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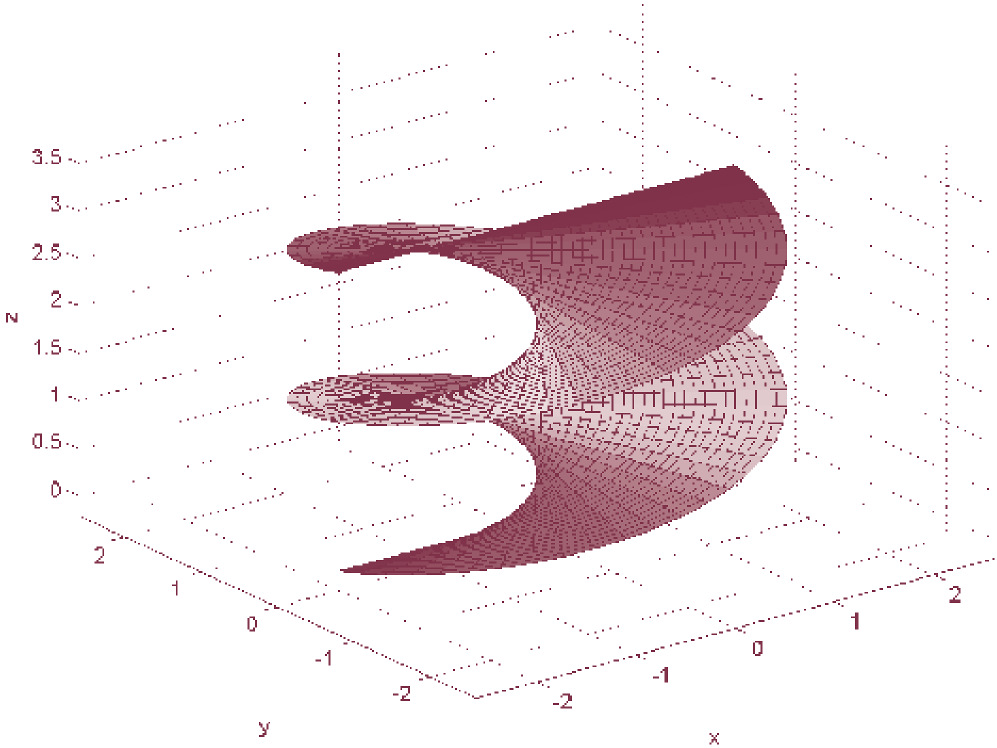

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Constructing large self-small modules

George Ciprian Modoi

Abstract. We give a method for constructing (possible large) self-small modules via some special homomorphisms of rings, called here weak epimorphisms.

Mathematics Subject Classification (2010): 16D99, 20K40.

Keywords: Self-small module, epimorphism of rings.

Various kinds of smallness appear naturally in the study of situations in which the covariant or contravariant hom functor induces an equivalence, respectively a duality between some categories of modules. For example Morita theory says that if R is an arbitrary ring, P is a progenerator in the category Mod-R of right R-modules and $E = \mathrm{End}_R(P)$ is its endomorphism ring, then the functor $\mathrm{Hom}_R(P, -) :$ Mod-$R \to$ Mod-E is an equivalence, with the inverse the tensor product $- \otimes_E P$. In these conditions, P has to be *small*, that is $\mathrm{Hom}_R(P, -)$ has to commute with arbitrary direct sums.

The smallness notion can be generalized in various ways, by imposing some restrictions to the class of direct sums which the covariant hom functor has to commute. In this note we deal with the following generalization: A *self-small* R-module is a module M such that $\mathrm{Hom}_R(M, M^{(I)}) \cong \mathrm{Hom}(M, M)^{(I)}$, naturally for every set I. Self-small abelian groups (that is, $\mathbb{Z}$-modules) were introduced by Arnold an Murley in [2]. The relevance of the study of self-small abelian groups is justified by many papers (see, for example, [1] and the references therein).

In this note we want to construct a module which is self-small but it is large in some sense. More precisely, we want this self-small module to be not small. Because finitely generated modules are always small, the modules we are looking for have to be infinitely generated. The method is inspired by the construction of the abelian group of p-adic integers $\mathbb{J}_p$, where p is a prime. In this case, $\mathbb{J}_p$ is uncountable, that is its cardinality is also larger than the cardinality of the ring of integers $\mathbb{Z}$.

Note that another way of constructing large self-small modules can be found in [8]. More precisely, from [8, Example 2.7] we learn that the direct product $\prod_p \mathbb{Z}/p$ is self-small, but the direct sum $\bigoplus_p \mathbb{Z}/p$ is not, where p runs over all primes and $Z/n = \mathbb{Z}/n\mathbb{Z}$ for every $n \in \mathbb{N}$. More generally, for a ring R let denote by $\mathcal{S}_R$ a representative set of simple modules. Then in [8, Theorem 2.5 and Corollary 1.3] we find some sufficient conditions for the direct product $\prod_{S\in\mathcal{S}_R} S$ to be, respectively to be not self-small.

In what follows we consider two rings with one R and J, and we denote by Mod-R and Mod-J the respective categories of modules (which by default are left modules). Let $\varphi : R \to J$ a unitary ring homomorphism. Thus J has a natural structure of $R-R$-bimodule and φ induces a pair of adjoint functors (the restriction and the induction of the scalars):

$$\varphi_* = \mathrm{Hom}_J(J,-) : \text{Mod-}J \rightleftarrows \text{Mod-}R : (J\otimes_R -) = \varphi^*.$$

The restriction functor φ_* acts as follows: $\varphi_*(M) = M$ and $ax = \varphi(a)x$ for all J-modules M and all $x \in M$ and $a \in R$. Henceforth it is obviously faithful, since it sends a J-linear map in itself, but seen as R-linear.

Recall that φ is called an *epimorphism of rings*, if for every two parallel homomorphisms of rings $\psi, \zeta : J \to J'$ we have

$$\psi\cdot\varphi = \zeta\cdot\varphi \Rightarrow \psi = \zeta.$$

By [7, Ch. XI, Proposition 1.2] this happens exactly if φ_* is full too, therefore if we have $\mathrm{Hom}_R(M,N) \cong \mathrm{Hom}_J(M,N)$ for all $M,N \in \text{Mod-}J$. Inspired by this, we call φ *weak epimorphism* if $\mathrm{Hom}_R(J,J) \cong \mathrm{Hom}_J(J,J)$, that is $\mathrm{Hom}_R(J,J) \cong J$.

Proposition 1. *If $\varphi : R \to J$ is a weak epimorphism of rings, then J is self-small as R-module.*

Proof. Let I be a set and denote by $\pi_i : J^{(I)} \to J$ the projection of the coproduct of copies of J into its i-th component ($i \in I$). If $f : J \to J^{(I)}$ is an arbitrary R-linear map, then $\pi_i f : J \to J$ is R-linear for all $i \in I$. According to our hypothesis it is J-linear too, therefore it is determined by $\pi_i f(1) \in J$. Because $\pi_i f(1) \neq 0$ only for a finite number of i's, we conclude that $\pi_I f = 0$ for almost all $i \in I$, hence f factors through a finite subcoprodct of $J^{(I)}$, what is the same as saying that $\mathrm{Hom}_R\left(J, J^{(I)}\right) \cong \mathrm{Hom}(J,J)^{(I)}$. □

Since epimorphisms of rings are obviously weak epimorphisms too we obtain:

Corollary 2. *If $\varphi : R \to J$ is an epimorphism of rings, then the R-module J is self-small.*

Example 3. The inclusion $\mathbb{Z} \to \mathbb{Q}$ is known to be an epimorphism of rings, namely one which is not surjective. Therefore Corollary 2 above gives us a new proof that the abelian group $\mathbb{Q}$ is self-small.

In the sequel we assume that the ring R is commutative. Thus Mod-R coincide to the category of right R-modules, and $\mathrm{Hom}_R(M,N)$ is an R-module for all $M,N \in$ Mod-R. In Mod-R consider an ascending chain of submodules

$$\text{(DS) } Z_1 \stackrel{\mu_1}{\to} Z_2 \stackrel{\mu_2}{\to} Z_3 \to \ldots,$$

of the module

$$Z_\infty = \lim_{\to} Z_n = \lim_{\to}(Z_1 \stackrel{\mu_1}{\to} Z_2 \stackrel{\mu_2}{\to} Z_3 \to \ldots) = \bigcup Z_n,$$

the morphisms μ_n being inclusions. Relative to the above chain consider the following conditions:

(1) All modules Z_m are finitely presented.
(2) $\mathrm{Hom}_R(Z_m, Z_n) \cong Z_m$ naturally, for all $1 \leq m \leq n$.
(3) The R-module Z_∞ is injective relative to all exact sequences

$$0 \to Z_m \overset{\mu_n}{\to} Z_{m+1} \to Z_{m+1}/Z_m \to 0,$$

with $m \geq 1$.
(4) Z_1 is simple, and denote by U the annihilator of Z_1 in R, that is U is a maximal ideal in R and there is a short exact sequence

$$0 \to U \to R \to Z_1 \to 0.$$

Moreover assume that $Z_{m+1}U = Z_n$, for all $m \in \mathbb{N}^*$.
(5) $Z_m \otimes_R Z_1 \cong Z_1$ naturally, for all $m \in \mathbb{N}^*$.

Note that the condition (3) is automatically satisfied, if we know that the R-module Z_∞ is injective. On the other hand we can replace (3) with a condition relative to the direct system (DS), rather than relative to its direct limit Z_∞, as in the the following:

(3') The R-module Z_n is injective relative to all exact sequences

$$0 \to Z_m \overset{\mu_n}{\to} Z_{m+1} \to Z_{m+1}/Z_m \to 0,$$

with $1 \leq m < n$.

Lemma 4. *If (1) and (3') are satisfied then (3) holds too.*

Proof. The condition (3') implies that the induced homomorphism

$$\mathrm{Hom}_R(Z_{m+1}, Z_n) \to \mathrm{Hom}_R(Z_m, Z_n)$$

is surjective for all $1 \leq m < n$. The condition (1) says that all Z_i, $i \geq 1$ are finitely generated, and this means the functors $\mathrm{Hom}_R(Z_i, -)$ commute with direct limits as we can see from [7, Ch. V, Proposition 3.4]. We deduce that the induced homomorphism

$$\begin{aligned}\mathrm{Hom}_R(Z_{m+1}, Z_\infty) &\cong \varinjlim \mathrm{Hom}_R(Z_{m+1}, Z_n)\\ &\to \varinjlim \mathrm{Hom}_R(Z_m, Z_n) \cong \mathrm{Hom}_R(Z_m, Z_\infty)\end{aligned}$$

is also surjective, therefore (3) holds. □

Lemma 5. *If (1) and (2) hold, we have for all $m \geq 1$ a natural isomorphism:*

$$\mathrm{Hom}_R(Z_m, Z_\infty) \cong Z_m.$$

Proof. Using again the property that $\mathrm{Hom}_R(Z_i, -)$ commutes with direct limits, for all $i \geq 1$, we get:

$$\mathrm{Hom}_R(Z_m, Z_\infty) = \mathrm{Hom}_R(Z_m, \varinjlim Z_n) \cong \varinjlim \mathrm{Hom}_R(Z_m, Z_n) \cong Z_m,$$

where the last isomorphisms follows from the fact that (2) implies that the direct system $\{\mathrm{Hom}_R(Z_m, Z_n)\}_{n\geq 1}$ looks like

$$\mathrm{Hom}_R(Z_1, Z_m) \to \ldots \to \mathrm{Hom}_R(Z_{m-1}, Z_m) \to Z_m \overset{=}{\to} Z_m \overset{=}{\to} Z_m \overset{=}{\to} \ldots,$$

that is it has a cofinal constant subsystem. □

Assume that (1) and (2) hold. For all $n \geq 1$, we denote δ_n the composed homomorphism

$$Z_{n+1} \stackrel{\cong}{\to} \mathrm{Hom}_R(Z_{n+1}, Z_\infty) \stackrel{(\mu_n)_*}{\to} \mathrm{Hom}_R(Z_n, Z_\infty) \stackrel{\cong}{\to} Z_n,$$

where the isomorphisms are coming from Lemma 5. We obtain an inverse system of R-modules

$$\text{(IS)}\ Z_1 \stackrel{\delta_1}{\leftarrow} Z_2 \stackrel{\delta_2}{\leftarrow} Z_3 \leftarrow \dots$$

Let now denote $J = \mathrm{End}_R(Z_\infty)$. Thus J is naturally an R-algebra, and let $\varphi : R \to J$ denote the structure homomorphism of this algebra.

Lemma 6. *If (1) and (2) hold, we have a natural isomorphism in* Mod-R*:*

$$J \cong \varprojlim Z_n = \varprojlim(Z_1 \stackrel{\delta_1}{\leftarrow} Z_2 \stackrel{\delta_2}{\leftarrow} Z_3 \leftarrow \dots).$$

Proof. The chain of isomorphisms (the last one coming from Lemma 5)

$$J = \mathrm{Hom}_R(Z_\infty, Z_\infty) \cong \mathrm{Hom}_R(\varinjlim Z_n, Z_\infty) \cong \varprojlim \mathrm{Hom}_R(Z_n, Z_\infty) \cong \varprojlim Z_n.$$

proves our lemma. □

For the inverse system (IS) we denote $\delta_{jj} = 1_{Z_j}$ and $\delta_{ji} = \delta_j \dots \delta_i$, for all $1 \leq j \leq i$. With these notations, the inverse system is called *Mittag-Leffler* if for each $k \geq 1$ there is $j > k$ such that $\mathrm{Im}(\delta_{ki}) = \mathrm{Im}(\delta_{kj})$ for all $j \leq i$. In particular this is always true, provided that the homomorphisms δ_i are surjective, for all $i \geq 1$.

Lemma 7. *If (1), (2) and (3) hold, then the inverse system (IS) is Mittag-Leffler.*

Proof. The homomorphism $(\mu_n)_*$ is surjective by (3), so the same property is true for δ_n, and the conclusion follows. □

From now on, we assume that all conditions (1)-(5) hold.

Lemma 8. *We have $Z_{n+m}/Z_m \cong Z_n$ for all $n, m \in \mathbb{N}^*$.*

Proof. First we will show that $Z_{n+1}/Z_n \cong Z_1$ for all $n \in \mathbb{N}^*$. Indeed, applying the functor $Z_{n+1} \otimes_R -$ to the short exact sequence $0 \to U \to R \to Z_1 \to 0$, keeping in the mind that $Z_n = Z_{n+1}U$ is the image of the map $Z_{n+1} \otimes_R U \to Z_{n+1} \otimes_R R$ and using condition (5) for the isomorphism in the last vertical arrow, we get a commutative diagram with exact rows

$$\begin{array}{ccccccccc}
 & & Z_{n+1} \otimes_R U & \longrightarrow & Z_{n+1} \otimes_R R & \longrightarrow & Z_{n+1} \otimes_R Z_1 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow \cong & & \downarrow \cong & & \\
0 & \longrightarrow & Z_n & \longrightarrow & Z_{n+1} & \longrightarrow & Z_1 & \longrightarrow & 0
\end{array}$$

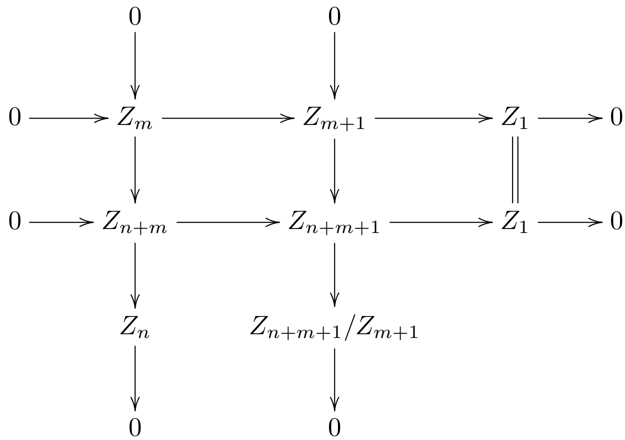
which proves our claim.

Fix $n \in \mathbb{N}^*$ and proceed by induction on m. For $m = 1$, we apply the functor $\mathrm{Hom}_R(-, Z_\infty)$ to the exact sequence from the second row of the last diagram. According to (3), we get an exact sequence too, which by Lemma 5 looks like:

$$0 \to Z_1 \to Z_{n+1} \to Z_n \to 0,$$

proving our desired isomorphism $Z_{n+1}/Z_1 \cong Z_n$.

Suppose now that $Z_{n+m}/Z_n \cong Z_m$. Then construct the diagram having exact rows and columns (the exactness of the rows is shown in the first part of this proof, the induction hypothesis gives exactness of the first column, and for the second column it is obvious):



Now the Ker-Coker lemma gives us the isomorphism $Z_{n+m+1}/Z_{m+1} \cong Z_n$. □

Remark 9. Puttig together above lemmas, we deduce that for all $n, m \in \mathbb{N}^*$ we have the short exact sequences

$$0 \to Z_n \to Z_{n+m} \to Z_m \to 0 \text{ and } 0 \to Z_n \to Z_{n+m} \to Z_m \to 0$$

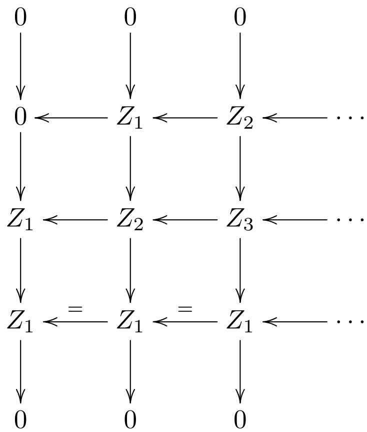
and the functor $\mathrm{Hom}_R(-, Z_\infty)$ sends them to each other.

Lemma 10. *There is a short exact sequence*

$$0 \to J \overset{u}{\to} J \to Z_1 \to 0$$

such that $\mathrm{Im}\, u = UJ$.

Proof. Consider the diagram with exact columns:



Note that the involved inverse systems are Mittag Leffler by Lemma 7, therefore the their inverse limits are exact by [4, Theorem 5]. Therefore the inverse limit gives us the desired short exact sequence.
By its construction the homomorphism u acts as follows: for all $(x_1, x_2, x_3, \ldots) \in J$ (that is $(x_1, x_2, x_3, \ldots) \in \prod_{n\geq 1} Z_n$ such that $\delta_n(x_{n+1}) = x_n$, for all n) we have $u(x_1, x_2, x_3, \ldots) = (0, x_1, x_2, \ldots)$, so $UZ_{n+1} = Z_n$, for all $n \geq 1$ implies $UJ = \operatorname{Im} u$. $\square$

Lemma 11. *The following sentences hold:*

(a) *For all $n \geq 1$ we have $Z_n \otimes_R J \cong Z_n$ as (left) J-modules.*
(b) *For all $n \geq 1$ we have $\operatorname{Hom}_R(J, Z_n) \cong Z_n$ as R-modules.*

Proof. Note first that $Z_n \cong \operatorname{Hom}_R(Z_n, Z_\infty)$ is a left $J = \operatorname{End}_R(Z_\infty)$-module.

(a). We proceed by induction on n. For $n = 1$ we apply the functor $Z_1 \otimes_R -$ to the short exact sequence $0 \to UJ \to J \to Z_1 \to 0$ coming from Lemma 10. Since U is the annihilator of Z_1 we deduce $Z_1 \otimes_R UJ = 0$, so we get an isomorphism $Z_1 \otimes_R J \overset{\cong}{\to} \mathbb{Z}_1 \otimes_R Z_1$, so $Z_1 \otimes_R J \cong Z_1$.

Now suppose $Z_n \otimes_R J \cong Z_n$. Starting from the short exact sequence $0 \to Z_n \to Z_{n+1} \to Z_1 \to 0$ given by Lemma 8) we construct the commutative diagram with exact rows:

$$\begin{array}{ccccccccc} & & Z_n \otimes_R J & \longrightarrow & Z_{n+1} \otimes_R J & \longrightarrow & Z_1 \otimes_R J & \longrightarrow & 0 \\ & & \downarrow \cong & & \downarrow & & \downarrow \cong & & \\ 0 & \longrightarrow & Z_n & \longrightarrow & Z_{n+1} & \longrightarrow & Z_1 & \longrightarrow & 0 \end{array}$$

whose vertical maps are obtained from the natural homomorphism

$$- \otimes_R J \cong \operatorname{Hom}_J(J, -) \otimes_R J = \varphi^* \cdot \varphi_* \to \mathbf{1}_{\text{Mod-}J},$$

the last arrow coming from the adjunction. Then the middle vertical arrow is an isomorphism too, proving the conclusion.

(b). Using first the (proof of the) point (a), and second the adjunction isomorphism we obtain an isomorphism of R-modules

$$\begin{aligned} \operatorname{Hom}_J(Z_n, Z_n) &\cong \operatorname{Hom}_J((\varphi^* \cdot \varphi_*)(Z_n), Z_n) \\ &\cong \operatorname{Hom}_R(\varphi_*(Z_n), \varphi_*(Z_n)) = \operatorname{Hom}_R(Z_n, Z_n) \cong Z_n. \end{aligned}$$

Combining it with the adjunction isomorphism between the functors

$$\operatorname{Hom}_J(Z_n, -) : \text{Mod-} \rightleftarrows \text{Mod-}R : Z_n \otimes_R -$$

and the isomorphism of part (a) we get the isomorphisms of R-modules:

$$\begin{aligned} \operatorname{Hom}_R(J, Z_n) &\cong \operatorname{Hom}_R(J, \operatorname{Hom}_J(Z_n, Z_n)) \\ &\cong \operatorname{Hom}_J(Z_n \otimes_R J, Z_n) \cong \operatorname{Hom}_J(Z_n, Z_n) \cong Z_n \end{aligned}$$

concluding the proof. $\square$

Theorem 12. *With the notations above, if all conditions (1)-(5) are true, then $\varphi : R \to J$ is a weak epimorphism of rings. Consequently J is a self-small R-module.*

Proof. Using the isomorphism from the point (b) of Lemma 11 we get

$$\mathrm{Hom}_R(J, J) = \mathrm{Hom}_R(J, \varprojlim Z_n) \cong \varprojlim \mathrm{Hom}_R(J, Z_n) \cong \varprojlim Z_n \cong J,$$

therefore the ring homomorphism $\varphi : R \to J$ is a weak epimorphism. Then J is self-small as R-module, by Proposition 1. □

Example 13. Let $R = \mathbb{Z}$ and let p be a prime. The direct system

$$\mathbb{Z}/p^1 \to \mathbb{Z}/p^2 \to \mathbb{Z}/p^3 \to \dots,$$

whose direct limit is the cocyclic abelian group $\mathbb{Z}/p^\infty$, satisfies the conditions (1)-(5). Thus Theorem 12 gives a proof that the group of p-adic integers $\mathbb{J}_p = \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/p^\infty, \mathbb{Z}/p^\infty)$ is self-small (for details, see also [5]).

Example 14. Let R be a Dedekind ring, and let $\mathfrak{m}$ be a maximal ideal. Put $Z_i = R/\mathfrak{m}^i$, for all $i \geq 1$. Then $S = Z_1$ is a simple R-module, and modules Z_i are indecomposable, uniserial, with the composition series of the form

$$0 \subseteq Z_1 \subseteq \dots \subseteq Z_{i-1} \subseteq Z_i$$

whose factors are all isomorphic to S. Moreover for every $i \geq 1$ there is an exact sequence

$$0 \to S \to Z_{i+1} \to Z_i \to 0.$$

For more details concerning these modules we refer to [6, 1.4]. Then we obtain a direct system (DS) satisfying the conditions (1)-(5), so its inverse limit, the so called $\mathfrak{m}$-*adic module*, $J = \varprojlim R/\mathfrak{m}^i$ is self-small as R-module.

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A topological representation of double Boolean lattices

Brigitte E. Breckner and Christian Săcărea

Abstract. Boolean Concept Logic has been introduced by R. Wille as a mathematical theory based on Formal Concept Analysis. Concept lattices are extended with two new operations, negation and opposition which then lead to algebras of protoconcepts which are equationally equivalent to double Boolean algebras. In this paper, we provide a topological representation for double Boolean algebras based on the so-called DB-topological contexts. A double Boolean algebra is then represented as the algebra of clopen protoconcepts of some DB-topological context.

Mathematics Subject Classification (2010): 18B35, 54B30, 68T30.

Keywords: Formal concept analysis, double boolean algebra, topological context.

1. Introduction

Formal Concept Analysis (FCA) is a prominent field of Applied Mathematics which is grounded on the mathematization of the notion of concept and concept hierarchy, having a wide range of applications in data analysis and knowledge discovery in databases. Topological FCA is an extension of FCA to topological spaces and is investigating issues related to the interplay between Topology and FCA.

Topological contexts were defined as an attempt to represent 0-1-lattices by open concepts of some topological context, i.e., formal concepts whose extents and intents are open sets. This theory was then completed to a categorical duality in a series of papers at the beginning of the 1990s, but for the sake of a more natural description, closed concepts were considered in order to represent 0-1-lattices ([3], [4]). Later on, a duality theory for 0-1 polarity lattices was developed in [5].

Categorical aspects in topological FCA have been studied in [1], especially for the metric case, while uniform contexts have been investigated in [8].

Bounded lattices, i.e., 0-1-lattices, have already been described by some topological representations by Stone [10], Priestley [7], and Urquhart [11]. Each of these representations is given via special topological spaces: compact totally disconnected spaces for Boolean lattices; spectral spaces or compact totally order-disconnected spaces for bounded distributive lattices; and the so-called L-spaces for arbitrary bounded lattices. In fact, when representing 0-1-lattices by standard topological contexts, one can recover all the above representations within the so-called double arrow space, i.e., a structure within the non-incidence of some topological context $\mathbb{K}^{\mathcal{T}}$.

Contextual Logic has been introduced by R. Wille as a logical extension of FCA "... with the aim to support knowledge representation and knowledge processing [...]. It is grounded on the traditional philosophical understanding of logic as the doctrine of the forms of thinking [...]" ([13].) While a logical extension of a formal context is quite straightforward, the same extension on concept lattices (which could be understood as the pattern counterpart of a context) is no longer straightforward, because of the semantics of the negation operator.

For introducing negations, two approaches were considered: a generalization of formal concepts to semiconcepts and protoconcepts, introducing the algebra of semiconcepts and the algebra of protoconcepts, respectively, which then leads to the notion of double Boolean algebra [6].

The main results of this paper are based on the representation theorem for double Boolean algebras from [6]. We define the notion of a DB-topological context in order to represent every double Boolean algebra as an algebra of clopen protoconcepts of a DB-topological context and show how this representation can be extended to a categorical duality.

2. Formal Concept Analysis

The basic structure FCA is using is a formal context. Using concept forming operators, formal concepts are extracted described as maximal patterns of incidences of a given binary relation. Concepts are ordered by the subconcept-superconcept relation and they form a conceptual hierarchy, i.e., a complete lattice which contains all knowledge patterns we can extract from a formal context. In Contextual Logic, a formal context is the mathematical structure in which the semantics of logical operators is declared, while for Conceptual Logic the same role is played by a concept lattice. Here we recall only some basic definitions. For more, we refer to [2].

Definition 2.1. A formal context is a triple $\mathbb{K} = (G, M, I)$, where G and M are sets and $I \subseteq G \times M$ is a binary relation. The set G is called set of objects, M is the set of attributes and I is called incidence relation.

For sets $A \subseteq G$ and $B \subseteq M$, we define concept forming operators by
$A' = \{m \in M \mid gIm \text{ for all } g \in A\}$ and $B' = \{g \in G \mid gIm \text{ for all } m \in B\}$.
These operators form a Galois connection on the power sets of G and M, respectively.

Definition 2.2. A formal concept of the context $\mathbb{K} = (G, M, I)$ is a pair (A, B) with $A \subseteq G$, $B \subseteq M$ and $A' = B, B' = A$. The set A is called extent and B is called the intent of the concept (A, B). The set of all concepts of $\mathbb{K}$ is denoted by $B(\mathbb{K})$.

On the set $B(\mathbb{K})$ of concepts we define the subconcept-superconcept relation by $(A_1, B_1) \le (A_2, B_2) \Leftrightarrow A_1 \subseteq A_2 (\Leftrightarrow B_1 \supseteq B_2)$.

Theorem 2.3 (Basic Theorem on Concept Lattices). *Let* $\mathbb{K} := (G, M, I)$ *be a formal context. The concept lattice* $(B(\mathbb{K}), \leq)$ *is a complete lattice in which infimum and supremum are given by:*

$$\bigwedge_{t\in T}(A_t, B_t) = \Big(\bigcap_{t\in T} A_t, \big(\bigcup_{t\in T} B_t\big)''\Big),$$

$$\bigvee_{t\in T}(A_t, B_t) = \Big(\big(\bigcup_{t\in T} A_t\big)'', \bigcap_{t\in T} B_t\Big).$$

A complete lattice V *is isomorphic to* $B(\mathbb{K})$ *if and only if there are mappings* $\tilde{\gamma} : G \to V$ *and* $\tilde{\mu} : M \to V$ *such that* $\tilde{\gamma}(G)$ *is supremum-dense in* V, $\tilde{\mu}(M)$ *is infimum-dense in* V *and* gIm *is equivalent to* $\tilde{\gamma}g \leq \tilde{\mu}m$ *for all* $g \in G$ *and all* $m \in M$. *In particular,* $V \simeq B(V, V, \leq)$.

Every object and every attribute can be recovered in the concept lattice of the given context. For an object $g \in G$, we write g' instead of $\{g\}'$ for the object intent $\{m \in M \mid gIm\}$ of the object g. Correspondingly, m' stands for the attribute extent $\{g \in G \mid gIm\}$ of the attribute m. Using the symbols from the Basic Theorem, we write γg for the object concept (g'', g') and μm for the attribute concept (m', m'').

Definition 2.4. A context (G, M, I) is called clarified, if for any objects $g, h \in G$ from $g' = h'$ always follows that $g = h$ and, correspondingly, $m' = n'$ implies $m = n$ for all $m, n \in M$.

Definition 2.5. A clarified context (G, M, I) is called row reduced, if every object concept is $\bigvee$-irreducible, and column reduced, if every attribute concept is $\bigwedge$-irreducible. A context which is row reduced and column reduced is called reduced.

If (G, M, I) is a context, $g \in G$ an object, and $m \in M$ an attribute, we write

$$g \swarrow m :\Leftrightarrow \begin{cases} g \not{I} m \quad \text{and} \\ \text{if} \quad g' \subseteq h' \quad \text{and} \quad g' \neq h', \quad \text{then} \quad hIm; \end{cases}$$

$$g \nearrow m :\Leftrightarrow \begin{cases} g \not{I} m \quad \text{and} \\ \text{if} \quad m' \subseteq n' \quad \text{and} \quad m' \neq n', \quad \text{then} \quad gIn; \end{cases}$$

$$g \swarrow\!\!\!\nearrow m :\Leftrightarrow g \swarrow m \quad \text{and} \quad g \nearrow m.$$

Thus, $g \swarrow m$ if and only if g' is maximal among all object intents which do not contain m. In other words: $g \swarrow m$ holds if and only if g does not have the attribute m, but m is contained in the intent of every proper subconcept of γg.

The relation $\swarrow\!\!\!\nearrow \subseteq G \times M$ is called the double-arrow space of the context (G, M, I).

3. Generalization of Concepts: Semi- and Protoconcepts

As stated in [13] and [14], the research in the field of Contextual Logic is structured in three major themes: Contextual Concept Logic, Contextual Judgement Logic, and Contextual Conclusion Logic. In order to develop a satisfactory theory of Contextual Concept Logic, there was necessary to introduce a suitable notion of negation and to overcome some difficulties due to the fact that the complement of an extent (intent) is generally not an extent (intent). The solution was a generalization of concepts to semiconcepts and protoconcepts. While formal concepts are structured in a complete lattice, protoconcepts will give rise to a structure called double Boolean algebra.

In the following, we give the basic facts and results about algebras of protoconcepts and double Boolean algebras, for a complete information see [6] and [13].

Definition 3.1. Let $\mathbb{K} := (G, M, I)$ be a formal context. A semiconcept of $\mathbb{K}$ is defined as a pair (A, B) with $A \subseteq G$ and $B \subseteq M$ such that $A' = B$ *or* $B' = A$. A pair (A, B) is called a **protoconcept** if $A'' = B'$ (or equivalently $A' = B''$). We denote by $\mathfrak{H}(\mathbb{K})$ the set of all semiconcepts of a formal context $\mathbb{K}$. The set of all protoconcepts of $\mathbb{K}$ is denoted by $\mathfrak{P}(\mathbb{K})$. Obviously, every semiconcept is a protoconcept, hence $\mathfrak{H}(\mathbb{K}) \subseteq \mathfrak{P}(\mathbb{K})$.

In the following, we consider mainly sets of protoconcepts, pointing out the differences when semiconcepts are involved.

The set $\mathfrak{P}(\mathbb{K})$ of all protoconcepts of $\mathbb{K}$ carries a natural order $\sqsubseteq$ which is defined by $(A_1, B_1) \sqsubseteq (A_2, B_2) :\Leftrightarrow A_1 \subseteq A_2$ and $B_1 \supseteq B_2$. This order does not generally yield a lattice structure on $\mathfrak{P}(\mathbb{K})$. However, there are natural operations on $\mathfrak{P}(\mathbb{K})$ which can be defined as follows:

$$\begin{aligned}
(A_1, B_1) \sqcap (A_2, B_2) &:= (A_1 \cap A_2, (A_1 \cap A_2)') \\
(A_1, B_1) \sqcup (A_2, B_2) &:= ((B_1 \cap B_2)', B_1 \cap B_2) \\
\neg(A, B) &:= (G \smallsetminus A, (G \smallsetminus A)') \\
\lrcorner(A, B) &:= ((M \smallsetminus B)', M \smallsetminus B) \\
\bot &:= (\varnothing, M) \\
\top &:= (G, \varnothing).
\end{aligned}$$

The set $\mathfrak{P}(\mathbb{K})$ together with the operations $\sqcap, \sqcup, \neg, \lrcorner, \bot$, and $\top$ is called **the algebra of protoconcepts** of $\mathbb{K}$ and is denoted by $\underline{\mathfrak{P}}(\mathbb{K})$. The operations are named **meet, join, negation, opposition, nothing** and **all**. The corresponding structure in the case of semiconcepts is the algebra of semiconcepts which will be denoted by $\underline{\mathfrak{H}}(\mathbb{K})$.

For an arbitrary element x in $\mathfrak{P}(\mathbb{K})$ we denote by $x_\sqcup := x \sqcup x$ and by $x_\sqcap := x \sqcap x$. Let $\mathfrak{P}(\mathbb{K})_\sqcap := \{(A, A') \mid A \subseteq G\} = \mathfrak{H}(\mathbb{K})_\sqcap$ and $\mathfrak{P}(\mathbb{K})_\sqcup := \{(B', B) \mid B \subseteq M\} = \mathfrak{H}(\mathbb{K})_\sqcup$. Until now, there is no obvious difference between algebras of semiconcepts and algebras of protoconcepts. But if we have a closer look to their structure, from the definition of semiconcepts, we have $\mathfrak{H}(\mathbb{K}) = \mathfrak{H}(\mathbb{K})_\sqcap \cup \mathfrak{H}(\mathbb{K})_\sqcup$, relation which does not hold for an algebra of protoconcepts. The main difference between algebras of semiconcepts and algebras of protoconcepts consists exactly in the existence of some "non-Boolean" elements whose influence on the structure and behaviour of such algebras is described in [6] and [13].

The formal concepts can be recovered in the meet of the two Boolean parts of $\mathfrak{P}(\mathbb{K})$, i.e., $\mathfrak{B}(\mathbb{K}) := \mathfrak{P}(\mathbb{K})_\sqcap \cap \mathfrak{P}(\mathbb{K})_\sqcup = \mathfrak{H}(\mathbb{K})_\sqcap \cap \mathfrak{H}(\mathbb{K})_\sqcup$, where $\underline{\mathfrak{B}}(\mathbb{K})$ is the complete lattice of the (formal) concepts of $\mathbb{K}$. The term "non-Boolean" elements, respectively

Boolean part of an algebra of protoconcepts (semiconcepts), is justified by the following additional operations defined as

$$x \sqcup\!\!\!\sqcup y := \neg(\neg x \sqcap \neg y) \ \text{ and } \ x \sqcap\!\!\!\sqcap y := \lrcorner(\lrcorner x \sqcup \lrcorner y),$$

$$\top := \neg\bot \ \text{ and } \ \bot := \lrcorner\top.$$

Obviously, $\mathfrak{P}(\mathbb{K})_\sqcap$ together with the restrictions of the operations $\sqcap, \sqcup\!\!\!\sqcup, \neg, \bot, \top$ is a Boolean algebra denoted $\underline{\mathfrak{P}}(\mathbb{K})_\sqcap$, which is isomorphic to the Boolean algebra of all subsets of G. Dually, $\mathfrak{P}(\mathbb{K})_\sqcup$ together with the restrictions of the operations $\sqcap\!\!\!\sqcap, \sqcup, \lrcorner, \bot, \top$ is a Boolean algebra denoted by $\underline{\mathfrak{P}}(\mathbb{K})_\sqcup$, and which is antiisomorphic to the Boolean algebra of all subsets of M.

Proposition 3.2. *The following equations are valid in* $\underline{\mathfrak{P}}(\mathbb{K})$*:*

1a) $(x \sqcap x) \sqcap y = x \sqcap y$	*1b)* $(x \sqcup x) \sqcup y = x \sqcup y$
2a) $x \sqcap y = y \sqcap x$	*2b)* $x \sqcup y = y \sqcup x$
3a) $x \sqcap (y \sqcap z) = (x \sqcap y) \sqcap z$	*3b)* $x \sqcup (y \sqcup z) = (x \sqcup y) \sqcup z$
4a) $x \sqcap (x \sqcup y) = x \sqcap x$	*4b)* $x \sqcup (x \sqcap y) = x \sqcup x$
5a) $x \sqcap (x \sqcup\!\!\!\sqcup y) = x \sqcap x$	*5b)* $x \sqcup (x \sqcap\!\!\!\sqcap y) = x \sqcup y$
6a) $x \sqcap (x \sqcup\!\!\!\sqcup y) = (x \sqcap y) \sqcup\!\!\!\sqcup (x \sqcap z)$	*6b)* $x \sqcup (y \sqcap\!\!\!\sqcap z) = (x \sqcup y) \sqcap\!\!\!\sqcap (x \sqcup z)$
7a) $\neg\neg(x \sqcap y) = x \sqcap y$	*7b)* $\lrcorner\lrcorner(x \sqcup y) = x \sqcup y$
8a) $\neg(x \sqcap x) = \neg x$	*8b)* $\lrcorner(x \sqcup x) = \lrcorner x$
9a) $x \sqcap \neg x = \bot$	*9b)* $x \sqcup \lrcorner x = \top$
10a) $\neg\bot = \top \sqcap \top$	*10b)* $\lrcorner\top = \bot \sqcup \bot$
11a) $\neg\top = \bot$	*11b)* $\lrcorner\bot = \top$

12) $(x \sqcap x) \sqcup (x \sqcap x) = (x \sqcup x) \sqcap (x \sqcup x)$.

Furthermore, the following condition holds in $\underline{\mathfrak{H}}(\mathbb{K})$*:*

13) $x = x \sqcap x$ *or* $x = x \sqcup x$.

4. Double Boolean Algebras

Double Boolean algebras are algebraic structures $\underline{D} := (D, \sqcap, \sqcup, \neg, \lrcorner, \bot, \top)$ of type $(2, 2, 1, 1, 0, 0)$ satisfying the equations 1a) to 11a), 1b) to 11b) and 12) of the precedent Proposition. If a double Boolean algebra satisfies also condition 13) of Proposition 3.3.1, it is called **pure**. An algebraic structure of the type $(2, 2, 1, 1, 0, 0)$ in which only equations 1a) to 11a) and 1b) to 11b) are valid is called **weak** double Boolean algebra. As one can easily see, algebras of protoconcepts are double Boolean algebras, while algebras of semiconcepts are pure double Boolean algebras.

We can define a quasiorder on a (weak) double Boolean algebra similar to that defined on the algebra of protoconcepts, namely

$$x \sqsubseteq y :\Leftrightarrow x \sqcap y = x \sqcap x \text{ and } x \sqcup y = y \sqcup y.$$

Lemma 4.1. [13] *In a (weak) double Boolean algebra the following conditions hold:*

(1) $x \sqcap y \sqsubseteq x \sqsubseteq x \sqcup y$,

(2) the mapping $x \mapsto x \sqcap y$ *preserves* $\sqsubseteq$ *and* $\sqcap$,

(3) the mapping $x \mapsto x \sqcup y$ *preserves* $\sqsubseteq$ *and* $\sqcup$.

For a (weak) double Boolean algebra $\underline{D} := (D, \sqcap, \sqcup, \neg, \lrcorner, \bot, \top)$ further operations are defined as in the preceding section by

$$x \Cup y := \neg(\neg x \sqcap y) \text{ and } x \Cap y := \lrcorner(\lrcorner x \sqcup \lrcorner y),$$

$$\top := \neg\bot \text{ and } \underline{\bot} := \lrcorner\top.$$

In addition, let $x_\sqcap := x \sqcap x$ and $x_\sqcup := x \sqcup x$. Lead again by the preceding section we define $D_\sqcap := \{x_\sqcap \mid x \in D\}$ and $D_\sqcup := \{x_\sqcup \mid x \in D\}$. Now, (weak) double Boolean algebras can be characterized as special ordered structures.

Proposition 4.2. [6] *Let $\underline{D} := (D, \sqcap, \sqcup, \neg, \lrcorner, \bot, \top)$ be a (weak) double Boolean algebra. Then the following conditions are satisfied.*

(1) $(D, \sqsubseteq)$ is a quasi-ordered set.

(2) $\underline{D_\sqcap} := (D_\sqcap, \sqcap, \Cup, \neg, \bot, \top)$ is a Boolean algebra whose order relation is the restriction of $\sqsubseteq$ to $D_\sqcap$.

(3) $\underline{D_\sqcup} := (D_\sqcup, \Cap, \sqcup, \lrcorner, \bot, \top)$ is a Boolean algebra whose order relation is the restriction of $\sqsubseteq$ to $D_\sqcup$.

(4) $y \sqsubseteq x_\sqcap \Leftrightarrow y \sqsubseteq x$ for $x \in D$ and $y \in D_\sqcap$.

(5) $x_\sqcup \sqsubseteq y \Leftrightarrow x \sqsubseteq y$ for $x \in D$ and $y \in D_\sqcup$.

(6) $x \sqsubseteq y \Leftrightarrow x_\sqcap \sqsubseteq y_\sqcap$ and $x_\sqcup \sqsubseteq y_\sqcup$ for $x, y \in D$.

How close weak double Boolean algebras are to double Boolean algebras is made clear by the following Proposition.

Proposition 4.3. [6] *Let $\underline{D} := (D, \sqcap, \sqcup, \neg, \lrcorner, \bot, \top)$ be a weak double Boolean algebra with $D = D_\sqcap \cup D_\sqcup$. Then $\underline{\sqsubset} := \{(x,x) \mid x \in D\} \cup \{(x_{\sqcap\sqcup}, x_{\sqcup\sqcap}) \mid x \in D\} \cup \{(x_{\sqcup\sqcap}, x_{\sqcap\sqcup}) \mid x \in D\}$ is a congruence relation of $\underline{D}$.*

5. Filters of Double Boolean Algebras

Let $\underline{D}$ be a double Boolean algebra. Our task is to give a topological representation of a double Boolean algebra as an algebra of protoconcepts of a suitable topological context. First, we have to remark that algebras of protoconcepts are ordered structures, so we will consider only **regular** double Boolean algebras, i.e., double Boolean algebras for which $\sqsubseteq$ is an order. There is no restriction of generality since every double Boolean algebra can be regularized by a suitable factorization.

Definition 5.1. Let $\underline{L}$ be an ordered set, F be a filter of $\underline{L}$, and I an ideal of $\underline{L}$. We say that F is I-maximal if F is a maximal filter which is disjoint from I. Dually, F-maximal ideals are defined. We obtain the following sets:

$\mathfrak{F}_0(\underline{L}) := \{F \subseteq L \mid \exists I \in \mathfrak{I}(\underline{L}) : F \text{ is } I \text{ maximal}\}$
$\mathfrak{I}_0(\underline{L}) := \{I \subseteq L \mid \exists I \in \mathfrak{F}(\underline{L}) : I \text{ is } F-\text{maximal}\}$
$\mathfrak{M}(\underline{L}) := \{(F, I) \in \mathfrak{F}_0(\underline{L}) \times \mathfrak{I}_0(\underline{L}) \mid F \text{ is } I-\text{maximal and } I \text{ is } F-\text{maximal}\}$.

The elements of $\mathfrak{M}(\underline{L})$ are called maximal filter-ideal pairs.

Definition 5.2. A filter of a double Boolean algebra $\underline{D}$ is defined to be to be a subset F of D satisfying

1. $x \in F$ and $x \sqsubseteq y$ in D imply $y \in F$;
2. If $x \in F$ and $y \in F$ then $x \sqcap y \in F$.

An ideal of $\underline{D}$ is defined dually. A subset F_0 is called a base of a filter F if $F = \{y \in D \mid x \sqsubseteq y$ *for some* $x \in F_0\}$, and we write $F = \uparrow F_0$. A base of an ideal is defined dually. If $\underline{D}$ is a regular double Boolean algebra, we denote by $\mathfrak{F}(\underline{D})$ the set of all filters of $\underline{D}$ and by $\mathfrak{I}(\underline{D})$ the set of all ideals of $\underline{D}$. A filter F of $\underline{D}$ is called prime if $F \cap \underline{D}_\sqcap$ is a prime filter in $D_\sqcap$; a prime ideal is defined dually. The prime filters and prime ideals are collected in $\mathfrak{F}_p(\underline{D})$ and $\mathfrak{I}_p(\underline{D})$, respectively.

Lemma 5.3. *The following holds true:*

1. *F is a filter of a regular double Boolean algebra $\underline{D}$ if and only if the characteristic function is an order homomorphism preserving $\sqcap$.*
2. *The filter F is prime if and only if the characteristic function of F preservers $\neg$ too.*

Proof. 1. Let $F \in \mathfrak{F}(D)$. We define the map $\phi : \underline{D} \to \underline{2}$ by

$$\phi(x) = \begin{cases} 1, & x \in F \\ 0, & x \notin F. \end{cases}$$

Let $x, y \in F$, then $x \sqcap y \in F$ and so $\phi(x \sqcap y) = \phi(x) \wedge \phi(y) = 1$. For $x \in F$ and $y \notin F$ or $x \notin F$ and $y \in F$ we have that $x \sqcap y \notin F$, hence $\phi(x \sqcap y) = \phi(x) \wedge \phi(y) = 0$.

Now, for $x, y \notin F$, we have that $x \sqcap y \notin F$ and, since $\top \in F$, we obtain that ϕ is a $\sqcap$-homomorphism. Define $F := \phi^{-1}(1)$, then $\phi(x) = \phi(y) = 1$ for $x, y \in F$ and so $\phi(x \sqcap y) = 1$, implying that $x \sqcap y \in F$. Let now $x \in F$ and $y \in D$ with $x \sqsubseteq y$. Then, by definition, $\phi(x) = 1$ and

$$x \sqsubseteq y \Leftrightarrow x \sqcap y = x \sqcap x \text{ and } x \sqcup y = y \sqcup y;$$

hence $\phi(x \sqcap y) = \phi(x) \wedge \phi(y) = \phi(x_\sqcap)$. Since $x \in F$, we have $\phi(x_\sqcap) = 1$. By Lemma 4.1, from $x \sqsubseteq y$ follows $x \sqcap y \sqsubseteq y \sqcap y$, and since ϕ was an order homomorphism, we have that $\phi(y_\sqcap) = 1$, i.e., $y_\sqcap \in F$. We conclude that $F_\sqcap$ is a filter of $\underline{D}_\sqcap$, hence F is a filter of $\underline{D}$ too.

2. Let F be a prime filter and $\phi : \underline{D} \to \underline{2}$ defined as above. As one can easily see, the only case where a proof deserves to be made is $x, y \notin F$. Consider $x, y \notin F$ arbitrary chosen, then $x_\sqcap, y_\sqcap \notin F$. But $F \cap \underline{D}_\sqcap$ is a prime filter of the Boolean algebra $\underline{D}_\sqcap$, hence $\neg x_\sqcap, \neg y_\sqcap \in F \cap \underline{D}_\sqcap$. It follows that $\phi(\neg x_\sqcap) = \phi(\neg x) = \neg\phi(x)$. For $x \in F$, we obviously have $\phi(\neg x) = \neg\phi(x)$; hence ϕ is a $\neg$ - homomorphism, since the same holds for $x \notin F$.

□

Remark 5.4. If $F \in \mathfrak{F}_p(\underline{D})$, then $F \cap \underline{D}_\sqcap$ is a prime filter of the Boolean algebra $\underline{D}_\sqcap$ and so there is an ideal I of $\underline{D}_\sqcap$, so that $F \cap \underline{D}_\sqcap$ is I-maximal and dually for prime ideals.

Let now F be a filter in $\underline{D}$ and $x \in F$. Then $x \sqcap x =: x_\sqcap \in F$ and so $F \cap \underline{D}_\sqcap \neq \emptyset$. For any $x_\sqcap \in F$ we have that $x \in F$ since $x \sqcap x \sqsubseteq x$. It follows that $x \in F \Leftrightarrow x_\sqcap \in F$. Hence $F \cap \underline{D}_\sqcap \in \mathfrak{F}(\underline{D}_\sqcap)$. In fact, $x_\sqcap \in F$ and $x_\sqcap \sqsubseteq y_\sqcap$ implies $y_\sqcap \in F$ and so $y_\sqcap \in F \cap \underline{D}_\sqcap$.

An easy calculation shows the validity of the following Lemma:

Lemma 5.5. *Let L and M be two regular double Boolean algebras and $f : L \to M$ an onto homomorphism. Then*

a) $F \in \mathfrak{F}(L) \Rightarrow f(F) \in \mathfrak{F}(M)$, and $I \in \mathfrak{I}(L) \Rightarrow f(I) \in \mathfrak{I}(M)$.

b) $E \in \mathfrak{F}_p(M) \Rightarrow f^{-1}(E) \in \mathfrak{F}_p(L)$ and $H \in \mathfrak{I}_p(M) \Rightarrow f^{-1}(H) \in \mathfrak{I}_p(L)$.

c) $(E, H) \in \mathfrak{M}(M) \Rightarrow (f^{-1}(E), f^{-1}(H)) \in \mathfrak{M}(L)$.

We conclude this section with a basic result from [6].

Lemma 5.6. *Let F be a filter of a double Boolean algebra $\underline{D}$.*

1. *$F \cap D_\sqcap$ and $F \cap D_\sqcup$ are filters of the Boolean algebra $\underline{D}_\sqcap$ and $\underline{D}_\sqcup$, respectively;*
2. *Each filter of the Boolean algebra $\underline{D}_\sqcap$ is a base of some filter of $\underline{D}$; in particular, $F \cap D_\sqcap$ is a base of F.*

6. Topological Representation

Double Boolean algebras play a substantial role in the development of the Concept Logic, a role which is in a certain way similar to that played by Boolean algebras in the classical Logic. In the following, we develop a topological representation theory for regular double Boolean algebras, i.e., double Boolean algebras for which the quasiorder $\sqsubseteq$ is an order relation.

A topological space is denoted by $(X, \mathcal{T})$ where $\mathcal{T}$ is the set of all closed subsets of X. Consider the context $\mathbb{K} := (G, M, I)$ and let $\mathcal{T}$ be a topology on G and τ be a topology on M. A clopen protoconcept is a pair $(A, B) \in \underline{\mathfrak{P}}(\mathbb{K})$ with $A \subseteq G$ clopen, and $B \subseteq M$ clopen too. We denote the set of all clopen protoconcepts by $\mathfrak{P}^{co}((G, \mathcal{T}), (M, \tau), I)$.

Definition 6.1. $\mathbb{K}^{DB} := ((G, \mathcal{T}), (M, \tau), I)$ is called a **DB-topological context** if:

(i) $(G, \mathcal{T})$ and (M, τ) are topological spaces and $I \subseteq G \times M$.

(ii) If $A \subseteq G$ is a clopen set then $A' \subseteq M$ is clopen too, and the same holds for clopen sets in M.

(iii) A subbasis for the closed and for the open sets in G is given by the extents of clopen protoconcepts of $\mathbb{K}^{DB}$ and, dually, a subbasis for the closed and for the open subsets of M is given by the intents of clopen protoconcepts of $\mathbb{K}^{DB}$.

Remark 6.2.

1. The set $\mathfrak{P}^{co}(\mathbb{K}^{DB})$ inherits the ordering from $\underline{\mathfrak{P}}(\mathbb{K}^{DB})$. We shall denote this ordered set by $\underline{\mathfrak{P}}^{co}(\mathbb{K}^{DB})$.
2. Using the same idea as in [3], we are able to represent the 0-1-lattice $\underline{D}_\sqcap \cap \underline{D}_\sqcup$ by the 0-1-lattice of clopen concepts of a suitable DB-topological context.

Remark 6.3. $\underline{\mathfrak{P}}^{co}(\mathbb{K}^{DB})$ is a subalgebra of $\underline{\mathfrak{P}}(\mathbb{K}^{DB})$. Moreover, it becomes now evident that it is necessary to consider clopen protoconcepts since the negation of a closed protoconcept would not be any longer a closed protoconcept.

Proposition 6.4. *For every DB-topological context $\mathbb{K}^{DB}$, the ordered set of clopen protoconcepts $\underline{\mathfrak{P}}^{co}(\mathbb{K}^{DB})$ is a regular double Boolean algebra.*

Proof. Remember that if (A,B) and (C,D) are clopen protoconcepts, then

$$\begin{aligned}(A,B)\sqcap(C,D)&=(A\cap C,(A\cap C)')\\(A,B)\sqcup(C,D)&=((B\cap D)',B\cap D)\\\neg(A,B)&=(G\setminus A,(G\setminus A)')\\\lrcorner(A,B)&=((M\setminus B)',M\setminus B)\\\bot&=(\varnothing,M)\\\top&=(G,\varnothing).\end{aligned}$$

Since the involved sets are all clopen, it follows that the restrictions of these operations to $\underline{\mathfrak{P}}^{co}(\mathbb{K}^{DB})$, i.e., meet, join, negation, opposition, nothing and all are well-defined. Since the ordering on $\underline{\mathfrak{P}}^{co}(\mathbb{K}^{DB})$ is that inherited from $\mathfrak{P}(\mathbb{K}^{DB})$, it follows that $\underline{\mathfrak{P}}^{co}(\mathbb{K}^{DB})$ is a regular double Boolean algebra. □

For a regular double Boolean algebra $\underline{D}$, we define the standard context of $\underline{D}$ as $\mathbb{K}(\underline{D}) := (\mathfrak{F}_p(\underline{D}),\mathfrak{I}_p(\underline{D}),\Delta)$, where $F\Delta I :\Leftrightarrow F\cap I\neq\varnothing$. On $\mathfrak{F}_p(\underline{D})$ consider the topology generated by the subbasis $\{F_x \mid x\in\underline{D}\}$ with $F_x := \{F\in\mathfrak{F}_p(\underline{D}) \mid x\in F\}$. Dually, on $\mathfrak{I}_p(\underline{D})$ consider the topology generated by the subbasis $\{I_x \mid x\in\underline{D}\}$ with $I_x := \{I\in\mathfrak{I}_p(\underline{D}) \mid x\in I\}$. We denote the context $\mathbb{K}(\underline{D})$ with the above topologies by $\mathbb{K}^{DB}(\underline{D})$ and we will prove that $\mathbb{K}^{DB}(\underline{D})$ is a DB-topological context.

Lemma 6.5. [6] *The derivations in* $\mathbb{K}^{DB}(\underline{D})$ *yield:*

1. $F_x' = I_x = I_{x_\sqcup}$ *for all* $x\in D_\sqcap$.
2. $I_y' = F_y = F_{y_\sqcap}$ *for all* $y\in D_\sqcup$.
3. $F_z' = I_{z_\sqcap} = I_{z_{\sqcap\sqcup}}$ *and* $I_z' = F_{z_\sqcup} = F_{z_{\sqcup\sqcap}}$ *for all* $z\in D\setminus(D_\sqcap\cap D_\sqcup)$.

The following result is true:

Theorem 6.6. *For every regular double Boolean algebra* $\underline{D}$*, the context* $\mathbb{K}^{DB}(\underline{D})$ *is a topological context.*

Proof. We first remark that, by Lemma 6.5, the pair (F_x,I_x) is a protoconcept of $\underline{\mathfrak{P}}^{co}(\mathbb{K}^{DB}(\underline{D}))$ for every $x\in\underline{D}$: For any $x\in\underline{D}_\sqcap$ we have $F_x' = I_x = I_{x_\sqcup}$ and so $F_x'' = I_{x_\sqcup}' = F_{x_\sqcup} = F_{x_{\sqcup\sqcap}}$ and $I_x' = I_{x_\sqcup}' = F_{x_\sqcup} = F_{x_{\sqcup\sqcap}}$. The same holds for $x\in\underline{D}_\sqcup$ and for the "non-Boolean" elements in $D\setminus(\underline{D}_\sqcap\cap\underline{D}_\sqcup)$. Moreover, the complement of an element of the given subbasis, $cF_x = \mathfrak{F}_p(\underline{D})\setminus F_x = \{F\in\mathfrak{F}_p(\underline{D}) \mid x\notin F\} = F_{\neg x}$, is again in that family, so every element of the subbasis is clopen. We want to prove now the second condition, namely that for a clopen $C\subseteq\mathfrak{F}_p(\underline{D})$, its derivation, C', is clopen too.

Let $C\subseteq\mathfrak{F}_p(\underline{D})$ be a clopen set, then C is closed and

$$C=\bigcap_{j\in J}\Big(\bigcup_{a\in A_j}F_a\Big)$$

where $J\neq\varnothing$ is an arbitrary index-set and $A_j\subseteq D$ are finite sets for every $j\in J$. Its complement

$$cC=\bigcup_{j\in J}\Big(\bigcap_{a\in A_j}cF_a\Big)=\bigcup_{j\in J}\Big(\bigcap_{a\in A_j}F_{\neg a}\Big)$$

can be expressed in terms of the given subbasis too, proving that the family $\{F_x \mid x \in \underline{D}\}$ is also a subbasis for the open subsets of $\mathfrak{F}_p(\underline{D})$. Hence an arbitrary closed set $C \subseteq \mathfrak{F}_p(\underline{D})$ will be expressed as $C = \bigcap_{j\in J}(\bigcup_{a\in A_j} F_a)$ and we denote the set of all finite subsets of J with $\mathcal{E}_J$. Then, for every $E \in \mathcal{E}_J$, the order ideal $\bigcap_{j\in E} \downarrow A_j$ has finitely many maximal elements. We denote the set of these maximal elements by max $\bigcap_{j\in E} \downarrow A_j$. Thus, the element

$$s_E := \bigsqcup \max \bigcap_{j\in E} \downarrow A_j$$

is well-defined for every $E \in \mathcal{E}_J$ and belongs to $\underline{D}$. Define $B_J := \{s_E \mid E \in \mathcal{E}_J\}$. We claim that

$$C' = \bigcup_{a\in B_J} I_a.$$

To prove this, let $a \in B_J$ and $I \in I_a$. It follows that there is an $E \in \mathcal{E}_J$ with $s_E = a$. The filter F belongs to C if and only if $F \cap A_j \neq \varnothing$ for all $j \in J$, and so $F \cap A_j \neq \varnothing$ for $j \in E$, concluding that $s_E \in F$. This statement is equivalent to $F\Delta I$ for all $F \in C$; hence $I \in C'$.

Consider $I \in \mathfrak{I}_p(\underline{D})$ with $a \notin I$ for all $a \in B_J$. For all $E \in \mathcal{E}_J$, there is a map

$$f_E : E \to \bigcup_{j\in J} A_j$$

with $f_E(j) \in A_j$ and $\sqcap f_E(E) \notin I$, hence $s_E \in (\uparrow \sqcap f_E(E)) \setminus I$. (Choose for $f_E(j)$ one of the maximal elements of $\downarrow A_j$ which appears in s_E. Then $s_E \in \uparrow \sqcap f_E(E)$, but $s_E \notin I$.)

Rado's Selection Theorem yields the existence of a map $f : J \to \bigcup_{j\in J} A_j$ defined by $f(j) \in A_j$ for every $j \in J$, so that, for all $E \in \mathcal{E}_J$, there is a filter $F \in \mathcal{E}_J$ with $E \subseteq F$ and $f_{|E} = f_{F|_E}$. It follows that the filter generated by $\{f(j) \mid j \in J\}$ contains B_J and is disjoint from I. Zorn's Lemma guarantees the existence of an I-maximal filter containing the given one and so $I \notin C'$. We just have proved the openness of the set C' in $\mathfrak{I}_p(\underline{D})$.

The set C is clopen and the family $\{F_x \mid x \in \underline{D}\}$ is a subbasis for the open sets in $\mathfrak{F}_p(\underline{D})$; hence $C = \bigcup_{k\in K}(\bigcap_{b\in B_k} F_b)$, where $K \neq \varnothing$ is an index-set and B_k are finite subsets of D for every $k \in K$. Since

$$\bigcap_{b\in B_k} F_b = \{F \in \mathfrak{F}_p(\underline{D}) \mid \forall b \in B_k : b \in F\} = F_{B_k},$$

we have that $C = \bigcup_{k\in K} F_{B_k} = \bigcup_{k\in K} F_{B_{k\sqcap}}$, where $B_{k\sqcap} := \{b_\sqcap \mid b \in B_k\}$.

The following holds:

$$C' = \bigcap_{k\in K} F'_{B_{k\sqcap}} = \bigcap_{k\in K} I_{B_{k\sqcap}} = \bigcap_{k\in K} I_{B_{k\sqcap\sqcup}} = \bigcap_{k\in K} \bigcap_{b\in B_{k\sqcap\sqcup}} I_b$$

concluding that C' is also closed and since it was also open, it is clopen. □

Now, we are able to give a representation theorem for regular double Boolean algebras in terms of Formal Concept Analysis.

Theorem 6.7. *Let $\underline{D}$ be a regular double Boolean algebra. Then*

$$\iota : \underline{D} \to \underline{\mathfrak{P}}^{co}(\mathbb{K}^{DB}(\underline{D})), \quad a \mapsto (F_a, I_a)$$

is an isomorphism.

Proof. By Lemma 6.5, (F_x, I_x) is a protoconcept of $\mathbb{K}(\underline{D})$ for all $x \in \underline{D}$. For $x \not\sqsubseteq y$ in $\underline{D}_\sqcap$, there is always an $F \in \mathfrak{F}_p(\underline{D})$ with $x \in F$ but $y \notin F$; hence $F_x \neq F_y$ and so $(F_x, I_x) \neq (F_y, I_y)$. Such inequality can be obtained dually for $y \not\sqsubseteq x$ in $\underline{D}_\sqcup$. If $x \not\sqsubseteq y$ in $\underline{D}$ with $x \notin D_\sqcup$ and $y \notin D_\sqcap$ then, because $\uparrow x := \{y \in D \mid x \sqsubseteq y\}$ is a filter of $\underline{D}_\sqcup$ by Lemma 4.1, there exists an $I \in \mathfrak{I}_p(\underline{D})$ with $x \in I$ but $y \notin I$; hence we have $(F_x, I_x) \neq (F_y, I_y)$ also in this case and its dual.

Using Theorem 2 from [13], we deduce that the map ι is a homomorphism, since $F_{x \sqcap y} = F_x \cap F_y$, $I_{x \sqcup y} = I_x \cap I_y$, $F_{\neg x} = \mathfrak{F}_p(\underline{D}) \setminus F_x$ and $I_{\lrcorner x} = \mathfrak{I}_p(\underline{D}) \setminus I_x$. These equalities result from the following equivalences and their duals: $F \in F_{x \sqcap y} \Leftrightarrow x \sqcap y \in F \Leftrightarrow x, y \in F \Leftrightarrow F \in F_x \cap F_y$ and $F \in F_{\neg x} \Leftrightarrow \neg x \in F \Leftrightarrow \neg(x \sqcap x) \in F \Leftrightarrow x \sqcap x \notin F \Leftrightarrow x \notin F \Leftrightarrow F \in \mathfrak{F}_p(\underline{D}) \setminus F_x$ (for a detailed proof of these equivalences, see [13]).

We want to prove that ι is onto. For this, let $(A, B) \in \underline{\mathfrak{P}}^{co}(\mathbb{K}(\underline{D}))$ be a clopen protoconcept, i.e., $A = \bigcap_{j \in J}(\bigcup_{a \in A_j} F_a)$ where J is an index set and A_j are finite subsets of D for every $j \in J$. But $\bigcup_{a \in A_j} F_a$ is a member of the given subbasis, since $c(\bigcup_{a \in A_j} F_a) = \bigcap_{a \in A_j} cF_a = \bigcap_{a \in A_j} F_{\neg a} = F_{\neg A_j} = F_{\sqcap \neg A_j}$. The set A_j is finite for every $j \in J$, so there is an $x_j \in D$ with $x_j = \sqcap \neg A_j$; hence $\bigcup_{a \in A_j} F_a = F_{\neg x_j}$. By Lemma 2 [13], there is a filter $X \in \mathfrak{F}(\underline{D})$ with $A = \bigcap_{j \in J}(\bigcup_{a \in A_j} F_a) = F_X$.

Let us denote by $A_\sqcap := \{F_\sqcap \in \mathfrak{F}_0(\underline{D}_\sqcap) \mid F_\sqcap = F \cap \underline{D}_\sqcap,\ F \in A\}$. Since $A = \bigcap_{j \in J}(\bigcup_{a \in A_j} F_a)$, we prove that $A_\sqcap = \bigcap_{j \in J}(\bigcup_{a_\sqcap \in A_{j_\sqcap}} (F_{a_\sqcap})_\sqcap)$. Let now $F_\sqcap \in A_\sqcap$, then there exists a filter $F \in A$, with $F_\sqcap = F \cap \underline{D}_\sqcap$. The following holds:

$$\begin{aligned} F \in A &\Leftrightarrow \forall j \in J\ \exists a \in A_j : a \in F \\ &\Rightarrow \forall j \in J\ \exists a_\sqcap \in A_{j_\sqcap} : a_\sqcap \in F \\ &\Rightarrow F_\sqcap \in \bigcap_{j \in J}\Big(\bigcup_{a_\sqcap \in A_{j_\sqcap}} F_{a_\sqcap}\Big). \end{aligned}$$

For the second inclusion, the following holds

$$\begin{aligned} &F_\sqcap \in \bigcap_{j \in J}\Big(\bigcup_{a_\sqcap \in A_{j_\sqcap}} F_{a_\sqcap}\Big) \\ &\Rightarrow \forall j \in J\ \exists a_\sqcap \in A_{j_\sqcap} : a_\sqcap \in F_\sqcap \\ &\Rightarrow \forall j \in J\ \exists a_\sqcap \in A_{j_\sqcap} : a_\sqcap \in F := \uparrow F_\sqcap \\ &\Rightarrow \exists F \in A : F_\sqcap = F \cap \underline{D}_\sqcap. \end{aligned}$$

This proves the closeness of $A_\sqcap$ in $\mathfrak{F}_0(\underline{D}_\sqcap)$. We have seen that $A_\sqcap = (F_{X_\sqcap})_\sqcap$. In order to prove that $A_\sqcap$ is a closed extent we use a similar approach to Proposition 5 from [3]. Hence there is an $a \in \underline{D}_\sqcap$ with $A_\sqcap = (F_{a_\sqcap})_\sqcap$, i.e., $A = F_a$ for a suitable $a \in \underline{D}$, which completes the proof. □

Remark 6.8. The topological representation of regular double Boolean algebras arises from the topological representation of bounded lattices, considering every filter as a basis of a filter $\mathfrak{P}^{co}(\mathbb{K}(\underline{D}))$ and dually for ideals.

With these considerations, we are now able to develop a duality for regular double Boolean algebras. The following simple observations, will be used repeteadly.

Lemma 6.9. *Consider $F \in \mathfrak{F}(\underline{D})$ and $I \in \mathfrak{I}(\underline{D})$.*

1. *If $F \cap I = \varnothing$, then there is an $\overline{F} \in \mathfrak{F}_p(\underline{D})$ with $F \subseteq \overline{F}$ and $\overline{F}$ is I-maximal.*

2. *If* $F \cap I = \emptyset$*, then there is an* $\overline{I} \in \mathfrak{I}_p(\underline{D})$ *with* $I \subseteq \overline{I}$ *and* $\overline{I}$ *is* F*-maximal.*
3. *If* $F \cap I = \emptyset$*, then there is an* $\overline{F} \in \mathfrak{F}_p(\underline{D})$ *and an* $\overline{I} \in \mathfrak{I}_p(\underline{D})$ *with* $F \subseteq \overline{F}$ *and* $I \subseteq \overline{I}$*, so that* F *is* I*-maximal and* I *is* F*-maximal.*

Remark 6.10. A filter $F \in \mathfrak{F}_p(\underline{D})$ if and only if there is an ideal $I \in \mathfrak{I}_p(\underline{D})$ such that F is I-maximal.

Lemma 6.11. *Let* $\underline{D}$ *be a regular double Boolean algebra,* $F \in \mathfrak{F}_p(\underline{D})$ *and* $I \in \mathfrak{I}_p(\underline{D})$*. Then:*

1. F *is* I*-maximal* $\Leftrightarrow F \swarrow I$ *in* $\mathbb{K}^{DB}(\underline{D})$*.*
2. I *is* F*-maximal* $\Leftrightarrow F \nearrow I$ *in* $\mathbb{K}^{DB}(\underline{D})$*.*
3. $(F, I) \in \mathfrak{M}(\underline{D}) \Leftrightarrow F \swarrow\!\!\!\nearrow I$ *in* $\mathbb{K}^{DB}(\underline{D})$*.*

Remark 6.12. By the previous Lemma, we observe that the context $\mathbb{K}^{DB}(\underline{D})$ is the context reduction of $(\mathfrak{F}(\underline{D}), \mathfrak{I}(\underline{D}), \Delta)$.

Definition 6.13. A DB-topological context is called standard if in addition the following hold:

(R) $\mathbb{K}^{DB}$ is reduced;

(S) $gIm \Rightarrow \exists (A, B) \in \underline{\mathfrak{P}}^{co}(\mathbb{K}^{DB}) : \ g \in A$ and $m \in B$;

(Q) $(cI, (\rho \times \sigma)_{|cI})$ is a quasicompact space where $cI := (G \times M)\backslash I$ and $\rho \times \sigma$ denotes the product topology on $G \times M$.

Remark 6.14. The topological context of a regular double Boolean algebra $\mathbb{K}^{DB}(\underline{D})$ is standard, due to the analogy to standard topological contexts as they have been defined in [4].

The next Theorem completes our representation and its proof is similar to that given by G. Hartung in [4] to the main representation theorem for standard topological contexts. We only have to modify some sections were concepts have to be replaced by protoconcepts, but this is an easy and routine job.

Theorem 6.15. *Let* $\mathbb{K}^{DB}$ *be a DB-standard topological context. The mappings*

$$\alpha : G \to \mathfrak{F}_p(\underline{\mathfrak{P}}^{co}(\mathbb{K}^{DB})), \quad g \mapsto \{(A, B) \in \mathfrak{P}^{co}(\mathbb{K}^{DB}) \mid g \in A\}$$

$$\beta : M \to \mathfrak{I}_p(\underline{\mathfrak{P}}^{co}(\mathbb{K}^{DB})), \quad m \mapsto \{(A, B) \in \mathfrak{P}^{co}(\mathbb{K}^{DB}) \mid m \in B\}$$

define an isomorphism between topological contexts.

We conclude our considerations with the following Representation Theorem:

Theorem 6.16 (Representation Theorem). *For every regular double Boolean algebra* $\underline{D}$ *the context* $\mathbb{K}^{DB}(\underline{D})$ *is a standard topological context with* $\underline{\mathfrak{P}}^{co}(\mathbb{K}^{DB}(\underline{D})) \simeq \underline{D}$*. For every standard topological context* $\mathbb{K}^{DB}$*,* $\underline{\mathfrak{P}}^{co}(\mathbb{K}^{DB})$ *is a regular double Boolean algebra and* $\mathbb{K}^{DB} \simeq \mathbb{K}^{DB}(\underline{\mathfrak{P}}^{co}(\mathbb{K}^{DB}))$*. Moreover, the set of all clopen concepts of the standard DB-topological context of every regular double Boolean algebra* $\underline{D}$ *is isomorphic to* $\underline{D}_\sqcap \cap \underline{D}_\sqcup$*.*

This representation can easily be extended to a categorical duality, with morphisms appropriately chosen. The methods used for this duality are widely described in [1].

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Ostrowski type inequalities for functions whose derivatives are strongly (α, m)-convex via k-Riemann-Liouville fractional integrals

Seth Kermausuor

Abstract. In this paper, we provide some Ostrowski type integral inequalities for functions whose derivatives in absolute value at some powers are strongly (α, m)-convex with modulus $\mu \geq 0$ via the k-Riemann-Liouville fractional integrals. Similar results related to (α, m)-convex functions are obtained as a particular case.

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1. Introduction

Recall that given an interval I in $\mathbb{R}$, a function $f : I \to \mathbb{R}$ is said to be convex if

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y),$$

for all $x, y \in I$ and $t \in [0, 1]$.

Several generalizations of convex functions including quasiconvex, s-convex, m-convex, MT-convex, h-convex, η-convex and so on has been provided over the years. Of particular interest is the generalization of convexity to (α, m)-convexity by Miheşan [20] as follows.

Definition 1.1. A function $f : [0, d] \subset \mathbb{R} \to \mathbb{R}, d > 0$ is said to be (α, m)-convex function where $(\alpha, m) \in [0, 1]^2$ if

$$f(tx + m(1-t)y) \leq t^{\alpha} f(x) + m(1 - t^{\alpha})f(y),$$

for all $x, y \in [0, d]$ and $t \in [0, 1]$.

Remark 1.2. If we choose $(\alpha, m) = (1, m)$, then we obtain m-convex functions and if $(\alpha, m) = (1, 1)$, then we have the ordinary convex functions.

For some recent generalizations and results related to (α, m)-convex functions, we refer the interested reader to the papers [6, 23, 24, 28, 29, 25, 14, 16].

In [26], Polyak gave the following extension of convex functions known as strongly convex functions.

Definition 1.3. Given an interval I in $\mathbb{R}$, a function $f : I \to \mathbb{R}$ is said to be strongly convex with modulus $\mu \geq 0$ if

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) - \mu t(1-t)(y-x)^2,$$

for all $x, y \in I$ and $t \in [0,1]$.

The introduction of the notion of strong convexity has led many authors to extend the other classes of convex functions in a similar way (see [17, 7, 14, 5, 4, 15]). In a similar way, we have the following definition of strongly (α, m)-convex functions.

Definition 1.4. A function $f : [0,d] \subset \mathbb{R} \to \mathbb{R}, d > 0$ is said to be strongly (α, m)-convex with modulus $\mu \geq 0$ for $(\alpha, m) \in [0,1]^2$ if

$$f(tx + m(1-t)y) \leq t^\alpha f(x) + m(1-t^\alpha)f(y) - \mu t(1-t)(y-x)^2,$$

for all $x, y \in [0,d]$ and $t \in [0,1]$.

Remark 1.5. If we choose $(\alpha, m) = (1, m)$ in Definition 1.4, then we obtain strongly m-convex functions and if $(\alpha, m) = (1,1)$, then we have the strongly convex functions.

In 1938, Ostrowski [22] obtained the following inequality which is known in the literature as Ostrowski inequality.

Theorem 1.6. *Let $f : [a,b] \to \mathbb{R}$ be continuous on $[a,b]$ and differentiable in (a,b) and its derivative $f' : (a,b) \to \mathbb{R}$ is bounded in (a,b). If $M := \sup_{t \in (a,b)} |f'(t)| < \infty$, then we have*

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(t)dt \right| \leq \left(\frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2} \right)(b-a)M,$$

for all $x \in [a,b]$. The inequality is sharp in the sense that the constant $\frac{1}{4}$ cannot be replaced by a smaller one.

Many authors have studied and generalized this inequality in several different ways. For more information about the Ostrowski inequality and its associates, we refer the reader to the papers [1, 2, 3, 8, 9, 10, 11, 12, 13, 18, 19, 27, 25]. The authors in [13, 18, 19, 27, 1, 8, 25] provided some Ostrowski type inequalities for some classes of convex functions.

Motivated by the above results, the main goal of this paper is to provide some Ostrowski type integral inequalities for functions whose derivatives at some powers are strongly (α, m)-convex via the k-Riemman-Liouville fractional integrals. We complete this section with the defiinition of the k-Riemann-Liouville fractional integrals.

Definition 1.7 (See [21]**).** The Riemann-Liouville k-fractional integrals of order $\beta > 0$, for a real-valued continuous function f are defined as

$${}_kJ^{\beta}_{a^+}f(x) = \frac{1}{k\Gamma_k(\beta)} \int_a^x (x-t)^{\frac{\beta}{k}-1} f(t)dt, \ x > a,$$

and

$$_kJ^{\beta}_{b^-}f(x) = \frac{1}{k\Gamma_k(\beta)} \int_x^b (t-x)^{\frac{\beta}{k}-1} f(t)dt, \ x < b,$$

where $k > 0$, and Γ_k is the k-gamma function given by

$$\Gamma_k(x) = \int_0^{\infty} t^{x-1} e^{-\frac{t^k}{k}} dt, \ Re(x) > 0$$

with the properties that $\Gamma_k(x+k) = x\Gamma_k(x)$ and $\Gamma_k(k) = 1$.

Remark 1.8. If $k = 1$ in Definition 1.7, then we have the classical Riemann-Liouville fractional integrals and if $\beta = k = 1$, then we obtain the classical Riemann integral.

2. Main results

To prove our results we need the following result which is given by Farid and Usman [13].

Lemma 2.1 ([13]). *Let $f : [a, b] \to \mathbb{R}$ be a differentiable function on (a, b) with $a < b$ such that $f' \in L_1([a, b])$, then for all $x \in [a, b]$ and $\beta, k > 0$, the following equality holds;*

$$\frac{(x-a)^{\frac{\beta}{k}} + (b-x)^{\frac{\beta}{k}}}{b-a} f(x) - \frac{\Gamma_k(\beta+k)}{b-a}\left[{}_kJ^{\beta}_{x^-}f(a) + {}_kJ^{\beta}_{x^+}f(b)\right]$$
$$= \frac{(x-a)^{\frac{\beta}{k}+1}}{b-a}\int_0^1 t^{\frac{\beta}{k}} f'\Big(tx + (1-t)a\Big)dt - \frac{(b-x)^{\frac{\beta}{k}+1}}{b-a}\int_0^1 t^{\frac{\beta}{k}} f'\Big(tx + (1-t)b\Big)dt.$$

Theorem 2.2. *Let $f : [0, \infty) \to \mathbb{R}$ be a differentiable function on $(0, \infty)$ such that $f' \in L_1([a, b])$ with $0 \le a < b$. If $|f'|$ is strongly (α, m)-convex with modulus $\mu \ge 0$ for $\alpha \in [0, 1]$ and $m \in (0, 1]$, then the inequality*

$$\left|\frac{(x-a)^{\frac{\beta}{k}} + (b-x)^{\frac{\beta}{k}}}{b-a} f(x) - \frac{\Gamma_k(\beta+k)}{b-a}\left[{}_kJ^{\beta}_{x^-}f(a) + {}_kJ^{\beta}_{x^+}f(b)\right]\right|$$
$$\le \frac{(x-a)^{\frac{\beta}{k}+1}}{b-a}\left[\frac{|f'(x)|}{\frac{\beta}{k}+\alpha+1} + \frac{\alpha m\left|f'\left(\frac{a}{m}\right)\right|}{\left(\frac{\beta}{k}+1\right)\left(\frac{\beta}{k}+\alpha+1\right)} - \frac{\mu\left(x-\frac{a}{m}\right)^2}{(\frac{\beta}{k}+2)(\frac{\beta}{k}+3)}\right]$$
$$+ \frac{(b-x)^{\frac{\beta}{k}+1}}{b-a}\left[\frac{|f'(x)|}{\frac{\beta}{k}+\alpha+1} + \frac{\alpha m\left|f'\left(\frac{b}{m}\right)\right|}{\left(\frac{\beta}{k}+1\right)\left(\frac{\beta}{k}+\alpha+1\right)} - \frac{\mu\left(\frac{b}{m}-x\right)^2}{(\frac{\beta}{k}+2)(\frac{\beta}{k}+3)}\right]$$

holds for all $x \in [a, b]$ and $\beta, k > 0$.

Proof. Using Lemma 2.1 and the strong (α, m)-convexity of $|f'|$, we have that

$$\begin{aligned}
&\left|\frac{(x-a)^{\frac{\beta}{k}}+(b-x)^{\frac{\beta}{k}}}{b-a}f(x)-\frac{\Gamma_k(\beta+k)}{b-a}\left[{}_kJ^{\beta}_{x-}f(a)+{}_kJ^{\beta}_{x+}f(b)\right]\right|\\
&\leq \frac{(x-a)^{\frac{\beta}{k}+1}}{b-a}\int_0^1 t^{\frac{\beta}{k}}\left|f'\Big(tx+(1-t)a\Big)\right|dt\\
&+\frac{(b-x)^{\frac{\beta}{k}+1}}{b-a}\int_0^1 t^{\frac{\beta}{k}}\left|f'\Big(tx+(1-t)b\Big)\right|dt\\
&\leq \frac{(x-a)^{\frac{\beta}{k}+1}}{b-a}\int_0^1 t^{\frac{\beta}{k}}\left(t^{\alpha}|f'(x)|+m(1-t^{\alpha})\left|f'\Big(\frac{a}{m}\Big)\right|-\mu t(1-t)\left(x-\frac{a}{m}\right)^2\right)dt\\
&+\frac{(b-x)^{\frac{\beta}{k}+1}}{b-a}\int_0^1 t^{\frac{\beta}{k}}\left(t^{\alpha}|f'(x)|+m(1-t^{\alpha})\left|f'\Big(\frac{b}{m}\Big)\right|-\mu t(1-t)\left(\frac{b}{m}-x\right)^2\right)dt.
\end{aligned}$$

That is,

$$\begin{aligned}
&\left|\frac{(x-a)^{\frac{\beta}{k}}+(b-x)^{\frac{\beta}{k}}}{b-a}f(x)-\frac{\Gamma_k(\beta+k)}{b-a}\left[{}_kJ^{\beta}_{x-}f(a)+{}_kJ^{\beta}_{x+}f(a)\right]\right|\\
&\leq \frac{(x-a)^{\frac{\beta}{k}+1}}{b-a}\left[|f'(x)|\int_0^1 t^{\frac{\beta}{k}+\alpha}dt+m\left|f'\Big(\frac{a}{m}\Big)\right|\int_0^1 t^{\frac{\beta}{k}}(1-t^{\alpha})dt\right.\\
&\qquad\left.-\mu\left(x-\frac{a}{m}\right)^2\int_0^1 t^{\frac{\beta}{k}+1}(1-t)dt\right]\\
&+\frac{(b-x)^{\frac{\beta}{k}+1}}{b-a}\left[|f'(x)|\int_0^1 t^{\frac{\beta}{k}+\alpha}dt+m\left|f'\Big(\frac{b}{m}\Big)\right|\int_0^1 t^{\frac{\beta}{k}}(1-t^{\alpha})dt\right.\\
&\qquad\left.-\mu\left(\frac{b}{m}-x\right)^2\int_0^1 t^{\frac{\beta}{k}+1}(1-t)dt\right]. \qquad (2.1)
\end{aligned}$$

The desired inequality follows from (2.1) by using the fact that

$$\int_0^1 t^{\frac{\beta}{k}+\alpha}dt=\frac{1}{\frac{\beta}{k}+\alpha+1}, \qquad (2.2)$$

$$\int_0^1 t^{\frac{\beta}{k}}(1-t^{\alpha})dt=\frac{\alpha}{\left(\frac{\beta}{k}+1\right)\left(\frac{\beta}{k}+\alpha+1\right)} \qquad (2.3)$$

and

$$\int_0^1 t^{\frac{\beta}{k}+1}(1-t)dt=\frac{1}{(\frac{\beta}{k}+2)(\frac{\beta}{k}+3)}. \qquad (2.4)$$

□

Corollary 2.3. *Let $f:[0,\infty)\to\mathbb{R}$ be a differentiable function on $(0,\infty)$ such that $f'\in L_1([a,b])$ with $0\leq a<b$. If $|f'|$ is (α,m)-convex for $\alpha\in[0,1]$ and $m\in(0,1]$,*

then the inequality

$$\left|\frac{(x-a)^{\frac{\beta}{k}}+(b-x)^{\frac{\beta}{k}}}{b-a}f(x)-\frac{\Gamma_k(\beta+k)}{b-a}\left[{}_kJ^{\beta}_{x^-}f(a)+{}_kJ^{\beta}_{x^+}f(b)\right]\right|$$
$$\leq\frac{(x-a)^{\frac{\beta}{k}+1}}{b-a}\left[\frac{|f'(x)|}{\frac{\beta}{k}+\alpha+1}+\frac{\alpha m\left|f'\left(\frac{a}{m}\right)\right|}{\left(\frac{\beta}{k}+1\right)\left(\frac{\beta}{k}+\alpha+1\right)}\right]$$
$$+\frac{(b-x)^{\frac{\beta}{k}+1}}{b-a}\left[\frac{|f'(x)|}{\frac{\beta}{k}+\alpha+1}+\frac{\alpha m\left|f'\left(\frac{b}{m}\right)\right|}{\left(\frac{\beta}{k}+1\right)\left(\frac{\beta}{k}+\alpha+1\right)}\right]$$

holds for all $x \in [a,b]$ and $\beta, k > 0$.

Proof. Take $\mu = 0$ in Theorem 2.2. □

Theorem 2.4. *Let $f : [0,\infty) \to \mathbb{R}$ be a differentiable function on $(0,\infty)$ such that $f' \in L_1([a,b])$ with $0 \leq a < b$. If $|f'|^q$ is strongly (α,m)-convex with modulus $\mu \geq 0$ for $q > 1, \alpha \in [0,1]$ and $m \in (0,1]$, then the inequality*

$$\left|\frac{(x-a)^{\frac{\beta}{k}}+(b-x)^{\frac{\beta}{k}}}{b-a}f(x)-\frac{\Gamma_k(\beta+k)}{b-a}\left[{}_kJ^{\beta}_{x^-}f(a)+{}_kJ^{\beta}_{x^+}f(b)\right]\right|$$
$$\leq\frac{(x-a)^{\frac{\beta}{k}+1}}{(b-a)(\frac{\beta}{k}p+1)^{\frac{1}{p}}}\left(\frac{|f'(x)|^q}{\alpha+1}+\frac{\alpha m\left|f'\left(\frac{a}{m}\right)\right|^q}{\alpha+1}-\frac{\mu\left(x-\frac{a}{m}\right)^2}{6}\right)^{\frac{1}{q}}$$
$$+\frac{(b-x)^{\frac{\beta}{k}+1}}{(b-a)(\frac{\beta}{k}p+1)^{\frac{1}{p}}}\left(\frac{|f'(x)|^q}{\alpha+1}+\frac{\alpha m\left|f'\left(\frac{b}{m}\right)\right|^q}{\alpha+1}-\frac{\mu\left(\frac{b}{m}-x\right)^2}{6}\right)^{\frac{1}{q}}$$

holds for all $x \in [a,b]$ and $\beta, k > 0$, for $\frac{1}{p}+\frac{1}{q}=1$.

Proof. Using Lemma 2.1, the Hölder's inequality and the strong (α,m)-convexity of $|f'|^q$, we have that

$$\left|\frac{(x-a)^{\frac{\beta}{k}}+(b-x)^{\frac{\beta}{k}}}{b-a}f(x)-\frac{\Gamma_k(\beta+k)}{b-a}\left[{}_kJ^{\beta}_{x^-}f(a)+{}_kJ^{\beta}_{x^+}f(b)\right]\right|$$
$$\leq\frac{(x-a)^{\frac{\beta}{k}+1}}{b-a}\int_0^1 t^{\frac{\beta}{k}}\left|f'\Big(tx+(1-t)a\Big)\right|dt$$
$$+\frac{(b-x)^{\frac{\beta}{k}+1}}{b-a}\int_0^1 t^{\frac{\beta}{k}}\left|f'\Big(tx+(1-t)b\Big)\right|dt$$
$$\leq\frac{(x-a)^{\frac{\beta}{k}+1}}{b-a}\left(\int_0^1 t^{\frac{\beta}{k}p}dt\right)^{\frac{1}{p}}\left(\int_0^1\left|f'\Big(tx+(1-t)a\Big)\right|^q dt\right)^{\frac{1}{q}}$$
$$+\frac{(b-x)^{\frac{\beta}{k}+1}}{b-a}\left(\int_0^1 t^{\frac{\beta}{k}p}dt\right)^{\frac{1}{p}}\left(\int_0^1|f'\Big(tx+(1-t)b\Big)\Big|^q dt\right)^{\frac{1}{q}}$$
$$\leq\frac{(x-a)^{\frac{\beta}{k}+1}}{b-a}\left(\int_0^1 t^{\frac{\beta}{k}p}dt\right)^{\frac{1}{p}}\left(\int_0^1\Big(t^{\alpha}|f'(x)|^q+m(1-t^{\alpha})\left|f'\left(\frac{a}{m}\right)\right|^q\right.$$

$$-\mu t(1-t)\left(x-\frac{a}{m}\right)^2\Big)dt\Big)^{\frac{1}{q}}$$

$$+\frac{(b-x)^{\frac{\beta}{k}+1}}{b-a}\Big(\int_0^1 t^{\frac{\beta}{k}p}dt\Big)^{\frac{1}{p}}\Big(\int_0^1\Big(t^\alpha|f'(x)|^q+m(1-t^\alpha)\Big|f'\Big(\frac{b}{m}\Big)\Big|^q$$

$$-\mu t(1-t)\left(\frac{b}{m}-x\right)^2\Big)dt\Big)^{\frac{1}{q}}$$

$$\leq\frac{(x-a)^{\frac{\beta}{k}+1}}{(b-a)(\frac{\beta}{k}p+1)^{\frac{1}{p}}}\left(\frac{|f'(x)|^q}{\alpha+1}+\frac{\alpha m\left|f'\left(\frac{a}{m}\right)\right|^q}{\alpha+1}-\frac{\mu\left(x-\frac{a}{m}\right)^2}{6}\right)^{\frac{1}{q}}$$

$$+\frac{(b-x)^{\frac{\beta}{k}+1}}{(b-a)(\frac{\beta}{k}p+1)^{\frac{1}{p}}}\left(\frac{|f'(x)|^q}{\alpha+1}+\frac{\alpha m\left|f'\left(\frac{b}{m}\right)\right|^q}{\alpha+1}-\frac{\mu\left(\frac{b}{m}-x\right)^2}{6}\right)^{\frac{1}{q}}.$$

This completes the proof. □

Corollary 2.5. *Let $f:[0,\infty)\to\mathbb{R}$ be a differentiable function on $(0,\infty)$ such that $f'\in L_1([a,b])$ with $0\leq a<b$. If $|f'|^q$ is (α,m)-convex for $q>1,\alpha\in[0,1]$ and $m\in(0,1]$, then the inequality*

$$\left|\frac{(x-a)^{\frac{\beta}{k}}+(b-x)^{\frac{\beta}{k}}}{b-a}f(x)-\frac{\Gamma_k(\beta+k)}{b-a}\left[{}_kJ^{\beta}_{x^-}f(a)+{}_kJ^{\beta}_{x^+}f(b)\right]\right|$$

$$\leq\frac{(x-a)^{\frac{\beta}{k}+1}}{(b-a)(\frac{\beta}{k}p+1)^{\frac{1}{p}}}\left(\frac{|f'(x)|^q}{\alpha+1}+\frac{\alpha m\left|f'\left(\frac{a}{m}\right)\right|^q}{\alpha+1}\right)^{\frac{1}{q}}$$

$$+\frac{(b-x)^{\frac{\beta}{k}+1}}{(b-a)(\frac{\beta}{k}p+1)^{\frac{1}{p}}}\left(\frac{|f'(x)|^q}{\alpha+1}+\frac{\alpha m\left|f'\left(\frac{b}{m}\right)\right|^q}{\alpha+1}\right)^{\frac{1}{q}}$$

holds for all $x\in[a,b]$ and $\beta,k>0$, for $\frac{1}{p}+\frac{1}{q}=1$.

Proof. Take $\mu=0$ in Theorem 2.4. □

Theorem 2.6. *Let $f:[0,\infty)\to\mathbb{R}$ be a differentiable function on $(0,\infty)$ such that $f'\in L_1([a,b])$ with $0\leq a<b$. If $|f'|^q$ is strongly (α,m)-convex with modulus $\mu\geq 0$ for $q>1,\alpha\in[0,1]$ and $m\in(0,1]$, then the inequality*

$$\left|\frac{(x-a)^{\frac{\beta}{k}}+(b-x)^{\frac{\beta}{k}}}{b-a}f(x)-\frac{\Gamma_k(\beta+k)}{b-a}\left[{}_kJ^{\beta}_{x^-}f(a)+{}_kJ^{\beta}_{x^+}f(b)\right]\right|$$

$$\leq\frac{(x-a)^{\frac{\beta}{k}+1}}{(b-a)(\frac{\beta}{k}+1)^{\frac{1}{p}}}\left(\frac{|f'(x)|^q}{\frac{\beta}{k}+\alpha+1}+\frac{\alpha m\left|f'\left(\frac{a}{m}\right)\right|^q}{\left(\frac{\beta}{k}+1\right)\left(\frac{\beta}{k}+\alpha+1\right)}-\frac{\mu\left(x-\frac{a}{m}\right)^2}{(\frac{\beta}{k}+2)(\frac{\beta}{k}+3)}\right)^{\frac{1}{q}}$$

$$+\frac{(b-x)^{\frac{\beta}{k}+1}}{(b-a)(\frac{\beta}{k}+1)^{\frac{1}{p}}}\left(\frac{|f'(x)|^q}{\frac{\beta}{k}+\alpha+1}+\frac{\alpha m\left|f'\left(\frac{b}{m}\right)\right|^q}{\left(\frac{\beta}{k}+1\right)\left(\frac{\beta}{k}+\alpha+1\right)}-\frac{\mu\left(\frac{b}{m}-x\right)^2}{(\frac{\beta}{k}+2)(\frac{\beta}{k}+3)}\right)^{\frac{1}{q}}$$

holds for all $x\in[a,b]$ and $\beta,k>0$, for $\frac{1}{p}+\frac{1}{q}=1$.

Proof. Using Lemma 2.1, the Hölder's inequality and the strong (α, m)-convexity of $|f'|^q$, we have that

$$\begin{aligned}
&\left|\frac{(x-a)^{\frac{\beta}{k}}+(b-x)^{\frac{\beta}{k}}}{b-a}f(x)-\frac{\Gamma_k(\beta+k)}{b-a}\Big[{}_kJ^{\beta}_{x^-}f(a)+{}_kJ^{\beta}_{x^+}f(b)\Big]\right|\\
&\le \frac{(x-a)^{\frac{\beta}{k}+1}}{b-a}\int_0^1 t^{\frac{\beta}{k}}\Big|f'\Big(tx+(1-t)a\Big)\Big|dt\\
&+\frac{(b-x)^{\frac{\beta}{k}+1}}{b-a}\int_0^1 t^{\frac{\beta}{k}}\Big|f'\Big(tx+(1-t)b\Big)\Big|dt\\
&\le \frac{(x-a)^{\frac{\beta}{k}+1}}{b-a}\Big(\int_0^1 t^{\frac{\beta}{k}}dt\Big)^{\frac{1}{p}}\Big(\int_0^1 t^{\frac{\beta}{k}}\Big|f'\Big(tx+(1-t)a\Big)\Big|^q dt\Big)^{\frac{1}{q}}\\
&+\frac{(b-x)^{\frac{\beta}{k}+1}}{b-a}\Big(\int_0^1 t^{\frac{\beta}{k}}dt\Big)^{\frac{1}{p}}\Big(\int_0^1 t^{\frac{\beta}{k}}|f'\Big(tx+(1-t)b\Big)\Big|^q dt\Big)^{\frac{1}{q}}\\
&\le \frac{(x-a)^{\frac{\beta}{k}+1}}{b-a}\Big(\int_0^1 t^{\frac{\beta}{k}}dt\Big)^{\frac{1}{p}}\Big(\int_0^1 t^{\frac{\beta}{k}}\Big(t^{\alpha}|f'(x)|^q+m(1-t^{\alpha})\Big|f'\Big(\frac{a}{m}\Big)\Big|^q\\
&\qquad\qquad -\mu t(1-t)\Big(x-\frac{a}{m}\Big)^2\Big)dt\Big)^{\frac{1}{q}}\\
&+\frac{(b-x)^{\frac{\beta}{k}+1}}{b-a}\Big(\int_0^1 t^{\frac{\beta}{k}}dt\Big)^{\frac{1}{p}}\Big(\int_0^1 t^{\frac{\beta}{k}}\Big(t^{\alpha}|f'(x)|^q+m(1-t^{\alpha})\Big|f'\Big(\frac{b}{m}\Big)\Big|^q\\
&\qquad\qquad -\mu t(1-t)\Big(\frac{b}{m}-x\Big)^2\Big)dt\Big)^{\frac{1}{q}}\\
&\le \frac{(x-a)^{\frac{\beta}{k}+1}}{(b-a)(\frac{\beta}{k}+1)^{\frac{1}{p}}}\Big(|f'(x)|^q\int_0^1 t^{\frac{\beta}{k}+\alpha}dt+m\Big|f'\Big(\frac{a}{m}\Big)\Big|^q\int_0^1 t^{\frac{\beta}{k}}(1-t^{\alpha})dt\\
&\qquad\qquad -\mu\Big(x-\frac{a}{m}\Big)^2\int_0^1 t^{\frac{\beta}{k}+1}(1-t)dt\Big)^{\frac{1}{q}}\\
&+\frac{(b-x)^{\frac{\beta}{k}+1}}{(b-a)(\frac{\beta}{k}+1)^{\frac{1}{p}}}\Big(|f'(x)|^q\int_0^1 t^{\frac{\beta}{k}+\alpha}dt+m\Big|f'\Big(\frac{b}{m}\Big)\Big|^q\int_0^1 t^{\frac{\beta}{k}}(1-t^{\alpha})dt\\
&\qquad\qquad -\mu\Big(\frac{b}{m}-x\Big)^2\int_0^1 t^{\frac{\beta}{k}+1}(1-t)dt\Big)^{\frac{1}{q}}. \qquad (2.5)
\end{aligned}$$

By using (2.2), (2.3), (2.4) and (2.5), we have the desired inequality. □

Corollary 2.7. *Let $f : [0, \infty) \to \mathbb{R}$ be a differentiable function on $(0, \infty)$ such that $f' \in L_1([a, b])$ with $0 \le a < b$. If $|f'|^q$ is (α, m)-convex for $q > 1, \alpha \in [0, 1]$ and $m \in (0, 1]$, then the inequality*

$$\left|\frac{(x-a)^{\frac{\beta}{k}}+(b-x)^{\frac{\beta}{k}}}{b-a}f(x)-\frac{\Gamma_k(\beta+k)}{b-a}\Big[{}_kJ^{\beta}_{x^-}f(a)+{}_kJ^{\beta}_{x^+}f(b)\Big]\right|$$

$$\leq \frac{(x-a)^{\frac{\beta}{k}+1}}{(b-a)(\frac{\beta}{k}+1)^{\frac{1}{p}}}\left(\frac{|f'(x)|^q}{\frac{\beta}{k}+\alpha+1}+\frac{\alpha m\left|f'\left(\frac{a}{m}\right)\right|^q}{\left(\frac{\beta}{k}+1\right)\left(\frac{\beta}{k}+\alpha+1\right)}\right)^{\frac{1}{q}}$$

$$+\frac{(b-x)^{\frac{\beta}{k}+1}}{(b-a)(\frac{\beta}{k}+1)^{\frac{1}{p}}}\left(\frac{|f'(x)|^q}{\frac{\beta}{k}+\alpha+1}+\frac{\alpha m\left|f'\left(\frac{b}{m}\right)\right|^q}{\left(\frac{\beta}{k}+1\right)\left(\frac{\beta}{k}+\alpha+1\right)}\right)^{\frac{1}{q}}$$

holds for all $x \in [a,b]$ *and* $\beta, k > 0$, *for* $\frac{1}{p}+\frac{1}{q}=1$.

Proof. Take $\mu = 0$ in Theorem 2.6. □

3. Conclusion

New Ostrowski type inequalities for functions whose derivatives in absolute value at certain powers are strongly (α, m)-convex and (α, m)-convex functions via the k-Riemman-Liouville fractional integrals has been provided. Similar results could be obtained for m and strongly m-convex functions as particular cases. Also several other interesting inequalities could be obtained by considering different values of the parameters β and k. For instance, if $k = 1$, then the results will be in terms of the classical Riemann-Liouville fractional integrals and if $\beta = k = 1$, then we have the results with the integrals in the classical Riemann sense. The details are left for the interested reader.

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Wirtinger type inequalities via fractional integral operators

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Abstract. In this study, we shall present Wirtinger type inequality in the fractional case with conformable fractional operators.

Mathematics Subject Classification (2010): 26A33, 26Dxx, 35A23.

Keywords: Fractional derivative, fractional integral, Wirtinger inequality.

1. Introduction

Fractional derivation and integration are as old as ordinary derivation and integration. The history of fractional calculus date back to 1695. In that time, L'Hospital asked to Leibniz "what would be the one-half derivative of x?" After this conversation, many mathematicians tried to give a coherent definition of fractional derivative and integral operators. By the beginning of 20th century, some definitions of fractional derivative are introduced called Riemann-Liouville, Caputo, and Grünwald-Letnikov derivatives and so on. Fractional derivatives and integrals are studied widely in different branches of sciences like engineering, physics etc. For more knowledge about the history and applications, we refer to [7, 9, 20].

The definitions we considered above mostly use the integral forms to define the fractional derivative. Riemann-Liouville and Caputo fractional derivatives use the Riemann-Liouville fractional integral defined by

$$J_a^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)dt, \; m-1 < \alpha < m, \; \alpha \in \mathbb{R}.$$

And so, Riemann-Liouville and Caputo fractional derivatives are defined as

$$D_a^\alpha f(x) = D^m J_a^{m-\alpha} f(x),$$

and

$${}^C D_a^\alpha f(x) = J_a^{m-\alpha} D^m f(x),$$

respectively, where $m = \lceil \alpha \rceil$, and in the right hand of the definitions operator D^m represents the ordinary derivative order m.

Apart from the linearity property, Riemann-Liouville or any of other fractional derivatives do not satisfy all properties of ordinary derivative. For example, Caputo derivative does not satisfy well-known formula of the product of two functions

$$D(f(t)g(t)) = g(t)Df(t) + f(t)Dg(t),$$

and Riemann-Liouville derivative does not satisfy

$$D[c] = 0, \ c \text{ is constant.}$$

Because of this facts, recently some mathematicians gave their efforts to give new definitions for fractional derivatives. To handle these difficulties, in 2014 Khalil et al. [15] gave a new definition of fractional derivative as

$$T_\alpha(f)(t) = \lim_{\varepsilon \to 0} \frac{f(t + \varepsilon t^{1-\alpha}) - f(t)}{\varepsilon}.$$

This definition, called conformable fractional derivative, satisfies many properties of ordinary derivatives like product rule, chain rule etc.

Because of inequalities were often used in the theoretical and applied mathematics, mathematicians studied about their extensions, generalizations and discretizations, see [2, 3, 12, 17, 18] and references cited therein. And in the last decade authors started to transfer those inequalities known in the classical settings into fractional settings, both continuous and discrete cases, to make contributions to the development of fractional calculus theory [4, 5, 8, 10, 11, 21].

In this paper, we shall give the fractional analogues of Wirtinger type inequalities given below:

Theorem 1.1 (Wirtinger's Inequality). *For any function $y \in C^1[0,1]$ such that*

$$y(0) = y(1) = 0,$$

we have

$$\int_0^1 (y'(t))^2 dt \geq \pi^2 \int_0^1 y^2(t) dt.$$

Remark 1.2. Although Fourier series are used for the proof of Theorem 1.1, this proof also can be made with Schwarz inequality. Then, in the second case, we have inequality

$$\int_0^1 (y'(t))^2 dt \geq \int_0^1 y^2(t) dt, \tag{1.1}$$

where condition $y(1) = 0$ is not needed.

In 1975, Hinton and Lewis [14] gave a generalized Wirtinger type inequality using Schwarz inequality:

Theorem 1.3. *For any positive $M \in C^1([a,b])$ with $M'(t) \neq 0$, and $y \in C^1([a,b])$ with $y(a) = y(b) = 0$, we have*

$$\int_a^b \frac{M^2(t)}{|M'(t)|}(y'(t))^2 dt \geq \frac{1}{4}\int_a^b |M'(t)|\, y^2(t) dt.$$

In 1999, Pena [19] gave the discrete analogue of the inequality established by Hinton and Lewis:

Theorem 1.4. *For a positive sequence $\{M_n\}_{0 \leq n \leq N+1}$ satisfying either $\triangle M > 0$ or $\triangle M < 0$ on $[0, N] \cap \mathbb{Z}$*

$$\sum_{n=0}^{N} \frac{M_n M_{n+1}}{|\triangle M_n|}(\triangle y_n)^2 \geq \frac{1}{\psi_J}\sum_{n=0}^{N} |\triangle M_n|\, y_{n+1}^2$$

holds for any sequence $\{y_n\}_{0 \leq n \leq N+1}$ with $y_0 = y_{N+1} = 0$, where

$$\psi_J = \left(\sup_{0 \leq n \leq N} \frac{M_n}{M_{n+1}}\right)\left[1 + \left(\sup_{0 \leq n \leq N} \frac{|\triangle M_n|}{|\triangle M_{n+1}|}\right)^{1/2}\right]^2.$$

For more discussion about the Wirtinger inequality, see [13, 16, 22] and references cited therein.

2. Preliminaries

In this section, we give basic definitions and fundamental results for conformable fractional operators, so the paper is self-contained.

Definition 2.1. The conformable fractional derivative of a function $f : [0, \infty) \to \mathbb{R}$ of order $0 \leq \alpha \leq 1$ is defined by

$$T_\alpha(f)(t) = \lim_{\varepsilon \to 0} \frac{f(t + \varepsilon t^{1-\alpha}) - f(t)}{\varepsilon}$$

for all $t > 0$.

We note that if the conformable fractional derivative of function f of order α exists, we say f is α-differentiable.

Theorem 2.2. *Let $\alpha \in (0,1]$ and functions f and g be α-differentiable at point $t > 0$. Then following properties are hold:*

(*i*) $T_\alpha(af + bg)(t) = aT_\alpha(f)(t) + bT_\alpha(g)(t)$, for all $a, b \in \mathbb{R}$.
(*ii*) $T_\alpha(t^m) = mt^{m-\alpha}$, for all $m \in \mathbb{R}$.
(*iii*) $T_\alpha(c) = 0$, for all constant functions $f(t) = c$.
(*iv*) $T_\alpha(fg)(t) = g(t)T_\alpha(f)(t) + f(t)T_\alpha(g)(t)$.
(*v*) $T_\alpha\left(\frac{f}{g}\right)(t) = \dfrac{g(t)T_\alpha(f)(t) - f(t)T_\alpha(g)(t)}{(g(t))^2}$
(*vi*) If, in addition, f is differentiable, then $T_\alpha(f)(t) = t^{1-\alpha}\dfrac{df}{dt}$.

Now, we give conformable fractional derivative of some functions:

(1) $T_\alpha(t^m) = mt^{m-\alpha}$, for all $m \in \mathbb{R}$.
(2) $T_\alpha(1) = 0$.
(3) $T_\alpha(e^{at}) = at^{1-\alpha}e^{at},\ a \in \mathbb{R}$.
(4) $T_\alpha(e^{\frac{1}{\alpha}t^\alpha}) = e^{\frac{1}{\alpha}t^\alpha}$.
(5) $T_\alpha(\sin at) = at^{1-\alpha}\cos at,\ a \in \mathbb{R}$.
(6) $T_\alpha(\cos at) = -at^{1-\alpha}\sin at,\ a \in \mathbb{R}$.
(7) $T_\alpha(\sin \frac{1}{\alpha}t^\alpha) = \cos \frac{1}{\alpha}t^\alpha$.
(8) $T_\alpha(\cos \frac{1}{\alpha}t^\alpha) = -\sin \frac{1}{\alpha}t^\alpha$.

Definition 2.3. The conformable fractional integral of a function $f : [0,\infty) \to \mathbb{R}$ of order $0 \leq \alpha \leq 1$ is defined by

$$I_\alpha^a(f)(t) = I_1^a(t^{\alpha-1}f)(t) = \int_a^t \frac{f(s)}{t^{1-\alpha}}ds,$$

where the integral is the usual Riemann improper integral, and $\alpha \in (0,1)$.

Theorem 2.4. *$T_\alpha I_\alpha^a(f)(t) = f(t)$, for $t \geq a$, where f is any continuous function in the domain of I_α.*

Example 2.5. For $a = 0$ and $\alpha = 1/2$, the conformable integral of function

$$f(t) = \sqrt{t}\cos t$$

is

$$I_{1/2}^0(\sqrt{t}\cos t) = \int_0^t \cos s ds = \sin t.$$

For more information and applications on conformable fractional operators, we refer to [1, 6, 15, 21] and papers cited therein.

3. Wirtinger type inequalities

In this section, we will state Wirtinger type inequalities using conformable fractional operators.

We start giving the fractional analogue of the inequality given in (1.1).

Theorem 3.1. *For any function $f \in C^\alpha([a,b])$ such that $f(a) = 0$, we have*

$$\int_a^b |T_\alpha f(t)|^2 d_\alpha t \geq \frac{\alpha^2}{(b^\alpha - a^\alpha)^2}\int_a^b |f(t)|^2 d_\alpha t, \tag{3.1}$$

where C^α represents the family of $\alpha-$differentiable functions, and $\int_a^t g(s)d_\alpha s$ denotes the conformable fractional integral.

Proof. From [1], we know

$$I_\alpha^a T_\alpha f(t) = f(t) - f(a).$$

Using the condition $f(a) = 0$, we have $f(t) = I_\alpha^a T_\alpha f(t)$, so on

$$|f(t)| \leq \int_a^t |T_\alpha f(s)| \, d_\alpha s. \tag{3.2}$$

Applying Schwarz inequality to the right side of (3.2), we find

$$\begin{aligned} |f(t)| &\leq \left(\int_a^t d_\alpha s \right)^{1/2} \left(\int_a^t |T_\alpha f(s)|^2 \, d_\alpha s \right)^{1/2} \\ &= \left(\frac{t^\alpha - a^\alpha}{\alpha} \right)^{1/2} \left(\int_a^t |T_\alpha f(s)|^2 \, d_\alpha s \right)^{1/2} \\ &\leq \frac{(b^\alpha - a^\alpha)^{1/2}}{\alpha^{1/2}} \left(\int_a^b |T_\alpha f(s)|^2 \, d_\alpha s \right)^{1/2}. \end{aligned} \tag{3.3}$$

After squaring the inequality (3.3) and taking its conformable integral from a to b, the desired result is obtained. □

Secondly, we state the fractional analogue of the inequality given in Theorem 1.3.

Theorem 3.2. *For any positive function $M \in C^\alpha([a,b])$ satisfying either $T_\alpha[M(t)] > 0$ or $T_\alpha[M(t)] < 0$ on $[a,b]$, we have*

$$\int_a^b \frac{M^2(t)}{|T_\alpha[M(t)]|} \left(T_\alpha[y(t)]\right)^2 d_\alpha t \geq \frac{1}{4} \int_a^b |T_\alpha[M(t)]| \, y^2(t) d_\alpha t, \tag{3.4}$$

for any function $y \in C^\alpha([a,b])$ with $y(a) = y(b) = 0$.

Proof. Suppose that $T_\alpha[M(t)] > 0$. Then we have

$$I_1 = \int_a^b T_\alpha[M(t)] \, y^2(t) d_\alpha t = M(t) y^2(t)\Big|_a^b - \int_a^b M(t) T_\alpha\left[y^2(t)\right] d_\alpha t$$

$$= M(b)y^2(b) - M(a)y^2(a) - 2\int_a^b M(t)y(t)T_\alpha[y(t)] d_\alpha t = -2\int_a^b M(t)y(t)T_\alpha[y(t)] d_\alpha t$$

$$\leq 2\int_a^b M(t) \, |y(t)| \, |T_\alpha[y(t)]| \, d_\alpha t = 2\int_a^b \sqrt{\frac{M^2(t)}{T_\alpha[M(t)]}} \, |T_\alpha[y(t)]| \sqrt{T_\alpha[M(t)]} \, |y(t)| \, d_\alpha t.$$

Using Schwarz inequality, we have

$$\begin{aligned} I_1 &= \int_a^b T_\alpha\left[M(t)\right] y^2(t) d_\alpha t \\ &\le 2\left(\int_a^b \frac{M^2(t)}{T_\alpha[M(t)]}\left(T_\alpha[y(t)]\right)^2 d_\alpha t\right)^{1/2}\left(\int_a^b T_\alpha[M(t)] y^2(t) d_\alpha t\right)^{1/2} \\ &= 2\sqrt{I_1 I_2}, \end{aligned}$$

where

$$I_2 = \int_a^b \frac{M^2(t)}{T_\alpha[M(t)]}\left(T_\alpha[y(t)]\right)^2 d_\alpha t.$$

Dividing both sides of the above inequality by $\sqrt{I_1}$, we obtain

$$\sqrt{I_1} \le 2\sqrt{I_2}.$$

Hence

$$I_2 \ge \frac{1}{4} I_1.$$

The proof is complete. □

Remark 3.3. If we take $\alpha = 1$ in (3.1), we have

$$\int_a^b \left|f'(t)\right|^2 dt \ge \frac{1}{(b-a)^2} \int_a^b \left|f(t)\right|^2 dt,$$

$a = 0,\ b = 1$ with $\alpha = 1$, we have

$$\int_0^1 \left|f'(t)\right|^2 dt \ge \int_0^1 \left|f(t)\right|^2 dt,$$

and this is the inequality given in (1.1).
Secondly, if we take $\alpha = 1$ in (3.4), we have

$$\int_a^b \frac{M^2(t)}{|M'(t)|}\left(y'(t)\right)^2 dt \ge \frac{1}{4} \int_a^b M'(t) y^2(t) dt,$$

i.e., we have the inequality given in Theorem 1.3.

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On some new integral inequalities concerning twice differentiable generalized relative semi-(m, h)-preinvex mappings

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Abstract. The authors first present some integral inequalities for Gauss-Jacobi type quadrature formula involving generalized relative semi-(m, h)-preinvex mappings. And then, a new identity concerning twice differentiable mappings defined on m-invex set is derived. By using the notion of generalized relative semi-(m, h)-preinvexity and the obtained identity as an auxiliary result, some new estimates with respect to Hermite-Hadamard type inequalities via conformable fractional integrals are established. These new presented inequalities are also applied to construct inequalities for special means.

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1. Introduction

The subsequent double inequality is known as Hermite-Hadamard inequality.

Theorem 1.1. *Let $f : I \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ be a convex mapping on an interval I of real numbers and $a, b \in I$ with $a < b$. Then The subsequent double inequality holds:*

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}. \tag{1.1}$$

For recent results concerning Hermite-Hadamard type inequalities through various classes of convex functions, please see [3]-[14], [19], [20], [18], [24], [26], [29], [38], [43], [44] and the references mentioned in these papers.
Let us evoke some definitions as follows.

Definition 1.2. [42] A set $M_\varphi \subseteq \mathbb{R}^n$ is named as a relative convex (φ-convex) set, if and only if, there exists a function $\varphi : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ such that,

$$t\varphi(x) + (1-t)\varphi(y) \in M_\varphi,\ \forall\, x, y \in \mathbb{R}^n : \varphi(x), \varphi(y) \in M_\varphi, t \in [0, 1]. \tag{1.2}$$

Definition 1.3. [42] A function f is named as a relative convex (φ-convex) function on a relative convex (φ-convex) set M_φ, if and only if, there exists a function $\varphi : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ such that,

$$f(t\varphi(x) + (1-t)\varphi(y)) \leq tf(\varphi(x)) + (1-t)f(\varphi(y)), \tag{1.3}$$

$\forall\, x, y \in \mathbb{R}^n : \varphi(x), \varphi(y) \in M_\varphi, t \in [0,1]$.

Definition 1.4. [7] A non-negative function $f : I \subseteq \mathbb{R} \longrightarrow [0, +\infty)$ is said to be P-function, if

$$f(tx + (1-t)y) \leq f(x) + f(y), \quad \forall x, y \in I,\ t \in [0,1].$$

Definition 1.5. [2] A set $K \subseteq \mathbb{R}^n$ is said to be invex respecting the mapping $\eta : K \times K \longrightarrow \mathbb{R}^n$, if $x + t\eta(y,x) \in K$ for every $x, y \in K$ and $t \in [0,1]$.

Definition 1.6. [25] Let $h : [0,1] \longrightarrow \mathbb{R}$ be a non-negative function and $h \neq 0$. The function f on the invex set K is said to be h-preinvex with respect to η, if

$$f\big(x + t\eta(y,x)\big) \leq h(1-t)f(x) + h(t)f(y) \tag{1.4}$$

for each $x, y \in K$ and $t \in [0,1]$ where $f(\cdot) > 0$.

Clearly, when putting $h(t) = t$ in Definition 1.6, f becomes a preinvex function, see [31]. If the mapping $\eta(y,x) = y - x$ in Definition (1.6), then the non-negative function f reduces to h-convex mappings, see [41].

Definition 1.7. [40] Let $f : K \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ be a non-negative function, a function $f : K \longrightarrow \mathbb{R}$ is said to be a tgs-convex function on K if the inequality

$$f\big((1-t)x + ty\big) \leq t(1-t)[f(x) + f(y)] \tag{1.5}$$

grips for all $x, y \in K$ and $t \in (0,1)$.

Definition 1.8. [5], [22] A function $f : I \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ is said to MT-convex functions, if f it is non-negative and $\forall\, x, y \in I$ and $t \in (0,1)$ satisfies the subsequent inequality:

$$f(tx + (1-t)y) \leq \frac{\sqrt{t}}{2\sqrt{1-t}} f(x) + \frac{\sqrt{1-t}}{2\sqrt{t}} f(y). \tag{1.6}$$

Definition 1.9. [27] A function: $I \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ is said to be m-MT-convex, if f is positive and for $\forall\, x, y \in I$, and $t \in (0,1)$, among $m \in [0,1]$, satisfies the following inequality

$$f\big(tx + m(1-t)y\big) \leq \frac{\sqrt{t}}{2\sqrt{1-t}} f(x) + \frac{m\sqrt{1-t}}{2\sqrt{t}} f(y). \tag{1.7}$$

Definition 1.10. [30] Let $K \subseteq \mathbb{R}$ be an open m-invex set respecting $\eta : K \times K \times (0,1] \longrightarrow \mathbb{R}$ and $h_1, h_2 : [0,1] \longrightarrow [0, +\infty)$. A function $f : K \longrightarrow \mathbb{R}$ is said to be generalized (m, h_1, h_2)-preinvex, if

$$f\big(mx + t\eta(y,x,m)\big) \leq mh_1(t)f(x) + h_2(t)f(y) \tag{1.8}$$

is valid for all $x, y \in K$ and $t \in [0,1]$, for some fixed $m \in (0,1]$.

We need the subsequent Riemann-Liouville fractional calculus background.

Definition 1.11. [23] Let $f \in L_1[a,b]$. The Riemann-Liouville integrals $J_{a+}^{\alpha} f$ and $J_{b-}^{\alpha} f$ of order $\alpha > 0$ with $a \geq 0$ are defined by

$$J_{a+}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)dt, \quad x > a$$

and

$$J_{b-}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t)dt, \quad b > x,$$

where $\Gamma(\alpha) = \displaystyle\int_0^{+\infty} e^{-u} u^{\alpha-1} du$. Here $J_{a+}^{0} f(x) = J_{b-}^{0} f(x) = f(x)$.
Note that $\alpha = 1$, the fractional integral reduces to the classical integral.

Due to the wide applications of Riemann-Liouville fractional integrals, many authors extended to research Riemann-Liouville fractional inequalities via different classes of convex mappings: for generalizations, variations and new inequalities for them, see for instance [23]-[32] and the references therein.
We also use here the subsequent conformable fractional integrals.

Definition 1.12. Let $\alpha \in (n, n+1]$ and set $\beta = \alpha - n$, then the left conformable fractional integral starting at a is defined by

$$\left(I_{\alpha}^{a} f\right)(t) = \frac{1}{n!} \int_a^t (t-x)^n (x-a)^{\beta-1} f(x)dx.$$

Analogously, the right conformable fractional integral is defined by

$$\left({}^{b}I_{\alpha} f\right)(t) = \frac{1}{n!} \int_t^b (x-t)^n (b-x)^{\beta-1} f(x)dx.$$

Notice that if $\alpha = n+1$, then $\beta = \alpha - n = n+1-n = 1$, where $n = 0, 1, 2, \ldots$, and hence $\left(I_{\alpha}^{a} f\right)(t) = \left(J_{n+1}^{a} f\right)(t)$.
In [33], Set et al. obtained a generalization of Hermite-Hadamard type inequality via conformable fractional integrals involving s-convex mappings.

Theorem 1.13. [33] *Enable $f : [a,b] \longrightarrow \mathbb{R}$ be a function with $0 \leq a < b$, $s \in (0,1]$, and $f \in L_1[a,b]$. If f is a convex mapping on $[a,b]$, then the coming inequalities for conformable fractional integrals clasp:*

$$\frac{\Gamma(\alpha-n)}{\Gamma(\alpha+1)} f\left(\frac{a+b}{2}\right) \leq \frac{1}{2^s (b-a)^{\alpha}} \left[\left(I_{\alpha}^{a} f\right)(b) + \left({}^{b}I_{\alpha} f\right)(a) \right]$$

$$\leq \left[\frac{\beta(n+s+1, \alpha-n) + \beta(n+1, \alpha-n+s)}{n!} \right] \frac{f(a)+f(b)}{2^s},$$

together $\alpha \in (n, n+1]$, $n \in \mathbb{N}$, $n = 0, 1, 2, \ldots$, where Γ is Euler gamma function.

In recent years, some researchers have studied bounds for Hermite-Hadamard inequality, Fejér type inequality and Ostrowski type inequality etc. via conformable fractional integrals. For more details about this topic, see [1], [15], [16], [34]-[37].

Let us recall the Gauss-Jacobi type quadrature formula as follows.

$$\int_a^b (x-a)^p(b-x)^q f(x)dx = \sum_{k=0}^{+\infty} B_{m,k} f(\gamma_k) + R_m^{\star}|f|, \tag{1.9}$$

for certain $B_{m,k}, \gamma_k$ and rest $R_m^{\star}|f|$, see [39].
In [21], Liu obtained integral inequalities for P-function related to the left-hand side of (1.9), and in [28], Özdemir et al. also presented several integral inequalities concerning the left-hand side of (1.9) via some kinds of convexity.
Motivated by the above literatures, the main objective of this article is to establish integral inequalities for Gauss-Jacobi type quadrature formula and some new estimates on Hermite-Hadamard type inequalities via conformable fractional integrals associated with generalized relative semi-(m,h)-preinvex mappings. These new obtained inequalities are also applied to construct inequalities for special means.
To end this section, let us consider the following special functions:

(1) The Beta function:

$$\beta(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} = \int_0^1 t^{x-1}(1-t)^{y-1}\mathrm{d}t, \quad x,y>0,$$

(2) The incomplete Beta function:

$$\beta_a(x,y) = \int_0^a t^{x-1}(1-t)^{y-1}\mathrm{d}t, \;\; 0<a<1, \;\; x,y>0.$$

2. Main results involving Gauss-Jacobi type quadrature formula

The following definitions will be used in this section.

Definition 2.1. [8] A set $K \subseteq \mathbb{R}^n$ is named as m-invex with respect to the mapping $\eta : K \times K \times (0,1] \longrightarrow \mathbb{R}^n$ for some fixed $m \in (0,1]$, if $mx + t\eta(y,mx) \in K$ grips for each $x,y \in K$ and any $t \in [0,1]$.

Remark 2.2. In Definition 2.1, under certain conditions, the mapping $\eta(y,mx)$ could reduce to $\eta(y,x)$. For example when $m=1$, then the m-invex set degenerates an invex set on K.

We next introduce generalized relative semi-(m,h)-preinvex mappings.

Definition 2.3. Let $K \subseteq \mathbb{R}$ be an open m-invex set with respect to the mapping $\eta : K \times K \times (0,1] \longrightarrow \mathbb{R}$, $h : [0,1] \longrightarrow [0,+\infty)$ and $\varphi : I \longrightarrow K$ are continuous functions. A mapping $f : K \longrightarrow \mathbb{R}$ is said to be generalized relative semi-(m,h)-preinvex, if

$$f\big(m\varphi(x) + t\eta(\varphi(y),\varphi(x),m)\big) \le mh(1-t)f(x) + h(t)f(y) \tag{2.1}$$

holds for all $x,y \in I$ and $t \in [0,1]$ for some fixed $m \in (0,1]$.

Remark 2.4. Let us discuss some special cases in Definition 2.3 as follows.
(I) Taking $h(t) = t$, then we get generalized relative semi-m-preinvex mappings.
(II) Taking $h(t) = t^s$ for $s \in (0,1]$, then we get generalized relative semi-(m,s)-Breckner-preinvex mappings.
(III) Taking $h(t) = t^{-s}$ for $s \in (0,1]$, then we get generalized relative semi-(m,s)-Godunova-Levin-Dragomir-preinvex mappings.
(IV) Taking $h(t) = 1$, then we get generalized relative semi-(m,P)-preinvex mappings.
(V) Taking $h(t) = t(1-t)$, then we get generalized relative semi-(m,tgs)-preinvex mappings.
(VI) Taking $h(t) = \frac{\sqrt{t}}{2\sqrt{1-t}}$, then we get generalized relative semi-m-MT-preinvex mappings.

It is worth to mention here that to the best of our knowledge all the special cases discussed above are new in the literature.
We claim the following integral identity.

Lemma 2.5. *Let $\varphi : I \longrightarrow K$ be a continuous function. Assume that $f : K = [m\varphi(a), m\varphi(a) + \eta(\varphi(b), \varphi(a), m)] \longrightarrow \mathbb{R}$ is a continuous mapping on K° with respect to $\eta : K \times K \times (0,1] \longrightarrow \mathbb{R}$, for $\eta(\varphi(b), \varphi(a), m) > 0$. Then for some fixed $m \in (0,1]$ and any fixed $p, q > 0$, we have*

$$\begin{aligned}&\int_{m\varphi(a)}^{m\varphi(a)+\eta(\varphi(b),\varphi(a),m)} (x - m\varphi(a))^p (m\varphi(a) + \eta(\varphi(b), \varphi(a), m) - x)^q f(x)dx \\ &= \eta^{p+q+1}(\varphi(b), \varphi(a), m) \int_0^1 t^p (1-t)^q f(m\varphi(a) + t\eta(\varphi(b), \varphi(a), m))dt.\end{aligned} \tag{2.2}$$

Proof. It is easy to observe that

$$\begin{aligned}&\int_{m\varphi(a)}^{m\varphi(a)+\eta(\varphi(b),\varphi(a),m)} (x - m\varphi(a))^p (m\varphi(a) + \eta(\varphi(b), \varphi(a), m) - x)^q f(x)dx \\ &= \eta(\varphi(b), \varphi(a), m) \int_0^1 (m\varphi(a) + t\eta(\varphi(b), \varphi(a), m) - m\varphi(a))^p \\ &\times (m\varphi(a) + \eta(\varphi(b), \varphi(a), m) - m\varphi(a) - t\eta(\varphi(b), \varphi(a), m))^q \\ &\times f(m\varphi(a) + t\eta(\varphi(b), \varphi(a), m))dt \\ &= \eta^{p+q+1}(\varphi(b), \varphi(a), m) \int_0^1 t^p (1-t)^q f(m\varphi(a) + t\eta(\varphi(b), \varphi(a), m))dt.\end{aligned}$$

This completes the proof of the lemma. □

With the help of Lemma 2.5, we have the following results.

Theorem 2.6. *Suppose $h : [0,1] \longrightarrow [0,+\infty)$ and $\varphi : I \longrightarrow K$ are continuous functions. Assume that $f : K = [m\varphi(a), m\varphi(a) + \eta(\varphi(b), \varphi(a), m)] \longrightarrow \mathbb{R}$ is a continuous mapping on K° respecting $\eta : K \times K \times (0,1] \longrightarrow \mathbb{R}$, for $\eta(\varphi(b), \varphi(a), m) > 0$. If $|f|^{\frac{k}{k-1}}$ for $k > 1$ is generalized relative semi-(m,h)-preinvex mapping on an open m-invex* *set K for some fixed $m \in (0,1]$, then for any fixed $p, q > 0$, we have*

$$\int_{m\varphi(a)}^{m\varphi(a)+\eta(\varphi(b),\varphi(a),m)} (x - m\varphi(a))^p (m\varphi(a) + \eta(\varphi(b),\varphi(a),m) - x)^q f(x)dx$$
$$\leq \eta^{p+q+1}(\varphi(b),\varphi(a),m)\beta^{\frac{1}{k}}(kp+1, kq+1) \tag{2.3}$$
$$\times \left[m|f(a)|^{\frac{k}{k-1}} + |f(b)|^{\frac{k}{k-1}}\right]^{\frac{k-1}{k}} \left(\int_0^1 h(t)dt\right)^{\frac{k-1}{k}}.$$

Proof. Since $|f|^{\frac{k}{k-1}}$ is generalized relative semi-(m,h)-preinvex on K, combining with Lemma 2.5, Hölder inequality and properties of the modulus, we get

$$\int_{m\varphi(a)}^{m\varphi(a)+\eta(\varphi(b),\varphi(a),m)} (x - m\varphi(a))^p (m\varphi(a) + \eta(\varphi(b),\varphi(a),m) - x)^q f(x)dx$$
$$\leq |\eta(\varphi(b),\varphi(a),m)|^{p+q+1}\left[\int_0^1 t^{kp}(1-t)^{kq}dt\right]^{\frac{1}{k}}$$
$$\times \left[\int_0^1 |f(m\varphi(a) + t\eta(\varphi(b),\varphi(a),m))|^{\frac{k}{k-1}}dt\right]^{\frac{k-1}{k}}$$
$$\leq \eta^{p+q+1}(\varphi(b),\varphi(a),m)\beta^{\frac{1}{k}}(kp+1, kq+1)$$
$$\times \left[\int_0^1 \left(mh(1-t)|f(a)|^{\frac{k}{k-1}} + h(t)|f(b)|^{\frac{k}{k-1}}\right)dt\right]^{\frac{k-1}{k}}$$
$$= \eta^{p+q+1}(\varphi(b),\varphi(a),m)\beta^{\frac{1}{k}}(kp+1, kq+1)$$
$$\times \left[m|f(a)|^{\frac{k}{k-1}} + |f(b)|^{\frac{k}{k-1}}\right]^{\frac{k-1}{k}} \left(\int_0^1 h(t)dt\right)^{\frac{k-1}{k}}.$$

So, the proof of this theorem is complete. □

We point out some special cases of Theorem 2.6.

Corollary 2.7. *In Theorem 2.6 for $h(t) = t^s$ where $s \in [0,1]$, we have the following inequality for generalized relative semi-(m,s)-Breckner-preinvex mappings:*

$$\int_{m\varphi(a)}^{m\varphi(a)+\eta(\varphi(b),\varphi(a),m)} (x - m\varphi(a))^p (m\varphi(a) + \eta(\varphi(b),\varphi(a),m) - x)^q f(x)dx$$
$$\leq \eta^{p+q+1}(\varphi(b),\varphi(a),m)\beta^{\frac{1}{k}}(kp+1, kq+1)\left[\frac{m|f(a)|^{\frac{k}{k-1}} + |f(b)|^{\frac{k}{k-1}}}{s+1}\right]^{\frac{k-1}{k}}.$$

Corollary 2.8. *In Theorem 2.6 for $h(t) = t^{-s}$ where $s \in [0,1)$, we get the following inequality for generalized relative semi-(m,s)-Godunova-Levin-Dragomir preinvex mappings:*

$$\int_{m\varphi(a)}^{m\varphi(a)+\eta(\varphi(b),\varphi(a),m)} (x - m\varphi(a))^p (m\varphi(a) + \eta(\varphi(b),\varphi(a),m) - x)^q f(x)dx$$

$$\leq \eta^{p+q+1}(\varphi(b),\varphi(a),m)\beta^{\frac{1}{k}}(kp+1,kq+1)\left[\frac{m|f(a)|^{\frac{k}{k-1}}+|f(b)|^{\frac{k}{k-1}}}{1-s}\right]^{\frac{k-1}{k}}.$$

Corollary 2.9. *In Theorem 2.6 for $h(t)=t(1-t)$, we obtain the following inequality for generalized relative semi-(m,tgs)-preinvex mappings:*

$$\int_{m\varphi(a)}^{m\varphi(a)+\eta(\varphi(b),\varphi(a),m)}(x-m\varphi(a))^p(m\varphi(a)+\eta(\varphi(b),\varphi(a),m)-x)^q f(x)dx$$

$$\leq \eta^{p+q+1}(\varphi(b),\varphi(a),m)\beta^{\frac{1}{k}}(kp+1,kq+1)\left[\frac{m|f(a)|^{\frac{k}{k-1}}+|f(b)|^{\frac{k}{k-1}}}{6}\right]^{\frac{k-1}{k}}.$$

Corollary 2.10. *In Theorem 2.6 for $h(t)=\frac{\sqrt{t}}{2\sqrt{1-t}}$, we deduce the following inequality for generalized relative semi-m-MT-preinvex mappings:*

$$\int_{m\varphi(a)}^{m\varphi(a)+\eta(\varphi(b),\varphi(a),m)}(x-m\varphi(a))^p(m\varphi(a)+\eta(\varphi(b),\varphi(a),m)-x)^q f(x)dx$$
$$\leq \left(\frac{\pi}{4}\right)^{\frac{k-1}{k}}\eta^{p+q+1}(\varphi(b),\varphi(a),m)\beta^{\frac{1}{k}}(kp+1,kq+1)$$
$$\times\left[m|f(a)|^{\frac{k}{k-1}}+|f(b)|^{\frac{k}{k-1}}\right]^{\frac{k-1}{k}}.$$

Theorem 2.11. *Suppose $h:[0,1]\longrightarrow[0,+\infty)$ and $\varphi:I\longrightarrow K$ are continuous functions. Assume that $f:K=[m\varphi(a),m\varphi(a)+\eta(\varphi(b),\varphi(a),m)]\longrightarrow\mathbb{R}$ is a continuous mapping on K° respecting $\eta:K\times K\times(0,1]\longrightarrow\mathbb{R}$, for $\eta(\varphi(b),\varphi(a),m)>0$. If $|f|^l$ for $l\geq 1$ is generalized relative semi-(m,h)-preinvex mapping on an open m-invex set K for some fixed $m\in(0,1]$, then for any fixed $p,q>0$, we have*

$$\int_{m\varphi(a)}^{m\varphi(a)+\eta(\varphi(b),\varphi(a),m)}(x-m\varphi(a))^p(m\varphi(a)+\eta(\varphi(b),\varphi(a),m)-x)^q f(x)dx$$
$$\leq \eta^{p+q+1}(\varphi(b),\varphi(a),m)\beta^{\frac{l-1}{l}}(p+1,q+1) \tag{2.4}$$
$$\times\left[m|f(a)|^l I(h(t);p,q)+|f(b)|^l I(h(t);q,p)\right]^{\frac{1}{l}},$$

where $I(h(t);p,q):=\displaystyle\int_0^1 t^p(1-t)^q h(1-t)dt.$

Proof. Since $|f|^l$ is generalized relative semi-(m,h)-preinvex on K, combining with Lemma 2.5, the well-known power mean inequality and properties of the modulus, we have

$$\begin{aligned}
&\int_{m\varphi(a)}^{m\varphi(a)+\eta(\varphi(b),\varphi(a),m)}(x-m\varphi(a))^p(m\varphi(a)+\eta(\varphi(b),\varphi(a),m)-x)^q f(x)dx\\
&=\eta^{p+q+1}(\varphi(b),\varphi(a),m)\\
&\times\int_0^1\left[t^p(1-t)^q\right]^{\frac{l-1}{l}}\left[t^p(1-t)^q\right]^{\frac{1}{l}}f(m\varphi(a)+t\eta(\varphi(b),\varphi(a),m))dt\\
&\leq|\eta(\varphi(b),\varphi(a),m)|^{p+q+1}\\
&\times\left[\int_0^1 t^p(1-t)^q dt\right]^{\frac{l-1}{l}}\left[\int_0^1 t^p(1-t)^q|f(m\varphi(a)+t\eta(\varphi(b),\varphi(a),m))|^l dt\right]^{\frac{1}{l}}\\
&\leq\eta^{p+q+1}(\varphi(b),\varphi(a),m)\beta^{\frac{l-1}{l}}(p+1,q+1)\\
&\times\left[\int_0^1 t^p(1-t)^q\left(mh(1-t)|f(a)|^l+h(t)|f(b)|^l\right)dt\right]^{\frac{1}{l}}\\
&=\eta^{p+q+1}(\varphi(b),\varphi(a),m)\beta^{\frac{l-1}{l}}(p+1,q+1)\\
&\times\left[m|f(a)|^l I(h(t);p,q)+|f(b)|^l I(h(t);q,p)\right]^{\frac{1}{l}}.
\end{aligned}$$

So, the proof of this theorem is complete. □

Let us discuss some special cases of Theorem 2.11.

Corollary 2.12. *In Theorem* 2.11 *for* $h(t)=t^s$ *with* $s\in[0,1]$*, one can get the following inequality for generalized relative semi-*(m,s)*-Breckner-preinvex mappings:*

$$\begin{aligned}
&\int_{m\varphi(a)}^{m\varphi(a)+\eta(\varphi(b),\varphi(a),m)}(x-m\varphi(a))^p(m\varphi(a)+\eta(\varphi(b),\varphi(a),m)-x)^q f(x)dx\\
&\leq\eta^{p+q+1}(\varphi(b),\varphi(a),m)\beta^{\frac{l-1}{l}}(p+1,q+1)\\
&\times\left[m|f(a)|^l\beta(p+1,q+s+1)+|f(b)|^l\beta(q+1,p+s+1)\right]^{\frac{1}{l}}.
\end{aligned}$$

Corollary 2.13. *In Theorem* 2.11 *for* $h(t)=t^{-s}$ *with* $s\in(0,1]$*, we deduce the following inequality for generalized relative semi-*(m,s)*-Godunova-Levin-Dragomir preinvex mappings:*

$$\begin{aligned}
&\int_{m\varphi(a)}^{m\varphi(a)+\eta(\varphi(b),\varphi(a),m)}(x-m\varphi(a))^p(m\varphi(a)+\eta(\varphi(b),\varphi(a),m)-x)^q f(x)dx\\
&\leq\eta^{p+q+1}(\varphi(b),\varphi(a),m)\beta^{\frac{l-1}{l}}(p+1,q+1)\\
&\times\left[m|f(a)|^l\beta(p+1,q-s+1)+|f(b)|^l\beta(q+1,p-s+1)\right]^{\frac{1}{l}}.
\end{aligned}$$

Corollary 2.14. *In Theorem* 2.11 *for* $h(t) = t(1-t)$, *one can obtain the following inequality for generalized relative semi-*(m, tgs)*-preinvex mappings:*

$$\int_{m\varphi(a)}^{m\varphi(a)+\eta(\varphi(b),\varphi(a),m)} (x-m\varphi(a))^p(m\varphi(a)+\eta(\varphi(b),\varphi(a),m)-x)^q f(x)dx$$
$$\leq \eta^{p+q+1}(\varphi(b),\varphi(a),m)\beta^{\frac{l-1}{l}}(p+1,q+1)\beta^{\frac{1}{l}}(p+2,q+2)\Big[m|f(a)|^l+|f(b)|^l\Big]^{\frac{1}{l}}.$$

Corollary 2.15. *In Theorem* 2.11 *for* $h(t) = \frac{\sqrt{t}}{2\sqrt{1-t}}$, *we derive the following inequality for generalized relative semi-m-MT-preinvex mappings:*

$$\int_{m\varphi(a)}^{m\varphi(a)+\eta(\varphi(b),\varphi(a),m)} (x-m\varphi(a))^p(m\varphi(a)+\eta(\varphi(b),\varphi(a),m)-x)^q f(x)dx$$
$$\leq \left(\frac{1}{2}\right)^{\frac{1}{l}}\eta^{p+q+1}(\varphi(b),\varphi(a),m)\beta^{\frac{l-1}{l}}(p+1,q+1)$$
$$\times\left[m|f(a)|^l\beta\left(p+\frac{1}{2},q+\frac{3}{2}\right)+|f(b)|^l\beta\left(q+\frac{1}{2},p+\frac{3}{2}\right)\right]^{\frac{1}{l}}.$$

3. Other results involving conformable fractional integrals

For establishing our main results regarding generalizations of Hermite-Hadamard type inequalities associated with generalized relative semi-(m,h)-preinvexity via conformable fractional integrals, we need the following lemma.

Lemma 3.1. *Let* $\varphi : I \longrightarrow K$ *be a continuous function. Suppose* $K \subseteq \mathbb{R}$ *be an open m-invex subset with respect to* $\eta : K\times K\times(0,1] \longrightarrow \mathbb{R}$ *for some fixed* $m \in (0,1]$ *and let* $\eta(\varphi(b),\varphi(a),m) > 0$. *Assume that* $f : K = [m\varphi(a), m\varphi(a)+\eta(\varphi(b),\varphi(a),m)] \longrightarrow \mathbb{R}$ *be a twice differentiable function on* K° *and* $f'' \in L_1[m\varphi(a), m\varphi(a)+\eta(\varphi(b),\varphi(a),m)]$. *Then for* $\alpha > 0$, *we have*

$$\frac{\eta^{\alpha+2}(\varphi(x),\varphi(a),m)}{\eta(\varphi(b),\varphi(a),m)}\Bigg\{\frac{\beta(n+2,\alpha-n)}{\eta(\varphi(x),\varphi(a),m)}\Big[f'(m\varphi(a)+\eta(\varphi(x),\varphi(a),m))-f'(m\varphi(a))\Big]$$
$$-\frac{\beta(n+2,\alpha-n)f'(m\varphi(a)+\eta(\varphi(x),\varphi(a),m))}{\eta(\varphi(x),\varphi(a),m)}+\frac{(n+1)!}{\eta^{\alpha+2}(\varphi(x),\varphi(a),m)}$$
$$\times\Bigg[\left({}^{m\varphi(a)+\eta(\varphi(x),\varphi(a),m)}I_\alpha f\right)(m\varphi(a))$$
$$-(\alpha-n-1)\left({}^{m\varphi(a)+\eta(\varphi(x),\varphi(a),m)}I_{\alpha-1} f\right)(m\varphi(a))\Bigg]\Bigg\}+\frac{\eta^{\alpha+2}(\varphi(x),\varphi(b),m)}{\eta(\varphi(b),\varphi(a),m)}$$
$$\times\Bigg\{\frac{\beta(n+2,\alpha-n)}{\eta(\varphi(x),\varphi(b),m)}\Big[f'(m\varphi(b)+\eta(\varphi(x),\varphi(b),m))-f'(m\varphi(b))\Big]$$
$$-\frac{\beta(n+2,\alpha-n)f'(m\varphi(b)+\eta(\varphi(x),\varphi(b),m))}{\eta(\varphi(x),\varphi(b),m)}+\frac{(n+1)!}{\eta^{\alpha+2}(\varphi(x),\varphi(b),m)}$$

$$\times\Bigg[\left({}^{m\varphi(b)+\eta(\varphi(x),\varphi(b),m)}I_{\alpha}f\right)(m\varphi(b))$$

$$-(\alpha-n-1)\left({}^{m\varphi(b)+\eta(\varphi(x),\varphi(b),m)}I_{\alpha-1}f\right)(m\varphi(b))\Bigg]\Bigg\}$$

$$=\frac{\eta^{\alpha+2}(\varphi(x),\varphi(a),m)}{\eta(\varphi(b),\varphi(a),m)}$$

$$\times\int_0^1(\beta(n+2,\alpha-n)-\beta_t(n+2,\alpha-n))f''(m\varphi(a)+t\eta(\varphi(x),\varphi(a),m))dt$$

$$+\frac{\eta^{\alpha+2}(\varphi(x),\varphi(b),m)}{\eta(\varphi(b),\varphi(a),m)}$$

$$\times\int_0^1(\beta(n+2,\alpha-n)-\beta_t(n+2,\alpha-n))f''(m\varphi(b)+t\eta(\varphi(x),\varphi(b),m))dt. \tag{3.1}$$

We denote

$$\begin{aligned}&I_{f,\eta,\varphi}(x;\alpha,n,m,a,b)\\&:=\frac{\eta^{\alpha+2}(\varphi(x),\varphi(a),m)}{\eta(\varphi(b),\varphi(a),m)}\\&\times\int_0^1(\beta(n+2,\alpha-n)-\beta_t(n+2,\alpha-n))f''(m\varphi(a)+t\eta(\varphi(x),\varphi(a),m))dt\\&+\frac{\eta^{\alpha+2}(\varphi(x),\varphi(b),m)}{\eta(\varphi(b),\varphi(a),m)}\\&\times\int_0^1(\beta(n+2,\alpha-n)-\beta_t(n+2,\alpha-n))f''(m\varphi(b)+t\eta(\varphi(x),\varphi(b),m))dt.\end{aligned} \tag{3.2}$$

Proof. A simple proof of the equality can be done by performing two integration by parts in the integrals from the right side of (3.2) and changing the variables. □

Using relation (3.2) and Lemma (3.1), we now state the following theorem.

Theorem 3.2. *Suppose $h:[0,1]\longrightarrow[0,+\infty)$ and $\varphi:I\longrightarrow K$ are continuous functions. Let $K\subseteq\mathbb{R}$ be an open m-invex subset with respect to $\eta:K\times K\times(0,1]\longrightarrow\mathbb{R}$ for some fixed $m\in(0,1]$ and let $\eta(\varphi(b),\varphi(a),m)>0$. Assume that $f:K=[m\varphi(a),m\varphi(a)+\eta(\varphi(b),\varphi(a),m)]\longrightarrow\mathbb{R}$ be a twice differentiable mapping on K° (the interior of K). If $|f''|^q$ is generalized relative semi-(m,h)-preinvex mapping on K, $q>1$, $p^{-1}+q^{-1}=1$, then for $\alpha>0$, we have*

$$\begin{aligned}\left|I_{f,\eta,\varphi}(x;\alpha,n,m,a,b)\right|&\le\frac{\delta^{\frac{1}{p}}(p,\alpha,n)}{\eta(\varphi(b),\varphi(a),m)}\left(\int_0^1h(t)dt\right)^{\frac{1}{q}}\\&\times\Bigg\{|\eta(\varphi(x),\varphi(a),m)|^{\alpha+2}\Big[m|f''(a)|^q+|f''(x)|^q\Big]^{\frac{1}{q}}\\&+|\eta(\varphi(x),\varphi(b),m)|^{\alpha+2}\Big[m|f''(b)|^q+|f''(x)|^q\Big]^{\frac{1}{q}}\Bigg\},\end{aligned} \tag{3.3}$$

where $\delta(p,\alpha,n) := \int_0^1 \big[\beta(n+2,\alpha-n) - \beta_t(n+2,\alpha-n)\big]^p dt.$

Proof. Suppose that $q>1$. Using relation (3.2), Hölder inequality, generalized relative semi-(m,h)-preinvexity of $|f''|^q$ on K° and properties of the modulus, we have

$$
\begin{aligned}
&\big|I_{f,\eta,\varphi}(x;\alpha,n,m,a,b)\big| \\
&\leq \frac{|\eta(\varphi(x),\varphi(a),m)|^{\alpha+2}}{|\eta(\varphi(b),\varphi(a),m)|} \\
&\times \int_0^1 |\beta(n+2,\alpha-n) - \beta_t(n+2,\alpha-n)||f''(m\varphi(a)+t\eta(\varphi(x),\varphi(a),m))|dt \\
&+ \frac{|\eta(\varphi(x),\varphi(b),m)|^{\alpha+2}}{|\eta(\varphi(b),\varphi(a),m)|} \\
&\times \int_0^1 |\beta(n+2,\alpha-n) - \beta_t(n+2,\alpha-n)||f''(m\varphi(b)+t\eta(\varphi(x),\varphi(b),m))|dt \\
&\leq \frac{|\eta(\varphi(x),\varphi(a),m)|^{\alpha+2}}{\eta(\varphi(b),\varphi(a),m)} \left(\int_0^1 \big[\beta(n+2,\alpha-n) - \beta_t(n+2,\alpha-n)\big]^p dt\right)^{\frac{1}{p}} \\
&\times \left(\int_0^1 |f''(m\varphi(a)+t\eta(\varphi(x),\varphi(a),m))|^q dt\right)^{\frac{1}{q}} \\
&+ \frac{|\eta(\varphi(x),\varphi(b),m)|^{\alpha+2}}{\eta(\varphi(b),\varphi(a),m)} \left(\int_0^1 \big[\beta(n+2,\alpha-n) - \beta_t(n+2,\alpha-n)\big]^p dt\right)^{\frac{1}{p}} \\
&\times \left(\int_0^1 |f''(m\varphi(b)+t\eta(\varphi(x),\varphi(b),m))|^q dt\right)^{\frac{1}{q}} \\
&\leq \frac{|\eta(\varphi(x),\varphi(a),m)|^{\alpha+2}}{\eta(\varphi(b),\varphi(a),m)} \left(\int_0^1 \big[\beta(n+2,\alpha-n) - \beta_t(n+2,\alpha-n)\big]^p dt\right)^{\frac{1}{p}} \\
&\times \left(\int_0^1 \Big[mh(1-t)|f''(a)|^q + h(t)|f''(x)|^q\Big] dt\right)^{\frac{1}{q}} \\
&+ \frac{|\eta(\varphi(x),\varphi(b),m)|^{\alpha+2}}{\eta(\varphi(b),\varphi(a),m)} \left(\int_0^1 \big[\beta(n+2,\alpha-n) - \beta_t(n+2,\alpha-n)\big]^p dt\right)^{\frac{1}{p}} \\
&\times \left(\int_0^1 \Big[mh(1-t)|f''(b)|^q + h(t)|f''(x)|^q\Big] dt\right)^{\frac{1}{q}} \\
&= \frac{\delta^{\frac{1}{p}}(p,\alpha,n)}{\eta(\varphi(b),\varphi(a),m)} \left(\int_0^1 h(t)dt\right)^{\frac{1}{q}} \\
&\times \Bigg\{ |\eta(\varphi(x),\varphi(a),m)|^{\alpha+2}\Big[m|f''(a)|^q + |f''(x)|^q\Big]^{\frac{1}{q}} \\
&+ |\eta(\varphi(x),\varphi(b),m)|^{\alpha+2}\Big[m|f''(b)|^q + |f''(x)|^q\Big]^{\frac{1}{q}} \Bigg\}.
\end{aligned}
$$

So, the proof of this theorem is complete. □

Corollary 3.3. *In Theorem* 3.2*, if choosing* $\alpha \in (n, n+1]$ *where* $n = 0, 1, 2, \ldots,$ *one can get the following inequality for conformable fractional integrals:*

$$\Bigg| \frac{-\eta^{\alpha+1}(\varphi(x), \varphi(a), m) f'(m\varphi(a)) - \eta^{\alpha+1}(\varphi(x), \varphi(b), m) f'(m\varphi(b))}{\eta(\varphi(b), \varphi(a), m)}$$

$$- \frac{(n+2-\alpha)(n+1)!}{\eta(\varphi(b), \varphi(a), m)}$$

$$\times \left[\left({}^{(m\varphi(a)+\eta(\varphi(x),\varphi(a),m))} I_\alpha f \right) (m\varphi(a)) + \left({}^{(m\varphi(b)+\eta(\varphi(x),\varphi(b),m))} I_\alpha f \right) (m\varphi(b)) \right] \Bigg|$$

$$\leq \frac{\delta^{\frac{1}{p}}(p, \alpha, n)}{\eta(\varphi(b), \varphi(a), m)} \left(\int_0^1 h(t)dt \right)^{\frac{1}{q}}$$

$$\times \left\{ |\eta(\varphi(x), \varphi(a), m)|^{\alpha+2} \Big[m|f''(a)|^q + |f''(x)|^q \Big]^{\frac{1}{q}} \right.$$

$$\left. + |\eta(\varphi(x), \varphi(b), m)|^{\alpha+2} \Big[m|f''(b)|^q + |f''(x)|^q \Big]^{\frac{1}{q}} \right\}.$$

Corollary 3.4. *In Corollary* 3.3*, if putting* $\alpha = n+1$ *where* $n = 0, 1, 2, \ldots$ *and* $|f''| \leq K$*, one can obtain the following inequality for fractional integrals:*

$$\Bigg| \frac{-\eta^{\alpha+1}(\varphi(x), \varphi(a), m) f'(m\varphi(a)) - \eta^{\alpha+1}(\varphi(x), \varphi(b), m) f'(m\varphi(b))}{(\alpha+1)\eta(\varphi(b), \varphi(a), m)}$$

$$+ \frac{\eta^{\alpha}(\varphi(x), \varphi(a), m) f(m\varphi(a) + \eta(\varphi(x), \varphi(a), m))}{\eta(\varphi(b), \varphi(a), m)}$$

$$+ \frac{\eta^{\alpha}(\varphi(x), \varphi(b), m) f(m\varphi(b) + \eta(\varphi(x), \varphi(b), m))}{\eta(\varphi(b), \varphi(a), m)}$$

$$- \frac{\Gamma(\alpha+1)}{\eta(\varphi(b), \varphi(a), m)}$$

$$\times \left[J^{\alpha}_{(m\varphi(a)+\eta(\varphi(x),\varphi(a),m))-} f(m\varphi(a)) + J^{\alpha}_{(m\varphi(b)+\eta(\varphi(x),\varphi(b),m))-} f(m\varphi(b)) \right] \Bigg|$$

$$\leq K(m+1)^{\frac{1}{q}} \left(\frac{\Gamma(p+1)\Gamma\left(\frac{1}{\alpha+1}\right)}{\Gamma\left(p+1+\frac{1}{\alpha+1}\right)} \right)^{\frac{1}{p}} \left(\int_0^1 h(t)dt \right)^{\frac{1}{q}}$$

$$\times \left[\frac{|\eta(\varphi(x), \varphi(a), m)|^{\alpha+2} + |\eta(\varphi(x), \varphi(b), m)|^{\alpha+2}}{\eta(\varphi(b), \varphi(a), m)} \right].$$

Theorem 3.5. *Suppose* $h : [0, 1] \longrightarrow [0, +\infty)$ *and* $\varphi : I \longrightarrow K$ *are continuous functions. Let* $K \subseteq \mathbb{R}$ *be an open* m*-invex subset with respect to* $\eta : K \times K \times (0, 1] \longrightarrow \mathbb{R}$ *for some fixed* $m \in (0, 1]$ *and let* $\eta(\varphi(b), \varphi(a), m) > 0$*. Assume that* $f : K =$

$[m\varphi(a), m\varphi(a)+\eta(\varphi(b),\varphi(a),m)] \longrightarrow \mathbb{R}$ *be a twice differentiable mapping on* K°. *If* $|f''|^q$ *is generalized relative semi-*(m,h)*-preinvex mapping on* K, $q \geq 1$, *then for* $\alpha > 0$, *we have*

$$\big|I_{f,\eta,\varphi}(x;\alpha,n,m,a,b)\big| \leq \frac{\beta^{1-\frac{1}{q}}(n+3,\alpha-n)}{\eta(\varphi(b),\varphi(a),m)}$$

$$\times\Bigg\{|\eta(\varphi(x),\varphi(a),m)|^{\alpha+2}\Big[m|f''(a)|^q A(h(t);\alpha,n)+|f''(x)|^q A(h(1-t);\alpha,n)\Big]^{\frac{1}{q}}$$

$$+|\eta(\varphi(x),\varphi(b),m)|^{\alpha+2}\Big[m|f''(b)|^q A(h(t);\alpha,n)+|f''(x)|^q A(h(1-t);\alpha,n)\Big]^{\frac{1}{q}}\Bigg\}, \tag{3.4}$$

where $A(h(t);\alpha,n) := \int_0^1 \big[\beta(n+2,\alpha-n)-\beta_t(n+2,\alpha-n)\big]h(1-t)dt.$

Proof. Suppose that $q \geq 1$. Using relation (3.2), the well-known power mean inequality, the generalized relative semi-(m,h)-preinvexity of $|f''|^q$ and properties of the modulus, we have

$$\big|I_{f,\eta,\varphi}(x;\alpha,n,m,a,b)\big| \leq \frac{|\eta(\varphi(x),\varphi(a),m)|^{\alpha+2}}{|\eta(\varphi(b),\varphi(a),m)|}$$

$$\times\int_0^1 |\beta(n+2,\alpha-n)-\beta_t(n+2,\alpha-n)||f''(m\varphi(a)+t\eta(\varphi(x),\varphi(a),m))|dt$$

$$+\frac{|\eta(\varphi(x),\varphi(b),m)|^{\alpha+2}}{|\eta(\varphi(b),\varphi(a),m)|}$$

$$\times\int_0^1 |\beta(n+2,\alpha-n)-\beta_t(n+2,\alpha-n)||f''(m\varphi(b)+t\eta(\varphi(x),\varphi(b),m))|dt$$

$$\leq \frac{|\eta(\varphi(x),\varphi(a),m)|^{\alpha+2}}{\eta(\varphi(b),\varphi(a),m)}\left(\int_0^1 \big[\beta(n+2,\alpha-n)-\beta_t(n+2,\alpha-n)\big]dt\right)^{1-\frac{1}{q}}$$

$$\times\left[\int_0^1 \big[\beta(n+2,\alpha-n)-\beta_t(n+2,\alpha-n)\big]|f''(m\varphi(a)+t\eta(\varphi(x),\varphi(a),m))|^q dt\right]^{\frac{1}{q}}$$

$$+\frac{|\eta(\varphi(x),\varphi(b),m)|^{\alpha+2}}{\eta(\varphi(b),\varphi(a),m)}\left(\int_0^1 \big[\beta(n+2,\alpha-n)-\beta_t(n+2,\alpha-n)\big]dt\right)^{1-\frac{1}{q}}$$

$$\times\left[\int_0^1 \big[\beta(n+2,\alpha-n)-\beta_t(n+2,\alpha-n)\big]|f''(m\varphi(b)+t\eta(\varphi(x),\varphi(b),m))|^q dt\right]^{\frac{1}{q}}$$

$$\leq \frac{|\eta(\varphi(x),\varphi(a),m)|^{\alpha+2}}{\eta(\varphi(b),\varphi(a),m)}\left(\int_0^1 \big[\beta(n+2,\alpha-n)-\beta_t(n+2,\alpha-n)\big]dt\right)^{1-\frac{1}{q}}$$

$$\times\left[\int_0^1 \big[\beta(n+2,\alpha-n)-\beta_t(n+2,\alpha-n)\big]\Big[mh(1-t)|f''(a)|^q+h(t)|f''(x)|^q\Big]dt\right]^{\frac{1}{q}}$$

$$+\frac{|\eta(\varphi(x),\varphi(b),m)|^{\alpha+2}}{\eta(\varphi(b),\varphi(a),m)}\left(\int_0^1\big[\beta(n+2,\alpha-n)-\beta_t(n+2,\alpha-n)\big]dt\right)^{1-\frac{1}{q}}$$

$$\times\left[\int_0^1\big[\beta(n+2,\alpha-n)-\beta_t(n+2,\alpha-n)\big]\Big[mh(1-t)|f''(b)|^q+h(t)|f''(x)|^q\Big]dt\right]^{\frac{1}{q}}$$

$$=\frac{\beta^{1-\frac{1}{q}}(n+3,\alpha-n)}{\eta(\varphi(b),\varphi(a),m)}$$

$$\times\Bigg\{|\eta(\varphi(x),\varphi(a),m)|^{\alpha+2}\Big[m|f''(a)|^qA(h(t);\alpha,n)+|f''(x)|^qA(h(1-t);\alpha,n)\Big]^{\frac{1}{q}}$$

$$+|\eta(\varphi(x),\varphi(b),m)|^{\alpha+2}\Big[m|f''(b)|^qA(h(t);\alpha,n)+|f''(x)|^qA(h(1-t);\alpha,n)\Big]^{\frac{1}{q}}\Bigg\}.$$

So, the proof of this theorem is complete. □

Corollary 3.6. *In Theorem* 3.5*, if the choice of* $\alpha\in(n,n+1]$ *where* $n=0,1,2,\dots,$ *we get the following inequality for conformable fractional integrals:*

$$\Bigg|\frac{-\eta^{\alpha+1}(\varphi(x),\varphi(a),m)f'(m\varphi(a))-\eta^{\alpha+1}(\varphi(x),\varphi(b),m)f'(m\varphi(b))}{\eta(\varphi(b),\varphi(a),m)}$$

$$-\frac{(n+2-\alpha)(n+1)!}{\eta(\varphi(b),\varphi(a),m)}$$

$$\times\left[\left({}^{(m\varphi(a)+\eta(\varphi(x),\varphi(a),m))}I_\alpha f\right)(m\varphi(a))+\left({}^{(m\varphi(b)+\eta(\varphi(x),\varphi(b),m))}I_\alpha f\right)(m\varphi(b))\right]\Bigg|$$

$$\leq\frac{\beta^{1-\frac{1}{q}}(n+3,\alpha-n)}{\eta(\varphi(b),\varphi(a),m)}$$

$$\times\Bigg\{|\eta(\varphi(x),\varphi(a),m)|^{\alpha+2}\Big[m|f''(a)|^qA(h(t);\alpha,n)+|f''(x)|^qA(h(1-t);\alpha,n)\Big]^{\frac{1}{q}}$$

$$+|\eta(\varphi(x),\varphi(b),m)|^{\alpha+2}\Big[m|f''(b)|^qA(h(t);\alpha,n)+|f''(x)|^qA(h(1-t);\alpha,n)\Big]^{\frac{1}{q}}\Bigg\}.$$

Corollary 3.7. *In Corollary* 3.6*, if the choice of* $\alpha=n+1$ *where* $n=0,1,2,\dots$ *and* $|f''|\leq K$*, we obtain the following inequality for fractional integrals:*

$$\Bigg|\frac{-\eta^{\alpha+1}(\varphi(x),\varphi(a),m)f'(m\varphi(a))-\eta^{\alpha+1}(\varphi(x),\varphi(b),m)f'(m\varphi(b))}{(\alpha+1)\eta(\varphi(b),\varphi(a),m)}$$

$$+\frac{\eta^{\alpha}(\varphi(x),\varphi(a),m)f(m\varphi(a)+\eta(\varphi(x),\varphi(a),m))}{\eta(\varphi(b),\varphi(a),m)}$$

$$+\frac{\eta^{\alpha}(\varphi(x),\varphi(b),m)f(m\varphi(b)+\eta(\varphi(x),\varphi(b),m))}{\eta(\varphi(b),\varphi(a),m)}-\frac{\Gamma(\alpha+1)}{\eta(\varphi(b),\varphi(a),m)}$$

$$\times\left[J^{\alpha}_{(m\varphi(a)+\eta(\varphi(x),\varphi(a),m))-}f(m\varphi(a))+J^{\alpha}_{(m\varphi(b)+\eta(\varphi(x),\varphi(b),m))-}f(m\varphi(b))\right]\Bigg|$$
$$\leq\frac{K}{(\alpha+2)^{1-\frac{1}{q}}}\left[mA(h(t);\alpha,n)+A(h(1-t);\alpha,n)\right]^{\frac{1}{q}}$$
$$\times\left[\frac{|\eta(\varphi(x),\varphi(a),m)|^{\alpha+2}+|\eta(\varphi(x),\varphi(b),m)|^{\alpha+2}}{|\eta(\varphi(b),\varphi(a),m)|}\right].$$

Remark 3.8. In Corollary 3.3 and Corollary 3.6, if taking $h(t)=t^{s}$, $h(t)=t^{-s}$, $h(t)=t(1-t)$ or $h(t)=\frac{\sqrt{t}}{2\sqrt{1-t}}$, then one can get some special conformable fractional integral inequalities for generalized relative semi-(m,s)-Breckner-preinvex functions, generalized relative semi-(m,s)-Godunova-Levin-Dragomir-preinvex functions, generalized relative semi-(m,tgs)-preinvex functions, and generalized relative semi-m-MT-preinvex functions, respectively. For Corollary 3.4 and Corollary 3.7, we also derive some similar Riemann-Liouville fractional integral inequalities for these functions.

4. Applications to special means

Let us begin this section by considering some particular means for two positive real numbers α,β $(\alpha\neq\beta)$ and for this aim we recall the following means:

1. The arithmetic mean:
$$A:=A(\alpha,\beta)=\frac{\alpha+\beta}{2}.$$
2. The geometric mean:
$$G:=G(\alpha,\beta)=\sqrt{\alpha\beta}.$$
3. The harmonic mean:
$$H:=H(\alpha,\beta)=\frac{2}{\frac{1}{\alpha}+\frac{1}{\beta}}.$$
4. The power mean:
$$P_r:=P_r(\alpha,\beta)=\left(\frac{\alpha^r+\beta^r}{2}\right)^{\frac{1}{r}},\quad r\geq 1.$$
5. The identric mean:
$$I:=I(\alpha,\beta)=\begin{cases}\frac{1}{e}\left(\frac{\beta^{\beta}}{\alpha^{\alpha}}\right), & \alpha\neq\beta;\\ \alpha, & \alpha=\beta.\end{cases}$$
6. The logarithmic mean:
$$L:=L(\alpha,\beta)=\frac{\beta-\alpha}{\ln(\beta)-\ln(\alpha)}.$$
7. The generalized log-mean:
$$L_p:=L_p(\alpha,\beta)=\left[\frac{\beta^{p+1}-\alpha^{p+1}}{(p+1)(\beta-\alpha)}\right]^{\frac{1}{p}};\ p\in\mathbb{R}\setminus\{-1,0\}.$$

8. The weighted p-power mean:

$$M_p\begin{pmatrix} \alpha_1, & \alpha_2, & \cdots & ,\alpha_n \\ u_1, & u_2, & \cdots & ,u_n \end{pmatrix} = \left(\sum_{i=1}^{n}\alpha_i u_i^p\right)^{\frac{1}{p}}$$

where $0 \le \alpha_i \le 1$, $u_i > 0$ $(i = 1, 2, \ldots, n)$ with $\sum_{i=1}^{n}\alpha_i = 1$.

It is well known that L_p is monotonic nondecreasing over $p \in \mathbb{R}$ with $L_{-1} := L$ and $L_0 := I$. In particular, we have $H \le G \le L \le I \le A$.
Now, let a and b be positive real numbers along with $a < b$. Consider the function $M := M(\varphi(x), \varphi(y)) : [\varphi(x), \varphi(x) + \eta(\varphi(y), \varphi(x))] \times [\varphi(x), \varphi(x) + \eta(\varphi(y), \varphi(x))] \longrightarrow \mathbb{R}_+$, which is one of the above mentioned means, $h : [0, 1] \longrightarrow [0, +\infty)$ and $\varphi : I \longrightarrow K$ are continuous mappings. Replace $\eta(\varphi(y), \varphi(x), m)$ with $\eta(\varphi(x), \varphi(y))$ and setting $\eta(\varphi(x), \varphi(y)) = M(\varphi(x), \varphi(y))$, $\forall x, y \in I$, together $m = 1$ in (3.3) and (3.4), one can obtain the subsequent interesting results involving means:

$$\left|I_{f,M(\cdot,\cdot),\varphi}(x;\alpha,n,1,a,b)\right|$$

$$= \Bigg| \frac{M^{\alpha+2}(\varphi(a),\varphi(x))}{M(\varphi(a),\varphi(b))}\Bigg\{ \frac{\beta(n+2,\alpha-n)}{M(\varphi(a),\varphi(x))}\Big[f'(\varphi(a)+M(\varphi(a),\varphi(x))) - f'(\varphi(a))\Big]$$

$$-\frac{\beta(n+2,\alpha-n)f'(\varphi(a)+M(\varphi(a),\varphi(x)))}{M(\varphi(a),\varphi(x))} + \frac{(n+1)!}{M^{\alpha+2}(\varphi(a),\varphi(x))}$$

$$\times\Bigg[\left({}^{\varphi(a)+M(\varphi(a),\varphi(x))}I_{\alpha}f\right)(\varphi(a)) - (\alpha-n-1)\left({}^{\varphi(a)+M(\varphi(a),\varphi(x))}I_{\alpha-1}f\right)(\varphi(a))\Bigg]\Bigg\}$$

$$+\frac{M^{\alpha+2}(\varphi(b),\varphi(x))}{M(\varphi(a),\varphi(b))}\Bigg\{ \frac{\beta(n+2,\alpha-n)}{M(\varphi(b),\varphi(x))}\Big[f'(\varphi(b)+M(\varphi(b),\varphi(x))) - f'(\varphi(b))\Big]$$

$$-\frac{\beta(n+2,\alpha-n)f'(\varphi(b)+M(\varphi(b),\varphi(x)))}{M(\varphi(b),\varphi(x))} + \frac{(n+1)!}{M^{\alpha+2}(\varphi(b),\varphi(x))}$$

$$\times\Bigg[\left({}^{\varphi(b)+M(\varphi(b),\varphi(x))}I_{\alpha}f\right)(\varphi(b)) - (\alpha-n-1)\left({}^{\varphi(b)+M(\varphi(b),\varphi(x))}I_{\alpha-1}f\right)(\varphi(b))\Bigg]\Bigg\}$$

$$\le \frac{\delta^{\frac{1}{p}}(p,\alpha,n)}{M(\varphi(a),\varphi(b))}\left(\int_0^1 h(t)dt\right)^{\frac{1}{q}}\Bigg\{M^{\alpha+2}(\varphi(a),\varphi(x))\Big[|f''(a)|^q + |f''(x)|^q\Big]^{\frac{1}{q}}$$

$$+M^{\alpha+2}(\varphi(b),\varphi(x))\Big[|f''(b)|^q + |f''(x)|^q\Big]^{\frac{1}{q}}\Bigg\}, \tag{4.1}$$

$$\begin{aligned}
&\left|I_{f,M(\cdot,\cdot),\varphi}(x;\alpha,n,1,a,b)\right| \\
&\leq \frac{\beta^{1-\frac{1}{q}}(n+3,\alpha-n)}{M(\varphi(a),\varphi(b))} \\
&\times\Bigg\{M^{\alpha+2}(\varphi(a),\varphi(x))\Big[|f''(a)|^q A(h(t);\alpha,n)+|f''(x)|^q A(h(1-t);\alpha,n)\Big]^{\frac{1}{q}} \\
&+M^{\alpha+2}(\varphi(b),\varphi(x))\Big[|f''(b)|^q A(h(t);\alpha,n)+|f''(x)|^q A(h(1-t);\alpha,n)\Big]^{\frac{1}{q}}\Bigg\}.
\end{aligned} \tag{4.2}$$

Letting $M(\varphi(x),\varphi(y)) := A, G, H, P_r, I, L, L_p, M_p, \forall\, x,y \in I$ in (4.1) and (4.2), we get the inequalities involving means for a particular choices of a twice differentiable generalized relative semi-$(1,h)$-preinvex mappings. The details are left to the interested reader.

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Some inequalities of the Turán type for confluent hypergeometric functions of the second kind

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Abstract. In the paper, by virtue of the Hölder integral inequality, the authors derive some inequalities of the Turán type for confluent hypergeometric functions of the second kind, for the Mellin transforms, and for the Laplace transforms, and improve some known inequalities of the Turán type.

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Keywords: Inequality of the Turán type, confluent hypergeometric function of the second kind, improvement, Mellin transform, Laplace transform, Hölder integral inequality.

1. Introduction

In 1950, P. Turán [16] proved that the Legendre polynomials $P_n(x)$ satisfy

$$P_n(x)P_{n+2}(x) - P_{n+1}^2(x) \leq 0$$

for $|x| \leq 1$ and $n = 0, 1, 2, \ldots$, where the equality holds only if $x = \pm 1$. An inequality of this kind is known as an inequality of the Turán type. This classical inequality has been extended to various special functions. For recent development on this classical inequality, please refer to [2, 3, 4, 6, 7, 11] and closely related reference therein.

It is known [1, p. 504-505] that confluent hypergeometric functions of the second kind $\psi(a, c, x)$ are also known as the Tricomi confluent hypergeometric functions, are a special solution of Kummer's differential equation

$$xy''(x) + (c - x)y'(x) - ay(x) = 0,$$

and have the integral representation

$$\psi(a, c, x) = \frac{1}{\Gamma(a)} \int_0^\infty t^{a-1}(1+t)^{c-a-1}e^{-xt} \,\mathrm{d}\, t \tag{1.1}$$

for $a > 0$, $c \in \mathbb{R}$, and $x > 0$, where

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} \,\mathrm{d}\, t, \quad \Re(z) > 0$$

is the classical Euler gamma function [12, 13, 14, 15].

The Laplace transform and the Mellin transform of a function $f(t)$ are respectively defined by

$$L(s) = \mathscr{L}(f)(s) = \int_0^\infty f(t) e^{-st} \,\mathrm{d}\, t, \quad s > 0$$

and

$$M(s) = \mathscr{M}(f)(s) = \int_0^\infty f(t) t^{s-1} \,\mathrm{d}\, t.$$

These transforms are widely-used integral transforms with many applications in physics and engineering.

In this paper, we will study some Turán type inequalities for confluent hypergeometric functions of the second kind $\psi(a, c, x)$.

2. Inequalities of the Turán type

We are now in a position to find some inequality of the Turán type for confluent hypergeometric functions of the second kind. These newly-founded inequalities improve existed inequality of the Turán type in [4, Theorem 2].

Theorem 2.1. *For $x > 0$, $a > 0$, and $c \in \mathbb{R}$, we have*

$$\psi^2(a+1, c, x) < \frac{a+1}{a} \psi(a, c, x) \psi(a+2, c, x). \tag{2.1}$$

Proof. The equality (1.1) can be reformulated as

$$f(a) \triangleq \psi(a, c, x)\Gamma(a) = \int_0^\infty \left(\frac{t}{1+t}\right)^a (1+t)^{c-1} e^{-xt} \frac{1}{t} \,\mathrm{d}\, t.$$

Replacing a by $\alpha p + (1-\alpha) q$ for $p, q > 0$, $p \ne q$, and $\alpha \in (0,1)$ and using the well-known Hölder integral inequality give

$$\begin{aligned} f(\alpha p + (1-\alpha) q) &= \int_0^\infty \left(\frac{t}{1+t}\right)^{\alpha p + (1-\alpha) q} (1+t)^{c-1} e^{-xt} \frac{1}{t} \,\mathrm{d}\, t \\ &= \int_0^\infty \left[t^{p-1} (1+t)^{c-p-1} e^{-xt}\right]^\alpha \left[t^{q-1} (1+t)^{c-q-1} e^{-xt}\right]^{1-\alpha} \\ &< \left[\int_0^\infty t^{p-1} (1+t)^{c-p-1} e^{-xt}\right]^\alpha \left[\int_0^\infty t^{q-1} (1+t)^{c-q-1} e^{-xt}\right]^{1-\alpha} \\ &= f^\alpha(p) f^{1-\alpha}(q). \end{aligned}$$

This implies that the function f is strictly logarithmically convex on $(0, \infty)$. Consequently, taking $\alpha = \frac{1}{2}$, $p = a$, and $q = a + 2$ leads to $f^2(a+1) < f(a) f(a+2)$ which is equivalent to (2.1). The proof of Theorem 2.1 is complete. □

Theorem 2.2. *For $x > 0$, $0 < a_1, a_2 < a$, and $c \in \mathbb{R}$, we have*

$$\psi^2(a,c,x) < \frac{\Gamma(a_1)\Gamma(a_2)}{\Gamma^2(a)}\psi(a_1,c,x)\psi(a_2,c,x). \tag{2.2}$$

Proof. We continue to adopt the notation $f(a)$ in the proof of Theorem 2.1. Then

$$\begin{aligned} f'(a) &= \frac{\mathrm{d}}{\mathrm{d}\,a}\left[\int_0^\infty \left(\frac{t}{1+t}\right)^a (1+t)^{c-1}\frac{1}{t}e^{-xt}\,\mathrm{d}\,t\right] \\ &= \int_0^\infty \left(\frac{t}{1+t}\right)^a \ln\left(\frac{t}{1+t}\right)(1+t)^{c-1}\frac{1}{t}e^{-xt}\,\mathrm{d}\,t \\ &< 0. \end{aligned}$$

Since $\ln\left(\frac{t}{1+t}\right) < 0$ for $t > 0$, the function $f(a)$ is decreasing on $(0,\infty)$ with respect to a. Accordingly, for $0 < a_1, a_2 < a$, we have $f(a) < f(a_1)$ and $f(a) < f(a_2)$. Consequently, it follows that $f^2(a) < f(a_1)f(a_2)$ which is equivalent to (2.2). The proof of Theorem 2.2 is complete. □

Theorem 2.3. *For $x > 0$, $a > 0$, and $c_1, c_2 < c \in \mathbb{R}$, we have*

$$\psi(a,c,x)\psi(a,c-1,x) < \psi^2(a,c+1,x) < \psi(a,c,x)\psi(a,c+2,x) \tag{2.3}$$

and

$$\psi(a,c_1,x)\psi(a,c_2,x) < \psi^2(a,c,x). \tag{2.4}$$

Proof. A straightforward computation yields

$$\begin{aligned} \psi'(c) &= \frac{1}{\Gamma(a)}\frac{\mathrm{d}}{\mathrm{d}\,c}\left[\int_0^\infty t^{a-1}(1+t)^{c-a-1}e^{-xt}\,\mathrm{d}\,t\right] \\ &= \frac{1}{\Gamma(a)}\int_0^\infty t^{a-1}(1+t)^{c-a-1}\ln(1+t)e^{-xt}\,\mathrm{d}\,t \\ &> 0. \end{aligned}$$

This means that the function $\psi(a,c,x)$ is increasing with respect to $c \in \mathbb{R}$. Hence, for $c_1, c_2 < c$, it follows that $\psi(a,c_1,x) < \psi(a,c,x)$ and $\psi(a,c_2,x) < \psi(a,c,x)$. Consequently, we obtain the inequality (2.4).

Taking $c_1 = c-1$ and $c_2 = c$ and replacing c by $c+1$ in (2.4) deduce that

$$\psi(a,c-1,x)\psi(a,c,x) < \psi^2(a,c+1,x).$$

From this inequality and the inequality $\psi^2(a,c+1,x) < \psi(a,c,x)\psi(a,c+2,x)$ in the paper [4], the inequality (2.3) follows immediately. The proof of Theorem 2.3 is complete. □

Theorem 2.4. *For $x, y > 0$, $a > 0$, $p, q > 0$ such that $\frac{1}{p} + \frac{1}{q} = 1$, and $c \in \mathbb{R}$, we have*

$$\psi\left(a,c,\frac{x}{p}+\frac{y}{q}\right) < \psi^{1/p}(a,c,x)\psi^{1/q}(a,c,y). \tag{2.5}$$

Proof. Applying the well-known Hölder integral inequality to the third variable x in $\psi(a,c,x)$ arrives at

$$\begin{aligned}\psi\left(a,c,\frac{x}{p}+\frac{y}{q}\right)&=\frac{1}{\Gamma(a)}\int_0^\infty t^{a-1}(1+t)^{c-a-1}e^{-(x/p+y/q)t}\,\mathrm{d}\,t\\&=\frac{1}{\Gamma(a)}\int_0^\infty \big[t^{a-1}(1+t)^{c-a-1}e^{-xt}\big]^{1/p}\big[t^{a-1}(1+t)^{c-a-1}e^{-yt}\big]^{1/q}\\&<\left[\frac{1}{\Gamma(a)}\int_0^\infty t^{a-1}(1+t)^{c-a-1}e^{-xt}\right]^{1/p}\left[\frac{1}{\Gamma(a)}\int_0^\infty t^{q-1}(1+t)^{c-q-1}e^{-yt}\right]^{1/q}\\&=\psi^{1/p}(a,c,x)\psi^{1/q}(a,c,x).\end{aligned}$$

Therefore, the inequality (2.5) is proved. The proof of Theorem 2.4 is complete. □

Theorem 2.5. *For $x,y>1$ such that $\frac{1}{x}+\frac{1}{y}\le 1$, $a>0$, $p,q>0$ such that $\frac{1}{p}+\frac{1}{q}=1$, and $c\in\mathbb{R}$, we have*

$$\psi\left(a,c,\frac{x^p}{p}+\frac{y^q}{q}\right)<\psi^{1/p}(a,c,px)\psi^{1/q}(a,c,qy). \tag{2.6}$$

Proof. Applying Young's inequality to the third variable x in $\psi(a,c,x)$ results in

$$\begin{aligned}\psi\left(\frac{x^p}{p}+\frac{y^q}{q}\right)&=\frac{1}{\Gamma(a)}\int_0^\infty t^{a-1}(1+t)^{c-a-1}e^{-(x^p/p+y^q/q)t}\,\mathrm{d}\,t\\&\le\frac{1}{\Gamma(a)}\int_0^\infty t^{a-1}(1+t)^{c-a-1}e^{-xyt}\\&\le\left[\frac{1}{\Gamma(a)}\int_0^\infty t^{a-1}(1+t)^{c-a-1}e^{-pxt}\right]^{1/p}\\&\quad\times\left[\frac{1}{\Gamma(a)}\int_0^\infty t^{q-1}(1+t)^{c-q-1}e^{-qyt}\right]^{1/q}\\&=\psi^{1/p}(a,c,px)\psi^{1/q}(a,c,qx).\end{aligned}$$

The inequality (2.6) is thus proved. The proof of Theorem 2.5 is complete. □

Theorem 2.6. *For $x,y>0$, $a>0$, and $c\in\mathbb{R}$, we have*

$$\psi^2(a,c,x+y)<\psi(a,c,x)\psi(a,c,y). \tag{2.7}$$

For $x>0$, $0<y<1$, $a>0$, and $c\in\mathbb{R}$, we have

$$\psi(a,c,x+y)<\psi(a,c,xy). \tag{2.8}$$

Proof. It is easy to see that the function $\psi(a,c,x)$ is decreasing with respect to $x\in(0,\infty)$. Since $x<x+y$ and $y<x+y$, it follows that $\psi(a,c,x+y)<\psi(a,c,x)$ and $\psi(a,c,x+y)<\psi(a,c,y)$. This means the inequality (2.7).
Similarly, the inequality (2.8) follows readily. The proof of Theorem 2.6 is complete. □

3. Inequalities of the Turán type for the Mellin transform

Now we discover an inequality of the Turán type for the Mellin transform.

Theorem 3.1. *For $s > 0$, the Mellin transform $M(s)$ satisfies*

$$F^2(s+1) \le F(s)F(s+2). \tag{3.1}$$

Proof. Applying the Hölder integral inequality finds that

$$\begin{aligned} M(\alpha p + (1-\alpha)q) &= \int_0^\infty f(t)t^{\alpha p+(1-\alpha)q-1}\,\mathrm{d}\,t \\ &= \int_0^\infty \left[f(t)t^{p-1}\right]^\alpha \left[f(t)t^{q-1}\right]^{1-\alpha}\,\mathrm{d}\,t \\ &\le \left[\int_0^\infty f(t)t^{p-1}\,\mathrm{d}\,t\right]^\alpha \left[\int_0^\infty f(t)t^{q-1}\,\mathrm{d}\,t\right]^{1-\alpha} \\ &= M^\alpha(p)M^{1-\alpha}(q). \end{aligned}$$

This means that the Mellin transform $M(s)$ is strictly logarithmically convex on $(0,\infty)$. Further letting $\alpha = \frac{1}{2}$, $p = s$, and $q = s+2$ in the above inequality leads to the inequality (3.1). The proof of Theorem 3.1 is complete. □

Example 3.2. Entry 17.43.26 in [9] states that

$$M_1(s) = \mathscr{M}(\operatorname{cosech}(x)) = 2(1-2^{-s})\Gamma(s)\zeta(s), \quad s > 1.$$

By Theorem 3.1, it follows readily that

$$M_1^2(s+1) \le M_1(s)M_1(s+2).$$

After some simplification we acquire

$$\zeta^2(s+1) \le \left(\frac{s+1}{s}\right)\left[\frac{(1-2^{-s})(1-2^{-s-2})}{(1-2^{-s-1})^2}\right]\zeta(s)\zeta(s+2), \quad s > 1$$

which improves the Turán type inequality for the zeta function in [11].

Example 3.3. Entry 6.3.8 in [10] states that

$$M_2(s) = \mathscr{M}\big(e^{-ax}(1-e^{-x})^{-1}\big) = \Gamma(s)\zeta(s,a), \quad s > 0, \quad a > 0.$$

By Theorem 3.1, we derive

$$M_2^2(s+1) \le M_2(s)M_2(s+2).$$

After some simplification we acquire

$$\zeta^2(s+1,a) \le \frac{s+1}{s}\zeta(s,a)\zeta(s+2,a), \quad s > 1, \quad a > 0. \tag{3.2}$$

When $a = 1$ in (3.2), we recover the Turán type inequality in [11].

4. Inequalities of the Turán type for the Laplace transform

Finally we find out an inequality of the Turán type for the Laplace transform.

Theorem 4.1. *The Laplace transform $L(s)$ satisfies*

$$L^2(s+1) \le L(s)L(s+2), \quad s>0. \tag{4.1}$$

Proof. By the Hölder integral inequality, we have

$$\begin{aligned} L(\alpha p+(1-\alpha)q) &= \int_0^\infty f(t)e^{-(\alpha p+(1-\alpha)q)t}\,\mathrm{d}\,t \\ &= \int_0^\infty \big[f(t)e^{-pt}\big]^\alpha \big[f(t)e^{-qt}\big]^{1-\alpha}\,\mathrm{d}\,t \\ &\le \left[\int_0^\infty f(t)e^{-pt}\,\mathrm{d}\,t\right]^\alpha \left[\int_0^\infty f(t)e^{-qt}\,\mathrm{d}\,t\right]^{1-\alpha} \\ &= L^\alpha(p)L^{1-\alpha}(q). \end{aligned}$$

In other words, the Laplace transform $L(s)$ is strictly logarithmically convex on $(0,\infty)$. Specially, setting $\alpha=\frac{1}{2}$, $p=s$, and $q=s+2$ in the above inequality leads to (4.1). The proof of Theorem 4.1 is complete. □

Example 4.2. Entry 4.15.29 in [10] states that

$$L_3(s)=\mathscr{L}\big((1-e^{-t})^{\nu/2}J_\nu(a(1-e^{-t})^{1/2})\big)=\Gamma(s)\left(\frac{2}{a}\right)^s J_{\nu+s}(a)$$

for $s>0$, $a>0$, $\nu>-1$, where $J_\mu(z)$ denotes Bessel's functions. By Theorem 4.1, it follows that

$$L_3^2(s+1)\le L_3(s)L_3(s+2)$$

which can be reformulated as

$$J_{\nu+s+1}^2(a)\le \frac{s+1}{s}J_{\nu+s}(a)J_{\nu+s+2}(a) \tag{4.2}$$

for $s>0$, $1>a>0$, and $\nu>-\frac{1}{2}$.

When taking $\nu=0$ and replacing s by $s-1$ for $s\ge 1$ in (4.2), we derive an upper bound of the Turán type inequality in [5, Eq. (2.3)] for $0<a<1$.

Example 4.3. Entry 4.3.11 in [10] reads that

$$L_4(s)=\mathscr{L}(t^2-a^2)^{\nu-1/2}=\frac{1}{\sqrt{\pi}}\Gamma\left(\nu+\frac{1}{2}\right)\left(\frac{2a}{s}\right)^\nu K_\nu(as)$$

for $s,a>0$ and $\nu>-\frac{1}{2}$, where $K_\mu(z)$ denotes modified Bessel's functions. By Theorem 4.1, it follows that

$$L_4^2(s+1)\le L_4(s)L_4(s+2)$$

which can be rewritten as

$$K_\nu^2(a(s+1))\le\left[\frac{s^2+2s+1}{s(s+2)}\right]^\nu K_\nu(as)K_\nu(a(s+2))$$

for $s,a>0$ and $\nu>-\frac{1}{2}$.

Remark 4.4. Many other Turán type inequalities can be obtained for functions whose Laplace and Mellin transforms exists. In particular, we can prove some Turán type inequalities for the gamma, beta, extended beta, hypergeometric, error, and compliment error functions.

Remark 4.5. This paper is a slightly revised version of the preprint [8].

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Subclasses of analytic functions of complex order defined by $q-$derivative operator

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Abstract. Using the q-derivative operator in conjunction with the principle of subordination between analytic functions, we introduce two subclasses of analytic functions in the open unit disk $\mathbb{U}$. We investigate convolution properties and coefficient estimates for these subclasses.

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1. Introduction

Recently, the theory of q-analysis has attracted a considerable effort of researchers due to its application in many branches of mathematics and physics, for example, in the areas of ordinary fractional calculus, optimal control problems, q-difference, q-integral equations and in q-transform analysis (see for instance [1, 9, 11, 19]). The main purpose of this paper is to introduce and study two subclasses of analytic functions in the open unit disk

$$\mathbb{U} = \{z : z \in \mathbb{C} \quad \text{and} \quad |z| < 1\}$$

by applying the q-derivative operator in conjunction with the principle of subordination between analytic functions.

Let $\mathcal{A}$ denote the class of functions $f(z)$ of the form:

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \tag{1.1}$$

which are analytic in $\mathbb{U}$. Also $\mathcal{S}$ be the subclass of all functions in $\mathcal{A}$, which are univalent in $\mathbb{U}$. Let $\mathcal{S}^*(\alpha)$ and $\mathcal{K}(\alpha)$ denote the subclasses of $\mathcal{S}$ consisting of starlike and convex functions of order α $(0 \leqq \alpha < 1)$. We note that

$$\mathcal{S}^*(0) = \mathcal{S}^* \qquad \text{and} \qquad \mathcal{K}(0) = \mathcal{K},$$

where $\mathcal{S}^*$ and $\mathcal{K}$ denote, respectively, the familiar subclasses of starlike and convex functions (see, for details, Srivastava and Owa [25]).

Let $\mathcal{K}[b; A, B]$ and $\mathcal{S}[b; A, B]$ $(b \in \mathbb{C}^* = \mathbb{C}\backslash\{0\},\ -1 \leqq B < A \leqq 1,\ z \in \mathbb{U})$ denote the subclasses of $\mathcal{A}$ and satisfy the following conditions:

$$\mathcal{K}[b; A, B] = \left\{ f : f(z) \in \mathcal{A} \quad \text{and} \quad 1 + \frac{1}{b}\frac{zf''(z)}{f'(z)} \prec \frac{1 + Az}{1 + Bz} \right\}$$

and

$$\mathcal{S}[b; A, B] = \left\{ f : f(z) \in \mathcal{A} \quad \text{and} \quad 1 + \frac{1}{b}\left[\frac{zf'(z)}{f(z)} - 1\right] \prec \frac{1 + Az}{1 + Bz} \right\},$$

where the symbol $\prec$ stands for subordination between analytic functions (see [13]) (see also [5] and [23]). The class $\mathcal{K}[b; A, B]$ was introduced and studied by Aouf *et al.* [3] and the class $\mathcal{S}[b; A, B]$ was introduced and studied by Sohi and Singh [21] (see also Aouf *et al.* [3] and [4]).

We note that

(i) $\mathcal{K}[b; 1, -1] = \mathcal{C}(b)$ (see Nasr and Aouf [15]).

(ii) $\mathcal{S}[b; 1, -1] = \mathcal{S}(b)$ (see Nasr and Aouf [18]).

For $f(z) \in \mathcal{A}$, the q-derivative $(0 < q < 1)$ of $f(z)$ is defined by (see Gasper and Rahman [9])

$$D_q f(z) = \begin{cases} \dfrac{f(qz) - f(z)}{(q-1)z} & (z \neq 0) \\ \\ f'(0) & (z = 0), \end{cases} \tag{1.2}$$

provided that $f'(0)$ exists. From (1.2), we deduce that

$$D_q f(z) = 1 + \sum_{k=2}^{\infty} [k]_q \ a_k z^{k-1} \quad (z \neq 0),$$

where

$$[k]_q = \frac{1 - q^k}{1 - q}.$$

As $q \to 1^-,\ [k]_q \to k$ and

$$\lim_{q \to 1^-} D_q f(z) = f'(z).$$

Also, the q-integral of a function $f(z)$ is defined by (see Gasper and Rahman [9])

$$\int_0^z f(t) d_q t = z(1 - q) \sum_{k=0}^{\infty} q^k f(zq^k). \tag{1.3}$$

It should be observed here that, as already pointed out by Srivastava and Bansal [24, p. 62], although the q-derivative operator in (1.2) was first applied to study a q-extension of the class $\mathcal{S}^*$ of starlike functions in $\mathbb{U}$, a firm footing of the usage of the q-calculus in the context of Geometric Function Theory was actually provided and the basic (or q-) hypergeometric functions were first used in Geometric Function Theory in a book chapter by Srivastava (see, for details, [22, pp. 347 *et seq.*]).

Making use of the q-derivative D_q given by (1.2), we introduce $\mathcal{K}_q[b; A, B]$ and $\mathcal{S}_q[b; A, B]$ of $\mathcal{A}$ for $b \in \mathbb{C}^*$, $0 < q < 1$ and $-1 \leqq B < A \leqq 1$ as follows:

$$\mathcal{K}_q[b; A, B] = \left\{ f(z) \in \mathcal{A} : 1 + \frac{1}{b}\left[\frac{D_q\left(zD_qf(z)\right)}{D_qf(z)} - 1\right] \prec \frac{1+Az}{1+Bz} \right\}, \tag{1.4}$$

and

$$\mathcal{S}_q[b; A, B] = \left\{ f(z) \in \mathcal{A} : 1 + \frac{1}{b}\left[\frac{zD_qf(z)}{f(z)} - 1\right] \prec \frac{1+Az}{1+Bz} \right\}. \tag{1.5}$$

From (1.4) and (1.5), we find that

$$f(z) \in \mathcal{S}_q[b; A, B] \Longleftrightarrow \int_0^z \frac{f(\zeta)}{\zeta} d_q\zeta \in \mathcal{K}_q[b; A, B]. \tag{1.6}$$

We also note that
(i) $\mathcal{K}_q[1; A, B] = \mathcal{K}_q[A, B]$ and $\mathcal{S}_q[1; A, B] = \mathcal{S}_q[A, B]$ (see Seoudy and Aouf [20]);
(ii) $\lim\limits_{q\to 1^-} \mathcal{K}_q[b; A, B] = \mathcal{K}[b; A, B]$ (see Aouf et al. [3]) and $\lim\limits_{q\to 1^-} \mathcal{S}_q[b; A, B] = \mathcal{S}[b; A, B]$ (see Sohi and Singh [21]) (see also Aouf *et al.* [3] and [4]);
(iii) $\mathcal{K}_q\left[b; 1, \frac{1-M}{M}\right] = \mathcal{G}_q(b, M)$

$$= \left\{ f(z) \in \mathcal{A} : \left| \frac{b - 1 + \frac{D_q(zD_qf(z))}{D_qf(z)}}{b} - M \right| < M \ \left(M > \frac{1}{2}\right) \right\};$$

and $\mathcal{S}_q\left[b; 1, \frac{1-M}{M}\right] = \mathcal{F}_q(b, M)$

$$= \left\{ f(z) \in \mathcal{A} : \left| \frac{b - 1 + \frac{zD_qf(z)}{f(z)}}{b} - M \right| < M \ \left(M > \frac{1}{2}\right) \right\};$$

(iv) $\lim\limits_{q\to 1^-} \mathcal{K}_q\left[b; 1, \frac{1-M}{M}\right] = \lim\limits_{q\to 1^-} \mathcal{G}_q(b, M) = \mathcal{G}(b, M)$ (see Nasr and Aouf [17]) and $\lim\limits_{q\to 1^-} \mathcal{S}_q\left[b; 1, \frac{1-M}{M}\right] = \lim\limits_{q\to 1^-} \mathcal{F}_q(b, M) = \mathcal{F}(b, M)$ (see Nasr and Aouf [16]);
(v) $\mathcal{G}_q\left(1 - m - M, \frac{M}{m+M-1}\right) = \mathcal{C}_q(m, M)$

$$= \left\{ f(z) \in \mathcal{A} : \left| \frac{D_q\left(zD_qf(z)\right)}{D_qf(z)} - m \right| < M \ \left(m = 1 - \frac{1}{M};\ M > \frac{1}{2}\right) \right\};$$

and $\mathcal{F}_q\left(1 - m - M, \frac{M}{m+M-1}\right) = \mathfrak{B}_q(m, M)$

$$= \left\{ f(z) \in \mathcal{A} : \left| \frac{zD_qf(z)}{f(z)} - m \right| < M \ \left(m = 1 - \frac{1}{M};\ M > \frac{1}{2}\right) \right\};$$

(vi) $\lim\limits_{q\to 1^-} \mathfrak{B}_q(m, M) = \mathfrak{B}(m, M)$ (see Jakubowski [10]);
(vii) $\mathcal{K}_q[b; 1, -1] = \mathcal{K}_q(b)$

$$= \left\{ f(z) \in \mathcal{A} : \operatorname{Re}\left(1 + \frac{1}{b}\left[\frac{D_q\left(zD_qf(z)\right)}{D_qf(z)} - 1\right]\right) > 0 \ (z \in \mathbb{U}) \right\};$$

and $\mathcal{S}_q[b;1,-1] = \mathcal{S}_q(b)$

$$= \left\{ f(z) \in \mathcal{A} : \operatorname{Re}\left(1 + \frac{1}{b}\left[\frac{zD_q f(z)}{f(z)} - 1\right]\right) > 0 \ (z \in \mathbb{U}) \right\};$$

(viii) $\lim\limits_{q\to 1^-} \mathcal{K}_q[b;1,-1] = \lim\limits_{q\to 1^-} \mathcal{C}_q(b) = \mathcal{C}(b)$ (see Nasr and Aouf [15]) and $\lim\limits_{q\to 1^-} \mathcal{S}_q[b;1,-1] = \lim\limits_{q\to 1^-} \mathcal{S}_q(b) = \mathcal{S}(b)$ (see Nasr and Aouf [18]);

(ix) $\mathcal{K}_q[e^{-i\lambda}\cos\lambda; A, B] = \mathcal{K}_q^{\lambda}[A, B]$

$$= \left\{ f(z) \in \mathcal{A} : e^{i\lambda} \frac{D_q\left(zD_q f(z)\right)}{D_q f(z)} \prec \cos\lambda \frac{1 + Az}{1 + Bz} + i \sin\lambda \ \left(|\lambda| < \frac{\pi}{2}\right) \right\};$$

and $\mathcal{S}_q[e^{-i\lambda}\cos\lambda; A, B] = \mathcal{S}_q^{\lambda}[A, B]$

$$= \left\{ f(z) \in \mathcal{A} : e^{i\lambda} \frac{zD_q f(z)}{f(z)} \prec \cos\lambda \frac{1 + Az}{1 + Bz} + i \sin\lambda \ \left(|\lambda| < \frac{\pi}{2}\right) \right\};$$

(x) $\lim\limits_{q\to 1^-} \mathcal{K}_q[e^{-i\lambda}\cos\lambda; A, B] = \mathcal{K}^{\lambda}[A, B] \ \left(|\lambda| < \frac{\pi}{2}\right)$ (see Bhoosnurmath and Devadas [7]) and $\lim\limits_{q\to 1^-} \mathcal{S}_q[e^{-i\lambda}\cos\lambda; A, B] = \mathcal{S}^{\lambda}[A, B] \ \left(|\lambda| < \frac{\pi}{2}\right)$ (see Dashrath and Shukla [8]) (see Bhoosnurmath and Devadas [6]; see also the more recent work by Xu et al. [26]);

(xi) $\mathcal{K}_q[e^{-i\lambda}\cos\lambda; A, B] = \mathcal{G}_{q,\lambda,M}$

$$= \left\{ f(z) \in \mathcal{A} : \left| \frac{e^{i\lambda} \frac{D_q(zD_q f(z))}{D_q f(z)} - i\sin\lambda}{\cos\lambda} - M \right| < M \ \left(|\lambda| < \frac{\pi}{2};\ M > \frac{1}{2}\right) \right\};$$

and $\mathcal{S}_q\left[e^{-i\lambda}\cos\lambda; 1, \frac{1-M}{M}\right] = \mathcal{F}_{q,\lambda,M}$

$$= \left\{ f(z) \in \mathcal{A} : \left| \frac{e^{i\lambda} \frac{zD_q f(z)}{f(z)} - i\sin\lambda}{\cos\lambda} - M \right| < M \ \left(|\lambda| < \frac{\pi}{2};\ M > \frac{1}{2}\right) \right\};$$

(xii) $\lim\limits_{q\to 1^-} \mathcal{K}_q\left[e^{-i\lambda}\cos\lambda; 1, \frac{1-M}{M}\right] = \lim\limits_{q\to 1^-} \mathcal{G}_{q,\lambda,M} = \mathcal{G}_{\lambda,M}$ *and* $\lim\limits_{q\to 1^-} \mathcal{S}_q\left[e^{-i\lambda}\cos\lambda; 1, \frac{1-M}{M}\right] = \lim\limits_{q\to 1^-} \mathcal{F}_{q,\lambda,M} = \mathcal{F}_{\lambda,M}$ *(see Kulshrestha* [12]*);*

(xiii) $\mathcal{K}_q\left[(1-\mu)e^{-i\lambda}\cos\lambda; 1, \frac{1-M}{M}\right] = \mathcal{G}_q[\lambda, \mu, M]$

$$= \left\{ f(z) \in \mathcal{A} : \left| \frac{e^{i\lambda} \frac{D_q(zD_q f(z))}{D_q f(z)} - \mu\cos\lambda - i\sin\lambda}{(1-\mu)\cos\lambda} - M \right| < M \right.$$
$$\left. \left(|\lambda| < \frac{\pi}{2};\ 0 \leqq \mu < 1;\ M > \frac{1}{2}\right) \right\};$$

and $\mathcal{S}_q\left[(1-\mu)e^{-i\lambda}\cos\lambda; 1, \frac{1-M}{M}\right] = \mathcal{F}_q[\lambda,\mu,M]$

$$= \left\{ f(z) \in \mathcal{A} : \left| \frac{e^{i\lambda}\frac{zD_qf(z)}{f(z)} - \mu\cos\lambda - i\sin\lambda}{(1-\mu)\cos\lambda} - M \right| < M \right.$$

$$\left. \left(|\lambda| < \frac{\pi}{2};\ 0 \leqq \mu < 1;\ M > \frac{1}{2} \right) \right\};$$

(xiv) $\lim\limits_{q\to 1^-} \mathcal{K}_q\left[(1-\mu)e^{-i\lambda}\cos\lambda; 1, \frac{1-M}{M}\right] = \lim\limits_{q\to 1^-} \mathcal{K}_q[\lambda,\mu,M] = \mathcal{K}[\lambda,\mu,M]$ *and* $\lim\limits_{q\to 1^-} \mathcal{S}_q\left[(1-\mu)e^{-i\lambda}\cos\lambda; 1, \frac{1-M}{M}\right] = \lim\limits_{q\to 1^-} \mathcal{F}_q[\lambda,\mu,M] = \mathcal{F}[\lambda,\mu,M]$ (see Aouf [2]).

2. Main results

Unless otherwise mentioned, we assume throughout this paper that $0 < q < 1$, $-1 \le B < A \le 1$, $b \in \mathbb{C}^*$, $z \in \mathbb{U}$ and $\theta \in [0, 2\pi)$.

Theorem 2.1. *If* $f(z) \in \mathcal{A}$, *then* $f(z) \in \mathcal{K}_q[b; A, B]$ *if and only if*

$$\frac{1}{z}\left[f(z) * \frac{z + [1 - (1 + M(\theta))(q+1)]qz^2}{(1-z)(1-qz)(1-q^2z)} \right] \neq 0, \tag{2.1}$$

where the symbol $*$ *stands for the convolution between two power series and*

$$M(\theta) = M^{b;A,B}(\theta) = \frac{1}{b}\left(\frac{e^{-i\theta} + B}{A - B} \right). \tag{2.2}$$

Proof. It is easy to verify that

$$zD_qf(z) * \frac{z}{1-z} = zD_qf(z) \text{ and } zD_qf(z) * \frac{z}{(1-z)(1-qz)} = zD_q\left(zD_qf(z)\right). \tag{2.3}$$

In order to prove that (2.1) holds we will write (1.4) by using the definition of the subordination, that is

$$1 + \frac{1}{b}\left[\frac{D_q\left(zD_qf(z)\right)}{D_qf(z)} - 1 \right] = \frac{1 + Aw(z)}{1 + Bw(z)}, \tag{2.4}$$

where $w(z)$ is Schwarz function, hence

$$\frac{1}{z}\left[zD_q\left(zD_qf(z)\right)\left(1 + Be^{i\theta}\right) - \left[1 + [B + b(A-B)]\,e^{i\theta}\right] zD_qf(z) \right] \neq 0. \tag{2.5}$$

Using (2.3), Eq. (2.5) may be written as

$$\frac{1}{z}\left[\left(1 + Be^{i\theta}\right)\left(zD_qf(z) * \frac{z}{(1-z)(1-qz)} \right) \right.$$

$$\left. - \left[1 + [B + b(A-B)]\,e^{i\theta}\right]\left(zD_qf(z) * \frac{z}{1-z} \right) \right] \neq 0,$$

which is equivalent to

$$\frac{1}{z}\left[zD_q f(z) * \frac{z-\left(1+\dfrac{e^{-i\theta}+B}{(A-B)b}\right)qz^2}{(1-z)(1-qz)}\cdot\Big[-(A-B)be^{i\theta}\Big]\right]\neq 0,$$

or

$$\frac{1}{z}\left[f(z) * zD_q\frac{z-\left(1+\dfrac{e^{-i\theta}+B}{(A-B)b}\right)qz^2}{(1-z)(1-qz)}\right]$$

$$=\frac{1}{z}\left[f(z) * \frac{z+\left[1-(q+1)\left(1+\dfrac{e^{-i\theta}+B}{(A-B)b}\right)\right]qz^2}{(1-z)(1-qz)(1-q^2z)}\right]\neq 0,$$

that is (2.1). Reversely, since, it was shown in the first part of the proof that the assumption (2.5) is equivalent to (2.1), we obtain that

$$\frac{D_q\left(zD_q f(z)\right)}{D_q f(z)}\neq\frac{1+[B+(A-B)b]e^{i\theta}}{1+Be^{i\theta}}. \tag{2.6}$$

Suppose that

$$\varphi(z)=\frac{D_q\left(zD_q f(z)\right)}{D_q f(z)}\quad\text{and}\quad\psi(z)=\frac{1+[B+(A-B)b]z}{1+Bz}.$$

The relation (2.6) means that

$$\varphi(\mathbb{U})\cap\psi(\partial\mathbb{U})=\emptyset.$$

Thus, the simply connected domain is included in a connected component of $\mathbb{C}\backslash\psi(\partial\mathbb{U})$. From this, using the fact that $\varphi(0)=\psi(0)$ and the univalence of the function ψ, it follows that $\varphi(z)\prec\psi(z)$, this implies that $f(z)\in\mathcal{K}_q[b;A,B]$. Thus, the proof is completed. □

Theorem 2.2. *If $f(z)\in\mathcal{A}$, then $f(z)\in\mathcal{S}_q[b;A,B]$ if and only if*

$$\frac{1}{z}\left[f(z) * \frac{z-(1+M(\theta))\,qz^2}{(1-z)(1-qz)}\right]\neq 0, \tag{2.7}$$

where $M(\theta)$ is given by (2.2).

Proof. From (1.6), it follows that $f\in\mathcal{S}_q[b;A,B]$ if and only if

$$\Phi_q(z)=\int_0^z\frac{f(\zeta)}{\zeta}d_q\zeta\in\mathcal{K}_q[b;A,B].$$

Then, according to Theorem 2.1, the function Φ_q belongs to $\mathcal{S}_q[b;A,B]$ if and only if

$$\frac{1}{z}\left[\Phi_q(z) * g(z)\right]\neq 0,\ \text{ for all } z\in\mathbb{U} \text{ and } \theta\in[0,2\pi), \tag{2.8}$$

where

$$g(z)=\frac{z+[1-(1+M(\theta))(q+1)]qz^2}{(1-z)(1-qz)(1-q^2z)}.$$

From (1.3), we have

$$\int_0^z \frac{g(\zeta)}{\zeta}d_q\zeta=\int_0^z \frac{1+[1-(1+M(\theta))(q+1)]q\zeta}{(1-\zeta)(1-q\zeta)(1-q^2\zeta)}d_q\zeta$$
$$=z(1-q)\sum_{k=0}^{\infty}\frac{q^k+[1-(1+M(\theta))(q+1)]zq^{2k+1}}{(1-zq^k)(1-zq^{k+1})(1-zq^{k+2})},$$

and therefore

$$\int_0^z \frac{g(\zeta)}{\zeta}d_q\zeta=\frac{z-(1+M(\theta))qz^2}{(1-z)(1-qz)}.$$

Using the above relation and the identity

$$\left[\int_0^z \frac{f(\zeta)}{\zeta}d_q\zeta\right]*g(z)=f(z)*\left[\int_0^z \frac{g(\zeta)}{\zeta}d_q\zeta\right],$$

it is easy to check that (2.8) is equivalent to (2.7). □

Theorem 2.3. *If $f(z)\in\mathcal{A}$, then $f(z)\in\mathcal{K}_q[b;A,B]$ if and only if*

$$1-\sum_{k=2}^{\infty}[k]_q\frac{([k]_q-1)(e^{-i\theta}+B)-(A-B)b}{(A-B)b}a_kz^{k-1}\neq 0 \text{ for all } \theta. \tag{2.9}$$

Proof. If $f(z)\in\mathcal{A}$, then from Theorem 2.1, we have $f(z)\in\mathcal{K}_q[b;A,B]$ if and only if (2.1) holds. Since

$$\frac{1}{(1-z)(1-qz)(1-q^2z)}=1+\left(1+q+q^2\right)z+\left(1+q+2q^2+q^3+q^4\right)z^2$$
$$+\left(1+q+2q^2+2q^3+2q^4+q^5+q^6\right)z^3+\ldots,$$

it follows that

$$\frac{z+[1-(1+M(\theta))(q+1)]qz^2}{(1-z)(1-qz)(1-q^2z)}=z+\sum_{k=2}^{\infty}[k]_q\Big(1-qM(\theta)[k-1]_q\Big)a_kz^k,$$

where $M(\theta)$ is given by (2.2) and so (2.1) may be written as

$$1-\sum_{k=2}^{\infty}[k]_q\frac{q[k-1]_q(e^{-i\theta}+B)-(A-B)b}{(A-B)b}a_kz^{k-1}\neq 0$$

that is (2.9). □

Theorem 2.4. *If $f(z)\in\mathcal{A}$, then $f(z)\in\mathcal{S}_q[b;A,B]$ if and only if*

$$1-\sum_{k=2}^{\infty}\frac{([k]_q-1)(e^{-i\theta}+B)-(A-B)b}{(A-B)b}a_kz^{k-1}\neq 0 \text{ for all } \theta. \tag{2.10}$$

Proof. If $f(z) \in \mathcal{A}$, then from Theorem 2.2, we have $f(z) \in \mathcal{S}_q[b; A, B]$ if and only if (2.7) holds. Since

$$\frac{1}{(1-z)(1-qz)} = 1 + (1+q)z + \left(1+q+q^2\right)z^2 + \left(1+q+q^2+q^3\right)z^3 + \ldots,$$

it follows that

$$\frac{z - (1+M(\theta))\,qz^2}{(1-z)(1-qz)} = z + \sum_{k=2}^{\infty}\Big(1 - qM(\theta)\,[k-1]_q\Big)a_k z^k,$$

where $M(\theta)$ is given by (2.2). Now, we may express (2.7) as

$$1 - \sum_{k=2}^{\infty}\frac{q[k-1]_q(e^{-i\theta}+B) - (A-B)b}{(A-B)b}a_k z^{k-1} \neq 0,$$

or equivalently, (2.10). □

Theorem 2.5. *If $f(z) \in \mathcal{A}$ satisfies the inequality*

$$\sum_{k=2}^{\infty}[k]_q\Big[([k]_q - 1)(1+|B|) + (A-B)\,|b|\Big]\,|a_k| \leq (A-B)\,|b|\,. \tag{2.11}$$

then $f(z) \in \mathcal{K}_q[b; A, B]$.

Proof. Since

$$\begin{aligned}
&\left|1 - \sum_{k=2}^{\infty}[k]_q\frac{([k]_q-1)(e^{-i\theta}+B)-(A-B)b}{(A-B)b}a_k z^{k-1}\right| \\
&\geq 1 - \left|\sum_{k=2}^{\infty}[k]_q\frac{([k]_q-1)(e^{-i\theta}+B)-(A-B)b}{(A-B)b}a_k z^{k-1}\right| \\
&\geq 1 - \sum_{k=2}^{\infty}[k]_q\frac{([k]_q-1)(1+|B|)+(A-B)\,|b|}{(A-B)\,|b|}\,|a_k| > 0.
\end{aligned}$$

Thus, the inequality (2.11) holds and our conclusion follows. □

By using arguments and analysis to those in the proof of Theorem 2.5, we can analogously derive Theorem 2.6.

Theorem 2.6. *If $f(z) \in \mathcal{A}$ satisfies*

$$\sum_{k=2}^{\infty}\Big[([k]_q - 1)(1+|B|) + (A-B)\,|b|\Big]\,|a_k| \leq (A-B)\,|b|\,.$$

then $f(z) \in \mathcal{S}_q[b; A, B]$.

Remark 2.7. (i) For different choices of q, b, A and B in Theorems 2.1 and 2.2, the results of Seoudy and Aouf (see [20, Theorems 1 and 5]), Nasr and Aouf (see [14, Theorems 1 and 2]) and Bhoosnurmath and Devadas (see [6] and [7]) follow.

(ii) For $b = 1$ in Theorems from 2.3 to 2.6, the results of Seoudy and Aouf (see [20, Theorems 9, 13, 17 and 21]) will follow.

(iii) For different choices of q, b, A and B in our results, we will obtain new results for different classes mentioned in the introduction.

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Some properties of a new subclass of analytic univalent functions defined by multiplier transformation

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Abstract. The purpose of the present paper is to study the integral operator of the form

$$\int_0^z \left\{ \frac{I_\mu^n f(t)}{t} \right\}^\delta dt$$

where f belongs to the subclass $C(n, \alpha, \beta, \mu)$ and δ is a real number. We obtain integral characterization for the subclass $C(n, \alpha, \beta, \mu)$ and also prove distortion, rotation and radii theorem for this class. Relevant connections of the results presented here with various known results are briefly indicated.

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1. Introduction

Let A denote the class of functions f of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \tag{1.1}$$

which are analytic in the open unit disk $U = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$ and satisfy the normalization condition $f(0) = f'(0) - 1 = 0$. Let S be the subclass of A consisting of functions of the form (1.1) which are also univalent in U.

A function f of S is said to be starlike of order $\alpha(0 \leq \alpha < 1)$, denoted by $f \in S^*(\alpha)$, if and only if

$$\Re \left\{ \frac{zf'(z)}{f(z)} \right\} > \alpha, \ z \in U,$$

and is said to be convex of order $\alpha(0 \leq \alpha < 1)$, denoted by $f \in K(\alpha)$, if and only if

$$\Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \alpha, \ z \in U.$$

The classes S^* and K of starlike and convex functions, respectively, are identified by $S^*(0) \equiv S^*$ and $K(0) \equiv K$.

These classes were first studied by Robertson [17].

In 2003 Cho and Srivastava [2], (see also [1]) introduced the multiplier transformation for functions f of the form (1.1) as follows

$$I_\mu^n f(z) = z + \sum_{k=2}^{\infty} \left(\frac{k+\mu}{1+\mu}\right)^n a_k z^k.$$

For $\mu = 1$, the operator $I_\mu^n \equiv I^n$ was studied by Uralegaddi and Somanatha [22] and for $\mu = 0$ the operator I_μ^n reduce to well-known Sălăgean operator introduced by Sălăgean [19].

Using the multiplier transformation we introduce the class $S(n, \alpha, \mu)$ of functions of the form (1.1) satisfying the following condition

$$\Re\left\{\frac{z\left(I_\mu^n f(z)\right)'}{I_\mu^n f(z)}\right\} > \alpha, \ z \in U. \tag{1.2}$$

It is worthy to note that for $\mu = 0$ the class $S(n, \alpha, \mu)$ reduce to the class $S(n, \alpha)$ was first introduced by Sălăgean [19] and further studied by Kadioğlu [4].

It should be worthy to note that $S(0, \alpha, 0) = S^*(\alpha)$ and $S(1, \alpha, 0) = K(\alpha)$.

A function f of A belongs to the class $C(n, \alpha, \beta, \mu)$ if there exists a function $F \in S^*(\alpha)$ such that

$$\left|\arg \frac{I_\mu^n f(z)}{F(z)}\right| < \frac{\beta\pi}{2}, \ z \in U,$$

where $n \in N_0, \ 0 \leq \alpha < 1, \ 0 < \beta \leq 1, \mu > -1$.

By specializing the parameters in $C(n, \alpha, \beta, \mu)$ we obtain the following known subclasses of A studied earlier by various researchers.

(1) $C(0, \alpha, \beta, 0) \equiv CS^*(\alpha, \beta)$ studied by Mishra [9].
(2) $C(1, \alpha, \beta, 0) \equiv C(\alpha, \beta)$ studied by Mishra [9].
(3) $C(0, 0, \beta, 0) \equiv CS^*(\beta)$ studied by Reade [16].
(4) $C(1, 0, \beta, 0) \equiv C(\beta)$ studied by Kaplan [5].
(5) $C(0, 0, 1, 0) \equiv S^*$ studied by Roberston [17], (see also [3], [21]).
(6) $C(1, 0, 1, 0) \equiv K$ studied by Roberston [17], (see also [3], [21]).

In the present paper, we study the integral operator

$$h(z) = \int_0^z \left\{\frac{I_\mu^n f(t)}{t}\right\}^\delta dt \tag{1.3}$$

where $n \in N_0$ and δ is a real number. For $n = 0$ and $n = 1$ this integral operator was studied by Kim [6], Merkes and Wright [8], Mishra [9], Nunokawa([10], [11]), Pfaltzgraff [13], Royster [18], Patil and Tahakare [12] and Shukla and Kumar [20], (see also [15]).

To prove our main results, we shall require the following definition and lemmas.

Definition 1.1. Let $P(\alpha)$ denote the class of functions of the form

$$P(z) = 1 + \sum_{k=1}^{\infty} p_k z^k$$

which are regular in U and satisfy $\Re\{P(z)\} > \alpha, \ z \in U$.

Lemma 1.2. *Let*

$$P(z) = 1 + \sum_{k=1}^{\infty} p_k z^k$$

be analytic in U. If $\Re\{P(z)\} > \alpha$ in U, then

$$\alpha(\theta_2 - \theta_1) < \int_{\theta_1}^{\theta_2} \Re\left\{P(re^{i\theta})\right\} d\theta < 2\pi(1-\alpha) + \alpha(\theta_2 - \theta_1), \tag{1.4}$$

where $0 \le \theta_1 < \theta_2 \le 2\pi$, $z = re^{i\theta}$ and $0 \le r < 1$.

Proof. Since

$$\Re\{P(z)\} > \alpha.$$

It is easy to see that

$$\left(\Re\{P(z)\} - \alpha\right)\big|_{z=0} = 1 - \alpha.$$

Then by mean value theorem, we have

$$0 \le \int_{\theta_1}^{\theta_2} \left(\Re\left\{P(re^{i\theta})\right\} - \alpha\right) d\theta \le \int_{0}^{2\pi} \left(\Re\left\{P(re^{i\theta})\right\} - \alpha\right) d\theta = 2\pi\left(1-\alpha\right).$$

or, equivalently

$$0 \le \int_{\theta_1}^{\theta_2} \left(\Re\left\{P(re^{i\theta})\right\}\right) d\theta - \alpha(\theta_2 - \theta_1) \le 2\pi\left(1-\alpha\right),$$

or

$$\alpha(\theta_2 - \theta_1) < \int_{\theta_1}^{\theta_2} \Re\left\{P(re^{i\theta})\right\} d\theta < 2\pi(1-\alpha) + \alpha(\theta_2 - \theta_1).$$

□

The following lemma is a direct consequence of Lemma 1.2, and improves a result of Patil and Thakare ([12], Lemma 2.2).

Lemma 1.3. *If $f \in S^*(\alpha)$, then*

$$\alpha(\theta_2 - \theta_1) < \int_{\theta_1}^{\theta_2} \Re\left\{\frac{zf'(z)}{f(z)}\right\} d\theta < 2\pi(1-\alpha) + \alpha(\theta_2 - \theta_1), \tag{1.5}$$

where $0 \le \theta_1 < \theta_2 \le 2\pi$, $z = re^{i\theta}$ and $0 \le r < 1$.

In the following lemma, we obtain integral characterization for the class $C(n, \alpha, \beta, \mu)$.

Lemma 1.4. *If* $f \in C(n,\alpha,\beta,\mu)$, *then*

$$-\beta\pi + \alpha(\theta_2 - \theta_1) < \int_{\theta_1}^{\theta_2} \Re\left\{\frac{z\left(I_\mu^n f(z)\right)'}{I_\mu^n f(z)}\right\} d\theta < \beta\pi + 2\pi(1-\alpha) + \alpha(\theta_2 - \theta_1), \quad (1.6)$$

where $0 \le \theta_1 < \theta_2 \le 2\pi$, $z = re^{i\theta}$ *and* $0 \le r < 1$. *Conversely, let* f *be analytic and satisfying* $I_\mu^n f(z) \neq 0$ *in* U, *if*

$$\int_{\theta_1}^{\theta_2} \Re\left\{\frac{z\left(I_\mu^n f(z)\right)'}{I_\mu^n f(z)}\right\} d\theta > -\beta\pi + \alpha(\theta_2 - \theta_1)$$

then $f \in C(n,\alpha,\beta,\mu)$.

Proof. $f \in C(n,\alpha,\beta,\mu)$ implies that there exists a function $F \in S^*(\alpha)$ such that

$$\left|\arg \frac{I_\mu^n f(z)}{F(z)}\right| < \frac{\beta\pi}{2}, \; z \in U.$$

Therefore

$$-\frac{1}{2}\beta\pi < \arg I_\mu^n f(z) - \arg F(z) < \frac{1}{2}\beta\pi.$$

Let $0 \le \theta_1 < \theta_2 \le 2\pi$. Then with $z = re^{i\theta_2}$, we have

$$-\frac{1}{2}\beta\pi < \arg I_\mu^n f(re^{i\theta_2}) - \arg F(re^{i\theta_2}) < \frac{1}{2}\beta\pi. \quad (1.7)$$

and with $z = re^{i\theta_1}$, we have

$$-\frac{1}{2}\beta\pi < -\arg I_\mu^n f(re^{i\theta_1}) + \arg F(re^{i\theta_1}) < \frac{1}{2}\beta\pi. \quad (1.8)$$

Combining (1.7) and (1.8), we obtain

$$\begin{aligned}-\beta\pi + \arg F(re^{i\theta_2}) - \arg F(re^{i\theta_1}) &< \arg I_\mu^n f(re^{i\theta_2}) - \arg I_\mu^n f(re^{i\theta_1}) \\ &< \beta\pi + \arg F(re^{i\theta_2}) - \arg F(re^{i\theta_1}),\end{aligned}$$

or

$$-\beta\pi + \int_{\theta_1}^{\theta_2} d\arg F(re^{i\theta}) < \int_{\theta_1}^{\theta_2} d\arg I_\mu^n f(re^{i\theta}) < \beta\pi + \int_{\theta_1}^{\theta_2} d\arg F(re^{i\theta}),$$

or

$$\begin{aligned}-\beta\pi + \int_{\theta_1}^{\theta_2} \Re\left\{\frac{zF'(z)}{F(z)}\right\} d\theta &< \int_{\theta_1}^{\theta_2} \Re\left\{\frac{z\left(I_\mu^n f(z)\right)'}{I_\mu^n f(z)}\right\} d\theta \\ &< \beta\pi + \int_{\theta_1}^{\theta_2} \Re\left\{\frac{zF'(z)}{F(z)}\right\} d\theta. \quad (1.9)\end{aligned}$$

But $F \in S^*(\alpha)$, then using Lemma 1.3 in (1.9), we have

$$-\beta\pi + \alpha(\theta_2 - \theta_1) < \int_{\theta_1}^{\theta_2} \Re\left\{\frac{z\left(I_\mu^n f(z)\right)'}{I_\mu^n f(z)}\right\} d\theta < \beta\pi + 2\pi(1-\alpha) + \alpha(\theta_2 - \theta_1)$$

and this completes the proof of direct part of the lemma.

To prove the converse part, we follow the techniques of Kaplan [5] and Patil and Thakare [12] and can obtain the desired result.

□

Remark 1.5. If we put $n = 1$, $\mu = 0$ in Lemma 1.4, we obtain the following result

If $f \in C(\alpha, \beta)$, then

$$-\beta\pi + \alpha(\theta_2 - \theta_1) < \int_{\theta_1}^{\theta_2} \Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} d\theta < \beta\pi + 2\pi(1-\alpha) + \alpha(\theta_2 - \theta_1), \quad (1.10)$$

where $0 \le \theta_1 < \theta_2 \le 2\pi$, $z = re^{i\theta}$ and $0 \le r < 1$. Conversely, let f be analytic and satisfying $f'(z) \ne 0$ in U, if

$$\int_{\theta_1}^{\theta_2} \Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} d\theta > -\beta\pi + \alpha(\theta_2 - \theta_1) \quad (1.11)$$

then $f \in C(\alpha, \beta)$.

2. Main results

Theorem 2.1. *If $f \in C(n, \alpha, \beta, \mu)$, then $h \in C(\eta, \gamma)$, provided*

$$\frac{-\gamma}{\beta + 2(1-\alpha)} \le \delta \le \frac{\gamma + 2(1-\eta)}{\beta + 2(1-\alpha)}. \quad (2.1)$$

The result is sharp when (i) $\gamma = 0$ (ii) $\eta = 0, \gamma = 1$.

Proof. From relation (1.3) we have

$$h'(z) = \left\{\frac{I_\mu^n f(z)}{z}\right\}^\delta.$$

Applying logarithmic differentiation and then taking real parts of both sides, we obtain

$$Re\left\{1 + \frac{zh''(z)}{h'(z)}\right\} = \delta Re\left\{\frac{z\left(I_\mu^n f(z)\right)'}{I_\mu^n f(z)}\right\} + (1-\delta).$$

For $\delta > 0$, using Lemma 1.4, we get

$$\begin{aligned}\int_{\theta_1}^{\theta_2} Re\left\{1 + \frac{zh''(z)}{h'(z)}\right\} d\theta &= \delta \int_{\theta_1}^{\theta_2} Re\left\{\frac{z\left(I_\mu^n f(z)\right)'}{I_\mu^n f(z)}\right\} d\theta + (1-\delta)(\theta_2 - \theta_1) \\ &> \delta[-\beta\pi + \alpha(\theta_2 - \theta_1)] + (1-\delta)(\theta_2 - \theta_1) \\ &= -\beta\delta\pi + [1 - (1-\alpha)\delta](\theta_2 - \theta_1).\end{aligned}$$

To prove that $h \in C(\eta, \gamma)$, we have to show that the right hand side of the above inequality is not less than $-\gamma\pi + \eta(\theta_2 - \theta_1)$, provided

$$0 \le \delta \le \frac{\gamma + 2(1-\eta)}{\beta + 2(1-\alpha)}. \quad (2.2)$$

Similarly, for $\delta < 0$, using Lemma 1.4, we get

$$\int_{\theta_1}^{\theta_2} \Re\left\{1 + \frac{zh''(z)}{h'(z)}\right\} d\theta > \delta\left[\beta\pi + 2(1-\alpha) + \alpha(\theta_2 - \theta_1)\right] + (1-\delta)(\theta_2 - \theta_1).$$

To show that $h \in C(\eta, \gamma)$, we have to prove that the right-hand side of the above inequality is not less than $-\gamma\pi + \eta(\theta_2 - \theta_1)$, provided

$$\frac{-\gamma}{\beta + 2(1-\alpha)} \leq \delta \leq 0. \tag{2.3}$$

Combining (2.2) and (2.3), we obtain (2.1).
Thus the proof of Theorem 2.1 is established.
To show the sharpness, let us take the function $f(z)$ defined by the relation

$$I_\mu^n f(z) = \frac{z}{(1-z)^{2(1-\alpha)+\beta}}, \tag{2.4}$$

then it is easy to see that this function belongs to $C(n, \alpha, \beta, \mu)$ with respect to the function $\frac{z}{(1-z)^{2(1-\alpha)}}$ belonging to $S^*(\alpha)$. Then

$$h(z) = \int_0^z \frac{dt}{(1-t)^{[2(1-\alpha)+\beta]\delta}} \tag{2.5}$$

and from condition (1.11) this functions belongs to $C(0, 1)$ if and only if

$$\frac{-1}{2(1-\alpha)+\beta} \leq \delta \leq \frac{3}{2(1-\alpha)+\beta}.$$

Again for $\gamma = 0$, from (2.5) we have

$$1 + \frac{zh''(z)}{h'(z)} = \frac{1 + \left[1 - 2\left(1 - \frac{\{2(1-\alpha)+\beta\}\delta}{2}\right)\right] z}{1-z}$$

and $\Re\left\{1 + \frac{zh''(z)}{h'(z)}\right\} > \eta$ if and only if

$$1 - \frac{\{2(1-\alpha)+\beta\}\,\delta}{2} \geq \eta \Rightarrow 0 \leq \delta \leq \frac{2(1-\eta)}{\beta + 2(1-\alpha)}. \qquad \square$$

Remark 2.2. The undermentioned results are particular cases of Theorem 2.1.

(i) If we put $n = 0$ and $n = 1$ with $\mu = 0$ in Theorem 2.1 we obtain the corresponding results of Mishra [9].

(ii) If we put $n = 1, \beta = 0, \gamma = 0$ with $\mu = 0$ we obtain a result of Patil and Thakare [12].

(iii) If we put $n = 1, \beta = 0, \eta = 0$ with $\mu = 0$ we obtain a result of Patil and Thakare [12].

(iv) If we put $n = 1, \alpha = 0, \eta = 0$ with $\mu = 0$ we obtain a result of Patil and Thakare [12].

(v) If we put $n = 0, \beta = 0, \eta = 0$ we obtain a result of Patil and Thakare [12].

(vi) If we put $n = 1, \alpha = 0, \beta = 0, \eta = 0$ and $\gamma = 1$ with $\mu = 0$ we obtain a result of Nunokawa [11] as well as that of Merkes and Wright [8].

(vii) If we put $n = 0, \alpha = 0, \beta = 0, \eta = 0$ and $\gamma = 1$ with $\mu = 0$ we obtain a result of Nunokawa [11] as well as that of Merkes and Wright [8].

(viii) If we put $n = 1, \alpha = 0, \beta = 1, \eta = 0$ and $\gamma = 1$ with $\mu = 0$ we obtain a result of Nunokawa [11] as well as that of Merkes and Wright [8].

(ix) If we put $n = 0, \alpha = 0, \eta = 0$ with $\mu = 0$ we obtain a result of Shukla and Kumar [20].

(x) If we put $n = 0, \alpha = 0, \beta = 1, \eta = 0$ and $\gamma = 1$ with $\mu = 0$ we obtain a result of Kim [6].

(xi) If we put $n = 0, \alpha = 1/2, \beta = 0, \eta = 0$ and $\gamma = 1$ with $\mu = 0$ we obtain a result of Nunokawa [11] as well as that of Merkes and Wright [8].

Theorem 2.3. *Let $f \in C(n, \alpha, \beta, \mu)$. Then for $|z| = r$*

$$\frac{r(1-r)^{\beta}}{(1+r)^{\beta+2(1-\alpha)}} \le |I_{\mu}^{n} f(z)| \le \frac{r(1+r)^{\beta}}{(1-r)^{\beta+2(1-\alpha)}}$$

The result is sharp.

Proof. By definition $f \in C(n, \alpha, \beta, \mu)$ if and only if there exists a function $P \in P(0)$ and $F(z) \in S^*(\alpha)$ such that

$$\frac{I_{\mu}^{n} f(z)}{F(z)} = [P(z)]^{\beta}.$$

Therefore

$$\left|I_{\mu}^{n} f(z)\right| = |P(z)|^{\beta} |F(z)|.$$

Now using the well-known inequalities (see [3])

$$\frac{1-r}{1+r} \le |P(z)| \le \frac{1+r}{1-r}$$

and

$$\frac{r}{(1+r)^{2(1-\alpha)}} \le |F(z)| \le \frac{r}{(1-r)^{2(1-\alpha)}},$$

we obtain the required inequalities.

Sharpness follows if we take $f(z)$ connected by the relation

$$I_{\mu}^{n} f(z) = \frac{z(1+z)^{\beta}}{(1-z)^{\beta+2(1-\alpha)}}$$

and

$$F(z) = \frac{z}{(1-z)^{2(1-\alpha)}}.$$

□

Theorem 2.4. *If $f \in C(n, \alpha, \beta, \mu)$, then*

$$\left|\arg \frac{I_{\mu}^{n} f(z)}{z}\right| \le \beta \sin^{-1} \frac{2r}{1+r^2} + 2(1-\alpha) \sin^{-1} r.$$

The result is sharp.

Proof. If $f \in C(n, \alpha, \beta, \mu)$, then

$$\frac{I_{\mu}^{n} f(z)}{F(z)} = [P(z)]^{\beta},$$

for some $P(z) \in P(0)$ and $F(z) \in S^*(\alpha)$.

Thus

$$\left|\arg \frac{I_\mu^n f(z)}{z}\right| \leq \alpha \left|\arg P(z)\right| + \left|\arg \frac{F(z)}{z}\right|. \tag{2.6}$$

Now using the well-known results

$$\left|\arg P(z)\right| \leq \sin^{-1} \frac{2r}{1+r^2} \tag{2.7}$$

and a result of Pinchuk [14]

$$\left|\arg \frac{F(z)}{z}\right| \leq 2(1-\alpha)\sin^{-1} r, \tag{2.8}$$

Using (2.7) and (2.8) in (2.6) we get the required result.
Sharpness follows if we take $f(z)$ to be the same as in Theorem 2.3. □

Theorem 2.5. *If $f \in C(n, \alpha, \beta, \mu)$, then $f \in S(n)$ for $|z| < r_0$, where*

$$r_0 = \frac{(1+\beta-\alpha) - \sqrt{\alpha^2 - 2\beta\alpha + \beta(2+\beta)}}{1-2\alpha}, \text{ when } \alpha \neq 1/2$$

and

$$r_0 = \frac{1}{1+2\beta}, \text{ when } \alpha = 1/2.$$

The result is sharp.

Proof. $f \in C(n, \alpha, \beta, \mu)$, if and only if there there exists a function $P \in P(0)$ and $F(z) \in S^*(\alpha)$ such that

$$\frac{I_\mu^n f(z)}{F(z)} = [P(z)]^\beta.$$

$$I_\mu^n f(z) = [P(z)]^\beta F(z). \tag{2.9}$$

Logarithmic differentiation of (2.9) yields

$$\frac{z(I_\mu^n f(z))'}{I_\mu^n f(z)} = \beta \frac{zP'(z)}{P(z)} + \frac{zF'(z)}{F(z)}.$$

Now by a result of MacGregor [7], we know that

$$\left|\frac{zP'(z)}{P(z)}\right| \leq \frac{2r}{1-r^2}.$$

Therefore

$$\begin{aligned}\Re\left\{\frac{z(I_\mu^n f(z))'}{I_\mu^n f(z)}\right\} &\geq \Re\left\{\frac{zF'(z)}{F(z)}\right\} - \beta\left|\frac{zP'(z)}{P(z)}\right| \\ &\geq \frac{1-(1-2\alpha)r}{1+r} - \beta\left(\frac{2r}{1-r^2}\right) \\ &= \frac{(1-2\alpha)r^2 - 2(1+\beta-\alpha)r + 1}{1-r^2}.\end{aligned}$$

The right hand side of the above inequality is not less than or equal to zero provided $|z| = r < r_0$, where r_0 is as given in the statement of theorem. Sharpness follows if we take $f(z)$ to be the same as in Theorem 2.3. □

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On multivalent starlike functions

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Abstract. We prove some new sufficient conditions for function to be p-valent, or p-valently starlike in the unit disc.

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Keywords: Analytic functions, univalent functions, Ozaki's condition.

1. Introduction

Let $\mathcal{H}(\mathbb{D})$ denote the class of all analytic functions in the open unit disk

$$\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}.$$

For n a positive integer and $a \in \mathbb{C}$, let

$$\mathcal{H}[a, n] = \left\{f \in \mathcal{H}(\mathbb{D}) : f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \cdots \right\}.$$

Especially, let $\mathcal{A}_p$ be the subclass of $\mathcal{H}(\mathbb{D})$ consisting of the functions of the form

$$f(z) = z^p + \sum_{n=1}^{\infty} a_{p+n} z^{p+n}$$

and $\mathcal{A} \equiv \mathcal{A}_1$. For more details see [1, 4, 8].

Further, a function $f \in \mathcal{A}_p$, $p = 2, 3, \ldots$, is said to be *p-valently (or multivalently) starlike of order* α, $0 \leq \alpha < p$, if

$$\mathfrak{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha \quad (z \in \mathbb{D}).$$

The class of all such functions is usually denoted by $\mathcal{S}_p^*(\alpha)$. For $p = 1$ we receive the well known class of normalized starlike univalent functions.

It is well known result from the theory of univalent functions due to Marx and Strohhacker ([3, 10])) that when $f \in \mathcal{A}$,

$$\mathfrak{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > 0 \quad (z \in \mathbb{D}) \qquad \Rightarrow \qquad \mathfrak{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \frac{1}{2} \quad (z \in \mathbb{D}).$$

In this paper we study the question, that naturally rises, about the relation between the expressions $\frac{zf^{(p)}(z)}{f^{(p-1)}(z)}$ and $\frac{zf^{(p-1)}(z)}{f^{(p-2)}(z)}$, i.e., between $\frac{zf^{(p)}(z)}{f^{(p-1)}(z)}$ and $\frac{zf'(z)}{f(z)}$. We close the paper with a sharp (necessary and sufficient) condition when the following implication holds:

$$\mathfrak{Re}\left\{\frac{zf^{(p)}(z)}{f^{(p-1)}(z)}\right\} > \alpha \quad (z \in \mathbb{D}) \qquad \Rightarrow \qquad \mathfrak{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \beta \quad (z \in \mathbb{D}),$$

for the case $\frac{p-1}{2} \leq \alpha < p$. We also give non-sharp result for $0 \leq \alpha < \frac{p-1}{2}$ and the sharp version in this case remains an open problem.

For the study we use two different methods, one based on a generalized Jack lemma, and other, based on a result from the theory of differential subordinations. The first method will bring stronger conclusion, but that conclusion will hold for smaller class than the corresponding conclusion received by the second method, which will even be sharp for $\frac{p-1}{2} \leq \alpha < p$. Further in the paper, this will be discussed again, in more details.

2. Results based on generalized Jack lemma

The famous Jack lemma was originally published in [2] and its generalization will be used for proving the main result in this section.

First, let denote by $\mathcal{Q}$ the class of functions f that are analytic and injective on $\overline{\mathbb{D}} \setminus E(f)$, where

$$E(f) := \{\zeta : \zeta \in \partial\mathbb{D} \ \text{ and } \ \lim_{z\to\zeta} f(z) = \infty\},$$

and are such that

$$f'(\zeta) \neq 0 \qquad (\zeta \in \partial(\mathbb{D}) \setminus E(f)).$$

Now, the generalization of the Jack lemma that we will need is the following

Lemma 2.1 ([4]). *Let $q \in \mathcal{Q}$ with $q(0) = a$ and let*

$$p(z) = a + a_n z^n + \cdots$$

be analytic in $\mathbb{D}$ with

$$p(z) \not\equiv a \ \text{ and } \ n \in \mathbb{N} = \{1, 2, 3, \cdots\}.$$

If p is not subordinate to q, then there exist points

$$z_0 = r_0 e^{i\theta} \in \mathbb{D} \ \text{ and } \ \zeta_0 \in \partial\mathbb{D} \setminus E(q),$$

for which

$$p(|z| < r_0) \subset q(\mathbb{D}),$$
$$p(z_0) = q(\zeta_0)$$

and

$$z_0 p'(z_0) = k\zeta_0 q'(\zeta_0)$$

for some $k \geq n$.

The next lemma will also be needed.

Lemma 2.2 **([7]).** *Let $p(z)$, $p(0)=1$, be analytic in $|z|<1$ and suppose that there exists a point z_0, $|z_0|<1$ such that*

$$\mathfrak{Re}\{p(z)\} > \beta \quad for \quad |z|<|z_0|$$

and

$$\mathfrak{Re}\{p(z_0)\} = \beta, \tag{2.1}$$

where $0 \le \beta < 1$ and $p(z_0) \neq \beta$. Then we have the following properties:
(1) *For the case* $\arg\{p(z_0)\} > 0$,

$$\begin{aligned} \frac{z_0 p'(z_0)}{p(z_0)-\beta} &= \left\{ \frac{-i}{|p(z)-\beta|} \frac{\mathrm{d}|p(z)-\beta|}{\mathrm{d}\theta} \right\}_{z=z_0} \\ &= ik, \end{aligned} \tag{2.2}$$

where k is a real number and $k \ge (a^2+1)/(2a)$, $p(z_0)-\beta = ia$ and $a>0$.
(2) *For the case* $\arg\{p(z_0)\} < 0$,

$$\begin{aligned} \frac{z_0 p'(z_0)}{p(z_0)-\beta} &= \left\{ \frac{-i}{|p(z)-\beta|} \frac{\mathrm{d}|p(z)-\beta|}{\mathrm{d}\theta} \right\}_{z=z_0} \\ &= -ik, \end{aligned} \tag{2.3}$$

where k is a real number and $k \ge (a^2+1)/(2a)$, $p(z_0)-\beta = -ia$ and $a>0$.

Before giving the main result of this section let prove the following lemma that we will also need.

Lemma 2.3. *Let $p(z)$, $p(0)=1$, be analytic in the unit disk $\mathbb{D}$ and suppose that there exists a point z_0, $|z_0|<1$, such that*

$$\mathfrak{Re}\{p(z)\} > \beta \quad for \quad |z|<|z_0| \tag{2.4}$$

and

$$\mathfrak{Re}\{p(z_0)\} = \beta, \quad a = |\mathfrak{Im}\{p(z_0)\}| > 0,$$

where $0 \le \beta < 1$. Then, there are the real numbers k and m such that

$$k \ge \frac{a^2+1}{2a}, \quad m \ge 1$$

and

$$z_0 p'(z_0) = \begin{cases} -ka & when \quad p(z_0) \neq \beta \\ -m(1-\beta)/2 & when \quad p(z_0) = \beta. \end{cases} \tag{2.5}$$

Proof. For the case $p(z_0) \neq \beta$, $0<\beta<1$, we may apply Lemma 2.2. If $\arg\{p(z_0)\} > 0$, then from (2.2) we have

$$\frac{z_0 p'(z_0)}{p(z_0)-\beta} = \frac{z_0 p'(z_0)}{ia} = ik,$$

hence $z_0 p'(z_0) = -ka$. For the case $\arg\{p(z_0)\} < 0$, from (2.3) we have

$$\frac{z_0 p'(z_0)}{p(z_0)-\beta} = \frac{z_0 p'(z_0)}{-ia} = -ik,$$

hence $z_0 p'(z_0) = -ka$.

For the case $p(z_0)=\beta$, $0\le\beta<1$, we will apply Lemma 2.1. If $p(z_0)=\beta$, then

$$p(z)\not\prec\frac{1+(1-2\beta)z}{1-z}:=g_\beta(z),$$

moreover

$$p(z_0)=\beta=g_\beta(\zeta_0),\ \zeta_0=-1.$$

From this and from (2.4), by Lemma 2.1, it follows that

$$z_0p'(z_0)=m\zeta_0g'_\beta(\zeta_0),\ \zeta_0=-1, \tag{2.6}$$

for some $m\ge1$. For the function g_β we have

$$g'_\beta(z)=\left.\frac{2(1-\beta)}{(1-z)^2}\right|_{z=\zeta_0}=\frac{1-\beta}{2}.$$

Applying this in (2.6) we obtain the second case in (2.5). □

Finally, we can formulate and prove the main result of this section.

Theorem 2.4. *Let $f\in\mathcal{A}_p$, $p\ge2$, be analytic in $\mathbb{D}$. Assume that*

$$\mathfrak{Re}\left\{\frac{zf^{(p)}(z)}{f^{(p-1)}(z)}\right\}>0\quad(z\in\mathbb{D}) \tag{2.7}$$

and

$$\frac{zf^{(p-1)}(z)}{2f^{(p-2)}(z)}\ne\frac{2}{3}\quad(z\in\mathbb{D}). \tag{2.8}$$

Then,

$$\mathfrak{Re}\left\{\frac{zf^{(p-1)}(z)}{f^{(p-2)}(z)}\right\}>\frac{4}{3}\quad(z\in\mathbb{D}). \tag{2.9}$$

Proof. Let us put

$$p(z)=\frac{zf^{(p-1)}(z)}{2f^{(p-2)}(z)},$$

such that $p(0)=1$. It follows that

$$\frac{zf^{(p)}(z)}{f^{(p-1)}(z)}=2p(z)+\frac{zp'(z)}{p(z)}-1. \tag{2.10}$$

If there exists a point $z_0\in\mathbb{D}$ such that

$$\mathfrak{Re}\{p(z_0)\}>\frac{2}{3}\ \text{ for }\ |z|<|z_0|$$

and

$$\mathfrak{Re}\{p(z_0)\}=\frac{2}{3},$$

then, from Lemma 2.2, we have

$$\frac{z_0p'(z_0)}{p(z_0)-2/3}=ik,\quad k\ge1, \tag{2.11}$$

where $k \geq (a^2+1)/(2a)$ if $p(z_0) = 2/3 + ia$ and $a > 0$, while $k \leq -(a^2+1)/(2a)$ if $p(z_0) = 2/3 - ia$ and $a > 0$. For the case $p(z_0) = 2/3 + ia$ and $a > 0$, from (2.10) and (2.11), we have

$$\begin{aligned} \frac{z_0 f^{(p)}(z_0)}{f^{(p-1)}(z_0)} &= 2p(z_0) + \frac{z_0 p'(z_0)}{p(z_0)} - 1 \\ &= 2p(z_0) + \frac{z_0 p'(z_0)}{p(z_0) - 2/3} \frac{p(z_0) - 2/3}{p(z_0)} - 1 \\ &= \frac{4}{3} + 2ia + ik\frac{ia}{2/3 + ia} - 1 \\ &= \frac{1}{3} - \frac{ak}{2/3 + ia}. \end{aligned} \tag{2.12}$$

Hence, for $a > 0$, we obtain

$$\begin{aligned} \mathfrak{Re}\left\{\frac{z_0 f^{(p)}(z_0)}{f^{(p-1)}(z_0)}\right\} &= \mathfrak{Re}\left\{\frac{1}{3} - \frac{ak}{2/3 + ia}\right\} \\ &= \frac{1}{3} - \frac{2ak/3}{4/9 + a^2} \\ &\leq \frac{1}{3} - \frac{6a\frac{a^2+1}{2a}}{4 + 9a^2} \\ &= \frac{1}{3} - \frac{3a^2 + 3}{4 + 9a^2} \\ &= \frac{4 - 9}{3(4 + 9a^2)} \\ &< 0. \end{aligned}$$

This contradicts with (2.7). For the case $p(z_0) = 2/3 - ia$, $k \leq -(a^2+1)/(2a)$ and $a > 0$, from (2.11) and (2.12), we have

$$\begin{aligned} \mathfrak{Re}\left\{\frac{z_0 f^{(p)}(z_0)}{f^{(p-1)}(z_0)}\right\} &= \mathfrak{Re}\left\{\frac{4}{3} - 2ia + ik\frac{-ia}{2/3 - ia} - 1\right\} \\ &= \mathfrak{Re}\left\{\frac{1}{3} + \frac{ak}{2/3 - ia}\right\} \\ &= \frac{1}{3} + \frac{2ak/3}{4/9 + a^2} \\ &\leq \frac{1}{3} - \frac{6a\frac{a^2+1}{2a}}{4 + 9a^2} \\ &= \frac{1}{3} - \frac{3a^2 + 3}{4 + 9a^2} \\ &= \frac{4 - 9}{3(4 + 9a^2)} \\ &< 0. \end{aligned}$$

This also contradicts with (2.7). Therefore,

$$\mathfrak{Re}\,\{p(z)\} > \frac{2}{3} \quad (z \in \mathbb{D})$$

which is equivalent to (2.9). □

The condition (2.8) in the above theorem is necessary in order to apply Lemma 2.2, i.e., without (2.8) we cannot obtain 4/3 in (2.9).

3. Results based on differential subordinations

In this section we will give sharp, i.e., necessary and sufficient, conditions when the following implications hold.

$$\mathfrak{Re}\left\{\frac{zf^{(p)}(z)}{f^{(p-1)}(z)}\right\} > \alpha \quad (z \in \mathbb{D}) \qquad \Rightarrow \qquad \mathfrak{Re}\left\{\frac{zf^{(p-1)}(z)}{f^{(p-2)}(z)}\right\} > \beta \quad (z \in \mathbb{D})$$

$$\mathfrak{Re}\left\{\frac{zf^{(p)}(z)}{f^{(p-1)}(z)}\right\} > \alpha \quad (z \in \mathbb{D}) \qquad \Rightarrow \qquad \mathfrak{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \beta \quad (z \in \mathbb{D})$$

For proving the main result we will use the following lemma from the theory of differential subordinations.

Lemma 3.1 (Theorem 2.3i(i), p. 35, [4]**).** *Let $\Omega \subset \mathbb{C}$ and suppose that the function $\psi : \mathbb{C}^2 \times \mathbb{D} \to \mathbb{C}$ satisfies $\psi(ix, y; z) \notin \Omega$ for all $x \in \mathbb{R}$, $y \leq -n(1 + x^2)/2$, and $z \in \mathbb{D}$. If $q \in H[1, n]$ and $\psi(q(z), zq'(z); z) \in \Omega$ for all $z \in \mathbb{D}$, then $\mathfrak{Re}\{q(z)\} > 0$, $z \in \mathbb{D}$.*

Theorem 3.2. *Let p be a positive integer, $p \geq 3$ and $\beta_1 \leq \beta < 2$, where $\beta_1 = \frac{1+\sqrt{17}}{4} = 1.280776\ldots$. Also, let*

$$\alpha \equiv \alpha(\beta) = -\frac{1}{2} + \beta - \frac{1}{\beta}. \tag{3.1}$$

If $f \in \mathcal{A}_p$ and

$$\mathfrak{Re}\left\{\frac{zf^{(p)}(z)}{f^{(p-1)}(z)}\right\} > \alpha \quad (z \in \mathbb{D}),$$

then

$$\mathfrak{Re}\left\{\frac{zf^{(p-1)}(z)}{f^{(p-2)}(z)}\right\} > \beta \quad (z \in \mathbb{D}). \tag{3.2}$$

This result is sharp, i.e. $\alpha(\beta)$ can not be replaced by a smaller number so that the implication to remains true.

Proof. In the view of Lemma 3.1, let define the functions

$$q(z) = \frac{1}{2-\beta}\left\{\frac{zf^{(p-1)}(z)}{f^{(p-2)}(z)} - \beta\right\} \quad (z \in \mathbb{D})$$

and

$$\psi(r, s; z) = \frac{s(2-\beta)}{r(2-\beta)+\beta} + r(2-\beta) + \beta - 1,$$

such that $q(z) \in \mathcal{H}[1,1]$ and

$$\psi(q(z), zq'(z); z) = \frac{zf^{(p)}(z)}{f^{(p-1)}(z)}.$$

Now, let consider

$$\psi(ix, y; z) = \frac{(2-\beta)y}{ix(2-\beta)+\beta} + ix(2-\beta) + \beta - 1,$$

where x is real number, $y \le -(1+x^2)/2$ and $z \in \mathbb{D}$. We have

$$\begin{aligned} \mathfrak{Re}\{\psi(ix, y; z)\} &= \beta - 1 + \frac{(2-\beta)\beta y}{(2-\beta)^2x^2+\beta^2} \\ &\le \beta - 1 - \frac{(2-\beta)\beta\frac{1+x^2}{2}}{(2-\beta)^2x^2+\beta^2} \equiv \varphi(\beta, x) \end{aligned}$$

and

$$\frac{\partial}{\partial x}\varphi(\beta, x) = \frac{4\beta(1-\beta)(2-\beta)x}{[(2-\beta)^2x^2+\beta^2]^2}.$$

Since $1 < \beta_1 \le \beta < 2$, we obtain that $\varphi(\beta, x)$ is a decreasing function of the variable x. So,

$$\varphi(\beta, x) \le \varphi(\beta, 0) = -\frac{1}{2} + \beta - \frac{1}{\beta} = \alpha(\beta)$$

and from Lemma 3.1 we conclude that

$$\mathfrak{Re}\left\{\frac{zf^{(p)}(z)}{f^{(p-1)}(z)}\right\} > \alpha(\beta) \quad (z \in \mathbb{D}) \qquad \Rightarrow \qquad \mathfrak{Re}\{q(z)\} > 0 \quad (z \in \mathbb{D}),$$

i.e.

$$\mathfrak{Re}\left\{\frac{zf^{(p-1)}(z)}{f^{(p-2)}(z)}\right\} > \beta \quad (z \in \mathbb{D}).$$

Now, we will show that the result is sharp using the extremal function $f_*(z)$ defined by

$$f_*^{(p-2)}(z) = \frac{p!}{2}z^2 \cdot (1-z)^{-2(2-\beta)},$$

(all powers are taken by their principal values) such that

$$\frac{zf_*^{(p-1)}(z)}{f_*^{(p-2)}(z)} - \beta = \frac{1+z}{1-z} \cdot (2-\beta) \equiv q_*(z).$$

First, let note that for $z = 0$, $\frac{zf_*^{(p-1)}(z)}{f_*^{(p-2)}(z)} = 2$ and that $f_*(z)$ is analytic in $\mathbb{D}$, i.e. that $f_*(z) \in \mathcal{A}_p$. Further, recall that $\frac{1+z}{1-z}$ maps the unit disk onto the right half of the complex plane and the unit circle onto the imaginary axis. So, $\frac{zf_*^{(p-1)}(z)}{f_*^{(p-2)}(z)}$ maps the unit disk onto $\{\omega : \mathfrak{Re}\{\omega\} > \beta\}$.

In order to prove the sharpness of the result it is enough to show that the boundary of the image of the unit disk by the function $\frac{zf_*^{(p)}(z)}{f_*^{(p-1)}(z)}$ touches the vertical line through $\alpha(\beta)$. Otherwise, it would be possible to find α larger than $\alpha(\beta)$ that implies the conclusion (3.2). In that direction, let note that for any $|z| = 1$, $z \neq 1$, there exists real number x such that

$$q_*(z) = \frac{1+z}{1-z} \cdot (2-\beta) = ix(2-\beta),$$

$$zq_*'(z) = \frac{2z(2-\beta)}{(1-z)^2} = \frac{1}{2}\left\{\left(\frac{1+z}{1-z}\right)^2 - 1\right\}(2-\beta) = -\frac{1}{2}(x^2+1)(2-\beta)$$

and

$$\begin{aligned} &\mathfrak{Re}\left\{\frac{zf_*^{(p)}(z)}{f_*^{(p-1)}(z)}\right\} \\ = \ &\mathfrak{Re}\left\{\frac{zq_*'(z)}{q_*(z)+\beta} + q_*(z) + \beta - 1\right\} \\ = \ &\mathfrak{Re}\left\{\frac{-\frac{1}{2}(x^2+1)(2-\beta)}{ix(2-\beta)+\beta} + ix(2-\beta) + \beta - 1\right\} \\ = \ &\beta - 1 - \frac{(2-\beta)\beta\frac{1+x^2}{2}}{(2-\beta)^2x^2+\beta^2} = \varphi(\beta, x). \end{aligned}$$

Therefore, for $z = -1$ we have

$$\mathfrak{Re}\left\{\frac{zf_*^{(p)}(z)}{f_*^{(p-1)}(z)}\right\} = \varphi(\beta, 0) = \beta - 1 - \frac{2-\beta}{2\beta} = \alpha(\beta).$$

So, $\frac{zf_*^{(p)}(z)}{f_*^{(p-1)}(z)}$ maps the unit disk onto a region that touches, from the right hand side, the vertical line through $\alpha(\beta)$. This completes the proof of the sharpness of the result. □

Remark 3.3. It can be verified that the function $\varphi(\beta, x)$ is negative if, and only if, $0 \leq \beta < \beta_1$ and this case is not of interest because it leads to negative values of α.

Theorem 3.2 can be rewritten in the following, equivalent form.

Theorem 3.4. *Let p be a positive integer such that $p \geq 3$. Also, let $0 \leq \alpha < 1$ and*

$$\beta \equiv \beta(\alpha) = \frac{1 + 2\alpha + \sqrt{(1+2\alpha)^2 + 16}}{4}. \tag{3.3}$$

If $f \in \mathcal{A}_p$ and

$$\mathfrak{Re}\left\{\frac{zf^{(p)}(z)}{f^{(p-1)}(z)}\right\} > \alpha \quad (z \in \mathbb{D}),$$

then

$$\mathfrak{Re}\left\{\frac{zf^{(p-1)}(z)}{f^{(p-2)}(z)}\right\} > \beta \quad (z \in \mathbb{D}).$$

This result is sharp, i.e. $\beta(\alpha)$ can not be replaced by a larger number so that the implication remains true.

Proof. For the function $\alpha(\beta)$ defined by (3.1) we have $\frac{\partial}{\partial\beta}\alpha(\beta) = 1 + \frac{1}{\beta^2} > 0$. So, $\alpha(\beta)$ is a strictly monotone function on the interval $[\beta_1, 2)$ and there exist its inverse function. It is not difficult to check that $\beta(\alpha)$ is that inverse function. □

Remark 3.5. For $\alpha = 0$ in the previous theorem we receive $\beta = \frac{1+\sqrt{17}}{4} = 1.280776\ldots$ which is smaller that $4/3$, received in Theorem 2.4. This does not mean that the result from Theorem 2.4 is stronger because it is obtained under the additional condition $\frac{zf^{(p-1)}(z)}{2f^{(p-2)}(z)} \neq \frac{2}{3}$ for all $z \in \mathbb{D}$.

For simplicity of the proofs of the next corollaries we will reformulate Theorem 3.4 again.

Theorem 3.6. *Let l be a positive integer such that $l \geq 3$. Also, let $0 \leq \alpha < 1$ and $\beta \equiv \beta(\alpha)$ be defined as in (3.3). If $g \in \mathcal{A}_l$ and*

$$\mathfrak{Re}\left\{\frac{zg^{(l)}(z)}{g^{(l-1)}(z)}\right\} > \alpha \quad (z \in \mathbb{D}),$$

then

$$\mathfrak{Re}\left\{\frac{zg^{(l-1)}(z)}{g^{(l-2)}(z)}\right\} > \beta \quad (z \in \mathbb{D}).$$

This result is sharp, i.e. $\beta(\alpha)$ can not be replaced by a larger number so that the implication remains true.

Now, as a corollary we obtain sharp information about the real part of $\frac{zf^{(k)}(z)}{f^{(k-1)}(z)}$ for any $2 \leq k \leq p-1$, having information about the real part of $\frac{zf^{(p)}(z)}{f^{(p-1)}(z)}$.

Corollary 3.7. *Let p and k be positive integers such that $p \geq 3$ and $2 \leq k \leq p-1$. Also, let $0 \leq \alpha < 1$ and let α_i, $i = 1, 2, \ldots$, be a sequence defined by:*

$$\alpha_1 = \beta(\alpha), \qquad \alpha_i = \beta(\alpha_{i-1} - 1),$$

where $\beta(\alpha)$ is defined as in (3.3). If $f \in \mathcal{A}_p$ and

$$\mathfrak{Re}\left\{\frac{zf^{(p)}(z)}{f^{(p-1)}(z)}\right\} > \alpha \quad (z \in \mathbb{D}),$$

then

$$\mathfrak{Re}\left\{\frac{zf^{(k)}(z)}{f^{(k-1)}(z)} - 1\right\} > \alpha_{p-k} \quad (z \in \mathbb{D}). \tag{3.4}$$

This result is sharp, i.e., α_{p-k} can not be replaced by a larger number so that implication remains true.

Proof. Applying Theorem 3.6 with $g(z) = f(z)$ and $l = p$ we receive

$$\mathfrak{Re}\left\{\frac{zf^{(p)}(z)}{f^{(p-1)}(z)}\right\} > \alpha \quad (z \in \mathbb{D}) \qquad \Rightarrow \qquad \mathfrak{Re}\left\{\frac{zf^{(p-1)}(z)}{f^{(p-2)}(z)}\right\} > \beta(\alpha) = \alpha_1 \quad (z \in \mathbb{D}).$$

Further, for the function $g(z) \in \mathcal{A}_{p-1}$ defined by $g^{(p-2)}(z) = \frac{2}{p} \cdot \frac{f^{(p-2)}(z)}{z}$ and $l = p-1$ we have

$$\mathfrak{Re}\left\{\frac{zg^{(l)}(z)}{g^{(l-1)}(z)}\right\} = \mathfrak{Re}\left\{\frac{zf^{(p-1)}(z)}{f^{(p-2)}(z)} - 1\right\} > \beta(\alpha) - 1 \quad (z \in \mathbb{D})$$

and from Theorem 3.6 we obtain

$$\mathfrak{Re}\left\{\frac{zf^{(p-2)}(z)}{f^{(p-3)}(z)} - 1\right\} = \mathfrak{Re}\left\{\frac{zg^{(l-1)}(z)}{g^{(l-2)}(z)}\right\} > \beta(\beta(\alpha) - 1) = \alpha_2 \quad (z \in \mathbb{D}).$$

Applying Theorem 3.6 recursively ($p-k$ times in total) we reach (3.4). The sharpness of the implication follows from the sharpness of Theorem 3.6. □

Remark 3.8.

(i) The sequence defined in Corollary 3.7 is increasing and bounded between β_1 and 2. Thus, the recursive application of Theorem 3.6 is possible.

(ii) For $\alpha = 0$ in Corollary 3.7 we receive an improvement of a result from [6] where the implication is proven with 0 on the place of α_{p-k} in (3.4).

For obtaining partially sharp information about the real part of $\frac{zf'(z)}{f(z)}$, having information about the real part of $\frac{zf^{(p)}(z)}{f^{(p-1)}(z)}$, we need the following two results.

Lemma 3.9 ([9])**.** *Let p be a positive integer and $\frac{p-1}{2} \le \alpha < p$. Also, let*

$$\beta \equiv \beta_1(\alpha, p) = \frac{p}{{}_2F_1(1, 2(p-\alpha); p+1; 1/2)}.$$

If $f \in \mathcal{A}_p$ and

$$\mathfrak{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \alpha \quad (z \in \mathbb{D}),$$

then

$$\mathfrak{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \beta \quad (z \in \mathbb{D}).$$

So, $\mathcal{K}_p(\alpha) \subset \mathcal{S}_p^*(\beta_1(\alpha, p))$, i.e. p-valently convex functions of order α, have $\beta_1(\alpha, p)$ order of p-valently starlikeness. This result is sharp, i.e., β can not be replaced by a number bigger than $\beta_1(\alpha, p)$ so that the implication remains true.

Lemma 3.10 ([5])**.** *Let p be a positive integer and $0 \le \alpha < p$. Also, let*

$$\beta \equiv \beta_2(\alpha, p) = \begin{cases} \frac{2(\alpha+p)-1+\sqrt{[2(\alpha+p)-1]^2-16\alpha p}}{4} & \text{if} \quad 0 \le \alpha \le \frac{p-1}{2} \\ \frac{2\alpha-1+\sqrt{(2\alpha-1)^2+8p}}{4} & \text{if} \quad \frac{p-1}{2} < \alpha < p \end{cases}.$$

If $f \in \mathcal{A}_p$ and

$$\mathfrak{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \alpha \quad (z \in \mathbb{D}),$$

then

$$\mathfrak{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \beta \quad (z \in \mathbb{D}).$$

So, $\mathcal{K}_p(\alpha) \subset \mathcal{S}_p^*(\beta_2(\alpha, p))$, i.e. p-valently convex functions of order α, have $\beta_2(\alpha, p)$ order of p-valently starlikeness.

Finally, here is a sharp solution of the general question studied in this paper.

Corollary 3.11. *Let p be a positive integer such that $p \geq 3$. Also, let $0 \leq \alpha < 1$, α_{p-2} is defined as in Corollary 3.7 and*

$$\beta \equiv \begin{cases} \beta_2(\alpha_{p-2} + 2, p), & 0 \leq \alpha < \frac{p-1}{2} \\ \beta_1(\alpha_{p-2} + 2, p), & \frac{p-1}{2} \leq \alpha < p \end{cases},$$

where functions $\beta_1(\alpha, p)$ and $\beta_2(\alpha, p)$ are defined as in Lemma 3.9 and Lemma 3.10. If $f \in \mathcal{A}_p$ and

$$\mathfrak{Re}\left\{\frac{zf^{(p)}(z)}{f^{(p-1)}(z)}\right\} > \alpha \quad (z \in \mathbb{D}),$$

then

$$\mathfrak{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \beta \quad (z \in \mathbb{D}).$$

For $\frac{p-1}{2} \leq \alpha < p$ the result is sharp.

Proof. For $k = 2$ in Corollary 3.7 we receive that

$$\mathfrak{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \alpha_{p-2} + 2 \quad (z \in \mathbb{D}).$$

The rest follows from Lemma 3.9 and Lemma 3.10. □

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Quasilinear parabolic equations with $p(x)$-Laplacian diffusion terms and nonlocal boundary conditions

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Abstract. In this study, we prove the existence of local solution for a quasi linear generalized parabolic equation with nonlocal boundary conditions for an elliptic operator involving the variable-exponent nonlinearities, using Faedo-Galerkin arguments and compactness method.

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Keywords: Nonlocal boundary conditions, quasi-linear parabolic equations, generalized Lebesgue space and Sobolev spaces with variable exponents.

1. Introduction

Let Ω be a bounded domain in $\mathbb{R}^n$, $n \geq 2$ with a smooth boundary $\Gamma = \partial\Omega$. We consider the following quasi linear parabolic equations with nonlocal boundary conditions:

$$\frac{\partial u}{\partial t} - \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(|u|^{p(x)-2}\frac{\partial u}{\partial x_i}\right) + |u|^{p(x)-2} u = f(x,t) \text{ in } Q_T = \Omega \times (0,T), \quad (1.1)$$

$$u(x,0) = u_0(x), \quad x \in \Omega, \quad (1.2)$$

$$u(x,t) = \int_\Omega K(x,y)\, u(y,t)\, dy, \quad x \in \Gamma, \ t \in (0,T), \quad (1.3)$$

where the exponent $p(\cdot)$ is a given measurable function on $\overline{\Omega}$ such that:

$$2 \leq n < p_1 \leq p(x) \leq p_2 \leq \infty, \quad (1.4)$$

where

$$p_2 = ess \sup_{x\in\Omega} p(x), \quad p_1 = ess \inf_{x\in\Omega} p(x).$$

We also assume that $p(\cdot)$ satisfies the following Zhikov-Fan uniform local continuity condition :

$$|p(x) - p(y)| \leq \frac{M}{|\log |x-y||}, \text{ for all } x,\ y \text{ in } \Omega \text{ with } |x-y| < \frac{1}{2},\ M > 0. \tag{1.5}$$

In recent years, many authors have paid attention to the study of nonlinear hyperbolic, parabolic and elliptic equations with nonstandard growth condition. For instance, modeling of physical phenomena such as flows of electro-rheological fluids or fluids with temperature-dependent viscosity, thermoelasticity, nonlinear viscoelasticity, filtration processes through a porous media and image processing. More details on these problems can be found in [5, 8, 1, 3, 4, 15, 17, 18] and references therein.

Constant exponent. In (1.1), when $p(\cdot) = p$ is constant, local, global existence and long-time behavior have been considered by many authors.

For instance, in the absence of the term $|u|^{p-2} u$ and when the kernel datum function $K(x, y) = 0$, using the compactness method and Faedo-Galerkin techniques, the existence and uniqueness of a weak solution has been proved see [16].

Baili Chen in [7] generalized the result of Lions to the situation when the presence of $|u|^{p-2} u$ and when $K(x, y) \neq 0$ in problem (1.1), applying exactly the same technique introduced in [16, Problème 12, page 140.], the author by constructing the approximate Galerkin solution, he proved the existence of generalized solution, the uniqueness questions are still open.

Problem (1.1)-(1.3) is the extension of the problems in Lion's book [16, p.140] in which the boundary conditions are homogeneous and in [7] in which the variable-exponent is constant. The uniqueness questions in problem (1.1)-(1.3) are more complicated than in [7] and are still open.

The main difficulty of this problem, concerns the weak converging approximate solution, is related to the presence of the quasilinear terms in (1.1) in the variable-exponent.

In this paper a class of quasi linear generalized parabolic equation with nonlocal boundary conditions for an elliptic operator involving the variable-exponent nonlinearities was considered. Hence by using Faedo-Galerkin arguments and compactness method as in [16], we will show the local existence of problem (1.1)-(1.3).

2. Preliminaries

In this section we list and recall some well-known results and facts from the theory of the Sobolev spaces with variable exponent. (For the details see [9, 11, 10, 12, 13, 14]).

Throughout the rest of the paper we assume that Ω is a bounded domain of $\mathbb{R}^n$, $n \geq 2$ with smooth boundary Γ, Let $p : \overline{\Omega} \to [1, \infty]$ be a measurable function. We denote by $L^{p(\cdot)}(\Omega)$ the set of measurable functions u on Ω such that

$$A_{p(\cdot)}(u) = \int_{\Omega} |u(x)|^{p(x)}\, dx < \infty.$$

The variable-exponent space $L^{p(\cdot)}(\Omega)$ equipped with the Luxemburg norm

$$\|u\|_{p(\cdot),\Omega} = \|u\|_{p(\cdot)} = \|u\|_{L^{p(\cdot)}(\Omega)} = \inf\left\{\lambda > 0,\ A_{p(\cdot)}\left(\frac{u}{\lambda}\right) \leq 1\right\}$$

is a Banach space.

In general, variable-exponent Lebesgue spaces are similar to classical Lebesgue spaces in many aspects, see the first discussed the $L^{p(x)}(\Omega)$ spaces and $W^{k,p(x)}(\Omega)$ spaces by Kovàcik and Rákosnik in [14].

Let us list some properties of the spaces $L^{p(\cdot)}(\Omega)$ which will be used in the study of the problem (1.1)-(1.3).

- It follows directly from the definition of the norm that (see [9]),

$$\min\left(\|u\|_{p(\cdot)}^{p_1}, \|u\|_{p(\cdot)}^{p_2}\right) \leq A_{p(\cdot)}(u) \leq \max\left(\|u\|_{p(\cdot)}^{p_1}, \|u\|_{p(\cdot)}^{p_2}\right).$$

- Let p, q, $s \geq 1$ be measurable functions defined on $\overline{\Omega}$ such that

$$\frac{1}{s(x)} = \frac{1}{p(x)} + \frac{1}{q(x)}, \text{ for a.e. } x \in \Omega.$$

  if $u \in L^{p(\cdot)}(\Omega)$, $v \in L^{q(\cdot)}(\Omega)$ then $u.v \in L^{s(\cdot)}(\Omega)$ and the following generalized Hölder inequality

$$\|uv\|_{s(\cdot)} \leq 2\|u\|_{p(\cdot)}\|v\|_{q(\cdot)}.$$

  holds.

  Let us consider the following variable-exponent Lebesgue Sobolev space (see [9]),

$$W^{1,p(\cdot)}(\Omega) = \left\{v \in L^{p(\cdot)}(\Omega) : \text{such that } |\nabla v| \text{ exists and } |\nabla v| \in L^{p(\cdot)}(\Omega)\right\}.$$

This space is a Banach space with respect to the norm

$$\|u\|_{W_0^{1,p(\cdot)}(\Omega)} = \|u\|_{p(\cdot)} + \sum_i \|\nabla u_i\|_{p(\cdot)}.$$

Furthermore, we set $W_0^{1,p(\cdot)}(\Omega)$, to be the closure of $C_0^\infty(\Omega)$ in $W^{1,p(\cdot)}(\Omega)$. Here we note that the space $W_0^{1,p(\cdot)}(\Omega)$ is usually defined in a different way for the variable exponent case. However (see Diening et al [9]), both definitions are equivalent under (1.5). The $\left(W_0^{1,p(\cdot)}(\Omega)\right)'$ is the dual space of $W_0^{1,p(\cdot)}(\Omega)$ with respect to the inner product in $L^2(\Omega)$ and is defined as $W^{-1,p'(\cdot)}(\Omega)$, in the same way as the classical Sobolev spaces, where $\frac{1}{p(.)} + \frac{1}{p'(\cdot)} = 1$, the function $p'(\cdot)$ is called the dual variable exponent of $p(\cdot)$.

- Let p, $q : \Omega \to [1, +\infty)$ be measurable functions satisfying condition (1.5). If $p(x) \leq q(x)$ almost everywhere in Ω, then the embedding $L^{q(\cdot)}(\Omega) \hookrightarrow L^{p(\cdot)}(\Omega)$ is continuous.

Lemma 2.1. ([9]) *Let Ω be a bounded domain in $\mathbb{R}^n$, $n \geq 1$ with a smooth boundary $\Gamma = \partial\Omega$, $p(\cdot)$ is a given measurable function on $\overline{\Omega}$ satisfy conditions* (1.5) *and $q = const \geq 1$. If $q \leq p(x)$ a.e. in Ω, then*

$$\|v\|_q \leq C_{q,\Omega}\|v\|_{p(\cdot)} \quad \textit{with the constant } C_{q,\Omega} = (1 + |\Omega|)^{\frac{1}{q}}. \tag{2.1}$$

3. Notations and preliminaries

In this article, on f, u_0 and $K(x,y)$ we make the following assumptions

$$f \in L^{p_2'}(0,T;L^{p_2'}(\Omega)),\ \frac{1}{p_2}+\frac{1}{p_2'}=1, \tag{3.1}$$

$$u_0 \in L^{\infty}(\Omega), \tag{3.2}$$

$$\text{for any } x\in\Gamma,\quad K(x)<\infty,\ \ K_i(x)<\infty, \tag{3.3}$$

$$\sum_{i=1}^{n}\int_{\Gamma}K(x)^{p_2-1}K_i(x)\,d\Gamma<\infty,\ \ \sum_{i=1}^{n}\int_{\Gamma}K(x)^{p_1-1}K_i(x)\,d\Gamma<\infty, \tag{3.4}$$

$$\gamma=\max\begin{pmatrix} C_{p_1,\Omega}^{p_2}\left(\sum_{i=1}^{n}\int_{\Gamma}K(x)^{p_2-1}K_i(x)\,d\Gamma\right), \\ \left(C_{p_1,\Omega}^{p_1}\sum_{i=1}^{n}\int_{\Gamma}K(x)^{p_1-1}K_i(x)\,d\Gamma\right)\end{pmatrix}\le\frac{p_1-1}{p_2}, \tag{3.5}$$

for any $x\in\Gamma$, where

$$K(x)=\left(\int_{\Omega}|K(x,y)|^{p_1'}\,dy\right)^{\frac{1}{p_1'}}:$$

$$\text{norm of } k(x,y) \text{ in } L^{p_1'}(\Omega) \text{ with respect to } y,\ \frac{1}{p_1}+\frac{1}{p_1'}=1$$

$$K_i(x)=\left(\sum_{i=1}^{n}\int_{\Omega}\left|\frac{\partial}{\partial x_i}K(x,y)\right|^{p_1'}dy\right)^{\frac{1}{p_1'}}:$$

$$\text{norm of } \frac{\partial}{\partial x_i}k(x,y) \text{ in } L^{p_1'}(\Omega) \text{ with respect to } y,\ \frac{1}{p_1}+\frac{1}{p_1'}=1$$

and $C_{p_1,\Omega}$ defined in (2.1).
Moreover, we assume that

$$r>\frac{n}{2}+2. \tag{3.6}$$

Let

$$\alpha_1=\left(\frac{p_2}{p_1}\left(\gamma+\frac{1}{p_2}\right)\right)^{\frac{p_2}{p_1-p_2}}. \tag{3.7}$$

We define the polynomial Q by

$$Q(\alpha)=\min\left(\alpha,\alpha^{\frac{p_1}{p_2}}\right)-\left(\gamma+\frac{1}{p_2}\right)\max\left(\alpha,\alpha^{\frac{p_1}{p_2}}\right)\quad\forall\alpha\in[0,+\infty].$$

Let

$$h(\alpha)=\alpha^{\frac{p_1}{p_2}}-\left(\gamma+\frac{1}{p_2}\right)\alpha.$$

Notice that $h(\alpha)=Q(\alpha)$, for $1\le\alpha\le\infty$. It is easy to check that the function $h(\alpha)$ is increasing for $1\le\alpha<\alpha_1$ and decreasing for $\alpha_1<\alpha\le+\infty$, where α_1 is its unique local maximum defined by (3.7). We will assume that:

$$1\le\|u_0\|_{p(\cdot)}^{p_2}=\alpha_0<\alpha_1 \tag{3.8}$$

and

$$\frac{1}{2}|u_0|^2 + C_{2,\Omega}^{\frac{p_2}{p_2-1}} \frac{p_2-1}{p_2} |\Omega|^{\frac{1}{2}} \int_0^T |f|_{p_2}^{\frac{p_2}{p_2-1}} dt < \int_0^T Q(\alpha_1)\, dt. \tag{3.9}$$

The classical formulation of the problem is as follows. Find a displacement field $u : \Omega \times (0,T) \to \mathbb{R}$, such that:

$$(u', v) - \left(\sum_{i=1}^{n} \frac{\partial}{\partial x_i} \left(|u|^{p(x)-2} \frac{\partial u}{\partial x_i} \right), v \right) + \left(|u|^{p(x)-2} u, v \right) = (f, v), \ \forall v \in V \tag{3.10}$$

$$u(x,0) = u_0(x), \quad x \in \Omega.$$

Where

$$V = \left\{ v \in H^r(\Omega) : v(x) = \int_\Omega K(x,y)\, v(y)\, dy \ \text{ for } x \in \Gamma \right\},$$

With assumption (1.4)-(3.6), using Sobelev embedding theorems, see [2], we have

$$H^r(\Omega) \hookrightarrow W^{2,p_2}(\Omega) \hookrightarrow W^{1,p_2}(\Omega) \hookrightarrow L^{p_2}(\Omega) \hookrightarrow L^2(\Omega)$$

It is easy to see that V is a subspace of $H^r(\Omega)$.

Whenever it doesn't cause a confusion, we use the following shorthand notations:

$L^q(\Omega)$: L^q space defined on Ω; $|.|_q = |.|_{q,\Omega}$: norm in $L^q(\Omega)$; $|.|_{q,\Gamma}$: norm in $L^q(\Gamma)$; $H^{-r}(\Omega)$: dual space of $H^r(\Omega)$; $|.|_{H^{-r}(\Omega)}$ norm in $H^{-r}(\Omega)$; C: nonnegative constant which may take different values on each occurrence.

4. Local existence

Theorem 4.1. *Under hypothesis* (1.4)-(3.9), *for any finite* $T > 0$, *the problem* (1.1)-(1.3) *admits a weak solution* u *such that*

$$u \in L^\infty\left(0,T; L^2(\Omega)\right) \cap C\left([0,T]; H^{-r}(\Omega)\right) \cap L^{p(\cdot)}(\Omega \times (0,T)), \tag{4.1}$$

$$\frac{\partial u}{\partial t} \in L^{p_2'}\left(0,T; H^{-r}(\Omega)\right), \tag{4.2}$$

$$|u|^{\frac{p(\cdot)-2}{2}} u \in \ L^2\left(0,T; H^1(\Omega)\right), \tag{4.3}$$

for all $v \in V$ *and a.e.* $t \in [0,T]$,

$$(u', v) - \left(\sum_{i=1}^{n} \frac{\partial}{\partial x_i} \left(|u|^{p(x)-2} \frac{\partial u}{\partial x_i} \right), v \right) + \left(|u|^{p(x)-2} u, v \right) = (f, v), \tag{4.4}$$

$$u(x,0) = u_0(x), \ x \in \Omega.$$

Proof. Since V is a subspace of $H^r(\Omega)$ which is separable. We can choose a countable set of distinct basis elements w_j $(j = 1, 2, ...)$ which generate V and are orthonormal in $L^2(\Omega)$. Let V_m be the subspace of V generated by the first m elements: $w_1, w_2, ..., w_m$. We search u of the form:

$$u_m(x,t) = \sum_{i=1}^{m} K_{im}(t) w_i(x), \tag{4.5}$$

satisfying:

$$\begin{cases} (u'_m, w_j) - \left(\sum_{i=1}^n \frac{\partial}{\partial x_i}\left(|u_m|^{p(x)-2}\frac{\partial u_m}{\partial x_i}\right), w_j\right) \\ + \left(|u_m|^{p(x)-2} u_m, w_j\right) = (f(t), w_j), \quad 1 \le j \le m, \\ u_m(0) = u_{0m}. \end{cases} \tag{4.6}$$

with

$$u_{0m} = \sum_{i=1}^m \alpha_{im} w_i \longrightarrow u_0 \text{ when } m \longrightarrow \infty \text{ in } L^{p(\cdot)}(\Omega). \tag{4.7}$$

Integrating by parts on the second term of left-hand side of (4.6), we have

$$\begin{cases} (u'_m, w_j) + \left(\sum_{i=1}^n \left(|u_m|^{p(x)-2}\frac{\partial u_m}{\partial x_i}\right), \frac{\partial}{\partial x_i} w_j\right) + \left(|u_m|^{p(x)-2} u_m, w_j\right) \\ = \int_\Gamma |u_m|^{p(x)-2}\frac{\partial u_m}{\partial x_i} w_j d\Gamma + (f(t), w_j), \quad 1 \le j \le m, \\ u_m(0) = u_{0m}. \end{cases} \tag{4.8}$$

By Peano's Theorem, for every finite m the problem (4.6), (4.8) has a solution on $(0, T_m)$ for each m. The following estimates permit us to confirm that T_m is independent of m.

a) A priori estimates

Multiplying the equation (4.8) by $K_{jm}(t)$, summing over $j = 1, ..., m$, we obtain

$$\frac{1}{2}\frac{d}{dt}|u_m(t)|^2 + \sum_{i=1}^n \int_\Omega \frac{4}{p^2(x)}\left(\frac{\partial}{\partial x_i}\left(|u_m|^{\frac{p(x)-2}{2}} u_m\right)\right)^2 dx + \int_\Omega |u_m|^{p(x)} dx \tag{4.9}$$

$$= \int_\Gamma |u_m|^{p(x)-2}\frac{\partial u_m}{\partial x_i} u_m(t) d\Gamma + (f(t), u_m)$$

Integrating on $(0, T)$ on both sides of (4.9), we get

$$\frac{1}{2}|u_m(T)|^2 + \int_0^T \sum_{i=1}^n \int_\Omega \frac{4}{p^2(x)}\left(\frac{\partial}{\partial x_i}\left(|u_m|^{\frac{p(x)-2}{2}} u_m\right)\right)^2 dxdt$$

$$+ \int_0^T \min\left(||u_m||_{p(\cdot)}^{p_2}, ||u_m||_{p(\cdot)}^{p_1}\right) dt$$

$$\le \int_0^T \int_\Gamma \left||u_m|^{p(x)-2}\frac{\partial u_m}{\partial x_i} u_m(t)\right| d\Gamma dt + \int_0^T |(f(t), u_m)| dt + \frac{1}{2}|u_{0m}|^2. \tag{4.10}$$

The second term in the right-hand side of (4.10) can be estimated as follows

$$|(f(t), u_m)| \le |f|_2 |u_m|_2 \le C_{2,\Omega} |f|_2 ||u_m||_{p(\cdot)} \text{ (holder's inequality) and (2.1)}$$

$$\le C_{2,\Omega}^{\frac{p_2}{p_2-1}} \frac{p_2 - 1}{p_2} |f|_2^{\frac{p_2}{p_2-1}} + \frac{1}{p_2} ||u_m||_{p(\cdot)}^{p_2} \text{ (Young's inequality)}$$

$$\le C_{2,\Omega}^{\frac{p_2}{p_2-1}} \frac{p_2 - 1}{p_2} |\Omega|^{\frac{1}{2}} |f|_{p_2}^{\frac{p_2}{p_2-1}} + \frac{1}{p_2} \max\left(\|u_m\|_{p(\cdot)}^{p_2}, \|u_m\|_{p(\cdot)}^{p_1}\right).$$

Next, we estimate first term in the right-hand side of (4.10) using (2.1): For $x \in \Gamma$, we have

$$|u_m(x, t)| \le K(x) |u_m|_{p_1} \le C_{p_1,\Omega} K(x) \|u_m\|_{p(\cdot)}. \tag{4.11}$$

Similarly, for $x \in \Gamma$ we have

$$\left|\sum_{i=1}^{n} \frac{\partial}{\partial x_i} u_m(x,t)\right| \leq K_i(x)\,|u_m|_{p_1} \leq C_{p_1,\Omega} K_i(x)\,\|u_m\|_{p(\cdot)} \tag{4.12}$$

Then using holder's inequality and assumptions (3.3) and (3.5), we have:

$$\begin{aligned}
\sum_{i=1}^{n}\int_{\Gamma}\left||u_m|^{p(x)-2}\frac{\partial u_m}{\partial x_i}u_m(t)\right|d\Gamma &\leq \sum_{i=1}^{n}\int_{\Gamma}|u_m|^{p(x)-1}\left|\frac{\partial u_m}{\partial x_i}\right|d\Gamma \\
&\leq \max\left(\sum_{i=1}^{n}\int_{\Gamma}|u_m|^{p_2-1}\left|\frac{\partial u_m}{\partial x_i}\right|d\Gamma, \sum_{i=1}^{n}\int_{\Gamma}|u_m|^{p_1-1}\left|\frac{\partial u_m}{\partial x_i}\right|d\Gamma\right) \\
&\leq \max\left(\begin{array}{c} C_{p_1,\Omega}^{p_2}\sum_{i=1}^{n}\int_{\Gamma}K(x)^{p_2-1}\|u_m\|_{p(\cdot)}^{p_2-1}K_i(x)\|u_m\|_{p(\cdot)}\,d\Gamma, \\ C_{p_1,\Omega}^{p_1}\sum_{i=1}^{n}\int_{\Gamma}K(x)^{p_1-1}\|u_m\|_{p(\cdot)}^{p_1-1}K_i(x)\|u_m\|_{p(\cdot)}\,d\Gamma \end{array}\right) \\
&= \max\left(\begin{array}{c} C_{p_1,\Omega}^{p_2}\left(\sum_{i=1}^{n}\int_{\Gamma}K(x)^{p_2-1}K_i(x)\,d\Gamma\right)\|u_m\|_{p(\cdot)}^{p_2}, \\ C_{p_1,\Omega}^{p_1}\left(\sum_{i=1}^{n}\int_{\Gamma}K(x)^{p_1-1}K_i(x)\,d\Gamma\right)\|u_m\|_{p(\cdot)}^{p_1} \end{array}\right) \\
&\leq \max\left(\begin{array}{c} C_{p_1,\Omega}^{p_2}\left(\sum_{i=1}^{n}\int_{\Gamma}K(x)^{p_2-1}K_i(x)\,d\Gamma\right), \\ \left(C_{p_1,\Omega}^{p_1}\sum_{i=1}^{n}\int_{\Gamma}K(x)^{p_1-1}K_i(x)\,d\Gamma\right) \end{array}\right) \\
&\quad\times\max\left(\|u_m\|_{p(\cdot)}^{p_2}, \|u_m\|_{p(\cdot)}^{p_1}\right)
\end{aligned}$$

This implies that

$$\begin{aligned}
\frac{1}{2}|u_m(t)|^2 + \int_0^T\sum_{i=1}^{n}\int_{\Omega}\frac{4}{p^2(x)}\left(\frac{\partial}{\partial x_i}\left(|u_m|^{\frac{p(x)-2}{2}}u_m\right)\right)^2 dxdt + \int_0^T Q\left(||u_m||_{p(\cdot)}^{p_2}\right)dt \\
\leq \frac{1}{2}|u_{0m}|^2 + C_{2,\Omega}^{\frac{p_2}{p_2-1}}\frac{p_2-1}{p_2}|\Omega|^{\frac{1}{2}}\int_0^T|f|_{p_2}^{\frac{p_2}{p_2-1}}dt,
\end{aligned} \tag{4.13}$$

at this step we will assume that $Q\left(||u_m||_{p(\cdot)}^{p_2}\right) \geq 0$, so from (3.9) and (4.13), we have the following a priori estimates:

$$|u_m| \leq C \text{ (C is independent of m);} \tag{4.14}$$

$$\int_0^T\sum_{i=1}^{n}\int_{\Omega}\frac{4}{p^2(x)}\left(\frac{\partial}{\partial x_i}\left(|u_m|^{\frac{p(x)-2}{2}}u_m\right)\right)^2 dxdt \leq C \text{ (C independent of m).} \tag{4.15}$$

So the solution $u_m(t)$ of problem (1.1)-(1.3) exists on $[0,T]$ for each m, and

$$u_m \quad \text{is bounded in } L^\infty\left(0,T;L^2(\Omega)\right); \tag{4.16}$$

$$|u_m|^{\frac{p(\cdot)-2}{2}}u_m \;\; \text{is bounded in } L^2\left(0,T;H^1(\Omega)\right)$$

Claim 4.2. There exists an integer N such that

$$||u_m||_{p(\cdot)}^{p_2} < \alpha_1 \quad \forall t \in [0,T_m) \qquad m > N. \tag{4.17}$$

Proof of the Claim. Suppose (4.17) false. Then for each $m > N$, there exists $t \in [0, T_m)$ such that $||u_m(t)||_{p(\cdot)}^{p_2} \geq \alpha_1$. We note that from (3.8) and (4.7) there exists N_0 such that

$$1 \leq ||u_m(0)||_{p(\cdot)}^{p_2} < \alpha_1 \quad \forall m > N_0$$

Then by continuity there exists a first $T_m^* \in (0, T_m)$ such that

$$||u_m(T_m^*)||_{p(\cdot)}^{p_2} = \alpha_1, \tag{4.18}$$

from where

$$Q\left(||u_m||_{p(\cdot)}^{p_2}\right) = h\left(||u_m(t)||_{p(\cdot)}^{p_2}\right) \geq 0 \quad \forall t \in [0, T_m^*].$$

Now from (3.9) and (4.13), there exist $N > N_0$ and $\beta \in (1; \alpha_1)$ such that

$$0 \leq \frac{1}{2}|u_m(t)|^2 + \int_0^t \sum_{i=1}^n \int_\Omega \frac{4}{p^2(x)} \left(\frac{\partial}{\partial x_i}\left(|u_m|^{\frac{p(x)-2}{2}} u_m\right)\right)^2 dxds$$
$$+ \int_0^t Q\left(||u_m||_{p(\cdot)}^{p_2}\right) ds \leq \int_0^t Q(\beta)\, ds \quad \forall t \in [0, T_m^*], \quad \forall m > N$$

Then the monotonicity of Q implies that

$$||u_m(t)||_{p(\cdot)}^{p_2} \leq \beta < \alpha_1 \quad \forall t \in [0, T_m^*]$$

and in particular, $||u_m(T_m^*)||_{p(\cdot)}^{p_2} < \alpha_1$, which is a contradiction to (4.18). And then the supposition $Q\left(||u_m||_{p(\cdot)}^{p_2}\right) \geq 0$ is true. □

From (4.17) the solution $u_m(t)$ of problem (1.1)-(1.3) satisfies other of (4.16),

$$u_m \text{ is bounded in } L^{p(\cdot)}(\Omega \times (0, T)). \tag{4.19}$$

Lemma 4.3. *Let u_m, constructed in* (4.5)*, be the approximate solution of* (1.1)-(1.3)*. Then*

$$\frac{\partial}{\partial t} u_m(t) \text{ is bounded in } L^{p_2'}(0, T; H^{-r}(\Omega)). \tag{4.20}$$

Proof. Let $v \in H^r(\Omega)$, from (4.6) we have

$$\left(\frac{\partial u_m(t)}{\partial t}, v\right) + \left(\sum_{i=1}^n \left(|u_m|^{p(x)-2} \frac{\partial u_m}{\partial x_i}\right), \frac{\partial}{\partial x_i} v\right) + \left(|u_m|^{p(x)-2} u_m, v\right) \tag{4.21}$$
$$= \sum_{i=1}^n \int_\Gamma |u_m|^{p(x)-2} \frac{\partial u_m}{\partial x_i} v d\Gamma + (f(t), v),$$

The last term in the left-hand side can be estimated as follows:

$$\left|\left(|u_m|^{p(x)-2} u_m, v\right)\right| \leq \left||u_m|^{p(x)-1}\right|_{p_2'} |v|_{p_2} \leq C \left||u_m|^{p(x)-1}\right|_{p'(\cdot)} |v|_{p_2} \quad (p_2' \leq p'(\cdot) \leq p_1')$$
$$\leq C \max\left(\left(\int_\Omega |u_m|^{p(x)} dx\right)^{\frac{1}{p_1'}}, \left(\int_\Omega |u_m|^{p(x)} dx\right)^{\frac{1}{p_2'}}\right) |v|_{p_2}$$
$$\leq C \max\left(\left(\int_\Omega |u_m|^{p(x)} dx\right)^{\frac{1}{p_1'}}, \left(\int_\Omega |u_m|^{p(x)} dx\right)^{\frac{1}{p_2'}}\right) |v|_{H^r}$$

Hence,

$$\left\| |u_m|^{p(\cdot)-2}\, u_m \right\|_{H^{-r}(\Omega)} \le C \max\left(\left(\int_\Omega |u_m|^{p(x)}\, dx \right)^{\frac{1}{p_1'}}, \left(\int_\Omega |u_m|^{p(x)}\, dx \right)^{\frac{1}{p_2'}} \right) < \infty.$$

The norm of $|u_m|^{p(\cdot)-2}\, u_m$ in $L^{p_2'}(0,T;H^{-r}(\Omega))$ is bounded by

$$C \left(\int_0^T \max\left(\left(\int_\Omega |u_m|^{p(x)}\, dx \right)^{\frac{p_2'}{p_1'}}, \int_\Omega |u_m|^{p(x)}\, dx \right) \right)^{\frac{1}{p_2'}} < \infty$$

Therefore, $|u_m|^{p(\cdot)-2}\, u_m$ is bounded in $L^{p_2'}(0,T;H^{-r}(\Omega))$.

Next, we consider the term $\sum_{i=1}^n \int_\Gamma |u_m|^{p(x)-2} \frac{\partial u_m}{\partial x_i} v d\Gamma$ in the left-hand side of (4.21):

$$\begin{aligned}
&\left| \sum_{i=1}^n \int_\Gamma |u_m|^{p(x)-2} \frac{\partial u_m}{\partial x_i} v d\Gamma \right| \le \left(\sum_{i=1}^n \left| |u_m|^{p(x)-2} \frac{\partial u_m}{\partial x_i} \right|_{p_1',\Gamma} \right) |v(t)|_{p_1,\Gamma} \\
&= \sum_{i=1}^n \left| \left| \int_\Omega K(x,y)\, u_m(y)\, dy \right|^{p(x)-2} \int_\Omega \frac{\partial}{\partial x_i} K_i(x,y)\, u_m(y)\, dy \right|_{p_1',\Gamma} \\
&\qquad \times \left| \int_\Omega K(x,y)\, v(y) dy \right|_{p_1,\Gamma} \\
&\le C \sum_{i=1}^n \left| K(x)^{p(x)-2} K_i(x) |u_m(y)|_{p_1}^{p(x)-1} \right|_{p_1',\Gamma} \left| K(x) |v(y)|_{p_1} \right|_{p_1,\Gamma} \\
&\le C \sum_{i=1}^n \left| K(x)^{p(x)-2} K_i(x) \right|_{p_1',\Gamma} |K(x)|_{p_1,\Gamma} |u_m(y)|_{p_1}^{p(x)-1} |v(y)|_{p_1} \\
&\le C \max\left(\sum_{i=1}^n \left| K(x)^{p_1-2} K_i(x) \right|_{p_1',\Gamma}, \sum_{i=1}^n \left| K(x)^{p_2-2} K_i(x) \right|_{p_1',\Gamma} \right) |K(x)|_{p_1,\Gamma} \\
&\qquad \times \max\left(|u_m(y)|_{p_1}^{p_1-1}, |u_m(y)|_{p_1}^{p_2-1} \right) |v(y)|_{H^r(\Omega)} \\
&\le C \max\left(\sum_{i=1}^n \left| K(x)^{p_1-2} K_i(x) \right|_{p_1',\Gamma}, \sum_{i=1}^n \left| K(x)^{p_2-2} K_i(x) \right|_{p_1',\Gamma} \right) |K(x)|_{p_1,\Gamma} \\
&\qquad \times \max\left(|u_m(y)|_{p(\cdot)}^{p_1-1}, |u_m(y)|_{p(\cdot)}^{p_2-1} \right) |v(y)|_{H^r}
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\left| \sum_{i=1}^n \int_\Gamma |u_m|^{p(x)-2} \frac{\partial u_m}{\partial x_i} d\Gamma \right|_{H^{-r}(\Omega)} \\
&\le C \max\left(\sum_{i=1}^n \left| K(x)^{p_1-2} K_i(x) \right|_{p_1',\Gamma}, \sum_{i=1}^n \left| K(x)^{p_2-2} K_i(x) \right|_{p_1',\Gamma} \right) \\
&\qquad \times \max\left(|u_m(y)|_{p(\cdot)}^{p_1-1}, |u_m(y)|_{p(\cdot)}^{p_2-1} \right) |K(x)|_{p_1,\Gamma} < \infty.
\end{aligned}$$

Then the norm of $\sum_{i=1}^{n}\int_{\Gamma}|u_m|^{p(x)-2}\frac{\partial u_m}{\partial x_i}d\Gamma$ in $L^{p_2'}(0,T;H^{-r}(\Omega))$ is bounded by

$$C\left(\begin{array}{c}\int_0^T\max\left(\sum_{i=1}^{n}\left|K(x)^{p_1-2}K_i(x)\right|_{p_1',\Gamma}^{p_2'},\sum_{i=1}^{n}\left|K(x)^{p_2-2}K_i(x)\right|_{p_1',\Gamma}^{p_2'}\right)\\ \times\max\left(|u_m(y)|_{p(\cdot)}^{(p_1-1)p_2'},|u_m(y)|_{p(\cdot)}^{(p_2-1)p_2'}\right)|K(x)|_{p_1,\Gamma}^{p_2'}dt\end{array}\right)^{\frac{1}{p_2'}}<\infty$$

Hence $\sum_{i=1}^{n}\int_{\Gamma}|u_m|^{p(x)-2}\frac{\partial u_m}{\partial x_i}d\Gamma$ is bounded in $L^{p_2'}(0,T;H^{-r}(\Omega))$.

Next, we consider the second term in the left-hand side of (4.21). Integrating by parts gives

$$\left(\sum_{i=1}^{n}\left(|u_m|^{p(x)-2}\frac{\partial u_m}{\partial x_i}\right),\frac{\partial v}{\partial x_i}\right)=\int_{\Omega}\sum_{i=1}^{n}\frac{1}{p(x)-1}\left(\frac{\partial}{\partial x_i}\left(|u_m|^{p(x)-2}u_m\right)\right)\frac{\partial v}{\partial x_i}dx \tag{4.22}$$

$$=\int_{\Gamma}\sum_{i=1}^{n}\frac{1}{p(x)-1}|u_m|^{p(x)-2}u_m\frac{\partial v}{\partial x_i}d\Gamma-\int_{\Omega}\frac{1}{p(x)-1}|u_m|^{p(x)-2}u_m\Delta v dx.$$

First, we have

$$\begin{aligned}\left|\int_{\Gamma}\sum_{i=1}^{n}\frac{1}{p(x)-1}|u_m|^{p(x)-2}u_m\frac{\partial v}{\partial x_i}d\Gamma\right|&\le\frac{1}{p_2-1}\sum_{i=1}^{n}\left||u_m|^{p(x)-2}u_m\right|_{p_1',\Gamma}\left|\frac{\partial v}{\partial x_i}\right|_{p_1,\Gamma}\\ =\frac{1}{p_2-1}\sum_{i=1}^{n}\left|\left(\int_{\Omega}K(x,y)u_m(y)dy\right)^{p(x)-1}\right|_{p_1',\Gamma}&\left|\int_{\Omega}\frac{\partial}{\partial x_i}K(x,y)v(y)dy\right|_{p_1,\Gamma}\\ &\le C\sum_{i=1}^{n}\left|K(x)^{p(x)-1}|u_m|_{p_1}^{p(x)-1}\right|_{p_1',\Gamma}\left|K_i(x)|v|_{p_1}\right|_{p_1,\Gamma}\\ &\le C\sum_{i=1}^{n}\left|K(x)^{p(x)-1}\right|_{p_1',\Gamma}|K_i(x)|_{p_1,\Gamma}|u_m|_{p_1}^{p(x)-1}|v|_{p_1}\\ &\le C\max\left(\sum_{i=1}^{n}\left|K(x)^{p_1-1}\right|_{p_1',\Gamma},\sum_{i=1}^{n}\left|K(x)^{p_2-1}\right|_{p_1',\Gamma}\right)\\ &\quad\times\max\left(|u_m|_{p_1}^{p_1-1},|u_m|_{p_1}^{p_2-1}\right)|K_i(x)|_{p_1,\Gamma}|v(y)|_{H^r}\end{aligned}$$

So we have

$$\begin{aligned}&\left|\int_{\Gamma}\sum_{i=1}^{n}\frac{1}{p(x)-1}|u_m|^{p(x)-2}u_m d\Gamma\right|_{H^{-r}(\Omega)}\\ &\le C\max\left(\sum_{i=1}^{n}\left|K(x)^{p_1-1}\right|_{p_1',\Gamma},\sum_{i=1}^{n}\left|K(x)^{p_2-1}\right|_{p_1',\Gamma}\right)\\ &\quad\times\max\left(|u_m|_{p_1}^{p_1-1},|u_m|_{p_1}^{p_2-1}\right)|K_i(x)|_{p_1,\Gamma}<\infty\end{aligned}$$

consequently, $\int_{\Gamma}\sum_{i=1}^{n}\frac{1}{p(x)-1}|u_m|^{p(x)-2}u_m d\Gamma$ is bounded in $L^{p_2'}(0,T;H^{-r}(\Omega))$.

Next, consider $\int_\Omega \frac{1}{p(x)-1}|u_m|^{p(x)-2} u_m \Delta v dx$, by the same manner, we have

$$\begin{aligned}\left|\int_\Omega \frac{1}{p(x)-1}|u_m|^{p(x)-2} u_m \Delta v dx\right| &\leq \frac{1}{p_1-1}\left||u_m|^{p(x)-1}\right|_{p_2'} |\Delta v|_{p_2} \\ &\leq C\left||u_m|^{p(x)-1}\right|_{p'(\cdot)} |\Delta v|_{p_2} \\ &\leq C\max\left(\left(\int_\Omega |u_m|^{p(x)} dx\right)^{\frac{1}{p_1'}}, \left(\int_\Omega |u_m|^{p(x)} dx\right)^{\frac{1}{p_2'}}\right)|v|_{H^r}\end{aligned}$$

therefore,

$$\int_\Omega \frac{1}{p(x)-1}|u_m|^{p(x)-2} u_m \Delta v dx$$

is bounded in $L^{p_2'}(0,T;H^{-r}(\Omega))$. Since $f \in L^{p_2'}(0,T;L^{p_2'}(\Omega)) \subset L^{p_2'}(0,T;H^{-r}(\Omega))$, from this discussion and (4.21) it yields that $\frac{\partial}{\partial t}u_m$ is bounded in $L^{p_2'}(0,T;H^{-r}(\Omega))$. □

Theorem 4.4. *Let B, B_1 be Banach spaces, and S be a set. Define*

$$M(v) = \max\left(\left(\sum_{i=1}^n \int_\Omega |v|^{p_1-2}\left(\frac{\partial v}{\partial x_i}\right)^2 dx\right)^{\frac{1}{p_1}}, \left(\sum_{i=1}^n \int_\Omega |v|^{p_2-2}\left(\frac{\partial v}{\partial x_i}\right)^2 dx\right)^{\frac{1}{p_2}}\right)$$

on S with:

a) $S \subset B \subset B_1$, and $M(v) \geq 0$ on S,

$$\begin{aligned}M(\lambda v) &= \max\left(\left(\sum_{i=1}^n \int_\Omega |v|^{p_1-2}\left(\frac{\partial v}{\partial x_i}\right)^2 dx\right)^{\frac{1}{p_1}}, \left(\sum_{i=1}^n \int_\Omega |v|^{p_2-2}\left(\frac{\partial v}{\partial x_i}\right)^2 dx\right)^{\frac{1}{p_2}}\right) \\ &= |\lambda| M(v)\end{aligned}$$

b) the set $\{v \mid v \in S,\ M(v) \leq 1\}$ is relatively compact in B.
Define the set

$$F = \left\{\begin{array}{c} v : v \ \text{ is locally summable on } [0,T] \text{ with value in } B_1; \\ \int_0^T (M(v(t)))^{q_0} dt \leq C,\ v' \text{ bounded in } L^{q_1}(0,T;B_1), \end{array}\right\}$$

where $1 < q_i < \infty$, $i = 0,1$. Then $F \subset L^{q_0}(0,T;B)$, and F is relatively compact in $L^{q_0}(0,T;B)$.

We need Theorem (4.4) to prove the following lemma (4.5).

Lemma 4.5. *Let u_m, constructed as in* (4.5)*, be the approximate solution of* (1.1)-(1.3)*, then $u_m \to u$ in $L^{p_2}(0,T;L^{p_2}(\Omega))$ strongly and almost everywhere.*

Proof. Let

$$S = \left\{v : \max\left(|v|^{\frac{p_1-2}{2}} v, |v|^{\frac{p_2-2}{2}} v\right) \in H^1(\Omega)\right\}$$

Since $H^1(\Omega)$ is compactly embedded in $L^2(\Omega)$, the proof of [16, Proposition 12.1,p. 143] also works for both $|v|^{\frac{p_1-2}{2}} v$ and $|v|^{\frac{p_2-2}{2}} v$, then (b) holds.

Let $B = L^{p_2}(\Omega)$, $B_1 = H^{-r}(\Omega)$, $q_0 = p_2$, $q_1 = p_2'$, we have

$$\int_0^T (M(v(t)))^{q_0}\,dt \le C\int_0^T \max\begin{pmatrix} \left(\sum_{i=1}^n \int_\Omega |v|^{p_1-2}\left(\frac{\partial v}{\partial x_i}\right)^2 dx\right)^{\frac{p_2}{p_1}}, \\ \left(\sum_{i=1}^n \int_\Omega |v|^{p_2-2}\left(\frac{\partial v}{\partial x_i}\right)^2 dx\right) \end{pmatrix} dt$$

$$\le C\int_0^T \max\begin{pmatrix} \left(\sum_{i=1}^n \int_\Omega \left(\frac{\partial}{\partial x_i}\left(|v|^{\frac{p_1-2}{2}} v\right)\right)^2 dx\right)^{\frac{p_2}{p_1}}, \\ \left(\sum_{i=1}^n \int_\Omega \left(\frac{\partial}{\partial x_i}\left(|v|^{\frac{p_2-2}{2}} v\right)\right)^2 dx\right) \end{pmatrix} dt < \infty$$

□

Now with Lemma (4.3) and a priori estimates, conclusion follows easily from application of Theorem (4.4).

Next, we prove that we can pass the limit in (4.21). Lemmas (4.6)-(4.10), below, show that we can pass the limit in each term in the left-hand side of (4.21)

Lemma 4.6. *Let u_m, constructed as in* (4.5), *be the approximate solution of* (1.1)-(1.3), *then* $\left(|u_m|^{p(x)-2} u_m, v\right) \to \left(|u|^{p(x)-2} u, v\right)$ *as* $m \to \infty$.

Proof. Since u_m is bounded in $L^{p(\cdot)}(\Omega \times (0,T))$ then $|u_m|^{p(\cdot)-2} u_m$ is bounded in $L^{\frac{p(\cdot)}{p(\cdot)-1}}(\Omega \times (0,T))$; hence, using same arguments as in [16, Lemma 1.3], we have

$$|u_m|^{p(\cdot)-2} u_m \to |u|^{p(\cdot)-2} u \text{ weakly in } L^{\frac{p(\cdot)}{p(\cdot)-1}}(\Omega \times (0,T)).$$ □

Lemma 4.7. *Let u_m, constructed as in* (4.5), *be the approximate solution of* (1.1)-(1.3), *then*

$$\int_\Gamma \sum_{i=1}^n \frac{1}{p(x)-1}\left(|u_m|^{p(x)-2}\frac{\partial}{\partial x_i}u_m\right) v\,d\Gamma \to \int_\Gamma \sum_{i=1}^n \frac{1}{p(x)-1}\left(|u|^{p(x)-2}\frac{\partial}{\partial x_i}u\right) v\,d\Gamma$$

as $m \to \infty$.

Proof. By a priori estimates, u_m is bounded in $L^{p(\cdot)}(\Omega)$ for almost every t, then there exists subsequence of u_m, still denoted as u_m, converges to u_m weak star in $L^{p(\cdot)}(\Omega)$ (Alaoglu's Theorem) for almost every $t \in [0,T]$. Under the assumption that for fixed $x \in \Gamma$, we have

$$\int_\Omega K(x,y)\,u_m(y)\,dy \to \int_\Omega K(x,y)\,u(y)\,dy \quad \text{as } m \to \infty$$

Similarly

$$\int_\Omega \frac{\partial}{\partial x_i}K(x,y)\,u_m(y)\,dy \to \int_\Omega \frac{\partial}{\partial x_i}K(x,y)\,u(y)\,dy \quad \text{as } m \to \infty$$

Therefore, for $x \in \Gamma$, we have

$$|u_m|^{p(\cdot)-2}\frac{\partial}{\partial x_i}u_m \to |u|^{p(\cdot)-2}\frac{\partial}{\partial x_i}u \quad \text{a.e.}$$

Since

$$\max\left(\int_\Gamma K^{p_1}(x)\,d\Gamma, \int_\Gamma K^{p_2}(x)\,d\Gamma\right) < \infty,$$
$$\text{and } \max\left(\int_\Gamma K_i^{p_1}(x)\,d\Gamma, \int_\Gamma K_i^{p_2}(x)\,d\Gamma\right) < \infty,$$

we have

$$|u_m|_{p(\cdot),\Gamma} \le C\max\left(\int_\Gamma K^{p_1}(x)\,d\Gamma, \int_\Gamma K^{p_2}(x)\,d\Gamma\right)\max\left(\|u_m\|_{p(\cdot)}^{p_1}, \|u_m\|_{p(\cdot)}^{p_2}\right) < \infty$$

and

$$\left|\frac{\partial}{\partial x_i}u_m\right|_{p(\cdot),\Gamma} \le C\max\left(\int_\Gamma K_i^{p_1}(x)\,d\Gamma, \int_\Gamma K_i^{p_2}(x)\,d\Gamma\right)\max\left(\|u_m\|_{p(\cdot)}^{p_1}, \|u_m\|_{p(\cdot)}^{p_2}\right) < \infty.$$

Then

$$\begin{aligned}
\left||u_m|^{p(\cdot)-2}\frac{\partial}{\partial x_i}u_m\right|_{p_2',\Gamma} &\le C\left||u_m|^{p(\cdot)-2}\frac{\partial}{\partial x_i}u_m\right|_{p'(\cdot),\Gamma} \quad \text{since } (p_2' \le p'(\cdot) \le p_1') \\
&\le \left||u_m|^{p(\cdot)-2}\right|_{\frac{p(\cdot)}{p(\cdot)-2},\Gamma}\left|\frac{\partial}{\partial x_i}u_m\right|_{p(\cdot),\Gamma} \quad \text{since } \left(\frac{1}{p'(\cdot)} = \frac{p(\cdot)-2}{p(\cdot)} + \frac{1}{p(\cdot)}\right) \\
&\le \max\left(\left(\int_\Gamma |u_m|^{p(x)}\,d\Gamma\right)^{\frac{1}{p_1}}, \left(\int_\Omega |u_m|^{p(x)}\,d\Gamma\right)^{\frac{1}{p_2}}\right) \\
&\times\max\left(\left(\int_\Omega \left|\frac{\partial}{\partial x_i}u_m\right|^{p(x)}d\Gamma\right)^{\frac{1}{p_1}}, \left(\int_\Gamma \left|\frac{\partial}{\partial x_i}u_m\right|^{p(x)}d\Gamma\right)^{\frac{1}{p_2}}\right) < \infty.
\end{aligned}$$

So, applying the same arguments as in [16, Lemma 1.3] to conclude that

$$|u_m|^{p(\cdot)-2}\frac{\partial}{\partial x_i}u_m \to |u|^{p(\cdot)-2}\frac{\partial}{\partial x_i}u \;\text{ weakly in } L^{p_2'}(\Gamma).$$

for a.e. $t \in [0,T]$. Since,

$$\max\left(\left(\int_\Omega \left|\frac{\partial}{\partial x_i}v\right|^{p(x)}d\Gamma\right)^{\frac{1}{p_1}}, \left(\int_\Omega \left|\frac{\partial}{\partial x_i}v\right|^{p(x)}d\Gamma\right)^{\frac{1}{p_2}}\right) < \infty,$$

the proof is complete. □

Lemma 4.8. *Let u_m, constructed as in* (4.5), *be the approximate solution of* (1.1)-(1.3), *then*

$$\int_\Gamma \sum_{i=1}^n \frac{1}{p(x)-1}|u_m|^{p(x)-2}u_m\frac{\partial v}{\partial x_i}d\Gamma \to \int_\Gamma \sum_{i=1}^n \frac{1}{p(x)-1}|u|^{p(x)-2}u\frac{\partial v}{\partial x_i}d\Gamma$$

as $m \to \infty$.

Proof. From the proof of Lemma (4.7), we have, for $x \in \Gamma$, $|u_m|^{p(\cdot)-2} u_m \to |u|^{p(\cdot)-2} u$ almost everywhere, and

$$\left| |u_m|^{p(\cdot)-2} u_m \right|_{p_2',\Gamma} \leq C \left| |u_m|^{p(\cdot)-2} u_m \right|_{p'(\cdot),\Gamma}$$
$$\leq \max \left(\left(\int_\Gamma |u_m|^{p(x)} d\Gamma \right)^{\frac{1}{p_1}}, \left(\int_\Gamma |u_m|^{p(x)} d\Gamma \right)^{\frac{1}{p_2}} \right) < \infty.$$

Therefore, by applying [16, Lemma 1.3] we conclude that

$$|u_m|^{p(\cdot)-2} u_m \to |u|^{p(\cdot)-2} u \text{ weakly in } L^{p_2'}(\Gamma).$$

Since $\frac{\partial v}{\partial x_i} \in L^{p_2'}(\Gamma)$, the proof is complete.

Lemma 4.9. *Let u_m, constructed as in* (4.5)*, be the approximate solution of* (1.1)-(1.3)*, then*

$$\int_\Omega \frac{1}{p(x)-1} \left(|u_m|^{p(x)-2} u_m \right) \Delta v dx \to \int_\Omega \frac{1}{p(x)-1} \left(|u|^{p(x)-2} u \right) \Delta v dx$$

as $m \to \infty$.

Proof. From lemma ((4.5)), we have $|u_m|^{p(\cdot)-2} u_m \to |u|^{p(\cdot)-2} u$ almost everywhere, for $x \in \Omega$, since

$$\left| |u_m|^{p(\cdot)-2} u_m \right|_{p_2',\Omega} \leq C \left| |u_m|^{p(\cdot)-2} u_m \right|_{p'(\cdot),\Omega}$$
$$\leq \max \left(\left(\int_\Omega |u_m|^{p(x)} dx \right)^{\frac{1}{p_1}}, \left(\int_\Omega |u_m|^{p(x)} dx \right)^{\frac{1}{p_2}} \right) < \infty.$$

by [16, Lemma 1.3], we have $|u_m|^{p(\cdot)-2} u_m \to |u|^{p(\cdot)-2} u$ weakly in $L^{p_2'}(\Omega)$. Since $\Delta v \in L^{p_2}(\Omega)$, the proof is complete. □

Lemma 4.10. *Let u_m, constructed as in* (4.5)*, be the approximate solution of* (1.1)-(1.3)*, then*

$$\left(\sum_{i=1}^n \left(|u_m|^{p(x)-2} \frac{\partial u_m}{\partial x_i} \right), \frac{\partial}{\partial x_i} v \right) \to \left(\sum_{i=1}^n \left(|u|^{p(x)-2} \frac{\partial u}{\partial x_i} \right), \frac{\partial}{\partial x_i} v \right)$$

as $m \to \infty$.

Proof. Replacing the results of (4.8) and (4.9) in (4.22), the proof is complete. □

Lemma 4.11. *Let u_m, constructed as in* (4.5)*. be the approximate solution of* (1.1)-(1.3)*, then $\left(\frac{\partial}{\partial t} u_m, v\right) \to \left(\frac{\partial}{\partial t} u, v\right)$ and $u(t)$ is continuous on* $[0, T]$.

Proof. Since $\frac{\partial}{\partial t} u_m(t)$ is bounded in $L^{p_2'}(0, T; H^{-r}(\Omega))$, by Alaoglu's theorem, there exists a subsequence, still denoted by $\frac{\partial}{\partial t} u_m(t)$, converging to χ weak star in $L^{p_2'}(0, T; H^{-r}(\Omega))$. By slightly modifying the proof of [6, Theorem 1] (with the space $L^{p_2'}(0, T; H^{-r}(\Omega))$ instead of $L^2(0, T; B_2^1(0, 1))$.), we have $\chi = u'$ and u is continuous on $[0, T]$. This ends the proof of Lemma (4.11). □

Combining all above results, the existence theorem (4.1) follows. □

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Properties of absolute-$*$-k-paranormal operators and contractions for $*$-$\mathcal{A}(k)$ operators

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Abstract. First, we see if T is absolute-$*$-k-paranormal for $k \geq 1$, then T is a normaloid operator. We also see some properties of absolute-$*$-k-paranormal operator and $*$-$\mathcal{A}(k)$ operator. Then, we will prove the spectrum continuity of the class $*$-$\mathcal{A}(k)$ operator for $k > 0$. Moreover, it is proved that if T is a contraction of the class $*$-$\mathcal{A}(k)$ for $k > 0$, then either T has a nontrivial invariant subspace or T is a proper contraction, and the nonnegative operator

$$D = \left(T^*|T|^{2k}T\right)^{\frac{1}{k+1}} - |T^*|^2$$

is a strongly stable contraction. Finally if $T \in *$-$\mathcal{A}(k)$ is a contraction for $k > 0$, then T is the direct sum of a unitary and $C_{\cdot 0}$ (c.n.u) contraction.

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Keywords: Class $*$-$\mathcal{A}(k)$ operators, absolute-$*$-k-paranormal operators, normaloid operators, continuity spectrum, contractions.

1. Introduction

Throughout this paper, let H and K be infinite dimensional separable complex Hilbert spaces with inner product $\langle \cdot, \cdot \rangle$. We denote by $L(H, K)$ the set of all bounded operators from H into K. To simplify, we put $L(H) := L(H, H)$. For $T \in L(H)$, we denote by $\ker(T)$ the null space and by $T(H)$ the range of T. The null operator and the identity on H will be denoted by O and I, respectively. If T is an operator, then T^* denotes its adjoint. We shall denote the set of all complex numbers by $\mathbb{C}$, the set of all non-negative integers by $\mathbb{N}$ and the complex conjugate of a complex number λ by $\overline{\lambda}$. The closure of a set M will be denoted by $\overline{M}$ and we shall henceforth shorten $T - \lambda I$ to $T - \lambda$. An operator $T \in L(H)$, is a positive operator, $T \geq O$, if $\langle Tx, x \rangle \geq 0$ for all $x \in H$. We write by $\sigma(T)$, $\sigma_p(T)$, and $\sigma_a(T)$ spectrum, point spectrum and approximate point spectrum respectively. Sets of isolated points and accumulation points of $\sigma(T)$ are denoted by $\text{iso}\sigma(T)$ and $\text{acc}\sigma(T)$, respectively. We write $r(T)$ for the spectral radius. It is well known that $r(T) \leq \|T\|$. The operator T is called normaloid if $r(T) = \|T\|$.

A contraction is an operator T such that $\|Tx\| \leq \|x\|$ for all $x \in H$. A proper contraction is an operator T such that $\|Tx\| < \|x\|$ for every nonzero $x \in H$. A strict contraction is an operator such that $\|T\| < 1$ (*i.e.*, $\sup_{x \neq 0} \frac{\|Tx\|}{\|x\|} < 1$). Obviously, every strict contraction is a proper contraction and every proper contraction is a contraction. An operator T is said to be completely non-unitary (c.n.u) if T restricted to every reducing subspace of H is non-unitary.

An operator T on H is uniformly stable, if the power sequence $\{T^m\}_{m=1}^{\infty}$ converges uniformly to the null operator (*i.e.*, $\|T^m\| \to O$). An operator T on H is strongly stable, if the power sequence $\{T^m\}_{m=1}^{\infty}$ converges strongly to the null operator (*i.e.*, $\|T^m x\| \to 0$, for every $x \in H$).

A contraction T is of class $C_{0\cdot}$ if T is strongly stable (*i.e.*, $\|T^m x\| \to 0$ and $\|Tx\| \leq \|x\|$ for every $x \in H$). If T^* is a strongly stable contraction, then T is of class $C_{\cdot 0}$. T is said to be of class $C_{1\cdot}$ if $\lim_{m\to\infty} \|T^m x\| > 0$ (equivalently, if $T^m x \not\to 0$ for every nonzero x in H). T is said to be of class $C_{\cdot 1}$ if $\lim_{m\to\infty} \|T^{*m} x\| > 0$ (equivalently, if $T^{*m} x \not\to 0$ for every nonzero x in H). We define the class $C_{\alpha\beta}$ for $\alpha, \beta = 0, 1$ by $C_{\alpha\beta} = C_{\alpha\cdot} \cap C_{\cdot\beta}$. These are the Nagy-Foiaş classes of contractions [21, p.72]. All combinations are possible leading to classes C_{00}, C_{01}, C_{10} and C_{11}. In particular, T and T^* are both strongly stable contractions if and only if T is a C_{00} contraction. Uniformly stable contractions are of class C_{00}.

For an operator $T \in L(H)$, as usual, $|T| = (T^*T)^{\frac{1}{2}}$. An operator T is said to be a normal operator if $T^*T = TT^*$ and T is said to be hyponormal, if $|T|^2 \geq |T^*|^2$. An operator $T \in L(H)$, is said to be paranormal [11], if $\|T^2 x\| \geq \|Tx\|^2$ for every unit vector x in H. Further, T is said to be $*$-paranormal [1], if $\|T^2 x\| \geq \|T^* x\|^2$ for every unit vector x in H.

In [13] authors Furuta, Ito and Yamazaki introduced the class $\mathcal{A}$ operator, respectively the class $\mathcal{A}(k)$ operator defined as follows: For each $k > 0$, an operator T is from class $\mathcal{A}(k)$ operator if

$$\left(T^*|T|^{2k}T\right)^{\frac{1}{k+1}} \geq |T|^2,$$

(for $k = 1$ it defines class $\mathcal{A}$ operator), and they showed that the class $\mathcal{A}$ is a subclass of paranormal operators.

In the same paper, authors introduced the absolute-k-paranormal operators as follows: For each $k > 0$, an operator T is absolute-k-paranormal if

$$\left\||T|^k Tx\right\| \geq ||Tx||^{k+1},$$

for every unit vector $x \in H$. In case where $k = 1$ it defines the paranormal operator. The class $\mathcal{A}(k)$ operator is included in the absolute-k-paranormal operator for any $k > 0$, [13, Theorem 2]).

B. P. Duggal, I. H. Jeon, and I. H. Kim [5], introduced $*$-class $\mathcal{A}$ operator. An operator $T \in L(H)$ is said to be a $*$-class $\mathcal{A}$ operator, if $|T^2| \geq |T^*|^2$. A $*$-class $\mathcal{A}$ is a generalization of a hyponormal operator, [5, Theorem 1.2], and $*$-class $\mathcal{A}$ is a subclass of the class of $*$-paranormal operators, [5, Theorem 1.3]. We denote the set of $*$-class

$\mathcal{A}$ by $\mathcal{A}^*$. An operator $T \in L(H)$ is said to be a k-quasi-$*$-class $\mathcal{A}$ operator [20], if

$$T^{*k}\left(|T^2| - |T^*|^2\right)T^k \geq O,$$

for a nonnegative integer k.

In [24] authors, S. Panayappan and A. Radharamani introduced the class $*$-$\mathcal{A}(k)$ operator and absolute-$*$-k-paranormal operator.

Definition 1.1. For each $k > 0$, an operator T is absolute-$*$-k-paranormal if

$$|||T|^k Tx|| \geq ||T^*x||^{k+1}$$

for every unit vector $x \in H$.

In case where $k = 1$ it defines the $*$-paranormal operator.

Definition 1.2. For each $k > 0$, an operator T is class $*$-$A(k)$, if

$$\left(T^*|T|^{2k}T\right)^{\frac{1}{k+1}} \geq |T^*|^2.$$

In case where $k = 1$ it defines the $\mathcal{A}^*$ class operators.

In this paper, we shall show behavior of the class $*$-$\mathcal{A}(k)$ operator and absolute-$*$-k-paranormal operator.

2. Properties of absolute-$*$-k-paranormal operator and $*$-$\mathcal{A}(k)$ operator

Theorem 2.1. *If T is an absolute-$*$-k-paranormal operator for $k > 0$, then T is a normaloid operator.*

Proof. Let T be an absolute-$*$-k-paranormal operator. In case where $k = 1$, T is a $*$-paranormal operator, then by [1, Theorem 1.1] it follows that T is a normaloid operator. Following, it will be proved that for $k > 1$ the operator T is a normaloid operator, because for $0 < k < 1$, it was proved in [3](Theorem 2.9). Without losing the generality, assume $\|T\| = 1$. Since T is an absolute-$*$-k-paranormal, then

$$\|T^*x\|^{k+1} \leq \||T|^k Tx\|\|x\|^k \leq \||T|^{k-1}\|\||T|Tx\|\|x\|^k \leq \|T^2x\|\|x\|^k$$

for all $x \in H$. Therefore,

$$\frac{\|T^*x\|^{k+1}}{\|x\|^k} \leq \|T^2x\| \leq \|x\| \tag{2.1}$$

for all $x \in H$.

By definition of $\|T^*\|$, there exists a sequence $\{x_i\}$ of unit vectors such that

$$\|T^*x_i\| \to \|T^*\| = \|T\| = 1. \tag{2.2}$$

Put $x = x_i$ in (2.1), then we have.

$$\frac{\|T^*x_i\|^{k+1}}{\|x_i\|^k} \leq \|T^2x_i\| \leq \|x_i\| = 1 \tag{2.3}$$

so, $\|T^2x_i\| \to 1$, by (2.2) and (2.3), that is

$$\|T^2\| = 1 = \|T\|^2.$$

Let us now suppose that

$$\|T^{n-1}x_i\| \to 1\ ,\ \|T^{n-2}x_i\| \to 1 \text{ and } \|T^{n-3}x_i\| \to 1 \text{ for } n \geq 3. \tag{2.4}$$

Put $x = T^{n-2}x_i$ in (2.1), then we have

$$\frac{\|T^*T^{n-2}x_i\|^{k+1}}{\|T^{n-2}x_i\|^k} \leq \|T^n x_i\| \leq \|T^{n-2}x_i\|. \tag{2.5}$$

From Cauchy-Schwarz inequality we have

$$\frac{\|T^{n-2}x\|^2}{\|T^{n-3}x\|} \leq \|T^*T^{n-2}x\|. \tag{2.6}$$

From relations (2.5) and (2.6) we have

$$\frac{\|T^{n-2}x_i\|^{k+2}}{\|T^{n-3}x_i\|^{k+1}} \leq \frac{\|T^*T^{n-2}x_i\|^{k+1}}{\|T^{n-2}x_i\|^k} \leq \|T^n x_i\| \leq \|T^{n-2}x_i\|.$$

respectively:

$$\frac{\|T^{n-1}x_i\|^{k+2}}{\|T^{n-3}x_i\|^{k+1}} \leq \frac{\|T^{n-2}x_i\|^{k+2}}{\|T^{n-3}x_i\|^{k+1}} \leq \frac{\|T^*T^{n-2}x_i\|^{k+1}}{\|T^{n-2}x_i\|^k} \leq \|T^n x_i\| \leq \|T^{n-2}x_i\|. \tag{2.7}$$

Hence, $\|T^n x_i\| \to 1$, by (2.4) and (2.7) that is $\|T^n\| = 1 = \|T\|^n$. Consequently

$$\|T^n\| = 1 = \|T\|^n,$$

for all positive integers n by induction. □

Example 2.2. An example of non-absolute-$*$-k-paranormal operator which is a normaloid operator. Let us denote by

$$T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

Then $||T^n|| = ||T||^n$ for all positive integers n. However, the relation

$$\left\||T|^k Tx\right\| \geq ||T^*x||^{k+1}$$

does not hold for the unit vector $e_3 = (0, 0, 1)$. With which was proved that T is a non-absolute-$*$-k-paranormal operator, but it is a normaloid operator.

It is known that there exists a linear operator T, so that T^n is compact operator for some $n \in \mathbb{N}$, but T itself is not compact. For instance, take any nilpotent noncompact operator (If $(e_n)_n$ is an orthonormal basis of H then the shift defined by $T(e_{2n}) = e_{2n+1}$ and $T(e_{2n+1}) = 0$ is not a compact operator for which $T^2 = O$).

In this context, we will show that in cases where an operator T is an absolute-$*$-k-paranormal operator and if its exponent T^n is compact, for some $n \in \mathbb{N}$, then T is compact too.

Theorem 2.3. *If T is an absolute-$*$-k-paranormal operator for $k > 0$ and if T^n is compact for some $n \in \mathbb{N}$, then it follows that T is compact too.*

Proof. Compactness of T^n implies countable spectrum (consisting of mutually orthogonal eigenvalues ([26], Theorem 6)), this then implies T^n normal compact, hence T is normal compact. $\square$

Corollary 2.4. *If T, R are absolute-$*$-k-paranormal operators for $k > 0$ and if T^n and R^m are compact for some $n, m \in \mathbb{N}$, then it follows that $T \oplus R$ is compact too.*

Corollary 2.5. *If T, R are absolute-$*$-k-paranormal operators for $k > 0$ and if T^n is a compact operator for some $n \in \mathbb{N}$ or R^m is a compact operator for some $m \in \mathbb{N}$, then it follows that $T \otimes R$ is compact too.*

Lemma 2.6. [16, Hansen Inequality] *If $A, B \in L(H)$, satisfying $A \geq O$ and $\|B\| \leq 1$, then*

$$(B^*AB)^\delta \geq B^*A^\delta B \text{ for all } \delta \in (0,1].$$

Lemma 2.7. [12, Löwner-Heinz Inequality] *If $A, B \in L(H)$, satisfying $A \geq B \geq O$, then $A^\delta \geq B^\delta$ for all $\delta \in [0,1]$.*

A subspace M of space H is said to be nontrivial invariant(alternatively, T-invariant) under T if $\{0\} \neq M \neq H$ and $T(M) \subseteq M$.

Theorem 2.8. *If T is a class $*$-$\mathcal{A}(k)$ operator for $0 < k \leq 1$ and M is its invariant subspace, then the restriction $T\mid_M$ of T to M is also a class $*$-$\mathcal{A}(k)$ operator.*

Proof. Since M is an invariant subspace of T, T has the matrix representation

$$T = \begin{pmatrix} A & B \\ O & C \end{pmatrix} \quad \text{on} \quad H = M \oplus M^\perp.$$

Let P be the projection of H onto M, where $A = T\mid_M$ and $\begin{pmatrix} A & O \\ O & O \end{pmatrix} = TP = PTP$. Since T is a class $*$-$\mathcal{A}(k)$ operator, we have

$$P\left(\left(T^*|T|^2T\right)^{\frac{1}{k+1}} - |T^*|^2\right)P \geq O.$$

By Hansen inequality, we have

$$\begin{aligned}\begin{pmatrix} |A^*|^2 & O \\ O & O \end{pmatrix} &\leq \begin{pmatrix} |A^*|^2 + |B^*|^2 & O \\ O & O \end{pmatrix} \\ &\leq \left(PT^*P|T|^{2k}PTP\right)^{\frac{1}{k+1}}.\end{aligned}$$

Since

$$P|T|^{2k}P \leq (P|T|^2P)^k,$$

then

$$PT^*P|T|^{2k}PTP \leq PT^*(P|T|^2P)^kTP.$$

By Löwner-Heinz inequality we have

$$\left(PT^*P|T|^{2k}PTP\right)^{\frac{1}{k+1}} \leq \left(PT^*(P|T|^2P)^kTP\right)^{\frac{1}{k+1}}.$$

So, we have

$$\begin{pmatrix} |A^*|^2 & O \\ O & O \end{pmatrix} \leq \begin{pmatrix} \left(A^*|A^*|^{2k}A\right)^{\frac{1}{k+1}} & O \\ O & O \end{pmatrix}.$$

Hence, A is a class $*$-$\mathcal{A}(k)$ operator on M. □

Theorem 2.9. *If T is a class $*$-$\mathcal{A}(k)$ operator, has the representation $T = \lambda \oplus A$ on $\ker(T-\lambda) \oplus (\ker(T-\lambda))^{\perp}$, where $\lambda \neq 0$ is an eigenvalue of T, then A is a class $*$-$\mathcal{A}(k)$ operator with $\ker(A-\lambda) = \{0\}$.*

Proof. Since $T = \lambda \oplus A$, then $T = \begin{pmatrix} \lambda & O \\ O & A \end{pmatrix}$ and we have:

$$\begin{aligned} \left(T^*|T|^{2k}T\right)^{\frac{1}{k+1}} - |T^*|^2 &= \begin{pmatrix} |\lambda|^{2(k+1)} & O \\ O & A^*|A|^{2k}A \end{pmatrix}^{\frac{1}{k+1}} - \begin{pmatrix} |\lambda|^2 & O \\ O & |A^*|^2 \end{pmatrix} \\ &= \begin{pmatrix} |\lambda|^2 & O \\ O & \left(A^*|A|^{2k}A\right)^{\frac{1}{k+1}} \end{pmatrix} - \begin{pmatrix} |\lambda|^2 & O \\ O & |A^*|^2 \end{pmatrix} \\ &= \begin{pmatrix} O & O \\ O & \left(A^*|A|^{2k}A\right)^{\frac{1}{k+1}} - |A^*|^2 \end{pmatrix} \end{aligned}$$

Since T is a class $*$-$\mathcal{A}(k)$ operator, then A is a class $*$-$\mathcal{A}(k)$ operator.
Let $x_2 \in \ker(A-\lambda)$. Then

$$(T-\lambda)\begin{pmatrix} 0 \\ x_2 \end{pmatrix} = \begin{pmatrix} O & O \\ O & A-\lambda \end{pmatrix}\begin{pmatrix} 0 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Hence $x_2 \in \ker(T-\lambda)$. Since $\ker(A-\lambda) \subseteq (\ker(T-\lambda))^{\perp}$, this implies $x_2 = 0$. Representation of T implies $A-\lambda$ is injective and by Theorem 2.8 A is $*$-$\mathcal{A}(k)$. □

3. Spectrum continuity on the set of class $*$-$\mathcal{A}(k)$ operator

Let $\{E_n\}_{n\in\mathbb{N}}$ be a sequence of compact subsets of $\mathbb{C}$. Let's define the inferior and superior limits of $\{E_n\}_{n\in\mathbb{N}}$, denoted respectively by $\liminf_{n\to\infty}\{E_n\}$ and $\limsup_{n\to\infty}\{E_n\}$ as it follows:

1) $\liminf_{n\to\infty}\{E_n\} = \{\lambda \in \mathbb{C} : \text{for every } \epsilon > 0, \text{there exists } N \in \mathbb{N} \text{ such that}$ $B(\lambda,\epsilon) \cap E_n \neq \emptyset \text{ for all } n > N\}$,

2) $\limsup_{n\to\infty}\{E_n\} = \{\lambda \in \mathbb{C} : \text{for every } \epsilon > 0, \text{there exists } J \subseteq \mathbb{N} \text{ infinite}$ $\text{such that } B(\lambda,\epsilon) \cap E_n \neq \emptyset \text{ for all } n \in J\}$.

If

$$\liminf_{n\to\infty}\{E_n\} = \limsup_{n\to\infty}\{E_n\},$$

then $\lim_{n\to\infty}\{E_n\}$ is said to exists and is equal to this common limit.

A mapping p, defined on $L(H)$, whose values are compact subsets on $\mathbb{C}$ is said to be upper semi-continuous at T, if $T_n \to T$ then $\limsup_{n\to\infty} p(T_n) \subset p(T)$, and lower semi-continuous at T, if $T_n \to T$ then $p(T) \subset \liminf_{n\to\infty} p(T_n)$. If p is both upper and lower semi-continuous at T, then it is said to be continuous at T and in this case $\lim_{n\to\infty} p(T_n) = p(T)$.

The spectrum $\sigma : T \to \sigma(T)$ is upper semi-continuous by [15, Problem 102], but it is not continuous in general, [25, Example 4.6]

We write $\alpha(T) = \text{dimker}(T)$, $\beta(T) = \text{dimker}(T^*)$. An operator $T \in L(H)$ is called an upper semi-Fredholm, if it has closed range and $\alpha(T) < \infty$, while T is called a lower semi-Fredholm if $\beta(T) < \infty$. However, T is called a semi-Fredholm operator if T is either an upper or a lower semi-Fredholm, and T is said to be a Fredholm operator if it is both an upper and a lower semi-Fredholm. If $T \in L(H)$ is semi-Fredholm, then the index is defined by

$$\text{ind}(T) = \alpha(T) - \beta(T).$$

An operator $T \in L(H)$ is said to be upper semi-Weyl operator if it is upper semi-Fredholm and $\text{ind}(T) \leq 0$, while $T \in L(H)$ is said to be lower semi-Weyl operator if it is lower semi-Fredholm and $\text{ind}(T) \geq 0$. An operator is said to be Weyl operator if it is Fredholm of index zero.

Lemma 3.1. [22] *If $\{T_n\} \subset L(H)$ and $T \in L(H)$ are such that T_n converges, according to the operator norm topology, to T then*

$$\text{iso}\sigma(T) \subseteq \liminf_{n\to\infty} \sigma(T_n).$$

Lemma 3.2. [2] *Let H be a complex Hilbert space. Then there exists a Hilbert space Y such that $H \subset Y$ and a map $\varphi : L(H) \to L(Y)$ with the following properties:*

1. *φ is a faithful $*$-representation of the algebra $L(H)$ on Y, so:*
$$\varphi(I_H) = I_Y \ , \ \varphi(T^*) = (\varphi(T))^* \ , \ \varphi(TS) = \varphi(T)\varphi(S)$$
$$\varphi(\alpha T + \beta S) = \alpha\varphi(T) + \beta\varphi(S) \text{ for any } T, S \in L(H) \text{ and } \alpha, \beta \in \mathbb{C},$$
2. *$\varphi(T) \geq 0$ for any $T \geq 0$ in $L(H)$,*
3. *$\sigma_a(T) = \sigma_a(\varphi(T)) = \sigma_p(\varphi(T))$ for any $T \in L(H)$,*
4. *If T is a positive operator, then $\varphi(T^\alpha) = |\varphi(T)|^\alpha$, for $\alpha > 0$,*

Lemma 3.3. *If T is a class $*$-$\mathcal{A}(k)$ operator, then $\varphi(T)$ is a class $*$-$\mathcal{A}(k)$ operator.*

Proof. Let $\varphi : L(H) \to L(K)$ be Berberian's faithful $*$-representation and let T be a class $*$-$\mathcal{A}(k)$ operator. Then, we have

$$\begin{aligned}\left((\varphi(T))^*|\varphi(T)|^{2k}\varphi(T)\right)^{\frac{1}{k+1}} - |(\varphi(T))^*|^2 &= \left(\varphi(T^*)\varphi(|T|^{2k})\varphi(T)\right)^{\frac{1}{k+1}} - |\varphi(T^*)|^2 \\ &= \left(\varphi(T^*|T|^{2k}T)^{\frac{1}{k+1}} - \varphi(|T^*|^2)\right) \\ &= \varphi\left((T^*|T|^{2k}T)^{\frac{1}{k+1}} - |T^*|^2\right) \geq 0\end{aligned}$$

thus $\varphi(T)$ is a class $*$-$\mathcal{A}(k)$ operator. □

Theorem 3.4. *The spectrum σ is continuous on the set of class $*$-$\mathcal{A}(k)$ operator for $k > 0$.*

Proof of the theorem is based in idea's given in the paper [6].

Proof. Since the function σ is upper semi-continuous, if $\{T_n\} \subset L(H)$ is a sequence which converges, to $T \in L(H)$, by operator norm topology. Then $\limsup_{n\to\infty} \sigma(T_n) \subset \sigma(T)$. Thus, to prove the theorem it would suffice to prove that if $\{T_n\}$ is a sequence of operators so that it belongs to class $*$-$\mathcal{A}(k)$ operator and $\lim_{n\to\infty} \|T_n - T\| = 0$ for some class $*$-$\mathcal{A}(k)$ operator T, then $\sigma(T) \subset \liminf_{n\to\infty} \sigma(T_n)$. From [25, Proposition

4.9] it would suffice to prove $\sigma_a(T) \subset \liminf_n \sigma(T_n)$. Since $\sigma(T) = \sigma(\varphi(T))$, $\sigma(T_n) = \sigma(\varphi(T)_n)$ and $\sigma_a(T) = \sigma_a(\varphi(T))$ we have

$$\sigma_a(T) \subset \liminf_{n\to\infty} \sigma(T_n) \Longleftrightarrow \sigma_a(\varphi(T)) \subset \liminf_{n\to\infty} \sigma(\varphi(T)_n).$$

Let $\lambda \in \sigma_a(\varphi(T))$. Then $\lambda \in \sigma_p(\varphi(T))$.
By Theorem 2.9, $\varphi(T)$ has a representation

$$\varphi(T) = \lambda \oplus A \text{ on } H = \ker(\varphi(T) - \lambda) \oplus (\ker(\varphi(T) - \lambda))^{\perp} \text{ and } \ker(A - \lambda) = \{0\}.$$

Therefore $A - \lambda$ is upper semi-Fredholm operator and $\alpha(A - \lambda) = 0$. There exists a $\epsilon > 0$ such that $A-(\lambda-\mu_0)$ is upper semi-Fredholm operator with $\text{ind}(A-(\lambda-\mu_0)) = \text{ind}(A - \lambda)$ and $\alpha(A - (\lambda - \mu_0)) = 0$ for every μ_0 such that $0 < |\mu_0| < \epsilon$. Let's set $\mu = \lambda - \mu_0$, and we have $\varphi(T) - \mu = (\lambda - \mu) \oplus (A - \mu)$ is upper semi-Fredholm operator, $\text{ind}(\varphi(T) - \mu) = \text{ind}(A - \mu)$ and $\alpha(\varphi(T) - \mu) = 0$.

Suppose the contrary, $\lambda \notin \liminf_{n\to\infty} \sigma(\varphi(T)_n)$. Then, there exists a $\delta > 0$, a neighborhood $\mathcal{D}_\delta(\lambda)$ of λ and a subsequence $\{\varphi(T)_{n_l}\}$ of $\{\varphi(T)_n\}$ such that $\sigma(\varphi(T)_{n_l}) \cap \mathcal{D}_\delta(\lambda) = \emptyset$ for every $l \geq 1$. This implies that $\varphi(T)_{n_l} - \mu$ is a Fredholm operator and $\text{ind}(\varphi(T)_{n_l} - \mu) = 0$ for every $\mu \in \mathcal{D}_\delta(\lambda)$ and

$$\lim_{n\to\infty} \|(\varphi(T)_{n_l} - \mu) - (\varphi(T) - \mu)\| = 0.$$

It follows from the continuity of the index that $\text{ind}(\varphi(T) - \mu) = 0$ and $\varphi(T) - \mu$ is a Fredholm operator. Since $\alpha(\varphi(T) - \mu) = 0$, $\mu \notin \sigma(\varphi(T))$ for every μ in a ϵ-neighborhood of λ. This contradicts Lemma 3.1, therefore we must have $\lambda \in \liminf_{n\to\infty} \sigma(\varphi(T)_n)$. □

It is well known **Index Product Theorem**: "If S and T are Fredholm operators then ST is a Fredholm operator and $\text{ind}(ST) = \text{ind}(S) + \text{ind}(T)$". The converse of this theorem is not true in general. To see this, we have operators on l_2:

$$T(x_1, x_2, x_3, ...) = (0, x_1, 0, x_2, 0, x_3, ...) \text{ and } S(x_1, x_2, x_3, ...) = (x_2, x_3, x_4, ...).$$

We see $ST = I$, so ST is a Fredholm operator, but S and T are not Fredholm operators. However, if S and T are commuting operators and if ST is a Fredholm operator then S and T are Fredholm operators. This fact is not true in general if S and T are Weyl operators, see [19, Remark 1.5.3].

Theorem 3.5. *If S and T are commuting class $*$-$\mathcal{A}(k)$ operators for $0 < k \leq 1$, then*

$$S, T \text{ are Weyl operators } \Longleftrightarrow ST \text{ is Weyl operator.}$$

Proof. If S and T are Weyl operators, by Index Product Theorem, we have that ST is a Weyl operator.

The converse, since $ST = TS$ then

$$\ker S \cup \ker T \subseteq \ker(ST) \text{ and } \ker S^* \cup \ker T^* \subseteq \ker(ST)^*,$$

then S and T are Fredholm operators.

Since S and T are class $*$-$\mathcal{A}(k)$ operators, from [26, theorem 2] S and T are class absolute-k^*-paranormal operators and by [26, theorem 6] we have $\text{ind}(S) \leq 0$ and $\text{ind}(T) \leq 0$. From

$$\text{ind}(S) + \text{ind}(T) = \text{ind}(ST) = 0,$$

we have $\operatorname{ind}(S) = 0$ and $\operatorname{ind}(T) = 0$, so S and T are Weyl operators. □

4. Contractions of the class $*$-$\mathcal{A}(k)$ operator

Definition 4.1. If the contraction T is a direct sum of the unitary and $C_{\cdot 0}$ (c.n.u) contractions, then we say that T has a **Wold-type decomposition**.

Definition 4.2. [9] An operator $T \in L(H)$ is said to have the Fuglede-Putnam commutativity property (**PF property** for short) if $T^*X = XJ$ for any $X \in L(K, H)$ and any isometry $J \in L(K)$ such that $TX = XJ^*$.

Lemma 4.3. [8, 23] *Let T be a contraction. The following conditions are equivalent:*

1. *For any bounded sequence $\{x_n\}_{n\in\mathbb{N}\cup\{0\}} \subset H$ such that $Tx_{n+1} = x_n$ the sequence $\{\|x_n\|\}_{n\in\mathbb{N}\cup\{0\}}$ is constant,*
2. *T has a* **Wold-type decomposition**,
3. *T has the* **PF property**.

Fugen Gao and Xiaochun Li [14] have proved that if a contraction $T \in \mathcal{A}^*$ has no nontrivial invariant subspace, then (a) T is a proper contraction and (b) The nonnegative operator $D = |T^2| - |T^*|^2$ is a strongly stable contraction. In [17] the authors proved: if T belongs to k-quasi-$*$-class $\mathcal{A}$ and is a contraction, then T has a Wold-type decomposition and T has the PF property. In this section we extend these results to contractions in class $*$-$\mathcal{A}(k)$.

Lemma 4.4. [4, Hölder-McCarthy inequality] *Let T be a positive operator. Then, the following inequalities hold for all $x \in H$:*

1. *$\langle T^r x, x\rangle \le \langle Tx, x\rangle^r \|x\|^{2(1-r)}$ for $0 < r < 1$,*
2. *$\langle T^r x, x\rangle \ge \langle Tx, x\rangle^r \|x\|^{2(1-r)}$ for $r \ge 1$.*

Proof of the theorems below is based in idea's given in the paper [7].

Theorem 4.5. *If T is a contraction of class $*$-$\mathcal{A}(k)$ operator, then the nonnegative operator*

$$D = \left(T^*|T|^{2k}T\right)^{\frac{1}{k+1}} - |T^*|^2$$

is a contraction whose power sequence $\{D^n\}_{n=1}^{\infty}$ converges strongly to a projection P and $T^*P = O$.

Proof. Suppose that T is a contraction of class $*$-$\mathcal{A}(k)$ operator. Then

$$D = \left(T^*|T|^{2k}T\right)^{\frac{1}{k+1}} - |T^*|^2 \ge O.$$

Let $R = D^{\frac{1}{2}}$ be the unique nonnegative square root of D. Then for every x in H and any nonnegative integer n, we have

$$\begin{aligned}
\langle D^{n+1}x, x\rangle &= \|R^{n+1}x\|^2 = \langle DR^n x, R^n x\rangle \\
&= \left\langle \left(T^*|T|^{2k}T\right)^{\frac{1}{k+1}} R^n x, R^n x\right\rangle - \langle |T^*|^2 R^n x, R^n x\rangle \\
&\le \langle T^*|T|^{2k}TR^n x, R^n x\rangle^{\frac{1}{k+1}} \|R^n x\|^{2(1-\frac{1}{k+1})} - \|T^* R^n x\|^2 \\
&= \||T|^k TR^n x\|^{\frac{2}{k+1}} \|R^n x\|^{2(1-\frac{1}{k+1})} - \|T^* R^n x\|^2 \\
&\le \||T|^k T\|^{\frac{2}{k+1}} \|R^n x\|^2 - \|T^* R^n x\|^2 \\
&\le \|R^n x\|^2 - \|T^* R^n x\|^2 \\
&\le \|R^n x\|^2 = \langle D^n x, x\rangle
\end{aligned}$$

Thus R (and so D) is a contraction (set $n = 0$), and $\{D^n\}_{n=1}^{\infty}$ is a decreasing sequence of nonnegative contractions. Then, $\{D^n\}_{n=1}^{\infty}$ converges strongly to a projection, say P. Moreover

$$\sum_{n=0}^{m} \|T^* R^n x\|^2 \le \sum_{n=0}^{m} \left(\|R^n x\|^2 - \|R^{n+1}x\|^2\right) = \|x\|^2 - \|R^{m+1}x\|^2 \le \|x\|^2,$$

for all nonnegative integers m and for every $x \in H$. Therefore $\|T^* R^n x\| \to 0$ as $n \to \infty$. Then, we have

$$T^* Px = T^* \lim_{n\to\infty} D^n x = \lim_{n\to\infty} T^* R^{2n} x = 0,$$

for every $x \in H$. So that $T^* P = O$. □

Theorem 4.6. *Let T be a contraction of class $*$-$\mathcal{A}(k)$ operator. If T has no nontrivial invariant subspace, then*

1) T is a proper contraction;

2) The nonnegative operator

$$D = \left(T^*|T|^{2k}T\right)^{\frac{1}{k+1}} - |T^*|^2$$

is a strongly stable contraction.

Proof. Suppose that T is a class $*$-$\mathcal{A}(k)$ operator.

1) From [18, Theorem 3.6] we have

$$T^*Tx = \|T\|^2 x \text{ if and only if } \|Tx\| = \|T\|\|x\| \text{ for every } x \in H.$$

Put $M = \{x \in H : \|Tx\| = \|T\|\|x\|\} = \ker(|T|^2 - \|T\|^2)$, which is a closed subspace of H. In the following, we shall show that M is a $T-$invariant subspace. For all $x \in M$, we have

$$\begin{aligned}
\|T(Tx)\|^2 &\le \|T\|^2\|Tx\|^2 = \|T\|^4\|x\|^2 = \|\|T\|^2 x\|^2 = \|T^*Tx\|^2 \\
&\le \| \left(T^*|T|^{2k}T\right)^{\frac{1}{k+1}} Tx\|\|Tx\| \le \| \left(T^*|T|^{2k}T\right)^{\frac{1}{k+1}} Tx\|\|T\|\|x\|.
\end{aligned}$$

So,

$$\|T\|^4\|x\|^2 \le \| \left(T^*|T|^{2k}T\right)^{\frac{1}{k+1}} Tx\|\|T\|\|x\|,$$

thus,

$$\|T\|^3\|x\| \leq \| \left(T^*|T|^{2k}T\right)^{\frac{1}{k+1}} Tx\|$$

and

$$\begin{aligned}
\left\|\left(T^*|T|^{2k}T\right)^{\frac{1}{k+1}} Tx\right\| &= \left\langle \left(T^*|T|^{2k}T\right)^{\frac{2}{k+1}} Tx, Tx\right\rangle^{\frac{1}{2}} \\
&\leq \left\langle \left(T^*|T|^{2k}T\right)^{2} Tx, Tx\right\rangle^{\frac{1}{2(k+1)}} \|Tx\|^{(1-\frac{1}{k+1})} \\
&= \left\|T^*|T|^{2k}TTx\right\|^{\frac{1}{k+1}} \|Tx\|^{(1-\frac{1}{k+1})} \\
&\leq \|T\|^{\frac{2k+3}{k+1}} \|x\|^{\frac{1}{k+1}} \|T\|^{\frac{k}{k+1}} \|x\|^{\frac{k}{k+1}} \\
&= \|T\|^3\|x\|.
\end{aligned}$$

Hence,

$$\|T\|^3\|x\| = \| \left(T^*|T|^{2k}T\right)^{\frac{1}{k+1}} Tx\|. \tag{4.1}$$

From relation (4.1) we have

$$\begin{aligned}
\|T\|^3\|x\| &= \left\|\left(T^*|T|^{2k}T\right)^{\frac{1}{k+1}} Tx\right\| \\
&= \left\langle \left(T^*|T|^{2k}T\right)^{\frac{2}{k+1}} Tx, Tx\right\rangle^{\frac{1}{2}} \\
&\leq \left\langle \left(T^*|T|^{2k}T\right)^{2} Tx, Tx\right\rangle^{\frac{1}{2(k+1)}} \|Tx\|^{(1-\frac{1}{k+1})} \\
&= \|T^*|T|^{2k}TTx\|^{\frac{1}{k+1}} \|Tx\|^{(1-\frac{1}{k+1})} \\
&\leq \|T^*|T|^{2k}\|^{\frac{1}{k+1}} \|T(Tx)\|^{\frac{1}{k+1}} \|Tx\|^{(1-\frac{1}{k+1})}
\end{aligned}$$

Then,

$$\|T\|^2\|x\| \leq \|T(Tx)\| \Longrightarrow \|T\|^2\|x\| = \|T(Tx)\|,$$

respectively,

$$\|T(Tx)\| = \|T\|^2\|x\| = \|T\|\|Tx\|.$$

Thus, M is a T-invariant subspace.

Now, let T be a contraction, i.e., $\|Tx\| \leq x$, for every $x \in H$. If $\|T\| < 1$, thus T is a strict contraction, then it is trivially a proper contraction. If $\|T\| = 1$, thus T is nonstrict contraction, then $M = \{x \in H : \|Tx\| = \|x\|\}$. Since T has no nontrivial invariant subspace, then the invariant subspace M is trivial: either $M = \{0\}$ or $M = H$. If $M = H$ then T is an isometry, and isometries have invariant subspaces. Thus $M = \{0\}$ so that $\|Tx\| < \|x\|$ for every nonzero $x \in H$. So T is proper contraction.

2) Let T be a contraction of class $*$-$\mathcal{A}(k)$ operator. By the above theorem, we have D is a contraction, $\{D^n\}_{n=1}^{\infty}$ converges strongly to a projection P, and $T^*P = O$. So, $PT = O$. Suppose T has no nontrivial invariant subspaces. Since $\ker P$ is a nonzero invariant subspace for T whenever $PT = O$ and $T \neq O$, it follows that $\ker P = H$. Hence $P = O$, and so $\{D^n\}_{n=1}^{\infty}$ converges strongly to null operator O, so D is a strongly stable contraction. Since D is self-adjoint, then $D \in C_{00}$. □

Corollary 4.7. *Let T be a contraction of the class $*$-$\mathcal{A}(k)$ operator. If T has no nontrivial invariant subspace, then both T and the nonnegative operator*

$$D = \left(T^*|T|^{2k}T\right)^{\frac{1}{k+1}} - |T^*|^2$$

are proper contractions.

Proof. Since a self−adjoint operator T is a proper contraction if and only if T is a C_{00} contraction. □

Theorem 4.8. *If T is a contraction and class $*$-$\mathcal{A}(k)$ operator for $k > 0$, then T has a Wold-type decomposition.*

Proof. Since T is a contraction operator, the decreasing sequence $\{T^n T^{*n}\}_{n=1}^{\infty}$ converges strongly to a nonnegative contraction. We denote by

$$S = \left(\lim_{n\to\infty} T^n T^{*n}\right)^{\frac{1}{2}}.$$

The operators T and S are related by $T^*S^2T = S^2$, $O \leq S \leq I$ and S is self-adjoint operator. By [10] there exists an isometry $V : \overline{S(H)} \to \overline{S(H)}$ such that $VS = ST^*$, and thus $SV^* = TS$, and $\|SV^m x\| \to \|x\|$ for every $x \in \overline{S(H)}$. The isometry V can be extended to an isometry on H, which we still denote by V.

For an $x \in \overline{S(H)}$, we can define $x_n = SV^n x$ for $n \in \mathbb{N} \cup \{0\}$. Then for all nonnegative integers m we have

$$T^m x_{n+m} = T^m SV^{m+n}x = SV^{*m}V^{m+n}x = SV^n x = x_n,$$

and for all $m \leq n$ we have

$$T^m x_n = x_{n-m}.$$

Since T is class $*$-$\mathcal{A}(k)$ operator for $k > 0$ and nontrivial $x \in \overline{A(H)}$ we have

$$\begin{aligned}
\|x_n\|^4 &= \|Tx_{n+1}\|^4 \leq \|T^*Tx_{n+1}\|^2\|x_{n+1}\|^2 \\
&\leq \|\left(T^*|T|^{2k}T\right)^{\frac{1}{k+1}} Tx_{n+1}\|\|Tx_{n+1}\|\|x_{n+1}\|^2 \\
&\leq \left\langle \left(T^*|T|^{2k}T\right)^2 Tx_{n+1}, Tx_{n+1}\right\rangle^{\frac{1}{2(k+1)}} \|Tx_{n+1}\|^{(1-\frac{1}{k+1})}\|Tx_{n+1}\|\|x_{n+1}\|^2 \\
&= \|T^*|T|^{2k}TTx_{n+1}\|^{\frac{1}{k+1}}\|x_n\|^{(1-\frac{1}{k+1})}\|x_n\|\|x_{n+1}\|^2 \\
&\leq \|TTx_{n+1}\|^{\frac{1}{k+1}}\|x_n\|^{\frac{2k+1}{k+1}}\|x_{n+1}\|^2 \\
&= \|x_{n-1}\|^{\frac{1}{k+1}}\|x_n\|^{\frac{2k+1}{k+1}}\|x_{n+1}\|^2
\end{aligned}$$

hence

$$\|x_n\| \leq \|x_{n-1}\|^{\frac{1}{2k+3}}\|x_{n+1}\|^{\frac{2k+2}{2k+3}} \leq \frac{1}{2k+3}\left(\|x_{n-1}\| + (2k+2)\|x_{n+1}\|\right).$$

Thus

$$(2k+2)\left(\|x_{n+1}\| - \|x_n\|\right) \geq \|x_n\| - \|x_{n-1}\|$$

Put, $b_n = \|x_n\| - \|x_{n-1}\|$, and we have

$$(2k+2)b_{n+1} \geq b_n. \tag{4.2}$$

Since $x_n = T(x_{n+1})$, then

$$\|x_n\| = \|Tx_{n+1}\| \leq \|x_{n+1}\| \text{ for every } n \in \mathbb{N},$$

then sequence $\{\|x_n\|\}_{n\in\mathbb{N}\cup\{0\}}$ is increasing. From

$$SV^n = SV^*V^{n+1} = TSV^{n+1}$$

we have

$$\|x_n\| = \|SV^n x\| = \|TSV^{n+1}x\| \leq \|SV^{n+1}x\| \leq \|x\|,$$

for every $x \in \overline{S(H)}$ and $n \in \mathbb{N}\cup\{0\}$. Then $\{\|x_n\|\}_{n\in\mathbb{N}\cup\{0\}}$ is bounded. From this we have $b_n \geq 0$ and $b_n \to 0$ as $n \to \infty$.

It remains to check that all b_n equal zero. Suppose that there exists an integer $i \geq 1$ such that $b_i > 0$. Using inequality (4.2) we get $b_{i+1} \geq \frac{b_i}{2k+2} > 0$, so $b_{i+1} > 0$. From that and using again inequality (4.2), we can show by induction that $b_n > 0$ for all $n > i$. This is contradictory with that $b_n \to 0$ as $n \to \infty$. So $b_n = 0$ for all $n \in \mathbb{N}$ and thus $\|x_{n-1}\| = \|x_n\|$ for all $n \geq 1$. Thus the sequence $\{\|x_n\|\}_{n\in\mathbb{N}\cup\{0\}}$ is constant. From Lemma 4.3, T has a **Wold-type decomposition**. □

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Book reviews

Derek K. Thomas, Nikola Tuneski, and Allu Vasudevarao; Univalent Functions. A Primer, De Gruyter Studies in Mathematics 69, De Gruyter, Berlin 2018, xii + 252 p., ISBN 978-3-11-056009-1/hbk; 978-3-11-056096-1/ebook.

Although there has been a continuing interest in the theory of univalent functions since its inception in the celebrated paper of Bieberbach in 1918, only a few books have appeared during this period. The seminal books by Hayman, Pommerenke and Duren, the last published in 1983, deal with the important fundamental properties, contains much material of an advanced nature, and also gives some information concerning subclasses.

In recent years, interest in univalent functions appears to have increased, particularly in the study of subclasses, and probably as a result of the changing nature of academic publishing, many more papers have appeared. The books of Goodman are primarily concerned with subclasses, but also published in 1983, are now in many ways out of date, and an update of present knowledge is now timely.

This book is directed at those new to research in the theory of univalent functions, and thus omits some material of a deeper nature. It is also of interest to those interested in updating what is currently known about a selection of problems in the important subclasses of univalent functions. In the preface, the authors set out their aims, which are to update current information on the important subclasses of univalent functions, concentrating on what they consider to be the more important problems, whilst at the same time providing examples of the use of the central ideas and methods involved in proving theorems. The book ends with a set of 50 open problems, many of which are related to the topics considered in the book.

The book contains 17 chapters. The first three chapters contain the elementary theory of univalent functions, basic definitions, and properties of the important subclasses, and a chapter laying out some fundamental lemmas which are used later in the book.

Chapter 4 gives an in-depth survey of the important results concerning starlike and convex functions. As elsewhere in the book, complete proofs of the theorems presented are given in most instances.

The next two chapters are concerned with starlike and convex functions of order α, and strongly starlike and convex functions. Here again, a thorough account of the often complicated proofs of many of the results is given.

Chapters 7 and 8 deal with the so-called α-convex functions and gamma-starlike functions, where again an up-to-date account is given, including a detailed proof of the sharp coefficient inequality for α-convex functions.

Chapter 9 contains a very full study of most of the important problems for close-to-convex functions. Amongst items discussed is an up-to-date treatment of finding the sharp bounds for the modulus of the logarithmic coefficients, in particular the third logarithmic coefficient, which remains an outstanding and significant unsolved problem.

The next two chapters deal with Bazilević functions $\mathcal{B}(\alpha)$, Chapter 10 with the case the $\alpha \geq 0$, and Chapter 11 with the so-called $\mathcal{B}_1(\alpha)$ functions, where the associated starlike function in the definition of $\mathcal{B}(\alpha)$ is the identity function.

Chapter 12 considers the class $\mathcal{U}(\lambda)$, and contains some results concerning conditions for univalence, and recent sharp coefficient estimates.

The main aim of the next chapter is to present a proof of the Pólya-Schoenberg Conjecture concerning convolutions.

Meromorphic univalent functions are dealt with in Chapter 14, where a detailed discussion is given to the determination of the Clunie-constant. Other results for the subclasses of starlike, close-to-convex and Bazilević meromorphic functions are also included

Chapter 15 gives a brief introduction to Loewner Theory with some applications, including a proof of the Bieberbach conjecture when $n = 3$, and the solution to the Fekete-Szegő problem.

In the next chapter a selection of topics not contained in the previous chapters are briefly introduced, and some basic properties are presented.

The book ends with a list of 50 open problems, most of which are connected with the material in the book.

There is an extensive bibliography, and a detailed index.

The book is a significant addition to the study of univalent functions, and will be particularly useful to those with an interest in subclasses.

Teodor Bulboacă

Wojbor A. Woyczynski, Geometry and martingales in Banach spaces, CRC Press, Boca Raton, FL, 2019, ISBN 978-1-138-61637-0/hbk; 978-0-4298-6883-2/ebook, xiii+315 p.

The study of Banach space valued random variables is tightly connected with the geometric properties of the underlying space. In particular, martingale theory is essential in the study of Radon-Nikodým property, finite tree property and super-reflexivity, and of the local properties of Banach spaces. The UMD spaces (meaning Banach spaces X for which X-valued martingale differences are unconditionally convergent in $L^p(X)$, $1 < p < \infty$) provide the correct framework for the development of the harmonic analysis for vector-valued functions. This is masterly illustrated in two recent books: G. Pisier, *Martingales in Banach spaces*, Cambridge University Press, Cambridge, 2016, and T. Hytönen, J. van Neerven, M. Veraar, L. Weis, *Analysis in*

Banach spaces. Vol. I. *Martingales and Littlewood-Paley theory*, Springer 2016; Vol. II. *Probabilistic methods and operator theory*, Springer, 2017.

As the author points out in Introduction:

> In this volume we are providing a compact exposition of the results explaining the interrelation existing between the metric geometry of Banach spaces and probability theory of random vectors with values in those Banach spaces. In particular martingales and random series of independent random vectors are studied.

The presentation is focussed on the remarkable results obtained in the 1970s (an effervescent period in this area) by reputed mathematicians as P. Assouad, D. L. Burkholder, S. D. Chatterji, G. Pisier, J. Hoffmann-Jorgensen, B. Maurey, and the author himself. A good idea on the topics treated in the book is given by the headings of the chapters: 1. *Preliminaries: Probability and geometry in Banach spaces*; 2. *Dentability, Radon-Nikodym Theorem, and Martingale Convergence Theorem*; 3. *Uniform Convexity and Uniform Smoothness*; 4. *Spaces that do not contain* c_0; 5. *Cotypes of Banach spaces*; 6. *Spaces of Rademacher and stable type*; 7. *Spaces of type* 2; 8. *Beck convexity* (B-convex spaces, meaning spaces of type > 1); 9. *Marcinkiewicz-Zygmund Theorem in Banach spaces.*

The author provides detailed proofs of all the results concerning the interplay between the geometry and martingales. For purely geometric or probabilistic results only references are given, the prerequisites being familiarity with basic facts of functional analysis and probability theory.

The book is of interest for researchers in Banach spaces, probability theory and their applications to the analysis of vector functions.

S. Cobzaş