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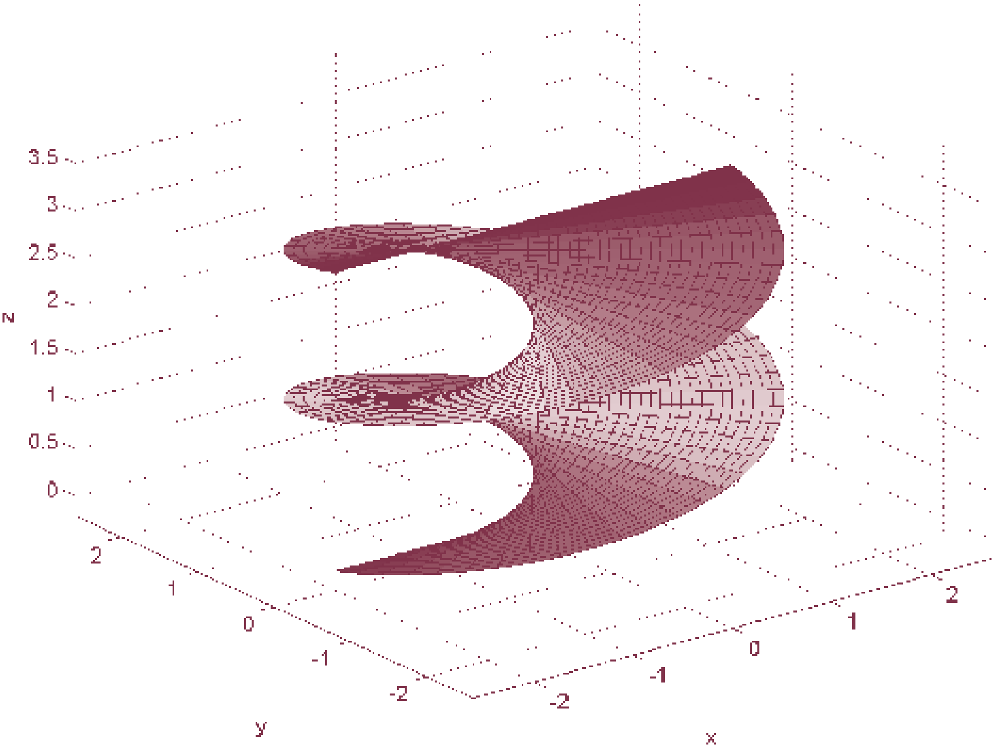

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General inequalities related Hermite-Hadamard inequality for generalized fractional integrals

Havva Kavurmacı-Önalan, Erhan Set and Abdurrahman Gözpınar

Abstract. In this article, we first establish a new general integral identity for differentiable functions with the help of generalized fractional integral operators introduced by Raina [8] and Agarwal *et al.* [1]. As a second, by using this identity we obtain some new fractional Hermite-Hadamard type inequalities for functions whose absolute values of first derivatives are convex. Relevant connections of the results presented here with those involving Riemann-Liouville fractional integrals are also pointed out.

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Keywords: Hermite-Hadamard inequality, Riemann-Liouville fractional integral, fractional integral operator.

1. Introduction and preliminaries

One of the most famous inequalities for convex functions is Hermite-Hadamard's inequality. This double inequality is stated as follows (see for example [3]).

Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a convex function on the interval I of real numbers and $a, b \in I$ with $a < b$. Then

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}. \tag{1.1}$$

Definition 1.1. The function $f : [a, b] \subset \mathbb{R} \to \mathbb{R}$ is said to be convex if the following inequality holds:

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y)$$

for all $x, y \in [a, b]$ and $\lambda \in [0, 1]$. We say that f is concave if $(-f)$ is convex.

Now, we will give some important definitions and mathematical preliminaries of fractional calculus theory which are used throughout of this paper.

Definition 1.2. [4] Let $f \in L_1[a,b]$. The Riemann-Liouville integrals $J_{a+}^{\alpha} f$ and $J_{b-}^{\alpha} f$ of order $\alpha > 0$ with $a \geq 0$ are defined by

$$J_{a+}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_{a}^{x} (x-t)^{\alpha-1} f(t)dt, \quad x > a$$

and

$$J_{b-}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_{x}^{b} (t-x)^{\alpha-1} f(t)dt, \quad x < b$$

respectively where

$$\Gamma(\alpha) = \int_{0}^{\infty} e^{-u} u^{\alpha-1} du.$$

Here is $J_{a+}^{0} f(x) = J_{b-}^{0} f(x) = f(x)$. In the case of $\alpha = 1$, the fractional integral reduces to the classical integral.

In [5], Iqbal *et al.* proved a new identity for differentiable convex functions via Riemann-Liouville fractional integrals.

Lemma 1.3. *Let f:$[a,b] \to \mathbb{R}$ be a differentiable function on (a,b). If $f' \in L'[a,b]$, then the following identity for Riemann-Liouville fractional integrals holds:*

$$f\left(\frac{a+b}{2}\right) - \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}} \left[J_{a+}^{\alpha} f(b) + J_{b-}^{\alpha} f(a)\right] = \sum_{k=1}^{\infty} I_k,$$

where

$$I_1 = \int_0^{\frac{1}{2}} t^{\alpha} f'(tb + (1-t)a)\, dt, \qquad I_2 = \int_0^{\frac{1}{2}} (-t^{\alpha}) f'(ta + (1-t)b)\, dt,$$

$$I_3 = \int_{\frac{1}{2}}^{1} (t^{\alpha} - 1) f'(tb + (1-t)a)\, dt, \quad I_4 = \int_{\frac{1}{2}}^{1} (1 - t^{\alpha}) f'(ta + (1-t)b)\, dt.$$

By using the above identity, the authors obtained left-sided of Hermite-Hadamard type inequalities for convex functions via Riemann-Liouville fractional integrals. Some other results related to those inequalities involving Riemann-Liouville fractional integrals can be found in the literature, for example, in [2, 7, 18, 16, 11] and the references therein.

In [8], Raina introduced a class of functions defined formally by

$$\mathcal{F}_{\rho,\lambda}^{\sigma}(x) = \mathcal{F}_{\rho,\lambda}^{\sigma(0),\, \sigma(1),\, \cdots}(x) = \sum_{k=0}^{\infty} \frac{\sigma(k)}{\Gamma(\rho k + \lambda)} x^k \quad (\rho, \lambda > 0; |x| < \mathbb{R}) \tag{1.2}$$

where the coefficients $\sigma(k)$, $(k \in \mathbb{N} = \mathbb{N} \cup \{0\})$, is a bounded sequence of positive real numbers and $\mathbb{R}$ is the set of real numbers. With the help of (1.2), Raina [8] and Agarwal *et al.* [1] defined the following left-sided and right-sided fractional integral operators respectively, as follows:

$$\left(\mathcal{J}_{\rho,\lambda,a+;w}^{\sigma} \varphi\right)(x) = \int_a^x (x-t)^{\lambda-1} \mathcal{F}_{\rho,\lambda}^{\sigma}[w(x-t)^{\rho}] \varphi(t) dt \ (x > a), \tag{1.3}$$

$$\left(\mathcal{J}^{\sigma}_{\rho,\lambda,b-;w}\varphi\right)(x) = \int_x^b (t-x)^{\lambda-1}\mathcal{F}^{\sigma}_{\rho,\lambda}[w(t-x)^{\rho}]\varphi(t)dt \ (x < b) \tag{1.4}$$

where $\lambda, \rho > 0$, $w \in \mathbb{R}$ and $\varphi(t)$ is such that the integral on the right side exits. It is easy to verify that $\left(\mathcal{J}^{\sigma}_{\rho,\lambda,a+;w}\varphi\right)(x)$ and $\left(\mathcal{J}^{\sigma}_{\rho,\lambda,b-;w}\varphi\right)(x)$ are bounded integral operators on $L(a,b)$, if

$$\mathfrak{M} := \mathcal{F}^{\sigma}_{\rho,\lambda+1}[w(b-a)^{\rho}] < \infty. \tag{1.5}$$

In fact, for $\varphi \in L(a,b)$, we have

$$||\mathcal{J}^{\sigma}_{\rho,\lambda,a+;w}\varphi(x)||_1 \leq \mathfrak{M}(b-a)^{\lambda}||\varphi||_1 \tag{1.6}$$

and

$$||\mathcal{J}^{\sigma}_{\rho,\lambda,b-;w}\varphi(x)||_1 \leq \mathfrak{M}(b-a)^{\lambda}||\varphi||_1 \tag{1.7}$$

where

$$||\varphi||_p := \left(\int_a^b |\varphi(t)|^p dt\right)^{\frac{1}{p}}.$$

Here, many useful fractional integral operators can be obtained by specializing the coefficient $\sigma(k)$. For instance the classical Riemann-Liouville fractional integrals J^{α}_{a+} and J^{α}_{b-} of order α follow easily by setting $\lambda = \alpha$, $\sigma(0) = 1$ and $w = 0$ in (1.3) and (1.4). Also, to see more results and generalizations for convex and some other several convex functions classes, as $Q(I), P(I), SX(h,I)$ and $r-$convex, involving generalized fractional integral operators, see [17, 14, 15, 10, 9, 13, 12, 19, 20] and references there in.

In this paper, we will prove a generalization of the identity given by Iqbal *et al.* in [5] by using generalized fractional integral operators. Then we will give some new Hermite-Hadamard type inequalities for fractional integral operators.

2. Main results

We start by giving a generalization of Lemma 1, [5]. We will use an abbreviation throughout of this study,

$$M_f(a,b;w;J) = F^{\sigma}_{\rho,\lambda+1}[w(b-a)^{\rho}]f\left(\frac{a+b}{2}\right) - \frac{1}{2(b-a)^{\lambda}}\left[\left(\mathcal{J}^{\sigma}_{\rho,\lambda,a+;w}f\right)(b) + \left(\mathcal{J}^{\sigma}_{\rho,\lambda,b-;w}f\right)(a)\right]$$

that is similar to the symbol "$L_f(a,b;w;J)$" in [17].

Lemma 2.1. *Let $f : [a,b] \to \mathbb{R}$ be a differentiable mapping on (a,b) with $a < b$ and $\lambda > 0$. If $f' \in L[a,b]$, then the following equality for generalized fractional integral operators holds:*

$$M_f(a,b;w;J) = \frac{b-a}{2}(I_1 + I_2 + I_3 + I_4)$$

where I_1, I_2, I_3 and I_4 given in the (2.1), (2.2), (2.3) and (2.4), respectively.

Proof. Integrating by parts, we get

$$I_1 = \int_0^{\frac{1}{2}} t^\lambda \mathcal{F}^\sigma_{\rho,\lambda+1}[w(b-a)^\rho t^\rho] f'(tb+(1-t)a)dt \tag{2.1}$$
$$= t^\lambda \mathcal{F}^\sigma_{\rho,\lambda+1}[w(b-a)^\rho t^\rho] \frac{f(tb+(1-t)a)}{b-a}\Big|_0^{\frac{1}{2}}$$
$$- \int_0^{\frac{1}{2}} t^{\lambda-1} \mathcal{F}^\sigma_{\rho,\lambda}[w(b-a)^\rho t^\rho] \frac{f(tb+(1-t)a)}{b-a}dt$$
$$= \frac{1}{b-a}\left(\frac{1}{2}\right)^\lambda \mathcal{F}^\sigma_{\rho,\lambda+1}\left[w\left(\frac{b-a}{2}\right)^\rho\right] f\left(\frac{a+b}{2}\right)$$
$$- \frac{1}{b-a}\int_0^{\frac{1}{2}} t^{\lambda-1} \mathcal{F}^\sigma_{\rho,\lambda}[w(b-a)^\rho t^\rho] f(tb+(1-t)a)dt.$$

Analogously:

$$I_2 = -\int_0^{\frac{1}{2}} t^\lambda \mathcal{F}^\sigma_{\rho,\lambda+1}[w(b-a)^\rho t^\rho] f'(ta+(1-t)b)dt \tag{2.2}$$
$$= \frac{1}{b-a}\left(\frac{1}{2}\right)^\lambda \mathcal{F}^\sigma_{\rho,\lambda+1} w\left[\left(\frac{b-a}{2}\right)^\rho\right] f\left(\frac{a+b}{2}\right)$$
$$- \frac{1}{b-a}\int_0^{\frac{1}{2}} t^{\lambda-1} \mathcal{F}^\sigma_{\rho,\lambda}[w(b-a)^\rho t^\rho] f(ta+(1-t)b)dt$$

and

$$I_3 = \int_{\frac{1}{2}}^1 \left[t^\lambda \mathcal{F}^\sigma_{\rho,\lambda+1}[w(b-a)^\rho t^\rho] - \mathcal{F}^\sigma_{\rho,\lambda+1}[w(b-a)^\rho]\right] f'(tb+(1-t)a)dt \tag{2.3}$$
$$= t^\lambda \mathcal{F}^\sigma_{\rho,\lambda+1}[w(b-a)^\rho t^\rho] \frac{f(tb+(1-t)a)}{b-a}\Big|_{\frac{1}{2}}^1$$
$$- \int_{\frac{1}{2}}^1 t^{\lambda-1} \mathcal{F}^\sigma_{\rho,\lambda}[w(b-a)^\rho t^\rho] \frac{f(tb+(1-t)a)}{b-a}dt$$
$$- \mathcal{F}^\sigma_{\rho,\lambda+1}[w(b-a)^\rho] \frac{f(tb+(1-t)a)}{b-a}\Big|_{\frac{1}{2}}^1$$
$$= \frac{1}{b-a}\mathcal{F}^\sigma_{\rho,\lambda+1}[w(b-a)^\rho] f\left(\frac{a+b}{2}\right)$$
$$- \frac{1}{b-a}\left(\frac{1}{2}\right)^\lambda \mathcal{F}^\sigma_{\rho,\lambda+1}\left[w\left(\frac{b-a}{2}\right)^\rho\right] f\left(\frac{a+b}{2}\right)$$
$$- \frac{1}{b-a}\int_{\frac{1}{2}}^1 t^{\lambda-1} \mathcal{F}^\sigma_{\rho,\lambda}[w(b-a)^\rho t^\rho] f(tb+(1-t)a)dt.$$

Analogously:

$$I_4 = \int_{\frac{1}{2}}^{1} \left[\mathcal{F}^{\sigma}_{\rho,\lambda+1}[w(b-a)^\rho] - t^\lambda \mathcal{F}^{\sigma}_{\rho,\lambda+1}[w(b-a)^\rho t^\rho]\right] f'(ta+(1-t)b)dt \tag{2.4}$$

$$= \frac{1}{b-a}\mathcal{F}^{\sigma}_{\rho,\lambda+1}[w(b-a)^\rho] f\left(\frac{a+b}{2}\right)$$

$$-\frac{1}{b-a}\left(\frac{1}{2}\right)^\lambda \mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[w\left(\frac{b-a}{2}\right)^\rho\right] f\left(\frac{a+b}{2}\right)$$

$$-\frac{1}{b-a}\int_{\frac{1}{2}}^{1} t^{\lambda-1}\mathcal{F}^{\sigma}_{\rho,\lambda}[w(b-a)^\rho t^\rho] f(ta+(1-t)b)dt.$$

Adding the resulting equalities, we obtain

$$\begin{aligned} I_1+I_2+I_3+I_4 &= \frac{2}{b-a}\mathcal{F}^{\sigma}_{\rho,\lambda+1}[w(b-a)^\rho] f\left(\frac{a+b}{2}\right) \\ &\quad -\frac{1}{b-a}\int_0^1 t^{\lambda-1}\mathcal{F}^{\sigma}_{\rho,\lambda}[w(b-a)^\rho t^\rho] f(ta+(1-t)b)dt \\ &\quad -\frac{1}{b-a}\int_0^1 t^{\lambda-1}\mathcal{F}^{\sigma}_{\rho,\lambda}[w(b-a)^\rho t^\rho] f(tb+(1-t)a)dt \\ &= \frac{2}{b-a}\mathcal{F}^{\sigma}_{\rho,\lambda+1}[w(b-a)^\rho] f\left(\frac{a+b}{2}\right) \\ &\quad -\frac{1}{(b-a)^{\lambda+1}}\left[\left(\mathcal{J}^{\sigma}_{\rho,\lambda,a^+;w} f\right)(b) + \left(\mathcal{J}^{\sigma}_{\rho,\lambda,b^-;w} f\right)(a)\right]. \end{aligned} \tag{2.5}$$

According to (1.3) and (1.4), changing variables with $x = tb + (1-t)a$, we get

$$\int_0^1 t^{\lambda-1}\mathcal{F}^{\sigma}_{\rho,\lambda}[w(b-a)^\rho t^\rho] f(tb+(1-t)a)dt = \frac{1}{(b-a)^\lambda}\left(\mathcal{J}^{\sigma}_{\rho,\lambda,a^+;w} f\right)(b)$$

and changing variables with $x = ta + (1-t)b$, we have

$$\int_0^1 t^{\lambda-1}\mathcal{F}^{\sigma}_{\rho,\lambda}[w(b-a)^\rho t^\rho] f(ta+(1-t)b)dt = \frac{1}{(b-a)^\lambda}\left(\mathcal{J}^{\sigma}_{\rho,\lambda,b^-;w} f\right)(a).$$

Thus multiplying both sides of (2.5) by $\frac{(b-a)}{2}$, we get desired result. □

Remark 2.2. Taking $\lambda = \alpha$, $\sigma(0) = 1$ and $w = 0$, then the above equality reduces to equality in Lemma 1, [5].

By using the above generalized new lemma, we obtain some new Hermite-Hadamard type inequalities via generalized fractional integral operators.

Theorem 2.3. *Let $f : [a,b] \to \mathbb{R}$ be a differentiable mapping on (a,b) with $a < b$. If $|f'|$ is convex on $[a,b]$, then the following inequality for generalized fractional integral operators holds:*

$$|M_f(a,b;w;J)| \le \frac{(b-a)}{2}\mathcal{F}^{\sigma_1}_{\rho,\lambda+1}[|w|(b-a)^\rho]\left[|f'(a)| + |f'(b)|\right]$$

where $\rho,\lambda>0,\ w\in\mathbb{R}$ *and* $\sigma_1(k)=\sigma(k)\left(\frac{1}{2}+\frac{\left(\frac{1}{2}\right)^{\lambda+\rho k}-1}{\lambda+\rho k+1}\right)$.

Proof. Using Lemma 2 and the convexity of $|f'|$, we have

$$\begin{aligned}
|M_f(a,b;w;J)| &\le \frac{b-a}{2}\{|I_1|+|I_2|+|I_3|+|I_4|\}\\
&= \frac{b-a}{2}\left\{\left|\int_0^{\frac{1}{2}} t^{\lambda}\mathcal{F}^{\sigma}_{\rho,\lambda+1}[w(b-a)^{\rho}t^{\rho}]f'(tb+(1-t)a)dt\right|\right.\\
&\quad+\left|\int_0^{\frac{1}{2}}\left(-t^{\lambda}\right)\mathcal{F}^{\sigma}_{\rho,\lambda+1}[w(b-a)^{\rho}t^{\rho}]f'(ta+(1-t)b)dt\right|\\
&\quad+\left|\int_{\frac{1}{2}}^{1}\left[t^{\lambda}\mathcal{F}^{\sigma}_{\rho,\lambda+1}[w(b-a)^{\rho}t^{\rho}]-\mathcal{F}^{\sigma}_{\rho,\lambda+1}[w(b-a)^{\rho}]\right]f'(tb+(1-t)a)dt\right|\\
&\quad\left.+\left|\int_{\frac{1}{2}}^{1}\left[\mathcal{F}^{\sigma}_{\rho,\lambda+1}[w(b-a)^{\rho}]-t^{\lambda}\mathcal{F}^{\sigma}_{\rho,\lambda+1}[w(b-a)^{\rho}t^{\rho}]\right]f'(ta+(1-t)b)dt\right|\right\}\\
&\le \frac{b-a}{2}\left\{\int_0^{\frac{1}{2}} t^{\lambda}\left|\mathcal{F}^{\sigma}_{\rho,\lambda+1}[w(b-a)^{\rho}t^{\rho}]\right||f'(tb+(1-t)a)|dt\right.\\
&\quad+\int_0^{\frac{1}{2}} t^{\lambda}\left|\mathcal{F}^{\sigma}_{\rho,\lambda+1}[w(b-a)^{\rho}t^{\rho}]\right||f'(ta+(1-t)b)|dt\\
&\quad+\int_{\frac{1}{2}}^{1}\left|t^{\lambda}\mathcal{F}^{\sigma}_{\rho,\lambda+1}[w(b-a)^{\rho}t^{\rho}]-\mathcal{F}^{\sigma}_{\rho,\lambda+1}[w(b-a)^{\rho}]\right||f'(tb+(1-t)a)|dt\\
&\quad\left.+\int_{\frac{1}{2}}^{1}\left|\mathcal{F}^{\sigma}_{\rho,\lambda+1}[w(b-a)^{\rho}]-t^{\lambda}\mathcal{F}^{\sigma}_{\rho,\lambda+1}[w(b-a)^{\rho}t^{\rho}]\right||f'(ta+(1-t)b)|dt\right\}\\
&\le \frac{b-a}{2}\sum_{k=0}^{\infty}\frac{\sigma(k)|w|^k(b-a)^{\rho k}}{\Gamma(\rho k+\lambda+1)}\times\left\{\int_0^{\frac{1}{2}} t^{\lambda+\rho k}\left[t|f'(b)|+(1-t)|f'(a)|\right]dt\right.\\
&\quad+\int_0^{\frac{1}{2}} t^{\lambda+\rho k}\left[t|f'(a)|+(1-t)|f'(b)|\right]dt\\
&\quad+\int_{\frac{1}{2}}^{1}\left[1-t^{\lambda+\rho k}\right]\left[t|f'(b)|+(1-t)|f'(a)|\right]dt\\
&\quad\left.+\int_{\frac{1}{2}}^{1}\left[1-t^{\lambda+\rho k}\right]\left[t|f'(a)|+(1-t)|f'(b)|\right]dt\right\}\\
&= \frac{b-a}{2}\sum_{k=0}^{\infty}\frac{\sigma(k)|w|^k(b-a)^{\rho k}}{\Gamma(\rho k+\lambda+1)}\\
&\quad\times\left\{|f'(a)|\left[\int_0^{\frac{1}{2}} t^{\lambda+\rho k}(1-t)dt+\int_0^{\frac{1}{2}} t^{\lambda+\rho k+1}dt\right.\right.
\end{aligned}$$

$$+\int_{\frac{1}{2}}^{1}\left[1-t^{\lambda+\rho k}\right](1-t)\,dt+\int_{\frac{1}{2}}^{1}\left[1-t^{\lambda+\rho k}\right]tdt\Bigg]$$

$$+\left|f'(b)\right|\left[\int_{0}^{\frac{1}{2}}t^{\lambda+\rho k+1}dt+\int_{0}^{\frac{1}{2}}t^{\lambda+\rho k}(1-t)\,dt\right.$$

$$\left.\left.+\int_{\frac{1}{2}}^{1}\left[1-t^{\lambda+\rho k}\right]tdt+\int_{\frac{1}{2}}^{1}\left[1-t^{\lambda+\rho k}\right](1-t)\,dt\right]\right\}$$

$$=\left(\frac{b-a}{2}\right)\mathcal{F}_{\rho,\lambda+1}^{\sigma}[w(b-a)^{\rho}]\left(\frac{1}{2}+\frac{\left(\frac{1}{2}\right)^{\lambda+\rho k}-1}{\lambda+\rho k+1}\right)\left[\left|f'(a)\right|+\left|f'(b)\right|\right]$$

where we used the facts that

$$\begin{aligned}
\int_{0}^{\frac{1}{2}}t^{\lambda+\rho k}(1-t)\,dt &= \frac{\left(\frac{1}{2}\right)^{\lambda+\rho k+1}}{\lambda+\rho k+1}-\frac{\left(\frac{1}{2}\right)^{\lambda+\rho k+2}}{\lambda+\rho k+2},\\
\int_{0}^{\frac{1}{2}}t^{\lambda+\rho k+1}dt &= \frac{\left(\frac{1}{2}\right)^{\lambda+\rho k+2}}{\lambda+\rho k+2},\\
\int_{\frac{1}{2}}^{1}\left[1-t^{\lambda+\rho k}\right](1-t)\,dt &= \frac{1}{8}+\frac{\left(\frac{1}{2}\right)^{\lambda+\rho k+1}-1}{\lambda+\rho k+1}+\frac{1-\left(\frac{1}{2}\right)^{\lambda+\rho k+2}}{\lambda+\rho k+2},\\
\int_{\frac{1}{2}}^{1}\left[1-t^{\lambda+\rho k}\right]tdt &= \frac{3}{8}+\frac{\left(\frac{1}{2}\right)^{\lambda+\rho k+2}-1}{\lambda+\rho k+2}.
\end{aligned}$$

The proof is completed. □

Corollary 2.4. *If we choose $\lambda=\alpha,\sigma(0)=1$ and $w=0$ in Theorem 2.1, we have*

$$\begin{aligned}
&\left|f\left(\frac{a+b}{2}\right)-\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[\mathcal{J}_{a^{+}}^{\alpha}f(b)+\mathcal{J}_{b^{-}}^{\alpha}f(a)\right]\right|\\
\leq\ &\frac{b-a}{4}\left(\frac{\alpha+2^{1-\alpha}-1}{\alpha+1}\right)\left[\left|f'(a)\right|+\left|f'(b)\right|\right].
\end{aligned}$$

Remark 2.5. The above inequality is better than one that was given in Theorem 2 of [5].

Remark 2.6. If we choose $\alpha=1$ in Corollary 1, we get the inequality in Theorem 2.2 in [6].

Theorem 2.7. *Let $f:[a,b]\to\mathbb{R}$ be a differentiable mapping on (a,b) with $a<b$. If $|f'|^{q}$ is convex on $[a,b]$ for some fixed $q>1$, then the following inequality for generalized fractional integral operators holds:*

$$\begin{aligned}
\left|M_{f}(a,b;w;J)\right| &\leq \frac{(b-a)\,\mathcal{F}_{\rho,\lambda+1}^{\sigma_{2}}[|w|\,(b-a)^{\rho}]}{2}\\
&\times\left\{\left(\frac{3\left|f'(a)\right|^{q}+\left|f'(b)\right|^{q}}{4}\right)^{\frac{1}{q}}+\left(\frac{\left|f'(a)\right|^{q}+3\left|f'(b)\right|^{q}}{4}\right)^{\frac{1}{q}}\right\}
\end{aligned}$$

where $\rho, \lambda > 0$, $w \in \mathbb{R}$,

$$\phi = \int_{\frac{1}{2}}^{1} \left(1 - t^{\lambda+\rho k}\right)^{p} dt$$

and

$$\sigma_2(k) = \sigma(k) \left[\left(\frac{\left(\frac{1}{2}\right)^{(\lambda+\rho k)p+1}}{(\lambda+\rho k)p+1} \right)^{\frac{1}{p}} + \phi^{\frac{1}{p}} \right].$$

Proof. By using Lemma 2 and properties of modulus, we have

$$|M_f(a,b;w;J)| \leq \frac{b-a}{2} \left[|I_1| + |I_2| + |I_3| + |I_4| \right]. \tag{2.6}$$

Then by using Hölder integral inequality and convexity of $|f'|^q$, we have

$$\begin{aligned} |I_1| &\leq \sum_{k=0}^{\infty} \frac{\sigma(k)|w|^k (b-a)^{\rho k}}{\Gamma(\rho k+\lambda+1)} \qquad (2.7)\\ &\quad \times \left(\int_0^{\frac{1}{2}} \left(t^{\lambda+\rho k}\right)^p dt \right)^{\frac{1}{p}} \left(\int_0^{\frac{1}{2}} \left[t|f'(b)|^q + (1-t)|f'(a)|^q \right] dt \right)^{\frac{1}{q}} \\ &= \sum_{k=0}^{\infty} \frac{\sigma(k)|w|^k (b-a)^{\rho k}}{\Gamma(\rho k+\lambda+1)} \left(\frac{\left(\frac{1}{2}\right)^{(\lambda+\rho k)p+1}}{(\lambda+\rho k)p+1} \right)^{\frac{1}{p}} \left(\frac{3|f'(a)|^q + |f'(b)|^q}{4} \right)^{\frac{1}{q}}, \end{aligned}$$

$$\begin{aligned} |I_2| &\leq \sum_{k=0}^{\infty} \frac{\sigma(k)|w|^k (b-a)^{\rho k}}{\Gamma(\rho k+\lambda+1)} \qquad (2.8)\\ &\quad \times \left(\int_0^{\frac{1}{2}} \left(t^{\lambda+\rho k}\right)^p dt \right)^{\frac{1}{p}} \left(\int_0^{\frac{1}{2}} \left[t|f'(a)|^q + (1-t)|f'(b)|^q \right] dt \right)^{\frac{1}{q}} \\ &= \sum_{k=0}^{\infty} \frac{\sigma(k)|w|^k (b-a)^{\rho k}}{\Gamma(\rho k+\lambda+1)} \left(\frac{\left(\frac{1}{2}\right)^{(\lambda+\rho k)p+1}}{(\lambda+\rho k)p+1} \right)^{\frac{1}{p}} \left(\frac{|f'(a)|^q + 3|f'(b)|^q}{4} \right)^{\frac{1}{q}}, \end{aligned}$$

$$\begin{aligned} |I_3| &\leq \sum_{k=0}^{\infty} \frac{\sigma(k)|w|^k (b-a)^{\rho k}}{\Gamma(\rho k+\lambda+1)} \qquad (2.9)\\ &\quad \times \left(\int_{\frac{1}{2}}^{1} \left(1-t^{\lambda+\rho k}\right)^p dt \right)^{\frac{1}{p}} \left(\int_{\frac{1}{2}}^{1} \left[t|f'(b)|^q + (1-t)|f'(a)|^q \right] dt \right)^{\frac{1}{q}} \\ &= \sum_{k=0}^{\infty} \frac{\sigma(k)|w|^k (b-a)^{\rho k}}{\Gamma(\rho k+\lambda+1)} \phi^{\frac{1}{p}} \left(\frac{|f'(a)|^q + 3|f'(b)|^q}{4} \right)^{\frac{1}{q}} \end{aligned}$$

and

$$
\begin{aligned}
|I_4| &\leq \sum_{k=0}^{\infty} \frac{\sigma(k)|w|^k (b-a)^{\rho k}}{\Gamma(\rho k+\lambda+1)} \qquad (2.10)\\
&\times \left(\int_{\frac{1}{2}}^{1} \left(1-t^{\lambda+\rho k}\right)^p dt\right)^{\frac{1}{p}} \left(\int_{\frac{1}{2}}^{1} \left[t\,|f'(a)|^q+(1-t)\,|f'(b)|^q\right] dt\right)^{\frac{1}{q}}\\
&= \sum_{k=0}^{\infty} \frac{\sigma(k)|w|^k (b-a)^{\rho k}}{\Gamma(\rho k+\lambda+1)} \phi^{\frac{1}{p}} \left(\frac{|f'(b)|^q+3\,|f'(a)|^q}{4}\right)^{\frac{1}{q}}
\end{aligned}
$$

where $\phi = \int_{\frac{1}{2}}^{1} \left(1-t^{\lambda+\rho k}\right)^p dt$.

If we use the inequalities (2.7), (2.8), (2.9) and (2.10) in the inequality (2.6), we get the desired result. So, the proof is completed. □

Corollary 2.8. *If we choose $\lambda=\alpha, \sigma(0)=1$ and $w=0$ in Theorem 2.2, we have*

$$
\begin{aligned}
&\left| f\left(\frac{a+b}{2}\right) - \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}} \left[\mathcal{J}_{a+}^{\alpha} f(b) + \mathcal{J}_{b-}^{\alpha} f(a)\right] \right|\\
&\leq \frac{b-a}{2} \left\{ \left(\frac{\left(\frac{1}{2}\right)^{\alpha p+1}}{\alpha p+1} \right)^{\frac{1}{p}} + \Omega^{\frac{1}{p}} \right\}\\
&\times \left\{ \left(\frac{3\,|f'(a)|^q + |f'(b)|^q}{4} \right)^{\frac{1}{q}} + \left(\frac{|f'(a)|^q + 3\,|f'(b)|^q}{4} \right)^{\frac{1}{q}} \right\}\\
&\leq \frac{b-a}{2} \left\{ \left(\frac{\left(\frac{1}{2}\right)^{\alpha p+1}}{\alpha p+1} \right)^{\frac{1}{p}} + \Omega^{\frac{1}{p}} \right\} \left(\frac{3^{\frac{1}{q}}+1}{4^{\frac{1}{q}}} \right) \left[|f'(a)| + |f'(b)| \right]
\end{aligned}
$$

where we used the fact that

$$
\sum_{i=1}^{n} (a_i+b_i)^r \leq \sum_{i=1}^{n} a_i^r + \sum_{i=1}^{n} b_i^r \qquad (2.11)
$$

for $0 \leq r < 1$, $a_1, a_2, a_{3,}..., a_n \geq 0$ *and* $b_1, b_2, b_3, ..., b_n \geq 0$. *Also,*

$$
\Omega = \int_{\frac{1}{2}}^{1} (1-t^{\alpha})^p \, dt.
$$

The following result is obtained by using the well-known power-mean integral inequality.

Theorem 2.9. *Let $f : [a,b] \to \mathbb{R}$ be a differentiable mapping on (a,b) with $a < b$. If $|f'|^{\frac{p}{p-1}}$ is convex on $[a,b]$ for some fixed $p > 1$ with $q = \frac{p}{p-1}$, then the following* *inequality for generalized fractional integral operators holds:*

$$\left|M_f(a,b;w;J)\right| \le \frac{b-a}{2}\mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[|w|(b-a)^{\rho}\right]\left(\left|f'(a)\right|+\left|f'(b)\right|\right) \tag{2.12}$$
$$\times\left\{\left(\frac{\left(\frac{1}{2}\right)^{\lambda+\rho k+1}}{\lambda+\rho k+1}\right)^{1-\frac{1}{q}}\mu_1+\left(\frac{1}{2}+\frac{\left(\frac{1}{2}\right)^{\lambda+\rho k+1}-1}{\lambda+\rho k+1}\right)^{1-\frac{1}{q}}\mu_2\right\}$$

$\rho,\lambda>0$, $w\in\mathbb{R}$ *and where*

$$\mu_1=\left(\frac{\left(\frac{1}{2}\right)^{\lambda+\rho k+2}}{\lambda+\rho k+2}\right)^{\frac{1}{q}}+\left(\frac{\left(\frac{1}{2}\right)^{\lambda+\rho k+1}}{\lambda+\rho k+1}-\frac{\left(\frac{1}{2}\right)^{\lambda+\rho k+2}}{\lambda+\rho k+2}\right)^{\frac{1}{q}}$$

and

$$\mu_2=\left(\frac{3}{8}+\frac{1-\left(\frac{1}{2}\right)^{\lambda+\rho k+2}}{\lambda+\rho k+2}\right)^{\frac{1}{q}}+\left(\frac{1}{8}+\frac{\left(\frac{1}{2}\right)^{\lambda+\rho k+1}-1}{\lambda+\rho k+1}+\frac{1-\left(\frac{1}{2}\right)^{\lambda+\rho k+2}}{\lambda+\rho k+2}\right)^{\frac{1}{q}}.$$

Proof. By using Lemma 2 and properties of modulus, we have

$$\left|M_f(a,b;w;J)\right| \le \frac{b-a}{2}\left\{|I_1|+|I_2|+|I_3|+|I_4|\right\}$$

Then by using the power mean-integral inequality for $p>1$, we have

$$|I_1| \le \sum_{k=0}^{\infty}\frac{\sigma(k)|w|^k(b-a)^{\rho k}}{\Gamma(\rho k+\lambda+1)} \tag{2.13}$$
$$\times\left(\int_0^{\frac{1}{2}}t^{\lambda+\rho k}dt\right)^{1-\frac{1}{q}}\left(\int_0^{\frac{1}{2}}t^{\lambda+\rho k}\left|f'(tb+(1-t)a)\right|^q dt\right)^{\frac{1}{q}}$$

and by using convexity of $|f'|^{\frac{p}{p-1}}$ in (2.13), we have

$$\int_0^{\frac{1}{2}}t^{\lambda+\rho k}\left|f'(tb+(1-t)a)\right|^q dt=\frac{\left(\frac{1}{2}\right)^{\lambda+\rho k+2}}{\lambda+\rho k+2}\left|f'(b)\right|^q$$
$$+\left(\frac{\left(\frac{1}{2}\right)^{\lambda+\rho k+1}}{\lambda+\rho k+1}-\frac{\left(\frac{1}{2}\right)^{\lambda+\rho k+2}}{\lambda+\rho k+2}\right)\left|f'(a)\right|^q.$$

If we use last equality in inequality of (2.13), then we get the following inequality as

$$|I_1| \le \sum_{k=0}^{\infty}\frac{\sigma(k)|w|^k(b-a)^{\rho k}}{\Gamma(\rho k+\lambda+1)}$$

$$\times\left(\frac{\left(\frac{1}{2}\right)^{\lambda+\rho k+1}}{\lambda+\rho k+1}\right)^{1-\frac{1}{q}}\left\{\frac{\left(\frac{1}{2}\right)^{\lambda+\rho k+2}}{\lambda+\rho k+2}\left|f'(b)\right|^q+\left(\frac{\left(\frac{1}{2}\right)^{\lambda+\rho k+1}}{\lambda+\rho k+1}-\frac{\left(\frac{1}{2}\right)^{\lambda+\rho k+2}}{\lambda+\rho k+2}\right)\left|f'(a)\right|^q\right\}.$$

As similar to computation of $|I_1|$, we can get $|I_2|$, $|I_3|$ and $|I_4|$ as following:

$$|I_2| \leq \sum_{k=0}^{\infty} \frac{\sigma(k)|w|^k (b-a)^{\rho k}}{\Gamma(\rho k+\lambda+1)}$$

$$\times \left(\frac{\left(\frac{1}{2}\right)^{\lambda+\rho k+1}}{\lambda+\rho k+1} \right)^{1-\frac{1}{q}} \left\{ \frac{\left(\frac{1}{2}\right)^{\lambda+\rho k+2}}{\lambda+\rho k+2} |f'(a)|^q + \left(\frac{\left(\frac{1}{2}\right)^{\lambda+\rho k+1}}{\lambda+\rho k+1} - \frac{\left(\frac{1}{2}\right)^{\lambda+\rho k+2}}{\lambda+\rho k+2} \right) |f'(b)|^q \right\}^{\frac{1}{q}},$$

$$|I_3| \leq \sum_{k=0}^{\infty} \frac{\sigma(k)|w|^k (b-a)^{\rho k}}{\Gamma(\rho k+\lambda+1)} \left(\frac{1}{2} + \frac{\left(\frac{1}{2}\right)^{\lambda+\rho k+1}-1}{\lambda+\rho k+1} \right)^{1-\frac{1}{q}}$$

$$\times \left\{ \left(\frac{3}{8} + \frac{1-\left(\frac{1}{2}\right)^{\lambda+\rho k+2}}{\lambda+\rho k+2} \right) |f'(b)|^q \right.$$

$$\left. + \left(\frac{1}{8} + \frac{\left(\frac{1}{2}\right)^{\lambda+\rho k+1}-1}{\lambda+\rho k+1} + \frac{1-\left(\frac{1}{2}\right)^{\lambda+\rho k+2}}{\lambda+\rho k+2} \right) |f'(a)|^q \right\}^{\frac{1}{q}}$$

and

$$|I_4| \leq \sum_{k=0}^{\infty} \frac{\sigma(k)|w|^k (b-a)^{\rho k}}{\Gamma(\rho k+\lambda+1)} \left(\frac{1}{2} + \frac{\left(\frac{1}{2}\right)^{\lambda+\rho k+1}-1}{\lambda+\rho k+1} \right)^{1-\frac{1}{q}}$$

$$\times \left\{ \left(\frac{3}{8} + \frac{1-\left(\frac{1}{2}\right)^{\lambda+\rho k+2}}{\lambda+\rho k+2} \right) |f'(a)|^q \right.$$

$$\left. + \left(\frac{1}{8} + \frac{\left(\frac{1}{2}\right)^{\lambda+\rho k+1}-1}{\lambda+\rho k+1} + \frac{1-\left(\frac{1}{2}\right)^{\lambda+\rho k+2}}{\lambda+\rho k+2} \right) |f'(b)|^q \right\}^{\frac{1}{q}}.$$

Then by using the fact (2.11) in the inequalities of $|I_1|$, $|I_2|$, $|I_3|$ and $|I_4|$ and by using necessary arrangement we get the desired result in (2.12). □

Corollary 2.10. *If we choose $\lambda = \alpha, \sigma(0) = 1$ and $w = 0$ in Theorem 2.3, we have*

$$\left| f\left(\frac{a+b}{2}\right) - \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha} \left[\mathcal{J}_{a+}^{\alpha} f(b) + \mathcal{J}_{b-}^{\alpha} f(a) \right] \right|$$

$$\leq \frac{b-a}{2} \left\{ \left(\frac{\left(\frac{1}{2}\right)^{\alpha+1}}{\alpha+1} \right)^{1-\frac{1}{q}} \eta_1 + \left(\frac{1}{2} + \frac{\left(\frac{1}{2}\right)^{\alpha+1}-1}{\alpha+1} \right)^{1-\frac{1}{q}} \eta_2 \right\} [|f'(a)| + |f'(b)|]$$

where

$$\eta_1 = \left(\frac{\left(\frac{1}{2}\right)^{\alpha+2}}{\alpha+2} \right)^{\frac{1}{q}} + \left(\frac{\left(\frac{1}{2}\right)^{\alpha+1}}{\alpha+1} - \frac{\left(\frac{1}{2}\right)^{\alpha+2}}{\alpha+2} \right)^{\frac{1}{q}}$$

and

$$\eta_2 = \left(\frac{3}{8} + \frac{1-\left(\frac{1}{2}\right)^{\alpha+2}}{\alpha+2} \right)^{\frac{1}{q}} + \left(\frac{1}{8} + \frac{\left(\frac{1}{2}\right)^{\alpha+1}-1}{\alpha+1} + \frac{1-\left(\frac{1}{2}\right)^{\alpha+2}}{\alpha+2} \right)^{\frac{1}{q}}.$$

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On subclasses of bi-convex functions defined by Tremblay fractional derivative operator

Sevtap Sümer Eker and Bilal Şeker

Abstract. We introduce and investigate new subclasses of analytic and bi-univalent functions defined by modified Tremblay operator in the open unit disk. Also we obtain upper bounds for the coefficients of functions belonging to these classes.

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1. Introduction

Let $\mathcal{A}$ denote the class of functions $f(z)$ which are *analytic* in the open unit disk $\mathbb{U} = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$ and normalized by the conditions $f(0) = f'(0) - 1 = 0$ and having the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1.1}$$

Also let $\mathcal{S}$ denote the subclass of functions in $\mathcal{A}$ which are univalent in $\mathbb{U}$ (for details, see [7]).

The Koebe One Quarter Theorem (e.g., see [7]) ensures that the image of $\mathbb{U}$ under every univalent function $f(z) \in \mathcal{A}$ contains the disk of radius $1/4$. Thus every univalent function f has an inverse f^{-1} satisfying

$$f^{-1}(f(z)) = z \quad (z \in \mathbb{U})$$

and

$$f\left(f^{-1}(w)\right) = w \quad \left(|w| < r_0(f),\ r_0(f) \geq \frac{1}{4}\right).$$

In fact, the inverse function f^{-1} is given by

$$\begin{aligned} g(w) = f^{-1}(w) &= w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2 a_3 + a_4)w^4 + \cdots \\ &= w + \sum_{n=2}^{\infty} b_n w^n. \end{aligned} \tag{1.2}$$

A function $f \in \mathcal{A}$ is said to be *bi-univalent* in $\mathbb{U}$ if both $f(z)$ and $f^{-1}(z)$ are univalent in $\mathbb{U}$. We denote by Σ the class of all bi-univalent functions in $\mathbb{U}$ given by the Taylor-Maclaurin series expansion (1.1).

For a brief history and examples of functions in the class Σ, see [19] (see also [5], [6], [11], [25]).

Coefficient bounds for various subclasses of bi-univalent functions were obtained by several authors including Ali *et al.* [2], Caglar et al. [3], Deniz [4], Kumar *et al.* [10], Magesh and Yamini [12], Srivastava *et al.* [17], [18], [22], Sümer Eker [1], [23], [24]. In fact, judging by the remarkable flood of papers on the subject, the pioneering work of Srivastava *et al.* [19] appears to have revived the study of analytic and bi-univalent functions in recent years.

The following definition of fractional derivative will be required in our investigation (see, for details, [13], [14], [20], [21]).

Definition 1.1. The fractional integral of order δ is defined, for a function f, by

$$D_z^{-\delta} f(z) = \frac{1}{\Gamma(\delta)} \int_0^z \frac{f(\xi)}{(z-\xi)^{1-\delta}} d\xi \,; \ (\delta > 0),$$

where f is an analytic function in a simply-connected region of complex z-plane containing the origin, and the multiplicity of $(z-\xi)^{\delta-1}$ is removed by requiring, $log(z-\xi)$ to be real when $z - \xi > 0$.

Definition 1.2. The fractional derivative of order δ is defined, for a function f , by

$$D_z^{\delta} f(z) = \frac{1}{\Gamma(1-\delta)} \frac{d}{dz} \int_0^z \frac{f(\xi)}{(z-\xi)^{\delta}} d\xi \ (0 \le \delta < 1),$$

where f is constrained, and the multiplicity of $(z-\xi)^{-\delta}$ is removed, as in Definition 1.1.

Definition 1.3. Under the hypotheses of Definition 2, the fractional derivative of order $(n+\delta)$ is defined by

$$D_z^{n+\delta} f(z) = \frac{d^n}{dz^n} D_z^{\delta} f(z) \quad (0 \le \delta < 1, n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\})$$

By virtue of Definitions 1.1, 1.2 and 1.3, we have

$$D_z^{-\delta} z^n = \frac{\Gamma(n+1)}{\Gamma(n+\delta+1)} z^{n+\delta} \qquad (n \in \mathbb{N}, \delta > 0)$$

and

$$D_z^{\delta} z^n = \frac{\Gamma(n+1)}{\Gamma(n-\delta+1)} z^{n-\delta} \qquad (n \in \mathbb{N}, 0 \le \delta < 1)$$

Tremblay [26] studied a fractional calculus operator defined in terms of the Riemann-Liouville fractional differential operator. Ibrahim and Jahangiri [9] extended and studied this operator in the complex plane.

Definition 1.4. The Tremblay fractional derivative operator $\mathrm{T}_z^{\mu,\gamma}$ of a function $f \in \mathcal{A}$ is defined, for all $z \in \mathbb{U}$, by

$$\mathrm{T}_z^{\mu,\gamma} f(z) = \frac{\Gamma(\gamma)}{\Gamma(\mu)} z^{1-\gamma} D_z^{\mu-\gamma} z^{\mu-1} f(z)$$
$$(0 < \mu \leq 1;\ 0 < \gamma \leq 1;\ \mu \geq \gamma;\ 0 \leq \mu - \gamma < 1).$$

It is clear that, for $\mu = \gamma = 1$, we have

$$\mathrm{T}_z^{1,1} f(z) = f(z).$$

Example 1.5. Let $f(z) = z^n$. The Tremblay Fractional Derivative of $f(z)$ is:

$$\mathrm{T}_z^{\mu,\gamma} f(z) = \frac{\Gamma(\gamma)}{\Gamma(\mu)} \frac{\Gamma(n+\mu)}{\Gamma(n+\gamma)} z^n,$$

and for $\mu = \gamma = 1$, we have $\mathrm{T}_z^{1,1}(z^n) = z^n$.

Recently in [8], Esa et al. defined modified of Tremblay operator of analytic functions in complex domain as follows:

Definition 1.6. Let $f(z) \in \mathcal{A}$. The modified Tremblay operator denoted by $\mathfrak{T}^{\mu,\gamma} : \mathcal{A} \to \mathcal{A}$ and defined such as:

$$\begin{aligned}
\mathfrak{T}^{\mu,\gamma} f(z) &= \frac{\gamma}{\mu} \mathrm{T}_z^{\mu,\gamma} f(z) \\
&= \frac{\Gamma(\gamma+1)}{\Gamma(\mu+1)} z^{1-\gamma} D_z^{\mu-\gamma} z^{\mu-1} f(z) \\
&= z + \sum_{n=2}^{\infty} \frac{\Gamma(\gamma+1)\Gamma(n+\mu)}{\Gamma(\mu+1)\Gamma(n+\gamma)} a_n z^n.
\end{aligned}$$

The object of the present paper is to introduce a new subclass of the function class Σ by using the modified Tremblay operator and find estimate on the coefficients $|a_2|$ and $|a_3|$ for functions in this class.

We begin by introducing the function class $\mathcal{C}_\Sigma^{\mu,\gamma}(\alpha)$ by means of the following definition.

2. Main results

Definition 2.1. A function $f(z)$ given by (1.1) is said to be in the class $\mathcal{C}_\Sigma^{\mu,\gamma}(\alpha)$ $(0 < \mu \leq 1;\ 0 < \gamma \leq 1;\ \mu \geq \gamma;\ 0 \leq \mu - \gamma < 1)$ if the following conditions are satisfied:

$$f \in \Sigma \text{ and } \left| arg\left(1 + \frac{z(\mathfrak{T}f)''(z)}{\mathfrak{T}f'(z)}\right)\right| < \frac{\alpha\pi}{2} \qquad (0 < \alpha \leq 1, z \in \mathbb{U}) \tag{2.1}$$

and

$$\left| arg\left(1 + \frac{w(\mathfrak{T}g)''(w)}{\mathfrak{T}g'(w)}\right)\right| < \frac{\alpha\pi}{2} \qquad (0 < \alpha \leq 1, w \in \mathbb{U}) \tag{2.2}$$

where the function $g(w)$ is given by (1.2).

We first state and prove the estimates on the coefficients $|a_2|$ and $|a_3|$ for functions in the class $\mathcal{C}_{\Sigma}^{\mu,\gamma}(\alpha)$.

Theorem 2.2. *If $f(z)$ given by (1.1) be in the class $\mathcal{C}_{\Sigma}^{\mu,\gamma}(\alpha)$, then*

$$\left|a_2\right| \leq \alpha(\gamma+1)\sqrt{\frac{(\gamma+2)}{(\mu+1)[3\alpha(\mu+2)(\gamma+1)+(1-3\alpha)(\mu+1)(\gamma+2)]}} \tag{2.3}$$

and

$$\left|a_3\right| \leq \frac{\alpha(\gamma+2)(\gamma+1)^2}{(\mu+1)(\mu\gamma-\mu+4\gamma+2)}. \tag{2.4}$$

Proof. For f given by (1.1), we can write from (2.1) and (2.2)

$$1+\frac{z(\mathfrak{T}f)''(z)}{\mathfrak{T}f'(z)} = [p(z)]^{\alpha} \tag{2.5}$$

$$1+\frac{w(\mathfrak{T}g)''(w)}{\mathfrak{T}g'(w)} = [q(w)]^{\alpha} \tag{2.6}$$

where $p(z)$ and $q(w)$ are in familiar Caratheódory Class $\mathcal{P}$ (see for details [7]) and have the following series representations:

$$p(z) = 1+p_1z+p_2z^2+p_3z^3+\cdots \tag{2.7}$$

and

$$q(w) = 1+q_1w+q_2w^2+q_3w^3+\cdots. \tag{2.8}$$

Now, equating the coefficients (2.5) and (2.6), we find that

$$2\frac{\mu+1}{\gamma+1}a_2 = \alpha p_1, \tag{2.9}$$

$$\frac{6(\mu+2)(\mu+1)}{(\gamma+2)(\gamma+1)}a_3 - 4\left(\frac{\mu+1}{\gamma+1}\right)^2 a_2^2 = \alpha p_2 + \frac{\alpha(\alpha-1)}{2}p_1^2, \tag{2.10}$$

$$-2\frac{\mu+1}{\gamma+1}a_2 = \alpha q_1 \tag{2.11}$$

and

$$\frac{6(\mu+2)(\mu+1)}{(\gamma+2)(\gamma+1)}(2a_2^2-a_3) - 4\left(\frac{\mu+1}{\gamma+1}\right)^2 a_2^2 = \alpha q_2 + \frac{\alpha(\alpha-1)}{2}q_1^2. \tag{2.12}$$

From (2.9) and (2.11), we get

$$p_1 = -q_1 \tag{2.13}$$

and

$$8\left(\frac{\mu+1}{\gamma+1}\right)^2 a_2^2 = \alpha^2(p_1^2+q_1^2). \tag{2.14}$$

Also from (2.10), (2.12) and 2.14, we get

$$a_2^2 = \frac{\alpha^2(p_2+q_2)(\gamma+2)(\gamma+1)^2}{4(\mu+1)\left[3\alpha(\mu+2)(\gamma+1)+(1-3\alpha)(\mu+1)(\gamma+2)\right]}. \tag{2.15}$$

According to the Caratheódory Lemma (see [7]), $|p_n| \le 2$ and $|q_n| \le 2$ for $n \in \mathbb{N}$. Now taking the absolute value of (2.15) and applying the Carathéodory Lemma for coefficients p_2 and q_2 we obtain

$$|a_2| \le \sqrt{\frac{\alpha^2(\gamma+2)(\gamma+1)^2}{(\mu+1)\left[3\alpha(\mu+2)(\gamma+1)+(1-3\alpha)(\mu+1)(\gamma+2)\right]}}.$$

This gives desired bound for $|a_2|$ as asserted in (2.3).

Now, in order to find the bound on $|a_3|$, from (2.12) and (2.10) and (2.13), we can write

$$\left\{\frac{72(\mu+2)^2(\mu+1)^2}{(\gamma+2)^2(\gamma+1)^2} - \frac{48(\mu+2)(\mu+1)^3}{(\gamma+2)(\gamma+1)^3}\right\} a_3 \tag{2.16}$$
$$= \alpha\left\{\left(\frac{12(\mu+2)(\mu+1)}{(\gamma+2)(\gamma+1)} - \frac{4(\mu+1)^2}{(\gamma+1)^2}\right)p_2 + \frac{4(\mu+1)^2}{(\gamma+1)^2}q_2\right\}$$
$$+\frac{6\alpha(\alpha-1)(\mu+2)(\mu+1)}{(\gamma+2)(\gamma+1)}p_1^2.$$

If $\alpha = 1$ then

$$|a_3| \le \frac{(\gamma+2)(\gamma+1)^2}{(\mu+1)(\mu\gamma-\mu+4\gamma+2)}.$$

Now, we consider the case $0 < \alpha < 1$. From (2.16), we can write

$$\left\{\frac{72(\mu+2)^2(\mu+1)^2}{(\gamma+2)^2(\gamma+1)^2} - \frac{48(\mu+2)(\mu+1)^3}{(\gamma+2)(\gamma+1)^3}\right\} Re(a_3) \tag{2.17}$$
$$= \alpha Re\left\{\left(\frac{12(\mu+2)(\mu+1)}{(\gamma+2)(\gamma+1)} - \frac{4(\mu+1)^2}{(\gamma+1)^2}\right)p_2 + \frac{4(\mu+1)^2}{(\gamma+1)^2}q_2\right\}$$
$$+Re\frac{6\alpha(\alpha-1)(\mu+2)(\mu+1)}{(\gamma+2)(\gamma+1)}p_1^2.$$

From Herglotz's Representation formula [15] for the functions $p(z)$ and $q(w)$, we have

$$p(z) = \int_0^{2\pi} \frac{1+ze^{-it}}{1-ze^{-it}}d\mu_1(t),$$

and

$$q(w) = \int_0^{2\pi} \frac{1+we^{-it}}{1-we^{-it}}d\mu_2(t),$$

where $\mu_i(t)$ are increasing on $[0, 2\pi]$ and $\mu_i(2\pi) - \mu_i(0) = 1$, $i = 1, 2$.

We also have

$$p_n = 2\int_0^{2\pi} e^{-int}d\mu_1(t), \qquad n = 1, 2, \dots$$
$$q_n = 2\int_0^{2\pi} e^{-int}d\mu_2(t), \qquad n = 1, 2, \dots.$$

Now (2.17) can be written as follows:

$$\left\{\frac{72(\mu+2)^2(\mu+1)^2}{(\gamma+2)^2(\gamma+1)^2} - \frac{48(\mu+2)(\mu+1)^3}{(\gamma+2)(\gamma+1)^3}\right\} Re(a_3)$$

$$= \alpha \left\{ \left(\frac{12(\mu+2)(\mu+1)}{(\gamma+2)(\gamma+1)} - \frac{4(\mu+1)^2}{(\gamma+1)^2} \right) 2 \int_0^{2\pi} \cos 2t d\mu_1(t) + \frac{8(\mu+1)^2}{(\gamma+1)^2} \int_0^{2\pi} \cos 2t d\mu_2(t) \right\}$$

$$- \frac{24\alpha(1-\alpha)(\mu+2)(\mu+1)}{(\gamma+2)(\gamma+1)} \left[\left(\int_0^{2\pi} \cos t d\mu_1 t \right)^2 - \left(\int_0^{2\pi} \sin t d\mu_1(t) \right)^2 \right]$$

$$\leq 2\alpha \left\{ \left(\frac{12(\mu+2)(\mu+1)}{(\gamma+2)(\gamma+1)} - \frac{4(\mu+1)^2}{(\gamma+1)^2} \right) \int_0^{2\pi} \cos 2t d\mu_1(t) + \frac{4(\mu+1)^2}{(\gamma+1)^2} \int_0^{2\pi} \cos 2t d\mu_2(t) \right\}$$

$$+ \frac{24\alpha(1-\alpha)(\mu+2)(\mu+1)}{(\gamma+2)(\gamma+1)} \left(\int_0^{2\pi} \sin t d\mu_1(t) \right)^2$$

$$= 2\alpha \left(\frac{12(\mu+2)(\mu+1)}{(\gamma+2)(\gamma+1)} - \frac{4(\mu+1)^2}{(\gamma+1)^2} \right) \int_0^{2\pi} (1 - 2\sin^2 t) d\mu_1(t)$$

$$+ \frac{8\alpha(\mu+1)^2}{(\gamma+1)^2} \int_0^{2\pi} (1 - 2\sin^2 t) d\mu_2(t) + \frac{24\alpha(1-\alpha)(\mu+2)(\mu+1)}{(\gamma+2)(\gamma+1)} \left(\int_0^{2\pi} \sin t d\mu_1(t) \right)^2.$$

By Jensen's inequality ([16]), we have

$$\left(\int_0^{2\pi} |\sin t| d\mu(t) \right)^2 \leq \int_0^{2\pi} \sin^2 t d\mu(t).$$

Hence

$$\left\{ \frac{72(\mu+2)^2(\mu+1)^2}{(\gamma+2)^2(\gamma+1)^2} - \frac{48(\mu+2)(\mu+1)^3}{(\gamma+2)(\gamma+1)^3} \right\} Re(a_3)$$

$$\leq 2\alpha \left(\frac{12(\mu+2)(\mu+1)}{(\gamma+2)(\gamma+1)} - \frac{4(\mu+1)^2}{(\gamma+1)^2} \right)$$

$$-4\alpha \left(\frac{12(\mu+2)(\mu+1)}{(\gamma+2)(\gamma+1)} - \frac{4(\mu+1)^2}{(\gamma+1)^2} \right) \int_0^{2\pi} \sin^2 t d\mu_1(t)$$

$$+ \frac{8\alpha(\mu+1)^2}{(\gamma+1)^2} - \frac{16\alpha(\mu+1)^2}{(\gamma+1)^2} \int_0^{2\pi} \sin^2 t d\mu_2(t)$$

$$+ \frac{24\alpha(1-\alpha)(\mu+2)(\mu+1)}{(\gamma+2)(\gamma+1)} \int_0^{2\pi} \sin^2 t d\mu_1(t)$$

and thus

$$Re(a_3) \leq \frac{\alpha(\gamma+2)(\gamma+1)^2}{(\mu+1)(\mu\gamma - \mu + 4\gamma + 2)}$$

which implies

$$|a_3| \leq \frac{\alpha(\gamma+2)(\gamma+1)^2}{(\mu+1)(\mu\gamma - \mu + 4\gamma + 2)}.$$

This completes the proof of theorem. □

If we take $\gamma = \mu$, in the Theorem 2.2, we obtain following corollary.

Corollary 2.3. *Let $f(z)$ given by (1.1) be in the class $\mathcal{C}_{\Sigma}^{\mu,\mu}(\alpha)$ $(0 < \alpha \leq 1)$. Then*

$$|a_2| \leq \alpha \qquad \text{and} \qquad |a_3| \leq \frac{2\alpha}{(\gamma+1)^2}.$$

3. Coefficient estimates for the function class $\mathcal{C}_{\Sigma}^{\mu,\gamma}(\beta)$

Definition 3.1. A function $f(z)$ given by (1.1) is said to be in the class $\mathcal{C}_{\Sigma}^{\mu,\gamma}(\beta)$ $(0<\mu\le 1;\ 0<\gamma\le 1;\ \mu\ge\gamma;\ 0\le\mu-\gamma<1)$ if the following conditions are satisfied:

$$f\in\Sigma \qquad \textit{and} \qquad Re\left\{1+\frac{z(\mathfrak{T}f)''(z)}{\mathfrak{T}f'(z)}\right\}>\beta \qquad (0\le\beta<1, z\in\mathbb{U}) \tag{3.1}$$

and

$$Re\left\{1+\frac{w(\mathfrak{T}g)''(w)}{\mathfrak{T}g'(w)}\right\}>\beta \qquad (0\le\beta<1, w\in\mathbb{U}) \tag{3.2}$$

where the function g is inverse of the function f given by (1.2).

For $\gamma=\mu$, the class of $\mathcal{C}_{\Sigma}^{\mu,\gamma}(\beta)$ is reduced to $C_{\Sigma}(\beta)$ of bi-convex of order β $(0\le\beta<1)$, which is introduced by Brannan and Taha [5], [6].

Theorem 3.2. *If $f(z)$ given by (1.1) be in the class $\mathcal{C}_{\Sigma}^{\mu,\gamma}(\beta)$, then*

$$|a_2|\le\sqrt{\frac{(1-\beta)(\gamma+1)^2(\gamma+2)}{(\mu+1)(\mu\gamma-\mu+4\gamma+2)}} \tag{3.3}$$

and

$$|a_3|\le\frac{(1-\beta)(\gamma+1)^2(\gamma+2)}{(\mu+1)(\mu\gamma-\mu+4\gamma+2)}. \tag{3.4}$$

Proof. The inequalities in (3.1) and (3.2) can be written in the following forms :

$$1+\frac{z(\mathfrak{T}f)''(z)}{\mathfrak{T}f'(z)}=\beta+(1-\beta)p(z) \tag{3.5}$$

and

$$1+\frac{w(\mathfrak{T}g)''(w)}{\mathfrak{T}g'(w)}=\beta+(1-\beta)q(w) \tag{3.6}$$

where $p(z)$ and $q(w)$ have the forms (2.7) and (2.8), respectively. As in the proof of Theorem 2.2, by equating coefficients (3.5) and (3.6) yields,

$$2\frac{\mu+1}{\gamma+1}a_2=(1-\beta)p_1, \tag{3.7}$$

$$\frac{6(\mu+2)(\mu+1)}{(\gamma+2)(\gamma+1)}a_3-4\left(\frac{\mu+1}{\gamma+1}\right)^2a_2^2=(1-\beta)p_2, \tag{3.8}$$

$$-2\frac{\mu+1}{\gamma+1}a_2=(1-\beta)q_1 \tag{3.9}$$

and

$$\frac{6(\mu+2)(\mu+1)}{(\gamma+2)(\gamma+1)}(2a_2^2-a_3)-4\left(\frac{\mu+1}{\gamma+1}\right)^2a_2^2=(1-\beta)q_2. \tag{3.10}$$

From (3.7) and (3.9) we get

$$p_1=-q_1 \tag{3.11}$$

and

$$8\left(\frac{\mu+1}{\gamma+1}\right)^2a_2^2=(1-\beta)^2(p_1^2+q_1^2). \tag{3.12}$$

Also from (3.8) and (3.10) we obtain

$$\frac{4(\mu+1)(\mu\gamma-\mu+4\gamma+2)}{(\gamma+1)^2(\gamma+2)}a_2^2=(1-\beta)(p_2+q_2). \tag{3.13}$$

Thus, clearly we have

$$|a_2|^2\leq\frac{(1-\beta)(\gamma+1)^2(\gamma+2)}{4(\mu+1)(\mu\gamma-\mu+4\gamma+2)}\left(|p_2|+|q_2|\right). \tag{3.14}$$

Applying the Carathéodory Lemma for the coefficients p_2 and q_2 we find the bound on $|a_2|$ as asserted in (3.3).
In order to find the bound on $|a_3|$, we multiply

$$\frac{12(\mu+2)(\mu+1)}{(\gamma+2)(\gamma+1)}-\frac{4(\mu+1)^2}{(\gamma+1)^2}\text{ and }\frac{4(\mu+1)^2}{(\gamma+1)^2}$$

to the relations (3.8) and (3.10) respectively and on adding them we obtain:

$$\left\{\frac{24(\mu+2)(\mu+1)^2(\mu\gamma-\mu+4\gamma+2)}{(\gamma+1)^3(\gamma+2)^2}\right\}a_3$$

$$=(1-\beta)\left\{\frac{4(\mu+1)(2\mu\gamma+\mu+5\gamma+4)}{(\gamma+2)(\gamma+1)^2}p_2+\frac{4(\mu+1)^2}{(\gamma+1)^2}q_2\right\}. \tag{3.15}$$

Taking the absolute value of (3.15) and applying the Carathéodory Lemma for the coefficients p_2, q_2 we find

$$|a_3|\leq\frac{(1-\beta)(\gamma+1)^2(\gamma+2)}{(\mu+1)(\mu\gamma-\mu+4\gamma+2)},$$

which is asserted in (3.4). □

If we take $\gamma=\mu$, in the Theorem 3.2, we obtain following corollary.

Corollary 3.3. [5], [6] *Let $f(z)$ given by (1.1) belong to $C_\Sigma(\beta)$ $(0\leq\beta<1)$. Then*

$$|a_2|\leq\sqrt{1-\beta}\qquad\textit{and}\qquad|a_3|\leq 1-\beta.$$

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Differential subordinations and superordinations for analytic functions defined by Sălăgean integro-differential operator

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Abstract. In this paper we consider the linear operator $\mathscr{L}^n : \mathcal{A} \to \mathcal{A}$,

$$\mathscr{L}^n f(z) = (1-\lambda)\,\mathscr{D}^n f(z) + \lambda I^n f(z),$$

where $\mathscr{D}^n$ is the Sălăgean differential operator and I^n is the Sălăgean integral operator. We give some results and applications for differential subordinations and superordinations for analytic functions and we will determine some properties on admissible functions defined with the new operator.

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1. Preliminaries

Let U be the unit disk in the complex plane:

$$U = \{z \in \mathbb{C} : |z| < 1\}.$$

Let $\mathcal{H} = \mathcal{H}(U)$ be the space of holomorphic functions in U and let

$$\mathcal{A}_n = \left\{f \in \mathcal{H}(U) : \ f(z) = z + a_{n+1}z^{n+1} + \cdots, z \in U\right\}$$

with $\mathcal{A}_1 = \mathcal{A}$. For $a \in \mathbb{C}$ and n a positive integer, let

$$\mathcal{H}[a, n] = \left\{f \in \mathcal{H}(U) : \ f(z) = a + a_n z^n + a_{n+1}z^{n+1} + \cdots, z \in U\right\}.$$

Denote by

$$K = \left\{ f \in \mathcal{A} : \Re \frac{zf''(z)}{f'(z)} + 1 > 0, z \in U \right\}$$

the class of normalized convex functions in U.

We denote by $\mathcal{Q}$ the set of functions f that are analytic and injective on $\overline{U} \setminus E(f)$, where

$$E(f) = \left\{ \zeta \in \partial U : \lim_{z \to \zeta} f(z) = \infty \right\}$$

and such that $f'(\zeta) \neq 0$ for $\zeta \in \partial U \setminus E(f)$.

Definition 1.1. ([9], Definition 3.5.1, [4]) Let $f, g \in \mathcal{H}$. We say that the function f is subordinate to the function g or g is superordinate to f, if there exists a function w, which is analytic in U and $w(0) = 0; |w(z)| < 1; z \in U$, such that $f(z) = g(w(z))$; $\forall z \in U$. We denote by $\prec$ the subordination relation. If g is univalent, then $f \prec g$ if and only if $f(0) = g(0)$ and $f(U) \subseteq g(U)$.

We omit the requirement $''z \in U''$ because the definition and conditions of the functions, in the unit disk U.

Let $\psi : \mathbb{C}^3 \times \overline{U} \to \mathbb{C}$ be a function and let h be univalent in U and $q \in \mathcal{Q}$. In article [6] it is studied the problem of determining conditions on admissible function ψ such that

$$\psi\left(p(z), zp'(z), z^2p''(z); z\right) \prec h(z), \ (z \in U) \tag{1.1}$$

(second-order) differential subordination, implies $p(z) \prec q(z)$, $\forall p \in \mathcal{H}[a,n]$. The univalent function q is called a dominant of the solution of the differential subordination, or more simply a dominant, if $p \prec q$ for all p satisfying (1.1).
A dominant $\widetilde{q}$, which is the "smallest" function with this property and satisfies $\widetilde{q} \prec q$ for all dominants q of (1.1) is said to be the best dominant of (1.1). The best dominant is unique up to a rotation of U.

Let $\varphi : \mathbb{C}^3 \times \overline{U} \to \mathbb{C}$ be a function and let $h \in \mathcal{H}$ and $q \in \mathcal{H}[a,n]$. If p and $\varphi\left(p(z), zp'(z), z^2p''(z); z\right)$ are univalent in U and satisfy the (second-order) differential superordination

$$h(z) \prec \varphi\left(p(z), zp'(z), z^2p''(z); z\right), \ (z \in U) \tag{1.2}$$

then p is called a solution of the differential superordination. In [7] the authors studied the dual problem of determining properties of functions p that satisfy the differential superordination (1.2). The analytic function q is called a subordinant of the solutions of the differential superordination, or more simply a subordinant, if $q \prec p$ for all p satisfying (1.2). An univalent subordinant $\widetilde{q}$ that satisfies $q \prec \widetilde{q}$ for all subordinants q of (1.2) is said to be the best subordinant of (1.2) and is the "largest" function with this property. The best subordinant is unique up to a rotation of U.

Definition 1.2. [11, 12] For $f \in \mathcal{A}, n \in \mathbb{N}_0$, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}, \mathbb{N} = \{1, 2, \ldots\}$, the Sălăgean differential operator $\mathscr{D}^n$ is defined by $\mathscr{D}^n : \mathcal{A} \to \mathcal{A}$,

$$\mathscr{D}^0 f(z) = f(z),$$
$$\mathscr{D}^{n+1} f(z) = z\left(\mathscr{D}^n f(z)\right)', \ z \in U.$$

Remark 1.3. If $f \in \mathcal{A}$ and $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$, then

$$\mathscr{D}^n f(z) = z + \sum_{k=2}^{\infty} k^n a_k z^k, z \in U.$$

Definition 1.4. [11] For $f \in \mathcal{A}, n \in \mathbb{N}_0$, the operator I^n is defined by

$$I^0 f(z) = f(z),$$

$$I^n f(z) = I\left(I^{n-1} f(z)\right), \ z \in U, \ n \geq 1.$$

Remark 1.5. If $f \in \mathcal{A}$ and $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$, then

$$I^n f(z) = z + \sum_{k=2}^{\infty} \frac{a_k}{k^n} z^k, \ z \in U, (n \in \mathbb{N}_0)$$

and $z\left(I^n f(z)\right)' = I^{n-1} f(z)$.

Definition 1.6. Let $\lambda \geq 0, n \in \mathbb{N}$. Denote by $\mathscr{L}^n$ the operator given by $\mathscr{L}^n : \mathcal{A} \to \mathcal{A}$,

$$\mathscr{L}^n f(z) = (1-\lambda)\mathscr{D}^n f(z) + \lambda I^n f(z), z \in U.$$

Remark 1.7. If $f \in \mathcal{A}$ and $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$, then

$$\mathscr{L}^n f(z) = z + \sum_{k=2}^{\infty} \left[k^n (1-\lambda) + \lambda \frac{1}{k^n}\right] a_k z^k, z \in U. \tag{1.3}$$

Lemma 1.8. [2] *Let q be an univalent function in U and $\gamma \in \mathbb{C}^* = \mathbb{C} \setminus \{0\}$ such that*

$$\Re\left\{1 + \frac{zq''(z)}{q'(z)}\right\} \geq \max\left\{0, -\Re\frac{1}{\gamma}\right\}.$$

If p is an analytic function in U, with $p(0) = q(0)$ and

$$p(z) + \gamma z p'(z) \prec q(z) + \gamma z q'(z), \tag{1.4}$$

then $p(z) \prec q(z)$ and q is the best dominant of (1.4).

Lemma 1.9. [2] *Let q be convex function in U, with $q(a) = 0$ and $\gamma \in \mathbb{C}$ such that $\Re\, \gamma > 0$. If $p \in \mathcal{H}[a,1] \cap \mathcal{Q}$ and $p(z) + \gamma z p'(z)$ is univalent in U, then*

$$q(z) + \gamma z q'(z) \prec p(z) + \gamma z p'(z) \Rightarrow q(z) \prec p(z)$$

and q is the best subordinant.

S. S. Miller and P. T. Mocanu obtained special results related to differential subordinations in [8] .

We follow Cotîrlă [3] and we generalise her results. Nechita obtained similar results in [10] for generalized Sălăgean differential operator (see also [1], [5]).

2. Main results

Theorem 2.1. *Let q be an univalent function in U with $q(0)=1$, $\gamma \in \mathbb{C}^*$ such that*

$$\Re\left\{1+\frac{zq''(z)}{q'(z)}\right\} \geq \max\left\{0, -\Re\frac{1}{\gamma}\right\}.$$

If $f \in \mathcal{A}$ and

$$\frac{\mathscr{L}^{n+1}f(z)}{\mathscr{L}^{n}f(z)} + \gamma\left\{1 - \frac{\mathscr{L}^{n+1}f(z)\left[(1-\lambda)\mathscr{D}^{n+1}f(z)+\lambda I^{n-1}f(z)\right]}{\left[\mathscr{L}^{n}f(z)\right]^2} + \right.$$
$$\left. + \frac{(1-\lambda)\left[\mathscr{D}^{n+2}f(z)-\mathscr{D}^{n}f(z)\right]}{\mathscr{L}^{n}f(z)}\right\} \prec q(z)+\gamma zq'(z), \tag{2.1}$$

then

$$\frac{\mathscr{L}^{n+1}f(z)}{\mathscr{L}^{n}f(z)} \prec q(z) \tag{2.2}$$

and q is the best dominant of (2.1).

Proof. We define the function

$$p(z) := \frac{\mathscr{L}^{n+1}f(z)}{\mathscr{L}^{n}f(z)}.$$

By calculating the logarithmic derivative of p, we obtain

$$\frac{zp'(z)}{p(z)} = z\frac{\left[\mathscr{L}^{n+1}f(z)\right]'}{\mathscr{L}^{n+1}f(z)} - z\frac{\left[\mathscr{L}^{n}f(z)\right]'}{\mathscr{L}^{n}f(z)}. \tag{2.3}$$

By using the identity

$$z\left[\mathscr{L}^{n+1}f(z)\right]' = (1-\lambda)\mathscr{D}^{n+2}f(z)+\lambda I^{n}f(z) \tag{2.4}$$

we obtain from (2.3) that

$$\frac{zp'(z)}{p(z)} = \frac{1}{p(z)} - \frac{(1-\lambda)\mathscr{D}^{n+1}f(z)+\lambda I^{n-1}f(z)}{\mathscr{L}^{n}f(z)}$$
$$+ \frac{(1-\lambda)\left(\mathscr{D}^{n+2}f(z)-\mathscr{D}^{n}f(z)\right)}{\mathscr{L}^{n+1}f(z)}$$

and

$$p(z)+\gamma zp'(z) = \frac{\mathscr{L}^{n+1}f(z)}{\mathscr{L}^{n}f(z)} + \gamma\left\{1 - \frac{\mathscr{L}^{n+1}f(z)\left[(1-\lambda)\mathscr{D}^{n+1}f(z)+\lambda I^{n-1}f(z)\right]}{\left[\mathscr{L}^{n}f(z)\right]^2}\right.$$
$$\left. + \frac{(1-\lambda)\left[\mathscr{D}^{n+2}f(z)-\mathscr{D}^{n}f(z)\right]}{\mathscr{L}^{n}f(z)}\right\}.$$

The subordination (2.1) becomes

$$p(z)+\gamma zp'(z) \prec q(z)+\gamma zq'(z).$$

We obtain the conclusion of our theorem by applying Lemma 1.8. □

In the particular case $\lambda = 0$ and $n = 0$ we obtain:

Corollary 2.2. *Let q be an univalent function in U with $q(0) = 1$, $\gamma \in \mathbb{C}^*$ such that*

$$\Re \left\{ 1 + \frac{zq''(z)}{q'(z)} \right\} \geq \max \left\{ 0, -\Re \frac{1}{\gamma} \right\}.$$

If $f \in \mathcal{A}$ and

$$(1+\gamma)\frac{zf'(z)}{f(z)} + \gamma \left[\frac{z^2 f''(z)}{f(z)} - \left(\frac{zf'(z)}{f(z)} \right)^2 \right] \prec q(z) + \gamma z q'(z)$$

then

$$\frac{zf'(z)}{f(z)} \prec q(z)$$

and q is the best dominant.

In the particular case $\lambda = 0$ and $n = 1$, we obtain:

Corollary 2.3. *Let q be an univalent function in U with $q(0) = 1$, $\gamma \in \mathbb{C}^*$ such that*

$$\Re \left\{ 1 + \frac{zq''(z)}{q'(z)} \right\} \geq \max \left\{ 0, -\Re \frac{1}{\gamma} \right\}.$$

If $f \in \mathcal{A}$ and

$$1 + (1+3\gamma)\frac{zf''(z)}{f'(z)} + \gamma \left[1 - \left(1 + \frac{zf''(z)}{f'(z)} \right)^2 + \frac{z^2 f'''(z)}{f'(z)} \right] \prec q(z) + \gamma z q'(z)$$

then

$$1 + \frac{zf''(z)}{f'(z)} \prec q(z)$$

and q is the best dominant.

When $\lambda = 1$ we get the Cotîrlă's result [3]:
We select in Theorem 2.1 a particular dominant q.

Corollary 2.4. *Let $A, B, \gamma \in \mathbb{C}, A \neq B$ such that $|B| \leq 1$ and $\Re\, \gamma > 0$. If for $f \in \mathcal{A}$*

$$\frac{\mathscr{L}^{n+1} f(z)}{\mathscr{L}^n f(z)} + \gamma \left\{ 1 - \frac{\mathscr{L}^{n+1} f(z) \left[(1-\lambda)\, \mathscr{D}^{n+1} f(z) + \lambda I^{n-1} f(z) \right]}{\left[\mathscr{L}^n f(z) \right]^2} \right.$$
$$\left. + \frac{(1-\lambda)\left[\mathscr{D}^{n+2} f(z) - \mathscr{D}^n f(z) \right]}{\mathscr{L}^n f(z)} \right\} \prec \frac{1+Az}{1+Bz} + \gamma \frac{(A-B)z}{(1+Bz)^2},$$

then

$$\frac{\mathscr{L}^{n+1} f(z)}{\mathscr{L}^n f(z)} \prec \frac{1+Az}{1+Bz}$$

and $q(z) = \dfrac{1+Az}{1+Bz}$ is the best dominant.

Theorem 2.5. *Let q be a convex function in U with $q(0) = 1$ and $\gamma \in \mathbb{C}$ such that $\Re\,\gamma > 0$. If $f \in \mathcal{A}$,*

$$\frac{\mathscr{L}^{n+1} f\left(z\right)}{\mathscr{L}^{n} f\left(z\right)} \in \mathcal{H}\left[1,1\right] \cap \mathcal{Q},$$

$$\frac{\mathscr{L}^{n+1} f\left(z\right)}{\mathscr{L}^{n} f\left(z\right)} + \gamma \left\{ 1 - \frac{\mathscr{L}^{n+1} f\left(z\right)\left[\left(1-\lambda\right)\mathscr{D}^{n+1} f\left(z\right) + \lambda I^{n-1} f\left(z\right)\right]}{\left[\mathscr{L}^{n} f\left(z\right)\right]^2} + \frac{\left(1-\lambda\right)\left[\mathscr{D}^{n+2} f\left(z\right) - \mathscr{D}^{n} f\left(z\right)\right]}{\mathscr{L}^{n} f\left(z\right)} \right\}$$

is univalent in U and

$$q(z) + \gamma z q'(z) \prec \frac{\mathscr{L}^{n+1} f\left(z\right)}{\mathscr{L}^{n} f\left(z\right)} + \gamma \left\{ 1 - \frac{\mathscr{L}^{n+1} f\left(z\right)\left[\left(1-\lambda\right)\mathscr{D}^{n+1} f\left(z\right) + \lambda I^{n-1} f\left(z\right)\right]}{\left[\mathscr{L}^{n} f\left(z\right)\right]^2} + \frac{\left(1-\lambda\right)\left[\mathscr{D}^{n+2} f\left(z\right) - \mathscr{D}^{n} f\left(z\right)\right]}{\mathscr{L}^{n} f\left(z\right)} \right\}, \tag{2.5}$$

then $q(z) \prec \dfrac{\mathscr{L}^{n+1} f\left(z\right)}{\mathscr{L}^{n} f\left(z\right)}$ *and q is the best subordinant .*

Proof. Let

$$p(z) := \frac{\mathscr{L}^{n+1} f\left(z\right)}{\mathscr{L}^{n} f\left(z\right)}.$$

If we proceed as in the proof of Theorem 2.1, the superordination (2.5) become

$$q(z) + \gamma z q'(z) \prec p(z) + \gamma z p'(z).$$

The conclusion of this theorem follows by applying the Lemma 1.9. □

From the combination of Theorem 2.1 and Theorem 2.5 we get the following "sandwich-type theorem".

Theorem 2.6. *Let q_1 and q_2 be convex functions in U with $q_1(0) = q_2(0) = 1$, $\gamma \in \mathbb{C}$ such that $\Re\,\gamma > 0$. If $f \in \mathcal{A}$,*

$$\frac{\mathscr{L}^{n+1} f\left(z\right)}{\mathscr{L}^{n} f\left(z\right)} \in \mathcal{H}\left[1,1\right] \cap \mathcal{Q},$$

$$\frac{\mathscr{L}^{n+1} f\left(z\right)}{\mathscr{L}^{n} f\left(z\right)} + \gamma \left\{ 1 - \frac{\mathscr{L}^{n+1} f\left(z\right)\left[\left(1-\lambda\right)\mathscr{D}^{n+1} f\left(z\right) + \lambda I^{n-1} f\left(z\right)\right]}{\left[\mathscr{L}^{n} f\left(z\right)\right]^2} + \frac{\left(1-\lambda\right)\left[\mathscr{D}^{n+2} f\left(z\right) - \mathscr{D}^{n} f\left(z\right)\right]}{\mathscr{L}^{n} f\left(z\right)} \right\}$$

is univalent in U *and*

$$q_1(z)+\gamma z q_1'(z) \prec \frac{\mathscr{L}^{n+1}f(z)}{\mathscr{L}^{n}f(z)}+\gamma\left\{1-\frac{\mathscr{L}^{n+1}f(z)\left[(1-\lambda)\mathscr{D}^{n+1}f(z)+\lambda I^{n-1}f(z)\right]}{\left[\mathscr{L}^{n}f(z)\right]^2}\right.$$

$$\left.+\frac{(1-\lambda)\left[\mathscr{D}^{n+2}f(z)-\mathscr{D}^{n}f(z)\right]}{\mathscr{L}^{n}f(z)}\right\} \prec q_2(z)+\gamma z q_2'(z), \tag{2.6}$$

then

$$q_1(z) \prec \frac{\mathscr{L}^{n+1}f(z)}{\mathscr{L}^{n}f(z)} \prec q_2(z),$$

q_1 *is the best subordinant and* $q_2(z)$ *is the best dominant.*

Theorem 2.7. *Let* q *be a convex function in* U *with* $q(0)=1$, $\gamma\in\mathbb{C}^*$ *such that*

$$\Re\left\{1+\frac{zq''(z)}{q'(z)}\right\}\geq \max\left\{0,-\Re\frac{1}{\gamma}\right\}.$$

If $f\in\mathcal{A}$ *and*

$$(1+\gamma)\,z\frac{\mathscr{L}^{n}f(z)}{\left[\mathscr{L}^{n+1}f(z)\right]^2}+\gamma z\frac{(1-\lambda)\mathscr{D}^{n+1}f(z)+\lambda I^{n-1}f(z)}{\left[\mathscr{L}^{n+1}f(z)\right]^2}$$

$$-2\gamma z\frac{\mathscr{L}^{n}f(z)\left[(1-\lambda)\mathscr{D}^{n+2}f(z)+\lambda I^{n}f(z)\right]}{\left[\mathscr{L}^{n+1}f(z)\right]^3} \prec q(z)+\gamma z q'(z), \tag{2.7}$$

then

$$z\frac{\mathscr{L}^{n}f(z)}{\left[\mathscr{L}^{n+1}f(z)\right]^2} \prec q(z),$$

q *is the best dominant.*

Proof. Let

$$p(z):=z\frac{\mathscr{L}^{n}f(z)}{\left[\mathscr{L}^{n+1}f(z)\right]^2}.$$

By calculating the logarithmic derivative of p, we obtain

$$\frac{zp'(z)}{p(z)}=1+\frac{(1-\lambda)\mathscr{D}^{n+1}f(z)+\lambda I^{n-1}f(z)}{\mathscr{L}^{n}f(z)}-2\frac{(1-\lambda)\mathscr{D}^{n+2}f(z)+\lambda I^{n}f(z)}{\mathscr{L}^{n+1}f(z)}. \tag{2.8}$$

It follows that

$$p(z)+\gamma zp'(z)=(1+\gamma)\,z\frac{\mathscr{L}^{n}f(z)}{\left[\mathscr{L}^{n+1}f(z)\right]^2}+\gamma z\frac{(1-\lambda)\mathscr{D}^{n+1}f(z)+\lambda I^{n-1}f(z)}{\left[\mathscr{L}^{n+1}f(z)\right]^2}$$

$$-2\gamma z\frac{\mathscr{L}^{n}f(z)\left[(1-\lambda)\mathscr{D}^{n+2}f(z)+\lambda I^{n}f(z)\right]}{\left[\mathscr{L}^{n+1}f(z)\right]^3}.$$

The subordination (2.7) becomes

$$p(z)+\gamma zp'(z) \prec q(z)+\gamma zq'(z).$$ □

We consider $n=0$ and $\lambda=0$.

Corollary 2.8. *Let q be univalent in U with $q(0)=1$, $\gamma \in \mathbb{C}^*$ such that*

$$\Re\left\{1+\frac{zq''(z)}{q'(z)}\right\} \geq \max\left\{0, -\Re\frac{1}{\gamma}\right\}.$$

If $f \in \mathcal{A}$ and

$$(1-\gamma)\frac{f(z)}{z\left[f'(z)\right]^2}+\gamma\left[\frac{1}{f'(z)}-\left(\frac{2f(z)\cdot f''(z)}{\left[f'(z)\right]^3}\right)^2\right] \prec q(z)+\gamma zq'(z)$$

then

$$\frac{f(z)}{z\left[f'(z)\right]^2} \prec q(z)$$

and q is the best dominant.

Corollary 2.9. *Let $A, B, \gamma \in \mathbb{C}, A \neq B$ such that $|B| \leq 1$ and $\Re\,\gamma > 0$. If for $f \in \mathcal{A}$*

$$(1+\gamma)\,z\frac{\mathscr{L}^n f(z)}{\left[\mathscr{L}^{n+1} f(z)\right]^2}+\gamma z\frac{(1-\lambda)\,\mathscr{D}^{n+1} f(z)+\lambda I^{n-1} f(z)}{\left[\mathscr{L}^{n+1} f(z)\right]^2}$$
$$-2\gamma z\frac{\mathscr{L}^n f(z)\left[(1-\lambda)\,\mathscr{D}^{n+2} f(z)+\lambda I^n f(z)\right]}{\left[\mathscr{L}^{n+1} f(z)\right]^3} \prec \frac{1+Az}{1+Bz}+\gamma\frac{(A-B)\,z}{(1+Bz)^2}, \tag{2.9}$$

then

$$z\frac{\mathscr{L}^n f(z)}{\left[\mathscr{L}^{n+1} f(z)\right]^2} \prec \frac{1+Az}{1+Bz}$$

and $q(z)=\dfrac{1+Az}{1+Bz}$ is the best dominant.

Theorem 2.10. *Let q be a convex function in U with $q(0)=1$, $\gamma \in \mathbb{C}$ such that $\Re\,\gamma > 0$. If $f \in \mathcal{A}$*

$$z\frac{\mathscr{L}^n f(z)}{\left[\mathscr{L}^{n+1} f(z)\right]^2} \in \mathcal{H}[1,1] \cap \mathcal{Q},$$

$$(1+\gamma)\,z\frac{\mathscr{L}^n f(z)}{\left[\mathscr{L}^{n+1} f(z)\right]^2}+\gamma z\frac{(1-\lambda)\,\mathscr{D}^{n+1} f(z)+\lambda I^{n-1} f(z)}{\left[\mathscr{L}^{n+1} f(z)\right]^2}$$
$$-2\gamma z\frac{\mathscr{L}^n f(z)\left[(1-\lambda)\,\mathscr{D}^{n+2} f(z)+\lambda I^n f(z)\right]}{\left[\mathscr{L}^{n+1} f(z)\right]^3}$$

is univalent in U and

$$q(z)+\gamma zq'(z) \prec (1+\gamma)\,z\frac{\mathscr{L}^n f(z)}{\left[\mathscr{L}^{n+1} f(z)\right]^2}+\gamma z\frac{(1-\lambda)\,\mathscr{D}^{n+1} f(z)+\lambda I^{n-1} f(z)}{\left[\mathscr{L}^{n+1} f(z)\right]^2}$$
$$-2\gamma z\frac{\mathscr{L}^n f(z)\left[(1-\lambda)\,\mathscr{D}^{n+2} f(z)+\lambda I^n f(z)\right]}{\left[\mathscr{L}^{n+1} f(z)\right]^3}, \tag{2.10}$$

then

$$q(z) \prec z\frac{\mathscr{L}^n f(z)}{\left[\mathscr{L}^{n+1} f(z)\right]^2},$$

q is the best subordinant.

From Theorem 2.7 and Theorem 2.10 we get the following "sandwich-type theorem".

Theorem 2.11. *Let q_1 and q_2 be convex functions in U with $q_1(0) = q_2(0) = 1$, $\gamma \in \mathbb{C}$ such that $\Re\,\gamma > 0$. If $f \in \mathcal{A}$*

$$z\frac{\mathscr{L}^n f(z)}{\left[\mathscr{L}^{n+1} f(z)\right]^2} \in \mathcal{H}[1,1] \cap \mathcal{Q},$$

$$(1+\gamma)\, z\frac{\mathscr{L}^n f(z)}{\left[\mathscr{L}^{n+1} f(z)\right]^2} + \gamma z\frac{(1-\lambda)\,\mathscr{D}^{n+1} f(z) + \lambda I^{n-1} f(z)}{\left[\mathscr{L}^{n+1} f(z)\right]^2}$$
$$-2\gamma z\frac{\mathscr{L}^n f(z)\left[(1-\lambda)\,\mathscr{D}^{n+2} f(z) + \lambda I^n f(z)\right]}{\left[\mathscr{L}^{n+1} f(z)\right]^3}$$

is univalent in U and

$$q_1(z) + \gamma z q_1'(z) \prec (1+\gamma)\, z\frac{\mathscr{L}^n f(z)}{\left[\mathscr{L}^{n+1} f(z)\right]^2} + \gamma z\frac{(1-\lambda)\,\mathscr{D}^{n+1} f(z) + \lambda I^{n-1} f(z)}{\left[\mathscr{L}^{n+1} f(z)\right]^2}$$
$$-2\gamma z\frac{\mathscr{L}^n f(z)\left[(1-\lambda)\,\mathscr{D}^{n+2} f(z) + \lambda I^n f(z)\right]}{\left[\mathscr{L}^{n+1} f(z)\right]^3} \prec q_2(z) + \gamma z q_2'(z), \tag{2.11}$$

then

$$q_1(z) \prec z\frac{\mathscr{L}^n f(z)}{\left[\mathscr{L}^{n+1} f(z)\right]^2} \prec q_2(z),$$

and q_1 is the best subordinant and $q_2(z)$ is the best dominant.

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Differential superordination for harmonic complex-valued functions

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Abstract. Let Ω and Δ be any sets in $\mathbb{C}$, and let $\varphi(r,s,t;z) : \mathbb{C}^3 \times U \to \mathbb{C}$. Let p be a complex-valued harmonic function in the unit disc U of the form $p(z) = p_1(z) + \overline{p_2(z)}$, where p_1 and p_2 are analytic in U. In [5] the authors have determined properties of the function p such that p satisfies the differential subordination

$$\varphi(p(z), Dp(z), D^2p(z); z) \subset \Omega \Rightarrow p(U) \subset \Delta.$$

In this article, we consider the dual problem of determining properties of the function p, such that p satisfies the second-order differential superordination

$$\Omega \subset \varphi(p(z), Dp(z), D^2p(z); z) \Rightarrow \Delta \subset p(U).$$

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1. Introduction and preliminaries

The theory of differential subordinations (or the method of admissible functions) for analytic functions was introduced by S.S. Miller and P.T. Mocanu in papers [6] and [7] and later developed in [1], [8], [10], [11], [12], [13].

The theory of differential subordinations has been extended from the analytic functions to the harmonic complex-valued functions in papers [2], [5], [14].

Let $U = \{z \in \mathbb{C} : |z| < 1\}$ be the open unit disc of the complex plane with

$$\overline{U} = \{z \in \mathbb{C} : |z| \leq 1\} \text{ and } \partial U = \{z \in \mathbb{C} : |z| = 1\}.$$

Denote by $\mathcal{H}(U)$ the class of holomorphic functions in the unit disc U, and

$$A_n = \{f \in \mathcal{H}(U) : f(z) = z + a_{n+1}z^{n+1} + \ldots\},\ A_1 = A.$$

A harmonic complex-valued mapping of the simply connected region Ω is a complex-valued function of the form

$$f(z) = h(z) + \overline{g(z)}, \tag{1.1}$$

where h and g are analytic in Ω, with $g(z_0) = 0$ for some prescribed point $z_0 \in \Omega$.

We call h and g analytic and co-analytic parts of f, respectively. If f is (locally) injective, then f is called (locally) univalent. The Jacobian and the second complex dilatation of f are given by

$$J_f(z) = |h'(z)|^2 - |g'(z)|^2$$

and

$$w(z) = g'(z)/h'(z),\ z \in \Omega,\ \text{respectively.}$$

A function $f \in C^2(\Omega)$, $f(z) = u(z) + iv(z)$ which satisfies

$$\Delta f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$$

or

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

and

$$\Delta v = \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$$

is called harmonic function.

By $Har(U)$ we denote the class of complex-valued, sense-preserving harmonic mappings in U. We note that each f of the form (1.1) is uniquely determined by coefficients of the power series expansion [2]

$$h(z) = a_0 + \sum_{n=1}^{\infty} a_n z^n,\ g(z) = b_0 + \sum_{n=1}^{\infty} b_n z^n,\ z \in U, \tag{1.2}$$

where $a_n \in \mathbb{C}$, $n = 0, 1, 2, \dots$ and $b_n \in \mathbb{C}$, $n = 0, 1, 2, \dots$

Several fundamental informations about harmonic mappings in the plane can also be found in [3].

For $f \in Har(U)$, let the differential operator D be defined as follows

$$Df = z\frac{\partial f}{\partial z} - \overline{z}\frac{\partial f}{\partial \overline{z}} = zh'(z) - \overline{zg'(z)}, \tag{1.3}$$

where $\frac{\partial f}{\partial z}$ and $\frac{\partial f}{\partial \overline{z}}$ are the formal derivatives of function f

$$\frac{\partial f}{\partial z} = \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right) \text{ and } \frac{\partial f}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right). \tag{1.4}$$

The conditions (1.4) are satisfied for any function $f \in C^1(\Omega)$ not necessarily harmonic, nor analytic.

Moreover, we define n-th order differential operator by recurrence relation

$$D^2 f = D(Df) = Df + z^2 h'' - \overline{z^2 g''},\ D^n f = D(D^{n-1} f). \tag{1.5}$$

Remark 1.1. If $f \in \mathcal{H}(U)$ (i.e. $g(z) = 0$) then $Df(z) = zf'(z)$.

Now we present several properties of the differential operator Df.

Proposition 1.1. *It is easy to prove that if $f, g \in Har(U)$, then the linear operator D satisfies the usual rules of differential calculus:*

a) $D(f \cdot g) = fDg + gDf$

b) $D\left(\frac{f}{g}\right) = \frac{gDf - fDg}{g^2}$

c) $D(f \circ g) = \frac{\partial f}{\partial g} \cdot Dg + \frac{\partial f}{\partial \overline{g}} \cdot D\overline{g}$

d) $D\overline{f} = -\overline{Df}$

e) $D\mathrm{Re}\, f = i\mathrm{Im}\, Df$

f) $D\mathrm{Im}\, f = -i\mathrm{Re}\, Df$

g) $D|f| = i|f| \cdot \mathrm{Im}\, \frac{Df}{f}$

h) $D \arg f = -i\mathrm{Re}\, \frac{Df}{f}$

If $z = re^{i\theta}$, then

a) $\frac{\partial f}{\partial \theta} = iDf,\ r\frac{\partial Df}{\partial r} = D^2 f$

b) $\frac{\partial}{\partial \theta} \arg f = \mathrm{Re}\, \frac{Df}{f} = \mathrm{Re}\, \frac{zh'(z) - \overline{zg'(z)}}{h(z) + \overline{g(z)}},\ (f(z) \neq 0)$

c) $\frac{\partial |f|}{\partial \theta} = -|f| \cdot \mathrm{Im}\, \frac{Df}{f}\ \ (f(z) \neq 0)$

In order to prove the main results of this paper, we use the following definitions and lemmas:

Definition 1.1. (Definition 2.2, [5]) By Q we denote the set of functions

$$q(z) = q_1(z) + \overline{q_2(z)},$$

harmonic complex-valued and univalent on $\overline{U} \setminus E(q)$, where

$$E(q) = \left\{\zeta \in \partial U;\ \lim_{z \to \zeta} f(z) = \infty\right\}.$$

Moreover, we assume that $D(q(\zeta)) \neq 0$, for $\zeta \in \partial U \setminus E(q)$. The set $E(q)$ is called an exception set. We note that the functions

$$q(z) = \overline{z},\ q(z) = \frac{1 + \overline{z}}{1 - \overline{z}}$$

are in Q, therefore Q is a non-empty set.

For the number $0 < r < 1$, we denote by $U_r = \{z \in \mathbb{C} :\ |z| < r\}$.

Lemma 1.1. (Lemma 2.2 [5]) *Let $p, q \in Har(U)$, $p(U)$ be simply connected and q be univalent in U. Also, let $p \in Q$ with $p(0) = q(0) = 1$, $q(z) \neq 1$. If q is not strongly subordinate to p, then there exist points $z_0 = r_0 e^{i\theta_0}$ and $\zeta_0 \in \partial U \setminus E(q)$ and a number $m \geq 1$ such that $q(U_{r_0}) \subset p(U)$, $q(z_0) = p(\zeta_0)$, and*

i) $Dq(z_0) = mDp(\zeta_0)$;

ii) $\operatorname{Re} \dfrac{D^2 q(z_0)}{Dq(z_0)} \geq m \operatorname{Re} \dfrac{D^2 p(\zeta_0)}{Dp(\zeta_0)}$.

2. Main results

In paper [9], S.S. Miller and P.T. Mocanu have introduced the dual notion of the differential superordination for analytic functions. In this paper we extend this notion for the harmonic complex-valued functions following the classical theory of differential superordination.

Definition 2.1. Let f and F be members of $Har(U)$. The function f is said to be **subordinate** to F, or F is said to be **superordinate** to f, if there exist a function w analytic in U, with $w(0) = 0$ and $|w(z)| < 1$ such that $f(z) = F(w(z))$. In such a case we write $f(z) \prec F(z)$. If F is univalent in U, then $f(z) \prec F(z)$ if and only if $f(0) = F(0)$ and $f(U) \subset F(U)$.

Let Ω and Δ be any sets in $\mathbb{C}$, let p be a harmonic complex-valued function in the unit disc U and let $\varphi(r, s, t; z) : \mathbb{C}^3 \times U \to \mathbb{C}$. In this paper we consider the problem of determining conditions on Ω, Δ and φ for which the following implication holds:

$$\Omega \subset \{\varphi(p(z), Dp(z), D^2 p(z); z) : \ z \in U\} \Rightarrow \Delta \subset p(U). \tag{2.1}$$

There are three distinct cases to consider in analyzing this implication, which we list as the following problems:

Problem 1. Given Ω and Δ, find conditions on the function φ so that (2.1) holds.

Problem 2. Given φ and Ω, find a set Δ such that (2.1) holds.
Furthermore, find the largest such Δ.

Problem 3. Given φ and Δ, find a set Ω, such that (2.1) holds.
Furthermore, find the smallest such Ω.

If either Ω or Δ in (2.1) is a simply connected domain, then it may be possible to rephrase (2.1) in terms of superordination. If p is harmonic univalent in U, and if Δ is a simply connected domain with $\Delta \neq \mathbb{C}$, then there is g a harmonic and univalent function, conformal mapping of U onto Δ, such that $q(0) = p(0)$.

In this case (2.1) can be rewritten as

$$\Omega \subset \{\varphi(p(z), Dp(z), D^2 p(z); z)\} \Rightarrow q(z) \prec p(z). \tag{2.1$'$}$$

If Ω is also a simply connected domain with $\Omega \neq \mathbb{C}$, then there is a conformal mapping h of U onto Ω, harmonic univalent function such that $h(0) = \varphi(p(0), 0, 0; 0)$.

If in addition, the function $\varphi(p(z), Dp(z), D^2p(z); z)$ is harmonic univalent in U, then (2.1) can be rewritten as

$$h(z) \prec \varphi(p(z), Dp(z), D^2p(z); z) \Rightarrow q(z) \prec p(z),\ z \in U. \tag{2.2}$$

In the special case when the set inclusion (2.1) can be replaced by the superordination for harmonic complex-valued function (2.2), we can reinterpret the three problems referred to above as follows:

Problem 1'. Given harmonic complex-valued functions h and q, find a class of admissible functions $\Phi[h, q]$ such that (2.2) holds.

Problem 2'. Given the differential superordination for harmonic complex-valued functions (2.2), find a subordinant q. Moreover, find the best subordinant.

Problem 3'. Given φ and subordinant q, find the largest class of harmonic complex-valued functions h such that (2.2) holds.

Remark 2.1. A function $f(z) = a\overline{z} + b$, $a \neq 0$, $a, b \in \mathbb{C}$, which is a harmonic function, is a conformal mapping of the complex plane into itself.

Let $z = x + iy$, $\overline{z} = x - iy$, $a = a_1 + ia_2$, $a_1 \neq 0$ or $a_2 \neq 0$, $b = b_1 + ib_2$. Then we let

$$f(z) = a_1x + a_2y + b_1 + i(a_2x - a_1y + b_2).$$

Denote by

$$P(x, y) = a_1x + a_2y + b_1,\ Q(x, y) = a_2x - a_1y + b_2.$$

The functions P and Q are continuous functions which admit partial derivatives with respect to x and y. We have

$$\frac{\partial P(x,y)}{\partial x} = a_1,\ \frac{\partial Q(x,y)}{\partial y} = -a_1,\ \frac{\partial P(x,y)}{\partial y} = a_2,\ \frac{\partial Q(x,y)}{\partial x} = a_2.$$

Since

$$\frac{\partial^2 P(x,y)}{\partial x^2} + \frac{\partial^2 P(x,y)}{\partial y^2} = 0 \text{ and } \frac{\partial^2 Q(x,y)}{\partial x^2} + \frac{\partial^2 Q(x,y)}{\partial y^2} = 0$$

we get that the function f is a harmonic function.

We now show that function f is a conformal mapping.

Let $a = |a|e^{i\phi}$, $|a| = R > 0$, $\phi = \arg a$, $\overline{z} = |\overline{z}|e^{i\theta} = |z|e^{i\theta}$, $\theta = \arg \overline{z}$.

Then $f(z) = Re^{i\phi}\overline{z} + b$ can be decomposed into three elementary substitutions:

(1) $z_1 = e^{i\phi}\overline{z} = |z|e^{i(\theta+\phi)}$, meaning that the point z_1 can be obtained by the rotation of the entire complex plane around the origin by a constant angle ϕ. Rotation preserves the angles of the rotated figures.

(2) $z_2 = Rz_1$, where $R > 0$, and a constant. This is a homothetic transformation. It is well-known that the homothetic transformation only changes the dimensions of the figures without changing the shape and it preserves the angles.

(3) $w = z_2 + b$, which is a translation of the complex-plane, characterized by b. Translation preserves dimensions and shape, hence it preserves the angles.

Since $f(z) = a\overline{z} + b$, is a combination between a rotation, a homothetic transformation and a translation, f preserves angles, hence it is a conformal mapping.

Definition 2.2. Let $\varphi : \mathbb{C}^3 \times U \to \mathbb{C}$ and let h be harmonic univalent in U. If p and $\varphi(p(z), Dp(z), D^2p(z))$ are harmonic univalent in U, and satisfy the second-order differential superordination for harmonic complex-valued functions

$$h(z) \prec \varphi(p(z), Dp(z), D^2p(z); z) \tag{2.3}$$

then p is called a solution of the differential superordination.

A harmonic univalent function q is called a subordinant of the solutions of the differential superordination for harmonic complex-valued functions, or more simply a subordinant if $q \prec p$, for all p satisfying (2.3). An univalent harmonic subordinant $\overline{q}$ that satisfies $q \prec \overline{q}$ for all subordinants q of (2.3) is said to be the best subordinant. The best subordinant is unique up to a rotation of U.

Remark 2.2. For Ω a set in $\mathbb{C}$, with φ and p as given in Definition 2.2, suppose (2.3) is replaced by

$$\Omega \subset \{\varphi(p(z), Dp(z), D^2p(z) : \ z \in U\}. \tag{2.4}$$

Although this more general situation is a differential containment, the condition in (2.4) will also be referred to as a differential superordination for harmonic complex-valued functions, and the definitions of solution, subordinant and best subordinant as given above can be extended to this generalization.

We next give the definition of the class of admissible function for harmonic complex-valued functions.

Definition 2.3. Let Ω be a set in $\mathbb{C}$ and let q be a harmonic univalent function. The class of admissible functions $\Phi[\Omega, q]$ consists of those functions $\varphi : \mathbb{C}^3 \times \overline{U} \to \mathbb{C}$ that satisfy the admissibility condition

$$(A) \qquad \varphi(r, s, t; \zeta) \in \Omega$$

where

$$r = q(z), \ s = \frac{Dq(z)}{m}, \ \operatorname{Re}\left(\frac{t}{s} + 1\right) \le \frac{1}{m}\operatorname{Re}\frac{D^2q(z)}{Dq(z)},$$

where $\zeta \in \partial U$, $z \in U$ and $m \ge 1$.

If $\varphi : \mathbb{C}^2 \times \overline{U} \to \mathbb{C}$, the admissibility condition (A) reduces to

$$(A') \qquad \varphi\left(q(z), \frac{Dq(z)}{m}; \zeta\right) \in \Omega,$$

where $z \in U$, $\zeta \in \partial U$ and $m \ge 1$.

In the special case when h is a harmonic complex-valued function conformal mapping of U onto $\Omega \ne \mathbb{C}$, we denote the class $\Phi[h(U), q]$ by $\Phi[h, q]$.

The following theorems are important results for the theory of differential superordinations for complex-valued harmonic functions.

Theorem 2.1. *Let $\Omega \subset \mathbb{C}$, let q be a harmonic and univalent function with $q(0) = 1$ and let $\varphi \in \Phi[\Omega, q]$. If $p \in Q$, $p(0) = 1$, $p(U)$ is simply connected and $\varphi(p(z), Dp(z), D^2p(z) : \ z \in U)$ is harmonic and univalent in U, then*

$$\Omega \subset \{\varphi(p(z), Dp(z), D^2p(z) : \ z \in U)\} \tag{2.5}$$

implies

$$q(z) \prec p(z), \ z \in U.$$

Proof. Assume $q \not\prec p$. From Lemma 1.1, there exist points

$$z_0 = r_0 e^{i\theta_0} \in U \text{ and } \zeta_0 \in \partial U \setminus E(q), \ m \geq 1,$$

that satisfy

$$q(z_0) = p(\zeta_0), \ Dq(z_0) = mDp(z_0), \ \mathrm{Re}\,\frac{D^2q(z_0)}{q(z_0)} \geq m\mathrm{Re}\,\frac{D^2p(\zeta_0)}{Dp(\zeta_0)}.$$

Let $r = p(\zeta_0)$, $s = Dp(\zeta_0)$, $t = D^2p(\zeta_0)$, and $\zeta = \zeta_0$, in Definition 2.3, then we obtain

$$\varphi(p(\zeta_0), Dp(\zeta_0), D^2p(\zeta_0); \zeta_0) \in \Omega.$$

Since this contradicts (2.5), we have that the assumption made is false, hence $q(z) \prec p(z)$, $z \in U$.

Remark 2.3. If h is a harmonic and univalent function in U, is a conformal mapping and $h(U) = \Omega \neq \mathbb{C}$, then the class $\Phi[h(U), q]$ is written as $\Phi[h, q]$ and the following result is an immediate consequence of Theorem 2.1.

Theorem 2.2. *Let q be a harmonic and univalent function in U, with $q(0) = 1$, let h be harmonic and univalent in U, with $p(0) = 1$, $p(U)$ is simply connected and $\varphi \in \Phi[h(U), q]$. If $p \in Q$ and $\varphi(p(z), Dp(z), D^2p(z); z)$ is harmonic and univalent in U, then*

$$h(z) \prec \varphi(p(z), Dp(z), D^2p(z); z) \tag{2.6}$$

implies

$$q(z) \prec p(z), \ z \in U.$$

From Theorem 2.1 and Theorem 2.2, we see that we can obtain subordinants of a differential superordination for harmonic complex-valued functions of the form (2.5) and (2.6), by simply checking that the function φ is an admissible function.

The following theorem proves the existence of the best subordinant of (2.6) for certain φ and also provides a method for finding the best subordinant.

Theorem 2.3. *Let h be a harmonic and univalent function in U, $h(U)$ is simply connected and let $\varphi : \mathbb{C}^3 \times \overline{U} \to \mathbb{C}$. Suppose that the differential equation*

$$\varphi(q(z), Dq(z), D^2q(z); z) = h(z) \tag{2.7}$$

has a solution $q \in Q$, harmonic and univalent in U. If $\varphi \in \Phi[h(U), q]$, $p \in Q$, $p(0) = 1$, $p(U)$ is simply connected and $\varphi(p(z), Dp(z), D^2p(z); z)$ is harmonic and univalent in U, then

$$h(z) \prec \varphi(p(z), Dp(z), D^2p(z); z) \tag{2.8}$$

implies

$$q(z) \prec p(z), \ z \in U,$$

and q is the best subordinant.

Proof. Since $\varphi \in \Phi[h(U), q]$ and is harmonic and univalent in U, by applying Theorem 2.2, we deduce that q is a subordinant of (2.8). Since q also satisfies (2.7), it is also a solution of the differential subordination (2.8) and therefore all subordinants of (2.8) will be subordinate to q. Hence q will be the best subordinant of (2.8).

From this theorem we see that the problem of finding the best subordinant of (2.8) essentially reduces to showing that differential equation (2.8) has an univalent solution and checking that $\varphi \in \Phi[h(U), q]$.

3. First-order differential superordinations for harmonic complex-valued functions

We can simplify Theorem 2.1, 2.2 and 2.3 for the case of first-order differential superordinations for harmonic complex-valued functions.

The following results are immediately obtained by using these theorems and admissibility condition (A').

Theorem 3.1. *Let $\Omega \subset \mathbb{C}$, let q be a harmonic and univalent function with $q(0) = 1$ and $\varphi \in \Phi[\Omega, q]$. If $p \in Q$ and $\varphi(p(z), Dp(z); z \in U)$ is harmonic and univalent in U, then*

$$\varphi(q(z), tDq(z); \zeta) \in \Omega \tag{3.1}$$

for $z \in U$, $\zeta \in \partial U$ and $0 < t \leq \dfrac{1}{m} \leq 1$, $m \geq 1$. If $p \in Q$, $p(0) = 1$, $p(U)$ is simply connected and $\varphi(p(z), Dp(z); z)$ is harmonic and is univalent in U, then

$$\Omega \subset \{\varphi(p(z), Dp(z)); z \in U\}$$

implies

$$q(z) \prec p(z),\ z \in U.$$

Theorem 3.2. *Let h, q be harmonic and univalent functions in U, $\varphi : \mathbb{C}^2 \times \overline{U} \to \mathbb{C}$, and suppose that*

$$\varphi(q(z), tDq(z); \zeta) \in h(U),$$

for $z \in U$, $\zeta \in \partial U$ and $0 < t \leq \dfrac{1}{m} \leq 1$, $m \geq 1$. If $p \in Q$, $p(0) = 1$, $p(U)$ is simply connected and $\varphi(p(z), Dp(z); z \in U)$ is harmonic and univalent in U, then

$$h(z) \prec \varphi(p(z), Dp(z); z)$$

implies

$$q(z) \prec p(z),\ z \in U.$$

Furthermore, if $\varphi(q(z), Dq(z); z) = h(z)$, has a univalent solution $q \in Q$, then q is the best subordinant.

We next give an example of finding the best subordinant of a differential superordination of harmonic functions.

Example 3.1. Let $q(z) = 1 + Mz + \overline{z}$, $z \in U$, $M > 0$ be a harmonic complex-valued function in the unit disc.
Let $z \in U$, $z = x + iy$, $\overline{z} = x - iy$.
Then

$$q(z) = 1 + x + Mx + i(My - y).$$

We denote

$$P(x, y) = 1 + x + Mx, \ Q(x, y) = My - y.$$

The functions P and Q are continuous functions in U which admit partial derivatives with respect to x and y. We have

$$\frac{\partial P(x,y)}{\partial x} = 1 + M, \ \frac{\partial P(x,y)}{\partial y} = 0, \ \frac{\partial Q(x,y)}{\partial x} = 0, \ \frac{\partial Q(x,y)}{\partial y} = M - 1.$$

Since $\dfrac{\partial^2 P(x,y)}{\partial x^2} + \dfrac{\partial^2 P(x,y)}{\partial y^2} = 0$ we have that $P(x, y)$ is a harmonic function.
Since $\dfrac{\partial^2 Q(x,y)}{\partial x^2} + \dfrac{\partial^2 Q(x,y)}{\partial y^2} = 0$ we have that $Q(x, y)$ is a harmonic function.
Hence, $f(z) = P(x, y) + iQ(x, y)$ is a harmonic function.
The function $q(z) = 1 + Mz + \overline{z}$ is the univalent harmonic solution of the equation

$$h(z) = q(z) + Dq(z) + D^2 q(z) = 1 + 3Mz + \overline{z}$$

which is an univalent harmonic function.
If

$$1 + 3Mz + \overline{z} \prec p(z) + Dp(z) + D^2 p(z),$$

then, using Theorem 2.3, we have that

$$1 + Mz + \overline{z} \prec p(z), \ z \in U$$

and $q(z) = 1 + Mz + \overline{z}$ is the best subordinant.

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On the periodicity of meromorphic functions when sharing two sets IM

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Abstract. In this paper, we have considered two sets sharing problems, and investigated on some sufficient conditions for the periodicity of meromorphic functions and obtained two results improving the result of *Bhoosnurmath-Kabbur* [6], *Qi-Dou-Yang* [17] and *Zhang* [20]. The results are:

Let $\mathcal{S}_1 = \left\{z : \int_0^{z-a} (t-a)^n (t-b)^4 dt + 1 = 0\right\}$ and $\mathcal{S}_2 = \left\{a, b\right\}$, where $n \geq 4 (n \geq 2)$ be an integer. *Let $f(z)$ be a non-constant meromorphic (entire) function satisfying $\overline{E}_{f(z)}(\mathcal{S}_j) = \overline{E}_{f(z+c)}(\mathcal{S}_j), (j = 1,\ 2)$ then $f(z) \equiv f(z+c)$.* Some examples have been exhibited to show that, the meromorphic functions, we have considered may be of infinite order, and also to show that the sets considered in the main results, can't be replace by some arbitrary sets. At the last section, we have posed a question for the future research in this direction.

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1. Introduction

We assume that the reader is familiar with the elementary Nevanlinna theory, see, e.g., [11, 13, 14, 18]. Meromorphic functions are always non-constant, unless otherwise specified. For such a function f and $a \in \overline{\mathbb{C}} =: \mathbb{C} \cup \{\infty\}$, each z with $f(z) = a$ will be called a-point of f. We will use here some standard definitions and basic notations from this theory. In particular by $N(r, a; f)$ ($\overline{N}(r, a; f)$) we denote the counting function (reduced counting function) of a-points of meromorphic functions f, $T(r, f)$ is the Nevanlinna characteristic function of f and $S(r, f)$ is used to denote each functions which is of smaller order than $T(r, f)$ when $r \to \infty$.

We also denote $\mathbb{C}^* := \mathbb{C} \setminus \{0\}$. As for the standard notation in the uniqueness theory of meromorphic functions, suppose that f and g are meromorphic. Denoting

$E_f(a)$ $(\overline{E}_f(a))$, the set of all a-points of f counting multiplicities (ignoring multiplicities). We say that two meromorphic functions f, g share the value a CM (IM) if $E_f(a) = E_g(a)$ $(\overline{E}_f(a) = \overline{E}_g(a))$.

The classical results in the uniqueness theory of meromorphic functions are the five-point, resp. four-point, theorems due to Nevanlinna [16]: If two meromorphic functions f, g share five distinct values in the extended complex plane IM, then $f \equiv g$. The beauty of this result lies in the fact that there is no counterpart of this result in the real function theory. Similarly, if two meromorphic functions f, g share four distinct values in the extended complex plane CM, then $f \equiv T \circ g$, where T is a Möbius transformation.

Clearly these results initiated the study of uniqueness of two meromorphic functions f and g. The study becomes more interesting if the function g is related with f.

Definition 1.1. For a non-constant meromorphic function f and any set $\mathcal{S} \subset \overline{\mathbb{C}}$, we define

$$E_f(\mathcal{S}) = \bigcup_{a\in\mathcal{S}} \Big\{(z,p) \in \mathbb{C}\times\mathbb{N} : f(z) = a, \text{ with multiplicity } p\Big\},$$

$$\overline{E}_f(\mathcal{S}) = \bigcup_{a\in\mathcal{S}} \Big\{(z,1) \in \mathbb{C}\times\{1\} : f(z) = a\Big\}.$$

If $E_f(\mathcal{S}) = E_g(\mathcal{S})$ $(\overline{E}_f(\mathcal{S}) = \overline{E}_g(\mathcal{S}))$ then we simply say f and g share $\mathcal{S}$ Counting Multiplicities(CM) (Ignoring Multiplicities(IM)).

Evidently, if $\mathcal{S}$ contains one element only, then it coincides with the usual definition of $CM(IM)$ sharing of values.

Definition 1.2. For a non-constant meromorphic function g and $a \in \mathbb{C}$, we define $\overline{N}_{(2}\left(r, \dfrac{1}{g-a}\right)$ the reduced counting function of those a-points of g of multiplicities ≥ 2.

In 1976, *Gross* [12] precipitated the research instigating the set sharing problem with a more general set up made tracks various direction of research for the uniqueness theory.

In connection with the question posed by *Gross* in[12], a sprinkling number of results have been obtained by many mathematicians [2, 3, 5, 9, 19, 21] concerning the uniqueness of meromorphic functions sharing two sets. But in most of the preceding results, in the direction, one set has always been kept fixed as the set of poles of a meromorphic function.

Recently set sharing corresponding to a function and its shift or difference operator have been given priority by the researchers than that of the introductory one.

In what follows, c always means a non-zero constant. For a non-constant meromorphic function, we define its shift and difference operator respectively by $f(z+c)$ and $\Delta_c f = f(z+c) - f(z)$.

Now-a-days among the researchers [1, 4, 6, 7, 8, 17, 20], an increasing amount of interest has been found to find the possible relationship between a meromorphic function $f(z)$ and its shift $f(z+c)$ or its difference $\Delta_c f$.

At the earlier stage, several authors were devoted to find uniqueness problems between two meromorphic functions f and g sharing two sets. But in this particular direction, the first inspection for uniqueness of a meromorphic function and its shift was due to *Zhang* [20].

In 2010, *Zhang* [20] obtained the following results.

Theorem A. [20] Let $m \geq 2$, $n \geq 2m+4$ with n and $n-m$ having no common factors. Let a and b be two non-zero constant such that the equation $w^n+aw^{n-m}+b=0$ has no multiple roots. Let $\mathcal{S}_1=\{w : w^n+aw^{n-m}+b=0\}$ and $\mathcal{S}_2=\{\infty\}$. Suppose that $f(z)$ is a non-constant meromorphic function of finite order. Then $E_{f(z)}(\mathcal{S}_j)=E_{f(z+c)}(\mathcal{S}_j)$ $(j=1,\ 2)$ imply that $f(z)\equiv f(z+c)$.

Remark 1.1. For meromorphic function, note that $\#(\mathcal{S}_1)=9$ when the nature of sharing is CM.

Theorem B. [20] Let $n \geq 5$ be an integer and let a, b be two non-zero constants such that the equation $w^n+aw^{n-1}+b=0$ has no multiple roots.
Denote $\mathcal{S}_1=\{w : w^n+aw^{n-1}+b=0\}$. Suppose that f is a non-constant entire function of finite order. Then $E_{f(z)}(\mathcal{S}_1)=E_{f(z+c)}(\mathcal{S}_1)$ implies $f(z)\equiv f(z+c)$.

Remark 1.2. For entire function, note that $\#(\mathcal{S}_1)=5$, when the nature of sharing is CM.

Thus we see that *Zhang* obtained the results for meromorphic function with the cardinality of main range set as 9 and for entire function as 5.

Later, *Qi-Dou-Yang* [17] studied the case for $m=1$ in *Theorem A* and with the aid of some extra supposition and got $\#(\mathcal{S}_1)=6$ when the nture of sharing is CM.

Afterworlds, *Bhoosnurmath-Kabbur* [6] improved *Theorem A* by reducing the lower bound of the cardinality of range set in a little different way and obtained the following result.

Theorem C. [6] Let $n \geq 8$ be an integer and $c(\neq 0,\ 1)$ is a constant such that the equation

$$P(w)=\frac{(n-1)(n-2)}{2}w^n-n(n-2)w^{n-1}+\frac{n(n-1)}{2}z^{n-2}-c.$$

Let us suppose that $S_1=\{w : P(w)=0\}$ and $S_2=\{\infty\}$. Suppose that $f(z)$ is a non-constant meromorphic function of finite order. Then $E_{f(z)}(\mathcal{S}_j)=E_{f(z+c)}(\mathcal{S}_j)$ $(j=1,\ 2)$ imply that $f(z)\equiv f(z+c)$.

Remark 1.3. For meromorphic function, we see that $\#(\mathcal{S}_1)=8$ when the nature of sharing is CM.

The worth noticing fact is that, the lower bound of the cardinality of the main range set for the meromorphic function has always been fixed to 8 without the help of any extra supposition.

So for the improvement of all the above mentioned results it is quite natural to investigate in this direction. *Theorems A, B, C* really motivates oneself for further study in this direction by solving the following question.

Question 1.1. Is it possible to diminish further the lower bound of the cardinalities of the main range sets in *Theorems* A, B and C ?

We also note that no attempts have so far been made by any researchers, till now to the best of our knowledge, to relax the nature of sharing the sets in connection with the periodicity of a meromorphic function when sharing sets. So the following question is inevitable.

Question 1.2. Can we relax the nature of sharing the sets from CM to IM in *Theorems* A, B and C ?

It would be interesting to know what happens if we replace the set of poles $\{\infty\}$ by new set in *Theorems A, B, C*.

In all the above mentioned results, the respective authors have considered meromorphic function with *finite ordered* and got their results. So a natural investigation is that: Are *Theorems A, B, C* not valid for infinite ordered meromorphic function ?

The following examples show that *Theorems A, B, C* are true for infinite ordered meromorphic functions also.

Example 1.1. Let

$$f(z)=\frac{\exp\left(\exp\left(\frac{2\pi iz}{c}\right)\right)}{\exp\left(\frac{2\pi iz}{c}\right)-1}.$$

Clearly $f(z)$ and $f(z+c)$ share the corresponding sets $\mathcal{S}_1$ and $\mathcal{S}_2$ in *Theorems A, B, C*, and $f(z)\equiv f(z+c)$.

Example 1.2. Let

$$f(z)=\frac{\exp\left(\sin\left(\frac{2\pi z}{c}\right)\right)}{\tan\left(\frac{\pi z}{c}\right)-1}.$$

Evidently, $f(z)$ and $f(z+c)$ share the corresponding sets $\mathcal{S}_1$ and $\mathcal{S}_2$ in *Theorems A, B, C*, and $f(z)\equiv f(z+c)$.

One can construct such examples plenty in numbers. Therefore, one natural question arises as follows:

Question 1.3. Can we get a corresponding results like *Theorems A, B, C* by omitting the term *finite ordered* ?

2. Main results

Answering all the questions affirmatively is the main motivation of writing this paper. Throughout the paper, for an integer $n\geq 4$, we will denote by

$$\mathcal{P}(z)=\int_0^{z-a}(t-a)^n(t-b)^4dt+1,\quad \text{where } a,b\in\mathbb{C} \text{ with } a\neq b.$$

Following are the two main result of this paper.

Theorem 2.1. Let $\mathcal{S}_1 = \{z : \mathcal{P}(z) = 0\}$ and $\mathcal{S}_2 = \left\{a, b\right\}$, where $a \in \mathbb{C}^*$, $n \geq 4$ be an integer. If $f(z)$ be a non-constant meromorphic function satisfying

$$\overline{E}_{f(z)}(\mathcal{S}_j) = \overline{E}_{f(z+c)}(\mathcal{S}_j),\ (j = 1,\ 2)$$

then $f(z) \equiv f(z + c)$.

Remark 2.1. For non-entire meromorphic function, one may observe that $\#(\mathcal{S}_1) = 9$ when the nature of sharing is IM.

Theorem 2.2. Let $\mathcal{S}_1 = \{z : \mathcal{P}(z) = 0\}$ and $\mathcal{S}_2 = \left\{a, b\right\}$, where $a \in \mathbb{C}^*$, $n \geq 2$ be an integer. If $f(z)$ be a non-constant entire function satisfying $\overline{E}_{f(z)}(\mathcal{S}_j) = \overline{E}_{f(z+c)}(\mathcal{S}_j)$, $(j = 1,\ 2)$, then $f(z) \equiv f(z + c)$.

Remark 2.2. For entire function, we see that $\#(\mathcal{S}_1) = 7$ when the nature of sharing is IM.

The following examples satisfy *Theorems* 2.1 and 2.2 for "entire" as well as "meromorphic" functions.

Example 2.1. Let us suppose that

$$f(z) = \frac{\tan\left(\frac{\pi z}{c}\right) + \alpha}{\tan\left(\frac{\pi z}{c}\right) - \beta} + \frac{\cos\left(\frac{2\pi z}{c}\right) + \gamma}{\sin\left(\frac{2\pi z}{c}\right) - \delta},$$

where $\alpha,\ \beta,\ \gamma,\ \delta,\ c \in \mathbb{C}^*$. It is clear that $\overline{E}_{f(z)}(\mathcal{S}_j) = \overline{E}_{f(z+c)}(\mathcal{S}_j)$, $(j = 1,\ 2)$ in *Theorem* 2.1 and note that $f(z) \equiv f(z + c)$.

Example 2.2. Let

$$f(z) = \frac{\alpha + \beta \sin^2\left(\frac{\pi z}{c}\right)}{\gamma - \delta \cos^2\left(\frac{\pi z}{c}\right)},$$

where p be an even positive integer, $\alpha,\ \beta,\ \gamma,\ \delta,\ c \in \mathbb{C}^*$.
It is clear that $\overline{E}_{f(z)}(\mathcal{S}_j) = \overline{E}_{f(z+c)}(\mathcal{S}_j)$, $(j = 1,\ 2)$ in *Theorem* 2.1 and note that $f(z) \equiv f(z + c)$.

Example 2.3. Let

$$f(z) = ae^{pz} + b\cos^2\left(\frac{\pi z}{c}\right),$$

where p be an even positive integer, $a,\ b\ , c \in \mathbb{C}^*$ with $e^c = -1$. It is clear that $\overline{E}_{f(z)}(\mathcal{S}_j) = \overline{E}_{f(z+c)}(\mathcal{S}_j)$, $(j = 1,\ 2)$ in *Theorem* 2.2 and note that $f(z) \equiv f(z + c)$.

The next examples shows that the set considered in *Theorem* 2.1 for "entire" and *Theorem* 2.2 for "meromorphic" functions respectively can not be replaced by arbitrary sets.

Example 2.4. Let us suppose that $\mathcal{S}_1 = \{\zeta : \zeta^9 - 1 = 0\}$ and $\mathcal{S}_2 = \{0, \infty\}$. Let

$$f(z) = \frac{ae^z}{b - d\sin^2\left(\frac{\pi z}{c}\right)}.$$

It is clear that $\overline{E}_{f(z)}(\mathcal{S}_j) = \overline{E}_{f(z+c)}(\mathcal{S}_j)$, $(j = 1,\ 2)$ in *Theorem* 2.1 with $e^c = \zeta$ and $a,\ b,\ c,\ d \in \mathbb{C}^*$ and note that $f(z) \not\equiv f(z+c)$.

Example 2.5. Let us suppose that $\mathcal{S}_1 = \{\zeta : \zeta^7 - 1 = 0\}$ and $\mathcal{S}_2 = \{0, 1\}$. Let

$$f(z) = \exp\left(\cos\left(\frac{\pi z}{c}\right)\right) \quad \text{or} \quad \exp\left(\sin\left(\frac{\pi z}{c}\right)\right).$$

Then $f(z+c) = \exp\left(-\cos\left(\frac{\pi z}{c}\right)\right)$ or $\exp\left(-\sin\left(\frac{\pi z}{c}\right)\right)$ respectively. It is clear that $\overline{E}_{f(z)}(\mathcal{S}_j) = \overline{E}_{f(z+c)}(\mathcal{S}_j)$, $(j = 1,\ 2)$ in *Theorem* 2.2 and note that $f(z) \not\equiv f(z+c)$.

Example 2.6. Let

$$\mathcal{S}_1 = \left\{ -1,\ 1,\ -i,\ 0,\ i,\ -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\}$$

and $\mathcal{S}_2 = \{-2,\ 2\}$. Let $f(z) = e^z$. It is clear that $\overline{E}_{f(z)}(\mathcal{S}_j) = \overline{E}_{f(z+c)}(\mathcal{S}_j)$, $(j = 1,\ 2)$ in *Theorem* 2.2 with $e^c = -1$, $c \in \mathbb{C}^*$ and note that $f(z) \not\equiv f(z+c)$.

3. Auxiliary definitions and some lemmas

It was *Fujimoto* [10], who first discovered a special property of a polynomial, reasonably called as critical injection property though initially *Fujimoto* [10] called it as property (H).

Definition 3.1. Let $\mathcal{P}(w)$ be a non-constant monic polynomial. We call $\mathcal{P}(w)$ a uniqueness polynomial if $\mathcal{P}(f) \equiv c\mathcal{P}(g)$ implies $f \equiv g$ for any non-constant meromorphic functions f and g and any non-zero constant c. We also call $\mathcal{P}(w)$ a uniqueness polynomial in a broad sense if $\mathcal{P}(f) \equiv \mathcal{P}(g)$ implies $f \equiv g$.

Next we recall here the property (H) and critically injective polynomial. Let $\mathcal{P}(w)$ be a monic polynomial without multiple zero whose derivative has mutually distinct k-zeros $e_1, e_2, \ldots, e_k$ with the multiplicities $q_1, q_2, \ldots, q_k$ respectively.

Now, the property $\mathcal{P}(e_l) \neq \mathcal{P}(e_m)$ for $1 \leq l < m \leq k$ is a known as property (H) and a polynomial $\mathcal{P}(w)$ satisfying this property is called critically injective polynomial.

Given meromorphic functions $f(z)$ and $f(z+c)$ we associate $\mathcal{F}$, $\mathcal{G}$ by

$$\mathcal{F} = \mathcal{P}(f), \ \ \mathcal{G} = \mathcal{P}(f(z+c)), \tag{3.1}$$

to $\mathcal{F}$, $\mathcal{G}$ we associate $\mathcal{H}$ and Φ by the following formulas

$$\mathcal{H} = \frac{\left(\frac{1}{\mathcal{F}}\right)''}{\left(\frac{1}{\mathcal{F}}\right)'} - \frac{\left(\frac{1}{\mathcal{G}}\right)''}{\left(\frac{1}{\mathcal{G}}\right)'} = \left(\frac{\mathcal{F}''}{\mathcal{F}'} - \frac{2\mathcal{F}'}{\mathcal{F}}\right) - \left(\frac{\mathcal{G}''}{\mathcal{G}'} - \frac{2\mathcal{G}'}{\mathcal{G}}\right), \tag{3.2}$$

$$\Phi = \frac{\mathcal{F}'}{\mathcal{F}} - \frac{\mathcal{G}'}{\mathcal{G}}. \tag{3.3}$$

Before proceeding to the actual proofs, we recall a few lemmas that take an important role in the reasoning.

Lemma 3.1. [15] Let g be a non-constant meromorphic function and let

$$\mathcal{R}^{\#}(g) = \frac{\sum_{i=1}^{n} a_i g^i}{\sum_{j=1}^{m} b_j g^j},$$

be an irreducible rational function in g with constant coefficients $\{a_i\}$, $\{b_j\}$, where $a_n \neq 0$ and $b_m \neq 0$. Then

$$T(r, \mathcal{R}^{\#}(g)) = \max\{n, m\}\, T(r, g) + S(r, g).$$

Lemma 3.2. [10] Let $\mathcal{P}(w)$ be a polynomial satisfying the property *(H)*. Then, $\mathcal{P}(w)$ is a uniqueness polynomial in a broad sense if and only if

$$\sum_{1 \le l < m \le k} q_l q_m > \sum_{l=1}^{k} q_l. \tag{3.4}$$

It can be easily verified that for the case $k \geq 4$, the condition (3.4) is always satisfied. Moreover, (3.4) holds when $\max\{q_1, q_2, q_3\} \geq 2$ for the case $k = 3$ and when $\min\{q_1, q_2\} \geq 2$ and $q_1 + q_2 \geq 5$ for the case $k = 2$.

4. Proofs of the theorems

In this section, we give the proofs of our main results.

Proof of Theorem 2.1**.** Let $f(z)$ and $f(z+c)$ be any two non-constant meromorphic functions. It is clear that

$$\mathcal{F}' = (f(z) - a)^n (f(z) - b)^4 f'(z) \text{ and}$$
$$\mathcal{G}' = (f(z+c) - a)^n (f(z+c) - b)^4 f'(z+c).$$

We now discuss the following two cases:
Case 1. There exists a $\lambda > 1$, $I \subset \mathbb{R}^+$ with measure of I as $+\infty$ such that

$$\begin{aligned} & 2\overline{N}\left(r, \frac{1}{f(z)-a}\right) + 2\overline{N}\left(r, \frac{1}{f(z)-b}\right) \\ \geq\ & \lambda\Big\{T(r, f(z)) + T(r, f(z+c))\Big\} + S(r, f(z)) + S(r, f(z+c)), \end{aligned} \tag{4.1}$$

where $r \to +\infty$, $r \in I$.

Let Φ is defined as in (3.3). Our aim is to show that $\Phi = 0$. Let if possible $\Phi \not\equiv 0$. Then since $n \geq 4$, so from the construction of Φ, we get

$$4\overline{N}\left(r, \frac{1}{f(z)-a}\right) + 4\overline{N}\left(r, \frac{1}{f(z)-b}\right) \leq N\left(r, \frac{1}{\Phi}\right). \tag{4.2}$$

The possible poles of Φ occur at the following points: (i) poles of $f(z)$, (ii) poles of $f(z+c)$, (iii) all the zeros of $\mathcal{F}$ of multiplicities ≥ 2 and (iv) all the zeros of $\mathcal{G}$ of multiplicities ≥ 2.

So we have

$$N(r,\Phi) \leq \overline{N}(r.f(z)) + \overline{N}_{(2}\left(r, \frac{1}{\mathcal{F}}\right) + \overline{N}(r.f(z+c)) + \overline{N}_{(2}\left(r, \frac{1}{\mathcal{G}}\right). \tag{4.3}$$

By using *First Fundamental Theorem* and (4.2), (4.3), we get

$$\begin{aligned} & 4\overline{N}\left(r, \frac{1}{f(z)-a}\right) + 4\overline{N}\left(r, \frac{1}{f(z)-b}\right) \qquad (4.4) \\ \leq\ & N\left(r, \frac{1}{\Phi}\right) \\ \leq\ & N(r,\Phi) \\ \leq\ & \overline{N}(r.f(z)) + \overline{N}_{(2}\left(r, \frac{1}{\mathcal{F}}\right) + \overline{N}(r.f(z+c)) + \overline{N}_{(2}\left(r, \frac{1}{\mathcal{G}}\right) \\ & +S(r,f(z)) + S(r,f(z+c)). \end{aligned}$$

Again since $\overline{E}_{f(z)}(\mathcal{S}_2) = \overline{E}_{f(z+c)}(\mathcal{S}_2)$, so we must have

$$\begin{aligned} & \overline{N}\left(r, \frac{1}{f(z)-a}\right) + \overline{N}\left(r, \frac{1}{f(z)-b}\right) \qquad (4.5) \\ =\ & \overline{N}\left(r, \frac{1}{f(z+c)-a}\right) + \overline{N}\left(r, \frac{1}{f(z+c)-b}\right). \end{aligned}$$

Adding $\overline{N}\left(r, \frac{1}{\mathcal{F}}\right) + \overline{N}\left(r, \frac{1}{\mathcal{G}}\right)$ on both sides of (4.4), we get

$$\begin{aligned} & 4\overline{N}\left(r, \frac{1}{f(z)-a}\right) + 4\overline{N}\left(r, \frac{1}{f(z)-b}\right) + \overline{N}\left(r, \frac{1}{\mathcal{F}}\right) \qquad (4.6) \\ & +\overline{N}\left(r, \frac{1}{\mathcal{G}}\right) \\ \leq\ & \overline{N}(r.f(z)) + N\left(r, \frac{1}{\mathcal{F}}\right) + \overline{N}(r.f(z+c)) + N\left(r, \frac{1}{\mathcal{G}}\right) \\ & +S(r,f(z)) + S(r,f(z+c)). \end{aligned}$$

Next using (4.5) in (4.6), we get

$$\begin{aligned}
&2\left\{\overline{N}\left(r,\frac{1}{f(z)-a}\right)+\overline{N}\left(r,\frac{1}{f(z)-b}\right)\right\} \qquad (4.7)\\
&\left\{\overline{N}\left(r,\frac{1}{f(z)-a}\right)+\overline{N}\left(r,\frac{1}{f(z)-b}\right)\right\}+\overline{N}\left(r,\frac{1}{\mathcal{F}}\right)+\overline{N}\left(r,\frac{1}{\mathcal{G}}\right)\\
&+\left\{\overline{N}\left(r,\frac{1}{f(z+c)-a}\right)+\overline{N}\left(r,\frac{1}{f(z+c)-b}\right)\right\}\\
\leq\; &\overline{N}(r.f(z))+N\left(r,\frac{1}{\mathcal{F}}\right)+\overline{N}(r.f(z+c))+N\left(r,\frac{1}{\mathcal{G}}\right)+S(r,f(z))\\
&+S(r,f(z+c)).
\end{aligned}$$

By applying *Second Fundamental Theorem*, we get

$$\begin{aligned}
&(n+5)\left\{T(r,f(z))+T(r,f(z+c))\right\} \qquad (4.8)\\
\leq\; &\overline{N}\left(r,\frac{1}{\mathcal{F}}\right)+\overline{N}\left(r,\frac{1}{f(z)-a}\right)+\overline{N}\left(r,\frac{1}{f(z)-b}\right)+\overline{N}\left(r,\frac{1}{\mathcal{G}}\right)\\
&+\overline{N}\left(r,\frac{1}{f(z+c)-a}\right)+\overline{N}\left(r,\frac{1}{f(z+c)-b}\right)+S(r,f(z))\\
&+S(r,f(z+c)).
\end{aligned}$$

Adding

$$2\overline{N}\left(r,\frac{1}{f(z)-a}\right)+2\overline{N}\left(r,\frac{1}{f(z)-b}\right)$$

both sides in (4.8) and using (4.7), we get

$$\begin{aligned}
&(n+5)\left\{T(r,f(z))+T(r,f(z+c))\right\}+2\overline{N}\left(r,\frac{1}{f(z)-a}\right)\\
&+2\overline{N}\left(r,\frac{1}{f(z)-b}\right)\\
\leq\; &N\left(r,\frac{1}{\mathcal{F}}\right)+N\left(r,\frac{1}{\mathcal{G}}\right)+\overline{N}(r,f(z))+\overline{N}(r,f(z+c))\\
&+S(r,f(z))+S(r,f(z+c))\\
\leq\; &(n+6)\left\{T(r,f(z))+T(r,f(z+c))\right\}.
\end{aligned}$$

i.e.,

$$2\overline{N}\left(r,\frac{1}{f(z)-a}\right)+2\overline{N}\left(r,\frac{1}{f(z)-b}\right)\leq\left\{T(r,f(z))+T(r,f(z+c))\right\},$$

which is not possible for $\lambda>1$ in view of (4.1).

Thus, we get $\Phi\equiv 0$. i.e., $\mathcal{F}\equiv\mathcal{A}\mathcal{G}$, for $\mathcal{A}\in\mathbb{C}\setminus\{0\}$. Using *Lemma* 3.1, we have

$$T(r,f(z))=T(r,f(z+c))+S(r,f(z)). \qquad (4.9)$$

Subcase 1.1. Let $\mathcal{A} \neq 1$.

So from the relation $\mathcal{F} \equiv \mathcal{A}\mathcal{G}$, we get

$$\mathcal{F} - \mathcal{A} \equiv \mathcal{A}(\mathcal{G} - 1). \tag{4.10}$$

A simple calculation shows that the polynomial $\mathcal{P}(z) - \mathcal{A}$ has all simple distinct roots and let them be σ_j $(j = 1, 2, \ldots, n+5)$ and all $\sigma_j \neq a,\ b$. Also we note that the polynomial $\mathcal{P}(z) - 1$ has roots as a of multiplicity $n+1$ and rest are δ_j $(j = 1, 2, 3, 4)$. Thus we see from (4.10) that

$$\begin{aligned} & \sum_{j=1}^{n+5} \overline{N}\left(r, \frac{1}{f(z) - \sigma_j}\right) \qquad (4.11) \\ = \ & \overline{N}\left(r, \frac{1}{f(z+c) - a}\right) + \sum_{j=1}^{4} \overline{N}\left(r, \frac{1}{f(z+c) - \delta_j}\right). \end{aligned}$$

By applying *Second Fundamental Theorem* and (4.9), we have

$$\begin{aligned} & (n+3)T(r, f(z)) \\ \leq \ & \sum_{j=1}^{n+5} \overline{N}\left(r, \frac{1}{f(z) - \sigma_j}\right) + S(r, f(z)) \\ \leq \ & \overline{N}\left(r, \frac{1}{f(z+c) - a}\right) + \sum_{j=1}^{4} \overline{N}\left(r, \frac{1}{f(z+c) - \delta_j}\right) + S(r, f(z)) \\ \leq \ & 5T(r, f(z)) + S(r, f(z)), \end{aligned}$$

which contradicts $n \geq 4$.

Subcase 1.2. Let $\mathcal{A} = 1$. i.e., we have $\mathcal{F} \equiv \mathcal{G}$. Thus we get $\mathcal{P}(f) \equiv \mathcal{P}(f(z+c))$. We see that the polynomial $\mathcal{P}(z) = \displaystyle\int_0^{z-a} (t-a)^n (t-b)^4 dt + 1$ satisfies the condition (H) and (3.4) since $\mathcal{P}'(z) = (z-a)^n (z-b)^4$, $k = 2$, $e_1 = a$, $e_2 = b$ and $q_1 = n \geq 4$, $q_2 = 4$. We next see that $\min\{q_1, q_2\} = \min\{n, 4\} \geq 2$ and $q_1 + q_2 = n + 4 \geq 5$. Therefore by *Lemma* 3.2, we see that the polynomial $\mathcal{P}(z)$ is a uniqueness polynomial in a broad sense. Hence the relation $\mathcal{P}(f) \equiv \mathcal{P}(f(z+c))$ implies $f(z) \equiv f(z+c)$.

Case 2. There exists $I \subset \mathbb{R}^+$ such that measure of I is $+\infty$ such that

$$\begin{aligned} & 2\overline{N}\left(r, \frac{1}{f(z) - a}\right) + 2\overline{N}\left(r, \frac{1}{f(z) - b}\right) \qquad (4.12) \\ \leq \ & \left(1 + \frac{1}{1000}\right)\left\{T(r, (z)f) + T(r, f(z+c))\right\} + S(r, (z)f) + S(r, f(z+c). \end{aligned}$$

We claim that $\mathcal{H} \equiv 0$. Suppose that $\mathcal{H} \not\equiv 0$. Next in view of the definition $\mathcal{H}$, we see that

$$\overline{N}_{1)}^{E}\left(r, \frac{1}{\mathcal{F}}\right) = \overline{N}_{1)}^{E}\left(r, \frac{1}{\mathcal{G}}\right) \leq N\left(r, \frac{1}{H}\right). \tag{4.13}$$

We see that the possible poles of $\mathcal{H}$ occur at the following points: (i) poles of $f(z)$, (ii) poles of $f(z+c)$, (iii) zeros of $f(z)$, (iv) 1-points of $f(z)$, (v) all those zeros of

$f'(z)$ which are not the zeros of $f(z)(f(z)-1)$ and (vi) all those zeros of $f'(z+c)$ which are not the zeros of $f(z+c)(f(z+c)-1)$. Thus we get

$$\begin{aligned} N(r,\mathcal{H}) &\leq \overline{N}(r,f(z))+\overline{N}\left(r,\frac{1}{f(z)-a}\right)+\overline{N}\left(r,\frac{1}{f(z)-b}\right) \\ &+ \overline{N}(r,f(z+c))+\overline{N}_0(r,0;f'(z))+\overline{N}_0(r,0;f'(z+c)), \end{aligned} \tag{4.14}$$

where $\overline{N}_0\left(r,\frac{1}{f'(z)}\right)$ is the reduced counting function of all those zeros of $f'(z)$ which are not the zeros of $(f(z)-a)(f(z)-b)$. Similarly $\overline{N}_0\left(r,\frac{1}{f'(z+c)}\right)$ is defined. Therefore using *First Fundamental Theorem*, we get

$$\begin{aligned} \overline{N}^E_{1)}\left(r,\frac{1}{\mathcal{F}}\right)r &\leq N\left(r,\frac{1}{\mathcal{H}}\right) \\ &\leq N(r,\mathcal{H}) \\ &\leq \overline{N}(r,f(z))+\overline{N}\left(r,\frac{1}{f(z)-a}\right)+\overline{N}\left(r,\frac{1}{f(z)-b}\right) \\ &+ \overline{N}(r,f(z+c))+\overline{N}_0(r,0;f'(z))+\overline{N}_0(r,0;f'(z+c)). \end{aligned} \tag{4.15}$$

We also note that

$$\overline{N}_{(2}\left(r,\frac{1}{\mathcal{F}}\right)\leq\overline{N}_0\left(r,\frac{1}{f'(z)}\right),\ \ \overline{N}_{(2}\left(r,\frac{1}{\mathcal{G}}\right)\leq\overline{N}_0\left(r,\frac{1}{f'(z+c)}\right).$$

We define

$$\Psi(z):=\frac{f'(z)}{[f(z)-a][(f(z)-b]}\frac{f'(z+c)}{[f(z+c)-a][f(z+c)-b}.$$

From the definition of Ψ and by using *First Fundamental Theorem* and (4.5), we get

$$\begin{aligned} & N_0\left(r,\frac{1}{f'(z)}\right)+N_0\left(r,\frac{1}{f'(z+c)}\right) \\ \leq\ & \overline{N}\left(r,\frac{1}{\Psi}\right) \\ \leq\ & \overline{N}(r,\Psi) \\ \leq\ & \overline{N}\left(r,\frac{1}{f(z)-a}\right)+\overline{N}\left(r,\frac{1}{f(z)-b}\right)+\overline{N}\left(r,\frac{1}{f(z+c)-a}\right) \\ & +\overline{N}\left(r,\frac{1}{f(z+c)-b}\right)+S(r,f(z))+S(r,f(z+c)) \\ \leq\ & 2\overline{N}\left(r,\frac{1}{f(z)-a}\right)+2\overline{N}\left(r,\frac{1}{f(z)-b}\right)+S(r,f(z))+S(r,f(z+c)). \end{aligned} \tag{4.16}$$

Adding

$$\overline{N}_{(2}\left(r,\frac{1}{\mathcal{F}}\right)+\overline{N}_{(2}\left(r,\frac{1}{\mathcal{G}}\right)+\overline{N}\left(r,\frac{1}{f(z)-a}\right)+\overline{N}\left(r,\frac{1}{f(z)-b}\right)$$

both sides of (4.15), we get

$$\begin{aligned}
&\overline{N}^{E}_{1)}\left(r,\frac{1}{\mathcal{F}}\right)+\overline{N}_{(2}\left(r,\frac{1}{\mathcal{F}}\right)+\overline{N}_{(2}\left(r,\frac{1}{\mathcal{G}}\right)+\overline{N}\left(r,\frac{1}{f(z)-a}\right) \qquad (4.17)\\
+\ &\overline{N}\left(r,\frac{1}{f(z)-b}\right)\\
\leq\ &\overline{N}(r,f(z))+2\overline{N}\left(r,\frac{1}{f(z)-a}\right)+2\overline{N}\left(r,\frac{1}{f(z)-b}\right)+\overline{N}(r,f(z+c))\\
+\ &2\overline{N}_0\left(r,\frac{1}{f'(z)}\right)+2\overline{N}_0\left(r,\frac{1}{f'(z+c)}\right).
\end{aligned}$$

i.e.,

$$\begin{aligned}
&\overline{N}\left(r,\frac{1}{\mathcal{F}}\right)+\overline{N}\left(r,\frac{1}{f(z)-a}\right)+\overline{N}\left(r,\frac{1}{f(z)-b}\right) \qquad (4.18)\\
\leq\ &\overline{N}(r,f(z))+6\overline{N}\left(r,\frac{1}{f(z)-a}\right)+6\overline{N}\left(r,\frac{1}{f(z)-b}\right)+\overline{N}(r,f(z+c))\\
+\ &S(r,f(z))+S(r,f(z+c)).
\end{aligned}$$

Similarly, we get

$$\begin{aligned}
&\overline{N}\left(r,\frac{1}{\mathcal{G}}\right)+\overline{N}\left(r,\frac{1}{f(z+c)-a}\right)+\overline{N}\left(r,\frac{1}{f(z+c)-b}\right) \qquad (4.19)\\
\leq\ &\overline{N}(r,f(z+c))+6\overline{N}\left(r,\frac{1}{f(z+c)-a}\right)+6\overline{N}\left(r,\frac{1}{f(z+c)-b}\right)\\
+\ &\overline{N}(r,f(z))+S(r,f(z))+S(r,f(z+c)).
\end{aligned}$$

By applying *Second Fundamental Theorem* and (4.12), (4.18) and (4.19), we get

$$\begin{aligned}
&(n+5)\Big\{T(r,f(z))+T(r,f(z+c))\Big\}\\
\leq\ &\overline{N}\left(r,\frac{1}{\mathcal{F}}\right)+\overline{N}(r,f(z))+\overline{N}\left(r,\frac{1}{f(z)-a}\right)+\overline{N}\left(r,\frac{1}{\mathcal{G}}\right)+\overline{N}(r,f(z+c))\\
&+\overline{N}\left(r,\frac{1}{f(z+c)-a}\right)+S(r,f(z))+S(r,f(z+c))\\
\leq\ &2\overline{N}(r,f(z))+2\overline{N}(r,f(z+c))+6\overline{N}\left(r,\frac{1}{f(z)-a}\right)\\
+\ &6\overline{N}\left(r,\frac{1}{f(z+c)-a}\right)+6\overline{N}\left(r,\frac{1}{f(z)-b}\right)+6\overline{N}\left(r,\frac{1}{f(z+c)-b}\right)\\
+\ &S(r,f(z))+S(r,f(z+c))\\
\leq\ &\left(8+\frac{6}{1000}\right)\Big\{T(r,f(z))+T(r,f(z+c))\Big\}++S(r,f(z))+S(r,f(z+c)),
\end{aligned}$$

which contradicts $n\geq 4$.

Therefore, we have $\mathcal{H} \equiv 0$. Thus we get

$$\frac{1}{\mathcal{F}} \equiv \frac{\mathcal{A}}{\mathcal{G}} + \mathcal{B}, \tag{4.20}$$

where $\mathcal{A}(\neq 0), \mathcal{B} \in \mathbb{C}$. In view of *Lemma* 3.1, we see from (4.20) that

$$T(r, f(z)) = T(r, f(z+c)) + S(r, f(z)). \tag{4.21}$$

Subcase 2.1. Let $\mathcal{B} \neq 0$. Thus we must have

$$\overline{N}(r, f(z)) = \overline{N}(r, \mathcal{F}) = \overline{N}\left(r, \frac{1}{\mathcal{G} + \frac{\mathcal{A}}{\mathcal{B}}}\right) \geq 3T(r, f(z+c)) + S(r, f(z+c)),$$

which is absurd in view of (4.21).

Subcase 2.2. So we have $\mathcal{B} = 0$. Therefore (4.20) reduces to $\mathcal{G} = \mathcal{A}\mathcal{F}$. Proceeding exactly same way as done in *Subcase 1.1*, we get $f(z) \equiv f(z+c)$. □

Proof of Theorem 2.2. Since $f(z)$ is a non-constant entire function, so we must have $\overline{N}(r, f(z)) = 0$ and hence $\overline{N}(r, f(z+c)) = 0$. Now keeping this in mind, the rest of the proof follows the proof of *Theorem* 2.1. □

5. An open question

Question 5.1. Is it possible to reduce the cardinalities further of two sets sharing problem (in case of IM sharing) for the periodicity of a meromorphic function f ?

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Analysis of fractional boundary value problem with non local flux multi-point conditions on a Caputo fractional differential equation

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Abstract. A brief analysis of boundary value problem of Caputo fractional differential equation with nonlocal flux multi-point boundary conditions has been done. The investigation depends on the Banach fixed point theorem, Krasnoselskii-Schaefer fixed point theorem due to Burton and Kirk, fixed point theorem due to O'Regan. Relevant examples illustrating the main results are also constructed.

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1. Introduction

In recent years, fractional differential equations are increasingly utilized to model many problems in biology, chemistry, engineering, physics, economic and other areas of applications. The fractional differential equations have become a useful tool for describing nonlinear phenomena of science and engineering models. Also, researchers found that fractional calculus was very suitable to describe long memory and hereditary properties of various materials and processes. we refer the reader to the texts [16]-[14], [8], [9]-[6], and the references cited therein.

Fractional differential equations have attracted considerable interest because of their ability to model complex artefacts. These equations capture non local relations in space and time with memory essentials. Due to extensive applications of FDEs in engineering and science, research in this area has grown significantly all around the world., for instance, see [18], [11], [15] and the references cited therein. Recently, much interest has been created in establishing the existence of solutions for various types of boundary value problem of fractional order with nonlocal multi-point boundary conditions. Nonlocal multi-point conditions involving Liouville-Caputo derivative, first of its kind was explored by Agarwal et.al. [1] on nonlinear fractional order boundary value problem. Ahmad et.al. [2]-[5], [3], [7] profound the idea of new kind of nonlocal multi-point boundary value problem of fractional integro-differential equations involving multi-point strips integral boundary conditions.

In this paper the existence and uniqueness of solutions for the below fractional differential equations with nonlocal multi-point boundary conditions are discussed. Consider the fractional differential equation

$$ {}^{C}\mathfrak{D}^{\delta} p(z) = k(z, p(z)), \qquad z \in \mathfrak{J} = [0,1], \quad n-1 < \delta \leq n, \tag{1.1}$$

supplemented with the nonlocal multi-point integral boundary conditions

$$\begin{aligned} &p(0) = \psi(p),\ p'(0) = \rho p'(\nu),\ p^{''}(0) = 0,\ p^{'''}(0) = 0, \cdots, p^{n-2}(0) = 0, \\ &p(1) = \lambda \int_0^{\varsigma} p(\sigma) d\sigma + \mu \sum_{j=1}^{m-2} \xi_j p(\zeta_j), \end{aligned} \tag{1.2}$$

where ${}^{C}\mathfrak{D}^{\delta}$ denote the Caputo fractional derivative and k: $\mathfrak{J} \times \mathbb{R}$ to $\mathbb{R}$ and ψ: $C(\mathfrak{J}, \mathbb{R})$ to $\mathbb{R}$, are given continuous functions, $0 < \nu < \varsigma < \zeta_1 < \zeta_2 < \cdots < \zeta_{m-2} < 1$, ξ_j, $j = 1, 2, \cdots, m-2$, ρ, λ, μ are positive real constants. The rest of the paper is organised as follows: The preliminaries section is devoted to some fundamental concepts of fractional calculus with basic lemma related to the given problem. In section 3, the existence and uniqueness of solutions are obtained based on Banach fixed point theorem, Krasnoselskii-Schaefer fixed point theorem due to Burton and Kirk, and fixed point theorem due to O'Regan and also the validation of the results is done by providing examples.

2. Preliminaries

In this section, we introduce some notations and definitions of fractional calculus.

Definition 2.1. The fractional integral of order δ with the lower limit zero for a function k is defined as

$$\mathfrak{I}^{\delta} k(z) = \frac{1}{\Gamma(\delta)} \int_0^z \frac{k(\sigma)}{(z-\sigma)^{1-\delta}} d\sigma, \quad z > 0,\ \delta > 0,$$

provided the right hand-side is point-wise defined on $[0, \infty)$, where $\Gamma(\cdot)$ is the gamma function, which is defined by $\Gamma(\delta) = \int_0^{\infty} z^{\delta-1} e^{-z} dz$.

Definition 2.2. The Riemann-Liouville fractional derivative of order $\delta > 0$, $n-1 < \delta < n$, $n \in \mathbb{N}$ is defined as

$$\mathfrak{D}^{\delta}_{0+} k(z) = \frac{1}{\Gamma(n-\delta)} \left(\frac{d}{dz}\right)^n \int_0^z (z-\sigma)^{n-\delta-1} k(\sigma) d\sigma,$$

where the function $k(z)$ has absolutely continuous derivative up to order $(n-1)$.

Definition 2.3. The Caputo derivative of order δ for a function $k : [0,\infty) \to \mathbb{R}$ can be written as

$$^{C}\mathfrak{D}^{\delta}k(z) \;=\; \mathfrak{D}^{\delta}_{0+}\Bigg(k(z) - \sum_{j=0}^{n-1}\frac{z^j}{j!}k^{(j)}(0)\Bigg), \quad z>0, \quad n-1<\delta<n.$$

Remark 2.4. If $k(z) \in C^n[0,\infty)$, then

$$\begin{aligned} ^{C}\mathfrak{D}^{\delta}k(z) &= \frac{1}{\Gamma(n-\delta)}\int_0^z \frac{k^n(\sigma)}{(z-\sigma)^{\delta+1-n}}d\sigma \\ &= \mathfrak{I}^{n-\delta}k^n(z), \quad z>0, n-1<\delta<n. \end{aligned}$$

Lemma 2.5. *For $\delta > 0$, the general solution of the fractional differential equation $^{C}\mathfrak{D}^{\delta}p(z) = 0$ is given by*

$$p(z) = a_0 + a_1 z + \cdots + a_{n-1}z^{n-1},$$

where $a_i \in \mathbb{R}$, $i = 1, 2, \ldots, n-1$ $(n = [\delta] + 1)$.

In view of Lemma 2.5, it follows that

$$\mathfrak{I}^{\delta C}\mathfrak{D}^{\delta}p(z) = p(z) + a_0 + a_1 z + \cdots + a_{n-1}z^{n-1},$$

for some $a_i \in \mathbb{R}$, $i = 1, 2, \ldots, n-1$ $(n = [\delta] + 1)$.

Next, we present an auxiliary lemma which plays a key role in the sequel.

Lemma 2.6. *For any $\hat{k} \in C(\mathfrak{J}, \mathbb{R})$, the solution of the linear fractional differential equation*

$$^{C}\mathfrak{D}^{\delta}p(z) = \hat{k}(z), \quad n-1<\delta\le n, \tag{2.1}$$

*supplemented with the boundary conditions (*1.2*) is given by*

$$\begin{aligned} p(z) &= \int_0^z \frac{(z-\sigma)^{\delta-1}}{\Gamma(\delta)}\hat{k}(\sigma)d\sigma \\ &+ \Big[1 + \frac{(z\upsilon_1 + z^{n-1}\varpi_1)}{\vartheta}(\lambda\delta + \mu\sum_{j=1}^{m-2}\xi_j - 1)\Big]\psi(p) \\ &+ \frac{\rho(z\upsilon_2 - z^{n-1}\varpi_2)}{\vartheta}\Big[\int_0^{\nu}\frac{(\nu-\sigma)^{\delta-2}}{\Gamma(\delta-1)}\hat{k}(\sigma)d\sigma\Big] \\ &+ \frac{(z\upsilon_1 + z^{n-1}\varpi_1)}{\vartheta}\Big[\lambda\int_0^{\varsigma}\Bigg(\int_0^{\sigma}\frac{(\sigma-\theta)^{\delta-1}}{\Gamma(\delta)}\hat{k}(\theta)d\theta\Bigg)d\sigma \\ &+ \mu\sum_{j=1}^{m-2}\xi_j\int_0^{\zeta_j}\frac{(\zeta_j-\sigma)^{\delta-1}}{\Gamma(\delta)}\hat{k}(\sigma)d\sigma - \int_0^1\frac{(1-\sigma)^{\delta-1}}{\Gamma(\delta)}\hat{k}(\sigma)d\sigma\Big] \end{aligned} \tag{2.2}$$

where

$$\varpi_1 = 1-\rho, \quad \varpi_2 = 1-\frac{\lambda\delta^2}{2}-\mu\sum_{j=1}^{m-2}\xi_j\zeta_j \tag{2.3}$$

$$\upsilon_1 = (n-1)\rho\delta^{n-2}, \quad \upsilon_2 = 1-\frac{\lambda\delta^n}{n}-\mu\sum_{j=1}^{m-2}\xi_j\zeta_j^{n-1} \tag{2.4}$$

$$\vartheta = \varpi_1\upsilon_2+\varpi_2\upsilon_1 \neq 0, \tag{2.5}$$

Proof. It is evident that the general solution of the fractional differential equations in (2.1) can be written as

$$p(z) = \int_0^z \frac{(z-\sigma)^{\delta-1}}{\Gamma(\delta)}\hat{k}(\sigma)d\sigma + a_0 + a_1 z + a_2 z^2 + \cdots + a_{n-1}z^{n-1} \tag{2.6}$$

where $a_i \in \mathbb{R}$, $(i = 0,1,2,...,(n-1))$ are arbitrary constants. Using the boundary conditions given by (1.2) in (2.6), we get $a_0 = \psi(p)$. On using the notations (2.3)-(2.5) along with (1.2) in (2.6), we get

$$a_1\varpi_1 - a_{n-1}\upsilon_1 = \rho\int_0^\nu \frac{(\nu-\sigma)^{\delta-2}}{\Gamma(\delta-1)}\hat{k}(\sigma)d\sigma \tag{2.7}$$

$$\begin{aligned} a_1\varpi_2 + a_{n-1}\upsilon_2 &= \lambda\int_0^\varsigma\left(\int_0^\sigma \frac{(\sigma-\theta)^{\delta-1}}{\Gamma(\delta)}\hat{k}(\theta)d\theta\right)d\sigma \\ &\quad +\mu\sum_{j=1}^{m-2}\xi_j\int_0^{\zeta_j}\frac{(\zeta_j-\sigma)^{\delta-1}}{\Gamma(\delta)}\hat{k}(\sigma)d\sigma \\ &\quad -\int_0^1\frac{(1-\sigma)^{\delta-1}}{\Gamma(\delta)}\hat{k}(\sigma)d\sigma. \end{aligned} \tag{2.8}$$

Solving the system (2.7) and (2.8) for a_1, a_{n-1}, we get

$$\begin{aligned} a_1 = \frac{1}{\vartheta}\Bigg[&\upsilon_2\left(\rho\int_0^\nu \frac{(\nu-\sigma)^{\delta-2}}{\Gamma(\delta-1)}\hat{k}(\sigma)d\sigma\right) \\ &+\upsilon_1\Bigg(\lambda\int_0^\varsigma\left(\int_0^\sigma \frac{(\sigma-\theta)^{\delta-1}}{\Gamma(\delta)}\hat{k}(\theta)d\theta\right)d\sigma \\ &+\mu\sum_{j=1}^{m-2}\xi_j\int_0^{\zeta_j}\frac{(\zeta_j-\sigma)^{\delta-1}}{\Gamma(\delta)}\hat{k}(\sigma)d\sigma + \psi(p)\Big(\lambda\delta+\mu\sum_{j=1}^{m-2}\xi_j - 1\Big) \\ &-\int_0^1\frac{(1-\sigma)^{\delta-1}}{\Gamma(\delta)}\hat{k}(\sigma)d\sigma\Bigg)\Bigg] \end{aligned} \tag{2.9}$$

$$a_{n-1} = \frac{-1}{\vartheta}\Bigg[\varpi_2\Bigg(\rho\int_0^{\nu}\frac{(\nu-\sigma)^{\delta-2}}{\Gamma(\delta-1)}\hat{k}(\sigma)d\sigma\Bigg)$$
$$+\varpi_1\Bigg(\lambda\int_0^{\varsigma}\Bigg(\int_0^{\sigma}\frac{(\sigma-\theta)^{\delta-1}}{\Gamma(\delta)}\hat{k}(\theta)d\theta\Bigg)d\sigma$$
$$+\mu\sum_{j=1}^{m-2}\xi_j\int_0^{\zeta_j}\frac{(\zeta_j-\sigma)^{\delta-1}}{\Gamma(\delta)}\hat{k}(\sigma)d\sigma+\psi(p)\Big(\lambda\delta+\mu\sum_{j=1}^{m-2}\xi_j-1\Big)$$
$$-\int_0^1\frac{(1-\sigma)^{\delta-1}}{\Gamma(\delta)}\hat{k}(\sigma)d\sigma\Bigg)\Bigg]. \tag{2.10}$$

Substituting the values of a_0, a_1, a_{n-1} in (2.6), we get the solution (2.2). This completes the proof.

3. Main results

We denote by $\mathfrak{G} = C(\mathfrak{J},\mathbb{R})$ be the Banach space of all continuous functions from $\mathfrak{J}\to\mathbb{R}$, equipped with the norm defined by

$$\|p\| = \sup_{z\in\mathfrak{J}}|p(z)|,\ \ z\in\mathfrak{J}\}.$$

Also by $\mathfrak{L}^1(\mathfrak{J},\mathbb{R})$, we denote the Banach space of measurable functions $p:\mathfrak{J}\to\mathbb{R}$ which are Lebesgue integral and normed by

$$\|p\|_{\mathfrak{L}^1} = \int_0^1 |p(z)|dz.$$

In view of Lemma 2.6, we define an operator $\mathfrak{T}:\mathfrak{G}\to\mathfrak{G}$ associated with problem (1.1) as

$$\begin{aligned}(\mathfrak{T}p)(z) \;=\; & \int_0^z\frac{(z-\sigma)^{\delta-1}}{\Gamma(\delta)}k(\sigma,p(\sigma))d\sigma\\ &+\Big[1+\frac{(zv_1+z^{n-1}\varpi_1)}{\vartheta}(\lambda\delta+\mu\sum_{j=1}^{m-2}\xi_j-1)\Big]\psi(p)\\ &+\frac{\rho(zv_2-z^{n-1}\varpi_2)}{\vartheta}\Big[\int_0^{\nu}\frac{(\nu-\sigma)^{\delta-2}}{\Gamma(\delta-1)}k(\sigma,p(\sigma))d\sigma\Big]\\ &+\frac{(zv_1+z^{n-1}\varpi_1)}{\vartheta}\Big[\lambda\int_0^{\varsigma}\Bigg(\int_0^{\sigma}\frac{(\sigma-\theta)^{\delta-1}}{\Gamma(\delta)}k(\theta,p(\theta))d\theta\Bigg)d\sigma\\ &+\mu\sum_{j=1}^{m-2}\xi_j\int_0^{\zeta_j}\frac{(\zeta_j-\sigma)^{\delta-1}}{\Gamma(\delta)}k(\sigma,p(\sigma))d\sigma\\ &-\int_0^1\frac{(1-\sigma)^{\delta-1}}{\Gamma(\delta)}k(\sigma,p(\sigma))d\sigma\Big]\end{aligned} \tag{3.1}$$

Let us define $\mathfrak{T}_1$,$\mathfrak{T}_2 : \mathfrak{G} \to \mathfrak{G}$ by

$$\begin{aligned}(\mathfrak{T}_1 p)(z) \;=\; & \int_0^z \frac{(z-\sigma)^{\delta-1}}{\Gamma(\delta)} k(\sigma, p(\sigma))d\sigma \\ & + \frac{\rho(z\upsilon_2 - z^{n-1}\varpi_2)}{\vartheta}\Big[\int_0^\nu \frac{(\nu-\sigma)^{\delta-2}}{\Gamma(\delta-1)} k(\sigma, p(\sigma))d\sigma\Big] \\ & + \frac{(z\upsilon_1 + z^{n-1}\varpi_1)}{\vartheta}\Big[\lambda \int_0^\varsigma \left(\int_0^\sigma \frac{(\sigma-\theta)^{\delta-1}}{\Gamma(\delta)} k(\theta, p(\theta))d\theta\right) d\sigma \\ & + \mu \sum_{j=1}^{m-2} \xi_j \int_0^{\zeta_j} \frac{(\zeta_j-\sigma)^{\delta-1}}{\Gamma(\delta)} k(\sigma, p(\sigma))d\sigma \\ & - \int_0^1 \frac{(1-\sigma)^{\delta-1}}{\Gamma(\delta)} k(\sigma, p(\sigma))d\sigma\Big] \end{aligned} \tag{3.2}$$

and

$$(\mathfrak{T}_2 p)(z) = \Big[1 + \frac{(z\upsilon_1 + z^{n-1}\varpi_1)}{\vartheta}(\lambda\delta + \mu\sum_{j=1}^{m-2}\xi_j - 1)\Big]\psi(p) \tag{3.3}$$

In the sequel, we use the notations:

$$\widehat{\eta} = \frac{1}{\Gamma(\delta+1)}\left[1 + \frac{\rho|(\upsilon_2 - \varpi_2)|\nu^{\delta-1}}{\vartheta\delta} + \frac{(\upsilon_1 + \varpi_1)}{\vartheta}\Big(\frac{\lambda\varsigma^{\delta+1}}{\delta+1} + \mu\sum_{j=1}^{m-2}\xi_j\zeta_j^\delta + 1\Big)\right] \tag{3.4}$$

and

$$\widehat{\omega} = 1 + \frac{(\upsilon_1 + \varpi_1)}{\vartheta}\Big(\lambda\delta + \mu\sum_{j=1}^{m-2}\xi_j + 1\Big) \tag{3.5}$$

Theorem 3.1. *The continuous function k defined from $\mathfrak{J} \times \mathbb{R}$ to $\mathbb{R}$. Let us speculate that*
$(\mathfrak{E}_1)$ $|k(z,p) - k(z,q)| \le \mathfrak{S}\|p - q\|, \ \forall z \in \mathfrak{J}, \mathfrak{S} > 0, p, q \in \mathbb{R}$.
$(\mathfrak{E}_2)$ *The continuous function ψ defined from $C(\mathfrak{J},\mathbb{R}) \to \mathbb{R}$ satisfying the condition:* $|\psi(v) - \psi(w)| \le \varepsilon\|v - w\|, \varepsilon\widehat{\omega} < 1, \ \forall\, v, w \in C(\mathfrak{J},\mathbb{R}), \varepsilon > 0$.
$(\mathfrak{E}_3)$ $\Theta := \mathfrak{S}\widehat{\eta} + \varepsilon\widehat{\omega} < 1$. *Then the boundary value problem (*1.1*)-(*1.2*) has unique solution on $\mathfrak{J}$.*

Proof. For $p, q \in \mathfrak{G}$ and for each $z \in \mathfrak{J}$, from the definition of $\mathfrak{T}$ and assumptions $(\mathfrak{E}_1)$ and $(\mathfrak{E}_2)$. We obtain

$$|(\mathfrak{T}p)(z) - (\mathfrak{T}q)(z)| \le \sup_{z\in\mathfrak{J}}\Bigg\{\int_0^z \frac{(z-\sigma)^{\delta-1}}{\Gamma(\delta)}|k(\sigma,p(\sigma)) - k(\sigma,q(\sigma))|d\sigma$$

$$+\Big|\Big[1 + \frac{(z\upsilon_1 + z^{n-1}\varpi_1)}{\vartheta}(\lambda\delta + \mu\sum_{j=1}^{m-2}\xi_j - 1)\Big]\Big||\psi(p) - \psi(q)|$$

$$+\Big|\frac{\rho(z\upsilon_2 - z^{n-1}\varpi_2)}{\vartheta}\Big|\Big[\int_0^\nu \frac{(\nu-\sigma)^{\delta-2}}{\Gamma(\delta-1)}|k(\sigma,p(\sigma)) - k(\sigma,q(\sigma))|d\sigma\Big]$$

$$+\Big|\frac{(z\upsilon_1+z^{n-1}\varpi_1)}{\vartheta}\Big|\Big[\lambda\int_0^{\varsigma}\bigg(\int_0^{\sigma}\frac{(\sigma-\theta)^{\delta-1}}{\Gamma(\delta)}|k(\theta,p(\theta))-k(\theta,q(\theta))|d\theta\bigg)d\sigma$$

$$+\mu\sum_{j=1}^{m-2}\xi_j\int_0^{\zeta_j}\frac{(\zeta_j-\sigma)^{\delta-1}}{\Gamma(\delta)}|k(\sigma,p(\sigma))-k(\sigma,q(\sigma))|d\sigma$$

$$+\int_0^1\frac{(1-\sigma)^{\delta-1}}{\Gamma(\delta)}|k(\sigma,p(\sigma))-k(\sigma,q(\sigma))|d\sigma\Big]\bigg\}$$

$$\leq\int_0^z\frac{(z-\sigma)^{\delta-1}}{\Gamma(\delta)}(\mathfrak{S}\|p-q\|)d\sigma$$

$$+\Big|\Big[1+\frac{(z\upsilon_1+z^{n-1}\varpi_1)}{\vartheta}(\lambda\delta+\mu\sum_{j=1}^{m-2}\xi_j-1)\Big]\Big||\psi(p)-\psi(q)|$$

$$+\Big|\frac{\rho(z\upsilon_2-z^{n-1}\varpi_2)}{\vartheta}\Big|\Big[\int_0^{\nu}\frac{(\nu-\sigma)^{\delta-2}}{\Gamma(\delta-1)}(\mathfrak{S}\|p-q\|)d\sigma\Big]$$

$$+\Big|\frac{(z\upsilon_1+z^{n-1}\varpi_1)}{\vartheta}\Big|\Big[\lambda\int_0^{\varsigma}\bigg(\int_0^{\sigma}\frac{(\sigma-\theta)^{\delta-1}}{\Gamma(\delta)}(\mathfrak{S}\|p-q\|)d\theta\bigg)d\sigma$$

$$+\mu\sum_{j=1}^{m-2}\xi_j\int_0^{\zeta_j}\frac{(\zeta_j-\sigma)^{\delta-1}}{\Gamma(\delta)}(\mathfrak{S}\|p-q\|)d\sigma+\int_0^1\frac{(1-\sigma)^{\delta-1}}{\Gamma(\delta)}(\mathfrak{S}\|p-q\|)d\sigma\Big]$$

$$\leq\frac{\mathfrak{S}}{\Gamma(\delta+1)}\left[1+\frac{\rho|(\upsilon_2-\varpi_2)|\nu^{\delta-1}}{\vartheta\delta}+\frac{(\upsilon_1+\varpi_1)}{\vartheta}\Big(\frac{\lambda\varsigma^{\delta+1}}{\delta+1}+\mu\sum_{j=1}^{m-2}\xi_j\zeta_j^{\delta}+1\Big)\right]\|p-q\|$$

$$+\left[1+\frac{(\upsilon_1+\varpi_1)}{\vartheta}\Big(\lambda\delta+\mu\sum_{j=1}^{m-2}\xi_j+1\Big)\right]\varepsilon\|p-q\|\leq(\mathfrak{S}\widehat{\eta}+\varepsilon\widehat{\omega})\|p-q\|.$$

Hence

$$\|(\mathfrak{T}p)-(\mathfrak{T}q)\| \quad\leq\quad \Theta\|p-q\|.$$

As $\Theta<1$ by $(\mathfrak{E}_3)$, the operator $\mathfrak{T}:\mathfrak{G}\to\mathfrak{G}$ is a contraction. Hence the conclusion of the theorem follows by the Banach fixed point theorem. □

Example 3.2. Consider the fractional differential equation given by

$$^{C}\mathfrak{D}^{\frac{7}{3}}p(z)=\sin z+\frac{e^{-z}\sin p(z)}{4\sqrt{z^6+16}},\quad z\in\mathfrak{J}, \tag{3.6}$$

subject to the boundary conditions

$$p(0)=\frac{1}{10}p(z),\ p'(0)=\frac{1}{4}x'\Big(\frac{1}{5}\Big)\ p(1)=\int_0^{\frac{1}{3}}p(\sigma)d\sigma+\sum_{j=1}^{4}\xi_jp(\zeta_j). \tag{3.7}$$

Here

$$2<\delta\leq3,\ \lambda=\mu=1,\ \rho=\frac{1}{4},\ \nu=\frac{1}{5},\ \varsigma=\frac{1}{3},$$
$$\xi_1=\frac{1}{5},\ \xi_2=\frac{1}{7},\ \xi_3=\frac{1}{6},\ \xi_4=\frac{1}{8},$$

$$\zeta_1 = \frac{1}{2},\ \zeta_2 = \frac{1}{4},\ \zeta_3 = \frac{1}{3},\ \zeta_4 = \frac{1}{5}.$$

Using the given data, we find that

$$|k(z, p(z))| = \sin z + \frac{e^{-z} \sin p(z)}{4\sqrt{z^6 + 16}}, \quad \psi(p) \ = \ \frac{1}{10} p(z).$$

Since

$$|k(z, p) - k(z, q) \leq \frac{1}{16} \|p - q\|,$$

$$|\psi(v) - \psi(w)| \leq \frac{1}{10} \|v - w\|,$$

therefore, $(\mathfrak{E}_1)$ and $(\mathfrak{E}_2)$ are respectively satisfied with $\mathfrak{S} = \frac{1}{16}$ and $\varepsilon = \frac{1}{10}$. With the given data, we find that $\widehat{\eta} = 5.18462$, $\widehat{\omega} = 2.62014$, it is found that

$$\Theta := \mathfrak{S}\widehat{\eta} + \varepsilon\widehat{\omega} \cong 0.586053 < 1.$$

Thus, the assumptions of Theorem 3.1 hold and the problem (3.6)-(3.7) has at most one solution on $\mathfrak{J}$.

Theorem 3.3. *Let $\mathfrak{Y}$ be a Banach space, and $\mathfrak{H}_1, \mathfrak{H}_2 : \mathfrak{Y} \to \mathfrak{Y}$ be two operators such that $\mathfrak{H}_1$ is a contraction and $\mathfrak{H}_2$ is completely continuous. Then either*
(i) the operator equation $u = \mathfrak{H}_1(u) + \mathfrak{H}_2(u)$ has a solution, or
(ii) the set $\mathfrak{F} = \{w \in \mathfrak{Y} : \kappa\mathfrak{H}_1(\frac{w}{\kappa}) + \kappa\mathfrak{H}_2(w) = w\}$ is unbounded for $\kappa \in (0, 1)$.

Theorem 3.4. *The continuous function k defined from $\mathfrak{J} \times \mathbb{R}$ to $\mathbb{R}$ and condition $(\mathfrak{E}_2)$ hold. Also let us understand that:*
$(\mathfrak{E}_4)$ $\psi(0) = 0$.
$(\mathfrak{E}_5)$ *there exists a function $x \in \mathfrak{L}^1(\mathfrak{J}, \mathbb{R}_+)$ such that $|k(z, v)| \leq x(z)$, for almost everywhere each $z \in \mathfrak{J}$, and each $v \in \mathbb{R}$.*
Then the problem (1.1)-(1.2) has at least one solution on $\mathfrak{J}$.

Proof. To transform the problem (1.1)-(1.2) into a fixed point problem. we consider the map $\mathfrak{T} : \mathfrak{G} \to \mathfrak{G}$ given by $(\mathfrak{T}p)(z) = (\mathfrak{T}_1 p)(z) + (\mathfrak{T}_2 p)(z),\ z \in \mathfrak{J}$, where $\mathfrak{T}_1$ and $\mathfrak{T}_2$ are defined by (3.2) and (3.3) respectively.
We shall show that the operators $\mathfrak{T}_1$ and $\mathfrak{T}_2$ satisfy all the conditions of Theorem 3.3.
Step 1. The operator $\mathfrak{T}_1$ defined by (3.2) is continuous.
Let $p_n \subset \mathfrak{B}_\theta = \{p \in \mathfrak{G} : \|p\| \leq \theta\}$ with $\|p_n - p\| \to 0$.
Then the limit $\|p_n(z) - p(z)\| \to 0$ is uniformly valid on $\mathfrak{J}$. From the uniform continuity of $k(z, p)$ on the compact set $\mathfrak{J} \times [-\theta, \theta]$, it follows that $\|k(z, p_n(z)) - k(z, p(z))\| \to 0$ uniformly on $\mathfrak{J}$. Hence $\|\mathfrak{T}_1 p_n - \mathfrak{T}_1 p\| \to 0$ as $n \to \infty$ which implies that the operator $\mathfrak{T}_1$ is continuous.
Step 2. The operator $\mathfrak{T}_1$ maps bounded sets into bounded sets in $\mathfrak{G}$.
It is indeed enough to show that for any $\theta > 0$ there exists a positive constant $\mathfrak{S}$ such that for each

$$p \in \mathfrak{B}_\theta = \{p \in \mathfrak{G} : \|p\| \leq \theta\},$$

we have

$$\|\mathfrak{T}_1 p\| \leq \mathfrak{Q}.$$

Let $p \in \mathfrak{B}_\theta$. Then

$$\begin{aligned}
\|\mathfrak{T}_1 p\| \leq & \int_0^z \frac{(z-\sigma)^{\delta-1}}{\Gamma(\delta)} |k(\sigma,p(\sigma))| d\sigma \\
& + \frac{\rho(z\upsilon_2 - z^{n-1}\varpi_2)}{\vartheta} \Big[\int_0^\nu \frac{(\nu-\sigma)^{\delta-2}}{\Gamma(\delta-1)} |k(\sigma,p(\sigma))| d\sigma \Big] \\
& + \frac{(z\upsilon_1 + z^{n-1}\varpi_1)}{\vartheta} \Big[\lambda \int_0^\varsigma \left(\int_0^\sigma \frac{(\sigma-\theta)^{\delta-1}}{\Gamma(\delta)} |k(\theta,p(\theta))| d\theta \right) d\sigma \\
& + \mu \sum_{j=1}^{m-2} \xi_j \int_0^{\zeta_j} \frac{(\zeta_j-\sigma)^{\delta-1}}{\Gamma(\delta)} |k(\sigma,p(\sigma))| d\sigma \\
& + \int_0^1 \frac{(1-\sigma)^{\delta-1}}{\Gamma(\delta)} |k(\sigma,p(\sigma))| d\sigma \Big] \\
\leq & \int_0^z \frac{(z-\sigma)^{\delta-1}}{\Gamma(\delta)} x(\sigma) d\sigma \\
& + \frac{\rho(z\upsilon_2 - z^{n-1}\varpi_2)}{\vartheta} \Big[\int_0^\nu \frac{(\nu-\sigma)^{\delta-2}}{\Gamma(\delta-1)} x(\sigma) d\sigma \Big] \\
& + \frac{(z\upsilon_1 + z^{n-1}\varpi_1)}{\vartheta} \Big[\lambda \int_0^\varsigma \left(\int_0^\sigma \frac{(\sigma-\theta)^{\delta-1}}{\Gamma(\delta)} x(\theta) d\theta \right) d\sigma \\
& + \mu \sum_{j=1}^{m-2} \xi_j \int_0^{\zeta_j} \frac{(\zeta_j-\sigma)^{\delta-1}}{\Gamma(\delta)} x(\sigma) d\sigma + \int_0^1 \frac{(1-\sigma)^{\delta-1}}{\Gamma(\delta)} x(\sigma) d\sigma \Big] \\
\leq & \frac{\|x\|}{\Gamma(\delta+1)} \Bigg[1 + \frac{\rho|(\upsilon_2 - \varpi_2)|\nu^{\delta-1}}{\vartheta\delta} \\
& + \frac{(\upsilon_1 + \varpi_1)}{\vartheta} \Big(\frac{\lambda\varsigma^{\delta+1}}{\delta+1} + \mu \sum_{j=1}^{m-2} \xi_j \zeta_j^\delta + 1 \Big) \Bigg] := \mathfrak{Q}
\end{aligned}$$

Step 3. The operator $\mathfrak{T}_1$ maps bounded sets into equicontinuous sets in $\mathfrak{G}$. Let $\varrho_1, \varrho_2 \in \mathfrak{J}$ with $\varrho_1 < \varrho_2$ and $p \in \mathfrak{B}_\theta$, we obtain

$$|(\mathfrak{T}_1 p)(\varrho_2) - (\mathfrak{T}_1 p)(\varrho_1)| \leq \left| \int_0^{\varrho_1} \frac{[(\varrho_2-\sigma)^{\delta-1} - (\varrho_1-\sigma)^{\delta-1}]}{\Gamma(\delta)} \times k(\sigma,p(\sigma)) d\sigma \right|$$

$$+ \left| \int_{\varrho_1}^{\varrho_2} \frac{(\varrho_2-\sigma)^{\delta-1}}{\Gamma(\delta)} k(\sigma,p(\sigma)) d\sigma \right|$$

$$+ \left| \frac{\rho((\varrho_2-\varrho_1)\upsilon_2 - (\varrho_2^{n-1} - \varrho_1^{n-1})\varpi_2)}{\vartheta} \right| \Big[\int_0^\nu \frac{(\nu-\sigma)^{\delta-2}}{\Gamma(\delta-1)} |k(\sigma,p(\sigma))| d\sigma \Big]$$

$$+ \frac{((\varrho_2-\varrho_1)\upsilon_1 + (\varrho_2^{n-1} - \varrho_1^{n-1})\varpi_1)}{\vartheta} \Big[\lambda \int_0^\varsigma \left(\int_0^\sigma \frac{(\sigma-\theta)^{\delta-1}}{\Gamma(\delta)} |k(\theta,p(\theta))| d\theta \right) d\sigma$$

$$+\mu\sum_{j=1}^{m-2}\xi_j\int_0^{\zeta_j}\frac{(\zeta_j-\sigma)^{\delta-1}}{\Gamma(\delta)}|k(\sigma,p(\sigma))|d\sigma+\int_0^1\frac{(1-\sigma)^{\delta-1}}{\Gamma(\delta)}|k(\sigma,p(\sigma))|d\sigma\Big]$$

$$\leq\left|\int_0^{\varrho_1}\frac{[(\varrho_2-\sigma)^{\delta-1}-(\varrho_1-\sigma)^{\delta-1}]}{\Gamma(\delta)}\times x(\sigma)d\sigma\right|+\left|\int_{\varrho_1}^{\varrho_2}\frac{(\varrho_2-\sigma)^{\delta-1}}{\Gamma(\delta)}x(\sigma)d\sigma\right|$$

$$+\left|\frac{\rho((\varrho_2-\varrho_1)\upsilon_2-(\varrho_2^{n-1}-\varrho_1^{n-1})\varpi_2)}{\vartheta}\right|\Big[\int_0^{\nu}\frac{(\nu-\sigma)^{\delta-2}}{\Gamma(\delta-1)}|x(\sigma)|d\sigma\Big]$$

$$+\frac{((\varrho_2-\varrho_1)\upsilon_1+(\varrho_2^{n-1}-\varrho_1^{n-1})\varpi_1)}{\vartheta}\Big[\lambda\int_0^{\varsigma}\left(\int_0^{\sigma}\frac{(\sigma-\theta)^{\delta-1}}{\Gamma(\delta)}|x(\theta)|d\theta\right)d\sigma$$

$$+\mu\sum_{j=1}^{m-2}\xi_j\int_0^{\zeta_j}\frac{(\zeta_j-\sigma)^{\delta-1}}{\Gamma(\delta)}|x(\sigma)|d\sigma+\int_0^1\frac{(1-\sigma)^{\delta-1}}{\Gamma(\delta)}|x(\sigma)|d\sigma\Big]$$

$$\leq\frac{\|x\|}{\Gamma(\delta+1)}\Bigg[[2(\varrho_2-\varrho_1)^{\delta}+(\varrho_2^{\delta}-\varrho_1^{\delta})]+\frac{\rho((\varrho_2-\varrho_1)\upsilon_2-(\varrho_2^{n-1}-\varrho_1^{n-1})\varpi_2)\nu^{\delta-1}}{\vartheta\delta}$$

$$+\frac{((\varrho_2-\varrho_1)\upsilon_1+(\varrho_2^{n-1}-\varrho_1^{n-1})\varpi_1)}{\vartheta}\Big(\frac{\lambda\varsigma^{\delta+1}}{\delta+1}+\mu\sum_{j=1}^{m-2}\xi_j\zeta_j^{\delta}+1\Big)\Bigg]$$

which is independent of p and tends to zero as $\varrho_2-\varrho_1\to 0$. Thus, $\mathfrak{T}_1$ is equicontinuous.
Step 4. The operator $\mathfrak{T}_2$ defined by (3.3) is continuous and Θ- contractive.
To show the continuity of $\mathfrak{T}_2$ for $z\in\mathfrak{J}$, let us consider a sequence p_n converging to p. Then we have

$$\begin{aligned}\|\mathfrak{T}_2p_n-\mathfrak{T}_2p\|&\leq\Big[1+\frac{(z\upsilon_1+z^{n-1}\varpi_1)}{\vartheta}(\lambda\delta+\mu\sum_{j=1}^{m-2}\xi_j-1)\Big]|\psi(p_n)-\psi(p)|\\&\leq\Big[1+\frac{(\upsilon_1+\varpi_1)}{\vartheta}\Big(\lambda\delta+\mu\sum_{j=1}^{m-2}\xi_j+1\Big)\Big]\varepsilon\|p_n-p\|,\end{aligned}$$

which, in view of $\mathfrak{E}_2$, implies that $\mathfrak{T}_2$ is continuous. Also is $\mathfrak{T}_2$ is Θ- contractive, since

$$\Theta=\Big[1+\frac{(\upsilon_1+\varpi_1)}{\vartheta}\Big(\lambda\delta+\mu\sum_{j=1}^{m-2}\xi_j+1\Big)\Big]\varepsilon=\widehat{\omega}\varepsilon<1.$$

Step 5. It remains to show that the set $\mathfrak{F}$ is bounded for every κ!. Let $p \in \mathfrak{F}$ be a solution of the integral equation

$$\begin{aligned}p(z) &= \int_0^z \frac{\kappa(z-\sigma)^{\delta-1}}{\Gamma(\delta)}k(\sigma,p(\sigma))d\sigma \\ &+\kappa\Big[1+\frac{(zv_1+z^{n-1}\varpi_1)}{\vartheta}(\lambda\delta+\mu\sum_{j=1}^{m-2}\xi_j-1)\Big]\psi(p) \\ &+\frac{\kappa\rho(zv_2-z^{n-1}\varpi_2)}{\vartheta}\Big[\int_0^\nu \frac{(\nu-\sigma)^{\delta-2}}{\Gamma(\delta-1)}k(\sigma,p(\sigma))d\sigma\Big] \\ &+\frac{\kappa(zv_1+z^{n-1}\varpi_1)}{\vartheta}\Big[\lambda\int_0^\varsigma\left(\int_0^\sigma \frac{(\sigma-\theta)^{\delta-1}}{\Gamma(\delta)}k(\theta,p(\theta))d\theta\right)d\sigma \\ &+\mu\sum_{j=1}^{m-2}\xi_j\int_0^{\zeta_j}\frac{(\zeta_j-\sigma)^{\delta-1}}{\Gamma(\delta)}k(\sigma,p(\sigma))d\sigma \\ &-\int_0^1\frac{(1-\sigma)^{\delta-1}}{\Gamma(\delta)}k(\sigma,p(\sigma))d\sigma\Big], \quad z\in\mathfrak{J}\end{aligned}$$

Then, for each $z \in \mathfrak{J}$, we have

$$\begin{aligned}|p(z)| \leq& \int_0^z \frac{(z-\sigma)^{\delta-1}}{\Gamma(\delta)}x(\sigma)d\sigma+\kappa\Big[1+\frac{(zv_1+z^{n-1}\varpi_1)}{\vartheta}(\lambda\delta+\mu\sum_{j=1}^{m-2}\xi_j-1)\Big] \\ \times&\left(\Big|\psi\Big(\frac{p(\sigma)}{\kappa}\Big)-\psi(0)\Big|+|\psi(0)|\right)+\frac{\rho(zv_2-z^{n-1}\varpi_2)}{\vartheta}\Big[\int_0^\nu\frac{(\nu-\sigma)^{\delta-2}}{\Gamma(\delta-1)}x(\sigma)d\sigma\Big] \\ &+\frac{(zv_1+z^{n-1}\varpi_1)}{\vartheta}\Big[\lambda\int_0^\varsigma\left(\int_0^\sigma\frac{(\sigma-\theta)^{\delta-1}}{\Gamma(\delta)}x(\theta)d\theta\right)d\sigma \\ &+\mu\sum_{j=1}^{m-2}\xi_j\int_0^{\zeta_j}\frac{(\zeta_j-\sigma)^{\delta-1}}{\Gamma(\delta)}x(\sigma)d\sigma+\int_0^1\frac{(1-\sigma)^{\delta-1}}{\Gamma(\delta)}x(\sigma)d\sigma\Big] \\ \leq& \int_0^z\frac{(z-\sigma)^{\delta-1}}{\Gamma(\delta)}x(\sigma)d\sigma+\frac{\rho(zv_2-z^{n-1}\varpi_2)}{\vartheta}\Big[\int_0^\nu\frac{(\nu-\sigma)^{\delta-2}}{\Gamma(\delta-1)}x(\sigma)d\sigma\Big] \\ &+\frac{(zv_1+z^{n-1}\varpi_1)}{\vartheta}\Big[\lambda\int_0^\varsigma\left(\int_0^\sigma\frac{(\sigma-\theta)^{\delta-1}}{\Gamma(\delta)}x(\theta)d\theta\right)d\sigma \\ &+\mu\sum_{j=1}^{m-2}\xi_j\int_0^{\zeta_j}\frac{(\zeta_j-\sigma)^{\delta-1}}{\Gamma(\delta)}x(\sigma)d\sigma+\int_0^1\frac{(1-\sigma)^{\delta-1}}{\Gamma(\delta)}x(\sigma)d\sigma\Big] \\ &+\Big[1+\frac{(v_1+\varpi_1)}{\vartheta}(\lambda\delta+\mu\sum_{j=1}^{m-2}\xi_j+1)\Big]\varepsilon\|p\|\end{aligned}$$

or

$$(1-\widehat{\omega}\varepsilon)\|p\| \leq \int_0^z\frac{(z-\sigma)^{\delta-1}}{\Gamma(\delta)}x(\sigma)d\sigma+\frac{\rho(zv_2-z^{n-1}\varpi_2)}{\vartheta}\Big[\int_0^\nu\frac{(\nu-\sigma)^{\delta-2}}{\Gamma(\delta-1)}x(\sigma)d\sigma\Big]$$

$$+\frac{(z\upsilon_1 + z^{n-1}\varpi_1)}{\vartheta}\Big[\lambda \int_0^{\varsigma}\left(\int_0^{\sigma}\frac{(\sigma-\theta)^{\delta-1}}{\Gamma(\delta)}x(\theta)d\theta\right)d\sigma$$
$$+\mu\sum_{j=1}^{m-2}\xi_j\int_0^{\zeta_j}\frac{(\zeta_j-\sigma)^{\delta-1}}{\Gamma(\delta)}x(\sigma)d\sigma + \int_0^1\frac{(1-\sigma)^{\delta-1}}{\Gamma(\delta)}x(\sigma)d\sigma\Big].$$

Consequently, we have

$$\begin{aligned}\|p\| \quad \leq \quad & \mathfrak{V} := \frac{1}{(1-\widehat{\omega}\varepsilon)}\Bigg\{\int_0^{z}\frac{(z-\sigma)^{\delta-1}}{\Gamma(\delta)}x(\sigma)d\sigma \\ & +\frac{\rho(z\upsilon_2 - z^{n-1}\varpi_2)}{\vartheta}\Big[\int_0^{\nu}\frac{(\nu-\sigma)^{\delta-2}}{\Gamma(\delta-1)}x(\sigma)d\sigma\Big] \\ & +\frac{(z\upsilon_1 + z^{n-1}\varpi_1)}{\vartheta}\Big[\lambda\int_0^{\varsigma}\left(\int_0^{\sigma}\frac{(\sigma-\theta)^{\delta-1}}{\Gamma(\delta)}x(\theta)d\theta\right)d\sigma \\ & +\mu\sum_{j=1}^{m-2}\xi_j\int_0^{\zeta_j}\frac{(\zeta_j-\sigma)^{\delta-1}}{\Gamma(\delta)}x(\sigma)d\sigma \\ & +\int_0^1\frac{(1-\sigma)^{\delta-1}}{\Gamma(\delta)}x(\sigma)d\sigma\Big]\Bigg\}\end{aligned}$$

which shows that the set $\mathfrak{F}$ is bounded, since $\widehat{\omega}\varepsilon < 1$. Hence, $\mathfrak{T}$ has a fixed point in $\mathfrak{J}$ by Theorem 3.3, and consequently the problem (1.1)-(1.2) has a solution. This completes the proof. □

Finally, we show that the existence of solutions for the boundary value problem (1.1)-(1.2) by applying a fixed poin theorem due to O'Regan.

Lemma 3.5. *Denote by $\mathfrak{X}$ an open set in a closed, convex set $\mathfrak{A}$ of a Banach space $\mathfrak{H}$. Assume $0 \in \mathfrak{X}$. Also assume that $\mathfrak{T}(\widehat{\mathfrak{X}})$ is bounded and that $\mathfrak{T} : \widehat{\mathfrak{X}} \to \mathfrak{A}$ is given by $\mathfrak{T} = \mathfrak{T}_1 + \mathfrak{T}_2$, in which $\mathfrak{T}_1 : \widehat{\mathfrak{X}} \to \mathfrak{H}$ is a nonlinear contraction (i.e., there exists a nonnegative nondecreasing function $\varphi : [0,\infty) \to [0,\infty)$ satisfying $\vartheta(y) < y$ for $y > 0$, such that $\|\mathfrak{T}_2(p) - \mathfrak{T}_2(q)\| \leq \vartheta(\|p-q\|)\ \forall\ p,q \in \widehat{\mathfrak{X}}$. Then, either*
($\mathfrak{W}_1$) *$\mathfrak{T}$ has a fixed point $x \in \widehat{\mathfrak{X}}$; or*
($\mathfrak{W}_2$) *there exist a point $x \in \partial\mathfrak{X}$ and $\kappa \in (0,1)$ with $x = \kappa\mathfrak{T}(x)$, where $\widehat{\mathfrak{X}}$ and $\partial\mathfrak{X}$, respectively, represent the closure and boundary of $\mathfrak{X}$.*

In the next result, we use the terminology:

$$\Delta_\theta = \{p \in \mathfrak{G} : \|p\| < \theta\},\ \mathfrak{V}_\theta = max\{|k(z,p)| : (z,p) \in \mathfrak{J}\times[\theta,-\theta]\}.$$

Theorem 3.6. *The continuous function k defined from $\mathfrak{J}\times\mathbb{R}$ to $\mathbb{R}$ and conditions ($\mathfrak{E}_1$),($\mathfrak{E}_2$),($\mathfrak{E}_4$) hold. Also let us understand that:*
($\mathfrak{E}_6$) *there exists a nonnegative function $x \in C(\mathfrak{J},\mathbb{R})$ and a nondecreasing function $\phi : [0,\infty) \to [0,\infty)$ such that $|k(z,v)| \leq x(z)\phi(\|v\|)$ for any $(z,v) \in \mathfrak{J}\times\mathbb{R}$;*
($\mathfrak{E}_7$) $\sup_{\theta\in(0,\infty)} \dfrac{\theta}{\widehat{\eta}\phi(\theta)\|x\|} > \dfrac{1}{1-\widehat{\omega}\varepsilon}$, *where $\widehat{\eta}$ and $\widehat{\omega}$ are defined in (3.4) and (3.5) respectively. Then the problem (1.1)-(1.2) has at least one solution on $\mathfrak{J}$.*

Proof. By the assumption ($\mathfrak{E}_7$), there exists a number $\widehat{\theta} > 0$ such that

$$\frac{\widehat{\theta}}{\widehat{\eta}\phi(\widehat{\theta})\|x\|} > \frac{1}{1-\widehat{\omega}\varepsilon} \tag{3.8}$$

We shall show that the operators $\mathfrak{T}_1$ and $\mathfrak{T}_2$ defined by (3.2) and (3.3) respectively, satisfy all the conditions of Lemma 3.5.

Step 1. The operator $\mathfrak{T}_1$ is continuous and completely continuous. We first show that $\mathfrak{T}_1(\Delta_{\widehat{\theta}})$ is bounded. For any $p \in \Delta_{\widehat{\theta}}$, we have

$$\begin{aligned}
\|\mathfrak{T}_1 p\| &\leq \int_0^z \frac{(z-\sigma)^{\delta-1}}{\Gamma(\delta)}|k(\sigma,p(\sigma))|d\sigma \\
&\quad + \frac{\rho(z\upsilon_2 - z^{n-1}\varpi_2)}{\vartheta}\Big[\int_0^\nu \frac{(\nu-\sigma)^{\delta-2}}{\Gamma(\delta-1)}|k(\sigma,p(\sigma))|d\sigma\Big] \\
&\quad + \frac{(z\upsilon_1 + z^{n-1}\varpi_1)}{\vartheta}\Big[\lambda\int_0^\varsigma\left(\int_0^\sigma \frac{(\sigma-\theta)^{\delta-1}}{\Gamma(\delta)}|k(\theta,p(\theta))|d\theta\right)d\sigma \\
&\quad + \mu\sum_{j=1}^{m-2}\xi_j\int_0^{\zeta_j}\frac{(\zeta_j-\sigma)^{\delta-1}}{\Gamma(\delta)}|k(\sigma,p(\sigma))|d\sigma \\
&\quad + \int_0^1 \frac{(1-\sigma)^{\delta-1}}{\Gamma(\delta)}|k(\sigma,p(\sigma))|d\sigma\Big] \\
&\leq \mathfrak{V}_\theta\int_0^z \frac{(z-\sigma)^{\delta-1}}{\Gamma(\delta)}x(\sigma)d\sigma \\
&\quad + \frac{\mathfrak{V}_\theta\rho(z\upsilon_2 - z^{n-1}\varpi_2)}{\vartheta}\Big[\int_0^\nu \frac{(\nu-\sigma)^{\delta-2}}{\Gamma(\delta-1)}x(\sigma)d\sigma\Big] \\
&\quad + \frac{\mathfrak{V}_\theta(z\upsilon_1 + z^{n-1}\varpi_1)}{\vartheta}\Big[\lambda\int_0^\varsigma\left(\int_0^\sigma \frac{(\sigma-\theta)^{\delta-1}}{\Gamma(\delta)}x(\theta)d\theta\right)d\sigma \\
&\quad + \mu\sum_{j=1}^{m-2}\xi_j\int_0^{\zeta_j}\frac{(\zeta_j-\sigma)^{\delta-1}}{\Gamma(\delta)}x(\sigma)d\sigma + \int_0^1 \frac{(1-\sigma)^{\delta-1}}{\Gamma(\delta)}x(\sigma)d\sigma\Big] \\
&\leq \frac{\|x\|\mathfrak{V}_\theta}{\Gamma(\delta+1)}\Bigg[1 + \frac{\rho|(\upsilon_2 - \varpi_2)|\nu^{\delta-1}}{\vartheta\delta} \\
&\quad + \frac{(\upsilon_1+\varpi_1)}{\vartheta}\left(\frac{\lambda\varsigma^{\delta+1}}{\delta+1} + \mu\sum_{j=1}^{m-2}\xi_j\zeta_j^\delta + 1\right)\Bigg] \\
&= \mathfrak{V}_\theta\|p\|\widehat{\eta}.
\end{aligned}$$

Thus the operator $\mathfrak{T}_1(\mathfrak{V}_{\widehat{\theta}})$ is uniformly bounded. Let $\varrho_1, \varrho_2 \in \mathfrak{J}$ with $\varrho_1 < \varrho_2$ and $p \in \mathfrak{B}_\theta$. Then

$$|(\mathfrak{T}_1 p)(\varrho_2) - (\mathfrak{T}_1 p)(\varrho_1)| \leq \mathfrak{V}_\theta\left|\int_0^{\varrho_1}\frac{[(\varrho_2-\sigma)^{\delta-1} - (\varrho_1-\sigma)^{\delta-1}]}{\Gamma(\delta)} \times x(\sigma)d\sigma\right|$$

$$+\mathfrak{V}_\theta\Big|\int_{\varrho_1}^{\varrho_2}\frac{(\varrho_2-\sigma)^{\delta-1}}{\Gamma(\delta)}x(\sigma)d\sigma\Big|$$
$$+\Big|\frac{\mathfrak{V}_\theta\rho((\varrho_2-\varrho_1)\upsilon_2-(\varrho_2^{n-1}-\varrho_1^{n-1})\varpi_2)}{\vartheta}\Big|\Big[\int_0^\nu\frac{(\nu-\sigma)^{\delta-2}}{\Gamma(\delta-1)}|x(\sigma)|d\sigma\Big]$$
$$+\frac{\mathfrak{V}_\theta((\varrho_2-\varrho_1)\upsilon_1+(\varrho_2^{n-1}-\varrho_1^{n-1})\varpi_1)}{\vartheta}\Big[\lambda\int_0^\varsigma\Big(\int_0^\sigma\frac{(\sigma-\theta)^{\delta-1}}{\Gamma(\delta)}|x(\theta)|d\theta\Big)d\sigma$$
$$+\mu\sum_{j=1}^{m-2}\xi_j\int_0^{\zeta_j}\frac{(\zeta_j-\sigma)^{\delta-1}}{\Gamma(\delta)}|x(\sigma)|d\sigma+\int_0^1\frac{(1-\sigma)^{\delta-1}}{\Gamma(\delta)}|x(\sigma)|d\sigma\Big]$$
$$\le\frac{\|x\|\mathfrak{V}_\theta}{\Gamma(\delta+1)}\Bigg[[2(\varrho_2-\varrho_1)^\delta+(\varrho_2^\delta-\varrho_1^\delta)]+\frac{\rho((\varrho_2-\varrho_1)\upsilon_2-(\varrho_2^{n-1}-\varrho_1^{n-1})\varpi_2)\nu^{\delta-1}}{\vartheta\delta}$$
$$+\frac{((\varrho_2-\varrho_1)\upsilon_1+(\varrho_2^{n-1}-\varrho_1^{n-1})\varpi_1)}{\vartheta}\Big(\frac{\lambda\varsigma^{\delta+1}}{\delta+1}+\mu\sum_{j=1}^{m-2}\xi_j\zeta_j^\delta+1\Big)\Bigg]$$

which is independent of p and tends to zero as $\varrho_2-\varrho_1\to 0$. Thus, $\mathfrak{T}_1$ is equicontinuous. Hence, by the Arzela-Ascoli Theorem. $\mathfrak{T}_1(\mathfrak{V}_{\widehat{\theta}})$ is a relatively compact set. Now, let $p_n\subset\mathfrak{V}_{\widehat{\theta}}$ with $\|p_n-p\|\to 0$. Then the $\|p_n(z)-p(z)\|\to 0$ is uniformly valid on $\mathfrak{J}$. From the uniform continuity of $k(z,p)$ on the compact set $\mathfrak{J}\times[\widehat{\theta},-\widehat{\theta}]$, it follows that

$$\|k(z,p_n(z))-k(z,p(z))|\to 0$$

uniformly on $\mathfrak{J}$. Hence $\|\mathfrak{T}_1p_n-\mathfrak{T}_1p\|\to 0$ as $n\to\infty$ which proves the continuity of $\mathfrak{T}_1$. This completes the proof Step 1.

Step 2. The operator $\mathfrak{T}_2:\mathfrak{V}_{\widehat{\theta}}\to C(\mathfrak{J},\mathbb{R})$ is contractive. This is a consequence of $(\mathfrak{E}_2)$.

Step 3. The set $\mathfrak{T}(\mathfrak{V}_{\widehat{\theta}})$ is bounded. The assumptions $(\mathfrak{E}_2)$ and $(\mathfrak{E}_4)$ imply that

$$\|\mathfrak{T}_2p\|\le\widehat{\omega}\varepsilon\widehat{\theta},$$

for any $p\in\mathfrak{V}_{\widehat{\theta}}$. This, with the boundedness of the set $\mathfrak{T}_1(\mathfrak{V}_{\widehat{\theta}})$ implies that the set $\mathfrak{T}(\mathfrak{V}_{\widehat{\theta}})$ is bounded.

Step 4. Finally, it will be shown that the case $\mathfrak{W}_2$ in Lemma 3.5 does not hold. On the contrary, we suppose that $\mathfrak{W}_2$ holds. Then, we have that there exist $\kappa\in(0,1)$ and $p\in\partial\mathfrak{V}_{\widehat{\theta}}$ such that $p=\kappa\mathfrak{T}p$.

So, we have $\|p\|=\widehat{\theta}$ and

$$p(z)=\int_0^z\frac{\kappa(z-\sigma)^{\delta-1}}{\Gamma(\delta)}k(\sigma,p(\sigma))d\sigma$$
$$+\kappa\Big[1+\frac{(z\upsilon_1+z^{n-1}\varpi_1)}{\vartheta}(\lambda\delta+\mu\sum_{j=1}^{m-2}\xi_j-1)\Big]\psi(p)$$
$$+\frac{\kappa\rho(z\upsilon_2-z^{n-1}\varpi_2)}{\vartheta}\Big[\int_0^\nu\frac{(\nu-\sigma)^{\delta-2}}{\Gamma(\delta-1)}k(\sigma,p(\sigma))d\sigma\Big]$$
$$+\frac{\kappa(z\upsilon_1+z^{n-1}\varpi_1)}{\vartheta}\Big[\lambda\int_0^\varsigma\Big(\int_0^\sigma\frac{(\sigma-\theta)^{\delta-1}}{\Gamma(\delta)}k(\theta,p(\theta))d\theta\Big)d\sigma$$

$$+\mu\sum_{j=1}^{m-2}\xi_j\int_0^{\zeta_j}\frac{(\zeta_j-\sigma)^{\delta-1}}{\Gamma(\delta)}k(\sigma,p(\sigma))d\sigma-\int_0^1\frac{(1-\sigma)^{\delta-1}}{\Gamma(\delta)}k(\sigma,p(\sigma))d\sigma\Big]\quad z\in\mathfrak{J}.$$

Using the assumptions $(\mathfrak{E}_4)$-$(\mathfrak{E}_6)$, we get

$$\begin{aligned}\widehat{\theta} &\leq \frac{\phi(\widehat{\theta})\|x\|}{\Gamma(\delta+1)}\Bigg[1+\frac{\rho|(\upsilon_2-\varpi_2)|\nu^{\delta-1}}{\vartheta\delta}\\ &\quad+\frac{(\upsilon_1+\varpi_1)}{\vartheta}\Big(\frac{\lambda\varsigma^{\delta+1}}{\delta+1}+\mu\sum_{j=1}^{m-2}\xi_j\zeta_j^{\delta}+1\Big)\Bigg]\\ &\quad+\widehat{\theta}\varepsilon\Big[1+\frac{(\upsilon_1+\varpi_1)}{\vartheta}\Big(\lambda\delta+\mu\sum_{j=1}^{m-2}\xi_j+1\Big)\Big].\end{aligned}$$

which yields

$$\widehat{\theta} \leq \widehat{\eta}\phi(\widehat{\theta})\|x\|+\widehat{\omega}\varepsilon.$$

Thus, we get a contradiction :

$$\frac{\widehat{\theta}}{\widehat{\eta}\phi(\widehat{\theta})\|x\|} \leq \frac{1}{1-\widehat{\omega}\varepsilon}.$$

Thus, the operators $\mathfrak{T}_1$ and $\mathfrak{T}_2$ satisfy all the conditions of Lemma 3.5. Hence, the operator $\mathfrak{T}$ has at least one fixed point $p\in\mathfrak{V}_{\widehat{\theta}}$, which is a solution of the problem (1.1)-(1.2). This completes the proof. □

Example 3.7. Consider the fractional differential equation given by

$$^{C}\mathfrak{D}^{\frac{5}{2}}p(z)=\frac{e^{-z}}{2\sqrt{z^2+16}}\Big(\frac{1}{2}+z\tan^{-1}(z)\Big),\quad z\in\mathfrak{J}, \tag{3.9}$$

supplemented with the boundary conditions of Example 3.2.
Observe that $|k(z,p)|\leq x(z)\phi(|p|)$ with

$$x(z)=\frac{e^{-z}}{4\sqrt{z^2+16}},\ \phi(|p|)=1+|p|$$

and $\psi(0)=0$, $\varepsilon=\frac{1}{10}$ as $|\psi(v)-\psi(w)|\leq\frac{1}{10}|v-w|$. With

$$\phi(\theta)=1+\theta,\ \|x\|=\frac{1}{16},\ \widehat{\eta}\cong 1.0683,\ \widehat{\omega}\cong 0.36416,$$

we have that $(\mathfrak{E}_7)$ holds, since

$$\frac{\widehat{\theta}}{\widehat{\eta}\phi(\widehat{\theta})\|x\|}\cong 14.9771>\ 1.03779\cong\frac{1}{1-\widehat{\omega}\varepsilon}.$$

Thus, all the conditions of Theorem 3.6 is satisfied and here the problem (3.9) with (3.7) has at least one solution on $\mathfrak{J}$.

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Statistical $e-$convergence of double sequences on probabilistic normed spaces

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Abstract. The concept of statistical convergence for double sequences on probabilistic normed spaces was presented by Karakus and Demirci in 2007. The purpose of this paper is to introduce the concept of statistical $e-$convergence for double sequences and study some fundamental properties of statistical $e-$convergence for double sequences on probabilistic normed spaces.

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1. Introduction

Statistical convergence which is a generalization of the notion of ordinary convergence was first introduced by Fast [4] and Steinhaus [21] in 1951. Then several generalizations and applications of this notion have been investigated by various authors [6], [11], [12], [14]. The concept of statistical convergence for double sequences was studied by Mursaleen and Edely [15]. Boos et al ([2], [3]) introduced and investigated the notion of $e-$convergence of double sequences which is essentially weaker than the Pringsheim convergence. Recently, Sever and Talo [19] have generalized the notion of $e-$convergence to statistical $e-$convergence for a double sequence [see also [20]].

The theory of probabilistic normed spaces [5] originated from the concept of statistical metric spaces which was introduced by Menger [13] and further studied by Schweizer and Sklar [17], [18]. It provides an important method of generalizing the deterministic results of normed linear spaces. It has also very useful applications in various fields, e.g., continuity properties [1], topological spaces [5], study of boundedness [7], convergence of random variables [8] etc.

The idea of statistical convergence of single sequences on probabilistic normed spaces was studied by Karakus in [9]. Then, Karakus and Demirci extended the concept of statistical convergence from single to double sequences in [10]. In this paper we introduce and study the concept of statistical $e-$convergence for double sequences on probabilistic normed spaces.

2. Background and preliminaries

First, we recall some notions and basic definitions those will be used in this paper. Throughout this paper, $\mathbb{N}$, $\mathbb{R}$ respectively denote the sets of positive integers and real numbers whereas $\mathbb{N} \times \mathbb{N}$ denotes the usual product set.

Definition 2.1. [5] A function $g : \mathbb{R} \to \mathbb{R}_0^+$ is called a distribution function if the following conditions holds:

a) it is non-decreasing,
b) it is left-continuous,
c) $\inf\limits_{t\in\mathbb{R}} g(t) = 0$ and $\sup\limits_{t\in\mathbb{R}} g(t) = 1$.

The set of all distribution functions will be denoted by E.

Definition 2.2. [18] A triangular norm or briefly $t-$norm is a continuous mapping $* : [0,1] \times [0,1] \to [0,1]$ such that $([0,1], *)$ is an abelian monoid with unit one and $p * q \geq m * n$ if $p \geq m$ and $q \geq n$ for all $m, n, p, q \in [0,1]$.

For example the $*$ operations

$$m * n = mn,\ m * n = \min\{m, n\} \text{ and } m * n = \max\{m + n - 1, 0\}$$

are $t-$norms on $[0,1]$.

Definition 2.3. [18] If D is a real vector space, η is a mapping from D into E (for $x \in D$ the distribution function $\eta(x)$ is denoted by η_x and $\eta_x(t)$ is the value of η_x at $t \in \mathbb{R}$) and $*$ is a $t-$norm satisfying the following conditions :

i) $\eta_x(0) = 0$,
ii) $\eta_x(t) = 1$ *for all* $t > 0$ *if and only if* $x = 0$,
iii) $\eta_{\alpha x}(t) = \eta_x\left(\frac{t}{|\alpha|}\right)$ *for all* $\alpha \in \mathbb{R} \setminus \{0\}$ *and for all* $t > 0$,
iv) $\eta_{x+y}(s+t) \geq \eta_x(s) * \eta_y(t)$ *for all* $x, y \in D$ *and* $s, t \in \mathbb{R}_0^+$,

then $(D, \eta, *)$ *is called a probabilistic normed space (briefly, a PNS).*

Definition 2.4. Let $(D, \eta, *)$ be a PNS. Then, a sequence (x_k) is said to be convergent to $L \in D$ with respect to the probabilistic norm η, that is $x_k \xrightarrow{\eta} L$ if for every $\varepsilon > 0$ and $\lambda \in (0,1)$ there exists a positive integer k_0 such that $\eta_{x_k - L}(\varepsilon) > 1 - \lambda$ whenever $k \geq k_0$. In this case we write $\eta - \lim x_k = L$ as $k \to \infty$.

Remark 2.5. Let $(D, \|.\|)$ be a real normed space and

$$\eta_x(t) = \frac{t}{t + \|x\|}$$

where $x \in D$ and $t \geq 0$ (standard $x-$norm induced by $\|.\|$). Then we can see that $x_k \xrightarrow{\|.\|} x$ if and only if $x_k \xrightarrow{\eta} x$.

3. Statistical $e-$convergence of double sequence on PNS

In this section we study the concept of statistical $e-$convergence for double sequences in probabilistic normed space. First, we recall the concept of statistical convergence.

Let $K \subseteq \mathbb{N}$. Then the asymptotic (or natural) density of K denoted by $\delta(K)$ is given by

$$\delta(K) := \lim_n \frac{1}{n} |\{k \leq n : k \in K\}|$$

whenever the limit exists, where the vertical bars denote the cardinality of the enclosed set.

A number sequence (x_k) is said to be statistically convergent to the number L if for each $\varepsilon > 0$ the set

$$K(\varepsilon) := \{k \in \mathbb{N} : |x_k - L| \geq \varepsilon\}$$

has asymptotic density zero, i.e.

$$\lim_n \frac{1}{n} |\{k \leq n : |x_k - L| \geq \varepsilon\}| = 0.$$

In this case we write $st - \lim x_k = L$ as $k \to \infty$.

So we give the concept of statistical convergence of double sequences.

By the convergence of a double sequence we mean the convergence in the Pringsheim sense that is, a double sequence (x_{kl}) has Pringsheim limit L provided that given $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that $|x_{kl} - L| < \varepsilon$ wherever $k, l > N$ [16]. We write this as $P - \lim\limits_{k,l} x_{kl} = L$. In case of this convergence, the row-index k and column-index l tend independently to infinity.

We can give the analogue of Definition 2.4 for a double sequence as follows:

Definition 3.1. [10] Let $(D, \eta, *)$ be a PNS. Then, a double sequence (x_{kl}) is said to be convergent to $L \in D$ with respect to the probabilistic norm η, that is $x_{kl} \xrightarrow{\eta} L$ if for every $\varepsilon > 0$ and $\lambda \in (0,1)$ there exists a positive integer k_0 such that $\eta_{x_{kl}-L}(\varepsilon) > 1 - \lambda$ whenever $k, l \geq k_0$. In this case we write $\eta_2 - \lim x_{kl} = L$ as $k, l \to \infty$.

Let $K \subseteq \mathbb{N} \times \mathbb{N}$ and $K(n,m)$ be the numbers of (i,j) in K such that $i \leq n$ and $j \leq m$. Then the two dimensional analog of natural density can be defined as follows:

$$\underline{\delta_2}(K) := \lim_{n,m} \frac{K(n,m)}{nm}.$$

For example, let $K = \left\{\left(i^2, j^2\right) : i, j \in \mathbb{N}\right\}$. Then the set K has double natural density zero.

Definition 3.2. [15] A double sequence (x_{kl}) is said to be statistically convergent to a number α if for each $\varepsilon > 0$ the set

$$\{(k,l), k \leq n, l \leq m : |x_{kl} - \alpha| \geq \varepsilon\}$$

has double natural density zero. We write this as $st_2 - \lim\limits_{k,l} x_{kl} = \alpha$.

Definition 3.3. [10] Let $(D, \eta, *)$ be a PNS. Then, a double sequence (x_{kl}) is said to be statistically convergent to $L \in D$ with respect to the probabilistic norm η if for every $\varepsilon > 0$ and $\lambda \in (0,1)$ the set

$$\{(k,l), k \leq n, l \leq m : \eta_{x_{kl}-\alpha}(\varepsilon) \leq 1 - \lambda\}$$

has double natural density zero. In this case we write $st_{\eta_2} - \lim\limits_{k,l} x_{kl} = \alpha$.

Boos, Leiger and Zeller [3] and Boos [2] introduced and investigated the notion of $e-$convergence of double sequences, which is essentially weaker than the Pringsheim convergence as follows:

Definition 3.4. A double sequence (x_{kl}) is said to be $e-$convergent to a number α if

$$\forall \varepsilon > 0, \ \exists l_0 \in \mathbb{N} \ \forall l \geq l_0, \ \exists k_l \in \mathbb{N} \ \forall k \geq k_l, \ |x_{kl} - \alpha| < \varepsilon.$$

We write this as $e - \lim\limits_{k,l} x_{kl} = \alpha$.

In contrast to the Pringsheim notion of convergence, $e-$convergence states that the row-index k depends on the column-index l whenever it tends to infinity.

Recently, Sever and Talo [19] have defined the concept of statistical $e-$convergence for a double sequence as follows:

Definition 3.5. [19] A double sequence (x_{kl}) is said to be statistically $e-$convergent to a number α if for every $\varepsilon > 0$ the set

$$\{l : \delta(\{k : |x_{kl} - \alpha| \geq \varepsilon\}) = 0\}$$

has natural density 1, that is

$$\delta(\{l : \delta(\{k : |x_{kl} - \alpha| \geq \varepsilon\}) = 0\}) = 1.$$

In this case, one writes $st_{(e)} - \lim_{k,l} x_{kl} = \alpha$.

Now we give the analogue of these definitions with respect to the probabilistic norm η.

Definition 3.6. Let $(D, \eta, *)$ be a PNS. A double sequence (x_{kl}) is said to be $e-$convergent to $\alpha \in D$ with respect to the probabilistic norm η provided that for every $\varepsilon > 0$ and $\lambda \in (0,1)$

$$\exists l_0 \in \mathbb{N}\ \forall l \geq l_0,\ \exists k_l \in \mathbb{N}\ \forall k \geq k_l,\ \eta_{x_{kl}-\alpha}(\varepsilon) > 1 - \lambda.$$

In this case, one writes $\eta_{(e)} - \lim_{k,l} x_{kl} = \alpha$. Also, the element α is called the $\eta_{(e)}-$limit of the double sequence (x_{kl}).

Definition 3.7. Let $(D, \eta, *)$ be a PNS. A double sequence (x_{kl}) is said to be statistically $e-$convergent to $\alpha \in D$ with respect to the probabilistic norm η provided that for every $\varepsilon > 0$ and $\lambda \in (0,1)$

$$\{l : \delta(\{k : \eta_{x_{kl}-\alpha}(\varepsilon) \leq 1 - \lambda\}) = 0\}$$

has natural density 1, that is

$$\delta(\{l : \delta(\{k : \eta_{x_{kl}-\alpha}(\varepsilon) \leq 1 - \lambda\}) = 0\}) = 1.$$

In this case, one writes $st_{\eta(e)} - \lim_{k,l} x_{kl} = \alpha$. Also, the element α is called the $st_{\eta(e)}-$limit of the double sequence (x_{kl}).

The following theorem gives the relation between $e-$convergence and statistical $e-$convergence on probabilistic normed spaces.

Lemma 3.8. *Let $(D, \eta, *)$ be a PNS. Then, for every $\varepsilon > 0$, $\alpha \in D$ and $\lambda \in (0,1)$ the following statements are equivalent:*

i) $st_{\eta(e)} - \lim_{k,l} x_{kl} = \alpha$.

ii) $\delta(\{l : \delta(\{k : \eta_{x_{kl}-\alpha}(\varepsilon) \leq 1 - \lambda\}) = 0\}) = 1$.

iii) $st_{(e)} - \lim_{k,l} \eta_{x_{kl}-\alpha}(\varepsilon) = 1$.

Proof. From Definition 3.7, the first two parts are equivalent.

(ii)$\Rightarrow$ (iii) Let $L = \{l : \delta(\{k : \eta_{x_{kl}-\alpha}(\varepsilon) \leq 1 - \lambda\}) = 0\}$. So $\delta(L) = 1$. Then for all $l \in L$,

$$\{k : |\eta_{x_{kl}-\alpha}(\varepsilon) - 1| \geq \lambda\} \subseteq \{k : \eta_{x_{kl}-\alpha}(\varepsilon) \geq 1 + \lambda\} \cup \{k : \eta_{x_{kl}-\alpha}(\varepsilon) \leq 1 - \lambda\}.$$

So, we get for $l \in L$,

$$\delta(\{k : |\eta_{x_{kl}-\alpha}(\varepsilon) - 1| \geq \lambda\}) = 0.$$

Then

$$\delta(\{l : \delta(\{k : |\eta_{x_{kl}-\alpha}(\varepsilon) - 1| \geq \lambda\}) = 0\}) = 1$$

which completes the proof. □

Theorem 3.9. *Let $(D, \eta, *)$ be a PNS and let (x_{kl}) be a double sequence whose terms are in the vector space D. If (x_{kl}) is statistically $e-$convergent with respect to the probabilistic norm η then its $st_{\eta(e)}-$limit is unique.*

Proof. Suppose that there exist α and β in D with $\alpha \neq \beta$ such that $st_{\eta(e)} - \lim_{k,l} x_{kl} = \alpha$ and $st_{\eta(e)} - \lim_{k,l} x_{kl} = \beta$. Let $\xi > 0$, choose $\lambda \in (0,1)$ such that

$$(1-\lambda) * (1-\lambda) \geq (1-\xi).$$

Let $\varepsilon > 0$ be given. Then we define the following sets:

$$\begin{aligned} L_{\eta,1}(\lambda, \varepsilon) &: = \{l : \delta(\{k : \eta_{x_{kl}-\alpha}(\varepsilon) \leq 1-\lambda\}) = 0\} \\ L_{\eta,2}(\lambda, \varepsilon) &: = \{l : \delta(\{k : \eta_{x_{kl}-\beta}(\varepsilon) \leq 1-\lambda\}) = 0\} \end{aligned}$$

and

$$\begin{aligned} K_{\eta,1}(\lambda, \varepsilon) &: = \{k : \eta_{x_{kl}-\alpha}(\varepsilon) \leq 1-\lambda\} \\ K_{\eta,2}(\lambda, \varepsilon) &: = \{k : \eta_{x_{kl}-\beta}(\varepsilon) \leq 1-\lambda\}. \end{aligned}$$

Since $st_{\eta(e)} - \lim_{k,l} x_{kl} = \alpha$ and $st_{\eta(e)} - \lim_{k,l} x_{kl} = \beta$ then we have $\delta(L_{\eta,1}(\lambda, \varepsilon)) = 1$, $\delta(L_{\eta,2}(\lambda, \varepsilon)) = 1$, $\delta(K_{\eta,1}(\lambda, \varepsilon)) = 0$ and $\delta(K_{\eta,2}(\lambda, \varepsilon)) = 0$, for all $\varepsilon > 0$. Let

$$\begin{aligned} K_{\eta}(\lambda, \varepsilon) &= K_{\eta,1}(\lambda, \varepsilon) \cap K_{\eta,2}(\lambda, \varepsilon) \\ L_{\eta}(\lambda, \varepsilon) &= L_{\eta,1}(\lambda, \varepsilon) \cap L_{\eta,2}(\lambda, \varepsilon). \end{aligned}$$

So we can see that $\delta(\mathbb{N} \backslash K_{\eta}(\lambda, \varepsilon)) = 1$ and $\delta(\mathbb{N} \backslash L_{\eta}(\lambda, \varepsilon)) = 0$.
If $(k,l) \in (\mathbb{N} \backslash K_{\eta}(\lambda, \varepsilon)) \times (\mathbb{N} \backslash L_{\eta}(\lambda, \varepsilon))$, then we have

$$\eta_{\alpha-\beta}(\varepsilon) \geq \eta_{x_{kl}-\alpha}\left(\frac{\varepsilon}{2}\right) * \eta_{x_{kl}-\beta}\left(\frac{\varepsilon}{2}\right) > (1-\lambda) * (1-\lambda) \geq (1-\xi).$$

Since $\xi > 0$ was arbitrary, we get $\eta_{\alpha-\beta}(\varepsilon) = 1$ for all $\varepsilon > 0$. So we get $\alpha = \beta$ from Definition 2.3 (ii). This completes the proof. □

Theorem 3.10. *Let $(D, \eta, *)$ be a PNS and let (x_{kl}) be a double sequence whose terms are in the vector space D. If there exists $M = K \times L \subset \mathbb{N} \times \mathbb{N}$ such that $\delta(K) = 1$ and $\delta(L) = 1$ and $\eta_{(e)} - \lim_{(k,l) \in M} x_{kl} = \alpha$ then $st_{\eta(e)} - \lim_{k,l} x_{kl} = \alpha$.*

Proof. Suppose that there exists $M = K \times L$ such that $\delta(K) = 1$ and $\delta(L) = 1$ and $\eta_{(e)} - \lim_{(k,l) \in M} x_{kl} = \alpha$. Then for each $\varepsilon > 0$ and $\lambda \in (0,1)$ there exists l_{ε} such that for each $l \geq l_{\varepsilon}$, $l \in L$ there exists k_l such that for each $k \geq k_l$, $k \in K$ we have $\eta_{x_{kl}-\alpha}(\varepsilon) > 1-\lambda$. So for such l we have

$$\{k : \eta_{x_{kl}-\alpha}(\varepsilon) \leq 1-\lambda\} \subseteq \mathbb{N} \backslash \{K \backslash \{k_1, k_2, ..., k_l\}\}.$$

Since $\delta(K) = 1$ we have $\delta(\{k : \eta_{x_{kl}-\alpha}(\varepsilon) \leq 1-\lambda\}) = 0$. On the other hand, this equation holds for each $l > l_{\varepsilon}$, $l \in L$. Therefore

$$L \backslash \{l_1, l_2, ..., l_{\varepsilon}\} \subseteq \{l : \delta(\{k : \eta_{x_{kl}-\alpha}(\varepsilon) \leq 1-\lambda\}) = 0\}.$$

So we have

$$\delta(\{l : \delta(\{k : \eta_{x_{kl}-\alpha}(\varepsilon) \leq 1-\lambda\}) = 0\}) = 1$$

which completes the proof. □

So, if a double sequence (x_{kl}) is $e-$convergent to $\alpha \in D$ with respect to the probabilistic norm η then it is statistically $e-$convergent to $\alpha \in D$ on the PNS. But the converse of this implication may not be true. The following examples show that the converse of Theorem 3.10 does not hold in general.

Example 3.11. Let $(\mathbb{R}, |.|)$ be a real normed space and $\eta_x(t) = \frac{t}{t+|x|}$ where $x \in \mathbb{R}$ and $t > 0$. In this case $(\mathbb{R}, \eta, |.|)$ is a PNS. Now we will give two examples in which our method of statistical $e-$convergence works but the other convergence methods do not work:

(i) Let (x_{kl}) be defined as

$$x_{kl} := \begin{cases} k+l, & k \leq l, \\ k, & k > l \text{ and } k \text{ is square,} \\ 0, & k > l \text{ and } k \text{ is not square.} \end{cases}$$

Then for every $\lambda \in (0,1)$ and for any $t > 0$,

$$\begin{aligned} \{k : \eta_{x_{kl}}(t) \leq 1-\lambda\} &= \left\{k : \frac{t}{t+|x_{kl}|} \leq 1-\lambda\right\} \\ &= \left\{k : |x_{kl}| \geq \frac{\lambda t}{1-\lambda} > 0\right\}. \end{aligned}$$

So we can get

$$\delta\left(\{l : \delta\left(\{k : \eta_{x_{kl}}(t) \leq 1-\lambda\}\right) = 0\}\right) = 1.$$

Also it is easy to see that $\eta_{(e)} - \lim_{k,l} x_{kl}$, $\eta_2 - \lim_{k,l} x_{kl}$, $st_2 - \lim_{k,l} x_{kl}$ and $st_{\eta_2} - \lim_{k,l} x_{kl}$ do not exist. On the other hand, we can see from the above equality that $st_{\eta(e)} - \lim_{k,l} x_{kl} = 0$.

(ii) Let (α_{kl}) be defined as follows:

$$\alpha_{kl} := \begin{cases} k, & k \leq l, \\ 1, & k > l \text{ and } k \text{ is square,} \\ 0, & k > l \text{ and } k \text{ is not square.} \end{cases}$$

Then we can see that $st_{\eta(e)} - \lim_{k,l} \alpha_{kl} = 0$. However $\eta_{(e)} - \lim_{k,l} \alpha_{kl}$, $\eta_2 - \lim_{k,l} \alpha_{kl}$, $st_2 - \lim_{k,l} \alpha_{kl}$ and $st_{\eta_2} - \lim_{k,l} \alpha_{kl}$ do not exist.

Now we will show that the concept of statistical $e-$convergence of a double sequences on a PNS has some basic properties.

Lemma 3.12. *Let $(D, \eta, *)$ be a PNS and let (x_{kl}) and (y_{kl}) be two double sequences on D.*

(i) If $st_{\eta(e)} - \lim_{k,l} x_{kl} = a$ and $st_{\eta(e)} - \lim_{k,l} y_{kl} = b$, then $st_{\eta(e)} - \lim_{k,l}(x_{kl} + y_{kl}) = a + b$.

(ii) If $st_{\eta(e)} - \lim_{k,l} x_{kl} = a$ and $\alpha \in \mathbb{R}$, then $st_{\eta(e)} - \lim_{k,l} \alpha \cdot x_{kl} = \alpha \cdot a$.

(iii) If $st_{\eta(e)} - \lim_{k,l} x_{kl} = a$ and $st_{\eta(e)} - \lim_{k,l} y_{kl} = b$, then $st_{\eta(e)} - \lim_{k,l}(x_{kl} - y_{kl}) = a - b$.

Proof. (i) Let $st_{\eta(e)} - \lim_{k,l} x_{kl} = a$ and $st_{\eta(e)} - \lim_{k,l} y_{kl} = b$, $\varepsilon > 0$ and $\xi \in (0,1)$. Choose $\lambda \in (0,1)$ such that $(1-\lambda) * (1-\lambda) \geq (1-\xi)$. Then we examine the following sets:

$$\begin{aligned} L_{\eta,1}(\lambda, \varepsilon) &: = \{l : \delta\left(\{k : \eta_{x_{kl}-a}(\varepsilon) \leq 1-\lambda\}\right) = 0\} \\ L_{\eta,2}(\lambda, \varepsilon) &: = \{l : \delta\left(\{k : \eta_{y_{kl}-b}(\varepsilon) \leq 1-\lambda\}\right) = 0\} \end{aligned}$$

and

$$\begin{aligned} K_{\eta,1}(\lambda, \varepsilon) &: = \{k : \eta_{x_{kl}-a}(\varepsilon) \leq 1-\lambda\} \\ K_{\eta,2}(\lambda, \varepsilon) &: = \{k : \eta_{y_{kl}-b}(\varepsilon) \leq 1-\lambda\}. \end{aligned}$$

Since the double sequences (x_{kl}) and (y_{kl}) are statistically $e-$convergent to a, b, respectively then we have $\delta\left(K_{\eta,1}\left(\lambda,\varepsilon\right)\right)=0$, $\delta\left(K_{\eta,2}\left(\lambda,\varepsilon\right)\right)=0$, $\delta\left(L_{\eta,1}\left(\lambda,\varepsilon\right)\right)=1$ and $\delta\left(L_{\eta,2}\left(\lambda,\varepsilon\right)\right)=1$ for all $\varepsilon>0$. Now let

$$\begin{aligned} K_{\eta}\left(\lambda,\varepsilon\right) &= K_{\eta,1}\left(\lambda,\varepsilon\right)\cap K_{\eta,2}\left(\lambda,\varepsilon\right) \\ L_{\eta}\left(\lambda,\varepsilon\right) &= L_{\eta,1}\left(\lambda,\varepsilon\right)\cap L_{\eta,2}\left(\lambda,\varepsilon\right). \end{aligned}$$

So, $\delta\left(\mathbb{N}\backslash K_{\eta}\left(\lambda,\varepsilon\right)\right)=1$ and $\delta\left(\mathbb{N}\backslash L_{\eta}\left(\lambda,\varepsilon\right)\right)=0$.
If $(k,l)\in\left(\mathbb{N}\backslash K_{\eta}\left(\lambda,\varepsilon\right)\right)\times\left(\mathbb{N}\backslash L_{\eta}\left(\lambda,\varepsilon\right)\right)$, then we have

$$\begin{aligned} \eta_{(x_{kl}+y_{kl})-(a+b)}\left(\varepsilon\right) &= \eta_{(x_{kl}-a)+(y_{kl}-b)}\left(\varepsilon\right) \\ &\geq \eta_{x_{kl}-a}\left(\frac{\varepsilon}{2}\right)*\eta_{y_{kl}-b}\left(\frac{\varepsilon}{2}\right) \\ &> (1-\lambda)*(1-\lambda)\geq(1-\xi). \end{aligned}$$

Then we see that

$$\delta\left(\left\{k:\eta_{(x_{kl}-a)+(y_{kl}-b)}\left(\varepsilon\right)\leq 1-\xi\right\}\right)=0$$

and

$$\delta\left(\left\{l:\delta\left(\left\{k:\eta_{(x_{kl}-a)+(y_{kl}-b)}\left(\varepsilon\right)\leq 1-\xi\right\}\right)=0\right\}\right)=1$$

so $st_{\eta(e)}-\lim\limits_{k,l}\left(x_{kl}+y_{kl}\right)=a+b$.

(ii) Case 1: Take $\alpha=0$ and let $st_{\eta(e)}-\lim\limits_{k,l}x_{kl}=a$. Let $\lambda\in(0,1)$ and $\varepsilon>0$. Then we can see that

$$\eta_{0\cdot x_{kl}-0\cdot a}\left(\varepsilon\right)=\eta_{0}\left(\varepsilon\right)=1>1-\lambda.$$

So we get

$$\delta\left(\left\{k:\eta_{0\cdot x_{kl}-0\cdot a}\left(\varepsilon\right)\leq 1-\lambda\right\}\right)=\delta\left(\left\{\varnothing\right\}\right)=0$$

and

$$\delta\left(\left\{l:\delta\left(\left\{k:\eta_{0\cdot x_{kl}-0\cdot a}\left(\varepsilon\right)\leq 1-\lambda\right\}\right)=0\right\}\right)=\delta\left(\mathbb{N}\right)=1.$$

Hence we obtain $st_{\eta(e)}-\lim\limits_{k,l}0\cdot x_{kl}=0$.

Case 2: Take $\alpha\neq 0$. Since $st_{\eta(e)}-\lim\limits_{k,l}x_{kl}=a$, so for every $\varepsilon>0$ and $\lambda\in(0,1)$, we define the sets:

$$L_{\eta}\left(\lambda,\varepsilon\right):=\left\{l:\delta\left(\left\{k:\eta_{x_{kl}-a}\left(\varepsilon\right)\leq 1-\lambda\right\}\right)=0\right\}$$

and

$$K_{\eta}\left(\lambda,\varepsilon\right):=\left\{k:\eta_{x_{kl}-a}\left(\varepsilon\right)\leq 1-\lambda\right\}.$$

Then we see that $\delta\left(K_{\eta}\left(\lambda,\varepsilon\right)\right)=0$ and $\delta\left(L_{\eta}\left(\lambda,\varepsilon\right)\right)=1$. So $\delta\left(\mathbb{N}\backslash K_{\eta}\left(\lambda,\varepsilon\right)\right)=1$ and $\delta\left(\mathbb{N}\backslash L_{\eta}\left(\lambda,\varepsilon\right)\right)=0$. If $(k,l)\in\left(\mathbb{N}\backslash K_{\eta}\left(\lambda,\varepsilon\right)\right)\times\left(\mathbb{N}\backslash L_{\eta}\left(\lambda,\varepsilon\right)\right)$ then

$$\begin{aligned} \eta_{\alpha\cdot x_{kl}-\alpha\cdot a}\left(\varepsilon\right) &= \eta_{x_{kl}-a}\left(\frac{\varepsilon}{|\alpha|}\right) \\ &\geq \eta_{x_{kl}-a}\left(\varepsilon\right)*\eta_{0}\left(\frac{\varepsilon}{|\alpha|}-\varepsilon\right) \\ &= \eta_{x_{kl}-a}\left(\varepsilon\right)*1 \\ &= \eta_{x_{kl}-a}\left(\varepsilon\right)>1-\lambda \end{aligned}$$

for $\alpha\in\mathbb{R}$ $(\alpha\neq 0)$. So

$$\delta\left(\left\{l:\delta\left(\left\{k:\eta_{\alpha\cdot x_{kl}-\alpha\cdot a}\left(\varepsilon\right)\leq 1-\lambda\right\}\right)=0\right\}\right)=1.$$

Hence we obtain $st_{\eta(e)}-\lim\limits_{k,l}\alpha\cdot x_{kl}=\alpha\cdot a$.

(iii) The proof of (iii) can be obtained from (i) and (ii). □

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Λ^2-statistical convergence and its application to Korovkin second theorem

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Abstract. In this paper, we use the notion of strong $(N, \lambda^2)-$summability to generalize the concept of statistical convergence. We call this new method a λ^2-statistical convergence and denote by S_{λ^2} the set of sequences which are λ^2-statistically convergent. We find its relation to statistical convergence and strong $(N, \lambda^2)-$summability. We will define a new sequence space and will show that it is Banach space. Also we will prove the second Korovkin type approximation theorem for λ^2-statistically summability and the rate of λ^2-statistically summability of a sequence of positive linear operators defined from $C_{2\pi}(\mathbb{R})$ into $C_{2\pi}(\mathbb{R})$.

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1. Introduction

By w, we denote the space of all real or complex valued sequences. If $x \in w$, then we simply write $x = (x_k)$ instead of $x = (x_k)_{k=1}^{\infty}$. Let $\Lambda = \{\lambda_k : k = 0, 1, \ldots\}$ be a nondecreasing sequence of positive numbers tending to ∞, as $k \to \infty$ and $\Delta^2 \lambda_n \geq 0$, for each $n \in \mathbb{N}$. The first difference is defined as follows: $\Delta\lambda_k = \lambda_k - \lambda_{k-1}$, where $\lambda_{-1} = \lambda_{-2} = 0$, and the second difference is defined as

$$\Delta^2(\lambda_k) = \Delta(\Delta(\lambda_k)) = \lambda_k - 2\lambda_{k-1} + \lambda_{k-2}.$$

Let $x = (x_k)$ be a sequence of complex numbers, such that $x_{-1} = x_{-2} = 0$. We will denote by

$$\Lambda^2(x) = \frac{1}{\lambda_n - \lambda_{n-1}} \sum_{k=0}^{n} (\lambda_k x_k - 2\lambda_{k-1} x_{k-1} + \lambda_{k-2} x_{k-2}). \tag{1.1}$$

A sequence $x = (x_k)$, is said to be strongly $(N, \lambda^2)-$ summable to a number L (see [8]) if

$$\lim_n \frac{1}{\lambda_n - \lambda_{n-1}} \sum_{k=0}^{n} |(\lambda_k x_k - 2\lambda_{k-1} x_{k-1} + \lambda_{k-2} x_{k-2}) - L| = 0.$$

Let us denote by

$$[N, \lambda^2] = \Big\{ x = (x_n) : \exists L \in \mathbb{C},$$

$$\lim_{n\to\infty} \frac{1}{\lambda_n - \lambda_{n-1}} \sum_{k=1}^{n} |(\lambda_k x_k - 2\lambda_{k-1} x_{k-1} + \lambda_{k-2} x_{k-2}) - L| = 0 \Big\}$$

for the sets of sequences $x = (x_n)$ which are strongly (N, λ^2) summable to L, i.e., $x_k \to L[N, \lambda^2]$. The idea of statistical convergence was introduced by Fast [12] and studied by various authors (see [10], [13], [20], [5], [6]). A sequence $x = (x_k)$ is said to be statistically convergent to the number L if for every $\varepsilon > 0$,

$$\lim_n \frac{1}{n} |\{k \le n : |x_k - L| \ge \varepsilon\}| = 0,$$

where the vertical bars indicate the number of elements in the enclosed set. In this case, we write $S - \lim_n x = L$ or $x_k \to L(S)$ and S denotes the set of all statistically convergent sequences. In this paper, we introduce and study the concept of λ^2-statistical convergence and determine how it is related to $[N, \lambda^2]$ and S.

Definition 1.1. A sequence $x = (x_n)$ is said to be λ^2-statistically convergent or $S_{\lambda^2}-$convergent to L if for every $\varepsilon > 0$

$$\lim_n \frac{1}{\lambda_n - \lambda_{n-1}} |\{k \le n : |(\lambda_k x_k - 2\lambda_{k-1} x_{k-1} + \lambda_{k-2} x_{k-2}) - L| \ge \varepsilon\}| = 0.$$

In this case we write $S_{\lambda^2} - \lim_n x_n = L$ or $x_n \to L(S_{\lambda^2})$, and

$$S_{\lambda^2} = \{x = (x_n) : \exists L \in \mathbb{C}, S_{\lambda^2} - \lim_n x_n = L\}.$$

Definition 1.2. A sequence $x = (x_n)$ is said to be λ^2-statistically Cauchy if for every $\varepsilon > 0$ exists a number $N = N(\varepsilon)$, such that

$$\lim_n \frac{1}{\lambda_n - \lambda_{n-1}} \left|\left\{k \le n : \left|\Delta^2 \lambda_k(x_k) - \Delta^2 \lambda_N(x_N)\right| \ge \varepsilon\right\}\right| = 0.$$

A sequence of positive integers $\theta = (k_r)$ is called lacunary if $k_0 = 0, 0 < k_r < k_{r+1}$ and $h_r = k_r - k_{r-1} \to \infty$ as $r \to \infty$. And with I_r we will denote the following interval:$I_r = (k_{r-1}, k_r]$, respectively q_r the ration: $\frac{k_r}{k_{r-1}}$.

Definition 1.3. A sequence $x = (x_n)$ is said to be lacunary λ^2-statistically convergent or $S^\theta_{\lambda^2}-$convergent to L if for every $\varepsilon > 0$

$$\lim_r \frac{1}{h_r} |\{k \in I_r : |(\lambda_k x_k - 2\lambda_{k-1} x_{k-1} + \lambda_{k-2} x_{k-2}) - L| \ge \varepsilon\}| = 0.$$

In this case we write $S^\theta_{\lambda^2} - \lim_n x_n = L$ or $x_n \to L(S^\theta_{\lambda^2})$, and

$$S^\theta_{\lambda^2} = \{x = (x_n) : \exists L \in \mathbb{C}, S^\theta_{\lambda^2} - \lim_n x_n = L\}.$$

Definition 1.4. A sequence $x = (x_n)$ is said to be lacunary λ^2-statistically Cauchy if for every $\varepsilon > 0$ exists a number $N = N(\varepsilon)$, such that

$$\lim_r \frac{1}{h_r} \left|\left\{k \in I_r : \left|\Delta^2\lambda_k(x) - \Delta^2\lambda_N(x)\right| \geq \varepsilon\right\}\right| = 0.$$

2. Some properties of $[N, \lambda^2]$ and S_{λ^2}

In this section we will show relation between $[N, \lambda^2]$ and S_{λ^2}.

Theorem 2.1. *Let* (λ_n) *be a sequence from* Λ, *then:*

1. $x_n \to L[N, \lambda^2]$, *then* $x_n \to L(S_{\lambda^2})$ *and the inclusion is proper.*
2. *If* $\Delta^2\lambda(x) \in l_\infty$ *and* $x_n \to L(S_{\lambda^2})$, *then* $x_n \to L[N, \lambda^2]$.
3. $S_{\lambda^2} \cap l_\infty = [N, \lambda^2] \cap l_\infty$.

Proof. (1) Let us suppose that $x_n \to L[N, \lambda^2]$. Then for every $\varepsilon > 0$ we have:

$$\sum_{k=1}^{n} |(\lambda_k x_k - 2\lambda_{k-1}x_{k-1} + \lambda_{k-2}x_{k-2}) - L|$$

$$\geq \sum_{\substack{k=1 \\ |(\lambda_k x_k - 2\lambda_{k-1}x_{k-1} + \lambda_{k-2}x_{k-2}) - L| \geq \varepsilon}}^{n} |(\lambda_k x_k - 2\lambda_{k-1}x_{k-1} + \lambda_{k-2}x_{k-2}) - L|$$

$$\geq \varepsilon \left|\{k \leq n : |(\lambda_k x_k - 2\lambda_{k-1}x_{k-1} + \lambda_{k-2}x_{k-2}) - L| \geq \varepsilon\}\right|.$$

Therefore $x_n \to L[N, \lambda^2] \Rightarrow x_n \to L(S_{\lambda^2})$. To prove the second part of the (1), we will show this.

Example 2.2. Let $x = x_n$ defined as follows:

$$x_n = \begin{cases} [\sqrt{\lambda_n - \lambda_{n-1}}], & 0 \leq k \leq n \\ 0, & \text{otherwise.} \end{cases}$$

Then $x = (x_n) \notin l_\infty$ and for every $\varepsilon > 0$, we get that

$$\lim_n \frac{1}{\lambda_n - \lambda_{n-1}} \left|\{k \leq n : |(\lambda_k x_k - 2\lambda_{k-1}x_{k-1} + \lambda_{k-2}x_{k-2}) - 0| \geq \varepsilon\}\right|$$

$$\leq \lim_n \frac{[\sqrt{\lambda_n - \lambda_{n-1}}]}{\lambda_n - \lambda_{n-1}} = 0.$$

On the other hand

$$\lim_{n\to\infty} \frac{1}{\lambda_n - \lambda_{n-1}} \sum_{k=1}^{n} |(\lambda_k x_k - 2\lambda_{k-1}x_{k-1} + \lambda_{k-2}x_{k-2}) - 0|$$

$$= \lim_n \frac{\lambda_n[\sqrt{\lambda_n - \lambda_{n-1}}] - 2\lambda_{n-1}[\sqrt{\lambda_{n-1} - \lambda_{n-2}}] + \lambda_{n-2}[\sqrt{\lambda_{n-2} - \lambda_{n-3}}]}{\lambda_n - 2\lambda_{n-1} + \lambda_{n-2}} = \infty.$$

(2) Let us suppose that $x_n \to L(S_{\lambda^2})$ and $\Delta^2\lambda(x) \in l_\infty$, then we can consider that

$$|\lambda_k x_k - 2\lambda_{k-1}x_{k-1} + \lambda_{k-2}x_{k-2} - L| \leq M.$$

For any given $\varepsilon > 0$ we get the following estimation:

$$\frac{1}{\lambda_n - \lambda_{n-1}} \sum_{k=1}^{n} |(\lambda_k x_k - 2\lambda_{k-1}x_{k-1} + \lambda_{k-2}x_{k-2}) - L|$$

$$= \frac{1}{\lambda_n - \lambda_{n-1}} \sum_{\substack{k=1 \\ |(\lambda_k x_k - 2\lambda_{k-1}x_{k-1} + \lambda_{k-2}x_{k-2}) - L| \geq \varepsilon}}^{n} |(\lambda_k x_k - 2\lambda_{k-1}x_{k-1} + \lambda_{k-2}x_{k-2}) - L|$$

$$+ \frac{1}{\lambda_n - \lambda_{n-1}} \sum_{\substack{k=1 \\ |(\lambda_k x_k - 2\lambda_{k-1}x_{k-1} + \lambda_{k-2}x_{k-2}) - L| \leq \varepsilon}}^{n} |(\lambda_k x_k - 2\lambda_{k-1}x_{k-1} + \lambda_{k-2}x_{k-2}) - L|$$

$$\leq \frac{M}{\lambda_n - \lambda_{n-1}} |\{k \leq n : |(\lambda_k x_k - 2\lambda_{k-1}x_{k-1} + \lambda_{k-2}x_{k-2}) - L| \geq \varepsilon\}| + \varepsilon,$$

which implies that $x_k \to L[N, \lambda^2]$.
(3) Follows immediately from (1) and (2). □

Proposition 2.3. *If $x = (x_n)$ is λ^2-statistically convergent to L, then it follows that x is λ^2-statistically Cauchy sequence.*

Proof. Let us suppose that x converges Λ^2-statistically to L and $\varepsilon > 0$. Then

$$\frac{1}{\lambda_n - \lambda_{n-1}} |\{k \leq n : |(\lambda_k x_k - 2\lambda_{k-1}x_{k-1} + \lambda_{k-2}x_{k-2}) - L| \geq \varepsilon\}| \leq \frac{\varepsilon}{2}$$

satisfies for almost all k, and if N is chosen such that

$$\frac{1}{\lambda_N - \lambda_{N-1}} |\{k \leq N : |(\lambda_N x_N - 2\lambda_{N-1}x_{N-1} + \lambda_{N-2}x_{N-2}) - L| \geq \varepsilon\}| \leq \frac{\varepsilon}{2},$$

then we have:

$$\frac{1}{\lambda_n - \lambda_{n-1}} \left|\{k \leq n : \left|\Delta^2\lambda_k(x) - \Delta^2\lambda_N(x)\right| \geq \varepsilon\}\right| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

for almost k. Hence x is λ^2-statistically Cauchy sequence. □

Proposition 2.4. *If $x = (x_n)$ is lacunary λ^2-statistically convergent to L, then it follows that x is λ^2-statistically lacunary Cauchy sequence.*

Proposition 2.5. *If $x = (x_n)$ is a sequence for which there is a λ^2-statistically convergent sequence $y = (y_n)$ such that $\Delta^2\lambda(x_k) = \Delta^2\lambda(y_k)$ for almost all k, then it follows that x is λ^2-statistically convergent sequence.*

Proof. Let us consider that $\Delta^2\lambda(x_k) = \Delta^2\lambda(y_k)$ for almost all k. And $y_k \to L(S_{\lambda^2})$. Then for each $\varepsilon > 0$ and for every n we have:

$$\left\{k \leq n : |\Delta^2\lambda(x_k) - L| \geq \varepsilon\right\}$$
$$\subset \left\{k \leq n : \Delta^2\lambda(x_k) \neq \Delta^2\lambda(y_k)\right\} \cup \left\{k \leq n : |\Delta^2\lambda(y_k) - L| \geq \varepsilon\right\}.$$

From fact that $y_k \to L(S_{\lambda^2})$, it follows that set $\{k \le n : |\Delta^2\lambda(y_k) - L| \ge \varepsilon\}$ has finite numbers which are not depended from n, hence

$$\frac{\left|\{k \le n : |\Delta^2\lambda(y_k) - L| \ge \varepsilon\}\right|}{\lambda_n - \lambda_{n-1}} \to 0, n \to \infty.$$

On the other hand, from $\Delta^2\lambda(x_k) = \Delta^2\lambda(y_k)$ for almost all k, we get:

$$\frac{\left|\{k \le n : \Delta^2\lambda(x_k) \ne \Delta^2\lambda(y_k) \ge \varepsilon\}\right|}{\lambda_n - \lambda_{n-1}} \to 0, n \to \infty.$$

From last two relations follows that:

$$\frac{\left|\{k \le n : |\Delta^2\lambda(x_k) - L| \ge \varepsilon\}\right|}{\lambda_n - \lambda_{n-1}} \to 0, n \to \infty.$$ □

Proposition 2.6. *If $x = (x_n)$ is a sequence for which there is a lacunary λ^2-statistically convergent sequence $y = (y_n)$ such that $\Delta^2\lambda(x_k) = \Delta^2\lambda(y_k)$ for almost all k, then it follows that x is lacunary λ^2-statistically convergent sequence.*

Theorem 2.7. *Let θ be a lacunary sequence, then*

1. $L(S_{\lambda^2}) \subset L(S^\theta_{\lambda^2})$ *if and only if* $\lim_r \inf q_r > 1$.
2. $L(S^\theta_{\lambda^2}) \subset L(S_{\lambda^2})$ *if and only if* $\lim_r \sup q_r < \infty$.
3. $L(S_{\lambda^2}) = L(S^\theta_{\lambda^2})$ *if and only if* $1 < \lim_r \inf q_r \le \lim_r \sup q_r < \infty$.

Proof. Proof of the Proposition is omitted, because it is similar to Lemmas 2,3 in [14]. □

We will denote by $\Lambda^2(X) = \{x = (x_n) \in w : \Lambda^2(x) \in X\}$. It is know that $(\Lambda^2(X), ||\cdot||_{\Lambda^2(X)})$ is a normed space where norm is given by (see [8]):

$$||x||_{\Lambda^2(X)} := \sup_{n\ge 0} \frac{1}{\lambda_n - \lambda_{n-1}} \sum_{k=0}^{n} |\lambda_k x_k - 2\lambda_{k-1}x_{k-1} + \lambda_{k-2}x_{k-2}|,$$

where $x = (x_k)$.

Theorem 2.8. $\Lambda^2(X)$ *is Banach space.*

Proof. Let (x_n) be any Cauchy sequence in $\Lambda^2(X)$, where $x^s = (x^s_1, x^s_2, \cdots, x^s_n, \cdots)$. Then there it follows that:

$$||x^s - x^t||_{\Lambda^2(X)} \to 0, s, t \to \infty.$$

From last relation we get:

$$\sup_{n\ge 0} \frac{1}{\lambda_n - \lambda_{n-1}} \sum_{k=0}^{n} \left|\lambda_k(x^s_k - x^t_k) - 2\lambda_{k-1}(x^s_{k-1} - x^t_{k-1}) + \lambda_{k-2}(x^s_{k-2} - x^t_{k-2})\right| \to 0,$$
$$t, s \to \infty.$$

Hence we obtain,

$$|x^t_k - x^s_k| \to 0, t, s \to \infty,$$

for every $k \in \mathbb{N}$. Therefore $(x_k^1, x_k^2, \cdots)$ is a Cauchy sequences in $\mathbb{C}$, the set of complex numbers. Since $\mathbb{C}$ is complete, it is convergent. Let us say

$$\lim_s x_k^s = x_k,$$

for every $k \in \mathbb{N}$. Since (x^s) is a Cauchy sequence, for each $\varepsilon > 0$, there exists a natural number $N = N(\varepsilon)$ such that

$$||x^s - x^t||_{\Lambda^2(X)} < \varepsilon$$

for all $s, t \geq N$ and for all $k \in \mathbb{N}$. Hence

$$\sup_{n\geq 0} \frac{1}{\lambda_n - \lambda_{n-1}} \sum_{k=0}^{n} \left|\lambda_k(x_k^s - x_k^t) - 2\lambda_{k-1}(x_{k-1}^s - x_{k-1}^t) + \lambda_{k-2}(x_{k-2}^s - x_{k-2}^t)\right| < \varepsilon,$$

for all $s, t \geq N$ and for all $k \in \mathbb{N}$. If we pass with limit, in the last relation, when $t \to \infty$, we get:

$$\lim_t \sup_{n\geq 0} \frac{1}{\lambda_n - \lambda_{n-1}} \sum_{k=0}^{n} \left|\lambda_k(x_k^s - x_k^t) - 2\lambda_{k-1}(x_{k-1}^s - x_{k-1}^t) + \lambda_{k-2}(x_{k-2}^s - x_{k-2}^t)\right|$$

$$= \sup_{n\geq 0} \frac{1}{\lambda_n - \lambda_{n-1}} \sum_{k=0}^{n} \left|\lambda_k(x_k^s - x_k) - 2\lambda_{k-1}(x_{k-1}^s - x_{k-1}) + \lambda_{k-2}(x_{k-2}^s - x_{k-2})\right| < \varepsilon,$$

for all $s \geq N$ and for all $k \in \mathbb{N}$. This implies that

$$||x^s - x||_{\Lambda^2(X)} < \varepsilon,$$

for all $s \geq N$, that is $x^s \to x$, as $s \to \infty$ where $x = (x_k)$.
Since

$$||x||_{\Lambda^2(x)} = \sup_{n\geq 0} \frac{1}{\lambda_n - \lambda_{n-1}} \sum_{k=0}^{n} |\lambda_k x_k - 2\lambda_{k-1}x_{k-1} + \lambda_{k-2}x_{k-2}|$$

$$= \sup_{n\geq 0} \frac{1}{\lambda_n - \lambda_{n-1}} \sum_{k=0}^{n} |\lambda_k(x_k - x_k^N + x_k^N) - 2\lambda_{k-1}(x_{k-1} - x_{k-1}^N + x_{k-1}^N)$$

$$+\lambda_{k-2}(x_{k-2} - x_{k-2}^N + x_{k-2}^N)|$$

$$\leq \sup_{n\geq 0} \frac{1}{\lambda_n - \lambda_{n-1}} \sum_{k=0}^{n} \left|\lambda_k(x_k - x_k^N) - 2\lambda_{k-1}(x_{k-1} - x_{k-1}^N) + \lambda_{k-2}(x_{k-2} - x_{k-2}^N)\right|$$

$$+ \sup_{n\geq 0} \frac{1}{\lambda_n - \lambda_{n-1}} \sum_{k=0}^{n} \left|\lambda_k x_k^N - 2\lambda_{k-1}x_{k-1}^N + \lambda_{k-2}x_{k-2}^N\right|$$

$$\leq ||x^N - x||_{\Lambda^2(X)} + ||x^N||_{\Lambda^2(X)} = O(1),$$

we obtain $x \in \Lambda^2(X)$. □

3. A Korovkin second type theorem

We say that the sequence (x_n) is Λ^2- summable to L if $\lim_n \Lambda^2 = L$.

Definition 3.1. We say that the sequence (x_n) is statistically summable to L by the weighted method determined by the sequence Λ^2 if $st - \lim_n \Lambda^2 = L$.

And we denote by $\Lambda^2(st)$ the set of all sequences which are statistically summable Λ^2. In the sequel we will use some notation related to the function spaces. With $F(\mathbb{R})$ we will denote the linear space of all real-valued functions defined in $\mathbb{R}$. And with $C(\mathbb{R})$ we will denote the space of all bounded and continuous functions defined in $\mathbb{R}$. It is know fact that $C(\mathbb{R})$ is a Banach space equipped with norm

$$||f||_\infty = \sup_{x\in\mathbb{R}} |f(x)|, f \in C(\mathbb{R}).$$

The space of all continuous and periodic functions with period 2π we will denote by $C_{2\pi}(\mathbb{R})$, which is a Banach space under norm given by

$$||f||_{2\pi} = \sup_{x\in\mathbb{R}} |f(x)|.$$

The classical Korovkin first and second theorems are given as follows (see [16, 17, 3]):

Theorem 3.2. *Let (T_n) be a sequence of positive linear operators from $C[0,1]$ into $F[0,1]$. Then*

$$\lim_{n\to\infty} ||T_n(f,x) - f(x)||_\infty = 0,$$

for all $f \in C[0,1]$ if and only if

$$\lim_{n\to\infty} ||T_n(f_i,x) - f_i(x)||_\infty = 0,$$

for $i \in \{0,1,2\}$ where $f_0(x) = 1$, $f_1(x) = x$ and $f_2(x) = x^2$.

Theorem 3.3. *Let (T_n) be a sequence of positive linear operators from $C_{2\pi}(\mathbb{R})$ into $F(\mathbb{R})$. Then*

$$\lim_{n\to\infty} ||T_n(f,x) - f(x)||_{2\pi} = 0,$$

for all $f \in C_{2\pi}(\mathbb{R})$ if and only if

$$\lim_{n\to\infty} ||T_n(f_i,x) - f_i(x)||_{2\pi} = 0,$$

for $i \in \{0,1,2\}$ where $f_0(x) = 1$, $f_1(x) = \cos x$ and $f_2(x) = \sin x$.

The Korovkin type theorems are investigated by several mathematicians in generalization of them in many ways and several settings such as function spaces, abstract Banach latices, Banach algebras, and so on. This theory is useful in real analysis, functional analysis, harmonic analysis, and so on. For more results related to the Korovkin type theorems see ([4, 11, 19, 21, 22, 24, 9, 7, 18, 2, 1, 23, 15]). In this paper we will prove the second Korovkin-type theorem with the help of Λ^2-statistically summability method which is a generalization of that given in [19] and [16, 17].

For given sequence of linear operators L_n we say that they are positive if $L_n(f(x)) \geq 0$ for all $f(x) \geq 0$, for given x.

Theorem 3.4. *Let (T_n) be a sequence of positive linear operators from $C_{2\pi}(\mathbb{R})$ into $C_{2\pi}(\mathbb{R})$. Then*

$$\Lambda^2(st)\text{-}\lim_{n\to\infty} ||T_n(f,x) - f(x)||_{2\pi} = 0, \ forall \ f \in C_{2\pi}(\mathbb{R}) \tag{3.1}$$

if and only if

$$\Lambda^2(st)\text{-}\lim_{n\to\infty} ||T_n(f_i,x) - f_i(x)||_{2\pi} = 0, \ i = 0,1,2, \tag{3.2}$$

where $f_0(x) = 1$, $f_1(x) = \cos x$ and $f_2(x) = \sin x$.

Proof. Let us consider that relation (3.1) is valid for all $f \in C_{2\pi}(\mathbb{R})$. Then it is valid especially for the $f(x) = 1, f(x) = \cos x$ and $f(x) = \sin x$, and condition (3.2) is valid. Now we will prove the contrary. Let us suppose that relations (3.2) is valid and we will prove that (3.1) is valid, too. Let $I = (a, a + 2\pi]$ any subinterval of length 2π from $\mathbb{R}$. Let us fix $x \in I$. By the conditions given for $f(x)$ it follows that:

$$(\forall \varepsilon > 0)(\exists \delta(\varepsilon) > 0) \to |f(t) - f(x)| < \varepsilon, \tag{3.3}$$

for all t, whenever $|t-x| < \delta$. If $|t-x| \geq \delta$. Let us consider that $t \in (x+\delta, 2\pi+x+\delta]$, then we get:

$$|f(t) - f(x)| \leq 2||f||_{2\pi} \leq \frac{2||f||_{2\pi}}{\sin^2 \frac{\delta}{2}} \psi(t) \tag{3.4}$$

where $\psi(t) = \sin^2\left(\frac{t-x}{2}\right)$. From relations (3.3) and (3.4) for any fixed $x \in I$ and for any t we obtain:

$$|f(t) - f(x)| \leq \frac{2||f||_{2\pi}}{\sin^2 \frac{\delta}{2}} \psi(t) + \varepsilon. \tag{3.5}$$

Respectively,

$$-\varepsilon - \frac{2||f||_{2\pi}}{\sin^2 \frac{\delta}{2}} \psi(t) < f(t) - f(x) < \frac{2||f||_{2\pi}}{\sin^2 \frac{\delta}{2}} \psi(t) + \varepsilon.$$

Applying the operator $T_n(1,x)$ in this inequality we have:

$$T_k(1,x)\left(-\varepsilon - \frac{2||f||_{2\pi}}{\sin^2 \frac{\delta}{2}} \psi(t)\right) < T_k(1,x)\left(f(t) - f(x)\right) < T_k(1,x)\left(\frac{2||f||_{2\pi}}{\sin^2 \frac{\delta}{2}} \psi(t) + \varepsilon\right).$$

Value of x is fixed, which means that $f(x)$ is a constant and above relation takes this form:

$$\begin{aligned} -\varepsilon T_k(1,x) - \frac{2||f||_{2\pi}}{\sin^2 \frac{\delta}{2}} T_k(\psi(t),x) &< T_k(f,x) - f(x)T_k(1,x) \\ &< \frac{2||f||_{2\pi}}{\sin^2 \frac{\delta}{2}} T_k(\psi(t),x) + \varepsilon T_k(1,x). \end{aligned} \tag{3.6}$$

On the other hand

$$T_k(f,x) - f(x) = T_k(f,x) - f(x)T_k(1,x) + f(x)[T_k(1,x) - 1]. \tag{3.7}$$

From relations (3.6) and (3.7) we have:

$$T_k(f,x) - f(x) < \frac{2||f||_{2\pi}}{\sin^2 \frac{\delta}{2}} T_k(\psi(t),x) + \varepsilon T_k(1,x) + f(x)[T_k(1,x) - 1]. \tag{3.8}$$

Let us now estimate the following expression:

$$T_k(\psi(t),x) = T_k\left(\sin^2\left(\frac{t-x}{2}\right),x\right) = T_k\left(\frac{1}{2}(1-\cos t\cos x-\sin t\sin x),x\right)$$
$$= \frac{1}{2}\left\{T_k(1,x)-\cos xT_k(\cos t,x)-\sin xT_k(\sin t,x)\right\}$$
$$= \frac{1}{2}\left\{[T_k(1,x)-1]-\cos x[T_k(\cos t,x)-\cos x]-\sin x[T_k(\sin t,x)-\sin x]\right\}.$$

Now, from the last relation and (3.8), we obtain that

$$T_k(f,x)-f(x) < \frac{2||f||_{2\pi}}{\sin^2\frac{\delta}{2}}\frac{1}{2}\Big\{[T_k(1,x)-1]-\cos x[T_k(\cos t,x)-\cos x]$$
$$-\sin x[T_k(\sin t,x)-\sin x]\Big\}+\varepsilon T_k(1,x)+f(x)[T_k(1,x)-1]$$
$$=\varepsilon+\varepsilon[T_k(1,x)-1]+f(x)[T_k(1,x)-1]+\frac{2||f||_{2\pi}}{\sin^2\frac{\delta}{2}}\frac{1}{2}\Big\{[T_k(1,x)-1]$$
$$-\cos x[T_k(\cos t,x)-\cos x]-\sin x[T_k(\sin t,x)-\sin x]\Big\}.$$

Therefore,

$$|T_k(f,x)-f(x)| \le \varepsilon+\left(\varepsilon+|f(x)|+\frac{2||f||_{2\pi}}{\sin^2\frac{\delta}{2}}\right)|T_k(1,x)-1|$$
$$+\frac{2||f||_{2\pi}}{\sin^2\frac{\delta}{2}}\Big\{|\cos x|\cdot|T_k(\cos t,x)-\cos x|$$
$$+|\sin x|\cdot|T_k(\sin t,x)-\sin x|\Big\}$$
$$\le \varepsilon+\left(\varepsilon+|f(x)|+\frac{2||f||_{2\pi}}{\sin^2\frac{\delta}{2}}\right)|T_k(1,x)-1|$$
$$+\frac{2||f||_{2\pi}}{\sin^2\frac{\delta}{2}}\Big\{|T_k(\cos t,x)-\cos x|+|T_k(\sin t,x)-\sin x|\Big\}.$$

Now taking the $\sup_{x\in I}$ in the above relation, we get:

$$||T_k(f,x)-f(x)||_{2\pi} \le \varepsilon+K\Big(||T_k(1,x)-1||_{2\pi}+||T_k(\cos t,x)-\cos x||_{2\pi}$$
$$+||T_k(\sin t,x)-\sin x||_{2\pi}\Big),$$

where

$$K=\max\left\{\varepsilon+||f||_{2\pi}+\frac{2||f||_{2\pi}}{\sin^2\frac{\delta}{2}},\frac{2||f||_{2\pi}}{\sin^2\frac{\delta}{2}}\right\}.$$

Now replacing $T_k(.,x)$ by

$$\Lambda^2(.,x)=\frac{1}{\lambda_n-\lambda_{n-1}}\sum_{k=0}^{n}(\lambda_kT_k(.,x)-2\lambda_{k-1}T_{k-1}(.,x)+\lambda_{k-2}T_{k-2}(.,x))$$

in the above inequality on both sides. For a given $r > 0$, we can choose ε_1 such that $\varepsilon_1 < r$. Now we will define the following sets:

$$D = \Big\{k \leq \mathbb{N} : ||\Lambda^2(f, x) - f(x)||_{2\pi} \geq r\Big\},$$

$$D_i = \Big\{k \leq \mathbb{N} : ||\Lambda^2(f_i, x) - f_i(x)||_{2\pi} \geq \frac{r - \varepsilon_1}{3K}\Big\},\ i = 0, 1, 2.$$

Then $D \subset \cup_{i=0}^{2} D_i$ and for their densities is satisfied relation:

$$\delta(D) \leq \delta(D_0) + \delta(D_1) + \delta(D_2).$$

Finally, from relations (3.2) and the above estimation we get:

$$\Lambda^2(st)\text{-}\lim_n ||\Lambda^2(f, x) - f(x)||_{2\pi} = 0,$$

which completes the proof. □

Remark 3.5. If we take $\lambda_n = n^2$, then our Theorem 3.4 reduce to Theorem 2.1 of [19].

4. Rate of Λ^2- statistically convergence

In this section we will show the rate of the Λ^2- statistical convergence of positive linear operators in $C_{2\pi}(\mathbb{R})$ spaces.

Definition 4.1. Let (a_n) be any positive, nondecreasing sequence of positive numbers. We say that sequence $x = (x_n)$ is Λ^2- statistical convergent to number L with rate of convergence $o(a_n)$, if for every $\varepsilon > 0$,

$$\lim_n \frac{1}{a_n} |\{m \leq n : |T_m - L| \geq \varepsilon\}| = 0.$$

In this case, we write $x_n - L = \Lambda^2(st) - o(a_n)$.

Lemma 4.2. *Let (a_n) and (b_n) be two positive nondecreasing positive numeric sequences. Let $x = (x_n)$ and $y = (y_n)$ be two sequences such that $x_n - L_1 = \Lambda^2(st) - o(a_n)$ and $y_n - L_2 = \Lambda^2(st) - o(b_n)$. Then*

1. *$\alpha(x_n - L) = \Lambda^2(st) - o(a_n)$, for any scalar α.*
2. *$(x_n - L_1) \pm (y_n - L_2) = \Lambda^2(st) - o(c_n)$.*
3. *$(x_n - L_1)(y_n - L_2) = \Lambda^2(st) - o(a_n b_n)$,*

where $c_n = \max\{a_n, b_n\}$.

Now let us recall the notion of the modules of continuity. The modulus of continuity for function $f(x) \in C_{2\pi}(\mathbb{R})$, is defined as follows:

$$\omega(f, \delta) = \sup_{|h|<\delta} |f(x + h) - f(x)|.$$

It is known that, for any value of the $|x - y|$, we get:

$$|f(x) - f(y)| \leq \omega(f, \delta)\left(\frac{|x - y|}{\delta} + 1\right). \tag{4.1}$$

We have the following result:

Theorem 4.3. *Let (T_n) be a sequence of positive linear operators from $C_{2\pi}(\mathbb{R})$ into $C_{2\pi}(\mathbb{R})$. Suppose that*

1. $||T_n(1,x)-1||_{2\pi} = \Lambda^2(st) - o(a_n)$.
2. $\omega(f,\lambda_k) = \Lambda^2(st) - o(b_n)$, *where* $\lambda_n = \sqrt{T_n(\phi_x,x)}$ *and* $\phi_x(y) = (y-x)^2$.

Then for all $f \in C_{2\pi}(\mathbb{R})$, we have:

$$||T_n(f,x) - f(x)||_{2\pi} = \Lambda^2(st) - o(c_n),$$

where $c_n = \max\{a_n, b_n\}$.

Proof. Let $f \in C_{2\pi}(\mathbb{R})$ and $x \in [-\pi,\pi]$. From relations (3.7) and (4.1) we get this estimation:

$$\begin{aligned}|T_n(f,x) - f(x)| &\le |T_n(|f(y)-f(x)|,x)| + |f(x)|\cdot|T_n(1,x)-1| \\ &\le T_n\left(\frac{|x-y|}{\delta}+1,x\right)\omega(f,\delta) + |f(x)|\cdot|T_n(1,x)-1| \\ &\quad \text{(by Cauchy-Schwartz inequality)} \\ &\le \frac{1}{\delta}(T_n((x-y)^2,x))^{\frac{1}{2}}(T_n(1,x))^{\frac{1}{2}}\omega(f,\delta) + |f(x)|\cdot|T_n(1,x)-1|.\end{aligned}$$

If we are putting $\delta = \lambda_n = \sqrt{T_n(\phi_x,x)}$ in the last relation we obtain:

$$\begin{aligned}||T_n(f,x)-f(x)||_{2\pi} &\le ||f||_{2\pi}||T_n(1,x)-1||_{2\pi} + 2\omega(f,\lambda_n) \\ &\quad + \omega(f,\lambda_n)||T_n(1,x)-1||_{2\pi} + \omega(f,\lambda_n)\sqrt{||T_n(1,x)-1||_{2\pi}} \\ &\le C\Big\{||T_n(1,x)-1||_{2\pi} + \omega(f,\lambda_n) + \omega(f,\lambda_n)||T_n(1,x)-1||_{2\pi} \\ &\quad + \omega(f,\lambda_n)\sqrt{||T_n(1,x)-1||_{2\pi}}\Big\},\end{aligned}$$

where $C = \max\big\{||f||_{2\pi}, 2\big\}$. Now replacing $T_k(.,x)$ by

$$\Lambda^2(.,x) = \frac{1}{\lambda_n - \lambda_{n-1}}\sum_{k=0}^{n}(\lambda_k T_k(.,x) - 2\lambda_{k-1}T_{k-1}(.,x) + \lambda_{k-2}T_{k-2}(.,x)),$$

we get

$$\begin{aligned}||\Lambda^2(f,x)-f(x)||_{2\pi} &\le C\Big\{||\Lambda^2(1,x)-1||_{2\pi} + \omega(f,\lambda_n) + \omega(f,\lambda_n)||\Lambda^2(1,x)-1||_{2\pi} \\ &\quad + \sqrt{\omega(f,\lambda_n)||\Lambda^2(1,x)-1||_{2\pi}}\Big\}.\end{aligned}$$

The proof follows from the conditions (1) and (2). □

In the following example we show that Theorem 3.4 is stronger than Theorem 3.3.

Example 4.4. For any $n \in \mathbb{N}$ we will denote by $S_n(f)$ the $n-$th partial sum of the Fourier series of f, i.e.,

$$S_n(f) = \frac{a_0}{2} + \sum_{k=1}^{n} a_k \cos kx + b_k \sin kx.$$

Let us consider the following expression:

$$\Lambda^2(f,x) = \frac{1}{\lambda_n - \lambda_{n-1}} \sum_{k=0}^{n} (\lambda_k - 2\lambda_{k-1} + \lambda_{k-2}) S_k(f).$$

We know that $\lim_{n\to\infty} \Lambda^2(f,x) = f$ (see [8]). Let us denote by $L_n : C_{2\pi}(\mathbb{R}) \to C_{2\pi}(\mathbb{R})$ defined by:

$$L_n(f,x) = (1 + x_n)\Lambda^2(f,x)$$

where (x_n) is defined as follow:

$$x_n := \begin{cases} 1 & (n \quad \text{odd}) \\ -1 & (n \quad \text{even}). \end{cases} \tag{4.2}$$

After some calculations we have:

$$\begin{aligned} &\Lambda^2(1,x) = 1, \\ &\Lambda^2(\cos t, x) = \cos x, \\ &\Lambda^2(\sin t, x) = \sin x. \end{aligned}$$

We see that conditions (3.2) are satisfied, and by Theorem 3.4, it follows that

$$\Lambda^2(st)\text{-}\lim_n ||L_n(f,x) - f||_{2\pi} = 0,$$

but Theorem 3.3 does't hold.

Remark 4.5. Based in the previous example and Remark 3.5, we show that our Theorem 3.4 is also stronger than Theorem 2.1 due to Mohiuddine and Alotaibi [19].

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The study of the solution of a Fredholm-Volterra integral equation by Picard operators

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Abstract. In this paper we will use the Picard operators technique, in order to establish the existence and uniqueness, data dependence and Gronwall-type results for the solutions of a Fredholm-Volterra functional-integral equation. The paper ends with a result of the Ulam-Hyers stability of this integral equation.

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Keywords: Picard operators, Fredholm integral equation, Volterra integral equation, data dependence, integral inequalities, Ulam-Hyers stability.

1. Introduction

The theory of integral equations has many applications in describing of numerous phenomena and problems from different research fields of the surrounding world, such as: mathematical physics, engineering, biology, economics and others. In what follows, we consider the following Fredholm-Volterra functional-integral equation:

$$x(t) = F(t, g(t, x(t)), I_{Fr}(t, s, a, b, x, K_1, h_1), I_{Vo}(t, s, a, x, K_2, h_2)), \tag{1.1}$$

where we denote:

$$I_{Fr}(t, s, a, b, x, K_1, h_1) = \int_a^b K_1(t, s) \cdot h_1(s, x(s), x(a), x(b))ds$$

$$I_{Vo}(t, s, a, x, K_2, h_2) = \int_a^t K_2(t, s) \cdot h_2(s, x(s), x(a))ds$$

and

$$F : [a, b] \times \mathbb{R}^3 \to \mathbb{R},\ K_1, K_2 : [a, b] \times [a, b] \to \mathbb{R},$$
$$h_1 : [a, b] \times \mathbb{R}^3 \to \mathbb{R},\ h_2 : [a, b] \times \mathbb{R}^2 \to \mathbb{R},\ g : [a, b] \times \mathbb{R} \to \mathbb{R},$$

and we will apply the Picard operators technique to prove the existence and uniqueness, data dependence, comparison and Gronwall-type results for the solution of the equation (1.1). Many authors have applied this technique to study the functional-integral equations of mixed type (see [1], [2], [6], [9], [19], [27], etc.). Also, many authors studied the functional-integral equations of Fredholm and Volterra type and we mention some of them (see [1], [3], [7], [8], [10], [11], [12], [13], [14], [16], [17], [18] [23], [24], [25], [26], [28], etc.).

In this paper we will use the notations from [22], [23] and [25] and we recall some of them.

Let (X, d) be a metric space and $A : X \to X$ an operator. We have:

$P(X) := \{Y \subset X \ / \ Y \neq \emptyset\} -$ the set of all nonempty subsets of X,

$I(A) := \{Y \in P(X) \ / \ A(Y) \subset Y\} -$ the family of the nonempty subsets of X, invariant for A,

$F_A := \{x \in X | A(x) = x\} -$ the fixed points set of A.

Also, we denote by $A^0 := 1_X, \ A^1 := A, \ A^{n+1} := A \circ A^n, \ n \in N$ – the iterate operators of A.

Below, we present the definitions of Picard operator, c-Picard operator and weakly Picard operator.

Definition 1.1. Let (X, d) be a metric space. An operator $A : X \to X$ is called Picard operator (briefly PO) if there exists $x^* \in X$ such that:
(a) $F_A = \{x^*\}$;
(b) the sequence $(A^n(x_0))_{n \in \mathbb{N}}$ converges to x^*, for all $x_0 \in X$.

Definition 1.2. Let (X, d) be a metric space and $c > 0$. An operator $A : X \to X$ is called c-Picard operator (briefly c-PO) if A is PO and

$$d(x, x^*) \leq c \cdot d(x, A(x)) \text{ for all } x \in X.$$

Definition 1.3. Let (X, d) be a metric space. An operator $A : X \to X$ is called weakly Picard operator (briefly WPO) if the sequence $(A^n(x_0))_{n \in \mathbb{N}}$ converges for all $x_0 \in X$ and the limit (which may depend on x_0) is a fixed point of A.

If A is a WPO, then it can be considered the operator $A^\infty : X \to X$, defined by

$$A^\infty(x) := \lim_{n \to \infty} A^n(x)$$

and we observe that $A^\infty(X) = F_A$.

In addition, if A is a PO and we denote by x^* its unique fixed point, then $A^\infty(x) = x^*$, for all $x \in X$.

In the second section we study the existence and uniqueness of the solution of the integral equation (1.1).

In order to obtain the presented results of this section, we applied the Picard operators technique and the Contraction Principle.

Theorem 1.4 (Contraction Principle). *Let (X, d) be a complete metric space and $A : X \to X$ an α-contraction ($\alpha < 1$). Under these conditions we have:*

(i) $F_A = \{x^*\}$;

(ii) $x^* = \lim_{n \to \infty} A^n(x_0)$, *for all* $x_0 \in X$;

(*iii*) $d(x^*, A^n(x_0)) \leq \frac{\alpha^n}{1-\alpha} d(x_0, A(x_0))$.

In order to obtain several Gronwall-type and comparison results for the solution of the integral equation (1.1), in the third section we will use the Abstract Comparison Lemma, the Abstract Gronwall Lemma and the Abstract Gronwall-Comparison Lemma, which we present below.

Lemma 1.5. (*see* [25]) *Let* $(X, d, \leq)$ *be an ordered metric space and* $A : X \to X$ *an operator. If:*

(*i*) *A is an increasing operator;*
(*ii*) *the operator A is a WPO,*

then the operator A^∞ is increasing.

Lemma 1.6 (Abstract Comparison Lemma). (*see* [22], [23], [25]) *Let* $A, B, C : X \to X$ *be three operators defined on the ordered metric space* $(X, d, \leq)$. *If:*

(*i*) $A \leq B \leq C$;
(*ii*) *A, B, C are WPOs;*
(*iii*) *the operator B is increasing,*

then

$$x \leq y \leq z \Rightarrow A^\infty(x) \leq B^\infty(y) \leq C^\infty(z).$$

Remark 1.7. Let A, B, C be the operators defined in the Abstract Comparison Lemma. In addition, we suppose that B is PO, i.e. $F_B = \{x_B^*\}$. Then we have

$$A^\infty(x) \leq x_B^* \leq C^\infty(x), \; for \; all \; x \in X.$$

But $A^\infty(X) = F_A$ and $C^\infty(X) = F_C$ and therefore $F_A \leq x_B^* \leq F_C$.

Lemma 1.8 (Abstract Gronwall Lemma). (*see* [22], [23], [25]) *Let* $A : X \to X$ *be an operator defined on the ordered metric space* $(X, d, \leq)$. *If:*

(*i*) *the operator A is PO and denote by x_A^* the unique fixed point of A;*
(*ii*) *A is an increasing operator,*

then

(*a*) $x \leq A(x) \Rightarrow x \leq x_A^*$;
(*b*) $x \geq A(x) \Rightarrow x \geq x_A^*$.

Lemma 1.9 (Abstract Gronwall-Comparison Lemma). (*see* [22], [23], [25]) *Let* $A_1, A_2 : X \to X$ *be two operators defined on the ordered metric space* $(X, d, \leq)$. *We assume that:*

(*i*) *A_1 is increasing;*
(*ii*) *A_1 and A_2 are POs;*
(*iii*) $A_1 \leq A_2$.

If we denote by x_2^* the unique fixed point of A_2, then

$$x \leq A_1(x) \Rightarrow x \leq x_2^*.$$

In the section 4 we prove a result of the continuous data dependence of the solution of the integral equation (1.1) using the General Data Dependence Theorem.

Theorem 1.10 (General Data Dependence Theorem). *Let (X, d) be a complete metric space, $A, B : X \to X$ two operators and suppose:*

(i) A is c-PO with respect to the metric d and $F_A = \{x_A^*\}$;
(ii) there exists $x_B^* \in F_B$;
(iii) there exists $\eta > 0$ such that $d(A(x), B(x)) \leq \eta$, for all $x \in X$.

Under these conditions we have:

$$d(x_A^*, x_B^*) \leq c \cdot \eta.$$

The last section of this paper contains a result concerning the Ulam-Hyers stability of the integral equation (1.1).

Definition 1.11. (I.A. Rus [21]) Let (X, d) be a metric space and $A : X \to X$ an operator. The equation of fixed point

$$x = A(x). \tag{1.2}$$

is Ulam-Hyers stable if there exists a real number $c_A > 0$ such that for each $\varepsilon > 0$ and each solution y^* of the inequation

$$d(y, A(y)) \leq \varepsilon,$$

there exists a solution x^* of equation (1.2) such that

$$d(y^*, x^*) \leq c_A \cdot \varepsilon.$$

Also, in this section we will use the Remark 2.1 from I.A. Rus [21], that you can find below.

Remark 1.12. (I.A. Rus [21], Remark 2.1) If A is a c-weakly Picard operator, then the fixed point equation (1.2) is Ulam-Hyers stable.

Indeed, let $\varepsilon > 0$ and y^* a solution of $d(y, A(y)) \leq \varepsilon$. Since A is c-weakly Picard operator, we have that

$$d(x, A^\infty(x)) \leq c \cdot d(x, A(x)), for\ all\ x \in X.$$

If we take $x := y^*$ and $x^* := A^\infty(y)$, then we have that $d(y^*, x^*) \leq c_A \cdot \varepsilon$ (see [20], [21]).

2. Existence and uniqueness

In this section we present several results of existence and uniqueness for the solution of the integral equation (1.1). These results were obtained by applying the known standard techniques as in [1], [2], [5], [6] for particular integral equations.

We suppose that the following conditions are fulfilled:

(a_1) $K_1, K_2 \in C([a, b] \times [a, b])$, $h_1 \in C([a, b] \times \mathbb{R}^3)$, $h_2 \in C([a, b] \times \mathbb{R}^2)$, $g \in C([a, b] \times \mathbb{R})$;
(a_2) $F \in C([a, b] \times \mathbb{R}^3)$.

Theorem 2.1. *We assume that the conditions (a_1) and (a_2) are satisfied. In addition we assume that:*

(*i*) *there exist* $\alpha, \beta, \gamma > 0$, *such that:*

$$|F(t,u_1,v_1,w_1) - F(t,u_2,v_2,w_2)| \le \alpha|u_1 - u_2| + \beta|v_1 - v_2| + \gamma|w_1 - w_2|,$$

for all $t \in [a,b], u_i, v_i, w_i \in \mathbb{R}, i = 1,2$;

(*ii*) *there exist* $L_1, L_2, L_3 > 0$ *such that:*

$$|h_1(s,u_1,u_2,u_3) - h_1(s,v_1,v_2,v_3)| \le L_1(|u_1 - v_1| + |u_2 - v_2| + |u_3 - v_3|),$$

for all $s \in [a,b], u_i, v_i \in \mathbb{R}, i = 1,2,3$;

$$|h_2(s,u_1,u_2) - h_2(s,v_1,v_2)| \le L_2(|u_1 - v_1| + |u_2 - v_2|),$$

for all $s \in [a,b], u_i, v_i \in \mathbb{R}, i = 1,2$;

$$|g(t,u) - g(t,v)| \le L_3|u - v|),$$

for all $t \in [a,b], u, v \in \mathbb{R}$;

(*iii*) $\alpha L_3 + (3\beta M_1 L_1 + 2\gamma M_2 L_2)(b-a) < 1$,
where we denoted by M_1 *and* M_2 *respectively, two positive constants, such that* $|K_1(t,s)| \le M_1$ *and* $|K_2(t,s)| \le M_2$, *for all* $t, s \in [a,b]$.

*Under these conditions the integral equation (*1.1*) has a unique solution* $x^* \in C[a,b]$, *that can be obtained by the successive approximations method starting at any element* $x_0 \in C[a,b]$.
In addition, if x_n *is the n-th successive approximation, then we have:*

$$\|x^* - x_n\|_C \le \frac{[\alpha L_3 + (3\beta M_1 L_1 + 2\gamma M_2 L_2)(b-a)]^n}{1 - \alpha L_3 - (3\beta M_1 L_1 + 2\gamma M_2 L_2)(b-a)} \cdot \|x_0 - x_1\|_C. \tag{2.1}$$

Proof. Let $X = (C[a,b], \|\cdot\|_C)$ be a Banach space, where $\|\cdot\|_C$ is the Chebyshev's norm

$$\|x\|_C = \max_{t\in[a,b]} |x(t)|, \text{ for all } x \in C[a,b].$$

Also, we consider the operator $A : X \to X$, defined by the relation:

$$A(x)(t) = F(t, g(t,x(t)), I_{Fr}(t,s,a,b,x,K_1,h_1), I_{Vo}(t,s,a,x,K_2,h_2)) \tag{2.2}$$

for all $t \in [a,b]$.

The set of the solutions of the integral equation (1.1) coincides with the set of fixed points of the operator A. From Contraction Principle it results that the operator A must be a contraction. We have:

$$\begin{aligned}|A(x)(t) - A(y)(t)| = |&F(t, g(t,x(t)), I_{Fr}(t,s,a,b,x,K_1,h_1), I_{Vo}(t,s,a,x,K_2,h_2)) \\ &- F(t, g(t,y(t)), I_{Fr}(t,s,a,b,y,K_1,h_1), I_{Vo}(t,s,a,y,K_2,h_2))|.\end{aligned}$$

From (i) and (ii) and using the Chebyshev's norm it results

$$\|A(x) - A(y)\|_{C[a,b]} \le [\alpha L_3 + (3\beta M_1 L_1 + 2\gamma M_2 L_2)(b-a)]\|x - y\|_{C[a,b]} \tag{2.3}$$

Consequently, from (iii) it results that the operator A is an L_A-contraction with the coefficient

$$L_A = \alpha L_3 + (3\beta M_1 L_1 + 2\gamma M_2 L_2)(b-a).$$

Now, from Contraction Principle it results that the operator A has a unique fixed point $F_A = \{x^*\}$ and consequently, the integral equation (1.1) has a unique solution $x^* \in C[a,b]$; this solution can be obtained by the successive approximations method starting at any element $x_0 \in C[a,b]$ and, if x_n is the n-th successive approximation, then the estimation (2.1) is true. The proof is complete. □

Remark 2.2. In order to obtain the Theorem 2.1, of existence and uniqueness of the solution of the integral equation (1.1) in the space $C[a,b]$, we reduced the problem of determination of the solutions of this integral equation to a fixed point problem. Under the conditions of the Theorem 2.1, the operator A, defined by (2.2), is PO.

Remark 2.3. If we consider the Banach space $X = (C[a,b], \|\cdot\|_B)$, where $\|\cdot\|_B$ is the Bielecki's norm:

$$\|x\|_B = \max_{t\in[a,b]} |x(t)| e^{-\tau(t-a)},$$

for all $x \in C[a,b]$, and $\tau > 0$ a parameter, and the operator $A : X \to X$, defined by (2.2), then we have another theorem of existence and uniqueness of the solution of the integral equation (1.1) in the space $C[a,b]$, that we present below.

Theorem 2.4. *We assume that the conditions* (a_1) *and* (a_2) *are satisfied and also, the conditions (i) and (ii) from Theorem 2.1 are fulfilled. Under these conditions the integral equation (*1.1*) has a unique solution* $x^* \in C[a,b]$.

Proof. We have

$$|A(x)(t) - A(y)(t)| \leq \alpha L_3 e^{\tau(t-a)} \|x-y\|_B + 3\frac{\beta M_1 L_1}{\tau} e^{\tau(t-a)} \|x-y\|_B + 2\frac{\gamma M_2 L_2}{\tau} e^{\tau(t-a+b-t)} \|x-y\|_B$$

and therefore, using the Bielecki's norm, we obtain:

$$\|A(x) - A(y)\|_B \leq [\alpha L_3 + 3\frac{\beta M_1 L_1}{\tau} + 2\frac{\gamma M_2 L_2}{\tau} e^{\tau(b-a)}] \|x-y\|_B. \tag{2.4}$$

It is clear that one can find a positive parameter τ, such that

$$\alpha L_3 + 3\frac{\beta M_1 L_1}{\tau} + 2\frac{\gamma M_2 L_2}{\tau} e^{\tau(b-a)} < 1,$$

and thus A is an L_A-contraction with

$$L_A = \alpha L_3 + 3\frac{\beta M_1 L_1}{\tau} + 2\frac{\gamma M_2 L_2}{\tau} e^{\tau(b-a)}$$

and the conclusion of theorem is obtained by applying the Contraction Principle (Theorem 1.4). □

Example 2.5. The following equation is a particular case of the integral equation (1.1), when $g(t, x(t)) = x(t)$:

$$x(t) = F(t, x(t), I_{Fr}(t, s, a, b, x, K_1, h_1), I_{Vo}(t, s, a, x, K_2, h_2)), \tag{2.5}$$

where we used the same notations for I_{Fr} and I_{Vo} as at the beginning of the first section.

Let us consider this integral equation in the following hypotheses:

(*i*) $F \in C([a,b] \times \mathbb{R}^3)$, $K_1, K_2 \in C([a,b] \times [a,b])$, $h_1 \in C([a,b] \times \mathbb{R}^3)$, $h_2 \in C([a,b] \times \mathbb{R}^2)$;

(ii) there exist $\alpha, \beta, \gamma > 0$, such that:

$$|F(t, u_1, v_1, w_1) - F(t, u_2, v_2, w_2)| \leq \alpha|u_1 - u_2| + \beta|v_1 - v_2| + \gamma|w_1 - w_2|,$$

for all $t \in [a, b], u_i, v_i, w_i \in \mathbb{R}, i = 1, 2$;

(iii) there exist $L_1, L_2 > 0$, such that:

$$|h_1(s, u_1, u_2, u_3) - h_1(s, v_1, v_2, v_3)| \leq L_1(|u_1 - v_1| + |u_2 - v_2| + |u_3 - v_3|),$$

for all $s \in [a, b], u_i, v_i \in \mathbb{R}, i = 1, 2, 3$;

$$|h_2(s, u_1, u_2) - h_1(s, v_1, v_2)| \leq L_2(|u_1 - v_1| + |u_2 - v_2|),$$

for all $s \in [a, b], u_i, v_i \in \mathbb{R}, i = 1, 2$;

(iv) $\alpha + (3\beta M_1 L_1 + 2\gamma M_2 L_2)(b - a) < 1$,
where we denoted by M_1 and M_2 respectively, two positive constants, such that $|K_1(t, s)| \leq M_1$ and $|K_2(t, s)| \leq M_2$, for all $t, s \in [a, b]$.

Then the integral equation (1.1) has a unique solution $x^* \in C[a, b]$, that can be obtained by the successive approximations method starting at any element $x_0 \in C[a, b]$. Moreover, if x_n is the n-th successive approximation, then we have:

$$\|x^* - x_n\|_C \leq \frac{[\alpha + (3\beta M_1 L_1 + 2\gamma M_2 L_2)(b - a)]^n}{1 - \alpha - (3\beta M_1 L_1 + 2\gamma M_2 L_2)(b - a)} \cdot \|x_0 - x_1\|_C. \tag{2.6}$$

In order to prove this result, we applied the Theorem 2.1 in particular case of

$$g(t, x(t)) = x(t).$$

Remark 2.6. A similar result can be obtained for the solution of integral equation

$$x(t) = F(t, x(a), I_{Fr}(t, s, a, b, x, K_1, h_1), I_{Vo}(t, s, a, x, K_2, h_2)), \tag{2.7}$$

by applying the Theorem 2.1 in particular case of $g(t, x(t)) = x(a)$.

Remark 2.7. In the paper [9] has been studied the existence and uniqueness of the solution of nonlinear Fredholm-Volterra functional-integral equation:

$$x(t) = F(t, x(a), \int_a^b K_1(t, s, x(g_1(s)))ds, \int_a^t K_2(t, s, x(g_2(s)))ds). \tag{2.8}$$

3. Comparison results and Gronwall lemmas

We present below a comparison result and two Gronwall-type lemmas for the solution of the integral equation (1.1). These results have been obtained by using the Picard operators technique and applying the Abstract Comparison Lemma, the Abstract Gronwall Lemma and the Abstract Gronwall-Comparison Lemma as in [4], [5], [15] for particular operatorial equations.

In order to obtain a comparison result, we consider the integral equations:

$$x(t) = F_i(t, g(t, x(t)), I^i_{Fr}(t, s, a, b, x, K_1, h^i_1), I^i_{Vo}(t, s, a, x, K_2, h^i_2)), \tag{3.1}$$

where we denoted:

$$I^i_{Fr}(t,s,a,b,x,K_1,h^i_1) = \int_a^b K_1(t,s)\cdot h^i_1(s,x(s),x(a),x(b))ds$$

$$I^i_{Vo}(t,s,a,x,K_2,h^i_2) = \int_a^t K_2(t,s)\cdot h^i_2(s,x(s),x(a))ds$$

where

$$F_i \in C([a,b]\times\mathbb{R}^3),\ g\in C([a,b]\times\mathbb{R}),$$
$$K_1,K_2\in C([a,b]\times[a,b],\mathbb{R}_+),\ h^i_1\in C([a,b]\times\mathbb{R}^3),$$
$$h^i_2\in C([a,b]\times\mathbb{R}^2),\ i=1,2,3.$$

We have:

Theorem 3.1. *Suppose that:*

(*i*) *the functions* $F_i, g, K_1, K_2, h^i_1, h^i_2$, $i=1,2,3$ *satisfy the conditions of Theorem 2.1, and let* x^*_i *be the unique solution of the integral equation* (3.1) *corresponding to* F_i, h^i_1, h^i_2, $i=1,2,3$*;*
(*ii*) *the functions* $F_2(t,\cdot,\cdot,\cdot), h^2_1(t,\cdot,\cdot,\cdot), h^2_2(t,\cdot,\cdot)$ *are increasing;*
(*iii*) $F_1\le F_2\le F_3$, $h^1_1\le h^2_1\le h^3_1$ *and* $h^1_2\le h^2_2\le h^3_2$.

Then

$$x^*_1\le x^*_2\le x^*_3.$$

Proof. We consider the Banach space $X=(C[a,b],\|\cdot\|_C)$ and the operators $A_i : X\to X$, defined by the relation (2.2) corresponding to functions $F_i, g, K_1, K_2, h^i_1, h^i_2$, $i=1,2,3$:

$$A_i(x)(t) = F_i(t, g(t,x(t)), I^i_{Fr}(t,s,a,b,x,K_1,h^i_1), I^i_{Vo}(t,s,a,x,K_2,h^i_2)).$$

From condition (i) it results that the operators $A_i : X\to X$, $i=1,2,3$ are PO's and therefore each of these operators has a unique fixed point, $F_{A_i}=\{x^*_i\}$.
From condition (ii) we deduce that the operator A_2 is increasing and from condition (iii) we obtain that $A_1\le A_2\le A_3$.
Now, applying the Abstract Comparison Lemma (Lemma 1.6), it results that

$$x_1\le x_2\le x_3 \implies A_1^\infty(x_1)\le A_2^\infty(x_2)\le A_3^\infty(x_3),$$

but A_1, A_2, A_3 are PO's and then by Remark 1.7, the conclusion of this theorem follows, i.e. $x^*_1\le x^*_2\le x^*_3$. The proof is complete. □

For the solution of the integral equation (1.1) we present below, the following two Gronwall-type lemmas.

Theorem 3.2. *We suppose that:*

(*i*) $F\in C([a,b]\times\mathbb{R}^3)$, $K_1,K_2\in C([a,b]\times[a,b],\mathbb{R}_+)$, $h_1\in C([a,b]\times\mathbb{R}^3)$, $h_2\in C([a,b]\times\mathbb{R}^2)$, $g\in C([a,b]\times\mathbb{R})$*;*
(*ii*) F, K_1, K_2, h_1, h_2, g *satisfy the conditions (i)-(iii) of Theorem 2.1, and denote by* $x^*\in C[a,b]$ *the unique solution of the integral equation* (1.1)*;*
(*iii*) $h_1(s,\cdot,\cdot,\cdot):\mathbb{R}^3\to\mathbb{R}, h_2(s,\cdot,\cdot):\mathbb{R}^2\to\mathbb{R}$ *are increasing functions for all* $s\in[a,b]$*;*

(*iv*) *$F(t,\cdot,\cdot,\cdot):\mathbb{R}^3\to\mathbb{R}$ is increasing function for all $t\in[a,b]$.*

Under these conditions, the following statements are true:

(*a*) *if x is a lower-solution of integral equation (1.1) then $x\le x^*$;*
(*b*) *if x is a upper-solution of integral equation (1.1) then $x\ge x^*$.*

Proof. We consider the operator $A:X\to X$, defined by (2.2). From conditions (i) and (ii) it results that this operator is PO and denote by x^* the unique fixed point of A. From the assumptions (i), (iii) and (iv) it results that the operator A is increasing.

Now, the conditions of the Abstract Gronwall Lemma (Lemma 1.8), being satisfied, it results that the conclusions of this theorem:

– if x is a lower-solution of the integral equation (1.1), i.e. $x\le A(x)$, then $x\le x^*$;
– if x is a upper-solution of the integral equation (1.1), i.e. $x\ge A(x)$, then $x\ge x^*$,

are true. The proof is complete. □

To obtain an effective Gronwall-type lemma, it can use the Abstract Gronwall-Comparison Lemma (Lemma 1.9), and we obtain a result that we present below.

Theorem 3.3. *We consider the integral equation (1.1) corresponding to F_i, g, K_1, K_2, h_1^i, h_2^i, for $i=1,2$. We assume that:*

(*i*) *$F_i\in C([a,b]\times\mathbb{R}^3)$, $K_1,K_2\in C([a,b]\times[a,b],\mathbb{R}_+)$, $h_1^i\in C([a,b]\times\mathbb{R}^3)$, $h_2^i\in C([a,b]\times\mathbb{R}^2)$, $g\in C([a,b]\times\mathbb{R})$, $i=1,2$;*
(*ii*) *$F_i,g,K_1,K_2,h_1^i,h_2^i$ satisfy the conditions (i)-(iii) of Theorem 2.1, for $i=1,2$;*
(*iii*) *$h_1^1(s,\cdot,\cdot,\cdot):\mathbb{R}^3\to\mathbb{R}, h_2^1(s,\cdot,\cdot):\mathbb{R}^2\to\mathbb{R}$ are increasing functions for all $s\in[a,b]$;*
(*iv*) *$F_1(t,\cdot,\cdot,\cdot):\mathbb{R}^3\to\mathbb{R}, g(t,\cdot):\mathbb{R}\to\mathbb{R}$ are increasing functions for all $t\in[a,b]$.*
(*v*) *$F_1\le F_2, h_1^1\le h_1^2$ and $h_2^1\le h_2^2$.*

If x is a solution of integral inequality

$$x(t)\le F_1(t,g(t,x(t)),I^1_{Fr}(t,s,a,b,x,K_1,h_1^1),I^1_{Vo}(t,s,a,x,K_2,h_2^1)), \tag{3.2}$$

where

$$I^1_{Fr}(t,s,a,b,x,K_1,h_1^1)=\int_a^b K_1(t,s)\cdot h_1^1(s,x(s),x(a),x(b))ds$$
$$I^1_{Vo}(t,s,a,x,K_2,h_2^1)=\int_a^t K_2(t,s)\cdot h_2^1(s,x(s),x(a))ds,$$

then $x\le x_2^*$, where x_2^* is the unique solution of integral equation (1.1) corresponding to $F_2,g,K_1,K_2,h_1^2,h_2^2$:

$$x(t)=F_2(t,g(t,x(t)),I^2_{Fr}(t,s,a,b,x,K_1,h_1^2),I^2_{Vo}(t,s,a,x,K_2,h_2^2)),$$

where

$$I^2_{Fr}(t,s,a,b,x,K_1,h_1^2)=\int_a^b K_1(t,s)\cdot h_1^2(s,x(s),x(a),x(b))ds$$
$$I^2_{Vo}(t,s,a,x,K_2,h_2^2)=\int_a^t K_2(t,s)\cdot h_2^2(s,x(s),x(a))ds.$$

Proof. We consider the operator A_1, A_2 defined by (2.2), corresponding to F_1, g, K_1, K_2, h_1^1, h_2^1 and F_2, g, K_1, K_2, h_1^2, h_2^2.

From Theorem 2.1 we have that A_1 and A_2 are POs, and we denote by x_i^* the unique fixed point of operator $A_i, i = 1, 2$.

From condition (ii) it results that A_1 is increasing and from condition (iii) we obtain that $A_1 \leq A_2$.

If x is a solution of (3.2), then $x \leq A_1(x)$.

Now, we apply the Abstract Gronwall-Comparison Lemma (Lemma 1.9), and we obtain the conclusion of the theorem. The proof is complete. □

4. Data dependence

In order to study the data dependence of the solution of the integral equation (1.1) we consider the following perturbed integral equation:

$$x(t) = F(t, g(t, x(t)), I_{Fr}(t, s, a, b, x, K_1, k_1), I_{Vo}(t, s, a, x, K_2, k_2)), \tag{4.1}$$

where

$$I_{Fr}(t, s, a, b, x, K_1, k_1) = \int_a^b K_1(t, s) \cdot k_1(s, x(s), x(a), x(b))ds$$
$$I_{Vo}(t, s, a, x, K_2, k_2) = \int_a^t K_2(t, s) \cdot k_2(s, x(s), x(a))ds$$

and

$$F : [a, b] \times \mathbb{R}^3 \to \mathbb{R},\ K_1, K_2 : [a, b] \times [a, b] \to \mathbb{R},$$
$$k_1 : [a, b] \times \mathbb{R}^3 \to \mathbb{R},\ k_2 : [a, b] \times \mathbb{R}^2 \to \mathbb{R},\ g : [a, b] \times \mathbb{R} \to \mathbb{R}.$$

We have the following data dependence theorem of the solution of the integral equation (1.1):

Theorem 4.1. *Suppose that:*

(i) F, K_1, K_2, h_1, h_2, g *satisfy the conditions of Theorem 2.1 and we denote by* $x^* \in C[a, b]$ *the unique solution of integral equation (1.1);*
(ii) $k_1 \in C([a, b] \times \mathbb{R}^3), k_2 \in C([a, b] \times \mathbb{R}^2)$*;*
(iii) there exists $\eta_1, \eta_2 > 0$ *such that*

$$|h_1(s, u, v, w) - k_1(s, u, v, w)| \leq \eta_1, \textit{for all } s \in [a, b], u, v, w \in \mathbb{R}, \textit{ and}$$
$$|h_2(s, u, v) - k_2(s, u, v)| \leq \eta_2, \textit{for all } s \in [a, b], u, v \in \mathbb{R}.$$

Under these conditions, if $y^* \in C[a, b]$ *is a solution of the integral equation (4.1), then we have:*

$$\|x^* - y^*\|_C \leq \frac{(M_1\eta_1 + M_2\eta_2)(b - a)}{1 - \alpha L_3 - (3\beta M_1 L_1 + 2\gamma M_2 L_2)(b - a)}. \tag{4.2}$$

Proof. We consider the operator from the proof of Theorem 2.1, $A : C[a,b] \to C[a,b]$, attached to integral equation (1.1) and defined by the relation (2.2):

$$A(x)(t) = F(t, g(t, x(t)), I_{Fr}(t, s, a, b, x, K_1, h_1), I_{Vo}(t, s, a, x, K_2, h_2)),$$

for all $t \in [a,b]$.

From condition (i) it results that the operator A is a L_A-contraction with the coefficient

$$L_A = \alpha L_3 + (3\beta M_1 L_1 + 2\gamma M_2 L_2)(b-a)$$

(Theorem 2.1) and therefore, A is c-PO with $c = \frac{1}{1-L_A}$.

Also, we attach to the integral equation (4.1) the operator $B : C[a,b] \to C[a,b]$, defined by the relation:

$$B(x)(t) = F(t, g(t, x(t)), I_{Fr}(t, s, a, b, x, K_1, k_1), I_{Vo}(t, s, a, x, K_2, k_2)) \qquad (4.3)$$

for all $t \in [a,b]$.

From conditions (i) and (ii) it results that the operator B is correctly defined. The set of the solutions of the perturbed integral equation (4.1) in the space $C[a,b]$ coincides with the fixed points set of the operator B defined by the relation (4.3). We have:

$$\begin{aligned} |A(x)(t) - B(x)(t)| &= |F(t, g(t, x(t)), I_{Fr}(t, s, a, b, x, K_1, h_1), I_{Vo}(t, s, a, x, K_2, h_2)) \\ &\quad - F(t, g(t, y(t)), I_{Fr}(t, s, a, b, y, K_1, k_1), I_{Vo}(t, s, a, y, K_2, k_2))| \end{aligned}$$

and from condition (iii) it results that

$$|A(x)(t) - B(x)(t)| \le (M_1\eta_1 + M_2\eta_2)(b-a), \ \textit{for all } t \in [a,b].$$

Now, using the Chebyshev's norm, we obtain:

$$\|A(x) - B(x)\|_C \le (M_1\eta_1 + M_2\eta_2)(b-a) \qquad (4.4)$$

and applying the General Data Dependence Theorem (Theorem 1.10), with

$$c = \frac{1}{1-L_A} \text{ and } \eta = (M_1\eta_1 + M_2\eta_2)(b-a),$$

it results the estimation (4.2). The proof is complete. □

5. Ulam-Hyers stability

Theorem 5.1. *Under the conditions of Theorem 2.1, the integral equation (*1.1*) is Ulam-Hyers stable, i.e. for $\varepsilon > 0$ and $y^* \in C[a,b]$ a solution of the inequation*

$$|y(t) - F(t, g(t, y(t)), I_{Fr}(t, s, a, b, y, K_1, h_1), I_{Vo}(t, s, a, y, K_2, h_2))| \le \varepsilon$$

*for all $t \in [a,b]$, there exists a solution of the integral equation (*1.1*), $x^* \in C([a,b]$, such that*

$$|y^*(t) - x^*(t)| \le \frac{1}{1-L_A}\varepsilon, \ \textit{for all } t \in [a,b],$$

where

$$L_A = \alpha L_3 + (3\beta M_1 L_1 + 2\gamma M_2 L_2)(b-a).$$

Proof. We consider the operator A, defined by the relation (2.2). Under the conditions of Theorem 2.1, it results that the operator A is a contraction and therefore, A is c-PO with the constant $c = \frac{1}{1-L_A}$,

$$L_A = \alpha L_3 + (3\beta M_1 L_1 + 2\gamma M_2 L_2)(b-a).$$

Now, the conclusion of this theorem is obtained as an application of the Remark 1.12 (I.A.Rus [21], Remark 2.1) anf the proof is complete. □

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Ascent, descent and additive preserving problems

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Abstract. Given an integer $n \geq 1$, we provide a complete description of all additive surjective maps, on the algebra of all bounded linear operators acting on a complex separable infinite-dimensional Hilbert space, preserving in both directions the set of all bounded linear operators with ascent (resp. descent) non-greater than n. In the context of Banach spaces, we consider the additive preserving problem for semi-Fredholm operators with ascent or descent non-greater than n.

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1. Introduction

Let X be an infinite-dimensional Banach space over the real or complex field $\mathbb{K}$, and let $\mathcal{B}(X)$ be the algebra of all bounded linear operators on X.

For a subset $\Lambda \subset \mathcal{B}(X)$, we say that a map Φ on $\mathcal{B}(X)$ *preserves* Λ *in both directions* (or, equivalently, that Φ is a *preserver of* Λ *in both directions*) if for every $T \in \mathcal{B}(X)$,

$$T \in \Lambda \text{ if and only if } \Phi(T) \in \Lambda.$$

For an operator $T \in \mathcal{B}(X)$, write $\ker(T)$ for its kernel, $\operatorname{ran}(T)$ for its range and T^* for its adjoint on the topological dual space X^*. The *ascent* $\mathrm{a}(T)$ and *descent* $\mathrm{d}(T)$ of $T \in \mathcal{B}(X)$ are defined by

$$\mathrm{a}(T) = \inf\{k \geq 0 : \ker(T^k) = \ker(T^{k+1})\}$$

and

$$\mathrm{d}(T) = \inf\{k \geq 0 : \operatorname{ran}(T^k) = \operatorname{ran}(T^{k+1})\},$$

where the infimum over the empty set is taken to be infinite (see [15, 19]). Clearly, a bounded linear operator is injective (resp. surjective) if and only if its ascent (resp. descent) is zero.

Over the last years, there has been a considerable interest in the so-called *linear preserver problems* that concern the question of determining the form of all linear, or additive, maps on $\mathcal{B}(X)$ that leave invariant certain subsets. The most linear preserver problems were solved in the finite-dimensional context, and extended later to the infinite-dimensional one. For excellent expositions on linear preserver problems, the reader is referred to [7, 11, 12, 13, 16] and the references therein.

One of the most famous problems in this direction is Kaplansky's problem [8], asking whether bijective unital linear maps Φ, between semi-simple Banach algebras, preserving in both directions invertibility, are Jordan isomorphisms (i.e. $\Phi(a^2) = \Phi(a)^2$ for all a). This problem was first solved in the finite-dimensional case [10], and it was later extended to von Neumann algebras [1]. In the case of the algebra $\mathcal{B}(X)$, A. A. Jafarian and A. R. Sourour established in [7] that every unital surjective linear map Φ on $\mathcal{B}(X)$, preserving in both directions invertibility, has one of the following two forms

$$T \mapsto ATA^{-1} \quad \text{or} \quad T \mapsto AT^*A^{-1}, \tag{1.1}$$

where A is a bounded linear operator between suitable spaces. Later, it was shown in [6] that every unital surjective additive preserver of injective operators or of surjective operators in both directions takes one of the two forms (1.1).

Since injective and surjective operators are precisely those operators with zero ascent and descent respectively, the following question arises: What can we say about surjective linear maps on $\mathcal{B}(X)$ preserving in both directions operators of finite ascent and descent, respectively?

Let H be a separable complex infinite-dimensional Hilbert space, and denote by $\mathcal{A}(H)$ (resp. $\mathcal{D}(H)$) the set of all operators in $\mathcal{B}(H)$ of finite ascent (resp. descent). In [11], the authors showed that a surjective additive continuous map $\Phi : \mathcal{B}(H) \to \mathcal{B}(H)$ preserves $\mathcal{A}(H)$ or $\mathcal{D}(H)$ in both directions if and only if

$$\Phi(T) = cATA^{-1} \quad \text{for all } \; T \in \mathcal{B}(H), \tag{1.2}$$

where c is a non-zero scalar and $A : H \to H$ is an invertible bounded linear, or conjugate linear, operator. An analog result was proved for $\mathcal{A}(H) \cup \mathcal{D}(H)$ by the same authors, see [12]. It should be noted that the question of removing the continuity condition or extending these results to the context of Banach spaces is still open.

The above results motivated us to continue the study of additive preservers involving the ascent and descent. This study may be considered as a key step towards a deeper understanding of operators with finite ascent or descent and their topological properties. In this paper, we will show that if we limit the variation of the ascent and the descent, then we obtain the same conclusion as in [11] without considering continuous preservers.

For each integer $n \geq 1$ let us introduce the following subsets of $\mathcal{B}(H)$:

1. $\mathcal{A}_n(H)$ the set of all operators $T \in \mathcal{B}(H)$ with $\mathrm{a}(T) \leq n$;
2. $\mathcal{D}_n(H)$ the set of all operators $T \in \mathcal{B}(H)$ with $\mathrm{d}(T) \leq n$.

Now, we summarize the first main result in the following theorem:

Theorem 1.1. *Let $\Phi : \mathcal{B}(H) \to \mathcal{B}(H)$ be an additive surjective map. Then the following assertions are equivalent:*

1. *Φ preserves $\mathcal{A}_n(H)$ in both directions;*
2. *Φ preserves $\mathcal{D}_n(H)$ in both directions;*

3. Φ *preserves* $\mathcal{A}_n(H) \cup \mathcal{D}_n(H)$ *in both directions;*
4. *there exist a non-zero scalar* c *and a bounded invertible linear, or conjugate linear, operator* $A : H \to H$ *such that*

$$\Phi(T) = cATA^{-1} \quad \text{for all } T \in \mathcal{B}(H).$$

Unfortunately, the approach used here does not allow us to obtain an analogue result in the context of Banach spaces. More precisely, one of the most important steps in the proof of the previous theorem consists in determining the topological interior of $\mathcal{A}_n(H)$, $\mathcal{D}_n(H)$, and $\mathcal{A}_n(H) \cup \mathcal{D}_n(H)$ using that of $\mathcal{A}(H) \cup \mathcal{D}(H)$, which is known only in the context of separable Hilbert spaces, see [12].

Recall that an operator $T \in \mathcal{B}(X)$ is called *upper* (resp. *lower*) *semi-Fredholm* if $\operatorname{ran}(T)$ is closed and $\dim\ker(T)$ (resp. $\operatorname{codim}\operatorname{ran}(T)$) is finite. The following properties will be used tacitly throughout the paper (see [15, Section 16]):

1. If the codimension of the range $\operatorname{ran}(T)$ of an operator $T \in \mathcal{B}(X)$ is finite, then $\operatorname{ran}(T)$ is automatically closed;
2. The composition of two upper (resp. lower) semi-Fredholm operators is an upper (resp. lower) semi-Fredholm operator;
3. If ST is an upper (resp. lower) semi-Fredholm operator, then T (resp. S) is upper (resp. lower) semi-Fredholm.

In [14], the authors studied all linear maps Φ on $\mathcal{B}(H)$ preserving in both directions semi-Fredholm operators. It has been shown that such maps Φ preserve in both directions the ideal of compact operators, and that the induced maps on the Calkin algebra are Jordan automorphisms. The problem of determining the structure of such maps on the whole space $\mathcal{B}(H)$ has remained open, and hence they conjectured that Φ is of the form $T \mapsto ATB + \Psi(T)$ where $A, B \in \mathcal{B}(H)$ are Fredholm operators and Ψ is a linear map on $\mathcal{B}(H)$ whose range is contained in the ideal of compact operators.

In this paper, we prove that if we limit the variation of the ascent (resp. descent) of upper (resp. lower) semi-Fredholm operators, then we obtain the complete description of all additive preservers of such operators in the context of Banach spaces. More precisely, we consider additive preservers of the following subsets of $\mathcal{B}(X)$:

1. $\mathcal{F}_n^+(X)$ the set of all upper semi-Fredholm operators $T \in \mathcal{B}(X)$ with $\mathrm{a}(T) \leq n$;
2. $\mathcal{F}_n^-(X)$ the set of all lower semi-Fredholm operators $T \in \mathcal{B}(X)$ with $\mathrm{d}(T) \leq n$;
3. $\mathcal{F}_n^\pm(X) = \mathcal{F}_n^+(X) \cup \mathcal{F}_n^-(X)$.

The second main result of the present paper is stated as follows:

Theorem 1.2. *Let* $\Phi : \mathcal{B}(X) \to \mathcal{B}(X)$ *be an additive surjective map preserving any one of the subsets* $\mathcal{F}_n^+(X)$, $\mathcal{F}_n^-(X)$ *or* $\mathcal{F}_n^\pm(X)$ *in both directions. Then there exist a non-zero scalar c, and either a bounded invertible linear, or conjugate linear, operator* $A : X \to X$ *such that*

$$\Phi(T) = cATA^{-1} \quad \text{for all } T \in \mathcal{B}(X),$$

or, a bounded invertible linear, or conjugate linear, operator $B : X^* \to X$ *such that*

$$\Phi(T) = cBT^*B^{-1} \quad \text{for all } T \in \mathcal{B}(X).$$

As an application of Theorem 1.2, we derive the following corollary:

Corollary 1.3. *Let $\Phi : \mathcal{B}(H) \to \mathcal{B}(H)$ be an additive surjective map. Then the following assertions are equivalent:*

1. *Φ preserves $\mathcal{F}_n^+(H)$ in both directions;*
2. *Φ preserves $\mathcal{F}_n^-(H)$ in both directions;*
3. *Φ preserves $\mathcal{F}_n^\pm(H)$ in both directions;*
4. *there exist a non-zero scalar c and a bounded invertible linear, or conjugate linear, operator $A : H \to H$ such that*

$$\Phi(T) = cATA^{-1} \quad \textit{for all } T \in \mathcal{B}(H).$$

The paper is organized as follows. In the second section, we give the topological interior of each of the subsets $\mathcal{A}_n(H)$, $\mathcal{D}_n(H)$, and $\mathcal{A}_n(H) \cup \mathcal{D}_n(H)$. The third section is devoted to establish some useful results on rank-one perturbations of these topological interiors. These results are needed for proving our theorems in the last section.

2. Topological interior of $\mathcal{A}_n(H)$, $\mathcal{D}_n(H)$, and $\mathcal{A}_n(H) \cup \mathcal{D}_n(H)$

Recall that the *hyper-kernel* and the *hyper-range* of an operator $T \in \mathcal{B}(X)$ are respectively the subspaces $\mathcal{N}^\infty(T) = \bigcup_{k \geq 0} \ker(T^k)$ and $\mathcal{R}^\infty(T) = \bigcap_{k \geq 0} \operatorname{ran}(T^k)$.

Let us introduce the following subsets of $\mathcal{B}(X)$:

1. $\mathcal{B}_n^+(X) = \{T \in \mathcal{B}(X) : \operatorname{ran}(T) \text{ is closed and } \dim \mathcal{N}^\infty(T) \leq n\}$;
2. $\mathcal{B}_n^-(X) = \{T \in \mathcal{B}(X) : \operatorname{codim} \mathcal{R}^\infty(T) \leq n\}$;
3. $\mathcal{B}_n^\pm(X) = \mathcal{B}_n^+(X) \cup \mathcal{B}_n^-(X)$.

One of the most important steps in the proof of our main theorems is to show that the maps we are dealing with preserve the subsets $\mathcal{B}_n^+(X)$, $\mathcal{B}_n^-(X)$ and $\mathcal{B}_n^\pm(X)$ in both directions. In order to prove this implication, we establish that the topological interior of $\mathcal{A}_n(H)$, $\mathcal{D}_n(H)$ and $\mathcal{A}_n(H) \cup \mathcal{D}_n(H)$ is respectively $\mathcal{B}_n^+(H)$, $\mathcal{B}_n^-(H)$ and $\mathcal{B}_n^\pm(H)$. Similar results are given for $\mathcal{F}_n^+(X)$, $\mathcal{F}_n^-(X)$ and $\mathcal{F}_n^\pm(X)$.

It should be noted that the ascent and the hyper-kernel of an operator $T \in \mathcal{B}(X)$ are related by the following inequality (see [17])

$$\mathrm{a}(T) \leq \dim \mathcal{N}^\infty(T). \tag{2.1}$$

Similarly, the descent is related to the hyper-range by

$$\mathrm{d}(T) \leq \operatorname{codim} \mathcal{R}^\infty(T). \tag{2.2}$$

Remark 2.1. For $T \in \mathcal{B}(X)$, it follows easily from the definition of the ascent and of the descent that:

1. $\dim \ker(T^{n+1}) \leq n$ if and only if $\dim \mathcal{N}^\infty(T) \leq n$;
2. $\operatorname{codim} \operatorname{ran}(T^{n+1}) \leq n$ if and only if $\operatorname{codim} \mathcal{R}^\infty(T) \leq n$.

Proposition 2.2. *$\mathcal{B}_n^+(X)$, $\mathcal{B}_n^-(X)$ and $\mathcal{B}_n^\pm(X)$ are open subsets of $\mathcal{F}_n^+(X)$, $\mathcal{F}_n^-(X)$ and $\mathcal{F}_n^\pm(X)$, respectively.*

Proof. It follows from (2.1) and (2.2) that $\mathcal{B}_n^+(X)$ and $\mathcal{B}_n^-(X)$ are subsets of $\mathcal{F}_n^+(X)$ and $\mathcal{F}_n^-(X)$ respectively, and so $\mathcal{B}_n^\pm(X)$ is a subset of $\mathcal{F}_n^\pm(X)$. Let $S \in \mathcal{B}_n^+(X)$. In particular, we have $\dim\ker(S^n) = \dim\ker(S^{n+1}) \leq n$ and S^{n+1} is an upper semi-Fredholm operator. Hence, it follows by [15, Theorem 16.11] that there exists $\eta > 0$ such that for $T \in \mathcal{B}(X)$ with $\| T - S^{n+1} \| < \eta$, we have that T is upper semi-Fredholm and

$$\dim\ker(T) \leq \dim\ker(S^{n+1}) \leq n. \tag{2.3}$$

On the other hand, since the function $T \mapsto T^{n+1}$ is continuous on $\mathcal{B}(X)$, there exists $\varepsilon > 0$ such that

$$\| T^{n+1} - S^{n+1} \| < \eta \quad \text{for all } T \in \mathcal{B}(X) \text{ with } \| T - S \| < \varepsilon. \tag{2.4}$$

Combining (2.4) and (2.3) we obtain that T^{n+1} is upper semi-Fredholm and

$$\dim\ker(T^{n+1}) \leq \dim\ker(S^{n+1}) \leq n,$$

and so $T \in \mathcal{B}_n^+(X)$ for all $T \in \mathcal{B}(X)$ with $\| T - S \| < \varepsilon$. This shows that $\mathcal{B}_n^+(X)$ is open.

Similarly, we prove that $\mathcal{B}_n^-(X)$ is open, and hence $\mathcal{B}_n^\pm(X)$ is also open. □

From [5, Lemma 1.1], given a non-negative integer d, we have

$$\mathrm{a}(T) \leq d \Leftrightarrow \ker(T^m) \cap \mathrm{ran}(T^d) = \{0\} \text{ for some } m \geq 1. \tag{2.5}$$

Remark 2.3. Let $T \in \mathcal{B}(X)$. Then the following assertions hold:

1. If T has finite ascent and descent then $\mathrm{a}(T) = \mathrm{d}(T)$ and $X = \ker(T^k) \oplus \mathrm{ran}(T^k)$, where $k = \mathrm{a}(T)$ and the direct sum is topological (see [15, Corollary 20.5]).
2. If $T = T_1 \oplus T_2$ with respect to any decomposition of X, then it follows from [18, Theorem 6.1] that

$$\mathrm{a}(T) = \max\{\mathrm{a}(T_1), \mathrm{a}(T_2)\} \quad \text{and} \quad \mathrm{d}(T) = \max\{\mathrm{d}(T_1), \mathrm{d}(T_2)\}.$$

The following example shows that $\mathcal{B}_n^+(X)$, $\mathcal{B}_n^-(X)$ and $\mathcal{B}_n^\pm(X)$ are proper subsets of $\mathcal{F}_n^+(X)$, $\mathcal{F}_n^-(X)$ and $\mathcal{F}_n^\pm(X)$, respectively, and that there exist operators with finite ascent and descent which are not semi-Fredholm.

Example 2.4. Let $Y \subset X$ be a closed subspace of dimension $n+1$, and write $X = Y \oplus Z$ where Z is a closed subspace of X. With respect to this decomposition, consider the operator $T = 0 \oplus I$. According to the previous remark, one can easily see that $\mathrm{a}(T) = \mathrm{d}(T) = 1$. Since $\mathcal{N}^\infty(T) = \ker(T) = Y$ and $\mathcal{R}^\infty(T) = \mathrm{ran}(T) = Z$, then T belongs to $\mathcal{F}_n^+(X) \cap \mathcal{F}_n^-(X)$ and not to $\mathcal{B}_n^\pm(X)$.

Similarly, for $S = I - T$, we have $\mathrm{a}(S) = \mathrm{d}(S) = 1$, $\ker(S) = Z$ and $\mathrm{ran}(S) = Y$. Thus, S is not a semi-Fredholm operator.

Recall that an operator $T \in \mathcal{B}(X)$ is called *upper* (resp. *lower*) *semi-Browder* if it is upper (resp. lower) semi-Fredholm of finite ascent (resp. descent). Clearly, every operator in $\mathcal{F}_n^+(X)$ (resp. $\mathcal{F}_n^-(X)$) is upper (resp. lower) semi-Browder.

Theorem 2.5. *Let $T \in \mathcal{B}(X)$ be non-zero. The following assertions are equivalent:*

1. *$T \in \mathcal{B}_n^\pm(X)$ (resp. $\mathcal{B}_n^+(X)$, $\mathcal{B}_n^-(X)$);*

2. *for every* $S \in \mathcal{B}(X)$ *there exists* $\varepsilon_0 > 0$ *such that* $T+\varepsilon S \in \mathcal{F}_n^{\pm}(X)$ *(resp.* $\mathcal{F}_n^{+}(X)$*,* $\mathcal{F}_n^{-}(X)$*), for all numbers (equivalently, rational numbers)* $|\varepsilon| < \varepsilon_0$*.*

Proof. (1) $\Rightarrow$ (2) follows immediately from the previous proposition.

(2) $\Rightarrow$ (1). Suppose that for every $S \in \mathcal{B}(X)$ there exists $\varepsilon_0 > 0$ such that $T + \varepsilon S \in \mathcal{F}_n^{\pm}(X)$ for all numbers $|\varepsilon| < \varepsilon_0$. In particular, we have $T \in \mathcal{F}_n^{\pm}(X)$, and so T is either upper semi-Browder or lower semi-Browder. It follows from [15, Theorem 20.10] that there exist two closed T-invariant subspaces X_1 and X_2 such that $X = X_1 \oplus X_2$, $\dim X_1 < \infty$, $T_1 = T_{|X_1}$ is nilpotent and $T_{|X_2}$ is either bounded below or onto, respectively. We claim that $\dim X_1 \le n$. Let $\{e_i : 0 \le i \le p\}$ be a basis of X_1 such that $Te_0 = 0$ and $Te_i = \varepsilon_i e_{i-1}$ for $1 \le i \le p$ where $\varepsilon_i \in \{0, 1\}$. With respect to the decomposition of X, consider the operator $S \in \mathcal{B}(X)$ given by $S = S_1 \oplus 0$ where $S_1 e_0 = 0$ and $S_1 e_i = e_{i-1}$ for $1 \le i \le p$. Clearly, for $\varepsilon \notin \{-1, 0\}$ we have

$$(T_1 + \varepsilon S_1)e_0 = 0 \quad \text{and} \quad (T_1 + \varepsilon S_1)e_i = (\varepsilon_i + \varepsilon)e_{i-1} \ \text{ for } \ 1 \le i \le p.$$

Hence $(T_1 + \varepsilon S_1)^p e_p = \lambda e_0 \neq 0$ where $\lambda = (\varepsilon_p + \varepsilon)\dots(\varepsilon_1 + \varepsilon)$.
Therefore $e_0 \in \ker(T_1 + \varepsilon S_1) \cap \operatorname{ran}(T_1 + \varepsilon S_1)^p$, and consequently

$$\mathrm{a}(T_1 + \varepsilon S_1) = \mathrm{d}(T_1 + \varepsilon S_1) \ge p + 1$$

by (2.5). But, we have also

$$\mathrm{a}(T_1 + \varepsilon S_1) \le \mathrm{a}(T + \varepsilon S) \quad \text{and} \quad \mathrm{d}(T_1 + \varepsilon S_1) \le \mathrm{d}(T + \varepsilon S).$$

Since $T + \varepsilon S \in \mathcal{F}_n^{\pm}(X)$, then $\mathrm{a}(T + \varepsilon S) \le n$ or $\mathrm{d}(T + \varepsilon S) \le n$. Thus $\dim X_1 \le n$. Now, if $T \in \mathcal{F}_n^{+}(X)$ (resp. $\mathcal{F}_n^{-}(X)$) then T is upper (resp. lower) semi-Browder, and so the space X_1 (resp. X_2) is uniquely determined and $X_1 = \mathcal{N}^{\infty}(T)$ (resp. $X_2 = \mathcal{R}^{\infty}(T)$) (see [15, Theorem 20.10]). This proves that $T \in \mathcal{B}_n^{+}(X)$ (resp. $\mathcal{B}_n^{-}(X)$). □

For a subset $\Gamma \subseteq \mathcal{B}(X)$, we write $\operatorname{Int}(\Gamma)$ for its interior. As a consequence of Theorem 2.5, we derive the following corollary.

Corollary 2.6. *We have* $\operatorname{Int}(\mathcal{F}_n^{+}(X)) = \mathcal{B}_n^{+}(X)$*,* $\operatorname{Int}(\mathcal{F}_n^{-}(X)) = \mathcal{B}_n^{-}(X)$ *and* $\operatorname{Int}(\mathcal{F}_n^{\pm}(X)) = \mathcal{B}_n^{\pm}(X)$*.*

Proof. Let us show that $\operatorname{Int}(\mathcal{F}_n^{+}(X)) = \mathcal{B}_n^{+}(X)$. Note that $\mathcal{B}_n^{+}(X) \subseteq \operatorname{Int}(\mathcal{F}_n^{+}(X))$ because $\mathcal{B}_n^{+}(X)$ is open. Let $T \notin \mathcal{B}_n^{+}(X)$, then Theorem 2.5 ensures the existence of an operator $S \in \mathcal{B}(X)$ and a sequence (ε_k) converging to zero such that $T+\varepsilon_k S \notin \mathcal{F}_n^{+}(X)$ for all $k \ge 0$. Consequently, $T \notin \operatorname{Int}(\mathcal{F}_n^{+}(X))$.

Similarly, we prove that $\operatorname{Int}(\mathcal{F}_n^{-}(X)) = \mathcal{B}_n^{-}(X)$ and $\operatorname{Int}(\mathcal{F}_n^{\pm}(X)) = \mathcal{B}_n^{\pm}(X)$. □

Theorem 2.7. *Let H be a separable complex infinite-dimensional Hilbert space and let $T \in \mathcal{B}(H)$. Then the following assertions are equivalent:*

1. $T \in \mathcal{B}_n^{\pm}(H)$ *(resp.* $\mathcal{B}_n^{+}(H)$*,* $\mathcal{B}_n^{-}(H)$*);*
2. *for every* $S \in \mathcal{B}(H)$ *there exists* $\varepsilon_0 > 0$ *such that* $T+\varepsilon S \in \mathcal{A}_n(H) \cup \mathcal{D}_n(H)$ *(resp.* $\mathcal{A}_n(H)$*,* $\mathcal{D}_n(H)$*), for all numbers (equivalently, rational numbers)* $|\varepsilon| < \varepsilon_0$*.*

Proof. (1) $\Rightarrow$ (2) follows immediately from Proposition 2.2.

(2) $\Rightarrow$ (1). Suppose that for every $S \in \mathcal{B}(H)$ there exists $\varepsilon_0 > 0$ such that $T + \varepsilon S \in \mathcal{A}_n(H) \cup \mathcal{D}_n(H)$ for all $|\varepsilon| < \varepsilon_0$. Then, using [12, Proposition 2.5], we get that T is a semi-Browder operator. The rest of the proof is similar to the proof of Theorem 2.5. $\square$

Using a similar proof of Corollary 2.6, we get the following result.

Corollary 2.8. *We have* $\mathrm{Int}(\mathcal{A}_n(H) \cup \mathcal{D}_n(H)) = \mathcal{B}_n^{\pm}(H)$*,* $\mathrm{Int}(\mathcal{A}_n(H)) = \mathcal{B}_n^{+}(H)$ *and* $\mathrm{Int}(\mathcal{D}_n(H)) = \mathcal{B}_n^{-}(H)$*.*

3. $\mathcal{B}_n^+(X)$, $\mathcal{B}_n^-(X)$ and $\mathcal{B}_n^{\pm}(X)$ under rank-one perturbations

Let $z \in X$ and let $f \in X^*$ be non-zero. We denote by $z \otimes f$ the rank-one operator defined by $(z \otimes f)(x) = f(x)z$ for all $x \in X$. Note that every rank-one operator in $\mathcal{B}(X)$ can be written in this form.

In [13], the authors proved that for a rank-one operator $F \in \mathcal{B}(X)$ and for $T \in \mathcal{B}(X)$ with $\dim\ker(T) \le n$, we have either $\dim\ker(T+F) \le n$ or $\dim\ker(T-F) \le n$. In the following, we extend this result to the setting of the hyper-kernel subspace.

Proposition 3.1. *Let* $T \in \mathcal{B}(X)$ *be such that* $\dim\mathcal{N}^\infty(T) \le n$*, and let* $F \in \mathcal{B}(X)$ *be a rank-one operator. Then either* $\dim\mathcal{N}^\infty(T+F) \le n$ *or* $\dim\mathcal{N}^\infty(T-F) \le n$*.*

Before giving the proof of this proposition, we need to establish some lemmas.
For $T, F \in \mathcal{B}(X)$, let

$$\mathrm{M}(T, F) = \{x \in \mathcal{N}^\infty(T) : FT^i x = 0 \text{ for all } i \ge 0\}.$$

Clearly, $\mathrm{M}(T, F)$ is a T-invariant subspace of $\mathcal{N}^\infty(T) \cap \ker(F)$. Furthermore, if T has a finite ascent, then $\mathrm{M}(T, F)$ is closed.

Lemma 3.2. *Let* $T \in \mathcal{B}(X)$ *be non-zero, and let* $F = z \otimes f$ *be a rank-one operator such that* $\ker(T) \cap \ker(F) = \{0\}$*. Assume that there exist an integer* $m \ge 0$ *and a vector* $x \in \ker(T+F)^{m+1} \setminus \ker(T+F)^m$ *such that* $x \notin \mathrm{M}(T, F)$*. Then* x *is a linear combination of linearly independent vectors* x_i*,* $0 \le i \le m$*, such that*

$$(T+F)x_0 = 0,\ (T+F)x_i = x_{i-1} \text{ for } 1 \le i \le m, \text{ and } f(x_i) = \delta_{i0} \text{ for } 0 \le i \le m.$$

Proof. Let $u_i = (T+F)^{m-i}x$ for $0 \le i \le m$. It follows that u_i, $0 \le i \le m$, are linearly independent vectors, $(T+F)u_0 = 0$ and $(T+F)u_i = u_{i-1}$ for $1 \le i \le m$. Since $\ker(T) \cap \ker(F) = \{0\}$, we infer that $f(u_0) \ne 0$. Without loss of generality we may assume that $f(u_0) = 1$. Consider the scalars $c_0, c_1, \ldots, c_{m-1}$ defined inductively by

$$\begin{aligned}
c_0 &= -f(u_1)\\
c_1 &= -c_0 f(u_1) - f(u_2)\\
c_2 &= -c_1 f(u_1) - c_0 f(u_2) - f(u_3)\\
&\vdots\\
c_{m-1} &= -c_{m-2} f(u_1) - \cdots - c_0 f(u_{m-1}) - f(u_m).
\end{aligned}$$

This means that we have

$$f(u_i)+\sum_{k=1}^{i}c_{i-k}f(u_{k-1})=0 \ \text{ for } 1\le i\le m. \tag{3.1}$$

Let $x_0=u_0$ and $x_i=u_i+\sum_{k=1}^{i}c_{i-k}u_{k-1}$ for $1\le i\le m$. Clearly, the vectors x_i, $0\le i\le m$, are linearly independent. Moreover, it follows from (3.1) that $f(x_i)=\delta_{i0}$ for $0\le i\le m$. Furthermore, we have $(T+F)x_0=(T+F)u_0=0$ and

$$(T+F)x_i=(T+F)u_i+\sum_{k=1}^{i}c_{i-k}(T+F)u_{k-1}=u_{i-1}+\sum_{k=2}^{i}c_{i-k}u_{k-2}=x_{i-1}$$

for $1\le i\le m$. Finally, we have

$$x=u_m\in \operatorname{Span}\{u_i:0\le i\le m\}=\operatorname{Span}\{x_i:0\le i\le m\}.$$

This completes the proof. □

The following lemma is a special case of Proposition 3.1, and it will be required for proving that proposition.

Lemma 3.3. *Let $T\in\mathcal{B}(X)$ be such that $\dim\mathcal{N}^\infty(T)\le n$, and let $F\in\mathcal{B}(X)$ be a rank-one operator such that $\ker(T)\cap\ker(F)=\{0\}$. Then either $\dim\mathcal{N}^\infty(T+F)\le n$ or $\dim\mathcal{N}^\infty(T-F)\le n$.*

Proof. Write $F=z\otimes f$ where $z\in X$ and $f\in X^*$ are non-zero. Clearly, if either $\ker(T+F)^{n+1}$ or $\ker(T-F)^{n+1}$ is contained in $\mathrm{M}(T,F)$, then either $\dim\mathcal{N}^\infty(T+F)\le n$ or $\dim\mathcal{N}^\infty(T-F)\le n$ respectively. Hence, we may assume that $\ker(T+F)^{n+1}\nsubseteq \mathrm{M}(T,F)$ and $\ker(T-F)^{n+1}\nsubseteq \mathrm{M}(T,F)$. Let $0\le m,p\le n$ be the biggest integers for which there exist $x\in\ker(T+F)^{m+1}\backslash\ker(T+F)^m$ and $y\in\ker(T-F)^{p+1}\backslash\ker(T-F)^p$ such that $x,y\notin \mathrm{M}(T,F)$. Without loss of generality we can assume that $m\le p$. We will show that $\dim\mathcal{N}^\infty(T+F)\le n$. Using the previous lemma, we infer that y is a linear combination of linearly independent vectors y_i, $0\le i\le p$, such that

$$(T-F)y_0=0,\ (T-F)y_i=y_{i-1} \text{ for } 1\le i\le p, \text{ and } f(y_i)=\delta_{i0} \text{ for } 0\le i\le p.$$

From this, one can easily see that $(T+F)y_0=2z$ and $(T+F)y_i=Ty_i=y_{i-1}$ for $1\le i\le p$, and so $(T+F)^k y_i=y_{i-k}$ for $0\le k\le i\le p$. Thus, we get easily that

$$I+\sum_{i=0}^{p}y_i\otimes f(T+F)^i=\prod_{i=0}^{p}\left(I+y_i\otimes f(T+F)^i\right).$$

Furthermore, since $f((T+F)^i y_i)=f(y_0)=1$ for $0\le i\le p$, the above equation defines an invertible operator denoted by S.

Let $u\in\ker(T+F)^{n+1}$ be an arbitrary non-zero vector, and let $0\le r\le n$ be such that $u\in\ker(T+F)^{r+1}\setminus\ker(T+F)^r$. If $u\in \mathrm{M}(T,F)$, then $f(T^i u)=0$, and so $(T+F)^i u=T^i u$ for every $i\ge 0$. Hence, $Su=u\in \mathrm{M}(T,F)\subseteq\mathcal{N}^\infty(T)$. Consider the case when $u \notin \mathrm{M}(T,F)$. Then, Lemma 3.2 asserts that u is a linear combination of linearly independent vectors x_i, $0 \le i \le r$, satisfying

$$(T+F)x_0 = 0,\ (T+F)x_i = x_{i-1} \text{ for } 1 \le i \le r, \text{ and } f(x_i) = \delta_{i0} \text{ for } 0 \le i \le r.$$

It follows that $(T+F)^k x_i = x_{i-k}$ for $k \ge 0$ and $0 \le i \le r$, where we set formally $x_j = 0$ for $j < 0$. Now, by the definition of m, we have $r \le m \le p$. This allows us to obtain easily that $Sx_i = x_i + y_i$ for $0 \le i \le r$. It follows that $T^i Sx_i = x_0 + y_0 \in \ker(T)$, and hence $Sx_i \in \mathcal{N}^\infty(T)$ for $0 \le i \le r$. Consequently, we get that $Su \in \mathcal{N}^\infty(T)$. The vector u was arbitrary, therefore $S \ker(T+F)^{n+1} \subseteq \mathcal{N}^\infty(T)$. So that $\dim \ker(T+F)^{n+1} \le n$. According to Remark 2.1, this completes the proof. □

For $T, F \in \mathcal{B}(X)$, we denote respectively by $\tilde{T}$ and $\tilde{F}$ the operators induced by T and F on $X/\mathrm{M}(T,F)$. Note that the hyper-kernels of $\tilde{T}+c\tilde{F}$ and $T+cF$ are related by the following relation (see [17, Lemma 2.9])

$$\mathcal{N}^\infty(\tilde{T}+c\tilde{F}) = \mathcal{N}^\infty(T+cF)/\mathrm{M}(T,F) \text{ for all } c \in \mathbb{K}. \tag{3.2}$$

Proof of Proposition 3.1. Firstly, if $\tilde{F} = 0$, then it follows from (3.2) that

$$\mathcal{N}^\infty(\tilde{T}+\tilde{F}) = \mathcal{N}^\infty(T+F)/\mathrm{M}(T,F) = \mathcal{N}^\infty(\tilde{T}) = \mathcal{N}^\infty(T)/\mathrm{M}(T,F).$$

So that $\dim \mathcal{N}^\infty(T+F) = \dim \mathcal{N}^\infty(T) \le n$.

Now, consider the case $\tilde{F} \ne 0$. Then $z \notin \mathrm{M}(T,F)$, and for every $x \in X$, we have

$$\begin{aligned} x + \mathrm{M}(T,F) \in \ker(\tilde{T}) \cap \ker(\tilde{F}) \quad &\Leftrightarrow \quad Tx \in \mathrm{M}(T,F) \text{ and } Fx = f(x)z \in \mathrm{M}(T,F) \\ &\Leftrightarrow \quad Tx \in \mathrm{M}(T,F) \text{ and } f(x) = 0 \\ &\Leftrightarrow \quad x \in \mathrm{M}(T,F). \end{aligned}$$

This implies that $\ker(\tilde{T}) \cap \ker(\tilde{F}) = \{0\}$.
Since $\dim \mathcal{N}^\infty(\tilde{T}) \le n-q$ where $q = \dim \mathrm{M}(T,F)$, the previous lemma ensures that either $\dim \mathcal{N}^\infty(\tilde{T}+\tilde{F}) \le n-q$ or $\dim \mathcal{N}^\infty(\tilde{T}-\tilde{F}) \le n-q$. Thus, we get that either $\dim \mathcal{N}^\infty(T+F) \le n$ or $\dim \mathcal{N}^\infty(T-F) \le n$. This completes the proof. □

Throughout the sequel, Λ will denote any of the subsets $\mathcal{B}_n^+(X)$, $\mathcal{B}_n^-(X)$ or $\mathcal{B}_n^\pm(X)$. Also, the subset $\mathcal{B}_n(X) = \mathcal{B}_n^+(X) \cap \mathcal{B}_n^-(X)$, introduced and studied in [17], will be used in the rest of this paper.

Recall that for a semi-Fredholm operator $T \in \mathcal{B}(X)$, the *index* is defined by

$$\mathrm{ind}(T) = \dim \ker(T) - \mathrm{codim}\,\mathrm{ran}(T),$$

and if the index is finite, T is said to be *Fredholm*. It should be noted that if $\mathrm{ind}(T) = 0$ then $\mathrm{a}(T) = \mathrm{d}(T)$ (see [12, Lemma 2.3]). Moreover, in this case

$$T \in \Lambda \;\Leftrightarrow\; T \in \mathcal{B}_n(X) \;\Leftrightarrow\; \dim \mathcal{N}^\infty(T) \le n.$$

Proposition 3.4. *Let $T \in \Lambda$ and let $F \in \mathcal{B}(X)$ be a rank-one operator. Then either $T+F \in \Lambda$ or $T-F \in \Lambda$.*

Before proving this proposition, a duality relation between $\mathcal{B}_n^+(X)$ and $\mathcal{B}_n^-(X)$ should be established first. For a subset $M \subseteq X$, we denote by $M^\perp = \{f \in X^* : M \subseteq \ker(f)\}$ its annihilator.

Lemma 3.5. *Let T be a bounded operator on X. Then :*

$$T \in \mathcal{B}_n^+(X) \text{ (resp. } \mathcal{B}_n^-(X)) \quad \Leftrightarrow \quad T^* \in \mathcal{B}_n^-(X^*) \text{ (resp. } \mathcal{B}_n^+(X^*)).$$

Proof. Suppose that $T \in \mathcal{B}_n^+(X)$. In particular, T is a semi-Fredholm operator, and so $\operatorname{ran}(T^k)$ is closed for every $k \geq 0$. Since $\operatorname{a}(T) \leq n$, it follows from [15, Corollary A.1.17] that

$$\ker(T^{n+1})^\perp = \ker(T^n)^\perp = \operatorname{ran}((T^*)^{n+1}) = \operatorname{ran}((T^*)^n).$$

Thus, $\operatorname{d}(T^*) \leq n$. Using [15, Theorem A.1.20] we get that

$$\operatorname{codim} \mathcal{R}^\infty(T^*) = \operatorname{codim} \operatorname{ran}((T^*)^n) = \dim \ker(T^n) = \dim \mathcal{N}^\infty(T) \leq n.$$

So that $T^* \in \mathcal{B}_n^-(X^*)$. The proofs of the converse and of the statement for $\mathcal{B}_n^-(X)$ are similar. □

Proof of Proposition 3.4. Let $T \in \Lambda$, and let $F \in \mathcal{B}(X)$ be a rank-one operator. It follows from [15, Theorem 16.16] that $T + F$ and $T - F$ are semi-Fredholm. If $T \in \mathcal{B}_n^+(X)$ then Proposition 3.1 implies that either $T+F \in \mathcal{B}_n^+(X)$ or $T-F \in \mathcal{B}_n^+(X)$.

The case when $T \in \mathcal{B}_n^-(X)$ follows from the first one by duality. □

The following theorem, will play a crucial role in proving the main results.

Theorem 3.6. *Let $F \in \mathcal{B}(X)$ be a non-zero operator. Then the following assertions hold:*

1. *There exists an invertible operator $T \in \mathcal{B}(X)$ such that $T + F \notin \Lambda$.*
2. *If $\dim \operatorname{ran}(F) \geq 2$, then there exists an invertible operator $T \in \mathcal{B}(X)$ such that $T + F \notin \Lambda$ and $T - F \notin \Lambda$.*

Proof. Suppose first that $\operatorname{ran}(F)$ has an infinite dimension. Then $\operatorname{codim} \ker(F) = \infty$, and hence there exist linearly independent vectors x_i, $0 \leq i \leq 2n+1$, that generate a subspace having trivial intersection with $\ker(F)$. It follows that the vectors Fx_i, $0 \leq i \leq 2n+1$, are linearly independent. Write

$$X = \operatorname{Span}\{x_i : 0 \leq i \leq 2n+1\} \oplus Y = \operatorname{Span}\{Fx_i : 0 \leq i \leq 2n+1\} \oplus Z,$$

where Y, Z are two closed subspaces and $Y = F^{-1}Z$. Then there exists an invertible operator $T \in \mathcal{B}(X)$ such that $TY = Z$, and $Tx_i = (-1)^i Fx_i$ for $0 \leq i \leq 2n+1$. Clearly, $x_{2i+1} \in \ker(T+F)$ and $x_{2i} \in \ker(T-F)$ for $0 \leq i \leq n$, and hence

$$\dim \ker(T \pm F) > n.$$

But, we have also

$$\operatorname{ran}(T+F) \subseteq \operatorname{Span}\{Fx_{2i} : 0 \leq i \leq n\} \oplus Z,$$

and

$$\operatorname{ran}(T-F) \subseteq \operatorname{Span}\{Fx_{2i+1} : 0 \leq i \leq n\} \oplus Z.$$

Then $\operatorname{codim} \operatorname{ran}(T \pm F) > n$, and so $T \pm F \notin \Lambda$. This establishes the assertions (1) and (2).

Assume now that F is finite-rank, and let $p = \min\{\dim \operatorname{ran}(F), 2\}$. It follows from [17, Proposition 2.12] that there exists an invertible operator $T \in \mathcal{B}(X)$ such that $T + F \notin \mathcal{B}_n(X)$ and $T - (-1)^p F \notin \mathcal{B}_n(X)$. But, $T + F$ and $T - (-1)^p F$ are

Fredholm operators of index zero, then $T+F \notin \Lambda$ and $T-(-1)^pF \notin \Lambda$. This completes the proof. □

4. Proofs of the main results

As a consequence of Theorem 3.6 and Proposition 3.4, we have the following result.

Lemma 4.1. *Let $\Phi : \mathcal{B}(X) \to \mathcal{B}(X)$ be an additive surjective map. If Φ preserves Λ in both directions, then Φ is injective and it preserves the set of rank-one operators in both directions.*

Proof. Suppose on the contrary that there exists $F \neq 0$ such that $\Phi(F) = 0$. Then, by Theorem 3.6, there exists an invertible operator $T \in \mathcal{B}(X)$ satisfying $T + F \notin \Lambda$. But, $\Phi(T + F) = \Phi(T) \in \Lambda$. This contradiction proves that Φ is injective.

Let $F \in \mathcal{B}(X)$ with $\dim \operatorname{ran}(F) \geq 2$. Then it follows again by Theorem 3.6 that there exists an invertible operator $T \in \mathcal{B}(X)$ such that $T+F$ and $T-F$ do not belong to Λ, and hence $\Phi(T + F)$ and $\Phi(T - F)$ do neither. Therefore, by Proposition 3.4, we obtain that $\dim \operatorname{ran}(\Phi(F)) \geq 2$. Since Φ is bijective and Φ^{-1} satisfies the same properties as Φ, we obtain that Φ preserves the set of rank-one operators in both directions. □

Recall that an operator $T \in \mathcal{B}(X)$ is said to be *algebraic* if there exists a non-zero complex polynomial P for which $P(T) = 0$. Such an operator T has finite ascent and descent (see [3, Theorem 2.7] and [4, Theorem 1.5]). Moreover, we have

$$T \in \Lambda \;\Leftrightarrow\; T \in \mathcal{B}_n(X) \;\Leftrightarrow\; \dim \mathcal{N}^{\infty}(T) \leq n.$$

Lemma 4.2. *Let $\Phi : \mathcal{B}(X) \to \mathcal{B}(X)$ be an additive surjective map preserving Λ in both directions. Then $\Phi(I) = cI$ where c is a non-zero scalar.*

Proof. We claim first that $S = \Phi(I)$ is an algebraic operator. Let $x \in X$ be non-zero. If the set $\{S^ix : 0 \leq i \leq 2n+1\}$ is linearly independent, then there exists a linear form $f \in X^*$ such that $f(S^ix) = -\delta_{i,2n+1}$ for $0 \leq i \leq 2n+1$. Let $T = S + S^{n+1}x \otimes fS^{n+1}$. It follows that

$$T(S^ix) = S^{i+1}x, \text{ for } 0 \leq i \leq n-1, \text{ and } T(S^nx) = 0.$$

Hence $\mathrm{a}(T) \geq n + 1$. On the other hand, we have

$$T^*(fS^i) = fS^{i+1}, \text{ for } 0 \leq i \leq n-1, \text{ and } T^*(fS^n) = 0.$$

Then $\mathrm{a}(T^*) \geq n + 1$, and so $\mathrm{d}(T) \geq n + 1$. Thus $T \notin \Lambda$. This contradiction shows that $\{S^ix : 0 \leq i \leq 2n + 1\}$ is a linearly dependent set. The vector x was arbitrary, therefore it follows from [2, Theorem 4.2.7] that S is algebraic.

Now assume, on the contrary, that S is not a scalar multiple of the identity. Then there exists $y_1 \in X$ such that the vectors y_1 and Sy_1 are linearly independent. Since $S \in \Lambda$, the subspace $\operatorname{ran}(S)$ has an infinite dimension, and hence there exists

$y_i \in X$, $2 \le i \le n$, such that $\{y_1, Sy_i : 1 \le i \le n\}$ is a linearly independent set. Consider linear forms $g_i \in X^*$ such that

$$g_i(y_1) = 0 \text{ and } g_i(Sy_j) = -\delta_{ij} \text{ for } 1 \le i, j \le n.$$

If we let $F = \sum_{i=1}^n S^2 y_i \otimes g_i$, we obtain easily that $Sy_j \in \ker(S+F)$, for $1 \le j \le n$, and $(S+F)y_1 = Sy_1 \in \ker(S+F)$. Consequently, $\dim \mathcal{N}^\infty(S+F) \ge n+1$. But, we have also that $S+F$ is an algebraic operator (see [4, Proposition 3.6]), therefore $S+F \notin \Lambda$. By Lemma 4.1, Φ is bijective and preserves rank-one operators in both directions. Hence, we obtain that $K = \Phi^{-1}(F)$ is of rank non-greater than n and $I + K \notin \Lambda$. However, $I+K$ is algebraic and $\ker((I+K)^{n+1}) \subseteq \operatorname{ran}(K)$, and so $I + K \in \Lambda$. This contradiction completes the proof. $\square$

Let τ be a field automorphism of $\mathbb{K}$. An additive map $A : X \to Y$ between two Banach spaces is called τ-*semi linear* if $A(\lambda x) = \tau(\lambda)Ax$ holds for all $x \in X$ and $\lambda \in \mathbb{K}$. Moreover, we say simply that A is *conjugate linear* when τ is the complex conjugation. Notice that if A is non-zero and bounded, then τ is continuous, and consequently, τ is either the identity or the complex conjugation (see [9, Theorem 14.4.2 and Lemma 14.5.1]). Moreover, in this case, the adjoint operator $A^* : Y^* \to X^*$, defined by $A^*(g) = \tau^{-1} \circ g \circ A$ for all $g \in Y^*$, is again τ-semi linear.

Lemma 4.3. *Let $\Phi : \mathcal{B}(X) \to \mathcal{B}(X)$ be an additive surjective map preserving Λ in both directions. Then there exists a non-zero scalar c, and either*

1. *there exists an invertible bounded linear, or conjugate linear, operator $A : X \to X$ such that $\Phi(F) = cAFA^{-1}$ for all finite-rank operators $F \in \mathcal{B}(X)$, or*
2. *there exists an invertible bounded linear, or conjugate linear, operator $B : X^* \to X$ such that $\Phi(F) = cBF^*B^{-1}$ for all finite-rank operators $F \in \mathcal{B}(X)$. In this case, X is reflexive.*

Proof. The existence of a non-zero scalar c such that $\Phi(I) = cI$ is ensured by Lemma 4.2. Clearly, we can suppose without loss of generality that $\Phi(I) = I$. Since Φ is bijective and preserves the set of rank-one operators in both directions, then by [16, Theorems 3.1 and 3.3], there exist a ring automorphism $\tau : \mathbb{K} \to \mathbb{K}$ and either two bijective τ-semi linear mappings $A : X \to X$ and $C : X^* \to X^*$ such that

$$\Phi(x \otimes f) = Ax \otimes Cf \quad \text{for all } x \in X \text{ and } f \in X^*, \tag{4.1}$$

or two bijective τ-semi linear mappings $B : X^* \to X$ and $D : X \to X^*$ such that

$$\Phi(x \otimes f) = Bf \otimes Dx \quad \text{for all } x \in X \text{ and } f \in X^*. \tag{4.2}$$

Suppose that Φ satisfies (4.1), and let us show that

$$C(f)(Ax) = \tau(f(x)) \quad \text{for all } x \in X \text{ and } f \in X^*. \tag{4.3}$$

Clearly, it suffices to establish that for all $x \in X$ and $f \in X^*$, $f(x) = -1$ if and only if $C(f)(Ax) = -1$. Let $x \in X$ and $f \in X^*$. We can choose linearly independent vectors

$z_1, \ldots, z_n$ in $\ker(f) \cap \ker(C(f)A)$. Then, it follows from [17, Lemma 3.8] that

$$\begin{aligned} f(x) = -1 \quad &\Leftrightarrow \quad \exists \{g_i\}_{i=1}^n \subseteq X^* : I + x \otimes f + \sum_{i=1}^n z_i \otimes g_i \notin \mathcal{B}_n(X) \\ &\Leftrightarrow \quad \exists \{g_i\}_{i=1}^n \subseteq X^* : I + x \otimes f + \sum_{i=1}^n z_i \otimes g_i \notin \Lambda \\ &\Leftrightarrow \quad \exists \{g_i\}_{i=1}^n \subseteq X^* : I + Ax \otimes Cf + \sum_{i=1}^n Az_i \otimes Cg_i \notin \Lambda \\ &\Leftrightarrow \quad \exists \{g_i\}_{i=1}^n \subseteq X^* : I + Ax \otimes Cf + \sum_{i=1}^n Az_i \otimes Cg_i \notin \mathcal{B}_n(X) \\ &\Leftrightarrow \quad C(f)(Ax) = -1. \end{aligned}$$

Thus, relation (4.3) holds, and arguing as in [16], we get that τ, A, C are continuous, τ is the identity or the complex conjugation, and $C = (A^{-1})^*$. Therefore, $\tau^{-1} = \tau$ and, for every $u \in X$, we have

$$\Phi(x \otimes f)u = \tau(fA^{-1}u)Ax = A(f(A^{-1}u)x) = A(x \otimes f)A^{-1}u.$$

Thus, $\Phi(x \otimes f) = A(x \otimes f)A^{-1}$ for all $x \in X$ and $f \in X^*$; that is, $\Phi(F) = AFA^{-1}$ for all finite-rank operators $F \in \mathcal{B}(X)$.

Now suppose that Φ satisfies (4.2), and let us show that

$$D(x)(Bf) = \tau(f(x)) \quad \text{for all } x \in X \text{ and } f \in X^*. \tag{4.4}$$

Let $x \in X$ and $f \in X^*$. Choose linearly independent linear forms $h_1, \ldots, h_n \in X^*$ such that $h_i(x) = 0$ and $D(x)(Bh_i) = 0$ for $1 \leq i \leq n$. Then, it follows from the surjectivity of D and from [17, Lemma 3.8] that

$$\begin{aligned} D(x)(Bf) = -1 \quad &\Leftrightarrow \quad \exists \{u_i\}_{i=1}^n \subseteq X : I + Bf \otimes Dx + \sum_{i=1}^n Bh_i \otimes Du_i \notin \mathcal{B}_n(X) \\ &\Leftrightarrow \quad \exists \{u_i\}_{i=1}^n \subseteq X : I + Bf \otimes Dx + \sum_{i=1}^n Bh_i \otimes Du_i \notin \Lambda \\ &\Leftrightarrow \quad \exists \{u_i\}_{i=1}^n \subseteq X : I + x \otimes f + \sum_{i=1}^n u_i \otimes h_i \notin \Lambda \\ &\Leftrightarrow \quad \exists \{u_i\}_{i=1}^n \subseteq X : I + x \otimes f + \sum_{i=1}^n u_i \otimes h_i \notin \mathcal{B}_n(X) \\ &\Leftrightarrow \quad \exists \{u_i\}_{i=1}^n \subseteq X : I + f \otimes \mathrm{J}x + \sum_{i=1}^n h_i \otimes \mathrm{J}u_i \notin \mathcal{B}_n(X^*) \\ &\Leftrightarrow \quad f(x) = -1, \end{aligned}$$

where $\mathrm{J} : X \to X^{**}$ is the natural embedding. Thus, relation (4.4) holds, and arguing as in [16], we get that τ, B, D are continuous, τ is the identity or the complex conjugation, and $D = (B^{-1})^*\mathrm{J}$. But, the operators D and $(B^{-1})^*$, and therefore also

J are bijections, which implies the reflexivity of X. Furthermore, $\tau^{-1} = \tau$ and, for every $u \in X$, we have

$$\begin{aligned}\Phi(x \otimes f)u &= (Bf \otimes (B^{-1})^* \mathrm{J}(x))u = (B^{-1})^* \mathrm{J}(x)(u) \cdot Bf \\ &= \tau(\mathrm{J}(x)(B^{-1}u)) \cdot Bf = B(\mathrm{J}(x)(B^{-1}u)f) \\ &= B(f \otimes \mathrm{J}(x))B^{-1}u = B(x \otimes f)^* B^{-1}u.\end{aligned}$$

Thus, $\Phi(x \otimes f) = B(x \otimes f)^* B^{-1}$ for all $x \in X$ and $f \in X^*$. Hence, $\Phi(F) = BF^*B^{-1}$ for all finite-rank operator $F \in \mathcal{B}(X)$. This completes the proof. □

Theorem 4.4. *Let $\Phi : \mathcal{B}(X) \to \mathcal{B}(X)$ be an additive surjective map preserving Λ in both directions. Then there exists a non-zero scalar c, and either*

1. *there exists an invertible bounded linear, or conjugate linear, operator $A : X \to X$ such that $\Phi(T) = cATA^{-1}$ for all $T \in \mathcal{B}(X)$, or*
2. *there exists an invertible bounded linear, or conjugate linear, operator $B : X^* \to X$ such that $\Phi(T) = cBT^*B^{-1}$ for all $T \in \mathcal{B}(X)$.*

Proof. Since Φ preserves Λ in both directions, it follows that Φ takes one of the two forms in Lemma 4.3.

Suppose that $\Phi(F) = cAFA^{-1}$ for all finite-rank operators $F \in \mathcal{B}(X)$. Let

$$\Psi(T) = c^{-1}A^{-1}\Phi(T)A \quad \text{for all } T \in \mathcal{B}(X).$$

Clearly, Ψ satisfies the same properties as Φ. Furthermore, $\Psi(I) = I$ and $\Psi(F) = F$ for all finite-rank operators $F \in \mathcal{B}(X)$. Let $T \in \mathcal{B}(X)$ and choose an arbitrary rational number λ such that $T - \lambda$ and $\Psi(T) - \lambda$ are invertible. Let $F \in \mathcal{B}(X)$ be a finite-rank operator. Since $T - \lambda + F$ and $\Psi(T) - \lambda + F$ are Fredholm of index zero, then

$$\begin{aligned}T - \lambda + F \in \mathcal{B}_n(X) &\Leftrightarrow T - \lambda + F \in \Lambda \Leftrightarrow \Psi(T) - \lambda + F \in \Lambda \\ &\Leftrightarrow \Psi(T) - \lambda + F \in \mathcal{B}_n(X).\end{aligned}$$

Hence, we get by [17, Proposition 2.17] that $\Psi(T) = T$.
This shows that $\Phi(T) = cATA^{-1}$ for all $T \in \mathcal{B}(X)$.
Now suppose that $\Phi(F) = cBF^*B^{-1}$ for all finite-rank operators $F \in \mathcal{B}(X)$. Then Lemma 4.3 ensures that X is reflexive. By considering

$$\Gamma(T) = c^{-1}\mathrm{J}^{-1}(B^{-1}\Phi(T)B)^*\mathrm{J} \quad \text{for all } T \in \mathcal{B}(X),$$

we get in a similar way that $\Gamma(T) = T$ for all $T \in \mathcal{B}(X)$. Thus, $\Phi(T) = cBT^*B^{-1}$ for all $T \in \mathcal{B}(X)$, as desired. This finishes the proof. □

With these results at hand, we are ready to prove our main results.

Proof of Theorem 1.1. (1) ⇒ (4). Suppose that Φ preserves $\mathcal{A}_n(H)$ in both directions. Using the fact that Φ is surjective, it follows by Theorem 2.7 that, for every $T \in \mathcal{B}(H)$,

$$\begin{aligned}T \in \mathcal{B}_n^+(H) &\Leftrightarrow \forall S \in \mathcal{B}(H), \exists \varepsilon_0 > 0 : \{T + \varepsilon S : \varepsilon \in \mathbb{Q} \text{ and } |\varepsilon| < \varepsilon_0\} \subseteq \mathcal{A}_n(H) \\ &\Leftrightarrow \forall S \in \mathcal{B}(H), \exists \varepsilon_0 > 0 : \{\Phi(T) + \varepsilon \Phi(S) : \varepsilon \in \mathbb{Q} \text{ and } |\varepsilon| < \varepsilon_0\} \subseteq \mathcal{A}_n(H) \\ &\Leftrightarrow \forall R \in \mathcal{B}(H), \exists \varepsilon_0 > 0 : \{\Phi(T) + \varepsilon R : \varepsilon \in \mathbb{Q} \text{ and } |\varepsilon| < \varepsilon_0\} \subseteq \mathcal{A}_n(H) \\ &\Leftrightarrow \Phi(T) \in \mathcal{B}_n^+(H).\end{aligned}$$

Thus Φ preserves $\mathcal{B}_n^+(H)$ in both directions. It follows that Φ takes one of the two forms in Theorem 4.4. Let us show that Φ cannot take the form

$$\Phi(T) = cBT^*B^{-1} \quad \text{for all } T \in \mathcal{B}(H). \tag{4.5}$$

Suppose on the contrary that Φ takes the form (4.5). Let $\{e_n : n \geq 0\}$ be an arbitrary orthonormal basis of H. Consider the weighted unilateral shift operator $U \in \mathcal{B}(H)$ given by

$$Ue_n = (n+1)^{-1}e_{n+1} \text{ for every } n \geq 0. \tag{4.6}$$

Clearly, U is an injective quasi-nilpotent operator.
Thus, $\mathrm{a}(U^*) = \mathrm{d}(U^*) = \infty$, $U \in \mathcal{B}_n^+(H)$ and $U^* \notin \mathcal{B}_n^{\pm}(H)$.
So that $\Phi(U) = cBU^*B^{-1} \notin \mathcal{B}_n^{\pm}(H)$, a contradiction.

(2) $\Rightarrow$ (4). Now, suppose that Φ preserves $\mathcal{D}_n(H)$ in both directions. As above, using Theorem 2.7 we infer that Φ preserves $\mathcal{B}_n^-(H)$ in both directions, and so Φ takes one of the two forms in Theorem 4.4. Consider the unilateral shift operator $S \in \mathcal{B}(H)$ given by

$$Se_0 = 0 \quad \text{and} \quad Se_n = e_{n-1} \text{ for } n \geq 1.$$

Clearly, S is surjective and $\mathrm{a}(S) = \infty$.
Thus, $\mathrm{d}(S^*) = \infty$, $S \in \mathcal{B}_n^-(H)$ and $S^* \notin \mathcal{B}_n^-(H)$. This contradiction shows that Φ cannot take the form (4.5).

(3) $\Rightarrow$ (4) is similar to the first implication with the same example (4.6).

(4) $\Rightarrow$ (1), (2) and (3) are obvious. □

Proof of Theorem 1.2. Follows from Theorems 2.5 and 4.4. □

Proof of Corollary 1.3. The proof is similar to the proof of Theorem 1.1. □

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A generalized Ekeland's variational principle for vector equilibria

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Abstract. In this paper, we establish an Ekeland-type variational principle for vector valued bifunctions defined on complete metric spaces with values in locally convex spaces ordered by closed convex cones. The main improvement consists in widening the class of bifunctions for which the variational principle holds. In order to prove this principle, a weak notion of continuity for vector valued functions is considered, and some of its properties are presented. We also furnish an existence result for vector equilibria in absence of convexity assumptions, passing through the existence of approximate solutions of an optimization problem.

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1. Introduction

Ekeland's variational principle (see [11]) has many applications in nonlinear analysis and optimization, see [1, 4, 2, 3, 5, 6], [7], [14], [19], [10] and the reference therein. Blum, Oettli [8] and Théra [18] showed that their existence result for a solution of an equilibrium problem is equivalent to Ekeland-type variational principle for bifunctions. Several authors have extended the Ekeland's variational principle to the case with a vector valued bifunction taking values in an ordered vector space, see [7], [2], [6], [15]. Araya et. al. [6] established a version of Ekeland's variational principle for vector valued bifunctions, which is expressed by the existence of a strict approximate minimizer for a weak vector equilibrium problem.

By a weak vector equilibrium problem we understand the problem of finding $\overline{x} \in X$ such that

$$f(\overline{x}, y) \notin -intK, \quad \text{for all } y \in X,$$

where $f : X \times X \to Y$ is a given bifunction, (X, d) is a complete metric space and (Y, K) is a Hausdorff topological vector space, ordered by the closed convex cone K.

Recall that $K \subseteq Y$ is said to be closed and convex cone if K is closed, $\alpha K \subseteq K$ for all $\alpha > 0$ and $K + K \subseteq K$.
The approach given in Araya et. al. [6] is based on the assumption that the equilibrium bifunction f satisfies the following triangle property:

$$f(x,y) + f(y,z) \in f(x,z) + K, \quad \text{for all } x,y,z \in X. \tag{1.1}$$

We stress the fact that (1.1) is a rather strong condition and it is rarely verified when the equilibrium problem is a variational inequality, see [10].
Motivated and inspired by [10], in this paper we shall give an improvement of Theorem 2.1 in Araya et. al. [6]. We widen the class of the vector bifunctions for which the Ekeland's variational principle is applicable. Further, some sufficient conditions for existence of equilibria which do not involve any convexity concept, neither for the domain nor for the bifunction are given, under a relaxed continuity concept for the vector functions.
The rest of the paper is organized as follows. In Section 2 we collect some definitions and results needed for further investigations. A weak notion of continuity for the vector valued functions is also studied and some of its properties are presented. Sections 3 and 4 are devoted to Ekeland's principles for the vector valued functions and bifunctions. Section 5 is devoted to an existence result for the weak vector equilibria where the vector bifunctions satisfy a property which generalizes the triangle inequality.

2. Preliminaries

Throughout this paper, unless otherwise specified, we assume that (X,d) is a complete metric space, (Y,K) is a locally convex Hausdorff topological vector space ordered by the nontrivial closed convex cone $K \subseteq Y$ with $intK \neq \emptyset$, where $intK$ denotes the topological interior of K, as follows:

$$x \leq_K y \Leftrightarrow y - x \in K.$$

We agree that any cone contains the origin, according to the following definition.

Definition 2.1. The set $K \subseteq Y$ is called a cone iff $\lambda x \in K$ for all $x \in K$ and $\lambda \geq 0$. The cone K is pointed iff $K \cap (-K) = \{0\}$; proper iff $K \neq Y$ and $K \neq \{0\}$.

Let $k_0 \in K \setminus (-K)$. The nonlinear scalarization function [20] (see also [16]) $z_{K,k_0} : Y \to [-\infty, \infty]$ is defined as

$$z_{K,k_0}(y) = \inf\{r \in \mathbb{R} \mid y \in rk_0 - K\}.$$

We present some properties of the scalarization function which will be used in the sequel.

Lemma 2.2. [16] *For each $r \in \mathbb{R}$ and $y \in Y$, the following statements are true:*

(i) z_{K,k_0} *is proper;*
(ii) z_{K,k_0} *is lower semicontinuous;*
(iii) z_{K,k_0} *is sublinear;*
(iv) z_{K,k_0} *is K monotone;*
(v) $z_{K,k_0}(y) \leq r \Leftrightarrow y \in rk_0 - K$;

(vi) $z_{K,k_0}(y) > r \Leftrightarrow y \notin rk_0 - K$;
(vii) $z_{K,k_0}(y) \geq r \Leftrightarrow y \notin rk_0 - intK$;
(viii) $z_{K,k_0}(y) < r \Leftrightarrow y \in rk_0 - intK$;
(ix) $z_{K,k_0}(y + \lambda k_0) = z_{K,k_0}(y) + \lambda$, *for every* $y \in Y$ *and* $\lambda \in \mathbb{R}$.

As a corollary of the lemma above, Göpfert et al. [13] presented the following nonconvex separation theorem, see also [16].

Lemma 2.3. [13] *Assume that* Y *is a topological vector space,* K *a closed solid convex and* $A \subset Y$ *a nonempty set such that* $A \cap (-intK) = \emptyset$. *Then* z_{K,k_0} *is a finite valued continuous function such that*

$$z_{K,k_0}(-y) < 0 \leq z_{K,k_0}(x) \quad \textit{for all } x \in A \quad \textit{and } y \in intK,$$

moreover $z_{K,k_0}(x) > 0$ *for all* $x \in intA$.

In the vector valued case there are several possible extensions of the scalar notion of *lower semicontinuity*, see [9]. We recall here the concept of (k_0, K)-lower semicontinuity introduced by Chr. Tammer [19] which will be used in the sequel. This concept is weaker than the K-*lower semicontinuity* which was introduced by Borwein et. al. [9] (see also [12], [17] and [21].)

Definition 2.4. [19] A function $\varphi : X \longrightarrow Y$ is said to be:
(i) (k_0, K)-*lower semicontinuous* if for all $r \in \mathbb{R}$, the set $\{x \in X : \varphi(x) \in rk_0 - K\}$ is closed;
(ii) (k_0, K)-*upper semicontinuous* if for all $r \in \mathbb{R}$, the set $\{x \in X : \varphi(x) \in rk_0 + K\}$ is closed;
(iii) (k_0, K)-*continuous* if it is both (k_0, K)-lower semicontinuous as well as (k_0, K)-upper semicontinuous.

The function $\varphi : X \longrightarrow Y$ is said to be K-*bounded* below if there exists $\overline{y} \in Y$ such that $\varphi(X) \subseteq \overline{y} + K$.
In [19], the following assertion was proved.

Lemma 2.5. [19]
(i) *If* φ *is* (k_0, K)-*lower semicontinuous, then* $z_{K,k_0} \circ \varphi$ *is lower semicontinuous;*
(ii) *If* φ *is* (k_0, K)-*upper semicontinuous, then* $z_{K,k_0} \circ \varphi$ *is upper semicontinuous.*

Remark 2.6. It is well known that the sum of two K-lower semicontinuous mappings is not a K-lower semicontinuous mapping in general, see [7]. Due to the following example, we can obtain a similar conclusion for the (k_0, K)-lower semicontinuity, i.e., if $\varphi : X \longrightarrow Y$ is (k_0, K)-lower semicontinuous, the function $\varphi(\cdot) - \varphi(x)$, where $x \in X$ is fixed, is not necessary (k_0, K)-lower semicontinuous.

Example 2.7. Let us consider $X = \mathbb{R}^2$, $Y = \mathbb{R}^2$ and $K = \mathbb{R}^2_+$. Define $\varphi : X \to Y$ as:

$$\varphi(x) = \begin{cases} (1, -2), & x_1 > 0, \ x_2 \in \mathbb{R}, \\ (x_1, x_1), & x_1 \leq 0, \ x_2 \in \mathbb{R}, \end{cases}$$

where $x = (x_1, x_2)$.
This function is (k_0, K)-lower semicontinuous with $k_0 = (1, 1)$. Now take $x = (1, 0)$.

We will prove that the function $\varphi(\cdot) - \varphi(x)$ is not (k_0, K)-lower semicontinuous. Take also $r = 1$ and consider the set

$$L = \{y \in X : \varphi(y) - \varphi(x) \in (1,1) - K\}.$$

It is easy to observe that $y_n = (\frac{1}{n}, \frac{2}{n}) \in L$, $n \in \mathbb{N}$, and $y_n \to y_0$, where $y_0 = (0,0)$. On the other hand,

$$\varphi(y_0) - \varphi(x) = (0,0) - (1,-2) = (-1,2) \notin (1,1) - K.$$

Hence $y_0 \notin L$, which shows that the set L is not closed, i.e., the conclusion.

In what follows, we will furnish some properties for this kind of continuity for the vector functions.

Proposition 2.8. *If $\varphi : X \longrightarrow Y$ is (k_0, K)-lower semicontinuous, then the function $-\varphi$ is (k_0, K)-upper semicontinuous.*

Theorem 2.9. *If $\varphi : X \longrightarrow Y$ is (k_0, K)-lower semicontinuous and*

$$\varphi(X) \subset \bigcup_{t \in \mathbb{R}} \{tk_0\},$$

then the function $\varphi(\cdot) - \varphi(x)$, where $x \in X$ is fixed, is (k_0, K)-lower semicontinuous.

Proof. Let us fix $x_0 \in X$ and consider the function $\delta : X \to Y$ defined by

$$\delta(y) = \varphi(y) - \varphi(x_0), \ y \in X.$$

Fix also $r \in \mathbb{R}$ and consider the set $S = \{y \in X : \varphi(y) - \varphi(x_0) \in rk_0 - K\}$.
We will prove that this set is closed.
Since $\varphi(X) \subset \bigcup_{t \in \mathbb{R}} \{tk_0\}$, it follows that, for $x_0 \in X$, there exists $t_0 \in \mathbb{R}$ such that $\varphi(x_0) = t_0 k_0$. We obtain

$$S = \{y \in X : \varphi(y) \in (r + t_0)k_0 - K\}.$$

Since $r, t_0 \in \mathbb{R}$ are fixed and φ is (k_0, K)-lower semicontinuous, it follows the set S is closed, i.e., the conclusion. □

Corollary 2.10. *If $\varphi : X \longrightarrow Y$ is (k_0, K)-lower semicontinuous and*

$$\varphi(X) \subset \bigcup_{t \in \mathbb{R}} \{tk_0\},$$

then the function $\varphi(x) - \varphi(\cdot)$, where $x \in X$ is fixed, is (k_0, K)-upper semicontinuous.

3. Ekeland's variational principle for the vector functions

This section deals with an Ekeland's variational principle for the vector valued functions. Inspired by the results obtained in Theorem 3.1 Araya [5], we are able to present our result when the vector function is (k_0, K)-lower semicontinuous.

Theorem 3.1. *If $\varphi : X \to Y$ is (k_0, K)-lower semicontinuous is such that*

(i) *for each $x \in X$, there exists $\overline{y} \in Y$ such that $(\varphi(X) - \varphi(x)) \cap (\overline{y} - intK) = \emptyset$;*

(ii) $\varphi(X) \subset \bigcup_{t\in\mathbb{R}} \{tk_0\}$,

then, for every given $\varepsilon > 0$ *and for every* $\widehat{x} \in X$ *there exists* $\overline{x} \in X$ *such that:*

(a) $\varphi(\overline{x}) - \varphi(\widehat{x}) + \varepsilon d(\overline{x}, \widehat{x})k_0 \in -K$;

(b) $\varphi(x) - \varphi(\overline{x}) + \varepsilon d(\overline{x}, x)k_0 \notin -K$, *for every* $x \in X, x \neq \overline{x}$.

Proof. Let us consider the functional

$$z_{K,k_0} : Y \to [-\infty, \infty],$$

defined by

$$z_{K,k_0}(y) = \inf\{r \in \mathbb{R} \mid y \in rk_0 - K\}.$$

For each $x \in X, \varepsilon > 0$ consider the set

$$S(x) = \{y \in X \mid \ y = x \text{ or } z_{K,k_0}(\varphi(y) - \varphi(x)) + \varepsilon d(x, y) \leq 0\}$$

It is obviously that $x \in S(x)$, therefore $S(x) \neq \emptyset$ for all $x \in X$. By Theorem 2.9, since φ is a (k_0, K)-lower semicontinuous function, then the function $\delta(\cdot) = \varphi(\cdot) - \varphi(x)$, where $x \in X$ is fixed, is also (k_0, K)-lower semicontinuous. From Lemma 2.5 it follows that $z_{K,k_0} \circ \delta$ is lower semicontinuous and $d(x, y)$ is continuous, therefore $S(x)$ is closed for every $x \in X$.

Now we show that $z_{K,k_0}(\varphi(X) - \varphi(x)) := \cup_{y\in X}\{z_{K,k_0}(\varphi(y) - \varphi(x)\})$ is bounded from below for all $x \in X$. By assumption (i) and Lemma 2.3 we have that

$$0 \leq z_{K,k_0}(\varphi(y) - \varphi(x) - \overline{y}), \text{ for all } y \in X.$$

Using (iii) of Lemma 2.2, we get

$$-\infty < -z_{K,k_0}(-\overline{y}) < z_{K,k_0}(\varphi(y) - \varphi(x)) \text{ for any } y \in X,$$

which implies that $z_{K,k_0}(\varphi(X) - \varphi(x))$ is bounded from below.

Let define the real valued function

$$v(x) = \inf_{y\in S(x)} z_{K,k_0}(\varphi(y) - \varphi(x)). \tag{3.1}$$

and set $x = \widehat{x} \in X$. Since $z \circ \delta$ is bounded below, we have

$$v(\widehat{x}) = \inf_{y\in S(\widehat{x})} z_{K,k_0}(\varphi(y) - \varphi(\widehat{x})) > -\infty.$$

Starting from $\widehat{x} \in X$, a sequence x_n of points of X can be defined such that $x_{n+1} \in S(x_n)$ such that

$$z_{K,k_0}(\varphi(x_{n+1}) - \varphi(x_n)) \leq v(x_n) + \frac{1}{n+1}.$$

Let us take $y \in S(x_{n+1}) \setminus \{x_{n+1}\}$. It follows that

$$z_{K,k_0}(\varphi(y) - \varphi(x_{n+1})) + \varepsilon d(x_{n+1}, y) \leq 0. \tag{3.2}$$

Since $x_{n+1} \in S(x_n)$, we also have

$$z_{K,k_0}(\varphi(x_{n+1}) - \varphi(x_n)) + \varepsilon d(x_{n+1}, x_n) \leq 0. \tag{3.3}$$

Adding (3.2) and (3.3) we obtain

$$z_{K,k_0}(\varphi(x_{n+1}) - \varphi(x_n)) + z_{K,k_0}(\varphi(y) - \varphi(x_{n+1})) + \varepsilon d(x_{n+1}, x_n) + \varepsilon d(x_{n+1}, y) \leq 0.$$

Using the triangle inequality for the distance and taking into account that z_{K,k_0} is sublinear, it follows that

$$z_{K,k_0}(\varphi(y) - \varphi(x_n)) + \varepsilon d(x_n, y) \leq 0 \Longleftrightarrow y \in S(x_n).$$

Therefore, $y \in S(x_n)$ implies that $S(x_{n+1}) \subseteq S(x_n)$. In particular,

$$\begin{aligned} v(x_{n+1}) &= \inf_{y \in S(x_{n+1})} z_{K,k_0}(\varphi(y) - \varphi(x_{n+1})) \geq \inf_{y \in S(x_n)} z_{K,k_0}(\varphi(y) - \varphi(x_n)) \\ &\geq \inf_{y \in S(x_n)} z_{K,k_0}(\varphi(y) - \varphi(x_n)) - z_{K,k_0}(\varphi(x_{n+1}) - \varphi(x_n)) \\ &= v(x_n) - z_{K,k_0}(\varphi(x_{n+1}) - \varphi(x_n)) \geq -\frac{1}{n+1} \end{aligned} \tag{3.4}$$

Thus, for $y \in S(x_{n+1}) \setminus \{x_{n+1}\}$, from (3.1), (3.2) and (3.4) we obtain

$$\varepsilon d(x_{n+1}, y) \leq -z_{K,k_0}(\varphi(y) - \varphi(x_{n+1})) \leq -v(x_n + 1) \leq \frac{1}{n+1} \to 0 \text{ as } n \to \infty,$$

which entails

$$diam(S(x_n)) \to 0 \text{ as } n \to \infty.$$

Since the sets $S(x_n)$ are closed and $S(x_{n+1}) \subseteq S(x_n)$ we obtain from this that the intersection of the sets $S(x_n)$ is a singleton $\{\overline{x}\}$ and $S(\overline{x}) = \{\overline{x}\}$. This implies that $\overline{x} \in S(\widehat{x})$, or equivalently

$$z_{K,k_0}(\varphi(\overline{x}) - \varphi(\widehat{x})) \leq -\varepsilon d(\widehat{x}, \overline{x}).$$

From Lemma 2.2 (v), it follows that

$$\varphi(\overline{x}) - \varphi(\widehat{x}) + \varepsilon d(\widehat{x}, \overline{x}) k_0 \in -K.$$

Therefore, (a) holds. Moreover, if $x \neq \overline{x}$, then $x \notin S(\overline{x})$, and we get

$$z_{K,k_0}(\varphi(x) - \varphi(\overline{x})) > -\varepsilon d(x, \overline{x}).$$

Using again Lemma 2.2 (vi) we have

$$\varphi(x) - \varphi(\overline{x}) \notin -\varepsilon d(x, \overline{x}) k_0 - K, \text{ for all } x \neq \overline{x}, \tag{3.5}$$

which is the conclusion (b) of our theorem. □

Remark 3.2. In Araya [5], an important assumption is

$$(H) \quad \{y \in X \mid \varphi(y) - \varphi(x) + d(x, y) k_0 \in -K\} \text{ is closed for every } x \in X.$$

On the other hand, we use the (k_0, K)-lower semicontinuity for the function φ.
Before going further, we spend some time discussing on the comparison between the condition (H) and the (k_0, K)-lower semicontinuity. Taking into account Example 2.7 we can observe that if the function φ is (k_0, K)-lower semicontinuous, not necessary satisfies condition (H).
However, if the function φ satisfies the condition (H) then is not necessary (k_0, K)-lower semicontinuous, as the following example shows.

Example 3.3. Let $X = [0,1]$, $Y = l_\infty$ and $\varphi : X \to Y$ defined as

$$\varphi(x) = \begin{cases} \left(\frac{1}{x+1}, \frac{1}{x+2}, \ldots, \frac{1}{x+n}, \ldots\right), & x \neq 0; \\ (2, \frac{1}{2}, \ldots, \frac{1}{n}, \ldots), & x = 0. \end{cases}$$

The ordering cone is $K_{l_\infty} = \{y \in l_\infty \mid y_i \geq 0 \text{ for all } i \in \mathbb{N}\}$ and has nonempty interior. Considering $k_0 = \left(1, \frac{1}{2}, \ldots, \frac{1}{n}, \ldots\right)$ and $r = 1$, by Definition 2.4, taking $x_n \to 0, x_n \in S$, it is easy to observe that the set

$$S = \{x \in X : \varphi(x) \in rk_0 - K\}$$

is not closed, $0 \notin S$. On the other hand, φ satisfies the condition (H). Concluding, no one implies the other.

4. Ekeland's variational principle for the vector bifunctions

Araya et al. [6] obtained a vectorial version of Ekeland's variational principle for the bifunctions related to an equilibrium problem. They used the triangle inequality in order to obtain the desired result. Further, instead the triangle inequality property a suitable approximation from below of the bifunction f is required.

Let $f : X \times X \to Y$ be a bifunction. Consider the following *property*: there exists $\varphi : X \to Y$ such that

$$(P) \quad f(x,y) \in \varphi(y) - \varphi(x) + K \text{ for all } x, y \in X.$$

Property (P) is more general than the triangle inequality:

$$(T) \quad f(x,z) + f(z,y) \in f(x,y) + K, \text{ for all } x, y, z \in X.$$

Indeed, take in triangle inequality, for example, $\varphi_{\widehat{x}} = f(\widehat{x}, \cdot)$, where $\widehat{x} \in X$ is fixed, and property (P) follows.

We illustrate that the property (P) is more general than the triangle inequality considering the following example.

Example 4.1. Let $X = [0,1]$ and $Y = l_\infty$ and $f : X \times X \to Y$ defined as:

$$f(x,y) = \begin{cases} y\left(\frac{1}{2}, \frac{1}{4}, \ldots, \frac{1}{2^n}, \ldots\right), & x \neq \frac{1}{2},\ y \neq \frac{1}{2}; \\ (0, 0, \ldots, 0, \ldots), & x = \frac{1}{2},\ y \neq \frac{1}{2}; \\ (1-x)\left(\frac{1}{2}, \frac{1}{4}, \ldots, \frac{1}{2^n}, \ldots\right), & x \neq \frac{1}{2},\ y = \frac{1}{2}; \\ \left(\frac{1}{2}, \frac{1}{4}, \ldots, \frac{1}{2^n}, \ldots\right), & x = \frac{1}{2},\ y = \frac{1}{2}. \end{cases}$$

The ordering cone is $K_{l_\infty} = \{y \in l_\infty \mid y_i \geq 0 \text{ for all } i \in \mathbb{N}\}$. The function f does not satisfy the triangle inequality; take $x = 1, y = \frac{1}{2}$ and $z = \frac{1}{4}$. We obtain

$$f\left(1, \frac{1}{2}\right) + f\left(\frac{1}{2}, \frac{1}{4}\right) \notin f\left(1, \frac{1}{4}\right) + K.$$

On the other hand, there exists $\varphi : X \to Y$, namely

$$\varphi(x) = \begin{cases} \left(\frac{x}{2}, \frac{x}{4}, \ldots, \frac{x}{2^n}, \ldots\right), & x \neq \frac{1}{2}; \\ \left(\frac{1}{2}, \frac{1}{4}, \ldots, \frac{1}{2^n}, \ldots\right), & x = \frac{1}{2}, \end{cases}$$

such that the property (P) is satisfied.

The following result extends Theorem 2.1 in [6].

Theorem 4.2. *Let $f : X \times X \to Y$ and assume that*

(i) *there exists $\varphi : X \to Y$ (k_0, K)-lower semicontinuous such that*

$$f(x, y) \in \varphi(y) - \varphi(x) + K, \text{ for all } x, y \in X;$$

(ii) *for each $x \in X$, there exists $\overline{y} \in Y$ such that $(\varphi(X) - \varphi(x)) \cap (\overline{y} - intK) = \emptyset$;*

(iii) *for each $x \in X$, $\{y \in X \mid (\varphi(y) - \varphi(x)) + d(x, y)k_0 \in -K\}$ is closed.*

Then, for every $\varepsilon > 0$ and for every $\widehat{x} \in X$, there exists $\overline{x} \in X$ such that

(a) $\varphi(\overline{x}) - \varphi(\widehat{x}) + \varepsilon d(\overline{x}, \widehat{x})k_0 \in -K$;

(b) $f(\overline{x}, x) + \varepsilon d(\overline{x}, y)k_0 \notin -K$, *for all $x \in X$, $x \neq \overline{x}$.*

Proof. The function φ satisfies all the assumptions of Theorem 3.1 in [5]. Then there exists $\overline{x} \in X$ such that item (a) is verified. From the property (P) we have

$$f(\overline{x}, x) - \varphi(x) + \varphi(\overline{x}) \in K, \text{ for all } x \in X,$$

and by item (iii) of Theorem 3.1 we get

$$\varphi(x) - \varphi(\overline{x}) + \varepsilon d(\overline{x}, x)k_0 \notin -K, \text{ for every } x \in X, x \neq \overline{x}.$$

Adding these two relations we obtain item (b) of the theorem. □

Remark 4.3. We have to remark the fact that we do not need the assumption

$$f(x, x) = 0,$$

see Theorem 2.1 in [6].

We present now the following vectorial form of equilibrium version of Ekeland-type variational principle, result which extends similar results from the literature, see [6], [7] and [2].

Theorem 4.4. *Let $f : X \times X \to Y$ such that*

(i) *there exists $\varphi : X \to Y$ (k_0, K)-lower semicontinuos such that*

$$f(x, y) \in \varphi(y) - \varphi(x) + K, \text{ for all } x, y \in X;$$

(ii) *for each $x \in X$, there exists $\overline{y} \in Y$ such that $(\varphi(X) - \varphi(x)) \cap (\overline{y} - intK) = \emptyset$;*

(iii) $\varphi(X) \subset \bigcup_{t \in \mathbb{R}} \{tk_0\}$.

Then, for every $\varepsilon > 0$ and for every $\widehat{x} \in X$, there exists $\overline{x} \in X$ such that

(a) $\varphi(\overline{x}) - \varphi(\widehat{x}) + \varepsilon d(\overline{x}, \widehat{x})k_0 \in -K$;

(b) $f(\overline{x}, x) + \varepsilon d(\overline{x}, x)k_0 \notin -K$, *for all $x \in X$, $x \neq \overline{x}$.*

Proof. The idea of the proof is like in Theorem 4.2 and is based on Theorem 3.1. □

There are many cases where Theorem 2.1 [6] cannot be applied but all the assumptions of Theorem 4.4 are satisfied.

Example 4.5. Let $X = [0, 2]$, $Y = \mathbb{R}^2$ and $f : X \times X \to Y$ defined as:

$$f(x, y) = \begin{cases} (y, 2y), & x > 0,\ y > 0; \\ (2 - x, 0), & x > 0,\ y = 0; \\ (y + 2, y), & x = 0,\ y > 0; \\ (0, 0), & x = 0,\ y = 0. \end{cases}$$

The ordering cone of Y is $K = \mathbb{R}^2_+$. The function f does not satisfy the triangle inequality; take $x = 2, y = 0$ and $z = 1$. We obtain

$$f(2, 0) + f(0, 1) \notin f(2, 1) + K.$$

On the other hand, there exists $\varphi : X \to Y$, namely

$$\varphi(x) = (x, 0),$$

such that φ is (k_0, K)-lower semicontinuous with $k_0 = (1, 0)$.
Moreover, $\varphi(X) \subset \bigcup_{t\in\mathbb{R}} \{tk_0\}$ and the property (P) is satisfied.
We notice that $x = 1$ is a solution for the weak equilibria.

5. Existence solutions for the weak equilibria

The settings for this section are the same like in the section before.
Using Theorem 3.1, we are able to show the nonemptiness of the solution set of the weak equilibria without any convexity requirements on the set X and the function f, going through the existence of approximate solutions of an optimization problem.
The next statement provides the existence of solution of an optimization problem when the domain is compact.

Theorem 5.1. *If C is a nonempty compact subset of X, $\varphi : C \to Y$ is (k_0, K)-lower semicontinuous such that*

(i) *for each $x \in C$, there exists $\overline{y} \in Y$ such that $(\varphi(C) - \varphi(x)) \cap (\overline{y} - intK) = \emptyset$;*
(ii) *$\varphi(C) \subset \bigcup_{t\in\mathbb{R}} \{tk_0\}$;*

then there exists $\overline{x} \in C$ such that $\varphi(y) - \varphi(\overline{x}) \notin -intK$, for every $y \in C$.

Proof. From Theorem 3.1, for each $n \in \mathbb{N}$, there exists $x_n \in C$ such that

$$\varphi(y) - \varphi(x_n) + \frac{1}{n} d(x_n, y)k_0 \notin -K, \quad \text{for all } y \in C,\ y \neq x_n.$$

By Lemma 2.2 (vi), we have

$$z_{K,k_0}(\varphi(y) - \varphi(x_n)) + \frac{1}{n} d(x_n, y) > 0, \quad \text{for all } y \in C,\ y \neq x_n \text{ and } n \in \mathbb{N}.$$

Since C is compact, we can choose a subsequence $\{x_{n_k}\}$ of x_n such that $x_{n_k} \to \overline{x} \in C$ as $k \to \infty$. Then, since $\varphi(y) - \varphi(\cdot)$, where $y \in C$ is fixed, is (k_0, K)-upper semicontinuous, we obtain that $z_{K,k_0}(\varphi(y) - \varphi(\cdot))$ is upper semicontinuous, see Lemma 2.5. Hence,

$$z_{K,k_0}(\varphi(y)-\varphi(\overline{x})) \geq \limsup_{k\to\infty}(z_{K,k_0}(\varphi(y)-\varphi(x_{n_k}))+\frac{1}{n_k}d(x_{n_k},x)) \geq 0, \quad \text{for all } y \in C.$$

Therefore, again by Lemma 2.2 (vii), it follows

$$\varphi(y) - \varphi(\overline{x}) \notin -intK, \quad \text{for all } y \in C,$$

and thus, $\overline{x}$ is a solution for an optimization problem. □

The next result gives sufficient conditions for the existence of solutions when we move to the wider class of bifunctions which satisfies the property (P).

Theorem 5.2. *Let C be a nonempty compact subset of X, $f : C \times C \to Y$ a bifunction which satisfies property (P) with respect to $\varphi : C \to Y$ which is (k_0, K)-lower semicontinuous. Assume that:*

(i) *for each $x \in C$, there exists $\overline{y} \in Y$ such that $(\varphi(C) - \varphi(x)) \cap (\overline{y} - intK) = \emptyset$;*

(ii) $\varphi(C) \subset \bigcup_{t\in\mathbb{R}}\{tk_0\}$,

Then there exists $\overline{x} \in C$ such that $f(\overline{x}, y) \notin -intK$, for every $y \in C$.

Proof. The proof is based on Theorem 5.1 taking into account the property (P). □

6. Concluding remarks

In this paper, we widen the class of vector bifunctions for which Ekeland's variational principle holds and obtain a result which improves the main result in Araya et. al [6]. In the literature, when dealing with vector equilibrium problems and the existence of their solutions, the most used assumptions are the convexity of the domain and the generalized convexity and monotonicity, together with some weak continuity assumptions of the vector function. In this paper, we focus on conditions that do not involve any convexity concept, neither for the domain nor for the bifunction involved. Sufficient conditions for the weak vector equilibria with bifunctions which satisfy property (P), in the absence of the convexity, are given for compact domains.

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Geometric characteristics and properties of a two-parametric family of Lie groups with almost contact B-metric structure of the smallest dimension

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Abstract. Almost contact B-metric manifolds of the lowest dimension 3 are constructed by a two-parametric family of Lie groups. Our purpose is to determine the class of considered manifolds in a classification of almost contact B-metric manifolds and their most important geometric characteristics and properties.

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1. Introduction

The study of the differential geometry of the almost contact B-metric manifolds has initiated in [5]. The geometry of these manifolds is a natural extension of the geometry of the almost complex manifolds with Norden metric [3, 6] in the case of odd dimension. Almost contact B-metric manifolds are investigated and studied for example in [5, 11, 12, 14, 15, 17, 18, 20].

Here, an object of special interest are the Lie groups considered as three-dimensional almost contact B-metric manifolds. For example of such investigation see [19].

The aim of the present paper is to make a study of the most important geometric characteristics and properties of a family of Lie groups with almost contact B-metric structure of the lowest dimension 3, belonging to the main vertical classes. These classes are $\mathcal{F}_4$ and $\mathcal{F}_5$, where the fundamental tensor F is expressed explicitly by the metric g, the structure (φ, ξ, η) and the vertical components of the Lee forms θ and θ^*, i.e. in this case the Lee forms are proportional to η at any point. These classes contain some significant examples as the time-like sphere of g and the light cone of the associated metric of g in the complex Riemannian space, considered in [5], as well as the Sasakian-like manifolds studied in [7].

The paper is organized as follows. In Sec. 2, we give some necessary facts about almost contact B-metric manifolds. In Sec. 3, we construct and study a family of Lie groups as three-dimensional manifolds of the considered type.

2. Almost contact manifolds with B-metric

Let $(M, \varphi, \xi, \eta, g)$ be a $(2n+1)$-dimensional *almost contact B-metric manifold*, i.e. (φ, ξ, η) is a triplet of a tensor (1,1)-field φ, a vector field ξ and its dual 1-form η called an almost contact structure and the following identities holds:

$$\varphi\xi = 0, \quad \varphi^2 = -\mathrm{Id} + \eta \otimes \xi, \quad \eta \circ \varphi = 0, \quad \eta(\xi) = 1,$$

where Id is the identity. The B-metric g is pseudo-Riemannian and satisfies

$$g(\varphi x, \varphi y) = -g(x, y) + \eta(x)\eta(y)$$

for arbitrary tangent vectors $x, y \in T_pM$ at an arbitrary point $p \in M$ [5].

Further, x, y, z, w will stand for arbitrary vector fields on M or vectors in the tangent space at an arbitrary point in M.

Let us note that the restriction of a B-metric on the contact distribution $H = \ker(\eta)$ coincides with the corresponding Norden metric with respect to the almost complex structure and the restriction of φ on H acts as an anti-isometry on the metric on H which is the restriction of g on H.

The associated metric $\tilde{g}$ of g on M is given by $\tilde{g}(x, y) = g(x, \varphi y) + \eta(x)\eta(y)$. It is a B-metric, too. Hence, $(M, \varphi, \xi, \eta, \tilde{g})$ is also an almost contact B-metric manifold. Both metrics g and $\tilde{g}$ are indefinite of signature $(n+1, n)$.

The structure group of $(M, \varphi, \xi, \eta, g)$ is $\mathcal{G} \times \mathcal{I}$, where $\mathcal{I}$ is the identity on span(ξ) and $\mathcal{G} = \mathcal{GL}(n; \mathbb{C}) \cap \mathcal{O}(n, n)$.

The (0,3)-tensor F on M is defined by $F(x, y, z) = g\big((\nabla_x \varphi)\, y, z\big)$, where ∇ is the Levi-Civita connection of g. The tensor F has the following properties:

$$F(x, y, z) = F(x, z, y) = F(x, \varphi y, \varphi z) + \eta(y)F(x, \xi, z) + \eta(z)F(x, y, \xi).$$

A classification of the almost contact B-metric manifolds is introduced in [5], where eleven basic classes $\mathcal{F}_i$ $(i = 1, 2, \ldots, 11)$ are characterized with respect to the properties of F. The special class $\mathcal{F}_0$ is defined by the condition $F(x, y, z) = 0$ and is contained in each of the other classes. Hence, $\mathcal{F}_0$ is the class of almost contact B-metric manifolds with ∇-parallel structures, i.e. $\nabla\varphi = \nabla\xi = \nabla\eta = \nabla g = \nabla\tilde{g} = 0$.

Let g_{ij}, $i, j \in \{1, 2, \ldots, 2n+1\}$, be the components of the matrix of g with respect to a basis $\{e_i\}_{i=1}^{2n+1} = \{e_1, e_2, \ldots, e_{2n+1}\}$ of T_pM at an arbitrary point $p \in M$, and g^{ij} – the components of the inverse matrix of (g_{ij}). The Lee forms associated with F are defined as follows:

$$\theta(z) = g^{ij}F(e_i, e_j, z), \quad \theta^*(z) = g^{ij}F(e_i, \varphi e_j, z), \quad \omega(z) = F(\xi, \xi, z).$$

In [12], the *square norm of* $\nabla\varphi$ is introduced by:

$$\|\nabla\varphi\|^2 = g^{ij}g^{ks}g\big((\nabla_{e_i}\varphi)\, e_k, \big(\nabla_{e_j}\varphi\big)\, e_s\big). \tag{2.1}$$

If $(M, \varphi, \xi, \eta, g)$ is an $\mathcal{F}_0$-manifold then the square norm of $\nabla\varphi$ is zero, but the inverse implication is not always true. An almost contact B-metric manifold satisfying the condition $\|\nabla\varphi\|^2 = 0$ is called an *isotropic-$\mathcal{F}_0$-manifold.* The square norms of $\nabla\eta$ and $\nabla\xi$ are defined in [13] by:

$$\|\nabla\eta\|^2 = g^{ij}g^{ks}\left(\nabla_{e_i}\eta\right)e_k\left(\nabla_{e_j}\eta\right)e_s, \quad \|\nabla\xi\|^2 = g^{ij}g\left(\nabla_{e_i}\xi, \nabla_{e_j}\xi\right). \tag{2.2}$$

Let R be the curvature tensor of type (1,3) of Levi-Civita connection ∇, i.e. $R(x, y)z = \nabla_x\nabla_y z - \nabla_y\nabla_x z - \nabla_{[x,y]}z$. The corresponding tensor of R of type (0,4) is defined by $R(x, y, z, w) = g(R(x, y)z, w)$.

The Ricci tensor ρ and the scalar curvature τ for R as well as their associated quantities are defined by the following traces $\rho(x, y) = g^{ij}R(e_i, x, y, e_j)$, $\tau = g^{ij}\rho(e_i, e_j)$, $\rho^*(x, y) = g^{ij}R(e_i, x, y, \varphi e_j)$ and $\tau^* = g^{ij}\rho^*(e_i, e_j)$, respectively.

An almost contact B-metric manifold is called *Einstein* if the Ricci tensor is proportional to the metric tensor, i.e. $\rho = \lambda g$, $\lambda \in \mathbb{R}$.

Let α be a non-degenerate 2-plane (section) in T_pM. It is known from [20] that the special 2-planes with respect to the almost contact B-metric structure are: a *totally real section* if α is orthogonal to its φ-image $\varphi\alpha$ and ξ, a *φ-holomorphic section* if α coincides with $\varphi\alpha$ and a *ξ-section* if ξ lies on α.

The sectional curvature $k(\alpha; p)(R)$ of α with an arbitrary basis $\{x, y\}$ at p regarding R is defined by

$$k(\alpha; p)(R) = \frac{R(x, y, y, x)}{g(x, x)g(y, y) - g(x, y)^2}. \tag{2.3}$$

It is known from [12] that a linear connection D is called a *natural connection* on an arbitrary manifold $(M, \varphi, \xi, \eta, g)$ if the almost contact structure (φ, ξ, η) and the B-metric g (consequently also $\tilde{g}$) are parallel with respect to D, i.e. $D\varphi = D\xi = D\eta = Dg = D\tilde{g} = 0$. In [18], it is proved that a linear connection D is natural on $(M, \varphi, \xi, \eta, g)$ if and only if $D\varphi = Dg = 0$. A natural connection exists on any almost contact B-metric manifold and coincides with the Levi-Civita connection if and only if the manifold belongs to $\mathcal{F}_0$.

Let T be the torsion tensor of D, i.e. $T(x, y) = D_x y - D_y x - [x, y]$. The corresponding tensor of T of type (0,3) is denoted by the same letter and is defined by the condition $T(x, y, z) = g(T(x, y), z)$.

In [15], it is introduced a natural connection $\dot{D}$ on $(M, \varphi, \xi, \eta, g)$ in all basic classes by

$$\dot{D}_x y = \nabla_x y + \tfrac{1}{2}\left\{(\nabla_x\varphi)\,\varphi y + (\nabla_x\eta)\,y \cdot \xi\right\} - \eta(y)\nabla_x\xi. \tag{2.4}$$

This connection is called a *φB-connection* in [16]. It is studied for the main classes $\mathcal{F}_1, \mathcal{F}_4, \mathcal{F}_5, \mathcal{F}_{11}$ in [15, 10, 11]. Let us note that the φB-connection is the odd-dimensional analogue of the B-connection on the almost complex manifold with Norden metric, studied for the class $\mathcal{W}_1$ in [4].

In [17], a natural connection $\ddot{D}$ is called a *φ-canonical connection* on (M,φ,ξ,η,g) if its torsion tensor $\ddot{T}$ satisfies the following identity:

$$\begin{aligned}&\ddot{T}(x,y,z)-\ddot{T}(x,z,y)-\ddot{T}(x,\varphi y,\varphi z)+\ddot{T}(x,\varphi z,\varphi y)\\ &=\eta(x)\left\{\ddot{T}(\xi,y,z)-\ddot{T}(\xi,z,y)-\ddot{T}(\xi,\varphi y,\varphi z)+\ddot{T}(\xi,\varphi z,\varphi y)\right\}\\ &+\eta(y)\left\{\ddot{T}(x,\xi,z)-\ddot{T}(x,z,\xi)-\eta(x)\ddot{T}(z,\xi,\xi)\right\}\\ &-\eta(z)\left\{\ddot{T}(x,\xi,y)-\ddot{T}(x,y,\xi)-\eta(x)\ddot{T}(y,\xi,\xi)\right\}.\end{aligned}$$

It is established that the φB-connection and the φ-canonical connection coincide if and only if (M,φ,ξ,η,g) is in the class $\mathcal{F}_1\oplus\mathcal{F}_2\oplus\mathcal{F}_4\oplus\mathcal{F}_5\oplus\mathcal{F}_6\oplus\mathcal{F}_8\oplus\mathcal{F}_9\oplus\mathcal{F}_{10}\oplus\mathcal{F}_{11}$.

In [8] it is determined the class of all three-dimensional almost contact B-metric manifolds. It is $\mathcal{F}_1\oplus\mathcal{F}_4\oplus\mathcal{F}_5\oplus\mathcal{F}_8\oplus\mathcal{F}_9\oplus\mathcal{F}_{10}\oplus\mathcal{F}_{11}$.

3. A family of Lie groups as three-dimensional $(\mathcal{F}_4\oplus\mathcal{F}_5)$-manifolds

In this section we study three-dimensional real connected Lie groups with almost contact B-metric structure. On a three-dimensional connected Lie group G we take a global basis of left-invariant vector fields $\{e_0,e_1,e_2\}$ on G.

We define an almost contact structure on G by

$$\begin{array}{l}\varphi e_0=o,\quad \varphi e_1=e_2,\quad \varphi e_2=-e_1,\quad \xi=e_0;\\ \eta(e_0)=1,\quad \eta(e_1)=\eta(e_2)=0,\end{array}\tag{3.1}$$

where o is the zero vector field and define a B-metric on G by

$$\begin{array}{l}g(e_0,e_0)=g(e_1,e_1)=-g(e_2,e_2)=1,\\ g(e_0,e_1)=g(e_0,e_2)=g(e_1,e_2)=0.\end{array}\tag{3.2}$$

We consider the Lie algebra $\mathfrak{g}$ on G, determined by the following non-zero commutators:

$$[e_0,e_1]=-be_1-ae_2,\quad [e_0,e_2]=ae_1-be_2,\quad [e_1,e_2]=0,\tag{3.3}$$

where $a,b\in\mathbb{R}$. We verify immediately that the Jacobi identity for $\mathfrak{g}$ is satisfied. Hence, G is a 2-parametric family of Lie groups with corresponding Lie algebra $\mathfrak{g}$.

Theorem 3.1. *Let (G,φ,ξ,η,g) be a three-dimensional connected Lie group with almost contact B-metric structure determined by* (3.1)*,* (3.2) *and* (3.3)*. Then it belongs to the class $\mathcal{F}_4\oplus\mathcal{F}_5$.*

Proof. The well-known Koszul equality for the Levi-Civita connection ∇ of g

$$2g\left(\nabla_{e_i}e_j,e_k\right)=g\left(\left[e_i,e_j\right],e_k\right)+g\left(\left[e_k,e_i\right],e_j\right)+g\left(\left[e_k,e_j\right],e_i\right)\tag{3.4}$$

implies the following form of the components $F_{ijk}=F(e_i,e_j,e_k)$ of F:

$$\begin{aligned}2F_{ijk}=g\left(\left[e_i,\varphi e_j\right]-\varphi\left[e_i,e_j\right],e_k\right)+g\left(\varphi\left[e_k,e_i\right]-\left[\varphi e_k,e_i\right],e_j\right)\\ +g\left(\left[e_k,\varphi e_j\right]-\left[\varphi e_k,e_j\right],e_i\right).\end{aligned}\tag{3.5}$$

Using (3.5) and (3.3) for the non-zero components F_{ijk}, we get:

$$\begin{aligned} F_{101} = F_{110} = -F_{202} = -F_{220} = a, \\ F_{102} = F_{120} = F_{201} = F_{210} = b. \end{aligned} \tag{3.6}$$

Immediately we establish that the components in (3.6) satisfy the condition $F = F^4 + F^5$ which means that the manifold belongs to $\mathcal{F}_4 \oplus \mathcal{F}_5$. Here, the components F^s of F in the basic classes $\mathcal{F}_s$ $(s = 4, 5)$ have the following form (see [8])

$$\begin{aligned} F_4(x,y,z) &= \tfrac{1}{2}\theta_0\Big\{x^1\left(y^0z^1 + y^1z^0\right) - x^2\left(y^0z^2 + y^2z^0\right)\Big\}, \\ &\tfrac{1}{2}\theta_0 = F_{101} = F_{110} = -F_{202} = -F_{220}; \\ F_5(x,y,z) &= \tfrac{1}{2}\theta^*_0\big\{x^1\left(y^0z^2 + y^2z^0\right) + x^2\left(y^0z^1 + y^1z^0\right)\big\}, \\ &\tfrac{1}{2}\theta^*_0 = F_{102} = F_{120} = F_{201} = F_{210}. \end{aligned} \tag{3.7}$$

where $\theta_0 = \theta(e_0)$ and $\theta^*_0 = \theta^*(e_0)$ are determined by $\theta_0 = 2a$, $\theta^*_0 = 2b$. Therefore, the induced three-dimensional manifold $(G, \varphi, \xi, \eta, g)$ belongs to the class $\mathcal{F}_4 \oplus \mathcal{F}_5$ from the mentioned classification. It is an $\mathcal{F}_0$-manifold if and only if $(a, b) = (0, 0)$ holds.

Obviously, $(G, \varphi, \xi, \eta, g)$ belongs to $\mathcal{F}_4$, $\mathcal{F}_5$ and $\mathcal{F}_0$ if and only if the parameters θ^*_0 vanishes if the manifold belongs to $\mathcal{F}_4$, and θ_0 vanishes if it belong to $\mathcal{F}_5$, and $\theta_0 = \theta^*_0$ vanishes if it belong to $\mathcal{F}_0$, respectively.

According to the above, the commutators in (3.3) take the form

$$\begin{gathered} [e_0, e_1] = -\tfrac{1}{2}(\theta^*_0 e_1 + \theta_0 e_2), \quad [e_0, e_2] = \tfrac{1}{2}(\theta_0 e_1 - \theta^*_0 e_2), \\ [e_1, e_2] = 0, \end{gathered} \tag{3.8}$$

in terms of the basic components of the Lee forms θ and θ^*. □

According to Theorem 3.1 and the consideration in [9], we can remark that the Lie algebra determined as above belongs to the type $Bia(VII_h)$, $h > 0$ of the Bianchi classification (see [1, 2]).

Using (3.4) and (3.3), we obtain the components of ∇:

$$\begin{aligned} \nabla_{e_1}e_0 = be_1 + ae_2, \quad \nabla_{e_1}e_1 = -be_0, \quad \nabla_{e_1}e_2 = ae_0, \\ \nabla_{e_2}e_0 = -ae_1 + be_2, \quad \nabla_{e_2}e_1 = ae_0, \quad \nabla_{e_2}e_2 = be_0. \end{aligned} \tag{3.9}$$

We denote by $R_{ijkl} = R(e_i, e_j, e_k, e_l)$ the components of the curvature tensor R, $\rho_{jk} = \rho(e_j, e_k)$ of the Ricci tensor ρ, $\rho^*_{jk} = \rho^*(e_j, e_k)$ of the associated Ricci tensor ρ^* and $k_{ij} = k(e_i, e_j)$ of the sectional curvature for ∇ of the basic 2-plane α_{ij} with a basis $\{e_i, e_j\}$, where $i, j \in \{0, 1, 2\}$. On the considered manifold $(G, \varphi, \xi, \eta, g)$ the basic 2-planes α_{ij} of special type are: a φ-holomorphic section — α_{12} and ξ-sections — α_{01}, α_{02}. Further, by (2.3), (3.2), (3.3) and (3.9), we compute

$$\begin{gathered} -R_{0101} = R_{0202} = \tfrac{1}{2}\rho_{00} = k_{01} = k_{02} = \tfrac{1}{4}(\theta_0^2 - \theta_0^{*2}), \\ R_{0102} = R_{0201} = -\rho_{12} = -\tfrac{1}{2}\rho^*_{00} = -\tfrac{1}{2}\tau^* = -\tfrac{1}{2}\theta_0\theta^*_0, \\ R_{1212} = \rho^*_{12} = k_{12} = -\tfrac{1}{4}(\theta_0^2 + \theta_0^{*2}), \quad \rho_{11} = -\rho_{22} = -\tfrac{1}{2}\theta_0^{*2}, \\ \tau = \tfrac{1}{2}(\theta_0^2 - 3\theta_0^{*2}). \end{gathered} \tag{3.10}$$

The rest of the non-zero components of R, ρ and ρ^* are determined by (3.10) and the properties $R_{ijkl} = R_{klij}$, $R_{ijkl} = -R_{jikl} = -R_{ijlk}$, $\rho_{jk} = \rho_{kj}$ and $\rho^*_{jk} = \rho^*_{kj}$.

Taking into account (2.1), (2.2), (3.1), (3.2) and (3.9), we have

$$\|\nabla\varphi\|^2 = -2\,\|\nabla\eta\|^2 = -2\,\|\nabla\xi\|^2 = \theta_0^2 - \theta_0^{*2}. \tag{3.11}$$

Proposition 3.2. *The following characteristics are valid for $(G, \varphi, \xi, \eta, g)$:*

1. *The φB-connection $\dot{D}$ (respectively, φ-canonical connection $\ddot{D}$) is zero in the basis $\{e_0, e_1, e_2\}$.*
2. *The manifold is an isotropic-$\mathcal{F}_0$-manifold if and only if the condition $\theta_0 = \pm\theta_0^*$ is valid.*
3. *The manifold is flat if and only if it belongs to $\mathcal{F}_0$.*
4. *The manifold is Ricci-flat (respectively, $*$-Ricci-flat) if and only if it is flat.*
5. *The manifold is scalar flat if and only if the condition $\theta_0 = \pm\sqrt{3}\,\theta_0^*$ holds.*
6. *The manifold is $*$-scalar flat if and only if it belongs to either $\mathcal{F}_4$ or $\mathcal{F}_5$.*

Proof. Using (2.4), (3.1) and (3.9), we get immediately the assertion (1). Equation (3.11) implies the assertion (2). The assertions (5), (3) and (6) hold, according to (3.10). On the three-dimensional almost contact B-metric manifold with the basis $\{e_0, e_1, e_2\}$, bearing in mind the definitions of the Ricci tensor ρ and the ρ^*, we have

$$\rho_{jk} = R_{0jk0} + R_{1jk1} - R_{2jk2} \qquad \rho^*_{jk} = R_{1kj2} + R_{2jk1}.$$

By virtue of the latter equalities, we get the assertion (4). □

According to (3.6) and (3.10) we establish the truthfulness of the following

Proposition 3.3. *The following properties are equivalent for the studied manifold $(G, \varphi, \xi, \eta, g)$:*

1. *it belongs to $\mathcal{F}_4$;*
2. *it is η-Einstein;*
3. *the Lee form θ^* vanishes.*

Using again (3.6) and (3.10)we establish the truthfulness of the following

Proposition 3.4. *The following properties are equivalent for the studied manifold $(G, \varphi, \xi, \eta, g)$:*

1. *it belongs to $\mathcal{F}_5$;*
2. *it is Einstein;*
3. *it is a hyperbolic space form with $k = -\frac{1}{4}\theta_0^{*2}$;*
4. *the Lee form θ vanishes.*

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Book reviews

Vijay Gupta, Michael Th. Rassias,
Moments of Linear Positive Operators and Approximation,
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In recent years the study of the linear methods of approximation became a strongly ingrained part of Approximation Theory. In the investigation of the linear positive operators the determination of their moments is extremely useful both in obtaining the convergence of the respective sequences in various function spaces and in establishing their asymptotic behavior.

The monograph is split into 3 chapters, each representing a specific direction aimed at studying the moments of some classes of operators. It offers coverage of classical and recent material on linear positive operators. In the first chapter the moments of 15 discrete type operators are established, among which Bernstein, Szász-Mirakjan, Baskakov, Stancu, Jain, Balázs-Szabados, Abel-Ivan, Chlodowsky operators. Further, integral operators are analyzed, such as Gamma, Post-Widder, Ismail-May, Phillips, Lupaş, Durrmeyer type operators. Also, the reader is acquainted with various mixed summation-integral operators. In the last chapter the authors approach approximation properties of certain operators, these including evaluations of the rate of convergence by using moduli of smoothness, preservation of some test functions through certain families of operators and the study of the difference between two approximation processes.

The presentation is distinguished by clarity and rigorous proofs. Also, it is essentially self-contained. The results are based on numerous published papers, the bibliography including over fifty works of the authors. The material offers information that put the reader at the forefront of current research and determines fruitful directions for future advanced study. It is addressed to researchers and graduate students specialized in pure and applied mathematics who are interested in Korovkin-type theory.

Octavian Agratini