uv\,dx = \int_\Omega f(x,u)v\,dx,\ \forall v \in E,$$

and by $(H1)$ and $(H2)$, one can verify that $\Phi \in C^1(E,\mathbb{R})$ and

$$\begin{aligned}\langle \Phi'(u), v\rangle = \Big[M(||u||^p)\Big]^{p-1}&\int_\Omega |\nabla u|^{p-2}\nabla u\nabla v\,dx + \int_\Omega \ell(x)|u|^{p-2}uv\,dx\\ &- \int_\Omega f(x,u)v\,dx,\ \forall u, v \in E;\end{aligned}$$

the weak solutions of the variational formulation are the critical points of Φ in E. Following the paper [3] and in order to obtain nonnegative and nonpositive solutions, we let $\widetilde{f}(x,s) = f(x,s) - m(x)|s|^{p-2}s$ and truncate $\widetilde{f}$ above or below $s = 0$, i.e., let

$$\widetilde{f}_+(x,s) = \begin{cases} \widetilde{f}(x,s), & s \geq 0,\\ 0, & s < 0,\end{cases} \qquad \text{and} \qquad \widetilde{f}_-(x,s) = \begin{cases} \widetilde{f}(x,s), & s \leq 0,\\ 0, & s > 0,\end{cases}$$

and $\widetilde{F}_+(x,s) = \int_0^s \widetilde{f}_+(x,t)\,dt$, $\widetilde{F}_-(x,s) = \int_0^s \widetilde{f}_-(x,t)\,dt$. Under $(H1)$ and $(H2)$, the functionals $\widetilde{\Phi}_+$ and $\widetilde{\Phi}_-$ defined as follows

$$\begin{aligned}\widetilde{\Phi}_+(u) &= \frac{1}{p}\widehat{M}(||u||^p) + \frac{1}{p}\int_\Omega \ell(x)|u|^p\,dx - \frac{1}{p}\int_\Omega m(x)|u|^p\,dx - \int_\Omega \widetilde{F}_+(x,u)\,dx,\\ &= \frac{1}{bp^2}\Big[(a + b||u||^p)^p - a^p\Big] + \frac{1}{p}\int_\Omega \ell(x)|u|^p\,dx\\ &\quad - \frac{1}{p}\int_\Omega m(x)|u|^p\,dx - \int_\Omega \widetilde{F}_+(x,u)\,dx,\\ \widetilde{\Phi}_-(u) &= \frac{1}{p}\widehat{M}(||u||^p) + \frac{1}{p}\int_\Omega \ell(x)|u|^p\,dx - \frac{1}{p}\int_\Omega m(x)|u|^p\,dx - \int_\Omega \widetilde{F}_-(x,u)\,dx,\\ &= \frac{1}{bp^2}\Big[(a + b||u||^p)^p - a^p\Big] + \frac{1}{p}\int_\Omega \ell(x)|u|^p\,dx - \frac{1}{p}\int_\Omega m(x)|u|^p\,dx\\ &\quad - \int_\Omega \widetilde{F}_-(x,u)\,dx,\end{aligned}$$

belong to $C^1(E, \mathbb{R})$ and

$$\langle \widetilde{\Phi}'_+(u), v\rangle = (a + b||u||^p)^{p-1} \int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla v\, dx + \int_\Omega \ell(x)|u|^{p-2}uv\, dx$$
$$- \int_\Omega m(x)|u|^{p-2}uv\, dx - \int_\Omega \widetilde{f}_+(x,u)v\, dx,$$
$$\langle \widetilde{\Phi}'_-(u), v\rangle = (a + b||u||^p)^{p-1} \int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla v\, dx + \int_\Omega \ell(x)|u|^{p-2}uv\, dx$$
$$- \int_\Omega m(x)|u|^{p-2}uv\, dx - \int_\Omega \widetilde{f}_-(x,u)v\, dx,$$

for all $u, v \in E$.

4. Proof of main results

We recall one critical point theorem which is the Mountain Pass lemma.

Theorem 4.1. *Let $(X, ||\cdot||_X)$ be a Banach space, suppose that $\Phi \in C^1(X, \mathbb{R})$ satisfies $\Phi(0) = 0$ and*

(i) *(the first geometric condition) there exist positive constants R_0 and α_0 such that*
$$\Phi(u) \geq \alpha_0 \ \text{ for all } u \in X \text{ with } ||u||_X = R_0,$$
(ii) *(the second geometric condition) there exists $e \in X$ with $||e||_X > R_0$ such that $\Phi(e) < 0$,*
(iii) *(the Palais-Smale condition) Φ satisfies the (C_c) condition, that is, for $c \in \mathbb{R}$, every sequence $(u_n) \subset X$ such that*
$$\Phi(u_n) \to c, \quad ||\Phi'(u_n)||\,(1 + ||u_n||) \to 0$$
has a convergent subsequence. Then $c := \inf_{\gamma\in\Gamma} \sup_{s\in[0,1]} \Phi(\gamma(s))$ is a critical value of Φ, where
$$\Gamma := \{\gamma \in C([0,1], X); \gamma(0) = 0, \gamma(1) = e\}.$$

We need also the following lemmas.

Lemma 4.1. *Assume that $N \geq 3$ and $v \in L^{\frac{N}{p}}(\Omega)$, then the functional*
$$\psi(u) := \int_\Omega v(x)|u|^p\, dx, \quad u \in W_0^{1,p}(\Omega)$$
is weakly continuous.

Proof. As in [8], the functional ψ is well defined by the Sobolev and Hölder inequalities. Assume that $u_n \rightharpoonup u$ in $W_0^{1,p}(\Omega)$ and consider an arbitrary subsequence (w_n) of (u_n). Since
$$w_n \to u \text{ in } L^p_{loc}(\Omega), \quad w_n^+ \to u^+ \text{ in } L^p_{loc}(\Omega) \quad \text{and} \quad w_n^- \to u^- \text{ in } L^p_{loc}(\Omega)$$
going if necessary to a subsequence, we can assume that
$$w_n \to u \text{ a.e. on } \Omega, \quad w_n^+ \to u^+ \text{ a.e. on } \Omega \quad \text{and} \quad w_n^- \to u^- \text{ a.e. on } \Omega.$$

Since both (w_n^+) and (w_n^-) are bounded in $L^{p^*}(\Omega)$, $((w_n^+)^p)$ and $((w_n^-)^p)$ are bounded in $L^{\frac{N}{N-p}}(\Omega)$. Hence $(w_n^+)^p \rightharpoonup (u^+)^p$ and $(w_n^-)^p \rightharpoonup (u^-)^p$ in $L^{\frac{N}{N-p}}(\Omega)$, and so

$$\int_\Omega v(x)|w_n|^p\,dx \to \int_\Omega v(x)|u|^p.$$

As a result, ψ is weakly continuous. □

Lemma 4.2. *Assume that* $m \in L^{\frac{N}{p}}(\Omega)$, *and there exists* $\Omega' \subset \Omega$ *with* $|\Omega'| > 0$ *such that*

$$m \leq \lambda_1 \text{ in } \Omega \text{ and } m < \lambda_1 \text{ in } \Omega'$$

then

$$d := \inf_{u\in W_0^{1,p}(\Omega), u\neq 0} \frac{\Big(a + b||u||^p\Big)^{p-1}||u||^p + \int_\Omega \ell(x)|u|^p\,dx - \int_\Omega m(x)|u|^p\,dx}{||u||^p}$$

is strictly positive $(d > 0)$.

Proof. Since $\lambda_1 = \inf_{u\in W_0^{1,p}(\Omega), u\neq 0} \dfrac{\Big(a + b||u||^p\Big)^{p-1}||u||^p + \int_\Omega \ell(x)|u|^p\,dx}{|u|_p^p}$ and from the assumption that $m \leq \lambda_1$ in Ω, we have that $d \geq 0$, because $m \leq \lambda_1$ implies that

$$-\int_\Omega m|u|^p\,dx \geq -\int_\Omega \lambda_1|u|^p\,dx,$$

and consequently, we have

$$\begin{aligned} & \frac{\Big(a + b||u||^p\Big)^{p-1}||u||^p + \int_\Omega \ell(x)|u|^p\,dx - \int_\Omega m(x)|u|^p\,dx}{||u||^p} \\ \geq\; & \frac{\Big(a + b||u||^p\Big)^{p-1}||u||^p + \int_\Omega \ell(x)|u|^p\,dx - \int_\Omega \lambda_1|u|^p\,dx}{||u||^p} \\ =\; & \frac{\Big(a + b||u||^p\Big)^{p-1}||u||^p + \int_\Omega \ell(x)|u|^p\,dx}{||u||^p} - \lambda_1\frac{\int_\Omega |u|^p\,dx}{||u||^p} \\ \geq\; & 0, \end{aligned}$$

by definition of λ_1. It remains to prove that $d \neq 0$; for that, we let

$$\begin{aligned} J(u) &:= \int_\Omega \ell(x)|u|^p\,dx, u \in W_0^{1,p}(\Omega), \\ K(u) &:= \int_\Omega m(x)|u|^p\,dx, u \in W_0^{1,p}(\Omega), \\ L(u) &:= \Big(a + b||u||^p\Big)^{p-1}||u||^p + J(u) - K(u), u \in W_0^{1,p}(\Omega). \end{aligned}$$

Supposing by contradiction that $d = 0$, it follows that there exists a sequence $(u_n)_n \subset W_0^{1,p}(\Omega)$ such that

$$||u_n|| = 1 \quad \text{and} \quad \lim_{n\to+\infty} L(u_n) = 0,$$

by the boundedness of $(u_n)_n$ in $W_0^{1,p}(\Omega)$, we can extract a subsequence such that $u_n \rightharpoonup u$ in $W_0^{1,p}(\Omega)$. Since J and K are weakly continuous, we have

$$\lim_{n\to+\infty} J(u_n) = J(u), \lim_{n\to+\infty} K(u_n) = K(u), \tag{4.1}$$

and from the weak lower semicontinuity of L, we obtain

$$0 \le L(u) \le \liminf_n L(u_n) = \lim_n L(u_n) = 0,$$

then we have

$$L(u) = \Big(a + b||u||^p\Big)^{p-1} ||u||^p + J(u) - K(u) = \lim_n L(u_n) = 0, \tag{4.2}$$

which implies that

$$\Big(a + b||u||^p\Big)^{p-1} ||u||^p + J(u) = K(u) = \int_\Omega m(x)|u|^p\,dx \le \lambda_1 \int_\Omega |u|^p\,dx,$$

so, we have

$$\Big(a + b||u||^p\Big)^{p-1} ||u||^p + J(u) \le \lambda_1 \int_\Omega |u|^p\,dx \le \Big(a + b||u||^p\Big)^{p-1} ||u||^p + J(u),$$

consequently

$$\Big(a + b||u||^p\Big)^{p-1} ||u||^p + J(u) = \lambda_1 \int_\Omega |u|^p\,dx. \tag{4.3}$$

If $u = 0$, from (4.1), (4.2), we have that

$$\lim_n L(u_n) = \lim_n \Big(\Big(a + b||u_n||^p\Big)^{p-1} ||u_n||^p\Big) + J(0) - K(0) = 0,$$

which implies that $\lim_{n\to+\infty} ||u_n|| = 0$, and this is a contradiction with $||u_n|| = 1$. So $u \neq 0$, then u is an eigenfunction corresponding to λ_1; since $m \le \lambda_1$ in Ω and $m < \lambda_1$ in Ω' with $|\Omega'| > 0$, it follows that,

$$\begin{aligned}
\Big(a + b||u||^p\Big)^{p-1} ||u||^p + J(u) = K(u) &= \int_\Omega m(x)|u|^p\,dx \\
&= \int_{\Omega'} m(x)|u|^p\,dx + \int_{\Omega\setminus\Omega'} m(x)|u|^p \\
&< \lambda_1 \int_{\Omega'} |u|^p\,dx + \int_{\Omega\setminus\Omega'} \lambda_1|u|^p\,dx \\
&= \int_\Omega \lambda_1|u|^p\,dx,
\end{aligned}$$

which is in contradiction with (4.3). Consequently, $d > 0$. □

Lemma 4.3. *Assume $(H1)$, $(H2)$ and $(H3)$ hold, then $\widetilde{\Phi}_+$ satisfies the first geometric condition.*

Proof. In the same way as in the paper [3], by $(H3)$ and for $\varepsilon \in \Big(0, \frac{dS_p^p}{2}\Big)$, there exists a positive constant $M_1 < 1$ such that

$$F_+(x, s) = F(x, s^+) \le \frac{1}{p}(m(x) + \varepsilon)(s^+)^p, \text{ for } |s| \le M_1 \text{ and a.e. } x \in \Omega \tag{4.4}$$

with $s^+ = \max(s,0)$; for the chosen ε and from $(H1)$ and $(H2)$, we have

$$\exists M_2 > 1, \exists L_{M_2} : |f_+(x,s)| = |f(x,s^+)| \le \varepsilon (s^+)^{p^*-1} + L_{M_2} \tag{4.5}$$

and

$$F_+(x,s) \le \frac{1}{p}(m(x)+\varepsilon)(s^+)^p + \Big(\frac{L_{M_2}M_2}{M_1^{p^*}} + \frac{\varepsilon}{p^*}\Big)(s^+)^{p^*}, \tag{4.6}$$

for $s \in \mathbb{R}$ and a.e. $x \in \Omega$. From (4.6), we have

$$\begin{aligned}
\widetilde{\Phi}_+(u) &= \frac{1}{bp^2}\Big[(a+b||u||^p)^p - a^p\Big] + \frac{1}{p}\int_\Omega \ell(x)|u|^p\,dx \\
&- \frac{1}{p}\int_\Omega m(x)|u|^p\,dx - \int_\Omega \widetilde{F}_+(x,u)\,dx \\
&= \frac{1}{bp^2}(a+b||u||^p)^p + \frac{1}{p}\int_\Omega \ell(x)|u|^p\,dx - \frac{1}{p}\int_\Omega m(x)|u|^p\,dx \\
&- \frac{a^p}{bp^2} - \int_\Omega \widetilde{F}_+(x,u)\,dx \\
&= \frac{1}{bp^2}(a+b||u||^p)^{p-1}(a+b||u||^p) + \frac{1}{p}\int_\Omega \ell(x)|u|^p\,dx - \frac{1}{p}\int_\Omega m(x)|u|^p\,dx \\
&- \frac{a^p}{bp^2} - \int_\Omega \widetilde{F}_+(x,u)\,dx \\
&\ge \frac{1}{bp^2}(a+b||u||^p)^{p-1}b||u||^p + \frac{1}{p}\int_\Omega \ell(x)|u|^p\,dx - \frac{1}{p}\int_\Omega m(x)|u|^p\,dx \\
&- \frac{a^p}{bp^2} - \int_\Omega \widetilde{F}_+(x,u)\,dx \\
&= \frac{1}{p^2}(a+b||u||^p)^{p-1}||u||^p + \frac{1}{p}\int_\Omega \ell(x)|u|^p\,dx - \frac{1}{p}\int_\Omega m(x)|u|^p\,dx \\
&- \frac{a^p}{bp^2} - \int_\Omega \widetilde{F}_+(x,u)\,dx, \\
&= \frac{1}{p}\Big[(a+b||u||^p)^{p-1}||u||^p + \int_\Omega \ell(x)|u|^p\,dx - \int_\Omega m(x)|u|^p\,dx\Big] \\
&- \frac{a^p}{bp^2} - \frac{1}{pp'}(a+b||u||^p)^{p-1}||u||^p - \int_\Omega \widetilde{F}_+(x,u)\,dx \\
&\ge \frac{d}{p}||u||^p - \int_\Omega \widetilde{F}_+(x,u)\,dx - \frac{a^p}{bp^2} - \frac{1}{pp'}(a+b||u||^p)^{p-1}||u||^p \\
&= \frac{d}{p}||u||^p - \int_\Omega F_+(x,u)\,dx + \frac{1}{p}\int_\Omega m(x)|u^+|^p\,dx \\
&- \frac{a^p}{bp^2} - \frac{1}{pp'}(a+b||u||^p)^{p-1}||u||^p \\
&\ge \frac{d}{p}||u||^p - \frac{1}{p}\int_\Omega (m(x)+\varepsilon)(u^+)^p\,dx - \int_\Omega \Big(\frac{L_{M_2}M_2}{M_1^{p^*}} + \frac{\varepsilon}{p^*}\Big)(u^+)^{p^*}\,dx \\
&+ \frac{1}{p}\int_\Omega m(x)|u^+|^p\,dx - \frac{a^p}{bp^2} - \frac{1}{pp'}(a+b||u||^p)^{p-1}||u||^p
\end{aligned}$$

$$
\begin{aligned}
&= \frac{d}{p}||u||^p - \frac{1}{p}\int_\Omega \varepsilon (u^+)^p\,dx - \int_\Omega \Big(\frac{L_{M_2}M_2}{M_1^{p^*}} + \frac{\varepsilon}{p^*}\Big)(u^+)^{p^*}\,dx \\
&- \frac{a^p}{bp^2} - \frac{1}{pp'}(a+b||u||^p)^{p-1}||u||^p \\
&\geq \frac{d}{p}||u||^p - \frac{\varepsilon}{pS_p^p}||u||^p - \Big(\frac{L_{M_2}M_2}{M_1^{2^*}} + \frac{\varepsilon}{2^*}\Big)\Big(\frac{1}{S_{p^*}}\Big)||u||^{p^*} \\
&- \frac{a^p}{bp^2} - \frac{1}{pp'}(a+b||u||^p)^{p-1}||u||^p \\
&= \frac{d}{2p}||u||^p - \Big(\frac{L_{M_2}M_2}{M_1^{p^*}} + \frac{\varepsilon}{p^*}\Big)\Big(\frac{1}{S_{p^*}}\Big)||u||^{p^*} - \frac{a^p}{bp^2} - \frac{1}{pp'}(a+b||u||^p)^{p-1}||u||^p, \\
&\forall u \in W_0^{1,p}(\Omega).
\end{aligned}
$$

Let $C_1 = \Big(\frac{L_{M_2}M_2}{M_1^{p^*}} + \frac{\varepsilon}{p^*}\Big)\Big(\frac{1}{S_{p^*}}\Big)$, we have that

$$
\widetilde{\Phi}_+(u) \geq \frac{d}{2p}||u||^p - C_1||u||^{p^*} - \frac{a^p}{bp^2} - \frac{1}{pp'}(a+b||u||^p)^{p-1}||u||^p, \quad \forall u \in W_0^{1,p}(\Omega).
$$

For R_0 sufficiently small, with $\|u\| = R_0$, one can have $||u||^{p^*} < ||u||^p$ and

$$
(a+b||u||^p)^{p-1}||u||^p < ||u||^p
$$

and

$$
\widetilde{\Phi}_+(u) \geq \frac{d}{2p}||u||^p - C_1||u||^{p^*} - \frac{a^p}{bp^2} - \frac{1}{pp'}(a+b||u||^p)^{p-1}||u||^p \geq \alpha_0 > 0, \ \forall u \in W_0^{1,p}(\Omega).
$$

Consequently, the first geometric condition is satisfied. □

Lemma 4.4. *Assume that* $(H1)$ *and* $(H4)$ *hold, then* $\widetilde{\Phi}_+$ *satisfies the second geometric condition.*

Proof. Note that, using the following standard inequality, for $\alpha, \beta \geq 0$ and $p \geq 1$, we have $(\alpha+\beta)^p \leq 2^{p-1}(\alpha^p + \beta^p)$, then,

$$
\begin{aligned}
\widehat{M}(||u||^p) &= \frac{1}{pb}\Big[(a+b||u||^p)^p - a^p\Big] \\
&\leq \frac{1}{pb}\Big[2^{p-1}\Big(a^p + b^p||u||^{p^2}\Big) - a^p\Big] \\
&\leq \frac{1}{pb}\Big[(2^{p-1}-1)a^p + 2^{p-1}b^p||u||^{p^2}\Big];
\end{aligned}
$$

let $c_1 = 2^{p-1} - 1$ and $c_2 = 2^{p-1}b^p$, we have

$$
\widehat{M}(||u||^p) \leq \frac{1}{pb}\Big[c_1 a^p + c_2||u||^{p^2}\Big]. \tag{4.7}
$$

From $(H1)$ and $(H4)$, we have

$$
\forall \Lambda > 0,\ \exists M_3 > 0,\ F_+(x,s) \geq \Lambda (s^+)^{p^2} - L_{M_3}M_3,
$$

for $s \in \mathbb{R}$ and a.e. $x \in \Omega$.

Then for $t > 0$, $\phi_1 > 0$, the first eigenfunction and using (4.7), we have

$$
\begin{aligned}
\widetilde{\Phi}_+(t\phi_1) &= \frac{1}{p}\widehat{M}(||t\phi_1||^p) + \frac{1}{p}\int_\Omega \ell(x)|t\phi_1|^p\,dx - \frac{1}{p}\int_\Omega m(x)|t\phi_1|^p\,dx - \int_\Omega \widetilde{F}_+(x,t\phi_1)\,dx \\
&\leq \frac{1}{p^2 b}\left[c_1 a^p + c_2||t\phi_1||^{p^2}\right] + \frac{1}{p}\int_\Omega \ell(x)|t\phi_1|^p\,dx - \frac{1}{p}\int_\Omega m(x)|t\phi_1|^p\,dx \\
&- \int_\Omega \widetilde{F}_+(x,t\phi_1)\,dx \\
&= \frac{1}{p^2 b}\left[c_1 a^p + c_2||t\phi_1||^{p^2}\right] + \frac{1}{p}\int_\Omega \ell(x)|t\phi_1|^p\,dx - \frac{1}{p}\int_\Omega m(x)|t\phi_1|^p\,dx \\
&+ \frac{1}{p}\int_\Omega m(x)((t\phi_1)^+)^p\,dx - \int_\Omega F(x,(t\phi_1)^+)\,dx \\
&\leq \frac{1}{p^2 b}\left[c_1 a^p + c_2||t\phi_1||^{p^2}\right] + \frac{1}{p}\int_\Omega \ell(x)|t\phi_1|^p\,dx - \Lambda\int_\Omega t^{p^2}\phi_1^{p^2}\,dx + L_{M_3}M_3|\Omega| \\
&= t^{p^2}\left[\frac{c_2}{p^2 b}||\phi_1||^{p^2} - \Lambda\int_\Omega \phi_1^{p^2}\,dx\right] + \frac{t^p}{p}\int_\Omega \ell(x)\phi_1^p\,dx + \frac{c_1 a^p}{p^2 b} + L_{M_3}M_3|\Omega| \\
&= At^{p^2} + Bt^p + C = P(t),
\end{aligned}
$$

where

$$
A = \frac{c_2}{p^2 b}||\phi_1||^{p^2} - \Lambda\int_\Omega \phi_1^{p^2}\,dx,\ B = \frac{1}{p}\int_\Omega \ell(x)\phi_1^p\,dx,\ \text{ and } C = \frac{c_1 a^p}{p^2 b} + L_{M_3}M_3|\Omega| > 0;
$$

by choosing

$$
\Lambda > \frac{c_2||\phi_1||^{p^2}}{p^2 b\int_\Omega \phi_1^{p^2}\,dx},
$$

we then have $A < 0$. For t_0 sufficiently large, we have that $P(t_0) < 0$ and then by taking

$$
e = t_0\phi_1 \in W_0^{1,p}(\Omega),
$$

we conclude that $\widetilde{\Phi}_+$ satisfies the second geometric condition. □

Lemma 4.5. *Assume that* $(H1)$, $(H2)$ *and* $(H5)$ *hold, then* $\widetilde{\Phi}_+$ *satisfies the Palais-Smale condition.*

Proof. Claim 1: Under the same hypotheses in the above lemma, any (C_c) sequence is bounded.
Indeed, for $c \in \mathbb{R}$, and $(u_n)_n \subset W_0^{1,p}(\Omega)$, such that

$$
\widetilde{\Phi}_+(u_n) \to c \text{ and } (1 + ||u_n||)\widetilde{\Phi}'_+(u_n) \to 0, \tag{4.8}
$$

we have for n large, that

$$\begin{aligned}
C_0 &\geq \widetilde{\Phi}_+(u_n) - \frac{1}{p^2}\widetilde{\Phi}'_+(u_n)u_n \\
&= \frac{1}{p}\widehat{M}(||u_n||^p) + \frac{1}{p}\int_\Omega \ell(x)|u_n|^p\,dx - \frac{1}{p}\int_\Omega m(x)|u_n|^p\,dx - \int_\Omega \widetilde{F}_+(x,u_n)\,dx \\
&- \frac{1}{p^2}\Big[\Big[M(||u_n||^p)\Big]^{p-1}||u_n||^p + \int_\Omega \ell(x)|u_n|^p\,dx - \int_\Omega m(x)|u_n|^p\,dx \\
&- \int_\Omega \widetilde{f}_+(x,u_n)u_n\,dx\Big] \\
&= \frac{1}{p}\widehat{M}(||u_n||^p) + \frac{1}{p}\int_\Omega \ell(x)|u_n|^p\,dx - \frac{1}{p}\int_\Omega m(x)|u_n|^p\,dx + \frac{1}{p}\int_\Omega m(x)(u_n^+)^p\,dx \\
&- \int_\Omega F_+(x,u_n)\,dx - \frac{1}{p^2}\Big[\Big[M(||u_n||^p)\Big]^{p-1}||u_n||^p + \int_\Omega \ell(x)|u_n|^p\,dx \\
&- \int_\Omega m(x)|u_n|^p\,dx + \int_\Omega m(x)(u_n^+)^p\,dx - \int_\Omega f_+(x,u_n)u_n\,dx\Big]
\end{aligned}$$

because,

$$\widetilde{F}_+(x,s) = F(x,s^+) - m(x)\frac{(s^+)^p}{p}$$

and

$$\widetilde{f}_+(x,s) = f(x,s^+)s^+ - m(x)(s^+)^p = f(x,s^+)s - m(x)(s^+)^p,$$

then

$$\begin{aligned}
C_0 \quad &\geq \quad \frac{1}{p}\widehat{M}(||u_n||^p) - \frac{1}{p^2}\Big[M(||u_n||^p)\Big]^{p-1}||u_n||^p + \Big(\frac{1}{p} - \frac{1}{p^2}\Big)\int_\Omega \ell(x)|u_n|^p\,dx \\
&\qquad - \Big(\frac{1}{p} - \frac{1}{p^2}\Big)\int_\Omega m(x)|u_n^-|^p\,dx - \int_\Omega F_+(x,u_n)\,dx + \frac{1}{p^2}\int_\Omega f_+(x,u_n)u_n\,dx \\
&= \quad \frac{1}{p}\widehat{M}(||u_n||^p) - \frac{1}{p^2}\Big[M(||u_n||^p)\Big]^{p-1}||u_n||^p + \Big(\frac{1}{p} - \frac{1}{p^2}\Big)\int_\Omega \ell(x)|u_n|^p\,dx \\
&\qquad - \Big(\frac{1}{p} - \frac{1}{p^2}\Big)\int_\Omega m(x)|u_n^-|^p\,dx + \int_\Omega \overline{F}(x,u_n^+)\,dx;
\end{aligned}$$

note that the quantity $\frac{1}{p}\widehat{M}(||u_n||^p) - \frac{1}{p^2}\Big[M(||u_n||^p)\Big]^{p-1}||u_n||^p$ is positive, so we obtain

$$\begin{aligned}
C_0 \quad &\geq \quad \widetilde{\Phi}_+(u_n) - \frac{1}{p^2}\widetilde{\Phi}'_+(u_n)u_n \\
&\geq \quad \Big(\frac{1}{p} - \frac{1}{p^2}\Big)\int_\Omega \ell(x)|u_n|^p\,dx - \Big(\frac{1}{p} - \frac{1}{p^2}\Big)\int_\Omega m(x)|u_n^-|^p\,dx + \int_\Omega \overline{F}(x,u_n^+)\,dx.
\end{aligned}$$

If $\ell(x) - m(x) \geq 0$ and since $(u^-)^p \leq |u|^p$, we have

$$\begin{aligned} C_0 &\geq \Big(\frac{1}{p} - \frac{1}{p^2}\Big) \int_\Omega \ell(x)|u_n|^p\,dx - \Big(\frac{1}{p} - \frac{1}{p^2}\Big) \int_\Omega m(x)|u_n|^p\,dx + \int_\Omega \overline{F}(x, u_n^+)\,dx \\ &= \Big(\frac{1}{p} - \frac{1}{p^2}\Big) \int_\Omega (\ell(x) - m(x))|u_n|^p\,dx + \int_\Omega \overline{F}(x, u_n^+)\,dx \\ &\geq \int_\Omega \overline{F}(x, u_n^+)\,dx. \end{aligned}$$

If $\ell(x) \geq 0$ and $a^{p-1} \geq \frac{|m|_{L^\infty}}{S_p^p}$ where $p \geq 2$, and using the fact that

$$\begin{aligned} \Big|\int_\Omega m(x)u_n^p\,dx\Big| &\leq |m|_{L^\infty}|u_n|_p^p \\ &\leq |m|_{L^\infty}\frac{1}{S_p^p}||u_n||^p, \end{aligned}$$

we have

$$\begin{aligned} C_0 &\geq \frac{1}{p}\widehat{M}(||u_n||^p) - \frac{1}{p^2}\Big[M(||u_n||^p)\Big]^{p-1}||u_n||^p + \Big(\frac{1}{p} - \frac{1}{p^2}\Big) \int_\Omega \ell(x)|u_n|^p\,dx \\ &\quad -\Big(\frac{1}{p} - \frac{1}{p^2}\Big) \int_\Omega m(x)|u_n|^p\,dx + \int_\Omega \overline{F}(x, u_n^+)\,dx \\ &\geq \frac{1}{p}\widehat{M}(||u_n||^p) - \frac{1}{p^2}\Big[M(||u_n||^p)\Big]^{p-1}||u_n||^p - \Big(\frac{1}{p} - \frac{1}{p^2}\Big)|m|_{L^\infty}\frac{1}{S_p^p}||u_n||^p \\ &\quad +\Big(\frac{1}{p} - \frac{1}{p^2}\Big) \int_\Omega \ell(x)|u_n|^p\,dx + \int_\Omega \overline{F}(x, u_n^+)\,dx \\ &\geq \int_\Omega \overline{F}(x, u_n^+)\,dx. \end{aligned}$$

So, in both cases, one can obtain

$$\begin{aligned} C_0 &\geq \widetilde{\Phi}_+(u_n) - \frac{1}{p^2}\widetilde{\Phi}'_+(u_n)u_n \\ &\geq \int_\Omega \overline{F}(x, u_n^+)\,dx; \end{aligned} \tag{4.9}$$

let suppose by contradiction that $||u_n|| \to +\infty$, and set

$$v_n = \frac{u_n}{||u_n||}.$$

Then $||v_n|| = 1$, and

$$|v_n|_s \leq \frac{1}{S_s}||v_n|| = \frac{1}{S_s},$$

for $s \in [1, p^*]$.

Observe that

$$\begin{aligned}\widetilde{\Phi}'_+(u_n)u_n &= (a+b||u_n||^p)^{p-1}||u_n||^p + \int_\Omega \ell(x)|u_n|^p\,dx - \int_\Omega m(x)|u_n|^p\,dx \\ &- \int_\Omega \widetilde{f}(x,u_n^+)u_n\,dx \\ &= (a+b||u_n||^p)^{p-1}||u_n||^p + \int_\Omega \ell(x)|u_n|^p\,dx - \int_\Omega m(x)|u_n|^p\,dx \\ &- \int_\Omega f(x,u_n^+)u_n\,dx + \int_\Omega m(x)(u_n^+)^p\,dx \\ &= (a+b||u_n||^p)^{p-1}||u_n||^p + \int_\Omega \ell(x)|u_n|^p\,dx - \int_\Omega m(x)(u_n^-)^p\,dx \\ &- \int_\Omega f(x,u_n^+)u_n\,dx \\ &= ||u_n||^{p^2}\Big(\frac{(a+b||u||^p)^{p-1}||u||^p}{||u_n||^{p^2}} + \frac{\int_\Omega \ell(x)|u_n|^p\,dx}{||u_n||^{p^2}} - \frac{\int_\Omega m(x)(u_n^-)^p\,dx}{||u_n||^{p^2}} \\ &- \frac{\int_\Omega f(x,u_n^+)v_n\,dx}{||u_n||^{p^2-1}}\Big).\end{aligned}$$

From (4.8), $\widetilde{\Phi}'_+(u_n) \to 0$ as $n \to +\infty$, so we have

$$\lim_n\Big(\frac{(a+b||u||^p)^{p-1}||u||^p}{||u_n||^{p^2}} + \frac{\int_\Omega \ell(x)|u_n|^p\,dx}{||u_n||^{p^2}} - \frac{\int_\Omega m(x)(u_n^-)^p\,dx}{||u_n||^{p^2}} - \frac{\int_\Omega f(x,u_n^+)v_n\,dx}{||u_n||^{p^2-1}}\Big) = 0.$$

Let's show that

$$\lim_n \frac{\int_\Omega \ell(x)|u_n|^p\,dx}{||u_n||^{p^2}} = 0$$

and

$$\lim_n \frac{\int_\Omega m(x)(u_n^-)^p\,dx}{||u_n||^{p^2}} = 0.$$

We have

$$\frac{\int_\Omega \ell(x)|u_n|^p\,dx}{||u_n||^{p^2}} = \frac{\int_\Omega \ell(x)|v_n|^p\,dx}{||u_n||^p}$$

and

$$\frac{\int_\Omega m(x)(u_n^-)^p\,dx}{||u_n||^{p^2}} = \frac{\int_\Omega m(x)(v_n^-)^p\,dx}{||u_n||^p};$$

since $v_n \to v$ in L^r $(r \in [1,p^*))$ and from Lemma 4.1, we deduce that

$$\lim_n \int_\Omega \ell(x)|v_n|^p\,dx = \int_\Omega \ell(x)|v|^p\,dx \text{ and } \lim_n \int_\Omega m(x)(v_n^-)^p\,dx = \int_\Omega m(x)(v^-)^p\,dx,$$

and since $||u_n|| \to +\infty$, we conclude that

$$\lim_n \frac{\int_\Omega \ell(x)|u_n|^p\,dx}{||u_n||^{p^2}} = \lim_n \frac{\int_\Omega m(x)(u_n^-)^p\,dx}{||u_n||^{p^2}} = 0.$$

We have also that

$$\lim_{n\to+\infty} \frac{(a+b||u_n||^p)^{p-1}||u||^p}{||u_n||^{p^2}} = \lim_{n\to+\infty} \frac{b^{p-1}||u||^{p(p-1)+p}}{||u_n||^{p^2}} = b^{p-1}.$$

Then

$$\lim_n \int_\Omega \frac{f(x,u_n^+)v_n\,dx}{||u_n||^{p^2-1}} = b^{p-1}. \tag{4.10}$$

Set for $r \geq 0$,

$$g(r) = \inf\{\overline{F}(x,u^+) : x \in \Omega \text{ and } u^+ \in \mathbb{R}_+ \text{ with } u^+ \geq r\}.$$

$(H5)$ implies that $g(r) > 0$ for all r large, and $g(r) \to +\infty$ as $r \to +\infty$. Set for $0 \leq \alpha < \beta \leq +\infty$,

$$\Lambda_n(\alpha,\beta) := \{x \in \Omega :\ \alpha \leq |u_n^+(x)| < \beta\}$$

and

$$\sigma_\alpha^\beta := \inf\Big\{\frac{\overline{F}(x,u^+)}{|u^+(x)|^p} : x \in \Omega \text{ and } u^+ \in \mathbb{R}_+ \text{ with } \alpha \leq u^+ < \beta\Big\}.$$

For large α, we have $\overline{F}(x,u^+) > 0$, $\sigma_\alpha^\beta > 0$ and

$$\overline{F}(x,u_n^+) \geq \sigma_\alpha^\beta |u_n^+|^p, \qquad \text{for } x \in \Lambda_n(\alpha,\beta).$$

It follows from (4.9) that

$$\begin{aligned} C_0 &\geq \int_{\Lambda_n(0,\alpha)} \overline{F}(x,u_n^+) + \int_{\Lambda_n(\alpha,\beta)} \overline{F}(x,u_n^+) + \int_{\Lambda_n(\beta,+\infty)} \overline{F}(x,u_n^+) \\ &\geq \int_{\Lambda_n(0,\alpha)} \overline{F}(x,u_n^+) + \sigma_\alpha^\beta \int_{\Lambda_n(\alpha,\beta)} |u_n^+|^p + g(\beta)\,|\Lambda_n(\beta,+\infty)|. \end{aligned}$$

Since $g(r) \to +\infty$ as $r \to +\infty$,

$$|\Lambda_n(\beta,+\infty)| \to 0, \qquad \text{as } \beta \to +\infty, \text{ uniformly in } n,$$

which implies that, by the Hölder inequality, that for any $s \in [1,p^*)$,

$$\begin{aligned} \int_{\Lambda_n(\beta,+\infty)} |v_n|^s &\leq \Big(\int_{\Lambda_n(\beta,+\infty)} (|v_n|^s)^{\frac{p^*}{s}}\Big)^{\frac{s}{p^*}} |\Lambda_n(\beta,+\infty)|^{\frac{p^*-s}{p^*}} \\ &\leq \frac{1}{S_{p^*}^s} |\Lambda_n(\beta,+\infty)|^{\frac{p^*-s}{p^*}} \\ &\to 0 \end{aligned} \tag{4.11}$$

as $\beta \to +\infty$ uniformly in n. Furthermore, for any fixed $0 < \alpha < \beta$,

$$\begin{aligned} \int_{\Lambda_n(\alpha,\beta)} |v_n^+|^p &= \frac{1}{||u_n||^p} \int_{\Lambda_n(\alpha,\beta)} |u_n^+|^p = \frac{1}{||u_n||^p} \int_{\Lambda_n(\alpha,\beta)} \frac{\sigma_\alpha^\beta |u_n^+|^p}{\sigma_\alpha^\beta} \\ &\leq \frac{1}{||u_n||^p \sigma_\alpha^\beta} \int_{\Lambda_n(\alpha,\beta)} \overline{F}(x,u_n^+) \\ &\leq \frac{C_0}{||u_n||^p \sigma_\alpha^\beta} \\ &\to 0. \end{aligned} \tag{4.12}$$

Set $0 < \eta < \frac{b^{p-1}}{3}$. From (4.5) (from $(H1)$ and $(H2)$), we have

$$\begin{aligned}
\int_{\Lambda_n(0,\alpha)} \frac{f(x,u_n^+)u_n}{||u_n||^{p^2}} &\leq \int_{\Lambda_n(0,\alpha)} \frac{|f(x,u_n^+)u_n|}{||u_n||^{p^2}} = \int_{\Lambda_n(0,\alpha)} \frac{|f(x,u_n^+)|\,(u_n^+ + u_n^-)}{||u_n||^{p^2}} \\
&= \int_{\Lambda_n(0,\alpha)} \frac{|f(x,u_n^+)|\,u_n^+}{||u_n||^{p^2}} + \int_{\Lambda_n(0,\alpha)} \frac{|f(x,u_n^+)|\,u_n^-}{||u_n||^{p^2}} \\
&\leq \int_{\Lambda_n(0,\alpha)} \frac{(\varepsilon(u_n^+)^{p^*} + L_{M_2}u_n^+)\,dx}{||u_n||^{p^2}} \\
&+ \int_{\Lambda_n(0,\alpha)} \frac{(\varepsilon(u_n^+)^{p^*-1} + L_{M_2})\,u_n^-\,dx}{||u_n||^{p^2}} \\
&\leq \int_{\Lambda_n(0,\alpha)} \frac{(\varepsilon\alpha^{p^*} + L_{M_2}\alpha)\,dx}{||u_n||^{p^2}} + \int_{\Lambda_n(0,\alpha)} \frac{L_{M_2}\,u_n^-\,dx}{||u_n||^{p^2}} \to 0,
\end{aligned}$$

because

$$\begin{aligned}
\int_{\Lambda_n(0,\alpha)} \frac{L_{M_2}\,u_n^-\,dx}{||u_n||^{p^2}} &= \frac{L_{M_2}}{||u_n||^{p^2-1}} \int_{\Lambda_n(0,\alpha)} v_n^-\,dx \\
&\leq \frac{L_{M_2}}{||u_n||^{p^2-1}} |\Omega|^{\frac{1}{p'}} |v_n^-|_{L^p} \\
&\leq \frac{L_{M_2}|\Omega|^{\frac{1}{p'}}}{||u_n||^{p^2-1}} |v_n|_{L^p} \\
&\leq \frac{L_{M_2}|\Omega|^{\frac{1}{p'}}}{S_p||u_n||^{p^2-1}} \to 0,
\end{aligned}$$

so there exists n_1, such that for $n > n_1$,

$$\int_{\Lambda_n(0,\alpha)} \frac{f(x,u_n^+)u_n}{||u_n||^{p^2}} < \eta. \tag{4.13}$$

Set $\sigma' = \frac{\sigma}{\sigma-1}$. Since $\sigma > \max\{1, \frac{N}{p}\}$, one can see that $p\sigma' \in (p, p^*)$. By $||u_n|| \to +\infty$, we take $n_2 > n_1$ such that $||u_n|| \geq 1$, if $n \geq n_2$, and by (4.11), $(H5)$ and Hölder inequality, one can take β large such that

$$\begin{aligned}
\int_{\Lambda_n(\beta,+\infty)} \frac{f(x,u_n^+)v_n}{||u_n||^{p^2-1}} &\leq \int_{\Lambda_n(\beta,+\infty)} \frac{f(x,u_n^+)}{|u_n|^{p-1}} v_n^p \\
&\leq \int_{\Lambda_n(\beta,+\infty)} \frac{f(x,u_n^+)}{|u_n^+|^{p-1}} v_n^p \qquad (4.14) \\
&\leq \Big(\int_{\Lambda_n(\beta,+\infty)} \Big|\frac{f(x,u_n^+)}{|u_n^+|^{p-1}}\Big|^{\sigma}\Big)^{\frac{1}{\sigma}} \Big(\int_{\Lambda_n(\beta,+\infty)} v_n^{p\sigma'}\Big)^{\frac{1}{\sigma'}} \\
&\leq \Big(\int_{\Lambda_n(\beta,+\infty)} C\overline{F}(x,u_n^+)\Big)^{\frac{1}{\sigma}} \Big(\int_{\Lambda_n(\beta,+\infty)} v_n^{p\sigma'}\Big)^{\frac{1}{\sigma'}} \\
&< \eta. \qquad (4.15)
\end{aligned}$$

Note that there is $C = C(\alpha, \beta)$ independent of n such that (because of the continuity of $(x, s) \mapsto \frac{f(x,s)}{s}$ on the compact $\overline{\Omega} \times [\alpha, \beta]$, so it is bounded)

$$|f(x, u_n^+)| \leq C u_n^+ \leq C|u_n|, \quad \text{for } x \in \Lambda_n(\alpha, \beta).$$

So by (4.12), there is $n_0 > n_2$ such that

$$\begin{aligned}
\int_{\Lambda_n(\alpha,\beta)} \frac{f(x, u_n^+) v_n}{||u_n||^{p^2-1}} &\leq \int_{\Lambda_n(\alpha,\beta)} \frac{C|u_n^+|\,|v_n|}{||u_n||^{p^2-1}} \\
&= \frac{C}{||u_n||^{p^2-2}} \int_{\Lambda_n(\alpha,\beta)} v_n^+ |v_n| \\
&= \frac{C}{||u_n||^{p^2-2}} \int_{\Lambda_n(\alpha,\beta)} v_n^+ (v_n^+ + v_n^-) \\
&= \frac{C}{||u_n||^{p^2-2}} \int_{\Lambda_n(\alpha,\beta)} (v_n^+)^2 \\
&\leq \frac{C}{||u_n||^{p^2-2}} \int_{\Lambda_n(\alpha,\beta)} (v_n)^2 \\
&\leq \frac{C}{||u_n||^{p^2-2}} \frac{1}{S_2^2} ||v_n||^2 \\
&= \frac{C}{||u_n||^{p^2-2}} \frac{1}{S_2^2} \\
&< \eta, \qquad (4.16)
\end{aligned}$$

for all $n > n_0$. Now, combining (4.13), (4.14) and (4.16), we obtain that for $n > n_0$,

$$\int_\Omega \frac{f(x, u_n^+) u_n}{||u_n||^{p^2}} < 3\eta < b^{p-1},$$

which contradicts (4.10). As a result, $(u_n)_n$ is bounded in $W_0^{1,p}(\Omega)$.
Claim 2: Assume the same hypotheses as in the last lemma, then any (C_c) condition has a convergent subsequence.
Indeed, let (u_n) be the (C_c) sequence such that

$$\widetilde{\Phi}_+(u_n) \to c, \quad (1 + ||u_n||)\widetilde{\Phi}'_+(u_n) \to 0.$$

We have

$$\begin{aligned}
\widetilde{\Phi}'_+(u_n)(u - u_n) = (a + b||u_n||^p)^{p-1} \int_\Omega |\nabla u_n|^{p-2} \nabla u_n \nabla (u - u_n)\, dx \\
- \int_\Omega f(x, u_n^+)(u - u_n)\, dx + \int_\Omega \ell(x)|u_n|^{p-2} u_n (u - u_n)\, dx \\
- \int_\Omega m(x)|u_n|^{p-2} u_n (u - u_n)\, dx + \int_\Omega m(x)|u_n^+|^{p-2} u_n^+ (u - u_n)\, dx.
\end{aligned}$$

Since (u_n) is bounded, one can extract a subsequence, named in the same way (u_n), that satisfies

$$\begin{array}{rcl} u_n & \rightharpoonup & u \text{ in } W_0^{1,p}(\Omega), \\ u_n & \to & u \text{ in } L^s(\Omega), \text{ for } 1 \leq s < p^*, \\ u_n & \to & u \text{ a.e. in } \Omega, \\ u_n^- & \rightharpoonup & u^- \text{ in } W_0^{1,p}(\Omega), \\ u_n^- & \to & u^- \text{ in } L^s(\Omega), \text{ for } 1 \leq s < p^*, \\ u_n^- & \to & u^- \text{ a.e. in } \Omega. \end{array} \tag{4.17}$$

For ε in (4.5), and from (4.17), there exists a positive constant $N(\varepsilon)$ such that

$$|u - u_n|_1 \leq \varepsilon, \forall n > N(\varepsilon); \tag{4.18}$$

from this, (4.5), Hölder inequality and $|u_n^+| \leq |u_n|$, it follows that for $n > N(\varepsilon)$,

$$\begin{array}{rcl} \Big|\displaystyle\int_\Omega f(x, u_n^+)(u - u_n)\, dx\Big| & = & \Big|\displaystyle\int_\Omega f_+(x, u_n)(u - u_n)\, dx\Big| \\ & \leq & \displaystyle\int_\Omega \Big(\varepsilon(u_n^+)^{p^*-1} + L_{M_2}\Big)|u - u_n|\, dx \\ & = & \displaystyle\int_\Omega \varepsilon(u_n^+)^{p^*-1}|u - u_n|\, dx + L_{M_2}|u - u_n|_1 \\ & \leq & \varepsilon|u_n|_{p^*}^{p^*-1}|u - u_n|_{p^*} + L_{M_2}\varepsilon, \end{array}$$

using the fact that $|u_n|_{p^*} \leq \frac{1}{S_{p^*}}||u_n||$, $|u - u_n|_{p^*} \leq \frac{1}{S_{p^*}}||u - u_n||$, also the boundedness of (u_n) in $W_0^{1,0}(\Omega)$ i.e. there exists $C_3 > 0$ such that $||u_n|| \leq C_3$ and the following inequality

$$\begin{array}{rcl} ||u - u_n|| & \leq & ||u|| + ||u_n|| \\ & \leq & \liminf\limits_n ||u_n|| + ||u_n|| \\ & \leq & 2C_3, \end{array}$$

we obtain

$$\Big|\int f(x, u_n^+)(u - u_n)\Big| \leq 2\varepsilon\Big(\frac{C_3}{S_{p^*}}\Big)^{p^*} + L_{M_2}\varepsilon;$$

this implies that $\lim\limits_n \displaystyle\int_\Omega f(x, u_n^+)(u - u_n)\, dx = 0$.

Also we have by Hölder inequality, that

$$\int_\Omega \ell(x)|u_n|^{p-2}u_n(u - u_n)\, dx \to 0,$$

$$\int_\Omega m(x)|u_n|^{p-2}u_n(u - u_n)\, dx \to 0$$

and

$$\int_\Omega m(x)|u_n^+|^{p-2}u_n^+(u - u_n)\, dx \to 0.$$

In addition, let $A(\nabla u_n) = |\nabla u_n|^{p-2}\nabla u_n$, then

$$\begin{aligned}&(a+b||u_n||^p)^{p-1}\int_\Omega |\nabla u_n|^{p-2}\nabla u_n\cdot\nabla(u-u_n)\,dx\\ &\quad=(a+b||u_n||^p)^{p-1}\int_\Omega A(\nabla u_n)\nabla(u-u_n)\,dx.\end{aligned}\tag{4.19}$$

Taking account all the previous estimations and limits, we obtain that

$$\int_\Omega A(\nabla u_n)\nabla(u-u_n)\,dx\to 0,\quad n\to+\infty.$$

From the fact that

$$\begin{aligned}\int_\Omega A(\nabla u_n)\nabla(u_n-u)\,dx&=\int_\Omega\Big(A(\nabla u_n)-A(\nabla u)\Big)\nabla(u_n-u)\,dx\\&\quad+\int_\Omega A(\nabla u)\nabla(u_n-u)\,dx\end{aligned}$$

and

$$\int_\Omega A(\nabla u)\nabla(u_n-u)\,dx\to 0,\quad n\to+\infty,$$

we deduce

$$\int_\Omega\Big(A(\nabla u_n)-A(\nabla u)\Big)\nabla(u_n-u)\,dx\to 0,\quad n\to+\infty.$$

From the following inequality

$$C_p\int_\Omega|\nabla(u-u_n)|^p\,dx\le\int_\Omega\Big(A(\nabla u_n)-A(\nabla u)\Big)\nabla(u_n-u)\,dx,$$

we deduce that $u_n\to u$ in $W_0^{1,p}(\Omega)$. □

Proof of Theorem 2.1. By Lemmas 4.3, 4.4 and 4.5 and by applying theorem 4.1, one can deduce that $\widetilde{\Phi}_+$ has a nontrivial critical point u, that is, for any v in E,

$$\begin{aligned}\langle\widetilde{\Phi}'_+(u),v\rangle&=(a+b||u||^p)^{p-1}\int_\Omega|\nabla u|^{p-2}\nabla u\cdot\nabla v\,dx+\int_\Omega\ell(x)|u|^{p-2}uv\,dx\\&\quad-\int_\Omega m(x)|u|^{p-2}uv\,dx-\int_\Omega\widetilde{f}_+(x,u)v\,dx=0.\end{aligned}$$

Taking as a test function $v=u^-$ in the precedent equation, we obtain

$$\begin{aligned}\langle\widetilde{\Phi}'_+(u),u^-\rangle&=(a+b||u||^p)^{p-1}\int_\Omega|\nabla u|^{p-2}\nabla u\cdot\nabla u^-\,dx+\int_\Omega\ell(x)|u|^{p-2}uu^-\,dx\\&\quad-\int_\Omega m(x)|u|^{p-2}uu^-\,dx-\int_\Omega\widetilde{f}_+(x,u)u^-\,dx\\&=0;\end{aligned}$$

from the definition of $\widetilde{f}_+$, we have $\int_\Omega \widetilde{f}_+(x,u)u^-\,dx = 0$, so

$$\begin{aligned}
\langle \widetilde{\Phi}'_+(u), u^-\rangle &= (a+b||u||^p)^{p-1}\int_\Omega |\nabla u|^{p-2}\nabla u\cdot\nabla u^-\,dx + \int_\Omega \ell(x)|u|^{p-2}uu^-\,dx\\
&- \int_\Omega m(x)|u|^{p-2}uu^-\,dx\\
&= (a+b||u||^p)^{p-1}\int_\Omega |\nabla u|^{p-2}(\nabla u^+ - \nabla u^-)\cdot\nabla u^-\,dx\\
&+ \int_\Omega \ell(x)|u|^{p-2}(u^+-u^-)u^-\,dx - \int_\Omega m(x)|u|^{p-2}(u^+-u^-)u^-\,dx\\
&= -(a+b||u||^p)^{p-1}\int_\Omega |\nabla u|^{p-2}|\nabla u^-|^2\,dx - \int_\Omega \ell(x)|u|^{p-2}|u^-|^2\,dx\\
&+ \int_\Omega m(x)|u|^{p-2}|u^-|^2\,dx\\
&= 0.
\end{aligned}$$

If $\ell(x)-m(x)\geq 0$, then, one can have that

$$(a+b||u||^p)^{p-1}\int_\Omega |\nabla u|^{p-2}|\nabla u^-|^2\,dx + \int_\Omega \Big(\ell(x)-m(x)\Big)|u|^{p-2}|u^-|^2\,dx = 0,$$

consequently, each term in the last equation is equal to zero, especially

$$\int_\Omega |\nabla u|^{p-2}|\nabla u^-|^2\,dx = 0,$$

since

$$\int_\Omega |\nabla u|^{p-2}|\nabla u^-|^2\,dx = \int_\Omega |\nabla u^-|^p\,dx,$$

one can deduce that $||u^-||=0$. If $\ell(x)\geq 0$ and $a^{p-1}\geq \frac{|m|_{L^\infty}}{S_p^p}$, then

$$\begin{aligned}
0 = -\langle \widetilde{\Phi}'_+(u), u^-\rangle &= (a+b||u||^p)^{p-1}\int_\Omega |\nabla u|^{p-2}|\nabla u^-|^2\,dx + \int_\Omega \ell(x)|u|^{p-2}|u^-|^2\,dx\\
&- \int_\Omega m(x)|u|^{p-2}|u^-|^2\,dx\\
&= (a+b||u||^p)^{p-1}\int_\Omega |\nabla u^-|^p\,dx + \int_\Omega \ell(x)|u^-|^p\,dx - \int_\Omega m(x)|u^-|^p\,dx\\
&\geq (a+b||u||^p)^{p-1}||u^-||^p - |m|_{L^\infty(\Omega)}\frac{1}{S_p^p}||u^-||^p + \int_\Omega \ell(x)|u^-|^p\,dx\\
&\geq 0,
\end{aligned}$$

as a result each term is equal to zero, consequently $||u^-||=0$.

So one can say that $u = u^+ \geq 0$. Then u is also a critical point of Φ_+, which means that,

$$\begin{aligned}\langle \Phi'_+(u), v\rangle &= (a + b||u||^p)^{p-1} \int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla v \, dx + \int_\Omega \ell(x)|u|^{p-2}uv \, dx \\ &\quad - \int_\Omega m(x)|u|^{p-2}uv \, dx - \int_\Omega f_+(x,u)v \, dx \\ &= 0, \ \forall v \in E.\end{aligned}$$

In addition, from $(H1)$, $(H2)$ and $\ell \in L^\infty(\Omega)$, we obtain that there exists a positive constant C_ε such that

$$|-a(x)u + f(x,u)| \leq C_\varepsilon\Big(1 + |u|^{p^*-1}\Big), \quad \text{for } s \in \mathbb{R} \text{ and a.e. } x \in \Omega.$$

Now, consider

$$b(x) := \frac{-\ell(x)|u(x)|^{p-2}u(x) + f(x,u(x))}{(a + b||u||^p)^{p-1}(1 + |u(x)|)},$$

then $b \in L^{\frac{N}{p}}(\Omega)$ and

$$-\Delta_p u = b(x)(1 + |u(x)|).$$

Remark 4.1. Following [7], we believe that one can obtain a positive and negative solutions for our problem. Note that, for the case $p = 2$ and using the same techniques as in [3], we have proved the existence of positive and negative solutions.
In a similar way, one can obtain a nonpositive solution for problem (1.4) by treating with $\widetilde{\Phi}_-$.

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A class of diffusion problem of Kirchhoff type with viscoelastic term involving the fractional Laplacian

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Abstract. This work is concerned with a class of diffusion problem of Kirchhoff type with viscoelastic term and nonlinear interior source in the setting of the fractional Laplacian. Under suitable conditions we prove the existence of global solutions and the exponential decay of the energy.

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1. Introduction

We consider the problem of finding $u = u(x,t)$ weak solutions to the following nonlinear heat equation of Kirchhoff type with variable exponent of nonlinearity, viscoelastic term and source term involving the fractional Laplacian

$$\begin{aligned} \left(1 + a|u|^{r(x)-2}\right) u_t \; + M(\|u\|^2_{w_0})(-\Delta)^s u - \int_0^t g(t-\tau)(-\Delta)^s u(\tau)d\tau & \\ = |u|^{\rho-1} \quad \text{in } \Omega\times]0,\infty[, & \\ u = 0 \quad \text{in } \left(\mathbb{R}^N \backslash \Omega\right) \times [0,\infty[, & \\ u(x,0) = u^0(x) \quad \text{in } \Omega, & \end{aligned} \tag{1.1}$$

where $\Omega \subseteq \mathbb{R}^N$ is a smooth bounded domain, $M(t) = t^{\alpha-1} + 1$, $t \geq 0$, $s \in]0,1[$, $2 < \rho < 2^*_s = \frac{2N}{N-2s}$, $2 < \frac{N}{s}$, $\alpha > 1$; $g : [0,\infty[\to]0,\infty[$ belongs to $C^1([0,\infty[)$, $g(0) > 0$, $l = 1 - \displaystyle\int_0^\infty g(\tau)d\tau > 0$, $g'(t) \leq 0$ and r is a given continuous function.

This type of problems without viscoelastic term (that is $g = 0$), $r(x) = constant$ and $M(t) = 1$ have been considered by many authors with the standard Laplace operator

$(-\triangle)^s, s = 1$ and can be seen as special case of doubly nonlinear parabolic type equations

$$(\varphi(u))_t - \Delta u = f(u),$$

which appear in the mathematical modelling of various physical processes such as flows of incompressible turbulent fluids or gases in pipes, processes of filtration in porous media, glaciology, see [3, 8, 7, 20, 33, 52] and the further references therein.
When $a = 0$, $M(t) = 1$ and $s = 1$, equation (1.1) is reduced to the following equation

$$u_t - \Delta u + \int_0^t g(t-\tau)\Delta u(\tau)d\tau = f(u). \tag{1.2}$$

This equation arises from the study of heat conduction in materials with memory. The questions of solvability and the long time behavior of solutions of the abstract evolutions equations of type

$$u_t - Bu + \int_0^t g(t-\tau)Au(\tau)d\tau = f(u),$$

where A and B are given operators, were studied in [12, 19, 36, 40].
Also, doubly nonlinear nonlocal parabolic equations

$$(\varphi(u))_t - div\sigma(\nabla u) = \int_0^t g(t-\tau)div\sigma(\nabla u(\tau))d\tau + f(x,t,u),$$

were studied in [9, 30, 47, 48, 49, 50].
On the other hand, many fractional and nonlocal operators are actively studied in the recent years. This type of operators arises in a quite natural way in many interesting applications, such as, finance, physics, game theory, Lévy stable diffusion processes, crystal dislocation, one can see [10, 35, 51] and their references. Some general motivations regarding the fractional Laplacian can be explicitly found in the recent monograph [17]. Nonlocal evolution equations of the form

$$u_t = \int_{\mathbb{R}^N} (u(y,t) - u(x,t))\, K(x-y)\, dy, \tag{1.3}$$

and variations of it, have been widely used to model diffusion processes, more precisely as stated in [26], if $u(x,t)$ is thought as a density of population at the point x at time t and $K(x-y)$ is thought of as the probability distribution of jumping from location y to location x, then $\int_{\mathbb{R}^N} (u(y,t)K(x-y))\, dy$ is the rate at which individuals are arriving at position x from all other places and $\int_{\mathbb{R}^N} (u(x,t)K(x-y))\, dy$ is the rate which they are leaving location x to travel to all other sites. So the density u satisfies (1.3). For recent references on nonlocal diffusion problems, see [5, 1, 29]. If we consider the effects of total population, then equation (1.3) becomes

$$u_t = M\left(\int\int_{\mathbb{R}^N} |u(y,t) - u(x,t)|^2 K(x-y)\, dx\, dy\right) \int_{\mathbb{R}^N} (u(y,t) - u(x,t))\, K(x-y)\, dy. \tag{1.4}$$

In particular, if $s \to 1^-$ and $K(x) = |x|^{-N-2s}$, then equation reduces to

$$u_t = -M\left(\int\int_{\mathbb{R}^N} |\nabla u|^2\, dx\right) \Delta u,$$

which is equation (1.2), with $M(t) = 1,\ g(t) = 0$ and $f(t) = 0$. Thus it is natural to consider equation (1.1) as a generalization of the model (1.4). The main feature of the equation (1.1) is that contains an integrodifferential operator usually called memory term or viscoelastic term, which can be used to represent the damping or memory effect on the diffusion process.

The research on nonlinear problems with variable exponent growth conditions is an an attractive topic, and these problems have many applications in nonlinear elastic electrorheological fluids and image restoration, see [2, 16, 18, 53].

The study of Kirchhoff type problems has been receiving considerable attention in more recent years, see [31, 38, 42, 41]. The interest arises from their contribution to the modeling of many physical and biological phenomena. We refer for example the reader to the bibliography [4, 6, 11, 32, 37] and references therein. The first result concerning fractional Kirchhoff problems was obtained in Fiscella and Valdinoci [27]. In this paper, the fractional Kirchhoff equation was first introduced and motivated.

In [42], by using the sub-differential approach, Pucci et al obtained the well-posedness of solutions for problem (1.1) with $f(x,t)$ instead of $|u|^{p-2}u$. Moreover, the large-time behavior and extinction of solutions also are considered. With the help of potential well theory, Fu and Pucci [28] studied the existence of global weak solutions and established the vacuum isolating and blow-up of strong solutions, provided that $M \equiv 1$ and $2 < p \leq 2_s^* = 2N/(N-2s)$. However, the Kirchhoff function M is assumed to satisfy the non-degenerate condition in the above papers. In [41], Pan et al investigated for the first time the existence of global weak solutions for degenerate Kirchhoff-type diffusion problems involving fractional p-Laplacian, by combining the Galerkin method with potential well theory, for the special function $M(t) = t$; Mingqi et al. [38] proved the local existence and blow-up of solutions for the similar equation with more general conditions on M which cover the degenerate case.

In the works mentioned above, there are few about the global existence and exponential decay rate for doubly nonlinear parabolic equation, involving variable exponent conditions, with viscoelastic term in the fractional setting. Motivated by it, we intend to study global existence for the problem (1.1) by using Galerkin's method and also give the exponential decay rate of the energy via the energy perturbation method.

The plan of the paper is the following. In Section 2, we give the preliminaries for our research. In Section 3, by using the Galerkin approximation method we are able to prove global existence and finally, we obtain the exponential decay under certain class of initial data.

2. Preliminaries

In this section, we present some materials and assumptions needed in the rest of this paper.

We denote: $Q = \mathbb{R}^{2n} \setminus (\mathcal{C}\Omega \times \mathcal{C}\Omega)$ and $\mathcal{C}\Omega := \mathbb{R}^n \setminus \Omega$,

$$W = \left\{ u : \mathbb{R}^N \to \mathbb{R} : u|_\Omega \in L^2(\Omega),\ \iint_Q \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\, dx\, dy < \infty \right\},$$

where $u|_\Omega$ represents the restriction to Ω of function $u(x)$. Also, we define the following linear subspace of W,

$$W_0 = \{ u \in W : u = 0 \text{ a.e. in } \mathbb{R}^N \setminus \Omega \}.$$

The linear space W is endowed with the norm

$$\|u\|_W := \|u\|_{L^2(\Omega)} + \Big(\iint_Q \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\, dx\, dy \Big)^{1/2}.$$

It is easily seen that $\|\cdot\|_W$ is a norm on W and $C_0^\infty(\Omega) \subseteq W_0$.
The functional

$$\|u\|_{W_0} = \left(\iint_Q \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}} dxdy \right)^{1/2},$$

is a equivalent norm on $W_0 = \{u \in W \ : \ u(x) = 0 \ a.e. \text{ in } \mathbb{R}^N \setminus \Omega\}$ which is a closed linear subspace of W. Furthermore $(W_0, \|\cdot\|_{W_0})$ is a Hilbert space with inner product

$$\langle u, v\rangle_{W_0} = \iint_Q \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+2s}} dxdy.$$

We review the main embedding results for the space W_0.

Lemma 2.1 ([44, 43, 46, 45])**.** *The embedding $W_0 \hookrightarrow L^r(\Omega)$ is continuous for any $r \in [1, 2_s^*]$, and compact for any $r \in [1, 2_s^*[$.*

Lemma 2.2 ([39, Lemma 2.1])**.** *Let $N \geq 1,\ 0 < s < 1,\ p > 1,\ q \geq 1,\ \tau > 0$ and $0 < \theta < 1$ be such that $\frac{1}{\tau} = \theta \left(\frac{1}{p} - \frac{s}{N} \right) + \frac{1-\theta}{q}$ then*

$$\|u\|_{L^\tau(\mathbb{R}^n)} \leq \|u\|^\theta_{W^{s,p}(\mathbb{R}^n)} \|u\|^{1-\theta}_{L^q(\mathbb{R}^n)}, \qquad \forall u \in C_0^1(\mathbb{R}^N).$$

Now, we recall some necessary background knowledge and propositions concerning the generalized Lebesgue-Sobolev spaces. We refer the reader to [21, 22, 25, 23] for details.
Set

$$C_+(\overline{\Omega}) = \{p(x) : p(x) \in C(\overline{\Omega}),\ p(x) > 1, \text{ for all } x \in \overline{\Omega}\}.$$

For any $p \in C_+(\overline{\Omega})$ we define

$$p^+ = \max\{p(x) : x \in \overline{\Omega}\}, \quad p^- = \min\{p(x); x \in \overline{\Omega}\};$$

$$L^{p(x)}(\Omega) = \{ u : u \text{ is a measurable real-valued function}, \int_\Omega |u(x)|^{p(x)} dx < \infty \},$$

with the norm

$$\|u\|_{p(x)} \equiv \|u\|_{L^{p(x)}(\Omega)} = \inf\left\{\lambda > 0 : \int_\Omega \left|\frac{u(x)}{\lambda}\right|^{p(x)} dx \leq 1\right\}$$

becomes a Banach space [34]. We also define the space

$$W^{1,p(x)}(\Omega) = \{u \in L^{p(x)}(\Omega) : |\nabla u| \in L^{p(x)}(\Omega)\},$$

equipped with the norm

$$\|u\|_{W^{1,p(x)}(\Omega)} = \|u(x)\|_{p(x)} + \|\nabla u(x)\|_{p(x)}.$$

We denote by $W_0^{1,p(x)}(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $W^{1,p(x)}(\Omega)$. Of course the norm $\|u\| = \|\nabla u\|_{p(x)}$ is an equivalent norm in $W_0^{1,p(x)}(\Omega)$.

Proposition 2.3 ([24, 25])**.** *(i) The conjugate space of $L^{p(x)}(\Omega)$ is $L^{p'(x)}(\Omega)$, where*

$$\frac{1}{p(x)} + \frac{1}{p'(x)} = 1.$$

For any $u \in L^{p(x)}(\Omega)$ and $v \in L^{p'(x)}(\Omega)$, we have

$$\int_\Omega |uv|dx \leq \Big(\frac{1}{p^-} + \frac{1}{p'^-}\Big)\|u\|_{p(x)}\|v\|_{p'(x)} \leq 2\|u\|_{p(x)}\|v\|_{p'(x)}.$$

(ii) If $p_1(x), p_2(x) \in C_+(\overline{\Omega})$ and $p_1(x) \leq p_2(x)$ for all $x \in \overline{\Omega}$, then $L^{p_2(x)}(\Omega) \hookrightarrow L^{p_1(x)}(\Omega)$ and the embedding is continuous.

Proposition 2.4 ([25])**.** *Set $\rho(u) = \int_\Omega |\nabla u(x)|^{p(x)}\,dx$, then for $u \in W_0^{1,p(x)}(\Omega)$ and $(u_k) \subset W_0^{1,p(x)}(\Omega)$, we have*

1. *$\|u\| < 1$ (respectively $= 1; > 1$) if and only if $\rho(u) < 1$ (respectively $= 1; > 1$);*
2. *for $u \neq 0$, $\|u\| = \lambda$ if and only if $\rho(\frac{u}{\lambda}) = 1$;*
3. *if $\|u\| > 1$, then $\|u\|^{p^-} \leq \rho(u) \leq \|u\|^{p^+}$;*
4. *if $\|u\| < 1$, then $\|u\|^{p^+} \leq \rho(u) \leq \|u\|^{p^-}$;*
5. *$\|u_k\| \to 0$ (respectively $\to \infty$) if and only if $\rho(u_k) \to 0$ (respectively $\to \infty$).*

For $x \in \Omega$, let us define

$$p^*(x) = \begin{cases} \frac{Np(x)}{N-p(x)} & \text{if } p(x) < N, \\ \infty & \text{if } p(x) \geq N. \end{cases}$$

Proposition 2.5 ([23])**.** *If $q \in C_+(\overline{\Omega})$ and $q(x) \leq p^*(x)$ $(q(x) < p^*(x))$ for $x \in \overline{\Omega}$, then there is a continuous (compact) embedding $W^{1,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega)$.*

Lemma 2.6. *Let $2 < r < \rho < 2_s^*$.For each $\varepsilon > 0$, there exists a positive constant C_ε such that*

$$\|v\|_\rho^\rho \leq \varepsilon\|v\|_{W_0}^2 + C_\varepsilon\|v\|_r^{kr},$$

for all $v \in W_0 \cap L^r(\Omega)$ where

$$k = \frac{2\rho(1-\theta)}{r(2-\rho\theta)} \quad , \quad \theta = \Big(\frac{1}{r} - \frac{1}{\rho}\Big)\Big(\frac{s}{N} - \frac{1}{2} + \frac{1}{r}\Big)^{-1}.$$

Proof. The conclusion of lemma immediately follows from Lemma 2.2 and Young's inequality. □

Lemma 2.7. [34, Theorem 1, pag 23] *Suppose that*

$$r \in L_+^{\infty}(\Omega),\ r^- \geq 2,\ w \in L^{r(x)}\left(\Omega\times]0,T[\right)\ \text{and}\ \frac{\partial}{\partial t}(|w|^{r(x)-2}w) \in L^{r'(x)}\left(\Omega\times]0,T[\right).$$

Then, for any $s,\tau \in [0,T]$, $s < \tau$ the following formula of integration by parts is correct:

$$\int_s^\tau \int_\Omega w\Big(\frac{1}{r(x)-1}|w|^{r(x)-2}w\Big)\,dx\,dt = \int_\Omega \frac{1}{r(x)}|w(\tau)|^{r(x)}\,dx - \int_\Omega \frac{1}{r(x)}|w(s)|^{r(x)}\,dx.$$

3. Global existence and exponential decay

In this section, we focus our attention on the global existence and exponential decay of the solution to problem (1.1).

Definition 3.1. Let $T > 0$. A weak solution of problem (1.1) is a function $u \in L^\infty(0,T;W_0)$, with $u_t \in L^2(0,T;L^2(\Omega))$ and $\left(|u|^{r(x)/2}\right)_t \in L^2\left(\Omega\times]0,T[\right)$ such that

$$\int_0^T \int_\Omega \Big(1+a|u|^{r(x)-2}\Big)\,u_t w\,dxdt + M(\|u\|^2_{w_0})\int_0^T \langle u,w\rangle_{W_0}\,dt$$
$$-\int_0^T\int_0^t g(t-\tau)\langle u(\tau),w\rangle_{W_0}d\tau dt = \int_0^T\int_\Omega |u|^{\rho-1}w\,dxdt,$$

for all $w \in L^2(0,T;W_0)$, and $u(x,0) = u^0(x) \in W_0$.

Theorem 3.2 (Local Solution). *Assume $u^0 \in W_0$, $2 < r^- < \rho < 2_s^*$, $\rho < 2 + \frac{2rs}{N}$, $r^+ \in]2, 2_s^*[$, then problem (1.1) has a unique weak solution u for T small enough.*

Proof. We prove the local existence of weak solutions by using the Faedo-Galerkin method benefited from the ideas of [14]. We choose a sequence $\{w_\nu\}_{\nu\in\mathbb{N}} \subseteq C_0^\infty(\Omega)$ such that $C_0^\infty(\Omega) \subseteq \overline{\bigcup_{\nu=1}^{\infty} V_m}^{C^1(\overline{\Omega})}$ and $\{w_\nu\}$ is a standard orthonormal basis with respect to the Hilbert space $L^2(\Omega)$ and an orthogonal basis in W_0, where

$$V_m = spam\{w_1, w_2, ...w_m\}.$$

Now, we construct approximate solutions u_m $(m = 1, 2, \cdots)$, of the problem (1.1), in the form

$$u_m(x,t) = \sum_{i=1}^m g_{jm}(t)w_j(x),$$

where the coefficient functions g_{jm} satisfy the system of ordinary differential equations

$$\begin{aligned}
&\int_\Omega \left(1 + a|u_m(t)|^{r(x)-2}\right) u_{mt}(t) w_j \, dx + M(\|u_m(t)\|^2_{w_0})\langle u_m(t), w_j\rangle_{W_0} \\
&\quad - \int_0^t g(t-\tau)\langle u_m(\tau), w_j\rangle_{W_0} d\tau dt = \int_\Omega |u_m(t)|^{\rho-1} w_j \, dx \qquad (3.1)\\
&\qquad j = 1, 2, \cdots m. \\
&\qquad\quad u_m(x,0) = u^0_m(x) \to u^0(x) \qquad \text{in} \quad W_0.
\end{aligned}$$

Let us show that the system (3.1) is locally solvable.
It is clear that (3.1) can be rewritten in the form

$$\frac{d}{dt}\Phi(g_m(t)) = -M\Big(\Big\|\sum_{i=1}^m g_{jm}(t)w_j(x)\Big\|^2_{W_0}\Big)Bg_m(t)+\int_0^t g(t-\tau)Bg_m(\tau)d\tau+F(g_m(t)), \tag{3.2}$$

where

$$g_m(t) = (g_{m1}(t), g_{m2}(t), \cdots, g_{mm}(t))^t, \qquad B = [\langle w_i, w_j\rangle]_{1\le i,j\le m},$$

$$\Phi(\eta) = (\Phi_1(\eta), \Phi_2(\eta), \cdots, \Phi_m(\eta))^t \qquad \text{with } \eta = (\eta_1, \eta_2, \cdots, \eta_m) \in \mathbb{R}^m,$$

$$\Phi_i(\eta) = \int_\Omega \Big\{\sum_{j=1}^m \eta_j w_j + \frac{a}{r(x)-1}\Big|\sum_{k=1}^m \eta_k w_k\Big|^{r(x)-2} \sum_{k=1}^m \eta_k w_k\Big\} w_i \, dx \qquad i = 1, 2, \cdots, m$$

and

$$F(\eta)=\left(\int_\Omega \Big|\sum_{k=1}^m \eta_j w_j\Big|^{\rho-1} w_1 \, dx, \int_\Omega \Big|\sum_{k=1}^m \eta_j w_j\Big|^{\rho-1} w_2 \, dx, \cdots, \int_\Omega \Big|\sum_{k=1}^m \eta_j w_j\Big|^{\rho-1} w_m dx\right)^t.$$

This system is equivalent to

$$\begin{aligned}
&\Phi(g_m(t)) = \Phi(g_m(0)) \\
&+\int_0^t \left[-M\left(\Big\|\sum_{i=1}^m g_{jm}(t)w_j(x)\Big\|^2_{W_0}\right)Bg_m(t) + \int_0^\xi g(\xi-\tau)Bg_m(\tau)d\tau + F(g_m(\xi))\right]d\xi.
\end{aligned}$$

If ζ, η are to arbitrary elements of $\mathbb{R}^m$, we get

$$(\Phi(\zeta) - \Phi(\eta), \zeta - \eta)_{\mathbb{R}^m} \ge C_m|\zeta - \eta|^2_{\mathbb{R}^m} \tag{3.3}$$

here C_m is a constant such that, for any g_m in $\mathbb{R}^m$

$$\int_\Omega |u_m|^2 \, dx \ge C_m|g_m|^2_{\mathbb{R}^m}.$$

Then Φ is monotone coercive. Also it is obviously continuous. So, by the Brouwer theorem Φ is onto. In view of (3.3), Φ^{-1} is locally Lipchitz continuous.

Consider the map $L : C(0,T,\mathbb{R}^m) \to C(0,T,\mathbb{R}^m)$, defined by

$$L(g_m)(t) = \Phi^{-1}\Bigg(\Phi(g_m(0)) + \int_0^t \left[-M\left(\left\|\sum_{i=1}^{m} g_{jm}(t)w_j(x)\right\|_{W_0}^2\right) Bg_m(t) + \int_0^{\xi} g(\xi-\tau)Bg_m(\tau)d\tau + F(g_m(\xi))\right] d\xi\Bigg),$$

$$t \in [0,T].$$

It is not hard to prove that L is completely continuous and also, there exist (sufficient small) $T_m > 0$ and (sufficient large) $R > 0$ such that $L(\overline{B_R}) \subseteq \overline{B_R}$, where $\overline{B_R}$ is the ball in $C(0,T_m,\mathbb{R}^m)$ with center the origin and radius R. Consequently, by Schauder's theorem, the operator L has a fixed point in $C(0,T_m,\mathbb{R}^m)$. This fixed point is a solution of (3.2).

So, we can obtain an approximate solution $u_m(t)$ of (3.1) in V_m over $[0,T_m[$ and it can be extended to the whole interval $[0,T]$, for all $T > 0$, as a consequence of the a priori estimates that shall be proven in the next step.

The First Estimate

Multiplying (3.1) by $g_{jm}(t)$ and adding in $j = 1;\cdots;m$, we have

$$\begin{aligned}\int_\Omega \left(1 + a|u_m(t)|^{r(x)-2}\right) u_{mt}(t)u_m(t)\,dx + M(\|u_m(t)\|_{w_0}^2)\langle u_m(t), u_m(t)\rangle_{W_0} \\ - \int_0^t g(t-\tau)\langle u_m(\tau), u_m(t)\rangle_{W_0} d\tau dt = \int_\Omega |u_m(t)|^{\rho-1}u_m(t)\,dx\end{aligned} \tag{3.4}$$

which implies, integrating with respect to the time variable from 0 to t on both sides, using Lemma 2.7 that

$$\begin{aligned}S_m(t) = S_m(0) + \int_0^t d\lambda \int_0^{\lambda} g(\lambda-\tau)\langle u_m(\tau), u_m(\lambda)\rangle_{W_0}\,d\tau \\ + \int_0^t \int_\Omega |u_m(t)|^{\rho-1}u_m(\tau)\,dx\,d\tau,\end{aligned} \tag{3.5}$$

where

$$S_m(t) = \int_\Omega |u_m(t)|^2\,dx + a\int_\Omega \frac{1}{r(x)}|u_m(t)|^{r(x)}\,dx + \int_0^t \left(\|u_m(\tau)\|_{W_0}^{2\alpha} + \|u_m(\tau)\|_{W_0}^2\right)\,d\tau.$$

Let us introduce the function $\Theta(\lambda) = \int_0^{\lambda} g(\lambda-\tau)\|u_m(\tau)\|_{W_0}$. Estimating the second term on right-hand side of (3.5) we have

$$\begin{aligned}\int_0^t d\lambda \int_0^{\lambda} g(\lambda-\tau)\langle u_m(\tau), u_m(\lambda)\rangle_{W_0}\,d\tau \le \frac{1}{2}\int_0^t \left(\|u_m(\tau)\|_{W_0}^{2\alpha} + \|u_m(\tau)\|_{W_0}^2\right)\,d\tau \\ + \frac{1}{2}\int_0^t \Theta^2(\lambda)\,d\lambda.\end{aligned} \tag{3.6}$$

But, using Young Inequality and noting that $\int_0^\infty g(\tau)d\tau < 1$, we get

$$\int_0^t \Theta^2(\lambda)d\lambda \le \int_0^\infty g(\tau)d\tau \int_0^t \left(\|u_m(\tau)\|_{W_0}^{2\alpha} + \|u_m(\tau)\|_{W_0}^2\right) d\tau. \tag{3.7}$$

Plugging (3.6)- (3.7)into (3.5), it follows that

$$S_m(t) \le S_m(0) + \frac{1}{2}\left(1 + \int_0^\infty g(\tau)d\tau\right)\int_0^t \left(\|u_m(\tau)\|_{W_0}^{2\alpha} + \|u_m(\tau)\|_{W_0}^2\right) d\tau$$
$$+ \int_0^t \|u_m(t)\|_\rho^\rho \, d\tau. \tag{3.8}$$

To estimate the last term in (3.8) we use Lemma 2.6,

$$\int_0^t \|u_m(t)\|_\rho^\rho \, d\tau \le \varepsilon \int_0^t \left(\|u_m(\tau)\|_{W_0}^{2\alpha} + \|u_m(\tau)\|_{W_0}^2\right) d\tau + c_0 \int_0^t S_m^k(\lambda)\, d\lambda, \tag{3.9}$$

where $k = \frac{2\rho(1-\theta)}{r^-(2-\rho\theta)} > 1$. Taking ε suitably small in (3.9), it follows from (3.5)-(3.9) that

$$S_m(t) \le \hat{C}_0 + \hat{C}_1 \int_0^t S_m^k(\lambda)\, d\lambda. \tag{3.10}$$

Hence, by employing Bihari-Langenhop's inequality (cf. [13]), there exists a constant T_0 such that

$$S_m(t) \le C_{T_0}, \qquad \forall t \in [0, T_0]. \tag{3.11}$$

The Second Estimate

Multiplying (3.1) by $g'_{jm}(t)$ and adding in $j = 1; \cdots ; m$, it holds that

$$\frac{d}{dt}\Big\{\frac{1}{2\alpha}\|u_m(t)\|_{W_0}^{2\alpha} + \frac{1}{2}\left(1 - \int_0^t g(\tau)d\tau\right)\|u_m(t)\|_{W_0}^2 + \frac{1}{2}(g \diamond u)(t)$$
$$-\frac{1}{\rho}\int_\Omega |u_m(t)|^{\rho-1}u_m(t)\, dx\Big\} + \|u_{mt}(t)\|_2^2 + a\int_\Omega |u_m(t)|^{r(x)-2}|u_{mt}(t)|^2\, dx$$
$$= \frac{1}{2}(g' \diamond u)(t) - \frac{1}{2}g(t)\|u_m(t)\|_{W_0}^2. \tag{3.12}$$

where $\quad (g \diamond u)(t) = \int_0^t g(t-\tau)\|u(t) - u(\tau)\|_{W_0}^2\, d\tau.$

Integrating (3.12) on $[0, t]$, $t \le T_0$ we get

$$\int_0^t \|u_{mt}(t)\|_2^2 + a\int_0^t\int_\Omega |u_m(t)|^{r(x)-2}|u_{mt}(t)|^2\, dx + \frac{1}{2\alpha}\|u_m(t)\|_{W_0}^{2\alpha} + \frac{l}{2}\|u_m(t)\|_{W_0}^2$$
$$\le \frac{1}{2\alpha}\|u_m(0)\|_{W_0}^{2\alpha} + \frac{1}{2}\|u_m(0)\||_{W_0}^2 - \frac{1}{\rho}\int_\Omega |u_m(0)|^{\rho-1}u_m(0)\, dx$$
$$+ \frac{1}{\rho}\int_\Omega |u_m(t)|^{\rho-1}u_m(t)\, dx.$$

From the assumptions on ρ and u^0, Lemma 2.6 and the estimate (3.11), it follows that

$$\int_0^t \|u_{mt}(t)\|_2^2 + a\int_0^t \int_\Omega |u_m(t)|^{r(x)-2}|u_{mt}(t)|^2\,dx + \frac{1}{2\alpha}\|u_m(t)\|_{W_0}^{2\alpha} + \frac{l}{2}\|u_m(t)\|_{W_0}^2 \leq M_1, \tag{3.13}$$

for some constant $M_1 > 0$.

By the above estimates (3.11) and (3.13), $\{u_m\}$ have subsequences still denoted by $\{u_m\}$ such that

$$u_m \to u \quad \text{weakly* in} \quad L^\infty(0,T_0;W_0), \tag{3.14}$$

$$u_{mt} \to u_t \quad \text{weakly in} \quad L^2(0,T_0;L^2(\Omega)), \tag{3.15}$$

$$\left(|u_m|^{r(x)/2}\right)_t \to \chi \quad \text{weakly in} \quad L^2(0,T_0;L^2(\Omega)). \tag{3.16}$$

Employing the same arguments as in [16] we can prove that

$$\chi = \left(|u|^{r(x)/2}\right)_t \quad |u_m|^{r(x)/2}u_{mt} \to |u|^{r(x)/2}u_t \text{ weakly in } L^2(\Omega\times]0,T_0[), \tag{3.17}$$

$$|u_m|^{\rho-1} \to |u|^{\rho-1} \quad \text{weakly in} \quad L^{\frac{\rho}{\rho-1}}(\Omega\times]0,T_0[). \tag{3.18}$$

Therefore, passing to the limit in (3.1) as $m \to +\infty$, by (3.14)– (3.18), we can show that u satisfies the initial condition $u(0) = u^0$ and

$$\int_0^T \int_\Omega \left(1 + a|u|^{r(x)-2}\right) u_t w\,dxdt + M(\|u\|_{w_0}^2)\int_0^T \langle u, w\rangle_{W_0}\,dt - \int_0^T \int_0^t g(t-\tau)\langle u(\tau), w\rangle_{W_0} d\tau dt = \int_0^T \int_\Omega |u|^{\rho-1}w\,dxdt,$$

for all $w \in L^2(0,T_0;W_0)$.

The uniqueness property of a solutions can be derived from [20, Theorem 3, p. 1095], observing that $\left(u + \frac{a}{r(x)-1}|u|^{r(x)-2}u\right) \in L^2(\Omega\times]0,T_0[)$ and $Au = M(\|u\|_{w_0}^2)(-\Delta)^s u$ is a monotone operator. We omit the details. □

Next, we consider the global existence and energy decay of solutions for problem (1.1). For this purpose we define the energy associated with problem (1.1) by

$$E(t) = \frac{1}{2\alpha}\|u(t)\|_{W_0}^{2\alpha} + \frac{1}{2}\left(1 - \int_0^t g(\tau)d\tau\right)\|u(t)\|_{W_0}^2 + \frac{1}{2}(g\diamond u)(t) - \frac{1}{\rho}\int_\Omega |u(t)|^{\rho-1}u(t)\,dx. \tag{3.19}$$

Then, we easily can check that

$$\frac{d}{dt}E(t) = \frac{1}{2}(g'\diamond u)(t) - \frac{1}{2}g(t)\|u(t)\|_{W_0}^2 - \|u_t(t)\|_2^2 - a\int_\Omega |u(t)|^{r(x)-2}u_t^2(t)\,dx \leq 0$$

for any regular solution. This remains valid for weak solutions by simple density argument. This shows that $E(t)$ is a nonincreasing function.

Let C_* be the optimal constant satisfying the Sobolev inequality $\|u\|_\rho \leq C_*\|u\|_{W_0}$, and $B_1 = \frac{C_*}{\sqrt{l}}$. We define the function $h(\lambda) = \frac{1}{2}\lambda^2 - \frac{B_1^\rho}{\rho}\lambda^\rho$. Then, we can verify that the function h is increasing in $]0, \lambda_1[$, decreasing in $]\lambda_1, \infty[$, $h(\lambda) \to -\infty$, as $\lambda \to \infty$ and h has a maximum at λ_1 with the maximum value

$$h(\lambda_1) = E_1 = \left(\frac{1}{2} - \frac{1}{\rho}\right) B_1^{-\frac{2\rho}{\rho-2}} = \frac{\rho-2}{2\rho} B_1^{-\frac{2\rho}{\rho-2}}.$$

where λ_1 is the first positive zero of the derivative function $h'(\lambda)$.
Here, note that

$$\begin{aligned} E(t) &\geq \frac{l}{2}\|u(t)\|_{W_0}^2 + \frac{1}{2}(g \diamond u)(t) - \frac{1}{\rho}\|u(t)\|_\rho^\rho \\ &\geq \frac{1}{2}\left(l\|u(t)\|_{W_0}^2 + (g \diamond u)(t)\right) - \frac{B_1^\rho l^{\rho/2}}{\rho}\|u(t)\|_{W_0}^\rho \\ &\geq h\left(\sqrt{l\|u(t)\|_{W_0}^2 + (g \diamond u)(t)}\right), \ \forall t \geq 0. \end{aligned} \tag{3.20}$$

Now, we are ready to state our result.

Theorem 3.3. *Assume that hypotheses of Theorem* 3.2 *are satisfied. Consider* $u_0 \in W_0$, *satisfying*

$$0 < l^{1/2}\|u_0\|_{W_0} < \lambda_1, \tag{3.21}$$

$$\frac{1}{2\alpha}\|u^0\|_{W_0}^{2\alpha} + \frac{1}{2}\|u^0\|_{W_0}^2 - \frac{1}{\rho}\int_\Omega |u^0|^{\rho-1}u^0\, dx < \left(\frac{\rho-2}{2\rho}\right) B_1^{-\frac{2\rho}{\rho-2}}. \tag{3.22}$$

Then problem admits a global weak solution in time. In addition, if there exists a constant $\xi_0 > 0$ *such that* $g'(t) \leq -\xi_0 g(t)$, *then this solution satisfies*

$$E(t) \leq L_0 e^{-\gamma t}, \forall t \geq 0, \tag{3.23}$$

where L_0 *and* γ *are positive constants.*

Proof. We will get global estimates for $u_m(t)$ solution of the approximate system (3.1) under the conditions (3.21)–(3.22) for u^0. For this, it suffices to show that

$$E_m(t) + \int_0^t \|u_{mt}(t)\|_2^2 + a\int_0^t \int_\Omega |u_m(t)|^{r(x)-2}|u_{mt}(t)|^2\, dx,$$

where $E_m(t)$ is defined in (3.19) with $u(t)$ replaced by $u_m(t)$, is bounded and independently of t. From (3.12) and the definition of energy, we have

$$E_m(t) + \int_0^t \|u_{mt}(t)\|_2^2 + a\int_0^t \int_\Omega |u_m(t)|^{r(x)-2}|u_{mt}(t)|^2\, dx \leq E_m(0). \tag{3.24}$$

Due to convergence $u_{0m} \to u^0$ in W_0 we see that $E_m(0) < \left(\frac{\rho-2}{2\rho}\right) B_1^{-\frac{2\rho}{\rho-2}}$ for sufficiently large m. We claim that there exists an integer ν_0 such that

$$\sqrt{l\|u_m(t)\|_{W_0}^2 + (g \diamond u_m)(t)} < \lambda_1 \qquad \forall t \in [0, T_m[, m \geq \nu_0. \tag{3.25}$$

Suppose the claim is proved. Then $h\left(\sqrt{l\|u_m(t)\|_{W_0}^2+(g\diamond u_m)(t)}\right)\geq 0$ and from (3.20), (3.24)–(3.25) we get

$$\|u_m(t)\|_{W_0}^{2\alpha}+\|u_m(t)\|_{W_0}^2+\int_0^t\|u_{mt}(t)\|_2^2+a\int_0^t\int_\Omega|u_m(t)|^{r(x)-2}|u_{mt}(t)|^2\,dx\leq C. \tag{3.26}$$

where C is a constant independent of m.Thus, we obtain the global existence.
Proof of Claim: Suppose (3.25) is not true. Thus, for each $m>\nu_0$,there exists $t_1\in[0,T_m[$ such that

$$\sqrt{l\|u_m(t_1)\|_{W_0}^2+(g\diamond u_m)(t_1)}\geq\lambda_1. \tag{3.27}$$

Here, we observe that, from (3.21) and the convergence $u_{0m}\to u^0$ in W_0 there exists ν_1 such that

$$l^{1/2}\|u_m(0)\|_{W_0}<\lambda_1\qquad\forall m>\nu_1.$$

Hence, by continuity there exists

$$t^*=\inf\{t\in[0,T_m[:\sqrt{l\|u_m(t)\|_{W_0}^2+(g\diamond u_m)(t)}\geq\lambda_1\},$$

such that

$$\sqrt{l\|u_m(t^*)\|_{W_0}^2+(g\diamond u_m)(t^*)}=\lambda_1. \tag{3.28}$$

By (3.20), we see that

$$E_m(t^*)\geq h\left(\sqrt{l\|u_m(t^*)\|_{W_0}^2+(g\diamond u_m)(t^*)}\right)=h(\lambda_1)=E_1 \tag{3.29}$$

which contradicts $E_m(t)\leq E_m(0)<E_1,\ \forall t\geq 0$. Therefore our claim is true.
The above estimates permit us to pass to the limit in the approximate equation.
To show the uniform decay of the solution we introduce the perturbed energy functional

$$F(t)=E(t)+\varepsilon\Phi(t), \tag{3.30}$$

where ε is a positive constant which shall be determined later, and

$$\Phi(t)=\int_\Omega(|u|^2+\frac{a}{r(x)}|u|^{r(x)})\,dx. \tag{3.31}$$

It is straightforward to see that $F(t)$ and $E(t)$ are equivalent in the sense that there exist two positive constants β_1 and β_2 depending on ε such that for $t\geq 0$

$$\beta_1E(t)\leq F(t)\leq\beta_2E(t). \tag{3.32}$$

By taking the time derivative of the function F defined above in (3.30), using (3.20), and performing several integration by parts, we get

$$\begin{aligned}\frac{d}{dt}F(t)&=\frac{1}{2}(g'\diamond u)(t)-\frac{1}{2}g(t)\|u(t)\|_{W_0}^2-\|u_t(t)\|_2^2-a\int_\Omega|u(t)|^{r(x)-2}u_t^2(t)\,dx\\&-\varepsilon\|u(t)\|_{W_0}^{2\alpha}-\varepsilon\|u(t)\|_{W_0}^2+\varepsilon\int_\Omega|u(t)|^{\rho-1}u(t)\,dx+\varepsilon\int_0^t g(t-\tau)\langle u(\tau),u(t)\rangle_{W_0}\,d\tau.\end{aligned} \tag{3.33}$$

On the other hand, we can easily see that the condition $E(0) < E_1$ is equivalent to the inequality:

$$B_1^\rho \left(\frac{2\rho}{\rho-2}E(0)\right)^{\frac{\rho-2}{2}} < 1. \tag{3.34}$$

From the assumption (3.21)–(3.22) and (3.24) we have

$$\begin{aligned} l\|u(t)\|_{W_0}^2 &\le \left(1-\int_0^t g(\tau)d\tau\right)\|u(t)\|_{W_0}^2 + (g\diamond u)(t) \\ &< \lambda_1^2 = B_1^{-\frac{2\rho}{\rho-2}}, \end{aligned}$$

which implies that

$$\begin{aligned} I(t) &= \left(1-\int_0^t g(\tau)d\tau\right)\|u(t)\|_{W_0}^2 + (g\diamond u)(t) - \int_\Omega |u(t)|^{\rho-1}u(t)\,dx \\ &\ge l\|u(t)\|_{W_0}^2 + (g\diamond u)(t) - \|u(t)\|_\rho^\rho \\ &\ge l\|u(t)\|_{W_0}^2 - C_*^\rho\|u(t)\|_{W_0}^\rho \ge 0. \end{aligned}$$

So, we have

$$\begin{aligned} \left(\frac{\rho-2}{2\rho}\right) l\|u(t)\|_{W_0}^2 &\le \frac{\rho-2}{2\rho}\left(1-\int_0^t g(\tau)d\tau\right)\|u(t)\|_{W_0}^2 \\ &\le \frac{\rho-2}{2\rho}\Big[\left(1-\int_0^t g(\tau)d\tau\right)\|u(t)\|_{W_0}^2 + (g\diamond u)(t)\Big] + \frac{1}{\rho}I(t) \\ &\le E(t) \le E(0), \end{aligned}$$

then

$$l\|u(t)\|_{W_0}^2 \le \frac{2\rho}{\rho-2}E(0). \tag{3.35}$$

Using the above inequality, we can deduce that

$$\begin{aligned} \left|\int_\Omega |u|^{\rho-1}u\right| &\le \|u(t)\|_\rho^\rho \\ &\le C_*^\rho\|u(t)\|_{W_0}^\rho \frac{C_*^\rho}{l}\left(\frac{2\rho}{l(\rho-2)}E(0)\right)^{\frac{\rho-2}{2}} l\|u(t)\|_{W_0}^2 \\ &\equiv \theta l\|u(t)\|_{W_0}^2. \end{aligned} \tag{3.36}$$

From the Young inequality and the fact that

$$\int_0^t g(\tau)\,d\tau \le \int_0^\infty g(\tau)\,d\tau = 1-l,$$

it follows that

$$\begin{aligned}
&\int_0^t g(t-\tau)\langle u(\tau), u(t)\rangle_{W_0}\, d\tau \\
&\le \frac{1}{2}\|u(t)\|_{W_0}^2 + \frac{1}{2}\Big\{ \int_0^t g(t-\tau)\left(\|u(\tau)-u(t)\|_{W_0} + \|u(t)\|_{W_0}\right) d\tau \Big\}^2 \\
&\le \frac{1}{2}\|u(t)\|_{W_0}^2 + \frac{1}{2}(1+\eta)\left(\int_0^t g(t-\tau)\|u(t)\|_{W_0}\, d\tau\right)^2 \\
&+ \frac{1}{2}(1+\frac{1}{\eta})\left(\int_0^t g(t-\tau)\|u(\tau)-u(t)\|_{W_0}\, d\tau\right)^2 \\
&\le \frac{1}{2}\|u(t)\|_{W_0}^2 + \frac{1}{2}(1+\eta)(1-l)^2\|u(t)\|_{W_0}^2 + \frac{1}{2}(1+\frac{1}{\eta})(1-l)(g\diamond u)(t). \qquad (3.37)
\end{aligned}$$

for any $\eta > 0$. Now, letting $\eta = \frac{l}{1-l} > 0$ then (3.37) yields

$$\int_0^t g(t-\tau)\langle u(\tau), u(t)\rangle_{W_0}\, d\tau \le \frac{2-l}{2}\|u(t)\|_{W_0}^2 + \frac{1-l}{2l}(g\diamond u)(t). \qquad (3.38)$$

Substituting (3.38) into (3.33), we obtain

$$\frac{d}{dt}F(t) \le -\frac{1}{2}\left(\xi_0 - \varepsilon\frac{1-l}{l}\right)(g\diamond u)(t) - \varepsilon\|u(t)\|_{W_0}^{2\alpha} - \frac{\varepsilon l}{2}\|u(t)\|_{W_0}^2 + \varepsilon\int_\Omega |u(t)|^{\rho-1}u(t)\, dx. \qquad (3.39)$$

Using the definition of $E(t)$ and (3.36) we have, for any positive constant M

$$\begin{aligned}
\frac{d}{dt}F(t) \le -M\varepsilon E(t) + \varepsilon\left(\frac{M}{2\alpha} - 1\right)\|u(t)\|_{W_0}^{2\alpha} + \frac{\varepsilon}{2}\left[M + 2\theta l(1-\frac{M}{\rho}) - l\right]\|u(t)\|_{W_0}^2 \\
+ \frac{1}{2}\left[\varepsilon(\frac{1-l}{l} + \frac{M}{2}) - \xi_0\right](g\diamond u)(t). \qquad (3.40)
\end{aligned}$$

At this point, we choose $1 > M > 0$ and $E(0)$ small sufficiently such that

$$\frac{M}{2\alpha} - 1 < 0 \qquad \text{and} \qquad M + 2\theta l\left(1 - \frac{M}{\rho}\right) - l < 0.$$

After M is fixed, we choose ε small enough such that

$$\varepsilon\left(\frac{1-l}{l} + \frac{M}{2}\right) - \xi_0 < 0.$$

Inequality (3.40) becomes

$$\frac{d}{dt}F(t) \le -M\varepsilon E(t).$$

By (3.32), we have

$$\frac{d}{dt}F(t) \le -M\beta_2\varepsilon F(t).$$

So $F(t) \le Ce^{-Kt}$ where $K = M\beta_2\varepsilon > 0$. Consequently, by using (3.32) once again, we conclude the result.
Thus, the proof of Theorem 3.3 is achieved. □

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Statistical Korovkin and Voronovskaya type theorem for the Cesáro second-order operator of fuzzy numbers

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Abstract. In this paper we define the Cesáro second-order summability method for fuzzy numbers and prove Korovkin type theorem, then as the application of it, we prove the rate of convergence. In the last section, we prove the kind of Voronovskaya type theorem and give some concluding remarks related to the obtained results.

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1. Introduction

The concepts of fuzzy sets and fuzzy set operations were first introduced by Zadeh [16] and subsequently, several authors have discussed various aspects of the theory and applications of fuzzy sets such as fuzzy topological spaces, similarity relations, and fuzzy orderings, fuzzy measures of fuzzy events, fuzzy mathematical programming. Matloka [13] introduced bounded and convergent sequences of fuzzy numbers studied some of their properties and showed that every convergent sequence of fuzzy numbers is bounded.

In the present paper, we will prove the Korovkin type theorem for statistical summability $(C, 2)$ and the rate of convergence. In this section, we give a brief overview of statistical convergence, fuzzy numbers, and sequences of fuzzy numbers. In section 2 we prove the main results of this paper. In section 3 we give results related to the rate of convergence.

The idea of statistical convergence depends upon the density of subsets of the set $\mathbb{N}$ of natural numbers. We shall denote by $\mathbb{N}$ the set of all natural numbers. Let $K \in \mathbb{N}$ and $K_n = \{k \leq n : k \in K\}$. Then the natural density of K is defined by

$d(K) = \lim_{n\to\infty} \frac{|K_n|}{n}$ if the limit exists, where the vertical bars indicate the number of elements in the enclosed set. The sequence $x = (x_k)$ is said to be statistically convergent to L([10]) if for every $\varepsilon > 0$, the set $K_\varepsilon = \{k \in \mathbb{N} : |x_k - L| \geq \varepsilon\}$ has natural density zero, i.e. for each $\varepsilon > 0$,

$$\lim_{n\to\infty} \frac{1}{n} |\{k \leq n : |x_k - L| \geq \varepsilon\}| = 0.$$

In this case, we write $st - \lim x = L$. Note that every convergent sequence is statistically convergent but not conversely.

In paper [6], was defined the second order Cesáro summability method as follows:

$$(C,2)_n = \sum_{n=1}^{\infty} \left(\frac{1}{(n+1)(n+2)} \sum_{k=0}^{n} (n+1-k)|x_k| \right).$$

The summability method $(C,2)_n$ is a regular. We say that the series $\sum_{n=1}^{\infty} x_n$ is $(C,2)_n-$ summable to L if

$$\lim_n \sum_{j=1}^{n} \left(\frac{1}{(j+1)(j+2)} \sum_{k=0}^{j} (j+1-k)|x_k| \right) = L.$$

In the present paper, we define Cesáro second-order summability method for sequences of fuzzy numbers and give Korovkin type theorem and rate of convergence. The theory of Korovkin type theorems was intensively investigated in recent years, see for example [3, 4, 1, 6, 7, 8, 9, 11, 12].

2. Preliminaries

Let $C(\mathbb{R})$ denote the family of all nonempty, compact, convex subsets of $\mathbb{R}$. Denote by

$$L(\mathbb{R}) = \{u : \mathbb{R} \to [0,1] : u \quad \text{satisfies} \quad (1) - (4) \quad \text{bellow}\}$$

where

1. u is normal, there exists an $x_0 \in \mathbb{R}$ such that $u(x_0) = 1$,
2. u is fuzzy convex, for any $x, y \in \mathbb{R}$ and $0 \leq \lambda \leq 1$, $u(\lambda x + (1-\lambda)y) \geq \min[u(x), u(y)]$,
3. u is upper semicontinuous,
4. the closure of $\{x \in \mathbb{R} : u(x) > 0\}$, denoted by $[u]_0$, is compact.

If $u \in L(\mathbb{R})$, then u is called fuzzy number, and $L(\mathbb{R})$ is said to be fuzzy number space. For $0 < \alpha \leq 1$, the $\alpha-$ level set $[u]_\alpha$ of u is defined by $[u]_\alpha = \{x \in \mathbb{R} : u(x) \geq \alpha\}$. Then from (1)-(4), it follows that the $\alpha-$level sets $[u]_\alpha \in C(\mathbb{R})$.

The set of real numbers can be embedded in $L(\mathbb{R})$, since each $r \in \mathbb{R}$ can be regarded as a fuzzy number $\overline{r}$ defined by

$$\overline{r} = \begin{cases} 1; & \text{if} \quad x = r, \\ 0; & \text{if} \quad x \neq r. \end{cases}$$

Let $u, v, w \in L(\mathbb{R})$ and $k \in \mathbb{R}$. Then the operations addition and scalar multiplications are defined in $L(\mathbb{R})$ as follows:

$$u+v=w \Leftrightarrow [w]_\alpha=[u]_\alpha+[v]_\alpha \quad \text{for all} \quad \alpha \in [0,1],$$

$$\Leftrightarrow w_\alpha^-=u_\alpha^-+v_\alpha^- \quad \text{and} \quad w_\alpha^+=u_\alpha^++v_\alpha^+ \quad \text{for all} \quad \alpha \in [0,1],$$

$$[ku]_\alpha=k[u]_\alpha \quad \text{for all} \quad \alpha \in [0,1].$$

Further details related to the structural properties of the fuzzy numbers, are given in [5]. Let us denote by W the set of all closed bounded intervals A of real numbers with endpoints $\underline{A}$ and $\overline{A}$, i.e., $A=[\underline{A},\overline{A}]$. Define the relation d on W by

$$d(A,B)=\max\{|\underline{A}-\underline{B}|,|\overline{A}-\overline{B}|\}.$$

Then it can be easily observed that d is a metric on W and (W,d) is a complete metric space, ([14]). Now, we may define the metric D on $L(\mathbb{R})$ by means of the Hausdorff metric d as follows

$$D(u,v)=\sup_{\alpha\in[0,1]} d([u]_\alpha,[v]_\alpha)=\sup_{\alpha\in[0;1]} \max\{|u^-(\alpha)-v^-(\alpha)|,|u^+(\alpha)-v^+(\alpha)|\}$$

and

$$D(u,0)=\sup_{\alpha\in[0;1]} \max\{|u^-(\alpha)|,|u^+(\alpha)|\}=\max\{|u^-(\alpha)|,|u^+(\alpha)|\}.$$

Let $f,g:[a,b]\to L(\mathbb{R})$, be fuzzy number valued functions. The parametric representation is as follows: $[f(x)]^r=[f_-^{(r)}(x),f_+^{(r)}(x)]$, for every $x\in[a,b]$ and every $r\in[0,1]$. Then, the distance between f and g is given by

$$D^*(f,g)=\sup_{x\in[a,b]}\sup_{r\in[0,1]}\max\left\{\left|f_-^{(r)}-g_-^{(r)}\right|,\left|f_+^{(r)}-g_+^{(r)}\right|\right\}.$$

Fuzzy function $f:[a,b]\to L(\mathbb{R})$, is continuous at $x_0\in[a,b]$ if for each $\varepsilon>0$ there is a $\delta>0$ such that $D(f(x),f(x_0))<\varepsilon$, whenever $x\in[a,b]$ with $|x-x_0|<\delta$. If f is continuous in each point on $[a,b]$, then we say that it is continuous whole $[a,b]$. The class of continuous function we will denote by $C_F[a,b]$.

A sequence $u=(u_k)$ of fuzzy numbers is a function u from the set $\mathbb{N}$, into the set $L(\mathbb{R})$. The fuzzy number u_k denotes the value of the function at $k\in\mathbb{N}$ and is called as the $k-$th term of the sequence. By $w(F)$, we denote the set of all sequences of fuzzy numbers. A sequence $(u_n)\in w(F)$ is said to be convergent to $u\in L(\mathbb{R})$, if for every $\varepsilon>0$ there exists an $n_0=n_0(\varepsilon)\in\mathbb{N}$ such that

$$D(u_n,u)<\varepsilon \quad \text{for all} \quad n>n_0.$$

Definition 2.1. Let $X=(X_k)$ be a sequence of fuzzy numbers. The sequence X is said to converge weighted statistically to a fuzzy number X_0, if for every $\varepsilon>0$

$$\lim_{n\to\infty}\frac{1}{(n+1)(n+2)}\left|\{k\le(n+1)(n+2):D(X_k,X_0)\ge\varepsilon\}\right|=0.$$

The above type of convergence will be denoted as

$$st_F-\lim_n X_n=X_0.$$

Definition 2.2. Let $X = (X_k)$ be a sequence of fuzzy numbers. The sequence X is said to be statistically Cesáro second order summable to a fuzzy number X_0 if the sequence

$$(C,2)_n(X) = \frac{1}{(n+1)(n+2)} \sum_{k=0}^{n} (n+1-k)X_k,$$

is statistically convergent to X_0, where the sum in $(C,2)_n(X)$ is usual addition of fuzzy real numbers through $\alpha-$ level sets. That is (X_k) is statistically Cesáro second order summable to the fuzzy number X_0, if for every $\varepsilon > 0$

$$\lim_{n\to\infty} \frac{1}{(n+1)(n+2)} |\{k \leq (n+1)(n+2) : D((C,2)_n, X_0) \geq \varepsilon\}| = 0.$$

The above type of convergence will be denoted as

$$st_{(C,2)} - \lim_n X_n = X.$$

3. Statistical fuzzy Korovkin type theorem

Let us denote by $C[a,b]$ the space of continuous function defined in the $[a,b]$. As we know, this space equipped with supremum norm

$$||f|| = \sup_{x\in[a,b]} |f(x)|,$$

is a complete metric space.

In this section we prove fuzzy Korovkin type theorem via the concept of statistical summability $(C,2)$. Let $f : [a,b] \to L(R)$ be fuzzy number valued functions. Then f is said to be fuzzy continuous at $x_0 \in [a,b]$ provided that whenever $x_n \to x_0$, then $D(f(x_n), f(x_0)) \to \infty$ as $n \to \infty$. Also, we say that f is fuzzy continuous on $[a,b]$ if it is fuzzy continuous at every point $x \in [a,b]$. The set of all fuzzy continuous functions on the interval $[a,b]$ is denoted by $C_F[a,b]$ (see, for instance, [3]).

Let $L : C_F[a,b] \to C_F[a,b]$ be an operator. Then L is said to be fuzzy linear, if for every $\alpha, \beta \in \mathbb{R}$, any $f, g \in C_F[a,b]$ and for every $x \in [a,b]$,

$$L(\alpha f + \beta g; x) = \alpha L(f;x) + \beta L(g;x),$$

holds. L is said to be fuzzy positive linear operator if it is fuzzy linear and the condition $L(f;x) \preceq L(g;x)$ is satisfied for any $f, g \in C_F[a,b]$ and for all $x \in [a,b]$ with $f(x) \preceq g(x)$. Last relation is fulfilled if and only if $f_-^{(r)}(x) \leq g_-^{(r)}(x)$ and $f_+^{(r)}(x) \leq g_+^{(r)}(x)$, where $[f(x)]^{(r)} = [f_-^{(r)}(x), f_+^{(r)}(x)]$. The fuzzy Korovkin type theorem was investigated by many authors(see [3, 4, 2]) and statistical version of the theorem, was given by [4], as follows.

Theorem 3.1. ([3]) *Let $\{L_n\}_{n\in\mathbb{N}} : C_F[a,b] \to C_F[a,b]$, be a sequence of fuzzy positive linear operators. Assume that there exists a corresponding sequence $\{\tilde{L_n}\}_{n\in\mathbb{N}} : C[a,b] \to C[a,b]$, of linear positive operators, with the property:*

$$\{L_n(f;x)\}_{\pm}^{(r)} = \tilde{L_n}(f_{\pm}^{(r)}; x) \tag{3.1}$$

for all $x \in [a,b], r \in [0,1], n \in \mathbb{N}$ *and* $f \in C_F[a,b]$. *Also assume that*

$$\lim_{n\to\infty} \left\| \tilde{L_n}(e_i) - e_i \right\| = 0, \quad \textit{for each} \quad i = 0,1,2,$$

where $e_i = x^i$. *Then, for all* $f \in C_F[a,b]$, *we have*

$$\lim_n D^*(L_n(f), f) = 0.$$

Later one, this result is extended to summability matrix as follows

Theorem 3.2. ([4]) *Let* $A = (a_{j_n})$ *be a non-negative regular summability method matrix and let* $\{L_n\}_{n\in\mathbb{N}}$ *be a sequence of fuzzy positive linear operators from* $C_F[a,b]$ *into itself. Assume that there exists a corresponding sequence* $\{\tilde{L_n}\}_{n\in\mathbb{N}}$ *of positive linear operators from* $C[a,b]$ *into itself with the property (*3.1*). Assume further that*

$$st_A - \lim_{n\to\infty} \left\| \tilde{L_n}(e_i) - e_i \right\| = 0, \quad \textit{for each} \quad i = 0,1,2,$$

where $e_i = x^i$. *Then, for all* $f \in C_F[a,b]$, *we have*

$$st_A - \lim_n D^*(L_n(f), f) = 0.$$

Now we prove the fuzzy Korovkin type theorem for statistical convergence, using the notion of the statistical summability method $(C,2)$.

Theorem 3.3. *Let* $\{L_n\}_{n\in\mathbb{N}}$ *be a sequence of fuzzy positive linear operators from* $C_F[a,b]$ *into itself. Assume that there exists a corresponding sequence* $\{\tilde{L_n}\}_{n\in\mathbb{N}}$ *of positive linear operators from* $C[a,b]$ *into itself with the property (*3.1*). Also assume that*

$$st_{(C,2)} - \lim_{n\to\infty} \left\| \tilde{L_n}(e_i) - e_i \right\| = 0, \quad \textit{for each} \quad i = 0,1,2, \tag{3.2}$$

where $e_i = x^i$. *Then, for all* $f \in C_F[a,b]$, *we have*

$$st_{(C,2)} - \lim_n D^*(L_n(f), f) = 0. \tag{3.3}$$

Proof. Let $f \in C_F[a,b]$ for $x \in [a,b]$ and $r \in [0,1]$. By hypothesis $f_\pm^{(r)} \in C[a,b]$, which means that for every $\varepsilon > 0$ there exists a $\delta(\varepsilon) > 0$, and for any $y \in [a,b]$ such that $|x-y| < \delta$ we obtain $|f_\pm^{(r)}(x) - f_\pm^{(r)}(y)| < \varepsilon$. From last relation and boundedness of function $f_\pm^{(r)}(x)$, we get

$$\left| f_\pm^{(r)}(x) - f_\pm^{(r)}(y) \right| \le \varepsilon + 2 \left\| f_\pm^{(r)} \right\| \frac{(x-y)^2}{\delta^2}.$$

Considering linearity and positivity of the operators $\tilde{L_n}$, we have for each $n \in \mathbb{N}$, that

$$\left| \tilde{L_n}\left(f_\pm^{(r)};x\right) - f_\pm^{(r)}(x) \right| \le \left| \tilde{L_n}\left(\left| f_\pm^{(r)}(x) - f_\pm^{(r)}(y) \right|;x \right) \right| + \left\| f_\pm^{(r)} \right\| \left| \tilde{L_n}(e_0;x) - e_0(x) \right| \le \varepsilon$$

$$+ \left(\varepsilon + \left\| f_\pm^{(r)} \right\| \right) \cdot \left| \tilde{L_n}(e_0;x) - e_0(x) \right| + \frac{2 \left\| f_\pm^{(r)} \right\|}{\delta^2} \left| \tilde{L_n}((x-y)^2;x) \right|,$$

if we put $M = \max\{|a|,|b|\}$, we have

$$\left| \tilde{L_n}\left(f_\pm^{(r)};x\right) - f_\pm^{(r)}(x) \right| \le \varepsilon + \left(\varepsilon + \left\| f_\pm^{(r)} \right\| + \frac{2x^2}{\delta^2} \left\| f_\pm^{(r)} \right\| \right) \cdot \left| \tilde{L_n}(e_0;x) - e_0(x) \right|$$

$$+\frac{4x}{\delta^2}\left\|f_{\pm}^{(r)}\right\|\left|\tilde{L_n}(e_1;x)-e_1(x)\right|+\frac{2}{\delta^2}\left\|f_{\pm}^{(r)}\right\|\left|\tilde{L_n}(e_2;x)-e_2(x)\right|\leq$$

$$\left|\tilde{L_n}\left(f_{\pm}^{(r)};x\right)-f_{\pm}^{(r)}(x)\right|\leq\varepsilon+\left(\varepsilon+\left\|f_{\pm}^{(r)}\right\|+\frac{2M^2}{\delta^2}\left\|f_{\pm}^{(r)}\right\|\right)\cdot\left|\tilde{L_n}(e_0;x)-e_0(x)\right|$$

$$+\frac{4M}{\delta^2}\left\|f_{\pm}^{(r)}\right\|\left|\tilde{L_n}(e_1;x)-e_1(x)\right|+\frac{2}{\delta^2}\left\|f_{\pm}^{(r)}\right\|\left|\tilde{L_n}(e_2;x)-e_2(x)\right|.$$

Let

$$M_{\pm}^{(r)}(\varepsilon)=\max\left\{\varepsilon+\left\|f_{\pm}^{(r)}\right\|+\frac{2M^2}{\delta^2}\left\|f_{\pm}^{(r)}\right\|,\frac{4M}{\delta^2}\left\|f_{\pm}^{(r)}\right\|,\frac{2}{\delta^2}\left\|f_{\pm}^{(r)}\right\|\right\}.$$

Taking supremum on the above inequality for $x\in[a,b]$, we obtain

$$\left\|\tilde{L_n}\left(f_{\pm}^{(r)}\right)-f_{\pm}^{(r)}\right\|\leq\varepsilon+M_{\pm}^{(r)}(\varepsilon)\left\{\left\|\tilde{L_n}(e_0)-e_0\right\|+\left\|\tilde{L_n}(e_1)-e_1\right\|+\left\|\tilde{L_n}(e_2)-e_2\right\|\right\}. \tag{3.4}$$

Now using into consideration relation (3.1) and relation (3.4), we have

$$D^*(f,g)=\sup_{x\in[a,b]}D(L_n(f;x),f(x))$$

$$=\sup_{x\in[a,b]}\sup_{r\in[0,1]}\max\left\{\left|\tilde{L_n}\left(f_{-}^{(r)};x\right)-f_{-}^{(r)}(x)\right|,\left|\tilde{L_n}\left(f_{+}^{(r)};x\right)-f_{+}^{(r)}(x)\right|\right\}$$

$$=\sup_{r\in[0,1]}\max\left\{\left\|\tilde{L_n}\left(f_{-}^{(r)}\right)-f_{-}^{(r)}\right\|,\left\|\tilde{L_n}\left(f_{+}^{(r)}\right)-f_{+}^{(r)}\right\|\right\}. \tag{3.5}$$

From relations (3.4) and (3.5), it yields

$$D^*(L_n(f),f)\leq\varepsilon+M(\varepsilon)\left\{\left\|\tilde{L_n}(e_0)-e_0\right\|+\left\|\tilde{L_n}(e_1)-e_1\right\|+\left\|\tilde{L_n}(e_2)-e_2\right\|\right\}, \tag{3.6}$$

where $M(\varepsilon)=\sup_{0\leq r\leq 1}\max\{M_{-}^{(r)}(\varepsilon),M_{+}^{(r)}(\varepsilon)\}$.
Let $\varepsilon_1>0$, we can choose $0<\varepsilon<\varepsilon_1$, and define sets:

$$A=\{n\in\mathbb{N}:D^*(L_n(f),f)\geq\varepsilon_1\},$$

$$A_1=\left\{n\in\mathbb{N}:\left\|\tilde{L_n}(e_0)-e_0\right\|\geq\frac{\varepsilon_1-\varepsilon}{3M(\varepsilon)}\right\},$$

$$A_2=\left\{n\in\mathbb{N}:\left\|\tilde{L_n}(e_1)-e_1\right\|\geq\frac{\varepsilon_1-\varepsilon}{3M(\varepsilon)}\right\},$$

$$A_3=\left\{n\in\mathbb{N}:\left\|\tilde{L_n}(e_2)-e_2\right\|\geq\frac{\varepsilon_1-\varepsilon}{3M(\varepsilon)}\right\}.$$

Then from relation (3.6), we have

$$A\subset A_1\cup A_2\cup A_3.$$

Now from last relation and relations (3.2), we get relation (3.3). □

Remark 3.4. Our theorem is generalization of the result given in theorem 3.1 and theorem 3.2, as it is shown on this.

Example 3.5. Take $A = (C,2) = (c_{jn})$, the Cesáro second order matrix and define the following sequence

$$(a_n) = \begin{cases} 0, & \text{if} \quad n \neq m^2, m = 1, 2, \cdots, \\ n^{\frac{3}{2}}, & \text{otherwise} \end{cases}$$

If we use into consideration the fuzzy Bernstein-type operators

$$B_n^F(f;x) = a_n \odot \bigoplus_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} \odot f_{\pm}^{(r)}\left(\frac{k}{n}\right),$$

where $f \in C_F[0,1]$, $x \in [0,1]$ and $n \in \mathbb{N}$. We can define

$$\left\{B_n^F(f;x)\right\}_{\pm}^{(r)} = \tilde{B_n}\left(f_{\pm}^{(r)};x\right) = a_n \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} f_{\pm}^{(r)}\left(\frac{k}{n}\right),$$

$f_{\pm}^{(r)} \in C[0,1]$.
Let us define the following operators

$$L_n(f;x) = (1+a_n)\tilde{B_n}(f;x). \tag{3.7}$$

Then we have:

$$L_n(e_0;x) = (1+a_n),$$

$$L_n(e_1;x) = x(1+a_n),$$

$$L_n(e_2;x) = \left(x^2 + \frac{x(1-x)}{n}\right)(1+a_n).$$

The limit $st_{(C,2)} - \lim a_n$, exist and it is:

$$\sum_{n:|a_n-0|\geq\varepsilon} c_{jn} = \sum_{n:|a_n-0|\geq\varepsilon} \frac{1}{(j+1)(j+2)} \leq \frac{j^{\frac{3}{2}}}{(j+1)(j+2)} \to 0, \quad \text{as} \quad j \to \infty,$$

which means that $st_{(C,2)} - \lim a_n = 0$.
From above relation we get

$$st_{(C,2)} - \lim_{n\to\infty} \|L_n(e_i) - e_i\| = 0, \quad \text{for each} \quad i = 0, 1, 2$$

and from theorem 3.3, we obtain

$$st_{(C,2)} - \lim_{n} D^*(L_n(f), f) = 0.$$

However, (a_n) is not convergent in usual sense, the sequence $\{B_n^F(f;x)\}$ is not fuzzy convergent to f.

4. Statistical fuzzy rate of convergence

In this section, we investigate the rate of the Cesáro second order operators, statistical convergence of positive linear operators in the space $C_F[a,b]$.

Definition 4.1. Let (a_n) be any nondecreasing sequence of positive numbers. We say that the sequence of functions $(f_n) \in C_F[a,b]$ is Cesáro second order statistical convergent to a function f with the rate of convergence given by $o(a_n)$, if, for each $\varepsilon > 0$,

$$\lim_{n\to\infty} \frac{1}{(n+1)(n+2)a_n} \left|\{m \leq (n+1)(n+2) \quad \text{and} \quad D^*((C,2)_m; f)) \geq \varepsilon\}\right| = 0.$$

In this case, we write

$$f_n - f = o(a_n)((C,2)_n - stat).$$

Lemma 4.2. *Let (a_n) and (b_n) be two nondecreasing sequences of positive numbers. Suppose also that the sequences (f_n) and (g_n) are constrained by*

$$f_n - f = o(a_n)((C,2)_n - stat) \qquad \textit{and} \qquad g_n - g = o(b_n)((C,2)_n - stat),$$

respectively. Then

1. $\alpha(f_n - f) = o(a_n)((C,2)_n - stat)$ *for any scalar* α;
2. $(f_n - f) \pm (g_n - g) = o(c_n)((C,2)_n - stat)$;
3. $(f_n - f)(g_n - g) = o(a_n b_n)((C,2)_n - stat)$,

where

$$c_n := \max\{a_n, b_n\}.$$

Now, by defining the modules of continuity, for a given function $f(x) \in C_F[a,b]$, as follows:

Definition 4.3. Let $f : [a,b] \to E$ be a fuzzy real number valued function. We define the modulus of continuity of f by

$$\omega_1^F(f,\delta) = \sup_{x,y\in[a,b]} D(f(x), f(y)),$$

for every $|x-y| \leq \delta$ and any $0 < \delta \leq b-a$.

We now state and prove the following result.

Theorem 4.4. *Let (L_n) be a sequence of fuzzy positive linear operators from $C_F[a,b]$ into $C_F[a,b]$. Assume that there exists a corresponding sequence $\{\tilde{L_n}\}_{n\in\mathbb{N}}$ of positive linear operators from $C[a,b]$ into itself with the property (3.1). Suppose that (a_n) and (b_n) are non-decreasing sequence and also that the operators $\tilde{L_n}$ satisfy the following conditions:*

1. $\left\|\tilde{L_n}(e_0) - e_0\right\| = (\mathrm{C},2)_\mathrm{n} - \mathrm{stat}\ o(a_n)$ *as* $n \to \infty$,
2. $\omega_1^F(f,\lambda_n) = (\mathrm{C},2)_\mathrm{n} - \mathrm{stat}\ o(b_n)$ *as* $n \to \infty$

where

$$\lambda_n = \sqrt{\left\|\tilde{L_n}(\varphi)\right\|} \qquad \textit{and} \qquad \varphi_y = (y-x)^2, \textit{for all} x \in [a,b].$$

Then, for all $f \in C_F[a,b]$, *we have*

$$\left\|\tilde{L_n}(f) - f\right\| = (\mathrm{C},2)_{\mathrm{n}} - \mathrm{stat}\; o(c_n), \, as \quad n \to \infty,$$

where $c_n = \max\{a_n, b_n\}$, *for each* $n \in \mathbb{N}$.

Proof. Let $f \in C_F[a,b]$. Then,

$$\tilde{L_k}(f_{\pm}^{(r)}, x) - f_{\pm}^{(r)}(x) = \tilde{L_k}(f_{\pm}^{(r)}, x) - f_{\pm}^{(r)}(x)\tilde{L_k}(1,x) + f_{\pm}^{(r)}(x)[\tilde{L_k}(1,x) - 1], \quad (4.1)$$

and

$$|f_{\pm}^{(r)}(x) - f_{\pm}^{(r)}(y)| \leq \omega_1^F(f,\delta)\left(\frac{|x-y|}{\delta} + 1\right), \quad (4.2)$$

in both cases, where $|x-y| \geq \delta$ and $|x-y| \leq \delta$.
By using the relations (4.1) and (4.2), we get the following estimate:

$$|\tilde{L_n}(f_{\pm}^{(r)}, x) - f_{\pm}^{(r)}(x)| \leq |\tilde{L_n}(|f_{\pm}^{(r)}(y) - f_{\pm}^{(r)}(x)|, x)| + |f_{\pm}^{(r)}(x)| \cdot |\tilde{L_n}(1,x) - 1|$$

$$\leq \tilde{L_n}\left(\frac{|x-y|}{\delta} + 1, x\right)\omega_1^F(f,\delta) + |f(x)| \cdot |\tilde{L_n}(1,x) - 1| \text{ (by Cauchy-Schwartz inequality)}$$

$$\leq \frac{1}{\delta}\tilde{L_n}\left((x-y)^2, x\right)^{\frac{1}{2}} \tilde{L_n}\left(1,x\right)^{\frac{1}{2}} \omega_1^F(f,\delta) + \tilde{L_n}(1,x)\omega_1^F(f,\delta)$$

$$+ |f_{\pm}^{(r)}(x)| \cdot |\tilde{L_n}(1,x) - 1| (\text{for } \delta = \lambda_n, \text{ we get})$$

$$\leq K\left|\tilde{L_n}(1,x) - 1\right| + 2\omega_1^F(f,\delta) + \omega_1^F(f,\delta)|\tilde{L_n}(1,x) - 1|$$

$$+ \omega_1^F(f,\delta)\sqrt{|\tilde{L_n}(1,x) - 1|},$$

where $K = \left\|f_{\pm}^{(r)}\right\|$. Now, by using relations (1) and (2) in the theorem and lemma 4.2, we complete proof of Theorem.

□

5. Statistical fuzzy Voronovskaya type theorem

In this section we show positive linear operators

$$D_n(f;x) = \frac{(1+b_n)}{n^2}\tilde{B_n}(f;x),$$

where sequence $n(b_n) = (a_n)$, and (a_n), is defined in example 3.5, satisfy a Voronovskaja type property in the $(C,2)-$ statistically convergence sense. We first prove the following lemma.

Lemma 5.1. *For* $x \in [a,b]$, *and* $\Phi(y) = y - x$ *then*

$$n^2 D_n(\Phi^4) \sim x^2(2x^2+1)(x-1)((C,2) - stat.) \quad on \quad [a,b].$$

Proof. After some calculations we get:

$$n^2D_n(\Phi^4) = (1+b_n)\left[\left(2-\frac{5}{n}+\frac{8}{n^2}-\frac{11}{n^3}+\frac{6}{n^4}\right)x^5+\left(-2+\frac{4}{n}-\frac{5}{n^2}+\frac{9}{n^3}-\frac{6}{n^4}\right)x^4\right.$$
$$\left.+\left(1-\frac{2}{n}+\frac{1}{n^3}\right)x^3-\left(1-\frac{2}{n}+\frac{3}{n^3}-\frac{2}{n^4}\right)x^2+\left(\frac{1}{n^2}-\frac{3}{n^3}+\frac{2}{n^4}\right)x\right].$$

Thus we obtain:

$$\left|n^2D_n(\Phi^4)-x^2(2x^2+1)(x-1)\right| \leq |(1+b_n)-1||(2x^5-2x^4+x^3-x^2)|$$
$$+\left|\left(-\frac{5}{n}+\frac{8}{n^2}-\frac{11}{n^3}+\frac{6}{n^4}\right)x^5\right|+\left|\left(\frac{4}{n}-\frac{5}{n^2}+\frac{9}{n^3}-\frac{6}{n^4}\right)x^4\right|+\left|\left(-\frac{2}{n}+\frac{1}{n^3}\right)x^3\right|$$
$$+\left|\left(-\frac{2}{n}+\frac{3}{n^3}-\frac{2}{n^4}\right)x^2\right|+\left|\left(\frac{1}{n^2}-\frac{3}{n^3}+\frac{2}{n^4}\right)x\right| \to 0((C,2)-stat.),$$

as $n \to \infty$, on $[a, b]$. This completes proof of the Lemma. □

In what follows we establish the following Voronovskaya fuzzy type theorem for operators D_n, defined as in above Lemma. Before given the main result of this section we will give some concepts related to the H-derivatives for the fuzzy functions.

A function $f : [x_0; x_0 + \alpha] \to R_F$, for $\alpha > 0$, is $H-$derivative at $x \in T$ if there exists a $f^{'}(x) \in R_F$ such that the limits

$$\lim_{h\to 0^+}\frac{f(x+h)-f(x)}{h}, \lim_{h\to 0^+}\frac{f(x)-f(x-h)}{h}$$

exists and are equal to $f^{'}(x)$.

We assume that the $H-$ differences $f(x+h)-f(x), f(x)-f(x-h) \in R_F$ in a neighborhood of x. We call $f^{'}(x)$ the derivative or $H-$ derivative of f at x (for more details see [15]). In paper [2], was given the Taylor formula for fuzzy functions as follows:

Theorem 5.2. *Let $T = [x_0, x_0 + \alpha] \subset \mathbb{R}$, and $\alpha > 0$. We assume that $f^{(i)} : T \to R_F$ are $H-$ differentiable for all $i \in \{0, 1, 2, 3, \cdots, n-1\}$, for any $x \in T$. (It means that there exists in R_F the $H-$ differences $f^{(i)}(x+h) - f^{(i)}(x)$, $f^{(i)}(x) - f^{(i)}(x-h)$, $i \in \{0, 1, 2, 3 \cdots, n-1\}$ for all h such that $0 < h < \alpha$. Furthermore there exists $f^{(i+1)}(x) \in R_F$ such that limits in $D-$ metrics exist and*

$$f^{(i+1)}(x) = \lim_{h\to 0^+}\frac{f^{(i)}(x+h)-f^{(i)}(x)}{h} = \lim_{h\to 0^+}\frac{f^{(i)}(x)-f^{(i)}(x-h)}{h},$$

for all $i \in \{0, 1, 2, 3 \cdots, n-1\}$.) Also we assume that $f^{(n)}$, is fuzzy continuous on T. Then for $s \geq a; s, a \in T$ we obtain

$$f(s) = f(a) + \frac{f^{'}(a)}{1!}(s-a) + \frac{f^{''}(a)}{2!}(s-a)^2 + \cdots + \frac{f^{(n-1)}(a)}{(n-1)!}(s-a)^{n-1} + R_n(s, a),$$

where

$$R_n(s,a) = \int_a^s \left(\int_a^{s_1} \cdots \left(\int_a^{s_{n-1}} f^{(n)}(s_n)ds_n\right) ds_{n-1} \cdots\right) ds_1,$$

above integration is in sense of Fuzzy-Riemann integral and $R_n(s,a)$ is fuzzy continuous on T as a function of s.

Theorem 5.3. *For every $f:[a,b]\to R_F$, we assume that there exists $f^{'}, f^{''}\in R_F$, then*

$$n\left[n^2 D_n(f)-f(x)\right]\sim\frac{1}{2}(x-x^2)f^{''}(x)((C,2)-stat.),$$

on $[a,b]$.

Proof. Let us suppose that $f^{'}, f^{''}\in R_F$ and $x\in[a,b]$. Define

$$\psi_x(y)=\begin{cases}\dfrac{f(y)-f(x)-(y-x)f^{'}(x)-\frac{1}{2}(y-x)^2f^{''}(x)}{(y-x)^2} & \text{for} \quad x\neq y\\ 0 & \text{for} \quad x=y.\end{cases}$$

Then $\psi_x(x)=0$ and $\psi_x\in C_F[a,b]$. By Taylor's formula, we get

$$f(y)=f(x)+(y-x)f^{'}(x)+\frac{1}{2}(y-x)^2f^{''}(x)+(y-x)^2\psi_x(y). \tag{5.1}$$

Knowing that

$$D_n(1,x)=\frac{(1+b_n)}{n^2}; D_n((y-x),x)=0 \quad \text{and} \quad D_n((y-x)^2,x)=(1+b_n)\frac{x-x^2}{n^3},$$

and after operating in the both sides of relation (5.1) by operator D_n, we obtain:

$$n^2D_n(f)=f(x)+b_nf(x)+\frac{f^{''}(x)}{2}\frac{x-x^2}{n}(1+b_n)+(1+b_n)D_n(\Phi^2\psi_x,x),$$

which yields

$$\left|n\left[n^2D_n(f)-f(x)\right]-\frac{f^{''}(x)}{2}(x-x^2)\right|\leq$$

$$nb_n|f(x)|+b_n\left|\frac{f^{''}(x)}{2}(x-x^2)\right|+n(1+b_n)\left|D_n(\Phi^2\psi_x,x)\right|,$$

respectively

$$\left|n\left[n^2D_n(f)-f(x)\right]-\frac{f^{''}(x)}{2}(x-x^2)\right|\leq nb_nM+n\left|D_n(\Phi^2\psi_x,x)\right|+nb_n\left|D_n(\Phi^2\psi_x,x)\right|, \tag{5.2}$$

where $\Phi(y)=y-x$ and $M=||f||_{C_F[a,b]}+||f^{''}||_{C_F[a,b]}$. After application of the Cauchy-Schwartz inequality in the terms of the right side of the relation (5.2), we obtain:

$$n\left|D_n(\Phi^2\psi_x,x)\right|\leq[n^2D_n(\Phi^4,x)]^{\frac{1}{2}}\cdot[D_n(\psi_x^2,x)]^{\frac{1}{2}}. \tag{5.3}$$

Putting $\eta_x(y)=(\psi_x(y))^2$, we get that $\eta_x(x)=0$ and $\eta_x(\cdot)\in C_F[a,b]$. Also

$$a_n\left|D_n(\Phi^2\psi_x,x)\right|\leq a_n[D_n(\Phi^4,x)]^{\frac{1}{2}}\cdot[D_n(\psi_x^2,x)]^{\frac{1}{2}}, \tag{5.4}$$

where $a_n\to 0((C,2)_n-stat)$.

Now from Theorem 3.3, it follows that

$$D_n(\eta_x)\to 0((C,2)_n-stat), \tag{5.5}$$

on [a,b]. Now, from relations (5.3), (5.5), (5.4) and Lemma 5.1, we have

$$n(1+b_n)D_n(\Phi^2\psi_x, x) \to 0((C,2)_n - stat), \tag{5.6}$$

on $[a,b]$. For a given $\varepsilon > 0$, we define the following sets:

$$A_n(x,\varepsilon) = \left|\left\{k : k \leq (n+1)(n+2) : \left|n\left[n^2 D_n(f) - f(x)\right] - \frac{f''(x)}{2}(x-x^2)\right| \geq \varepsilon\right\}\right|,$$

$$A_{1,n}(x,\varepsilon) = \left|\left\{k : k \leq (n+1)(n+2) : |kb_k| \geq \frac{\varepsilon}{2M}\right\}\right|,$$

and

$$A_{2,n}(x,\varepsilon) = \left|\left\{k : k \leq (n+1)(n+2) : |k(1+b_k)D_k(\Phi^2\psi_x, x)| \geq \frac{\varepsilon}{2}\right\}\right|.$$

From last relation we have

$$\frac{A_n(x,\varepsilon)}{(n+1)(n+2)a_n} \leq \frac{A_{1,n}(x,\varepsilon)}{(n+1)(n+2)a_n} + \frac{A_{2,n}(x,\varepsilon)}{(n+1)(n+2)a_n}. \tag{5.7}$$

From definition of the sequence (b_n), we get

$$nb_n \to 0((C,2)_n - stat), \tag{5.8}$$

on [a,b]. Now from relations (5.6) and (5.8), the right hand side of the relation (5.7), tends to zero as $n \to \infty$. Therefore, we have

$$\lim_{n\to\infty} \frac{A_n(x,\varepsilon)}{(n+1)(n+2)a_n} = 0,$$

which proves that

$$n\left[n^2 D_n(f) - f(x)\right] \sim \frac{1}{2}(x-x^2)f''(x)((C,2)_n - stat),$$

on $[a,b]$. □

6. Concluding remarks

In this section, we will give some remarks related to the results obtained in this paper and their relationship with other results.

Remark 6.1. Suppose that we replace the conditions (1) and (2) in Theorem 4.4 by the following condition:

$$\tilde{L_n}(x_i) - x_i = o(a_{n_i})((C,2)_n - stat) \quad \text{on} \quad [a,b](i=0,1,2). \tag{6.1}$$

Then, since

$$\tilde{L_n}(\psi^2; x) = \tilde{L_n}(t^2, x) - 2x\tilde{L_n}(t,x) + x^2\tilde{L_n}(1,x),$$

we may write

$$\tilde{L_n}(\psi^2, x) \leq K[|\tilde{L_n}(1,x) - 1| + |\tilde{L_n}(t,x) - t| + |\tilde{L_n}(t^2,x) - t^2|],$$

where

$$K = 1 + 2||t|| + ||t^2||.$$

Now it follows from above relations and Lemma 4.2 that

$$\delta_n = \sqrt{\tilde{L_n}(\psi^2)} = o(d_n)((C,2)_n - stat)$$

on $[a,b]$, where $d_n = \min\{a_{n_0}, a_{n_1}, a_{n_2}\}$. Hence

$$\omega(f, d_n) = o(d_n)((C,2)_n - stat)$$

on $[a,b]$. If those conditions which are given here we can use in Theorem 3.3, we can thus see that, for all $f \in C_F[a,b]$,

$$\tilde{L_n}(f) - f = o(d_n)((C,2)_n - stat)$$

on $[a,b]$. Therefore, if we use the condition (6.1) in Theorem 4.4 instead of the conditions (1) and (2), then we obtain the rates of $(C,2)_n - stat$ convergent of the sequence of positive linear operators in Theorem 3.3.

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Approximation by a generalization of Szász-Mirakjan type operators

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Abstract. In the present paper we propose a new generalization of Szász-Mirakjan-type operators. We discuss their weighted convergence and rate of convergence via weighted modulus of continuity. We also give an asymptotic estimate through Voronovskaja type result for these operators.

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1. Introduction

In [7] Rempulska et al. introduced the following operators of Szász-Mirakjan type

$$L_n(f;x) = \sum_{k=0}^{\infty} p_{n,k}(x) f\left(\frac{2k}{n}\right), \tag{1.1}$$

with

$$p_{n,k}(x) = \frac{1}{\cosh nx} \frac{(nx)^{2k}}{(2k)!}, \ k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}, \tag{1.2}$$

where $f \in C_B$ and C_B is the space of real-valued functions uniformly continuous and bounded on $\mathbb{R}^+ = [0, \ \infty)$ and the norm in C_B is given as

$$\|f\| = \sup_{x \in \mathbb{R}^+} |f(x)|.$$

In [8, 9] a Voronovskaja-type theorem was proved for these operators.

In 2014, Aral et al. [1] introduced a very interesting generalization of the Szász-Mirakjan operators [10] using a function ρ as

$$\begin{aligned} S_n^{\rho}(f;x) =& e^{-n\rho(x)} \sum_{k=0}^{\infty} \left(f \circ \rho^{-1}\right) \left(\frac{k}{n}\right) \frac{(n\rho(x))^k}{k!} \\ =& (S_n(f \circ \rho^{-1}) \circ \rho)(x) \\ =& e^{-n\rho(x)} \sum_{k=0}^{\infty} f\left(\rho^{-1}\left(\frac{k}{n}\right)\right) \frac{(n\rho(x))^k}{k!}, \end{aligned} \tag{1.3}$$

where the function ρ satisfies following properties:
(ρ_1) ρ is continuously differentiable on $\mathbb{R}^+$,
(ρ_2) $\rho(0) = 0$, $\inf\limits_{x\in\mathbb{R}^+} \rho'(x) \geq 1$.

We propose a similar generalization of the operators (1.1) as follows

$$L_n^{\rho}(f;x) = \frac{1}{\cosh\left(n\rho(x)\right)} \sum_{k=0}^{\infty} (f \circ \rho^{-1}) \left(\frac{2k}{n}\right) \frac{(n\rho(x))^{2k}}{(2k)!}, \tag{1.4}$$

where $x \in \mathbb{R}^+$, $n \in \mathbb{N}$, $k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and function ρ satisfies conditions (ρ_1) and (ρ_2).

We see that these new operators are positive linear operators. For $\rho(x) = x$, these operators (1.4) reduce to the operators (1.1). Also from conditions (ρ_1) and (ρ_2) we can draw out that
(i) $\lim\limits_{x\in\mathbb{R}^+} \rho(x) = \infty$,
(ii) $|t - x| \leq |\rho(t) - \rho(x)|$ for all x, $t \in \mathbb{R}^+$.

In this paper we study some approximation properties of these new operators. Firstly we prove a theorem for the weighted convergence of $L_n^{\rho} f$ to f with the help of a weighted Korovkin-type theorem [4], [3]. Then we determine an estimate of the rate of the weighted convergence using weighted modulus of continuity defined in [5]. At the end we prove a Voronovskaja type result for these new operators.

2. Weighted convergence of $L_n^{\rho}(f;x)$

From the definition of the operators L_n^{ρ} one can easily derive the following results.

Lemma 2.1. *For the operators defined in* (1.4) *we have*

$$L_n^{\rho}(1;x) = 1, \tag{2.1}$$

$$L_n^{\rho}(\rho;x) = \rho(x)\tanh(n\rho(x)), \tag{2.2}$$

$$L_n^{\rho}(\rho^2;x) = \rho^2(x) + \frac{\rho(x)}{n}\tanh(n\rho(x)), \tag{2.3}$$

$$L_n^{\rho}(\rho^3;x) = \rho^3(x)\tanh(n\rho(x)) + \frac{3\rho^2(x)}{n} + \frac{\rho(x)}{n}\tanh(n\rho(x)), \tag{2.4}$$

$$L_n^{\rho}(\rho^4;x) = \rho^4(x) + \frac{6\rho^3(x)}{n}\tanh(n\rho(x)) + 7\frac{\rho^2(x)}{n^2} + \frac{\rho(x)}{n^3}\tanh(n\rho(x)). \tag{2.5}$$

Lemma 2.2. *For the operators defined in* (1.4) *we have*

$$\begin{aligned} L_n^\rho(\rho(t)-\rho(x);x) =& \rho(x)(\tanh(n\rho(x))-1),\\ L_n^\rho((\rho(t)-\rho(x))^2;x) =& \left(2\rho^2(x)-\frac{\rho(x)}{n}\right)(1-\tanh(n\rho(x)))+\frac{\rho(x)}{n},\\ L_n^\rho((\rho(t)-\rho(x))^4;x) =& \left(8\rho^4(x)-\frac{12\rho^3(x)}{n}+\frac{4\rho^2(x)}{n^2}-\frac{\rho(x)}{n^3}\right)\\ &\times(1-\tanh(n\rho(x)))+\frac{3\rho^2(x)}{n^2}+\frac{\rho(x)}{n^3}. \end{aligned}$$

Now we give a very useful lemma.

Lemma 2.3. *For the operators defined in* (1.4) *we have*

$$\begin{aligned} \lim_{n\to\infty} nL_n^\rho(\rho(t)-\rho(x);x) &= 0,\\ \lim_{n\to\infty} nL_n^\rho((\rho(t)-\rho(x))^2;x) &= \rho(x). \end{aligned}$$

Proof. From Lemma 2.2

$$\begin{aligned} nL_n^\rho(\rho(t)-\rho(x);x) =& n\rho(x)(\tanh(n\rho(x))-1)\\ =& \frac{-2n\rho(x)}{e^{2n\rho(x)}+1}. \end{aligned}$$

Thus

$$\lim_{n\to\infty} nL_n^\rho(\rho(t)-\rho(x);x) = 0.$$

Again from Lemma 2.2

$$\begin{aligned} nL_n^\rho((\rho(t)-\rho(x))^2;x) &= \left(2\rho(x)-\frac{1}{n}\right)n\rho(x)(1-\tanh(n\rho(x)))+\rho(x)\\ &= \left(2\rho(x)-\frac{1}{n}\right)\left(\frac{2n\rho(x)}{e^{2n\rho(x)}+1}\right)+\rho(x). \end{aligned}$$

Thus we have

$$\lim_{n\to\infty} nL_n^\rho((\rho(t)-\rho(x))^2;x) = \rho(x). \qquad \square$$

We prove the convergence theorem using weighted Korovkin type theorem. Korovkin's theorem [6] was extended to unbounded intervals and a weighted Korovkin type theorem in a subspace of continuous functions on the real axis $\mathbb{R}$ was proved in [4], [3]. It was shown that the test functions 1, x, x^2 of original Korovkin's theorem can be replaced by 1, ρ, ρ^2 under certain additional conditions on the function ρ. We recall some notations and results given in [1], [4], [3].

Let $\varphi(x) = 1+\rho^2(x)$, where ρ satisfies conditions (ρ_1) and (ρ_2). Thus we see that ρ is continuous and strictly increasing function on positive real axis. We will consider following weighted space:

$$B_\varphi(\mathbb{R}^+) = \{f : \mathbb{R}^+ \to \mathbb{R} :\ |f(x)| \leq M_f \varphi(x),\ x \in \mathbb{R}^+\},$$

where M_f is positive constant depending only on f. $B_\varphi(\mathbb{R}^+)$ is a normed space with the norm

$$||f||_\varphi = \sup_{x\in\mathbb{R}^+} \frac{|f(x)|}{\varphi(x)}.$$

We denote the subspace of all continuous function in $B_\varphi(\mathbb{R}^+)$ by $C_\varphi(\mathbb{R}^+)$. $C^k_\varphi(\mathbb{R}^+)$ denotes the subspace of all functions $f \in C_\varphi(\mathbb{R}^+)$ with the property

$$\lim_{x\to\infty} \frac{|f(x)|}{\varphi(x)} = k_f,$$

where k_f is a constant depending on f. $U_\varphi(\mathbb{R}^+)$ be the subspace of all functions f in $C_\varphi(\mathbb{R}^+)$ such that $\frac{f(x)}{\varphi(x)}$ is uniformly continuous. Then obviously

$$C^k_\varphi(\mathbb{R}^+) \subset U_\varphi(\mathbb{R}^+) \subset C_\varphi(\mathbb{R}^+) \subset B_\varphi(\mathbb{R}^+).$$

Lemma 2.4 ([4, 3])**.** *The linear positive operators L_n, $n \geq 1$, act from $C_\varphi(\mathbb{R}^+)$ to $B_\varphi(\mathbb{R}^+)$ if and only if*

$$|L_n(\varphi; x)| \leq K\varphi(x),$$

where $x \in \mathbb{R}^+$, $\varphi(x)$ is the weight function and K is a positive constant.

Theorem 2.5 ([4, 3])**.** *Let $(L_n)_{n\geq1}$ be the sequence of positive linear operators which act from $C_\varphi(\mathbb{R}^+)$ to $B_\varphi(\mathbb{R}^+)$ satisfying the conditions*

$$\lim_{n\to\infty} ||L_n(\rho^i) - \rho^i||_\varphi = 0,\ i = 0, 1, 2.$$

then for any function $f \in C^k_\varphi(\mathbb{R}^+)$

$$\lim_{n\to\infty} ||L_n(f) - f||_\varphi = 0.$$

Lemma 2.6. *The linear positive operators L^ρ_n, $n \in \mathbb{N}$, act from $C_\varphi(\mathbb{R}^+)$ to $B_\varphi(\mathbb{R}^+)$, where $\varphi(x) = 1 + \rho^2(x)$ is the weight function.*

Proof. In view of (2.1) and (2.3) we see that operators L^ρ_n, $n \in \mathbb{N}$ satisfy the condition of the Lemma 2.4. Thus the result follows. □

In [8] the following inequality was proved

$$0 \leq x^r(1 - tanh(nx)) \leq 2^{1-r}r!n^{-r},\ n, r \in N \ and\ x \geq 0.$$

Similarly for $\rho(x)$ satisfying ρ_1 and ρ_2 and $n, r \in N$ we can get the following inequality

$$0 \leq \rho^r(x)(1 - tanh(n\rho(x)) \leq 2^{1-r}r!n^{-r}. \tag{2.6}$$

Now we prove the convergence theorem for the operators (L^ρ_n).

Theorem 2.7. *Let $(L_n^\rho)_{n\in\mathbb{N}}$ be the sequence of linear positive operators defined by* (1.4)*. Then for any $f \in C_\varphi^k(\mathbb{R}^+)$ we have*

$$\lim_{n\to\infty} ||L_n^\rho(f) - f||_\varphi = 0.$$

Proof. Using Theorem 2.5 we see that in order to prove the theorem, it is sufficient to prove the following three conditions

$$\lim_{n\to\infty} ||L_n^\rho(\rho^v) - \rho^v||_\varphi = 0, \ v = 0, 1, 2.$$

Now from (2.1) we have

$$\lim_{n\to\infty} ||L_n^\rho(1) - 1||_\varphi = 0.$$

From (2.2) we get

$$||L_n^\rho(\rho) - \rho||_\varphi \le \sup_{x\in\mathbb{R}^+} \frac{\rho(x)}{1+\rho^2(x)}(1 - \tanh(n\rho(x)),$$

so using (2.6) for $r = 1$ we have

$$||L_n^\rho(\rho) - \rho||_\varphi \le \frac{1}{n}.$$

This leads to

$$\lim_{n\to\infty} ||L_n^\rho(\rho) - \rho||_\varphi = 0.$$

Again from (2.3)

$$\begin{aligned} L_n^\rho(\rho^2) - \rho^2 =& \frac{\rho(x)}{n}\tanh(n\rho(x)) \\ =& \frac{\rho(x)}{n} - \frac{\rho(x)}{n}(1 - \tanh(n\rho(x))), \end{aligned}$$

thus

$$||L_n^\rho(\rho^2) - \rho^2||_\varphi \le \sup_{x\in\mathbb{R}^+} \left[\frac{\rho(x)}{n(1+\rho^2(x))} + \frac{\rho(x)}{n(1+\rho^2(x))}(1 - \tanh(n\rho(x)))\right]$$

and using (2.6) we get

$$||L_n^\rho(\rho^2) - \rho^2||_\varphi \le \left[\frac{1}{n} + \frac{1}{n}\right] = \frac{2}{n}. \tag{2.7}$$

So we have

$$\lim_{n\to\infty} ||L_n^\rho(\rho^2) - \rho^2||_\varphi = 0.$$

This completes the proof. □

3. Rate of convergence via weighted modulus of continuity

In this section we compute the rate of convergence of the operators defined in (1.4) in terms of weighted modulus of continuity. In [5] Holhoş defined for all $f \in C_\varphi(\mathbb{R}^+)$ and for every $\delta \geq 0$, the weighted modulus of continuity as

$$\omega_\rho(f,\delta) = \sup_{x\in\mathbb{R}^+,\ |\rho(t)-\rho(x)|\leq\delta} \frac{|f(t)-f(x)|}{\varphi(t)+\varphi(x)}.$$

We see that $\omega_\rho(f,0)=0$ for all $f \in C_\varphi(\mathbb{R}^+)$ and also that $\omega_\rho(f,\delta)$ is a nonnegative and nondecreasing function with respect to δ. The properties of weighted modulus of continuity were discussed in [5]. The following results were given by Holhos [5].

Lemma 3.1 ([5]). *For every $f \in U_\varphi(\mathbb{R}^+)$, $\lim_{\delta\to 0}\omega_\rho(f,\delta)=0$.*

Theorem 3.2 ([5]). *Let $(L_n)_{n\geq 1}$ be a sequence of linear positive operators acting from $C_\varphi(\mathbb{R}^+)$ to $B_\varphi(\mathbb{R}^+)$ with*

$$||L_n(\rho^0)-\rho^0||_{\varphi^0} = a_n,$$
$$||L_n(\rho)-\rho||_{\varphi^{\frac{1}{2}}} = b_n,$$
$$||L_n(\rho^2)-\rho^2||_{\varphi} = c_n,$$
$$||L_n(\rho^3)-\rho^3||_{\varphi^{\frac{3}{2}}} = d_n,$$

where a_n, b_n, c_n and d_n tend to zero as n goes to infinity. Then

$$||L_n(f)-f||_{\varphi^{\frac{3}{2}}} \leq (7+4a_n+2c_n)\omega_\rho(f,\delta_n)+||f||_\varphi a_n$$

for all $f \in C_\varphi(\mathbb{R}^+)$, where

$$\delta_n = 2\sqrt{(a_n+2b_n+c_n)(1+a_n)}+a_n+3b_n+3c_n+d_n.$$

Theorem 3.3. *For all $f \in C_\varphi(\mathbb{R}^+)$, we have*

$$||L_n^\rho(f)-f||_{\varphi^{\frac{3}{2}}} \leq \left(7+\frac{4}{n}\right)\omega_\rho(f,\delta_n),$$

where

$$\delta_n = \frac{4}{\sqrt{n}}+\frac{15}{n}.$$

Proof. From (2.1) and (2.2) we see that

$$||L_n(\rho^0)-\rho^0||_{\varphi^0} = a_n = 0,$$

$$b_n = ||L_n(\rho)-\rho||_{\varphi^{\frac{1}{2}}} \leq \sup_{x\in\mathbb{R}^+}\frac{\rho(x)}{\sqrt{1+\rho^2(x)}}(1-\tanh(n\rho(x))$$

and using (2.6) we get

$$b_n = ||L_n(\rho)-\rho||_{\varphi^{\frac{1}{2}}} \leq \frac{1}{n}.$$

From (2.7) we have

$$c_n = ||L_n(\rho^2)-\rho^2||_\varphi \leq \frac{2}{n}.$$

Again from (2.4) we obtain

$$\begin{aligned} d_n =&||L_n(\rho^3)-\rho^3||_{\varphi^{\frac{3}{2}}} \\ =& \sup_{x\in\mathbb{R}^+}\frac{1}{(1+\rho^2(x))^{\frac{3}{2}}}\Big|\rho^3(x)\tanh(n\rho(x))-\rho^3(x)+\frac{3\rho^2(x)}{n} \\ &+\frac{\rho(x)}{n^2}\tanh(n\rho(x))\Big| \\ \le& \sup_{x\in\mathbb{R}^+}\frac{1}{(1+\rho^2(x))^{\frac{3}{2}}}\Big|\rho^3(x)(1-\tanh(n\rho(x)))+\frac{\rho(x)}{n^2}(1-\tanh(n\rho(x))) \\ &+\frac{3\rho^2(x)}{n}+\frac{\rho(x)}{n^2}\Big| \\ \le&\frac{1}{n}+\frac{1}{n^2}+\frac{3}{n}+\frac{1}{n^2}. \end{aligned}$$

Using (2.6) and by the fact that $\frac{1}{n^2}\le\frac{1}{n}$ we obtain

$$d_n=||L_n(\rho^3)-\rho^3||_{\varphi^{\frac{3}{2}}}\le\frac{6}{n}.$$

Thus we see that a_n, b_n, c_n and d_n tend to zero as n goes to infinity. So on applying Theorem 3.2, we get

$$||L_n^\rho(f)-f||_{\varphi^{\frac{3}{2}}}\le\Big(7+\frac{4}{n}\Big)\omega_\rho(f,\delta_n),$$

where

$$\delta_n=\frac{4}{\sqrt{n}}+\frac{15}{n}.$$

This completes the proof. □

Remark 3.4. We see from Theorem 3.3 that as $n\to\infty$, $\delta_n\to 0$. Thus, using Lemma 3.1, we have

$$\lim_{n\to\infty}||L_n^\rho(f)-f||_{\varphi^{\frac{3}{2}}}=0$$

for every $f\in U_\varphi(\mathbb{R}^+)$.

4. Voronovskaja type theorem

Now we give a Voronovskaja-type result using the technique of Cárdenas-Morales et al. [2].

Theorem 4.1. *Let $f\in C_\varphi(\mathbb{R}^+)$, $x\in\mathbb{R}^+$ and suppose that the first and second derivatives of $f\circ\rho^{-1}$ exist at $\rho(x)$. If the second derivative of $f\circ\rho^{-1}$ is bounded on $\mathbb{R}^+$, then we have*

$$\lim_{n\to\infty}n[L_n^\rho(f;x)-f(x)]=\frac{\rho(x)}{2}(f\circ\rho^{-1})''(\rho(x)).$$

Proof. By the Taylor expansion of $f\circ\rho^{-1}$ at the point $\rho(x)\in\mathbb{R}^+$, there exists ξ lying between x and t such that

$$\begin{aligned}f(t) =&(f\circ\rho^{-1})(\rho(t)) = (f\circ\rho^{-1})(\rho(x)) + (f\circ\rho^{-1})'(\rho(x))(\rho(t)-\rho(x))\\ &+\frac{1}{2}(f\circ\rho^{-1})''(\rho(x))(\rho(t)-\rho(x))^2 + h(t;x)(\rho(t)-\rho(x))^2,\end{aligned}$$

where

$$h(t;x) = \frac{(f\circ\rho^{-1})''(\rho(\xi)) - (f\circ\rho^{-1})''(\rho(x))}{2}. \tag{4.1}$$

On applying the operator (1.4)

$$\begin{aligned}&n[L_n^\rho(f;x) - f(x)]\\ &\quad=(f\circ\rho^{-1})'(\rho(x))nL_n^\rho(\rho(t)-\rho(x);x) + \frac{1}{2}(f\circ\rho^{-1})''(\rho(x))\\ &\qquad\times nL_n^\rho((\rho(t)-\rho(x))^2;x) + nL_n^\rho(h(t;x)(\rho(t)-\rho(x))^2;x).\end{aligned} \tag{4.2}$$

Now using Lemma 2.3 in (4.2) we get

$$\begin{aligned}&\lim_{n\to\infty} n[L_n^\rho(f;x) - f(x)]\\ &\quad=\frac{\rho(x)}{2}(f\circ\rho^{-1})''(\rho(x)) + \lim_{n\to\infty} nL_n^\rho(h(t;x)(\rho(t)-\rho(x))^2;x).\end{aligned} \tag{4.3}$$

From the hypothesis of the theorem we have $|h(t;x)|\le M$ and

$$\lim_{t\to x} h(t;x) = 0.$$

Thus, for any $\varepsilon>0$ there exist a $\delta>0$ such that

$$|h(t;x)| < \varepsilon \ for\ |t-x|<\delta.$$

But from the condition (ρ_2) we have

$$|t-x| \le |\rho(t)-\rho(x)|.$$

Therefore, if $|\rho(t)-\rho(x)|<\delta$, then

$$|h(t;x)(\rho(t)-\rho(x))^2| < \varepsilon(\rho(t)-\rho(x))^2$$

and if

$$|\rho(t)-\rho(x)|\ge\delta,$$

then

$$|h(t;x)(\rho(t)-\rho(x))^2| < \frac{M}{\delta^2}(\rho(t)-\rho(x))^4.$$

Hence

$$\begin{aligned}&L_n^\rho(h(t;x)(\rho(t)-\rho(x))^2;x)\\ &\qquad< \varepsilon L_n^\rho((\rho(t)-\rho(x))^2;x) + \frac{M}{\delta^2}L_n^\rho((\rho(t)-\rho(x))^4;x).\end{aligned}$$

From Lemma 2.2 we see that

$$L_n^\rho((\rho(t)-\rho(x))^4;x) = O\Big(\frac{1}{n^2}\Big).$$

Thus we get

$$\lim_{n\to\infty} nL_n^{\rho}(h(t;x)(\rho(t)-\rho(x))^2;x)=0.$$

On applying this to (4.3) we get the desired result. □

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Perturbations of local C-cosine functions

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Abstract. We show that $A+B$ is a closed subgenerator of a local C-cosine function $T(\cdot)$ on a complex Banach space X defined by

$$T(t)x = \sum_{n=0}^{\infty} B^n \int_0^t j_{n-1}(s) j_n(t-s) C(|t-2s|)x ds$$

for all $x \in X$ and $0 \leq t < T_0$, if A is a closed subgenerator of a local C-cosine function $C(\cdot)$ on X and one of the following cases holds: (i) $C(\cdot)$ is exponentially bounded, and B is a bounded linear operator on $\overline{D(A)}$ so that $BC = CB$ on $\overline{D(A)}$ and $BA \subset AB$; (ii) B is a bounded linear operator on $\overline{D(A)}$ which commutes with $C(\cdot)$ on $\overline{D(A)}$ and $BA \subset AB$; (iii) B is a bounded linear operator on X which commutes with $C(\cdot)$ on X. Here $j_n(t) = \frac{t^n}{n!}$ for all $t \in \mathbb{R}$, and

$$\int_0^t j_{-1}(s) j_0(t-s) C(|t-2s|)x ds = C(t)x$$

for all $x \in X$ and $0 \leq t < T_0$.

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1. Introduction

Let X be a complex Banach space with norm $\|\cdot\|$, and let $L(X)$ denote the set of all bounded linear operators on X. For each $0 < T_0 \leq \infty$ and each injection $C \in L(X)$, a family $C(\cdot)$ $(= \{C(t) \,|\, 0 \leq t < T_0\})$ in $L(X)$ is called a local C-cosine function on X if it is strongly continuous, $C(0) = C$ on X and satisfies

$$2C(t)C(s) = C(t+s)C + C(|t-s|)C \tag{1.1}$$

on X for all $0 \leq t, s, t+s < T_0$ (see [5], [7], [14], [15], [21], [23], [25]). In this case, the generator of $C(\cdot)$ is a closed linear operator A in X defined by

$$D(A) = \{x \in X \mid \lim_{h \to 0^+} 2(C(h)x - Cx)/h^2 \in R(C)\}$$

and $Ax = C^{-1}\lim_{h\to 0^+} 2(C(h)x - Cx)/h^2$ for $x \in D(A)$. Moreover, we say that $C(\cdot)$ is locally Lipschitz continuous, if for each $0 < t_0 < T_0$ there exists a $K_{t_0} > 0$ such that

$$\|C(t+h) - C(t)\| \le K_{t_0}h \tag{1.2}$$

for all $0 \le t, h, t+h \le t_0$; exponentially bounded, if $T_0 = \infty$and there exist $K, \omega \ge 0$ such that

$$\|C(t)\| \le Ke^{\omega t} \tag{1.3}$$

for all $t \ge 0$; exponentially Lipschitz continuous, if $T_0 = \infty$ and there exist $K, \omega \ge 0$ such that

$$\|C(t+h) - C(t)\| \le Khe^{\omega(t+h)} \tag{1.4}$$

for all $t, h \ge 0$. In general, a local C-cosine function is also called a C-cosine function if $T_0 = \infty$ (see [2], [12], [14], [16]) or a cosine function if $C = I$ (identity operator on X) (see [1], [4], [6]), and a C-cosine function may not be exponentially bounded (see [16]). Moreover, a local C-cosine function is not necessarily extendable to the half line $[0, \infty)$ (see [21]) except for $C = I$ (see [1], [4], [6]) and the generator of a C-cosine function may not be densely defined (see [2]). Perturbations of local C-cosine functions have been extensively studied by many authors appearing in [1], [2], [4], [9], [11], [17], [18], [19]. Some interesting applications of this topic are also illustrated there. In particular, a classical perturbation result of cosine functions shows that if A is the generator of a C-cosine function $C(\cdot)$ on X, and B a bounded linear operator on X, then $A + B$ is the generator of a C-cosine function on X when $C = I$, but the conclusion may not be true when C is arbitrary, and is still unknown until now even though B and $C(\cdot)$ are commutable, which can be completely solved in this paper and several new additive perturbation theorems concerning local C-cosine functions are also established as results in [20] for the case of C-semigroup and in [8], [13] for the case of local C-semigroup. A new representation of the perturbation of a local C-cosine function is given in (1.5) below. We show that if $C(\cdot)$ is an exponentially bounded C-cosine function on X with closed subgenerator A and B a bounded linear operator on $\overline{D(A)}$ such that $BC = CB$ on $\overline{D(A)}$ and $BA \subset AB$, then $A + B$ is a closed subgenerator of an exponentially bounded C-cosine function $T(\cdot)$ on X defined by

$$T(t)x = \sum_{n=0}^{\infty} B^n \int_0^t j_{n-1}(s)j_n(t-s)C(|t-2s|)xds \tag{1.5}$$

for all $x \in X$ and $0 \le t < T_0$ (see Theorem 2.6 below). Here $j_n(t) = \frac{t^n}{n!}$ for all $t \in \mathbb{R}$, and

$$\int_0^t j_{-1}(s)j_0(t-s)C(|t-2s|)xds = C(t)x$$

for all $x \in X$ and $0 \le t < T_0$. Moreover, $T(\cdot)$ is also exponentially Lipschitz continuous or norm continuous if $C(\cdot)$ is. We then show that the exponential boundedness of $T(\cdot)$ can be deleted and C-cosine functions can be extended to the context of local C-cosine functions when the assumption of $BC(\cdot) = C(\cdot)B$ on $\overline{D(A)}$ is added (see Theorem 2.7 below). Moreover, $T(\cdot)$ is locally Lipschitz continuous or norm continuous if $C(\cdot)$ is. We also show that $A + B$ is a closed subgenerator of a local C-cosine function $T(\cdot)$ on X if A is a closed subgenerator of a local C-cosine function $C(\cdot)$ on X and

B a bounded linear operator on X such that $BC(\cdot) = C(\cdot)B$ on X (see Theorem 2.8 below). A simple illustrative example of these results is presented in the final part of this paper.

2. Perturbation theorems

In this section, we first note some basic properties of a local C-cosine function with its subgenerator and generator.

Definition 2.1. (see [10], [14]) Let $C(\cdot)$ be a strongly continuous family in $\mathrm{L}(X)$. A linear operator A in X is called a subgenerator of $C(\cdot)$ if

$$C(t)x - Cx = \int_0^t \int_0^s C(r)Axdrds$$

for all $x \in \mathrm{D}(A)$ and $0 \le t < T_0$, and

$$\int_0^t \int_0^s C(r)xdrds \in \mathrm{D}(A) \quad \text{and } A\int_0^t \int_0^s C(r)xdrds = C(t)x - Cx$$

for all $x \in X$ and $0 \le t < T_0$. A subgenerator A of $C(\cdot)$ is called the maximal subgenerator of $C(\cdot)$ if it is an extension of each subgenerator of $C(\cdot)$ to $D(A)$.

Proposition 2.2. (see [4], [5], [10], [14], [21]) *Let A be the generator of a local C-cosine function $C(\cdot)$ on X. Then*

$$C(t)x \in D(A) \quad \text{and } C(t)Ax = AC(t)x \tag{2.1}$$

for all $x \in D(A)$ and $0 \le t < T_0$;

$$C^{-1}AC = A \quad \text{and } R(C(t)) \subset \overline{D(A)} \tag{2.2}$$

for all $0 \le t < T_0$;

$$x \in D(A) \text{ and } Ax = y_x \text{ if and only if } C(t)x - Cx = \int_0^t \int_0^s C(r)y_x drds \tag{2.3}$$

for all $0 \le t < T_0$;

$$A_0 \text{ is closable and } C^{-1}\overline{A_0}C = A \tag{2.4}$$

for each subgenerator A_0 of $C(\cdot)$;

$$A \text{ is the maximal subgenerator of } C(\cdot). \tag{2.5}$$

From now on, we always assume that $A : D(A) \subset X \to X$ is a closed linear operator so that $CA \subset AC$.

Theorem 2.3. (see [10], [16]) *A strongly continuous family $C(\cdot)$ in $L(X)$ satisfying (1.3) is a C-cosine function on X with subgenerator A if and only if $CC(\cdot) = C(\cdot)C$, $\lambda^2 \in \rho_C(A)$, and $\lambda(\lambda^2 - A)^{-1}C = L_\lambda$ on X for all $\lambda > \omega$. Here*

$$L_\lambda x = \int_0^\infty e^{-\lambda t} C(t)xdt \text{ for } x \in X.$$

Lemma 2.4. (see [1]) *Let* $C(\cdot)(=\{C(t)\,|\,0\le t<T_0\})$ *be a strongly continuous family in* $L(X)$*. We set* $C(-t)=C(t)$ *for* $0\le t<T_0$*. Then* $C(\cdot)$ *is a local* C*-cosine function on* X *if and only if* $2C(t)C(s)=C(t+s)C+C(t-s)C$ *on* X *for all* $|t|,|s|,|t-s|,|t+s|<T_0$*. In this case,*

$$S(-t)=-S(t) \tag{2.6}$$

for all $0\le t<T_0$*;*

$$S(t+s)C=S(t)C(s)+C(t)S(s) \text{ on } X \tag{2.7}$$

for all $|t|,|s|,|t+s|<T_0$*. Here* $S(t)=j_0*C(t)$ *for all* $|t|<T_0$*.*

By slightly modifying the proof of [3, Lemma 2], the next lemma is also attained.

Lemma 2.5. *Let* $C(\cdot)(=\{C(t)\,|\,0\le t<T_0\})$ *be a local* C*-cosine function on* X*, and* $C(-t)=C(t)$ *for* $0\le t<T_0$*. Assume that* S^{*n+1} *denotes the* $(n+1)$*-fold convolution of* S *for* $n\in\mathbb{N}\cup\{0\}$*, that is*

$$S^{*2}(t)x=\int_0^t S(t-s)S(s)xds$$

and

$$S^{*n+1}(t)x=\int_0^t S^{*n}(t-s)S(s)xds.$$

Then

$$S^{*n+1}(t)=\int_0^t j_{n-1}(s)j_n(t-s)S(t-2s)C^n ds=\int_0^t j_n(s)j_n(t-s)C(t-2s)C^n ds$$

on X *for all* $|t|<T_0$*. Here* $S(t)=j_0*C(t)$ *and*

$$\int_0^t j_{-1}(s)j_0(t-s)S(t-2s)C^0 ds=S(t)=S^{*1}(t)$$

for all $|t|<T_0$*.*

Proof. It is easy to see that

$$\begin{aligned}S^{*n+1}(t)&=\int_0^t j_{n-1}(s)j_n(t-s)S(t-2s)C^n ds\\&=\int_0^t j_n(s)j_n(t-s)C(t-2s)C^n ds\end{aligned}$$

on X for $n = 0$. By induction, we have

$$\begin{aligned}
S^{*n+1}(t)x &= \int_0^t S^{*n}(s)S(t-s)xds \\
&= \int_0^t \int_0^s j_{n-1}(r)j_{n-1}(s-r)C(s-2r)C^{n-1}S(t-s)xdrds \\
&= \frac{1}{2}\int_0^t \int_0^s j_{n-1}(r)j_{n-1}(s-r)\big[S(t-2r)+S(t+2r-2s)\big]C^n xdrds \\
&= \int_0^t \int_0^s j_{n-1}(r)j_{n-1}(s-r)S(t-2r)C^n xdrds \\
&= \int_0^t \int_r^t j_{n-1}(r)j_{n-1}(s-r)S(t-2r)C^n xdsdr \\
&= \int_0^t j_{n-1}(r)j_n(t-r)S(t-2r)C^n xdr \\
&= \frac{1}{2}\int_0^t \big[j_{n-1}(r)j_n(t-r)-j_n(r)j_{n-1}(t-r)\big]S(t-2r)C^n xdr \\
&= \frac{1}{2}\int_0^t \frac{d}{dr}[j_n(r)j_n(t-r)]S(t-2r)C^n xdr \\
&= \int_0^t j_n(r)j_n(t-r)C(t-2r)C^n xdr
\end{aligned}$$

for all $n \in \mathbb{N}$, $x \in X$ and $|t| < T_0$. □

Applying Theorem 2.3 we can obtain the next perturbation theorem concerning exponentially bounded C-cosine functions just as a corollary of [11, Corollary 2.6.6].

Theorem 2.6. *Let A be a subgenerator of an exponentially bounded C-cosine function $C(\cdot)$ on X. Assume that $B \in L(\overline{D(A)}), BC = CB$ on $\overline{D(A)}$ and $BA \subset AB$. Then $A + B$ is a closed subgenerator of an exponentially bounded C-cosine function $T(\cdot)$ on X given as in (1.5). Moreover, $T(\cdot)$ is also exponentially Lipschitz continuous or norm continuous if $C(\cdot)$ is.*

Proof. It is easy to see that

$$(\lambda^2 - A - B)^{-1}C = \sum_{n=0}^{\infty} B^n(\lambda^2 - A)^{-n-1}C$$

for $\lambda > \omega$, and the boundedness of $\{\|C(t)\| \mid 0 \le t \le t_0\}$ for each $t_0 > 0$ and the strong continuity of $C(\cdot)$ imply that the right-hand side of (1.5) converges uniformly on compact subsets of $[0, \infty)$. In particular, $T(\cdot)$ is a strongly continuous family in $L(X)$. For simplicity, we may assume that $\|C(t)\| \le Ke^{\omega t}$ for all $t \ge 0$ and for some fixed $K, \omega \geq 0$. Then $\|T(t)\| \leq Ke^{(\omega+\sqrt{\|B\|})t}$ for all $t \geq 0$, and

$$\begin{aligned}(\lambda^2 - A - B)^{-1}Cx &= \sum_{n=0}^{\infty} B^n \int_0^{\infty} e^{-\lambda t} \int_0^t j_{n-1}(s) j_n(t-s) S(t-2s) x ds dt \\ &= \int_0^{\infty} \sum_{n=0}^{\infty} B^n e^{-\lambda t} \int_0^t j_{n-1}(s) j_n(t-s) S(t-2s) x ds dt \\ &= \int_0^{\infty} e^{-\lambda t} j_0 * T(t) x dt\end{aligned}$$

for $\lambda > \omega$ and $x \in X$ or equivalently,

$$\lambda(\lambda^2 - A - B)^{-1}Cx = \int_0^{\infty} e^{-\lambda t} T(t) x dt$$

for $\lambda > \omega$ and $x \in X$. Here

$$\int_0^t j_{-1}(s) j_0(t-s) S(t-2s) x ds = S(t)x \text{ for } t \geq 0.$$

Applying Theorem 2.3, we get that $T(\cdot)$ is an exponentially bounded C-cosine function on X with closed subgenerator $A + B$. Since

$$\begin{aligned}&\int_0^t j_{n-1}(r) j_n(t-r) C(t-2r) x dr \\ &- \int_0^s j_{n-1}(r) j_n(s-r) C(s-2r) x dr \\ &= \int_s^t j_{n-1}(r) j_n(t-r) C(t-2r) x dr \\ &+ \int_0^s j_{n-1}(r) [j_n(t-r) C(t-2r) - j_n(s-r) C(s-2r)] x dr\end{aligned} \tag{2.8}$$

and

$$\begin{aligned}&\int_0^s j_{n-1}(r) [j_n(t-r) C(t-2r) - j_n(s-r) C(s-2r)] x dr \\ &= \int_0^s j_{n-1}(r) j_n(s-r) [C(t-2r) - C(s-2r)] x dr \\ &+ \int_0^s j_{n-1}(r) [j_n(t-r) - j_n(s-r)] C(t-2r) x dr \\ &= \int_0^s j_{n-1}(r) j_n(s-r) [C(|t-2r|) - C(|s-2r|)] x dr \\ &+ \int_0^s j_{n-1}(r) [j_n(t-r) - j_n(s-r)] C(|t-2r|) x dr\end{aligned} \tag{2.9}$$

for all $n \in \mathbb{N}$, $x \in X$ and $t \geq s \geq 0$, we observe from (1.5) that $T(\cdot)$ is also exponentially Lipschitz continuous or norm continuous if $C(\cdot)$ is. $\square$

Next we deduce a new perturbation theorem concerning local C-cosine functions. In particular, the exponential boundedness of $T(\cdot)$ in Theorem 2.6 can be deleted when the assumption of $BC(\cdot) = C(\cdot)B$ on $\overline{D(A)}$ is added.

Theorem 2.7. *Let A be a subgenerator of a local C-cosine function $C(\cdot)$ on X. Assume that B is a bounded linear operator on $\overline{D(A)}$ such that $BC(\cdot) = C(\cdot)B$ on $\overline{D(A)}$ and $BA \subset AB$. Then $A + B$ is a closed subgenerator of a local C-cosine function $T(\cdot)$ on X given as in (1.5). Moreover, $T(\cdot)$ is also locally Lipschitz continuous or norm continuous if $C(\cdot)$ is.*

Proof. Just as in the proof of Theorem 2.6, we observe from (2.8)-(2.9) and (1.5) that $T(\cdot)$ is also locally Lipschitz continuous or norm continuous if $C(\cdot)$ is. Since

$$R(C(t)) \subset \overline{D(A)} \text{ and } BC(\cdot) = C(\cdot)B \text{ on } \overline{D(A)},$$

we have

$$CT(\cdot) = T(\cdot)C \text{ on } X.$$

Let $x \in X$ and $0 \le t \le r < T_0$ be fixed. Then

$$\int_0^t j_{n-1}(s)j_n(t-s)S(t-2s)xds = \frac{1}{2}[j_1(t)\widetilde{S}(t) - \int_0^t \widetilde{S}(t-2s)xds]$$

for $n = 1$, and

$$\int_0^t j_{n-1}(s)j_n(t-s)S(t-2s)xds$$
$$= \frac{1}{2}\int_0^t [j_{n-2}(s)j_n(t-s) - j_{n-1}(s)j_{n-1}(t-s)]\widetilde{S}(t-2s)xds$$

for all $n \ge 2$. Here

$$\widetilde{S}(\cdot) = j_0 * S(\cdot).$$

Since $BA \subset AB$ and

$$\widetilde{S}(r)x = \int_0^r \int_0^t C(s)xdsdt \in D(A),$$

we have

$$\begin{aligned}
&AB\int_0^r \big[j_1(t)\widetilde{S}(t)x - \int_0^t \widetilde{S}(t-2s)xds\big]dt \\
&= BA\int_0^r \big[j_1(t)\widetilde{S}(t)x - \int_0^t \widetilde{S}(t-2s)xds\big]dt \\
&= B\int_0^r \big(j_1(t)[C(t)x - Cx] - \int_0^t [C(t-2s)x - Cx]ds\big)dt \\
&= B\int_0^r j_1(t)C(t)xdt - B\int_0^r \int_0^t C(t-2s)xdsdt \\
&= B\int_0^r j_1(t)C(t)xdt - B\int_0^r S(t)xdt.
\end{aligned}$$

Since

$$\int_0^r j_1(t)C(t)xdt = xj_1(r)S(r)x - \widetilde{S}(r)x$$

and

$$j_1(r)S(r)x = 2\int_0^r j_1(r-s)C(r-2s)xds,$$

we also have

$$\begin{aligned} &AB\int_0^r \big[j_1(t)\widetilde{S}(t)x - \int_0^t \widetilde{S}(t-2s)xds\big]dt \\ &= 2B\int_0^r j_1(r-s)C(r-2s)xds - 2B\int_0^r\int_0^t C(s)xdsdt. \end{aligned} \tag{2.10}$$

Let $n \geq 2$ be fixed.
Using integration by parts, we have

$$\begin{aligned} &\int_0^t j_{n-1}(s)j_n(t-s)S(t-2s)xds \\ &= \frac{1}{2}\int_0^t \big[j_{n-2}(s)j_n(t-s) - j_{n-1}(s)j_{n-1}(t-s)\big]\widetilde{S}(t-2s)xds. \end{aligned}$$

Since

$$\int_0^r\int_0^t j_{n-2}(s)j_n(t-s)Cxdsdt = \int_0^r\int_0^t j_{n-1}(s)j_{n-1}(t-s)Cxdsdt,$$

we have

$$\begin{aligned} &A\int_0^r\int_0^t j_{n-1}(s)j_n(t-s)S(t-2s)xdsdt \\ &= \frac{1}{2}\Big[\int_0^r\int_0^t j_{n-2}(s)j_n(t-s)A\widetilde{S}(t-2s)xdsdt \\ &- \int_0^r\int_0^t j_{n-1}(s)j_{n-1}(t-s)A\widetilde{S}(t-2s)xdsdt\Big] \\ &= \frac{1}{2}\Big[\int_0^r\int_0^t j_{n-2}(s)j_n(t-s)(C(t-2s)x - Cx)dsdt \\ &- \int_0^r\int_0^t j_{n-1}(s)j_{n-1}(t-s)(C(t-2s)x - Cx)dsdt\Big] \\ &= \frac{1}{2}\int_0^r\int_0^t j_{n-2}(s)j_n(t-s)C(t-2s)xdsdt \\ &- \frac{1}{2}\int_0^r\int_0^t j_{n-1}(s)j_{n-1}(t-s)C(t-2s)xdsdt. \end{aligned} \tag{2.11}$$

Since

$$\begin{aligned}
&\int_0^r \int_0^t j_{n-2}(s)j_n(t-s)C(t-2s)xdsdt \\
=&\int_0^r \int_s^r j_{n-2}(s)j_n(t-s)C(t-2s)xdtds \\
=&\int_0^r j_{n-2}(s)\big[j_n(r-s)S(r-2s)x \\
-&\int_s^r j_{n-1}(t-s)S(t-2s)xdt\big]ds \\
=&\int_0^r j_{n-2}(s)j_n(r-s)S(r-2s)xds \\
-&\int_0^r j_{n-2}(s)\int_s^r j_{n-1}(t-s)S(t-2s)xdtds,
\end{aligned} \tag{2.12}$$

$$\begin{aligned}
&\int_0^r j_{n-2}(s)j_n(r-s)S(r-2s)xds \\
=&\int_0^r j_{n-1}(s)j_{n-1}(r-s)S(r-2s)xds \\
&+2\int_0^r j_{n-1}(s)j_n(r-s)C(r-2s)xds \\
=&2\int_0^r j_{n-1}(s)j_n(r-s)C(r-2s)xds
\end{aligned} \tag{2.13}$$

and

$$\begin{aligned}
&\int_0^r \int_s^r j_{n-2}(s)j_{n-1}(t-s)S(t-2s)xdtds \\
=&\int_0^r \int_0^t j_{n-2}(s)j_{n-1}(t-s)S(t-2s)xdsdt,
\end{aligned}$$

we have

$$\begin{aligned}
&\int_0^r \int_0^t j_{n-2}(s)j_n(t-s)C(t-2s)xdsdt \\
=&\ 2\int_0^r j_{n-1}(s)j_n(r-s)C(r-2s)xds \\
-&\ \int_0^r \int_0^t j_{n-2}(s)j_{n-1}(t-s)S(t-2s)xdsdt.
\end{aligned} \tag{2.14}$$

By Lemma 2.5, we have

$$\begin{aligned}
&\int_0^r \int_0^t j_n(s)j_n(t-s)C(t-2s)xdsdt \\
=&\int_0^r \int_0^t j_{n-1}(s)j_n(t-s)S(t-2s)xdsdt.
\end{aligned} \tag{2.15}$$

Combining (1.11) with (2.14) and (2.15), we have

$$\begin{aligned} &A\int_0^r\int_0^t j_{n-1}(s)j_n(t-s)S(t-2s)xdsdt \\ &=\int_0^r j_{n-1}(s)j_n(r-s)C(r-2s)xds \\ &-\int_0^r\int_0^t j_{n-2}(s)j_{n-1}(t-s)S(t-2s)xdsdt. \end{aligned} \tag{2.16}$$

It follows from (2.10)and (2.16) that we have

$$\begin{aligned} &A\int_0^r\sum_{n=0}^{\infty}B^n\int_0^t j_{n-1}(s)j_n(t-s)S(t-2s)xdsdt \\ &=A\sum_{n=0}^{\infty}B^n\int_0^r\int_0^t j_{n-1}(s)j_n(t-s)S(t-2s)xdsdt \\ &=\sum_{n=0}^{\infty}AB^n\int_0^r\int_0^t j_{n-1}(s)j_n(t-s)S(t-2s)xdsdt \\ &=A\int_0^r\int_0^t C(s)xdsdt+AB\int_0^r\int_0^t j_1(t-s)S(t-2s)xdsdt \\ &+\sum_{n=2}^{\infty}B^nA\int_0^r\int_0^t j_{n-1}(s)j_n(t-s)S(t-2s)xdsdt \\ &=\big[C(r)x-Cx\big]+B\left[\int_0^r j_1(r-s)C(r-2s)xds-\int_0^r\int_0^t C(s)xdsdt\right] \\ &+\sum_{n=2}^{\infty}B^n\bigg[\int_0^r j_{n-1}(s)j_n(r-s)C(r-2s)xds \\ &-\int_0^r\int_0^t j_{n-2}(s)j_{n-1}(t-s)S(t-2s)xdsdt\bigg] \\ &=\sum_{n=0}^{\infty}B^n\int_0^r j_{n-1}(s)j_n(r-s)C(r-2s)xds-Cx-B\int_0^r\int_0^t C(s)xdsdt \\ &-\int_0^r\sum_{n=1}^{\infty}B^{n+1}\int_0^t j_{n-1}(s)j_n(t-s)S(t-2s)xdsdt \\ &=\sum_{n=0}^{\infty}B^n\int_0^r j_{n-1}(s)j_n(r-s)C(r-2s)xds-Cx \\ &-B\int_0^r\sum_{n=0}^{\infty}B^n\int_0^t j_{n-1}(s)j_n(t-s)S(t-2s)xdsdt \end{aligned} \tag{2.17}$$

for all $x\in X$ and $0\le r<T_0$ or equivalently,

$$(A+B)\int_0^r\int_0^t T(s)xdsdt=T(r)x-Cx$$

for all $x \in X$ and $0 \le r < T_0$. Since $AB^n = B^n A$ and $B^n C(t) = C(t)B^n$ on $D(A)$, we have

$$\int_0^r \int_0^t T(s)(A+B)xdsdt = (A+B)\int_0^r \int_0^t T(s)xdsdt = T(r)x - Cx$$

for all $x \in D(A)$ and $0 \le r < T_0$. It follows from [14, Theorem 2.5] that $T(\cdot)$ is a local C-cosine function on X with closed subgenerator $A+B$, and is also locally Lipschitz continuous or norm continuous if $C(\cdot)$ is. □

By slightly modifying the proof of Theorem 2.7 we also obtain the next perturbation theorem concerning local C-cosine functions which is still new even though $T_0 = \infty$.

Theorem 2.8. *Let A be a subgenerator of a local C-cosine function $C(\cdot)$ on X. Assume that B is a bounded linear operator on X such that $BC(\cdot) = C(\cdot)B$ on X. Then $A+B$ is a closed subgenerator of a local C-cosine function $T(\cdot)$ on X satisfying*

$$T(t)x = \sum_{n=0}^{\infty} \int_0^t j_{n-1}(s)j_n(t-s)C(|t-2s|)B^n xds \tag{2.18}$$

for all $x \in X$ and $0 \le t < T_0$. Moreover, $T(\cdot)$ is also locally Lipschitz continuous or norm continuous if $C(\cdot)$ is.

Proof. Suppose that B is a bounded linear operator on X which commutes with $C(\cdot)$ on X. Then

$$T(t)x = \sum_{n=0}^{\infty} \int_0^t j_{n-1}(s)j_n(t-s)C(|t-2s|)B^n xds$$

for all $x \in X$ and $0 \le t < T_0$. Since the assumption of $BA \subset AB$ in the proof of Theorem 2.7 is only used to show that (2.10) and (2.17) hold, but both are automatically satisfied if $BA \subset AB$ is replaced by assuming that B is a bounded linear operator on X which commutes with $C(\cdot)$ on X. Therefore, the conclusion of this theorem is true. □

We end this paper with a simple illustrative example.

Example 2.9. Let $C(\cdot)$ $(= \{C(t)|0 \le t < 1\})$ be a family of bounded linear operators on c_0 (family of all convergent sequences in $\mathbb{C}$ with limit 0), defined by

$$C(t)x = \{x_n e^{-n} \cosh nt\}_{n=1}^{\infty}$$

for all $x = \{x_n\}_{n=1}^{\infty} \in c_0$ and $0 \le t < 1$, then $C(\cdot)$ is a local C-cosine function on c_0 with generator A defined by $Ax = \{n^2 x_n\}_{n=1}^{\infty}$ for all $x = \{x_n\}_{n=1}^{\infty} \in c_0$ with $\{n^2 x_n\}_{n=1}^{\infty} \in c_0$. Here $C = C(0)$. Let B be a bounded linear operator on c_0 defined by $Bx = \{x_n e^{-n} \cosh n\}_{n=1}^{\infty}$ for all $x = \{x_n\}_{n=1}^{\infty} \in D(A)$, then $C(\cdot)B = BC(\cdot)$ on c_0. Applying Theorem 2.8, we get that $A+B$ generates a local C-cosine function $T(\cdot)$ on c_0 satisfying (1.5).

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On the viscoelastic equation with Balakrishnan-Taylor damping and nonlinear boundary/interior sources with variable-exponent nonlinearities

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Abstract. This work is devoted to the study of a nonlinear viscoelastic Kirchhoff equation with Balakrishnan-Taylor damping and nonlinear boundary interior sources with variable exponents. Under appropriate assumptions, we establish a uniform decay rate of the solution energy in terms of the behavior of the nonlinear feedback and the relaxation function, without setting any restrictive growth assumptions on the damping at the origin and weakening the usual assumptions on the relaxation function.

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1. Introduction

In this paper, we study the following viscoelastic problem with Balakrishnan-Taylor damping and nonlinear boundary interior sources involving the variable-exponent nonlinearities

$$\frac{\partial^2 u}{\partial t^2} - M\left(\left|\nabla u\left(t\right)\right|^2\right)\Delta u + \int_0^t g\left(t-s\right)\Delta u\left(s\right)\mathrm{d}s = \left|u\right|^{p(x)-1} u \text{ in } \Omega\times\left(0,+\infty\right), \tag{1.1}$$

$$u = 0 \text{ on } \Gamma_0\times\left(0,+\infty\right), \tag{1.2}$$

$$M\left(\left|\nabla u\left(t\right)\right|^2\right)\frac{\partial u}{\partial\nu} - \int_0^t g\left(t-s\right)\frac{\partial}{\partial\nu}u\left(s\right)\mathrm{d}s + h\left(u_t\right) = \left|u\right|^{k(x)-1} u \text{ on } \Gamma_1\times\left(0,+\infty\right), \tag{1.3}$$

$$u(x,0) = u_0(x),\;\; u_t(x,0) = u_1(x),\;\; x\in\Omega, \tag{1.4}$$

where $M(r)$ is a locally Lipschitz function in r, $g > 0$ is a memory kernel and $\Omega \subset \mathbb{R}^n$ $(n \geq 2)$, be a bounded domain with a smooth boundary $\Gamma = \partial\Omega = \Gamma_0 \cup \Gamma_1$. The boundary Γ of Ω is assumed to be regular and is divided by two closed and disjoint parts Γ_0, Γ_1, Here, $\Gamma_0 \neq \emptyset$. $(.)'$ denotes the derivative with respect to time t thus $u_t = \frac{\partial u}{\partial t}$, $u_{tt} = \frac{\partial^2 u}{\partial t^2}$, Δ stands for the Laplacian with respect to the spatial variables, respectively. Let ν be the outward normal to Γ. The exponents $k(.)$ and $p(.)$ are given measurable functions on Ω satisfying

$$\begin{cases} 1 < p^- \leq p(x) \leq p^+ < \infty, \\ 1 < k^- \leq k(x) \leq k^+ < \infty, \end{cases} \tag{1.5}$$

where

$$\begin{cases} p^+ = ess \sup_{x\in\Omega} p(x), \quad p^- = ess \inf_{x\in\Omega} p(x), \\ k^+ = ess \sup_{x\in\Omega} k(x), \quad k^- = ess \inf_{x\in\Omega} k(x). \end{cases} \tag{1.6}$$

We also assume that k satisfies the following Zhikov-Fan uniform local continuity condition:

$$|k(x) - k(y)| \leq \frac{M}{|\log|x-y||}, \text{ for all } x,\ y \text{ in } \Omega \text{ with } |x-y| < \frac{1}{2},\ M > 0.$$

In recent years, many authors have paid attention to the study of nonlinear hyperbolic, parabolic and elliptic equations with nonstandard growth condition. For instance, modeling of physical phenomena such as flows of electro-rheological fluids or fluids with temperature-dependent viscosity, thermoelasticity, nonlinear viscoelasticity, filtration processes through a porous media and image processing. More details on these problems can be found in [5, 7, 1, 2, 3, 26, 34, 35] and references therein.

Constant exponent. In (1.1)-(1.4), when $g \geq 0$ and k, p are constants, this equation has its origin in the nonlinear vibration of an elastic string, were the source term $|u|^{p-2} u$ forces the negative-energy solutions to explode in finite time. While, the dissipation term $h(u_t)$ assures the existence of global solutions for any initial data, local, global existence and long-time behavior have been considered by many authors (see for example [40, 31, 19, 41] and references therein). It is well known that Kirchhoff first investigated the following nonlinear vibration of an elastic string for $f = g = 0$:

$$\rho h \frac{\partial^2 u}{\partial t^2} + \delta \frac{\partial u}{\partial t} + g\left(\frac{\partial u}{\partial t}\right) = \left\{p_0 + \frac{Eh}{2L}\int_0^L \left(\frac{\partial u}{\partial x}\right)^2 \mathrm{d}x\right\} \frac{\partial^2 u}{\partial x^2} + f(u)$$

for $0 < x < L$, $t \geq 0$, where $u(x,t)$ is the lateral displacement, E the Young modulus, ρ the mass density, h the cross-section area, L the length, p_0 the initial axial tension, δ the resistance modulus, and f and g the external forces. The above equation is described by the second order hyperbolic equation (1.1) and it is seemed to be important and natural that the equation with external forces is considered for analyzing phenomena in real world. The equations in (1.1)-(1.4) with $M \equiv 1$ form a class of nonlinear viscoelastic equations used to investigate the motion of viscoelastic materials. As these materials have a wide application in the natural sciences, their dynamics are interesting and of great importance. Hence, questions related to the behavior of the solutions for the wave equation with Dirichlet's boundary condition has attracted considerable the attention of many authors. In particular, there are many results of proving the nonexistence and blow-up of solutions with negative initial energy (see [24, 25, 22, 38, 32, 9] and a list of references therein) also these results were obtained with convexity method. However much less is known when the initial energy is positive (cf. [8, 21, 33, 42]) and these results used several, for example, contradiction method, decomposition method and so on. The equations in (1.1) with $M(r) = a + br$ and $a > 0$, $b > 0$ is the model to describe the motion of deformable solids as hereditary effect is incorporated, which was first studied by Torrejon and Yong [37]. They proved the existence of weakly asymptotic stable solution for the large analytical datum. Later, Rivera [30] showed the existence of global solutions for small datum and the total energy decays to zero exponentially under some restrictions. Problem (1.1)-(1.4) is the extension of the problems in which the variable-exponent are constants and $g \geq 0$. The main difficulty of this problem is related to the presence of the quasilinear terms in (1.1)-(1.4) in the variable-exponent. In this paper a class of a weakly damped wave equation of generalized Kirchhoff type with nonlinear damping and source terms involving the variable-exponent nonlinearities were considered. Hence by using the Faedo-Galerkin arguments and compactness method as in [27], together with the Banach fixed point theorem, we will show the local existence of the problem (1.1)-(1.4). The purpose of this paper is to generalize the existence and uniform decay theorems of local solutions due to the constant-exponents. In other words we prove the existence and uniform decay rate of local solutions to weakly damped degenerate wave equations of Kirchhoff type (1.1)-(1.4) with nonlinear damping and source terms. This paper consists of 3 sections in addition to the introduction. In Section 2, we recall the definitions of the variable-exponent Lebesgue spaces $L^{p(\cdot)}(\Omega)$, the Sobolev spaces $W^{1,p(\cdot)}(\Omega)$, and some of their properties. In Section 3, we state, with the proof, existence and uniqueness result of weak solution for (1.1)-(1.4) by employing Faedo–Galerkin's together with the Banach fixed point theorem and compactness methods. In Section 4, the statement and the proof of our global existence and a stability theorem for certain solutions with positive initial energy. To the best of our knowledge, this problem has not been studied previously. In addition, our method of determining these results, because the presence of the exponents $m(.)$ and $p(.)$, is somewhat different.

2. Preliminaries. Function spaces

In this section, we list and recall some well-known results and facts from the theory of the Sobolev spaces with variable exponent. (On the basic properties of the spaces $W^{1,p(x)}(\Omega)$ and $L^{p(x)}(\Gamma)$ we refer to [10, 11, 15, 17, 23]).

Throughout the rest of the paper we assume that Ω is a bounded domain of $\mathbb{R}^n$, $n \geq 1$ with smooth boundary Γ and assume that $p(.)$ is a measurable function on $\overline{\Omega}$ such that:

$$1 < p^- \leq p(x) \leq p^+ < \infty, \tag{2.1}$$

where

$$p^+ = ess \sup_{x\in\Omega} p(x), \quad p^- = ess \inf_{x\in\Omega} p(x).$$

We also assume that p satisfies the following Zhikov–Fan uniform local continuity condition:

$$|p(x)-p(y)| \leq \frac{M}{|\log|x-y||}, \text{ for all } x,\ y \text{ in } \Omega \text{ with } |x-y|<\frac{1}{2},\ M>0. \tag{2.2}$$

Given a function $p:\overline{\Omega}\rightarrow[p^-,\ p^+]\subset(1,\infty)$, $p^{\pm}=\text{const}$, we define the set

$$L^{p(.)}(\Omega)=\left\{\begin{array}{c} v:\Omega\rightarrow\mathbb{R}: v \text{ measurable functions on } \Omega, \\ \varrho_{p(.),\,\Omega}(v)=\int_{\Omega}|v(x)|^{p(x)}\,\mathrm{d}x<\infty. \end{array}\right\}$$

The variable-exponent space $L^{p(.)}(\Omega)$ equipped with the Luxemburg norm

$$\|u\|_{p(.),\Omega}=\|u\|_{p(.)}=\inf\left\{\lambda>0,\ \int_{\Omega}\left|\frac{u}{\lambda}\right|^{p(x)}\mathrm{d}x\leq 1\right\},$$

becomes a Banach space.

In general, variable-exponent Lebesgue spaces are similar to classical Lebesgue spaces in many aspects, see the first discussed the $L^{p(x)}$ spaces and $W^{k,p(x)}$ spaces by Kovàcik and Rákosnik in [23].

Let us list some properties of the spaces $L^{p(.)}(\Omega)$ which will be used in the study of the problem (1.1)-(1.4).

- If $p(x)$ is measurable and $1<p^-\leq p(x)\leq p^+<\infty$ in Ω, then $L^{p(.)}(\Omega)$ is a reflexive and separable Banach space and $C_0^{\infty}(\Omega)$ is dense in $L^{p(.)}(\Omega)$.
- If condition (2.2) is fulfilled, and Ω has a finite measure and p, q are variable exponents so that $p(x)\leq q(x)$ almost everywhere in Ω, the inclusion $L^{q(.)}(\Omega)\subset L^{p(.)}(\Omega)$ is continuous and
$$\forall v\in L^{q(.)}(\Omega)\quad \|u\|_{p(.)}\leq C\,\|u\|_{q(.)}\,;\quad C=C\left(|\Omega|\,,p^{\pm}\right) \tag{2.3}$$
- The variable Sobolev space $W^{1,p(.)}(\Omega)$ is defined as the closure of $C_0^{\infty}(\Omega)$ with respect to the norm
$$\|u\|_{W_0^{1,p(.)}(\Omega)}=\|u\|_{p(.),\Omega}+\|\nabla u\|_{p(.),\Omega}\,.$$
It is known that for the elements of $W_0^{1,p(.)}(\Omega)$ the Poincar´e inequality holds,
$$\|u\|_{p(.),\Omega}\leq C\,(n,\Omega)\,\|\nabla u\|_{p(.),\Omega}\,, \tag{2.4}$$
and an equivalent norm of $W_0^{1,p(.)}(\Omega)$ can be defined by
$$\|u\|_{W_0^{1,p(.)}(\Omega)}=\|\nabla u\|_{p(.),\Omega}\,.$$
According to (2.2) $W_0^{1,p(.)}(\Omega)\subset W_0^{1,p^-}(\Omega)$. If $p^->\frac{2n}{n+2}$, then the embedding $W_0^{1,p^-}(\Omega)\subset L^2(\Omega)$ is compact.
- It is known that $C_0^{\infty}(\Omega)$ is dense in $W_0^{1,p(.)}(\Omega)$ according to (2.2) if $p(x)\in C_{\log}(\overline{\Omega})$, that is, the variable exponent $p(x)$ is continuous in $\overline{\Omega}$ with the logarithmic module of continuity.
- It follows directly from the definition of the norm that
$$\min\left(\|u\|_{p(.)}^{p^-},\|u\|_{p(.)}^{p^+}\right)\leq\varrho_{p(.)}(u)\leq\max\left(\|u\|_{p(.)}^{p^-},\|u\|_{p(.)}^{p^+}\right). \tag{2.5}$$

- The following generalized Hölder inequality

$$\int_{\Omega} |u(x)\, v(x)|\, \mathrm{d}x \leq \left(\frac{1}{p^-} + \frac{1}{(p^-)'}\right) \|u\|_{p(x)} \|v\|_{p'(x)} \leq 2 \|u\|_{p(x)} \|v\|_{p'(x)}$$

  holds, for all $u \in L^{p(.)}(\Omega)$, $v \in L^{p'(.)}(\Omega)$ with $p(x) \in (1, \infty)$, $p'(x) = \frac{p(x)}{p(x)-1}$.
- If $p : \Omega \to [p^-,\ p^+] \subset [1, +\infty)$ is a measurable function and $p_* > ess \sup_{\{x\in\Omega\}} p(x)$ with $p_* \leq \frac{2n}{n-2}$, then the embedding $H_0^1(\Omega) = W_0^{1,2}(\Omega) \hookrightarrow L^{p(.)}(\Omega)$ is continuous and compact.

Lemma 2.1. ([10]) *Let Ω be a bounded domain of $\mathbb{R}^n$, $p(.)$ and $m(.)$ satisfies (1.5) and (2.2), then*

$$B_0 \|\nabla u\|_{m(.)} \geq \|u\|_{p(.)} \quad \text{for all } u \in W_0^{1,m(.)}(\Omega). \tag{2.6}$$

where the optimal constant of Sobolev embedding B_0 is depends on $p^{\pm}$ and $|\Omega|$.

Lemma 2.2 (Poincaré's Inequality). ([10]) *Let Ω be a bounded domain of $\mathbb{R}^n$ and $p(.)$ satisfies (2.2), then*

$$D_0 \|\nabla u\|_{p(.)} \geq \|u\|_{p(.)} \quad \text{for all } u \in W_0^{1,p(.)}(\Omega). \tag{2.7}$$

where the optimal constant of Sobolev embedding D_0 is depends on $p^{\pm}$ and $|\Omega|$.

Proposition 2.3. (See [16, 14, 15, 12, 13]) *Let Ω be a bounded domain in $\mathbb{R}^n$, $p \in C^{0,1}\left(\overline{\Omega}\right)$, $1 < p^- \leq p(x) \leq p^+ < n$. Then For any $q \in C(\Gamma)$ with $1 \leq q(x) \leq \frac{(n-1)p(x)}{n-p(x)}$, there is a continuous trace $W^{1,p(x)}(\Omega) \to L^{q(x)}(\Gamma)$, when $1 \leq q(x) << \frac{(n-1)p(x)}{n-p(x)}$, the trace is compact, in particulary the continuous trace $W^{1,p(x)}(\Omega) \to L^{p(x)}(\Gamma)$ is compact.*

There are many proprieties of the theory of Lebesgue and Sobolev spaces with variable exponent, see the detailed exposition given in the monograph [4, Ch.1.].

Lemma 2.4 (Modified Gronwall inequality). *Let ϕ and h be nonnegative functions on $[0, +\infty)$ satisfying*

$$0 \leq \phi \leq K + \int_0^t h(s)\, \phi(s)^{r+1}\, \mathrm{d}s,$$

with $K > 0$ and $r > 0$. Then

$$\phi(t) \leq \left(K^{-r} - r \int_0^t h(s)\, \mathrm{d}s\right)^{\frac{-1}{r}}.$$

as long as the right-hand side exists.

2.1. Mathematical hypotheses

We begin this section by introducing some hypotheses and our main result. Throughout this paper, we use standard functional spaces and denote that $\|.\|_{p(.)}$, $\|.\|_{p(.),\Gamma_1}$ are $L^{p(.)}(\Omega)$ norm and $L^{p(.)}(\Gamma_1)$ norm, respectively, such that:

$$\|u\|_{p(.),\Gamma_1} = \int_{\Gamma_1} |u(\eta)|^{p(\eta)} \mathrm{d}\eta = \int_{\Gamma_1} |u(x)|^{p(x)} \mathrm{d}\Gamma;$$
$$\|.\|_{q,\Gamma_1} = \int_{\Gamma_1} |u(x)|^{q} \mathrm{d}\Gamma.$$

Also, we define $(u,v) = \int_\Omega u(x)\, v(x)\,\mathrm{d}x$ and $(u,v)_{\Gamma_1} = \int_{\Gamma_1} u(x)\, v(x)\,\mathrm{d}\Gamma$.

The inner products and norms in $L^2(\Omega)$ and $H_0^1(\Omega)$ are represented by $(.,.)$, $\|.\|$ respectively and they are given by:

$$(u,v) = \int_\Omega u(x)\, v(x)\,\mathrm{d}x \text{ and } \|u\|^2_{L^2(\Omega)} = |u|^2 = \int_\Omega u^2 \mathrm{d}x;$$
$$\|u\|^2_{H_0^1(\Omega)} = \|u\|^2 = |\nabla u|^2 = \int_\Omega |\nabla u|^2 \mathrm{d}x.$$

Next, we state the assumptions for problem (1.1)-(1.4).

(H1) Hypotheses on M**.** Let $M \in C([0,+\infty), \mathbb{R}_+)$ be a nonnegative locally Lipschitz function and for positive constant $m > 0$, we have

$$M(s) \geq m_3 > 0, \; s \geq 0 \tag{2.8}$$

(H2) Hypotheses on g**.** $g : [0,\infty) \to (0,\infty)$ is a bounded C^1 function satisfying

$$g(0) > 0, \; m_3 - \int_0^\infty g(s)\,\mathrm{d}s = l > 0, \tag{2.9}$$

and there exists a non-increasing positive differentiable function ζ such that

$$g'(t) \leq -\zeta(t)\, g(t) \text{ for all } t \geq 0. \tag{2.10}$$

(H3) Hypotheses on h**.** $h : \mathbb{R} \to \mathbb{R}$ is a Lipschitz non-decreasing function with

$$h(s)s \geq 0 \text{ for all } s \in \mathbb{R} \tag{2.11}$$

(H4) Hypotheses on $p(.)$, $k(.)$**.** Let $m(.)$ and $p(.)$ are given measurable functions on $\overline{\Omega}$ satisfying the following conditions,

$$\begin{array}{l} 1 < p^- \leq p(x) \leq p^+ \leq \frac{n}{n-2}, \; n > 2 \text{ and } 1 \leq p^- \leq p^+ < \infty \text{ if } n = 2, \\ 1 < k^- \leq k(x) \leq k^+ < \frac{n-1}{n-2}, \; n > 2 \text{ and } 1 \leq k^- \leq k^+ < \infty \text{ if } n = 2 \end{array} \tag{2.12}$$

According to (2.12), we have

$$\|u\|_{p^++1} \leq B\,|\nabla u| \quad \forall u \in H^1_{\Gamma 0}(\Omega). \tag{2.13}$$

where

$$H^1_{\Gamma_0}(\Omega) = \left\{ u \in H^1(\Omega) : u\,|_{\Gamma_0} = 0 \right\}$$

endow with the Hilbert structure induced by $H^1(\Omega)$, is a Hilbert space and $B > 0$ be the optimal constant of Sobolev embedding $H^1_{\Gamma_0}(\Omega) \hookrightarrow L^{p^++1}(\Omega)$ which satisfies the inequality (2.13) and we use the trace-Sobolev imbedding:

$H^1_{\Gamma_0}(\Omega) \hookrightarrow L^{k^+ +1}(\Gamma_1)$, $1 < k^+ < \frac{n-1}{n-2}$. In this case, the imbedding constant is denoted by B_*, i.e.,

$$\|u\|_{k^+ +1,\Gamma_1} \leq B_* |\nabla u| \quad \forall u \in H^1_{\Gamma_0}(\Omega). \tag{2.14}$$

(H5) Assumptions on u_0, u_1**.** Assume that $(u_0, u_1) \in H^1_{\Gamma_0}(\Omega) \cap H^2(\Omega) \times H^1_{\Gamma_0}(\Omega)$ satisfying the compatibility conditions

$$M\left(|\nabla u_0|^2\right) \frac{\partial u_0}{\partial \nu} + h(u_1) = |u_0|^{k(.)-1} u_0 \text{ on } \Gamma_1. \tag{2.15}$$

3. Main result

This section first presents the local existence and uniqueness of the solution for problem (1.1)-(1.4) with a degenerated second order equation on Γ_1. Our method of proof by perturbing the boundary equation is based on the combination of the Faedo-Galerkin approximation and the compactness method together with the Banach fixed point theorem with the ones from [36].

3.1. Existence of local solutions

In this section, under the assumptions (H_1)-(H_5), we prove the existence of the local solution to the wave equation of Kirchhoff type (1.1)-(1.4) for any initial value $(u_0, u_1) \in H^1_{\Gamma_0}(\Omega) \cap H^2(\Omega) \times H^1_{\Gamma_0}(\Omega)$. First we need the local existence and uniqueness of the solution to the following wave equation:

$$\begin{gathered}
\frac{\partial^2 u}{\partial t^2} - M\left(|\nabla \varphi(t)|^2\right) \Delta u + \int_0^t g(t-s) \Delta u(s)\, \mathrm{d}s = |u|^{p(x)-1} u \text{ in } \Omega \times (0, +\infty), \\
u = 0 \text{ on } \Gamma_0 \times (0, +\infty), \\
M\left(|\nabla \varphi(t)|^2\right) \frac{\partial u}{\partial \nu} - \int_0^t g(t-s) \frac{\partial}{\partial \nu} u(s)\, \mathrm{d}s + h(u_t) = |u|^{k(x)-1} u \text{ on } \Gamma_1 \times (0, +\infty), \\
u(x,0) = u_0(x), \;\; u_t(x,0) = u_1(x), \;\; x \in \Omega,
\end{gathered} \tag{P4}$$

where $\varphi : [0, T] \to H^1_{\Gamma_0}(\Omega)$ is a continuous function. So we first prove the existence and uniqueness of the local solution to (P4). Let (w_j), $j = 1, 2, ...$, be a completely orthonormal system in $L^2(\Omega)$ having the following properties:
* $\forall j; w_j \in H^1_{\Gamma_0}(\Omega)$;
* The family $\{w_1, w_2, ..., w_m\}$ is linearly independent;
* V_m the space generated by $\{w_1, w_2, ..., w_m\}$, $\underset{m}{\cup} V_m$ is dense in $H^1_{\Gamma_0}(\Omega) \cap H^2(\Omega)$. We construct approximate solutions, for each $\eta \in (0, 1)$, $u^{\eta m}$ $(m = 1, 2, 3, ...)$ in V_m in the form:

$$u^{\eta m}(t) = \sum_{i=1}^{m} K_{jm}(t) w_i, \;\; m = 1, 2, ..., \tag{3.1}$$

where $K_{jm}(t)$ are determined by the following ordinary differential perturbed equation:

$$\begin{aligned}(u_{tt}^{\eta m}(t), w_j) + M\left(|\nabla\varphi(t)|^2\right)(\nabla u^{\eta m}, \nabla w_j) - \left(\int_0^t g(t-s)\nabla u^{\eta m}(s)\,\mathrm{d}s, \nabla w_j\right) \\ + \left(h\left(u_t^{\eta m}\right), w_j\right)_{\Gamma_1} + \eta\left(u_t^{\eta m}(t), w_j\right)_{\Gamma_1} \\ = \left(|u^{\eta m}(t)|^{k(x)-1} u^{\eta m}(t), w_j\right)_{\Gamma_1} + \left(|u^{\eta m}(t)|^{p(x)-1} u^{\eta m}(t), w_j\right),\ \ j = 1, 2, ...,\end{aligned}$$

and will be completed by the following initial conditions $u^{\eta m}(0)$, $u_t^{\eta m}(0)$ which satisfies:

$$\left\{\begin{aligned} u^{\eta m}(0) = u_0^{\eta m} = \sum_{i=1}^{m}\alpha_{im}w_i \longrightarrow u_0(x) \text{ when } m \longrightarrow \infty \ \text{ in } H^1_{\Gamma_0}(\Omega) \cap H^2(\Omega), \\ u_t^{\eta m}(0) = u_1^{\eta m} = \sum_{i=1}^{m}\beta_{im}w_i \longrightarrow u_1(x) \text{ when } m \longrightarrow \infty \ \text{ in } H^1_{\Gamma_0}(\Omega). \end{aligned}\right. \tag{3.2}$$

Then it holds that for any given $v \in Span\{w_1, w_2, ..., w_m\}$,

$$\begin{aligned}(u_{tt}^{\eta m}(t), v) + M\left(|\nabla\varphi(t)|^2\right)(\nabla u^{\eta m}, \nabla v) - \left(\int_0^t g(t-s)\nabla u^{\eta m}(s)\,\mathrm{d}s, \nabla v\right) \\ + \left(h\left(u_t^{\eta m}\right), v\right)_{\Gamma_1} + \eta\left(u_t^{\eta m}(t), v\right)_{\Gamma_1} \\ = \left(|u^{\eta m}(t)|^{k(x)-1} u^{\eta m}(t), v\right)_{\Gamma_1} + \left(|u^{\eta m}(t)|^{p(x)-1} u^{\eta m}(t), v\right).\end{aligned} \tag{3.3}$$

By virtue of the theory of ordinary differential equations, system (3.1), (3.2) and (3.3) has a unique local solution which is extended to a maximal intervals $[0, t_m[$.

A solution u to the problem (1.1)-(1.4) on some interval $[0, t_m[$ will be obtain as the limit of $u^{\eta m}$ as $m \to \infty$ and $\eta \to 0$. Then, this solution can be extended to the whole interval $[0, T]$, for all $T > 0$, as a consequence of the a priori estimates that shall be proven in the next step. In this paper, ε, $C(\varepsilon)$, C_ε, C, $C(m_3)$, c, c^* or c_* denote a various positive constant which changes from line to line and are independent of natural number n and depends only (possibly) on the initial value.

Let us first recall a useful identity for the memory term who play an important role in the sequel. By denoting

$$(g \diamond \nabla(u))(t) = \int_0^t g(t-s)\int_\Omega |\nabla u(s) - \nabla u(t)|^2\,\mathrm{d}x\mathrm{d}s,$$

it is easy, by differentiating the term $(\beta \diamond \nabla(u))(t)$ with respect to t, to show that

$$\begin{aligned}\int_\Omega\int_0^t g(t-s)\nabla u(s)\nabla u_t(t)\,\mathrm{d}x\mathrm{d}s \\ = -\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\{(g \diamond \nabla u)(t) - |\nabla(u(t))|^2\int_0^t g(s)\,\mathrm{d}s\right\} \\ +\frac{1}{2}(g' \diamond \nabla u)(t) - \frac{1}{2}g(t)|\nabla u(t)|^2.\end{aligned} \tag{3.4}$$

We prove by the Galerkin method the following lemma on the existence and uniqueness of local solution to (P4) in time.

Lemma 3.1. *Let $M(r)$ be a nonnegative locally Lipschitz function. Let*

$$(u_0, u_1) \in H^1_{\Gamma_0}(\Omega) \cap H^2(\Omega) \times H^1_{\Gamma_0}(\Omega).$$

Assume that the differentiable function $\varphi(t)$ satisfies

$$\varphi(0) = u_0,\ \varphi'(0) = u_1.$$

Assume that the following condition is satisfied

$$1 < k^+ < \frac{n-1}{n-2} \text{ and } 1 < p^+ \leq \frac{n}{n-2} \text{ if } n \geq 3,$$
$$1 \leq k^- \leq k^+ < \infty \text{ and } 1 \leq p^- \leq p^+ < \infty \text{ if } n = 2.$$

Then there exists a time $T_0 = T_0(u_0, u_1, m_1, m_2, m_3) > 0$ such that if there exist $m_1, m_2, m_3 > 0$ and $T > 0$ satisfying

$$|\nabla\varphi(t)| \leq m_1,\ |\nabla\varphi'(t)| \leq m_2,\ M\left(|\nabla\varphi(t)|^2\right) \geq m_3 > 0$$

for all $t \in [0, T]$, then there exists a unique local weak solution in time $u(t)$ to (P4) with the initial value (u_0, u_1) on $[0, T_0]$, where $T_0 \leq T$ satisfying:

$$u(t) \in C([0, T_0] : H^1_{\Gamma_0}(\Omega)),$$
$$u_t(t) \in C([0, T_0] : L^2(\Omega)) \cap C([0, T] : H^1_{\Gamma_0}(\Omega)),$$
$$u_{tt}(t) \in C([0, T_0] : L^2(\Omega)).$$

Proof. <u>The first estimate</u> (Estimates on $u_t^{\eta m}$):

By taking $v = u_t^{\eta m}(t)$ in (3.3), we have that

$$\begin{aligned}(u_{tt}^{\eta m}(t), u_t^{\eta m}) &+ M\left(|\nabla\varphi(t)|^2\right)(\nabla u^{\eta m}, \nabla u_t^{\eta m}) - \left(\int_0^t g(t-s)\nabla u^{\eta m}(s)\,\mathrm{d}s, \nabla u_t^{\eta m}\right)\\ &+\eta\,||u_t^{\eta m}(t)||^2_{2,\Gamma_1} + (h(u_t^{\eta m}), u_t^{\eta m})_{\Gamma_1} = \left(|u^{\eta m}(t)|^{k(x)-1}u^{\eta m}(t), u_t^{\eta m}\right)_{\Gamma_1}\\ &+ \left(|u^{\eta m}(t)|^{p(x)-1}u^{\eta m}(t), u_t^{\eta m}\right).\end{aligned}$$

Since it holds that

$$\begin{aligned}\int_0^t M\left(|\nabla\varphi(t)|^2\right)(\nabla u^{\eta m}, \nabla u_t^{\eta m})\,\mathrm{d}s &= \frac{1}{2}\int_0^t M\left(|\nabla\varphi(s)|^2\right)\frac{\mathrm{d}}{\mathrm{d}t}|\nabla u^{\eta m}(s)|^2\,\mathrm{d}s\\ &\geq \left[\frac{1}{2}M\left(|\nabla\varphi(s)|^2\right)|\nabla u^{\eta m}|^2\right]_0^t - \frac{1}{2}\int_0^t\left[\frac{\mathrm{d}^+}{\mathrm{d}s}M\left(|\nabla\varphi(s)|^2\right)\right]|\nabla u^{\eta m}|^2\,\mathrm{d}s,\end{aligned}$$

$$\frac{\mathrm{d}^+}{\mathrm{d}s}M\left(|\nabla\varphi(s)|^2\right) \leq 2\left(\frac{\mathrm{d}^+}{\mathrm{d}r}M(r)\right)|\nabla\varphi(s)|\,|\nabla\varphi'(s)| \leq 2Lm_1m_2,\ \ s \in [0, T_1].$$

where $L = L(m_1)$ is a local Lipschitz constant for $M(r)$, we have for $t \in (0, t_m)$

$$\begin{aligned} &\frac{1}{2}\left|u_t^{\eta m}\right|^2 + \frac{1}{2}\left(M\left(\left|\nabla\varphi(t)\right|^2\right) - \int_0^t g(s)\,\mathrm{d}s\right)\left|\nabla u^{\eta m}\right|^2 + \left(g \diamond \nabla u^{\eta m}\right)(t) \\ &\quad -\frac{1}{2}\int_0^t \left(g' \diamond \nabla u^{\eta m}\right)(s)\,\mathrm{d}s + \eta\int_0^t \left\|u_t^{\eta m}(s)\right\|_{2,\Gamma_1}^2\mathrm{d}s \qquad (3.5) \\ &\quad +\frac{1}{2}\int_0^t g(s)\left|\nabla u^{\eta m}\right|^2\mathrm{d}s + \int_0^t \left(h\left(u_t^{\eta m}\right), u_t^{\eta m}(s)\right)_{\Gamma_1}\mathrm{d}s \\ &\leq Lm_1m_2\int_0^t \left|\nabla u^{\eta m}\right|^2\mathrm{d}s + \frac{1}{2}M\left(\left|\nabla\varphi(0)\right|^2\right)\left|\nabla u_0\right|^2 + \frac{1}{2}\left|u_1\right|^2 \\ &\quad = \int_0^t \left(\left|u^{\eta m}(s)\right|^{k(x)-1}u^{\eta m}(s), u_t^{\eta m}\right)_{\Gamma_1}\mathrm{d}s \\ &\quad + \int_0^t \left(\left|u^{\eta m}(s)\right|^{p(x)-1}u^{\eta m}(s), u_t^{\eta m}\right)\mathrm{d}s. \end{aligned}$$

Young's inequality gives

$$\begin{aligned} \left|\int_\Omega \left|u^{\eta m}(t)\right|^{p(x)-1}u^{\eta m}(t)\,u_t^{\eta m}(t)\mathrm{d}x\right| &\leq \int_\Omega \left|u^{\eta m}(t)\right|^{p(x)-1}\left|u^{\eta m}(t)\right|\left|u_t^{\eta m}(t)\right|\mathrm{d}x \qquad (3.6) \\ &\leq \left|u^{\eta m}(t)\right|^{p(x)-1}\left|u^{\eta m}(t)\right|\left|u_t^{\eta m}(t)\right| \\ \leq \frac{1}{2}C_\varepsilon \max\left(\int_\Omega \left|u^{\eta m}(t)\right|^{2p^+}\mathrm{d}x, \int_\Omega \left|u^{\eta m}(t)\right|^{2p^-}\mathrm{d}x\right) &+ \frac{1}{2}\varepsilon\int_\Omega \left|u_t^{\eta m}(t)\right|^2\mathrm{d}x \\ \leq \frac{1}{2}C_\varepsilon\left(\left|\nabla u^{\eta m}\right|^{2p^+} + \left|\nabla u^{\eta m}\right|^{2p^-}\right) &+ \frac{1}{2}\varepsilon\left|u_t^{\eta m}(t)\right|^2 \end{aligned}$$

Also

$$\begin{aligned} &\left|\int_{\Gamma_1} \left|u^{\eta m}(t)\right|^{k(x)-1}u^{\eta m}(t)\,u_t^{\eta m}(t)\mathrm{d}\Gamma\right| \\ &\leq \frac{1}{2}C_\varepsilon \max\left(\left\|u^{\eta m}\right\|_{2k^+,\Gamma_1}^{2k^+}, \left\|u^{\eta m}\right\|_{2k^-,\Gamma_1}^{2k^-}\right) + \frac{1}{2}\varepsilon\int_{\Gamma_1}\left|u_t^{\eta m}(t)\right|^2\mathrm{d}\Gamma \qquad (3.7) \\ &\frac{1}{2}C_\varepsilon\left(\left|\nabla u^{\eta m}\right|^{2k^+} + \left|\nabla u^{\eta m}\right|^{2k^-}\right) + \frac{1}{2}\varepsilon\left\|u_t^{\eta m}(t)\right\|_{2,\Gamma_1}^2, \end{aligned}$$

consequently, taking (2.8) and (2.9) into account

$$\left(M\left(\left|\nabla\varphi(t)\right|^2\right) - \int_0^t g(s)\,\mathrm{d}s\right) \geq m_3 - \int_0^\infty g(s)\,\mathrm{d}s = l > 0$$

Combining above results, and observing that $g > 0$ and $g' \leq 0$, we deduce

$$\frac{1}{2}\left|u_t^{\eta m}\right|^2 + \frac{1}{2}l\left|\nabla u^{\eta m}\right|^2 + \left(g \diamond \nabla u^{\eta m}\right) - \frac{1}{2}\int_0^t \left(g' \diamond \nabla u^{\eta m}\right)(s)\,\mathrm{d}s$$

$$+\eta\int_0^t \left|\left|u_t^{\eta m}(s)\right|\right|_{2,\Gamma_1}^2 \mathrm{d}s + \frac{1}{2}\int_0^t g(s)\left|\nabla u^{\eta m}\right|^2 \mathrm{d}s + \int_0^t \left(h\left(u_t^{\eta m}\right), u_t^{\eta m}(s)\right)_{\Gamma_1} \mathrm{d}s$$

$$\leq Lm_1m_2\int_0^t \left|\nabla u^{\eta m}\right|^2 \mathrm{d}s + \frac{1}{2}M\left(\left|\nabla\varphi(0)\right|^2\right)\left|\nabla u_0\right|^2 + \frac{1}{2}\left|u_1\right|^2$$

$$+C_\varepsilon\int_0^t \left(\left|\nabla u^{\eta m}\right|^{2k^+} + \left|\nabla u^{\eta m}\right|^{2k^-} + \left|\nabla u^{\eta m}\right|^{2p^+} + \left|\nabla u^{\eta m}\right|^{2p^-}\right)\mathrm{d}s$$

$$+\frac{1}{2}\varepsilon\int_0^t \left|u_t^{\eta m}(s)\right|^2 \mathrm{d}s + \varepsilon\int_0^t \left|\left|u_t^{\eta m}(s)\right|\right|_{2,\Gamma_1}^2 \mathrm{d}s,$$

Choosing $\varepsilon = \frac{\eta}{2}$, we arrive at

$$\frac{1}{2}\left|u_t^{\eta m}\right|^2 + \frac{1}{2}l\left|\nabla u^{\eta m}\right|^2 + \left(g \diamond \nabla u^{\eta m}\right)(t) - \frac{1}{2}\int_0^t \left(g' \diamond \nabla u^{\eta m}\right)(s)\,\mathrm{d}s$$

$$+\frac{\eta}{2}\int_0^t \left|\left|u_t^{\eta m}(s)\right|\right|_{2,\Gamma_1}^2 \mathrm{d}s + \frac{1}{2}\int_0^t g(s)\left|\nabla u^{\eta m}(s)\right|^2 \mathrm{d}s + \int_0^t \left(h\left(u_t^{\eta m}\right), u_t^{\eta m}(s)\right)_{\Gamma_1} \mathrm{d}s$$

$$\leq \frac{\eta}{2}\int_0^t \left|u_t^{\eta m}(s)\right|^2 \mathrm{d}s + Lm_1m_2\int_0^t \left|\nabla u^{\eta m}\right|^2 \mathrm{d}s$$

$$+C_\varepsilon\int_0^t \left(\left|\nabla u^{\eta m}\right|^{2k^+} + \left|\nabla u^{\eta m}\right|^{2k^-} + \left|\nabla u^{\eta m}\right|^{2p^+} + \left|\nabla u^{\eta m}\right|^{2p^-}\right)\mathrm{d}s$$

$$+\frac{1}{2}Lm_1\left|\nabla u_0\right|^2 + \frac{1}{2}\left|u_1\right|^2 + C_\varepsilon. \tag{3.8}$$

Thus, there exist $B > 0$, $\beta > 0$ and $r > 0$ such that

$$\left|\nabla u^{\eta m}\right|^2 + \left|u_t^{\eta m}\right|^2 \leq B + \beta\int_0^t \left[1 + \left(\left|\nabla u^{\eta m}(s)\right|^2 + \left|u_t^{\eta m}(s)\right|^2\right)^{r+1}\right]\mathrm{d}s$$

where we note that B and β are independent of m and r. Since $r > 0$, there exists an enough small time $T_0 := T_0(u_0, u_1, m_3) \in (0, T_1)$ satisfying

$$\left(B + \beta T_0\right)^{-r} - r\beta T_0 > 0$$

Thus, we have by the modified Gronwall lemma 2.4

$$\left|\nabla u^{\eta m}\right|^2 + \left|u_t^{\eta m}\right|^2 \leq \left(\left(B + \beta T_0\right)^{-r} - r\beta T_0\right)^{\frac{-1}{r}}$$

Therefore, there exist constants $c_i = c_i(u_0, u_1, m_3) > 0$ $(i = 1, 2, 3)$ such that for any $t \in [0, T_0]$

$$\left|\nabla u^{\eta m}\right|^2 \leq C_1 \text{ and } \left|u_t^{\eta m}\right|^2 \leq C_2. \tag{3.9}$$

Furthermore, by (3.8) it follows that

$$(g \diamond \nabla u^{\eta m})(t) - \frac{1}{2}\int_0^t (g' \diamond \nabla u^{\eta m})(s)\,\mathrm{d}s + \int_0^t \|u_t^{\eta m}(s)\|_{2,\Gamma_1}^2\,\mathrm{d}s \qquad (3.10)$$
$$+ \int_0^t g(s)\,|\nabla u^{\eta m}(s)|^2\,\mathrm{d}s + \int_0^t (h(u_t^{\eta m}), u_t^{\eta m}(s))_{\Gamma_1}\,\mathrm{d}s \le C_3$$

where C_i are a positive constants which are independent of m, η and t. Thus, the solution can be extended to $[0,T)$ and, in addition, we have

$$(u^{\eta m}) \text{ is bounded sequence in } L^\infty\left(0,T;H^1_{\Gamma_0}(\Omega)\right),$$
$$(u_t^{\eta m}) \text{ is bounded sequence in } L^\infty\left(0,T;L^2(\Omega)\right),$$
$$(h(u_t^{\eta m}).u_t^{\eta m}) \text{ is bounded sequence in } L^1\left(0,T;L^1(\Gamma_1)\right).$$

The second estimate (Estimates on $u_{tt}^{\eta m}$):

First of all, we are going to estimate $u_{tt}^{\eta m}(0)$. By taking $t=0$ in (3.3), taking (2.15) into account, we get

$$\begin{aligned} |u_{tt}^{\eta m}(0)|^2 &\le c\left|M\left(|\nabla u_0|^2\right)\right|^2 |\Delta u_0|^2 + c\int_\Omega |u_0|^{2p(x)}\,\mathrm{d}x \\ &\le cL\,|\nabla u_0|^4\,|\Delta u_0|^2 + c\max\left(|\nabla u_0|^{2p^+}, |\nabla u_0|^{2p^-}\right) \le c^* \end{aligned} \qquad (3.11)$$

Now, by differentiating (3.3) with respect to t and substituting $w_j = u_{tt}^{\eta m}(t)$, we have

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|u_{tt}^{\eta m}|^2 + 2M'\left(|\nabla\varphi(t)|^2\right)(\nabla\varphi,\nabla\varphi')(\nabla u^{\eta m},\nabla u_{tt}^{\eta m}) \\ &\quad + M\left(|\nabla\varphi(t)|^2\right)(\nabla u_t^{\eta m},\nabla u_{tt}^{\eta m}) \\ &+ (h'(u_t^{\eta m})u_{tt}^{\eta m}, u_{tt}^{\eta m})_{\Gamma_1} + \eta\,\|u_{tt}^{\eta m}(s)\|_{2,\Gamma_1}^2 = \left(k(x)\,|u^{\eta m}(t)|^{k(x)-1} u_t^{\eta m}(t), u_{tt}^{\eta m}\right)_{\Gamma_1} \\ &+ \left(p(x)\,|u^{\eta m}(t)|^{p(x)-1} u_t^{\eta m}(t), u_{tt}^{\eta m}\right) + g(0)\frac{\mathrm{d}}{\mathrm{d}t}(\nabla u^{\eta m}(t),\nabla u_t^{\eta m}) - g(0)\,|\nabla u_t^{\eta m}|^2 \\ &+ \frac{\mathrm{d}}{\mathrm{d}t}\left(\int_0^t g'(t-s)\nabla u^{\eta m}(s)\,\mathrm{d}s, \nabla u_t^{\eta m}\right) - g'(0)(\nabla u^{\eta m},\nabla u_t^{\eta m}(t)) \\ &- \left(\int_0^t g''(t-s)\nabla u^{\eta m}(s)\,\mathrm{d}s,\nabla u_t^{\eta m}\right). \end{aligned} \qquad (3.12)$$

To analyze the term $2M'\left(|\nabla\varphi(t)|^2\right)(\nabla\varphi,\nabla\varphi')(\nabla u^{\eta m},\nabla u_{tt}^{\eta m}(t))$, we multiplying both sides of (3.3) by

$$f(t) = \frac{2M'\left(|\nabla\varphi(t)|^2\right)(\nabla\varphi,\nabla\varphi')}{M\left(|\nabla\varphi(t)|^2\right)}\left(\le \frac{2Lm_1m_2}{m_3}\right)$$

and replacing $v = u_{tt}^{\eta m}(t)$, we have

$$\begin{aligned}
2M'\left(|\nabla\varphi(t)|^2\right)(\nabla\varphi,\nabla\varphi')\left(\nabla u^{\eta m},\nabla u_{tt}^{\eta m}\right) &= -f(t)\left|u_{tt}^{\eta m}\right|^2 \\
&+f(t)\left(\int_0^t g(t-s)\nabla u^{\eta m}(s)\,\mathrm{d}s,\nabla u_{tt}^{\eta m}\right) \\
&-f(t)\left(h\left(u_t^{\eta m}\right),u_{tt}^{\eta m}\right)_{\Gamma_1}-\eta f(t)\left(u_t^{\eta m}(t),u_{tt}^{\eta m}\right)_{\Gamma_1} \\
&+f(t)\left(\left|u^{\eta m}(t)\right|^{k(x)-1}u^{\eta m}(t),u_{tt}^{\eta m}\right)_{\Gamma_1} \\
&+f(t)\left(\left|u^{\eta m}(t)\right|^{p(x)-1}u^{\eta m}(t),u_{tt}^{\eta m}\right)
\end{aligned}$$

By replacing above equality in (3.12), we have

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left|u_{tt}^{\eta m}\right|^2+f(t)\left(\int_0^t g(t-s)\nabla u^{\eta m}(s)\,\mathrm{d}s,\nabla u_{tt}^{\eta m}\right) \\
&\quad+f(t)\left(\left|u^{\eta m}(t)\right|^{k(x)-1}u^{\eta m}(t),u_{tt}^{\eta m}\right)_{\Gamma_1} \\
&\quad+f(t)\left(\left|u^{\eta m}(t)\right|^{p(x)-1}u^{\eta m}(t),u_{tt}^{\eta m}\right) \\
&+\eta\left\|u_{tt}^{\eta m}(s)\right\|_{2,\Gamma_1}^2+M\left(|\nabla\varphi(t)|^2\right)\left(\nabla u_t^{\eta m},\nabla u_{tt}^{\eta m}\right)+\left(h'\left(u_t^{\eta m}\right)u_{tt}^{\eta m},u_{tt}^{\eta m}\right)_{\Gamma_1} \\
&\quad= f(t)\left|u_{tt}^{\eta m}\right|^2+\left(k(x)\left|u^{\eta m}(t)\right|^{k(x)-1}u_t^{\eta m}(t),u_{tt}^{\eta m}\right)_{\Gamma_1} \\
&\quad+\left(p(x)\left|u^{\eta m}(t)\right|^{p(x)-1}u_t^{\eta m}(t),u_{tt}^{\eta m}\right) \\
&\quad+\eta f(t)\left(u_t^{\eta m}(t),u_{tt}^{\eta m}\right)_{\Gamma_1}+g(0)\frac{\mathrm{d}}{\mathrm{d}t}\left(\nabla u^{\eta m}(t),\nabla u_t^{\eta m}\right) \\
&\quad-g(0)\left|\nabla u_t^{\eta m}\right|^2+f(t)\left(h\left(u_t^{\eta m}\right),u_{tt}^{\eta m}\right)_{\Gamma_1} \\
&\quad-\left(\int_0^t g''(t-s)\nabla u^{\eta m}(s)\,\mathrm{d}s,\nabla u_t^{\eta m}\right) \\
&+\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_0^t g'(t-s)\nabla u^{\eta m}(s)\,\mathrm{d}s,\nabla u_t^{\eta m}\right)-g'(0)\left(\nabla u^{\eta m},\nabla u_t^{\eta m}(t)\right).
\end{aligned} \tag{3.13}$$

Next, we are going to analyze the term on the right-hand side of (3.13), taking in mind the estimates (3.9) and (3.10).
Estimate for I_1 :

$$|I_1| = \left|f(t)\left(h\left(u_t^{\eta m}\right),u_{tt}^{\eta m}\right)_{\Gamma_1}\right| \le \frac{\eta}{8}\left\|u_{tt}^{\eta m}(s)\right\|_{2,\Gamma_1}^2+\frac{4Lm_1m_2}{\eta m_3}C_h\left\|u_t^{\eta m}\right\|_{2,\Gamma_1}^2 \tag{3.14}$$

Estimate for I_2 :

$$|I_2| = \left|-\int_\Omega h'\left(u_t^{\eta m}\right)u_t^{\eta m}(t)u_{tt}^{\eta m}(t)\mathrm{d}x\right| \le \frac{C_hC_1}{2}+\frac{C_h}{2}\left|u_{tt}^{\eta m}(t)\right|^2 \tag{3.15}$$

Estimate for I_3 : From the generalized Hölder's inequality, Young's inequality and the conditions (2.14), we have

$$|I_3| = \left| f(t) \left(|u^{\eta m}(t)|^{k(x)-1} u^{\eta m}(t), u_{tt}^{\eta m} \right)_{\Gamma_1} \right| \tag{3.16}$$
$$\leq \left(\frac{2Lm_1m_2}{m_3} \right)^2 C(\varepsilon) \max \left(\int_{\Gamma_1} |u^{\eta m}|^{2k^+} \mathrm{d}\Gamma, \int_{\Gamma_1} |u^{\eta m}|^{2k^-} \mathrm{d}\Gamma \right) + \varepsilon \, ||u_{tt}^{\eta m}||_{2,\Gamma_1}^2$$
$$\leq \left(\frac{2Lm_1m_2}{m_3} \right)^2 C(\varepsilon) \max \left(|\nabla u^{\eta m}|^{2k^+}, |\nabla u^{\eta m}|^{2k^-} \right) + \varepsilon \, ||u_{tt}^{\eta m}(t)||_{2,\Gamma_1}^2$$
$$\leq C_\varepsilon + \varepsilon \, ||u_{tt}^{\eta m}(t)||_{2,\Gamma_1}^2 .$$

Estimate for I_4 : From the generalized Hölder's inequality, it hold that

$$|I_4| = \left| \left(k(x) |u^{\eta m}(t)|^{k(x)-1} u_t^{\eta m}(t), u_{tt}^{\eta m} \right)_{\Gamma_1} \right| \tag{3.17}$$
$$\leq k^+ \max \left(\int_{\Gamma_1} |u^{\eta m}|^{k^+-1} |u_t^{\eta m}| \, |u_{tt}^{\eta m}(t)| \, \mathrm{d}\Gamma, \int_{\Gamma_1} |u^{\eta m}|^{k^--1} |u_t^{\eta m}| \, |u_{tt}^{\eta m}(t)| \, \mathrm{d}\Gamma \right)$$
$$\leq k^+ \max \begin{pmatrix} ||u^{\eta m}(t)||_{2k^+,\Gamma_1}^{k^+-1} ||u_t^{\eta m}(t)||_{2k^+,\Gamma_1} ||u_{tt}^{\eta m}(t)||_{2,\Gamma_1}, \\ ||u^{\eta m}(t)||_{2k^-,\Gamma_1}^{k^--1} ||u_t^{\eta m}(t)||_{2k^-,\Gamma_1} ||u_{tt}^{\eta m}(t)||_{2,\Gamma_1} \end{pmatrix}$$
$$\leq k^+ \max \left(|\nabla u^{\eta m}|^{k^+-1}, |\nabla u^{\eta m}|^{k^--1} \right) |\nabla u_t^{\eta m}| \, ||u_{tt}^{\eta m}(t)||_{2,\Gamma_1}$$
$$\leq C(\varepsilon) |\nabla u_t^{\eta m}|^2 + \varepsilon \, ||u_{tt}^{\eta m}(t)||_{2,\Gamma_1}^2$$

Estimate for I_5 :

$$|I_5| = \left| \left(p(x) |u^{\eta m}(t)|^{p(x)-1} u_t^{\eta m}(t), u_{tt}^{\eta m} \right) \right|$$
$$\leq p^+ \max \left(\int_\Omega |u^{\eta m}|^{p^+-1} |u_t^{\eta m}| \, |u_{tt}^{\eta m}(t)| \, \mathrm{d}x, \int_\Omega |u^{\eta m}|^{p^--1} |u_t^{\eta m}| \, |u_{tt}^{\eta m}(t)| \, \mathrm{d}x \right)$$
$$\leq p^+ \max \begin{pmatrix} ||u^{\eta m}(t)||_{2p^+}^{p^+-1} ||u_t^{\eta m}(t)||_{2p^+} |u_{tt}^{\eta m}(t)|, \\ ||u^{\eta m}(t)||_{2p^-}^{p^--1} ||u_t^{\eta m}(t)||_{2p^-} |u_{tt}^{\eta m}(t)| \end{pmatrix} \tag{3.18}$$
$$\leq p^+ \max \left(|\nabla u^{\eta m}|^{p^+-1}, |\nabla u^{\eta m}|^{p^--1} \right) |\nabla u_t^{\eta m}| \, |u_{tt}^{\eta m}(t)|$$
$$\leq C(\varepsilon) |\nabla u_t^{\eta m}|^2 + \varepsilon \, |u_{tt}^{\eta m}(t)|^2 .$$

Estimate for I_6 :

$$|I_6| = \left| f(t) \left(|u^{\eta m}(t)|^{p(x)-1} u^{\eta m}(t), u_{tt}^{\eta m} \right) \right|$$
$$\leq \frac{2Lm_1m_2}{m_3} \max \left(\int_\Omega |u^{\eta m}|^{p^+} |u_{tt}^{\eta m}(t)| \, \mathrm{d}x, \int_\Omega |u^{\eta m}|^{p^-} |u_{tt}^{\eta m}(t)| \, \mathrm{d}x \right)$$
$$\leq \max \left(|\nabla u^{\eta m}|^{p^+}, |\nabla u^{\eta m}|^{p^-} \right) |u_{tt}^{\eta m}(t)| \leq C_\varepsilon + \varepsilon \, |u_{tt}^{\eta m}(t)|^2$$

Estimate for I_7 :

$$I_7 = \left|\eta f(t)\left(u_t^{\eta m}(t), u_{tt}^{\eta m}\right)_{\Gamma_1}\right| \le \frac{\eta}{8}\left\|u_{tt}^{\eta m}(t)\right\|_{2,\Gamma_1}^2 + 2\eta\left(\frac{2Lm_1m_2}{m_3}\right)^2 \left\|u_t^{\eta m}(t)\right\|_{2,\Gamma_1}^2$$

Estimate for I_8 :

$$I_8 = \left|-g'(0)\left(\nabla u^{\eta m}, \nabla u_t^{\eta m}(t)\right)\right| \le C_\varepsilon + C(\varepsilon)\left|\nabla u_t^{\eta m}\right|^2$$

Estimate for I_9 :

$$\begin{aligned}
I_9 = \left|-\left(\int_0^t g''(t-s)\nabla u^{\eta m}(s)\,\mathrm{d}s, \nabla u_t^{\eta m}\right)\right| &\le \left|\nabla u_t^{\eta m}\right|\int_0^t g''(t-s)\left|\nabla u^{\eta m}\right|\mathrm{d}s \\
&\le C(\varepsilon)\left|\nabla u_t^{m}\right|^2 + \varepsilon\left\|g''\right\|_{L^1}\int_0^t \left|g''(t-s)\right|\left|\nabla u^{\eta m}\right|^2\mathrm{d}s \\
&\le C(\varepsilon)\left|\nabla u_t^{\eta m}\right|^2 + \left(\varepsilon\left\|g''\right\|_{L^1}^2 + \varepsilon\right)\int_0^t \left|\nabla u^{\eta m}\right|^2\mathrm{d}s \\
&\le C(\varepsilon)\left|\nabla u_t^{\eta m}\right|^2 + C_\varepsilon \sup_{(0,T)}\left\|u^{\eta m}(t)\right\|_{L^\infty\left(0,T;H_0^1(\Omega)\right)}^2 .
\end{aligned}$$

Estimate for I_{10} :

$$\begin{aligned}
I_{10} &= \left(\int_0^t g'(t-s)\nabla u^{\eta m}(s)\,\mathrm{d}s, \nabla u_t^{\eta m}\right) \\
&\le \frac{m_3}{8}\left|\nabla u^{\eta m}\right|^2 + \frac{2\xi(0)\left\|g\right\|_{L^1}\left\|g\right\|_{L^\infty}}{m_3}\left|\nabla u_t^{\eta m}\right|^2 \\
&\le \frac{m_3}{8}\left|\nabla u_t^{\eta m}\right|^2 + C(m_3)\sup_{(0,T)}\left\|u^{\eta m}(t)\right\|_{L^\infty\left(0,T;H_0^1(\Omega)\right)}^2 .
\end{aligned}$$

By replacing (3.14)-(3.17) in (3.13) and choosing $\varepsilon = \frac{\eta}{4}$, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left|u_{tt}^{\eta m}\right|^2 + \frac{1}{2}M\left(\left|\nabla\varphi(t)\right|^2\right)\frac{\mathrm{d}}{\mathrm{d}t}\left|\nabla u_t^{\eta m}(t)\right|^2 \\
&\quad + g(0)\left|\nabla u_t^{\eta m}\right|^2 + \frac{\eta}{2}\left\|u_{tt}^{\eta m}(t)\right\|_{2,\Gamma_1}^2 \qquad (3.19) \\
&\le -f(t)\left(\int_0^t g(t-s)\nabla u^{\eta m}(s)\,\mathrm{d}s, \nabla u_{tt}^{\eta m}\right) + g(0)\frac{\mathrm{d}}{\mathrm{d}t}\left(\nabla u^{\eta m}(t), \nabla u_t^{\eta m}\right) \\
&\quad + 3C_\varepsilon + f(t)\left|u_{tt}^{\eta m}\right|^2 + 3C(\varepsilon)\left|\nabla u_t^{\eta m}\right|^2 + 2\varepsilon\left|u_{tt}^{\eta m}(t)\right|^2 \\
&\quad + 2\eta\left(\frac{2Lm_1m_2}{m_3}\right)^2\left\|u_t^{\eta m}(t)\right\|_{2,\Gamma_1}^2 \\
&\quad + \frac{2Lm_1m_2}{m_3}\left|u_{tt}^{\eta m}(t)\right|^2 + \frac{4Lm_1m_2}{\eta m_3}C_h\left\|u_t^{\eta m}\right\|_{2,\Gamma_1}^2 \\
&\quad + \frac{C_hC_1}{2} + \frac{C_h}{2}\left|u_{tt}^{\eta m}(t)\right|^2 + \frac{\mathrm{d}}{\mathrm{d}t}\left(\int_0^t g'(t-s)\nabla u^{\eta m}(s)\,\mathrm{d}s, \nabla u_t^{\eta m}\right).
\end{aligned}$$

Employing Hölder's inequality, Young's inequality, integrating by parts on $(0,t)$, the first and second terms on the right-hand side and the first term on the left-hand side of (3.19) can be estimated as follows, for

$$\left|\int_0^t -f(\zeta)\left(\int_0^\zeta g(\zeta-s)\nabla u^{\eta m}(s)\,\mathrm{d}s, \nabla u_{tt}^{\eta m}\right)\mathrm{d}\zeta\right|$$

$$\leq \frac{2Lm_1m_2}{m_3}\left|\int_0^t\left(\int_0^\zeta g(\zeta-s)\nabla u^{\eta m}(s)\,\mathrm{d}s, \nabla u_{tt}^{\eta m}(\zeta)\right)\mathrm{d}\zeta\right|$$

$$\leq \frac{2Lm_1m_2}{m_3}\left|\left(\int_0^t g(t-s)\nabla u^{\eta m}(s)\,\mathrm{d}s, \nabla u_t^{\eta m}(t)\right)\right|$$

$$+\frac{2Lm_1m_2}{m_3}g(0)\left|\int_0^t\left(\nabla u^{\eta m}(s), \nabla u_t^{\eta m}(s)\right)\mathrm{d}s\right|$$

$$\leq C+\frac{m_3}{8}\left|\nabla u_t^{\eta m}\right|^2+\frac{Lm_1m_2}{m_3}g(0)\left(\int_0^t\left|\nabla u_t^{\eta m}\right|^2\mathrm{d}s+\int_0^t\left|\nabla u^{\eta m}\right|^2\mathrm{d}s\right)$$

$$\leq C+\frac{m_3}{8}\left|\nabla u_t^{\eta m}\right|^2+\frac{Lm_1m_2}{m_3}g(0)\int_0^t\left|\nabla u_t^{\eta m}\right|^2\mathrm{d}s$$

$$+\frac{Lm_1m_2}{m_3}g(0)\sup_{(0,T)}\left|\left|u^{\eta m}(t)\right|\right|^2_{L^\infty\left(0,T;H_0^1(\Omega)\right)}$$

because, from estimate (3.9) we have

$$\frac{2Lm_1m_2}{m_3}\int_\Omega\nabla u_t^{\eta m}(t)\int_0^t g(t-s)\nabla u^{\eta m}(s)\,\mathrm{d}s\mathrm{d}x\leq C\left|\nabla u_t^{\eta m}\right|\|g\|_{L^1(\mathbb{R}_+)}$$

$$\leq C+\frac{m_3}{8}\left|\nabla u_t^{\eta m}\right|^2,$$

and

$$g(0)\int_0^t\frac{\mathrm{d}}{\mathrm{d}t}\left(\nabla u^{\eta m}(t), \nabla u_t^{\eta m}\right)\mathrm{d}s$$

$$\leq\frac{m_3}{8}\left|\nabla u_t^{\eta m}\right|^2+\frac{2}{m_3}g(0)^2\left|\nabla u^{\eta m}\right|^2+g(0)\left|\nabla u_0\right|\left|\nabla u_1\right|$$

$$\leq\frac{m_3}{8}\left|\nabla u_t^{\eta m}\right|^2+\frac{2}{m_3}g(0)^2\sup_{(0,T)}\left|\left|u^{\eta m}(t)\right|\right|^2_{L^\infty\left(0,T;H_0^1(\Omega)\right)}+g(0)\left|\nabla u_0\right|\left|\nabla u_1\right|$$

and

$$\frac{1}{2}\int_0^t M\left(\left|\nabla\varphi(s)\right|^2\right)\frac{\mathrm{d}}{\mathrm{d}t}\left|\nabla u_t^{\eta m}(s)\right|^2\mathrm{d}s$$

$$\geq\left[\frac{1}{2}M\left(\left|\nabla\varphi(s)\right|^2\right)\left|\nabla u_t^{\eta m}\right|^2\right]_0^t-\frac{1}{2}\int_0^t\left[\frac{\mathrm{d}^+}{\mathrm{d}s}M\left(\left|\nabla\varphi(s)\right|^2\right)\right]\left|\nabla u_t^{\eta m}\right|^2\mathrm{d}s$$

$$\geq\left[\frac{1}{2}M\left(\left|\nabla\varphi(s)\right|^2\right)\left|\nabla u_t^{\eta m}\right|^2\right]_0^t-Lm_1m_2\int_0^t\left|\nabla u_t^{\eta m}\right|^2\mathrm{d}s,\quad s\in[0,T_1].$$

Combining, we get

$$\begin{aligned}
&\frac{1}{2}\left|u_{tt}^{\eta m}\right|^{2}+\frac{m_{3}}{8}\left|\nabla u_{t}^{\eta m}\right|^{2}+g\left(0\right)\int_{0}^{t}\left|\nabla u_{t}^{\eta m}\right|^{2}\mathrm{d}s+\eta\int_{0}^{t}\left|\left|u_{tt}^{\eta m}\left(s\right)\right|\right|_{2,\Gamma_{1}}^{2}\\
&\leq\left(\frac{2Lm_{1}m_{2}}{m_{3}}+2\varepsilon+\frac{C_{h}}{2}\right)\int_{0}^{t}\left|u_{tt}^{\eta m}(s)\right|^{2}\mathrm{d}s\\
&+\left(\frac{Lm_{1}m_{2}}{m_{3}}g\left(0\right)+Lm_{1}m_{2}+2C\left(\varepsilon\right)\right)\int_{0}^{t}\left|\nabla u_{t}^{\eta m}\right|^{2}\mathrm{d}s\\
&+\left(\frac{2}{m_{3}}g\left(0\right)^{2}+C\left(m_{3}\right)+\frac{Lm_{1}m_{2}}{m_{3}}g\left(0\right)+C_{\varepsilon}\right)\sup_{(0,T)}\left|\left|u^{\eta m}\left(t\right)\right|\right|_{L^{\infty}\left(0,T;H_{\Gamma_{0}}^{1}(\Omega)\right)}^{2}\\
&+\left(\frac{4Lm_{1}m_{2}}{\eta m_{3}}C_{h}+2\eta\left(\frac{2Lm_{1}m_{2}}{m_{3}}\right)^{2}\right)\int_{0}^{t}\left|\left|u_{t}^{\eta m}\left(t\right)\right|\right|_{2,\Gamma_{1}}^{2}\mathrm{d}s+C_{5}
\end{aligned}$$

where

$$C_{5}=\left(C,C_{h},C_{1},u_{1},u_{0},C_{\varepsilon},T,g\left(0\right),\frac{Lm_{1}m_{2}}{m_{3}}\right).$$

Choosing $\varepsilon=\frac{\eta}{4}$, therefore, by using estimates (3.10), (3.5) and Gronwall's lemma, we arrive at

$$\left|u_{tt}^{\eta m}\right|^{2}+\left|\nabla u_{t}^{\eta m}\right|^{2}+\int_{0}^{t}\left|\nabla u_{t}^{\eta m}\right|^{2}\mathrm{d}s+\int_{0}^{t}\left|\left|u_{tt}^{\eta m}\left(s\right)\right|\right|_{2,\Gamma_{1}}^{2}\mathrm{d}s\leq C_{6} \tag{3.20}$$

where C_6 is a positive constant which is independent of m, η and t.

Thanks to (3.10) and (3.20), we obtain

$$\left(u^{\eta m}\right)\text{ is a bounded sequence in }L^{\infty}\left(0,T_{0};H_{\Gamma_{0}}^{1}\left(\Omega\right)\right), \tag{3.21}$$

$$\left(u_{t}^{\eta m}\right)\text{ is a bounded sequence in }L^{\infty}\left(0,T_{0};H_{\Gamma_{0}}^{1}\left(\Omega\right)\right)\cap L^{2}\left(0,T_{0};L^{2}(\Omega)\right), \tag{3.22}$$

$$\left(u_{tt}^{\eta m}\right)\text{ is bounded in }L^{\infty}\left(0,T_{0};L^{2}(\Omega)\right), \tag{3.23}$$

$$\left(u_{t}^{\eta m}\right)\text{ is a bounded sequence in }L^{2}\left(0,T_{0};L^{2}(\Gamma_{1})\right), \tag{3.24}$$

$$\left(u_{tt}^{\eta m}\right)\text{ is bounded in }L^{2}\left(0,T_{0};L^{2}(\Gamma_{1})\right),$$

By (2.11), (3.22) and (3.24), we have

$$h\left(u_{t}^{\eta m}\right)\text{ is bounded in }L^{2}\left(0,T_{0};L^{2}(\Gamma_{1})\right). \tag{3.25}$$

From (3.21)-(3.24), there exists a subsequence of $(u^{\eta m})$, still denote by $(u^{\eta m})$, such that such that

$$\begin{aligned}
u^{\eta m} &\longrightarrow u^{\eta}\text{ weak star in }L^{\infty}\left(0,T_{0};H_{\Gamma_{0}}^{1}\left(\Omega\right)\right), &(3.26)\\
u_{t}^{\eta m} &\longrightarrow u_{t}^{\eta}\text{ weak star in }L^{\infty}\left(0,T_{0};H_{\Gamma_{0}}^{1}\left(\Omega\right)\right), &(3.27)\\
u_{tt}^{\eta m} &\longrightarrow u_{tt}^{\eta}\text{ weak star in }L^{\infty}\left(0,T_{0};L^{2}(\Omega)\right), &(3.28)\\
u_{t}^{\eta m} &\longrightarrow u_{t}^{\eta}\text{ weakly in }L^{2}\left(0,T_{0};L^{2}(\Gamma_{1})\right), &(3.29)\\
u_{tt}^{\eta m} &\longrightarrow u_{tt}^{\eta}\text{ weakly in }L^{2}\left(0,T_{0};L^{2}(\Gamma_{1})\right), &(3.30)\\
u_{t}^{\eta m} &\longrightarrow u_{t}^{\eta}\text{ weak star in }L^{\infty}\left(0,T_{0};H^{\frac{1}{2}}(\Gamma_{1})\right), &(3.31)
\end{aligned}$$

Since $H^{\frac{1}{2}}(\Gamma_1) \hookrightarrow L^2(\Gamma_1)$ and $H^1(\Gamma_0) \hookrightarrow L^2(\Omega)$ are compact and from Aubin–Lions theorem, we deduce that

$$\begin{aligned}
u^{\eta m} &\longrightarrow u^{\eta} \text{ strongly in } L^2\left(0,T_0;L^2(\Omega)\right),\\
u^{\eta m} &\longrightarrow u^{\eta} \text{ strongly in } L^2\left(0,T_0;L^2(\Gamma_1)\right),\\
u_t^{\eta m} &\longrightarrow u_t^{\eta} \text{ strongly in } L^2\left(0,T_0;L^2(\Omega)\right)\\
u_t^{\eta m} &\longrightarrow u_t^{\eta} \text{ strongly in } L^2\left(0,T_0;L^2(\Gamma_1)\right),
\end{aligned}$$

Consequently, by making use of Lions' Lemma [27, Lemma 1.3.], we have

$$\begin{aligned}
\left|u^{\eta m}(t)\right|^{p(.)-1}u^{\eta m}(t) &\to \left|u^{\eta}(t)\right|^{p(.)-1}u^{\eta}(t) \text{ weakly in } L^2\left(0,T_0;L^2(\Omega)\right)\\
\left|u^{\eta m}(t)\right|^{k(.)-1}u^{\eta m}(t) &\to \left|u^{\eta}(t)\right|^{k(.)-1}u^{\eta}(t) \text{ weakly in } L^2\left(0,T_0;L^2(\Gamma_1)\right).
\end{aligned}$$

From (3.28) and (3.29) and since the injection of $H^{\frac{1}{2}}(\Gamma_1)$ in $L^2(\Gamma_1)$ is compact, there exists a subsequence of $(u^{\eta m})$, still denote by $(u^{\eta m})$, such that

$$u_t^{\eta m} \longrightarrow u_t^{\eta} \text{ a.e. in } Q_0,$$

where $Q_0 = \Gamma_1 \times]0,T_0[$. Then by (2.11), we have

$$h\left(u_t^{\eta m}\right) \to h\left(u_t^{\eta}\right) \text{ a.e. in } Q_0, \tag{3.32}$$

From (3.25) and (3.32) and by using Lions' lemma, we conclude that

$$h\left(u_t^{\eta m}\right) \to h\left(u_t^{\eta}\right) \text{ weakly in } L^2\left(0,T_0;L^2(\Gamma_1)\right) \tag{3.33}$$

The convergences (3.26), (3.28), (3.31), (4.16) and (3.33) permit us to pass to the limit in the (3.3). Since (w_j) is a basis of $H^1_{\Gamma_0}(\Omega) \cap H^2(\Omega)$ and V_m is dense in $H^1_{\Gamma_0}(\Omega) \cap H^2(\Omega)$, after passing to the limit, we obtain

$$\begin{aligned}
&\int_0^{T_0} \left(u_{tt}^{\eta}(t), v\right)\theta(t)\,\mathrm{d}t + \int_0^{T_0} M\left(\left|\nabla\varphi(t)\right|^2\right)\left(\nabla u^{\eta},\nabla v\right)\theta(t)\,\mathrm{d}t\\
&-\int_0^{T_0}\left(\int_0^t g(t-s)\nabla u^{\eta}(s)\,\mathrm{d}s,\nabla v\right)\theta(t)\,\mathrm{d}t + \int_0^{T_0}\left(h\left(u_t^{\eta}\right),v\right)_{\Gamma_1}\theta(t)\,\mathrm{d}t\\
&+\eta\int_0^{T_0}\left(u_t^{\eta}(t),v\right)_{\Gamma_1}\theta(t)\,\mathrm{d}t = \int_0^{T_0}\left(\left|u^{\eta}(t)\right|^{k(x)-1}u^{\eta}(t),v\right)_{\Gamma_1}\theta(t)\,\mathrm{d}t\\
&+\int_0^{T_0}\left(\left|u^{\eta}(t)\right|^{p(x)-1}u^{\eta}(t),v\right)\theta(t)\,\mathrm{d}t,
\end{aligned} \tag{3.34}$$

for all $\theta \in D(0,T)$, and for all $v \in H^1_{\Gamma_0}(\Omega) \cap H^2(\Omega)$.

We can see that the estimates (3.10) and (3.21) are also independent of η. Therefore, by the same argument used to obtain u^{η} from $u^{\eta m}$, we can pass to the limit when

$\eta \to 0$ in u^η, obtaining a function u such that

$$u^\eta \longrightarrow u \text{ weak star in } L^\infty\left(0,T_0;H^1_{\Gamma_0}(\Omega)\right), \tag{3.35}$$

$$u^\eta_t \longrightarrow u_t \text{ weak star in } L^\infty\left(0,T_0;H^1_{\Gamma_0}(\Omega)\right),$$

$$u^\eta_{tt} \longrightarrow u_{tt} \text{ weak star in } L^\infty\left(0,T_0;L^2(\Omega)\right), \tag{3.36}$$

$$u^\eta_t \longrightarrow u_t \text{ weak star in } L^\infty\left(0,T_0;H^{\frac{1}{2}}(\Gamma_1)\right),$$

$$h\left(u^\eta_t\right) \to h\left(u_t\right) \text{ weakly in } L^2\left(0,T_0;L^2(\Gamma_1)\right), \tag{3.37}$$

$$\left|u^\eta(t)\right|^{p(.)-1}u^\eta(t) \to \left|u(t)\right|^{p(.)-1}u(t) \text{ weakly in } L^2\left(0,T_0;L^2(\Omega)\right),$$

$$\left|u^\eta(t)\right|^{k(.)-1}u^\eta(t) \to \left|u(t)\right|^{k(.)-1}u(t) \text{ weakly in } L^2\left(0,T_0;L^2(\Gamma_1)\right)$$

From the above convergence in (3.10) and by observing that V_m is dense in $H^1_{\Gamma_0}(\Omega)\cap H^2(\Omega)$, we have

$$\begin{aligned}
&\int_0^{T_0}(u_{tt}(t),v)\,\theta(t)\,\mathrm{d}t+\int_0^{T_0}M\left(\left|\nabla\varphi(t)\right|^2\right)(\nabla u,\nabla v)\,\theta(t)\,\mathrm{d}t\\
&-\int_0^{T_0}\left(\int_0^t g(t-s)\nabla u(s)\,\mathrm{d}s,\nabla v\right)\theta(t)\,\mathrm{d}t+\int_0^{T_0}\left(h\left(u_t\right),v\right)_{\Gamma_1}\theta(t)\,\mathrm{d}t\\
&=\int_0^{T_0}\left(\left|u(t)\right|^{k(x)-1}u(t),v\right)_{\Gamma_1}\theta(t)\,\mathrm{d}t+\int_0^{T_0}\left(\left|u(t)\right|^{p(x)-1}u(t),v\right)\theta(t)\,\mathrm{d}t,
\end{aligned} \tag{3.38}$$

for all $v\in H^1_{\Gamma_0}(\Omega)$ and for all $\theta\in D(0,T_0)$.
By taking $v\in D(\Omega)$, we get that

$$\frac{\partial^2u}{\partial t^2}-M\left(\left|\nabla\varphi(t)\right|^2\right)\Delta u+\int_0^t g(t-s)\,\Delta u(s)\,\mathrm{d}s=|u|^{p(x)-1}u \text{ in } D'(\Omega).$$

Therefore, by (3.36) and (3.37), we obtain

$$\frac{\partial^2u}{\partial t^2}-M\left(\left|\nabla\varphi(t)\right|^2\right)\Delta u+\int_0^t g(t-s)\,\Delta u(s)\,\mathrm{d}s=|u|^{p(x)-1}u \text{ in } L^2\left(0,T_0;L^2(\Omega)\right). \tag{3.39}$$

From the hypotheses of M, g and (3.35), we conclude that

$$g(t-s)\,u,\ M\left(\left|\nabla\varphi(t)\right|^2\right)u\in L^\infty\left(0,T_0;H^1_{\Gamma_0}(\Omega)\right),$$

and by (3.39),

$$-\Delta\left(M\left(\left|\nabla\varphi(t)\right|^2\right)u-\int_0^t g(t-s)\,u(s)\,\mathrm{d}s\right)\in L^2\left(0,T_0;L^2(\Omega)\right)$$

Then

$$M\left(\left|\nabla\varphi(t)\right|^2\right)\frac{\partial u}{\partial\nu}-\int_0^t g(t-s)\,\frac{\partial}{\partial\nu}u(s)\,\mathrm{d}s\in L^2\left(0,T_0;H^{-\frac{1}{2}}(\Gamma_1)\right)$$

according to Miranda [29] is established. By taking (3.39) into account and making use of the generalized Green formula, we deduce

$$M\left(|\nabla\varphi(t)|^2\right)\frac{\partial u}{\partial\nu}-\int_0^t g(t-s)\frac{\partial}{\partial\nu}u(s)\,\mathrm{d}s+h(u_t)=|u|^{k(x)-1}u$$

in $D'\left(0,T_0;H^{-\frac{1}{2}}(\Gamma_1)\right)$, and as $h(u_t)$, $|u|^{k(.)-1}u\in L^2\left(0,T_0;L^2(\Gamma_1)\right)$, we infer

$$M\left(|\nabla\varphi(t)|^2\right)\frac{\partial u}{\partial\nu}-\int_0^t g(t-s)\frac{\partial}{\partial\nu}u(s)\,\mathrm{d}s+h(u_t)=|u|^{k(x)-1}u \text{ in } L^2\left(0,T_0;L^2(\Gamma_1)\right). \tag{3.40}$$

Prove the uniqueness of the local solution. To this end let $u(t)$ and $v(t)$ be two local solutions to (3.3) with the same initial value. Let $w(t)=u(t)-v(t)$. Then $w(0)=0$, $w_t(0)=0$ for all $t\in[0,T_0]$ and

$$\begin{aligned}(w''(t),\psi)+M\left(|\nabla\varphi(t)|^2\right)(\nabla w,\nabla\psi)-\left(\int_0^t g(t-s)\nabla w(s)\,\mathrm{d}s,\nabla\psi\right)\\+(h(u_t)-h(v_t),\psi)_{\Gamma_1}=\left(|u(t)|^{k(x)-1}u(t)-|v(t)|^{k(x)-1}v(t),\psi\right)_{\Gamma_1}\\+\left(|u(t)|^{p(x)-1}u(t)-|v(t)|^{p(x)-1}v(t),\psi\right)\end{aligned} \tag{3.41}$$

for all $\psi\in H^1_{\Gamma_0}(\Omega)$. By replacing $\psi=w_t(t)$ in (3.41) and observing that $(h(u_t)-h(v_t),\psi)_{\Gamma_1}\geq 0$, it hold that

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|w_t(t)|^2+\frac{1}{2}\frac{\mathrm{d}^+}{\mathrm{d}t}\left(\left(M\left(|\nabla\varphi(t)|^2\right)-\int_0^t g(s)\,\mathrm{d}s\right)|\nabla w(t)|^2\right)\\+\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(g\diamond\nabla w)(t)-\frac{1}{2}(g'\diamond\nabla w)(t)+\frac{1}{2}g(t)|\nabla w(t)|^2\\\leq\frac{1}{2}\left(\frac{\mathrm{d}^+}{\mathrm{d}t}M\left(|\nabla\varphi(t)|^2\right)\right)|\nabla w|^2+\left(|u(t)|^{k(x)-1}u(t)-|v(t)|^{k(x)-1}v(t),w_t(t)\right)_{\Gamma_1}\\+\left(|u(t)|^{p(x)-1}u(t)-|v(t)|^{p(x)-1}v(t),w_t(t)\right)\end{aligned} \tag{3.42}$$

From the generalized Hölder's and Young's inequalities and estimates (3.21)-(3.24), it hold that

$$\begin{aligned}&\left|\left(|u(t)|^{k(x)-1}u(t)-|v(t)|^{k(x)-1}v(t),w_t\right)\right|\\&\leq c\max\left(\begin{array}{c}\left(\|u(t)\|_{2k^-}^{k^--1}+\|v(t)\|_{2k^-}^{k^--1}\right)\|u(t)-v(t)\|_{2k^-}\|w_t\|_2,\\\left(\|u(t)\|_{2k^+}^{k^+-1}+\|v(t)\|_{2k^+}^{k^+-1}\right)\|u(t)-v(t)\|_{2k^+}\|w_t\|_2\end{array}\right)\\&\leq cc_*\max\left(\begin{array}{c}\left(|\nabla u(t)|^{k^--1}+|\nabla v(t)|^{k^--1}\right),\\\left(|\nabla u(t)|^{k^+-1}+|\nabla v(t)|^{k^+-1}\right)\end{array}\right)|\nabla w||w_t|\\&\leq c|\nabla w|^2+c|w_t|^2.\end{aligned}$$

By the same manner

$$\left|\left(|u(t)|^{p(x)-1}u(t)-|v(t)|^{p(x)-1}v(t),w_t\right)\right|$$
$$\leq c\max\begin{pmatrix}\left(\|u(t)\|_{2p^-}^{p^--1}+\|v(t)\|_{2p^-}^{p^--1}\right)\|u(t)-v(t)\|_{2p^-}\|w_t\|_2\,,\\ \left(\|u(t)\|_{2p^+}^{p^+-1}+\|v(t)\|_{2p^+}^{p^+-1}\right)\|u(t)-v(t)\|_{2p^+}\|w_t\|_2\end{pmatrix}$$
$$\leq cc_*\max\begin{pmatrix}\left(|\nabla u(t)|^{p^--1}+|\nabla v(t)|^{p^--1}\right),\\ \left(|\nabla u(t)|^{p^+-1}+|\nabla v(t)|^{p^+-1}\right)\end{pmatrix}|\nabla w|\,|w_t|$$
$$\leq c|\nabla w|^2+c|w_t|^2.$$

Substituting the last two inequalities in (3.42) and integrating the results over $(0,t)$, it holds

$$\frac{1}{2}|w_t(t)|^2+\frac{1}{2}l|\nabla w(t)|^2\leq C\int_0^t\left(|\nabla w|^2+|w_t|^2\right)\mathrm{d}s$$

Thus, employing Gronwall's lemma, we conclude that $|w_t(t)|^2=|\nabla w(t)|^2=0$. Consequently this completes the proof of the lemma. □

We are concerned with the existence and uniqueness of local solution in time to degenerate wave equation (1.1)-(1.4). So by using Lemma 3.1 we prove the existence and uniqueness of local solution in time to (1.1)-(1.4) by the Banach fixed point theorem.

Theorem 3.2. *Assume that $M(r)>0$ is a locally Lipschitz function and assume that the following condition is satisfied*

$$1<k^+<\frac{n-1}{n-2}\text{ and }1<p^+\leq\frac{n}{n-2}\text{ if }n\geq 3,$$
$$1\leq k^-\leq k^+<\infty\text{ and }1\leq p^-\leq p^+<\infty\text{ if }n=2.$$

Let $(u_0,u_1)\in H^1_{\Gamma_0}(\Omega)\cap H^2(\Omega)\times H^1_{\Gamma_0}(\Omega)$ with $|\nabla u_1|\neq 0$ or $|\nabla u_0|\neq 0$. Assume that $M\left(|\nabla u_0|^2\right)>0$. Then there exists a time $T_0>0$ and a unique local weak solution $u(t)$ to (1.1)-(1.4) with the initial value (u_0,u_1) satisfying

$$u(t)\in C([0,T_0]:H^1_{\Gamma_0}(\Omega)),$$
$$u_t(t)\in C([0,T_0]:L^2(\Omega))\cap C([0,T_0]:H^1_{\Gamma_0}(\Omega)),$$
$$u_{tt}(t)\in C([0,T_0]:L^2(\Omega)).$$

Proof. Since $M\left(|\nabla u_0|^2\right)>0$, there exists a positive real number m_3 such that $0<m_3<M\left(|\nabla u_0|^2\right)$. Assume that

$$0<m_3-\int_0^{+\infty}g(t)\,\mathrm{d}t<1.$$

Let R_0 be a positive real number such that

$$R_0 = \sqrt{\frac{2}{l}\left(|\nabla u_1|^2 + M\left(|\nabla u_0|^2\right)|\nabla u_0|^2\right)}$$

Since $M\left(|\nabla u_0|^2\right) > 0$, for sufficiently small time $T > 0$, we define the space $B_T(R_0)$ by

$$B_T(R_0) = \left\{ \begin{array}{c} \phi(t) \in C([0,T] : H^1_{\Gamma_0}(\Omega)) \cap C([0,T] : H^1_{\Gamma_0}(\Omega)), \\ \phi'(t) \in C([0,T] : L^2(\Omega)) \cap C([0,T] : H^1_{\Gamma_0}(\Omega)), \\ \phi''(t) \in C([0,T] : L^2(\Omega)), \\ M\left(|\nabla\phi(t)|^2\right) \geq m_3,\ \ |\nabla\phi'(t)|^2 + |\nabla\phi(t)|^2 \leq R_0^2 \text{ on } [0,T], \\ \phi(0) = u_0,\ \phi'(0) = u_1. \end{array} \right\}$$

We introduce the metric d on the space $B_T(R_0)$ by

$$\mathrm{d}(u,v) = \sup_{0\leq t\leq T}\left(|u_t(t) - v_t(t)|^2 + |\nabla u(t) - \nabla v(t)|^2\right) \text{ for } u,\ v \in B_T(R_0).$$

Then the space $B_T(R_0)$ is the complete metric space. Let $\phi \in B_T(R_0)$.
Then $|\nabla\phi(t)| \leq R_0$, $|\nabla\phi'(t)| \leq R_0$ and $M\left(|\nabla\phi(t)|^2\right) \geq m_3$ for all $t \in [0,T]$. Thus thanks to Lemma 3.1 we obtain a unique local weak solution $u(t)$ on $[0,T_1]$ with $T_1 \leq T$ to the following wave equation:

$$\begin{aligned} (u_{tt}(t), v) + M\left(|\nabla\varphi(t)|^2\right)(\nabla u, \nabla v) - \left(\int_0^t g(t-s)\nabla u(s)\,\mathrm{d}s, \nabla v\right) + (h(u_t), v)_{\Gamma_1} \\ = \left(|u(t)|^{k(x)-1}u(t), v\right)_{\Gamma_1} + \left(|u(t)|^{p(x)-1}u(t), v\right) \\ \text{in } L^2\left(0,T_1;H^{-1}(\Omega)\right) \cap L^2\left(0,T_1;H^{-\frac{1}{2}}(\Gamma_1)\right). \end{aligned} \tag{3.43}$$

Let $T = T_1$ without loss of generality. Define the mapping Φ by

$$\Phi(\varphi) = u$$

Then we have that

$$\Phi(\varphi) = u \in B_T(R_0) \text{ for } \varphi \in B_T(R_0), \tag{3.44}$$

$$\Phi : B_T(R_0) \to B_T(R_0) \text{ is a contractive mapping.} \tag{3.45}$$

For showing (3.44), posing $v = u_t$ in (3.43) and taking

$$(h(u_t), u_t)_{\Gamma_1} - \frac{1}{2}(g' \diamond \nabla u)(t) + \frac{1}{2}g(t)|\nabla u(t)|^2 \geq 0,$$

into account we have that:

$$\frac{1}{2}\frac{\mathrm{d}^+}{\mathrm{d}t}\left(|u_t(t)|^2+\left(\left(M\left(|\nabla\varphi(t)|^2\right)-\int_0^t g(s)\,\mathrm{d}s\right)|\nabla u(t)|^2\right)+\frac{1}{2}(g\diamond\nabla u)(t)\right)$$
$$\leq\frac{1}{2}\left(\frac{\mathrm{d}^+}{\mathrm{d}t}M\left(|\nabla\varphi(t)|^2\right)\right)|\nabla u|^2$$
$$+\left(|u(t)|^{k(x)-1}u(t),u_t\right)_{\Gamma_1}+\left(|u(t)|^{p(x)-1}u(t),u_t\right)=I_1+I_2+I_3.$$

And so we estimates I_1 and I_2 as follows

$$I_1=\frac{1}{2}\left(\frac{\mathrm{d}^+}{\mathrm{d}t}M\left(|\nabla\varphi(t)|^2\right)\right)|\nabla u|^2\leq L|\nabla\varphi(t)||\nabla\varphi'(t)||\nabla u|^2\leq\frac{LR_0^2}{l}\psi_\varphi u(t)$$

Taking estimates (4.9) into account

$$|I_2|=\left|\left(k(x)|u(t)|^{k(x)-1}u(t),u_t\right)_{\Gamma_1}\right|$$
$$\leq k^+\max\left(\int_{\Gamma_1}|u|^{k^+}|u_t(t)|\,\mathrm{d}\Gamma,\int_{\Gamma_1}|u|^{k^-}|u_t|\,\mathrm{d}\Gamma\right)$$
$$\leq k^+\max\left(||u(t)||_{2k^+,\Gamma_1}^{k^+},||u(t)||_{2k^-,\Gamma_1}^{k^-}\right)||u_t(t)||_{2,\Gamma_1}$$
$$\leq k^+\max\left(B_*^{k^+}|\nabla u|^{k^+},B_*^{k^-}|\nabla u|^{k^-}\right)||u_t(t)||_{2,\Gamma_1}$$
$$\leq k^+\max\left((B_*R_0)^{k^+},(B_*R_0)^{k^-}\right)||u_t(t)||_{2,\Gamma_1}\leq C_2$$

similarly

$$|I_3|=\left|\left(p(x)|u(t)|^{p(x)-1}u(t),u_{tt}^{\eta m}\right)\right|$$
$$\leq p^+\max\left(\int_\Omega|u|^{p^+}|u_t(t)|\,\mathrm{d}x,\int_\Omega|u|^{p^-}|u_t(t)|\,\mathrm{d}x\right)$$
$$\leq p^+\max\left(||u(t)||_{2p^+}^{p^+},||u(t)||_{2p^-}^{p}\right)|u_t(t)|$$
$$\leq p^+\max\left(B^{p^+}|\nabla u|^{p^+},B^{p^-}|\nabla u|^{p^-}\right)|u_t(t)|$$
$$\leq p^+\max\left((BR_0)^{p^+},(BR_0)^{p^-}\right)|u_t(t)|\leq C_3\psi_\varphi u(t)^{\frac{1}{2}}$$

because $||u_t(t)||_{2,\Gamma_1}\leq C|\nabla u_t(t)|$ is bounded on $[0,T]$ by Lemma 3.1. Thus

$$\frac{\mathrm{d}^+}{\mathrm{d}t}\psi_\varphi u(t)\leq 2C_2+2C_1\psi_\varphi u(t)+2C_3\psi_\varphi u(t)^{\frac{1}{2}}$$

where

$$\psi_\varphi u(t)=|u_t(t)|^2+\left(\left(M\left(|\nabla\varphi(t)|^2\right)-\int_0^t g(s)\,\mathrm{d}s\right)|\nabla u(t)|^2\right)+(g\diamond\nabla u)(t),$$

and $C_1 = \frac{LR_0^2}{l}$. Gronwall inequality yields

$$\begin{array}{rcl}\psi_\varphi u(t) & \leq & (\psi_\varphi u(0) + 2C_2T_2)\, e^{(2C_1+2C_3)T_2} \\ & < & lR_0^2, \quad 0 \leq t \leq T_2,\end{array}$$

for sufficiently small $0 < T_2 \leq T_1$. Thus

$$lR_0^2 > |u_t(t)|^2 + \left(\left(M\left(|\nabla\varphi(t)|^2\right) - \int_0^t g(s)\,\mathrm{d}s\right)|\nabla u(t)|^2\right) + (g \diamond \nabla u)(t)$$
$$> |u_t(t)|^2 + l\,|\nabla u(t)|^2, \quad (l < 1)$$

We have that

$$R_0^2 > |u_t(t)|^2 + |\nabla u(t)|^2, \ 0 \leq t \leq T_2,$$

Let $T = T_2$ be modified. Thus (3.44) is satisfied. Rest to show (3.45). Let $w = u_1 - u_2$, where $u_1 = \Phi(\varphi_1)$, $u_2 = \Phi(\varphi_2)$ with $\varphi_1, \varphi_2 \in B_T(R_0)$. Then we have that

$$\begin{aligned}(w_{tt}(t), v) + M\left(|\nabla\varphi_1(t)|^2\right)(\nabla w, \nabla v) + (h(u_{1t}) - h(u_{2t}), v)_{\Gamma_1} \\ = \left(M\left(|\nabla\varphi_2(t)|^2\right) - M\left(|\nabla\varphi_1(t)|^2\right)\right)(\nabla u_2, \nabla v) \\ + \left(\int_0^t g(t-s)\,\nabla w(s)\,\mathrm{d}s, \nabla v\right) \\ = \left(|u_1(t)|^{k(x)-1} u_1(t) - |u_2(t)|^{k(x)-1} u_2(t), v\right)_{\Gamma_1} \\ + \left(|u_1(t)|^{p(x)-1} u_1(t) - |u_2(t)|^{p(x)-1} u_2(t), v\right) \text{ in } L^2\left(0, T_1; H^{-1}(\Omega)\right).\end{aligned} \tag{3.46}$$

Set

$$\beta_{\varphi_1}(w)(t) = |w_t(t)|^2 + \left(\left(M\left(|\nabla\varphi_1(t)|^2\right) - \int_0^t g(s)\,\mathrm{d}s\right)|\nabla w(t)|^2\right)$$

Since $0 < l = m_3 - \int_0^\infty g(s)\,\mathrm{d}s < 1$, we have that

$$\beta_{\varphi_1}(w)(t) \geq l\left(|w_t(t)|^2 + |\nabla w(t)|^2\right)$$

By replacing v in (3.46) by w_t we have that

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}^+}{\mathrm{d}t}\left(|w_t(t)|^2 + \left(\left(M\left(|\nabla\varphi_1(t)|^2\right) - \int_0^t g(s)\,\mathrm{d}s\right)|\nabla w(t)|^2\right)\right) \\ + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(g \diamond \nabla w)(t) - \frac{1}{2}(g' \diamond \nabla w)(t) + \frac{1}{2}g(t)\,|\nabla u(t)|^2 \\ \leq \frac{1}{2}\left(\frac{\mathrm{d}^+}{\mathrm{d}t}M\left(|\nabla\varphi_1(t)|^2\right)\right)|\nabla w|^2 \\ + \left(M\left(|\nabla\varphi_2(t)|^2\right) - M\left(|\nabla\varphi_1(t)|^2\right)\right)(\nabla u_2, \nabla w_t) \\ + \left(|u_1(t)|^{k(x)-1} u_1(t) - |u_2(t)|^{k(x)-1} u_2(t), w_t\right)_{\Gamma_1} \\ + \left(|u_1(t)|^{p(x)-1} u_1(t) - |u_2(t)|^{p(x)-1} u_2(t), w_t\right) = I_4 + I_5 + I_6 + I_7\end{aligned}$$

Then

$$|I_4| = \left|\frac{1}{2}\left(\frac{\mathrm{d}^+}{\mathrm{d}t}M\left(|\nabla\varphi_1(t)|^2\right)\right)|\nabla w|^2\right| \le LR_0^2|\nabla w|^2$$
$$\le \frac{LR_0^2}{l}\beta_{\varphi_1}(w)(t) := \xi_4\beta_{\varphi_1}(w)(t)$$

and

$$|I_5| = \left|\left(M\left(|\nabla\varphi_2(t)|^2\right) - M\left(|\nabla\varphi_1(t)|^2\right)\right)(\nabla u_2, \nabla w_t)\right|$$
$$\le LR_0^2\mathrm{d}(\varphi_1,\varphi_2)^{\frac{1}{2}}|\nabla u_2||\nabla w_t| \le \frac{2LR_0^2}{\sqrt{l}}\mathrm{d}(\varphi_1,\varphi_2)^{\frac{1}{2}}\beta_{\varphi_1}(w)(t)^{\frac{1}{2}}$$
$$:= \xi_5\mathrm{d}(\varphi_1,\varphi_2)^{\frac{1}{2}}\beta_{\varphi_1}(w)(t)^{\frac{1}{2}}.$$

Since

$$|I_6| = \left|\left(|u_1(t)|^{k(x)-1}u_1(t) - |u_2(t)|^{k(x)-1}u_2(t), w_t\right)_{\Gamma_1}\right|$$
$$\le c\max\begin{pmatrix} \left(||u_1(t)||_{2k^-}^{k^--1} + ||u_2(t)||_{2k^-}^{k^--1}\right)||u_1(t)-u_2(t)||_{2k^-}\|w_t\|_2, \\ \left(||u_1(t)||_{2k^+}^{k^+-1} + ||u_2(t)||_{2k^+}^{k^+-1}\right)||u_1(t)-u_2(t)||_{2k^+}\|w_t\|_2 \end{pmatrix}$$
$$\le cc_*\max\begin{pmatrix} \left(|\nabla u_1(t)|^{k^--1} + |\nabla u_2(t)|^{k^--1}\right), \\ \left(|\nabla u_1(t)|^{k^+-1} + |\nabla u_2(t)|^{k^+-1}\right) \end{pmatrix}|\nabla w||w_t|$$
$$\le 2cc_*\left(\sqrt{C_1^{k^--1}} + \sqrt{C_1^{k^+-1}}\right)|\nabla w||w_t|$$
$$\le cc_*\frac{1}{l}\left(\sqrt{C_1^{k^--1}} + \sqrt{C_1^{k^+-1}}\right)\beta_{\varphi_1}(w)(t) := \zeta_6\beta_{\varphi_1}(w)(t)$$

and

$$|I_7| = \left|\left(|u_1(t)|^{p(x)-1}u_1(t) - |u_2(t)|^{p(x)-1}u_2(t), w_t\right)\right|$$
$$\le c\max\begin{pmatrix} \left(||u_1(t)||_{2p^-}^{p^--1} + ||u_2(t)||_{2p^-}^{p^--1}\right)||u_1(t)-u_2(t)||_{2p^-}\|w_t\|_2, \\ \left(||u_1(t)||_{2p^+}^{p^+-1} + ||u_2(t)||_{2p^+}^{k^+-1}\right)||u_1(t)-u_2(t)||_{2p^+}\|w_t\|_2 \end{pmatrix}$$
$$\le cc_*\max\begin{pmatrix} \left(|\nabla u_1(t)|^{p^--1} + |\nabla u_2(t)|^{p^--1}\right), \\ \left(|\nabla u_1(t)|^{p^+-1} + |\nabla u_2(t)|^{p^+-1}\right) \end{pmatrix}|\nabla w||w_t|$$
$$\le 2cc_*\left(\sqrt{C_1^{p^--1}} + \sqrt{C_1^{p^+-1}}\right)|\nabla w||w_t|$$
$$\le cc_*\frac{1}{l}\left(\sqrt{C_1^{p^--1}} + \sqrt{C_1^{p^+-1}}\right)\beta_{\varphi_1}(w)(t) := \zeta_7\beta_{\varphi_1}(w)(t)$$

It follows that

$$\beta_{\varphi_1}(w)(t) \le (\xi_4 + \zeta_6 + \zeta_7)\int_0^t \beta_{\varphi_1}(w)(s)\,\mathrm{d}s + \xi_5\int_0^t \mathrm{d}(\varphi_1,\varphi_2)^{\frac{1}{2}}\beta_{\varphi_1}(w)(s)^{\frac{1}{2}}\,\mathrm{d}s$$

Gronwall's lemma gives

$$\mathrm{d}\left(u_1, u_2\right) \leq \frac{\xi_5^2 T}{l} \mathrm{d}\left(\varphi_1, \varphi_2\right) e^{(1+\xi_4+\zeta_6+\zeta_7)T}.$$

Choose a $0 < T_3 \leq T$ small enough which satisfies that

$$\frac{\xi_5^2}{l} T_3 e^{(1+\xi_4+\zeta_6+\zeta_7)T_3} < 1.$$

Thus by the Banach contraction mapping theorem there exists a fixed point

$$u = \Phi(u) \in B_{T_3}(R_0),$$

which is a unique local weak solution in time to (1.1)-(1.4). This completes the proof of the theorem. □

4. Uniform decay rates

In this section, we shall prove the general decay rates of solution for system (1.1)-(1.4).

In this section we assume that

$$\begin{gathered} M\left(|\nabla u|^2\right) = m_3 + b\,|\nabla u|^2 + \sigma \int_\Omega \nabla u \nabla u_t \mathrm{d}x, \\ m_3 > 0,\ b > 0,\ \sigma : \text{positive and small enough.} \end{gathered} \tag{4.1}$$

and providing that h satisfies:

(H′3) Hypotheses on h**.** $h : \mathbb{R} \to \mathbb{R}$ is a non-decreasing function with $h(s)s \geq 0$ for all $s \in \mathbb{R}$ and there exists a convex and increasing function $H : \mathbb{R}_+ \to \mathbb{R}_+$ of class $C^1(\mathbb{R}_+) \cap C^2((0,\infty))$ satisfying $H(0) = 0$ and H is linear on $[0, r]$ or $H'(0) = 0$ and $H'' > 0$ on $(0, r]$ $(r > 0)$ such that

$$\begin{gathered} m_1\,|s| \leq |h\,(s)| \leq M_1\,|s| \ \text{ if } \ |s| \geq r, \\ h^2\,(s) \leq H^{-1}\,(sh\,(s)) \ \text{ if } \ |s| \leq r, \end{gathered} \tag{4.2}$$

where r, m_1 and M_1 are positive constants.

For formulate our results it is convenient to introduce the energy of the system

$$E\,(t) = \frac{1}{2}\,|u_t(t)|^2 + J\,(u\,(t)) \ \text{ for } u \in H^1_{\Gamma_0}(\Omega) \tag{4.3}$$

where

$$\begin{aligned} J\,(u\,(t)) = {} & \frac{1}{2}\left(m_3 - \int_0^t g\,(s)\,\mathrm{d}s\right) \|\nabla u(t)\|_2^2 + \frac{b}{4}\,\|\nabla u\|_2^4 + \frac{1}{2}\,(g \circ \nabla\,(u))\,(t) \\ & - \int_\Omega \frac{1}{p\,(x)+1}\,|u|^{p(x)+1}\,\mathrm{d}x - \int_{\Gamma_1} \frac{1}{k\,(x)+1}\,|u|^{k(x)+1}\,\mathrm{d}\Gamma, \end{aligned} \tag{4.4}$$

so, we have

$$\begin{aligned}
J(u(t)) \geq & \frac{1}{2}\left(m_3-\int_0^t g(s)\,\mathrm{d}s\right)\|\nabla u(t)\|_2^2+\frac{b}{4}\|\nabla u\|_2^4+\frac{1}{2}(g\circ\nabla(u))(t) \\
& -\frac{1}{p^-+1}\max\left(\int_\Omega |u|^{p^++1}\,\mathrm{d}x, \int_\Omega |u|^{p^-+1}\,\mathrm{d}x\right) \\
& -\frac{1}{k^-+1}\max\left(\int_{\Gamma_1} |u|^{k^++1}\,\mathrm{d}\Gamma, \int_{\Gamma_1} |u|^{k^-+1}\,\mathrm{d}\Gamma\right) \\
\geq & \frac{1}{2}l\|\nabla u(t)\|_2^2+\frac{b}{4}\|\nabla u\|_2^4+\frac{1}{2}(g\circ\nabla(u))(t) \\
& -\left(\frac{1}{p^-+1}\int_\Omega |u|^{p^++1}\,\mathrm{d}x+\frac{1}{k^-+1}\int_{\Gamma_1}|u|^{k^++1}\,\mathrm{d}\Gamma\right) \\
& -\left(\frac{1}{p^-+1}\int_\Omega |u|^{p^-+1}\,\mathrm{d}x+\frac{1}{k^-+1}\int_{\Gamma_1}|u|^{k^-+1}\,\mathrm{d}\Gamma\right),
\end{aligned} \tag{4.5}$$

then

$$E'(t)=-\sigma\left(\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla u(t)\|_2^2\right)^2-\int_{\Gamma_1} u_t h(u_t)\,\mathrm{d}\Gamma+\frac{1}{2}(g'\circ\nabla(u))(t)-\frac{1}{2}g(t)\|\nabla u(t)\|_2^2\leq 0, \tag{4.6}$$

so the energy $E(t)$ is nonincreasing function.

Next, with some modifications, we define a functionals $F_{1,2}$ introduced by Cavalcanti et al. [28], which helps in establishing desired results. Setting

$$F_1(x)=\frac{1}{4}x^2-\frac{K_{-,\Omega}^{p^-+1}}{p^-+1}x^{p^-+1}-\frac{K_{-,\Gamma}^{k^-+1}}{k^-+1}x^{k^-+1},\ x>0 \tag{4.7}$$

$$F_2(x)=\frac{1}{4}x^2-\frac{K_{+,\Omega}^{p^++1}}{p^-+1}x^{p^++1}-\frac{K_{+,\Gamma}^{k^++1}}{k^-+1}x^{k^++1},\ x>0, \tag{4.8}$$

where

$$0<K_{+,\Omega}=\sup_{u\in H^1_{\Gamma_0}(\Omega),\ u\neq 0}\left(\frac{\|u\|_{p^++1}}{\sqrt{l\|\nabla u\|_2^2+\frac{b}{2}\|\nabla u\|_2^4}}\right)<\infty, \tag{4.9}$$

$$0<K_{-,\Omega}=\sup_{u\in H^1_{\Gamma_0}(\Omega),\ u\neq 0}\left(\frac{\|u\|_{p^-+1}}{\sqrt{l\|\nabla u\|_2^2+\frac{b}{2}\|\nabla u\|_2^4}}\right)<\infty. \tag{4.10}$$

and

$$0<K_{+,\Gamma}=\sup_{u\in H^1_{\Gamma_0}(\Omega),\ u\neq 0}\left(\frac{\|u\|_{k^++1,\Gamma_1}}{\sqrt{l\|\nabla u\|_2^2+\frac{b}{2}\|\nabla u\|_2^4}}\right)<\infty, \tag{4.11}$$

$$K_{-,\Gamma}=\sup_{u\in H^1_{\Gamma_0}(\Omega),\ u\neq 0}\left(\frac{\|u\|_{k^-+1,\Gamma_1}}{\sqrt{l\|\nabla u\|_2^2+\frac{b}{2}\|\nabla u\|_2^4}}\right)<\infty. \tag{4.12}$$

Remark 4.1. **(i).** As in [28], we can verify that the functional F_1 is increasing in $(0,\lambda_1)$, decreasing in (λ_1,∞), and F_1 has a maximum at λ_1 with the maximum value

$$\mathrm{d}_1 = F_1(\lambda_1) = \frac{1}{4}\lambda_1^2 - \frac{K_{-,\Omega}^{p^-+1}}{p^-+1}\lambda_1^{p^-+1} - \frac{K_{-,\Gamma}^{k^-+1}}{k^-+1}\lambda_1^{k^-+1}, \tag{4.13}$$

also, for F_2 is increasing in $(0,\lambda_2)$, decreasing in (λ_2,∞), and F_2 has a maximum at λ_2 with the maximum value

$$\mathrm{d}_2 = F_2(\lambda_2) = \frac{1}{4}\lambda_2^2 - \frac{K_{+,\Omega}^{p^++1}}{p^-+1}\lambda_2^{p^++1} - \frac{K_{+,\Gamma}^{k^++1}}{k^-+1}\lambda_2^{k^++1}, \tag{4.14}$$

λ_1 and λ_2 are the first positive zero of the derivative functions $F_1'(x)$ and $F_2'(x)$, respectively.

(ii). From (4.3), (4.5), (2.9), (2.12) and the definition of F_1 and F_2 we have

$$\begin{aligned} E(t) \geq J(t) \geq \frac{1}{4}\gamma(t)^2 - \frac{K_{-,\Omega}^{p^-+1}}{p^-+1}\gamma(t)^{p^-+1} - \frac{K_{-,\Gamma}^{k^-+1}}{k^-+1}\gamma(t)^{k^-+1} \\ +\frac{1}{4}\gamma(t)^2 - \frac{K_{+,\Omega}^{p^++1}}{p^-+1}\gamma(t)^{p^++1} - \frac{K_{+,\Gamma}^{k^++1}}{k^-+1}\gamma(t)^{k^++1} = F_1(\gamma(t)) + F_2(\gamma(t)),\ t \geq 0, \end{aligned} \tag{4.15}$$

where

$$\gamma(t) = \sqrt{l\left\|\nabla u\right\|_2^2 + \frac{b}{2}\left\|\nabla u\right\|_2^4 + (g \circ \nabla(u))(t)}$$

Now, if one considers $\gamma(t) < \lambda_0 = \min(\lambda_1,\lambda_2)$, then, from (4.15), we get

$$\begin{aligned} E(t) &\geq F_1(\gamma(t)) + F_2(\gamma(t)) \\ &> \gamma(t)^2\left(\frac{1}{4} - \frac{K_{-,\Omega}^{p^-+1}}{p^-+1}\gamma(t)^{p^--1} - \frac{K_{-,\Gamma}^{k^-+1}}{k^-+1}\gamma(t)^{k^--1}\right) \\ &+\gamma(t)^2\left(\frac{1}{4} - \frac{K_{+,\Omega}^{p^++1}}{p^-+1}\gamma(t)^{p^+-1} - \frac{K_{+,\Gamma}^{k^++1}}{k^-+1}\gamma(t)^{k^+-1}\right),\ t \geq 0, \end{aligned}$$

which together with the identities

$$\frac{1}{2} - K_{-,\Omega}^{p^-+1}\gamma(t)^{p^--1} - K_{-,\Gamma}^{k^-+1}\gamma(t)^{k^--1} = 0, \text{ and} \tag{4.16}$$

$$\frac{1}{2} - \frac{p^++1}{p^-+1}K_{+,\Omega}^{p^++1}\gamma(t)^{p^+-1} - \frac{k^++1}{k^-+1}K_{+,\Gamma}^{k^++1}\gamma(t)^{k^+-1} = 0 \tag{4.17}$$

give

$$F_1(\gamma(t)) > c_0\gamma(t)^2,\ c_0 = \begin{cases} \frac{p^--1}{4(p^-+1)} \text{if } k^- \geq p^- \\ \frac{k^--1}{4(k^-+1)} \text{if } p^- \geq k^- \end{cases},$$

also, since $\frac{p^{+}+1}{p^{-}+1}>1$ and $\frac{k^{+}+1}{k^{-}+1}>1$ and from (4.17) we deduce that

$$\begin{aligned} 0 &= \frac{1}{2}-\frac{p^{+}+1}{p^{-}+1}K_{+,\Omega}^{p^{+}+1}\gamma\left(t\right)^{p^{+}-1}-\frac{k^{+}+1}{k^{-}+1}K_{+,\Gamma}^{k^{+}+1}\gamma\left(t\right)^{k^{+}-1} \\ &\leq \frac{1}{2}-K_{+,\Omega}^{p^{+}+1}\gamma\left(t\right)^{p^{+}-1}-K_{+,\Gamma}^{k^{+}+1}\gamma\left(t\right)^{k^{+}-1}, \end{aligned}$$

therefore

$$-K_{+,\Omega}^{p^{+}+1}\gamma\left(t\right)^{p^{+}-1}-K_{+,\Gamma}^{k^{+}+1}\gamma\left(t\right)^{k^{+}-1}\geq -\frac{1}{2},$$

and consequently,

$$\begin{aligned} F_{2}\left(t\right) &> \gamma\left(t\right)^{2}\left(\frac{1}{4}-\frac{K_{+,\Omega}^{p^{+}+1}}{p^{-}+1}\gamma\left(t\right)^{p^{+}-1}-\frac{K_{+,\Gamma}^{k^{+}+1}}{k^{-}+1}\gamma\left(t\right)^{k^{+}-1}\right) \\ &> c_{0}\gamma\left(t\right)^{2}, \quad c_{0}=\left\{\begin{array}{l} \frac{p^{-}-1}{4(p^{-}+1)}\text{if } k^{-}\geq p^{-} \\ \frac{k^{-}-1}{4(k^{-}+1)}\text{if } p^{-}\geq k^{-} \end{array}\right., \end{aligned}$$

consequently

$$E\left(t\right)\geq F_{1}\left(\gamma\left(t\right)\right)+F_{2}\left(\gamma\left(t\right)\right)=F\left(\gamma\left(t\right)\right)\geq 2c_{0}\gamma\left(t\right)^{2} \tag{4.18}$$

and identities (4.16), (4.17) are derived because λ_1 and λ_2 are the first positive zero of the derivative function $F_1'(x)$ and $F_2'(x)$ respectively.

Lemma 4.2. *Let $(u_0,u_1)\in H_{\Gamma_0}^{1}\left(\Omega\right)\times H_{\Gamma_0}^{1}\left(\Omega\right)$ and hypotheses (H_1)-(H_3) hold. Assume further that $\gamma\left(0\right)=\sqrt{l\left\Vert \nabla u_0\right\Vert_2^2+\frac{b}{2}\left\Vert \nabla u_0\right\Vert_2^4}<\lambda_0$ and $E\left(0\right)<\mathrm{d}=\min\left(\mathrm{d}_1,\mathrm{d}_2\right)$. Then*

$$\gamma\left(t\right)=\sqrt{l\left\Vert \nabla u\right\Vert_2^2+\frac{b}{2}\left\Vert \nabla u\right\Vert_2^4+\left(g\circ\nabla\left(u\right)\right)\left(t\right)}<\lambda_0, \tag{4.19}$$

for all $t\in[0,T)$.

Proof. Using (4.15) and considering $E(t)$ is a non-increasing function, we obtain

$$F\left(\gamma\left(t\right)\right)=F_1\left(\gamma\left(t\right)\right)+F_2\left(\gamma\left(t\right)\right)\leq E\left(t\right)\leq E\left(0\right)<\mathrm{d},\ t\in[0,T) \tag{4.20}$$

In addition, from Remark 4.1 (i), we see that F is increasing in $(0,\lambda_0)$, decreasing in $(\max\left(\lambda_1,\lambda_2\right),\infty)$, and $F\to-\infty$ as $\max\left(\lambda_1,\lambda_2\right)\to\infty$. Thus, as $E(0)<\mathrm{d}$, there exist $0\leq\lambda_3'\leq\lambda_0\leq\lambda_3$ such that $F\left(\lambda_3'\right)=F\left(\lambda_3\right)=E\left(0\right)$. Besides, through the assumption $\gamma\left(0\right)<\lambda_0$, we observe for $t=0$ that

$$F\left(\gamma\left(0\right)\right)\leq E\left(0\right)=F\left(\lambda_3'\right).$$

This implies that $\gamma\left(0\right)\leq\lambda_3'$. Next, we will prove that

$$\gamma\left(t\right)\leq\lambda_3',\ t\in[0,T). \tag{4.21}$$

To establish (4.21), we reason by absurd. Suppose that (4.21) does not hold, then there exists $t^*\in(0,T)$ such that $\gamma(t^*)>\lambda_3'$.
Case 1. If $\lambda_3'<\gamma(t^*)<\lambda_0$, then

$$F\left(\gamma(t^*)\right)>F\left(\lambda_3'\right)=E\left(0\right)\geq E\left(t^*\right).$$

This contradicts (4.20).

Case 2. If $\gamma(t^*) \geq \lambda_0$, then by continuity of $\gamma(t)$, there exists $0 < t_1 < t^*$ such that

$$\lambda_3' < \gamma(t_1) < \lambda_0,$$

then

$$F(\gamma(t_1)) > F(\lambda_3') = E(0) \geq E(t_1).$$

This is also a contradiction of (4.20). Thus, we have proved (4.21). □

Theorem 4.3. *Under the hypotheses of Lemma* 4.2 *the problem (*1.1*)-(*1.4*) have a global solution.*

Proof. It follows from (4.19), (4.18) and (4.15) that

$$\frac{1}{2}|u_t|^2 + 2c_0\gamma(t)^2 \leq \frac{1}{2}|u_t|^2 + F(\gamma(t)) \leq \frac{1}{2}|u_t|^2 + J(t) = E(t) < E(0) < \mathrm{d}. \quad (4.22)$$

Thus, we establish the boundedness of u_t in $L^2(\Omega)$ and the boundedness of u in $H^1_{\Gamma_0}$. Moreover, from (2.13), (2.14) and (4.22), we also obtain

$$\begin{aligned}
&\int_\Omega \frac{1}{p(x)+1}|u|^{p(x)+1}\mathrm{d}x + \int_{\Gamma_1}\frac{1}{k(x)+1}|u|^{k(x)+1}\mathrm{d}\Gamma \\
&\quad\leq \frac{1}{p^-+1}\max\left(\int_\Omega |u|^{p^++1}\mathrm{d}x, \int_\Omega |u|^{p^-+1}\mathrm{d}x\right) \\
&\quad\quad+\frac{1}{k^-+1}\max\left(\int_{\Gamma_1}|u|^{k^++1}\mathrm{d}\Gamma, \int_{\Gamma_1}|u|^{k^-+1}\mathrm{d}\Gamma\right) \\
&\leq \frac{1}{p^-+1}\max\left(B^{p^++1}|\nabla u|^{p^+-1}, B^{p^-+1}|\nabla u|^{p^--1}\right)|\nabla u|^2 \\
&\quad+\frac{1}{k^-+1}\max\left(B_*^{k^++1}|\nabla|^{k^+-1}, B_*^{k^-+1}|\nabla u|^{k^--1}\right)|\nabla u|^2 \\
&\quad\quad\leq Ll|\nabla|^2 \leq \frac{L}{2c_0}E(t) < \frac{L}{2c_0}E(0) < \frac{L}{2c_0}\mathrm{d}
\end{aligned}$$

which implies that the boundedness of u in $L^{p(.)+1}(\Omega)$ and in $L^{k(.)+1}(\Gamma_1)$ with

$$\begin{aligned}
L = &\frac{1}{l}\left(\frac{1}{p^-+1}\max\left(B^{p^++1}\left(\frac{E(0)}{2lc_0}\right)^{p^+-1}, B^{p^-+1}\left(\frac{E(0)}{2lc_0}\right)^{p^--1}\right)\right) \\
&+\frac{1}{l}\left(\frac{1}{k^-+1}\max\left(B_*^{k^++1}\left(\frac{E(0)}{2lc_0}\right)^{k^+-1}, B_*^{k^-+1}\left(\frac{E(0)}{2lc_0}\right)^{k^--1}\right)\right).
\end{aligned}$$

Hence, it must have $T = \infty$. □

Now, we shall investigate the asymptotic behavior of the energy function $E(t)$. First, let us define the perturbed modified energy by

$$G(t) = ME(t) + \varepsilon\Phi(t) + \Psi(t) \quad (4.23)$$

where

$$\Phi(t) = \int_\Omega u_t u\mathrm{d}x + \frac{\sigma}{4}\|\nabla u(t)\|_2^4, \quad (4.24)$$

$$\Psi(t)=\int_{\Omega}u_t\int_0^t g(t-s)(u(s)-u(t))\,\mathrm{d}s\mathrm{d}x, \tag{4.25}$$

and M, ε are some positive constants to be specified later.

In order to prove the main theorem, we recall the following lemmas.

Lemma 4.4. *There exist two positive constants β_1 and β_2 such that*

$$\beta_1 E(t)\le G(t)\le \beta_2 E(t) \tag{4.26}$$

relation holds, for $\varepsilon>0$ small enough while $M>0$ is large enough.

Proof. By Hölder's and Young's inequalities, (2.9) and (2.12), we deduce that

$$|G(t)-ME(t)|\le\varepsilon|\Phi(t)|+|\Psi(t)|$$

$$\le\frac{\varepsilon+1}{2}|u_t|^2+\frac{\varepsilon B^2}{2}|\nabla u|^2+\frac{\sigma\varepsilon}{4}|\nabla u|^4+\frac{1}{2}\int_{\Omega}\left(\int_0^t g(t-s)(u(s)-u(t))\,\mathrm{d}s\right)^2\mathrm{d}x$$

$$\le\frac{\varepsilon+1}{2}|u_t|^2+\frac{\varepsilon B^2}{2}|\nabla u|^2+\frac{\sigma\varepsilon}{4}|\nabla u|^4+\frac{B^2(m_3-l)}{2}(g\diamond\nabla u)(t)$$

$$\le c_1\left(\frac{1}{2}|u_t|^2+2c_0\left(l|\nabla u|^2+(g\diamond\nabla u)(t)+\frac{b}{2}|\nabla u|^4\right)\right),$$

where

$$c_1=\max\left(\varepsilon+1,\frac{\varepsilon B^2}{8c_0 l},\frac{B^2(m_3-l)}{8c_0 l},\frac{\sigma\varepsilon}{8bc_0}\right).$$

Employing (4.22) and choosing $\varepsilon>0$ small enough and M sufficiently large, there exist two positive constants β_1 and β_2 such that

$$\beta_1 E(t)\le G(t)\le\beta_2 E(t). \qquad\square$$

Lemma 4.5. *Assume that the hypotheses of Lemma 4.2 be fulfilled. Furthermore, if $E(0)$ is small enough, then, for any $t_0>0$, the functional $G(t)$ verifies, along solution of (1.1)-(1.4) and for $t\ge t_0$,*

$$G'(t)\le-\alpha_1E(t)+\alpha_2(g\diamond\nabla u)(t)+\alpha_3\int_{\Gamma_1}h^2(u_t)\,\mathrm{d}\Gamma-\alpha_4E(0)E'(t) \tag{4.27}$$

where α_i, $i=1,...,4$ are some positive constants.

Proof. In the following, we estimate the derivative of $G(t)$. From (4.24) and (1.1)-(1.4), we have

$$\Phi'(t)=|u_t|^2-\left(m_3+b|\nabla u|^2\right)+\int_{\Omega}\nabla u(t)\int_0^t g(t-s)\nabla u(s)\,\mathrm{d}s\mathrm{d}x-\int_{\Gamma_1}uh(u_t)\,\mathrm{d}\Gamma$$
$$+\int_{\Omega}|u|^{p(x)+1}\mathrm{d}x+\int_{\Gamma_1}|u|^{k(x)+1}\mathrm{d}\Gamma. \tag{4.28}$$

Employing Hölder's inequality, Young's inequality, (2.14) and (2.9), the third and fourth terms on the right-hand side of (4.28) can be estimated as follows, for η, $\delta>0$,

$$\left|\int_{\Omega}\nabla u(t)\int_0^t g(t-s)\nabla u(s)\,\mathrm{d}s\mathrm{d}x\right|\le(\eta+m_3-l)|\nabla u|^2+\frac{(m_3-l)}{4\eta}(g\diamond\nabla u)(t), \tag{4.29}$$

and

$$\left|\int_{\Gamma_1} uh\left(u_t\right)\mathrm{d}\Gamma\right| \leq \delta B_*^2 \left|\nabla u\right|^2 + \frac{1}{4\delta}\int_{\Gamma_1} h^2\left(u_t\right)\mathrm{d}\Gamma. \tag{4.30}$$

A substitution of (4.29)-(4.30) into (4.28) yields

$$\Phi'\left(t\right) = \left|u_t\right|^2 - \left(-\eta + l - \delta B_*^2\right)\left|\nabla u\right|^2 + \frac{\left(m_3 - l\right)}{4\eta}\left(g \diamond \nabla u\right)\left(t\right) - \int_{\Gamma_1} uh\left(u_t\right)\mathrm{d}\Gamma$$
$$+\frac{1}{4\delta}\int_{\Gamma_1} h^2\left(u_t\right)\mathrm{d}\Gamma + \int_\Omega \left|u\right|^{p(x)+1}\mathrm{d}x + \int_{\Gamma_1}\left|u\right|^{k(x)+1}\mathrm{d}\Gamma.$$

Letting $\eta = \frac{l}{2} > 0$ and $\delta = \frac{l}{4B_*^2}$ in above inequality, we obtain

$$\Phi'\left(t\right) \leq \left|u_t\right|^2 - \frac{l}{4}\left|\nabla u\right|^2 + \frac{\left(m_3 - l\right)}{2l}\left(g \diamond \nabla u\right)\left(t\right) - \int_{\Gamma_1} uh\left(u_t\right)\mathrm{d}\Gamma \tag{4.31}$$
$$+\frac{B_*^2}{l}\int_{\Gamma_1} h^2\left(u_t\right)\mathrm{d}\Gamma + \int_\Omega \left|u\right|^{p(x)+1}\mathrm{d}x + \int_{\Gamma_1}\left|u\right|^{k(x)+1}\mathrm{d}\Gamma.$$

For estimate $\Psi'(t)$, taking the derivative of $\Psi(t)$ in (4.25) and using (1.1)-(1.4), we obtain

$$\begin{aligned}
\Psi'\left(t\right) &= \int_\Omega \left(m_3 + b\left|\nabla u\right|^2\right)\nabla u\left(t\right)\int_0^t g\left(t-s\right)\left(\nabla u\left(t\right) - \nabla u\left(s\right)\right)\mathrm{d}s\mathrm{d}x \\
&+ \int_\Omega \left(\sigma\int_\Omega \nabla u \nabla u_t \mathrm{d}x\right)\nabla u\left(t\right)\int_0^t g\left(t-s\right)\left(\nabla u\left(t\right) - \nabla u\left(s\right)\right)\mathrm{d}s\mathrm{d}x \\
&- \int_\Omega \left(\int_0^t g\left(t-s\right)\nabla u\left(s\right)\mathrm{d}s\right)\left(\int_0^t g\left(t-s\right)\left(\nabla u\left(t\right) - \nabla u\left(s\right)\right)\mathrm{d}s\right)\mathrm{d}x \\
&+ \int_{\Gamma_1} h\left(u_t\right)\int_0^t g\left(t-s\right)\left(u\left(t\right) - u\left(s\right)\right)\mathrm{d}s\mathrm{d}\Gamma \\
&- \int_{\Gamma_1}\left|u\right|^{k(x)-1}u\int_0^t g\left(t-s\right)\left(u\left(t\right) - u\left(s\right)\right)\mathrm{d}s\mathrm{d}\Gamma \\
&- \int_\Omega \left|u\right|^{p(x)-1}u\int_0^t g\left(t-s\right)\left(u\left(t\right) - u\left(s\right)\right)\mathrm{d}s\mathrm{d}x \\
&- \int_\Omega u_t\int_0^t g'\left(t-s\right)\left(u\left(t\right) - u\left(s\right)\right)\mathrm{d}s\mathrm{d}x - \left(\int_0^t g\left(s\right)\mathrm{d}s\right)\left|u_t\right|^2.
\end{aligned} \tag{4.32}$$

Similar to deriving (4.31), in what follows we will estimate the right-hand side of (4.32). Using Young's inequality, Hölder's inequality,

$$\left|\nabla u\right|^2 \leq \frac{E\left(0\right)}{2lc_0} \text{ by (4.22)},$$

$$E'\left(t\right) \leq -\sigma\left(\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|\nabla u(t)\right\|_2^2\right)^2 \text{ by (4.6)},$$

and applying (2.14) and (2.9), we have, for $\delta > 0$,

$$\begin{aligned}
&\left|\int_\Omega \left(m_3 + b\left|\nabla u\right|^2\right)\nabla u\left(t\right)\int_0^t g\left(t-s\right)\left(\nabla u\left(t\right)-\nabla u\left(s\right)\right)\mathrm{d}s\mathrm{d}x\right| \\
\leq\ & \left|\int_\Omega \left(m_3 + \frac{b}{2c_0}E\left(0\right)\right)\nabla u\left(t\right)\int_0^t g\left(t-s\right)\left(\nabla u\left(t\right)-\nabla u\left(s\right)\right)\mathrm{d}s\mathrm{d}x\right| \\
\leq\ & \delta\left|\nabla u\right|^2 + \frac{m_3 - l}{4\delta}\left(m_3 + \frac{b}{2c_0}E\left(0\right)\right)^2\left(g\diamond\nabla u\right)\left(t\right),
\end{aligned} \tag{4.33}$$

$$\begin{aligned}
&\left|\int_\Omega \left(\sigma\int_\Omega \nabla u\nabla u_t \mathrm{d}x\right)\nabla u\left(t\right)\int_0^t g\left(t-s\right)\left(\nabla u\left(t\right)-\nabla u\left(s\right)\right)\mathrm{d}s\mathrm{d}x\right| \\
&\leq \sigma^2\left(\int_\Omega \nabla u\nabla u_t\mathrm{d}x\right)^2 l\left|\nabla u\right|^2 + \frac{1}{4l}\int_\Omega\left(\int_0^t g\left(t-s\right)\left(\nabla u\left(t\right)-\nabla u\left(s\right)\right)\mathrm{d}s\right)^2\mathrm{d}x \\
&\leq \frac{-\sigma}{2c_0}E\left(0\right)E'\left(t\right) + \frac{m_3 - l}{4\delta}\left(g\diamond\nabla u\right)\left(t\right),
\end{aligned} \tag{4.34}$$

$$\begin{aligned}
&\left|\int_\Omega\left(\int_0^t g\left(t-s\right)\nabla u\left(s\right)\mathrm{d}s\right)\left(\int_0^t g\left(t-s\right)\left(\nabla u\left(t\right)-\nabla u\left(s\right)\right)\mathrm{d}s\right)\mathrm{d}x\right| \\
\leq\ & 2\delta\left(m_3 - l\right)^2\left|\nabla u\right|^2 + \left(2\delta + \frac{1}{4\delta}\right)\left(m_3 - l\right)\left(g\diamond\nabla u\right)\left(t\right),
\end{aligned} \tag{4.35}$$

and

$$\begin{aligned}
&\left|\int_{\Gamma_1} h\left(u_t\right)\int_0^t g\left(t-s\right)\left(u\left(t\right)-u\left(s\right)\right)\mathrm{d}s\mathrm{d}\Gamma\right| \\
\leq\ & \frac{1}{2}\int_{\Gamma_1} h^2\left(u_t\right)\mathrm{d}\Gamma + \frac{\left(m_3 - l\right)B_*^2}{2}\left(g\diamond\nabla u\right)\left(t\right).
\end{aligned} \tag{4.36}$$

As for the the fifth and sixth terms on the right-hand side of (4.32), utilizing Hölder's inequality, Young's inequality, (2.9), (2.13), (2.14) and (4.22), we obtain,

$$\begin{aligned}
&\left|\int_{\Gamma_1}\left|u\right|^{k(x)-1}u\int_0^t g\left(t-s\right)\left(u\left(t\right)-u\left(s\right)\right)\mathrm{d}s\mathrm{d}\Gamma\right| \\
&\leq \delta\max\left(\left\|u\right\|_{2k^+,\Gamma_1}^{2k^+}, \left\|u\right\|_{2k^-,\Gamma_1}^{2k^-}\right) + \frac{\left(m_3 - l\right)B_*^2}{4\delta}\left(g\diamond\nabla u\right)\left(t\right) \\
&\leq \delta\max\left(B_*^{2k^+}\left|\nabla u\right|^{2k^+}, B_*^{2k^-}\left|\nabla u\right|^{2k^-}\right) + \frac{\left(m_3 - l\right)B_*^2}{4\delta}\left(g\diamond\nabla u\right)\left(t\right) \\
&\leq \delta\max\left(B_*^{2k^+}\left(\frac{E\left(0\right)}{2lc_0}\right)^{k^+-1}, B_*^{2k^-}\left(\frac{E\left(0\right)}{2lc_0}\right)^{k^--1}\right)\left|\nabla u\right|^2 + \frac{\left(m_3 - l\right)B_*^2}{4\delta}\left(g\diamond\nabla u\right)\left(t\right)
\end{aligned} \tag{4.37}$$

and

$$\begin{aligned} &\left|\int_\Omega |u|^{p(x)-1} u \int_0^t g(t-s)(u(t)-u(s))\,\mathrm{d}s\mathrm{d}x\right| \qquad (4.38)\\ &\le \delta \max\left(B^{2p^+}|\nabla u|^{2p^+}, B^{2p^-}|\nabla u|^{2p^-}\right) + \frac{(m_3-l)B^2}{4\delta}(g\diamond\nabla u)(t)\\ &\le \delta \max\left(B^{2p^+}\left(\frac{E(0)}{2lc_0}\right)^{p^+-1}, B^{2p^-}\left(\frac{E(0)}{2lc_0}\right)^{p^--1}\right)|\nabla u|^2 + \frac{(m_3-l)B^2}{4\delta}(g\diamond\nabla u)(t). \end{aligned}$$

Exploiting Hölder's inequality, Young's inequality and (H_1) to estimate the seventh term, we have

$$\begin{aligned} &\left|\int_\Omega u_t \int_0^t g'(t-s)(u(t)-u(s))\,\mathrm{d}s\mathrm{d}x - \left(\int_0^t g(s)\,\mathrm{d}s\right)|u_t|^2\right| \qquad (4.39)\\ &\le \delta |u_t|^2 - \frac{g(0)B^2}{4\delta}(g'\diamond\nabla u)(t). \end{aligned}$$

Then, combining these estimates (4.33)-(4.39), (4.32) becomes

$$\begin{aligned} \Psi'(t) \le & -\left(\int_0^t g(s)\,\mathrm{d}s - \delta\right)|u_t|^2 + c_2\delta|\nabla u|^2 + c_3(g\diamond\nabla u)(t) \qquad (4.40)\\ & -\frac{g(0)B^2}{4\delta}(g'\diamond\nabla u)(t) + \frac{1}{2}\int_{\Gamma_1} h^2(u_t)\,\mathrm{d}\Gamma - \frac{\sigma}{2c_0}E(0)E'(t), \end{aligned}$$

where

$$\begin{aligned} c_2 = & 1 + 2(m_3-l)^2 + \max\left(B_*^{2k^+}\left(\frac{E(0)}{2lc_0}\right)^{k^+-1}, B_*^{2k^-}\left(\frac{E(0)}{2lc_0}\right)^{k^--1}\right)\\ & + \max\left(B^{2p^+}\left(\frac{E(0)}{2lc_0}\right)^{p^+-1}, B^{2p^-}\left(\frac{E(0)}{2lc_0}\right)^{p^--1}\right), \end{aligned}$$

and

$$c_3 = (m_3-l)\left(\frac{1+\left(m_3+\frac{bE(0)}{2lc_0}\right)^2}{4\delta} + 2\delta + \frac{1}{4l} + \frac{B_*^2}{2} + \frac{B^2+B_*^2}{4\delta}\right).$$

Since g is continuous and $g(0) > 0$, then there exists $t_0 > 0$ such that,

$$\int_0^t g(s)\,\mathrm{d}s \ge \int_0^{t_0} g(s)\,\mathrm{d}s = g_0,\ \forall t \ge t_0.$$

Hence, we conclude from (4.23), (4.6), (4.31), and (4.40) that

$$\begin{aligned} G'(t) &= ME'(t) + \varepsilon\Phi'(t) + \Psi'(t) \\ &\le -\left(\frac{M}{2} - \frac{g(0)B^2}{4\delta}\right)\left(-\left(g' \diamond \nabla u\right)(t)\right) - \left(g_0 - \delta - \varepsilon\right)\left|u_t\right|^2 \\ &+ \left(c_2\delta - \frac{\varepsilon l}{4}\right)\left|\nabla u\right|^2 + \left(c_3 + \frac{\left(m_3 - l\right)\varepsilon}{2l}\right)\left(g \diamond \nabla u\right)(t) \\ &+ \left(\frac{1}{2} + \frac{2B_*^2\varepsilon}{l}\right)\int_{\Gamma_1} h^2\left(u_t\right)\mathrm{d}\Gamma - \frac{\sigma}{2c_0}E(0)E'(t) \\ &+\varepsilon\left(\int_\Omega |u|^{p(x)+1}\,\mathrm{d}x + \int_{\Gamma_1}|u|^{k(x)+1}\,\mathrm{d}\Gamma\right). \end{aligned} \tag{4.41}$$

At this point, we choose $\varepsilon > 0$ small enough so that Lemma4.4 holds and $\varepsilon < \frac{g_0}{2}$. Once ε is fixed, we choose δ to satisfy

$$\delta < \min\left(\frac{g_0}{4}, \frac{\varepsilon l}{8c_2}\right)$$

and then pick M sufficiently large such that $M > \frac{g(0)B^2}{2\delta}$. Thus, for all $t \ge t_0$, we arrive at

$$\begin{aligned} G'(t) &\le -\frac{\varepsilon l}{8}\left|\nabla u\right|^2 - \frac{g_0}{4}\left|u_t\right|^2 + c_4\left(g \diamond \nabla u\right)(t) + c_5\int_{\Gamma_1} h^2\left(u_t\right)\mathrm{d}\Gamma \\ &\quad -c_6E(0)E'(t) + \varepsilon\left(\int_\Omega |u|^{p(x)+1}\,\mathrm{d}x + \int_{\Gamma_1}|u|^{k(x)+1}\,\mathrm{d}\Gamma\right) \\ &\le -\frac{\varepsilon l}{4\left(m_3 - g_0\right)}\frac{1}{2}\left(m_3 - \int_0^t g(s)\,\mathrm{d}s\right)\left|\nabla u\right|^2 - \frac{g_0}{4}\left|u_t\right|^2 \\ &\quad +c_4\left(g \diamond \nabla u\right)(t) + c_5\int_{\Gamma_1} h^2\left(u_t\right)\mathrm{d}\Gamma - c_6E(0)E'(t) \\ &\quad +\varepsilon\left(\int_\Omega |u|^{p(x)+1}\,\mathrm{d}x + \int_{\Gamma_1}|u|^{k(x)+1}\,\mathrm{d}\Gamma\right). \end{aligned}$$

with some positive constants $c_i, i = 4, 5, 6$. Additionally, observing the fact that $\frac{\varepsilon l}{4(m_3 - g_0)} < g_0$ due to $\varepsilon < g_0$ and $\frac{l}{(m_3 - g_0)} < 1$ and employing the definition of $E(t)$ by (4.3) and using $\left|\nabla u\right|^2 \le \frac{E(0)}{2lc_0}$ by (4.22), we deduce that

$$\begin{aligned} G'(t) &\le -c_7E(t) + \frac{c_7 b}{4}\left|\nabla u\right|^4 + \left(c_4 + \frac{c_7}{2}\right)\left(g \diamond \nabla u\right)(t) \\ &+c_5\int_{\Gamma_1} h^2\left(u_t\right)\mathrm{d}\Gamma - c_6E(0)E'(t) + \varepsilon c_8\left(\int_\Omega |u|^{p(x)+1}\,\mathrm{d}x + \int_{\Gamma_1}|u|^{k(x)+1}\,\mathrm{d}\Gamma\right) \\ &\le -\alpha_1E(t) + \left(c_4 + \frac{c_7}{2}\right)\left(g \diamond \nabla u\right)(t) + c_5\int_{\Gamma_1} h^2\left(u_t\right)\mathrm{d}\Gamma - c_6E(0)E'(t), \end{aligned}$$

where

$$c_7 = \frac{\varepsilon l}{4\left(m_3 - g_0\right)},$$

$$c_8 = \max\left(1 - \frac{l}{4\left(p^- + 1\right)\left(m_3 - g_0\right)}, 1 - \frac{l}{4\left(k^- + 1\right)\left(m_3 - g_0\right)}\right) > 0$$

and

$$\alpha_1 = c_7 - \left(\frac{c_7 b}{8l^2 c_0} E(0) + \varepsilon \frac{c_8}{2lc_0}\left(\begin{array}{c} \max\left(B^{p^+ + 1}\left(\frac{E(0)}{2lc_0}\right)^{\frac{p^+ - 1}{2}}, B^{p^- + 1}\left(\frac{E(0)}{2lc_0}\right)^{\frac{p^- - 1}{2}}\right), \\ \max\left(B_*^{k^+ + 1}\left(\frac{E(0)}{2lc_0}\right)^{\frac{k^+ - 1}{2}}, B_*^{k^- + 1}\left(\frac{E(0)}{2lc_0}\right)^{\frac{k^- - 1}{2}}\right)\end{array}\right)\right).$$

Hence, if $E(0)$ is small enough, then not only the condition $E(0) < \mathrm{d}$ is satisfied, but also $\alpha_1 > 0$ is assured. Therefore, we have, for $t \geq t_0$,

$$G'(t) \leq -\alpha_1 E(t) + \alpha_2 (g \diamond \nabla u)(t) + \alpha_3 \int_{\Gamma_1} h^2(u_t)\, \mathrm{d}\Gamma - \alpha_4 E(0) E'(t), \qquad (4.42)$$

where α_i, $i = 1, ..., 4$ are all positive constants. This completes the proof. □

Before stating our main result, we need to recall that if φ is a proper convex function from $\mathbb{R}$ to $\mathbb{R} \cup \{\infty\}$, then its convex conjugate φ^* is defined as

$$\varphi^*(y) = \sup_{x \in \mathbb{R}} \{xy - \varphi(x)\} \qquad (4.43)$$

Now, we are in a position to state our main result by adopting and modifying the arguments in [18, 39, 20]. We consider the following partition of Γ_1

$$\Gamma_1^+ = \{x \in \Gamma_1 \mid |u_t| > r\}, \; \Gamma_1^- = \{x \in \Gamma_1 \mid |u_t| \leq r\}.$$

Theorem 4.6. *Assume that the conditions of* 4.5 *are valid, then, for each* $t_0 > 0$ *and* k_1, k_2 *and* ε_0 *are positive constants, the solution energy of* (1.1)*-*(1.4) *satisfies*

$$E(t) \leq k_2 H_1^{-1}\left(k_1 \int_0^t \zeta(s)\, \mathrm{d}s\right), \; t \geq t_0 \qquad (4.44)$$

where

$$H_1(t) = \int_t^1 \frac{1}{H_2(s)} \mathrm{d}s \qquad (4.45)$$

and

$$H_2(t) = \begin{cases} t, & \text{if } H \text{ is linear on } [0, r], \\ tH'(\varepsilon_0 t), & \text{if } H'(0) = 0 \text{ and } H'' > 0 \text{ on } (0, r]. \end{cases} \qquad (4.46)$$

Proof. The global existence of solution u of (1.1)-(1.4) is guaranteed directly by Theorem 4.3. Next, we consider the following two cases: (i) H is linear on $[0, r]$ and (ii) $H'(0) = 0$ and $H'' > 0$ on $(0, r]$.

Case 1. H is linear on $[0, r]$. In this case, there exists $\alpha_1' > 0$ such that $|h(s)| \leq \alpha_1' |s|$, for all $s \in \mathbb{R}$. By (4.6), we have

$$\int_{\Gamma_1} h^2(u_t)\, \mathrm{d}\Gamma \leq \alpha_1' \int_{\Gamma_1} u_t h(u_t)\, \mathrm{d}\Gamma \leq -\alpha_1' E'(t),$$

which together with (4.42) implies that

$$(G(t) + c_9 E(t))' \leq -\alpha_1 H_2(E(t)) + \alpha_2 (g \diamond \nabla u)(t), \qquad (4.47)$$

where $H_2(s) = s$ and $c_9 = \alpha_1'\alpha_3 + \alpha_4 E(0)$.
Case 2. $H'(0) = 0$ and $H'' > 0$ on $(0, r]$. In this case, we first estimate $\int_{\Gamma_1} h^2(u_t)\,d\Gamma$ on the right-hand side of (4.42). Given (4.2), noting that H^{-1} is concave and increasing and using the well-known Jensen's inequality and (4.6), we deduce that

$$\begin{aligned}\int_{\Gamma_1} h^2(u_t)\,d\Gamma &= \int_{\Gamma_1} h^2(u_t)\,d\Gamma + \int_{\Gamma_1} h^2(u_t)\,d\Gamma \\ &\leq M_1 \int\limits_{\Gamma_1^+} u_t h(u_t)\,d\Gamma + \int\limits_{\Gamma_1^-} h^2(u_t)\,d\Gamma \\ &\leq -M_1 E'(t) + \int\limits_{\Gamma_1^-} H^{-1}(u_t h(u_t))\,d\Gamma \\ &\leq -M_1 E'(t) + \frac{1}{c_{10}} H^{-1}\left(c_{10} \int\limits_{\Gamma_1^-} (u_t h(u_t))\,d\Gamma \right) \\ &\leq -M_1 E'(t) + \frac{1}{c_{10}} H^{-1}(-c_{10} E'(t)),\end{aligned}$$

where $c_{10} = \frac{1}{\left|\Gamma_1^-\right|}$. Hence, (4.42) becomes

$$G_1'(t) \leq -\alpha_1 E(t) + \alpha_3 \left|\Gamma_1^-\right| H^{-1}(-c_{10} E'(t)) + \alpha_2 (g \diamond \nabla u)(t), \ \forall t \geq t_0, \tag{4.48}$$

where

$$G_1(t) = G(t) + (M_1\alpha_3 + \alpha_4 E(0)) E(t). \tag{4.49}$$

Now, we define

$$G_2(t) = H'(\varepsilon_0 E(t)) G_1(t) + \beta E(t), \tag{4.50}$$

where $\varepsilon_0 > 0$ and $\beta > 0$ to be determined later. Then, using $E'(t) \leq 0$, $H''(t) \geq 0$, and (4.48), we obtain

$$\begin{aligned}G_2'(t) &= \varepsilon_0 E'(t) H''(\varepsilon_0 E(t)) G_1(t) + H'(\varepsilon_0 E(t)) G_1'(t) + \beta E'(t) \\ &\leq -\alpha_1 H'(\varepsilon_0 E(t)) E(t) + \alpha_2 H'(\varepsilon_0 E(t)) (g \diamond \nabla u)(t) \\ &\quad + c_{11} H'(\varepsilon_0 E(t)) H^{-1}(-c_{10} E'(t)) + \beta E'(t).\end{aligned} \tag{4.51}$$

To estimate the fourth term in the right hand side of (4.51), let H^* be the conjugate function of the convex function H defined by (4.43), then

$$ab \leq H^*(a) + H(b) \ \text{ for } a, b \geq 0. \tag{4.52}$$

Moreover, due to the argument given in [6], it holds that

$$H^*(s) = s\left(H'\right)^{-1}(s) - H\left(\left(H'\right)^{-1}(s)\right) \ \text{ for } s \geq 0. \tag{4.53}$$

Further, using (4.53) and noting that $H'(0) = 0$, $\left(H'\right)^{-1}$ is increasing and H is also increasing yield

$$H^*(s) \leq s\left(H'\right)^{-1}(s), \ s \geq 0. \tag{4.54}$$

Taking $H'(\varepsilon_0 E(t)) = a$ and $H^{-1}(-c_{10}E'(t)) = b$ in (4.51), applying (4.54) and (4.52), and noting that $0 \leq H'(\varepsilon_0 E(t)) \leq H'(\varepsilon_0 E(0))$ due to H' is increasing, to obtain

$$\begin{aligned} G_2'(t) &\leq -\alpha_1 H'(\varepsilon_0 E(t)) E(t) + c_{11} H^*(H'(\varepsilon_0 E(t))) \\ &\quad + c_{13}(g \diamond \nabla u)(t) + (\beta - c_{12}) E'(t) \\ &\leq -(\alpha_1 - c_{11}\varepsilon_0) H'(\varepsilon_0 E(t)) E(t) + c_{13}(g \diamond \nabla u)(t) + (\beta - c_{12}) E'(t) \end{aligned}$$

with $c_{12} = c_{10}c_{11}$ and $c_{13} = \alpha_2 H'(\varepsilon_0 E(0)) > 0$. Thus, choosing $0 < c_{11}\varepsilon_0 < \alpha_1$, $\beta > c_{12}$ and using $E'(t) \leq 0$ by (4.6), we have

$$G_2'(t) \leq -c_{14} H'(\varepsilon_0 E(t)) E(t) + c_{13}(g \diamond \nabla u)(t) = -c_{14} H_2(E(t)) + c_{13}(g \diamond \nabla u)(t), \tag{4.55}$$

where $H_2(s) = sH'(\varepsilon_0 s)$ and c_{14} is a positive constant.

Let

$$F_1(t) = \begin{cases} G(t) + c_9 E(t), \text{if } H \text{ is linear on } [0, r], \\ G_2(t), \text{if } H'(0) = 0 \text{ and } H'' > 0 \text{ on } (0, r]. \end{cases}$$

Then, by Lemma 4.4 and the definition of G_2 by (4.49)-(4.50), there exist β_1', $\beta_2' > 0$ such that

$$\beta_2' E(t) \leq F_1(t) \leq \beta_1' E(t), \tag{4.56}$$

which is equivalent to $E(t)$, and from (4.47) and (4.55), we have

$$F_1'(t) \leq -c_{15} H_2(E(t)) + c_{16}(g \diamond \nabla u)(t), \ t \geq t_0, \tag{4.57}$$

where c_{15} and c_{16} denote some positive constants. In addition, using (4.56) and $\xi(t) \leq \xi(0)$ by (H_2) and for $l_1 = \beta_1'\xi(0) + 2c_{16} > 0$, we see that

$$\xi(t) F_1(t) + 2c_{16} E(t) \leq l_1 E(t), \ t \geq t_0, \tag{4.58}$$

Now, we define

$$G_3(t) = \varepsilon_1 [\xi(t) F_1(t) + 2c_{16} E(t)], \ 0 < l_1 \varepsilon_1 < r, \tag{4.59}$$

which is equivalent to $E(t)$ by (4.56). Thanks to (4.57), (2.10) and (4.6), we arrive at

$$\begin{aligned} G_3'(t) &= \varepsilon_1 [\xi'(t) F_1(t) + \xi(t) F_1'(t) + 2c_{16} E'(t)] \\ &\leq -c_{15}\varepsilon_1 H_2(E(t)) \xi(t) + c_{16}\varepsilon_1 \xi(t)(g \diamond \nabla u)(t) + 2c_{16}\varepsilon_1 E'(t) \\ &\leq -c_{15}\varepsilon_1 H_2(E(t)) \xi(t) - c_{16}\varepsilon_1 (g' \diamond \nabla u)(t) + 2c_{16}\varepsilon_1 E'(t) \\ &\leq -c_{15}\varepsilon_1 H_2(E(t)) \xi(t). \end{aligned}$$

Exploiting the fact that H_2 is increasing, using (4.58) and using the fact that $0 < l_1 \varepsilon_1 < r$ by (4.59), we obtain

$$\begin{aligned} G_3'(t) &\leq -c_{15}\varepsilon_1 \xi(t) H_2\left(\frac{1}{l_1}(\xi(t) F_1(t) + 2c_{16} E(t))\right) \\ &\leq -c_{15}\varepsilon_1 \xi(t) H_2(\varepsilon_1(\xi(t) F_1(t) + 2c_{16} E(t))) = -c_{15}\varepsilon_1 \xi(t) H_2(G_3(t)). \end{aligned}$$

Using that $H_1'(t) H_2(t) = -1$ (see (4.45)), we see that

$$G_3'(t) H_1'(G_3(t)) \geq c_{15}\varepsilon_1 \xi(t), \ t \geq t_0.$$

Integrating this over (t_0, t) which implies, having in mind that H_1^{-1} is decreasing on $(0, r]$, that

$$G_3(t) \leq H_1^{-1}\left(H_1(G_3(0)) + c_{15}\varepsilon_1 \int_0^t \xi(s)\mathrm{d}s - c_{15}\varepsilon_1 \int_0^{t_0} \xi(s)\mathrm{d}s\right)$$
$$\leq H_1^{-1}\left(c_{15}\varepsilon_1 \int_0^t \xi(s)\mathrm{d}s\right),$$

where we need $\varepsilon_1 > 0$ sufficiently small so that $H_1(G_3(0)) - c_{15}\varepsilon_1 \int_0^{t_0} \xi(s)\mathrm{d}s > 0$.

Consequently, from the equivalent relation of G_3 and E yields

$$E(t) \leq k_2 H_1^{-1}\left(k_1 \int_0^t \xi(s)\,\mathrm{d}s\right), \quad t \geq t_0,$$

where k_1 and k_2 are positive constants. Hence, this completes the proof. □

Remark 4.7. Because $\lim\limits_{t\to 0} H_1(t) = \infty$ (see (4.46)), thus, if $\int_0^\infty \xi(s)\,\mathrm{d}s = \infty$, we get the stability of system (1.1)-(1.4), in the other words, $\lim\limits_{t\to+\infty} E(t) = 0$.

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Gradient-type deformations of cycles in EPH geometries

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Abstract. The aim of this paper is to study the cycles of EPH geometries through their homogeneous gradient-type deformations recently introduced by the author. A special topic is the orthogonality between a given cycle C and its deformations as well as between C and its rotated version $R(C)$.

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1. Introduction

It is well-known that up to isomorphisms there are three 2-dimensional real algebras: $\mathbb{C} = \mathbb{R}[X]/(x^2 + 1)$, $\mathbb{D} = \mathbb{R}[X]/(x^2)$ and $\mathbb{A} = \mathbb{R}[X]/(x^2 - 1)$. The theory of the first algebra is richer than the following two, a fact corresponding to the field property of $\mathbb{C}$. Inspired by the terminology of [6, p. 1458] or [7, p. 2] we call *EPH geometries* these spaces and a common image consists in $A(\sigma) := \mathbb{R}[X]/(x^2 - \sigma)$ with $\sigma := i^2 \in \{-1, 0, 1\}$ respectively and i the corresponding imaginary unit.

The recent papers [2] and [5], devoted to Finsler geometry, start with a deformation of a conic Γ obtained by deforming the gradient vector field for the quadratic form defining Γ. These deformations are inspired by the scaling (linear) transformation of Computer Graphics: $(x, y) \in \mathbb{R}^2 \to (\lambda_x \cdot x, \lambda_y \cdot y) \in \mathbb{R}^2$, following [8, p. 136]. Using the well-known invariants from the Euclidean geometry of conics we obtained the classifications of the new conics which depends on two scalars denoted α and β, having the role of λ_x, λ_y. The new conic of [2], denoted $\tilde{\Gamma}$, is a degenerate one and we could interpret the map $\Gamma \to \tilde{\Gamma}$ as a "curve shortening" transformation. The same fact holds for the new conic of [5], denoted Γ^m, if the initial conic Γ does not have linear terms.

In this note we use these classes of gradient-type deformation to a main object of EPH geometries, called *cycle*, which is a particular case of conic sections, invariant under the action of the group $SL(2, \mathbb{R})$ through Mobiüs transformations. A detailed analysis of the deformed cycles depends on the vanishing or not of σ as well as the vanishing or not of a parameter k separating the circles to lines. Also, we discuss the transformation of a square matrix associated to any cycle C.

Moreover, we treat these deformations in terms of $A(\sigma)$-numbers. In the second section we study the orthogonality of a given cycle C with its deformations restricting to the $\sigma \neq 0$ case. In the last section we introduce a natural rotation R in $A(\sigma)$ and we study the relationships between a given C and its rotated cycle $R(C)$.

2. The cycles of EPH geometries and their gradient-type deformations

In the two-dimensional Euclidean space $\mathbb{R}^2$ let us consider the conic Γ implicitly defined by $f \in C^\infty(\mathbb{R}^2)$ as: $\Gamma = \{(x,y) \in \mathbb{R}^2 \mid f(x,y) = 0\}$ where f is a quadratic function of the form $f(x,y) = r_{11}x^2 + 2r_{12}xy + r_{22}y^2 + 2r_{10}x + 2r_{20}y + r_{00}$ with $r_{11}^2 + r_{12}^2 + r_{22}^2 \neq 0$ for the non-degenerate conics.

It is well-known that the gradient vector field of f, namely

$$\nabla f = \left(f_x = \frac{\partial f}{\partial x}, f_y = \frac{\partial f}{\partial y}\right),$$

gives important properties of Γ; for example, the centers of Γ are exactly the critical points of ∇f. Inspired by this fact we introduced recently:

Definition 2.1. Fix the scalars α, β with $\alpha\beta \neq 0$.
i) ([2, p. 86-87], [3, p. 60]) The (α, β)-*deformation* of Γ is the conic:

$$\tilde{\Gamma} = \Gamma_{\alpha,\beta} : \alpha \left[\frac{1}{2} f_x\right]^2 + \beta \left[\frac{1}{2} f_y\right]^2 = 0. \tag{2.1}$$

ii) ([5, p. 102]) The (α, β)-*mixed deformation* of Γ is the conic:

$$\Gamma^m = \Gamma^m_{\alpha,\beta} : \alpha y \left[\frac{1}{2} f_x\right] + \beta x \left[\frac{1}{2} f_y\right] = 0. \tag{2.2}$$

A main object in EPH geometries is given in [6, p. 1459], [7, p. 4]:

Definition 2.2. The common name *cycle* will be used to denote circles, parabolas and hyperbolas (as well as straight lines as their limits) in the respective EPH geometry.

An analytical study of a cycle can be done via the general equation given in [6, p. 1460] or [7, p. 6]:

$$C : f(u,v) := k(u^2 - \sigma v^2) - 2lu - 2nv + m = 0 \tag{2.3}$$

and hence C is a conic section completely defined by the data $(k, l, n, m) \in \mathbb{P}^3$. As usual, if $k = 0$ then C can be called *a degenerate cycle*. In fact, in the cited works C is identified with the matrix:

$$C^s_{\breve{\sigma}} := \begin{pmatrix} l + \breve{\imath}sn & -m \\ k & -l + \breve{\imath}sn \end{pmatrix} \tag{2.4}$$

where s is a new parameter, usually equal to ± 1, and a new imaginary unit $\breve{\imath}$. Its square $\breve{\sigma} := \breve{\imath}^2$ belongs again to $\{-1, 0, 1\}$ but independently of σ.

Since C is a conic section we can apply the ideas of Definition 2.1 to introduce the gradient-type deformations of a cycle:

$$\begin{cases} \tilde{C} = C_{\alpha,\beta} : \alpha(ku - l)^2 + \beta(k\sigma v + n)^2 = 0, \\ C^m : \alpha v(ku - l) - \beta u(k\sigma v + n) = 0 \end{cases} \tag{2.5}$$

which yields immediately:

Proposition 2.3. *Since $\alpha \neq 0$ we have:*
i) $\tilde{C}$ is a cycle if and only if $\sigma(\alpha + \sigma\beta) = 0$,
ii) C^m is a cycle if and only if $k(\alpha - \beta\sigma) = 0$. In this case C^m is the straight line:

$$(\beta n)u + (\alpha l)v = 0.$$

Example 2.4. In the following we discuss the remarkable particular cases of the result above.
i) Suppose $\sigma = 0$. Then $\tilde{C}$ is the cycle:

$$\tilde{C} : (ku - l)^2 + \frac{\beta}{\alpha} n^2 = 0 \tag{2.6}$$

with the matrix:

$$\tilde{C}^s_{\breve{\sigma}} = \begin{pmatrix} kl & -(l^2 + \frac{\beta}{\alpha}n^2) \\ k^2 & -kl \end{pmatrix}. \tag{2.7}$$

The degenerate case of an initial line i.e. $k = 0$ is possible if and only if $\alpha l^2 + \beta n^2 = 0$ which is relation (2.19) below. If $k \neq 0$ then, due to the projective character of the coefficients of a cycle, we get the matrix:

$$\tilde{C}^s_{\breve{\sigma}} = \begin{pmatrix} l & -\frac{1}{k}(l^2 + \frac{\beta}{\alpha}n^2) \\ k & -l \end{pmatrix}. \tag{2.8}$$

If $\frac{\beta}{\alpha} > 0$ then $\tilde{C}$ is a void set for $n \neq 0$ while $n = 0$ gives the deformation:

$$C : ku^2 - 2lu + m = 0 \to \tilde{C} : ku = l \ (line : k \neq 0). \tag{2.9}$$

If $\frac{\beta}{\alpha} < 0$ then we have the lines:

$$\tilde{C} : ku - l = \pm\sqrt{-\frac{\beta}{\alpha}}n. \tag{2.10}$$

C^m is a cycle if and only if $k = 0$ which means that we have the mixed deformation:

$$C : 2lu + 2nv - m = 0 \ (line) \to C^m : (\beta n)u + (\alpha l)v = 0 \ (line). \tag{2.11}$$

If $\beta = -\alpha$ then these two lines are Euclidean orthogonal. From the matrix point of view the deformation (2.11) means:

$$C^s_{\breve{\sigma}} = \begin{pmatrix} l + \breve{\imath}sn & -m \\ 0 & -l + \breve{\imath}sn \end{pmatrix} \to C^{m,s}_{\breve{\sigma}} = \begin{pmatrix} -\beta n + \breve{\imath}s(-\alpha l) & 0 \\ 0 & \beta n + \breve{\imath}s(-\alpha l) \end{pmatrix}. \tag{2.12}$$

ii) For $\sigma \neq 0$ we have that $\tilde{C}$ is a cycle only for $\beta = -\frac{\alpha}{\sigma} = -\sigma\alpha$ and then:

$$\tilde{C} : \left[k(u + iv) - l + \frac{n}{i}\right]\left[k(u - iv) - l - \frac{n}{i}\right] = 0. \tag{2.13}$$

Hence, if $k \neq 0$ then $\tilde{C}$ consists in a single point: $M = (\frac{l}{k}, -\frac{n}{k\sigma})$. Let us point out that for $\sigma \neq 0$ we have $\frac{1}{\sigma} = \sigma$ and hence $M = (\frac{l}{k}, -\sigma\frac{n}{k})$ which is exactly the *e/h-center* of the initial cycle C, as it is introduced in formula (7) of [6, p. 1460] or [7, p. 7]. In conclusion, for $\sigma \cdot k \neq 0$ we have the deformation:

$$C \to \tilde{C} = its\ center. \tag{2.14}$$

The matrix corresponding to $\tilde{C}$ is:

$$\tilde{C}^s_{\breve{\sigma}} = \begin{pmatrix} k(l + \breve{\imath}sn) & n^2\sigma - l^2 \\ k^2 & k(-l + \breve{\imath}sn) \end{pmatrix} \tag{2.15}$$

which for $k = 0$ becomes:

$$\tilde{C}^s_{\breve{\sigma}} = \begin{pmatrix} 0 & n^2\sigma - l^2 \\ 0 & 0 \end{pmatrix} \tag{2.16}$$

while for $k \neq 0$, due to the projective character of the parameters of a cycle:

$$\tilde{C}^s_{\breve{\sigma}} = \begin{pmatrix} l + \breve{\imath}sn & \frac{1}{k}(n^2\sigma - l^2) \\ k & -l + \breve{\imath}sn \end{pmatrix}. \tag{2.17}$$

The same case $\sigma \cdot k \neq 0$ for ii) of proposition above gives $\beta = \frac{\alpha}{\sigma} = \sigma\alpha$ and C^m is the line:

$$C^m : nu + (\sigma l)v = 0. \tag{2.18}$$

For elliptic geometry the condition $\beta = -\frac{\alpha}{\sigma} = -\sigma\alpha$ becomes the equality $\alpha = \beta$ discussed in [2, p. 89] and [3, p. 62]; it can be called *the diagonal case*. Remark that the elliptic center $\tilde{C}$ of (2.14) is obtained in [6, p. 1461] or [7, p. 8] from the vanishing condition $detC^s_{-1} = 0$.

Remark 2.5. The cycle C^m contains the origin $(u, v) = (0, 0) = O$. This fact holds for $\tilde{C}$ if and only if:

$$\alpha l^2 + \beta n^2 = 0. \tag{2.19}$$

With the discussion of above particular cases it results:
i) for $\sigma = 0$ the only available case is $\frac{\beta}{\alpha} < 0$ yielding:

$$l_{\pm} = \pm\sqrt{-\frac{\beta}{\alpha}}n. \tag{2.20}$$

ii) for $\sigma \neq 0$ since $\beta = -\frac{\alpha}{\sigma} = -\sigma\alpha$ we get that for the elliptic geometry the only possible case is $O = M$ the center of C while for the hyperbolic geometry:

$$l_{\pm} = \pm n. \tag{2.21}$$

The gradient-type deformation of a standard (i.e. Euclidean) ellipse is discussed in example 2.2i) of [2, p. 87]. Let us point out that (2.20) and (2.21) coincide for $\beta = -\alpha$ which for the case ii) correspond to the hyperbolic geometry. Hence the above cases i) and ii) are completely different, both from σ and the sign of $\frac{\beta}{\alpha}$ points of view.

Returning to the general case of α and β we treat the considered deformations within $A(\sigma)$ following the model of [3] and [5]. More precisely, with the usual notation $z = u + iv \in A(\sigma)$ we derive the expression of C:

$$C : F(z, \bar{z}) := kz\bar{z} + Bz + \bar{B}\bar{z} + m = 0, \quad B := -l - \frac{n}{\sigma}i \in A(\sigma)\ (\sigma \neq 0). \tag{2.22}$$

For $\sigma = 0$ we have: $B = -l - \frac{n}{i}$. The inverse relationship between f and F is:

$$l = -\Re B, \quad n = -\sigma \Im B \tag{2.23}$$

with $\Re$ and $\Im$ respectively the real and imaginary part. By replacing in (2.5) the usual relations:

$$u = \frac{1}{2}(z + \bar{z}), \quad v = \frac{1}{2i}(z - \bar{z}) \tag{2.24}$$

we get:

$$\begin{cases} \tilde{C} : \alpha[k(z+\bar{z}) - 2l]^2 + \beta[ki(z-\bar{z}) + 2n]^2 = 0, \\ C^m : \alpha(z-\bar{z})[k(z+\bar{z}) - 2l] - \beta(z+\bar{z})[k\sigma(z-\bar{z}) + 2ni] = 0. \end{cases} \tag{2.25}$$

For the case $\sigma \neq 0$ we follow the discussion of Example 2.4ii and then:

$$\begin{cases} \tilde{C} : [k(z+\bar{z}) - 2l]^2 - \sigma[ki(z-\bar{z}) + 2n]^2 = 0, \\ C^m : (z-\bar{z})[k(z+\bar{z}) - 2l] - \sigma(z+\bar{z})[k\sigma(z-\bar{z}) + 2ni] = 0. \end{cases} \tag{2.26}$$

The second equation (2.26) reduces to:

$$C^m : Bz - \bar{B}\bar{z} = 0 \leftrightarrow Bz \in \mathbb{R} \tag{2.27}$$

and hence, for $B \neq 0$ we have the line: $z = \bar{B} \cdot \mathbb{R}$.
We finish this section by applying to the cycle C (not containing the origin, hence $m \neq 0$) the *inversion* $J : z \in A(\sigma)^* \to \frac{1}{z} = w$. We get a new cycle, expressed in w:

$$J(C) : mw\bar{w} + \bar{B}w + B\bar{w} + k = 0 \tag{2.28}$$

which means $J : (k, l, n, m) \to (m, l, -n, k)$. With (2.26)-(2.27) its gradient deformations for $\sigma \neq 0$ are:

$$\begin{cases} \widetilde{J(C)} : [m(w+\bar{w}) - 2l]^2 - \sigma[mi(w-\bar{w}) - 2n]^2 = 0, \\ J(C)^m : B\bar{w} - \bar{B}w = 0 \leftrightarrow \bar{B}w \in \mathbb{R}. \end{cases} \tag{2.29}$$

Again, if $B \neq 0$ then the second cycle from from above is the line: $w = B \cdot \mathbb{R}$.

3. Orthogonality in the geometry of cycles

In [6, p. 1462] or [7, p. 2] a Möbius-invariant (indefinite) inner product (depending on $\breve{\sigma}$) is defined on the set of cycles through:

$$< C^s_{\breve{\sigma}}, \hat{C}^s_{\breve{\sigma}} > := Tr(C^s_{\breve{\sigma}} \cdot \overline{\hat{C}^s_{\breve{\sigma}}}) \tag{3.1}$$

which yields an associated $\breve{\sigma}$-orthogonality. Here, the bar means the conjugation with respect to $\breve{\imath}$.

For our setting we derive firstly the norms of a cycle and its gradient-type deformations for $k\sigma \neq 0$:

$$\begin{cases} \|C^s_{\breve{\sigma}}\|^2 = 2(l^2 - km - \breve{\sigma}n^2) = \|J(C)^s_{\breve{\sigma}}\|^2, \\ \|\tilde{C}^s_{\breve{\sigma}}\|^2 = 2(\sigma - \breve{\sigma})n^2, \quad \|C^{m,s}_{\breve{\sigma}}\|^2 = \frac{1}{2}(n^2 - \breve{\sigma}l^2). \end{cases} \tag{3.2}$$

Let us remark that:

$$det\ C^s_{\breve{\sigma}} = km + \breve{\sigma}n^2 - l^2 \to \|C^s_{\breve{\sigma}}\|^2 = \|J(C)^s_{\breve{\sigma}}\|^2 = -2det\ C^s_{\breve{\sigma}}. \tag{3.3}$$

Secondly, we study all the possible cases of orthogonality for our setting:

Theorem 3.1. *Let $\sigma \neq 0$ and the cycle C with $k \neq 0$. Then:*
1) C is $\breve{\sigma}$-orthogonal to its gradient deformation $\tilde{C}$ if and only if:

$$l^2 - km + (\sigma - 2\breve{\sigma})n^2 = 0. \tag{3.4}$$

2) C is $\breve{\sigma}$-orthogonal to its mixed-gradient deformation C^m if and only if:

$$(1 - \sigma\breve{\sigma})nl = 0. \tag{3.5}$$

3) $\tilde{C}$ is $\breve{\sigma}$-orthogonal to C^m if and only if (3.4) holds.
4) Suppose also $m \neq 0$. Then C is $\breve{\sigma}$-orthogonal to $J(C)$ if and only if:

$$2(l^2 + \breve{\sigma}n^2) - k^2 - m^2 = 0. \tag{3.6}$$

Proof. 1) A straightforward computation gives:

$$< C^s_{\breve{\sigma}}, \tilde{C}^s_{\breve{\sigma}} >= l^2 - km + (\sigma - 2\breve{\sigma})n^2. \tag{3.7}$$

2) The matrix of C^m from (2.18) is:

$$C^{m,s}_{\breve{\sigma}} = \frac{1}{2}\begin{pmatrix} n + \breve{\imath}s\sigma l & 0 \\ 0 & -n + \breve{\imath}s\sigma l \end{pmatrix} \tag{3.8}$$

and then:

$$< C^s_{\breve{\sigma}}, C^{m,s}_{\breve{\sigma}} >= (1 - \sigma\breve{\sigma})nl. \tag{3.9}$$

3) The same computation as above.
4) The matrix of $J(C)$ is:

$$J(C)^s_{\breve{\sigma}} := \begin{pmatrix} l - \breve{\imath}sn & -k \\ m & -l - \breve{\imath}sn \end{pmatrix} \tag{3.10}$$

and:

$$< C^s_{\breve{\sigma}}, JC^s_{\breve{\sigma}} >= 2(l^2 + \breve{\sigma}n^2) - m^2 - k^2. \tag{3.11}$$

which gives the conclusion. □

Example 3.2. Suppose $\sigma = \breve{\sigma}$. Then $1 - \sigma\breve{\sigma} = 0$ since $\sigma^2 = 1$ and then C^m is both orthogonally on C and $\tilde{C}$. In this case C is orthogonally to $\tilde{C}$ if and only if $l^2 - km - \breve{\sigma}n^2 = 0$ but from the first equation (3.2) this means that $\|C\| = 0$ i.e. C is also self-orthogonal.

Returning to the Möbius-type study of cycles we continue this section considering some transformation of cycles. The first one is inspired by [1, p. 2706]. Let $\alpha \in A(\sigma)$ with module $|\alpha| \neq 1$ and consider the map $T_\alpha : A(\sigma) \to A(\sigma)$:

$$T_\alpha(z) = z + \alpha\bar{z} := w. \tag{3.12}$$

It follows directly that T_α is a bijective map with the inverse:

$$z := T^{-1}_\alpha(w) = \frac{1}{1 - |\alpha|^2}(w - \alpha\bar{w}). \tag{3.13}$$

Replacing this expression of z in (2.22) we find the image of cycle C through T_α:

$$T_\alpha(C) : k|w - \alpha\bar{w}|^2 + (1 - |\alpha|^2)[(B - \bar{\alpha}\bar{B})w + (\bar{B} - \alpha B)\bar{w} + (1 - |\alpha|^2)m] = 0 \tag{3.14}$$

but this curve is not a cycle for $\alpha \cdot k \neq 0$.

The second transformation is a Blaschke factor B_a defined by $a \in A(\sigma)$ with module $|a| < 1$:

$$w := B_a(z) = \frac{z-a}{1-\bar{a}z}, \tag{3.15}$$

having the inverse:

$$z = B_{-a}(w) = \frac{w+a}{1+\bar{a}w}. \tag{3.16}$$

The Blaschke transformation of the cycle (2.22) is again a cycle:

$$B_a(C) : b_a(k)w\bar{w} + b_a(B)w + \overline{b_a(B)}\bar{w} + b_a(m) = 0 \tag{3.17}$$

with:

$$\begin{cases} b_a(k) = k + m|a|^2 + 2\Re(Ba), \\ b_a(B) = (k+m)\bar{a} + B + \bar{B}\bar{a}^2, \\ b_a(m) = m + k|a|^2 + 2\Re(Ba). \end{cases} \tag{3.18}$$

Example 3.3. Suppose that $|B| < 1$ and let $a = \bar{B}$. Then the Blaschke transformation of the coefficients is:

$$\begin{cases} b_{\bar{B}}(k) = k + (m+2)|B|^2, \\ b_{\bar{B}}(B) = (k+m+1+|B|^2)B, \\ b_{\bar{B}}(m) = m + (k+2)|B|^2. \end{cases} \tag{3.19}$$

The last transformation is a similarity defined by $a, b \in A(\sigma)$ with $a \neq 0$:

$$w := S_{a,b}(z) = az + b, \tag{3.20}$$

having the inverse:

$$z = \frac{1}{a}(w-b) = S_{\frac{1}{a},\frac{-b}{a}}(w). \tag{3.21}$$

The similarity transformation of the cycle (2.22) is again a cycle:

$$S_{a,b}(C) : kw\bar{w} + (B\bar{a} - k\bar{b})w + (\bar{B}a - kb)\bar{w} + m|a|^2 + k|b|^2 - 2\Re(Bb\bar{a}) = 0. \tag{3.22}$$

If the initial cycle C is non-degenerate then we restrict to the case $k = 1$ due to the projective character of the coefficients of C. Then a non-degenerate C is called *decomposable* if it is a product of lines:

$$C : (z - B)(\bar{z} - \bar{B}) = 0 \tag{3.23}$$

which means that $m = |B|^2 = l^2 - \sigma n^2$. A similarity preserves the class of decomposable cycles since its image is:

$$S_{a,b}(C) : (w - b + a\bar{B})(\bar{w} - \bar{b} + \bar{a}B). \tag{3.24}$$

From (3.3) it follows that a decomposable cycle has:

$$det\ C^s_{\breve{\sigma}} = (\breve{\sigma} - \sigma)n^2. \tag{3.25}$$

4. The rotation of a cycle

In this section we suppose that $\sigma \neq 0$. In $A(\sigma)$ we introduce the rotation map $R : (u, v) \to i \cdot (u, v) = (\sigma v, u)$; then its square is: $R^2 = \sigma I$. It follows that a given cycle C has an associated rotation cycle $R(C)$ with equation:

$$R(C) : k(\sigma^2 v^2 - \sigma u^2) - 2l\sigma v - 2nu + m = 0. \tag{4.1}$$

A short computation gives a more simple form:

$$R(C) : k(u^2 - \sigma v^2) + 2(\sigma n)u + 2lv - \sigma m = 0 \tag{4.2}$$

and then we have the deformation:

$$C = (k, l, n, m) \to R(C) = (k, -\sigma n, -l, -\sigma m). \tag{4.3}$$

The general rotation of conics is treated in [4].

Remark 4.1. Concerning the compositions $J \circ R$ and $R \circ J$ we have:

$$J \circ R(C) = (-\sigma m, -\sigma n, l, k), \quad R \circ J(C) = (m, \sigma n, -l, -\sigma k) \tag{4.4}$$

and then J and R anti-commutes in the hyperbolic setting respectively J and R commutes if and only if $l = 0$ in the complex setting: $\sigma = -1$.

In terms of associated matrix we have:

$$R(C)^s_{\breve{\sigma}} = \begin{pmatrix} -\sigma n - \breve{\imath} sl & \sigma m \\ k & \sigma n - \breve{\imath} sl \end{pmatrix}, \|R(C)^s_{\breve{\sigma}}\|^2 = 2(n^2 + \breve{\sigma} l^2 + \sigma km). \tag{4.5}$$

Then R preserves the norm of C if and only if:

$$(\sigma + 1)km + (\breve{\sigma} - 1)l^2 + (1 - \breve{\sigma})n^2 = 0. \tag{4.6}$$

Also, recall from section 2 that the e/h-center of C is $M(\frac{l}{k}, -\sigma\frac{n}{k})$ and hence its rotation is $R(M) = (-\frac{n}{k}, \frac{l}{k})$. But the center of $R(C)$ is $\bar{M} = (-\frac{\sigma n}{k}, \frac{\sigma l}{k})$ and then $\bar{M} = \sigma R(M)$; these points coincide for $\sigma = 1$.

Concerning the orthogonality of this new cycle with the previous three cycles we have:

Proposition 4.2. *Let C be a cycle with $k \neq 0$. Then:*
i) C is $\breve{\sigma}$-orthogonal to its rotated cycle $R(C)$ if and only if:

$$(\breve{\sigma} - \sigma)nl + (\sigma - 1)km = 0. \tag{4.7}$$

ii) $\tilde{C}$ is $\breve{\sigma}$-orthogonal to $R(C)$ if and only if:

$$2(\breve{\sigma} - \sigma)nl + \sigma(n^2 + km) - l^2 = 0. \tag{4.8}$$

iii) C^m is $\breve{\sigma}$-orthogonal to $R(C)$ if and only if:

$$\breve{\sigma} l^2 = n^2. \tag{4.9}$$

Proof. A straightforward computation gives:

$$< C^s_{\breve{\sigma}}, R(C)^s_{\breve{\sigma}} >= 2[(\breve{\sigma} - \sigma)nl + (\sigma - 1)km], \tag{4.10}$$

$$< \tilde{C}^s_{\breve{\sigma}}, R(C)^s_{\breve{\sigma}} >= 2(\breve{\sigma} - \sigma)nl + \sigma(n^2 + km) - l^2, \tag{4.11}$$

$$< C^{m,s}_{\breve{\sigma}}, R(C)^s_{\breve{\sigma}} >= 2\sigma(\breve{\sigma} l^2 - n^2] \tag{4.12}$$

which yields the conclusion. □

Example 4.3. Suppose that $\sigma = \breve{\sigma} = 1$. Then $R(C)$ is orthogonal to C and:
a) is orthogonal to $\tilde{C}$ if and only if: $l^2 = n^2 + km$; for $k = 1$ this means that C is decomposable,
b) is orthogonal to C^m if and only if: $l_{\pm} = \pm n$, which is exactly the relation (2.21).

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The size of some vanishing and critical sets

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Abstract. We prove that the vanishing sets of all top forms on a non-orientable manifold are at least 1-dimensional in the general case and at most 1-codimensional in the compact case. We apply these facts to show that the critical sets of some differentiable maps are at least 1-dimensional in the general case and at most 1-codimensional when the source manifold is compact.

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1. Introduction

It is well-known that the orientability of a manifold is characterized by the existence of a top differential form which never vanishes. Therefore it is natural to investigate the size of the vanishing sets $V(\theta) := \{p \in M : \theta_p = 0\}$ of the top forms $\theta \in \Omega^m(M)$ towards a measure of the *deviation from orientability* of the involved non-orientable manifold M. Indeed, the complement of every vanishing set of a top form is orientable and the smallest such vanishing sets are good candidates to measure this deviation. In this paper we show that the top forms of non-orientable manifolds cannot have arbitrarily small vanishing sets and apply this fact to show that some maps cannot have arbitrarily small critical sets. For instance the zero dimensional subsets of the non-orientable manifolds are neither vanishing sets of the top differentiable forms, nor critical sets of any differentiable function with orientable regular set, for the orientable option of the target manifold. Similar lower bounds for the size of the branch locus arise due to Church and Timourian [5, 6] in the codimension cases 0, -1 and -2. On the other hand, the critical set of a zero codimensional differentiable map was treated before in [17], where the critical set is realised as the vanishing set of the pull-back of a volume form on the oriented target manifold.

Note that the other extreme is well represented in the recent years, as quite some effort oriented towards the maps with finite critical sets has been done, not only for one dimensional, but also for higher dimensional target manifolds [1, 2, 3, 8, 9, 10].

The paper is organized as follows: In the second and third sections we quickly review the tools and emphasize the preparatory results needed to prove the main results of the paper, which are also stated here. In the fourth section we prove the main results of the paper, the first of which concerns the surjectivity of the group homomorphism induced, at the level of fundamental groups, by the inclusion $M\setminus A \hookrightarrow M$, where M^m $(m \geq 2)$ is a manifold and $A \subset M$ is a closed zero dimensional set. As a consequence we observe that the dimension of the critical set of a zero or lower codimensional map, whose target manifold is orientable and the source manifold is non-orientable, is at least 1-dimensional. Relying, all over this paper, on the inductive definition of the '*dimension*' [7, 13], we prove that the dimension of the critical set of a zero or lower codimensional map, whose target manifold is compact orientable and the source manifold M^n is compact non-orientable, is at least $(n-1)$-dimensional. Recall however that the small and large inductive dimensions are equal to each other and both are equal with the covering dimension whenever the evaluated space is separable [7, p. 65]. Since differential manifolds are metrizable metric spaces, it follows that the inductive dimensions of a certain subset of a given manifold are equal to each other and both are equal with the covering dimension of that subset.

2. Main results

In order to achieve such results we rely on the characterization of orientability of a connected differential manifold M by means of the *orientation character*, i.e. the group homomorphism $w_{_M} : \pi_1(M) \longrightarrow C_2 := \{-1, 1\}$ defined by

$$w_{_M}([\gamma]) = \begin{cases} 1 & \text{if } \tilde{\gamma}(1) = \tilde{x}_1 \\ -1 & \text{if } \tilde{\gamma}(1) = \tilde{x}_{-1}, \end{cases}$$

where $\tilde{\gamma} : [0,1] \longrightarrow \hat{M}$ is the lift of the loop $\gamma : [0,1] \longrightarrow M$, $\gamma(0) = \gamma(1) = x$, with $\tilde{\gamma}(0) = \tilde{x}_1$, $p : \hat{M} \longrightarrow M$ is the orientable double cover of M and $p^{-1}(x) = \{\tilde{x}_1, \tilde{x}_{-1}\}$. Indeed, M is orientable if and only if the orientation character is trivial. Equivalently, M is non-orientable if and only if $w_{_M}$ is onto. Taking into account that the orientation double cover of O is $p\big|_{p^{-1}(O)} : p^{-1}(O) \longrightarrow O$, we deduce that the orientation character of a connected open set $O \subseteq M$ can be decomposed as

$$\omega_{_O} = w_{_M} \circ \pi_1(i_{_O}), \quad \text{where } \pi_1(i_{_O}) : \pi_1(O) \longrightarrow \pi_1(M)$$

is the group homomorphism induced by the inclusion map $i_{_O} : O \hookrightarrow M$. Consequently the open connected subset O of a non-orientable manifold M remains non-orientable whenever $\pi_1(i_{_O})$ is surjective. Note that the orientation character $\omega_{_M}$ of M coincides with $w_1(M) \circ \rho$, where $\rho : \pi_1(M) \longrightarrow H_1(M, \mathbb{Z})$ stands for the Hurewicz homomorphism and $w_1(M)$ for the first Stiefel-Whitney class regarded as a homomorphism via the homomorphism of the universal coefficient Theorem

$$H^1(M; \mathbb{Z}_2) \longrightarrow \operatorname{Hom}(H_1(M; \mathbb{Z}), \mathbb{Z}_2)$$

and C_2 is identified with $\mathbb{Z}_2$.

Remark 2.1. Let M^m is a connected non-orientable manifold.

1. If the 1-skeleton M^1 of a certain CW-decomposition of M is a strong deformation retract of some of its open neighbourhood U, then the complement $M \setminus U$ cannot be the vanishing set of any top form on M, as the group homomorphism

$$\pi_1(i_{M\setminus U}) : \pi_1(M \setminus U) \longrightarrow \pi_1(M)$$

is onto [11, p. 39].
2. The $m-2$ and lower dimensional submanifolds of M^m cannot be the vanishing sets of any top forms on M, as the group homomorphism

$$\pi_1(i_{M\setminus X}) : \pi_1(M \setminus X) \longrightarrow \pi_1(M)$$

is an isomorphism for $m \geq 3$ and an epimorphism for $m = 2$, whenever X is such a submanifold of M. In particular the discrete subsets of M cannot be the vanishing sets of any top forms on M [16, Proposition 2.3]. By using the same type of arguments one can actually show that no countable subset of M can be the vanishing sets of any top form on M. In other words the vanishing set of every top form on M is uncountable. In fact the zero dimensional subsets of M cannot be the vanishing sets of any top forms on M, as we shall see in the Theorem 2.1 and Corollary 2.2.

Theorem 2.1. *If M^m is a smooth connected manifold ($m \geq 2$) and $A \subseteq M$ is a closed zero dimensional set, then $M \setminus A$ is also connected and the group homomorphism*

$$\pi_1(i) : \pi_1(M \setminus A) \longrightarrow \pi_1(M),$$

induced by the inclusion $i : M \setminus A \hookrightarrow M$, is onto, i.e. $\pi_1(M, M \setminus A) = 0$.

Corollary 2.2. *If M^m is a non-orientable manifold, then $\dim V(\omega) \geq 1$ for every differentiable form $\omega \in \Omega^m(M)$.*

Proof. Assume that $\dim V(\omega) = 0$ for some differentiable form $\omega \in \Omega^m(M)$. According to Theorem 2.1, the complement $M \setminus V(\theta)$ of the vanishing set is also connected and the group homomorphism

$$\pi_1(i) : \pi_1(M \setminus V(\theta)) \longrightarrow \pi_1(M)$$

is onto. The non-orientability of M shows that the orientation character w_M is onto. Consequently the orientaion character $\omega_{M\setminus V(\theta)} = w_M \circ \pi_1(i_{M\setminus V(\theta)})$, of $M \setminus V(\theta)$, is also onto, due to Theorem 2.1.

On the other hand the restriction $\theta|_{M\setminus V(\theta)}$ is a nowhere vanishing top form of $M \setminus V(\theta)$, which shows that $M \setminus V(\theta)$ is an orientable open submanifold of M. In other words, the orientation character $\omega_{M\setminus V(\theta)} = w_M \circ \pi_1(i_{M\setminus V(\theta)})$ is trivial, which implies that either the orientation character w_M is not onto or the induced group homomorphism $\pi_1(i_{M\setminus V(\theta)}) : \pi_1(M\setminus V(\theta)) \longrightarrow \pi_1(M)$ is not onto, which is absurd. □

In the compact non-orientable case we can provide, by using some different type of arguments, a much larger lower bound for the vanishing sets of all top forms.

Theorem 2.3. *If M^m is a compact connected non-orientable manifold, then $\dim V(\omega) \geq m-1$ for every differentiable form $\omega \in \Omega^m(M)$.*

Remark 2.2. The estimate, provided by Corollary 2.2 is sometimes sharp. Indeed, by removing a suitable circle out of Klein bottle we obtain a cylinder, which is orientable. Also by removing a suitable copy of the $(2n-1)$-dimensional real projective space, out of the $2n$-dimensional real projective space we obtain a $2n$-disc, which is also orientable. In both casses the removed submanifolds have, due to Corollary 2.2 and Theorem 2.3, the smallest possible dimension in order to get orientability on their complements.

3. Preliminary results

3.1. Vanishing sets of differentiable forms

If ω is a k-differential on M, recall that the *vanishing set* $V(\omega)$ of ω is the collection of points $z \in U$ at which ω vanishes, i.e.

$$V(\omega) := \{z \in M : \omega_z(v_1, \ldots, v_k) = 0 \text{ for all } v_i \in T_z(M)\}.$$

We shall only use in this paper the vanishing sets of the top differential forms of M.

In this subsection we investigate the size of critical sets of maps between two manifolds with the same dimension via the vanishing set of the pull-back form of a volume form on the target manifold.

Remark 3.1. If $f : M^n \to N^n$ is a local diffeomorphism and $\theta \in \Omega^k(N)$, then $V(f^*\theta) = f^{-1}(V(\theta))$. If f is additionally surjective, then this equality can be rewritten as $f(V(f^*\theta)) = V(\theta)$, which shows, by means of Hodel [12],

$$\dim(V(f^*\theta)) = \dim V(\theta) \tag{3.1}$$

whenever $V(f^*\theta)$ is compact.

Theorem 3.1. ([17]) *If $M^m, N^n, m \geq n$ are differential manifolds with N orientable and $f : M \longrightarrow N$ is a differential map, then $C(f) = V\big(f^*\mathrm{vol}_N\big)$, where vol_N is a volume form on N.*

Corollary 3.2. *Let M^n, N^n be differential manifolds. If N is orientable and M is non-orientable then $\dim C(f) \geq 1$ for every differentiable function $f : M \to N$.*

Proof. Let vol_N be a volume form on N. Combining Theorem 3.1 with Corollary 2.2 we deduce that $\dim C(f) = \dim V(f^*\mathrm{vol}_N) \geq 1$. □

In addition to the usefulness of the vanishing sets of differentiable forms in evaluating the size of the critical sets, they are also useful in evaluating the size of the tangency sets [4].

3.2. Zero dimensional subsets of manifolds

Lemma 3.3. *If C is a closed subset of a smooth manifold M^n, then there exists a smooth nonnegative function $f : M \longrightarrow \mathbb{R}$ such that $f^{-1}(0) = C$.*

Proof. We first consider an embedding $j : M \hookrightarrow \mathbb{R}^{2n+1}$, whose existence is ensured by Whitney's embedding theorem.
If $K \subseteq \mathbb{R}^{2n+1}$ is a closed subset such that $j(C) = K \cap j(M)$, i.e. $j^{-1}(K) = C$, then the required function is $f = g \circ j$, where $g : \mathbb{R}^{2n+1} \longrightarrow \mathbb{R}$ is a smooth positive function such that $g^{-1}(0) = K$, whose existence is ensured by the Whitney theorem ([18, Théorème 1, p. 17]). □

Proposition 3.4. *If A is a closed zero dimensional subset of a smooth manifold M^n, then for each $x \in M$ and every neighbourhood U of x, there exists an open neighbourhood V of x such that $V \subseteq U$, $\partial V \cap A = \emptyset$ and ∂V is smooth.*

Proof. If $x \notin A$, then the existence of V is immediate. Assume now that $a \in A$ and consider an open and relatively compact neighbourhood V' of a such that $V' \subseteq U$ and $\partial V' \cap A = \emptyset$. We may assume that V' is actually connected, as otherwise we reduce V' to its connected component containing a. If $\varphi : M \longrightarrow \mathbb{R}$ is a smooth nonnegative function such that $\varphi^{-1}(0) = A$, whose existence is ensured by Lemma 3.3, observe that $m := \min\{\varphi(x) \mid x \in \partial V'\} > 0$, since the compact set $A \cap cl(V') = A \cap V'$ has no common points with the compact boundary $\partial V'$. If $y \in (0, m)$ is a regular value of $\varphi|_{V'} : V' \to \mathbb{R}$, then $(\varphi|_{V'})^{-1}(y)$ is a compact hypersurface in V', as $(\varphi|_{V'})^{-1}(y) = \varphi^{-1}(y) \cap cl(V')$. Indeed, the inclusion $(\varphi|_{V'})^{-1}(y) \subseteq \varphi^{-1}(y) \cap cl(V')$ is obvious. If $x \in \varphi^{-1}(y) \cap cl(V')$, then $\varphi(x) = y$ and $x \in cl(V') = V' \cup \partial V'$. But since $y > 0$, it follows that $x \notin \partial V'$, which shows that $x \in V'$ and $x \in (\varphi|_{V'})^{-1}(y)$ as well. Because $y < m$, it follows that $(\varphi|_{V'})^{-1}(y) \cap A = \emptyset$.

Finally, we consider a regular value $y \in (0, m)$ of $\varphi|_{V'} : V' \to \mathbb{R}$ and observe that the inverse image $\left(\varphi|_{V'}\right)^{-1}(-\infty, y) \subseteq V'$ is an open neighbourhood of A and

$$\partial \left[\left(\varphi|_{V'}\right)^{-1}(-\infty, y)\right] = \left(\varphi|_{V'}\right)^{-1}(y),$$

which shows that $\partial \left[\left(\varphi|_{V'}\right)^{-1}(-\infty, y)\right] \cap A = \emptyset$. If V is the connected component of the inverse image $\left(\varphi|_{V'}\right)^{-1}(-\infty, y)$ containing a, then its boundary is a collection of connected components of $\left(\varphi|_{V'}\right)^{-1}(y)$ and therefore $\partial V \cap A = \emptyset$. □

Remark 3.2. If A is a closed zero dimensional subset of a smooth surface Σ, then for each $x \in \Sigma$ and every neighbourhood U of x, there exists an open disk D such that $x \in D \subseteq U$, $\partial D \cap A = \emptyset$ and ∂D is a smooth circle. Indeed, we consider, via Proposition 3.4, a local chart (W, ψ) of Σ at x as well as a connected neighbourhood V of x with smooth boundary such that $x \in V$, $\mathrm{cl}(V) \subseteq W \subseteq U$, $\psi(W) = D^2$ and $\partial V \cap A = \emptyset$. Note that the boundary of $\psi(V)$ is a union of pairwise disjoint circles, as the circle is the only compact boundaryless one dimensional manifold. One of these circles, say C, is the boundary of the unbounded component of $\mathbb{R}^2 \setminus \psi(V)$. The bounded component of $\mathbb{R}^2 \setminus C$ is completely contained in D^2, contains $\psi(V)$ and we may choose its inverse image through ψ to play the role of D.

3.3. Deformations of punctured manifolds

Since the deformations of the punctured Euclidean space and the punctured manifolds [16] will be repeatedly used in what follows, we shall review them shortly.

For $r > 0$ and $n \in \mathbb{N}^*$ denote by D_r^n and S_r^{n-1} the open disk and the sphere respectively, both of them having the center at the origin of the space $\mathbb{R}^n$ and radius r. D_1^n and S_1^{n-1} will be simply denoted by D^n and S^{n-1} respectively. For $x \in D^n$, consider the map $h_x : \mathbb{R}^n \setminus \{x\} \longrightarrow \mathbb{R}^n \setminus \{x\}$ defined to be the identity outside the open disc D^n and $h_x(y) = S^{n-1} \cap]xy$ for every $y \in D^n \setminus \{x\}$, where $]xy$ stands for the half line $\{(1-s)x + sy \ : \ s > 0\}$. In particular $h_x(y) = y$, $\forall y \in S^{n-1}$.

Let N be an n-dimensional manifold and $c = (U, \varphi)$ be a local chart of N such that $\mathrm{cl}(D^n) \subseteq \varphi(U)$. Denote by D_φ and S_φ the sets $\varphi^{-1}(D^n)$ and $\varphi^{-1}(S^{n-1})$ respectively. For $x \in D_\varphi$ we define the continuous map $h_{c,x} : N \setminus \{x\} \longrightarrow N \setminus \{x\}$ by

$$h_{c,x}(y) = \begin{cases} y & \text{if } y \in M \setminus D_\varphi \\ \varphi^{-1}\left(h_{\varphi(x)}\left(\varphi(y)\right)\right) & \text{if } y \in U \setminus \{x\}. \end{cases}$$

Note that $h_{c,x}(D_\varphi \setminus \{x\}) = S_\varphi$ and $h_{c,x}(y) = y$, $\forall y \in S_\varphi$.

Remark 3.3. 1. $h_x(D^n \setminus \{x\}) = S^{n-1}$ and $h_x \simeq_{H_x} id_{\mathbb{R}^n \setminus \{x\}}(\mathrm{rel}\, \mathbb{R}^n \setminus D^n)$, where

$$H_x : \mathbb{R}^n \setminus \{x\} \times [0,1] \to \mathbb{R}^n \setminus \{x\},\ H_x(y,t) = (1-t)y + th_x(y).$$

2. $h_{c,x} \simeq_{H_{c,x}} id_{M \setminus \{x\}}$, where $H_{c,x} : (M \setminus \{x\}) \times [0,1] \to M \setminus \{x\}$,

$$H_{c,x}(y,t) = \begin{cases} y & \text{if } y \in M \setminus D_\varphi \\ \varphi^{-1}\left(H_{\varphi(x)}(\varphi(y),t)\right) & \text{if } y \in U \setminus \{x\}. \end{cases}$$

If P is a given manifold and $f : P \longrightarrow M$ is a continuous map whose image avoids the point x, then $f \simeq h_{c,x} \circ f$ and a homotopy between f and $h_{c,x} \circ f$ is $H_{c,x}(\cdot, t) \circ f$. We shall refere to each $h_{c,x} \circ f$ and $H_{c,x}(\cdot, t) \circ f$ as the *punctured deformation* of f from x onto S_φ.

4. The proofs of theorems 2.1 and 2.3

Proof of Theorem 2.1. Consider a homotopy class of curves in $\pi_1(M, M \setminus A)$ represented by a continuous curve $\alpha : [0,1] \longrightarrow M$, $\alpha(0), \alpha(1) \in M \setminus A$ and deform α $\mathrm{rel}\{0,1\}$ to some differentiable curve β with non vanishing tangent vector field. The immersion β might actually be chosen to be a geodesic from $\alpha(0)$ to $\alpha(1)$ with respect to some Riemannian metric on M (see e.g. [14, Theorem 1.4.6, p. 24]). Obviously $\dim\left(A \cap \mathrm{Im}(\beta)\right) \leq \dim(A) = 0$ and $\dim \mathrm{Im}(\beta) = 1$.

From this point we continue the proof by induction with respect to the dimension m of the manifold M. First assume that $m = 2$ and observe that for each $t \in \beta^{-1}(A)$ there exists, via Remark 3.2, a two dimensional disc $D_t \subseteq M$ with circular boundary, neighbourhood of $\beta(t)$, such that its circular boundary C_t has no common points with A. Since β is locally an embedding, D_t might be chosen inside the domain U_t of a coordinate chart $c_t = (U_t, \varphi_t)$ in such a way that $D_t = D_{\varphi_t}$, $C_t = S_{\varphi_t}$, $\alpha(0) = \beta(0)$, $\alpha(1) = \beta(1) \in M \setminus \mathrm{cl}(D_t)$, $J_t := \beta^{-1}(D_t)$ is an open interval and $\varphi_t\left(D_t \cap \mathrm{Im}(\beta|_{J_t})\right) = \varphi(D_t) \cap \mathbb{R}$. Since $\{D_t \mid t \in \beta^{-1}(A)\}$ is an open covering of the compact set $\mathrm{Im}(\beta) \cap A$, we may extract a finite open cover, say $D_{t_1}, \ldots, D_{t_s}$. We may assume that $D_{t_i} \setminus \mathrm{cl}(D_{t_j}) \neq \emptyset$ whenever $i \neq j$. Since $\dim(\mathrm{Im}(\beta) = 1$ it follows that $\mathrm{Im}(\beta)$ cannot fill the open set

$$D_{t_i} \setminus \bigcup_{\substack{j=1\\ j\neq i}}^{r} \mathrm{cl}(D_{t_j}).$$

For each $i \in \{1, \ldots, r\}$, consider a point

$$x_i \in D_{t_i} \setminus \Big(\mathrm{Im}(\beta) \cup \bigcup_{\substack{j=1\\ j\neq i}}^{r} \mathrm{cl}(D_{t_j})\Big)$$

and the maps $g_{c_{t_i},x_i,} : M \setminus \{x_1, \ldots, x_s\} \longrightarrow M \setminus \{x_1, \ldots, x_s\}$ given by

$$g_{c_{t_i},x_i}(y) = h_{c_{t_i},x_i}(y).$$

Note that $\mathrm{id}_{M\setminus\{x_1,\ldots,x_s\}} \simeq g_{c_{t_1},x_1} \circ \cdots \circ g_{c_{t_s},x_s}$ (rel $M \setminus (D_{t_1} \cup \cdots \cup D_{t_s})$) as each of the maps $g_{c_{t_1},x_1}, \ldots, g_{c_{t_s},x_s}$ is homotopic to $\mathrm{id}_{M\setminus\{x_1,\ldots,x_s\}}$ relative to $M \setminus (D_{t_1} \cup \cdots \cup D_{t_s})$. Thus

$$\beta \simeq (g_{c_{t_1},x_1} \circ \cdots \circ g_{c_{t_s},x_s} \circ \beta)(\mathrm{rel}\{0, 1\})$$

and $\mathrm{Im}(g_{c_{t_1},x_1} \circ \cdots \circ g_{c_{t_s},x_s} \circ \beta) \subseteq M \setminus A$, as

$$(g_{c_{t_1},x_1} \circ \cdots \circ g_{c_{t_s},x_s}) \left((D_{t_1} \cup \cdots \cup D_{t_s}) \setminus \{x_1, \ldots, x_s\}\right) \subseteq C_{t_1} \cup \cdots \cup C_{t_s} \subseteq M \setminus A$$

and $\beta^{-1} \left(M \setminus (D_{t_1} \cup \cdots \cup D_{t_s})\right) \subseteq \beta^{-1}(M \setminus A)$.

We next assume that the statement holds for $(m-1)$-dimensional manifolds and we shall prove it for the m-dimensional manifold M. In this respect we consider a partition $0 = t_0 < t_1 < \cdots < t_r = 1$ of the interval $[0, 1]$ with small enough norm such that:

1. $\beta([t_0, t_1]) \cap A = \beta([t_{r-1}, t_r]) \cap A = \emptyset$ and $\beta(t_1), \ldots, \beta(t_{r-1}) \in M \setminus A$.
2. there are small enough open discs $D_1 = D_{\varphi_1}, \ldots, D_{r-2} = D_{\varphi_{r-2}}$ with spherical boundaries $S_1 = S_{\varphi_1}, \ldots, S_{r-2} = S_{\varphi_{r-2}}$, for some charts $c_1 = (U_1, \varphi_1), \ldots, c_r = (U_{r-2}, \varphi_{r-2})$, with the following properties:
   (a) $\beta^{-1}(D_i)$ is the open interval (t_i, t_{i+1}) and the restriction $(t_i, t_{i+1}) \longrightarrow D_i$, $t \mapsto \beta(t)$ is an embedding, for every $i = \overline{1, r-2}$;
   (b) $\mathrm{cl}(D_i) \cap \mathrm{Im}(\beta) = S_i \cap \mathrm{Im}(\beta) = \{\beta(t_i), \beta(t_{i+1})\}$ and $D_i \cap D_{i+1} = \emptyset$ while $\mathrm{cl}(D_i) \cap \mathrm{cl}(D_{i+1}) = S_i \cap S_{i+1} = \{\beta(t_{i+1})\}$, for every $i = \overline{1, r-3}$.

Note that $\mathrm{Im}(\beta) \cap A \subset D_1 \cup \cdots \cup D_{r-2}$. For every $i \in \{1, \ldots, r-2\}$, consider $x_i \in D_i \setminus \mathrm{Im}(\beta)$ and observe that $\beta|_{[x_i,x_{i+1}]} \simeq h_{c_{t_i},x_i} \circ \beta|_{[x_i,x_{i+1}]}(\mathrm{rel}(\{x_i, x_{i+1}\}))$. By applying the inductive hypothesis to the punctured deformation $h_{c_{t_i},x_i} \circ \beta|_{[x_i,x_{i+1}]}(\mathrm{rel}(\{x_i, x_{i+1}\}))$ of $\beta|_{[x_i,x_{i+1}]}$ from x_i onto S_i, whose image is contained in the $(m-1)$-dimensional sphere S_i, one can conclude that $h_{c_{t_i},x_i} \circ \beta|_{[x_i,x_{i+1}]}$ is homotopic $\mathrm{rel}(\{x_i, x_{i+1}\})$ to some continuous curve $\gamma_i : [x_i, x_{i+1}] \longrightarrow S_i$ whose image avoids the set A, i.e. $\gamma_i([x_i, x_{i+1}]) \subseteq S_i \setminus A$. Thus $h_{c_{t_i},x_i} \circ \beta$ is homotopic $\mathrm{rel}(\{0, 1\})$ to the continuous curve $\gamma : [0, 1] \longrightarrow M \setminus A$ defined by $\gamma|_{[x_0,x_1]} = \beta|_{[x_0,x_1]}$, $\gamma|_{[x_i,x_{i+1}]} = \gamma_i$ for $1 \le i \le r-2$ and $\gamma|_{[x_{r-1},x_r]} = \beta|_{[x_{r-1},x_r]}$. □

Proof of Theorem 2.3. We first observe that every top-form $\omega \in \Omega^m(M)$ is exact, as the top de Rham cohomology group $H^m_{dR}(M)$ is trivial [15, Th. 15.21, p. 405], i.e.

$\omega = d\theta$ for some $\theta \in \Omega^{m-1}(M)$. If $p : \tilde{M} \longrightarrow M$ is the orientable double cover, then $p^*\omega = p^*(d\theta) = d(p^*\theta)$, which shows that $\int_{\tilde{M}} p^*\omega = 0$ and that $\dim(V(p^*\omega)) \geq m-1$ therefore. Thus

$$\dim(V(\omega)) = \dim(p(V(p^*\omega))) = \dim(V(p^*\omega)) \geq m-1,$$

as $p(V(p^*\omega)) = V(\omega)$. □

Corollary 4.1. *Let M^n, N^n be differential manifolds. If N is orientable and M is compact and non-orientable then* $\dim C(f) \geq n-1$ *for every differentiable function* $f : M \to N$.

Proof. Let vol_N be a volume form on N. Combining Theorem 3.1 with Theorem 2.3 we deduce that $\dim C(f) = \dim V(f^*\mathrm{vol}_N) \geq n-1$, for every differentiable function $f : M \longrightarrow N$. □

A proof of Corollary 4.1 of similar flavor appears in [17, Theorem 2.4.(b)].

Remark 4.1. Corollaries 3.2 and 4.1 rely on the orientability of the regular set $R(f) = M \setminus C(f)$ in the $0 = \dim(N) - \dim(M)$ codimension case which is a priori ensured by the nowhere vanishing restricted top form $f^*\mathrm{vol}_N\big|_{R(f)}$ on $R(f)$. In the lower codimension case $\dim(M) > \dim(N)$, the lack of orientability of the regular set is obvious, even for the orientable option of the target manifold N. We stress this by the example of the projection of a product $M = N \times X$ on the first factor, when N is orientable and X is non-orientable. The critical set of this projection is obviously empty, but its regular set is the whole non-orientable product $M = N \times X$.

However, the orientability of the regular set $R(f)$ ensure similar lower bounds even in the lower codimensional context. More precisely, if N^n is orientable and M^m $(m > n)$ is connected non-orientable and $f : M \longrightarrow N$ is a differentiable function with orientable regular set $R(f)$, then $\dim(C(f)) \geq 1$. The proof of this statement works along the same lines with the one of Corollary 2.2, the role of the vanishing set $V(\theta)$ is played here by the critical set $C(f)$.

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