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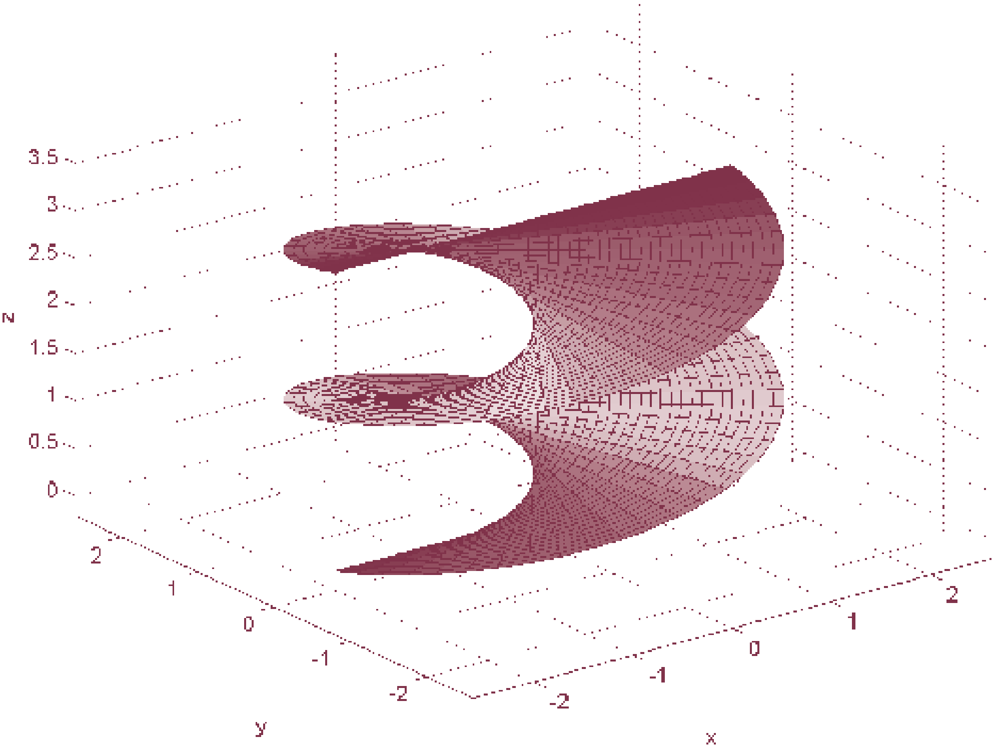

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Strong inequalities for the iterated Boolean sums of Bernstein operators

Li Cheng and Xinlong Zhou

Dedicated to Professor Heiner Gonska on the occasion of his 70th anniversary.

Abstract. In this paper we investigate the approximation properties for the iterated Boolean sums of Bernstein operators. The approximation behaviour of those operators is presented by the so-called strong inequalities. Moreover, such strong inequalities are valid for any individual continuous function on $[0,1]$. The obtained estimate covers global direct, inverse and saturation results.

Mathematics Subject Classification (2010): 41A05, 41A25, 41A40.

Keywords: Approximation rate, Bernstein operator, Boolean sum, strong inequality.

1. Introduction

For $f \in C[0,1]$ the classical Bernstein operators is given by

$$B_n(f,x) := \sum_{k=0}^{n} f\left(\frac{k}{n}\right)\binom{n}{k} x^k (1-x)^{n-k}.$$

Clearly, $B_n(f,\cdot)$ is of degree at most n.

There are many papers dealt with the global approximation degree of Bernstein operators. The final estimate is obtained in [7]. Denote $||\cdot||$ the maximal norm on $[0,1]$. There exists a constant $C>0$ such that for all $f \in C[0,1]$ and all $n = 1,2,\dots$ the following strong inequalities are true:

$$C^{-1}\omega_\varphi^2\left(f,\frac{1}{\sqrt{n}}\right) \le ||f - B_n(f)|| \le C\omega_\varphi^2\left(f,\frac{1}{\sqrt{n}}\right), \tag{1.1}$$

where

$$\varphi(x) = \sqrt{x(1-x)}$$

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and $\omega_\varphi^2(f,\cdot)$ is the second-order modulus of continuity of the function $f \in C[0,1]$ given by

$$\omega_\varphi^\ell(f,t) := \sup_{0\le\eta\le t} ||\Delta_{\eta\varphi}^\ell f||,\ \ell = 1,2,\dots.$$

It is well-known (see e.g. [2]) that this modulus is equivalent to the K-functional $K_\varphi^\ell(f,\cdot)$:

$$K_\varphi^\ell(f,t) := \inf_{g\in C^\ell[0,1]}\{||f-g|| + t^\ell||\varphi^l g^{(\ell)}||\}.$$

Thus, the approximation behaviour of Bernstein operators can be completely characterised by (1.1). In particular the maximal approximation degree can only be $\mathcal{O}(1/n)$, i.e. the Bernstein operator is saturated with saturation degree $1/n$. There are many methods to increase the approximation degree of this operator. One of them is the so-called Boolean sum. Let P, Q be operators, $P, Q : X \longrightarrow X$ for some linear space X. Then the Boolean sum of P and Q is defined to be

$$P \oplus Q := P + Q - PQ.$$

For Bernstein operator B_n we will be concerned with iterated Boolean sums of the form $B_n \oplus B_n \oplus \cdots \oplus B_n$, and will denote such an ℓ-fold Boolean sum of the Bernstein operator by $\oplus^\ell B_n$. The easiest way to see that $\oplus^\ell B_n$ is indeed an approximation operator is to look at the error operator representation: with the identity operator I one has

$$I - \oplus^\ell B_n = (I - B_n)^\ell,$$

that can be easily verified by induction. From the last equality we obtain

$$\oplus^\ell B_n = I - (I - B_n)^\ell.$$

The right hand side of this equality represents really a linear combination of a fixed Bernstein operator. Such combination were investigated in the past. The earliest reference in regard to such an approach which we were able to located is [11] (see also [10]).

From the numerical point of view, this combination appears to be of interest, since in the case of discretely defined operators, it uses only the data required by the original operators, in the case of B_n this is just the set of numbers

$$\left\{f(0), f\left(\frac{1}{n}\right), \dots, f\left(\frac{n-1}{n}\right), f(1)\right\}.$$

The operator $\oplus^\ell B_n$ was introduced independently in [1, 4, 8, 9] and investigated, e.g. in [3, 5] .

In 1994 Gonska and the second author of this paper (see [6]) obtain the following result for $\oplus^\ell B_n$:

Theorem 1.1. *Let $\ell \ge 1$ be fixed. Then there is constant $C > 0$ such that for any $f \in C[0,1]$ and all $n = 1,2,\dots$*

$$||f - \oplus^\ell B_n(f)|| \le C\left\{\omega_\varphi^{2\ell}\left(f, \frac{1}{\sqrt{n}}\right) + ||f||n^{-\ell}\right\}. \tag{1.2}$$

Furthermore, there holds the Steckin-type inequality

$$\omega_\varphi^{2\ell}\left(f, \frac{1}{\sqrt{n}}\right) \le \frac{C}{n^{\ell+1/2}} \sum_{k=1}^{n} k^{\ell-1/2} ||f - \oplus^\ell B_k(f)||. \tag{1.3}$$

The o-saturation class is described as follows:

$$||f - \oplus^\ell B_n(f)|| = o\left(\frac{1}{n^\ell}\right) \iff f \textit{ is a linear function.}$$

It follows immediately from (1.2) and (1.3) that for all $0 < \alpha \le 2l$

$$||f - \oplus^\ell B_n(f)|| = \mathcal{O}(n^{-\alpha/2}) \iff \omega_\varphi^{2\ell}(f, t) = \mathcal{O}(t^\alpha).$$

Thus, Theorem 1.1 covers global direct, inverse and saturation results for the Boolean sum of Bernstein operator B_n. In this paper we will show that like (1.1) we have also the strong inequalities for $\oplus^\ell B_n$ in some weak form. To this end, denote $E_n(f)$ to be the best approximation constant of f via algebraic polynomials p_n of degree n, i.e.

$$E_n(f) := \min_{p_n} ||f - p_n||.$$

We have

Theorem 1.2. *Let $\ell \ge 1$ be fixed. Then there are constants $C > 0$ and $A \ge 1$ such that for any $f \in C[0,1]$ and all $n = 1, 2, \ldots$*

$$\begin{aligned} C^{-1}\left\{\omega_\varphi^{2\ell}\left(f, \frac{1}{\sqrt{n}}\right) + E_1(f)n^{-\ell}\right\} &\le \max_{n \le k \le An}(||f - \oplus^\ell B_k(f)|| + E_1(f)k^{-\ell}) \\ &\le \max_{k \ge n}(||f - \oplus^\ell B_k(f)|| + E_1(f)k^{-\ell}) \\ &\le C\left\{\omega_\varphi^{2\ell}\left(f, \frac{1}{\sqrt{n}}\right) + E_1(f)n^{-\ell}\right\}. \end{aligned} \tag{1.4}$$

Moreover, if f is not an algebraic polynomial of degree less than 2ℓ, then for some constants $D, A > 0$ and all $n = 1, 2, \ldots$ there holds

$$\begin{aligned} D^{-1}\omega_\varphi^{2\ell}\left(f, \frac{1}{\sqrt{n}}\right) &\le \max_{n \le k \le An} ||f - \oplus^\ell B_k(f)|| \\ &\le \max_{k \ge n} ||f - \oplus^\ell B_k(f)|| \le D\omega_\varphi^{2\ell}\left(f, \frac{1}{\sqrt{n}}\right). \end{aligned} \tag{1.5}$$

We prove this result in the next section.

2. Proof of Theorem 1.2

Proof of Theorem 1.2. First we note that $\oplus^l B_n$ is invariant for linear functions. Hence we conclude from (1.2)

$$||f - \oplus^\ell B_n(f)|| \le C\left\{\omega_\varphi^{2\ell}\left(f, \frac{1}{\sqrt{n}}\right) + E_1(f)n^{-\ell}\right\}. \tag{2.1}$$

Let $0 < \delta_1 < \delta_2 < 1/2$. We obtain from (1.3) for $i = 1, 2$

$$\begin{aligned}\omega_\varphi^{2\ell}\left(f, \frac{1}{\sqrt{n}}\right) + E_1(f)n^{-\ell} &\le \frac{C}{n^{\ell+1/2}}\sum_{k=1}^{n} k^{\ell-1/2}(\|f - \oplus^\ell B_k(f)\| + E_1(f)k^{-\ell}) \\ &\le Cn^{-\delta_i-\ell}\max_{1\le k\le n} k^{\ell+\delta_i}(\|f - \oplus^\ell B_k(f)\| + E_1(f)k^{-\ell}).\end{aligned}$$

Noticing $\omega_\varphi^{2\ell}(f,t_1)/t_1^{2\ell} \le C\omega_\varphi^{2\ell}(f,t_2)/t_2^{2\ell}$ for $0 \le t_2 \le t_1$, we conclude from (2.1) for $1 \le k \le n$

$$\|f - \oplus^\ell B_k(f)\| + E_1(f)k^{-\ell} \le C\frac{n^\ell}{k^\ell}\left(\omega_\varphi^{2\ell}\left(f, \frac{1}{\sqrt{n}}\right) + n^{-\ell}E_1(f)\right).$$

It follows from the last two estimates that for $i = 1, 2$

$$\begin{aligned}\omega_\varphi^{2\ell}\left(f, \frac{1}{\sqrt{n}}\right) + E_1(f)n^{-\ell} &\le Cn^{-\delta_i-\ell}\max_{1\le k\le n} k^{\ell+\delta_i}(\|f - \oplus^\ell B_k(f)\| + E_1(f)k^{-\ell}) \\ &\le C_1\left(\omega_\varphi^{2\ell}\left(f, \frac{1}{\sqrt{n}}\right) + E_1(f)n^{-\ell}\right).\end{aligned}$$

Consequently, for some constant $C > 0$

$$\frac{1}{n^{\delta_1+\ell}}\max_{1\le k\le n} k^{\ell+\delta_1}(\|f - \oplus^\ell B_k(f)\| + E_1(f)k^{-\ell})$$
$$\le \frac{C}{n^{\delta_2+\ell}}\max_{1\le k\le n} k^{\ell+\delta_2}(\|f - \oplus^\ell B_k(f)\| + E_1(f)k^{-\ell}).$$

Let the maximum on the right hand side be reached at n_0. So we have

$$n^{-\delta_1-\ell}n_0^{\ell+\delta_1}(\|f - \oplus^\ell B_{n_0}(f)\| + E_1(f)n_0^{-\ell})$$
$$\le Cn^{-\delta_2-\ell}n_0^{\ell+\delta_2}(\|f - \oplus^\ell B_{n_0}(f)\| + E_1(f)n_0^{-\ell}).$$

In other words, for some $c' > 0$ there holds $c'n \le n_0$. Therefore,

$$\begin{aligned}\omega_\varphi^{2\ell}\left(f, \frac{1}{\sqrt{n}}\right) + E_1(f)n^{-\ell} &\le Cn^{-\delta_2-\ell}n_0^{\ell+\delta_2}(\|f - \oplus^\ell B_{n_0}(f)\| + E_1(f)n_0^{-\ell}) \\ &\le C\max_{c'n\le k\le n}(\|f - \oplus^\ell B_k(f)\| + E_1(f)k^{-\ell}) \\ &\le C\max_{k\ge c'n}(\|f - \oplus^\ell B_k(f)\| + E_1(f)k^{-\ell}).\end{aligned}$$

Or for some constant $A > 0$

$$\begin{aligned}\omega_\varphi^{2\ell}\left(f, \frac{1}{\sqrt{n}}\right) + E_1(f)n^{-\ell} &\le C\max_{n\le k\le An}(\|f - \oplus^\ell B_k(f)\| + E_1(f)k^{-\ell}) \\ &\le C\max_{k\ge n}(\|f - \oplus^\ell B_k(f)\| + E_1(f)k^{-\ell}).\end{aligned}$$

Clearly, by (2.1)

$$\max_{k\ge n}(\|f - \oplus^\ell B_k(f)\| + E_1(f)k^{-\ell}) \le C\left(\omega_\varphi^{2\ell}\left(f, \frac{1}{\sqrt{n}}\right) + E_1(f)n^{-\ell}\right).$$

The first assertion (1.4) is proved.

It remains to show the assertion (1.5). To this end we note that f is not an algebraic polynomial of degree less than 2ℓ. Hence, $\omega_\varphi^{2\ell}(f,1) \neq 0$. On the other hand, as we mention at the beginning of this paper $\omega_\varphi^{2\ell}(f,\cdot)$ is equivalent to the $K-$functional $K_\varphi^{2\ell}(f,\cdot)$. Thus, for $0 \le t \le 1$ we have with some constant $C > 0$ the inequality

$$\omega_\varphi^{2\ell}(f,1)/1^{2\ell} \le C\omega_\varphi^{2\ell}(f,t)/t^{2\ell}.$$

But $E_1(f) \le C\omega_\varphi^{2\ell}(f,1)$. Therefore,

$$E_1(f)\frac{1}{n^\ell} \le C\omega_\varphi^{2\ell}\left(f, \frac{1}{\sqrt{n}}\right).$$

Thus, (2.1) can be written as

$$||f - \oplus^\ell B_n(f)|| \le C\omega_\varphi^{2\ell}\left(f, \frac{1}{\sqrt{n}}\right).$$

Combining this estimate with (1.3) and using the same approach as above we obtain (1.5). □

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On two modified Phillips operators

Gancho Tachev

Dedicated to Professor Heiner Gonska on the occasion of his 70th anniversary.

Abstract. In this note we introduce two new modified Phillips operators G_n^1 and G_n^2. We obtain direct estimates for approximation of bounded continuous functions, defined on $[0,\infty)$ by G_n^1, as well as for approximation of unbounded continuous functions by G_n^2. We improve some previous results on this topic.

Mathematics Subject Classification (2010): 41A25, 41A36.

Keywords: Phillips operators, exponential functions, quantitative results.

1. Introduction

The Phillips operator [20] is defined as

$$S_n(f;x) = n\sum_{k=1}^{\infty} e^{-nx}\frac{(nx)^k}{k!}\int_0^{\infty} e^{-nt}\frac{(nt)^{k-1}}{(k-1)!}f(t)\,dt + e^{-nx}f(0).$$

These operators preserve constant as well linear functions. Some approximation results on these operators- commutativity, direct and strong converse inequalities, inverse estimates, linear combinations etc. have been discussed in [8, 21, 22, 11, 14, 23, 24, 12, 15, 13] . Usually we denote the basis functions by

$$s_{n,k}(x) = e^{-nx}\frac{(nx)^k}{k!}. \tag{1.1}$$

In the last decade a lot of papers appeared devoted to such modifications of classical positive linear operators, which preserve certain exponential type functions. For the new modified Szász–Mirakyan operators we refer the reader to recent papers [1, 2, 3, 5, 4].In the same way , Gupta and Tachev [12] considered Phillips type operators fixing e^{-t} and e^{At}, $A\in\mathbb{R}$, but not both together. Very recently Gupta and Lopez-Moreno in [13] defined a modification of the Phillips so as to fix both e^{at} and e^{bt} for any two real numbers, different or not , $a,b\in\mathbb{R}$ and studied their approximation behaviour. In most of the results, mentioned above - the modification consists of modelling the basis functions (for example $s_{n,k}(x)$ in case of Szász-Mirakyan operator) such that certain exponential functions are reproduced. The second approach is to modify the argument of the function, to be approximated. The start of this method was given by the work of King [17] on the classical Bernstein operator and further developed for the Szász-Mirakyan operator in [7]. To combine the both methods Aral, Inoan and Raşa generalized Szász-Mirakyan operator in [3] introducing the function $\rho(x)$ in both places - in basis functions, as well as in the argument of the approximated function. Our method is different from all , mentioned above and we modify simultaneously the basis functions $s_{n,k}(x)$ and multiply the approximated function f by $e^{At}, A=-1,1$. The case of arbitrary $A \in \mathbb{R}$ is similar and we omit the details. In [12] the following modification of Phillips operator was introduced

$$P_n\left(f;x\right)=n\sum_{k=1}^{\infty}e^{-n\alpha_n(x)}\frac{\left(n\alpha_n\left(x\right)\right)^k}{k!}\int_0^{\infty}e^{-nt}\frac{\left(nt\right)^{k-1}}{\left(k-1\right)!}f\left(t\right)dt+e^{-n\alpha_n(x)}f(0). \tag{1.2}$$

By Lemma 1 in [12] it follows for $A \in \mathbb{R}$

$$P_n(e^{At};x)=e^{\frac{An\alpha_n(x)}{n-A}}. \tag{1.3}$$

It was shown in [12] that if we choose

$$\alpha_n(x)=\frac{x(n+1)}{n},\ x\in[0,\infty), \tag{1.4}$$

then

$$P_n(e^{-t};x)=e^{-x}.$$

If we choose

$$\alpha_n(x)=x(1-\frac{A}{n}),\ A>0 \tag{1.5}$$

then from [12] we have

$$P_n(e^t;x)=e^x. \tag{1.6}$$

Further we restrict ourselves in (1.5) for $A=1$. Now we define our two modified Phillips operators as follows:

$$G_n^1\left(f;x\right)=e^{-x}\left(n\sum_{k=1}^{\infty}s_{n,k}(\alpha_n(x))\int_0^{\infty}s_{n,k-1}(t)f(t)e^t\,dt+e^{-n\alpha_n(x)}f(0)\right), \tag{1.7}$$

where $\alpha_n(x)=x(1-\frac{1}{n})$. The second modification is given by

$$G_n^2\left(f;x\right)=e^{x}\left(n\sum_{k=1}^{\infty}s_{n,k}(\alpha_n(x))\int_0^{\infty}s_{n,k-1}(t)f(t)e^{-t}\,dt+e^{-n\alpha_n(x)}f(0)\right), \tag{1.8}$$

where

$$\alpha_n(x)=x\left(1+\frac{1}{n}\right).$$

Further we adopt that 2 in definition of G_n^2 serves as an index and not as a power factor. It is clear that

$$G_n^1(f;x)=e^{-x}P_n\left(f(t)e^t,x\right), \tag{1.9}$$

with $\alpha_n(x)$ from (1.7) and

$$G_n^2(f;x) = e^x P_n\left(f(t)e^{-t}, x\right), \tag{1.10}$$

with $\alpha_n(x)$ from (1.8). We apply the operator G_n^1 to approximate bounded continuous function $f \in C^\star[0,\infty)$-the subspace of real-valued continuous functions , which possess finite limit at infinity, endowed with the uniform norm. The operator G_n^2 will be used to approximate unbounded continuous $f \in C[0,\infty)$. In Section 2 we study the approximation order of G_n^1 and compare our direct estimate with previous known results on this topic. In Section 3 using some ideas from [2] we discuss an uniform error estimate for the operator G_n^2 measured in weighted norm.

2. Estimate for G_n^1.

In Boyanov, Veselinov -[6] the uniform convergence of linear positive operators was established. Later Holhoş in [16] established the following quantitative estimate for a sequence of linear positive operators:

Theorem A. [16] If a sequence of linear positive operators $L_n : C^*[0,\infty) \to C^*[0,\infty)$ satisfy the equalities

$$\begin{aligned} &||L_n e_0 - 1||_{[0,\infty)} = \alpha_n \\ &||L_n(e^{-t}) - e^{-x}||_{[0,\infty)} = \beta_n \\ &||L_n(e^{-2t}) - e^{-2x}||_{[0,\infty)} = \gamma_n \end{aligned}$$

then

$$||L_n f - f||_{[0,\infty)} \leq \|f\|_\infty \cdot \alpha_n + (2+\alpha_n)\cdot \omega^*(f, \sqrt{\alpha_n + 2\beta_n + \gamma_n}), f \in C^*[0,\infty).$$

The modulus of continuity used in the above theorem is defined as:

$$\omega^*(f,\delta) := \sup_{\substack{|e^{-x}-e^{-t}|\leq \delta \\ x,t\geq 0}} |f(t) - f(x)|.$$

Our first result states the following

Theorem 2.1. *For $f \in C^*[0,\infty)$ we have*

$$\|G_n^1 f - f\|_{C[0,\infty)} \leq 2\omega^*\left(f, \sqrt{\gamma_n}\right), \tag{2.1}$$

where

$$\gamma_n = \|G_n^1(e^{-2t};x) - e^{-2x}\|_{C[0,\infty)} = \frac{1}{n+1}\left(\frac{n+1}{n}\right)^{-n} < \frac{1}{2(n+1)}. \tag{2.2}$$

Proof. From definition of the operator G_n^1-(1.9) and (1.4) it follows that

$$G_n^1(1;x) = 1,\ G_n^1(e^{-t};x) = e^{-x},$$

i.e. $\alpha_n = \beta_n = 0$. Simple calculations and (1.3) imply (2.2). Now the proof follows immediately from Theorem A. □

Our further uniform estimate is based on the suitable transformation which reduces the uniform approximation problem on $C^*[0,\infty)$ to that one on $C[0,1]$. This observation was developed by Gonska in [10] and by Paltanea in [18] and quantitative results were obtained in [19]. In [1] it was shown that the spaces $\big(C^*[0,\infty),\, \|\cdot\|_{C[0,\infty)}\big)$ and $\big(C[0,1],\, \|\cdot\|_{C[0,1]}\big)$ are isometrically isomorphic. If we define

$$\psi(t) = e^{-t},\ t \in [0,\infty)$$

and $S^* : C^*[0,\infty) \to C^*[0,\infty)$ is a positive linear operator, reproducing the constant functions , then the following statement was proved in [1] (see Theorem 9 there):

Theorem B. [1] If $S^* : C^*[0,\infty) \to C^*[0,\infty)$, then for all $f^* \in C^*[0,\infty)$ and $0 < h \leq \frac{1}{2}$, the following inequality holds

$$\|S^*f^* - f^*\|_{C[0,\infty)} \leq \frac{1}{h}\|S^*(\psi) - \psi\|_{C[0,\infty)}\omega_1(f,h)$$
$$+\left[1 + \frac{1}{2h^2}\left(\|S^*(\psi^2) - \psi^2\|_{C[0,\infty)} + 2\|S^*(\psi) - \psi\|_{C[0,\infty)}\right)\right]\omega_2(f,h). \tag{2.3}$$

Here $f = f^* \circ \psi^{-1}$, i.e. $f \in C[0,1]$ and ω_1, ω_2 are the usual first and second order moduli of continuity. Our second result states the following:

Theorem 2.2. *For $f^* \in C^*[0,\infty)$ we have*

$$\|G_n^1 f^* - f^*\|_{C[0,\infty)} \leq \frac{5}{4}\omega_2\left(f, \frac{1}{\sqrt{n+1}}\right), \tag{2.4}$$

where $f = f^* \circ \psi^{-1}$.

Proof. The proof follows directly from Theorem B, (2.2) and the fact that G_n^1 reproduces constant and $e^{-t} = \psi(t)$. □

3. Estimates for G_n^2

In this section we study the approximation of unbounded functions, satisfying certain exponential growth by the operator G_n^2 with

$$\alpha_n(x) = x\left(1 + \frac{1}{n}\right).$$

Set $\varphi(x) = 1 + e^x$, $x \in \mathbb{R}^+$ and consider the following weighted spaces

$$B_\varphi(\mathbb{R}^+) = \left\{f : \mathbb{R}^+ \to \mathbb{R} : |f(x)| \leq M_f \cdot \varphi(x),\, x \geq 0\right\},$$
$$C_\varphi(\mathbb{R}^+) = C(\mathbb{R}^+) \cap B_\varphi(\mathbb{R}^+),$$
$$C_\varphi^k(\mathbb{R}^+) = \left\{f \in C_\varphi(\mathbb{R}^+) : \lim_{x\to\infty} \frac{f(x)}{\varphi(x)} = k_f \text{ exists and it is finite,}\right\}$$

where M_f, k_f are constants depending on f. All three spaces are normed with the norm

$$\|f\|_\varphi = \sup\ \frac{|f(x)|}{\varphi(x)}.$$

Obviously we have

$$G_n^2(1;x) = 1, \text{ and } G_n^2(e^t;x) = e^x. \tag{3.1}$$

Further from (1.10) we obtain

$$\begin{aligned} G_n^2(f;x) &= e^x P_n\left(\frac{f(t)}{1+e^t}\cdot(1+e^t)e^{-t};x\right) \\ &\le e^x\|f\|_\varphi P_n\left(e^{-t}+1;\alpha_n(x)\right) \\ &= e^x\|f\|_\varphi\left(e^{-x}+1\right) \\ &= \|f\|_\varphi\left(1+e^x\right). \end{aligned}$$

Consequently we have

$$\|G_n^2 f\|_\varphi \le \|f\|_\varphi$$

and we conclude that G_n^2 maps $C_\varphi(\mathbb{R}^+)$ to $C_\varphi(\mathbb{R}^+)$. Following the general result obtained by Gadziev -[9] if we choose as a weight function $\varphi(x) = 1 + e^{2ax},\ a = 1$, instead of $\varphi(x) = 1 + e^x$, to conclude that for each function $f \in C_\varphi^k(\mathbb{R}^+)$

$$\lim_{n\to\infty} \|G_n^2 f - f\|_\varphi = 0$$

it is enough to verify the three conditions

$$\lim_{n\to\infty} \|G_n^2 e^{it} - e^{ix}\|_\varphi = 0,\ i = 0,1,2.$$

For $i = 0,1$ this follows from (3.1). Unfortunately the condition for $i = 2$ is not satisfied. Indeed we have

$$\begin{aligned} G_n^2\left(e^{2t};x\right) &= e^x P_n\left(e^{2t}e^{-t};x\right) = e^x P_n\left(e^t;\alpha_n(x)\right) \\ &= e^x \cdot e^{\frac{n}{n-1}x\frac{(n+1)}{n}} = e^{x\frac{2n}{n-1}}. \end{aligned}$$

Therefore

$$\frac{G_n^2\left(e^{2t};x\right) - e^{2x}}{1+e^{2x}} = \frac{e^{2x\frac{n}{n-1}} - e^{2x}}{1+e^{2x}} \to \infty,$$

when $x \to \infty$ for $n \ge 2$. This is the reason to choose $\varphi(x) = 1 + e^x$ instead of $\varphi(x) = 1 + e^{2x}$. Now , according to the Korovkin- type theorem, established in [9] we need to verify

$$\lim_{n\to\infty} \|G_n^2\left(e^{\frac{t}{2}};x\right) - e^{\frac{x}{2}}\|_\varphi = 0, \tag{3.2}$$

with

$$\alpha_n(x) = \frac{x(n+1)}{n}$$

and using (1.3) we calculate

$$\begin{aligned} G_n^2\left(e^{\frac{t}{2}};x\right) &= e^x P_n\left(e^{-\frac{t}{2}};\alpha_n(x)\right) = e^x \cdot e^{-\frac{1}{2}\frac{n}{(n+\frac{1}{2})}\frac{n+1}{n}x} \\ &= e^{x\cdot\frac{n}{2n+1}}. \end{aligned}$$

Therefore

$$\frac{G_n^2\left(e^{\frac{t}{2}};x\right) - e^{\frac{x}{2}}}{1+e^x} = \frac{e^{x\cdot\frac{n}{2n+1}} - e^{\frac{x}{2}}}{1+e^x} = \frac{y^{\frac{2n}{2n+1}} - y}{1+y^2} := g(y), \tag{3.3}$$

where we set $e^{\frac{x}{2}} = y \in [1,\infty)$.

Hence

$$g(y)=\frac{y^2}{(1+y^2)}\left[y^{-\frac{2n+2}{2n+1}}-y^{-1}\right].$$

Consequently

$$\begin{aligned}\|g\|_{C[1,\infty)} &\leq \|\frac{y^2}{1+y^2}\|_{C[1,\infty)}\|y^{-\frac{2n+2}{2n+1}}-y^{-1}\|_{C[1,\infty)}\\ &\leq \|y^{-\frac{2n+2}{2n+1}}-y^{-1}\|_{C[1,\infty)}. \end{aligned}\tag{3.4}$$

Simple computations imply

$$\begin{aligned}\|y^{-\frac{2n+2}{2n+1}}-y^{-1}\|_{C[1,\infty)} &= \frac{1}{(2n+1)}\left(1-\frac{1}{2n+2}\right)^{2n+2}\\ &< \frac{1}{2(2n+1)},\end{aligned}\tag{3.5}$$

where we used

$$\lim_{n\to\infty}\left(1-\frac{1}{2n+2}\right)^{2n+2}=e^{-1}<\frac{1}{2}.$$

It is clear that (3.2) follows from (3.3), (3.4), (3.5). Our next statement is:

Theorem 3.1. *For each function $f\in C^k_\varphi(\mathbb{R}^+)$, $\varphi(x)=1+e^x$ we have*

$$\lim_{n\to\infty}\|G^2_nf-f\|_\varphi=0.$$

Proof. The proof is straightforward corollary from (3.1) and (3.2). □

Remark 3.2. If in the definition of G^2_n from (1.10) instead of

$$\alpha_n(x)=x\left(1+\frac{1}{n}\right)$$

we use

$$\alpha_n(x)=x\left(1-\frac{1}{n}\right)$$

then (1.3) implies in this case

$$G^2_n(e^t;x)=e^x,\ G^2_n(e^{2t};x)=e^{2x}.$$

But for the third test function of Korovkin-type theorem $e^{0t}=1$ simple calculations show

$$\lim_{n\to\infty}\|G^2_n(1;x)-1\|_\varphi=\infty,$$

with $\varphi(x)=1+e^{2x}$. Similar examples for nonconvergence in weighted norm can be found for Phillips operator in [13].

Remark 3.3. Further results for the operators G^1_n, G^2_n, like representation of moments, central moments, images of monomials, quantitative Voronovskaja-type and direct estimates etc. will be subject of another paper.

Remark 3.4. In [13] Gupta and Lopez-Moreno considered for Phillips operators different Korovkin test system $\{1, t, e^{bt}(1+t^r)\}$. But in their settings we need to verify the weighted approximation for two test functions with more complicated calculations, than presented in our note.

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Approximation theorems for multivariate Taylor-Abel-Poisson means

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Dedicated to Professor Heiner Gonska on the occasion of his 70th anniversary.

Abstract. We obtain direct and inverse approximation theorems of functions of several variables by Taylor-Abel-Poisson means in the integral metrics. We also show that norms of multipliers in the spaces $L_{p,Y}(\mathbb{T}^d)$ are equivalent for all positive integers d.

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1. Introduction

It is well-known that any function $f \in L_p(\mathbb{T}^1)$ that is different from a constant can be approximated by its Abel-Poisson means $f(\varrho, \cdot)$ with a precision not better than $1-\varrho$. It relates to the so-called saturation property of this approximation method. From this property, it follows that for any $f \in L_p(\mathbb{T}^1)$, the relation

$$\|f - f(\varrho, \cdot)\|_p = o(1-\varrho), \ \ \varrho \to 1-,$$

only holds in the trivial case when f is a constant function. Therefore, any additional restrictions on the smoothness of functions do not give us any order of approximation better than $1-\varrho$. In this connection, a natural question is to find a linear operator, constructed similarly to the Poisson operator, which takes into account the smoothness properties of functions and at the same time, for a given functional class, is the best in a certain sense. In [19], for classes of convolutions whose kernels were generated by some moment sequences, the authors proposed a general method of construction of similar operators that take into account properties of such kernels and hence, the smoothness of functions from corresponding classes. One example of such operators are the operators $A_{\varrho,r}$, which are the main subject of study in this paper.

The operators $A_{\varrho,r}$ were first studied in [15] where, in the terms of these operators, the author gave the structural characteristic of Hardy-Lipschitz classes $H_p^r \operatorname{Lip}\alpha$ of functions of one variable, holomorphic on the unit disc of the complex plane. In [17], in terms of approximation estimates of such operators in some spaces S^p of Sobolev type, the authors give a constructive description of classes of functions of several variables whose generalized derivatives belong to the classes S^pH_ω. In [13], direct and inverse approximation theorems of 2π-periodic functions by the operators $A_{\varrho,r}$ were given in the terms of K–functionals of functions generated by their radial derivatives.

Approximations of functions of one variable by similar operators of polynomial type were studied in [11], [4], [7], [10], [12], [6] etc. In particular, in [7], the authors found the degree of convergence of the well-known Euler and Taylor means to the functions f from some subclasses of the Lipschitz classes $\operatorname{Lip}\alpha$ in the uniform norm. In [12], the analogous results for Taylor means were obtained in the L_p–norm.

In the present paper, we continue the study of approximative properties of the operators $A_{\varrho,r}$. In particular, we extend the results of the paper [13] to the multivariate case and prove direct and inverse approximation theorems of functions of several variables by the operators $A_{\varrho,r}$ in the integral metrics. We also show that norms of multipliers in the spaces $L_{p,Y}(\mathbb{T}^d)$ are equivalent for all positive integers d.

2. Preliminaries

Let d be an integer, let $\mathbb{R}^d$, $\mathbb{R}^d_+$ and $\mathbb{Z}^d$ be the sets of all vectors $\mathbf{k} := (k_1, \ldots, k_d)$ with real, real non-negative and integer coordinates respectively. Set $\mathbb{T}^d := \mathbb{R}^d/2\pi\mathbb{Z}^d$. Further, let $L_p(\mathbb{T}^d)$, $1 \le p \le \infty$, be the space of all functions $f(\mathbf{x}) = f(x_1, \ldots, x_d)$ defined on $\mathbb{R}^d$, 2π-periodic in each variable with the finite norm

$$\|f\|_p = \|f\|_{L_p(\mathbb{T}^d)} := \begin{cases} \Big(\displaystyle\int_{\mathbb{T}^d} |f(\mathbf{x})|^p \mathrm{d}\sigma(\mathbf{x})\Big)^{\frac{1}{p}}, & 1 \le p < \infty, \\ \operatorname{ess\,sup}_{\mathbf{x}\in\mathbb{T}^d} |f(\mathbf{x})|, & p = \infty, \end{cases} \tag{2.1}$$

where σ is the normalized Lebesgue measure on $\mathbb{T}^d$.

Let $(\mathbf{x}, \mathbf{y}) := x_1y_1 + \ldots + x_dy_d$ denote the inner product of the elements $\mathbf{x}, \mathbf{y} \in \mathbb{R}^d$. Let us set $\mathrm{e}_{\mathbf{k}} := \mathrm{e}_{\mathbf{k}}(\mathbf{x}) = \mathrm{e}^{\mathrm{i}(\mathbf{k},\mathbf{x})}$, $\mathbf{k} \in \mathbb{Z}^d$, and for any function $f \in L_1(\mathbb{T}^d)$, define its Fourier coefficients by

$$\widehat{f}_{\mathbf{k}} := \int_{\mathbb{T}^d} f(\mathbf{x})\overline{\mathrm{e}}_{\mathbf{k}}(\mathbf{x})\mathrm{d}\sigma(\mathbf{x}), \quad \mathbf{k} \in \mathbb{Z}^d,$$

where $\overline{z}$ is the complex-conjugate number of z.

Set $|\mathbf{k}|_1 := \sum_{j=1}^d |k_j|$, and for any function $f \in L_1(\mathbb{T}^d)$ with the Fourier series of the form

$$S[f](\mathbf{x}) = \sum_{\mathbf{k}\in\mathbb{Z}^d} \widehat{f}_{\mathbf{k}}\mathrm{e}_{\mathbf{k}}(\mathbf{x}) = \sum_{\nu=0}^{\infty} \sum_{|\mathbf{k}|_1=\nu} \widehat{f}_{\mathbf{k}}\mathrm{e}_{\mathbf{k}}(\mathbf{x}), \tag{2.2}$$

denote by $f(\boldsymbol{\varrho}, \mathbf{x})$ its Poisson integral (the Poisson operator), i.e.,

$$f(\boldsymbol{\varrho}, \mathbf{x}) := \int_{\mathbb{T}^d} f(\mathbf{x}+\mathbf{s})P(\boldsymbol{\varrho}, \mathbf{s})\mathrm{d}\sigma(\mathbf{s}), \tag{2.3}$$

where $\boldsymbol{\varrho} \in \mathbb{R}_+^d$, $\mathbf{x} \in \mathbb{R}^d$, the function $P(\boldsymbol{\varrho}, \mathbf{x}) := \sum_{\mathbf{k}\in\mathbb{Z}^d} \boldsymbol{\varrho}^{|\mathbf{k}|} \mathrm{e}_{\mathbf{k}}(\mathbf{x})$ is the Poisson kernel,

$$\boldsymbol{\varrho}^{|\mathbf{k}|} := \varrho_1^{|k_1|} \cdots \varrho_d^{|k_d|}.$$

In what follows, the expression $f(\varrho, \mathbf{x})$ means the Poisson integral, where $\boldsymbol{\varrho}$ is a vector with the same coordinates, i.e., $\boldsymbol{\varrho} = (\varrho, \dots, \varrho)$. In such case, we have

$$P(\varrho, \mathbf{x}) := \sum_{\nu=0}^{\infty} \varrho^{\nu} \sum_{|\mathbf{k}|_1=\nu} \mathrm{e}_{\mathbf{k}}(\mathbf{x}).$$

Let $f \in L_1(\mathbb{T}^d)$. For $\varrho \in [0,1)$ and $r \in \mathbb{N}$, we set

$$A_{\varrho,r}(f)(\mathbf{x}) := \sum_{\nu=0}^{\infty} \lambda_{\nu,r}(\varrho) \sum_{|\mathbf{k}|_1=\nu} \widehat{f}_{\mathbf{k}} \mathrm{e}_{\mathbf{k}}(\mathbf{x}), \tag{2.4}$$

where for $\nu = 0, 1, \dots, r-1$, the numbers $\lambda_{\nu,r}(\varrho) \equiv 1$ and for $\nu = r, r+1, \dots,$

$$\lambda_{\nu,r}(\varrho) := \sum_{j=0}^{r-1} \binom{\nu}{j} (1-\varrho)^j \varrho^{\nu-j} = \sum_{j=0}^{r-1} \frac{(1-\varrho)^j}{j!} \frac{d^j}{d\varrho^j} \varrho^{\nu}. \tag{2.5}$$

The transformation $A_{\varrho,r}$ can be considered as a linear operator on $L_1(\mathbb{T}^d)$ into itself. Indeed, $\lambda_{\nu,r}(0) = 0$ and for all $\nu = r, r+1, \dots$ and $\varrho \in (0,1)$,

$$\sum_{j=0}^{r-1} \binom{\nu}{j} (1-\varrho)^j \varrho^{\nu-j} \le r q^{\nu} \nu^{r-1}, \text{ where } 0 < q := \max\{1-\varrho, \varrho\} < 1.$$

Therefore, for any function $f \in L_1(\mathbb{T}^d)$ and for any $0 < \varrho < 1$, the series on the right-hand side of (2.4) is majorized by the convergent series $2r\|f\|_1 \sum_{\nu=r}^{\infty} q^{\nu} \nu^{r-1}$.

Leis [11] considered for $f \in L_p(\mathbb{T}^1)$, $1 < p < \infty$, the transformation

$$L_{\varrho,r}(f)(x) := \sum_{k=0}^{r-1} \frac{\mathrm{d}^k f(x)}{\mathrm{d}n^k} \cdot \frac{(1-\varrho)^k}{k!}, \quad r \in \mathbb{N},$$

where

$$\frac{\mathrm{d}f(x)}{\mathrm{d}n} = -\frac{\partial f(\varrho, x)}{\partial \varrho}\bigg|_{\varrho=1}$$

is the normal derivative of the function f. He showed that if $1 < p < \infty$ and

$$\|f(\varrho, \cdot) - L_{\varrho,r}(f)(\cdot)\|_p = \mathcal{O}\Big((1-\varrho)^r / r!\Big), \quad \varrho \to 1-,$$

then $\mathrm{d}^r f / \mathrm{d}n^r \in L_p(\mathbb{T}^1)$.

Butzer and Sunouchi [4] considered for $f \in L_p(\mathbb{T}^1)$, $1 \le p < \infty$, the transformation

$$B_{\varrho,r}(f)(x) := \sum_{k=0}^{r-1} (-1)^{\frac{k+1}{2}} f^{\{k\}}(x) \frac{(-\ln \varrho)^k}{k!},$$

where $f^{\{k\}} := f^{(k)}$ for $k \in 2\mathbb{Z}_+$ and $f^{\{k\}} := \widetilde{f}^{(k)}$ for $k-1 \in 2\mathbb{Z}_+$, where

$$\widetilde{f}(x) = \lim_{\varepsilon\to 0} -\frac{1}{\pi}\int\limits_{\varepsilon}^{\pi}(f(x+u)-f(x-u))\frac{1}{2}\cot\frac{u}{2}\mathrm{d}u.$$

They proved the following theorem:

Theorem A. [4]. *Assume that $f \in L_p(\mathbb{T}^1)$, $1 \le p < \infty$.*

i) If the derivatives $f^{\{j\}}$, $j = 0,1,\dots,r-1$, are absolutely continuous and $f^{\{r\}} \in L_p(\mathbb{T}^1)$, then

$$\|f(\varrho,\cdot) - B_{\varrho,r}(f)(\cdot)\|_p = \mathcal{O}\Big((-\ln\varrho)^r/r!\Big), \quad \varrho \to 1- \ . \tag{2.6}$$

ii) If the derivatives $f^{\{j\}}$, $j = 0,1,\dots,r-2$, $r \ge 2$, are absolutely continuous, $f^{\{r-1\}} \in L_p(\mathbb{T}^1)$, $1 < p < \infty$, and relation (2.6) holds, then $\widetilde{f}^{\{r-1\}}$ is absolutely continuous and $\widetilde{f}^{\{r\}} \in L_p(\mathbb{T}^1)$.

These results summarize the approximation behaviour of the operators $L_{\varrho,r}$ and $B_{\varrho,r}$ in the space $L_p(\mathbb{T}^1)$. In particular, Leis's result and the statement *ii*) of Theorem A represent the so-called inverse theorems and the statement *i*) is the so-called direct theorem. Direct and inverse theorems are among the main theorems of approximation theory. They were studied by many authors. Here, we mention only the books [3], [8], [18] which contain fundamental results in this subject. The result of Leis and Theorem A are based on the investigations in the papers [5], [2], where the authors find the direct and inverse approximation theorems for the one-parameter semi-groups of bounded linear transformations $\{T(t)\}$ of some Banach space X into itself by the "Taylor polynomial" $\sum_{k=0}^{r-1}(t^k/k!)A^k f$, where Af is the infinitesimal operator of a semi–group $\{T(t)\}$.

The transformations $A_{\varrho,r}$ considered in this paper are similar to the transformations $L_{\varrho,r}$ and $B_{\varrho,r}$ as they are also based on the "Taylor polynomials". The relation between the operators $A_{\varrho,r}$ and the "Taylor polynomials" is shown in the following statement.

Lemma 2.1. *Assume that $f \in L_1(\mathbb{T}^d)$. Then for any numbers $r \in \mathbb{N}$, $\varrho \in [0,1)$ and $\mathbf{x} \in \mathbb{T}^d$,*

$$A_{\varrho,r}(f)(\mathbf{x}) = \sum_{j=0}^{r-1}\frac{\partial^j f\,(\varrho,\mathbf{x})}{\partial\varrho^j}\cdot\frac{(1-\varrho)^j}{j!}. \tag{2.7}$$

Proof. With respect to the variable ϱ, let us differentiate the decomposition of the Poisson integral into the uniformly convergent series

$$f\,(\varrho,\mathbf{x}) = \sum_{\nu=0}^{\infty}\varrho^{\nu}\sum_{|\mathbf{k}|_1=\nu}\widehat{f}_{\mathbf{k}}\mathrm{e}_{\mathbf{k}}(\mathbf{x}), \quad \varrho \in [0,1),\ \mathbf{x} \in \mathbb{T}^d. \tag{2.8}$$

We see that for any $j = 0,1,\dots$

$$\frac{\partial^j f\,(\varrho,\mathbf{x})}{\partial\varrho^j} = \sum_{\nu=j}^{\infty}\frac{\nu!}{(\nu-j)!}\varrho^{\nu-j}\sum_{|\mathbf{k}|_1=\nu}\widehat{f}_{\mathbf{k}}\mathrm{e}_{\mathbf{k}}(\mathbf{x}). \tag{2.9}$$

Since

$$\sum_{j=0}^{\nu}\binom{\nu}{j}(1-\varrho)^j\varrho^{\nu-j}=\big((1-\varrho)+\varrho\big)^{\nu}=1,\ \nu=0,1,\dots,$$

then

$$\sum_{j=0}^{r-1}\frac{\partial^j f(\varrho,\mathbf{x})}{\partial\varrho^j}\cdot\frac{(1-\varrho)^j}{j!}=\sum_{\nu=0}^{r-1}\sum_{j=0}^{\nu}\binom{\nu}{j}(1-\varrho)^j\varrho^{\nu-j}\sum_{|\mathbf{k}|_1=\nu}\widehat{f}_{\mathbf{k}}\mathrm{e}_{\mathbf{k}}(\mathbf{x})$$

$$+\sum_{\nu=r}^{\infty}\sum_{j=0}^{r-1}\binom{\nu}{j}(1-\varrho)^j\varrho^{\nu-j}\sum_{|\mathbf{k}|_1=\nu}\widehat{f}_{\mathbf{k}}\mathrm{e}_{\mathbf{k}}(\mathbf{x})=A_{\varrho,r}(f)(\mathbf{x}).\qquad\square$$

3. Direct and inverse approximation theorems

3.1. Radial derivatives and K-functionals

If for a function $f\in L_1(\mathbb{T}^d)$ and for a positive integer n there exists the function $g\in L_1(\mathbb{T}^d)$ such that

$$\widehat{g}_{\mathbf{k}}=\begin{cases}0, & \text{if}\quad |\mathbf{k}|_1=\nu<n,\\ \dfrac{\nu!}{(\nu-n)!}\widehat{f}_{\mathbf{k}}, & \text{if}\quad |\mathbf{k}|_1=\nu\ge n,\end{cases}\qquad \mathbf{k}\in\mathbb{Z}^d,\quad \nu=0,1,\dots,$$

then we say that for the function f, there exists the radial derivative g of order n for which we use the notation $f^{[n]}$.

Let us note that if the function $f^{[r]}\in L_1(\mathbb{T}^d)$, then its Poisson integral can be presented as

$$f^{[r]}(\varrho,\mathbf{x})=(f(\varrho,\cdot))^{[r]}(\mathbf{x})=\varrho^r\frac{\partial^r f(\varrho,\mathbf{x})}{\partial\varrho^r}\quad \varrho\in[0,1),\ \forall\,\mathbf{x}\in\mathbb{T}^d.\tag{3.1}$$

In the space $L_p(\mathbb{T}^d)$, the K–functional of a function f (see, for example, [8, Chap. 6]) generated by the radial derivative of order n is the following quantity:

$$K_n(\delta,f)_p:=\inf\Big\{\,\|f-h\|_p+\delta^n\|h^{[n]}\|_p : h^{[n]}\in L_p(\mathbb{T}^d)\Big\},\quad \delta>0.$$

3.2. Main results

Let $\mathbb{Z}^d_-$ denote the set of all vectors $\mathbf{k}:=(k_1,\dots,k_d)$ with negative integer coordinates, $\mathbb{Z}^d_+:=\mathbb{Z}^d\cap\mathbb{R}^d_+$ and $Y:=\mathbb{Z}^d_+\cup\mathbb{Z}^d_-$. Let also $L_{p,Y}(\mathbb{T}^d)$ be the set of all functions f from $L_p(\mathbb{T}^d)$ such that the Fourier coefficients $\widehat{f}_{\mathbf{k}}=0$ for all $\mathbf{k}\in\mathbb{Z}^d\setminus Y$. Further, we consider the functions $\omega(t)$, $t\in[0,1]$, satisfying the following conditions 1)–4): **1)** $\omega(t)$ is continuous on $[0,1]$; **2)** $\omega(t)$ is monotonically increasing; **3)** $\omega(t)\neq 0$ for all $t\in(0,1]$; **4)** $\omega(t)\to 0$ as $t\to 0$; and the well-known Zygmund–Bari–Stechkin conditions (see, for example, [1]):

$$(\mathcal{Z}):\int_0^{\delta}\frac{\omega(t)}{t}\mathrm{d}t=\mathcal{O}(\omega(\delta)),\quad (\mathcal{Z}_n):\int_{\delta}^{1}\frac{\omega(t)}{t^{n+1}}\mathrm{d}t=\mathcal{O}\Big(\frac{\omega(\delta)}{\delta^n}\Big),\ n\in\mathbb{N},\ \delta\to 0+.$$

The main results of this paper are contained in the following two statements:

Theorem 3.1. *Assume that $f \in L_{p,Y}(\mathbb{T}^d)$, $1 \le p \le \infty$, $n, r \in \mathbb{N}$, $n \le r$ and the function $\omega(t)$, $t \in [0,1]$, satisfies conditions 1)–4) and $(\mathcal{Z})$. If*

$$f^{[r-n]} \in L_p(\mathbb{T}^d) \quad \textit{and} \quad K_n\left(\delta, f^{[r-n]}\right)_p = \mathcal{O}(\omega(\delta)),\ \delta \to 0+, \tag{3.2}$$

then

$$\|f - A_{\varrho,r}(f)\|_p = \mathcal{O}\left((1-\varrho)^{r-n}\omega(1-\varrho)\right), \quad \varrho \to 1-. \tag{3.3}$$

Theorem 3.2. *Assume that $f \in L_{p,Y}(\mathbb{T}^d)$, $1 \le p \le \infty$, $n, r \in \mathbb{N}$, $n \le r$ and the function $\omega(t)$, $t \in [0,1]$, satisfies conditions 1)–4), $(\mathcal{Z})$ and $(\mathcal{Z}_n)$. If relation (3.3) holds, then relations (3.2) hold as well.*

Remark 3.3. For a given $n \in \mathbb{N}$, from condition $(\mathcal{Z}_n)$ it follows that

$$\lim_{\delta \to 0+} \inf(\delta^{-n}\omega(\delta)) > 0$$

or, equivalently, that

$$(1-\varrho)^{r-n}\omega(1-\varrho) \gg (1-\varrho)^r \quad \text{as} \quad \varrho \to 1-.$$

Therefore, if condition $(\mathcal{Z}_n)$ is satisfied, then the quantity on the right-hand side of (3.3) decreases to zero as $\varrho \to 1-$ not faster than the function $(1-\varrho)^r$. Also note that the relation $\|f - A_{\varrho,r}(f)\|_p = o((1-\varrho)^r))$, $\varrho \to 1-$, only holds in the trivial case when

$$f(\mathbf{x}) = \sum_{\nu=0}^{n-1} \sum_{|\mathbf{k}|_1=\nu} \widehat{f}_{\mathbf{k}} \mathrm{e}_{\mathbf{k}}(\mathbf{x}),$$

and in such case, the theorems are easily true. This fact is related to the so-called saturation property of the approximation method generated by the operator $A_{\varrho,r}$. In particular, in [15], it was shown that the operator $A_{\varrho,r}$ generates the linear approximation method of holomorphic functions which is saturated in the space H_p with the saturation order $(1-\varrho)^r$ and the saturation class $H_p^{r-1}\,\mathrm{Lip}\,1$.

3.3. Norms of multipliers in the spaces $L_{p,Y}(\mathbb{T}^d)$

Before proving Theorems 3.1 and 3.2, let us give some auxiliary results. In particular, the following Lemma 3.4 shows that norms of multipliers in the spaces $L_{p,Y}(\mathbb{T}^d)$ are equivalent for all d. In our opinion, such a result is interesting in itself.

Let $\mathrm{M} = \{\mu_\nu\}_{\nu=0}^{\infty}$ be a sequence of arbitrary complex numbers. If, for any function $f \in L_{1,Y}(\mathbb{T}^d)$ with Fourier series of the form (2.2), there exists a function $g \in L_{1,Y}(\mathbb{T}^d)$ with Fourier series of the form

$$S[g](\mathbf{x}) = \sum_{\nu=0}^{\infty} \mu_\nu \sum_{\mathbf{k}\in\mathbf{Y}:\,|\mathbf{k}|_1=\nu} \widehat{f}_{\mathbf{k}} \mathrm{e}_{\mathbf{k}}(\mathbf{x}),$$

then we say that in the space $L_{1,Y}(\mathbb{T}^d)$ the multiplier M is defined. In this case we use the notation $g = \mathrm{M}(f)$.

Let $B_{p,Y}$, $1 \le p \le \infty$, be a unit ball of the space $L_{p,Y}(\mathbb{T}^d)$, that is, the set of all functions $f \in L_{p,Y}(\mathbb{T}^d)$ such that $\|f\|_p \le 1$.

If $\mathrm{M}: L_{p,Y}(\mathbb{T}^d) \to L_{p,Y}(\mathbb{T}^d)$, then the norm of the operator M is the number

$$\|\mathrm{M}\|_{L_{p,Y}(\mathbb{T}^d)\to L_{p,Y}(\mathbb{T}^d)} = \sup_{f\in B_{p,Y}} \|\mathrm{M}(f)\|_p = \sup_{\substack{f\in L_{p,Y}(\mathbb{T}^d),\\ f\neq 0}} \frac{\|\mathrm{M}(f)\|_p}{\|f\|_p}.$$

We also denote by $\|\mathrm{M}\|_{L_p(\mathbb{T}^1)\to L_p(\mathbb{T}^1)}$ the norm of the operator $\mathrm{M}: L_p(\mathbb{T}^1) \to L_p(\mathbb{T}^1)$.

Let us note that if M is a continues operator from $L_{p,Y}(\mathbb{T}^d)$ to $L_{p,Y}(\mathbb{T}^d)$, then M is called the multiplier of series of the form (2.2) of (p,p)-type (see, for example, [9, Ch. 16]).

In [16], the authors proved that the norms of the multipliers M, which are defined in a similar way, for the Hardy spaces $H_p(\mathbb{D}^d)$ and $H_p(\mathbb{D}^1)$ are equivalent for all $d \in \mathbb{N}$. Without going into the details, we note that the space $H_p(\mathbb{D}^d)$ can be considered as the space of all complex-valued functions $f : \mathbb{T}^d \to \mathbb{C}$ such that $|f| \in L_p(\mathbb{T}^d)$ and $\widehat{f}(\mathbf{k}) = 0$ for all $\mathbf{k} \in \mathbb{Z}^d \setminus \mathbb{Z}^d_+$ (see, for example, Theorem 2.1.4 [14]). Here, we complement the result of [16] and show that the norms of the multipliers $\mathrm{M} : L_{p,Y}(\mathbb{T}^d) \to L_{p,Y}(\mathbb{T}^d)$ are equal as well.

Lemma 3.4. *Assume that $1 \le p \le \infty$, $d \in \mathbb{N}$ and* M *is a multiplier generated by a sequence of complex numbers $\{\mu_\nu\}_{\nu=0}^\infty$. Then*

$$\|\mathrm{M}\|_{L_{p,Y}(\mathbb{T}^d)\to L_{p,Y}(\mathbb{T}^d)} = \|\mathrm{M}\|_{L_p(\mathbb{T}^1)\to L_p(\mathbb{T}^1)}. \tag{3.4}$$

Proof. Let $f \in L_{p,Y}(\mathbb{T}^d)$. Note that for almost all $\mathbf{x} \in \mathbb{T}^d$, the multiplier M can be defined by the following rule:

$$\mathrm{M}(f)(\mathbf{x}) = \lim_{\varrho\to 1-} \mathrm{M}(f)(\varrho, \mathbf{x}), \tag{3.5}$$

where for $0 < \varrho < 1$ and $\mathbf{x} \in \mathbb{T}^d$,

$$\mathrm{M}(f)(\varrho, \mathbf{x}) = \sum_{\nu=0}^{\infty} \lambda_\nu \varrho^\nu \sum_{\mathbf{k}\in Y:\, |\mathbf{k}|_1=\nu} \widehat{f}_{\mathbf{k}} \mathrm{e}_{\mathbf{k}}(\mathbf{x}).$$

If $f \in L_p(\mathbb{T})$, then this rule has the form

$$\mathrm{M}(f)(\varrho, t) = \lim_{\varrho\to 1-} \sum_{n\in\mathbb{Z}} \mu_{|n|} \varrho^{|n|} \widehat{f}_n \mathrm{e}^{\mathrm{i}nt}.$$

For any $f \in L_{p,Y}(\mathbb{T}^d)$, we set $\mathrm{M}(f)(\mathbf{z}) = \mathrm{M}(f)(\boldsymbol{\varrho}, \mathbf{x})$, where for $0 < \varrho_j < 1$ and $\mathbf{x} \in \mathbb{T}^d$, the point $\mathbf{z} := (\varrho_1 \mathrm{e}^{\mathrm{i}x_1}, \ldots, \varrho_d \mathrm{e}^{\mathrm{i}x_d})$ belongs to the unit polydisc

$$\mathbb{D}^d := \{\mathbf{z} \in \mathbb{C}^d : \max_{1\le j\le d} |z_j| < 1\}.$$

Therefore, the function $\mathrm{M}(f)(\mathbf{z})$ is a d–harmonic function in $\mathbb{D}^d$ and according to the assertion (c) of Theorem 2.1.3 [14], we have $\|\mathrm{M}(f)(\varrho\cdot)\|_p \le \|\mathrm{M}(f)\|_p$. On the other hand, by virtue of Fatou's lemma,

$$\|\mathrm{M}(f)\|_p \le \liminf_{\varrho\to 1-} \|\mathrm{M}(f)(\varrho\cdot)\|_p,$$

hence, for $1 \le p < \infty$,

$$\|\mathrm{M}(f)\|_p = \lim_{\varrho \to 1-} \|\mathrm{M}(f)(\varrho, \cdot)\|_p. \tag{3.6}$$

If $p = \infty$, then instead of the last relation we have

$$\int\limits_{\mathbb{T}^d} \mathrm{M}(f)(\mathbf{w}) g(\mathbf{w}) \mathrm{d}\sigma(\mathbf{w}) = \lim_{\varrho \to 1-} \int\limits_{\mathbb{T}^d} \mathrm{M}(f)(\varrho, \mathbf{w}) g(\mathbf{w}) \mathrm{d}\sigma(\mathbf{w})$$

for any function $g \in L_1(\mathbb{T}^d)$, i.e., we have convergence in the weak L_1–topology of space $L_\infty(\mathbb{T}^d)$.

Let $f \in L_{p,Y}(\mathbb{T}^d)$, $f \not\equiv 0$, $\mathbf{z}$ be a fixed point in $\overline{\mathbb{D}}^d$ and $0 \le \varrho < 1$. In the disc $\mathbb{D}^1$, consider the function $u_{\varrho\mathbf{z}}(\omega) := f(\varrho, \mathbf{z}\omega)$. Applying Lemma 3.3.2 [14], we consistently have the following equality and estimate for the integral of $|\mathrm{M}(f)(\varrho\,\cdot)|^p$ for $0 \le \varrho < 1$ and $1 \le p < \infty$:

$$\begin{aligned}
&\int\limits_{\mathbb{T}^d} |\mathrm{M}(f)(\varrho, \mathbf{w})|^p \mathrm{d}\sigma(\mathbf{w}) = \int\limits_{\mathbb{T}^d} \mathrm{d}\sigma(\mathbf{w}) \int\limits_{\mathbb{T}^1} |\mathrm{M}(u_{\varrho\mathbf{w}})(\omega)|^p d\omega \\
= \;& \int\limits_{\mathbb{T}^d} \|\mathrm{M}(u_{\varrho\mathbf{w}})\|_p^p \mathrm{d}\sigma(\mathbf{w}) = \int\limits_{\mathbb{T}^d} \|u_{\varrho\mathbf{w}}\|_p^p \frac{\|\mathrm{M}(u_{\varrho\mathbf{w}})\|_p^p}{\|u_{\varrho\mathbf{w}}\|_p^p} \mathrm{d}\sigma(\mathbf{w}) \\
\le \;& \max_{\mathbf{w} \in \mathbb{T}^d} \frac{\|\mathrm{M}(u_{\varrho\mathbf{w}})\|_p^p}{\|u_{\varrho\mathbf{w}}\|_p^p} \int\limits_{\mathbb{T}^d} \|u_{\varrho\mathbf{w}}\|_p^p \mathrm{d}\sigma(\mathbf{w}) \\
\le \;& \|\mathrm{M}\|^p_{L_p(\mathbb{T}^1) \to L_p(\mathbb{T}^1)} \int\limits_{\mathbb{T}^d} \|u_{\varrho\mathbf{w}}\|_p^p \mathrm{d}\sigma(\mathbf{w}) \\
= \;& \|\mathrm{M}\|^p_{L_p(\mathbb{T}^1) \to L_p(\mathbb{T}^1)} \int\limits_{\mathbb{T}^d} |f(\varrho, \mathbf{w})|^p \mathrm{d}\sigma(\mathbf{w}).
\end{aligned} \tag{3.7}$$

In the case $p = \infty$, we similarly obtain the estimate

$$\begin{aligned}
&|\mathrm{M}(f)(\varrho, \omega\mathbf{w})| = |\mathrm{M}(u_{\varrho\mathbf{w}})(\omega)| \\
= \;& \lim_{\rho \to 1-} |\mathrm{M}(u_{\varrho\mathbf{w}})(\rho\omega)| \le \max_{\omega \in \mathbb{T}^1} |\mathrm{M}(u_{\varrho\mathbf{w}})(\omega)| \\
\le \;& \|\mathrm{M}\|_{L_{\infty,Y}(\mathbb{T}^d) \to L_{\infty,Y}(\mathbb{T}^d)} \max_{\omega \in \mathbb{T}^1} |f(\varrho, \omega\mathbf{w})|.
\end{aligned} \tag{3.8}$$

From (3.7) and (3.8) in view of (3.5) it follows that for $1 \le p \le \infty$,

$$\begin{aligned}
\|\mathrm{M}\|_{L_{p,Y}(\mathbb{T}^d) \to L_{p,Y}(\mathbb{T}^d)} &= \lim_{\varrho \to 1-} \sup_{f \in L_{p,Y}(\mathbb{T}^d)} \frac{\|\mathrm{M}(f)(\varrho, \cdot)\|_p}{\|f(\varrho, \cdot)\|_p} \\
&\le \|\mathrm{M}\|_{L_p(\mathbb{T}^1) \to L_p(\mathbb{T}^1)}.
\end{aligned} \tag{3.9}$$

To prove the reverse inequality let us consider the continuation operator Q, given on $L_p(\mathbb{T}^1)$, $1 \le p \le \infty$, by the formula

$$Q(g)(w_1, \mathbf{w}^1) = g(w_1),$$

where $w_1 \in \mathbb{T}^1$, $\mathbf{w}^1 = (w_2, \ldots, w_d) \in \mathbb{T}^{d-1}$.

It is easy to show that the continuation operator Q is a linear isometry of the space $L_p(\mathbb{T}^1)$ in $L_p(\mathbb{T}^d)$. Therefore, taking into account the relation $Q\big(\mathrm{M}(f)\big) = \mathrm{M}\big(Q(f)\big)$, which is satisfied for any function $f \in L_p(\mathbb{T}^1)$, we obtain

$$\begin{aligned}\|\mathrm{M}(f)\|_p &= \|Q\big(\mathrm{M}(f)\big)\|_p = \|\mathrm{M}\big(Q(f)\big)\|_p \\ &\le \|\mathrm{M}\|_{L_{p,Y}(\mathbb{T}^d)\to L_{p,Y}(\mathbb{T}^d)} \|Q(f)\|_p \\ &= \|\mathrm{M}\|_{L_{p,Y}(\mathbb{T}^d)\to L_{p,Y}(\mathbb{T}^d)} \|f\|_p.\end{aligned}$$

This implies the estimate

$$\|\mathrm{M}\|_{L_p(\mathbb{T}^1)\to L_p(\mathbb{T}^1)} \le \|\mathrm{M}\|_{L_{p,Y}(\mathbb{T}^d)\to L_{p,Y}(\mathbb{T}^d)},$$

which in combination with (3.9) gives the relation (3.4).

3.4. Auxiliary statements

Let

$$\mathcal{P}(\varrho, \mathbf{x}) := \prod_{j=1}^{d} \frac{1}{1-\varrho \mathrm{e}^{\mathrm{i}x_j}} + \prod_{j=1}^{d} \frac{1}{1-\varrho \mathrm{e}^{-ix_j}} - 1. \tag{3.10}$$

Lemma 3.5. *Assume that $f \in L_{1,Y}(\mathbb{T}^d)$, $0 \le \varrho < 1$ and $\mathbf{x} \in \mathbb{T}^d$. Then*

$$f\,(\varrho, \mathbf{x}) = \int_{\mathbb{T}^d} f(\mathbf{x}+\mathbf{s})\mathcal{P}(\varrho, \mathbf{s})\mathrm{d}\sigma(\mathbf{s}). \tag{3.11}$$

Proof. By virtue of the definition of the set $L_{1,Y}(\mathbb{T}^d)$, we have

$$f(\varrho, \mathbf{x}) = \sum_{\nu=0}^{\infty} \varrho^{\nu} \sum_{\mathbf{k}\in Y:\, |\mathbf{k}|_1=\nu} \widehat{f}_{\mathbf{k}} \mathrm{e}_{\mathbf{k}}(\mathbf{x}). \tag{3.12}$$

On the other hand

$$\begin{aligned}\mathcal{P}(\varrho, \mathbf{x}) &= \sum_{k_1=0}^{\infty} \dots \sum_{k_d=0}^{\infty} \varrho^{k_1+\ldots+k_d} \Big(\mathrm{e}^{\mathrm{i}(k_1x_1+\ldots+k_dx_d)} + \mathrm{e}^{-\mathrm{i}(k_1x_1+\ldots+k_dx_d)}\Big) - 1 \\ &= 1 + \sum_{\nu=1}^{\infty} \varrho^{\nu} \sum_{\mathbf{k}\in Y:\, |\mathbf{k}|_1=\nu} \mathrm{e}_{\mathbf{k}}(\mathbf{x}). \end{aligned}\tag{3.13}$$

Therefore, the right-hand side of (3.11) is equivalent to the right-hand side of (3.12). □

Lemma 3.6. *Assume that $f \in L_{p,Y}(\mathbb{T}^d)$, $1 \le p \le \infty$, $r = 0, 1, \ldots$ and $\varrho \in [0, 1)$. Then the following relations are true:*

$$\Big\|\frac{\partial^r f\,(\varrho, \cdot)}{\partial \varrho^r}\Big\|_p \le C_1(r)\frac{\|f\|_p}{(1-\varrho)^r} \tag{3.14}$$

and

$$\|A_{\varrho,r}^{[r]}(f)\|_p \le C_2(r)\frac{\|f\|_p}{(1-\varrho)^r}, \tag{3.15}$$

where the constants $C_1(r)$ and $C_2(r)$ depend only on r.

Proof. It is easy to see that the function $\partial^r f(\varrho, \mathbf{x}) / \partial \varrho^r$ can be considered as the image $\mathrm{M}_1(f)(\mathbf{x})$ of the multiplier generated by the sequence $\{\mu_{1,\nu}\}_{\nu=0}^{\infty}$, where $\mu_{1,\nu} = 0$ for $\nu = 0, 1, \ldots, r-1$ and $\mu_{1,\nu} = \nu \cdot (\nu - 1) \cdot \ldots \cdot (\nu - r + 1)\varrho^{\nu - r}$ for $\nu \geq r$. Similarly, the function $A_{\varrho,r}^{[r]}(f)(\mathbf{x})$ can be considered as the image $\mathrm{M}_2(f)(\mathbf{x})$ of the multiplier generated by the sequence $\{\mu_{2,\nu}\}_{\nu=0}^{\infty}$ such that $\mu_{2,\nu} = 0$ for $\nu = 0, 1, \ldots, r-1$ and $\mu_{2,\nu} = \nu! \cdot \lambda_{\nu,r}(\varrho)/(\nu - r)!$ for $\nu \geq r$. Therefore, to prove estimates (3.14) and (3.15) it is sufficient to apply Lemma 3.4 and the estimates (23) and (22) for the norms of the corresponding multipliers in the space $L_p(\mathbb{T}^1)$ from [13]. □

For any $f \in L_p(\mathbb{T}^d)$, $1 \leq p \leq \infty$, $0 \leq \varrho < 1$ and $r = 0, 1, 2, \ldots$, we set

$$M_p(\varrho, f, r) := \varrho^r \Big\| \frac{\partial^r f(\varrho, \cdot)}{\partial \varrho^r} \Big\|_p = \Big\| (f(\varrho, \cdot))^{[r]} \Big\|_p. \tag{3.16}$$

Lemma 3.7. *Assume that $f \in L_{p,Y}(\mathbb{T}^d)$, $1 \leq p \leq \infty$. Then for any numbers $n \in \mathbb{N}$ and $\varrho \in [0, 1)$,*

$$\begin{aligned} & C_3(n)(1-\varrho)^n M_p(\varrho, f, n) \leq K_n(1-\varrho, f)_p \\ \leq\ & C_4(n)\Big(\|f - A_{\varrho,n}(f)\|_p + (1-\varrho)^n M_p(\varrho, f, n) \Big), \end{aligned} \tag{3.17}$$

where the constants $C_3(n)$ and $C_4(n)$ depend only on n.

Proof. First, let us note that the statement of Lemma 3.7 is trivial in the case, when f is a polynomial of the form

$$f(\mathbf{x}) = \sum_{\nu=0}^{n-1} \sum_{|\mathbf{k}|_1 = \nu} \widehat{f}_{\mathbf{k}} \mathrm{e}_{\mathbf{k}}(\mathbf{x}),$$

as well as in the case, when $\varrho = 0$. Therefore, further in the proof, we exclude these two cases.

Let g be a function such that $g^{[n]} \in L_p(\mathbb{T}^d)$. Using Lemma 3.6, we get

$$\begin{aligned} \Big\| \frac{\partial^n f(\varrho, \cdot)}{\partial \varrho^n} \Big\|_p &= \Big\| \frac{\partial^n (f-g)(\varrho, \cdot)}{\partial \varrho^n} + \frac{\partial^n g(\varrho, \cdot)}{\partial \varrho^n} \Big\|_p \\ &\leq C_1(n) \frac{\|f-g\|_p}{(1-\varrho)^n} + \Big\| \frac{\partial^n g(\varrho, \cdot)}{\partial \varrho^n} \Big\|_p. \end{aligned}$$

Setting $C_3(n) = \min\{1, 1/C_1(n)\}$ and taking into account relations (3.1), (3.16) and the inequality $\|g^{[n]}(\varrho, \cdot)\|_p \leq \|g^{[n]}\|_p$, we see that

$$C_3(n)(1-\varrho)^n M_p(\varrho, f, n) \leq \|f-g\|_p + (1-\varrho)^n \|g^{[n]}\|_p.$$

Considering the infimum over all functions g such that $g^{[n]} \in L_p(\mathbb{T}^d)$, we conclude that

$$C_3(n)(1-\varrho)^n M_p(\varrho, f, n) \leq K_n(1-\varrho, f)_p.$$

On the other hand, from the definition of the K–functional, it follows that

$$K_n(1-\varrho, f)_p \leq \|f - A_{\varrho,n}(f)\|_p + (1-\varrho)^n \Big\| A_{\varrho,n}^{[n]}(f) \Big\|_p. \tag{3.18}$$

According to (2.7) and (3.1), we have

$$\begin{aligned} A^{[n]}_{\varrho,n}(f)(\mathbf{x}) &= \Big(\sum_{k=0}^{n-1} \frac{(f(\varrho,\cdot))^{[k]}(\cdot)}{\varrho^k k!}(1-\varrho)^k \Big)^{[n]}(\mathbf{x}) \\ &= \sum_{k=0}^{n-1} \frac{((f(\varrho,\cdot))^{[k]}(\cdot))^{[n]}(\mathbf{x})}{\varrho^k k!}(1-\varrho)^k. \end{aligned}$$

Since for any nonnegative integers k and n

$$((f(\varrho,\cdot))^{[n]}(\cdot))^{[k]}(\mathbf{x}) = ((f(\varrho,\cdot))^{[k]}(\cdot))^{[n]}(\mathbf{x}), \tag{3.19}$$

we obtain

$$A^{[n]}_{\varrho,n}(f)(\mathbf{x}) = \sum_{k=0}^{n-1} \frac{((f(\varrho,\cdot))^{[n]}(\cdot))^{[k]}(\mathbf{x})}{\varrho^k k!}(1-\varrho)^k.$$

This yields

$$\|A^{[n]}_{\varrho,n}(f)\|_p \le \sum_{k=0}^{n-1} \frac{\|((f(\varrho,\cdot))^{[n]}(\cdot))^{[k]}\|_p}{\varrho^k k!}(1-\varrho)^k, \tag{3.20}$$

where by virtue of Lemma 3.6 and (3.16)

$$\|((f(\varrho,\cdot))^{[n]}(\cdot))^{[k]}\|_p \le M_p(\varrho,f,n)\frac{C_1(k)\varrho^k}{(1-\varrho)^k}. \tag{3.21}$$

Therefore,

$$\|A^{[n]}_{\varrho,n}(f)\|_p \le M_p(\varrho,f,n)\sum_{k=0}^{n-1}\frac{C_1(k)}{k!}. \tag{3.22}$$

Setting

$$C_4(n) = \max\{1, \sum_{k=0}^{n-1} C_1(k)/k!\}$$

and combining relations (3.18) and (3.22), we obtain the right-hand inequality in (3.17). □

Lemma 3.8. *Assume that* $f \in L_p(\mathbb{T}^d)$, $1 \le p \le \infty$, $0 \le \varrho < 1$ *and* $r = 2, 3, \dots$ *such that*

$$\int_\varrho^1 \left\| \frac{\partial^r f(\zeta,\cdot)}{\partial \zeta^r} \right\|_p (1-\zeta)^{r-1} d\zeta < \infty. \tag{3.23}$$

Then for almost all $\mathbf{x} \in \mathbb{T}^d$,

$$f(\mathbf{x}) - A_{\varrho,r}(f)(\mathbf{x}) = \frac{1}{(r-1)!}\int_\varrho^1 \frac{\partial^r f(\zeta,\mathbf{x})}{\partial\zeta^r}(1-\zeta)^{r-1}d\zeta. \tag{3.24}$$

Proof. For fixed $r = 2, 3, \dots$ and $0 \le \varrho < 1$, the integral on the right-hand side of (3.24) defines a certain function $F(\mathbf{x})$. By virtue of (3.23) and the integral Minkowski inequality, we conclude that the function F belongs to the space $L_p(\mathbb{T}^d)$. Let us find the Fourier coefficients of F and compare them with the Fourier coefficients of the function $G := f - A_{\varrho,r}(f)$.

Since for any $\nu = r, r+1\ldots$,

$$\frac{1}{(r-1)!\cdot(\nu-r)!}\int_{\varrho}^{\varrho_1}\zeta^{\nu-r}(1-\zeta)^{r-1}d\zeta = \sum_{j=0}^{r-1}\frac{\varrho_1^{\nu-j}(1-\varrho_1)^j-\varrho^{\nu-j}(1-\varrho)^j}{j!\cdot(\nu-j)!},$$

then in view of (2.9) for a fixed $\varrho_1\in(\varrho,1)$, we have

$$\begin{aligned}
&\frac{1}{(r-1)!}\int_{\varrho}^{\varrho_1}\frac{\partial^r f(\zeta,\mathbf{x})}{\partial\zeta^r}(1-\zeta)^{r-1}\mathrm{d}\zeta\\
=\ &\sum_{\nu=r}^{\infty}\sum_{|\mathbf{k}|_1=\nu}\frac{\nu!\cdot\widehat{f}_{\mathbf{k}}\cdot\mathrm{e}_{\mathbf{k}}(\mathbf{x})}{(r-1)!\cdot(\nu-r)!}\int_{\varrho}^{\varrho_1}\zeta^{\nu-r}(1-\zeta)^{r-1}\mathrm{d}\zeta\\
=\ &\sum_{\nu=r}^{\infty}\sum_{|\mathbf{k}|_1=\nu}\widehat{f}_{\mathbf{k}}\mathrm{e}_{\mathbf{k}}(\mathbf{x})\sum_{j=0}^{r-1}\binom{\nu}{j}\Big(\varrho_1^{\nu-j}(1-\varrho_1)^j-\varrho^{\nu-j}(1-\varrho)^j\Big). \qquad (3.25)
\end{aligned}$$

Now if in relation (3.25), the value ϱ_1 tends to $1-$, then we see that the Fourier coefficients $\widehat{F}_{\mathbf{k}}$ of the function F are equivalent to zero when $|\mathbf{k}|_1=\nu<r$ and for $|\mathbf{k}|_1\ge r$,

$$\widehat{F}_{\mathbf{k}}=\widehat{f}_{\mathbf{k}}\cdot\Big(1-\sum_{j=0}^{r-1}\binom{\nu}{j}(1-\varrho)^j\varrho^{\nu-j}\Big)=(1-\lambda_{\nu,r}(\varrho))\widehat{f}_{\mathbf{k}}. \qquad (3.26)$$

Therefore, for all $\mathbf{k}\in\mathbb{Z}^d$ we have $\widehat{F}_{\mathbf{k}}=(1-\lambda_{\nu,r}(\varrho))\widehat{f}_{\mathbf{k}}=\widehat{G}_{\mathbf{k}}$. Hence, for almost all $\mathbf{x}\in\mathbb{T}^d$, relation (3.24) holds. □

3.5. Proof of main results

Proof of Theorem 3.1. Assume that the function f is such that $f^{[r-n]}\in L_{p,Y}(\mathbb{T}^d)$ and relation (3.2) is satisfied. Let us apply the first inequality of Lemma 3.7 to the function $f^{[r-n]}$. In view of (3.1) and (3.16), we obtain

$$C_3(n)(1-\varrho)^n M_p(\varrho,f,r)\le K_n\left(1-\varrho,f^{[r-n]}\right)_p.$$

This yields

$$M_p(\varrho,f,r)=\mathcal{O}(1)(1-\varrho)^{-n}\omega(1-\varrho),\quad \varrho\to1-. \qquad (3.27)$$

Using relations (3.16), (3.27), ($\mathcal{Z}$) and the integral Minkowski inequality, we obtain

$$\int_{\varrho}^{1}\left\|\frac{\partial^r f(\zeta,\cdot)}{\partial\zeta^r}\right\|_p(1-\zeta)^{r-1}\mathrm{d}\zeta\le\int_{\varrho}^{1}M_p(\zeta,f,r)\frac{(1-\zeta)^{r-1}}{\zeta^r}\mathrm{d}\zeta$$

$$\le C_1(1-\varrho)^{r-n}\int_{\varrho}^{1}\frac{\omega(1-\zeta)}{1-\zeta}\mathrm{d}\zeta=\mathcal{O}\left((1-\varrho)^{r-n}\omega(1-\varrho)\right),\ \varrho\to1-. \qquad (3.28)$$

Therefore, for almost all $\mathbf{x}\in\mathbb{T}^d$, relation (3.24) holds. Hence, by virtue of (3.24), using the integral Minkowski inequality and (3.28), we finally get (3.3):

$$\begin{aligned}
\|f-A_{\varrho,r}(f)\|_p&\le\frac{1}{(r-1)!}\int_{\varrho}^{1}M_p(\zeta,f,r)\frac{(1-\zeta)^{r-1}}{\zeta^r}\mathrm{d}\zeta\\
&=\mathcal{O}\left((1-\varrho)^{r-n}\omega(1-\varrho)\right),\quad\varrho\to1-.
\end{aligned}$$

□

Proof of Theorem 3.2. First, let us note that for any function $f \in L_p(\mathbb{T}^d)$ and all fixed numbers $s, r \in \mathbb{N}$ and $\varrho \in (0,1)$, we have

$$\begin{aligned}\|A_{\varrho,r}^{[s]}(f)\|_p &= \Big\| \sum_{\nu=s}^{\infty} \frac{\nu!}{(\nu-s)!} \lambda_{\nu,r}(\varrho) \sum_{|\mathbf{k}|_1=\nu} \widehat{f}_{\mathbf{k}} \mathrm{e}_{\mathbf{k}} \Big\|_p \\ &\le 2r\|f\|_p \Bigg(\sum_{\nu=s}^{\max\{s,r\}-1} \frac{\nu!}{(\nu-s)!} + \sum_{\nu \ge \max\{s,r\}} q^{\nu} \nu^{s+r-1} \Bigg) < \infty,\end{aligned}$$

where $0 < q = \max\{1-\varrho, \varrho\} < 1$. In the case where $s \ge r$, the sum $\sum\limits_{\nu=s}^{s-1}$ is set equal to zero. Put $\varrho_k := 1 - 2^{-k}$, $k \in \mathbb{N}$, and $A_k := A_k(f) := A_{\varrho_k, r}(f)$. For any $\mathbf{x} \in \mathbb{T}^d$ and $s \in \mathbb{N}$, consider the series

$$A_0^{[s]}(f)(\mathbf{x}) + \sum_{k=1}^{\infty} (A_k^{[s]}(f)(\mathbf{x}) - A_{k-1}^{[s]}(f)(\mathbf{x})). \tag{3.29}$$

According to the definition of the operator $A_{\varrho,r}$, we see that for any $\varrho_1, \varrho_2 \in [0,1)$ and $r \in \mathbb{N}$,

$$A_{\varrho_1,r}\left(A_{\varrho_2,r}(f)\right) = A_{\varrho_2,r}\left(A_{\varrho_1,r}(f)\right).$$

By virtue of Lemma 3.6 and relation (3.3), for any $k \in \mathbb{N}$ and $s \in \mathbb{N}$, we have

$$\begin{aligned} \left\| A_k^{[s]} - A_{k-1}^{[s]} \right\|_p &= \left\| A_k^{[s]}(f - A_{k-1}(f)) - A_{k-1}^{[s]}(f - A_k(f)) \right\|_p \\ &\le \left\| A_k^{[s]}(f - A_{k-1}(f)) \right\|_p + \left\| A_{k-1}^{[s]}(f - A_k(f)) \right\|_p \\ &\le C_2(s) \frac{\|f - A_{k-1}(f)\|_p}{(1-\varrho_k)^s} + C_2(s) \frac{\|f - A_k(f)\|_p}{(1-\varrho_{k-1})^s} \\ &= \mathcal{O}\left(\frac{\omega(1-\varrho_{k-1})}{(1-\varrho_k)^{s-r+n}} \right) + \mathcal{O}\left(\frac{\omega(1-\varrho_k)}{(1-\varrho_{k-1})^{s-r+n}} \right), \quad k \to \infty. \end{aligned} \tag{3.30}$$

Therefore, for any $s \le r - n$,

$$\left\| A_k^{[s]} - A_{k-1}^{[s]} \right\|_p = \mathcal{O}\left(\omega(1-\varrho_{k-1})\right) = \mathcal{O}\left(\omega(2^{-(k-1)})\right), \quad k \to \infty. \tag{3.31}$$

Consider the sum $\sum\limits_{k=1}^{N} \omega(2^{1-k})$, $N \in \mathbb{N}$. Taking into account the monotonicity of the function ω and $(\mathcal{Z})$, we see that for all $N \in \mathbb{N}$,

$$\sum_{k=1}^{N} \omega(2^{1-k}) \le \omega(1) + \int_1^N \omega(2^{1-t})\mathrm{d}t = \omega(1) + \int_{2^{1-N}}^{1} \frac{\omega(\tau)\, d\tau}{\tau \ln 2} < \infty.$$

Combining the last relation and (3.31), we conclude that the series in (3.29) converges in the norm of the space $L_p(\mathbb{T}^d)$, $1 \le p \le \infty$. Hence, by virtue of the Banach–Alaoglu theorem, for any $s = 0, 1, \dots, r-n$, there exists the subsequence

$$S^{[s]}_{N_j}(\mathbf{x}) = A^{[s]}_0(f)(\mathbf{x}) + \sum_{k=1}^{N_j}(A^{[s]}_k(f)(\mathbf{x}) - A^{[s]}_{k-1}(f)(\mathbf{x})), \quad j = 1, 2, \dots \tag{3.32}$$

of partial sums of this series, converging to a certain function $g \in L_p(\mathbb{T}^d)$ almost everywhere on $\mathbb{T}^d$ as $j \to \infty$.

Let us show that $g = f^{[s]}$. For this, let us find the Fourier coefficients of the function g. For any fixed $\mathbf{k} \in \mathbb{Z}^d$ and all $j = 1, 2, \dots$, we have

$$\widehat{g}_{\mathbf{k}} := \int_{\mathbb{T}^d} S^{[s]}_{N_j}(\mathbf{x})\bar{\mathrm{e}}_{\mathbf{k}}(\mathbf{x})\mathrm{d}\sigma(\mathbf{x}) + \int_{\mathbb{T}^d} (g(\mathbf{x}) - S^{[s]}_{N_j}(\mathbf{x}))\bar{\mathrm{e}}_{\mathbf{k}}(\mathbf{x})\mathrm{d}\sigma(\mathbf{x}).$$

Since the sequence $\{S^{[s]}_{N_j}\}_{j=1}^{\infty}$ converges almost everywhere on $\mathbb{T}^d$ to the function g, the second integral on the right-hand side of the last equality tends to zero as $j \to \infty$. By virtue of (3.32) and the definition of the radial derivative, for $|\mathbf{k}|_1 = \nu < s$ the first integral is equal to zero, and for all $|\mathbf{k}|_1 = \nu \geq s$,

$$\int_{\mathbb{T}^d} S^{[s]}_{N_j}(\mathbf{x})\bar{\mathrm{e}}_{\mathbf{k}}(\mathbf{x})\mathrm{d}\sigma(\mathbf{x}) = \lambda_{\nu,r}(1 - 2^{-N_j})\frac{\nu!}{(\nu-s)!}\widehat{f}_{\mathbf{k}} \underset{j\to\infty}{\longrightarrow} \frac{\nu!}{(\nu-s)!}\widehat{f}_{\mathbf{k}}.$$

Therefore, the equality $g = f^{[s]}$ is true.
Hence, for the function f and all $s = 0, 1, \dots, r-n$, there exists the derivative $f^{[s]}$ and $f^{[s]} \in L_p(\mathbb{T}^d)$.
Now, let us prove the estimate (3.27). By virtue of (3.16), (3.30), for any $k \in \mathbb{N}$ and $\varrho \in (0, 1)$, we have

$$\begin{aligned} M_p(\varrho, A_k - A_{k-1}, r) &\leq \left\| A^{[r]}_k - A^{[r]}_{k-1} \right\|_p \\ &= \mathcal{O}\left(\frac{\omega(1-\varrho_{k-1})}{(1-\varrho_k)^n}\right) + \mathcal{O}\left(\frac{\omega(1-\varrho_k)}{(1-\varrho_{k-1})^n}\right) \\ &= \mathcal{O}\left(2^{kn}\omega(2^{-k+1}) + 2^{(k-1)n}\omega(2^{-k})\right) \\ &= \mathcal{O}\left(2^{(k-1)n}\omega(2^{-(k-1)})\right), \quad k \to \infty. \end{aligned} \tag{3.33}$$

By virtue of (3.16), (3.14) and (3.3), for any $r \in \mathbb{N}$ and $\varrho \in (0, 1)$, we obtain

$$M_p(\varrho, f - A_{\varrho,r}(f), r) = \mathcal{O}(1)\frac{\|f - A_{\varrho,r}(f)\|_p}{(1-\varrho)^r} = \mathcal{O}\left(\frac{\omega(1-\varrho)}{(1-\varrho)^n}\right), \quad \varrho \to 1-.$$

Therefore, for $N \to \infty$,

$$M_p(\varrho_N, f - A_N(f), r) = \mathcal{O}\left(\frac{\omega(1-\varrho_N)}{(1-\varrho_N)^n}\right) = \mathcal{O}\left(2^{Nn}\omega(2^{-N})\right). \tag{3.34}$$

Consider the sum $\sum_{k=1}^{N} 2^{(k-1)n}\omega(2^{-(k-1)})$, $N \in \mathbb{N}$. Since the function ω satisfies the condition $(\mathcal{Z}_n)$, the function $\omega(t)/t^n$ almost decreases on $(0, 1]$, i.e., there exists the number $C > 0$ such that $\omega(t_1)/t_1^n \geq C\omega(t_2)/t_2^n$ for any $0 < t_1 < t_2 \leq 1$ (see, for example [1]). Therefore,

$$
\begin{aligned}
&\sum_{k=1}^{N} 2^{(k-1)n}\omega(2^{-(k-1)}) \\
\leq\ & C\Big(2^{(N-1)n}\omega(2^{-(N-1)}) + \int_1^N 2^{(t-1)n}\omega(2^{-(t-1)})\mathrm{d}t\Big) \\
\leq\ & C\Big(2^{(N-1)n}\omega(2^{-(N-1)}) + \int_{2^{-N+1}}^1 \frac{\omega(\tau)\,d\tau}{\tau^{n+1}\ln 2}\Big) \\
=\ & \mathcal{O}\Big(2^{(N-1)n}\omega(2^{-(N-1)})\Big) = \mathcal{O}\Big(2^{Nn}\omega(2^{-N})\Big), \quad N \to \infty. \qquad (3.35)
\end{aligned}
$$

Putting $\varrho = \varrho_N$ and taking into account relations (3.33), (3.34), (3.35) and

$$A_0(\mathbf{x}) = S_{r-1}(f)(\mathbf{x}) = \sum_{|\mathbf{k}|_1 \leq r-1} \widehat{f}_{\mathbf{k}} \mathrm{e}_{\mathbf{k}}(\mathbf{x}),$$

we get

$$
\begin{aligned}
& M_p\,(\varrho_N, f, r) = M_p\,(\varrho_N, f - S_{r-1}(f), r) \\
=\ & M_p\Big(\varrho_N, f - A_{\varrho_N} + \sum_{k=1}^{N}(A_k - A_{k-1}), r\Big) = \mathcal{O}\Big(\sum_{k=1}^{N} 2^{(k-1)n}\omega(2^{-(k-1)})\Big) \\
=\ & \mathcal{O}\left(2^{Nn}\omega(2^{-N})\right) = \mathcal{O}\left((1-\varrho_N)^{-n}\omega(1-\varrho_N)\right), \quad N \to \infty. \qquad (3.36)
\end{aligned}
$$

If the function ω satisfies the condition $(\mathcal{Z}_n)$, then $\sup\limits_{t\in[0,1]}\Big(\omega(2t)/\omega(t)\Big) < \infty$ (see, for example [1]). Furthermore, for all $\varrho \in [\varrho_{N-1}, \varrho_N]$, we have $1-\varrho_N \leq 1-\varrho \leq 2(1-\varrho_N)$. Hence, relation (3.36) yields the estimate (3.27).
Now, applying the second inequality in Lemma 3.7 to the function $f^{[r-n]}$, we get

$$
\begin{aligned}
K_n\left(1-\varrho, f^{[r-n]}\right)_p \ \leq\ & C_4(n)\Big(\|f^{[r-n]} - A_{\varrho,n}(f^{[r-n]})\|_p \\
+\ & (1-\varrho)^n M_p(\varrho, f, r)\Big). \qquad (3.37)
\end{aligned}
$$

By virtue of (3.16) and (3.27), we see that for $\varrho \in [1/2, 1)$,

$$
\begin{aligned}
& \int_\varrho^1 \left\|\frac{\partial^n f^{[r-n]}(\zeta,\cdot)}{\partial\zeta^n}\right\|_p (1-\zeta)^{n-1}\mathrm{d}\zeta \\
=\ & \int_\varrho^1 \left\|(f(\zeta,\cdot))^{[r]}\right\|_p \frac{(1-\zeta)^{n-1}}{\zeta^n}\mathrm{d}\zeta \\
=\ & \int_\varrho^1 M_p\,(\zeta, f, r)\,\frac{(1-\zeta)^{n-1}}{\zeta^n}\mathrm{d}\zeta \\
\leq\ & C_1\int_\varrho^1 \frac{\omega(1-\zeta)}{1-\zeta}\mathrm{d}\zeta = \mathcal{O}\left(\omega(1-\varrho)\right), \quad \varrho \to 1-. \qquad (3.38)
\end{aligned}
$$

Therefore, we can apply Lemma 3.8 to the function $f^{[r-n]}$. Taking into account (3.16), we obtain

$$f^{[r-n]}(\mathbf{x}) - A_{\varrho,n}(f^{[r-n]})(\mathbf{x}) = \frac{1}{(n-1)!}\int_{\varrho}^{1}(f(\zeta,\cdot))^{[r]}(\mathbf{x})\frac{(1-\zeta)^{n-1}}{\zeta^n}\mathrm{d}\zeta.$$

Using the integral Minkowski inequality and (3.38), we conclude

$$\begin{aligned} \|f^{[r-n]} - A_{\varrho,n}(f^{[r-n]})\|_p &\le \frac{1}{(n-1)!}\int_{\varrho}^{1} M_p\left(\zeta,f,r\right)\frac{(1-\zeta)^{n-1}}{\zeta^n}\mathrm{d}\zeta \\ &= \mathcal{O}\left(\omega(1-\varrho)\right), \quad \varrho\to 1-. \end{aligned} \tag{3.39}$$

Combining relations (3.37), (3.27) and (3.39), we finally get (3.2).

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Iterates of positive linear operators on Bauer simplices

Mădălina Dancs and Sever Hodiş

Dedicated to Professor Heiner Gonska on the occasion of his 70th anniversary.

Abstract. We consider positive linear operators acting on $C(K)$, where K is a metrizable Bauer simplex. For such an operator L we investigate the limit of the iterates L^m, when $m \to \infty$. Qualitative results and rates of convergence are obtained. The general results are illustrated by examples involving classical operators.

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1. Introduction

Iterates of positive linear operators were investigated in many papers and from several points of view. General criteria for their convergence can be found in [1], [2], [13], [14], [20], [21], [23]. Rates of convergence of the iterates were established in [6], [10], [16], [17], [18], [20], [21], [28]. The relationship with Korovkin theory is presented in [6], [7], [8], [22]. Iterates are essentially used for representing some strongly continuous semigroups of operators: see [7], [8], [17], [28]. Iterates for q-Bernstein operators are studied in [24]; the case of complex operators is considered in [11]. In the above papers analytical methods were used and also methods from probability theory. Results based on spectral theory can be found in [9]; fixed point theory is used in [3], [4], [30], [31], [32], [33].

This paper is devoted to the study of iterates of positive linear operators on Bauer simplices. General definitions and results are presented in this introduction; see also [5], [7], [8], [25].

Section 2 is devoted to the iterates of operators preserving affine functions. An example concerning a finite dimensional simplex is discussed in Section 3. Other examples are presented in Section 4.

Throughout the paper we shall use the following notions.

Let E be a real locally convex Hausdorff space and K a convex compact subset of E. Let $C(K)$ be the space of all continuous real-valued functions on K, endowed with the usual ordering and the supremum norm. By Hervé's theorem [5, Th.I.4.3], [7, p.57], $C(K)$ contains a strictly convex function if and only if K is metrizable. Throughout the paper we shall suppose that K is metrizable.

The set of all probability Radon measures on K will be denoted by $M_1^+(K)$. For each $x \in K$, ε_x stands for the probability Radon measure concentrated on $\{x\}$.

The *Choquet-Meyer* ordering $<$ on $M_1^+(K)$ is defined as follows: for every $\mu, \nu \in M_1^+(K)$, $\mu < \nu$ if $\mu(f) \leq \nu(f)$ for every convex function $f \in C(K)$. A measure μ which is maximal with respect to $<$ will be simply called a *maximal measure*.

Let $A(K)$ be the set of all affine functions for all $h \in C(K)$. The *barycenter* of $\mu \in M_1^+(K)$ is the point $r \in K$ for which $\mu(h) = h(r),\ h \in A(K)$; in this case $\mu(f) \geq f(r)$ for each convex function $f \in C(K)$.

There are several equivalent properties defining a Choquet simplex. We need the following one:

K is called a *Choquet simplex* if for every $x \in K$ there exists a unique maximal measure $\mu_x \in M_1^+(K)$ having x as barycenter.

The set of the extreme points of K will be denoted by $ex(K)$.

A Choquet simplex K such that $ex(K)$ is closed will be called a *Bauer simplex*. In this case μ_x is supported by $ex(K)$; moreover, if $\mu_x = \varepsilon_x$, then $x \in ex(K)$.

If K is a Bauer simplex, then the operator $P : C(K) \longrightarrow A(K)$ defined by

$$Pf(x) = \mu_x(f),\ f \in C(K),\ x \in K,$$

is linear, positive, and $Ph = h$ for all $h \in A(K)$.

P is called the *canonical projection* associated with the Bauer simplex K.

Let $L : C(K) \longrightarrow C(K)$ be a positive linear operator such that $Lh = h$, for every $h \in A(K)$. For each $x \in K$ let $\nu_x(f) := Lf(x),\ f \in C(K)$. Then $\nu_x \in M_+^1(K)$ and x is the barycenter of ν_x. In particular, if $x \in ex(K)$ then $\nu_x = \varepsilon_x$, so that

$$Lf(x) = f(x),\ x \in ex(K),\ f \in C(K). \tag{1.1}$$

Moreover, if $g \in C(K)$ is convex, then $\nu_x(g) \geq g(x),\ x \in K$, i.e.,

$$Lg \geq g. \tag{1.2}$$

We shall need the following result [26], [27], [7, Th.1.5.2].

Lemma 1.1. *Let $\mu \in M_1^+(K)$ with barycenter r and let u be a strictly convex function. If $\mu(u) = u(r)$, then $\mu = \varepsilon_r$.*

2. Iterates of positive linear operators preserving the affine functions

In the sequel, K will be a metrizable Bauer simplex.

Theorem 2.1. *Let $L : C(K) \longrightarrow C(K)$ be a positive linear operator such that $Lh = h$, $h \in A(K)$. Let $u \in C(K)$ be a strictly convex function. If*

$$\lim_{m\to\infty} L^m f = Pf,\ f \in C(K), \tag{2.1}$$

then

$$Lu(x) > u(x),\ x \in K \setminus ex(K). \tag{2.2}$$

Proof. Let $x \in K$. As in the preceding section, let $\nu_x(f) := Lf(x),\ f \in C(K)$. By (1.2), $Lu(x) \geq u(x)$. Suppose that $Lu(x) = u(x)$. Then $\nu_x(u) = u(x)$, and Lemma 1.1 yields $\nu_x = \varepsilon_x$, i.e., $Lf(x) = f(x),\ f \in C(K)$. By induction, $L^m f(x) = f(x)$, $f \in C(K)$. Now (2.1) shows that $Pf(x) = f(x),\ f \in C(K)$. This means that $\mu_x = \varepsilon_x$, which entails $x \in ex(K)$. □

For $K = [0,1]$ the above result was obtained in [29] and [12, Corollary 2].

We shall prove that the converse of Th. 2.1 is also true. Having applications in mind, let us consider a sequence of positive linear operators $L_n : C(K) \longrightarrow C(K)$ preserving the affine functions, i.e.,

$$L_n h = h,\ h \in A(K),\ n \geq 1. \tag{2.3}$$

Fix a strictly convex function $u \in C(K)$.
For $n \geq 1$ and $s \in (0, +\infty)$ define

$$a_n(s) := \max_K (Pu - u - ns(L_n u - u)). \tag{2.4}$$

For $x \in ex(K)$ we have $Pu(x) - u(x) = L_n u(x) - u(x) = 0$, so that $a_n(s) \geq 0$.

Lemma 2.2. *If* $ns \geq 1,\ m \geq 1$, *then*

$$0 \leq Pu - L_n^m u \leq a_n(s)\mathbf{1} + \left(1 - \frac{1}{ns}\right)^m (Pu - u), \tag{2.5}$$

where $\mathbf{1}$ *is the constant function of value* 1 *defined on* K.

Proof. Since P preserves the affine functions, we have $u \leq Pu$ by virtue of (1.2). Moreover, $Pu \in A(K)$, and so $L_n u \leq L_n(Pu) = Pu$. By induction we get $L_n^m u \leq Pu$, and this is the first inequality in (2.5).
From (2.4) we derive

$$a_n(s)\mathbf{1} \geq Pu - u - ns(L_n u - u).$$

This implies

$$\frac{1}{ns}(Pu - a_n(s)\mathbf{1}) + \left(1 - \frac{1}{ns}\right) u \leq L_n u.$$

Since $1 - \dfrac{1}{ns} \geq 0$, iterating over $m \geq 1$

$$\left(1 - \left(1 - \frac{1}{ns}\right)^m\right)(Pu - a_n(s)\mathbf{1}) + \left(1 - \frac{1}{ns}\right)^m u \leq L_n^m u.$$

This leads immediately to the second inequality in (2.5), and the lemma is proved. □

Lemma 2.3. *Let* $n \geq 1$ *be fixed, and suppose that for a given strictly convex function* $u \in C(K)$ *one has*

$$L_n u(x) > u(x),\ x \in K \setminus ex(K). \tag{2.6}$$

Then $\lim\limits_{s \to \infty} a_n(s) = 0$.

Proof. Since $a_n \geq 0$ and a_n is decreasing on $(0, +\infty)$, we have $l := \lim\limits_{s\to\infty} a_n(s) \geq 0$. Suppose that $l > 0$. Let

$$A_s := \{x \in K : \ Pu(x) - u(x) - ns(L_n u(x) - u(x)) \geq l\}.$$

The sets A_s are closed and the family $(A_s)_{s>0}$ is descending. For each $s > 0$, A_s and $ex(K)$ are disjoint, so that (2.6) implies $\bigcap\limits_{s>0} A_s = \emptyset$. Since K is compact, there exists $t > 0$ such that $A_t = \emptyset$. Then $a_n(t) < l$, a contradiction. □

Theorem 2.4. *(i) Let* $0 < c < 1$. *Then*

$$0 \leq Pu - L_n^m u \leq a_n(m^c)\mathbf{1} + \left(1 - \frac{1}{nm^c}\right)^m (Pu - u), \tag{2.7}$$

for all $m, n \geq 1$.
(ii) If (2.6) holds for a given $n \geq 1$, *then*

$$\lim_{m\to\infty} L_n^m f = Pf, \ \ f \in C(K). \tag{2.8}$$

Proof. (i) is a consequence of (2.5), with $s = m^c$. From (2.7) and Lemma 2.2 we infer that $\lim\limits_{m\to\infty} L_n^m u = Pu$. This fact, combined with Corollary 3.3.4 of [7], leads to (2.8). □

In the sequel we shall suppose that the limit

$$T(t)f := \lim_{n\to\infty} L_n^{k(n)} f$$

exists in $C(K)$ for each $f \in C(K)$, each $t \geq 0$, and each sequence of positive integers $(k(n))_{n\geq 1}$ such that $\lim\limits_{n\to\infty} \dfrac{k(n)}{n} = t$.

Denote $a(s) = \sup\{a_n(s) : n \geq 1\}, \ s > 0$.

Theorem 2.5. *(i) Let* $0 < c < 1$. *Then for all* $t > 0$,

$$0 \leq Pu - T(t)u \leq a(t^c)\mathbf{1} + (Pu - u)\exp(-t^{1-c}). \tag{2.9}$$

(ii) If $\lim\limits_{s\to\infty} a(s) = 0$, *then*

$$\lim_{t\to\infty} T(t)f = Pf, \ \ f \in C(K). \tag{2.10}$$

Proof. Let $t > 0$ be fixed. If $nt^c \geq 1$, from (2.5) we get

$$0 \leq Pu - L_n^{k(n)} u \leq a(t^c)\mathbf{1} + \left(1 - \frac{1}{nt^c}\right)^{k(n)} (Pu - u).$$

Choosing $k(n)$ such that $\lim\limits_{n\to\infty} \dfrac{k(n)}{n} = t$, we obtain (2.9).
If $\lim\limits_{s\to\infty} a(s) = 0$, (2.9) yields

$$\lim_{t\to\infty} T(t)u = Pu.$$

Another application of [7, Cor. 3.3.4] concludes the proof. □

3. An example and a quantitative result

Let K be the canonical simplex of $\mathbb{R}^d$, that is

$$K = \{x \in \mathbb{R}^d : x_1, \dots x_d \geq 0,\ x_1 + \dots + x_d \leq 1\}.$$

The canonical projection associated with K is defined, for every $f \in C(K)$ and $x \in K$, by

$$Pf(x) = (1 - x_1 - \dots - x_d)f(0) + x_1 f(v_1) + \dots + x_d f(v_d), \tag{3.1}$$

where $0,\ v_1 := (1, 0, \dots, 0), \dots, v_d := (0, \dots, 0, 1)$ are the vertices of K.
Let $f \in C(K)$; suppose that there exists a constant $Q_f > 0$ such that for all $x \in K$,

$$|f(x) - f(0)| \leq Q_f \sum_{i=1}^{d} x_i, \tag{3.2}$$

$$|f(x) - f(v_j)| \leq Q_f \left(1 - 2x_j + \sum_{i=1}^{d} x_i\right),\ j = 1, \dots, d. \tag{3.3}$$

Then, for $x \in K$,

$$\begin{aligned}
|f(x) - Pf(x)| &= |f(x) - \left(1 - \sum_{i=1}^{d} x_i\right) f(0) - \sum_{i=1}^{d} x_i f(v_i)| \\
&= \left| \left(1 - \sum_{i=1}^{d} x_i\right)(f(x) - f(0)) + \sum_{i=1}^{d} x_i (f(x) - f(v_i)) \right| \\
&\leq Q_f \left(\left(1 - \sum_{i=1}^{d} x_i\right) \sum_{i=1}^{d} x_i + \sum_{i=1}^{d} x_i \left(1 - 2x_i + \sum_{j=1}^{d} x_j\right)\right) \\
&= 2Q_f \left(\sum_{i=1}^{d} x_i - \sum_{i=1}^{d} x_i^2 \right).
\end{aligned}$$

Consider the strictly convex function $u \in C(K),\ u(x) = \sum_{i=1}^{d} x_i^2,\ x \in K$. Then

$$Pu(x) = \sum_{i=1}^{d} x_i,$$

so that for the above function f we have

$$|f(x) - Pf(x)| \leq 2Q_f(Pu(x) - u(x)),\ x \in K. \tag{3.4}$$

Let $L_n : C(K) \longrightarrow C(K)$ be a positive linear operator preserving affine functions. From (3.4) we get

$$|L_n^m f - Pf| \leq 2Q_f(Pu - L_n^m u). \tag{3.5}$$

Finally, (3.5) and (2.7) yield

$$|L_n^m f - Pf| \leq 2Q_f \left(a_n(m^c)\mathbf{1} + \left(1 - \frac{1}{nm^c}\right)^m (Pu - u)\right),$$

i.e.,

$$|L_n^m f(x) - Pf(x)| \le 2Q_f \left[a_n(m^c) + \left(1 - \frac{1}{nm^c}\right)^m \sum_{i=1}^d x_i(1-x_i) \right]. \tag{3.6}$$

Moreover, in the context of Theorem 2.3 we derive from (3.4):

$$|T(t)f - Pf| \le 2Q_f(Pu - T(t)u). \tag{3.7}$$

Combined with (2.9), this gives

$$|T(t)f(x) - Pf(x)| \le 2Q_f \left[a(t^c) + (\exp(-t^{1-c})) \sum_{i=1}^d x_i(1-x_i) \right]. \tag{3.8}$$

Remark 3.1. If $f \in C^1(K)$, i.e., f has continuous partial derivatives on the interior of K which can be continuously extended on K, then (3.2) and (3.3) are satisfied with

$$Q_f := \max \left\{ \left\| \frac{\partial f}{\partial x_i} \right\|_\infty \; : \; i = 1, \dots, d \right\}.$$

4. Examples

In this section we present examples of sequences $(L_n)_{n\ge1}$ of operators preserving affine functions and satisfying the fundamental condition (2.6).

Example 4.1. Let $B_n,\ n \ge 1$, be the *Bernstein-Schnabl* operators associated with the canonical projection P and the arithmetic mean *Toeplitz* matrix (see [7, p. 381]). Let $u \in C(K)$ be a strictly convex function. Suppose that for a given $n \ge 1$ and a given $x \in K$ one has $B_n u(x) = u(x)$. According to *Lemma 1.1*, we infer that $B_n f(x) = f(x)$, for every $f \in C(K)$. In particular, $B_n h^2(x) = h^2(x)$, for all $h \in A(K)$. Now [7, (6.1.16)] leads to $P(h^2)(x) = h^2(x),\ h \in A(K)$. From [7, Cor. 3.3.4 and Remark to Prop. 3.3.2] we deduce that $x \in ex(K)$. So (2.6) is satisfied for the operators B_n.

Example 4.2. Let $U_n,\ n \ge 1$, be the genuine *Bernstein-Durrmeyer* operators on a simples K in $\mathbb{R}^d$ (see [34], [19], [35]). If $u \in C(K)$ is strictly convex, then $U_n u \ge B_n u \ge u$ [19, Th.8]. If $U_n u(x) = u(x)$, then $B_n u(x) = u(x)$, and from Ex. 4.1 we know that $x \in ex(K)$. So (2.6) is satisfied for the operators U_n.

Example 4.3. It was proved in [28, Example 2.4] that (2.6) is satisfied for the classical Meyer-König and Zeller operators on $C[0,1]$.

Example 4.4. The case of the Bernstein-Schnabl operators on the unit interval, associated with a continuous selection of probability Borel measures on $[0,1]$, is considered in [28, Example 3.3]. The operators satisfy (2.6).

For all the operators presented in the above examples one can apply Lemma 2.2 and, consequently, one can obtain the corresponding quantitative results derived from Theorems 2.2 and 2.3.

Other examples and quantitative results can be found in [18], [28], [29].

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Choquet boundary for some subspaces of continuous functions

Laura Hodiş and Alexandra Măduţa

Dedicated to Professor Heiner Gonska on the occasion of his 70th anniversary.

Abstract. We investigate the Choquet boundary for subspaces of parabolic functions and for linearly separating subspaces of continuous functions. The relation of the Choquet boundary with the set of peak points is also investigated.

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Keywords: Choquet boundary, parabolic functions, linearly separating subspaces, peak points.

1. Introduction

It is well known that Choquet Theory provides a unified approach to integral representations in several areas of mathematics: potential theory, probability, function algebras, operator theory, group representations, ergodic theory (see, e.g., [1-3, 10, 14]). Particularly, the Choquet boundary is an essential tool in Korovkin approximation theory (see, e.g., [2, 3, 5, 9, 11, 12]).

In this paper we are concerned with the Choquet boundary for subspaces of parabolic functions and linearly separating subspaces of continuous functions. For other results concerning boundaries, parabolic functions, linearly separating subspaces see, e.g., [1-4, 6, 7, 10, 12, 14] and the references therein.

Section 2 is devoted to subspaces of parabolic functions. We recall some known results about the Choquet boundary of such a subspace, motivated by their relations with Korovkin theory. The relation between the Choquet boundary and the set of peak points is also investigated.

In Section 3 we study the Choquet boundary for linearly separating subspaces. Important results in this direction were obtained in [6,13], and the references therein. Our main result is Theorem 3.1. We start with Proposition 48 from [13] and add a supplementary hypothesis; then we construct an example showing that without this hypothesis the conclusion in not generally true.

Throughout the paper we use the following definitions and notations.

For other definitions and notations see, e.g., [1-3], [10].

For a compact Hausdorff space X, let $C(X)$ denote the Banach space of real-valued continuous functions on X, equipped with the supremum norm. Let S be a subset of $C(X)$.

A subset B of X is called a boundary for S if for each $f \in S$ there exists $b \in B$ such that $f(b) = \min_X f$.

Let $M(X)$ be the space of all Radon measures on X and $M_+(X)$ the set of all Radon positive measures on X. Let

$$M_+^1(X) = \{\mu \in M_+(X) : \mu(1) = 1\}.$$

For x in X let e_x be the corresponding Dirac measure on X.
If S is a subset of $C(X)$ and μ, γ are in $M^+(X)$, we write $\mu \prec_S \gamma$ (or, simply, $\mu \prec \gamma$) if $\mu(s) \leq \gamma(s)$ for all s in S.
The Choquet boundary of X with respect to S is the set

$$Ch(S) = \{x \in X : \mu \in M_+^1(X), \mu \prec e_x \Rightarrow \mu = e_x\}.$$

If S separates the points of X, then $Ch(S)$ is a boundary for S (see [4]).
Let us consider the set of peak points with respect to S (see [1, p. 39]):
$P(S) = \{x \in X : \exists f \in S, f(x) < f(y) \quad \text{for all} \quad y \in X \setminus \{x\}\}$.
It is easily seen that $P(S) \subset Ch(S)$.
If S is a linear subspace of $C(X)$, then

$$Ch(S) = \{x \in X : \mu \in M_+^1(X), \mu_{|S} = e_{x|S} \Rightarrow \mu = e_x\}.$$

2. The Choquet boundary for subspaces of parabolic functions

Let E be a locally convex Hausdorff space over $\mathbb{R}$, and K a compact metrizable convex subset of E. We denote by $A(K)$ the set of all continuous real-valued affine functions on K and by $\mathrm{ex}K$ the set of all extreme points of K.

Theorem 2.1. ([1, Proposition I.4.3]) *The Choquet boundary of the subspace $A(K)$ coincides with* $\mathrm{ex}K$.

We shall see that the Choquet boundary of the linear subspace of $C(K)$ generated by $A(K)$ and $f \in C(K)$, coincides with K.

Let $f \in C(K)$ be convex. Then, it is known that f has a right Gateaux derivative, given by

$$Df(x; y) = \lim_{t \downarrow 0} \frac{f(x+ty) - f(x)}{t} = \inf_{t>0} \frac{f(x+ty) - f(x)}{t}$$

for all x, y such that $x \in K, x + y \in K$.

We will say that f is *smooth* provided that for all $x \in K$ the mapping

$$a_x : K \to \mathbb{R}, \quad a_x(y) = Df(x; y - x) \quad \text{is in} \quad A(K).$$

Now let $f \in C(K)$ be strictly convex. Note that such a function exists since K is metrizable (see [8]).

Let $S(f)$ be the subspace of $C(K)$ spanned by $A(K)$ and f. The functions belonging to $S(f)$ are called *parabolic functions*. These subspaces were studied by C.A. Micchelli [9]. In particular, in [9, Proposition 3.1] he proved that, under the assumption that f is strictly convex and smooth, then $e_x \in U(S(f))$ for all $x \in K$, where

$$U(S(f)) = \{\mu \in M_+^1(K) : \gamma \in M_+^1(K), \gamma_{|_{S(f)}} = \mu_{|_{S(f)}} \Rightarrow \gamma = \mu\}.$$

This implies

Theorem 2.2. *If $f \in C(K)$ is strictly convex and smooth, then $Ch(S(f)) = K$.*

In [11, Proposition 2] it was shown that the results due to C.A. Micchelli remain true if we omit the hypothesis that f is smooth. Then we get the result.

Theorem 2.3. *If $f \in C(K)$ is strictly convex, then $Ch(S(f)) = K$.*

From this it follows that if $f \in C(K)$ is strictly convex, then the subspace of parabolic functions $S(f)$ is a Korovkin subspace of $C(K)$. This result was proved in [5] in the case when K is a compact convex subset of $\mathbb{R}^n$ and in [9] in the general case under the hypothesis that f is smooth.

As far as the peak point set of $S(f)$ is concerned, we state the following result.

Theorem 2.4. *Let K be metrizable, and $f \in C(K)$ be strictly convex and smooth. Then $P(S(f)) = K$.*

Proof. Let $x \in K$ and consider the function

$$s : K \to \mathbb{R}, s(y) = f(y) - f(x) - a_x(y) \quad \text{for all} \quad y \in K.$$

Then $s \in S(f), s(x) = 0, s(y) > 0$ for all $y \in K \setminus \{x\}$. Thus $x \in P(S(f))$. $\square$

Remark 2.5. If $f \in C(K)$ is strictly convex but is not smooth, it is possible to have $P(S(f)) \neq K$.

This is shown in:

Example 2.6. Let $K = [-1, 1] \times [-1, 1]$ and let

$$f : K \to \mathbb{R}, \quad f(x, y) = x^2 + y^2 - \sqrt{1 - y^2} \quad \text{for all} \quad (x, y) \in K.$$

Then f is strictly convex on K. By Theorem 2.3 we have $Ch(S(f)) = K$.
But $P(S(f)) = K \setminus \{(x, \pm 1) : |x| < 1\}$.

3. The Choquet boundary for linearly separating subspaces of $C(X)$

Let H be a linear subspace of $C(X)$ which separates the points of X. H^* denotes the dual of H, equipped with the weak *- topology.
Let us consider the map

$$\Phi : X \to H^*, \Phi(x)(h) = h(x) \quad \text{for all} \quad x \in X, \quad h \in H.$$

Φ is easily seen to be a homeomorphism between X and $\Phi(X)$. Now set

$$Y = \overline{co}(\Phi(X)).$$

Then Y is a compact convex subset of H^*. We have (see [10,13])

$$\mathrm{ex}Y = \Phi(Ch(H)). \tag{3.1}$$

Let us denote

$$H^+ = \{h \in H : h \geq 0\}.$$
$$(H^*)^+ = \{h^* \in H^* : h^*(h) \geq 0 \quad \text{for all} \quad h \in H^+\}.$$

Proposition 3.1. (see, [Prop. 46, 13]). *Let us consider the following assertions:*
(1) $H = H^+ - H^+$,
(2) $(H^*)^+ \cap (-(H^*)^+) = \{0\}$,
(3) $0 \notin \Phi(X)$,
(4) $0 \notin Y$,
(5) *For all* $x \in X$ *there exists* $h \in H$ *such that* $h(x) \neq 0$,
(6) *There exists* $h_0 \in H$ *such that* $h_0 > 0$,
(7) $(H^*)^+$ *has a compact base.*
Then we have:
(3) $\Longleftrightarrow$ (5) $\Longleftarrow$ (4) $\Longleftrightarrow$ (6) $\Longrightarrow$ (1) $\Longrightarrow$ (2), (6) $\Longrightarrow$ (7), (2) *and* (5) $\Longleftrightarrow$ (6).

Let F be a subset of $C(X)$ and set

$$\partial(F) = \{x \in X : \mu \in M_+(X), \mu \prec_F e_x \Longrightarrow \mu = e_x\}.$$

We need the following lemma.

Lemma 3.2. *If* F *is a subset of* $C(X)$, *the following properties hold:*

(i) $\partial(F) = \partial(gF)$ *for all* $g \in C(X), g > 0$.

(ii) Suppose that there exists $f_0 \in F, f_0 > 0$ *and for all* $x \in X$ *there exists* $f \in F$ *such that* $f(x) < 0$.

Then $\partial(F) = \cap\{Ch(gF) : g \in C(X), g > 0\}$.

(iii) If there exists $f_0 \in F$ *such that* $f_0 > 0$ *and* $-f_0 \in F$, *then* $\partial(F) = Ch\left(\frac{F}{f_0}\right)$.

Proof. (i) Fix $g \in C(X), g > 0$ and $y \in \partial(F)$. Let $\mu \in M_+(X)$ be such that $\mu \prec_{gF} e_y$. Then

$$\int_X f(x)\frac{g(x)}{g(y)}d\mu(x) \leq f(y) \quad \text{for all} \quad f \in F. \tag{3.2}$$

Let us define $\gamma \in M_+(X)$ by

$$d\gamma(x) = \frac{g(x)}{g(y)}d\mu(x).$$

From (3.2) it follows that $\gamma \prec_F e_y$, and hence

$$\gamma = e_y. \tag{3.3}$$

Let now $t \in C(X)$, and set

$$h = \frac{g(y)}{g}t.$$

From (3.3) we obtain $\gamma(h) = h(y)$, i.e. $\mu(t) = t(y)$ for all $t \in C(X)$. Thus $\mu = e_y$. This means that y belongs to $\partial(gF)$. So we have $\partial(F) \subset \partial(gF)$.
Now $\partial(gF) \subset \partial\left(\frac{1}{g}gF\right) = \partial(F)$, i.e. $\partial(F) = \partial(gF)$.

(ii) We have $\partial(F) = \partial(gF) \subset Ch(gF)$ for all $g \in C(X),\ g > 0$. This yields

$$\partial(F) \subset \cap\{Ch(gF) : g \in C(X), g > 0\}.$$

Let now $x \in X$, $x \notin \partial(F)$. Then there exists $\mu \in M_+(X)$ such that

$$\mu \prec_F e_x, \tag{3.4}$$

$$\mu \neq e_x. \tag{3.5}$$

From hypothesis there exists $f \in F$ such that $f(x) < 0$.
Then (3.4) implies $\mu(f) \leq f(x) < 0$. So

$$\mu \neq 0. \tag{3.6}$$

From (3.5) and (3.6) we deduce that there exists a compact subset K of X such that $x \notin K$,

$$\mu(K) > 0. \tag{3.7}$$

From Urysohn's Lemma we see that there exists a continuous function $k : X \longrightarrow [0,1]$ such that

$$k(x) = 0, \quad k|_K = 1. \tag{3.8}$$

Then (3.7) and (3.8) imply

$$\mu(k) > 0.$$

From hypothesis there exists $f_0 \in F$ such that $f_0 > 0$. By (3.4) we have

$$\mu(f_0) \leq f_0(x).$$

Let us consider the function $v \in C(X)$ given by

$$v = f_0 + \frac{f_0(x) - \mu(f_0)}{\mu(k)} k.$$

We have

$$v > 0, \quad \mu(v) = v(x). \tag{3.9}$$

Now we define $\gamma \in M_+(X)$ by

$$d\gamma(y) = \frac{v(y)}{v(x)} d\mu(y). \tag{3.10}$$

From (3.9) we deduce $\gamma(1) = 1$; thus

$$\gamma \in M_+^1(X). \tag{3.11}$$

Let f be arbitrarily chosen in F. Then we have

$$\gamma\left(\frac{1}{v}f\right) = \frac{1}{v(x)}\mu(f) \leq \frac{1}{v(x)} f(x).$$

This implies

$$\gamma \prec_{\frac{1}{v}F} e_x. \tag{3.12}$$

Suppose now that $\gamma = e_x$, i.e. $\gamma(g) = g(x)$ for all $g \in C(X)$. Let t be arbitrarily chosen in $C(X)$. Let us denote

$$g = \frac{v(x)}{v} t.$$

From $\gamma(g) = g(x)$ and from (3.10) we deduce $\mu(t) = t(x)$. This means that $\mu = e_x$, which contradicts (3.5). Thus we have

$$\gamma \neq e_x. \tag{3.13}$$

Now (3.11), (3.12) and (3.13) imply

$$x \notin Ch\left(\frac{1}{v}F\right).$$

So $x \notin \cap\{Ch(gF) : g \in C(X), g > 0\}$. This completes the proof of (ii).
(iii) Let $f_0 \in F$ be such that $f_0 > 0$ and $-f_0 \in F$. Then the constant functions 1 and -1 belong to $\frac{F}{f_0}$; hence $\partial\left(\frac{F}{f_0}\right) = Ch\left(\frac{F}{f_0}\right)$. From (i) we deduce $\partial(F) = \partial\left(\frac{F}{f_0}\right)$ and so $\partial(F) = Ch\left(\frac{F}{f_0}\right)$.

Thus Lemma 3.1 is completely proved. □

In what follows we need the following definition. A subset F of $C(X)$ is called *linearly separating* (see [6, p. 55]) if for all $x, y \in X, x \neq y$ there exist $f, g \in F$ such that

$$\begin{vmatrix} f(x) & f(y) \\ g(x) & g(y) \end{vmatrix} \neq 0.$$

Remark 3.3. It is easily seen that F is linearly separating if and only if for all $x, y \in X, x \neq y$, and for all $c \in \mathbb{R}$ there exists $f \in F$ such that $f(x) \neq cf(y)$ (see [13]).

Remark 3.4. If F separates the points of X and $f + 1$ belongs to F for all $f \in F$, then F is linearly separating.

Remark 3.5. If F is linearly separating and $h \in C(X), h(x) \neq 0$ for all $x \in X$, then the set hF is linearly separating.

Remark 3.6. A linear subspace H of $C(X)$ is linearly separating if and only if for all $x, y \in X, x \neq y$, we have $\Phi(x) \neq 0$ and $\Phi(y)$ does not belong to the line generated in H^* by 0 and $\Phi(x)$.

Remark 3.7. Let H be a linear subspace of $C[0, 1]$, $\dim H = 2$. Then H is linearly separating if and only if H is a Tchebycheff subspace. If H is linearly separating, then there exists $h_0 \in H$ such that $h_0 > 0$.

The following result is essentially contained in [Proposition 48, 13]. Here we introduce at 3^0 the additional hypothesis that there exists $h_0 \in H$, $h_0 > 0$. We shall construct an example in which, without this hypothesis, 3^0 does not hold, that is

$$\emptyset = \partial(H) \subsetneqq \cap\{Ch(fH) : f \in C(X), f > 0\} \subsetneqq Ch(H).$$

Theorem 3.8. *Let H be a linear subspace of $C(X)$. Then:*
1^0 *$\partial(H) \subset Ch(H)$. If H contains the constant functions, then $\partial(H) = Ch(H)$.*
2^0 *$\partial(H) = \partial(fH)$ for all $f \in C(X), f > 0$.*
3^0 *If there exists $h_0 \in H, h_0 > 0$, then*

$$\partial(H) = \cap\{Ch(fH) : f \in C(X), f > 0\} = Ch\left(\frac{H}{h_0}\right).$$

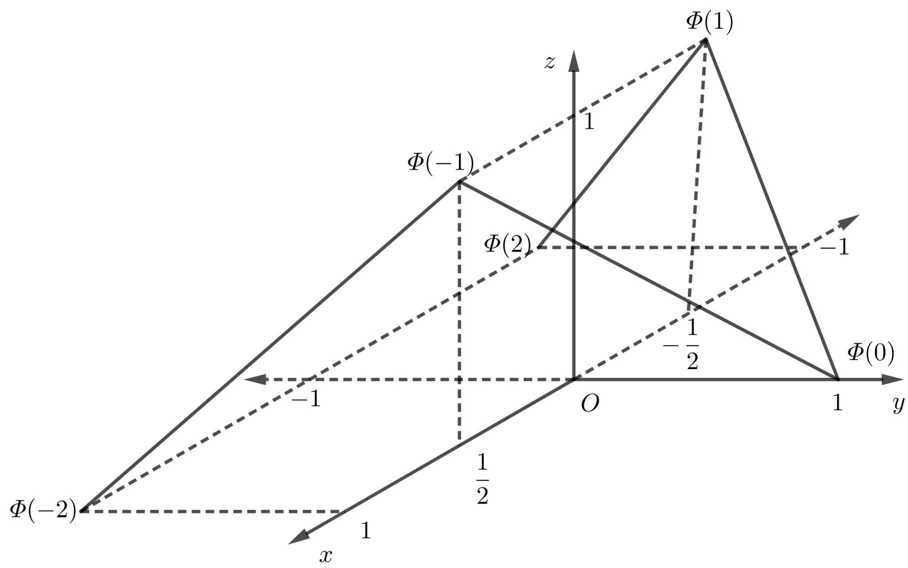
4^0 *If H is linearly separating, then the following assertions are equivalent:*
a) $\partial(H) \neq \emptyset$,
b) there exists $h_0 \in H, h_0 > 0$,
c) $H = H^+ - H^+$,
d) $(H^*)^+ \cap (-(H^*)^+) = \{0\}$.

Proof. 1^0 is obvious.
2^0 and 3^0 follow from Lemma 3.1.
4^0 is a consequence of Proposition 3.1. □

Example 3.9. Let $X = [-2, 2]$. Consider the functions h_1, h_2, h_3, belonging to $C[-2, 2]$ and defined for all $x \in [-2, 2]$ as

$$h_1(x) = -\frac{1}{2}x, \quad h_2(x) = 1 - |x|,$$
$$h_3(x) = \begin{cases} 1 - |x+1|, & \text{if} \quad x \in [-2, 0], \\ 1 - |x-1|, & \text{if} \quad x \in (0, 2]. \end{cases}$$

Let us denote by H the linear subspace of $C[-2, 2]$ generated by h_1, h_2, h_3.
We identify the functional $\varphi \in H^*$ with the vector $(\varphi(h_1), \varphi(h_2), \varphi(h_3))$; so, we identify H^* with R^3.
$\Phi([-2, 2])$ is the following curve in R^3:



From Remark 3.6 we have that H is linearly separating. Since $0 \in Y = \overline{co}(\Phi([-2,2]))$, from Proposition 3.1 we deduce that H does not contain strictly positive functions (this fact can be easily proved directly).
By Theorem 3.1 we have

$$\partial(H) = \emptyset.$$

From (3.1) we deduce

$$Ch(H) = \{-2,-1,0,1,2\}.$$

Let us prove that

$$\cap\{Ch(fH) : f \in C[-2,2], f > 0\} = \{-2,0,2\}. \tag{3.14}$$

Let $t \in \{-2,0,2\}$ and $f \in C[-2,2], f > 0$. Let $\mu \in M_+^1([-2,2])$ be such that

$$\mu|_{fH} = e_t|_{fH}. \tag{3.15}$$

Then we have $\mu(fh_3) = f(t)h_3(t) = 0$. This yields

$$\text{supp}\mu \subset \{-2,0,2\}.$$

Hence there exist $a, b, c \in [0,1]$ such that

$$a + b + c = 1, \tag{3.16}$$

$$\mu = ae_{-2} + be_0 + ce_2.$$

From (3.15) and (3.16) we obtain

$$\begin{cases} f(-2)a - f(2)c & = f(t)h_1(t) \\ -f(-2)a + f(0)b - f(2)c & = f(t)h_2(t) \\ a + b + c & = 1. \end{cases}$$

It is easily seen that this system has a unique solution, and we deduce $\mu = e_t$. This means that $t \in Ch(fH)$. So we have

$$\{-2,-0,2\} \subset \cap\{Ch(fH) : f \in C[-2,2], f > 0\} \subset Ch(H) = \{-2,-1,0,1,2\}. \tag{3.17}$$

Let us consider now the functions $f_1, f_2 \in C[-2,2]$ defined by

$$f_1(x) = \begin{cases} \frac{1}{3}(1 + 2|x+1|), & x \in [-2,0], \\ 1, & x \in (0,2]. \end{cases}$$

$$f_2(x) = \begin{cases} 1, & x \in [-2,0], \\ \frac{1}{3}(1 + 2|x-1|), & x \in (0,2]. \end{cases}$$

It is easy to verify that

$$\frac{1}{3}(e_{-2} + e_0 + e_1)|_{f_1H} = e_{-1}|_{f_1H},$$

$$\frac{1}{3}(e_{-2} + e_0 + e_2)|_{f_2H} = e_1|_{f_2H}.$$

This means that $-1 \notin Ch(f_1H), 1 \notin Ch(f_2H)$. Hence -1 and 1 do not belong to $\cap\{Ch(fH) : f \in C[-2,2], f > 0\}$. From (3.17) we deduce (3.14).

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An elliptic Diophantine equation from the study of partitions

Dorin Andrica and George C. Ţurcaş

Abstract. We present the elliptic equation $X^3 + 2 = Y^2$ as the first in a sequence of Diophantine equations arising from some new results in the theory of partitions of multisets with equal sums. Two proofs for Theorem 2.3, showing that the only integer solutions to this equation are $(-1, 1)$ and $(-1, -1)$, are given.

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1. Introduction and motivation

For a positive integer $k \geq 2$ and an arbitrary positive integer n, in the papers [2] and [1] the authors introduced the sequence $(Q_k(n))_{n\geq 1}$,

$$Q_k(n) = \frac{1}{2\pi}\int_0^{2\pi}\prod_{s=1}^{n}(k - 2 + 2\cos st)\,\mathrm{d}t. \tag{1.1}$$

An enumerative formula for $Q_k(n)$ is given by the number of ordered partitions of $[n] = \{1, \ldots, n\}$ into k disjoint sets $A_1, \ldots, A_k$ with the property that $\sigma(A_1) = \sigma(A_k)$, where $\sigma(A)$ denotes the sum of all elements in A.

Clearly, $Q_k(n)$ is a monic polynomial of degree n in $k - 2$. Moreover, in the paper [2] is proved that

$$Q_k(n) = \sum_{d=0}^{n} N(d, n)(k - 2)^{n-d}, \tag{1.2}$$

where for each $d = 0, \ldots, n$, the coefficient $N(d, n)$ is the number of ordered partitions of $[n]$ into 3 subsets A, B, C such that $|B| = d$ and $\sigma(A) = \sigma(C)$, where $|B|$ is the cardinality of B.

Therefore, $Q_k(n)$ has non-negative integer coefficients, and each coefficient has a combinatorial meaning in terms of partitions of the set $[n]$. A simple direct computation of the

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integral (1.1) shows that for $n = 3, 4, 5, 6$ and $k \geq 2$, we have

$$\begin{aligned}
Q_k(3) &= (k-2)^3 + 2;\\
Q_k(4) &= (k-2)^4 + 4(k-2) + 2;\\
Q_k(5) &= (k-2)^5 + 8(k-2)^2 + 6(k-2);\\
Q_k(6) &= (k-2)^6 + 12(k-2)^3 + 16(k-2)^2 + 6(k-2).
\end{aligned}$$

The sequence $Q_k(3)$ is indexed as A084380 in OEIS [10], where it is mentioned that it does not contain any perfect squares, i.e. the elliptic equation $X^3 + 2 = Y^2$ has no solutions in positive integers. This is linked to a Catalan-type conjecture related to Pillai's equation $X^U - Y^V = m$, with $X, Y, U, V \geq 2$ integers. The conjecture states that for any given integer m, there are finitely many perfect powers whose difference is m (see [13], Conjecture 1.6). For $m = 2$, it was computationally checked that the only solution involving perfect powers smaller than 10^{18} is $2 = 3^3 - 5^2$. The number of such solutions is linked to A076427 in OEIS.

Motivated by the property that the sequence $Q_k(3)$ does not contain any perfect squares, in the papers [2] and [1], the authors suggested the following problems: study if the sequence $Q_k(n)$ contains any $n-1$ powers, where $n = 4, 5$ or 6. These are equivalent to the study of the following Diophantine equations:

$$\begin{aligned}
X^4 + 4X + 2 &= Y^3;\\
X^5 + 8X^2 + 6X &= Y^4;\\
X^6 + 12X^3 + 16X^2 + 6X &= Y^5.
\end{aligned}$$

Using effective methods for identifying integral points on curves, we will discuss these equations and variations of them in a following series of papers.

In Theorem 2.3 of the present paper we prove that the equation $X^3 + 2 = Y^2$ has only integer solutions $(-1, 1)$ and $(-1, -1)$. We give two proofs for this statement. In the first we use the fact that $\mathbb{Q}(\sqrt{2})$ has trivial class group, property that allows us to pass from factorisations of ideals to nice factorisations in the ring $\mathbb{Z}[\sqrt{2}]$. The second proof uses the geometry of the elliptic curve defining the equation.

2. The equation $Y^2 = X^3 + 2$

Although the family of Mordell equations $Y^2 = X^3 + D$, where $D \in \mathbb{Z} \setminus \{0\}$ (see [7]) was extensively studied, we were unable to find in the literature an explicit solution for the case $D = 2$. In this section, we give two different solutions to the problem of finding all integral x, y satisfying the aforementioned equation. In the first one we combine factorisations in the ring of integers of $\mathbb{Q}(\sqrt{2})$ with an elementary solution to a particular cubic Thue equation. Our second solution relies on the geometric structure of the elliptic curve defined by the given affine equation.

Before going further, let us make a few remarks about the finiteness of the set of integral points on various curves. For any bivariate polynomial $f \in \mathbb{Z}[X, Y]$, let $C_f := \{(x, y) \in \overline{\mathbb{Q}}^2 : f(x, y) = 0\}$ be an affine algebraic curve. The points of C_f with coordinates in $\mathbb{Q}$ are called rational and, in general, for any $S \subseteq \overline{\mathbb{Q}}$, we denote by $C_f(S) = C_f \cap S^2$. Curves can be classified by their genus, a non-negative integer associated to their projectivization. The genus is a geometric invariant. A classical result in number theory is the following theorem

Theorem 2.1 (Siegel, 1929). *If $f \in \mathbb{Z}[X, Y]$ defines an irreducible curve C_f of genus $g(C_f) > 0$, then $C_f(\mathbb{Z})$ is finite.*

If additionally $g_f(C_f) \geq 2$, this result is superseded by the notorious Falting's theorem, which says that $C_f(\mathbb{Q})$ is also finite. Although both Siegels' and Faltings' theorems are milestones in number theory, they are "ineffective" results, meaning that their proof does not even allow one to control the size of the sets known to be finite. Therefore, they cannot be used to explicitly determine $C_f(\mathbb{Z})$ or $C_f(\mathbb{Q})$.

Effectively finding rational points on curves is an incredible difficult task and a very active topic of research. The toolbox for determining $C_f(\mathbb{Z})$ became a lot richer starting with the monumental work of Baker on linear forms in logarithms. As one of the first applications to his theory, Baker proved the following result.

Theorem 2.2 (Baker, 1969). *Suppose $f(X,Y) = Y^2 - a_nX^n - a_{n-1}X^{n-1} - \cdots - a_0 \in \mathbb{Z}[X,Y]$, the polynomial $a_nX^n + \cdots + a_0$ is irreducible in $\mathbb{Z}[X]$, $a_n \neq 0$ and $n \geq 5$. Let $H = \max\{|a_0|, \ldots, |a_n|\}$. Then, any integral point $(x,y) \in C_f(\mathbb{Z})$ satisfies*

$$\max(|x|,|y|) \leq \exp\exp\exp\{(n^{10n}H)^{n^2}\}.$$

Bounds on such solutions have been improved by many authors, but they remain astronomical and often involve inexplicit constants. Let us proceed to the resolution of our Diophantine equations.

To settle the conjecture posed by Andrica and Bagdasar in [2] and [1] which inferred that X^3+2 does not contain perfect squares when X runs through the set of positive integers, we prove the following theorem.

Theorem 2.3. *The only solutions of $X^3 + 2 = Y^2$ in the set of integer numbers are $(-1,1)$ and $(-1,-1)$.*

A few remarks are in order before giving the proof of this theorem. Since the genus of (the projectivization of) the curve determined by this equation is 1, we can use Siegel's theorem to deduce that there are finitely many points with integer coordinates. By Theorem 2.2, we know that if $(x,y) \in \mathbb{Z}^2$ is a point lying on this curve, then

$$\max(|x|,|y|) \leq \exp\exp\exp((3^{30} \cdot 2)^{3^2}).$$

Although theoretically one could now run a for loop through all possible values of x and check for which $x^3 + 2$ is a perfect square, the triple exponential bound presented above is astronomical and way out of the current computational limitations. In practice, one could check values of x up to 10^{18}, but could not hope to even get close to the aforementioned triple exponential. We proceed with the first proof of for our theorem.

3. Proof to Theorem 2.3

We will make use of the following proposition.

Proposition 3.1. *The only solution $(a,b) \in \mathbb{Z}^2$ to the equation*

$$a^3 + 3a^2b + 6ab^2 + 2b^3 = 1 \tag{3.1}$$

is $(a,b) = (1,0)$.

Proof. Write $f(X) = X^3 + 3X^2 + 6X + 2 \in \mathbb{Q}[X]$. It is an irreducible polynomial and let $\theta \in \overline{\mathbb{Q}}$ be any root of f. Denote by $L = \mathbb{Q}(\theta)$, the number field obtained by adjoining θ to $\mathbb{Q}$ and write $\mathcal{O}_L$ for its ring of integers. L is a degree 3 extension over $\mathbb{Q}$ and has signature $(1,1)$. We are going to denote by $\sigma_1, \sigma_2, \sigma_3 : L \hookrightarrow \mathbb{C}$ its three different complex embeddings.

It can be checked that ring of integers $\mathcal{O}_L$ is $\mathbb{Z}[\theta, \theta^2]$ and, making use of Dirichelt's unit theorem, one can compute the group of units

$$\mathcal{O}_L^\times = \langle \pm 1 \rangle \cdot \langle -\theta^2 - 3\theta - 1 \rangle \cong (\mathbb{Z}/2\mathbb{Z}) \cdot \mathbb{Z}.$$

The element $\mu := -\theta^2 - 3\theta - 1$ is a fundamental unit, $\mathrm{Norm}_{L/\mathbb{Q}}(\mu) = 1$ and $\mathrm{Norm}_{L/\mathbb{Q}}(-1) = -1$. The equation (3.1) can be written as

$$\mathrm{Norm}_{L/\mathbb{Q}}(a - b \cdot \theta) = \prod_{i=1}^{3}(a - b \cdot \sigma_i(\theta)) = 1, \text{ where } a, b \in \mathbb{Z}.$$

The above implies that $a - b\theta$ is a unit of norm 1 in O_L, hence

$$a - b\theta = \mu^n \text{ for some } n \in \mathbb{Z}. \tag{3.2}$$

We are going to use p-adic analysis to solve this last equation. We first need a local field $\mathbb{Q}_p$ into which there are three distinct embeddings of L, equivalently a prime number p such that the polynomial $X^3 + 3X^2 + 6X + 2$ has three distinct roots in $\mathbb{Q}_p$. We find $p = 79$ to be such a prime and the distinct roots are

$$\begin{array}{ll} \theta_1 = 19 - 32 \cdot 79 \pmod{79^2} & \\ \theta_2 = 20 - 7 \cdot 79 \pmod{79^2} & \in \mathbb{Q}_{79}. \\ \theta_3 = 37 + 38 \cdot 79 \pmod{79^2} & \end{array}$$

The root θ of f is mapped to r_1, r_2 and r_3 respectively, under the embeddings of L into $\mathbb{Q}_{79}$. Under the same embeddings, the fundamental unit $\mu = -\theta^2 - 3\theta - 1$ maps to

$$\begin{array}{ll} \mu_1 = 55 - 37 \cdot 79 \pmod{79^2} & \\ \mu_2 = 13 - 21 \cdot 79 \pmod{79^2} & \in \mathbb{Q}_{79}. \\ \mu_3 = 20 - 22 \cdot 79 \pmod{79^2} & \end{array}$$

By embedding the equation (3.2) into $\mathbb{Q}_{79}$, we obtain that $a - b \cdot \theta_i = \mu_i^n$ and hence $a = \mu_i^n + b \cdot \theta_i$ for $i = 1, 2$ and 3. One obtains the equality

$$(\theta_3 - \theta_2) \cdot \mu_1^n + (\theta_1 - \theta_3) \cdot \mu_2^n + (\theta_2 - \theta_1) \cdot \mu_3^n = 0$$

and since $\mu_1\mu_2\mu_3 = \mathrm{Norm}(\mu) = 1$, we can rewrite this as

$$(\theta_3 - \theta_2) + (\theta_1 - \theta_3) \cdot (\mu_2^2\mu_3)^n + (\theta_2 - \theta_1) \cdot (\mu_2\mu_3^2)^n = 0. \tag{3.3}$$

Now $\mu_2^2\mu_3 \equiv 62 \pmod{79}$ and $\mu_2\mu_3^2 \equiv 65 \pmod{79}$. Since the left hand side of (3.3) must be equal to zero modulo 79, we can check that n is divisible by 13. Hence $n = 13 \cdot m$ for some $m \in \mathbb{Z}$.

We have that $(\mu_2^2\mu_3)^{13} \equiv 1 + 8 \cdot 79 \pmod{79^2}$ and $(\mu_2\mu_3^2)^{13} \equiv 1 + 36 \cdot 79 \pmod{79^2}$. We can now use Lemma 5.2 in [5] to expand

$$(\theta_3 - \theta_2) + (\theta_1 - \theta_3) \cdot (\mu_2^2\mu_3)^{13 \cdot m} + (\theta_2 - \theta_1) \cdot (\mu_2\mu_3^2)^{13 \cdot m} = \sum_{k=1}^{\infty} a_k \cdot m^k,$$

with $\lim_{k \to \infty} \|a_k\|_{79} = 0$ and it can be checked that $\|a_1\|_{79} = 79^{-1}$ and $\|a_k\|_{79} \leq 79^{-2}$ for every $k \geq 2$. Using Strassmann's theorem (see Theorem 4.1 in [5]), we obtain that the only value of m for which $\sum_{k=1}^{\infty} a_k \cdot m^k$ vanishes is $m = 0$.

This proves that $n = 0$ and replacing in (3.2) we obtain $(a, b) = (1, 0)$ is the only solution to the equation in the statement, as claimed. $\square$

Remark 3.2. We have used the computer algebra package **Sage** [12] for basic modular arithmetic computations. The equation (3.1) is a Thue equation. It was proved that the latter have finitely many solutions and algorithms that find all of them have been implemented in various computer algebra packages. One can consult [3] for a very efficient such algorithm. The known methods for solving general Thue equations are involved, making use of Baker's bounds for linear forms in complex and of complicated reduction methods such as the one in described in loc. cit. In the above proof, we made essential use of the fact that the right hand side of (3.1) is 1 and that the ring $\mathcal{O}_L$ has only one fundamental unit to apply p-adic analysis techniques successfully.

We now return to the proof of our theorem. Let $K = \mathbb{Q}(\sqrt{2})$ and denote by $\mathcal{O}_K = \mathbb{Z}[\sqrt{2}]$ its ring of integers. The later is a Dedekind domain, i.e. it is Noetherian, integrally closed in its field of fractions $\mathrm{Frac}(\mathcal{O}_K) = K$ and all its non-zero prime ideals are maximal. For any element $o \in \mathcal{O}_K$, we are going to denote by $(o) \subseteq O_K$ the principal ideal o generates.

Suppose that $x, y \in \mathbb{Z} \setminus \{0\}$ are such that $y^2 = x^3 + 2$. Therefore, in $\mathcal{O}_K$ we have the factorization $(y - \sqrt{2}) \cdot (y + \sqrt{2}) = x^3$ and the same holds for the ideals generated by these factors. It is known that ideals of $\mathcal{O}_K$ factor uniquely into prime ideals. Suppose the prime ideal $\mathfrak{p} \subset \mathcal{O}_K$ divides both of the non-zero ideals $(y - \sqrt{2})$ and $(y + \sqrt{2})$. Then, $\mathfrak{p}$ must divide the ideal generated by the difference $y + \sqrt{2} - y + \sqrt{2} = 2\sqrt{2} = \sqrt{2^3}$. As $(\sqrt{2}) \subset \mathcal{O}_K$ is the only prime ideal of $\mathcal{O}_K$ that lies above 2, we must have $\mathfrak{p} = (\sqrt{2})$. Hence, the ideals $(y - \sqrt{2})$ and $(y + \sqrt{2})$ are coprime outside of $(\sqrt{2})$. From the previous factorization, we deduce that for every prime ideal $\mathfrak{p} \neq (\sqrt{2})$, if $\mathfrak{p}$ divides $(y - \sqrt{2})$, then $\mathfrak{p}^3$ divides the same ideal.

To see what happens in the case $\mathfrak{p} = (\sqrt{2})$, let $\mu \in \mathrm{Gal}(K/\mathbb{Q})$ be the non-trivial $\mathbb{Q}$-automorphism of K. Given a rational prime p, $\mathrm{Gal}(K/\mathbb{Q})$ acts naturally on the ideals $\mathfrak{p}$ of $\mathcal{O}_K$ that lie above p. Write $\mathfrak{p}^\mu$ for the ideal obtained from $\mathfrak{p}$ by applying μ to every element in $\mathfrak{p}$. As $\mu(\sqrt{2}) = -\sqrt{2}$, we note that $(\sqrt{2})^\mu = (-\sqrt{2}) = (\sqrt{2})$, i.e. μ stabilises the ideal above 2. Notice that $(y - \sqrt{2})^\mu = (y + \sqrt{2})$, hence the powers of $(\sqrt{2})$ that divide the ideals $(y - \sqrt{2})$ and $(y + \sqrt{2})$ are equal. Since the product $(y - \sqrt{2}) \cdot (y + \sqrt{2})$ is a third power, we conclude that the power of $(\sqrt{2})$ dividing $(y - \sqrt{2})$ must be divisible by 3.

It is an easy exercise, using for example the Minkowski bound, to prove that the class group of K is trivial. In particular, this means that every ideal of $\mathcal{O}_K$ is principal. Considering the remarks above, we have

$$(y - \sqrt{2}) = (x_0)^3 = (x_0^3), \text{ as ideals, where } x_0 \in \mathcal{O}_K.$$

We deduce that $y - \sqrt{2}$ and x_0^3 are the same up to a unit in the ring $\mathcal{O}_K$, that is there exists a unit $u \in U(\mathcal{O}_K)$ such that $y - \sqrt{2} = u \cdot x_0^3$.

By Dirichlet unit's theorem we know that $U(\mathcal{O}_K)$ is isomorphic to $T \cdot \mathbb{Z}$, where T is the finite group formed by the roots of unity that lie in K. It is an easy exercise to verify that $U(\mathcal{O}_K) = \langle -1 \rangle \cdot \langle 1 - \sqrt{2} \rangle$, so $1 - \sqrt{2}$ is the fundamental unit of $\mathcal{O}_K$. Observing that every element $u \in U(U_K)$ can be written as $u = (1 - \sqrt{2})^i \cdot (u_0)^3$ where $i \in \{-1, 0, 1\}$ and $u_0 \in U(\mathcal{O}_K) \subseteq \mathcal{O}_K$, we derive that

$$y - \sqrt{2} = (1 - \sqrt{2})^i \cdot x_1^3,$$

for some $i \in \{-1, 0, 1\}$ and $x_1 \in \mathcal{O}_K$. The element x_1 is of the form $a + b\sqrt{2}$ for $a, b \in \mathbb{Z}$. For each choice of $i \in \{-1, 0, 1\}$, by equating the coefficients of $\sqrt{2}$ in the left and right hand side of the above equation, we obtain an equality of the form

$$f(a, b) = -1 \tag{3.4}$$

where $f \in \mathbb{Z}[x, y]$ is a homogeneous cubic polynomial. When f is reducible (3.4) can be easily solved using factorization in $\mathbb{Z}$. If this is not the case and f is irreducible, the equation (3.4) is a cubic Thue equation. It is known (see for example [3]) that the latter have finitely many integral solutions and routines for determining them have been implemented in various computer algebra packages. We will appeal to Proposition 3.1 to find the solutions of the latter type of equations that arise here.

Let us analyse each of the three cases.

Case 1. $i = -1 \Rightarrow y - \sqrt{2} = (1 - \sqrt{2})^{-1} \cdot (a + b\sqrt{2})^3$.

Hence,

$$y - \sqrt{2} = -a^3 - 6a^2b - 6ab^2 - 4b^3 + \sqrt{2}\left(-a^3 - 3a^2b - 6ab^2 - 2b^3\right).$$

Using that $1, \sqrt{2}$ are linearly independent over $\mathbb{Q}$, we obtain the following two equations:

$$y = -a^3 - 6a^2b - 6ab^2 - 4b^3 \tag{3.5}$$

and

$$1 = a^3 + 3a^2b + 6ab^2 + 2b^3. \tag{3.6}$$

The variable y is an indeterminate and every solution (a, b) to (3.6) will determine a value for y. From Proposition 3.1, we know that the only solution in integers to the last equation is $a = 1$ and $b = 0$. Substituting, we see that this corresponds to $y = -1$, which implies that $x = -1$.

Case 2. $i = 0 \Rightarrow y - \sqrt{2} = (a + b\sqrt{2})^3$.

Expanding the right hand side, we see that

$$y - \sqrt{2} = a^3 + 6ab^2 + \sqrt{2}\left(3a^2b + 2b^3\right)$$

and since $1, \sqrt{2}$ are linearly independent over $\mathbb{Q}$ we must have

$$-1 = b \cdot (3a^2 + 2b^2).$$

Trying $b = \pm 1$, we see that $3a^2 + 2 = \mp 1$ is not solvable. Hence this case does not give us any solutions.

Case 3. $i = 1 \Rightarrow y - \sqrt{2} = \left(1 - \sqrt{2}\right) \cdot (a + b\sqrt{2})^3$.

This gives us

$$y - \sqrt{2} = a^3 - 6a^2b + 6ab^2 - 4b^3 + \sqrt{2}\left(-a^3 + 3a^2b - 6ab^2 + 2b^3\right),$$

which implies that

$$1 = a^3 - 3a^2b + 6ab^2 - 2b^3.$$

By making the substitution $t := -b$ in the last equation we obtain the one discussed in **Case 1**. Therefore, using Proposition 3.1 once again we find $a = 1$, $b = 0$ and hence $y = 1$. Using that $y^2 = x^3 + 2$, we get that $x = -1$. The proof of our theorem is now complete.

In the proof above we made explicit use of the fact that $\mathbb{Q}(\sqrt{2})$ has trivial class group, information that allowed us to pass from factorisations of ideals to nice factorisations of elements in the ring $\mathbb{Z}[\sqrt{2}]$. In general, for $D \in \mathbb{Z}$ the ideal class group of $\mathbb{Q}(\sqrt{D})$ can be arbitrary large so our first strategy will not work for more general Mordell equations. The second proof of our theorem can be adapted to find all the integral solutions of $Y^2 = X^3 + D$ for any fixed $D \in \mathbb{Z}$.

The given problem is one of explicitly determining the integral points on the affine curve given by $Y^2 = X^3 + 2$. These can be found by exploiting its rich geometric structure, as presented below.

4. Alternate proof to Theorem 2.3

The geometry of the curve is better captured by its projectivization

$$E := Y^2Z = X^3 + 2Z^3 \in \mathbb{P}^2(\mathbb{C}), \tag{4.1}$$

a non-singular projective curve of genus 1, which contains the point $\mathcal{O} = [0 : 1 : 0] \in \mathbb{P}^2(\mathbb{Q})$, commonly called "the point at infinity". The point at infinity is the only one on the projective curve that does not naturally project on our chosen affine model. The set of complex points on E can be given an abelian group structure for which the distinguished point $\mathcal{O}$ acts as the identity element. The group law is given by chord-tangent formulas and therefore it is easy to see that $E(\mathbb{Q})$ is a subgroup of $E(\mathbb{C})$. By a famous theorem of Mordell, we know that $E(\mathbb{Q}) \cong T \times \mathbb{Z}^r$ (as abstract abelian groups) where T is a finite group, commonly called *the torsion subgroup* and r is a positive integer called the *rank*.

Using the Lutz-Nagell theorem (see Corollary 7.2 in [11]), it is easy to deduce that T is included in $\{\mathcal{O}, P, -P\}$, where $P = [-1 : 1 : 1]$ and $-P = [-1 : -1 : 1]$ are inverses of each other under the group law. Using the formulae for addition on the elliptic curve, we compute all the values of $2 \cdot P, \ldots, 12 \cdot P$ and observe that none of them is equal to the origin $\mathcal{O}$. For example, $5 \cdot P = [108305279/48846121 : 1226178094681/341385539669 : 1] \neq \mathcal{O}$, and the larger multiples of P involve denominators that are too big to fit in one line. In his seminal article [9], Mazur gave a classification of all the possible isomorphisms types for the torsion group of an elliptic curve defined over $\mathbb{Q}$. From there, we see that the order of any torsion point is at most 12 and therefore we can conclude that P has infinite order.

The non-torsion part of $E(\mathbb{Q})$ is in general extremely difficult to compute. Even computing the rank of a given elliptic curve is, in general, a notorious problem. The latter quantity features in the famous Birch and Swinnerton-Dyer conjecture, one of the Millenium Problems. There are implementations of algorithms that succeed most of the times in computing the rank and finding generators. By running one such, namely John Cremona's **mwrank** algorithm implemented in **Sage** [12], we prove unconditionally that $r = 1$ and P is the generator of $E(\mathbb{Q})$. Just to sum up,

$$E(\mathbb{Q}) = \langle P \rangle \cong \mathbb{Z},$$

so all the points with rational coordinates on the projective curve are of the form $k \cdot P$, for $k \in \mathbb{Z}$. By computing with the group law, one can observe that $2 \cdot P = [17/4 : -71/8 : 1]$ and $-2 \cdot P = [17/4 : 71/8 : 1]$. As $|k| \geq 2$, the experiments suggest that the coordinates of $k \cdot P$ have denominators that grow extremely fast. We should remark that we always set the last coordinate $Z = 1$, as we are interested in the image of these points on the affine curve.

Suspecting that P and $-P$ are the only points with integral affine coordinates, we will use the program **integral_points** implemented in **Sage** by Cremona to prove it. The algorithm behind **integral_points** is described in Section 8.7 of [6]. We will mention briefly that this algorithm relies on a deep generalisation of Baker's theorem due to David and Hirata-Köhno [8], which if applied to our setup proves that if $|k| > e^{100}$ then $k \cdot P$ does not have integral coordinates on our affine model. Additionally, the aforementioned algorithm includes a clever application of the **LLL** reduction algorithm to reduce the bound e^{100} to 13, in our case. After this reduction, the program tests which of $k \cdot P$ are integral, when $k \leq 13$. The **Sage** program **integral_points** requires as input our elliptic curve E and a list of generators for the Mordell-Weil group $E(\mathbb{Q})$. It returns as output all the points in $E(\mathbb{Z})$. We refer the reader to Section 8.7 of [6] for a deeper understanding of **integral_points** and of the **Sage** output below, which proves our theorem.

```
sage: E = EllipticCurve([0,2]);
sage: E
Elliptic Curve defined by y^2 = x^3 + 2 over Rational Field
sage: P = E(-1,1)
sage: E.integral_points(mw_base = [P], both_signs = True, verbose = True)
Using mw_basis  [(-1 : 1 : 1)]
e1,e2,e3:  0.629960524947437 - 1.09112363597172*I,
 0.629960524947437 + 1.09112363597172*I, -1.25992104989487
Minimal and maximal eigenvalues of height pairing matrix:
 0.754576903181227,0.754576903181227
x-coords of points on non-compact component with  -1 <=x<= 2
[-1]
starting search of remaining points using coefficient bound 4 and
 |x| bound 184648.204428771
x-coords of extra integral points:
[-1]
```

```
Total number of integral points: 2
[(-1 : -1 : 1), (-1 : 1 : 1)]
```

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Approximation with Riemann-Liouville fractional derivatives

George A. Anastassiou

Abstract. In this article we study quantitatively with rates the pointwise convergence of a sequence of positive sublinear operators to the unit operator over continuous functions. This takes place under low order smothness, less than one, of the approximated function and it is expressed via the left and right Riemann-Liouville fractional derivatives of it. The derived related inequalities in their right hand sides contain the moduli of continuity of these fractional derivatives and they are of Shisha-Mond type. We give applications to Bernstein Max-product operators and to positive sublinear comonotonic operators connecting them to Choquet integral.

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Keywords: Riemann-Liouville fractional derivative, positive sublinear operators, modulus of continuity, comonotonic operator, Choquet integral.

1. Introduction

In this paper among others we are motivated by the following results:

First by P.P. Korovkin [9], (1960), p. 14: Let $[a,b]$ be a closed interval in $\mathbb{R}$ and $(L_n)_{n\in\mathbb{N}}$ be a sequence of positive linear operators mapping $C([a,b])$ into itself. Suppose that $(L_n f)$ converges uniformly to f for the three test functions $f = 1, x, x^2$. Then $(L_n f)$ converges uniformly to f on $[a,b]$ for all functions $f \in C([a,b])$.

Let $f \in C([a,b])$ and $0 \leq h \leq b-a$. The first modulus of continuity of f at h is given by

$$\omega_1(f,h) = \sup_{\substack{x,y\in[a,b] \\ |x-y|\leq h}} |f(x) - f(y)|.$$

If $h > b-a$, then we define $\omega_1(f,h) = \omega_1(f,b-a)$.

Another motivation is the following:

By Shisha and Mond [12], (1968): Let $[a,b] \subset \mathbb{R}$ a closed interval. Let $\{L_n\}_{n\in\mathbb{N}}$ be a sequence of positive linear operators acting on $C([a,b])$ into itself. For $n = 1, ...,$ suppose $L_n(1)$ is bounded. Let $f \in C([a,b])$. Then for $n = 1, 2, ...,$ we have

$$\|L_n f - f\|_\infty \leq \|f\|_\infty \|L_n 1 - 1\|_\infty + \|L_n 1 + 1\|_\infty \omega_1(f, \mu_n),$$

where

$$\mu_n = \left\| L_n\left((t-x)^2\right)(x)\right\|_\infty^{\frac{1}{2}}$$

and $\|\cdot\|_\infty$ stands for the sup-norm over $[a,b]$.

One can easily see, for $n = 1, 2, ...$

$$\mu_n^2 \leq \left\|L_n\left(t^2; x\right) - x^2\right\|_\infty + 2c\|L_n(t;x) - x\|_\infty + c^2 \|L_n(1;x) - 1\|_\infty,$$

where $c = \max(|a|, |b|)$.

Thus, given the Korovkin assumptions, as $n \to \infty$, we get $\mu_n \to 0$, and $\|L_n f - f\|_\infty \to 0$ for any $f \in C([a,b])$. That is one derives the Korovkin conclusion in a quantitative way and with rates of convergence.

We continue this type as research here for positive sublinear operators over continuous functions with existing left and right Riemann-Liouville fractional derivatives of order less than one. We give applications.

Other motivations come from author's monographs [2], [3] and [4].

2. Main results

We mention

Definition 2.1. ([10, pp. 68, 89]) Let $x, x' \in [a,b]$, $f \in C([a,b])$. The Riemann-Liouville (R-L) fractional derivative of a function f of order q $(0 < q < 1)$ is defined as

$$D_x^q f(x') = \left\{ \begin{array}{ll} D_{x+}^q f(x'), & x' > x, \\ D_{x-}^q f(x'), & x' < x \end{array} \right\}$$

$$= \frac{1}{\Gamma(1-q)} \left\{ \begin{array}{ll} \frac{d}{dx'} \int_x^{x'} (x'-t)^{-q} f(t)\,dt, & x' > x, \\ -\frac{d}{dx'} \int_{x'}^{x} (t-x')^{-q} f(t)\,dt, & x' < x, \end{array} \right. \tag{2.1}$$

the left and right R-L fractional derivatives, respectively, where Γ is the gamma function.

We need

Lemma 2.2. ([1], [10], pp. 71, 75) *Let $x, x' \in [a,b]$, $f \in C([a,b])$, $0 < q < 1$. Assume that $D_{x+}^q(f(\cdot) - f(x)) \in C([x,b])$, $D_{x-}^q(f(\cdot) - f(x)) \in C([a,x])$, where x is fixed. Then*

$$f(x') - f(x) = \frac{1}{\Gamma(q)} \int_x^{x'} (x'-z)^{q-1} D_{x+}^q(f(z) - f(x))\,dz, \tag{2.2}$$

all $x < x' \leq b$, and

$$f(x') - f(x) = \frac{1}{\Gamma(q)} \int_{x'}^{x} (z-x')^{q-1} D_{x-}^q(f(z) - f(x))\,dz, \tag{2.3}$$

all $a \leq x' < x$.

We accept $0 \cdot \infty = 0$ and we notice that $D_{x+}^{q} 0 = D_{x-}^{q} 0 = 0$.
We need

Definition 2.3. Let $f \in C([a,b])$. The first modulus of continuity is given by

$$\omega_1(f,\delta) := \sup_{\substack{x,y \in [a,b] \\ |x-y| \leq \delta}} |f(x) - f(y)|, \quad \delta > 0. \tag{2.4}$$

We need

Definition 2.4. Denote by $D_x^q (f(\cdot) - f(x))$ any of $D_{x\pm}^q (f(\cdot) - f(x))$, and $\delta > 0$. We set

$$\omega_1 \left(D_x^q (f(\cdot) - f(x)), \delta \right) \tag{2.5}$$
$$:= \max \left\{ \omega_1 \left(D_{x+}^q (f(\cdot) - f(x)), \delta \right)_{[x,b]}, \omega_1 \left(D_{x-}^q (f(\cdot) - f(x)), \delta \right)_{[a,x]} \right\}.$$

We give

Theorem 2.5. *Here $f \in C([a,b])$, $0 < q < 1$, $\delta > 0$; $x, x' \in [a,b]$. Assume that $D_{x+}^q (f(\cdot) - f(x)) \in C([x,b])$, and $D_{x-}^q (f(\cdot) - f(x)) \in C([a,x])$, where x is fixed. Then*

$$|f(x') - f(x)| \leq \frac{\omega_1 \left(D_x^q (f(\cdot) - f(x)), \delta \right)}{\Gamma(q+1)} \left[|x' - x|^q + \frac{|x' - x|^{q+1}}{(q+1)\delta} \right], \tag{2.6}$$

$\forall \; x' \in [a,b]$.

Proof. Obviously $D_{x+}^q (f(x) - f(x)) = 0$. We estimate:
i) Case of $x < x' \leq b$:

$$|f(x') - f(x)| \leq \frac{1}{\Gamma(q)} \int_x^{x'} (x' - z)^{q-1} \left| D_{x+}^q (f(z) - f(x)) \right| dz$$

$$= \frac{1}{\Gamma(q)} \int_x^{x'} (x' - z)^{q-1} \left| D_{x+}^q (f(z) - f(x)) - D_{x+}^q (f(x) - f(x)) \right| dz \tag{2.7}$$

$$\overset{(\delta_1 > 0)}{\leq} \frac{1}{\Gamma(q)} \int_x^{x'} (x' - z)^{q-1} \omega_1 \left(D_{x+}^q (f(\cdot) - f(x)), \frac{\delta_1 (z - x)}{\delta_1} \right)_{[x,b]} dz$$

$$\leq \frac{1}{\Gamma(q)} \omega_1 \left(D_{x+}^q (f(\cdot) - f(x)), \delta_1 \right)_{[x,b]} \left(\int_x^{x'} (x' - z)^{q-1} \left(1 + \frac{z - x}{\delta_1} \right) dz \right)$$

$$= \frac{\omega_1 \left(D_{x+}^q (f(\cdot) - f(x)), \delta_1 \right)_{[x,b]}}{\Gamma(q)} \left[\frac{(x' - x)^q}{q} + \frac{1}{\delta_1} \int_x^{x'} (x' - z)^{q-1} (z - x)^{2-1} dz \right]$$

$$= \frac{\omega_1 \left(D_{x+}^q (f(\cdot) - f(x)), \delta_1 \right)_{[x,b]}}{\Gamma(q)} \left[\frac{(x' - x)^q}{q} + \frac{1}{\delta_1} \frac{\Gamma(q)\Gamma(2)}{\Gamma(q+2)} (x' - x)^{q+1} \right] \tag{2.8}$$

$$= \frac{\omega_1 \left(D_{x+}^q (f(\cdot) - f(x)), \delta_1 \right)_{[x,b]}}{\Gamma(q)} \left[\frac{(x' - x)^q}{q} + \frac{1}{\delta_1} \frac{\Gamma(q)}{\Gamma(q+2)} (x' - x)^{q+1} \right]$$

$$= \frac{\omega_1\left(D^q_{x+}\left(f\left(\cdot\right)-f\left(x\right)\right),\delta_1\right)_{[x,b]}}{\Gamma\left(q\right)}\left[\frac{\left(x'-x\right)^q}{q}+\frac{1}{\delta_1}\frac{\left(x'-x\right)^{q+1}}{q\left(q+1\right)}\right] \tag{2.9}$$

$$= \frac{\omega_1\left(D^q_{x+}\left(f\left(\cdot\right)-f\left(x\right)\right),\delta_1\right)_{[x,b]}}{\Gamma\left(q+1\right)}\left[\left(x'-x\right)^q+\frac{1}{\delta_1}\frac{\left(x'-x\right)^{q+1}}{\left(q+1\right)}\right].$$

When $x < x' \leq b$, we have proved that

$$\left|f\left(x'\right)-f\left(x\right)\right| \leq$$

$$\frac{\omega_1\left(D^q_{x+}\left(f\left(\cdot\right)-f\left(x\right)\right),\delta_1\right)_{[x,b]}}{\Gamma\left(q+1\right)}\left[\left(x'-x\right)^q+\frac{\left(x'-x\right)^{q+1}}{\left(q+1\right)\delta_1}\right], \tag{2.10}$$

where $0 < q < 1$, $\delta_1 > 0$.

ii) Case of $a \leq x' < x$ (here $D^q_{x-}\left(f\left(x\right)-f\left(x\right)\right)=0$):

$$\left|f\left(x'\right)-f\left(x\right)\right| \leq \frac{1}{\Gamma\left(q\right)}\int_{x'}^{x}\left(z-x'\right)^{q-1}\left|D^q_{x-}\left(f\left(z\right)-f\left(x\right)\right)\right|dz$$

$$= \frac{1}{\Gamma\left(q\right)}\int_{x'}^{x}\left(z-x'\right)^{q-1}\left|D^q_{x-}\left(f\left(z\right)-f\left(x\right)\right)-D^q_{x-}\left(f\left(x\right)-f\left(x\right)\right)\right|dz \tag{2.11}$$

$$\overset{(\delta_2>0)}{\leq} \frac{1}{\Gamma\left(q\right)}\int_{x'}^{x}\left(z-x'\right)^{q-1}\omega_1\left(D^q_{x-}\left(f\left(\cdot\right)-f\left(x\right)\right),\frac{\delta_2\left(x-z\right)}{\delta_2}\right)_{[a,x]}dz$$

$$\leq \frac{\omega_1\left(D^q_{x-}\left(f\left(\cdot\right)-f\left(x\right)\right),\delta_2\right)_{[a,x]}}{\Gamma\left(q\right)}\left(\int_{x'}^{x}\left(z-x'\right)^{q-1}\left(1+\frac{x-z}{\delta_2}\right)dz\right)$$

$$= \frac{\omega_1\left(D^q_{x-}\left(f\left(\cdot\right)-f\left(x\right)\right),\delta_2\right)_{[a,x]}}{\Gamma\left(q\right)}\left[\frac{\left(x-x'\right)^q}{q}+\frac{1}{\delta_2}\int_{x'}^{x}\left(x-z\right)^{2-1}\left(z-x'\right)^{q-1}dz\right]$$

$$= \frac{\omega_1\left(D^q_{x-}\left(f\left(\cdot\right)-f\left(x\right)\right),\delta_2\right)_{[a,x]}}{\Gamma\left(q\right)}\left[\frac{\left(x-x'\right)^q}{q}+\frac{1}{\delta_2}\frac{\Gamma\left(2\right)\Gamma\left(q\right)}{\Gamma\left(q+2\right)}\left(x-x'\right)^{q+1}\right] \tag{2.12}$$

$$= \frac{\omega_1\left(D^q_{x-}\left(f\left(\cdot\right)-f\left(x\right)\right),\delta_2\right)_{[a,x]}}{\Gamma\left(q\right)}\left[\frac{\left(x-x'\right)^q}{q}+\frac{\left(x-x'\right)^{q+1}}{q\left(q+1\right)\delta_2}\right]$$

$$= \frac{\omega_1\left(D^q_{x-}\left(f\left(\cdot\right)-f\left(x\right)\right),\delta_2\right)_{[a,x]}}{\Gamma\left(q+1\right)}\left[\left(x-x'\right)^q+\frac{\left(x-x'\right)^{q+1}}{\left(q+1\right)\delta_2}\right].$$

When $a \leq x' < x$, we have proved that

$$\left|f\left(x'\right)-f\left(x\right)\right| \leq$$

$$\frac{\omega_1\left(D^q_{x-}\left(f\left(\cdot\right)-f\left(x\right)\right),\delta_2\right)_{[a,x]}}{\Gamma\left(q+1\right)}\left[\left(x-x'\right)^q+\frac{\left(x-x'\right)^{q+1}}{\left(q+1\right)\delta_2}\right], \tag{2.13}$$

where $0 < q < 1$, $\delta_2 > 0$.

Finally choose: $\delta_1 = \delta_2 =: \delta > 0$. The theorem is proved. $\square$

We need

Definition 2.6. Here $C_+([a,b]) := \{f : [a,b] \to \mathbb{R}_+, \text{ continuous functions}\}$.
Let $L_N : C_+([a,b]) \to C_+([a,b])$, operators, $\forall\ N \in \mathbb{N}$, such that

(i) $L_N(\alpha f) = \alpha L_N(f)$, $\forall \alpha \geq 0, \forall f \in C_+([a,b])$,

(ii) if $f, g \in C_+([a,b]) : f \leq g$, then $L_N(f) \leq L_N(g)$, $\forall N \in \mathbb{N}$,

(iii) $L_N(f+g) \leq L_N(f) + L_N(g)$, $\forall\ f, g \in C_+([a,b])$.

We call $\{L_N\}_{N \in \mathbb{N}}$ positive sublinear operators.

We make

Remark 2.7. Let $f, g \in C_+([a,b])$, then it holds

$$|L_N(f)(x) - L_N(g)(x)| \leq L_N(|f-g|)(x), \ \ \forall\ x \in [a,b]. \tag{2.14}$$

Furthermore, we also have

$$|L_N(f)(x) - f(x)| \leq L_N(|f(\cdot) - f(x)|)(x) + |f(x)|\,|L_N(e_0)(x) - 1|, \tag{2.15}$$

$\forall\ x \in [a,b]$; $e_0(t) = 1$.

From now on we assume that $L_N(1) = 1$. Hence

$$|L_N(f)(x) - f(x)| \leq L_N(|f(\cdot) - f(x)|)(x), \ \ \forall x \in [a,b]. \tag{2.16}$$

We give

Theorem 2.8. *Let $f \in C_+([a,b])$, $0 < q < 1$, $D_{x+}^q(f(\cdot) - f(x)) \in C([x,b])$, $D_{x-}^q(f(\cdot) - f(x)) \in C([a,x])$, x is fixed, where $x \in [a,b]$. Then*

$$|f(\cdot) - f(x)| \leq \frac{\omega_1\left(D_x^q(f(\cdot) - f(x)), \delta\right)}{\Gamma(q+1)} \left[|\cdot - x|^q + \frac{|\cdot - x|^{q+1}}{(q+1)\delta}\right], \ \ \delta > 0. \tag{2.17}$$

We present:

Theorem 2.9. *Let $f \in C_+([a,b])$, $D_{x+}^q(f(\cdot) - f(x)) \in C([x,b])$, $D_{x-}^q(f(\cdot) - f(x)) \in C([a,x])$, where $x \in [a,b]$ is fixed, $0 < q < 1$, $\delta > 0$. Let $L_N : C_+([a,b]) \to C_+([a,b])$, be positive sublinear operators, such that $L_N(1) = 1$, $\forall\ N \in \mathbb{N}$. Then*

$$|L_N(f)(x) - f(x)| \tag{2.18}$$

$$\leq \frac{\omega_1\left(D_x^q(f(\cdot) - f(x)), \delta\right)}{\Gamma(q+1)} \left[L_N(|\cdot - x|^q)(x) + \frac{L_N\left(|\cdot - x|^{q+1}\right)(x)}{(q+1)\delta}\right],$$

$\forall\ N \in \mathbb{N}$.

We need Hölder's inequality for positive sublinear operators:

Lemma 2.10. ([5], p. 6) *Let $L : C_+([a,b]) \to C_+([a,b])$, be a positive sublinear operator and $f, g \in C_+([a,b])$, furthermore let $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$. Assume that $L((f(\cdot))^p)(s_*)$, $L((g(\cdot))^q)(s_*) > 0$ for some $s_* \in [a,b]$. Then*

$$L(f(\cdot)f(\cdot))(s_*) \leq \left(L((f(\cdot))^p)(s_*)\right)^{\frac{1}{p}} \left(L((g(\cdot))^q)(s_*)\right)^{\frac{1}{q}}. \tag{2.19}$$

We make

Remark 2.11. In Theorem 2.9 we assumed $L_N(1) = 1$, $\forall\ N \in \mathbb{N}$. We further assume that $L_N\left(|\cdot - x|^{q+1}\right)(x) > 0$, $\forall\ N \in \mathbb{N}$, for the fixed $x \in [a,b]$.

Then, by (2.19), we obtain

$$L_N\left(|\cdot - x|^{q}\right)(x) \leq \left(L_N\left(|\cdot - x|^{q+1}\right)(x)\right)^{\frac{q}{q+1}}, \quad \forall N \in \mathbb{N}. \tag{2.20}$$

We give

Theorem 2.12. *All as in Theorem 2.9, plus* $L_N\left(|\cdot - x|^{q+1}\right)(x) > 0$, $\forall\ N \in \mathbb{N}$, *for a fixed* $x \in [a,b]$. *Then*

$$|L_N(f)(x) - f(x)| \leq \frac{\omega_1\left(D_x^q(f(\cdot) - f(x)), \delta\right)}{\Gamma(q+1)}$$

$$\cdot\left(L_N\left(|\cdot - x|^{q+1}\right)(x)\right)^{\frac{q}{q+1}}\left[1 + \frac{\left(L_N\left(|\cdot - x|^{q+1}\right)(x)\right)^{\frac{1}{q+1}}}{(q+1)\delta}\right], \tag{2.21}$$

$\forall\ N \in \mathbb{N}$.

Next we choose $\delta := \left(L_N\left(|\cdot - x|^{q+1}\right)(x)\right)^{\frac{1}{q+1}} > 0$, to obtain:

Theorem 2.13. *All as in Theorem 2.9, plus* $L_N\left(|\cdot - x|^{q+1}\right)(x) > 0$, $\forall\ N \in \mathbb{N}$; $x \in [a,b]$ *is fixed. Then*

$$|L_N(f)(x) - f(x)| \leq \frac{(q+2)}{\Gamma(q+2)}$$

$$\cdot\omega_1\left(D_x^q(f(\cdot) - f(x)), \left(L_N\left(|\cdot - x|^{q+1}\right)(x)\right)^{\frac{1}{q+1}}\right)\left(L_N\left(|\cdot - x|^{q+1}\right)(x)\right)^{\frac{q}{q+1}}, \tag{2.22}$$

$\forall\ N \in \mathbb{N}$.

Application 2.14. The max-product Bernstein operators are defined by

$$B_N^{(M)}(f)(x) := \frac{\vee_{k=0}^{N} p_{N.k}(x) f\left(\frac{k}{N}\right)}{\vee_{k=0}^{N} p_{N.k}(x)}, \quad \forall\ N \in \mathbb{N}, \tag{2.23}$$

where $\vee$ stands for maximum, and $p_{N,k}(x) = \binom{N}{k} x^k (1-x)^{N-k}$, and $f : [0,1] \to \mathbb{R}_+$ is a continuous function, see [6], p. 10.

These are positive sublinear operators mapping $C_+([0,1])$ into itself. Notice $B_N^{(M)}(1) = 1$, $\forall\ N \in \mathbb{N}$.

In [5], p. 76, we proved that

$$B_N^{(M)}\left(|\cdot - x|^{1+\beta}\right)(x) \leq \frac{6}{\sqrt{N+1}}, \quad \forall\ x \in [0,1], \tag{2.24}$$

$\forall\ N \in \mathbb{N}$, $\forall\ \beta > 0$.

Furthermore, clearly it holds that

$$B_N^{(M)}\left(|\cdot - x|^{1+\beta}\right)(x) > 0, \quad \forall\ N \in \mathbb{N},\ \forall\ \beta \geq 0, \tag{2.25}$$

and any $x \in (0,1)$.

We present

Theorem 2.15. *Let $f \in C_+([0,1])$, $D_{x+}^q(f(\cdot) - f(x)) \in C([x,1])$, $D_{x-}^q(f(\cdot) - f(x)) \in C([0,x])$, where $x \in (0,1)$, $0 < q < 1$. Then*

$$\left|B_N^{(M)}(f)(x) - f(x)\right|$$

$$\leq \frac{(q+2)}{\Gamma(q+2)}\omega_1\left(D_x^q(f(\cdot) - f(x)), \left(\frac{6}{\sqrt{N+1}}\right)^{\frac{1}{q+1}}\right)\left(\frac{6}{\sqrt{N+1}}\right)^{\frac{q}{q+1}}, \tag{2.26}$$

$\forall\ N \in \mathbb{N}$.

As $N \to +\infty$, we get $B_N^{(M)}(f)(x) \to f(x)$.

Proof. By (2.23), (2.24), (2.25) and Theorem 2.13. □

One can give many examples like in Theorem 2.15, but we choose to omit it this task.

Choquet integral has become very important in statistical mechanics, potential theory, non-additive measure theory, and lately in economics. For the definition and properties of Choquet integral read [7], [8], [13].

We denote it by $(C)\int$.

Next we talk about representations of positive sublinear operators by Choquet integrals:

We need

Definition 2.16. Let Ω be a set, and let $f, g : \Omega \to \mathbb{R}$ be bounded functions. We say that f and g are comonotonic, if for every $\omega, \omega' \in \Omega$,

$$(f(\omega) - f(\omega'))(g(\omega) - g(\omega')) \geq 0. \tag{2.27}$$

We also need the famous Schmeidler's Representation Theorem (Schmeidler 1986).

Theorem 2.17. ([11]) *Denote with $\mathcal{L}_\infty(\mathcal{A})$ the vector space of $\mathcal{A}$-measurable bounded real valued functions on Ω, where $\mathcal{A} \subset 2^\Omega$ is a σ-algebra. Given a real functional $\Gamma : \mathcal{L}_\infty(\mathcal{A}) \to \mathbb{R}$, assume that for $f, g \in \mathcal{L}_\infty(\mathcal{A})$:*

(i) $\Gamma(cf) = c\Gamma(f)$, $\forall\ c > 0$,

(ii) $f \leq g$, implies $\Gamma(f) \leq \Gamma(g)$,

and

(iii) $\Gamma(f+g) = \Gamma(f) + \Gamma(g)$, for any comonotonic f, g.

Then $\gamma(A) := \Gamma(1_A)$, $\forall\ A \in \mathcal{A}$, defines a finite monotone set function on $\mathcal{A}$, and Γ is the Choquet integral with respect to γ, i.e.

$$\Gamma(f) = (C)\int_\Omega f(t)\,d\gamma(t), \quad \forall\ f \in \mathcal{L}_\infty(\mathcal{A}). \tag{2.28}$$

Above 1_A denotes the characteristic function on A.

We make

Remark 2.18. Consider here $[a,b] \subset \mathbb{R}$, $\mathcal{B} = \mathcal{B}([a,b])$ is the Borel σ-algebra on $[a,b]$, and $\mathcal{L}_\infty(\mathcal{B})$ is the vector space of $\mathcal{B}$-measurable bounded real valued functions on $[a,b]$. Let $(L_N)_{N\in\mathbb{N}}$ be a sequence of positive sublinear operators from $\mathcal{L}_\infty(\mathcal{B})$ into $C_+([a,b])$, and $x \in [a,b]$. That is here L_N fulfills the positive homogenuity, monotonicity and subadditivity properties, see Definition 2.6.

Assume $L_N(1) = 1$, $\forall\ N \in \mathbb{N}$. Clearly here $\mathcal{L}_\infty(\mathcal{B}) \supset C_+([a,b])$. In particular we treat $L_N|_{C_+([a,b])}$, just denoted for simplicity by L_N, $\forall\ N \in \mathbb{N}$.

It is clear that $L_N(\cdot)(x) : \mathcal{L}_\infty(\mathcal{B}) \to \mathbb{R}$ is a functional, $\forall\ N \in \mathbb{N}$. It has the properties:

(i)

$$L_N(cf)(x) = cL_N(f)(x), \ \forall\ c > 0, \ \forall\ f \in \mathcal{L}_\infty(\mathcal{B}), \tag{2.29}$$

(ii)

$$f \leq g, \text{ implies } L_N(f)(x) \leq L_N(g)(x), \quad \text{where } f, g \in \mathcal{L}_\infty(\mathcal{B}), \tag{2.30}$$

and

(iii)

$$L_N(f+g)(x) \leq L_N(f)(x) + L_N(g)(x), \quad \forall\ f, g \in \mathcal{L}_\infty(\mathcal{B}). \tag{2.31}$$

For comonotonic $f, g \in \mathcal{L}_\infty(\mathcal{B})$, we further assume that

$$L_N(f+g)(x) = L_N(f)(x) + L_N(g)(x). \tag{2.32}$$

In that case L_N is called comonotonic.

By Theorem 2.17 we get that:

$$\gamma_{N,x}(A) := L_N(1_A)(x), \ \forall\ A \in \mathcal{B}, \forall\ N \in \mathbb{N}, \tag{2.33}$$

defines a finite monotone set function on $\mathcal{B}$, and

$$L_N(f)(x) = (C)\int_a^b f(t)\, d\gamma_{N,x}(t), \tag{2.34}$$

$\forall\ f \in \mathcal{L}_\infty(\mathcal{B})$, $\forall\ N \in \mathbb{N}$.

In particular (2.34) is valid for any $f \in C_+([a,b])$. Furthermore $\gamma_{N,x}$ is normalized, that is $\gamma_{N,x}([a,b]) = 1$, $\forall\ N \in \mathbb{N}$.

We give

Theorem 2.19. *Let $f \in C_+([a,b])$, $D^q_{x+}(f(\cdot) - f(x)) \in C([x,b])$,*

$$D^q_{x-}(f(\cdot) - f(x)) \in C([a,x]),$$

where $x \in [a,b] \subset \mathbb{R}$ is fixed, $0 < q < 1$. Let $L_N : \mathcal{L}_\infty(\mathcal{B}([a,b])) \to C_+([a,b])$, be positive sublinear comonotonic operators, such that $L_N(1) = 1$, $\forall\ N \in \mathbb{N}$. Assume that

$$(C)\int_a^b |t-x|^{q+1}\, d\gamma_{N,x}(t) > 0, \ \forall N \in \mathbb{N}.$$

Then

$$|L_N(f)(x) - f(x)| \leq \frac{(q+2)}{\Gamma(q+2)} \omega_1 \left(D_x^q (f(\cdot) - f(x)), \left((C) \int_a^b |t-x|^{q+1} d\gamma_{N,x}(t) \right)^{\frac{1}{q+1}} \right)$$

$$\cdot \left((C) \int_a^b |t-x|^{q+1} d\gamma_{N,x}(t) \right)^{\frac{q}{q+1}}, \ \forall \ N \in \mathbb{N}. \tag{2.35}$$

If

$$(C) \int_a^b |t-x|^{q+1} d\gamma_{N,x}(t) \to 0,$$

then $L_N(f)(x) \to f(x)$*, as* $N \to \infty$.

Proof. By Theorem 2.13. □

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Some results on a question of Li, Yi and Li

Abhijit Banerjee and Sujoy Majumder

Abstract. The purpose of this paper is to study the uniqueness problems of certain difference polynomials of meromorphic functions sharing a nonzero polynomial. The results of this paper improve and generalize some recent results due to Li, Yi and Li [11]. Some examples have been exhibited to show that some conditions used in the paper are sharp.

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1. Introduction, definitions and results

In this paper by meromorphic functions we shall always mean meromorphic functions in the complex plane.

We adopt the standard notations of value distribution theory (see [6]). For a nonconstant meromorphic function f, we denote by $T(r,f)$ the Nevanlinna characteristic of f and by $S(r,f)$ any quantity satisfying $S(r,f)=o\{T(r,f)\}$ as $r\to\infty$ possibly outside a set of finite linear measure. We denote by $T(r)$ the maximum of $T(r,f)$ and $T(r,g)$. The notation $S(r)$ denotes any quantity satisfying $S(r)=o(T(r))$ as $r\longrightarrow\infty$, outside of a possible exceptional set of finite linear measure.

A meromorphic function $a(z)$ is called a small function with respect to f, provided that $T(r,a)=S(r,f)$. The order of f is defined by

$$\sigma(f)=\limsup_{r\longrightarrow\infty}\frac{\log T(r,f)}{\log r}.$$

For $a\in\mathbb{C}\cup\{\infty\}$, we define

$$\Theta(a;f)=1-\limsup_{r\longrightarrow\infty}\frac{\overline{N}(r,a;f)}{T(r,f)}$$

and

$$\delta(a;f)=1-\limsup_{r\longrightarrow\infty}\frac{N(r,a;f)}{T(r,f)}.$$

Let $f(z)$ and $g(z)$ be two non-constant meromorphic functions. Let $a(z)$ be a small function with respect to $f(z)$ and $g(z)$. We say that $f(z)$ and $g(z)$ share $a(z)$ CM (counting multiplicities) if $f(z)-a(z)$ and $g(z)-a(z)$ have the same zeros with the same multiplicities, we say that $f(z)$, $g(z)$ share $a(z)$ IM (ignoring multiplicities) if we do not consider the multiplicities.

We say that a finite value z_0 is called a fixed point of f if $f(z_0)=z_0$ or z_0 is a zero of $f(z)-z$.

Let $f(z)$ be a transcendental meromorphic function and $n\in\mathbb{N}$. Many authors have investigated the value distributions of $f^n f'$. At the starting point, we recall the result of Hayman (see [5], Corollary of Theorem 9). In 1959, Hayman proved the following theorem.

Theorem A. [5] *Let $f(z)$ be a transcendental meromorphic function and $n\in\mathbb{N}$ such that $n\geq 3$. Then $f^n(z)f'(z)=1$ has infinitely many solutions.*

The case $n=2$ was settled by Mues [15] in 1979. Bergweiler and Eremenko [1] showed that $f(z)f'(z)-1$ has infinitely many zeros.

For an analogue of the above results Laine and Yang [10] investigated the value distribution of difference products of entire functions in the following manner.

Theorem B. [10] *Let $f(z)$ be a transcendental entire function of finite order, and c be a non-zero complex constant. Then for $n\geq 2$, $f^n(z)f(z+c)$ assumes every non-zero value $a\in\mathbb{C}$ infinitely often.*

The following example shows that Theorem B does not remain valid if $n=1$.

Example 1.1. [10] Let $f(z)=1+e^z$. Then $f(z)f(z+\pi i)-1=-e^{2z}$ has no zeros.

The following example shows that Theorem B does not remain valid if $f(z)$ is of infinite order.

Example 1.2. [13] Let $f(z)=e^{-e^z}$. Then $f^2(z)f(z+c)-2=-1$ and $\rho(f)=\infty$, where c is a non-zero constant satisfying $e^c=-2$. Clearly $f^2(z)f(z+c)-2$ has no zeros.

It is to be mentioned that in the meantime Chen, Huan and Zheng [2] obtained some results a part of which related to the content of the present paper.

In 2009, Liu and Yang [13] further improved Theorem B and obtained the next result.

Theorem C. [13] *Let $f(z)$ be a transcendental entire function of finite order, and c be a non-zero complex constant. Then, for $n\geq 2$, $f^n(z)f(z+c)-p(z)$ has infinitely many zeros, where $p(z)$ is a non-zero polynomial.*

The following example shows that the condition "$\rho(f)<\infty$" in Theorem C is necessary.

Example 1.3. [13] Let $f(z)=e^{-e^z}$. Then $f^n(z)f(z+c)-P(z)=1-P(z)$ and $\rho(f)=\infty$, where c is a non-zero constant satisfying $e^c=-n$, $P(z)$ is a non-constant polynomial, n is a positive integer. Clearly $f^n(z)f(z+c)-P(z)$ has finitely many zeros.

In 2010, Qi, Yang and Liu [16] studied the uniqueness of the difference monomials and obtained the following result.

Theorem D. [16] *Let $f(z)$ and $g(z)$ be two transcendental entire functions of finite order, and $c \in \mathbb{C} \setminus \{0\}$; let $n \in \mathbb{N}$ such that $n \geq 6$. If $f^n(z)f(z+c)$ and $g^n(z)g(z+c)$ share z CM, then $f(z) \equiv t_1 g(z)$ for a constant t_1 that satisfies $t_1^{n+1} = 1$.*

Theorem E. [16] *Let $f(z)$ and $g(z)$ be two transcendental entire functions of finite order, and $c \in \mathbb{C} \setminus \{0\}$; let $n \in \mathbb{N}$ such that $n \geq 6$. If $f^n(z)f(z+c)$ and $g^n(z)g(z+c)$ share 1 CM, then $f(z)g(z) \equiv t_2$ or $f(z) \equiv t_3 g(z)$ for some constants t_2 and t_3 that satisfy $t_2^{n+1} = 1$ and $t_3^{n+1} = 1$.*

In 2014, Li, Yi and Li [11] improved Theorems C, D and E to meromorphic functions and obtained a number of results as follows.

Theorem F. [11] *Let $f(z)$ be a transcendental meromorphic function such that its order $\rho(f) < \infty$, let c be a non-zero complex number, and let $n \geq 6$ be an integer. Suppose that $P(z) \not\equiv 0$ is a polynomial. Then $f^n(z)f(z+c) - P(z)$ has infinitely many zeros.*

Theorem G. [11] *Let $f(z)$ be a transcendental meromorphic function such that its order $\rho(f) < \infty$ and $\delta(\infty; f(z)) > 0$, let c be a non-zero complex number, and let $n \geq 5$ be an integer. Suppose that $P(z) \not\equiv 0$ is a polynomial. Then $f^n(z)f(z+c) - P(z)$ has infinitely many zeros.*

Theorem H. [11] *Let $f(z)$ and $g(z)$ be two distinct transcendental meromorphic functions of finite order, let c be a non-zero complex number, let $n \geq 14$ be an integer and let $P(z) \not\equiv 0$ be a polynomial such that $2\deg(P) < n-1$. Suppose that $f^n(z)f(z+c) - P(z)$ and $g^n(z)g(z+c) - P(z)$ share 0 CM. Then*

- (I) *if $n \geq 10$ and if $f^n(z)f(z+c)/P(z)$ is a Möbius transformation of $g^n(z)g(z+c)/P(z)$, then one of the following two cases will hold:*
  - (i) *$f(z) \equiv tg(z)$, where $t \neq 1$ is a constant satisfying $t^{n+1} = 1$.*
  - (ii) *$f(z)g(z) = t$, where $P(z)$ reduces to a non-zero constant c_1, say, and t is a constant such that $t^{n+1} = c_1^2$.*
- (II) *if $n \geq 14$, then one of the two cases (I) (i) and (I) (ii) will hold.*

Theorem I. [11] *Let $f(z)$ and $g(z)$ be two distinct transcendental meromorphic functions of finite order, let c be a non-zero complex number, let $n \geq 12$ be an integer and let $P(z) \not\equiv 0$ be a polynomial such that $2\deg(P) < n+1$. Suppose that $f(z)$ and $g(z)$ share ∞ IM, $f^n(z)f(z+c) - P(z)$ and $g^n(z)g(z+c) - P(z)$ share $0, \infty$ CM. Then*

- (I) *if $n \geq 10$ and if $f^n(z)f(z+c)/P(z)$ is a Möbius transformation of $g^n(z)g(z+c)/P(z)$, then one of the following two cases will hold:*
  - (i) *$f(z) \equiv tg(z)$, where $t \neq 1$ is a constant satisfying $t^{n+1} = 1$.*
  - (ii) *$f(z) = e^{Q(z)}$ and $g(z) = te^{-Q(z)}$, where $P(z)$ reduces to a non-zero constant c_1, say, and t is a constant such that $t^{n+1} = c_1^2$, $Q(z)$ is a non-constant polynomial.*
- (II) *if $n \geq 12$, then one of the two cases (I) (i) and (I) (ii) will hold.*

Theorem J. [11] *Let $f(z)$ and $g(z)$ be two distinct non-constant meromorphic functions of finite order. Suppose that c is a non-zero complex number and $n \geq 17$ is an integer. If $f^n(z)f(z+c) - z$ and $g^n(z)g(z+c) - z$ share 0 CM, then $f(z) \equiv tg(z)$, where $t \neq 1$ is a constant satisfying $t^{n+1} = 1$.*

Theorem K. [11] *Let $f(z)$ and $g(z)$ be two distinct non-constant meromorphic functions of finite order, c be a non-zero complex number and $n \geq 13$ be an integer. Suppose that $f(z)$ and $g(z)$ share ∞ IM, $f^n(z)f(z+c)-z$ and $g^n(z)g(z+c)-z$ share $0, \infty$ CM. Then $f(z) \equiv tg(z)$, where $t \neq 1$ is a constant satisfying $t^{n+1} = 1$.*

At the end of [11] the following open problem was posed by the authors.
Open problem. What can be said about the conclusion of Corollary 1.1 [11] if we replace the condition "$n \geq 6$" with "$2 \leq n \leq 5$"?

One of our objective to write this paper is to solve this open problem.

Next we recall the notion of weighted sharing [9] as it will render an useful tool to relax the nature of sharing.

Definition 1.1. [9] Let $k \in \mathbb{N} \cup \{0\} \cup \{\infty\}$. For $a \in \mathbb{C} \cup \{\infty\}$ we denote by $E_k(a; f)$ the set of all a-points of f where an a-point of multiplicity m is counted m times if $m \leq k$ and $k+1$ times if $m > k$. If $E_k(a; f) = E_k(a; g)$, we say that f, g share the value a with weight k.

We write f, g share (a, k) to mean that f, g share the value a with weight k. Clearly if f, g share (a, k) then f, g share (a, p) for any integer p, $0 \leq p < k$. Also we note that f, g share a value a IM or CM if and only if f, g share $(a, 0)$ or (a, ∞) respectively.

Next observing the above results the following questions are inevitable.
Question 1. Can the lower bound of n be further reduced in Theorem I?
Question 2. Can one replaced the condition $\delta(\infty; f) > 0$ of Theorem G by weaker one?
Question 3. Can"CM" sharing in Theorems H, I, J, K be reduced to finite weight sharing?

In this paper we want to investigate the above situations. We now present the following theorems which are the main results of the paper.

Theorem 1.1. *Let $f(z)$ be a transcendental meromorphic function of finite order, $c \in \mathbb{C} \setminus \{0\}$ be fixed, $n \in \mathbb{N}$ such that $n > 1$ and let $a(z)(\not\equiv 0, \infty)$ be a small function of $f(z)$. If*

$$\Theta(0; f) + \Theta(\infty; f) > \frac{5-n}{2},$$

then $f^n(z)f(z+c) - a(z)$ has infinitely many zeros.

Theorem 1.2. *Let $f(z)$ and $g(z)$ be two distinct transcendental meromorphic functions of finite order, c be a non-zero complex number, $n \geq 14$ be an integer and $p(z) \not\equiv 0$ be a polynomial such that $2\deg(p) < n-1$. Suppose that $f^n(z)f(z+c) - p(z)$ and $g^n(z)g(z+c) - p(z)$ share $(0, 2)$. Then*

(I) *if $n \geq 10$ and if $f^n(z)f(z+c)/p(z)$ is a Möbius transformation of $g^n(z)g(z+c)/p(z)$, then one of the following two cases will hold:*

(i) $f(z) \equiv tg(z)$, *where* $t \neq 1$ *is a constant satisfying* $t^{n+1} = 1$.
(ii) $f(z)g(z) \equiv t$, *where* $p(z)$ *reduces to a nonzero constant* c_1, *say, and* t *is a constant such that* $t^{n+1} = c_1^2$.

(II) *if* $n \geq 14$, *then one of the two cases (I) (i) or (I) (ii) will hold.*

Theorem 1.3. *Let* $f(z)$ *and* $g(z)$ *be two distinct transcendental meromorphic functions of finite order,* c *be a non-zero complex number,* $n \geq 12$ *be an integer and* $p(z) \not\equiv 0$ *be a polynomial such that* $2\deg(p) < n+1$. *Suppose that* f *and* g *share* $(\infty, 0)$, $f^n(z)f(z+c) - p(z)$ *and* $g^n(z)g(z+c) - p(z)$ *share* $(0,2)$ *and* (∞, ∞). *Then*

(I) *if* $n \geq 8$ *and if* $f^n(z)f(z+c)/p(z)$ *is a Möbius transformation of* $g^n(z)g(z+c)/p(z)$, *then one of the following two cases will hold:*
(i) $f(z) \equiv tg(z)$, *where* $t \neq 1$ *is a constant satisfying* $t^{n+1} = 1$.
(ii) $f(z) = e^{Q(z)}$ *and* $g(z) = te^{-Q(z)}$, *where* $p(z)$ *reduces to a non-zero constant* c_1, *say, and* t *is a constant such that* $t^{n+1} = c_1^2$, $Q(z)$ *is a non-constant polynomial.*

(II) *if* $n \geq 12$, *then one of the two cases (I) (i) or (I) (ii) will hold.*

Theorem 1.4. *Let* $f(z)$ *and* $g(z)$ *be two distinct non-constant meromorphic functions of finite order and let* $p(z)$ *be a non-constant polynomial such that* $\deg(p) = l$. *Suppose that* c *is a non-zero complex number and* $n \geq 14 + 3l$ *is an integer.*
If $f^n(z)f(z+c) - p(z)$ *and* $g^n(z)g(z+c) - p(z)$ *share* $(0,2)$, *then* $f(z) \equiv tg(z)$, *where* $t \neq 1$ *is a constant satisfying* $t^{n+1} = 1$.

Remark 1.1. It is easy to see that the conditions $f(z)$ and $g(z)$ as well as

$$f^n(z)f(z+c) - p(z) \text{ and } g^n(z)g(z+c) - p(z)$$

have common poles in Theorem 1.3 are sharp by the following examples.

Example 1.4. Let

$$P_1(z) = \frac{1}{e^z + 1} \text{ and } Q_1(z) = \frac{1}{e^z - 1}.$$

Let c be a non-zero constant satisfying $e^c = -1$. Clearly $P_1(z)$ and $Q_1(z)$ are transcendental meromorphic functions of finite order. Let t be a nonzero constant such that $t^{n+1} = 1$ and let

$$f(z) = \frac{P_1(z)}{Q_1(z)}, \quad g(z) = t\frac{Q_1(z)}{P_1(z)}.$$

Then $f(z)$ and $g(z)$ are transcendental meromorphic functions of finite order. Note that neither $f(z)$ and $g(z)$ nor $f^n(z)f(z+c) - 1$ and $g^n(z)g(z+c) - 1$ have common poles. Clearly $f^n(z)f(z+c) - 1$ and $g^n(z)g(z+c) - 1$ share $(0, \infty)$, but neither $f(z) \equiv tg(z)$ nor $f(z) = e^{Q(z)}$ and $g(z) = t_1 e^{-Q(z)}$, where t_1 is a nonzero constant and $Q(z)$ is a non-constant polynomial.

Example 1.5. Let

$$f(z) = p(z)\frac{e^z - 1}{e^z + 1} \text{ and } g(z) = p(z)\frac{e^z + 1}{e^z - 1},$$

where $p(z)$ is a non-zero polynomial.
Let c be a non-zero constant satisfying $e^c = -1$. Clearly $f(z)$ and $g(z)$ are transcendental meromorphic functions of finite order. Note that neither $f(z)$ and $g(z)$ nor $f^n(z)f(z+c) - p^n(z)p(z+c)$ and $g^n(z)g(z+c) - p^n(z)p(z+c)$ have common poles. Clearly $f^n(z)f(z+c) - p^n(z)p(z+c)$ and $g^n(z)g(z+c) - p^n(z)p(z+c)$ share $(0, \infty)$, but neither $f(z) \equiv tg(z)$ nor $f(z) = e^{Q(z)}$ and $g(z) = t_1 e^{-Q(z)}$, where t_1 is a non-zero constant and $Q(z)$ is a non-constant polynomial.

Example 1.6. Let

$$P_1(z) = \sum_{n=0}^{\infty} e^{-n^2} z^n + 2 \ \ and \ \ Q_1(z) = \sum_{n=0}^{\infty} e^{-n^3} z^{2n} + 3.$$

Clearly $P_1(z)$ and $Q_1(z)$ are transcendental entire functions with zero order. Let t be a non-zero constant such that $t^{n+1} = 1$ and let

$$f(z) = \frac{P_1(z)}{Q_1(z)}, \quad g(z) = t\frac{Q_1(z)}{P_1(z)}.$$

Then $f(z)$ and $g(z)$ are transcendental meromorphic functions with zero order. Note that neither $f(z)$ and $g(z)$ nor $f^n(z)f(z+c) - 1$ and $g^n(z)g(z+c) - 1$ have common poles. Clearly $f^n(z)f(z+c) - 1$ and $g^n(z)g(z+c) - 1$ share $(0, \infty)$, but neither $f(z) \equiv tg(z)$ nor $f(z) = e^{Q(z)}$ and $g(z) = t_1 e^{-Q(z)}$, where t_1 is a non-zero constant and $Q(z)$ is a non-constant polynomial.

We now explain following definitions and notations which are used in the paper.

Definition 1.2. [8] Let $a \in \mathbb{C} \cup \{\infty\}$. For $p \in \mathbb{N}$ we denote by $N(r, a; f \mid\leq p)$ the counting function of those a-points of f (counted with multiplicities) whose multiplicities are not greater than p. By $\overline{N}(r, a; f \mid\leq p)$ we denote the corresponding reduced counting function.

In an analogous manner we can define $N(r, a; f \mid\geq p)$ and $\overline{N}(r, a; f \mid\geq p)$.

Definition 1.3. [9] Let $k \in \mathbb{N} \cup \{\infty\}$. We denote by $N_k(r, a; f)$ the counting function of a-points of f, where an a-point of multiplicity m is counted m times if $m \leq k$ and k times if $m > k$. Then $N_k(r, a; f) = \overline{N}(r, a; f) + \overline{N}(r, a; f \mid\geq 2) + \ldots + \overline{N}(r, a; f \mid\geq k)$. Clearly $N_1(r, a; f) = \overline{N}(r, a; f)$.

2. Lemmas

Let F and G be two non-constant meromorphic functions. Henceforth we shall denote by H the following function.

$$H = \left(\frac{F''}{F'} - \frac{2F'}{F-1}\right) - \left(\frac{G''}{G'} - \frac{2G'}{G-1}\right). \tag{2.1}$$

Lemma 2.1. [18] *Let $f(z)$ be a non-constant meromorphic function and let $a_n(z)$ $(\not\equiv 0)$, $a_{n-1}(z)$, $\ldots$, $a_0(z)$ be meromorphic functions such that $T(r, a_i(z)) = S(r, f)$ for $i = 0, 1, 2, \ldots, n$. Then*

$$T(r, a_n f^n + a_{n-1} f^{n-1} + \ldots + a_1 f + a_0) = nT(r, f) + S(r, f).$$

Lemma 2.2. [3] *Let $f(z)$ be a meromorphic function of finite order σ, and let $c \in \mathbb{C}\backslash\{0\}$ be fixed. Then for each $\varepsilon > 0$, we have*

$$m\left(r, \frac{f(z+c)}{f(z)}\right) + m\left(r, \frac{f(z)}{f(z+c)}\right) = O(r^{\sigma-1+\varepsilon}).$$

Lemma 2.3. [4] *Let $f(z)$ be a meromorphic function of finite order σ, and let $c \in \mathbb{C}\backslash\{0\}$ be fixed. Then for each $\varepsilon > 0$, we have*

$$T(r, f(z+c)) = T(r, f(z)) + O(r^{\sigma-1+\varepsilon}) + O(\log r)$$

and

$$\sigma(f(z+c)) = \sigma(f(z)).$$

The following lemma has little modifications of the original version (Theorem 2.1 of [3]).

Lemma 2.4. *Let $f(z)$ be a transcendental meromorphic function of finite order, $c \in \mathbb{C} \setminus \{0\}$ be fixed. Then*

$$T(r, f(z+c)) = T(r, f) + S(r, f).$$

Lemma 2.5. [7] *Let $f(z)$ be a non-constant meromorphic function of finite order and $c \in \mathbb{C}$. Then*

$$N(r, 0; f(z+c)) \le N(r, 0; f(z)) + S(r, f), \quad N(r, \infty; f(z+c)) \le N(r, \infty; f) + S(r, f),$$
$$\overline{N}(r, 0; f(z+c)) \le \overline{N}(r, 0; f(z)) + S(r, f), \quad \overline{N}(r, \infty; f(z+c)) \le \overline{N}(r, \infty; f) + S(r, f).$$

Lemma 2.6. *Let $f(z)$ be a non-constant meromorphic function of finite order σ, $c \in \mathbb{C} \setminus \{0\}$ be fixed and let $\Phi(z) = f^n(z)f(z+c)$, where $n \in \mathbb{N}$ such that $n > 1$. Then for each $\varepsilon > 0$, we have*

$$(n-1)\, T(r, f) \le T(r, \Phi) + O(r^{\sigma-1+\varepsilon}) + S(r, f).$$

Proof. The proof of lemma follows from Lemmas 2.6 [14] and 2.2. □

Lemma 2.7. *Let $f(z)$ be a non-constant meromorphic function of finite order σ, $c \in \mathbb{C} \setminus \{0\}$ be fixed and let $n \in \mathbb{N}$ with $n > 1$. Then $S(r, f^n(z)f(z+c)) = S(r, f)$.*

Proof. By Lemmas 2.1 and 2.3 we have

$$\begin{aligned} T(r, f^n(z)f(z+c)) &\le T(r, f^n) + T(r, f(z+c)) \\ &\le T(r, f^n) + T(r, f) + O(r^{\sigma-1+\varepsilon}) + O(\log r) + S(r, f) \\ &\le (n+1)\, T(r, f) + O(r^{\sigma-1+\varepsilon}) + O(\log r) + S(r, f), \end{aligned}$$

for all $\varepsilon > 0$. This shows that $T(r, f^n(z)f(z+c)) = O(T(r, f))$.
Also by Lemma 2.6 we have $T(r, f) = O(T(r, f^n(z)f(z+c)))$. Thus we have

$$S(r, f^n(z)f(z+c)) = S(r, f).$$

This completes the proof. □

Lemma 2.8. [9] *Let f and g be two non-constant meromorphic functions sharing $(1, 2)$. Then one of the following holds:*

(i) $T(r, f) \le N_2(r, 0; f) + N_2(r, 0; g) + N_2(r, \infty; f) + N_2(r, \infty; g) + S(r, f) + S(r, g)$,

(ii) $fg \equiv 1$,
(iii) $f \equiv g$.

Lemma 2.9. [20] *Let H be defined as in (2.1). If $H \equiv 0$ and*

$$\limsup_{r \longrightarrow \infty} \frac{\overline{N}(r,0;F)+\overline{N}(r,0;G)+\overline{N}(r,\infty;F)+\overline{N}(r,\infty;G)}{T(r)} < 1, \quad r \in I,$$

where I is a set of infinite linear measure, then $F \equiv G$ or $F \cdot G \equiv 1$.

Lemma 2.10. [[19], Lemma 7.1] *Let F and G be two non-constant meromorphic functions such that G is a Möbius transformation of F. Suppose that there exists a subset $I \subset \mathbb{R}^+$ with its measure $mesI = +\infty$ such that*

$$\overline{N}(r,0;F)+\overline{N}(r,0;G)+\overline{N}(r,\infty;F)+\overline{N}(r,\infty;G) < (\lambda + o(1))T(r,F),$$

as $r \in I$ and $r \to \infty$, where $\lambda < 1$, then $F \equiv G$ or $F \cdot G \equiv 1$.

Lemma 2.11. [Hadamard Factorization Theorem] *Let f be an entire function of finite order σ with zeros $a_1, a_2, \ldots$, each zeros is counted as often as its multiplicity. Then f can be expressed in the form*

$$f(z) = \beta(z)e^{\alpha(z)},$$

where $\alpha(z)$ is a polynomial of degree not exceeding $[\sigma]$ and $\beta(z)$ is the canonical product formed with the zeros of f.

Lemma 2.12. *Let $f(z)$, $g(z)$ be two non-constant meromorphic functions of finite order σ, $c \in \mathbb{C} \setminus \{0\}$ and $n \in \mathbb{N}$ such that $n \geq 2$. If*

$$f^n(z)f(z+c) \equiv g^n(z)g(z+c),$$

then $f(z) \equiv tg(z)$ for some constant $t \neq 1$ such that $t^{n+1} = 1$.

Proof. Suppose

$$f^n(z)f(z+c) \equiv g^n(z)g(z+c). \tag{2.2}$$

Let $h = \frac{f}{g}$. Then from (2.2) we have

$$h^n(z) \equiv \frac{1}{h(z+c)}. \tag{2.3}$$

Now by Lemmas 2.1, 2.2 and 2.5 we get

$$\begin{aligned} nT(r,h) &= T(r,h^n) + S(r,h) = T\left(r, \frac{1}{h(z+c)}\right) + S(r,h) \\ &\leq N(r,0;h(z+c)) + m\left(r, \frac{1}{h(z+c)}\right) + S(r,h) \\ &\leq N(r,0;h(z)) + m\left(r, \frac{h(z)}{h(z+c)}\right) + m\left(r, \frac{1}{h(z)}\right) + S(r,h) \\ &\leq T(r,h) + O(r^{\sigma-1+\varepsilon}) + S(r,h), \end{aligned}$$

which is a contradiction since $n \geq 2$. Hence h must be a constant, which implies that $h^{n+1} = 1$, where $h \neq 1$, thus $f(z) = tg(z)$ for some constant $t \neq 1$ such that $t^{n+1} = 1$. This completes the the proof. □

Lemma 2.13. *Let $f(z)$, $g(z)$ be two non-constant meromorphic functions of finite order σ, $c \in \mathbb{C} \setminus \{0\}$ and $n \in \mathbb{N}$ such that $n \geq 2$. Let $p(z)$ be a nonzero polynomial such that $2\deg(p) < n-1$. Suppose*

$$f^n(z)f(z+c)g^n(z)g(z+c) \equiv p^2(z).$$

Then $f(z)g(z) \equiv t$, where $p(z)$ reduces to a nonzero constant c_1, say, and t is a constant such that $t^{n+1} = c_1^2$.
In particular when f and g share $(\infty, 0)$ and $2\deg(p) < n+1$, then

$$f(z) = e^{Q(z)} \text{ and } g(z) = te^{-Q(z)},$$

where $p(z)$ reduces to a nonzero constant c_1, say, and t is a constant such that

$$t^{n+1} = c_1^2,$$

$Q(z)$ is a non-constant polynomial.

Proof. Suppose

$$f^n(z)f(z+c)g^n(z)g(z+c) \equiv p^2(z). \tag{2.4}$$

Let $h_1 = fg$. Then from (2.4) we have

$$h_1^n(z) \equiv \frac{p^2(z)}{h_1(z+c)}. \tag{2.5}$$

First we suppose that $h_1(z)$ is a non-constant meromorphic function. We now consider following two cases.
Case 1. Let $h_1(z)$ be a transcendental meromorphic function.
Now by Lemmas 2.1, 2.2 and 2.5 we get

$$\begin{aligned} nT(r, h_1) = T(r, h_1^n) + S(r, h_1) &= T\left(r, \frac{p^2}{h_1(z+c)}\right) + S(r, h_1) \\ &\leq N(r, 0; h_1(z+c)) + m\left(r, \frac{1}{h_1(z+c)}\right) + S(r, h_1) \\ &\leq N(r, 0; h_1(z)) + m\left(r, \frac{1}{h_1(z)}\right) + O(r^{\sigma-1+\varepsilon}) + S(r, h_1) \\ &\leq T(r, h_1) + O(r^{\sigma-1+\varepsilon}) + S(r, h_1), \end{aligned}$$

which is a contradiction.
Case 2. Let $h_1(z)$ be a rational function.
Let

$$h_1 = \frac{h_2}{h_3}, \tag{2.6}$$

where h_2 and h_3 are two nonzero relatively prime polynomials. From (2.6) we have

$$T(r, h_1) = \max\{\deg(h_2), \deg(h_3)\} \log r + O(1). \tag{2.7}$$

Now from (2.5), (2.6) and (2.7) we have

$$\begin{aligned} & n\ \max\{\deg(h_2), \deg(h_3)\}\ \log r \\ = \ & T(r, h_1^n) + O(1) \\ \leq \ & T(r, h_1(z+c)) + 2\,T(r,p) + O(1) \\ = \ & \max\{\deg(h_2), \deg(h_3)\}\ \log r + 2\ \deg(p)\ \log r + O(1). \end{aligned} \tag{2.8}$$

We see that

$$\max\{\deg(h_2), \deg(h_3)\} \geq 1.$$

Now from (2.8) we deduce that

$$n - 1 \leq 2\deg(p),$$

which contradicts our assumption that $2\deg(p) < n-1$.
Hence $h_1(z)$ is a non-zero constant. Let $h_1 = t \in \mathbb{C} \setminus \{0\}$. Therefore in this case $p(z)$ reduces to a non-zero constant. Let $p(z) = c_1 \in \mathbb{C} \setminus \{0\}$. So from (2.5) we see that

$$h_1^{n+1} \equiv c_1^2, \ \text{ i.e., } \ t^{n+1} \equiv c_1^2.$$

Therefore

$$f(z)g(z) \equiv t,$$

where t is a constant such that $t^{n+1} = c_1^2$.

In particular, suppose $f(z)$ and $g(z)$ share $(\infty, 0)$. Now from (2.4) one can easily say that $f(z)$ and $g(z)$ are non-constant entire functions.
Let $h_1 = fg$. First we suppose that h_1 is non-constant.

Now from Case 1, one can easily say that h_1 can not be a transcendental entire function. Hence h_1 is a non-constant polynomial. Since $2\deg(p) < n+1$, from (2.4), we arrive at a contradiction. Hence h_1 is a nonzero constant, say t. Therefore in this case $p(z)$ reduces to a non-zero constant. Let $p(z) = c_1 \in \mathbb{C} \setminus \{0\}$.

Clearly 0 is a Picard exceptional value of both $f(z)$ and $g(z)$. Consequently both $f(z)$ and $g(z)$ are transcendental entire functions.

Now by Lemma 2.11, $f(z)$ and $g(z)$ take the forms

$$f(z) = e^{Q(z)} \text{ and } g(z) = te^{-Q(z)},$$

where t is a constant such that $t^{n+1} = c_1^2$ and $Q(z)$ is a non-constant polynomial. This completes the proof. □

Lemma 2.14. *Let $f(z)$, $g(z)$ be two non-constant meromorphic functions of finite order σ, $c \in \mathbb{C} \setminus \{0\}$ and $n \in \mathbb{N}$ such that $n \geq 2$. Let $p(z)$ be a non-constant polynomial such that $2\deg(p) < n-1$. Then*

$$f^n(z)f(z+c)g^n(z)g(z+c) \not\equiv p^2(z).$$

Proof. The proof of lemma follows from Lemma 2.13. □

3. Proofs of the theorems

Proof of Theorem 1.1. Let $\Phi(z) = f^n(z)f(z+c)$. Now in view of Lemmas 2.1, 2.6 and the second theorem for small functions (see [17]), we get

$$
\begin{aligned}
&(n-1)T(r,f)\\
&\leq T(r,\Phi) + O(r^{\sigma-1+\varepsilon}) + S(r,f)\\
&\leq \overline{N}(r,0;\Phi) + \overline{N}(r,\infty;\Phi) + \overline{N}(r,a(z);\Phi) + O(r^{\sigma-1+\varepsilon}) + \left(\frac{\varepsilon}{3} + o(1)\right)T(r,f)\\
&\leq \overline{N}(r,0;f^n) + \overline{N}(r,0;f(z+c)) + \overline{N}(r,\infty;f^n) + \overline{N}(r,\infty;f(z+c)) + \overline{N}(r,a(z);\Phi)\\
&+ O(r^{\sigma-1+\varepsilon}) + \left(\frac{\varepsilon}{3} + o(1)\right)T(r,f)\\
&\leq 2\overline{N}(r,0;f) + 2\overline{N}(r,\infty;f) + \overline{N}(r,a(z);\Phi) + O(r^{\sigma-1+\varepsilon}) + \left(\frac{\varepsilon}{3} + o(1)\right)T(r,f)\\
&\leq \left(4 - 2\Theta(0;f) - 2\Theta(\infty;f) + \frac{2\varepsilon}{3}\right)T(r,f) + \overline{N}(r,a(z);\Phi)\\
&+ O(r^{\sigma-1+\varepsilon}) + \left(\frac{\varepsilon}{3} + o(1)\right)T(r,f)\\
&\leq (4 - 2\Theta(0;f) - 2\Theta(\infty;f) + \varepsilon)T(r,f) + \overline{N}(r,a(z);\Phi) + O(r^{\sigma-1+\varepsilon}) + o(T(r,f)),
\end{aligned}
$$

for all $\varepsilon > 0$. Take $\varepsilon < 2\Theta(0;f) + 2\Theta(\infty;f)$. Since $\Theta(0;f) + \Theta(\infty;f) > \frac{5-n}{2}$, from above one can easily say that $\Phi(z) - a(z)$ has infinitely many zeros. This completes the proof. □

Proof of Theorem 1.2. Let

$$F(z) = \frac{f^n(z)f(z+c)}{p(z)} \text{ and } G(z) = \frac{g^n(z)g(z+c)}{p(z)}.$$

Then F and G share $(1,2)$. We now consider following two cases.
Case 1. Suppose F is a Möbius transformation of G.
By Valiron-Mokhon'ko Lemma, we see that $T(r,F) = T(r,G) + O(1)$. Clearly $S(r,F) = S(r,G)$. Now in view of Lemmas 2.5 and 2.6, we get

$$
\begin{aligned}
&\overline{N}(r,0;F) + \overline{N}(r,0;G) + \overline{N}(r,\infty;F) + \overline{N}(r,\infty;G)\\
&= \overline{N}(r,0;f) + \overline{N}(r,0;f(z+c)) + \overline{N}(r,0;g) + \overline{N}(r,0;g(z+c))\\
&+ \overline{N}(r,\infty;f) + \overline{N}(r,\infty;f(z+c)) + \overline{N}(r,\infty;g) + \overline{N}(r,\infty;g(z+c))\\
&+ S(r,f) + S(r,g)\\
&= 2\overline{N}(r,0;f) + 2\overline{N}(r,\infty;f) + 2\overline{N}(r,0;g) + 2\overline{N}(r,\infty;g) + S(r,f) + S(r,g)\\
&\leq 4T(r,f) + 4\,T(r,g) + O(r^{\sigma-1+\varepsilon}) + O(\log r) + S(r,f) + S(r,g)\\
&\leq \frac{4}{n-1}T(r,F) + \frac{4}{n-1}T(r,G) + O(r^{\sigma-1+\varepsilon}) + O(\log r) + S(r,F) + S(r,G)\\
&\leq \frac{8}{n-1}T(r,F) + O(r^{\sigma-1+\varepsilon}) + O(\log r) + S(r,F),
\end{aligned}
$$

for all $\varepsilon > 0$. Since $n \geq 10$, we must have

$$\overline{N}(r,0;F) + \overline{N}(r,0;G) + \overline{N}(r,\infty;F) + \overline{N}(r,\infty;G) < (\lambda + o(1))T(r,F),$$

where $\lambda < 1$ and so by Lemma 2.10, we have either $F \equiv G$ or $F \cdot G \equiv 1$.
We now consider following two sub-cases.
Sub-case 1.1. $F \equiv G$.
Then by Lemma 2.12, we have $f(z) \equiv tg(z)$ for some constant $t \neq 1$ such that

$$t^{n+1} = 1.$$

Sub-case 1.2. $F \cdot G \equiv 1$.
Then

$$f^n(z)f(z+c)g^n(z)g(z+c) \equiv p^2(z)$$

and so by Lemma 2.13, we have $f(z)g(z) \equiv t$, where $p(z)$ reduces to a nonzero constant c_1, say and t is a constant such that

$$t^{n+1} = c_1^2.$$

Case 2. Suppose $n \geq 14$.
Now applying Lemma 2.8, we see that one of the following three sub-cases holds.
Sub-case 2.1. Suppose

$$T(r,F) \leq N_2(r,0;F) + N_2(r,0;G) + N_2(r,\infty;F) + N_2(r,\infty;G) + S(r,F) + S(r,G).$$

Now by applying Lemmas 2.1 and 2.5, we have

$$\begin{aligned}
&T(r,F)\\
&\leq N_2(r,0;F) + N_2(r,0;G) + N_2(r,\infty;F) + N_2(r,\infty;G) + S(r,f) + S(r,g)\\
&= N_2(r,0;f^n f(z+c)) + N_2(r,0;g^n g(z+c))\\
&+ N_2(r,\infty;f^n f(z+c)) + N_2(r,\infty;g^n g(z+c)) + S(r,f) + S(r,g)\\
&\leq N_2(r,0;f^n) + N_2(r,0;f(z+c)) + N_2(r,0;g^n) + N_2(r,0;g(z+c)) + N_2(r,\infty;f^n)\\
&+ N_2(r,\infty;f(z+c)) + N_2(r,\infty;g^n) + N_2(r,\infty;g(z+c)) + S(r,f) + S(r,g)\\
&\leq 2N(r,0;f) + N(r,0;f(z+c)) + 2N(r,0;g) + N(r,0;g(z+c)) + 2N(r,\infty;f)\\
&+ N(r,\infty;f(z+c)) + 2N(r,\infty;g) + N(r,\infty;g(z+c)) + S(r,f) + S(r,g)\\
&\leq 4T(r,f) + N(r,0;f) + N(r,\infty;f) + 4T(r,g)\\
&+ N(r,0;g) + N(r,\infty;g) + O(r^{\sigma-1+\varepsilon}) + O(\log r) + S(r,f) + S(r,g)\\
&\leq 6T(r,f) + 6T(r,g) + O(r^{\sigma-1+\varepsilon}) + O(\log r) + S(r,f) + S(r,g),
\end{aligned}$$

for all $\varepsilon > 0$. From Lemma 2.6, we have

$$\begin{aligned}
(n-1)T(r,f) &\leq 6T(r,f) + 6T(r,g) + O(r^{\sigma-1+\varepsilon}) + O(\log r) + S(r,f) + S(r,g)\\
&\leq 12T(r) + O(r^{\sigma-1+\varepsilon}) + O(\log r) + S(r). \qquad (3.1)
\end{aligned}$$

Similarly we have

$$(n-1)\,T(r,g) \leq 12\,T(r) + O(r^{\sigma-1+\varepsilon}) + O(\log r) + S(r). \qquad (3.2)$$

Combining (3.1) and (3.2), we get

$$(n-1)\,T(r) \leq 12\,T(r) + O(r^{\sigma-1+\varepsilon}) + O(\log r) + S(r),$$

which contradicts with $n \geq 14$.
Sub-case 2.2. $F \equiv G$.
Then by Lemma 2.12, we have $f(z) \equiv tg(z)$ for some constant $t \neq 1$ such that

$$t^{n+1} = 1.$$

Sub-case 2.3. $F \cdot G \equiv 1$.
Then by Lemma 2.13, we have $f(z)g(z) \equiv t$, where $p(z)$ reduces to a nonzero constant c_1, say and t is a constant such that $t^{n+1} = c_1^2$. This completes the proof. □

Proof of Theorem 1.3. Let

$$F(z) = \frac{f^n(z)f(z+c)}{p(z)} \text{ and } G(z) = \frac{g^n(z)g(z+c)}{p(z)}.$$

Also F, G share $(1,2)$ and (∞,∞) except for zeros of $p(z)$. We now consider following two cases.
Case 1. Suppose F is a Möbius transformation of G.
Let

$$F \equiv \frac{AG+B}{CG+D}, \tag{3.3}$$

where A, B, C, D are constants and $AD - BC \neq 0$. Again

$$T(r,F) = T(r,G) + O(1). \tag{3.4}$$

Clearly $S(r,F) = S(r,G)$. We now consider the following sub-cases:
Sub-case 1.1. Let $AC \neq 0$. Since F, G share (∞,∞), it follows from (3.3) that

$$N(r,\infty;F) = S(r,F) \text{ and } N(r,\infty;G) = S(r,F).$$

Again since

$$F \equiv \frac{A + \frac{B}{G}}{C + \frac{D}{G}},$$

it follows that

$$N(r,\frac{A}{C};F) = S(r,F).$$

So in view of Lemma 2.6 and using the second fundamental theorem, we get

$$\begin{aligned}(n-1)T(r,f) &\leq T(r,f^n(z)f(z+c)) + O(r^{\sigma-1+\varepsilon}) + S(r,f)\\ &\leq T(r,F) + O(r^{\sigma-1+\varepsilon}) + S(r,f)\\ &\leq \overline{N}(r,0;F) + \overline{N}(r,\infty;F) + \overline{N}\left(r,\frac{A}{C};F\right) + O(r^{\sigma-1+\varepsilon}) + S(r,f)\\ &\leq 2\overline{N}(r,0;f) + 2\overline{N}(r,\infty;f) + O(r^{\sigma-1+\varepsilon}) + S(r,f)\\ &\leq 4T(r,f) + O(r^{\sigma-1+\varepsilon}) + S(r,f),\end{aligned}$$

for all $\varepsilon > 0$, which is impossible since $n \geq 6$.
Sub-case 1.2. Let $A \neq 0$ and $C = 0$. Then $F \equiv \alpha G + \beta$, where

$$\alpha = \frac{A}{D} \neq 0 \text{ and } \beta = \frac{B}{D}.$$

Sub-case 1.2.1. Let $\beta = 0$. Then we get $F \equiv \alpha G$. Since $n \geq 6$, it follows that $F - 1$ and $G - 1$ have infinitely many zeros. Clearly 1 can not be a Picard exceptional value of F and G. Since F, G share $(1, \infty)$, it follows that $\alpha = 1$ and so $F \equiv G$, i.e.,

$$f^n(z)f(z+c) \equiv g^n(z)g(z+c).$$

Now by Lemma 2.12, we have $f(z) \equiv tg(z)$ for some constant $t \neq 1$ such that $t^{n+1} = 1$.
Sub-case 1.2.2. Let $\beta \neq 0$. Clearly $\alpha \neq 1$, as F, G share $(1, \infty)$. So in view of Lemmas 2.5 and 2.6 and using the second fundamental theorem, we get

$$\begin{aligned}
&(n-1)T(r,f)\\
&\leq T(r, f^n(z)f(z+c)) + O(r^{\sigma-1+\varepsilon}) + S(r,f)\\
&\leq T(r,F) + O(r^{\sigma-1+\varepsilon}) + S(r,f)\\
&\leq \overline{N}(r,\infty;F) + \overline{N}(r,0;F) + \overline{N}(r,\beta;F) + O(r^{\sigma-1+\varepsilon}) + S(r,f)\\
&\leq \overline{N}(r,\infty;F) + \overline{N}(r,0;F) + \overline{N}(r,0;G) + S(r,f)\\
&\leq 2\overline{N}(r,\infty;f) + 2\overline{N}(r,0;f) + 2\overline{N}(r,0;g) + O(r^{\sigma-1+\varepsilon}) + S(r,f) + S(r,g)\\
&\leq 4T(r,g) + 2T(r,f) + O(r^{\sigma-1+\varepsilon}) + O(\log r) + S(r,f) + S(r,g),
\end{aligned}$$

for all $\varepsilon > 0$. Without loss of generality, we suppose that there exists a set I with infinite measure such that $T(r,f) \leq T(r,g)$ for $r \in I$.
So for $r \in I$, we have

$$(n-7)\, T(r,g) \leq O(r^{\sigma-1+\varepsilon}) + O(\log r) + S(r,g),$$

for all $\varepsilon > 0$, which is a contradiction since $n \geq 8$.
Case 1.3. Let $A = 0$ and $C \neq 0$. Then $F \equiv \frac{1}{\gamma G+\delta}$, where $\gamma = \frac{C}{B} \neq 0$ and $\delta = \frac{D}{B}$.
Sub-case 1.3.1. Let $\delta = 0$. Then $F \equiv \frac{1}{\gamma G}$. Since F, G share $(1, \infty)$, it follows that $\gamma = 1$ and then $FG \equiv 1$, i.e., $f^n(z)f(z+c)g^n(z)g(z+c) \equiv p^2(z)$. Now by Lemma 2.13, we have $f(z) = e^{Q(z)}$ and $g(z) = te^{-Q(z)}$, where $p(z)$ reduces to a nonzero constant c_1, say and t is a constant such that $t^{n+1} = c_1^2$ and $Q(z)$ is a non-constant polynomial.
Sub-case 1.3.2. Let $\delta \neq 0$. Clearly $\gamma \neq 1$, as F, G share $(1, \infty)$. Since F, G share (∞, ∞), it follows that $N(r,\infty;F) = S(r,F)$ and $N(r,\infty;G) = S(r,F)$. Consequently

$$N(r, -\frac{\delta}{\gamma}; G) = S(r,F).$$

So in view of Lemma 2.6 and using the second fundamental theorem, we get

$$\begin{aligned}
&(n-1)T(r,g)\\
&\leq T(r, g^n(z)g(z+c)) + O(r^{\sigma-1+\varepsilon}) + S(r,g)\\
&\leq T(r,G) + O(r^{\sigma-1+\varepsilon}) + S(r,g)\\
&\leq \overline{N}(r,0;G) + \overline{N}(r,\infty;G) + \overline{N}\left(r, \frac{-\delta}{\gamma}; G\right) + O(r^{\sigma-1+\varepsilon}) + S(r,g)\\
&\leq 2\overline{N}(r,0;g) + 2\overline{N}(r,\infty;g) + O(r^{\sigma-1+\varepsilon}) + S(r,g)\\
&\leq 4T(r,g) + O(r^{\sigma-1+\varepsilon}) + S(r,g),
\end{aligned}$$

for all $\varepsilon > 0$, which is impossible since $n \geq 6$.

Case 2. Suppose $n \geq 12$.
We now consider following two sub-cases.

Sub-case 2.1. Let $H \not\equiv 0$.
From (2.1) it can be easily calculated that the possible poles of H occur at (i) multiple zeros of F and G, (ii) those 1 points of F and G whose multiplicities are different, (iii) those poles of F and G whose multiplicities are different, (iv) zeros of $F'(G')$ which are not the zeros of $F(F-1)(G(G-1))$.
Since H has only simple poles we get

$$\begin{aligned} & N(r,\infty;H) \\ \leq\; & \overline{N}_*(r,\infty;F,G)+\overline{N}_*(r,1;F,G)+\overline{N}(r,0;F|\geq 2)+\overline{N}(r,0;G|\geq 2) \\ +\; & \overline{N}_0(r,0;F')+\overline{N}_0(r,0;G')+S(r,f)+S(r,g), \end{aligned} \tag{3.5}$$

where $\overline{N}_0(r,0;F')$ is the reduced counting function of those zeros of F' which are not the zeros of $F(F-1)$ and $\overline{N}_0(r,0;G')$ is similarly defined.
Let z_0 be a simple zero of $F-1$ but $p(z_0)\neq 0$. Then z_0 is a simple zero of $G-1$ and a zero of H. So

$$N(r,1;F|=1) \leq N(r,0;H) \leq N(r,\infty;H)+S(r,f)+S(r,g). \tag{3.6}$$

Note that

$$\overline{N}_*(r,\infty;F,G)=S(r,f).$$

Now using (3.5) and (3.6) we get

$$\begin{aligned} & \overline{N}(r,1;F) \\ \leq\; & N(r,1;F|=1)+\overline{N}(r,1;F|\geq 2) \\ \leq\; & \overline{N}_*(r,1;F,G)+\overline{N}(r,0;F|\geq 2)+\overline{N}(r,0;G\,|\geq 2) \\ +\; & \overline{N}(r,1;F\,|\geq 2)+\overline{N}_0(r,0;F')+\overline{N}_0(r,0;G')+S(r,f)+S(r,g). \end{aligned} \tag{3.7}$$

Now in view of Lemma 2.3 we get

$$\begin{aligned} & \overline{N}_0(r,0;G')+\overline{N}(r,1;F\,|\geq 2)+\overline{N}_*(r,1;F,G) \\ \leq\; & \overline{N}_0(r,0;G')+\overline{N}(r,1;F\,|\geq 2)+\overline{N}(r,1;F\,|\geq 3) \\ =\; & \overline{N}_0(r,0;G')+\overline{N}(r,1;G\,|\geq 2)+\overline{N}(r,1;G\,|\geq 3) \\ \leq\; & \overline{N}_0(r,0;G')+N(r,1;G)-\overline{N}(r,1;G) \\ \leq\; & N(r,0;G'\mid G\neq 0) \leq \overline{N}(r,0;G)+\overline{N}(r,\infty;G)+S(r,g). \end{aligned} \tag{3.8}$$

Hence using (3.7), (3.8) and Lemma 2.5, we get from the second fundamental theorem that

$$\begin{aligned}
&T(r,F)\\
&\leq \overline{N}(r,0;F)+\overline{N}(r,\infty;F)+\overline{N}(r,1;F)-N_0(r,0;F')+S(r,f)\\
&\leq \overline{N}(r,0;F)+\overline{N}(r,\infty;F)+\overline{N}(r,0;F\mid\geq 2)+\overline{N}(r,0;G\mid\geq 2)+\overline{N}_*(r,1;F,G)\\
&+\overline{N}(r,1;F\mid\geq 2)+\overline{N}_0(r,0;G')+S(r,f)+S(r,g)\\
&\leq N_2(r,0;F)+N_2(r,0;G)+\overline{N}(r,\infty;F)+\overline{N}(r,\infty;G)+S(r,f)+S(r,g)\\
&= N_2(r,0;f^nf(z+c))+N_2(r,0;g^ng(z+c))\\
&+\overline{N}(r,\infty;f^nf(z+c))+\overline{N}(r,\infty;g^ng(z+c))+S(r,f)+S(r,g)\\
&\leq N_2(r,0;f^n)+N_2(r,0;f(z+c))+N_2(r,0;g^n)+N_2(r,0;g(z+c))+\overline{N}(r,\infty;f^n)\\
&+\overline{N}(r,\infty;f(z+c))+\overline{N}(r,\infty;g^n)+\overline{N}(r,\infty;g(z+c))+S(r,f)+S(r,g)\\
&\leq 2\;N(r,0;f)+N(r,0;f(z+c))+2\;N(r,0;g)+N(r,0;g(z+c))+N(r,\infty;f)\\
&+N(r,\infty;f(z+c))+N(r,\infty;g)+N(r,\infty;g(z+c))+S(r,f)+S(r,g)\\
&\leq 3T(r,f)+N(r,0;f)+N(r,\infty;f)+3T(r,g)\\
&+N(r,0;g)+N(r,\infty;g)+O(r^{\sigma-1+\varepsilon})+O(\log r)+S(r,f)+S(r,g)\\
&\leq 5T(r,f)+5T(r,g)+O(r^{\sigma-1+\varepsilon})+O(\log r)+S(r,f)+S(r,g),
\end{aligned}$$

for all $\varepsilon>0$. From Lemma 2.6, we have

$$\begin{aligned}
(n-1)T(r,f) &\leq 5T(r,f)+5T(r,g)+O(r^{\sigma-1+\varepsilon})+O(\log r)+S(r,f)+S(r,g)\\
&\leq 10T(r)+O(r^{\sigma-1+\varepsilon})+O(\log r)+S(r).
\end{aligned} \tag{3.9}$$

Similarly we have

$$(n-1)\;T(r,g)\leq 10\;T(r)+O(r^{\sigma-1+\varepsilon})+O(\log r)+S(r). \tag{3.10}$$

Combining (3.9) and (3.10), we get

$$(n-1)\;T(r)\leq 10\;T(r)+O(r^{\sigma-1+\varepsilon})+O(\log r)+S(r),$$

which contradicts with $n\geq 12$.

Case 2.2. Let $H\equiv 0$.

Here in view of Lemmas 2.5, 2.6 and proceeding in the same way as done in Theorem 1.2, we get

$$\begin{aligned}
&\overline{N}(r,0;F)+\overline{N}(r,0;G)+\overline{N}(r,\infty;F)+\overline{N}(r,\infty;G)\\
\leq\;&\frac{8}{n-1}\;T(r,F)+O(r^{\sigma-1+\varepsilon})+S(r,F),
\end{aligned}$$

for all $\varepsilon>0$. Since $n\geq 10$, we must have

$$\overline{N}(r,0;F)+\overline{N}(r,0;G)+\overline{N}(r,\infty;F)+\overline{N}(r,\infty;G)<(\lambda+o(1))T(r,F),$$

where $\lambda<1$ and so by Lemma 2.9, we have either $F\equiv G$ or $F\cdot G\equiv 1$.

We now consider following two sub-cases.

Sub-case 2.2.1. $F \equiv G$.
Then by Lemma 2.12, we have $f(z) \equiv tg(z)$ for some constant $t \neq 1$ such that $t^{n+1} = 1$.
Sub-case 2.2.2. $F \cdot G \equiv 1$.
Then $f^n(z)f(z+c)g^n(z)g(z+c) \equiv p^2(z)$ and so by Lemma 2.13, we have $f(z) = e^{Q(z)}$ and $g(z) = te^{-Q(z)}$, where $p(z)$ reduces to a nonzero constant c_1, say and t is a constant such that $t^{n+1} = c_1^2$, $Q(z)$ is a non-constant polynomial. This completes the proof. □

Proof of Theorem 1.4. Let

$$F(z) = \frac{f^n(z)f(z+c)}{p(z)} \text{ and } G(z) = \frac{g^n(z)g(z+c)}{p(z)}.$$

Then F and G share $(1,2)$ except for zeros of $p(z)$. Note that

$$T(r, f^n(z)f(z+c)) \leq T(r,F) + l\log r \text{ and } T(r, g^n(z)g(z+c)) \leq T(r,G) + l\log r.$$

Also we see that $T(r,f) \geq \log r + O(1)$ and $T(r,g) \geq \log r + O(1)$.
Now applying Lemma 2.8, we see that one of the following three cases holds.
Case 1. Suppose

$$T(r,F) \leq N_2(r,0;F) + N_2(r,0;G) + N_2(r,\infty;F) + N_2(r,\infty;G) + S(r,F) + S(r,G).$$

Now by applying Lemmas 2.1 and 2.5, we have

$$\begin{aligned}
& T(r,F) \\
\leq\ & N_2(r,0;F) + N_2(r,0;G) + N_2(r,\infty;F) + N_2(r,\infty;G) + S(r,f) + S(r,g) \\
=\ & N_2(r,0;f^n f(z+c)) + N_2(r,0;g^n g(z+c)) \\
& + N_2(r,\infty;f^n f(z+c)) + N_2(r,\infty;g^n g(z+c)) + 2l\log r + S(r,f) + S(r,g) \\
\leq\ & N_2(r,0;f^n) + N_2(r,0;f(z+c)) + N_2(r,0;g^n) + N_2(r,0;g(z+c)) + N_2(r,\infty;f^n) \\
& + N_2(r,\infty;f(z+c)) + N_2(r,\infty;g^n) + N_2(r,\infty;g(z+c)) + 2l\log r + S(r,f) + S(r,g) \\
\leq\ & 2N(r,0;f) + N(r,0;f(z+c)) + 2N(r,0;g) + N(r,0;g(z+c)) + 2N(r,\infty;f) \\
& + N(r,\infty;f(z+c)) + 2N(r,\infty;g) + N(r,\infty;g(z+c)) + 2l\log r + S(r,f) + S(r,g) \\
\leq\ & 4T(r,f) + N(r,0;f) + N(r,\infty;f) + 4T(r,g) \\
& + N(r,0;g) + N(r,\infty;g) + 2l\log r + O(r^{\sigma-1+\varepsilon}) + O(\log r) + S(r,f) + S(r,g) \\
\leq\ & 6T(r,f) + 6T(r,g) + 2l\log r + O(r^{\sigma-1+\varepsilon}) + O(\log r) + S(r,f) + S(r,g),
\end{aligned}$$

for all $\varepsilon > 0$. From Lemma 2.6, we have

$$\begin{aligned}
& (n-1)T(r,f) \qquad (3.11) \\
\leq\ & T(r, f^n(z)f(z+c)) + O(r^{\sigma-1+\varepsilon}) \\
\leq\ & T(r,F) + l\log r + O(r^{\sigma-1+\varepsilon}) \\
\leq\ & 6T(r,f) + 6T(r,g) + 3l\log r + O(r^{\sigma-1+\varepsilon}) + O(\log r) + S(r,f) + S(r,g) \\
\leq\ & (12+3l)T(r) + O(r^{\sigma-1+\varepsilon}) + O(\log r) + S(r).
\end{aligned}$$

Similarly we have

$$(n-1)\, T(r,g) \leq (12+3l)\, T(r) + O(r^{\sigma-1+\varepsilon}) + O(\log r) + S(r). \qquad (3.12)$$

Combining (3.11) and (3.12), we get

$$(n-1)\,T(r) \leq (12+3l)\,T(r) + O(r^{\sigma-1+\varepsilon}) + O(\log r) + S(r),$$

which contradicts with $n \geq 14+3l$.

Sub-case 2.2. $F \equiv G$.

Then by Lemma 2.12 we have $f(z) \equiv tg(z)$ for some constant $t \neq$ such that $t^{n+1} = 1$.

Sub-case 2.3. $F \cdot G \equiv 1$.

Then we have $f^n(z)f(z+c)g^n(z)g(z+c) \equiv p^2(z)$. But this is impossible by Lemma 2.14. This completes the proof. □

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On (h, k, μ, ν)-trichotomy of evolution operators in Banach spaces

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Abstract. The paper considers some concepts of trichotomy with different growth rates for evolution operators in Banach spaces. Connections between these concepts and characterizations in terms of Lyapunov- type norms are given.

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Keywords: Evolution operator, trichotomy.

1. Introduction

In the qualitative theory of evolution equations, exponential dichotomy, essentially introduced by O. Perron in [16] is one of the most important asymptotic properties and in last years it was treated from various perspective.

For some of the most relevant early contributions in this area we refer to the books of J.L. Massera and J.J. Schaffer [11], Ju. L. Dalecki and M.G. Krein [8] and W.A. Coppel [6]. We also refer to the book of C. Chichone and Yu. Latushkin [5].

In some situations, particularly in the nonautonomous setting, the concept of uniform exponential dichotomy is too restrictive and it is important to consider more general behaviors. Two different perspectives can be identify for to generalize the concept of uniform exponential dichotomy: on one hand one can define dichotomies that depend on the initial time (and therefore are nonuniform) and on the other hand one can consider growth rates that are not necessarily exponential.

The first approach leads to concepts of nonuniform exponential dichotomies and can be found in the works of L. Barreira and C. Valls [1] and in a different form in the works of P. Preda and M. Megan [20] and M. Megan, L. Sasu and B. Sasu [13].

The second approach is present in the works of L. Barreira and C. Valls [2], A.J.G. Bento and C.M. Silva [3] and M. Megan [12].

A more general dichotomy concept is introduced by M. Pinto in [19] called (h, k)-dichotomy, where h and k are growth rates. The concept of $(h, k)-$ dichotomy has a great generality and it permits the construction of similar notions for systems with dichotomic behaviour which are not described by the classical theory of J.L. Massera [11].

As a natural generalization of exponential dichotomy (see [2], [7], [9], [21], [22] and the references therein), exponential trichotomy is one of the most complex asymptotic properties of dynamical systems arising from the central manifold theory (see [4]). In the study of the trichotomy the main idea is to obtain a decomposition of the space at every moment into three closed subspaces: the stable subspace, the unstable subspace and the central manifold.

Two concepts of trichotomy have been introduced: the first by R.J. Sacker and G.L. Sell [21] (called (S,S)-trichotomy) and the second by S. Elaydi and O. Hayek [9] (called (E,H)-trichotomy).

The existence of exponential trichotomies is a strong requirement and hence it is of considerable interest to look for more general types of trichotomic behaviors.

In previous studies of uniform and nonuniform trichotomies, the growth rates are always assumed to be the same type functions. However, the nonuniformly hyperbolic dynamical systems vary greatly in forms and none of the nonuniform trichotomy can well characterize all the nonuniformly dynamics. Thus it is necessary and reasonable to look for more general types of nonuniform trichotomies.

The present paper considers the general concept of nonuniform $(h, k, \mu, \nu)-$ trichotomy, which not only incorporates the existing notions of uniform or nonuniform trichotomy as special cases, but also allows the different growth rates in the stable subspace, unstable subspace and the central manifold.

We give characterizations of nonuniform $(h, k, \mu, \nu)-$ trichotomy using families of norms equivalent with the initial norm of the states space. Thus we obtain a characterization of the nonuniform $(h, k, \mu, \nu)-$trichotomy in terms of a certain type of uniform $(h, k, \mu, \nu)-$trichotomy.

As an original reference for considering families of norms in the nonuniform theory we mention Ya. B. Pesin's works [17] and [18]. Our characterizations using families of norms are inspired by the work of L. Barreira and C. Valls [2] where characterizations of nonuniform exponential trichotomy in terms of Lyapunov functions are given.

2. Preliminaries

Let X be a Banach space and $\mathcal{B}(X)$ the Banach algebra of all linear and bounded operators on X. The norms on X and on $\mathcal{B}(X)$ will be denoted by $\|\cdot\|$. The identity operator on X is denoted by I. We also denote by $\Delta = \{(t, s) \in \mathbb{R}_+^2 : t \geq s \geq 0\}$. We recall that an application $U : \Delta \to \mathcal{B}(X)$ is called *evolution operator* on X if

(e_1) $U(t, t) = I$, for every $t \geq 0$

and

(e_2) $U(t, t_0) = U(t, s)U(s, t_0)$, for all $(t, s), (s, t_0) \in \Delta$.

Definition 2.1. A map $P : \mathbb{R}_+ \to \mathcal{B}(X)$ is called

(i) *a family of projectors* on X if

$$P^2(t) = P(t), \text{ for every } t \geq 0;$$

(ii) *invariant* for the evolution operator $U : \Delta \to \mathcal{B}(X)$ if

$$U(t,s)P(s)x = P(t)U(t,s)x,$$

for all $(t,s,x) \in \Delta \times X$;

(iii) *strongly invariant* for the evolution operator $U : \Delta \to \mathcal{B}(X)$ if it is invariant for U and for all $(t,s) \in \Delta$ the restriction of $U(t,s)$ on Range $P(s)$ is an isomorphism from Range $P(s)$ to Range $P(t)$.

Remark 2.2. It is obvious that if P is strongly invariant for U then it is also invariant for U. The converse is not valid (see [15]).

Remark 2.3. If the family of projectors $P : \mathbb{R}_+ \to \mathcal{B}(X)$ is strongly invariant for the evolution operator $U : \Delta \to \mathcal{B}(X)$ then ([10]) there exists a map $V : \Delta \to \mathcal{B}(X)$ with the properties:

(v_1) $V(t,s)$ is an isomorphism from Range $P(t)$ to Range $P(s)$,
(v_2) $U(t,s)V(t,s)P(t)x = P(t)x$,
(v_3) $V(t,s)U(t,s)P(s)x = P(s)x$,
(v_4) $V(t,t_0)P(t) = V(s,t_0)V(t,s)P(t)$,
(v_5) $V(t,s)P(t) = P(s)V(t,s)P(t)$,
(v_6) $V(t,t)P(t) = P(t)V(t,t)P(t) = P(t)$,

for all $(t,s), (s,t_0) \in \Delta$ and $x \in X$.

Definition 2.4. Let $P_1, P_2, P_3 : \mathbb{R} \to \mathcal{B}(X)$ be three families of projectors on X. We say that the family $\mathcal{P} = \{P_1, P_2, P_3\}$ is

(i) *orthogonal* if
  o_1) $P_1(t) + P_2(t) + P_3(t) = I$ for every $t \geq 0$
  and
  o_2) $P_i(t)P_j(t) = 0$ for all $t \geq 0$ and all $i, j \in \{1,2,3\}$ with $i \neq j$;

(ii) *compatible* with the evolution operator $U : \Delta \to \mathcal{B}(X)$ if
  c_1) P_1 is invariant for U
  and
  c_2) P_2, P_3 are strongly invariant for U.

In what follows we shall denote by $V_j(t,s)$ the isomorphism (given by Remark 2.3) from Range $P_j(t)$ to Range $P_j(s)$ and $j \in \{2,3\}$, where $\mathcal{P} = \{P_2, P_2, P_3\}$ is compatible with U.

Definition 2.5. We say that a nondecreasing map $h : \mathbb{R}_+ \to [1, \infty)$ is a *growth rate* if

$$\lim_{t \to \infty} h(t) = \infty.$$

As particular cases of growth rates we remark:

(r_1) *exponential rates*, i.e. $h(t) = e^{\alpha t}$ with $\alpha > 0$;
(r_2) *polynomial rates*, i.e. $h(t) = (t+1)^{\alpha}$ with $\alpha > 0$.

Let $\mathcal{P} = \{P_1, P_2, P_3\}$ be an orthogonal family of projectors which is compatible with the evolution operator $U : \Delta \to \mathcal{B}(X)$ and $h, k, \mu, \nu : \mathbb{R}_+ \to [1, \infty)$ be four growth rates.

Definition 2.6. We say that the pair $(U,\mathcal{P})$ is (h,k,μ,ν)*-trichotomic* (and we denote $(h,k,\mu,\nu)-t)$ if there exists a nondecreasing function $N:\mathbb{R}_+\to[1,\infty)$ such that
(ht_1) $h(t)\|U(t,s)P_1(s)x\|\le N(s)h(s)\|P_1(s)x\|$
(kt_1) $k(t)\|P_2(s)x\|\le N(t)k(s)\|U(t,s)P_2(s)x\|$
(μt_1) $\mu(s)\|U(t,s)P_3(s)x\|\le N(s)\mu(t)\|P_3(s)x\|$
(νt_1) $\nu(s)\|P_3(s)x\|\le N(t)\nu(t)\|U(t,s)P_3(s)x\|$,
for all $(t,s,x)\in\Delta\times X$.

In particular, if the function N is constant then we obtain the *uniform* (h,k,μ,ν)-*trichotomy* property, denoted by $u-(h,k,\mu,\nu)-t$.

Remark 2.7. As important particular cases of (h,k,μ,ν)-trichotomy we have:

(i) *(nonuniform) exponential trichotomy* (*et*) and respectively *uniform exponential trichotomy* (*uet*) when the rates h,k,μ,ν are exponential rates;
(ii) *(nonuniform) polynomial trichotomy* (*pt*) and respectively *uniform polynomial trichotomy* (*upt*) when the rates h,k,μ,ν are polynomial rates;
(iii) *(nonuniform)* $(h,k)-$*dichotomy* $((h,k)-d)$ respectively uniform $(h,k)-$dichotomy $(u-(h,k)-d)$ for $P_3=0$;
(iv) *(nonuniform) exponential dichotomy* (*ed*) and respectively *uniform exponential dichotomy* (*ued*) when $P_3=0$ and the rates h,k are exponential rates;
(v) *(nonuniform) polynomial dichotomy* (p.d.) and respectively *uniform polynomial dichotomy* (*upd*) when $P_3=0$ and the rates h,k are polynomial rates;

It is obvious that if the pair $(U,\mathcal{P})$ is $u-(h,k,\mu,\nu)-t$ then it is also $(h,k,\mu,\nu)-t$ In general, the reverse of this statement is not valid, phenomenon illustrated by

Example 2.8. Let $U:\Delta\to\mathcal{B}(X)$ be the evolution operator defined by

$$U(t,s)=\frac{u(s)}{u(t)}\left(\frac{h(s)}{h(t)}P_1(s)+\frac{k(t)}{k(s)}P_2(s)+\frac{\mu(t)}{\mu(s)}\frac{\nu(s)}{\nu(t)}P_3(s)\right)\tag{2.1}$$

where $u,h,k,\mu,\nu:\mathbb{R}_+\to[1,\infty)$ are growth rates and $P_1,P_2,P_3:\mathbb{R}_+\to\mathcal{B}(X)$ are projectors families on X with the properties:

(i) $P_1(t)+P_2(t)+P_3(t)=I$ for every $t\ge 0$;
(ii) $P_i(t)P_j(s)=\begin{cases}0 & \text{if } i\neq j\\ P_i(s), & \text{if } i=j,\end{cases}$ for all $(t,s)\in\Delta$.
(iii) $U(t,s)P_i(s)=P_i(t)U(t,s)$ for all $(t,s)\in\Delta$ and all $i\in\{1,2,3\}$.

For example if P_1,P_2,P_3 are constant and orthogonal then the conditions (i), (ii) and (iii) are satisfied.

We observe that

$$h(t)\|U(t,s)P_1(s)x\|=\frac{u(s)h(s)}{u(t)}\|P_1(s)x\|\le u(s)h(s)\|P_1(s)x\|$$
$$u(t)k(s)\|U(t,s)P_2(s)x\|=u(s)k(s)\|P_2(s)x\|\ge k(t)\|P_2(s)x\|$$
$$\mu(s)\|U(t,s)P_3(s)x\|=\frac{u(s)\mu(t)\nu(s)}{u(t)\nu(t)}\|P_3(s)x\|\le u(s)\mu(t)\|P_3(s)x\|$$
$$u(t)\nu(t)\|U(t,s)P_3(s)x\|=\frac{u(s)\nu(s)\mu(t)}{\mu(s)}\|P_3(s)x\|\ge\nu(s)\|P_3(s)x\|$$

for all $(t, s, x) \in \Delta \times X$.

Thus the pair $(U, \mathcal{P})$ is $(h, k, \mu, \nu) - t$.

If we assume that the pair $(U, \mathcal{P})$ is $u - (h, k, \mu, \nu) - t$ then there exists a real constant $N \geq 1$ such that

$$Nu(s) \geq u(t), \text{ for all } (t, s) \in \Delta.$$

Taking $s = 0$ we obtain a contradiction.

Remark 2.9. The previous example shows that for all four growth rates h, k, μ, ν there exits a pair $(U, \mathcal{P})$ which is $(h, k, \mu, \nu) - t$ and is not $u - (h, k, \mu, \nu) - t$.

In the particular case when $\mathcal{P}$ is compatible with U a characterization of $(h, k, \mu, \nu) - t$ is given by

Proposition 2.10. *If* $\mathcal{P} = \{P_1, P_2, P_3\}$ *is compatible with the evolution operator* $U : \Delta \to \mathcal{B}(X)$ *then the pair* $(U, \mathcal{P})$ *is* (h, k, μ, ν)*-trichotomic if and only if there exists a nondecreasing function* $N_1 : \mathbb{R}_+ \to [1, \infty)$ *such that*
(ht_2) $h(t)\|U(t,s)P_1(s)x\| \leq N_1(s)h(s)\|x\|$
(kt_2) $k(t)\|V_2(t,s)P_2(t)x\| \leq N_1(t)k(s)\|x\|$
(μt_2) $\mu(s)\|U(t,s)P_3(s)x\| \leq N_1(s)\mu(t)\|x\|$
(νt_2) $\nu(s)\|V_3(t,s)P_3(t)x\| \leq N_1(t)\nu(t)\|x\|$
for all $(t, s, x) \in \Delta \times X$*, where* $V_j(t, s)$ *for* $j \in \{2, 3\}$ *is the isomorphism from Range* $P_j(t)$ *to Range* $P_j(s)$*.*

Proof. Necessity. By Remark 2.3 and the Definition 2.6 we obtain

$$\begin{aligned}
(ht_2)\quad & h(t)\|U(t,s)P_1(s)x\| \leq N(s)h(s)\|P_1(s)x\| \leq N(s)\|P_1(s)\|h(s)\|x\| \\
& \leq N_1(s)h(s)\|x\| \\
(kt_2)\quad & k(t)\|V_2(t,s)P_2(t)x\| = k(t)\|P_2(s)V_2(t,s)P_2(t)x\| \\
& \leq N(t)k(s)\|U(t,s)P_2(s)V_2(t,s)P_2(t)x\| \\
& = N(t)k(s)\|P_2(t)x\| \leq N(t)\|P_2(t)\|k(s)\|x\| \leq N_1(t)k(s)\|x\| \\
(\mu t_2)\quad & \mu(s)\|U(t,s)P_3(s)x\| \leq N(s)\mu(t)\|P_3(s)x\| \leq N(s)\|P_3(s)\|\mu(t)\|x\| \\
& \leq N_1(s)\mu(t)\|x\| \\
(\nu t_2)\quad & \nu(s)\|V_3(t,s)P_3(t)x\| = \nu(s)\|P_3(s)V_3(t,s)P_3(t)x\| \\
& \leq N(t)\nu(t)\|U(t,s)P_3(s)V_3(t,s)P_3(t)x\| \\
& = N(t)\nu(t)\|P_3(t)x\| \leq N(t)\|P_3(t)\|\nu(t)\|x\| \leq N_1(t)\nu(t)\|x\|,
\end{aligned}$$

for all $(t, s, x) \in \Delta \times X$, where

$$N_1(t) = \sup_{s \in [0,t]} N(s)(\|P_1(s)\| + \|P_2(s)\| + \|P_3(s)\|).$$

Sufficiency. The implications $(ht_2) \Rightarrow (ht_1)$ and $(\mu t_2) \Rightarrow (\mu t_1)$ result by replacing x with $P_1(s)x$ respectively by $P_3(s)x$.

For the implications $(kt_2) \Rightarrow (kt_1)$ and $(\nu t_2) \Rightarrow (\nu t_1)$ we have (by Remark 2.3)

$$k(t)\|P_2(s)x\| = k(t)\|V_2(t,s)U(t,s)P_2(s)x\| \leq N(t)k(s)\|U(t,s)P_2(s)x\|$$

and

$$\nu(s)\|P_3(s)x\| = \nu(s)\|V_3(t,s)U(t,s)P_3(s)x\| \leq N(t)\nu(t)\|U(t,s)P_3(s)x\|,$$

for all $(t,s,x) \in \Delta \times X$. □

A similar characterization for the $u-(h,k,\mu,\nu)-t$ concept results under the hypotheses of boundedness of the projectors P_1, P_2, P_3. A characterization with compatible family of projectors without assuming the boundedness of projectors is given by

Proposition 2.11. *If $\mathcal{P} = \{P_1, P_2, P_3\}$ is compatible with the evolution operator $U : \Delta \to \mathcal{B}(X)$ then the pair $(U, \mathcal{P})$ is uniformly$-(h,k,\mu,\nu)-$ trichotomic if and only if there exists a constant $N \geq 1$ such that*
(uht_1) $h(t)\|U(t,s)P_1(s)x\| \leq Nh(s)\|P_1(s)x\|$
(ukt_1) $k(t)\|V_2(t,s)P_2(t)x\| \leq Nk(s)\|P_2(t)x\|$
$(u\mu t_1)$ $\mu(s)\|U(t,s)P_3(s)x\| \leq N\mu(t)\|P_3(s)x\|$
$(u\nu t_1)$ $\nu(s)\|V_3(t,s)P_3(t)x\| \leq N\nu(t)\|P_3(t)x\|$
for all $(t,s,x) \in \Delta \times X$, where $V_j(t,s)$ for $j \in \{2,3\}$ is the isomorphism from Range $P_j(t)$ to Range $P_j(s)$.

Proof. It is similar to the proof of Proposition 2.10. □

3. The main results

In this section we give a characterization of $(h,k,\mu,\nu)-$trichotomy in terms of a certain type of uniform $(h,k,\mu,\nu)-$trichotomy using families of norms equivalent with the norms of X. Firstly we introduce

Definition 3.1. A family $\mathcal{N} = \{\|\cdot\|_t : t \geq 0\}$ of norms on the Banach space X (endowed with the norm $\|\cdot\|$) is called *compatible* with the norm $\|\cdot\|$ if there exists a nondecreasing map $C : \mathbb{R}_+ \to [1,\infty)$ such that

$$\|x\| \leq \|x\|_t \leq C(t)\|x\|, \tag{3.1}$$

for all $(t,x) \in \mathbb{R}_+ \times X$.

Proposition 3.2. *If the pair $(U, \mathcal{P})$ is $(h,k,\mu,\nu)-t$ then the family of norms*

$$\mathcal{N}_1 = \{\|\cdot\|_t : t \geq 0\}$$

given by

$$\begin{aligned} \|x\|_t = &\sup_{\tau \geq t} \frac{h(\tau)}{h(t)}\|U(\tau,t)P_1(t)x\| + \sup_{r \leq t} \frac{k(t)}{k(r)}\|V_2(t,r)P_2(t)x\| \\ &+ \sup_{\tau \geq t} \frac{\mu(t)}{\mu(\tau)}\|U(\tau,t)P_3(t)x\| \end{aligned} \tag{3.2}$$

is compatible with $\|\cdot\|$.

Proof. For $\tau = t = r$ in (3.2) we obtain that

$$\|x\|_t \geq \|P_1(t)x\| + \|P_2(t)x\| + \|P_3(t)x\| \geq \|x\|$$

for all $t \geq 0$.

If the pair $(U, \mathcal{P})$ is $(h,k,\mu,\nu) - t$ then by Proposition 2.10 there exits a nondecreasing function $N_1 : \mathbb{R}_+ \to \mathcal{B}(X)$ such that

$$\|x\|_t \leq 3N_1(t)\|x\|, \text{ for all } (t,x) \in \mathbb{R}_+ \times X.$$

Finally we obtain that $\mathcal{N}_1$ is compatible with $\|\cdot\|$. □

Proposition 3.3. *If the pair $(U, \mathcal{P})$ is $(h,k,\mu,\nu) - t$ then the family of norms*

$$\mathcal{N}_2 = \{|||\cdot|||_t, t \geq 0\}$$

defined by

$$|||x|||_t = \sup_{\tau \geq t} \frac{h(\tau)}{h(t)} \|U(\tau,t)P_1(t)x\| + \sup_{r \leq t} \frac{k(t)}{k(r)} \|V_2(t,r)P_2(t)x\|$$
$$+ \sup_{r \leq t} \frac{\nu(r)}{\nu(t)} \|V_3(t,r)P_3(t)x\| \tag{3.3}$$

is compatible with $\|\cdot\|$.

Proof. If the pair $(U, \mathcal{P})$ is $(h,k,\mu,\nu) - t$ then by Proposition 2.10 there exits a nondecreasing function $N_1 : \mathbb{R}_+ \to \mathcal{B}(X)$ such that

$$|||x|||_t \leq 3N_1(t)\|x\|, \text{ for all } (t,x) \in \mathbb{R}_+ \times X.$$

On the other hand, for $\tau = t = r$ in the definition of $|||\cdot|||_t$ we obtain

$$|||x|||_t \geq \|P_1(t)x\| + \|P_2(t)x\| + \|P_3(t)x\| \geq \|x\|.$$

In consequence, by Definition 3.1 it results that the family of norms $\mathcal{N}_2$ is compatible to $\|\cdot\|$. □

The main result of this paper is

Theorem 3.4. *If $\mathcal{P} = \{P_1, P_2, P_3\}$ is compatible with the evolution operator $U : \Delta \to \mathcal{B}(X)$ then the pair $(U, \mathcal{P})$ is (h,k,μ,ν)-trichotomic if and only if there exist two families of norms $\mathcal{N}_1 = \{\|\cdot\|_t : t \geq 0\}$ and $\mathcal{N}_2 = \{|||\cdot|||_t : t \geq 0\}$ compatible with the norm $\|\cdot\|$ such that the following take place*
(ht_3) $h(t)\|U(t,s)P_1(s)x\|_t \leq h(s)\|P_1(s)x\|_s$
(kt_3) $k(t)|||V_2(t,s)P_2(t)x|||_s \leq k(s)|||P_2(t)x|||_t$
(μt_3) $\mu(s)\|U(t,s)P_3(s)x\|_t \leq \mu(t)\|P_3(s)x\|_s$
(νt_3) $\nu(s)|||V_3(t,s)P_3(t)x|||_s \leq \nu(t)|||P_3(t)x|||_t$
for all $(t,s,x) \in \Delta \times X$.

Proof. Necessary. If the pair $(U, \mathcal{P})$ is (h,k,μ,ν)-trichotomic then by Propositions 3.2 and 3.3 there exist the families of norms

$$\mathcal{N}_1 = \{\|\cdot\|_t : t \geq 0\} \text{ and } \mathcal{N}_2 = \{|||\cdot|||_t : t \geq 0\}$$

compatible with $\|\cdot\|$.

$\boldsymbol{(ht_1) \Rightarrow (ht_3)}$. We have that

$$\begin{aligned} h(t)\|U(t,s)P_1(s)x\|_t &= h(t)\|P_1(t)U(t,s)P_1(s)x\|_t \\ &= h(t)\sup_{\tau\geq t}\frac{h(\tau)}{h(t)}\|U(\tau,t)P_1(t)U(t,s)P_1(s)x\| \\ &\leq h(s)\sup_{\tau\geq s}\frac{h(\tau)}{h(s)}\|U(\tau,s)P_1(s)x\| = h(s)\|P_1(s)\|_s, \end{aligned}$$

for all $(t,s,x)\in\Delta\times X$.

$\boldsymbol{(kt_2) \Rightarrow (kt_3)}$. If (kt_2) holds then

$$\begin{aligned} k(t)|||V_2(t,s)P_2(t)x|||_s &= k(t)|||P_2(s)V_2(t,s)P_2(t)x|||_s \\ &= k(t)\sup_{r\leq s}\frac{k(s)}{k(r)}\|V_2(s,r)P_2(s)V_2(t,s)P_2(t)x\| \\ &\leq k(s)\sup_{r\leq t}\frac{k(t)}{k(r)}\|V_2(t,r)P_2(t)x\| = k(s)|||P_2(t)|||_t \end{aligned}$$

for all $(t,s,x)\in\Delta\times X$.

$\boldsymbol{(\mu t_1) \Rightarrow (\mu t_3)}$. If $(U,\mathcal{P})$ is $(h,k,\mu,\nu)-$ trichotomic then by (μt_1) it results

$$\begin{aligned} \mu(s)\|U(t,s)P_3(s)x\|_t &= \mu(s)\|P_3(t)U(t,s)P_3(s)x\|_t \\ &= \mu(s)\sup_{\tau\geq t}\frac{\mu(t)}{\mu(\tau)}\|U(\tau,t)P_3(t)U(t,s)P_3(s)x\| \\ &= \mu(s)\sup_{\tau\geq t}\frac{\mu(t)}{\mu(\tau)}\|U(\tau,s)P_3(s)x\| \leq \mu(t)\sup_{\tau\geq s}\frac{\mu(s)}{\mu(\tau)}\|U(\tau,s)P_3(s)x\| \\ &= \mu(t)\|P_3(s)x\|_s, \end{aligned}$$

for all $(t,s,x)\in\Delta\times X$.

$\boldsymbol{(\nu t_2) \Rightarrow (\nu t_3)}$. Using Proposition 3.1 we obtain

$$\begin{aligned} \nu(s)|||V_3(t,s)P_3(t)x|||_s &= \nu(s)|||P_3(s)V_3(t,s)P_3(t)x|||_s \\ &= \nu(s)\sup_{r\leq s}\frac{\nu(r)}{\nu(s)}\|V_3(s,r)P_3(s)V_3(t,s)P_3(t)x\| \\ &\leq \nu(t)\sup_{r\leq t}\frac{\nu(r)}{\nu(t)}\|V_3(t,r)P_3(t)x\| = \nu(t)|||P_3(t)x|||_t, \end{aligned}$$

for all $(t,s,x)\in\Delta\times X$.

Sufficiency. We assume that there are two families of norms

$$\mathcal{N}_1 = \{\|\cdot\|_t : t\geq 0\} \text{ and } \mathcal{N}_2 = \{|||\cdot|||_t : t\geq 0\}$$

compatible with the norm $\|\cdot\|$ such that the inequalities $(ht_3)-(\nu t_3)$ take place. Let $(t,s,x)\in\Delta\times X$.

$\boldsymbol{(ht_3) \Rightarrow (ht_2)}$. The inequality (ht_3) and Definition 3.1 imply that

$$\begin{aligned} h(t)\|U(t,s)P_1(s)x\| &\leq \|U(t,s)P_1(s)x\|_t \leq h(s)\|P_1(s)x\|_s \\ &\leq h(s)C(s)\|P_1(s)x\| \leq C(s)\|P_1(s)\|h(s)\|x\|. \end{aligned}$$

$\boldsymbol{(kt_3)} \Rightarrow \boldsymbol{(kt_2)}$. Similarly,

$$\begin{aligned} k(t)\|V_2(t,s)P_2(t)x\| &\le k(t)|||V_2(t,s)P_2(t)x|||_s \le k(s)|||P_2(t)|||_t \\ &\le k(s)C(t)\|P_2(t)x\| \le C(t)\|P_2(t)\|k(s)\|x\|. \end{aligned}$$

$\boldsymbol{(\mu t_3)} \Rightarrow \boldsymbol{(\mu t_2)}$. From Definition 3.1 and inequality (μt_3) we have

$$\begin{aligned} \mu(s)\|U(t,s)P_3(s)x\| &\le \mu(s)\|U(t,s)P_3(s)x\|_t \le \mu(t)\|P_3(s)x\|_s \\ &\le C(s)\mu(t)\|P_3(s)x\| \le C(s)\|P_3(s)\|\mu(t)\|x\|. \end{aligned}$$

$\boldsymbol{(\nu t_3)} \Rightarrow \boldsymbol{(\nu t_2)}$. Similarly,

$$\begin{aligned} \nu(s)\|V_3(t,s)P_3(t)x\|| &\le \nu(s)|||V_3(t,s)P_3(s)x|||_s \le \nu(t)|||P_3(t)x|||_t \\ &\le C(t)\nu(t)\|P_3(t)x\| \le C(t)\|P_3(t)\|\nu(t)\|x\|. \end{aligned}$$

If we denote by

$$N(t) = \sup_{s\in[0,t]} C(s)(\|P_1(s)\| + \|P_2(s)\| + \|P_3(s)\|)$$

then we obtain that the inequalities $(ht_2), (kt_2), (\mu t_2), (\nu t_2)$ are satisfied.
By Proposition 2.10 it follows that $(U,\mathcal{P})$ is $(h,k,\mu,\nu)-t$. $\square$

As a particular case, we obtain a characterization of (nonuniform) exponential trichotomy given by

Corollary 3.5. *If $\mathcal{P} = \{P_1, P_2, P_3\}$ is compatible with the evolution operator $U : \Delta \to \mathcal{B}(X)$ then the pair $(U,\mathcal{P})$ is exponential trichotomic if and only if there are four real constants $\alpha, \beta, \gamma, \delta > 0$ and two families of norms*

$$\mathcal{N}_1 = \{\|\cdot\|_t : t \ge 0\} \text{ and } \mathcal{N}_2 = \{|||\cdot|||_t : t \ge 0\}$$

compatible with the norm $\|\cdot\|$ such that
(et_1) $\|U(t,s)P_1(s)x\|_t \le e^{-\alpha(t-s)}\|P_1(s)x\|_s$
(et_2) $|||V_2(t,s)P_2(t)x|||_s \le e^{-\beta(t-s)}|||P_2(t)x|||_t$
(et_3) $\|U(t,s)P_3(s)x\|_t \le e^{\gamma(t-s)}\|P_3(s)x\|_s$
(et_4) $|||V_3(t,s)P_3(t)x|||_s \le e^{\delta(t-s)}|||P_3(t)x|||_t$,
for all $(t,s,x) \in \Delta \times X$.

Proof. It results from Theorem 3.4 for

$$h(t) = e^{\alpha t}, k(t) = e^{\beta t}, \nu(t) = e^{\gamma t}, \nu(t) = e^{\delta t},$$

with $\alpha, \beta, \gamma, \delta > 0$. $\square$

If the growth rates are of polynomial type then we obtain a characterization of (nonuniform) polynomial trichotomy given by

Corollary 3.6. *Let $\mathcal{P} = \{P_1, P_2, P_3\}$ is compatible with the evolution operator $U : \Delta \to \mathcal{B}(X)$. Then $(U,\mathcal{P})$ is nonuniform polynomial trichotomic if and only if there exist two families of norms*

$$\mathcal{N}_1 = \{\|\cdot\|_t : t \ge 0\} \text{ and } \mathcal{N}_2 = \{|||\cdot|||_t : t \ge 0\}$$

compatible with the norm $\|\cdot\|$ and four real constants $\alpha, \beta, \gamma, \delta > 0$ such that
(pt_1) $(t+1)^\alpha\|U(t,s)P_1(s)x\|_t \le (s+1)^\alpha\|P_1(s)x\|_s$

(pt_2) $(t+1)^{\beta}|||V_2(t,s)P_2(t)x|||_s \leq (s+1)^{\beta}|||P_2(t)x|||_t$
(pt_3) $(s+1)^{\gamma}\|U(t,s)P_3(s)x\|_t \leq (t+1)^{\gamma}\|P_3(s)x\|_s$
(pt_4) $(s+1)^{\delta}|||V_3(t,s)P_3(t)x|||_s \leq (t+1)^{\delta}|||P_3(t)x|||_t$,
for all $(t,s,x) \in \Delta \times X$.

Proof. It results from Theorem 3.4 for

$$h(t) = (t+1)^{\alpha}, k(t) = (t+1)^{\beta}, \mu(t) = (t+1)^{\gamma}, \nu(t) = (t+1)^{\delta},$$

with $\alpha, \beta, \gamma, \delta > 0$. □

Definition 3.7. A family of norms $\mathcal{N} = \{\|\cdot\|_t, t \geq 0\}$ is *uniformly compatible* with the norm $\|\cdot\|$ if there exits a constant $c > 0$ such that

$$\|x\| \leq \|x\|_t \leq c\|x\|, \text{ for all } (t,x) \in \mathbb{R}_+ \times X. \tag{3.4}$$

Remark 3.8. From the proofs of Propositions 3.2, 3.3 it results that if the pair $(U, \mathcal{P})$ is uniformly $(h,k,\mu,\nu)-$ trichotomic then the families of norms

$$\mathcal{N}_1 = \{\|\cdot\|_t : t \geq 0\} \text{ and } \mathcal{N}_2 = \{|||\cdot|||_t : t \geq 0\}$$

(given by (3.2) and (3.3)) are uniformly compatible with the norm $\|\cdot\|$.

A characterization of the uniform$-(h,k,\mu,\nu)-$trichotomy is given by

Theorem 3.9. *Let $\mathcal{P} = \{P_1, P_2, P_3\}$ be compatible with the evolution operator $U : \Delta \to \mathcal{B}(X)$. Then the pair $(U, \mathcal{P})$ is uniformly$-(h,k,\mu,\nu)-$trichotomic if and only if there exist two families of norms*

$$\mathcal{N}_1 = \{\|\cdot\|_t : t \geq 0\} \textit{ and } \mathcal{N}_2 = \{|||\cdot|||_t : t \geq 0\}$$

uniformly compatible with the norm $\|\cdot\|$ such that the inequalities $(ht_3), (kt_3), (\mu t_3)$ and (νt_3) are satisfied.

Proof. It results from the proof of Theorem 3.4 (via Proposition 2.11). □

Remark 3.10. Similarly as in Corollaries 3.5, 3.6 one can obtain characterizations for uniform exponential trichotomy respectively uniform polynomial trichotomy.

Another characterization of the $(h,k,\mu,\nu)-$trichotomy is given by

Theorem 3.11. *If $\mathcal{P} = \{P_1, P_2, P_3\}$ is compatible with the evolution operator $U : \Delta \to \mathcal{B}(X)$ then the pair $(U, \mathcal{P})$ is (h,k,μ,ν)-trichotomic if and only if there exist two families of norms*

$$\mathcal{N}_1 = \{\|\cdot\|_t, t \geq 0\} \textit{ and } \mathcal{N}_2 = \{|||\cdot|||_t : t \geq 0\}$$

compatible with the family of projectors $\mathcal{P} = \{P_1, P_2, P_3\}$ such that
(ht_4) $h(t)\|U(t,s)P_1(s)x\|_t \leq h(s)\|x\|_s$
(kt_4) $k(t)|||V_2(t,s)P_2(t)x|||_s \leq k(s)|||x|||_t$
(μt_4) $\mu(s)\|U(t,s)P_3(s)x\|_t \leq \mu(t)\|x\|_s$
(νt_4) $\nu(s)|||V_3(t,s)P_3(t)x|||_s \leq \nu(t)|||x||||_t$
for all $(t,s,x) \in \Delta \times X$.

Proof. Necessity. It results from Theorem 3.4 and inequalities

$$\|P_i(t)x\|_t \le \|x\|_t \quad \text{and} \quad |||P_i(t)x|||_t \le |||x|||_t,$$

for all $(t,x) \in \mathbb{R}_+ \times X$ and $i = \{1,2,3\}$.
Sufficiency. It results replacing x by $P_1(s)x$ in (ht_4), x by $P_2(t)x$ in (kt_4), x by $P_3(s)x$ in (μt_4) and x by $P_3(t)x$ in (νt_4). □

The variant of the previous theorem for uniform $(h,k,\mu,\nu)-$trichotomy is given by

Theorem 3.12. *If $\mathcal{P} = \{P_1, P_2, P_3\}$ is compatible with the evolution operator $U : \Delta \to \mathcal{B}(X)$ then the pair $(U, \mathcal{P})$ is uniformly $-(h,k,\mu,\nu)-$ trichotomic if and only if there exist two families of norms*

$$\mathcal{N}_1 = \{\|\cdot\|_t : t \ge 0\} \text{ and } \mathcal{N}_2 = \{|||\cdot|||_t : t \ge 0\}$$

uniformly compatible with the family of projectors $\mathcal{P} = \{P_1, P_2, P_3\}$ such that the inequalities $(ht_4), (kt_4), (\mu t_4)$ and (νt_4) are satisfied.

Proof. It is similar with the proof of Theorem 3.4. □

Remark 3.13. If the growth rates are exponential respectively polynomial then we obtain characterizations for exponential trichotomy, uniform exponential trichotomy and uniform polynomial trichotomy.

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A note on a transmission problem for the Brinkman system and the generalized Darcy-Forchheimer-Brinkman system in Lipschitz domains in $\mathbb{R}^3$

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Abstract. The purpose of this paper is to treat a nonlinear transmission-type problem for a generalized version of the Darcy-Forchheimer-Brinkman system and the classical Brinkman system in complementary Lipschitz domains in $\mathbb{R}^3$. First of all, we define the required spaces in which we seek our solution. Next, we describe the generalized Brinkman and the generalized Darcy-Forchheimer-Brinkman systems. Further, we give important lemmas that allow us to introduce the trace and conormal derivative operators that appear in the formulation of our transmission problem. We invoke a result regarding the well-posedness of the (linear) transmission problem for the generalized and classical Brinkman systems in complementary Lipschitz domains in $\mathbb{R}^3$. The above mentioned well-posedness result in the linear case combined with Banach's fixed point theorem will allow us to establish the main result of the paper, the well-posedness of the transmission problem for the Brinkman system and the nonlinear generalized Darcy-Forchheimer-Brinkman system in Lipschitz domains in $\mathbb{R}^3$.

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1. Introduction

Transmission problems that appear in the field of fluid mechanics are important to researchers nowadays, due to their practical applications, such as environmental problems with free air flow interacting with evaporation from soils or transvascular exchange between blood flow in vessel and the surrounding tissue as porous material. Another relevant example is the geophysical model of flow of water or other viscous fluids, which pass through porous rocks or porous soil (see [8], [14] and the references therein, and see also [21], [5]). We also mention the important role of the partial differential equations that model different types of flow, such as the Brinkman equations or the Darcy-Forchheimer-Brinkman equations (for additional details, see [21]).

Escauriaza and Mitrea in [4] have established the well-posedness of the transmission problem for the Laplace operator across a Lipschitz interface, for data in Lebesgue and Hardy spaces on the boundary.

Medkova in [17] has studied the transmission problem for the Stokes system with constant coefficients in $\mathbb{R}^3$ using the integral equation method.

Mitrea and Wright in [20] have given well-posedness results for transmission problems for the Stokes systems in arbitrary Lipschitz domains in Euclidean setting and in L^p, Sobolev and Besov spaces.

Groşan, Kohr and Wendland in [6] have studied the Dirichlet problem for the generalized Brinkman system in a bounded Lipschitz domain in $\mathbb{R}^n$, $n \geq 2$ and the Dirichlet problem for the generalized Darcy-Forchheimer-Brinkman system in a bounded Lipschitz domain in $\mathbb{R}^n$, $n = 2, 3$.

Kohr, Lanza de Cristoforis and Wendland in [13] have treated Poisson problems for a semilinear and a generalized Brinkman system on a bounded Lipschitz domain in $\mathbb{R}^n$, $n \geq 2$, with Dirichlet or Robin boundary conditions and data given in L^2-based Sobolev spaces.

Kohr, Lanza de Cristoforis and Wendland in [10] have used a layer potential method and a fixed point theorem to show the existence of a solution of the Robin problem for the standard Darcy-Forchheimer-Brinkman system in a bounded Lipschitz domain in $\mathbb{R}^n$, $n = 2, 3$.

Kohr, Lanza de Cristoforis and Wendland in [9] have studied the Robin problem for the Brinkman and the Darcy-Forchheimer-Brinkman systems with constant coefficients. They also proceeded to study mixed boundary value problems for the Brinkman system and Darcy-Forchheimer-Brinkman system. respectively. Moreover, they have proved a well-posedness result for a boundary problem of mixed Dirichlet-Robin and transmission type for two Brinkman systems.

Medkova in [16] has tackled the transmission problem for the Brinkman system and also the Robin-transmission and the Dirichlet-transmission problems for the Brinkman system in the setting of a bounded Lipschitz domain in $\mathbb{R}^n$, $n > 2$. In each of these problems, the systems have constant coefficients.

Kohr, Lanza de Cristoforis and Wendland in [11] have studied nonlinear Neumann-Transmission problems for the (linear) Stokes and Brinkman systems with a nonlinear Neumann condition.

Kohr, Lanza de Cristoforis, Mikhailov and Wendland in [8] have treated transmission problems for the nonlinear Darcy-Forchheimer-Brinkman system and the linear Stokes system in complementary Lipschitz domains in $\mathbb{R}^3$.

Kohr, Lanza de Cristoforis and Wendland in [12] have obtained a well-posendess result for the nonlinear Robin-transmission problem for the nonlinear Navier-Stokes and Darcy-Forchheimer-Brinkman systems in the setting of bounded Lipschitz domains in $\mathbb{R}^n$, $n = 2, 3$.

Kohr, Mikhailov and Wendland in [14] have obtained well-posedness results for transmission problems for the Navier-Stokes and Darcy-Forchheimer-Brinkman systems in Lipschitz domains on compact Riemannian manifolds of dimension $m = 2, 3$. The coefficients of these systems of partial differential equations are smooth due to the smoothness of the Riemannian metric tensor.

Mitrea, Mitrea and Shi in [19] have studied variable coefficient transmission problems and singular integral operators on non-smooth manifolds.

Transmission problems for Stokes and Navier-Stokes systems with nonsmooth coefficients (L^∞-coefficients) on compact Riemannian manifolds have been recently treated by Kohr and Wendland in [15].

The paper is structured as follows. In the second section, we define the Sobolev spaces in which we seek our solutions. There, we describe the generalized versions of the Brinkman system and of the Darcy-Forchheimer-Brinkman system. These systems of PDEs contain L^∞ coefficients. We give a result that allows us to consider the trace operator in the setting of Sobolev spaces (Lemma 2.5). In addition, we mention a result that allows us to consider the conormal derivative operator for the generalized Brinkman system (Lemma 2.6). We end this section with two results. The first of them is related to the growth conditions at infinity of a pair $(\mathbf{w}, r)$ that satisfy the homogeneous Brinkman equation with constant coefficients in an exterior Lipschitz domain in $\mathbb{R}^3$ (Lemma 2.7). The second result gives mapping properties of a nonlinear operator related with our nonlinear transmission problem (Lemma 2.8). In the third section, we state the well-posedness result for the linear transmission problem for the classical and generalized Brinkman systems in complementary Lipschitz domains in $\mathbb{R}^3$ (Theorem 3.1). Using this well-posedness result and the Banach fixed point theorem, we obtain the well-posedness result for the main nonlinear problem considered in this paper, which is the transmission problem for the Brinkman and the generalized Darcy-Forchheimer-Brinkman systems in complementary Lipschitz domains in $\mathbb{R}^3$, namely Theorem 3.2.

2. Preliminaries

In this paper, by the superscript $'$ we refer to the topological dual of a given space. Also, we use the notation $\langle \cdot, \cdot \rangle_A$ to denote the duality pairing of two dual Sobolev spaces defined on A, where A is either an open set or a surface in $\mathbb{R}^3$. Also, denote by $\mathrm{E}(\mathbf{w})$ the symmetric part of $\nabla \mathbf{w}$ (where $\mathbf{w}$ is a given field),

$$\mathrm{E}(\mathbf{w}) := \frac{1}{2}(\nabla \mathbf{w} + \nabla \mathbf{w}^t),$$

and the superscript t refers to the transpose. Also, by $\mathring{E}$, we denote the operator of extension by zero outside our considered bounded Lipschitz domain $\Omega \subset \mathbb{R}^3$.

Next, we introduce the Sobolev spaces in which we seek the solution of our transmission problem. Also, we describe the generalized version of the Darcy-Forchheimer-Brinkman system and we give the results that allow us to introduce the trace and conormal derivative operators, operators that appear in the boundary conditions of our problem.

To this end, let $\Omega_+ := \Omega \subset \mathbb{R}^3$ be a bounded Lipschitz domain (an open, connected set, whose boundary is locally the graph of a Lipschitz function) with connected boundary (denoted by $\partial\Omega$) and by $\Omega_- := \mathbb{R}^3 \setminus \overline{\Omega}$, we denote the corresponding complementary Lipschitz set.

In the latter Ω_0 denotes either one of the following sets: Ω_+, Ω_- or $\mathbb{R}^3$.

Recall that $\mathcal{D}(\Omega_0)$ is the space of compactly supported smooth functions $C_0^\infty(\Omega_0)$ and by $\mathcal{D}'(\Omega_0)$ we denote its dual, the space of distributions on Ω_0. Note that $\mathcal{D}(\Omega_0)$ is endowed with the inductive limit topology and $\mathcal{D}'(\Omega_0)$ is endowed with the weak-star topology.

Definition 2.1. Let $p \in [1, \infty)$. Then, the Lebesgue space $L^p(\mathbb{R}^3)$ is the space of all (equivalence classes of) measurable functions $f : \mathbb{R}^3 \to \mathbb{R}$ with the property that:

$$\int_{\mathbb{R}^3} |f(x)|^p \mathrm{d}x < \infty.$$

We also denote by $\mathcal{F}$ the Fourier transform and by $\mathcal{F}^{-1}$ its inverse acting on functions from $L^1(\mathbb{R}^3)$. We shall consider their generalization to the space of tempered distributions.

We have the following definition (see, e.g., [8, (2.1)-(2.3)], [1, Ch. 1], [7, Ch. 4]).

Definition 2.2. Let $s \in \mathbb{R}$. Introduce the L^2-based (Bessel potential) Sobolev spaces by:

$$\begin{aligned}
&H^s(\mathbb{R}^3) := \{\mathcal{F}^{-1}(1+|\xi|^2)^{-\frac{s}{2}}\mathcal{F}u : u \in L^2(\mathbb{R}^3)\},\\
&H^s(\Omega_0) := \{u \in \mathcal{D}'(\Omega_0) : \exists\, U \in H^s(\mathbb{R}^3) \text{ such that } U|_{\Omega_0} = u\},\\
&\widetilde{H}^s(\Omega_0) := \overline{\mathcal{D}(\Omega_0)}^{||\cdot||_{H^s(\mathbb{R}^3)}},
\end{aligned}$$

hence $\widetilde{H}^s(\Omega_0)$ is the closure of $\mathcal{D}(\Omega_0)$ in $H^s(\mathbb{R}^3)$.

One may also introduce the vector-valued spaces component-wise.

We also have the following definition (see also [7, p. 169]).

Definition 2.3. Let $s \in (0,1)$. Then, the boundary Sobolev spaces $H^s(\partial\Omega)$ is defined by:

$$\begin{aligned}
H^s(\partial\Omega) := \Big\{ u \in L^2(\partial\Omega) : &||u||_{H^s(\partial\Omega)} = ||u||_{L^2(\partial\Omega)} + \\
&\int_{\partial\Omega}\int_{\partial\Omega} \frac{|u(x)-u(y)|}{|x-y|^{2+2s}} d\sigma_x d\sigma_y < \infty \Big\}.
\end{aligned}$$

The following duality

$$H^{-s}(\partial\Omega) := (H^s(\partial\Omega))',$$

allows us to introduce the boundary Sobolev space with negative order $H^{-s}(\partial\Omega)$. Again, the vector-valued spaces are introduced component-wise. Note that, all these L^2-based Sobolev spaces are Hilbert spaces (see, e.g., [1]).

We also describe here the space $\mathfrak{M}(\Omega_0)$ (see [8, p. 23]), i.e., the space in which we shall seek the unknown pressure field in the exterior Lipschitz domain in our transmission problem.

Let us consider the weight function:

$$\rho(\mathbf{x}) := (1 + |\mathbf{x}|^2)^{\frac{1}{2}}, \ \forall \ x \in \mathbb{R}^3.$$

Then, the weighted Lebesgue space $L^2(\rho^{-1}; \Omega_0)$ is the set of all functions v with the property that $\rho^{-1} u \in L^2(\Omega_0)$.

Moreover, the space $\mathfrak{M}(\Omega_0)$ is defined by:

$$\mathfrak{M}(\Omega_0) := \{\mathfrak{g} \in L^2(\rho^{-1}, \Omega_0) : \nabla \mathfrak{g} \in H^{-1}_{\text{curl}}(\Omega_0)^3\},$$

where, by $H^{-1}_{\text{curl}}(\Omega_0)^3$ we understand the space:

$$H^{-1}_{\text{curl}}(\Omega_0)^3 := \{\mathfrak{h} \in H^{-1}(\Omega_0)^3 : \ \text{curl } \mathfrak{h} = \nabla \times \mathfrak{h} = 0\}.$$

By denoting $\mathfrak{M}'(\Omega_0)$ the dual of $\mathfrak{M}(\Omega_0)$, we have the very suggestive chain of continuous embeddings (cf. [8, (A.24)]):

$$L^2(\rho, \Omega_0) \subset \mathfrak{M}'(\Omega_0) \subset L^2(\Omega_0) \subset \mathfrak{M}(\Omega_0) \subset L^2(\rho^{-1}, \Omega_0) \subset L^2_{\text{loc}}(\Omega_0).$$

The generalized version of the Brinkman system of PDEs, is given by (see, e.g., [13, Relation (2.14)]):

$$\Delta \mathbf{w} - \mathcal{P}\mathbf{w} - \nabla p = \mathfrak{F} \text{ in } \Omega_+, \ \text{div } \mathbf{w} = 0 \text{ in } \Omega_+, \tag{2.1}$$

where $\mathcal{P} \in L^\infty(\Omega_+)^{3\times 3}$ satisfies the condition:

$$\langle \mathcal{P}\mathbf{v}, \mathbf{v}\rangle_{\Omega_+} \geq c_{\mathcal{P}} ||\mathbf{v}||^2_{L^2(\Omega_+)^3}, \ \forall \ \mathbf{v} \in L^2(\Omega_+)^3, \tag{2.2}$$

with some constant $c_{\mathcal{P}} > 0$ is a constant.

The important generalization that we consider in this work is the generalized version of the Darcy-Forchheimer-Brinkman system:

$$\Delta \mathbf{w} - \mathcal{P}\mathbf{w} - k|\mathbf{w}|\mathbf{w} - \beta(\mathbf{w} \cdot \nabla)\mathbf{w} - \nabla p = \mathfrak{F} \text{ in } \Omega_+, \ \text{div } \mathbf{w} = 0 \text{ in } \Omega_+, \tag{2.3}$$

with $\mathcal{P} \in L^\infty(\Omega_+)^{3\times 3}$ as above, $k, \beta : \Omega_+ \to \mathbb{R}_+$ are functions such that $k, \beta \in L^\infty(\Omega_+)$, i.e., essentially bounded, non-negative functions defined on Ω_+.

Remark 2.4. (i) If we let $\mathcal{P} = \alpha\mathbb{I}$ where $\alpha > 0$ is a constant in (2.1), we get the classical Brinkman system.

(ii) If $\mathcal{P} = 0$ in (2.1), we get the well-known Stokes system.

(iii) If $\mathcal{P} \equiv \alpha\mathbb{I}$, where $\alpha > 0$ is a constant and $k, \beta > 0$ are also constants in (2.3), one obtains the classical Darcy-Forchheimer-Brinkman system.

(iv) If $\mathcal{P} \equiv 0$, $k = 0$ and $\beta > 0$ is a constant in (2.3), we recover the Navier-Stokes system.

Next, we introduce the following result that allows us to define the trace operator (see, e.g., [18, Theorem 2.3]).

Lemma 2.5. (**Gagliardo Trace Lemma**) *Let $\Omega_+ := \Omega \subset \mathbb{R}^3$ be a bounded Lipschitz domain with connected boundary $\partial\Omega$ and denote by $\Omega_- := \mathbb{R}^3 \setminus \overline{\Omega}$ the complementary Lipschitz set. Then, there exist linear, continuous trace operators $Tr^\pm : H^1(\Omega_\pm) \to H^{\frac{1}{2}}(\partial\Omega)$, such that*

$$Tr^\pm u = u|_{\partial\Omega}, \quad \forall v \in C^\infty(\overline{\Omega_\pm}). \tag{2.4}$$

Moreover, these operators are surjective, having (non-unique) linear and continuous right inverse operators $Z^\pm : H^{\frac{1}{2}}(\partial\Omega) \to H^1(\Omega_\pm)$.

We have the following result that allows us to consider the conormal derivative for the generalized Brinkman system (see, e.g., [13, Lemma 2.3]).

Lemma 2.6. *Let $\Omega_+ := \Omega \subset \mathbb{R}^3$, be a bounded Lipschitz domain with connected boundary $\partial\Omega$. Let $\mathcal{P} \in L^\infty(\Omega_+)^{3\times 3}$. Consider the following space:*

$$\begin{aligned}\boldsymbol{H}^1(\Omega_+, \mathcal{B}_\mathcal{P}) :=& \{(\mathbf{w}, p, \mathfrak{F}) \in H^1(\Omega_+)^3 \times L^2(\Omega_+) \times \widetilde{H}^{-1}(\Omega_+)^3 :\\ & \mathcal{B}_\mathcal{P}(\mathbf{w}, p) := \Delta w - \mathcal{P}\mathbf{w} - \nabla p = \mathfrak{F}|_{\Omega_+}\\ & \textit{and } \operatorname{div} \mathbf{w} = 0 \textit{ in } \Omega_+\}.\end{aligned}$$

Define the conormal derivative operator for the generalized Brinkman system,

$$\mathbf{t}^+_{\mathcal{P},\nu} : \boldsymbol{H}^1(\Omega_+, \mathcal{B}_\mathcal{P}) \to H^{-\frac{1}{2}}(\partial\Omega)^3, \tag{2.5}$$

by the following relation:

$$\begin{aligned}\langle \mathbf{t}^+_{\mathcal{P},\nu}(\mathbf{w}, p, \mathfrak{F}), \boldsymbol{\phi}\rangle_{\partial\Omega} :=& 2\langle \mathrm{E}(\mathbf{w}), \mathrm{E}(Z^+\boldsymbol{\phi})\rangle_{\Omega_+} + \langle \mathcal{P}\mathbf{w}, Z^+\boldsymbol{\phi}\rangle_{\Omega_+}\\ & - \langle p, \operatorname{div}\,(Z^+\boldsymbol{\phi})\rangle_{\Omega_+}\\ & + \langle \mathfrak{F}, Z^+\boldsymbol{\phi}\rangle_{\Omega_+}, \quad \forall \boldsymbol{\phi} \in H^{-\frac{1}{2}}(\partial\Omega)^3,\end{aligned} \tag{2.6}$$

where Z^+ is a right inverse of the trace operator $Tr^+ : H^1(\Omega_+)^3 \to H^{\frac{1}{2}}(\partial\Omega)^3$. The operator $\mathbf{t}^+_{\mathcal{P},\nu}$ is linear, bounded and does not depend on the choice of the right inverse Z^+ of the trace operator Tr^+.

Moreover, the following Green formula holds:

$$\begin{aligned}\langle \mathbf{t}^+_{\mathcal{P},\nu}(\mathbf{w}, p, \mathfrak{F}), Tr^+\boldsymbol{\psi}\rangle_{\partial\Omega} =& 2\langle \mathrm{E}(\mathbf{w}), \mathrm{E}(\boldsymbol{\psi})\rangle_{\Omega_+} + \langle \mathcal{P}\mathbf{w}, \boldsymbol{\psi}\rangle_{\Omega_+}\\ & - \langle p,\ \operatorname{div} \boldsymbol{\psi}\rangle_{\Omega_+} + \langle \mathfrak{F}, \boldsymbol{\psi}\rangle_{\Omega_+},\end{aligned} \tag{2.7}$$

for all $(\mathbf{w}, p, \mathfrak{F}) \in \boldsymbol{H}^1(\Omega_+, \mathcal{B}_\mathcal{P})$ and for any $\boldsymbol{\psi} \in H^1(\Omega_+)^3$.

Similarly, one may introduce the conormal derivative operator for the classical Brinkman system which is denoted by $\mathbf{t}^\pm_{\alpha,\nu}$ where $\alpha > 0$ is a constant. The statement of the lemma for the introduction of the above described operator is omitted for the sake of brevity, but we refer the reader to [8, Lemma 2.5].

Next, we are concerned with the behavior at infinity of a solution of the classical homogeneous Brinkman system in the unbounded domain Ω_-. We have the following lemma (cf. [2, Lemma A.2]).

Lemma 2.7. *Let $\alpha > 0$ be a constant. Let $\Omega_+ := \Omega \subset \mathbb{R}^3$ be a bounded Lipschitz domain with connected boundary and let $\Omega_- := \mathbb{R}^3 \setminus \overline{\Omega}$. If the pair $(\mathbf{w}, r) \in H^1(\Omega_-)^3 \times \mathfrak{M}(\Omega_-)$ satisfy the Brinkman equations:*

$$\Delta\mathbf{w} - \alpha\mathbf{w} - \nabla r = 0, \quad \operatorname{div} \mathbf{w} = 0, \quad \textit{in } \Omega_-, \tag{2.8}$$

then

$$\mathbf{w}(\mathbf{x}) = O(|\mathbf{x}|^{-2}), \quad \nabla\mathbf{w}(\mathbf{x}) = O(|\mathbf{x}|^{-1}), \quad r(\mathbf{x}) = O(|\mathbf{x}|^{-1}), \quad \textit{as } |\mathbf{x}| \to \infty. \tag{2.9}$$

The proof of this lemma can be consulted in [2].

Finally, we mention a lemma that gives an important characterization of the following nonlinear operator that appears in the nonlinear transmission problem:

$$\mathcal{J}_{k,\beta,\Omega_+}(\mathbf{v}) := \mathring{E}(k|\mathbf{v}|\mathbf{v} + \beta(\mathbf{v}\cdot\nabla)\mathbf{v}).$$

The mapping and other properties of this operator are provided below (see, e.g., [3, Lemma 3.1] and [8, Lemma 5.1] in the case $\mathcal{P} = \alpha\mathbb{I}$, where $\alpha > 0$ is a constant and $k, \beta > 0$ are constants).

Lemma 2.8. *Let $\Omega_+ := \Omega \subset \mathbb{R}^3$ be a bounded Lipschitz domain with connected boundary. Let $k, \beta : \Omega_+ \to \mathbb{R}_+$ such that $k, \beta \in L^\infty(\Omega_+)$ and let*

$$\mathcal{J}_{k,\beta,\Omega_+}(\mathbf{v}) := \mathring{E}(k|\mathbf{v}|\mathbf{v} + \beta(\mathbf{v}\cdot\nabla)\mathbf{v}). \tag{2.10}$$

Then, the nonlinear operator $\mathcal{J}_{k,\beta,\Omega_+} : H^1_{\mathrm{div}}(\Omega_+)^3 \to \widetilde{H}^{-1}(\Omega_+)^3$ is continuous, positively homogeneous of order 2, and bounded, in the sense that there is a constant $c_0 = c_0(\Omega_+, k, \beta) > 0$ such that

$$||\mathcal{J}_{k,\beta,\Omega_+}(\mathbf{v})||_{\widetilde{H}^{-1}(\Omega_+)^3} \le c_0||\mathbf{v}||^2_{H^1(\Omega_+)^3}. \tag{2.11}$$

Moreover, the following Lipschitz-like relation holds:

$$\begin{aligned} &||\mathcal{J}_{k,\beta,\Omega_+}(\mathbf{v}) - \mathcal{J}_{k,\beta,\Omega_+}(\mathbf{w})||_{\widetilde{H}^{-1}(\Omega_+)^3} \\ &\qquad \le c_0(||\mathbf{v}||_{H^1(\Omega_+)^3} + ||\mathbf{w}||_{H^1(\Omega_+)^3})||\mathbf{v}-\mathbf{w}||_{H^1(\Omega_+)^3}, \end{aligned} \tag{2.12}$$

with $c_0 = c_0(\Omega_+, k, \beta) > 0$ is the same constant as in relation (2.11).

Proof. We provide here the main ideas that lead to the statement of the lemma (for additional details, see the proof of Lemma 5.1 in [8]).

First, we have the following continuous embeddings, due to the Sobolev embedding theorem (see, e.g., [7, Theorem 4.1.5, Theorem 4.1.6]):

$$H^1(\Omega_+)^3 \hookrightarrow L^q(\Omega_+)^3, \quad L^{q'}(\Omega_+)^3 \hookrightarrow \widetilde{H}^{-1}(\Omega_+)^3, \tag{2.13}$$

where $2 \le q \le 6$, and $\frac{1}{q} + \frac{1}{q'} = 1$.

Using relations (2.13) and Hölder's inequality, one may show that $|\mathbf{v}|\mathbf{w} \in L^2(\Omega_+)^3$ and $(\mathbf{v}\cdot\nabla)\mathbf{w} \in L^{\frac{3}{2}}(\Omega_+)^3$, for all $\mathbf{v}, \mathbf{w} \in H^1(\Omega_+)^3$.

Let us now consider the operators

$$\begin{aligned} b_1 &: H^1(\Omega_+)^3 \times H^1(\Omega_+)^3 \to \widetilde{H}^{-1}(\Omega_+)^3, \\ b_2 &: H^1(\Omega_+)^3 \times H^1(\Omega_+)^3 \to \widetilde{H}^{-1}(\Omega_+)^3, \end{aligned} \tag{2.14}$$

given by

$$\begin{aligned} b_1(\mathbf{v},\mathbf{w}) &:= \mathring{E}(k|\mathbf{v}|\mathbf{w}), \\ b_2(\mathbf{v},\mathbf{w}) &:= \mathring{E}(\beta(\mathbf{v}\cdot\nabla)\mathbf{w}). \end{aligned} \tag{2.15}$$

Using the embeddings (2.13) and again Hölder's inequality, one may show that there are two constants $c_* = c_*(\Omega_+, k) > 0$ and $c^* = c^*(\Omega_+, \beta) > 0$ such that the following relations hold:

$$\begin{aligned} ||b_1(\mathbf{v},\mathbf{w})||_{\widetilde{H}^{-1}(\Omega_+)^3} &\le c_*||\mathbf{v}||_{H^1(\Omega_+)^3}||\mathbf{w}||_{H^1(\Omega_+)^3}, \\ ||b_2(\mathbf{v},\mathbf{w})||_{\widetilde{H}^{-1}(\Omega_+)^3} &\le c^*||\mathbf{v}||_{H^1(\Omega_+)^3}||\mathbf{w}||_{H^1(\Omega_+)^3}, \end{aligned} \tag{2.16}$$

which show the continuity of the operators b_1 and b_2.

Note that, the operator $\mathcal{J}_{k,\beta,\Omega_+}$ can be written as:

$$\mathcal{J}_{k,\beta,\Omega_+}(\mathbf{v}) = b_1(\mathbf{v},\mathbf{v}) + b_2(\mathbf{v},\mathbf{v}), \tag{2.17}$$

and by employing the relations (2.16), we have that $\mathcal{J}_{k,\beta,\Omega_+}$ satisfies (2.11) with $c_0 = c_* + c^*$, as asserted.

Also, by using similar arguments to those in the proof of [8, Lemma 5.1] and again relations (2.16), one shows that the operator $\mathcal{J}_{k,\beta,\Omega_+}$ satisfies the Lipschitz-like condition (2.12).

The full argument is omitted for the sake of brevity. □

3. The main result

We introduce the following spaces:

$$\begin{aligned}
&H^1_{\mathrm{div}}(\Omega_+)^3 := \{\mathbf{w} \in H^1(\Omega_+)^3 : \ \mathrm{div}\ \mathbf{w} = 0 \text{ in } \Omega_+\},\\
&H^1_{\mathrm{div}}(\Omega_-)^3 := \{\mathbf{w} \in H^1(\Omega_-)^3 : \ \mathrm{div}\ \mathbf{w} = 0 \text{ in } \Omega_-\},\\
&\mathfrak{X} := H^1_{\mathrm{div}}(\Omega_+)^3 \times L^2(\Omega_+) \times H^1_{\mathrm{div}}(\Omega_-)^3 \times L^2(\Omega_-),\\
&\mathfrak{Y} := \widetilde{H}^{-1}(\Omega_+)^3 \times \widetilde{H}^{-1}(\Omega_-)^3 \times H^{\frac{1}{2}}(\partial\Omega)^3 \times H^{-\frac{1}{2}}(\partial\Omega)^3.
\end{aligned}$$

Let $\boldsymbol{L} \in L^\infty(\partial\Omega)^{3\times 3}$ be a symmetric matrix-valued function, which satisfies the following positivity condition:

$$\langle \boldsymbol{L}\mathbf{v}, \mathbf{v}\rangle_{\partial\Omega} \geq 0, \quad \forall\ \mathbf{v} \in L^2(\partial\Omega)^3. \tag{3.1}$$

Before we state the main result of this paper, we invoke an auxiliary property which refers to the well-posedness result of the Poisson problem of transmission-type for the generalized Brinkman system and classical Brinkman system in complementary Lipschitz domains in $\mathbb{R}^3$ and in the space $\mathfrak{X}$. Such a result is useful to obtain the existence of a solution (and its uniqueness) in the space $\mathfrak{X}$ for the nonlinear transmission problem concerning the classical Brinkman and the generalized Darcy-Forchheimer-Brinkman systems in complementary Lipschitz domains in $\mathbb{R}^3$.

The result is as follows (cf. [2, Theorem 3.3]).

Theorem 3.1. *Let $\alpha > 0$ be a given constant. Let $\Omega_+ := \Omega \subset \mathbb{R}^3$ be a bounded Lipschitz domain with connected boundary and let $\Omega_- := \mathbb{R}^3 \setminus \overline{\Omega}$ the complementary Lipschitz set. Let $\mathcal{P} \in L^\infty(\Omega_+)^{3\times 3}$ be such that condition* (2.2) *holds. Let $\boldsymbol{L} \in L^\infty(\partial\Omega)^{3\times 3}$ be a symmetric matrix-valued function that satisfied condition* (3.1). *Then, for given data $(\mathfrak{F}_+, \mathfrak{F}_-, \mathfrak{G}_0, \mathfrak{H}_0) \in \mathfrak{Y}$, the Poisson problem of transmission-type for the classical Brinkman and the generalized Brinkman systems:*

$$\begin{cases}
\Delta\mathbf{w}_+ - \mathcal{P}\mathbf{w}_+ - \nabla p_+ = \mathfrak{F}|_{\Omega_+} \ \textit{in } \Omega_+,\\
\Delta\mathbf{w}_- - \alpha\mathbf{w}_- - \nabla p_- = \mathfrak{F}|_{\Omega_-} \ \textit{in } \Omega_-,\\
Tr^+\mathbf{w}_+ - Tr^-\mathbf{w}_- = \mathfrak{G}_0 \ \textit{on } \partial\Omega,\\
\mathbf{t}^+_{\mathcal{P},\nu}(\mathbf{w}_+, p_+, \mathfrak{F}_+) - \mathbf{t}^-_{\alpha,\nu}(\mathbf{w}_-, p_-, \mathfrak{F}_-) + \boldsymbol{L}Tr^+\mathbf{w}_+ = \mathfrak{H}_0 \ \textit{on } \partial\Omega,
\end{cases} \tag{3.2}$$

has a unique solution $(\mathbf{w}_+, p_+, \mathbf{w}_-, p_-) \in \mathfrak{X}$*. In addition, the 'solution' operator:*

$$\mathcal{T} : \mathfrak{Y} \to \mathfrak{X}, \tag{3.3}$$

that maps the given data $(\mathfrak{F}_+, \mathfrak{F}_-, \mathfrak{G}_0, \mathfrak{H}_0) \in \mathfrak{Y}$ *to the solution* $(\mathbf{w}_+, p_+, \mathbf{w}_-, p_-) \in \mathfrak{X}$ *of the transmission problem* (3.2)*, is well-defined, linear and continuous.*

Hence, there is a constant $C \equiv C(\Omega_+, \Omega_-, \mathcal{P}, \boldsymbol{L}) > 0$ *such that:*

$$||(\mathbf{w}_+, p_+, \mathbf{w}_-, p_-)||_{\mathfrak{X}} \leq C||(\mathfrak{F}_+, \mathfrak{F}_-, \mathfrak{G}_0, \mathfrak{H}_0)||_{\mathfrak{Y}}. \tag{3.4}$$

The proof of this result can be consulted in [2].

The main result of this paper which is the well-posedness result for the transmission problem for the generalized Darcy-Forchheimer-Brinkman system and classical Brinkman system in complementary Lipschitz domains in $\mathbb{R}^3$. We aim to determine the unknown fields $(\mathbf{w}_+, p_+, \mathbf{w}_-, p_-) \in \mathfrak{X}$ such that:

$$\begin{cases} \Delta\mathbf{w}_+ - \mathcal{P}\mathbf{w}_+ - k|\mathbf{w}_+|\mathbf{w}_+ - \beta(\mathbf{w}_+ \cdot \nabla)\mathbf{w}_+ \\ \quad - \nabla p_+ = \mathfrak{F}|_{\Omega_+} \text{ in } \Omega_+, \\ \Delta\mathbf{w}_- - \alpha\mathbf{w}_- - \nabla p_- = \mathfrak{F}|_{\Omega_-} \text{ in } \Omega_-, \\ Tr^+\mathbf{w}_+ - Tr^-\mathbf{w}_- = \mathfrak{G}_0 \text{ on } \partial\Omega, \\ \mathbf{t}^+_{\mathcal{P},\nu}(\mathbf{w}_+, p_+, \mathfrak{F}_+ + \mathring{E}(k|\mathbf{w}_+|\mathbf{w}_+ + \beta(\mathbf{w}_+ \cdot \nabla)\mathbf{w}_+)) \\ \quad - \mathbf{t}^-_{\alpha,\nu}(\mathbf{w}_-, p_-, \mathfrak{F}_-) + \boldsymbol{L}Tr^+\mathbf{w}_+ = \mathfrak{H}_0 \text{ on } \partial\Omega, \end{cases} \tag{3.5}$$

where $\alpha > 0$ is a given constant.

The main result of the paper reads as follows (see e.g., [3, Theorem 3.2], and [8, Theorem 5.2] in the case $\mathcal{P} = \alpha\mathbb{I}$, where $\alpha, k, \beta > 0$ are constants).

Theorem 3.2. *Let* $\alpha > 0$ *be a given constant. Let* $\Omega_+ := \Omega \subset \mathbb{R}^3$ *be a bounded Lipschitz domain with connected boundary and let* $\Omega_- := \mathbb{R}^3 \setminus \overline{\Omega}$ *the complementary Lipschitz set. Let* $\mathcal{P} \in L^\infty(\Omega_+)^{3\times 3}$ *be such that condition* (2.2) *holds. Let* $\boldsymbol{L} \in L^\infty(\partial\Omega)^{3\times 3}$ *be a symmetric matrix-valued function that satisfies condition* (3.1)*. Then, there exist two constants* $\xi = \xi(\Omega_+, \Omega_-, \mathcal{P}, k, \beta, \boldsymbol{L}) > 0$ *and* $\lambda = \lambda(\Omega_+, \Omega_-, \mathcal{P}, k, \beta, \boldsymbol{L}) > 0$*, such that for all given data* $(\mathfrak{F}_+, \mathfrak{F}_-, \mathfrak{G}_0, \mathfrak{H}_0) \in \mathfrak{Y}$ *that satisfy the condition*

$$||(\mathfrak{F}_+, \mathfrak{F}_-, \mathfrak{G}_0, \mathfrak{H}_0)||_{\mathfrak{Y}} \leq \xi, \tag{3.6}$$

the transmission problem (3.5) *has a unique solution* $(\mathbf{w}_+, p_+, \mathbf{w}_-, p_-) \in \mathfrak{X}$ *such that*

$$||\mathbf{w}_+||_{H^1_{\mathrm{div}}(\Omega_+)^3} \leq \lambda. \tag{3.7}$$

In addition, the solution depends continuously on the given data, which means that there exists a given constant $C_0 = C_0(\Omega_+, \Omega_-, \mathcal{P}, \boldsymbol{L}) > 0$ *such that:*

$$||(\mathbf{w}_+, p_+, \mathbf{w}_-, p_-)||_{\mathfrak{X}} \leq C_0||(\mathfrak{F}_+, \mathfrak{F}_-, \mathfrak{G}_0, \mathfrak{H}_0)||_{\mathfrak{Y}}. \tag{3.8}$$

Proof. We provide here only the main ideas of the proof (for additional details, see [8, Theorem 5.2]).

We start with the existence part of the proof.

Let us write the problem (3.5) in the equivalent form:

$$\begin{cases} \Delta \mathbf{w}_+ - \mathcal{P}\mathbf{w}_+ - \nabla p_+ = \mathfrak{F}|_{\Omega_+} + \mathcal{J}_{k,\beta,\Omega_+}(\mathbf{w}_+) \text{ in } \Omega_+, \\ \Delta \mathbf{w}_- - \alpha \mathbf{w}_- - \nabla p_- = \mathfrak{F}|_{\Omega_-} \text{ in } \Omega_-, \\ Tr^+\mathbf{w}_+ - Tr^-\mathbf{w}_- = \mathfrak{G}_0 \text{ on } \partial\Omega, \\ \mathbf{t}^+_{\mathcal{P},\nu}(\mathbf{w}_+, p_+, \mathfrak{F}_+ + \mathcal{J}_{k,\beta,\Omega_+}(\mathbf{w}_+)) \\ \qquad - \mathbf{t}^-_{\alpha,\nu}(\mathbf{w}_-, p_-, \mathfrak{F}_-) + \boldsymbol{L}Tr^+\mathbf{w}_+ = \mathfrak{H}_0 \text{ on } \partial\Omega, \end{cases} \tag{3.9}$$

where $\mathcal{J}_{k,\beta,\Omega_+}(\mathbf{w}_+) \in \widetilde{H}^{-1}(\Omega_+)^3$ is given by Lemma 2.8.

Now, we fix $\mathbf{w}_+ \in H^1_{\mathrm{div}}(\Omega_+)^3$ and consider the following linear transmission problem for the generalized and classical Brinkman systems with the unknowns $(\mathbf{w}^0_+, p^0_+, \mathbf{w}^0_-, p^0_-)$:

$$\begin{cases} \Delta \mathbf{w}^0_+ - \mathcal{P}\mathbf{w}^0_+ - \nabla p^0_+ = \mathfrak{F}|_{\Omega_+} + \mathcal{J}_{k,\beta,\Omega_+}(\mathbf{w}_+) \text{ in } \Omega_+, \\ \Delta \mathbf{w}^0_- - \alpha \mathbf{w}^0_- - \nabla p^0_- = \mathfrak{F}|_{\Omega_-} \text{ in } \Omega_-, \\ Tr^+\mathbf{w}^0_+ - Tr^-\mathbf{w}^0_- = \mathfrak{G}_0 \text{ on } \partial\Omega, \\ \mathbf{t}^+_{\mathcal{P},\nu}(\mathbf{w}^0_+, p^0_+, \mathfrak{F}_+ + \mathcal{J}_{k,\beta,\Omega_+}(\mathbf{w}_+)) \\ \qquad - \mathbf{t}^-_{\alpha,\nu}(\mathbf{w}^0_-, p^0_-, \mathfrak{F}_-) + \boldsymbol{L}Tr^+\mathbf{w}^0_+ = \mathfrak{H}_0 \text{ on } \partial\Omega. \end{cases} \tag{3.10}$$

By applying Theorem 3.1 we deduce that the problem (3.10) has a unique solution $(\mathbf{w}^0_+, p^0_+, \mathbf{w}^0_-, p^0_-)$ in $\mathfrak{X}$ given by

$$\begin{aligned} (\mathbf{w}^0_+, p^0_+, \mathbf{w}^0_-, p^0_-) &= (\mathcal{W}_+(\mathbf{w}_+), \mathfrak{P}_+(\mathbf{w}_+), \mathcal{W}_-(\mathbf{w}_+), \mathfrak{P}_-(\mathbf{w}_+)) \\ &:= \mathcal{T}(\mathfrak{F}_+|_{\Omega_+} + \mathcal{J}_{k,\beta,\Omega_+}(\mathbf{w}_+)|_{\Omega_+}, \mathfrak{F}_-|_{\Omega_-}, \mathfrak{G}_0, \mathfrak{H}_0) \in \mathfrak{X}, \end{aligned} \tag{3.11}$$

where $\mathcal{T}$ is the solution operator introduced in Theorem 3.1.

Note that, for fixed given data $\mathfrak{F}_\pm, \mathfrak{G}_0, \mathfrak{H}_0$, the nonlinear operators

$$\mathcal{W}_\pm, \mathfrak{P}_\pm : H^1_{\mathrm{div}}(\Omega_+)^3 \to \mathfrak{X}, \tag{3.12}$$

are bounded, in the sense that there exists a constant $d \equiv d(\Omega_+, \Omega_-, \mathcal{P}, \boldsymbol{L}) > 0$, a constant, such that

$$\begin{aligned} ||(\mathcal{W}_+(\mathbf{w}_+), &\mathfrak{P}_+(\mathbf{w}_+), \mathcal{W}_-(\mathbf{w}_+), \mathfrak{P}_-(\mathbf{w}_+))||_{\mathfrak{X}} \\ &\leq d||(\mathfrak{F}_+|_{\Omega_+} + \mathcal{J}_{k,\beta,\Omega_+}(\mathbf{u}_+)|_{\Omega_+}, \mathfrak{F}_-|_{\Omega_-}, \mathfrak{G}_0, \mathfrak{H}_0)||_{\mathfrak{Y}} \\ &\leq d||(\mathfrak{F}_+|_{\Omega_+}, \mathfrak{F}_-|_{\Omega_-}, \mathfrak{G}_0, \mathfrak{H}_0)||_{\mathfrak{Y}} + dc_0||\mathbf{w}_+||^2_{H^1(\Omega_+)^3}, \end{aligned} \tag{3.13}$$

for all $\mathbf{w}_+ \in H^1_{\mathrm{div}}(\Omega_+)^3$. Indeed, such an inequality is provided by Lemma 2.8 and $c_0 > 0$ is the constant involved in Lemma 2.8.

Next, we rewrite the problem (3.10) in terms of the operators $\mathcal{W}_\pm, \mathfrak{P}_\pm$, as

$$\begin{cases} \Delta \mathcal{W}_+(\mathbf{w}_+) - \mathcal{P}\mathcal{W}_+(\mathbf{w}_+) - \nabla \mathfrak{P}_+(\mathbf{w}_+) = \\ \quad \mathfrak{F}|_{\Omega_+} + \mathcal{J}_{k,\beta,\Omega_+}(\mathbf{w}_+) \text{ in } \Omega_+, \\ \Delta \mathcal{W}_-(\mathbf{w}_+) - \alpha \mathcal{W}_-(\mathbf{w}_+) - \nabla \mathfrak{P}_+(\mathbf{w}_+) = \mathfrak{F}|_{\Omega_-} \text{ in } \Omega_-, \\ Tr^+\mathcal{W}_+(\mathbf{w}_+) - Tr^-\mathcal{W}_-(\mathbf{w}_+) = \mathfrak{G}_0 \text{ on } \partial\Omega, \\ \mathbf{t}^+_{\mathcal{P},\nu}(\mathcal{W}_+(\mathbf{w}_+), \mathfrak{P}_+(\mathbf{w}_+), \mathfrak{F}_+ + \mathcal{J}_{k,\beta,\Omega_+}(\mathbf{w}_+)) \\ \qquad - \mathbf{t}^-_{\alpha,\nu}(\mathcal{W}_-(\mathbf{w}_+), \mathfrak{P}_-(\mathbf{w}_+), \mathfrak{F}_-) + \boldsymbol{L}Tr^+\mathcal{W}_+(\mathbf{w}_+) = \mathfrak{H}_0 \text{ on } \partial\Omega. \end{cases} \tag{3.14}$$

The next step is to show that the nonlinear operator $\mathcal{W}_+$ has a unique fixed point. If we are able to show this property, then the fixed point $\mathbf{w}_+ \in H^1_{\text{div}}(\Omega_+)^3$, together with the fields $\mathbf{w}_- = \mathcal{W}_-(\mathbf{w}_+)$ and with $p_\pm = \mathfrak{P}_\pm(\mathbf{w}_+)$ will give a solution of our nonlinear problem (3.9).

Now, we use similar ideas to those in the proof of Theorem 5.2 in [8], to show that $\mathcal{W}_+$ maps a closed ball $\mathbf{B}_\lambda$ to the same closed ball in $H^1_{\text{div}}(\Omega_+)^3$ and that $\mathcal{W}_+$ is a contraction on that ball.

We make the following choice of constants

$$\xi := \frac{3}{16c_0d^2} > 0, \quad \lambda := \frac{1}{4c_0d} > 0, \tag{3.15}$$

and we introduce the closed ball

$$\mathbf{B}_\lambda := \{\mathbf{v}_+ \in H^1_{\text{div}}(\Omega_+)^3 : ||\mathbf{v}_+||_{H^1(\Omega_+)^3} \le \lambda\}. \tag{3.16}$$

We impose the following condition on the given data:

$$||(\mathfrak{F}_+|_{\Omega_+}, \mathfrak{F}_-|_{\Omega_-}, \mathfrak{G}_0, \mathfrak{H}_0)||_{\mathfrak{Y}} \le \xi. \tag{3.17}$$

Then, by using relations (3.13), (3.15), (3.16), (3.17), one may show that

$$||(\mathcal{W}_+(\mathbf{w}_+), \mathfrak{P}_+(\mathbf{w}_+), \mathcal{W}_-(\mathbf{w}_+), \mathfrak{P}_-(\mathbf{w}_+))||_{\mathfrak{X}} \le \lambda, \tag{3.18}$$

for all $\mathbf{w}_+ \in \mathbf{B}_\lambda$ and hence $||\mathcal{W}_+(\mathbf{w}_+)||_{H^1(\Omega_+)^3} \le \lambda$ for all $\mathbf{w}_+ \in \mathbf{B}_\lambda$, that is, $\mathcal{W}_+$ maps the ball $\mathbf{B}_\lambda$ to itself.

In order to show that $\mathcal{W}_+$ is Lipschitz continuous on $\mathbf{B}_\lambda$, we fix the given data $(\mathfrak{F}_+|_{\Omega_+}, \mathfrak{F}_-|_{\Omega_-}, \mathfrak{G}_0, \mathfrak{H}_0)$ and we consider two arbitrary functions $\mathbf{w}_+, \mathbf{v}_+ \in \mathbf{B}_\lambda$. Then, we get

$$\begin{aligned} &||\mathcal{W}_+(\mathbf{w}_+) - \mathcal{W}_+(\mathbf{v}_+)||_{H^1(\Omega_+)^3} \\ &\le d||\mathcal{J}_{k,\beta,\Omega_+}(\mathbf{w}_+) - \mathcal{J}_{k,\beta,\Omega_+}(\mathbf{v}_+)||_{\widetilde{H}^{-1}(\Omega_+)^3} \\ &\le dc_0(||\mathbf{w}_+||_{H^1(\Omega_+)^3} + ||\mathbf{v}_+||_{H^1(\Omega_+)^3})||\mathbf{w}_+ - \mathbf{v}_+||_{H^1(\Omega_+)^3} \\ &\le 2dc_0||\mathbf{w}_+ - \mathbf{v}_+||_{H^1(\Omega_+)^3} = \frac{1}{2}||\mathbf{w}_+ - \mathbf{v}_+||_{H^1(\Omega_+)^3}, \end{aligned} \tag{3.19}$$

for all $\mathbf{w}_+, \mathbf{v}_+ \in \mathbf{B}_\lambda$, where we have take into account the continuity of the operator $\mathcal{T}$ and inequality (2.12) and the constants d and c_0 are the same constants as in relation (3.13). Based on the above considerations, we deduce that $\mathcal{W}_+ : \mathbf{B}_\lambda \to \mathbf{B}_\lambda$ is a $\frac{1}{2}$-contraction.

By applying Banach's fixed point theorem we deduce that there is a unique fixed point $\mathbf{w}_+ \in \mathbf{B}_\lambda$ of the operator $\mathcal{W}_+$, which, together with the fields given by $\mathbf{w}_- = \mathcal{W}_-(\mathbf{w}_+)$ and $p_\pm = \mathfrak{P}_\pm(\mathbf{w}_+)$, determines a solution of the transmission problem (3.9).

Now, we use the fact that the field $\mathbf{w}_+ \in \mathbf{B}_\lambda$ in order to deduce that

$$dc_0||\mathbf{w}_+||_{H^1(\Omega_+)^3} \le \frac{1}{4},$$

and by using inequality (3.13), we obtain

$$\begin{aligned} &||\mathbf{w}_+||_{H^1(\Omega_+)^3} + ||p_+||_{L^2(\Omega_+)} + ||\mathbf{w}_-||_{H^1(\Omega_-)^3} + ||p_-||_{\mathfrak{M}(\Omega_-)} \\ &\qquad \le d||(\mathfrak{F}_+|_{\Omega_+}, \mathfrak{F}_-|_{\Omega_-}, \mathfrak{G}_0, \mathfrak{H}_0)||_{\mathfrak{Y}} + \frac{1}{4}||\mathbf{w}_+||_{H^1(\Omega_+)^3}, \end{aligned} \tag{3.20}$$

and we obtain that

$$||\mathbf{w}_+||_{H^1(\Omega_+)^3} \leq \frac{4}{3}d||(\mathfrak{F}_+|_{\Omega_+}, \mathfrak{F}_-|_{\Omega_-}, \mathfrak{G}_0, \mathfrak{H}_0)||_{\mathfrak{Y}}. \tag{3.21}$$

By substituting relation (3.21) into (3.20) we obtain the desired estimate (3.8) where $C_0 = \frac{4}{3}d$.

For other details, we refer to the proof of Theorem 5.2 in [8].

As for the uniqueness part, the ideas are as follows.

Assume that we have two solutions of the transmission problem (3.5), say $(\mathbf{w}_+^1, p_+^1, \mathbf{w}_-^1, p_-^1)$ and $(\mathbf{w}_+^2, p_+^2, \mathbf{w}_-^2, p_-^2)$. Note that these solutions belong to the space $\mathfrak{X}$ and both satisfy inequality (3.7).

We obtain the linear, homogeneous transmission problem for the classical and generalized Brinkman systems in Lipschitz domains in $\mathbb{R}^3$ with the unknowns $(\mathcal{W}_+(\mathbf{w}_+^2) - \mathbf{w}_+^2, \mathfrak{P}_+(\mathbf{w}_+^2) - p_+^2, \mathcal{W}_-(\mathbf{w}_+^2) - \mathbf{w}_-^2, \mathfrak{P}_-(\mathbf{w}_+^2) - p_-^2)$ and Theorem 3.1 shows that this problem has only the trivial solution in $\mathfrak{X}$. It follows that $\mathcal{W}_+(\mathbf{w}_+^2) - \mathbf{w}_+^2 = 0$, that is, $\mathbf{w}_+^2$ is a fixed point of the nonlinear operator $\mathcal{W}_+$. Recall that $\mathcal{W}_+ : \mathbf{B}_\lambda \to \mathbf{B}_\lambda$ is a $\frac{1}{2}$-contractions, and hence, there is a unique fixed point $\mathbf{w}_+^1$ in $\mathbf{B}_\lambda$. Consequently, $\mathbf{w}_+^2 = \mathbf{w}_+^1$, $\mathbf{w}_-^2 = \mathbf{w}_-^1$ and $p_\pm^2 = p_\pm^1$.

This concludes the uniqueness argument.

This concludes the proof. □

Remark 3.3. (i) If $k = 0$ and $\beta : \Omega_+ \to \mathbb{R}_+$ such that $\beta \in L^\infty(\Omega_+)$, then we get the well-posedness result for the nonlinear transmission problem for the generalized Navier-Stokes and Brinkman systems in complementary Lipschitz domains in $\mathbb{R}^3$.

(ii) If $k : \Omega_+ \to \mathbb{R}_+$ such that $k \in L^\infty(\Omega_+)$ and $\beta = 0$, then we get the well-posedness result for a semilinear transmission problem for a generalized semilinear Darcy-Forchheimer-Brinkman system and the Brinkman system in complementary Lipschitz domains in $\mathbb{R}^3$.

All these problems are important for their practical applications (see, e.g., [21], [5]).

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On the Rockafellar function associated to a non-cyclically monotone mapping

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Abstract. In an earlier paper, we have given a definition of the Rockafellar integration function associated to a cyclically monotone mapping considering only systems of distinct elements in its domain. Thus, this function can be proper for certain non-cyclically monotone mappings. In this paper we establish general properties of Rockafellar function if the graph of mapping does not contain finite set of accumulation elements where the mapping is not cyclically monotone. Also, some dual properties are given.

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1. Introduction

Let X be a real linear normed space and let X^* be its dual. Given a function $f : X \to \overline{\mathbb{R}}$ its subdifferential is the (multivalued) mapping $\partial f : X \to X^*$ defined by

$$\partial f(x) = \{x^* \in X^*; x^*(u - x) \leq f(u) - f(x), \text{ for all } u \in X\}, x \in X, \tag{1.1}$$

where $\overline{\mathbb{R}} = [-\infty, \infty]$. We also suppose the usual extension in convex analysis of addition by condition $\infty - \infty = \infty$. Here, we consider only proper function f, that is, its domain

$$\operatorname{Dom} f = \{x \in X; f(x) < \infty\} \tag{1.2}$$

is a nonvoid set and $f(u) > -\infty$ for all $u \in X$.

It is well known that any proper convex lower-semicontinuous function is subdifferentiable, that is

$$\operatorname{Dom} \partial f = \{x \in X; \partial f(x) \neq \emptyset\} \neq \emptyset. \tag{1.3}$$

The problem of integration with respect to this subdifferential was studied by many authors. In this line, a remarkable result was established by Rockafellar [7],

[8]: any maximal cyclically monotone mapping can be represented as the subdifferential of a proper convex lower-semicontinuous function. This function is unique up to an additive constant function. Also, the subdifferential of a proper convex lower-semicontinuous function is a maximal cyclically monotone mapping. We recall that a mapping $T : X \to X^*$ is cyclically monotone if

$$\sum_{i=0}^{n} x_i^*(x_i - x_{i+1}) \geq 0 \text{ for } \text{ all } (x_i, x_i^*) \in \operatorname{Graph} T, i = \overline{0, n}, \tag{1.4}$$

where $x_{n+1} = x_0$, $n = 1, 2, \ldots$

If (1.4) is fulfilled for $n = 1$, then the mapping T is called monotone.

In the proof of Rockafellar's result it is used the following function associated to a cyclically monotone mapping $T : X \to X^*$

$$\begin{aligned} f_{x_0;T}(x) = \sup \Big\{ \sum_{i=1}^{n} x_i^*(x_{i+1} - x_i); (x_i, x_i^*) \in \operatorname{Graph} T, \\ i = \overline{0, n}, n = 1, 2, \ldots, x_{n+1} = x \Big\}, \end{aligned} \tag{1.5}$$

for any $x \in X$, where x_0 is a fixed element in $\operatorname{Dom} T$.

We mention that in the papers [1,3,4,5,9] are given special properties using different concepts of subdifferential.

Obviously, Rockafellar (integration) function $f_{x_0;T}$ is convex and lower-semicontinuous. In fact, this function can be defined for any mapping T, but it is a proper function only in the case of cyclically monotone mappings. In an earlier paper [6] we established the following result:

Theorem 1.1. *Let us consider a mapping $T : X \to X^*$. The following statements are equivalent:*

(i) *T is a cyclically monotone mapping;*

(ii) *$f_{x_0;T}(x_0) = 0$ for any (one) element $x_0 \in \operatorname{Dom} T$;*

(iii) *$\operatorname{Dom} f_{x_0;T} \neq \emptyset$ for any (one) element $x_0 \in \operatorname{Dom} T$.*

Indeed, if T is not cyclically monotone, there exist $x_0, x_1, \ldots, x_n \in X$ such that

$$\sum_{i=0}^{n} x_i^*(x_i - x_{i+1}) = a < 0, \quad x_{n+1} = x_0.$$

Now, if we consider the finite system of elements $x_0, x_1, x_2, \ldots x_{k(n+1)} \in X$, where $x_i = x_{i+j(n+1)}$, $i = \overline{0, n}$, $j = \overline{0, k-1}$, then

$$\begin{aligned} f_{x_0,T}(x) &\geq -ka + x_{k(n+1)}^*(x - x_{k(n+1)}) - x_{k(n+1)}^*(x_0 - x_{k(n+1)}) \\ &= -ka + x_{k(n+1)}^*(x - x_0) = -ka + x_n^*(x - x_0), \text{ for all } k = 1, 2, \ldots \end{aligned}$$

and so $f_{x_0;T}(x) = \infty$, for all $x \in X$.

Taking into account the equivalence (i)$\Leftrightarrow$(iii) we can give an other equivalent definition for Roackafellar function $f_{T;x_0}$ such that its domain is also nonvoid for some non cyclically monotone mappings (see also [6]).

In this paper we give special cases when Rockafellar function is proper. Also, we establish a subdifferential property and some dual inequalities.

2. A new definition of Rockafellar function

Let $T : X \to X^*$ be a proper multivalued mapping. Considering only systems of distinct elements in $\operatorname{Dom} T$ we obtain the following slight modification of the Rockafellar function associated to T as follows:

$$g_{x_0;T}(x) = \sup \Big\{ \sum_{i=0}^{n} x_i^*(x_{i+1} - x_i); (x_i, x_i^*) \in \operatorname{Graph} T, \tag{2.1}$$
$$x_i \neq x_j \text{ for any } i \neq j, i, j = \overline{0,n}, n = 1, 2, \ldots, x_{n+1} = x \Big\}, \quad x \in X,$$

where x_0 is an fixed element of $\operatorname{Dom} T$.

This function is also convex and lower-semicontinuous.

Proposition 2.1. *If T is a cyclically monotone mapping then*

$$g_{x_0;T} = f_{x_0;T} \text{ for all } x_0 \in \operatorname{Dom} T.$$

Proof. Obviously, $f_{x_0;T} \geq g_{x_0;T}$. On the other hand, if we consider a system $(x_i, x_i^*) \in \operatorname{Graph} T, i = \overline{0,n}$, which contains two equal elements $x_k = x_{k+l}$, $l > 0$, $k, k+l = \overline{0,n}$, then by (1.4) we have

$$\sum_{i=0}^{n} x_i^*(x_{i+1} - x_i) = \sum_{i=0}^{k-1}(x_{i+1} - x_i) + \sum_{i=k}^{k+l-1} x_i^*(x_{i+1} - x_i) + \sum_{i=k+l}^{n} x_i^n(x_{i+1} - x_i)$$
$$\leq \sum_{i=0}^{k-1} x_i^*(x_{i+1} - x_1) + \sum_{i=k+l}^{n} x_i^*(x_{i+1} - x_i).$$

Therefore, in the definition (1.4) of Rockafellar function $f_{x_0;T}$ we can omit the systems $x_i \in \operatorname{Dom} T$, $i = \overline{1,n}$ which contain equal elements.

Example 2.2. Let $T : \mathbb{R} \to \mathbb{R}$ defineed by

$$T(x) = \begin{cases} x, & \text{if } x \in [0,1] \\ \alpha, & \text{if } x = 2, \\ \phi, & \text{if } x \overline{\in} [0,1] \cup \{2\}, \end{cases} \tag{2.2}$$

where $\alpha \in \mathbb{R}$. If $\alpha \geq 1$ obviously T is a cyclically monotone mapping, and so, in this case $f_{x_0;T} = g_{x_0;T}$ for any $x_0 \in \operatorname{Dom} T$. But, if $\alpha < 1$, then T is not cyclically monotone. Thus, $f_{x_0;T}(x) = \infty$ for all $x \in \mathbb{R}$. On the other hand, we remark that T is cyclically monotone on $[0,1]$. We denote by T_0 this mapping (the identity mapping of $[0,1]$. Now, by a standard calculus we obtain

$$f_{x_0;T_0}(x) = \begin{cases} -\frac{x_0^2}{2}, & \text{for } x < 0, \\ -\frac{x_0^2}{2} + \dfrac{x^2}{2}, & \text{for } x \in [0,1], \\ -\frac{x_0^2}{2} + x - \frac{1}{2}, & \text{for } x > 1. \end{cases} \tag{2.3}$$

It is obvious that the sums in the definitions of $g_{x_0;T}$ and $f_{x_0;T_0}$ are distinct only for systems of elements which contain the element $x = 2$. Thus, $f_{x_0;T}(x) \neq f_{x_0;T_0}(x)$ if and only if for a system $(x_i, x_i^*) \in \operatorname{Graph} T$, $i = \overline{0, n}$, which contains the element $x = 2$, the corresponding sum in (2.1) has a greater value than the sum where the element $x = 2$ is omitted. For example if $\alpha \geq 1$ and $x_0 \in [0, 1]$, then $g_{x_0;T}(x) = f_{x_0;T_0}(x)$ for all $x \in (-\infty, 2)$, while $g_{x_0;T} > f_{x_0;T_0}(x)$ for any $x > 2$.

Also, if $\alpha < 1$, then $g_{x_0;T} > f_{x_0;T_0}(x)$. On the other hand, concerning the integration property we get that on $[0, 1]$ we have the following equality

$$\partial g_{x_0;T} = \partial f_{x_0;T_0} = T_0.$$

Generally, $\operatorname{Dom} g_{x_0;T} \neq \emptyset$ only if T is cyclically monotone excepting a "small" subsets of its domain. In this line we give the following two results.

Theorem 2.3. *If there exists a system of accumulation elements $(x_i, x_i^*) \in \operatorname{Graph} T$, $i = \overline{1, k}$, $x_{k+1} = x_1$, such that*

$$\sum_{i=1}^{k} x_i^*(x_i - x_{i+1}) < 0,$$

then $g_{x_0;T}(x) = \infty$ for any $x \in X$.

Proof. Let us denote $\sum_{i=1}^{k} x_i^*(x_{i+1} - x_i) = \alpha > 0$. Now, given $\varepsilon > 0$, $M > 0$ we inductively define the sequence $(x_{i,n}, x_{i,n}^*) \in \operatorname{Graph} T$, $i = \overline{1, k}$ and $n = 1, 2, \ldots$ such that $x_{i,n} \neq x_{j,m}$ for any $i \neq j$, $n = 2, 3 \ldots$

$$\|x_{i,n} - x_i\| < \frac{\varepsilon}{2^{n+2}kM}, \quad \|x_{i,n}^* - x_i^*\| < \frac{\varepsilon}{2^{n+2}kM},$$

where $\|x_{i,n}\| \leq M$, $\|x_{i,n}^*\| \leq M$. Then, we have

$$\begin{aligned} &|x_{i,n}^*(x_{i+1,n} - x_{i,n}) - x_i^*(x_{i+1} - x_i)| \leq \|x_{i,n}^*\| \|(x_{i+1,n} - x_{i+1}) + (x_i - x_{i,n})\| \\ &+ \|x_i - x_{i+1}\| \|x_{i,n}^* - x_i^*\| \leq \frac{\varepsilon}{2^n k}, \text{ for any } i = \overline{1, k}, n = 1, 2, \ldots \end{aligned}$$

Let x_0 be a given element in $\operatorname{Dom} T$ and $x \in X$. By hypotheses we can suppose that $x_0 \neq x_{i;m}$, for any $i = \overline{1, k}$, $m = 1, 2, \ldots$

Now, we consider the system of distinct elements $\{x_{1,1}, x_{1,2}, \ldots, x_{1,k}, \ldots, x_{1,n}, x_{2,n}, \ldots, x_{k,n}\}$ and corresponding sum of right hand of formula (2.1). We have

$$\begin{aligned} &x_0^*(x_{1,1} - x_0) + \sum_{i=1}^{k} \sum_{m=1}^{n} x_{i,m}^*(x_{i+1,m} - x_{i,m}) + x_{k,n}^*(x - x_{k,n}) \\ &= x_0^*(x_{1,1} - x_0) + \sum_{i=1}^{k} \sum_{m=1}^{n} x_{i,m}^*(x_{i+1,m} - x_{i,m}) + x_{k,n}^*(x - x_{k,n}) + n\alpha \\ &- n\sum_{i=1}^{k} x_i^*(x_{i+1} - x_i) = x_0^*(x_{1,1} - x_0) + n\alpha + \sum_{i=1}^{k} \sum_{m=1}^{n} [x_{i,m}^*(x_{i+1,m} - x_{i,m}) \\ &- x_i^*(x_{i+1} - x_i)] + x_{k,n}^*(x - x_{k,n}) \geq x_0^*(x_{1,1} - x_0) - k\sum_{m=1}^{n} \frac{\varepsilon}{k2^m} + n\alpha \\ &+ x_{k,n}^*(x - x_{k,n}) \geq -\varepsilon + x_0^*(x_{1,1} - x_0) + x_{k,n}^*(x - x_{k,n}) + n\alpha, \end{aligned}$$

for any $n = 1, 2, \ldots$ Since the sequence $(x_{i,n}, x^*_{i,n}) \in \mathrm{Graph}\, T$, $i = 1, k$ and $n = 1, 2, \ldots$ is bounded, according to the definition (2.1) we get $g_{x_0,T}(x) = \infty$ for all $x \in X$, as claimed.

The following result establishes a sufficient condition such that $\mathrm{Dom}\, g_{x_0;T} \neq \emptyset$.

Theorem 2.4. *Let $T_0 : X \to X^*$ be a cyclically monotone mapping. If $T : X \to X^*$ such that $Graph\, T = Graph\, T_0 \cup \{(u_i, u^*_i) \in X \times X^*, i = \overline{1, k}\}$ and there exists $M > 0$ such that*

$$(u^*_i - u^*)(u - u_i) \leq M, \text{ for all } i = \overline{1, k}, u \in Dom\, T_0, u^* \in T_0(X), \tag{2.4}$$

then $Dom\, g_{x_0;T} \supset Dom\, f_{x_0;T_0}$ for any $x_0 \in Dom\, T_0$.

Proof. Let x_0 be an element in $\mathrm{Dom}\, T_0$ and let $\{x_1, x_2, \ldots, x_n\} \subset \mathrm{Dom}\, T_0$ be a system of distinct elements. If we add only an element u_j, $j = \overline{1, k}$, then

$$\begin{aligned}
&x^*_0(x_1 - x_0) + x^*_1(x_2 - x_1) + \ldots + x^*_{i_0}(u_j - x_{i_0}) \\
&+ u^*_j(x_{i_0+1} - u_j) + x^*_{i_0+1}(x_{i_0+2} - x_{i_0+1}) \\
&+ \ldots + x^*_{n-1}(x_n - x_{n-1}) + x^*_n(x - x_n) \\
&= \sum_{i=0}^{n-1} x^*_i(x_{i+1} - x_i) + x^*_n(x - x_n) + x^*_{i_0}(u_j - x_{i_0}) \\
&+ u^*_j(x_{i_0+1} - u_j) - x^*_{i_0}(x_{i_0+1} - x_{i_0}) \leq f_{x_0;T_0}(x) + (u^*_j - x^*_{i_0})(x_{i_0+1} - u_j) \\
&\leq f_{x_0;T_0}(x) + M, \text{ for any } x \in \mathrm{Dom}\, f_{x_0,T_0}.
\end{aligned}$$

Therefore, according to the definition (2.1) of $g_{x_0;T}$ it follows that

$$g_{x_0;T}(x) \leq f_{x_0;T_0}(x) + kM, \text{ for all } x \in \mathrm{Dom}\, f_{x_0;T_0}, \text{ as claimed.}$$

Remark 2.5. Obviously, if $M = 0$ the mapping T is cyclically monotone,. Also, the convex function $g_{x_0;T}$ can be proper in some special case when $(\mathrm{Dom}\, T)\backslash(\mathrm{Dom}\, T_0)$ is an infinite set. Generally, the following inequality

$$f_{x_0;T_0} \leq g_{x_0;T} \tag{2.5}$$

holds.

In the next result we prove that the integration property of a mapping $T : X \to X^*$ can be generated by the subdifferential of $g_{x_0;T}$.

Theorem 2.6. *Let $T : X \to X^*$ be an extension of a cyclically monotone mapping T_0. If there exist $x_0 \in Dom\, T_0$ and $\overline{x} \in Dom\, f_{x_0;T_0}$ such that $f_{x_0;T_0}(\overline{x}) = g_{x_0;T}(\overline{x})$, then $\partial f_{x_0;T_0}(\overline{x}) \subset \partial g_{x_0;T}(\overline{x})$.*

Proof. If $x^* \in \partial f_{x_0;T_0}(\overline{x})$, using the inequality (2.5) we have

$$x^*(x - \overline{x}) \leq f_{x_0;T_0}(x) - f_{x_0;T_0}(\overline{x}) = f_{x_0;T_0}(x) - g_{x_0;T}(\overline{x}) \leq g_{x_0;T}(x) - g_{x_0;T}(\overline{x}),$$

for all $x \in X$, and so $x^* \in \partial g_{x_0;T}(\overline{x})$.

Remark 2.7. Let T_0 be a maximal cyclically monotone mapping such that $\mathrm{Graph}\, T_0 \subset \mathrm{Graph}\, T$. Then on the set $\{x \in X; f_{x_0;T_0}(x) = g_{x_0;T}(x)\}$ the function $f_{x_0;T_0}$ can be regarded as an integral of mapping T.

3. Some dual properties

Firstly, we recall some fundamental dual concepts in convex analysis (see, for example [2]). Given a function $f : X \to (-\infty, \infty]$, its conjugate $f^* : X^* \to (-\infty, \infty]$ is defined by

$$f^*(x^*) = \sup\{x^*(x) - f(x); x \in X\}, x^* \in X^*. \tag{3.1}$$

If $A \subset X$ we define the support function

$$s_A(x^*) = \sup\{x^*(x); x \in A\}. \tag{3.2}$$

Similarly, we define the support function associated of a subset of X^*.

Now, we give an other equivalent form for the Rockafellar function $g_{x_0;T}$ with respect to dual space X^* namely

$$\begin{aligned} g_{x_0;T}(x) = \sup \Big\{ &\sum_{i=1}^{n} (x_{i-1}^* - x_i^*)(x_i) + x_n^*(x) - x_0^*(x_0); \\ &(x_i, x_i^*) \in \mathrm{Graph}\, T, x_i \neq x_j \text{ for } i \neq j, i,j = \overline{1,n}, n = 1,2,\dots \Big\},\ x \in X, \end{aligned} \tag{3.3}$$

where $x_0 \in \mathrm{Dom}\, T$.

This formula leads to consider the Rockafellar function associated to the mapping T^{-1}. Indeed, if $x_0^* \in \mathrm{Dom}\, T^{-1}$ by (2.1) we have

$$\begin{aligned} g_{x_0^*;T^{-1}}(x^*) = \sup \Big\{ &\sum_{i=1}^{n+1} (x_i^* - x_{i-1}^*)(x_{i-1}); (x_i^*, x_i) \in \mathrm{Graph}\, T^{-1}, \\ &x_i^* \neq x_j^*, \text{ for } i \neq j, i,j = \overline{0,n}, x_{n+1}^* = x^*, n = 1,2,\dots \Big\},\ x^* \in X^*. \end{aligned} \tag{3.4}$$

But, if the system $\{x_1^*, x_2^*, \dots, x_n^*\}$ is replaced by the system $\{x_n^*, x_{n-1}^*, \dots, x_1^*\}$ we obtain the equivalent formula

$$\begin{aligned} g_{x_0^*;T^{-1}}(x^*) = \sup \Big\{ &\sum_{i=2}^{n} (x_{i-1}^* - x_i^*)(x_i) + (x_n^* - x_0^*)(x_0) + (x^* - x_1^*)(x_1); x_i^* \neq x \\ &(x_i^*, x_i) \in \mathrm{Graph}\, T^{-1}, x_i^* \neq x_j^* \text{ for } i \neq j, i,j = \overline{1,n}, n = 1,2\dots \Big\},\ x^* \in X^*. \end{aligned} \tag{3.5}$$

Generally, if $\{x_1, x_2, \dots, x_n\}$ is a system of distinct elements of $\mathrm{Dom}\, T$ then the corresponding system $\{x_1^*, x_2^*, \dots, x_n^*\}$, $(x_i, x_i^*) \in \mathrm{Graph}\, T$, can have equal elements. Thus, in the following results we need to suppose that T is injective, that is $u^* \neq v^*$ whenever $u \neq v$, $(u, u^*), (v, v^*) \in \mathrm{Graph}\, T$.

Theorem 3.1. *Let $T : X \to X^*$ be an injective mapping. If $(x_0, x_0^*) \in Graph T$, then*

$$g_{x_0;T}(x) \leq g_{x_0^*;T^{-1}}(x^*) + s_{Dom\, T}(x_0^0 - x^*) + s_{Dom\, T^{-1}}(x - x_0), \tag{3.6}$$

for all $(x, x^*) \in X \times X^*$.

Proof. Let (x_0, x_0^*) be an element in $\operatorname{Graph} T$. By hypothesis, if $\{u_1, u_2, \ldots, u_m\}$ is a system of distinct elements in $\operatorname{Dom} T$, then a corresponding system $\{u_1^*, u_2^*, \ldots, u_m^*\}$, where $(u_j, u_j^*) \in \operatorname{Graph} T$, $j = \overline{1,m}$, also contains only distinct elements in $\operatorname{Dom} T^{-1}$. According to (3.5), taking $u_0^* = x_0^*$, we obtain

$$\sum_{i=1}^{m}(u_{j-1}^* - u_j^*)(u_j) + u_m^*(x) - x_0^*(x_0) \le \sup\Big\{\sum_{i=2}^{n}[(x_{i-1}^* - x_i^*)(x_i) + (x_n^* - x_0^*)(x_0)$$
$$+ (x^* - x_1^*)(x_1)] + [(x_0^* - x_1^*)(x_1)]$$
$$+ x_n^*(x) - x_0^*(x_0) - (x_n^* - x_0^*)(x_0) - (x^* - x_1^*)(x_1);$$
$$(x_i, x_i^*) \in \operatorname{Graph} T, x_i \neq x_j, \text{ for } i \neq j, i, j = \overline{1,n}\Big\}$$
$$\le \sup\Big\{\sum_{i=2}^{n}(x_{i-1}^* - x_i^*)(x_i) + (x_0^* - x^*)(x_1) + x_n^*(x - x_0); (x_i, x_i^*) \in \operatorname{Graph} T,$$
$$x_i \neq x_j \text{ for } i \neq j, i, j = \overline{1,n}\Big\} + \sup\{(x_0^* - x^*)(x_1); x_1 \in \operatorname{Dom} T\}$$
$$+ \sup\{x_n^*(x - x_0); x_n^* \in \operatorname{Dom} T^{-1}\} = g_{x_0^*;T^{-1}}(x^*) + s_{\operatorname{Dom} T}(x_0^* - x^*)$$
$$+ s_{\operatorname{Dom} T^{-1}}(x - x_0), \text{ for all } u_j, u_j^* \in \operatorname{Graph} T, u_i \neq u_j \text{ for } i \neq j,$$
$$i, j = \overline{1,n},\ (x, x^*) \in X \times X^*, n = 1, 2, \ldots$$

Now, passing to the supremum with respect to systems $\{u_1, u_2, \ldots, u_m\}$, $m = 1, 2, \ldots$ we obtain the inequality (3.6).

Remark 3.2. If T^{-1} is injective we have a converse inequality

$$g_{x_0^*;T^{-1}}(x^*) \le g_{x_0;T}(x) + s_{\operatorname{Dom} T}(x^* - x_0^*) + s_{\operatorname{Dom} T^{-1}}(x - x_0), \tag{3.7}$$

for all $(x, x^*) \in X \times X^*$. Consequently, if T is an one to one mapping, then we can obtain an estimation for $|g_{x_0;T}(x) - g_{x_0^*,T^{-1}}(x^*)|$, $(x, x^*) \in X \times X^*$. If T is a cyclically monotone mapping the inequality (3.6) was given in [6].

Concerning the conjugate of the Rockafellar function associated to a mapping T and the Rockafellar function associated to the mapping T^{-1} we have the following result.

Theorem 3.2. *Let $T : X \to X^*$ be an injective mapping. If $Tx_0 = \{x_0^*\}$, $x_0 \in DomT$, then*

$$g_{x_0;T}^*(x^*) \le x_0^*(x_0) - g_{x^*;T^{-1}}(x_0^*), \text{ for all } x^* \in \operatorname{Dom} T^{-1}. \tag{3.8}$$

Proof. By formula (3.3) we obtain

$$g_{x^*;T^{-1}(x_0^*)} = \sup\sum_{i=1}^{n-1}(x_{i-1}^* - x_i^*)(x_i) + (x_{n-1}^* - x^*)(x), x \in T^{-1}(x^*), x_i \neq x_j, \tag{3.9}$$
$$(x_i, x_i^*) \in \operatorname{Graph} T, i, j \in \overline{1, n-1}, n = 1, 2, \ldots\}, \text{ for any } x^* \in \operatorname{Dom} T^{-1}.$$

On the other hand, by definition (3.1) of the conjugate we have

$$g^*_{x_0;T}(x^*) = \sup\{x^*(x) - g_{x_0;T}(x); x \in X\}$$

$$= \sup\{x^*(x) - \sup\Big\{\sum_{i=1}^{n}(x^*_{i-1} - x^*_i)(x_i) + x^*_n(x) - x^*_0(x_0); (x_i, x^*_i) \in \operatorname{Graph} T,$$

$$x_i \neq x_j, \text{ for } i \neq j, i, j \in \overline{1,n}, n = 1, 2, \ldots\Big\}; x \in X\} = x^*_0(x_0)$$

$$+ \sup\inf\Big\{(x^* - x^*_n)(x) - \sum_{i=1}^{n-1}(x^*_{i-1} - x^*_i)(x_i) + (x^*_{n-1} - x^*_n)(x_n); (x_i, x^*_i) \in \operatorname{Graph} T,$$

$$x_i \neq x_j, i \neq j, i, j = \overline{1,n}, n = 1, 2, \ldots\Big\}.$$

Now, if we take $(x_n, x^*_n) = (x, x^*)$, according to (3.9) we obtain

$$g^*_{x_0;T}(x^*) \leq x^*_0(x_0) - \sup\sum_{i=1}^{n-1}\{(x^*_{i-1} - x^*_i)(x_i) + (x^*_{n-1} - x^*)(x); (x_i, x^*_i) \in \operatorname{Graph} T,$$

$$x_i \neq x_j, \text{ for } i \neq j, i, j = \overline{1, n-1}, n = 2, 3, \ldots\} = x^*_0(x_0) - g_{x^*;T^{-1}}(x^*_0),$$

for all $x^* \in \operatorname{Dom} T^{-1}$, as claimed.

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Ball comparison for three optimal eight order methods under weak conditions

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Abstract. We considered three optimal eighth order method for solving nonlinear equations. In earlier studies Taylors expansions and hypotheses reaching up to the eighth derivative are used to prove the convergence of these methods. These hypotheses restrict the applicability of the methods. In our study we use hypotheses on the first derivative. Numerical examples illustrating the theoretical results are also presented in this study.

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1. Introduction

In this paper we are concerned with the problem of approximating a solution x^* of the equation

$$F(x) = 0, \tag{1.1}$$

where $F : D \subseteq S \to T$ is a Fréchet-differentiable operator defined on a convex set D, where S, T are subsets of $\mathbb{R}$ or $\mathbb{C}$.

Equation of the form (1.1) is used to study problems in Computational Sciences and other disciplines [3, 7, 14, 16, 20]. Newton-like iterative methods [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23] are famous for approximating a solution of the equation (1.1).

In this paper, we study the local convergence analysis of the methods defined for each $n = 0, 1, 2, \cdots$ by Siyyam et al. [19]

$$y_n = x_n - \frac{1}{F'(x_n)} F(x_n),$$

$$\begin{aligned} z_n &= x_n + (1+\beta)\frac{1}{F'(x_n)}(F(x_n)+F(y_n)), \\ &-\frac{1}{F'(x_n)}F(x_n)(F(x_n)-F(y_n))^{-1}F(x_n) \\ &-\beta(\frac{1}{F'(x_n)}F(x_n)+(F'(x_n)+F^2(x_n)F'(x_n))^{-1}F(y_n)) \qquad (1.2) \\ x_{n+1} &= z_n - A_n^{-1}F(z_n), \end{aligned}$$

where $x_0 \in D$ is an initial point, $\beta \in S$,

$$A_n = F'(x_n) + ([x_n, y_n, z_n; F] + [x_n, x_n, y_n; F])(z_n - x_n) + 2([x_n, y_n, z_n; F] - [x_n, x_n, y_n; F])(z_n - y_n)$$

and $[\cdot,\cdot,\cdot; F]$ denotes a divided difference of order two for function F on D. The second and third method are due to Wang et. al. [23] and are defined, respectively as

$$\begin{aligned} y_n &= x_n - \frac{1}{F'(x_n)}F(x_n), \\ z_n &= x_n - \frac{1}{F'(x_n)}F(x_n)(F(x_n)-2F(y_n))^{-1}(F(x_n)-F(y_n)), \\ x_{n+1} &= z_n - \frac{1}{F'(x_n)}F(z_n) \qquad (1.3) \\ &\times\left[\frac{1}{2}+\frac{1+\frac{8F(y_n)}{5F(x_n)}+\frac{2}{5}(\frac{F(y_n)}{F(x_n)})^2}{1-\frac{12}{5}\frac{1}{F'(x_n)}F(y_n)}(1+F'(y_n)^{-1}F(z_n))\right], \end{aligned}$$

and

$$\begin{aligned} y_n &= x_n - \frac{1}{F'(x_n)}F(x_n), \\ z_n &= x_n - \frac{1}{F'(x_n)}F(x_n)(F(x_n)-2F(y_n))^{-1}(F(x_n)-F(y_n)), \\ x_{n+1} &= z_n - F(x_n)^{-1}F(x_n)\left[\frac{1-\frac{2}{5}\frac{1}{F'(x_n)}F(y_n)+\frac{1}{5}(F(x_n)^{-1}F(y_n))^2}{1-\frac{12}{5}\frac{1}{F'(x_n)}F(y_n)}\right. \\ &\left.+(1+4\frac{1}{F'(x_n)}F(y_n))F'(y_n)^{-1}F(z_n)\right]. \qquad (1.4) \end{aligned}$$

Convergence ball of high convergence order methods is usually very small and in general decreases as the convergence order increases. The approach in this paper establishes the local convergence result under hypotheses only on the first derivative and give a larger convergence ball than the earlier studies, under weaker hypotheses. Notice that in earlier studies [19, 23] the convergence is shown under hypotheses on the eighth derivative. The same technique can be used to other methods. As a motivational example, define function f on $D = [-\frac{1}{2}, \frac{3}{2})$ by

$$f(x) = \begin{cases} x^3 \ln x^2 + x^5 - x^4, & x \neq 0 \\ 0, & x = 0 \end{cases} \qquad (1.5)$$

Choose $x^* = 1$. We also have that

$$f'(x) = 3x^2 \ln x^2 + 5x^4 - 4x^3 + 2x^2,$$
$$f''(x) = 6x \ln x^2 + 20x^3 + 12x^2 + 10x$$

and

$$f'''(x) = 6 \ln x^2 + 60x^2 - 24x + 22.$$

Notice that $f'''(x)$ is unbounded on D. Hence, the results in [19, 23], cannot apply to show the convergence of method (1.2) (see also the numerical examples).

The rest of the paper is organized as follows. In Section 2 we present the local convergence analysis of methods (1.2)–(1.4). The numerical examples are given in the concluding Section 3.

2. Local convergence

The local convergence of method (1.2), method (1.3) and method (1.4) is based on some functions and parameters. Let $K_0 > 0, K > 0, L_0 > 0, L > 0$, $M \geq 1$ and $\beta \in S$ be given parameters. Let g_1, p_1, h_{p_1}, p_2 and h_{p_2} be functions defined on the interval $[0, \frac{1}{L_0})$ by

$$\begin{aligned} g_1(t) &= \frac{Lt}{2(1-L_0t)} \\ p_1(t) &= \frac{L_0t}{2} + Mg_1(t) \\ h_{p_1}(t) &= p_1(t) - 1, \\ p_2(t) &= L_0t + \frac{M^2t^2}{1-L_0t} \\ h_{p_2}(t) &= p_2(t) - 1 \end{aligned}$$

and parameter r_1 by

$$r_1 = \frac{2}{2L_0 + L}. \tag{2.1}$$

We have that $g_1(r_1) = 1$ and for each $t \in [0, r_1) : \ 0 \leq g_1(t) < 1$. We also get that $h_{p_1}(0) = h_{p_2}(0) = -1 < 0$ and $h_{p_1}(t) \to +\infty, h_{p_2}(t) \to +\infty$ as $t \to \frac{1}{L_0}^{-}$. It then follows from the intermediate value theorem that functions p_1 and p_2 have zeros in the interval $(0, \frac{1}{L_0})$. Denote by r_{p_1} and r_{p_2} the smallest such zeros of functions h_{p_1} and r_{p_2}, respectively. Let $\bar{r} = \min\{r_{p_1}, r_{p_2}\}$. Define functions g_2 and h_2 on the interval $[0, \bar{r})$ by

$$\begin{aligned} g_2(t) = {} & \frac{Lt}{2(1-L_0t)} + \frac{2M^2g_1(t)}{(1-L_0t)(1-p_1(t))} \\ & + \frac{|1+\beta|Mg_1(t)}{1-L_0t} + \frac{M|\beta|g_1(t)}{1-p_2(t)} \end{aligned}$$

and $h_2(t) = g_2(t) - 1$. We have that $h_2(0) = -1 < 0$ and $h_2(t) \to +\infty$ as $t \to \bar{r}^{-}$. Denote by r_2 the smallest zero of function h_2 in the interval $(0, \bar{r})$. Moreover, define functions q and h_q on the interval $[0, \bar{r})$ by $q(t) = L_0 t + (K + K_0)(1 + g_2(t))t + 2(K_0 + K)(g_1(t) + g_2(t))t$ and $h_q(t) = q(t) - 1$. We get that $h_q(0) = -1 < 0$ and $h_q(t) \to +\infty$ as $t \to \bar{r}_-$. Denote by r_q the smallest zero of function h_q on the interval $(0, \bar{r})$. Let $\bar{r}_0 = \min\{\bar{r}, r_q\}$.

Finally, define functions g_3 and h_3 on the interval $[0, \bar{r}_0)$ by

$$g_3(t) = (1 + \frac{M}{1 - q(t)})g_2(t)$$

and $h_3(t) = g_3(t) - 1$. We get that $h_3(0) = -1 < 0$ and $h_3(t) \to +\infty$ as $t \to \bar{r}_0^-$. Denote by r_3 the smallest zero of function h_3 on the interval $(0, \bar{r}_0)$. Define the radius of convergence r by

$$r = \min\{r_i\},\ i = 1, 2, 3. \tag{2.2}$$

Then, we have that

$$0 < r < r_1 < \frac{1}{L_0} \tag{2.3}$$

and for each $t \in [0, r)$

$$0 \le g_i(t) < 1,\ i = 1, 2, 3 \tag{2.4}$$

$$0 \le p_j(t) < 1,\ j = 1, 2 \tag{2.5}$$

and

$$0 \le q(t) < 1. \tag{2.6}$$

Let us denote by $U(v, \rho)$, $\overline{U}(v, \rho)$ the open and closed balls in S with center $v \in S$ and of radius $\rho > 0$.

Next, we present the local convergence analysis of method (1.2) using the preceding notation.

Theorem 2.1. *Let $F : D \subset S \to T$ be a differentiable function. Let also $[.,.,.;F]$ denote a divided difference of order two for function F on D. Suppose that there exist $x^* \in D$*

$$F(x^*) = 0,\ F'(x^*) \neq 0 \tag{2.7}$$

and

$$\|F'(x^*)^{-1}(F'(x) - F'(x^*))\| \le L_0 \|x - x^*\|. \tag{2.8}$$

Moreover, suppose that there exist $L > 0$ and $M \ge 1$ and $K > 0$ such that for each $x, y, z \in D_0 = D \cap U(x^*, \frac{1}{L_0})$

$$\|F'(x^*)^{-1}(F'(x) - F'(y))\| \le L \|x - y\|, \tag{2.9}$$

$$\|F'(x^*)^{-1} F'(x)\| \le M, \tag{2.10}$$

$$\|F'(x^*)^{-1}[x, x, y; F]\| \le K_0,\ \|F'(x^*)^{-1}[x, y, z; F]\| \le K \tag{2.11}$$

and

$$\bar{U}(x^*, r) \subseteq D, \tag{2.12}$$

where the radius of convergence r is defined by (2.2). Then, the sequence $\{x_n\}$ generated for $x_0 \in U(x^*, r) - \{x^*\}$ is well defined in $U(x^*, r)$, remains in $U(x^*, r)$ for each $n = 0, 1, 2, \cdots$, and converges to x^*. Moreover, the following estimates hold

$$\|y_n - x^*\| \le g_1(\|x_n - x^*\|)\|x_n - x^*\| \le \|x_n - x^*\| < r, \tag{2.13}$$

$$\|z_n - x^*\| \leq g_2(\|x_n - x^*\|)\|x_n - x^*\| \leq \|x_n - x^*\| \tag{2.14}$$

and

$$\|x_{n+1} - x^*\| \leq g_3(\|x_n - x^*\|)\|x_n - x^*\| \leq \|x_n - x^*\|, \tag{2.15}$$

where the "g" functions are defined previously. Furthermore, for $T \in [r, \frac{2}{L_0})$ the limit point x^* is the only solution of the equation $F(x) = 0$ in $D_1 = D \cap \bar{U}(x^*, T)$.

Proof. We shall show that method (1.2) is well defined in $U(x^*, r)$ remains in $U(x^*, r)$ for each $n = 0, 1, 2, \ldots$, and converges to x^* so that estimates (2.13)–(2.15) are satisfied. Using hypothesis $x_0 \in U(x^*, r) - \{x^*\}$, (2.3) and (2.8), we have that

$$\|F'(x^*)^{-1}(F'(x_0) - F'(x^*))\| \leq L_0\|x_0 - x^*\| \leq L_0 r < 1. \tag{2.16}$$

It follows from (2.16) and the Banach Lemma on invertible functions [3, 7, 14] that $F'(x_0) \neq 0$ and

$$\|F'(x_0)^{-1}F'(x^*)\| \leq \frac{1}{1 - L_0\|x_0 - x^*\|}. \tag{2.17}$$

Hence, y_0 is well defined. By the first sub-step of method (1.2) for $n = 0$, (2.3), (2.4), (2.7), (2.9) and (2.17), we get in turn that

$$\begin{aligned}
\|y_0 - x^*\| &= \|x_0 - x^* - F'(x_0)^{-1}F'(x_0)\| \\
&\leq \|F'(x_0)^{-1}F'(x^*)\| \\
&\quad \times \|\int_0^1 F'(x^*)^{-1}(F'(x^* + \theta(x_0 - x^*)) - F'(x_0))(x_0 - x^*)d\theta\| \\
&\leq \frac{L\|x_0 - x^*\|^2}{2(1 - L_0\|x_0 - x^*\|)} \\
&\leq g_1(\|x_0 - x^*\|)\|x_0 - x^*\| < \|x_0 - x^*\| < r,
\end{aligned} \tag{2.18}$$

which shows (2.13) for $n = 0$ and $y_0 \in U(x^*, r)$.

We can write by (2.7) that

$$F(y_0) = F(y_0) - F(x^*) = \int_0^1 F'(x^* + \theta(y_0 - x^*))(y_0 - x^*)d\theta. \tag{2.19}$$

Notice that $\|x^* + \theta(y_0 - x^*) - x^*\| = \theta\|y_0 - x^*\| < r$, so $x^* + \theta(y_0 - x^*) \in U(x^*, r)$ for each $\theta \in [0, 1]$. Then, by (2.10), (2.18) and (2.19), we get that

$$\|F(y_0)F'(x^*)^{-1}\| \leq M\|y_0 - x^*\| \leq Mg_1(\|x_0 - x^*\|)\|x_0 - x^*\|. \tag{2.20}$$

We must show in turn that $F(x_0) - F(y_0) \neq 0$ and $F'(x_0) + \frac{F^2(x_0)}{F'(x_0)} \neq 0$. We have by (2.3), (2.5), (2.8) and (2.20) that

$$\begin{aligned}
&\|(F'(x^*)(x_0 - x^*))^{-1}(F(x) - F(x^*) - F'(x^*)(x_0 - x^*) - F(y_0))\| \\
\leq\ & \|x_0 - x^*\|^{-1}(\frac{L_0}{2}\|x_0 - x^*\|^2 + M\|y_0 - x^*\|) \\
\leq\ & p_1(\|x_0 - x^*\|) < p_1(r) < 1,
\end{aligned} \tag{2.21}$$

so

$$\|(F(x_0) - F(y_0))^{-1}F'(x^*)\| \leq \frac{1}{\|x_0 - x^*\|(1 - p_1(\|x_0 - x^*\|)}. \tag{2.22}$$

Similarly, by (2.3), (2.5), (2.8) and (2.20) (for $x_0 = y_0$) that

$$\begin{aligned}
&\|F'(x^*)^{-1}(F'(x_0) - F'(x^*) + \frac{F^2(x_0)}{F'(x_0)})\| \\
\leq\ & L_0\|x_0 - x^*\| + \frac{M^2\|x_0 - x^*\|^2}{1 - L_0\|x_0 - x^*\|} = p_2(\|x_0 - x^*\|) \\
<\ & p_2(r) < 1,
\end{aligned} \tag{2.23}$$

so

$$\|(F'(x_0) + \frac{F^2(x_0)}{F'(x_0)})^{-1}F'(x^*)\| \leq \frac{1}{1 - p_2(\|x_0 - x^*\|)}. \tag{2.24}$$

and z_0 is well defined. Using the second substep of method (1.2), (2.3), (2.17), (2.18), (2.20), (2.22) and (2.24) we obtain in turn that

$$\begin{aligned}
z_0 - x^* &= x_0 - x^* - F'(x_0)^{-1}F(x_0) + (2+\beta)F'(x_0)^{-1}F(x_0) \\
&\quad +(1+\beta)F'(x_0)^{-1}F(y_0) - 2\frac{F^2(x_0)}{F'(x_0)(F(x_0) - F(y_0))} \\
&\quad -\beta F'(x_0)F(x_0) - \beta\frac{F(y_0)}{F'(x_0) + \frac{F^2(x_0)}{F'(x_0)}} \\
&= y_0 - x^* - 2[F'(x^*)^{-1}F(x_0)][F'(x_0)^{-1}F'(x^*)] \\
&\quad \times[(F(x_0) - F(y_0))^{-1}F'(x^*)][F'(x^*)^{-1}F(y_0)] \\
&\quad +(1+\beta)[F'(x_0)^{-1}F'(x^*)][F'(x^*)^{-1}F(y_0)] \\
&\quad -\beta[F'(x^*)^{-1}F(y_0)][(F'(x_0) + \frac{F^2(x_0)}{F'(x_0)})^{-1}F'(x^*)],
\end{aligned} \tag{2.25}$$

so

$$\begin{aligned}
\|z_0 - x^*\| &\leq \|y_0 - x^*\| \\
&\quad + \frac{2M^2\|y_0 - x^*\|\|x_0 - x^*\|}{\|x_0 - x^*\|(1 - L_0\|x_0 - x^*\|)(1 - p_1(\|x_0 - x^*\|)} \\
&\quad + \frac{|1+\beta|M\|y_0 - x^*\|}{1 - L_0\|x_0 - x^*\|} + \frac{|\beta|M\|y_0 - x^*\|}{1 - p_2(\|x_0 - x^*\|)} \\
&= g_2(\|x_0 - x^*\|)\|x_0 - x^*\| \leq \|x_0 - x^*\| < r,
\end{aligned} \tag{2.26}$$

which shows (2.14) for $n = 0$ and $z_0 \in U(x^*, r)$. Next, we must show that $A_0 \neq 0$. Using (2.3), (2.6), (2.8), (2.11), (2.18) and (2.26), we get in turn that

$$\begin{aligned}
&\|F'(x^*)^{-1}(A_0 - F'(x^*))\| \\
\leq\ & L_0\|x_0 - x^*\| \\
&+(K_0 + K)[\|z_0 - x^*\| + \|x_0 - x^*\|] + 2(K_0 + K)[\|z_0 - x^*\| + \|y_0 - x^*\|] \\
\leq\ & L_0\|x_0 - x^*\| + (K_0 + K)(1 + g_2(\|x_0 - x^*\|)\|x_0 - x^*\| \\
& 2(K_0 + K)(g_1(\|x_0 - x^*\|) + g_2(\|x_0 - x^*\|))\|x_0 - x^*\| \\
=\ & q(\|x_0 - x^*\|) < q(r) < 1,
\end{aligned}$$

so

$$\|A_0^{-1}F'(x^*)\| \leq \frac{1}{1-q(\|x_0-x^*\|)} \tag{2.27}$$

and x_1 is well defined. Then, from (2.3), (2.4), (2.18), (2.20) (for $y_0 = z_0$), (2.27), and the last substep of method (1.2) for $n = 0$, we have that

$$\begin{aligned} \|x_1 - x^*\| &\leq \|z_0 - x^*\| + \frac{M\|z_0 - x^*\|}{1-q(\|x_0-x^*\|)} \\ &= g_3(\|x_0 - x^*\|)\|x_0 - x^*\| \leq \|x_0 - x^*\| < r, \end{aligned} \tag{2.28}$$

which implies (2.15) holds for $n = 0$ and $x_1 \in U(x^*, r)$. By simply replacing x_0, y_0, z_0, x_1 by x_k, y_k, z_k, x_{k+1} in the preceding estimates we arrive at (2.13)–(2.15). Using the estimate $\|x_{k+1} - x^*\| \leq c\|x_k - x^*\|, c = g_3(\|x_0 - x^*\|) \in [0, 1)$, we deduce that $\lim_{k\to\infty} x_k = x^*$ and $x_{k+1} \in U(x^*, r)$. The proof of the uniqueness part is standard [5]. □

Next, we introduce the needed functions as the corresponding ones above Theorem 2.1 but for method (1.3). Define functions $\varphi_1, \varphi_2, \varphi_3, h_{\varphi_1}, h_{\varphi_2}, h_{\varphi_3}$ on the interval $[0, \frac{1}{L_0})$ by

$$\varphi_1(t) = \frac{12}{5}\frac{Mg_1(t)}{1-\frac{L_0}{2}t},\ h_{\varphi_1}(t) = \varphi_1(t) - 1,$$

$$\varphi_2(t) = \frac{L_0}{2}t + 2Mg_1(t),\ h_{\varphi_2}(t) = \varphi_2(t) - 1,$$

$$\varphi_3(t) = \frac{L_0}{2}g_1(t)t \text{ and } h_{\varphi_3}(t) = \varphi_3(t) - 1.$$

We have that $h_{\varphi_1}(0) = h_{\varphi_2}(0) = h_{\varphi_3}(0) = -1 < 0$ and $h_{\varphi_1}(t) \to +\infty$, $h_{\varphi_2}(t) \to +\infty$, $h_{\varphi_3}(t) \to +\infty$ as $t \to \frac{1}{L_0}^-$. Denote by $r_{\varphi_1}, r_{\varphi_2}, r_{\varphi_3}$ the smallest zero of functions $h_{\varphi_1}, h_{\varphi_2}, h_{\varphi_3}$, respectively on the interval $(0, \frac{1}{L_0})$. Moreover, define functions g_2 and h_2 on the interval $[0, r_{\varphi_2})$ by

$$g_2(t) = (1 + \frac{M^2}{(1-L_0t)(1-\varphi_2(t))})g_1(t)$$

and $h_2(t) = g_2(t) - 1$. We get that $h_2(0) = -1 < 0$ and $h_2(t) \to +\infty$ as $t \to r_{\varphi_2}$. Denote by r_2 the smallest such zero. Finally, for

$$\bar{r} = \min\{r_{\varphi_1}, r_{\varphi_2}, r_{\varphi_3}\}$$

define functions g_3 and h_3 on the interval $[0, \bar{r})$ by

$$g_3(t) = \left[1 + \frac{M}{1-L_0t}\left(\frac{1}{2} + \frac{1 + \frac{8Mg_1(t)}{5(1-\frac{L_0}{2}t)} + \frac{2}{5}\left(\frac{Mg_1(t)}{1-\frac{L_0}{2}t}\right)^2}{1-\varphi_1(t)}\right)\left(\frac{1}{2} + \frac{M\bar{g}_2(t)}{1-\frac{L_0}{2}t}\right)g_2(t)\right],$$

$$h_3(t) = g_3(t) - 1$$

and

$$\bar{g}_2(t) = 1 + \frac{M^2}{(1-L_0t)(1-\varphi_2(t))}.$$

We have that $h_3(0) = -1 < 0$ and $h_3(t) \to +\infty$ as $t \to \bar{r}^-$. Denote by r_3 the smallest zero of function g_3 on the interval $(0, \bar{r})$. Define the radius of convergence ρ_1 by

$$\rho_1 = \min\{r_i\},\ i = 1, 2, 3. \tag{2.29}$$

Finally, for method (1.4), define functions g_1 and g_2 as in method (1.3) but define function g_3 and h_3 by

$$g_3(t) = \left[1 + \frac{M}{1 - L_0 t} \frac{1 + \frac{2Mg_1(t)}{5\left(1 - \frac{L_0}{2}t\right)} + \frac{1}{5}\left(\frac{Mg_1(t)}{1 - \frac{L_0}{2}t}\right)^2}{1 - \varphi_1(t)} \left(1 + \frac{4Mg_1(t)}{1 - \frac{L_0}{2}t}\right) \frac{M\bar{g}_2(t)}{1 - \varphi_3(t)}\right] g_2(t),$$

$h_3(t) = g_3(t) - 1$

and radius of convergence ρ_2 by

$$\rho_2 = \min\{r_i\}, i = 1, 2, 3. \tag{2.30}$$

Next, drop the hypotheses on the divided differences and K from Theorem 2.1 and exchange the "g" functions and r with the corresponding "g" functions for method (1.3), ρ_1 and method (1.4), ρ_2. Call the resulting hypotheses (C) and (H), respectively. Then, we obtain the corresponding results.

Theorem 2.2. *Under the (C) hypotheses the conclusions of Theorem* 2.1 *hold for method* (1.3) *with* ρ_1 *replacing* r.

Theorem 2.3. *Under the (H) hypotheses the conclusions of Theorem* 2.1 *hold for method* (1.4) *with* ρ_2 *replacing* r.

Remark 2.4. (a) The radius r_1 was obtained by Argyros in [2] as the convergence radius for Newton's method under condition (2.13)-(2.15). Notice that the convergence radius for Newton's method given independently by Rheinboldt [18] and Traub [21] is given by

$$\rho = \frac{2}{3L} < r_1.$$

As an example, let us consider the function $f(x) = e^x - 1$. Then $x^* = 0$.
Set $D = U(0, 1)$. Then, we have that $L_0 = e - 1 < l = e$, so

$$\rho = 0.24252961 < r_1 = 0.3827.$$

Moreover, the new error bounds [2, 3, 6, 7] are:

$$\|x_{n+1} - x^*\| \leq \frac{L}{1 - L_0\|x_n - x^*\|}\|x_n - x^*\|^2,$$

whereas the old ones [14, 16]

$$\|x_{n+1} - x^*\| \leq \frac{L}{1 - L\|x_n - x^*\|}\|x_n - x^*\|^2.$$

Clearly, the new error bounds are more precise, if $L_0 < L$. Clearly, we do not expect the radius of convergence of method (1.2) given by r to be larger than r_1 (see (2.4)) .

(b) The local results can be used for projection methods such as Arnoldi's method, the generalized minimum residual method(GMREM), the generalized conjugate method(GCM) for combined Newton/finite projection methods and in connection to the mesh independence principle in order to develop the cheapest and most efficient mesh refinement strategy [2, 3, 6, 7].

(c) The results can be also be used to solve equations where the operator F' satisfies the autonomous differential equation [3, 7, 14, 16]:

$$F'(x) = p(F(x)),$$

where p is a known continuous operator. Since $F'(x^*) = p(F(x^*)) = p(0)$, we can apply the results without actually knowing the solution x^*. Let as an example $F(x) = e^x - 1$. Then, we can choose $p(x) = x + 1$ and $x^* = 0$.

(d) It is worth noticing that method (1.2) are not changing if we use the new instead of the old conditions [23]. Moreover, for the error bounds in practice we can use the computational order of convergence (COC)

$$\xi = \frac{\ln \frac{\|x_{n+2}-x_{n+1}\|}{\|x_{n+1}-x_n\|}}{\ln \frac{\|x_{n+1}-x_n\|}{\|x_n-x_{n-1}\|}}, \qquad \text{for each } n = 1, 2, \ldots$$

or the approximate computational order of convergence (ACOC)

$$\xi^* = \frac{\ln \frac{\|x_{n+2}-x^*\|}{\|x_{n+1}-x^*\|}}{\ln \frac{\|x_{n+1}-x^*\|}{\|x_n-x^*\|}}, \qquad \text{for each } \ n = 0, 1, 2, \ldots$$

instead of the error bounds obtained in Theorem 2.1. Related work on convergence orders can be found in [8].

(e) In view of (2.9) and the estimate

$$\begin{aligned} \|F'(x^*)^{-1}F'(x)\| &= \|F'(x^*)^{-1}(F'(x) - F'(x^*)) + I\| \\ &\leq 1 + \|F'(x^*)^{-1}(F'(x) - F'(x^*))\| \leq 1 + L_0\|x - x^*\| \end{aligned}$$

condition (2.11) can be dropped and M can be replaced by

$$M(t) = 1 + L_0 t$$

or

$$M(t) = M = 2,$$

since $t \in [0, \frac{1}{L_0})$.

3. Numerical Example

We present a numerical example in this section.

Example 3.1. Returning back to the motivation example at the introduction on this paper, we have $L_0 = L = 96.662907$, $M = 1.0631, K = K_0 = \frac{L}{2}, \beta = -1$. Then, the parameters for method (1.2) are

$$r_1 = 0.0069,\ r_2 = 0.0051 = r,\ r_3 = 0.1217.$$

We have $ACOC = 1.7960$ and $COC = 1.8371$.

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A dynamic Tresca's frictional contact problem with damage for thermo elastic-viscoplastic bodies

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Abstract. We consider a dynamic contact problem between an elastic-viscoplastic body and a rigid obstacle. The contact is frictional and bilateral, the friction is modeled with Tresca's law with heat exchange. We employ the elastic-viscoplastic with damage constitutive law for the material. The evolution of the damage is described by an inclusion of parabolic type. We establish a variational formulation for the model and we prove the existence of a unique weak solution to the problem. The proof is based on a classical existence and uniqueness result on parabolic inéqualities, differentiel equations and fixed point argument.

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1. Introduction

The modelization of a contact phenomenon is determined by a set of assumptions influencing on the form and structure of partial differential equations system or on boundary conditions of the associated mathematical model.

Among the assumptions influencing the partial differential equations system: *Hypothesis about the geometry of the deformation* (small deformation or others), *Hypothesis about the mechanical process* (quasi-static or dynamic), *Hypothesis about the laws of material behavior* (elastic, viscoelastic,...).

The model equations can be influenced by additional phenomena (thermal, piezo-electric,...).

The boundary conditions on the contact surface are described in both normal direction and in the tangential plane, these are called boundary conditions of friction.

In the direction of normal, we have unilateral and bilateral contact (when there is no separation between the body and the obstacle). The normal compliance (when the obstacle is deformable).

The boundary conditions are also influenced by several phenomena accompanying the contact with friction, such as adhesion, wear, thermal effects, friction threshold dependence with respect to sliding or the sliding speed.

The contact between deformable bodies are very common in the industry and everyday life, contact of braking pads with wheels, tires with roads, pistons with skirts or the complex metal.

Recently we investigated a number of problems related to quasistatic contact for thermo mechancical models coupled or uncoupled. In particular, models uncoupled thermo viscoplastic were considered in [10]. In this case the consitutive equation law depends on two parameters θ, χ, where θ be interpreted as absolute temperature.

Different models have been developed to describe the interaction between the thermal and mechanical field see [3, 11]. A thermo elastic-viscoplastic body is considered in [6, 11].

Initial and boundary value problems for termo mechanical models were studied by many authors. So, existence and uniqueness result concerning the uncoupled thermo viscoelastic was obtained in [10] using a monotony method.

A quasistatic contact problem with friction and adhesion has been analized in [12] for viscoelastic body with long memory. The constitutive laws with internal states variables has been used in various publications see for example [4, 5, 7].

The damage is one of the internal state variable considered by many authors, we can see [1, 3, 6, 9].

In this paper we consider the processes frictional contact between a termo elastic viscoplastic body with damage. We assume that the process is dynamic.

This article is organized as follows. In Section 2 we describe the mathematical model for the problem. In Section 3 we introduce some notation, list the assumptions on the problem's data, and derive the variational formulation of the model. Finally in Section 4 we state our main existence and uniqueness result which is based on classical result of nonlinear first order evolution inequalities, equations with monotone operators and the fixed point arguments.

For the mathematical problem we consider a rate-type constitutive equation for bodies of the form

$$\boldsymbol{\sigma} = \mathcal{A}\varepsilon(\dot{\mathbf{u}})+\mathcal{G}\big(\boldsymbol{\varepsilon}(\mathbf{u}),\xi\big) + \int_0^t \mathcal{F}\big(\boldsymbol{\sigma}\,(s) - \mathcal{A}\varepsilon(\dot{\mathbf{u}}\,(s)\,),\boldsymbol{\varepsilon}(\mathbf{u}\,(s))\big)ds - C_e\theta, \tag{1.1}$$

in which:
$\mathbf{u}$, $\boldsymbol{\sigma}$ represent, respectively, the displacement field and the stress field where the dot above denotes the derivative with respect to the time variable;
ξ, θ represent the damage, and the temperature;
$\mathcal{A}$, $\mathcal{G}$ and $\mathcal{F}$ are, respectively, nonlinear operators describing the purely viscous, the elastic and the viscoplastic properties of the material;
$C_e = (c_{ij})$ represents the thermal expansion tensor.
The differential inclusion used for the evolution of the damage field is

$$\dot{\xi} - k_1\Delta\xi + \partial\varphi_F(\xi) \ni S(\varepsilon(\mathbf{u}),\xi), \quad \text{in } \Omega\times(0,T), \tag{1.2}$$

where $\varphi_F(\xi)$ denotes the subdifferential of the indicator function of the set F of admissible damage functions defined by

$$F = \{\xi \in H^1(\Omega); 0 \leq \xi \leq 1, \text{ a.e.in } \Omega\}$$

and S are given constitutive functions which describe the sources of the damage in the system. When $\xi = 0$ the material is completely damaged, when $\xi = 1$ the material is undamaged, and for $0 < \xi < 1$ there is partial damage.

The evolution of the temperature field θ is governed by the heat equation, obtained from the conservation of energy and defined by the following differential equation for the temperature

$$\dot{\theta} - divK(\Delta\theta) = r(\dot{\mathbf{u}}, \xi) + \mathbf{q}$$

K represent the thermal conductivity tensor, $q(t)$ represent the density of volume heat source and r is non linear function of velocity.

2. Problem statement

We consider an elasto-viscoplastic body which occupies a bounded domain Ω of the space $\mathbb{R}^d(d = 2, 3)$. For Ω, the boundary Γ is assumed to be Lipschitz continuous, and is partitioned into three disjoint measurable parts Γ_1, Γ_2 and Γ_3, such that $meas\Gamma_1 > 0$. Let $T > 0$ and let $[0, T]$ denotes the time interval of interest. The body Ω is clamped on $\Gamma_1 \times (0, T)$, and therfore, the displacement field vanishes there. Surface traction of density $\mathbf{f}_2$ act on $\Gamma_2 \times (0, T)$ and a body force of density $\mathbf{f}_0$ acts on $\Omega \times (0, T)$. Morever the process is dynamic, and thus the inertial terms are included in the equation of motion. The material is assumed to behave according to the general elasto-viscoplastic constitutive law with damage and thermal effects given by (1.1)

With the assumption above, the classical formulation of a dynamic contact between an elasto-viscoplastic body and an obstacle with damage and thermal effects is the following.

Problem P. Find a displacement field $\mathbf{u} : \Omega \times (0, T) \to \mathbb{R}^d$, a stress field $\boldsymbol{\sigma} : \Omega \times (0, T) \to \mathbb{S}^d$, a temperature $\theta : \Omega \times (0, T) \to \mathbb{R}$, and the damage field $\xi : \Omega \times [0, T] \to \mathbb{R}$ such that

$$\boldsymbol{\sigma} = \mathcal{A}\varepsilon(\dot{\mathbf{u}}) + \mathcal{G}\big(\boldsymbol{\varepsilon}(\mathbf{u}), \xi\big) + \int_0^t \mathcal{F}\big(\boldsymbol{\sigma}(s) - \mathcal{A}\varepsilon(\dot{\mathbf{u}}(s)), \boldsymbol{\varepsilon}(\mathbf{u}(s))\big)ds - C_e\theta \tag{2.1}$$

$$\dot{\theta} - divK(\Delta\theta) = r(\dot{\mathbf{u}}, \xi) + \mathbf{q}, \quad \text{on } \Omega \times (0, T), \tag{2.2}$$

$$\dot{\xi} - k_1\Delta\xi + \partial\varphi_F(\xi) \ni S(\varepsilon(\mathbf{u}), \xi), \quad \text{in } \Omega \times (0, T), \tag{2.3}$$

$$div\boldsymbol{\sigma} + \mathbf{f}_0 = \rho\ddot{\mathbf{u}} \quad \text{in } \Omega \times (0, T), \tag{2.4}$$

$$\mathbf{u} = 0 \quad \text{on } \Gamma_1 \times (0, T), \tag{2.5}$$

$$\boldsymbol{\sigma} \cdot \nu = \mathbf{f}_2 \quad \text{on } \Gamma_2 \times (0, T), \tag{2.6}$$

$$-k_{ij}\frac{\partial\theta}{\partial x_i}n_j = k_e(\theta - \theta_R) + h_\tau(|\dot{\mathbf{u}}_\tau|), \quad \text{on } \Gamma_3 \times (0, T), \tag{2.7}$$

$$\frac{\partial\xi}{\partial\nu} = 0 \quad \text{on } \Gamma \times (0, T), \tag{2.8}$$

$$\begin{cases} \mathbf{u}_\nu = 0, \quad |\boldsymbol{\sigma}_\tau| \le g \\ |\boldsymbol{\sigma}_\tau| < g \Rightarrow \dot{\mathbf{u}}_\tau = 0, \quad \text{on } \Gamma_3 \times (0,T), \\ |\boldsymbol{\sigma}_\tau| = g \Rightarrow \ \exists \lambda \ge 0 \text{ such that } \boldsymbol{\sigma}_\tau = -\lambda \dot{\mathbf{u}}_\tau \end{cases} \tag{2.9}$$

$$\theta = 0, \quad \text{on } (\Gamma_1 \cup \Gamma_2) \times (0,T), \tag{2.10}$$

$$\mathbf{u}(0) = \mathbf{u}_0, \dot{\mathbf{u}}(0) = \mathbf{v}_0, \xi(0) = \xi_0, \ \theta(0) = \theta_0, \quad , \quad \text{in } \Omega, \tag{2.11}$$

First, equations (2.1), (2.2) and (2.3) represent the elastic-viscoplastic constitutive law with damage and thermal effects, equation (2.4) represents the equation of motion where ρ represents the mass density. Equations (2.5) and (2.6) represent the displacement and traction boundary condition, respectively. (2.7), (2.8) represent, respectively on Γ, a Fourier boundary condition for the temperature and an homogeneous Neumann boundary condition for the damage field on Γ. We assume that the contact is bilateral, therfore, the normal displacement $\mathbf{u}_\nu$ vanishes on $\Gamma_3 \times (0,T)$. We involve the friction process with Tresca's friction law, where the friction yield limit is g, which is assumed to depend only on each point of Γ_3, $\dot{\mathbf{u}}_\tau$ denotes the tangential velocity and $\boldsymbol{\sigma}_\tau$ represent the tangential stress. The strong inequality holds in stick zone and the equality in slip zone. To simplify the notation, we do not indicate explcitely the dependence of various functions on the variable $x \in \Omega \cup \Gamma$ and $t \in [0,T]$. Equation (2.10) means that the temperature vanishes on $(\Gamma_1 \cup \Gamma_2) \times (0,T)$. The functions $\mathbf{u}_0$, $\mathbf{v}_0$, $\xi_0$and θ_0 in (2.11) are the initial data.

3. Variational formulation and preliminaries

In this section, we list the assumptions on the data and derive a variational formulation for the contact problem. To this end, we need to introduce some notations and preliminary material. For more details, we refer the reader to [2, 8]. We denote by $\mathbb{S}^d$ the space of second order symmetric tensors on $\mathbb{R}^d$ $(d = 2,3)$, while $\|\cdot\|$ denotes the Euclidean norm.

Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with Lipschitz boundary Γ and let ν denote the unit outer normal on $\partial\Omega = \Gamma$. We shall use the notations

$$H = L^2(\Omega)^d = \left\{\mathbf{u} = (u_i) : u_i \in L^2(\Omega)\right\}, \quad \mathcal{H} = \left\{\boldsymbol{\sigma} = (\sigma_{ij}) : \sigma_{ij} = \sigma_{ji} \in L^2(\Omega)\right\},$$
$$H^1(\Omega)^d = \{\mathbf{u} = (u_i) \in H : u_i \in H^1(\Omega)\}, \quad \mathcal{H}_1 = \{\boldsymbol{\sigma} \in \mathcal{H} : div\boldsymbol{\sigma} \in H\}.$$

Here $\varepsilon : H^1(\Omega)^d \to \mathcal{H}$ and $div : \mathcal{H}_1 \to H$ are the deformation and divergence operators, respectively, defined by

$$\varepsilon(\mathbf{u}) = (\varepsilon_{ij}(\mathbf{u})), \quad \varepsilon_{ij}(\mathbf{u}) = \frac{1}{2}(u_{i,j} + u_{j,i}), \quad div\boldsymbol{\sigma} = (\sigma_{ij,j}).$$

Here and below, the indices i and j run from 1 to d, the summation convention over repeated indices is used and the index that follows a comma indicates a partial derivative with respect to the corresponding component of the independent variable. The spaces H, $\mathcal{H}$, $H^1(\Omega)^d$ and $\mathcal{H}_1$ are real Hilbert spaces endowed with the canonical inner products given by:

$$(\mathbf{u},\mathbf{v})_H=\int_\Omega u_iv_idx,\quad \mathbf{u},\mathbf{v}\in H,\quad (\boldsymbol{\sigma},\boldsymbol{\tau})_{\mathcal{H}}=\int_\Omega \sigma_{ij}\tau_{ij}dx,\quad \forall\boldsymbol{\sigma},\boldsymbol{\tau}\in\mathcal{H}$$

$$(\mathbf{u},\mathbf{v})_{H^1(\Omega)^d}=\int_\Omega \mathbf{u}.\mathbf{v}dx+\int_\Omega \nabla\mathbf{u}.\nabla\mathbf{v}dx\quad \forall\mathbf{u},\mathbf{v}\in H^1(\Omega)^d,$$

where

$$\nabla\mathbf{v}=(v_{i,j}),\quad \forall\mathbf{v}\in H^1(\Omega)^d.$$

$$(\boldsymbol{\sigma},\boldsymbol{\tau})_{\mathcal{H}_1}=(\boldsymbol{\sigma},\boldsymbol{\tau})_{\mathcal{H}}+(div\boldsymbol{\sigma},div\boldsymbol{\tau})_H\ \forall\boldsymbol{\sigma},\boldsymbol{\tau}\in\mathcal{H}_1,$$

The associated norms are denoted by $\|\cdot\|_H$, $\|\cdot\|_{\mathcal{H}}$, $\|\cdot\|_{H^1}$ and $\|\cdot\|_{\mathcal{H}_1}$, respectively. Let $H_\Gamma=(H^{1/2}(\Gamma))^d$ and $\gamma:H^1(\Gamma))^d\to H_\Gamma$ be the trace map. For every element $\mathbf{v}\in H^1(\Omega)^d$, we also use the notation $\mathbf{v}$ to denote the trace map $\gamma\mathbf{v}$ of $\mathbf{v}$ on Γ, and we denote by v_ν and $\mathbf{v}_\tau$ the normal and tangential components of $\mathbf{v}$ on Γ given by

$$v_\nu=\mathbf{v}\cdot\nu,\quad \mathbf{v}_\tau=\mathbf{v}-v_\nu\nu. \tag{3.1}$$

Similarly, for a regular (say $\mathcal{C}^1$) tensor field $\boldsymbol{\sigma}:\Omega\to\mathbb{S}^d$ we define its normal and tangential components by

$$\boldsymbol{\sigma}_\nu=(\boldsymbol{\sigma}\nu)\cdot\boldsymbol{\nu},\quad \boldsymbol{\sigma}_\tau=\boldsymbol{\sigma}\nu-\boldsymbol{\sigma}_\nu\boldsymbol{\nu},$$

and for all $\boldsymbol{\sigma}\in\mathcal{H}_1$ the following Green's formula holds

$$(\boldsymbol{\sigma},\varepsilon(\mathbf{v}))_{\mathcal{H}}+(div\boldsymbol{\sigma},\mathbf{v})_H=\int_\Gamma \boldsymbol{\sigma}\nu.\mathbf{v}da\quad \forall\mathbf{v}\in H^1(\Omega)^d. \tag{3.2}$$

Finally, for any real Hilbert space X, we use the classical notation for the spaces $L^p(0,T;X)$ and $W^{k,p}(0,T;X)$, where $1\leqslant p\leqslant\infty$ and $k>1$. For $T>0$ we denote by $\mathcal{C}(0,T;X)$ and $\mathcal{C}^1(0,T;X)$ the space of continuous and continuously differentiable functions from $[0,T]$ to X, respectively, with the norms

$$\|\mathbf{f}\|_{\mathcal{C}(0,T;X)}=\max_{t\in[0,T]}\|\mathbf{f}\,(t)\,\|_X,$$

$$\|\mathbf{f}\|_{\mathcal{C}^1(0,T;X)}=\max_{t\in[0,T]}\|\mathbf{f}\,(t)\,\|_X+\max_{t\in[0,T]}\|\dot{\mathbf{f}}\,(t)\,\|_X,$$

respectively. Moreover, we use the dot above to indicate the derivative with respect to the time variable and if X_1 and X_2 are real Hilbert spaces then $X_1\times X_2$ denotes the product Hilbert space endowed with the canonical inner product $(\cdot,\cdot)_{X_1\times X_2}$. Now, let E denote the closed subspace of $H^1(\Omega)$ given by

$$E=\{\gamma\in H^1(\Omega):\gamma=0\quad\text{on}\quad\Gamma_1\cup\Gamma_2\}$$

Let V denote the closed subspace of H_1 defined by

$$V=\{v\in H_1:v=0\quad\text{on}\quad\Gamma_1\quad\text{and}\quad v_\nu=0\quad\text{on}\quad\Gamma_3\}$$

Since $meas\Gamma_1>0$, the following Korn's inequality holds:

$$\|\varepsilon(\mathbf{v})\|_{\mathcal{H}}\geq c_K\|\mathbf{v}\|_{H_1}\quad \forall\mathbf{v}\in V, \tag{3.3}$$

where the constant c_K denotes a positive constant which may depends only on Ω, Γ_1 Over the space V we consider the inner product given by

$$(\mathbf{u},\mathbf{v})_V=(\varepsilon(\mathbf{u}),\varepsilon(\mathbf{v}))_{\mathcal{H}},\quad \forall\mathbf{u},\mathbf{v}\in V. \tag{3.4}$$

Let $\|\cdot\|_V$ be the associated norm. It follows from Korn's inequality (3.3) that the norms $\|\cdot\|_{H_1}$ and $\|\cdot\|_V$ are equivalent on V. Then $(V, \|\cdot\|_V)$ is a real Hilbert space. Moreover, by the Sobolev trace theorem and (3.3), there exists a constant $c_0 > 0$, depending only on Ω, Γ_1 and Γ_3 such that

$$\|\mathbf{v}\|_{L^2(\Gamma_3)^d} \leq c_0 \|\mathbf{v}\|_V \quad \forall \mathbf{v} \in V. \tag{3.5}$$

The mechanical problem may be formulated as follows.
In the study of the Problem $\mathbf{P}$, we consider the following assumptions:
The *viscosity function* $\mathcal{A} : \Omega \times \mathbb{S}^d \to \mathbb{S}^d$ satisfies:

$$\begin{aligned}
&\text{(a) There exists } L_{\mathcal{A}} > 0 \text{ such that}\\
&|\mathcal{A}(\boldsymbol{x}, \varepsilon_1) - \mathcal{A}(\boldsymbol{x}, \varepsilon_2)| \leq L_{\mathcal{A}} |\varepsilon_1 - \varepsilon_2| \text{ for all } \varepsilon_1, \varepsilon_2 \in \mathbb{S}^d, \text{ a.e. } \boldsymbol{x} \in \Omega.\\
&\text{(b) There exists } m_{\mathcal{A}} > 0 \text{ such that}\\
&(\mathcal{A}(\boldsymbol{x}, \varepsilon_1) - \mathcal{A}(\boldsymbol{x}, \varepsilon_2)) \cdot (\varepsilon_1 - \varepsilon_2) \geq m_{\mathcal{A}} |\varepsilon_1 - \varepsilon_2|^2 \text{ for all } \varepsilon_1, \varepsilon_2 \in \mathbb{S}^d,\\
&\text{a.e. } \boldsymbol{x} \in \Omega.\\
&\text{(c) The mapping } \boldsymbol{x} \mapsto \mathcal{A}(\boldsymbol{x}, \varepsilon) \text{ is Lebesgue measurable on } \Omega,\\
&\text{for any } \varepsilon \in \mathbb{S}^d.\\
&\text{(d) The mapping } \boldsymbol{x} \mapsto \mathcal{A}(\boldsymbol{x}, \mathbf{0}) \text{ is continuous on } \mathbb{S}^d, \text{ a.e. } \boldsymbol{x} \in \Omega.
\end{aligned} \tag{3.6}$$

The *elasticity operator* $\mathcal{G} : \Omega \times \mathbb{S}^d \times \mathbb{R} \to \mathbb{S}^d$ satisfies:

$$\begin{aligned}
&\text{(a) There exists } L_{\mathcal{G}} > 0 \text{ such that}\\
&|\mathcal{G}(\boldsymbol{x}, \varepsilon_1, \boldsymbol{\xi}_1) - \mathcal{G}(\boldsymbol{x}, \varepsilon_2, \boldsymbol{\xi}_2)| \leq L_{\mathcal{G}} \big(|\varepsilon_1 - \varepsilon_2| + |\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2| \big),\\
&\text{for all } \varepsilon_1, \varepsilon_2 \in \mathbb{S}^d, \text{ for all } \boldsymbol{\xi}_1, \boldsymbol{\xi}_2 \in \mathbb{R}, \text{ a.e. } \boldsymbol{x} \in \Omega.\\
&\text{(b) The mapping } \boldsymbol{x} \mapsto \mathcal{G}(\boldsymbol{x}, \varepsilon, \boldsymbol{\xi}) \text{ is Lebesgue measurable on } \Omega,\\
&\text{for any } \varepsilon \in \mathbb{S}^d, \text{ and for all } \boldsymbol{\xi} \in \mathbb{R}.\\
&\text{(c) The mapping } \boldsymbol{x} \mapsto \mathcal{G}(\boldsymbol{x}, \mathbf{0}, \mathbf{0}) \text{ belongs to } \mathcal{H}.
\end{aligned} \tag{3.7}$$

The *visco-plasticity operator* $\mathcal{F} : \Omega \times \mathbb{S}^d \times \mathbb{S}^d \to \mathbb{S}^d$ satisfies:

$$\begin{aligned}
&\text{(a) There exists } L_{\mathcal{F}} > 0 \text{ such that}\\
&|\mathcal{F}(\boldsymbol{x}, \boldsymbol{\sigma}_1, \varepsilon_1) - \mathcal{F}(\boldsymbol{x}, \boldsymbol{\sigma}_2, \varepsilon_2)| \leq L_{\mathcal{F}} \big(|\boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2| + |\varepsilon_1 - \varepsilon_2| \big),\\
&\text{for all } \boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2, \varepsilon_1, \varepsilon_2 \in \mathbb{S}^d \text{ a.e. } \boldsymbol{x} \in \Omega,\\
&\text{(b) The mapping } \boldsymbol{x} \mapsto \mathcal{F}(\boldsymbol{x}, \boldsymbol{\sigma}, \varepsilon) \text{ is Lebesgue measurable on } \Omega,\\
&\text{for any } \boldsymbol{\sigma}, \varepsilon \in \mathbb{S}^d\\
&\text{(c) The mapping } \boldsymbol{x} \mapsto \mathcal{F}(\boldsymbol{x}, \mathbf{0}, \mathbf{0}) \text{ belongs to } \mathcal{H}.
\end{aligned} \tag{3.8}$$

The *damage source function* $S : \Omega \times \mathbb{S}^d \times \mathbb{R} \to \mathbb{S}^d$ satisfies:

$$\begin{aligned}
&\text{(a) There exists } L_S > 0 \text{ such that}\\
&|S(\boldsymbol{x}, \boldsymbol{\varepsilon}_1, \boldsymbol{\xi}_1) - S(\boldsymbol{x}, \boldsymbol{\varepsilon}_2, \boldsymbol{\xi}_2)| \leq L_S \big(|\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2| + |\boldsymbol{\zeta}_1 - \boldsymbol{\zeta}_2| \big),\\
&\text{for all } \boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2 \in \mathbb{S}^d, \ \boldsymbol{\xi}_1, \boldsymbol{\xi}_2 \in \mathbb{R}, \text{ a.e. } \boldsymbol{x} \in \Omega.\\
&\text{(b) The mapping } \boldsymbol{x} \mapsto S(\boldsymbol{x}, \boldsymbol{\varepsilon}, \boldsymbol{\xi}) \text{ is Lebesgue measurable on } \Omega,\\
&\text{for any } \boldsymbol{\varepsilon} \in \mathbb{S}^d, \text{and for all } \boldsymbol{\xi} \in \mathbb{R}.\\
&\text{(c) The mapping } \boldsymbol{x} \mapsto S(\boldsymbol{x}, \mathbf{0}, \mathbf{0}) \text{ belongs to } \mathcal{H}.
\end{aligned} \tag{3.9}$$

The *thermal expansion operator* $C_e : \Omega \times \mathbb{R} \to \mathbb{R}$ satisfies:

$$\left\{\begin{array}{l}\text{(a) There exists } L_{C_e} > 0 \text{ such that}\\ |C_e(\boldsymbol{x},\boldsymbol{\theta}_1) - C_e(\boldsymbol{x},\boldsymbol{\theta}_2)| \leq L_{C_e}|\boldsymbol{\theta}_1 - \boldsymbol{\theta}_2| \text{ for all } \boldsymbol{\theta}_1,\boldsymbol{\theta}_2 \in \mathbb{R},\\ \text{a.e. } \boldsymbol{x} \in \Omega.\\ \text{(b) } C_e = (c_{ij}),\ c_{ij} = c_{ji} \in L^\infty(\Omega).\\ \text{(c) The mapping } \boldsymbol{x} \mapsto C_e(\boldsymbol{x},\boldsymbol{\theta}) \text{ is Lebesgue measurable on } \Omega,\\ \text{for any } \boldsymbol{\theta} \in \mathbb{R}.\\ \text{(d) The mapping } \boldsymbol{x} \mapsto C_e(\boldsymbol{x},\mathbf{0}) \in \mathcal{H}.\end{array}\right. \tag{3.10}$$

The *thermal conductivity operator* $K : \Omega \times \mathbb{R} \to \mathbb{R}$ satisfies:

$$\left\{\begin{array}{l}\text{(a) There exists } L_K > 0 \text{ such that}\\ |K(\boldsymbol{x},r_1) - K(\boldsymbol{x},r_2)| \leq L_K|r_1 - r_2|, \text{ for all } r_1,r_2 \in \mathbb{R}, \text{ a.e. } \boldsymbol{x} \in \Omega.\\ \text{(b) } k_{ij} = k_{ji} \in L^\infty(\Omega),\ k_{ij}\alpha_i\alpha_j \leq c_k\alpha_i\alpha_j \text{ for some } c_k > 0,\\ \text{for all } (\alpha_i) \in \mathbb{R}.\\ \text{(c) The mapping } \boldsymbol{x} \mapsto k(\boldsymbol{x},\mathbf{0}) \text{ belongs to } L^2(\Omega).\end{array}\right. \tag{3.11}$$

We assume that the *tangential function* $h_\tau : \Gamma_3 \times \mathbb{R} \to \mathbb{R}_+$ satisfies:

$$\left\{\begin{array}{l}\text{(a) There exists } L_\tau > 0 \text{ such that}\\ |h_\tau(\boldsymbol{x},r_1) - h_\tau(\boldsymbol{x},r_2)| \leq L_\tau|r_1 - r_2| \text{ for all } r_1,r_2 \in \mathbb{R}_+, \text{ a.e. } \boldsymbol{x} \in \Omega.\\ \text{(b) The mapping } \boldsymbol{x} \mapsto h_\tau(\boldsymbol{x},r) \text{ is Lebesgue measurable on } \Gamma_3 \text{ for}\\ \text{all } r \in \mathbb{R}_+.\\ \text{(c) The mapping } \boldsymbol{x} \mapsto h_\tau(\boldsymbol{x},0) \text{ belongs to } L^2(\Gamma_3).\end{array}\right. \tag{3.12}$$

A concrete example of a tangential function h_τ is given by

$$h_\tau(x,r) = \lambda(x)\, r,\ \forall r \in \mathbb{R}_+,\ \text{a.e } x \in \Gamma_3,$$

where $\lambda \in L^\infty(\Gamma_3, \mathbb{R}_+)$ represents some rate coefficient for the gradient of the temperature.
The masse density satisfies

$$\rho \in L^\infty(\Omega), \text{there exists } \rho^* > 0 \text{ such that } \rho(x) \geq \rho^*,\ \text{a.e } x \in \Omega \tag{3.13}$$

and

$$g \in L^\infty(\Gamma_3), \quad g \geq 0,\ \text{a.e. on } \Gamma_3 \tag{3.14}$$

We also suppose the following regularities

$$\mathbf{f}_0 \in L^2(0,T;H), \quad \mathbf{f}_2 \in L^2(0,T;L^2(\Gamma_2)^d),\ \mathbf{q} \in L^2\left(0,T;L^2(\Omega)\right). \tag{3.15}$$

The boundary and initial data satisfy

$$\mathbf{u}_0 \in V, \mathbf{v}_0 \in H \tag{3.16}$$

$$\xi_0 \in F \tag{3.17}$$

$$\theta_0 \in E \tag{3.18}$$

$$\theta_R \in L^2\left(0,T;L^2(\Gamma_3)\right) \tag{3.19}$$

$$k_e \in L^\infty(\Omega, \mathbb{R}_+) \tag{3.20}$$

The function $r : V \to L^2(\Omega)$ satisfies that there exists a constant $L_r > 0$ such that

$$\begin{aligned} |r(\mathbf{v}_1, \xi_1) - r(\mathbf{v}_2, \xi_2)|_{L^2(\Omega)} &\leq L_r \left(|\mathbf{v}_1 - \mathbf{v}_2|_V + |\xi_1 - \xi_2)| \right) \\ \forall \quad \mathbf{v}_1, \mathbf{v}_2 &\in V, \quad \xi_1, \xi_2 \in \mathbb{R} \end{aligned} \tag{3.21}$$

We use a modified inner product on $H = L^2(\Omega)^d$ given by

$$((\mathbf{u}, \mathbf{v}))_H = (\rho \mathbf{u}, \mathbf{v})_H, \quad \forall \mathbf{u}, \mathbf{v} \in H$$

that is, it is weighted with ρ. We let $\|.\|_H$ be the associated norm, i.e

$$\|\mathbf{v}\|_H = (\rho \mathbf{v}, \mathbf{v})_H^{\frac{1}{2}}, \quad \forall \mathbf{v} \in H$$

The notation $(\cdot, \cdot)_{V' \times V}$ represent the duality pairing between V' and V.
Then, we have

$$(\mathbf{u}, \mathbf{v})_{V' \times V} = ((\mathbf{u}, \mathbf{v}))_H, \quad \forall \mathbf{u} \in H, \quad \forall \mathbf{v} \in V$$

It follows from assumption (3.13) that $\|.\|_H$ and $|.|_H$ are equivalent norms on H, and also the inclusion mapping of $(V, |.|_V)$ into $(H, \|.\|_H)$ is continuous and dense. We denote by V' the dual space of V. Identifying H with its own dual, we can write the Gelfand triple

$$V \subset H \subset V'.$$

From assumption (3.15) we define $\mathbf{f}(t) \in V$ for a.e. $t \in (0, T)$ by

$$(\mathbf{f}(t), \mathbf{v})_{V' \times V} = \int_\Omega \mathbf{f}_0(t) \cdot \mathbf{v}\, dx + \int_{\Gamma_2} \mathbf{f}_2(t) \cdot \mathbf{v}\, da \quad \forall \mathbf{v} \in \boldsymbol{V}, \tag{3.22}$$

and note that

$$\mathbf{f} \in L^2(0, T; V).$$

We define the bilinear form $j : H^1(\Omega) \times H^1(\Omega) \to \mathbb{R}$

$$a(\varsigma, \zeta) = \kappa \int_\Omega \nabla \varsigma \cdot \nabla \zeta dx. \tag{3.23}$$

Next we define the functional $j : V \to \mathbb{R}$ by

$$j(\mathbf{v}) = \int_{\Gamma_3} g\, |\mathbf{v}_\tau|\, da, \quad \forall \mathbf{v} \in \boldsymbol{V}.$$

By using a standard arguments, we obtain the following variational formulation of the mechanical problem (2.1)–(2.11).

Problem PV. Find a displacement field $\mathbf{u} : [0,T] \to \boldsymbol{V}$, a stress field $\boldsymbol{\sigma} : [0,T] \to \mathcal{H}$, a temperature $\theta : [0,T] \to E$, a damage $\xi : [0,T] \to E_1$, such that for a.e. $t \in (0,T)$

$$\begin{aligned}\boldsymbol{\sigma}(t) &= \mathcal{A}\varepsilon(\dot{\mathbf{u}}(t)) + \mathcal{G}\big(\boldsymbol{\varepsilon}(\mathbf{u}(t), \xi(t))\big) \\ &+ \int_0^t \mathcal{F}\big(\boldsymbol{\sigma}(s) - \mathcal{A}\varepsilon(\dot{\mathbf{u}}(s)), \boldsymbol{\varepsilon}(\mathbf{u}(s))\big)ds - C_e\theta(t)\end{aligned} \tag{3.24}$$

$$\begin{aligned}&(\ddot{\mathbf{u}}(t), w - \dot{\mathbf{u}}(t))_{V'\times V} + (\boldsymbol{\sigma}(t), \varepsilon(w - \dot{\mathbf{u}}(\mathbf{t})))_{\mathcal{H}} \\ &+ j(w) - j(\dot{\mathbf{u}}(t)) \geq (\mathbf{f}(t), w - \dot{\mathbf{u}}(t))_{V'\times V}, \quad \forall w \in V\end{aligned} \tag{3.25}$$

$$\dot{\theta}(t) + K\theta(t) = R\dot{\mathbf{u}}(t) + Q(t), \quad \text{in } E' \tag{3.26}$$

$$\begin{aligned}&(\dot{\xi}(t), \zeta - \xi(t))_{L^2(\Omega)} + a(\xi(t), \zeta - \xi(t)) \\ &\geq (S(\boldsymbol{\varepsilon}(\mathbf{u}(t)), \xi(t)), \zeta - \xi(t))_{L^2(\Omega)} \\ &\text{for all } \xi(t) \in F, \zeta \in F \text{ and } t \in (0,T)\end{aligned} \tag{3.27}$$

$$\mathbf{u}(0) = \mathbf{u}_0, \dot{\mathbf{u}}(0) = \mathbf{v}_0, \theta(0) = \theta_0, \xi(0) = \xi_0, \tag{3.28}$$

where $Q : [0,T] \to E'$, $K : E \to E'$, and $\mathcal{R} : V \to E'$ are given by

$$(Q(t), \eta)_{E'\times E} = \int_{\Gamma_3} k_e\theta_R(t)\eta da + \int_\Omega q(t)\eta dx, \tag{3.29}$$

$$(K\tau, \eta)_{E'\times E} = \sum_{i,j=1}^{d} \int_\Omega k_{ij}\frac{\partial\tau}{\partial x_j}\frac{\partial\eta}{\partial x_i}dx + \int_{\Gamma_3} k_e\tau\eta da, \tag{3.30}$$

$$(Rv, \eta)_{E'\times E} = \int_\Omega r(v)\eta dx + \int_{\Gamma_3} h_r(|v_r|)\eta da, \tag{3.31}$$

for all $v \in V, \eta, \tau \in E$.

We notice that the variational Problem **PV** is formulated in terms of a displacement field, a stress field, a temperature, and damage. The existence of the unique solution of problem **PV** is stated and proved in the next section.

4. Existence and uniqueness result

The main results are stated by the following theorems.

Theorem 4.1. *Assume that* (3.6)–(3.21) *hold and, then there exists a unique solution* $\{\mathbf{u}, \boldsymbol{\sigma}, \theta, \xi\}$ *to problem PV. Moreover, the solution has the regularity*

$$\mathbf{u} \in W^{1,2}(0,T;V) \cap \mathcal{C}^1(0,T;H) \cap W^{2,2}(0,T;V'), \tag{4.1}$$

$$\boldsymbol{\sigma} \in L^2(0,T;\mathcal{H}), div\boldsymbol{\sigma} \in L^2(0,T;V'), \tag{4.2}$$

$$\theta \in \mathcal{C}(0,T;L^2(\Omega)) \cap L^2(0,T;E) \cap W^{1,2}(0,T;E'), \tag{4.3}$$

$$\xi \in W^{1,2}(0,T;L^2(\Omega)) \cap L^2(0,T;H^1(\Omega)). \tag{4.4}$$

We conclude that under the assumptions, the mechanical problem has a unique weak solution with the regularity.

The proof of this theorem will be carried out in several steps. It is based on arguments of first order evolution nonlinear inequalities, evolution equations, and fixed point arguments.

Let $\boldsymbol{\eta} \in L^2(0,T;V')$ be given, in the first step, we consider the following variational problem.

Problem PV1$_\eta$. Find a displacement field $\mathbf{u}_\eta : [0,T] \to \boldsymbol{V}$, such that

$$\begin{aligned}&(\ddot{\mathbf{u}}_\eta(t), w - \dot{\mathbf{u}}_\eta(t))_{V'\times V} + (\mathcal{A}\varepsilon(\dot{\mathbf{u}}_\eta(\mathbf{t})), \varepsilon(w - \dot{\mathbf{u}}_\eta(\mathbf{t})))_{\mathcal{H}} + \\ &(\boldsymbol{\sigma}(t), \varepsilon(w - \dot{\mathbf{u}}(\mathbf{t})))_{\mathcal{H}} + j(w) - j(\dot{\mathbf{u}}_\eta(t)) + (\eta(t), w - \dot{\mathbf{u}}_\eta(t))_{V'\times V} \\ &\geq (\mathbf{f}(t), w - \dot{\mathbf{u}}_\eta(t))_{V'\times V}, \quad \forall w \in V\end{aligned} \tag{4.5}$$

$$\mathbf{u}_\eta(0) = \mathbf{u}_0, \dot{\mathbf{u}}_\eta(0) = \mathbf{v}_0 \tag{4.6}$$

We define $\mathbf{f}_\eta(t) \in V'$ for $a.e.t \in [0,T]$ by

$$(\mathbf{f}_\eta(t), w)_{V'\times V} = (\mathbf{f}(t) - \eta(t), w)_{V'\times V}. \tag{4.7}$$

we deduce that

$$\mathbf{f}_\eta \in L^2(0,T;V'). \tag{4.8}$$

We define the operator $\mathbf{A} : V \to V'$ by

$$(\mathbf{A}\mathbf{v}, w)_{V'\times V} = (\mathcal{A}\varepsilon(\mathbf{v}), \varepsilon(w))_{\mathcal{H}}, \quad \forall \mathbf{v}, w \in V. \tag{4.9}$$

We consider the following variational inequality.

Problem QV$_\eta$. Find a displacement field $\mathbf{v}_\eta : [0,T] \to \boldsymbol{V}$, such that

$$\begin{aligned}&(\dot{\mathbf{v}}_\eta(t), w - \mathbf{v}_\eta(t))_{V'\times V} + (\mathcal{A}\mathbf{v}_\eta(\mathbf{t}), w - \mathbf{v}_\eta(\mathbf{t}))_{V'\times V} + j(w) - j(\mathbf{v}_\eta(t)) \\ &\geq (\mathbf{f}_\eta(t), w - \mathbf{v}_\eta(t))_{V'\times V} \quad \forall w \in V, a.e.t \in [0,T],\end{aligned} \tag{4.10}$$

$$\mathbf{v}_\eta(0) = \mathbf{v}_0. \tag{4.11}$$

In the study of Problem $\mathbf{QV}_\eta$, we have the following result.

Lemma 4.2. *For all $\eta \in L^2(0,T;V')$, $\mathbf{QV}_\eta$ has a unique solution with the regularity*

$$\mathbf{v}_\eta \in \mathcal{C}(0,T;H) \cap L^2(0,T;V) \cap W^{1,2}(0,T;V'),$$

Proof. We begin by the step of regularization (see[8]). We define

$$h(t) = \mathbf{f}_\eta(t), \quad t \in [0,T]$$

and for all $\varepsilon > 0$

$$j_\varepsilon(w) = \int_{\Gamma_3} g\sqrt{|w_r|^2 + \varepsilon^2}\, da, \quad \forall w \in V.$$

After some algebra, for all $\varepsilon > 0$, j_ε is convex and $\mathcal{C}^1$ on V, and its Fréchet derivative satisfies

$$\exists C > 0, \quad \forall w \in V, |j'_\varepsilon(w)|_V \leq C\,|g|_{L^2(\Gamma_3)}.$$

From (3.6) and the monotonicity of j'_ε, it follows from classical first order evolution equation that

$$\forall \varepsilon > 0, \mathbf{v}_\eta^\varepsilon \in L^2(0,T;V) \cap W^{1,2}(0,T;V')$$

such that

$$\begin{cases} \dot{\mathbf{v}}_\eta^\varepsilon(t) + \left(\mathcal{A}\mathbf{v}_\eta^\varepsilon(t) + j'_\varepsilon\left(\mathbf{v}_\eta^\varepsilon(t)\right)\right) = h(t) \ \text{in } V', \quad a.e.t \in [0,T], \\ \mathbf{v}_\eta^\varepsilon(0) = 0 \end{cases} \tag{4.12}$$

Then, we obtain

$$\begin{aligned}&\left(\dot{\mathbf{v}}_\eta^\varepsilon(t), w-\mathbf{v}_\eta^\varepsilon(t)\right)_{V'\times V}+\left(\mathcal{A}\mathbf{v}_\eta^\varepsilon(t), w-\mathbf{v}_\eta^\varepsilon(t)\right)_{V'\times V}+j_\varepsilon(w)-j_\varepsilon\left(\mathbf{v}_\eta^\varepsilon(t)\right)\\&\geq\left(h(t), w-\mathbf{v}_\eta^\varepsilon(t)\right)_{V'\times V}, \quad \forall w\in V, a.e.t\in[0,T]\end{aligned}\tag{4.13}$$

From (4.12), we have

$$\begin{aligned}&\left(\dot{\mathbf{v}}_\eta^\varepsilon(t), \mathbf{v}_\eta^\varepsilon(t)\right)_{V'\times V}+\left(\mathcal{A}\mathbf{v}_\eta^\varepsilon(t), \mathbf{v}_\eta^\varepsilon(t)\right)_{V'\times V}+\left(j_\varepsilon\left(\mathbf{v}_\eta^\varepsilon(t)\right), \mathbf{v}_\eta^\varepsilon(t)\right)_{V'\times V}\\&=\left(h(t), \mathbf{v}_\eta^\varepsilon(t)\right)_{V'\times V}, \quad a.e.t\in[0,T]\end{aligned}$$

Using (3.6), and the monotony of j'_ε, we deduce that

$$\exists C>0,\ \forall t\in[0,T],\ \left|\mathbf{v}_\eta^\varepsilon(t)\right|_H\leq C,\ \int_0^T\left|\mathbf{v}_\eta^\varepsilon(t)\right|_V^2dt\leq C,\ \int_0^T\left|\dot{\mathbf{v}}_\eta^\varepsilon(t)\right|_V^2dt\leq C.$$

Using a subsequence to find that

$$\left\{\begin{array}{l}\mathbf{v}_\eta^\varepsilon\to\mathbf{v}_\eta \text{ weakly in } L^2(0,T;V) \text{ and star weakly in } L^2(0,T;H),\\ \dot{\mathbf{v}}_\eta^\varepsilon\to\dot{\mathbf{v}}_\eta \text{ star weakly in } L^2(0,T;V\prime).\end{array}\right.\tag{4.14}$$

It follows that

$$\mathbf{v}_\eta\in\mathcal{C}(0,T;H) \text{ and } \mathbf{v}_\eta^\varepsilon(t)\to\mathbf{v}_\eta(t) \text{ weakly in } H,\ \forall t\in[0,T]\tag{4.15}$$

Integrating (4.13), we have $\forall w\in L^2(0,T;V)$

$$\begin{aligned}&\textstyle\int_0^T\left(\dot{\mathbf{v}}_\eta^\varepsilon, w\right)_{V'\times V}dt+\int_0^T\left(\mathcal{A}\mathbf{v}_\eta^\varepsilon, w\right)_{V'\times V}dt+\int_0^T j_\varepsilon(w)\,dt\\&\geq\textstyle\int_0^T\left(\dot{\mathbf{v}}_\eta^\varepsilon, \mathbf{v}_\eta^\varepsilon\right)_{V'\times V}dt+\int_0^T\left(\mathcal{A}\mathbf{v}_\eta^\varepsilon, \mathbf{v}_\eta^\varepsilon\right)_{V'\times V}dt\\&+\textstyle\int_0^T j_\varepsilon\left(\mathbf{v}_\eta^\varepsilon\right)dt+\int_0^T\left(h, w-\mathbf{v}_\eta^\varepsilon\right)_{V'\times V}dt\\&\geq\textstyle\frac{1}{2}\left|\mathbf{v}_\eta^\varepsilon(T)\right|_H^2-\frac{1}{2}\left|\mathbf{v}_\eta^\varepsilon(0)\right|_H^2+\int_0^T\left(\mathcal{A}\mathbf{v}_\eta^\varepsilon, \mathbf{v}_\eta^\varepsilon\right)_{V'\times V}dt\\&+\textstyle\int_0^T j_\varepsilon\left(\mathbf{v}_\eta^\varepsilon\right)dt+\int_0^T\left(h, w-\mathbf{v}_\eta^\varepsilon\right)_{V'\times V}dt\end{aligned}\tag{4.16}$$

From (4.14), (4.15) and the weak lower semicontinuity, we obtain that for all $w\in L^2(0,T;V)$:

$$\begin{aligned}&\textstyle\int_0^T\left(\dot{\mathbf{v}}_\eta, w-\mathbf{v}_\eta\right)_{V'\times V}dt+\int_0^T\left(\mathcal{A}\mathbf{v}_\eta, w-\mathbf{v}_\eta\right)_{V'\times V}dt+\int_0^T j(w)-j(\mathbf{v}_\eta)\,dt\\&\geq\textstyle\int_0^T\left(h, w-\mathbf{v}_\eta\right)_{V'\times V}.\end{aligned}$$

The previous inequality implies (see [8]) that

$$\begin{aligned}&\left(\dot{\mathbf{v}}_\eta(t), w-\mathbf{v}_\eta(t)\right)_{V'\times V}+\left(\mathcal{A}\mathbf{v}_\eta(t), w-\mathbf{v}_\eta(t)\right)_{V'\times V}+j(w)-j(\mathbf{v}_\eta(t))\\&\geq\left(h(t), w-\mathbf{v}_\eta(t)\right)_{V'\times V}, \quad \forall w\in V, a.e.t\in[0,T].\end{aligned}$$

We conclude that Problem $\mathbf{QV}_\eta$ has at least a solution $\mathbf{v}_\eta\in\mathcal{C}(0,T;H)\cap L^2(0,T;V)\cap W^{1,2}(0,T;V')$. For the uniqueness, let $\mathbf{v}_\eta^1$, $\mathbf{v}_\eta^2$ be two solutions of $\mathbf{QV}_\eta$. We use (4.10) to obtain for $a.e.t\in[0,T]$,

$$\left(\dot{\mathbf{v}}_\eta^2(t)-\dot{\mathbf{v}}_\eta^1(t), \mathbf{v}_\eta^2(t)-\mathbf{v}_\eta^1(t)\right)_{V'\times V}+\left(\mathcal{A}\mathbf{v}_\eta^2(t)-\mathcal{A}\mathbf{v}_\eta^1(t), \mathbf{v}_\eta^2(t)-\mathbf{v}_\eta^1(t)\right)_{V'\times V}\leq 0$$

Integrating the previous inequality, using (3.6)and (4.9), we find

$$\frac{1}{2}\left|\mathbf{v}_\eta^2(t)-\mathbf{v}_\eta^1(t)\right|_H^2+m_{\mathcal{A}}\int_0^t\left|\mathbf{v}_\eta^2(s)-\mathbf{v}_\eta^1(s)\right|_V^2 ds\leq 0$$

which implies

$$\mathbf{v}_\eta^1=\mathbf{v}_\eta^2.$$

□

Let now $\mathbf{u}_\eta:[0,T]\to V$ be the function defined by

$$\mathbf{u}_\eta(t)=\int_0^t\mathbf{v}_\eta(s)\,ds+\mathbf{u}_0,\qquad\forall t\in[0,T]\,. \tag{4.17}$$

In the study of Problem $\mathbf{PV1}_\eta$, we have the following result.

Lemma 4.3. $\mathbf{PV1}_\eta$ *has a unique solution satisfying the regularity expressed in (4.1)*

Proof. The proof of Lemma 4.3 is a consequence of Lemma 4.2 and the relation (4.17). □

In the second step, we use the displacement field $\mathbf{u}_\eta$ obtained in Lemma 4.3 to consider the following variational problem.
Problem $\mathbf{PV2}_\eta$. Find a temperature field $\theta_\eta:[0,T]\to\boldsymbol{E}$, such that

$$\dot{\theta}_\eta(t)+K\theta_\eta(t)=R\dot{\mathbf{u}}_\eta(t)+Q(t)\,,\quad\text{in } E',\quad a.e.t\in[0,T] \tag{4.18}$$

$$\theta_\eta(0)=\theta_0 \tag{4.19}$$

In the study of Problem $\mathbf{PV2}_\eta$, we have the following result.

Lemma 4.4. $PV2_\eta$ *has a unique solution satisfying*

$$\theta_\eta\in C\left(0,T;L^2(\Omega)\right)\cap L^2(0,T;E)\cap W^{1,2}(0,T;E')\,. \tag{4.20}$$

Moreover, $\exists C>0$ *such that* $\forall\eta_1,\eta_2\in L^2(0,T;V')$

$$|\theta_{\eta_1}(t)-\theta_{\eta_2}(t)|_{L^2(\Omega)}^2\leq C\int_0^t|\eta_1(s)-\eta_2(s)|_V^2\,ds,\quad\forall t\in[0,T]\,. \tag{4.21}$$

Proof. The result follows from classical first order evolution equation given in [2]. Here the Gelfand triple is given by

$$E\subset L^2(\Omega)=\left(L^2(\Omega)\right)'\subset E'.$$

The operator K is linear and coercive. By Korn's inequality, we have

$$(K\tau,\tau)_{E'\times E}\geq C\,|\tau|_E^2\,.$$

Here and below, $C>0$ denotes a generic constant whose value may change from line to line. □

Let $\eta \in \mathcal{C}\left(0,T;L^2\left(\Omega\right)\right)$ be given and consider the following variational problem for the damage filed.

Problem PV3$_\eta$. Find the damage field $\xi_\eta : [0,T] \to H^1(\Omega)$ such that $\xi_\eta(t) \in F$ and

$$\begin{aligned}&(\dot{\xi}_\eta\left(t\right),\zeta-\xi_\eta\left(t\right))_{L^2(\Omega)}+a\left(\xi_\eta\left(t\right),\zeta-\xi_\eta\left(t\right)\right)\\&\geq\left(S\left(\boldsymbol{\varepsilon}\left(\mathbf{u}\left(t\right)\right),\xi_\eta\left(t\right)\right),\zeta-\xi_\eta\left(t\right)\right)_{L^2(\Omega)}\end{aligned}\tag{4.22}$$

$$\xi_\eta(0)=\xi_0\tag{4.23}$$

for all $\xi\left(t\right) \in F, \zeta \in F$ and $t \in (0,T)$

Note that if $f \in H$ then

$$(f,v)_{V'\times V}=(f,v)_H\,,\forall v\in H.$$

Theorem 4.5. *Let $V \subset H \subset V'$ be a Gelfand triple. Let K be a nonempty, closed, and convex set of V. Assume that $a\left(\cdot,\cdot\right) : V\times V \to \mathbb{R}$ is a continuous and symmetric bilinear form such that for some constants $\zeta > 0$ and c_0,*

$$a\left(v,v\right)=c_0\left\|v\right\|_H^2\geqslant\zeta\left\|v\right\|_V^2\ ,\forall v\in H.$$

Then, for every $u_0 \in K$ and $f \in L^2\left(0,T;H\right)$, there exists a unique function $u \in H^1\left(0,T;H\right) \cap L^2\left(0,T;V\right)$ such that $u\left(0\right) = u_0$, $u\left(t\right) \in K$ for all $t \in [0,T]$, and for almost all $t \in (0,T)$,

$$\left(\dot{u}\left(t\right),v-u\left(t\right)\right)_{V'\times V}+a\left(u\left(t\right),v-u\left(t\right)\right)\geqslant\left(f\left(t\right),v-u\left(t\right)\right)_H,\forall v\in K,$$

We apply this theorem to Problem **PV3**$_\eta$.

Lemma 4.6. *There exists a unique solution ξ_η to the auxiliary problem* **PV3**$_\eta$ *such that:*

$$\xi_\eta\in W^{1,2}\left(0,T;L^2\left(\Omega\right)\right)\cap L^2\left(0,T;H^1\left(\Omega\right)\right).\tag{4.24}$$

The above lemma follows from a standard result for parabolic variational inequalities.

Proof. The inclusion mapping of $(H^1\left(\Omega\right),\|\cdot\|_{H^1(\Omega)})$ into $(L^2\left(\Omega\right),\|\cdot\|_{L^2(\Omega)}$ is continuous and its range is dense. We denote by $\left(H^1\left(\Omega\right)\right)'$ the dual space of $H^1\left(\Omega\right)$ and, identifying the dual of $L^2\left(\Omega\right)$ with itself, we can write the Gelfand triple

$$H^1\left(\Omega\right)\subset L^2\left(\Omega\right)\subset\left(H^1\left(\Omega\right)\right)'.$$

We use the notation $(\cdot,\cdot)_{(H^1(\Omega))'\times H^1(\Omega)}$ to represent the duality pairing between $\left(H^1\left(\Omega\right)\right)'$ and $H^1\left(\Omega\right)$. we have

$$(\xi,\beta)_{(H^1(\Omega))'\times H^1(\Omega)}=(\xi,\beta)_{L^2(\Omega)}\,,\forall\xi\in L^2\left(\Omega\right),\beta\in H^1\left(\Omega\right)$$

and we note that F is a closed convex set in $H^1\left(\Omega\right)$. Then, using the definition (3.23) of the bilinear form a, and the fact that $\xi_\eta \in F$. □

In the fourth step, we use the displacement field u_η obtained in Lemma 4.3, θ_η obtained in Lemma 4.4 and the damage ξ_η obtained in Lemma 4.6 to construct the following Cauchy problem for the stress field.

Problem PV4$_\eta$. Find a stress field $\boldsymbol{\sigma}_\eta : [0,T] \to \mathcal{H}$ such that

$$\begin{aligned}\boldsymbol{\sigma}_\eta(t) &= \mathcal{G}\big(\boldsymbol{\varepsilon}(\mathbf{u}_\eta(t), \xi_\eta(t))\big) + \int_0^t \mathcal{F}\big(\boldsymbol{\sigma}_\eta(s), \varepsilon(\mathbf{u}_\eta(s))\big)ds - C_e\theta(t) \\ \forall t &\in [0,T].\end{aligned} \tag{4.25}$$

In the study of Problem **PV4$_\eta$**, we have the following result.

Lemma 4.7. **PV4$_\eta$***has a unique solutions* $\boldsymbol{\sigma}_\eta \in W^{1,2}(0,T;\mathcal{H})$. *Moreover, if* $\boldsymbol{\sigma}_i, \mathbf{u}_i, \theta_i$ *and* ξ_i *represent the solutions of Problems* $PV4_\eta$, $PV1_\eta$, $PV2_\eta$ *and,* $PV3_\eta$ *respectively, for* $\eta_i \in L^2(0,T;V'), i = 1,2$ *then there exists* $C > 0$ *such that*

$$\begin{aligned}|\boldsymbol{\sigma}_1(\mathbf{t}) - \boldsymbol{\sigma}_2(\mathbf{t})|^2_{\mathcal{H}} &\le C(|\mathbf{u}_1(t) - \mathbf{u}_2(t)|^2_V + |\theta_1(t) - \theta_2(t)|^2_{L^2(\Omega)} + \\ &|\xi_1(t) - \xi_2(t)|^2_{L^2(\Omega)} + \int_0^T |\mathbf{u}_1(s) - \mathbf{u}_2(s)|^2_V\, ds)\ \forall t \in [0,T]\end{aligned} \tag{4.26}$$

Proof. Let $\Lambda_\eta : L^2(0,T;\mathcal{H}) \to L^2(0,T;\mathcal{H})$ be the operator given by

$$\Lambda_\eta \boldsymbol{\sigma}(\mathbf{t}) = \mathcal{G}\big(\boldsymbol{\varepsilon}(\mathbf{u}_\eta(t), \xi_\eta(t))\big) + \int_0^t \mathcal{F}\big(\boldsymbol{\sigma}(s), \varepsilon(\mathbf{u}_\eta(s))\big)ds - C_e\theta(t) \tag{4.27}$$

for all $\boldsymbol{\sigma}_\eta \in L^2(0,T;\mathcal{H})$ and $t \in [0,T]$. For $\boldsymbol{\sigma}_1,\boldsymbol{\sigma}_2 \in L^2(0,T;\mathcal{H})$, we use (4.27) and (3.8) to obtain for all $t \in [0,T]$:

$$|\Lambda_\eta\boldsymbol{\sigma}_1(\mathbf{t}) - \Lambda_\eta\boldsymbol{\sigma}_2(\mathbf{t})|_{\mathcal{H}} \le L_{\mathcal{F}}\, |\boldsymbol{\sigma}_1(\mathbf{s}) - \boldsymbol{\sigma}_2(\mathbf{s})|_{\mathcal{H}}\, ds.$$

It follows from this inequality that for large p enough, the operator Λ_η^p is a contraction on the Banach space $L^2(0,T;\mathcal{H})$, and therefore there exists a unique element $\boldsymbol{\sigma}_\eta \in L^2(0,T;\mathcal{H})$ such that $\Lambda_\eta\boldsymbol{\sigma}_\eta(\mathbf{t}) = \boldsymbol{\sigma}_\eta$. Moreover, $\boldsymbol{\sigma}_\eta$ is the unique solution of Problem **PV4$_\eta$**, and using (4.25), the regularity of $\mathbf{u}_\eta$, the regularity of ξ_η, the regularity of θ_η, and the properties of the operators $\mathcal{G}$, $\mathcal{F}$, and C_e, it follows that $\boldsymbol{\sigma}_\eta \in W^{1,2}(0,T;V')$. Consider now $\boldsymbol{\eta}_1,\boldsymbol{\eta}_2 \in L^2(0,T;V')$ and for $i = 1,2$ denote $\mathbf{u}_{\eta_i} = \mathbf{u}_i$, $\sigma_{\eta_i} = \sigma_i$, $\xi_{\eta_i} = \xi_i$ and $\theta_{\eta_i} = \theta_i$. We have

$$\boldsymbol{\sigma}_i(\mathbf{t}) = \mathcal{G}\big(\boldsymbol{\varepsilon}(\mathbf{u}_i(t), \xi_i(t))\big) + \int_0^t \mathcal{F}\big(\boldsymbol{\sigma}_i(s), \varepsilon(\mathbf{u}_i(s))\big)ds - C_e\theta_i(t)$$

and using the properties (3.7), (3.8), (3.10) and of $\mathcal{G}$, $\mathcal{F}$ and C_e we find

$$\begin{aligned}&|\boldsymbol{\sigma}_1(\mathbf{t}) - \boldsymbol{\sigma}_2(\mathbf{t})|^2_{\mathcal{H}} \le C(|\mathbf{u}_1(t) - \mathbf{u}_2(t)|^2_V + |\theta_1(t) - \theta_2(t)|^2_{L^2(\Omega)} + |\xi_1(t) - \xi_2(t)|^2_{L^2(\Omega)} \\ &+ \textstyle\int_0^T |\boldsymbol{\sigma}_1(\mathbf{s}) - \boldsymbol{\sigma}_2(\mathbf{s})|^2_{\mathcal{H}}\, ds) + \int_0^T |\mathbf{u}_1(s) - \mathbf{u}_2(s)|^2_V\, ds, \qquad \forall t \in [0,T].\end{aligned}$$

We use Gronwall argument in the previous inequality to deduce (4.26), which concludes the proof of Lemma 4.7. □

Finally, we define the operator

$$\Lambda : L^2(0,T;V') \to L^2(0,T;V')$$

by

$$\begin{aligned}&(\Lambda\eta(\mathbf{t}), w)_{V' \times V} = (\mathcal{G}\boldsymbol{\varepsilon}(\mathbf{u}_\eta(t), \xi_\eta(t))), \varepsilon(w))_{\mathcal{H}} \\ &+ \big(\textstyle\int_0^t \mathcal{F}\big(\boldsymbol{\sigma}_\eta(s), \varepsilon(\mathbf{u}_\eta(s))\big)ds - C_e\theta_\eta(t), \varepsilon(w)\big)_{\mathcal{H}}, \quad \forall t \in [0,T]\end{aligned} \tag{4.28}$$

Here, for every $\boldsymbol{\eta} \in L^2(0,T;V')$ $\mathbf{u}_\eta$, θ_η, ξ_η and σ_η represent the displacement field, the temperature field, the damage and the stress field obtained in Lemmas 4.3, 4.4, 4.6 and 4.7 respectively. We have the following result.

Lemma 4.8. *The operator Λ has a unique fixed point $\boldsymbol{\eta} \in L^2(0,T;V')$ such that $\Lambda\eta = \eta$.*

Proof. Let now $\boldsymbol{\eta}_1$,$\boldsymbol{\eta}_2 \in L^2(0,T;V')$. We use the notation that $\mathbf{u}_{\eta_i} = \mathbf{u}_i$, $\dot{\mathbf{u}}_{\eta_i} = \mathbf{v}_{\eta_i} = \mathbf{v}_i$, $\sigma_{\eta_i} = \sigma_i$,$\xi_{\eta_i} = \xi_i$ and $\theta_{\eta_i} = \theta_i$, for $i = 1, 2$. Using (3.4),(3.6),(3.8), (3.15), and (4.28) to find

$$|\Lambda\boldsymbol{\eta}_1(\mathbf{t}) - \Lambda\boldsymbol{\eta}_2(\mathbf{t})|^2_{V'} \le C(|\mathbf{u}_1(t) - \mathbf{u}_2(t)|^2_V + |\theta_1(t) - \theta_2(t)|^2_{L^2(\Omega)} + |\xi_1(t) - \xi_2(t)|^2_{L^2(\Omega)} \\ + \int_0^T |\boldsymbol{\sigma}_1(\mathbf{s}) - \boldsymbol{\sigma}_2(\mathbf{s})|^2_{\mathcal{H}}\, ds + \int_0^T |\mathbf{u}_1(s) - \mathbf{u}_2(s)|^2_V\, ds) \tag{4.29}$$

We use the estimate (4.26) to obtain

$$|\Lambda\boldsymbol{\eta}_1(\mathbf{t}) - \Lambda\boldsymbol{\eta}_2(\mathbf{t})|^2_{V'} \le C(|\mathbf{u}_1(t) - \mathbf{u}_2(t)|^2_V + |\theta_1(t) - \theta_2(t)|^2_{L^2(\Omega)} + |\xi_1(t) - \xi_2(t)|^2_{L^2(\Omega)} \\ + \int_0^T |\mathbf{u}_1(s) - \mathbf{u}_2(s)|^2_V + \int_0^T |\theta_1(s) - \theta_2(s)|^2_{L^2(\Omega)}\, ds) \tag{4.30}$$

Moreover, from (4.10) we obtain

$$(\dot{\mathbf{v}}_1 - \dot{\mathbf{v}}_2, \mathbf{v}_1 - \mathbf{v}_2)_{V'\times V} + (\mathcal{A}\mathbf{v}_1 - \mathcal{A}\mathbf{v}_2, \mathbf{v}_1 - \mathbf{v}_2)_{V'\times V} \\ \le -(\eta_1 - \eta_2, \mathbf{v}_1 - \mathbf{v}_2)_{V'\times V}$$

We integrate this equality with respect to time.
We use the initial conditions $\mathbf{v}_1(0) = \mathbf{v}_2(0) = \mathbf{v}_0$, the relation (4.9) and (3.6) to find that

$$m_{\mathcal{A}} \int_0^T |\mathbf{v}_1(s) - \mathbf{v}_2(s)|^2_V\, ds \le C \int_0^T |\boldsymbol{\eta}_1(\mathbf{t}) - \boldsymbol{\eta}_2(\mathbf{t})|_V\, |\mathbf{v}_1(s) - \mathbf{v}_2(s)|_V\, ds$$

For all $t \in [0,T]$. Then, using the inequality $2ab \le \frac{a^2}{m_{\mathcal{A}}} + m_{\mathcal{A}} b^2$ we obtain

$$\int_0^T |\mathbf{v}_1(s) - \mathbf{v}_2(s)|^2_V\, ds \le C \int_0^T |\boldsymbol{\eta}_1(\mathbf{s}) - \boldsymbol{\eta}_2(\mathbf{s})|_V\, ds, \qquad \forall t \in [0,T] \tag{4.31}$$

Since $\mathbf{u}_1(0) = \mathbf{u}_2(0) = \mathbf{u}_0$ we have

$$|\mathbf{u}_1(t) - \mathbf{u}_2(t)|^2_V \le C \int_0^T |\mathbf{v}_1(s) - \mathbf{v}_2(s)|^2_V\, ds$$

We use the previous inequality and (4.30) to obtain

$$|\Lambda\boldsymbol{\eta}_1(\mathbf{t}) - \Lambda\boldsymbol{\eta}_2(\mathbf{t})|^2_{V'} \le C(\int_0^T |\mathbf{v}_1(s) - \mathbf{v}_2(s)|^2_V\, ds + \\ |\theta_1(t) - \theta_2(t)|^2_{L^2(\Omega)} + |\xi_1(t) - \xi_2(t)|^2_{L^2(\Omega)} + \int_0^T |\theta_1(s) - \theta_2(s)|^2_{L^2(\Omega)}\, ds)$$

The estimates (4.31) and (4.21) imply that

$$|\Lambda\boldsymbol{\eta}_1(\mathbf{t}) - \Lambda\boldsymbol{\eta}_2(\mathbf{t})|^2_{V'} \le \int_0^T C\, |\boldsymbol{\eta}_1(\mathbf{s}) - \boldsymbol{\eta}_2(\mathbf{s})|^2_{V'}\, ds$$

Reiterating this inequality m times leads to

$$|\Lambda^m \boldsymbol{\eta}_1 - \Lambda^m \boldsymbol{\eta}_2|^2_{L^2(0,T;V')} \leq \frac{C^m T^m}{m!} |\boldsymbol{\eta}_1 - \boldsymbol{\eta}_2|^2_{L^2(0,T;V')}$$

For m sufficiently large, Λ^m is a contraction on the Banach space $L^2(0,T;V')$, and so Λ has a unique fixed point. □

Now, we have all the ingredients needed to prove Theorem 4.1.

Proof. Let $\boldsymbol{\eta}^* \in L^2(0,T;V')$ be the fixed point of Λ defined by (4.28) and denote

$$\mathbf{u} = \mathbf{u}_{\boldsymbol{\eta}^*}, \theta = \theta_{\boldsymbol{\eta}^*}, \xi = \xi_{\boldsymbol{\eta}^*}, \sigma = \sigma_{\boldsymbol{\eta}^*} \tag{4.32}$$

$$\boldsymbol{\sigma} = \mathcal{A}\varepsilon(\dot{\mathbf{u}}) + \boldsymbol{\sigma}^* \tag{4.33}$$

We prove that $(\mathbf{u}, \sigma, \xi, \theta)$, satisfies (3.24)-(3.28) and (4.1)-(4.4). Indeed, we write (4.25) for $\boldsymbol{\eta} = \boldsymbol{\eta}^*$ and use (4.32)-(4.33), we obtain that (3.24) is satisfied. We consider (4.5) for $\boldsymbol{\eta} = \boldsymbol{\eta}^*$ and use the first equality in (4.32) to find

$$\begin{aligned} &(\ddot{\mathbf{u}}(t), w - \dot{\mathbf{u}}(t))_{V' \times V} + (\mathcal{A}\varepsilon(\dot{\mathbf{u}}), \varepsilon(w - \dot{\mathbf{u}}(\mathbf{t}))_{\mathcal{H}} + j(w) - j(\dot{\mathbf{u}}(t)) \\ &+ (\eta^*(t), w - \dot{\mathbf{u}}(\mathbf{t}))_{V' \times V} \geq (\mathbf{f}(t), w - \dot{\mathbf{u}}(t))_{V' \times V}, \quad \forall w \in V \end{aligned} \tag{4.34}$$

Equation $\Lambda \boldsymbol{\eta}^* = \boldsymbol{\eta}^*$ combined with (4.28), (4.32) and (4.33) shows that

$$\begin{aligned} &(\eta^*(t), w)_{V' \times V} = \mathcal{G}(\boldsymbol{\varepsilon}(\mathbf{u}(t)), \boldsymbol{\varepsilon}(w))_{\mathcal{H}} + \\ &\left(\int_0^t \mathcal{F}\big(\boldsymbol{\sigma}(s) - \mathcal{A}\varepsilon(\dot{\mathbf{u}}(s)), \boldsymbol{\varepsilon}(\mathbf{u}(s))\big) ds - C_e \theta(t), \boldsymbol{\varepsilon}(w) \right) \quad \forall w \in V \end{aligned} \tag{4.35}$$

We now substitute (4.35) into (4.34) and use (4.33) to see that (3.25) is satisfied. We write (4.18) for $\boldsymbol{\eta} = \boldsymbol{\eta}^*$ and use (4.32) to find that (3.26) is also satisfied. Next, (3.28) is satisfied when the regularities (4.1) and (4.4) follow from Lemmas 4.3 and 4.4. The regularity $\boldsymbol{\sigma} \in L^2(0,T;\mathcal{H})$ follows from Lemmas 4.3 and 4.4, the assumptions (3.6) and (4.33). Finally (3.25) implies that

$$div\boldsymbol{\sigma} + \mathbf{f}_0(t) = \rho \ddot{\mathbf{u}}(t) \quad \text{in } V', \qquad a.e.t \in [0,T]$$

and therefore by (3.13) and (3.15), we find $div\boldsymbol{\sigma} \in L^2(0,T;V')$. We deduce that the regularity (4.3) holds which concludes the existence part of Theorem 4.1. The uniqueness of Theorem 4.1 is a consequence of the uniqueness of the fixed point of the operator Λ defined by (4.28) and the unique solvability of Problems $\mathbf{PV1}_\eta$, $\mathbf{PV2}_\eta$, $\mathbf{PV3}_\eta$ and $\mathbf{PV4}_\eta$. □

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