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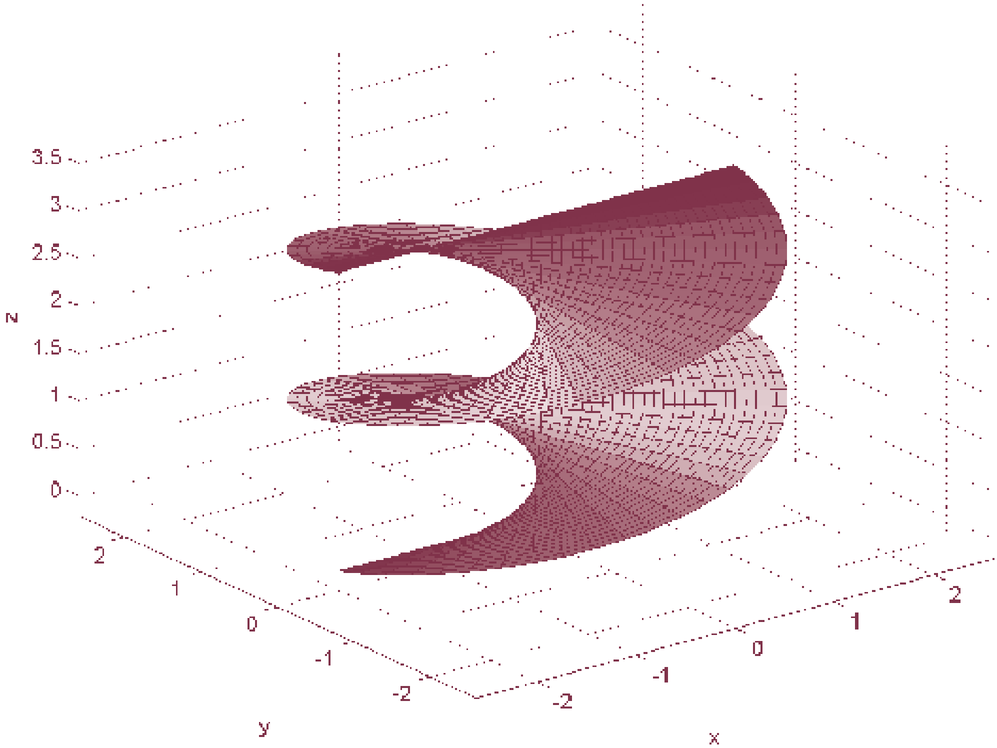

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On Fejér type inequalities for products convex and s-convex functions

Hüseyin Budak and Yonca Bakış

Abstract. In this paper, we first obtain some new Fejér type inequalities for products of convex and s-convex mappings. Then, some Fejér type inequalities for products of two s-convex function are established.

Mathematics Subject Classification (2010): 26D07, 26D10, 26D15, 26A33.

Keywords: Fejér type inequalities, convex function, integral inequalities.

1. Introduction

The inequalities discovered by C. Hermite and J. Hadamard for convex functions are considerable significant in the literature (see, e.g., [6], [14, p. 137]). These inequalities state that if $f : I \to \mathbb{R}$ is a convex function on the interval I of real numbers and $a, b \in I$ with $a < b$, then

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}. \tag{1.1}$$

Both inequalities hold in the reversed direction if f is concave. We note that Hermite-Hadamard inequality may be regarded as a refinement of the concept of convexity and it follows easily from Jensen's inequality. Over the years, many studies have focused on to establish generalization of the inequality (1.1) and to obtain new bounds for left hand side and right hand side of the inequality (1.1).

The overall structure of the paper takes the form of five sections including introduction. The remainder of this work is organized as follows: we first give some Hermite-Hadamard and Fejér type inequalities.. Moreover, we give some Hermite-Hadamard type inequalities for products two convex functions. In Section 2 and Section 3, we obtain some integral inequalities of Hermite-Hadamard-Fejér type for products convex and s-convex functions and for products two s-convex functions. We give also some special cases of these inequalities. Finally, conclusions and future directions of research are discussed in Section 4.

The weighted version of the inequalities (1.1), so-called Hermite-Hadamard-Fejér inequalities, was given by Fejer in [7] as follow:

Theorem 1.1. *$f:[a,b]\to\mathbb{R}$, be a convex function, then the inequality*

$$f\left(\frac{a+b}{2}\right)\int_a^b w(x)dx \leq \int_a^b f(x)w(x)dx \leq \frac{f(a)+f(b)}{2}\int_a^b w(x)dx \tag{1.2}$$

holds, where $w:[a,b]\to\mathbb{R}$ is non-negative, integrable, and symmetric about $x=\frac{a+b}{2}$ (i.e. $w(x)=w(a+b-x)$).

In [13], Pachpatte established the Hermite-Hadamard type inequalities for products of two convex functions.

Theorem 1.2. *Let f and g be real-valued, non-negative and convex functions on $[a,b]$. Then we have*

$$\frac{1}{b-a}\int_a^b f(x)g(x)dx \leq \frac{1}{3}M(a,b)+\frac{1}{6}N(a,b), \tag{1.3}$$

and

$$2f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)g(x)dx+\frac{1}{6}M(a,b)+\frac{1}{3}N(a,b) \tag{1.4}$$

where $M(a,b)=f(a)g(a)+f(b)g(b)$ and $N(a,b)=f(a)g(b)+f(b)g(a)$.

In recent years, the generalized versions of inequalities (1.3) and (1.4) for several convexity have been proved. For some of them please refer to ([4]-[5], [8], [16], [17]). Kirmaci et al. gave the proved inequalities (1.3) and (1.4) for products of convex and s-convex functions in [9]. On the other hand, Budak and Bakış [1] proved the weighted versions of the inequalities (1.3) and (1.4) which generalize the several obtained inequalities. Moreover in [10], Latif and Alomari proved some inequalities for product of two co-ordinated convex function. Furthermore in [11] and [12], Ozdemir et al. gave some generalizations of results given by Latif and Alomari using the product of two coordinated s-convex mappings and product of two coordinated h-convex mappings, respectively. In [2], Budak and Sarıkaya proved Hermite-Hadamard type inequalities for products of two co-ordinated convex mappings via fractional integrals.

2. Fejér type inequalities for products convex and s-convex functions

In this section, we present some Fejér type inequalities for products convex and s-convex functions.

Theorem 2.1. *Suppose that $w:I\to\mathbb{R}$ is non-negative, integrable, and symmetric about $x=\frac{a+b}{2}$ (i.e. $w(x)=w(a+b-x)$). If $f:I\to\mathbb{R}$ is a real-valued, non-negative* *and convex functions on I and if $g : I \to \mathbb{R}$ is a s-convex on I for some fixed $s \in (0,1]$, then for any $a, b \in I$, we have*

$$\int_a^b f(x)g(x)w(x)dx \leq \frac{M(a,b)}{(b-a)^{s+1}} \int_a^b (b-x)^{s+1} w(x)dx + \frac{N(a,b)}{(b-a)^{s+1}} \int_a^b (b-x)(x-a)^s w(x)dx \tag{2.1}$$

where

$$M(a,b) = f(a)g(a) + f(b)g(b) \text{ and } N(a,b) = f(a)g(b) + f(b)g(a).$$

Proof. Since f is convex and g is s-convex functions on $[a,b]$, then we have

$$f(ta + (1-t)b) \leq tf(a) + (1-t)f(b) \tag{2.2}$$

and

$$g(ta + (1-t)b) \leq t^s g(a) + (1-t)^s g(b). \tag{2.3}$$

By adding the inequalities (2.2) and (2.3), we get

$$\begin{aligned} & f(ta + (1-t)b)\, g(ta + (1-t)b) \\ \leq\ & t^{s+1} f(a)g(a) + (1-t)^{s+1} f(b)g(b) \\ +\ & t(1-t)^s f(a)g(b) + t^s (1-t) f(b)g(a). \end{aligned} \tag{2.4}$$

Multiplying both sides of (2.4) by $w(ta + (1-t)b)$, then integrating the resulting inequality with respect to t from 0 to 1, we obtain

$$\begin{aligned} & \int_0^1 f(ta + (1-t)b)\, g(ta + (1-t)b)\, w(ta + (1-t)b)\, dt \\ \leq\ & f(a)g(a) \int_0^1 t^{s+1} w(ta + (1-t)b)\, dt \\ & + f(b)g(b) \int_0^1 (1-t)^{s+1} w(ta + (1-t)b)\, dt \\ & + f(a)g(b) \int_0^1 t(1-t)^s w(ta + (1-t)b)\, dt \\ & + f(b)g(a) \int_0^1 t^s (1-t)\, w(ta + (1-t)b)\, dt. \end{aligned} \tag{2.5}$$

By change of variable $x = ta + (1-t)b$ with $dx = -(b-a)dt$, we get

$$\int_0^1 f(ta+(1-t)b)\, g(ta+(1-t)b)\, w(ta+(1-t)b)\, dt \tag{2.6}$$
$$= \frac{1}{b-a}\int_a^b f(x)g(x)w(x)dx.$$

Moreover, it is easily observe that

$$\int_0^1 t^{s+1} w(ta+(1-t)b)\, dt = \frac{1}{(b-a)^{s+2}}\int_a^b (b-x)^{s+1} w(x)dx \tag{2.7}$$

and since w is symmetric about $\frac{a+b}{2}$, we have

$$\int_0^1 (1-t)^{s+1} w(ta+(1-t)b)\, dt = \frac{1}{(b-a)^{s+2}}\int_a^b (x-a)^{s+1} w(x)dx \tag{2.8}$$
$$= \frac{1}{(b-a)^{s+2}}\int_a^b (b-u)^{s+1} w(a+b-u)du.$$
$$= \frac{1}{(b-a)^{s+2}}\int_a^b (b-u)^{s+1} w(u)du.$$

We also have

$$\int_0^1 t(1-t)^s w(ta+(1-t)b)\, dt = \frac{1}{(b-a)^{s+2}}\int_a^b (b-x)(x-a)^s w(x)dx \tag{2.9}$$

and

$$\int_0^1 t^s (1-t)\, w(ta+(1-t)b)\, dt \tag{2.10}$$
$$= \frac{1}{(b-a)^{s+2}}\int_a^b (b-x)^s (x-a)\, w(x)dx$$
$$= \frac{1}{(b-a)^{s+2}}\int_a^b (b-u)(u-a)^s w(a+b-u)du$$
$$= \frac{1}{(b-a)^{s+2}}\int_a^b (b-u)(u-a)^s w(u)du.$$

By substituting the equalities (2.6)-(2.10) in (2.5), then we have the following inequality

$$\begin{aligned} &\frac{1}{b-a}\int_a^b f(x)g(x)w(x)dx \qquad (2.11)\\ &\leq \frac{[f(a)g(a)+f(b)g(b)]}{(b-a)^{s+2}}\int_a^b (b-x)^{s+1}w(x)dx \\ &+\frac{f(a)g(b)+f(b)g(a)}{(b-a)^{s+2}}\int_a^b (b-x)(x-a)^s w(x)dx. \end{aligned}$$

If we multiply both sides of (2.11) by $(b-a)$, then we obtain the desired result. □

Remark 2.2. If we choose $w(x)=1$ for all $x\in[a,b]$ in Theorem 2.1, then we have the following inequality

$$\frac{1}{b-a}\int_a^b f(x)g(x)dx \leq \frac{1}{s+2}M(a,b)+\frac{1}{(s+1)(s+2)}N(a,b)$$

which is proved by Kırmacı et al. in [9].

Remark 2.3. If we choose $s=1$ in Theorem 2.1, then we have the following inequality

$$\begin{aligned} \int_a^b f(x)g(x)w(x)dx &\leq \frac{M(a,b)}{(b-a)^2}\int_a^b (b-x)^2 w(x)dx \\ &+\frac{N(a,b)}{(b-a)^2}\int_a^b (b-x)(x-a)w(x)dx \end{aligned}$$

which is proved by Budak and Bakış in [1].

Remark 2.4. If we choose $f(x)=1$ for all $x\in[a,b]$ in Theorem 2.1, then we have the following inequality

$$\int_a^b g(x)w(x)dx \leq \frac{g(a)+g(b)}{2(b-a)^s}\int_a^b \left[(b-x)^s+(x-a)^s\right]w(x)dx$$

which is proved by Sarıkata et al. in [15, for $h(t)=t^s$].

Proof. From the inequality (2.1) for $f(x)=1$ for all $x\in[a,b]$, we have

$$
\begin{aligned}
&\int_a^b g(x)w(x)dx \\
\leq\ & \frac{g(a)+g(b)}{(b-a)^{s+1}}\int_a^b (b-x)^{s+1}w(x)dx \qquad (2.12)\\
&+\frac{g(a)+g(b)}{(b-a)^{s+1}}\int_a^b (b-x)(x-a)^{s}w(x)dx \\
=\ & \frac{g(a)+g(b)}{(b-a)^{s+1}}\left[\int_a^b (b-x)^{s+1}w(x)dx+\int_a^b (b-x)(x-a)^{s}w(x)dx\right].
\end{aligned}
$$

Since w is symmetric about $\frac{a+b}{2}$, we have

$$\int_a^b (b-x)^{s+1}w(x)dx=\int_a^b (x-a)^{s+1}w(x)dx.$$

Using this equality in (2.12), we get

$$
\begin{aligned}
&\int_a^b g(x)w(x)dx \\
\leq\ & \frac{g(a)+g(b)}{(b-a)^{s+1}}\left[\int_a^b (x-a)^{s+1}w(x)dx+\int_a^b (b-x)(x-a)^{s}w(x)dx\right] \\
=\ & \frac{g(a)+g(b)}{(b-a)^{s}}\int_a^b (x-a)^{s}w(x)dx \\
=\ & \frac{g(a)+g(b)}{2(b-a)^{s}}\int_a^b \left[(x-a)^{s}+(b-x)^{s}\right]w(x)dx
\end{aligned}
$$

which completes the proof. □

Theorem 2.5. *Suppose that conditions of Theorem* 2.1 *hold, then we have the following inequality*

$$2^s f\left(\frac{a+b}{2}\right) g\left(\frac{a+b}{2}\right) \int_a^b w(x)\,dx \tag{2.13}$$

$$\leq \int_a^b f(x)g(x)w(x)dx + \frac{M(a,b)}{(b-a)^{s+1}} \int_a^b (x-a)^s (b-x)\, w(x)dx$$

$$+\frac{N(a,b)}{(b-a)^{s+1}} \int_a^b (b-x)^{s+1} w(x)dx.$$

where $M(a,b)$ and $N(a,b)$ are defined as in Theorem 2.1.

Proof. For $t \in [0,1]$, we can write

$$\frac{a+b}{2} = \frac{(1-t)a+tb}{2} + \frac{ta+(1-t)b}{2}.$$

Using the convexity of f and s-convexity of g, we have

$$\begin{aligned}
& f\left(\frac{a+b}{2}\right) g\left(\frac{a+b}{2}\right) \\
= \;& f\left(\frac{(1-t)a+tb}{2} + \frac{ta+(1-t)b}{2}\right) g\left(\frac{(1-t)a+tb}{2} + \frac{ta+(1-t)b}{2}\right) \\
\leq \;& \frac{1}{2^{s+1}} \left[f((1-t)a+tb) + f(ta+(1-t)b)\right] \\
& \times \left[g((1-t)a+tb) + g(ta+(1-t)b)\right] \\
= \;& \frac{1}{2^{s+1}} \left[f((1-t)a+tb)g((1-t)a+tb) + f(ta+(1-t)b)g(ta+(1-t)b)\right] \\
& +\frac{1}{2^{s+1}} \left[f((1-t)a+tb)g(ta+(1-t)b) + f(ta+(1-t)b)g((1-t)a+tb)\right].
\end{aligned}$$

By using again the convexity of f and s-convexity of g, we obtain

$$f\left(\frac{a+b}{2}\right) g\left(\frac{a+b}{2}\right) \tag{2.14}$$

$$\begin{aligned}
\leq \;& \frac{1}{2^{s+1}} \left[f((1-t)a+tb)g((1-t)a+tb) + f(ta+(1-t)b)g(ta+(1-t)b)\right] \\
& +\frac{1}{2^{s+1}} \left[t^s(1-t) + t(1-t)^s\right] \left[f(a)g(a) + f(b)g(b)\right] \\
& +\frac{1}{2^{s+1}} \left[t^{s+1} + (1-t)^{s+1}\right] \left[f(a)g(b) + f(b)g(a)\right].
\end{aligned}$$

Multiplying both sides of (2.14) by $w\left((1-t)\,a+tb\right)$, then integrating the resulting inequality with respect to t from 0 to 1, we obtain

$$
\begin{aligned}
& f\left(\frac{a+b}{2}\right) g\left(\frac{a+b}{2}\right) \int_0^1 w\left((1-t)\,a+tb\right) dt \qquad (2.15)\\
\leq \;& \frac{1}{2^{s+1}} \int_0^1 \left[f((1-t)a+tb)g((1-t)a+tb) \right.\\
& \left. +f(ta+(1-t)b)g(ta+(1-t)b) \right] w\left((1-t)\,a+tb\right) dt \\
& +\frac{M(a,b)}{2^{s+1}} \int_0^1 \left[t^s\,(1-t)+t(1-t)^s\right] w\left((1-t)\,a+tb\right) dt \\
& +\frac{N(a,b)}{2^{s+1}} \int_0^1 \left[t^{s+1}+(1-t)^{s+1}\right] w\left((1-t)\,a+tb\right) dt.
\end{aligned}
$$

Using the change of variable, we have

$$
\int_0^1 w\left((1-t)\,a+tb\right) dt = \frac{1}{b-a} \int_a^b w\left(x\right) dx, \qquad (2.16)
$$

$$
\begin{aligned}
& \int_0^1 f((1-t)a+tb)g((1-t)a+tb)w\left((1-t)\,a+tb\right) dt \qquad (2.17)\\
& +\int_0^1 f(ta+(1-t)b)g(ta+(1-t)b)w\left((1-t)\,a+tb\right) dt \\
= \;& \frac{1}{b-a} \int_a^b f(x)g(x)w(x)dx + \frac{1}{b-a} \int_a^b f(x)g(x)w(a+b-x)dx \\
= \;& \frac{2}{b-a} \int_a^b f(x)g(x)w(x)dx,
\end{aligned}
$$

$$
\begin{aligned}
& \int_0^1 \left[t^s\,(1-t)+t(1-t)^s\right] w\left((1-t)\,a+tb\right) dt \qquad (2.18)\\
= \;& \int_0^1 \left[t^s\,(1-t)\,w\left((1-t)\,a+tb\right)+t(1-t)^s w\left((1-t)\,a+tb\right)\right] dt
\end{aligned}
$$

$$= \frac{1}{(b-a)^{s+2}} \int_a^b (x-a)^s (b-x) w(x) dx$$

$$+ \frac{1}{(b-a)^{s+2}} \int_a^b (x-a)^s (b-x) w(a+b-x) dx$$

$$= \frac{2}{(b-a)^{s+2}} \int_a^b (x-a)^s (b-x) w(x) dx$$

and

$$\int_0^1 \left[t^{s+1} + (1-t)^{s+1}\right] w\left((1-t)a + tb\right) dt \tag{2.19}$$

$$= \int_0^1 \left[t^{s+1} w\left((1-t)a + tb\right) + (1-t)^{s+1} w\left((1-t)a + tb\right)\right] dt$$

$$= \frac{1}{(b-a)^{s+2}} \int_a^b (b-x)^{s+1} w(a+b-x) dx$$

$$+ \frac{1}{(b-a)^{s+2}} \int_a^b (b-x)^{s+1} w(x) dx$$

$$= \frac{2}{(b-a)^{s+2}} \int_a^b (b-x)^{s+1} w(x) dx.$$

If we substitute the equalities (2.16)-(2.19) in (2.15), then we have the following inequality

$$f\left(\frac{a+b}{2}\right) g\left(\frac{a+b}{2}\right) \frac{1}{b-a} \int_a^b w(x)\, dx \tag{2.20}$$

$$\leq \frac{1}{2^s (b-a)} \int_a^b f(x) g(x) w(x) dx + \frac{M(a,b)}{2^s (b-a)^{s+2}} \int_a^b (x-a)^s (b-x) w(x) dx$$

$$+ \frac{N(a,b)}{2^s (b-a)^{s+2}} \int_a^b (b-x)^{s+1} w(x) dx.$$

By multiplying the both sides of (2.20) by $2^s(b-a)$ then we obtain the desired result (2.13). $\square$

Remark 2.6. If we choose $w(x) = 1$ for all $x \in [a,b]$ in Theorem 2.5, then we have the following inequality

$$2^s f\left(\frac{a+b}{2}\right) g\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)g(x)dx + \frac{M(a,b)}{(s+1)(s+2)} + \frac{N(a,b)}{s+2}$$

which is proved by Kırmacı et al. in [9].

Remark 2.7. If we choose $s = 1$ in Theorem 2.1, then we have the following inequality

$$\begin{aligned} 2f\left(\frac{a+b}{2}\right) g\left(\frac{a+b}{2}\right) \int_a^b w(x)\,dx \le & \int_a^b f(x)g(x)w(x)dx \\ & + \frac{M(a,b)}{(b-a)^2}\int_a^b (x-a)(b-x)\,w(x)dx \\ & + \frac{N(a,b)}{(b-a)^2}\int_a^b (b-x)^2\, w(x)dx. \end{aligned}$$

which is proved by Budak and Bakış in [1].

Corollary 2.8. *If we choose $f(x) = 1$ for all $x \in [a,b]$ in Theorem 2.5, then we have the following the following Fejér type inequality*

$$2^s g\left(\frac{a+b}{2}\right)\int_a^b w(x)\,dx \le \int_a^b g(x)w(x)dx + \frac{g(a)+g(b)}{2(b-a)^s}\int_a^b \left[(x-a)^s + (b-x)\right]^s w(x)dx.$$

Proof. From inequality (2.13) for $f(x) = 1$ for all $x \in [a,b]$, we have

$$\begin{aligned} 2g\left(\frac{a+b}{2}\right)\int_a^b w(x)\,dx \le & \int_a^b g(x)w(x)dx + \frac{g(a)+g(b)}{(b-a)^{s+1}}\int_a^b (x-a)^s (b-x)\,w(x)dx \\ & + \frac{g(a)+g(b)}{(b-a)^{s+1}}\int_a^b (b-x)^{s+1}\, w(x)dx \\ = & \int_a^b g(x)w(x)dx \\ & + \frac{g(a)+g(b)}{(b-a)^{s+1}}\left[\int_a^b (x-a)^s(b-x)w(x)dx + \int_a^b (b-x)^{s+1}w(x)dx\right] \\ = & \int_a^b g(x)w(x)dx + \frac{g(a)+g(b)}{2(b-a)^s}\int_a^b \left[(x-a)^s + (b-x)^s\right] w(x)dx. \end{aligned}$$

This completes the proof. □

3. Fejér type inequalities for products two s-convex functions

In this section, we present some Fejér type inequalities for products two s-convex functions which generalize the results in Section 2.

Theorem 3.1. *Suppose that $w : I \to \mathbb{R}$ is non-negative, integrable, and symmetric about $x = \dfrac{a+b}{2}$ (i.e. $w(x) = w(a+b-x)$). If $f : I \to \mathbb{R}$ is s_1-convex functions on I and if $g : I \to \mathbb{R}$ is s_2-convex on I for some fixed $s_1, s_2 \in (0,1]$, then for any $a, b \in I$, we have*

$$\begin{aligned} \int_a^b f(x)g(x)w(x)dx \quad &\leq \quad \frac{M(a,b)}{(b-a)^{s_1+s_2}} \int_a^b (b-x)^{s_1+s_2} w(x)dx \\ &\quad + \frac{N(a,b)}{(b-a)^{s_1+s_2}} \int_a^b (b-x)^{s_1} (x-a)^{s_2} w(x)dx. \end{aligned} \tag{3.1}$$

where $M(a,b)$ and $N(a,b)$ are defined as in Theorem 2.1.

Proof. Since f is s_1-convex and g is s_2-convex functions on $[a,b]$, then we have

$$f(ta + (1-t)b) \leq t^{s_1} f(a) + (1-t)^{s_1} f(b) \tag{3.2}$$

and

$$g(ta + (1-t)b) \leq t^{s_2} g(a) + (1-t)^{s_2} g(b). \tag{3.3}$$

By (3.2) and (3.3), we have

$$\begin{aligned} & f(ta + (1-t)b)\, g(ta + (1-t)b) \\ \leq \quad & t^{s_1+s_2} f(a)g(a) + (1-t)^{s_1+s_2} f(b)g(b) \\ & + t^{s_1} (1-t)^{s_2} f(a)g(b) + t^{s_2} (1-t)^{s_1} f(b)g(a). \end{aligned} \tag{3.4}$$

Multiplying both sides of (3.4) by $w\left(ta+(1-t)b\right)$, then integrating the resulting inequality with respect to t from 0 to 1, we obtain

$$\int_0^1 f\left(ta+(1-t)b\right) g\left(ta+(1-t)b\right) w\left(ta+(1-t)b\right) dt \tag{3.5}$$

$$\begin{aligned}
\leq\ & f(a)g(a)\int_0^1 t^{s_1+s_2} w\left(ta+(1-t)b\right) dt \\
& +f(b)g(b)\int_0^1 (1-t)^{s_1+s_2} w\left(ta+(1-t)b\right) dt \\
& +f(a)g(b)\int_0^1 t^{s_1}(1-t)^{s_2} w\left(ta+(1-t)b\right) dt \\
& +f(b)g(a)\int_0^1 t^{s_2}(1-t)^{s_1} w\left(ta+(1-t)b\right) dt.
\end{aligned}$$

By change of variable $x=ta+(1-t)b$, we get

$$\int_0^1 t^{s_1+s_2} w\left(ta+(1-t)b\right) dt = \frac{1}{(b-a)^{s_1+s_2+1}}\int_a^b (b-x)^{s_1+s_2} w(x)dx \tag{3.6}$$

and since w is symmetric about $\dfrac{a+b}{2}$, we have

$$\begin{aligned}
\int_0^1 (1-t)^{s_1+s_2} w\left(ta+(1-t)b\right) dt &= \frac{1}{(b-a)^{s_1+s_2+1}}\int_a^b (x-a)^{s_1+s_2} w(x)dx \\
&= \frac{1}{(b-a)^{s_1+s_2+1}}\int_a^b (b-u)^{s_1+s_2} w(a+b-u)du. \\
&= \frac{1}{(b-a)^{s_1+s_2+1}}\int_a^b (b-u)^{s_1+s_2} w(u)du.
\end{aligned}$$

We also have

$$\int_0^1 t^{s_1}(1-t)^{s_2} w(ta+(1-t)b)dt = \frac{1}{(b-a)^{s_1+s_2+1}}\int_a^b (b-x)^{s_1}(x-a)^{s_2} w(x)dx \tag{3.7}$$

and

$$\begin{aligned}
& \int_0^1 t^{s_2}(1-t)^{s_1} w(ta+(1-t)b)\,dt && (3.8)\\
= & \frac{1}{(b-a)^{s_1+s_2+1}} \int_a^b (b-x)^{s_2}(x-a)^{s_1} w(x)dx \\
= & \frac{1}{(b-a)^{s_1+s_2+1}} \int_a^b (b-u)^{s_1}(u-a)^{s_2} w(a+b-u)du \\
= & \frac{1}{(b-a)^{s_1+s_2+1}} \int_a^b (b-u)^{s_1}(u-a)^{s_2} w(u)du
\end{aligned}$$

By substituting the equalities (3.6)-(3.8) in (3.5), then we have the following inequality

$$\begin{aligned}
& \frac{1}{b-a}\int_a^b f(x)g(x)w(x)dx && (3.9)\\
& \leq \frac{f(a)g(a)+f(b)g(b)}{(b-a)^{s_1+s_2+1}} \int_a^b (b-x)^{s_1+s_2} w(x)dx \\
& + \frac{f(a)g(b)+f(b)g(a)}{(b-a)^{s_1+s_2+1}} \int_a^b (b-x)^{s_1}(x-a)^{s_2} w(x)dx.
\end{aligned}$$

If we multiply both sides of (3.9) by $(b-a)$, then we obtain the desired result. □

Remark 3.2. If we choose $w(x)=1$ for all $x \in [a,b]$ in Theorem 3.1, then we have the following inequality

$$\frac{1}{b-a}\int_a^b f(x)g(x)dx \leq \frac{1}{s_1+s_2+1}M(a,b) + B(s_1+1, s_2+1)N(a,b)$$

which is proved by Kırmacı et al. in [9]. Here $B(x,y)$ is the Beta Euler function.

Remark 3.3. If we choose $s_1 = 1$ and $s_2 = s$ in Theorem 3.1, then the inequality (3.1) reduces to the inequality (2.1).

Corollary 3.4. *If we choose $s_1 = s_2 = s$ in Theorem 3.1, then we have the following inequality*

$$\int_a^b f(x)g(x)w(x)dx \leq \frac{M(a,b)}{(b-a)^{2s}} \int_a^b (b-x)^{2s} w(x)dx + \frac{N(a,b)}{(b-a)^{2s}} \int_a^b (b-x)^s (x-a)^s w(x)dx.$$

Theorem 3.5. *Suppose that conditions of Theorem 3.1 hold, then we have the following inequality*

$$\begin{aligned} & 2^{s_1+s_2-1} f\left(\frac{a+b}{2}\right) g\left(\frac{a+b}{2}\right) \int_a^b w(x)\,dx \qquad (3.10) \\ \leq & \int_a^b f(x)g(x)w(x)dx + \frac{M(a,b)}{(b-a)^{s_1+s_2}} \int_a^b (x-a)^{s_1} (b-x)^{s_2} w(x)dx \\ & + \frac{N(a,b)}{(b-a)^{s_1+s_2}} \int_a^b (b-x)^{s_1+s_2} w(x)dx. \end{aligned}$$

where $M(a,b)$ and $N(a,b)$ are defined as in Theorem 2.1.

Proof. Using the s_1-convexity of f and s_2-convexity of g, we have

$$\begin{aligned} & f\left(\frac{a+b}{2}\right) g\left(\frac{a+b}{2}\right) \\ = & f\left(\frac{(1-t)a+tb}{2} + \frac{ta+(1-t)b}{2}\right) g\left(\frac{(1-t)a+tb}{2} + \frac{ta+(1-t)b}{2}\right) \\ \leq & \frac{1}{2^{s_1+s_2}} \left[f((1-t)a+tb) + f(ta+(1-t)b)\right] \\ & \times \left[g((1-t)a+tb) + g(ta+(1-t)b)\right] \\ = & \frac{1}{2^{s_1+s_2}} \left[f((1-t)a+tb)g((1-t)a+tb) + f(ta+(1-t)b)g(ta+(1-t)b)\right] \\ & + \frac{1}{2^{s_1+s_2}} \left[f((1-t)a+tb)g(ta+(1-t)b) + f(ta+(1-t)b)g((1-t)a+tb)\right]. \end{aligned}$$

By using again the s_1-convexity of f and s_2-convexity, we obtain

$$f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right) \tag{3.11}$$

$$\leq \frac{1}{2^{s_1+s_2}}\left[f((1-t)a+tb)g((1-t)a+tb)+f(ta+(1-t)b)g(ta+(1-t)b)\right]$$

$$+\frac{1}{2^{s_1+s_2}}\left[t^{s_1}(1-t)^{s_2}+t^{s_2}(1-t)^{s_1}\right]\left[f(a)g(a)+f(b)g(b)\right]$$

$$+\frac{1}{2^{s_1+s_2}}\left[t^{s_1+s_2}+(1-t)^{s_1+s_2}\right]\left[f(a)g(b)+f(b)g(a)\right].$$

Multiplying both sides of (3.11) by $w((1-t)a+tb)$, then integrating the resulting inequality with respect to t from 0 to 1, we obtain

$$f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right)\int_0^1 w((1-t)a+tb)\,dt \tag{3.12}$$

$$\leq \frac{1}{2^{s_1+s_2}}\int_0^1 [f((1-t)a+tb)g((1-t)a+tb)$$

$$+f(ta+(1-t)b)g(ta+(1-t)b)]\,w((1-t)a+tb)\,dt$$

$$+\frac{M(a,b)}{2^{s_1+s_2}}\int_0^1\left[t^{s_1}(1-t)^{s_2}+t^{s_2}(1-t)^{s_1}\right]w((1-t)a+tb)\,dt$$

$$+\frac{N(a,b)}{2^{s_1+s_2}}\int_0^1\left[t^{s_1+s_2}+(1-t)^{s_1+s_2}\right]w((1-t)a+tb)\,dt.$$

Using the change of variable, we have

$$\int_0^1\left[t^{s_1}(1-t)^{s_2}+t^{s_2}(1-t)^{s_1}\right]w((1-t)a+tb)\,dt \tag{3.13}$$

$$=\int_0^1\left[t^{s_1}(1-t)^{s_2}w((1-t)a+tb)+t^{s_2}(1-t)^{s_1}w((1-t)a+tb)\right]dt$$

$$=\frac{1}{(b-a)^{s_1+s_2+1}}\int_a^b (x-a)^{s_1}(b-x)^{s_2}w(x)dx$$

$$+\frac{1}{(b-a)^{s_1+s_2+1}}\int_a^b (x-a)^{s_1}(b-x)^{s_2}w(a+b-x)dx$$

$$= \frac{2}{(b-a)^{s_1+s_2+1}} \int_a^b (x-a)^{s_1} (b-x)^{s_2} w(x)dx$$

and

$$\int_0^1 \left[t^{s_1+s_2} + (1-t)^{s_1+s_2} \right] w\left((1-t)a+tb\right)dt \tag{3.14}$$

$$= \int_0^1 \left[t^{s_1+s_2} w\left((1-t)a+tb\right) + (1-t)^{s_1+s_2} w\left((1-t)a+tb\right) \right] dt$$

$$= \frac{1}{(b-a)^{s_1+s_2+1}} \int_a^b (b-x)^{s_1+s_2} w(a+b-x)dx$$

$$+ \frac{1}{(b-a)^{s_1+s_2+1}} \int_a^b (b-x)^{s_1+s_2} w(x)dx$$

$$= \frac{2}{(b-a)^{s_1+s_2+1}} \int_a^b (b-x)^{s_1+s_2} w(x)dx.$$

If we substitute the equalities (2.16), (2.17), (3.13) and (3.14) in (3.12), then we have the following inequality

$$f\left(\frac{a+b}{2}\right) g\left(\frac{a+b}{2}\right) \frac{1}{b-a} \int_a^b w(x)\,dx \tag{3.15}$$

$$\leq \frac{1}{2^{s_1+s_2-1}(b-a)} \int_a^b f(x)g(x)w(x)dx$$

$$+ \frac{M(a,b)}{2^{s_1+s_2-1}(b-a)^{s_1+s_2+1}} \int_a^b (x-a)^{s_1} (b-x)^{s_2} w(x)dx$$

$$+ \frac{N(a,b)}{2^{s_1+s_2-1}(b-a)^{s_1+s_2+1}} \int_a^b (b-x)^{s_1+s_2} w(x)dx.$$

By multiplying the both sides of (3.15) by $2^{s_1+s_2-1}(b-a)$ then we obtain the desired result (3.10). $\square$

Corollary 3.6. *If we choose $w(x) = 1$ for all $x \in [a,b]$ in Theorem 3.5, then we have the following inequality*

$$
\begin{aligned}
&2^{s_1+s_2-1} f\left(\frac{a+b}{2}\right) g\left(\frac{a+b}{2}\right) \\
\leq\ & \frac{1}{b-a}\int_a^b f(x)g(x)dx + B(s_1+1, s_2+1)M(a,b) + \frac{1}{s_1+s_2+1}N(a,b).
\end{aligned}
$$

Remark 3.7. If we choose $s_1 = 1$ and $s_2 = s$ in Theorem 3.5, then the inequality (3.10) reduces to the inequality (2.13).

Corollary 3.8. *If we choose $s_1 = s_2 = s$ in Theorem 3.5, then we have the following inequality*

$$
\begin{aligned}
&2^{2s-1} f\left(\frac{a+b}{2}\right) g\left(\frac{a+b}{2}\right) \int_a^b w(x)\,dx \\
\leq\ & \int_a^b f(x)g(x)w(x)dx \\
& + \frac{M(a,b)}{(b-a)^{2s}} \int_a^b (x-a)^s (b-x)^s w(x)dx + \frac{N(a,b)}{(b-a)^{2s}} \int_a^b (b-x)^{2s} w(x)dx.
\end{aligned}
$$

4. Concluding remarks

In this paper, we present some Hermite-Hadamard-Fejér type inequalities for products convex and s-convex functions. For further investigations we propose to consider the Fejér type inequalities for products other type convex functions or for fractional integral operators.

References

[1] Budak, H., Bakış, Y., *On Fejer type inequalities for products two convex functions*, Note di Matematica, in press.

[2] Budak, H., Sarıkaya, M.Z., *Hermite-Hadamard type inequalities for products of two co-ordinated convex mappings via fractional integrals*, International Journal of Applied Mathematics and Statistics, **58**(2019), no. 4, 11-30.

[3] Chen, F., *A note on Hermite-Hadamard inequalities for products of convex functions*, Journal of Applied Mathematics, vol. 2013, Art. ID 935020, 5 pages.

[4] Chen, F., *A note on Hermite-Hadamard inequalities for products of convex functions via Riemann-Liouville fractional integrals*, Ital. J. Pure Appl. Math., **33**(2014), 299-306.

[5] Chen, F., Wu, S., *Several complementary inequalities to inequalities of Hermite-Hadamard type for s-convex functions*, J. Nonlinear Sci. Appl., **9**(2016), 705-716.

[6] Dragomir, S.S., Pearce, C.E.M., *Selected Topics on Hermite-Hadamard Inequalities and Applications*, RGMIA Monographs, Victoria University, 2000.

[7] Fejer, L., *Über die Fourierreihen,* (Hungarian), II. Math. Naturwiss. Anz Ungar. Akad. Wiss., **24**(1906), 369-390.

[8] Hue, N.N., Huy, D.Q., *Some inequalities of the Hermite-Hadamard type for product of two functions*, Journal of New Theory, 2016, 26-37.

[9] Kırmacı, U.S., Bakula, M.K., Özdemir, M.E., Pečarić, J., *Hadamard-tpye inequalities for s-convex functions*, Appl. Math. Comput., **193**(2007), 26-35.

[10] Latif, M.A., Alomari, M., *Hadamard-type inequalities for product two convex functions on the co-ordinates*, Int. Math. Forum, **47**(2009), no. 4, 2327-2338.

[11] Ozdemir, M.E., Latif, M.A., Akdemir, A.O., *On some Hadamard-type inequalities for product of two s-convex functions on the co-ordinates*, J. Inequal. Appl., **21**(2012), 1-13.

[12] Ozdemir, M.E., Latif, M.A., Akdemir, A.O., *On some Hadamard-type inequalities for product of two h-convex functions on the co-ordinates*, Turkish Journal of Science, **1**(2016), 41-58.

[13] Pachpatte, B.G., *On some inequalities for convex functions*, RGMIA Res. Rep. Coll., **6**(E)(2003).

[14] Pečarić, J.E., Proschan, F., Tong, Y.L., *Convex Functions, Partial Orderings and Statistical Applications*, Academic Press, Boston, 1992.

[15] Sarikaya, M.Z., Set, E., Ozdemir, M.E., *On some new inequalities of Hadamard type involving h-convex functions*, Acta Math. Univ. Comenian. (N.S.), **79**(2010), no. 2, 265-272.

[16] Set, E., Özdemir, M.E., Dragomir, S.S., *On the Hermite-Hadamard inequality and other integral inequalities involving two functions*, J. Inequal. Appl., **9**(2010), Art. ID 148102.

[17] Yin, H.-P., Qi, F. *Hermite-Hadamard type inequalities for the product of (α, m)-convex functions*, J. Nonlinear Sci. Appl., **8**(2015), 231-236.

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Generalization of weighted Ostrowski–Grüss type inequality by using Korkine's identity

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Abstract. We obtain generalized weighted Ostrowski-Grüss type inequality with parameters for differentiable functions by using the weighted Korkine's identity, and we then apply these obtained inequalities to probability density functions. Also, we discuss some applications of numerical quadrature rules.

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1. Introduction

Inequalities are frequently used in different areas of sciences. Inequalities play a major role in numerical analysis for error estimation of bounds. In numerical analysis, inequalities help us to find out the best bounds. In the last few years, the mid-point, trapezoid and Simpson's type rules have been examined with the perspective of getting bounds for the quadrature rules. By using modern theory of inequalities and weighted Peano kernel approach, present article is devoted to investigate several refinements of inequalities for weighted Ostrowski-Grüss type inequality and to deduce explicit bounds for the numerical quadrature rules in terms of variety of norms.

In 1935, Grüss gave a celebrated integral inequality known as Grüss inequality [4] which provides a bound on Čebyšev inequality (see [8], p.297) which establishes a relation between the integral of the product of two functions and the product of the integral of two functions. To highlight its importance, these inequalities are discussed in detail by D. S. Mitrinović, J. E. Pečarić and A. M. Fink in their books "Classical and New Inequalities in Analysis" [8] and "Inequalities involving Functions and their Integrals and Derivatives" [9]. The Grüss inequality is stated as:

Proposition 1.1. *Let ψ, $\varphi : [b_0, b_1] \to \mathbb{R}$ be two integrable functions, satisfying the conditions*

$$m \le \psi(\eta) \le M, \quad n \le \varphi(\eta) \le N,$$

for each $\eta \in [b_0, b_1]$, where m, M, n, N are given real constants. Further, let the Čebyšev functional be defined as

$$T(\psi,\varphi) = \frac{1}{b_1-b_0}\int_{b_0}^{b_1}\psi(\eta)\varphi(\eta)d\eta - \frac{1}{(b_1-b_0)^2}\int_{b_0}^{b_1}\psi(\eta)d\eta\int_{b_0}^{b_1}\varphi(\eta)d\eta.$$

Then,

$$|T(\psi,\varphi)| \leq \frac{1}{4}(M-m)(N-n),$$

where the constant $\frac{1}{4}$ is the best possible.

In 1997, using the Grüss inequality S. S. Dragomir and S. Wang [3] verified the following Ostrowski-Grüss type integral inequality:

Proposition 1.2. *Let $\psi : I \to \mathbb{R}$, where $I \subseteq \mathbb{R}$ is an interval, be a mapping differentiable in the interior I^0 of I, and let $b_0, b_1 \in I^0$ with $b_0 < b_1$. If $\nu \leq \psi'(\eta) \leq \mu$, $\eta \in [b_0, b_1]$ for some constants $\nu, \mu \in \mathbb{R}$. Then*

$$\left|\psi(\eta) - \frac{1}{b_1-b_0}\int_{b_0}^{b_1}\psi(s)ds - \frac{\psi(b_1)-\psi(b_0)}{b_1-b_0}\left(\eta - \frac{b_0+b_1}{2}\right)\right| \leq \frac{1}{4}(b_1-b_0)(\mu-\nu) \tag{1.1}$$

for all $\eta \in [b_0, b_1]$.

Relation (1.1) generates a link between the Ostrowski inequality [10] and the Grüss inequality [8].
In 2000, Proposition 1.2 was improved by M. Matić, J. E. Pečarić and N. Ujević [7].

Proposition 1.3. *Let the assumptions of Proposition* 1.2 *be true. Then*

$$\left|\psi(\eta) - \frac{1}{b_1-b_0}\int_{b_0}^{b_1}\psi(s)ds - \frac{\psi(b_1)-\psi(b_0)}{b_1-b_0}\left(\eta - \frac{b_0+b_1}{2}\right)\right| \leq \frac{1}{4\sqrt{3}}(\mu-\nu)(b_1-b_0) \tag{1.2}$$

for all $\eta \in [b_0, b_1]$.

In the same year, N. S. Barnett, S. S. Dragomir and A. Sofo [2] worked upon inequality (1.2). The improved version of the inequality states that:

Proposition 1.4. *Let $\psi : I \to \mathbb{R}$ be an absolutely continuous function whose derivative $\psi' \in L_2[b_0, b_1]$, if $\nu \leq \psi'(\eta) \leq \mu$, $\eta \in [b_0, b_1]$ for some constants $\nu, \mu \in \mathbb{R}$. Then*

$$\begin{aligned}\left|\psi(\eta) - \frac{1}{b_1-b_0}\int_{b_0}^{b_1}\psi(s)ds - \frac{\psi(b_1)-\psi(b_0)}{b_1-b_0}\left(\eta - \frac{b_0+b_1}{2}\right)\right| &\leq \frac{(b_1-b_0)}{2\sqrt{3}}\left[\frac{1}{b_1-b_0}\|\psi'\|_2^2 - \left(\frac{\psi(b_1)-\psi(b_0)}{b_1-b_0}\right)^2\right]^{\frac{1}{2}} \\ &\leq \frac{1}{4\sqrt{3}}(\mu-\nu)(b_1-b_0).\end{aligned} \tag{1.3}$$

In 2006, B. G. Pachpatte discussed inequalities by using Montgomery's identity. In 2010, F. Zafar and N. A. Mir in [12] introduced some parameters in the Peano kernel, defined as

$$K(\eta, s) = \begin{cases} s - \left(b_0 + h\dfrac{b_1 - b_0}{2}\right), & \text{if} \quad s \in [b_0, \eta], \\ \\ s - \left(b_1 - h\dfrac{b_1 - b_0}{2}\right), & \text{if} \quad s \in (\eta, b_1] \end{cases}$$

and generalized the inequality (1.3) in the next proposition:

Proposition 1.5. *Let the assumptions of Proposition* 1.4 *be valid. Then the inequality of Ostrowski-Grüss type is*

$$\begin{aligned} & \left|(1-h)\left[\psi(\eta) - \frac{\psi(b_1)-\psi(b_0)}{b_1-b_0}\left(\eta - \frac{b_0+b_1}{2}\right)\right]\right. \\ & \left. +h\frac{\psi(b_0)+\psi(b_1)}{2} - \frac{1}{b_1-b_0}\int_{b_0}^{b_1}\psi(s)ds\right| \\ \leq\ & \left[\frac{(b_1-b_0)^2}{12}(3h^2-3h+1) + h(1-h)\left(\eta - \frac{b_0+b_1}{2}\right)^2\right]^{\frac{1}{2}} \\ & \times \left[\frac{1}{b_1-b_0}\|\psi'\|_2^2 - \left(\frac{\psi(b_1)-\psi(b_0)}{b_1-b_0}\right)^2\right]^{\frac{1}{2}} \\ \leq\ & \frac{1}{2}(\mu-\nu)\left[\frac{(b_1-b_0)^2}{12}(3h^2-3h+1) + h(1-h)\left(\eta - \frac{b_0+b_1}{2}\right)^2\right]^{\frac{1}{2}} \end{aligned} \tag{1.4}$$

for all $\eta \in \left[b_0 - h\frac{b_1-b_0}{2}, b_1 + h\frac{b_1-b_0}{2}\right]$ *and* $h \in [0,1]$.

In this article, we generalized inequality (1.4) for differentiable functions in terms of weights and parameters. The generalization of Grüss inequality will be established by introducing weighted Peano kernel. The parameters and weights can be adjusted to recapture the previous results. Our first section is based on Introduction and Propositions. In the second section, we would state results related to weighted Ostrowski-Grüss inequality by using the technique of weighted Korkine's identity. In the third section, we apply our established results to probability density functions. Fourth section is based on applications of numerical quadrature rules. Our last section concludes the article.

2. Main result

We need following two lemmas from [1] to prove our main result.

Lemma 2.1. *(Weighted Korkine's Identity) Let $p, \psi, \varphi : [b_0, b_1] \to \mathbb{R}$ be the measurable mapping for which the integrals involved in the following identity exist and finite. Then*

$$\int_{b_0}^{b_1} p(s)ds \int_{b_0}^{b_1} p(s)\psi(s)\varphi(s)ds - \int_{b_0}^{b_1} p(s)\psi(s)ds \int_{b_0}^{b_1} p(s)\varphi(s)ds$$
$$= \frac{1}{2}\int_{b_0}^{b_1}\int_{b_0}^{b_1} p(s)p(t)\left(\psi(s) - \psi(t)\right)\left(\varphi(s) - \varphi(t)\right) dsdt. \tag{2.1}$$

Lemma 2.2. *Let the assumptions of Lemma 2.1 be valid. Then we have the following inequality*

$$0 \le \int_{b_0}^{b_1} p(s)\psi^2(s)ds - \left(\int_{b_0}^{b_1} p(s)\psi(s)ds\right)^2 \le \frac{1}{4}(M-m)^2 \tag{2.2}$$

where $m \le \psi(s) \le M$ a.e. on $[b_0, b_1]$.

Throughout the paper $\alpha = b_0 + h\frac{b_1 - b_0}{2}$ and $\beta = b_1 - h\frac{b_1 - b_0}{2}$ where $h \in [0, 1]$.

Theorem 2.3. *Let the assumptions of Proposition 1.4 be valid. Then we get the inequality*

$$\Bigg|\psi(\eta)\int_{\alpha}^{\beta} p(u)du + \psi(b_0)\int_{b_0}^{\alpha} p(u)du + \psi(b_1)\int_{\beta}^{b_1} p(u)du$$
$$- \left(\eta\int_{\alpha}^{\beta} p(u)du + b_0\int_{b_0}^{\alpha} p(u)du + b_0\int_{\beta}^{b_1} p(u)du - \int_{b_0}^{b_1} p(s)sds\right)$$
$$\times \left(\int_{b_0}^{b_1} p(s)\psi'(s)ds\right) - \int_{b_0}^{b_1} p(s)\psi(s)ds\Bigg|$$
$$\le \left[\int_{b_0}^{b_1} \frac{K_p^2(\eta, s)ds}{p(s)} - \left(\int_{b_0}^{b_1} K_p(\eta, s)ds\right)^2\right]^{\frac{1}{2}}$$
$$\times \left[\int_{b_0}^{b_1} p(s)[\psi'(s)]^2 ds - \left(\int_{b_0}^{b_1} p(s)\psi'(s)ds\right)^2\right]^{\frac{1}{2}}$$
$$\le \frac{1}{2}(\mu - \nu)H_p(\eta, s) \tag{2.3}$$

where

$$H_p(\eta, s) = \left[\int_{b_0}^{b_1} \frac{K_p^2(\eta, s)ds}{p(s)} - \left(\int_{b_0}^{b_1} K_p(\eta, s)ds\right)^2\right]^{\frac{1}{2}}$$

and $p : [b_0, b_1] \to [0, \infty)$ *is some probability density function satisfying*

$$\int_{b_0}^{b_1} p(s)ds = 1$$

for all $\eta \in [\alpha, \beta]$ *and* $h \in [0, 1]$.

Proof. We have the kernel as defined in [5], $K_p(\eta, s) : [b_0, b_1]^2 \to \mathbb{R}$

$$K_p(\eta, s) = \begin{cases} \displaystyle\int_{\alpha}^{s} p(u)du, & \text{if} \quad s \in [b_0, \eta], \\ \displaystyle\int_{\beta}^{s} p(u)du, & \text{if} \quad s \in (\eta, b_1]. \end{cases}$$

From (2.1), we get the Korkine's identity in the form of

$$\begin{aligned} &\int_{b_0}^{b_1} K_p(\eta, s)\psi'(s)ds - \int_{b_0}^{b_1} K_p(\eta, s)ds \int_{b_0}^{b_1} p(s)\psi'(s)ds \\ &= \frac{1}{2} \int_{b_0}^{b_1} \int_{b_0}^{b_1} p(s)p(t) \left(\frac{K_p(\eta, s)}{p(s)} - \frac{K_p(\eta, t)}{p(t)} \right) (\psi'(s) - \psi'(t))\, dsdt. \end{aligned} \tag{2.4}$$

From [5], we have

$$\begin{aligned} \int_{b_0}^{b_1} K_p(\eta, s)\psi'(s)ds &= \psi(\eta) \int_{\alpha}^{\beta} p(u)du + \psi(b_0) \int_{b_0}^{\alpha} p(u)du \\ &\quad + \psi(b_1) \int_{\beta}^{b_1} p(u)du - \int_{b_0}^{b_1} p(s)\psi(s)ds \end{aligned} \tag{2.5}$$

and

$$\begin{aligned} \int_{b_0}^{b_1} K_p(\eta, s)ds &= \eta \int_{\alpha}^{\beta} p(u)du + b_0 \int_{b_0}^{\alpha} p(u)du \\ &\quad + b_1 \int_{\beta}^{b_1} p(u)du - \int_{b_0}^{b_1} p(s)sds. \end{aligned} \tag{2.6}$$

By putting (2.5) and (2.6) in (2.4), we get

$$\begin{aligned} &\psi(\eta) \int_{\alpha}^{\beta} p(u)du + \psi(b_0) \int_{b_0}^{\alpha} p(u)du + \psi(b_1) \int_{\beta}^{b_1} p(u)du \\ &- \left(\eta \int_{\alpha}^{\beta} p(u)du + b_0 \int_{b_0}^{\alpha} p(u)du + b_1 \int_{\beta}^{b_1} p(u)du - \int_{b_0}^{b_1} p(s)sds \right) \\ &\times \left(\int_{b_0}^{b_1} p(s)\psi'(s)ds \right) - \int_{b_0}^{b_1} p(s)\psi(s)ds \\ &= \frac{1}{2} \int_{b_0}^{b_1} \int_{b_0}^{b_1} p(s)p(t) \left(\frac{K_p(\eta, s)}{p(s)} - \frac{K_p(\eta, t)}{p(t)} \right) (\psi'(s) - \psi'(t))\, dsdt, \end{aligned} \tag{2.7}$$

$\forall\ \eta \in [\alpha, \beta]$.

Using Cauchy-Schwartz inequality for double integrals, we get

$$\begin{aligned}
&\left|\frac{1}{2}\int_{b_0}^{b_1}\int_{b_0}^{b_1} p(s)p(t)\left(\frac{K_p(\eta,s)}{p(s)}-\frac{K_p(\eta,t)}{p(t)}\right)(\psi'(s)-\psi'(t))\,dsdt\right| \\
\le\;& \left(\frac{1}{2}\int_{b_0}^{b_1}\int_{b_0}^{b_1} p(s)p(t)\left(\frac{K_p(\eta,s)}{p(s)}-\frac{K_p(\eta,t)}{p(t)}\right)^2 dsdt\right)^{\frac{1}{2}} \\
&\times\left(\frac{1}{2}\int_{b_0}^{b_1}\int_{b_0}^{b_1} p(s)p(t)\left(\psi'(s)-\psi'(t)\right)^2 dsdt\right)^{\frac{1}{2}}. \qquad (2.8)
\end{aligned}$$

By using (2.4), we get the following identities

$$\begin{aligned}
&\frac{1}{2}\int_{b_0}^{b_1}\int_{b_0}^{b_1} p(s)p(t)\left(\frac{K_p(\eta,s)}{p(s)}-\frac{K_p(\eta,t)}{p(t)}\right)^2 dsdt \\
&\qquad = \int_{b_0}^{b_1}\frac{K_p^2(\eta,s)ds}{p(s)}-\left(\int_{b_0}^{b_1}K_p(\eta,s)ds\right)^2 \qquad (2.9)
\end{aligned}$$

and

$$\begin{aligned}
&\frac{1}{2}\int_{b_0}^{b_1}\int_{b_0}^{b_1} p(s)p(t)\left(\psi'(s)-\psi'(t)\right)^2 dsdt \\
&\qquad = \int_{b_0}^{b_1} p(s)[\psi'(s)]^2ds-\Big(\int_{b_0}^{b_1}p(s)\psi'(s)ds\Big)^2. \qquad (2.10)
\end{aligned}$$

Using weighted Ostrowski Grüss inequality (2.2), if $\nu \le \psi'(s) \le \mu$ and $s \in (b_0, b_1)$, we get

$$0 \le \int_{b_0}^{b_1} p(s)\big(\psi'(s)\big)^2ds-\Big(\int_{b_0}^{b_1}p(s)\psi'(s)ds\Big)^2 \le \frac{1}{4}(\mu-\nu)^2. \qquad (2.11)$$

Using (2.7) – (2.11), we obtain

$$\begin{aligned}
&\Bigg|\psi(\eta)\int_{\alpha}^{\beta}p(u)du+\psi(b_0)\int_{b_0}^{\alpha}p(u)du+\psi(b_1)\int_{\beta}^{b_1}p(u)du \\
&-\left(\eta\int_{\alpha}^{\beta}p(u)du+b_0\int_{b_0}^{\alpha}p(u)du+b_1\int_{\beta}^{b_1}p(u)du-\int_{b_0}^{b_1}p(s)sds\right) \\
&\times\left(\int_{b_0}^{b_1}p(s)\psi'(s)ds\right)-\int_{b_0}^{b_1}p(s)\psi(s)ds\Bigg| \\
\le\;& \left[\int_{b_0}^{b_1}\frac{K_p^2(\eta,s)ds}{p(s)}-\left(\int_{b_0}^{b_1}K_p(\eta,s)ds\right)^2\right]^{\frac{1}{2}}
\end{aligned}$$

$$
\begin{aligned}
& \times\left[\int_{b_0}^{b_1} p(s)[\psi'(s)]^2 ds-\left(\int_{b_0}^{b_1} p(s)\psi'(s)ds\right)^2\right]^{\frac{1}{2}} \\
\leq\ & \frac{1}{2}(\mu-\nu)\left[\int_{b_0}^{b_1} \frac{K_p^2(\eta,s)ds}{p(s)}-\left(\int_{b_0}^{b_1} K_p(\eta,s)ds\right)^2\right] \\
=\ & \frac{1}{2}(\mu-\nu)H_p(\eta,s)
\end{aligned}
$$

which proves our result (2.3). □

We can state some special cases of (2.3).

Remark 2.4. If we put $p(s)\equiv\dfrac{1}{b_1-b_0}$ in (2.3), then we get the result (1.4) of [12].

Remark 2.5. If we put $h=0$ in (2.3), then $\alpha=b_0$ and $\beta=b_1$, then following inequality holds

$$
\begin{aligned}
& \left|\psi(\eta)-\left(\eta-\int_{b_0}^{b_1} p(s)sds\right)\left(\int_{b_0}^{b_1} p(s)\psi'(s)ds\right)-\int_{b_0}^{b_1} p(s)\psi(s)ds\right| \\
\leq & \left[\int_{b_0}^{b_1} \frac{K_p^2(\eta,s)ds}{p(s)}-\left(\int_{b_0}^{b_1} K_p(\eta,s)ds\right)^2\right]^{\frac{1}{2}} \\
& \times\left[\int_{b_0}^{b_1} p(s)[\psi'(s)]^2 ds-\left(\int_{b_0}^{b_1} p(s)\psi'(s)ds\right)^2\right]^{\frac{1}{2}} \\
\leq & \frac{1}{2}(\mu-\nu)H_p(\eta,s).
\end{aligned} \tag{2.12}
$$

The above inequality is Theorem 1 of paper [6].

Remark 2.6. If we put $p(s)\equiv\dfrac{b_1-b_0}{2}$ in (2.12), then we get the inequality (1.3) of [2].

Remark 2.7. If we put $h=1$ in (2.3), then $\alpha=\beta=\dfrac{b_0+b_1}{2}$, then following inequality holds

$$
\begin{aligned}
& \Bigg|\psi(b_0)\int_{b_0}^{\frac{b_0+b_1}{2}} p(u)du+\psi(b_1)\int_{b_1}^{\frac{b_0+b_1}{2}} p(u)du-\int_{b_0}^{b_1} p(s)\psi(s)ds \\
& -\left(b_0\int_{b_0}^{\frac{b_0+b_1}{2}} p(u)du+b_1\int_{\frac{b_0+b_1}{2}}^{b_1} p(u)du-\int_{b_0}^{b_1} p(s)sds\right)\left(\int_{b_0}^{b_1} p(s)\psi'(s)ds\right)\Bigg|
\end{aligned}
$$

$$
\begin{aligned}
&\leq \left[\int_{b_0}^{b_1}\frac{K_p^2(\eta,s)ds}{p(s)}-\left(\int_{b_0}^{b_1}K_p(\eta,s)ds\right)^2\right]^{\frac{1}{2}}\\
&\times\left[\int_{b_0}^{b_1}p(s)[\psi'(s)]^2ds-\left(\int_{b_0}^{b_1}p(s)\psi'(s)ds\right)^2\right]^{\frac{1}{2}}\\
\leq&\frac{1}{2}(\mu-\nu)H_p(\eta,s). \qquad (2.13)
\end{aligned}
$$

Remark 2.8. If we put $p(s)\equiv\dfrac{1}{b_1-b_0}$ in (2.13), then trapezoidal inequality holds

$$
\begin{aligned}
&\left|\frac{\psi(b_0)+\psi(b_1)}{2}-\frac{1}{b_1-b_0}\int_{b_0}^{b_1}\psi(s)ds\right|\\
\leq\;& \frac{b_1-b_0}{2\sqrt{3}}\left[\frac{1}{b_1-b_0}\|\psi'\|_2^2-\left(\frac{\psi(b_1)-\psi(b_0)}{b_1-b_0}\right)^2\right]^{\frac{1}{2}}\\
\leq\;& \frac{1}{4\sqrt{3}}(\mu-\nu)(b_1-b_0).
\end{aligned}
$$

The above inequality is Corollary 1 (Part 1) of [12].

Corollary 2.9. *If we put $\eta=\dfrac{b_0+b_1}{2}$ in (2.3), then following inequality holds*

$$
\begin{aligned}
&\Bigg|\psi\left(\frac{b_0+b_1}{2}\right)\int_{\alpha}^{\beta}p(u)du+\psi(b_0)\int_{b_0}^{\alpha}p(u)du+\psi(b_1)\int_{\beta}^{b_1}p(u)du\\
&-\left(\frac{b_0+b_1}{2}\int_{\alpha}^{\beta}p(u)du+b_0\int_{b_0}^{\alpha}p(u)du+b_1\int_{\beta}^{b_1}p(u)du-\int_{b_0}^{b_1}p(s)sds\right)\\
&\times\left(\int_{b_0}^{b_1}p(s)\psi'(s)ds\right)-\int_{b_0}^{b_1}p(s)\psi(s)ds\Bigg|\\
&\leq\left[\int_{b_0}^{b_1}\frac{K_p^2\left(\frac{b_0+b_1}{2},s\right)ds}{p(s)}-\left(\int_{b_0}^{b_1}K_p\left(\frac{b_0+b_1}{2},s\right)ds\right)^2\right]^{\frac{1}{2}}\\
&\times\left[\int_{b_0}^{b_1}p(s)[\psi'(s)]^2ds-\left(\int_{b_0}^{b_1}p(s)\psi'(s)ds\right)^2\right]^{\frac{1}{2}}\\
&\leq\frac{1}{2}(\mu-\nu)H_p\left(\frac{b_0+b_1}{2},s\right). \qquad (2.14)
\end{aligned}
$$

Remark 2.10. If we put $p(s) \equiv \dfrac{1}{b_1 - b_0}$ in (2.14), then the bound of average midpoint and trapezoidal inequality holds

$$
\begin{aligned}
&\left|(1-h)\psi\left(\frac{b_0+b_1}{2}\right) + h\frac{\psi(b_0)+\psi(b_1)}{2} - \frac{1}{b_1-b_0}\int_{b_0}^{b_1}\psi(s)ds\right| \\
\leq \quad & \frac{b_1-b_0}{2\sqrt{3}}\sqrt{3h^2-3h+1}\left[\frac{1}{b_1-b_0}\|\psi'\|_2 - \left(\frac{\psi(b_1)-\psi(b_0)}{b_1-b_0}\right)^2\right]^{\frac{1}{2}} \\
\leq \quad & \frac{(\mu-\nu)(b_1-b_0)}{4\sqrt{3}}\sqrt{3h^2-3h+1}.
\end{aligned}
$$

Remark 2.11. If we put $h = 1$, then $\alpha = \beta = \dfrac{b_0+b_1}{2}$ in (2.14), then following inequality holds

$$
\begin{aligned}
&\left|\psi(b_0)\int_{b_0}^{\frac{b_0+b_1}{2}} p(u)du + \psi(b_1)\int_{\frac{b_0+b_1}{2}}^{b_1} p(u)du\right. \\
&+ \left(b_0\int_{b_0}^{\frac{b_0+b_1}{2}} p(u)du + b_1\int_{\frac{b_0+b_1}{2}}^{b_1} p(u)du - \int_{b_0}^{b_1} p(s)sds\right) \\
&\left.\times\left(\int_{b_0}^{b_1} p(s)\psi'(s)ds\right) - \int_{b_0}^{b_1} p(s)\psi(s)ds\right| \\
\leq &\left[\int_{b_0}^{b_1} \frac{K_p^2\left(\frac{b_0+b_1}{2},s\right)ds}{p(s)} - \left(\int_{b_0}^{b_1} K_p\left(\frac{b_0+b_1}{2},s\right)ds\right)^2\right]^{\frac{1}{2}} \\
&\times\left[\int_{b_0}^{b_1} p(s)[\psi'(s)]^2ds - \left(\int_{b_0}^{b_1} p(s)\psi'(s)ds\right)^2\right]^{\frac{1}{2}} \\
\leq &\frac{1}{2}(\mu-\nu)H_p\left(\frac{b_0+b_1}{2},s\right).
\end{aligned}
\tag{2.15}
$$

Remark 2.12. If we put $p(s) \equiv \dfrac{1}{b_1 - b_0}$ in (2.15), then trapezoidal inequality holds as we achieve in (2.14).

Remark 2.13. If we put, $h = 0$, then $\alpha = b_0$ and $\beta = b_1$ in (2.14), then weighted midpoint inequality holds

$$
\left|\psi\left(\frac{b_0+b_1}{2}\right) - \left(\frac{b_0+b_1}{2} - \int_{b_0}^{b_1} p(s)sds\right)\left(\int_{b_0}^{b_1} p(s)\psi'(s)ds\right) - \int_{b_0}^{b_1} p(s)\psi(s)ds\right|
$$

$$
\begin{aligned}
&\leq \left[\int_{b_0}^{b_1} \frac{K_p^2\left(\frac{b_0+b_1}{2}, s\right) ds}{p(s)} - \left(\int_{b_0}^{b_1} K_p\left(\frac{b_0+b_1}{2}, s\right) ds\right)^2\right]^{\frac{1}{2}} \\
&\quad \times \left[\int_{b_0}^{b_1} p(s)[\psi'(s)]^2 ds - \left(\int_{b_0}^{b_1} p(s)\psi'(s) ds\right)^2\right]^{\frac{1}{2}} \\
&\leq \frac{1}{2}(\mu-\nu) H_p\left(\frac{b_0+b_1}{2}, s\right). \qquad (2.16)
\end{aligned}
$$

The above inequality is Corollary 1 of [6].

Remark 2.14. If we put $p(s) = \dfrac{1}{b_1 - b_0}$ in (2.16), then midpoint inequality holds

$$
\begin{aligned}
&\left|\psi\left(\frac{b_0+b_1}{2}\right) - \frac{1}{b_1-b_0}\int_{b_0}^{b_1} \psi(s) ds\right| \\
\leq\ & \frac{(b_1-b_0)}{2\sqrt{3}} \left[\frac{1}{b_1-b_0}||\psi'||_2^2 - \left(\frac{\psi(b_1)-\psi(b_0)}{b_1-b_0}\right)^2\right]^{\frac{1}{2}} \\
\leq\ & \frac{1}{4\sqrt{3}}(\mu-\nu)(b_1-b_0).
\end{aligned}
$$

The above inequality is the Corollary 1 (Part 2) of [12].

Remark 2.15. If we put $h = \dfrac{1}{2}$, then $\alpha = \dfrac{3b_0+b_1}{4}$ and $\beta = \dfrac{b_0+3b_1}{4}$ in (2.14), then following inequality holds

$$
\begin{aligned}
&\left|\psi\left(\frac{b_0+b_1}{2}\right)\int_{\frac{3b_0+b_1}{4}}^{\frac{b_0+3b_1}{4}} p(u)du + \psi(b_0)\int_{b_0}^{\frac{3b_0+b_1}{4}} p(u)du + \psi(b_1)\int_{\frac{b_0+3b_1}{4}}^{b_1} p(u)du\right. \\
&- \left(\frac{b_0+b_1}{2}\int_{\frac{3b_0+b_1}{4}}^{\frac{b_0+3b_1}{4}} p(u)du + b_0\int_{b_0}^{\frac{3b_0+b_1}{4}} p(u)du + b_1\int_{\frac{b_0+3b_1}{4}}^{b_1} p(u)du - \int_{b_0}^{b_1} p(s)sds\right) \\
&\left.\times\left(\int_{b_0}^{b_1} p(s)\psi'(s)ds\right) - \int_{b_0}^{b_1} p(s)\psi(s)ds\right|
\end{aligned}
$$

$$
\begin{aligned}
&\leq \left[\int_{b_0}^{b_1} \frac{K_p^2\left(\frac{b_0+b_1}{2}, s\right) ds}{p(s)} - \left(\int_{b_0}^{b_1} K_p\left(\frac{b_0+b_1}{2}, s\right) ds\right)^2\right]^{\frac{1}{2}} \\
&\quad \times \left[\int_{b_0}^{b_1} p(s)[\psi'(s)]^2 ds - \left(\int_{b_0}^{b_1} p(s)\psi'(s) ds\right)^2\right]^{\frac{1}{2}} \\
&\leq \frac{1}{2}(\mu-\nu) H_p\left(\frac{b_0+b_1}{2}, s\right). \qquad (2.17)
\end{aligned}
$$

Remark 2.16. If we put $p(s) = \dfrac{1}{b_1 - b_0}$ in (2.17), then the bound of average midpoint and trapezoidal inequality holds

$$\begin{aligned}
&\left| \frac{1}{2}\left[\frac{\psi(b_0)+\psi(b_1)}{2} + \psi\left(\frac{b_0+b_1}{2}\right)\right] - \frac{1}{b_1-b_0}\int_{b_0}^{b_1}\psi(s)ds \right| \\
\leq\ & \frac{(b_1-b_0)}{4\sqrt{3}}\left[\frac{1}{b_1-b_0}||\psi'||_2^2 - \left(\frac{\psi(b_1)-\psi(b_0)}{b_1-b_0}\right)^2\right]^{\frac{1}{2}} \\
\leq\ & \frac{1}{8\sqrt{3}}(\mu-\nu)(b_1-b_0).
\end{aligned}$$

The above inequality is Corollary 1 (Part 3) of [12].

Remark 2.17. If we put, $h = \dfrac{1}{3}$, then $\alpha = \dfrac{5b_0+b_1}{6}$ and $\beta = \dfrac{b_0+5b_1}{6}$ in (2.14), then following inequality holds

$$\begin{aligned}
&\Bigg| \psi\left(\frac{b_0+b_1}{2}\right)\int_{\frac{5b_0+b_1}{6}}^{\frac{b_0+5b_1}{6}} p(u)du + \psi(b_0)\int_{b_0}^{\frac{5b_0+b_1}{6}} p(u)du + \psi(b_1)\int_{\frac{b_0+5b_1}{6}}^{b_1} p(u)du \\
&- \left(\frac{b_0+b_1}{2}\int_{\frac{5b_0+b_1}{6}}^{\frac{b_0+5b_1}{6}} p(u)du + b_0\int_{b_0}^{\frac{5b_0+b_1}{6}} p(u)du + b_1\int_{\frac{b_0+5b_1}{6}}^{b_1} p(u)du - \int_{b_0}^{b_1} p(s)sds\right) \\
&\times \left(\int_{b_0}^{b_1} p(s)\psi'(s)ds\right) - \int_{b_0}^{b_1} p(s)\psi(s)ds \Bigg| \\
\leq & \left[\int_{b_0}^{b_1} \frac{K_p^2\left(\frac{b_0+b_1}{2}, s\right)ds}{p(s)} - \left(\int_{b_0}^{b_1} K_p\left(\frac{b_0+b_1}{2}, s\right)ds\right)^2\right]^{\frac{1}{2}} \\
&\times \left[\int_{b_0}^{b_1} p(s)[\psi'(s)]^2 ds - \left(\int_{b_0}^{b_1} p(s)\psi'(s)ds\right)^2\right]^{\frac{1}{2}} \\
\leq & \frac{1}{2}(\mu-\nu)H_p\left(\frac{b_0+b_1}{2}, s\right).
\end{aligned} \tag{2.18}$$

Remark 2.18. If we put $p(s) = \dfrac{1}{b_1 - b_0}$ in (2.18), then bound of $\dfrac{1}{3}$ Simpson's rule holds

$$\begin{aligned}
&\left| \frac{1}{3}\left[\frac{\psi(b_0)+\psi(b_1)}{2} + 2\psi\left(\frac{b_0+b_1}{2}\right)\right] - \frac{1}{b_1-b_0}\int_{b_0}^{b_1}\psi(s)ds \right| \\
\leq\ & \frac{(b_1-b_0)}{6}\left[\frac{1}{b_1-b_0}||\psi'||_2^2 - \left(\frac{\psi(b_1)-\psi(b_0)}{b_1-b_0}\right)^2\right]^{\frac{1}{2}} \\
\leq\ & \frac{1}{12}(\mu-\nu)(b_1-b_0).
\end{aligned}$$

The above inequality is Corollary 1 (Part 4) of [12].

Remark 2.19. If we put $\eta = b_0$ or $\eta = b_1$ and $p(s) \equiv \dfrac{1}{b_1 - b_0}$ in (2.3), then trapezoidal inequality holds which is independent of the value of h

$$\begin{aligned}
&\left| \frac{\psi(b_0) + \psi(b_1)}{2} - \frac{1}{b_1 - b_0} \int_{b_0}^{b_1} \psi(s)ds \right| \\
\leq\ & \frac{b_1 - b_0}{2\sqrt{3}} \left[\frac{1}{b_1 - b_0} \|\psi'\|_2 - \left(\frac{\psi(b_1) - \psi(b_0)}{b_1 - b_0} \right)^2 \right]^{\frac{1}{2}} \\
\leq\ & \frac{(b_1 - b_0)(\mu - \nu)}{4\sqrt{3}}.
\end{aligned}$$

In Sections 3 and 4, we are going to present applications involving probability density function and numerical quadrature rules respectively.

3. Application to probability density functions

From [6], let X be a continuous random variable having the probability density function $\psi : [b_0, b_1] \to \mathbb{R}_+$ and the cumulative distribution function $\Psi : [b_0, b_1] \to [0, 1]$, i.e.,

$$\Psi(\eta) = \int_{b_0}^{\eta} \psi(s)ds, \quad \eta \in [\alpha, \beta] \subset [b_0, b_1],$$

$$E(X) = \int_{b_0}^{b_1} s\psi(s)ds,$$

and weighted expectation would be

$$E_p(X) = \int_{b_0}^{b_1} p(s)s\psi(s)ds$$

on the interval $[b_0, b_1]$. Then we have the following theorem.

Theorem 3.1. *Let the assumptions of Theorem* 2.3 *be valid and if probability density function belongs to* $L_2[b_0, b_1]$ *space, then following inequality holds*

$$\begin{aligned}
&\Bigg| \Psi(\eta) \int_{\alpha}^{\beta} p(u)du + \int_{\beta}^{b_1} p(u)du - b_1 p(b_1) + E_p(X) \\
&- \left(\eta \int_{\alpha}^{\beta} p(u)du + b_0 \int_{b_0}^{\alpha} p(u)du + b_1 \int_{\beta}^{b_1} p(u)du - \int_{b_0}^{b_1} p(s)sds \right) \\
&\times \left(p(b_1) - \int_{b_0}^{b_1} p'(s)\Psi(s)ds \right) + \int_{b_0}^{b_1} p'(s)s\Psi(s)ds \Bigg| \\
&\leq \frac{1}{2}(\mu - \nu) H_p(\eta, s) \qquad (3.1)
\end{aligned}$$

for all $\eta \in [\alpha, \beta]$.

Proof. Put $\psi = \Psi$ in (2.3) and by using these two identities mention below, we get (3.1),

$$\int_{b_0}^{b_1} p(s)\Psi(s)ds = b_1 p(b_1) - E_p(X) - \int_{b_0}^{b_1} p'(s)s\Psi(s)ds$$

and

$$\int_{b_0}^{b_1} p(s)\Psi'(s)ds = p(b_1) - \int_{b_0}^{b_1} p'(s)\Psi(s)ds. \qquad \square$$

Remark 3.2. Let the assumptions of Theorem 3.1 be valid, if we substitute $p(s) \equiv \dfrac{1}{b_1 - b_0}$ in (3.1), then following inequality holds

$$\begin{aligned}
&\left|(1-h)\left[\Psi(\eta) - \frac{1}{b_1-b_0}\left(\eta - \frac{b_0+b_1}{2}\right)\right] + \frac{h}{2} - \frac{b_1 - E(X)}{b_1 - b_0}\right| \\
\le\ & \frac{1}{b_1-b_0}\left[\frac{1}{12}(3h^2-3h+1) + h(1-h)\left(\eta - \frac{b_0+b_1}{2}\right)^2\right]^{\frac{1}{2}} \\
& \times \left[(b_1-b_0)\|\psi\|_2^2 - 1\right]^{\frac{1}{2}} \\
\le\ & \frac{\mu-\nu}{2(b_1-b_0)}\left[\frac{1}{12}(3h^2-3h+1) + h(1-h)\left(\eta - \frac{b_0+b_1}{2}\right)^2\right]^{\frac{1}{2}}.
\end{aligned}$$

4. Applications to numerical quadrature rules

Let $I_n : b_0 = z_0 < z_1 < \ldots < z_n = b_1$ be a partition of the interval $[b_0, b_1]$ and let $\Delta z_k = z_{k+1} - z_k$, $k \in \{0, 1, 2, \ldots, n-1\}$. Then

$$\int_{b_0}^{b_1} p(s)\psi(s)ds = Q_n(I_n, \psi, p) + R_n(I_n, \psi, p) \tag{4.1}$$

where $Q_n(I_n, \psi, p)$ is a quadrature formula, define as

$$\begin{aligned}
Q_n(I_n,\psi,p) = \sum_{k=0}^{n-1}\Bigg[&\psi(\eta)\int_{\alpha_k}^{\beta_k} p(u)du + \psi(z_k)\int_{z_k}^{\alpha_k} p(u)du + \psi(z_{k+1})\int_{\beta_k}^{z_{k+1}} p(u)du \\
&- \left(\eta_k\int_{\alpha_k}^{\beta_k} p(u)du + z_k\int_{z_k}^{\alpha_k} p(u)du + z_{k+1}\int_{\beta_k}^{z_{k+1}} p(u)du - \int_{z_k}^{z_{k+1}} p(s)sds\right) \\
&\times \left(\int_{z_k}^{z_{k+1}} p(s)\psi'(s)ds\right)\Bigg]
\end{aligned} \tag{4.2}$$

for all $\eta_k \in [z_k, z_{k+1}]$.

Theorem 4.1. *Let ψ as be defined in Theorem 2.3. Then (4.1) holds where $Q_n(I_n, \psi, p)$ is given by formula (4.2) and the remainder $R_n(I_n, \psi, p)$ satisfies the estimates*

$$|R_n(I_n, \psi, p)| \leq \sum_{k=0}^{n-1} \frac{(\mu - \nu)}{2} H_p(\eta_k, s) \tag{4.3}$$

for all $\eta_k \in [z_k, z_{k+1}]$.

Proof. Using (2.3) on $[z_k, z_{k+1}]$,

$$\begin{aligned} R_k(I_k, \psi, p) = &\int_{z_k}^{z_{k+1}} p(s)\psi(s)ds - \psi(\eta_k) \int_{\alpha_k}^{\beta_k} p(u)du - \psi(z_k) \int_{z_k}^{\alpha_k} p(u)du \\ &- \psi(z_{k+1}) \int_{\beta_k}^{z_{k+1}} p(u)du + \left(\eta_k \int_{\alpha_k}^{\beta_k} p(u)du + z_k \int_{z_k}^{\alpha_k} p(u)du \right. \\ &\left. + z_{k+1} \int_{\beta_k}^{z_{k+1}} p(u)du - \int_{z_k}^{z_{k+1}} p(s)sds \right) \times \left(\int_{z_k}^{z_{k+1}} p(s)\psi'(s)ds \right). \end{aligned}$$

Summing over k from 0 to $n-1$. This yields

$$\begin{aligned} R_n(I_k, \psi, p) = &\int_{b_0}^{b_1} p(s)\psi(s)ds - \sum_{k=0}^{n-1} \left[\psi(\eta_k) \int_{\alpha_k}^{\beta_k} p(u)du + \psi(z_k) \int_{z_k}^{\alpha_k} p(u)du \right. \\ &+ \psi(z_{k+1}) \int_{\beta_k}^{z_{k+1}} p(u)du - \left(\eta_k \int_{\alpha_k}^{\beta_k} p(u)du + z_k \int_{z_k}^{\alpha_k} p(u)du \right. \\ &\left.\left. + z_{k+1} \int_{\beta_k}^{z_{k+1}} p(u)du - \int_{z_k}^{z_{k+1}} p(s)sds \right) \times \left(\int_{z_k}^{z_{k+1}} p(s)\psi'(s)ds \right) \right]. \end{aligned}$$

Applying absolute property on the above identity, we get

$$\begin{aligned} |R_n(I_k, \psi, p)| = &\left| \int_{b_0}^{b_1} p(s)\psi(s)ds - \sum_{k=0}^{n-1} \left[\psi(\eta_k) \int_{\alpha_k}^{\beta_k} p(u)du + \psi(z_k) \int_{z_k}^{\alpha_k} p(u)du \right. \right. \\ &+ \psi(z_{k+1}) \int_{\beta_k}^{z_{k+1}} p(u)du - \left(\eta_k \int_{\alpha_k}^{\beta k} p(u)du + z_k \int_{z_k}^{\alpha_k} p(u)du \right. \\ &\left.\left.\left. + z_{k+1} \int_{\beta_k}^{z_{k+1}} p(u)du - \int_{z_k}^{z_{k+1}} p(s)sds \right) \times \left(\int_{z_k}^{z_{k+1}} p(s)\psi'(s)ds \right) \right] \right| \\ &\leq \frac{1}{2}(\mu - \nu) H_p(\eta_k, s). \end{aligned}$$

□

5. Conclusion

Main objective of this article is to generalize the results of [6] and [12]. By introducing the weighted kernel as defined in [5], we have obtained generalization of Ostrowski-Grüss integral inequality for first differentiable functions in terms of weights. By using appropriate substitution we get different previously published results. At the end, we have also discussed some applications for probability density functions and numerical quadrature rules.

References

[1] Barnett, N.S., Cerone, P., Dragomir, S.S., Roumeliotis, J., *Some inequalities for the dispersion of a random variable whose pdf is defined on a finite interval*, J. Inequal. Pure and Appl. Math., **2**(2001), no. 1, Art. 1, 18 pp.

[2] Barnett, N.S., Dragomir, S.S., Sofo, A., *Better bounds for an inequality of Ostrowski type with applications*, Demonstratio Math., **34**(2001), no. 3, 533-542. Preprint RGMIA Res. Rep. Coll., **3**(2000), no. 1, Art. 11.

[3] Dragomir, S.S., Wang, S., *An inequality of Ostrowski-Grüss type and its applications to the estimation of error bounds for some special means and for some numerical quadrature rules*, Comput. Math. Appl., **33**(1997), 15-20.

[4] Grüss, G., *Über das Maximum des absoluten Betrages von*, Math. Z., **39**(1935), 215-226.

[5] Imtiaz, M., Irshad, N., Khan, A.R., *Generalization of weighted Ostrowski integral inequality for twice differentiable mappings*, Adv. Inequal. Appl., **20**(2016), Art. 20.

[6] Irshad, N., Khan, A.R., *On weighted Ostrowski-Grüss inequality with applications*, T.J.M.M, **10**(2018), no. 1, 15-20.

[7] Matić, M., Pečarić, J.E., Ujević, N., *Improvement and further generalization of some inequalities of Ostrowski-Grüss type*, Comput. Math. Appl., **39**(2000), no. 3/4, 161-175.

[8] Mitrinović, D.S., Pečarić, J.E., Fink, A.M., *Classical and New Inequalities in Analysis*, Kluwer Academic Publishers, Dordrecht, 1991.

[9] Mitrinović, D.S., Pečarić, J.E., Fink, A.M., *Inequalities for Functions and their Integrals and Derivatives*, Kluwer Academic, Dordrecht, 1991.

[10] Ostrowski, A.M., *Über die absolutabweichung einer differentiebaren funktion von ihren integralmittelwert*, Comment. Math. Helv., **10**(1938), 226-227.

[11] Pachpatte, B.G., *On Čebyšev-Gruss type inequalities via Pecaric's extention of the Montgomery identity*, J. Inequal. Pure and Appl. Math., **7**(2006), no. 1, Art. 11.

[12] Zafar, F., Mir, N.A., *A generalization of Ostrowski-Grüss type inequality for first differentiable mappings*, Tamsui Oxford J. Math. Sci., **26**(2010), no. 1, 61-76.

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Application of Ruscheweyh q-differential operator to analytic functions of reciprocal order

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Abstract. The core object of this paper is to define and study new class of analytic function using Ruscheweyh q-differential operator. We also investigate a number of useful properties such as inclusion relation, coefficient estimates, subordination result,for this newly subclass of analytic functions.

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1. Introduction

Quantum calculus (q-calculus) is simply the study of classical calculus without the notion of limits. The study of q-calculus attracted the researcher due to its applications in various branches of mathematics and physics, see detail [8]. Jackson [10, 12] was the first to give some application of q-calculus and introduced the q-analogue of derivative and integral. Later on Aral and Gupta [5, 6, 7] defined the q-Baskakov Durrmeyer operator by using q-beta function while the author's in [2, 3, 4] discussed the q-generalization of complex operators known as q-Picard and q-Gauss-Weierstrass singular integral operators. Recently, Kanas and Răducanu [13] defined q-analogue of Ruscheweyh differential operator using the concepts of convolution and then studied some of its properties. The application of this differential operator was further studied by Mohammed and Darus [1] and Mahmood and Sokół [14]. The aim of the current paper is to define a new class of analytic functions of reciprocal order involving q-differetial operator.

Let $\mathcal{A}$ be the class of functions having the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

which are analytic in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$. Let $\mathcal{M}(\alpha)$ denote a subclass of $\mathcal{A}$ consisting of functions which satisfy the inequality

$$\mathfrak{Re}\frac{zf'(z)}{f(z)} < \alpha \quad (z \in \mathbb{U}),$$

for some $\alpha\,(\alpha > 1)$. And let $\mathcal{N}(\alpha)$ be the subclass of $\mathcal{A}$ consisting of functions f which satisfy the inequality:

$$\mathfrak{Re}\frac{(zf'(z))'}{f'(z)} < \alpha \quad (z \in \mathbb{U}),$$

for some $\alpha\,(\alpha > 1)$. These classes were studied by Owa et al. [16, 18]. Shams et al. [20] have introduced the k-uniformly starlike $\mathcal{SD}\,(k, \alpha)$ and k-uniformly convex $\mathcal{CD}\,(k, \alpha)$ of order α, for some $k\,(k \geq 0)$ and $\alpha\,(0 \leq \alpha < 1)$. Using these ideas in above defined classes, Junichi et al. [17] introduced the following classes.

Definition 1.1. Let $f \in \mathcal{A}$. Then f is said to be in class $\mathcal{MD}\,(k, \alpha)$ if it satisfies

$$\mathfrak{Re}\frac{zf'(z)}{f(z)} < k\left|\frac{zf'(z)}{f(z)} - 1\right| + \alpha \quad (z \in \mathbb{U}),$$

for some $\alpha\,(\alpha > 1)$ and $k\,(k \leq 0)$.

Definition 1.2. An analytic function f of the form (1.1) belongs to the class $\mathcal{ND}\,(k, \alpha)$, if and only if

$$\mathfrak{Re}\frac{(zf'(z))'}{f'(z)} < k\left|\frac{(zf'(z))'}{f'(z)} - 1\right| + \alpha \quad (z \in \mathbb{U}),$$

for some $\alpha\,(\alpha > 1)$ and $k\,(k \leq 0)$.

If f and g are analytic in $\mathbb{U}$, we say that f is subordinate to g, written as $f \prec g$ or $f(z) \prec g(z)$, if there exists a Schwarz function w, which is analytic in $\mathbb{U}$ with $w\,(0) = 0$ and $|w(z)| < 1$ such that $f(z) = g(w(z))$. Furthermore, if the function $g(z)$ is univalent in $\mathbb{U}$, then we have the following equivalence holds, see [11, 15].

$$f(z) \prec g(z) \quad (z \in \mathbb{U}) \quad \Longleftrightarrow \quad f(0) = g(0) \quad \text{and} \quad f(\mathbb{U}) \subset g(\mathbb{U}).$$

For two analytic functions

$$f(z) = \sum_{n=1}^{\infty} a_n z^n \quad g(z) = \sum_{n=1}^{\infty} b_n z^n \quad (z \in \mathbb{U}),$$

For $t \in \mathbb{R}$ and $q > 0$, $q \neq 1$, the number $[t, q]$ is defined in [14] as

$$[t, q] = \frac{1 - q^t}{1 - q}, \quad [0, q] = 0.$$

For any non-negative integer n the q-number shift factorial is defined by

$$[n, q]! = [1, q]\,[2, q]\,[3, q] \cdots [n, q]\,, \quad ([0, q]! = 1)\,.$$

We have $\lim\limits_{q \to 1} [n, q] = n$. Throughout in this paper we will assume q to be fixed number between 0 and 1.

The q-derivative operator or q-difference operator for $f \in \mathcal{A}$ is defined as

$$\partial_q f(z) = \frac{f\,(qz) - f(z)}{z\,(q - 1)}, \quad z \in \mathbb{U}.$$

It can easily be seen that for $n \in \mathbb{N} := \{1, 2, 3, \ldots\}$ and $z \in \mathbb{U}$

$$\partial_q z^n = [n, q]\, z^{n-1}, \quad \partial_q \left\{ \sum_{n=1}^{\infty} a_n z^n \right\} = \sum_{n=1}^{\infty} [n, q]\, a_n z^{n-1}.$$

The q-generalized Pochhammer symbol for $t \in \mathbb{R}$ and $n \in \mathbb{N}$ is defined as

$$[t, q]_n = [t, q]\, [t+1, q]\, [t+2, q] \cdots [t+n-1, q]\,,$$

and for $t > 0$, let q-gamma function is defined as

$$\Gamma_q\,(t+1) = [t, q]\, \Gamma_q\,(t) \ \text{ and } \Gamma_q\,(1) = 1.$$

Definition 1.3. [14] For a function $f(z) \in \mathcal{A}$, the Ruscheweyh q-differential operator is defined as

$$\mathfrak{D}_q^{\mu} f(z) = \phi\,(q, \mu+1; z) * f(z) = z + \sum_{n=2}^{\infty} \Phi_{n-1} a_n z^n, \qquad (z \in \mathbb{U} \text{ and } \mu > -1)\,, \quad (1.2)$$

where

$$\phi\,(q, \mu+1; z) = z + \sum_{n=2}^{\infty} \Phi_{n-1} z^n, \quad (1.3)$$

and

$$\Phi_{n-1} = \frac{\Gamma_q\,(\mu+n)}{[n-1, q]!\Gamma_q\,(\mu+1)} = \frac{[\mu+1, q]_{n-1}}{[n-1, q]!}. \quad (1.4)$$

From (1.2), it can be seen that

$$L_q^0 f(z) = f(z) \text{ and } L_q^1 f(z) = z \partial_q f(z),$$

and

$$L_q^m f(z) = \frac{z \partial_q^m \left(z^{m-1} f(z) \right)}{[m, q]!}, \qquad (m \in \mathbb{N})\,.$$

$$\lim_{q \to 1^-} \phi\,(q, \mu+1; z) = \frac{z}{(1-z)^{\mu+1}},$$

and

$$\lim_{q \to 1^-} \mathfrak{D}_q^{\mu} f(z) = f(z) * \frac{z}{(1-z)^{\mu+1}}.$$

This shows that in case of $q \to 1^-$, the Ruscheweyh q-differential operator reduces to the Ruscheweyh differential operator $D^{\delta}\,(f(z))$ (see [19]). From (1.2) the following identity can easily be derived.

$$z \partial \mathfrak{D}_q^{\mu} f(z) = \left(1 + \frac{[\mu, q]}{q^{\mu}} \right) \mathfrak{D}_q^{\mu} f(z) - \frac{[\mu, q]}{q^{\mu}} \mathfrak{D}_q^{\mu} f(z). \quad (1.5)$$

If $q \to 1^-$, then

$$z \left(\mathfrak{D}_q^{\mu} f(z) \right)' = (1+\mu)\, \mathfrak{D}_q^{\mu} f(z) - \mu \mathfrak{D}_q^{\mu} f(z).$$

Now using the Ruscheweyh q-differential operator, we define the following class.

Definition 1.4. Let $f \in \mathcal{A}$. Then f is in the class $\mathcal{KD}_q(k, \alpha, \gamma)$ if

$$\mathfrak{Re}\left\{1 + \frac{1}{\gamma}\left(\frac{z\partial_q \mathfrak{D}_q^\mu f(z)}{\mathfrak{D}_q^\mu f(z)} - 1\right)\right\} < k\left|\frac{1}{\gamma}\left(\frac{z\partial_q \mathfrak{D}_q^\mu f(z)}{\mathfrak{D}_q^\mu f(z)} - 1\right)\right| + \alpha,$$

for some $k\,(k \leq 0)$, $\alpha\,(\alpha > 1)$ and for some $\gamma \in \mathbb{C} \setminus \{0\}$.

We note that $\mathcal{LD}_2^0(1, 1, \alpha) = \mathcal{M}(\alpha)$ and $\mathcal{LD}_1^0(1, 1, \alpha) = \mathcal{N}(\alpha)$, the classes introduced by Owa et al. [16, 18]. When we take $\gamma = 1, 2$, $c = 1$, and $a = 1$ the class $\mathcal{KD}_q(k, \alpha, \gamma)$ reduces to the classes $\mathcal{MD}(k, \alpha)$ and $\mathcal{ND}(k, \alpha)$ (see [17]). For $1 < \alpha < 4/3$ the classes $\mathcal{M}(\alpha)$ and $\mathcal{N}(\alpha)$ were investigated by Uralegaddi et al. [21].

2. Preliminary results

Lemma 2.1. [9]*For a positive integer t, we have*

$$\sigma \sum_{j=1}^{t} \frac{(\sigma)_{j-1}}{(j-1)!} = \frac{(\sigma)_t}{(t-1)!}. \tag{2.1}$$

Proof. Consider

$$\begin{aligned}
& \sigma \sum_{j=1}^{t} \frac{(\sigma)_{j-1}}{(j-1)!} \\
= \;& \sigma\left(1 + \frac{\sigma}{1} + \frac{(\sigma)_2}{2!} + \frac{(\sigma)_3}{3!} + \frac{(\sigma)_4}{4!} + \cdots + \frac{(\sigma)_{t-1}}{(t-1)!}\right) \\
= \;& \sigma(1+\sigma)\left(1 + \frac{\sigma}{2} + \frac{\sigma(\sigma+2)}{2\times 3} + \cdots + \frac{\sigma(\sigma+2)\cdots(\sigma+t-2)}{2\times\cdots\times(t-1)}\right) \\
= \;& \sigma(1+\sigma)\frac{(\sigma+2)}{2}\left(1 + \frac{\sigma}{3} + \cdots + \frac{\sigma(\sigma+3)\cdots(\sigma+t-2)}{3\times 4\times\cdots\times(t-1)}\right) \\
= \;& \sigma(1+\sigma)\frac{(\sigma+2)}{2}\frac{(\sigma+3)}{3}\left(1 + \frac{\sigma}{4} + \cdots + \frac{\sigma(\sigma+4)\cdots(\sigma+t-2)}{4\times\cdots\times(t-1)}\right) \\
= \;& \sigma(1+\sigma)\frac{(\sigma+2)}{2}\frac{(\sigma+3)}{3}\frac{(\sigma+4)}{4}\left(1 + \frac{\sigma}{5} + \cdots + \frac{\sigma\cdots(\sigma+t-2)}{5\times 6\times\cdots\times(t-1)}\right) \\
= \;& \sigma(1+\sigma)\frac{(\sigma+2)}{2}\frac{(\sigma+3)}{3}\frac{(\sigma+4)}{4}\cdots\left(1 + \frac{\sigma}{t-1}\right) \\
= \;& \sigma(1+\sigma)\frac{(\sigma+2)}{2}\frac{(\sigma+3)}{3}\frac{(\sigma+4)}{4}\cdots\left(\frac{\sigma+(t-1)}{t-1}\right) \\
= \;& \frac{(\sigma)_t}{(t-1)!}.
\end{aligned}$$

□

3. Main results

With the help of the definition of $\mathcal{KD}_q(k,\alpha,\gamma)$, we prove the following results.

Theorem 3.1. *If* $f(z)\in\mathcal{KD}_q(k,\alpha,\gamma)$, *then*

$$f(z)\in\mathcal{KD}_q\left(0,\frac{\alpha-k}{1-k},\gamma\right).$$

Proof. Because $k\leq 0$, we have

$$\begin{aligned}\mathfrak{Re}\left\{1+\frac{1}{\gamma}\left(\frac{z\partial_q\mathfrak{D}_q^\mu f(z)}{\mathfrak{D}_q^\mu f(z)}-1\right)\right\} &< k\left|\frac{1}{\gamma}\left(\frac{z\partial_q\mathfrak{D}_q^\mu f(z)}{\mathfrak{D}_q^\mu f(z)}-1\right)\right|+\alpha,\\ &\leq k\mathfrak{Re}\left(\frac{1}{\gamma}\left(\frac{z\partial_q\mathfrak{D}_q^\mu f(z)}{\mathfrak{D}_q^\mu f(z)}-1\right)\right)+\alpha-k,\end{aligned}$$

which implies that

$$(1-k)\,\mathfrak{Re}\frac{1}{\gamma}\left(\frac{z\partial_q\mathfrak{D}_q^\mu f(z)}{\mathfrak{D}_q^\mu f(z)}-1\right)<\alpha-k.$$

After simplification, we obtain

$$\mathfrak{Re}\left[1+\frac{1}{\gamma}\left(\frac{z\partial_q\mathfrak{D}_q^\mu f(z)}{\mathfrak{D}_q^\mu f(z)}-1\right)\right]<\frac{\alpha-k}{1-k},(k\leq 0,\ \alpha>1\text{ and }). \tag{3.1}$$

This completes the proof. $\square$

Theorem 3.2. *If* $f(z)\in\mathcal{KD}_q(k,\alpha,\gamma)$ *and if* $f(z)$ *has the form* (1.1), *then*

$$|a_n|\leq\frac{(\sigma)_{n-1}}{(n-1)!\Phi_{n-1}}, \tag{3.2}$$

where

$$\sigma=\frac{2|\gamma|(\alpha-1)}{q(1-k)}. \tag{3.3}$$

Proof. Let us define a function

$$p(z)=\frac{(\alpha-k)-(1-k)\left[1+\frac{1}{\gamma}\left(\frac{z\partial_q\mathfrak{D}_q^\mu f(z)}{\mathfrak{D}_q^\mu f(z)}-1\right)\right]}{\alpha-1}. \tag{3.4}$$

Then $p(z)$ is analytic in $\mathbb{U}$, $p(0)=1$ and $\mathfrak{Re}\{p(z)\}>0$ for $z\in\mathbb{U}$. We can write

$$\left[1+\frac{1}{\gamma}\left(\frac{z\partial_q\mathfrak{D}_q^\mu f(z)}{\mathfrak{D}_q^\mu f(z)}-1\right)\right]=\frac{(\alpha-k)-(\alpha-1)p(z)}{1-k} \tag{3.5}$$

If we take $p(z)=1+\sum\limits_{n=1}^{\infty}p_nz^n$, then (3.5) can be written as

$$z\partial_q\mathfrak{D}_q^\mu f(z)-\mathfrak{D}_q^\mu f(z)=-\frac{\gamma(\alpha-1)}{1-k}\left(\mathfrak{D}_q^\mu f(z)\right)\left(\sum_{n=1}^{\infty}p_nz^n\right).$$

this implies that

$$\left[\sum_{n=2}^{\infty}q\,[n-1]\,\Phi_{n-1}a_nz^n\right]=-\frac{\gamma(\alpha-1)}{1-k}\left(\sum_{n=1}^{\infty}\Phi_{n-1}a_nz^n\right)\left(\sum_{n=1}^{\infty}p_nz^n\right).$$

Using Cauchy product $\left(\sum_{n=1}^{\infty} x_n\right) \cdot \left(\sum_{n=1}^{\infty} y_n\right) = \sum_{j=1}^{\infty} \sum_{k=1}^{j} x_k y_{k-j}$, we obtain

$$q\,[n-1]\,\Phi_{n-1} a_n z^n = -\frac{\gamma(\alpha-1)}{1-k} \sum_{n=2}^{\infty} \left(\sum_{j=1}^{n-1} \Phi_{j-1} a_j p_{n-j} \right) z^n.$$

Comparing the coefficients of nth term on both sides, we obtain

$$a_n = \frac{-\gamma(\alpha-1)}{q\,[n-1]\,\Phi_{n-1}\,(1-k)} \sum_{j=1}^{n-1} \Phi_{j-1} a_j p_{n-j}.$$

By taking absolute value and applying triangle inequality, we get

$$|a_n| \le \frac{|\gamma|\,(\alpha-1)}{q\,[n-1]\,\Phi_{n-1}\,(1-k)} \sum_{j=1}^{n-1} \Phi_{j-1}\,|a_j|\,|p_{n-j}|\,.$$

Applying the coefficient estimates $|p_n| \le 2$ $(n \ge 1)$ for Caratheodory functions [11], we obtain

$$\begin{aligned} |a_n| \quad &\le \quad \frac{2\,|\gamma|\,(\alpha-1)}{q\,[n-1]\,\Phi_{n-1}(1-k)} \sum_{j=1}^{n-1} \Phi_{j-1}\,|a_j| \\ &= \quad \frac{\sigma}{[n-1]\,\Phi_{n-1}} \sum_{j=1}^{n-1} \psi_{j-1}\,|a_j|\,, \end{aligned} \tag{3.6}$$

where $\sigma = 2|\gamma|(\alpha-1)/q(1-k)$. To prove (3.2) we apply mathematical induction. So for $n = 2$, we have from (3.6)

$$|a_2| \le \frac{\sigma}{\Phi_1} = \frac{(\sigma)_{2-1}}{[2-1]!\Phi_{2-1}}, \tag{3.7}$$

which shows that (3.2) holds for $n = 2$. For $n = 3$, we have from (3.6)

$$|a_3| \le \frac{\sigma}{[3-1]\,\Phi_{3-1}} \left\{1 + \Phi_1\,|a_2|\right\},$$

using (3.7), we have

$$|a_3| \le \frac{\sigma}{[2]\,\Phi_2}\,(1+\sigma) = \frac{(\sigma)_{3-1}}{[3-1]\,\Phi_{3-1}},$$

which shows that (3.2) holds for $n = 3$. Let us assume that (3.2) is true for $n \le t$, that is,

$$|a_t| \le \frac{(\sigma)_{t-1}}{[t-1]!\Phi_{t-1}} \quad j = 1, 2, \ldots, t. \tag{3.8}$$

Using (3.6) and (3.8), we have

$$\begin{aligned} |a_{t+1}| &\leq \frac{\sigma}{t\Phi_t}\sum_{j=1}^{t}\Phi_{j-1}\,|a_j| \\ &\leq \frac{\sigma}{t\Phi_t}\sum_{j=1}^{t}\psi_{j-1}\frac{(\sigma)_{j-1}}{[j-1]!\Phi_{j-1}} \\ &= \frac{\sigma}{t\Phi_t}\sum_{j=1}^{t}\frac{(\sigma)_{j-1}}{[j-1]!}. \end{aligned}$$

Applying (2.1), we have

$$\begin{aligned} |a_{t+1}| &\leq \frac{1}{t\Phi_t}\frac{(\sigma)_t}{[t-1]!} \\ &= \frac{1}{\Phi_t}\frac{(\sigma)_t}{[t]!}. \end{aligned}$$

Consequently, using mathematical induction, we have proved that (3.2) holds true for all n, $n \geq 2$. This completes the proof. □

Theorem 3.3. *If a function $f \in \mathcal{KD}_q\,(k,\alpha,\gamma)$, then*

$$\frac{z\partial_q\mathfrak{D}_q^{\mu}f(z)}{\mathfrak{D}_q^{\mu}f(z)} \prec 1 + 2\,(\alpha_1 - 1) - \frac{2\,(\alpha_1-1)}{1-z} \quad (z \in \mathbb{U}), \tag{3.9}$$

$$\alpha_1 = \frac{\alpha - k}{1-k}. \tag{3.10}$$

Proof. If $f(z) \in \mathcal{KD}_q\,(k,\alpha,\gamma)$, then by (3.1)

$$\mathfrak{Re}\left\{1 + \frac{1}{\gamma}\left(\frac{z\partial_q\mathfrak{D}_q^{\mu}f(z)}{\mathfrak{D}_q^{\mu}f(z)} - 1\right)\right\} < \alpha_1. \tag{3.11}$$

Then there exists a Schwarz function $w(z)$ such that

$$\frac{\alpha_1 - \left\{1 + \frac{1}{\gamma}\left(\frac{z\partial_q\mathfrak{D}_q^{\mu}f(z)}{\mathfrak{D}_q^{\mu}f(z)} - 1\right)\right\}}{\alpha_1 - 1} = \frac{1+w(z)}{1-w(z)}, \tag{3.12}$$

and

$$\mathfrak{Re}\left\{\frac{1+w(z)}{1-w(z)}\right\} > 0, \quad (z \in \mathbb{U}).$$

Therefore, from (3.12), we obtain

$$\frac{z\partial_q\mathfrak{D}_q^{\mu}f(z)}{\mathfrak{D}_q^{\mu}f(z)} = 1 + \gamma\,(\alpha_1 - 1)\left(1 - \frac{1+w(z)}{1-w(z)}\right).$$

This gives

$$\frac{z\partial_q\mathfrak{D}_q^{\mu}f(z)}{\mathfrak{D}_q^{\mu}f(z)} = 1 + 2\gamma\,(\alpha_1 - 1) - \frac{2\gamma\,(\alpha_1-1)}{1-w(z)}$$

and hence

$$\frac{z\partial_q\mathfrak{D}_q^{\mu}f(z)}{\mathfrak{D}_q^{\mu}f(z)} \prec 1+2\gamma\left(\alpha_1-1\right)-\frac{2\gamma\left(\alpha_1-1\right)}{1-z} \quad (z\in\mathbb{U}).$$

which was required in (3.9). □

Theorem 3.4. *If function $f\in\mathcal{KD}_q\left(k,\alpha,\gamma\right)$, then we have*

$$\frac{1-\left[1+2\gamma(\alpha_1-1)\right]r}{1-r} \leq \mathfrak{Re}\left\{\frac{z\partial_q\mathfrak{D}_q^{\mu}f(z)}{\mathfrak{D}_q^{\mu}f(z)}\right\} \leq \frac{1+\left[1+2\gamma(\alpha_1-1)\right]r}{1+r}, \tag{3.13}$$

for $|z|=r<1$ and α_1 is defined by (3.10).

Proof. By the virtue of Theorem (3.3), let us take the function $\phi(z)$ defined by

$$\phi(z)=1+2\gamma\left(\alpha_1-1\right)-\frac{2\gamma(\alpha_1-1)}{1-z} \quad \left(z\in\mathbb{U}\right).$$

Letting $z=re^{i\theta}(0\leq r<1)$, we see that

$$\mathfrak{Re}\phi(z)=1+2\gamma\left(\alpha_1-1\right)+\frac{2\gamma\left(1-\alpha_1\right)\left(1-r\cos\theta\right)}{1+r^2-2r\cos\theta}.$$

Let us define

$$\psi(t)=\frac{1-rt}{1+r^2-2rt} \quad \left(t=\cos\theta\right).$$

Since $\psi'(t)=\dfrac{r\left(1-r^2\right)}{\left(1+r^2-2rt\right)^2}\geq 0$, because $r<1$. Therefore we get

$$1+2\gamma\left(\alpha_1-1\right)-\frac{2\gamma\left(\alpha_1-1\right)}{1-r} \leq \mathfrak{Re}\phi(z) \leq 1+2\gamma\left(\alpha_1-1\right)-\frac{2\gamma\left(\alpha_1-1\right)}{1+r}.$$

After simplification, we have

$$\frac{1-\left[1+2\gamma\left(\alpha_1-1\right)\right]r}{1-r} \leq \mathfrak{Re}\phi(z) \leq \frac{1+\left[1+2\gamma\left(\alpha_1-1\right)\right)\right]r}{1+r}.$$

Since we note that $\dfrac{z\partial_q\mathfrak{D}_q^{\mu}f(z)}{\mathfrak{D}_q^{\mu}f(z)} \prec \phi(z),(z\in\mathbb{U})$ by Theorem 3.3 and $\phi(z)$ is analytic in $\mathbb{U}$, we proved the inequality (3.13). □

Theorem 3.5. *If $f\in\mathcal{A}$ satisfies*

$$\left|\frac{z\partial_q\mathfrak{D}_q^{\mu}f(z)}{\mathfrak{D}_q^{\mu}f(z)}-1\right| < \frac{(\alpha-1)|\gamma|}{(1-k)} \quad z\in\mathbb{U}, \tag{3.14}$$

for some $k\,(k\leq 0)$, $\alpha\,(\alpha>1)$ and $\gamma\in\mathbb{C}\setminus\{0\}$. Then $f\in\mathcal{KD}_q(k,\alpha,\gamma)$.

Proof.

$$
\begin{aligned}
& \left|\frac{z\partial_q \mathfrak{D}_q^\mu f(z)}{\mathfrak{D}_q^\mu f(z)} - 1\right| < \frac{(\alpha-1)|\gamma|}{(1-k)} \\
\Rightarrow\quad & \left|\frac{1}{\gamma}\left(\frac{z\partial_q \mathfrak{D}_q^\mu f(z)}{\mathfrak{D}_q^\mu f(z)} - 1\right)\right| < \frac{\alpha-1}{1-k} \\
\Rightarrow\quad & (1-k)\left|\frac{1}{\gamma}\left(\frac{z\partial_q \mathfrak{D}_q^\mu f(z)}{\mathfrak{D}_q^\mu f(z)} - 1\right)\right| + 1 < \alpha \\
\Rightarrow\quad & \left|\frac{1}{\gamma}\left(\frac{z\partial_q \mathfrak{D}_q^\mu f(z)}{\mathfrak{D}_q^\mu f(z)} - 1\right)\right| + 1 < k\left|\frac{1}{\gamma}\left(\frac{z\partial_q \mathfrak{D}_q^\mu f(z)}{\mathfrak{D}_q^\mu f(z)} - 1\right)\right| + \alpha \\
\Rightarrow\quad & \mathfrak{Re}\left\{1 + \frac{1}{\gamma}\left(\frac{z\partial_q \mathfrak{D}_q^\mu f(z)}{\mathfrak{D}_q^\mu f(z)} - 1\right)\right\} + 1 < k\left|\frac{1}{\gamma}\left(\frac{z\partial_q \mathfrak{D}_q^\mu f(z)}{\mathfrak{D}_q^\mu f(z)} - 1\right)\right| + \alpha \\
\Rightarrow\quad & f \in \mathcal{LD}_b^k(a, c, \beta)
\end{aligned}
$$

□

Corollary 3.6. *Let $f \in \mathcal{A}$ be of the form (1.1) and satisfies*

$$
\left|\frac{\sum_{n=2}^{\infty}[n-1]\,\Phi_{n-1}a_n z^{n-1}}{1+\sum_{n=2}^{\infty}\Phi_{n-1}a_n z^{n-1}}\right| < \frac{(\alpha-1)|\gamma|}{q(1-k)} \quad z \in \mathbb{U}, \tag{3.15}
$$

for some $k\,(k \leq 0)$, $\beta\,(\beta > 1)$ and for some $b \in \mathbb{C}\setminus\{0\}$. Then $f \in \mathcal{KD}_q(k,\alpha,\gamma)$..

Proof. We have

$$
\mathfrak{D}_q^\mu f(z) = z + \sum_{n=2}^{\infty}\Phi_{n-1}a_n z^n
$$

and by (1.5)

$$
z\partial\mathfrak{D}_q^\mu f(z) = z + \sum_{n=2}^{\infty}[n]\,\Phi_{n-1}a_n z^n.
$$

Therefore, (3.14) follows immediately (3.15). □

Theorem 3.7. *Let $f \in \mathcal{A}$ be of the form (1.1) and satisfies*

$$
\sum_{n=2}^{\infty}([n-1]+y)\,|\Phi_{n-1}||a_n| < y \quad z \in \mathbb{U}, \tag{3.16}
$$

for some $k\,(k \leq 0)$, $\beta\,(\beta > 1)$ and for some $b \in \mathbb{C}\setminus\{0\}$ and where

$$
y = \frac{(\alpha-1)|\gamma|}{q(1-k)} > 0.
$$

Then $f \in \mathcal{KD}_q(k,\alpha,\gamma)$.

Proof. We have

$$
\begin{aligned}
&\sum_{n=2}^{\infty}([n-1]+y)\,|\Phi_{n-1}||a_n|<y\\
\Rightarrow\quad &\sum_{n=2}^{\infty}([n-1]+y)\,|\Phi_{n-1}||a_n|<y-y\sum_{n=2}^{\infty}|\Phi_{n-1}||a_n|\\
\Rightarrow\quad &0<y-y\sum_{n=2}^{\infty}|\Phi_{n-1}||a_n|\\
\Rightarrow\quad &0<y-y\sum_{n=2}^{\infty}|\Phi_{n-1}||a_n||z^{n-1}|\\
\Rightarrow\quad &0<y\left|1+\sum_{n=2}^{\infty}\Phi_{n-1}a_nz^{n-1}\right| \qquad (3.17)
\end{aligned}
$$

We have

$$
\begin{aligned}
&\sum_{n=2}^{\infty}([n-1]+y)\,|\Phi_{n-1}||a_n|<y\\
\Rightarrow\quad &\sum_{n=2}^{\infty}([n-1]+y)\,|\Phi_{n-1}||a_n||z^{n-1}|<y\\
\Rightarrow\quad &\sum_{n=2}^{\infty}[n-1]\,|\Phi_{n-1}||a_n||z^{n-1}|<y-y\sum_{n=2}^{\infty}|\Phi_{n-1}||a_n||z^{n-1}|\\
\Rightarrow\quad &\left|\sum_{n=2}^{\infty}[n-1]\,\Phi_{n-1}a_nz^{n-1}\right|<y\left|1+\sum_{n=2}^{\infty}\Phi_{n-1}a_nz^{n-1}\right|\\
\Rightarrow\quad &\left|\frac{\sum_{n=2}^{\infty}[n-1]\,\Phi_{n-1}a_nz^{n-1}}{1+\sum_{n=2}^{\infty}\Phi_{n-1}a_nz^{n-1}}\right|<y,
\end{aligned}
$$

because of (3.17). By (3.15) it follows $f\in\mathcal{LD}_b^k(a,c,\beta)$. □

References

[1] Aldweby, H., Darus, M., *Some subordination results on q-analogue of Ruscheweyh differential operator,* Abstr. Appl. Anal., Vol. 2014, Art. ID 958563, 6 pages.

[2] Anastassiu, G.A., Gal, S.G., *Geometric and approximation properties of some singular integrals in the unit disk,* J. Inequal. Appl., Vol. 2006, Art. ID 17231, 19 pages.

[3] Anastassiu, G.A., Gal, S.G., *Geometric and approximation properties of generalized singular integrals,* J. Korean Math. Soc., **23**(2006), no. 2, 425-443.

[4] Aral, A., *On the generalized Picard and Gauss Weierstrass singular integrals,* J. Comput. Anal. Appl., **8**(2006), no. 3, 249-261.

[5] Aral, A., Gupta, V., *On q-Baskakov type operators*, Demonstr. Math., **42**(2009), no. 1, 109-122.

[6] Aral, A., Gupta, V., *On the Durrmeyer type modification of the q-Baskakov type operators,* Nonlinear Anal. Theory, Methods and Appl., **72**(2010), no. 3-4, 1171-1180.

[7] Aral, A., Gupta, V., *Generalized q-Baskakov operators,* Math. Slovaca, **61**(2011), no. 4, 619-634.

[8] Aral, A., Gupta, V., Agarwal, R.P., *Applications of q-Calculus in Operator Theory,* Springer, New York, NY, USA, 2013.

[9] Arif, M., Mahmood, S., Sokół, J., Dziok, J., *New subclass of analytic functions in conical domain associated with a linear operator,* Acta Math. Sci., **36**B(2016), no. 3, 1-13.

[10] Jackson, F.H., *On q-functions and a certain difference operator,* Trans. Roy. Soc. Edinburgh, **46**(1909), no. 2, 253-281.

[11] Goodman, A.W., *Univalent Functions*, Vol. I, II, Polygonal Publishing House, Washington, New Jersey, 1983.

[12] Jackson, F.H., *On q-definite integrals,* Quart. J. Pure Appl. Math., **41**(1910), 193-203.

[13] Kanas, S., Raducanu, D., *Some class of analytic functions related to conic domains,* Math. Slovaca, **64**(2014), no. 5, 1183-1196.

[14] Mahmmod, S., Sokół, J., *New subclass of analytic functions in conical domain associated with Ruscheweyh q-differential operator,* J. Results Math., **71**(2017), 1345-1357.

[15] Miller, S.S., Mocanu, P.T., *Differential Subordinations Theory and Applications*, Marcel Decker Inc., New York, 2000.

[16] Nishiwaki, J., Owa, S., *Coefficient estimates for certain classes of analytic functions*, J. Inequal. Pure Appl. Math., **3**(2002), 1-5.

[17] Nishiwaki, J., Owa, S., *Certain classes of analytic functions concerned with uniformly starlike and convex functions*, Appl. Math. Comput., **187**(2007), 350-355.

[18] Owa, S., Srivastava, H.M., *Some generalized convolution properties associated with cerain subclasses of analytic functions*, J. Inequal. Pure Appl. Math., **3**(2002), no. 3, 1-13.

[19] Ruscheweyh,St., *New criteria for univalent functions*, Proc. Amer. Math. Soc., **49**(1975), 109-115.

[20] Shams, S., Kulkarni, S.R., Jahangiri, J.M., *Classes of uniformly starlike and convex functions*, Int. J. Math. Math. Sci., **55**(2004), 2959-2961.

[21] Uralegaddi, B.A., Ganigi, M.D., Sarangi, S.M., *Univalent functions with positive coefficients*, Tamkang J. Math., **25**(1994), 225-230.

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Inclusion properties of hypergeometric type functions and related integral transforms

Lateef Ahmad Wani and Swaminathan Anbhu

Abstract. In this work, conditions on the parameters a, b and c are given so that the normalized Gaussian hypergeometric function $zF(a, b; c; z)$, where

$$F(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n (1)_n} z^n, \quad |z| < 1,$$

is in certain class of analytic functions. Using Taylor coefficients of functions in certain classes, inclusion properties of the Hohlov integral transform involving $zF(a, b; c; z)$ are obtained. Similar inclusion results of the Komatu integral operator related to the generalized polylogarithm are also obtained. Various results for the particular values of these parameters are deduced and compared with the existing literature.

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1. Introduction

Let $\mathcal{A}$ denote the class of functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \tag{1.1}$$

analytic in the open unit disk $\mathbb{D} = \{z : |z| < 1\}$, and $\mathcal{S}$ denote the subclass of $\mathcal{A}$ that contains functions univalent in $\mathbb{D}$. A function $f \in \mathcal{A}$ is called starlike, denoted by $f \in \mathcal{S}^*$, if $tw \in f(\mathbb{D})$ whenever $w \in f(\mathbb{D})$ and $t \in [0, 1]$. The class of all convex functions, denoted by $\mathcal{C}$, consists of the functions $f \in \mathcal{A}$ such that zf' is starlike. A function $f \in \mathcal{A}$ is said to be *close-to-convex* with respect to a fixed starlike function $g \in S^*$ if and only if $\mathrm{Re}\left(e^{i\lambda} \dfrac{zf'(z)}{g(z)}\right) > 0$ for $z \in \mathbb{D}$ and $\lambda \in \mathbb{R}$. . Let $\mathcal{K}$ denote the subclass of all such close-to-convex functions, where $\lambda = 0$. Various generalization of these classes and various other subclasses of S exist in the literature. For example the class of starlike functions of order σ, denoted by $S^*(\sigma)$, $0 \le \sigma < 1$, which has the analytic characterization $\mathrm{Re}\,\dfrac{zf'(z)}{f(z)} > \sigma$, is the generalization of the class $S^*(0) = S^*$. Note that $C(\sigma)$, the class of convex functions of order σ contains all functions $f \in S$ for which $zf' \in S^*(\sigma)$.

We introduce the class $R^{\tau}_{\gamma,\alpha}(\beta)$, with $0 \le \gamma < 1$, $0 \le \alpha \le 1$, $\tau \in \mathbb{C}\backslash\{0\}$ and $\beta < 1$ as

$$R^{\tau}_{\gamma,\alpha}(\beta) := \left\{ f \in \mathcal{A} : \left| \frac{(1-\alpha+2\gamma)\frac{f}{z} + (\alpha-2\gamma)f' + \gamma z f'' - 1}{2\tau(1-\beta) + (1-\alpha+2\gamma)\frac{f}{z} + (\alpha-2\gamma)f' + \gamma z f'' - 1} \right| < 1,\ z \in \mathbb{D} \right\}. \tag{1.2}$$

Note that few particular cases of this class discussed in the literature.

1. The class $R^{\tau}_{\gamma,\alpha}(\beta)$ for $\alpha = 2\gamma + 1$, was considered in [16], where references about other particular cases in this direction are provided.
2. The class $R^{\tau}_{\gamma,\,\alpha}(\beta)$ for $\tau = e^{i\eta}\cos\eta$, where $-\pi/2 < \eta < \pi/2$ is considered in [1] (see also [2, 3]), and the properties of certain integral transforms of the type
$$V_\lambda(f) = \int_0^1 \lambda(t)\frac{f(tz)}{t}dt, \quad f \in R^{(e^{i\eta}\cos\eta)}_{0,\gamma}(\beta) \tag{1.3}$$
with $\beta < 1$, $\gamma < 1$ and $|\eta| < \pi/2$, under suitable restriction on $\lambda(t)$ was discussed using duality techniques for various values of γ in [1]. For other interesting cases, we refer to [3, 16] and references therein.
3. The class $R^{\tau}_{0,1}(0)$ with $\tau = e^{i\eta}\cos\eta$ was considered in [10] with reference to the univalency of partial sums.

It is clear that the geometric properties of certain integral transforms under duality techniques, which is one of recent research interest (for example, see [1, 3] and references therein), cannot be proved easily as the results involve certain multiple integrals and it is difficult to check the conditions given for the existence of the inclusion results for these integral transforms. For this purpose, the inclusion properties of certain special functions to be in the analytic subclasses like $R^{(e^{i\eta}\cos\eta)}_{\gamma,\,\alpha}(\beta)$ are studied using techniques other than duality methods which motivates this work.

Among various results related to the integral operator (1.3) available in the literature, an important and interesting result is application of the operator (1.3) when $\lambda(t)$ is related to the function $zF(a, b; c; z)$. Here by $F(a, b; c; z)$ we mean the well-known Gaussian hypergeometric function

$$F(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n(1)_n} z^n \tag{1.4}$$

$z \in \mathbb{D}$, with $(\lambda)_n$ being the Pochhammer symbol given by $(\lambda)_n = \lambda(\lambda+1)_{n-1}$, $(\lambda)_0 = 1$. Also, there has been considerable interest to find conditions on the parameters a, b, and c such that the normalized hypergeometric functions $(c/ab)\,(F(a, b; c; z) - 1)$ or $zF(a, b; c; z)$ belong to one of the known subclasses of $\mathcal{S}$. For more details on the basic ideas of Gaussian hypergeometric functions, we refer to [11] and on the applications related to geometric function theory, we refer to [1, 14, 15, 16] and references therein.

Related to $F(a,b;c;z)$ is the Hohlov operator $H_{a,b,c}(f)(z) = zF(a,b;c;z)*f(z)$, where $*$ denotes the well-known Hadamard product or convolution. This operator is particular case of a general integral transform studied in [5]. To be more specific, the properties of certain integral transforms of the type

$$V_\lambda(f) = \int_0^1 \lambda(t)\frac{f(tz)}{t}dt, \quad f \in R_{\gamma,\alpha}^{(e^{i\eta}\cos\eta)}(\beta) \tag{1.5}$$

under suitable restriction on $\lambda(t)$ was discussed by many authors [1, 3, 5]. In particular, if

$$\lambda(t) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}t^{b-1}(1-t)^{c-b-1}$$

then $V_\lambda(f) = \mathcal{L}(b,c)(f)(z)$ which is the well-known Carlson-Schaffer operator. Note that $H_{1,b,c}(f)(z) = \mathcal{L}(b,c)(f)(z)$. The following lemma exhibits the relation between the integral operator in discussion with the Hohlov operator.

Lemma 1.1. *If $f \in \mathcal{A}$ and $c-a+1>b>0$, then*

$$V_\lambda(f)(z) = H_{a,b,c}(f)(z)$$

where

$$H_{a,b,c}(f)(z) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)}\int_0^1 \frac{(1-t)^{c-a-b}}{\Gamma(c-a-b+1)}t^{b-2}F(c-a,1-a;c-a-b+1;1-t)f(tz)dt.$$

The Komatu operator $K_a^p : \mathcal{A} \to \mathcal{A}$ [9] is defined as

$$K_a^p[f](z) = \frac{(1+a)^p}{\Gamma(p)}\int_0^1 \left(\log(\frac{1}{t})\right)^{p-1}t^{a-1}f(tz)dt,$$

where $a>-1$ and $p \geq 0$. It has a series representation as

$$K_a^p[f](z) = z + \sum_{n=2}^{\infty}\frac{(1+a)^p}{(n+a)^p}a_n z^n$$

and in terms of convolution, we can write

$$K_a^p[f](z) = \mathcal{K}_a^p(z) * f(z),$$

where $\mathcal{K}_a^p(z) = z + \sum_{n=2}^{\infty}\frac{(1+a)^p}{(n+a)^p}z^n$.

In this paper we study the operators $H_{a,b,c}(f)(z)$ and $K_a^p[f](z)$ for various choices of the function f.

The paper is organized as follows. In Section 2, some preliminary results about the Gaussian hypergeometric function $F(a,b;c;z)$ and conditions on the Taylor coefficients of $f \in R_{\gamma,\alpha}^\tau(\beta)$ are given which are used in the subsequent sections. Conditions on the triplets a,b,c are obtained so that in Section 3 inclusion properties of $F(a,b;c;z)$ and its normalized case to be in the class $R_{\gamma,\alpha}^\tau(\beta)$ are discussed and in Section 4, inclusion properties of $zF(a,b;c;z)*f(z)$ for f in various subclasses of S are discussed. Similar type of inclusion results for the Komatu operator is discussed in Section 5. In the last section, certain remarks are given to provide motivation for further research in this direction.

2. Preliminary results

The following result is available in [16], which can also be easily verified by simple computation.

Lemma 2.1. *Let $F(a,b;c;z)$ be the Gaussian hypergeometric function as given in* (1.4). *Then we have the following*

(i) *For* $\mathrm{Re}\,(c-a-b)>0$ *and* $c\neq 0,-1,-2,\ldots$,

$$F(a,b;c;1)=\frac{\Gamma(c-a-b)\Gamma(c)}{\Gamma(c-a)\Gamma(c-b)} \tag{2.1}$$

(ii) *For* $a,b>0,\ c>a+b+1$,

$$\sum_{n=0}^{\infty}\frac{(n+1)(a)_n(b)_n}{(c)_n(1)_n}=F(a,b;c;1)\left[\frac{ab}{c-1-a-b}+1\right]. \tag{2.2}$$

(iii) *For* $a\neq 1,\ b\neq 1$ *and* $c\neq 1$ *with* $c>\max\{0,a+b-1\}$,

$$\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n(1)_{n+1}}=\frac{(c-1)}{(a-1)(b-1)}\left[F(a-1,b-1;c-1;1)-1\right]. \tag{2.3}$$

(iv) *For* $a\neq 1$ *and* $c\neq 1$ *with* $c>\max\{0,2\mathrm{Re}\,a-1\}$,

$$\sum_{n=0}^{\infty}\frac{|(a)_n|^2}{(c)_n(1)_{n+1}}=\frac{(c-1)}{|a-1|^2}\left[F(a-1,\bar{a}-1;c-1;1)-1\right]. \tag{2.4}$$

Proof. Part (i) is the well-known Gauss summation formula. Part (ii) follows from splitting the left hand side into two parts and applying (2.1). For part (iii), using the fact that $\lambda(\lambda+1)_m=(\lambda)_{m+1}$, in place of a, b and c, the required result follows. Part (iv) is nothing but Part (iii) with $b=\bar{a}$. □

In order to obtain the objective, we need conditions on the Taylor coefficients of $R^{\tau}_{\gamma,\alpha}(\beta)$ which is given in the following results.

Lemma 2.2. *Let $f(z)\in\mathcal{S}$ and is of the form* (1.1). *If $f(z)$ is in $R^{\tau}_{\gamma,\alpha}(\beta)$, then*

$$|a_n|\leq\frac{2|\tau|(1-\beta)}{1+(n-1)(\alpha-2\gamma+\gamma n)},\quad n=2,3,\ldots. \tag{2.5}$$

Equality holds for the function

$$f(z)=\frac{1}{z^{(1/\nu)-1}}\frac{1}{\mu\nu}\int_0^z\frac{1}{t^{\frac{1}{\mu}-\frac{1}{\nu}+1}}\int_0^t w^{\frac{1}{1-\frac{1}{\mu}}}\left(1+\frac{2(1-\beta)\tau w^{n-1}}{1-w^{n-1}}\right)dw, \tag{2.6}$$

where $\mu+\nu=\alpha-\gamma$ and $\mu\nu=\gamma$.

Proof. Clearly $f \in R^{\tau}_{\gamma,\alpha}(\beta)$ is equivalent to

$$1 + \frac{1}{\tau}\left((1-\alpha+2\gamma)\frac{f}{z} + (\alpha-2\gamma)f' + \gamma z f'' - 1\right) = \frac{1+(1-2\beta)w(z)}{1-w(z)},$$

where $w(z)$ is analytic in $\mathbb{D}$ and satisfies the condition $w(0)=0$, $|w(z)|<1$ for $z \in \mathbb{D}$. Hence we have

$$\frac{1}{\tau}\left((1-\alpha+2\gamma)\frac{f}{z} + (\alpha-2\gamma)f' + \gamma z f'' - 1\right)$$
$$= w(z)\left(2(1-\beta) + \frac{1}{\tau}\left((1-\alpha+2\gamma)\frac{f}{z} + (\alpha-2\gamma)f' + \gamma z f'' - 1\right)\right).$$

Using (1.1) and $w(z) = \sum_{n=1}^{\infty} b_n z^n$ we have

$$\left[2(1-\beta) + \frac{1}{\tau}\left(\sum_{n=2}^{\infty}[1+(\alpha-2\gamma+\gamma n)(n-1)]a_n z^{n-1}\right)\right]\left[\sum_{n=1}^{\infty} b_n z^n\right]$$
$$= \frac{1}{\tau}\sum_{n=2}^{\infty}[1+(\alpha-2\gamma+\gamma n)(n-1)]a_n z^{n-1}.$$

Equating the coefficients of the powers of z^{n-1} on both sides of the above equation, it is easy to observe that the coefficient a_n in right hand side of the above expression depends only on $a_2,\ldots,a_{n-1}$ and the left hand side of the above expression. Hence, for $n \geq 2$ this gives

$$\left[2(1-\beta) + \frac{1}{\tau}\left(\sum_{n=2}^{k-1}[1+(\alpha-2\gamma+\gamma n)(n-1)]a_n z^{n-1}\right)\right] w(z)$$
$$= \frac{1}{\tau}\sum_{n=2}^{k}[1+(\alpha-2\gamma+\gamma n)(n-1)]a_n z^{n-1} + \sum_{n=k+1}^{\infty} d_n z^{n-1}.$$

Using $|w(z)|<1$, this reduces to the inequality

$$\left|2(1-\beta) + \frac{1}{\tau}\left(\sum_{n=2}^{k-1}[1+(\alpha-2\gamma+\gamma n)(n-1)]a_n z^{n-1}\right)\right|$$
$$> \left|\frac{1}{\tau}\sum_{n=2}^{k}[1+(\alpha-2\gamma+\gamma n)(n-1)]a_n z^{n-1} + \sum_{n=k+1}^{\infty} d_n z^{n-1}\right|.$$

Squaring the above inequality and integrating around $|z|=r$, $0<r<1$, we get

$$4(1-\beta)^2 + \frac{1}{|\tau|^2}\left(\sum_{n=2}^{k-1}[1+(\alpha-2\gamma+\gamma n)(n-1)]^2|a_n|^2 r^{2(n-1)}\right)$$
$$> \frac{1}{|\tau|^2}\sum_{n=2}^{k}[1+(\alpha-2\gamma+\gamma n)(n-1)]^2|a_n|^2 r^{2(n-1)} + \sum_{n=k+1}^{\infty}|d_n|^2 r^{2(n-1)}.$$

and letting $r \to 1$ we obtain

$$4(1-\beta)^2 \geq \frac{1}{|\tau|^2}[1+(\alpha-2\gamma+\gamma n)(n-1)]^2|a_n|^2$$

which gives the desired result. For sharpness, consider the function

$$(1-\alpha+2\gamma)\frac{f}{z}+(\alpha-2\gamma)f'+\gamma z f'' = 1+\frac{2(1-\beta)\tau z^{n-1}}{1-z^{n-1}} := p(z).$$

Simplifying and using the fact $\mu+\nu=\alpha-\gamma$ and $\mu\nu=\gamma$ gives (2.6). □

Remark 2.3. The condition given in (2.5) is equivalent to the condition

$$|a_n| \leq \frac{2|\tau|(1-\beta)}{1+\alpha(n-1)+\gamma(n-1)(n-2)}, \quad n=2,3,\ldots, \tag{2.7}$$

which will be used in the sequel.

Lemma 2.4. *Let $f(z)$ be of the of the form* (1.1)*. Then a sufficient condition for $f(z)$ to be in $R^{\tau}_{\gamma,\alpha}(\beta)$ is*

$$\sum_{n=2}^{\infty}[1+(n-1)(\alpha-2\gamma+\gamma n)]|a_n| \leq |\tau|(1-\beta). \tag{2.8}$$

This condition is also necessary if $\eta=0$ in (1.2) *and $a_n<0$ in* (1.1)*.*

Proof. Using (1.1) it is easy to see that

$$\begin{aligned}
&\operatorname{Re} e^{i\eta}\left((1-\alpha+2\gamma)\frac{f}{z}+(\alpha-2\gamma)f'+\gamma z f''-\beta\right)\\
&= (1-\beta)\cos\eta+\operatorname{Re} e^{i\eta}\sum_{n=2}^{\infty}\Big(1+(\alpha-2\gamma+\gamma n)(n-1)\Big)a_n z^{n-1}\\
&\geq (1-\beta)\cos\eta-\sum_{n=2}^{\infty}\Big|\Big(1+(\alpha-2\gamma+\gamma n)(n-1)\Big)\Big|\,|a_n| \geq 0,
\end{aligned}$$

using (2.8). The resultant obtained above is equivalent to the analytic characterization of $f \in R^{\tau}_{\gamma,\alpha}(\beta)$ and the proof is complete. □

3. Inclusion results for $zF(a,b;c;z)$

Theorem 3.1. *Let a,b,c and γ satisfy any one of the following conditions such that $T_i(a,b,c,\gamma) \leq |\tau|(1-\beta)$ for $i=1,2,3$.*

(i) *$a,b>0$, $c>a+b+2$ and*

$$T_1(a,b,c,\gamma) = F(a,b;c;1)+\alpha\frac{a\,b}{c}F(a+1,b+1;c+1;1)+\gamma\frac{(a)_2\,(b)_2}{(c)_2}F(a+2,b+2;c+2;1)-1.$$

(ii) *$a,b \in \mathbb{C}\backslash\{0\}$, $|a|\neq 1, |b|\neq 1$, $c>|a|+|b|+2$ and*
$T_2(a,b,c,\gamma)$

$$= F(|a|+1,|b|+1;c+1;1)\left(\alpha\frac{|ab|}{c}+\gamma\frac{(|a|)_2(|b|)_2}{(c)(c-|a|-|b|-2)}+\frac{c-|a|-|b|-1}{c}\right)-1.$$

(iii) $-1 < a < 0$, $-1 < b < 0$, $c > 0$ *and*
$$T_3(a,b,c,\gamma)$$
$$= F(a+1,b+1;c+1;1)\left(\alpha\frac{ab}{c}+\gamma\frac{(a)_2(b)_2}{(c)(c-a-b-2)}+\frac{c-a-b-1}{c}\right)-1.$$

Then $zF(a,b;c;z)$ is in $R^{\tau}_{\gamma,\alpha}(\beta)$.

Proof. Clearly $zF(a,b;c;z)$ has the series representation of the form (1.1) where
$$a_n = \frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}}. \tag{3.1}$$
Using Lemma 2.4, it suffices to prove that
$$\sum_{n=2}^{\infty}[1+(n-1)(\alpha-2\gamma+\gamma n)]|a_n| \le |\tau|(1-\beta),$$
which is equivalent in writing
$$\sum_{n=2}^{\infty}[1+\alpha(n-1)+\gamma(n-1)(n-2)]|a_n| \le |\tau|(1-\beta) \Longrightarrow f\in R^{\tau}_{\gamma,\alpha}(\beta). \tag{3.2}$$
Case (i): Let $a,b>0$ and $c>a+b+2$. Then the series in the left hand side of (3.2) can be written as
$$\begin{aligned}S &:= \sum_{n=2}^{\infty}\left(1+\alpha(n-1)+\gamma(n-1)(n-2)\right)\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}}\\ &= \sum_{n=1}^{\infty}\frac{(a)_n(b)_n}{(c)_n(1)_n}+\alpha\frac{ab}{c}\sum_{n=2}^{\infty}\frac{(a+1)_{n-2}(b+1)_{n-2}}{(c+1)_{n-2}(1)_{n-2}}+\gamma\frac{(a)_2(b)_2}{(c)_2}\sum_{n=3}^{\infty}\frac{(a+2)_{n-3}(b+2)_{n-3}}{(c+2)_{n-3}(1)_{n-3}}.\end{aligned}$$
An easy computation by using the hypothesis of the theorem and applying (2.1), we get the required result.

Case (ii): Let $a,b\in\mathbb{C}\setminus\{0\}$, $c>|a|+|b|+2$. Since $|(a)_n|\le(|a|)_n$, we have from (3.2),
$$\begin{aligned}S &:= \sum_{n=2}^{\infty}\left(1+\alpha(n-1)+\gamma(n-1)(n-2)\right)|a_n|\\ &\le \frac{|ab|}{c}\sum_{n=0}^{\infty}\frac{(|a|+1)_n(|b|+1)_n}{(c+1)_n(1)_{n+1}}+\alpha\sum_{n=0}^{\infty}\frac{(|a|)_{n+1}(|b|)_{n+1}}{(c)_{n+1}(1)_n}+\gamma\sum_{n=1}^{\infty}(n-1)\frac{(|a|)_n(|b|)_n}{(c)_n(1)_{n-1}}.\end{aligned} \tag{3.3}$$
Note that the third sum in the right hand side of (3.3) is equivalent to
$$\begin{aligned}&\sum_{n=0}^{\infty}n\frac{(|a|)_{n+1}(|b|)_{n+1}}{(c)_{n+1}(1)_n}\\ &\quad= \sum_{n=0}^{\infty}(n+1)\frac{(|a|)_{n+1}(|b|)_{n+1}}{(c)_{n+1}(1)_n}-\sum_{n=0}^{\infty}\frac{(|a|)_{n+1}(|b|)_{n+1}}{(c)_{n+1}(1)_n}\\ &\quad= \frac{|ab|}{c}\sum_{n=0}^{\infty}(n+1)\frac{(|a|+1)_n(|b|+1)_n}{(c+1)_n(1)_n}-\frac{|ab|}{c}\sum_{n=0}^{\infty}\frac{(|a|+1)_n(|b|+1)_n}{(c+1)_n(1)_n}.\end{aligned}$$

Using the above value in (3.3) we get that the inequality (3.3) is equivalent to

$$S \leq \frac{|ab|}{c}\sum_{n=0}^{\infty}\frac{(|a|+1)_n(|b|+1)_n}{(c+1)_n(1)_{n+1}}$$

$$+(\alpha-\gamma)\frac{|ab|}{c}\sum_{n=0}^{\infty}\frac{(|a|+1)_n(|b|+1)_n}{(c+1)_n(1)_n}+\gamma\frac{|ab|}{c}\sum_{n=0}^{\infty}(n+1)\frac{(|a|+1)_n(|b|+1)_n}{(c+1)_n(1)_n}. \quad (3.4)$$

Now applying (2.3) and the hypothesis of the theorem in the first sum of (3.4) gives

$$\left(\frac{c-|a|-|b|-1}{c}F(|a|+1,|b|+1;c+1;1)-1\right). \quad (3.5)$$

Similarly applying (2.2) and the hypothesis of the theorem in the third sum of (3.4) gives

$$\frac{|ab|}{c}\left(F(|a|+1,|b|+1;c+1,1)\left(\frac{(|a|+1)(|b|+1)}{c-|a|-|b|-2}+1\right)\right). \quad (3.6)$$

Clearly the second sum of (3.4) is related to (2.1) which gives

$$\frac{|ab|}{c}F(|a|+1,|b|+1;c+1;1).$$

Now substituting this resultant and (3.5) and (3.6) in (3.4) gives the required result.
Case (iii): Let $-1<a<0$, $-1<b<0$ and $c>0$. The result follows by proceeding in a similar way to the previous case. □

Since the substitution $a=\bar{b}$ in Theorem 3.1 is useful in characterizing polynomials with positive coefficients when b is some negative integer, we give the corresponding result independently, wherein only the second case can be applied.

Corollary 3.2. *Let $c>2\operatorname{Re}b+2$ and $T_4(b,c,\gamma)\leq|\tau|(1-\beta)$ where*

$$T_4(b,c,\gamma)=F(\overline{b+1},b+1;c+1;1)\left(\alpha\frac{|b|^2}{c}+\gamma\frac{(|b|)_2{}^2}{c(c-2\operatorname{Re}b-2)}+\frac{c-2\operatorname{Re}b-1}{c}\right)-1.$$

Then $zF(\bar{b},b;c;z)$ is in $R^{\tau}_{\gamma,\alpha}(\beta)$.

Note that the results in Corollary 3.2 can also be obtained directly by using (2.4) instead of (2.3), as used in Theorem 3.1.

Further, if we set $\alpha=1$ and $\gamma=0$, then by choosing $\beta=0$ and $\tau=e^{i\eta}\cos\eta$ with $-\pi/2<\eta<\pi/2$, we get the functions in the class $R^{\tau}_{\gamma,\alpha}(\beta)$ satisfying the analytic criterion $\operatorname{Re}f'>0$ which implies that $f(z)$ is close-to-convex with respect to the starlike function $g(z)=z$. Hence the following result is immediate.

Corollary 3.3. *Let $c>2|b-1|+3$ and*

$$F(\bar{b},b;c;1)\leq\frac{2(c-1)}{|b-2|^2+c-3}, \quad (3.7)$$

then $zF(\bar{b},b;c;z)$ is close-to-convex with respect to the starlike function $g(z)=z$.

Remark 3.4. Corollary 3.3, with the absence of α, β, γ and τ, is much useful, in particular, for extracting polynomials with positive coefficients, which is the main idea behind choosing $a = \bar{b}$. Moreover, if we take $b = -m$, then (3.7) gives

$$F(-m, -m; c; 1)\left(\frac{m^2 + 4m + c + 1}{2(c-1)}\right) \leq 1.$$

But, when m is sufficiently large, c has to be chosen so large to have the value in the left side bounded by 1. This is given by the condition that $c > 2m + 5$. In the case of $m = 2$, c need to be larger than 9 and should satisfy $c^3 - 18c^2 - 75c - 104 \geq 0$ so that the corresponding polynomial $1 + \frac{4}{c}z + \frac{2}{c(c+1)}z^2$ is close-to-convex. It is easy to see that the condition is satisfied for c more than $21.68057259\ldots$, which is obtained using mathematical software. Hence if m is chosen as a larger negative integer then this result is true for polynomials having their coefficients very small, which is not interesting.

Instead, if we consider, Theorem 3.1, with either $a = -m$ or $b = -m$ we can still extract polynomials that can have smaller values of c, with coefficients having alternate signs, that satisfy the hypothesis given in Theorem 3.1.

In Theorem 3.1, if we take $a = 1$, we get the result for the incomplete beta function $zF(1, b; c; z)$. Since the incomplete beta function plays an important role in geometric function theory (for example, see [15]), we give the result for the incomplete beta function independently as

Theorem 3.5. *Let b, c and γ satisfy any one of the following conditions such that $T_i(b, c, \gamma) \leq |\tau|(1-\beta)$ for $i = 1, 2$.*

(i) $b > 0$, $c > b + 3$ *and*

$$T_1(b, c, \gamma) = F(1, b; c; 1) + \alpha\frac{b}{c}F(2, b+1; c+1; 1) + \gamma\frac{2\,(b)_2}{(c)_2}F(3, b+2; c+2; 1) - 1.$$

(ii) $b \in \mathbb{C}\backslash\{0\}$, $c > |b| + 3$ *and*

$$T_2(b, c, \gamma) = F(2, |b|+1; c+1; 1)\left(\alpha\frac{|b|}{c} + \gamma\frac{2(|b|)_2}{(c)(c-|b|-3)} + \frac{c-|b|-2}{c}\right) - 1.$$

Then the incomplete beta function $\phi(b; c; z) := zF(1, b; c; z)$ is in $R^{\tau}_{\gamma,\alpha}(\beta)$.

Remark 3.6. Note that at $\alpha = 1$, $\gamma = 0$, $\beta = 0$ and $\tau = e^{i\eta}\cos\eta$ with $-\pi/2 < \eta < \pi/2$ the above result reduces to $c > b + 3$, $b > 0$. Under these conditions, the normalized incomplete beta function $zF(1, b; c; z)$ is close-to-convex with respect to the starlike function $g(z) = z$.

Consider the operator of the form $G(a, b; c; z) := \int_0^z F(a, b; c; t)dt$. Then we have

$$G(a, b; c; z) := z + \sum_{n=2}^{\infty}\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_n}z^n = z + \sum_{n=2}^{\infty}\frac{a_n}{n}z^n,$$

where a_n is given as in (3.1). This is the normalized form of the hypergeometric function $F(a, b; c; z)$ which has many interesting properties. Note that a function may fail to inherit its geometric properties under such normalization. For example, $1+z$ is convex univalent in $\mathbb{D}$, whereas its normalized form $z(1+z)$ is not even univalent.

Theorem 3.7. *Let $a, b \in \mathbb{C}\backslash\{0\}$ with $|a| \neq 1$, $|b| \neq 1$ and $|c| > |a|+|b|+1$ such that $T(a,b,c,\gamma) \leq |\tau|(1-\beta)$ where*
$T(a,b,c,\gamma)$

$$= F(a,b;c;1)\left(\frac{\gamma\, ab}{c-a-b-1} + \alpha + \frac{(1-\alpha+2\gamma)(c-a-b)}{(a-1)(b-1)}\right) - \frac{(1-\alpha+2\gamma)(c-1)}{(a-1)(b-1)}.$$

Then $G(a,b;c;z)$ is in $R^{\tau}_{\gamma,\alpha}(\beta)$.

Proof. We have $G(a,b;c;z) = z + \sum_{n=2}^{\infty} \frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_n} z^n$. So it is sufficient to prove that

$$\sum_{n=2}^{\infty} [1+\alpha(n-1)+\gamma(n-1)(n-2)]\,|a_n| \leq |\tau|(1-\beta).$$

The left hand side of the above inequality can be expressed as

$$\sum_{n=1}^{\infty} \frac{(a)_n(b)_n}{(c)_n(1)_{n+1}} + \alpha \sum_{n=1}^{\infty} n \frac{(a)_n(b)_n}{(c)_n(1)_{n+1}} + \gamma \sum_{n=1}^{\infty} n(n-1)\frac{(a)_n(b)_n}{(c)_n(1)_{n+1}}. \tag{3.8}$$

For the third part (3.8), writing $n(n-1) = n(n+1) - 2(n+1) + 2$ and adding with the second part of (3.8) gives

$$(1-\alpha+2\gamma)\sum_{n=1}^{\infty} \frac{(a)_n(b)_n}{(c)_n(1)_{n+1}} + (\alpha-2\gamma)\sum_{n=1}^{\infty} \frac{(a)_n(b)_n}{(c)_n(1)_n} + \frac{\gamma ab}{c}\sum_{n=0}^{\infty} \frac{(a+1)_n(b+1)_n}{(c+1)_n(1)_n}. \tag{3.9}$$

Now, using the hypothesis and comparing the first part of (3.9) with (2.3), second and third part of (3.9) with (2.1) gives the required result upon simplification. □

Check, if at $a = \bar{b}$ in the above result gives the following Corollary.

Corollary 3.8. *Let $a = \bar{b}$, $0 < b \neq 1$ and $c > 2\mathrm{Re\, b}+1$ such that $T(\bar{b}, b, c, \gamma) \leq |\tau|(1-\beta)$ where*
$T(\bar{b}, b, c, \gamma)$

$$= F(b,\bar{b};c;1)\left(\frac{\gamma\,|b|^2(\alpha-2\gamma)}{c-2\mathrm{Re\,b}-1} + \frac{(1-\alpha+2\gamma)(c-2\mathrm{Reb})}{|b-1|^2}\right) - \left(\frac{(1-\alpha+2\gamma)(c-1)}{|b-1|^2}\right)$$

Then $G(\bar{b}, b; c; z)$ is in $R^{\tau}_{\gamma,\alpha}(\beta)$.

4. Inclusion properties of $H_{a,\,b\,c}(f)(z)$

Our next interest is to find the inclusion properties of

$$H_{a,\,b\,c}(f)(z) = zF(a,b;c;z) * f(z),$$

where $f(z)$ is in certain subclass of S. For this, we recall certain subclasses that are necessary for further discussion. We begin with the following definition.

Definition 4.1. [4] Let $f \in \mathcal{A}$, $0 \leq k < \infty$, and $0 \leq \sigma < 1$. Then $f \in k - UCV(\sigma)$ if and only if

$$\operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) \geq k \left|\frac{zf''(z)}{f'(z)}\right| + \sigma. \tag{4.1}$$

This class generalizes various other classes which are worthy to mention here. The class $k-UCV(0)$, called as k-uniformly convex is due to [8], and has the geometric characterization that for $0 \leq k < \infty$, the function $f \in \mathcal{A}$ is said to be k-uniformly convex in $\mathbb{D}$, if f is convex in $\mathbb{D}$, and the image of every circular arc γ contained in $\mathbb{D}$, with center ζ, where $|\zeta| \leq k$, is convex.

The class $1 - UCV(0) = UCV$ [7] (see also [12]) describes geometrically the domain of values of the expression $p(z) = 1 + \dfrac{zf''(z)}{f'(z)}$, $z \in \mathbb{D}$, as $f \in UCV$ if and only if p is in the conic region

$$\Omega = \{\omega \in \mathbb{C} : (\mathrm{Im}\omega)^2 < 2 \operatorname{Re}\omega - 1\}.$$

Using Alexander transform a related class $k - \mathcal{S}_p(\sigma)$ is obtained as $f \in k - UCV(\sigma)$ $\Longleftrightarrow zf' \in k - \mathcal{S}_p(\sigma)$. Results for the condition on the Taylor coefficients of functions in these classes are available in the literature. Among them, we mention the results that serve our purpose.

Lemma 4.2. [4] *A function $f \in \mathcal{A}$ is in $k - UCV(\sigma)$ if it satisfies the condition*

$$\sum_{n=2}^{\infty} n\left[n(1+k) - (k+\sigma)\right]|a_n| \leq 1 - \sigma. \tag{4.2}$$

It was also found that the condition (4.2) is necessary if $f \in \mathcal{A}$ given by (1.1) has $a_n < 0$. Further that the condition

$$\sum_{n=2}^{\infty} \left[n(1+k) - (k+\sigma)\right]|a_n| \leq 1 - \sigma. \tag{4.3}$$

is sufficient for f to be in $k - \mathcal{S}_p(\sigma)$ and turns out to be also necessary if $f \in \mathcal{A}$ given by (1.1) has $a_n < 0$.

Theorem 4.3. *Let $f \in \mathcal{A}$ be defined as in* (1.1)*. Suppose that $a, b \in \mathbb{C}\backslash\{0\}$, $c > |a| + |b| + 1$ be such that, for $k \geq 0$, $0 \leq \sigma < 1$,*
$F(|a|+1, |b|+1; c+1; 1)$

$$(|ab|(1+k) + (1-\sigma)(c - |a| - |b| - 1) \leq c(1-\sigma)\left(1 + \frac{\alpha - 3\gamma}{2|\tau|(1-\beta)}\right). \tag{4.4}$$

Then, for $f \in R^{\tau}_{\gamma,\alpha}(\beta)$, $0 \leq \gamma \leq 1$, $0 \leq \alpha \leq 1$ and $0 \leq \beta < 1$, $H_{a,\,b\,c}(f)(z) \in k - UCV(\sigma)$.

Proof. Let $f \in \mathcal{A}$ be defined as in Theorem 4.3. Considering (4.2), from Lemma 2.2, we need to prove that if $f \in \mathcal{A}$ satisfies (2.5), then

$$\sum_{n=2}^{\infty} n\left(\, n(1+k) - (k+\sigma)\right)|A_n| \leq 1 - \sigma, \tag{4.5}$$

where
$$A_n = \frac{(a,n-1)(b,n-1)}{(c,n-1)(1,n-1)}a_n, \qquad n \geq 2.$$
Since $1+\alpha(n-1)+\gamma(n-1)(n-2) \geq n(\alpha-3\gamma)$ for $0 \leq \gamma \leq 1$ and $n \geq 2$, using $|(a,n)| \leq (|a|,n)$ it is enough if we prove that
$$T := \sum_{n=2}^{\infty} n\frac{(n)(1+k)-(k+\sigma)}{n}\frac{(|a|,n-1)(|b|,n-1)}{(|c|,n-1)(1,n-1)} \leq \frac{(1-\sigma)(\alpha-3\gamma)}{2|\tau|(1-\beta)}.$$
Using $(n+2)(1+k)-(k+\sigma) = (n+1)(1+k)+(1-\sigma)$ and
$$F(a,b;c;1) = \sum_{n=0}^{\infty}\frac{(a,n)(b,n)}{(c,n)(1,n)} = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}, \qquad \mathrm{Re}\,(c-a-b) > 0,$$
we get
$$\begin{aligned} T &= (1+k)\sum_{n=0}^{\infty}(n+1)\frac{(|a|,n+1)(|b|,n+1)}{(c,n+1)(1,n+1)} + (1-\sigma)\sum_{n=0}^{\infty}\frac{(|a|,n+1)(|b|,n+1)}{(c,n+1)(1,n+1)} \\ &= (1+k)\frac{ab}{c}\left(\frac{\Gamma(c-|a|-|b|-1)\Gamma(c+1)}{\Gamma(c-|a|)\Gamma(c-|b|)}\right) + (1-\sigma)\left(\frac{\Gamma(c-|a|-|b|)\Gamma(c)}{\Gamma(c-|a|)\Gamma(c-|b|)} - 1\right) \\ &= \left(\frac{\Gamma(c-|a|-|b|-1)\Gamma(c)}{\Gamma(c-|a|)\Gamma(c-|b|)}\right)\Big(|ab|(1+k)+(1-\sigma)(c-|a|-|b|-1)\Big) - (1-\sigma), \end{aligned}$$
which by using the hypothesis, gives the required result. □

Another sufficient condition for the class $k-UCV$ is also given in [8] by the following result.

Lemma 4.4. [8] *Let $f \in \mathcal{S}$ and has the form (1.1). If for some k, $0 \leq k < \infty$, the inequality*
$$\sum_{n=2}^{\infty} n(n-1)|a_n| \leq \frac{1}{k+2}, \tag{4.6}$$
holds, then $f \in k-UCV$. The number $1/(k+2)$ cannot be increased.

It is interesting to observe that, even though σ is not involved in this sufficient condition, this condition holds for $f \in k-UCV(\sigma)$, by the method of proof given for Lemma 4.4 in [8]. Also that, using the Alexander transform, a result for $f \in k-S_p(\sigma)$ analogous to (4.6) cannot be obtained by replacing a_n by a_n/n as in many other situations.

To compare the results we are interested in giving a theorem equivalent to Theorem 4.3, by using (4.6) instead of (4.2). Since σ is not involved in (4.6), we present this result for the case $\sigma = 0$ only. The proof of this theorem is similar to Theorem 4.3 and we omit details.

Theorem 4.5. *Let $f \in \mathcal{A}$ be defined as in (1.1). Suppose that $a, b \in \mathbb{C}\backslash\{0\}$, $c > |a|+|b|+1$ be such that, for $k \geq 0$, $0 \leq \alpha < 1$,*
$$F(|a|+1,|b|+1;c+1;1)\frac{|ab|}{c} \leq \frac{(\alpha-3\gamma)}{2|\tau|(1-\beta)(k+2)}. \tag{4.7}$$

Then, for $f \in R^{\tau}_{\gamma,\alpha}(\beta)$, $0 \le \gamma \le 1$,$0 \le \alpha \le 1$ *and* $0 \le \beta < 1$, $H_{a,\,b,\,c}(f)(z) \in k\!-\!UCV$.

If we let $a = \bar{b}$ in $F(a, b; c; z)$ we get polynomials with positive coefficients when b is some negative integer. Hence the above Theorems are useful in characterizing convex polynomials and we give the corresponding results independently.

Corollary 4.6. *Let* $f \in \mathcal{A}$ *be defined as in* (1.1)*. Suppose that* $b > 0$, $c > 2\mathrm{Re}b + 1$ *and* b, c *satisfy*

$$\begin{aligned} F(b+1, \bar{b}+1; c+1; 1)(|b|^2(1+k) + (1-\sigma)(c - 2\mathrm{Re}b - 1) \\ \le c(1-\sigma)\left(1 + \frac{\alpha - 3\gamma}{2|\tau|(1-\beta)}\right). \end{aligned} \tag{4.8}$$

Then, for $f \in R^{\tau}_{\gamma,\alpha}(\beta)$, $0 \le \gamma \le 1$, $0 \le \alpha \le 1$ *and* $0 \le \beta < 1$, $H_{\bar{b},\,b\,c}(f)(z) \in k - UCV(\sigma)$.

Corollary 4.7. *Let* $f \in \mathcal{A}$ *be defined as in* (1.1)*. Suppose that* $b > 0$, $c > 2\mathrm{Re}b + 1$ *be such that*

$$F(b+1, \bar{b}+1; c+1; 1)\frac{|b|^2}{c} \le \frac{(\alpha - 3\gamma)}{2|\tau|(1-\beta)(k+2)}. \tag{4.9}$$

Then, for $f \in R^{\tau}_{\gamma,\alpha}(\beta)$, $0 \le \gamma \le 1$,$0 \le \alpha \le 1$ *and* $0 \le \beta < 1$, $H_{\bar{b},\,b,\,c}(f)(z) \in k\!-\!UCV$ *where* $k \ge 0$.

The Hohlov operator $H_{a,\,b,\,c}(f)(z)$ reduces to the Carlson-Shaffer operator $\mathcal{L}(b, c)(f)(z)$ if $a = 1$. Hence we give the statement of the following results.

Corollary 4.8. *Let* $f \in \mathcal{A}$ *be defined as in* (1.1)*. Suppose that* $b > 0$, $c > b+2$ *are such that, for* $k \ge 0$, $0 \le \sigma < 1$ *and*

$$\frac{(c-1)}{(c-b-1)(c-b-2)}(|b|(1+k) + (1-\sigma)(c-b-2) \le (1-\sigma)\left(1 + \frac{\alpha - 3\gamma}{2|\tau|(1-\beta)}\right). \tag{4.10}$$

Then, for $f \in R^{\tau}_{\gamma,\alpha}(\beta)$, $0 \le \gamma \le 1$, $0 \le \alpha \le 1$ *and* $0 \le \beta < 1$, $\mathcal{L}(b, c)(f)(z) \in k - UCV(\sigma)$.

Corollary 4.9. *Let* $f \in \mathcal{A}$ *be defined as in* (1.1)*. Suppose that* $b > 0$, $c > b+2$ *are such that, for* $k \ge 0$, $0 \le \sigma < 1$ *and*

$$\frac{(\alpha - 3\gamma)}{2|\tau|(1-\beta)(k+2)}\Big(\, (c-1)^2 + (2b+1)(c-1) + b(b+1)\Big) - b(c-1) \ge 0. \tag{4.11}$$

Then, for $f \in R^{\tau}_{\gamma,\alpha}(\beta)$, $0 \le \gamma \le 1$, $0 \le \alpha \le 1$ *and* $0 \le \beta < 1$, $\mathcal{L}(b, c)(f)(z) \in k\!-\!UCV$.

Let $\mathcal{S}^*_{\lambda}$ $(\lambda > 0)$, denotes the class of functions in $\mathcal{S}$ such that $\left|\dfrac{zf'(z)}{f(z)} - 1\right| < \lambda$.

A sufficient condition for $f \in \mathcal{A}$ of the form (1.1) to be in $\mathcal{S}^*_1 \subset \mathcal{S}^*$, is given by $\displaystyle\sum_{n=2}^{\infty} n|a_n| \le 1$, and is proved by many authors. For example, see [6]. A particular extension of this, due to [13], is

$$\sum_{n=2}^{\infty}(n+\lambda-1)|a_n| \leq \lambda \Longrightarrow f \in \mathcal{S}^*_\lambda. \tag{4.12}$$

Theorem 4.10. *Let $a, b > 0$ or $a \in \mathbb{C}\backslash\{0\}$ with $a = \bar{b}$. Further, let $|a| \neq 1$, $|b| \neq 1$, and $0 \neq c > a + b$ be such that*

$$F(a,b;c;1)\left(1+\frac{(\lambda-1)(c-|a|-|b|)}{(|a|-1)(|b|-1)}\right) \leq \frac{(\lambda-1)(c-1)}{(|a|-1)(|b|-1)} + \lambda\left(1+\frac{(\alpha-3\gamma)}{2|\tau|(1-\beta)}\right). \tag{4.13}$$

Suppose that $f \in \mathcal{A}$ be defined as in (1.1). Then, for $f \in R^\tau_{\gamma,\alpha}(\beta)$, $0 \leq \gamma \leq 1$, $0 \leq \alpha \leq 1$, $0 \leq \beta < 1$, and $\lambda > 0$, $H_{a,\,b,\,c}(f)(z) \in \mathcal{S}^*_\lambda$.

Proof. Let $f(z)$ be of the form (1.1). In view of (4.12), it suffices to prove that

$$\sum_{n=2}^{\infty}(n+\lambda-1)|A_n| \leq \lambda, \tag{4.14}$$

where

$$A_n = \frac{(a,n-1)(b,n-1)}{(c,n-1)(1,n-1)}\, a_n, \qquad n \geq 2.$$

Since $f \in R^\tau_{\gamma,\alpha}(\beta)$, using (2.5) and $1+\alpha(n-1)+\gamma(n-1)(n-2) \geq n(\alpha-3\gamma)$, we need only to show that

$$T := \sum_{n=2}^{\infty}\frac{(|a|,n-1)(|b|,n-1)}{(c,n-1)(1,n-1)} + (\lambda-1)\sum_{n=2}^{\infty}\frac{(|a|,n-1)(|b|,n-1)}{(c,n-1)(1,n)} \leq \frac{\lambda\,(\alpha-3\gamma)}{2|\tau|(1-\beta)}.$$

But this last inequality is true by the hypothesis of the theorem and (2.3). □

5. Inclusion properties of $K^p_a[f](z)$

Theorem 5.1. *Let $f \in \mathcal{A}$ be as in (1.1). Suppose $a > -1, p \geq 0$ and*

$$\sum_{n=2}^{\infty}[n(1+k)-(k+\sigma)]B_n(a,p) \leq \frac{(1-\sigma)(\alpha-3\gamma)}{2|\tau|(1-\beta)}, \tag{5.1}$$

where $B_n(a,p) = \dfrac{(1+a)^p}{(n+a)^p}$. Then for $f \in \mathcal{R}^\tau_{\gamma,\alpha}(\beta), 0 \leq \gamma \leq 1, 0 \leq \alpha \leq 1$ and $0 \leq \beta < 1$, we have $K^p_a[f](z) \in k - UCV(\sigma)$.

Proof. Since $f \in \mathcal{R}^\tau_{\gamma,\alpha}(\beta)$, we have from Lemma 2.2 and the fact

$$1+\alpha(n-1)+\gamma(n-1)(n-2) \geq n(\alpha-3\gamma), n \geq 2$$

that

$$|a_n| \leq \frac{2|\tau|(1-\beta)}{n(\alpha-3\gamma)}.$$

Now using Lemma 4.2, it is enough to show that

$$\sum_{n=2}^{\infty} n\left[n(1+k)-(k+\sigma)\right]|A_n| \leq 1-\sigma,$$

where $A_n = B_n(a,p)a_n$. Clearly, the above inequality is true if (5.1) holds. $\square$

It is easy to see that, for all $n \geq 2$,

$$B_n(a,p) = \frac{(1+a)^p}{(n+a)^p} < 1, \qquad a > -1, \quad p \geq 0$$

which leads to

Corollary 5.2. *Let $f \in \mathcal{A}$ be as in* (1.1). *Suppose $a > -1, p \geq 0$ and*

$$\sum_{n=2}^{\infty}[n(1+k)-(k+\sigma)] \leq \frac{(1-\sigma)(\alpha-3\gamma)}{2|\tau|(1-\beta)}.$$

Then for $f \in \mathcal{R}^{\tau}_{\gamma,\alpha}(\beta)$, $0 \leq \gamma \leq 1$, $0 \leq \alpha \leq 1$ and $0 \leq \beta < 1$, we have $K^p_a[f](z) \in k-UCV(\sigma)$.

Theorem 5.3. *Let $p \geq 0$, $a > -1$ and $f \in \mathcal{A}$ be as in* (1.1). *Suppose that*

$$\sum_{n=2}^{\infty}[n+\lambda-1]\frac{B_n(a,p)}{n} \leq \frac{\lambda(\alpha-3\gamma)}{2|\tau|(1-\beta)}, \tag{5.2}$$

where $B_n(a,p) = \dfrac{(1+a)^p}{(n+a)^p}$. Then for $f \in \mathcal{R}^{\tau}_{\gamma,\alpha}(\beta), 0 \leq \gamma \leq 1, 0 \leq \alpha \leq 1$ and $0 \leq \beta < 1$, we have $K^p_a[f](z) \in \mathcal{S}^*_\lambda$.

Proof. Since $f \in \mathcal{R}^{\tau}_{\gamma,\alpha}(\beta)$, Lemma 2.2 gives

$$|a_n| \leq \frac{2|\tau|(1-\beta)}{1+\alpha(n-1)+\gamma(n-1)(n-2)}.$$

Using the fact that $1+\alpha(n-1)+\gamma(n-1)(n-2) \geq n(\alpha-3\gamma), n \geq 2$, we obtain

$$|a_n| \leq \frac{2|\tau|(1-\beta)}{n(\alpha-3\gamma)}. \tag{5.3}$$

Now $K^p_a[f](z) \in \mathcal{S}^*_\lambda$ if

$$\begin{aligned}
&\sum_{n=2}^{\infty}[n+\lambda-1]\left|\frac{(1+a)^p}{(n+a)^p}a_n\right| \leq \lambda \\
\Longrightarrow \quad &\sum_{n=2}^{\infty}[n+\lambda-1]\frac{(1+a)^p}{(n+a)^p}|a_n| \leq \lambda \\
\Longrightarrow \quad &\sum_{n=2}^{\infty}[n+\lambda-1]\frac{(1+a)^p}{(n+a)^p}\frac{2|\tau|(1-\beta)}{n(\alpha-3\gamma)} \leq \lambda, \qquad \text{using (5.3)} \\
\Longrightarrow \quad &\sum_{n=2}^{\infty}[n+\lambda-1]\frac{(1+a)^p}{(n+a)^p}\frac{1}{n} \leq \frac{\lambda(\alpha-3\gamma)}{2|\tau|(1-\beta)},
\end{aligned}$$

which is the hypothesis and the proof is complete. □

6. Concluding remarks

Remark 6.1. If $k = 0$ then it is clear from the analytic characterization that $k - UCV(\sigma)$ reduces to the class of Convex functions of order σ, denoted by $\mathcal{C}(\sigma)$. Similarly, (using Alexander transform), $k - \mathcal{S}_p(\sigma)$ reduces to the class of Starlike functions of order σ, $(\mathcal{S}^*(\sigma))$. For results regarding to these classes we refer to [6]. Further results on the restriction $k = 0$ can be found in the literature, e.g. see [8].

Remark 6.2. We note that Theorem 4.3 and Theorem 4.5 are not sharp. In particular, for a, b real with $\eta = 0$, $k = 0$ and $\sigma = 0$, we get from (2.5),

$$F(|a|+1, |b|+1; c+1; 1)\frac{|ab|}{c} + (c - |a| - |b| - 1) \leq 1 + \frac{\alpha - 3\gamma}{2(1-\beta)}. \tag{6.1}$$

This inequality for $\alpha = 1$ and $\gamma = 0$ further reduces to

$$F(|a|+1, |b|+1; c+1; 1)\frac{|ab|}{c} + (c - |a| - |b| - 1) \leq 1 + \frac{1}{2(1-\beta)}. \tag{6.2}$$

Similarly, (4.7) reduces to

$$F(|a|+1, |b|+1; c+1; 1)\frac{|ab|}{c} \leq \frac{1}{4(1-\beta)}. \tag{6.3}$$

From (6.2) and (6.3), it is easy to see that Theorem 4.5 is better for all c lying between $|a| + |b| + 1$ and $|a| + |b| + \frac{3}{2}$ and for all other values of c satisfying $c > |a| + |b| + \frac{3}{2}$, Theorem 4.3 is better.

Note that, in Theorem 4.10, $|a| \neq 1$ and $|b| \neq 1$. Hence Theorem 4.10 cannot be reduced to the important transforms such as Carlson-Schaffer integral operator, which leads to the following.

Problem 6.3. To find conditions on b and c such that the Carlson-Schaffer operator $\mathcal{L}(b, c)(f)(z)$ maps the class $R^{\tau}_{\gamma,\alpha}(\beta)$ onto S^*_{λ}.

Note that, for $p = 1$, the results given in Section 5 for the Komatu operator $K^p_a[f](z)$ reduce to the results for the Bernardi integral operator and coincide with the results of Section 4 for particular values of a, b and c. However, for no values of p or a, the Komatu operator $K^p_a[f](z)$ can be reduced to the Carlson-Schaffer operator $\mathcal{L}(b, c)(f)(z)$. Hence Problem 6.3 gains further significance.

References

[1] Ali, R.M., Badghaish, A.O., Ravichandran, V., Swaminathan, A., *Starlikeness of integral transforms and duality*, J. Math. Anal. Appl., **385**(2012), no. 2, 808-822.

[2] Ali, R.M., Lee, S.K., Subramanian, K.G., Swaminathan, A., *A third-order differential equation and starlikeness of a double integral operator*, Abstr. Appl. Anal., Article in press, 10 pages, DOI: 10.1155/2011/901235.

[3] Ali, R.M., Nargesi, M.M., Ravichandran, V., *Convexity of integral transforms and duality*, Complex Variables and Elliptic Equations: An International Journal, DOI: 10.1080/17476933.2012.693483.

[4] Bharati, R., Parvatham, R., Swaminathan, A., *On subclasses of uniformly convex functions and corresponding class of starlike functions*, Tamkang J. Math., **28**(1997), no. 1, 17-32.

[5] Fournier, R., Ruscheweyh, S., *On two extremal problems related to univalent functions*, Rocky Mountain J. Math., **24**(1994), no. 2, 529-538.

[6] Goodman, A.W., *Univalent functions and nonanalytic curves*, Proc. Amer. Math. Soc., **8**(1957), 598-601.

[7] Goodman, A.W., *On uniformly convex functions*, Ann. Polon. Math., **56**(1991), no. 1, 87-92.

[8] Kanas, S., Wisniowska, A., *Conic regions and k-uniform convexity*, Continued fractions and geometric function theory (CONFUN) (Trondheim, 1997), J. Comput. Appl. Math., **105**(1999), no. 1-2, 327-336.

[9] Komatu, Y., *On a family of integral operators related to fractional calculus*, Kodai Math. J., **10**(1987), no. 1, 20-38.

[10] Li, J.L., *On some classes of analytic functions*, Math. Japon., **40**(1994), no. 3, 523-529.

[11] Rainville, E., *Special Functions*, The Macmillan Co., New York 1960, xii+365 pp.

[12] Ronning, F., *Uniformly convex functions and a corresponding class of starlike functions*, Proc. Amer. Math. Soc., **118**(1993), no. 1, 189-196.

[13] Silverman, H., *Univalent functions with negative coefficients*, Proc. Amer. Math. Soc., **51**(1975), 109-116.

[14] Swaminathan, A., *Inclusion theorems of convolution operators associated with normalized hypergeometric functions*, J. Comput. Appl. Math., **197**(2006), no. 1, 15-28.

[15] Swaminathan, A., *Convexity of the incomplete beta functions*, Integral Transforms Spec. Funct., **18**(2007), no. 7-8, 521-528.

[16] Swaminathan, A., *Sufficient conditions for hypergeometric functions to be in a certain class of analytic functions*, Comput. Math. Appl., **59**(2010), no. 4, 1578-1583.

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Existence and multiplicity of solutions to the Navier boundary value problem for a class of $(p(x), q(x))$-biharmonic systems

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Abstract. In this article, we study the following problem with Navier boundary conditions.

$$\begin{cases} \Delta(a(x, \Delta u)) = F_u(x, u, v), & \text{in } \Omega \\ \Delta(a(x, \Delta v)) = F_v(x, u, v), & \text{in } \Omega, \\ u = v = \Delta u = \Delta v = 0 & \text{on } \partial\Omega. \end{cases}$$

By using the Mountain Pass Theorem and the Fountain Theorem, we establish the existence of weak solutions of this problem.

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1. Introduction

In recent years, the study of differential equations and variational problems with $p(x)$-growth conditions was an interesting topic, which arises from nonlinear electrorheological fluids and elastic mechanics. In that context we refer the reader to Ruzicka [15], Zhikov [20] and the reference therein; see also [4, 7, 8, 5].

Fourth-order equations appears in many context. Some of theses problems come from different areas of applied mathematics and physics such as Micro Electro-Mechanical systems, surface diffusion on solids, flow in Hele-Shaw cells (see [10]). In addition, this type of equations can describe the static from change of beam or the sport of rigid body.

In [1] the authors studied a class of $p(x)$-biharmonic of the form

$$\Delta(|\Delta u|^{p(x)-2}\Delta u) = \lambda|u|^{q(x)-2}u \quad \text{in } \Omega,$$
$$u = \Delta u = 0 \quad \text{on } \partial\Omega,$$

where Ω is a bounded domain in $\mathbb{R}^N$, with smooth boundary $\partial\Omega$, $N \geq 1$, $\lambda \geq 0$.

In [3], A. El Amrouss and A. Ourraoui considered the below problem and using variational methods, by the assumptions on the Carathéodory function f, they establish the existence of Three solutions the problem of the form

$$\Delta(|\Delta u|^{p(x)-2}\Delta u) + a(x)|u|^{p(x)-2}u = f(x,u) + \lambda g(x,u) \quad \text{in } \Omega,$$
$$Bu = Tu = 0 \quad \text{on } \partial\Omega.$$

Inspired by the above references, the work of L. Li [11]and [14], the aim of this article is to study the existence and multiplicity of weak solutions for $(p(x), q(x))-$biharmonic type system

$$\begin{cases} \Delta(a(x,\Delta u)) = F_u(x,u,v), & \text{in } \Omega \\ \Delta(a(x,\Delta v)) = F_v(x,u,v), & \text{in } \Omega, \\ u = \Delta u = 0, v = \Delta v = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$, $N \geq 1$,

$$\Delta^2_{p(x)} u := \Delta(|\Delta u|^{p(x)-2}\Delta u),$$

is the $p(x)$-biharmonic operator, p,q are continuous functions on $\overline{\Omega}$ with

$$\inf_{x\in\overline{\Omega}} p(x) > \max\left\{1, \frac{N}{2}\right\}, \ \inf_{x\in\overline{\Omega}} q(x) > \max\left\{1, \frac{N}{2}\right\}$$

and $F : \Omega \times \mathbb{R}^2 \to \mathbb{R}$ is a function such that $F(.,s,t)$ is continuous in $\overline{\Omega}$, for all $(s,t) \in \mathbb{R}^2$, $F(x,.,.)$ is C^1 in $\mathbb{R}^2$ for every $x \in \Omega$, and F_u, F_v denote the partial derivative of F, with respect to u, v respectively such that

(F_1) For all $(x,s,t) \in \Omega \times \mathbb{R}^2$, we assume

$$\lim_{|s|\to 0} \frac{F_s(x,s,t)}{|s|^{p(x)-1}} = 0, \lim_{|t|\to 0} \frac{F_t(x,s,t)}{|s|^{q(x)-1}} = 0.$$

(F_2) For all $(x,s,t) \in \Omega \times \mathbb{R}^2$, we assume

$$F(x,s,t) = o(|s|^{p(x)-1} + |t|^{q(x)-1}) \ as \ |(s,t)| \to \infty.$$

(F_3) There exists $\underline{u} > 0$, $\underline{v} > 0$ such that $F(x,\underline{u},\underline{v}) > 0$ for a.e $x \in \Omega$

(F_4) There exist $\lambda > 0$ such that $F(x,s,t) \geq \lambda(|s|^{\alpha(x)} - |t|^{\beta(x)})$ for all $(s,t) \in \mathbb{R}^2$, with

$$\alpha^- > r^+, \ 1 < \beta^- \leq \beta^+ < r^-.$$

(F_5) For all $(x,s,t) \in \Omega \times \mathbb{R}^2$ $F(x,-s,-t) = -F(x,s,t)$.

Let $a : \Omega \times \mathbb{R}^N \to \mathbb{R}^N$ to be a continuous potential derivative with respect to ξ of the mapping $A : \overline{\Omega} \times \mathbb{R}^N \to \mathbb{R}^N$ where $a = DA = A^{'}$, with the assumption as below

(A_1) $A(x,0) = 0$, for all $x \in \Omega$.

(A_2) $a(x,\xi) \leq C_1(1 + |\xi|^{r(x)-1})$, $C_1 > 0$ and $r^- > p^+, r^- > q^+$.

(A_3) A is $r(x)$-uniformly convex: there exists a constant $k > 0$ such that

$$A\left(x, \frac{\xi+\eta}{2}\right) \leq \frac{1}{2}A(x,\xi) + \frac{1}{2}A(x,\eta) - k|\xi-\eta|^{r(x)},$$

for all $x \in \Omega, \quad \xi, \eta \in \mathbb{R}^N$.

(A_4) A is $r(x)$-subhomogenuous, for all $(x,\xi) \in \Omega \times \mathbb{R}^N$,

$$|\xi|^{r(x)} \leq a(x,\xi) \leq r(x)A(x,\xi).$$

(A_5) For all $(x,s) \in \Omega \times \mathbb{R}^N$ $a(x,-s) = -a(x,s)$.

The main results of this paper are the following theorems.

Theorem 1.1. *Assume that* $(A_1)-(A_4)$ *and* $(F_1)-(F_3)$ *hold. Then the problem* (1.1) *has two weak solutions.*

Theorem 1.2. *Assume that* $(A_1)-(A_5)$ *and* $(F_1)-(F_5)$ *hold. Then the problem* (1.1) *has a sequence of weak solutions such that* $\phi(\pm(u_k, v_k)) \to +\infty$, *as* $k \to +\infty$ *with* ϕ *is a energy associated of the problem* (1.1) *defined in* (2.2).

This paper is organized as three sections. In Section 2, we recall some basic properties of the variable exponent Lebegue-Sobolev spaces. In Section3 we give the proof of main results.

2. Preliminaries

To study $p(x)$)-Laplacian problems, we need some results on the spaces $L^{p(x))}(\Omega)$ and $W^{k,p(x))}(\Omega)$, and properties of $p(x)$)-Laplacian, which we use later. Let Ω be a bounded domain of $\mathbb{R}^N$, denote

$$C_+(\overline{\Omega}) = \{h(x); h(x) \in C(\overline{\Omega}), h(x) > 1, \forall x \in \overline{\Omega}\}.$$

For any $h \in C_+(\overline{\Omega})$, we define

$$h^+ = \max\{h(x);\ x \in \overline{\Omega}\}, \quad h^- = \min\{h(x);\ x \in \overline{\Omega}\};$$

For any $p \in C_+(\overline{\Omega})$, we define the *variable exponent Lebesgue space*

$$L^{p(x))}(\Omega) = \Big\{u; u \text{ is a measurable real-valued function such that} \\ \int_\Omega |u(x)|^{p(x))}dx < \infty\Big\},$$

endowed with the so-called *Luxemburg norm*

$$|u|_{p(x))} = \inf\left\{\mu > 0;\ \int_\Omega \left|\frac{u(x)}{\mu}\right|^{p(x))} dx \leq 1\right\}.$$

Then $(L^{p(x))}(\Omega), |\cdot|_{p(x))})$ becomes a Banach space.

Proposition 2.1 ([9]). *The space $(L^{p(x))}(\Omega), |\cdot|_{p(x))})$ is separable, uniformly convex, reflexive and its conjugate space is $L^{q(x)}(\Omega)$ where $q(x)$ is the conjugate function of $p(x))$, i.e.,*

$$\frac{1}{p(x))} + \frac{1}{q(x)} = 1,$$

for all $x \in \Omega$. For $u \in L^{p(x))}(\Omega)$ and $v \in L^{q(x)}(\Omega)$, we have

$$\left|\int_\Omega uvdx\right| \leq \Big(\frac{1}{p^-} + \frac{1}{q^-}\Big)|u|_{p(x))}|v|_{q(x)} \leq 2|u|_{p(x))}|v|_{q(x)}.$$

The Sobolev space with variable exponent $W^{k,p(x))}(\Omega)$ is defined as

$$W^{k,p(x))}(\Omega) = \{u \in L^{p(x))}(\Omega) : D^\alpha u \in L^{p(x))}(\Omega), |\alpha| \leq k\},$$

where

$$D^\alpha u = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \dots \partial x_N^{\alpha_N}} u,$$

with $\alpha = (\alpha_1, \dots, \alpha_N)$ is a multi-index and $|\alpha| = \sum_{i=1}^N \alpha_i$. The space $W^{k,p(x))}(\Omega)$ equipped with the norm

$$\|u\|_{k,p(x))} = \sum_{|\alpha|\leq k} |D^\alpha u|_{p(x))},$$

also becomes a separable and reflexive Banach space. For more details, we refer the reader to [6, 9, 13]. Denote

$$p_k^*(x) = \begin{cases} \frac{Np(x)}{N-kp(x)} & \text{if } kp(x) < N, \\ +\infty & \text{if } kp(x) \geq N \end{cases}$$

for any $x \in \overline{\Omega}$, $k \geq 1$.

Proposition 2.2 ([9]). *For $p, r \in C_+(\overline{\Omega})$ such that $r(x) \leq p_k^*(x)$ for all $x \in \overline{\Omega}$, there is a continuous embedding*

$$W^{k,p(x))}(\Omega) \hookrightarrow L^{r(x)}(\Omega).$$

If we replace $\leq$ with $<$, the embedding is compact.

We denote by $W_0^{k,p(x)}(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $W^{k,p(x)}(\Omega)$. Then the function space $\left(\left(W^{2,p(x)}(\Omega) \cap W_0^{1,p(x)}(\Omega)\right), \|u\|_{p(x)}\right)$ is a separable and reflexive Banach space, where

$$\|u\|_{p(x)} = \inf\Big\{\mu > 0 : \int_\Omega \Big(\Big|\frac{\Delta u(x)}{\mu}\Big|^{p(x)} \leq 1\Big\}.$$

Remark 2.3. According to [[18] Theorem 4.4.], the norm $\|\cdot\|_{2,p(x)}$ is equivalent to the norm $\|\cdot\|_{p(x)}$ in the space X. Consequently, the norms $\|\cdot\|_{2,p(x)}$, $\|\cdot\|$ and $\|\cdot\|_{p(x)}$ are equivalent.

Proposition 2.4 ([2]). *If we denote $\rho(u) = \int_\Omega |\Delta u|^{p(x)}dx$, then for $u, u_n \in X$, we have*

(1) $\|u\|_p < 1$ *(respectively=1; > 1)* $\iff \rho(u) < 1$ *(respectively $= 1$; > 1);*

(2) $\|u\|_p \le 1 \Rightarrow \|u\|_p^{p^+} \le \rho(u) \le \|u\|_p^{p^-}$;
(3) $\|u\|_p \ge 1 \Rightarrow \|u\|_p^{p^-} \le \rho(u) \le \|u\|_p^{p^+}$;
(4) $\|u_n\|_p \to 0$ *(respectively* $\to \infty$*)* $\Longleftrightarrow$ $\rho(u_n) \to 0$ *(respectively* $\to \infty$*).*

Note that the weak solutions of problem (1.1) are considered in the generalized Sobolev space

$$X = \left(W^{2,p(x)}(\Omega) \cap W_0^{1,p(x)}(\Omega)\right) \times \left(W^{2,q(x)}(\Omega) \cap W_0^{1,q(x)}(\Omega)\right)$$

equipped with the norm

$$\|(u,v)\| = \max\{\|u\|_{p(x)}, \|u\|_{q(x)}\}.$$

Remark 2.5 (see [19]**).** As the Sobolev space X is a reflexive and separable Banach space, there exist $(e_n)_{n\in\mathbb{N}^*} \subseteq X$ and $(f_n)_{n\in\mathbb{N}^*} \subseteq X^*$ such that $f_n(e_l) = \delta_{nl}$ for any $n, l \in \mathbb{N}^*$ and

$$X = \overline{\operatorname{span}\{e_n : n \in \mathbb{N}^*\}}, \quad X^* = \overline{\operatorname{span}\{f_n : n \in \mathbb{N}^*\}}^{w^*}.$$

For $k \in \mathbb{N}^*$, denote by

$$X_k = \operatorname{span}\{e_k\},\ Y_k = \oplus_{j=1}^k X_j, \quad Z_k = \overline{\oplus_k^\infty X_j}.$$

For every $m > 1$, $u, v \in L^m(\Omega)$, we define

$$|(u,v)|_m := \max\{|u|_m, |v|_m\}.$$

Lemma 2.6 (See [8]**).** *Define*

$$\beta_k := \sup\{|(u,v)|_m; \|(u,v)\| = 1, (u,v) \in Z_k\},$$

where $m := \max\limits_{x\in\overline{\Omega}}(p(x), q(x))$*. Then, we have*

$$\lim_{k\to\infty} \beta_k = 0.$$

2.1. Existence and multiplicity of weak solutions

Definition 2.7. We say that $(u,v) \in X$ is weak solution of (1.1) if

$$\int_\Omega a(x,\Delta u)\Delta\varphi dx + \int_\Omega a(x,\Delta v)\Delta\varphi dx = \int_\Omega F_u(x,u,v)\varphi dx + \int_\Omega F_v(x,u,v)\varphi dx, \tag{2.1}$$

for all $\varphi \in X$.

The functional associated to (1.1) is given by

$$\phi(u,v) = \int_\Omega A(x,\Delta u)dx + \int_\Omega A(x,\Delta v)dx - \int_\Omega F(x,u,v)dx. \tag{2.2}$$

It should be noticed that under the condition $(F_1) - (F_2)$ the functional ϕ is of class $C^1(X,\mathbb{R})$ and

$$\begin{aligned} \phi^{'}(u,v)(\psi,\varphi) = &\int_\Omega a(x,\Delta u)\Delta\psi dx + \int_\Omega a(x,\Delta v)\Delta\varphi dx \\ &- \int_\Omega F_u(x,u,v)\psi dx - \int_\Omega F_v(x,u,v)\varphi dx,\ \forall(\psi,\varphi) \in X. \end{aligned} \tag{2.3}$$

Then, we know that the weak solution of (1.1) corresponds to critical point of the functional ϕ.

Definition 2.8. We say that

(1) The C^1-functional ϕ satisfies the Palais-Smale condition (in short (PS) condition) if any sequence $(u_n)_{n\in\mathbb{N}} \subseteq X$ for which, $(\phi(u_n))_{n\in\mathbb{N}} \subseteq \mathbb{R}$ is bounded and $\phi^{'}(u_n) \to 0$ as $n \to \infty$, has a convergent subsequence.
(2) The C^1-functional ϕ satisfies the Palais-Smale condition at the level c (in short $(PS)_c$ condition) for $c \in \mathbb{R}$ if any sequence $(u_n)_{n\in\mathbb{N}} \subseteq X$ for which, $\phi(u_n) \to c$ and $\phi^{'}(u_n) \to 0$ as $n \to \infty$, has a convergent subsequence.
(3) The C^1-functional ϕ satisfies the $(PS)^*_c$ condition for $c \in \mathbb{R}$ if any sequence $(u_n)_{n\in\mathbb{N}} \subseteq X$ for which, $u_n \in Y_n$ for each $n \in \mathbb{N}$, $\phi(u_n) \to c$ and $\phi^{'}_{|Y_n})(u_n) \to 0$ as $n \to \infty$ with $Y_n,\ n \in \mathbb{N}$ as defined in Remark 2.5, has a subsequence convergent to a critical point of ϕ.

Remark 2.9. It is easy to see that if ϕ satisfies the (PS) condition, then ϕ satisfies the $(PS)_c$ condition for every $c \in \mathbb{R}$.

Proof of Theorem 1.1. To prove Theorem 1.1, we shall use the Mountain Pass theorem [16]. We first start with the following lemmas.

Lemma 2.10. *Under the assumptions* (F_1)-(F_3) *and* (A_1)-(A_3) ϕ *is sequentially weakly lower semi continuous and coercive .*

Proof. By (F_1)-(F_2), we see that

$$|F(x,s,t)| \leq C_3(1 + |s|^{p(x)} + |t|^{q(x)}),\ \forall (s,t) \in \mathbb{R}^2. \tag{2.4}$$

By the compact embeddings

$$X \hookrightarrow L^{p(x)}(\Omega),\ X \hookrightarrow L^{q(x)}(\Omega),$$

we deduce that $w \mapsto \int_\Omega F(x,w)dx$ is sequentially lower semi continuous $\forall w \in \mathbb{R}^2$. Since

$$w \mapsto \int_\Omega A(x,\Delta u)dx + \int_\Omega A(x,\Delta v)dx$$

is convex uniformly, so it is sequentially lower semi continuous.
Now we prove that ϕ is coercive. From (F_2) for ε small enough, there exist $\delta > 0$ such that

$$\mid F(x,s,t) \mid \leq \varepsilon(|s|^{p(x)} + |t|^{q(x)}),\ \text{for } |(s,t)| > \delta,$$

and thus we have

$$\mid F(x,s,t) \mid \leq \varepsilon(|s|^{p(x)} + |t|^{q(x)}) + \max_{|(s,t)|\leq\delta} \mid F(x,s,t) \mid ||(s,t)|, \forall (s,t) \in \mathbb{R}^2,$$

for a.e $x \in \Omega$. Consequently, for $\|(u,v)\| > 1$ we obtain

$$\begin{aligned}
\phi(u,v) &\geq \int_\Omega A(x,\Delta u)dx + \int_\Omega A(x,\Delta v)dx \\
&- \varepsilon \int_\Omega |u|^{p(x)}dx - \varepsilon \int_\Omega |v|^{q(x)}dx - \max_{|(u,v)|\leq\delta} \mid F(x,u,v) \mid \int_\Omega |(u,v)|dx \\
&\geq \int_\Omega \frac{1}{r(x)}|\Delta u|^{r(x)}dx + \int_\Omega \frac{1}{r(x)}|\Delta v|^{r(x)}dx \\
&- C\varepsilon \int_\Omega |u|^{p(x)}dx - C\varepsilon \int_\Omega |v|^{q(x)})dx - \max_{|(u,v)|\leq\delta} \mid F(x,u,v) \mid \int_\Omega |(u,v)|dx \\
&\geq \frac{1}{r^+} \max\left(\|u\|^{r^-}_{r(x)}, \|v\|^{r^-}_{r(x)}\right) - 2C\varepsilon \max\left(\|u\|^{p^+}_{p(x)}, \|v\|^{q^+}_{q(x)}\right) \\
&\quad -C\varepsilon|\Omega| \max_{|(u,v)|\leq\delta} \mid F(x,u,v) \mid \max\left(\|u\|^{p^+}_{p(x)}, \|v\|^{q^+}_{q(x)}\right).
\end{aligned}$$

Therefore, ϕ is coercive and has a global minimizer $(\overline{u_1}, \overline{v_1})$ which is a nontrivial because by (F_3)

$$\phi(\overline{u_1}, \overline{v_1}) \leq \phi(\underline{u}, \underline{v}) < 0.$$

Lemma 2.11. *Under the assumptions* (F_1)-(F_3) *and* (A_1)-(A_4)*. Then* ϕ *satisfies the Palais-smale condition.*

Proof. Let $w_n = (u_n, v_n) \subset X$ be a Palais-smale sequence, then

$$\phi^{'}(w_n) \to 0 \text{ in } X^*, \ \ \phi(w_n) \to l \in \mathbb{R}.$$

We show that (w_n) is bounded. By (A_5) we have

$$\begin{aligned}
\phi(w_n) &= \int_\Omega A(x,\Delta u_n)dx + \int_\Omega A(x,\Delta v_n)dx - \int_\Omega F(x,u_n,v_n)dx \\
&\geq \int_\Omega \frac{1}{r(x)}|\Delta u_n|^{r(x)}dx + \int_\Omega \frac{1}{r(x)}|\Delta v_n|^{r(x)}dx - \int_\Omega F(x,u_n,v_n)dx,
\end{aligned}$$

and we get

$$\begin{aligned}
&\phi^{'}(u_n,v_n)(u_n,v_n) \\
&= \int_\Omega a(x,\Delta u_n)\Delta u_n dx + \int_\Omega a(x,\Delta v_n)\Delta v_n dx \\
&- \int_\Omega F_{u_n}(x,u_n,v_n)u_n dx - \int_\Omega F_{v_n}(x,u_n,v_n)v_n dx \\
&\leq \int_\Omega r(x)A(x,\Delta u_n)dx + \int_\Omega r(x)A(x,\Delta v_n)dx \\
&- \int_\Omega F_{u_n}(x,u_n,v_n)u_n dx - \int_\Omega F_{v_n}(x,u_n,v_n)v_n dx.
\end{aligned}$$

Using the fact that $F_s, F_t \in C(\Omega \times \mathbb{R}^2, \mathbb{R})$ and with $(F_1)-(F_2)$, for $\varepsilon > 0$ there exists $\delta > 0$ and $\eta > 0$ such that

$$|F_s(x,s,t)| \leq \varepsilon|s|^{p(x)-1}, \ \ |F_t(x,s,t)| \leq \varepsilon|t|^{q(x)-1},$$

and

$$|F(x,s,t)| \leq \varepsilon(|s|^{p(x)} + |t|^{q(x)}),$$

for all $|s,t)| \leq \delta$, and for all $|s,t)| \geq \eta$.
Then we have

$$|F_s(x,s,t)s| \leq \varepsilon|s|^{p(x)},\ |F_t(x,s,t)t| \leq \varepsilon|t|^{q(x)}, \tag{2.5}$$

and

$$|F(x,s,t)| \leq \varepsilon(|s|^{p(x)} + |t|^{q(x)}),$$

for all $|s,t)| \leq \delta$, and for all $|s,t)| \geq \eta$.
It yields,

$$\begin{aligned}
& -\frac{1}{2r^+}\phi^{'}(u_n,v_n)(u_n,v_n) \\
\geq\ & -\frac{1}{2r^+}\int_\Omega r(x)A(x,\Delta u_n)dx - \frac{1}{2r^+}\int_\Omega r(x)A(x,\Delta v_n)dx \\
+\ & \frac{1}{2r^+}\left[\int_\Omega F_{u_n}(x,u_n,v_n)u_n dx + \int_\Omega F_{v_n}(x,u_n,v_n)v_n dx\right] \\
\geq\ & -\frac{1}{2r^+}\int_\Omega r(x)A(x,\Delta u_n)dx - \frac{1}{2r^+}\int_\Omega r(x)A(x,\Delta v_n)dx \\
+\ & \frac{1}{2r^+}\left[\int_\Omega F_{u_n}(x,u_n,v_n)u_n dx + \int_\Omega F_{v_n}(x,u_n,v_n)v_n dx\right].
\end{aligned}$$

Thus,

$$\begin{aligned}
& \phi(u_n,v_n) - \frac{1}{2r^+}\phi^{'}(u_n,v_n)(u_n,v_n) \\
\geq\ & \int_\Omega A(x,\Delta u_n)dx + \int_\Omega A(x,\Delta v_n)dx - \int_\Omega F(x,u_n,v_n)dx \\
-\ & \frac{1}{2r^+}\int_\Omega r(x)A(x,\Delta u_n)dx - \frac{1}{2r^+}\int_\Omega r(x)A(x,\Delta v_n)dx \\
-\ & \int_\Omega F(x,u_n,v_n)dx + \frac{1}{2r^+}\left[\int_\Omega F_{u_n}(x,u_n,v_n)u_n dx + \int_\Omega F_{v_n}(x,u_n,v_n)v_n dx\right] \\
\geq\ & \frac{1}{2}\left[\int_\Omega |\Delta u_n|^{r(x)}dx + \int_\Omega |\Delta v_n|^{r(x)}dx\right] - \int_\Omega F(x,u_n,v_n)dx \\
+\ & \frac{1}{2r^+}\left[\int_\Omega F_{u_n}(x,u_n,v_n)u_n dx + \int_\Omega F_{v_n}(x,u_n,v_n)v_n dx\right] \\
\geq\ & \frac{1}{2}\max\left(\|u_n\|^{r^+}_{r(x)}, \|v_n\|^{r^+}_{r(x)}\right) - (C\varepsilon+\varepsilon)\int_\Omega |u_n|^{p(x)}dx - (C\varepsilon+\varepsilon)\int_\Omega |v_n|^{q(x)})dx.
\end{aligned}$$

Since $r^- > p^+ > 1,\ r^- > q^+ > 1$, by the compact embeddings

$$X \hookrightarrow L^{p(x)}(\Omega),\ X \hookrightarrow L^{q(x)}(\Omega),$$

we deduce

$$\begin{aligned} & \phi(u_n,v_n) - \frac{1}{2r^+}\phi^{'}(u_n,v_n)(u_n,v_n) \\ \geq\; & \frac{1}{2}\max\left(\|u_n\|_{r(x)}^{r^+},\|v_n\|_{r(x)}^{r^+}\right) - 2(C^{'}\varepsilon+\varepsilon)\|(u_n,v_n)\| \\ \geq\; & \left[\frac{1}{2} - 2(C^{'}\varepsilon+\varepsilon)\right]\|(u_n,v_n)\|, \end{aligned}$$

where $C^{'}$ is positive constant.

For ε small enough with $R = \frac{1}{2} - 2(C^{'}\varepsilon+\varepsilon) > 0$, we get

$$\|(u_n,v_n)\| \leq \frac{1}{R}\left(\phi(u_n,v_n) - \frac{1}{2r^+}\phi^{'}(u_n,v_n)(u_n,v_n)\right).$$

Since $\phi(u_n,v_n)$ is bounded and $\phi^{'}(u_n,v_n)(u_n,v_n) \to 0$ as $n\to\infty$, then (u_n,v_n) is bounded in X, passing to a subsequence, so $(u_n,v_n) \rightharpoonup (u,v)$ in X and $(u_n,v_n) \to L^{p(x)}(\Omega)\times L^{q(x)}(\Omega)$. We show that $(u_n,v_n)\to(u,v)$ in X.

$$\begin{aligned} & \phi^{'}(u_n,v_n)\left((u_n,v_n)-(u,v)\right) \\ =\; & \int_\Omega a(x,\Delta u_n)\Delta(u_n-u)dx + \int_\Omega a(x,\Delta v_n)\Delta(v_n-v)dx \\ -\; & \int_\Omega F_{u_n}(x,u_n,v_n)(u_n-u)dx - \int_\Omega F_{v_n}(x,u_n,v_n)(v_n-v)dx. \end{aligned}$$

Since

$$\begin{aligned} & \left|\int_\Omega a(x,\Delta u_n)\Delta(u_n-u)dx + \int_\Omega a(x,\Delta v_n)\Delta(v_n-v)dx\right| \\ =\; & |\phi^{'}(u_n,v_n)\left((u_n,v_n)-(u,v)\right) + \int_\Omega F_{u_n}(x,u_n,v_n)(u_n-u)dx \\ +\; & \int_\Omega F_{v_n}(x,u_n,v_n)(v_n-v)dx| \\ \leq\; & \|\phi^{'}(u_n,v_n)\|_{X^\star}\|(u_n,v_n)-(u,v)\| \\ +\; & \int_\Omega |F_{u_n}(x,u_n,v_n)||(u_n-u)|dx + \int_\Omega |F_{v_n}(x,u_n,v_n)||(v_n-v)|dx. \end{aligned}$$

By (2.5), we have

$$\begin{aligned} & \int_\Omega |F_{u_n}(x,u_n,v_n)||(u_n-u)|dx + \int_\Omega |F_{v_n}(x,u_n,v_n)||(v_n-v)|dx \\ \leq\; & \varepsilon\int_\Omega\left(|u_n-u|^{p(x)} + |v_n-v|^{q(x)}\right)dx, \end{aligned}$$

we get

$$\limsup_{n\to+\infty}\left(\int_\Omega a(x,\Delta u_n)\Delta(u_n-u)dx + \int_\Omega a(x,\Delta v_n)\Delta(v_n-v)dx\right) \leq 0.$$

Since $a(x,\xi)$ is of (S_+) type, we see that $(u_n,v_n)\to(u,v)$ in X.

Now, we verified the conditions of Mountain Pass Theorem. By Hölder's inequality, from (F_1) there exists $\delta > 0$ such that

$$\begin{aligned}|F(x,u,v)| &\leq \left|\int_0^u F_s(x,s,v)dx + \int_0^v F_t(x,0,t)dx + F(x,0,0)\right| \\ &\leq \varepsilon\left|\int_0^u |s|^{p(x)-1}dx + \int_0^v |t|^{q(x)-1}dx\right| + |F(x,0,0)| \\ &\leq \varepsilon(|u|^{p(x)} + |v|^{q(x)}) + M,\end{aligned}$$

for all $|u,v)| \leq \delta$, with $M := \max\limits_{x\in\overline{\Omega}} F(x,0,0)$ and by (F_2), there exists $M(\delta) > 0$ such that

$$| F(x,u,v) | \leq M(\delta)(|u|^{p(x)} + |v|^{q(x)}), \text{ for } |(u,v)| > \delta.$$

Therefore, for $\|(u,v)\| = \varrho$ small enough, we have

$$\begin{aligned}\phi(u,v) &\geq \int_\Omega A(x,\Delta u)dx + \int_\Omega A(x,\Delta v)dx - \varepsilon\int_{|(u,v)|<\delta}\left(|u|^{p(x)} + |v|^{q(x)}\right)dx \\ &- M(\delta)\int_{|(u,v)|>\delta}(|u|^{p(x)} + |v|^{q(x)}) - Mmeas\{|(u,v)| < \delta\} \\ &\geq \frac{1}{r^+}\max\left(\|u\|_{r(x)}^{r^+}, \|v\|_{r(x)}^{r^+}\right) \\ &- \min(\varepsilon C, M(\delta)C^{'})\max\left(\|u\|_{p(x)}^{p^-}, \|v\|_{q(x)}^{q^-}\right) - Mmeas\{|(u,v)| < \delta\} \\ &= g(\varrho).\end{aligned}$$

There exists $\theta > 0$ such that $g(\varrho) > \theta > 0$. Since $\phi(0,0) = 0$, we conclude that ϕ satisfies the conditions of Mountain Pass Theorem. Then there exists $(\overline{u_2}, \overline{v_2})$ such that $\phi^{'}(\overline{u_2}, \overline{v_2}) = 0$.

Proof of Theorem 1.2. To prove Theorem 1.2, above, will be based on a variational approach, using the critical points theory, we shall prove that the C^1-functional ϕ has a sequence of critical values. The main tools for this end are "Fountain theorem" (see Willem [16, Theorem 6.5]) which we give below.

Theorem 2.12 ("Fountain theorem", [16]**).** *Let X be a reflexive and separable Banach space, $\phi \in C^1(X,\mathbb{R})$ be an even functional and the subspaces X_k, Y_k, Z_k as defined in remark* 2.5. *If for each $k \in \mathbb{N}^*$ there exist $\rho_k > r_k > 0$ such that*

(1) $\inf_{x\in Z_k, \|x\|=r_k} \phi(x) \to \infty$ *as* $k \to \infty$,
(2) $\max_{x\in Y_k, \|x\|=\rho_k} \phi(x) \leq 0$,
(3) *I satisfies the $(PS)_c$ condition for every $c > 0$.*

Then I has a sequence of critical values tending to $+\infty$.

According to Lemma 2.6, (F_5) and (A_5), $\Phi \in \mathcal{C}^1(X,\mathbb{R})$ is an even functional. We will prove that if k is large enough, then there exist $\rho_k > \nu_k > 0$ such that

$$b_k := \inf\{\Phi(u)/u \in Z_k, \|u\| = \nu_k\} \to +\infty \quad \text{as } k \to +\infty; \tag{2.6}$$

$$a_k := \max\{\Phi(u)/u \in Y_k, \|u\| = \rho_k\} \to 0 \quad \text{as } k \to +\infty. \tag{2.7}$$

For any $(u,v) \in Z_k$, $\|v\|_{q(x)} > 1$, $\|u\|_{p(x)} > 1$ and $\|(u,v)\| = \eta_k$, (η_k will be specified later), by (2.4) we have

$$\begin{aligned}
\phi(u,v) &= \int_\Omega A(x,\Delta u)dx + \int_\Omega A(x,\Delta v)dx - \int_\Omega F(x,u,v)dx \\
&\geq \frac{1}{r^+}\max\left(\|u\|^{r^-}_{r(x)}, \|v\|^{r^-}_{r(x))}\right) - \int_\Omega C_3(1+|u|^{p(x)}+|v|^{q(x)})dx \\
&\geq \frac{1}{r^+}\max\left(\|u\|^{r^-}_{r(x)}, \|v\|^{r^-}_{r(x))}\right) - C_3\int_\Omega dx - C_3\int_\Omega |u|^{p(x)}dx - C_3\int_\Omega |v|^{q(x)}dx \\
&\geq \frac{1}{r^+}\|(u,v)\|^{r^-} - C_3(\beta_k\|(u,v)\|)^{p^+} - C_3(\beta_k\|(u,v)\|)^{q^+} - C_3|\Omega| \\
&\geq \frac{1}{r^+}\|(u,v)\|^{r^-} - C_4\beta_k\|(u,v)\|^m - C_3|\Omega|,
\end{aligned}$$

where m is defined in Lemma 2.6. We fix

$$\eta_k = \left(\frac{1}{r^+C_4\beta_k^b}\right)^{\frac{1}{m-r^-}} \to +\infty \text{ as } k \to +\infty.$$

Consequently

$$\phi(u,v) \geq \eta_k\left[\frac{1}{r^+}\eta_k^{r^- -1} - C_4\beta_k^b\eta_k^{m-1}\right] - C_3|\Omega|.$$

Then,

$$\phi(u,v) \to +\infty \text{ as } k \to +\infty.$$

Proof of (2.7). From (F_4), there exists $\lambda > 0$ such that

$$F(x,s,t) \geq \lambda(|s|^{\alpha(x)} - |t|^{\beta(x)}),$$

with $\alpha^- > r^+, \beta^+ < r^-$.

Therefore, by Lemma 2.1 [12] and Lemma 3.1 [17], for any $\omega := (u,v) \in Y_k$ with $\|\omega\| = 1$ and $1 < t = \rho_k$, we have

$$\begin{aligned}
\phi(t\omega) &= \int_\Omega A(x,t\Delta u)dx + \int_\Omega A(x,t\Delta v)dx - \int_\Omega F(x,t\omega)dx \\
&\leq \int_\Omega t^{r(x)}A(x,\Delta u)dx + \int_\Omega t^{r(x)}A(x,\Delta v)dx \\
&\quad - \lambda\int_\Omega |tu|^{\alpha(x)}dx + \lambda\int_\Omega |tv|^{\beta(x)}dx \\
&\leq t^{r^+}\left[\int_\Omega A(x,\Delta u)dx + \int_\Omega A(x,\Delta v)dx\right] \\
&\quad - \lambda t^{\alpha^-}\int_\Omega |u|^{\alpha(x)}dx + \lambda t^{\beta^-}\int_\Omega |v|^{\beta(x)}dx.
\end{aligned}$$

By $\alpha^- > r^+ > \beta^-$ and $dim Y_k < \infty$, we conclude that $\phi(tu,tv) \to -\infty$ as $\|t\omega\| \to +\infty$ for $\omega \in Y_k$. By applying the fountain Theorem, we achieved the proof of Theorem 1.2.

References

[1] Ayoujil, A., El Amrouss, A., *Continuous spectrum of a fourth order nonhomogeneous elliptic equation with variable exponent*, *Electron. J. Differ. Equ.*, Vol. **2011**(2011), 1-24.

[2] El Amrouss, A., Moradi, F., Moussaoui, M., *Existence of solutions for fourth-order PDEs with variable exponents*, Electron. J. Differ. Equ., **2009**(2009), no. 153, 1-13.

[3] El Amrouss, A., Ourraoui, A., *Existence of solutions for a boundary problem involving $p(x)-$biharmonic operator*, Bol. Soc. Parana. Mat., **31**(2013), no. 1, 179-192.

[4] El Hamidi, A., *Existence results to elliptic systems with nonstandard growth conditions*, J. Math. Anal. Appl., **300**(2004), 30-42.

[5] Fan, X.L., *Solutions for $p(x)$-Laplacian Dirichlet problems with singular coefficients*, J. Math. Anal. Appl., **312**(2005), 464-477.

[6] Fan, X.L., Fan, X., *A Knobloch-type result for $p(t)$ Laplacian systems*, J. Math. Anal. Appl., **282**(2003), 453-464.

[7] Fan, X.L., Han, X.Y., *Existence and multiplicity of solutions for $p(x)$-Laplacian equations in $\mathbb{R}^N$*, Nonlinear Anal., **59**(2004), 173-188.

[8] Fan, X.L., Zhang, Q.H., *Existence of solutions for $p(x)$-Laplacian Dirichlet problems*, Nonlinear Anal., **52**(2003), 1843-1852.

[9] Fan, X.L., Zhao, D., *On the spaces $L^{p_1(x)}$ and $W^{m,p_1(x)}$*, J. Math. Anal. Appl., **263**(2001), 424-446.

[10] Ferrero, A., Warnault, G., *On a solutions of second and fourth order elliptic with power type nonlinearities*, Nonlinear Anal., **70**(2009), 2889-2902.

[11] Li, L., Tang, C. L., *Existence and multiplicity of solutions for a class of $p(x)$-biharmonic equations*, Acta Math. Sci. Ser. B., **33**(2013), 155-170.

[12] Mashiyev, R.A., Cekic, B., Avci, M., Yücedag, Z., *Existence and multiplicity of weak solutions for nonuniformly elliptic equations with nonstandard growth condition*, Complex Var. Elliptic Equ., **57**(2012), no. 5, 579-595.

[13] Mihăilescu, M., *Existence and multiplicity of solutions for a Neumann problem involving the $p(x)$-Laplace operator*, Nonlinear Anal., **67**(2007), 1419-1425.

[14] Ourraoui, A., *On $\overrightarrow{p}(x)-$anisotropic problems with Neumann boundary conditions*, Int. J. Differ. Equ, vol. 2015, Art. ID 238261, 7 pages, https://doi.org/10.1155/2015/238261.

[15] Ruzicka, M., *Electrorheological Fluids: Modeling and Mathematical Theory*, Lecture Notes in Math 1748, Springer-Verlag, Berlin, 2000.

[16] Willem, M., *M*inimax theorems, Birkhäuser, Boston, 1996.

[17] Yucedag, Z., *Existence and multiplicity of solutions for $p(x)-$Kirchhoff-type problem*, An. Univ. Craiova Ser. Mat. Inform., **44**(2017), no. 1, 21-29.

[18] Zang, A., Fu, Y. *Interpolation inequalities for derivatives in variable exponent Lebesgue-Sobolev spaces*, Nonlinear Anal., **69**(2008), 3629-3636.

[19] Zhao, J.F., *S*tructure Theory of Banach Spaces, (Chinese), Wuhan Univ. Press, Wuhan, 1991.

[20] Zhikov, V.V., *Averaging of functionals of the calculus of variations and elasticity theory*, Math. USSR. Izv., **9**(1987), 33-66.

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Korovkin type approximation on an infinite interval via generalized matrix summability method using ideal

Sudipta Dutta and Rima Ghosh

Abstract. Following the notion of $A^{\mathcal{I}}$-summability method for real sequences [24] we establish a Korovkin type approximation theorem for positive linear operators on $UC_*[0,\infty)$, the Banach space of all real valued uniform continuous functions on $[0,\infty)$ with the property that $\lim\limits_{x\to\infty} f(x)$ exists finitely for any $f \in UC_*[0,\infty)$. In the last section, we extend the Korovkin type approximation theorem for positive linear operators on $UC_*([0,\infty)\times[0,\infty))$. We then construct an example which shows that our new result is stronger than its classical version.

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1. Introduction and background

Throughout the paper $\mathbb{N}$ will denote the set of all positive integers. For a sequence $\{L_n\}_{n\in\mathbb{N}}$ of positive linear operators on $C(X)$, the space of real valued continuous functions on a compact subset X of real numbers, Korovkin [17] first established the necessary and sufficient conditions for the uniform convergence of $\{L_n(f)\}_{n\in\mathbb{N}}$ to a function f by using the test functions $e_1 = 1,\ e_2 = x,\ e_3 = x^2$ [1]. The study of the Korovkin type approximation theory has a long history and is a well-established area of research. In recent years, using the concept of uniform statistical convergence various statistical approximation results have been proved ([9]). Erkuş and Duman [13] studied a Korovkin type approximation theorem via A-statistical convergence in the space $H_w(I^2)$ where $I^2 = [0,\infty)\times[0,\infty)$ which was extended for double sequences of positive linear operators of two variables in A-statistical sense by Demirci and Dirik in [6, 8]. Further it was extended for double sequences of positive linear operators of two variables in $A_2^{\mathcal{I}}$-statistical sense and in the sense of $A_2^{\mathcal{I}}$-summability method, by Dutta et. al. [11, 10].

Our primary interest, in this paper, is to obtain general Korovkin type approximation theorem for positive linear operators on the space $UC_*(D)$, the Banach space of all real valued uniform continuous functions on $D := [0, \infty)$ with the property that $\lim_{x\to\infty} f(x)$ exists and finite, endowed with the supremum norm $\|f\|_* = \sup_{x\in D} \mid f(x) \mid$ for $f \in UC_*(D)$, using the concept of $A^{\mathcal{I}}$-summability method for real sequences and test functions $1,\ e^{-x},\ e^{-y}$. In the last section, we extend the Korovkin-type approximation theorem for double sequence of positive linear operators on $UC_*\left([0,\infty)\times[0,\infty)\right)$. We also construct an example which shows that our new result is stronger than its classical version.

The concept of convergence of a sequence of real numbers was extended to statistical convergence by Fast [14]. Further investigations started in this area after the pioneering works of Šalát [22] and Fridy [15]. The notion of $\mathcal{I}$-convergence of real sequences was introduced by Kostyrko et. al. [18] as a generalization of statistical convergence using the notion of ideals. On the other hand statistical convergence was generalized to A-statistical convergence by Kolk ([16]). Later a lot of works have been done on matrix summability and A-statistical convergence (see [2, 3, 5, 12, 16, 19, 23]). In particular, in [25, 24] the very general notion of $A^{\mathcal{I}}$-statistical convergence and $A^{\mathcal{I}}$-summability was introduced and studied.

Recall that a real double sequence $\{x_{mn}\}_{m,n\in\mathbb{N}}$ is said to be convergent to L in Pringsheim's sense if for every $\varepsilon > 0$ there exists $N(\varepsilon) \in \mathbb{N}$ such that $|x_{mn} - L| < \varepsilon$ for all $m, n > N(\varepsilon)$ and denoted by $\lim\limits_{m,n} x_{mn} = L$. A double sequence is called bounded if there exists a positive number M such that $|x_{mn}| \le M$ for all $(m, n) \in \mathbb{N}\times\mathbb{N}$. A real double sequence $\{x_{mn}\}_{m,n\in\mathbb{N}}$ is statistically convergent to L if for every $\varepsilon > 0$,

$$\lim_{j,k} \frac{|\{m \le j, n \le k : |x_{mn} - L| \ge \varepsilon\}|}{jk} = 0 \ [20].$$

Recall that a family $\mathcal{I} \subset 2^Y$ of subsets of a nonempty set Y is said to be an ideal in Y if $(i) A, B \in \mathcal{I}$ implies $A \cup B \in \mathcal{I}$; $(ii) A \in \mathcal{I}, B \subset A$ implies $B \in \mathcal{I}$, while an admissible ideal $\mathcal{I}$ of Y further satisfies $\{x\} \in \mathcal{I}$ for each $x \in Y$. If $\mathcal{I}$ is a non-trivial proper ideal in Y (i.e. $Y \notin \mathcal{I}, \mathcal{I} \neq \{\emptyset\}$) then the family of sets $F(\mathcal{I}) = \{M \subset Y :$ there exists $A \in \mathcal{I} : M = Y\setminus A\}$ is a filter in Y. It is called the filter associated with the ideal $\mathcal{I}$. A non-trivial ideal $\mathcal{I}$ of $\mathbb{N}\times\mathbb{N}$ is called strongly admissible if $\{i\}\times\mathbb{N}$ and $\mathbb{N}\times\{i\}$ belong to $\mathcal{I}$ for each $i \in \mathbb{N}$. It is evident that a strongly admissible ideal is admissible also. Let $\mathcal{I}_0 = \{A \subset \mathbb{N}\times\mathbb{N} :$ there is $m(A) \in \mathbb{N}$ such that $i, j \ge m(A) \Longrightarrow (i, j) \notin A\}$. Then $\mathcal{I}_0$ is a non-trivial strongly admissible ideal [4].

2. A Korovkin type approximation for a sequence of positive linear operators of single variable

Throughout this section $\mathcal{I}$ denotes the non-trivial admissible ideal on $\mathbb{N}$. If $\{x_k\}_{k\in\mathbb{N}}$ is a sequence of real numbers and $A = (a_{nk})_{n,k=1}^{\infty}$ is an infinite matrix, then Ax is the sequence whose n-th term is given by

$$A_n(x) = \sum_{k=1}^{\infty} a_{nk} x_k.$$

A matrix A is called regular if $A \in (c, c)$ and

$$\lim_{k\to\infty} A_k(x) = \lim_{k\to\infty} x_k \text{ for all } x = \{x_k\}_{k\in\mathbb{N}} \in c$$

when c, as usual, stands for the set of all convergent sequences. It is well-known that the necessary and sufficient conditions for A to be regular are

$$R1) \quad \|A\| = \sup_n \sum_k |a_{nk}| < \infty;$$

$$R2) \quad \lim_n a_{nk} = 0, \text{ for each } k;$$

$$R3) \quad \lim_n \sum_k a_{nk} = 1.$$

We first recall the following definition

Definition 2.1. [25] Let $A = (a_{nk})$ be a non-negative regular summability matrix. Then a real sequence $x = \{x_k\}_{k\in\mathbb{N}}$ is said to be $A^{\mathcal{I}}$-summable to a number L if for every $\varepsilon > 0$, $\{n \in \mathbb{N} : |A_n(x) - L| \geq \varepsilon\} \in \mathcal{I}$ where $A_n(x) = \sum_{k=1}^{\infty} a_{nk} x_k$.

Thus $x = \{x_k\}_{k\in\mathbb{N}}$ is $A^{\mathcal{I}}$-summable to a number L if and only if $\{A_n(x)\}_{n\in\mathbb{N}}$ is $\mathcal{I}$-convergent to L. In this case, we write $\mathcal{I} - \lim_n \sum_{k\in\mathbb{N}} a_{nk} x_k = L$.

It should be noted that for $\mathcal{I} = \mathcal{I}_d$, the set of all subsets of $\mathbb{N}$ with natural density zero, $A^{\mathcal{I}}$-summability reduces to statistical A-summability [12].

We now establish a Korovkin type approximation theorem for positive linear operators on $UC_*[0, \infty)$, the Banach space of all real valued uniform continuous functions on $[0, \infty)$ with the property that $\lim_{x\to\infty} f(x)$ exists finitely for any $f \in UC_*[0, \infty)$. If L be a positive linear operator then $L(f) \geq 0$ for any positive function f. Also we denote the value of $L(f)$ at a point $x \in [0, \infty)$ by $L(f; x)$.

Theorem 2.2. *Let $\{L_n\}$ be a sequence of positive linear operators from $UC_*[0, \infty)$ into itself and let, $A = (a_{jn})$ be a non-negative regular summability matrix then for all $f \in UC_*[0, \infty)$*

$$\mathcal{I} - \lim_n \left\| \sum_{k=1}^{\infty} a_{nk} L_k(f) - f \right\|_* = 0$$

if and only if the following statements hold

$$\mathcal{I} - \lim_n \left\| \sum_{k=1}^{\infty} a_{nk} L_k(e^{-pt}) - e^{-px} \right\|_* = 0, \; p = 0, 1, 2.$$

Proof. Since the necessity is clear, then it is enough to proof sufficiency. Our objective is to show that for given $\varepsilon > 0$ there exist constants C_0 , C_1 , C_2 (depending on $\varepsilon > 0$) such that

$$\begin{aligned}\left\|\sum_{k=1}^{\infty} a_{nk} L_k(f) - f\right\|_* &\leq \varepsilon + C_2 \left\|\sum_{k=1}^{\infty} a_{nk} L_k(e^{-2t}) - e^{-2x}\right\|_* \\ &+ C_1 \left\|\sum_{k=1}^{\infty} a_{nk} L_k(e^{-t}) - e^{-x}\right\|_* \\ &+ C_0 \left\|\sum_{k=1}^{\infty} a_{nk} L_k(1) - 1\right\|_* .\end{aligned}$$

If this is done then our hypothesis implies that for any $\varepsilon > 0$,

$$\left\{n \in \mathbb{N} : \left\|\sum_{k=1}^{\infty} a_{nk} L_k(f) - f\right\| \geq \varepsilon\right\} \in \mathcal{I}.$$

Let $f \in UC_*[0,\infty)$ then $\exists$ a constant M such that $\mid f(x) \mid \leq M$ for each $x \in [0,\infty)$. Let ε be an arbitrary positive number. By hypothesis we may find $\delta := \delta(\varepsilon) > 0$ such that for every $t, x \in [0,\infty)$, $\mid e^{-t} - e^{-x} \mid < \delta$ implies $\mid f(t) - f(x) \mid < \varepsilon$. We can write $\mid f(t) - f(x) \mid < 2M \ \forall\ t, x \in [0,\infty)$. Also if $\mid e^{-t} - e^{-x} \mid \geq \delta$ then

$$\mid f(t) - f(x) \mid < \frac{2M}{\delta^2}(e^{-t} - e^{-x})^2.$$

Then for all $t, x \in [0,\infty)$,

$$\mid f(t) - f(x) \mid < \varepsilon + \frac{2M}{\delta^2}(e^{-t} - e^{-x})^2.$$

Then for $n \in \mathbb{N}$, using the linearity and the positivity of the operators L_n,

$$\begin{aligned}\left|\sum_{k=1}^{\infty} a_{nk} L_k(f(t);x) - f(x)\right| &\leq \sum_{k=1}^{\infty} a_{nk} L_k(\mid f(t) - f(x) \mid; x) \\ &+ \mid f(x) \mid \left|\sum_{k=1}^{\infty} a_{nk} L_k(1;x) - 1\right| \\ \leq \sum_{k=1}^{\infty} a_{nk} L_k(\varepsilon + \frac{2M}{\delta^2}(e^{-t} - e^{-x})^2; x) &+ \mid f(x) \mid \left|\sum_{k=1}^{\infty} a_{nk} L_k(1;x) - 1\right| \\ \leq \varepsilon + (\varepsilon + M)\left|\sum_{k=1}^{\infty} a_{nk} L_k(1;x) - 1\right| &+ \frac{2M}{\delta^2}\sum_{k=1}^{\infty} a_{nk} L_k((e^{-t} - e^{-x})^2; x) \\ \leq \varepsilon + (\varepsilon + M)\left|\sum_{k=1}^{\infty} a_{nk} L_k(1;x) - 1\right| &+ \frac{2M}{\delta^2} \mid e^{-2x} \mid \left|\sum_{k=1}^{\infty} a_{nk} L_k(1;x) - 1\right| \\ + \frac{2M}{\delta^2}\left|\sum_{k=1}^{\infty} a_{nk} L_k(e^{-2t};x) - e^{-2x}\right| &+ \frac{4M}{\delta^2} \mid e^{-x} \mid \left|\sum_{k=1}^{\infty} a_{nk} L_k(e^{-t};x) - e^{-x}\right|\end{aligned}$$

where $| e^{-kt} | \leq 1 \; \forall \; t \in [0,\infty)$ and $k \in \mathbb{N}$.
Then taking supremum over $x \in [0,\infty)$ we have

$$\left\| \sum_{k=1}^{\infty} a_{nk} L_k(f) - f \right\|_* \leq \varepsilon + K \left\{ \left\| \sum_{k=1}^{\infty} a_{nk} L_k(1) - 1 \right\|_* + \left\| \sum_{k=1}^{\infty} a_{nk} L_k(e^{-t}) - e^{-x} \right\|_* + \left\| \sum_{k=1}^{\infty} a_{nk} L_k(e^{-2t}) - e^{-2x} \right\|_* \right\}$$

where

$$K = \max \left\{ \varepsilon + M + \frac{2M}{\delta^2}, \frac{2M}{\delta^2}, \frac{4M}{\delta^2} \right\}.$$

For a given $r > 0$ choose $\varepsilon > 0$ such that $\varepsilon < r$ let us define the following sets

$$D = \left\{ n \in \mathbb{N} : \left\| \sum_{k=1}^{\infty} a_{nk} L_k(f) - f \right\|_* \geq r \right\}$$

$$D_1 = \left\{ n \in \mathbb{N} : \left\| \sum_{k=1}^{\infty} a_{nk} L_k(1) - 1 \right\|_* \geq \frac{r-\varepsilon}{3K} \right\}$$

$$D_2 = \left\{ n \in \mathbb{N} : \left\| \sum_{k=1}^{\infty} a_{nk} L_k(e^{-t}) - e^{-x} \right\|_* \geq \frac{r-\varepsilon}{3K} \right\}$$

$$D_3 = \left\{ n \in \mathbb{N} : \left\| \sum_{k=1}^{\infty} a_{nk} L_k(e^{-2t}) - e^{-2x} \right\|_* \geq \frac{r-\varepsilon}{3K} \right\}.$$

It follows that $D \subset D_1 \cup D_2 \cup D_3$. Since from hypotheses $D_1,\ D_2,\ D_3$ are belong to $\mathcal{I}$ so $D \in \mathcal{I}$ i.e.

$$\left\{ n \in \mathbb{N} : \left\| \sum_{k=1}^{\infty} a_{nk} L_k(f) - f \right\| \geq \varepsilon \right\} \in \mathcal{I}$$

and this completes the proof. □

3. A Korovkin type approximation for a sequence of positive linear operators of two variables

Throughout this section $\mathcal{I}$ denotes the non-trivial strongly admissible ideal on $\mathbb{N} \times \mathbb{N}$. Let $A = (a_{jkmn})$ be a four dimensional summability matrix. For a given double sequence $\{x_{mn}\}_{m,n\in\mathbb{N}}$, the A-transform of x, denoted by $Ax := ((Ax)_{jk})$, is given by

$$(Ax)_{jk} = \sum_{(m,n)\in\mathbb{N}^2} a_{jkmn} x_{mn}$$

provided the double series converges in Pringsheim sense for every $(j,k) \in \mathbb{N}^2$. In 1926, Robison [21] presented a four dimensional analog of the regularity by considering an additional assumption of boundedness. This assumption was made because a convergent double sequence is not necessarily bounded.

Recall that a four dimensional matrix $A = (a_{jkmn})$ is said to be RH-regular if it maps every bounded convergent double sequence into a convergent double sequence with the same limit. The Robison-Hamilton conditions state that a four dimensional matrix $A = (a_{jkmn})$ is RH-regular if and only if

(i) $\lim\limits_{j,k} a_{jkmn} = 0$ for each $(m, n) \in \mathbb{N}^2$,

(ii) $\lim\limits_{j,k} \sum\limits_{(m,n)\in\mathbb{N}^2} a_{jkmn} = 1$,

(iii) $\lim\limits_{j,k} \sum\limits_{m\in\mathbb{N}} |a_{jkmn}| = 0$ for each $n \in \mathbb{N}$,

(iv) $\lim\limits_{j,k} \sum\limits_{n\in\mathbb{N}} |a_{jkmn}| = 0$ for each $m \in \mathbb{N}$,

(v) $\sum\limits_{(m,n)\in\mathbb{N}^2} |a_{jkmn}|$ is convergent for each $(j, k) \in \mathbb{N}^2$,

(vi) there exist finite positive integers M_0 and N_0 such that $\sum\limits_{m,n>N_0} |a_{jkmn}| < M_0$

holds for every $(j, k) \in \mathbb{N}^2$.

Let $A = (a_{jkmn})$ be a nonnegative RH-regular summability matrix and let $K \subset \mathbb{N}^2$. Then the A-density of K is given by

$$\delta_A^{(2)}\{K\} = \lim_{j,k} \sum_{(m,n)\in K} a_{jkmn}.$$

Recall the following definition

Definition 3.1. [10] Let $A = (a_{jkmn})$ be a nonnegative RH-regular summability matrix. Then a real double sequence $x = \{x_{mn}\}_{m,n\in\mathbb{N}}$ is said to be $A_2^{\mathcal{I}}$-summable to a number L if for every $\varepsilon > 0$, $\left\{(j, k) \in \mathbb{N}^2 : |(Ax)_{j,k} - L| \geq \varepsilon\right\} \in \mathcal{I}$.

Thus $x = \{x_{mn}\}_{m,n\in\mathbb{N}}$ is $A_2^{\mathcal{I}}$-summable to a number L if and only if $(Ax)_{j,k}$ is $\mathcal{I}$-convergent to L. In this case, we write $\mathcal{I}_2 - \lim\limits_{j,k} \sum\limits_{(m,n)\in\mathbb{N}^2} a_{jkmn}x_{mn} = L$.

It should be noted that, if we take $\mathcal{I} = \mathcal{I}_d$, the set of all subsets of $\mathbb{N} \times \mathbb{N}$ with natural density zero, then $A_2^{\mathcal{I}}$-summability reduces to the notion of statistical A-summability for double sequence [2].

We now establish the Korovkin type approximation theorem for a double sequence of positive linear operators on $UC_*([0, \infty) \times [0, \infty))$, the Banach space of all real valued uniform continuous functions defined on $[0, \infty) \times [0, \infty)$ with the property that $\lim\limits_{(x,y)\to(\infty,\infty)} f(x, y)$ exists finitely for any $f \in UC_*([0, \infty) \times [0, \infty))$ endowed with the supremum norm $||f||_* = \sup\limits_{x,y\in[0,\infty)} |f(x, y)|$, in $A_2^{\mathcal{I}}$-summability method. If L be a positive linear operator then $L(f) \geq 0$ for any positive function f. Also we denote the value of $L(f)$ at a point $(x, y) \in [0, \infty) \times [0, \infty)$ by $L(f; x, y)$.

Theorem 3.2. *Assume $\mathcal{K} := [0, \infty) \times [0, \infty)$ and let $\{L_{mn}\}_{m,n\in\mathbb{N}}$ be a sequence of positive linear operators on $UC_*(\mathcal{K})$, the Banach space of all real valued uniform continuous functions defined on $\mathcal{K}$ with the property that $\lim\limits_{(x,y)\to(\infty,\infty)} f(x, y)$ exists* *finitely for any $f \in UC_*(\mathcal{K})$ and let $A = (a_{jkmn})$ be a non-negative RH-regular summability matrix. Then for any $f \in UC_*(\mathcal{K})$,*

$$\mathcal{I}_2 - \lim_{j,k} \left\| \sum_{(m,n)\in\mathbb{N}^2} a_{jkmn} L_{mn}(f) - f \right\|_* = 0$$

is satisfied if the following hold

$$\mathcal{I}_2 - \lim_{j,k} \left\| \sum_{(m,n)\in\mathbb{N}^2} a_{jkmn} L_{mn}(f_i) - f_i \right\|_* = 0, \;\; i = 0,1,2,3 \tag{3.1}$$

where $f_0 = 1,\; f_1 = e^{-x},\; f_2 = e^{-y}, f_3 = e^{-2x} + e^{-2y}$.

Proof. Assume that (3.1) holds. Let $f \in UC_*(\mathcal{K})$. Our objective is to show that for given $\varepsilon > 0$ there exist constants $C_0,\; C_1,\; C_2,\; C_3$ (depending on $\varepsilon > 0$) such that

$$\left\| \sum_{(m,n)\in\mathbb{N}^2} a_{jkmn} L_{mn}(f) - f \right\|_* \le \varepsilon + \sum_{i=0}^{3} C_i \left\| \sum_{(m,n)\in\mathbb{N}^2} a_{jkmn} L_{mn}(f_i) - f_i \right\|_* .$$

If this is done then our hypothesis implies that for any $\varepsilon > 0$,

$$\{(j,k) \in \mathbb{N}^2 : \| \sum_{(m,n)\in\mathbb{N}^2} a_{jkmn} L_{mn}(f) - f\|_* \ge \varepsilon\} \in \mathcal{I}.$$

To this end, start by observing that for each $(u,v) \in \mathcal{K}$ the function $0 \le g_{uv} \in UC_*(\mathcal{K})$ defined by

$$g_{uv}(s,t) = (e^{-s} - e^{-u})^2 + ((e^{-t} - (e^{-v})^2$$

satisfies

$$g_{uv} = (e^{-x})^2 + (e^{-y})^2 - 2e^{-u}e^{-x} - 2e^{-v}e^{-y} + (e^{-u})^2 + (e^{-v})^2.$$

Since each L_{mn} is a positive operator, $L_{mn} g_{uv}$ is a positive function. In particular, we have for each $(u,v) \in \mathcal{K}$,

$$0 \le \sum_{(m,n)\in\mathbb{N}^2} a_{jkmn} L_{mn}(g_{uv})(u,v)$$

$$= \left[\sum_{(m,n)\in\mathbb{N}^2} a_{jkmn} L_{mn} \left(\left(e^{-x}\right)^2 + \left(e^{-y}\right)^2 - 2e^{-u}e^{-x} - 2e^{-v}e^{-y} + \left(e^{-u}\right)^2 + \left(e^{-v}\right)^2 ; u,v \right) \right]$$

$$= \left[\sum_{(m,n)\in\mathbb{N}^2} a_{jkmn} L_{mn} \left(\left(e^{-x}\right)^2 + \left(e^{-y}\right)^2 ; u,v \right) - \left(e^{-u}\right)^2 - \left(e^{-v}\right)^2 \right]$$

$$-2e^{-u} \left[\sum_{(m,n)\in\mathbb{N}^2} a_{jkmn} L_{mn} \left(e^{-x}; u,v\right) - e^{-u} \right]$$

$$-2e^{-v} \left[\sum_{(m,n)\in\mathbb{N}^2} a_{jkmn} L_{mn} \left(e^{-y}; u,v\right) - e^{-v} \right]$$

$$+\left\{\left(e^{-u}\right)^2+\left(e^{-v}\right)^2\right\}\left[\sum_{(m,n)\in\mathbb{N}^2} a_{jkmn}L_{mn}(f_0)-f_0\right]$$

$$\leq \left\|\sum_{(m,n)\in\mathbb{N}^2} a_{jkmn}L_{mn}(f_3)-f_3\right\|_* + 2e^{-u}\left\|\sum_{(m,n)\in\mathbb{N}^2} a_{jkmn}L_{mn}(f_1)-f_1\right\|_*$$

$$+2e^{-v}\left\|\sum_{(m,n)\in\mathbb{N}^2} a_{jkmn}L_{mn}(f_2)-f_2\right\|_*$$

$$+\left\{\left(e^{-u}\right)^2+\left(e^{-v}\right)^2\right\}\left\|\sum_{(m,n)\in\mathbb{N}^2} a_{jkmn}L_{mn}(f_0)-f_0\right\|_* .$$

Let $f \in UC_*(\mathcal{K})$. Then there exists a constant M such that $|f(x,y)| \leq M$ for each $(x,y) \in \mathcal{K}$. Let $\varepsilon > 0$ be arbitrary. Then by the uniform continuity of f on $\mathcal{K}$ there exists a $\delta = \delta(\varepsilon) > 0$ such that if $\mid e^{-x} - e^{-u} \mid < \delta$ and $\mid e^{-y} - e^{-v} \mid < \delta$ then

$$\mid f(x,y) - f(u,v) \mid < \varepsilon + \frac{2M}{\delta^2}\left[\left(e^{-x}-e^{-u}\right)^2+\left(e^{-y}-e^{-v}\right)^2\right]$$

for all $(x,y),(u,v) \in \mathcal{K}$.
Since each L_{mn} is positive and linear it follows that

$$-\varepsilon \sum_{(m,n)\in\mathbb{N}^2} a_{jkmn}L_{mn}(f_0) - \frac{2M}{\delta^2}\sum_{(m,n)\in\mathbb{N}^2} a_{jkmn}L_{mn}(g_{uv})$$

$$\leq \sum_{(m,n)\in\mathbb{N}^2} a_{jkmn}L_{mn}(f) - f(u,v)L_{mn}(f_0)$$

$$\leq \varepsilon \sum_{(m,n)\in\mathbb{N}^2} a_{jkmn}L_{mn}(f_0) + \frac{2M}{\delta^2}\sum_{(m,n)\in\mathbb{N}^2} a_{jkmn}L_{mn}(g_{uv}).$$

Therefore

$$\left|\sum_{(m,n)\in\mathbb{N}^2} a_{jkmn}L_{mn}(f;u,v) - f(u,v)L_{mn}(f_0;u,v)\right|$$

$$\leq \varepsilon + \varepsilon\left[\sum_{(m,n)\in\mathbb{N}^2} a_{jkmn}L_{mn}(f_0;u,v) - f_0(u,v)\right] + \frac{2M}{\delta^2}\sum_{(m,n)\in\mathbb{N}^2} a_{jkmn}L_{mn}(g_{uv})$$

$$\leq \varepsilon + \varepsilon\left\|\sum_{(m,n)\in\mathbb{N}^2} a_{jkmn}L_{mn}(f_0) - f_0\right\| + \frac{2M}{\delta^2}\sum_{(m,n)\in\mathbb{N}^2} a_{jkmn}L_{mn}(g_{uv}).$$

In particular, note that

$$\left|\sum_{(m,n)\in\mathbb{N}^2} a_{jkmn}L_{mn}(f;u,v) - f(u,v)\right|$$

$$\leq \left| \sum_{(m,n)\in\mathbb{N}^2} a_{jkmn} L_{mn}(f;u,v) - f(u,v) \sum_{(m,n)\in\mathbb{N}^2} a_{jkmn} L_{mn}(f_0;u,v) \right|$$
$$+ \left| \sum_{(m,n)\in\mathbb{N}^2} a_{jkmn} f(u,v) \right| \left| \sum_{(m,n)\in\mathbb{N}^2} a_{jkmn} L_{mn}(f_0;u,v) - f_0(u,v) \right|$$
$$\leq \varepsilon + (M+\varepsilon) \left\| \sum_{(m,n)\in\mathbb{N}^2} a_{jkmn} L_{mn}(f_0) - f_0 \right\|_* + \frac{2M}{\delta^2} \sum_{(m,n)\in\mathbb{N}^2} a_{jkmn} L_{mn}(g_{uv})$$

which implies

$$\begin{aligned} \left\| \sum_{(m,n)\in\mathbb{N}^2} a_{jkmn} L_{mn}(f) - f \right\|_* \leq \varepsilon &+ C_3 \left\| \sum_{(m,n)\in\mathbb{N}^2} a_{jkmn} L_{mn}(f_3) - f_3 \right\|_* \\ &+ C_2 \left\| \sum_{(m,n)\in\mathbb{N}^2} a_{jkmn} L_{mn}(f_2) - f_2 \right\|_* \\ &+ C_1 \left\| \sum_{(m,n)\in\mathbb{N}^2} a_{jkmn} L_{mn}(f_1) - f_1 \right\|_* \\ &+ C_0 \left\| \sum_{(m,n)\in\mathbb{N}^2} a_{jkmn} L_{mn}(f_0) - f_0 \right\|_* \end{aligned}$$

where there exist such A and B such that

$$C_0 = \left[\frac{2M}{\delta^2} \{(e^{-A})^2 + (e^{-B})^2\} + M + \varepsilon \right],\ C_1 = \frac{4M}{\delta^2} e^{-A},$$
$$C_2 = \frac{4M}{\delta^2} e^{-B} \text{ and } C_3 = \frac{2M}{\delta^2}.$$

i.e.

$$\left\| \sum_{(m,n)\in\mathbb{N}^2} a_{jkmn} L_{mn}(f) - f \right\|_* \leq \varepsilon + C \sum_{i=0}^{3} \left\| \sum_{(m,n)\in\mathbb{N}^2} a_{jkmn} L_{mn}(f_i) - f_i \right\|_*,\ i = 0,1,2,3$$

where $C = \max\{C_0,\ C_1,\ C_2,\ C_3\}$.

For a given $\gamma > 0$, choose $\varepsilon > 0$ such that $\varepsilon < \gamma$. Now let

$$U = \left\{ (j,k) \in \mathbb{N}^2 : \left\| \sum_{(m,n)\in\mathbb{N}^2} a_{jkmn} L_{mn}(f) - f \right\|_* \geq \gamma \right\}$$

and

$$U_i = \left\{ (j,k) \in \mathbb{N}^2 : \left\| \sum_{(m,n)\in\mathbb{N}^2} a_{jkmn} L_{mn}(f_i) - f_i \right\|_* \geq \frac{\gamma - \varepsilon}{4C} \right\},\ i = 0,1,2,3.$$

It follows that $U \subset \bigcup_{i=0}^{3} U_i$. By hypotheses each $U_i \in \mathcal{I}$, $i = 0, 1, 2, 3$ and consequently $U \in \mathcal{I}$ i.e.

$$\left\{(j,k) \in \mathbb{N}^2 : \left\| \sum_{(m,n)\in\mathbb{N}^2} a_{jkmn} L_{mn}(f) - f \right\|_* \geq \gamma \right\} \in \mathcal{I}.$$

This completes the proof of the theorem. □

Remark 3.3. We now show that our theorem is stronger than the statistical A-summable version [7] (and so the classical version). Let $\mathcal{I}$ be a non-trivial strongly admissible ideal of $\mathbb{N} \times \mathbb{N}$. Choose an infinite subset $C = \{(p_i, q_i) : i \in \mathbb{N}\}$ (where $p_i \neq q_i$, $p_1 < p_2 < ...$, and $q_1 < q_2 < ...$) from $\mathcal{I} \setminus \mathcal{I}_d$ where $\mathcal{I}_d$ denotes the set of all subsets of $\mathbb{N} \times \mathbb{N}$ with natural density zero. Let $\{u_{mn}\}_{m,n\in\mathbb{N}}$ be given by

$$u_{mn} = \begin{cases} 1 & \text{if } m, n \text{ are even} \\ 0 & \text{otherwise.} \end{cases}$$

Let $A = (a_{jkmn})$ be given by

$$a_{jkmn} = \begin{cases} 1 & \text{if } j = p_i, k = q_i, m = 2p_i, n = 2q_i \text{ for some } i \in \mathbb{N} \\ 1 & \text{if } (j,k) \neq (p_i, q_i), \text{ for any } i, m = 2j+1, n = 2k+1 \\ 0 & \text{otherwise.} \end{cases}$$

Now

$$y_{j,k} = \sum_{(m,n)\in\mathbb{N}^2} a_{jkmn} u_{mn} = \begin{cases} 1 & \text{if } j = p_i, k = q_i \text{ for some } i \in \mathbb{N} \\ 0 & \text{if } (j,k) \neq (p_i, q_i), \text{ for any } i \in \mathbb{N}. \end{cases}$$

Let $\varepsilon > 0$ be given. Then $\{(j,k) \in \mathbb{N}^2 : |y_{j,k} - 0| \geq \varepsilon\} = C \in \mathcal{I}$. Then the sequence $\{u_{mn}\}_{m,n\in\mathbb{N}}$ is $A_2^{\mathcal{I}}$-summable to 0. Evidently this sequence is not statistically A-summable to 0.

Let $\mathcal{K} = [0, \infty) \times [0, \infty)$. We consider the following Baskakov operators

$$B_{mn} : UC_*(\mathcal{K}) \to UC_*(\mathcal{K})$$

defined by

$$B_{mn}(f; x, y) = \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} f\left(\frac{j}{n}, \frac{k}{n}\right) \binom{m-1+j}{j} \binom{n-1+k}{k} (1+x)^{-m-j} (1+y)^{-n-k} x^j y^k.$$

We now consider the double sequence $\{L_{mn}\}_{m,n\in\mathbb{N}}$ of positive linear operators defined by

$$L_{mn}(f; x, y) = (1 + u_{mn}) B_{mn}(f; x, y).$$

Then observe that

$$L_{mn}(f_0;x,y) = (1+u_{mn})f_0(x,y),$$
$$L_{mn}(f_1;x,y) = (1+u_{mn})\Big(1+x-xe^{-\frac{1}{m}}\Big)^{-m},$$
$$L_{mn}(f_2;x,y) = (1+u_{mn})\Big(1+y-ye^{-\frac{1}{n}}\Big)^{-n},$$
$$L_{mn}(f_3;x,y) = (1+u_{mn})\left[\Big(1+x-xe^{-\frac{1}{m}}\Big)^{-m} + \Big(1+y-ye^{-\frac{1}{n}}\Big)^{-n}\right].$$

Now as A is a nonnegative RH-regular summability matrix and $\{u_{mn}\}_{m,n\in\mathbb{N}}$ is $A_2^{\mathcal{I}}$-summable to 0 then for any $\varepsilon > 0$,

$$\left\{(j,k)\in\mathbb{N}^2 : ||\sum_{(m,n)\in\mathbb{N}^2} a_{jkmn}L_{mn}(f_i) - f_i||_* \geq \varepsilon\right\} \in \mathcal{I},\ i=0,1,2,3.$$

Therefore by previous theorem

$$\left\{(j,k)\in\mathbb{N}^2 : ||\sum_{(m,n)\in\mathbb{N}^2} a_{jkmn}L_{mn}(f) - f||_* \geq \varepsilon\right\} \in \mathcal{I}.$$

But since $\{u_{mn}\}_{m,n\in\mathbb{N}}$ is not usual convergent and statistical A-summable so we can say that the classical version and statistical A-summable version of the previous theorem do not work for the operator defined above.

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References

[1] Aliprantis, C.D., Burkinshaw, O., *Principles of Real Analysis*, Academic Press, New York, 1998.

[2] Belen, C., Mursaleen, M., Yildirim, M., *Statistical A-summability of double sequences and a Korovkin type approximation theorem*, Bull. Korean Math. Soc., **49**(2012), no. 4, 851-861.

[3] Connor, J., *The Statistical and strong p-Cesaro convergence of sequences*, Analysis, **8**(1988), 47-63.

[4] Das, P., Kostyrko, P., Wilczyński, W., Malik, P., *$\mathcal{I}$ and $\mathcal{I}^*$-convergence of double sequences*, Math. Slovaca, **58**(2008), no. 5, 605-620.

[5] Demirci, K., *Strong A-summabilty and A-statistical convergence*, Indian J. Pure Appl. Math., **27**(1996), 589-593.

[6] Demirci, K., Dirik, F., *A Korovkin type approximation theorem for double sequences of positive linear operators of two variables in A-statistical sense*, Bull. Korean Math. Soc., **47**(2010), no. 4, 825-837.

[7] Demirci, K., Karakuş, S., *A-statistical Korovkin type approximation theorem for functions of two variables on an infinite interval*, Acta Math. Univ. Comenian.(N.S.), **81**(2012), no. 2, 151-157.

[8] Demirci, K., Karakuş, S., *Korovkin type approximation theorem for double sequences of positive linear operators via statistical A-summability*, Results Math., **63**(2013), 1-13.

[9] Duman, O., Khan, M.K., Orhan, C., *A-statistical convergence of approximating operators*, Math. Inequal. Appl., **6**(2003), no. 4, 689-699.

[10] Dutta, S., Akdağ, S., Das, P., *Korovkin type approximation theorem via* $A_2^{\mathcal{I}}$*-summability method*, Filomat, **30**(2016), no. 10, 2663-2672.

[11] Dutta, S., Das, P., *Korovkin type approximation theorem in* $A_2^{\mathcal{I}}$*-statistical sense*, Mat. Vesnik, **67**(2015), no. 4, 288-300.

[12] Edely, O.H.H., Mursaleen, M., *On statistical A-summability*, Math. Comp. Model., **49**(2009), no. 8, 672-680

[13] Erkuş, E., Duman, O., *A-statistical extension of the Korovkin type approximation theorem*, Proc. Indian Acad. Sci. Math. Sci., **115**(2005), no. 4, 499-508.

[14] Fast, H., *Sur la convergence statistique*, Colloq. Math., **2**(1951), 241-244.

[15] Fridy, J.A., *On statistical convergence*, Analysis, **5**(1985), 301-313.

[16] Kolk, E., *Matrix summability of statistically convergent sequences*, Analysis, **13**(1993), 77-83.

[17] Korovkin, P.P., *Linear Operators and Approximation Theory*, Delhi, Hindustan Publ. Co., 1960.

[18] Kostyrko, P., Šalát, T., Wilczyński, W., $\mathcal{I}$*-convergence*, Real Anal. Exchange, **26**(2000/2001), no. 2, 669-685.

[19] Mursaleen, M., Alotaibi, A., *Korovkin type approximation theorem for functions of two variables through statistical A-summability*, Advances in Difference Equations, https://doi.org/10.1186/1687-1847-2012-65.

[20] Mursaleen, M., Edely, O.H.H., *Statistical convergence of double sequences*, J. Math. Anal. Appl., **288**(2003), 223-331.

[21] Robison, G.M., *Divergent double sequences and series*, Trans. Amer. Math. Soc., **28**(1926), no. 1, 50-73.

[22] Šalát, T., *On Statistically convergent sequences of real numbers*, Math. Slovaca, **30**(1980), 139-150.

[23] Savas, E., Das, P., *A generalized statistical convergence via ideals*, Appl. Math. Lett., **24**(2011), 826-830.

[24] Savas, E., Das, P., Dutta, S., *A note on strong matrix summability via ideals*, Appl. Math. Lett., **25**(2012), 733-738.

[25] Savas, E., Das, P., Dutta, S., *A note on some generalized summability methods*, Acta Math. Univ. Comenian. (N.S.), **82**(2013), no. 2, 297-304.

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$\mathcal{A}-$Summation process in the space of locally integrable functions

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Abstract. In this paper, using the concept of summation process, we give a Korovkin type approximation theorem for a sequence of positive linear operators acting from $L_{p,q}(loc)$, the space of locally integrable functions, into itself. We also study rate of convergence of these operators.

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Keywords: Summation process, positive linear operators, locally integrable functions, Korovkin type theorem, modulus of continuity, rate of convergence.

1. Introduction

Approximation theory has many connections with theory of polynomial approximation, functional analysis, numerical solutions of differential and integral equations, summability theory, measure theory and probability theory ([1], [14], [7]).

A Korovkin type theorem for positive linear operators acting from $L_p(a,b)$ to $L_p(a,b)$ was studied in [5], [8], [11] and [20]. Note that all the results just mentioned are devoted to the case of a finite interval (a,b). Roughly speaking a Korovkin type approximation theorem provides conditions for whether a given sequence of positive linear operators converges strongly to the identity operator [1], [12] and [14]. These theorems exhibit a variety of test functions which guarantee that convergence property holds on the whole space provided it holds on them ([1], [14]). If the sequence of positive linear operators does not converge, then it might be useful to use matrix summability methods. The main aim of using summability methods has always been to make a non-convergent sequence to converge. This was the motivation behinde Fejer's famous theorem showing that Cesàro method being effective in making the Fourier series of a continuous periodic function to converge ([22]). Summability methods are also considered in physics ([6]) to make a nonconvergent sequence to converge.

In this paper, using matrix summability methods which includes both convergence and almost convergence, we obtain a Korovkin type approximation theorem of a function f in $L_{p,q}(loc)$. We also give rate of convergence in $L_{p,q}(loc)$ approximation by means of the modulus of continuity. We recall that some results concerning the approximation in $L_{p,q}(loc)$ may be found in [9], [10], [18], [19], [21]. Also $L_{p,q}$ approximation via Abel convergence has been studied in [4]. We remark that matrix summability methods are quite effective, in summing sequences of nonlinear integral operators ([2]).

First of all, we recall some notation and basic definitions used in this paper.

Let $q(x) = 1 + x^2$; $-\infty < x < \infty$. For $h > 0$, by $L_{p,q}(loc)$ we will denote the space of measurable functions f satisfying the inequality,

$$\left(\frac{1}{2h} \int_{x-h}^{x+h} |f(t)|^p \, dt \right)^{1/p} \leq M_f \; q(x), -\infty < x < \infty \tag{1.1}$$

where $p \geq 1$ and M_f is a positive constant which depends on the function f.

It is known [13] that $L_{p,q}(loc)$ is a linear normed space with norm,

$$\|f\|_{p,q} = \sup_{-\infty<x<\infty} \frac{\left(\frac{1}{2h} \int_{x-h}^{x+h} |f(t)|^p \, dt \right)^{1/p}}{q(x)}. \tag{1.2}$$

where $\|f\|_{p,q}$ may also depend on $h > 0$. To simplify the notation, we need the following. For any real numbers a and b put

$$\|f; L_p(a,b)\|_{p,q} := \left(\frac{1}{b-a} \int_a^b |f(t)|^p \, dt \right)^{1/p},$$

$$\|f; L_{p,q}(a,b)\|_{p,q} = \sup_{a<x<b} \frac{\|f; L_p(x-h, x+h)\|_{p,q}}{q(x)},$$

$$\|f; L_{p,q}(|x| \geq a)\|_{p,q} = \sup_{|x| \geq a} \frac{\|f; L_p(x-h, x+h)\|_{p,q}}{q(x)}.$$

With this notation the norm in $L_{p,q}(loc)$ may be written in the form

$$\|f\|_{p,q} = \sup_{x \in \mathbb{R}} \frac{\|f; L_p(x-h, x+h)\|}{q(x)}.$$

It is known [13] that $L^k_{p,q}(loc)$ is the subspace of all functions $f \in L_{p,q}(loc)$ for which there exists a constant k_f such that

$$\lim_{|x|\to\infty} \frac{\|f - k_f q; L_p(x-h, x+h)\|}{q(x)} = 0.$$

As usual, if T is a positive linear operator from $L_{p,q}(loc)$ into $L_{p,q}(loc)$, then the operator norm $\|T\|$ is given by $\|T\| := \sup_{f \neq 0} \frac{\|Tf\|_{p,q}}{\|f\|_{p,q}}$.

Let $\mathcal{A} := \{A^{(n)}\} = \left\{ a^{(n)}_{kj} \right\}$ be a sequence of infinite matrices with nonnegative real entries. A sequence $\{T_j\}$ of positive linear operators from $L_{p,q}(loc)$ into itself is called a *strong* $\mathcal{A}-$*summation process* in $L_{p,q}(loc)$ if $\{T_j f\}$ is strongly $\mathcal{A}-$*summable* to f for every $f \in L_{p,q}(loc)$, i.e.,

$$\lim_k \sum_j a_{kj}^n \left\| T_j f - f \right\|_{p,q} = 0, \quad \text{uniformly in } n.$$

Some results concerning strong summation processes in $L_{p,q}(loc)$ may be found in [3].

2. $\mathcal{A}-$summation process in $L_{p,q}(loc)$

The main aim of the present work is to study a Korovkin type approximation theorem for a sequence of positive linear operators acting on the space $L_{p,q}(loc)$ by using matrix summability method which includes both convergence and almost convergence. We also present an example of positive linear operators which verifies our Theorem 2.6 but does not verify the classical one (see Theorem 2.2 below).

Let $\mathcal{A} :=\{A^{(n)}\} = \left\{a_{kj}^{(n)}\right\}$ be a sequence of infinite matrices with nonnegative real entries. A sequence $\{T_j\}$ of positive linear operators from $L_{p,q}(loc)$ into itself is called an $\mathcal{A}-$*summation process* in $L_{p,q}(loc)$ if $\{T_j f\}$ is $\mathcal{A}-$*summable* to f for every f in $L_{p,q}(loc)$, i.e.,

$$\lim_k \left\| \sum_j a_{kj}^n T_j f - f \right\|_{p,q} = 0, \quad \text{uniformly in } n, \tag{2.1}$$

where it is assumed that the series converges for each k, n and f. Some results concerning summation processes on some other spaces may be found in [16], [17] and [20].

The next result establishes a relationship between strong summation process and summation process in $L_{p,q}(loc)$.

Proposition 2.1. *Let* $\mathcal{A} :=\{A^{(n)}\} = \left\{a_{kj}^{(n)}\right\}$ *be a sequence of infinite matrices with nonnegative real entries and assume that*

$$\lim_k \sup_n \sum_j a_{kj}^{(n)} = 1.$$

Let $\{T_j\}$ *be a sequence of positive linear operators from* $L_{p,q}(loc)$ *into itself If* $\{T_j\}$ *is a strong* $\mathcal{A}-$*summation process in* $L_{p,q}(loc)$ *then* $\{T_j\}$ *is an* $\mathcal{A}-$*summation process in* $L_{p,q}(loc)$.

Proof. The proof may be obtained by using the idea given in [16].

Throughout the paper let

$$B_k^{(n)}(f) = B_k^{(n)}(f;x) := \sum_j a_{kj}^n T_j(f;x)$$

where we assume that the series on the right is convergent for each $k, n \in \mathbb{N}$ and $f \in L_{p,q}(loc)$.

We recall the following result of [13] that we need in the sequel.

Theorem 2.2. *Let $\{T_j\}$ be a sequence of positive linear operators from $L_{p,q}(loc)$ into itself and satisfy the conditions*

i) The sequence (T_j) is uniformly bounded, that is, $\|T_j\| \le C < \infty$, where C is a constant independent of j,

ii) For $f_i(y) = y^i$, $i = 0, 1, 2$;

$$\lim_j \|T_j(f_i; x) - f_i(x)\|_{p,q} = 0.$$

Then

$$\lim_j \|T_j f - f\|_{p,q} = 0$$

for each function $f \in L^k_{p,q}(loc)$, (see [13]).

We show that the Korovkin type theorem holds in the subspace $L^k_{p,q}(loc)$. First we give the following

Lemma 2.3. *Let $\mathcal{A} := \{A^{(n)}\} = \left\{a^{(n)}_{kj}\right\}$ be a sequence of infinite matrices with nonnegative real entries. Let $\{T_j\}$ be a sequence of positive linear operators from $L_{p,q}(loc)$ into itself satisfying the condition*

$$\lim_k \sup_n \left\| B^{(n)}_k(f_i; x) - f_i(x) \right\|_{p,q} = 0.$$

Then, for any continuous and bounded function f on the real axis, we have

$$\lim_k \sup_n \left\| B^{(n)}_k(f; x) - f(x); L_{p,q}(a, b) \right\| = 0$$

where a and b are any real numbers.

Proof. By the uniform continuity of f on the interval $[a, b]$ and by the positivity and linearity of T_j, we may write that

$$\begin{aligned} \left\| B^{(n)}_k(f(t); x) - f(x); L_{p,q}(a, b) \right\| &\le \left\| B^{(n)}_k(f(t) + f(x) - f(x); x) - f(x) \right\|_{p,q} \\ &\le \left\| B^{(n)}_k(|f(t) - f(x)|; x) \right\|_{p,q} \\ &+ |f(x)| \left\| B^{(n)}_k(1; x) - 1 \right\|_{p,q} \\ &< \varepsilon + \frac{2M}{\delta^2} \left\| B^{(n)}_k(t^2; x) - x^2 \right\|_{p,q} \\ &+ \frac{4Mc}{\delta^2} \left\| B^{(n)}_k(t; x) - x \right\|_{p,q} \\ &+ \left(\frac{2Mc^2}{\delta^2} + \varepsilon + M \right) \left\| B^{(n)}_k(1; x) - 1 \right\|_{p,q}. \end{aligned}$$

Hence the proof is completed.

Theorem 2.4. *Let $\mathcal{A} := \{A^{(n)}\} = \left\{a^{(n)}_{kj}\right\}$ be a sequence of infinite matrices with nonnegative real entries. Let $\{T_j\}$ be a sequence of positive linear operators from*

$L_{p,q}(loc)$ into itself. Assume that

$$H := \sup_{n,k} \sum_j a_{k,j}^n \|T_j\| < \infty. \tag{2.2}$$

Then $\{T_j\}$ is an $\mathcal{A}-$ summation process in $L_{p,q}^k(loc)$, i.e.,for any function $f \in L_{p,q}^k(loc)$

$$\lim_k \sup_n \left\| B_k^{(n)}(f;x) - f(x) \right\|_{p,q} = 0$$

if and only if

$$\lim_k \sup_n \left\| B_k^{(n)}(f_i;x) - f_i(x) \right\|_{p,q} = 0$$

where $f_i(y) = y^i$ for $i = 0, 1, 2$.

Proof. We follow [13] up to a certain stage. If $f \in L_{p,q}^k(loc)$ then $f - k_f.q \in L_{p,q}^0(loc)$. So it is sufficient to prove the theorem for the function $f \in L_{p,q}^0(loc)$. For $\varepsilon > 0$, there exists a point x_0 such that the inequality

$$\left(\frac{1}{2h} \int_{x-h}^{x+h} |f(t)|^p \, dt \right)^{1/p} < \varepsilon q(x) \tag{2.3}$$

holds for all x, $|x| \geq x_0$. By the well known Lusin theorem, there exists a continuous function φ on the finite interval $[-x_0 - h, x_0 + h]$ such that the inequality

$$\|f - \varphi; L_p(-x_0, x_0)\| < \varepsilon \tag{2.4}$$

is fulfilled. Setting

$$\delta < \min\left\{ \frac{2h\varepsilon^p}{M^p(x_0)}, h \right\}, \tag{2.5}$$

where $M(x_0) = \max\left\{ \max\limits_{|x| \leq x_0 + h} |\varphi(x)|, 1 \right\}$, we define a continuous function g by

$$g(x) = \begin{cases} \varphi(x), & \text{if } |x| \leq x_0 + h \\ 0, & \text{if } |x| \geq x_0 + h + \delta \\ \text{linear}, & \text{otherwise.} \end{cases}$$

Then by (2.3), (2.4), (2.5) and the Minkowski inequality, we obtain

$$\|f - g\|_{p,q} < \varepsilon \tag{2.6}$$

for any $\varepsilon > 0$ (see [13]).
Now we can find a point $x_1 > x_0$ such that

$$q(x_1) > \frac{M(x_0)}{\varepsilon} \text{ and } g(x) = 0 \text{ for } |x| > x_1, \tag{2.7}$$

where $M(x_0)$ is defined above. Then by (2.4), (2.5), (2.6) and by Lemma 2.3 we get

$$\begin{aligned}
\left\|B_k^{(n)}(f;x)-f(x)\right\|_{p,q} &\le \left\|B_k^{(n)}(f-g)\right\|_{p,q}+\left\|B_k^{(n)}g-g\right\|_{p,q}+\|f-g\|_{p,q} \\
&\le \varepsilon\sum_j a_{kj}^{(n)}\|T_j\|_{p,q}+\varepsilon+\left\|B_k^{(n)}g-g\right\|_{p,q} \\
&\le \varepsilon\left(\sum_j a_{kj}^{(n)}\|T_j\|_{p,q}+1\right)+\left\|B_k^{(n)}g-g;L_{p,q}(-x_1,x_1)\right\| \\
&+\left\|B_k^{(n)}g-g;L_{p,q}(|x|\ge x_1)\right\| \\
&\le \varepsilon\left(\sum_j a_{kj}^{(n)}\|T_j\|_{p,q}+2\right)+\left\|B_k^{(n)}g;L_{p,q}(|x|\ge x_1)\right\|.
\end{aligned} \tag{2.8}$$

Since $|g(x)|\le M(x_0)$ for all $x\in\mathbb{R}$, we can write

$$\begin{aligned}
\left\|B_k^{(n)}g;L_{p,q}(|x|\ge x_1)\right\|_{p,q} &\le M(x_o)\left\|B_k^{(n)}1;L_{p,q}(|x|\ge x_1)\right\| \\
&\le M(x_o)\left\|B_k^{(n)}1-1;L_{p,q}(|x|\ge x_1)\right\| \\
&+M(x_o)\|1;L_{p,q}(|x|\ge x_1)\| \\
&\le M(x_o)\left\|B_k^{(n)}1-1\right\|_{p,q}+\frac{M(x_o)}{q(x_1)}.
\end{aligned}$$

Considering hypothesis and (2.7) we get by (2.8) that

$$\lim_k \sup_n \left\|B_k^{(n)}f-f\right\|_{p,q}=0.$$

The next result shows that Korovkin type theorem does not hold in the whole space $L_{p,q}(loc)$.

Theorem 2.5. *Let $\mathcal{A} := \{A^{(n)}\} = \left\{a_{kj}^{(n)}\right\}$ be a sequence of infinite matrices with nonnegative real entries. Let $\{T_j\}$ be a sequence of positive linear operators from $L_{p,q}(loc)$ into itself satisfying*

$$\lim_k \sup_n \left\|\sum_j a_{kj}^{(n)}T_j(f_i;x)-f_i(x)\right\|_{p,q}=0.$$

Then there exists a function f^* in $L_{p,q}(loc)$ for which

$$\lim_k \sup_n \left\|\sum_j a_{kj}^{(n)}T_jf^*-f^*\right\|_{p,q}\ge 2^{1-\frac{1}{p}}. \tag{2.9}$$

Proof. We consider the sequence of operators T_j given in [13] $(for\ j = 1, 2, \ldots)$:

$$T_j(f; x) = \begin{cases} \dfrac{x^2}{(x+h)^2} f(x+h), & x \in [(2j-1)h, (2j+1)h) \\ f(x), & \text{otherwise.} \end{cases}$$

As observed in [13] that $T_j : L_{p,q}(loc) \to L_{p,q}(loc)$. Assume now that

$$\mathcal{A} := \left\{A^{(n)}\right\} = \left\{a_{kj}^{(n)}\right\}$$

is a sequence of infinite matrices defined by

$$a_{kj}^{(n)} = \begin{cases} \dfrac{1}{k+1}, & n \le j \le n+k \\ 0, & \text{otherwise.} \end{cases}$$

It is shown in [13] that

$$\|T_j f_i - f_i\|_{p,q} \to 0, \ \ (as\ j \to \infty).$$

Now it is easy to verify that, for each $i = 0, 1, 2$

$$\begin{aligned} \left\| \frac{1}{k+1} \sum_{j=n}^{k+n} T_j f_i - f_i \right\|_{p,q} &= \left\| \frac{1}{k+1} \sum_{j=n}^{k+n} T_j f_i - \frac{1}{k+1} \sum_{j=n}^{k+n} f_i \right\|_{p,q} \\ &= \left\| \frac{1}{k+1} \sum_{j=n}^{k+n} (T_j f_i - f_i) \right\|_{p,q} \\ &\le \frac{1}{k+1} \sum_{j=n}^{k+n} \|T_j f_i - f_i\|_{p,q} \\ &\to 0 \ (k \to \infty, \ \text{uniformly in } n). \end{aligned}$$

Consider the following function f^* given in [13] :

$$f^*(x) = \begin{cases} x^2, & \text{if} \quad x \in \bigcup\limits_{k=1}^{\infty} [(2k-1)h, 2kh) \\ -x^2, & \text{if} \quad x \in \bigcup\limits_{k=0}^{\infty} [2kh, (2k+1)h) \\ 0, & \text{if} \quad x < 0. \end{cases}$$

Then $f^* \in L_{p,q}(loc)$ and we get

$$\begin{aligned}
\left\| \frac{1}{k+1} \sum_{j=n}^{k+n} T_j f^* - f^* \right\|_{p,q} &\geq \sup_{x \in [(2n-1)h, 2(n+k)h]} \frac{\left(\frac{1}{2h} \int\limits_{x-h}^{x+h} \left| \frac{1}{k+1} \sum\limits_{j=n}^{k+n} T_j f^* - f^* \right|^p dt \right)^{\frac{1}{p}}}{q(x)} \\
&\geq \frac{\left(\frac{1}{2h} \int\limits_{2nh-h}^{2nh+h} \left| \frac{1}{k+1} \sum\limits_{j=n}^{k+n} \frac{\xi^2}{(\xi+h)^2} f^*(\xi+h) - f^*(\xi) \right|^p d\xi \right)^{\frac{1}{p}}}{q(2nh)} \\
&> \frac{\left(\frac{1}{2h} 2^p \left((2n-1)h\right)^{2p} h \right)^{\frac{1}{p}}}{1 + 4n^2 h^2} \\
&= \frac{2^{1-\frac{1}{p}} (2n-1)^2 h^2}{1 + 4n^2 h^2}.
\end{aligned}$$

On applying the operator $\lim\limits_{k} \sup\limits_{n}$ on both sides one can see that

$$\lim_{k} \sup_{n} \left\| \frac{1}{k+1} \sum_{j=n}^{k+n} T_j f^* - f^* \right\|_{p,q} \geq 2^{1-1/p}$$

Therefore the theorem is proved.
In the whole space $L_{p,q}(loc)$ we have the following

Theorem 2.6. *Let* $\mathcal{A} := \left\{ A^{(n)} \right\} = \left\{ a_{kj}^{(n)} \right\}$ *be a sequence of infinite matrices with nonnegative real entries for which (2.2) holds. Let* $\{T_j\}$ *be a sequence of positive linear operators from* $L_{p,q}(loc)$ *into itself. Assume that*

$$\lim_{k} \sup_{n} \left\| B_k^{(n)}(f_i; x) - f_i(x) \right\|_{p,q} = 0$$

where $f_i(y) = y^i$ *for* $i = 0, 1, 2$. *Then for any functions* $f \in L_{p,q}(loc)$ *we have*

$$\lim_{k} \sup_{n} \left(\sup_{x \in \mathbb{R}} \frac{\left\| B_k^{(n)} f - f; L_p(x-h, x+h) \right\|}{q^*(x)} \right) = 0$$

where q^* *is a weight function such that* $\lim\limits_{|x| \to \infty} \frac{1+x^2}{q^*(x)} = 0$.

Proof. By hypothesis, given $\varepsilon > 0$, there exists x_0 such that for all x with $|x| \geq x_0$ we have

$$\frac{1+x^2}{q^*(x)} < \varepsilon. \tag{2.10}$$

Let $f \in L_{p,q}(loc)$. Then, for all n, k we get

$$\begin{aligned}
\gamma_k^{(n)} &:= \left\| B_k^{(n)} f - f; L_p\left(|x| > x_0\right) \right\| \\
&\leq \left\| B_k^{(n)} f - f \right\|_{p,q} \\
&\leq \sum_j a_{k,j}^{(n)} \left\| T_j f \right\|_{p,q} + \|f\|_{p,q} \\
&\leq \|f\|_{p,q} \left(\sum_j a_{k,j}^{(n)} \left\| T_j \right\|_{p,q} + 1 \right) < N, \text{ say.}
\end{aligned}$$

Hence we have $\sup\limits_{n,k} \gamma_k^{(n)} < \infty$. By Lusin's theorem we can find a continuous function φ on $[-x_0 - h, x_0 + h]$ such that

$$\|f - \varphi; L_p(-x_0 - h, x_0 + h)\| < \varepsilon. \tag{2.11}$$

Now we consider the function G defined in [13] given by

$$G(x) := \begin{cases} \varphi(-x_0 - h), & x \leq -x_0 - h \\ \varphi(x_0), & |x| < x_0 + h \\ \varphi(x_0 + h), & x \geq x_0 + h. \end{cases}$$

We see that G is continuous and bounded on the whole real axis.
Now let $f \in L_{p,q}(loc)$. Then we get for all n, k that

$$\begin{aligned}
\beta_k^{(n)} &:= \left\| \sum_j a_{k,j}^{(n)} T_j f - f; L_{p,q}(-x_0, x_0) \right\| \\
&\leq \|f - G; L_{p,q}(-x_0 - h, x_0 + h)\| \left(\sum_j a_{k,j}^{(n)} \left\| T_j \right\|_{p,q} + 1 \right) \\
&\quad + \left\| \sum_j a_{k,j}^{(n)} T_j G - G; L_{p,q}(-x_0, x_0) \right\|.
\end{aligned}$$

Hence by the hypothesis and Lemma 2.3 we have

$$\lim_k \sup_n \beta_k^{(n)} = 0. \tag{2.12}$$

On the other hand, a simple calculation shows that

$$\begin{aligned}
u_k^{(n)} &:= \left\| \sum_j a_{k,j}^{(n)} T_j f - f \right\|_{p,q^*} \\
&< \beta_k^{(n)} \sup_{|x| < x_0} \frac{q(x)}{q^*(x)} + \gamma_k^{(n)} \sup_{|x| \geq x_0} \frac{q(x)}{q^*(x)} \\
&< \beta_k^{(n)} M + \varepsilon \gamma_k^{(n)}, \text{ (for some } M > 0). \qquad (2.13)
\end{aligned}$$

It follows from (2.10), (2.11), (2.12), (2.13) and Lemma 2.3 that

$$\begin{aligned} u_k^{(n)} &< q(x_0)\,\|f-G; L_{p,q}(-x_0-h, x_0+h)\| \left(\sum_j a_{k,j}^{(n)} \|T_j\|_{p,q} + 1\right) \\ &+ q(x_0)\left\|\sum_j a_{k,j}^{(n)} T_j G - G; L_{p,q}(-x_0, x_0)\right\| + \varepsilon N \\ &= K\varepsilon + q(x_0)\left\|\sum_j a_{k,j}^{(n)} T_j G - G; L_{p,q}(-x_0, x_0)\right\| \end{aligned}$$

where $K := Mq(x_0) + N$ and $M := H + 1$. By Lemma 2.3 we get

$$\lim_k \sup_n \left(\sup_{x\in\mathbb{R}} \frac{\left\|B_k^{(n)} f - f; L_p(x-h, x+h)\right\|_{p,q}}{q^*(x)}\right) = 0.$$

Remark 2.7. We now present an example of a sequence of positive linear operators which satisfies Theorem 2.6 but does not satisfy Theorem 2.2. Assume now that $\mathcal{A} := \left\{A^{(n)}\right\} = \left\{a_{kj}^{(n)}\right\}$ is a sequence of infinite matrices defined by

$$a_{k,j}^{(n)} = \begin{cases} \frac{1}{k+1}, & n \le j \le n+k \\ 0, & \text{otherwise.} \end{cases}$$

In this case $\mathcal{A}-$summability method reduces to almost convergence, ([15]).

Let $T_j : L_{p,q}(loc) \to L_{p,q}(loc)$ be given by

$$T_j(f;x) = \begin{cases} \frac{x^2}{(x+h)^2} f(x+h), & x \in [(2j-1)h, (2j+1)h) \\ f(x), & \text{otherwise.} \end{cases}$$

The sequence $\{T_j\}$ satisfies Theorem 1 in [13]. It is also shown in [13] that for all $j \in \mathbb{N}$, $\|T_j f\|_{p,q} \le 4\|f\|_{p,q}$. Hence $\{T_j\}$ is an uniformly bounded sequence of positive linear operators from $L_{p,q}(loc)$ into itself. Also

$$\lim_k \sup_n \left\|B_k^{(n)}(f_i;x) - f_i(x)\right\|_{p,q} = 0$$

where $f_i(y) = y^i$ for $i = 0, 1, 2$. Now we define $\{P_j\}$ by

$$P_j(f;x) = (1+u_j)\,T_j(f;x)$$

where

$$u_j = \begin{cases} 1, & j = 2^n,\ n \in \mathbb{N} \\ 0, & \text{otherwise.} \end{cases}$$

It is easy to see that $\{u_j\}$ is almost convergent to zero. Therefore the sequence of positive linear operators $\{P_j\}$ satisfies Theorem 2.6 but does not satisfy Theorem 2.2.

3. Rates of convergence for $\mathcal{A}-$summation process in $L_{p,q}(loc)$

In this section, using the modulus of continuity, we study rates of convergence of operators given in Theorem 2.6.

We consider the following modulus of continuity:

$$w(f,\delta) = \sup_{|x-y|\le\delta} |f(y) - f(x)|$$

where δ is a positive constant, $f \in L_{p,q}(loc)$. It is easy to see that, for any $c > 0$ and all $f \in L_{p,q}(loc)$,

$$w(f, c\delta) \le (1 + [c])\, w(f,\delta), \tag{3.1}$$

where $[c]$ is defined to be the greatest integer less than or equal to c.

We first need the following lemma

Lemma 3.1. *Let $\mathcal{A} := \{A^{(n)}\} = \left\{a_{kj}^{(n)}\right\}$ be a sequence of infinite matrices with non-negative real entries. Let $\{T_j\}$ be a sequence of positive linear operators from $L_{p,q}(loc)$ into itself. Then for each $j \in \mathbb{N}$ and $\delta > 0$, and for every function f that is continuous and bounded on the whole real axis, we have*

$$\begin{aligned}\left\|B_k^{(n)} f - f; L_{p,q}(a,b)\right\| &\le w(f;\delta)\left\|B_k^{(n)} f_0 - f_0\right\|_{p,q} \\ &\quad + 2w(f;\delta) + C_1\left\|B_k^{(n)} f_0 - f_0\right\|_{p,q}\end{aligned}$$

where $f_0(t) = 1, \varphi_x(t) := (t-x)^2, C_1 = \sup_{a\le x\le b} |f(x)|$ and

$$\delta := \alpha_k^{(n)} = \sqrt{\left\|B_k^{(n)}\varphi_x\right\|_{p,q}}.$$

Proof. Let f be any continuous and bounded function on the real axis, and let $x \in [a,b]$ be fixed. Using linearity and monotonicity of T_j and for any $\delta > 0$, by (3.1), we get

$$\begin{aligned}\left|B_k^{(n)}(f;x) - f(x)\right| &\le B_k^{(n)}(|f(t) - f(x)|;x) + |f(x)|\left|B_k^{(n)}(f_0;x) - f_0(x)\right| \\ & B_k^{(n)}\left(w\left(f, \frac{|t-x|}{\delta}\delta\right), x\right) \\ &+ |f(x)|\left|B_k^{(n)}(f_0;x) - f_0(x)\right| \\ &\le w(f,\delta)\, B_k^{(n)}\left(1 + \left[\frac{t-x}{\delta}\right], x\right) \\ &+ |f(x)|\left|B_k^{(n)}(f_0;x) - f_0(x)\right| \\ &\le w(f,\delta)\left|B_k^{(n)}(f_0;x) - f_0(x)\right| + w(f,\delta) \\ &+ \frac{w(f,\delta)}{\delta^2}\left|B_k^{(n)}\varphi_x\right| + |f(x)|\left|B_k^{(n)}(f_0;x) - f_0(x)\right|.\end{aligned}$$

Now let $C_1 = \sup_{a<x<b} |f(x)|$ and $\delta := \alpha_k^{(n)} = \sqrt{\left\| B_k^{(n)} \varphi_x \right\|_{p,q}}$. Then we have

$$\left\| B_k^{(n)} f - f \right\|_{p,q} \leq w(f,\delta) \left\| B_k^{(n)} (f_0;x) - f_0(x) \right\|_{p,q} + 2w(f,\delta) + C_1 \left\| B_k^{(n)} (f_0;x) - f_0(x) \right\|_{p,q}.$$

Theorem 3.2. *Let* $\mathcal{A} : = \{A^{(n)}\} = \left\{a_{kj}^{(n)}\right\}$ *be a sequence of infinite matrices with nonnegative real entires. Let* $\{T_j\}$ *be a sequence of positive linear operators from* $L_{p,q}(loc)$ *into itself. Assume that for each continuous and bounded function* f *on the real line, the following conditions hold:*

(i) $\lim_k \sup_n \left\| B_k^{(n)} (f_0;x) - f_0(x) \right\|_{p,q} = 0,$

(ii) $\lim_k \sup_n w(f,\delta) = 0.$

Then we have

$$\lim_k \sup_n \left\| B_k^{(n)} f - f \right\|_{p,q} = 0.$$

Proof. Using Lemma 3.1 and considering (i) and (ii), we have

$$\lim_k \sup_n \left\| B_k^{(n)} f - f; L_{p,q}(a,b) \right\| = 0$$

for all continuous and bounded functions on the real axis.

Theorem 3.3. *Let* $\mathcal{A} : = \{A^{(n)}\} = \left\{a_{kj}^{(n)}\right\}$ *be a sequence of infinite matrices with nonnegative real entries for which* (2.2) *holds. Let* $\{T_j\}$ *be a sequence of positive linear operators from* $L_{p,q}(loc)$ *into itself. For a given* $f \in L_{p,q}(loc)$ *assume that*

$$\lim_k \sup_n \left\| B_k^{(n)} (f_i;x) - f_i(x) \right\|_{p,q} = 0$$

where $f_i(y) = y^i$ *for* $i = 0,1,2.$ *If*

(i) $\lim_k \sup_n \left\| B_k^{(n)} (f_0;x) - f_0(x) \right\|_{p,q} = 0,$

(ii) $\lim_k \sup_n w(G,\delta) = 0$

where G *is given as in the proof of Theorem* 2.6. *Then we have*

$$\lim_k \sup_n \left(\sup_{x\in\mathbb{R}} \frac{\left\| B_k^{(n)} f - f; L_p(x-h,x+h) \right\|}{q^*(x)} \right) = 0$$

where q^* *is a weight function such that*

$$\lim_{|x|\to\infty} \frac{1+x^2}{q^*(x)} = 0.$$

Proof. It is known from Theorem 2.6 that

$$\begin{aligned} u_k^{(n)} &< q(x_0) \|f - G; L_{p,q}(-x_0 - h, x_0 + h)\| \left(\sum_j a_{kj}^{(n)} \|T_j\|_{p,q} + 1 \right) \\ &+ q(x_0) \left\| B_k^{(n)} G - G; L_{p,q}(-x_0, x_0) \right\| + \varepsilon N \\ &= K\varepsilon + q(x_0) \left\| B_k^{(n)} G - G; L_{p,q}(-x_0, x_0) \right\| \end{aligned}$$

where $K := Mq(x_0) + N$ and $M := H + 1$. Then by Lemma 3.1 and Theorem 3.2 we get

$$\begin{aligned} u_k^{(n)} &\leq K\varepsilon + q(x_0) w(G; \delta) \left\| B_k^{(n)}(f_0; x) - f_0(x) \right\|_{p,q} + 2q(x_0) w(G; \delta) \\ &+ q(x_0) C_1' \left\| B_k^{(n)}(f_0; x) - f_0(x) \right\|_{p,q} \end{aligned}$$

where $C_1' := \sup\limits_{-x_0 < x < x_0} |G(x)|$ and the proof is completed.

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References

[1] Altomare, F., Campiti, M., *Korovkin Type Approximation Theory and its Applications*, de Gruyter, Berlin, 1994.

[2] Aslan, I., Duman, O., *Summability on Mellin-Type nonlinear integral operators*, Integral Transforms and Special Function, **30**(2019), no. 6, 492-511.

[3] Bayram, N.Ş., *Strong summation process in locally integrable function spaces*, Hacet. J. Math. Stat., **45**(2016), no. 3, 683-694.

[4] Bayram, N.Ş., Orhan, C., *Abel convergence of the sequence of positive linear operators in $L_{p,q}$ (loc)*, Bulletin of the Belgian Mathematical Society, Simon Stevin, **26**(2019), 71-83.

[5] Bernau, S.J., *Theorems of Korovkin type for Lp spaces*, Pacific J. Math., **53**(1974), 11-19.

[6] Costin, O., Dunne, G.V., *Convergence from divergence*, J. Phys. A, **51**(2018), no. 4, 10 pp.

[7] Devore, R.A., *The Approximation of Continuous Functions by Positive Linear Operators*, Lecture Notes in Mathematics, Springer-Verlag, Berlin, 293, 1972.

[8] Donner, K., *Korovkin theorems in Lp spaces*, J. Functional Analysis, **42**(1981), 12-28.

[9] Duman, O., Orhan, C., *Statistical approximation in the space of locally integrable functions*, Publ. Math. Debrecen, **63**(2003), 134-144.

[10] Duman, O., Orhan, C., *Rates of A-statistical convergence of operators in the space of locally integrable functions*, Appl. Math. Letters, (2008), 431-435.

[11] Dzyadik, V.K., *On the approximation of functions by linear positive operators and singular integrals*, (Russian), Mat. Sbornik, **70**(112)(1966), 508-517.

[12] Gadjiev, A.D., *On P.P. Korovkin type theorems*, Math. Zametki, **20**(1976).

[13] Gadjiev, A.D., Efendiyev, R.O., İbikli, E., *On Korovkin's type theorem in the space of locally integrable functions*, Czech. Math. J., **53**(128)(2003), 45-53.

[14] Korovkin, P.P., *Linear Operators and the Theory of Approximation*, Hindustan Publ. Co. Delhi, 1960.

[15] Lorentz, G.G., *A contribution to the theory of divergent sequences*, Acta. Math., **80**(1948), 167-190.

[16] Orhan, C., Sakaoğlu, İ., *Rate of convergence in Lp approximation*, Periodica Mathematica Hungarica, **68**(2014), 176-184.

[17] Sakaoğlu, İ., Orhan, C., *Strong summation process in Lp spaces*, Nonlinear Analysis, **86**(2013), 89-94.

[18] Stepanets, A.I., *Approximations in spaces of locally integrable functions*, Ukrainian Math. J., **46**(1994), no. 5, 638-670.

[19] Stepanets, A.I., *Approximations in spaces of locally integrable functions*, (Russian), Akad. Nauk Ukrainy Inst. Mat. Preprint no. **18**(1993), 47 pp.

[20] Swetits, J.J., Wood, B., *On degree of L_p-approximation with positive linear operators*, Journal of Approximation Theory, **87**(1996), 239-241.

[21] Taberski, R., *Aprroximation properties of the integral Bernstein operators and their derivatives in some classes of locally integrable functions*, Funct. Approx. Comment. Math., **21**(1992), 85-96.

[22] Zygmund, A., *Trigonometric Series*, Cambridge University Press, 1979.

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Constrained visualisation using Shepard-Bernoulli interpolation operator

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Abstract. We consider Shepard-Bernoulli operator in order to solve a problem of interpolation of scattered data that is constrained to preserve positivity, using the technique described by K.W. Brodlie, M.R. Asim and K. Unsworth (2005). We also give some numerical examples.

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1. Introduction

The interpolation operators and the radial basis functions are the usual tools used for approximating scattered data. Sometimes we may have data that have to preserve some constraints, subject to certain physical laws (e.g., the densities, percentage mass concentrations in a chemical reaction, volume and mass are always positive, see [1], [2]); such cases require to impose some special conditions to the interpolants.

The Shepard method is a well suited method for multivariate interpolation of very large scattered data sets, but it does not guarantee to preserve positivity.

In [3] and [4] there have been introduced some combined Shepard operators of Bernoulli type which diminish the drawbacks of the Shepard operator. In [4] the combined operators are obtained using the classical and the modified Shepard operator, introduced, respectively, in [12] and [8]. They preserve the advantages and improve the reproduction qualities, have better accuracy and better computational efficiency.

We recall some results from [6]. Bernoulli polynomials are defined by:

$$\begin{cases} B_0(x) = 1, \\ B_n'(x) = nB_{n-1}(x), & n \geq 1, \\ \displaystyle\int_0^1 B_n(x)dx = 0, & n \geq 1. \end{cases} \tag{1.1}$$

For $f \in C^m[a,b]$, the univariate Bernoulli interpolant is given by

$$(B_m f)(x) := B_m[f;a,b] = f(a) + \sum_{i=1}^{m} S_i\left(\frac{x-a}{h}\right)\frac{h^{i-1}}{i!}\Delta_h f^{(i-1)}(a), \tag{1.2}$$

where $h = b-a$ and

$$S_i\left(\frac{x-a}{h}\right) = B_i\left(\frac{x-a}{h}\right) - B_i, \qquad i \geq 1, \tag{1.3}$$
$$\Delta_h f^{(i-1)}(a) = f^{(i-1)}(b) - f^{(i-1)}(a), \;\; 1 \leq i \leq m.$$

Let $X = [a,b]\times[c,d]$. Denote $h := b-a$, $k := d-c$ and consider the operators:

$$\begin{aligned}
&\Delta_{(h,0)} f(x,y) := f(x+h,y) - f(x,y), \\
&\Delta_{(0,k)} f(x,y) := f(x,y+k) - f(x,y), \\
&\Delta_{(h,k)} f(x,y) := \Delta_{(h,0)}\Delta_{(0,k)} f(x,y) = \Delta_{(0,k)}\Delta_{(h,0)} f(x,y).
\end{aligned} \tag{1.4}$$

For $f \in C^{m,n}(X)$, the Bernoulli interpolant on the rectangle is [6]:

$$\begin{aligned}
(B_{m,n}f)(x,y) :=& f(a,c) + \sum_{i=1}^{m}\Delta_{(h,0)} f^{(i-1,0)}(a,c)\frac{h^{i-1}}{i!}S_i\left(\frac{x-a}{h}\right) \\
&+ \sum_{j=1}^{n}\Delta_{(0,k)} f^{(0,j-1)}(a,c)\frac{k^{j-1}}{j!}S_j\left(\frac{y-c}{k}\right) \\
&+ \sum_{i=1}^{m}\sum_{j=1}^{n}\Delta_{(h,k)} f^{(i-1,j-1)}(a,c)\frac{h^{i-1}k^{j-1}}{i!j!}S_i\left(\frac{x-a}{h}\right)S_j\left(\frac{y-c}{k}\right),
\end{aligned} \tag{1.5}$$

where S_k, $k > 1$ are given in (1.3). The polynomial from (1.5) satisfies the following interpolation conditions [6]:

$$\begin{aligned}
(B_{m,n}f)(a,c) &= f(a,c), \\
(\Delta_{(h,0)}B_{m,n}f)^{(i,0)}(a,c) &= \Delta_{(h,0)} f^{(i,0)}(a,c), \quad 0 \leq i \leq m-1, \\
(\Delta_{(0,k)}B_{m,n}f)^{(0,j)}(a,c) &= \Delta_{(0,k)} f^{(0,j)}(a,c), \quad 0 \leq j \leq n-1, \\
(\Delta_{(h,k)}B_{m,n}f)^{(i,j)}(a,c) &= \Delta_{(h,k)} f^{(i,j)}(a,c), \quad 0 \leq i \leq m-1, 0 \leq j \leq n-1.
\end{aligned} \tag{1.6}$$

The Shepard method, introduced in [12], is a well suited method for multivariate interpolation of very large scattered data sets. The bivariate Shepard operator is given by

$$(Sf)(x,y) = \sum_{i=0}^{N} A_{i,\mu}(x,y) f(x_i,y_i), \tag{1.7}$$

where

$$A_{i,\mu}(x,y) = \frac{\prod\limits_{\substack{j=0\\ j\neq i}}^{N} r_j^{\mu}(x,y)}{\sum\limits_{k=0}^{N}\prod\limits_{\substack{j=0\\ j\neq k}}^{N} r_j^{\mu}(x,y)}, \tag{1.8}$$

with $\mu > 0$ and $r_i\,(x,y)$ are the distances between (x,y) and the given points (x_i,y_i), $i = 0,1,...,N$.

In [4] we have introduced the bivariate Shepard-Bernoulli operator that preserve the advantages and improve the reproduction qualities, have better accuracy and computational efficiency:

$$(S_B f)(x,y) = \sum_{i=0}^{N} A_{i,\mu}(x,y)(B^i_{m,n}f)(x,y), \qquad \mu > 0, \tag{1.9}$$

where $B^i_{m,n}f$ denotes the Bernoulli interpolant $B_{m,n}[f;(x_i,y_i),(h_i,k_i)]$ in the rectangle with opposite vertices (x_i,y_i), (x_{i+1},y_{i+1}), given by (1.5), having $h_i = x_{i+1} - x_i$, $k_i = y_{i+1} - y_i$, $i = 0,...,N$.

Remark 1.1. The operator S_B has the following interpolation properties:

$$(S_B f)(x_p,y_p) = f(x_p,y_p),\ 0 \le p \le N; \mu > m+n-2$$

and its degree of exactness is (m,n).

There are flat spots at each data point and the accuracy tends to decrease in the areas where the interpolation nodes are sparse. This can be improved using the local version of Shepard interpolation, introduced by Franke and Nielson in [8] and improved in [7], [10], [11]:

$$(Sf)\,(x,y) = \frac{\sum\limits_{i=0}^{N} W_i\,(x,y)\,f\,(x_i,y_i)}{\sum\limits_{i=0}^{N} W_i\,(x,y)}, \tag{1.10}$$

with

$$W_i\,(x,y) = \left[\frac{(R_w - r_i)_+}{R_w r_i}\right]^2,$$

where R_w is a radius of influence about the node (x_i,y_i) and it is varying with i. This is taken as the distance from node i to the jth closest node to (x_i,y_i) for $j > N_w$ (N_w is a fixed value) and j as small as possible within the constraint that the jth closest node is significantly more distant than the $(j-1)$st closest node (see, e.g. [10]). As it is mentioned in [9], this modified Shepard method is one of the most powerful software tools for the multivariate interpolation of large scattered data sets.

With these assumptions, for $f \in C^{(m,n)}(X)$ and distinct points $(x_i,y_i) \in X$, $i = 0,...,N$, the Shepard-Bernoulli operator, given in (1.9), becomes (see [4]):

$$(S^w_B f)\,(x,y) := \frac{\sum\limits_{i=0}^{N} W_i\,(x,y)\,(B^i_{m,n}f)(x,y)}{\sum\limits_{i=0}^{N} W_i\,(x,y)}. \tag{1.11}$$

2. Constraints of the Shepard-Bernoulli operator

There are two most important classes of interpolation methods of very large scattered data sets: radial basis functions and Shepard type methods. Both are widely used in practice.

There could be cases when the data are inherently positive. We will make the modified Shepard-Bernoulli operator to preserve positivity by forcing the quadratic basis functions to be positive, using the method introduced in [2].

The modified Shepard-Bernoulli operator preserves the advantages of the classical Shepard operator and improves the reproduction qualities, have better accuracy and computational efficiency. There are cases when we have additional information to take into account in reconstruction by interpolation as the case when the information are the subject to certain physical laws that constrain their behavior. In [2] there are mentioned the case when the information refer to some densities and the case when data values are specified as fractions of a whole. In the first case the reconstruction must be positive and in the second must be within $[0,1]$ to be realistic.

The classical Shepard operator S, given in (1.7) satisfies the following property:

$$\min\{f\,(x_i,y_i)\} \leq (Sf)(x,y) \leq \max\{f\,(x_i,y_i)\}, \qquad i=0,...,N. \tag{2.1}$$

A consequence of this property is that a positive interpolant is guaranteed if the data values are positive.

The modified Shepard operator, given in (1.10), has superior qualities but it does not satisfy the property (2.1).

For a function $f \in C^{(m,n)}(X)$, $X=[a,b]\times[c,d]$ and a set of $N+1$ distinct points $(x_i,y_i)\in X,\ i=0,...,N$, we consider Shepard-Bernoulli operator given by (1.9). We will impose constraints to positivity, using the method from [2].

We will use as a basis function a linear transformation of the old one, namely the function

$$(C_B^i f)(x,y) = \alpha(B_{m,n}^i f)(x,y)+\beta, \qquad i=0,...,N \tag{2.2}$$

where α and β are chosen as

$$\alpha = \frac{f(x_i,y_i)}{f(x_i,y_i)-\displaystyle\min_{(x,y)\in[x_i,x_{i+1}]\times[y_i,y_{i+1}]}\{B_{m,n}^i f(x,y)\}} \in [0,1]$$

$$\beta = (1-\alpha)f(x_i,y_i), \qquad \text{for } i=1,...N.$$

Remark 2.1. $B_{m,n}^i f$, $i=0,...,N$ could have negative values but the constrained function $C_B^i f$, $i=0,...,N$ have just positive values.

Theorem 2.2. *If (x_A,y_A) and (x_B,y_B) are two points such that*

$$(B_{m,n}^i f)(x_A,y_A) \leq (B_{m,n}^i f)(x_B,y_B)$$

then

$$(C_B^i f)(x_A,y_A) \leq (C_B^i f)(x_B,y_B).$$

Proof. The proof follows directly taking into account the form (2.2). □

Remark 2.3. The method can be extended to handle other types of constraints, for example, in the interval $[0,1]$ or, furthermore, in any arbitrary interval $[a,b]$, $a > b$, $a,b \in \mathbb{R}$.

The constrained Shepard-Bernoulli operator of first kind is given by

$$(S_B^c f)(x,y) = \sum_{i=0}^{N} A_{i,\mu}(x,y)(C_B^i f)(x,y), \qquad \mu > 0, \tag{2.3}$$

with $A_{i,\mu}(x,y)$ given in (1.8).

Theorem 2.4. *For $f \in C^{(m,n)}(X)$ the operator S_B has the following interpolation properties:*

$$(S_B^c f)(x_p, y_p) = f(x_p, y_p),$$

for $0 \le p \le N$ and $\mu > m + n - 2$.

Proof. We have

$$\begin{aligned}(S_B^c f)(x_p, y_p) &= \sum_{i=0}^{N} A_{i,\mu}(x_p,y_p)(C_B^i f)(x_p,y_p) \\ &= \alpha \sum_{i=0}^{N} A_{i,\mu}(x_p,y_p)(B_{m,n}^i f)(x_p,y_p) + \beta \sum_{i=0}^{N} A_{i,\mu}(x_p,y_p)\end{aligned}$$

Taking into account that $\sum\limits_{i=0}^{N} A_{i,\mu}(x_p,y_p) = 1$, we get

$$\begin{aligned}(S_B^c f)(x_p, y_p) &= \alpha \sum_{i=0}^{N} A_{i,\mu}(x_p,y_p)(B_{m,n}^i f)(x_p,y_p) + \beta \\ &= \alpha (S_B f)(x,y)(x_p,y_p) + \beta\end{aligned}$$

and by the interpolation properties of S_B (given in [4]) the conclusion follows. □

Theorem 2.5. *The degree of exactness of the operator S_B^c is (m,n).*

Proof. The proof follows considering the form of

$$(C_B^i f)(x,y) = \alpha (B_{m,n}^i f)(x,y) + \beta$$

and the property that degree of exactness of the operator S_B is (m,n), (as it was proved in [4]). □

We consider also the modified Shepard-Bernoulli operator given by (1.11). We will keep the benefits of the modified Shepard-Bernoulli interpolation and impose constraints, using the previous method (see [2]).

The constrained Shepard-Bernoulli operator of second kind is given by

$$(S_B^{c,w} f)(x,y) := \frac{\sum_{i=0}^{N} W_i(x,y)\,(C_B^i f)(x,y)}{\sum_{i=0}^{N} W_i(x,y)}, \tag{2.4}$$

Remark 2.6. If $f(x_i, y_i) = 0$ for any i then $\alpha = 0$ and $\beta = f(x_i, y_i)$ therefore the interpolants becomes the classical Shepard interpolants.

3. Numerical examples

We consider the following test functions ([7], [10], [11]):

$$f_1(x,y) = \exp\left[-\frac{81}{16}((x-0.5)^2 + (y-0.5)^2)\right]/3, \qquad \text{(Gentle)}$$

$$f_2(x,y) = \sqrt{64 - 81((x-0.5)^2 + (y-0.5)^2)}/9 - 0.5 \quad \text{(Sphere)}$$

Table 1 contains the maximum errors for approximating by the Shepard, Shepard-Bernoulli, the modified Shepard-Bernoulli and the coresponding constrained methods, (2.3) and (2.4), considering 52 random generated nodes in the unit square, $m = n = 2$ and $\mu = 2$. By numerical experiments we have obtained that for these data the optimal value for N_w is $N_w = 8$. We compare the obtained numerical results with some combined Shepard operators known in the literature, namely with the combined Shepard operators of Lagrange, Hermite and Taylor type, denoted respectively by S_L, S_H and S_T.

In Figures 1 and 2 we plot the graphs of f_i, $S_B f_i$, $S_B^w f_i$, $S_B^c f_i$, $S_B^{c,w} f_i$, for $i = 1, 2$.

Table 1. Maximum approximation errors.

	f_1	f_2
Sf	0.1870	0.2374
$S_B f$	0.0905	0.0274
$S_B^w f$	0.0628	0.0187
$S_B^c f$	0.2975	0.3563
$S_B^{c,w} f$	0.3085	0.3767
$S_L f$	0.5353	0.5798
$S_H f$	0.4646	0.0883
$S_T f$	0.2110	0.3635

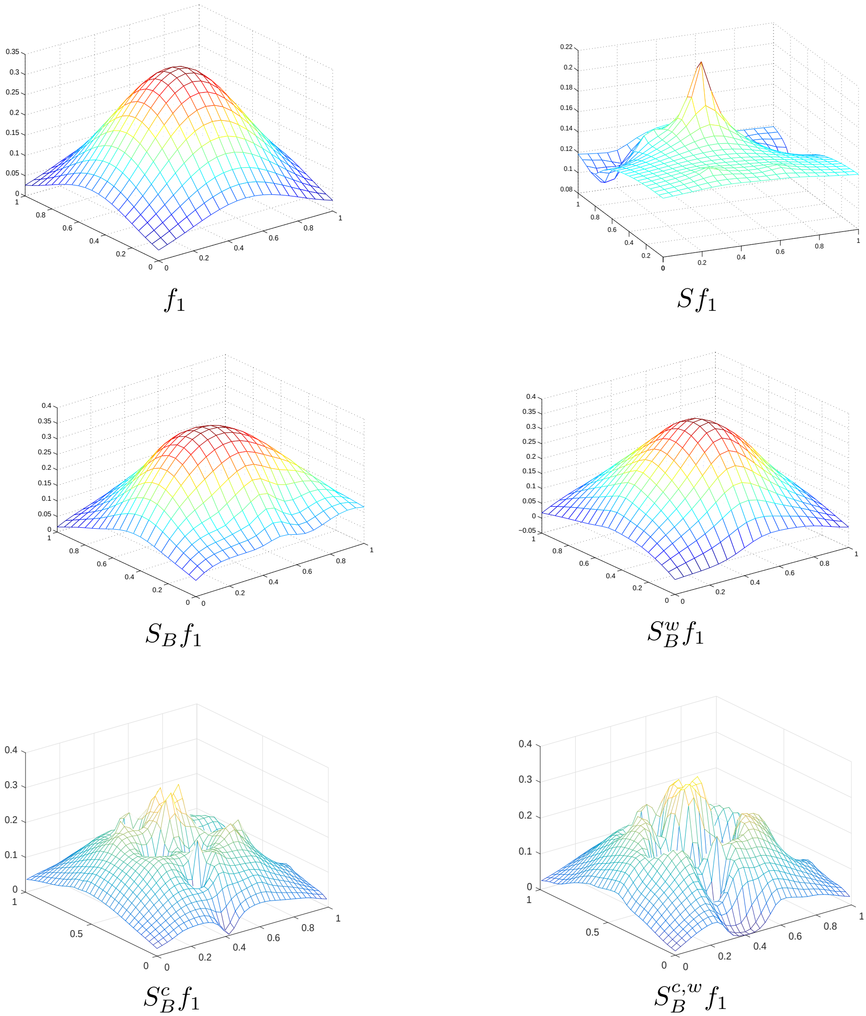

Figure 1. Graphs for the function f_1.

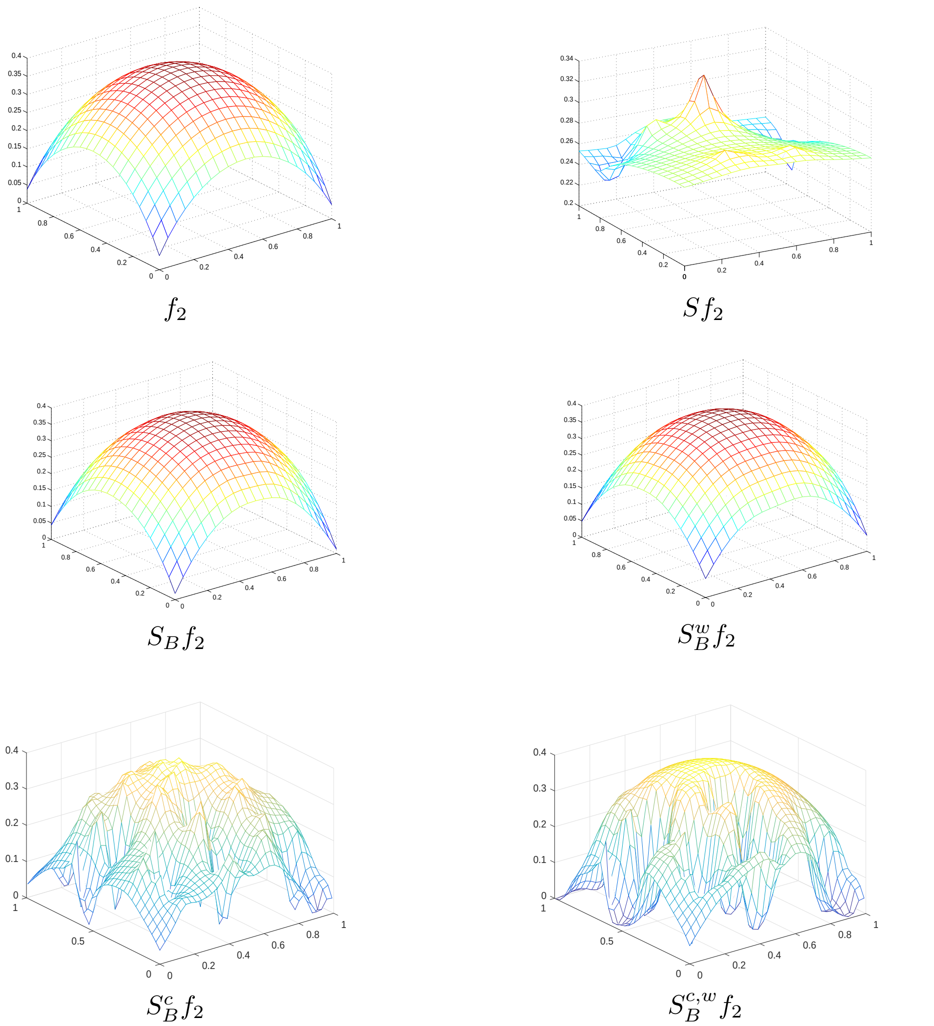

Figure 2. Graphs for the function f_2.

References

[1] Asim, M.R., Mustafa, G., Brodlie, K.W., *Constrained Visualization of 2D Positive Data using Modified Quadratic Shepard Method*, WSCG, 2004.

[2] Brodlie, K.W., Asim, M.R., Unsworth, K., *Constrained Visualization Using the Shepard Interpolation Family,* Computer Graphics Forum, **24**(2005), 809-820.

[3] Caira, R., Dell'Accio, F., *Shepard-Bernoulli operators,* Math. Comp., **76**(2007), 299-321.

[4] Cătinaş, T., *The bivariate Shepard operator of Bernoulli type,* Calcolo, **44**(2007), 189-202.

[5] Coman, Gh., *The remainder of certain Shepard type interpolation formulas*, Stud. Univ. Babeş-Bolyai Math., **32**(1987), no. 4, 24-32.

[6] Costabile, F.A., Dell'Accio, F., *Expansion Over a Rectangle of Real Functions in Bernoulli Polynomials and Applications*, BIT, **41**(2001), no. 3, 451-464.

[7] Franke, R., *Scattered data interpolation: tests of some methods,* Math. Comp., **38**(1982), 181-200.

[8] Franke, R., Nielson, G., *Smooth interpolation of large sets of scattered data,* Int. J. Numer. Meths. Engrg., **15**(1980), 1691-1704.

[9] Lazzaro, D., Montefusco, L.B., *Radial basis functions for multivariate interpolation of large scattered data sets,* J. Comput. Appl. Math., **140**(2002), 521-536.

[10] Renka, R.J., *Multivariate interpolation of large sets of scattered data,* ACM Trans. Math. Software, **14**(1988), 139-148.

[11] Renka, R.J., Cline, A.K., *A triangle-based C^1 interpolation method,* Rocky Mountain J. Math., **14**(1984), 223-237.

[12] Shepard, D., *A two dimensional interpolation function for irregularly spaced data,* Proc. 23rd Nat. Conf. ACM, 1968, 517-523.

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King-type operators related to squared Szász-Mirakyan basis

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Abstract. In this paper we study some approximation properties of a sequence of positive linear operators defined by means of the squared Szász-Mirakyan basis and prove that these operators behave better than the classical Szász-Mirakyan operators.

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1. Introduction

The operators defined by

$$S_n(f,x)=\sum_{k=0}^{\infty} s_{n,k}(x) f\left(\frac{k}{n}\right), \quad x\in[0,\infty),\ n=1,2,\dots,$$

where $s_{n,k}$ are

$$s_{n,k}(x)=e^{-nx}\frac{(nx)^k}{k!},$$

were introduced and studied independently by Mirakyan [14], Favard [3] and Szász [17]. They usually are referred to as Szász-Mirakyan operators and the functions $s_{n,k}$ form the Szász-Mirakyan basis or the Poisson distribution.

Motivated by the article of Gavrea and Ivan [4] we study the following operators

$$A_n(f,x)=\frac{\displaystyle\sum_{k=0}^{\infty} s_{n,k}^2(x) f\left(\frac{k}{n}\right)}{\displaystyle\sum_{k=0}^{\infty} s_{n,k}^2(x)}, \quad x\ge 0,\ n=1,2,\dots \tag{1.1}$$

Herzog [5] introduced and studied the following sequence of positive linear operators

$$A_n^\nu(f,x)=\begin{cases}\dfrac{1}{I_\nu(nx)}\displaystyle\sum_{k=0}^{\infty}\frac{\left(\frac{nx}{2}\right)^{2k+\nu}}{\Gamma(k+1)\Gamma(k+1+\nu)}\cdot f\left(\frac{2k}{n}\right), & x>0\\ f(0), & x=0\end{cases}$$

where I_ν is the modified Bessel function defined by

$$I_\nu(t)=\sum_{k=0}^{\infty}\frac{\left(\frac{nt}{2}\right)^{2k+\nu}}{\Gamma(k+1)\Gamma(k+1+\nu)}.$$

For $\nu=0$ the operators A_n^ν can be written in terms of the operators (1.1) by

$$A_n^0(f,x)=A_n(f\circ g^{-1},g(x)),$$

where g is the function defined by $g(x)=x/2$, $x\geq 0$.

The author of [5] studied the operators A_n^ν in polynomial and exponential weight spaces (see also [6]), but did not point out how well behave these operators compared to the Szász-Mirakyan operators.

In this paper, we show that A_n are King-type operators [12] preserving the functions e_0 and e_2 and so extending the class of Szász-Mirakyan type operators which preserve some polynomial functions [2, 18]. We also prove that the error of approximation of a function f by A_nf is smaller than the error of approximation by the classical Szász-Mirakyan operators. In the final part of the paper, we present some approximation properties of (A_n), showing what functions can be uniformly approximated by these operators and what is the order of the convergence by giving a quantitative Voronovskaya theorem. A similar study for Bernstein operators was done recently in [4, 9] and for Baskakov operators in [10].

2. Some properties of the operators

Let us notice first that the operators A_n preserve the functions e_0 and e_2 (we denote as usual $e_k(x)=x^k$). From the relation (1.1) we can easily see that

$$A_n(e_0,x)=e_0(x)=1.$$

From the following relation

$$\sum_{k=0}^{\infty}s_{n,k}^2(x)\cdot\frac{k^2}{n^2}=e^{-2nx}\sum_{k=0}^{\infty}\frac{(nx)^{2k}}{(k!)^2}\cdot\frac{k^2}{n^2}=x^2e^{-2nx}\sum_{k=1}^{\infty}\frac{(nx)^{2k-2}}{[(k-1)!]^2}$$
$$=x^2e^{-2nx}\sum_{i=0}^{\infty}\frac{(nx)^{2i}}{(i!)^2}=x^2\sum_{i=0}^{\infty}s_{n,i}^2(x).$$

we deduce that $A_n(e_2,x)=e_2(x)=x^2$, for every $x\geq 0$. In fact, only for $\nu=0$, the general operators A_n^ν do preserve the function e_2. This can be seen from the following relation obtained in [5]

$$A_n^\nu(e_2,x) = x^2 \cdot \frac{I_{\nu+2}(nx)}{I_\nu(nx)} + \frac{2x}{n} \cdot \frac{I_{\nu+1}(nx)}{I_\nu(nx)}$$

and the recurrence relation (9.6.26) of [1]

$$I_{\nu-1}(t) - I_{\nu+1}(t) = \frac{2\nu}{t} I_\nu(t).$$

We have

$$A_n^\nu(e_2,x) = x^2 - \frac{2x\nu}{n} \cdot \frac{I_{\nu+1}(nx)}{I_\nu(nx)}.$$

So, $A_n^\nu(e_2) = e_2$ if and only if $\nu = 0$.

Let us denote

$$\mu_{n,k}(x) = A_n((e_1 - x)^k, x), \quad k = 0, 1, 2, \ldots$$

the central moments of the operators A_n, which will be very important in our study.

Next let us observe that

$$\mu_{n,2}(x) = -2x\mu_{n,1}(x). \tag{2.1}$$

Indeed,

$$\mu_{n,2}(x) = A_n(e_2,x) - 2xA_n(e_1,x) + x^2A_n(e_0,x) = 2x^2 - 2xA_n(e_1,x) = -2x\mu_{n,1}(x).$$

Lemma 2.1. *For every $x \in (0,\infty)$ we have*

$$\lim_{n\to\infty} 4n \cdot \mu_{n,1}(x) = -1 \tag{2.2}$$

$$\lim_{n\to\infty} 2n \cdot \mu_{n,2}(x) = x. \tag{2.3}$$

Proof. Because of the relation (2.1) it suffices to prove (2.2).
Let us denote

$$K_n(x) = \sum_{k=0}^{\infty} s_{n,k}^2(x). \tag{2.4}$$

The function K_n was expressed [15] in terms of the modified Bessel function I_0 by

$$K_n(x) = e^{-2nx} I_0(2nx). \tag{2.5}$$

Using the well-known relation

$$s'_{n,k}(x) = s_{n,k}(x) \cdot \frac{k - nx}{x}$$

we have

$$2n \cdot \mu_{n,1}(x) = 2n \left(\frac{\sum_{k=0}^{\infty} s_{n,k}^2(x) \cdot \frac{k}{n}}{K_n(x)} - x \right) = \frac{2\sum_{k=0}^{\infty} s_{n,k}^2(x)(k-nx)}{K_n(x)}$$

$$= \frac{2x\sum_{k=0}^{\infty} s_{n,k}(x)s'_{n,k}(x)}{K_n(x)} = \frac{xK'_n(x)}{K_n(x)} = \frac{2nx[I'_0(2nx) - I_0(2nx)]}{I_0(2nx)}.$$

We have obtained a formula expressing the central moment of order 1 in terms of the modified Bessel function I_0:

$$\mu_{n,1}(x) = x\left(\frac{I'_0(2nx)}{I_0(2nx)} - 1\right). \tag{2.6}$$

For every $x \in (0,\infty)$ the quantity $t = 2nx$ grows to infinity when n tends to infinity. Using the asymptotic relations (9.7.1) and (9.7.3) from Abramowitz and Stegun [1]

$$\begin{aligned} I_0(t) &\sim \frac{e^t}{\sqrt{2\pi t}}\left(1 + \frac{1}{8t} + \frac{9}{2(8t)^2} + \dots\right) \quad (t\to\infty) \\ I'_0(t) &\sim \frac{e^t}{\sqrt{2\pi t}}\left(1 - \frac{3}{8t} - \frac{15}{2(8t)^2} - \dots\right) \quad (t\to\infty) \end{aligned} \tag{2.7}$$

we obtain

$$\mu_{n,1}(x) \sim -\frac{1}{4n} - \frac{1}{32n^2x} - \frac{15}{1024n^3x^2} - \dots \quad (n\to\infty)$$

which proves (2.2). □

Lemma 2.2. *The sequence $(n \cdot \mu'_{n,1}(x))$ converges to zero for every $x > 0$.*

Proof. Computing the derivative of $\mu_{n,1}$ we obtain

$$\mu'_{n,1}(x) = \frac{I'_0(2nx)}{I_0(2nx)} - 1 + 2nx \cdot \frac{I''_0(2nx)I_0(2nx) - [I'_0(2nx)]^2}{[I_0(2nx)]^2}.$$

Using the relation $tI''_0(t) + I'_0(t) - tI_0(t) = 0$ (see (9.6.1) from [1]), we have

$$\mu'_{n,1}(x) = 2nx - 1 - 2nx\frac{[I'_0(2nx)]^2}{[I_0(2nx)]^2}.$$

The asymptotic relations (2.7) show that

$$\mu'_{n,1}(x) \sim -\frac{29}{128(2nx)^2} + \frac{31}{1024(2nx)^3} + \dots \quad (n\to\infty)$$

and this proves the assertion stated in the lemma. □

Lemma 2.3. *For every $x \geq 0$ we have*

$$\mu_{n,2}(x) \leq S \cdot \frac{x}{n}, \tag{2.8}$$

where S is defined by

$$S = \sup_{x>0} \left(x - \frac{x^2}{\frac{1}{2} + \sqrt{x^2 + \frac{9}{4}}} \right) = 0.67038\ldots$$

Proof. Using (2.6) and (2.1) the central moment of order 2 can be expressed by

$$\mu_{n,2}(x) = 2x^2 \left(1 - \frac{I_0'(2nx)}{I_0(2nx)} \right).$$

To prove (2.8) it is enough to prove that

$$t \left(1 - \frac{I_0'(t)}{I_0(t)} \right) < S, \quad t > 0.$$

Using inequality (73) of [16] we have

$$\frac{tI_0'(t)}{I_0(t)} > \frac{t^2}{\frac{1}{2} + \sqrt{\frac{9}{4} + t^2}}.$$

But this proves that

$$t \left(1 - \frac{I_0'(t)}{I_0(t)} \right) < t - \frac{t^2}{\frac{1}{2} + \sqrt{\frac{9}{4} + t^2}} \le S. \qquad \square$$

Remark 2.4. Because the second central moment of the usual Szász-Mirakyan operators is $\frac{x}{n}$, inequality (2.8) proves that the central moment of order 2 of the operators (1.1) is smaller than the classical Szász-Mirakyan operators. In addition, we use the estimation

$$|L_n(f,x) - f(x)| \le (1 + n\mu_{n,2}(x)) \cdot \omega \left(f, \frac{1}{\sqrt{n}} \right),$$

which is valid for every sequence of positive linear operators (L_n) preserving constants functions and for every uniformly continuous function f. This estimation proves that the error by approximating f with $A_n f$ is smaller than the error of approximation by the classical Szász-Mirakyan operators.

We prove in the next Lemma that A_n satisfy a differential equation. This equation is similar to the relation satisfied by the so called exponential type operators (see [13, 11]).

Lemma 2.5. *For every $f \in C[0,1]$ and $x \in (0,1)$ we have*

$$(A_n(f,x))' = \frac{2n}{x} \left[A_n(f \cdot (e_1 - xe_0), x) - A_n(e_1 - xe_0, x) \cdot A_n(f,x) \right]. \tag{2.9}$$

Proof. Using again

$$s_{n,k}'(x) = s_{n,k}(x) \cdot \frac{k - nx}{x}$$

we get

$$\left(\frac{s_{n,k}^2(x)}{\sum_{i=0}^n s_{n,i}^2(x)}\right)' = \frac{2s_{n,k}(x)s'_{n,k}(x)}{\sum_{i=0}^n s_{n,i}^2(x)} - \frac{2s_{n,k}^2(x)\sum_{i=0}^n s_{n,i}(x)s'_{n,i}(x)}{\left(\sum_{i=0}^n s_{n,i}^2(x)\right)^2}$$

$$= \frac{2s_{n,k}^2(x)}{\sum_{i=0}^n s_{n,i}^2(x)} \cdot \left(\frac{k-nx}{x} - \frac{\sum_{i=0}^n s_{n,i}^2(x)\frac{i-nx}{x}}{\sum_{i=0}^n s_{n,i}^2(x)}\right)$$

$$= \frac{2s_{n,k}^2(x)}{\sum_{i=0}^n s_{n,i}^2(x)} \cdot \left(\frac{k}{x} - \frac{\sum_{i=0}^n s_{n,i}^2(x)\frac{i}{x}}{\sum_{i=0}^n s_{n,i}^2(x)}\right)$$

$$= \frac{2n}{x} \cdot \frac{s_{n,k}^2(x)}{\sum_{i=0}^n s_{n,i}^2(x)} \cdot \left(\frac{k}{n} - \frac{\sum_{i=0}^n s_{n,i}^2(x)\frac{i}{n}}{\sum_{i=0}^n s_{n,i}^2(x)}\right).$$

We obtain

$$(A_n(f,x))' = \frac{2n}{x} \cdot A_n(f \cdot (e_1 - A_n(e_1,x)),x)$$

which is equivalent with (2.9). □

Lemma 2.6. *We have for every $x > 0$*

$$\lim_{n\to\infty} (2n)^2 \cdot \mu_{n,4}(x) = 3x^2.$$

Proof. Using Lemma 2.2 and (2.1) the following limit holds true for every $x > 0$

$$\lim_{n\to\infty} 2n \cdot \mu'_{n,2}(x) = \lim_{n\to\infty} -4n\mu_{n,1}(x) - 4nx\mu'_{n,1}(x) = 1.$$

In relation (2.9) we take $f = (e_1 - xe_0)^k$ and we obtain the recurrence relation

$$(\mu_{n,k}(x))' + k \cdot \mu_{n,k-1}(x) = \frac{2n}{x} \cdot \left[\mu_{n,k+1}(x) - \mu_{n,1}(x) \cdot \mu_{n,k}(x)\right], \tag{2.10}$$

which is similar to the relation (2.7) of Ismail and May [11]. Using (2.10) we get

$$2n\mu_{k+1}(x) = x\mu'_{n,k}(x) + kx\mu_{n,k-1}(x) + 2n\mu_{n,1}(x)\mu_{n,k}(x), \quad k = 1,2,\ldots$$

For $k = 2$ we have

$$2n\mu_3(x) = x\mu'_{n,2}(x) + 2x\mu_{n,1}(x) + 2n\mu_{n,1}(x)\mu_{n,2}(x).$$

Multiplying this equality with $2n$ and using the relations (2.2) and (2.3), we have for every x

$$\lim_{n\to\infty} 4n^2 \cdot \mu_{n,3}(x) = -\frac{x}{2}.$$

For $k = 3$, the recurrence (2.10) becomes

$$\mu'_{n,3}(x) + 3\mu_{n,2}(x) = \frac{2n}{x} \cdot [\mu_{n,4}(x) - \mu_{n,1}(x)\mu_{n,3}(x)]\,.$$

Multiplying with $2n$ and letting n tend to infinity we get

$$\lim_{n\to\infty} 4n^2 \cdot \mu_{n,4}(x) = 3x^2,$$

for every $x > 0$, if $2n\mu'_{n,3}(x) \to 0$. We prove this convergence.

Applying the derivative to the relation (2.10) for $k = 2$ we get

$$\begin{aligned} 2n\mu'_{n,3}(x) &= 2n\mu_{n,1}(x)\mu'_{n,2}(x) + 2n\mu_{n,2}(x)\mu'_{n,1}(x) \\ &\quad + \mu'_{n,2}(x) + x\mu''_{n,2}(x) + 2x\mu'_{n,1}(x) + 2\mu_{n,1}(x). \end{aligned}$$

It remains to prove that $\mu''_{n,2}$ converges to zero.

Applying the derivative twice to the relation (2.1), the sequence $(\mu''_{n,2})$ converges to zero if and only if the sequence $\mu''_{n,1}$ converges to zero. But applying the derivative to the relation (2.10) for $k = 1$ we obtain

$$2n\mu'_{n,2}(x) = 4n\mu_{n,1}(x)\mu'_{n,1}(x) + \mu'_{n,1}(x) + x\mu''_{n,1}(x) + 1.$$

Using that $2n\mu'_{n,2}(x) \to 1$ we obtain that $\mu''_{n,1} \to 0$ and the lemma is proved. □

3. Some approximation results

In order to give some approximation results for the operators A_n, let us introduce some notation.

For $\alpha \geq 0$, we denote by $C_{\theta,\alpha}$ the space of all continuous functions defined on the positive half-line $f : (0,\infty) \to \mathbb{R}$ with the property that exists a constant $M > 0$ such that $|f(x)| \leq Me^{\alpha\theta(x)}$, for every $x > 0$. We denote with C_θ the union of all spaces $C_{\theta,\alpha}$.

Let us observe that for $\theta(x) = x$, the functions $A_n f$ exist for every $f \in C_{\theta,\alpha}$. To prove this, it is enough to prove that $A_n(e^{\alpha t})$ exist. We will prove more in the next lemma.

Lemma 3.1. *The sequence $A_n(e^{\alpha t}, x)$ converges pointwise to the function $e^{\alpha x}$.*

Proof. We have

$$A_n(e^{\alpha t}, x) = \frac{I_0\left(2nxe^{\frac{\alpha}{2n}}\right)}{I_0(2nx)}.$$

For a fixed $x \in (0,\infty)$ we use the asymtotic relation (2.7) and we obtain

$$A_n(e^{\alpha t}, x) \sim \frac{e^{2nxe^{\frac{\alpha}{2n}}}}{\sqrt{2\pi \cdot 2nxe^{\frac{\alpha}{2n}}}} \cdot \frac{\sqrt{2\pi \cdot 2nx}}{e^{2nx}} \sim e^{2nx(e^{\frac{\alpha}{2n}}-1)} \sim e^{\alpha x} \quad (n \to \infty).$$ □

Remark 3.2. The Lemma 3.1 implies that for a fixed $x>0$ we have

$$A_n(\max(e^{\alpha t},e^{\alpha x}),x)\le M_\alpha(x), \tag{3.1}$$

for every $n\in\mathbb{N}$. Indeed, for $x>0$, there is n_0 such that

$$\left|A_n(e^{\alpha t},x)-e^{\alpha x}\right|\le 1,\qquad \text{for every } n\ge n_0.$$

We obtain for every $n\ge n_0$

$$A_n(\max(e^{\alpha t},e^{\alpha x}),x)\le A_n(e^{\alpha t}+e^{\alpha x},x)\le 1+2e^{\alpha x}.$$

The inequality (3.1) is true for

$$M_\alpha(x)=1+2e^{\alpha x}+\max_{n\le n_0}A_n(\max(e^{\alpha t},e^{\alpha x}),x).$$

Remark 3.3. As was pointed out in Remark 7.2.1 of [6], the function A_nf does not necessarily belong to the space $C_{\theta,\alpha}$ when f belong to the space $C_{\theta,\alpha}$, for $\theta(x)=x$. We prove that for $\theta(x)=\sqrt{x}$, this condition is satisfied as in the case of the classical Szász-Mirakyan operators (see [7]).

Lemma 3.4. *There is a constant $M_\alpha>0$ not depending on n or x such that*

$$A_n(e^{\alpha\sqrt{t}},x)\le M_\alpha e^{\alpha\sqrt{x}}, \tag{3.2}$$

for every $x>0$, $\alpha\ge 0$ and $n\in\mathbb{N}$.

Proof. We need to prove that $A_n(e^{\alpha(\sqrt{t}-\sqrt{x})},x)$ is bounded.
Starting from the inequality

$$\sqrt{t}-\sqrt{x}=\frac{t-x}{\sqrt{t}+\sqrt{x}}\le\frac{t-x}{\sqrt{x}},\quad x>0 \tag{3.3}$$

we obtain that

$$A_n(e^{\alpha(\sqrt{t}-\sqrt{x})},x)\le A_n(e^{\frac{\alpha(t-x)}{\sqrt{x}}},x)=\frac{A_n(e^{\frac{\alpha t}{\sqrt{x}}},x)}{e^{\alpha\sqrt{x}}}=\frac{I_0\left(2nxe^{\frac{\alpha}{2n\sqrt{x}}}\right)}{I_0(2nx)\cdot e^{\alpha\sqrt{x}}}.$$

Using again (2.7) we deduce the existence of a constant $t_0>0$ such that

$$\frac{e^t}{2\sqrt{2\pi t}}<I_0(t)<\frac{3e^t}{2\sqrt{2\pi t}},\qquad\text{for every }\ t>t_0.$$

So, for $x>\frac{t_0}{2n}$ and $n\in\mathbb{N}$

$$\begin{aligned}A_n(e^{\alpha(\sqrt{t}-\sqrt{x})},x)&\le 3\frac{e^{2nxe^{\frac{\alpha}{2n\sqrt{x}}}}}{\sqrt{2\pi\cdot 2nxe^{\frac{\alpha}{2n\sqrt{x}}}}}\cdot\frac{\sqrt{2\pi\cdot 2nx}}{e^{2nx}\cdot e^{\alpha\sqrt{x}}}\\&\le 3\exp\left(2nx(e^{\frac{\alpha}{2n\sqrt{x}}}-1)-\alpha\sqrt{x}\right).\end{aligned}$$

Using the inequality $e^u-1\le u+u^2e^u$, $u\ge 0$, we obtain

$$\begin{aligned}A_n(e^{\alpha(\sqrt{t}-\sqrt{x})},x)&\le 3\exp\left(2nx\cdot\frac{\alpha}{2n\sqrt{x}}+2nx\cdot\frac{\alpha^2}{4n^2x}e^{\frac{\alpha}{2n\sqrt{x}}}-\alpha\sqrt{x}\right)\\&=3\exp\left(\frac{\alpha^2}{2n}e^{\frac{\alpha}{2n\sqrt{x}}}\right)\le 3\exp\left(\frac{\alpha^2}{2}e^{\frac{\alpha}{\sqrt{2t_0}}}\right).\end{aligned}$$

Consider now the case when x is smaller than $\frac{t_0}{2n}$. In this case, we need only prove that $A_n(e^{\alpha\sqrt{t}}, x)$ is bounded. Because $\sqrt{k} \leq k$, for every $k = 0, 1, 2, \ldots$ and $I_0(2nx) \geq 1$ we obtain

$$A_n(e^{\alpha\sqrt{t}}, x) \leq A_n(e^{t\alpha\sqrt{n}}, x) = \frac{I_0(2nxe^{\frac{\alpha}{2\sqrt{n}}})}{I_0(2nx)} \leq I_0\left(2nxe^{\frac{\alpha}{2\sqrt{n}}}\right) \leq I_0\left(t_0 e^{\frac{\alpha}{2}}\right). \qquad \square$$

We need the following general result.

Theorem 3.5 ([8])**.** *Let m be a nonnegative integer and let $f \in C_{\theta,\alpha}$ such that f is m times continuously differentiable with $f^{(m)} \in C_{\theta,\alpha}$. Then*

$$\left| L_n(f, x) - \sum_{k=0}^{m} \frac{f^{(k)}(x)}{k!} \cdot \mu_{n,k}(x) \right| \leq \frac{1}{m!}\left(A_{n,m}(x) + \frac{B_{n,m}(x)}{\delta_n}\right)\omega_{\varphi,\theta,\alpha}\left(f^{(m)}, \delta_n\right)$$

where

$$A_{n,m}(x) = L_n\left(\max\left(e^{\alpha\theta(t)}, e^{\alpha\theta(x)}\right)|t - x|^m, x\right)$$

$$B_{n,m}(x) = L_n\left(\max\left(e^{\alpha\theta(t)}, e^{\alpha\theta(x)}\right)|t - x|^m \cdot |\varphi(t) - \varphi(x)|, x\right)$$

$$\omega_{\varphi,\theta,\alpha}(f, \delta) = \sup_{\substack{x,t \in I \\ |\varphi(t)-\varphi(x)| \leq \delta}} \frac{|f(t) - f(x)|}{\max\left(e^{\alpha\theta(t)}, e^{\alpha\theta(x)}\right)}$$

and φ is a continuous and strictly increasing function on I such that $\theta \circ \varphi^{-1}$ is uniformly continuous on $\varphi(I)$.

Theorem 3.6. *Let $\theta(x) = \varphi(x) = \sqrt{x}$. For every $f \in C_{\theta,\alpha}$ there is a constant $M > 0$ independent of n and x such that*

$$|A_n(f, x) - f(x)| \leq Me^{\alpha\sqrt{x}} \cdot \omega_{\varphi,\theta,\alpha}\left(f, \frac{1}{\sqrt{n}}\right),$$

for every $x > 0$ and $n \in \mathbb{N}$.

Proof. We apply Theorem 3.5 for $m = 0$ and $\delta_n = \frac{1}{\sqrt{n}}$. Using inequality (3.2) we easily obtain that $A_{n,0}(x) \leq C_1 e^{\alpha\sqrt{x}}$, for every $x > 0$, for some constant $C_1 > 0$. Using the Cauchy-Schwarz inequality for positive linear operators the quantity $B_{n,0}(x)$ is bounded by

$$B_{n,0}(x) \leq \sqrt{A_n(\max(e^{2\alpha\sqrt{t}}, e^{2\alpha\sqrt{x}}), x)} \cdot \sqrt{A_n(|\varphi(t) - \varphi(x)|^2, x)}.$$

Using inequalities (3.3) and (2.8) we have for $x > 0$

$$A_n(|\varphi(t) - \varphi(x)|^2, x) \leq \frac{1}{x} \cdot \mu_{n,2}(x) \leq \frac{S}{n}.$$

Using again (3.2), the inequality

$$\sqrt{n} \cdot B_{n,0}(x) \leq C_2$$

is true for every $x > 0$ and $n \geq 1$, where C_2 is some constant independent of n and x. $\square$

Corollary 3.7. *For every function f such that $g(x) = e^{-x} f(x^2)$ is uniformly continuous on $(0, \infty)$ we have*

$$\lim_{n\to\infty} \sup_{x>0} e^{-\alpha\sqrt{x}} |A_n(f, x) - f(x)| = 0.$$

Proof. Because g is uniformly continuous, $\omega_{\varphi,\theta,\alpha}(f, 1/\sqrt{n}) \to 0$ when $n \to \infty$ (see [8]). □

Theorem 3.8. *For $\alpha \geq 0$, $\theta(x) = x$ and $\varphi(x) = x$ let $f \in C_{\theta,\alpha}$ be a twice continuously differentiable function such that $f'' \in C_{\theta,\alpha}$. Then*

$$\left| A_n(f, x) - f(x) - \mu_{n,1}(x) f'(x) - \frac{\mu_{n,2}(x)}{2} f''(x) \right|$$
$$\leq \frac{1}{2} \left(\sqrt{\mu_{n,4}(x) M_{2\alpha}(x)} + \sqrt{n} \cdot \sqrt[4]{M_{4\alpha}(x)} \cdot \sqrt[4]{[\mu_{n,4}(x)]^3} \right) \cdot \omega_{\varphi,\theta,\alpha} \left(f'', \frac{1}{\sqrt{n}} \right),$$

for every $x > 0$ and $n \in \mathbb{N}$.

Proof. We use Theorem 3.5 for $m = 2$ and $\delta_n = \frac{1}{\sqrt{n}}$. We have

$$A_{n,2}(x) \leq \sqrt{A_n(\max(e^{2\alpha t}, e^{2\alpha x}), x)} \cdot \sqrt{A_n(|t - x|^4, x)} \leq \sqrt{\mu_{n,4}(x) M_{2\alpha}(x)}.$$

Using Hölder inequality for $p = 4$ and $q = 4/3$ we obtain

$$\begin{aligned} B_{n,2}(x) &= A_n(\max(e^{\alpha t}, e^{\alpha x}) |t - x|^3, x) \\ &\leq \left(A_n(\max(e^{4\alpha t}, e^{4\alpha x}), x)\right)^{\frac{1}{4}} \cdot \left(A_n\left(|t - x|^4, x\right)\right)^{\frac{3}{4}} \\ &\leq \sqrt[4]{M_{4\alpha}(x)} \cdot \sqrt[4]{[\mu_{n,4}(x)]^3}. \end{aligned}$$

□

Corollary 3.9. *For every $f \in C_{\theta,\alpha}$, with $\theta(x) = x$ such that f'' exists and*

$$g(x) = e^{-x} f''(x)$$

is uniformly continuous on $(0, \infty)$ and for every $x > 0$

$$\lim_{n\to\infty} n[A_n(f, x) - f(x)] = -\frac{1}{4} \cdot f'(x) + \frac{x}{4} \cdot f''(x).$$

Proof. Because g is uniformly continuous on $(0, \infty)$, the quantity $\omega_{\varphi,\theta,\alpha} \left(f'', \frac{1}{\sqrt{n}} \right)$ tends to zero as n goes to infinity. We multiply with n the inequality proved in Theorem 3.8 and we take the limit as n tends to infinity, using Lemma 2.6 and the relations (2.2) and (2.3). The right-hand side of this inequality is 0. □

Problem 3.10. We propose the reader to study the general operators

$$L_n(f, x) = \frac{\sum_{k=0}^{\infty} g(s_{n,k}(x)) f\left(\frac{k}{n}\right)}{\sum_{k=0}^{\infty} g(s_{n,k}(x))}, \quad x \geq 0, \ n = 1, 2, \ldots.$$

For $g(x) = x$ we obtain the classical Szász-Mirakyan operators. For $g(x) = x^2$ we have the operators studied in this paper. It would be interesting to study the operators for $g(x) = x^m$, related to the Rényi entropy and for $g(x) = x \ln x$, related to the Shannon entropy.

References

[1] Abramowitz, M., Stegun, I.A., *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, Dover Publications, Inc., New York, 1970.

[2] Duman, O., Özarslan, M.A., *Szász-Mirakjan type operators providing a better error estimation*, Appl. Math. Lett., **20**(2007), 1184-1188.

[3] Favard, J., *Sur les multiplicateurs d'interpolation*, J. Math. Pures Appl., **23**(1944), 219-247.

[4] Gavrea, I., Ivan, M., *On a new sequence of positive linear operators related to squared Bernstein polynomials*, Positivity, **23**(2019), no. 3, 571-580.

[5] Herzog, M., *Approximation theorems for modified Szasz-Mirakjan operators in polynomial weight spaces*, Le Mathematiche, **54**(1999), 77-90.

[6] Herzog, M., *Approximation of continuous and unbounded functions*, In: Selected Topics in Modern Mathematics Edition 2014, G. Gancarzewicz and M. Skrzyński (eds.), Publishing House AKAPIT, Kraków (2014), 99-114.

[7] Holhoş, A., *Uniform weighted approximation by positive linear operators*, Stud. Univ. Babeş-Bolyai Math., **56**(2011), no. 3, 135-146.

[8] Holhoş, A., *Quantitative estimates of Voronovskaya type in weighted spaces*, Results Math., **73**:53(2018).

[9] Holhoş, A., *Voronovskaya theorem for a sequence of positive linear operators related to squared Bernstein polynomials*, Positivity, (2018), https://doi.org/10.1007/s11117-018-0625-y.

[10] Holhoş, A., *A sequence of positive linear operators related to powered Baskakov basis*, Carpathian J. Math., **35**(2019), no. 1, 51-58.

[11] Ismail, M.E.H., May, C.P., *On a family of approximation operators*, J. Math. Anal. Appl., **63**(1978), 446-462.

[12] King, J.P., *Positive linear operators which preserve x^2*, Acta Math Hungar., **99**(2003), no. 3, 203-208.

[13] May, C.P., *Saturation and inverse theorems for combinations of a class of exponential-type operators*, Canad. J. Math., **28**(1976), 1224-1250.

[14] Mirakjan, G., *Approximation des fonctions continues au moyen de polynomes de la forme $e^{-nx} \sum_{k=0}^{m} C_{k,n} x^k$*, Dokl. Akad. Nauk, **31**(1941), 201-205.

[15] Raşa, I., *Entropies and Heun functions associated with positive linear operators*, Appl. Math. Comput., **268**(2015), 422-431.

[16] Segura, J., *Bounds for ratios of modified Bessel functions and associated Turán-type inequalities*, J. Math. Anal. Appl., **374**(2011), 516-528.

[17] Szász, O., *Generalization of S. Bernstein's polynomials to the infinite interval*, Journal of Research of the National Bureau of Standards, **45**(1950), 239-245.

[18] Yilmaz, Ö.G., Aral, A., Yeşildal, F.T., *On Szász-Mirakyan type operators preserving polynomials*, J. Numer. Anal. Approx. Theory, **46**(2017), 93-106.

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Modifying an approximation process using non-Newtonian calculus

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Abstract. In the present note we modify a linear positive Markov process of discrete type by using so called multiplicative calculus. In this framework, a convergence property and the error of approximation are established. In the final part some numerical examples are delivered.

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1. Introduction

The study of the linear methods of approximation, which are given by sequences of linear and positive operators, has become a firmly rooted part of an old area of mathematical research called Approximation Theory. It has a great potential for applications to a wide variety of issues.

The starting point of this note is a general approximation process of discrete type which acts on the real valued functions defined on a compact interval $K \subset \mathbb{R}$. Since a linear substitution maps any compact interval $[a, b]$ into $[0, 1]$, we will only consider functions defined on $[0, 1]$. Each operator L_n of the class to which we refer, uses an equidistant network with a flexible step of the form $\Delta_n = (k\lambda_n)_{0\le k\le n}$, where $(\lambda_n)_{n\ge1}$ is a strictly decreasing sequence of real numbers with the property

$$0 < \lambda_n \le \frac{1}{n},\ n \in \mathbb{N}. \tag{1.1}$$

The operators we are referring to are designed as follows

$$(L_n f)(x) = \sum_{k=0}^{n} a_k(\lambda_n; x) f(k\lambda_n),\ n \in \mathbb{N},\ x \in [0, 1], \tag{1.2}$$

where the function $a_k(\lambda_n;\cdot) : [0,1] \to \mathbb{R}_+$ is continuous for each $n \in \mathbb{N}$ and $k \in \{0,1,\ldots,n\}$. The condition (1.2) guarantees that Δ_n is indeed a network on the compact $[0,1]$.

Typically, the operators described by (1.2) satisfy the condition of reproducing constants. Being linear operators, this property is involved in achieving the following identity

$$\sum_{k=0}^{n} a_k(\lambda_n;x) = 1,\ \ x \in [0,1]. \tag{1.3}$$

Throughout the paper, we consider this as a working hypothesis. Note that such operators are called Markov operators.

At this point we refer to non-Newtonian calculus also called as *multiplicative calculus*. In the 1970s, Michael Grossman and Robert Katz [5] have developed this type of calculation moving the roles of subtraction and addition to division and multiplication. See also Dick Stanley's paper [8]. This type of calculus was also called *geometric calculus* in order to emphasize that changes in function arguments are measured by differences, while changes in values are measured in ratios. Recently, Bashirov et al. [1] have given the complete mathematical description of multiplicative calculus. In the last decade there have been extensions of this notion in different directions of mathematics, even if it has a relatively restrictive area of applications than the classical calculus of Newton and Leibniz covering only positive quantities. Of the area where this type of approach has proven efficacy, we mention the theory of economic growth, see, for example, [4].

In the present paper our aim is to bring up multiplicative calculus to the attention of researchers in the branch of positive approximation processes. We could find no references that treat any kind of multiplicative calculus to the above mentioned directions.

In the next two sections we present basic elements of multiplicative calculus and new results as regards the positive approximation processes. A particular case is treated at the end of the paper.

2. Preliminaries

We introduce the second central moment of L_n, $n \in \mathbb{N}$, operators, i.e.,

$$\mathcal{M}_2(L_n;x) := (L_n\varphi_x^2)(x), \quad \text{where} \quad \varphi_x(t) = |t-x|,$$

$(t,x) \in [0,1] \times [0,1]$. Taking into account Bohman-Korovkin criterion, since (1.3) is fulfilled, in order the sequence $(L_n)_{n\geq 1}$ to become an approximation process on the space $C([0,1])$, it is necessary and sufficient to take place the following relation

$$\lim_{n\to\infty} \mathcal{M}_2(L_n;x) = 0, \quad \text{uniformly in} \quad x \in [0,1]. \tag{2.1}$$

We also consider that this identity is achieved.

Set $\mathbb{R}_+^* = (0,\infty)$. Also, $B^+([0,1])$ stands for all strictly positive real valued functions defined on $[0,1]$ and

$$C^+([0,1]) = \{f \in B^+([0,1]) :\ f \text{ continuous on } [0,1]\}.$$

We collected some information about multiplicative calculus. Here, the symbol $\ominus$ represents the difference in non-Newtonian geometric calculus which means the division in the classical calculus. Consequently, $a \ominus b$ means a/b, provided that a/b makes sense.

In non-Newtonian geometric calculus, the multiplicative absolute value (or modulus) of an element $x \in (0, \infty)$ be a number $|x|^*$ such that

$$|x|^* = \begin{cases} x & , \quad x > 1 \\ 1 & , \quad x = 1 \\ 1/x & , \quad x < 1. \end{cases}$$

Owing to the definition of multiplicative absolute value, the multiplicative distance between two elements $x, y \in (0, \infty)$ is given by

$$|x \ominus y|^* = \left|\frac{x}{y}\right|^* = \begin{cases} x/y & , \quad x/y > 1 \\ 1 & , \quad x = y \\ y/x & , \quad x/y < 1. \end{cases}$$

From the definition of the multiplicative absolute value, it is obvious that $|x|^* \geq 1$ for all $x \in (0, \infty)$.

For a closed interval $I \subseteq \mathbb{R}$, denoting $\{f \mid f : I \to \mathbb{R}_+^*\} = \mathcal{F}^+(I)$, we present the following

Definition 2.1. A function $f \in \mathcal{F}^+(I)$ is said to tend to the limit $L > 0$ as x tends to $a \in I$, if, corresponding to any arbitrary chosen number $\varepsilon > 1$, there exists a positive number $\delta > 1$ such that

$$|f(x) \ominus L|^* < \varepsilon,$$

for all values of x for which

$$1 < |x \ominus a|^* < \delta.$$

Here

$$1 < |x \ominus a|^* < \delta \Longleftrightarrow \frac{a}{\delta} < x < a\delta$$

and

$$|f(x) \ominus L|^* < \varepsilon \Longleftrightarrow \frac{L}{\varepsilon} \leq f(x) < L\varepsilon.$$

We use the notation $\lim\limits_{x \to a} f(x) \overset{m}{=} L$ or $f(x) \xrightarrow{m} L$, $x \to a$.

Definition 2.2. A function $f \in \mathcal{F}^+(I)$ is said to be multiplicative continuous at $x = a \in I$, if

$$\lim_{x \to a} f(x) \overset{m}{=} f(a)$$

holds.

In other words, a function $f \in \mathcal{F}^+(I)$ is said to be multiplicative continuous at $x = a \in I$, if, corresponding to any arbitrary chosen number $\varepsilon > 1$, there exists a positive number $\delta > 1$ such that

$$|f(x) \ominus f(a)|^* < \varepsilon,$$

for all values of x for which

$$|x \ominus a|^* < \delta.$$

Similar to the classic modulus of smoothness, can be defined the modulus of multiplicative smoothness.

Definition 2.3. Let $I \subseteq \mathbb{R}_+$ be an interval and $f \in B^+(I)$. The modulus of multiplicative smoothness of f is denoted by $\omega^{\langle m \rangle}(f;\cdot)$ and is defined as follows

$$\omega^{\langle m \rangle}(f;\delta) = \sup_{\substack{1 \le |x \ominus t| \le \delta \\ x,t \in I}} |f(x) \ominus f(t)|^*, \ \delta \ge 1. \tag{2.2}$$

Remark 2.4. Examining relation (2.2) we deduce

i) $\omega^{\langle m \rangle}(f;1) = 1$;

ii) if $1 \le \delta_1 < \delta_2$, then $\omega^{\langle m \rangle}(f;\delta_1) \le \omega^{\langle m \rangle}(f;\delta_2)$,

consequently $\omega^{\langle m \rangle}(f;\cdot)$ is a non-decreasing function.

We associate to the operators defined by (1.2) with the fulfillment of hypotheses (1.3) and (2.1), the following operators

$$(L_n^{\langle m \rangle} f)(x) = \prod_{k=0}^{n} (f(k\lambda_n))^{a_k(\lambda_n;x)}, \ x \in [0,1], \tag{2.3}$$

for each function $f \in B^+([0,1])$. This new class of operators loses the linearity property.

We also notice that it keeps the constants. Indeed, by virtue of property (1.3), if $f(x) = c > 0$, $x \in [0,1]$, then $(L_n^{\langle m \rangle} c)(x) = c$, $x \in [0,1]$.

Further on, our goal is to highlight approximation properties of the sequence $(L_n^{\langle m \rangle})_{n \ge 1}$.

3. Results

Theorem 3.1. *Let $f \in B^+([0,1])$ and the operators $L_n^{\langle m \rangle}$, $n \in \mathbb{N}$, be defined by (2.3). The following relation*

$$\lim_{n \to \infty} (L_n^{\langle m \rangle} f)(x_0) \overset{m}{=} f(x_0) \tag{3.1}$$

holds at each point $x_0 \in (0,1]$ of multiplicative continuity of f.

Proof. Let $\varepsilon > 1$ be arbitrarily fixed. In order to prove the theorem we have to show

$$\left|(L_n^{\langle m \rangle} f)(x_0) \ominus f(x_0)\right|^* < \varepsilon$$

holds true at each point $x_0 \in (0,1]$ of multiplicative continuity of $f \in B^+([0,1])$.

If $f \in B^+([0,1])$ is a constant function then one has

$$(L_n^{\langle m \rangle} f)(x_0) = f(x_0),$$

and hence

$$\left|(L_n^{\langle m \rangle} f)(x_0) \ominus f(x_0)\right|^* = 1 < \varepsilon$$

holds true at every point $x_0 \in (0,1]$. This proves (3.1) for constant functions.

Now, we assume that $f \in B^+([0,1])$ is not a constant function. Since the multiplicative absolute value is always greater than or equal to $1\big(|\bullet|^* \geq 1\big)$, it is sufficient only to show that the inequality

$$\left|(L_n^{\langle m\rangle} f)(x_0) \ominus f(x_0)\right|^* < \varepsilon \tag{3.2}$$

is valid for $n \geq N$, N being a certain rank. By using (2.3) and (1.3) we can write

$$\left|(L_n^{\langle m\rangle} f)(x_0) \ominus f(x_0)\right|^* = \left|\prod_{k=0}^{n} \left(\frac{f(k\lambda_n)}{f(x_0)}\right)^{a_k(\lambda_n;x_0)}\right|^*. \tag{3.3}$$

Since $\lim\limits_{x\to x_0} f(x) \overset{m}{=} f(x_0)$, in accordance with Definition 2.2, there exists a positive number $\delta > 1$ such that

$$|f(x) \ominus f(x_0)|^* < \varepsilon, \tag{3.4}$$

for all values of x for which

$$|x \ominus x_0|^* < \delta. \tag{3.5}$$

We split up the set $J = \{0, 1, \ldots, n\}$ as follows

$$\begin{aligned} J_0 &= \{0\}, \\ J_1 &= \{k \in J\backslash J_0 : \ |k\lambda_n \ominus x_0|^* < \delta\}, \\ J_2 &= \{k \in J\backslash J_0 : \ |k\lambda_n \ominus x_0|^* \geq \delta\}. \end{aligned}$$

Returning at (3.2) we break down the product as follows

$$\begin{aligned} &\left|\prod_{k=0}^{n} \left(\frac{f(k\lambda_n)}{f(x_0)}\right)^{a_k(\lambda_n;x_0)}\right|^* \\ \leq\ &\left(\left|\frac{f(0)}{f(x_0)}\right|^*\right)^{a_0(\lambda_n;x_0)} \prod_{k\in J_1}\left(\left|\frac{f(k\lambda_n)}{f(x_0)}\right|^*\right)^{a_k(\lambda_n;x_0)} \prod_{k\in J_2}\left(\left|\frac{f(k\lambda_n)}{f(x_0)}\right|^*\right)^{a_k(\lambda_n;x_0)}. \end{aligned} \tag{3.6}$$

The first product can be increased in the following way

$$\prod_{k\in J_1}\left(\left|\frac{f(k\lambda_n)}{f(x_0)}\right|^*\right)^{a_k(\lambda_n;x_0)} < \varepsilon^{\sum\limits_{k\in J_1} a_k(\lambda_n;x_0)} \leq \varepsilon,$$

see (3.3) and (1.3). The relation $k \in J_0 \cup J_2$ involves

$$\begin{aligned} \sum_{k\in J_0\cup J_2} a_k(\lambda_n;x_0) &\leq \sum_{k\in J_0\cup J_2} \frac{(k\lambda_n - x_0)^2}{\delta^2} a_k(\lambda_n;x_0) \\ &\leq \frac{1}{\delta^2}\sum_{k=0}^{n}(k\lambda_n - x_0)^2 a_k(\lambda_n;x_0) \\ &= \frac{1}{\delta^2}\mathcal{M}_2(L_n;x_0). \end{aligned} \tag{3.7}$$

Based on (2.1), for any $\mu > 0$, there exists a rank $N \in \mathbb{N}$ such that

$$\mathcal{M}_2(L_n;x_0) < \mu, \ \text{for every } n \geq N. \tag{3.8}$$

Setting

$$\sup_{x\in[0,1]} \left| \frac{f(x)}{f(x_0)} \right|^* = M, \tag{3.9}$$

we can evaluate the last part from (3.6)

$$\prod_{k\in J_0\cup J_2} \left(\left| \frac{f(k\lambda_n)}{f(x_0)} \right|^* \right)^{a_k(\lambda_n;x_0)} \le M^{\sum\limits_{k\in J_0\cup J_2}(\lambda_n;x_0)} \le M^{\mu/\delta^2},\ n \ge N, \tag{3.10}$$

see (3.7) and (3.8). Returning at (3.6), we get

$$\forall\ \mu > 0,\ \exists\ N \in \mathbb{N},\ \forall\ n \ge N,\ \prod_{k=0}^{n} \left| \frac{f(k\lambda_n)}{f(x_0)} \right|^{a_k(\lambda_n;x_0)} < \varepsilon M^{\mu/\delta^2}.$$

If $M = 1$, then we obtain exactly the inequality (3.2). Otherwise ($M > 1$), choosing $\mu = \delta^2 \log_M \varepsilon > 0$, we obtain the same inequality from (3.2) with $\varepsilon := \varepsilon^2$ which does not alter the statement.
In this moment, the proof of (3.2) is completed, consequently (3.1) takes place. □

Next, we establish an upper bound of the error of approximation by using the modulus of multiplicative smoothness.

Theorem 3.2. *Let $f \in B^+([0,1])$ and the operators $L_n^{\langle m\rangle}$, $n \in \mathbb{N}$, be defined by (2.3). For n large enough, the following relation*

$$\left| (L_n^{\langle m\rangle} f)(x) \ominus f(x_0) \right|^* \le M\omega^{\langle m\rangle}(f;\delta),\ \delta \ge 1, \tag{3.11}$$

holds at each point $x_0 \in (0,1]$ of multiplicative continuity of f. The constant M is defined at (3.9).

Proof. We use the identity (3.3) and the decomposition of that product according to the relation (3.6). Based both on the definition of $\omega^{\langle m\rangle}(f;\cdot)$, see (2.2), and the inequalities set out in (3.10) that are valid for any $\mu > 0$ and n sufficiently large, we can write

$$\begin{aligned} \left| (L_n^{\langle m\rangle} f)(x_0) \ominus f(x_0) \right|^* &\le \left(\prod_{|k\lambda_n \ominus x_0|^* \le \delta} \omega^{\langle m\rangle}(f;\delta)^{a_k(\lambda_n;x_0)} \right) M^{\mu/\delta^2} \\ &\le M^{\mu/\delta^2} \omega^{\langle m\rangle}(f;\delta). \end{aligned}$$

Choosing $\mu = \delta^2$, we obtain the desired result. □

4. A special case

In this section, we give a particular example of operators satisfying the assumptions employed in the previous sections.

4.1. Multiplicative (geometric) Bernstein operators

By choosing $\lambda_n = \frac{1}{n}$, $a_k(\lambda_n; x) = p_{n,k}(x)$, where $p_{n,k}(x) = \left(\begin{array}{c} n \\ k \end{array} \right) x^k (1-x)^{n-k}$ is the Bernstein basis, then we obtain a special case of the operators (2.3), namely multiplicative (geometric) version of the celebrated Bernstein operators. More precisely, $B_n^{\langle m \rangle} : B^+[0,1] \to C^+[0,1]$ $(n \geq 1)$ is given by

$$\left(B_n^{\langle m \rangle} f \right)(x) = \prod_{k=0}^{n} \left[f\left(\frac{k}{n}\right) \right]^{p_{n,k}(x)}, \quad x \in [0,1]. \tag{4.1}$$

As a consequence of Theorem 3.1 for functions $f \in B^+([0,1])$ and for multiplicative (geometric) Bernstein operators $B_n^{\langle m \rangle}$, we have the following direct estimate:

Corollary 4.1. *Let $f \in B^+([0,1])$ be a function. Let the operators $B_n^{\langle m \rangle}$, $n \in \mathbb{N}$, be defined by (4.1). The following relation*

$$\lim_{n\to\infty} (B_n^{\langle m \rangle} f)(x_0) \overset{m}{=} f(x_0)$$

holds at each point $x_0 \in (0,1]$ of multiplicative continuity of f.

4.2. Graphical and Numerical Representations

In the recent period, many operators have been investigated that generalize the classical approximation operators and the theoretical approach is usually accompanied by illustrations of convergence properties of particular functions. The included graphics are realized using software programs. Among such papers, we randomly quote [3], [7], [2], [6], the last three appeared in the years 2019-2020.

Following this line, we give some graphical and numerical examples to illustrate the approximation results for multiplicative (geometric) Bernstein operators obtained in the present paper.

We note that in the Figures 1, 2 and 3, the graph with the red line belongs to the original function, the graph with the green line to the operators with $n = 2$, and finally the graph consisting of blue line to the operators with $n = 10$.

Example 4.2. Let us consider the function $f(x) = x^3+1$, and we take its corresponding multiplicative Bernstein operator $(B_n^{\langle m \rangle} f)(x)$ (4.1), that one has for $n = 2$ and for $n = 10$.

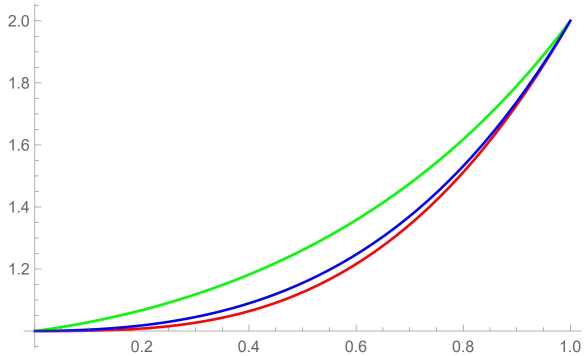

FIGURE 1. Approximation of $f(x) = x^3 + 1$ by multiplicative Bernstein operators (4.1), for $n = 2$ and $n = 10$.

In the following tables, by using the Wolfram Mathematica 11 Program, we compute the error of approximation numerically at certain points for $n = 100, 300$ and 500;

$x = 0.2$	$n = 100$	$n = 300$	$n = 500$
$\left(B_n^{\langle m\rangle} f\right)(0.2)$	1.00896	1.00832	1.00819
$f(0.2)$	1.008	1.008	1.008
The Error	0.000955622	0.000316981	0.000189999

,

$x = 0.5$	$n = 100$	$n = 300$	$n = 500$
$\left(B_n^{\langle m\rangle} f\right)(0.5)$	1.12811	1.12604	1.12562
$f(0.5)$	1.125	1.125	1.125
The Error	0.00310983	0.00103996	0.000624384

,

and finally

$x = 0.8$	$n = 100$	$n = 300$	$n = 500$
$\left(B_n^{\langle m\rangle} f\right)(0.8)$	1.5139	1.51263	1.51238
$f(0.8)$	1.512	1.512	1.512
The Error	0.00189809	0.000630805	0.000378252

.

Example 4.3. Let us consider the function $f(x) = \sin(x + 1)$, and we take its corresponding multiplicative Bernstein operator $(B_n^{\langle m\rangle} f)(x)$ (4.1), that one has for $n = 2$ and for $n = 10$.

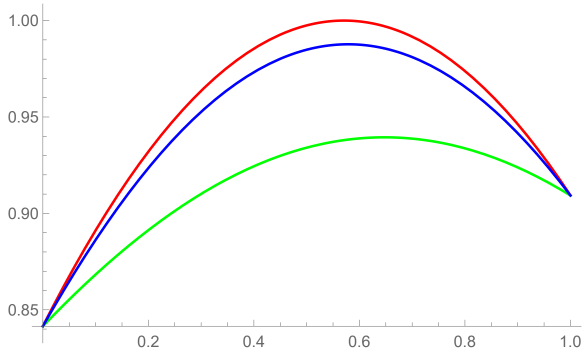

FIGURE 2. Approximation of $f(x) = \sin(x+1)$ by multiplicative Bernstein operators (4.1), for $n = 2$ and $n = 10$.

In the following tables, we compute the error of approximation numerically at certain points for $n = 100, 300$ and 500;

$x = 0.2$	$n = 100$	$n = 300$	$n = 500$
$\left(B_n^{\langle m \rangle} f\right)(0.2)$	0.931181	0.931753	0.931867
$f(0.2)$	0.932039	0.932039	0.932039
The Error	0.000857595	0.000286029	0.000171637

,

$x = 0.5$	$n = 100$	$n = 300$	$n = 500$
$\left(B_n^{\langle m \rangle} f\right)(0.5)$	0.996241	0.997077	0.997244
$f(0.5)$	0.997495	0.997495	0.997495
The Error	0.00125394	0.000417802	0.00025066

,

and finally

$x = 0.8$	$n = 100$	$n = 300$	$n = 500$
$\left(B_n^{\langle m \rangle} f\right)(0.8)$	0.973026	0.973574	0.973683
$f(0.8)$	0.973848	0.973848	0.973848
The Error	0.000821133	0.000273789	0.000164283

Example 4.4. Let us consider the function $f(x) = \sin x + 1$, and we take its corresponding multiplicative Bernstein operator $(B_n^{\langle m \rangle} f)(x)$ (4.1), that one has for $n = 2$ and for $n = 10$.

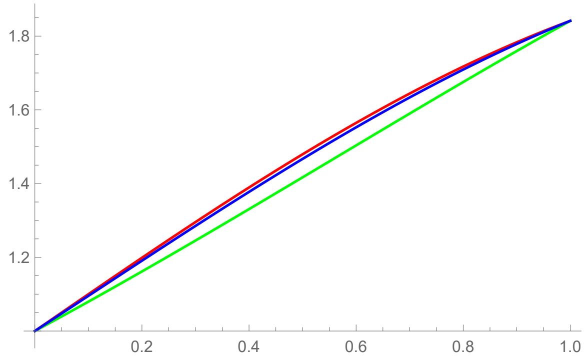

FIGURE 3. Approximation of $f(x) = \sin x + 1$ by multiplicative Bernstein operators (4.1), for $n = 2$ and $n = 10$.

In the following tables, we compute the error of approximation numerically at certain points for $n = 100, 300$ and 500;

$x = 0.2$	$n = 100$	$n = 300$	$n = 500$
$\left(B_n^{\langle m\rangle} f\right)(0.2)$	1.19787	1.1984	1.19851
$f(0.2)$	1.19867	1.19867	1.19867
The Error	0.000798903	0.000266545	0.000159956

,

$x = 0.5$	$n = 100$	$n = 300$	$n = 500$
$\left(B_n^{\langle m\rangle} f\right)(0.5)$	1.47818	1.47901	1.47918
$f(0.5)$	1.47943	1.47943	1.47943
The Error	0.00125027	0.000416697	0.000250011

,

and finally

$x = 0.8$	$n = 100$	$n = 300$	$n = 500$
$\left(B_n^{\langle m\rangle} f\right)(0.8)$	1.71656	1.71709	1.7172
$f(0.8)$	1.71736	1.71736	1.71736
The Error	0.000800703	0.000266745	0.000160028

Remark 4.5. Unlike error evaluation for linear and positive operators, from (3.11) we cannot deduce the convergence property of the sequence $(L_n^{\langle m\rangle} f)_{n\geq 1}$ to f. This note should be regarded as a pioneering activity in order to introduce multiplicative calculus in the field promoted by Korovkin type theory.

References

[1] Bashirov, A.E., Kurpinar, E.M., Özyapici, A., *Multiplicative calculus and its applications*, J. Math. Anal. Appl., **337**(2008), 36-48.

[2] Cai, Q.-Bo, Zhou, G., Li, J., *Statistical approximation properties of λ-Bernstein operators based on q-integers*, Open Math., **17**(2019), no. 1, 487-498.

[3] Cătinaş, T., *Some classes of surfaces generated by Nielson and Marshall type operators on the triangle with one curved side*, Stud. Univ. Babeş-Bolyai Math., **61**(2016), no. 3, 305-314.

[4] Filip, D.A., Piatecki, C., *A non-newtonian examination of the theory of exogenous economic growth*, Mathematica Aeterna, **4**(2014), no. 2, 101-117.

[5] Grossman, M., Katz, R., *Non-Newtonian Calculus*, Lee Press, Pigeon Cove, MA, 1972.

[6] Gupta, V., Agrawal, G., *Approximation for link Ismail-May operators*, Ann. Funct. Anal., https://doi.org/10.1007/s43034-019-00051-y, online: 13 January 2020.

[7] Mursaleen, M., Rahman, S., Ansari, K.J., *Approximation by Jakimovski-Leviatan-Stancu-Durrmeyer type operators*, Filomat, **33**(2019), no. 6, 1517-1530.

[8] Stanley, D., *A multiplicative calculus*, Primus: Problems, Resources, and Issues in Mathematics Undergraduate Studies, **9**(1999), no. 4, 310-326.

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Ulam stability of Volterra integral equation on a generalized metric space

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Abstract. The aim of this paper is to give some Ulam-Hyers stability results for Volterra integral equations on a generalized metric space. In this case we consider the Volterra integral equation in the Krasnoselski-Krein and Naguno-Perron-Van Kampen conditions. Here we present only Ulam-Hyers stability for the Volterra integral equation.

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Keywords: Volterra integral equations, Ulam-Hyers stability, generalized metric space, Krasnoselski-Krein conditions, Naguno-Perron-Van Kampen conditions.

1. Introduction

The Ulam stability is an important concept in the theory of Volterra integral equations. This problem has been studied by L.P. Castro and A. Ramos [1], N. Cădariu and V. Radu [2], S.M. Jung [3], I.A. Rus [9], [10], I.A. Rus and N. Lungu [11]. But, on a generalized metric spaces this problem has been studied in the papers [1] and [10]. In what follows we shall present Ulam-Hyers stability of a Volterra integral equation on a generalized metric space, N. Lungu [5]. Here, we consider a Volterra integral equation in the Krasnoselski-Krein and Naguno-Perron-Van Kampen conditions. In the present work we consider a generalized metric space (X, d), where $d(x, y) \in \mathbb{R}_+ \cup \{+\infty\}$ is a generalized metric on X. For these we need some notions and results from the generalized metric spaces theory.

Let (X, d) be a generalized metric space. On X we have the following equivalence relation:

$$x \sim y \Leftrightarrow d(x, y) < \infty,\ \forall\ x, y \in X.$$

Let $X = \bigcup\limits_{\lambda \in \Lambda} X_\lambda$ be the canonical decomposition of X after this equivalence relation. We denote

$$d_\lambda(x, y) = d(x, y)\Big|_{X_\lambda \times X_\lambda}$$

and we have that (X_λ, d_λ) is a metric space ([7]).

In this paper we need the following two theorems (see W.A.J. Luxemburg [6], I.A. Rus [7], [8]):

Theorem 1.1. *Let (X, d) be a generalized complete metric space and $A : X \to X$ an operator with the property:*

$$\exists\ \alpha \in [0,1] \text{ such that } d(x,y) < \infty \Rightarrow d(A(x), A(y)) \le \alpha d(x,y)$$

for all $x, y \in X$.

If there exists $x_0 \in X$ such that $d(x_0, A(x_0)) < +\infty$, then the operator A has at least one fixed point.

Theorem 1.2. (Luxemburg-Jung). *Let (X, d) be a generalized complete metric space and his canonical decomposition $X = \bigcup X_\lambda$. If $A : X \to X$ ia a contraction, then the operator A have in every X_λ, for which exists u_λ such that*

$$d(u_\lambda, A(u_\lambda)) < +\infty,$$

a unique fixed point.

2. Ulam-Hyers stability in the generalized Krasnoselski-Krein conditions

In what follows we shall consider the following integral equation

$$u(x,y) = h(x,y) + \int_0^x \int_0^y f(s,t,u(s,t))dsdt \tag{2.1}$$

$$f : [0,a) \times [0,b) \times \mathbb{R} \to \mathbb{R},\ \ h : [0,a) \times [0,b) \to \mathbb{R},$$
$$f \in C([0,a) \times [0,b) \times \mathbb{R}, \mathbb{R}),$$
$$h \in C([0,a) \times [0,b), \mathbb{R}),\ \ u \in C([0,a) \times [0,b), \mathbb{R}),$$
$$(x,y) \in [0,a) \times [0,b),\ \ D = [0,a) \times [0,b).$$

Let X be the set:

$$X = C(D) \tag{2.2}$$

and the generalized metrics:

$$d : X \times X \to \mathbb{R}_+ \cup \{+\infty\}$$
$$d(u_1, u_2) := \sup_D \frac{|u_1(x,y) - u_2(x,y)|}{(xy)^{p\sqrt{k}}} \tag{2.3}$$

for all $u_1, u_2 \in X$, $p > 1$, $k > 0$.

It is known that the space (X, d) is a generalized complete metric space.

Let $a, b \in (0, \infty]$ and $\varepsilon > 0$. In what follows we denote by A the operator

$$A : X \to X$$

$$A(u)(x,y) := \text{the second part of (2.1)}.$$

Then the equation (2.1) becomes

$$u(x,y) = A(u)(x,y). \tag{2.4}$$

For the fixed point equation (2.4) we have:

Definition 2.1. ([10]) The equation (2.4) is Ulam-Hyers stable if there exists the positive real number $C_f > 0$ such that, for each $\varepsilon \in \mathbb{R}_+^*$ and each solution v of the inequation

$$d(v, Av) \leq \varepsilon \tag{2.5}$$

there exists a solution $u \in X$ of (2.4) such that

$$d(u, v) \leq C_f \cdot \varepsilon.$$

In this case we have

Theorem 2.2. *We suppose that:*
(i) $f : E \to \mathbb{R}$ is continuous and bounded on E, $E = D \times \mathbb{R}$;
(ii) $a < \infty$, $b < \infty$;
(iii) f verifies the generalized Krasnoselski-Krein conditions ([4])*:*

$$|f(x, y, u_1) - f(x, y, u_2)| \leq \frac{k}{xy}|u_1 - u_2|,\ k > 0 \tag{2.6}$$

$$|f(x, y, u_1) - f(x, y, u_2)| \leq \frac{c}{(xy)^\beta}|u_1 - u_2|^\alpha,\ c > 0 \tag{2.7}$$

$$\alpha \in (0, 1),\ \beta < \alpha,\ k(1-\alpha)^2 < (1-\beta)^2,\ \beta < p\sqrt{k},\ xy \neq 0,$$
$$p^2 k(1-\alpha)^2 < (1-\beta)^2,\ \textit{for all}\ (x, y, u) \in E.$$

Then the equation (2.4) is Ulam-Hyers stable.

Proof. We consider $X = C(D)$ and $X = \bigcup\limits_{\lambda \in \Lambda} X_\lambda$. Let v be a solution of the inequation (2.5) and there exists $\lambda \in \Lambda$ such that $v \in X_\lambda$. By Luxemburg-Jung theorem (Theorem 1.2), the equation (2.4) has a unique solution u in X_λ.

From (2.1), (2.5), (2.6) and (2.7), we have:

$$\begin{aligned} |v(x,y) - u(x,y)| \leq &\left| v(x,y) - h(x,y) - \int_0^x \int_0^y f(s,t,v(s,t))dsdt \right| \\ &+ \int_0^x \int_0^y |f(s,t,v(s,t)) - f(s,t,u(s,t))|dsdt. \end{aligned} \tag{2.8}$$

Hence, from (2.4), (2.6) and (2.7), we have

$$|v(x,y) - u(x,y)| \leq |v(x,y) - A(v)(x,y)| + \int_0^x \int_0^y \frac{k}{st}|v(s,t) - u(s,t)|dsdt,$$

or

$$|v(x,y) - u(x,y)| \leq |v(x,y) - A(v)(x,y)| + \int_0^x \int_0^y kd(u,v)(st)^{p\sqrt{k}-1}dsdt,$$

and

$$|v(x,y) - u(x,y)| \leq |v(x,y) - A(v)(x,y)| + kd(u,v)\frac{(xy)^{p\sqrt{k}}}{p^2 k},$$

from where we have

$$d(u,v) \leq \varepsilon + \frac{1}{p^2}d(u,v)$$

and

$$d(u,v) \leq \frac{p^2}{p^2-1}\varepsilon \tag{2.9}$$

then

$$d(u,v) \leq C_f \cdot \varepsilon$$

where

$$C_f = \frac{p^2}{p^2-1}.$$

So, from Definition 2.1, the equation (2.4) is Ulam-Hyers stable.

Example 2.3. Let us consider the equation (2.1) in the Krasnoselski-Krein conditions (2.6)+(2.7) and

$$f(x,y,u) = u(x,y)xye^{-x^2y^2}, \ h(x,y) = x^2y^2,$$

then $\alpha = \frac{1}{2}$, $\beta = \frac{1}{3}$, $k = 1$, $p = 2$.

In this case we have $c_f = \frac{p^2}{p^2-1}$ and for $p = 2$, $c_f = \frac{4}{3}$, hence the equation (2.1) is Ulam-Hyers stable.

3. Ulam-Hyers stability in the generalized Naguno-Perron-Van Kampen conditions

In this case we consider the integral equation (2.1) in the same conditions. Let $X = C(D)$ and the generalized metrics

$$d : X \times X \to \mathbb{R}_+ \cup \{+\infty\}$$

$$d(u_1,u_2) = \sup_D \frac{|u_1(x,y) - u_2(x,y)|}{(xy)^{p+1}} \tag{3.1}$$

for all $u_1, u_2 \in X$, $p > -1$.

It is known that the space (X, d) is a generalized complete metric space. Here, we consider the stability of the equation (2.4) in the generalized Naguno-Perron-Van Kampen conditions.

Theorem 3.1. *If we have*

(i) $f : E \to \mathbb{R}$ is continuous and bounded on E;

(ii) $a < +\infty$, $b < +\infty$;

(iii) f verifies the generalized Naguno-Perron-Van Kampen conditions ([12])*:*

$$|f(x,y,u)| \leq \alpha(xy)^p, \ p > -1, \ \alpha > 0. \tag{3.2}$$

$$|f(x,y,u_1) - f(x,y,u_2)| \leq \frac{c}{(xy)^r}|u_1 - u_2|^q, \ q \geq 1, \ c > 0, \tag{3.3}$$

$$pq + q - r = p, \ xy \neq 0, \ \rho = \frac{c(2\alpha)^{q-1}}{(p+1)^{2q}} < 1, \ \textit{for all } (x,y,u) \in E.$$

Then the equation (2.4) is Ulam-Hyers stable.

Proof. Evidently, in the conditions Naguno-Perron-Van Kampen, by Luxemburg-Jung theorem (Theorem 1.2), the equation (2.4) has a unique solution u in X_λ.

First we observe that

$$|v(x,y) - u(x,y)| \leq \frac{2\alpha}{(p+1)^2}(xy)^{p+1}. \tag{3.4}$$

From (2.1), (2.5), (3.2), (3.3) we have

$$|v(x,y) - u(x,y)| \leq \left| v(x,y) - h(x,y) - \int_0^x \int_0^y f(s,t,v(s,t))dsdt \right| + \int_0^x \int_0^y |f(s,t,v(s,t)) - f(s,t,u(s,t))|dsdt.$$

From (3.3) we have

$$|v(x,y) - u(x,y)| \leq |v(x,y) - A(v)(x,y)| + \int_0^x \int_0^y \frac{c}{(st)^r}|v(s,t) - u(s,t)|^q dsdt$$

$$\leq |v(x,y) - A(v)(x,y)| + \int_0^x \int_0^y \frac{c}{(st)^r} \cdot \frac{|v(s,t) - u(s,t)|}{(st)^{p+1}} \cdot \frac{|v(s,t) - u(s,t)|^{q-1}}{(st)^{-p-1}} dsdt$$

$$\leq |v(x,y) - A(v)(x,y)| + cd(u,v) \int_0^x \int_0^y \frac{(2\alpha)^{q-1}}{(p+1)^{2(q-1)}}(st)^{pq+q-r} dsdt.$$

Then we have

$$d(u,v) \leq d(v, A(v)) + \rho d(u,v) \tag{3.5}$$

and

$$d(u,v) \leq \frac{\varepsilon}{1-\rho},$$

then

$$d(u,v) \leq C_f \cdot \varepsilon$$

where

$$C_f = \frac{1}{1-\rho}.$$

From Definition 2.1, the equation (2.4) is Ulam-Hyers stable.

Remark 3.2. For every $\lambda \in \Lambda$ there exists at least a solution v of (2.5) in X_λ and for each v exists a unique solution u of (2.4) which is Ulam-Hyers stable.

Remark 3.3. It is possible that the inequation (2.5) do not have a solution, but in this case the equation (2.4) is Ulam-Hyers stable.

Example 3.4. Let us consider the equation (2.1) in the Naguno-Perron-Van Kampen conditions (3.2)+(3.3), $p > -1$, $r = 1$, $q \geq 1$.

In this case $c_f = \dfrac{1}{1-\rho}$, where $\rho = \dfrac{c(2\alpha)^{q-1}}{(p+1)^{2q}}$ and the equation (2.1) is Ulam-Hyers stable. If $\rho = 1$ then the equation (2.1) is Ulam-Hyers instable.

References

[1] Castro, L.P., Ramos, A., *Hyers-Ulam-Rassias stability for a class of nonlinear Volterra integral equations*, Banach J. Math. Anal., **3**(2009), no. 1, 36-43.

[2] Cădariu, L., Radu, V., *The fixed points method for the stability of some functional equations*, Carpathian J. Math., **23**(2007), no. 1-2, 63-72.

[3] Jung, S.M., *A fixed point approach to the stability of a Volterra integral equation*, Fixed Point Theory and Applications, 2007, Art. ID57064, 9 pages.

[4] Krasnoselski, M.A., Krein, S.G., *Ob odnom classe theorem edinstvenosti dlia uravnenia* $y' = f(x, y)$, Uspehi Math. Nauk, 1(67), **11**(1956), 209-213.

[5] Lungu, N., *Hyperbolic-type inequalities on generalized metric spaces*, Carpathian J. Math., **24**(2008), no. 3, 341-347.

[6] Luxemburg, W.A.J., *On the convergence of successive approximations in the theory of ordinary differential equations II*, Indag. Math., **20**(1958), 540-546.

[7] Rus, A.I., *Principii şi aplicaţii ale teoriei punctului fix*, Ed. Dacia, Cluj-Napoca, 1979.

[8] Rus, A.I., *Ulam stability of ordinary differential equations*, Stud. Univ. Babeş-Bolyai Math., **54**(2009), no. 4, 125-133.

[9] Rus, A.I., *Gronwall lemma approach to the Hyers-Ulam-Rassias stability of an integral equation*, in: Nonlinear Analysis and Variational Problems, Springer, 2009, 147-152.

[10] Rus, A.I., *Remarks on Ulam stability of the operatorial equations*, Fixed Point Theory, **10**(2009), no. 2, 305-320.

[11] Rus, A.I., Lungu, N., *Ulam stability of a nonlinear hyperbolic partial differential equation*, Carpathian J. Math., **24**(2008), no. 3, 403-408.

[12] Wong, I.S.W., *On the convergence of successive approximations in the Darboux problem*, Ann. Polon. Math., **18**(1966), 329-336.

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Nonlinear economic growth dynamics in the context of a military arms race

Daniel Metz and Adrian Viorel

Abstract. In the present contribution, we propose and analyze a dynamical economic growth model for two rival countries that engage an arms race. Under natural assumptions, we prove that global solutions exist and discuss their asymptotic long-time behavior. The results of our stability analysis support the recurring hypothesis in Cold War political science that engaging in an arms race with a technologically superior and hence faster growing adversary has damaging economic consequences. Numerical findings illustrate our claims.

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Keywords: Solow-Swan model, arms race, asymptotic behavior.

1. Introduction

The Cold War has ended three decades ago and regional antagonisms have replaced the previous colossal struggle. Nevertheless, our understanding of the United States vs. USSR Arms Race and its wider economic consequences remains far from complete (see [7]).

A plausible, and often repeated explanation attributes the Eastern Block's collapse to an economic crisis triggered by unsustainable military ambitions. Given the ever growing and ever more visible gap in technological and economic capabilities, matching American military development was possible only at the expense of economic growth and stability.

The aim of the present contribution is to examine this hypothesis from an analytic point of view by developing a model that, at least qualitatively, reproduces economic stagnation caused by a prolonged military rivalry with a faster developing adversary.

Arms races have a long history that goes back far beyond the Cold War Era. The ancient Greeks and Romans built fleets to match their Persian and Carthaginian rivals, but the naval race that followed the 1889 Naval Defense Act calling for the Royal

Navy to be as strong as the world's next two largest navies combined, is probably the most intensively studied predecessor as it led to World War I (WWI).

For decades, Arms races have been a topical subject in Political Science, such that the scarcity of treatments from a mathematical perspective comes as a real surprise - all the more so given L. F. Richardson's pioneering contributions to the field synthesized in *Arms and Insecurity: A Mathematical Study of the Causes and Origins of War* [12] and *Statistics of Deadly Quarrels* [13].

The classical Richardson model, which relies on a system of two coupled linear differential equations, has dominated theoretical debates for more than half a century. If $x(t)$ and $y(t)$ denote the levels of arms for two rival states, with rates of change driven by the sum of a positive reaction to the other country's arms, a negative 'fatigue' reaction to own military level and a constant 'grievance' term, then the time evolution is described by

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t}(t) = -\beta_{11}x(t) + \beta_{12}y(t) + \gamma_1, \\ \dfrac{\mathrm{d}y}{\mathrm{d}t}(t) = \quad \beta_{21}x(t) - \beta_{22}y(t) + \gamma_2. \end{cases} \tag{1.1}$$

The unique equilibrium point of the system, which exists provided that the two straight lines defined by the right hand side of (1.1) are not parallel, may be unstable and Richardson related exponentially diverging solutions with the outbreak of war. Nonlinear extensions of the classical Richardson model have been considered by Hill [6].

We take a similar approach but augment the model by adding an economic dimension described in terms of Solow-Swan dynamics discussed below based on [3]. A different line of thought, that we don't pursue here, deals with arms races or, more generally, strategic interactions from a game theoretical perspective. Two or more actors play a (repeated) game in which the strategies that they can choose from are to arm or not to arm (see, for example [10]). For a recent contribution that is somewhat pertaining to the present work, we refer to [9]. The direction contrary to our study, that is disarmament models has also been pursued (cf. [4]), while a strongly misleading use of the term arms race in a biological context has been rendered popular by Dawkins and Krebs in [5].

The Solow-Swan model, originating from the independent works [14] and [15], explains long-run economic growth in a neoclassical framework by relating capital, labor and technology. The model relies on three fundamental assumptions. The first assumption is an **exponential population (or labor) growth**

$$L(t) = L_0 e^{nt} \quad \text{(equivalently } \frac{\mathrm{d}L}{\mathrm{d}t} = nL\text{)}.$$

The second assumption concerns a **Cobb-Douglas production function** connecting the economic output Y to the labor L, capital K and the level of technology A

$$Y = AK^{\alpha}L^{1-\alpha}.$$

Here, $\alpha \in [0, 1]$ is the returns to scale constant.

The third assumption of the model asserts that **change in capital** $K(t)$ is due to the positive capital output saving (with saving rate $\sigma \in (0,1)$) and to the negative capital depreciation (at a rate δ)

$$\frac{\mathrm{d}K}{\mathrm{d}t} = \sigma Y - \delta K.$$

By combining these three assumptions and expressing them in terms of the capital intensity

$$k(t) = \frac{K(t)}{L(t)}$$

one obtains

$$\frac{\mathrm{d}k}{\mathrm{d}t}\, L + k\, nL = \frac{\mathrm{d}K}{\mathrm{d}t} = \sigma A(kL)^{\alpha} L^{1-\alpha} - \delta(kL)$$

and deduces the fundamental equation of the Solow-Swan model

$$\frac{\mathrm{d}k}{\mathrm{d}t} = \sigma A k^{\alpha} - (n+\delta)k. \tag{1.2}$$

Observe that here we have reached an explicitly solvable Bernoulli equation which defines a dynamical system with two equilibria, $k^* = 0$ being unstable in contrast to

$$k^* = \left(\frac{\sigma A}{n+\delta}\right)^{1/1-\alpha}$$

which is asymptotically stable (attractor).

2. An economic growth model with arms race military expenses

The simplicity and lack of specificity proved to be both a strength and weakness of the classical Richardson model which has become a cornerstone of strategic thinking despite the somewhat imprecise concept of arms not allowing rigorous fitting to measurable data. It turns out that replacing weapon quantities by an abstract 'security' concept that can be linked to economic factors is more lucrative. Loosely following discrete models in both Krabs [8] and Larrosa [9] we consider an augmented arms race model in underlying economic growth context

$$\begin{cases} \dfrac{\mathrm{d}s_1}{\mathrm{d}t}(t) = -k_2(t)s_1(t) + k_1(t)s_2(t), \\ \dfrac{\mathrm{d}s_2}{\mathrm{d}t}(t) = \quad k_2(t)s_1(t) - k_1(t)s_2(t), \\ \dfrac{\mathrm{d}k_1}{\mathrm{d}t}(t) = a_1 k_1(t)^{\alpha} - bk_1(t) - cs_2(t)k_1(t), \\ \dfrac{\mathrm{d}k_2}{\mathrm{d}t}(t) = a_2 k_2(t)^{\alpha} - bk_2(t) - cs_1(t)k_2(t). \end{cases} \tag{2.1}$$

Here, $s_i(t)$ describes the level of security of the state i at time t, $k_i(t)$ being the country's capital intensity. Security levels obey Richardson-type equations but with time-varying coefficients. The competitive nature of the model is reflected in the fact that an increase in one actor's security is its adversary's security loss as

$$\frac{\mathrm{d}s_1}{\mathrm{d}t}(t) = -\frac{\mathrm{d}s_2}{\mathrm{d}t}(t). \tag{2.2}$$

In other words the total security is constant $s_1(t) + s_2(t) = \text{const}$ and imposing $s_1(0) + s_2(0) = 1$ will assure, as we will see in the next section, $0 \leq s_1(t), s_2(t) \leq 1$ meaning that security levels range from 0 (totally insecure) to 1 (totally secure).

On the other hand, both economies grow according to a Solow-Swan model with an additional term explicitly accounting for military expenses. These exchange terms including the adversary's security might look surprising at first glance, but in view of (2.2) one country's security is the other's insecurity $s_j = 1 - s_i$ and military expenses are proportional precisely to the insecurity $1-s_i$. The coefficient $c \in (0,1)$ represents a budget constraint and expresses the percentual limit which military spendings cannot exceed in a functional peacetime economy.

Returning to the security equations, one can now see that the right hand side terms are actually proportional to military costs, insecurity rising based on rival spending and decreasing based on own spendings.

The parameters α, b and a_i retain their original Solow model meaning and only a_1, a_2 differ from country to country. In view of (1.2), this difference is essential to our model and accounts for the technological gap separating the two economies.

3. Analysis of the model

We start our analysis by discussing an uncoupled Solow-Swan model with variable military expenditures. Quite naturally, **the best and worst case scenarios**, namely zero or maximal military spending, provide upper and lower bounds for the dynamics.

Lemma 3.1 (upper and lower bounds). *Let us consider the initial value problem*

$$\frac{\mathrm{d}k}{\mathrm{d}t} = ak^\alpha - bk - cs(t)k, \quad k(0) = k_0 \tag{3.1}$$

with coefficients $a > 0, \alpha \in (0,1), b > 0, c \in [0,1]$ and $s : [0,\infty) \to \mathbb{R}$ a given smooth function with $s(t) \in [0,1]$ for any $t \geq 0$. If $k_0 > 0$ then the solution of (3.1) *exists, is positive and satisfies for all times*

a) $k(t) \leq \overline{k}(t)$, *where $\overline{k}$ is the solution of*

$$\frac{\mathrm{d}\overline{k}}{\mathrm{d}t} = a\overline{k}^\alpha - b\overline{k}, \quad \overline{k}(0) = k_0; \tag{3.2}$$

a) $\underline{k}(t) \leq k(t)$, *where $\underline{k}$ is the solution of*

$$\frac{\mathrm{d}\underline{k}}{\mathrm{d}t} = a\underline{k}^\alpha - b\underline{k} - c\underline{k}, \quad \underline{k}(0) = k_0. \tag{3.3}$$

Proof. In (3.1) we are dealing with a Bernoulli equation which is exactly solvable, hence the global existence using the usual change substitution $z(t) = k(t)^{1-\alpha}$. Using the variation of constants formula one has the desired positivity from

$$k(t)^{1-\alpha} = z(t) = e^{-\int_0^t \frac{b+cs(\rho)}{1-\alpha}\mathrm{d}\rho} \left[k_0^{1-\alpha} + \frac{a}{1-\alpha} \int_0^t e^{\int_0^\tau \frac{b+cs(\rho)}{1-\alpha}\mathrm{d}\rho}\,\mathrm{d}\tau \right]. \tag{3.4}$$

To obtain both the upper and lower bounds, one can rely on standard sub and super-solution arguments. Since $0 \le s(t) \le 1$ and $k(t) \ge 0$

$$ak^\alpha - bk - ck \le \frac{\mathrm{d}k}{\mathrm{d}t} \le ak^\alpha - bk$$

and the conclusion follows. □

Remark 3.2. From a dynamical systems point of view, both autonmous equations in Lemma 3.1 are Solow-Swan equations and have the same stability behavior albeit with different nonzero asymptotically stable equilibria namely

$$\overline{k}^* = \left(\frac{a}{b}\right)^{\frac{1}{1-\alpha}} \quad \text{and} \quad \underline{k}^* = \left(\frac{a}{b+c}\right)^{\frac{1}{1-\alpha}}$$

respectively.

After this helpful preliminaries we are in position to prove the existence of global solutions to (2.1).

Theorem 3.3 (global existence). *Let us consider the growth under arms race rivalry model* (2.1) *with* $a_1, a_2 > 0, \alpha \in (0,1), b > 0$ *and* $c \in [0,1]$. *Then for any initial conditions* $k_1(0), k_2(0) > 0$ *and* $s_1(0), s_2(0) > 0$ *with* $s_1(0) + s_2(0) = 1$ *there exists a unique classical solution of the initial value problem associated to the system* (2.1) *which remains bounded for all* $t \ge 0$.

Proof. We divide the proof in several steps.
Step 1. Local existence. As the right hand side of the system has good regularity (only local Lipschitz continuity is actually required), a standard Banach fixed point argument guarantees the existence of local in time solutions, defined on a maximal interval $t \in [0,T)$, $T = T(s_1(0), s_2(0), k_1(0), k_2(0)$.
Step 2. Positivity of k_1 and k_2. Based on the representation formula (3.4) which holds on their maximal interval of existence $t \in [0,T)$, one can see that for positive initial states $k_1(0), k_2(0) > 0$, both $k_1(t)$ and $k_2(t)$ must be positive for $t \in [0,T)$.
Step 3. Positivity of s_1 and s_2. Using the fact that

$$s_1(t) + s_2(t) = 1 \quad \text{for all} \quad t \in [0,T), \tag{3.5}$$

one can rewrite the evolution equations for s_1 and s_2 as

$$\begin{aligned} \frac{\mathrm{d}s_1}{\mathrm{d}t} &= k_1(t) - (k_1(t) + k_2(t))s_1, \\ \frac{\mathrm{d}s_2}{\mathrm{d}t} &= k_2(t) - (k_1(t) + k_2(t))s_2, \end{aligned} \tag{3.6}$$

such that aplying the variation of constants formula one again

$$s_1(t) = e^{-\int_0^t (k_1(\rho)+k_2(\rho))\mathrm{d}\rho} \left[s_1(0) + \int_0^t e^{\int_0^\tau (k_1(\rho)+k_2(\rho))\mathrm{d}\rho} k_1(\tau)\,\mathrm{d}\tau \right] \tag{3.7}$$

which is positive for positive k_1. Similarly, one can show that s_2 has the same property. In view of (3.5) and the positivity of s_1, s_2 we have

$$0 \le s_1(t), s_2(t) \le 1 \quad \text{for all} \quad t \in [0, T). \tag{3.8}$$

Step 4. Boundedness of k_1, k_2 and global solutions. The estimates in (3.8) not only assure the boundedness for s_1, s_2 but also allow us to apply Lemma 3.1, more precisely the upper bound in a), and hence deduce the boundedness of k_1 and k_2. A classical result (see Barbu [2]) now assures that the local in time solutions can be extend to arbitrary positive times. □

4. Asymptotic behavior of the model

We start by determining the equilibrium points of the system, that is the solutions of

$$\begin{cases} 0 = -k_2^* s_1^* + k_1^* s_2^*, \\ 0 = \quad k_2^* s_1^* - k_1^* s_2^*, \\ 0 = a_1 (k_1^*)^\alpha - b k_1^* - c s_2^* k_1^*, \\ 0 = a_2 (k_2^*)^\alpha - b k_2^* - c s_1^* k_2^*. \end{cases} \tag{4.1}$$

One can reduce this to a 3 by 3 nonlinear system by assuming that (3.5) holds. The resulting equilibrium equations are

$$\begin{cases} 0 = k_1^* - (k_1^* + k_2^*) s_1^*, \\ 0 = a_1 (k_1^*)^\alpha - b k_1^* - c(1-) s_1^*) k_1^*, \\ 0 = a_2 (k_2^*)^\alpha - b k_2^* - c s_1^* k_2^*. \end{cases} \tag{4.2}$$

Trivial equilibria, that is with $k_i^* = 0$, exist but are not interesting from a modeling perspective as they would indicate the disappearance of an economy. Nevertheless, we note without going into details, that all such equilibria are unstable, as a natural consequence of the lower bound b) in Lemma 3.1 means that both k_1 and k_2 are pushed away from zero even when starting arbitrarily close.

However, there exists also a nontrivial equilibrium point.

4.1. The unique nontrivial equilibrium

In terms of the convenient notation

$$R^* = \frac{s_1^*}{s_2^*} = \frac{k_1^*}{k_2^*},$$

from (3.5) we have

$$s_1^* = \frac{R^*}{1 + R^*} \quad \text{and} \quad s_2^* = \frac{1}{1 + R^*}. \tag{4.3}$$

Such that inserting this in the 3^{rd} and 4^{th} equation of (4.1) leads to

$$\left(\frac{k_1^*}{k_2^*}\right)^{1-\alpha} = \frac{a_1}{a_2} \cdot \frac{b + (b+c) R^*}{(b+c) + b R^*},$$

that is,

$$(R^*)^{1-\alpha} = \frac{a_1}{a_2} F(R^*) \tag{4.4}$$

with

$$F(R) = \frac{b + (b+c)R}{(b+c) + bR}.$$

Hence, finding the nontrivial equilibrium reduces to solving the coincidence problem (4.4) (or equivalently the fixed point problem $R^* = \frac{a_1}{a_2} F(R^*)^{1/1-\alpha}$). The existence of a unique nontrivial coincidence point R^* follows from geometric considerations.

Lemma 4.1. *The coincidence problem* (4.4) *with* $a_1 \geq a_2$ *has a unique solution* $R^* \geq 1$ *provided that* $\alpha \in \left[0, \frac{1}{2}\right)$ *and* $\frac{c}{2b+c} \leq 1 - \alpha$. *If* $a_1 > a_2$ *then* $R^* > \frac{a_1}{a_2}$.

Proof. We start by noting some geometric properties of F. One can easily check that

$$F(0) = \frac{b}{b+c}, \quad F(1) = 1 \quad \text{and} \quad F(\infty) = \frac{b+c}{b}$$

while

$$F'(R) = \frac{(2b+c)c}{((b+c)+bR)^2} > 0 \quad \text{and} \quad F''(R) < 0.$$

In other words, F is monotonically increasing, convex and bounded from above by $b + c/b$. Consequently, the range of $R \mapsto \frac{a_1}{a_2} F(R)$ is $[\frac{a_1}{a_2}\frac{b}{b+c}, \frac{a_1}{a_2}\frac{b+c}{b}]$.

Now, observe that for $\frac{a_1}{a_2} = 1$, $R = 1$ is a solution of

$$R^{1-\alpha} = F(R).$$

For the moment, we assume that there are no other solutions in $(0,1)$ and later give a sufficient condition for this to hold true.

If $R^{1-\alpha} = F(R)$ has no solutions is $(0,1)$, that is, $F(R) > R^{1-\alpha}$ for $R \in (0,1)$ due to $F(0) = b/(b+c) > 0$, then $\frac{a_1}{a_2} R^{1-\alpha} = F(R)$ has no solutions in $(0, a_1/a_2)$. Indeed, on one hand

$$R^{1-\alpha} < F(R) \leq \frac{a_1}{a_2} F(R) \quad \text{for} \quad 0 < R < 1$$

while on the other hand

$$R^{1-\alpha} < R < \frac{a_1}{a_2} < \frac{a_1}{a_2} F(R) \quad \text{for} \quad 1 < R < \frac{a_1}{a_2}.$$

As $R \mapsto R^{1-\alpha}$ is increasing and unbounded while F is increasing and bounded, the two curves will cross at a unique point $R^* > \frac{a_1}{a_2}$.

The necessary and sufficient condition for $R^{1-\alpha} = F(R)$ to have no solutions in $(0,1)$ is again geometric in nature. Actually, the slope of F at $R = 1$ must not exceed that of $G(R) = R^{1-\alpha}$, that is precisely

$$\frac{c}{2b+c} \leq 1 - \alpha. \qquad \square$$

Remark 4.2. Both essential conditions for the uniqueness of the coincidence point

$$\alpha < \frac{1}{2} \quad \text{and} \quad \frac{c}{2b+c} \leq 1-\alpha \tag{4.5}$$

are natural and in accordance with econometric data. The returns to scale constant is generally considered to be $\alpha \approx 1/3$ while the depreciation constant is $b \approx 0.05$ (see Acemoglu [1]). On the other hand, even at the hight of the Cold War, according to the World Bank[1] military expenses have not exceeded 10% of GDP, so roughly $c \approx 2b$, which satisfies the coincidence condition.

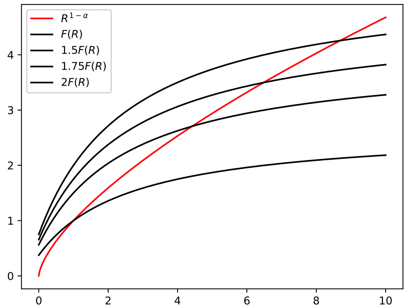

FIGURE 1. Qualitative behavior of the coincidence problem under the uniqueness assumptions (4.5).

In the sequel, we analyze the stability of the equilibrium point corresponding to this unique R^*, that is of (4.3) together with

$$k_1^* = \left(\frac{a_1}{b + \frac{c}{1+R^*}}\right)^{\frac{1}{1-\alpha}} \quad \text{and} \quad k_2^* = \left(\frac{a_2}{b + \frac{cR^*}{1+R^*}}\right)^{\frac{1}{1-\alpha}}.$$

To this end, we compute the Jacobi matrix of the (3 by 3) system which is

$$J(s_1^*, k_1^*, k_2^*) = \begin{bmatrix} -(k_1^* + k_2^*) & 1 - s_1^* & -s_1^* \\ ck_1^* & T_1 & 0 \\ -ck_2^* & 0 & T_2 \end{bmatrix} \tag{4.6}$$

with

$$T_1 = (\alpha - 1)\frac{b + c + bR^*}{1 + R^*} \quad \text{and} \quad T_2 = (\alpha - 1)\frac{b + (b+c)R^*}{1 + R^*}.$$

We discuss the eigenvalues of this matrix in two different, parameter-dependent cases.

[1]https://data.worldbank.org/indicator/MS.MIL.XPND.GD.ZS?locations=US

4.2. The catch-up scenario $a_1 = a_2 = a$

This is the simpler yet less realistic situation in which there exists no difference between the parameters describing the two countries, this especially means that both economies have the same technology level, and only their initial states may differ.

Returning to (4.4), one can see that it reduces to the simpler

$$(R^*)^{1-\alpha} = F(R^*)$$

which has the unique coincidence point $R^* = 1$. As a consequence

$$s_1^* = s_2^* = \frac{1}{2} \quad \text{and} \quad k_1^* = k_2^* = k^* = \left(\frac{a}{b + \frac{1}{2}c}\right)^{\frac{1}{1-\alpha}}$$

and straightforward but rather tedious computation show that all three eigenvalues of the Jacobian $J(\frac{1}{2}, k^*, k^*)$ have negative real part, so the equilibrium is locally asymptotically stable.

From a modeling perspective, this describes a catch-up evolution in which the country with the initially weaker economy will recover the deficit in the long-run and stabilize at the same level as its rival, as depicted in Figure 2.

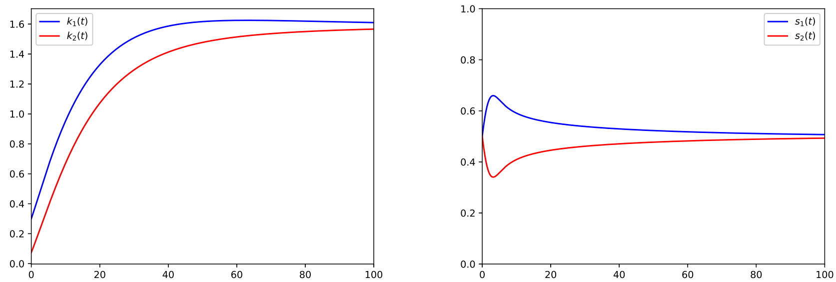

FIGURE 2. The catch-up scenario $a_1 = a_2$. The two countries experience a convergent economic growth with the initialy weaker economy catching up to the stronger (Left panel). Security levels also converge towards a balanced stationary state (Right panel). Simulations correspond to $a_1 = a_2 = 0.15, \alpha = 1/3, b = 0.06, c = 0.1$ and $s_1(0) = s_2(0) = 1/2, k_1(0) = 0.3, k_2(0) = k_1(0)/4$.

4.3. The increasing gap scenario $a_1 > a_2$

From our point of view, the more interesting and realistic situation is that of unequal coefficients $a_1 > a_2$. This describes a technological gap between the two contenders, and we will show that the quotient a_1/a_2 plays a decisive role in the long-term dynamics as its affects the equilibrium quotient R^* of the two economies (see Figure 3).

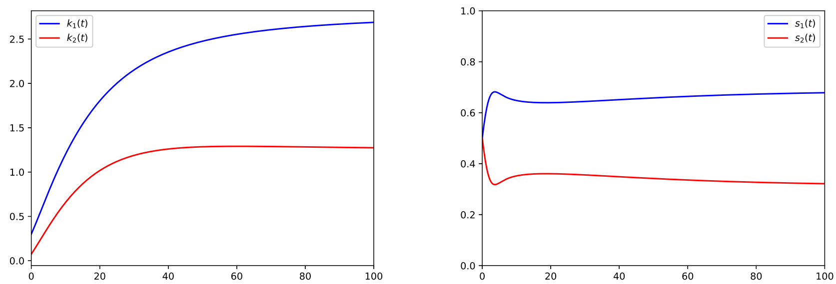

FIGURE 3. The increasing gap scenario $a_1 > a_2$. Despite growth for both countries, the gap separating them widens as excessive military spending harms the slower developing one (Left panel). Security levels also separate in the stronger economy's favour (Right panel). Simulations correspond to $a_1 = 0.18, a_2 = 0.15, \alpha = 1/3, b = 0.06, c = 0.1$ and $s_1(0) = s_2(0) = 1/2, k_1(0) = 0.3, k_2(0) = k_1(0)/4$.

Indeed, from the proof of Lemma 4.1, we know that the coincidence point R^* must lie above the $\frac{a_1}{a_2}$ threshold. This means that the equlibirum quotient exceeds the quotient of coefficients

$$\frac{k_1^*}{k_2^*} = R^* > \frac{a_1}{a_2}.$$

Again, the eigenvalues of the Jacobian at the equilibrium point corresponding to $R^* > \frac{a_1}{a_2}$ have negative real part and hence the equilibrium point is asymptotically stable. We omit the details of this technical computation, but in order to strike a balance between the abstract and the concrete level, we mention that given the realistic values

$$a_1 = 0.18, a_2 = 0.15, \alpha = 1/3, b = 0.06, c = 0.1$$

for the parameters, one obtains

$$R^* = 2.208, \quad k_1^* = 2.773 \quad \text{and} \quad k_2^* = 1.256$$

such that the eigenvalues of the Jacobi matrix are all negative

$$\lambda_{1,2,3} = -4.081, \ -0.076 \ \text{ and } \ -0.019.$$

5. Conclusions

In order to describe the economic implications of a prolonged military rivalry, we have constructed a nonlinear dynamical model that merges the classical Richardson arms race evolution with economic growth in the sense of Solow's pioneering work.

The ensuing nonstandard model turns out to be well-posed and in accordance with both political and economic intuitions. More precisely, when considering different levels of technology for the two competing powers, the model predicts that in the long run, due to nonlinear effects, the sizes of the two economies will be separated by a gap that exceeds the technology gap.

The reality of Cold War dynamics has been far more complex than the relatively simple model that we propose can describe. Many extensions are possible and, actually desirable. The most natural extension would be to consider the augmented human capital version of the Solow-Swan model due to Mankiw, Romer and Weil [11] not the Solow-Swan economic growth model itself.

Furthermore, the Cold War arms race is just a prototype for more general economic rivalry phenomena. Trade or economic wars provide very interesting challenges form a modeling perspective.

References

[1] Acemoglu, D., *Introduction to Modern Economic Growth*, Princeton University Press, 2011.

[2] Barbu, V., *Differential Equations*, Springer Undergraduate Mathematics Series, Springer, Cham, 2016. Translated from the 1985 Romanian original by Liviu Nicolaescu.

[3] Barro, R.J., Sala-i Martin, X., *Economic Growth*, MIT Press, 2004.

[4] Bueno de Mesquita, B., Lalman, D., *Arms races and the opportunity for peace*, Formal Analysis in International Relations, **76**(1988), 263-283.

[5] Dawkins, R., Krebs, J.R., *Arms races between and within species*, Proceedings of the Royal Society of London, Series B, Biological Sciences, **205**(1161)(1979), 489-511.

[6] Hill, W.W., *Several sequential augmentations of Richardson's arms race model*, Math. Comput. Modelling, **6**(8-9)(1992).

[7] Intriljgator, M.D., Brito, D.L., *Arms races*, Defence and Peace Economics, **11**(2000), no. 1, 45-54.

[8] Krabs, W., *Dynamische Systeme: Steuerbarkeit und Chaotisches Verhalten*, B. G. Teubner, Stuttgart, 1998.

[9] Larrosa, J.M.C., *Arms build-up and arms race in optimal economic growth*, Int. J. Econ. Theory, **12**(2016), no. 2, 167-182.

[10] Majeski, S.J., *Arms races as iterated prisoner's dilemma games*, Math. Social Sci., **7**(3)(1984), 253-266.

[11] Mankiw, N.G., Romer, D., Weil, D.N., *A contribution to the empirics of economic growth*, The Quarterly Journal of Economics, **107**(1992), no. 2, 407-437.

[12] Richardson, L.F., *Arms and Insecurity: A Mathematical Study of the Causes and Origins of War*, Edited by Nicolas Rashevsky and Ernesto Trucco, The Boxwood Press, Pittsburgh, Pa.; Quadrangle Books, Chicago, Ill., 1960.

[13] Richardson, L.F., *Statistics of Deadly Quarrels*, Boxwood Press, 1960.

[14] Solow, R.M., *A contribution to the theory of economic growth*, The Quarterly Journal of Economics, **70**(1956), no. 1, 65-94.

[15] Swan, T.W., *Economic growth and capital accumulation*, Economic Record, **32**(1956), no. 2, 334-361.

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Book reviews

Teodor Bulboacă, Santosh B. Joshi, and Pranay Goswami, Complex Analysis: Theory and Applications, de Gruyter, 2019, xii + 409 pages, ISBN 978-3-11-065782-1, e-ISBN (PDF) 978-3-11-065786-9, e-ISBN (EPUB) 978-3-11-065803-3

The present book provides a modern presentation of various classical topics in complex analysis. As it is mentioned in the preface, the level of difficulty of the material increases gradually from chapter to chapter.

The book is divided into seven chapters, as follows.

The first chapter is an introductory chapter, in which there are reviewed the complex numbers, the topological and metric structures of the complex plane $\mathbb{C}$, and the topological structure of the extended complex plane $\widehat{\mathbb{C}}$.

The second chapter is devoted to the notion of holomorphy in the case of functions of one complex variable. There are presented basic properties regarding the derivative of a complex function, including the fundamental Cauchy-Riemann theorem of characterization the complex differentiability. Also, there are included useful examples of elementary entire functions. A special attention is paid to Möbius transformations and their basic properties (the invariance of the cross ratio, and the preservation of circles in $\mathbb{C}_\infty$ onto circles in $\mathbb{C}_\infty$).

The third chapter is concerned with the theory of the complex integral. For this aim, first there are defined the notions of paths, homotopy of two paths, simply connected domain, etc. Then there are presented the notion of the complex integral and basic properties regarding this notion. Among them, we mention here the fundamental Cauchy integral theorem for holomorphic functions, and the Cauchy integral formulas with its important consequences and applications. The notion of the index of a path, and the analytic branches of multi-valued functions are also discussed.

In the fourth chapter, the authors are concerned with the local properties of analytic functions in terms of the power series expansions. The chapter begins with a review of locally uniformly convergence of sequences of holomorphic functions, and continues with important properties of power series, the treatment of zeros of holomorphic functions, followed by the maximum modulus theorem and the Schwarz lemma. Finally, the Laurent series, the notion of an isolated singular point, and basic properties of meromorphic functions are also presented.

The fifth chapter deals with the residue theory and various applications of the fundamental residue theorem in the computation of complex integrals as well as on some real integrals. The residue theory is then applied to study the number of zeros and poles of meromorphic functions. There are proved the argument principle and Rouché's theorem with its main consequences and applications in the theory of holomorphic functions.

The sixth chapter is one of the main chapters of this book. In the first section, there is proved the classical result of Montel concerning the equivalence between the notions of locally uniformly convergence and normal families in the case of holomorphic functions. In the second section, there is studied the notion of univalence for holomorphic functions, and there are obtained a necessary condition of univalence (the non-vanishing of the first derivative of a univalent function) and the Hurwitz theorem concerning the locally uniformly convergence of sequences of univalent functions. In the third section, there is treated a fundamental problem in the theory of univalent mappings, namely the conformal (biholomorphic) equivalence between simply connected domains $D \subsetneq \mathbb{C}$ and the open unit disc $\mathbb{U}$. In the fourth section, there is proved the famous Riemann mapping theorem, and there are deduced some consequences of this significant result in complex analysis.

Each chapter contains a useful collections of exercises of different level. The solutions to these exercises are carefully presented in the seventh chapter of this book.

The present book is clearly written, in an accessible style, and the proofs of the main results are rigorous. The examples and exercises help the reader to become acquainted with the theory of functions of one complex variable. It is recommended to undergraduate and graduate students, and to all researchers that are interested in classical and advanced topics of complex analysis.

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