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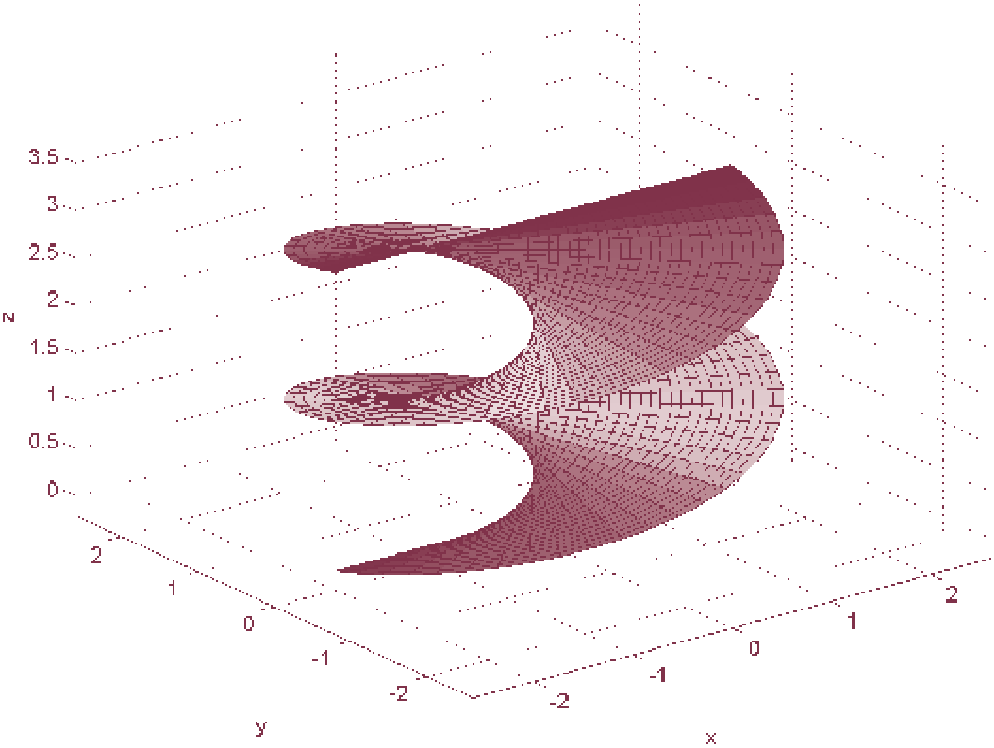

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CONTENTS

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A refinement of an inequality due to Ankeny and Rivlin

Dinesh Tripathi

Abstract. Let $p(z) = \sum_{\nu=0}^{n} a_\nu z^\nu$ be a polynomial of degree n,

$$M(p, R) := \max_{|z|=R\geq 0} |p(z)|, \text{ and } M(p, 1) := M(p).$$

Then by well-known result due to Ankeny and Rivlin [1], we have

$$M(p.R) \leq \left(\frac{R^n + 1}{2}\right) M(p), \ R \geq 1.$$

In this paper, we sharpen and generalizes the above inequality by using a result due to Govil [5].

Mathematics Subject Classification (2010): 15A18, 30C10, 30C15, 30A10.

Keywords: Inequalities, polynomials, maximum modulus.

1. Introduction

Let $\mathcal{P}_n := \left\{ p(z); p(z) = \sum_{\nu=0}^{n} a_\nu z^\nu \right\}$ be a class of polynomial of degree n. Let $\max_{|z|=R} |p(z)| = M(p, R)$ and $M(p, 1) = M(p)$.Then from maximum modulus principle, $M(p, R)$ is a strictly increasing function and for $0 \leq R < \infty$. Also, it is a simple deduction from the maximum modulus principle (see [10, p. 158, Problem 269]) that for $R \geq 1$,

$$M(p, R) \leq R^n M(p). \tag{1.1}$$

The result is best possible and equality holds if and only if $p(z) = \lambda z^n$, where λ being a complex number.
For $p \in \mathcal{P}_n$ not vanishing in the interior of unit circle, Ankeny and Rivlin [1] sharpened inequality (1.1), by proving following result.

Theorem 1.1. *If $p \in \mathcal{P}_n$ and $p(z) \neq 0$ for $|z| < 1$, then for $R \geq 1$,*

$$M(p,R) \leq \left(\frac{R^n + 1}{2}\right) M(p), \;\; R \geq 1. \tag{1.2}$$

The above inequality is sharp and equality holds for polynomial

$$p(z) = \alpha + \beta z^n, \;\; |\alpha| = |\beta|.$$

Since the equality in (1.2) holds only for $p(z) = \alpha + \beta z^n$, which satisfy

$$|\beta| = \frac{1}{2} M(p), \tag{1.3}$$

therefore it should possible to improve the bound (1.2) for the polynomial not satisfying (1.3). Govil [5] solve this problem by proving the following result.

Theorem 1.2. *If $p \in \mathcal{P}_n$ and $p(z) \neq 0$ for $|z| < 1$, then for $R \geq 1$,*

$$\begin{aligned} M(p,R) \;\; &\leq \;\; \left(\frac{R^n + 1}{2}\right) M(p) - \frac{n}{2}\left(\frac{M(p)^2 - 4|a_n|^2}{M(p)}\right)\left\{\frac{(R-1)M(p)}{M(p) + 2|a_n|}\right. \\ &\qquad \left. - \ln\left(1 + \frac{(R-1)M(p)}{M(p) + 2|a_n|}\right)\right\}. \end{aligned} \tag{1.4}$$

The result is best possible and the equality holds for $p(z) = (\lambda + \mu z^n)$, λ and μ being complex numbers with $|\lambda| = |\mu|$.

The other extension and generalization of Theorem 1.1 has been mentioned in the various article,e.g Aziz [2], Aziz and Mohammad [3], Milovanović, Mitrinović and Rassias [8], Govil [6], Govil, Qazi and Rahman [7] and Rahman and Schmeisser [12], Tripathi [13] etc.

2. Main results

In this paper, we prove the following improved generalization of Theorem 1.2 for the class of Lacunary type of polynomial

$$p(z) = a_0 + \sum_{\nu=\mu}^{n} a_\nu z^\nu.$$

Theorem 2.1. *If $p(z) = a_0 + \sum_{\nu=\mu}^{n} a_\nu z^\nu$ is a polynomial of degree n and $p(z) \neq 0$ for $|a| < k, k \geq 1$, then for $R > r \geq 1$,*

$$\begin{aligned} |\{p(Re^{i\theta})\}^s| \leq \frac{(R^{ns} - r^{ns})}{1 + k^\mu}\{M(p)\}^s - \frac{n}{1+k^\mu}\{M(p)\}^s \left(1 - \frac{(1+k^\mu)|a_n|}{M(p)}\right) h(n) \\ + |\{p(re^{i\theta})\}^s|, \end{aligned} \tag{2.1}$$

where

$$h(n) = \left(\frac{R^n - r^n}{n}\right) + \sum_{k=1}^{n-1}\left(\frac{R^{n-k} - r^{n-k}}{n-k}\right)(-1)^k\left(\frac{(1+k^\mu)|a_n|}{M(p)}+1\right)\left(\frac{(1+k^\mu)|a_n|}{M(p)}\right)^{k-1}$$
$$+ (-1)^n\left(\frac{(1+k^\mu)|a_n|}{M(p)}+1\right)\left(\frac{(1+k^\mu)|a_n|}{M(p)}\right)^{n-1}\ln\left(\frac{R(M(p)) + (1+k^\mu)|a_n|}{r(M(p)) + (1+k^\mu)|a_n|}\right)$$

for $n \geq 1$ *and* $h(0) = 0$.

On taking $s = 0$, $\mu = 1$, $r = 1$ and $k = 1$, we have the following application of above Theorem 2.1.

Corollary 2.2. *If* $p \in \mathcal{P}_n$ *and* $p(z) \neq 0$ *for* $|z| < 1$, *then for* $R \geq 1$,

$$|p(Re^{i\theta})| \leq \frac{(R^n+1)}{2}M(p) - \frac{n}{2}M(p)\left(1 - \frac{2|a_n|}{M(p)}\right)h(n), \tag{2.2}$$

where

$$h(n) = \left(\frac{R^n - 1}{n}\right) + \sum_{k=1}^{n-1}\left(\frac{R^{n-k} - 1}{n-k}\right)(-1)^k\left(\frac{2|a_n|}{M(p)}+1\right)\left(\frac{2|a_n|}{M(p)}\right)^{k-1}$$
$$+ (-1)^n\left(\frac{2|a_n|}{M(p)}+1\right)\left(\frac{2|a_n|}{M(p)}\right)^{n-1}\ln\left(1 + \frac{(R-1)M(p)}{M(p)+2|a_n|}\right)$$

for $n \geq 1$ *and* $h(0) = 0$.

Remark 2.3. From Lemma 3.7, we get $0 \leq h(n)$. Using this in Corollary 2.2, we get

$$|p(Re^{i\theta})| \leq \frac{(R^n+1)}{2}M(p) - \frac{n}{2}M(p)\left(1 - \frac{2|a_n|}{M(p)}\right)h(n) \leq \frac{(R^n+1)}{2}M(p),$$

which shows that Corollary 2.2, clearly refines Theorem 1.1 due to Ankeny and Rivlin [1].

Remark 2.4. From Lemma 3.7, we have $h(1) \leq h(n)$. Using this inequality in Corollary 2.2, we get

$$\begin{aligned} |p(Re^{i\theta})| &\leq \frac{(R^n+1)}{2}M(p) - \frac{n}{2}M(p)\left(1 - \frac{2|a_n|}{M(p)}\right)h(n) \\ &\leq \frac{(R^n+1)}{2}M(p) - \frac{n}{2}M(p)\left(1 - \frac{2|a_n|}{M(p)}\right)h(1), \end{aligned} \tag{2.3}$$

and,

$$h(1) = (R-1) - \left(1 + \frac{2|a_n|}{M(p)}\right)\ln\left(1 + \frac{(R-1)M(p)}{M(p)+2|a_n|}\right). \tag{2.4}$$

Substitute the value of $h(1)$ in (2.3), we get

$$\begin{aligned} |p(Re^{i\theta})| \leq & \left(\frac{R^n+1}{2}\right)M(p) - \frac{n}{2}\left(\frac{M(p)^2 - 4|a_n|^2}{M(p)}\right)\left\{\frac{(R-1)M(p)}{M(p)+2|a_n|}\right. \\ & \left. - \ln\left(1 + \frac{(R-1)M(p)}{M(p)+2|a_n|}\right)\right\}, \end{aligned}$$

which is Theorem 1.2 due to Govil [5].

By taking $\mu = 1$ in inequality (2.1), we obtain the following results.

Corollary 2.5. *If $p \in \mathcal{P}_n$ and $p(z) \neq 0$ for $|z| < k, k \geq 1$, then for $R > r \geq 1$,*

$$|\{p(Re^{i\theta})\}^s| \leq \frac{(R^{ns} - r^{ns})}{1+k}\{M(p)\}^s - \frac{n}{1+k}\{M(p)\}^s\left(1 - \frac{(1+k)|a_n|}{M(p)}\right)h(n) + |\{p(re^{i\theta})\}^s|, \tag{2.5}$$

where

$$h(n) = \left(\frac{R^n - r^n}{n}\right) + \sum_{k=1}^{n-1}\left(\frac{R^{n-k} - r^{n-k}}{n-k}\right)(-1)^k\left(\frac{(1+k)|a_n|}{M(p)} + 1\right)\left(\frac{(1+k)|a_n|}{M(p)}\right)^{k-1}$$
$$+ (-1)^n\left(\frac{(1+k)|a_n|}{M(p)} + 1\right)\left(\frac{(1+k)|a_n|}{M(p)}\right)^{n-1}\ln\left(\frac{R(M(p)) + (1+k)|a_n|}{r(M(p)) + (1+k)|a_n|}\right)$$

for $n \geq 1$ and $h(0) = 0$.

Remark 2.6. We also have some other application Theorem 2.1, by taking $s = 0$, $k = 1$ and $r = 1$ respectively.

3. Lemmas

For the proof of theorem, we need the following lemmas. Our first lemma is a well-known generalization of Schwarz's lemma (see for example [9, p. 167]).

Lemma 3.1. *If $f(z)$ is analytic inside and on the circle $|z| = 1$, $f(0) = a$, where $|a| < f$, then*

$$|f(z)| \leq M(f)\left(\frac{M(f)|z| + |a|}{|a||z| + M(f)}\right). \tag{3.1}$$

Lemma 3.2. *If $p(z) = \sum_{v=0}^{n} a_v z^v$ is a polynomial of degree n, then for $|z| = R \geq 1$,*

$$|p(z)| \leq \left(\frac{|a_n|R + M(p)}{M(p)R + |a_n|}\right)M(p)R^n. \tag{3.2}$$

The proof follows easily on applying Lemma 3.1 to the function $T(z) = z^n p(1/z)$ and noting that $M(T) = M(p)$ (for details see [12, Lemma 2]).

From Lemma 3.2, one immediately gets:

Lemma 3.3. *If $p(z) = \sum_{v=0}^{n} a_v z^v$ is a polynomial of degree n, then for $|z| = R \geq 1$,*

$$|p(z)| \leq \left(1 - \frac{(M(p) - |a_n|)(R-1)}{M(p)R + |a_n|}\right)M(p)R^n. \tag{3.3}$$

The following result is due to Chan and Malik [4].

Lemma 3.4. *If* $p(z) = a_0 + \sum_{v=\mu}^{n} a_v z^v$ *is a polynomial of degree* n*, and* $p(z) \neq 0$ *for* $|z| < k, k \geq 1$*, then*

$$\max_{|z|=1} |p'(z)| \leq \frac{n}{1+k^\mu} M(p). \tag{3.4}$$

Lemma 3.5. *If* $p(z) = \sum_{v=0}^{n} a_v z^v$ *is a polynomial of degree* n*, and let* $r \geq 1$*, then*

$$\left(1 - \frac{(x-|a_n|)(r-1)}{rx + n|a_n|}\right) x \tag{3.5}$$

is an increasing function of x*, for* $x > 0$*.*

The proof of above lemma is straight forward using derivative test, so we omit the detail proof.

Lemma 3.6. *Let*

$$h(n) = \int_r^R \frac{(t-1)(t^{n-1})}{t+a} dt \text{ for } n \geq 1.$$

Then

$$\begin{aligned} h(n) = &\left(\frac{R^n - r^n}{n}\right) + \sum_{k=1}^{n-1} \left(\frac{R^{n-k} - r^{n-k}}{n-k}\right)(-1)^k (a+1) a^{k-1} \\ &+ (-1)^n (a+1) a^{n-1} \ln\left(\frac{R+a}{r+a}\right). \end{aligned}$$

Proof. We define the function $f(n) = \int_r^R \frac{t^n}{t+a} dt$ for $n \geq 0$. It is easy to see that

$$h(n) = f(n) - f(n-1) \text{ for } n \geq 1.$$

We can obtain

$$\begin{aligned} f(n) + af(n-1) &= \int_r^R \frac{t^n + at^{n-1}}{t+a} dt \\ &= \int_r^R \frac{t^{n-1}(t+a)}{t+a} dt = \frac{R^n - r^n}{n} = g(n), \quad \text{(say)}. \end{aligned}$$

Then

$$f(n) = g(n) - af(n-1). \tag{3.6}$$

Solving the recurrence relation (3.6), we get

$$f(n) = \sum_{k=0}^{n-1} g(n-k)(-1)^k a^k + (-1)^n a^n f(0), \tag{3.7}$$

where

$$f(0) = \int_1^R \frac{1}{r+a} dr = \ln\left(\frac{R+a}{r+a}\right).$$

Now, Substituting the value of $f(0)$ in (3.7), we get

$$f(n) \quad = \quad \sum_{k=0}^{n-1} g(n-k)(-1)^k a^k + (-1)^n a^n \ln\left(\frac{R+a}{r+a}\right), n \geq 0. \tag{3.8}$$

Using $h(n) = f(n) - f(n-1)$ and value of $g(n)$, we have Lemma 3.6 for $n \geq 1$. $\square$

Lemma 3.7. *Let*

$$h(n) = \int_r^R \frac{(t-1)(t^{n-1})}{t+a} dt \text{ for } n \geq 1.$$

Then $h(n)$ is a non-negative increasing function of n for $n \geq 1$.

Proof. Let

$$f(n) = \int_r^R \frac{r^n}{r+a} dr \text{ for } n \geq 0.$$

It is easy to see that $h(n) = f(n) - f(n-1)$ for $n \geq 1$. For $n \geq 1$,

$$f(n) - f(n-1) = \int_1^R \frac{(r-1)(r^{n-1})}{r+a} dr \geq \int_1^R \frac{(r-1)(r^{n-2})}{r+a} dr = f(n-1) - f(n-2)$$

as $r^{n-1} \geq r^{n-2}$ for $r \geq 1$. Therefore,

$$h(n) = f(n) - f(n-1) \geq f(n-1) - f(n-2) = h(n-1).$$

Therefore, $h(n)$ is an increasing function of n for $n \geq 1$.
Also, $h(n) = f(n) - f(n-1) \geq 0$ for $n \geq 0$ as

$$\int_r^R \frac{(t-1)(t^{n-1})}{t+a} dr \geq 0$$

for $n \geq 1$ and $h(0) = 0$. Therefore, $h(n) \geq 0$ and is an increasing function of n for $n \geq 0$. $\square$

4. Proof of the Theorem

Proof of Theorem 2.1. For each θ, $0 \leq \theta < 2\pi$, we have

$$\begin{aligned} |\{p(Re^{i\theta})\}^s - \{p(re^{i\theta})\}^s| &= \left| \int_r^R \frac{d}{dt} \{p(te^{i\theta})\}^s dt \right| \leq \int_r^R s|\{p(te^{i\theta})\}^{s-1}||p'(te^{i\theta})|dt, \\ &\leq \{M(p)\}^{s-1} \int_r^R t^{ns-n} s|p'(te^{i\theta})|dt \end{aligned}$$

$$\begin{aligned} &|\{p(Re^{i\theta})\}^s - \{p(re^{i\theta})\}^s| \\ &\leq \{M(p)\}^{s-1} \int_r^R st^{ns-1} \left\{1 - \frac{(M(p') - n|a_n|)(t-1)}{n|a_n| + tM(p')}\right\} M(p')dt, \end{aligned} \tag{4.1}$$

by using Lemma 3.3 for the polynomial $p'(z)$, which is of degree $n-1$. We can see, from Lemma 3.5, the integrand in (4.1) is an increasing function of $M(p')$.

Now, applying Lemma 3.4 to inequality (4.1), we get for $0 \leq \theta < 2\pi$,

$$\begin{aligned}
&|\{p(Re^{i\theta})\}^s - \{p(re^{i\theta})\}^s| \\
&\leq \{M(p)\}^{s-1} \int_r^R st^{sn-1} \left\{1 - \frac{(\frac{n}{1+k^\mu}M(p) - n|a_n|)(t-1)}{n|a_n| + t\frac{n}{1+k^\mu}M(p)}\right\} \frac{n}{1+k^\mu} M(p) dt \\
&= \frac{(R^{ns} - r^{ns})}{1+k^\mu}\{M(p)\}^s - \frac{n}{1+k^\mu}\{M(p)\}^s(1-a)\int_r^R \frac{(t-1)(t^{n-1})}{t+a} dt, \qquad (4.2)
\end{aligned}$$

by taking $a = \dfrac{(1+k^\mu)|a_n|}{M(p)}$.

Using Lemma 3.6 in inequality (4.2), and substituting the value of a, we get

$$\begin{aligned}
|\{p(Re^{i\theta})\}^s| \quad \leq \quad & \frac{(R^{ns} - r^{ns})}{1+k^\mu}\{M(p)\}^s - \frac{n}{1+k^\mu}\{M(p)\}^s \left(1 - \frac{(1+k^\mu)|a_n|}{M(p)}\right) h(n) \\
& + |\{p(re^{i\theta})\}^s|, \qquad (4.3)
\end{aligned}$$

where

$$\begin{aligned}
h(n) = & \left(\frac{R^n - r^n}{n}\right) + \sum_{k=1}^{n-1}\left(\frac{R^{n-k} - r^{n-k}}{n-k}\right)(-1)^k \left(\frac{(1+k^\mu)|a_n|}{M(p)} + 1\right)\left(\frac{(1+k^\mu)|a_n|}{M(p)}\right)^{k-1} \\
& + (-1)^n \left(\frac{(1+k^\mu)|a_n|}{M(p)} + 1\right)\left(\frac{(1+k^\mu)|a_n|}{M(p)}\right)^{n-1} \ln\left(\frac{R(M(p)) + (1+k^\mu)|a_n|}{r(M(p)) + (1+k^\mu)|a_n|}\right)
\end{aligned}$$

for $n \geq 1$ and $h(0) = 0$. □

5. Computation

For the polynomial $p(z) = (z-2)^2$, $p(z) \neq 0$ for $|z| < 1$ and $M(p) = 9$. Then, for $R = 3$, exact value of $M(p, R)$ is 25. Using Theorem 1.2,

$$M(p, R) \leq 45 - 7 * (2 - 11/9 \log(29/11)) = 39.29 \qquad (5.1)$$

Using Corollary 2.2 of Theorem 2.1,

$$M(p, R) \leq 45 - 7 * (4 - 22/9 + 22/81 \log(29/11)) = 32.26 \qquad (5.2)$$

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Unit exchange elements in rings

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Abstract. Replacing left principal ideals by cosets in the monoid $(R, \cdot)$ of a unital ring R, we say that an element $a \in R$ is *left unit exchange (or suitable*) if there is an idempotent $e \in R$ such that $e - a \in U(R)(a - a^2)$ where $U(R)$ denotes the set of units. Unit-regular and clean elements are left (and right) unit suitable, and left (or right) unit suitable elements are exchange (suitable). The paper studies the multiple facets of this new notion.

Mathematics Subject Classification (2010): 16U99, 16U60, 13G99.

Keywords: Clean element, unit-regular element, exchange (suitable) element, unit suitable element, matrix rings.

1. Introduction

First recall that an element a in a ring R is *clean* if it is a sum of an idempotent and unit and *strongly clean* if these two commute. For an idempotent $e \in R$ we denote by $\overline{e} = 1 - e$ the complementary idempotent. The set of units of a (unital) ring R is denoted by $U(R)$.

An element a in a ring R was defined as (see [5] for this numbering) *left suitable* (or exchange) by any of the following equivalent conditions:

(1) there is an idempotent $e \in R$ such that $e - a \in R(a - a^2)$.

(3) there is an idempotent $e \in R$ such that $e \in Ra$ and $Re + R(1 - a) = R$.

(4) there is an idempotent $e \in R$ such that $e \in Ra$ and $1 - e \in R(1 - a)$.

Replacing R by $U(R)$, we introduce (similar to (1)) the following definition.

Definition 1.1. An element $a \in R$ is *left unit suitable* if there exists $e^2 = e \in R$ such that $e - a \in U(R)(a - a^2)$. When we intend to emphasize the idempotent, a will be called *e-left unit suitable*.

For an idempotent e and a unit u we consider the equation (called left *eu-equation*)

$$P_{u,e}(x) := x^2 - (1 + u)x + eu = 0.$$

It is readily seen that $x \in R$ is a solution of this equation iff

$$\exists e^2 = e \in R, \exists u \in U(R) : u^{-1}eu - x = u^{-1}(x - x^2).$$

Therefore *the left unit suitable elements in a ring are exactly the solutions of such eu-equations* (more precisely, $e - a = u(a - a^2)$ is equivalent to $P_{u^{-1},ueu^{-1}}(a) = 0$). By computation $P_{u,e}(u + \overline{e}) = 0$, so

$$\{\text{clean elements}\} \subseteq \{\text{left unit suitable elements}\} \subseteq \{\text{suitable elements}\}.$$

Examples in Section 3 will show that both inclusions may be proper.

However, it is easy to see that the left inclusion is equality in the following cases:

(i) R has no zero divisors;

(ii) R is a clean ring and in particular, is a matrix ring $\mathcal{M}_n(R)$ over any clean ring R;

(iii) R is Abelian and in particular commutative.

(iv) R is Artinian and in particular finite.

Right unit suitable elements are defined symmetrically and similarly *clean elements are right unit suitable and right unit suitable elements are suitable.*

As an easy first example (which is largely generalized further), any square-zero element is trivially left and right unit suitable, since since $0 - a = (-1)(a - a^2)$ holds whenever $a^2 = 0$.

In the first section, we give some useful characterizations for (left) unit suitable elements, for clean elements and for unit-regular elements, since such elements turn out to be left (or right) unit suitable. Since eu-equations are of degree two, some hints are given on the (possible) not clean solution.

The second section is devoted to results on left unit suitable 2×2 matrices. We show that over any commutative domain left unit suitable 2×2 matrices are also right unit suitable, we characterize left unit suitable zero lower row integral 2×2 matrices, trace 1 left (or right) unit suitable integral matrices via Diophantine equations and diagonal 2×2 left unit suitable matrices over any commutative domain. The matrix $\begin{bmatrix} 3 & 9 \\ 7 & -2 \end{bmatrix}$, already used in [1] as nil-clean but not clean, turns out to be suitable but not left (or right) unit suitable. Finally, a characterization of 2×2 unit-suitable matrices over any commutative domain is given, in connection again with unit-regular elements.

2. Basic properties

As already mentioned above, (left) suitable elements were defined in [5], by four equivalent conditions. Our definition corresponds to (1). Here is an equivalent definition corresponding to (4).

Proposition 2.1. *An element $a \in R$ is left unit suitable iff there exist $e^2 = e \in R$ and $b, c \in R$ such that $e = ba$, $1 - e = c(1 - a)$ and $b - c \in U(R)$.*

Proof. If $e - a = u(a - a^2)$ with $u \in U(R)$ then $e = [1 + u(1 - a)]a \in Ra$ and $1 - e = (1 - ua)(1 - a) \in R(1 - a)$ and $b - c = 1 + u(1 - a) - (1 - ua) = u$. Conversely, if $e = ba$, $1 - e = c(1 - a)$ and $b - c = u \in U(R)$ then $1 - e = (b - u)(1 - a)$ and so $1-ba = (b-u)(1-a) = b-u-ba+ua$ gives $b = 1+u-ua$ and $e-a = (b-1)a = u(1-a)a$. Here $b = 1 + u - ua$ and $c = 1 - ua$. □

A symmetric result holds for right unit suitable elements:

Proposition 2.2. *An element $a \in R$ is right unit suitable iff there exist $e^2 = e \in R$ and $b, c \in R$ such that $e = ab \in aR$, $1 - e = (1 - a)c \in (1 - a)R$ and $b - c \in U(R)$.*

Here $b = 1 + u - au$, $c = 1 - au$ and again $b - c = u$.

Corollary 2.3. *Left (or right) unit suitable elements have the "complement property", that is, if α is left (or right) unit suitable, so is $1 - \alpha$.*

In [7], another class of rings, intermediate between clean and suitable rings is introduced, under the name of *weakly clean* rings (and elements). Recall that weakly clean elements do not have the "complement property" (see Remark 4.7 (ii) in [8]), so these are different elements compared to left (or right) unit suitable elements (by the previous corollary).

"For any $a, b \in R$, $1 - ab$ is a unit iff $1 - ba$ is a unit" is known as *Jacobson's lemma* for units.

Since this lemma fails for clean elements and for suitable elements but holds for (unit) regular elements, we could ask whether it holds or fails for left (and/or right) unit suitable elements. *Actually it fails*: in [4] an example of clean (and so also left and right unit suitable) matrix $CD \in \mathcal{M}_2(\mathbf{Z})$ is given, for which DC is not suitable (and so nor left or right unit suitable). It remains to use the previous corollary.

The set of left unit suitable elements in a ring also includes the *unit regular* elements. More, we can prove the following *characterization* (with above notations)

Proposition 2.4. *A left unit suitable element a is unit regular iff, with the notations in the previous proposition, $c^2 = c$ and $ac = 0$.*

Proof. Suppose $a = aua$ with $u \in U(R)$. Then ua and so $c = 1 - ua$ are both idempotents and $ac = 0$. Take $b = c + u$. Then $ba - a = u(1 - a)a$ and $baba = ba$ shows that a is left unit suitable (one can also check $1 - ba = c(1 - a)$). Conversely, assume $c^2 = c$, $ac = 0$, $1 - ba = c(1 - a)$ and $b - c = u \in U(R)$. By left multiplication with a we get $a - aba = ac - aca = 0$ so $aba = a$. Hence $aua = a(b - c)a = aba = a$, as desired. □

A symmetric result holds for *right unit suitable elements*.

In particular, unit regular elements are left and right unit suitable and so elements which are both *left and right suitable need not be clean* (see the example after Theorem 3.3, in the next section).

An elementary trick, more or less always used in the context of exchange rings, is the following: for a ring R and elements $a, e \in R$, if $e \in Ra$ is an idempotent, an element $b \in R$ can be chosen such that $e = ba$ and $bab = b$. Note that such an element b is regular.

Recall (see Introduction) that an element $a \in R$ was called *left suitable* (or exchange) if there is an idempotent $e \in R$ such that $e \in Ra$ and $1 - e \in R(1 - a)$.

Using the previous observation, (regular) elements $b, c \in R$ can be chosen such that $e = ba$, $bab = b$, $1 - e = c(1 - a)$ and $c(1 - a)c = c$.

Coming back to our initial definition, we could consider elements $a \in R$ such that the elements b, c above can be chosen *with* $b - c \in U(R)$.

This restriction is too strong because it is easy to prove the following characterization (already noticed in [2])

Proposition 2.5. *An element a in a ring R is clean iff there exist $b, c \in R$ such that $bab = b$, $c(1 - a)c = c$, $1 - ba = c(1 - a)$ and $b - c \in U(R)$.*

Notice that if *only one* from the conditions $bab = b$, $c(1 - a)c = c$ holds, the statement above is no longer valid. An example is given after Theorem 3.3.

Rephrasing, *a left unit suitable element a is clean iff for the regular elements b, c emphasized above with $bab = b$ and $c(1 - a)c = c$, $b - c$ is a unit.*

Remark 2.6. Since we already noticed that $x = u + \overline{e}$ is a solution of the eu-equation $P_{u,e}(x) = 0$, we could wonder when this degree two polynomial factors into two degree one polynomials. Denoting by a the second solution it is easy to show that $P_{u,e}(x) = x^2 - (1 + u)x + eu = (x - u - \overline{e})(x - a)$ iff $(u + \overline{e} - x)a = e(u - x)$. If $ue = eu$ then $a = e$ is a clean solution.

3. Unit suitable 2×2 matrices

Recall that a ring R is *Dedekind finite* (DF for short) if, for every $a, b \in R$, $ab = 1$ implies $ba = 1$. A ring R is *stably finite* if the matrix rings $\mathcal{M}_n(R)$ are Dedekind finite for all natural numbers n.

We first point out some simple but general results.

Proposition 3.1. *(a) In any DF ring, the 0-left unit suitable elements are clean.*

(b) In any DF ring, the 0-left unit suitable elements are also right unit suitable.

(c) For any positive integer n, and any stably finite ring k, 0_n-left unit suitable matrices in $\mathcal{M}_n(k)$ are clean.

(d) In any DF ring, the 1-left unit suitable elements are units and so clean.

(e) In any DF ring, the 1-left unit suitable elements are also right unit suitable.

(f) For any positive integer n, and any stably finite ring k, I_n-left unit suitable matrices in $\mathcal{M}_n(k)$ are clean.

Proof. (a) First notice that $e - a = u(a - a^2)$ is equivalent to $1 - e = (1 - ua)(1 - a)$. Taking $e = 0$ yields $1 = (1 - ua)(1 - a)$ and if the ring is DF, $1 - a$ is a unit. Hence a is clean.

(b), (c) follow from (a).

(d) We just notice that $1 = a + u(a - a^2)$ is now equivalent to $1 = [1 + u(1 - a)]a$, so a is a unit.

(e), (f) follow from (d). □

If k is a commutative domain, then $E \in \mathcal{M}_2(k)$ is an idempotent iff $E = 0_2$, $E = I_2$ or $\det(E) = 0$ and $\mathrm{Tr}(A) = 1$.

Theorem 3.2. *Left unit suitable* 2×2 *matrices over a commutative domain* k *are also right unit suitable.*

Proof. Suppose $A \in \mathcal{M}_2(k)$ is left unit suitable, i.e. $E = A + U(A - A^2)$ with a unit U and an idempotent E.

If $E = 0_2$ or $E = I_2$ the result follows from Proposition 3.1 (b) and (e), respectively.

In the remaining case, assume $\det(E) = 0$ and $\mathrm{Tr}(E) = 1$. Letting

$$F = A + (A - A^2)U,$$

we have

$$\mathrm{Tr}(F) = \mathrm{Tr}(A) + \mathrm{Tr}((A - A^2)U) = \mathrm{Tr}(A) + \mathrm{Tr}(U(A - A^2)) = \mathrm{Tr}(E) = 1.$$

Moreover, $\det(E) = 0$ gives $\det(I_2 + U(I_2 - A)) \det(A) = 0$, hence

$$\det(I_2 + U(I_2 - A)) = 0 \quad \text{or} \quad \det(A) = 0.$$

Notice that for any 2×2 matrix M over a commutative ring,

$$\det(I_2 + M) = \det(M) + \mathrm{Tr}(M) + 1.$$

Therefore,

$$\begin{aligned} \det[I_2 + U(I_2 - A)] &= \det[U(I_2 - A)] + \mathrm{Tr}[U(I_2 - A)] + 1 \\ &= \det[(I_2 - A)U] + \mathrm{Tr}[(I_2 - A)U] + 1 \\ &= \det[I_2 + (I_2 - A)U] \end{aligned}$$

and so $\det[I_2 + U(I_2 - A)] = 0$ iff $\det[I_2 + (I_2 - A)U] = 0$. It follows that

$$\det(I_2 + (I_2 - A)U) = 0 \quad \text{or} \quad \det(A) = 0,$$

hence $\det(F) = \det(I_2 + (I_2 - A)U) \det(A) = 0$. This shows that $\mathrm{Tr}(F) = 1$ and $\det(F) = 0$, proving that F is an idempotent. Hence A is right unit suitable (in this case, with respect to the same unit). $\square$

In contrast with suitable elements, it is unlikely that the set of left unit suitable rings and the set of right unit suitable coincide. As seen above, the always good source of examples, $\mathcal{M}_2(\mathbf{Z})$, cannot be used when searching for a left unit suitable element which is not right unit suitable.

Since $\mathbf{Z}$ is not exchange (and so nor clean), in searching for unit suitable elements which are not clean, it is worth trying with $\mathcal{M}_2(\mathbf{Z})$. In searching for *left unit suitable elements which are not clean*, or *suitable elements which are not left unit suitable* we first prove the following

Theorem 3.3. *For a matrix* $A = \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} \in \mathcal{M}_2(\mathbf{Z})$, *the following are equivalent:*

1. *A is unit-regular or clean in $\mathcal{M}_2(\mathbf{Z})$.*
2. *A is left unit suitable in $\mathcal{M}_2(\mathbf{Z})$.*
3. *A is suitable in $\mathcal{M}_2(\mathbf{Z})$.*
4. *(a, b) is a unimodular row or $a \in \{0, 2\}$.*

Proof. $1 \Rightarrow 2 \Rightarrow 3$ follow from earlier results in the paper (and hold for any ring element).

$3 \Rightarrow 4$: Suppose that A is suitable and $\gcd(a,b) = n > 1$ (i.e. (a,b) is not unimodular). By the suitable property, $E = RA$ and $I_2 - E = S(I_2 - A)$ for some $R, S, E \in \mathcal{M}_2(\mathbf{Z})$ with $E^2 = E$. Since $A \in n\mathcal{M}_2(\mathbf{Z})$, it follows that $E \in n\mathcal{M}_2(\mathbf{Z})$, hence $E = 0$ as $E = E^2$.

Hence $I_2 = S(I_2 - A)$ so that $I_2 - A$ is a unit. The determinant of this matrix is $1 - a$. Hence $1 - a \in \{\pm 1\}$, i.e. $a \in \{0, 2\}$.

$4 \Rightarrow 1$: If (a,b) is unimodular then A is clearly unit-regular, and if $a \in \{0,2\}$ then $I_2 - A$ is a unit so that A is clean. $\square$

Recall from [3] (Theorem 4.7) that if (a,b) is a *reduced unimodular* row (i.e. $|a| \geq 2|b|$ and a, b generate the unit ideal), $\begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix}$ is clean iff $a \equiv \pm 1 (\mathrm{mod} b)$.

Therefore, again according to [3], the reduced unimodular rows (12, 5), (13, 5), (12, 7), (13, 8), (17, 5), (16, 7), (18, 5), (17, 7) yield *not clean* unit-regular matrices *which are left unit suitable.*

Example 3.4. For $A = \begin{bmatrix} 12 & 5 \\ 0 & 0 \end{bmatrix}$, $U = \begin{bmatrix} 19 & 8 \\ 7 & 3 \end{bmatrix}$ and $E = \begin{bmatrix} 8 & -8 \\ 7 & -7 \end{bmatrix}$ one checks the eu-equation

$$X^2 - (I_2 + U)X + EU = 0_2.$$

This example also suits as a left unit suitable element with

$$C(I_2 - A)C = C = \begin{bmatrix} -35 & -15 \\ 84 & 36 \end{bmatrix}$$

(that is, regular C) but

$$BAB = \begin{bmatrix} -32 & 32 \\ 77 & -77 \end{bmatrix} \neq \begin{bmatrix} -32 & -23 \\ 77 & 55 \end{bmatrix} = B$$

(so not regular B). Here

$$B - C = \begin{bmatrix} 3 & -8 \\ -7 & 19 \end{bmatrix}$$

is a unit, $AU^{-1}A = A$ and both

$$BA = F = \begin{bmatrix} -384 & -160 \\ 924 & 385 \end{bmatrix}$$

(here $F - A = U^{-1}(A - A^2)$), $AB = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$ are idempotents.

Remark 3.5. The nice idempotent obtained in the previous example,

$$AB = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$$

is not specific. It is the result for any matrix

$$A = \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix}$$

with coprime a, b. As seen in the previous proof,

$$U = A + \begin{bmatrix} z & u_{22} \\ z & u_{22} \end{bmatrix}$$

with $au_{22} - bz = 1$. Then

$$U^{-1} = \begin{bmatrix} u_{22} & -u_{22} \\ -z & z \end{bmatrix} + \begin{bmatrix} 0 & -b \\ 0 & a \end{bmatrix} = S + T$$

and indeed

$$AS = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad AT = 0_2.$$

Hence

$$AB = A + AU^{-1}(I_2 - A) = AU^{-1} = AS = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}.$$

Obviously AB is idempotent and in general

$$BAB = \begin{bmatrix} r & s \\ m & n \end{bmatrix} AB = \begin{bmatrix} r & s \\ m & n \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} r & -s \\ m & -n \end{bmatrix} \neq \begin{bmatrix} r & s \\ m & n \end{bmatrix} = B$$

unless $n = s = 0$.

Since unipotent elements (i.e. sums $1 + t$ with nilpotent t) are special units in rings, we could ask whether (left) *unipotent suitable* elements are not clean. First notice (e.g. from the proof of the previous theorem), that in general there is no uniqueness for the unit in the definition of the (left) unit suitable elements. So a better rephrased question would be: if for a (left) unit suitable element, the unit can be chosen as an unipotent, is the element clean? The answer is no as shows the next example.

Example 3.6. Take the second unimodular row in the list above (from [3]). For the matrix $A = \begin{bmatrix} 13 & 5 \\ 0 & 0 \end{bmatrix}$, $U = \begin{bmatrix} 5 & 2 \\ -8 & -3 \end{bmatrix}$ and $E = \begin{bmatrix} -7 & 7 \\ -8 & 8 \end{bmatrix}$ one checks the eu-equation

$$X^2 - (I_2 + U)X + EU = 0_2.$$

Therefore A is left unipotent suitable but not clean (it is readily checked that unipotents in $\mathcal{M}_2(\mathbf{Z})$ are the matrices of trace $= 2$ and determinant $= 1$).

As mentioned above, the same matrix satisfies the eu-equation with $U = \begin{bmatrix} 18 & 7 \\ 5 & 2 \end{bmatrix}$ and $E = \begin{bmatrix} 6 & -6 \\ 5 & -5 \end{bmatrix}$ where U is no more unipotent.

Similar results (on zero upper row, or columns) may be obtained using conjugation by $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ or transposes.

The ring $\mathcal{M}_2(\mathbf{Z})$ contains many suitable elements that are not left unit suitable. Matrices with trace 1 are definitely one example, but there are also other classes that are easier to characterize. One of those classes are the diagonal matrices. Below is given a characterization of suitable and left unit suitable *diagonal* matrices.

Lemma 3.7. *Let* $A = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$ *be a* 2×2 *diagonal matrix over a commutative domain* k *and let* $d \in k$. *The following are equivalent:*

1. There exists $X \in \mathcal{M}_2(k)$ *with* $\det(X) = d$ *such that* $E = A + X(A - A^2)$ *is a nontrivial idempotent (i.e.* $E \neq 0_2, I_2$*).*

2. There exists $t \in k$ *such that* $t(a - b) - d(a - a^2) = 1$.

Proof. $1 \Rightarrow 2$: Suppose that $E = A + X(A - A^2)$ is a nontrivial idempotent and $\det(X) = d$. Write $\alpha = a - a^2$, $\beta = b - b^2$ and $X = \begin{bmatrix} x & y \\ z & w \end{bmatrix}$. Then

$$E = \begin{bmatrix} a + x\alpha & y\beta \\ z\alpha & b + w\beta \end{bmatrix}$$

and since E is nontrivial $\mathrm{Tr}(E) = 1$, hence

$$a + b + x\alpha + w\beta = 1. \tag{3.1}$$

If $b = 0$ then this equation gives $a + x\alpha = 1$, whence $a \in U(k)$. Hence, taking $t = (1 + d\alpha)a^{-1}$ we get

$$t(a - b) - d\alpha = ta - d\alpha = (1 + d\alpha) - d\alpha = 1,$$

which proves the claim. Similarly if $b = 1$ then (3.1) gives $a + x\alpha = 0$.
Hence $1 + x(1 - a)a = 0$ which implies $1 - a \in U(k)$. Hence taking $t = (1 + d\alpha)(a - 1)^{-1}$ we get $t(a - b) - d\alpha = t(a - 1) - d\alpha = (1 + d\alpha) - d\alpha = 1$, as desired. Thus we may assume $b \neq 0, 1$ and so $\beta \neq 0$.
The determinant condition $\det(E) = 0$ gives

$$ab + aw\beta + bx\alpha + \alpha\beta d = 0. \tag{3.2}$$

Now (3.1) and (3.2) together give

$$b = b(a + b + x\alpha + w\beta) - (ab + aw\beta + bx\alpha + \alpha\beta d) = b^2 + (b - a)w\beta - \alpha\beta d,$$

hence $\beta = (b - a)w\beta - \alpha\beta d$. Cancelling β we obtain $1 = (b - a)w - \alpha d$ and so $t = -w$ fulfills the desired condition.

$2 \Rightarrow 1$: Take t with $t(a - b) - d(a - a^2) = 1$. Letting

$$X = \begin{bmatrix} t + d(a + b - 1) & -1 \\ t(t + d(a + b - a)) & -t \end{bmatrix}$$

it is easy to see that $\det(X) = d$ and $E = A + X(A - A^2)$ satisfies $\mathrm{Tr}(E) = 1$ and $\det(E) = 0$, which proves the claim. ☐

Theorem 3.8. *For a* 2×2 *matrix* $A = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$ *over a commutative domain* k, *the following are equivalent:*

1. A *is suitable.*

2. Either A *is a unit or* $I_2 - A$ *is a unit or* $t(a - b) + s(a - a^2) = 1$ *for some* $t, s \in k$.

Proof. $1 \Rightarrow 2$: Let $E = A + X(A - A^2)$ with idempotent E and $X \in \mathcal{M}_2(k)$. If $E = I_2$ then A is a unit, and if $E = 0_2$ then $I_2 - A = XA(I_2 - A) + I_2$, so that $I_2 - A$ is a unit. Therefore we may assume that E is a nontrivial idempotent. By the previous lemma we obtain $t \in k$ such that $t(a-b) - \det(X)(a - a^2) = 1$. Hence t and $s = -\det(X)$ satisfy the required condition.

$2 \Rightarrow 1$: If A or $I_2 - A$ is invertible then there is nothing to prove. Thus let $t(a-b) + s(a-a^2) = 1$ with $t, s \in k$. Using again the previous lemma we get a matrix X with $\det(X) = -s$ such that $E = A + X(A - A^2)$ is a nontrivial idempotent, as desired. □

Theorem 3.9. *For a 2×2 matrix $A = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$ over a commutative domain k, the following are equivalent:*

1. A is left unit suitable.

2. A is clean.

3. Either A is a unit or $I_2 - A$ is a unit or $t(a-b) + s(a-a^2) = 1$ for some $t \in k$ and $s \in U(k)$.

Proof. $2 \Rightarrow 1$ is clear

$1 \Rightarrow 3$: Suppose A is left unit suitable, i.e. $E = A + U(A - A^2)$ with idempotent E and unit U. As before, if $E = I_2$ then A is a unit, and if $E = 0_2$ then $I_2 - A$ is a unit. Therefore we may assume E is nontrivial. Hence by Lemma 3.7 there exists $t \in k$ with $t(a-b) - \det(U)(a-a^2) = 1$, so that t and $s = -\det(U)$ satisfy the desired condition.

$3 \Rightarrow 2$: If A or $I_2 - A$ is invertible there is nothing to prove. Thus let

$$t(a-b) + s(a-a^2) = 1$$

for some $t \in k$ and $s \in U(k)$. We can check directly that

$$E = \begin{bmatrix} a - s^{-1}t & 1 + t(a+b-1-s^{-1}t) \\ s^{-1} & 1 - a + s^{-1}t \end{bmatrix}$$

is an idempotent (with $\mathrm{Tr}(E) = 1$ and $\det(E) = 0$) and $U = A - E$ is a unit (with $\det(U) = s^{-1}$). □

Example 3.10. Let $a = 2$ and $b = -3$. Then, taking $t = 1$ and $s = 2$ we get

$$t(a-b) + s(a-a^2) = 1 \cdot 5 + 2 \cdot (-2) = 1,$$

so that $A = \begin{bmatrix} 2 & 0 \\ 0 & -3 \end{bmatrix}$ is suitable in $\mathcal{M}_2(\mathbf{Z})$ by Theorem 3.8. However, the equation $t \cdot 5 + s \cdot (-2) = 1$ clearly has no solution in $\mathbf{Z}$ if $s \in U(\mathbf{Z})$, hence A is not left unit suitable in $\mathcal{M}_2(\mathbf{Z})$ by Theorem 3.9.

Next we prove another result which connects unit-suitable 2×2 matrices with unit-regular ones.

Theorem 3.11. *For a 2×2 matrix A over a commutative domain k, the following are equivalent:*

1. A is unit-suitable.

2. Either A is unit-regular or $I_2 - A$ is unit-regular or there exists a unit U such that the pairs $(\mathrm{Tr}(U), \det(UA))$ *and* $(\mathrm{Tr}(UA) - 1, \det(U)(\mathrm{Tr}(A) - 1))$ *have the same sum and the same product.*

Proof. For any 2×2 matrix, Cayley-Hamilton's theorem gives

$$A - A^2 = \det(A)I_2 - (\mathrm{Tr}(A) - 1)A.$$

Hence, a matrix A is (left) unit-suitable iff there is a unit U such that

$$E = (I_2 + U(I_2 - A))A = A + \det(A)U - (\mathrm{Tr}(A) - 1)UA \tag{3.3}$$

is an idempotent.

As mentioned in the proof of Theorem 3.2, if $E = 0_2$ then $I_2 - A$ is a unit and if $E = I_2$ then A is a unit (and units are unit-regular). In the remaining case, assume $\mathrm{Tr}(E) = 1$ and $\det(E) = 0$.

Notice that equivalently $\det(E) = 0$ gives $\det(I_2 + U(I_2 - A))\det(A) = 0$, hence

$$\det(I_2 + U(I_2 - A)) = 0 \quad \text{or} \quad \det(A) = 0.$$

By (3.3), $\mathrm{Tr}(E) = 1$ is equivalent to $\det(A)\mathrm{Tr}(U) = (\mathrm{Tr}(A) - 1)(\mathrm{Tr}(UA) - 1)$.
Case 1. If $\det(A) = 0$ then $\mathrm{Tr}(A) = 1$ or $\mathrm{Tr}(UA) = 1$ and since

$$\det(UA) = \det(U)\det(A) = 0,$$

A or UA is a (nontrivial) idempotent. As well-known, in both cases A is unit-regular.
Case 2. If $\det(A) \neq 0$ then $\det(I_2 + U(I_2 - A)) = 0$. Notice that for any 2×2 matrix B,

$$\det(I_2 + B) = 1 + \mathrm{Tr}(B) + \det(B),$$

so the previous condition amounts to $1 + \mathrm{Tr}(U(I_2 - A)) + \det(U(I_2 - A)) = 0$. Equivalently,

$$1 + \mathrm{Tr}(U) - \mathrm{Tr}(UA) + \det(U)[1 - \mathrm{Tr}(A) + \det(A)] = 0$$

or

$$1 + \mathrm{Tr}(U) + \det(UA) = \mathrm{Tr}(UA) + \det(U)(\mathrm{Tr}(A) - 1).$$

Multiplying $\det(A)\mathrm{Tr}(U) = (\mathrm{Tr}(A) - 1)(\mathrm{Tr}(UA) - 1)$ by $\det(U)$ shows that the pairs $(\mathrm{Tr}(U), \det(UA))$ and $(\mathrm{Tr}(UA) - 1, \det(U)(\mathrm{Tr}(A) - 1))$ have the same sum and the same product. □

Corollary 3.12. *Let A be a 2×2 matrix over a commutative domain k, and* $\det(A) = 0$. *Then A is unit-suitable iff either A is unit-regular or $I_2 - A$ is unit-regular.*

Proof. One way follows from the previous proof, and the converse follows from Corollary 2.3, since unit-regular elements are unit-suitable. □

For integral matrices we can say more.

Corollary 3.13. *Let A be an integral 2×2 matrix. Then A is unit-suitable iff either A is unit-regular or $I_2 - A$ is unit-regular or there exists a unit U such that*

$$\mathrm{Tr}(UA) = 1 + \det(UA), \quad \mathrm{Tr}(U) = \det(U)(\mathrm{Tr}(A) - 1).$$

Proof. Indeed, since the pairs in the previous theorem are roots of the same degree two (solvable) equation we have

$$\{\mathrm{Tr}(U), \det(UA)\} = \{\mathrm{Tr}(UA) - 1, \det(U)(\mathrm{Tr}(A) - 1)\}.$$

Therefore

$$\mathrm{Tr}(UA) = 1 + \det(UA), \quad \mathrm{Tr}(U) = \det(U)(\mathrm{Tr}(A) - 1),$$

or

$$\mathrm{Tr}(UA) = 1 + \mathrm{Tr}(U), \quad \det(UA) = \det(U)(\mathrm{Tr}(A) - 1).$$

In the second case, we can show that $I_2 - A$ is a unit-regular element.

Recall that we are working in the hypothesis of Case 2 (the proof of the previous theorem), that is,

$$\det(I_2 + U(I_2 - A)) = 0.$$

Since

$$\mathrm{Tr}(I_2 + U(I_2 - A)) = 2 + \mathrm{Tr}(U) - \mathrm{Tr}(UA) = 1$$

it follows that $E := I_2 + U(I_2 - A)$ is an idempotent. Hence

$$I_2 - A = -U^{-1}(I_2 - E)$$

is unit-regular (indeed, an element $b \in R$ is unit-regular iff there are a unit u and an idempotent e such that $b = ue$). $\square$

Trace 1 left (or right) unit suitable integral matrices can be characterized via Diophantine equations. We just mention the following

Proposition 3.14. *(i) A trace 1, 2×2 integral matrix $A = \begin{bmatrix} a+1 & b \\ c & -a \end{bmatrix}$ with $b \neq 0$ is (left or right) unit suitable iff*

$$bx^2 - (2a+1)xy - cy^2 + (1 + \det(A))y + b = 0 \quad (1)$$

and

$$b \text{ divides } 1 + \det(A) - (2a+1)x - cy \quad (2)$$

or else

$$bx^2 - (2a+1)xy - cy^2 + (1 - \det(A))y - b = 0 \quad (3)$$

and

$$b \text{ divides } 1 - \det(A) - (2a+1)x - cy \quad (4)$$

(ii) The matrix $C = \begin{bmatrix} 3 & 9 \\ -7 & -2 \end{bmatrix}$ is a 2×2 suitable (exchange) matrix which is not left nor right unit suitable in $\mathcal{M}_2(\mathbf{Z})$.

Remark 3.15. This shows that C, our example in [1], can be used to improve its initial purpose: this is *a nil-clean matrix which is not unit suitable* (not only not clean). As for now, the problem of finding an *example of nil-clean element which is not suitable* (exchange) remains open.

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Various results in relation with the hypergeometric equations and the hypergeometric functions in the complex plane

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Abstract. The main purpose of this investigation is to specify an extensive relation between the hypergeometric functions and the hypergeometric equations in the complex plane and then to point various implications of our main result, conclusion and also recommendations out.

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1. Introduction, definitions and motivation

In this section, we first introduce the (Gauss) hypergeometric function, the (Gauss) hypergeometric differential equation and certain well-known relationship between them. We then focus on revealing some possible interesting results associating with certain novel and extensive relationships between them. In the light of the alleged results, we also remark that there are a number of important implications of our results between various (differential) equations and special functions in the complex plane.

The well-known hypergeometric functions (and also the hypergeometric differential equations) with complex (or real) variable have been attracting much more attention in the literature. This interest is due to its importance as solutions (or applications) of many applied problems in mathematics given by the references in [1]-[3], [8], [7], [12], [13], [14], [18], [30], [38], [39], [44]-[45] and [49], in Statistics and Probability given by [2], [11], [14], [21], [33] and [38], in physics [2], [3], [7], [11], [19],

[25] and [38], and also in the majority of engineering sciences given by [2], [5], [7], [9], [14], [17], [22], [24], [25], [33], [34], [38], [40], [43], [48], [50] and [51].

As is known, the hypergeometric functions (and also the functions being the solutions of hypergeometric differential equations) constitute a wide and important class of special functions with complex (or real) variable. In particularly, a great number of special functions of mathematical physics turn out to be hypergeometric function. In addition, multivariate hypergeometric functions can be introduced as solutions to certain overdetermined systems of linear partial differential equations with polynomial coefficients. In general, such systems of equations are of substantial independent interest and appear in several applications. The simplest ordinary differential equation of this kind is the Gauss hypergeometric equation in the literature. Any second-order linear differential equation with three regular singularities in the Riemann sphere can be also reduced to the Gauss equation by the help of a suitable change of the variables. For their details, it can be also checked the works given in [2], [5]-[6], [9], [10], [16], [17], [20], [23], [31], [32], [38]-[37], [41], [43]-[48], [50] and [51].

Since our main purpose in this scientific work is to reveal certain *novel* and/or *non-linear* relationships between the (Gauss, Gaussian or ordinary) hypergeometric functions and certain special functions in the complex plane, primarily, we have to remember certain basic information about the functions and the (differential) equations which are related to the mentioned topics and the related ones. In mathematics, we note that the hypergeometric function is a special function represented by series (or integral), which includes many other special functions as specific or limiting cases. This function is a solution of a second-order linear ordinary differential equation that every second-order linear ordinary differential equation with three singular points can be transformed into the related differential equation. For their details, let us now start by recalling (or introducing) the following information.

First of them, here and throughout this present work, firstly, we note that the well-known notations:

$$\mathbb{N} \quad , \quad \mathbb{Z}^- \quad , \quad \mathbb{R} \quad , \quad \mathbb{C} \quad \text{and} \quad \mathbb{U}$$

denote the set of *natural* numbers, the set of *negative integers*, the set of *real* numbers, the set of *complex* numbers and the *open unit disk* in the *complex* plane, respectively.

The first important topic is related to the function set by the series, so let me know about it now. For this, the following functional series in the complex variable z, called the (Gauss, Gausian or ordinary) hypergeometric function, is denoted by any one of the notations: ${}_2F_1(\alpha,\beta;\gamma;z)$, $F(\alpha,\beta;\gamma;z)$ and $\mathbf{F}(\alpha,\beta;\gamma;z)$ and also defined by

$$\begin{aligned} F(\alpha,\beta;\gamma;z) &= 1 + \frac{\alpha\beta}{\gamma}z + \frac{\alpha(\alpha+1)\beta(\beta+1)}{\gamma(\gamma+1)}\frac{z^2}{2!} \\ &\quad + \frac{\alpha(\alpha+1)(\alpha+2)\beta(\beta+1)(\beta+2)}{\gamma(\gamma+1)(\gamma+2)}\frac{z^3}{3!} + \cdots \\ &= \sum_{n=0}^{\infty} \frac{(\alpha)_n(\beta)_n}{(\gamma)_n}\frac{z^n}{n!}, \end{aligned} \tag{1.1}$$

where $\alpha \in \mathbb{C} - \mathbb{Z}_0^-$, $\beta \in \mathbb{C} - \mathbb{Z}_0^-$, $\gamma \in \mathbb{C} - \mathbb{Z}_0^-$ and $z \in \mathbb{C}$, and, in terms of the Gamma function $\Gamma(z)$, the (*rising*) *Pochhammer* symbol, i.e., the symbol $(v)_n$ is also defined by

$$(v)_n = \begin{cases} 1 & \big(n = 0\big) \\ v(v+1)\cdots(v+n-1) & \big(n \in \mathbb{N}\big) \end{cases}$$
$$= \frac{\Gamma(v+n)}{\Gamma(v)}, \tag{1.2}$$

where $n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$ and $v \in \mathbb{C} - \mathbb{Z}_0^-$.

We here note that, as certain characteristic properties of the series in (1.1) together with (1.2), it is absolutely and uniformly convergent in the disk $\mathbb{U}$. The convergence also extends over the unit circle when $\Re e\big(\alpha+\beta-\gamma\big) < 0$, it converges at all the points of the unit circle except the point $z = 1$ when $0 \le \Re e\big(\alpha+\beta-\gamma\big) < 1$. Nevertheless, there exists an analytic continuation of the hypergeometric function in (1.1) to the exterior $|z| > 1$ of the unit disk with the slit $(1, \infty)$. The function defined by the series in (1.1), namely, $F(\alpha, \beta; \gamma; z)$ is an univalent-analytic function in the complex plane with slit $(1, \infty)$. When α or β are zero or negative integers, the series given (1.1) terminates after a finite number terms and the hypergeometric function is a polynomial in z. Further, when $n \in \mathbb{Z}_0^-$, the function given by (1.1) is not defined but the limit can be considered there.

It follows from (1.1) that

$$F\big(\alpha, \beta; \gamma; z\big) = F\big(\beta, \alpha; \gamma; z\big), \tag{1.3}$$

and, in the light of the identity (1.2), it is easily shown that

$$\frac{d}{dz}\Big(F\big(\alpha, \beta; \gamma; z\big)\Big) = \frac{\alpha\beta}{\gamma} F(\alpha, \beta; \gamma; z), \tag{1.4}$$

and, more generally,

$$\frac{d^n}{dz^n}\Big(F(\alpha, \beta; \gamma; z)\Big) = \frac{(\alpha)_n(\beta)_n}{(\gamma)_n} F\big(\alpha+n, \beta+n; \gamma+n; z\big) \tag{1.5}$$

for all $n \in \mathbb{N}_0$. In particular, we note that the properties relating to the derivative given in (1.4) (or (1.5) will be useful in the simpler expression of complex statements for equations (or inequalities) stated by derivative(s).

The second important issue is associated with a homogenous differential equation in the complex plane. So, there is a need to present about it. For this, the function, given by the series in the form (1.1), is a solution of the following homogeneous differential equation given by:

$$z\,(1-z)\,\frac{d^2w}{dz^2} + \big[\gamma - \big(\alpha+\beta+1\big)z\big]\frac{dw}{dz} - \alpha\beta w(z) = 0\,, \tag{1.6}$$

where $\alpha \in \mathbb{C} - \mathbb{Z}_0^-$, $\beta \in \mathbb{C} - \mathbb{Z}_0^-$, $\gamma \in \mathbb{C} - \mathbb{Z}_0^-$ and $z \in \mathbb{C} - [1, \infty)$.

The differential equation just above is also known as the (Euler's) hypergeometric differential equation in the literature. By a simple focusing, it can be easy seen that, for the equation in (1.6), there have three singular points, which are 0, 1 and ∞.

Because of the important relation between the function given by (1.1) and the equation (1.6), there are a large number of the possible special functions specified by the function given (1.1) and, naturally, they will also include many significant relations with the differential equation in (1.6). We can now continue to determine those relations (or properties) in the second section.

2. Main result, comment and recommendations

As is known, the proof technique used in the proof of the theories is very important in the theoretical studies. A few examples given as in [27]-[29] are some of the proofs used in the complex functions theory, which is appropriate for the purpose of this study. As a different proof method will be used in this paper, in order to prove our main result, initially, we need to recall the following-well-known assertion (see [35] and [36]).

Lemma 2.1. *Let a function $p(z)$ in the form:*

$$p(z) = 1 + e_n z^n + e_{n+1} z^{n+1} + e_{n+2} z^{n+2} + \cdots \tag{2.1}$$

be analytic in the open set:

$$\mathbb{U} = \left\{ z : z \in \mathbb{C} \text{ and } |z| < 1 \right\},$$

where $n \in \mathbb{N}$ and $e_n \in \mathbb{C}$.

If the function $p(z)$ is not with positive real part in $\mathbb{U}$, then there is a point $z_0 \in \mathbb{U}$ such that

$$p(z)\Big|_{z=z_0} = i\lambda \quad \text{and} \quad z \frac{d}{dz}\Big(p(z)\Big)\Big|_{z=z_0} = \mu\,, \tag{2.2}$$

where

$$\lambda \in \mathbb{R} - \{0\} \ , \ \mu \in \mathbb{R} \quad \text{and} \quad \mu \leq -\,n\,\frac{1+\lambda^2}{2} \quad \big(n \in \mathbb{N}\big)\,. \tag{2.3}$$

Let us now introduce a third-order differential equations with (complex) variable coefficients in the complex plane, which will play an important role in our main result, as in the following form:

$$\begin{aligned} (1-z)\,z^2\,\frac{d^3\omega}{dz^3} \;+\; z\left[1+\gamma - \big(3+\alpha+\beta\big)\,z\right]\frac{d^2\omega}{dz^2} & \\ -\,\big(1+\alpha+\beta\big)\,z\,\frac{d\omega}{dz} \;= \alpha\,\beta\;\Phi(z)\,, & \end{aligned} \tag{2.4}$$

where $z \in \mathbb{U}$, $\alpha \in \mathbb{C} - \mathbb{Z}_0^-$, $\beta \in \mathbb{C} - \mathbb{Z}_0^-$ and $\gamma \in \mathbb{C} - \mathbb{Z}_0^-$. In special, note that, since the differential equation in (2.4) is the derivative of both sides of the equation in (1.6), clearly, both the function $w := F(\alpha, \beta; \gamma; z)$ is the solution for the (complex) differential equation in (2.4) and the (complex) function $\Phi(z)$ is analytic in $\mathbb{U}$.

Theorem 2.2. *Let the functions $\omega := \omega(z)$ and $\Phi(z)$ be in the forms defined by (1.1) and (2.4), respectively. For any $z \in \mathbb{U}$ and for some $\alpha \in \mathbb{C} - \mathbb{Z}_0^-$ and $\beta \in \mathbb{C} - \mathbb{Z}_0^-$, if* *any one of the cases of the following inequality:*

$$\Re e\Big(\alpha\beta\,\Phi(z)\Big)\begin{cases} > -\frac{\Re e(\alpha\beta)}{2} & if \quad \Re e\big(\alpha\beta\big)\geq 0 \\ < -\frac{\Re e(\alpha\beta)}{2} & if \quad \Re e\big(\alpha\beta\big)\leq 0 \end{cases}, \tag{2.5}$$

is satisfied, then

$$\Re e\Big(\omega(z)\Big) > 0 \quad \big(z\in\mathbb{U}\big) \tag{2.6}$$

is also satisfied

Proof. Since the function $\omega \equiv \omega(z)$ has the form given by (1.1), it is a particular solution for the differential equation given by (1.6). By taking into account this fact, let us take $p(z)$ as

$$p(z) = \omega(z)\ \big(\equiv F(\alpha,\beta;\gamma;z)\big) \tag{2.7}$$

to show that $\Re e\big(p(z)\big) > 0$ for all $z\in\mathbb{U}$.

It is clear that the function $\omega(z)$ both has the series form given by (1.6) and is analytic in the open set $\mathbb{U}$ of the complex plane. Therefore, the function $p(z)$ is also satisfies the conditions $p(0)=1$ and $n=1$, accentuated in Lemma 2.1.

It easily follows from (2.7) that

$$\frac{d}{dz}\Big(p(z)\Big) = \frac{d\omega}{dz} \tag{2.8}$$

and in consideration of the equation determined in (2.4), the following relationships:

$$\begin{aligned}(1-z)\,z^2\,\frac{d^3\omega}{dz^3} + z\left[1+\gamma-\big(3+\alpha+\beta\big)\,z\right]\frac{d^2\omega}{dz^2}& \\ -\big(1+\alpha+\beta\big)\,z\,\frac{d\omega}{dz} &= \alpha\,\beta\,z\omega'(z) \\ &\equiv \alpha\,\beta\,\Phi(z) \qquad (say)\end{aligned} \tag{2.9}$$

is easily identified, where $z\in\mathbb{U}$, $\alpha\in\mathbb{C}-\mathbb{Z}_0^-$ and $\beta\in\mathbb{C}-\mathbb{Z}_0^-$.

Suppose now that the related function $p(z)$ is not *with positive real part* in the domain $\mathbb{U}$. In the circumstances, under the conditions (2.2) and (2.3) of Lemma 2.1, there is a point $z_0\in\mathbb{U}$ such that

$$p(z)\big|_{z=z_0} = p\big(z_0\big) = i\lambda \quad \text{and} \quad z\frac{d}{dz}\left[p(z)\right]\Big|_{z=z_0} = \mu\,,$$

where

$$n=1\ \ ,\ \ \lambda\in\mathbb{R}-\{0\}\ \ ,\ \ \mu\in\mathbb{R} \quad \text{and} \quad \mu\leq -\frac{1+\lambda^2}{2}\,.$$

Then, under favour of the assumptions above, from (2.9), it follows that

$$\Re e\Big(\alpha\beta\,\Phi\big(z_0\big)\Big) = \mu\,\Re e\big(\alpha\beta\big)\begin{cases} \leq -\frac{\Re e(\alpha\beta)}{2} & if \quad \Re e(\alpha\beta)\geq 0 \\ \geq -\frac{\Re e(\alpha\beta)}{2} & if \quad \Re e(\alpha\beta)\leq 0 \end{cases},$$

where

$$\alpha\in\mathbb{C}-\mathbb{Z}_0^-\ \ ,\ \ \beta\in\mathbb{C}-\mathbb{Z}_0^- \quad \text{and} \quad \mu\leq -\frac{1+\lambda^2}{2}\ \ \big(\lambda\in\mathbb{R}-\{0\}\big)\,.$$

But, these cases above are contradictions, respectively, with the cases of the inequality, given in (2.5). Therefore, the function $p(z)$, defined by (2.7), immediately yields the the inequality given by (2.6). So, this completes the proof of Theorem 2.1. $\square$

As we indicated in the first section, the function $F(\alpha,\beta;\gamma;z)$ defined by (1.1) plays very important roles in mathematical analysis and its applications. Specially, it also enables us to solve many important and interesting problems, such as conformal mapping of triangular domains bounded by line segments or circular arcs and various problems of quantum mechanics. Most of the functions that occur during analysis (or searches) can be expressed with nearly special forms of the hypergeometric functions.

In order to emphasize the importance of our main result, we here think useful to provide some information again. Especially, the series representation in (1.1) gives researchers much more motivations for their investigations; that is, the fact that the elementary functions and several other important functions in mathematics can be stated in terms of hypergeometric functions. Moreover, hypergeometric functions can be described as solutions of special second order linear differential equations that we pointed out as the hypergeometric differential equations given as in (1.6). Afterwards, Riemann was the first to raise this idea and introduce a special symbol to classify hypergeometric functions by singularities and exponents of differential equations. As we have also noted in the section 1, the hypergeometric function is a solution of the hypergeometric differential equation given in (1.6). The generalization of this equation to three arbitrary regular singular points is given by Riemanns differential equation. Any second order differential equation with three regular singular points can be transformed to the hypergeometric differential equation by changing of its variable. For more information, see the works given by the references in [2], [5]-[6], [9], [10], [16], [17], [20], [23], [31], [32], [38]-[41] and [43]-[51]. Therefore, as a requirement of the above explanations, in view of the above those relationships between the function given in (1.1) and the equation given in (1.6) will be important for our novel investigation. For this reason, our main result and their implications have various *novel and/or nonlinear* relations between them. Moreover, the desired research can be further expanded, taking into consideration the derivatives mentioned in (1.4) (or (1.5) for all the possible results that can be obtained. Accordingly, to determine all those results will be determined by the related elementary and also special functions, we need first recall some extra information in relation with the related definitions in (1.1) and (1.6), which are in the following forms.

(i) Some of Elementary Functions:

$$(1+z)^n = F\big(-n,1;1;-z\big) \quad \big(z\in\mathbb{C}\big), \tag{2.10}$$

$$\frac{1}{1-z} = F\big(1,1;1;z\big) \quad \big(z\in\mathbb{C}-\{0\}\big), \tag{2.11}$$

$$\cos(z) = F\big(1/2,-1/2;1/1;\sin^2 z\big) \quad \big(z\in\mathbb{C}\big), \tag{2.12}$$

$$\ln(1+z) = zF\big(1,1;2;-z\big) \quad \big(z\in\mathbb{C}-\{-1\}\big), \tag{2.13}$$

$$\ln\left(\frac{1+z}{1-z}\right) = 2zF\big(1/2,1;3/2;z^2\big) \quad \big(z\in\mathbb{C}-\{\pm1\}\big), \tag{2.14}$$

$$\exp(z) = \lim_{\beta \to \infty} F\big(1, \beta; 1; z/\beta\big) \quad \big(z \in \mathbb{C}\big)\,, \tag{2.15}$$

$$\arcsin(z) = zF\big(1/2, 1/2; 3/2; z^2\big) \quad \big(z \in \mathbb{C}\big) \tag{2.16}$$

and

$$\arctan(z) = zF\big(1/2, 1; 3/2; -z^2\big) \quad \big(z \in \mathbb{C}\big)\,. \tag{2.17}$$

(ii) The Complete Elliptic Integrals of the First and Second Kinds:

$$K(z) = \frac{\pi}{2}\, F\big(1/2, 1/2; 1; -z^2\big) \quad \big(z \in \mathbb{C}\big) \tag{2.18}$$

and

$$E(z) = \frac{\pi}{2}\, F\big(-1/2, 1/2; 1; -z^2\big) \quad \big(z \in \mathbb{C}\big)\,. \tag{2.19}$$

(iii) The Adjoint Legendre Functions:

$$P_m^n(z) = \frac{(z+1)^{n/2}}{(z-1)^{m/2}} \frac{1}{\Gamma(1-n)} F\bigg(-n, n+1; 1-m; \frac{1-z}{2}\bigg)\,. \tag{2.20}$$
$$\big(-n, m \in \mathbb{N}_0; z \in \mathbb{C}\big)$$

(iv) The Chebyshev Polynomials:

$$T_n(z) = F\bigg(-n, n; \frac{1}{2}; \frac{1-z}{2}\bigg) \quad \big(n \in \mathbb{N}_0; z \in \mathbb{C}\big)\,. \tag{2.21}$$

(v) The Legendre Polynomials:

$$P_n(z) = F\bigg(-n, n+1; 1; \frac{1-z}{2}\bigg) \quad \big(n \in \mathbb{N}_0; z \in \mathbb{C}\big)\,. \tag{2.22}$$

(vi) The Gegenbauer (Ultraspherical) Polynomials:

$$C_n^{\alpha}(z) = \frac{(1+\alpha)_n}{\Gamma(n+1)} F\bigg(-n, n+2\alpha; \alpha+\frac{1}{2}; \frac{1-z}{2}\bigg) \tag{2.23}$$
$$\big(n \in \mathbb{N}_0; \alpha \in \mathbb{Z}_0^-; z \in \mathbb{C}\big).$$

(vii) The Jacobi Polynomials:

$$P_n^{\alpha,\beta}(z) = \frac{(1+\alpha)_n}{\Gamma(n+1)} F\bigg(-n, 1+n+\alpha+\beta; \alpha+1; \frac{1-z}{2}\bigg)\,. \tag{2.24}$$
$$\big(n \in \mathbb{N}_0; \alpha \in \mathbb{Z}_0^-; \beta \in \mathbb{Z}_0^-; z \in \mathbb{C}\big).$$

(viii) The Confluent Hypergeometric Function ${}_1F_1(\alpha; \beta; z)$:

$${}_1F_1(\alpha; \beta; z) = \lim_{\gamma \to \infty} F\bigg(\alpha, \gamma; \beta; \frac{z}{\gamma}\bigg) \quad \big(\alpha, \beta, \gamma \in \mathbb{Z}_0^-; z \in \mathbb{C}\big). \tag{2.25}$$

(ix) The Error Functions:

$$\operatorname{erf}(z) = \frac{2z}{\sqrt{\pi}}\, {}_1F_1\big(1/2; 3/2; -z^2\big) \quad \big(z \in \mathbb{C}\big) \tag{2.26}$$

and

$$\operatorname{erfc}(z)=\frac{1}{\sqrt{\pi}}\,e^{-z^2}\,{}_1F_1\big(1/2;1/2;z^2\big)\quad\big(z\in\mathbb{C}\big) \tag{2.27}$$

and so on. For the details of those functions (or series representations), one may refer to the works given by the references in [2], [5], [15], [6], [12], [18], [20], [34], [38], [40] and [43]-[51].

As certain special implications of our main result, when considering the important relationships signified by (2.10)-(2.27), it is naturally easy to determine a number of special results, which are related to the main result, namely, Theorem 2.2. In order to determine both appropriate sampling and possible special results, we would like to leave the researchers with detailed research and emphasize only two of them, as examples.

As one of the special implications, we would like to point out a comprehensive result in relation with the confluent hypergeometric function that we mentioned as in (2.25). For this, we would like to remind researchers of some detailed information about this particular results again.

The following functional series with the complex variable z:

$$\begin{aligned}{}_1F_1\big(\alpha;\beta;z\big)&=1+\frac{\alpha}{\beta}z+\frac{\alpha(\alpha+1)}{\beta(\beta+1)}\frac{z^2}{2!}+\ \dots\\&=\sum_{k=0}^{\infty}\frac{(\alpha)_k}{(\beta)_k}\frac{z^k}{k!}\end{aligned} \tag{2.28}$$

$$\big(z\in\mathbb{U};\alpha\in\mathbb{C}-\mathbb{Z}_0^-;\beta\in\mathbb{C}-\mathbb{Z}_0^-\big),$$

where $(\alpha)_k$ and $(\beta)_k$ are the Pochhammer symbols defined by (1.2), in generally, is called as the confluent hypergeometric function in the literature. Clearly, it defines an analytic function for all finite z, is closely connected the hypergeometric function given by (1.1), and is then obtained as a limit of $F(\alpha,\beta;\gamma;z/\beta)$ when β tends to ∞ as it was indicated in (2.25). It is clear that the confluent hypergeometric function is a degenerate form of the hypergeometric function ${}_2F_1\big(\alpha;\beta;\gamma;z\big)$ which arises as a solution of the confluent hypergeometric differential equation given by above.

Since the confluent hypergeometric function is any of the solutions of the following second-order ordinary linear differential equation:

$$z\,\frac{d^2\omega}{dz^2}\;+\;\big(\alpha-z\big)\,\frac{d\omega}{dz}\;-\;\beta\,\omega\;=\;0 \tag{2.29}$$

$$\big(z\in\mathbb{C};\alpha\in\mathbb{C}-\mathbb{Z}_0^-;\beta\in\mathbb{C}-\mathbb{Z}_0^-\big),$$

this differential equation is also called as the confluent hypergeometric differential equation in the literature.

The first special implication of our main result is contained in the following proposition below.

Proposition 2.3. *Let the function $\omega := \omega(z)$ be in the form given as (2.28) and also let any one of the cases of the following inequality:*

$$\Re e\left\{z^2\,\frac{d^3\omega}{dz^3} + \big[1+(\alpha-z)\big]\,z\,\frac{d^2\omega}{dz^2} - z\,\frac{d\omega}{dz}\right\}$$

$$\begin{cases} > -\frac{\Re e(\beta)}{2} & if \quad \Re e\big(\beta\big) \geq 0 \\ < -\frac{\Re e(\beta)}{2} & if \quad \Re e\big(\beta\big) \leq 0 \end{cases} \tag{2.30}$$

be provided for any $z \in \mathbb{U}$ and for some $\alpha \in \mathbb{C} - \mathbb{Z}_0^-$ and $\beta \in \mathbb{C} - \mathbb{Z}_0^-$. In the present case,

$$\Re e\Big(\omega(z)\Big) > 0 \quad \big(z \in \mathbb{U}\big) \tag{2.31}$$

is also provided.

Proof. By means of the information presented as in (2.28) and (2.29), and also in consideration of the proof of Theorem 2.2, if one takes the function $p(z)$, defined as in (2.7), namely, define it in the form:

$$p(z) = \omega(z)\ \Big(\equiv {}_1F_1\big(\alpha;\beta;z\big)\Big)$$

$$\big(z \in \mathbb{U}; \alpha \in \mathbb{C} - \mathbb{Z}_0^-; \beta \in \mathbb{C} - \mathbb{Z}_0^-\big),$$

and then the related steps used (in the proof of Theorem 2.2) are again followed, the desired proof can be easily obtained. Here, its details are left to the researchers.

Through the instrument of the relation between the hypergeometric function and the complex error function in (2.26) together with (2.27), as second implication of our main result, certain special results can be also obtained between the various inequalities associated with error functions in the complex plane. For those, in (2.26), (2.28) and (2.29), respectively, by choosing the suitable values of the parameters α and β, one can derive some of them. For example, by setting

$$\alpha := \frac{1}{2} \qquad \text{and} \qquad \beta := \frac{3}{2}$$

in (2.26), the following results:

$${}_1F_1\big(1/2;3/2;-z^2\big) = \frac{\sqrt{\pi}}{2}\,\frac{\mathrm{erf}\,(z)}{z} \tag{2.32}$$

$$\big(z \in \mathbb{D} := \mathbb{U} - \{0\}\big),$$

$${}_1F_1\big(1/2;3/2;z\big) = 2\sum_{k=0}^{\infty}\frac{(-z)^k}{k!} \quad \big(z \in \mathbb{U}\big) \tag{2.33}$$

and

$$2\,z\,\frac{d^2\omega}{dz^2} + (1-2z)\,\frac{d\omega}{dz} - 3\,\omega = 0 \quad \big(z \in \mathbb{U}\big) \tag{2.34}$$

are easily obtained.

So, as we have informed above, the following-special function:

$${}_1F_1\big(1/2;3/2;-z^2\big) \quad \big(z \in \mathbb{U}\big),$$

which is given by (2.32) (or (2.33)), is a solution for the second-order linear differential equation given by (2.34). After these explanations, the following proposition, i.e., Proposition 2.4 below, can be easily proved within the scope of the rationale of the main result (or Proposition 2.3). The detail of the related proof has been left to the researchers again.

Proposition 2.4. *For any* $z \in \mathbb{D}$ $\big($*or,* $z \in \mathbb{U}\big)$, *if the inequality:*

$$\Re e\left\{2z^2\frac{d^3}{dz^3}\left(\frac{erf(z)}{z}\right)+\big[2+\big(1-2z\big)\big]z\frac{d^2}{dz^2}\left(\frac{erf(z)}{z}\right)-2z\frac{d}{dz}\left(\frac{erf(z)}{z}\right)\right\}>-\frac{3}{\sqrt{\pi}}$$

is ensured, then

$$\Re e\left(\frac{erf(z)}{z}\right)>0$$

is also ensured.

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Notes on the norm of pre-Schwarzian derivatives of certain analytic functions

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Abstract. In this paper, we obtain sharp bounds for the norm of pre-Schwarzian derivatives of certain analytic functions. Initially this problem was handled by H. Rahmatan, Sh. Najafzadeh and A. Ebadian [Stud. Univ. Babeş-Bolyai Math. **61**(2016), no. 2, 155-162]. We pointed out that their proofs are incorrect and present correct proofs.

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1. Introduction

Let $\Delta = \{z \in \mathbb{C} : |z| < 1\}$ be the open unit disc on the complex plane $\mathbb{C}$. Let $\mathcal{H}$ be the family of all analytic functions and $\mathcal{A} \subset \mathcal{H}$ be the family of all normalized functions in Δ. We denote by $\mathcal{U}$ the class of all univalent functions in Δ and denote by $\mathcal{LU} \subset \mathcal{H}$ the class of all locally univalent functions in Δ. For a $f \in \mathcal{LU}$, we consider the following norm

$$||f|| = \sup_{z\in\Delta}(1 - |z|^2)\left|\frac{f''(z)}{f'(z)}\right|,$$

where the quantity f''/f is often referred to as pre-Schwarzian derivative of f such that in the theory of Teichmüller spaces is considered as element of complex Banach spaces. We remark that $||f|| < \infty$ if, and only if, f is uniformly locally univalent in Δ. We also notice that, $||f|| \leq 6$ if f is univalent in Δ and, conversely, f is univalent in Δ if $||f| \leq 1$. Both of these bounds are sharp, see [1]. For more geometric properties of the function f relating the norm, see [2, 4, 9] and the references therein.

We say that a function f is subordinate to g, written by $f(z) \prec g(z)$ or $f \prec g$ where f and g belonging to the class $\mathcal{A}$, if there exists a Schwarz function $w(z)$ is analytic in Δ with

$$w(0) = 0 \quad \text{and} \quad |w(z)| < 1 \quad (z \in \Delta),$$

such that $f(z) = g(w(z))$ for all $z \in \Delta$.

Here are two certain subclasses of analytic and normalized functions $\mathcal{A}$ functions defined. First, we say that a function $f \in \mathcal{A}$ belongs to the class $\mathcal{S}(\alpha, \beta)$ if it satisfies the following two-sided inequality

$$\alpha < \mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\right\} < \beta \quad (z \in \Delta),$$

where $0 \le \alpha < 1$ and $\beta > 1$. The class $\mathcal{S}(\alpha, \beta)$ was introduced by Kuroki and Owa (cf. [7]) and generalized by Kargar et al. [6]. We also say that a function $f \in \mathcal{A}$ belongs to the class $\mathcal{V}(\alpha, \beta)$ if

$$\alpha < \mathrm{Re}\left\{\left(\frac{z}{f(z)}\right)^2 f'(z)\right\} < \beta \quad (z \in \Delta).$$

The class $\mathcal{V}(\alpha, \beta)$ was first introduced by Kargar et al., see [5].
Since the convex univalent function

$$P_{\alpha,\beta}(z) = 1 + \frac{(\beta-\alpha)i}{\pi}\log\left(\frac{1-e^{i\phi}z}{1-z}\right) \quad (z \in \Delta), \tag{1.1}$$

where

$$\phi := \frac{2\pi(1-\alpha)}{\beta-\alpha}, \tag{1.2}$$

maps Δ onto the domain $\Omega = \{\omega : \alpha < \mathrm{Re}\{\omega\} < \beta\}$ conformally, thus we have.

Lemma 1.1. ([7, Lemma 1.3]) *Let $\alpha \in [0, 1)$ and $\beta \in (1, \infty)$. Then $f \in \mathcal{S}(\alpha, \beta)$ if, and only if,*

$$\frac{zf'(z)}{f(z)} \prec 1 + \frac{(\beta-\alpha)i}{\pi}\log\left(\frac{1-e^{i\phi}z}{1-z}\right) \quad (z \in \Delta),$$

where ϕ is defined in (1.2).

Lemma 1.2. ([5, Lemma 1.1]) *Let $\alpha \in [0, 1)$ and $\beta \in (1, \infty)$. Then $f \in \mathcal{V}(\alpha, \beta)$ if, and only if,*

$$\left(\frac{z}{f(z)}\right)^2 f'(z) \prec 1 + \frac{(\beta-\alpha)i}{\pi}\log\left(\frac{1-e^{i\phi}z}{1-z}\right) \quad (z \in \Delta),$$

where ϕ is defined in (1.2).

Rahmatan, Najafzadeh and Ebadian (see [10]) estimated the norm of pre-Schwarzian derivatives of the function f where f belongs to the classes $\mathcal{S}(\alpha, \beta)$ and $\mathcal{V}(\alpha, \beta)$. Both their estimates and proofs are incorrect. Indeed, the results that were wrongly proven by them are as follows:

Theorem A. *For $0 \le \alpha < 1 < \beta$, if $f \in \mathcal{S}(\alpha, \beta)$, then*

$$||f|| \le \frac{2(\beta-\alpha)}{\pi}\left(1 - e^{2\pi i\frac{1-\alpha}{\beta-\alpha}}\right).$$

Theorem B. *For $0 \le \alpha < 1 < \beta$, if $f \in \mathcal{V}(\alpha, \beta)$, then*

$$||f|| \le \frac{3(\beta-\alpha)}{\pi}\left(1 - e^{2\pi i\frac{1-\alpha}{\beta-\alpha}}\right).$$

We first note that both the above bounds are complex numbers!

In this paper we give the best estimate for $||f||$ when $f \in \mathcal{S}(\alpha,\beta)$ and disprove the Theorem B. However, we show that $||f|| < \infty$ when $f \in \mathcal{V}(\alpha,\beta)$.

2. Main results

The correct version of Theorem A is as follows.

Theorem 2.1. *Let $\alpha \in [0,1)$ and $\beta \in (1,\infty)$. If a function f belongs to the class $\mathcal{S}(\alpha,\beta)$, then*

$$||f|| \leq \frac{2(\beta-\alpha)}{\pi}\sqrt{4\sin^2(\phi/2)+2\pi^2} - \frac{4\sin(\phi/2)}{\sqrt{4\sin^2(\phi/2)+2\pi^2}}, \tag{2.1}$$

where ϕ is defined in (1.2). *The result is sharp.*

Proof. Let that $\alpha \in [0,1)$, $\beta \in (1,\infty)$ and ϕ be given by (1.2). If $f \in \mathcal{S}(\alpha,\beta)$, by Lemma 1.1, then we have

$$\frac{zf'(z)}{f(z)} \prec 1 + \frac{(\beta-\alpha)i}{\pi}\log\left(\frac{1-e^{i\phi}z}{1-z}\right) \quad (z \in \Delta). \tag{2.2}$$

The above subordination relation (2.2) implies that

$$\frac{zf'(z)}{f(z)} = 1 + \frac{(\beta-\alpha)i}{\pi}\log\left(\frac{1-e^{i\phi}w(z)}{1-w(z)}\right) \quad (z \in \Delta),$$

or equivalently

$$\log\left\{\frac{zf'(z)}{f(z)}\right\} = \log\left\{1 + \frac{(\beta-\alpha)i}{\pi}\log\left(\frac{1-e^{i\phi}w(z)}{1-w(z)}\right)\right\} \quad (z \in \Delta), \tag{2.3}$$

where $w(z)$ is the well-known Schwarz function. From (2.3), differentiating on both sides, after simplification, we obtain

$$\begin{aligned}\frac{f''(z)}{f'(z)} \quad = \quad & \frac{(\beta-\alpha)i}{\pi}\left[\frac{1}{z}\log\left(\frac{1-e^{i\phi}w(z)}{1-w(z)}\right)\right. \\ & \left. + \frac{(1-e^{i\phi})w'(z)}{(1-w(z))(1-e^{i\phi}w(z))\left(1+\frac{(\beta-\alpha)i}{\pi}\log\left(\frac{1-e^{i\phi}w(z)}{1-w(z)}\right)\right)}\right].\end{aligned} \tag{2.4}$$

It is well-known that $|w(z)| \leq |z|$ (cf. [3]) and also by the Schwarz-Pick lemma, for a Schwarz function the following inequality

$$|w'(z)| \leq \frac{1-|w(z)|^2}{1-|z|^2} \quad (z \in \Delta), \tag{2.5}$$

holds (see [8]). We also know that if log is the principal branch of the complex logarithm, then we have

$$\log z = \ln|z| + i\arg z \quad (z \in \Delta \setminus \{0\}, -\pi < \arg z \leq \pi). \tag{2.6}$$

Therefore, by the above equation (2.6), it is well-known that if $|z| \geq 1$, then

$$|\log z| \leq \sqrt{|z-1|^2+\pi^2}, \tag{2.7}$$

while for $0<|z|<1$, we have

$$|\log z| \le \sqrt{\left|\frac{z-1}{z}\right|^2+\pi^2}. \tag{2.8}$$

Thus, it is natural to distinguish the following cases.
Case 1. $\left|\frac{1-e^{i\phi}w(z)}{1-w(z)}\right| \ge 1$.
By (2.7), we have

$$\begin{aligned}\left|\log\left(\frac{1-e^{i\phi}w(z)}{1-w(z)}\right)\right| &\le \sqrt{\left|\frac{1-e^{i\phi}w(z)}{1-w(z)}-1\right|^2+\pi^2} \\ &= \frac{\sqrt{|1-e^{i\phi}|^2|w(z)|^2+\pi^2|1-w(z)|^2}}{|1-w(z)|} \\ &\le \frac{\sqrt{4\sin^2(\phi/2)|w(z)|^2+\pi^2(1+|w(z)|^2)}}{1-|w(z)|} \\ &\le \frac{\sqrt{4\sin^2(\phi/2)|z|^2+\pi^2(1+|z|^2)}}{1-|z|} \end{aligned} \tag{2.9}$$

for all $z\in\Delta$. We note that the above inequality is well defined also for $z=0$. Thus from (2.4), (2.5) and (2.9), we get

$$\begin{aligned} &\left|\frac{f''(z)}{f'(z)}\right| \\ = &\left|\frac{(\beta-\alpha)i}{\pi}\left[\frac{1}{z}\log\left(\frac{1-e^{i\phi}w(z)}{1-w(z)}\right)\right.\right. \\ &\left.\left.+\frac{(1-e^{i\phi})w'(z)}{(1-w(z))(1-e^{i\phi}w(z))\left(1+\frac{(\beta-\alpha)i}{\pi}\log\left(\frac{1-e^{i\phi}w(z)}{1-w(z)}\right)\right)}\right]\right| \\ \le &\frac{(\beta-\alpha)}{\pi}\left[\frac{1}{|z|}\left|\log\left(\frac{1-e^{i\phi}w(z)}{1-w(z)}\right)\right|\right. \\ &\left.+\frac{\left|1-e^{i\phi}\right||w'(z)|}{|1-w(z)|\,|1-e^{i\phi}w(z)|\left(1-\frac{(\beta-\alpha)}{\pi}\left|\log\left(\frac{1-e^{i\phi}w(z)}{1-w(z)}\right)\right|\right)}\right] \\ \le &\frac{(\beta-\alpha)}{\pi}\left[\frac{1}{|z|}\left\{\frac{\sqrt{4\sin^2(\phi/2)|z|^2+\pi^2(1+|z|^2)}}{1-|z|}\right\}\right. \\ &\left.+\frac{2\sin(\phi/2)}{1-|z|-\frac{(\beta-\alpha)}{\pi}\sqrt{4\sin^2(\phi/2)|z|^2+\pi^2(1+|z|^2)}}\cdot\frac{1+|z|}{1-|z|^2}\right]. \end{aligned}$$

However, we obtain

$$\begin{aligned}
||f|| &= \sup_{z\in\Delta}(1-|z|^2)\left|\frac{f''(z)}{f'(z)}\right| \\
&\leq \sup_{z\in\Delta}\left\{\frac{(\beta-\alpha)}{\pi}\left[\frac{1+|z|}{|z|}\sqrt{4\sin^2(\phi/2)|z|^2+\pi^2(1+|z|^2)}\right.\right. \\
&\quad \left.\left.+\frac{2\sin(\phi/2)(1+|z|)}{1-|z|-\frac{(\beta-\alpha)}{\pi}\sqrt{4\sin^2(\phi/2)|z|^2+\pi^2(1+|z|^2)}}\right]\right\} \\
&= \frac{2(\beta-\alpha)}{\pi}\sqrt{4\sin^2(\phi/2)+2\pi^2}-\frac{4\sin(\phi/2)}{\sqrt{4\sin^2(\phi/2)+2\pi^2}}
\end{aligned}$$

concluding the inequality (2.1).

Case 2. $\left|\frac{1-e^{i\phi}w(z)}{1-w(z)}\right|<1.$

By (2.8), we have

$$\begin{aligned}
\left|\log\left(\frac{1-e^{i\phi}w(z)}{1-w(z)}\right)\right| &\leq \sqrt{\left|\frac{\frac{1-e^{i\phi}w(z)}{1-w(z)}-1}{\frac{1-e^{i\phi}w(z)}{1-w(z)}}\right|^2+\pi^2} \\
&= \frac{\sqrt{|1-e^{i\phi}|^2|w(z)|^2+\pi^2|1-e^{i\phi}w(z)|^2}}{|1-e^{i\phi}w(z)|} \\
&\leq \frac{\sqrt{4\sin^2(\phi/2)|w(z)|^2+\pi^2(1+|w(z)|^2)}}{1-|w(z)|} \qquad (|e^{i\phi}|=1) \\
&\leq \frac{\sqrt{4\sin^2(\phi/2)|z|^2+\pi^2(1+|z|^2)}}{1-|z|}.
\end{aligned}$$

Since in both cases 1 and 2 we have the equal estimates for

$$\left|\log\left(\frac{1-e^{i\phi}w(z)}{1-w(z)}\right)\right|,$$

therefore, in this case also, the desired result will be achieved. For the sharpness, consider the function $f_{\alpha,\beta}(z)$ as follows

$$\begin{aligned}
f_{\alpha,\beta}(z) &= z\exp\left\{\frac{(\beta-\alpha)i}{\pi}\int_0^z\frac{1}{\xi}\log\left(\frac{1-e^{i\phi}\xi}{1-\xi}\right)\mathrm{d}\xi\right\} \\
&= z+\frac{(\beta-\alpha)i}{\pi}\left(1-e^{i\phi}\right)z^2+\cdots,
\end{aligned}$$

where ϕ is defined in (1.2), $0\leq\alpha<1$ and $\beta>1$. A simple calculation, gives us

$$\frac{zf'_{\alpha,\beta}(z)}{f_{\alpha,\beta}(z)}=1+\frac{(\beta-\alpha)i}{\pi}\log\left(\frac{1-e^{i\phi}z}{1-z}\right) \quad (z\in\Delta)$$

and thus $f_{\alpha,\beta}(z) \in \mathcal{S}(\alpha,\beta)$. With the same proof as above we get the desired result. The result also is sharp for a rotation of the function $f_{\alpha,\beta}(z)$ as follows:

$$\mathfrak{f}_{\alpha,\beta}(z) = z \exp\left\{\frac{(\beta-\alpha)i}{\pi}\int_0^z \frac{1}{\xi}\log\left(\frac{1-e^{i\phi}\xi}{1-e^{-i\phi}\xi}\right)\mathrm{d}\xi\right\}.$$

This is the end of proof. □

Remark 2.2. In Theorem B, the authors of [10] estimated the norm $||f||$ when $f \in \mathcal{V}(\alpha,\beta)$. But in the proof of this theorem [10, p. 160], wrongly, they used from the following equation

$$\frac{zf'(z)}{f(z)} = P_{\alpha,\beta}(w(z)),$$

where $P_{\alpha,\beta}$ is defined in (1.1). This means that f, simultaneously, belonging to the class $\mathcal{S}(\alpha,\beta)$ and $\mathcal{V}(\alpha,\beta)$.
Next, we show that the best estimate for $||f||$ when $f \in \mathcal{V}(\alpha,\beta)$ does not exist.

Theorem 2.3. *Let $\alpha \in [0,1)$ and $\beta \in (1,\infty)$. If a function f belongs to the class $\mathcal{V}(\alpha,\beta)$, then $||f|| < \infty$.*

Proof. Let $\alpha \in [0,1)$ and $\beta \in (1,\infty)$ and $f \in \mathcal{V}(\alpha,\beta)$. Then by Lemma 1.2 and by use of definition of subordination, we have

$$\left(\frac{z}{f(z)}\right)^2 f'(z) = P_{\alpha,\beta}(w(z)) = 1 + \frac{(\beta-\alpha)i}{\pi}\log\left(\frac{1-e^{i\phi}w(z)}{1-w(z)}\right), \tag{2.10}$$

where w is Schwarz function and ϕ is defined in (1.2). Taking logarithm on both sides of (2.10) and differentiating, we get

$$\begin{aligned}\frac{f''(z)}{f'(z)} &= 2\left(\frac{f'(z)}{f(z)} - \frac{1}{z}\right) + \frac{(\beta-\alpha)i}{\pi} \qquad (2.11)\\ &\times\left[\frac{(1-e^{i\phi})w'(z)}{(1-w(z))(1-e^{i\phi}w(z))\left(1+\frac{(\beta-\alpha)i}{\pi}\log\left(\frac{1-e^{i\phi}w(z)}{1-w(z)}\right)\right)}\right].\end{aligned}$$

With a simple calculation, (2.10) implies that

$$\left(\frac{f'(z)}{f(z)} - \frac{1}{z}\right) = \frac{f(z)}{z}\left(\frac{P_{\alpha,\beta}(w(z))}{z} - 1\right). \tag{2.12}$$

Combining (2.11) and (2.12), give us

$$\begin{aligned}\frac{f''(z)}{f'(z)} &= 2\left(\frac{f(z)}{z}\left(\frac{P_{\alpha,\beta}(w(z))}{z} - 1\right)\right)\\ &+ \frac{(\beta-\alpha)i}{\pi}\left[\frac{(1-e^{i\phi})w'(z)}{(1-w(z))(1-e^{i\phi}w(z))\left(1+\frac{(\beta-\alpha)i}{\pi}\log\left(\frac{1-e^{i\phi}w(z)}{1-w(z)}\right)\right)}\right]\end{aligned}$$

It was proved in ([5, Theorem 2.2]) that if $f \in \mathcal{V}(\alpha,\beta)$ where $0 < \alpha \le 1/2$ and $\beta > 1$, then

$$1 - \frac{1}{\alpha} < \mathrm{Re}\left\{\frac{f(z)}{z}\right\} < \infty \quad (z \in \Delta).$$

Since $\mathrm{Re}\{z\} \leq |z|$, the last two-sided inequality means that $|f(z)/z| < \infty$ when $f \in \mathcal{V}(\alpha, \beta)$. Thus from the above we deduce that

$$\left| \frac{f''(z)}{f'(z)} \right| < \infty \quad (z \in \Delta)$$

concluding the proof. □

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On a certain class of harmonic functions and the generalized Bernardi-Libera-Livingston integral operator

Grigore Ştefan Sălăgean and Ágnes Orsolya Páll-Szabó

Abstract. In this paper we examine the closure properties of the class $\mathcal{V}_{\mathcal{H}}(F;\gamma)$ under the generalized Bernardi-Libera-Livingston integral operator $\mathcal{L}_c(f)$, $(c > -1)$ which is defined by $\mathcal{L}_c(f) = \mathcal{L}_c(h) + \overline{\mathcal{L}_c(g)}$, $f = h + \overline{g}$, h and g are analytic functions, where

$$\mathcal{L}_c(h)(z) = \frac{c+1}{z^c}\int_0^z (t^{c-1}h(t)dt \quad \text{and} \quad \mathcal{L}_c(g)(z) = \frac{c+1}{z^c}\int_0^z (t^{c-1}g(t)dt.$$

The obtained results are sharp and they improve known results.

Mathematics Subject Classification (2010): 30C45, 30C50.

Keywords: Harmonic univalent functions, extreme points, varying arguments, Hadamard product, integral operator.

1. Preliminaries

A continuous function $f = u + iv$ is a complex-valued harmonic function in a complex domain $\mathcal{G}$ if both u and v are real and harmonic in $\mathcal{G}$. In any simply-connected domain $D \subset \mathcal{G}$, we can write $f = h + \overline{g}$, where h and g are analytic in D. We call h the analytic part and g the co-analytic part of f. A necessary and sufficient condition for f to be locally univalent and orientation preserving in D is that $|h'(z)| > |g'(z)|$ in D (see [3]).

Denote by $\mathcal{H}$ the family of functions

$$f = h + \overline{g} \tag{1.1}$$

which are harmonic, univalent and orientation preserving in the open unit disc $\mathcal{U} = \{z : |z| < 1\}$ so that f is normalized by $f(0) = h(0) = f'_z(0) - 1 = 0$. Thus, for $f = h + \overline{g} \in \mathcal{H}$, the functions h and g analytic in $\mathcal{U}$ can be expressed in the following forms:

$$h(z) = z + \sum_{m=2}^{\infty} a_m z^m, \quad g(z) = \sum_{m=1}^{\infty} b_m z^m \quad (0 \leq b_1 < 1),$$

and $f(z)$ is then given by

$$f(z) = z + \sum_{m=2}^{\infty} a_m z^m + \overline{\sum_{m=1}^{\infty} b_m z^m} \quad (0 \leq b_1 < 1). \tag{1.2}$$

For functions $f \in \mathcal{H}$ given by (1.2) and $F \in \mathcal{H}$ given by

$$F(z) = H(z) + \overline{G(z)} = z + \sum_{m=2}^{\infty} A_m z^m + \overline{\sum_{m=1}^{\infty} B_m z^m}, \ (0 \leq B_1 \leq 1), \tag{1.3}$$

we recall the Hadamard product (or convolution) of f and F by

$$(f * F)(z) = z + \sum_{m=2}^{\infty} a_m A_m z^m + \overline{\sum_{m=1}^{\infty} b_m B_m z^m} \quad (z \in \mathcal{U}). \tag{1.4}$$

In terms of the Hadamard product (or convolution), we choose F as a fixed function in $\mathcal{H}$ such that $(f * F)(z)$ exists for any $f \in \mathcal{H}$, and for various choices of F we get different linear operators which have been studied in recent past.
In [8] a subclass of $\mathcal{H}$ denoted by $\mathcal{S}_{\mathcal{H}}(F;\gamma)$, for $0 \leq \gamma < 1$, is defined and studied and it consists of functions of the form (1.1) satisfying the inequality:

$$\frac{\partial}{\partial \theta}\left(arg\left[(f * F)(z)\right]\right) > \gamma \tag{1.5}$$

$0 \leq \theta < 2\pi$ and $z = re^{i\theta}$. Equivalently

$$Re\left\{\frac{z\left(h\left(z\right) * H\left(z\right)\right)' - \overline{z\left(g\left(z\right) * G\left(z\right)\right)'}}{h\left(z\right) * H\left(z\right) + \overline{g\left(z\right) * G\left(z\right)}}\right\} \geq \gamma \tag{1.6}$$

where $z \in \mathcal{U}$. We also let $\mathcal{V}_{\mathcal{H}}(F;\gamma) = S_{\mathcal{H}}(F;\gamma) \bigcap V_{\mathcal{H}}$ where $V_{\mathcal{H}}$ is the class of harmonic functions with varying arguments introduced by Jahangiri and Silverman [6], consisting of functions f of the form (1.1) in $\mathcal{H}$ for which there exists a real number ϕ such that

$$\eta_m + (m-1)\,\phi \equiv \pi \,(\text{mod } 2\pi), \quad \delta_m + (m+1)\,\phi \equiv 0 \,(\text{mod } 2\pi), \quad m \geq 2, \tag{1.7}$$

where $\eta_m = arg\,(a_m)$ and $\delta_m = arg\,(b_m)$.
Some of the function classes emerge from the function class $S_{\mathcal{H}}(F;\gamma)$ defined above. Indeed, if we specialize the function $F(z)$ we can obtain, respectively, (see [8]) the class of functions defined using: the Wright's generalized operator on harmonic functions ([9], [13]), the Dziok-Srivastava operator on harmonic functions ([1]), the Carlson-Shaffer operator ([2]), the Ruscheweyh derivative operator on harmonic functions ([5], [7], [10]) , the Srivastava-Owa fractional derivative operator ([12]), the Sălăgean derivative operator for harmonic functions ([4], [11]).

In the following we suppose that $F(z)$ is of the form

$$F(z) = H(z) + \overline{G(z)} = z + \overline{z} + \sum_{m=2}^{\infty} C_m \left(z^m + \overline{z^m}\right), \tag{1.8}$$

where $C_m \geq 0\,(m \geq 2)$.
In [8] the following characterization theorem is proved

Theorem 1.1. *Let $f = h + \overline{g}$ be given by (1.2) with restrictions (1.7) and $0 \leq b_1 < \frac{1-\gamma}{1+\gamma}$, $0 \leq \gamma < 1$. Then $f \in \mathcal{V}_{\mathcal{H}}(F;\gamma)$ if and only if the inequality*

$$\sum_{m=2}^{\infty} \left(\frac{m-\gamma}{1-\gamma} |a_m| + \frac{m+\gamma}{1-\gamma} |b_m| \right) C_m \leq 1 - \frac{1+\gamma}{1-\gamma} b_1 \tag{1.9}$$

holds true.

Theorem 1.2. [8] *Set $\lambda_m = \frac{1-\gamma}{(m-\gamma)C_m}$ and $\mu_m = \frac{1-\gamma}{(m+\gamma)C_m}$. Then for b_1 fixed, $0 \leq b_1 < \frac{1-\gamma}{1+\gamma}$ the extreme points for $\mathcal{V}_{\mathcal{H}}(F;\gamma)$, $0 \leq \gamma < 1$ are*

$$\left\{ z + \lambda_m x z^m + \overline{b_1 z} \right\} \cup \left\{ z + \overline{b_1 z + \mu_m x z^m} \right\}$$

where $m \geq 2$ and $x = 1 - \frac{1+\gamma}{1-\gamma} b_1$.

2. Main result

Now, we will examine the closure properties of the class $\mathcal{V}_{\mathcal{H}}(F;\gamma)$ under the generalized Bernardi-Libera-Livingston integral operator $\mathcal{L}_c(f)$, $(c > -1)$ which is defined by $\mathcal{L}_c(f) = \mathcal{L}_c(h) + \overline{\mathcal{L}_c(g)}$ where

$$\mathcal{L}_c(h)(z) = \frac{c+1}{z^c} \int_0^z (t^{c-1} h(t) dt \quad \text{and} \quad \mathcal{L}_c(g)(z) = \frac{c+1}{z^c} \int_0^z (t^{c-1} g(t) dt.$$

Theorem 2.1. *Let $f \in \mathcal{V}_{\mathcal{H}}(F;\gamma)$. Then $\mathcal{L}_c(f) \in \mathcal{V}_{\mathcal{H}}(F;\delta(\gamma))$ where*

$$\delta(\gamma) = \frac{(2+\gamma)(c+2)(1-b_1) - 2(c+1)\left[(1-\gamma) - (1+\gamma) b_1\right]}{(2+\gamma)(c+2)(1+b_1) + (c+1)\left[(1-\gamma) - (1+\gamma) b_1\right]} > \gamma.$$

The result is sharp.

Proof. Since $f \in \mathcal{V}_{\mathcal{H}}(F;\gamma)$ we have

$$\frac{\sum_{m=2}^{\infty} \left(\frac{m-\gamma}{1-\gamma} |a_m| + \frac{m+\gamma}{1-\gamma} |b_m| \right) C_m}{1 - \frac{1+\gamma}{1-\gamma} b_1} \leq 1. \tag{2.1}$$

We know from Theorem 1.1 that $\mathcal{L}_c(f) \in \mathcal{V}_{\mathcal{H}}(F;\delta(\gamma))$ if and only if

$$\frac{\sum_{m=2}^{\infty}\left(\frac{m-\delta(\gamma)}{1-\delta(\gamma)}\frac{c+1}{c+m}|a_m|+\frac{m+\delta(\gamma)}{1-\delta(\gamma)}\frac{c+1}{c+m}|b_m|\right)C_m}{1-\frac{1+\delta(\gamma)}{1-\delta(\gamma)}b_1}\leq 1. \tag{2.2}$$

We note that the inequalities

$$\frac{\sum_{m=2}^{\infty}\left(\frac{m-\delta(\gamma)}{1-\delta(\gamma)}\frac{c+1}{c+m}|a_m|+\frac{m+\delta(\gamma)}{1-\delta(\gamma)}\frac{c+1}{c+m}|b_m|\right)C_m}{1-\frac{1+\delta(\gamma)}{1-\delta(\gamma)}b_1}$$

$$\leq \frac{\sum_{m=2}^{\infty}\left(\frac{m-\gamma}{1-\gamma}|a_m|+\frac{m+\gamma}{1-\gamma}|b_m|\right)C_m}{1-\frac{1+\gamma}{1-\gamma}b_1} \tag{2.3}$$

imply (2.2). It is sufficient to determine $\delta(\gamma)$ such that

$$\frac{\frac{m-\delta(\gamma)}{1-\delta(\gamma)}\frac{c+1}{c+m}}{1-\frac{1+\delta(\gamma)}{1-\delta(\gamma)}b_1}\leq\frac{\frac{m-\gamma}{1-\gamma}}{1-\frac{1+\gamma}{1-\gamma}b_1} \tag{2.4}$$

and

$$\frac{\frac{m+\delta(\gamma)}{1-\delta(\gamma)}\frac{c+1}{c+m}}{1-\frac{1+\delta(\gamma)}{1-\delta(\gamma)}b_1}\leq\frac{\frac{m+\gamma}{1-\gamma}}{1-\frac{1+\gamma}{1-\gamma}b_1}. \tag{2.5}$$

holds true. (2.4) is equivalent to

$$\frac{m-\delta(\gamma)}{1-\delta(\gamma)-b_1-\delta(\gamma)b_1}\frac{c+1}{c+m}\leq\frac{m-\gamma}{(1-\gamma)-(1+\gamma)b_1}$$

$$\delta(\gamma)\leq\frac{(m-\gamma)(c+m)(1-b_1)-m(c+1)[(1-\gamma)-(1+\gamma)b_1]}{(m-\gamma)(c+m)(1+b_1)-(c+1)[(1-\gamma)-(1+\gamma)b_1]}. \tag{2.6}$$

Relation (2.5) is equivalent to

$$\frac{m+\delta(\gamma)}{1-\delta(\gamma)-b_1-\delta(\gamma)b_1}\frac{c+1}{c+m}\leq\frac{m+\gamma}{(1-\gamma)-(1+\gamma)b_1}$$

$$\delta(\gamma)\leq\frac{(m+\gamma)(c+m)(1-b_1)-m(c+1)[(1-\gamma)-(1+\gamma)b_1]}{(m+\gamma)(c+m)(1+b_1)+(c+1)[(1-\gamma)-(1+\gamma)b_1]}. \tag{2.7}$$

From (2.6) and (2.7) we choose the smaller one:

$$\frac{(m-\gamma)(c+m)(1-b_1)-m(c+1)[(1-\gamma)-(1+\gamma)b_1]}{(m-\gamma)(c+m)(1+b_1)-(c+1)[(1-\gamma)-(1+\gamma)b_1]}$$
$$>\frac{(m+\gamma)(c+m)(1-b_1)-m(c+1)[(1-\gamma)-(1+\gamma)b_1]}{(m+\gamma)(c+m)(1+b_1)+(c+1)[(1-\gamma)-(1+\gamma)b_1]}$$

or equivalently

$$\frac{2(c+1)\Delta^2 m(m-1)}{[(m-\gamma)(c+m)(1+b_1)-(c+1)\Delta][(m+\gamma)(c+m)(1+b_1)+(c+1)\Delta]}>0,$$

where $\Delta=[(1-\gamma)-(1+\gamma)b_1]>0$ which is true. So

$$\delta(\gamma)\le\frac{(m+\gamma)(c+m)(1-b_1)-m(c+1)[(1-\gamma)-(1+\gamma)b_1]}{(m+\gamma)(c+m)(1+b_1)+(c+1)[(1-\gamma)-(1+\gamma)b_1]}. \tag{2.8}$$

Let us consider the function $E:[2;\infty)\to\mathbb{R}$

$$E(x)=\frac{(x+\gamma)(c+x)(1-b_1)-x(c+1)[(1-\gamma)-(1+\gamma)b_1]}{(x+\gamma)(c+x)(1+b_1)+(c+1)[(1-\gamma)-(1+\gamma)b_1]};$$

then its derivative is:

$$E'(x)=\frac{(c+1)[(1-\gamma)-(1+\gamma)b_1]\left[(1+b_1)x^2+2x(1-b_1)+2\gamma+b_1-1\right]}{\{(x+\gamma)(c+x)(1+b_1)+(c+1)[(1-\gamma)-(1+\gamma)b_1]\}^2}>0.$$

$E(x)$ is an increasing function. In our case we need $\delta(\gamma)\le E(m),\forall m\ge 2$ and for this reason we choose

$$\delta(\gamma)=E(2)=\frac{(2+\gamma)(c+2)(1-b_1)-2(c+1)[(1-\gamma)-(1+\gamma)b_1]}{(2+\gamma)(c+2)(1+b_1)+(c+1)[(1-\gamma)-(1+\gamma)b_1]}.$$

We must check $\delta(\gamma)>\gamma$ that is equivalent to

$$\frac{[(1-\gamma)-(1+\gamma)b_1](2+\gamma)[(c+2)-(c+1)]}{(2+\gamma)(c+2)(1+b_1)+(c+1)[(1-\gamma)-(1+\gamma)b_1]}>0$$

which is true.
The result is sharp, because if

$$f(z)=z+\overline{b_1 z+\frac{1-\gamma}{(2+\gamma)C_2}\left(1-\frac{1+\gamma}{1-\gamma}b_1\right)z^2}$$

then

$$\mathcal{L}_c(f)(z)=z+\overline{b_1 z+\frac{1-\gamma}{(2+\gamma)C_2}\left(1-\frac{1+\gamma}{1-\gamma}b_1\right)z^2\frac{c+1}{c+2}}$$
$$=z+\overline{b_1 z+\frac{1-\delta(\gamma)}{(2+\delta(\gamma))C_2}\left(1-\frac{1+\delta(\gamma)}{1-\delta(\gamma)}b_1\right)z^2}$$
$$\Leftrightarrow\frac{1-\gamma}{(2+\gamma)}\frac{c+1}{c+2}\frac{1-\gamma-(1+\gamma)b_1}{1-\gamma}=\frac{1-\delta(\gamma)}{(2+\delta(\gamma))}\frac{1-\delta(\gamma)-(1+\delta(\gamma))b_1}{1-\delta(\gamma)}$$
$$\Leftrightarrow\delta(\gamma)=\frac{(2+\gamma)(c+2)(1-b_1)-2(c+1)[(1-\gamma)-(1+\gamma)b_1]}{(2+\gamma)(c+2)(1+b_1)+(c+1)[(1-\gamma)-(1+\gamma)b_1]}$$

this is the (2.7) inequality. □

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Meromorphic close-to-convex functions satisfying a differential inequality

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Abstract. In the present paper, we study the differential inequality

$$-\Re\left[(1-\alpha)z^2 f'(z)+\alpha\left(1+\frac{zf''(z)}{f'(z)}\right)\right]>\beta,(z\in\mathbb{E})$$

where $f\in\Sigma$ and notice that the members of class Σ which satisfy the above inequality are meromorphic close-to-convex.

Mathematics Subject Classification (2010): 30C45, 30C80.

Keywords: Meromorphic function, meromorphic starlike function, meromorphic close-to-convex function.

1. Introduction

Let Σ denote the class of meromorphic functions of the form

$$f(z)=\frac{1}{z}+\sum_{n=1}^{\infty}a_n z^n,$$

which are analytic in the punctured open unit disc $\mathbb{E}_0=\mathbb{E}\setminus\{0\}$, where

$$\mathbb{E}=\{z\in\mathbb{C}:|z|<1\}.$$

A function $f\in\Sigma$ is said to be meromorphic starlike of order α if and only if

$$-\Re\left(\frac{zf'(z)}{f(z)}\right)>\alpha,(z\in\mathbb{E})$$

for some real α $(0\le\alpha<1)$. The class of such functions is denoted by $\mathcal{MS}^*(\alpha)$. Write $\mathcal{MS}^*=\mathcal{MS}^*(0)$, the class of meromorphic starlike functions i.e. meromorphic functions which satisfy the condition

$$-\Re\left(\frac{zf'(z)}{f(z)}\right)>0,(z\in\mathbb{E}).$$

A function $f \in \Sigma$ is said to be meromorphic close-to-convex of order α if there exists a meromorphic starlike function $g \in \mathcal{MS}^*$ such that

$$-\Re\left(\frac{zf'(z)}{g(z)}\right) > \alpha, (z \in \mathbb{E}).$$

The class of such functions is denoted by $\mathcal{MC}(\alpha)$. Write $\mathcal{MC} = \mathcal{MC}(0)$, the class of meromorphic close-to-convex functions i.e. meromorphic functions which satisfy the condition

$$-\Re\left(\frac{zf'(z)}{g(z)}\right) > 0, (z \in \mathbb{E}) \tag{1.1}$$

where $g \in \mathcal{MS}^*$.

A little calculation yields that the function $g(z) = \dfrac{1}{z}$ is a member of class $\mathcal{MS}^*$. Therefore, the condition (1.1) reduces to the following condition

$$-\Re(z^2 f'(z)) > 0, (z \in \mathbb{E}).$$

Therefore, $f \in \mathcal{MC}$ if $-\Re(z^2 f'(z)) > 0$.

In the literature of meromorphic functions, many authors obtained the conditions for meromorphic close-to-convex functions. Some of the results from literature are given below:

Jing and Li [4] have proved the following results:

Theorem 1.1. *For any $f \in \Sigma$, suppose that for arbitrary α, f satisfies $-z^2 f'(z) \neq \alpha$ and the following inequalities:*

(i) *For the case $0 < \alpha < \frac{1}{2}$*

$$2 + \Re\left(\frac{zf''(z)}{f'(z)}\right) < \frac{\alpha}{2(1-\alpha)},$$

(ii) *For the case $\frac{1}{2} \leq \alpha < 1$*

$$2 + \Re\left(\frac{zf''(z)}{f'(z)}\right) < \frac{1-\alpha}{2\alpha},$$

then $f \in \mathcal{MC}(\alpha)$.

Theorem 1.2. *Let $f \in \Sigma$, suppose that for arbitrary α, f satisfies $-z^2 f'(z) \neq \alpha$ and the following inequality:*

$$1 + \Re\left(\frac{zf''(z)}{f'(z)}\right) \geq \frac{3\alpha - 2}{2(1-\alpha)},$$

then $f \in \mathcal{MC}(\alpha)$.

Goyal and Prajapat [1] proved the following results:

Theorem 1.3. *If $f \in \Sigma$ satisfies the following inequality*

$$\left|\frac{zf''(z)}{f'(z)} - z^2 f'(z) + 1\right| < \frac{(1-\alpha)(3-\alpha)}{2-\alpha} \qquad (0 \leq \alpha < 1),$$

then $f \in \mathcal{MC}(\alpha)$.

Theorem 1.4. *If $f \in \Sigma$ satisfies the following inequality*

$$\left|\frac{zf''(z)}{f'(z)} - z^2f'(z) + 1\right| < \frac{3}{2},$$

then $f \in \mathcal{MC}$.

Theorem 1.5. *If $f \in \Sigma$ satisfies the following inequality*

$$\Re[z^2\{f'(z)(z^2f'(z) - 1) - zf''(z)\}] > -\frac{1}{2},$$

then $f \in \mathcal{MC}$.

Recently Wang and Guo [3] proved the following results:

Theorem 1.6. *Let $f \in \Sigma$ and suppose that there exists a meromorphic starlike function g such that*

$$\Re\left\{\frac{zf'(z)}{g(z)}\left(1 + \frac{zf''(z)}{f'(z)} - \frac{zg'(z)}{g(z)}\right)\right\} > \frac{1}{2}\left(1 + \left|\frac{zf'(z)}{g(z)}\right|^2\right),$$

then $f \in \mathcal{MC}$.

Theorem 1.7. *Let $f \in \Sigma$ and suppose that there exists a meromorphic starlike function g such that*

$$\Re\left\{\frac{zf'(z)}{g(z)}\left(-1 - \frac{zf''(z)}{f'(z)} + \frac{zg'(z)}{g(z)}\right)\right\} > -\frac{1}{4}\left(1 + \left|\frac{zf'(z)}{g(z)}\right|^2\right),$$

then $f \in \mathcal{MC}(\frac{1}{2})$.

Theorem 1.8. *For $f \in \Sigma$, suppose that there exists a meromorphic starlike function g such that*

$$\Re\left\{\frac{zf'(z)}{g(z)}\left(-1 - \frac{zf''(z)}{f'(z)} + \frac{zg'(z)}{g(z)}\right)\right\} > -\frac{1}{2}(1-\alpha), (0 \le \alpha < 1)$$

then $f \in \mathcal{MC}(\alpha)$.

2. Preliminaries

We shall need the following lemma of Miller and Mocanu [2] to prove our main result.

Lemma 2.1. *Let $\mathbb{D}$ be a subset of $\mathbb{C} \times \mathbb{C}$ ($\mathbb{C}$ is the complex plane) and let $\phi : \mathbb{D} \to \mathbb{C}$ be a complex function. For $u = u_1 + iu_2$, $v = v_1 + iv_2$ (u_1, u_2, v_1, v_2 are reals), let ϕ satisfy the following conditions:*
(i) $\phi(u, v)$ is continuous in $\mathbb{D}$;
(ii) $(1, 0) \in \mathbb{D}$ and $\Re\phi(1, 0) > 0$; and
(iii) $\Re\ \{\phi(iu_2, v_1)\} \le 0$ for all $(iu_2, v_1) \in \mathbb{D}$ such that $v_1 \le -(1 + u_2^2)/2$.
Let $p(z) = 1+p_1z+p_2z^2+\ldots$ be regular in the unit disc $\mathbb{E}$ such that $(p(z), zp'(z)) \in \mathbb{D}$ for all $z \in \mathbb{E}$. If

$$\Re[\phi(p(z), zp'(z))] > 0, z \in \mathbb{E},$$

then $\Re\ p(z) > 0$, $z \in \mathbb{E}$.

3. Main theorem

Theorem 3.1. *Let α and β be real numbers such that $\alpha \leq \beta < 1$. If $f \in \Sigma$ satisfies*

$$-\Re\left[(1-\alpha)z^2 f'(z) + \alpha\left(1 + \frac{zf''(z)}{f'(z)}\right)\right] > \beta, z \in \mathbb{E}, \tag{3.1}$$

then $-\Re(z^2 f'(z)) > 0$ in $\mathbb{E}$. So, f is meromorphic close-to-convex in $\mathbb{E}$. The result is sharp in the sense that the constant β on the right hand side of (3.1) cannot be replaced by a real number smaller than α.

Proof. Define a function p by $p(z) = -z^2 f'(z)$ where p is analytic in $\mathbb{E}$. Then,

$$\begin{aligned} &-\left[(1-\alpha)z^2 f'(z) + \alpha\left(1 + \frac{zf''(z)}{f'(z)}\right)\right] \\ &= -\left[(1-\alpha)(-p(z)) + \alpha\left(-1 + \frac{zp'(z)}{p(z)}\right)\right] \end{aligned} \tag{3.2}$$

Thus, condition (3.1) is equivalent to

$$\Re\left[\frac{1-\alpha}{1-\beta}p(z) - \frac{\alpha}{1-\beta}\frac{zp'(z)}{p(z)} + \frac{\alpha-\beta}{1-\beta}\right] > 0, z \in \mathbb{E}. \tag{3.3}$$

If $\mathbb{D} = (\mathbb{C} \setminus \{0\}) \times \mathbb{C}$, define $\phi(u, v) : \mathbb{D} \to \mathbb{C}$ as under:

$$\phi(u, v) = \frac{1-\alpha}{1-\beta}u - \frac{\alpha}{1-\beta}\frac{v}{u} + \frac{\alpha-\beta}{1-\beta}.$$

Then $\phi(u, v)$ is continuous in $\mathbb{D}$, $(1, 0) \in D$ and $\Re(\phi(1, 0)) = 1 > 0$. Further, in view of (3.3),

$$\Re[\phi(p(z), zp'(z))] > 0, \ z \in \mathbb{E}.$$

Let $u = u_1 + iu_2, v = v_1 + iv_2 (u_1, u_2, v_1, v_2$ are real numbers). Then, for $(iu_2, v_1) \in \mathbb{D}$, with $v_1 \leq -\dfrac{1+u_2^2}{2}$, we have

$$\Re[\phi(iu_2, v_1)] = \Re\left[\frac{1-\alpha}{1-\beta}iu_2 - \frac{\alpha}{1-\beta}\frac{v_1}{iu_2} + \frac{\alpha-\beta}{1-\beta}\right] = \frac{\alpha-\beta}{1-\beta} \leq 0.$$

In view of Lemma 2.1, proof now follows.
To show that the constant β on the right side of (3.1) cannot be replaced by a real number smaller than α, we consider the function

$$f_0(z) = \frac{-z - 2\log(1-z)}{z^2},$$

which belongs to the class Σ. A simple calculation gives

$$\begin{aligned} &-\left[(1-\alpha)z^2 f_0'(z) + \alpha\left(1 + \frac{zf_0''(z)}{f_0'(z)}\right)\right] \\ &\qquad = -(1-\alpha)\left[\frac{-z^2 + 3z + 4(1-z)\log(1-z)}{z(1-z)}\right] \\ &\qquad -\alpha\left[\frac{-z^3 + 10z^2 - 7z - 8(1-z)^2\log(1-z)}{z^3 - 4z^2 + 3z + 4(1-z)^2\log(1-z)}\right] \end{aligned}$$

Using Mathematica 7.0, we plot in Figure 3.1, the image of the unit disc $\mathbb{E}$ under the operator

$$-\left[(1-\alpha)z^2 f_0'(z) + \alpha\left(1 + \frac{zf_0''(z)}{f_0'(z)}\right)\right]$$

taking $\alpha = -1$.

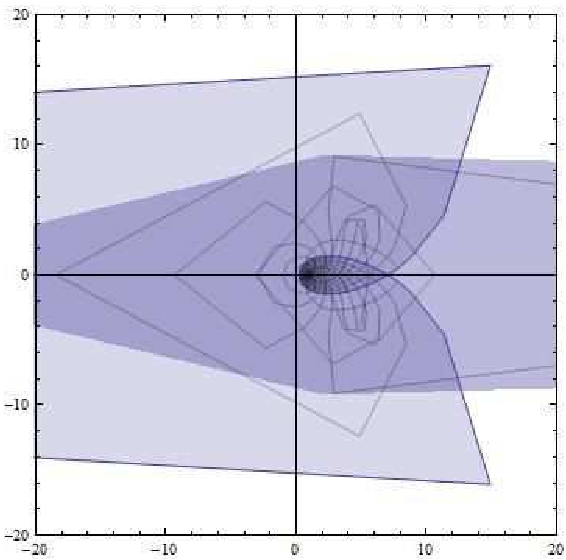

Figure 3.1

From Figure 3.1, we observe that minimum real part of

$$-\left[(1-\alpha)z^2 f_0'(z) + \alpha\left(1 + \frac{zf_0''(z)}{f_0'(z)}\right)\right] \text{ for } \alpha = -1$$

is smaller than -1 (the chosen value of α).

In Figure 3.2, we plot the image of unit disc $\mathbb{E}$ under the function $-z^2 f_0'(z)$.

It is obvious that $-\Re(z^2 f_0'(z)) \not> 0$ for all z in $\mathbb{E}$.

Moreover, the point $z = 0.9$ is an interior point of $\mathbb{E}$, but at this point

$$-\Re(z^2 f_0'(z)) = -10.766... < 0.$$

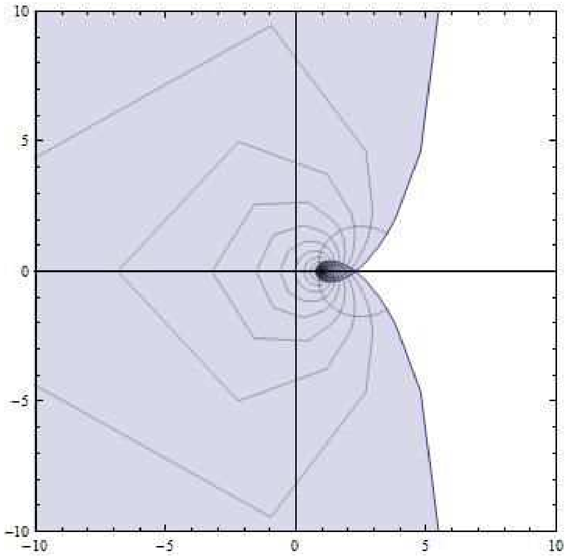

Figure 3.2

This justifies our claim. □

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Inequalities involving Mittag-Leffler type q-Konhauser polynomial

Bharti Vishandas Nathwani

Abstract. In the present work, we propose generalized structure of the q-Konhauser polynomial suggested by a generalized q-Mittag-Leffler function. For this polynomial, we obtain its difference equation and several other properties involving inequalities are also derived which yield as the particular cases, q-analogues of the generating function relations and finite summation formulas.

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Keywords: q-Mittag-Leffler function, q-Konhauser polynomial, series inequalities, difference equation, generating function relation, series inequality relations.

1. Introduction

In 1903, Mittag-Leffler [20] proposed a function $E_\alpha(z)$ defined by

$$E_\alpha(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + 1)},$$

where z is a complex variable and $\alpha \in \mathbb{C}$, $\Re(\alpha) > 0$. Later on this function was referred to as Mittag-Leffler function. The Mittag-Leffler function is direct generalization of the exponential function to which it reduces for $\alpha = 1$. This function has some interesting properties which later became essential for the description of many problems arising in applications. Nowadays the Mittag-Leffler function and its numerous generalizations have acquired a new life. The recent notable increased interest in the study of their relevant properties is due to the close connection to the Fractional Calculus and its application to the study of Differential and Integral Equations. Many modern models of fractional type have recently been proposed in Probability Theory, Mechanics, Mathematical Physics, Chemistry, Biology, Mathematical Economics, Engineering and Applied Sciences etc. There are many applications of Mittag-Leffler function and its generalizations in Astrophysics problems (see [17]). One application of Mittag-Leffler function is described below.

In a reaction-diffusion process if $N(t)$ is the number density at a time t and if the production rate is proportional to original number, then

$$\frac{d}{dt}N(t) \quad = \quad \lambda N(t), \quad \lambda > 0 \tag{1.1}$$

where λ is the rate of production. If the consumption or destruction rate is also proportional to the original number then

$$\frac{d}{dt}N(t) \quad = \quad -\mu N(t), \quad \mu > 0 \tag{1.2}$$

where μ is the destruction rate. Then the residual part is given by

$$\frac{d}{dt}N(t) \quad = \quad -cN(t), \quad c = \mu - \lambda. \tag{1.3}$$

If c is free of t then the solution is exponential model

$$N(t) \quad = \quad N_0 e^{-ct}, \quad N_0 = N(t) \text{ at } t = t_0 \tag{1.4}$$

where t_0 is the starting time. Instead of total derivative in (1.1) to (1.3) if the fractional derivative or fractional nature of reactions is considered, that is, an equation of the form

$$N(t) - N_0 \quad = \quad -c^v \,{}_0D_t^{-v}\, N(t) \tag{1.5}$$

is considered where ${}_0D_t^{-v}$ is the standard Riemann-Liouville fractional integral operator, then the solution

$$N(t) = N_0 \sum_{n=0}^{\infty} \frac{(-1)^k\ (ct)^{vk}}{\Gamma(vk+1)} = N_0\ E_v(-(ct)^v), \tag{1.6}$$

involves $E_v(.)$ which is nothing but the Mittag-Leffler function.
The well known Mittag-Leffler function [20]

$$E_\alpha(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n+1)}, \tag{1.7}$$

where z is a complex variable and $\alpha \in \mathbb{C},\ \Re(\alpha) > 0$ was generalized by Wiman [37] in 1905 in the form:

$$E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n+\beta)}, \quad \Re(\alpha,\beta) > 0, \tag{1.8}$$

which is known as Wiman's function or generalized Mittag-Leffler function.

Note 1.1. $E_{\alpha,1}(z) = E_\alpha(z)$.

In 1971, Prabhakar [25] introduced its extension:

$$E^{\gamma}_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_n}{\Gamma(\alpha n+\beta)} \frac{z^n}{n!}, \tag{1.9}$$

wherein $\Re(\alpha,\beta,\gamma) > 0$.

Note 1.2. $E^{1}_{\alpha,\beta}(z) = E_{\alpha,\beta}(z)$.

In 2007, Shukla and Prajapati [34] introduced the function:

$$E_{\alpha,\beta}^{\gamma,q}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_{qn}}{\Gamma(\alpha n+\beta)} \frac{z^n}{n!}, \tag{1.10}$$

in which $\alpha, \beta, \gamma \in \mathbb{C}$; $\Re(\alpha,\beta,\gamma) > 0$ and $q \in (0,1) \cup \mathbb{N}$.

Note 1.3. $E_{\alpha,\beta}^{\gamma,1}(z) = E_{\alpha,\beta}^{\gamma}(z)$.

As a continuation of these studies, Nathwani, Dave and Prajapati [27, 28, 29, 23, 22, 24] introduced the following function:

Definition 1.4. For $\alpha, \beta, \gamma, \lambda \in \mathbb{C}$, $\Re(\alpha,\beta,\gamma,\lambda) > 0$, $\delta, \mu > 0$, $r \in \{-1,0\} \cup \mathbb{N}$, $s \in \mathbb{N} \cup \{0\}$

$$E_{\alpha,\beta,\lambda,\mu}^{\gamma,\delta}(z;s,r) = \sum_{n=0}^{\infty} \frac{[(\gamma)_{\delta n}]^s}{\Gamma(\alpha n+\beta)\ [(\lambda)_{\mu n}]^r\ n!} z^n. \tag{1.11}$$

Note 1.5. $E_{\alpha,\beta,\lambda,\mu}^{\gamma,q}(z;1,0) = E_{\alpha,\beta}^{\gamma,q}(z)$.

The objective of constructing this function is to
(i) include certain existing generalizations of Mittag-Leffler function,
(ii) also include the functions such as Bessel Maitland function, Dotsenko function, Bessel function, generalized Bessel Maitland function, Lommel function etc. especially by means of parameters r, γ, λ (Table-1 below)
(iii) obtain inverse inequality relations and some other inequalities by means of the integer $'s'$.
Since the time of Wiman (1905), many researchers have proposed and studied various generalizations of the Mittag-Leffler function [20] (see [38], [25], [34], [14], [15], [18], [21], [27], [29], [32], [33], [10]).
The q-analogue of the above generalized Mittag-Leffler function (1.11) is given by Nathwani and Dave [22, 24]:

$$E_{\alpha,\,\beta,\,\lambda,\,\mu}^{\gamma,\delta}(z;s,r|q) \quad = \quad \sum_{n=0}^{\infty} \frac{(-1)^{pn}\ q^{pn(n-1)/2}\ [\Gamma_q(\gamma+\delta n)]^s}{\Gamma_q(\beta+\alpha n)\ [\Gamma_q(\lambda+\mu n)]^r\ (q;q)_n} z^n, \tag{1.12}$$

where $\alpha, \beta, \gamma, \lambda \in \mathbf{C}$ with $\Re(\alpha,\beta,\gamma,\lambda) > 0$, $\delta, \mu > 0$, $r \in \{-1,0\} \cup \mathbf{N}$, $s \in \mathbf{N} \cup \{0\}$, $p = \alpha^2 + r\mu^2 - s\delta^2 + 1$ with $\Re(p) > 0$.
The aim of the present work is to extend the well known q-Konhauser Polynomial [6]

$$Z_m^{\beta}(x;k|q) = \frac{[q^{\beta+1}]_{km}}{(q^k;q^k)_m} \sum_{n=0}^{m} \frac{q^{kn(kn-1)/2+kn(m+\beta+1)}(q^{-mk};q^k)_n\ x^{kn}}{[q^{\beta+1}]_{kn}\ (q^k;q^k)_n},$$

and hence the generalized q-Laguerre polynomial [30]:

$$L_m^{(\beta)}(x|q) = \frac{[q^{\beta+1}]_m}{(q;q)_m} \sum_{n=0}^{m} \frac{q^{n(n+1)/2+n(m+\beta)}(q^{-m};q)_n\ x^n}{[q^{\beta+1}]_n\ (q;q)_n}, \tag{1.13}$$

where $\Re(\beta) > -1$ suggested by the generalized q-Mittag-Leffler function (1.12).
The following definitions and formulas will be used in this work.

For $a \in \mathbf{C}$, and $0 < |q| < 1$, the q-shifted factorial is defined by [13, Eq.(1.2.15), p.3 and Eq.(1.2.30), p.6]

$$(a;q)_n = \begin{cases} 1 & if \quad n = 0 \\ (1-a)(1-aq)\cdots(1-aq^{n-1}) & if \quad n \in \mathbf{N} \\ \dfrac{(q;q)_\infty}{(aq^n;q)_\infty} & if \quad n \in \mathbf{C}, \end{cases} \tag{1.14}$$

where

$$(a;q)_\infty = \prod_{k=0}^{\infty}(1-aq^k)\ , \quad |q| < 1.$$

A well-known extension of the q-shifted factorial is given by [12]

$$[t-|a]_n = (t-a)(t-aq)(t-aq^2)\cdots(t-aq^{n-1}). \tag{1.15}$$

A finite series-product identity [12]

$$\sum_{k=0}^{n} q^{k(k-1)/2} \begin{bmatrix} n \\ k \end{bmatrix} x^k = \prod_{k=1}^{n}(1+xq^{k-1}). \tag{1.16}$$

The q-derivative of a function $f(x)$ is defined by [13, Ex.1.12, p.22]

$$D_q f(x) = \frac{f(x)-f(xq)}{x(1-q)} \tag{1.17}$$

Definition 1.6. A q-Gamma function is defined as ([16], [35]):

$$\Gamma_q(\alpha) \ = \ \frac{(q;q)_\infty \ (1-q)^{1-\alpha}}{(q^\alpha;q)_\infty}, \tag{1.18}$$

where $\alpha \ \neq \ 0,-1,-2,...,$ and $0 < q < 1$.

The q-analogue of Stirling's asymptotic formula [19, Eq.(2.25), p.482] for the q-Gamma function is

$$\Gamma_q(x) \sim (1+q)^{\frac{1}{2}} \ \Gamma_{q^2}\left(\frac{1}{2}\right) \ (1-q)^{\frac{1}{2}-x} \ e^{\mu_q(x)}, \tag{1.19}$$

where $\mu_q(x) = \dfrac{\theta \, q^x}{1-q-q^x}$, $0 < \theta < 1$.

A q-Beta function $\mathbf{B}_q(x,y)$ is expressible in different ways [13].

$$\mathbf{B}_q(x,y) = \int_0^1 t^{x-1}(tq)_{y-1} \, \mathrm{d_q t}, \tag{1.20}$$

$$\mathbf{B}_q(x,y) = \frac{(1-q) \ (q)_\infty \ (q^{x+y})_\infty}{(q^x)_\infty \ (q^y)_\infty}, \tag{1.21}$$

and

$$\mathbf{B}_q(x,y) = \int_0^1 t^{x-1} \frac{(tq;q)_\infty}{(tq^y;q)_\infty} \, \mathrm{d_q t} \tag{1.22}$$

in which $y \neq 0, -1, -2, ...,$ $\Re(x) > 0$.
The q-Euler (Beta) transform is [13]:

$$\mathbf{B}\{f(z) : a,\, b|q\} = \int_0^1 u^{\beta-1} \frac{(uq;q)_\infty}{(uq^\eta;q)_\infty} f(z)\, \mathrm{d_q u}. \tag{1.23}$$

The q-Laplace transform.
Hahn [16] defined the q-analogues of the well known classical Laplace transform:

$$F(S) \;=\; \phi(S) \;=\; \int_0^\infty e^{-St}\, f(t)\, \mathrm{dt},$$

by means of the following two integral equations.

$$\mathcal{L}_q\{f(t)\} \;=\; \frac{1}{(1-q)} \int_0^{S^{-1}} E_q(qSt)\, f(t)\, \mathrm{d_q t}, \tag{1.24}$$

and

$$\mathcal{L}_q\{f(t)\} \;=\; \frac{1}{(1-q)} \int_0^\infty e_q(-St)\, f(t)\, \mathrm{d_q t}, \tag{1.25}$$

where $\Re(S) > 0$.
The q-analogue of Riemann-Liouville fractional integral operator ([4], [39], [31]) is given by

$${}_qI^{\mu}_{a+} f(x) \;=\; \frac{1}{\Gamma_q(\mu)} \int_a^x (x - |yq)_{\mu-1}\, f(y)\, \mathrm{d}_q y, \tag{1.26}$$

where μ is an arbitrary order of integration with $\Re(\mu) > 0$.
In particular, for $f(x) = x^{\nu-1}$, the equation (1.26) reduces to

$${}_qI^{\mu}_{0+} f(x)[x^{\nu-1}] \;=\; \frac{\Gamma_q(\nu)}{\Gamma_q(\nu+\mu)} x^{\nu+\mu-1}. \tag{1.27}$$

The fractional q-differential operator of arbitrary order α, is defined as ([5], [31]):

$$\left({}_qD^{\alpha}_{0+} f\right)(x) \;=\; \frac{1}{\Gamma_q(-\alpha)} \int_0^x (x - |yq)_{-\alpha-1}\, f(y)\, \mathrm{d}_q y, \tag{1.28}$$

in which $\Re(\alpha) < 0,\ 0 < |q| < 1$.
It is to be noted that $\left({}_qD^{\alpha}_{0+} f\right)(x) = D^{\alpha}_{x,q} f(x)$. In this context, we have

$$\left({}_qD^{\alpha}_{a+} f\right)(x) = \left(\frac{d_q}{d_q x}\right)^n \left({}_qI^{n-\alpha}_{a+} f\right)(x). \tag{1.29}$$

If $f(x) = x^{\mu-1}$, then (1.28) reduces to

$${}_qD^{\alpha}_{0+}[x^{\mu-1}] \;=\; \frac{\Gamma_q(\mu)}{\Gamma_q(\mu-\alpha)} x^{\mu-\alpha-1}. \tag{1.30}$$

The generalized Konhauser polynomial due to Prajapati, Ajudia and Agarwal is given by [26]:

$$L_{m^*}^{(k,\sigma)}(x^k) = Z_{m^*}^{\sigma}(x;k) = \frac{\Gamma(km+\sigma+1)}{\Gamma(m+1)} \sum_{n=0}^{m^*} \frac{(-m)_{\delta n}}{\Gamma(kn+\sigma+1)} \frac{x^{kn}}{n!}. \tag{1.31}$$

We propose here generalization of q-Konhauser polynomial.

Definition 1.7. For $\alpha, \beta, \lambda > 0,\ m, \delta, \mu, k, s \in \mathbf{N},\ r \in \mathbf{N} \cup \{0\},\ m^* = [\frac{m}{\delta}]$, the integral part of $\frac{m}{\delta}$, define

$$\begin{aligned} B_{m^*}^{(\alpha,\beta,\lambda,\mu)}(x^k; s, r|q) &= \frac{(q^{\beta+1};q)_{\alpha m}}{[(q^k;q^k)_m]^s} \sum_{n=0}^{m^*} \frac{q^{sk\delta n(m+(\delta nk-1)/2)}\ q^{\delta n(\alpha(\beta+1)+r\mu\lambda)}}{(q^{\beta+1};q)_{\alpha n}\ [(q^\lambda;q)_{\mu n}]^r} \\ &\quad \times \frac{[(q^{-mk};q^k)_{\delta n}]^s\ x^{kn}}{(q^k;q^k)_n}. \end{aligned} \tag{1.32}$$

The polynomial in (1.32) will be referred to as q-GKP.

2. Generalized q-Konhauser polynomial

If $\alpha = k \in \mathbf{N}, s = 1, r = 0$ then (1.32) reduces to q-analogue of another generalization of the Konhauser polynomial (1.31) in the form considered by [26]:

$$\begin{aligned} B_{m^*}^{(k,\beta,\lambda,\mu)}(x^k; 1, 0|q) &= \frac{(q^{\beta+1};q)_{km}}{(q^k;q^k)_m} \sum_{n=0}^{m^*} \frac{q^{k\delta n(k\delta n-1)/2+\delta nm}\ q^{\delta nk(\beta+1)}}{(q^{\beta+1};q)_{\alpha n}} \\ &\quad \times \frac{(q^{-mk};q^k)_{\delta n}\ x^{kn}}{(q^k;q^k)_n} \\ &= Z_{m^*}^{\beta}(x;k|q). \end{aligned} \tag{2.1}$$

A q-analogue of the classical Konhauser polynomial (1.13) is obtained from (2.1) by taking $\delta = 1$, that is

$$\begin{aligned} B_{m}^{(k,\beta,\lambda,\mu)}(x^k; 1, 0|q) &= \frac{(q^{\beta+1};q)_{km}}{(q^k;q^k)_m} \sum_{n=0}^{m} \frac{q^{kn(kn-1)/2+kn(m+\beta+1)}(q^{-mk};q^k)_n\ x^{kn}}{(q^{\beta+1};q)_{kn}\ (q^k;q^k)_n} \\ &= Z_m^{\beta}(x;k|q). \end{aligned} \tag{2.2}$$

Further, with $k = 1$,

$$\begin{aligned} B_{m}^{(1,\beta,\lambda,\mu)}(x; 1, 0|q) &= \frac{(q^{\beta+1};q)_{m}}{(q;q)_m} \sum_{n=0}^{m} \frac{q^{n(n+1)/2+n(m+\beta)}(q^{-m};q)_n\ x^{n}}{(q^{\beta+1};q)_{n}\ (q;q)_n} \\ &= L_m^{(\beta)}(x|q) \end{aligned} \tag{2.3}$$

is a q-analogue of the generalized Laguerre polynomial.

Theorem 2.1. *Let*

$$B_{m^*}^{(\alpha,\beta,\lambda,\mu)}(x^k;s,r|q) = \frac{(q^{\beta+1};q)_{\alpha m}}{[(q^k;q^k)_m]^s}\sum_{n=0}^{m^*}\frac{q^{s(k\delta n(k\delta n-1)/2+k\delta nm)}\ q^{\delta n(\alpha(\beta+1)+r\mu\lambda)}}{(q^{\beta+1};q)_{\alpha n}\ [(q^\lambda;q)_{\mu n}]^r} \times\frac{[(q^{-mk};q^k)_{\delta n}]^s\ x^{kn}}{(q^k;q^k)_n}. \tag{2.4}$$

Then as limit $m\to\infty$, $B_{m^*}^{(\alpha,\beta,\lambda,\mu)}(x^k;s,r|q)$ approaches to the entire function

$$B_{\infty}^{(\alpha,\beta,\lambda,\mu)}(x^k;s,r|q) = \frac{(q^{\beta+1};q)_{\infty}}{[(q^k;q^k)_\infty]^s}\sum_{n=0}^{\infty}\frac{(-1)^{s\delta n}\ q^{s(k\delta n(k\delta n-1)/2+k\delta n(\delta n-1)/2)}}{(q^{\beta+1};q)_{\alpha n}\ [(q^\lambda;q)_{\mu n}]^r} \times\frac{q^{\delta n(\alpha(\beta+1)+r\mu\lambda)}\ x^{kn}}{(q^k;q^k)_n} \tag{2.5}$$

in any bounded domain.

Proof. It will be shown first that the series in (2.5) has an infinite radius of convergence.
Taking

$$\begin{aligned}
\upsilon_n &= \frac{(-1)^{s\delta n}\ q^{s(k\delta n(k\delta n-1)/2+k\delta n(\delta n-1)/2)}}{(q^{\beta+1};q)_{\alpha n}\ [(q^\lambda;q)_{\mu n}]^r}\frac{q^{\delta n(\alpha(\beta+1)+r\mu\lambda)}}{(q^k;q^k)_n}\\
&= \frac{(-1)^{s\delta n}\ q^{s(k\delta n(k\delta n-1)/2+k\delta n(\delta n-1)/2)}\ q^{\delta n(\alpha(\beta+1)+r\mu\lambda)}}{(q^{\beta+1};q)_{\infty}\ [(q^\lambda;q)_{\infty}]^r}\\
&\quad\times\frac{(q^{\alpha n+\beta+1};q)_\infty\ [(q^{\mu n+\lambda};q)_\infty]^r}{(q^k;q^k)_n}.
\end{aligned}$$

Then using D'Albert's Ratio test, the radius of convergence R is given by

$$\begin{aligned}
R &= \lim_{n\to\infty}\left|\frac{\upsilon_n}{\upsilon_{n+1}}\right|\\
&= \lim_{n\to\infty}\left|\frac{(-1)^{s\delta n}\ q^{s(k\delta n(k\delta n-1)/2+k\delta n(\delta n-1)/2)}\ q^{\delta n(\alpha(\beta+1)+r\mu\lambda)}}{(q^{\beta+1};q)_\infty\ [(q^\lambda;q)_\infty]^r\ (q^k;q^k)_n}\right.\\
&\quad\times\frac{(q^{\beta+1};q)_\infty\ [(q^\lambda;q)_\infty]^r\ (q^{\alpha n+\beta+1};q)_\infty\ [(q^{\mu n+\lambda};q)_\infty]^r}{(-1)^{s\delta(n+1)}\ q^{sk\delta(n+1)(\delta(n+1)-1)/2+(k\delta(n+1)-1)/2}}\\
&\quad\left.\times\frac{(q^k;q^k)_{n+1}}{q^{\delta(n+1)(\alpha(\beta+1)+r\mu\lambda)}\ (q^{\alpha(n+1)+\beta+1};q)_\infty\ [(q^{\mu(n+1)+\lambda};q)_\infty]^r}\right|\\
&= \lim_{n\to\infty}\left|\frac{q^{sk\delta-sk^2\delta^2}\ (1-q^{(n+1)k})}{q^{ns\delta^2(k(k+1))}\ q^{\delta(\alpha(\beta+1)+r\mu\lambda)}}\right.\\
&\quad\left.\times\frac{(1-q^{\alpha n+\beta+1})\ (1-q^{\alpha n+\beta+2})\dots(1-q^{\alpha n+\beta+\alpha})}{[(1-q^{\mu n+\lambda+1})\ (1-q^{\mu n+\lambda+2})\dots(1-q^{\mu n+\lambda+\mu})]^{-r}}\right|\\
&= \infty.
\end{aligned}$$

Here it suffices to show that for m sufficiently large,

$$\sum_{n=0}^{m^*} \frac{q^{s(k\delta n(k\delta n-1)/2+k\delta nm)}\ q^{\delta n(\alpha(\beta+1)+r\mu\lambda)}}{(q^{\beta+1};q)_{\alpha n}\ [(q^{\lambda};q)_{\mu n}]^r} \frac{[(q^{-mk};q^k)_{\delta n}]^s\ x^{kn}}{(q^k;q^k)_n} \tag{2.6}$$

tends to

$$\sum_{n=0}^{\infty} \frac{(-1)^{s\delta n}\ q^{s(k\delta n(k\delta n-1)/2+k\delta n(\delta n-1)/2)}}{(q^{\beta+1};q)_{\alpha n}\ [(q^{\lambda};q)_{\mu n}]^r} \frac{q^{\delta n(\alpha(\beta+1)+r\mu\lambda)}\ x^{kn}}{(q^k;q^k)_n}. \tag{2.7}$$

In fact,

$$\begin{aligned}
&\Bigg|\sum_{n=0}^{\infty} \frac{(-1)^{s\delta n}\ q^{s(k\delta n(k\delta n-1)/2+k\delta n(\delta n-1)/2)}}{(q^{\beta+1};q)_{\alpha n}\ [(q^{\lambda};q)_{\mu n}]^r} \frac{q^{\delta n(\alpha(\beta+1)+r\mu\lambda)}\ x^{kn}}{(q^k;q^k)_n} \\
&-\sum_{n=0}^{m^*} \frac{q^{s(k\delta n(k\delta n-1)/2+k\delta nm)}\ q^{\delta n(\alpha(\beta+1)+r\mu\lambda)}}{(q^{\beta+1};q)_{\alpha n}\ [(q^{\lambda};q)_{\mu n}]^r} \frac{[(q^{-mk};q^k)_{\delta n}]^s\ x^{kn}}{(q^k;q^k)_n}\Bigg| \\
= &\Bigg|\sum_{n=0}^{m^*} \left\{q^{sk\delta n(\delta n-1)/2} - [(q^{-mk};q^k)_{\delta n}]^s\ q^{sk\delta nm}\ (-1)^{s\delta n}\right\} \\
&\times \frac{q^{sk\delta n(k\delta n-1)/2}\ q^{\delta n(\alpha(\beta+1)+r\mu\lambda)}\ (-1)^{s\delta n}\ x^{kn}}{(q^{\beta+1};q)_{\alpha n}\ [(q^{\lambda};q)_{\mu n}]^r\ (q^k;q^k)_n}\Bigg| \\
= &\Bigg|\sum_{n=0}^{m^*} \Bigg\{q^{sk\delta n(\delta n-1)/2} - \Big[(1-q^{-mk})\ (1-q^{-mk+k})\ (1-q^{-mk+2k})\dots \\
&\times(1-q^{-mk+(\delta n-1)k})\Big]^s\ q^{sk\delta nm}\ (-1)^{s\delta n}\Bigg\} \\
&\times \frac{q^{sk\delta n(k\delta n-1)/2}\ q^{\delta n(\alpha(\beta+1)+r\mu\lambda)}\ (-1)^{s\delta n}\ x^{kn}}{(q^{\beta+1};q)_{\alpha n}\ [(q^{\lambda};q)_{\mu n}]^r\ (q^k;q^k)_n}\Bigg| \\
= &\Bigg|\sum_{n=0}^{m^*} \Bigg\{q^{sk\delta n(\delta n-1)/2} - \Big[(q^{-mk}-1)\ (q^{-mk+k}-1)\ (q^{-mk+2k}-1)\dots \\
&\times(q^{-mk+(\delta n-1)k}-1)\Big]^s\ q^{sk\delta nm}\Bigg\} \\
&\times \frac{q^{sk\delta n(k\delta n-1)/2}\ q^{\delta n(\alpha(\beta+1)+r\mu\lambda)}\ (-1)^{s\delta n}\ x^{kn}}{(q^{\beta+1};q)_{\alpha n}\ [(q^{\lambda};q)_{\mu n}]^r\ (q^k;q^k)_n}\Bigg|
\end{aligned}$$

$$
\begin{aligned}
&= \Bigg|\sum_{n=0}^{m^*}\Big\{q^{sk\delta n(\delta n-1)/2}-\Big[(1-q^{mk})\ (1-q^{mk-k})\ (1-q^{mk-2k})\ldots \\
&\quad \times(1-q^{mk-(\delta n-1)k})\Big]^s\ q^{sk\delta nm}\ q^{sk\delta n(\delta n-1)/2-sk\delta nm}\Big\} \\
&\quad \times\frac{q^{sk\delta n(k\delta n-1)/2}\ q^{\delta n(\alpha(\beta+1)+r\mu\lambda)}\ (-1)^{s\delta n}\ x^{kn}}{(q^{\beta+1};q)_{\alpha n}\ [(q^\lambda;q)_{\mu n}]^r\ (q^k;q^k)_n}\Bigg| \\
&\le \sum_{n=0}^{m^*}\Bigg|q^{sk\delta n(\delta n-1)/2}-\Big[(1-q^{mk})\ (1-q^{mk-k})\ (1-q^{mk-2k})\ldots \\
&\quad \times(1-q^{mk-(\delta n-1)k})\Big]^s\ q^{sk\delta n(\delta n-1)/2}\Bigg| \\
&\quad \times\frac{q^{sk\delta n(k\delta n-1)/2}\ q^{\delta n(\alpha(\beta+1)+r\mu\lambda)}\ |x|^{kn}}{(q^{\beta+1};q)_{\alpha n}\ [(q^\lambda;q)_{\mu n}]^r\ (q^k;q^k)_n}.
\end{aligned}
\tag{2.8}
$$

The absolute difference may be simplified with the aid of the inequality

$$\prod_{j=1}^{k}(1-x_j)\ge 1-\sum_{j=1}^{k}x_j,\quad 0\le x_j\le 1,\ j=1,2,\ldots,k,$$

to get

$$
\begin{aligned}
&\Bigg|q^{sk\delta n(\delta n-1)/2}-\Big[(1-q^{mk})\ (1-q^{mk-k})\ (1-q^{mk-2k})\ldots \\
&\times(1-q^{mk-(\delta n-1)k})\Big]^s\ q^{sk\delta n(\delta n-1)/2}\Bigg| \\
=\ & q^{sk\delta n(\delta n-1)/2} \\
&\times\Bigg|1-\Big[(1-q^{mk})\ (1-q^{mk-k})\ (1-q^{mk-2k})\ldots(1-q^{mk-(\delta n-1)k})\Big]^s\Bigg| \\
=\ & q^{sk\delta n(\delta n-1)/2}\left|1-\left[\prod_{j=1}^{\delta n}(1-q^{mk-jk+k})\right]^s\right| \\
\le\ & q^{sk\delta n(\delta n-1)/2}\left|1-\left(1-\sum_{j=1}^{\delta n}q^{mk-jk+k}\right)^s\right| \\
\le\ & q^{sk\delta n(\delta n-1)/2}\left|\sum_{j=1}^{\delta n}q^{mk-jk+k}\right|^s
\end{aligned}
$$

$$
\begin{aligned}
&= q^{sk\delta n(\delta n-1)/2}\left(\sum_{j=1}^{\delta n} q^{mk-jk+k}\right)^s \\
&= q^{sk\delta n(\delta n-1)/2+skm}\left(\sum_{j=0}^{\delta n-1} q^{-jk}\right)^s \\
&= q^{sk\delta n(\delta n-1)/2+skm}\frac{(1-q^{-\delta nk})^s}{(1-q^{-k})^s} \\
&= q^{sk\delta n(\delta n-1)/2-sk\delta n+smk+sk}\frac{(1-q^{\delta nk})^s}{(1-q^{k})^s} \\
&\leq \frac{q^{sk\delta n(\delta n-1)/2-sk\delta n+smk+sk}}{(1-q^k)^s},
\end{aligned}
$$

because $\delta n \leq m$. Therefore,

$$
\begin{aligned}
&\Big| q^{sk\delta n(\delta n-1)/2} - \Big[(1-q^{mk})\,(1-q^{mk-k})\,(1-q^{mk-2k})\ldots \\
&\times(1-q^{mk-(\delta n-1)k})\Big]^s\, q^{sk\delta n(\delta n-1)/2}\Big| \\
\leq\;& \frac{q^{sk\delta n(\delta n-1)/2-sn\delta k+smk+sk}}{(1-q^k)^s}. \qquad (2.9)
\end{aligned}
$$

This last inequality is valid for all non negative values of δn. Substituting this into (2.8), one gets

$$
\begin{aligned}
&\Bigg| \sum_{n=0}^{\infty} \frac{(-1)^{s\delta n}\, q^{s(k\delta n(k\delta n-1)/2+k\delta n(\delta n-1)/2)}}{(q^{\beta+1};q)_{\alpha n}\,[(q^{\lambda};q)_{\mu n}]^r}\frac{q^{\delta n(\alpha(\beta+1)+r\mu\lambda)}\, x^{kn}}{(q^k;q^k)_n} \\
&-\sum_{n=0}^{m^*} \frac{q^{s(k\delta n(k\delta n-1)/2+k\delta nm)}\, q^{\delta n(\alpha(\beta+1)+r\mu\lambda)}}{(q^{\beta+1};q)_{\alpha n}\,[(q^{\lambda};q)_{\mu n}]^r}\frac{[(q^{-mk};q^k)_{\delta n}]^s\, x^{kn}}{(q^k;q^k)_n}\Bigg| \\
\leq\;& \frac{q^{smk+sk}}{(1-q^k)^s}\sum_{n=0}^{\infty}\frac{q^{sk\delta n(\delta n-1)/2-sn\delta k}q^{sk\delta n(k\delta n-1)/2}\, q^{\delta n(\alpha(\beta+1)+r\mu\lambda)}\, |x|^{kn}}{(q^{\beta+1};q)_{\alpha n}\,[(q^{\lambda};q)_{\mu n}]^r\,(q^k;q^k)_n} \qquad (2.10)
\end{aligned}
$$

Thus the last series (2.10) has an infinite radius of convergence and is therefore bounded in every bounded domain. It follows that the left hand side in (2.8) $\to 0$ as $n \to \infty$ uniformly in any bounded domain. Hence the series (2.6) converges to (2.7) uniformly on any bounded domain. □

3. Difference equations

The operators considered in this section are listed below.

$$
\Lambda_q f(x) = f(x) - f(xq^{-1}), \quad \Theta f(x) = f(x) - f(xq),
$$

$$\mathcal{D}_q\ f(x) = (1-q)\ D_q f(x) := (1-q)\ \frac{f(x)-f(xq)}{x-xq} = \frac{f(x)-f(xq)}{x},$$

$$\frac{\left\{\prod_{u=0}^{a-1}\prod_{v=0}^{a-1}[\Theta + c^{-u}q^{1-(b+v)/a} - 1]^m\right\}}{\left\{\prod_{u=0}^{a-1}\prod_{v=0}^{a-1}[c^{-u}q^{1-(b+v)/a}]^m\right\}} = \Phi_{u,v}^{(a,b,c;m)}$$

and

$$\frac{\left\{\prod_{u=0}^{a-1}\prod_{v=0}^{a-1}[\Theta + c^{-u}q^{(b+v)/a} - 1]^m\right\}}{\left\{\prod_{u=0}^{a-1}\prod_{v=0}^{a-1}[c^{-u}q^{-(b+v)/a}]^m\right\}} = \Psi_{u,v}^{(a,b,c;m)}.$$

In the notations of these operators, the difference equation satisfied by the polynomial $B_{m^*}^{(\alpha,\beta,\lambda,\mu)}(x^k; s, r|q)$ is derived in the following theorem.

Theorem 3.1. *Let $\alpha,\ \beta,\ \lambda,\ m,\ \delta,\ \mu,\ k,\ s \in \mathbf{N}$, $r \in \mathbf{N}\cup\{0\}$, $m^* = [\frac{m}{\delta}]$ then $B_{m^*}^{(\alpha,\beta,\lambda,\mu)}(x^k; s, r|q)$ satisfies the equation*

$$\begin{aligned}&\left[\Phi_{\ell,\kappa}^{(\mu,\lambda,\eta;r)}\ \Phi_{h,g}^{(\alpha,\beta+1,\sigma;1)}\ \Theta\right] B_{m^*}^{(\alpha,\beta,\lambda,\mu)}(x^k; s, r|q)\\ &-x^k\ q^{s(k\delta(k\delta-1)/2)+sk\delta m}\ \Psi_{j,i}^{(\delta k,-mk,\chi;s)}\ B_{m^*}^{(\alpha,\beta,\lambda,\mu)}(x^k q^{s(k\delta)^2}; s, r|q) = 0, \quad (3.1)\end{aligned}$$

where χ is $(\delta k)^{th}$ root of unity, η is μ^{th} root of unity, σ is α^{th} root of unity.

Proof. The coefficient of x^{nk} in (1.32) will be first expressed in q-factorial notation with the aid of the formulas [13, Appendix I]:

$$(a;q)_{kn} = (a, aq, \ldots, aq^{k-1}; q^k)_n,$$

$$(a^k;q^k)_n = (a, a\omega_k, \ldots, a\omega_k^{k-1}; q^k)_n\ \ ;\omega_k = e^{(2\pi i)/k},$$

$$(A;q^n)_{\nu k} = (A^{1/n};q)_{\nu k}(A^{1/n}\omega;q)_{\nu k}\ldots(A^{1/n}\omega^{n-1};q)_{\nu k},\ \omega^n = 1,$$

and

$$(q^\gamma;q^\delta)_n = (q^{\gamma/\delta};q)_n\ (\varpi q^{\gamma/\delta};q)_n \ldots (\varpi^{\delta-1}q^{\gamma/\delta};q)_n = \prod_{i=0}^{\delta-1}(\varpi^i q^{\gamma/\delta};q)_n,\ \varpi^\delta = 1.$$

Now if

$$\begin{aligned}&\sum_{n=0}^{[n/\delta]}\left\{\prod_{j=0}^{\delta k-1}\ \prod_{i=0}^{\delta k-1}[(\chi^j\ q^{(-mk+i)/(\delta k)};q)_n]^s\right\}\left\{\prod_{\ell=0}^{\mu-1}\prod_{\kappa=0}^{\mu-1}[(\eta^\ell\ q^{(\lambda+\kappa)/\mu};q)_n]^r\right\}^{-1}\\ &\times\frac{q^{s(k\delta n(k\delta n-1)/2+k\delta nm)}\ q^{\delta n(\alpha(\beta+1)+r\mu\lambda)}}{\left\{\prod_{h=0}^{\alpha-1}\prod_{g=0}^{\alpha-1}(\sigma^h\ q^{(\beta+g)/\alpha};q)_n\right\}}\frac{x^{nk}}{(q^k;q^k)_n} = \mathcal{Y}, \quad (3.2)\end{aligned}$$

$$\prod_{j=0}^{\delta k-1}\prod_{i=0}^{\delta k-1}[(\chi^j\ q^{(-mk+i)/(\delta k)};q)_n]^s=\mathcal{H}_n,\ \prod_{\ell=0}^{\mu-1}\prod_{k=0}^{\mu-1}[(\eta^\ell\ q^{(\lambda+k)/\mu};q)_n]^r=\mathcal{B}_n,$$

$$\prod_{h=0}^{\alpha-1}\prod_{g=0}^{\alpha-1}(\sigma^h\ q^{(\beta+g)/\alpha};q)_n=\mathcal{C}_n,\ q^{s(k\delta n(k\delta n-1)/2+k\delta nm)}\ q^{\delta n(\alpha(\beta+1)+r\mu\lambda)}=\mathcal{G}_n,$$

then (3.2) will assume the elegant form:

$$\mathcal{Y}=\sum_{n=0}^{[n/\delta]}\frac{\mathcal{H}_n\ \mathcal{G}_n}{\mathcal{B}_n\ \mathcal{C}_n\ (q^k;q^k)_n}x^{nk}.$$

Now

$$\Theta\mathcal{Y}\ =\ \sum_{n=0}^{[n/\delta]}\frac{\mathcal{H}_n\ \mathcal{G}_n}{\mathcal{B}_n\ \mathcal{C}_n\ (q^k;q^k)_n}\Theta x^{nk}=\sum_{n=1}^{[n/\delta]}\frac{\mathcal{H}_n\ \mathcal{G}_n}{\mathcal{B}_n\ \mathcal{C}_n}\frac{x^{nk}}{(q^k;q^k)_{n-1}}.$$

Next operating by $\Phi_{h,g}^{(\alpha,\beta,\sigma;1)}$, one gets

$$\begin{aligned}\Phi_{h,g}^{(\alpha,\beta,\sigma;1)}\Theta\mathcal{Y}\ &=\ \sum_{n=1}^{[n/\delta]}\frac{\mathcal{H}_n\ \mathcal{G}_n}{\mathcal{B}_n\ (q^k;q^k)_{n-1}}\frac{\left\{\prod_{h=0}^{\alpha-1}\prod_{g=0}^{\alpha-1}(\Theta+\sigma^{-h}q^{1-(\beta+g)/\alpha}-1)\right\}}{\left\{\prod_{h=0}^{\alpha-1}\prod_{g=0}^{\alpha-1}(\sigma^{-h}q^{1-(\beta+g)/\alpha})\right\}}\\&\quad\times\left\{\prod_{h=0}^{\alpha-1}\prod_{g=0}^{\alpha-1}(\sigma^h q^{(\beta+g)/\alpha};q)_n\right\}^{-1}x^{nk}\\&=\ \sum_{n=1}^{[n/\delta]}\frac{\mathcal{H}_n\ \mathcal{G}_n}{\mathcal{B}_n\ (q^k;q^k)_{n-1}}\frac{\left\{\prod_{h=0}^{\alpha-1}\prod_{g=0}^{\alpha-1}(1-\sigma^h q^{n-1+(\beta+g)/\alpha})\right\}}{\left\{\prod_{h=0}^{\alpha-1}\prod_{g=0}^{\alpha-1}(\sigma^h q^{(\beta+g)/\alpha};q)_n\right\}}x^{nk}\\&=\ \sum_{n=1}^{[n/\delta]}\frac{\mathcal{H}_n\ \mathcal{G}_n}{\mathcal{B}_n\ \mathcal{C}_{n-1}\ (q^k;q^k)_{n-1}}x^{nk}.\end{aligned}$$

Finally,

$$\begin{aligned}&\Phi_{\ell,\kappa}^{(\mu,\lambda,\eta;r)}\Phi_{h,g}^{(\alpha,\beta,\sigma;1)}\Theta\mathcal{Y}\\&=\ \sum_{n=1}^{[n/\delta]}\frac{\mathcal{H}_n\ \mathcal{G}_n}{\mathcal{C}_{n-1}\ (q^k;q^k)_{n-1}}\frac{\left\{\prod_{\ell=0}^{\mu-1}\prod_{\kappa=0}^{\mu-1}[(\Theta+\eta^{-\ell}q^{1-(\lambda+\kappa)/\mu}-1)]^r\right\}}{\left\{\prod_{\ell=0}^{\mu-1}\prod_{\kappa=0}^{\mu-1}[(\eta^{-\ell}q^{1-(\lambda+\kappa)/\mu})]^r\right\}}\\&\quad\times\left\{\prod_{\ell=0}^{\mu-1}\prod_{\kappa=0}^{\mu-1}[(\eta^\ell\ q^{(\lambda+\kappa)/\mu};q)_n]^r\right\}^{-1}x^{nk}\end{aligned}$$

$$= \sum_{n=1}^{[n/\delta]} \frac{\mathcal{H}_n\ \mathcal{G}_n}{\mathcal{C}_{n-1}\ (q^k;q^k)_{n-1}} \frac{\left\{\prod_{\ell=0}^{\mu-1}\prod_{\kappa=0}^{\mu-1}[(-q^n+\eta^{-\ell}q^{1-(\lambda+\kappa)/\mu})]^r\right\}}{\left\{\prod_{\ell=0}^{\mu-1}\prod_{\kappa=0}^{\mu-1}[(\eta^{-\ell}q^{1-(\lambda+\kappa)/\mu})]^r\right\}}$$

$$\times\left\{\prod_{\ell=0}^{\mu-1}\prod_{\kappa=0}^{\mu-1}[(\eta^\ell\ q^{(\lambda+\kappa)/\mu};q)_n]^r\right\}^{-1} x^{nk}$$

$$= \sum_{n=1}^{[n/\delta]} \frac{\mathcal{H}_n\ \mathcal{G}_n}{\mathcal{B}_{n-1}\ \mathcal{C}_{n-1}\ (q^k;q^k)_{n-1}} x^{nk}.$$

Thus,

$$\Phi_{\ell,\kappa}^{(\mu,\lambda,\eta;r)}\Phi_{h,g}^{(\alpha,\beta,\sigma;1)}\Theta\mathcal{Y} = \sum_{n=0}^{[n/\delta]} \frac{\mathcal{H}_{n+1}\ \mathcal{G}_{n+1}}{\mathcal{B}_n\ \mathcal{C}_n\ (q^k;q^k)_n} x^{nk+k}. \tag{3.3}$$

Further,

$$\Psi_{j,i}^{(\delta k,-mk,\chi;s)}\ B_{m^*}^{(\alpha,\beta,\lambda,\mu)}(x^k q^{s(k\delta)^2};s,r|q)$$

$$= \sum_{n=0}^{[n/\delta]} \frac{\mathcal{H}_n\ \mathcal{G}_n\ q^{s(k\delta)^2 n}}{\mathcal{B}_n\ \mathcal{C}_n\ (q^k;q^k)_n} \frac{\left\{\prod_{j=0}^{\delta k-1}\prod_{i=0}^{\delta k-1}[(\Theta+\chi^{-j}q^{-(-mk+i)/(\delta k)}-1)]^s\right\}}{\left\{\prod_{j=0}^{\delta k-1}\prod_{i=0}^{\delta k-1}[(\chi^{-j}q^{-(-mk+i)/(\delta k)})]^s\right\}} x^{nk}$$

$$= \sum_{n=0}^{[n/\delta]} \frac{\mathcal{G}_n\ q^{s(k\delta)^2}}{\mathcal{B}_n\ \mathcal{C}_n\ (q^k;q^k)_n} \left\{\prod_{j=0}^{\delta k-1}\prod_{i=0}^{\delta k-1}[(\chi^j\ q^{(-mk+i)/(\delta k)};q)_n]^s\right\}$$

$$\times\left\{\prod_{j=0}^{\delta k-1}\prod_{i=0}^{\delta k-1}[(1-\chi^j q^{n+(-mk+i)/(\delta k)})]^s\right\} x^{nk},$$

and hence

$$x^k\ q^{s(k\delta(k\delta-1)/2)+sk\delta m}\ \Psi_{j,i}^{(\delta k,-mk,\chi;s)}\ B_{m^*}^{(\alpha,\beta,\lambda,\mu)}(x^k q^{s(k\delta)^2};s,r|q)$$

$$= \sum_{n=0}^{[n/\delta]} \frac{\mathcal{H}_{n+1}\ \mathcal{G}_{n+1}}{\mathcal{B}_n\ \mathcal{C}_n\ (q^k;q^k)_n} x^{nk+k}. \tag{3.4}$$

The equation (3.1) now follows by comparing (3.3) and (3.4). □

4. Generating function inequality

With the motivation of work done by ([36], [9], [7], [8], [1], [2], [3]) on inequalities, in this section certain inequalities containing q-GKP are obtained.

Theorem 4.1. *If $\alpha, \beta, \lambda > 0$, $m, \delta, \mu, k, s \in \mathbf{N}$, $r \in \mathbf{N} \cup \{0\}$, $0 < st < 1$ then the following series inequality holds.*

$$\sum_{m=0}^{\infty} \frac{B_{m^*}^{(\alpha,\beta,\lambda,\mu)}(x^k; s, r|q)}{(q^{\beta+1}; q)_{\alpha m}}\, t^{ms} \leq \left(e_{q^k}(t)\right)^s \frac{[(q^k; q^k)_\infty]^s}{(q^{\beta+1}; q)_\infty} \times B_{\infty}^{(\alpha,\beta,\lambda,\mu)}(x^k\, t^{s\delta}; s, r|q). \tag{4.1}$$

Proof. From left hand side of (4.1),

$$\begin{aligned}
&\sum_{m=0}^{\infty} \frac{B_{m^*}^{(\alpha,\beta,\lambda,\mu)}(x^k; s, r|q)}{(q^{\beta+1}; q)_{\alpha m}}\, t^{ms} \\
&= \sum_{m=0}^{\infty} \frac{1}{(q^{\beta+1}; q)_{\alpha m}} \frac{(q^{\beta+1}; q)_{\alpha m}}{[(q^k; q^k)_m]^s} \\
&\quad \times \sum_{n=0}^{m^*} \frac{q^{sk\delta n(m+(\delta nk-1)/2)}\, q^{\delta n(\alpha(\beta+1)+r\mu\lambda)}\, [(q^{-mk}; q^k)_{\delta n}]^s\, x^{kn}}{(q^{\beta+1}; q)_{\alpha n}\, [(q^\lambda; q)_{\mu n}]^r\, (q^k; q^k)_n}\, t^{ms} \\
&= \sum_{m=0}^{\infty} \sum_{n=0}^{m^*} \frac{q^{sk\delta n(m+(\delta nk-1)/2)}\, q^{\delta n(\alpha(\beta+1)+r\mu\lambda)}\, x^{kn}}{[(q^k; q^k)_m]^s\, (q^{\beta+1}; q)_{\alpha n}\, [(q^\lambda; q)_{\mu n}]^r\, (q^k; q^k)_n} \\
&\quad \times \frac{(-1)^{s\delta n}\, q^{sk\delta n(\delta n-1)/2 - skm\delta n}\, [(q^k; q^k)_m]^s}{[(q^k; q^k)_{(m-\delta n)}]^s}\, t^{ms} \\
&= \sum_{m=0}^{\infty} \sum_{n=0}^{m^*} \frac{(-1)^{s\delta n}\, q^{sk\delta n(k\delta n-1)/2 + sk\delta n(\delta n-1)/2}\, q^{\delta n(\alpha(\beta+1)+r\mu\lambda)}\, x^{kn}}{(q^{\beta+1}; q)_{\alpha n}\, [(q^\lambda; q)_{\mu n}]^r\, (q^k; q^k)_n} \\
&\quad \times \frac{t^{ms}}{[(q^k; q^k)_{(m-\delta n)}]^s} \\
&= \sum_{m=0}^{\infty} \frac{t^{ms}}{[(q^k; q^k)_m]^s} \sum_{n=0}^{\infty} \frac{(-1)^{s\delta n}\, q^{sk\delta n(k\delta n-1)/2 + sk\delta n(\delta n-1)/2}\, q^{\delta n(\alpha(\beta+1)+r\mu\lambda)}}{(q^{\beta+1}; q)_{\alpha n}\, [(q^\lambda; q)_{\mu n}]^r} \\
&\quad \times \frac{t^{s\delta n}\, x^{kn}}{(q^k; q^k)_n}
\end{aligned}$$

Here the inner sum is obtained by making limit $m \to \infty$ in

$$\begin{aligned}
&B_{m^*}^{(\alpha,\beta,\lambda,\mu)}(x^k\, t^{s\delta}; s, r|q) \\
&= \frac{(q^{\beta+1}; q)_{\alpha m}}{[(q^k; q^k)_m]^s} \sum_{n=0}^{m^*} \frac{t^{s\delta n} q^{sk\delta n(m+(\delta nk-1)/2)}\, q^{\delta n(\alpha(\beta+1)+r\mu\lambda)}\, [(q^{-mk}; q^k)_{\delta n}]^s x^{kn}}{(q^{\beta+1}; q)_{\alpha n}\, [(q^\lambda; q)_{\mu n}]^r\, (q^k; q^k)_n},
\end{aligned}$$

and since $0 < t < 1$,

$$\begin{aligned}
\sum_{m=0}^{\infty} \frac{B_{m^*}^{(\alpha,\beta,\lambda,\mu)}(x^k; s, r|q)}{(q^{\beta+1}; q)_{\alpha m}}\, t^{ms} &= \sum_{m=0}^{\infty} \frac{t^{ms}}{[(q^k; q^k)_m]^s} \frac{[(q^k; q^k)_\infty]^s}{(q^{\beta+1}; q)_\infty}\, B_{\infty}^{(\alpha,\beta,\lambda,\mu)}(x^k\, t^{s\delta}; s, r|q) \\
&\leq \left(\sum_{m=0}^{\infty} \frac{t^m}{(q^k; q^k)_m}\right)^s \frac{[(q^k; q^k)_\infty]^s}{(q^{\beta+1}; q)_\infty}\, B_{\infty}^{(\alpha,\beta,\lambda,\mu)}(x^k\, t^{s\delta}; s, r|q)
\end{aligned}$$

$$= \big(e_{q^k}\,(t)\big)^s\; \frac{[(q^k;q^k)_\infty]^s}{(q^{\beta+1};q)_\infty}\; B_\infty^{(\alpha,\beta,\lambda,\mu)}(x^k\,t^{s\delta};s,r|q). \qquad \square$$

4.1. Special cases - Generating function relations

For $s=1$, the series inequality relations in Theorem 4.1 will yield the generating function relation. Their various specializations are deduced here.
(i) Taking $\alpha = k \in \mathbf{N},\ r=0$ in (4.1) leads to

$$\sum_{m=0}^{\infty} \frac{L_{m^*}^{(k,\beta)}(x^k|q)}{(q^{\beta+1};q)_{km}}\, t^m \;=\; e_{q^k}\,(t)\; \frac{(q^k;q^k)_\infty}{(q^{\beta+1};q)_\infty}\; L_\infty^{(k,\beta)}(x^k\,t^\delta|q).$$

Further the case $\delta=1$, gives

$$\sum_{m=0}^{\infty} \frac{Z_m^{\beta}(x;k|q)}{(q^{\beta+1};q)_{km}}\, t^m \;=\; e_{q^k}\,(t)\; \frac{(q^k;q^k)_\infty}{(q^{\beta+1};q)_\infty}\; Z_\infty^{\beta}(x\,t^{\frac{1}{k}};k|q).$$

Finally, for $k=1$ this reduces to (cf. [30, Eq. (1), p. 201])

$$\sum_{m=0}^{\infty} \frac{L_m^{(\beta)}(x|q)}{(q^{\beta+1};q)_m}\, t^m \;=\; e_q\,(t)\; \frac{(q;q)_\infty}{(q^{\beta+1};q)_\infty}\; L_\infty^{(\beta)}(xt|q).$$

4.2. Special cases-Inequalities

If $\alpha = k \in \mathbf{N}$ in (4.1) then

$$\sum_{m=0}^{\infty} \frac{B_{m^*}^{(k,\beta,\lambda,\mu)}(x^k;s,r|q)}{(q^{\beta+1};q)_{km}}\, t^{ms} \le \big(e_{q^k}\,(t)\big)^s\, \frac{[(q^k;q^k)_\infty]^s}{(q^{\beta+1};q)_\infty}\; B_\infty^{(k,\beta,\lambda,\mu)}(x^k\,t^{s\delta};s,r|q).$$

This will be used in the next section.
Further for $\delta=1, r=0$, this reduces to

$$\sum_{m=0}^{\infty} \frac{Z_{m,s}^{\beta}(x^k|q)}{(q^{\beta+1};q)_{km}}\, t^{ms} \;\le\; \big(e_{q^k}\,(t)\big)^s\; \frac{[(q^k;q^k)_\infty]^s}{(q^{\beta+1};q)_\infty}\; Z_\infty^{\beta}(x^k\,t^s;|q).$$

Consequently, the generalized Laguerre polynomial case $k=1$, is

$$\sum_{m=0}^{\infty} \frac{L_{m,s}^{(\beta)}(x|q)}{(q^{\beta+1};q)_m}\, t^{ms} \;\le\; (e_q\,(t))^s\; \frac{[(q;q)_\infty]^s}{(q^{\beta+1};q)_\infty}\; L_{\infty,s}^{(\beta)}(x\,t^s|q).$$

Here

$$\begin{aligned} Z_{m,s}^{\beta}(x^k|q) \;&=\; \frac{(q^{\beta+1};q)_{km}}{[(q^k;q^k)_m]^s} \sum_{n=0}^{m} \frac{q^{s(kn(kn-1)/2+knm)}\; q^{n(k(\beta+1))}}{(q^{\beta+1};q)_{kn}} \\ &\quad \times \frac{[(q^{-mk};q^k)_n]^s\; x^{kn}}{(q^k;q^k)_n}, \quad \Re(\beta) > -1, \end{aligned} \tag{4.2}$$

is q-extended Konhauser polynomial. And

$$L_{m,s}^{(\beta)}(x|q) \;=\; Z_{m,s}^{\beta}(x|q).$$

is q-extended Laguerre polynomial.

5. Finite q-series inequality

In this section, the inequality below involves finite q-series and q-GKP.

Theorem 5.1. *If $\beta, \lambda > 0,\ m, \delta, \mu, k, s \in \mathbf{N},\ r \in \mathbf{N} \cup \{0\},$ then*

$$B_{m^*}^{(k,\beta,\lambda,\mu)}(x^k; s, r|q) \leq (q^{\beta+1}; q)_{km} \left(\frac{x}{y}\right)^{\frac{km}{\delta}} \sum_{j=0}^{m} \frac{(-1)^j\ q^{kj(j-1)/2}}{(q^k; q^k)_j} \times \left(\left(\frac{y}{x}\right)^{\frac{k}{\delta}} q^{k(-j-s+1)}; q^k\right)_j \frac{B_{(m-j)^*}^{(k,\beta,\lambda,\mu)}(y^k; s, r|q)}{(q^{\beta+1}; q)_{k(m-j)}}. \tag{5.1}$$

Proof. From the inequality (4.2), one gets

$$\sum_{m=0}^{\infty} \frac{B_{m^*}^{(k,\beta,\lambda,\mu)}(x^k; s, r|q)}{(q^{\beta+1}; q)_{km}}\ t^{ms} \quad \leq \quad \left(e_{q^k}(t)\right)^s \frac{[(q^k; q^k)_\infty]^s}{(q^{\beta+1}; q)_\infty}\ B_\infty^{(k,\beta,\lambda,\mu)}(x^k\ t^{s\delta}; s, r|q).$$

With $t = \left(\frac{y}{k}\right)^{\frac{k}{s\delta}} w$, it gives

$$\sum_{m=0}^{\infty} \frac{B_{m^*}^{(k,\beta,\lambda,\mu)}(x^k; s, r|q)}{(q^{\beta+1}; q)_{km}} \left(\frac{y}{k}\right)^{\frac{km}{\delta}} w^{ms} \leq \left(e_{q^k}\left(\left(\frac{y}{k}\right)^{\frac{k}{s\delta}} w\right)\right)^s \frac{[(q^k; q^k)_\infty]^s}{(q^{\beta+1}; q)_\infty}\ B_\infty^{(k,\beta,\lambda,\mu)}\left(\left(\frac{xy}{k}\right)^k w^{s\delta}; s, r|q\right).$$

Hence,

$$\sum_{m=0}^{\infty} \frac{B_{m^*}^{(k,\beta,\lambda,\mu)}(x^k; s, r|q)}{(q^{\beta+1}; q)_{km}} \left(\frac{y}{k}\right)^{\frac{km}{\delta}} w^{ms} \left(e_{q^k}\left(\left(\frac{y}{k}\right)^{\frac{k}{s\delta}} w\right)\right)^{-s} \leq \frac{[(q^k; q^k)_\infty]^s}{(q^{\beta+1}; q)_\infty}\ B_\infty^{(k,\beta,\lambda,\mu)}\left(\left(\frac{xy}{k}\right)^k w^{s\delta}; s, r|q\right). \tag{5.2}$$

Now interchanging the role of x and y in (5.2), it yields

$$\sum_{m=0}^{\infty} \frac{B_{m^*}^{(k,\beta,\lambda,\mu)}(y^k; s, r|q)}{(q^{\beta+1}; q)_{km}} \left(\frac{x}{k}\right)^{\frac{km}{\delta}} w^{ms} \left(e_{q^k}\left(\left(\frac{x}{k}\right)^{\frac{k}{s\delta}} w\right)\right)^{-s} \leq \frac{[(q^k; q^k)_\infty]^s}{(q^{\beta+1}; q)_\infty}\ B_\infty^{(k,\beta,\lambda,\mu)}\left(\left(\frac{xy}{k}\right)^k w^{s\delta}; s, r|q\right). \tag{5.3}$$

Here from (5.2) and (5.3), either

$$\sum_{m=0}^{\infty} \frac{B_{m^*}^{(k,\beta,\lambda,\mu)}(x^k; s, r|q)}{(q^{\beta+1}; q)_{km}} \left(\frac{y}{k}\right)^{\frac{km}{\delta}} w^{ms} \left(e_{q^k}\left(\left(\frac{y}{k}\right)^{\frac{k}{s\delta}} w\right)\right)^{-s} \leq \sum_{m=0}^{\infty} \frac{B_{m^*}^{(k,\beta,\lambda,\mu)}(y^k; s, r|q)}{(q^{\beta+1}; q)_{km}} \left(\frac{x}{k}\right)^{\frac{km}{\delta}} w^{ms} \left(e_{q^k}\left(\left(\frac{x}{k}\right)^{\frac{k}{s\delta}} w\right)\right)^{-s}$$

or

$$\begin{aligned}
&\sum_{m=0}^{\infty} \frac{B_{m^*}^{(k,\beta,\lambda,\mu)}(y^k;s,r|q)}{(q^{\beta+1};q)_{km}} \left(\frac{x}{k}\right)^{\frac{km}{\delta}} w^{ms} \left(e_{q^k}\left(\left(\frac{x}{k}\right)^{\frac{k}{s\delta}} w\right)\right)^{-s} \\
\leq\ &\sum_{m=0}^{\infty} \frac{B_{m^*}^{(k,\beta,\lambda,\mu)}(x^k;s,r|q)}{(q^{\beta+1};q)_{km}} \left(\frac{y}{k}\right)^{\frac{km}{\delta}} w^{ms} \left(e_{q^k}\left(\left(\frac{y}{k}\right)^{\frac{k}{s\delta}} w\right)\right)^{-s}
\end{aligned}$$

Now rewriting the inequality (5.4) in the form

$$\begin{aligned}
&\sum_{m=0}^{\infty} \frac{B_{m^*}^{(k,\beta,\lambda,\mu)}(x^k;s,r|q)}{(q^{\beta+1};q)_{km}} \left(\frac{y}{k}\right)^{\frac{km}{\delta}} w^{ms} \\
\leq\ &\sum_{m=0}^{\infty} \frac{B_{m^*}^{(k,\beta,\lambda,\mu)}(y^k;s,r|q)}{(q^{\beta+1};q)_{km}} \left(\frac{x}{k}\right)^{\frac{km}{\delta}} w^{ms} \left(e_{q^k}\left(\left(\frac{y}{k}\right)^{\frac{k}{s\delta}} w\right)\right)^{s} \\
&\times \left(e_{q^k}\left(\left(\frac{x}{k}\right)^{\frac{k}{s\delta}} w\right)\right)^{-s}
\end{aligned}$$

and using the easily verifiable identities and inequalities ($sx, sy \in (0,1),\ s \in \mathbf{N}$), ([11], [13]):

$$\begin{aligned}
e_q(x)E_q(-x) &= 1, \\
(E_q(-x))^s &\leq E_q(-x^s), \\
(1+x)\, E_q(qx) &= E_q(x), \\
e_{q^{-1}}(x) &= E_q(-xq), \\
(1-x)\, e_q(x) &= e_q(qx), \\
\left(e_{q^{-1}}(-xq^{-1})\right)^{-s} &\leq e_q(x^s q^{-s}), \\
(e_q(-x))^s &\leq e_q(-x^s), \\
\left(e_{q^{-1}}(-xq^{-1})\right)^{s} &\leq e_q(-x^s q^{-s}),
\end{aligned}$$

the above inequality can easily be written as

$$\begin{aligned}
&\sum_{m=0}^{\infty} \frac{B_{m^*}^{(k,\beta,\lambda,\mu)}(x^k;s,r|q)}{(q^{\beta+1};q)_{km}} \left(\frac{y}{k}\right)^{\frac{km}{\delta}} w^{ms} \\
\leq\ &\sum_{m=0}^{\infty} \frac{B_{m^*}^{(k,\beta,\lambda,\mu)}(y^k;s,r|q)}{(q^{\beta+1};q)_{km}} \left(\frac{x}{k}\right)^{\frac{km}{\delta}} w^{ms} \left(E_{q^k}\left(-\left(\frac{x}{k}\right)^{\frac{k}{s\delta}} w\right)\right)^{s} \\
&\times \left(E_{q^k}\left(-\left(\frac{y}{k}\right)^{\frac{k}{s\delta}} w\right)\right)^{-s} \\
\leq\ &\sum_{m=0}^{\infty} \frac{B_{m^*}^{(k,\beta,\lambda,\mu)}(y^k;s,r|q)}{(q^{\beta+1};q)_{km}} \left(\frac{x}{k}\right)^{\frac{km}{\delta}} w^{ms}\ E_{q^k}\left(-\left(\frac{x}{k}\right)^{\frac{k}{\delta}} w^s\right) \\
&\times \left(E_{q^k}\left(-\left(\frac{y}{k}\right)^{\frac{k}{s\delta}} w\right)\right)^{-s}
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{m=0}^{\infty} \frac{B_{m^*}^{(k,\beta,\lambda,\mu)}(y^k;s,r|q)}{(q^{\beta+1};q)_{km}} \left(\frac{x}{k}\right)^{\frac{km}{\delta}} \ w^{ms}\ E_{q^k}\left(-\left(\frac{x}{k}\right)^{\frac{k}{\delta}} w^s\right) \\
&\quad \times \left(1-\left(\frac{y}{k}\right)^{\frac{k}{s\delta}} w\right)^{-s} \ \left(E_{q^k}\left(-q^k\ \left(\frac{y}{k}\right)^{\frac{k}{s\delta}} w\right)\right)^{-s} \\
&= \sum_{m=0}^{\infty} \frac{B_{m^*}^{(k,\beta,\lambda,\mu)}(y^k;s,r|q)}{(q^{\beta+1};q)_{km}} \left(\frac{x}{k}\right)^{\frac{km}{\delta}} \ w^{ms}\ E_{q^k}\left(-\left(\frac{x}{k}\right)^{\frac{k}{\delta}} w^s\right) \\
&\quad \times \left(1-\left(\frac{y}{k}\right)^{\frac{k}{s\delta}} w\right)^{-s} \ \left(e_{q^{-k}}\left(\left(\left(\frac{y}{k}\right)^{\frac{k}{s\delta}} w\right)\right)\right)^{-s} \\
&= \sum_{m=0}^{\infty} \frac{B_{m^*}^{(k,\beta,\lambda,\mu)}(y^k;s,r|q)}{(q^{\beta+1};q)_{km}} \left(\frac{x}{k}\right)^{\frac{km}{\delta}} \ w^{ms}\ E_{q^k}\left(-\left(\frac{x}{k}\right)^{\frac{k}{\delta}} w^s\right) \\
&\quad \times \left(e_{q^{-k}}\left(q^{-k}\ \left(\frac{y}{k}\right)^{\frac{k}{s\delta}} w\right)\right)^{-s} \\
&\leq \sum_{m=0}^{\infty} \frac{B_{m^*}^{(k,\beta,\lambda,\mu)}(y^k;s,r|q)}{(q^{\beta+1};q)_{km}} \left(\frac{x}{k}\right)^{\frac{km}{\delta}} \ w^{ms}\ E_{q^k}\left(-\left(\frac{x}{k}\right)^{\frac{k}{\delta}} w^s\right) \\
&\quad \times e_{q^k}\left(q^{-sk}\ \left(\frac{y}{k}\right)^{\frac{k}{\delta}} w^s\right) \\
&= \sum_{m=0}^{\infty} \frac{B_{m^*}^{(k,\beta,\lambda,\mu)}(y^k;s,r|q)}{(q^{\beta+1};q)_{km}} \left(\frac{x}{k}\right)^{\frac{km}{\delta}} \ w^{ms} \sum_{j=0}^{\infty} \frac{(-1)^j\ q^{kj(j-1)/2}}{(q^k;q^k)_j} \left(\frac{x}{k}\right)^{\frac{kj}{\delta}} \ w^{sj} \\
&\quad \times \sum_{i=0}^{\infty} \frac{\left(\frac{y}{k}\right)^{\frac{ki}{\delta}}\ q^{-ski}\ w^{si}}{(q^k;q^k)_i} \\
&= \sum_{m=0}^{\infty} \frac{B_{m^*}^{(k,\beta,\lambda,\mu)}(y^k;s,r|q)}{(q^{\beta+1};q)_{km}} \left(\frac{x}{k}\right)^{\frac{km}{\delta}} \ w^{ms} \sum_{j=0}^{\infty}\sum_{i=0}^{j} \frac{(-1)^{j-i}\ q^{k(j-i)(j-i-1)/2}}{(q^k;q^k)_{j-i}} \\
&\quad \times \left(\frac{x}{k}\right)^{\frac{k(j-i)}{\delta}} \ \frac{w^{s(j-i)}\ \left(\frac{y}{k}\right)^{\frac{ki}{\delta}}\ q^{-ski}\ w^{si}}{(q^k;q^k)_i} \\
&= \sum_{m=0}^{\infty} \frac{B_{m^*}^{(k,\beta,\lambda,\mu)}(y^k;s,r|q)}{(q^{\beta+1};q)_{km}} \left(\frac{x}{k}\right)^{\frac{km}{\delta}} \ w^{ms} \sum_{j=0}^{\infty} \frac{(-1)^j\ q^{kj(j-1)/2}\ w^{sj}}{(q^k;q^k)_j} \left(\frac{x}{k}\right)^{\frac{kj}{\delta}} \\
&\quad \times \sum_{i=0}^{j}(-1)^i\ q^{ki(i-1)/2} \begin{bmatrix} j \\ i \end{bmatrix}_k \left(\frac{y}{x}\right)^{\frac{ki}{\delta}} \ q^{(1-j-s)ki} \\
&= \sum_{m=0}^{\infty} \frac{B_{m^*}^{(k,\beta,\lambda,\mu)}(y^k;s,r|q)}{(q^{\beta+1};q)_{km}} \left(\frac{x}{k}\right)^{\frac{km}{\delta}} \ w^{ms} \sum_{j=0}^{\infty} \frac{(-1)^j\ q^{kj(j-1)/2}\ w^{sj}}{(q^k;q^k)_j} \left(\frac{x}{k}\right)^{\frac{kj}{\delta}} \\
&\quad \times \prod_{i=1}^{j}\left(1-\left(\frac{y}{x}\right)^{\frac{k}{\delta}} q^{k(i-j-s)}\right)
\end{aligned}
$$

$$= \sum_{m=0}^{\infty}\sum_{j=0}^{m} \frac{B^{(k,\beta,\lambda,\mu)}_{(m-j)^*}(y^k;s,r|q)}{(q^{\beta+1};q)_{k(m-j)}} \left(\frac{x}{k}\right)^{\frac{k(m-j)}{\delta}} w^{(m-j)s} \frac{q^{kj(j-1)/2}\ w^{sj}}{(q^k;q^k)_j} \left(\frac{x}{k}\right)^{\frac{kj}{\delta}}$$
$$\times \prod_{i=1}^{j}\left(1-\left(\frac{y}{x}\right)^{\frac{k}{\delta}} q^{k(i-j)}\right)$$
$$= \sum_{m=0}^{\infty}\sum_{j=0}^{m} \frac{B^{(k,\beta,\lambda,\mu)}_{(m-j)^*}(y^k;s,r|q)}{(q^{\beta+1};q)_{k(m-j)}} \frac{q^{kj(j-1)/2}}{(q^k;q^k)_j} \left(\frac{x}{k}\right)^{\frac{km}{\delta}} \left(\left(\frac{y}{x}\right)^{\frac{k}{\delta}} q^{k(-j-s+1)};q^k\right)_j w^{ms}.$$

Now comparing the coefficients of w^{ms} both the sides, one arrives at (5.1). □

5.1. Special cases

(i) From (5.1), one gets finite summation formulas for $s=1$:

$$B^{(k,\beta,\lambda,\mu)}_{m^*}(x^k;1,r|q) = (q^{\beta+1};q)_{km} \left(\frac{x}{y}\right)^{\frac{km}{\delta}} \sum_{j=0}^{m} \frac{(-1)^j\ q^{kj(j-1)/2}}{(q^k;q^k)_j}$$
$$\times \left(\left(\frac{y}{x}\right)^{\frac{k}{\delta}} q^{-kj};q^k\right)_j \frac{B^{(k,\beta,\lambda,\mu)}_{(m-j)^*}(y^k;1,r|q)}{(q^{\beta+1};q)_{k(m-j)}} \tag{5.4}$$

From (5.4), with $r=0$, the following summation formula involving the generalized Laguerre polynomial (1.31) occurs.

$$L^{(k,\beta)}_{m^*}(x^k|q) = (q^{\beta+1};q)_{km} \left(\frac{x}{y}\right)^{\frac{km}{\delta}} \sum_{j=0}^{m} \frac{(-1)^j\ q^{kj(j-1)/2}}{(q^k;q^k)_j} \left(\left(\frac{y}{x}\right)^{\frac{k}{\delta}} q^{-kj};q^k\right)_j$$
$$\times \frac{L^{(k,\beta)}_{(m-j)^*}(y^k|q)}{(q^{\beta+1};q)_{k(m-j)}}.$$

Further, $\delta=1$ in (5.5) provides

$$Z^{\beta}_{m}(x;k|q) = (q^{\beta+1};q)_{km} \left(\frac{x}{y}\right)^{km} \sum_{j=0}^{m} \frac{(-1)^j\ q^{kj(j-1)/2}}{(q^k;q^k)_j} \left(\left(\frac{y}{x}\right)^{k} q^{-kj};q^k\right)_j$$
$$\times \frac{Z^{\beta}_{(m-j)}(y;k|q)}{(q^{\beta+1};q)_{k(m-j)}}.$$

The Laguerre polynomial case follows immediately with $k=1$ in the form:

$$L^{(\beta)}_{m}(x|q) = (q^{\beta+1};q)_{m} \left(\frac{x}{y}\right)^{m} \sum_{j=0}^{m} \frac{(-1)^j\ q^{j(j-1)/2}}{(q;q)_j} \left(\left(\frac{y}{x}\right) q^{-j};q\right)_j$$
$$\times \frac{L^{(\beta)}_{(m-j)}(y|q)}{(q^{\beta+1};q)_{(m-j)}}.$$

6. Mixed relation

Theorem 6.1. *For $\beta, \lambda > 0$, $m, \delta, \mu, k, s \in \mathbf{N}$, $r \in \mathbf{N} \cup \{0\}$ there hold the mixed relations:*

$$\begin{aligned} & (1-q^{\beta})\ B_{m^*}^{(k,\beta,\lambda,\mu)}(x^k; s, r|q) + (1-q)\ q^{\beta}\ x\ D_q B_{m^*}^{(k,\beta,\lambda,\mu)}(x^k; s, r|q) \\ = \ & (1-q^{\beta+km}) B_{m^*}^{(k,\beta-1,\lambda,\mu)}((xq^{\delta})^k; s, r|q), \end{aligned} \tag{6.1}$$

where

$$D_q f(x) = \frac{f(x) - f(xq)}{x - xq}.$$

Proof. Here

$$\begin{aligned} l.h.s. \ = \ & (1-q^{\beta})\ B_{m^*}^{(k,\beta,\lambda,\mu)}(x^k; s, r|q) + (1-q)\ q^{\beta}\ x\ D_q B_{m^*}^{(k,\beta,\lambda,\mu)}(x^k; s, r|q) \\ = \ & (1-q^{\beta}) \frac{(q^{\beta+1}; q)_{km}}{[(q^k; q^k)_m]^s} \sum_{n=0}^{m^*} \frac{q^{s(k\delta n(k\delta n-1)/2 + k\delta nm)}\ q^{\delta n(k(\beta+1)+r\mu\lambda)}}{(q^{\beta+1}; q)_{kn}\ [(q^{\lambda}; q)_{\mu n}]^r} \\ & \times \frac{[(q^{-mk}; q^k)_{\delta n}]^s\ x^{kn}}{(q^k; q^k)_n} + (1-q)\ q^{\beta}\ x\ D_q \frac{(q^{\beta+1}; q)_{\alpha m}}{[(q^k; q^k)_m]^s} \\ & \times \sum_{n=0}^{m^*} \frac{q^{s(k\delta n(k\delta n-1)/2 + k\delta nm)}\ q^{\delta n(\alpha(\beta+1)+r\mu\lambda)}}{(q^{\beta+1}; q)_{kn}\ [(q^{\lambda}; q)_{\mu n}]^r} \frac{[(q^{-mk}; q^k)_{\delta n}]^s\ x^{kn}}{(q^k; q^k)_n} \\ = \ & (1-q^{\beta}) \frac{(q^{\beta+1}; q)_{km}}{[(q^k; q^k)_m]^s} \sum_{n=0}^{m^*} \frac{q^{s(k\delta n(k\delta n-1)/2 + k\delta nm)}\ q^{\delta n(k(\beta+1)+r\mu\lambda)}}{(q^{\beta+1}; q)_{kn}\ [(q^{\lambda}; q)_{\mu n}]^r} \\ & \times \frac{[(q^{-mk}; q^k)_{\delta n}]^s\ x^{kn}}{(q^k; q^k)_n} + (1-q)\ q^{\beta} x \frac{(q^{\beta+1}; q)_{\alpha m}}{[(q^k; q^k)_m]^s} \\ & \times \sum_{n=0}^{m^*} \frac{q^{s(k\delta n(k\delta n-1)/2 + k\delta nm)}\ q^{\delta n(\alpha(\beta+1)+r\mu\lambda)}}{(q^{\beta+1}; q)_{kn}\ [(q^{\lambda}; q)_{\mu n}]^r} \frac{[(q^{-mk}; q^k)_{\delta n}]^s}{(q^k; q^k)_n} D_q(x^{kn}) \\ = \ & (1-q^{\beta}) \frac{(q^{\beta+1}; q)_{km}}{[(q^k; q^k)_m]^s} \sum_{n=0}^{m^*} \frac{q^{s(k\delta n(k\delta n-1)/2 + k\delta nm)}\ q^{\delta n(k(\beta+1)+r\mu\lambda)}}{(q^{\beta+1}; q)_{kn}\ [(q^{\lambda}; q)_{\mu n}]^r} \\ & \times \frac{[(q^{-mk}; q^k)_{\delta n}]^s\ x^{kn}}{(q^k; q^k)_n} + (1-q)\ q^{\beta} x \frac{(q^{\beta+1}; q)_{\alpha m}}{[(q^k; q^k)_m]^s} \\ & \times \sum_{n=0}^{m^*} \frac{q^{s(k\delta n(k\delta n-1)/2 + k\delta nm)}\ q^{\delta n(\alpha(\beta+1)+r\mu\lambda)}}{(q^{\beta+1}; q)_{kn}\ [(q^{\lambda}; q)_{\mu n}]^r} \frac{[(q^{-mk}; q^k)_{\delta n}]^s}{(q^k; q^k)_n} \frac{(1-q^{kn})}{x(1-q)} \end{aligned}$$

$$\begin{aligned}
&= (1-q^{\beta})\frac{(q^{\beta+1};q)_{km}}{[(q^k;q^k)_m]^s}\sum_{n=0}^{m^*}\frac{q^{s(k\delta n(k\delta n-1)/2+k\delta nm)}\ q^{\delta n(k(\beta+1)+r\mu\lambda)}}{(q^{\beta+1};q)_{kn}\ [(q^{\lambda};q)_{\mu n}]^r}\\
&\quad\times\frac{[(q^{-mk};q^k)_{\delta n}]^s\ x^{kn}}{(q^k;q^k)_n}+(q^{\beta}-q^{kn+\beta})\frac{(q^{\beta+1};q)_{\alpha m}}{[(q^k;q^k)_m]^s}\\
&\quad\times\sum_{n=0}^{m^*}\frac{q^{s(k\delta n(k\delta n-1)/2+k\delta nm)}\ q^{\delta n(\alpha(\beta+1)+r\mu\lambda)}}{(q^{\beta+1};q)_{kn}\ [(q^{\lambda};q)_{\mu n}]^r}\frac{[(q^{-mk};q^k)_{\delta n}]^s}{(q^k;q^k)_n}\\
&\quad\frac{(1-q^{\beta+km})(q^{\beta};q)_{km}}{[(q^k;q^k)_m]^s}\sum_{n=0}^{m^*}\frac{q^{s(k\delta n(k\delta n-1)/2+k\delta nm)}\ q^{\delta n(k(\beta+1)+r\mu\lambda)}}{[q^{\beta}]_{kn}\ [(q^{\lambda};q)_{\mu n}]^r}\\
&\quad\times\frac{[(q^{-mk};q^k)_{\delta n}]^s\ x^{kn}}{(q^k;q^k)_n}.\\
&= (1-q^{\beta+km})B_{m^*}^{(k,\beta-1,\lambda,\mu)}((xq^{\delta})^k;s,r|q)\\
&= r.h.s.
\end{aligned}$$

□

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A class of differential systems of even degree with exact non-algebraic limit cycles

Abdelkrim Kina, Aziza Berbache and Ahmed Bendjeddou

Abstract. Up until now all the polynomial differential systems for which non-algebraic limit cycles are known explicitly have degree odd. Here we show that that there are polynomial systems of even degree with explicit no-algebraic limit cycles. To our knowledge, there are no such type of examples in the literature.

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Keywords: Planar polynomial differential system, first integral, non-algebraic limit cycle.

1. Introduction and statement of the main results

We consider a polynomial differential system of the form

$$\begin{cases} \dot{x} = P(x,y), \\ \dot{y} = Q\left(x,y\right), \end{cases} \tag{1.1}$$

where P and Q are real polynomials in the variables x and y. The degree of the system (1.1) is the maximum of the degrees of the polynomials P and Q. As usual the dot denotes derivative with respect to the independent variable t.

A limit cycle of system (1.1) is an isolated periodic solution in the set of all periodic solutions of system (1.1). If a limit cycle is contained in the zero level set of a polynomial function, see for example, [[1], [4], [5], [9], [11]], then we say that it is algebraic, otherwise it is called non–algebraic see for example ([2], [4], [8], [10]). The topic of limit cycles is interesting both in mathematics and in science and many models from physics, engineering, chemistry, biology, economics,..., were displayed as differential systems with limit cycles.

An important problem of the qualitative theory of differential equations is to determine the limit cycles of a system of form (1.1). We usually only ask for the number of such limit cycles, but their location as orbits of the system is also an interesting problem. And an even more difficult problem is to give an explicit expression of them.

In the chronological order the first examples where explicit non-algebraic limit cycles appeared are those of A. Gasull and all [8] and J. Gine and M. Grau [10] and by Al-Dosary, Khalil I. T.[2] for $n = 5$. In [6], an example of an explicit limit cycle which is not algebraic is given for $n = 3$. Bendjeddou in [3] provide a class of polynomial differential system of degree odd with explicit limit cycle non-algebraic.

In this paper, we consider the family of the polynomial differential system of the form

$$\begin{cases} \dot{x} = x\left(l + wx + vy\right)^{n+1} + n\left(vx^2 - vy^2 - 2ly - 2wxy\right)\left(x^2+y^2\right)^n \\ \quad + x\left(l + wx + vy\right)\left(a\left(x^2+y^2\right) + 2c\left(x^2-y^2\right) - 4bxy\right)\left(x^2+y^2\right)^{n-1} \\ \dot{y} = y\left(l + wx + vy\right)^{n+1} + n\left(wx^2 - wy^2 + 2lx + 2vxy\right)\left(x^2+y^2\right)^n \\ \quad + y\left(l + wx + vy\right)\left(a\left(x^2+y^2\right) + 2c\left(x^2-y^2\right) - 4bxy\right)\left(x^2+y^2\right)^{n-1}, \end{cases} \tag{1.2}$$

where a, b, c, w, v, n and l are real constants, n is strictly positive integer ($n \in \mathbb{N}^*$). We prove that these systems are Liouville integrable. Moreover, we determine sufficient conditions for a polynomial differential system (1.2) to possess an explicit non-algebraic limit cycle.

It remains the open question to determine if the polynomial differential systems of degree 2 can exhibit explicit non-algebraic limit cycles (this question is due to Benterki and Llibre [6]).

Thus, our main result is the following one.

Theorem 1.1. *Consider a multi-parameter polynomial differential system* (1.2). *Then the following statements hold.*
(a) System (1.2) *is Darboux integrable with the Liouvillian first integral*

$$\begin{aligned} H\left(x,y\right) &= \left(\frac{x^2+y^2}{wx+vy+l}\right)^n e^{-\left(a\left(\arctan\frac{y}{x}\right) + \frac{bx^2+2cxy-by^2}{x^2+y^2}\right)} \\ &\quad - \int_0^{\arctan\frac{y}{x}} e^{-as-b\cos 2s - c\sin 2s}\, ds. \end{aligned}$$

(b) If $a < 0$, $w \geq 0$, $l > 0$ and $2a\pi + b \neq 0$ then system (1.2) *has an explicit non algebraic limit cycles, given in polar coordinates (r, θ) by*

$$r^*(\theta) = \frac{1}{2}\left(g\left(\theta\right)\rho^*\left(\theta\right)^{\frac{1}{n}} + \sqrt{\left(g\left(\theta\right)\rho^*\left(\theta\right)^{\frac{1}{n}}\right)^2 + 4l\rho^*\left(\theta\right)^{\frac{1}{n}}}\right),$$

where

$$g\left(\theta\right) = w\cos\theta + v\sin\theta,$$

$$f\left(\theta\right) = \int_0^{\theta} e^{-as-b\cos 2s-c\sin 2s}\, ds,$$

$$\rho^*\left(\theta\right) = e^{a\theta+b\cos 2\theta+c\sin 2\theta}\left(\frac{e^{2\pi a} f\left(2\pi\right)}{1-e^{2\pi a}} + f\left(\theta\right)\right).$$

Moreover, this limit cycle is hyperbolic.

2. Proof of Theorem 1.1

Firstly, we have

$$x\dot{y} - y\dot{x} = n\left(2l + wx + vy\right)\left(x^2 + y^2\right)^{n+1},$$

thus, the equilibrium points of system (1.2) are present in the equation curve's

$$\left(x^2 + y^2\right)^{n+1}\left(2l + wx + vy\right) = 0, \tag{2.1}$$

we deduce that the origin is an equilibrium point, and any other, if exists must lies on the straight line

$$(\Delta) : (2l + wx + vy) = 0.$$

Let $(x_0, y_0) \neq (0, 0)$ be such a point. Then form the remark above, x_0 and y_0 must satisfy

$$\begin{cases} x_0(-l)^{n+1} + n(vx_0^2 - wx_0y_0)(x_0^2 + y_0^2) + x_0(-l)(a(x_0^2 + y_0^2) \\ \quad +2c(x_0^2 - y_0^2) - 4bx_0y_0) = 0, \\ y_0(-l)^{n+1} + n(vx_0y_0 - wy_0^2)(x_0^2 + y_0^2) + y_0(-l)(a(x_0^2 + y_0^2) \\ \quad +2c(x_0^2 - y_0^2) - 4bx_0y_0) = 0, \\ vy_0 + wx_0 + 2l = 0, \end{cases}$$

$$\begin{cases} (-l)^{n+1} + n\left(vx_0 - wy_0\right)\left(x_0^2 + y_0^2\right) + (-l)\left(a\left(x_0^2 + y_0^2\right) + 2c\left(x_0^2 - y_0^2\right) - 4bx_0y_0\right) = 0, \\ y_0 = -\frac{1}{v}\left(2l + wx_0\right), \end{cases}$$

this system can be written as

$$\begin{aligned} &-l\left(av^3 + 2cv^3 - 6nw^3 + avw^2 + 4bv^2w - 2cvw^2 - 6nv^2w\right)x_0^2 \\ &-4l^2\left(2bv^2 - nv^2 - 3nw^2 + avw - 2cvw\right)x_0 + n\left(v^2 + w^2\right)^2 x_0^3 \\ &-\left(4al^3v - (-l)^{n+1}v^3 - 8cl^3v - 8l^3nw\right) = 0, \end{aligned} \tag{2.2}$$

then, the equilibrium points of system (1.2) are $\left\{(0,0), \left(x_0, -\frac{1}{v}\left(2l + wx_0\right)\right)\right\}$, where x_0 is a real root of the equation (2.2).

Note that, the origin of coordinates which is an unstable node because its eigenvalues are $l^{n+1} > 0$ with multiplicity two, for more details see for instance [[7], Theorem 2.15].

Proof of statement (a).

To prove our results (a) and (b) we write the polynomial differential system (1.2) in polar coordinates (r, θ), defined by $x = r\cos\theta$ and $y = r\sin\theta$. Then the system (1.2) become

$$\begin{cases} \dot{r} = r\left(l + wr\cos\theta + vr\sin\theta\right)^{n+1} + l\left(a + 2c\cos 2\theta - 2b\sin 2\theta\right)r^{2n+1} \\ \quad + \left(n\left(v\cos\theta - w\sin\theta\right) + \left(w\cos\theta + v\sin\theta\right)\left(a + 2c\cos 2\theta - 2b\sin 2\theta\right)\right)r^{2n+2}, \\ \dot{\theta} = 2lnr^{2n} + n\left(v\sin\theta + w\cos\theta\right)r^{2n+1}. \end{cases}$$

Taking θ as an independent variable, we obtain the equation

$$\begin{aligned} \frac{dr}{d\theta} &= \frac{\left(l + wr\cos\theta + vr\sin\theta\right)^{n+1} r + l\left(a + 2c\cos 2\theta - 2b\sin 2\theta\right)r^{2n+1}}{2lnr^{2n} + n\left(v\sin\theta + w\cos\theta\right)r^{2n+1}} \\ &+ \frac{\left(n\left(v\cos\theta - w\sin\theta\right) + \left(w\cos\theta + v\sin\theta\right)\left(a + 2c\cos 2\theta - 2b\sin 2\theta\right)\right)r^{2n+2}}{2lnr^{2n} + n\left(v\sin\theta + w\cos\theta\right)r^{2n+1}}. \end{aligned} \tag{2.3}$$

Via the change of variables

$$\rho = \frac{r^{2n}}{\left(\left(w\cos\theta + v\sin\theta\right)r + l\right)^{n}},$$

the equation (2.3) is transformed into the linear differential equation

$$\frac{d\rho}{d\theta} = \left(a + 2c\cos 2\theta - 2b\sin 2\theta\right)\rho + 1. \tag{2.4}$$

The general solution of linear equation (2.4) is

$$\rho\left(\theta, k\right) = e^{a\theta + b\cos 2\theta + c\sin 2\theta}\left(k + \int_0^{\theta} e^{-as - b\cos 2s - c\sin 2s}ds\right), \tag{2.5}$$

where $k \in \mathbb{R}$. Going back through the changes of variables we obtain the first integral of the statement (a) of Theorem 1. Since this first integral is a function that can be expressed by quadratures of elementary functions, it is a Liouvillian function, and consequently system (1.2) is Darboux integrable.

Proof of statement (b) of Theorem 1.

In (2.5) let $\theta \to \rho(\theta, k^*)$ be the solution taking the value of $k^* \in \mathbb{R}$ for $\theta = 0$. To be a periodic solution, it must satisfy at first the condition

$$\rho\left(0, k^*\right) = \rho\left(2\pi, k^*\right),$$

providing the value of k^* is

$$k^* = \frac{e^{2\pi a} f\left(2\pi\right)}{1 - e^{2\pi a}} > 0,$$

because $a < 0$ and $f\left(\theta\right) = \displaystyle\int_0^{\theta} e^{-as - b\cos 2s - c\sin 2s}ds > 0$ for all $\in \mathbb{R}$.

After the substitution of the value k^* into $\rho\left(\theta, k\right)$ we obtain

$$\rho\left(\theta, k^*\right) = \rho^*\left(\theta\right) = e^{a\theta + b\cos 2\theta + c\sin 2\theta}\left(k^* + \int_0^{\theta} e^{-as - b\cos 2s - c\sin 2s}ds\right). \tag{2.6}$$

Note that, since

$$f\left(\theta\right) = \int_0^{\theta} e^{-as - b\cos 2s - c\sin 2s}ds > 0$$

for all $\in \mathbb{R}$ and $k^* > 0$, consequently,$\rho^*\left(\theta\right) > 0$ for all $\theta \in \mathbb{R}$.
Note that, since $\rho^*\left(\theta\right) > 0$ for all $\theta \in \mathbb{R}$, from the expression of the change of variable that transform (2.3) into (2.4), one gets a unique $r^*\left(\theta\right) > 0$ for all $\theta \in \mathbb{R}$ and it has the expression

$$r^*\left(\theta\right) = \frac{1}{2}\left(g\left(\theta\right)\rho^*\left(\theta\right)^{\frac{1}{n}} + \sqrt{\left(g\left(\theta\right)\rho^*\left(\theta\right)^{\frac{1}{n}}\right)^2 + 4l\rho^*\left(\theta\right)^{\frac{1}{n}}}\right). \tag{2.7}$$

Moreover, since $l > 0$ and $\rho^*\left(\theta\right) > 0$ for all $\theta \in \mathbb{R}$, then $r^*\left(\theta\right) > 0$, one can see that it is $2\pi-$periodic, since g and ρ^* are $2\pi-$ periodic.

In order to prove that the periodic orbit is hyperbolic limit cycles, we consider (2.6), and introduce the Poincaré return map

$$\lambda \mapsto \Pi(2\pi,\lambda) = \rho(\theta,\lambda).$$

Therefore, a limit cycle of system (1.2) is hyperbolic if and only if

$$\left.\frac{d\rho(2\pi,\lambda)}{d\lambda}\right|_{\lambda=k^*} \neq 1.$$

An easy computation shows that:

$$\left.\frac{d\rho(2\pi,\lambda)}{d\lambda}\right|_{\lambda=k^*} = \frac{d\rho^*(\theta)}{dk^*} = e^{2\pi a+b} \neq 1.$$

Therefore the limit cycle of the differential equation (2.4) is hyperbolic, for more details see [12]. Consequently 2.7 is hyperbolic limit cycle of the differential equation (2.3).

Clearly the curve $(r(\theta)\cos\theta, r(\theta)\sin\theta)$ in the (x,y) plane with

$$\frac{r^{2n}}{(g(\theta)r+l)^n} - e^{a\theta+b\cos 2\theta+c\sin 2\theta}\left(\tfrac{e^{2\pi a}f(2\pi)}{1-e^{2\pi a}} + f(\theta)\right) = 0, \tag{2.8}$$

is not algebraic, due to the expression $\frac{e^{2\pi a}f(2\pi)}{1-e^{2\pi a}}e^{a\theta+b\cos 2\theta+c\sin 2\theta}$. More precisely, in Cartesian coordinates $r^2 = x^2+y^2$, $\theta = \arctan\frac{y}{x}$, the curve defined by this limit cycle is

$$\begin{aligned} F(x,y) &= \left(\frac{x^2+y^2}{wx+vy+l}\right)^n - e^{a\left(\arctan\frac{y}{x}\right)+\frac{bx^2+2cxy-by^2}{x^2+y^2}} \\ &\quad\times\left(\frac{e^{2\pi a}f(2\pi)}{1-e^{2\pi a}} + \int_0^{\arctan\frac{y}{x}} e^{-as-b\cos 2s-c\sin 2s}\,ds\right). \end{aligned}$$

If the limit cycle is algebraic this curve must be given by a polynomial, but a polynomial $F(x,y)$ in the variables x and y satisfies that there is a positive integer n such that $\frac{\partial^{(n)}F}{\partial x^n} = 0$, and this is not the case because in the derivative

$$\begin{aligned} \frac{d}{dx}F(x,y) &= n\frac{\left(wx^2+2vxy+2lx-wy^2\right)\left(\frac{x^2+y^2}{l+vy+wx}\right)^{n-1}}{(l+vy+wx)^2} \\ &\quad - \frac{y}{x^2+y^2}\left(\begin{array}{c} 1+\left(\frac{ax^2+ay^2+2cx^2-2cy^2-4bxy}{x^2+y^2}\right)e^{a\left(\arctan\frac{y}{x}\right)+\frac{bx^2+2cxy-by^2}{x^2+y^2}} \\ \times\left(\frac{e^{2\pi a}f(2\pi)}{1-e^{2\pi a}} + \int_0^{\arctan\frac{y}{x}} e^{-as-b\cos 2s-c\sin 2s}\,ds\right) \end{array}\right), \end{aligned}$$

it appears again the expression

$$e^{a\left(\arctan\frac{y}{x}\right)+\frac{bx^2+2cxy-by^2}{x^2+y^2}}\left(\frac{e^{2\pi a}f(2\pi)}{1-e^{2\pi a}} + \int_0^{\arctan\frac{y}{x}} e^{-as-b\cos 2s-c\sin 2s}\,ds\right),$$

which already appears in $F(x,y)$, and this expression will appear in the partial derivative at any order. This completes the proof of theorem.

3. Example

If we take $b = \frac{1}{2}, c = 0, a = -1, v = w = l = 1$, then system (1.2) reads

$$\begin{cases} \dot{x} = x\left(1+x+y\right)^{n+1} + n\left(x^2 - y^2 - 2y - 2xy\right)\left(x^2+y^2\right)^n \\ \qquad + x\left(1+x+y\right)\left(-\left(x^2+y^2\right) + -2xy\right)\left(x^2+y^2\right)^{n-1} \\ \dot{y} = y\left(1+x+y\right)^{n+1} + n\left(x^2 - y^2 + 2x + 2xy\right)\left(x^2+y^2\right)^n \\ \qquad + y\left(1+x+y\right)\left(-\left(x^2+y^2\right) + -2xy\right)\left(x^2+y^2\right)^{n-1}, \end{cases} \tag{3.1}$$

has a non-algebraic limit cycle whose expression in polar coordinates (r, θ) is

$$r^*\left(\theta\right) = \frac{1}{2}\left(\left(\cos\theta + \sin\theta\right)\rho^*\left(\theta\right)^{\frac{1}{n}} + \sqrt{\left(\cos\theta+\sin\theta\right)^2\rho^*\left(\theta\right)^{\frac{2}{n}} + 4l\rho^*\left(\theta\right)^{\frac{1}{n}}}\right),$$

where

$$\rho^*\left(\theta\right) = e^{-\theta+\frac{1}{2}\cos 2\theta}\left(\frac{e^{-2\pi}f(2\pi)}{1-e^{-2\pi}} + f\left(\theta\right)\right) \text{ and } f\left(\theta\right) = \int_0^\theta e^{s-\frac{1}{2}\cos 2s}\,ds.$$

For $n = 1$**:** The system (3.1) is a quartic system and that has a non algebraic limit cycle whose expression in polar coordinates (r, θ) is

$$r^*(\theta) = \frac{1}{2}\left(\left(\cos\theta+\sin\theta\right)\rho^*\left(\theta\right) + \sqrt{\left(\cos\theta+\sin\theta\right)^2\rho^*\left(\theta\right)^2 + 4l\rho^*\left(\theta\right)}\right).$$

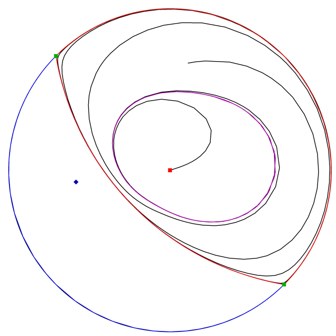

FIGURE 1. Limit cycle of system (3.1) for $n = 1$

For $n = 2$**:** The system (3.1) is of degree 6 and that has a non algebraic limit cycle whose expression in polar coordinates (r, θ) is

$$r^*(\theta) = \frac{1}{2}\left(\left(\cos\theta+\sin\theta\right)\rho^*\left(\theta\right)^{\frac{1}{2}} + \sqrt{\left(\left(\cos\theta+\sin\theta\right)\rho^*\left(\theta\right)^{\frac{1}{2}}\right)^2 + 4l\rho^*\left(\theta\right)^{\frac{1}{2}}}\right).$$

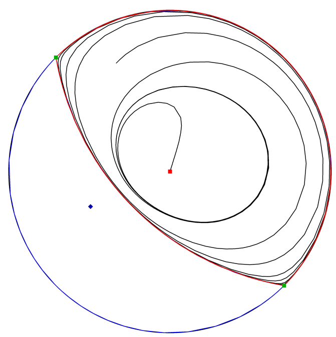

FIGURE 2. Limit cycle of system (3.1) for $n = 2$

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Some remarks on linear set-valued differential equations

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Abstract. The article discusses various definitions of the derivative of a set-valued mapping and their properties. Also, a linear set-valued differential equation is considered and the existence of solutions for this equation with Hukuhara derivative, Plotnikov-Skripnik derivative and Bede-Gal derivative is investigated.

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1. Introduction

The set-valued differential, integral and discrete-time equations and inclusions are an important part of the theory of set-valued analysis, and they are high-valued for the control theory and its applications, as well as for fuzzy differential equations. They were first studied in 1969 by F.S. de Blasi and F. Iervolino [5]. Later, set-valued differential equations have been studied by many scientists due to their applications in many areas. A lot of results on the theory of set-valued differential, integral and discrete-time equations and inclusions can be found in the following books and articles [6, 10, 12, 13, 14, 15, 16, 17, 22, 23, 24, 25, 26, 27, 31, 36, 30, 38, 41, 42, 44] and references therein.

In this article first we consider some definitions of the derivative of a set-valued mapping (Hukuhara derivative [11], Plotnikov-Skripnik derivative [32] and Bede-Gal derivative [1, 19, 20, 46, 47]) and some of their properties. Next, we consider a linear set-valued differential equation with different derivatives that were previously discussed and study the existence of solutions for these equations.

2. Preliminaries

Let R be the set of real numbers and let R^n denote the n-dimensional Euclidean space ($n \geq 2$). We denote by $comp(R^n)$ and $conv(R^n)$ the set of nonempty compact subsets of R^n and the set of nonempty convex and compact subsets of R^n, respectively. For two given sets $X, Y \in comp(R^n)$ and $\lambda \in R$, the Minkowski sum and scalar multiple are defined by

$$X + Y = \{x + y \,|\, x \in X, y \in Y\} \quad \text{and} \quad \lambda X = \{\lambda x \,|\, x \in X\}.$$

We consider the Hausdorff distance $h : comp(R^n) \times comp(R^n) \to R_+ \bigcup \{0\}$ given by

$$h(X, Y) = \min\{r \geq 0 \,|\, X \subset Y + B_r(0), Y \subset X + B_r(0)\},$$

where $B_r(0) = \{x \in R^n \,|\, \|x\| \leq r\}$ is the closed ball with radius r centered at the origin ($\|x\|$ denotes the Euclidean norm).

Lemma 2.1. [39, 40] *The following properties hold:*

1) $(conv(R^n), h)$ *is a complete metric space,*

2) $h(A + C, B + C) = h(A, B)$*,*

3) $h(\lambda A, \lambda B) = |\lambda| h(A, B)$ *for all* $A, B, C \in conv(R^n)$ *and* $\lambda \in R$*.*

However, $comp(R^n)$ and $conv(R^n)$ are not linear spaces since they do not contain inverse elements for the addition, and therefore difference is not well defined, i.e. if $A \in comp(R^n)$ and $A \neq \{a\}$, then $A + (-1)A \neq \{0\}$. As a consequence, alternative formulations for difference have been suggested [7, 11, 28, 39]. One of these alternatives is the Hukuhara difference [11].

Definition 2.2. [11] Let $X, Y \in conv(R^n)$. A set $Z \in conv(R^n)$ such that $X = Y + Z$ is called a Hukuhara difference (H-difference) of the sets X and Y and is denoted by $X \overset{H}{-} Y$.

In this case $X \overset{H}{-} X = \{0\}$ and also $(A + B) \overset{H}{-} B = A$ for any $A, B \in conv(R^n)$. Also, we note that $X \overset{H}{-} Y \neq X + (-1)Y$.

Remark 2.3. Let $A, B \in conv(R^n)$. Then the following statements are true:

1) if the H-difference $A \overset{H}{-} B$ exists, then $diam(A) \geq diam(B)$;
2) if $n = 1$ and $diam(A) \geq diam(B)$, then the H-difference $A \overset{H}{-} B$ exists;
3) if $n \geq 2$ and $diam(A) \geq diam(B)$, then the H-difference $A \overset{H}{-} B$ may not exist. For example, if $A = \{a \in R^n \,|\, |a_i| \leq 2,\, i = \overline{1, n}\}$ and $B = \{b \in R^n \,|\, \|b\| \leq 1\}$, then $A \overset{H}{-} B$ does not exist.

The properties of this difference are studied in detail in [11, 15, 16, 22, 31, 30, 39].

M. Hukuhara introduced the concept of H-differentiability [11] for set-valued functions by using the H-difference.

Let $X : [0, T] \to conv(R^n)$ be a set-valued mapping; $(t_0 - \Delta, t_0 + \Delta) \subset [0, T]$ be a Δ- neighborhood of a point $t_0 \in [0, T]; \Delta > 0$.

For any $t \in (t_0 - \Delta, t_0 + \Delta)$ consider the following Hukuhara differences $X(t) \overset{h}{-} X(t_0)$, $t \geq t_0$, and $X(t_0) \overset{h}{-} X(t)$, $t \geq t_0$ if these differences exist.

Definition 2.4. [11] We say that the mapping $X : [0, T] \to conv(R^n)$ has Hukuhara derivative (H-derivative) $D_H X(t_0)$ at a point $t_0 \in [0, T]$, if there exists an element $D_H X(t_0) \in conv(R^n)$ such that the limits

$$\lim_{t \downarrow t_0} \frac{1}{t - t_0}(X(t) \overset{h}{-} X(t_0)) \quad \text{and} \quad \lim_{t \uparrow t_0} \frac{1}{t_0 - t}(X(t_0) \overset{h}{-} X(t)) \tag{2.1}$$

exist in the topology of $conv(R^n)$ and are equal to $D_H X(t_0)$.

The properties of Hukuhara derivative are studied in detail in [8, 11, 15, 22, 31, 30, 39]. Here, we mention some of them.

Theorem 2.5. [11] *If the mapping $X : [0, T] \to conv(R^n)$ is H-differentiable on $[0, T]$, then*

$$X(t) = X(0) + \int_0^t D_H X(s) ds,$$

where the integral is understood in the sense of [11].

Corollary 2.6. *If the mapping $X(\cdot)$ is H-differentiable on $[0, T]$, then $diam(X(\cdot))$ is a non-decreasing function on $[0, T]$.*

Remark 2.7. The inverse statement is not true. For, example. Let $X(\cdot) : [0, 1] \to conv(R^2)$ be such that $X(t) = A(t)C(t)$, where $A(t) = \begin{pmatrix} \cos(t) & -\sin(t) \\ \sin(t) & \cos(t) \end{pmatrix}$ is a rotation matrix, $C(t) = \{x \in R^2 \,|\, |x_i| \leq t, i = 1, 2\}$ is square. Obviously, $diam(X(t)) = \sqrt{2}t$. However, the mapping $X(\cdot)$ is not H-differentiable on $[0, 1]$.

Corollary 2.8. *If the function $diam(X(\cdot))$ is a decreasing function on $[0, T]$, then the mapping $X(\cdot)$ is not H-differentiable on $[0, T]$.*

In order to overcome these shortcomings of this approach, other types of derivatives for set-valued functions have been explored.

The first alternative of the derivative for set-valued mappings have been introduced by H.T. Banks, M.Q. Jacobs [7] and J.N.Tyurin [45]. According to the Radströms embedding theorem [40] there is a real normed linear space $\mathcal{B}$ and an isometric mapping $\pi : conv(R^n) \to \mathcal{B}$. $\mathcal{B}$ is a space of equivalence classes (see [7, 39, 40]). Then, taking advantage of this embedding theorem, a set-valued mapping $X(\cdot)$ is said to be π-differentiable at t_0 if $\pi \circ X(\cdot)$ is differentiable at t_0. Some properties of this derivative and its connection with other derivatives for set-valued mappings can be found in [7, 9, 18, 21, 37, 39]. However, the π-derivative of a set-valued mapping $X(\cdot)$ may be an element of the space $\mathcal{B}$, which does not have a comparable set in the space $conv(R^n)$ (examples, see [15, 22, 31, 30]).

In [28, 31, 30] the definition of the T-derivative that generalizes the H-derivative and reminds outwardly the π - derivative was introduced. However, its use had difficulty when writing the corresponding set-valued differential equation.

Later, A.V. Plotnikov and N.V. Skripnik took advantage of some approaches that were used in [28] and introduced a new definition of a derivative.

Definition 2.9. [32] Let $X : [0,T] \to conv(R^n)$ and $t \in [0,T]$. We say that $X(\cdot)$ has a Plotnikov-Skripnik derivative (PS-derivative) $D_{ps}X(t) \in conv(R^n)$ at $t \in (0,T)$, if for all $\Delta > 0$ that are sufficiently close to 0, the H-differences and the limits exist in at least one of the following expressions:
(i) $\lim\limits_{\Delta\to 0} \Delta^{-1}(X(t+\Delta) \overset{H}{-} X(t)) = \lim\limits_{\Delta\to 0} \Delta^{-1}(X(t) \overset{H}{-} X(t-\Delta)) = D_{ps}X(t)$
or
(ii) $\lim\limits_{\Delta\to 0} \Delta^{-1}(X(t) \overset{H}{-} X(t+\Delta)) = \lim\limits_{\Delta\to 0} \Delta^{-1}(X(t-\Delta) \overset{H}{-} X(t)) = D_{ps}X(t)$
or
(iii) $\lim\limits_{\Delta\to 0} \Delta^{-1}(X(t+\Delta) \overset{H}{-} X(t)) = \lim\limits_{\Delta\to 0} \Delta^{-1}(X(t-\Delta) \overset{H}{-} X(t)) = D_{ps}X(t)$
or
(iv) $\lim\limits_{\Delta\to 0} \Delta^{-1}(X(t) \overset{H}{-} X(t+\Delta)) = \lim\limits_{\Delta\to 0} \Delta^{-1}(X(t) \overset{H}{-} X(t-\Delta)) = D_{ps}X(t)$.

The properties of this derivative were obtained in [32, 33, 34, 35]. Here, we mention some of them.

Remark 2.10. If the set-valued mapping $X(\cdot)$ is H-differentiable then it is PS-differentiable and $D_{ps}X(t) = D_H X(t)$.

Remark 2.11. If the set-valued mapping $X(\cdot)$ is PS-differentiable on I and $diam X(\cdot)$ is a non-decreasing function on $[0,T]$ then the set-valued mapping $X(\cdot)$ is H-differentiable and $D_{ps}X(t) = D_H X(t)$.

Remark 2.12. There exist set-valued mappings that are PS-differentiable but not H-differentiable.

Example 2.13. The set-valued mapping $X(t) = B_{|t|}(0)$ is PS-differentiable on R and its PS-derivative $D_{ps}X(t) \equiv B_1(0)$. It is obvious that the given set-valued mapping is H-differentiable only on the interval $(0,+\infty)$ and $D_H X(t) = B_1(0)$. On the interval $(-\infty, 0)$ it is not H-differentiable as its diameter on this interval decreases.

Theorem 2.14. [32] *If the mapping $X : [0,T] \to conv(R^n)$ is PS-differentiable on $[0,T]$, then for all $t \in [0,T]$*

(i) if function $diam(X(t))$ is a non-decreasing function on $[0,T]$, then

$$X(t) = X(0) + \int_0^t D_{ps}X(s)ds;$$

(ii) if function $diam(X(t))$ is a decreasing function on $[0,T]$, then

$$X(t) = X(0) \overset{H}{-} \int_0^t D_{ps}X(s)ds.$$

Later, M.T. Malinowski [19, 20], H. Vu and L.S. Dong [46], H. Vu and N. Van Hoa [47] and Ş.E. Amrahov, A. Khastan, N. Gasilov and A.G. Fatullayev [1] adapted the concept of the Bede-Gal derivative [3, 4, 10, 43] for interval-valued mappings on set-valued mappings, that is, such that $X : [0,T] \to conv(R^n)$, and studied its properties [47].

Definition 2.15. [1, 46] Let $X : [0,T] \to conv(R^n)$ and $t \in [0,T]$. We say that $X(\cdot)$ has a Bede-Gal derivative (BG-derivative) $D_{bg}X(t) \in conv(R^n)$ at $t \in (0,T)$, if for all $\Delta > 0$ that are sufficiently close to 0, the H-differences and the limits exist in at least one of the following expressions:
(i) $\lim\limits_{\Delta\to 0} \Delta^{-1}(X(t+\Delta) \overset{H}{-} X(t)) = \lim\limits_{\Delta\to 0} \Delta^{-1}(X(t) \overset{H}{-} X(t-\Delta)) = D_{bg}X(t)$
or
(ii) $\lim\limits_{\Delta\to 0} (-\Delta)^{-1}(X(t) \overset{H}{-} X(t+\Delta)) = \lim\limits_{\Delta\to 0} (-\Delta)^{-1}(X(t-\Delta) \overset{H}{-} X(t)) = D_{bg}X(t)$
or
(iii) $\lim\limits_{\Delta\to 0} \Delta^{-1}(X(t+\Delta) \overset{H}{-} X(t)) = \lim\limits_{\Delta\to 0} (-\Delta)^{-1}(X(t-\Delta) \overset{H}{-} X(t)) = D_{bg}X(t)$
or
(iv) $\lim\limits_{\Delta\to 0} (-\Delta)^{-1}(X(t) \overset{H}{-} X(t+\Delta)) = \lim\limits_{\Delta\to 0} \Delta^{-1}(X(t) \overset{H}{-} X(t-\Delta)) = D_{bg}X(t)$.

Remark 2.16. In the article [19, 20] M.T. Malinowski considered set-valued mappings that satisfy condition (ii) and called this derivative a second type Hukuhara derivative.

Remark 2.17. If the set-valued mapping $X(\cdot)$ is H-differentiable on $[0,T]$ it is BG-differentiable on $[0,T]$ and $D_{bg}X(t) = D_H X(t)$.

Remark 2.18. If the set-valued mapping $X(\cdot)$ is BG-differentiable on $[0,T]$ and $diam X(\cdot)$ is a non-decreasing function on $[0,T]$ then the set-valued mapping $X(\cdot)$ is H-differentiable and $D_{bg}X(t) = D_H X(t)$.

Remark 2.19. There exist set-valued mappings that are BG-differentiable but not H-differentiable.

Example 2.20. [1] The set-valued mapping $X(t) = B_{|t|}(0)$ is BG-differentiable on R and its BG-derivative $D_{bg}X(t) \equiv B_1(0)$. It is obvious that the given set-valued mapping is H-differentiable only on the interval $(0,+\infty)$ and $D_H X(t) = B_1(0)$. On the interval $(-\infty, 0)$ it is not H-differentiable as its diameter on this interval decreases.

Theorem 2.21. [1] *If the mapping $X : [0,T] \to conv(R^n)$ is BG-differentiable on $[0,T]$, then for all $t \in [0,T]$*

(i) if function $diam(X(t))$ is a non-decreasing function on $[0,T]$, then

$$X(t) = X(0) + \int\limits_0^t D_{bg}X(s)ds;$$

(ii) if function $diam(X(t))$ is a decreasing function on $[0,T]$, then

$$X(t) = X(0) \overset{H}{-} (-1) \int\limits_0^t D_{bg}X(s)ds.$$

Remark 2.22. By Remarks 2.10 and 2.17, if the set-valued mapping $X(\cdot)$ is H-differentiable on $[0,T]$ then it is BG-differentiable on $[0,T]$ and PS-differentiable on $[0,T]$ as well as $D_H X(t) = D_{ps}X(t) = D_{bg}X(t)$.

Remark 2.23. By Remarks 2.13 and 2.20, we see that the set-valued mapping $X(t) = B_{|t|}(0)$ is BG-differentiable on R and PS-differentiable on R as well as $D_{bg}X(t) \equiv D_{ps}X(t) \equiv B_1(0)$ for all $t \in R$.

Remark 2.24. There exist set-valued mappings $X(\cdot)$ such that $D_{bg}X(t) \neq D_{ps}X(t)$ for any t.

Example 2.25. Let $X : [0,2] \to conv(R^2)$ and $X(t) = B_{|1-t|}(g(t))$, where $g(t) = (t+1, t+1)^T$ (see Figure 1).

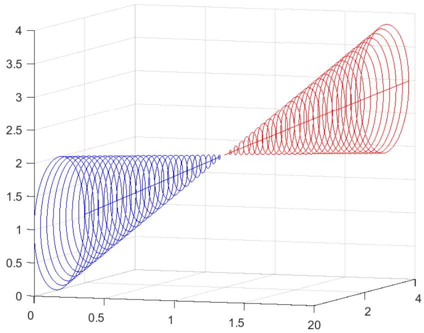

Figure 1: $X(t)$, $t \in [0,2]$

The set-valued mapping $X(\cdot)$ is BG-differentiable on $(0,2)$ and its BG-derivative $D_{bg}X(t) \equiv B_1(a)$, where $a = (1,1)^T$. However, the set-valued mapping $X(\cdot)$ is PS-differentiable on $(0,1)$ and its PS-derivative $D_{ps}X(t) \equiv B_1(b) \neq D_{bg}X(t)$, where $b = (-1,-1)^T$. Also, the set-valued mapping $X(\cdot)$ is PS-differentiable on $(1,2)$ and its PS-derivative $D_{ps}X(t) \equiv B_1(a) = D_{bg}X(t)$, where $a = (1,1)^T$. As well as the PS-derivative $D_{ps}X(t)$ at the point $t = 1$ does not exist (see Figure 2 and Figure 3).

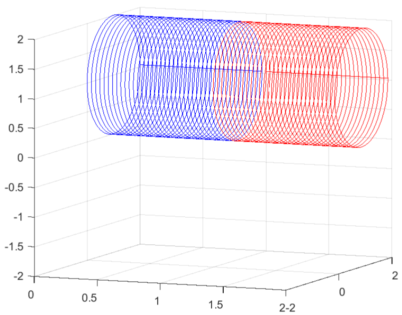

Figure 2: $D_{bg}X(t),\ t \in [0,2]$

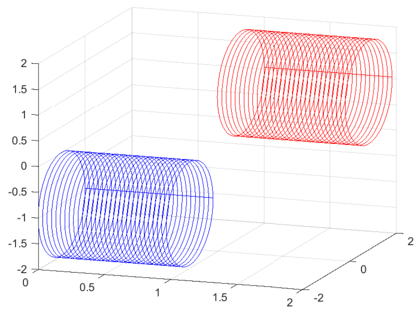

Figure 3: $D_{ps}X(t)$, $t \in [0,2]$

Example 2.26. Let $X : [0,2] \to conv(R^2)$ such that

$$X(t) = \begin{cases} \{x \in R^2 \,|\, x_1^2 + x_2^2 \leq t,\ x_2 \geq 0\}, & t \in [0,1], \\ \{x \in R^2 \,|\, x_1^2 + x_2^2 \leq 2 - t,\ x_2 \geq 0\}, & t \in (1,2] \end{cases}$$

(see Figure 4).

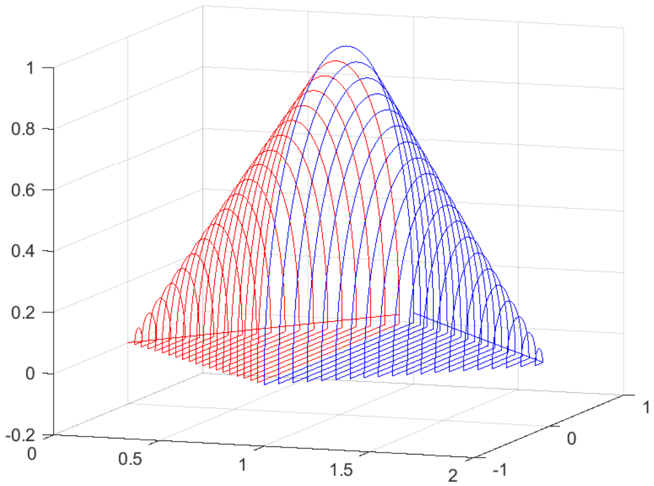

Figure 4: $X(t)$, $t \in [0,2]$

The set-valued mapping $X(\cdot)$ is PS-differentiable on $(0,2)$ and its PS-derivative $D_{ps}X(t) \equiv \{x \in R^2 \,|\, x_1^2 + x_2^2 \leq 1, x_2 \geq 0\}$. However, the set-valued mapping $X(\cdot)$ is BG-differentiable on $(0,1)$ and its BG-derivative $D_{bg}X(t) \equiv D_{ps}X(t)$. Also, the set-valued mapping $X(\cdot)$ is BG-differentiable on $(1,2)$ and its BG-derivative $D_{bg}X(t) \equiv (-1)D_{ps}X(t)$. As well as the BG-derivative $D_{bg}X(t)$ at the point $t = 1$ does not exist (see Figure 5 and Figure 6).

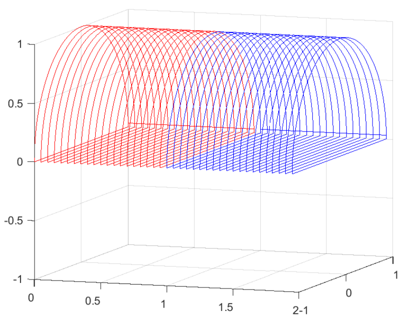

Figure 5: $D_{ps}X(t)$, $t \in [0,2]$

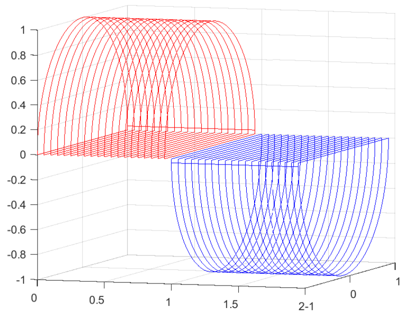

Figure 6: $D_{bg}X(t)$, $\; t \in [0,2]$

3. Linear set-valued differential equations

In this section, we consider linear set-valued differential equations

$$DX(t) = aX(t), \quad X(0) = X_0, \tag{3.1}$$

where $a \in R$, $X : [0,T] \to conv(R^n)$ is a set-valued mapping, $DX(t)$ is one of the previously considered derivatives $(D_H X(t), D_{ps}X(t), D_{bg}(t))$ of the set-valued mapping $X(t)$.

Definition 3.1. A set-valued mapping $X(\cdot)$ is called a solution of (3.1) if it is continuously differentiable and satisfies system (3.1) everywhere on $[0,T]$.

As known, linear Hukuhara differential equation

$$D_H X(t) = aX(t), \quad X(0) = X_0, \tag{3.2}$$

has a unique solution on the interval $[0,T]$ [22, 31]. It's also obvious that function $diam(X(t))$ is a non-decreasing function on $[0,T]$.

Remark 3.2. [5, 22, 31] If $a \geq 0$ then $X(t) = e^{at}X_0$ for all $t \in [0,T]$.

Remark 3.3. [38] System (3.2) may not be equivalent to the following system of interval-valued differential equations with the Hukuhara derivative

$$\begin{cases} D_H X_1(t) = aX_1(t), & X_1(0) = X_{01}, \\ \quad\quad \dots & \quad\quad \dots \\ D_H X_n(t) = aX_n(t), & X_n(0) = X_{0n}, \end{cases} \tag{3.3}$$

where $X_i : [0,T] \to conv(R)$ is a interval-valued mapping, X_{0i} is the projection of the set X_0 on the axis $0x_i$, $i = \overline{1,n}$.

If $X(\cdot)$ is a solution of (3.2) and $X_i(\cdot)$, $i = \overline{1,n}$ are solutions of (3.3), then $X(t) \subset X_1(t) \times \ldots \times X_n(t)$ for all $t \in [0,T]$.

If $X_0 = X_{01} \times \ldots \times X_{0n}$ then system (3.2) is equivalent to system (3.3).

We demonstrate this by the following example.

Example 3.4. Let

$$D_H X(t) = X(t),\ X(0) = B_1(0),\ t \in [0,1], \tag{3.4}$$

and

$$\begin{cases} D_H X_1(t) = X_1(t), & X_1(0) = X_{01} = [-1,1], \\ D_H X_2(t) = X_2(t), & X_2(0) = X_{02} = [-1,1], \end{cases} \tag{3.5}$$

where $X : [0,1] \to conv(R^2)$ is a set-valued mapping, $X_i : [0,1] \to conv(R)$ is an interval-valued mapping, X_{0i} is the projection of the set X_0 on the axis $0x_i$, $i = \overline{1,2}$.

The set-valued mapping $X(t) = B_{e^t}(0)$ is a solution of Hukuhara differential equation (3.4). The interval-valued mappings $X_i(t) = [-e^t, e^t]$, $i = 1,2$ are solutions of the system of Hukuhara differential equations (3.5). It's obvious that $X(t) \subset X_1(t) \times X_2(t)$ for all $t \in [0,1]$ (see Figure 7). However, if $X(0) = \{x \in R^2 \,|\, |x_i| \leq 1,\ i = 1,2\}$ is a square, then $X_0 \equiv X_{01} \times X_{02}$ and $X(t) \equiv X_1(t) \times X_2(t)$ for all $t \in [0,1]$ (see Figure 8).

Now, we consider linear differential equation (3.1) with PS-derivative and BG-derivative. By [1, 32, 33, 34, 35], this set-valued differential equation (3.1) has at least one solution. Moreover, one of these solutions (the one whose diameter is a non-decreasing function) coincides with the solution of the corresponding differential equation (3.2).

We will show it by the following example.

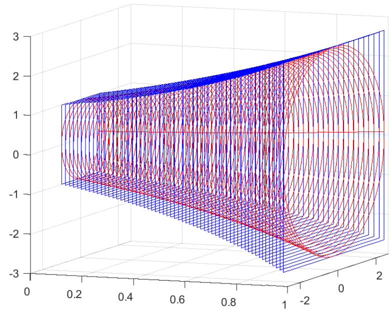

Figure 7:
$X(t) \subset X_1(t) \times X_2(t)$, $t \in [0,1]$

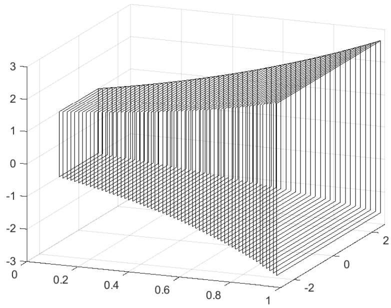

Figure 8:
$X(t) \equiv X_1(t) \times X_2(t),\ t \in [0,1]$

Example 3.5. Let

$$DX(t) = X(t),\ X(0) = B_1(0),\ t \in [0,1], \tag{3.6}$$

where $X : [0,1] \to conv(R^2)$ is a set-valued mapping, $DX(t)$ is one of the previously considered derivatives $(D_H X(t), D_{ps}X(t), D_{bg}(t))$ of the set-valued mapping $X(t)$.

The set-valued mapping $X(t) = B_{e^t}(0)$ is a solution of Hukuhara differential equation (3.6) (see Figure 9).

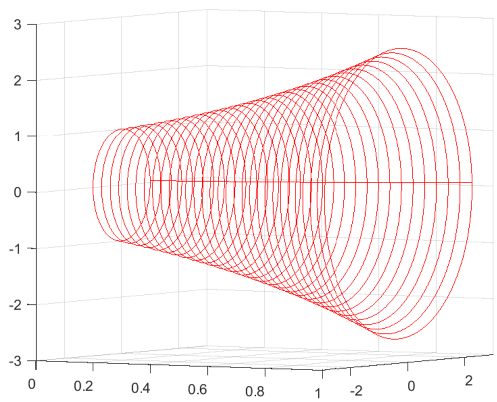

Figure 9: $X(t)$, $t \in [0,1]$

Set-valued mappings $X_1(t) = B_{e^t}(0)$ and $X_2(t) = B_{e^{-t}}(0)$ are solutions of differential equation (3.6) with PS-derivative and BG-derivative (see Figure 10 and Figure 11).

In this case, solutions of differential equations with PS-derivative will be solutions of the differential equation with BG-derivative and vice versa. For the first solution $X_1(\cdot)$ the function $diam(X_1(t))$ is an increasing function on $[0,1]$. For the second solution $X_2(\cdot)$ the function $diam(X_2(t))$ is a decreasing function on $[0,1]$. Also, the

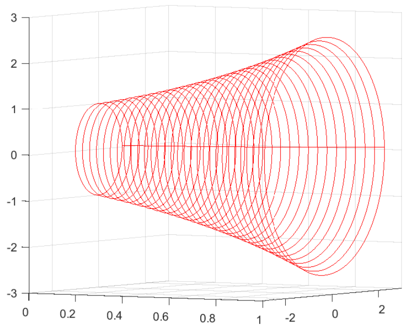

Figure 10: $X_1(t)$, $t \in [0,1]$

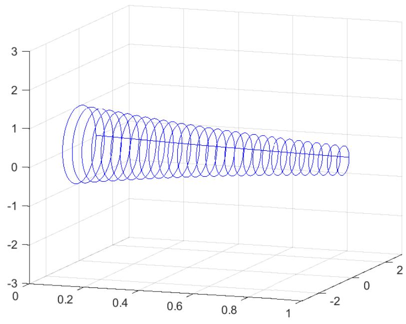

Figure 11: $X_2(t)$, $t \in [0,1]$

first solution $X_1(\cdot)$ is the solution of the Hukuhara differential equation, i.e. $X(t) = X_1(t)$ for all $t \in [0,1]$.

Solutions $X_1(\cdot)$ and $X_2(\cdot)$ will be called basic solutions.

We also note that set-valued mappings

$$Y_1(t) = \begin{cases} B_{e^t}(0), & t \in [0,0.5] \\ B_{e^{1-t}}(0), & t \in [0.5,1] \end{cases} \quad Y_2(t) = \begin{cases} B_{e^{-t}}(0), & t \in [0,0.5] \\ B_{e^{t-1}}(0), & t \in [0.5,1] \end{cases}$$

are the solutions of differential equation (3.6) with PS-derivative and BG-derivative (see Figure 12 and Figure 13).

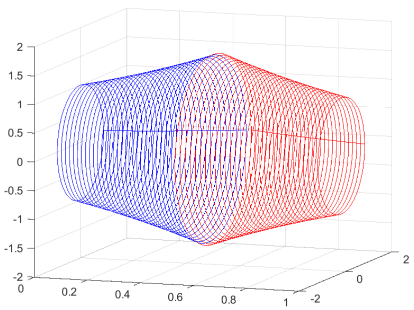

Figure 12: $Y_1(t)$, $t \in [0,1]$

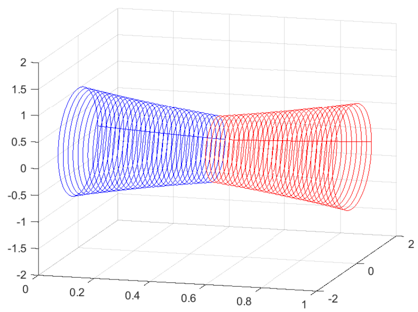

Figure 13: $Y_2(t)$, $t \in [0,1]$

It is obvious that in this example such solutions can be built infinitely many. These solutions will be called mixed solutions. For these mixed solutions $Y(\cdot)$, the diameter function $diam(Y(\cdot))$ is not increasing or decreasing over the entire interval. We also note that the shape of the cross section of solutions corresponds to the shape of the initial set.

Later in this article we will consider only the basic solutions.

The question arises: Do such equations always have two basic solutions?

Consider the following examples when $a = 1$ $(a > 0)$.

Example 3.6. Let
$$D_{ps}X(t) = X(t),\ X(0) = K,\ t \in [0,1], \tag{3.7}$$
where $X : [0,1] \to conv(R^2)$ is a set-valued mapping,
$$K = \{x \in R^2 \,|\, x_1^2 + x_2^2 \le 1,\ x_2 \ge 0\}.$$
This differential equation with PS-derivative has two basic solutions $X_1(\cdot)$ and $X_2(\cdot)$ (see Figure 14 and Figure 15).

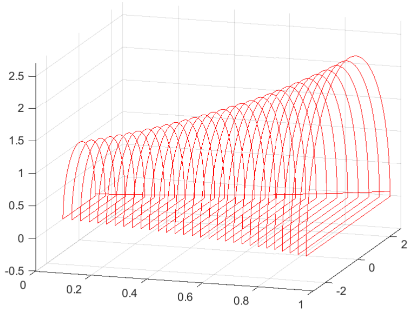

Figure 14: $X_1(t),\, t \in [0,1]$

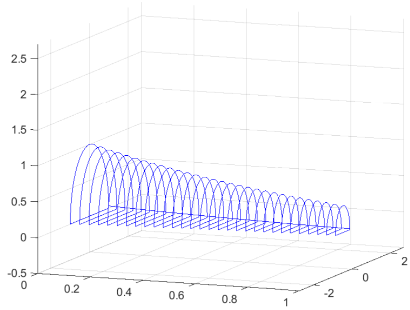

Figure 15: $X_2(t),\ t \in [0,1]$

Example 3.7. Let
$$D_{bg}X(t) = X(t),\ X(0) = K,\ t \in [0,1]. \tag{3.8}$$
This differential equation with BG-derivative has only one basic solution, which coincides with the solution of the Hukuhara differential equation and the first basic solution $X_1(\cdot)$ of the differential equation with the PS-derivative (see Figure 14).

There will be no second solution because there is no set-valued mapping that satisfies the corresponding integral equation (since the set K is not a centrally symmetric set, the Hukuhara difference does not exist)
$$X(t) = K \overset{H}{\text{—}} (-1)\int_0^t D_{bg}X(s)ds = K \overset{H}{\text{—}} (-1)\int_0^t X(s)ds.$$

Now, we consider the same examples when $a = -1$ $(a < 0)$.

Example 3.8. Let
$$D_{bg}X(t) = (-1)X(t),\ X(0) = K,\ t \in [0,1], \tag{3.9}$$
where $X : [0,1] \to conv(R^2)$ is a set-valued mapping,
$$K = \{x \in R^2 \,|\, x_1^2 + x_2^2 \le 1,\ x_2 \ge 0\}.$$
This differential equation with BG-derivative has two basic solutions $X_1(\cdot)$ and $X_2(\cdot)$ (see Figure 16 and Figure 17).

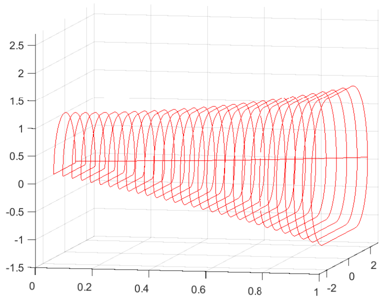

Figure 16: $X_1(t)$, $t \in [0,1]$

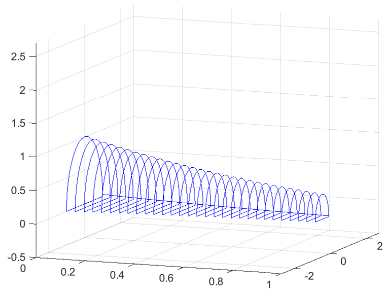

Figure 17: $X_2(t)$, $t \in [0,1]$

Example 3.9. Let

$$D_{ps}X(t) = (-1)X(t),\ X(0) = K,\ t \in [0,1]. \tag{3.10}$$

This differential equation with PS-derivative has only one basic solution, which coincides with the solution of the Hukuhara differential equation and the first basic solution $X_1(\cdot)$ of the differential equation with the BG-derivative.

There will be no second basic solution because there is no set-valued mapping that satisfies the corresponding integral equation (the Hukuhara difference does not exist) $X(t) = K \overset{H}{-} \int\limits_0^t D_{ps}X(s)ds = K \overset{H}{-} (-1)\int\limits_0^t X(s)ds.$

Next, we consider the same examples when X_0 is such that H-difference $X_0 \overset{H}{-} (-1)X_0$ exists (X_0 is centrally symmetric set [7]).

Example 3.10. Let

$$D_{bg}X(t) = X(t),\ X(0) = P,\ t \in [0,1], \tag{3.11}$$

$$D_{ps}X(t) = X(t),\ X(0) = P,\ t \in [0,1], \tag{3.12}$$

where $X : [0,1] \to conv(R^2)$ is set-valued mapping, $P = \{x \in R^2 \,|\, 0 \le x_1 - 2 \le 4,\ 1 \le x_2 - 2 \le 3\}$.

Each differential equation will have two basic solutions $X_1^{bg}(\cdot)$, $X_2^{bg}(\cdot)$ and $X_1^{ps}(\cdot)$, $X_2^{ps}(\cdot)$ (see Figures 18,19 and Figures 20,21).

Example 3.11. Let

$$D_{bg}X(t) = (-1)X(t),\ X(0) = P,\ t \in [0,1], \tag{3.13}$$

$$D_{ps}X(t) = (-1)X(t),\ X(0) = P,\ t \in [0,1]. \tag{3.14}$$

Also, each differential equation will have two basic solutions $X_1^{bg}(\cdot)$, $X_2^{bg}(\cdot)$ and $X_1^{ps}(\cdot)$, $X_2^{ps}(\cdot)$ (see Figures 22, 23 and Figures 24, 25).

Remark 3.12. It's obvious that the basic solution $X_2^{ps}(\cdot)$ of differential equation (3.12) coincides with the basic solution $X_2^{bg}(\cdot)$ of differential equation (3.13). Also, the basic solution $X_2^{bg}(\cdot)$ of differential equation (3.11) coincides with the basic solution $X_2^{ps}(\cdot)$

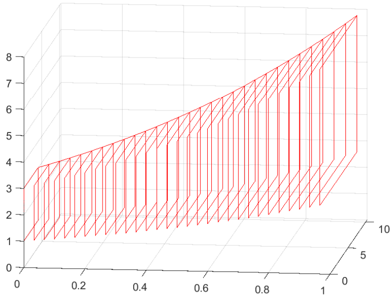

Figure 18: $X_1^{bg}(t)$, $t \in [0,1]$

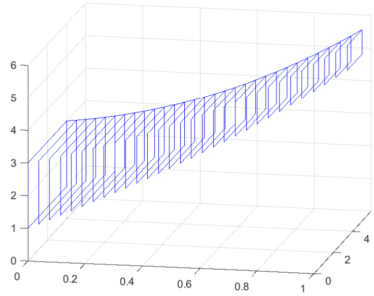

Figure 19: $X_2^{bg}(t)$, $t \in [0,1]$

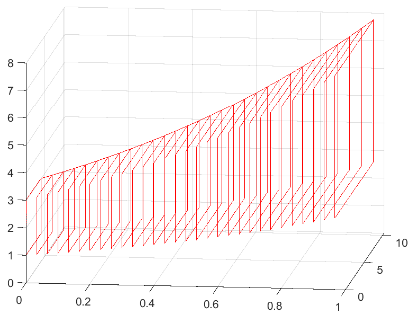

Figure 20: $X_1^{ps}(t)$, $t \in [0,1]$

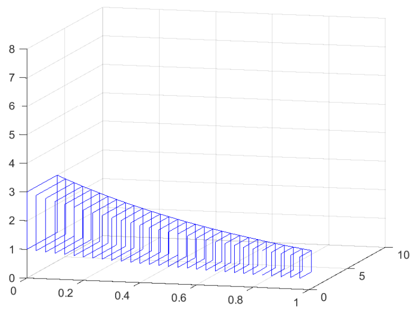

Figure 21: $X_2^{ps}(t)$, $t \in [0,1]$

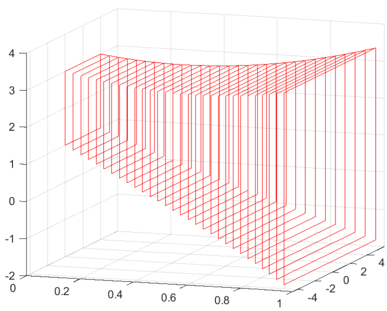

Figure 22: $X_1^{bg}(t)$, $t \in [0,1]$

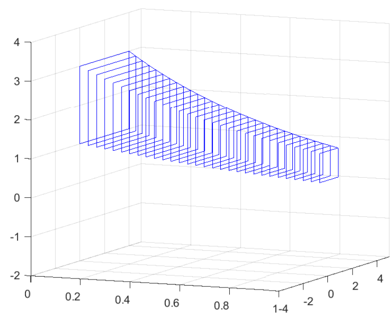

Figure 23: $X_2^{bg}(t)$, $t \in [0,1]$

of differential equation (3.14). This is confirmed by integral equations that correspond to differential equations (3.11), (3.12), (3.13) and (3.14):

$$X_2^{bg}(t) = P\frac{H}{}(-1)\int_0^t X_2^{bg}(s)ds, \tag{3.15}$$

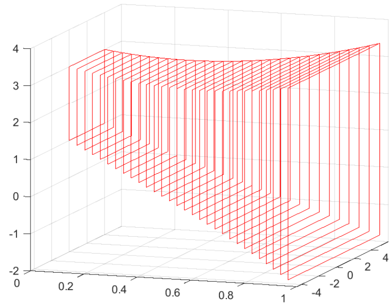

Figure 24: $X_1^{ps}(t)$, $t \in [0,1]$

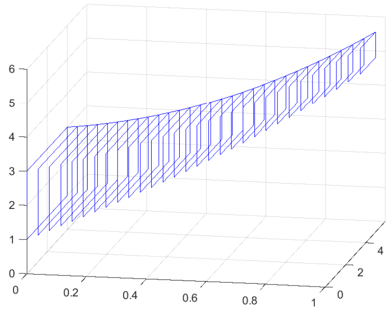

Figure 25: $X_2^{ps}(t)$, $t \in [0,1]$

$$X_2^{ps}(t) = P\overset{H}{\text{---}}\int_0^t X_2^{ps}(s)ds, \tag{3.16}$$

$$X_2^{bg}(t) = P\overset{H}{\text{---}}(-1)\int_0^t (-1)X_2^{bg}(s)ds = P\overset{H}{\text{---}}\int_0^t X_2^{bg}(s)ds, \tag{3.17}$$

$$X_2^{ps}(t) = P\overset{H}{\text{---}}\int_0^t (-1)X_2^{ps}(s)ds = P\overset{H}{\text{---}}(-1)\int_0^t X_2^{ps}(s)ds. \tag{3.18}$$

Remark 3.13. If the differential equation with the PS-derivative (BG-derivative) has two basic solutions and we write the corresponding system of interval-valued differential equations the PS-derivative (BG-derivative) similar to (3.3), then Remark 3.3 will be satisfied. However, we note that this system will always have two basic solutions (even when the original equation has only one basic solution).

Based on all above stated, we can make the following proposition.

Proposition 3.14. *For system* (3.1) *the following statements are true:*

1) if H-difference $X_0 \overset{H}{\text{---}} (-1)X_0$ *exists, then differential equation* (3.1) *with PS(BG)-derivative has two basic solutions;*

2) if H-difference $X_0 \overset{H}{\text{---}} (-1)X_0$ *does not exist, then*

a) if $a > 0$, *then differential equation* (3.1) *with PS-derivative has two basic solutions and differential equation* (3.1) *with BG-derivative has one basic solution;*

a) if $a < 0$, *then differential equation* (3.1) *with BG-derivative has two basic solutions and differential equation* (3.1) *with PS-derivative has one basic solution.*

4. Conclusion

In the article it is shown that linear set-valued differential equations have significant differences from ordinary and interval-valued linear differential equations. In these equations, the number of solutions may depend on the form (shape) of the initial set, the considered derivative and the coefficient in the right-hand side. We also note that in articles [32, 33, 34, 35, 42], the authors considered a new type of differential equations with PS-derivative, in which no more than one solution can exist.

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Exponential decay of the viscoelastic wave equation of Kirchhoff type with a nonlocal dissipation

Mohamed Mellah and Ali Hakem

Abstract. The following viscoelastic wave equation of Kirchhoff type with nonlinear and nonlocal damping

$$u_{tt}-\psi\left(\|\nabla u\|_2^2\right)\Delta u-\alpha\Delta u_t+\int_0^t g(t-\tau)\Delta u(\tau)d\tau+M\left(\|\nabla u\|_2^2\right)u_t=f(u),$$

where $M(r)$ is a $C^1([0,\infty))$ -function satisfying $M(r)\geq m_1>0$ for $r\geq 0$, is considered in a bounded domain Ω of $\mathbb{R}^N$. The existence of global solutions and decay rates of the energy are proved.

Mathematics Subject Classification (2010): 35L05, 35L70, 93D15.

Keywords: Kirchhoff equation, nonlocal damping, global solution, energy decay, relaxation function.

1. Introduction

In this paper, we shall consider the initial boundary value problem for the following integro-differential problem

$$\begin{cases} u_{tt}-\psi\left(\|\nabla u\|_2^2\right)\Delta u-\alpha\Delta u_t+\displaystyle\int_0^t g(t-\tau)\Delta u(\tau)d\tau \\ \qquad +M\left(\|\nabla u\|_2^2\right)u_t=f(u), \quad \text{in} \quad \Omega\times(0,T), \\ u(x,0)=u_0(x), \quad u_t(x,0)=u_1(x), \quad x\in\Omega, \\ u(x,t)=0, \quad x\in\partial\Omega,\ t>0, \end{cases} \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^N$ $(N\geq 1)$ with smooth boundary $\partial\Omega$ so that the divergence theorem can be applied. $\psi(r)$ is a positive locally Lipschitz function satisfying $\psi(r)\geq m_0>0$, for $r\geq 0$ like $\psi(r)=m_0+br^\gamma$, $b\geq 0$ and $\gamma\geq 1$. $M(r)$ is a $C^1[0,\infty)$ -function satisfying $M(r)\geq m_1>0$ for $r\geq 0$, the scalar function $g(s)$ (so-called relaxation kernel) is assumed to satisfy (2.1) and f is a non-linear function as similar to $|u|^{p-2}u$, $p>2$. Here $\alpha \geq 0$. The motivation for this problem comes from the following original equation

$$\rho h \frac{\partial^2 u}{\partial t^2} + \delta \frac{\partial u}{\partial t} = \left\{ p_0 + \frac{Eh}{L} \int_0^L \left(\frac{\partial u}{\partial x} \right)^2 dx \right\} \frac{\partial^2 u}{\partial x^2} + f, \tag{1.2}$$

where $0 \leq x \leq L$, $t \geq 0$ and $u = u(x,t)$ is the lateral displacement at the space coordinate x and the time t, ρ the mass density, h the cross-section area, L the length, E the Youngs modulus, p_0 the initial axial tension, δ the resistance modulus and f the external force. When $\delta = f = 0$, the equation (1.2) was first introduced by Kirchhoff [2].

In the absence of the term $M\left(\|\nabla u\|_2^2\right)u_t$. Wu and Tsai [7] studied (1.1) with $\alpha = 1$. The authors established the global existence and energy decay under the assumption $g'(t) \leq -rg(t)$, $\forall t \geq 0$ for some $r > 0$. Recently, this decay estimate of the energy function was improved by Wu in [6] under a weaker condition on g i.e. $g'(t) \leq 0$, $\forall t \geq 0$.

If we consider (1.1) with $[\psi \equiv 1, f = \alpha = 0]$ and the bi-harmonic instead of Laplace operator one we get the model

$$u_{tt} + \Delta^2 u - \int_0^t g(t-\tau)\Delta^2 u(\tau) d\tau + M\left(\|\nabla u\|_2^2\right) u_t = 0. \tag{1.3}$$

Cavalcanti et al. [1] investigated the global existence, uniqueness and stabilization of energy. By taking a bounded or unbounded open set Ω, the authors showed that the energy goes to zero exponentially provided that g goes to zero at the same form.

The main interest of the present paper is to examine whether there exists a global solution u to (1.3) under the presence of the nonlinear and nonlocal dissipation represented by $M\left(\int_\Omega |\nabla u(x,t)|^2 dx\right) u_t$ and the real-value function $M : [0,+\infty) \to [m_1,+\infty)$, where $m_1 > 0$ will be considered of class C^1.

This kind of damping effect was firstly introduced by H. Lange and G. Perla Menzala [3] for the beam equation where the following model was considered

$$u_{tt} + \Delta^2 u + M\left(\int_\Omega |\nabla u(x,t)|^2 dx\right) u_t = 0 \quad \text{in} \quad \mathbb{R}^N \times \mathbb{R}^+. \tag{1.4}$$

The nonlocal nonlinearity $M\left(\int_\Omega |\nabla u(x,t)|^2 dx\right) u_t$ is indeed a damping term. It models a frictional mechanism acting on the body that depends on the average of u itself. Moreover, if such u does exist, we intend to investigate its asymptotic behavior as $t \to \infty$.

In this paper we show that under some conditions the solution is global in time and the energy decays exponentially. We first use Faedo-Galerkin method to study the existence of the simpler problem (3.1). Then, we obtain the local existence Theorem 3.2 by using contraction mapping principle. We obtain global existence of the solutions of (1.1) given in Theorem 4.4. Our technique of proof is similar to the one in [7] with some necessary modifications due to the nature of the problem treated here. Moreover, the asymptotic behavior of global solutions is investigated under some assumptions on the initial data.

2. Preliminaries

In this section we present some assumptions, notations and Lemmas. We first make the following hypotheses.
(A1) $g : \mathbb{R}^+ \to \mathbb{R}^+$ is a bounded C^1 function satisfying

$$\begin{gathered}\int_0^\infty g(\tau)d\tau = l_1 > 0, \quad g(0) - K_1 \int_0^\infty g(\tau)d\tau = l_2 > 0, \\ -K_1 g(t) \le g'(t) \le -K_2 g(t),\end{gathered} \tag{2.1}$$

here K_1 and K_2 are positive constants.
(A2) $f(0) = 0$ and there is a positive constant K_3 such that

$$\big|f(u) - f(v)\big| \le K_3 \big|u - v\big| \Big(\big|u\big|^{p-2} + \big|v\big|^{p-2}\Big) \quad \text{for} \quad u, v \in \mathbb{R}, \tag{2.2}$$

and

$$2 < p < \infty \quad \text{if} \quad N = 1, 2 \quad \text{and} \quad 2 < p \le \frac{2(N-1)}{N-2} \quad \text{if} \quad N \ge 3. \tag{2.3}$$

(A3) The function $M(r)$ for $r \ge 0$ belongs to the class $C^1[0, \infty)$ and satisfies

$$M(r) \ge m_1 > 0 \quad \text{for} \quad r \ge 0. \tag{2.4}$$

For functions $u(x,t)$, $v(x,t)$ defined on Ω, we introduce

$$(u, v) = \int_\Omega uv dx, \qquad \|u\|_2 = \left(\int_\Omega |u|^2 dx\right)^{\frac{1}{2}}, \qquad \|u\|_\infty = ess \sup_{x \in \Omega} |u(x)|,$$

$$\|u\|_p = \left(\int_\Omega |u|^p dx\right)^{\frac{1}{p}}, \qquad \|u\|_{H^m} = \left(\sum_{|\beta| \le m} \|D^\beta\|_2^2\right)^{\frac{1}{2}}.$$

Lemma 2.1. (*Sobolev-Poincaré inequality* [5]) *If* $2 \le p \le \frac{2N}{N-2}$, *then*

$$\|u\|_p \le B_1 \|\nabla u\|_2, \tag{2.5}$$

for $u \in H_0^1(\Omega)$ *holds with some constant* B_1.

3. Local existence of solution

In this section, we shall discuss the local existence of solutions for (1.1) by using contraction mapping principle. An important step in the proof of local existence Theorem 3.2 below is the study of the following simpler problem:

$$\begin{cases} u_{tt} - \mu(t)\Delta u - \alpha \Delta u_t + \displaystyle\int_0^t g(t-\tau)\Delta u(\tau) d\tau \\ \qquad + \chi(t) u_t = f_1(x,t), \quad \text{in} \quad \Omega \times (0, T), \\ u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x), \quad x \in \Omega, \\ u(x,t) = 0, \quad x \in \partial\Omega, \ t \ge 0. \end{cases} \tag{3.1}$$

Here, $T > 0$, $\alpha \geq 1$, f_1 is a fixed forcing term in $\Omega \times (0, T)$, $\mu(t)$ is a positive locally Lipschitz function on $[0, \infty)$ with $\mu(t) \geq m_0 > 0$ for $t \geq 0$ and $\chi(t)$ is C^1-function on $[0, \infty)$ such that $\chi(t) \geq 0$ for $t \geq 0$.

Lemma 3.1. *Suppose that* $(A1)$ *holds, and that* $u_0 \in H^2(\Omega) \cap H_0^1(\Omega)$, $u_1 \in H_0^1(\Omega)$ *and* $f_1 \in L^2\big([0, T]; L^2(\Omega)\big)$ *be given. Then the problem* (3.1) *admits a unique solution* u *such that*

$$u \in C([0, T]; H^2(\Omega) \cap H_0^1(\Omega)), \quad u_t \in C([0, T]; L^2(\Omega)) \cap L^2([0, T]; H_0^1(\Omega)),$$

$$u_{tt} \in L^2([0, T]; L^2(\Omega)).$$

Proof. Let $(\omega_n)_{n \in \mathbb{N}}$ be a basis in $H^2(\Omega) \cap H_0^1(\Omega)$ and V^n be the space generated by $\omega_1, \ldots., \omega_n$, $n = 1, 2, \cdots$. Let us consider

$$u^n(t) = \sum_{k=1}^{n} d_k^n(t) w_k,$$

be the weak solution of the following approximate problem corresponding to (3.1)

$$\begin{aligned} &\int_\Omega u_{tt}^n(t) \omega dx + \mu(t) \int_\Omega \nabla u^n(t) \cdot \nabla \omega dx \\ &- \int_0^t g(t-\tau) \int_\Omega \nabla u^n(\tau) \cdot \nabla \omega dx d\tau + \alpha \int_\Omega \nabla u_t^n(t) \cdot \nabla \omega dx \\ &+ \chi(t) \int_\Omega u_t^n(t) \omega dx = \int_\Omega f_1(x, t) w dx \quad \text{ for } \ \omega \in V^n, \end{aligned} \tag{3.2}$$

with initial conditions

$$u^n(0) = u_0^n = \sum_{k=1}^{n} \int_\Omega u_0 w_k dx w_k \longrightarrow u_0 \ \text{ in } \ \ H^2(\Omega) \cap H_0^1(\Omega), \tag{3.3}$$

$$u_t^n(0) = u_1^n = \sum_{k=1}^{n} \int_\Omega u_1 w_k dx w_k \longrightarrow u_1 \ \text{ in } \ \ H_0^1(\Omega). \tag{3.4}$$

By standard methods in differential equations, we prove the existence of solutions to $(3.2) - (3.4)$ on some interval $[0, t_n)$, $0 < t_n < T$. In order to extend the solution of $(3.2) - (3.4)$ to the whole interval $[0, T]$, we need the following a priori estimate.

Step 1. (The first priori estimate) Replacing w by $2u_t^n(t)$ in (3.2), we have

$$\begin{aligned} &\tfrac{d}{dt} \left[\|u_t^n(t)\|_2^2 + \mu(t) \|\nabla u^n(t)\|_2^2 \right] + 2\alpha \|\nabla u_t^n(t)\|_2^2 + 2\chi(t) \|u_t^n(t)\|_2^2 \\ &\quad = \mu'(t) \|\nabla u^n(t)\|_2^2 + 2 \int_0^t g(t-\tau) \int_\Omega \nabla u^n(\tau) \cdot \nabla u_t^n(t) dx d\tau \\ &\quad +2 \int_\Omega f_1(x, t) u_t^n(t) dx \leq \mu'(t) \|\nabla u^n(t)\|_2^2 + \|\nabla u_t^n(t)\|_2^2 \\ &\quad + \|g\|_{L^1} \int_0^t g(t-\tau) \|\nabla u^n(\tau)\|_2^2 d\tau + \|f_1\|_2^2 + \|u_t^n(t)\|_2^2. \end{aligned} \tag{3.5}$$

Then, integrating (3.5) from 0 to t, we get

$$\begin{aligned} &\|u_t^n(t)\|_2^2 + \mu(t)\|\nabla u^n(t)\|_2^2 + (2\alpha - 1)\int_0^t \|\nabla u_\tau^n(\tau)\|_2^2 d\tau \le c_1 \\ &+ \int_0^t \left[1 + \frac{1}{\mu(\tau)}\left(|\mu'(\tau)| + \|g\|_{L^1}^2\right)\right]\left[\|u_\tau^n(\tau)\|_2^2 + \mu(\tau)\|\nabla u^n(\tau)\|_2^2\right] d\tau, \end{aligned} \tag{3.6}$$

where

$$c_1 = \|u_1^n\|_2^2 + \mu(0)\|\nabla u_0^n\|_2^2 + \int_0^t \|f_1\|_2^2 dt.$$

Taking into account (3.3) and (3.4), we obtain from Gronwall's Lemma the first priori estimate

$$\|u_t^n(t)\|_2^2 + \mu(t)\|\nabla u^n(t)\|_2^2 + \int_0^t \|\nabla u_t^n(\tau)\|_2^2 d\tau \le L_1, \tag{3.7}$$

for all $t \in [0,T]$. Here L_1 is a positive constant independent of $n \in \mathbb{N}$ and $t \in [0,T]$.
Step 2. (The second priori estimate) Replacing ω by $u_{tt}^n(t)$ in (3.2), we have

$$\begin{aligned} &\tfrac{d}{dt}\left[\mu(t)\int_\Omega \nabla u^n(t)\cdot\nabla u_t^n(t)dx + \frac{\alpha}{2}\|\nabla u_t^n(t)\|_2^2 + \frac{\chi(t)}{2}\|u_t^n(t)\|_2^2\right] \\ &\quad +\|u_{tt}^n(t)\|_2^2 = \mu'(t)\int_\Omega \nabla u^n(t)\cdot\nabla u_t^n(t)dx + \mu(t)\|\nabla u_t^n(t)\|_2^2 \\ &\quad +\tfrac{\chi'(t)}{2}\|u_t^n(t)\|_2^2 + \tfrac{d}{dt}\left(\int_0^t g(t-\tau)\int_\Omega \nabla u^n(\tau)\cdot\nabla u_t^n(t)dxd\tau\right) \\ &\quad -g(0)\int_\Omega \nabla u^n(t)\cdot\nabla u_t^n(t)dx + \int_\Omega f_1(x,t)u_{tt}^n(t))dx \\ &\quad -\int_0^t g'(t-\tau)\int_\Omega \nabla u^n(\tau)\cdot\nabla u_t^n(t)dxd\tau. \end{aligned} \tag{3.8}$$

By $(A1)$, Hölder's inequality and Young's inequality, one has than we have

$$\begin{aligned} -\int_0^t g'(t-\tau)\int_\Omega \nabla u^n(\tau)\cdot\nabla u_t^n(t)dxd\tau &\le \frac{1}{2}\|\nabla u_t^n(t)\|_2^2 \\ &+\tfrac{\xi_1^2\|g\|_{L^1}}{2}\int_0^t g(t-\tau)\|\nabla u^n(\tau)\|_2^2 d\tau. \end{aligned} \tag{3.9}$$

Since $\mu(t) \ge m_0$ and from (3.7) we obtain

$$\begin{aligned} -g(0)\int_\Omega \nabla u^n(t)\cdot\nabla u_t^n(t)dx &\le \frac{1}{2}\|\nabla u_t^n(t)\|_2^2 + \frac{g(0)^2}{2}\|\nabla u^n(t)\|_2^2 \\ &\le \tfrac{1}{2}\|\nabla u_t^n(t)\|_2^2 + \tfrac{g(0)^2 L_1}{2m_0}. \end{aligned} \tag{3.10}$$

Since $\chi(t)$ is C^1-function on $[0,\infty)$ and using (3.7) we infer that

$$\frac{\chi'(t)}{2}\|u_t^n(t)\|_2^2 \le \frac{A_1}{2}\|u_t^n(t)\|_2^2 \le \frac{A_1}{2}L_1. \tag{3.11}$$

Moreover,

$$\begin{aligned} \left|\mu'(t)\int_\Omega \nabla u^n(t)\cdot\nabla u_t^n(t)dx\right| &\le \tfrac{1}{2}\|\nabla u_t^n(t)\|_2^2 + \tfrac{M_1^2}{2}\|\nabla u^n(t)\|_2^2 \\ &\le \tfrac{1}{2}\|\nabla u_t^n(t)\|_2^2 + \tfrac{M_1^2 L_1}{2m_0}, \end{aligned} \tag{3.12}$$

where $M_1 = \sup_{0\le t\le T}\{|\mu'(t)|\}$ and $A_1 = \max_{0\le t\le T}\{|\chi'(t)|\}$. Then, by using (3.9) – (3.12), we obtain from (3.8)

$$\begin{aligned}
&\tfrac{d}{dt}\left[\mu(t)\int_\Omega \nabla u^n(t)\cdot\nabla u_t^n(t)dx + \frac{\alpha}{2}\|\nabla u_t^n(t)\|_2^2 + \frac{\chi(t)}{2}\|u_t^n(t)\|_2^2\right]\\
&\quad +\tfrac{1}{2}\|u_{tt}^n(t)\|_2^2 \le c_2 + \tfrac{\xi_1^2\|g\|_{L^1}}{2}\int_0^t g(t-\tau)\|\nabla u^n(\tau)\|_2^2 d\tau\\
&\quad +\tfrac{d}{dt}\left(\int_0^t g(t-\tau)\int_\Omega \nabla u^n(\tau)\cdot\nabla u_t^n(t)dxd\tau\right)\\
&\quad +\left(\tfrac{3}{2}+M_2\right)\|\nabla u_t^n(t)\|_2^2,
\end{aligned} \tag{3.13}$$

where $c_2 = \left(\frac{g(0)^2+M_1^2+A_1m_0}{2m_0}\right)L_1 + \frac{1}{2}\|f_1\|_2^2$ and $M_2 = \sup_{0\le t\le T}\{|\mu(t)|\}$. Thus, integrating (3.13) over $(0,t)$, we obtain

$$\begin{aligned}
&\tfrac{\alpha}{2}\|\nabla u_t^n(t)\|_2^2 + \tfrac{1}{2}\int_0^t \|u_{\tau\tau}^n(\tau)\|_2^2 dt + \frac{\chi(t)}{2}\|u_t^n(t)\|_2^2\\
&\le c_3 + \mu(t)\left|\int_\Omega \nabla u^n(t)\cdot\nabla u_t^n(t)dx\right| + \mu(0)\left|\int_\Omega \nabla u_0^n\cdot\nabla u_1^n dx\right|\\
&+\left(M_2+\tfrac{3}{2}\right)\int_0^t \|\nabla u_\tau^n(\tau)\|_2^2 d\tau + \int_0^t g(t-\tau)\int_\Omega \nabla u^n(\tau)\cdot\nabla u_t^n(t)dxd\tau,
\end{aligned} \tag{3.14}$$

where $\left(c_3 = c_2 + \xi_1^2\|g\|_{L^1}^2 L_1\right)T + \frac{\alpha}{2}\|\nabla u_1^n\|_2^2 + \frac{\chi(0)}{2}\|u_1^n\|_2^2$. We note that using the inequality $ab \le \frac{1}{4\eta}a^2 + \eta b^2$, where $\eta > 0$ is arbitrary, it follows that

$$\begin{aligned}
&\int_0^t g(t-\tau)\int_\Omega \nabla u^n(\tau)\cdot\nabla u_t^n(t)dxd\tau \le \eta\|\nabla u_t^n(t)\|_2^2\\
&\quad +\tfrac{1}{4\eta}\|g\|_{L^1(0,\infty)}\|g\|_{L^\infty(0,\infty)}\int_0^t \|\nabla u^n(\tau)\|_2^2 d\tau \le \eta\|\nabla u_t^n(t)\|_2^2\\
&\quad +\frac{\|g\|_{L^1(0,\infty)}\|g\|_{L^\infty(0,\infty)}}{4\eta m_0}L_1 T,
\end{aligned} \tag{3.15}$$

and

$$\begin{aligned}
\mu(t)\left|\int_\Omega \nabla u^n(t)\cdot\nabla u_t^n(t)dx\right| &\le \eta\|\nabla u_t^n(t)\|_2^2 + \frac{M_2^2}{4\eta}\|\nabla u^n(t)\|_2^2\\
&\le \eta\|\nabla u_t^n(t)\|_2^2 + \frac{M_2^2}{4\eta m_0}L_1.
\end{aligned} \tag{3.16}$$

By plugging (3.15) and (3.16) into (3.14) with $0 < \eta \le \frac{\alpha}{4}$, we obtain from $\chi(t) \ge 0$ that

$$\begin{aligned}
\left(\tfrac{\alpha}{2}-2\eta\right)\|\nabla u_t^n(t)\|_2^2 \quad &+\tfrac{1}{2}\int_0^t \|u_{\tau\tau}^n(\tau)\|_2^2 dt \le c_4\\
&+\left(M_2+\tfrac{3}{2}\right)\int_0^t \|\nabla u_\tau^n(\tau)\|_2^2 d\tau,
\end{aligned} \tag{3.17}$$

where

$$c_4 = c_3 + \mu(0)\|\nabla u_0^n\|_2\|\nabla u_1^n\|_2 + \frac{M_2^2}{4\eta m_0}L_1 + \frac{\|g\|_{L^1}\|g\|_{L^\infty}}{4\eta m_0}L_1 T. \tag{3.18}$$

Taking into account $(3.3)-(3.4)$, we obtain from Gronwall's Lemma the second priori estimate

$$\|\nabla u_t^n(t)\|_2^2 + \int_0^t \|u_{\tau\tau}^n(\tau)\|_2^2 d\tau \le L_2, \tag{3.19}$$

for all $t \in [0,T]$. Here L_2 is a positive constant independent of $n \in \mathbb{N}$ and $t \in [0,T]$.
Step 3. (The third priori estimate) Replacing ω by $-\Delta u^n(t)$ in (3.2), we have

$$\begin{aligned}
&\tfrac{d}{dt}\left[-\int_\Omega u_t^n(t)\Delta u^n(t)dx + \frac{\alpha}{2}\|\Delta u^n(t)\|_2^2 + \frac{\chi(t)}{2}\|\nabla u^n(t)\|_2^2\right] \\
&\quad -\|\nabla u_t^n(t)\|_2^2 + \mu(t)\|\Delta u^n(t)\|_2^2 \\
&\quad = \tfrac{\chi'(t)}{2}\|\nabla u^n(t)\|_2^2 + \int_0^t g(t-\tau)\int_\Omega \Delta u^n(\tau)\cdot\Delta u^n(t)dxd\tau \\
&\quad + \int_\Omega f_1(x,t)(-\Delta u^n(t))dx \le \frac{A_1}{2}\|\nabla u^n(t)\|_2^2 + 2\eta\|\Delta u^n(t)\|_2^2 \\
&\quad + \tfrac{\|g\|_{L^1}}{4\eta}\int_0^t g(t-\tau)\|\Delta u^n(\tau)\|_2^2 d\tau + \frac{1}{4\eta}\|f_1\|_2^2,
\end{aligned} \tag{3.20}$$

where $0 < \eta \le \frac{m_0}{2}$ is some positive constant. From $\mu(t) \ge m_0 > 0$, we deduce by integration

$$\begin{aligned}
&\tfrac{\alpha}{2}\|\Delta u^n(t)\|_2^2 + (m_0 - 2\eta)\int_0^t \|\Delta u^n(\tau)\|_2^2 d\tau + \frac{\chi(t)}{2}\|\nabla u^n(t)\|_2^2 \\
&\le \int_0^t \|\nabla u_\tau^n(\tau)\|_2^2 dt + \frac{A_1}{2}\int_0^t \|\nabla u^n(\tau)\|_2^2 d\tau + \left|\int_\Omega u_t^n(t)\Delta u^n(t)dx\right| \\
&\quad + \left|\int_\Omega u_t^n(0)\Delta u^n(0)dx\right| + \tfrac{1}{4\eta}\int_0^t \|f_1\|_2^2 dt \\
&\quad + \tfrac{\alpha}{2}\|\Delta u_0^n\|_2^2 + \tfrac{\chi(0)}{2}\|\nabla u_0^n\|_2^2 + \tfrac{\|g\|_{L^1}^2}{4\eta}\int_0^t \|\Delta u^n(\tau)\|_2^2 d\tau \\
&\quad \le c_5 + \left|\int_\Omega u_t^n(t)\Delta u^n(t)dx\right| + \tfrac{\|g\|_{L^1}^2}{4\eta}\int_0^t \|\Delta u^n(\tau)\|_2^2 d\tau,
\end{aligned} \tag{3.21}$$

where

$$c_5 = \|u_1^n\|_2\|\Delta u_0^n\|_2 + \frac{\alpha}{2}\|\Delta u_0^n\|_2^2 + \frac{1}{4\eta}\int_0^t \|f_1\|_2^2 d\tau + \frac{\chi(0)}{2}\|\nabla u_0^n\|_2^2 + \left(\frac{A_1}{m_0}L_1 + L_2\right)T.$$

We note that using the inequality $ab \le \frac{1}{4}a^2 + b^2$, it follows that

$$\int_\Omega u_t^n(t)\Delta u^n(t)dx \le \frac{1}{4}\|\Delta u^n(t)\|_2^2 + \|u_t^n(t)\|_2^2. \tag{3.22}$$

Plugging (3.22) into (3.21), we obtain from $\chi(t) \ge m_1 > 0$ that

$$\begin{aligned}
&\left(\tfrac{\alpha}{2} - \tfrac{1}{4}\right)\|\Delta u^n(t)\|_2^2 + (m_0 - 2\eta)\int_0^t \|\Delta u^n(\tau)\|_2^2 d\tau \\
&\qquad + \tfrac{\chi(t)}{2}\|\nabla u^n(t)\|_2^2 \le c_6 + \tfrac{\|g\|_{L^1}^2}{4\eta}\int_0^t \|\Delta u_m(\tau)\|_2^2 d\tau,
\end{aligned} \tag{3.23}$$

where

$$c_6 = c_5 + L_1.$$

Taking into account (3.3) – (3.4), we obtain from Gronwall's Lemma the third priori estimate,

$$\|\Delta u^n(t)\|_2^2 + \int_0^t \|\Delta u^n(\tau)\|_2^2 d\tau \leq L_3, \tag{3.24}$$

for all $t \in [0,T]$ and L_3 is a positive constant independent of $n \in \mathbb{N}$ and $t \in [0,T]$.

Step 4. Let $p \geq n$ be two natural numbers, and consider $z^n = u^p - u^n$. Then, applying the same way as in the estimate step 1 and step 3 and observing that $\{u_0^n\}$ and $\{u_1^n\}$ are Cauchy sequence in $H_0^1(\Omega) \cap H^2(\Omega)$ and $H_0^1(\Omega)$, respectively, we deduce for all $t \in [0,T]$

$$\|z_t^n(t)\|_2^2 + \mu(t)\|\nabla z^n(t)\|_2^2 + \int_0^t \|\nabla z_\tau^n(\tau)\|_2^2 d\tau \to 0, \tag{3.25}$$

and

$$\|\Delta z^n(t)\|_2^2 + \int_0^t \|\Delta z^n(\tau)\|_2^2 d\tau \to 0, \quad \text{as} \quad n \to \infty. \tag{3.26}$$

Therefore, (3.7), (3.19), (3.24), (3.25) and (3.26), we see that

$$u^n \to u \ \text{ strongly in } \ C(0,T;H_0^1(\Omega)), \tag{3.27}$$

$$u_t^n \to u_t \ \text{ strongly in } \ C(0,T;L^2(\Omega)). \tag{3.28}$$

$$u_t^n \to u_t \ \text{ strongly in } \ L^2(0,T;H_0^1(\Omega)), \tag{3.29}$$

$$u_{tt}^n \to u_{tt} \ \text{ weakly in } \ L^2(0,T;L^2(\Omega)). \tag{3.30}$$

Then (3.27) – (3.30) are sufficient to pass the limit in (3.2) to obtain in $L^2(0,T;H^{-1}(\Omega))$

$$u_{tt} - \mu(t)\Delta u + \int_0^t g(t-\tau)\Delta u(\tau)d\tau - \alpha\Delta u_t + \chi(t)u_t = f_1(x,t). \tag{3.31}$$

Next, we want to show the uniqueness of (3.1). Let $u^{(1)}$ and $u^{(2)}$ be two solutions of (3.1). Then $y = u^{(1)} - u^{(2)}$ satisfies for $\omega \in H_0^1(\Omega)$

$$\begin{aligned}
&\mu(t)\int_\Omega \nabla y(t)\cdot\nabla\omega dx - \int_0^t g(t-\tau)\int_\Omega \nabla y(\tau)\cdot\nabla\omega dx d\tau \\
&+ \int_\Omega y_{tt}(t)\omega dx + \alpha\int_\Omega \nabla y_t(t)\cdot\nabla\omega dx + \chi(t)\int_\Omega y_t(t)\omega dx = 0, \\
&y(x,0) = 0, \quad y_t(x,0) = 0, \quad x \in \Omega, \\
&y(x,t) = 0, \quad x \in \partial\Omega, \ t \geq 0.
\end{aligned} \tag{3.32}$$

Setting $w = 2y_t(t)$ in (3.32), then as in deriving (3.7), we see that

$$\begin{aligned}
&\|y_t(t)\|_2^2 + \mu(t)\|\nabla y(t)\|_2^2 + (2\alpha - 1)\int_0^t \|\nabla y_\tau(\tau)\|_2^2 d\tau \\
&\leq \int_0^t \left[1 + \frac{1}{\mu(\tau)}\left(|\mu'(\tau)| + \|g\|_{L^1}^2\right)\right]\left[\|y_\tau(\tau)\|_2^2 + \mu(\tau)\|\nabla y(\tau)\|_2^2\right] d\tau.
\end{aligned} \tag{3.33}$$

Thus employing Gronwall's Lemma, we conclude that

$$\|y_t(t)\|_2 = \|\nabla y(t)\|_2 = 0 \quad \text{for all} \quad t \in [0,T]. \tag{3.34}$$

Therefore, we have the uniqueness. This finishes the proof of Lemma 3.1. □

Now, let us prove the local existence of the problem (1.1).

Theorem 3.2. *Assume that (A1), (A2) and (A3) are fulfilled. Suppose that* $u_0 \in H^2(\Omega) \cap H_0^1(\Omega)$, $u_1 \in H_0^1(\Omega)$ *be given. Then there exists a unique solution* u *of* (1.1) *satisfying* $u \in C([0,T]; H^2(\Omega) \cap H_0^1(\Omega))$ *and* $u_t \in C([0,T]; L^2(\Omega)) \cap L^2(0,T; H_0^1(\Omega))$, *and at least one of the following statements is valid:*

$$\begin{aligned} &(i)\ \ T = \infty, \\ &(ii)\ e\big(u(t)\big) \equiv \|u_t(t)\|_2^2 + \|\Delta u(t)\|_2^2 \to \infty \quad as \quad t \to T^-. \end{aligned} \tag{3.35}$$

Proof. Define the following two-parameter space:

$$X_{T,R_0} = \left\{ \begin{array}{l} v \in C([0,T]; H_0^1(\Omega) \cap H^2(\Omega)), \\ v_t \in C([0,T]; L^2(\Omega)) \cap L^2(0,T; H_0^1(\Omega)) : \\ e\big(v(t)\big) \le R_0^2,\ t \in [0,T], \quad \text{with} \quad v(0) = u_0, \quad v_t(0) = u_1. \end{array} \right\},$$

for $T > 0$, $R_0 > 0$. Then X_{T,R_0} is a complete metric space with the distance

$$d(y,z) = \sup_{0 \le t \le T} e\left(y(t) - z(t)\right)^{\frac{1}{2}}, \tag{3.36}$$

where $y, z \in X_{T,R_0}$. Given $v \in X_{T,R_0}$, we consider the following problem

$$\left\{ \begin{array}{l} u_{tt} - \psi\left(\left\|\nabla v\right\|_2^2\right) \Delta u - \alpha \Delta u_t + \displaystyle\int_0^t g(t-\tau) \Delta u(\tau) d\tau \\ \qquad +M\left(\left\|\nabla v\right\|_2^2\right) u_t = f(v), \ \text{ in } \ \ \Omega \times (0,T), \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad x \in \Omega, \\ u(x,t) = 0, \quad x \in \partial\Omega,\ t \ge 0. \end{array} \right. \tag{3.37}$$

By $(A2)$, we see that $f(v) \in L^2\big(0,T; L^2(\Omega)\big)$. Thus, by Lemma 3.1, we derive that problem (3.37) admits a unique solution $u \in C([0,T]; H^2(\Omega) \cap H_0^1(\Omega))$ and $u_t \in C([0,T]; L^2(\Omega)) \cap L^2(0,T; H_0^1(\Omega))$. Then, we define the nonlinear mapping $Sv = u$, and we would like to show that there exist $T > 0$ and $R_0 > 0$ such that S is a contraction mapping from X_{T,R_0} into itself. For this, we multiply the first equation of (3.37) by $2u_t$ and integrate it over Ω to get

$$\begin{aligned} &\tfrac{d}{dt}\left[\left(\psi\Big(\|\nabla v\|_2^2\Big) - \int_0^t g(\tau) d\tau\right) \|\nabla u(t)\|_2^2 + \big(g \circ \nabla u\big)(t)\right] \\ &\qquad + \tfrac{d}{dt}\left[\|u_t(t)\|_2^2\right] + 2\alpha \|\nabla u_t(t)\|_2^2 + 2M\Big(\left\|\nabla v\right\|_2^2\Big) \|u_t(t)\|_2^2 \\ &\qquad -\big(g' \circ \nabla u\big)(t) + g(t) \|\nabla u(t)\|_2^2 \\ &\qquad = \Big(\tfrac{d}{dt}\psi\Big(\left\|\nabla v\right\|_2^2\Big)\Big) \|\nabla u(t)\|_2^2 + 2\int_\Omega f(v) u_t dx. \end{aligned} \tag{3.38}$$

The equality in (3.38) is obtained, because

$$\begin{aligned} &-2\int_0^t \int_\Omega g(t-\tau) \nabla u(\tau) \cdot \nabla u_t(t) dx d\tau = -\big(g' \circ \nabla u\big)(t) \\ &+g(t) \|\nabla u(t)\|_2^2 + \tfrac{d}{dt}\left[\big(g \circ \nabla u\big)(t) - \int_0^t g(\tau) \|\nabla u(t)\|_2^2 d\tau\right], \end{aligned} \tag{3.39}$$

where

$$\big(g \circ \nabla u\big)(t) = \int_0^t g(t-\tau) \int_\Omega |\nabla u(\tau) - \nabla u(t)|^2 dx d\tau.$$

Next, multiplying the first equation of (3.37) by $-2\Delta u$, and integrating it over Ω, we have

$$\begin{aligned}
&\tfrac{d}{dt}\left[\alpha\|\Delta u(t)\|_2^2 - 2\int_\Omega u_t\Delta u dx + M\Big(\big\|\nabla v\big\|_2^2\Big)\|\nabla u(t)\|_2^2\right]\\
&+2\psi\Big(\big\|\nabla v\big\|_2^2\Big)\|\Delta u(t)\|_2^2 - 2\|\nabla u_t(t)\|_2^2\\
&= \Big(\tfrac{d}{dt}M\Big(\big\|\nabla v\big\|_2^2\Big)\Big)\|\nabla u(t)\|_2^2 - 2\int_\Omega f(v)\Delta u dx\\
&+2\int_0^t g(t-\tau)\Delta u(\tau)\cdot\Delta u(t) dx d\tau.
\end{aligned} \tag{3.40}$$

Multiplying (3.40) by ϵ, $0 \le \epsilon \le 1$, adding (3.38) together and taking into account $(A1)$ and $(A3)$, we obtain

$$\frac{d}{dt}e^*\big(u(t)\big) + 2(\alpha-\epsilon)\|\nabla u_t(t)\|_2^2 + 2\epsilon\psi\Big(\big\|\nabla v\big\|_2^2\Big)\|\Delta u(t)\|_2^2 \le I_1 + I_2 + I_3, \tag{3.41}$$

where

$$\begin{aligned}
e^*\big(u(t)\big) = \|u_t(t)\|_2^2 &+ \left(\psi\Big(\|\nabla v\|_2^2\Big) - \int_0^t g(\tau)d\tau\right)\|\nabla u(t)\|_2^2\\
&+\big(g\circ\nabla u\big)(t) + \epsilon\alpha\|\Delta u(t)\|_2^2 - 2\epsilon\int_\Omega u_t\Delta u dx\\
&+\epsilon M\Big(\big\|\nabla v\big\|_2^2\Big)\|\nabla u(t)\|_2^2.
\end{aligned} \tag{3.42}$$

$$I_1 = 2\int_\Omega f(v)\big(u_t - \epsilon\Delta u\big)dx,$$

$$I_2 = \left(\frac{d}{dt}\psi\Big(\big\|\nabla v\big\|_2^2\Big) + \epsilon\frac{d}{dt}M\Big(\big\|\nabla v\big\|_2^2\Big)\right)\|\nabla u(t)\|_2^2,$$

and

$$I_3 = 2\epsilon\int_0^t g(t-\tau)\Delta u(\tau)\cdot\Delta u(t)dxd\tau.$$

Estimate for $I_1 = 2\int_\Omega f(v)\big(u_t - \epsilon\Delta u\big)dx$. From $(A2)$ and making use of Hölder's inequality and Lemma 2.1, we have

$$\begin{aligned}
I_1 &= 2\int_\Omega f(v)\big(u_t - \epsilon\Delta u\big)dx\\
&\le 2\int_\Omega \Big|f(v)u_t\Big|dx + 2\epsilon\int_\Omega\Big|f(v)\Delta u\Big|dx\\
&\le 2K_3\int_\Omega \big|v\big|^{p-1}\big|u_t\big|dx + 2\epsilon K_3\int_\Omega\big|v\big|^{p-1}\big|\Delta u\big|dx\\
&\le 2K_3B_1^{2(p-1)}\|\Delta v\|_2^{p-1}\|u_t\|_2 + 2\epsilon K_3B_1^{2(p-1)}\|\Delta v\|_2^{p-1}\|\Delta u\|_2\\
&\le 2K_3B_1^{2(p-1)}R_0^{p-1}e\big(u(t)\big)^{\frac12} + 2\epsilon K_3B_1^{2(p-1)}R_0^{p-1}e\big(u(t)\big)^{\frac12}\\
&= 2K_3(1+\epsilon)B_1^{2(p-1)}R_0^{p-1}e\big(u(t)\big)^{\frac12}.
\end{aligned} \tag{3.43}$$

Estimate for $I_2 = \Big(\frac{d}{dt}\psi\Big(\big\|\nabla v\big\|_2^2\Big) + \epsilon\frac{d}{dt}M\Big(\big\|\nabla v\big\|_2^2\Big)\Big)\|\nabla u(t)\|_2^2$. First of all, we observe that

$$\begin{aligned}
\tfrac{d}{dt}\psi\Big(\big\|\nabla v\big\|_2^2\Big) &= 2\psi'\Big(\big\|\nabla v\big\|_2^2\Big)\int_\Omega \nabla v\cdot\nabla v_t dx\\
&\le 2M_3\|\Delta v\|_2\|v_t\|_2 \le 2M_3R_0^2,
\end{aligned} \tag{3.44}$$

where $M_3 = \sup\left\{|\psi'(s)|;\ 0 \le s \le B_1^2 R_0^2\right\}$, and

$$\begin{aligned} \epsilon \tfrac{d}{dt} M\Big(\big\|\nabla v\big\|_2^2\Big) &= 2\epsilon M'\Big(\big\|\nabla v\big\|_2^2\Big) \int_\Omega \nabla v \cdot \nabla v_t dx \\ &\le 2\epsilon A_2 \|\Delta v\|_2 \|v_t\|_2 \le 2\epsilon A_2 R_0^2, \end{aligned} \tag{3.45}$$

where $A_2 = \max\left\{|M'(s)|;\ 0 \le s \le B_1^2 R_0^2\right\}$. Then, from (3.44) ,(3.45) and using (3.35) we arrive at

$$I_2 \le 2B_1^2 R_0^2 (M_3 + \epsilon A_2) e\big(u(t)\big). \tag{3.46}$$

Estimate for $I_3 = 2\epsilon \int_0^t g(t-\tau)\Delta u(\tau)\cdot\Delta u(t) dx d\tau$. Using the inequality $ab \le \frac{1}{4\eta}a^2 + \eta b^2$, where $\eta > 0$ is arbitrary, we get

$$\begin{aligned} I_3 &= 2\epsilon \int_0^t g(t-\tau) \int_\Omega \Delta u(\tau)\cdot\Delta u(t) dx d\tau \\ &\le 2\epsilon\eta \|\Delta u(t)\|_2^2 + \epsilon \tfrac{\|g\|_{L^1}}{2\eta} \int_0^t g(t-\tau)\|\Delta u(\tau)\|_2^2 d\tau. \end{aligned} \tag{3.47}$$

Combining these inequalities with $0 < \eta < \frac{\|g\|_{L^1}}{2}$, we get

$$\begin{aligned} &\tfrac{d}{dt} e^*\big(u(t)\big) + 2(\alpha - \epsilon)\|\nabla u_t(t)\|_2^2 + 2\epsilon\left(\psi\Big(\|\nabla v\|_2^2\Big) - \eta\right)\|\Delta u(t)\|_2^2 \\ &\quad \le 2B_1^2 R_0^2 (M_3 + \epsilon A_2) e\big(u(t)\big) + 2K_3(1+\epsilon) B_1^{2(p-1)} R_0^{p-1} e\big(u(t)\big)^{\frac{1}{2}} \\ &\quad + \epsilon \tfrac{\|g\|_{L^1}}{2\eta} \int_0^t g(t-\tau)\|\Delta u(\tau)\|_2^2 d\tau. \end{aligned} \tag{3.48}$$

When we take $\epsilon = 0$ in (3.48), we see that

$$\begin{aligned} &\tfrac{d}{dt}\left[\left(\psi\Big(\|\nabla v\|_2^2\Big) - \int_0^t g(\tau)d\tau\right)\|\nabla u(t)\|_2^2 + \big(g\circ\nabla u\big)(t)\right] \\ &\quad + \tfrac{d}{dt}\left[\|u_t(t)\|_2^2\right] + 2\alpha\|\nabla u_t(t)\|_2^2 \\ &\quad \le 2B_1^2 R_0^2 M_3 e\big(u(t)\big) + 2K_3 B_1^{2(p-1)} R_0^{p-1} e\big(u(t)\big)^{\frac{1}{2}}. \end{aligned} \tag{3.49}$$

By Young's inequality, we get

$$2\epsilon \int_\Omega u_t \Delta u dx \le 2\epsilon\|u_t\|_2^2 + \frac{\epsilon}{2}\|\Delta u(t)\|_2^2.$$

Hence

$$\begin{aligned} &e^*\big(u(t)\big) \ge (1-2\epsilon)\|u_t\|_2^2 + \epsilon\left(\alpha - \tfrac{1}{2}\right)\|\Delta u(t)\|_2^2 + \big(g\circ\nabla u\big)(t) \\ &+ \epsilon M\Big(\big\|\nabla v\big\|_2^2\Big)\|\nabla u(t)\|_2^2 + \left(\psi\Big(\|\nabla v\|_2^2\Big) - \int_0^t g(\tau)d\tau\right)\|\nabla u(t)\|_2^2. \end{aligned} \tag{3.50}$$

Choosing $\epsilon = \frac{2}{5}$ and taking into account $(A1)$ and $(A3)$, we have

$$e^*\big(u(t)\big) \ge \frac{1}{5} e\big(u(t)\big), \tag{3.51}$$

and

$$\begin{aligned} &e^*\big(u_0\big) \le (1+2\epsilon)\|u_1\|_2^2 + \epsilon\left(\alpha + \tfrac{1}{2}\right)\|\Delta u_0\|_2^2 + \psi\Big(\|\nabla u_0\|_2^2\Big)\|\nabla u_0\|_2^2 \\ &+ \epsilon M\Big(\big\|\nabla u_0\big\|_2^2\Big)\|\nabla u_0\|_2^2 \le 2\|u_1\|_2^2 + \left(\alpha + \tfrac{1}{2}\right)\|\Delta u_0\|_2^2 \\ &+ \psi\Big(\|\nabla u_0\|_2^2\Big)\|\nabla u_0\|_2^2 + M\Big(\big\|\nabla u_0\big\|_2^2\Big)\|\nabla u_0\|_2^2 = c^*. \end{aligned} \tag{3.52}$$

Integrating (3.48) over $(0,t)$, we get

$$\begin{aligned} &e^*\big(u(t)\big) + \tfrac{4}{5}\left(m_0 - \eta - \tfrac{\|g\|_{L^1}^2}{4\eta}\right)\int_0^t \|\Delta u(\tau)\|_2^2 d\tau \\ &\leq e^*\big(u_0\big) + \int_0^t \left[C_1 e^*\big(u(\tau)\big) + C_2 e^*\big(u(\tau)\big)^{\frac{1}{2}}\right] d\tau, \end{aligned} \tag{3.53}$$

where $C_1 = 10B_1^2 R_0^2 (M_3 + \frac{2}{5}A_2)$ and $C_2 = \frac{14\sqrt{5}}{5} K_1 B_1^{2(p-1)} R_0^{p-1}$. Taking $\eta = \frac{\|g\|_{L^1}}{2\eta}$ in (3.53), then from $(A1)$, we deduce

$$\begin{aligned} e^*\big(u(t)\big) &\leq e^*\big(u_0\big) + \int_0^t \left[C_1 e^*\big(u(\tau)\big) + C_2 e^*\big(u(\tau)\big)^{\frac{1}{2}}\right] d\tau \\ &\leq c^* + \int_0^t \left[C_1 e^*\big(u(\tau)\big) + C_2 e^*\big(u(\tau)\big)^{\frac{1}{2}}\right] d\tau. \end{aligned} \tag{3.54}$$

Hence, by Gronwall's inequality, we have

$$e^*\big(u(t)\big) \leq \left(\sqrt{c^*} + \frac{C_2}{2}T\right)^2 e^{C_1 T}. \tag{3.55}$$

Then, by (3.51), we obtain

$$e\big(u(t)\big) \leq 5\left(\sqrt{c^*} + \frac{C_2}{2}T\right)^2 e^{C_1 T}. \tag{3.56}$$

for any $t \in (0,T]$. Therefore, we see that for the parameters T and R_0 satisfy

$$5\left(\sqrt{c^*} + \frac{C_2}{2}T\right)^2 e^{C_1 T} \leq R_0^2. \tag{3.57}$$

That means S maps X_{T,R_0} into itself. Moreover, by Lemma 3.1,

$$u \in C^0([0,T]; H^2(\Omega) \cap H_0^1(\Omega)) \cap C^1([0,T]; L^2(\Omega)).$$

On the other hand, it follows from (3.49) and (3.56) that

$$u_t \in L^2(0,T; H_0^1(\Omega)).$$

Next, we shall verify that S is a contraction mapping with respect to the metric $d(\cdot,\cdot)$. We take $v_1, v_2 \in X_{T,R_0}$, and denote $u^{(1)} = Sv_1$ and $u^{(2)} = Sv_2$. Hereafter we suppose that (3.57) is valid, thus $u^{(1)}, u^{(2)} \in X_{T,R_0}$. Putting $w(t) = \big(u^{(1)} - u^{(2)}\big)(t)$, then w satisfies

$$\begin{cases} w_{tt} - \psi\left(\|\nabla v_1\|_2^2\right)\Delta w + \displaystyle\int_0^t g(t-\tau)\Delta w(\tau)d\tau - \alpha\Delta w_t \\ \qquad + M\left(\|\nabla v_1\|_2^2\right) w_t = f(v_1) - f(v_2) \\ \qquad + \left[\psi\left(\|\nabla v_1\|_2^2\right) - \psi\left(\|\nabla v_2\|_2^2\right)\right]\Delta u^{(2)} \\ \qquad + \left[M\left(\|\nabla v_2\|_2^2\right) - M\left(\|\nabla v_1\|_2^2\right)\right] u_t^{(2)}, \\ w(0) = 0, \quad w_t(0) = 0, \\ w(x,t) = 0, \quad x \in \partial\Omega,\ t \geq 0. \end{cases} \tag{3.58}$$

We multiply the first equation of (3.58) by $2w_t$ and integrate it over Ω to get

$$\begin{aligned}&\tfrac{d}{dt}\left[\left(\psi\Big(\|\nabla v_1\|_2^2\Big) - \int_0^t g(\tau)d\tau\right)\|\nabla w(t)\|_2^2 + (g\circ\nabla w)(t)\right]\\ &\quad +\tfrac{d}{dt}\left[\|w_t(t)\|_2^2\right] + 2\alpha\|\nabla w_t(t)\|_2^2 \le I_4 + I_5 + I_6 + I_7.\end{aligned} \tag{3.59}$$

We now estimate I_4-I_7 (defined as below), respectively.

$$I_4 = \left(\frac{d}{dt}\psi\left(\|\nabla v_1\|_2^2\right)\right)\|\nabla w(t)\|_2^2 \le 2M_3B_1^2R_0^2 e\big(w(t)\big), \tag{3.60}$$

$$\begin{aligned}I_5 &= 2\int_\Omega \Big[f(v_1) - f(v_2)\Big]w_t dx\\ &\le 2K_3\int_\Omega\Big(|v_1|^{p-2} + |v_2|^{p-2}\Big)|v_1 - v_2|w_t dx\\ &\le 2K_3\Big[\|v_1\|_{N(p-2)}^{p-2} + \|v_2\|_{N(p-2)}^{p-2}\Big]\|v_1 - v_2\|_{\frac{2N}{N-2}}\|w_t\|_2\\ &\le 4K_3B_1^{2(p-1)}R_0^{p-2}e\big(v_1 - v_2\big)^{\frac12}e\big(w(t)\big)^{\frac12},\end{aligned} \tag{3.61}$$

$$\begin{aligned}I_6 &= 2\left[\psi\left(\|\nabla v_1\|_2^2\right) - \psi\left(\|\nabla v_2\|_2^2\right)\right]\int_\Omega \Delta u^{(2)}w_t dx\\ &\le 2L\Big(\|\nabla v_1\|_2 + \|\nabla v_2\|_2\Big)\|\nabla v_1 - \nabla v_2\|_2\left\|\Delta u^{(2)}\right\|_2\|w_t\|_2\\ &\le 4LB_1^2R_0^2 e\big(v_1 - v_2\big)^{\frac12}e\big(w(t)\big)^{\frac12},\end{aligned} \tag{3.62}$$

where $L = L(R)$ is the Lipschitz constant of $\psi(s)$ in $[0, R_0]$.
Estimate for $I_7 = 2\left[M\left(\|\nabla v_2\|_2^2\right) - M\left(\|\nabla v_1\|_2^2\right)\right]\int_\Omega u_t^{(2)}w_t dx$. Assumption $(A3)$ gives

$$\begin{aligned}\left|M\left(\|\nabla v_2\|_2^2\right) - M\left(\|\nabla v_1\|_2^2\right)\right| &= \left|\int_{\|\nabla v_1\|_2^2}^{\|\nabla v_2\|_2^2} M'(r)dr\right|\\ &\le \int_{\|\nabla v_1\|_2^2}^{\|\nabla v_2\|_2^2}|M'(r)|\,dr \le C_*\left|\|\nabla v_2\|_2^2 - \|\nabla v_1\|_2^2\right|\\ &\le C_*\Big(\|\nabla v_1\|_2 + \|\nabla v_1\|_2\Big)\|\nabla v_2 - \nabla v_1\|_2,\end{aligned} \tag{3.63}$$

where C_* is a positive constant. From (3.63) and (3.35), we have

$$\begin{aligned}I_7 &= 2\left[M\left(\|\nabla v_2\|_2^2\right) - M\left(\|\nabla v_1\|_2^2\right)\right]\int_\Omega u_t^{(2)}w_t dx\\ &\le 2C_*\Big(\|\nabla v_1\|_2 + \|\nabla v_1\|_2\Big)\|\nabla(v_2 - v_1)\|_2\left\|u_t^{(2)}\right\|_2\|w_t\|_2\\ &\le 2C_*B_1^2R_0^2 e\big(v_1 - v_2\big)^{\frac12}e\big(w(t)\big)^{\frac12}.\end{aligned} \tag{3.64}$$

Inserting (3.60) – (3.64) in (3.59), we get

$$\begin{aligned}&\tfrac{d}{dt}\left[\left(\psi\Big(\|\nabla v_1\|_2^2\Big) - \int_0^t g(\tau)d\tau\right)\|\nabla w(t)\|_2^2 + (g\circ\nabla w)(t)\right]\\ &\quad\tfrac{d}{dt}\left[\|w_t(t)\|_2^2\right] + 2\alpha\|\nabla w_t(t)\|_2^2\\ &\quad\le C_3 e\big(w(t)\big) + C_4 e\big(v_1 - v_2\big)^{\frac12}e\big(w(t)\big)^{\frac12},\end{aligned} \tag{3.65}$$

where $C_3 = 2M_3B_1^2R_0^2$ and $C_4 = 4K_3B_1^{2(p-1)}R_0^{p-2} + 4LB_1^2R_0^2 + 2C_*B_1^2R_0^2$.
On the other hand, multiplying the first equation in (3.58) by $-2\Delta w$, and integrating it over Ω, we get

$$\begin{aligned}&\tfrac{d}{dt}\left\{\alpha\|\Delta w(t)\|_2^2 - 2\int_\Omega w_t\Delta w dx + M\Big(\big\|\nabla v_1\big\|_2^2\Big)\|\nabla w(t)\|_2^2\right\}\\ &+2\psi\Big(\big\|\nabla v_1\big\|_2^2\Big)\|\Delta w(t)\|_2^2 - 2\|\nabla w_t\|_2^2 = I_8 + I_9 + I_{10} + I_{11} + I_{12}.\end{aligned} \tag{3.66}$$

We now estimate I_8-I_{11} (defined as below), respectively.
Applying the similar arguments as in estimating I_i, $i = 2, 3, 5, 6, 7$, we observe that

$$I_8 = \left(\frac{d}{dt}M\Big(\big\|\nabla v_1\big\|_2^2\Big)\right)\|\nabla w(t)\|_2^2 \leq 2A_2R_0^2B_1^2e\big(w(t)\big), \tag{3.67}$$

$$\begin{aligned}I_9 &= -2\int_\Omega\Big[f(v_1) - f(v_2)\Big]\Delta w dx\\ &\leq 4K_3B_1^{2(p-1)}R_0^{p-2}e\big(v_1 - v_2\big)^{\frac{1}{2}}e\big(w(t)\big)^{\frac{1}{2}},\end{aligned} \tag{3.68}$$

$$\begin{aligned}I_{10} &= 2\left[\psi\left(\|\nabla v_1\|_2^2\right) - \psi\left(\|\nabla v_2\|_2^2\right)\right]\int_\Omega \Delta u^{(2)}\Delta w dx\\ &\leq 4LB_1^2R_0^2e\big(v_1 - v_2\big)^{\frac{1}{2}}e\big(w(t)\big)^{\frac{1}{2}},\end{aligned} \tag{3.69}$$

$$\begin{aligned}I_{11} &= 2\left[M\left(\|\nabla v_2\|_2^2\right) - M\left(\|\nabla v_1\|_2^2\right)\right]\int_\Omega \Delta u^{(2)}\Delta w dx\\ &\leq 2C_*B_1^2R_0^2e\big(v_1 - v_2\big)^{\frac{1}{2}}e\big(w(t)\big)^{\frac{1}{2}},\end{aligned} \tag{3.70}$$

and

$$\begin{aligned}I_{12} &= 2\int_0^t g(t-\tau)\int_\Omega \Delta w(\tau)\cdot\Delta w(t)dxd\tau\\ &\leq 2\eta\|\Delta w(t)\|_2^2 + \tfrac{\|g\|_{L^1}}{2\eta}\int_0^t g(t-\tau)\|\Delta w(\tau)\|_2^2d\tau,\end{aligned} \tag{3.71}$$

where $\eta > 0$ is arbitrary. Combining these inequalities with $0 < \eta \leq \frac{\|g\|_{L^1}}{2}$, we get

$$\begin{aligned}&\tfrac{d}{dt}\left\{\alpha\|\Delta w(t)\|_2^2 - 2\int_\Omega w_t\Delta w dx + M\Big(\big\|\nabla v_1\big\|_2^2\Big)\|\nabla w(t)\|_2^2\right\}\\ &\quad+2\left(\psi\Big(\big\|\nabla v_1\big\|_2^2\Big) - 2\eta\right)\|\Delta w(t)\|_2^2 \leq C_4e\big(v_1 - v_2\big)^{\frac{1}{2}}e\big(w(t)\big)^{\frac{1}{2}}\\ &\quad+\tfrac{\|g\|_{L^1}}{2\eta}\int_0^t g(t-\tau)\|\Delta w(\tau)\|_2^2d\tau + 2\|\nabla w_t\|_2^2 + C_5e\big(w(t)\big),\end{aligned} \tag{3.72}$$

where $C_5 = 2A_2B_1^2R^2$. Multiplying (3.72) by ϵ, $0 < \epsilon \leq 1$, and adding (3.65) together, we obtain

$$\begin{aligned}&\tfrac{d}{dt}e^{**}\big(w(t)\big) + 2(\alpha - \epsilon)\|\nabla w_t\|_2^2 + 2\epsilon\left(\psi\Big(\big\|\nabla v_1\big\|_2^2\Big) - 2\eta\right)\|\Delta w(t)\|_2^2\\ &\quad\leq (C_3 + \epsilon C_5)e\big(w(t)\big) + (1+\epsilon)C_4e\big(v_1 - v_2\big)^{\frac{1}{2}}e\big(w(t)\big)^{\frac{1}{2}}\\ &\quad+\epsilon\tfrac{\|g\|_{L^1}}{2\eta}\int_0^t g(t-\tau)\|\Delta w(\tau)\|_2^2d\tau,\end{aligned} \tag{3.73}$$

where

$$\begin{aligned} e^{**}\big(w(t)\big) = {} & \|w_t(t)\|_2^2 + \left(\psi\Big(\|\nabla v_1\|_2^2\Big) - \int_0^t g(\tau)d\tau\right)\|\nabla w(t)\|_2^2 \\ & +(g\circ\nabla w)(t) + \epsilon\alpha\|\Delta w(t)\|_2^2 - 2\epsilon\int_\Omega w_t\Delta w dx \\ & +\epsilon M\Big(\big\|\nabla v_1\big\|_2^2\Big)\|\nabla w(t)\|_2^2. \end{aligned} \tag{3.74}$$

By using Young's inequality on the fifth term of right hand side of (3.74), we get

$$\begin{aligned} e^{**}\big(w(t)\big) \geq {} & (1-2\epsilon)\|w_t(t)\|_2^2 + \epsilon\left(\alpha - \tfrac{1}{2}\right)\|\Delta w(t)\|_2^2 \\ & + \left(\psi\Big(\|\nabla v_1\|_2^2\Big) - \int_0^t g(\tau)d\tau\right)\|\nabla w(t)\|_2^2 \\ & +(g\circ\nabla w)(t) + \epsilon M\Big(\big\|\nabla v_1\big\|_2^2\Big)\|\nabla w(t)\|_2^2. \end{aligned} \tag{3.75}$$

Choosing $\epsilon = \frac{2}{5}$ and by (2.1), (2.4), we have

$$e^{**}\big(w(t)\big) \geq \frac{1}{5}e\big(w(t)\big). \tag{3.76}$$

Then, applying the some way as in obtained (3.53) and taking $\eta = \frac{\|g\|_{L^1}}{2\eta}$, we deduce

$$\begin{aligned} e^{**}\big(w(t)\big) \leq {} & \int_0^t \left[5\left(C_3 + \frac{2}{5}C_5\right)e^{**}\big(w(t)\big)\right. \\ & \left. +\tfrac{7\sqrt{5}}{5}C_4 e\big(v_1 - v_2\big)^{\frac{1}{2}} e^{**}\big(w(t)\big)^{\frac{1}{2}}\right]d\tau + e^{**}\big(w(0)\big). \end{aligned} \tag{3.77}$$

Thus, applying Gronwall's Lemma and noting that $e^{**}\big(w(0)\big) = 0$, we have

$$e^{**}\big(w(t)\big) \leq \frac{49}{20}C_4^2T^2e^{5\left(C_3+\frac{2}{5}C_5\right)T}\sup_{0\leq t\leq T} e(v_1 - v_2). \tag{3.78}$$

By (3.36) and (3.76), we have

$$d(u^{(1)}, u^{(2)}) \leq C(T, R_0)^{\frac{1}{2}} d(v_1, v_2), \tag{3.79}$$

where

$$C(T, R_0)^{\frac{1}{2}} = \frac{49}{4}C_4^2T^2e^{5\left(C_3+\frac{2}{5}C_5\right)T}. \tag{3.80}$$

Hence, under inequality (3.57), S is a contraction mapping if $C(T, R_0) < 1$. Indeed, we choose R_0 sufficient large and T sufficient small so that (3.57) and (3.79) are satisfied at the same time. By applying Banach fixed point theorem, we obtain the local existence result.

The second statement of the theorem is proved by a standard continuation argument. Indeed, let $[0, T)$ be a maximal existence interval on which the solution of (1.1) exists. Suppose that $T < \infty$ and $\lim\limits_{t\to T^-}\left(\|u_t(t)\|_2^2 + \|\Delta u(t)\|_2^2\right) < \infty$. Then, there are a sequence $\{t_n\}$ and a constant $K > 0$ such that $t_n \to T^-$ as $n \to \infty$ and $\|u_t(t_n)\|_2^2 + \|\Delta u(t_n)\|_2^2 \leq K$, $n = 1, 2, \ldots$ Since for all $n \in \mathbb{N}$, there exists a unique solution of (1.1) with initial data $(u(t_n), u_t(t_n))$ on $[t_n, t_{n+\rho}]$, $\rho > 0$ depending on K and independent of $n \in \mathbb{N}$. Thus, we can get $T < t_n + \rho$ for $n \in \mathbb{N}$ large enough. It contradicts to the maximality of T. The proof of Theorem 3.2 is now completed. $\square$

4. Global existence and energy decay

In this section, we consider the global existence and energy decay of solutions for a kind of the problem (1.1):

$$\begin{cases} u_{tt} - \psi\left(\left\|\nabla u\right\|_2^2\right)\Delta u - \alpha\Delta u_t + \displaystyle\int_0^t g(t-\tau)\Delta u(\tau)d\tau \\ \quad +M\left(\left\|\nabla u\right\|_2^2\right)u_t = |u|^{p-2}u, \quad x\in\Omega,\ t>0, \\ u(x,0)=u_0(x), \quad u_t(x,0)=u_1(x), \quad x\in\Omega, \\ u(x,t)=0, \quad x\in\partial\Omega,\ t>0, \end{cases} \tag{4.1}$$

where $2 < p \leq \frac{2(N-1)}{N-2}$, $\alpha \geq 1$ and $\psi(r) = 1 + br^\gamma$, $b \geq 0$, $\gamma \geq 1$ and $r \geq 0$.
To obtain the results of this section, we now define some functionals as follows:

$$I_1(t) = I_1\big(u(t)\big) = \left(1 - \int_0^t g(\tau)d\tau\right)\|\nabla u(t)\|_2^2 + (g\circ\nabla u)(t) - \|u(t)\|_p^p, \tag{4.2}$$

$$I_2(t) = I_2\big(u(t)\big) = I_1(t) + b\|\nabla u(t)\|_2^{2(\gamma+1)}, \tag{4.3}$$

$$\begin{aligned} J(t) = J\big(u(t)\big) &= \tfrac{1}{2}\left(1 - \int_0^t g(\tau)d\tau\right)\|\nabla u(t)\|_2^2 + \tfrac{1}{2}(g\circ\nabla u)(t) \\ &\quad + \tfrac{b}{2(\gamma+1)}\|\nabla u(t)\|_2^{2(\gamma+1)} - \tfrac{1}{p}\|u(t)\|_p^p. \end{aligned} \tag{4.4}$$

We define the energy of the solution u of (4.1) by

$$\begin{aligned} E(t) = E\big(u(t)\big) &= \tfrac{1}{2}\|u_t(t)\|_2^2 + J\big(u(t)\big) = \tfrac{1}{2}\|u_t(t)\|_2^2 \\ &\quad + \tfrac{1}{2}\left(1 - \int_0^t g(\tau)d\tau\right)\|\nabla u(t)\|_2^2 + \tfrac{1}{2}(g\circ\nabla u)(t) \\ &\quad + \tfrac{b}{2(\gamma+1)}\|\nabla u(t)\|_2^{2(\gamma+1)} - \tfrac{1}{p}\|u(t)\|_p^p. \end{aligned} \tag{4.5}$$

Lemma 4.1. *$E(t)$ is a non-increasing function for $t \geq 0$, that is*

$$\begin{aligned} E'(t) \leq -\Big[& m_1\|u_t(t)\|_2^2 + \alpha\|\nabla u_t(t)\|_2^2 + \tfrac{K_2}{2}\big(g\circ\nabla u\big)(t) \\ & + \tfrac{1}{2}g(t)\|\nabla u(t)\|_2^2\Big] \leq 0, \quad \textit{for all} \quad t > 0. \end{aligned} \tag{4.6}$$

Proof. Multiplying the differential equation in (4.1) by u_t, integrating by parts over Ω and using $(A3)$, we obtain

$$\begin{aligned} & \tfrac{d}{dt}\left[\tfrac{1}{2}\|u_t(t)\|_2^2 + \tfrac{1}{2}\|\nabla u(t)\|_2^2 + \tfrac{b}{2(\gamma+1)}\|\nabla u(t)\|_2^{2(\gamma+1)} - \tfrac{1}{p}\|u(t)\|_p^p\right] \\ & \quad = -\alpha\|\nabla u_t(t)\|_2^2 - M\Big(\|\nabla u\|_2^2\Big)\|u_t(t)\|_2^2 \\ & \quad + \int_0^t\int_\Omega g(t-\tau)\nabla u(\tau)\cdot\nabla u_t(t)dxd\tau \\ & \leq -\alpha\|\nabla u_t(t)\|_2^2 - m_1\|u_t(t)\|_2^2 + \int_0^t\int_\Omega g(t-\tau)\nabla u(\tau)\cdot\nabla u_t(t)dxd\tau. \end{aligned}$$

Exploiting (3.39) on the third term on the right hand side of the above inequality and using $(A1)$, we have the result. $\square$

Lemma 4.2. *Let u be the solution of* (4.1)*. Assume the conditions of Theorem 3.2 hold. If $I_1(0) > 0$ and*

$$\sigma = \frac{B_1^p}{l_1}\left(\frac{2p}{l_1(p-2)}E(0)\right)^{\frac{p-2}{2}} < 1, \tag{4.7}$$

then $I_2(t) > 0$, for all $t \geq 0$.

Proof. Since $I_1(0) > 0$, it follows from the continuity of $u(t)$ that

$$I_1(t) > 0, \tag{4.8}$$

for some interval near $t = 0$. Let $t_{max} > 0$ be a maximal time (possibly $t_{max} = T$), when (4.8) holds on $[0, t_{max})$. From (4.2) and (4.4), we have

$$\begin{aligned} J(t) &\geq \tfrac{1}{2}\left(1 - \int_0^t g(\tau)d\tau\right)\|\nabla u\|_2^2 + \tfrac{1}{2}\big(g \circ \nabla u\big)(t) - \tfrac{1}{p}\|u\|_p^p \\ &\geq \tfrac{p-2}{2p}\left[\left(1 - \int_0^t g(\tau)d\tau\right)\|\nabla u\|_2^2 + (g \circ \nabla u)(t)\right] + \tfrac{1}{p}I_1(t) \\ &\geq \tfrac{p-2}{p}\left(1 - \int_0^t g(\tau)d\tau\right)\|\nabla u\|_2^2 \geq \left(\tfrac{p-2}{2p}\right) l_1\|\nabla u\|_2^2. \end{aligned} \tag{4.9}$$

Using (4.9), (4.5) and $E(t)$ is non-increasing by (4.6), we get

$$l_1\|\nabla u\|_2^2 \leq \frac{2p}{p-2}J(t) \leq \frac{2p}{p-2}E(t) \leq \frac{2p}{p-2}E(0). \tag{4.10}$$

Exploiting Lemma 2.1 and (4.7), we obtain from (4.10) on $[0, t_{max})$

$$\begin{aligned} \|u\|_p^p &\leq B_1^p\|\nabla u\|_2^p = B_1^p\|\nabla u\|_2^{p-2}\|\nabla u\|_2^2 \\ &\leq \tfrac{B_1^p}{l_1}\left(\tfrac{2p}{l_1(p-2)}E(0)\right)^{\frac{p-2}{2}} l_1\|\nabla u\|_2^2 = \sigma l_1\|\nabla u\|_2^2 \\ &< \left(1 - \int_0^t g(\tau)d\tau\right)\|\nabla u\|_2^2. \end{aligned}$$

Thus on $[0, t_{max})$, we have

$$I_1(t) = \left(1 - \int_0^t g(\tau)d\tau\right)\|\nabla u(t)\|_2^2 + \big(g \circ \nabla u\big)(t) - \|u(t)\|_p^p > 0. \tag{4.11}$$

This implies that we can take $t_{max} = T$. But, from (4.2) and (4.3), we see that

$$I_2(t) \geq I_1(t) > 0, \quad t \in [0, T]. \tag{4.12}$$

Therefore, we have $I_2(t) > 0$, $t \in [0, T]$.
Next, we want to show that $T = \infty$. Multiplying the first equation in (4.1) by $-2\Delta u$, and integrating it over Ω, we get

$$\begin{aligned} &\tfrac{d}{dt}\left\{\alpha\|\Delta u\|_2^2 - 2\int_\Omega u_t\Delta u dx + M\Big(\big\|\nabla u\big\|_2^2\Big)\|\nabla u\|_2^2\right\} \\ &\quad + \Big(2\psi\Big(\big\|\nabla u\big\|_2^2\Big) - 2\eta\Big)\|\Delta u\|_2^2 \leq 2\|\nabla u_t\|_2^2 - 2\int_\Omega |u|^{p-2}u\Delta u dx \\ &\quad + \tfrac{\|g\|_{L^1}}{2\eta}\int_0^t g(t-\tau)\|\Delta u(\tau)\|_2^2 d\tau + \left(\frac{d}{dt}M\Big(\big\|\nabla u\big\|_2^2\Big)\right)\|\nabla u\|_2^2, \end{aligned} \tag{4.13}$$

where $0 < \eta \leq \frac{\|g\|_{L^1}}{2}$. On the other hand, multiplying the first equation in (4.1) by $2u_t$, and integrating it over Ω, we get

$$\begin{aligned} \tfrac{d}{dt}\left(2E(t)\right) + 2\alpha\|\nabla u_t\|_2^2 = &\left(g' \circ \nabla u\right)(t) - g(t)\|\nabla u(t)\|_2^2 \\ &-2M\left(\left\|\nabla u\right\|_2^2\right)\|u_t\|_2^2. \end{aligned} \tag{4.14}$$

Multiplying (4.13) by ϵ, $0 < \epsilon \leq 1$, and adding (4.14) together, we obtain

$$\begin{aligned} \tfrac{d}{dt}E^*(t) \quad &+2(\alpha - \epsilon)\|\nabla u_t\|_2^2 + 2\epsilon\left(\psi\left(\left\|\nabla u\right\|_2^2\right) - 2\eta\right)\|\Delta u\|_2^2 \\ &\leq -2\epsilon\int_\Omega |u|^{\alpha-2}u\Delta u dx + \epsilon\left(\frac{d}{dt}M\left(\left\|\nabla u\right\|_2^2\right)\right)\|\nabla u\|_2^2 \\ &+2\epsilon\tfrac{\|g\|_{L^1}}{2\eta}\int_0^t g(t-\tau)\|\Delta u(\tau)\|_2^2 d\tau, \end{aligned} \tag{4.15}$$

where

$$E^*(t) = 2E(t) - 2\epsilon\int_\Omega u_t\Delta u dx + \epsilon\alpha\|\Delta u\|_2^2 + \epsilon M\left(\left\|\nabla u\right\|_2^2\right)\|\nabla u\|_2^2. \tag{4.16}$$

By young's inequality, we get

$$\left|2\epsilon\int_\Omega u_t\Delta u dx\right| \leq 2\epsilon\|u_t\|_2^2 + \frac{\epsilon}{2}\|\Delta u\|_2^2. \tag{4.17}$$

Hence, choosing $\epsilon = \frac{2}{5}$ and by (4.11), we see that

$$E^*(t) \geq \frac{1}{5}\left(\|u_t\|_2^2 + \|\Delta u\|_2^2\right). \tag{4.18}$$

Let us estimate $I_{13} = \left(\frac{d}{dt}M\left(\left\|\nabla u\right\|_2^2\right)\right)\|\nabla u\|_2^2$. Since $M \in C^1\left([0,\infty\right)$, using (4.10) and (4.18) we infer that

$$\begin{aligned} I_{13} &= \left(\tfrac{d}{dt}M\left(\left\|\nabla u\right\|_2^2\right)\right)\|\nabla u\|_2^2 \\ &= 2M'\left(\left\|\nabla u\right\|_2^2\right)\left(\int_\Omega \nabla u \cdot \nabla u_t dx\right)\|\nabla u\|_2^2 \\ &\leq 2A_3\|\Delta u\|_2\|u_t\|_2\|\nabla u\|_2^2 \leq 10A_3\left(\tfrac{2p}{l_1(p-2)}\right)E(0)E^*(t) = c_7E^*(t), \end{aligned} \tag{4.19}$$

where $c_7 = 10A_3(\frac{2p}{l_1(p-2)})E(0)$ and $A_3 = \max\{M'(r),\ 0 \leq r \leq (\frac{2p}{l_1(p-2)})E(0)\}$. Moreover, we note that

$$\begin{aligned} 2\left|\int_\Omega |u|^{p-2}u\Delta u dx\right| &\leq 2(p-1)\int_\Omega |u|^{p-2}|\nabla u|^2 dx \\ &\leq 2(p-1)\|u\|_{(p-2)\theta_1}^{p-2}\|\nabla u\|_{2\theta_2}^2, \end{aligned} \tag{4.20}$$

where $\frac{1}{\theta_1} + \frac{1}{\theta_2} = 1$, so that, we put $\theta_1 = 1$ and $\theta_2 = \infty$, if $N = 1$; $\theta_1 = 1 + \epsilon_1$ (for arbitrary small $\epsilon_1 > 0$), if $N = 2$; and $\theta_2 = \frac{N}{N-2}$, if $N \geq 3$. Then, by Lemma 2.1, (4.10) and (4.18), we have

$$2\left|\int_\Omega |u|^{p-2}u\Delta u dx\right| \leq 2B_1^p(p-1)\|\nabla u\|_2^{p-2}\|\Delta u\|_2^2 \leq c_8E^*(t), \tag{4.21}$$

where $c_8 = 10B_1^p(p-1)\left(\frac{2p}{l_1(p-2)}E(0)\right)^{\frac{p-2}{2}}$. Inserting (4.19) and (4.21) into (4.15), and then integrating it over $(0,t)$, we obtain

$$\begin{aligned} E^*(t) \quad & +\tfrac{4}{5}\left(m_0 - \eta - \tfrac{\|g\|_{L^1}^2}{4\eta}\right)\int_0^t \|\Delta u(\tau)\|_2^2 d\tau \\ & \leq E^*(0) + \int_0^t c_9 E^*(\tau)d\tau, \end{aligned} \tag{4.22}$$

where $c_9 = c_7 + c_8$. Taking $\eta = \frac{\|g\|_{L^1}}{2}$ in (4.22), and by Gronwall's Lemma, we deduce

$$E^*(t) \leq E^*(0)e^{c_9 t}, \tag{4.23}$$

for any $t \geq 0$. Therefore by Theorem 3.2, we have $T = \infty$. □

Lemma 4.3. *If u satisfies the assumptions of Lemma 4.2, then there exists $B > 0$ such that*

$$\|u\|_p^p \leq BE(t). \tag{4.24}$$

Proof. Using Lemma 2.1 and (4.10), we have

$$\begin{aligned} \|u\|_p^p \quad & \leq \quad B_1^p\|\nabla u\|_2^p = B_1^p\|\nabla u\|_2^{p-2}\|\nabla u\|_2^2 \\ & \leq \quad \frac{B_1^p}{l_1}\left(\frac{2p}{l_1(p-2)}E(0)\right)^{\frac{p-2}{2}} l_1\|\nabla u\|_2^2 = \sigma l_1\|\nabla u\|_2^2 \\ & \leq \quad \sigma\left(\frac{2p}{p-2}\right)E(t). \end{aligned}$$

Let $B = \sigma\left(\frac{2p}{p-2}\right)$, then we have (4.24). □

Theorem 4.4. *(Global existence and Energy decay) Suppose that (A1) and (A3) hold. Assume $I_1(u_0) > 0$ and (4.7) holds, then the problem (4.1) admits a global solution u if $u_0 \in H^2(\Omega) \cap H_0^1(\Omega)$ and $u_1 \in H_0^1(\Omega)$. Moreover, we have the following decay estimates*

$$E(t) \leq ce^{-\kappa\epsilon t}, \quad \forall t \geq 0 \quad \text{and} \quad \epsilon \in (0, \epsilon_1],$$

where c, κ and ϵ_1 are positive constants.

Proof. Defining the perturbed energy by

$$E_\epsilon(t) = E(t) + \epsilon\varphi(t), \tag{4.25}$$

where

$$\varphi(t) = \int_\Omega u(t)u_t(t)dx, \tag{4.26}$$

we can show that for ϵ small enough, there exist two positive constants β_1 and β_2 such that

$$\beta_1 E(t) \leq E_\epsilon(t) \leq \beta_2 E(t). \tag{4.27}$$

In fact

$$\begin{aligned} E_\epsilon(t) \quad & \leq E(t) + \tfrac{\epsilon}{2}\|u_t\|_2^2 + \tfrac{\epsilon}{2}\|u\|_2^2 \leq (1+\epsilon)E(t) + \tfrac{\epsilon}{2}B_1^2\|\nabla u\|_2^2 \\ & \leq (1+\epsilon)E(t) + \tfrac{\epsilon}{2}B_1^2\left(\tfrac{2p}{l_1(p-2)}\right)E(t) \leq \beta_2 E(t), \end{aligned} \tag{4.28}$$

and

$$E_\epsilon(t) \geq E(t) - \frac{\epsilon}{4\delta}\|u_t\|_2^2 - \epsilon\delta\|u\|_2^2 \geq E(t) - \frac{\epsilon}{4\delta}\|u_t\|_2^2 - \epsilon\delta B_1^2\|\nabla u\|_2^2. \tag{4.29}$$

By choosing δ small enough, we have

$$E_\epsilon(t) \geq E(t) - \frac{\epsilon}{4\delta}\|u_t\|_2^2 \geq J\big(u(t)\big) + \left(\frac{1}{2} - \frac{\epsilon}{4\delta}\right)\|u_t\|_2^2. \tag{4.30}$$

Once δ is chosen, we take ϵ so small that

$$E_\epsilon(t) \geq J\big(u(t)\big) + \frac{\beta_1}{2}\|u_t\|_2^2 \geq \beta_1 E(t), \tag{4.31}$$

where $\frac{\beta_1}{2} \leq \frac{1}{2} - \frac{\epsilon}{4\delta}$. Now taking the derivative of $\varphi(t)$ defined in (4.26) and substituting

$$\begin{aligned} u_{tt} \quad &= \psi\Big(\big\|\nabla u\big\|_2^2\Big)\nabla u + \alpha\Delta u_t - \int_0^t g(t-\tau)\Delta u(\tau)d\tau \\ &-M\Big(\big\|\nabla u\big\|_2^2\Big)u_t + |u|^{p-2}u, \end{aligned} \tag{4.32}$$

in the obtained expression, it results that

$$\begin{aligned} \varphi'(t) \quad &= \|u_t\|_2^2 - \|\nabla u\|_2^2 - b\|\nabla u\|_2^{2(\gamma+1)} \\ &+ \int_0^t g(t-\tau)\int_\Omega \nabla u(\tau)\cdot\nabla u(t)dxd\tau - \alpha\big(\nabla u_t, \nabla u\big) \\ &-M\Big(\big\|\nabla u\big\|_2^2\Big)\big(u_t, u\big) + \|u\|_p^p. \end{aligned} \tag{4.33}$$

Adding and subtracting $2E(t)$, and taking (4.5) into account, from (4.33) we infer

$$\begin{aligned} \varphi'(t) \quad &= -2E(t) + 2\|u_t\|_2^2 - \left(\int_0^t g(\tau)d\tau\right)\|\nabla u(t)\|_2^2 \\ &+\big(g\circ\nabla u\big)(t) - b\left(1 - \tfrac{1}{\gamma+1}\right)\|\nabla u\|_2^{2(\gamma+1)} \\ &+\left(1 - \tfrac{2}{p}\right)\|u\|_p^p - \alpha\big(\nabla u_t, \nabla u\big) - M\Big(\big\|\nabla u\big\|_2^2\Big)\big(u_t, u\big) \\ &+\int_0^t g(t-\tau)\int_\Omega \nabla u(\tau)\cdot\nabla u(t)dxd\tau. \end{aligned} \tag{4.34}$$

Estimate for $J_1 = \alpha\big(\nabla u_t, \nabla u\big)$. Considering Cauchy-Schwartz inequality, we have

$$|J_1| \leq \frac{\alpha^2}{2}\|\nabla u_t(t)\|_2^2 + \frac{1}{2}\|\nabla u(t)\|_2^2. \tag{4.35}$$

Let us estimate $J_2 = M\Big(\big\|\nabla u\big\|_2^2\Big)\big(u_t, u\big)$. Noting that $\|\nabla u(t)\|_2^2 \leq \frac{2p}{l_1(p-2)}E(0) = \beta_3$ for all $t \geq 0$, we have that

$$M\Big(\big\|\nabla u\big\|_2^2\Big) \leq \xi, \quad \forall t \geq 0, \tag{4.36}$$

where $\xi = \max\{M(r);\ r \in [0, \beta_3]\}$. From (4.36) we conclude that

$$|J_2| \leq \frac{\xi^2}{2}\|u_t(t)\|_2^2 + \frac{1}{2}\|u(t)\|_2^2 \leq \frac{\xi^2}{2}\|u_t(t)\|_2^2 + \frac{1}{2}B_1^2\|\nabla u(t)\|_2^2. \tag{4.37}$$

Estimate $J_3 = \displaystyle\int_0^t g(t-\tau)\int_\Omega \nabla u(\tau)\cdot\nabla u(t)dxd\tau$. From assumption $(A1)$ and making use of the Cauchy-Schwarz inequality, we have

$$
\begin{aligned}
J_3 &= \int_0^t g(t-\tau)\int_\Omega \nabla u(\tau)\cdot\nabla u(t)dxd\tau\\
&= \int_0^t g(t-\tau)\int_\Omega \Big[\nabla u(\tau)-\nabla u(t)+\nabla u(t)\Big]\cdot\nabla u(t)dxd\tau\\
&\le \int_0^t g(t-\tau)\big|\nabla u(t)-\nabla u(\tau)\big|\big|\nabla u(t)\big|dxd\tau\\
&+\left(\int_0^t g(\tau)d\tau\right)\|\nabla u(t)\|_2^2\\
&\le \|\nabla u(t)\|_2^2\int_0^t g(t-\tau)\|\nabla u(t)-\nabla u(\tau)\|_2^2d\tau\\
&+\left(\int_0^t g(\tau)d\tau\right)\|\nabla u(t)\|_2^2\\
&\le \tfrac{1}{2}\|\nabla u(t)\|_2^2+\tfrac{1}{2}\|g\|_{L^1(0,\infty)}\big(g\circ\nabla u\big)(t)+\left(\int_0^t g(\tau)d\tau\right)\|\nabla u(t)\|_2^2\\
&\le \tfrac{1}{2}\|\nabla u(t)\|_2^2+\tfrac{1}{2}\big(g\circ\nabla u\big)(t)+\left(\int_0^t g(\tau)d\tau\right)\|\nabla u(t)\|_2^2.
\end{aligned}
\tag{4.38}
$$

Utilizing Lemma 4.3 and inserting (4.35), (4.38) and (4.37) in (4.34), we have

$$
\begin{aligned}
\varphi'(t) &\le \left(\tfrac{\xi^2}{2}+2\right)\|u_t\|_2^2+\left(1+\tfrac{B_1^2}{2}\right)\|\nabla u\|_2^2\\
&+\left[\left(1-\tfrac{2}{p}\right)B-2\right]E(t)-b\left(1-\tfrac{1}{\gamma+1}\right)\|\nabla u\|_2^{2(\gamma+1)}\\
&+\tfrac{\alpha^2}{2}\|\nabla u_t(t)\|_2^2+\tfrac{3}{2}\Big(g\circ\nabla u\Big)(t).
\end{aligned}
\tag{4.39}
$$

Then, from (4.6), (4.25), (4.26) and (4.39) we arrive at

$$
\begin{aligned}
E_\epsilon'(t) &= E'(t)+\epsilon\varphi'(t)\le-\Big(m_1-\lambda_1\epsilon\Big)\|u_t\|_2^2+\lambda_2\epsilon\|\nabla u\|_2^2\\
&-\left(\tfrac{K_2}{2}-\tfrac{3}{2}\epsilon\right)\Big(g\circ\nabla u\Big)(t)-\left(\alpha-\tfrac{\alpha^2}{2}\epsilon\right)\|\nabla u_t(t)\|_2^2\\
&-\epsilon(-\lambda_3)E(t)-b\epsilon\left(1-\tfrac{1}{\gamma+1}\right)\|\nabla u\|_2^{2(\gamma+1)}-\tfrac{1}{2}g(t)\|\nabla u(t)\|_2^2,
\end{aligned}
\tag{4.40}
$$

where

$$\lambda_1=\frac{\xi^2}{2}+2>0,\ \lambda_2=\frac{B_1^2}{2}+1>0$$

and

$$\lambda_3=\left(1-\frac{2}{p}\right)B-2=\left(1-\frac{2}{p}\right)\left(\frac{2p}{p-2}\right)\sigma-2=2\sigma-2<0.$$

On the other hand, since

$$\int_0^t g'(\tau)d\tau=g(t)-g(0),$$

then

$$-g(t)\|\nabla u(t)\|_2^2=-g(0)\|\nabla u(t)\|_2^2-\left(\int_0^t g'(\tau)d\tau\right)\|\nabla u(t)\|_2^2.$$

From $(A1)$ the last inequality yields

$$-\frac{1}{2}g(t)\|\nabla u(t)\|_2^2 \leq -\frac{1}{2}g(0)\|\nabla u(t)\|_2^2 + \frac{K_1}{2}\|g\|_{L^1(0,\infty)}\|\nabla u(t)\|_2^2. \tag{4.41}$$

Combining (4.40) and (4.41) we conclude that

$$\begin{aligned} E'_\epsilon(t) &\leq -\Big(m_1 - \lambda_1\epsilon\Big)\|u_t\|_2^2 - \left(\tfrac{K_2}{2} - \tfrac{3}{2}\epsilon\right)\Big(g \circ \nabla u\Big)(t) \\ &- \Big(\alpha - \tfrac{\alpha^2}{2}\epsilon\Big)\|\nabla u_t(t)\|_2^2 - b\epsilon\Big(1 - \tfrac{1}{\gamma+1}\Big)\|\nabla u\|_2^{2(\gamma+1)} - \epsilon(-\lambda_3)E(t) \\ &-\tfrac{1}{2}\left[g(0) - K_1\|g\|_{L^1(0,\infty)} - 2\lambda_2\epsilon\right]\|\nabla u(t)\|_2^2. \end{aligned} \tag{4.42}$$

From (2.1) we have $l_2 = g(0) - K_1\|g\|_{L^1(0,\infty)} > 0$. Defining

$$\epsilon_1 = \min\left\{\frac{m_1}{\lambda_1}, \frac{K_2}{3}, \frac{2}{\alpha}, \frac{l_2}{2\lambda_2}\right\}, \tag{4.43}$$

we conclude by taking $\epsilon \in (0, \epsilon_1]$ in (4.42) that

$$E'_\epsilon(t) \leq -\epsilon(-\lambda_3)E(t). \tag{4.44}$$

Thus, we see that $\forall t \geq 0$ and $\epsilon \in (0, \epsilon_1]$

$$E'_\epsilon(t) \leq -\epsilon(-\lambda_3)E(t) \leq -\frac{-\lambda_3}{\beta_2}\epsilon E_\epsilon(t). \tag{4.45}$$

By the Gronwall inequality, we see that

$$E_\epsilon(t) \leq E_\epsilon(0)e^{-\kappa\epsilon t}, \quad \forall t \geq 0 \text{ and } \quad \epsilon \in (0, \epsilon_1], \tag{4.46}$$

where $\kappa = \frac{-\lambda_3}{\beta_2}$. Combining with (4.27), we obtain

$$\beta_1 E(t) \leq E_\epsilon(t) \leq E_\epsilon(0)e^{-\kappa\epsilon t}, \quad \forall t \geq 0 \text{ and } \quad \epsilon \in (0, \epsilon_1], \tag{4.47}$$

and

$$E(t) \leq ce^{-\kappa\epsilon t}, \quad \forall t \geq 0 \text{ and } \quad \epsilon \in (0, \epsilon_1], \tag{4.48}$$

where $c = \frac{E_\epsilon(0)}{\beta_1}$. Thus, the proof of the theorem is completed. □

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A smooth approximation for non-linear second order boundary value problems using composite non-polynomial spline functions

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Abstract. A different amalgamation of non-polynomial splines is used to find the approximate solution of linear and non-linear second order boundary value problems. Cubic spline functions are assembled with exponential and trigonometric functions to develop the different orders of numerical schemes. Free parameter k of the non-polynomial part is also used to form a new scheme, which elevates the accuracy of the solution. Numerical illustrations are given to validate the applicability and feasibility of the present methods and also depicted in the graphs.

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1. Introduction

To demonstrate the basic concept and idea of our technique, we consider the following general non-linear second order two point boundary value problems (BVPs), which arise in a wide variety of engineering applications

$$u^{(2)}(x) = f(x, u), \ -\infty \leqslant a \leqslant x \leqslant b \leqslant \infty \tag{1.1}$$

with the boundary conditions (BCs)

$$u(a) = A_1, \ u(b) = A_2, \tag{1.2}$$

where A_i, i=1, 2 are arbitrary finite real constants and $-\infty < u < \infty$. The function $f(x, u(x))$ is a continuous function of two variables with $f_u \geqslant 0$ on $[a, b]$. DE (1.1) with BC (1.2) has a unique solution, whose existence and uniqueness can be studied in [24]. For the linear case, $f(x, u) = p(x)u + g(x)$ with $p(x)$ and $g(x)$ continuous functions on the interval $[a, b]$.

It is well acknowledged that numerous real-life phenomena in physics and engineering sciences often convert to boundary value problems for second order differential equations such as in heat transfer, optimal control, deflection in cables and plates, vibration of springs, electric circuits and in a number of other scientific applications [19]. Most of the BVPs are essentially solved using numerical approaches as those are not explained enough using existing analytical approaches. Consequently, some useful numerical schemes were being promoted, most notably spline-based' schemes. Spline functions were applied by many authors to establish the accurate and efficient numerical schemes for the solution of boundary value problems [4]. An exploration of the literature on a number of polynomial and non-polynomial spline techniques to solve the second order BVPs can be comprehended as quadratic spline method [8, 26, 32, 42, 49], cubic spline method [2-3, 5, 9-12, 15, 20-23, 27-28, 30-34, 36-38, 40-41, 50], quartic spline method [6, 13-14, 29, 47], quintic spline method [7, 16, 43, 48] and others [39, 46]. Voluminous research work have been contributed to this field but we are mainly concerned on those papers which have implemented non-polynomial splines for the solution of second order BVPs with various types of boundary conditions.

For instance, Rashidinia *et al.* [40] built up a technique based on cubic non-polynomial spline functions of the form

$$T_n = Span\{1, x, \sin(\tau x), \cos(\tau x)\}, \tag{1.3}$$

They applied their scheme to acquire the numerical solution of the following form of second order two point BVPs

$$-\frac{d}{dx}\left[p(x)\frac{du}{dx}\right] = g(x);\ u(a) = u(b) = 0. \tag{1.4}$$

Here, authors employed direct method to simplify the obtained system and facilitated the smooth approximations to linear second order BVPs. Similar approach was exercised by Islam and Tirmizi [27] to find the approximate solution of the system of two-point second order BVPs with Dirichlet BCs (1.2). They established the consistency equations to attain the desired results and solved linear second order equations to show the feasibility of their method. Khan and Aziz [34] proposed the parametric cubic spline functions with a parameter for attaining approximations to the solutions of the system of BVP. They presented improved results while comparing with some existing methods. Former approach [35] was yet again instituted by Khan in [33] to solve the following second order linear BVPs

$$y^{(2)}(x) = f(x)y(x) + g(x);\ a \leqslant x \leqslant b \tag{1.5}$$

with Dirichlet BCs (1.2). Here, the author developed the method of order four for specific values of parameters, or else his method was of order two. Over again, Zahra et al. [50] used cubic non-polynomial spline function space (1.3) to compute approximation to the solution of above linear BVPs (1.5) but with Neumann BCs. Kalyani and Rao [31] also adopted similar approach demonstrated by [27, 40, 50] to solve the following BVP of second order

$$-\frac{d}{dx}\left[p(x)\frac{du}{dx}\right] + v(x)u(x) = g(x);\ u(a) = u(b) = 0. \tag{1.6}$$

They solved many linear and non-linear examples to study the performance of their method. Cubic non-polynomial spline scheme was once more deliberated by Justine and Sulaiman [30] to solve the general linear second order BVPs subject to Dirichlet BCs. To solve the obtained linear system, they used successive over relaxation in conjunction with Gauss-Seidel method. However, to establish the result, here authors considered the total number of iterations, execution times along with maximum absolute error (MAE).

Above, we have summarized numerous contributions that are made to deal with the solution of various types of second order BVPs choosing non-polynomial splines. The present research could contribute remarkably to this field as it includes some novel methods to solve non-linear second order BVPs with significant results. Our method is based on distinctive exponential and trigonometric spline function space given as

$$\begin{aligned} T_3 &= Span\{1, x, e^{kx}, \sin(kx)\} \\ &= Span\Big\{1, x, (\frac{2}{k^2})\left(e^{kx} - kx - 1\right), (\frac{6}{k^3})\left(kx - \sin(kx)\right)\Big\}, \end{aligned} \tag{1.7}$$

where k is the frequency of trigonometric and exponential part of the spline function, which can be real or pure imaginary. It follows that if $k \to 0$, T_3 reduces to $Span\big(1, x, x^2, x^3\big)$. In this paper, we have developed different order methods along with a modified k-dependent method based on the angular frequency of the non-polynomial part for smooth approximation of the second order linear and non-linear BVPs. We have solved several examples using our developed methods and also shown comparisons of our results with some known methods like collocation, finite difference, Galerkin, Adomian decomposition and other spline methods. Our spline method solution and comparisons demonstrate that our algorithm performs comparatively better with more precise results.

Now, the paper is organized as follows: section 2 shows the formulation of our schemes and section 3 describes the solution of BVPs using the developed scheme. Section 4 deliberates the convergence of the schemes, while in section 5 some examples are solved using our developed spline methods. Paper is concluded in section 6.

2. Derivation of the method

In this section, we develop a numerical method to approximate the solution of second order BVP (1.1)-(1.2). To do that, we first set a framework of $N+1$ equally spaced points x_i of an interval [a, b] and divide them into N equal sections such that $x_i = a + ih$, $i = 0, 1, 2, \ldots, N$ where $x_0 = a, x_N = b$ and $h = \frac{(b-a)}{N}$. Then, our spline function $P_i(x)$ holds the following structure in every section of the interval

$$P_i(x) = a_i \sin k(x - x_i) + b_i e^{k(x-x_i)} + c_i(x - x_i) + d_i;\; i = 0, 1, 2 \ldots, N, \tag{2.1}$$

where a_i, b_i, c_i and d_i are constants and k is free parameter, which can be real or purely imaginary and will be used to raise the accuracy of the method. The function $P_i(x)$, which interpolates $S(x)$ at the mesh points x_i and reduces to cubic spline as $k \to 0$, where $S(x)$ is the approximate solution of (1.1). Let $u(x)$ be the exact solution

and S_i be an approximation to $u_i = u(x_i)$ obtained by the segment $P_i(x)$ of the spline function passing through the points (x_i, S_i) and (x_{i+1}, S_{i+1}). Then the mixed spline defined by the function $S(x) = P_i(x)$.
Now, we assume

$$P_i(x_i) = S_i, \quad P_i(x_{i+1}) = S_{i+1}, \quad P_i^{(2)}(x_i) = M_i, \quad P_i^{(2)}(x_{i+1}) = M_{i+1},$$

to get the following value of coefficients

$$a_i = \frac{1}{k^2 \sin(\theta)}[e^\theta M_i - M_{i+1}], \quad b_i = \frac{1}{k^2}[M_i],$$

$$c_i = \frac{S_{i+1} - S_i}{h} + \frac{M_{i+1} + M_i}{k^2 h} - \frac{2e^\theta M_i}{k^2 h}, \quad d_i = S_i - \frac{1}{k^2}[M_i],$$

whereby $\theta = kh$ and $i = 0, 1, 2, ..., N$.

Next, use the continuity condition of the first derivative and substitute the value of coefficients a_i, b_i, c_i and d_i. After some algebraic manipulations, we can obtain the following main relation

$$S_{i-1} - 2S_i + S_{i+1} = h^2[\alpha M_{i-1} + \beta M_i + \gamma M_{i+1}]; \; i = 1, 2, ...N - 1, \tag{2.2}$$

where,

$$\alpha = \frac{\theta e^\theta \{\sin(\theta) + \cos(\theta)\} + \sin(\theta)(1 - 2e^\theta)}{\theta^2 \sin(\theta)},$$

$$\beta = \frac{2e^\theta \sin(\theta) - \theta e^\theta - \theta \{\sin(\theta) + \cos(\theta)\}}{\theta^2 \sin(\theta)},$$

$$\gamma = \frac{\theta - \sin(\theta)}{\theta^2 \sin(\theta)}$$

and $M_i = S^{(2)}(x_i) = f(x, u)$, by discretizing the considered DE (1.1) at the nodal point x_i. As $k \to 0$, $\alpha = 1/6, \beta = 4/6$ and $\gamma = 1/6$, our scheme (2.2) reduces to ordinary cubic spline scheme [5] and then, it is evidently second order convergent.

Accordingly, equation (2.2) provides a system of $N - 1$ non-linear algebraic equations in the $N - 1$ unknowns $S_i, i = 1, 2, \ldots, N - 1$, which by discretizing can be written as

$$\big(S_{i-1} - \alpha h^2 f(x_{i-1}, S_{i-1})\big) - \big(2S_i + \beta h^2 f(x_i, S_i)\big) + \big(S_{i+1} - \gamma h^2 f(x_{i+1}, S_{i+1})\big) + t_i = 0. \tag{2.3}$$

Then, the local truncation error t_i, $i = 1, 2, ..., N - 1$, can be written as

$$\begin{aligned} t_i = \{1 - (\alpha + \beta + \gamma)\} h^2 u_i^{(2)} + (\alpha - \gamma) h^3 u_i^{(3)} + \left\{\frac{1}{12} - \frac{1}{2}(\alpha + \gamma)\right\} h^4 u_i^{(4)} \\ + \frac{1}{6}(\alpha - \gamma) h^5 u_i^{(5)} + \left\{\frac{1}{360} - \frac{1}{24}(\alpha + \gamma)\right\} h^6 u_i^{(6)} + O(h^7). \end{aligned} \tag{2.4}$$

Thus, our schemes (2.2) and (2.4) give rise to a family of methods of different orders as follows:

2.1. Different order of methods

Case (i). First order method

For $\alpha+\beta+\gamma=1$, $\alpha\neq\gamma$. Here,

$$t_i=(\alpha-\gamma)h^3u_i^{(3)}+O(h^4),$$
$$\|T\|=|(\alpha-\gamma)|h^3M_3,\;\; M_3=max|u^{(3)}(x)|. \tag{2.5}$$

Case (ii). Second order method

For $\alpha+\beta+\gamma=1$, $\alpha=\gamma$ and $\alpha+\gamma\neq\frac{1}{6}$. Here,

$$t_i=\left\{\frac{1}{12}-\frac{1}{2}(\alpha+\gamma)\right\}h^4u_i^{(4)}+O(h^5),$$
$$\|T\|=\left|\frac{1}{12}-\frac{1}{2}(\alpha+\gamma)\right|h^4M_4,\;\; M_4=max|u^{(4)}(x)|. \tag{2.6}$$

Case (iii). Fourth order method

For $\alpha+\beta+\gamma=1$, $\alpha=\gamma$ and $\alpha+\gamma=\frac{1}{6}$. Here,

$$t_i=\left\{\frac{1}{360}-\frac{1}{24}(\alpha+\gamma)\right\}h^6u_i^{(6)}+O(h^7),$$
$$\|T\|=\left|\frac{1}{360}-\frac{1}{24}(\alpha+\gamma)\right|h^6M_6,\;\; M_6=max|u^{(6)}(x)|. \tag{2.7}$$

where $\|\cdot\|$ represents the ∞ norm in matrix vector.

2.2. Modified k-dependent method

In this section, we will use the parameter k to raise the order of accuracy of the obtained scheme (2.2). To do this, we first rearrange the terms in equation (2.4) in the following manner

$$\begin{aligned}
t_i =&h^4\left[\frac{1}{\theta^2}+\frac{(e^\theta-1)(1-\cos(\theta))+\sin(\theta)(1+e^\theta)}{\theta^3\sin(\theta)}\right](k^2u_i^{(2)}-u_i^{(4)})\\
&+h^5\left[\frac{e^\theta(\sin(\theta)+\cos(\theta))-1}{\theta^3\sin(\theta)}+\frac{2(1-e^\theta)}{\theta^4}\right]k^2u_i^{(3)}\\
&+h^6\left[\frac{1}{12\theta^2}-\frac{1+e^\theta(\sin(\theta)+\cos(\theta))}{2\theta^3\sin(\theta)}+\frac{(1+e^\theta)}{\theta^4}\right]k^2u_i^{(4)}\\
&+h^6\left[\frac{(\sin(\theta)+\cos(\theta))+1+e^\theta(\sin(\theta)-\cos(\theta)-1)}{\theta^5\sin(\theta)}\right]k^2u_i^{(4)}\\
&+h^6\left[\left\{\frac{1}{360}+\frac{-e^\theta(\sin(\theta)+\cos(\theta))}{24\theta\sin(\theta)}+\frac{(2e^\theta-1)}{24\theta^2}\right\}u_i^{(6)}(\eta_1)+\left\{\frac{1}{24\theta^2}-\frac{1}{24\theta\sin(\theta)}\right\}u_i^{(6)}(\eta_2)\right]\\
&+h^7\left[\frac{e^\theta(\sin(\theta)+\cos(\theta)-1)}{6\theta^3\sin(\theta)}+\frac{(1-e^\theta)}{3\theta^4}\right]k^2u_i^{(5)}+\cdots
\end{aligned}$$

Equating the coefficient of the leading term in the above equation to zero, we can get the equation in k_i as

$$k_i^2=\frac{u_i^{(4)}}{u_i^{(2)}}=\frac{f''(x_i,u_i)}{f(x_i,u_i)} \tag{2.8}$$

For the linear case, $f(x_i, u_i) = p_i u_i + g_i$. Then,

$$k_i^2 = \frac{(p_i'' + p_i^2)u_i + 2p_i'u_i' + p_i g_i + g_i''}{p_i u_i + g_i} \tag{2.9}$$

Thus, from above we see that calculation of k_i requires the approximations for u_i and u_i'. Approximation for u_i can be obtained by means of our developed scheme (2.2) for $k = 0$ and for u_i', following steps can be adapted:

(i) Differentiating equation (2.1) at $x = x_i$, to get

$$\begin{aligned} P_i'(x) = {} & \frac{1}{k\sin(\theta)}\left\{(\sin(\theta) + e^{\theta})M_i - M_{i+1}\right\} \\ & + \frac{(S_{i+1} - S_i)}{h} + \frac{1}{k^2 h}\left\{(1 - 2e^{\theta})M_i + M_{i+1}\right\}, \end{aligned}$$

(ii) If the limit k going to zero in the above equation, we obtain

$$P_i'(x) = -\frac{h}{6}f(x_{i+1}, u_{i+1}) - \frac{h}{3}f(x_i, u_i) + \frac{(S_{i+1} - S_i)}{h}; \; i = 0, 1, ..., N. \tag{2.10}$$

3. Composite non-polynomial spline solution

To develop the approximation to the solution of BVP (1.1)-(1.2) based on our developed spline method, we write our scheme (2.2) in the following standard matrix form:

$$A_0 S^{(1)} - h^2 B f^{(1)}\left(S^{(1)}\right) = C^{(1)}, \tag{3.1}$$

where A_0 and B are three-band square matrices of order $N - 1$, given by

$$A_0 = \begin{bmatrix} -2 & 1 & & & & & & & & \\ 1 & -2 & 1 & & & & & & & \\ & 1 & -2 & 1 & & & & & & \\ & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ & & & & & & & 1 & -2 & 1 \\ & & & & & & & & 1 & -2 \end{bmatrix}$$

$$B = \begin{bmatrix} \beta & \gamma & & & & & & & & \\ \alpha & \beta & \gamma & & & & & & & \\ & \alpha & \beta & \gamma & & & & & & \\ & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ & & & & & & & \alpha & \beta & \gamma \\ & & & & & & & & \alpha & \beta \end{bmatrix}$$

Matrix: $f^{(1)}(S^{(1)}) = f(x_i, S_i^{(1)})$, $S^{(1)} = [S_1, S_2, ..., S_{N-1}]^t$ and

$$C^{(1)} = \begin{cases} -A_1 + h^2\alpha f(x_0, A_1), & i = 1, \\ 0, & i = 2, 3, ...N - 2, \\ -A_2 + h^2\gamma f(x_N, A_2), & N - 1. \end{cases}$$

Likewise,

$$A_0U^{(1)} - h^2Bf^{(1)}(U^{(1)}) = C^{(1)} + T^{(1)}, \tag{3.2}$$

where the vector $U^{(1)} = u(x_i)$ is the exact solution with truncation error $T^{(1)} = (t_i^{(1)})$, for $i = 1, 2, \ldots, N-1$.
From (3.1) and (3.2), we have

$$[A_0 - h^2BQ]E^{(1)} = T^{(1)} \tag{3.3}$$

where

$$E^{(1)} = U^{(1)} - S^{(1)} = [e_1^{(1)}, e_2^{(1)}, \ldots\ldots, e_{N-1}^{(1)}]^T$$

and

$$Q = diag\left(\frac{\partial f_i^{(1)}}{\partial u_i^{(1)}}\right),\ i = 1, 2, \ldots, N-1$$

is the diagonal matrix of order $N-1$, whereas for the linear case, Q $=diag(f_i^{(1)})$.

Thus, the equations (3.1)-(3.3) demonstrate our scheme, using which one can obtain the approximate solution of non-linear DE (1.1) with the BC (1.2). We shall use Newton's method to obtain the solution of the non-linear system (2.2), which converge to the solution of (1.1)-(1.2) for all sufficiently small values of h [24, 46].

4. Convergence analysis

Now, we will derive a bound on $\left\|E^{(1)}\right\|$. From equation (3.3), we get

$$AE^{(1)} = T^{(1)},$$

where, A $=[A_0 - h^2BQ]$ is a tri-diagonal matrix. The elements of A are given by

$$a_{ij} = \begin{cases} -2 - h^2\beta f_u(x_i, u_i), & i = j, \\ 1 - h^2\alpha f_u(x_i, u_i), & i - j = 1, \\ 1 - h^2\gamma f_u(x_i, u_i), & j - i = 1, \\ 0, & |i - j| > 1. \end{cases}$$

From above, we have

$$\left\|E^{(1)}\right\| \leq \left\|A^{-1}\right\| \left\|T^{(1)}\right\|.$$

(See [24]) $\left\|A^{-1}\right\| \leq (b-a)^2/8h^2$ and so, we can infer the following convergent schemes:

Case 4.1. First order convergent method

For (α, β, γ) =(75/1920,1755/1920,90/1920), $\left\|T^{(1)}\right\|_\infty = \frac{1}{128}h^3M_3$.
Then from equation (2.5), we get

$$\left\|E^{(1)}\right\| \leq K_1h \cong O(h^1). \tag{4.1}$$

This relation (4.1) shows that the method is first order convergent.

Case 4.2. Second order convergent method

For $\alpha = \gamma = \frac{3}{38}$ and $\beta = \frac{32}{38}$, $\left\|T^{(1)}\right\|_\infty = \frac{1}{128}h^4M_4$.

Then it follows from (2.6) that

$$\left\| E^{(1)} \right\| \leq K_2 h^2 \cong O(h^2). \tag{4.2}$$

The relation (4.2) confirms second order convergence of the method.

Case 4.3. Fourth order convergent method

For $\alpha = \gamma = \frac{1}{12}$ and $\beta = \frac{10}{12}$, $\left\|T^{(1)}\right\|_\infty = \frac{1}{240}h^6 M_6$.
Then from equation (2.7), we have

$$\left\| E^{(1)} \right\| \leq K_3 h^4 \cong O(h^4). \tag{4.3}$$

which confirms fourth order convergence of the method.

5. Numerical illustration

To illuminate the use of our developed methods, we have considered several linear and non-linear examples of second order BVPs and also compared our results with other existing methods.

Problem 5.1. Consider the linear BVP

$$u^{(2)}(x) = \frac{2}{x^2}u - \frac{1}{x}; \quad 2 < x < 3; \qquad u(2) = u(3) = 0. \tag{5.1}$$

The theoretical (exact) solution of (5.1) is

$$u(x) = \frac{1}{38}(-5x^2 + 19x - \frac{36}{x}). \tag{5.2}$$

Comparing the given equation (5.1) with (1.1) at $x = x_i$, we have

$$f(x_i, u_i) = \frac{2}{x_i^2}u_i - \frac{1}{x_i}.$$

Table 1. Absolute error for the solution of Problem 5.1 at different value of x for $N = 8$

x	Our method for $k = 0$	Our k-based method	Value of k
17/8	2.36×10^{-5}	4.28×10^{-6}	1.0674
18/8	3.66×10^{-5}	6.31×10^{-6}	0.9581
19/8	4.16×10^{-5}	6.86×10^{-6}	0.8623
20/8	4.07×10^{-5}	6.45×10^{-6}	0.7781
21/8	3.52×10^{-5}	5.38×10^{-6}	0.7040
22/8	2.61×10^{-5}	3.87×10^{-6}	0.6387
23/8	1.42×10^{-5}	2.05×10^{-6}	0.5809

For the linear case, $f(x,u) = p(x)u + g(x)$, so $p_i = p(x_i) = 2/x_i^2; g_i = g(x_i) = -1/x_i$ and equation(3.1) is changed to $AS = C$, where $A = A_0 - h^2BQ$; $Q = diag(f_i)$. By substituting these values, we get system of linear equations for Problem 5.1 that can be solved using any suitable method. Absolute errors at different point of x are summarized in Table 1 for $k = 0$, i.e. $(\alpha, \beta, \gamma) = (1/6, 4/6, 1/6)$ and k-based method, when $h = 1/8$. Results indicate that the modified k-dependent method provides better results than the method for $k = 0$. The value of parameter k at different value of x is also listed in Table 1 (col. IV).

Table 2 reports the MAE at different value of N for second order schemes together with k-based technique. Table indicates that k-based method is a third order convergent method. Comparison of numerical results with other existing methods is also included in this table. Fourth order method solution when $(\alpha, \beta, \gamma) = (1/12, 10/12, 1/12)$ of Problem 5.1 for N=10 is presented in Table 3, along with comparison with Galerkin method.

Table 2. Comparison of maximum absolute errors for Problem 5.1

Our method	N = 4	N = 8	N = 16
Our second order methods			
$(\alpha = \gamma = 3/38, \beta = 32/38)$	5.94×10^{-6}	2.00×10^{-6}	5.37×10^{-7}
$(\alpha = \gamma = 1/13, \beta = 11/13)$	9.88×10^{-6}	3.01×10^{-6}	7.90×10^{-7}
Our method for $k = 0$	1.65×10^{-4}	4.16×10^{-5}	1.04×10^{-5}
Our k-based Method	5.05×10^{-5}	6.86×10^{-6}	8.61×10^{-7}
Quadratic spline [9]	1.60×10^{-4}	2.66×10^{-5}	5.58×10^{-6}
Centered Difference method [10]	2.79×10^{-4}	5.42×10^{-5}	1.19×10^{-5}
Quadratic spline [42]	7.93×10^{-5}	2.06×10^{-5}	5.20×10^{-6}
Cubic spline [10]	5.49×10^{-5}	1.87×10^{-5}	5.07×10^{-6}
Cubic non-poly. spline [33]	2.05×10^{-5}	5.74×10^{-6}	1.47×10^{-6}
Discrete cubic spline [21]	1.77×10^{-5}	5.00×10^{-6}	1.29×10^{-6}

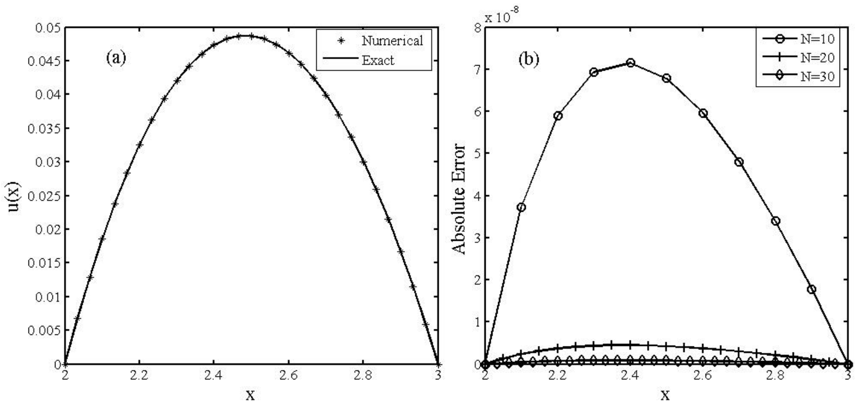

Figure 1. (a) Comparison of approximate and exact values for Problem 5.1. (b) Error graph for Problem 5.1 at different values of N (Table 3).

Table 3. Comparison of MAE for the solution of Problem 5.1 (Fourth order method)

x	2.1	2.2	2.3	2.4	2.5
Our method	3.73×10^{-8}	5.89×10^{-8}	6.92×10^{-8}	7.15×10^{-8}	6.78×10^{-8}
Galerkin method [25]	2.52×10^{-7}	1.15×10^{-6}	6.73×10^{-7}	6.90×10^{-7}	1.24×10^{-6}
x	2.6	2.7	2.8	2.9	
Our method	5.96×10^{-8}	4.81×10^{-8}	3.39×10^{-8}	1.77×10^{-8}	
Galerkin method [25]	4.51×10^{-7}	7.90×10^{-7}	9.70×10^{-7}	3.17×10^{-7}	

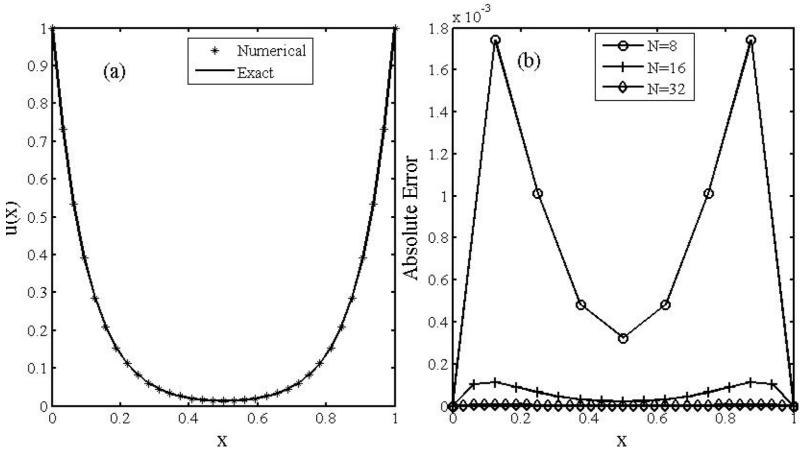

Figure 2. (a) Comparison of approximate and exact values for Problem 5.2. (b) Error graph for Problem 5.2 at different values of N (Table 4).

Problem 5.2. Consider the linear BVP

$$u^{(2)}(x) = 100u; \quad 0 < x < 1; \qquad u(0) = u(1) = 1. \tag{5.3}$$

The theoretical solution of (5.3) is

$$u(x) = \frac{\cosh(10x-5)}{\cosh\ 5}. \tag{5.4}$$

Problem 5.3. Consider the linear BVP

$$u^{(2)}(x) = u + \cos(x), \quad 0 < x < 1; \qquad u(0) = u(1) = 1. \tag{5.5}$$

The theoretical solution of (5.5) is

$$\begin{aligned} u(x) =& \frac{-3\cosh(1) + 3\sinh(1) + \cos(1) + 2}{4\sinh(1)} e^{x} \\ &+ \frac{3\cosh(1) + 3\sinh(1) - \cos(1) - 2}{4\sinh(1)} e^{-x} - \frac{\cos(x)}{2} \end{aligned} \tag{5.6}$$

Table 4. Comparison of maximum absolute errors for Problem 5.2

Our method	N = 16	N = 32	N = 20	N = 40
$\alpha=\gamma=3/38, \beta$=32/38	1.95×10^{-4}	7.15×10^{-5}	1.54×10^{-4}	4.75×10^{-5}
$\alpha=\gamma=1/13, \beta$=11/13	3.37×10^{-4}	1.07×10^{-4}	2.47×10^{-4}	7.08×10^{-5}
Our method for $k=0$	6.10×10^{-3}	1.50×10^{-3}	3.90×10^{-3}	9.65×10^{-4}
Our k-based method	1.16×10^{-2}	1.11×10^{-3}	5.40×10^{-3}	5.57×10^{-4}
Our fourth-order method	1.12×10^{-4}	7.28×10^{-6}	4.75×10^{-5}	2.99×10^{-6}
Cubic non-poly. spline [33]	7.22×10^{-4}	2.06×10^{-4}	5.00×10^{-4}	1.34×10^{-4}
Discrete cubic spline [21]	6.18×10^{-4}	1.80×10^{-4}	4.32×10^{-4}	1.17×10^{-4}
Quadratic spline [42]	3.06×10^{-3}	7.58×10^{-4}	—	—
Collocation method [32]	—	—	1.80×10^{-3}	4.70×10^{-4}
Cubic spline [10]	2.27×10^{-3}	6.84×10^{-4}	1.57×10^{-3}	4.53×10^{-4}

Maximum absolute errors at the different values of N are tabulated in Table 4 for Problem 5.2 and in Table 5 for Problem 5.3. Fourth order method solution and error graphs at different values of N are also given in Figures 1-3 respectively for Problems 5.1-5.3.

Table 5. Comparison of maximum absolute errors for the solution of Problem 5.3

x	Our method for $k=0$	Our k-based method	Our fourth order method	Standard Tau-method [45]	Perturbed Tau-method [45]	EADM [17]	EFM [44]
1/8	5.24×10^{-4}	7.13×10^{-6}	8.97×10^{-8}	1.00×10^{-4}	2.10×10^{-4}	4.37×10^{-7}	6.88×10^{-5}
2/8	9.69×10^{-4}	1.17×10^{-5}	1.50×10^{-7}	0	1.10×10^{-4}	8.07×10^{-7}	4.93×10^{-5}
3/8	1.26×10^{-3}	1.43×10^{-5}	1.84×10^{-7}	1.00×10^{-4}	7.51×10^{-5}	1.05×10^{-6}	3.21×10^{-5}
4/8	1.37×10^{-3}	1.50×10^{-5}	1.93×10^{-7}	1.00×10^{-4}	6.25×10^{-5}	1.14×10^{-6}	2.63×10^{-5}
5/8	1.26×10^{-3}	1.39×10^{-5}	1.79×10^{-7}	2.00×10^{-4}	4.31×10^{-5}	1.05×10^{-6}	2.16×10^{-5}
6/8	9.69×10^{-4}	1.11×10^{-5}	1.42×10^{-7}	2.00×10^{-4}	2.43×10^{-5}	8.07×10^{-7}	1.09×10^{-5}
7/8	5.24×10^{-4}	6.56×10^{-6}	8.32×10^{-8}	2.00×10^{-4}	1.13×10^{-5}	4.37×10^{-7}	1.01×10^{-5}

Abbreviations: EADM: Extended Adomian Decomposition Method; EFM: Exponential fitting method

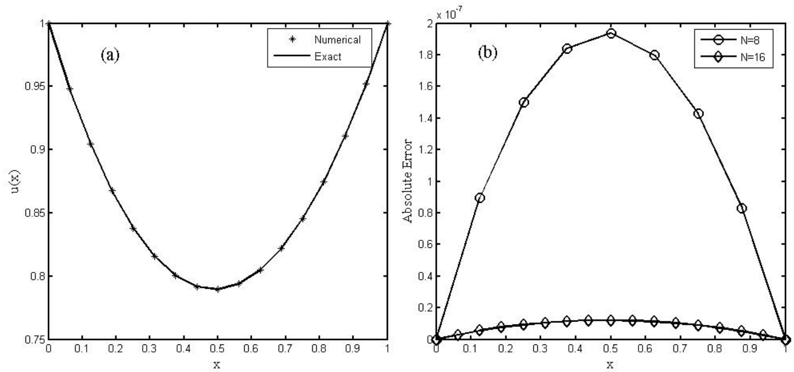

Figure 3. (a) Comparison of approximate and exact values for Problem 5.3. (b) Error graph for Problem 5.3 at different values of N (Table 5).

Problem 5.4. Consider the non-linear BVP

$$u^{(2)}(x) = 2(u(x))^3, \quad -1 < x < 0; \qquad u(-1) = 1/2, u(0) = 1/3. \tag{5.7}$$

The theoretical solution of equation (5.7) is

$$u(x) = \frac{1}{(x+3)} \tag{5.8}$$

To solve non-linear BVP (Problem 5.4), compare the equation (5.7) with equation (1.1) at $x = x_i$ and we have

$$f(x_i, u_i) = 2(u(x_i))^3;$$

Using equation (3.1), we obtain a system of non-linear equations that have been solved using Newton's method. Results are verified with MATLAB builtin solver(*fsolve*) command. Tables 6 and 7 show the maximum absolute errors, in case of k=0, modified k-dependent method and fourth order method solution. Tables clearly indicate that our developed methods produce the better accuracy than some other specified methods. We have also listed the value of parameter k at different value of x in Table 8.

Table 6. Comparison of MAE at N=10 for the solution of Problem 5.4

Our method for $k = 0$	Our k-based method	Our fourth order method	Quintic spline [7]	Cubic spline[20]	Quartic spline [6]
2.65×10^{-5}	8.08×10^{-6}	3.23×10^{-7}	8.82×10^{-6}	1.68×10^{-5}	4.67×10^{-6}

Table 7. Maximum absolute errors at different value of N for Problem 5.4

Our method	N = 4	N = 8	N = 16
Our method for $k = 0$	1.63×10^{-4}	4.13×10^{-5}	1.03×10^{-5}
Our k-based method	1.28×10^{-4}	1.53×10^{-5}	6.83×10^{-6}
Our fourth-order method	2.56×10^{-6}	1.64×10^{-7}	1.08×10^{-8}

Table 8. The value of k at different value of x for the solution of Problem 5.4

x	-0.9	-0.8	-0.7	-0.6	-0.5	-0.4	-0.3	-0.2	-0.1
k	1.6499	1.5748	1.5062	1.4433	1.3855	1.3321	1.2827	1.2368	1.1941

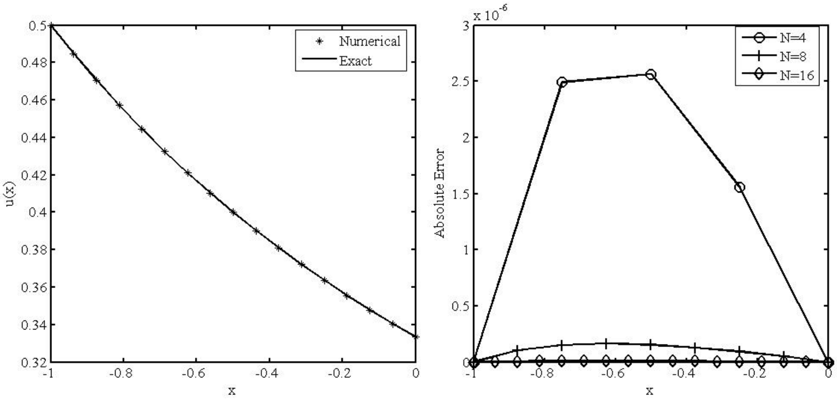

Figure 4. (a) Comparison of approximate values and exact values for Problem 5.4. (b) Error graph for Problem 5.4 at different values of N (Table 7).

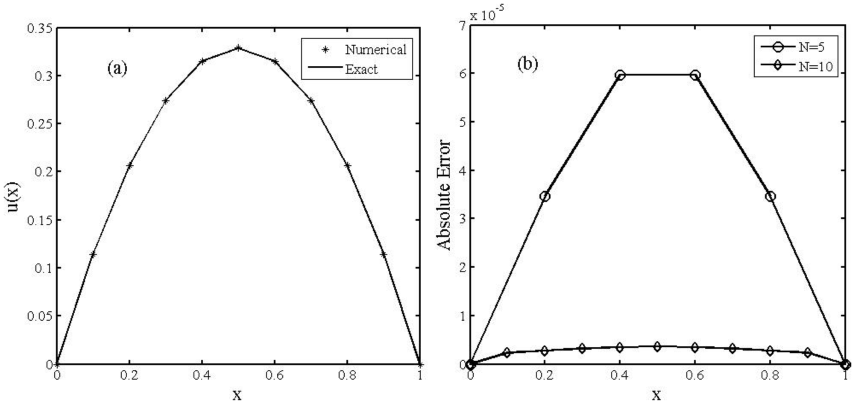

Figure 5. (a) Comparison of approximate values and exact values for Problem 5.5. (b) Error graph for Problem 5.5 at different values of N (Table 9, col. III).

Problem 5.5. Consider the non-linear BVP (Bratu Problem)

$$u^{(2)}(x) + 2e^{u(x)} = 0, \quad 0 < x < 1; \qquad u(0) = u(1) = 0. \tag{5.9}$$

The theoretical solution of (5.9) is

$$u(x) = -2\ln(\cosh(1.17878\ (x-0.5)))/\cosh\ (0.589388). \tag{5.10}$$

Table 9. Comparison of MAE for the solution of Problem 5.5 at $N = 10$

Our method for $k = 0$	Our k-based method	Our fourth order method	LGSM [1]	Quintic spline [7]
8.83×10^{-4}	3.56×10^{-5}	3.64×10^{-6}	5.7×10^{-6}	6.22×10^{-6}
B-Spline method [18]	Quartic spline method [6]	Cubic spline[20]	LADM [35]	ADM [22]
5.29×10^{-5}	1.10×10^{-4}	6.26×10^{-4}	1.24×10^{-2}	1.52×10^{-2}

Abbreviations: ADM: Adomian Decomposition Method;
LGSM: Lie-group shooting method;
LADM: Laplace Adomian Decomposition Method

Problem 5.6. Consider the non-linear BVP

$$u^{(2)}(x) = \frac{1}{2}(1 + x + u)^3, \quad 0 < x < 1; \qquad u(0) = u(1) = 0. \tag{5.11}$$

The theoretical solution of (5.11) is

$$u(x) = \frac{2}{(2 - x)} - x - 1. \tag{5.12}$$

The other non-linear BVPs mentioned in Problems 5.5 and 5.6, are also solved just like Problem 5.4 using Newton's method. Obtained results show the efficiency and accuracy of our proposed methods. Maximum absolute errors at the nodal points with a comparison with other methods are summarized in Table 9 for Problem 5.5 and in Table 10 for Problem 5.6, respectively. Figures 4-6 demonstrate the fourth order method solution and error graphs for nonlinear Problems 5.4-5.6 respectively with comparison of errors at the nodal points.

Table 10. Comparison of MAE for Problem 5.6 with Approaching spline method at $N = 5$

x *values*	0	0.2	0.4	0.6	0.8	1
Our method for $k = 0$	0	1.30×10^{-3}	2.40×10^{-3}	3.10×10^{-3}	2.80×10^{-3}	0
Our k-based method	0	2.70×10^{-5}	5.25×10^{-5}	7.19×10^{-5}	6.49×10^{-5}	0
Our fourth order method	0	3.80×10^{-5}	7.26×10^{-5}	9.92×10^{-5}	9.96×10^{-5}	0
Approaching spline [31]	0	1.40×10^{-4}	2.60×10^{-4}	3.20×10^{-4}	2.70×10^{-4}	0

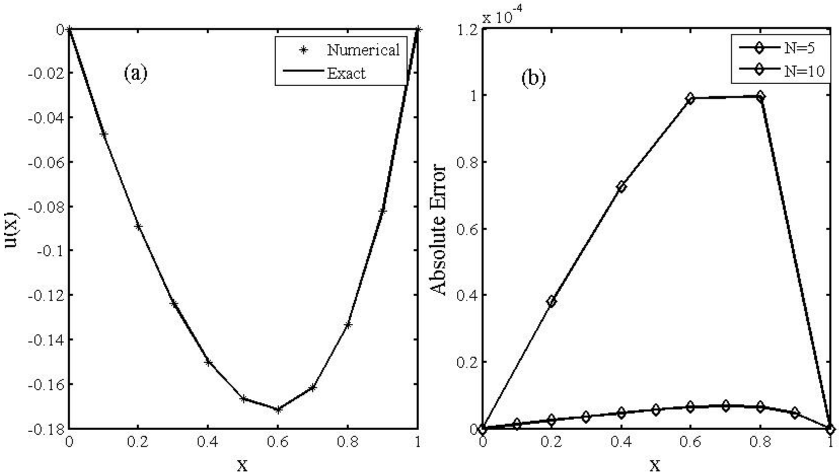

Figure 6. (a) Comparison of approximate values and exact values for Problem 5.6. (b) Error graph for Problem 5.6 at different values of N (Table 10).

6. Conclusion

A unique approach based on a different combination of non-polynomial cubic splines is used to develop various orders methods for solving linear and non-linear second order BVPs. We have also developed a parameter k-based method for smooth approximation of these BVPs. The convergence of the developed method is also established. Competence of the demonstrated technique can also be weighed through comparisons with the literature given in tables, which show that our results are comparatively better with more precise result. Graphs are plotted at different values of N for all the problems, which clearly show that absolute errors decrease rapidly as step size N increases.

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Optimal decay rates for the acoustic wave motions with boundary memory damping

Khalida Benomar and Abbes Benaissa

Abstract. A linear wave equation with acoustic boundary conditions (ABC) on a portion of the boundary and Dirichlet conditions on the rest of the boundary is considered. The (ABC) contain a memory damping with respect to the normal displacement of the boundary point. In this paper, we establish polynomial energy decay rates for the wave equation by using resolvent estimates.

Mathematics Subject Classification (2010): 93D15, 35B40, 47D03, 74D05.

Keywords: Wave equation, acoustic boundary conditions, boundary memory.

1. Introduction

In this paper we investigate the existence and decay properties of solutions for the initial boundary value problem of the wave equation of the type

$$\begin{cases} y_{tt}(x,t) - y_{xx}(x,t) = 0 & \text{in } (0,L)\times(0,+\infty), \\ y(0,t) = 0 & \text{in } (0,+\infty), \\ y_x(L,t) = z_t(t) & \text{in } (0,+\infty), \\ y_t(L,t) + mz(t) + \gamma\partial_t^{\alpha,\eta} z(t) = 0 & \text{in } (0,+\infty), \\ y(x,0) = y_0(x), \quad y_t(x,0) = y_1(x) & \text{in } (0,L), \end{cases} \tag{P}$$

where $(x,t) \in (0,L)\times(0,+\infty), m > 0, \gamma > 0, \eta \geq 0$ and the initial data are taken in suitable spaces. The notation $\partial_t^{\alpha,\eta}$ stands for the generalized Caputo's fractional derivative of order α, $0 < \alpha < 1$, with respect to the time variable (see Choi and MacCamy [9]). It is defined as follows

$$\partial_t^{\alpha,\eta} w(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} e^{-\eta(t-s)} \frac{dw}{ds}(s)\, ds, \quad \eta \geq 0.$$

The problem (P) describes sound wave propagation in a domain which is full of some kind of medium and with a portion of boundary made of light-weight viscoelastic material.

Acoustic model was proposed by Morse and Ingard [15], and improved in a rigorous mathematical way by Beale and Rosencrans [5]. Under the assumption that each local-reacting boundary point acts as a spring, the author analyzed the model in both bounded and exterior domains in [3], [4]. Uniform energy decay rates were studied in [7], [16] for acoustic wave systems with both internal and boundary memory damping terms. To our knowledge, there has been few work about the decay rates of acoustic wave energies when only one memory damping acting on the acoustic boundary.

Recently, In [11], the authors considered the following initial boundary value problem with memory type acoustic boundary conditions,

$$\begin{cases} y_{tt}(x,t) - \Delta y(x,t) = 0 & \text{in } \Omega \times (0,+\infty), \\ y(x,t) = 0 & \text{in } \Gamma_0 \times (0,+\infty), \\ \frac{\partial y}{\partial \nu}(x,t) = z_t(x,t) & \text{in } \Gamma_1 \times (0,+\infty), \\ y_t(x,t) + mz(x,t) + \gamma \partial_t^{\alpha,0} z(x,t) = 0 & \text{in } \Gamma_1 \times (0,+\infty), \\ y(x,0) = y_0(x), \quad y_t(x,0) = y_1(x) & \text{in } (0,L). \end{cases} \tag{P}$$

They proved well-posedness and strong stability of the system (P) without giving an energy decay rate. Very Recently, in [12] the authors proved that the energy is polynomially stable but without obtaining the precise exponent.

The aim of the present paper is to obtain more precise rates of decay. This can be achieved via some theorems about operator semigroups. We provide a standard method of going from resolvent estimates for a suitable PDE to rates of decay of classical (strong) solutions.

We should mention here that the approach in [11] and [12], which is based on Laplace transform is different from ours. By redescribing the fractional derivative term by means of a suitable diffusion equation as in [14], the original model is transformed into an augmented system which can be more easily tackled by the energy method.

2. Augmented model

This section is concerned with the reformulation of the model (P) into an augmented system. For that, we need the following claims.

Theorem 2.1 (see [14]). *Let μ be the function:*

$$\mu(\xi) = |\xi|^{(2\alpha-1)/2}, \quad -\infty < \xi < +\infty,\ 0 < \alpha < 1. \tag{2.1}$$

Then the relationship between the 'input' U and the 'output' O of the system

$$\partial_t \phi(\xi,t) + (\xi^2 + \eta)\phi(\xi,t) - U(t)\mu(\xi) = 0, \quad -\infty < \xi < +\infty, \eta \geq 0, t > 0, \tag{2.2}$$

$$\phi(\xi,0) = 0, \tag{2.3}$$

$$O(t) = (\pi)^{-1} \sin(\alpha\pi) \int_{-\infty}^{+\infty} \mu(\xi)\phi(\xi,t)\, d\xi \tag{2.4}$$

is given by

$$O = I^{1-\alpha,\eta} U = D^{\alpha,\eta} U, \tag{2.5}$$

where

$$[I^{\alpha,\eta}f](t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}e^{-\eta(t-\tau)}f(\tau)\,d\tau.$$

Lemma 2.2 (see [1]). *If* $\lambda \in D_\eta = \mathbb{C}\backslash]-\infty,-\eta]$ *then*

$$\int_{-\infty}^{+\infty} \frac{\mu^2(\xi)}{\lambda+\eta+\xi^2}\,d\xi = \frac{\pi}{\sin\alpha\pi}(\lambda+\eta)^{\alpha-1}.$$

We are now in a position to reformulate system (P). Indeed, by using Theorem 2.1, system (P) may be recast into the augmented model:

$$\begin{cases} y_{tt}(x,t) - y_{xx}(x,t) = 0 & \text{in } (0,L)\times(0,+\infty),\\ \partial_t\phi(\xi,t) + (\xi^2+\eta)\phi(\xi,t) - z_t(t)\mu(\xi) = 0 & \text{in } (-\infty,+\infty)\times(0,+\infty),\\ y(0,t) = 0 & \text{in } (0,+\infty),\\ y_x(L,t) = z_t(t) & \text{in } (0,+\infty),\\ y_t(L,t) + mz(t) + \zeta\displaystyle\int_{-\infty}^{+\infty}\mu(\xi)\phi(\xi,t)\,d\xi = 0 & \text{in } (0,+\infty),\\ y(x,0) = y_0(x), \quad y_t(x,0) = y_1(x) & \text{in } (0,L),\\ \phi(\xi,0) = 0 & \text{in } (-\infty,+\infty). \end{cases} \tag{P'}$$

We define the energy associated to the solution of the problem (P') by the following formula:

$$E(t) = \frac{1}{2}\|y_t\|_2^2 + \frac{1}{2}\|y_x\|_2^2 + \frac{m}{2}|z(t)|^2 + \frac{\zeta}{2}\int_{-\infty}^{+\infty}|\phi(\xi,t)|^2\,d\xi. \tag{2.6}$$

Lemma 2.3. *Let* (y,ϕ) *be a solution of the problem* (P')*. Then, the energy functional defined by* (2.6) *satisfies*

$$E'(t) = -\zeta\int_{-\infty}^{+\infty}(\xi^2+\eta)|\phi(\xi,t)|^2\,d\xi \le 0. \tag{2.7}$$

Proof. Multiplying the first equation in (P') by $\overline{y}_t$, integrating over $(0,L)$ and using integration by parts, we get

$$\frac{1}{2}\frac{d}{dt}\|y_t\|_2^2 - \Re\int_0^L y_{xx}\overline{y}_t dx = 0.$$

Then

$$\frac{d}{dt}\left(\frac{1}{2}\|y_t\|_2^2 + \frac{1}{2}\|y_x\|_2^2\right) + \Re z_t(t)\left(m\overline{z}(t) + \zeta\int_{-\infty}^{+\infty}\mu(\xi)\overline{\phi}(\xi,t)\,d\xi\right) = 0. \tag{2.8}$$

Multiplying the second equation in (P') by $\zeta\overline{\phi}_t$ and integrating over $(-\infty,+\infty)$, to obtain:

$$\frac{\zeta}{2}\frac{d}{dt}\|\phi\|_2^2 + \zeta\int_{-\infty}^{+\infty}(\xi^2+\eta)|\phi(\xi,t)|^2\,d\xi - \zeta\Re z_t(t)\int_{-\infty}^{+\infty}\mu(\xi)\overline{\phi}(\xi,t)\,d\xi = 0. \tag{2.9}$$

From (2.6), (2.8) and (2.9) we get

$$E'(t) = -\zeta\int_{-\infty}^{+\infty}(\xi^2+\eta)|\phi(\xi,t)|^2\,d\xi.$$

This completes the proof of the lemma. □

3. Well-posedness

The energy space associated to system (P) is

$\mathcal{H} = H_L^1(0,L) \times L^2(0,L) \times L^2(-\infty,+\infty) \times \mathbb{C}, \quad H_L^1(0,L) = \{y \in H^1(0,L),\ y(0) = 0\}$

equipped with the inner product

$$< U, \tilde{U} >_{\mathcal{H}} = \int_\Omega \left(v\bar{\tilde{v}} + y_x \bar{\tilde{y}}_x \right) dx + mz\bar{\tilde{z}} + \zeta \int_{-\infty}^{+\infty} \phi\bar{\tilde{\phi}} d\xi,$$

where $U = (y, v, \phi, z)^T, \tilde{U} = (\tilde{y}, \tilde{v}, \tilde{\phi}, \tilde{z})^T \in \mathcal{H}$.

Let $U = (y, y_t, \phi, z)^T$ and rewrite (P') as

$$\begin{cases} U' = \mathcal{A}U, \\ U(0) = (y_0, y_1, \phi_0, z_0), \end{cases} \tag{3.1}$$

where the operator $\mathcal{A}$ is defined by

$$\mathcal{A} \begin{pmatrix} y \\ v \\ \phi \\ z \end{pmatrix} = \begin{pmatrix} v \\ y_{xx} \\ -(\xi^2 + \eta)\phi + y_x(L)\mu(\xi) \\ y_x(L) \end{pmatrix} \tag{3.2}$$

with domain

$$D(\mathcal{A}) = \left\{ \begin{array}{l} (y, v, \phi, z)^T \text{ in } \mathcal{H} : y \in H^2(0,L) \cap H_L^1(0,L), \\ v \in H_L^1(0,L), z \in \mathbb{C}, \\ -(\xi^2 + \eta)\phi + y_x(L)\mu(\xi) \in L^2(-\infty,+\infty), \\ v(L) + mz + \zeta \displaystyle\int_{-\infty}^{\infty} \mu(\xi)\phi(\xi)\, d\xi = 0, \\ |\xi|\phi \in L^2(-\infty,+\infty) \end{array} \right\}. \tag{3.3}$$

Now, we will give well-posedness results for problem (P) using semigroup theory. We show that the operator $\mathcal{A}$ generates a C_0- semigroup in $\mathcal{H}$. We prove that $\mathcal{A}$ is a maximal dissipative operator (see [8]). For this purpose we need the following two lemmas.

Lemma 3.1. *The operator $\mathcal{A}$ is dissipative and satisfies, for any $U \in D(\mathcal{A})$,*

$$\Re\langle \mathcal{A}U, U \rangle_{\mathcal{H}} = -\zeta \int_{-\infty}^{+\infty} (\xi^2 + \eta)|\phi(\xi)|^2\, d\xi. \tag{3.4}$$

Proof. For any $U = (y, v, \phi, z)^T \in D(A)$, Using (3.1) and the fact that

$$\|(y, y_t, \phi, z)\|_{\mathcal{H}}^2 = \|U\|_{\mathcal{H}}^2, \tag{3.5}$$

estimate (3.4) easily follows. □

Lemma 3.2. *The operator $\lambda I - \mathcal{A}$ is surjective for all $\lambda > 0$.*

Proof. We need to show that for all $F = (f_1, f_2, f_3, f_4)^T \in \mathcal{H}$, there exists

$$U = (y, u, \phi, v)^T \in D(\mathcal{A})$$

such that

$$\lambda U - \mathcal{A}U = F, \tag{3.6}$$

that is

$$\begin{cases} \lambda y - v = f_1, \\ \lambda v - y_{xx} = f_2, \\ \lambda\phi + (\xi^2+\eta)\phi - y_x(L)\mu(\xi) = f_3, \\ \lambda z - y_x(L) = f_4. \end{cases} \tag{3.7}$$

Suppose that we have found y. Therefore, the first equation in (3.7) gives

$$v = \lambda y - f_1. \tag{3.8}$$

It is clear that $u \in H^1_L(0,L)$. Furthermore, by (3.7) we can find ϕ as

$$\phi = \frac{f_3(\xi) + \mu(\xi)y_x(L)}{\xi^2+\eta+\lambda}. \tag{3.9}$$

By using (3.7) and (3.8) the function y satisfying the following system

$$\lambda^2 y - y_{xx} = f_2 + \lambda f_1. \tag{3.10}$$

Solving system (3.10) is equivalent to finding $y \in H^2 \cap H^1_L(0,L)$ such that

$$\int_0^L (\lambda^2 y\overline{w} - y_{xx}\overline{w})\,dx = \int_0^L (f_2+\lambda f_1)\overline{w}\,dx, \tag{3.11}$$

for all $w \in H^1_L(0,L)$. By using (3.11) and (3.9) the function y satisfying the following system

$$\begin{cases} \displaystyle\int_0^L (\lambda^2 y\overline{w} + y_x\overline{w}_x)\,dx + \frac{\lambda^2}{m+\gamma\lambda(\lambda+\eta)^{\alpha-1}}y(L)\overline{w}(L) \\ \displaystyle\qquad = \int_0^L (f_2+\lambda f_1)\overline{w}\,dx + \frac{1}{m+\gamma\lambda(\lambda+\eta)^{\alpha-1}}(\lambda f_1(L) \\ \displaystyle\qquad\qquad -\zeta\lambda\int_{-\infty}^{+\infty}\frac{\mu(\xi)}{\xi^2+\eta+\lambda}f_3(\xi)\,d\xi - mf_4\overline{w}(L). \end{cases} \tag{3.12}$$

Consequently, problem (3.12) is equivalent to the problem

$$a(y,w) = L(w), \tag{3.13}$$

where the sesquilinear form $a : H^1_L(0,L)\times H^1_L(0,L) \to \mathbb{C}$ and the antilinear form $L : H^1_L(0,L) \to \mathbb{C}$ are defined by

$$a(y,w) = \int_0^L (\lambda^2 y\overline{w} + y_x\overline{w}_x)\,dx + \frac{\lambda^2}{m+\gamma\lambda(\lambda+\eta)^{\alpha-1}}y(L)\overline{w}(L)$$

and

$$L(w) = \int_0^L (f_2+\lambda f_1)\overline{w}\,dx + \frac{1}{m+\gamma\lambda(\lambda+\eta)^{\alpha-1}}(\lambda f_1(L)$$
$$-\zeta\lambda\int_{-\infty}^{+\infty}\frac{\mu(\xi)}{\xi^2+\eta+\lambda}f_3(\xi)\,d\xi - mf_4\overline{w}(L).$$

It is easy to verify that a is continuous and coercive, and L is continuous. So applying the Lax-Milgram theorem, we deduce that for all $w \in H^1_L(0,L)$ problem (3.13) admits a unique solution $y \in H^1_L(0,L)$. Applying the classical elliptic regularity, it follows from (3.12) that $y \in H^2(0,L)$. Therefore, the operator $\lambda I - \mathcal{A}$ is surjective for any $\lambda > 0$. $\square$

Consequently, using Hille-Yosida Theorem, we have the following well-posedness result:

Theorem 3.3 (Existence and uniqueness).

(1) *If $U_0 \in D(\mathcal{A})$, then system* (3.1) *has a unique strong solution*
$$U \in C^0(\mathbb{R}_+, D(\mathcal{A})) \cap C^1(\mathbb{R}_+, \mathcal{H}).$$

(1) *If $U_0 \in \mathcal{H}$, then system* (3.1) *has a unique weak solution*
$$U \in C^0(\mathbb{R}_+, \mathcal{H}).$$

4. Lack of exponential stability

In order to state and prove our stability results, we need the following well known theorems.

Theorem 4.1 ([17]-[10])**.** *Let $S(t) = e^{\mathcal{A}t}$ be a C_0-semigroup of contractions on Hilbert space $\mathcal{H}$. Then $S(t)$ is exponentially stable if and only if*
$$\rho(\mathcal{A}) \supseteq \{i\beta : \beta \in \mathbb{R}\} \equiv i\mathbb{R}$$
and
$$\overline{\lim_{|\beta|\to\infty}} \|(i\beta I - \mathcal{A})^{-1}\|_{\mathcal{L}(\mathcal{H})} < \infty.$$

Theorem 4.2 ([6])**.** *Let $S(t) = e^{\mathcal{A}t}$ be a C_0-semigroup on a Hilbert space $\mathcal{H}$. If*
$$i\mathbb{R} \subset \rho(\mathcal{A}) \text{ and } \sup_{|\beta|\geq 1} \frac{1}{\beta^\delta} \|(i\beta I - \mathcal{A})^{-1}\|_{\mathcal{L}(\mathcal{H})} < M$$
for some $\delta > 0$, then there exist c such that
$$\|e^{\mathcal{A}t}U_0\|^2 \leq \frac{c}{t^{\frac{2}{\delta}}} \|U_0\|^2_{D(\mathcal{A})}.$$

Theorem 4.3 ([2]-[13])**.** *Let $\mathcal{A}$ be the generator of a uniformly bounded C_0- semigroup $\{S(t)\}_{t\geq 0}$ on a Hilbert space $\mathcal{H}$. If:*

(i) *$\mathcal{A}$ does not have eigenvalues on $i\mathbb{R}$.*

(ii) *The intersection of the spectrum $\sigma(\mathcal{A})$ with $i\mathbb{R}$ is at most a countable set,*

then the semigroup $\{S(t)\}_{t\geq 0}$ is asymptotically stable, i.e, $\|S(t)z\|_{\mathcal{H}} \to 0$ as $t \to \infty$ for any $z \in \mathcal{H}$.

Our main first result is

Theorem 4.4. *The semigroup generated by the operator $\mathcal{A}$ is not exponentially stable.*

Proof. We will examine two cases.

• **Case 1.** $\eta = 0$: We shall show that $i\lambda = 0$ is not in the resolvent set of the operator $\mathcal{A}$. Indeed, noting that $(0, 0, 0, \cos L)^T \in \mathcal{H}$, and denoting by $(y, v, \phi, z)^T$ the image of $(0, 0, 0, \cos L)^T$ by $\mathcal{A}^{-1}$, we see that $\phi(\xi) = |\xi|^{\frac{2\alpha-5}{2}} \cos L$. But, then $\phi \notin L^2(-\infty, +\infty)$, since $\alpha \in]0, 1[$. Hence $(y, v, \phi, z)^T \notin D(\mathcal{A})$.

• **Case 2.** $\eta \neq 0$: We aim to show that an infinite number of eigenvalues of $\mathcal{A}$ approach the imaginary axis which prevents the wave system (P) from being exponentially stable. Indeed we first compute the characteristic equation that gives the eigenvalues of $\mathcal{A}$. Let λ be an eigenvalue of $\mathcal{A}$ with associated eigenvector $U = (y, v, \phi, z)^T$. Then $\mathcal{A}U = \lambda U$ is equivalent to

$$\begin{cases} \lambda y - v = 0, \\ \lambda v - y_{xx} = 0, \\ \lambda\phi + (\xi^2 + \eta)\phi - y_x(L)\mu(\xi) = 0, \\ \lambda z - y_x(L) = 0, \\ v(L) + mz + \zeta \displaystyle\int_{-\infty}^{+\infty} \mu(\xi)\phi(\xi)\, d\xi = 0. \end{cases} \tag{4.1}$$

From $(4.1)_1 - (4.1)_2$ for such λ, we find

$$\lambda^2 y - y_{xx} = 0. \tag{4.2}$$

Since $v = \lambda y(L)$, using $(4.1)_3$ and $(4.1)_4$, we get

$$\begin{cases} y(0) = 0, \\ \lambda^2 y(L) + (m + \gamma\lambda(\lambda+\eta)^{\alpha-1})y_x(L) = 0. \end{cases} \tag{4.3}$$

The solution y is given by

$$y(x) = \sum_{i=1}^{2} c_i e^{t_i x}, \tag{4.4}$$

where

$$t_1(\lambda) = \lambda, \quad t_2(\lambda) = -\lambda.$$

Thus the boundary conditions may be written as the following system:

$$M(\lambda)C(\lambda) = \begin{pmatrix} 1 & 1 \\ h(t_1)e^{t_1 L} & h(t_2)e^{t_2 L} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{4.5}$$

where we have set

$$h(r) = (m + \gamma\lambda(\lambda+\eta)^{\alpha-1})r + \lambda^2.$$

Hence a non-trivial solution y exists if and only if the determinant of $M(\lambda)$ vanishes. Set $f(\lambda) = det M(\lambda)$, thus the characteristic equation is $f(\lambda) = 0$.

Our purpose is to prove, thanks to Rouché's Theorem, that there is a subsequence of eigenvalues for which their real part tends to 0. □

In the sequel, since $\mathcal{A}$ is dissipative, we study the asymptotic behavior of the large eigenvalues λ of $\mathcal{A}$ in the strip $-\alpha_0 \le \mathcal{R}(\lambda) \le 0$, for some $\alpha_0 > 0$ large enough and for such λ, we remark that $e^{t_i}, i = 1, 2$ remains bounded.

Lemma 4.5. *There exists $N \in \mathbb{N}$ such that*

$$\{\lambda_k\}_{k\in\mathbb{Z}^*, |k|\ge N} \subset \sigma(\mathcal{A}) \tag{4.6}$$

where

$$\lambda_k = i\frac{k\pi}{L} + \frac{\tilde{\alpha}}{k^{1-\alpha}} + \frac{\beta}{k^{1-\alpha}} + o\left(\frac{1}{k^{1-\alpha}}\right), k \ge N, \tilde{\alpha} \in i\mathbb{R}, \beta \in \mathbb{R}, \beta < 0,$$

$$\lambda_k = \overline{\lambda_{-k}} \text{ if } k \le -N.$$

Moreover for all $|k| \ge N$, the eigenvalues λ_k are simple.

Proof. **Step 1.**

$$\begin{aligned} f(\lambda) &= e^{t_2}h(t_2) - e^{t_1}h(t_1) \\ &= -e^{-\lambda L}((m+\gamma\lambda(\lambda+\eta)^{\alpha-1})+\lambda)\left(e^{2\lambda L} - \frac{\lambda-(m+\gamma\lambda(\lambda+\eta)^{\alpha-1})}{\lambda+(m+\gamma\lambda(\lambda+\eta)^{\alpha-1})}\right) \\ &= -e^{-\lambda L}((m+\gamma\lambda(\lambda+\eta)^{\alpha-1})+\lambda)\left(e^{2\lambda L} - 1 + 2\frac{m+\gamma\lambda(\lambda+\eta)^{\alpha-1}}{m+\lambda+\gamma\lambda(\lambda+\eta)^{\alpha-1}}\right). \end{aligned} \tag{4.7}$$

We set

$$\begin{aligned} \tilde{f}(\lambda) &= e^{2\lambda L} - 1 + 2\frac{m+\gamma\lambda(\lambda+\eta)^{\alpha-1}}{m+\lambda+\gamma\lambda(\lambda+\eta)^{\alpha-1}} \\ &= f_0(\lambda) + \tfrac{f_1(\lambda)}{\lambda^{1-\alpha}} + o\left(\tfrac{1}{\lambda^{1-\alpha}}\right) \end{aligned} \tag{4.8}$$

where

$$f_0(\lambda) = e^{2\lambda L} - 1, \tag{4.9}$$

$$f_1(\lambda) = 2\gamma. \tag{4.10}$$

Note that f_0 and f_1 remain bounded in the strip $-\alpha_0 \leq \mathcal{R}(\lambda) \leq 0$.

Step 2. We look at the roots of f_0. From (4.9), f_0 has one familie of roots that we denote λ_k^0.

$$f_0(\lambda) = 0 \Leftrightarrow e^{2\lambda L} = 1.$$

Hence

$$2\lambda L = i2k\pi, \quad k \in \mathbb{Z},$$

i.e.,

$$\lambda_k^0 = \frac{ik\pi}{L}, \quad k \in \mathbb{Z}.$$

Now with the help of Rouché's Theorem, we will show that the roots of $\tilde{f}$ are close to those of f_0. Changing in (4.8) the unknown λ by $u = 2\lambda L$ then (4.8) becomes

$$\tilde{f}(u) = (e^u - 1) + O\left(\frac{1}{u}\right) = f_0(u) + O\left(\frac{1}{u}\right).$$

The roots of f_0 are $u_k = \frac{ik}{L}\pi, k \in \mathbb{Z}$, and setting $u = u_k + re^{it}, t \in [0, 2\pi]$, we can easily check that there exists a constant $C > 0$ independent of k such that $|e^u - 1| \geq Cr$ for r small enough. This allows to apply Rouché's Theorem. Consequently, there exists a subsequence of roots of $\tilde{f}$ which tends to the roots u_k of f_0. Equivalently, it means that there exists $N \in \mathbb{N}$ and a subsequence $\{\lambda_k\}_{|k|\geq N}$ of roots of $f(\lambda)$, such that $\lambda_k = \lambda_k^0 + o(1)$ which tends to the roots $\frac{ik}{L}\pi$ of f_0. Finally for $|k| \geq N, \lambda_k$ is simple since λ_k^0 is.

Step 3. From Step 2, we can write

$$\lambda_k = i\frac{1}{L}k\pi + \varepsilon_k. \tag{4.11}$$

Using (4.11), we get

$$e^{2\lambda_k L} = 1 + 2L\varepsilon_k + 2L^2\varepsilon_k^2 + o(\varepsilon_k^2). \tag{4.12}$$

Substituting (4.12) into (4.8), using that $\tilde{f}(\lambda_k) = 0$, we get:

$$\begin{aligned} \tilde{f}(\lambda_k) &= 2L\varepsilon_k + \frac{2\gamma}{(\frac{k\pi i}{L} + \varepsilon_k)^{(1-\alpha)}} + o(\varepsilon_k) + o(1/k) \\ &= 2L\varepsilon_k + \frac{2\gamma}{(\frac{k\pi}{L} i)^{(1-\alpha)}} + o\left(\frac{1}{k}\right) = 0 \end{aligned} \tag{4.13}$$

and hence

$$\varepsilon_k = -\frac{\gamma}{L^\alpha} \frac{1}{(k\pi)^{(1-\alpha)}} \left(\cos(1-\alpha)\frac{\pi}{2} - i\sin(1-\alpha)\frac{\pi}{2}\right) + o\left(\frac{1}{k^{1-\alpha}}\right) \text{ for } k \succeq 0.$$

From (4.13) we have in that case $|k|^{1-\alpha}\mathcal{R}\lambda_k \sim \beta$, with

$$\beta = -\frac{\gamma}{L^\alpha \pi^{1-\alpha}} \cos(1-\alpha)\frac{\pi}{2}.$$

The operator $\mathcal{A}$ has a non exponential decaying branch of eigenvalues. Thus the proof is complete. □

5. Polynomial stability and optimality (for $\eta \neq 0$)

In the previous section,we have shown that the transmission wave system is not exponentially stable. In this section, we prove that it is polynomially stable with an optimal rate of decay when $\eta > 0$. To achieve this, we use a recent result by Borichev and Tomilov [6]. Accordingly, if we consider a bounded C_0-semigroup $S(t) = e^{\mathcal{A}t}$ on a Hilbert space. If

$$i\mathbb{R} \subset \rho(\mathcal{A}) \text{ and } \overline{\lim}_{|\beta|\to\infty} \frac{1}{\beta^\delta} \|(i\beta I - \mathcal{A})^{-1}\|_{\mathcal{L}(\mathcal{H})} < \infty$$

for some $\delta > 0$, then there exist c such that

$$\|e^{\mathcal{A}t}U_0\|^2 \leq \frac{c}{t^{\frac{2}{\delta}}} \|U_0\|^2_{D(\mathcal{A})}.$$

Our main result is as follows.

Theorem 5.1. *The semigroup $S_{\mathcal{A}}(t)_{t\geq 0}$ is polynomially stable and*

$$E(t) = \|S_{\mathcal{A}}(t)U_0\|^2_{\mathcal{H}} \leq \frac{1}{t^{2/(1-\alpha)}} \|U_0\|^2_{D(\mathcal{A})}.$$

Moreover, the rate of energy decay $t^{-2/(1-\alpha)}$ is optimal for any initial data in $D(\mathcal{A})$.

Proof. We will need to study the resolvent equation $(i\lambda - \mathcal{A})U = F$, for $\lambda \in \mathbb{R}$, namely

$$\begin{cases} i\lambda y - v = f_1, \\ i\lambda v - y_{xx} = f_2, \\ i\lambda\phi + (\xi^2 + \eta)\phi - y_x(L)\mu(\xi) = f_3, \\ i\lambda z - y_x(L) = f_4 \end{cases} \tag{5.1}$$

with the boundary condition

$$v(L) + mz + \zeta \int_{-\infty}^{+\infty} \mu(\xi)\phi(\xi)\, d\xi = 0. \tag{5.2}$$

We divide the proof into three steps, as follows:
Step 1. Inserting $(5.1)_1$ into $(5.1)_2$, we get

$$\lambda^2 y + y_{xx} = -(f_2 + i\lambda f_1).$$

As $y(0) = 0$, then

$$y(x) = c_1 \sin \lambda x - \frac{1}{\lambda}\int_0^x (f_2(\sigma) + i\lambda f_1(\sigma)) \sin \lambda(x-\sigma)\, d\sigma, \tag{5.3}$$

and hence

$$y_x(x) = c_1 \lambda \cos \lambda x - \int_0^x (f_2(\sigma) + i\lambda f_1(\sigma)) \cos \lambda(x-\sigma)\, d\sigma. \tag{5.4}$$

Step 2. With the third equation of (5.1), we get

$$\phi(\xi) = \frac{y_x(L)\mu(\xi) + f_3(\xi)}{i\lambda + \xi^2 + \eta}. \tag{5.5}$$

Inserting (5.5) in the boundary condition (5.2), we easy to check that

$$-\lambda^2 y(L) + (m + \gamma i\lambda(i\lambda+\eta)^{\alpha-1}) y_x(L) = i\lambda f_1(L) - m f_4 - \zeta i\lambda \int_{-\infty}^{+\infty} \frac{\mu(\xi) f_3(\xi)}{i\lambda + \xi^2 + \eta}\, d\xi. \tag{5.6}$$

Using (5.3) and (5.4), we can rewrite (5.6) as an equation in the unknown c_1,

$$\begin{aligned} c_1(-\lambda^2 \sin \lambda L + \lambda(m + \gamma i\lambda(i\lambda+\eta)^{\alpha-1}) \cos \lambda L) \\ = i\lambda f_1(L) - m f_4 - \zeta i\lambda \int_{-\infty}^{+\infty} \frac{\mu(\xi) f_3(\xi)}{i\lambda + \xi^2 + \eta}\, d\xi - \lambda \int_0^L (f_2(\sigma) + i\lambda f_1(\sigma)) \sin \lambda(L-\sigma)\, d\sigma \\ + (m + \gamma i\lambda(i\lambda+\eta)^{\alpha-1}) \int_0^L (f_2(\sigma) + i\lambda f_1(\sigma)) \cos \lambda(L-\sigma)\, d\sigma. \end{aligned} \tag{5.7}$$

Step 3. We set

$$g(\lambda) = -\lambda \sin \lambda L + (m + \gamma i\lambda(i\lambda+\eta)^{\alpha-1}) \cos \lambda L. \tag{5.8}$$

As $f_1 \in H_L^1(0,L)$ and $f_2 \in L^1(0,L)$, we have

$$\left| \int_0^L (f_2(\sigma) + i\lambda f_1(\sigma)) \sin \lambda(L-\sigma)\, d\sigma \right| \le c(\|f_2\|_{L^2(0,L)} + \|f_1\|_{H^1(0,L)}).$$

$$\left| \int_0^L (f_2(\sigma) + i\lambda f_1(\sigma)) \cos \lambda(L-\sigma)\, d\sigma \right| \le c(\|f_2\|_{L^2(0,L)} + \|f_1\|_{H^1(0,L)}).$$

As $g(\lambda) \neq 0$ for all λ (if $\eta = 0$ then for all $\lambda \neq 0$) , then c_1 is uniqueley determined by (5.7). Hence the operator $i\lambda - \mathcal{A}$ is surjective for all λ (if $\eta = 0$ then for all $\lambda \neq 0$). Moreover, taking account of Lemma 4.5, the operator $i\lambda - \mathcal{A}$ is injective for all λ. Then $i\mathbb{R} \subset \rho(\mathcal{A})$ (if $\eta = 0$ then $i\mathbb{R}^* \subset \rho(\mathcal{A})$).

Moreover, we can easily prove that

$$|g(\lambda)| \ge c|\lambda|^\alpha \text{ for } \lambda \text{ large}.$$

Hence

$$|c_1| \le c|\lambda|^{-\alpha} \text{ for } \lambda \text{ large}.$$

Then, we deduce that

$$\|y_x\|_{L^2(0,L)} \leq c|\lambda|^{1-\alpha} \text{ for } \lambda \text{ large.}$$
$$\|v\|_{L^2(0,L)} \leq c|\lambda|^{1-\alpha} \text{ for } \lambda \text{ large.}$$
$$|z| \leq c|\lambda|^{-\alpha} \text{ for } \lambda \text{ large.}$$

Moreover from (3.4), we have

$$\|\phi\|^2_{L^2(-\infty,\infty)} \leq \int_{-\infty}^{+\infty} (\xi^2+\eta)|\phi(\xi)|^2\, d\xi \leq c\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}}.$$

Thus, we conclude that

$$\|(i\lambda I - \mathcal{A})^{-1}\|_{\mathcal{H}} \leq c|\lambda|^{1-\alpha} \quad \text{as } |\lambda| \to \infty. \tag{5.9}$$

The conclusion then follows by applying the Theorem 4.2. Besides, we prove that the decay rate is optimal. Indeed, the decay rate is consistent with the asymptotic expansion of eigenvalues which show a behavior of the real part like $k^{-(1-\alpha)}$. □

Remark 5.2. The method developed in this paper is direct and very flexible; it can be applied to various dissipative problems. In particular, we will consider in the future more general acoustic wave motions and also multidimensional cases under some geometric control conditions.

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Book reviews

Tom Richmond, General topology. An introduction, de Gruyter, 2020,
ISBN 978-3-11-068656-2/pbk; 978-3-11-068657-9/ebook, xi + 314 pages.

The specific feature of this introductory text on general topology is the insistence on the relations between topology and order, with emphasis on the so called asymmetric topologies, meaning topologies in the non T_1 setting. Since the only T_1 topology on a finite set is the discrete one, studies in computer science, where one works with finite sets of points (pixels), require the use of non T_1 topologies. The specialization order on a topological space (X, τ) is defined by $x \leq_\tau y$ if $x \in \mathrm{cl}\,\{y\}$. Since in a T_1 space it becomes the equality relation, it is relevant only in non T_1 setting. Actually, there exists a bijective correspondence between quasi-orders (reflexive and transitive relations) and topologies, done through the Alexandroff topologies, meaning topologies for which the intersection of an arbitrary family of open sets is open. All these are presented in the second part of the book, Chapters 8 to 10. In Ch. 11 one discusses some typical examples of asymmetric topologies given by extended distances – pseudometrics, quasi-metrics (the symmetry of the distance is broken), partial-metrics (it is possible that $d(x, x) > 0$ for some x). Uniform spaces are discussed in Chapter 12, including a brief presentation of quasi-uniform spaces, the asymmetric analogs of uniform spaces, where the opposite U^{-1} of an entourage U is not necessarily an entourage. The last chapter of the book, Chapter 13. *Continuous deformation of sets and curves*, contains a quick introduction to some topics in algebraic topology, laying the groundwork for further study in this area.

The first chapter, Chapter 0. *Preliminaries*, contains some notions and results from set theory, logic and ordered sets. Chapters 1 to 7 provide an introduction to classical general topology, culminating with connectedness, separation axioms and compactness (Tychonoff theorem), the presentation being based on motivation by examples and intuition. For instance, the quotient topology is exemplified on the circle obtained by the identification of the endpoints of a segment, the cylinder and the Möbius strip obtained in a similar way from a rectangle and the torus from a cylinder.

Another specific of the book is the rich supply of exercises (over than 740) spread through the book, completing the main text with further examples and applications as well as suggesting areas for continued investigation.

This is a well written introductory course on general topology. The numerous examples (illustrated by figures) and the intuitive approach adopted by the author makes it appealing to students in mathematics and related areas. Students in computer science will find a carefully motivated presentation of some topics in asymmetric topology (tightly connected with discrete mathematics) they may encounter in their study.

S. Cobzaş