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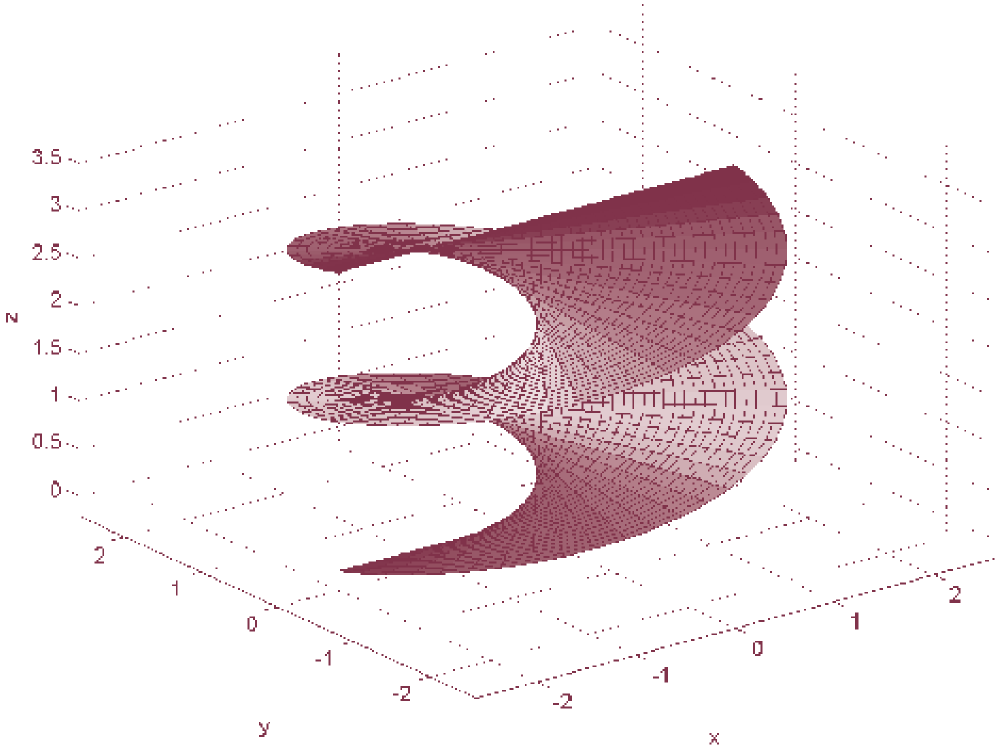

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Degenerate Hermite poly-Bernoulli numbers and polynomials with q-parameter

Waseem A. Khan, Idrees A. Khan and Musharraf Ali

Abstract. In this paper, we introduce a new class of degenerate Hermite poly-Bernoulli polynomials with q-parameter and give some identities of these polynomials related to the Stirling numbers of the second kind. Some implicit summation formulae and general symmetry identities are derived by using different analytical means and applying generating functions. These results extend some known summations and identities of degenerate Hermite poly-Bernoulli numbers and polynomials.

Mathematics Subject Classification (2010): 11B68, 11B73, 11B75, 33C45.

Keywords: Hermite polynomials, degenerate q-poly-Bernoulli polynomials, degenerate Hermite q-poly-Bernoulli polynomials, summation formulae, symmetric identities.

1. Introduction

The special polynomials of more than one variable provide new means of analysis for the solution of wide class of partial differential equations often encountered in physical problems. The importance of multi-variable Hermite polynomials has been recognized [6] and these polynomials have been exploited to deal with quantum mechanical and optical beam transport problems.

It happens very often that the solution of a given problem in physics or applied mathematics requires the evaluation of infinite sums, involving special functions. Problems of this type arise, for example, in the computation of the higher-order moments of a distribution or to evaluate transition matrix elements in quantum mechanics. In [7], [8], [9], [19], [20], [21], [22], it has been shown that the summation formulae of special functions, encountered in applications ranging from electromagnetic process to combinatorics can be written in terms of Hermite polynomials of more than one variable.

The 2-variable Kampe de Feriet generalization of the Hermite polynomials [3] and [8] are defined as

$$H_n(x,y) = n!\sum_{r=0}^{[\frac{n}{2}]} \frac{y^r x^{n-2r}}{r!(n-2r)!}. \tag{1.1}$$

These polynomials are specified by the generating function:

$$e^{xt+yt^2} = \sum_{n=0}^{\infty} H_n(x,y)\frac{t^n}{n!} \tag{1.2}$$

and reduce to the ordinary Hermite polynomials $H_n(x)$ (see [1]) when $y=-1$ and x is replaced by $2x$.

In (2016), Khan [13] introduced the degenerate Hermite polynomials of two variables by means of the following generating function:

$$(1+\lambda t)^{\frac{x}{\lambda}}(1+\lambda t^2)^{\frac{y}{\lambda}} = \sum_{n=0}^{\infty} H_n(x,y;\lambda)\frac{t^n}{n!}, \tag{1.3}$$

where $\lambda \neq 0$. Since $(1+\lambda t)^{\frac{1}{\lambda}} \longrightarrow e^t$ as $\lambda \longrightarrow 0$, it is evident that (1.3) reduces to (1.2). That is $H_n(x,y)$ limiting case of $H_n(x,y;\lambda)$ when $\lambda \longrightarrow 0$.

The following representation of degenerate Hermite polynomials $H_n(x,y;\lambda)$ is given by

$$H_n(x,y;\lambda) = n!\sum_{r=0}^{[\frac{n}{2}]} \frac{(-\frac{x}{\lambda})_{n-2r}(-\frac{y}{\lambda})_r(-\lambda)^{n-r}}{r!(n-2r)!}. \tag{1.4}$$

Since $\lim\limits_{\lambda\longrightarrow 0} H_n(x,y;\lambda) = H_n(x,y)$, (1.1) is a limiting case of (1.4).
For $\lambda \in \mathbb{C}$, Carlitz introduced the degenerate Bernoulli polynomials by means of the following generating function:

$$\frac{t}{(1+\lambda t)^{\frac{1}{\lambda}}-1}(1+\lambda t)^{\frac{x}{\lambda}} = \sum_{n=0}^{\infty} \beta_n(x;\lambda)\frac{t^n}{n!}, \quad \text{(see [4], [18], [17])} \tag{1.5}$$

so that

$$\beta_n(x;\lambda) = \sum_{n=0}^{m} \binom{n}{m} \beta_m(\lambda)\left(\frac{x}{\lambda}\right)_{n-m}. \tag{1.6}$$

When $x=0$, $\beta_n(\lambda) = \beta_n(0;\lambda)$ are called the degenerate Bernoulli numbers.
From (1.5), we note that

$$\sum_{n=0}^{\infty} \lim_{\lambda\longrightarrow 0} \beta_n(x;\lambda)\frac{t^n}{n!} = \lim_{\lambda\longrightarrow 0} \frac{t}{(1+\lambda t)^{\frac{1}{\lambda}}-1}(1+\lambda t)^{\frac{x}{\lambda}}$$

$$= \frac{t}{e^t-1}e^{xt} = \sum_{n=0}^{\infty} B_n(x)\frac{t^n}{n!}, \tag{1.7}$$

where $B_n(x)$ are called the Bernoulli polynomials (see [1]-[25]).

The classical polylogarithm function $\mathrm{Li}_k(z)$ is

$$\mathrm{Li}_k(z)=\sum_{m=1}^{\infty}\frac{z^m}{m^k},(k\in\mathbb{Z})\quad \text{(see [13], [14], [16])} \tag{1.8}$$

so for $k\leq 1$,

$$\mathrm{Li}_k(z)=-\ln(1-z),\ \mathrm{Li}_0(z)=\frac{z}{1-z},\ \mathrm{Li}_{-1}(z)=\frac{z}{(1-z)^2},\dots$$

The poly-Bernoulli polynomials are given by

$$\frac{\mathrm{Li}_k(1-e^{-t})}{e^t-1}e^{xt}=\sum_{n=0}^{\infty}B_n^{(k)}(x)\frac{t^n}{n!},\quad \text{(see [2], [12], [13])}. \tag{1.9}$$

For $k=1$ in (1.9), we have

$$\frac{\mathrm{Li}_1(1-e^{-t})}{e^t-1}e^{xt}=\frac{t}{e^t-1}e^{xt}=\sum_{n=0}^{\infty}B_n(x)\frac{t^n}{n!}. \tag{1.10}$$

From (1.7) and (1.10), we have

$$B_n^{(1)}(x)=B_n(x).$$

Very recently, Khan [13] introduced the degenerate Hermite poly-Bernoulli polynomials of two variables ${}_H\beta_{n,q}^{(k)}(x,y;\lambda)$ by means of the following generating function:

$$\left(\frac{\mathrm{Li}_k(1-e^{-t})}{(1+\lambda t)^{\frac{1}{\lambda}}-1}\right)^{\alpha}(1+\lambda t)^{\frac{x}{\lambda}}(1+\lambda t^2)^{\frac{y}{\lambda}}=\sum_{n=0}^{\infty}{}_H\beta_n^{(k)}(x,y;\lambda)\frac{t^n}{n!}, \tag{1.11}$$

so that

$${}_HB_n^{(k)}(x,y;\lambda)=\sum_{m=0}^{n}\binom{n}{m}\beta_{n-m}^{(k)}(\lambda)H_m(x,y;\lambda). \tag{1.12}$$

The Stirling number of the first kind is given by

$$(x)_n=x(x-1)\cdots(x-n+1)=\sum_{l=0}^{n}S_1(n,l)x^l,(n\geq 0), \tag{1.13}$$

and the Stirling number of the second kind is defined by generating function:

$$(e^t-1)^n=n!\sum_{l=n}^{\infty}S_2(l,n)\frac{t^l}{l!}. \tag{1.14}$$

A generalized falling factorial sum $\sigma_k(n;\lambda)$ can be defined by the generating function [25]

$$\sum_{k=0}^{\infty}\sigma_k(n;\lambda)\frac{t^k}{k!}=\frac{(1+\lambda t)^{\frac{(n+1)}{\lambda}}-1}{(1+\lambda t)^{\frac{1}{\lambda}}-1}. \tag{1.15}$$

Note that

$$\lim_{\lambda\longrightarrow 0}\sigma_k(n;\lambda)=S_k(n).$$

The object of this paper as follows, we first give definition of the degenerate Hermite poly-Bernoulli polynomials ${}_H\beta_n^{(k)}(x,y;\lambda)$ with q-parameter and then extend and illustrate how, a connection between Hermite and Bernoulli polynomials can yield new expansions and representations. Some implicit summation formulae and general symmetry identities are derived. These results establish a link between these families of polynomials (namely degenerate Hermite and degenerate q-poly-Bernoulli polynomials).

2. q-analogue of degenerate Hermite poly-Bernoulli polynomials

In this section, we introduce q-analogue of degenerate of Hermite-poly-Bernoulli numbers and polynomials and its properties.

Definition 2.1. For $\lambda \in \mathbb{C}$, $k \in \mathbb{Z}$ and $n \geq 0$, $0 \leq q < 1$, we introduce q-analogue of degenerate Hermite poly-Bernoulli polynomials by means of the following generating function:

$$\frac{\mathrm{Li}_{k,q}(1-e^{-t})}{(1+\lambda t)^{\frac{1}{\lambda}}-1}(1+\lambda t)^{\frac{x}{\lambda}}(1+\lambda t^2)^{\frac{y}{\lambda}} = \sum_{n=0}^{\infty} {}_H\beta_{n,q}^{(k)}(x,y;\lambda)\frac{t^n}{n!}, \tag{2.1}$$

where $\mathrm{Li}_{k,q}(t) = \sum_{n=0}^{\infty} \frac{t^n}{[n]_q^k!}$ is the k-th q-polylogarithm function (see [6], [10], [23]).

When $x = y = 0$ in (2.1), $\beta_n^{(k)}(\lambda) = {}_H\beta_n^{(k)}(0,0;\lambda)$ are called the q-analogue of degenerate poly-Bernoulli numbers.

Note that

$${}_H\beta_{n,q}^{(1)}(x,y;\lambda) = {}_H\beta_{n,q}(x,y;\lambda)$$

and

$$\lim_{\lambda \longrightarrow 0} {}_H\beta_{n,q}^{(k)}(x,y;\lambda) = {}_HB_{n,q}^{(k)}(x,y).$$

Remark 2.2. For $y = 0$ in (2.1), the result reduces to the q-analogue of degenerate poly-Bernoulli polynomials of Jung and Ryoo [10, p. 32, Eq. (2.1)] defined as

$$\frac{\mathrm{Li}_{k,q}(1-e^{-t})}{(1+\lambda t)^{\frac{1}{\lambda}}-1}(1+\lambda t)^{\frac{x}{\lambda}} = \sum_{n=0}^{\infty} \beta_{n,q}^{(k)}(x;\lambda)\frac{t^n}{n!}, (k \in \mathbb{Z}). \tag{2.2}$$

Theorem 2.3. *For $\lambda \in \mathbb{C}$, $k \in \mathbb{Z}$ and $n \geq 0$, $0 \leq q < 1$, we have*

$${}_H\beta_{n,q}^{(k)}(x,y;\lambda) = \sum_{l=0}^{n} \binom{n}{l} \beta_{l,q}^{(k)}(\lambda)H_{n-l}(x,y;\lambda). \tag{2.3}$$

Proof. By using definition (2.1), we have

$$\sum_{n=0}^{\infty} {}_H\beta_{n,q}^{(k)}(x,y;\lambda)\frac{t^n}{n!} = \frac{\mathrm{Li}_{k,q}(1-e^{-t})}{(1+\lambda t)^{\frac{1}{\lambda}}-1}(1+\lambda t)^{\frac{x}{\lambda}}(1+\lambda t^2)^{\frac{y}{\lambda}}$$

$$= \sum_{l=0}^{\infty} \beta_{l,q}^{(k)}(\lambda)\frac{t^l}{l!} \sum_{n=0}^{\infty} H_n(x,y;\lambda)\frac{t^n}{n!}$$

$$\sum_{n=0}^{\infty} {}_H\beta_{n,q}^{(k)}(x,y;\lambda)\frac{t^n}{n!} = \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\binom{n}{l}\beta_{l,q}^{(k)}(\lambda)H_n(x,y;\lambda)\right)\frac{t^n}{n!}.$$

Comparing the coefficients of $\frac{t^n}{n!}$ in both sides, we get (2.3). □

Theorem 2.4. *For $n \geq 0$, we have*

$$ {}_H\beta_{n,1}^{(2)}(x,y;\lambda) = \sum_{m=0}^{n}\binom{n}{m}\frac{B_m}{m+1}{}_H\beta_{n-m}(x,y;\lambda). \quad (2.4)$$

Proof. Consider equation(2.1), we have

$$\sum_{n=0}^{\infty} {}_H\beta_{n,1}^{(k)}(x,y;\lambda)\frac{t^n}{n!} = \frac{\mathrm{Li}_{k,1}(1-e^{-t})}{(1+\lambda t)^{\frac{1}{\lambda}}-1}(1+\lambda t)^{\frac{x}{\lambda}}(1+\lambda t^2)^{\frac{y}{\lambda}}$$

$$= \frac{(1+\lambda t)^{\frac{x}{\lambda}}(1+\lambda t^2)^{\frac{y}{\lambda}}}{(1+\lambda t)^{\frac{1}{\lambda}}-1}\int_0^t \underbrace{\frac{1}{e^z-1}\int_0^t\frac{1}{e^z-1}\cdots\frac{1}{e^z-1}\int_0^t\frac{z}{e^z-1}}_{(k-1)-\text{times}}dz\cdots dz \quad (2.5)$$

For $k=2$ in (2.5), we have

$$\sum_{n=0}^{\infty} {}_H\beta_{n,1}^{(2)}(x,y;\lambda)\frac{t^n}{n!} = \frac{(1+\lambda t)^{\frac{x}{\lambda}}(1+\lambda t^2)^{\frac{y}{\lambda}}}{(1+\lambda t)^{\frac{1}{\lambda}}-1}\int_0^t\frac{z}{e^z-1}dz$$

$$= \left(\sum_{m=0}^{\infty}\frac{B_m}{m+1}\frac{t^m}{m!}\right)\frac{(1+\lambda t)^{\frac{x}{\lambda}}(1+\lambda t^2)^{\frac{y}{\lambda}}}{(1+\lambda t)^{\frac{1}{\lambda}}-1}$$

$$= \left(\sum_{m=0}^{\infty}\frac{B_m}{m+1}\frac{t^m}{m!}\right)\left(\sum_{n=0}^{\infty}{}_H\beta_n(x,y;\lambda)\frac{t^n}{n!}\right).$$

Replacing n by $n-m$ in above equation, we have

$$= \sum_{n=0}^{\infty}\sum_{m=0}^{n}\binom{n}{m}\frac{B_m}{m+1}{}_H\beta_{n-m}(x,y;\lambda)\frac{t^n}{n!}.$$

On equating the coefficients of the like powers of t in the above equation, we get the result (2.4). □

Theorem 2.5. *For $n \geq 0$, we have*

$${}_H\beta_{n,q}^{(k)}(x,y;\lambda) = \sum_{p=0}^{n}\binom{n}{p}\left(\sum_{l=1}^{p+1}\frac{(-1)^{l+p+1}l!S_2(p+1,l)}{[l]_q^k(p+1)}\right){}_H\beta_{n-p}(x,y;\lambda). \quad (2.6)$$

Proof. From equation (2.1), we have

$$\sum_{n=0}^{\infty}{}_H\beta_{n,q}^{(k)}(x,y;\lambda)\frac{t^n}{n!} = \left(\frac{\mathrm{Li}_{k,q}(1-e^{-t})}{t}\right)\left(\frac{t(1+\lambda t)^{\frac{x}{\lambda}}(1+\lambda t^2)^{\frac{y}{\lambda}}}{(1+\lambda t)^{\frac{1}{\lambda}}-1}\right). \quad (2.7)$$

Now

$$\frac{1}{t}\mathrm{Li}_{k,q}(1-e^{-t}) = \frac{1}{t}\sum_{l=1}^{\infty}\frac{(1-e^{-t})^l}{[l]_q^k} = \frac{1}{t}\sum_{l=1}^{\infty}\frac{(-1)^l}{l^k}(1-e^{-t})^l$$

$$= \frac{1}{t}\sum_{l=1}^{\infty}\frac{(-1)^l}{[l]_q^k}l!\sum_{p=l}^{\infty}(-1)^p S_2(p,l)\frac{t^p}{p!}$$

$$= \frac{1}{t}\sum_{p=1}^{\infty}\sum_{l=1}^{p}\frac{(-1)^{l+p}}{[l]_q^k}l!S_2(p,l)\frac{t^p}{p!}$$

$$= \sum_{p=0}^{\infty}\left(\sum_{l=1}^{p+1}\frac{(-1)^{l+p+1}}{[l]_q^k}l!\frac{S_2(p+1,l)}{p+1}\right)\frac{t^p}{p!}. \tag{2.8}$$

From equations (2.7) and (2.8), we have

$$\sum_{n=0}^{\infty}{}_H\beta_{n,q}^{(k)}(x,y;\lambda)\frac{t^n}{n!}$$

$$= \sum_{p=0}^{\infty}\left(\sum_{l=1}^{p+1}\frac{(-1)^{l+p+1}}{[l]_q^k}l!\frac{S_2(p+1,l)}{p+1}\right)\frac{t^p}{p!}\left(\sum_{n=0}^{\infty}{}_H\beta_n(x,y;\lambda)\frac{t^n}{n!}\right).$$

Replacing n by $n-p$ in the r.h.s of above equation and comparing the coefficients of t^n, we get the result (2.6). □

Theorem 2.6. *For $n \geq 1$, we have*

$${}_H\beta_{n,q}^{(k)}(x+1,y;\lambda) - {}_H\beta_n^{(k)}(x,y;\lambda)$$

$$= \sum_{p=1}^{n}\binom{n}{p}\left(\sum_{l=0}^{p-1}\frac{(-1)^{l+p+1}}{[l+1]_q^k}(l+1)!S_2(p,l+1)\right)H_{n-p}(x,y;\lambda). \tag{2.9}$$

Proof. Using the Definition (2.1), we have

$$\sum_{n=0}^{\infty}{}_H\beta_{n,q}^{(k)}(x+1,y;\lambda)\frac{t^n}{n!} - \sum_{n=0}^{\infty}{}_H\beta_{n,q}^{(k)}(x,y;\lambda)\frac{t^n}{n!}$$

$$= \frac{\mathrm{Li}_{k,q}(1-e^{-t})}{(1+\lambda t)^{\frac{1}{\lambda}}-1}(1+\lambda t)^{\frac{x+1}{\lambda}}(1+\lambda t^2)^{\frac{y}{\lambda}} - \frac{\mathrm{Li}_{k,q}(1-e^{-t})}{(1+\lambda t)^{\frac{1}{\lambda}}-1}(1+\lambda t)^{\frac{x}{\lambda}}(1+\lambda t^2)^{\frac{y}{\lambda}}$$

$$= \mathrm{Li}_{k,q}(1-e^{-t})(1+\lambda t)^{\frac{x}{\lambda}}(1+\lambda t^2)^{\frac{y}{\lambda}}$$

$$= \sum_{l=0}^{\infty}\frac{(1-e^{-t})^{l+1}}{[l+1]_q^k}(1+\lambda t)^{\frac{x}{\lambda}}(1+\lambda t^2)^{\frac{y}{\lambda}}$$

$$= \sum_{p=1}^{\infty}\left(\sum_{l=0}^{p-1}\frac{(-1)^{l+p+1}}{[l+1]_q^k}(l+1)!S_2(p,l+1)\right)\frac{t^p}{p!}(1+\lambda t)^{\frac{x}{\lambda}}(1+\lambda t^2)^{\frac{y}{\lambda}}$$

$$= \left(\sum_{p=1}^{\infty}\left(\sum_{l=0}^{p-1}\frac{(-1)^{l+p+1}}{[l+1]_q^k}(l+1)!S_2(p,l+1)\right)\frac{t^p}{p!}\right)\left(\sum_{n=0}^{\infty}H_n(x,y;\lambda)\frac{t^n}{n!}\right).$$

Replacing n by $n-p$ in the above equation and comparing the coefficients of t^n, we get the result (2.9). □

Theorem 2.7. *For $n \geq 0$, $d \in \mathbb{N}$ and $k \in \mathbb{Z}$, we have*

$$ {}_H\beta^{(k)}_{n,q}(x,y;\lambda) $$

$$ = \sum_{a=0}^{d-1}\sum_{l=0}^{n}\sum_{p=1}^{l+1}\left(\begin{array}{c} n \\ l \end{array}\right) d^{n-l-1}\frac{(-1)^{l+p+1}p!S_2(l+1,p)}{p^k[l+1]_q^k}\,{}_H\beta_{n-l}\left(\frac{l+x}{d},y;\frac{\lambda}{d}\right). \quad (2.10) $$

Proof. From equation (2.1), we can be written as

$$ \sum_{n=0}^{\infty}{}_H\beta^{(k)}_{n,q}(x,y;\lambda)\frac{t^n}{n!} = \frac{\mathrm{Li}_{k,q}(1-e^{-t})}{(1+\lambda t)^{\frac{1}{\lambda}}-1}(1+\lambda t)^{\frac{x}{\lambda}}(1+\lambda t^2)^{\frac{y}{\lambda}} $$

$$ = \frac{\mathrm{Li}_{k,q}(1-e^{-t})}{(1+\lambda t)^{\frac{d}{\lambda}}-1}\sum_{a=0}^{d-1}(1+\lambda t)^{\frac{l+x}{\lambda}}(1+\lambda t^2)^{\frac{y}{\lambda}} $$

$$ = \left(\frac{\mathrm{Li}_{k,q}(1-e^{-t})}{t}\right)\frac{1}{d}\sum_{a=0}^{d-1}\frac{dt}{(1+\lambda t)^{\frac{d}{\lambda}}-1}(1+\lambda t)^{\frac{l+x}{\lambda}}(1+\lambda t^2)^{\frac{y}{\lambda}} $$

$$ = \left(\sum_{l=0}^{\infty}\left(\sum_{p=1}^{l+1}\frac{(-1)^{l+p+1}}{p^k}p!\frac{S_2(l+1,p)}{[l+1]_q^k}\right)\frac{t^l}{l!}\right)\left(\sum_{n=0}^{\infty}d^{n-1}\sum_{a=0}^{d-1}{}_H\beta_n\left(\frac{l+x}{d},y;\frac{\lambda}{d}\right)\frac{t^n}{n!}\right). $$

Replacing n by $n-l$ in above equation and comparing the coefficient of t^n, we get the result (2.10). □

3. Summation formulae for degenerate Hermite poly-Bernoulli polynomials with q-parameter

For the derivation of implicit formulae involving degenerate q-poly-Bernoulli polynomials $\beta^{(k)}_{n,q}(x;\lambda)$ and degenerate Hermite poly-Bernoulli polynomials ${}_H\beta^{(k)}_{n,q}(x,y;\lambda)$ the same considerations as developed for the ordinary Hermite and related polynomials in Khan [14] and Hermite-Bernoulli polynomials in Pathan and Khan [19], [20], [21], [22] holds as well. First we prove the following results involving degenerate Hermite poly-Bernoulli polynomials with q-parameter ${}_H\beta^{(k)}_{n,q}(x,y;\lambda)$.

Theorem 3.1. *The following implicit summation formulae involving degenerate Hermite polynomials ${}_H\beta^{(k)}_{n,q}(\lambda,\mu;x,y)$ holds true:*

$$ {}_H\beta^{(k)}_{m+n,q}(x,y;\lambda) $$

$$ = \sum_{r,s=0}^{m,n}\left(\begin{array}{c} m \\ r \end{array}\right)\left(\begin{array}{c} n \\ s \end{array}\right)(x-v)^{r+s}\left[\frac{1}{\lambda}\log(1+\lambda)\right]^{r+s}{}_H\beta^{(k)}_{m+n-r-s,q}(v,y;\lambda). \quad (3.1) $$

Proof. Replacing t by $t+u$ in (2.1) and rewrite the generating function (2.1) as

$$ \frac{\mathrm{Li}_{k,q}(1-e^{-(t+u)})}{(1+\lambda(t+u))^{\frac{1}{\lambda}}-1}e^{\frac{y(t+u)^2}{\lambda}(\log(1+\lambda))} $$

$$= e^{-\frac{x(t+u)}{\lambda}(\log(1+\lambda)} \sum_{m,n=0}^{\infty} {}_H\beta_{m+n,q}^{(k)}(x,y;\lambda)\frac{t^m}{m!}\frac{t^n}{n!}. \tag{3.2}$$

Upon replacing x by v in the above equation, it is not difficult to observe that

$$\sum_{m,n=0}^{\infty} {}_H\beta_{m+n,q}^{(k)}(\lambda;x,y)\frac{t^m}{m!}\frac{t^n}{n!} = e^{\frac{x(t+u)(x-v)}{\lambda}\log(1+\lambda)} \sum_{m,n=0}^{\infty} {}_H\beta_{m+n,q}^{(k)}(\lambda;v,y)\frac{t^m}{m!}\frac{t^n}{n!}$$

$$\sum_{m,n=0}^{\infty} {}_H\beta_{m+n,q}^{(k)}(\lambda,\mu;x,y)\frac{t^p}{p!}\frac{t^q}{q!}$$

$$= \sum_{N=0}^{\infty} \frac{[\frac{x(t+u)(x-v)}{\lambda}\log(1+\lambda)]^N}{N!} \sum_{p,q=0}^{\infty} {}_H\beta_{p+q}^{(k)}(\lambda,\mu;v,y)\frac{t^p}{p!}\frac{t^q}{q!}.$$

Now, by applying the following known series identity [24, p. 52, Eq. 1.6(2)]:

$$\sum_{N=0}^{\infty} f(N)\frac{(x+y)^N}{N!} = \sum_{p,q=0}^{\infty} f(n+m)\frac{x^p}{p!}\frac{y^q}{q!}$$

$$\sum_{m,n=0}^{\infty} {}_H\beta_{m+n,q}^{(k)}(\lambda;x,y)\frac{t^m}{m!}\frac{t^n}{n!}$$

$$= \sum_{r,s=0}^{\infty} (x-v)^{r+s}[\frac{1}{\lambda}\log(1+\lambda)]^{r+s}\frac{t^r}{r!}\frac{u^s}{s!} \sum_{m,n=0}^{\infty} {}_H\beta_{m+n,q}^{(k)}(\lambda;v,y)\frac{t^m}{m!}\frac{t^n}{n!}.$$

On replacing m by $m-r$ and n by $n-s$ in above equation, we get

$$\sum_{m,n=0}^{\infty} {}_H\beta_{m+n,q}^{(k)}(\lambda;x,y)\frac{t^m}{m!}\frac{t^n}{n!}$$

$$= \sum_{m,n=0}^{\infty}\sum_{r,s=0}^{p,q} (x-v)^{r+s}\left[\frac{1}{\lambda}\log(1+\lambda)\right]^{r+s} {}_H\beta_{m+n-r-s,q}^{(k)}(\lambda;v,y)\frac{t^m}{(m-r)!r!}\frac{t^n}{(n-s)!s!}.$$

Comparing the coefficients of $\frac{t^m}{m!}$ and $\frac{t^n}{n!}$, we get the result (3.1). □

Theorem 3.2. *For $x, y \in \mathbb{R}$ and $n \geq 0$. Then*

$${}_H\beta_{n,q}^{(k)}(x+u,y+w;\lambda) = \sum_{m=0}^{n}\binom{n}{m} {}_H\beta_{n-m,q}^{(k)}(x,y;\lambda)H_m(u,w;\lambda). \tag{3.3}$$

Proof. By the definition of q-degenerate poly-Bernoulli polynomials and the definition (1.3), we have

$$\frac{\mathrm{Li}_{k,q}(1-e^{-t})}{(1+\lambda t)^{\frac{1}{\lambda}}-1}(1+\lambda t)^{\frac{x+u}{\lambda}}(1+\lambda t^2)^{\frac{y+w}{\lambda}}$$

$$= \left(\sum_{n=0}^{\infty} {}_H\beta_{n,q}^{(k)}(x,y;\lambda)\frac{t^n}{n!}\right)\left(\sum_{m=0}^{\infty} H_m(u,w;\lambda)\frac{t^m}{m!}\right).$$

Now replacing n by $n-m$ and comparing the coefficients of t^n, we get the result (3.3). □

Theorem 3.3. *For $x, y \in \mathbb{R}$ and $n \geq 0$. Then*

$$ {}_H\beta_{n,q}^{(k)}(x,y;\lambda) = \sum_{m=0}^{n-2j} \sum_{j=0}^{[\frac{n}{2}]} \beta_{m,q}^{(k)}(\lambda) \Big(-\frac{x}{\lambda}\Big)_{n-m-2j} (-\lambda)^{n-m-j} \Big(-\frac{y}{\lambda}\Big)_j \frac{n!}{m!j!(n-2j-m)!}. \tag{3.4} $$

Proof. Applying the definition (2.1) to the term $\frac{\mathrm{Li}_{k,q}(1-e^{-t})}{(1+\lambda t)^{\frac{1}{\lambda}}-1}$ and expanding the function $(1+\lambda t)^{\frac{x}{\lambda}}(1+\lambda t^2)^{\frac{y}{\lambda}}$ at $t=0$ yields

$$ \frac{\mathrm{Li}_{k,q}(1-e^{-t})}{(1+\lambda t)^{\frac{1}{\lambda}}-1}(1+\lambda t)^{\frac{x}{\lambda}}(1+\lambda t^2)^{\frac{y}{\lambda}} $$

$$ = \left(\sum_{m=0}^{\infty} \beta_{m,q}^{(k)}(\lambda)\frac{t^m}{m!}\right)\left(\sum_{n=0}^{\infty} \Big(-\frac{x}{\lambda}\Big)_n \frac{(-\lambda t)^n}{n!}\right)\left(\sum_{j=0}^{\infty} \Big(-\frac{y}{\lambda}\Big)_j \frac{(-\lambda t^2)^j}{j!}\right) $$

$$ = \sum_{n=0}^{\infty} \left(\sum_{m=0}^{n} \binom{n}{m} \beta_{m,q}^{(k)}(\lambda) \Big(-\frac{x}{\lambda}\Big)_{n-m} (-\lambda)^{n-m}\right) \frac{t^n}{n!} \left(\sum_{j=0}^{\infty} \Big(-\frac{y}{\lambda}\Big)_j \frac{(-\lambda t^2)^j}{j!}\right). $$

Replacing n by $n-2j$, we have

$$ \sum_{n=0}^{\infty} {}_H\beta_{n,q}^{(k)}(x,y;\lambda)\frac{t^n}{n!} $$

$$ = \sum_{n=0}^{\infty} \left(\sum_{m=0}^{n-2j} \sum_{j=0}^{[\frac{n}{2}]} \binom{n-2j}{m} \beta_{m,q}^{(k)}(\lambda) \Big(-\frac{x}{\lambda}\Big)_{n-m-2j} (-\lambda)^{n-m-j} \Big(-\frac{y}{\lambda}\Big)_j \right) \frac{t^n}{(n-2j)!j!}. \tag{3.5} $$

Equating their coefficients of t^n, we get the result (3.4). □

Theorem 3.4. *For $x, y \in \mathbb{R}$ and $n \geq 0$. Then*

$$ {}_H\beta_{n,q}^{(k)}(x,y;\lambda) = \sum_{m=0}^{n} \binom{n}{m} \Big(-\frac{z}{\lambda}\Big)_{n-m} (-\lambda)^{n-m} {}_H\beta_{m,q}^{(k)}(x-z,y;\lambda). \tag{3.6} $$

Proof. By exploiting the generating function (2.1), we can write the equation

$$ \sum_{n=0}^{\infty} {}_H\beta_{n,q}^{(k)}(x,y;\lambda)\frac{t^n}{n!} = \frac{\mathrm{Li}_{k,q}(1-e^{-t})}{(1+\lambda t)^{\frac{1}{\lambda}-1}}(1+\lambda t)^{\frac{x-z}{\lambda}}(1+\lambda t^2)^{\frac{y}{\lambda}}(1+\lambda t)^{\frac{z}{\lambda}}. \tag{3.7} $$

$$ = \left(\sum_{m=0}^{\infty} {}_H\beta_{m,q}^{(k)}(x-z,y;\lambda)\frac{t^m}{m!}\right)\left(\sum_{n=0}^{\infty} (-\frac{z}{\lambda})_n \frac{(-\lambda t)^n}{n!}\right). $$

Replacing n by $n-m$ in above equation and equating their coefficients of t^n leads to formula (3.6). □

Theorem 3.5. *The following implicit summation formula involving degenerate Hermite poly-Bernoulli polynomials with q-parameter* ${}_H\beta^{(k)}_{n,q}(x,y;\lambda)$ *holds true:*

$$ {}_H\beta^{(k)}_{n,q}(x+1,y;\lambda)=\sum_{r=0}^{n}\binom{n}{r}\left(-\frac{1}{\lambda}\right)_r(-\lambda)^r\,{}_H\beta^{(k)}_{n-r,q}(x,y;\lambda). \tag{3.8}$$

Proof. By the definition of degenerate Hermite poly-Bernoulli polynomials with q-parameter, we have

$$\sum_{n=0}^{\infty}{}_H\beta^{(k)}_{n,q}(x+1,y;\lambda)\frac{t^n}{n!}+\sum_{n=0}^{\infty}{}_H\beta^{(k)}_{n,q}(x,y;\lambda)\frac{t^n}{n!}$$

$$=\frac{\mathrm{Li}_{k,q}(1-e^{-t})}{(1+\lambda t)^{\frac{1}{\lambda}}-1}(1+\lambda t)^{\frac{x}{\lambda}}(1+\lambda t^2)^{\frac{y}{\lambda}}((1+\lambda t)^{\frac{1}{\lambda}}+1)$$

$$=\left(\sum_{n=0}^{\infty}{}_H\beta^{(k)}_{n,q}(x,y;\lambda)\frac{t^n}{n!}\right)\left(\sum_{r=0}^{\infty}\left(-\frac{1}{\lambda}\right)_r\frac{(-\lambda t)^r}{r!}\right)+\sum_{n=0}^{\infty}{}_H\beta^{(k)}_{n,q}(x,y;\lambda)\frac{t^n}{n!}$$

$$=\sum_{n=0}^{\infty}\sum_{r=0}^{n}{}_H\beta^{(k)}_{n-r,q}(x,y;\lambda)\left(-\frac{1}{\lambda}\right)_r(-\lambda)^r\frac{t^n}{(n-r)!r!}+\sum_{n=0}^{\infty}{}_H\beta^{(k)}_{n,q}(x,y;\lambda)\frac{t^n}{n!}.$$

Finally, equating the coefficients of the like powers of t^n, we get (3.8). □

4. General symmetry identities

In this section, we establish symmetry identities for the q-degenerate poly-Bernoulli polynomials $\beta^{(k)}_{n,q}(x;\lambda)$ and the degenerate Hermite poly-Bernoulli polynomials with q-parameter ${}_H\beta^{(k)}_{n,q}(x,y;\lambda)$ by applying the generating function(2.1) and (2.2). The results extend some known identities of Khan [13], [14], [15], [16], Pathan and Khan [19], [20], [21], [22].

Theorem 4.1. *Let* $a,b>0$ *and* $a\neq b$. *For* $x,y\in\mathbb{R}$, $n\geq 0$, $0\leq q<1$, *then the following identity holds true:*

$$\sum_{m=0}^{n}\binom{n}{m}b^m a^{n-m}\,{}_H\beta^{(k)}_{n-m,q}(bx,b^2y;\lambda)\,{}_H\beta^{(k)}_{m,q}(ax,a^2y;\lambda)$$

$$=\sum_{m=0}^{n}\binom{n}{m}a^m b^{n-m}\,{}_H\beta^{(k)}_{n-m,q}(ax,a^2y;\lambda)\,{}_H\beta^{(k)}_{m,q}(bx,b^2y;\lambda). \tag{4.1}$$

Proof. Start with

$$G(t)=\left(\frac{\mathrm{Li}_{k,q}(1-e^{-at})\mathrm{Li}_{k,q}(1-e^{-bt})}{((1+\lambda t)^{\frac{a}{\lambda}}-1)((1+\lambda t)^{\frac{b}{\lambda}}-1)}\right)(1+\lambda t)^{\frac{abx}{\lambda}}(1+\lambda t^2)^{\frac{a^2b^2y}{\lambda}}. \tag{4.2}$$

Then the expression for $G(t)$ is symmetric in a and b and we can expand $G(t)$ into series in two ways to obtain

$$G(t)=\sum_{n=0}^{\infty}{}_H\beta^{(k)}_{n,q}(bx,b^2y;\lambda)\frac{(at)^n}{n!}\sum_{m=0}^{\infty}{}_H\beta^{(k)}_{m,q}(ax,a^2y;\lambda)\frac{(bt)^m}{m!}$$

$$= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\binom{n}{m} a^{n-m}b^{m}{}_{H}\beta^{(k)}_{n-m,q}(bx,b^2y;\lambda){}_{H}\beta^{(k)}_{m,q}(ax,a^2y;\lambda)\right)\frac{t^n}{n!}.$$

On the similar lines we can show that

$$G(t) = \sum_{n=0}^{\infty}{}_{H}\beta^{(k)}_{n,q}(ax,a^2y;\lambda)\frac{(bt)^n}{n!}\sum_{m=0}^{\infty}{}_{H}\beta^{(k)}_{m,q}(bx,b^2y;\lambda)\frac{(at)^m}{m!}$$

$$= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\binom{n}{m} a^{m}b^{n-m}{}_{H}\beta^{(k)}_{n-m,q}(ax,a^2y;\lambda){}_{H}\beta^{(k)}_{m,q}(bx,b^2y;\lambda)\right)\frac{t^n}{n!}.$$

Comparing the coefficients of $\frac{t^n}{n!}$ on the right hand sides of the last two equations, we arrive at the desired result. □

Remark 4.2. For $b = 1$, Theorem 4.1 reduces to

$$\sum_{m=0}^{n}\binom{n}{m} a^{n-m}{}_{H}\beta^{(k)}_{n-m,q}(x,y;\lambda){}_{H}\beta^{(k)}_{m,q}(ax,a^2y;\lambda)$$

$$= \sum_{m=0}^{n}\binom{n}{m} a^{m}{}_{H}\beta^{(k)}_{n-m,q}(ax,a^2y;\lambda){}_{H}\beta^{(k)}_{m,q}(x,y;\lambda). \tag{4.3}$$

Theorem 4.3. *For all integers $a > 0, b > 0$ and $n \geq 0$, $0 \leq q < 1$, the following identity holds true:*

$$\sum_{m=0}^{n}\binom{n}{m} a^{n-m}b^{m}{}_{H}\beta^{(k)}_{n-m,q}\left(bx,b^2z;\lambda\right)\sum_{i=0}^{m}\binom{m}{i}\sigma_i(a-1;\lambda)\beta^{(k)}_{m-i,q}(ay;\lambda)$$

$$= \sum_{m=0}^{n}\binom{n}{m} a^{m}b^{n-m}{}_{H}\beta^{(k)}_{n-m,q}\left(ax,a^2z;\lambda\right)\sum_{i=0}^{m}\binom{m}{i}\sigma_i(b-1;\lambda)\beta^{(k)}_{m-i,q}(by;\lambda), \tag{4.4}$$

where generalized falling factorial sum $\sigma_k(n;\lambda)$ is given by (1.15).

Proof. We now use

$$H(t) = \frac{\mathrm{Li}_{k,q}(1-e^{-at})\mathrm{Li}_{k,q}(1-e^{-bt})((1+\lambda t)^{\frac{ab}{\lambda}}-1)(1+\lambda t)^{\frac{ab(x+y)}{\lambda}}(1+\lambda t^2)^{\frac{a^2b^2z}{\lambda}}}{((1+\lambda t)^{\frac{a}{\lambda}}-1)((1+\lambda t)^{\frac{b}{\lambda}}-1)^2}$$

to find that

$$g(t) = \left(\frac{\mathrm{Li}_{k,q}(1-e^{-at})}{(1+\lambda t)^{\frac{a}{\lambda}}-1}\right)(1+\lambda t)^{\frac{abx}{\lambda}}(1+\lambda t^2)^{\frac{a^2b^2z}{\lambda}}\left(\frac{(1+\lambda t)^{\frac{ab}{\lambda}}-1}{(1+\lambda t)^{\frac{b}{\lambda}}-1}\right)$$

$$\times\left(\frac{\mathrm{Li}_{k,q}(1-e^{-bt})}{(1+\lambda t)^{\frac{b}{\lambda}}-1}\right)(1+\lambda t)^{\frac{aby}{\lambda}}$$

$$= \sum_{n=0}^{\infty}{}_{H}\beta^{(k)}_{n,q}\left(bx,b^2z;\lambda\right)\frac{(at)^n}{n!}\sum_{n=0}^{\infty}\sigma_n(a-1;\lambda)\frac{(bt)^n}{n!}\sum_{n=0}^{\infty}\beta^{(k)}_{n,q}(ay;\lambda)\frac{(bt)^n}{n!}. \tag{4.5}$$

Using a similar plan, we get

$$g(t) = \sum_{n=0}^{\infty} {}_H\beta_{n,q}^{(k)}\left(ax, a^2z; \lambda\right) \frac{(bt)^n}{n!} \sum_{n=0}^{\infty} \sigma_n(b-1; \lambda) \frac{(at)^n}{n!} \sum_{n=0}^{\infty} \beta_{n,q}^{(k)}(by; \lambda) \frac{(at)^n}{n!}. \quad (4.6)$$

By comparing the coefficients of t^n on the right hand sides of the last two equations, we arrive at the desired result. □

5. Conclusion

The degenerate Hermite-poly-Bernoulli polynomials with q-parameter ${}_H\beta_{n,q}^{(k)}(x, y; \lambda)$ plays a major role in obtaining new expansions, identities and representations. We can introduce and study a class of related generalized polynomials by defining degenerate Gould-Hopper-poly-Bernoulli polynomials with q-parameter

$$\frac{\mathrm{Li}_{k,q}(1-e^{-t})}{(1+\lambda t)^{\frac{1}{\lambda}} - 1}(1+\lambda t)^{\frac{x}{\lambda}}(1+\lambda t^r)^{\frac{y}{\lambda}} = \sum_{n=0}^{\infty} {}_H\beta_{n,q}^{(k,r)}(x, y; \lambda) \frac{t^n}{n!}. \quad (5.1)$$

The equation (2.1) may be derived from (5.1) for $r = 2$.

This process can easily be extended to establish degenerate multi-variable Hermite-poly-Bernoulli polynomials with q-parameter and Apostle type Bernoulli polynomials.

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Choquet integral analytic inequalities

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Abstract. Based on an amazing result of Sugeno [15], we are able to transfer classic analytic integral inequalities to Choquet integral setting. We give Choquet integral inequalities of the following types: fractional-Polya, Ostrowski, fractional Ostrowski, Hermite-Hadamard, Simpson and Iyengar. We provide several examples for the involved distorted Lebesgue measure.

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1. Background

We need the following fractional calculus background:

Let $\alpha > 0$, $m = [\alpha]$ ($[\cdot]$ is the integral part), $\beta = \alpha - m$, $0 < \beta < 1$, $f \in C([a,b])$, $[a,b] \subset \mathbb{R}$, $x \in [a,b]$. The gamma function Γ is given by $\Gamma(\alpha) = \int_0^\infty e^{-t} t^{\alpha-1} dt$. We define the left Riemann-Liouville integral

$$\left(J_\alpha^{a+} f\right)(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)\, dt, \tag{1.1}$$

$a \leq x \leq b$. We define the subspace $C_{a+}^\alpha([a,b])$ of $C^m([a,b])$:

$$C_{a+}^\alpha([a,b]) = \left\{ f \in C^m([a,b]) : J_{1-\beta}^{a+} f^{(m)} \in C^1([a,b]) \right\}. \tag{1.2}$$

For $f \in C_{a+}^\alpha([a,b])$, we define the left generalized α-fractional derivative of f over $[a,b]$ as

$$D_{a+}^\alpha f := \left(J_{1-\beta}^{a+} f^{(m)} \right)', \tag{1.3}$$

see [1], p. 24. Canavati first in [5] introduced the above over $[0,1]$.

Notice that $D_{a+}^\alpha f \in C([a,b])$.

Furthermore we need:

Let again $\alpha > 0$, $m = [\alpha]$, $\beta = \alpha - m$, $f \in C([a,b])$, call the right Riemann-Liouville fractional integral operator by

$$\left(J_{b-}^{\alpha} f\right)(x) := \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t)\,dt, \tag{1.4}$$

$x \in [a,b]$, see also [2], [9], [14]. Define the subspace of functions

$$C_{b-}^{\alpha}([a,b]) := \left\{ f \in C^m([a,b]) : J_{b-}^{1-\beta} f^{(m)} \in C^1([a,b]) \right\}. \tag{1.5}$$

Define the right generalized α-fractional derivative of f over $[a,b]$ as

$$D_{b-}^{\alpha} f = (-1)^{m-1} \left(J_{b-}^{1-\beta} f^{(m)} \right)', \tag{1.6}$$

see [2]. We set $D_{b-}^{0} f = f$. Notice that $D_{b-}^{\alpha} f \in C([a,b])$.

We need the following fractional Polya type (see [12], [13], p. 62) integral inequality without any boundary conditions.

Theorem 1.1. ([4], p. 4) *Let* $0 < \alpha < 1$, $f \in C([a,b])$. *Assume* $f \in C_{a+}^{\alpha}\left(\left[a, \frac{a+b}{2}\right]\right)$ *and* $f \in C_{b-}^{\alpha}\left(\left[\frac{a+b}{2}, b\right]\right)$. *Set*

$$M(f) := \max\left\{ \left\| D_{a+}^{\alpha} f \right\|_{\infty,\left[a,\frac{a+b}{2}\right]}, \left\| D_{b-}^{\alpha} \right\|_{\infty,\left[\frac{a+b}{2},b\right]} \right\}. \tag{1.7}$$

Then

$$\left| \int_a^b f(x)\,dx \right| \leq \int_a^b |f(x)|\,dx \leq M(f) \frac{(b-a)^{\alpha+1}}{\Gamma(\alpha+2)\,2^{\alpha}}. \tag{1.8}$$

*Inequality (*1.8*) is sharp, namely it is attained by*

$$f_*(x) = \left\{ \begin{array}{l} (x-a)^{\alpha},\ x \in \left[a, \frac{a+b}{2}\right], \\ (b-x)^{\alpha},\ x \in \left[\frac{a+b}{2}, b\right] \end{array} \right\},\ 0 < \alpha < 1. \tag{1.9}$$

The famous Ostrowski ([11]) inequality motivates this work and has as follows:

Theorem 1.2. *It holds*

$$\left| \frac{1}{b-a} \int_a^b f(y)\,dy - f(x) \right| \leq \left(\frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2} \right) (b-a) \left\| f' \right\|_{\infty}, \tag{1.10}$$

where $f \in C^1([a,b])$, $x \in [a,b]$, *and it is a sharp inequality.*

Another motivation is author's next fractional result, see [3], p. 44:

Theorem 1.3. *Let* $[a,b] \subset \mathbb{R}$, $\alpha > 0$, $m = \lceil \alpha \rceil$ *(*$\lceil \cdot \rceil$ *ceiling of the number),* $f \in AC^m([a,b])$ *(i.e.* $f^{(m-1)}$ *is absolutely continuous), and* $\left\| \overline{D}_{x_0-}^{\alpha} f \right\|_{\infty,[a,x_0]}$, $\left\| \overline{D}_{*x_0}^{\alpha} f \right\|_{\infty,[x_0,b]} < \infty$ *(where* $\overline{D}_{x_0-}^{\alpha} f, \overline{D}_{*x_0}^{\alpha} f$ *are the right ([2]) and left ([8], p. 50) Caputo fractional derivatives of* f *of order* α*, respectively),* $x_0 \in [a,b]$. *Assume* $f^{(k)}(x_0) = 0$, $k = 1, ..., m-1$. *Then*

$$\left| \frac{1}{b-a} \int_a^b f(x)\,dx - f(x_0) \right| \leq \frac{1}{(b-a)\,\Gamma(\alpha+2)}$$

$$\cdot\left\{\left\|\overline{D}_{x_0-}^{\alpha}f\right\|_{\infty,[a,x_0]}(x_0-a)^{\alpha+1}+\left\|\overline{D}_{*x_0}^{\alpha}f\right\|_{\infty,[x_0,b]}(b-x_0)^{\alpha+1}\right\}$$
$$\leq\frac{1}{\Gamma(\alpha+2)}\max\left\{\left\|\overline{D}_{x_0-}^{\alpha}f\right\|_{\infty,[a,x_0]},\left\|\overline{D}_{*x_0}^{\alpha}f\right\|_{\infty,[x_0,b]}\right\}(b-a)^{\alpha}. \quad (1.11)$$

In the next assume that $(X,\mathcal{F})$ is a measurable space and $(\mathbb{R}^+)$ $\mathbb{R}$ is the set of all (nonnegative) real numbers.

We recall some concepts and some elementary results of capacity and the Choquet integral [6, 7].

Definition 1.4. A set function $\mu:\mathcal{F}\to\mathbb{R}^+$ is called a non-additive measure (or capacity) if it satisfies

(1) $\mu(\varnothing)=0$;

(2) $\mu(A)\leq\mu(B)$ for any $A\subseteq B$ and $A,B\in\mathcal{F}$.

The non-additive measure μ is called concave if

$$\mu(A\cup B)+\mu(A\cap B)\leq\mu(A)+\mu(B), \quad (1.12)$$

for all $A,B\in\mathcal{F}$. In the literature the concave non-additive measure is known as submodular or 2-alternating non-additive measure. If the above inequality is reverse, μ is called convex. Similarly, convexity is called supermodularity or 2-monotonicity, too.

First note that the Lebesgue measure λ for an interval $[a,b]$ is defined by $\lambda([a,b])=b-a$, and that given a distortion function m, which is increasing (or non-decreasing) and such that $m(0)=0$, the measure $\mu(A)=m(\lambda(A))$ is a distorted Lebesgue measure. We denote a Lebesgue measure with distortion m by $\mu=\mu_m$. It is known that μ_m is concave (convex) if m is a concave (convex) function.
The family of all the nonnegative, measurable function $f:(X,\mathcal{F})\to(\mathbb{R}^+,\mathcal{B}(\mathbb{R}^+))$ is denoted as L_∞^+, where $\mathcal{B}(\mathbb{R}^+)$ is the Borel σ-field of $\mathbb{R}^+$. The concept of the integral with respect to a non-additive measure was introduced by Choquet [6].

Definition 1.5. Let $f\in L_\infty^+$. The Choquet integral of f with respect to non-additive measure μ on $A\in\mathcal{F}$ is defined by

$$(C)\int_A fd\mu:=\int_0^\infty\mu(\{x:f(x)\geq t\}\cap A)\,dt, \quad (1.13)$$

where the integral on the right-hand side is a Riemann integral.

Instead of $(C)\int_X fd\mu$, we shall write $(C)\int fd\mu$. If $(C)\int fd\mu<\infty$, we say that f is Choquet integrable and we write

$$L_C^1(\mu)=\left\{f:(C)\int fd\mu<\infty\right\}.$$

The next lemma summarizes the basic properties of Choquet integrals [7].

Lemma 1.6. *Assume that* $f,g\in L_C^1(\mu)$.

(1) $(C)\int 1_A d\mu=\mu(A)$, $A\in\mathcal{F}$.

(2) (Positive homogeneity) For all $\lambda \in \mathbb{R}^+$, *we have*

$$(C)\int \lambda f d\mu = \lambda \cdot (C)\int f d\mu.$$

(3) (Translation invariance) For all $c \in \mathbb{R}$, *we have*

$$(C)\int (f + c)\, d\mu = (C)\int f d\mu + c.$$

(4) (Monotonicity in the integrand) If $f \leq g$, *then we have*

$$(C)\int f d\mu \leq (C)\int g d\mu.$$

(Monotonicity in the set function) If $\mu \leq \nu$, *then we have*

$$(C)\int f d\mu \leq (C)\int f d\nu.$$

(5) (Subadditivity) If μ *is concave, then*

$$(C)\int (f + g)\, d\mu \leq (C)\int f d\mu + (C)\int g d\mu.$$

(Superadditivity) If μ *is convex, then*

$$(C)\int (f + g)\, d\mu \geq (C)\int f d\mu + (C)\int g d\mu.$$

(6) (Comonotonic additivity) If f *and* g *are comonotonic, then*

$$(C)\int (f + g)\, d\mu = (C)\int f d\mu + (C)\int g d\mu,$$

where we say that f *and* g *are comonotonic, if for any* $x, x' \in X$, *then*

$$(f(x) - f(x'))\,(g(x) - g(x')) \geq 0.$$

We next mention the amazing result from [15], which permits us to compute the Choquet integral when the non-additive measure is a distorted Lebesgue measure.

Theorem 1.7. *Let* f *be a nonnegative and measurable function on* $\mathbb{R}^+$ *and* $\mu = \mu_m$ *be a distorted Lebesgue measure. Assume that* $m(x)$ *and* $f(x)$ *are both continuous and* $m(x)$ *is differentiable. When* f *is an increasing (non-decreasing) function on* $\mathbb{R}^+$, *the Choquet integral of* f *with respect to* μ_m *on* $[0, t]$ *is represented as*

$$(C)\int_{[0,t]} f d\mu_m = \int_0^t m'(t - x)\, f(x)\, dx, \tag{1.14}$$

however, when f *is a decreasing (non-increasing) function on* $\mathbb{R}^+$, *the Choquet integral of* f *is*

$$(C)\int_{[0,t]} f d\mu_m = \int_0^t m'(x)\, f(x)\, dx. \tag{1.15}$$

2. Main results

From now on we assume that $f : \mathbb{R}^+ \to \mathbb{R}^+$ is a monotone continuous function, and $\mu = \mu_m$ i.e. $\mu(A) = m(\lambda(A))$, denotes a distorted Lebesgue measure where m is such that $m(0) = 0$, m is increasing (non-decreasing) and continuously differentiable.

By Theorem 1.7 and mean value theorem for integrals we get:

i) If f is an increasing (non-decreasing) function on $\mathbb{R}^+$, we have

$$(C)\int_{[0,t]} f d\mu_m \overset{(1.14)}{=} \int_0^t m'(t-x) f(x) dx$$

$$= m'(t-\xi)\int_0^t f(x) dx, \text{ where } \xi \in (0,t). \tag{2.1}$$

ii) If f is a decreasing (non-increasing) function on $\mathbb{R}^+$, we have

$$(C)\int_{[0,t]} f d\mu_m \overset{(1.15)}{=} \int_0^t m'(x) f(x) dx = m'(\xi)\int_0^t f(x) dx, \tag{2.2}$$

where $\xi \in (0,t)$.

We denote by

$$\gamma(t,\xi) := \begin{cases} m'(t-\xi), \text{ when } f \text{ is increasing (non-decreasing)} \\ m'(\xi), \text{ when } f \text{ is decreasing (non-increasing),} \end{cases} \tag{2.3}$$

for some $\xi \in (0,t)$ per case.

We give the following Choquet-fractional-Polya inequality:

Theorem 2.1. *Let $0 < \alpha < 1$, $f = f|_{[0,t]}$, $t \in \mathbb{R}^+$, all considered as above in this section. Assume further that $f \in C^{\alpha}_{0+}\left(\left[0, \frac{t}{2}\right]\right)$ and $f \in C^{\alpha}_{t-}\left(\left[\frac{t}{2}, t\right]\right)$. Set*

$$M^*(f)(t) := \max\left\{\left\|D^{\alpha}_{0+} f\right\|_{\infty,\left[0,\frac{t}{2}\right]}, \left\|D^{\alpha}_{t-} f\right\|_{\infty,\left[\frac{t}{2},t\right]}\right\}. \tag{2.4}$$

Then

$$(C)\int_{[0,t]} f d\mu_m \le \gamma(t,\xi) M^*(f)(t) \frac{t^{\alpha+1}}{\Gamma(\alpha+2) 2^{\alpha}}. \tag{2.5}$$

Proof. By Theorem 1.1 and earlier comments. □

Usual Polya inequality with ordinary derivative requires boundary conditions making a Choquet-Polya inequality impossible.

We give applications:

Remark 2.2. i) If $m(t) = \frac{t}{1+t}$, $t \in \mathbb{R}^+$, then $m(0) = 0$, $m(t) \ge 0$, $m'(t) = \frac{1}{(1+t)^2} > 0$, and m is increasing. Then $\gamma(t,\xi) \le 1$.

ii) If $m(t) = 1 - e^{-t} \ge 0$, $t \in \mathbb{R}^+$, then $m(0) = 0$, $m'(t) = e^{-t} > 0$, and m is increasing. Then $\gamma(t,\xi) \le 1$.

iii) If $m(t) = e^t - 1 \ge 0$, $t \in \mathbb{R}^+$, $m(0) = 0$, $m'(t) = e^t > 0$, and m is increasing. Then $\gamma(t,\xi) \le e^t$.

iv) If $m(t) = \sin t$, for $t \in \left[0, \frac{\pi}{2}\right]$, we get $m(0) = 0$, $m'(t) = \cos t \ge 0$, and m is increasing. Then $\gamma(t,\xi) \le 1$.

We continue with the Choquet-Ostrowski type inequalities:

Theorem 2.3. *Here $f:\mathbb{R}^+\to\mathbb{R}^+$ is a monotone continuous function, μ_m is a distorted Lebesgue measure, where m is such that $m(0)=0$, m is increasing and is twice continuously differentiable on $\mathbb{R}^+$. Here $0\le x_0\le t\in\mathbb{R}^+$. Then*

1)

$$\left|\frac{1}{t}(C)\int_{[0,t]} f d\mu_m - m'(t-x_0) f(x_0)\right|$$
$$\le\left(\frac{1}{4}+\frac{\left(x_0-\frac{t}{2}\right)^2}{t^2}\right) t\left\|\left(m'(t-\cdot) f\right)'\right\|_{\infty,[0,t]}, \tag{2.6}$$

if f is an increasing function on $\mathbb{R}^+$,
and

2)

$$\left|\frac{1}{t}(C)\int_{[0,t]} f d\mu_m - m'(x_0) f(x_0)\right|$$
$$\le\left(\frac{1}{4}+\frac{\left(x_0-\frac{t}{2}\right)^2}{t^2}\right) t\left\|\left(m' f\right)'\right\|_{\infty,[0,t]}, \tag{2.7}$$

if f is a decreasing function on $\mathbb{R}^+$.

Proof. By (1.10) we have that ($x_0\in[0,t]$)

$$\left|\frac{1}{t}(C)\int_{[0,t]} f d\mu_m - m'(t-x_0) f(x_0)\right|$$
$$\overset{(1.14)}{=}\left|\frac{1}{t}\int_0^t m'(t-x) f(x)\,dx - m'(t-x_0) f(x_0)\right|$$
$$\le\left(\frac{1}{4}+\frac{\left(x_0-\frac{t}{2}\right)^2}{t^2}\right) t\left\|\left(m'(t-\cdot) f\right)'\right\|_{\infty,[0,t]}, \tag{2.8}$$

when f is an increasing function on $\mathbb{R}^+$.

Also we have that

$$\left|\frac{1}{t}(C)\int_{[0,t]} f d\mu_m - m'(x_0) f(x_0)\right|$$
$$\overset{(1.15)}{=}\left|\frac{1}{t}\int_0^t m'(x) f(x)\,dx - m'(x_0) f(x_0)\right|$$
$$\overset{(1.10)}{\le}\left(\frac{1}{4}+\frac{\left(x_0-\frac{t}{2}\right)^2}{t^2}\right) t\left\|\left(m' f\right)'\right\|_{\infty,[0,t]}, \tag{2.9}$$

when f is a decreasing function on $\mathbb{R}^+$. □

We make

Remark 2.4. (continuing from Remark 2.2) Assuming m is twice continuously differentiable is quite natural. Indeed:

i) If $m(t) = \frac{t}{1+t}$, $t \in \mathbb{R}^+$, then $m'(t) = (1+t)^{-2}$, $m''(t) = -2(1+t)^{-3}$, $m^{(3)}(t) = 6(1+t)^{-4}$, $m^{(4)}(t) = -24(1+t)^{-5}$, etc, all higher order derivatives exist and are continuous.

ii) If $m(t) = 1 - e^{-t}$, $t \in \mathbb{R}^+$, then $m'(t) = e^{-t}$, $m''(t) = -e^{-t}$, $m^{(3)}(t) = e^{-t}$, $m^{(4)}(t) = -e^{-t}$, etc, all higher order derivatives exist and are continuous.

iii) If $m(t) = e^t - 1$, $t \in \mathbb{R}^+$, then $m^{(i)}(t) = e^t$, $i = 1, 2, ...$, all derivatives exist and are continuous.

iv) If $m(t) = \sin t$, $t \in \left[0, \frac{\pi}{2}\right]$, then $m'(t) = \cos t$, $m''(t) = -\sin t$, $m^{(3)}(t) = -\cos t$, $m^{(4)}(t) = \sin t$, etc, all derivatives exist and are continuous.

We continue with fractional Choquet-Ostrowski type inequalities.

Theorem 2.5. *Here $f : \mathbb{R}^+ \to \mathbb{R}^+$ is an increasing continuous function, μ_m is a distorted Lebesgue measure and $0 \leq x_0 \leq t \in \mathbb{R}^+$.*
Let $\alpha > 0$, $m = \lceil \alpha \rceil$, $(m'(t-\cdot) f) \in AC^m([0,t])$, and $\left\| \overline{D}_{x_0-}^{\alpha} (m'(t-\cdot) f) \right\|_{\infty,[0,x_0]}$, $\left\| \overline{D}_{*x_0}^{\alpha} (m'(t-\cdot) f) \right\|_{\infty,[x_0,t]} < \infty$. Assume $(m'(t-\cdot) f)^{(k)}(x_0) = 0$, $k = 1, ..., m-1$. Then

$$\left| \frac{1}{t} (C) \int_{[0,t]} f d\mu_m - m'(t-x_0) f(x_0) \right|$$
$$\leq \frac{1}{t\Gamma(\alpha+2)} \left\{ \left\| \overline{D}_{x_0-}^{\alpha} (m'(t-\cdot) f) \right\|_{\infty,[0,x_0]} x_0^{\alpha+1} + \left\| \overline{D}_{*x_0}^{\alpha} (m'(t-\cdot) f) \right\|_{\infty,[x_0,t]} (t-x_0)^{\alpha+1} \right\} \tag{2.10}$$
$$\leq \frac{t^{\alpha}}{\Gamma(\alpha+2)} \max \left\{ \left\| \overline{D}_{x_0-}^{\alpha} (m'(t-\cdot) f) \right\|_{\infty,[0,x_0]}, \left\| \overline{D}_{*x_0}^{\alpha} (m'(t-\cdot) f) \right\|_{\infty,[x_0,t]} \right\}.$$

Proof. By Theorem 1.3. □

Theorem 2.6. *Here $f : \mathbb{R}^+ \to \mathbb{R}^+$ is a decreasing continuous function, μ_m is a distorted Lebesgue measure and $0 \leq x_0 \leq t \in \mathbb{R}^+$. Let $\alpha > 0$, $m = \lceil \alpha \rceil$, $(m'f) \in AC^m([0,t])$, and $\left\| \overline{D}_{x_0-}^{\alpha} (m'f) \right\|_{\infty,[0,x_0]}$, $\left\| \overline{D}_{*x_0}^{\alpha} (m'f) \right\|_{\infty,[x_0,t]} < \infty$. Assume $(m'f)^{(k)}(x_0) = 0$, $k = 1, ..., m-1$. Then*

$$\left| \frac{1}{t} (C) \int_{[0,t]} f d\mu_m - m'(x_0) f(x_0) \right|$$
$$\leq \frac{1}{t\Gamma(\alpha+2)} \left\{ \left\| \overline{D}_{x_0-}^{\alpha} (m'f) \right\|_{\infty,[0,x_0]} x_0^{\alpha+1} + \left\| \overline{D}_{*x_0}^{\alpha} (m'f) \right\|_{\infty,[x_0,t]} (t-x_0)^{\alpha+1} \right\}$$
$$\leq \frac{t^{\alpha}}{\Gamma(\alpha+2)} \max \left\{ \left\| \overline{D}_{x_0-}^{\alpha} (m'f) \right\|_{\infty,[0,x_0]}, \left\| \overline{D}_{*x_0}^{\alpha} (m'f) \right\|_{\infty,[x_0,t]} \right\}. \tag{2.11}$$

Proof. By Theorem 1.3. □

We need the well-known Hermite-Hadamard inequality:

Theorem 2.7. *Let $f:[a,b]\to\mathbb{R}$ be a continuous convex function, $[a,b]\subset\mathbb{R}$. Then*

$$f\left(\frac{a+b}{2}\right)\leq\frac{1}{b-a}\int_a^b f(x)\,dx\leq\frac{f(a)+f(b)}{2}. \tag{2.12}$$

We give the following Choquet-Hermite-Hadamard inequalities:

Theorem 2.8. *Here $f:\mathbb{R}^+\to\mathbb{R}^+$ is a monotone continuous convex function, μ_m is a distorted Lebesgue measure, where m is such that $m(0)=0$, m is increasing and continuously differentiable on $\mathbb{R}^+$. Here $[a,b]\subseteq\mathbb{R}^+$. Then*

i) If f is decreasing, we have that

$$m'(\xi)\,f\left(\frac{a+b}{2}\right)\leq\frac{1}{b-a}(C)\int_{[a,b]}f(x)\,d\mu_m(x)\leq m'(\xi)\frac{f(a)+f(b)}{2}, \tag{2.13}$$

for some $\xi\in(0,b-a)$.

ii) If f is increasing, we have that

$$m'(b-a-\psi)\,f\left(\frac{a+b}{2}\right)\leq\frac{1}{b-a}(C)\int_{[a,b]}f(x)\,d\mu_m(x)$$
$$\leq m'(b-a-\psi)\frac{f(a)+f(b)}{2}, \tag{2.14}$$

for some $\psi\in(0,b-a)$.

Proof. Let f be a convex function from $[a,b]\subset\mathbb{R}^+$ into $\mathbb{R}^+$. Let $t_1,t_2\in[0,b-a]$, these are of the form $t_1=x-a$, $t_2=y-a$, where $x,y\in[a,b]$.

Consider $(\lambda\in(0,1))$

$$\begin{aligned}f(a+\lambda t_1+(1-\lambda)t_2)&=f(a+\lambda(x-a)+(1-\lambda)(y-a))\\&=f(\lambda x+(1-\lambda)y)\leq\lambda f(x)+(1-\lambda)f(y)\\&=\lambda f(a+x-a)+(1-\lambda)f(a+y-a)\\&=\lambda f(a+t_1)+(1-\lambda)f(a+t_2),\end{aligned}$$

proving that $f(a+\cdot)$ is convex over $[0,b-a]$.

Also it holds

$$(C)\int_{[a,b]}f(x)\,d\mu_m(x)=(C)\int_{[0,b-a]}f(a+x)\,d\mu_m(x). \tag{2.15}$$

Clearly, if f is increasing over $[a,b]$, then $f(a+\cdot)$ is increasing on $[0,b-a]$, and vice verca. And if f is decreasing over $[a,b]$, then $f(a+\cdot)$ is decreasing on $[0,b-a]$, and vice verca.

i) If f is decreasing, then

$$(C)\int_{[0,b-a]}f(a+x)\,d\mu_m(x)\overset{(1.15)}{=}\int_0^{b-a}m'(x)\,f(a+x)\,dx$$
$$=m'(\xi)\int_0^{b-a}f(a+x)\,dx,\quad\text{for some }\xi\in(0,b-a). \tag{2.16}$$

By (2.12) we get

$$f\left(\frac{a+b}{2}\right)\leq\frac{1}{b-a}\int_{0}^{b-a}f\left(a+x\right)dx\leq\frac{f\left(a\right)+f\left(b\right)}{2}, \tag{2.17}$$

and then

$$f\left(\frac{a+b}{2}\right)m'\left(\xi\right)\leq\frac{m'\left(\xi\right)}{b-a}\int_{0}^{b-a}f\left(a+x\right)dx\leq\left(\frac{f\left(a\right)+f\left(b\right)}{2}\right)m'\left(\xi\right). \tag{2.18}$$

That is we proved (by (2.15), (2.16))

$$f\left(\frac{a+b}{2}\right)m'\left(\xi\right)\leq\frac{(C)\int_{[a,b]}f\left(x\right)d\mu_{m}\left(x\right)}{b-a}\leq\left(\frac{f\left(a\right)+f\left(b\right)}{2}\right)m'\left(\xi\right), \tag{2.19}$$

for some $\xi\in(0,b-a)$.

ii) If f is increasing, then

$$(C)\int_{[0,b-a]}f\left(a+x\right)d\mu_{m}\left(x\right)\overset{(1.14)}{=}\int_{0}^{b-a}m'\left(b-a-x\right)f\left(a+x\right)dx$$

$$=m'\left(b-a-\psi\right)\int_{0}^{b-a}f\left(a+x\right)dx,\quad\text{for some }\psi\in(0,b-a). \tag{2.20}$$

Again by (2.12) we get

$$f\left(\frac{a+b}{2}\right)\leq\frac{1}{b-a}\int_{0}^{b-a}f\left(a+x\right)dx\leq\frac{f\left(a\right)+f\left(b\right)}{2}, \tag{2.21}$$

and

$$f\left(\frac{a+b}{2}\right)m'\left(b-a-\psi\right)\leq\frac{m'\left(b-a-\psi\right)}{b-a}\int_{0}^{b-a}f\left(a+x\right)dx$$

$$\leq\left(\frac{f\left(a\right)+f\left(b\right)}{2}\right)m'\left(b-a-\psi\right). \tag{2.22}$$

That is we proved (by (2.15), (2.20))

$$f\left(\frac{a+b}{2}\right)m'\left(b-a-\psi\right)\leq\frac{(C)\int_{[a,b]}f\left(x\right)d\mu_{m}\left(x\right)}{b-a}$$

$$\leq\left(\frac{f\left(a\right)+f\left(b\right)}{2}\right)m'\left(b-a-\psi\right), \tag{2.23}$$

for some $\psi\in(0,b-a)$. □

We need the well-known Simpson inequality:

Theorem 2.9. *If $f:[a,b]\to\mathbb{R}$ is four times continuously differentiable on (a,b) and*

$$\left\|f^{(4)}\right\|_{\infty}=\sup_{x\in(a,b)}\left|f^{(4)}\left(x\right)\right|<\infty,$$

then the Simpson inequality holds:

$$\left|\frac{1}{b-a}\int_a^b f(x)\,dx-\frac{1}{3}\left[\frac{f(a)+f(b)}{2}+2f\left(\frac{a+b}{2}\right)\right]\right|\leq\frac{1}{2880}\left\|f^{(4)}\right\|_\infty(b-a)^4. \tag{2.24}$$

We give the corresponding Choquet-Simpson inequalities:

Theorem 2.10. *Here $f:\mathbb{R}^+\to\mathbb{R}^+$ is a monotone function which is four times continuously differentiable on $\mathbb{R}^+$, μ_m is a distorted Lebesgue measure, where m is such that $m(0)=0$, m is increasing and five times continuously differentiable on $\mathbb{R}^+$, $t\in\mathbb{R}^+$. Then*

i) if f is increasing, we have that

$$\left|\frac{1}{t}(C)\int_{[0,t]}fd\mu_m-\frac{1}{3}\left[\frac{m'(t)f(0)+m'(0)f(t)}{2}+2m'\left(\frac{t}{2}\right)f\left(\frac{t}{2}\right)\right]\right|$$
$$\leq\frac{1}{2880}\left\|\left(m'(t-\cdot)f\right)^{(4)}\right\|_{\infty,[0,t]}t^4, \tag{2.25}$$

and

ii) if f is decreasing, we have that

$$\left|\frac{1}{t}(C)\int_{[0,t]}fd\mu_m-\frac{1}{3}\left[\frac{m'(0)f(0)+m'(t)f(t)}{2}+2m'\left(\frac{t}{2}\right)f\left(\frac{t}{2}\right)\right]\right|$$
$$\leq\frac{1}{2880}\left\|\left(m'f\right)^{(4)}\right\|_{\infty,[0,t]}t^4. \tag{2.26}$$

Proof. i) If f is increasing, then

$$\left|\frac{1}{t}(C)\int_{[0,t]}fd\mu_m-\frac{1}{3}\left[\frac{m'(t)f(0)+m'(0)f(t)}{2}+2m'\left(\frac{t}{2}\right)f\left(\frac{t}{2}\right)\right]\right|$$
$$\overset{(1.14)}{=}\left|\frac{1}{t}\int_0^t m'(t-x)f(x)\,dx-\frac{1}{3}\left[\frac{m'(t)f(0)+m'(0)f(t)}{2}+2m'\left(\frac{t}{2}\right)f\left(\frac{t}{2}\right)\right]\right|$$
$$\overset{(2.24)}{\leq}\frac{1}{2880}\left\|\left(m'(t-\cdot)f\right)^{(4)}\right\|_{\infty,[0,t]}t^4. \tag{2.27}$$

ii) If f is decreasing, then

$$\left|\frac{1}{t}(C)\int_{[0,t]}fd\mu_m-\frac{1}{3}\left[\frac{m'(0)f(0)+m'(t)f(t)}{2}+2m'\left(\frac{t}{2}\right)f\left(\frac{t}{2}\right)\right]\right|$$
$$\overset{(1.15)}{=}\left|\frac{1}{t}\int_0^t m'(x)f(x)\,dx-\frac{1}{3}\left[\frac{m'(0)f(0)+m'(t)f(t)}{2}+2m'\left(\frac{t}{2}\right)f\left(\frac{t}{2}\right)\right]\right|$$
$$\overset{(2.24)}{\leq}\frac{1}{2880}\left\|\left(m'f\right)^{(4)}\right\|_{\infty,[0,t]}t^4. \tag{2.28}$$

□

We need the famous Iyengar inequality [10] coming next:

Theorem 2.11. *Let f be a differentiable function on $[a,b] \subset \mathbb{R}$ and $|f'(x)| \leq M_1$. Then*

$$\left| \int_a^b f(x)\,dx - \frac{1}{2}(b-a)(f(a)+f(b)) \right| \leq \frac{M_1 (b-a)^2}{4} - \frac{(f(b)-f(a))^2}{4M_1}. \tag{2.29}$$

We present the corresponding Choquet-Iyengar inequalities:

Theorem 2.12. *Here $f : \mathbb{R}^+ \to \mathbb{R}^+$ is a monotone differentiable function on $\mathbb{R}^+$, μ_m is a distorted Lebesgue measure, where m is such that $m(0) = 0$, m is increasing and twice continuously differentiable on $\mathbb{R}^+$, $t \in \mathbb{R}^+$. Then*

i) if f is increasing and $\left|(m'(t-\cdot)f)'(x)\right| \leq M_2$, $\forall\, x \in [0,t]$, $M_2 > 0$, we have that

$$\left| (C)\int_{[0,t]} f(x)\,d\mu_m(x) - \frac{t}{2}(m'(t)f(0)+m'(0)f(t)) \right|$$
$$\leq \frac{M_2 t^2}{4} - \frac{(m'(0)f(t)-m'(t)f(0))^2}{4M_2}. \tag{2.30}$$

ii) if f is decreasing and $\left|(m'f)'(x)\right| \leq M_3$, $\forall\, x \in [0,t]$, $M_3 > 0$, we have that

$$\left| (C)\int_{[0,t]} f(x)\,d\mu_m(x) - \frac{t}{2}(m'(0)f(0)+m'(t)f(t)) \right|$$
$$\leq \frac{M_3 t^2}{4} - \frac{(m'(t)f(t)-m'(0)f(0))^2}{4M_3}. \tag{2.31}$$

Proof. i) If f is increasing and $\left|(m'(t-\cdot)f)'(x)\right| \leq M_2$, $\forall\, x \in [0,t]$, then

$$\left| (C)\int_{[0,t]} f(x)\,d\mu_m(x) - \frac{t}{2}(m'(t)f(0)+m'(0)f(t)) \right|$$
$$\overset{(1.14)}{=} \left| \int_0^t m'(t-x)f(x)\,dx - \frac{t}{2}(m'(t)f(0)+m'(0)f(t)) \right|$$
$$\overset{(2.29)}{\leq} \frac{M_2 t^2}{4} - \frac{(m'(0)f(t)-m'(t)f(0))^2}{4M_2}. \tag{2.32}$$

ii) If f is decreasing and $\left|(m'f)'(x)\right| \leq M_3$, $\forall\, x \in [0,t]$, then

$$\left| (C)\int_{[0,t]} f(x)\,d\mu_m(x) - \frac{t}{2}(m'(0)f(0)+m'(t)f(t)) \right|$$
$$\overset{(1.15)}{=} \left| \int_0^t m'(x)f(x)\,dx - \frac{t}{2}(m'(0)f(0)+m'(t)f(t)) \right|$$
$$\overset{(2.29)}{\leq} \frac{M_3 t^2}{4} - \frac{(m'(t)f(t)-m'(0)f(0))^2}{4M_3}.$$
□

Note 2.13. One can transfer many analytic integral classic inequalities to Choquet integral setting but we choose to stop here.

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Existence and stability of fractional differential equations involving generalized Katugampola derivative

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Abstract. The present article deals with the existence and stability results for a class of fractional differential equations involving generalized Katugampola derivative. Some fixed point theorems are used to obtain the results and enlightening examples of obtained result are also given.

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1. Introduction

Fractional calculus has proven to be an useful tool in the description of various complex phenomena in the real world problems. During the theoretical development of the calculus of arbitrary order, numerous fractional integral and differential operators are emerged and/or used by timely mathematicians, see [1, 3]-[7],[10]-[18],[23]-[28]. Although the well-developed theory and many more applications of Wyel, Liouville, Riemann-Liouville, Hadamard operators, still this is a spotlight area of research in applied sciences.

U. Katugampola in [24, 25] generalized the above mentioned fractional integral and differential operators. In the same work, he obtained boundedness of generalized fractional integral in an extended Lebesgue measurable space. Further he studied existence and uniqueness of solution of initial value problem (IVP) for a class of generalized fractional differential equations (FDEs) in [26]. R. Almeida, *et al.* [7] studied these results with its Caputo counterpart. R. Almeida [6] discussed certain problems of calculus of variations dependent on Lagrange function with the same approach for first and second order. In 2015, D. Anderson *et al.* [8] studied properties of Katugampola fractional derivative with potential application in quantum mechanics as well constructed a Hamiltonian from its self adjoint operator and applied to the particle in a box model. Recently, D. S. Oliveira, *et al.* [29] proposed a generalization of Katugampola and Caputo-Katugampola fractional derivatives with the name Hilfer-Katugampola fractional derivative. This new fractional derivative interpolates the well-known fractional derivatives: Hilfer [20], Hilfer-Hadamard [23], Katugampola [25], Caputo-Katugampola [7], Riemann-Liouville [27], Hadamard [28], Caputo [27], Caputo-Hadamard [4], Liouville, Wyel as its particular cases. Following the results of [20], they further obtained existence and uniqueness of solution of nonlinear FDEs involving this generalized Katugampola derivative with initial condition [29].

The stability of functional equations was first posed by Ulam [30], thereafter, this type of stability evolved as an interesting field of research. The concept of stability of functional equations arises when the functional equation is being replaced by an inequality which acts as a perturbation of the functional equation, see the monograph [22] and the references cited therein. The considerable attention paid to recent development of stability results for FDEs can be found in [2, 3, 14, 12, 9, 20, 22, 30, 31, 32]. The present work deals with following two IVPs.
Problem I:

$$\begin{cases} \big({}^\rho D^{\alpha,\beta}_{a+}x\big)(t) &= f\big(t, x(t), ({}^\rho D^{\alpha,\beta}_{a+}x)(t)\big); \qquad t \in \Omega, \\ \big({}^\rho I^{1-\gamma}_{a+}x\big)(a) &= c_1, \qquad c_1 \in \mathbb{R}, \gamma = \alpha + \beta(1-\alpha), \end{cases} \tag{1.1}$$

where $\alpha \in (0,1), \beta \in [0,1], \rho > 0, \Omega = [a,b], f : \Omega \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is given function and $a > 0$.
Problem II:

$$\begin{cases} \big({}^\rho D^{\alpha,\beta}_{a+}x\big)(t) &= f(t, x(t)); \qquad t \in \Omega, \\ \big({}^\rho I^{1-\gamma}_{a+}x\big)(a) &= c_2, \qquad c_2 \in \mathbb{R}, \gamma = \alpha + \beta(1-\alpha), \end{cases} \tag{1.2}$$

where $\alpha \in (0,1), \beta \in [0,1], \rho > 0, f : \Omega \times \mathbb{R} \to \mathbb{R}$ is given function. The operators ${}^\rho D^{\alpha,\beta}_{a+}$ and ${}^\rho I^{1-\gamma}_{a+}$ involved herein are the generalized Katugampola fractional derivative (of order α and type β) and Katugampola fractional integral (of order $1-\gamma$) respectively.

The rest of paper is organized as follows: Section 2 introduces some preliminary facts that we need in the sequel. Section 3 presents our main results on existence and stability of considered problems. As an application of main results, two illustrative examples are given in section 4. Concluding remarks are given in last section.

2. Preliminaries

Let $\Omega = [a,b] (0 < a < b < \infty)$. As usual C denotes the Banach space of all continuous functions $x : \Omega \to E$ with the superemum (uniform) norm

$$\|x\|_\infty = \sup_{t\in\Omega} \|x(t)\|_E$$

and $AC(\Omega)$ be the space of absolutely continuous functions from Ω into E. Denote $AC^1(\Omega)$ the space defined by

$$AC^1(\Omega) = \Big\{ x : \Omega \to E | \frac{d}{dt}x(t) \in AC(\Omega) \Big\}.$$

Throughout the paper, let $\delta_\rho = t^{\rho-1}\frac{d}{dt}, n = [\alpha]+1$, and mention $[\alpha]$ as integer part of α. Define the space

$$AC^n_{\delta_\rho} = \big\{x : \Omega \to E | \delta^{n-1}_\rho x(t) \in AC(\Omega)\big\}, \quad n \in \mathbb{N}.$$

Here we define the weighted space of continuous functions g on $\Omega^* = (a, b]$ by

$$C_{\gamma,\rho}(\Omega) = \left\{ g : \Omega^* \to \mathbb{R} | \left(\frac{t^\rho - a^\rho}{\rho}\right)^{1-\gamma} g(t) \in C(\Omega) \right\}, \quad 0 < \gamma \leq 1,$$

with the norm

$$\|g\|_{C_{\gamma,\rho}} = \left\| \left(\frac{t^\rho - a^\rho}{\rho}\right)^{1-\gamma} g(t) \right\|_C = \max_{t\in\Omega} \left| \left(\frac{t^\rho - a^\rho}{\rho}\right)^{1-\gamma} g(t) \right|$$

and

$$C^1_{\delta_\rho,\gamma}(\Omega) = \{g \in C(\Omega) : \delta_\rho g \in C_{\gamma,\rho}(\Omega)\}$$

with the norms

$$\|g\|_{C^1_{\delta_\rho,\gamma}} = \|g\|_C + \|\delta_\rho g\|_{C_{\gamma,\rho}} \quad \text{and} \quad \|g\|_{C^1_{\delta_\rho}} = \sum_{k=0}^{1} \max_{t\in\Omega} \left|\delta^k_\rho g(t)\right|.$$

Note that $C^0_{\delta_\rho,\gamma}(\Omega) = C_{\delta_\rho,\gamma}(\Omega)$, $C_{0,\rho}(\Omega) = C(\Omega)$ and $C_{\gamma,\rho}(\Omega)$ is a complete metric space [29].

Now we introduce some preliminaries from fractional calculus. For more details, we refer the readers to [3, 24, 25, 29]:

Definition 2.1. [24] [Katugampola fractional integral] Let $\alpha \in \mathbb{R}_+, c \in \mathbb{R}$ and $g \in X^p_c(a, b)$, where $X^p_c(a, b)$ is the space of Lebesgue measurable functions. The Katugampola fractional integral of order α is defined by

$$({}^\rho I^\alpha_{a+} g)(t) = \int_a^t s^{\rho-1} \left(\frac{t^\rho - s^\rho}{\rho}\right)^{\alpha-1} \frac{g(s)}{\Gamma(\alpha)} ds, \qquad t > a, \rho > 0,$$

where $\Gamma(\cdot)$ is a Euler's gamma function.

Definition 2.2. [25] [Katugampola fractional derivative] Let $\alpha \in \mathbb{R}_+ \setminus \mathbb{N}$ and $\rho > 0$. The Katugampola fractional derivative ${}^\rho D^{\alpha,\beta}_{a+}$ of order α is defined by

$$\begin{aligned}({}^\rho D^\alpha_{a+} g)(t) &= \delta^n_\rho ({}^\rho I^{n-\alpha}_{a+} g)(t) \\ &= \left(t^{1-\rho}\frac{d}{dt}\right)^n \int_a^t s^{\rho-1} \left(\frac{t^\rho - s^\rho}{\rho}\right)^{n-\alpha-1} \frac{g(s)}{\Gamma(n-\alpha)} ds.\end{aligned}$$

Definition 2.3. [29] [Generalized Katugampola fractional derivative] The generalized Katugampola fractional derivative of order $\alpha \in (0, 1)$ and type $\beta \in [0, 1]$ with respect to t and is defined by

$$({}^\rho D^{\alpha,\beta}_{a\pm} g)(t) = (\pm {}^\rho I^{\beta(1-\alpha)}_{a\pm} \delta_\rho {}^\rho I^{(1-\beta)(1-\alpha)}_{a\pm} g)(t), \quad \rho > 0 \tag{2.1}$$

for the function for which right hand side expression exists.

Remark 2.4. The generalized Katugampola operator ${}^\rho D_{a+}^{\alpha,\beta}$ can be written as

$${}^\rho D_{a+}^{\alpha,\beta} = {}^\rho I_{a+}^{\beta(1-\alpha)} \delta_\rho {}^\rho I_{a+}^{1-\gamma} = {}^\rho I_{a+}^{\beta(1-\alpha)} {}^\rho D_{a+}^{\gamma}, \quad \gamma = \alpha + \beta - \alpha\beta.$$

Lemma 2.5. [24] *[Semigroup property] If $\alpha, \beta > 0, 1 \leq p \leq \infty, 0 < a < b < \infty$ and $\rho, c \in \mathbb{R}$ for $\rho \geq c$. Then, for $g \in X_c^p(a,b)$ the following relation hold:*

$$({}^\rho I_{a+}^{\alpha}\, {}^\rho I_{a+}^{\beta} g)(t) = ({}^\rho I_{a+}^{\alpha+\beta} g)(t).$$

Lemma 2.6. [29] *Let $t > a$, ${}^\rho I_{a+}^{\alpha}$ and ${}^\rho D_{a+}^{\alpha}$ are as in Definition* 2.1 *and Definition* 2.2, *respectively. Then the following hold:*

(i) $\left({}^\rho I_{a+}^{\alpha} \left(\frac{s^\rho - a^\rho}{\rho}\right)^\sigma\right)(t) = \frac{\Gamma(\sigma+1)}{\Gamma(\sigma+\alpha+1)} \left(\frac{t^\rho - a^\rho}{\rho}\right)^{\sigma+\alpha}, \quad \alpha \geq 0, \sigma > 0,$

(ii) for $\sigma = 0$, $\left({}^\rho I_{a+}^{\alpha} \left(\frac{s^\rho - a^\rho}{\rho}\right)^\sigma\right)(t) = \left({}^\rho I_{a+}^{\alpha} 1\right)(t) = \frac{\left(\frac{t^\rho - a^\rho}{\rho}\right)^\alpha}{\Gamma(\alpha+1)}, \quad \alpha \geq 0,$

(iii) for $0 < \alpha < 1$, $\left({}^\rho D_{a+}^{\alpha} \left(\frac{s^\rho - a^\rho}{\rho}\right)^{\alpha-1}\right)(t) = 0.$

3. Main results

In this section, we present the results on existence, attractivity and Ulam stability of solutions for fractional differential equations involving generalized Katugampola fractional derivatives.

Denote $BC = BC(I)$, $I = [a, \infty)$. Let $D \subset BC$ is a nonempty set and let $G : D \to D$. Consider the solutions of equation

$$(Gx)(t) = x(t). \tag{3.1}$$

We define the attractivity of solutions for equation (3.1) as follows:

Definition 3.1. A solutions of equation (3.1) are locally attractive if there exists a ball $B(x_0, \mu)$ in the space BC such that, for arbitrary solutions $y(t)$ and $z(t)$ of equation (3.1) belonging to $B(x_0, \mu) \cap D$, we have

$$\lim_{t\to\infty} (y(t) - z(t)) = 0. \tag{3.2}$$

Whenever limit (3.2) is uniform with respect to $B(x_0, \mu) \cap D$, solutions of equation (3.1) are said to be uniformly locally attractive.

Lemma 3.2. [14] *If $X \subset BC$. Then X is relatively compact in BC if following conditions hold:*
1. X is uniformly bounded in BC,
2. The functions belonging to X are almost equicontinuous on $\mathbb{R}_+$, i.e. equicontinuous on every compact of $\mathbb{R}_+$,
3. The functions from X are equiconvergent, i.e. given $\epsilon > 0$ there corresponds $T(\epsilon) > 0$ such that $|x(t) - \lim_{t\to\infty} x(t)| < \epsilon$ for any $t \geq T(\epsilon)$ and $x \in X$.

Now we discuss the existence and attractivity of solutions of IVP (1.1). Throughout the work, we mean $BC_{\gamma,\rho} = BC_{\gamma,\rho}(I)$ is a weighted space of all bounded and continuous functions defined by

$$BC_{\gamma,\rho} = \left\{ x : (a,\infty] \to \mathbb{R} \mid \left(\frac{t^\rho - a^\rho}{\rho}\right)^{1-\gamma} x(t) \in BC \right\}$$

with the norm

$$\|x\|_{BC_{\gamma,\rho}} = \sup_{t\in\mathbb{R}_+} \left| \left(\frac{t^\rho - a^\rho}{\rho}\right)^{1-\gamma} x(t) \right|.$$

Theorem 3.3. [21]*[Schauder fixed point theorem] Let E be a Banach space and Q be a nonempty bounded convex and closed subset of E and $\Lambda : Q \to Q$ is compact, and continuous map. Then Λ has at least one fixed point in Q.*

Definition 3.4. A solution of IVP (1.1) is a measurable function $x \in BC_{\gamma,\rho}$ satisfying initial value $({}^\rho I_{a+}^{1-\gamma} x)(a^+) = c_1$ and differential equation

$$\left({}^\rho D_{a+}^{\alpha,\beta} x\right)(t) = f\left(t, x(t), ({}^\rho D_{a+}^{\alpha,\beta} x)(t)\right)$$

on I.

From ([29], Theorem 3 pp. 9), we conclude the following lemma.

Lemma 3.5. *Let $\gamma = \alpha + \beta(1-\alpha)$, where $0 < \alpha < 1, 0 \le \beta \le 1$ and $\rho > 0$. Let $f : \Omega \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be such that $f(\cdot, x(\cdot), y(\cdot)) \in BC_{\gamma,\rho}$ for any $x, y \in BC_{\gamma,\rho}$. Then IVP* (1.1) *is equivalent to Volterra integral equation*

$$x(t) = \frac{c_1}{\Gamma(\gamma)} \left(\frac{t^\rho - a^\rho}{\rho}\right)^{\gamma-1} + \int_a^t s^{\rho-1} \left(\frac{t^\rho - s^\rho}{\rho}\right)^{\alpha-1} \frac{g(s)}{\Gamma(\alpha)} ds,$$

where $g(\cdot) \in BC_{\gamma,\rho}$ such that

$$g(t) = f\left(t,\ \frac{c_1}{\Gamma(\gamma)} \left(\frac{t^\rho - a^\rho}{\rho}\right)^{\gamma-1} + ({}^\rho I_{a+}^\alpha g)(t),\ g(t)\right). \tag{3.3}$$

We use the following hypotheses in the sequel:

(H_1). Function $t \longmapsto f(t,x,y)$ is measurable on I for each $x, y \in BC_{\gamma,\rho}$, and functions $x \longmapsto f(t,x,y)$ and $y \longmapsto f(t,x,y)$ are continuous on $BC_{\gamma,\rho}$ for a.e. $t \in I$;

(H_2). There exists a continuous function $p : I \to \mathbb{R}_+$ such that

$$|f(t,x,y)| \le \frac{p(t)}{1 + |x| + |y|}, \text{ for a.e. } t \in I, \text{ and each } x, y \in \mathbb{R}.$$

Moreover, assume that

$$\lim_{t\to\infty} \left(\frac{t^\rho - a^\rho}{\rho}\right)^{1-\gamma} \left({}^\rho I_{a+}^\alpha p\right)(t) = 0.$$

Set

$$p^* = \sup_{t\in I} \left(\frac{t^\rho - a^\rho}{\rho}\right)^{1-\gamma} \left({}^\rho I_{a+}^\alpha p\right)(t).$$

Theorem 3.6. *Suppose that* (H_1) *and* (H_2) *hold. Then IVP* (1.1) *has at least one solution on* I*. Moreover, solutions of IVP* (1.1) *are locally attractive.*

Proof. For any $x \in BC_{\gamma,\rho}$, define the operator Λ such that

$$(\Lambda x)(t) = \frac{c_1}{\Gamma(\gamma)}\left(\frac{t^\rho - a^\rho}{\rho}\right)^{\gamma-1} + \int_a^t s^{\rho-1}\left(\frac{t^\rho - s^\rho}{\rho}\right)^{\alpha-1}\frac{g(s)}{\Gamma(\alpha)}ds, \tag{3.4}$$

where $g \in BC_{\gamma,\rho}$ given by (3.3). The operator Λ is well defined and maps $BC_{\gamma,\rho}$ into $BC_{\gamma,\rho}$. Indeed, the map $\Lambda(x)$ is continuous on I for any $x \in BC_{\gamma,\rho}$, and for each $t \in I$, we have

$$\begin{aligned}\left|\left(\frac{t^\rho - a^\rho}{\rho}\right)^{1-\gamma}(\Lambda x)(t)\right| &\le \frac{|c_1|}{\Gamma(\gamma)} + \left(\frac{t^\rho - a^\rho}{\rho}\right)^{1-\gamma}\int_a^t s^{\rho-1}\left(\frac{t^\rho - s^\rho}{\rho}\right)^{\alpha-1}\frac{|g(s)|}{\Gamma(\alpha)}ds \\ &\le \frac{|c_1|}{\Gamma(\gamma)} + \frac{\left(\frac{t^\rho - a^\rho}{\rho}\right)^{1-\gamma}}{\Gamma(\alpha)}\int_a^t s^{\rho-1}\left(\frac{t^\rho - s^\rho}{\rho}\right)^{\alpha-1} p(s)ds \\ &\le \frac{|c_1|}{\Gamma(\gamma)} + \left(\frac{t^\rho - a^\rho}{\rho}\right)^{1-\gamma}\left({}^\rho I^\alpha_{a+}p\right)(t).\end{aligned}$$

Thus

$$\|\Lambda(x)\|_{BC_{\gamma,\rho}} \le \frac{|c_1|}{\Gamma(\gamma)} + p^* := M. \tag{3.5}$$

Hence, $\Lambda(x) \in BC_{\gamma,\rho}$. This proves that operator Λ maps $BC_{\gamma,\rho}$ into itself.

By Lemma 3.5, the IVP of finding the solutions of IVP (1.1) is reduced to the finding solution of the operator equation $\Lambda(x) = x$. Equation (3.5) implies that Λ transforms the ball $B_M := B(0, M) = \{x \in BC_{\gamma,\rho} : \|x\|_{BC_{\gamma,\rho}} \le M\}$ into itself.

Now we show that the operator Λ satisfies all the assumptions of Theorem 3.3. The proof is given in following steps:

Step 1: Λ is continuous.

Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence such that $x_n \to x$ in B_M. Then, for each $t \in I$, we have

$$\begin{aligned}&\left|\left(\frac{t^\rho - a^\rho}{\rho}\right)^{1-\gamma}(\Lambda x_n)(t) - \left(\frac{t^\rho - a^\rho}{\rho}\right)^{1-\gamma}(\Lambda x)(t)\right| \\ &\qquad \le \frac{\left(\frac{t^\rho - a^\rho}{\rho}\right)^{1-\gamma}}{\Gamma(\alpha)}\int_a^t s^{\rho-1}\left(\frac{t^\rho - s^\rho}{\rho}\right)^{\alpha-1}|g_n(s) - g(s)|ds,\end{aligned} \tag{3.6}$$

where $g_n, g \in BC_{\gamma,\rho}$,

$$g_n(t) = f\left(t,\ \frac{c_1}{\Gamma(\gamma)}\left(\frac{t^\rho - a^\rho}{\rho}\right)^{\gamma-1} + ({}^\rho I^\alpha_{a+}g_n)(t),\ g_n(t)\right)$$

and g is defined by (3.3). If $t \in I$, then from (3.6), we obtain

$$\left|\left(\frac{t^\rho - a^\rho}{\rho}\right)^{1-\gamma}(\Lambda x_n)(t) - \left(\frac{t^\rho - a^\rho}{\rho}\right)^{1-\gamma}(\Lambda x)(t)\right| \le 2\frac{\left(\frac{t^\rho - a^\rho}{\rho}\right)^{1-\gamma}}{\Gamma(\alpha)}\int_a^t s^{\rho-1}\left(\frac{t^\rho - s^\rho}{\rho}\right)^{\alpha-1} p(s)ds. \quad (3.7)$$

Since $x_n \to x$ as $n \to \infty$ and $\left(\frac{t^\rho - a^\rho}{\rho}\right)^{1-\gamma}\left({}^\rho I^\alpha_{a+}p\right)(t) \to 0$ as $t \to \infty$ then (3.7) implies that

$$\|\Lambda(x_n) - \Lambda(x)\|_{BC_{\gamma,\rho}} \to 0 \quad \text{as } n \to \infty.$$

Step 2: $\Lambda(B_M)$ is uniformly bounded.

This is clear since $\Lambda(B_M) \subset B_M$ and B_M is bounded.

Step 3: $\Lambda(B_M)$ is equicontinuous on every compact subset $[a, T]$ of I, $T > a$.

Let $t_1, t_2 \in [a, T], t_1 < t_2$ and $x \in B_M$. We have

$$\begin{aligned}&\left|\left(\frac{{t_2}^\rho - a^\rho}{\rho}\right)^{1-\gamma}(\Lambda x)(t_2) - \left(\frac{{t_1}^\rho - a^\rho}{\rho}\right)^{1-\gamma}(\Lambda x)(t_1)\right| \\ &\le \left|\frac{\left(\frac{{t_2}^\rho - a^\rho}{\rho}\right)^{1-\gamma}}{\Gamma(\alpha)}\int_a^{t_2} s^{\rho-1}\left(\frac{{t_2}^\rho - s^\rho}{\rho}\right)^{\alpha-1} g(s)ds \right.\\ &\left. - \frac{\left(\frac{{t_1}^\rho - a^\rho}{\rho}\right)^{1-\gamma}}{\Gamma(\alpha)}\int_a^{t_1} s^{\rho-1}\left(\frac{{t_1}^\rho - s^\rho}{\rho}\right)^{\alpha-1} g(s)ds\right|,\end{aligned}$$

with $g(\cdot) \in BC_{\gamma,\rho}$ given by (3.3). Thus we get

$$\begin{aligned}&\left|\left(\frac{{t_2}^\rho - a^\rho}{\rho}\right)^{1-\gamma}(\Lambda x)(t_2) - \left(\frac{{t_1}^\rho - a^\rho}{\rho}\right)^{1-\gamma}(\Lambda x)(t_1)\right| \\ &\le \frac{\left(\frac{{t_2}^\rho - a^\rho}{\rho}\right)^{1-\gamma}}{\Gamma(\alpha)}\int_{t_1}^{t_2} s^{\rho-1}\left(\frac{{t_2}^\rho - s^\rho}{\rho}\right)^{\alpha-1} |g(s)|ds \\ &+ \int_a^{t_1}\left|\left[\left(\frac{{t_2}^\rho - a^\rho}{\rho}\right)^{1-\gamma} s^{\rho-1}\left(\frac{{t_2}^\rho - s^\rho}{\rho}\right)^{\alpha-1} \right.\right.\\ &\left.\left. - \left(\frac{{t_1}^\rho - a^\rho}{\rho}\right)^{1-\gamma} s^{\rho-1}\left(\frac{{t_1}^\rho - s^\rho}{\rho}\right)^{\alpha-1}\right]\right|\frac{|g(s)|}{\Gamma(\alpha)}ds \\ &\le \frac{\left(\frac{{t_2}^\rho - a^\rho}{\rho}\right)^{1-\gamma}}{\Gamma(\alpha)}\int_{t_1}^{t_2} s^{\rho-1}\left(\frac{{t_2}^\rho - s^\rho}{\rho}\right)^{\alpha-1} p(s)ds \\ &+ \int_a^{t_1}\left|\left[\left(\frac{{t_2}^\rho - a^\rho}{\rho}\right)^{1-\gamma} s^{\rho-1}\left(\frac{{t_2}^\rho - s^\rho}{\rho}\right)^{\alpha-1} \right.\right.\\ &\left.\left. - \left(\frac{{t_1}^\rho - a^\rho}{\rho}\right)^{1-\gamma} s^{\rho-1}\left(\frac{{t_1}^\rho - s^\rho}{\rho}\right)^{\alpha-1}\right]\right|\frac{p(s)}{\Gamma(\alpha)}ds.\end{aligned}$$

Thus, for $p_* = \sup\limits_{t\in[a,T]} p(t)$ and from the continuity of the function p, we obtain

$$\begin{aligned}
&\left|\left(\frac{{t_2}^\rho - a^\rho}{\rho}\right)^{1-\gamma}(\Lambda x)(t_2) - \left(\frac{{t_1}^\rho - a^\rho}{\rho}\right)^{1-\gamma}(\Lambda x)(t_1)\right| \\
&\qquad \leq p_* \frac{\left(\frac{T^\rho - a^\rho}{\rho}\right)^{1-\gamma+\alpha}}{\Gamma(\alpha+1)}\left(\frac{t_2^\rho - t_1^\rho}{\rho}\right)^\alpha \\
&\qquad + \frac{p_*}{\Gamma(\alpha)}\int_a^{t_1}\left|\left(\frac{{t_2}^\rho - a^\rho}{\rho}\right)^{1-\gamma} s^{\rho-1}\left(\frac{{t_2}^\rho - s^\rho}{\rho}\right)^{\alpha-1}\right. \\
&\qquad \left. - \left(\frac{{t_1}^\rho - a^\rho}{\rho}\right)^{1-\gamma} s^{\rho-1}\left(\frac{{t_1}^\rho - s^\rho}{\rho}\right)^{\alpha-1}\right| ds.
\end{aligned}$$

As $t_1 \to t_2$, the right hand side of the above inequation tends to zero.

Step 4: $\Lambda(B_M)$ is equiconvergent.

Let $t \in I$ and $x \in B_M$, then we have

$$\left|\left(\frac{t^\rho - a^\rho}{\rho}\right)^{1-\gamma}(\Lambda x)(t)\right| \leq \frac{|c_1|}{\Gamma(\gamma)} + \frac{\left(\frac{t^\rho - a^\rho}{\rho}\right)^{1-\gamma}}{\Gamma(\alpha)}\int_a^t s^{\rho-1}\left(\frac{t^\rho - s^\rho}{\rho}\right)^{\alpha-1}|g(s)|ds$$

where $g(\cdot) \in BC_{\gamma,\rho}$ is given by (3.3). Thus we get

$$\begin{aligned}
\left|\left(\frac{t^\rho - a^\rho}{\rho}\right)^{1-\gamma}(\Lambda x)(t)\right| &\leq \frac{|c_1|}{\Gamma(\gamma)} + \frac{\left(\frac{t^\rho - a^\rho}{\rho}\right)^{1-\gamma}}{\Gamma(\alpha)}\int_a^t s^{\rho-1}\left(\frac{t^\rho - s^\rho}{\rho}\right)^{\alpha-1}p(s)ds \\
&\leq \frac{|c_1|}{\Gamma(\gamma)} + \left(\frac{t^\rho - a^\rho}{\rho}\right)^{1-\gamma}\left({}^\rho I^\alpha_{a+}p\right)(t).
\end{aligned}$$

Since $\left(\frac{t^\rho - a^\rho}{\rho}\right)^{1-\gamma}\left({}^\rho I^\alpha_{a+}p\right)(t) \to 0$ as $t \to \infty$, then, we get

$$|(\Lambda x)(t)| \leq \frac{|c_1|}{\left(\frac{t^\rho - a^\rho}{\rho}\right)^{1-\gamma}\Gamma(\gamma)} + \frac{\left(\frac{t^\rho - a^\rho}{\rho}\right)^{1-\gamma}\left({}^\rho I^\alpha_{a+}p\right)(t)}{\left(\frac{t^\rho - a^\rho}{\rho}\right)^{1-\gamma}} \to 0 \text{ as } t \to \infty.$$

Hence

$$|(\Lambda x)(t) - (\Lambda x)(+\infty)| \to 0 \text{ as } t \to \infty.$$

In view of Lemma 3.2 and immediate consequence of Steps 1 to 4, we conclude that $\Lambda : B_M \to B_M$ is continuous and compact. Theorem 3.3 implies that Λ has a fixed point x which is a solution of IVP (1.1) on I.

Step 5: Local attactivity of solutions.

Let x_0 is a solution of IVP (1.1). Taking $x \in B(x_0, 2p^*)$, we have

$$\begin{aligned}\Big|\Big(\frac{t^\rho - a^\rho}{\rho}\Big)^{1-\gamma}(\Lambda x)(t) - \Big(\frac{t^\rho - a^\rho}{\rho}\Big)^{1-\gamma} x_0(t)\Big| \\ &= \Big|\Big(\frac{t^\rho - a^\rho}{\rho}\Big)^{1-\gamma}(\Lambda x)(t) - \Big(\frac{t^\rho - a^\rho}{\rho}\Big)^{1-\gamma}(\Lambda x_0)(t)\Big| \\ &\le \frac{\big(\frac{t^\rho - a^\rho}{\rho}\big)^{1-\gamma}}{\Gamma(\alpha)}\int_a^t s^{\rho-1}\Big(\frac{t^\rho - s^\rho}{\rho}\Big)^{\alpha-1}|f(s,g(s)) - f(s,g_0(s))|ds,\end{aligned}$$

where $g_0 \in BC_{\gamma,\rho}$ and

$$g_0(t) = f\big(t,\ \frac{c_1}{\Gamma(\gamma)}\Big(\frac{t^\rho - a^\rho}{\rho}\Big)^{\gamma-1} + ({}^\rho I^\alpha_{a+} g_0)(t),\ g_0(t)\big).$$

Then

$$\begin{aligned}\Big|\Big(\frac{t^\rho - a^\rho}{\rho}\Big)^{1-\gamma}(\Lambda x)(t) - \Big(\frac{t^\rho - a^\rho}{\rho}\Big)^{1-\gamma} x_0(t)\Big| \\ &\le 2\frac{\big(\frac{t^\rho - a^\rho}{\rho}\big)^{1-\gamma}}{\Gamma(\alpha)}\int_a^t s^{\rho-1}\Big(\frac{t^\rho - s^\rho}{\rho}\Big)^{\alpha-1} p(s)ds.\end{aligned}$$

We obtain

$$\|(\Lambda(x) - x_0\|_{BC_{\gamma,\rho}} \le 2p^*.$$

Hence Λ is a continuous function such that $\Lambda(B(x_0, 2p^*)) \subset B(x_0, 2p^*)$.
Moreover, if x is a solution of IVP (1.1), then

$$\begin{aligned}|x(t) - x_0(t)| &= |(\Lambda x)(t) - (\Lambda x_0)(t)| \\ &\le \frac{1}{\Gamma(\alpha)}\int_a^t s^{\rho-1}\Big(\frac{t^\rho - a^\rho}{\rho}\Big)^{\alpha-1}|g(s) - g_0(s)|ds \\ &\le 2({}^\rho I^\alpha_{a+} p)(t).\end{aligned}$$

Thus

$$|x(t) - x_0(t)| \le 2\frac{\big(\frac{t^\rho - a^\rho}{\rho}\big)^{1-\gamma}({}^\rho I^\alpha_{a+} p)(t)}{\big(\frac{t^\rho - a^\rho}{\rho}\big)^{1-\gamma}}. \tag{3.8}$$

With the fact that $\lim\limits_{t\to\infty}\Big(\frac{t^\rho - a^\rho}{\rho}\Big)^{1-\gamma}({}^\rho I^\alpha_{a+} p)(t) = 0$ and inequation (3.8), we obtain

$$\lim_{t\to\infty}|x(t) - x_0(t)| = 0.$$

Consequently, all solutions of IVP (1.1) are locally attractive. □

Now onwards in this section, we deal with existence of solutions and Ulam stability for IVP (1.2).

Lemma 3.7. *Let $\gamma = \alpha + \beta(1-\alpha)$, where $0 < \alpha < 1, 0 \le \beta \le 1$ and $\rho > 0$. If $f : \Omega \times \mathbb{R} \to \mathbb{R}$ be such that $f(\cdot, x(\cdot)) \in C_{\gamma,\rho}(\Omega)$ for any $x \in C_{\gamma,\rho}(\Omega)$. Then IVP (1.2) is equivalent to the Volterra integral equation*

$$x(t) = \frac{c_2}{\Gamma(\gamma)}\Big(\frac{t^\rho - a^\rho}{\rho}\Big)^{\gamma-1} + ({}^\rho I^\alpha_{a+} f(\cdot, x(\cdot)))(t).$$

Let $\epsilon > 0, \Phi : \Omega \to [0, \infty)$ be a continuous function and consider the following inequalities:

$$|\big({}^\rho D^{\alpha,\beta}_{a+} x\big)(t) - f(t, x(t))| \le \epsilon; \qquad t \in \Omega, \tag{3.9}$$

$$|\big({}^\rho D^{\alpha,\beta}_{a+} x\big)(t) - f(t, x(t))| \le \Phi(t); \qquad t \in \Omega, \tag{3.10}$$

$$|\big({}^\rho D^{\alpha,\beta}_{a+} x\big)(t) - f(t, x(t))| \le \epsilon\Phi(t); \qquad t \in \Omega. \tag{3.11}$$

Definition 3.8. [1] IVP (1.2) is Ulam-Hyers stable if there exists a real number $\psi > 0$ such that for each $\epsilon > 0$ and for each solution $x \in C_{\gamma,\rho}$ of inequality (3.9) there exists a solution $\bar{x} \in C_{\gamma,\rho}$ of IVP (1.2) with

$$|x(t) - \bar{x}(t)| \le \epsilon\psi; \qquad t \in \Omega.$$

Definition 3.9. [1] IVP (1.2) is generalized Ulam-Hyers stable if there exists $\Psi : C([0, \infty), [0, \infty))$ with $\Psi(0) = 0$ such that for each $\epsilon > 0$ and for each solution $x \in C_{\gamma,\rho}$ of inequality (3.9) there exists a solution $\bar{x} \in C_{\gamma,\rho}$ of IVP (1.2) with

$$|x(t) - \bar{x}(t)| \le \Psi(\epsilon); \qquad t \in \Omega.$$

Definition 3.10. [1] IVP (1.2) is Ulam-Hyers-Rassias stable with respect to Φ if there exists a real number $\psi_\phi > 0$ such that for each $\epsilon > 0$ and for each solution $x \in C_{\gamma,\rho}$ of inequality (3.11) there exists a $\bar{x} \in C_{\gamma,\rho}$ of IVP (1.2) with

$$|x(t) - \bar{x}(t)| \le \epsilon\psi_\phi\Phi(t); \qquad t \in \Omega.$$

Definition 3.11. [1] IVP (1.2) is generalized Ulam-Hyers-Rassias stable with respect to Φ if there exists a real number $\psi_\phi > 0$ such that for each solution $x \in C_{\gamma,\rho}$ of inequality (3.10) there exists a $\bar{x} \in C_{\gamma,\rho}$ of IVP (1.2) with

$$|x(t) - \bar{x}(t)| \le \psi_\phi\Phi(t); \qquad t \in \Omega.$$

Remark 3.12. It is clear that

- **(i).** If the IVP (1.2) is Ulam-Hyers stable then, for a real number $\psi > 0$ as a continuous function defined in Definition 3.9, it is generalized Ulam-Hyers stable.
- **(ii).** If the IVP (1.2) is Ulam-Hyers-Rassias stable then it is generalized Ulam-Hyers-Rassias stable.
- **(iii).** If the IVP (1.2) is Ulam-Hyers-Rassias stable with respect to Φ then, for $\Phi(\cdot) = 1$, it is Ulam-Hyers stable.

Definition 3.13. A solution of IVP (1.2) is a measurable function $x \in C_{\gamma,\rho}$ that satisfies initial condition $\big({}^\rho I^{1-\gamma}_{a+} x\big)(a) = c_2$ and differential equation $\big({}^\rho D^{\alpha,\beta}_{a+} x\big)(t) = f(t, x(t))$ on Ω.

Consider the following hypotheses:

(H_3). Function $t \longmapsto f(t,x)$ is measurable on Ω for each $x \in C_{\gamma,\rho}$ and function $x \longmapsto f(t,x)$ is continuous on $C_{\gamma,\rho}$ for a.e. $t \in \Omega$,

(H_4). There exists a continuous function $p : \Omega \to [0,\infty)$ such that

$$|f(t,x)| \leq \frac{p(t)}{1+|x|}|x|, \text{ for a.e. } t \in \Omega, \text{ and each } x \in \mathbb{R}.$$

Set $p^* = \sup_{t\in\Omega} p(t)$. Now we shall give the existence theorem in the following:

Theorem 3.14. *Assume that* (H_3) *and* (H_4) *hold. Then IVP* (1.2) *has at least one solution defined on* Ω.

Proof. Consider the operator $\Lambda : C_{\gamma,\rho} \to C_{\gamma,\rho}$ such that

$$(\Lambda x)(t) = \frac{c_2}{\Gamma(\gamma)}\left(\frac{t^\rho - a^\rho}{\rho}\right)^{\gamma-1} + \int_a^t s^{\rho-1}\left(\frac{t^\rho - s^\rho}{\rho}\right)^{\alpha-1}\frac{f(s,x(s))}{\Gamma(\alpha)}ds. \tag{3.12}$$

Clearly, the fixed points of this operator equation $(\Lambda x)(t) = x(t)$ are solutions of IVP (1.2). For any $x \in C_{\gamma,\rho}$ and each $t \in \Omega$, we have

$$\begin{aligned}
\left|\left(\frac{t^\rho - a^\rho}{\rho}\right)^{1-\gamma}(\Lambda x)\right| &\leq \frac{|c_2|}{\Gamma(\gamma)} + \left(\frac{t^\rho - a^\rho}{\rho}\right)^{1-\gamma}\int_a^t s^{\rho-1}\left(\frac{t^\rho - s^\rho}{\rho}\right)^{\alpha-1}\frac{|f(s,x)|}{\Gamma(\alpha)}ds \\
&\leq \frac{|c_2|}{\Gamma(\gamma)} + \frac{\left(\frac{t^\rho - a^\rho}{\rho}\right)^{1-\gamma}}{\Gamma(\alpha)}\int_a^t s^{\rho-1}\left(\frac{t^\rho - s^\rho}{\rho}\right)^{\alpha-1}p(s)ds \\
&\leq \frac{|c_2|}{\Gamma(\gamma)} + \frac{p^*}{\Gamma(\alpha)}\left(\frac{t^\rho - a^\rho}{\rho}\right)^{1-\gamma}\int_a^t s^{\rho-1}\left(\frac{t^\rho - s^\rho}{\rho}\right)^{\alpha-1}ds \\
&\leq \frac{|c_2|}{\Gamma(\gamma)} + \frac{p^*}{\Gamma(\alpha+1)}\left(\frac{t^\rho - a^\rho}{\rho}\right)^{1-\gamma}\left(\frac{t^\rho - a^\rho}{\rho}\right)^{\alpha} \\
&\leq \frac{|c_2|}{\Gamma(\gamma)} + \frac{p^*}{\Gamma(\alpha+1)}\left(\frac{T^\rho - a^\rho}{\rho}\right)^{\alpha+1-\gamma}.
\end{aligned}$$

Thus

$$\|\Lambda x\|_C \leq \frac{|c_2|}{\Gamma(\gamma)} + \frac{p^*}{\Gamma(\alpha+1)}\left(\frac{T^\rho - a^\rho}{\rho}\right)^{\alpha+1-\gamma} := N. \tag{3.13}$$

Thus Λ transforms the ball $B_N = B(0,N) = \{z \in C_{\gamma,\rho} : \|z\|_C \leq N\}$ into itself. We shall show that the operator $\Lambda : B_N \to B_N$ satisfies all the conditions of Theorem 3.16. The proof is given in following several steps.

Step 1: $\Lambda : B_N \to B_N$ is continuous.

Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence such that $x_n \to x$ in B_N. Then, for each $t \in I$, we have

$$\begin{aligned}
&\left|\left(\frac{t^\rho - a^\rho}{\rho}\right)^{1-\gamma}(\Lambda x_n)(t) - \left(\frac{t^\rho - a^\rho}{\rho}\right)^{1-\gamma}(\Lambda x)(t)\right| \\
&\qquad\leq \frac{\left(\frac{t^\rho - a^\rho}{\rho}\right)^{1-\gamma}}{\Gamma(\alpha)}\int_a^t s^{\rho-1}\left(\frac{t^\rho - s^\rho}{\rho}\right)^{\alpha-1}|f(s,x_n(s)) - f(s,x(s))|ds.
\end{aligned} \tag{3.14}$$

Since $x_n \to x$ as $n \to \infty$ and f is continuous, then by Lebesgue dominated convergence theorem, inequality (3.14) implies $\|\Lambda(x_n) - \Lambda(x)\|_C \to 0$ as $n \to \infty$.
Step 2: $\Lambda(B_N)$ is uniformly bounded.

Since $\Lambda(B_N) \subset B_N$ and B_N is bounded. Hence, $\Lambda(B_N)$ is uniformly bounded.
Step 3: $\Lambda(B_N)$ is equicontinuous.

Let $t_1, t_2 \in \Omega, t_1 < t_2$ and $x \in B_N$. We have

$$\begin{aligned}
&\left|\left(\frac{{t_2}^\rho - a^\rho}{\rho}\right)^{1-\gamma}(\Lambda x)(t_2) - \left(\frac{{t_1}^\rho - a^\rho}{\rho}\right)^{1-\gamma}(\Lambda x)(t_1)\right| \\
&\qquad \leq \left|\frac{\left(\frac{{t_2}^\rho - a^\rho}{\rho}\right)^{1-\gamma}}{\Gamma(\alpha)}\int_a^{t_2} s^{\rho-1}\left(\frac{{t_2}^\rho - s^\rho}{\rho}\right)^{\alpha-1} f(s,x(s))ds\right. \\
&\qquad\quad \left. - \frac{\left(\frac{{t_1}^\rho - a^\rho}{\rho}\right)^{1-\gamma}}{\Gamma(\alpha)}\int_a^{t_1} s^{\rho-1}\left(\frac{{t_1}^\rho - s^\rho}{\rho}\right)^{\alpha-1} f(s,x(s))ds\right| \\
&\qquad \leq \frac{\left(\frac{{t_2}^\rho - a^\rho}{\rho}\right)^{1-\gamma}}{\Gamma(\alpha)}\int_{t_1}^{t_2} s^{\rho-1}\left(\frac{{t_2}^\rho - s^\rho}{\rho}\right)^{\alpha-1} |f(s,x(s))|ds \\
&\qquad\quad + \int_a^{t_1}\left|\left[\left(\frac{{t_2}^\rho - a^\rho}{\rho}\right)^{1-\gamma} s^{\rho-1}\left(\frac{{t_2}^\rho - s^\rho}{\rho}\right)^{\alpha-1}\right.\right. \\
&\qquad\quad \left.\left. - \left(\frac{{t_1}^\rho - a^\rho}{\rho}\right)^{1-\gamma} s^{\rho-1}\left(\frac{{t_1}^\rho - s^\rho}{\rho}\right)^{\alpha-1}\right]\right|\frac{|f(s,x(s))|}{\Gamma(\alpha)}ds \\
&\qquad \leq \frac{\left(\frac{{t_2}^\rho - a^\rho}{\rho}\right)^{1-\gamma}}{\Gamma(\alpha)}\int_{t_1}^{t_2} s^{\rho-1}\left(\frac{{t_2}^\rho - s^\rho}{\rho}\right)^{\alpha-1} p(s)ds \\
&\qquad\quad + \int_a^{t_1}\left|\left[\left(\frac{{t_2}^\rho - a^\rho}{\rho}\right)^{1-\gamma} s^{\rho-1}\left(\frac{{t_2}^\rho - s^\rho}{\rho}\right)^{\alpha-1}\right.\right. \\
&\qquad\quad \left.\left. - \left(\frac{{t_1}^\rho - a^\rho}{\rho}\right)^{1-\gamma} s^{\rho-1}\left(\frac{{t_1}^\rho - s^\rho}{\rho}\right)^{\alpha-1}\right]\right|\frac{p(s)}{\Gamma(\alpha)}ds.
\end{aligned}$$

Thus, for $p_* = \sup_{t\in\Omega} p(t)$ and from the continuity of the function p, we obtain

$$\begin{aligned}
&\left|\left(\frac{{t_2}^\rho - a^\rho}{\rho}\right)^{1-\gamma}(\Lambda x)(t_2) - \left(\frac{{t_1}^\rho - a^\rho}{\rho}\right)^{1-\gamma}(\Lambda x)(t_1)\right| \\
&\qquad \leq \frac{p_*}{\Gamma(\alpha+1)}\left(\frac{T^\rho - a^\rho}{\rho}\right)^{1-\gamma+\alpha}\left(\frac{t_2^\rho - t_1^\rho}{\rho}\right)^{\alpha} \\
&\qquad\quad + \frac{p_*}{\Gamma(\alpha)}\int_a^{t_1}\left|\left(\frac{{t_2}^\rho - a^\rho}{\rho}\right)^{1-\gamma} s^{\rho-1}\left(\frac{{t_2}^\rho - s^\rho}{\rho}\right)^{\alpha-1}\right. \\
&\qquad\quad \left. - \left(\frac{{t_1}^\rho - a^\rho}{\rho}\right)^{1-\gamma} s^{\rho-1}\left(\frac{{t_1}^\rho - s^\rho}{\rho}\right)^{\alpha-1}\right|ds.
\end{aligned}$$

As $t_1 \to t_2$, the right hand side of the above inequality tends to zero.

As a consequence of Steps 1 to 3 together with Arzela-Ascoli Theorem, we can conclude that Λ is continuous and compact. By applying the Schauder fixed point theorem, we conclude that Λ has a fixed point x which is a solution of IVP (1.2). □

Theorem 3.15. *Assume that* (H_3), (H_4) *and the following hypotheses hold:*

(H_5). *There exists* $\lambda_\phi > 0$ *such that for each* $t \in \Omega$, *we have*

$$({}^\rho I^\alpha_{a+}\Phi(t)) \leq \lambda_\phi \Phi(t);$$

(H_6). *There exists* $q \in C(\Omega, [0,\infty))$ *such that for each* $t \in \Omega$,

$$p(t) \leq q(t)\Phi(t).$$

Then, IVP (1.2) *is generalized Ulam-Hyers-Rassias stable.*

Proof. Consider the operator $\Lambda : C_{\gamma,\rho} \to C_{\gamma,\rho}$ defined in (3.12). Let x be a solution of inequality (3.10), and let us assume that $\bar{x}$ is a solution of IVP (1.2). Thus

$$\bar{x}(t) = \frac{c_2}{\Gamma(\gamma)}\left(\frac{t^\rho - a^\rho}{\rho}\right)^{\gamma-1} + \int_a^t s^{\rho-1}\left(\frac{t^\rho - s^\rho}{\rho}\right)^{\alpha-1}\frac{f(s,\bar{x}(s))}{\Gamma(\alpha)}ds.$$

From inequality (3.10), for each $t \in \Omega$, we have

$$\left|x(t) - \frac{c_2}{\Gamma(\gamma)}\left(\frac{t^\rho - a^\rho}{\rho}\right)^{\gamma-1} - \int_a^t s^{\rho-1}\left(\frac{t^\rho - s^\rho}{\rho}\right)^{\alpha-1}\frac{f(s,x(s))}{\Gamma(\alpha)}ds\right| \leq \Phi(t).$$

Set $q^* = \sup_{t\in\Omega} q(t)$. From the hypotheses (H5) and (H6), for each $t \in \Omega$, we get

$$\begin{aligned}
\left|x(t) - \bar{x}(t)\right| &\leq \Big|x(t) - \frac{c_2}{\Gamma(\gamma)}\left(\frac{t^\rho - a^\rho}{\rho}\right)^{\gamma-1} - \int_a^t s^{\rho-1}\left(\frac{t^\rho - s^\rho}{\rho}\right)^{\alpha-1}\frac{f(s,x(s))}{\Gamma(\alpha)}ds\Big| \\
&\quad + \int_a^t s^{\rho-1}\left(\frac{t^\rho - s^\rho}{\rho}\right)^{\alpha-1}\frac{|f(s,x(s)) - f(s,\bar{x}(s))|}{\Gamma(\alpha)}ds \\
&\leq \Phi(t) + \int_a^t s^{\rho-1}\left(\frac{t^\rho - s^\rho}{\rho}\right)^{\alpha-1}\frac{2q^*\Phi(s)}{\Gamma(\alpha)}ds \\
&\leq \Phi(t) + 2q^*({}^\rho I^\alpha_{a+}\Phi)(t) \\
&\leq \Phi(t) + 2q^*\lambda_\phi\Phi(t) \\
&= [1 + 2q^*\lambda_\phi]\Phi(t).
\end{aligned}$$

Thus

$$|x(t) - \bar{x}(t)| \leq \psi_\phi \Phi(t).$$

Hence, IVP (1.2) is generalized Ulam-Hyers-Rassias stable. □

Define the metric

$$d(x,y) = \sup_{t\in\Omega}\frac{\left(\frac{t^\rho - a^\rho}{\rho}\right)^{1-\gamma}|x(t) - y(t)|}{\Phi(t)}$$

in the space $C_{\gamma,\rho}(\Omega)$. Following fixed point theorem is used in our further result.

Theorem 3.16. [19] *Let $\Theta : C_{\gamma,\rho} \to C_{\gamma,\rho}$ be a strictly contractive operator with a Lipschitz constant $L < 1$. There exists a nonnegative integer k such that*

$$d(\Theta^{k+1}x, \Theta^k x) < \infty$$

for some $x \in C_{\gamma,\rho}$, then the following propositions hold true:
(A1) The sequence $\{\Theta^k x\}_{n\in\mathbb{N}}$ converges to a fixed point x^* of Θ;
(A2) x^* is a unique fixed point of Θ in $X = \{y \in C_{\gamma,\rho}(\Omega) : d(\Theta^k x, y) < \infty\}$;
(A3) If $y \in X$, then $d(y, x^*) \leq \frac{1}{1-L} d(y, \Theta x)$.

Theorem 3.17. *Assume that (H_5) and the following hypothesis hold:*

(H_7). *There exists $\phi \in C(\Omega, [0, \infty))$ such that for each $t \in \Omega$, and all $x, \bar{x} \in \mathbb{R}$, we have*

$$|f(t, x) - f(t, \bar{x})| \leq \left(\frac{t^\rho - a^\rho}{\rho}\right)^{1-\gamma} \phi(t)\Phi(t)|x - \bar{x}|.$$

If

$$L = \left(\frac{T^\rho - a^\rho}{\rho}\right)^{1-\gamma} \phi^* \lambda_\phi < 1,$$

where $\phi^* = \sup_{t\in\Omega} \phi(t)$, then there exists a unique solution x_0 of IVP (1.2), *and IVP* (1.2) *is generalized Ulam-Hyers-Rassias stable. Furthermore, we have*

$$|x(t) - \bar{x}(t)| \leq \frac{\Phi(t)}{1 - L}.$$

Proof. Let $\Lambda : C_{\gamma,\rho} \to C_{\gamma,\rho}$ be the operator defined in (3.12). Apply Theorem 3.16, we have

$$\begin{aligned}
|(\Lambda x)(t) - (\Lambda \bar{x})(t)| &\leq \int_a^t s^{\rho-1}\left(\frac{t^\rho - s^\rho}{\rho}\right)^{\alpha-1} \frac{|f(s, x(s)) - f(s, \bar{x}(s))|}{\Gamma(\alpha)} ds \\
&\leq \int_a^t s^{\rho-1}\left(\frac{t^\rho - s^\rho}{\rho}\right)^{\alpha-1} \phi(s) \\
&\quad \times \Phi(s)\frac{\left|\left(\frac{s^\rho - a^\rho}{\rho}\right)^{1-\gamma} x(s) - \left(\frac{s^\rho - a^\rho}{\rho}\right)^{1-\gamma} \bar{x}(s)\right|}{\Gamma(\alpha)} ds \\
&\leq \int_a^t s^{\rho-1}\left(\frac{t^\rho - s^\rho}{\rho}\right)^{\alpha-1} \phi^*(s)\Phi(s)\frac{\|x - \bar{x}\|_C}{\Gamma(\alpha)} ds \\
&\leq \phi^*({}^\rho I^\alpha_{a+})\Phi(t)\|x - \bar{x}\|_C \\
&\leq \phi^* \lambda_\phi \Phi(t)\|x - \bar{x}\|_C.
\end{aligned}$$

Thus

$$\left|\left(\frac{t^\rho - a^\rho}{\rho}\right)^{1-\gamma}(\Lambda x) - \left(\frac{t^\rho - a^\rho}{\rho}\right)^{1-\gamma}(\Lambda \bar{x})\right| \leq \left(\frac{T^\rho - a^\rho}{\rho}\right)^{1-\gamma} \phi^* \lambda_\phi \Phi(t)\|x - \bar{x}\|_C.$$

Hence

$$d(\Lambda(x), \Lambda(\bar{x})) = \sup_{t\in\Omega} \frac{\|(\Lambda x)(t) - (\Lambda \bar{x})(t)\|_C}{\Phi(t)} \leq L\|x - \bar{x}\|_C$$

from which we conclude the theorem. ∎

4. Examples

In this section we present some examples to illustrate our main results.

Example 4.1. Consider the following IVP involving generalized Katugampola fractional derivative:

$$\begin{cases} \left({}^\rho D_{a+}^{\frac{1}{2},\frac{1}{2}} x\right)(t) &= f(t,x,y); \qquad t \in [a,b], \\ \left({}^\rho I_{a+}^{\frac{1}{4}} x\right)(a) &= (1-a), \end{cases} \tag{4.1}$$

where $\alpha = \frac{1}{2}, \beta = \frac{1}{2}, \rho > 0, \gamma = \frac{3}{4}, 0 < a < b \leq e$, and

$$\begin{cases} f(t,x,y) &= \frac{\theta(t-a)^{-\frac{1}{4}} \sin(t-a)}{64(1+\sqrt{t-a})(1+|x|+|y|)}; \quad t \in (a,b],\ x,y \in \mathbb{R}, \\ f(a,x,y) &= 0; \qquad x,y \in \mathbb{R}. \end{cases}$$

Clearly, function f is continuous for each $x, y \in \mathbb{R}$ and (H_2) is satisfied with

$$\begin{cases} p(t) &= \frac{\theta(t-a)^{-\frac{1}{4}} |\sin(t-a)|}{64(1+\sqrt{t-a})}; \quad 0 < \theta \leq 1,\ t \in (a,+\infty), \\ p(a) &= 0. \end{cases}$$

Thus, all the conditions of Theorem 3.6 are satisfied. Hence, IVP (4.1) has at least one solution defined on $[a, +\infty)$.

Also, we have

$$\begin{aligned} \left(\frac{t^\rho - a^\rho}{\rho}\right)^{1-\gamma} ({}^\rho I_{a+}^{\frac{1}{2}} p)(t) &= \left(\frac{t^\rho - a^\rho}{\rho}\right)^{\frac{1}{4}} \int_a^t s^{\rho-1} \left(\frac{t^\rho - s^\rho}{\rho}\right)^{-\frac{1}{2}} \frac{p(s)}{\Gamma(\frac{1}{2})} ds \\ &\leq \frac{1}{8} \left(\frac{t^\rho - a^\rho}{\rho}\right)^{-\frac{1}{4}} \to 0 \quad \text{as } t \to +\infty. \end{aligned}$$

This implies that solutions of IVP (4.1) are locally asymptotically stable.

Example 4.2. Consider the following IVP involving generalized Katugampola derivative:

$$\begin{cases} \left({}^\rho D_{a+}^{\frac{1}{2},\frac{1}{2}} x\right)(t) &= f(t,x); \qquad t \in [a,b], \\ \left({}^\rho I_{a+}^{\frac{1}{4}} x\right)(a) &= (1-a), \end{cases} \tag{4.2}$$

where $\alpha = \frac{1}{2}, \beta = \frac{1}{2}, \rho > 0, \gamma = \frac{3}{4}, 0 < a < b \leq e$, and

$$\begin{cases} f(t,x) &= \frac{\theta(t-a)^{-\frac{1}{4}} \sin(t-a)}{64(1+\sqrt{t-a})(1+|x|)}; \quad t \in (a,b],\ x \in \mathbb{R}, \\ f(a,x) &= 0; \qquad x \in \mathbb{R}. \end{cases}$$

Clearly, function f is continuous for all $x \in \mathbb{R}$ and (H_4) is satisfied with

$$\begin{cases} p(t) &= \frac{\theta(t-a)^{-\frac{1}{4}} |\sin(t-a)|}{64(1+\sqrt{t-a})}; \quad 0 < \theta \leq 1,\ t \in (a,b],\ x \in \mathbb{R}, \\ p(a) &= 0. \end{cases}$$

Hence, Theorem 3.14 implies that IVP (4.2) has at least one solution defined on $[a, b]$. Also, one can see that (H_5) is satisfied with

$$\Phi(t) = e^3, \quad \text{and} \quad \lambda_\phi = \frac{1}{\Gamma(\frac{3}{2})}.$$

Consequently, Theorem 3.15 implies that IVP (4.2) is generalized Ulam-Hyers-Rassias stable.

5. Concluding remarks

In this article, two IVPs involving generalized Katugampola fractional derivative are considered. The existence and local attractivity of solution is obtained for first IVP while Ulam-type stability of second IVP is obtained by using fixed point theorems. Both the results are supported with suitable illustrative examples.

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New subclasses of bi-univalent functions defined by multiplier transformation

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Abstract. In the present paper, we introduce new subclasses of the function class $\sum$ of bi-univalent functions by using multiplier transformation. Furthermore, we obtain estimates on the coefficients $|a_2|, |a_3|$ and $|a_4|$ for functions of this class. Relevant connections of the results presented here with various well-known results are briefly indicated.

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1. Introduction

Let $\mathcal{A}$ denote the class of functions f of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \tag{1.1}$$

which are analytic in the open unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$ and satisfy the normalization condition $f(0) = f'(0) - 1 = 0$. Let $\mathcal{S}$ be the subclass of $\mathcal{A}$ consisting of functions of the form (1.1) which are also univalent in U.

It is well known that every $f \in \mathcal{S}$ has an inverse f^{-1}, defined by

$$f^{-1}(f(z)) = z, \ (z \in U)$$

and

$$f\left(f^{-1}(\omega)\right) = \omega, \ \left(|\omega| < r_0(f), \ r_0(f) \geq \frac{1}{4}\right),$$

where

$$f^{-1}(\omega) = \omega - a_2\omega^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2a_3 + a_4)\omega^4 + \dots$$

A function $f(z) \in \mathcal{A}$ is said to be bi-univalent in U if both $f(z)$ and $f^{-1}(z)$ are univalent in U. Let Σ denote the class of bi-univalent functions in U given by the Taylor-Maclaurin series expansion (1.1). Examples of functions in the class Σ are $\frac{z}{1-z}$, $-\log(1-z)$, $\frac{1}{2}\log\left(\frac{1+z}{1-z}\right)$ and so on.

However, the familiar Koebe function is not a member of Σ. Other common examples of functions in $\mathcal{S}$ such as $z-\frac{z^2}{2}$ and $\frac{z}{1-z^2}$ are also not members of Σ.

Lewin [7] first investigated the bi-univalent function class Σ showed that

$$|a_2| < 1.5.$$

Subsequently, Brannan and Clunie [2] conjectured that $|a_2| \le \sqrt{2}$.

Netanyahu [9], on the other hand, showed that $max_{f\in\Sigma}|a_2| = \frac{4}{3}$.

The coeffiecient estimate problems for bi-univalent function class Σ is an interesting problem of Geometric Function Theory. Several researchers e.g. (see [1], [6], [11], [12], [14], [16], [17]), introduced the various new subclasses of the bi-univalent function class Σ and obtained non-sharp bounds on the first two coefficients $|a_2|$ and $|a_3|$. Recently, Mishra and Soren [8] obtain coefficient bounds for bi-starlike analytic functions and improve results of [3].

In order to prove our main results, we shall require the following lemma due to [10].

Lemma 1.1. *If $h \in P$ then $|c_k| \le 2$ for each k, where P is the family of all functions h analytic in U for which $\Re\{h(z)\} > 0$,*

$$h(z) = 1 + c_1 z + c_2 z^2 + c_3 z^3 + \dots \ \textit{for } z \in U.$$

2. Coefficient bounds for the function class $B_\Sigma(n,\beta,\lambda,\mu)$

Cho and Srivastava [5], (see also [4]), introduced the operator $I_\lambda^n : \mathcal{A} \to \mathcal{A}$ defined for the function $f(z)$ of the form (1.1) as

$$I_\lambda^n f(z) = z + \sum_{k=2}^{\infty}\left(\frac{k+\lambda}{1+\lambda}\right)^n a_k z^k,$$

where $n \in N_0 = N \cup \{0\}$ and $-1 < \lambda \le 1$. For $\lambda = 1$, the operator $I_\lambda^n \equiv I^n$ was studied by Uralegaddi and Somanatha [15] and for $\lambda = 0$ the operator I_λ^n reduces to well-known Sălăgean operator introduced by Sălăgean [13].

Definition 2.1. A function $f(z)$ given by (1.1) is said to be in the class $B_\Sigma(n,\beta,\lambda,\mu)$, $(n \in N_0, 0 < \beta \le 1, \mu \ge 1, -1 < \lambda < 1)$, if the following conditions are satisfied:

$$f \in \Sigma \text{ and } \left|\arg\left\{\frac{(1-\mu)I_\lambda^n f(z) + \mu I_\lambda^{n+1} f(z)}{z}\right\}\right| < \frac{\beta\pi}{2},\ (z \in U) \tag{2.1}$$

and

$$\left|\arg\left\{\frac{(1-\mu)I_\lambda^n g(\omega) + \mu I_\lambda^{n+1} g(\omega)}{\omega}\right\}\right| < \frac{\beta\pi}{2},\ (\omega \in U), \tag{2.2}$$

where the function g is given by

$$g(\omega) = \omega - a_2\omega^2 + (2a_2^2 - a_3)\omega^3 - (5a_2^3 - 5a_2a_3 + a_4)\omega^4 + \dots. \tag{2.3}$$

Throught this paper, we shall frequently use the notation

$$M_k = \left[(1-\mu)\left(\frac{k+\lambda}{1+\lambda}\right)^n + \mu\left(\frac{k+\lambda}{1+\lambda}\right)^{n+1}\right], \ k = 2, 3, \ldots .$$

By specializing the parameters in the class $B_\Sigma(n, \beta, \lambda, \mu)$, we obtain the following known subclasses studied earlier by various researchers.

1. $B_\Sigma(n, \beta, 0, \mu) \equiv B_\Sigma(n, \beta, \mu)$ studied by Porwal and Darus [11].
2. $B_\Sigma(0, \beta, 0, \mu) \equiv B_\Sigma(\beta, \mu)$ studied by Frasin and Aouf [6].
3. $B_\Sigma(0, \beta, 0, 1) \equiv B_\Sigma(\beta)$ studied by Srivastava *et al.* [14].

Theorem 2.2. *Let the function $f(z)$ given by (1.1) be in the class $B_\Sigma(n, \beta, \lambda, \mu)$, $n \in N_0, 0 < \beta \leq 1$ and $\mu \geq 1, -1 < \lambda \leq 1$. Then*

$$|a_2| \leq \frac{2\beta}{\sqrt{2M_3\beta + M_2^2(1-\beta)}}, \tag{2.4}$$

$$|a_3| \leq \frac{2\beta}{M_3}, \tag{2.5}$$

and

$$|a_4| \leq \begin{cases} \frac{\beta}{M_4}\left[2 + \frac{4(1-\beta)}{3}\left\{\frac{2M_3\beta+(1-2\beta){M_2}^2}{2M_3\beta+{M_2}^2(1-\beta)}\right\}\right], & \left(0 < \beta \leq \frac{1}{2}\right) \\ \frac{\beta}{M_4}\left[2 + \frac{4(1-\beta)}{3}\left\{\frac{2M_3\beta+(2\beta-1){M_2}^2}{2M_3\beta+{M_2}^2(1-\beta)}\right\}\right], & \left(\frac{1}{2} \leq \beta \leq 1\right) \end{cases}. \tag{2.6}$$

Proof. It follows from (2.1) and (2.2) that

$$\frac{(1-\mu)I_\lambda^n f(z) + \mu I_\lambda^{n+1} f(z)}{z} = [p(z)]^\beta, \tag{2.7}$$

and

$$\frac{(1-\mu)I_\lambda^n g(\omega) + \mu I_\lambda^{n+1} g(\omega)}{\omega} = [q(\omega)]^\beta, \tag{2.8}$$

where $p(z)$ and $q(\omega)$ in P and have the forms

$$p(z) = 1 + p_1 z + p_2 z^2 + p_3 z^3 + \ldots \tag{2.9}$$

and

$$q(\omega) = 1 + q_1\omega + q_2\omega^2 + q_3\omega^3 + \ldots \tag{2.10}$$

Now, equating the coefficients in (2.7) and (2.8), we obtain

$$M_2 a_2 = \beta p_1 \tag{2.11}$$

$$M_3 a_3 = \beta p_2 + \frac{\beta(\beta-1)}{2}p_1^2, \tag{2.12}$$

$$M_4 a_4 = \beta p_3 + \beta(\beta-1)p_1 p_2 + \frac{\beta(\beta-1)(\beta-2)}{6}p_1^3, \tag{2.13}$$

$$-M_2 a_2 = \beta q_1, \tag{2.14}$$

$$M_3(2a_2^2 - a_3) = \beta q_2 + \frac{\beta(\beta-1)}{2}q_1^2, \tag{2.15}$$

$$-M_4(5a_2^3 - 5a_2 a_3 + a_4) = \beta q_3 + \beta(\beta-1)q_1 q_2 + \frac{\beta(\beta-1)(\beta-2)}{6}q_1^3, \tag{2.16}$$

where

$$M_k = \left[(1-\mu)\left(\frac{k+\lambda}{1+\lambda}\right)^n + \mu\left(\frac{k+\lambda}{1+\lambda}\right)^{n+1}\right].$$

From (2.11) and (2.14), we obtain

$$p_1 = -q_1. \tag{2.17}$$

We shall obtain a estimates on $|p_1|$ for use in the estimates of $|a_2|$, $|a_3|$ and $|a_4|$. For this purpose we first add (2.12) and (2.15), we get

$$2M_3a_2^2 = \beta(p_2+q_2) + \frac{\beta(\beta-1)}{2}(p_1^2+q_1^2).$$

Using (2.17) in last equation, we have

$$2M_3a_2^2 = \beta(p_2+q_2) + \beta(\beta-1)p_1^2.$$

Putting $a_2 = \frac{\beta p_1}{M_2}$ from (2.11), we have after simplification

$$p_1^2 = \frac{(p_2+q_2)\,M_2^2}{2M_3\beta + M_2^2(1-\beta)}. \tag{2.18}$$

By applying $|p_2| \leq 2$ and $|q_2| \leq 2$, we get

$$|p_1| \leq \frac{2M_2}{\sqrt{2M_3\beta + M_2^2(1-\beta)}}.$$

Therefore

$$|a_2| \leq \frac{2\beta}{\sqrt{2M_3\beta + M_2^2(1-\beta)}}.$$

To find a bound on $|a_3|$ we may express a_3 in terms of p_1, p_2, q_1 and q_2. For this purpose we first subtract (2.15) from (2.12), we get

$$2M_3(a_3 - a_2^2) = \beta(p_2-q_2) + \frac{\beta(\beta-1)}{2}(p_1^2-q_1^2).$$

Using (2.17) in last equations, we have

$$2M_3a_3 = 2M_3a_2^2 + \beta(p_2-q_2). \tag{2.19}$$

Putting $a_2 = \frac{\beta p_1}{M_2}$ from (2.11) and using (2.18), we get

$$\begin{aligned} 2M_3a_3 &= 2M_3\left(\frac{\beta p_1}{M_2}\right)^2 + \beta(p_2-q_2) \\ &= \frac{2M_3\beta^2\,(p_2+q_2)}{2M_3\beta + M_2^2(1-\beta)} + \beta(p_2-q_2) \\ &= \beta\left[\frac{\left(4M_3\beta + M_2^2(1-\beta)\right)p_2 - M_2^2(1-\beta)q_2}{2M_3\beta + M_2^2(1-\beta)}\right]. \end{aligned}$$

Using the inequalities $|p_2| \leq 2$ and $|q_2| \leq 2$ and after simple calculation, we have

$$|a_3| \leq \frac{2\beta}{M_3}.$$

We shall next find an estimates on $|a_4|$. At first we shall derive a relation connecting p_1, p_2, p_3, q_1, q_2 and q_3. To this end, first we add the equations (2.13) and (2.16), we get

$$-M_4\left(5a_2^3 - 5a_2a_3\right) = \beta(p_3+q_3) + \beta(\beta-1)(p_1p_2+q_1q_2) + \frac{\beta(\beta-1)(\beta-2)}{6}(p_1^3+q_1^3).$$

Using (2.17) and (2.19), we have

$$\frac{5M_4\beta(p_2-q_2)a_2}{2M_3} = \beta(p_3+q_3) + \beta(\beta-1)p_1(p_2-q_2).$$

Using (2.11) and after simple calculation we have

$$p_1(p_2-q_2) = \frac{2M_2M_3(p_3+q_3)}{5M_4\beta + 2M_2M_3(1-\beta)}. \tag{2.20}$$

We wish to express $|a_4|$ in terms of p_1, p_2, p_3, q_1, q_2 and q_3. For this purpose subtracting equation (2.16) from (2.13), we get

$$M_4\left(2a_4 + 5a_2^3 - 5a_2a_3\right) = \beta(p_3-q_3) + \beta(\beta-1)p_1(p_2+q_2) + \frac{\beta(\beta-1)(\beta-2)}{6}(p_1^3-q_1^3).$$

Using (2.17), (2.19) and after simple calculation we have

$$2M_4a_4 - \frac{5M_4a_2\beta}{2M_3}(p_2-q_2) = \beta(p_3-q_3) + \beta(\beta-1)p_1(p_2+q_2) + \frac{\beta(\beta-1)(\beta-2)}{3}p_1^3.$$

Again using equations (2.11) and (2.18), we get

$$2M_4a_4 = \frac{5M_4\beta}{2M_3}\frac{\beta p_1}{M_2}(p_2-q_2) + \beta(p_3-q_3) + \beta(\beta-1)p_1(p_2+q_2) + \frac{\beta(\beta-1)(\beta-2)}{3}p_1\frac{(p_2+q_2)\,M_2^2}{2M_3\beta + M_2^2(1-\beta)}.$$

Using (2.20) we have

$$\begin{aligned} 2M_4a_4 &= \beta\left[\frac{5M_4\beta(p_3+q_3)}{5M_4\beta + 2M_2M_3(1-\beta)} + p_3 - q_3 + (\beta-1)p_1(p_2+q_2) + \frac{(\beta-1)(\beta-2)}{3}p_1\frac{(p_2+q_2)\,M_2^2}{2M_3\beta + M_2^2(1-\beta)}\right] \\ &= \beta\left[\frac{\left(10M_4\beta + 2M_2M_3(1-\beta)\right)p_3 - 2M_2M_3(1-\beta)q_3}{5M_4\beta + 2M_2M_3(1-\beta)} + (1-\beta)p_1(p_2+q_2)\left\{\frac{(2-\beta)M_2^2}{3(2M_3\beta + M_2^2(1-\beta))} - 1\right\}\right] \end{aligned}$$

Using Lemma 1.1 and after simple calculation, we have

$$|a_4| \le \begin{cases} \frac{\beta}{M_4}\left[2 + \frac{4(1-\beta)}{3}\left\{\frac{2M_3\beta + (1-2\beta)\,{M_2}^2}{2M_3\beta + {M_2}^2\,(1-\beta)}\right\}\right], & \left(0 < \beta \le \frac{1}{2}\right) \\ \frac{\beta}{M_4}\left[2 + \frac{4(1-\beta)}{3}\left\{\frac{2M_3\beta + (2\beta-1)\,{M_2}^2}{2M_3\beta + {M_2}^2\,(1-\beta)}\right\}\right], & \left(\frac{1}{2} \le \beta \le 1\right). \end{cases} \qquad \square$$

3. Coefficient bounds for the function class $H_\Sigma(n,\gamma,\lambda,\mu)$

Definition 3.1. A function $f(z)$ given by (1.1) is said to be in the class $H_\Sigma(n,\gamma,\lambda,\mu)$, $(n \in N_0, 0 \leq \gamma < 1, \mu \geq 1, -1 < \lambda \leq 1)$, if the following conditions are satisfied:

$$f \in \Sigma \text{ and } \Re\left\{\frac{(1-\mu)I_\lambda^n f(z) + \mu I_\lambda^{n+1} f(z)}{z}\right\} > \gamma, \ (z \in U) \tag{3.1}$$

and

$$\Re\left\{\frac{(1-\mu)I_\lambda^n g(\omega) + \lambda I_\lambda^{n+1} g(\omega)}{\omega}\right\} > \gamma, \ (\omega \in U), \tag{3.2}$$

where the function g is defined by (2.3).

By specializing the parameters in the class $H_\Sigma(n,\gamma,\lambda,\mu)$, we obtain the following known subclasses studied earlier by various researchers

1. $H_\Sigma(n,\gamma,0,\mu) \equiv H_\Sigma(n,\gamma,\mu)$ studied by Porwal and Darus [11].
2. $H_\Sigma(0,\gamma,0,\mu) \equiv H_\Sigma(\gamma,\mu)$ studied by Frasin and Aouf [6].
3. $H_\Sigma(0,\gamma,0,1) \equiv H_\Sigma(\gamma)$ studied by Srivastava *et al.* [14].

Theorem 3.2. *Let the function $f(z)$ given by* (1.1) *be in the class $H_\Sigma(n,\gamma,\lambda,\mu)$, $n \in N_0, 0 \leq \gamma < 1, \mu \geq 1, -1 < \lambda \leq 1$. Then*

$$|a_2| \leq \sqrt{\frac{2(1-\gamma)}{M_3}}, \tag{3.3}$$

$$|a_3| \leq \frac{2(1-\gamma)}{M_3}, \tag{3.4}$$

and

$$|a_4| \leq \frac{2(1-\gamma)}{M_4}. \tag{3.5}$$

Proof. It follows from (3.1) and (3.2) that there exists $p(z) \in P$ and $q(\omega) \in P$ such that

$$\frac{(1-\mu)I_\lambda^n f(z) + \mu I_\lambda^{n+1} f(z)}{z} = \gamma + (1-\gamma)p(z) \tag{3.6}$$

and

$$\frac{(1-\mu)I_\lambda^n g(\omega) + \mu I_\lambda^{n+1} g(\omega)}{\omega} = \gamma + (1-\gamma)q(\omega), \tag{3.7}$$

where $p(z)$ and $q(\omega)$ have the forms (2.9) and (2.10), respectively. Equating coefficients in (3.6) and (3.7) yields

$$M_2 a_2 = (1-\gamma)p_1, \tag{3.8}$$

$$M_3 a_3 = (1-\gamma)p_2, \tag{3.9}$$

$$M_4 a_4 = (1-\gamma)p_3, \tag{3.10}$$

$$-M_2 a_2 = (1-\gamma)q_1, \tag{3.11}$$

$$M_3(2a_2^2 - a_3) = (1-\gamma)q_2 \tag{3.12}$$

and

$$-M_4(5a_2^3 - 5a_2a_3 + a_4) = (1-\gamma)q_3. \tag{3.13}$$

From (3.8) and (3.11), we have

$$p_1 = -q_1. \tag{3.14}$$

Adding equation (3.9) and (3.14), we get

$$2M_3 a_2^2 = (1-\gamma)(p_2+q_2).$$

Putting $a_2 = \frac{(1-\gamma)p_1}{M_2}$ from (3.8), we have

$$p_1^2 = \frac{(p_2+q_2)M_2^2}{2M_3(1-\gamma)}. \tag{3.15}$$

By applying the inequalities $|p_2| \leq 2$ and $|q_2| \leq 2$, we get

$$|p_1| \leq \sqrt{\frac{2}{M_3(1-\gamma)}} M_2.$$

Therefore

$$|a_2| \leq \sqrt{\frac{2(1-\gamma)}{M_3}}.$$

To obtain a bound on $|a_3|$ we wish express in terms of p_1, p_2, q_1 and q_2. For this purpose subtracting (3.12) from (3.9), we get

$$2M_3(a_3 - a_2^2) = (1-\gamma)(p_2 - q_2). \tag{3.16}$$

Using (3.8), (3.15) and after simple calculation, we have

$$2M_3 a_3 = 2(1-\gamma)p_2.$$

Using $|p_2| \leq 2$ we have

$$|a_3| \leq \frac{2(1-\gamma)}{M_3}.$$

We shall next find an estimate on $|a_4|$. At first we shall derive a relation connecting p_1, p_2, p_3, q_1, q_2 and q_3. To this end, we first add the equations (3.10) and (3.13), we get

$$-M_4(5a_2^3 - 5a_2a_3) = (1-\gamma)(p_3+q_3).$$

Using (3.8), (3.16) and after simple calculation, we get

$$p_1(p_2 - q_2) = \frac{2M_2M_3(p_3+q_3)}{5M_4(1-\gamma)}. \tag{3.17}$$

We wish to express a_4 in terms of p_1, p_2, p_3, q_1, q_2 and q_3. Now subtracting (3.13) from (3.10), we get

$$M_4(2a_4 + 5a_2^3 - 5a_2a_3) = (1-\gamma)(p_3 - q_3).$$

Using (3.8), (3.16), (3.17) and after simple calculation

$$2M_4 a_4 = 2(1-\gamma)p_3.$$

Using inequality $|p_3| \leq 2$, we have

$$|a_4| \leq \frac{2(1-\gamma)}{M_4}. \qquad \square$$

Remark 3.3. If we put $\lambda = 0$ in Theorems 2.2 and 3.2, then our estimate on $|a_3|$ improves the corresponding results of Porwal and Darus [11].

Remark 3.4. If we put $n = 0$, $\lambda = 0$ in Theorems 2.2 and 3.2, then our estimate on $|a_3|$ improves the corresponding results due to Frasin and Aouf [6].

Remark 3.5. If we put $n = 0$, $\lambda = 0$, $\mu = 1$ in Theorems 2.2 and 3.2, then our estimate on $|a_3|$ improves the corresponding results due to Srivastava *et al.* [14].

Remark 3.6. Sharp estimates for the coefficients $|a_2|$, $|a_3|$ and other coefficients of functions belonging to the classes investigated in this paper are yet open problems. Indeed it would be of interest even to find estimates (not necessarily sharp) for $|a_n|,\ n \geq 5$.

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Coefficient estimates for a subclass of meromorphic bi-univalent functions defined by subordination

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Abstract. In this work, we use the Faber polynomial expansion by a new method to find upper bounds for $|b_n|$ coefficients for meromorphic bi-univalent functions class Σ' which is defined by subordination. Further, we generalize and improve some of the previously published results.

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1. Introduction and preliminaries

Let $\mathcal{A}$ be a class of analytic functions in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1.1}$$

Also denote by $\mathcal{S}$ the class of all functions in $\mathcal{A}$ which are univalent and normalized by the conditions $f(0) = 0 = f'(0) - 1$. It is well known that every function $f \in \mathcal{S}$ has an inverse f^{-1}, which is defined by

$$f^{-1}(f(z)) = z \quad (z \in \mathbb{U}) \quad \text{and} \quad f(f^{-1}(w)) = w \quad \left(|w| < r_0(f);\ r_0(f) \geq \frac{1}{4}\right).$$

So, if F is the inverse of a function $f \in \mathcal{S}$, then F has the following representation

$$F(w) = f^{-1}(w) = w + \sum_{n=2}^{\infty} \tilde{a}_n w^n \tag{1.2}$$

which is valid in some neighborhood of the origin.

In 1936, Robertson [23] introduced the concept of starlike functions of order α for $0 \leq \alpha < 1$. A function $f \in \mathcal{A}$ is said to be starlike of order α if

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > \alpha \qquad (z \in \mathbb{U}).$$

This class is denoted by $\mathcal{ST}(\alpha)$. Note that $\mathcal{ST}(0) = \mathcal{ST}$.

Definition 1.1. [8] For two functions f and g which are analytic in $\mathbb{U}$, we say that the function f is subordinate to g in $\mathbb{U}$, and write

$$f(z) \prec g(z) \qquad (z \in \mathbb{U}),$$

if there exists a Schwarz function ω, which is analytic in $\mathbb{U}$ with

$$\omega(0) = 0 \qquad \text{and} \qquad |\omega(z)| < 1 \quad (z \in \mathbb{U}),$$

such that

$$f(z) = g(\omega(z)) \quad (z \in \mathbb{U}).$$

In particular, if the function g is univalent in $\mathbb{U}$, then $f \prec g$ if and only if $f(0) = g(0)$ and $f(\mathbb{U}) \subseteq g(\mathbb{U})$.

Ma and Minda [20] have given a unified treatment of various subclass consisting of starlike functions by replacing the superordinate function $q(z) = \frac{1+z}{1-z}$ by a more general analytic function. For this purpose, they considered an analytic function φ with positive real part on $\mathbb{U}$, satisfying $\varphi(0) = 1$, $\varphi'(0) > 0$ and φ maps the unit disk $\mathbb{U}$ onto a region starlike with respect to 1, symmetric with respect to the real axis. The class $\mathcal{ST}(\varphi)$ of Ma-Minda starlike functions consists of functions $f \in \mathcal{S}$ satisfying

$$\frac{zf'(z)}{f(z)} \prec \varphi(z), \quad \text{for } z \in \mathbb{U}.$$

It is clear that for special choices of φ, this class envelop several well-known subclasses of univalent function as special cases. The idea of subordination was used for defining many of classes of functions studied in the Geometric Function Theory, for example see [7, 21].

Let Σ' denote the class of meromorphic univalent functions g defined in $\Delta := \{z \in \mathbb{C} : 1 < |z| < \infty\}$ of the form

$$g(z) = z + \sum_{n=0}^{\infty} \frac{b_n}{z^n}. \tag{1.3}$$

Since $g \in \Sigma'$ is univalent, it has an inverse $g^{-1} = G$ that satisfy

$$g^{-1}(g(z)) = z \quad (z \in \Delta) \quad \text{and} \quad g(g^{-1}(w)) = w \quad (M < |w| < \infty,\ M > 0).$$

Furthermore, the inverse function $g^{-1} = G$ has a series expansion of the form

$$G(w) = g^{-1}(w) = w + \sum_{n=0}^{\infty} \frac{\tilde{b}_n}{w^n} \qquad (M < |w| < \infty). \tag{1.4}$$

A simple calculation shows that the inverse function $g^{-1} = G$, is given by

$$G(w) = g^{-1}(w) = w - b_0 - \frac{b_1}{w} - \frac{b_2 + b_0 b_1}{w^2} + \cdots. \tag{1.5}$$

Let $(\mathcal{ST})'(\varphi)$ denote the class of functions $g \in \Sigma'$ which satisfy

$$\frac{1}{z}\frac{g'(1/z)}{g(1/z)} \prec \varphi(z), \quad \text{for } z \in \mathbb{U}.$$

The mapping $f(z) \mapsto g(z) := 1/f(1/z)$ establishes a one-to-one correspondence between functions in the classes $\mathcal{S}$ and Σ' and also between functions in the classes $\mathcal{ST}(\varphi)$ and $(\mathcal{ST})'(\varphi)$ because (see for more details [5])

$$\frac{zg'(z)}{g(z)} = \frac{z(1/f(1/z))'}{1/f(1/z)} = \frac{1}{z}\frac{f'(1/z)}{f(1/z)}, \quad \text{for } |z| > 1.$$

Noth that if $g \in (\mathcal{ST})'(\varphi)$, then there exists a unique function $f \in \mathcal{ST}(\varphi)$ such that $g(z) = 1/f(1/z)$. Also, it can be easily verified that $G(w) = 1/F(1/w)$, where $F(w)$ is the inverse of $f(z)$.

Analogous to the bi-univalent analytic functions, a function $g \in \Sigma'$ is said to be meromorphic bi-univalent if $g^{-1} \in \Sigma'$. Examples of the meromorphic bi-univalent functions are

$$z + \frac{1}{z}, \qquad z - 1, \qquad -\frac{1}{\log\left(1 - \frac{1}{z}\right)}.$$

Determination of the sharp coefficient estimates of inverse functions in various subclasses of the class of analytic and univalent functions is an interesting problem in geometric function theory. Schiffer [24] obtained the estimate $|b_2| \leq \frac{2}{3}$ for meromorphic univalent functions $g \in \Sigma'$ with $b_0 = 0$ and Duren [8] gave an elementary proof of the inequality $|b_n| \leq \frac{2}{n+1}$ on the coefficient of meromorphic univalent functions $g \in \Sigma'$ with $b_k = 0$ for $1 \leq k < \frac{n}{2}$. But the interest on coefficient estimates of the meromorphic univalent functions keep on by many researchers, see for example, [18, 19, 25, 26]. Several authors by using Faber polynomial expansions obtained coefficient estimates $|a_n|$ for classes meromorphic bi-univalent functions and bi-univalent functions, see for example [10, 12, 13, 14, 15, 16, 17, 28, 27]. First we recall some definitions and lemmas that used in this work.

Faber [9] introduced the Faber polynomials which play an important role in various areas of mathematical sciences, especially in geometric function theory. By using the Faber polynomial expansion of functions $g \in \Sigma'$ of the form (1.3), the coefficients of its inverse map $g^{-1} = G$ defined in (1.5) may be expressed, (see for details [2] and [3]),

$$G(w) = g^{-1}(w) = w - b_0 - \sum_{n\geq 1} \frac{1}{n} K_{n+1}^{n} \frac{1}{w^n}, \tag{1.6}$$

where

$$\begin{aligned} K_{n+1}^{n} &= nb_0^{n-1}b_1 + n(n-1)b_0^{n-2}b_2 + \frac{n(n-1)(n-2)}{2}b_0^{n-3}(b_3 + b_1^2) \\ &\quad + \frac{n(n-1)(n-2)(n-3)}{3!}b_0^{n-3}(b_4 + 3b_1b_2) + \sum_{j\geq 5} b_0^{n-j}V_j, \end{aligned}$$

such that V_j with $5 \leq j \leq n$ is a homogeneous polynomial in the variables $b_1, b_2, \cdots, b_n$, (see for details [3]).

Definition 1.2. [4] Let φ is an analytic function with positive real part in the unit disk $\mathbb{U}$, satisfying $\varphi(0) = 1,\ \varphi'(0) > 0$, φ maps the unit disk $\mathbb{U}$ onto a region starlike with respect to 1, symmetric with respect to the real axis. Such a function has series expansion of the form

$$\varphi(z) = 1 + B_1 z + B_2 z^2 + \cdots \qquad (B_1 > 0). \tag{1.7}$$

Lemma 1.3. [8] *Let $u(z)$ and $v(z)$ be two analytic functions in the unit disk $\mathbb{U}$ with*

$$u(0) = v(0) = 0 \qquad \text{and} \qquad \max\{|u(z)|\,,\ |v(z)|\} < 1.$$

We suppose also that

$$u(z) = \sum_{n=1}^{\infty} p_n z^n \qquad \text{and} \qquad v(z) = \sum_{n=1}^{\infty} q_n z^n \quad (z \in \mathbb{U}). \tag{1.8}$$

Then

$$|p_1| \le 1, \quad |p_2| \le 1 - |p_1|^2, \quad |q_1| \le 1, \quad |q_2| \le 1 - |q_1|^2. \tag{1.9}$$

Lemma 1.4. [1, 2] *Let the function $f \in \mathcal{A}$ be given by (1.1). Then for any $p \in \mathbb{Z}$, there are the polynomials K_n^p, such that*

$$(1 + a_2 z + a_3 z^2 + \cdots + a_k z^{k-1} + \cdots)^p = 1 + \sum_{n=1}^{\infty} K_n^p(a_2, a_3, \cdots, a_{n+1}) z^n,$$

where

$$K_n^p(a_2, \cdots, a_{n+1}) = p a_{n+1} + \frac{p(p-1)}{2} D_n^2 + \frac{p!}{(p-3)!3!} D_n^3 + \cdots + \frac{p!}{(p-n)!(n)!} D_n^n,$$

and

$$D_n^m(a_2, a_3, \cdots, a_n) = \sum_{n=2}^{\infty} \frac{m!(a_2)^{\mu_1} \cdots (a_n)^{\mu_n}}{\mu_1! \cdots \mu_n!}, \text{ for } m \in \mathbb{N} = \{1, 2, \ldots\} \text{ and } m \le n,$$

the sum is taken over all nonnegative integers $\mu_1, ..., \mu_n$ satisfying

$$\begin{cases} \mu_1 + \mu_2 + \cdots + \mu_n = m, \\ \mu_1 + 2\mu_2 + \cdots + n\mu_n = n. \end{cases}$$

It is clear that $D_n^n(a_2, a_3, \cdots, a_n) = a_2^n$. In particular,

$$\begin{aligned} K_n^1 &= a_{n+1}, \qquad K_1^2 = 2a_2, \qquad K_2^2 = 2a_3 + a_2^2, \\ K_3^2 &= 2a_4 + 2a_2 a_3, \qquad K_4^2 = 2a_5 + 2a_2 a_4 + a_3^2. \end{aligned}$$

Lemma 1.5. [2, 3] and [6, page 52] *Let the function $g \in \Sigma'$ be given by (1.3). Then we have the following expansion*

$$\frac{zg'(z)}{g(z)} = 1 + \sum_{n=0}^{\infty} F_{n+1}(b_0, b_1, \cdots b_n) \frac{1}{z^{n+1}},$$

where

$$F_{n+1}(b_0, b_1, \cdots b_n) = \sum_{i_1 + 2i_2 + \cdots + (n+1)i_{n+1} = n+1} A(i_1, i_2, \cdots, i_{n+1})(b_0^{i_1} b_1^{i_2} \cdots b_n^{i_{n+1}}),$$

and

$$A(i_1, i_2, \cdots, i_{n+1}) := (-1)^{(n+1)+2i_1+\cdots+(n+2)i_{n+1}} \frac{(i_1 + i_2 + \cdots + i_{n+1} - 1)!(n+1)}{i_1!i_2! \cdots i_{n+1}!}.$$

The first four terms of the Faber polynomials F_n are given by

$$F_1 = -b_0, \qquad F_2 = b_0^2 - 2b_1, \qquad F_3 = -b_0^3 + 3b_1b_0 - 3b_2,$$
$$F_4 = b_0^4 - 4b_0^2b_1 + 4b_0b_2 + 2b_1^2 - 4b_3.$$

In this work, by using the Faber polynomial expansion we find upper bounds for $|b_n|$ coefficients by a new method for meromorphic bi-univalent functions class Σ' which is defined by subordination. Further, we generalize and improve some of the previously published results.

2. Main results

In this section, first we obtain estimates of coefficients $|b_n|$ of meromorphic bi-univalent functions in the class $(\mathcal{ST})'(\varphi)$. Next we obtain an improvement of the bounds $|b_0|$ and $|b_1|$ for special choices of φ.

Theorem 2.1. *Let the function g given by* (1.3) *and its inverse map $g^{-1} = G$ given by* (1.4) *be in the class $(\mathcal{ST})'(\varphi)$, where φ is given by Definition* 1.2*. If $b_k = 0$ for $0 \le k \le n-1$, then*

$$|b_n| \le \frac{B_1}{n+1}.$$

Proof. From $g \in (\mathcal{ST})'(\varphi)$, we obtain

$$\frac{1}{z}\frac{g'(1/z)}{g(1/z)} = \frac{1 - b_1z^2 - 2b_2z^3 - \cdots}{1 + b_0z + b_1z^2 + \cdots} = 1 - b_0z + (b_0^2 - 2b_1)z^2 + \cdots. \tag{2.1}$$

Similar to Lemma 1.5, for function $g \in (\mathcal{ST})'(\varphi)$ and for its inverse map $g^{-1} = G$, we have

$$\frac{1}{z}\frac{g'(1/z)}{g(1/z)} = 1 + \sum_{n=0}^{\infty} F_{n+1}(b_0, b_1, \cdots b_n)z^{n+1}, \tag{2.2}$$

$$\frac{1}{w}\frac{G'(1/w)}{G(1/w)} = 1 + \sum_{n=0}^{\infty} F_{n+1}(\tilde{b}_0, \tilde{b}_1, \cdots \tilde{b}_n)w^{n+1}, \tag{2.3}$$

respectively, where $\tilde{b}_0 = -b_0$, $\tilde{b}_n = \frac{1}{n}K_{n+1}^n$.

On the other hand, since $g,\ G \in (\mathcal{ST})'(\varphi)$, by the Definition 1.1, there exist two Schwarz functions $u, v : \mathbb{U} \to \mathbb{U}$ where $u,\ v$ are given by (1.8), so that

$$\frac{1}{z}\frac{g'(1/z)}{g(1/z)} = \varphi(u(z)) = 1 + \sum_{n=1}^{\infty}\sum_{k=1}^{n} B_kD_n^k(p_1, p_2, \cdots, p_n)z^n, \tag{2.4}$$

and

$$\frac{1}{w}\frac{G'(1/w)}{G(1/w)} = \varphi(v(w)) = 1 + \sum_{n=1}^{\infty}\sum_{k=1}^{n} B_kD_n^k(q_1, q_2, \cdots, q_n)w^n. \tag{2.5}$$

Comparing the corresponding coefficients of (2.2) and (2.4), we get that

$$F_{n+1}(b_0, b_1, \cdots b_n) = \sum_{k=1}^{n+1} B_k D_{n+1}^k(p_1, p_2, \cdots, p_{n+1}). \tag{2.6}$$

Similarly, by comparing the corresponding coefficients of (2.3) and (2.5), we get that

$$F_{n+1}(\tilde{b}_0, \tilde{b}_1, \cdots \tilde{b}_n) = \sum_{k=1}^{n+1} B_k D_{n+1}^k(q_1, q_2, \cdots, q_{n+1}). \tag{2.7}$$

Note that $b_k = 0$ for $0 \le k \le n-1$, yields $\tilde{b}_n = -b_n$ and hence from (2.6) and (2.7), respectively, we get

$$-(n+1)b_n = B_1 p_{n+1},$$

and

$$-[-(n+1)]b_n = B_1 q_{n+1}.$$

By solving either of the above two equations for b_n and applying $|p_{n+1}| \le 1, |q_{n+1}| \le 1$, we obtain

$$|b_n| \le \frac{B_1}{n+1},$$

this completes the proof. □

Corollary 2.2. *Let the function g given by* (1.3) *and its inverse map $g^{-1} = G$ given by* (1.4) *be in the class $(\mathcal{ST})'\left(\left(\frac{1+z}{1-z}\right)^{\alpha}\right)$. If $b_k = 0$ for $0 \le k \le n-1$, then*

$$|b_n| \le \frac{2\alpha}{n+1} \qquad (0 < \alpha \le 1).$$

Corollary 2.3. [13] *Let the function g given by* (1.3) *and its inverse map $g^{-1} = G$ given by* (1.4) *be in the class $(\mathcal{ST})'\left(\frac{1+(1-2\beta)z}{1-z}\right)$. If $b_k = 0$ for $0 \le k \le n-1$, then*

$$|b_n| \le \frac{2(1-\beta)}{n+1} \qquad (0 \le \beta < 1).$$

Corollary 2.4. *Let the function f given by* (1.1) *and its inverse map $f^{-1} = F$ given by* (1.2) *be in the class $\mathcal{ST}(\varphi)$. If $a_k = 0$ for $2 \le k \le n-1$, then*

$$|a_n| \le \frac{B_1}{n-1}.$$

Proof. Setting $f(1/z) := 1/g(z)$ and $F(1/w) = 1/G(w)$ in Theorem 2.1 we obtain the result and this completes the proof. □

Corollary 2.5. ([14, Theorem 2.1]) *Let the function f given by* (1.1) *and its inverse map $f^{-1} = F$ given by* (1.2) *be in the class $\mathcal{ST}\left(\frac{1+Az}{1+Bz}\right)$, where A and B are real numbers so that $-1 \le B < A \le 1$. If $a_k = 0$ for $2 \le k \le n-1$, then*

$$|a_n| \le \frac{A-B}{n-1}.$$

Theorem 2.6. *Let the function g given by* (1.3) *and its inverse map $g^{-1} = G$ given by* (1.4) *be in the class $(\mathcal{ST})'(\varphi)$, where φ is given by Definition* 1.2. *Then*

$$|b_0| \leq \frac{B_1\sqrt{B_1}}{\sqrt{|B_1^2 - B_2| + B_1}} \tag{2.8}$$

and

$$|b_1| \leq \frac{B_1}{2}. \tag{2.9}$$

Proof. The equations (2.6) and (2.7) for $n = 0$ and $n = 1$, respectively, imply

$$-b_0 = B_1 p_1, \tag{2.10}$$
$$b_0^2 - 2b_1 = B_1 p_2 + B_2 p_1^2, \tag{2.11}$$
$$b_0 = B_1 q_1, \tag{2.12}$$
$$b_0^2 + 2b_1 = B_1 q_2 + B_2 q_1^2. \tag{2.13}$$

From (2.10) and (2.12), we have

$$p_1 = -q_1 \tag{2.14}$$

and

$$2b_0^2 = B_1^2\left(p_1^2 + q_1^2\right). \tag{2.15}$$

Also by adding (2.11) and (2.13), and considering (2.15) we have

$$\begin{aligned} 2b_0^2 &= B_1\left(p_2 + q_2\right) + B_2\left(p_1^2 + q_1^2\right) \\ &= B_1\left(p_2 + q_2\right) + \frac{2B_2 b_0^2}{B_1^2}. \end{aligned}$$

So we obtain

$$b_0^2 = \frac{B_1^3\left(p_2 + q_2\right)}{2\left(B_1^2 - B_2\right)}.$$

By (1.9), (2.10), (2.14) and the above equality give

$$\begin{aligned} |b_0|^2 &\leq \frac{B_1^3\left(1 - |p_1|^2\right)}{|B_1^2 - B_2|} \\ &\leq \frac{B_1^3}{|B_1^2 - B_2|}\left(1 - \frac{|b_0|^2}{B_1^2}\right). \end{aligned}$$

Therefore we obtain

$$|b_0|^2 \leq \frac{B_1^3}{|B_1^2 - B_2| + B_1}, \tag{2.16}$$

which is the desired estimate on the coefficient $|b_0|$ as asserted in (2.8).
On the other hand, by subtracting (2.13) from (2.11) and considering (2.14) we get

$$-4b_1 = B_1\left(p_2 - q_2\right).$$

Taking the absolute values and considering (1.9) we obtain the desired estimate on the coefficient $|b_1|$ as asserted in (2.9). This completes the proof. □

Theorem 2.7. *Let the function* g *given by* (1.3) *and its inverse map* $g^{-1} = G$ *given by* (1.4) *be in the class* $(\mathcal{ST})'\left(\left(\frac{1+z}{1-z}\right)^{\alpha}\right)$. *Then*

$$|b_0| \leq \frac{2\alpha}{\sqrt{\alpha+1}},$$

and

$$|b_1| \leq \alpha.$$

Remark 2.8. Theorem 2.7 is an refinement of estimate for $|b_0|$ obtained by Panigrahi [22, Corollary 2.3]. Also, for $|b_1|$ if $\frac{1}{\sqrt{5}} < \alpha \leq 1$ and $|b_0|$, Theorem 2.7 is an refinement of estimates obtained by Halim *et al.* [11, Theorem 2].

Theorem 2.9. *Let the function* g *given by* (1.3) *and its inverse map* $g^{-1} = G$ *given by* (1.4) *be in the class* $(\mathcal{ST})'\left(\frac{1+(1-2\beta)z}{1-z}\right)$. *Then*

$$|b_0| \leq \begin{cases} \sqrt{2(1-\beta)} & , \quad 0 \leq \beta \leq \frac{1}{2} \\ \frac{\sqrt{2}(1-\beta)}{\sqrt{\beta}} & , \quad \frac{1}{2} \leq \beta < 1 \end{cases}$$

and

$$|b_1| \leq 1-\beta.$$

Remark 2.10. Theorem 2.9 is an improvement of the estimates obtained by Panigrahi [22, Corollary 3.3] and also obtained by Halim et al. [11, Theorem 1].

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Some properties of a linear operator involving generalized Mittag-Leffler function

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Abstract. This paper introduces a new class $T^{\gamma}_{\alpha,\beta,k}(\eta)$ of analytic functions which is defined by means of a linear operator involving generalized Mittag-Leffler function $\mathcal{H}^{\gamma}_{\alpha,\beta,k}(f)$. The results investigated in this paper include, an inclusion relation for functions in the class $T^{\gamma}_{\alpha,\beta,k}(\eta)$ and also some subordination results of the linear operator $\mathcal{H}^{\gamma}_{\alpha,\beta,k}(f)$. Several consequences of our results are also pointed out.

Mathematics Subject Classification (2010): 33E12, 30C45.

Keywords: Analytic functions, univalent functions, Mittag-Leffler function, differential subordination, convex function.

1. Introduction

Let $\mathcal{A}$ denote the class of the normalized functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n,$$

which are analytic in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$. Further, let f and g be analytic functions in $\mathbb{U}$, then we say that f is subordinate to g, written $f \prec g$ or $f(z) \prec g(z)$, if there exists an analytic function w on $\mathbb{U}$ such that $w(0) = 0$, $|w(z)| < 1$ and $f(z) = g(w(z))$ for all $z \in \mathbb{U}$. In particular, if g is univalent in $\mathbb{U}$, then we have

$$f(z) \prec g(z) \Leftrightarrow f(0) = g(0) \text{ and } f(\mathbb{U}) \subset g(\mathbb{U}).$$

Let $E_\alpha(z)$ be the Mittag-Leffler function [11] defined by

$$E_\alpha(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + 1)}, \quad (z, \alpha \in \mathbb{C}; \mathrm{Re}(\alpha) > 0\ ). \tag{1.1}$$

A more general function $E_{\alpha,\beta}$ generalizing $E_\alpha(z)$ was introduced by Wiman [14] and defined by

$$E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n+\beta)}, \quad (z, \alpha, \beta \in \mathbb{C}; \mathrm{Re}(\alpha) > 0\ ). \tag{1.2}$$

Moreover, Srivastava and Tomovski [13] introduced the function $E_{\alpha,\beta}^{\gamma,k}(z)$ as

$$E_{\alpha,\beta}^{\gamma,k}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_{nk} z^n}{\Gamma(\alpha n+\beta) n!}, (\alpha, \beta, \gamma \in \mathbb{C}; \mathrm{Re}(\alpha) > \max\{0, \mathrm{Re}(k)-1\}; \mathrm{Re}(k) > 0),$$

where $(\gamma)_n$ is Pochhammer symbol (or the shifted factorial, since $(1)_n = n!$) is given in term of the Gamma functions can be written as

$$(\gamma)_n = \frac{\Gamma(\gamma+n)}{\Gamma(\gamma)} = \begin{cases} 1, & \text{if } n = 0; \\ \gamma(\gamma+1)...(\gamma+n-1), & \text{if } n \in \mathbb{N}. \end{cases} \tag{1.3}$$

The Mittag-Leffler function arises naturally in the solution of fractional order differential and integral equations, and especially in the investigations of fractional generalization of kinetic equation, random walks, Lévy flights, super-diffusive transport and in the study of complex systems. Several properties of Mittag-Leffler function and generalized Mittag-Leffler function can be found e.g. in [2, 3, 4, 6, 7, 8, 9, 11, 12, 13].

In [1], Attiya defined the operator $\mathcal{H}_{\alpha,\beta,k}^{\gamma}(f) : \mathcal{A} \to \mathcal{A}$ by

$$\mathcal{H}_{\alpha,\beta,k}^{\gamma}(f)(z) = Q_{\alpha,\beta,k}^{\gamma}(z) * f(z), \qquad (z \in \mathbb{U}),$$

where

$$\begin{aligned} &Q_{\alpha,\beta,k}^{\gamma}(z) = \frac{\Gamma(\alpha+\beta)}{(\gamma)_k}\left(E_{\alpha,\beta}^{\gamma,k}(z) - \frac{1}{\Gamma(\beta)}\right), \qquad (z \in \mathbb{U}), \\ &(\alpha, \beta, \gamma \in \mathbb{C}; \mathrm{Re}(\alpha) > \max\{0, \mathrm{Re}(k)-1\}; \mathrm{Re}(k) > 0; \\ &\mathrm{Re}(\alpha) = 0 \text{ when } \mathrm{Re}(k) = 1 \text{ with } \beta \neq 0), \end{aligned}$$

and the symbol $(*)$ denotes the Hadamard product (or convolution).
We note that,

$$\mathcal{H}_{\alpha,\beta,k}^{\gamma}(f)(z) = z + \sum_{n=2}^{\infty} \frac{\Gamma(\gamma+nk)\Gamma(\alpha+\beta)}{\Gamma(\gamma+k)\Gamma(\beta+\alpha n) n!} a_n z^n. \tag{1.4}$$

It can be easily verified from (1.4) that

$$z\left(\mathcal{H}_{\alpha,\beta,k}^{\gamma}(f)(z)\right)' = \left(\frac{\gamma+k}{k}\right)(\mathcal{H}_{\alpha,\beta,k}^{\gamma+1}(f)(z)) - \frac{\gamma}{k}(\mathcal{H}_{\alpha,\beta,k}^{\gamma}(f)(z)). \tag{1.5}$$

Also we have

$$\mathcal{H}_{0,\beta,1}^{1}(f)(z) = f(z), \mathcal{H}_{0,\beta,1}^{2}(f)(z) = \frac{1}{2}\left(f(z) + zf'(z)\right) \text{ and } \mathcal{H}_{0,\beta,1}^{0}(f)(z) = \int_0^z \frac{1}{t} f(t)dt.$$

Definition 1.1. We say that the function $f \in \mathcal{A}$ is in the class $T^{\gamma}_{\alpha,\beta,k}(\eta), \eta \in [0,1)$, if f satisfies the condition

$$\mathrm{Re}\left[\mathcal{H}^{\gamma}_{\alpha,\beta,k}(f)(z)\right]' > \eta, \qquad (z \in \mathbb{U}). \tag{1.6}$$

The object of this paper is to investigate an inclusion relation for functions in the class $T^{\gamma}_{\alpha,\beta,k}(\eta)$ and obtain some subordination results for functions defined by the linear operator $\mathcal{H}^{\gamma}_{\alpha,\beta,k}(f)$. Several consequences of our results are also discussed.

The following results will be required in our investigation.

Lemma 1.2. ([5]) *If $p(z) = 1 + p_1 z + p_2 z^2 + \cdots$ is analytic in $\mathbb{U}$ and $h(z)$ is convex function in $\mathbb{U}$ with $h(0) = 1$ and μ is a complex constant such that* $\mathrm{Re}\mu > 0$, *then*

$$p(z) + \frac{zp'(z)}{\mu} \prec h(z), \tag{1.7}$$

implies

$$p(z) \prec q(z) \prec h(z),$$

where

$$q(z) = \frac{\mu}{z^{\mu}} \int_0^z h(t) t^{\mu-1} dt,$$

and $q(z)$ is the best dominant.

Lemma 1.3. ([10]) *Let q be a convex function in $\mathbb{U}$ and let*

$$h(z) = q(z) + \alpha z q'(z),$$

where $\alpha > 0$. If

$$p(z) = q(0) + p_1 z + \cdots$$

and

$$p(z) + \alpha z p'(z) \prec h(z),$$

then

$$p(z) \prec q(z),$$

and this result is sharp.

2. Inclusion relation

We begin by showing the following inclusion relation.

Theorem 2.1. *If $\eta \in [0,1)$, then*

$$T^{\gamma+1}_{\alpha,\beta,k}(\eta) \subset T^{\gamma}_{\alpha,\beta,k}(\delta), \tag{2.1}$$

where

$$\delta = \delta(\eta,\gamma,k) = 2\eta - 1 + \frac{2(1-\eta)(\gamma+k)}{k}\mathbf{B}\left(\frac{\gamma+k}{k}\right), \tag{2.2}$$

B *being the Beta function defined by*

$$\mathbf{B}(x) = \int_0^1 \frac{t^{x-1}}{t+1} dt. \tag{2.3}$$

Proof. Let $f \in T^{\gamma+1}_{\alpha,\beta,k}(\eta)$ and define the function $p(z)$ by

$$p(z) = \left(\mathcal{H}^{\gamma}_{\alpha,\beta,k}(f)(z)\right)'. \tag{2.4}$$

Making use the identity (1.5), we get

$$\left(\mathcal{H}^{\gamma+1}_{\alpha,\beta,k}(f)(z)\right)' = p(z) + \frac{k}{\gamma+k} z p'(z), \qquad (z \in \mathbb{U}). \tag{2.5}$$

Since $f \in T^{\gamma+1}_{\alpha,\beta,k}(\eta)$, from Definition 1.1 we have

$$\operatorname{Re}\left(\mathcal{H}^{\gamma+1}_{\alpha,\beta,k}(f)(z)\right)' > \eta, \qquad (z \in \mathbb{U}).$$

Using (2.5) we get

$$\operatorname{Re}\left(p(z) + \frac{k}{\gamma+k} z p'(z)\right) > \eta,$$

which is equivalent to

$$p(z) + \frac{k}{\gamma+k} z p'(z) \prec \frac{1+(2\eta-1)z}{1+z} \equiv h(z).$$

By using Lemma 1.2, with $\mu = \frac{\gamma+k}{k}$ we have

$$p(z) \prec q(z) \prec h(z),$$

where

$$\begin{aligned}
q(z) &= \frac{\gamma+k}{k z^{\frac{\gamma+k}{k}}} \int_0^z \frac{1+(2\eta-1)t}{1+t} t^{\frac{\gamma+k}{k}-1} dt \\
&= \frac{\gamma+k}{k z^{\frac{\gamma+k}{k}}} \int_0^z \left[2\eta - 1 + 2(1-\eta)\right] \frac{1}{1+t} t^{\frac{\gamma+k}{k}-1} dt \\
&= \frac{\gamma+k}{k z^{\frac{\gamma+k}{k}}} \int_0^z (2\eta-1) t^{\frac{\gamma+k}{k}-1} dt + \frac{2(1-\eta)(\gamma+k)}{k z^{\frac{\gamma+k}{k}}} \int_0^z \frac{t^{\frac{\gamma+k}{k}-1}}{1+t} dt \\
&= 2\eta - 1 + \frac{2(1-\eta)(\gamma+k)}{k z^{\frac{\gamma+k}{k}}} \int_0^z \frac{t^{\frac{\gamma+k}{k}-1}}{1+t} dt.
\end{aligned}$$

The function q is convex and is the best dominant.
Since $p(z) \prec q(z)$, we get

$$\operatorname{Re}\left[\mathcal{H}^{\gamma}_{\alpha,\beta,k}(f)(z)\right]' > q(1) = \delta, \tag{2.6}$$

where

$$\delta = \delta(\eta, \gamma, k) = 2\eta - 1 + \frac{2(1-\eta)(\gamma+k)}{k}\mathbf{B}\left(\frac{\gamma+k}{k}\right).$$

From (2.6) we deduce that $T^{\gamma+1}_{\alpha,\beta,k}(\eta) \subset T^{\gamma}_{\alpha,\beta,k}(\delta)$. □

3. Subordination results

With the help of Lemma 1.3, we obtain the following result.

Theorem 3.1. *Let $q(z)$ be convex univalent in $\mathbb{U}$ with $q(0) = 1$ and let h be a function such that*

$$h(z) = q(z) + \frac{k}{\gamma+k} zq'(z). \tag{3.1}$$

If $f \in \mathcal{A}$ and verifies the differential subordination

$$\left(\mathcal{H}^{\gamma+1}_{\alpha,\beta,k}(f)(z)\right)' \prec h(z), \tag{3.2}$$

then

$$\left(\mathcal{H}^{\gamma}_{\alpha,\beta,k}(f)(z)\right)' \prec q(z), \tag{3.3}$$

and the result is sharp.

Proof. From (2.5) and (3.2) we obtain

$$p(z) + \frac{k}{\gamma+k} zp'(z) \prec q(z) + \frac{k}{\gamma+k} zq'(z) \equiv h(z),$$

then, by using Lemma 1.3 we get

$$p(z) \prec q(z),$$

that is,

$$\left(\mathcal{H}^{\gamma}_{\alpha,\beta,k}(f)(z)\right)' \prec q(z), \qquad (z \in \mathbb{U}),$$

and this result is sharp. □

Theorem 3.2. *Let $h \in \mathcal{A}$ with $h(0) = 1$ and $h'(0) \neq 0$, which verifies the inequality*

$$\operatorname{Re}\left[1 + \frac{zh''(z)}{h'(z)}\right] > -\frac{1}{2}, \qquad (z \in \mathbb{U}). \tag{3.4}$$

If $f \in \mathcal{A}$ and verifies the differential subordination

$$\left(\mathcal{H}^{\gamma+1}_{\alpha,\beta,k}(f)(z)\right)' \prec h(z), \tag{3.5}$$

then

$$\left(\mathcal{H}^{\gamma}_{\alpha,\beta,k}(f)(z)\right)' \prec q(z), \tag{3.6}$$

where

$$q(z) = \frac{\gamma+k}{kz^{\frac{\gamma+k}{k}}} \int_0^z h(t) t^{\frac{\gamma+k}{k}-1} dt.$$

The function q is convex and is the best dominant.

Proof. If we let

$$p(z) = \left(\mathcal{H}^{\gamma}_{\alpha,\beta,k}(f)(z)\right)',$$

and using the identity (1.5), we obtain

$$\left(\mathcal{H}^{\gamma+1}_{\alpha,\beta,k}(f)(z)\right)' = p(z) + \frac{k}{\gamma+k} zp'(z), \qquad (z \in \mathbb{U}).$$

Therefore, (3.5) becomes

$$p(z) + \frac{k}{\gamma+k} zp'(z) \prec h(z).$$

By using Lemma 1.2, we get

$$p(z) \prec q(z) = \frac{\gamma+k}{kz^{\frac{\gamma+k}{k}}} \int_0^z h(t) t^{\frac{\gamma+k}{k}-1} dt,$$

that is,

$$\left(\mathcal{H}^{\gamma}_{\alpha,\beta,k}(f)(z)\right)' \prec q(z), \qquad (z \in \mathbb{U}).$$ □

Theorem 3.3. *Let $q(z)$ be convex univalent in $\mathbb{U}$ with $q(0) = 1$. And let h be a function such that*

$$h(z) = q(z) + zq'(z), \qquad (z \in \mathbb{U}). \tag{3.7}$$

If $f \in A$ and verifies the differential subordination

$$\left(\mathcal{H}^{\gamma}_{\alpha,\beta,k}(f)(z)\right)' \prec h(z), \tag{3.8}$$

then

$$\frac{\mathcal{H}^{\gamma}_{\alpha,\beta,k}(f)(z)}{z} \prec q(z), \tag{3.9}$$

and the result is sharp.

Proof. Let the function $p(z)$ be defined by

$$p(z) = \frac{\mathcal{H}^{\gamma}_{\alpha,\beta,k}(f)(z)}{z}. \tag{3.10}$$

Then, by differentiating (3.10), we get

$$\left(\mathcal{H}^{\gamma}_{\alpha,\beta,k}(f)(z)\right)' = p(z) + zp'(z), \qquad (z \in \mathbb{U}). \tag{3.11}$$

Thus (3.8) becomes

$$p(z) + zp'(z) \prec q(z) + zq'(z) \equiv h(z),$$

and from Lemma 1.3 we get (3.9). □

Theorem 3.4. *Let $h \in \mathcal{A}$ with $h(0) = 1$ and $h'(0) \neq 0$, which verifies the inequality (3.4). If $f \in \mathcal{A}$ and verifies the differential subordination*

$$\left(\mathcal{H}^{\gamma}_{\alpha,\beta,k}(f)(z)\right)' \prec h(z), \qquad (z \in \mathbb{U}), \tag{3.12}$$

then

$$\frac{\mathcal{H}^{\gamma}_{\alpha,\beta,k}(f)(z)}{z} \prec q(z), \qquad (z \in \mathbb{U}, z \neq 0), \tag{3.13}$$

where

$$q(z) = \frac{1}{z}\int_0^z h(t)dt.$$

The function q is convex and is the best dominant.

Proof. Let the function $p(z)$ be defined as in (3.10). Then from (3.11) and (3.12),we have

$$p(z) + zp'(z) \prec h(z).$$

By using Lemma 1.2, we get

$$p(z) \prec q(z) = \frac{1}{z}\int_0^z h(t)dt,$$

and q is convex and is the best dominant. □

If we set $\gamma = 1$, $\alpha = 0$ and $k = 1$, in Theorems 3.1-3.4, we immediately have the following special cases.

Corollary 3.5. *Let $q(z)$ be convex univalent in $\mathbb{U}$ with $q(0) = 1$ and let h be a function such that*

$$h(z) = q(z) + \frac{1}{2}zq'(z). \tag{3.14}$$

If $f \in \mathcal{A}$ and verifies the differential subordination

$$f'(z) + \frac{1}{2}zf''(z) \prec h(z), \tag{3.15}$$

then

$$f'(z) \prec q(z), \tag{3.16}$$

and the result is sharp.

Corollary 3.6. *Let $h \in \mathcal{A}$ with $h(0) = 1$ and $h'(0) \neq 0$, which verifies the inequality (3.4). If $f \in \mathcal{A}$ and verifies the differential subordination*

$$f'(z) + \frac{1}{2}zf''(z) \prec h(z), \tag{3.17}$$

then

$$f'(z) \prec q(z), \tag{3.18}$$

where

$$q(z) = \frac{2}{z^2}\int_0^z h(t)tdt.$$

The function q is convex and is the best dominant.

Corollary 3.7. *Let $q(z)$ be convex univalent in $\mathbb{U}$ with $q(0) = 1$ and let h be a function such that*

$$h(z) = q(z) + zq'(z), \qquad (z \in \mathbb{U}). \tag{3.19}$$

If $f \in \mathcal{A}$ and verifies the differential subordination

$$f'(z) \prec h(z), \tag{3.20}$$

then

$$\frac{f(z)}{z} \prec q(z), \tag{3.21}$$

and the result is sharp.

Corollary 3.8. *Let $h \in \mathcal{A}$ with $h(0) = 1$ and $h'(0) \neq 0$, which verifies the inequality (3.4). If $f \in \mathcal{A}$ and verifies the differential subordination*

$$f'(z) \prec h(z), \qquad (z \in \mathbb{U}), \tag{3.22}$$

then

$$\frac{f(z)}{z} \prec q(z), \qquad (z \in \mathbb{U}, z \neq 0), \tag{3.23}$$

where

$$q(z) = \frac{1}{z}\int_0^z h(t)dt.$$

The function q is convex and is the best dominant.

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The critical point of a sigmoidal curve

Ayse Humeyra Bilge and Yunus Ozdemir

Abstract. Let $y(t)$ be a monotone increasing curve with $\lim_{t\to\pm\infty} y^{(n)}(t) = 0$ for all n and let t_n be the location of the global extremum of the nth derivative $y^{(n)}(t)$. Under certain assumptions on the Fourier and Hilbert transforms of $y(t)$, we prove that the sequence $\{t_n\}$ is convergent. This implies in particular a preferred choice of the origin of the time axis and an intrinsic definition of the even and odd components of a sigmoidal function. In the context of phase transitions, the limit point has the interpretation of the critical point of the transition as discussed in previous work [3].

Mathematics Subject Classification (2010): 34A99.

Keywords: Sigmoidal curve, critical point, Fourier transform, Hilbert transform.

1. Introduction

A sigmoidal function $y(t)$ is a monotone increasing function with horizontal asymptotes as $t \to \pm\infty$. Such functions occur in probability theory and in a variety of applications that represent the passage between two stable states, in particular in phase transitions.

In previous work [5] we have modeled the sol-gel transition of the polyacrylamide-sodium alginate composite in terms of the Susceptible-Infected-Removed (SIR) epidemic model that represents the spread of an epidemic in a closed society. This model was shown to be in good agreement with the aforementioned gelation phenomena and we tried to take advantage of an exact mathematical model to search for the exact instant of onset of the sol-gel transition [3, 4]. We computed higher derivatives of the sigmoidal curve representing the phase transition, up to orders 20 to 30 and we observed that the points where their reach their absolute extrema seemed to have a limit point, as shown in Fig. 1.

This point agreed qualitatively with the gel point of the polyacrylamide-sodium alginate composite and we proposed to define the "critical point of a sigmoidal curve"

$y(t)$ as the limit of the sequence of points where the higher derivatives $y^{(n)}(t)$ reach their absolute extreme values [3].

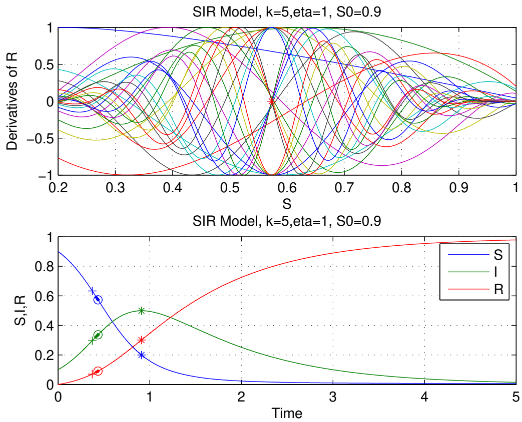


FIGURE 1. The Susceptible-Infected-Removed dynamical system $S' = -kSI$, $I' = kSI - \eta I$, $R' = \eta I$, as a model for the gelation phenomena. (a) The first 24 derivatives of $R(t)$ normalized to 1 for $k = 5$, $\eta = 1$ and $S_0 = 0.9$ are plotted against $S(t)$, which is a monotone decreasing function of t. The phase transition point is indicated by $(*)$. (b) The time domain plots of the solution curves S, I, R. The phase transition point t_c indicated by (o) is located between the maximum of I, t_m denoted by $(*)$ and the inflection point of I, t_a denoted by $(+)$. The derivatives of the sigmoidal function $R(t)$ are plotted versus $S(t)$, which is a monotone function of time.

Referring to Fig. 1(a), we first note that there *seems* to be a gap in the zero set of the derivatives; that is, the normalized absolute values of the odd derivatives agglomerate quickly near the point shown by $(*)$, while the absolute extrema of the even derivatives approach this point much more slowly. On the other hand, the work of Polya [9] on the zeros of the set of derivatives of an analytic function applied to a smooth sigmoidal curve implies that there should be no gap in the set of zeros. Due to the computational limitations of the SIR system, we worked with the logistic growth function and we could in fact see that the gap closes when derivatives up to order 200 are included, as shown in Fig. 2. Nevertheless, despite the strong evidence for the existence of a critical point, we were unable to prove even the simplest observed fact that the *absolute* extreme values of the odd derivatives of the logistic growth curve are located at $t = 0$.

The aim of the present work is the study of the existence and the location of the "critical point of a sigmoidal curve" as described Section 2, Definition 1. In earlier stages of this study, the existence of a critical point was thought to be a peculiarity of the SIR system, but later on after working with numerous examples we came up

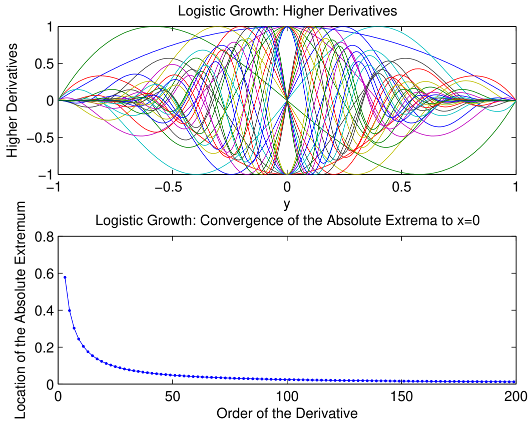


FIGURE 2. The normalized derivatives of the logistic growth $y' = (1 - y^2)$ (a) The plot of derivatives up to order 30 with respect to y. All odd derivatives reach their absolute extremum at the origin but there seems to be a gap in the zero set. (b) The distance of the first zero of the nth derivative to the origin. The apparent gap seems to close after $n = 200$.

with the belief that it is a consequence of the general properties of sigmoidal curves. Intuitively, the critical point of an odd sigmoidal curve is expected to be $t = 0$, but there was no guess on where the critical point of curves with no symmetry would be located.

The main results of the paper are presented in Section 3. In Section 3.1, we prove the existence of critical point for the general case (Proposition 2), then in Section 3.2, we consider odd sigmoidal curves and use milder assumptions to prove that $t = 0$ is the critical point. For the case with no symmetry, the location of the critical point is crucially related to an appropriate choice of the origin of the time axis, in such a way that the phase of the Fourier transform of the first derivative is asymptotically constant. This leads to an intrinsic choice of origin, hence an intrinsic definition of the even and odd components, provided that the relevant assumptions are satisfied. With this choice of origin, $t = 0$ turns out to be the critical point, provided that it exists.

The plan of the paper is as follows. The definitions and theorems necessary for subsequent derivations are presented in Section 2. In Section 3.1, we first prove the existence of the critical point for the general case, then, in Section 3.2, we give alternative proofs for sigmoidal curves with symmetry, using weaker assumptions. Basic properties of the Fourier and Hilbert transforms are given in the Appendix.

2. Preliminaries

In Section 2.1, we illustrate the existence and non-existence of critical points for certain sigmoidal curves. We define the critical point of a sigmoidal curve and present our basic assumptions in Section 2.2. In Section 2.2, we define the intrinsically even and odd components and in Section 2.3, we prove certain results related to the properties of the envelope.

2.1. The existence and non-existence of the critical point

We will denote the sigmoidal curve as $y(t)$ and its first derivative that is a localized hump by $f(t)$, hoping that there will be no confusion when we refer to even and odd derivatives of y or f. Let t_n be the point where the nth derivative $y^{(n)}(t)$ reaches its extreme value and let $y_n = y^{(n)}(t_n)$. Based on our observations we expect that the subsequences $\{t_{2k}\}$ and $\{t_{2k+1}\}$ converge at different rates. For example, in the case of an odd sigmoidal curve, $t_{2k+1} = 0$ for each k, while $\{t_{2k}\}$ converges slowly as seen from Fig. 2(b).

The standard and generalized logistic growth curves provide examples to the existence of the critical point for symmetrical and asymmetrical growth. The standard logistic growth curve is the sigmoidal curve $y(t) = \tanh(t)$, while the generalized logistic growth curve with horizontal asymptotes at -1 and 1 is given by $y(t) = -1 + 2\left[1 + ke^{-\beta t}\right]^{-1/\nu}$, where $k > 0$, $\beta > 0$ and $\nu > 0$. The parameter k can be adjusted by a time shift, β corresponds to a scaling of time and ν is the key parameter that determines the shape of the growth. For $k = 1$ the critical point is located at $t = 0$ (see Example 2). In Fig. 3(a), all even derivatives of the sigmoidal function are zero at $t = 0$; the apparent gap is still discernable despite a much higher number of derivatives are plotted. The behavior of the generalized logistic growth (Fig. 3(b)) is more or less the same except that the zeros of even derivatives are not fixed but they agglomerate near $t = 0$.

The Gompertz function with the same asymptotes is given by $y(t) = -1 + 2\exp(-e^{-\beta t})$. This function can be expressed as the limit of the generalized logistic family for $k = 1/n$, $\nu = 1/n$, as $n \to \infty$ and provides an example to the non-existence of the critical point. As seen in Fig. 3(c), the normalized derivatives do not accumulate and the critical point seems to have moved to negative infinity. The derivatives of the Gompertz function of orders 20, 30 and 40 are presented in Fig. 4, in order to display the shift of the wave packets towards minus infinity.

The standard logistic growth curve occurs as the solution of the "Susceptible-Infected-Susceptible" (SIS) model. In a study of the formation of (reversible) physical gels, [6], we showed that that generalized logistic growth curves are solutions to a modified form of the (SIS) model and we used the results of Example 2, in Section 3, to determine the gel point directly, using the approximation of experimental results by generalized logistic growth curves, known also as "5-point sigmoids".

In [2] we expressed sufficient conditions for the existence of a critical point of a sigmoidal curve in terms of the Fourier transform of the first derivative. For the sigmoidal curves that arise as solutions of the SIR model, we could only give numerical evidence for the existence of the critical point. But for the solutions of the SIS model expressed in terms of generalized logistic growth functions, we could express the location of the critical point in terms of the parameters of the generalized logistic growth curve [6] where we used without proof the expression of the Fourier transform of its first derivative.

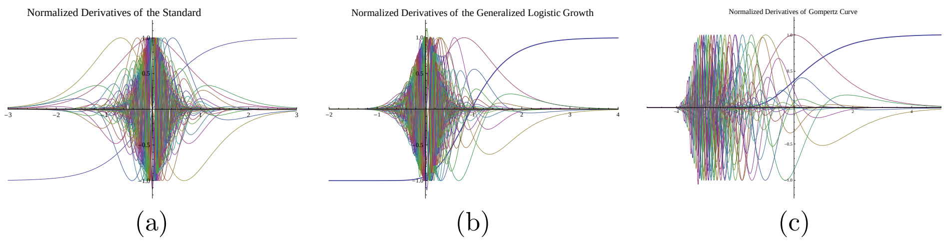


FIGURE 3. (a) Normalized derivatives of the standard logistic growth; (b) Normalized derivatives of the generalized logistic growth ($\beta = 1, k = 1, \nu = 1/5$) up to order 30. The apparent gap is still discernable in both figures, despite the large number of derivatives plotted. The behavior of the generalized logistic is more or less the same except that the zeros of even derivatives are not fixed. (c) Normalized derivatives of the Gompertz function. The Gompertz function is the limit of the generalized logistic family, the critical point seems to have moved to negative infinity.

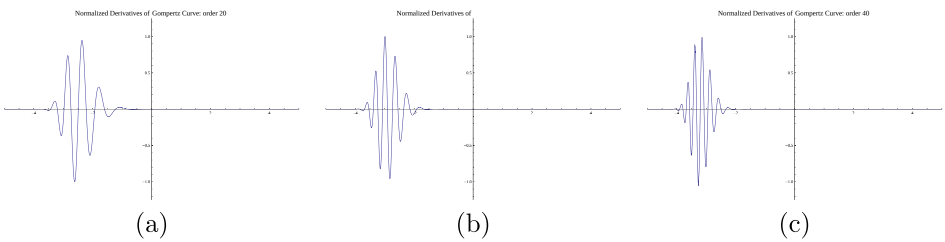


FIGURE 4. The time domain plots of the 20th, 30th and 40th derivatives of the Gompertz function respectively in (a), (b) and (c). The wave packets are shifted to left towards minus infinity.

2.2. Basic definitions

A sigmoidal function $y(t)$ is a monotone increasing function with horizontal asymptotes y_1 and y_2 as $t \to \pm\infty$ and with $\lim_{t\to\pm\infty} y^{(n)}(t) = 0$ for all $n \geq 1$. We propose the following definition for the "critical point".

Definition 2.1. Let $y(t)$ be a sigmoidal curve and assume that the set of points where the even derivatives and the odd derivatives reach their absolute extremum converge to the same point. The common limit of these derivatives, if it exists, is called the critical point of the sigmoidal curve.

The location of the critical point brings into consideration an intrinsic definition of being even or odd, hence a preferred choice of origin of the time axis. If $f(t)$ is even, then its Fourier transform $F(\omega)$ is real and even, but if $f(t)$ is shifted in time, then $F(\omega)$ is no longer real, but its phase is linear. The other way around we can recognize an intrinsically even function by looking to the phase of its Fourier transform. From this point of view, the preferred origin for the time axis is given by the shift that will make $F(\omega)$ real.

In the general case where $f(t)$ has no symmetry, $F(\omega)$ has a nonzero phase. But if the phase has an oblique asymptote $\alpha\omega + \phi_0$, then after a time shift, it will be asymptotically constant. Thus, the preferred origin of time is obtained by the time shift that makes the phase of $F(t)$ asymptotically constant, provided that the phase has an oblique asymptote.

We use this property to define the *intrinsically even* and *intrinsically odd* functions via their Fourier transform.

Definition 2.2. Let $f(t)$ be a function whose Fourier transform $F(\omega)$ exists. $f(t)$ is called **intrinsically even** if there is real number α such that $e^{-i\alpha\omega}F(\omega)$ is real. $f(t)$ is called **intrinsically odd** if there is real number α such that $e^{-i\alpha\omega}F(\omega)$ is pure imaginary.

If there is no real number α such that $f(t-\alpha)$ is neither even or nor odd, then we define its *intrinsically even and odd components* provided that the phase of $F(\omega)$ is asymptotically linear, i.e, $F(\omega) = |F(\omega)|e^{i\phi(\omega)}$ where $\phi(\omega)$ has an oblique asymptote with slope α, as $\omega \to \pm\infty$.

Definition 2.3. Let $f(t)$ be a function such that the Fourier transform $F(\omega)$ exists and the phase of $F(\omega)$ has an oblique asymptote with slope α as $\omega \to \pm\infty$. Then, the **intrinsically even and odd components of** $f(t)$ are the inverse Fourier transforms of the real and imaginary parts of $e^{-i\alpha\omega}F(\omega)$.

2.3. The envelope of the derivatives

Let $f(t)$ be a derivative of a sigmoidal function. We will prove that the magnitude of the analytic representation $f_A(t)$ (as defined in the Appendix) gives the envelope of $f(t)$ in the sense that $f(t)$ touches $|f_A(t)|$ between any two consecutive zeros.

Proposition 2.4. *Let $f(t)$ be a real function whose Fourier and Hilbert transforms exit. If t_1 and t_2 are any two consecutive zeros of $f(t)$, then there is a t_3 such that $t_1 < t_3 < t_2$ and $f(t_3) = \pm|f_A(t)|$.*

Proof. Writing $f_A(t) = A(t)e^{i\varphi(t)}$, we can express $f(t)$ and $f_h(t)$ as

$$f(t) = |f_A(t)| \, \cos(\varphi(t)), \quad f_h(t) = A(t)\sin(\varphi(t)).$$

If t_1 and t_2 are two consecutive zeros of $f(t)$, then we should have $\varphi(t_1) = \frac{\pi}{2} + k\pi$ and $\varphi(t_2) = \frac{\pi}{2} + (k+1)\pi$. Thus, provided that $\varphi(t)$ is continuous, there will be a time t_3, $t_1 < t_3 < t_2$, such that $\varphi(t_3) = (k+1)\pi$, hence, $f_h(t_3) = 0$. Thus the zeros of $f(t)$ and $f_h(t)$ alternate. It follows that $f(t_3) = \pm|f_A(t_3)|$ for some $t_1 < t_3 < t_2$. The local extremum of $f(t)$ in between t_1 and t_2 is denoted by t_4. If $f(t)$ is positive (negative) on (t_1, t_2) and $f'(t_3)$ is negative (positive), then $t_4 < t_3$, while if $f(t)$ is positive (negative) on (t_1, t_2) but $f'(t_3)$ is positive (negative), then $t_3 < t_4$. □

3. The existence and the location of the critical point of a sigmoidal curve

For the case with symmetry, i.e, for an odd sigmoidal curve $y(t)$, the Fourier transform of the first derivative, $F(\omega)$ is real. All odd derivatives of $y(t)$ have a local maximum at $t = 0$. We expect the local maximum to be located at $t = 0$. For a sigmoidal curve with no symmetry, we will show that the location of the critical point is given by the linear phase factor of $F(\omega)$.

The motivation for the choice of the assumptions in Proposition 2 is based on the following observation. In the examples studied, higher derivatives of the sigmoidal curve look like wave packets, hence in the frequency domain, the Fourier transform of the derivatives should have a nearly band bass spectrum. If the Fourier transform of all derivatives have the same (constant) phase, then in the time domain, the wave packets are centered at the same point. As this condition is not satisfied for sigmoidal curves without symmetry, the best that we can expect is that all derivatives have asymptotically a constant phase (possibly after a shift of time) and the band pass spectrum moves to infinity as the order of differentiation increases.

Before proceeding to the proofs, we start by examples to illustrate the behavior of curves with no symmetry.

Example 3.1. If $f(t)$ is an even localized hump, then its even derivatives are even and its odd derivatives are odd functions. By adding these with appropriate multiples one can generate positive pulses with no symmetry. As an example, the first derivative of the standard logistic growth, $\operatorname{sech}^2(t)$, is an even pulse. We obtain a positive pulse with no symmetry by adding a multiple of its second derivative

$$f(t) = y^{(1)}(t) = \operatorname{sech}^2(t) - \lambda\left[-2\operatorname{sech}^2(t)\tanh(t)\right] \qquad (0 < \lambda < 0.5),$$

the peak being located at the right of the point $t = 0$. The Fourier transform of $f(t)$ is

$$F(\omega) = (1 - i\lambda\omega)\sqrt{\frac{2}{\pi}}\frac{\pi\omega/2}{\sinh(\pi\omega/2)}.$$

The phase of $F(\omega)$ is $\phi(\omega) = \arctan\left(\operatorname{Im}(F)/\operatorname{Re}(F)\right) = \arctan(-\lambda\omega)$. As $\omega \to \pm\infty$, $\phi(\omega)$ approaches $\mp\pi/2$, hence $F(\omega)$ has asymptotically constant phase.

Example 3.2. The first derivative of the generalized logistic growth is

$$f(t) = \frac{2k\beta}{\nu}e^{-\beta t}\left(1 + ke^{-\beta t}\right)^{-1-1/\nu}.$$

The Fourier transform of $f(t)$ is [1]

$$F(\omega) = \sqrt{\frac{2}{\pi}}\,k^{-\frac{i\omega}{\beta}}\,\frac{\Gamma(1 + \frac{i\omega}{\beta})\Gamma(\frac{1}{\nu} - \frac{iw}{\beta})}{\Gamma(\frac{1}{\nu})}.$$

Since $k^{-\frac{i\omega}{\beta}} = e^{-i\left(\frac{\ln k}{\beta}\right)\omega}$, it follows for $k \neq 1$, there is a linear phase factor in $F(\omega)$. Let $k = 1$ and $\beta = 1$. For $1/\nu = n$, one can use the property $\Gamma(\overline{z}) = \overline{\Gamma(z)}$ repeatedly to see that

$$F(\omega) = \frac{1}{\Gamma(n)}(1 - i\omega)(2 - i\omega)\dots(n - 1 - i\omega)\ F_s(\omega),$$

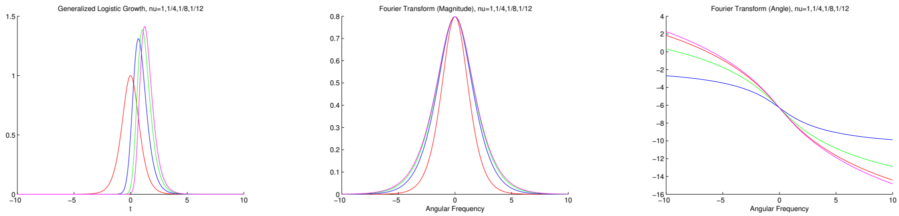


FIGURE 5. a) Time domain plots of the first derivative of the generalized logistic growth for $1/\nu = 1, 1/4, 1/8, 1/12$. b) The magnitude of the Fourier transform, c) The angle of the Fourier transform.

where $F_s(\omega)$ is the Fourier transform of the standard logistic growth. The multiplicative factor is a complex polynomial; in particular the zeros of its real part are bounded, hence as $\omega \to \pm\infty$, the phase goes to a multiple of $\pi/2$. It follows that, for each n, the phase is asymptotically constant (provided that $k = 1$). The time domain plot, the magnitude and the phase of the Fourier transforms of the generalized logistic family for $\nu = 1$, $1/4$, $1/8$ and $1/12$ are presented in Fig. 5. We note that as $1/\nu$ increases, the phase approaches to its horizontal asymptotes more and more slowly.

3.1. Existence of the critical point: The general case

We will now prove that if $f(t)$ satisfies asymptotically constant phase and band-pass hypotheses, to be specified below, then the critical point is located at $t = 0$. Note that the asymptotically constant phase condition is trivially satisfied when $f(t)$ is even.

Proposition 3.3. *Let $f(t)$ be the first derivative of a sigmoidal curve $y(t)$ and $f^{(n)}(t)$ be its nth derivative. If*
(i) the Fourier transform of $f(t)$ has the form $F(\omega) = |F(\omega)|e^{-i\alpha\omega}e^{i\psi(\omega)}$ where α is a constant and $\psi(\omega)$ has horizontal asymptotes,
(ii) for $\omega > 0$, $\omega^n|F(\omega)|$ has a single maximum at ω_n and the ω_n's are unbounded,
(iii) the spectrum is localized in the sense that there are constants ω_a and ω_b (depending on n), such that

$$\lim_{n\to\infty} \int_{|\omega|<\omega_a} \omega^n|F(\omega)|\, d\omega = \lim_{n\to\infty} \int_{|\omega|>\omega_b} \omega^n|F(\omega)|\, d\omega \; = 0,$$

then the sigmoidal curve $y(t)$ has a critical point located at $t = \alpha$.

Proof. For simplicity assume that $\alpha = 0$. If the Fourier transform of $f(t)$ is $F(\omega)$, then the Fourier transform of $f(t-\alpha)$ is $e^{-i\alpha\omega}F(\omega)$. We will express $|f^{(n)}(t)|$ using the Fourier inversion formula and compare it with $|f^{(n)}(0)|$. The assumption (iii) implies that for all ε, there is $N > 0$ such that for $n > N$, there are constants ω_a and ω_b (depending on n), such that

$$\int_{|\omega|<\omega_a} \omega^n|F(\omega)|\, d\omega \; < \; \frac{\varepsilon}{8} \quad \text{and} \quad \int_{|\omega|>\omega_b} \omega^n|F(\omega)|\, d\omega \; < \; \frac{\varepsilon}{8}.$$

Thus for n large, the contribution from low and high frequencies are negligible, hence the main contribution comes from a neighborhood of ω_n. By assumption (ii) the ω_n's are unbounded and using the asymptotically constant phase assumption we obtain the estimates below:

$$\int_{\omega_a \leq |\omega| \leq \omega_b} \omega^n |F(\omega)|(1 - e^{i\psi(\omega)})\ d\omega < \frac{\varepsilon}{2}$$

Letting I to be the set defined by $\omega_a \leq |\omega| \leq \omega_b$, we obtain upper bounds for $f^{(n)}(t)$ as below:

$$\begin{aligned}
\sqrt{2\pi}|f^{(n)}(t)| &= \left| \int_{-\infty}^{\infty} \omega^n F(\omega) e^{i\omega t}\ d\omega \right| \leq \int_{-\infty}^{\infty} \omega^n |F(\omega)|\ d\omega \\
&= \int_{|\omega|<\omega_a} \omega^n |F(\omega)|\ d\omega\ + \int_{|\omega|>\omega_b} \omega^n |F(\omega)|\ d\omega\ + \int_{\omega \in I} \omega^n |F(\omega)|\ d\omega \\
&= \frac{\varepsilon}{4} + \int_{\omega \in I} \omega^n |F(\omega)|\ d\omega
\end{aligned}$$

We estimate the integral above as

$$\begin{aligned}
\int_{\omega \in I} \omega^n |F(\omega)|\ d\omega &= \left| \int_{\omega \in I} \omega^n |F(\omega)|\ d\omega \right| \\
&= \left| \int_{\omega \in I} \omega^n |F(\omega)| \left(1 - e^{i\psi(\omega)}\right)\ d\omega + \int_{\omega \in I} \omega^n |F(\omega)| e^{i\psi(\omega)}\ d\omega \right| \\
&\leq \left| \int_{\omega \in I} \omega^n |F(\omega)| \left(1 - e^{i\psi(\omega)}\right)\ d\omega \right| + \left| \int_{\omega \in I} \omega^n |F(\omega)| e^{i\psi(\omega)}\ d\omega \right| \\
&\leq \frac{\varepsilon}{2} + \left| \int_{\omega \in I} \omega^n |F(\omega)| e^{i\phi_0} e^{i\psi(\omega)}\ d\omega \right| \\
&\leq \frac{\varepsilon}{2} + \left| \int_{\omega \in I} \omega^n F(\omega)\ d\omega \right|.
\end{aligned}$$

Finally we estimate the last term as

$$\begin{aligned}
\left| \int_{\omega \in I} \omega^n F(\omega)\ d\omega \right| &= \left| \int_{-\infty}^{\infty} \omega^n F(\omega)\ d\omega - \int_{|\omega|<\omega_a} \omega^n F(\omega)\ d\omega - \int_{|\omega|>\omega_b} \omega^n F(\omega)\ d\omega \right| \\
&\leq \left| \int_{-\infty}^{\infty} \omega^n F(\omega)\ d\omega \right| + \left| \int_{|\omega|<\omega_a} \omega^n F(\omega)\ d\omega \right| + \left| \int_{|\omega|>\omega_b} \omega^n F(\omega)\ d\omega \right| \\
&\leq \left| \int_{-\infty}^{\infty} \omega^n F(\omega)\ d\omega \right| + \int_{|\omega|<\omega_a} \omega^n |F(\omega)|\ d\omega + \int_{|\omega|>\omega_b} \omega^n |F(\omega)|\ d\omega \\
&\leq \frac{\varepsilon}{4} + \sqrt{2\pi} \left| f^{(n)}(0) \right|.
\end{aligned}$$

It follows that $f^{(n)}(t) \leq f^{(n)}(0) + \varepsilon$ hence as $t = 0$ is the critical point. □

The sequence ω_n can be obtained by maximizing $\omega_n |F(\omega)|$, i.e, by equating its first derivative to zero. We present in Fig. 6, a graphical display of the corresponding equality $\omega/n = F(\omega)/F'(\omega)$ for the standard logistic growth. The comparison of the

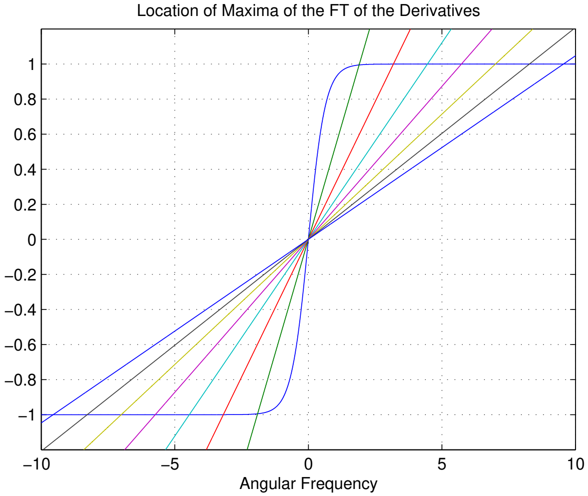


FIGURE 6. Graphical solution of the equation $\omega/n = F(\omega)/F'(\omega)$ for the standard logistic growth.

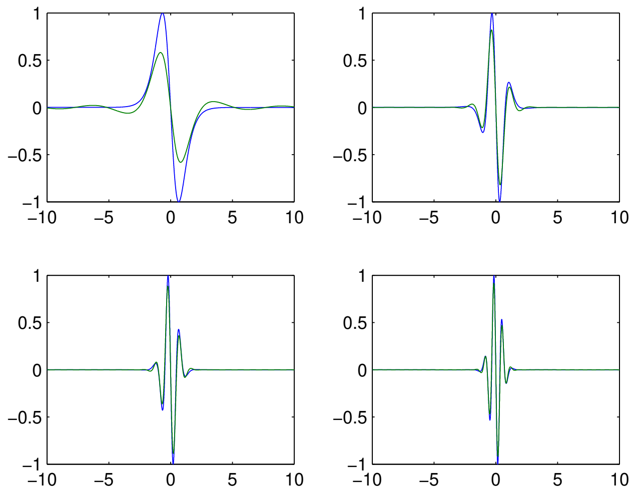


FIGURE 7. Comparison of the derivatives $f^{(n)}(t)$ of the standard logistic function with sinusoids of frequency ω_n modulating $|f_A^{(n)}(t)|$.

higher derivatives with sinusoids of frequency ω_n modulating the amplitude of the corresponding analytical representation are shown in Fig. 7.

3.2. The existence of the critical point: Odd sigmoidal curves

In this section we will prove the existence of the critical point of an odd sigmoidal curve under a different set of assumptions. We first prove that the odd derivatives of an odd sigmoidal function $y(t)$ reach their global extreme values at $t = 0$ (Proposition 3.4). Then we prove that the global extreme value of the even derivatives of $y(t)$ is the local extreme value that is closest to $t = 0$ (Proposition 3.5) and the sequence of points where the even derivatives of $y(t)$ reach their global extreme values converge to $t = 0$ (Corollary 3.6). The time domain plots and the magnitude of the Fourier and Hilbert transforms of the standard logistic growth are displayed in Fig. 8.

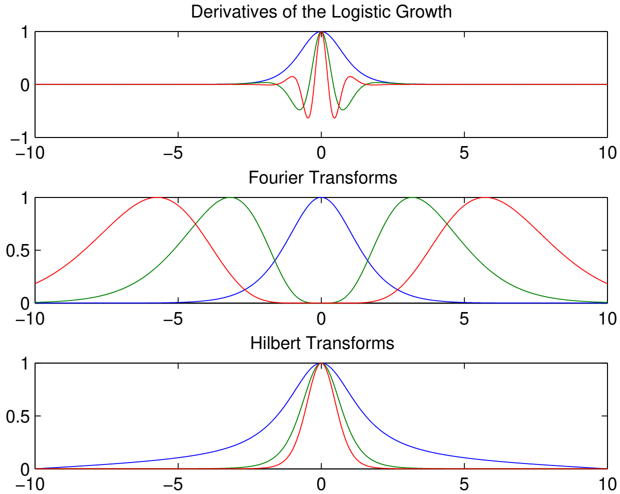


FIGURE 8. Normalized graphs of the time domain variations, the magnitude of the Fourier transform and the magnitude of the analytic representation for $y(t)$, $y^{(4)}(t)$ and $y^{(8)}(t)$. As the order of differentiation increases, the time domain pulses get narrower while the frequency spectrum spreads out.

The simplest property that has to be proved is the fact that if $f(t)$ is an even, the local extremum at $t = 0$ is the global one. We prove this by requiring the "monotonicity" of the envelope of $y^{(n)}(t)$, expressed in terms of its analytic representation.

Proposition 3.4. *Let $f(t)$ be a real, even function and assume that that the magnitude of its analytical representation $|f_A(t)|$ has a single local maximum. Then*

$$|f(t)| \leq |f(0)|.$$

Proof. Recall that $f_A(t) = f(t) + if_h(t)$, where $f_h(t)$ is the Hilbert transform of $f(t)$. Since $f_A^2(t) = f^2(t) + f_h^2(t)$, $|f(t)| \leq |f_A(t)|$. When $f(t)$ is even, $f_h(t)$ is odd, hence, $f(0) = f_A(0)$. Since $|f_A(t)|$ is monotone decreasing, $|f(t)| \leq |f_A(0)| = |f(0)|$. □

We next prove that the global extremum of the even derivatives of an odd sigmoidal curve is the one closest to $t = 0$.

Proposition 3.5. *Let $f(t)$ be a real, odd function such that the magnitude of its analytical representation $|f_A(t)|$ has a single local maximum. Then $|f(t)| \leq |f(t_0)|$ where t_0 is the location of the first local extremum of $f(t)$ for $t > 0$.*

Proof. Since $f(t)$ is odd, $f(0) = 0$. For simplicity assume that $f'(0) > 0$. Let t_1 be the first zero of $f(t)$ for $t > 0$. By the alternation of roots, there is a point $t_2 < t_1$ at which $f(t_2) = |f_A(t_2)|$ and since $|f_A(t)|$ is decreasing, $f(t)$ is reaching its first local maximum at some $t_0 < t_2$. It follows that $|f(t)| < |f(t_0)|$. □

Finally we prove that the global extreme values of the derivatives converge to $t = 0$, i.e, there is no gap between the maximum of $|y_A^{(2k)}(t)|$ and the global extremum of $y^{(2k)}(t)$.

Corollary 3.6. *Let $y(t)$ be an odd sigmoidal function and assume that for each n, $\left|y_A^{(n)}(t)\right|$ has a single local maximum. Then $t = 0$ is the limit point of the global extreme values of $y^{(2k)}(t)$.*

Proof. For $n = 2k + 1$, Proposition 3.4 implies that the global extremum is at $t = 0$. For $n = 2k$, we will prove that the global extremum of $y^{(2k+2)}$ occurs earlier than the global extremum of $y^{(2k)}$. By Proposition 3.5 above, the global extremum is the first local extremum. Let $y^{(2k)}(0) = 0$, assume that $y^{(2k+1)}(0) > 0$ and let t_1 be the first intersection of $y^{(2k)}$ with its envelope, $y^{(2k)}(t_1) = \left|y_A^{(2k)}(t_1)\right|$. The global extreme value of $y^{(2k)}$ is at some $t_2 < t_1$, since the envelope is decreasing. Then, $y^{(2k+1)}(t_2) = 0$, $y_h^{(2k+1)}(t_2)$ is tangent to the envelope, hence it has its global extremum at $t_3 < t_2$. Finally, $y_h^{(2k+2)}(t_3) = 0$, hence $y^{(2k+2)}(t)$ is tangent to the envelope at this point and it has its global extremum at some $t_4 < t_3$. Hence the global extreme values form a decreasing sequence that converge to $t = 0$. □

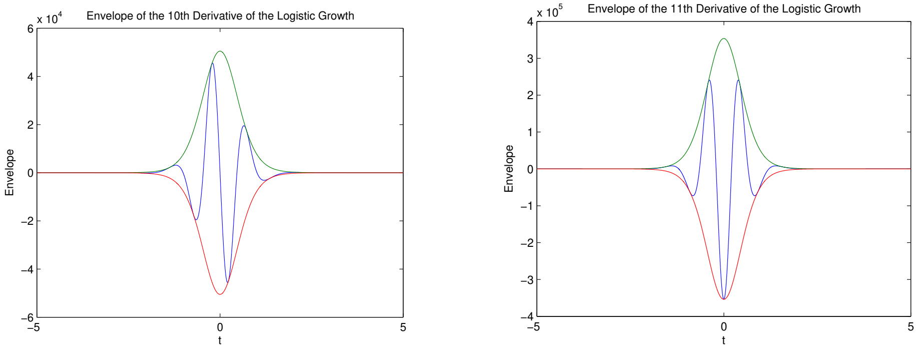
Typical examples of the even and odd derivatives of the standard logistic growth and their envelopes are presented in Fig. 9, as to illustrate how the proofs work.

For the case with no symmetry, we again need to assume that $|y_A^{(n)}(t)|$ has a single maximum, but the location of this maximum, that we denote as $t_{n,*}$ changes with n. The comparison of $|y_A^{(n)}(t)|$ for sigmoidal curves with and without symmetry is given in Fig. 10.

Appendix A. The Fourier and Hilbert transforms

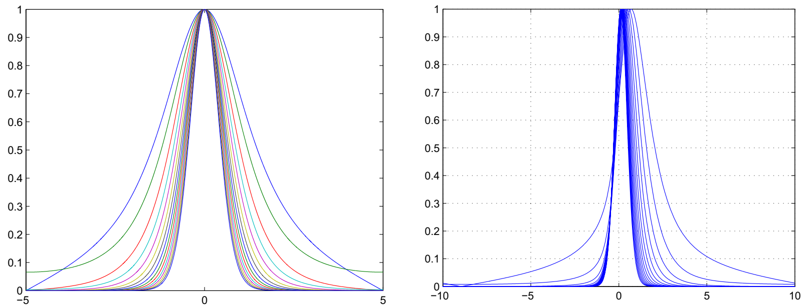
The Fourier transform: The Fourier transform of a function $f(t)$, $\mathcal{F}(f) = F(\omega)$ is defined as

$$F(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, e^{-i\omega t}\, dt,$$



(a) (b)

FIGURE 9. The envelope of the 10th (a) and of 11th (b) derivative of the standard logistic growth function.



(a) (b)

FIGURE 10. Envelopes of normalized derivatives up to order 15: The standard logistic growth function (a) and generalized logistic growth function with $\beta = 2$, $k = 1$, $\nu = 1/5$ (b).

provided that the integral exists in the sense of Cauchy principal value [8]. If $f(t)$ is in L^1, then its Fourier transform exists. Since a sigmoidal function is finite as $t \to \infty$, its first derivative is in L^1. We can recover $f(t)$ from the inverse transform by

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(\omega)\, e^{i\omega t}\, d\omega.$$

The Fourier transform of the odd sigmoidal function exists in the sense of Cauchy principal value.

The Hilbert transform and the analytic representation: For our purposes, the simplest description of the Hilbert transform is given by its relation to the Fourier transform

[7]. Given $f(t)$ and its Fourier transform $F(\omega)$, we define the function $F_A(\omega)$ by

$$F_A(\omega) = \begin{cases} 2F(\omega) & , \quad \omega > 0 \\ 0 & , \quad \omega < 0 \end{cases} .$$

The inverse transform is a complex function that is called the "analytic representation" $f_A(t)$ of $f(t)$. The imaginary part of $f_A(t)$ is the Hilbert transform $f_h(t)$ of $f(t)$.

$$f_A(t) = f(t) + i f_h(t) = \sqrt{\frac{2}{\pi}} \int_0^\infty F(\omega)\, e^{i\omega t}\, d\omega.$$

The existence of the Hilbert transform necessitates that $f(t)$ be in L^p for $1 < p < \infty$; for L^1 functions, it exists in $L^{1,weak}$ ([10], Lemma V.2.8).

Symmetry properties: The property, $\mathcal{F}\left(\overline{f(t)}\right) = \overline{F(-\omega)}$ implies that if $f(t)$ is real then $F(-\omega) = \overline{F(\omega)}$. Thus if $f(t)$ is real, $F(\omega)$ will be real provided that $F(-\omega) = F(\omega)$. The scaling property $f(at) \rightarrow \frac{1}{|a|} F\left(\frac{\omega}{a}\right)$ implies that $f(-t) \rightarrow F(-\omega)$, hence if $f(t)$ is real and even, then $F(\omega)$ is real and even. Similarly, if $f(t)$ is real and odd, then $F(\omega)$ is pure imaginary and odd. We note that if $f(t)$ is even (odd), its Hilbert transform is odd (even).

Differentiation: The effect of differentiation in the time domain is multiplication by $i\omega$ in the frequency domain. Thus

$$f^{(n)}(t) \rightarrow (i\omega)^n F(\omega).$$

Convolution and modulation: There is a correspondence between products and convolutions in the time and frequency domains; multiplication in the time domain leads to convolution in the frequency domain, i.e,

$$f(t)\, g(t) \rightarrow \frac{1}{\sqrt{2\pi}}\, F(\omega) * G(\omega).$$

The "modulation" of a low frequency signal in the time domain is the multiplication of this signal by a sinusoidal function of fixed (usually high) angular frequency ω_0. In the frequency domain, the Fourier transform of the low frequency function is convolved with the Fourier transform of the sinusoid. The Fourier transform of a pure sinusoid is not defined in the usual sense, but it is represented as the Dirac δ functions occurring at $\pm\omega_0$ and convolution carries the spectrum of the low frequency signal to the frequencies $\pm\omega_0$. Since the Fourier transform of a complex exponential is a δ-function, we have the correspondence below:

$$f(t) e^{i\omega_0 t} \rightarrow F(\omega - \omega_0).$$

Time shift: As an analogue of multiplication with a complex exponential in the time domain, the multiplication of a function in the frequency domain by a linear phase factor leads to a shift in the time domain:

$$e^{-i\alpha\omega} F(\omega) \rightarrow f(t - \alpha).$$

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Integrodifferential evolution systems with nonlocal initial conditions

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Abstract. The paper deals with systems of abstract integrodifferential equations subject to general nonlocal initial conditions. In order to allow the nonlinear terms of the equations to behave independently as much as possible, we use a vector approach based on matrices, vector-valued norms and a vector version of Krasnoselskii's fixed point theorem for a sum of two operators. The assumptions take into account the support of the nonlocal initial conditions and the hybrid character of the system. Two examples are given to illustrate the theory.

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1. Introduction

In this paper, we are concerned with the existence of solutions to the semilinear system of abstract integrodifferential equations with nonlocal initial conditions, of the type

$$\begin{cases} u_i'(t) + A_i u_i(t) = \int_0^t K_i(t-s, u_s)ds + F_i(t, u_t), & t \in [0,T] \\ u_i(t) = \alpha_i(u)(t), \quad t \in [-\tau, 0], \quad i = 1, ..., n. \end{cases} \tag{1.1}$$

Here $n \geq 1$, and for each $i \in I := \{1, ..., n\}$, the linear operator $-A_i : D(A_i) \subseteq X_i \to X_i$ generates a C_0-semigroup of contractions $\{S_i(t); t \geq 0\}$ on a Banach space $\left(X_i, |.|_{X_i}\right)$, $\tau \geq 0$, $u \in C([-\tau, T], X)$, where $X = X_1 \times ... \times X_n$, $u = (u_1, ..., u_n)$, and for each t, u_t is the restriction of u to $[t-\tau, t]$ shifted to the interval $[-\tau, 0]$, i.e., $u_t \in C([-\tau, 0], X)$ and

$$u_t(s) = u(t+s), \quad s \in [-\tau, 0]. \tag{1.2}$$

The nonlinear perturbations in equations are given by the continuous mappings F_i from $[0,T]\times C([-\tau,0],X)$ to X_i, K_i from $[0,T]\times C([-\tau,0],X)$ to X_i, and the nonlocal initial conditions are expressed by the continuous mappings α_i from $C([-\tau,T],X)$ to $C([-\tau,0],X_i)$.

We note that the nonlocal initial conditions include in particular:

- *the initial condition*:

$$u_i(t)=\varphi_i(t),\quad t\in[-\tau,0],\quad i=1,...,n$$

  where $\varphi=(\varphi_1,...,\varphi_n)\in C([-\tau,0],X)$ is given;
- *linear multi-point conditions* (linear nonlocal initial conditions of discrete type):

$$u_i(t)=\varphi_i(t)+\sum_{j=1}^{m_i}a_{ij}(t)\,u_i(t+t_{ij}),\quad t\in[-\tau,0],\ i=1,...,n, \tag{1.3}$$

  where $0<t_{ij}<t_{i,j+1}\leq T$ for $j=1,...,m_i$ and $i=1,...,n$. The linear multi-point conditions include in particular the initial condition, and the *periodicity condition*

$$u_i(t)=u_i(T+t),\quad t\in[-\tau,0],\quad i=1,...,n;$$

- *linear nonlocal initial conditions of continuous type*, given by integrals:

$$\begin{aligned} u_i(t) &= \varphi_i(t)+\int_0^T k_i(t,s)\,u_i(t+s)\,ds \\ &= \varphi_i(t)+\int_t^{T+t} k_i(t,s-t)\,u_i(s)\,ds,\quad t\in[-\tau,0],\quad i=1,...,n. \end{aligned}$$

Starting with Volterra's pioneering works on integrodifferential equations with delayed effects in population dynamics and materials with memory, the theory of delay differential equations has progressed continuously following the development of functional analysis and being stimulated by numerous applications in physics, chemistry, biology, medicine, economy, etc., see e.g., [23]), aimed to described evolution processes whose future states depend not only on the present, but also on the past history.

As concerns differential equations with nonlocal initial conditions of multi-point or integral type, we mention as some pioneering contributions, the papers of Cioranescu [15], Whyburn [42] and Conti [16]). Among further developments, we refer the readers to the works [2], [3], [7], [17], [21], [28], [29], [41], to the recent survey paper [35], and the references therein.

Parabolic problems with nonlocal initial conditions were considered in the papers of Kerefov [22], Vabishchevich [36], Chabrowski [14], Pao [33], Olmstead and Roberts [31], and Chapter 10 in [26], as nonlocal versions of some deterministic models from physics, mechanics, biology and medicine. Abstract evolution equations with nonlocal initial conditions were considered by Byszewski [11], Jackson [20], Lin and Liu [24]. For more recent contributions, we refer the readers to the papers [4], [6], [8], [10], [12], [19], [24], [25], [27], [30], [32], [39] and the recent monograph [9].

This paper has a double motivation. First, it is motivated by the second author's recent paper [5], which mainly inspires the operator technique of proof, and secondly, by the paper of Webb [40] for the class of integrodifferential equations.

There are several aspects in the present paper which are mixed together requiring a laboured technique of proof and yielding to a very general result:

► **The use of the notion of support of a nonlocal initial condition and of a corresponding split norm.** Throughout the paper, by $[-\tau, T_0]$ we shall denote the *support* of the nonlocal initial condition, that is the smallest subinterval $[-\tau, T_0]$ of $[-\tau, T]$ with $T_0 \geq 0$ such that

$$\begin{aligned} \alpha_i(u) &= \alpha_i(v), \ i = 1, ..., n, \ \text{ for every } \ u, v \in C([-\tau, T], X) \\ \text{with } \ u|_{[-\tau, T_0]} &= v|_{[-\tau, T_0]}. \end{aligned}$$

Here by $u|_{[-\tau, T_0]}$ we mean the restriction of the function u to the interval $[-\tau, T_0]$. Physically, this means that the evolution of a process is subjected to some constraints until a given moment of time T_0, and becomes free of any constraints after that moment.

The notion of support of a nonlocal initial condition was first used in the papers [7] and [8], and used after in [29], [2], [12], [4], [5]. As explained in these papers, and as we shall see in the following, stronger conditions on nonlinearities have to be asked on the support subinterval, compared to those required on the rest of the interval. Mathematically, the integral equation equivalent to the nonlocal initial problem is of Fredholm type on the support interval, and of Volterra type on the rest of the interval. This makes useful to consider a split norm on the functional space where the problem is studied. Thus, in connection with the delay system (1.1) and with the support $[-\tau, T_0]$ of the nonlocal initial condition, on a space of the type $C([-\tau, T], E)$, where $(E, |\cdot|_E)$ is a Banach space, we shall consider the *split norm*

$$|u|_\tau = \max\left\{ |u|_{C([-\tau, T_0], E)}, \ \ |u|_{C_\theta([T_0 - \tau, T], E)} \right\}, \tag{1.4}$$

where $|u|_{C([-\tau, T_0], E)}$ is the usual max norm

$$|u|_{C([-\tau, T_0], E)} = \max_{t \in [-\tau, T_0]} |u(t)|_E,$$

while for any $\theta > 0$, $|u|_{C_\theta([T_0 - \tau, T], E)}$ is the Bielecki type norm on $C([T_0 - \tau, T], E)$,

$$\begin{aligned} |u|_{C_\theta([T_0 - \tau, T], E)} &= \max_{t \in [T_0, T]} \left(|u_t|_{C([-\tau, 0], E)} \, e^{-\theta(t - T_0)} \right) \\ &= \max_{t \in [T_0, T]} \left(|u|_{C([t - \tau, t], E)} \, e^{-\theta(t - T_0)} \right). \end{aligned}$$

In particular, when there is no a delay, i.e., when $\tau = 0$, the norm (1.4) reduces to the split norm previously considered in [7], [2], [28] and [29].

► **The hybrid character of the system.** The system is split into to subsystems: the first m equations for which Lipschitz conditions are assumed to guarantee that the corresponding integral operators are contractive, and the last $n - m$ equations $(0 \leq m \leq n)$ for which only at most linear growth conditions are required on the nonlinear terms, but in return, the compactness of the semigroups of operators is assumed to insure the compactness of the integral operators. In this way the proof will be a perfect illustration of Krasnoselskii's fixed point theorem for a sum of a compact map and a contraction, more exactly of its vector version of Viorel [37].

► **The presence of integral terms.** There is not only the bounded delay in the equations of system (1.1), but also cumulative integral terms which bring into the equations the whole history of the process. Such kind of equations arise from mathematical modeling of many real processes with memory from physics, biology and economics. These cumulative terms play a special role in the split analysis on two intervals as discussed previously.

2. Preliminaries

For the treatment of systems we use the vector approach based on vector-valued metrics and norms, and matrices instead of constants.

Let us make the convention that the elements of $\mathbb{R}^n$ are seen as column vectors. By a *vector-valued metric* on a set E we mean a mapping $d : E \times E \to \mathbb{R}^n_+$ such that $d(x, y) = 0$ if and only if $x = y$; $d(x, y) = d(y, x)$ for all $x, y \in E$ and $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in E$. Here by $\leq$ we mean the natural componentwise order relation of $\mathbb{R}^n$, more exactly, if $r, s \in \mathbb{R}^n$, $r = (r_1, ..., r_n)$, $s = (s_1, ..., s_n)$, then by $r \leq s$ one means that $r_i \leq s_i$ for $i = 1, ..., n$. A set E together with a vector-valued metric d is called a *generalized metric space*. For such a space, the notions of Cauchy sequence, convergence, completeness, open and closed set, are similar to those in usual metric spaces.

Similarly, a *vector-valued norm* on a linear space E, is defined as being a mapping $\|\cdot\| : E \to \mathbb{R}^n_+$ with $\|x\| = 0$ only for $x = 0$; $\|\lambda x\| = |\lambda| \, \|x\|$ for $x \in E$, $\lambda \in \mathbb{R}$, and $\|x + y\| \leq \|x\| + \|y\|$ for every $x, y \in E$. To any vector-valued norm $\|.\|$ one can associate the vector-valued metric $d\,(x, y) := \|x - y\|$. A linear space E endowed with a vector-valued norm $\|\cdot\|$ is called a *generalized Banach space* if E is complete with respect to the associated vector-valued metric d.

If (E, d) is a generalized metric space with d taking values in $\mathbb{R}^n$, we say that a mapping $\Gamma : E \to E$ is a *generalized contraction* (in Perov's sense) if there exists a square matrix M of size n with nonnegative entries such that its powers M^k tend to the zero matrix 0 as $k \to \infty$, and

$$d(\Gamma(x), \Gamma(y)) \leq M d(x, y) \;\text{ for all }\; x, y \in E.$$

Such a matrix is said to be a *Lipschitz matrix*. Notice that for a matrix M the property $M^k \to 0$ as $k \to \infty$ is equivalent to the fact that the spectral radius $\rho\,(M)$ of the matrix M is less than one. The role of matrices with spectral radius less than one in the study of operator systems was pointed out in [34], in connection with several abstract principles from nonlinear functional analysis.

For generalized contractions, the following extension of Banach's contraction principle holds.

Theorem 2.1 (Perov). *If (E, d) is a complete generalized metric space, then any generalized contraction $\Phi : E \to E$ with the Lipschitz matrix M has a unique fixed point x^*, and*

$$d(\Phi^k(x), x^*) \leq M^k(J - M)^{-1} d(x, \Phi(x)),$$

for all $x \in E$ and $k \in \mathbb{N}$ (where J stands for the identity matrix of the same size as M).

In this paper we use the following generalization of Theorem 2.1, a vector version of Krasnoselskii's fixed point theorem for a sum of two operators, owed to Viorel [37].

Theorem 2.2. *Let $(E, \|\cdot\|)$ be a generalized Banach space, $D \subset E$ a nonempty bounded closed convex set and $\Gamma : D \to E$ a mapping such that*

- **(i).** *$\Gamma = \Phi + \Psi$ with $\Phi : D \to E$ a generalized contraction in Perov's sense, and $\Psi : D \to E$ a compact operator;*
- **(ii).** *$\Phi(u) + \Psi(v) \in D$ for every $u, v \in D$.*

Then Γ has at least one fixed point in D.

The following obvious proposition will be used in the proof of the main result.

Proposition 2.3. (a) *If $M \in \mathcal{M}_{n\times n}(\mathbb{R}_+)$ is a matrix with $\rho(M) < 1$, then $\rho\left(\widetilde{M}\right) < 1$ for every matrix $\widetilde{M} \in \mathcal{M}_{n\times n}(\mathbb{R}_+)$ whose elements are close enough to the corresponding elements of M.*

(b) *If $M \in \mathcal{M}_{n\times n}(\mathbb{R}_+)$ is a matrix with $\rho(M) < 1$, then $\rho\left(\widehat{M}\right) < 1$ for every matrix $\widehat{M} \in \mathcal{M}_{n\times n}(\mathbb{R}_+)$ such that $\widehat{M} \leq M$ componentwise.*

We conclude this preliminary section by a result about the compactness of the solution operator associated to a non-homogenous evolution equation [1].

Lemma 2.4 (Baras-Hassan-Veron). *Let $A : D(A) \subset E \to E$ be the generator of a compact C_0-semigroup $\{S(t); t \geq 0\}$. Then for every uniformly integrable family of functions $\mathcal{F} \subset L^1(0, T; E)$, the set of functions*

$$\left\{ \int_0^t S(t-s) f(s)\, ds : \; f \in \mathcal{F} \right\}$$

is relatively compact in $C([0, T], E)$.

For other basic notions and results of semigroup theory we mention the books [13], [18] and [38].

3. Main result

Looking for mild solutions to the problem (1.1), with $u_i \in C([-\tau, T], X_i)$ for $i = 1, ..., n$ we are led in a standard way to the following integral system

$$\begin{cases} u_i(t) &= \alpha_i(u)(t), \quad t \in [-\tau, 0], \\ u_i(t) &= S_i(t)\alpha_i(u)(0) + \int_0^t S_i(t-s)\int_0^s K_i(s-\sigma, u_\sigma)\, d\sigma ds \\ &\quad + \int_0^t S_i(t-s) F_i(s, u_s)\, ds, \quad t \in [0, T], \quad i = 1, ..., n. \end{cases} \tag{3.1}$$

Our assumptions are given differently for two sets of indices,

$$I_1 := \{1, ..., m\} \quad \text{and} \quad I_2 := \{m+1, ..., n\},$$

where $0 \leq m \leq n$, and it is understood that $I_1 = \emptyset$ if $m = 0$, and $I_2 = \emptyset$ if $m = n$. Let $p > 1$ be any fixed number.

The hypotheses are:

(H$_0$) (a) For each $i \in I_1$, the linear operator $-A_i : D(A_i) \subset X_i \to X_i$ generates a C_0-semigroup of contractions on the Banach space X_i.

(b) For each $i \in I_2$, the linear operator $-A_i : D(A_i) \subset X_i \to X_i$ generates a compact C_0-semigroup of contractions on the Banach space X_i.

(H$_1$) (a) For each $i \in I_1$, $K_i : [0,T] \times C([-\tau,0], X) \to X_i$, is continuous, and there exist $a_{ij} \in C([0,T], \mathbb{R}_+)$ for $j \in I$, such that

$$|K_i(t,u) - K_i(t,v)|_{X_i} \leq \sum_{j=1}^{n} a_{ij}(t)\, |u_j - v_j|_{C([-\tau,0],X_j)}$$

for all $u, v \in C([-\tau, 0], X)$ and $t \in [0, T]$.

(b) For each $i \in I_2$, $K_i : [0,T] \times C([-\tau,0], X) \to X_i$, is continuous, and there exist d_i, $a_{ij} \in C([0,T], \mathbb{R}_+)$ for all $j \in I$, such that

$$|K_i(t,u)|_{X_i} \leq \sum_{j=1}^{n} a_{ij}(t)\, |u_j|_{C([-\tau,0],X_j)} + d_i(t)$$

for all $u \in C([-\tau, 0], X)$ and $t \in [0, T]$.

(H$_2$) (a) For each $i \in I_1$, $F_i : [0,T] \times C([-\tau,0], X) \to X_i$ is continuous and there exists $b_{ij} \in C([0,T], \mathbb{R}_+)$ for all $j \in I$, such that

$$|F_i(t,u) - F_i(t,v)|_{X_i} \leq \sum_{j=1}^{n} b_{ij}(t)\, |u_j - v_j|_{C([-\tau,0],X_j)}$$

for $u, v \in C([-\tau, 0], X)$ and $t \in [0, T]$.

(b) For each $i \in I_2$, $F_i : [0,T] \times C([-\tau,0], X_i) \to X_i$ is continuous and there exist f_i, $b_{ij} \in C([0,T], \mathbb{R}_+)$ for all $j \in I$, such that

$$|F_i(t,u)|_{X_i} \leq \sum_{j=1}^{n} b_{ij}(t)\, |u_j|_{C([-\tau,0],X_j)} + f_i(t)$$

for all $u \in C([-\tau, 0], X)$ and $t \in [0, T]$.

(H$_3$) For each $i \in I$, $\alpha_i : C([-\tau, T], X) \to C([-\tau, 0], X_i)$ and there exist $c_{ij} \in \mathbb{R}_+$ for all $j \in I$, such that

$$|\alpha_i(u) - \alpha_i(v)|_{C([-\tau,0],X_i)} \leq \sum_{j=1}^{n} c_{ij} |u_j - v_j|_{C([-\tau,T_0],X_j)}$$

for all $u, v \in C([-\tau, T], X)$.

Theorem 3.1. *Assume that the conditions* (H$_0$)-(H$_3$) *hold. In addition assume that the spectral radius of the $n \times n$ square matrix $M = [m_{ij}]$, where*

$$m_{ij} = T_0 |a_{ij}|_{L^1(0,T_0)} + \overline{a}_{ij} + |b_{ij}|_{L^1(0,T_0)} + c_{ij} \quad \text{for } i, j \in I, \tag{3.2}$$

and

$$\overline{a}_{ij} = \int_{T_0}^{T} d\xi \int_0^{T_0} a_{ij}(\xi - \sigma) \, d\sigma,$$

is less than one.
Then the problem (1.1) *has at least one mild solution $u \in C([-\tau, T], X)$. In case that $m = n$, the solution u is unique.*

Proof. The integral system (3.1) can be seen as a fixed point equation $u = \Gamma(u)$ in $C([-\tau, T], X)$ for the nonlinear operator Γ from the space $C([-\tau, T], X)$ to itself, $\Gamma = (\Gamma_1, ..., \Gamma_n)$, where $\Gamma_i : C([-\tau, T], X) \to C([-\tau, T], X_i)$ are defined by

$$\begin{cases} \Gamma_i(u)(t) & = \alpha_i(u)(t), \quad t \in [-\tau, 0], \\ \Gamma_i(u)(t) & = S_i(t) \alpha_i(u)(0) + \int_0^t S_i(t-s) \int_0^s K_i(s-\sigma, u_\sigma) \, d\sigma ds \\ & \quad + \int_0^t S_i(t-s) F_i(s, u_s) \, ds, \quad t \in [0, T]. \end{cases} \tag{3.3}$$

Clearly, the operator Γ admits the representation $\Gamma = \Phi + \Psi$, where

$$\Phi = (\Gamma_1, ..., \Gamma_m, \Phi_{m+1}, ..., \Phi_n), \quad \Psi = (0, ..., 0, \Psi_{m+1}, ..., \Psi_n),$$

where for $i \in J_2$,

$$\Phi_i(u)(t) = \begin{cases} \alpha_i(u)(t), & t \in [-\tau, 0], \\ S_i(t) \alpha_i(u)(0), & t \in [0, T], \end{cases}$$

and

$$\Psi_i(u)(t) = \begin{cases} 0, \ t \in [-\tau, 0], \\ \int_0^t S_i(t-s) \int_0^s K_i(s-\sigma, u_\sigma) d\sigma ds + \int_0^t S_i(t-s) F_i(s, u_s) \, ds, \quad t \in [0, T]. \end{cases}$$

We shall apply the vector version of Krasnoselskii's fixed point theorem to the operator Γ on the space

$$E := C([-\tau, T], X) = C([-\tau, T], X_1) \times ... \times C([-\tau, T], X_n)$$

endowed with the vector-valued norm

$$\|u\| = (|u_1|_\tau, ..., |u_n|_\tau)^{tr},$$

where for each i, by $|u_i|_\tau$ we mean the norm in $C([-\tau,T],X_i)$ given by (1.4), with $\theta>0$ large enough chosen below, and to a bounded closed convex subset D of the form

$$\begin{aligned} D &= \{u=(u_1,...,u_n)\in C([-\tau,T],X):\ |u_i|_\tau \le R_i \ \text{ for } i\in I\} \\ &= \{u\in C([-\tau,T],X):\ \|u\|\le R\} \end{aligned}$$

with conveniently chosen radii $R_i,\ i\in I$. Here the notation R stands for the vector column $(R_1,...,R_n)^{tr}$. The result will follow from Theorem 2.2 once the following lemmas have been proved: □

Lemma 3.2. *There exists* $R\in\mathbb{R}^n_+$ *such that* $\|\Phi(u)+\Psi(v)\|\le R$ *for all* $u,v\in C([-\tau,T],X)$ *satisfying* $\|u\|,\|v\|\le R$.

Lemma 3.3. *The operator* Φ *is a generalized contraction in Perov's sense on* $C([-\tau,T],X)$.

Lemma 3.4. *The operator* Ψ *is completely continuous on* $C([-\tau,T],X)$.

Proof of Lemma 3.2. Let $R\in\mathbb{R}^n_+$. The result will follow once we have proved that

$$\|\Phi(u)+\Psi(v)\|\le \widetilde{M}R+\Lambda, \tag{3.4}$$

for all $u,v\in C([-\tau,T],X)$ with $\|u\|,\|v\|\le R$, and some vector $\Lambda\in\mathbb{R}^n_+$ and matrix $\widetilde{M}$ close enough M such that $\rho\left(\widetilde{M}\right)<1$. Indeed, in this case, we can find a vector $R\in\mathbb{R}^n_+$ such that

$$\widetilde{M}R+\Lambda\le R,$$

that is $\left(J-\widetilde{M}\right)R\ge\Lambda$, for example, the vector $R=\left(J-\widetilde{M}\right)^{-1}\Lambda$. The vector R belongs to $\mathbb{R}^n_+$ since the matrix $J-\widetilde{M}$ is inverse-positive as a consequence of the fact that $\rho\left(\widetilde{M}\right)<1$ (see, e.g., [34]).

Thus, in order to obtain (3.4) we need estimates of the norms $|\Phi_i(u)+\Psi_i(v)|_\tau$. Clearly, $\Phi_i(u)+\Psi_i(v)=\Gamma_i(u)$ for $i\in I_1$.

First note that from $(H_1)(a)$, for $v=0$,

$$|K_i(t,u)|_{X_i}\le\sum_{j=1}^n a_{ij}(t)\,|u_j|_{C([-\tau,0],X_j)}+|K_i(t,0)|_{X_i},$$

hence the inequality in $(H_1)(b)$ also holds for $i\in I_1$, with $d_i(t)=|K_i(t,0)|_{X_i}$. Similarly, the inequality in $(H_2)(b)$ holds for $i\in I_1$ with $f_i=|F_i(0)|_{C([-\tau,0],X_i)}$. Also, from (H$_3$), one has

$$|\alpha_i(u)|_{C([-\tau,0],X_i)}\le\sum_{j=1}^n c_{ij}\,|u_j|_{C([-\tau,T_0],X_j)}+h_i$$

for all $i \in I$ with $h_i = |\alpha_i(0)|_{C([-\tau,0],X_i)}$.
For $t \in [-\tau, 0]$, we have

$$\begin{aligned} |\alpha_i(u)(t)|_{X_i} &\leq |\alpha_i(u)|_{C([-\tau,0],X_i)} \leq \sum_{j=1}^{n} c_{ij} |u_j|_{C([-\tau,T_0],X_j)} + h_i \\ &\leq \sum_{j=1}^{n} c_{ij} |u_j|_\tau + h_i \leq \sum_{j=1}^{n} c_{ij} R_j + h_i \end{aligned} \tag{3.5}$$

For $t \in [0, T_0]$ and $i \in I_1$, since the semigroups are of contractions,

$$\begin{aligned} |\Gamma_i(u)(t)|_{X_i} \leq |\alpha_i(u)(0)|_{X_i} &+ \int_0^t \int_0^s |K_i(s-\sigma, u_\sigma)|_{X_i} \, d\sigma ds \\ &+ \int_0^t |F_i(s, u_s)|_{X_i} \, ds. \end{aligned} \tag{3.6}$$

From (3.5), the first term is estimated as above, that is

$$|\alpha_i(u)(0)|_{X_i} \leq \sum_{j=1}^{n} c_{ij} R_j + h_i, \tag{3.7}$$

while the integrals are estimated as follows:

$$\begin{aligned} &\int_0^t \int_0^s |K_i(s-\sigma, u_\sigma)|_{X_i} \, d\sigma ds \\ \leq\ &\int_0^t \int_0^s \left(\sum_{j=1}^{n} a_{ij}(s-\sigma) \left| (u_j)_\sigma \right|_{C([-\tau,0],X_j)} + d_i(s-\sigma) \right) d\sigma ds \\ =\ &\sum_{j=1}^{n} \int_0^t \int_0^s a_{ij}(s-\sigma) |u_j|_{C([\sigma-\tau,\sigma],X_j)} \, d\sigma ds + T_0 |d_i|_{L^1(0,T_0)} \end{aligned} \tag{3.8}$$

and

$$\begin{aligned} \int_0^t |F_i(s, u_s)|_{X_i} \, ds &\leq \int_0^t \left(\sum_{j=1}^{n} b_{ij}(s) \left| (u_j)_s \right|_{C([-\tau,0],X_j)} + f_i(s) \right) ds \\ &= \sum_{j=1}^{n} \int_0^t b_{ij}(s) |u_j|_{C([s-\tau,s],X_j)} \, ds + T_0 |f_i|_{L^1(0,T_0)}. \end{aligned} \tag{3.9}$$

Since $0 \leq s \leq t \leq T_0$, one has $|u_j|_{C([s-\tau,s],X_j)} \leq |u_j|_{C([-\tau,T_0],X_j)} \leq |u_j|_\tau$.

Then (3.8) and (3.9) give

$$\begin{aligned}\int_0^t\int_0^s |K_i\left(s-\sigma,u_\sigma\right)|_{X_i}\, d\sigma ds &\leq \sum_{j=1}^n \int_0^t\int_0^s a_{ij}\left(s-\sigma\right)|u_j|_\tau\, d\sigma ds + T_0|d_i|_{L^1(0,T_0)}\\ &= \sum_{j=1}^n |u_j|_\tau \int_0^t\int_0^s a_{ij}(s-\sigma)d\sigma ds + T_0|d_i|_{L^1(0,T_0)}\\ &\leq T_0\sum_{j=1}^n |a_{ij}|_{L^1(0,T_0)}R_j + T_0|d_i|_{L^1(0,T_0)} \qquad (3.10)\end{aligned}$$

and

$$\int_0^t |F_i\left(s,u_s\right)|_{X_i}\, ds \leq \sum_{j=1}^n |b_{ij}|_{L^1(0,T_0)}\, R_j + |f_i|_{L^1(0,T_0)}\,. \qquad (3.11)$$

Hence for $t\in[-\tau,T_0]$ and all $i\in I_1$, from (3.7), (3.10) and (3.11), we deduce that

$$\begin{aligned}|\Gamma_i\left(u\right)\left(t\right)|_{X_i} &\leq \sum_{j=1}^n \left(T_0\left|a_{ij}\right|_{L^1(0,T_0)} + |b_{ij}|_{L^1(0,T_0)} + c_{ij}\right)|u_j|_\tau + \lambda_i\\ &= \sum_{j=1}^n \left(m_{ij}-\overline{a}_{ij}\right)|u_j|_\tau + \lambda_i \qquad (3.12)\end{aligned}$$

where $\lambda_i = T_0\left|d_i\right|_{L^1(0,T_0)} + |f_i|_{L^1(0,T_0)} + h_i$. Therefore

$$|\Gamma_i\left(u\right)|_{C([-\tau,T_0],X_i)} \leq \sum_{j=1}^n \left(m_{ij}-\overline{a}_{ij}\right)|u_j|_\tau + \lambda_i. \qquad (3.13)$$

Next we estimate

$$|\Gamma_i\left(u\right)|_{C_\theta([T_0-\tau,T],X_i)} = \max_{t\in[T_0,T]}\left(|\Gamma_i\left(u\right)|_{C([t-\tau,t],X_i)}\, e^{-\theta(t-T_0)}\right)\ \left(i\in I_1\right).$$

To do this, take any $t\in[T_0,T]$ and $s\in[t-\tau,t]$. For $s\leq T_0$, we already have the estimate given by (3.13). Let $s\in[T_0,t]$. Then

$$\begin{aligned}&\Gamma_i\left(u\right)\left(s\right)\\ =\ &\Gamma_i\left(u\right)\left(T_0\right) + \int_{T_0}^s S_i\left(s-\xi\right)F_i\left(\xi,u_\xi\right)d\xi + \int_{T_0}^s S_i\left(s-\xi\right)\int_0^\xi K_i\left(\xi-\sigma,u_\sigma\right)d\sigma d\xi\\ =\ &\Gamma_i\left(u\right)\left(T_0\right) + \int_{T_0}^s S_i\left(s-\xi\right)F_i\left(\xi,u_\xi\right)d\xi + \int_{T_0}^s S_i\left(s-\xi\right)\int_0^{T_0} K_i\left(\xi-\sigma,u_\sigma\right)d\sigma d\xi\\ &+\int_{T_0}^s S_i\left(s-\xi\right)\int_{T_0}^\xi K_i\left(\xi-\sigma,u_\sigma\right)d\sigma d\xi.\end{aligned}$$

Using (H_1)(b), one has

$$\left|\int_{T_0}^s S_i\left(s-\xi\right)\int_0^{T_0} K_i\left(\xi-\sigma,u_\sigma\right)d\sigma d\xi\right|_{X_i} \leq \sum_{j=1}^n \overline{a}_{ij}\left|u_j\right|_\tau + |f_i|_{L^1(0,T_0)}\,,$$

where

$$\overline{a}_{ij} = \int_{T_0}^{T} d\xi \int_0^{T_0} a_{ij}(\xi - \sigma)\, d\sigma.$$

Furthermore

$$\begin{aligned}
&\left| \int_{T_0}^{s} S_i(s-\xi) \int_{T_0}^{\xi} K_i(\xi - \sigma, u_\sigma)\, d\sigma d\xi \right|_{X_i} \\
&\leq \int_{T_0}^{s} \int_{T_0}^{\xi} \left(\sum_{j=1}^{n} a_{ij}(\xi - \sigma) \left| (u_j)_\sigma \right|_{C([-\tau,0],Z_j)} e^{-\theta(\sigma - T_0)} e^{\theta(\sigma - T_0)} + d_i(\xi - \sigma) \right) d\sigma d\xi \\
&\leq \sum_{j=1}^{n} |u_j|_\tau \int_{T_0}^{s} \int_{T_0}^{\xi} a_{ij}(\xi - \sigma)\, e^{\theta(\sigma - T_0)} d\sigma d\xi + (T - T_0) |d_i|_{L^1(0,T-T_0)}.
\end{aligned}$$

Next using Holder's inequality gives

$$\begin{aligned}
\left| \int_{T_0}^{s} S_i(s-\xi) \int_{T_0}^{\xi} K_i(\xi - \sigma, u_\sigma)\, d\sigma d\xi \right|_{X_i} &\leq \frac{1}{\theta (q\theta)^{1/q}} e^{\theta(t - T_0)} \sum_{j=1}^{n} |a_{ij}|_{L^p(0,T-T_0)} |u_j|_\tau \\
&\quad + (T - T_0) |d_i|_{L^1(0,T-T_0)}.
\end{aligned}$$

Similar arguments yield

$$\left| \int_{T_0}^{s} S_i(s-\xi) F_i(\xi, u_\xi)\, d\xi \right|_{X_i} \leq \frac{1}{\theta} e^{\theta(t-T_0)} \sum_{j=1}^{n} |b_{ij}|_{L^p(T_0,T)} |u_j|_\tau + |f_i|_{L^1(T_0,T)}$$

It follows that

$$|\Gamma_i(u)(s)|_{X_i} \leq \sum_{j=1}^{n} \widetilde{m_{ij}} |u_j|_\tau e^{\theta(t-T_0)} + \Lambda_i \quad \text{for} \;\; s \in [t - \tau, t],$$

where

$$\widetilde{m_{ij}} = m_{ij} + \frac{1}{\theta (q\theta)^{1/q}} |a_{ij}|_{L^1(0,T-T_0)} + \frac{1}{\theta} |b_{ij}|_{L^p(T_0,T)},$$
$$\Lambda_i = \lambda_i + |f_i|_{L^1(0,T)} + (T - T_0) |d_i|_{L^1(0,T-T_0)}.$$

This gives the estimate

$$|\Gamma_i(u)|_{C_\theta([T_0-\tau,T],X_i)} \leq \sum_{j=1}^{n} \widetilde{m_{ij}} R_j + \Lambda_i.$$

Also taking into account (3.13), we may conclude that

$$|\Phi_i(u) + \Psi_i(v)|_\tau = |\Gamma_i(u)|_\tau \leq \sum_{j=1}^{n} \widetilde{m_{ij}} R_j + \Lambda_i \quad \text{for} \;\; i \in I_1.$$

Since for $i \in I_2$, the structure of $\Phi_i(u) + \Psi_i(v)$ is analogue to that of Γ_i, we easily see that we also have

$$|\Phi_i(u) + \Psi_i(v)|_\tau \leq \sum_{j=1}^{n} \widetilde{m_{ij}} R_j + \Lambda_i \quad \text{for} \;\; i \in I_2.$$

Hence (3.4) holds with $\widetilde{M}=[\widetilde{m_{ij}}]$ and $\Lambda=(\Lambda_1,...,\Lambda_n)^{tr}$. Clearly, the matrix $\widetilde{M}$ is close enough to M if θ is sufficiently large. □

Proof of Lemma 3.3. Similar estimations to those in the proof of Lemma 3.2 give for $i\in I_1$ and any $u,v\in C([-\tau,T],X)$,

$$|\Gamma_i(u)-\Gamma_i(v)|_{C([-\tau,T_0],X_i)}\leq\sum_{j=1}^n m_{ij}|u_j-v_j|_{C([-\tau,T_0],X_j)}$$

and

$$|\Gamma_i(u)-\Gamma_i(v)|_{C_\theta([T_0-\tau,T],X_i)}\leq\sum_{j=1}^n \widetilde{m}_{ij}|u_j-v_j|_\tau.$$

Hence

$$|\Gamma_i(u)-\Gamma_i(v)|_\tau\leq\sum_{j=1}^n \widetilde{m}_{ij}|u_j-v_j|_\tau\quad(i\in I_1).$$

For $i\in I_2$, from (H_3), we obtain

$$|\Phi_i(u)-\Phi_i(v)|_\tau\leq\sum_{j=1}^n c_{ij}|u_j-v_j|_\tau\quad(i\in I_2).$$

Consequently,

$$\|\Phi(u)-\Phi(v)\|\leq\widehat{M}\,\|u-v\|,\tag{3.14}$$

where $\widehat{M}$ is the $n\times n$ square matrix $[\widehat{m}_{ij}]$, with

$$\widehat{m}_{ij}=\begin{cases}\widetilde{m}_{ij} & \text{for } i\in I_1,\ j\in I\\ c_{ij} & \text{for } i\in I_2,\ j\in I.\end{cases}$$

Clearly $\widehat{M}\leq\widetilde{M}$, hence according to Proposition 2.3, the spectral radius of $\widehat{M}$ is less than one. Then (3.14) shows that Φ is a generalized contraction in Perov's sense. □

Proof of Lemma 3.4. The first components of Ψ for $i\in I_1$ are zero, so compact. The growth conditions for F_i and K_i $(i\in I_2)$ and the boundedness of D guarantee the uniform integrability of the set $\{\Psi_i(u):\ u\in D\}$. Since in addition for $i\in I_2$, the semigroups generated by A_i are compact, we may apply the compactness criterion from Lemma 2.4 to conclude that the operator Ψ_i is compact on D for every $i\in I_2$. □

Remark 3.5. It is useful to analyze the elements of the matrix M to conclude about the contributions of the nonlinear terms to the sufficient condition for the existence of solutions. They show that $b_{ij}(t)$ can be however large for $T_0<t\leq T$. The same happens for $a_{ij}(t)$ $(t\in[0,T])$ and $b_{ij}(t)$ $(t\in[0,T_0])$ provided that T_0 is sufficiently small. Also note the special contribution of $\overline{a}_{ij}$ in connection with the "convolution type" integral term of problem (1.1), which is null if $T_0=0$ or $T_0=T$.

We conclude by two examples illustrating our main result.

Example 3.6. Consider the semilinear integrodifferential equation

$$\frac{\partial}{\partial t}u(t,x)-\Delta u(t,x)=\int_0^t\kappa(t-s,u(s,x))ds+\mu(t)u(t-\tau,x),\quad t\in[0,T],\ x\in\Omega,$$

subject to the Dirichlet condition $u(t,x) = 0$ for $x \in \partial\Omega$, and to the nonlocal initial condition

$$u(t,x) = \lambda u(t+T,x), \quad \textit{for} \quad x \in \Omega,\ t \in [-\tau, 0].$$

Here $\Omega \subset \mathbb{R}^N$ is a smooth bounded domain, $\tau \geq 0$, $0 < \lambda < 1$, $\kappa : [0,T] \times \mathbb{R} \to \mathbb{R}$ and $\mu : [0,T] \to \mathbb{R}$ are continuous functions. The problem is of type (1.1), where $n = m = 1$, $X = L^2(\Omega)$, $A = -\Delta$, $D(A) = H^2(\Omega) \cap H_0^1(\Omega)$, K, F, α are defined as follows:

$$\begin{aligned} K, F &: [0,T] \times C([-\tau,0], L^2(\Omega)) \to L^2(\Omega), \\ K(t,v) &= \kappa(t, v(0)), \qquad v \in C\left[[-\tau,0], L^2(\Omega)\right] \\ F(t,v) &= \mu(t)\, v(-\tau); \end{aligned}$$

$$\alpha : C\left([-\tau,T], L^2(\Omega)\right) \to C\left([-\tau,0], L^2(\Omega)\right), \quad \alpha(v)(t) = \lambda v(t+T).$$

It is clear that $T_0 = T$ and (H_2) and (H_3) hold with $b_{11}(t) = \mu(t)$ and $c_{11} = \lambda$. Also (H_1) holds if there is a function $\gamma \in C([0,T], \mathbb{R}_+)$ such that

$$|\kappa(t,y) - \kappa(t,z)| \leq \gamma(t)\, |y - z| \qquad \textit{for all} \quad t \in [0,T] \quad \textit{and} \quad y, z \in \mathbb{R}.$$

It is easy to check that $a_{11}(t) = \gamma(t)$. Also $\overline{a}_{11} = 0$. Therefore, Theorem 3.1 yields the following conclusion: If

$$T\, |\gamma|_{L^1(0,T)} + |\mu|_{L^1(0,T)} < 1 - \lambda,$$

then the problem has a unique mild solution $u \in C\left([-\tau,T], L^2(\Omega)\right)$.

Example 3.7. Let us consider a semilinear reaction-diffusion integrodifferential system with Neumann boundary conditions and multi-point nonlocal initial conditions

$$\begin{cases} \dfrac{\partial u}{\partial t}(t,x) - \varkappa_1 \Delta u(t,x) = \displaystyle\int_0^t \kappa_1(t-s, u(s,x))ds - \lambda_1 u(t,x) + \mu_1(t) v(t-\tau,x), & \text{in } Q, \\ \dfrac{\partial v}{\partial t}(t,x) - \varkappa_2 \Delta v(t,x) = \displaystyle\int_0^t \kappa_2(t-s, v(s,x))ds + \mu_2(t) u(t-\tau,x) - \lambda_2 v(t,x), & \text{in } Q, \\ \dfrac{\partial}{\partial \nu} u(t,x) = \dfrac{\partial}{\partial \nu} v(t,x) = 0, & \text{on } \Sigma, \\ u(t,x) = \varphi(t)(x) + \displaystyle\sum_{k=1}^{p_1} \beta_{1k} u(t_{1k} + t, x), & \text{in } Q_\tau, \\ v(t,x) = \psi(t)(x) + \displaystyle\sum_{k=1}^{p_2} \beta_{2k} v(t_{2k} + t, x), & \text{in } Q_\tau, \end{cases} \tag{3.15}$$

where $Q = [0,T] \times \Omega$, $\Sigma = [0,T] \times \partial\Omega$, $Q_\tau = [-\tau, 0] \times \Omega$, $\Omega \subset R^N$ is a smooth bounded domain, $\varkappa_1, \varkappa_2, \lambda_1, \lambda_2 > 0$, $\tau \geq 0$ and $0 < t_{i1} < \ldots < t_{ip_i} \leq T$ for $i = 1, 2$. We assume that κ_1, $\kappa_2 : [0,T] \times \mathbb{R} \to R$ are continuous; φ, $\psi \in C\left([-\tau,0], L^2(\Omega)\right)$, and $\mu_i \in C([0,T]; \mathbb{R}_+)$, $i = 1, 2$.

We apply Theorem 3.1 with $X_1 = X_2 = L^2(\Omega)$, and to the operators $A_i : D(A_i) \to L^2(\Omega)$ $(i = 1, 2)$ given by

$$\begin{aligned} D(A_i) &= \left\{ u \in H^2(\Omega) : \frac{\partial u}{\partial \nu} = 0 \ on\ \partial\Omega \right\}, \\ A_i u &= \varkappa_i \Delta u - \lambda_i u, \end{aligned}$$

which generate compact semigroups [9, Theorem 1.11.8].
Here $I_1 = \emptyset$ and $I_2 = I = \{1, 2\}$,

$$c_{ii} = \sum_{k=1}^{p_i} |\beta_{ik}| \ \ (i = 1, 2),$$

$$c_{12} = c_{21} = 0, \ b_{11} = b_{22} = 0$$

and

$$b_{12}(t) = \mu_1(t), \ b_{21}(t) = \mu_2(t).$$

Also $T_0 = \max\{t_{ij} : \ j = 1, \ldots, p_i; \ i = 1, 2\}$.
Assume that the functions κ_1 and κ_2 are bounded, i.e.,

$$|\kappa_i(t, y)| \le d_i, \quad i = 1, 2, \quad for\ all \quad t \in [0, T] \quad and \quad y \in \mathbb{R}.$$

Then $a_{ij} = 0$ for $i, j = 1, 2$. Therefore, according to Theorem 3.1, if the spectral radius of the matrix

$$M = \begin{bmatrix} \sum_{k=1}^{p_1} |\beta_{1k}| & |\mu_1|_{L^1(0,T_0)} \\ |\mu_2|_{L^1(0,T_0)} & \sum_{k=1}^{p_2} |\beta_{2k}| \end{bmatrix}$$

is less than one, then the problem (3.15) has at least one mild solution in $C([-\tau, T], L^2(\Omega) \times L^2(\Omega))$.

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Existence and multiplicity of positive radial solutions to the Dirichlet problem for nonlinear elliptic equations on annular domains

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Abstract. In this paper, we study the existence and nonexistence of monotone positive radial solutions of elliptic boundary value problems on bounded annular domains subject to local boundary condition. By using Krasnoselskii's fixed point theorem of cone expansion-compression type we show that there exists $\lambda^* \geq \lambda_* > 0$ such that the elliptic equation has at least two, one and no radial positive solutions for $0 < \lambda \leq \lambda_*$, $\lambda_* < \lambda \leq \lambda^*$ and $\lambda > \lambda^*$ respectively. We include an example to illustrate our results.

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1. Introduction

In this paper, we are interested in the existence of radial positive solutions to the following, boundary value problem BVP

$$\begin{cases} -\triangle u(x) = \lambda f(|x|, u(x)), & x \in \Omega, \\ u(x) = 0, & x \in \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega = \left\{x \in \mathbb{R}^N : a < |x| < b\right\}$ with $1 < a < b$ is an annulus in $\mathbb{R}^N$ $(N \geq 3)$, $f \in C([a,b] \times [0,\infty), [0,\infty))$ and λ is a positive parameter.

The study of such problems is motivated by a lot of physical applications starting from the well-known Poisson-Boltzmann equation (see [2, 20, 30]), also they serve as models for some phenomena which arise in fluid mechanics, such as the exothermic chemical reactions or autocatalytic reactions (see [27, Section 5.11.1]). The nonlinearity f in applications always has a special form and here we assume only the continuity of f and some inequalities at some points for the values of this function. However, we know that in the integrand should stay a superposition of u with a given function (usually the exponent of u in applications) instead of u alone, but we treat this paper as the first step in this direction. The method we use is typical for local boundary value problems. We shall formulate an equivalent fixed point problem and look for its solution in the cone of nonnegative function in an appropriate Banach space. The most popular fixed point theorem in a cone is the cone-compression and cone-expansion theorem due to M. Krasnosel'skii [19] which we use in the form taken from [16]. We also point out the fact that problems of type (1.1) when equation does not contain parameter λ, are connected with the classical boundary value theory of Bernstein [1] (see also the studies of Granas, Gunther and Lee [15] for some extensions to nonlinear problems).

The existence and uniqueness of positive radial solutions for equations of type (1.1) when equation does not contain paramete λ, were obtained in [5], [21], [32]. Wang [32] proved that if $f : (0,\infty) \to (0,\infty)$ satisfies $\lim\limits_{z\to 0} \frac{f(z)}{z} = \infty$ and $\lim\limits_{z\to\infty} \frac{f(z)}{z} = 0$ then problem (1.1) when equation does not contain paramete λ, has a positive radial solution in $\Omega = \left\{x \in \mathbb{R}^N,\ N > 2\right\}$. That result was extended for the systems of elliptic equations by Ma [23]. We quote also the research of Ovono and Rougirel [28] where the diffusion at each point depends on all the values of the solutions in a neighborhood of this point and Chipot et al. [11], [12]. For example in [11] considered the solvability of a class of nonlocal problems which admit a formulation in term of quasi-variational inequalities. There is a wide literature that deals with existence multiplicity results for various second-order, fourth-order and higher-order boundary value problems by different approaches, see [5], [8], [6], [7], [10], [17], [25], [22].

In 2011, Bohneure et al. [4] Studied the existence of positive increasing radial solutions for superlinear Neumann problem in the unit ball B in $\mathbb{R}^N,\ N \geq 2$,

$$\begin{cases} -\Delta u + u = a\left(|x|\right) f\left(u\right), & \text{in} \quad B, \\ u > 0, & \text{in} \quad B, \\ \partial_t u = 0, & \text{on} \quad \partial B, \end{cases}$$

where $a \in C^1\left([0,1], \mathbb{R}\right),\ a(0) > 0$ is nondecreasing, $f \in C^1\left([0,1], \mathbb{R}\right),\ f(0) = 0$,

$$\lim_{s\to 0^+} \frac{f(s)}{s} = 0 \text{ and } \lim_{s\to +\infty} \frac{f(s)}{s} > \frac{1}{a(0)}.$$

In 2011, Hakimi and Zertiti, [17] studied the nonexistence of radial positive solutions for a nonpositone problem when the nonlinearity is superlinear and has more than one zero,

$$\begin{cases} -\triangle u(x) = \lambda f\left(u(x)\right), & x \in \Omega, \\ u(x) = 0, & x \in \partial\Omega, \end{cases}$$

where $f \in C\left([0,+\infty), \mathbb{R}\right)$.

In 2014, Sfecci [31], obtained the existence result by introduced the lim sup and lim inf types of nonresonance condition below the first positive eigenvalue for the following Neumann problems defined on the ball $B_R = \left\{x \in \mathbb{R}^N,\ |x| < R\right\}$,

$$\begin{cases} -\triangle u(x) = f\left(u(x)\right) + e\left(|x|\right), & \text{in} \quad B_R, \\ u(x) = 0, & \text{on} \quad \partial B_R, \end{cases}$$

where $f \in C(\mathbb{R}, \mathbb{R})$ and $e \in C([0, R], \mathbb{R})$.
In 2014, Butler et. al, [9] studied the positive radial solutions to the boundary value problem

$$\begin{cases} -\Delta u + u = \lambda a(|x|) f(u), & x \in \Omega, \\ \frac{\partial u}{\partial \eta} + \bar{c}(u) u = 0, & |x| = r_0, \\ u(x) \to 0, & |x| \to \infty, \end{cases}$$

where $f \in C([0, \infty), \mathbb{R}), \Omega = \{x \in \mathbb{R}^N : N > 2, |x| > r_0 \text{ with } r_0 > 0\}$, λ is a positive parameter, $a \in C([r_0, \infty), \mathbb{R}^+)$ such that $\lim_{r \to \infty} a(r) = 0$, $\frac{\partial}{\partial u}$ is the outward normal derivative and $\bar{c} \in C([0, \infty), (0, \infty))$.
Instead of working directly with (1.1), we note that the change of variable

$$u(x) = u(|x|), \ t = |x|$$

transforms (1.1) into the following boundary value problem (for details, see [14]:

$$\begin{cases} -u''(t) - \frac{N-1}{t} u(t) = \lambda f(t, u(t)), \ t \in (a, b), \\ u(a) = u(b) = 0, \end{cases}$$

where $\lambda \geq 0$ is a positive parameter and $f \in C([a, b] \times [0, \infty), [0, \infty))$.

Inspired and motivated by the works mentioned above, we deal with existence and nonexistence of radial positive solutions to the BVP (1.1) i.e., an equivalant problem (2.1) by using of the fixed point theorem together with the properties of Green's function and we impose certain conditions on f. The paper is organized as follows. In Section 2, we present that a nontrivial and nonnegative solution of BVP (2.1) is monotone positive solution. In Section 3, we obtain some results of the existence, multiplicity and nonexistence positive solutions for BVP (2.1) depends on the parameter λ and we give an example to illustrate our results.

2. Preliminaries

We shall consider the Banach space $E = C[a, b]$ equipped with sup norm

$$\|u\| = \max_{a \leq t \leq b} |u(t)|,$$

and $C^+[a, b]$ is the cone of nonnegative functions in $C[a, b]$, where $1 < a < b$.

Definition 2.1. A nonempty closed and convex set $P \subset E$ is called a cone of E if it satisfies
(i) $u \in P, \ r > 0$ implies $ru \in P$,
(ii) $u \in P, \ -u \in P$ implies $u = \theta$, where θ denote the zero element of E.

Definition 2.2. A cone P is said to be normal if there exists a positive number N called the normal constant of P, such that $\theta \leq u \leq v$ implies $\|u\| \leq N \|v\|$.

We are interested in finding radial solutions for problem (1.1). We proceed as in introduction, setting $u(x) = u(|x|)$, $t = |x|$, we have the following equivalent boundary value problem

$$\begin{cases} -u''(t) - \frac{N-1}{t} u(t) = \lambda f(t, u(t)), \ t \in (a,b), \\ u(a) = u(b) = 0. \end{cases} \tag{2.1}$$

We observe that the existence and nonexistence of radial positive solutions of (1.1) is equivalent to the existence and nonexistence of positive solutions of the problem (2.1).

In arriving our results, we need the following six preliminary lemmas. The first one is well known.

Lemma 2.3. (see [13]) *Let $y(\cdot) \in C[a,b]$. If $u \in C^4[a,b]$, then the BVP*

$$\begin{cases} -u''(t) - \frac{N-1}{t} u(t) = y(t), \ t \in (a,b), \\ u(a) = u(b) = 0, \end{cases}$$

has a unique solution

$$u(t) = \int_a^b s^{N-1} G(t,s) y(s) \, ds, \ N > 2,$$

where

$$G(t,s) = \begin{cases} \frac{\left(1-\left(\frac{a}{s}\right)^{N-2}\right)\left(\left(\frac{b}{t}\right)^{N-2}-1\right)}{(N-2)(b^{N-2}-a^{N-2})}, & a \le t \le s \le b, \\ \frac{\left(1-\left(\frac{a}{t}\right)^{N-2}\right)\left(\left(\frac{b}{s}\right)^{N-2}-1\right)}{(N-2)(b^{N-2}-a^{N-2})}, & a \le s \le t \le b. \end{cases} \tag{2.2}$$

Lemma 2.4. *For any $(t,s) \in [a,b] \times [a,b]$, we have*

$$\frac{\left(1-\left(\frac{a}{t}\right)^{N-2}\right)}{(N-2)(b^{N-2}-a^{N-2})} \le G(t,s) \le \frac{\left(\left(\frac{b}{t}\right)^{N-2}-1\right)}{(N-2)(b^{N-2}-a^{N-2})}, \tag{2.3}$$

and

$$0 \le \frac{\partial G}{\partial t}(t,s) \le \frac{\left(\left(\frac{b}{s}\right)^{N-2}-1\right)\left(\frac{(N-2)b}{a^{N-1}}\right)}{(N-2)(b^{N-2}-a^{N-2})}, \ (t,s) \in [a,b] \times [a,b]. \tag{2.4}$$

Proof. The proof is evident, we omit it. □

Lemma 2.5. (see [10]) *For $y(\cdot) \in C^+[a,b]$. Then the unique solution $u(t)$ of BVP*

$$\begin{cases} -u''(t) - \frac{N-1}{t} u(t) = y(t), \ t \in (a,b), \\ u(a) = u(b) = 0. \end{cases}$$

is nonnegative and satisfies

$$\min_{a_1 \le t \le b_1} u(t) \ge c \|u\|,$$

where $c = \frac{\min\left\{\left(\frac{b}{b_1}\right)^{N-2}-1, 1-\left(\frac{a}{a_1}\right)^{N-2}\right\}}{\max\left\{\left(\frac{b}{a}\right)^{N-2}-1, 1-\left(\frac{a}{b}\right)^{N-2}\right\}}$ *and $a_1, b_1 \in (a,b)$ with $a_1 < b_1$.*

If we let

$$P=\left\{u\in C^{+}\left[a,b\right]:\quad \min_{a_1\le t\le b_1}u\left(t\right)\ge c\left\|u\right\|\right\},\tag{2.5}$$

then it is easy to see that P is a cone in $C\left[a,b\right]$. It is evident that BVP (2.1) has an integral formulation given by

$$u\left(t\right)=\lambda\int_a^b s^{N-1}G\left(t,s\right)f\left(s,u\left(s\right)\right)ds,$$

where G defined in (2.2).
Now, we define an integral operator $T_\lambda : P\to C\left[a,b\right]$ by

$$\left(T_\lambda u\right)\left(t\right)=\lambda\int_a^b s^{N-1}G\left(t,s\right)f\left(s,u\left(s\right)\right)ds.$$

Lemma 2.6. *Let $y\in C^{+}\left[a,b\right]$. If $u\in C^{2}\left[a,b\right]$ satisfies*

$$\begin{cases} -u''\left(t\right)-\frac{N-1}{t}\ u(t)=y\left(t\right),\ t\in(a,b),\\ u\left(a\right)=0,\ u\left(b\right)=0,\end{cases}$$

then
(i) $u\left(t\right)\ge 0$ for $t\in\left[a,b\right]$,
(ii) $u'\left(t\right)\ge 0$ for $t\in\left[a,b\right]$.

Proof. From Lemma 2.4, we obtain $u\left(t\right)\ge 0$ and $u'\left(t\right)\ge 0$ for $t\in\left[a,b\right]$. □

Lemma 2.7. *$T_\lambda\left(P\right)\subset P$.*

Proof. For any $u\in P$, we have

$$\begin{aligned}\min_{a_1\le t\le b_1}T_\lambda u\left(t\right)&=\frac{\lambda}{\left(N-2\right)\left(b^{N-2}-a^{N-2}\right)}\min_{a_1\le t\le b_1}\left\{\int_a^t\left(1-\left(\frac{a}{s}\right)^{N-2}\right)s^{N-1}f\left(s,u\left(s\right)\right)\right.\\
&\quad\left.\times\left(\left(\frac{b}{t}\right)^{N-2}-1\right)ds+\int_t^b\left(1-\left(\frac{a}{t}\right)^{N-2}\right)\left(\left(\frac{b}{s}\right)^{N-2}-1\right)s^{N-1}f\left(s,u\left(s\right)\right)ds\right\}\\
&\ge\frac{\lambda}{\left(N-2\right)\left(b^{N-2}-a^{N-2}\right)}\min_{a_1\le t\le b_1}\left\{\int_a^t\left(1-\left(\frac{a}{s}\right)^{N-2}\right)\left(\left(\frac{b}{b_1}\right)^{N-2}-1\right)\right.\\
&\quad\left.\times s^{N-1}f\left(s,u\left(s\right)\right)ds+\int_t^b\left(1-\left(\frac{a}{a_1}\right)^{N-2}\right)\left(\left(\frac{b}{s}\right)^{N-2}-1\right)s^{N-1}f\left(s,u\left(s\right)\right)ds\right\}\\
&\ge\frac{\lambda\min\left\{\left(\frac{b}{b_1}\right)^{N-2}-1,1-\left(\frac{a}{a_1}\right)^{N-2}\right\}}{\left(N-2\right)\left(b^{N-2}-a^{N-2}\right)}\min_{a_1\le t\le b_1}\left\{\int_a^t s^{N-1}f\left(s,u\left(s\right)\right)\right.\end{aligned}$$

$$\times \left(1-\left(\frac{a}{s}\right)^{N-2}\right) ds + \int_t^b \left(\left(\frac{b}{s}\right)^{N-2} - 1\right) s^{N-1} f(s,u(s))\, ds \Bigg\}$$

$$= \frac{\lambda \min\left\{\left(\frac{b}{b_1}\right)^{N-2} - 1, 1-\left(\frac{a}{a_1}\right)^{N-2}\right\}}{(N-2)\left(b^{N-2}-a^{N-2}\right)} \min_{a_1 \le t \le b_1} \Bigg\{ \int_a^t \frac{\left(\frac{b}{s}\right)^{N-2}-1}{\left(\frac{b}{s}\right)^{N-2}-1} s^{N-1} f(s,u(s))$$

$$\times \left(1-\left(\frac{a}{s}\right)^{N-2}\right) ds + \int_t^b \frac{1-\left(\frac{a}{s}\right)^{N-2}}{1-\left(\frac{a}{s}\right)^{N-2}} \left(\left(\frac{b}{s}\right)^{N-2} - 1\right) s^{N-1} f(s,u(s))\, ds \Bigg\}$$

$$\ge \frac{\lambda \min\left\{\left(\frac{b}{b_1}\right)^{N-2} - 1, 1-\left(\frac{a}{a_1}\right)^{N-2}\right\}}{(N-2)\left(b^{N-2}-a^{N-2}\right)} \min_{a_1 \le t \le b_1} \Bigg\{ \int_a^t \frac{\left(\frac{b}{s}\right)^{N-2}-1}{\left(\frac{b}{a}\right)^{N-2}-1} s^{N-1} f(s,u(s))$$

$$\times \left(1-\left(\frac{a}{s}\right)^{N-2}\right) ds + \int_t^b \frac{1-\left(\frac{a}{s}\right)^{N-2}}{1-\left(\frac{a}{b}\right)^{N-2}} \left(\left(\frac{b}{s}\right)^{N-2} - 1\right) s^{N-1} f(s,u(s))\, ds \Bigg\}$$

$$\ge \frac{c\lambda}{(N-2)\left(b^{N-2}-a^{N-2}\right)} \min_{a_1 \le t \le b_1} \Bigg\{ \int_a^t \left(\left(\frac{b}{s}\right)^{N-2} - 1\right)\left(1-\left(\frac{a}{s}\right)^{N-2}\right)$$

$$s^{N-1} f(s,u(s))\, ds + \int_t^b \left(1-\left(\frac{a}{s}\right)^{N-2}\right)\left(\left(\frac{b}{s}\right)^{N-2} - 1\right) s^{N-1} f(s,u(s))\, ds \Bigg\}$$

$$= \frac{c\lambda}{(N-2)\left(b^{N-2}-a^{N-2}\right)} \int_a^b \left(\left(\frac{b}{s}\right)^{N-2} - 1\right)\left(1-\left(\frac{a}{s}\right)^{N-2}\right) s^{N-1} f(s,u(s)\, ds)$$

$$\ge \frac{c\lambda}{(N-2)\left(b^{N-2}-a^{N-2}\right)} \max_{a \le t \le b} \Bigg\{ \int_a^t \left(\left(\frac{b}{s}\right)^{N-2} - 1\right)\left(1-\left(\frac{a}{s}\right)^{N-2}\right)$$

$$\times s^{N-1} f(s,u(s))\, ds + \int_t^b \left(1-\left(\frac{a}{s}\right)^{N-2}\right)\left(\left(\frac{b}{s}\right)^{N-2} - 1\right) s^{N-1} f(s,u(s))\, ds \Bigg\}$$

$$\ge \frac{c\lambda}{(N-2)\left(b^{N-2}-a^{N-2}\right)} \max_{a \le t \le b} \Bigg\{ \int_a^t \left(\left(\frac{b}{t}\right)^{N-2} - 1\right)\left(1-\left(\frac{a}{s}\right)^{N-2}\right)$$

$$\times s^{N-1} f(s,u(s))\, ds + \int_t^b \left(1-\left(\frac{a}{t}\right)^{N-2}\right)\left(\left(\frac{b}{s}\right)^{N-2} - 1\right) s^{N-1} f(s,u(s))\, ds \Bigg\}$$

$$= c \max_{a \le t \le b} T_\lambda u(t) = c \left\| T_\lambda u \right\|.$$

In other words, we find,

$$\max_{a_1 \le t \le b_1} T_\lambda u(t) = \|T_\lambda u\|, \ \forall u \in P.$$

Thus, we get that $T_\lambda : P \to P$ is well defined. Moreover, it is easy to show that T_λ is completely continuous. □

If we let

$$K = \{u \in P / u(t) \ \textit{is nondecreasing}\},$$

then, it is easy to show that $K \subset P$ is also a cone in E.

Lemma 2.8. $T_\lambda(P) \subset K$.

Proof. It follows from Lemma 2.6 (ii) and Lemma 2.7. □

Lemma 2.9. $T_\lambda : K \to K$ *is completely continuous.*

Proof. Let $D \subset K$ is a bounded subset. Then there exists a positive constanty M_1 such that

$$\|u\| \le M_1, \ \forall u \in D$$

Now, we shall prove that $T_\lambda(D)$ is relatively compact in K.
Suppose that $(y_k)_{k\in\mathbb{N}^\star} \subset T_\lambda(D)$. Then there exist $(x_k)_{k\in\mathbb{N}^\star} \subset D$, such that

$$y_k = Ax_k$$

Let $M_2 = \sup_{a\le t\le b} |f(t, u(t))|$ for all $(t, u) \in [a, b] \times [0, M_1]$. For any $k \in \mathbb{N}^*$, by Lemma 2.2, we have

$$|y_k(t)| = |(T_n x_k)(t)| = \lambda \left| \int_a^b s^{N-1} G(t,s) f(s, x_k(s)) \, ds \right|$$

$$\le \lambda M_2 \int_a^b s^{N-1} G(t,s) \, ds$$

$$\le \frac{1}{(N-2)(b^{N-2} - a^{N-2})} \lambda M_2 \left(\left(\frac{b}{t}\right)^{N-2} - 1 \right) \int_a^b s^{N-1} ds$$

$$\le \frac{b^N - a^N}{N(N-2)(b^{N-2} - a^{N-2})} \lambda M_2 \left(\left(\frac{b}{a}\right)^{N-2} - 1 \right),$$

which implies that $(y_k(t))_{k\in\mathbb{N}^\star}$ is uniformly bounded.

Now, we show that T_λ is equicontinuous. For any $u \in K$, $n \ge 2$, and $t_1, t_2 \in [a, b]$ with $|t_1 - t_2| < \delta$, we have

$$|y_k(t_1) - y_k(t_2)| = |T_\lambda u(t_1) - T_\lambda u(t_2)|$$

$$\le \left| \lambda \int_a^b s^{N-1} (G(t_1, s) - G(t_2, s)) f(s, x_k(s)) \, ds \right|$$

$$\leq \lambda M_2 \int_a^b s^{N-1} \left| G(t_1, s) - G(t_2, s) \right| ds.$$

It follows from the uniform continuity of Green's function G on $[a, b] \times [a, b]$, that for any $\varepsilon > 0$, we have

$$\left| G(t_1, s) - G(t_2, s) \right| \leq \frac{\varepsilon N}{\lambda (b^N - a^N) M_2}, \quad for\ t_1, t_2, s \in [a, b],\ |t_1 - t_2| < \delta.$$

Then

$$\begin{aligned} |y_k(t_1) - y_k(t_2)| &= |T_\lambda u(t_1) - T_\lambda u(t_2)| \\ &\leq \lambda M_2 \int_a^b s^{N-1} \left| G(t_1, s) - G(t_2, s) \right| ds \\ &\leq \varepsilon. \end{aligned}$$

Therefore, T_λ is equicontinuous. By the Ascoli-Arzela Theorem, we know that T_λ is completely continuous. □

By Lemmas 2.8 and 2.9, we know that if $u \in P \setminus \theta$ is solution for BVP (2.1), then u is positive solution for BVP (2.1) and it is obvious from Lemma 2.8 that if $u \in P \setminus \{\theta\}$ is a solution for BVP (2.1) then $u \in K \setminus \{\theta\}$.

3. Existence and nonexistence results

In this section we will apply theorem due Krasnoselskii to study the existence, multiplicity and nonexistence of solutions for BVP (2.1) in $K \setminus \{\theta\}$.

Theorem 3.1. (see [19]) *Let E be a Banach space and $K \subset E$ be a cone in E. Assume Ω_1 and Ω_2 are open subset of E with $0 \in \Omega_1$ and $\overline{\Omega_1} \subset \Omega_2$, $T : K \cap \left(\bar{\Omega}_2 \setminus \Omega_1\right) \to K$ be a completely continuous operator such that*
(A) $\|Tu\| \leq \|u\|$, $\forall u \in K \cap \partial\Omega_1$ and $\|Tu\| \geq \|u\|$, $\forall u \in K \cap \partial\Omega_2$; or
(B) $\|Tu\| \geq \|u\|$, $\forall u \in K \cap \partial\Omega_1$ and $\|Tu\| \leq \|u\|$, $\forall u \in K \cap \partial\Omega_2$
Then T has a fixed point in $K \cap \left(\bar{\Omega}_2 \setminus \Omega_1\right)$.

We adopt the following assumptions:
(H_1) $f(t, u(t)) \in C((a, b), [0, \infty))$ is nondecreasing in $u \in [0, \infty)$ for fixed $t \in [a, b]$.
(H_2) $F_a = \displaystyle\int_a^b s^{N-1} f(s, 0)\, ds > 0,$
(H_3) $f_\infty = \lim\limits_{u \to \infty} \min\limits_{t \in \left[\frac{a}{a+b}, b\right]} \frac{f(t,u)}{u} = +\infty.$
Set

$$\Lambda = \left\{ \lambda > 0 / there\ exists\ u_\lambda \in K \setminus \{\theta\}\ such\ that\ T_\lambda u_\lambda = u_\lambda \right\},$$

and

$$\lambda^* = sup \Lambda.$$

Lemma 3.2. *Suppose that $(H_1) - (H_3)$ hold. If $\lambda' \in \Lambda$, then $(0, \lambda'] \subset \Lambda$.*

Proof. $\lambda^{'} \in \Lambda$ means that there exists $u_{\lambda'} \in K\backslash\{\theta\}$ such that $T_{\lambda'}u_{\lambda'} = u_{\lambda'}$. Therefore, for any $\lambda \in (0, \lambda']$ we have

$$T_\lambda u_{\lambda'} \le T_{\lambda'} u_{\lambda'} = u_{\lambda'},$$

Set

$$w_0 = u_{\lambda'},\ w_n = T_\lambda w_{n-1},\ n = 1, 2, ...$$

From (H_1), we obtain

$$w_0\left(t\right) \ge w_1\left(t\right) \ge ... \ge w_n\left(t\right) \ge ... \ge \frac{F_a\lambda}{\left(N-2\right)\left(b^{N-2}-a^{N-2}\right)}\left(1-\left(\frac{a}{t}\right)^{N-2}\right),$$

by Lemma 2.9 and (H_2), $\{w_n\}$ converges to fixed point of T_λ in $K \setminus \{\theta\}$. Thus $(0, \lambda'] \subset \Lambda$. The proof is complete. □

Let

$$\lambda_* < \frac{\left(b^{N-2}-a^{N-2}\right)}{F_b}, \quad F_b = \int_a^b s^{N-1} f\left(s, \left(\frac{b}{a}\right)^{N-2} - 1\right) ds,$$

$$u_0\left(t\right) = \frac{\lambda F_a}{\left(N-2\right)\left(b^{N-2}-a^{N-2}\right)}\left(1-\left(\frac{a}{t}\right)^{N-2}\right),\ v_0\left(t\right) = \left(\left(\frac{b}{t}\right)^{N-2} - 1\right),$$

and

$$F_\infty = \lim_{u\to\infty} sup \max_{a\le t\le b} \frac{f\left(t,u\right)}{u}.$$

Theorem 3.3. *Suppose that $(H_1) - (H_3)$ hold. Then T_λ has minimal and maximal fixed point in $[u_0, v_0]$ for $\lambda \in (0, \lambda_*]$. Moreover, there exists $\lambda^* \ge \lambda_* > 0$ such that T_λ has at least one and has no fixed points in $K \setminus \{\theta\}$ for $0 < \lambda < \lambda^*$ and $\lambda > \lambda^*$, respectively.*

Proof. From $(H_1) - (H_3)$ and (2.3), we have $\lambda_* > 0$. For any $\lambda \in (0, \lambda_*]$, we obtain

$$\begin{aligned}
\left(T_\lambda u_0\right)\left(t\right) &= \lambda \int_a^b s^{N-1} G\left(t,s\right) f\left(s, u_0\left(s\right)\right) ds \\
&\ge \lambda \int_a^b s^{N-1} G\left(t,s\right) f\left(s, u_0\left(a\right)\right) ds \\
&\ge \frac{\lambda}{\left(N-2\right)\left(b^{N-2}-a^{N-2}\right)}\left(1-\left(\frac{a}{t}\right)^{N-2}\right)\int_a^b s^{N-1} f\left(s,0\right) ds \\
&\ge \frac{\lambda F_a}{\left(N-2\right)\left(b^{N-2}-a^{N-2}\right)}\left(1-\left(\frac{a}{t}\right)^{N-2}\right) = u_0\left(t\right),
\end{aligned}$$

and

$$\left(T_\lambda v_0\right)\left(t\right) = \lambda \int_a^b s^{N-1} G\left(t,s\right) f\left(s, v_0\left(s\right)\right) ds$$

$$\leq \lambda_* \int_a^b s^{N-1} G(t,s) f(s, v_0(b))\, ds$$

$$\leq \frac{\lambda_*}{(N-2)(b^{N-2} - a^{N-2})} \left(\left(\frac{b}{t} \right)^{N-2} - 1 \right) \int_a^b s^{N-1} f(s, v_0(b))\, ds$$

$$\leq \frac{\lambda_* F_b}{(N-2)(b^{N-2} - a^{N-2})} \left(\left(\frac{b}{t} \right)^{N-2} - 1 \right) \leq v_0(t),$$

Set

$$u_n = T_\lambda u_{n-1},\ v_n = T_\lambda v_{n-1},\ n = 1, 2, ...,$$

then from (H_1), we have

$$u_0(t) \leq u_1(t) \leq ... \leq u_n(t) \leq ... \leq v_1(t) \leq v_0(t). \tag{3.1}$$

Lemma 2.9 implies that $\{u_n\}$ and $\{v_n\}$ converge to fixed points u_λ and v_λ of T_λ, respectively.
From (3.1) it is evident that $u_\lambda, v_\lambda \in K \setminus \{\theta\}$ are the minimal fixed point and maximal fixed point of T_λ in $[u_0, v_0]$, respectively.
By the definition of λ^*, there exists a nondecreasing sequence $\{\lambda_n\}_1^{+\infty}$ such that $\lim\limits_{n \to +\infty} \lambda_n = \lambda^*$. Let $\{u_{\lambda_n}\}_1^{+\infty}$ is bounded subset in K. Then there exists a constant $M > 0$ such that

$$\|u_{\lambda_n}\| \leq M,\ for\ n \in \mathbb{N}^*,$$

which implies that $\{u_{\lambda_n}\}_1^{+\infty}$ is uniformly bounded.
Now, we show that $\{u_{\lambda_n}\}_1^{+\infty}$ is equicontinuous. For any $u_{\lambda_n} \in K$, $n \in \mathbb{N}^*$ and $t_1, t_2 \in [a,b]$, with $|t_1 - t_2| < \delta$, we have

$$|x_{\lambda_n}(t_1) - x_{\lambda_n}(t_2)| \leq \lambda^* \int_a^b s^{N-1} |G(t_1,s) - G(t_2,s)| f(s, M)\, ds$$

$$\leq \lambda^* \int_a^b s^{N-1} |G(t_1,s) - G(t_2,s)| f(s, M)\, ds,$$

which implies that $\{x_{\lambda_n}\}_1^{+\infty}$ is equicontinuous subset in K. Consequently, by an application of the Arzela-Ascoli theorem we conclude that $\{x_{\lambda_n}\}_1^{+\infty}$ is a relatively compact set in K. So, there exists a subsequence $\left\{x_{\lambda_{n_i}}\right\} \subset \{x_{\lambda_n}\}$ converging to $x^* \in K$. Note that

$$\left(x_{\lambda_{n_i}}\right)(t) = \lambda_{n_i} \int_a^b s^{N-1} G(t,s) f\left(s, x_{\lambda_{n_i}}(s)\right) ds.$$

By taking the limit we have $x^*(t) = (T_{\lambda^*} x^*)(t)$. Therefore T_λ has at least one fixed point for $0 < \lambda < \lambda^*$. Finaly, for T_λ has no fixed point for $\lambda > \lambda^*$. The proof is complete. □

Theorem 3.4. *Suppose that* $(H_1), (H_3)$ *and* (2.3) *hold. If* $(F_{+\infty} < +\infty)$, *then when* $F_\infty > 0$, *there exists* $\lambda^* \geq \frac{N(N-2)\left(b^{N-2}-a^{N-2}\right)\left(b^N-a^N\right)}{F_\infty} > 0$ *such that* T_λ *has at least one and has no fixed points in* $K \setminus \{\theta\}$ *for* $0 < \lambda < \lambda^*$ *and* $\lambda > \lambda^*$, *respectively. When* $F_\infty = 0$, T_λ *has at least one fixed points in* $K \setminus \{\theta\}$ *for* $\lambda > 0$.

Proof. Since $F_\infty < \infty$, for any $\varepsilon > 0$, there exists $N_0 > 0$ such that

$$f(t,u) \leq (F_\infty + \varepsilon) u$$

for $u > N_0,\ t \in [a,b]$.
Let $w_0(t) = N_0\left(\left(\frac{b}{t}\right)^{N-2} - 1\right)$ and $\lambda_0 = \frac{N(N-2)\left(b^{N-2}-a^{N-2}\right)\left(b^N-a^N\right)}{\left(\left(\frac{b}{a}\right)^{N-2}-1\right)(F_\infty+\varepsilon)}$, then $\lambda_0 > 0$ and

$$\begin{aligned}
(T_{\lambda_0} w_0)(t) &= \lambda_0 \int_a^b s^{N-1} G(t,s) f(s, w_0(s))\, ds \\
&\leq \frac{\lambda_0}{(N-2)\left(b^{N-2}-a^{N-2}\right)} \left(\left(\frac{b}{t}\right)^{N-2} - 1\right) \int_a^b s^{N-1} (F_\infty + \epsilon) w_0(t)\, ds \\
&\leq \frac{\lambda_0 w_0(t)(F_\infty + \epsilon)}{(N-2)\left(b^{N-2}-a^{N-2}\right)} \left(\left(\frac{b}{t}\right)^{N-3} - 1\right) \int_a^b s^{N-1} ds \\
&\leq \frac{\lambda_0 w_0(t)(F_\infty + \epsilon)}{N(N-2)\left(b^{N-2}-a^{N-2}\right)\left(b^N-a^N\right)} \left(\left(\frac{b}{a}\right)^{N-2} - 1\right) \\
&\leq w_0(t),
\end{aligned}$$

Now, set $w_0(t) = N_0\left(\left(\frac{b}{t}\right)^{N-2} - 1\right)$,

$$w_n = T_{\lambda_{n-1}} w_{n-1},\ n = 1, 2, ...$$

From (H_1), we obtain

$$w_0(t) \geq w_1(t) \geq ... \geq w_n(t) \geq ... \geq \frac{F_a \lambda}{(N-2)\left(b^{N-2}-a^{N-2}\right)} \left(1 - \left(\frac{a}{t}\right)^{N-2}\right). \quad (3.2)$$

Therefore, the sequence $\{w_n\}$ is bounded in $K \setminus \{\theta\}$. By Lemma 2.9 and the definition of λ^*, the operator T_{λ_n} completely continuous. Hence the sequence $\{w_n\}$ is compact in $K \setminus \{\theta\}$, its also monotone. Then it is uniformly convergent to fixed points u^* of T_{λ_n} in $K \setminus \{\theta\}$. When we pass to the limit we get

$$u^* = T_{\lambda^*} u^*$$

For $\lambda > \lambda^*$, there exists $\{\lambda_n\}_1^\infty$, *with* $\lim_{n\to\infty} \lambda_n = \lambda$, we prove that problem has no positive solution. suppose the contrary that the problem has a positive solution x_{λ_n}, then we get

$$\|u_{\lambda_n}\| = (T_{\lambda_n} u_{\lambda_n}) \left(\left(\frac{b}{a}\right)^{N-2} - 1\right)$$

$$\leq \frac{\lambda_n}{(N-2)\left(b^{N-2}-a^{N-2}\right)}\left(\left(\frac{b}{a}\right)^{N-2}-1\right)\int_a^b s^{N-1}f\left(s,u_{\lambda_n}(s)\right)ds$$

$$\leq \frac{\lambda_n}{(N-2)\left(b^{N-2}-a^{N-2}\right)}\left(\left(\frac{b}{a}\right)^{N-2}-1\right)\int_a^b s^{N-1}\left(F_\infty+\epsilon\right)u_{\lambda_n}(b)\,ds$$

$$\leq \frac{\lambda_n\left(b^N-a^N\right)}{N(N-2)\left(b^{N-2}-a^{N-2}\right)}\left(\left(\frac{b}{a}\right)^{N-2}-1\right)\left(F_\infty+\epsilon\right)u_{\lambda_n}(b)$$

$$\leq \frac{\lambda_n\left(b^N-a^N\right)}{N(N-2)\left(b^{N-2}-a^{N-2}\right)}\left(\left(\frac{b}{a}\right)^{N-2}-1\right)\left(F_\infty+\epsilon\right)\left\|u_{\lambda_n}\right\| < \left\|u_{\lambda^*}\right\|.$$

Taking the limit we obtain

$$\left\|u_\lambda\right\| < \left\|u_\lambda\right\|,$$

which is a contradiction. The proof is complete. □

Lemma 3.5. *Assume that* $(H_1),(H_2)$ *and* (H_3) *hold. If* Λ *is nonempty, then*
(i) Λ *is bounded from above, that* $\lambda^* < +\infty$.
(ii) $\lambda^* \in \Lambda$.

Proof. Suppose to the contrary that there exists an increasing sequence $\{\lambda_n\}_1^{+\infty} \subset \Lambda$ such that $\lim_{n\to+\infty}\lambda_n = +\infty$. Set $x_{\lambda_n} \in K/\{\theta\}$ is a fixed point of T_{λ_n} that is ,

$$T_{\lambda_n}u_{\lambda_n} = u_{\lambda_n}.$$

There are two cases to be considered.
Case 1. $\{u_{\lambda_n}\}_1^{+\infty}$ is bounded, that is there exists a constant $M>0$ such that

$$\left\|u_{\lambda_n}\right\| \leq M,\ for\ n=1,2,\ldots.$$

Hence, from $(H_1),(H_2),$ *and* (H_3) and Lemma 2.3, we have

$$M \geq \left\|u_{\lambda_n}\right\| \geq \left(T_{\lambda_n}u_{\lambda_n}\right)(t)$$

$$\geq \frac{\lambda_n}{(N-2)\left(b^{N-2}-a^{N-2}\right)}\left(1-\left(\frac{a}{b}\right)^{N-2}\right)\int_a^b s^{N-1}f(s,0)\,ds$$

$$= \frac{\lambda_n}{(N-2)\left(b^{N-2}-a^{N-2}\right)}\left(1-\left(\frac{a}{b}\right)^{N-2}\right)F_a \to +\infty,$$

which is a contradiction.
Case 2. $\{u_{\lambda_n}\}_1^{+\infty}$ is unbounded, that is there exists subsequence of $\{u_{\lambda_n}\}_1^{+\infty}$ still denoted by $\{u_{\lambda_n}\}_1^{+\infty}$ such that $\lim_{n\to+\infty}\left\|u_{\lambda_n}\right\| = +\infty$.
When (H_3), take

$$L > \frac{N(N-2)\left(b^{N-2}-a^{N-2}\right)}{\left(1-\left(\frac{a}{b}\right)^{N-2}\right)\lambda_1}$$

there exists $N_1 > 0$ such that $f(t,u) \geq Lu,\ for\ u \geq N_1,\ t \in [a,b]$. Choose n_1 such that $\|u_{\lambda_{n_1}}\| > NN_1$.
Thus, for $t \in [a,b]$, we have

$$f\left(t, \frac{1}{N}\|u_{\lambda_{n_1}}\|\right) \geq \frac{1}{N} L \|u_{\lambda_{n_1}}\|.$$

Moreover, from (H_1) and the definition of K, we have

$$\|x_{\lambda_{n_1}}\| \geq (T_{\lambda_{n_1}} u_{\lambda_{n_1}})(t)$$

$$\geq \frac{\lambda_{n_1}}{(N-2)(b^{N-2}-a^{N-2})}\left(1-\left(\frac{a}{b}\right)^{N-2}\right)\int_a^b s^{N-1} f(s, u_{\lambda_{n_1}}(s))\,ds$$

$$\geq \frac{\lambda_{n_1}}{(N-2)(b^{N-2}-a^{N-2})}\left(1-\left(\frac{a}{b}\right)^{N-2}\right)\int_a^b s^{N-1} f\left(s, \frac{1}{6}\|u_{\lambda_{n_1}}(s)\|\right) ds$$

$$= \frac{\lambda_{n_1} L\left(1-\left(\frac{a}{b}\right)^{N-2}\right)}{N(N-2)(b^{N-2}-a^{N-2})}\|u_{\lambda_{n_1}}\| > \|u_{\lambda_{n_1}}\|,$$

which is a contradiction.
Consequently, we find that Λ is bounded from above.

(ii) From the definition of λ^*, there exists a nondecreasing sequence $\{\lambda_n\}_1^{+\infty}$ such that $\lim_{n\to+\infty} \lambda_n = \lambda^*$. Let $\{u_{\lambda_n}\}_1^{+\infty} \in K \setminus \{\theta\}$ be a fixed point of T_{λ_n}. Arguing similarly as above in Case 2, we can show that $\{u_{\lambda_n}\}_1^{+\infty}$ is bounded subset in K, that is there exists a constant $M > 0$. Hence from $(H_1), (H_2),\ and\ (H_3)$, we have

$$\|u_{\lambda_n}\| = (T_{\lambda_n} u_{\lambda_n})\left(\left(\frac{b}{a}\right)^{N-2} - 1\right)$$

$$\leq \frac{\lambda_n}{(N-2)(b^{N-2}-a^{N-2})}\left(\left(\frac{b}{a}\right)^{N-2}-1\right)\int_a^b s^{N-1} f(s, u_{\lambda_n}(s))\,ds$$

$$\leq \frac{\lambda_n}{(N-2)(b^{N-2}-a^{N-2})}\left(\left(\frac{b}{a}\right)^{N-2}-1\right)\int_a^b s^{N-1} f(s, u_{\lambda_n}(b))\,ds$$

$$\leq \frac{\lambda_n}{(N-2)(b^{N-2}-a^{N-2})}\left(\left(\frac{b}{a}\right)^{N-2}-1\right)\int_a^b s^{N-1} f(s, v_{\lambda_n}(b))\,ds$$

$$= \frac{\lambda_n}{(N-2)(b^{N-2}-a^{N-2})}\left(\left(\frac{b}{a}\right)^{N-2}-1\right)\int_a^b s^{N-1} f(s, 0)\,ds$$

$$= \frac{\lambda_n F_a}{(N-2)(b^{N-2}-a^{N-2})}\left(\left(\frac{b}{a}\right)^{N-2}-1\right) \to \frac{\lambda_* F_a\left(\left(\frac{b}{a}\right)^{N-2}\right)}{(N-2)(b^{N-2}-a^{N-2})} = M,$$

as $n \to \infty$.

Therefore

$$\|u_{\lambda_n}\| \le M,\ n = 1, 2, ...$$

which shows that $\{u_{\lambda_n}\}_1^{+\infty}$ is uniformly bounded.
From the proof of Theorem 3.3 we know that $\{u_{\lambda_n}\}_1^{+\infty}$ is equicontinuous subset in K and by an application of the Arzela-Ascoli theorem we conclude that $\{u_{\lambda_n}\}_1^{+\infty}$ is a relatively compact set in K. So, there exists a subsequence $\left\{u_{\lambda_{n_i}}\right\} \subset \{u_{\lambda_n}\}$ converging to $u^* \in K$. Note that

$$\left(u_{\lambda_{n_i}}\right)(t) = \lambda_{n_i} \int_0^1 s^{N-1} G(t,s) f\left(s, u_{\lambda_{n_i}}(s)\right) ds.$$

By taking the limit we have

$$u^*(t) = (T_{\lambda^*} u^*)(t) \ge \frac{\lambda_1 F_a}{(N-2)\left(b^{N-2} - a^{N-2}\right)} \left(1 - \left(\frac{a}{t}\right)^{N-2}\right),$$

that is $\lambda^* \in \Lambda$. The proof is complete. □

Theorem 3.6. *Suppose that* $(H_1) - (H_3)$ *holds. Then there exists* $\lambda^* \ge \lambda_* > 0$ *such that* BVP (2.1) *has at least two, one and no positive solutions for* $0 < \lambda \le \lambda_*,\ \lambda_* < \lambda \le \lambda^*$ *and* $\lambda > \lambda^*$ *respectively.*

Proof. From $(H_1), (H_2)$ and (H_3) we have $(0, \lambda_*] \subset \Lambda$. So $\lambda^* \ge \lambda_* > 0$.
From Lemma 3.2 and 3.5, we have $(0, \lambda^*] = \Lambda$. Therefore, from the definition of λ^* we only to prove that T_λ has at least two fixed points in $K \setminus \{\theta\}$ for $\lambda \in (0, \lambda_*]$.
Now, given $\lambda \in (0, \lambda_*]$. Theorem 3.3 means that T_λ has at least one fixed point $u_{\lambda,1} \in K \setminus \{\theta\}$ which satisfies $\|u_{\lambda,1}\| \le \left(\frac{b}{a}\right)^{N-2} - 1$.
Let

$$K_1 = \left\{ x \in K \mid \|u\| < \left(\frac{b}{a}\right)^{N-2} - 1 \right\}.$$

For $t \in [a, b]$, so for $u \in K$ with $\|u\| = \left(\frac{b}{a}\right)^{N-2} - 1$, i.e $u \in \partial K_1$, we have

$$\|u\| = \|T_\lambda u\| = (T_\lambda u)\left(\left(\frac{b}{a}\right)^{N-2} - 1\right)$$

$$\le \frac{\lambda}{(N-2)\left(b^{N-2} - a^{N-2}\right)} \left(\left(\frac{b}{a}\right)^{N-2} - 1\right) \int_a^b s^{N-1} f(s, u(s))\, ds$$

$$\le \frac{\lambda_*}{(N-2)\left(b^{N-2} - a^{N-2}\right)} \left(\left(\frac{b}{a}\right)^{N-2} - 1\right) \int_a^b s^{N-1} f\left(s, \left(\frac{b}{a}\right)^{N-2} - 1\right) ds$$

$$< \frac{\left(\left(\frac{b}{a}\right)^{N-2} - 1\right)}{N-2} < \|u\|. \tag{3.3}$$

When (H_3), take

$$L > \frac{N(N-2)\left(b^{N-2}-a^{N-2}\right)}{\left(1-\left(\frac{a}{b}\right)^{N-2}\right)\lambda_1}$$

there exists $N_1 > 0$ such that $f(t,u) \geq Lu,\ for\ u \geq N_1,\ t \in [a,b]$.
Set $K_2 = \{u : \|u\| < NN_1\}$. Then $\overline{K_1} \subset K_2$. If $u \in \partial K_2$, we have

$$\|u\| = \|T_\lambda u\| = (T_\lambda u)\left(\left(\frac{b}{a}\right)^{N-2} - 1\right)$$

$$\geq \frac{\lambda}{(N-2)\left(b^{N-2}-a^{N-2}\right)}\left(1-\left(\frac{a}{b}\right)^{N-2}\right)\int_a^b s^{N-1} f(s, u(s))\, ds$$

$$\geq \frac{\lambda}{(N-2)\left(b^{N-2}-a^{N-2}\right)}\left(1-\left(\frac{a}{b}\right)^{N-2}\right)\int_a^b s^{N-1} f\left(s, \frac{1}{N}\|u\|\right) ds$$

$$\geq \frac{\lambda}{(N-2)\left(b^{N-2}-a^{N-2}\right)}\left(1-\left(\frac{a}{b}\right)^{N-2}\right)\int_a^b s^{N-1} f\left(s, \frac{1}{N}\|u(s)\|\right) ds$$

$$\geq \frac{\lambda L\left(1-\left(\frac{a}{b}\right)^{N-2}\right)}{N(N-2)\left(b^{N-2}-a^{N-2}\right)}\|u\| > \|u\|.$$

Consequently, Applying Theorem 3.1 that T_λ has a fixed point $u_{\lambda,2} \in \overline{K_2} \setminus K_1$.
Equation (3.3) implies that T_λ has no fixed point in ∂K_1. In conclusion, for $\lambda \in (0, \lambda_*]$, T_λ has at least two fixed points $u_{\lambda,1}$ and $u_{\lambda,2}$ in K. The proof is complete. □

We present an example to illustrate the applicability of the results shown before.

Example 3.7. Consider in $\mathbb{R}^3$ the elliptic boundary value problem

$$\begin{cases} -\triangle u(x) = \lambda\left(|x| + u + \ln(1+u)\right), & x \in \Omega, \\ u(x) = 0, & x \in \partial\Omega, \end{cases} \tag{3.4}$$

To the system (3.4) we associate the the second order boundary value problem

$$\begin{cases} -u''(t) - \dfrac{2}{t}\, u(t) = \lambda\left(t + u + \ln(1+u)\right),\ t \in (a,b), \\ u(a) = u(b) = 0, \end{cases}$$

By direct computation, we have

$$F_\infty = 2,\ F_0 = \frac{1}{4},\ F_1 = \frac{1}{2} + \frac{2}{3}(1+\ln(2)) \text{ and } \lambda_* = \frac{48-9\pi}{6+8(1+\ln(2))}.$$

So, the assumptions (H_1), (H_2) and (H_3) are satisfied, it follows from Theorem 3.4 there exists $\lambda^* = 3 \geq \lambda_*$ such that boundary value problem (3.4) has at least one positive solution for $0 < \lambda \leq 3$ and has no positive solution for $\lambda > \lambda^*$.

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Ball convergence for combined three-step methods under generalized conditions in Banach space

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Abstract. We give a local convergence analysis for an eighth-order convergent method in order to approximate a locally unique solution of nonlinear equation for Banach space valued operators. In contrast to the earlier studies using hypotheses up to the seventh Fréchet-derivative, we only use hypotheses on the first-order Fréchet-derivative and Lipschitz constants. Therefore, we not only expand the applicability of these methods but also provide the computable radius of convergence of these methods. Finally, numerical examples show that our results apply to solve those nonlinear equations but earlier results cannot be used.

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1. Introduction

One of the most basic and important problems in Numerical Analysis concerns with approximating a locally unique solution x^* of the equation of the form

$$F(x) = 0, \tag{1.1}$$

where $F : \mathbb{D} \subset \mathbb{X} \to \mathbb{Y}$ is a Fréchet-differentiable operator, $\mathbb{X},\ \mathbb{Y}$ are Banach spaces and $\mathbb{D}$ is a convex subset of $\mathbb{X}$. Let us also denote $L(\mathbb{X},\ \mathbb{Y})$ as the space of bounded linear operators from $\mathbb{X}$ to $\mathbb{Y}$.

Approximating x^* is very important, since numerous problems can be reduced to equation (1.1) using mathematical modeling [4, 7, 12, 9, 16, 21, 23, 24]. However, it is not always possible to find the solution x^* in a closed form. Therefore, most of the methods are iterative to solve such type of problems. The convergence analysis of iterative methods is usually divided into two categories: semi-local and local convergence analysis. The semi-local convergence matter is, based on the information around an initial point, to give criteria ensuring the convergence of iteration procedures. A very important problem in the study of iterative procedures is the convergence domain. Therefore, it is very important to propose the radius of convergence of the iterative methods.

We study the local convergence of the three step eighth-order convergent method defined for each $n = 0,\ 1\ 2,\ \ldots$ by

$$\begin{aligned} y_n &= x_n - F'(x_n)^{-1}F(x_n),\\ z_n &= \phi(x_n,\ F(x_n),\ F'(x_n),\ F'(y_n),\\ x_{n+1} &= z_n - \beta A_n^{-1}F(z_n), \end{aligned} \tag{1.2}$$

where $x_0 \in \mathbb{D}$ is an initial point, for $\alpha,\ \beta \in S,\ A_n = (\beta - \alpha)F'(x_n) + \alpha F'(y_n),\ (S = \mathbb{R}$ or $S = \mathbb{C})$ and the second sub step represents any iterative method, in which the order of convergence is at least $m = 1,\ 2,\ 3,\ \ldots$. It was shown in [9] using Taylor series expansions when $\mathbb{X} = \mathbb{Y} = \mathbb{R}$ that method (1.2) is of order at least $2m$, if $m < 3$ and of order at least $m + 3$, if $m \geq 3$ provided that F is eighth times differentiable. The hypotheses on the derivatives of F restrict the applicability of method (1.2). As a motivational example, define function F on $\mathbb{X} = \mathbb{Y} = \mathbb{R}$, $D = [-\frac{3}{2}, \frac{1}{2}]$ by

$$F(x) = \begin{cases} x^3 lnx^2 + x^5 - x^4, & x \neq 0,\\ 0, & x = 0. \end{cases}$$

Then, we have that

$$F'(x) = 3x^2 lnx^2 + 5x^4 - 4x^3 + 2x^2,$$
$$F''(x) = 6x lnx^2 + 20x^3 - 12x^2 + 10x$$

and

$$F'''(x) = 6lnx^2 + 60x^2 - 24x + 22.$$

Then, obviously the third-order derivative of the involved function $F'''(x)$ is not bounded on $\mathbb{D}$. Notice that, in particular there is a plethora of iterative methods for approximating solutions of nonlinear equations [2, 1, 3, 4, 5, 7, 8, 6, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24]. These results show that initial guess should be close to the required root for the convergence of the corresponding methods. But, how close initial guess should be required for the convergence of the corresponding method? These local results give no information on the radius of the ball convergence for the corresponding method. We address this question for method (1.2) in the next section 2.

In the present study, we expand the applicability of method (1.2) by using only hypotheses on the first-order derivative of function F and generalized Lipschitz conditions. Moreover, we we will avoid to use Taylor series expansions and use Lipschitz parameters. In this way, there is no need to use the higher-order derivatives to show the convergence of the scheme (1.2).

The rest of the paper is organized as follows: in section 2 contains the local convergence analysis of method (1.2). The numerical examples appear in the concluding Section 3.

2. Local convergence

The local convergence uses some scalars functions and parameters. Let $v,\ w_0,\ w,\ \bar{g}_2 : [0,\ +\infty) \to [0,\ +\infty)$ be continuous, increasing functions with $w_0(0) = w(0) = 0$ and $\alpha,\ \beta \in S$. Define parameter r_0 by

$$r_0 = \sup\{t \geq 0 : w_0(t) < 1\}. \tag{2.1}$$

Moreover, define functions $g_1,\ h_1,\ p$ and h_p on the interval $[0,\ r_0)$ by

$$g_1(t) = \frac{\int_0^1 w((1-\theta)t)d\theta}{1-w_0(t)},$$

$$h_1 = g_1(t) - 1,$$

$$p(t) = |\beta|^{-1}[|\beta - \alpha|w_0(t) + |\alpha|w_0(g_1(t)t)],\ \beta \neq 0,$$

and

$$h_p = p(t) - 1.$$

We have by (2.2) that $h_1(0) = h_p(0) = -1 < 0$ and $h_1(t) \to +\infty$, $h_q(t) \to +\infty$ as $t \to r_0^-$. Then, by the intermediate value theorem, we know that the functions h_1 and h_p have zeros in the interval $(0,\ r_0)$. Denote by r_1 and r_p, respectively the smallest such zeros of the function h_1 and h_p. Furthermore, define functions g_2 and h_2 on the interval $(0,\ r_0)$ by

$$g_2(t) = \bar{g}_2(t)t^{m-1},$$

and

$$h_2(t) = g_2(t) - 1.$$

Suppose that

$$\bar{g}_2(0) < 1,\ \text{if}\ m = 1 \tag{2.2}$$

and

$$g_2(t) \to a \text{ a number greater than one or } +\infty \tag{2.3}$$

as $t \to \bar{r}_0^-$ for some $\bar{r}_0 \leq r_0$. Then, we have again by the intermediate value theorem that function h_2 has zeros in the interval $(0,\ \bar{r}_0)$. Denote by r_2 the smallest such zero. Notice that, if $m > 1$ condition (2.2) is not needed to show $h_2(0) < 0$, since in this case $h_2(0) = g_2(0) - 1 = 0 - 1 = -1 < 0$. Finally, define functions g_3 and h_3 on the interval $[0,\ \bar{r}_p)$ by $\bar{r}_p = \min\{r_p,\ r_2\}$,

$$g_3(t) = \left(1 + \frac{\int_0^1 v(\theta g_2(t)t)d\theta}{1-p(t)}\right) g_2(t),$$

and

$$h_3(t) = g_3(t) - 1.$$

Suppose that

$$(1 + v(0))\bar{g}_2(0) < 1,\ \text{if}\ m = 1, \tag{2.4}$$

we get by (2.4) that $h_3(0) = (1 + v(0))\bar{g}_2(0) - 1 < 0$ and $h_3(t) \to +\infty$ or positive number as $t \to \bar{r}_p^-$. Denote by r_3 the smallest zero of function h_3 in the interval $(0,\ r_p)$. Define the radius of convergence r by

$$r = \min\{r_1,\ r_3\}. \tag{2.5}$$

Then, we have that for each $t \in [0,\ r)$

$$0 \le g_i(t) < 1,\ i = 1,\ 2,\ 3. \tag{2.6}$$

Let $U(z,\ \rho)$, $\bar{U}(z,\ \rho)$, stand respectively for the open and closed balls in $\mathbb{X}$ with center $z \in \mathbb{X}$ and of radius $\rho > 0$. Next, we present the local convergence analysis of method (1.2) using the preceding notations.

Theorem 2.1. *Let $F : \mathbb{D} \subseteq \mathbb{X} \to \mathbb{Y}$ be a continuously Fréchet-differentiable operator. Let $v,\ w_0,\ w,\ \bar{g}_2 : [0,\ \infty) \to [0,\ \infty)$ be increasing continuous functions with $w_0(0) = w(0) = 0$ and let $r_0 \in [0,\ \infty)$, $\alpha \in S$, $\beta \in S - \{0\}$, $m \ge 1$ and r_0 be defined by (2.1) so that (2.1) and (2.2) are satisfied. Suppose that there exists $x^* \in \mathbb{D}$ such that for each $x \in \mathbb{D}$ parameter r_0 be defined by (2.1).*

$$F(x^*) = 0, \quad F'(x^*)^{-1} \in L(\mathbb{Y},\ \mathbb{X}) \tag{2.7}$$

and

$$\|F'(x^*)^{-1}(F'(x) - F'(x^*)\| \le w_0(\|x - x^*\|). \tag{2.8}$$

Moreover, suppose that for each $x,\ y \in \mathbb{D}_0 := \mathbb{D} \cap U(x^*,\ r_0)$

$$\|F'(x^*)^{-1}\big(F'(x) - F'(y)\big)\| \le w(\|x - y\|), \tag{2.9}$$

$$\|F'(x^*)^{-1}F'(x)\| \le v(\|x - x^*\|), \tag{2.10}$$

$$\|\phi(x,\ F(x),\ F'(x),\ F'(y))\| \le \bar{g}_2(\|x - x^*\|)\|x - x^*\|^m \tag{2.11}$$

and

$$\bar{U}(x^*, r) \subseteq \mathbb{D}, \tag{2.12}$$

where the radius of convergence r is defined by (2.3). Then, sequence $\{x_n\}$ generated for $x_0 \in U(x^*,\ r) - \{x^*\}$ by method (1.2) is well defined, remains in $U(x^*,\ r)$ for each $n = 0, 1, 2, \ldots$ and converges to x^*. Moreover, the following estimates hold

$$\|y_n - x^*\| \le g_1(\|x_n - x^*\|)\|x_n - x^*\| \le \|x_n - x^*\| < r, \tag{2.13}$$

$$\|z_n - x^*\| \le g_2(\|x_n - x^*\|)\|x_n - x^*\| \le \|x_n - x^*\| \tag{2.14}$$

and

$$\|x_{n+1} - x^*\| \le g_3(\|x_n - x^*\|)\|x_n - x^*\| \le \|x_n - x^*\|, \tag{2.15}$$

where the functions g_i, $i = 1,\ 2,\ 3$ are defined above the Theorem. Furthermore, if

$$\int_0^1 w_0(\theta R)d\theta < 1,\ \text{for } R \ge r, \tag{2.16}$$

then the point x^* is the only solution of equation $F(x) = 0$ in $\mathbb{D}_1 := \mathbb{D} \cap \bar{U}(x^*,\ R)$.

Proof. We shall show using mathematical induction that the sequences $\{x_n\}$ is well defined in $U(x^*,\ r)$ and converges to x^*. By the hypothesis $x_0 \in U(x^*, r) - \{x^*\}$, (2.1), (2.3) and (2.10), we have that

$$\|F'(x^*)^{-1}(F'(x_0) - F'(x^*))\| \leq w_0(\|x_0 - x^*\|) < w_0(r) < 1. \tag{2.17}$$

In view of (2.17) and the Banach Lemma on invertible operators [4, 7] that $F'(x_0)^{-1} \in L(\mathbb{Y},\ \mathbb{X})$, y_0 is well defined by the first two sub steps of method (1.2) and

$$\|F'(x_0)^{-1}F'(x^*)\| \leq \frac{1}{1 - w_0(\|x_0 - x^*\|)}. \tag{2.18}$$

We get by (2.1), (2.5), (2.6) (for $i = 1$), (2.7) and (2.18) that

$$\begin{aligned} \|y_0 - x^*\| &= \|(x_0 - x^* - F'(x_0)^{-1}F(x_0))\| \\ &\leq \|F'(x_0)^{-1}F(x^*)\| \left\| \int_0^1 F'(x^*)^{-1}(F'(x^* + \theta(x_0 - x^*)) \right. \\ &\left. - F'(x_0))(x_0 - x^*)d\theta \right\| \\ &\leq \frac{\int_0^1 w((1-\theta)\|x_0 - x^*\|)d\theta\|x_0 - x^*\|}{1 - w_0(\|x_0 - x^*\|)} \\ &\leq g_1(\|x_0 - x^*\|)\|x_0 - x^*\| \leq \|x_0 - x^*\| < r, \end{aligned} \tag{2.19}$$

which implies (2.13) for $n = 0$ and $y_0 \in U(x^*,\ r)$. By (2.5), (2.6) (for $i = 2$) and (2.11), we obtain in turn that

$$\begin{aligned} \|z_0 - x^*\| &= \|\phi(x_0,\ F(x_0),\ F'(x_0),\ F'(y_0))\| \\ &\leq \bar{g}_2(\|x_0 - x^*\|)\|x_0 - x^*\|^m \\ &= g_2(\|x_0 - x^*\|)\|x_0 - x^*\| \leq \|x_0 - x^*\| < r, \end{aligned} \tag{2.20}$$

which shows (2.14) for $n = 0$ and $z_0 \in U(x^*,\ r)$. We must show that x_1 exists. Using (2.1), (2.5) and (2.8), we obtain in turn that

$$\begin{aligned} &\left\|(\beta F'(x^*))^{-1}\left[(\beta - \alpha)(F'(x_0) - F'(x^*)) + \alpha(F'(y_0) - F'(x^*))\right]\right\| \\ &\qquad \leq |\beta|^{-1}\left[|\beta - \alpha|w_0(\|x_0 - x^*\|) + |\alpha|w_0(\|y_0 - x^*\|)\right] \\ &\qquad \leq |\beta|^{-1}\left[|\beta - \alpha|w_0(\|x_0 - x^*\|) + |\alpha|w_0(g_1(\|x_0 - x^*\|)\|x_0 - x^*\|)\right] \\ &\qquad = p(\|x_0 - x^*\|) \leq p(r) < 1, \end{aligned} \tag{2.21}$$

so

$$\|((\beta - \alpha)F'(x_0) + \alpha F'(y_0))^{-1}F'(x^*)\| \leq \frac{1}{1 - p(\|x_0 - x^*\|)}. \tag{2.22}$$

Then, from the last sub step of method (2.1), (2.5), (2.6) (for $i = 3$), (2.10), (2.19), (2.20) and (2.21), we get in turn that

$$\begin{aligned} \|x_1 - x^*\| &= \|z_0 - x^*\| + |\beta| \int_0^1 v(\theta\|z_0 - x^*\|)d\theta \|x_0 - x^*\| \\ &\leq \left(1 + \frac{|\beta| \int_0^1 v(\theta g_2(\|x_0 - x^*\|))d\theta}{|\beta|(1 - p(\|x_0 - x^*\|))}\right) g_2(\|x_0 - x^*\|)\|x_0 - x^*\| \\ &= g_3(\|x_0 - x^*\|)\|x_0 - x^*\| \leq \|x_0 - x^*\| < r, \end{aligned} \tag{2.23}$$

which shows (2.15) and $x_1 \in U(x^*, \ r)$. By simply replacing $x_0, \ y_0, \ x_1$ by $x_k, \ y_k, \ x_{k+1}$ in the preceding estimates we arrive at (2.15) and (2.16). Then, in view of the estimates

$$\|x_{k+1} - x^*\| \leq c\|x_k - x^*\| < r, \ c = g_2(\|x_0 - x^*\|) \in [0, \ 1), \tag{2.24}$$

we deduce that $\lim_{k\to\infty} x_k = x^*$ and $x_{k+1} \in U(x^*, \ r)$. Finally, to show the uniqueness part, let $y^* \in D_1$ with $F(y^*) = 0$. Define $Q = \int_0^1 F'(x^* + \theta(x^* - y^*))d\theta$. Using (2.5) and (2.12), we get that

$$\begin{aligned} \|F'(x^*)^{-1}(Q - F'(x^*))\| &\leq \| \int_0^1 w_0(\theta\|y^* - x^*\|)d\theta \\ &\leq \int_0^1 w_0(\theta R)d\theta < 1. \end{aligned} \tag{2.25}$$

It follows from (2.25) that Q is invertible. Then, in view of the identity

$$0 = F(x^*) - F(y^*) = Q(x^* - y^*), \tag{2.26}$$

we conclude that $x^* = y^*$. □

Remark 2.2. (a) It follows from (2.10) that condition (2.12) can be dropped and be replaced by

$$v(t) = 1 + w_0(t) \text{ or } v(t) = 1 + w_0(r_0), \tag{2.27}$$

since,

$$\begin{aligned} \|F'(x^*)^{-1}\left[\left(F'(x) - F'(x^*)\right) + F'(x^*)\right]\| &= 1 + \|F'(x^*)^{-1}(F'(x) - F'(x^*))\| \\ &\leq 1 + w_0(\|x - x^*\|) \\ &= 1 + w_0(t) \quad \text{for } \ \|x - x^*\| \leq r_0. \end{aligned} \tag{2.28}$$

(b) If the function w_0 is strictly increasing, then we can choose

$$r_0 = w_0^{-1}(1) \tag{2.29}$$

instead of (2.1).

(c) If $w_0, \ w, \ v$ are constants functions (the proof of Theorem 2.1 goes through too in this case), then

$$r_1 = \frac{2}{2w_0 + w} \tag{2.30}$$

and

$$r \leq r_1. \tag{2.31}$$

Therefore, the radius of convergence r can be larger than the radius of convergence r_1 for Newton's method

$$x_{n+1} = x_n - F'(x_n)^{-1}F(x_n). \tag{2.32}$$

Notice also that the earlier radius of convergence given independently by Rheindoldt [22] and Traub [24] is

$$r_{TR} = \frac{2}{3w_1} \tag{2.33}$$

and by Argyros [4, 7]

$$r_A = \frac{2}{2w_0 + w_1}, \tag{2.34}$$

where w_1 is the Lipschitz constant for (2.6) on D. But, we have

$$w \leq w_1,\ w_0 \leq w_1, \tag{2.35}$$

so

$$r_{TR} \leq r_A \leq r_1 \tag{2.36}$$

and

$$\frac{r_{TR}}{r_A} \to \frac{1}{3} \quad \text{as} \quad \frac{w_0}{w} \to 0. \tag{2.37}$$

The radius of convergence q used in [9] is smaller than the radius r_{DS} given by Dennis and Schabel [4]

$$q < r_{SD} = \frac{1}{2w_1} < r_{TR}. \tag{2.38}$$

However, q can not be computed using the Lipschitz constants.

(d) The results obtained here can be used for operators F satisfying the autonomous differential equation [4, 7] of the form

$$F'(x) = P(F(x)) \tag{2.39}$$

where P is a known continuous operator. Since $F'(x^*) = P(F(x^*)) = P(0)$, we can apply the results without actually knowing the solution x^*. Let as an example $F(x) = e^x - 1$. Then, we can choose $P(x) = x + 1$.

(e) Let us show how to choose functions $\phi,\ \bar{g}_2,\ g_2$ and m. In addition, we assume that $\mathbb{X} = \mathbb{Y} = \mathbb{R}$. Define function ϕ on $\mathbb{R}^4$ by

$$\phi(x_n,\ F(x_n),\ F'(x_n),\ F'(y_n)) = y_n - F'(y_n)^{-1}F(y_n). \tag{2.40}$$

Then, we can choose

$$g_2(t) = \frac{\int_0^1 w((1-\theta)g_1(t)t)d\theta g_1(t)}{1 - w_0(g_1(t)t)}. \tag{2.41}$$

If $w_0,\ w,\ v$ are given in particular by $w_0(t) = L_0 t,\ w(t) = Lt$ and $v(t) =<$ for some $L > 0,\ L > 0$ and $M \geq 1$, then we have that

$$\bar{g}_2(t) = \frac{\frac{L^2}{8(1-L_0 t)^2}}{1 - \frac{L_0 L t^2}{2(1-L_0 t)}}, \qquad g_2(t) = \bar{g}_2(t)t^3 \text{ and } m = 4. \tag{2.42}$$

(f) If $\beta = 0$, we can obtain the results for the two-step method

$$\begin{aligned} y_n &= x_n - F'(x_n)^{-1}F(x_n), \\ x_{n+1} &= \phi(x_n,\ F(x_n),\ F'(x_n),\ F'(y_n)) \end{aligned} \tag{2.43}$$

by setting $z_n = x_{n+1}$ in Theorem 2.1.

3. Numerical examples and applications

In this section, we shall demonstrate the theoretical results which we have proposed in the section 2. Therefore, we consider four numerical examples in this section, which are defined as follows:

Example 3.1. Let $X = Y = C[0,\ 1]$ and consider the nonlinear integral equation of the mixed Hammerstein-type [13, 16], defined by

$$x(s) = \int_0^1 G(s,\ t)\left(x(t)^{\frac{3}{2}} + \frac{x(t)^2}{2}\right) dt \tag{3.1}$$

where the kernel G is the Green's function defined on the interval $[0,1] \times [0,1]$ by

$$F(s,t) = \begin{cases} (1-s)t, & t \le s, \\ s(1-t), & s \le t. \end{cases} \tag{3.2}$$

The solution $x^*(s) = 0$ is the same as the solution of equation (1.1), where $F :\subseteq C[0,\ 1] \to C[0,\ 1]$ defined by

$$F(x)(s) = x(s) - \int_0^t G(s,\ t)\left(x(t)^{\frac{3}{2}} + \frac{x(t)^2}{2}\right) dt. \tag{3.3}$$

Notice that

$$\left\| \int_0^t G(s,\ t)dt \right\| \le \frac{1}{8}. \tag{3.4}$$

Then, we have that

$$F'(x)y(s) = y(s) - \int_0^t G(s,\ t)\left(\frac{3}{2}x(t)^{\frac{1}{2}} + x(t)\right) dt,$$

so since $F'(x^*(s)) = I$,

$$\left\|F'(x^*)^{-1}\big(F'(x) - F'(y)\big)\right\| \le \frac{1}{8}\left(\frac{3}{2}\|x-y\|^{\frac{1}{2}} + \|x-y\|\right). \tag{3.5}$$

Therefore, we can choose

$$w_0(t) = w(t) = \frac{1}{8}\left(\frac{3}{2}t^{\frac{1}{2}} + t\right)$$

and by Remark 2.2(a)

$$v(t) = 1 + w_0(t).$$

The results in [16, 9] can not be used to solve this problem, since F' is not Lipschitz. However, our results can apply.

Example 3.2. Suppose that the motion of an object in three dimensions is governed by system of differential equations

$$\begin{aligned} f_1'(x) - f_1(x) - 1 = 0 \\ f_2'(y) - (e-1)y - 1 = 0 \\ f_3'(z) - 1 = 0 \end{aligned} \tag{3.6}$$

with $x,\ y,\ z \in \Omega$ for $f_1(0) = f_2(0) = f_3(0) = 0$. then, the solution of the system is given for $w = (x,\ y,\ z)^T$ by function $F := (f_1,\ f_2,\ f_3) : \Omega \to \mathbb{R}^3$ defined by

$$F(v) = \left(e^x - 1,\ \frac{e-1}{2}y^2 + y,\ z\right)^T. \tag{3.7}$$

Then the Fréchet-derivative is given by

$$F'(v) = \begin{bmatrix} e^x & 0 & 0 \\ 0 & (e-1)y+1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Then, we have that $w_0(t) = L_0 t,\ w(t) = Lt,\ w_1(t) = L_1 t, w_0 = L_0,\ w_1 = L_1$ and $v(t) = M$, where $L_0 = e - 1 < L = e^{\frac{1}{L_0}} = 1.789572397,\ L_1 = e$ and $M = e^{\frac{1}{L_0}} = 1.7896$. Then, we get

$$r = 0.0039782.$$

Example 3.3. Let $A_1 = A_2 = C[0,\ 1]$, be the space of continuous functions defined on the interval $[0,\ 1]$ and be equipped with max norm. Let $\Omega = \bar{U}(0,\ 1)$ and $B(x) = F''(x)$ for each $x \in \Omega$. Define F on Ω

$$F(\varphi)(x) = \phi(x) - 5\int_0^1 x\theta\varphi(\theta)^3 d\theta. \tag{3.8}$$

We have that

$$F'(\varphi(\xi))(x) = \xi(x) - 15\int_0^1 x\theta\varphi(\theta)^2\xi(\theta)d\theta, \text{ for each } \xi \in \Omega. \tag{3.9}$$

Then, we have that $x^* = 0,\ L_0 = 7.5,\ L_1 = L = 15$ and $M = 2$. Using method (1.2) for $w_0(t) = L_0 t,\ v(t) = 2 = M,\ w(t) = Lt,\ w_1 = L$ and $w_0 = L_0$, we get

$$r = 0.0013404.$$

Example 3.4. Returning back to the motivation example at the introduction on this paper, we have $L = L_0 = 96.662907$ and $M = 2$. Using method (1.2) for $w_0(t) = L_0 t,\ v(t) = 2 = M,\ w(t) = Lt,\ w_1(t) = L$ and $w_0 = L_0$, we can choose

$$r = 0.00085.$$

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Viscous dissipative free convective flow from a vertical cone with heat generation/absorption, MHD in the presence of radiative non-uniform surface heat flux

Bapuji Pullepu and Rayampettai Munisamy Kannan

Abstract. The problem of combined effect of heat generation/absorption and thermal radiation on unsteady, laminar, natural convective movement with MHD, viscous dissipation over a vertical cone in the presence of variable heat flux is considered. The converted FDE's of the flow which is of partial natured, unsteady, united and non-linear are solved numerically subject to proper boundary conditions by Crank-Nicholson scheme which is an efficient, correct and absolutely stable FDM. The velocity and thermal profiles are obtained and analyze to expose the outcome of heat generation/absorption and thermal radiation at different values of the MHD, Prandtl numeral, viscous dissipation and the exponent in the power law difference of the surface heat flux. The local as well as average shear stress and heat transfer rate are accessible and analyzed. The present outcome is compared by available outcome in literature and is originate to exist in excellent conformity.

Mathematics Subject Classification (2010): 65M06, 76R10.

Keywords: Cone, finite difference method, heat generation/absorption, MHD, thermal radiation, viscous dissipation variable heat flux.

Nomenclature:

$F_0''(0)$	–	Shear-Stress co-efficient in Ref: [12]
Gr_L	–	Grashof number
a	–	Constant
g	–	Rate of change of velocity due to gravity
k	–	Thermal conductivity
$k*$	–	Mean sink co-efficient
L	–	Reference span
M	–	Magnetic constraint

n	–	Exponent in power law variation in surface temperature
Nu_x	–	Local Nusselt number
Nu_X	–	Dimensionless Local Nusselt numeral
$\overline{Nu}$	–	Dimensionless average Nusselt numeral
Pr	–	Prandtl number
q_w	–	Variable heat flux per unit area
R	–	Non- dimensional local radius of the cone
r	–	Local radius of the cone
T'	–	Temperature
T	–	Non-dimensional temperature
t'	–	Time
t	–	Non-dimensional time
U	–	Non-dimensional velocity in X-direction
u	–	Velocity component in x-direction
V	–	Non-dimensional velocity in Y-direction
v	–	Rate component in y-direction
X	–	Non-dimensional spatial co-ordinate
x	–	Spatial coefficient along cone generator
Y	–	Non-dimensional spatial coefficient along the normal
	–	to the cone generator
y	–	Spatial coefficient along the normal to the cone generator

Greek Symbols:

α	–	Thermal diffusivity
β	–	Volumetric thermal expansion
σ	–	Electrical conductivity
$\sigma*$	–	Stefan-Boltzmann constant
Δ	–	Non-dimensional heat source/sink constraint
Δt	–	Non-dimensional time step
ΔX	–	Non-dimensional finite difference grid size in X-direction
ΔY	–	Non-dimensional finite difference grid size in Y-direction
ϵ	–	Viscous dissipation parameter
ϕ	–	Semi vertical angle of the cone
μ	–	Dynamic viscosity
γ	–	Kinematic viscosity
ρ	–	Density
τ_x	–	Local skin friction
τ_X	–	Non-dimensional local skin friction
τ	–	Non-dimensional average skin friction

Subscripts:

w	–	Condition on the wall
∞	–	Free stream condition

1. Introduction

Free convection boundary layer flow and thermal transport over a vertical cone has been the subject of attention of several investigators because these phenomenons arise regularly in environment, as good as in industrialized and technical applications. In the field of engineering, MHD has two areas of applications namely, Magneto hydrodynamic propulsion and power generation. MHD generator is based on the concept of using ionized gases as the moving conductor. The effect of thermal diffusion on the MHD free convection and mass transfer flow has important role in isotopes separation and in mixtures between gases. MHD flowing in an electrically conducting fluid are encountered in many industrial applications, such as purification of molten metals, non-metallic intrusion, liquid metal, plasma studies, geothermal energy extraction, nuclear reactor and the boundary layer control in the field of aerodynamics and aeronautic. Anjalidevi and Kandasamy [3] have analyzed the effects of chemical reaction, temperature and mass transport on non-linear MHD laminar boundary layer flow over a wedge among suction and injection. Analytical solution for the largely thermal and mass transport on MHD movement of an unvaryingly expanded vertical permeable surface with the effects of thermal source/sink and chemical response were obtainable by Chamkha [5].

Hakeem et al. [1] reported the methodical solutions for thermal and mass transport by laminar movement of an electrically performing fluid on a continuously vertical porous surface in the occurrence of a radiation and chemical response effect. Elbashbeshy et al. [10] investigated the laminar natural convection from a vertical circular cone with uneven surface heat flux in the existence of the pressure effort. As the difference between the surface temperature and the ambient temperature is large. The radiation upshots become vital. During the part of convection radiation, Viskanta and Grosh [25] considered the effect of temperature radiation on the thermal sharing and the thermal transport in an absorbing and emitting medium pouring in excess of a wedge by using the Rosseland diffusion estimate. These guesses leads to a significant generalization in the expression for the radiant flux. Muthucumarasamy and Ganesan [18] discussed heat effects on flow past an on impulsively established infinite vertical plate with uneven temperatures using the Laplace transform method.

Alam et al. [2] investigated the crisis of laminar free convective flow and thermal transport from a vertical permeable round cone retained at a non-uniform surface temperature with the force effort governed by the exponent law variation by the span from the convection boundary-layer flow of a micro-polar fluid above a vertical permeable cone with a changeable wall temperature. Cheng [9] analyzed a free convection boundary layer flow of a micro glacial fluid over a vertical permeable cone with a non-uniform surface temperature. Chen et al. [8] explored the free convection on parallel, inclined and vertical plates by dissimilar combination of non-uniform surface heat or changeable heat flux. Kabeir et al [11] used perturbation scheme to study the outcome of temperature and mass transport on natural convection flow with an unvarying suction and injection over a cone in a micro polar fluid. Makinde [14] analyzed hydromagnetic mixed convection flow and mass transport over a vertical porous plate by stable heat flux surrounded in a porous medium. Chamkha and Khaled [6] considered hydromagnetic combined heat and mass transport by free convection from a porous vertical plate embedded in a liquid saturated porous medium in the existence of temperature production or assimilation. Patra et al. [20] investigated the outcome of radiation on natural convection movement of a viscous and incompressible fluid close to a vertical flat plate with inclined temperature. They compared the effects of Radiative temperature transmit on free convection flow near a ramped temperature plate by the flow close to a uniform plate.

Mohamed et al. [15] investigated transient MHD natural convection temperature and mass transport boundary layer flow of viscous, incompressible, optically fat and electrically conducting fluid throughout a permeable medium along an on impulsively moving heat vertical plate in the existence of uniform chemical response of first order and heat reliant temperature drop. They obtained analytical answer of the leading equation in closed form by Laplace transform method. Mosa et al. [16] considered the Bouger numeral effects or glowing MHD Ekman flow on a permeable platter, present closed shape solution for together the optically lean and optically-thick case (achieve when Bouger numeral $>> 1$, so that the mean free pathway of the emission is much lesser than the distinguishing measurement (i.e.) the scatter case) performance that for permanent magnetic field, the temperature distributions are powerfully affect by radiative fluctuation.

Beg et al. [4] focused the temperature source/sink effects on oscillatory magneto convection in a permeable medium using hypergeometrices. Chamkha et al. [7] considered combined temperature generation/absorption, emission and magnetic field effects on forced convection temperature transfer over a wedge with stress work effects. Muralidharan and Muthucumarasamy [17] Radiative heat transport effects on transient movement of viscous non-compressible fluid past an unvaryingly accelerated never-ending vertical plate by changeable heat and homogeneous mass flux have been investigated.

Sharma and Varshney [24] discussed the effects of temperature distribution and viscous dissipation on the transient flow of a viscous incompressible dirty gas throughout a hexagonal channel of regular cross-section under the influence of a magnetic field and moment-dependent pressure gradient.

2. Mathematical analysis

A 2-dimensional uniform, consisting of laminar free convection of thermal through a liquid or gas caused by molecular motion flow of a incapable of being compressed viscous fluid over a vertical cone with non-isothermal surface temperature under the determination of reaction due to the presence of electrically conducting and radiating liquid past a vertical cone with non-homogeneous heat flux by talking into an account the effects of viscous dissipation is viewed or the sensation caused by heat energy is carefully weighed with the following assumptions.

(i) The system is axi-symmetrical.
(ii) The Joule heating of the fluid (magnetic dissipation) is neglected.
(iii) The co-efficient of electrical conductivity is a stable throughout the fluid.

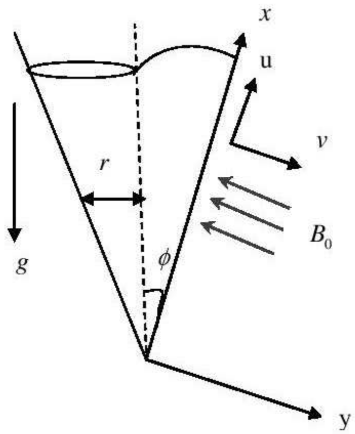

FIGURE 1. Physical Model and Coordinate System

(iv) Transverse uniform magnetic field is applied perpendicular to the surface of the cone.
(v) The attractive Reynolds number is little so that the induce magnetic field is neglect and consequently, do not alter the magnetic field.
(vi) Thermal radiation is current in the appearance a unidirectional flux in the y direction
(vii) The Radiative thermal flux into the x direction is regard as unimportant in comparison with that in the y direction.
(viii) The magnetic field equation is the common electromagnetic and hydro magnetic equation, but the communication between the flow and the attractive field is taken in to account.
(ix) Maxwell's dislocation current is ignored, so as to electric current is regard as flow in closed circuits.

The co-ordinate system is chosen such to establish position is give an exhibition of to an interested study has be consider as x-axis is in use on the surface of the cone from the vertex $x = 0$ and y denotes the distance normally outward. The fluid belongings are taken to be unvarying and take exception to density divergences which stimulate perkiness strength term in the velocity equation and it maneuver key factor the discussion. Here ϕ is the half vertical angle of the cone and $r(x)$ is the local radius of the cone.

Primarily at $t^{'} \leq 0$, it is moreover considered that the cone surface and the enclosing fluid, which is at rest, have the same temperature $T^{'}_{\infty}$. Then at $t^{'} > 0$, the temperature of cone surface is suddenly raised to inconsistently $q_w(x) = ax^n$. It is considered that the fluid properties are non-varying except for density variations, which induce buoyancy force term in the momentum equation. The governing boundary layer equations of continuity", an impelling force or strength and a thermodynamic

quantity equivalent to the capacity of a physical system to do work which was proved by an approximation given by Boussinesq are stated below:
Equation of continuity:

$$\frac{\partial}{\partial x}(ru) + \frac{\partial}{\partial y}(rv) = 0 \tag{2.1}$$

Equation of momentum:

$$\frac{\partial u}{\partial t'} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = g\beta\cos\phi(T^{'} - T^{'}_{\infty}) + \nu\frac{\partial^2 u}{\partial y^2} - \frac{\sigma B_0^2 u}{\rho} \tag{2.2}$$

Equation of energy:

$$\frac{\partial T^{'}}{\partial t'} + u\frac{\partial T^{'}}{\partial x} + v\frac{\partial T^{'}}{\partial y} = \alpha\frac{\partial^2 T^{'}}{\partial y^2} + \frac{\mu}{\rho C_P}\left(\frac{\partial u}{\partial y}\right)^2 + \frac{Q_0}{\rho C_P}(T^{'} - T^{'}_{\infty}) - \frac{1}{\rho C_P}\frac{\partial q_r}{\partial y} \tag{2.3}$$

The primary and boundary condition are prescribed as

$$\left.\begin{aligned} &t^{'} \leq 0 : u = 0, v = 0, T^{'} = T^{'}_{\infty} \text{ for all } x \text{ and } y \\ &t^{'} > 0 : u = 0, v = 0, \frac{\partial T^{'}}{\partial y} = \frac{-q_w(x)}{k} \text{ at } y = 0 \\ &u = 0, T^{'} = T^{'}_{\infty} \text{ at } x = 0 \\ &u \to 0, T^{'} \to T^{'}_{\infty} \text{ as } y \to \infty \end{aligned}\right\} \tag{2.4}$$

Using the Roseland estimate for radiation [23], Radiative heat flux is reduced.

$$q_r = \frac{-4\sigma^*}{3k^*}\frac{\partial T^{'4}}{\partial y} \tag{2.5}$$

where σ^* the Stefan-Boltzmann is stable and k^* is the represent mean absorption coefficient. It must be prominent that by using the Roseland approximation, the current study is restricted to optically substantial fluid. If temperature difference within the flow is adequately small, then equation (2.5) can be linearized by expanding $T^{'4}$ in Taylor series about $T^{'}_{\infty}$ which after forgetting upper order conditions take the form:

$$T^{'4} \cong 4T^{'3}_{\infty}T^{'} - 3T^{'4}_{\infty} \tag{2.6}$$

Using equations (2.5) and (2.6), the energy equation (2.3) becomes

$$\frac{\partial T^{'}}{\partial t'} + u\frac{\partial T^{'}}{\partial x} + v\frac{\partial T^{'}}{\partial y} = \alpha\frac{\partial^2 T^{'}}{\partial y^2} + \frac{\mu}{\rho C_P}\left(\frac{\partial u}{\partial y}\right)^2 + \frac{Q_0}{\rho C_P}(T^{'} - T^{'}_{\infty}) + \frac{1}{\rho C_P}\frac{16\sigma^* T^{'3}_{\infty}}{3k^*}\frac{\partial^2 T^{'}}{\partial y^2} \tag{2.7}$$

Local skin-friction and local Nusselt number has been given by

$$\tau^{'}_x = \mu\left(\frac{\partial u}{\partial y}\right)_{y=0}; Nu^{'}_x = \frac{-x\left(\frac{\partial T^{'}}{\partial y}\right)_{y=0}}{T^{'}_w - T^{'}_{\infty}} \tag{2.8}$$

Further, we introducing the subsequent dimensionless quantities:

$$\left.\begin{aligned}
&X=\frac{x}{L}, Y=\frac{y}{L}(Gr_L)^{\frac{1}{5}}, R=\frac{r}{L}\ \text{where}\ r=x\sin\phi,\\
&t=\frac{\nu t^{'}}{L^2}(Gr_L)^{\frac{2}{5}}, T=\frac{T^{'}-T^{'}_{\infty}}{\frac{q_w(L)}{k^{**}}}(Gr_L)^{\frac{1}{5}}, \varepsilon=\frac{g\beta L}{C_P},\\
&U=\frac{uL}{\nu}(Gr_L)^{\frac{-2}{5}}, V=\frac{vL}{\nu}(Gr_L)^{\frac{-1}{5}},\\
&\gamma=\frac{\mu}{\rho}, \Delta=\frac{Q_0L^2}{C_P\mu}(Gr_L)^{\frac{-1}{5}}, Pr=\frac{\nu}{\alpha},\\
&M=\frac{\sigma B_0^2L^2}{\mu}(Gr_L)^{\frac{-2}{5}}, R_d=\frac{k^*k}{4\sigma^*T^{'3}_{\infty}},\\
&Gr_L=\frac{g\beta\frac{q_w(L)}{k^{**}}L^4\cos\phi}{\nu^2}
\end{aligned}\right\} \tag{2.9}$$

Equations (2.1) to (2.3) can be written in the subsequent dimensionless form:

$$\frac{\partial}{\partial X}(UR)+\frac{\partial}{\partial Y}(VR)=0 \tag{2.10}$$

$$\frac{\partial U}{\partial t}+U\frac{\partial U}{\partial X}+V\frac{\partial U}{\partial Y}=T-MU+\frac{\partial^2 U}{\partial Y^2} \tag{2.11}$$

$$\frac{\partial T}{\partial t}+U\frac{\partial T}{\partial X}+V\frac{\partial T}{\partial Y}=\frac{1}{Pr}\left(1+\frac{4}{3R_d}\right)\frac{\partial^2 T}{\partial Y^2}+\Delta T+\varepsilon\left(\frac{\partial U}{\partial Y}\right)^2 \tag{2.12}$$

The following dimensionless primary and boundary condition are

$$\left.\begin{aligned}
&t\le 0: U=0, V=0, T=0\ \text{for all}\ X and Y\\
&t>0: U=0, V=0, \frac{\partial T}{\partial Y}=-X^n\ \text{at}\ Y=0\\
&U=0, T=0\ \text{at}\ X=0\\
&U\to 0, T\to 0\ \text{as}\ Y\to\infty
\end{aligned}\right\} \tag{2.13}$$

Shear stress and heat transfer rate in dimensionless form are given by

$$\tau_X=(Gr_L)^{\frac{3}{5}}\left(\frac{\partial U}{\partial Y}\right)_{Y=0} \tag{2.14}$$

$$Nu_X=\frac{X}{T_{Y=0}}\left(\frac{-\partial T}{\partial Y}\right)_{Y=0}(Gr_L)^{\frac{1}{5}} \tag{2.15}$$

Also, the dimensionless average shear stress $\overline{\tau}$ and the average heat transfer rate $\overline{Nu}$ are able to write as

$$\overline{\tau}=2Gr_L^{\frac{3}{5}}\int_0^1 X\left(\frac{\partial U}{\partial Y}\right)_{Y=0}dX \tag{2.16}$$

$$\overline{Nu}=2Gr_L^{\frac{1}{5}}\int_0^1 \frac{X}{T_{Y=0}}\left(\frac{-\partial T}{\partial Y}\right)_{Y=0}dX \tag{2.17}$$

3. Method of solution

The transient, non-linear, coupled PDE (2.10) to (2.12) with (2.13) are worked out by using Crank-Nicholson method. After applying the method the dimensionless equation converted to the system of tri-diagonal equations. We work out the scheme of equations by use well known Thomas algorithm by which we attain the desired solution with convergence of this algorithm occurring in a brief period of time and also it is unconditionally stable to change as discussed Bapuji et al. [22]. The integral area is treated as a rectangle with $X_{\max} = 1$ and $Y_{\max} = 26$ (the value for Y is taken to be ∞) by analyzing in detail and considered in order to satisfy the ultimate and penultimate conditions of (2.13) and we observed that it is fulfilled with accuracy up to 10^{-5}.

The mesh sizes has been permanent as $\Delta X = 0.05, \Delta Y = 0.05$ with time step $\Delta t = 0.01$. The computations are carried out first by reducing the spatial mesh sizes by 50% in one way, and afterward in both direction by 50%. The computations are conceded out first by reducing the spatial net sizes by 50%. The results are compared. It is revealed that in all cases, the outcomes differ simply in the fifth decimal place. Hence, the choice of the mesh sizes seem to be suitable.

The co-efficient of $U^k_{i,j}$ and $V^k_{i,j}$ appearing in the FDE's are treated as constants at any one-time step. Here i designates the grid point along the X direction, j along the Y direction and k along the time t. The values of V, U and T are known at all grid points when t=0 from the initial conditions.

The computations of U, V and T at a time level $(k + 1)$, using the values at previous time level k are carried out as follows. The FDE (2.12) at every internal nodal point on a particular i - level constitutes a tri-diagonal scheme of equations and is solved by Thomas algorithm as discussed in Bapuji et al [22]. Thus, the values of T are found out at every lattice point at a particular i at (k+1)th time level. Similarly, the values of U are calculated from equation (2.11), and finally the values of V are calculated explicitly by using equation (2.10) at every mesh point on a particular i-level at (k+1)th time level. In a similar, manner computations are carried out by moving along i direction. Subsequent to calculating values corresponding to each i at a time level, the values at the next time level are determined in a similar manner. Computations are repeated until steady state is reached. The steady state solution is assumed to have been reached when the absolute difference between the values of the rate U, as well as temperature T at two successive time steps are less than 10^{-5} at all grid points.

4. Results and discussions

This segment provides the behavior of a range of parameter involved in the expressions velocity and temperature the near result in stable condition at $X = 1.0$ is established with accessible resemblance solution in literature. The momentum and thermal boundary layer profile of the cone with isothermal surface heat flux when $Pr = 0.72$, $Rd = 2.0$, $M = 2.0$ and surface temperature power law exponent $n = 0.5$ the numerical values of shear stress τ_X and rate of heat transfer Nu_x for distinct value

of Prandtl number revealed in table (1) are examined with resemblance solution of Lin [13] in steady state using a appropriate conversion.
(i.e.) $Y = (20/9)^{1/5}\eta$, $T = (20/9)^{1/5}[-\theta(0)]$, $U = (20/9)^{1/5} f'(\eta)$,
$\tau_X = (20/9)f''(0)$.
In adding up, the local shear stress τ_X and rate of heat transfer Nu_x, for distinct numerical quantities of Pr when heat flux gradient $n = 0.50$ and $M = 0$ at $X = 1.00$ instable situation are compared with the non-similarity result of Hossain and Paul [12] in table 2. It is noticed that the results are in good agreement with each other. It is also noticed that the current result concur well with those of Pop and Watanable [21], Na and Chiou [19] (as pointed out in table 1).

The transitory velocity and temperature profile at $X = 1.0$ for dissimilar values of Pr and M be plot in figs. 2 and 3. The viscous force increases and the thermal diffusivity decrease by a rising Pr, which cause a decline in the velocity and temperature. Also, the influence of magnetic constraint M against span-wise spatial distribution of velocity and temperature are depicted. Application of attractive meadow usual to the flow of an electrically conduct fluid give increase to an opposite force. These opposite powers tend to slow downward the movement of the fluid along the cone and cause a raise in its hotness and a decline in velocity as M increase. An increases in M from 1 although 2, 3 obviously reduce flow-wise velocity together in the close to-wall area and distant-field regime of the boundary layer, while the surface temperature increase as the bigger values of M.

Table 1: Relationship of steady - state shear stress and temperature values at $X = 1.0$ with those of Lin [13] for isothermal surface heat flux.

$*$ Values taken from Pop and Watanabe [21] when suction/injection is zero.
$**$ Values taken from Na and Chiou [19] when solutions for flow over a full cone.

Pr	Temperature			Local skin friction		
	Lin [13] results		Present values	Lin [13] results		Present values
	$-\theta(0)$	$-\left(\frac{20}{9}\right)^{1/5}\theta(0)$	T	$-f''(0)$	$\left(\frac{20}{9}\right)^{2/5} f''(0)$	T
0.72	1.522878	1.7864	1.7714	0.88930, 0.88930$*$	1.224	1.2105
1	1.39174	1.6327, 1.6329$**$	1.6182	0.78446	1.0797	1.0669
2	1.16209	1.3633	1.3499	0.60252	0.8293	0.8182
4	0.98095	1.1508	1.1385	0.46307	0.6373	0.6275
6	0.89195	1.0464	1.0344	0.39688	0.5462	0.5371
8	0.83497	0.9796	0.9677	0.35563	0.4895	0.4808
10	0.79388	0.9314	0.9196	0.32655	0.4494	0.4411
100	0.48372	0.5675	0.5531	0.13371	0.184	0.1778

Table 2: Comparison of steady-state shear stress and heat transfer rate values at $X = 1.0$ with those of Hossain and Paul [12] for different values of Pr when $n = 0.5$ and $M = 0$ suction is zero.

Pr	Temperature		Local Nusselt number	
	Hossain and Paul [12]	Present values	Hossain and Paul [12]	Present values
	$F_0^{''}(0)$	$\frac{\tau_X}{Gr_L^{3/5}}$	$\frac{1}{\phi_0(0)}$	$\frac{Nu_X}{Gr_L^{1/5}}$
0.01	5.1345	5.1155	0.14633	0.1458
0.05	2.93993	2.9297	0.26212	0.2630
0.1	2.29051	2.2838	0.33174	0.3324

Figs. 4 and 5 shows the effect of different heat generation/absorption parameter Δ and radiation constraint Rd on the dimensionless velocity and dimensionless thermal for the Prandtl number $Pr = 0.72, n = 0.5, M = 2$ and $\varepsilon = 0.5$. It is revealed that the velocity and temperature profile are increase with rising quantities of Δ. In raising *Rd* cause a considerable decreases in velocity with detachment into the boundary layer (i.e.) decelerate the flow. We too note down that with growing values of *Rd* the time taken to achieve the stable state is abridged. As estimated temperature value are also considerably abridged with increases in *Rd*.

Figs. 6 and 7 represents the velocity and temperature profiles for dissimilar values of n and viscous dissipative constraint ε. It is seen that the impulsive forces are decreased by the side of the cone surface close to the apex with increases in 'n'. Owing to this, the dissimilarity involving the sequential utmost and stable state quantities decrease, the velocity with temperature decreases, the time requisite for the achievement of a stable state increase and velocity and the temperature boundary layer turn out to be thinner at lower values of n.

An increase viscous dissipative thermal cause an increase in the temperature, so as to the momentum boundary layer and thermal boundary layer increases with ε. It is as well see to facilitate the variation connecting the temporal utmost and stable state values are abridged, even as the moment necessary for the achievement of a steady state as well as the velocity and thermal boundary layer increases by ε.

Once, if temperature and velocity are found, it is interest to study local as well as average shear stress and heat transfer rate distributions in transitory state. The derivative in the equations (2.14) - (2.17) is acquired with the use of five-point approximation formula and the integrals be calculated via the Newton-Cotes closed integration method. Figures 8 and 9 illustrate local Nusselt number and shear stress for various values of Pr and M. It is demonstrates that both quantity increases with decreasing Pr and decrease with distance from the cone vertex. An increasing magnetic parameter M leads to decreasing both local shear stress and local heat transfer rate. Figs. 10 and 11 indicates the outcome of heat absorption/ generation constraint Δ and radiation constraint *Rd*. The local shear stress increases for the higher values of Δ. But the trend is reversed the heat transfer rate case. Also, stronger thermal radiations accelerate the flow but reduce Nusselt number hence the local skin-friction got decreased due to the presence of radiation. Consequently, heat transfer rate increases for lower value of Rd. Figs. 12 and 13 depicts the effect of viscous dissipative constraint and the exponent in the power rule difference n on the local shear stress coefficient and Nusselt number are analyzed. From these graphs, it is able to be seeing that increasing ε clearly boosts the wall shear stresses τ_X, which grow powerfully from the directing boundary downstream alongside the cone surface.

Consequently, it enhances the viscous dissipative heats direct to a decline in the local heat transfer rate Nu_X (i.e.) with a substantial increase in ε a strong reduce in the surface heat transfer rate. In figs. 14 and 15 shows the influence of Prandtl numeral and magnetic parameter over the average local skin friction and average heat transfer rate are observed. The average shear stress decreases for smaller value of Pr and larger value of magnetic parameter M. Also, it is noticed that the average heat transfer rate decreases for increasing values of Prandtl numeral Pr and M.

Figs. 16 and 17 demonstrates the results of heat source/sink constraint and radiation parameter Rd. Stronger thermal radiation accelerate the flow but reduces average Nusselt number hence the average skin-friction got decreased due to the existence of magnetic field and radiation where as it boost for the higher values of thermal absorption/ generation constraint Δ.

In figs. 18 and 19 depicts the variations of average shear stress and average heat transfer rate for controlling parameter n and viscous dissipation parameter ε. Average shear stress is more for lower values of n and average heat transfer rate is almost negligible. Also, it boost in ε the viscous dissipative heat leads to boost in the average skin-friction. But the trend is reversed in average Nusselt number cases.

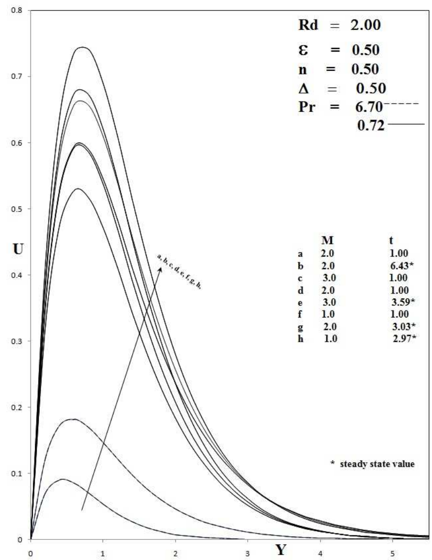

FIGURE 2. Transient velocity profiles at $X = 1.0$ for different values of Pr and M

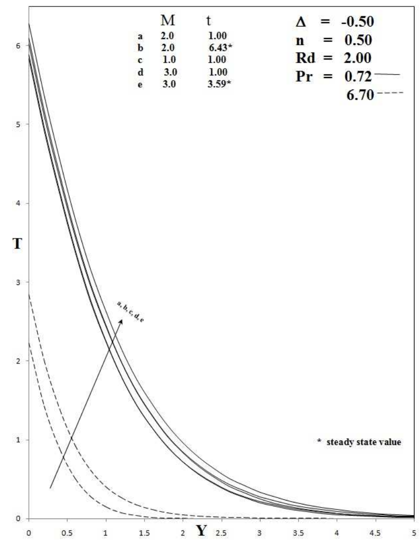

FIGURE 3. Transient temperature profiles at $X = 1.0$ for different values of Pr and M

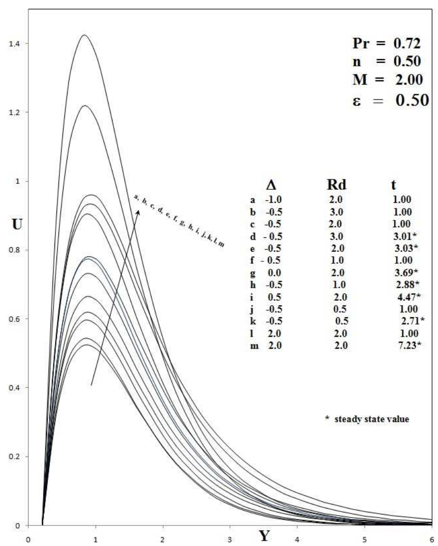

FIGURE 4. Transient velocity profiles at $X = 1.0$ for different values of Δ and Rd

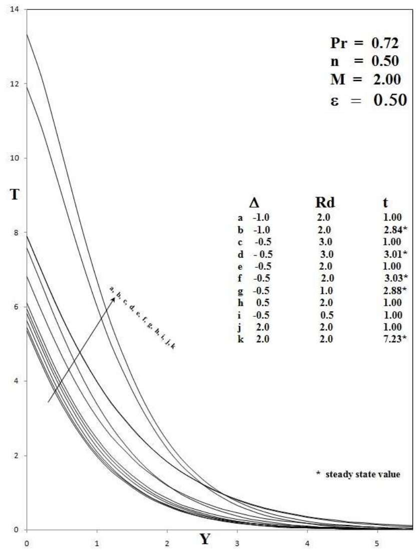

FIGURE 5. Transient temperature profiles at $X = 1.0$ for different values of Δ and Rd

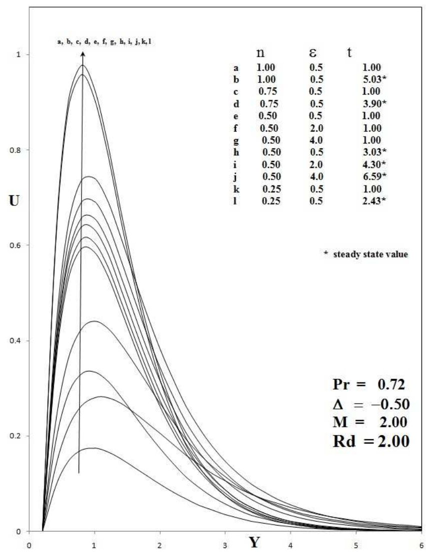

FIGURE 6. Transient velocity profiles at $X = 1.0$ for different values of n and ε

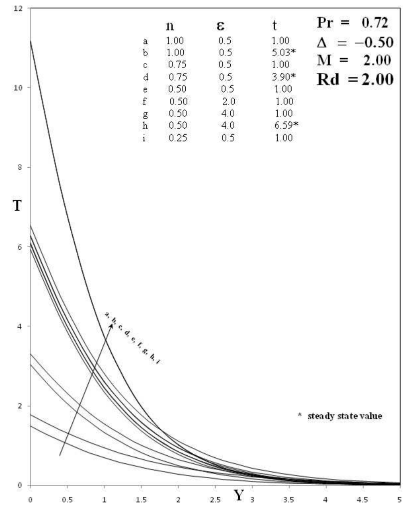

FIGURE 7. Transient velocity profiles at $X = 1.0$ for different values of n and ε

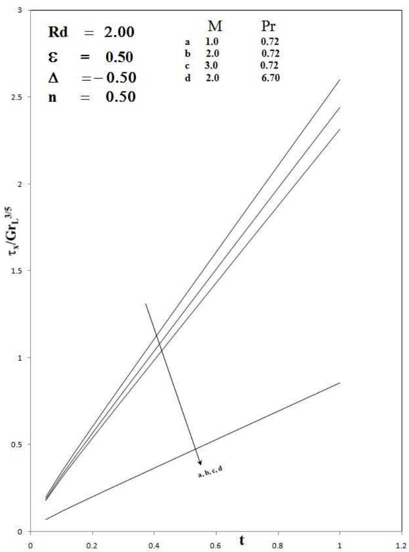

FIGURE 8. Local skin friction for different values of Pr and M in transient period

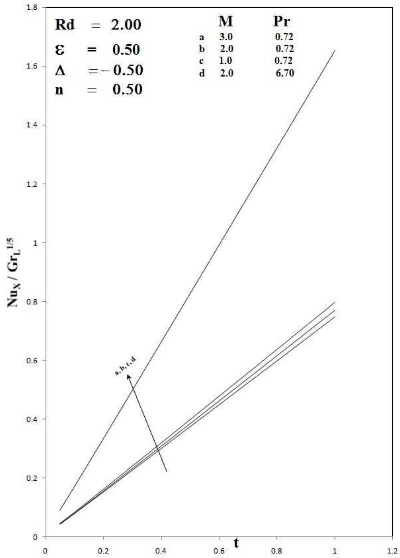

FIGURE 9. Local Nusselt number for different values of Pr and M in transient period

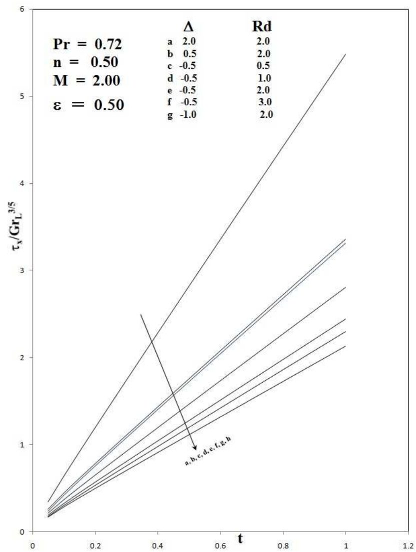

FIGURE 10. Local skin friction for different values of Δ and Rd in transient period

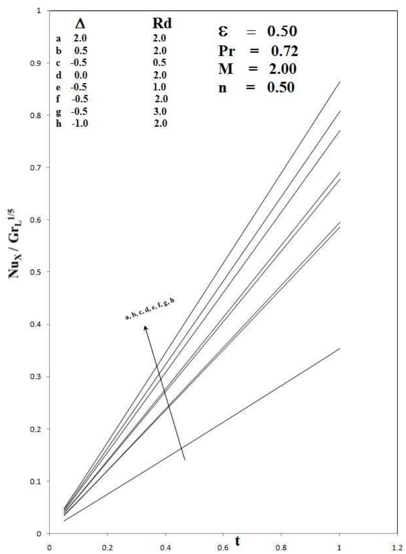

FIGURE 11. Local Nusselt number for different values of Δ and Rd in transient period

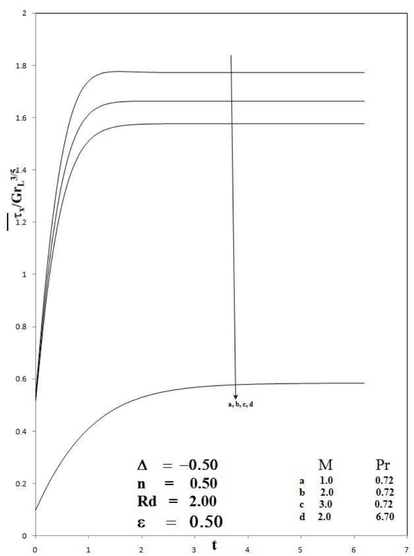

FIGURE 12. Average skin friction for different values of Pr and M in transient period

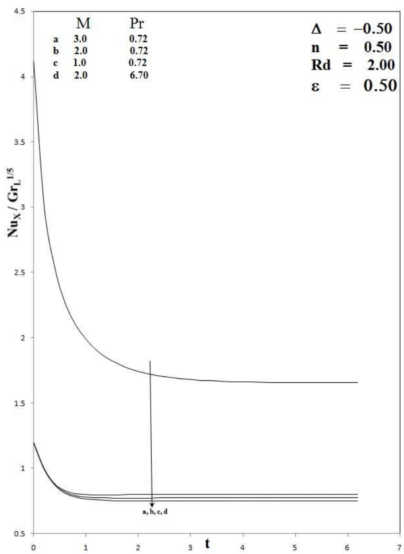

FIGURE 13. Average Nusselt Number for different values of Pr and M in transient period

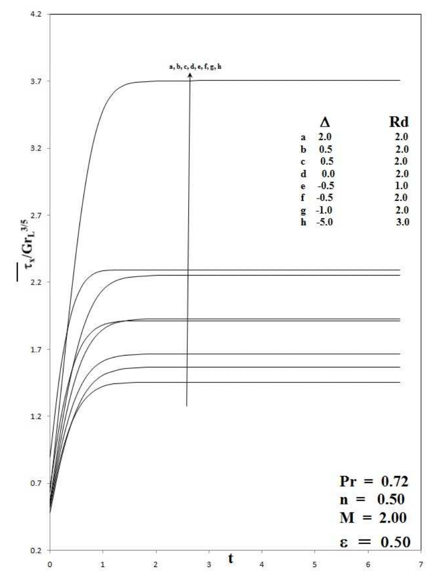

FIGURE 14. Average skin friction for different values of Δ and Rd in transient period

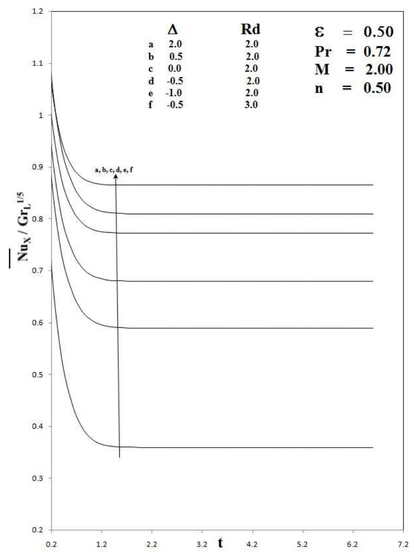

FIGURE 15. Average Nusselt number for different values of Δ and Rd in transient period

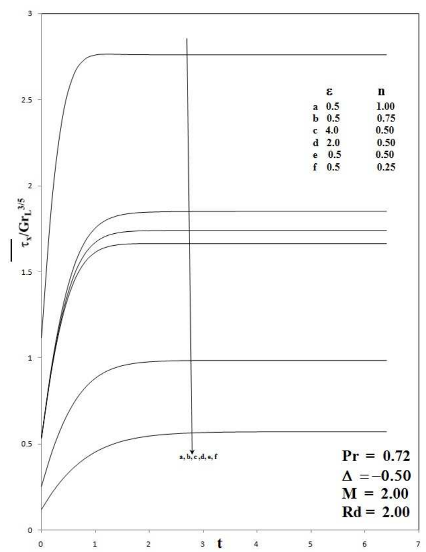

FIGURE 16. Average skin friction for different values of n and ε in transient period

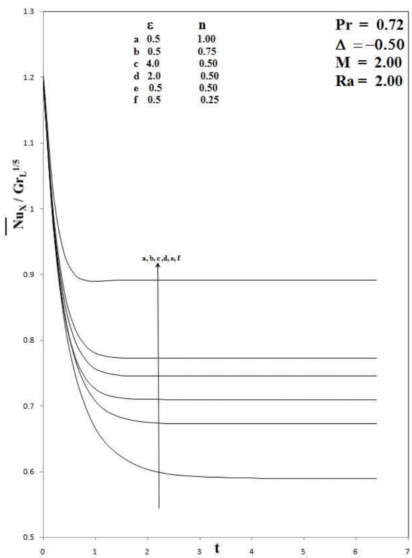

FIGURE 17. Average Nusselt number for different values of n and ε in transient period

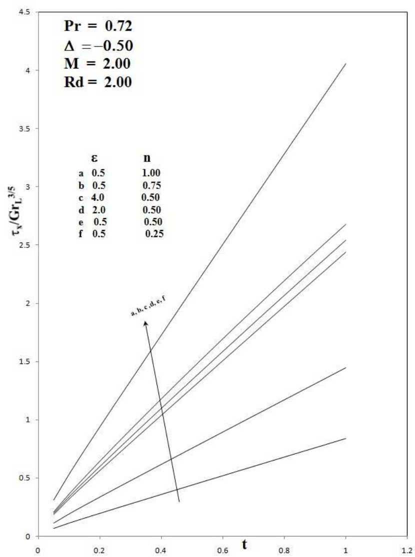

FIGURE 18. Local skin friction for different values of n and ε in transient period

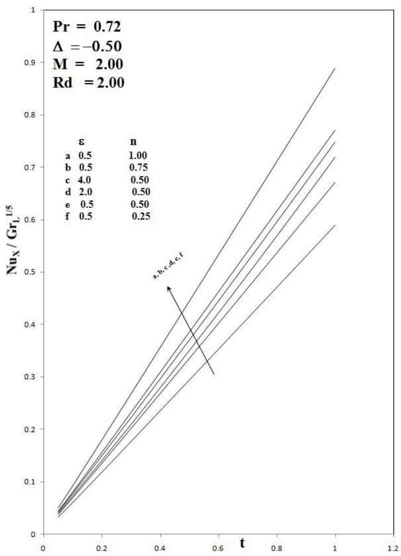

FIGURE 19. Local Nusselt number for different values of n and ε in transient period

5. Conclusion

A mathematical study of the flow past a variable vertical cone has been studied. The family of leading partial differential equations are solved via an implicit finite difference scheme of Crank-Nicholson type. The subsequent conclusions are drawn:

- The computations have shown that the velocity and temperature allocation decrease with growing the values of Pr, n and while the velocity and temperature allocation increase with increasing the value of Δ, ε.
- The velocity increases and temperature decreases when the controlling parameter M and radiation parameter Rd.
- The shear stress τ_X and heat transfer rate Nu_X values decrease as M, n, Rd and Pr increases.
- Smaller values of heat generation /absorption parameter Δ lead to decline in the values of the shear stress coefficient while the local Nusselt number increases.
- The average shear stress is more for larger values of Δ, ε and smaller values of Pr, n, Rd and M.
- The average heat transfer rate increases for bigger value of Pr, n and ε for lesser value of Δ, Rd and M.

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Book reviews

John R. Graef, Johnny Henderson, Abdelghani Ouahab; Topological Methods for Differential Equations and Inclusions, CRC Press, Taylor & Francis Group, Boca Raton, FL, USA, 2019, xiv + 360 pp., ISBN 9781138332294 - CAT# K393112.

In the last 30 years, Nonlinear Analysis became a topic with a flourishing development. Part of this field, Multi-valued Analysis has experienced a spectacular growth, generating new research directions in various classical areas of mathematics. In this respect, the study of integral and differential inclusions gets a strong evolution. The present book brings an important contribution to these fields, by presenting, in a unified and exhaustive manner, many interesting results from the theory of Differential Equations and Inclusions, via Multi-valued Analysis and Topological Fixed Point Theory.

The contents of this monograph is organized in the following chapters: 1. *Background in Multi-valued Analysis*; 2. *Hausdorff-Pompeiu Metric Topology*; 3. *Measurable Multi-functions*; 4. *Continuous Selection Theorems*; 5. *Linear Multivalued Operators*; 6. *Fixed Point Theorems*; 7. *Generalized Metric and Banach Spaces*; 8. *Fixed Point Theorems in Vector Metric and Banach Spaces*; 9. *Random Fixed Point Theorem*; 10. *Semi-groups*, 11. *Systems of Impulsive Differential Equations on the Half-line*; 12. *Differential Inclusions*; 13. *Random Systems of Differential Equations*; 14. *Random Fractional Differential Equations via Hadamard Fractional Derivative*; 15. *Existence Theory for Systems of Discrete Equations*; 16. *Discrete Inclusions*; 17. *Semi-linear System of Discrete Equations*; 18. *Discrete Boundary Value Problems*; 19. *Appendix.*

The monograph is well written, the concepts and the results are presented in a clear and rigorous way. The material is based on numerous papers and books previously published by the authors. The bibliography includes 296 titles, most of them from the last decades. The book will be very useful for graduate students, professors and researchers interested in the field of integral and differential equations and inclusions, via topological methods of nonlinear functional analysis.

Adrian Petruşel

Julian Havil; Curves for the Mathematically Curious: An Anthology of the Unpredictable, Historical, Beautiful, and Romantic, Princeton University Press, 2019, xviii+280 p. ISBN 978-0-691-18005-2/hbk; 978-0-691-19778-4/ebook.

The book contains ten chapters, each one describing a famous curve: 1. *The Euler curves*; 2. *The Weierstrass curve*; 3. *Bézier curves*; 4. *The rectangular hyperbola*; 5. *The quadratrix of Hippias*; 6. *The space filling curves*; 7. *Curves of constant width*; 8. *The normal curve*; 9. *The catenary*, and 10. *Elliptic curves.*

Of course, this choice reflects author's taste and ideas, an important omission being that of conic sections (briefly mentioned in Appendix B as solutions of a differential equation), but as the author says "not every anthology of poems contains works by Shakespeare".

The book is dedicated to a mathematically inclined large audience, so it is written in a didactic style with a lot of mathematical details, historical detours and witty remarks of the author. As he writes in the Preface:

> We invite the reader to join us in this particular and eclectic mathematical adventure, with stories bringing us into glancing contact with (among others) Pablo Picasso, George II, Queen Victoria's consort (Prince Albert), the Inquisition, the Holy Roman Emperor (Frederick II) and many mathematicians who existed over millennia.

The author is well-known for his popular books on various topics in mathematics: *Gamma: Exploring Euler's Constant* (2003); *Nonplussed! Mathematical Proof of Implausible Ideas* (2007); *Impossible? Surprising Solutions to Counterintuitive Conundrums* (2008); *The Irrationals: A Story of the Numbers You Can't Count On* (2014), and *John Napier: Life, Logarithms, and Legacy* (2014). All these books were published with Princeton and each of them knew several editions (we quoted the year of the first one). Two of them were translated into German.

Undoubtedly that this new one, written in the same entertaining unmistakable style of the author and containing a lot of information – mathematical, historical and general – will attract, as the previous ones, a large audience.

S. Cobzaş