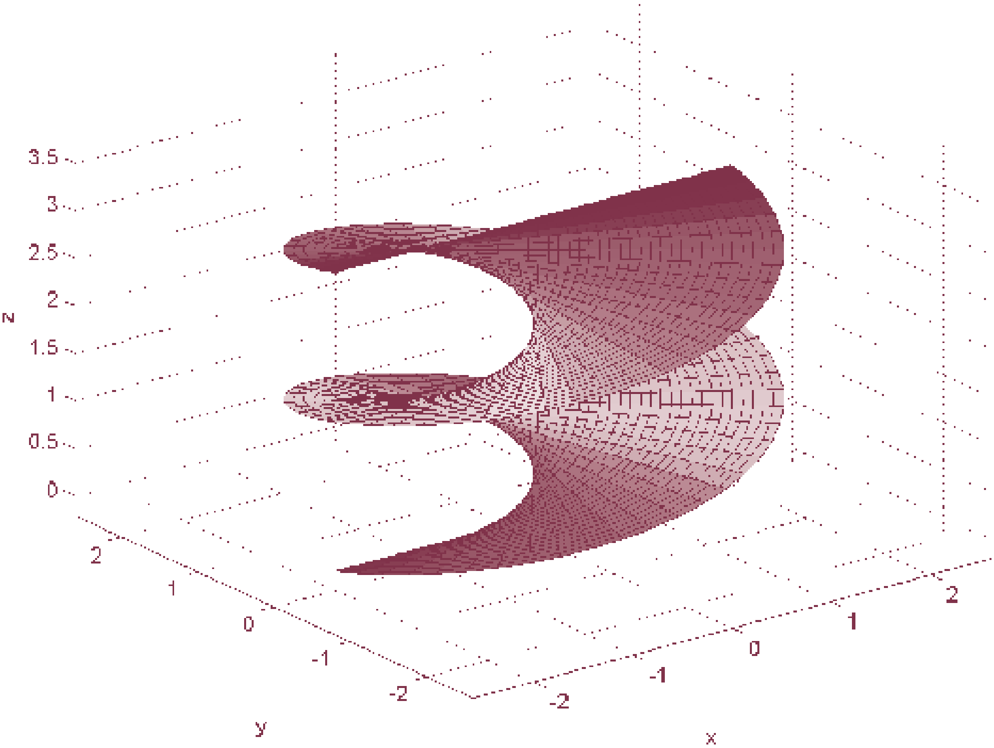
STUDIA UNIVERSITATIS
BABEŞ-BOLYAI



MATHEMATICA

4/2018

STUDIA
UNIVERSITATIS BABEŞ-BOLYAI
MATHEMATICA

4/2018

EDITORIAL BOARD OF
STUDIA UNIVERSITATIS BABEŞ-BOLYAI MATHEMATICA

EDITORS:

Radu Precup, Babeş-Bolyai University, Cluj-Napoca, Romania (Editor-in-Chief)
Octavian Agratini, Babeş-Bolyai University, Cluj-Napoca, Romania
Simion Breaz, Babeş-Bolyai University, Cluj-Napoca, Romania
Csaba Varga, Babeş-Bolyai University, Cluj-Napoca, Romania

MEMBERS OF THE BOARD:

Ulrich Albrecht, Auburn University, USA
Francesco Altomare, University of Bari, Italy
Dorin Andrica, Babeş-Bolyai University, Cluj-Napoca, Romania
Silvana Bazzoni, University of Padova, Italy
Petru Blaga, Babeş-Bolyai University, Cluj-Napoca, Romania
Wolfgang Breckner, Babeş-Bolyai University, Cluj-Napoca, Romania
Teodor Bulboacă, Babeş-Bolyai University, Cluj-Napoca, Romania
Gheorghe Coman, Babeş-Bolyai University, Cluj-Napoca, Romania
Louis Funar, University of Grenoble, France
Ioan Gavrea, Technical University, Cluj-Napoca, Romania
Vijay Gupta, Netaji Subhas Institute of Technology, New Delhi, India
Gábor Kassay, Babeş-Bolyai University, Cluj-Napoca, Romania
Mirela Kohr, Babeş-Bolyai University, Cluj-Napoca, Romania
Iosif Kolumbán, Babeş-Bolyai University, Cluj-Napoca, Romania
Alexandru Kristály, Babeş-Bolyai University, Cluj-Napoca, Romania
Andrei Mărcuş, Babeş-Bolyai University, Cluj-Napoca, Romania
Waclaw Marzantowicz, Adam Mickiewicz, Poznan, Poland
Giuseppe Mastroianni, University of Basilicata, Potenza, Italy
Mihail Megan, West University of Timişoara, Romania
Gradimir V. Milovanović, Megatrend University, Belgrade, Serbia
Boris Mordukhovich, Wayne State University, Detroit, USA
András Némethi, Rényi Alfréd Institute of Mathematics, Hungary
Rafael Ortega, University of Granada, Spain
Adrian Petruşel, Babeş-Bolyai University, Cluj-Napoca, Romania
Cornel Pintea, Babeş-Bolyai University, Cluj-Napoca, Romania
Patrizia Pucci, University of Perugia, Italy
Ioan Purdea, Babeş-Bolyai University, Cluj-Napoca, Romania
John M. Rassias, National and Capodistrian University of Athens, Greece
Themistocles M. Rassias, National Technical University of Athens, Greece
Ioan A. Rus, Babeş-Bolyai University, Cluj-Napoca, Romania
Grigore Sălăgean, Babeş-Bolyai University, Cluj-Napoca, Romania
Mircea Sofonea, University of Perpignan, France
Anna Soós, Babeş-Bolyai University, Cluj-Napoca, Romania
András Stipsicz, Rényi Alfréd Institute of Mathematics, Hungary
Ferenc Szenkovits, Babeş-Bolyai University, Cluj-Napoca, Romania
Michel Théra, University of Limoges, France

BOOK REVIEWS:

Ştefan Cobzaş, Babeş-Bolyai University, Cluj-Napoca, Romania

SECRETARIES OF THE BOARD:

Teodora Cătinaş, Babeş-Bolyai University, Cluj-Napoca, Romania
Hannelore Lisei, Babeş-Bolyai University, Cluj-Napoca, Romania

TECHNICAL EDITOR:

Georgeta Bonda, Babeş-Bolyai University, Cluj-Napoca, Romania

YEAR **(LXIII) 2018**
MONTH **DECEMBER**
ISSUE **4**

S T U D I A
UNIVERSITATIS BABEŞ-BOLYAI

MATHEMATICA
4

Redacţia: 400084 Cluj-Napoca, str. M. Kogălniceanu nr. 1
Telefon: 0264 405300

CONTENTS

HARI MOHAN SRIVASTAVA, SHAHID KHAN, QAZI ZAHOOR AHMAD, NAZAR KHAN and SAQIB HUSSAIN, The Faber polynomial expansion method and its application to the general coefficient problem for some subclasses of bi-univalent functions associated with a certain q-integral operator 419
TRUONG CONG QUYNH and SERAP ŞAHINKAYA, Goldie absolute direct summand rings and modules 437
MOUFFAK BENCHOHRA, SOUFYANE BOURIAH and JUAN J. NIETO, Existence and stability results for nonlocal initial value problems for differential equations with Hilfer fractional derivative 447
GHULAM FARID, UDITA N. KATUGAMPOLA and MUHAMMAD USMAN, Ostrowski-type fractional integral inequalities for mappings whose derivatives are h-convex via Katugampola fractional integrals 465
ESZTER SZATMARI, Differential subordinations obtained by using a fractional operator 475
ANDRIY BANDURA and OLEH SKASKIV, Sufficient conditions of boundedness of **L**-index and analog of Hayman's Theorem for analytic functions in a ball 483
TAYEB BLOUHI and MOHAMED FERHAT, Existence and topological structure of solution sets for φ-Laplacian impulsive stochastic differential systems 503
GÜLEN BAŞCANBAZ-TUNCA, MURAT BODUR and DILEK SÖYLEMEZ, On Lupaş-Jain operators 525

Tatyana A. Komleva, Liliya I. Plotnikova and Andrej V. Plotnikov,
Partial averaging of discrete-time set-valued systems 539
Traian Ionuţ Luca and Dorel I. Duca, Approximations of bi-criteria
optimization problem 549
Book reviews 561

Stud. Univ. Babeş-Bolyai Math. 63(2018), No. 4, 419–436
DOI: 10.24193/subbmath.2018.4.01

The Faber polynomial expansion method and its application to the general coefficient problem for some subclasses of bi-univalent functions associated with a certain q-integral operator

Hari Mohan Srivastava, Shahid Khan, Qazi Zahoor Ahmad, Nazar Khan and Saqib Hussain

Abstract. In our present investigation, we first introduce several new subclasses of analytic and bi-univalent functions by using a certain q-integral operator in the open unit disk $\mathbb{U} = \{z : z \in \mathbb{C}\text{and}\, |z| < 1\}$. By applying the Faber polynomial expansion method as well as the q-analysis, we then determine bounds for the nth coefficient in the Taylor-Maclaurin series expansion for functions in each of these newly-defined analytic and bi-univalent function classes subject to a gap series condition. We also highlight some known consequences of our main results.

Mathematics Subject Classification (2010): 05A30, 30C45, 11B65, 47B38.

Keywords: Analytic functions, univalent functions, Taylor-Maclaurin series representation, Faber polynomials, bi-inivalent functions, q-derivative operator, q-hypergeometric functions, q-integral operators.

1. Introduction and definitions

Let $\mathcal{A}$ be the class of all functions f which are analytic in the open unit disk

$$\mathbb{U} = \{z : z \in \mathbb{C} \quad \text{and} \quad |z| < 1\}$$

and normalized by

$$f(0) = 0 = f'(0) - 1.$$

Thus, clearly, the function $f \in \mathcal{A}$ has the following Taylor-Maclaurin series representation:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \qquad (z \in \mathbb{U}). \tag{1.1}$$

Further, by $\mathcal{S} \subset \mathcal{A}$ we shall denote the class of all functions which are univalent in $\mathbb{U}$.

For two functions $f, g \in \mathcal{A}$, the function f is said to be subordinate to the function g in $\mathbb{U}$, denoted by

$$f(z) \prec g(z) \qquad (z \in \mathbb{U}),$$

if there exists a function

$$w \in \mathbb{B}_0 := \{w : w \in \mathcal{A}, \quad w(0) = 0 \quad \text{and} \quad |w(z)| < 1 \quad (z \in \mathbb{U})\}$$

such that

$$f(z) = g\big(w(z)\big) \qquad (z \in \mathbb{U}).$$

In the case when the function g is univalent in $\mathbb{U}$, we have the following equivalence:

$$f(z) \prec g(z) \quad (z \in \mathbb{U}) \Longleftrightarrow f(0) = g(0) \qquad \text{and} \qquad f(\mathbb{U}) \subset g(\mathbb{U}).$$

Next, for a function $f \in \mathcal{A}$ given by (1.1) and another function $g \in \mathcal{A}$ given by

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n \qquad (z \in \mathbb{U}),$$

the convolution (or the Hadamard product) of the functions f and g is defined by

$$(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n = (g * f)(z). \tag{1.2}$$

It is well known that every univalent function f has an inverse f^{-1}, defined by

$$f^{-1}\big(f(z)\big) = z = f\left(f^{-1}(z)\right) \qquad (z \in \mathbb{U})$$

and

$$f\left(f^{-1}(w)\right) = w \qquad \left(|w| < r_0(f);\ r_0(f) \geqq \frac{1}{4}\right),$$

where

$$f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2 a_3 + a_4)w^4 + \cdots. \tag{1.3}$$

A function $f \in \mathcal{A}$ is said to be bi-univalent in $\mathbb{U}$ if both f and f^{-1} are univalent in $\mathbb{U}$. We denote the class of all such functions by Σ. In recent years, the pioneering work of Srivastava *et al.* [22] essentially revived the investigation of various subclasses of the analytic and bi-univalent function class Σ. In fact, in a remarkably large number of sequels to the pioneering work of Srivastava *et al.* [22], several different subclasses of the analytic and bi-univalent function class Σ were introduced and studied analogously by the many authors (see, for example, [5], [7], [9], [23], [24], [25], [28] and [29]). However, only non-sharp estimates on the initial coefficients $|a_2|$ and $|a_3|$ in the Taylor-Maclaurin series expansion (1.1) were obtained in these recent papers.

The Faber polynomials introduced by Faber [11] play an important rôle in various areas of mathematical sciences, especially in Geometric Function Theory of Complex Analysis (see, for details, [27]). Recently, several authors (see, for example, [13] and [26]; see also [6], [8], [12] and [20]) investigated some interesting and useful properties for analytic functions by applying the Faber polynomial expansion method. Motivated by these and other recent works (see, for example, [1], [14] and [30]), here we make use of the q-analysis in order to define new subclasses of analytic and bi-univalent functions in $\mathbb{U}$ and (by means of the Faber polynomial expansion method) we determine estimates for the general coefficient $|a_n|$ $(n \geqq 3)$ in the Taylor-Maclaurin series expansion (1.1) of functions in each of these subclasses.

We begin by recalling here some basic definitions and other concept details of the q-calculus $(0 < q < 1)$, which will be used in this paper.

Definition 1.1. Let $q \in (0,1)$ and define the q-number $[\kappa]_q$ by

$$[\kappa]_q = \begin{cases} \dfrac{1-q^\kappa}{1-q} & (\kappa \in \mathbb{C}) \\ \sum\limits_{k=0}^{n-1} q^k = 1 + q + q^2 + \cdots + q^{n-1} & (\kappa = n \in \mathbb{N}), \end{cases}$$

where $\mathbb{N}$ denotes the set of positive integers and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$.

Definition 1.2. Let $q \in (0,1)$ and define the q-factorial $[n]_q!$ by

$$[n]_q! = \begin{cases} 1 & (n = 0) \\ \prod\limits_{k=1}^{n} [k]_q & (n \in \mathbb{N}). \end{cases}$$

Definition 1.3. (see [15] and [16]) The q-derivative (or the q-difference) $D_q f$ of a function f is defined, in a given subset of $\mathbb{C}$, by

$$\big(D_q f\big)(z) = \begin{cases} \dfrac{f(z) - f(qz)}{(1-q)z} & (z \neq 0) \\ f'(0) & (z = 0), \end{cases} \tag{1.4}$$

provided that $f'(0)$ exists.

We note from Definition 1.3 that

$$\lim_{q \to 1-} \big(D_q f\big)(z) = \lim_{q \to 1-} \frac{f(z) - f(qz)}{(1-q)z} = f'(z)$$

for a function f which is differentiable in a given subset of $\mathbb{C}$. It is readily deduced from (1.1) and (1.4) that

$$\big(D_q f\big)(z) = 1 + \sum_{n=2}^{\infty} [n]_q \, a_n z^{n-1}.$$

Definition 1.4. The q-Pochhammer symbol $[\kappa]_{n,q}$ $(\kappa \in \mathbb{C};\ n \in \mathbb{N}_0)$ is defined as follows:

$$[\kappa]_{n,q} = \frac{(q^\kappa; q)_n}{(1-q)^n} := \begin{cases} 1 & (n = 0) \\ [\kappa]_q [\kappa+1]_q [\kappa+2]_q \cdots [\kappa+n-1]_q & (n \in \mathbb{N}). \end{cases}$$

Moreover, the q-gamma function $\Gamma_q(z)$ is defined by the following recurrence relation:

$$\Gamma_q(z+1) = [z]_q \, \Gamma_q(z) \qquad \text{and} \qquad \Gamma_q(1) = 1.$$

Definition 1.5. [17] For $f \in \mathcal{A}$, let the Ruscheweyh q-derivative operator be defined as follows:

$$\mathcal{I}_q^{\lambda} f(z) = f(z) * \mathcal{F}_{q,\lambda+1}(z) \qquad (z \in \mathbb{U};\ \lambda > -1),$$

where

$$\mathcal{F}_{q,\lambda+1}(z) = z + \sum_{n=2}^{\infty} \frac{\Gamma_q(\lambda+n)}{[n-1]_q!\Gamma_q(\lambda+1)}\, z^n = z + \sum_{n=2}^{\infty} \frac{[\lambda+1]_{q,n-1}}{[n-1]_q!}\, z^n$$

in terms the Hadamard product (or convolution) given by (1.2).

We next define a certain q-integral operator by using the same technique as that used by Noor [19].

Definition 1.6. For $f \in \mathcal{A}$, let the q-integral operator $\mathcal{F}_{q,\lambda}$ be defined by

$$\mathcal{F}_{q,\lambda+1}^{-1}(z) * \mathcal{F}_{q,\lambda+1}(z) = zD_q f(z).$$

Then

$$\begin{aligned} \mathcal{I}_q^{\lambda} f(z) &= f(z) * \mathcal{F}_{q,\lambda+1}^{-1}(z) \\ &= z + \sum_{n=2}^{\infty} \Psi_{n-1}\, a_n z^n \qquad (z \in \mathbb{U};\ \lambda > -1), \end{aligned} \tag{1.5}$$

where

$$\mathcal{F}_{q,\lambda+1}^{-1}(z) = z + \sum_{n=2}^{\infty} \Psi_{n-1}\, z^n$$

and

$$\Psi_{n-1} = \frac{[n]_q!\Gamma_q(\lambda+1)}{\Gamma_q(\lambda+n)} = \frac{[n]_q!}{[\lambda+1]_{q,n-1}}.$$

Clearly, we have

$$\mathcal{I}_q^0 f(z) = zD_q f(z) \qquad \text{and} \qquad \mathcal{I}_q^1 f(z) = f(z).$$

We note also that, in the limit case when $q \to 1-$, the q-integral operator $\mathcal{F}_{q,\lambda}$ given by Definition 1.6 would reduce to the integral operator which was studied by Noor [18].

The following identity can be easily verified:

$$zD_q\left(\mathcal{I}_q^{\lambda+1} f(z)\right) = \left(1 + \frac{[\lambda]_q}{q^{\lambda}}\right) \mathcal{I}_q^{\lambda} f(z) - \frac{[\lambda]_q}{q^{\lambda}} \mathcal{I}_q^{\lambda+1} f(z). \tag{1.6}$$

When $q \to 1-$, this last identity (1.6) implies that

$$z\left(\mathcal{I}^{\lambda+1} f(z)\right)' = (1+\lambda)\mathcal{I}^{\lambda} f(z) - \lambda \mathcal{I}^{\lambda+1} f(z),$$

which is the well-known recurrence relation for the above-mentioned integral operator which studied by Noor [18].

The above-defined q-calculus provides valuable tools that have been extensively used in order to examine several subclasses of $\mathcal{A}$. Even though Ismail *et al.* [14] were the first to use the q-derivative operator D_q in order to study a certain q-analogue of the class $\mathcal{S}^*$ of starlike functions in $\mathbb{U}$, yet a rather significant usage of the q-calculus in the context of Geometric Function Theory of Complex Analysis was basically furnished and the basic (or q-) hypergeometric functions were first used in Geometric Function Theory in a book chapter by Srivastava (see, for details, [21, pp. 347 *et seq.*]; see also [23]).

We now introduce the following subclasses of the analytic and bi-univalent function class Σ.

Definition 1.7. For a function $f \in \Sigma$, we say that

$$f \in \mathcal{R}_q(\Sigma, \alpha, \gamma) \quad (0 \leqq \alpha < 1;\ \gamma \geqq 0)$$

if and only if

$$\left| D_q f(z) + \gamma z D_q^2 f(z) - \frac{1-\alpha q}{1-q} \right| < \frac{1-\alpha}{1-q} \qquad (z \in \mathbb{U})$$

and

$$\left| D_q g(w) + \gamma w D_q^2 g(w) - \frac{1-\alpha q}{1-q} \right| < \frac{1-\alpha}{1-q} \qquad (w \in \mathbb{U}).$$

Equivalently, by using the principle of subordination between analytic functions, we can write the above conditions as follows (see, for details, [30]):

$$D_q f(z) + \gamma z D_q^2 f(z) \prec \frac{1 + [1 - \alpha(1+q)]\ z}{1 - qz} \qquad (z \in \mathbb{U})$$

and

$$D_q g(w) + \gamma w D_q^2 g(w) \prec \frac{1 + [1 - \alpha(1+q)]\, w}{1 - qw} \qquad (w \in \mathbb{U}),$$

respectively, where $g(w) = f^{-1}(w)$ is given by (1.3).

Definition 1.8. For a function $f \in \Sigma$, we say that

$$f \in \mathcal{R}_q(\Sigma, \alpha, \gamma, \lambda) \quad (0 \leqq \alpha < 1;\ \gamma \geqq 0;\ \lambda \geqq 0)$$

if and only if

$$D_q \mathcal{I}_q^\lambda f(z) + \gamma z D_q^2 \mathcal{I}_q^\lambda f(z) \prec \frac{1 + [1 - \alpha(1+q)]\, z}{1 - qz} \qquad (z \in \mathbb{U})$$

and

$$D_q \mathcal{I}_q^\lambda g(w) + \lambda w D_q^2 \mathcal{I}_q^\lambda g(w) \prec \frac{1 + [1 - \alpha(1+q)]\, w}{1 - qw} \qquad (w \in \mathbb{U}),$$

where $g(w) = f^{-1}(w)$ is given by (1.3).

2. The Faber polynomial expansion method and its applications

In this section, by using the Faber polynomial expansion of a function $f \in \mathcal{A}$ of the form (1.1), we observe that the coefficients of its inverse map $g = f^{-1}$ may be expressed as follows (see [4]; see also [13] and [26]):

$$g(w) = f^{-1}(w) = w + \sum_{n=2}^{\infty} \frac{1}{n} K_{n-1}^{-n}(a_2, a_3, \cdots, a_n)\, w^n, \tag{2.1}$$

where

$$\begin{aligned}K_{n-1}^{-n} = &\frac{(-n)!}{(-2n+1)!(n-1)!}a_2^{n-1} + \frac{(-n)!}{(2(-n+1))!(n-3)!}a_2^{n-3}a_3 \\ &+ \frac{(-n)!}{(-2n+3)!(n-4)!}a_2^{n-4}a_4 \\ &+ \frac{(-n)!}{(2(-n+2))!(n-5)!}a_2^{n-5}\left[a_5 + (-n+2)a_3^2\right] \\ &+ \frac{(-n)!}{(-2n+5)!(n-6)!}a_2^{n-6}\left[a_6 + (-2n+5)a_3a_4\right] \\ &+ \sum_{j\geqq 7} a_2^{n-j}V_j. \end{aligned} \tag{2.2}$$

Here, and in what follows, such expressions as (for example) $(-n)!$ occurring in (2.2) are to be interpreted *symbolically* by

$$(-n)! \equiv \Gamma(1-n) := (-n)(-n-1)(-n-2)\cdots \qquad \big(n \in \mathbb{N}_0\big)$$

and V_j $(7 \leqq j \leqq n)$ is a homogeneous polynomial in the variables $a_2, a_3, \cdots, a_n$. In particular, the first three terms of K_{n-1}^{-n} are given below:

$$K_1^{-2} = -2a_2, \qquad K_2^{-3} = 3\left(2a_2^2 - a_3\right)$$

and

$$K_3^{-4} = -4\left(5a_2^3 - 5a_2a_3 + a_4\right).$$

In general, an expansion of $K_n^{\mathfrak{p}}$ is given by (see, for details, [3])

$$K_n^{\mathfrak{p}} = \mathfrak{p}a_n + \frac{\mathfrak{p}(\mathfrak{p}-1)}{2}\,E_n^2 + \frac{\mathfrak{p}!}{(-3)!3!}\,E_n^3 + \cdots + \frac{\mathfrak{p}!}{(\mathfrak{p}-n)!n!}\,E_n^n \qquad (\mathfrak{p} \in \mathbb{Z}),$$

where $\mathbb{Z} := \{0, \pm 1, \pm 2, \cdots\}$ and

$$E_n^{\mathfrak{p}} = E_n^{\mathfrak{p}}\left(a_2, a_3, \cdots\right).$$

It is clearly seen that

$$E_n^n(a_1, a_2, \cdots, a_n) = a_1^n.$$

and

$$E_{n-1}^m(a_2, \cdots, a_n) = \sum_{n=2}^{\infty} \frac{m!(a_2)^{\mu_1}\cdots(a_n)^{\mu_{n-1}}}{\mu_{1!}, \cdots, \mu_{n-1}!} \qquad (m \leqq n).$$

We also have (see [2])

$$E_{n-1}^{n-1}\left(a_2, \cdots, a_n\right) = a_2^{n-1}$$

and

$$E_n^m(a_1, a_2, \cdots, a_n) = \sum \left(\frac{m!}{\mu_1! \cdots \mu_n!}\right) a_1^{\mu_1} \cdots a_n^{\mu_n} \qquad (m \leqq n),$$

where $a_1 = 1$ and the sum is taken over all non-negative integers $\mu_1, \cdots, \mu_n$ satisfying the following conditions:

$$\mu_1 + \mu_2 + \cdots + \mu_n = m$$

and

$$\mu_1 + 2\mu_2 + \cdots + n\mu_n = n.$$

By a similar argument, we note that

$$E_n^n(a_1, \cdots, a_n) = E_1^n$$

and that the first and the last polynomials are given by

$$E_n^n = a_1^n \qquad \text{and} \qquad E_n^1 = a_n.$$

We now state and prove our main results. Throughout our discussion, the parameters $\mathcal{L}$ and $\mathcal{M}$ are given by

$$\mathcal{L} := [1-\alpha(1+q)] \qquad \text{and} \qquad \mathcal{M} := -q.$$

Theorem 2.1. *For $0 \leqq \alpha < 1$ and $\gamma \geqq 0$, let $f \in \mathcal{R}_q(\Sigma, \alpha, \gamma)$. If*

$$a_m = 0 \qquad (2 \leqq m \leqq n-1),$$

then

$$|a_n| \leqq \frac{|1-\alpha+q(1-\alpha)|}{[n]_q + \gamma [n]_q [n-1]_q} \qquad (n \geqq 3). \tag{2.3}$$

Proof. For the function $f \in \mathcal{R}_q(\Sigma, \alpha, \gamma)$ of the form (1.1), we have

$$D_q f(z) + \gamma z D_q^2 f(z) = 1 + \sum_{n=2}^{\infty} \left([n]_q + \gamma [n]_q [n-1]_q \right) a_n z^{n-1} \tag{2.4}$$

and, for its inverse map $g = f^{-1}$, we get

$$D_q g(w) + \gamma w D_q^2 g(w) = 1 + \sum_{n=2}^{\infty} \left([n]_q + \gamma [n]_q [n-1]_q \right) b_n w^{n-1}, \tag{2.5}$$

where

$$b_n = \frac{1}{[n]_q} K_{n-1}^{-n}(a_2, a_3, \cdots, a_n).$$

Since both the function f and its inverse map $g = f^{-1}$ are in $\mathcal{R}_q(\Sigma, \alpha, \gamma)$, by the definition of subordination, there exist two Schwarz functions $p(z)$ and $q(w)$ given by

$$p(z) = \sum_{n=1}^{\infty} c_n z^n \qquad \text{and} \qquad q(w) = \sum_{n=1}^{\infty} d_n w^n \qquad (z, w \in \mathbb{U}),$$

so that we have

$$\begin{aligned} D_q f(z) + \gamma z D_q^2 f(z) &= \frac{1+\mathcal{L}p(z)}{1+\mathcal{M}p(z)} \\ &= 1 - \sum_{n=1}^{\infty} (\mathcal{L}-\mathcal{M}) K_n^{-1}(c_1, c_2, \cdots, c_n, \mathcal{M})\, z^n \end{aligned} \tag{2.6}$$

and

$$\begin{aligned} D_q g(w) + \gamma w D_q^2 g(w) &= \frac{1+\mathcal{L}q(w)}{1+\mathcal{M}q(w)} \\ &= 1 - \sum_{n=1}^{\infty} (\mathcal{L}-\mathcal{M}) K_n^{-1}(d_1, d_2, \cdots, d_n, \mathcal{M})\, w^n. \end{aligned} \tag{2.7}$$

In general, for any $\mathfrak{p} \in \mathbb{N}$ and $n \geqq 2$, we have the following expansion of $K_n^{\mathfrak{p}}(k_1, k_2, \cdots, k_n, \mathcal{M})$ (see [3] and [4]):

$$\begin{aligned} K_n^{\mathfrak{p}}&(k_1, k_{2,} \cdots, k_n, \mathcal{M}) \\ &= \frac{\mathfrak{p}!}{(\mathfrak{p}-n)!n!}\, k_1^n \mathcal{M}^{n-1} + \frac{\mathfrak{p}!}{(\mathfrak{p}-n+1)!(n-2)!}\, k_1^{n-2} k_2 \mathcal{M}^{n-2} \\ &\quad + \frac{\mathfrak{p}!}{(\mathfrak{p}-n+2)!(n-3)!} \cdot k_1^{n-3} k_3 \mathcal{M}^{n-3} \\ &\quad + \frac{\mathfrak{p}!}{(\mathfrak{p}-n+3)!(n-4)!}\, k_1^{n-4} \left[k_4 \mathcal{M}^{n-4} + \frac{\mathfrak{p}-n+3}{2}\, k_3^2 \mathcal{M} \right] \\ &\quad + \frac{\mathfrak{p}!}{(\mathfrak{p}-n+4)!(n-5)!}\, k_1^{n-5} \left[k_5 \mathcal{M}^{n-5} + (\mathfrak{p}-n+4) k_3 k_4 \mathcal{M} \right] \\ &\quad + \sum_{j \geqq 6} k_1^{n-1} X_j, \end{aligned} \tag{2.8}$$

where X_j is a homogeneous polynomial of degree j in the variables $k_1, k_2, \cdots, k_n$. For the coefficients of the Schwarz functions $p(z)$ and $q(w)$, we have (see [10])

$$|c_n| \leqq 1 \qquad \text{and} \qquad |d_n| \leqq 1.$$

Thus, upon comparing with the corresponding coefficients in (2.4) and (2.6), we find that

$$\left([n]_q + \gamma\, [n]_q\, [n-1]_q \right) a_n = -(\mathcal{L} - \mathcal{M}) K_{n-1}^{-1}(c_1, c_2, \cdots, c_{n-1}, \mathcal{M}). \tag{2.9}$$

Similarly, in view of the corresponding coefficients in (2.5) and (2.7), we have

$$\left([n]_q + \gamma\, [n]_q\, [n-1]_q \right) b_n = -(\mathcal{L} - \mathcal{M}) K_n^{-1}(d_1, d_2, \cdots, d_n, \mathcal{M}). \tag{2.10}$$

We note for

$$a_m = 0 \quad (2 \leqq m \leqq n-1) \qquad \text{and} \qquad b_n = -a_n,$$

that

$$\left([n]_q + \gamma\, [n]_q\, [n-1]_q \right) a_n = -(\mathcal{L} - \mathcal{M}) c_{n-1} \tag{2.11}$$

and

$$-\left([n]_q + \gamma\, [n]_q\, [n-1]_q \right) a_n = -(\mathcal{L} - \mathcal{M}) d_{n-1}. \tag{2.12}$$

Taking the moduli in (2.11) and (2.12), we thus obtain

$$\begin{aligned} |a_n| &\leqq \frac{|\mathcal{L} - \mathcal{M}|}{[n]_q + \gamma\, [n]_q\, [n-1]_q}\ |c_{n-1}| \\ &= \frac{|\mathcal{L} - \mathcal{M}|}{[n]_q + \gamma\, [n]_q\, [n-1]_q}\, |d_{n-1}|\,. \end{aligned}$$

Therefore, we have

$$|a_n| \leqq \frac{|1 - \alpha + q(1-\alpha)|}{[n]_q + \gamma\, [n]_q\, [n-1]_q} \qquad (n \geqq 3),$$

which completes the proof of the assertion (2.3) of Theorem 2.1. □

If we let $q \to 1-$ in Theorem 2.1 above, we obtain the following known result given by Srivastava *et al.* [26].

Corollary 2.2. (see [26]) Let f given by (1.1) be in the class

$$\mathcal{R}_{\Sigma}^{\alpha,\gamma} \quad (0 \leqq \alpha < 1;\ \gamma \geqq 0).$$

If

$$a_m = 0 \qquad (2 \leqq m \leqq n-1),$$

then

$$|a_n| \leqq \frac{2\,(1-\alpha)}{n\,[1+\gamma(n-1)]} \qquad (n \in \mathbb{N} \setminus \{1,2\}).$$

Theorem 2.3. *For* $0 \leqq \alpha < 1$ *and* $0 \leqq \gamma$, *let* $f \in \mathcal{R}_q(\Sigma, \alpha, \gamma)$. *Then*

$$|a_2| \leqq \min\left\{ \frac{|1-\alpha+q(1-\alpha)|}{[2]_q + \gamma\,[2]_q\,[1]_q},\ \sqrt{\frac{2(1+q)\,|1-\alpha+q(1-\alpha)|}{[2]_q\left([3]_q + \gamma\,[3]_q\,[2]_q\right)}} \right\},$$

$$|a_3| \leqq \min\left\{ \frac{|1-\alpha+q(1-\alpha)|}{[1]_q + [1]_q}\left(\frac{[2]_q\,|1-\alpha+q(1-\alpha)|}{\left([2]_q + \gamma\,[2]_q\,[1]_q\right)^2} + \frac{2}{[3]_q + \gamma\,[3]_q\,[2]_q} \right),\right.$$
$$\left. \frac{2\,(q+2)\,|1-\alpha+q(1-\alpha)|}{\left([1]_q + [1]_q\right)\left([3]_q + \gamma\,[3]_q\,[2]_q\right)} \right\},$$

$$\left| a_3 - [2]_q\,a_2^2 \right| \leqq \frac{(1+q)\,|1-\alpha+q(1-\alpha)|}{[3]_q + \gamma\,[3]_q\,[2]_q}$$

and

$$\left| a_3 - \frac{[2]_q}{[1]_q + [1]_q}a_2^2 \right| \leqq \frac{2\,|1-\alpha+q(1-\alpha)|}{\left|\left([1]_q + [1]_q\right)\left([3]_q + \gamma\,[3]_q\,[2]_q\right)\right|}.$$

Proof. Upon setting $n = 2$ and $n = 3$ in (2.9) and (2.10), respectively, we get

$$\left([2]_q + \gamma\,[2]_q\,[1]_q\right) a_2 = -(\mathcal{L}-\mathcal{M})c_1, \tag{2.13}$$

$$\left([3]_q + \gamma\,[3]_q\,[2]_q\right) a_3 = -(\mathcal{L}-\mathcal{M})(\mathcal{M}c_1^2 - c_2), \tag{2.14}$$

$$-\left([2]_q + \gamma\,[2]_q\,[1]_q\right) a_2 = -(\mathcal{L}-\mathcal{M})d_1 \tag{2.15}$$

and

$$\left([3]_q + \gamma\,[3]_q\,[2]_q\right)\left([2]_q\,a_2^2 - a_3\right) = -(\mathcal{L}-\mathcal{M})(\mathcal{M}d_1^2 - d_2). \tag{2.16}$$

From (2.13) and (2.15), we have

$$\begin{aligned} |a_2| &\leqq \frac{|\mathcal{L}-\mathcal{M}|}{[2]_q+\gamma\,[2]_q\,[1]_q}\,|c_1| \\ &= \frac{|\mathcal{L}-\mathcal{M}|}{[2]_q+\gamma\,[2]_q\,[1]_q}\,|d_1| \\ &\leqq \frac{|1-\alpha+q(1-\alpha)|}{[2]_q+\gamma\,[2]_q\,[1]_q}. \end{aligned} \tag{2.17}$$

Adding (2.14) and (2.16), we find that

$$[2]_q\left([3]_q+\gamma\,[3]_q\,[2]_q\right)a_2^2=-(\mathcal{L}-\mathcal{M})\left[\mathcal{M}\left(c_1^2+d_1^2\right)-(c_2+d_2)\right], \tag{2.18}$$

which, upon taking the moduli on both sides, yields

$$|a_2|^2=\frac{2\,|\mathcal{L}-\mathcal{M}|\,(|\mathcal{M}|+1)}{[2]_q\left([3]_q+\gamma\,[3]_q\,[2]_q\right)}.$$

This last equation can be written as follows:

$$|a_2| \leqq \sqrt{\frac{2(1+q)\,|1-\alpha+q(1-\alpha)|}{[2]_q\left([3]_q+\gamma\,[3]_q\,[2]_q\right)}}. \tag{2.19}$$

Now, in order to find $|a_3|$, by subtracting (2.16) from (2.14), we obtain

$$a_3=\frac{(\mathcal{L}-\mathcal{M})\left[\mathcal{M}\left(d_1^2-c_1^2\right)-(c_2-d_2)\right]}{([1]_q+[1]_q)\left([3]_q+\gamma\,[3]_q\,[2]_q\right)}+\frac{[2]_q}{([1]_q+[1]_q)}\,a_2^2. \tag{2.20}$$

Taking the moduli in (2.20) and using the fact that $d_1^2=c_1^2$, we have

$$|a_3| \leqq \frac{2\,|\mathcal{L}-\mathcal{M}|}{([1]_q+[1]_q)\left([3]_q+\gamma\,[3]_q\,[2]_q\right)}+\frac{[2]_q}{[1]_q+[1]_q}\;|a_2|^2. \tag{2.21}$$

Using (2.17) in (2.21), we obtain

$$\begin{aligned} |a_3| \leqq{}& \frac{|1-\alpha+q(1-\alpha)|}{[1]_q+[1]_q} \\ &\cdot\left(\frac{[2]_q\,|1-\alpha+q(1-\alpha)|}{\left([2]_q+\gamma\,[2]_q\,[1]_q\right)^2}+\frac{2}{[3]_q+\gamma\,[3]_q\,[2]_q}\right). \end{aligned} \tag{2.22}$$

Again, by using the equation (2.19) in (2.21), we have

$$|a_3| \leqq \frac{2\,(q+2)\,|1-\alpha+q(1-\alpha)|}{([1]_q+[1]_q)\left([3]_q+\gamma\,[3]_q\,[2]_q\right)}. \tag{2.23}$$

We also find from (2.16) that

$$\left|a_3-[2]_q\,a_2^2\right| \leqq \frac{(1+q)\,|1-\alpha+q(1-\alpha)|}{[3]_q+\gamma\,[3]_q\,[2]_q}.$$

From (2.20) and using the fact that $d_1^2 = c_1^2$, we have

$$a_3 - \frac{[2]_q}{[1]_q + [1]_q} a_2^2 = \frac{(\mathcal{L} - \mathcal{M})(c_2 - d_2)}{([1]_q + [1]_q)\left([3]_q + \gamma [3]_q [2]_q\right)}. \tag{2.24}$$

Finally, by taking the moduli in (2.24), we finally obtain

$$\left| a_3 - \frac{[2]_q}{[1]_q + [1]_q} a_2^2 \right| \leqq \frac{2\,|1 - \alpha + q(1-\alpha)|}{\left|([1]_q + [1]_q)\left([3]_q + \gamma [3]_q [2]_q\right)\right|}.$$

The proof of Theorem 2.3 is thus completed. □

In the limit case when $q \to 1-$, Theorem 2.3 yields the following bounds on $|a_2|$ and $|a_3|$ given by Srivastava *et al.* [26].

Corollary 2.4. (see [26]) Let f given by (1.1) be in the class

$$\mathcal{R}_\Sigma^{\alpha,\gamma} \quad (0 \leqq \alpha < 1;\ \gamma \geqq 0).$$

Then

$$a_2 \leqq \begin{cases} \sqrt{\dfrac{2(1-\alpha)}{3(1+2\gamma)}} & \left(0 \leqq \alpha \leqq \dfrac{1+2\gamma-2\gamma^2}{3(1+2\gamma)}\right) \\[2ex] \dfrac{1-\alpha}{1+\gamma} & \left(\dfrac{1+2\gamma-2\gamma^2}{3(1+2\gamma)} \leqq \alpha < 1\right) \end{cases}$$

and

$$a_3 \leqq \frac{2(1-\alpha)}{3(1+2\gamma)}.$$

Theorem 2.5. *For $0 \leqq \alpha < 1$ and $0 \leqq \gamma$, let $f \in \mathcal{R}_q(\Sigma, \alpha, \gamma, \lambda)$. If*

$$a_m = 0 \qquad (2 \leqq m \leqq n-1),$$

then

$$|a_n| \leqq \frac{|1 - \alpha + q(1-\alpha)|\,[\lambda+1]_{q,n-1}}{\left([n]_q + \gamma [n]_q [n-1]_q\right)[n]_q!} \qquad (n \geqq 3). \tag{2.25}$$

Proof. For the function $f \in \mathcal{R}_q(\Sigma, \alpha, \gamma, \lambda)$ of the form (1.1), we have

$$D_q \mathcal{I}_q^\lambda f(z) + \gamma z D_q^2 \mathcal{I}_q^\lambda f(z) = 1 + \sum_{n=2}^{\infty} \left([n]_q + \gamma [n]_q [n-1]_q\right) \Psi_{n-1} a_n z^{n-1}. \tag{2.26}$$

Also, for its inverse mapping $g = f^{-1}$, we have

$$D_q \mathcal{I}_q^\lambda g(w) + \gamma w D_q^2 \mathcal{I}_q^\lambda g(w) = 1 + \sum_{n=2}^{\infty} \left([n]_q + \gamma [n]_q [n-1]_q\right) \Psi_{n-1} b_n w^{n-1}, \tag{2.27}$$

where

$$b_n = \frac{1}{[n]_q} K_{n-1}^{-n}\left(a_2, a_3, \cdots, a_n\right).$$

Since, both f and its inverse $g = f^{-!}$ are in the function class $\mathcal{R}_q\left(\Sigma, \alpha, \gamma, \lambda\right)$, by the definition of subordination, there exist two Schwarz functions $p(z)$ and $q(w)$ given by

$$p(z) = \sum_{n=1}^{\infty} c_n z^n \qquad \text{and} \qquad q(w) = \sum_{n=1}^{\infty} d_n w^n \qquad (z, w \in \mathbb{U}),$$

so that we have

$$\begin{aligned} D_q \mathcal{I}_q^{\lambda} f(z) &+ \gamma z D_q^2 \mathcal{I}_q^{\lambda} f(z) \\ &= \frac{1 + \mathcal{L}p(z)}{1 + \mathcal{M}p(z)} \\ &= 1 - \sum_{n=1}^{\infty} (\mathcal{L} - \mathcal{M}) K_n^{-1}\left(c_1, c_2, \cdots, c_n, \mathcal{M}\right) z^n \end{aligned} \tag{2.28}$$

and

$$\begin{aligned} D_q \mathcal{I}_q^{\lambda} g(w) &+ \gamma w D_q^2 \mathcal{I}_q^{\lambda} g(w) \\ &= \frac{1 + \mathcal{L}q(w)}{1 + \mathcal{M}q(w)} \\ &= 1 - \sum_{n=1}^{\infty} (\mathcal{L} - \mathcal{M}) K_n^{-1}\left(d_1, d_2, \cdots, d_n, \mathcal{M}\right) w^n. \end{aligned} \tag{2.29}$$

In general, for any $\mathfrak{p} \in \mathbb{N}$ and $n \geqq 2$, an expansion of

$$K_n^{\mathfrak{p}}\left(k_1, k_2, \cdots, k_n, \mathcal{M}\right)$$

is given by (2.8) (see [3] and [4]). Moreover, the coefficients of the Schwarz functions $p(z)$ and $q(w)$ are constrained by (see [10])

$$|c_n| \leqq 1 \qquad \text{and} \qquad |d_n| \leqq 1.$$

Thus, upon comparing the corresponding coefficients in (2.26) and (2.28), we find that

$$\begin{aligned} \Big([n]_q &+ \gamma [n]_q [n-1]_q\Big) \Psi_{n-1} a_n \\ &= -(\mathcal{L} - \mathcal{M}) K_{n-1}^{-1}\left(c_1, c_2, \cdots, c_{n-1}, \mathcal{M}\right). \end{aligned} \tag{2.30}$$

Similarly, by comparing the corresponding coefficients in (2.27) and (2.29), we have

$$\begin{aligned} \Big([n]_q &+ \gamma [n]_q [n-1]_q\Big) \Psi_{n-1} b_n \\ &= -(\mathcal{L} - \mathcal{M}) K_n^{-1}\left(d_1, d_2, \cdots, d_n, \mathcal{M}\right). \end{aligned} \tag{2.31}$$

We note also that, for

$$a_m = 0 \quad (2 \leqq m \leqq n-1) \qquad \text{and} \qquad b_n = -a_n,$$

we have

$$\Big([n]_q + \gamma [n]_q [n-1]_q\Big) \Psi_{n-1} a_n = -(\mathcal{L} - \mathcal{M}) c_{n-1} \tag{2.32}$$

and

$$-\left([n]_q+\gamma\,[n]_q\,[n-1]_q\right)\Psi_{n-1}a_n=-(\mathcal{L}-\mathcal{M})d_{n-1}. \tag{2.33}$$

Finally, by taking the moduli in (2.32) and (2.33), we obtain

$$\begin{aligned}|a_n| &\leqq \frac{|\mathcal{L}-\mathcal{M}|}{\left([n]_q+\gamma\,[n]_q\,[n-1]_q\right)\Psi_{n-1}}\,|c_{n-1}| \\ &= \frac{|\mathcal{L}-\mathcal{M}|}{\left([n]_q+\gamma\,[n]_q\,[n-1]_q\right)\Psi_{n-1}}\,|d_{n-1}|\,.\end{aligned}$$

Consequently, we have

$$|a_n| \leqq \frac{|1-\alpha+q(1-\alpha)|\,[\lambda+1]_{q,n-1}}{\left([n]_q+\gamma\,[n]_q\,[n-1]_q\right)[n]_q!} \qquad (n\geqq 3),$$

which completes the proof of the assertion (2.25) of Theorem 2.5. □

Theorem 2.6. *For* $0 \leqq \alpha < 1$ *and* $\gamma \geqq 0$, *let* $f \in \mathcal{R}_q(\Sigma,\alpha,\gamma,\lambda)$. *Then*

$$|a_2| \leqq \min\left\{\frac{|1-\alpha+q(1-\alpha)|\,[\lambda+1]_{q,1}}{\left([2]_q+\gamma\,[2]_q\,[1]_q\right)[2]_q!},\ \sqrt{\frac{2(1+q)\,|1-\alpha+q(1-\alpha)|\,[\lambda+1]_{q,2}}{[2]_q\left([3]_q+\gamma\,[3]_q\,[2]_q\right)[3]_q!}}\right\}, \tag{2.34}$$

$$|a_3| \leqq \min\left\{\frac{|1-\alpha+q(1-\alpha)|}{[1]_q+[1]_q}\left(\frac{\left([\lambda+1]_{q,1}\right)^2[2]_q\,|1-\alpha+q(1-\alpha)|}{([2]_q!)^2\left([2]_q+\gamma\,[2]_q\,[1]_q\right)^2}+\frac{2[\lambda+1]_{q,2}}{\left([3]_q+\gamma\,[3]_q\,[2]_q\right)[3]_q!}\right),\ \frac{2\,(q+2)\,|1-\alpha+q(1-\alpha)|\,[\lambda+1]_{q,2}}{([1]_q+[1]_q)\left([3]_q+\gamma\,[3]_q\,[2]_q\right)[3]_q!}\right\}, \tag{2.35}$$

$$\left|a_3-[2]_q\,a_2^2\right| \leqq \frac{(1+q)\,|1-\alpha+q(1-\alpha)|\,[\lambda+1]_{q,2}}{\left([3]_q+\gamma\,[3]_q\,[2]_q\right)[3]_q!} \tag{2.36}$$

and

$$\left|a_3-\left(\frac{[2]_q}{[1]_q+[1]_q}\right)a_2^2\right| \leqq \frac{2\,|1-\alpha+q(1-\alpha)|\,[\lambda+1]_{q,2}}{\left|([1]_q+[1]_q)\left([3]_q+\gamma\,[3]_q\,[2]_q\right)\right|[3]_q!}. \tag{2.37}$$

Proof. Upon setting $n = 2$ and $n = 3$ in (2.30) and (2.31), respectively, we have

$$\left([2]_q + \gamma [2]_q [1]_q\right) \Psi_1 a_2 = -(\mathcal{L} - \mathcal{M})c_1, \tag{2.38}$$

$$\left([3]_q + \gamma [3]_q [2]_q\right) \Psi_2 a_3 = -(\mathcal{L} - \mathcal{M})(\mathcal{M}c_1^2 - c_2), \tag{2.39}$$

$$-\left([2]_q + \gamma [2]_q [1]_q\right) \Psi_1 a_2 = -(\mathcal{L} - \mathcal{M})d_1 \tag{2.40}$$

and

$$\left([3]_q + \gamma [3]_q [2]_q\right) \Psi_2 \left([2]_q a_2^2 - a_3\right) = -(\mathcal{L} - \mathcal{M})(\mathcal{M}d_1^2 - d_2). \tag{2.41}$$

Making use of (2.38) and (2.40), we find that

$$\begin{aligned} |a_2| &\leqq \frac{|\mathcal{L} - \mathcal{M}|}{\left([2]_q + \gamma [2]_q [1]_q\right) \Psi_1} |c_1| \\ &= \frac{|\mathcal{L} - \mathcal{M}|}{\left([2]_q + \gamma [2]_q [1]_q\right) \Psi_1} |d_1| \\ &\leqq \frac{|1 - \alpha + q(1 - \alpha)| \, [\lambda + 1]_{q,1}}{\left([2]_q + \gamma [2]_q [1]_q\right) [2]_q!}. \end{aligned} \tag{2.42}$$

Also, by adding (2.39) and (2.41), we have

$$[2]_q \left([3]_q + \gamma [3]_q [2]_q\right) \Psi_2 a_2^2 = -(\mathcal{L} - \mathcal{M}) \left[\mathcal{M}\left(c_1^2 + d_1^2\right) - (c_2 + d_2)\right]. \tag{2.43}$$

Now, if we take the moduli in both sides of (2.43), we obtain

$$|a_2|^2 = \frac{2\,|\mathcal{L} - \mathcal{M}|\,(|\mathcal{M}| + 1)}{[2]_q \left([3]_q + \gamma [3]_q [2]_q\right) \Psi_2},$$

so that

$$|a_2| \leqq \sqrt{\frac{2(1+q)\,|1 - \alpha + q(1 - \alpha)|\,[\lambda + 1]_{q,2}}{[2]_q \left([3]_q + \gamma [3]_q [2]_q\right) [3]_q!}}. \tag{2.44}$$

In order to find $|a_3|$, we subtract (2.41) from (2.39), We thus obtain

$$a_3 = \frac{(\mathcal{L} - \mathcal{M}) \left[\mathcal{M}\left(d_1^2 - c_1^2\right) - (c_2 - d_2)\right]}{([1]_q + [1]_q) \left([3]_q + \gamma [3]_q [2]_q\right) \Psi_2} + \left(\frac{[2]_q}{([1]_q + [1]_q)}\right) a_2^2, \tag{2.45}$$

which, after taking the moduli and using the fact that

$$d_1^2 = c_1^2,$$

yields

$$|a_3| \leqq \frac{2\,|\mathcal{L} - \mathcal{M}|}{([1]_q + [1]_q) \left([3]_q + \gamma [3]_q [2]_q\right) \Psi_2} + \left(\frac{[2]_q}{[1]_q + [1]_q}\right) |a_2|^2. \tag{2.46}$$

Using (2.42) in (2.46), we have

$$|a_3| \leqq \frac{|1-\alpha+q(1-\alpha)|}{[1]_q+[1]_q}\left(\frac{\left([\lambda+1]_{q,1}\right)^2[2]_q\,|1-\alpha+q(1-\alpha)|}{([2]_q!)^2\left([2]_q+\gamma\,[2]_q\,[1]_q\right)^2} + \frac{2[\lambda+1]_{q,2}}{\left([3]_q+\gamma\,[3]_q\,[2]_q\right)[3]_q!}\right). \tag{2.47}$$

Again, by using (2.44) in (2.46), we get

$$|a_3| \leqq \frac{2\,(q+2)\,|1-\alpha+q(1-\alpha)|\,[\lambda+1]_{q,2}}{([1]_q+[1]_q)\left([3]_q+\gamma\,[3]_q\,[2]_q\right)[3]_q!}.$$

It follows from (2.41) that

$$\left|a_3-[2]_q\,a_2^2\right| \leqq \frac{(1+q)\,|1-\alpha+q(1-\alpha)|\,[\lambda+1]_{q,2}}{\left([3]_q+\gamma\,[3]_q\,[2]_q\right)[3]_q!}.$$

Using the fact that

$$d_1^2=c_1^2$$

in (2.45), we have

$$a_3-\left(\frac{[2]_q}{[1]_q+[1]_q}\right)a_2^2=\frac{(\mathcal{L}-\mathcal{M})\,(c_2-d_2)}{([1]_q+[1]_q)\left([3]_q+\gamma\,[3]_q\,[2]_q\right)\Psi_2}. \tag{2.48}$$

By taking the moduli on both sides of (2.48), we finally obtain

$$\left|a_3-\left(\frac{[2]_q}{([1]_q+[1]_q)}\right)a_2^2\right| \leqq \frac{2\,|1-\alpha+q(1-\alpha)|\,[\lambda+1]_{q,2}}{\left|([1]_q+[1]_q)\left([3]_q+\gamma\,[3]_q\,[2]_q\right)\right|[3]_q!},$$

which completes the proof of Theorem 2.6. □

3. Concluding remarks and observations

Here, in our present investigation, we have successfully applied the Faber polynomial expansion method as well as the q-analysis in our study of several new subclasses of analytic and bi-univalent functions by using a certain q-integral operator in the open unit disk $\mathbb{U}$. We have derived bounds for the nth coefficient in the Taylor-Maclaurin series expansion for functions in each of these newly-defined analytic and bi-univalent function classes subject to a gap series condition. By means of corollaries of our main theorems, we have also highlighted some known consequences of our main results, which were given recently by Srivastava *et al.* [26].

References

[1] Agrawal, S., Sahoo, S.K., *A generalization of starlike functions of order α*, Hokkaido Math. J., **46**(2017), 15–27.

[2] Airault, H., *Symmetric sums associated to the factorization of Grunsky coefficients*, in "Proceedings of the Conference on Groups and Symmetries", Montréal, Canada, 2007.

[3] Airault, H., *Remarks on Faber polynomials*, Internat. Math. Forum, **3**(2008), 449–456.

[4] Airault, H., Bouali, A., *Differential calculus on the Faber polynomials*, Bull. Sci. Math., **130**(2006), 179–222.

[5] Altınkaya, Ş., Yalçın, S., *Faber polynomial coefficient bounds for a subclass of bi-univalent functions*, C.R. Acad. Sci. Paris Sér. I, **353**(2015), 1075–1080.

[6] Altınkaya, Ş., Yalçın, S., *Faber polynomial coefficient bounds for a subclass of bi-univalent functions*, Stud. Univ. Babeş-Bolyai Math., **61**(2016), 37–44.

[7] Bulut, S., *Faber polynomial coefficient estimates for a comprehensive subclass of analytic bi-univalent functions*, C. R. Acad. Sci. Paris Sér. I, **352**(2014), 479–484.

[8] Bulut, S., *Faber polynomial coefficient estimates for certain subclasses of meromorphic bi-univalent functions*, C.R. Acad. Sci. Paris Sér. I, **353**(2015), 113–116.

[9] Çağlar, M., Deniz, E., Srivastava, H.M., *Second Hankel determinant for certain subclasses of bi-univalent functions*, Turkish J. Math., **41**(2017), 694–706.

[10] Duren, P.L., *Univalent Functions*, Grundlehren der Mathematischen Wissenschaften, Band **259**, Springer-Verlag, New York, Berlin, Heidelberg and Tokyo, 1983.

[11] Faber, G., *Über polynomische Entwicklungen*, Math. Ann., **57**(1903), 389–408.

[12] Hamidi, S.G., Jahangiri, J.M., *Faber polynomial coefficient estimates for analytic bi-close-to-convex functions*, C.R. Acad. Sci. Paris Sér. I, **352**(2014), 17–20.

[13] Hamidi, S.G., Jahangiri, J.M., *Faber polynomial coefficients of bi-subordinate functions*, C.R. Acad. Sci. Paris Sér. I, **354**(2016), 365–370.

[14] Ismail, M.E.H., Merkes, E., Styer, D., *A generalization of starlike functions*, Complex Variables Theory Appl., **14**(1990), 77–84.

[15] Jackson, F.H., *On q-definite integrals*, Quart. J. Pure Appl. Math., **41**(1910), 193–203.

[16] Jackson, F.H., *q-difference equations*, Amer. J. Math., **32**(1910), 305–314.

[17] Kanas, S., Răducanu, D., *Some class of analytic functions related to conic domains*, Math. Slovaca, **64**(2014), 1183–1196.

[18] Noor, K. I., *On new classes of integral operators*, J. Natur. Geom., **16**(1999), 71–80.

[19] Noor, K.I., Noor, M.A., *On certain classes of analytic functions defined by Noor integral operator*, J. Math. Anal. Appl., **281**(2003), 244–252.

[20] Sharma, P., *Faber polynomial coefficient estimates for a class of analytic bi-univalent functions involving a certain differential operator*, Asian-European J. Math., **10**(2017), Article ID 1750016, 1–11.

[21] Srivastava, H.M., *Univalent functions, fractional calculus, and associated generalized hypergeometric functions*, in "Univalent Functions, Fractional Calculus, and Their Applications" (Srivastava, H.M. and Owa, S., Editors), Halsted Press (Ellis Horwood Limited, Chichester), pp. 329–354, John Wiley and Sons, New York, Chichester, Brisbane and Toronto, 1989.

[22] Srivastava, H.M., Mishra, A.K., Gochhayat, P., *Certain subclasses of analytic and bi-univalent functions*, Appl. Math. Lett., **23**(2010), 1188–1192.

[23] Srivastava, H.M., Bansal, D., *Close-to-convexity of a certain family of q-Mittag-Leffler functions*, J. Nonlinear Var. Anal., **1**(2017), 61–69.

[24] Srivastava, H.M., Bulut, S., Çağlar, M., Yağmur, N., *Coefficient estimates for a general subclass of analytic and bi-univalent functions*, Filomat, **27**(2013), 831–842.

[25] Srivastava, H.M., Gaboury, S., Ghanim, F., *Coefficient estimates for some general subclasses of analytic and bi-univalent functions*, Afrika Mat., **28**(2017), 693–706.

[26] Srivastava, H.M., Sümer Eker, S., Ali, R.M., *Coefficient bounds for a certain class of analytic and bi-univalent functions*, Filomat, **29**(2015), 1839–1845.

[27] Todorov, P.G., *On the Faber polynomials of the univalent functions of class* Σ, J. Math. Anal. Appl., **162**(1991), 268–276.

[28] Xu, Q.-H., Gui, Y.-C., Srivastava, H.M., *Coefficient estimates for a certain subclass of analytic and bi-univalent functions*, Appl. Math. Lett., **25**(2012), 990–994.

[29] Xu, Q.-H., Xiao, H.-G., Srivastava, H.M., *A certain general subclass of analytic and bi-univalent functions and associated coefficient estimate problems*, Appl. Math. Comput., **218**(2012), 11461–11465.

[30] Uçar, H.E.Ö., *Coefficient inequality for q-starlike functions*, Appl. Math. Comput., **276**(2016), 122–126.

Hari Mohan Srivastava
Department of Mathematics and Statistics
University of Victoria
Victoria, British Columbia V8W 3R4, Canada
and
Department of Medical Research
China Medical University Hospital
China Medical University, Taichung 40402
Taiwan, Republic of China
e-mail: harimsri@math.uvic.ca

Shahid Khan
Department of Mathematics
Riphah International University
Islamabad, Pakistan
e-mail: shahidmath761@gmail.com

Qazi Zahoor Ahmad
Department of Mathematics
Abbottabad University of Science and Technology
Abbottabad, Pakistan
e-mail: zahoorqazi5@gmail.com

Nazar Khan
Department of Mathematics
Abbottabad University of Science and Technology
Abbottabad, Pakistan
e-mail: nazarmaths@gmail.com

Saqib Hussain
Department of Mathematics
COMSATS Institute of Information Technology
Abbottabad, Pakistan
e-mail: saqibmath@yahoo.com

Stud. Univ. Babeş-Bolyai Math. 63(2018), No. 4, 437–445
DOI: 10.24193/subbmath.2018.4.02

Goldie absolute direct summand rings and modules

Truong Cong Quynh and Serap Şahinkaya

Abstract. In the present paper, we introduce and study Goldie ADS modules and rings, which subsume two generalizations of Goldie extending modules due to Akalan et al. [3] and ADS-modules due to Alahmadi et al. [7]. A module M will be called a Goldie ADS module if for every decomposition $M = S \oplus T$ of M and every complement T' of S, there exists a submodule D of M such that $T' \beta D$ and $M = S \oplus D$. Various properties concerning direct sums of Goldie ADS modules are established.

Mathematics Subject Classification (2010): 16D40, 16E50, 16N20.

Keywords: Goldie extending modules, ADS modules, CS modules.

1. Introduction

The purpose of the present paper is to introduce and study Goldie ADS modules, which allow us to give a unified approach of Goldie extending modules and ADS-modules, introduced by E. Akalan et al. [3] and A. Alahmadi et al. [7], respectively. We define a Goldie ADS module by the property that for every decomposition $M = S \oplus T$ of M and every complement T' of S, there exists a submodule D of M such that $T' \beta D$ and $M = S \oplus D$. We study these modules, generalizing several results both on Goldie extending modules and ADS-modules. We show that a non-singular Goldie ADS module is an ADS module. We emphasize that our properties are of the same type as those for Goldie extending modules and ADS-modules, sharing similar limitations in studying certain properties, such as the closure of the respective class of modules under direct sums. We also analyze when a direct summand of Goldie ADS modules is a Goldie ADS and also when a direct sum of Goldie ADS module is Goldie ADS, by using the concepts of relative ejectivity. In the last section, we look at Goldie ADS property of some ring extensions.

2. Definitions and notions

In this paper, R will present an associative ring with identity and all modules over R are unitary right modules. We also write M_R to indicate that M is a right R-module. We shall denote the fact that a submodule N is essential in a module M by $N \leq_e M$. The following generalization of relative injectivity is introduced in [3, Definition 2.1]. Let N and M be modules. N is called *M-ejective* if, for each $K \leq M$ and each homomorphism $f : K \to N$, there exist a homomorphism $\overline{f} : M \to N$ and a $X \leq_e K$ such that $\overline{f}(x) = f(x)$, for all $x \in X$. M and N is called *mutually ejective* if M is N-ejective and N is M-ejective. A submodule K of M is called *fully invariant* if $f(K) \subseteq K$ for every $f \in End_R(M)$. Clearly 0 and M are fully invariant submodules of M. The right R-module M is called a *duo module* provided every submodule of M is fully invariant. The singular submodule of a module M will be denoted by $Z(M) = \{m \in M : mI = 0 \text{ for some } I \leq_e R_R\}$. A module M is called *singular* (respectively *non-singular*) if $Z(M) = M$ (respectively $Z(M) = 0$).

(CS): Every complement submodule of M is a direct summand of M.

(C2): Every submodule of M that is isomorphic to a direct summand of M is itself a direct summand of M.

(C3): For any two direct summands A and B of M with $A \cap B = 0$, the sum $A + B$ is a direct summand of M.

A module M is called is called *continuous* (respectively, *quasi continuous*) if M satisfies (CS) and (C2) (respectively, (CS) and (C3)).

Let M be an R-module and $X, Y \leq M$. In [3], the notion of β relation on submodules X, Y of M, denoted by $X\beta Y$, is defined such as $X\beta Y$ if and only if $X \cap A = 0$ implies $Y \cap A = 0$ and $Y \cap B = 0$ implies $X \cap B = 0$ for all $A, B \leq M$. A right module M is *Goldie extending* if for each $X \leq M$, there exists a direct summand D of M such that $X\beta D$. M is Goldie extending if and only if for each closed submodule C of M there is a direct summand D of M such that $C\beta D$.

Another notion generalizing extending property, ADS (Absolute Direct Summand) modules, was recently considered in [7]. It was introduced by Fuchs [10] for abelian groups and for general modules by Alahmadi, Jain and Leroy [7]. As the authors pointed out in [7], if R is commutative then every cyclic R-module is ADS. Also every right quasi-continuous module is ADS, but the converse is not true. However, a right ADS module which is also CS is quasi-continuous. Also in [14], Quynh and Koşan proved that every ADS module satisfies (C3). Hence, this is a class of modules between quasi-continuous modules and modules satisfying the (C3) condition. Quynh and Koşan gave also different characterizations of ADS modules and showed how to characterize semisimple modules and semisimple artinian rings using the ADS. The SC and SI rings were also characterized by the ADS notion in [14].

3. Goldie absolute direct summand modules

A module M is called *Goldie absolute direct summand* (Goldie ADS) if, for every decomposition $M = S \oplus T$ of M and every complement T' of S, there exists a submodule D of M such that $T'\beta D$ and $M = S \oplus D$.

A ring R right Goldie ADS if the (right) R-module R is Goldie ADS. We know that extending modules are Goldie extending, but need not be ADS. Similarly, Goldie extending modules do not necessarily satisfy Goldie ADS. Hence, the notions extending, Goldie extending, ADS and the property Goldie ADS are not directly related.

Example 3.1. Let $R = \mathbb{Z}_2[x_1, x_2, ...]$, where x_i are commuting indeterminants satisfying the relations: $x_i^3 = 0$ for all i, $x_i x_j = 0$ for all $i \neq j$, and $x_i^2 = x_j^2$ for all i and j. Then R is a commutative, semiprimary ring with simple essential socle. But R is not a self-injective ring (see [13, Example 5.45]). On the other hand, R_R is soc-R_R-injective by [8, Example 5.7]. It follows that R_R is soc-$(R_R \oplus R_R)$-injective by [8, Theorem 2.2(4)]. We have $Soc(R_R \oplus R_R) = Soc(R_R) \oplus Soc(R_R)$ is finitely generated. Therefore $R_R \oplus R_R$ is soc-$(R_R \oplus R_R)$-injective by [8, Theorem 2.10]. Since $Soc(R_R \oplus R_R)$ is essential in $R_R \oplus R_R$, then $R_R \oplus R_R$ is self-ejective by [3, Corollary 2.5(iii)]. It follows that for each decomposition $R_R \oplus R_R = A \oplus B$, A and B are mutually ejective by Lemma 3.11(2). It shows that $R_R \oplus R_R$ is Goldie ADS by Lemma 3.6. On the other hand, $R_R \oplus R_R$ is not ADS. Indeed, if $R_R \oplus R_R$ is ADS, then R must be self-injective, a contradiction.

Example 3.2. Let R be a triangle matrix ring over a field K. Then R_R is CS. Note that R_R is non-singular. Since R_R is not a C3-module, R_R is not ADS. It follows that R_R is not Goldie ADS by Corollary 3.7.

Example 3.3 ([14, Example 2.10]**).** Let K be a field and let $R = K[x, y]/\langle x^2, xy, y^2\rangle$. Assume that S is any simple injective R-module. Let $M = R \oplus S$. Then M is not a CS-module (since R is indecomposable and not uniform). On the other hand, R, S are relatively injective, and any two decompositions of M are isomorphic (since R and $End(S)$ are local rings). Hence M is an ADS module.

Let us mention the following equivalent conditions for Goldie ADS modules.

Lemma 3.4. *The following conditions are equivalent for a module M.*

1. *M is Goldie ADS.*
2. *For every decomposition $M = S \oplus T$ of M and every complement T' of S, there exists a submodule D of M and X of M such that $X \leq_e T'$, $X \leq_e D$ and $M = S \oplus D$.*

Proof. (1) $\Rightarrow$ (2). Assume that M is Goldie ADS and M has a decomposition $M = S \oplus T$. Let T' be a complement of S. Then there exists a submodule D of M such that $T' \beta D$ and $M = S \oplus D$. Let $X = T' \cap D$. Hence $X \leq_e T'$, $X \leq_e D$.
(2) $\Rightarrow$ (1) is obvious. □

It is well-known that in general the class of extending modules is not closed under direct sums, and this behavior is also carried on by Goldie extending modules and ADS-modules. Finding necessary and sufficient conditions for ensuring the closure of such classes under direct sums has been one of the most important open problems in the theory of extending modules and their generalizations. In the next parts of our work, we shall deal with such a problem for Goldie ADS modules.

In order to obtain when a direct sum of two Goldie ADS modules has the same property, the following concept generalizing relative injectivity will be useful.

Lemma 3.5 ([3, Theorem 2.7]). *Let M_1 and M_2 be modules such that $M = M_1 \oplus M_2$. Then M_1 is M_2-ejective if and only if for every $K \leq M$ such that $K \cap M_1 = 0$, there exists $M_3 \leq M$ such that $M = M_1 \oplus M_3$ and $K \cap M_3 \leq_e K$.*

In [7, Lemma 3.1], it is shown that an R-module M is ADS if and only if for each decomposition $M = A \oplus B$, A and B are mutually injective.

Lemma 3.6. *An R-module M is Goldie ADS if and only if for each decomposition $M = A \oplus B$, A and B are mutually ejective.*

Proof. Suppose $M = A \oplus B$ is Goldie ADS. We will show that A is B-ejective. Let K be a submodule of M such that $K \cap A = 0$. So K is contained in a complement, say C, of A. Then, by hypothesis, there exists $D \leq M$ such that $C\beta D$ and $M = A \oplus D$. It is easy to see that $K \cap D \leq_e K$. Thus, we have A is B-ejective by Lemma 3.5.

Conversely, suppose for each decomposition $M = A \oplus B$, A and B are mutually ejective. Let C be a complement of A. By Lemma 3.5, there exists and $D \leq M$ such that $M = A \oplus D$ and $C \cap D \leq_e C$. So, $A \oplus (C \cap D) \leq^e M$. It follows that $C \cap D \leq_e D$. So we are done by Lemma 3.4. □

Let M_1 and M_2 be modules with $Z(M_1) = 0$ and $M = M_1 \oplus M_2$. In [3, Corollary 2.8], it is shown that M_1 is M_2-injective if and only if M_1 is M_2-ejective. As the authors pointed out in [3], if $Z(M) = 0$ and M is R-ejective, then M is injective (because of the Baer criterion).

Corollary 3.7. *A non-singular Goldie ADS module is ADS.*

Proof. By Lemma 3.6 and [3, Corollary 2.8]. □

We collect, in the following theorem, some fundamental properties of Goldie ADS modules.

Theorem 3.8. *Assume that M is Goldie ADS. Then the following statements hold.*

1. *Every direct summand of M is Goldie ADS.*
2. *M satisfies (C3) condition on fully invariant summands.*
3. *For any decomposition $M = A \oplus B$ and any $b \in B$, A is bR-ejective.*

Proof. (1) Assume that A is a direct summand of M, i.e., $M = A \oplus B$ for some $B \leq M$. Let $A = A_1 \oplus A_2$ and K be a complement of A_1 in A. Then we have $M = A_1 \oplus (A_2 \oplus B)$. First we show that $K \oplus B$ is a complement of A_1 in M. Let $C \leq M$ such that $K \oplus B \leq C$ and $C \cap A_1 = 0$. Then $K \leq C \cap A$ and $(C \cap A) \cap A_1 = 0$. Since K is a complement of A_1 in A, we can obtain that $K = C \cap A$. It follows that

$$K \oplus B = (C \cap A) \oplus B = C \cap (A \oplus B) = C.$$

Since M is Goldie ADS, there exists a submodule D of M such that $(K \oplus B)\beta D$ and $M = A_1 \oplus D$. Hence $A = A_1 \oplus (D \cap A)$. It is easy to see that $K\beta(D \cap A)$.

(2) Let A and B be fully invariant direct summands of M such that $A \cap B = 0$. We shall show that $A \oplus B$ is a direct summand of M. Write $M = A \oplus A'$ and $M = B \oplus B'$

for some submodules A', B' of M. By Lemma 3.6, A is A'-ejective. Hence there exists $M' \leq M$ such that $M = M' \oplus A$ and $B \cap M' \leq_e B$ by Lemma 3.5. Inasmuch as B is a fully invariant submodule of M, $B = (M' \cap B) \oplus (A \cap B) = M' \cap B$. It follows that $B \leq M'$. Then $M' = B \oplus (M' \cap B')$ and so $M = A \oplus B \oplus (M' \cap B')$.

(3) Suppose M has a decomposition $M = A \oplus B$. By Lemma 3.6, the module A is B-ejective. Let $K = A \oplus bR$ and X be a submodule of K such that $X \cap A = 0$. Since A is B-ejective, there exists $C \leq M$ such that $M = A \oplus C$ and $X \cap C \leq_e X$ by Lemma 3.5. Note that $K = A \oplus (C \cap K)$. It follows that $(C \cap K) \cap X = X \cap C \leq_e X$. Now A is bR-ejective for any $b \in B$ by Lemma 3.5. ☐

A module M_R is called *Goldie quasi continuous* if M is Goldie extending and satisfies (C3) (see [3]).

Proposition 3.9. *Every Goldie quasi continuous module is Goldie ADS.*

Proof. Assume that M has a decomposition $M = S \oplus T$ and T' is a complement of S in M. Since M is Goldie extending, there exists a direct summand D of M such that $T'\beta D$. Since $T' \cap S = 0$ we have $D \cap S = 0$ by the equivalence relation β. We have $S \oplus (T' \cap D) \leq^e M$ and obtain that $S \oplus D \leq^e M$. So, by (C3) property, we obtain that $M = S \oplus D$. ☐

In [11], Kuratomi defined the GQC (generalized quasi continuous) modules by using Goldie extending modules. M is said to be *GQC* if for every submodule X_1 and X_2 of M with $X_1 \cap X_2 = 0$ there exists an essential submodules $Y_i \leq_e X_i$ and a decomposition $M = A_1 \oplus A_2$ such that Y_i is a submodule of A_i for $i = 1, 2$. Let $\{M_i : i \in I\}$ be a family of modules. The direct sum decomposition $M = \oplus_I M_i$ is said to be *exchangeable* if, for any direct summand X of M, there exists $\overline{M}_i \leq M_i$ $(i \in I)$ such that $M = X \oplus (\oplus_I \overline{M}_i)$. A module M is said to have the *finite internal exchange property* (FIEF) if, any finite direct sum decomposition $M = M_1 \oplus M_2 \oplus \cdots \oplus M_n$ is exchangeable.

Corollary 3.10. *The following statements are equivalent for a duo module M:*

1. *M is Goldie extending and Goldie ADS.*
2. *M is Goldie quasi continuous.*
3. *M is GQC with FIEP.*

Proof. By Theorem 3.8, Proposition 3.9 and [11, Theorem 3.4]. ☐

We have the following direct sum decomposition theorem for Goldie extending submodules.

Lemma 3.11 ([11, Proposition 2.1])**.** *Let M, N, M_i and N_i be modules.*

1. *If N is M_1-ejective and M_2-ejective, then N is $M_1 \oplus M_2$-ejective.*
2. *Let M_1 be a direct summand of M and N_1 a direct summand of N. If M is N-ejective, then M_1 is N_1-ejective.*
3. *If M_1 and M_2 are N-ejective modules then so is $M_1 \oplus M_2$.*

The proof of the following proposition uses the similar argument as in [14].

Proposition 3.12. *Let $M = \bigoplus_{i=1}^{n} M_i$ be direct sum of fully invariant submodules M_i's . Then M is Goldie ADS if and only if each M_i is Goldie ADS and M_i is M_j-ejective for all $i, j = 1, 2, \ldots, n$ and $i \neq j$.*

Proof. $\Longrightarrow$: By Theorem 3.8 (1), M_i's are Goldie ADS. Again by Theorem 3.8 (2), $M_i \oplus M_j$, which is a direct summand of M for $i \neq j$, is Goldie ADS. But by Lemma 3.6, M_i is M_j-ejective for $i \neq j$.

$\Longleftarrow$: Let $M = A \oplus B$. We claim that A is B-ejective. Since each M_i is fully invariant, we can obtain that $M_i = (A \cap M_i) \oplus (B \cap M_i)$ for all $i = 1, 2, \ldots, n$ by [2]. It follows that $A = \bigoplus_{i=1}^{n} (A \cap M_i)$ and $B = \bigoplus_{i=1}^{n} (B \cap M_i)$. Since M_i is Goldie ADS, $(A \cap M_i)$ is $(B \cap M_i)$-ejective. Since M_i is M_j-ejective then $(A \cap M_i)$ is $(B \cap M_j)$-ejective for all $i, j = 1, 2, \ldots, n$ by Lemma 3.11(2). It follows that $A \cap M_i$ is B-ejective for all $i = 1, 2, \ldots, n$ by Lemma 3.11(1). Thus A is B-ejective by Lemma 3.11(3). □

$E(-)$ denotes the injective hull for a module.

Theorem 3.13. *The following conditions are equivalent for a module M:*

1. *M is Goldie ADS.*
2. *For every decomposition $M = A \oplus B$, for all $f \in Hom(E(B), E(A))$, there exists $D \leq M$ such that $M = A \oplus D$ and $D\beta X$, where $X = \{b + f(b) | \ b \in B, f(b) \in A\}$.*

Proof. (1) $\Rightarrow$ (2) We show that $X = \{b + f(b) | b \in B, f(b) \in A\}$ is a complement of A in M. First, we note that that $A \cap X = 0$. Let L be a submodule of M such that $L \cap A = 0$ and $X \leq L$. Consider the the natural projections π_A and π_B of M onto A and B, respectively.
Claim: $\pi_A(x) = f\pi_B(x)$ for all $x \in L$. Assume that there exists $x \in L$ such that $(\pi_A - f\pi_B)(x) \neq 0$. Since $A \leq_e E(A)$, there exists $r \in R$ such that $0 \neq (\pi_A - f\pi_B)(xr) \in A$. But $\pi_A(xr) - f\pi_B(xr) = xr - (\pi_B(xr) + f\pi_B(xr)) \in A \cap L = 0$, a contradiction. Thus $\pi_A(x) = f\pi_B(x)$ for all $x \in L$.

Now, let $x \in L$. Hence $x = a + b$, where $a \in A$ and $b \in B$. Then $\pi_A(x) = a$. By the claim, we can obtain $\pi_A(x) = a = f\pi_B(x)$ for all $x \in L$. Therefore, $x = a + b = f\pi_B(x) + b \in X$. It follows that $L = X$. The rest is clear from the definition of Goldie ADS.
(2) $\Rightarrow$ (1) Let $M = A \oplus B$, and T be a complement of A in M.
Then $T = \{k + f(k) | \ k \in K\}$ for some $K \leq B$ and $f \in Hom(E(B), E(A))$. In fact, let $\pi_B : A \oplus B \to B$ be the canonical projection. There exists $f : E(B) \to E(A)$ such that $f\pi_B(t) = t - \pi_B(t)$ for all $t \in T$. Thus $T = \{k + f(k) | \ k \in \pi_B(T)\}$. By (2), there exists $D \leq M$ such that $M = A \oplus D$ and $D\beta X$, where $X = \{b + f(b) | \ b \in B, f(b) \in A\}$. Note that $f(\pi_B(T)) \leq A$ and so $D \cap T \leq_e T$. Now we show that $T \cap D \leq_e D$. Let $d \in D$, $d \neq 0$. Assume $T \cap dR = 0$. If $(T \oplus dR) \cap A \neq 0$, write $a = c + dr \neq 0$ for some $a \in A, c \in T$ and $r \in R$. We have $c \neq 0$ and obtain that there exists $r' \in R$ such that $cr' \in T \cap D$ and $cr' \neq 0$. Therefore $ar' = cr' + drr' \in A \cap D = 0$, which implies $cr' + drr' = 0$. It follows that $cr' = -drr' \in T \cap dR = 0$ hence $cr' = 0$, a contradiction. Thus $(T \oplus dR) \cap A = 0$ and then $T = T \oplus dR$ by the maximality of

T. It follows that $d = 0$, again a contradiction. Thus we get $T \cap dR \neq 0$. Hence there exists $y \in R$ such that $dy \in T \cap D$ and $dy \neq 0$. So $D \cap T \leq_e D$. □

4. Goldie ADS rings

A ring R is called a right Goldie ADS ring if R_R is a Goldie ADS module.
We start this section with the following ring extension.

Theorem 4.1. *Let M be a $S - R$-bimodule. Assume that*

$$T = \begin{pmatrix} S & M \\ 0 & R \end{pmatrix}$$

is right Goldie ADS. Then

1. *R is right Goldie ADS*
2. *M_R is Goldie ADS.*

Proof. (1) Let $R_R = A \oplus B$, $I \leq A$ and $f : I \to B$ an R-homomorphism. Let

$$\bar{A} = \begin{pmatrix} 0 & 0 \\ 0 & A \end{pmatrix}, \; \bar{B} = \begin{pmatrix} 0 & 0 \\ 0 & B \end{pmatrix} \text{ and } \bar{I} = \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix}.$$

It is easy to see that $\bar{A} \oplus \bar{B}$ is a direct summand of T_T. We define $\theta : \bar{I} \to \bar{B}$ via

$$\theta\left(\begin{pmatrix} 0 & 0 \\ 0 & r \end{pmatrix}\right) = \begin{pmatrix} 0 & 0 \\ 0 & f(r) \end{pmatrix}.$$

Then θ is a T-homomorphism. By the hypothesis, there exists a T-homomorphism $\phi : \bar{A} \to \bar{B}$ and $\bar{J} \leq_e \bar{I}$ such that $\phi(\bar{j}) = \theta(\bar{j})$ for every $\bar{j} \in \bar{J}$, where

$$\bar{J} = \begin{pmatrix} 0 & 0 \\ 0 & J \end{pmatrix}.$$

It is clear that ϕ is an R-homomorphism. Let $\iota : A \to \bar{A}$ via

$$\iota(a) = \begin{pmatrix} 0 & 0 \\ 0 & a \end{pmatrix}$$

and $\pi : \bar{B} \to B$ via

$$\pi\left(\begin{pmatrix} 0 & 0 \\ 0 & b \end{pmatrix}\right) = b.$$

Then ι and π are R-homomorphisms. Since $\bar{J} \leq_e \bar{I}$ then $J \leq_e I$. Let say $\bar{f} := \pi\phi\iota$. So

$$\bar{f}(j) = \pi\phi\iota(j) = \pi\phi\left(\begin{pmatrix} 0 & 0 \\ 0 & j \end{pmatrix}\right) = \pi\theta\left(\begin{pmatrix} 0 & 0 \\ 0 & j \end{pmatrix}\right) = \pi\left(\begin{pmatrix} 0 & 0 \\ 0 & f(j) \end{pmatrix}\right) = f(j)$$

for every $j \in J$ so we are done by Lemma 3.6.
(2) Assume that that $M_R = M_1 \oplus M_2$, $N \leq M_1$ and $f : N \to M_2$ is an R-homomorphism. Let

$$\bar{M_1} = \begin{pmatrix} S & M_1 \\ 0 & 0 \end{pmatrix}, \; \bar{M_2} = \begin{pmatrix} 0 & M_2 \\ 0 & R \end{pmatrix} \text{ and } \bar{N} = \begin{pmatrix} 0 & N \\ 0 & 0 \end{pmatrix}.$$

It is easy to see that $T_T = \bar{M_1} \oplus \bar{M_2}$. We define $\theta : \bar{N} \to \bar{M_2}$ via

$$\theta\left(\begin{pmatrix} 0 & n \\ 0 & 0 \end{pmatrix}\right) = \begin{pmatrix} 0 & f(n) \\ 0 & 0 \end{pmatrix}.$$

Then θ is a T-homomorphism. By the hypothesis, there exists a T-homomorphism $\phi : \bar{M_1} \to \bar{M_2}$ and $\bar{J} \leq_e \bar{N}$ such that $\phi(\bar{j}) = \theta(\bar{j})$ for all $\bar{j} \in \bar{J}$. Then ϕ is an R-homomorphism. Let $\iota : M_1 \to \bar{M_1}$ via

$$\iota(m) = \begin{pmatrix} 0 & m \\ 0 & 0 \end{pmatrix}$$

and $\pi : \bar{M_2} \to M_2$ via

$$\pi\left(\begin{pmatrix} 0 & m \\ 0 & r \end{pmatrix}\right) = m.$$

Then ι and π are R-homomorphisms. Since $\bar{J} \leq_e \bar{N}$ then $J \leq_e N$. Say $\bar{f} := \pi\phi\iota$ so $\bar{f}(j) = f(j)$. □

We recall the following useful lemma proved in [5, Lemma 5].

Lemma 4.2. *Let M be a right R-module, and let L be a submodule of M, where $R = ReR$ for some $e^2 = e \in R$ and $S = eRe$. Then:*

1. *L is essential in M if and only if Le is essential in $(Me)_S$;*
2. *L is a complement in M if and only if Le is a complement in $(Me)_S$;*
3. *L is a direct summand of M if and only if Le is a direct summand of $(Me)_S$.*

Proposition 4.3. *Let M be a right R-module, where $R = ReR$ for some $e^2 = e \in R$ and $S = eRe$. Then:*

1. *$(Me)_S$ is a Goldie ADS module if and only if M_R is a Goldie ADS module.*
2. *$(Re)_S$ is a Goldie ADS if and only if R_R is Goldie ADS.*

Proof. (1) Assume that $(Me)_S$ is a Goldie ADS module. Let $M_R = X \oplus Y$ and Z be a submodule of M with $Z \cap X = 0$. Then $Me = Xe \oplus Ye$ and, by Lemma 4.2, $Ze \cap Xe = 0$. Since $(Me)_S$ is a Goldie ADS module, there exists a submodule D of Me such that $Ze \cap D$ is essential in Ze and $Me = Xe \oplus D$. Hence $M_R = X \oplus DR$ and $Z \cap DR$ is essential in Z. They imply that X is Y-ejective by Lemma 3.5, and hence M_R is a Goldie ADS module by Lemma 3.6.

Assume that M_R is a Goldie ADS module. Let $Me = D \oplus T$ and K be a submodule of $(Me)_S$ with $K \cap D = 0$. Then $M = DR \oplus TR$ and, by Lemma 4.2, $KR \cap DR = 0$. Since M_R is a Goldie ADS module, there exists a submodule X of M such that $KR \cap X$ is essential in KR and $M = DR \oplus X$. Hence $Me = (DR)e \oplus XS$ and $(KR)e \cap XeR \leq XS$. Now $DRe = DeRe = D$ and $KRe = KeRe = K$. This implies that $Me = D \oplus XS$ and $K \cap XS$ is essential in K. Thus D is T-ejective by Lemma 3.5, and hence $(Me)_S$ is a Goldie ADS module by Lemma 3.6.
(2) It is a direct consequence of (1). □

Theorem 4.4. *$M_n(R)$ is Goldie ADS if and only if $(\oplus_{i=1}^n R_i)_R$ is Goldie ADS, where each $R_i = R$.*

Proof. It is easy to obtain that $M_n(R) = M_n(R)eM_n(R)$, where e is the matrix unit with 1 in the $(1, 1)$th position and zero elsewhere. The rest is follows by Proposition 4.3. □

References

[1] Abdioglu, C., Zemlicka, J., *On type-ADS modules and rings*, Journal of Algebra and Its Applications, https://doi.org/10.1142/S0219498818500834.

[2] Agayev, N., Leghwel, A., Koşan, M.T., Harmanci, A., *Duo modules and duo rings*, Far East J. Math. Sci, **20**(2005), no. 3, 341-346.

[3] Akalan, E., Birkenmeier, G.F, Tercan, A., *Goldie extending modules* , Commun. Algebra, **37**(2009), no. 3, 663-683.

[4] Akalan, E., Birkenmeier, G.F., Tercan, A., *Corrigendum to: Goldie extending modules*, Commun. Algebra, **38**(2010), 4747-4748.

[5] Akalan, E., Birkenmeier, G.F., Tercan, A., *Goldie extending rings*, Comm. Algebra, **40**(2012), no. 3, 423-428.

[6] Akalan, E., Birkenmeier, G.F., Tercan, A., *Corrigendum to: Goldie extending modules*, Commun. Algebra, **41**(2013), no. 4, 2005-2005.

[7] Alahmadi, A., Jain, S.K., Leroy, A., *ADS modules*, J. Algebra, **352**(2012), 215-222.

[8] Amin, I., Yousif, M., Zeyada, N., *Soc-Injective Rings and Modules*, Comm. Algebra, **33**(11)(2005), 4229-4250.

[9] Dung, N.V., Huynh, D.V., Smith, P.F., Wisbauer, R., *Extending modules*, Pitman Research Notes in Math. Longman, 1994, 313.

[10] Fuchs, L., *Infinite Abelian Groups*, vol. I, Pure Appl. Math. Ser. Monogr. Textb. vol. 36, Academic Press, New York, San Francisco, London 1970.

[11] Kuratomi, Y., *Goldie extending modules and generalizations of quasi-continuous modules*, Tsukuba J. Math, **38**(2014), no. 1, 25-37.

[12] Mohammed, S.H., Müller, B.J., *Continous and Discrete Modules*, London Math. Soc. LN. Cambridge Univ. Press, 1990, 147.

[13] Nicholson, W.K., Yousif, M.F., *Quasi-Frobenius Rings*, Cambridge Univ. Press, 2003.

[14] Quynh, T.C., Koşan, M.T., *On ADS modules*, Commun. Algebra, **42**(2014), 3541-3551.

Truong Cong Quynh
Department of Mathematics, Danang University
459 Ton Duc Thang, DaNang city, Vietnam
e-mail: tcquynh@dce.udn.vn

Serap Şahinkaya
Department of Mathematics, Faculty of Science
Gebze Technical University
Kocaeli, Turkey
e-mail: srpsahinkaya@gmail.com

Stud. Univ. Babeş-Bolyai Math. 63(2018), No. 4, 447–464
DOI: 10.24193/subbmath.2018.4.03

Existence and stability results for nonlocal initial value problems for differential equations with Hilfer fractional derivative

Mouffak Benchohra, Soufyane Bouriah and Juan J. Nieto

Abstract. In this paper, we establish sufficient conditions for the existence and stability of solutions for a class of nonlocal initial value problems for differential equations with Hilfer's fractional derivative, The arguments are based upon the Banach contraction principle. Two examples are included to show the applicability of our results.

Mathematics Subject Classification (2010): 26A33.

Keywords: Hilfer's fractional derivative, Caputo and Riemann-Liouville sense, existence, uniqueness, fixed point, Ulam stability.

1. Introduction

In our paper, we study the following nonlocal initial value problem

$$D_{0^+}^{\alpha,\beta} y(t) = f(t, y(t), D_{0^+}^{\alpha,\beta} y(t)), \quad \text{for every } t \in (0,T], T > 0, \tag{1.1}$$

$$I_{0^+}^{1-\gamma} y(0^+) = \sum_{i=1}^{m} \lambda_i y(\tau_i), \quad \tau_i \in (0,T], \tag{1.2}$$

where $0 < \alpha < 1, 0 \le \beta \le 1, \gamma = \alpha+\beta-\alpha\beta, f : (0,T]\times\mathbb{R}\times\mathbb{R} \longrightarrow \mathbb{R}$, $\tau_i, i = 1, 2, \ldots, m$ are pre-fixed points satisfying $0 < \tau_1 \le \cdots \le \tau_m < T$, λ_i are real numbers and

$$\sum_{i=1}^{m} \lambda_i \tau_i^{\gamma-1} \neq \Gamma(\gamma),$$

$D_{0^+}^{\alpha,\beta}$ denotes the generalized Riemann-Liouville derivative operator introduced by Hilfer in [7].

In the recent years, fractional calculus has gained much interest mainly thanks to the increasing presence of research works in the applied sciences considering models based on fractional operators see for example [1, 2, 6, 9, 12, 16, 17, 21]. Beside that, the mathematical study of fractional calculus has proceeded, leading to intersections with other mathematical fields such as probability and the study of stochastic processes. In the literature, several different definitions of fractional integrals and derivatives are present. Some of them such as the Riemann-Liouville integral, the Caputo and the Riemann-Liouville derivatives are thoroughly studied and actually used in applied models. Hilfer has introduced a generalized form of the Riemann-Liouville fractional derivative [7]. In short, Hilfer fractional derivative $D_{0^+}^{\alpha,\beta}$ is an interpolation between the Riemann-Liouville and Caputo fractional derivatives, see [10, 13, 17, 19]. It has many applications in fractional evolutions equations [8], and physical problems [15]. Also in the theoretical simulation of dielectric relaxation in glass forming materials. In [4], Furati *et al.* considered an initial value problem for a class of nonlinear fractional differential equations involving Hilfer fractional derivative. In [3], the authors consider the Ulam stability for nonlinear implicit fractional differential equations with Hadamard derivative. In [18], the solution of a fractional diffusion equation with a Hilfer time fractional derivative was obtained in terms of Mittag-Leffler functions and Fox's H-function. To the best of our knowledge, there has no results about the stability of differential equations with Hilfer fractional derivative.

Motivated by the works [3, 19], we prove in this paper the existence, uniqueness and stability for the non-linear nonlocal problem (1.1)-(1.2) in a weighted space of continuous functions. The present work is organized as follows. In Section 2, some notations are introduced and we recall some concepts of preliminaries about fractional calculus and auxiliary results. The proof for the main results is presented in Section 3 by applying the Banach fixed point theorem. In Section 4, the Ulam stability of our problem will be study. Finally, in the last section, we give two examples to illustrate the applicability of our main results.

2. Preliminaries

In this section, we recall some basic definitions and results concerning the fractional calculus, that we will use in the next sections .

Let $J := [0, T]$. By $C(J), AC(J)$ and $C^n(J)$ we denote the spaces of continuous, absolutely continuous and n times continuously differentiable functions on J, respectively. We denote by $L^p(J), p \geq 0$, the space of Lebesgue integrable functions on J.

We consider the weighted spaces of continuous functions

$$C_\gamma(J) = \{y : (0, T] \to \mathbb{R} : t^\gamma y(t) \in C(J)\},\ 0 \leq \gamma < 1,$$

$$C_\gamma^n(J) = \{y \in C^{n-1}(J) : y^{(n)} \in C_\gamma(J)\},\ n \in \mathbb{N},$$

$$C_\gamma^0(J) = C_\gamma(J),$$

with the norms

$$\|y\|_{C_\gamma} = \|t^\gamma y(t)\|_\infty = \sup_{t \in J} |t^\gamma y(t)|$$

and

$$\|y\|_{C_\gamma^n} = \sum_{k=0}^{n-1} \|y^{(k)}\|_\infty + \|y^{(n)}\|_{C_\gamma}.$$

These spaces satisfy the following properties.

- $C_0(J) = C(J)$.
- $C_\gamma^n(J) \subset AC^n(J)$.
- $C_{\gamma_1}(J) \subset C_{\gamma_2}(J),\ 0 \le \gamma_1 < \gamma_2 < 1$.

Definition 2.1. ([10, 13]). The fractional (arbitrary) order integral of the function $h \in L^1([0,T],\mathbb{R}_+)$ of order $\alpha \in \mathbb{R}_+$ called the left-sided Riemann-Liouville integral of the function h is defined by

$$(I_{0^+}^\alpha h)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s)ds,$$

where Γ is the Euler gamma function defined by $\Gamma(\alpha) = \displaystyle\int_0^\infty t^{\alpha-1}e^{-t}dt,\ \alpha > 0.$

The following lemmas provide some mapping properties of $I_{0^+}^\alpha$. The proofs can be found in [11].

Lemma 2.2. *For $\alpha > 0, I_{0^+}^\alpha$ maps $C(J)$ into $C(J)$.*

Lemma 2.3. *Let $\alpha > 0$ and $0 \le \gamma < 1$. Then, $I_{0^+}^\alpha$ is bounded from $C_\gamma(J)$ into $C_\gamma(J)$*

Lemma 2.4. *Let $\alpha > 0$ and $0 \le \gamma < 1$. If $\gamma \le \alpha$, then $I_{0^+}^\alpha$ is bounded from $C_\gamma(J)$ into $C(J)$.*

Lemma 2.5. [4] *Let $0 \le \gamma < 1$ and $y \in C_\gamma(J)$. Then*

$$I_{0^+}^\alpha y(0) := \lim_{t\to 0^+} I_{0^+}^\alpha y(t) = 0, \quad 0 \le \gamma < \alpha.$$

Definition 2.6. [5] The Riemann-Liouville left-sided fractional derivative $D_{0^+}^\alpha$ of order α is defined by

$$(D_{0^+}^\alpha y)(t) = D(I_{0^+}^{1-\alpha} y)(t), \quad \left(t > 0,\ 0 < \alpha < 1,\ D = \frac{d}{dt}\right),$$

that is

$$(D_{0^+}^\alpha y)(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t (t-s)^\alpha y(s)ds,$$

when $\alpha = 1$ we have $(D_{0^+}^\alpha y) = Dy$. In particular, when $\alpha = 0, (D_{0^+}^0 y) = y$.

Lemma 2.7. [4] *For $t > 0$, we have*

$$[I_{0^+}^\alpha t^{\beta-1}](t) = \frac{\Gamma(\beta)}{\Gamma(\alpha+\beta)} t^{\beta+\alpha-1}, \quad \alpha \ge 0, \beta > 0.$$

$$[D_{0^+}^\alpha t^{\alpha-1}](t) = 0, \quad 0 < \alpha < 1.$$

The following lemmas follows by direct calculations using Dirichlet formula

Lemma 2.8. *Let $\alpha > 0, \beta > 0$ and $y \in L^1(J)$. Then*

$$I_{0^+}^{\alpha} I_{0^+}^{\beta} y(t) = I_{0^+}^{\alpha+\beta} y(t), \quad a.e.\ t \in J.$$

In particular, if $y \in C_\gamma(J)$ or $y \in C(J)$, then equality holds at every $t \in (0,T]$ or $t \in [0,T]$, respectively.

Lemma 2.9. *Let $\alpha > 0, 0 \leq \gamma < 1$ and $y \in C_\gamma(J)$. Then*

$$D_{0^+}^{\alpha} I_{0^+}^{\alpha} y(t) = y(t), \quad \text{for all} \ \ t \in (0,T].$$

Lemma 2.10. [17] *Let $0 < \alpha < 1, 0 \leq \gamma < 1$. If $y \in C_\gamma(J)$ and $I_{0^+}^{1-\alpha} y \in C_\gamma^1(J)$, then*

$$I_{0^+}^{\alpha} D_{0^+}^{\alpha} y(t) = y(t) - \frac{I_{0^+}^{1-\alpha} y(0)}{\Gamma(\alpha)} t^{\alpha-1}, \quad \text{for all} \ \ t \in (0,T].$$

Let $\alpha \in (0,1), \beta \in [0,1]$ and $y \in L^1(J, \mathbb{R}^n)$. We say that the function y possesses the left-sided generalized Riemann-Liouville derivative (so called Hilfer derivative) $D_{0^+}^{\alpha,\beta}$ of order α and type β, if the function $I_{0^+}^{(1-\alpha)(1-\beta)} y$ is absolutely continuous on J and then

$$(D_{0^+}^{\alpha,\beta} y)(t) := \left(I_{0^+}^{\beta(1-\alpha)} D I_{0^+}^{(1-\alpha)(1-\beta)} y \right)(t), \quad a.e.\ t \in J. \tag{2.1}$$

The operator $D_{0^+}^{\alpha,\beta} y$, given by (2.1), was introduced by Hilfer in [7].

Remark 2.11. [4]

1. The Hilfer derivative $D_{0^+}^{\alpha,\beta} y$ can be written as

   $$(D_{0^+}^{\alpha,\beta} y)(t) := \left(I_{0^+}^{\beta(1-\alpha)} D I_{0^+}^{(1-\gamma)} y \right)(t) = (I_{0^+}^{\beta(1-\alpha)} D_{0^+}^{\gamma} y)(t) = (I_{0^+}^{\gamma-\alpha} D_{0^+}^{\gamma} y)(t)$$

   for a.e. $t \in J$, where $\gamma = \alpha + \beta - \alpha\beta$.
2. The $D_{0^+}^{\alpha,\beta} y$ derivative is considered as an interpolator between the Riemann-Liouville and Caputo derivative since

   $$D_{0^+}^{\alpha,\beta} y = \begin{cases} D_{0^+}^{\alpha} y, & \beta = 0 \\ {}^C D_{0^+}^{\alpha} y, & \beta = 1. \end{cases} \tag{2.2}$$

3. The parameter γ satisfies

   $$0 < \gamma \leq 1, \quad \gamma \geq \alpha, \quad \gamma > \beta, \quad 1 - \gamma < 1 - \beta(1-\alpha).$$

We introduce the spaces

$$C_{1-\gamma}^{\alpha,\beta}(J) = \{ y \in C_{1-\gamma}(J), D_{0^+}^{\alpha,\beta} y \in C_{1-\gamma}(J) \},$$

and

$$C_{1-\gamma}^{\gamma}(J) = \{ y \in C_{1-\gamma}(J), D_{0^+}^{\gamma} y \in C_{1-\gamma}(J) \}.$$

Since $D_{0^+}^{\alpha,\beta} y = I_{0^+}^{\beta(1-\alpha)} D_{0^+}^{\gamma} y$, it follows from Lemma 2.3 that

$$C_{1-\gamma}^{\gamma}(J) \subset C_{1-\gamma}^{\alpha,\beta}(J) \subset C_{1-\gamma}(J)$$

The following lemma follows directly from semigroup property in Lemma 2.8.

Lemma 2.12. *Let* $0<\alpha<1, 0\le\beta\le 1$, *and* $\gamma=\alpha+\beta-\alpha\beta$. *If* $y\in C^{\gamma}_{1-\gamma}(J)$, *then*

$$I^{\gamma}_{0^+}D^{\gamma}_{0^+}y=I^{\alpha}_{0^+}D^{\alpha,\beta}_{0^+}y,$$

and

$$D^{\gamma}_{0^+}I^{\alpha}_{0^+}y=D^{\beta(1-\alpha)}_{0^+}y.$$

For the proof of the following lemmas, we can see [4].

Lemma 2.13. *Let* $y\in L^1(J)$. *If* $D^{\beta(1-\alpha)}_{0^+}y$ *exists and in* $L^1(J)$ *then*

$$D^{\alpha,\beta}_{0^+}I^{\alpha}_{0^+}y=I^{\beta(1-\alpha)}_{0^+}D^{\beta(1-\alpha)}_{0^+}y.$$

Lemma 2.14. *Let* $0<\alpha<1, 0\le\beta\le 1$, *and* $\gamma=\alpha+\beta-\alpha\beta$. *If* $y\in C_{1-\gamma}(J)$ *and* $I^{1-\beta(1-\alpha)}_{0^+}y\in C^{1}_{1-\gamma}(J)$ *then* $D^{\alpha,\beta}_{0^+}I^{\alpha}_{0^+}y$ *exists in* $(0,T]$ *and*

$$D^{\alpha,\beta}_{0^+}I^{\alpha}_{0^+}y(t)=y(t),\quad t\in(0,T].$$

Lemma 2.15. *(*[20]*) Let* $\upsilon:[0,T]\longrightarrow[0,+\infty)$ *be a real function and* $\omega(\cdot)$ *is a non-negative, locally integrable function on* $[0,T]$. *Assume that there are constants* $a>0$ *and* $0<\alpha\le 1$ *such that*

$$\upsilon(t)\le\omega(t)+a\int_0^t(t-s)^{-\alpha}\upsilon(s)ds,$$

then, there exists a constant $K=K(\alpha)$ *such that*

$$\upsilon(t)\le\omega(t)+Ka\int_0^t(t-s)^{-\alpha}\omega(s)ds,\ \textit{for every}\ t\in[0,T].$$

For the implicit fractional-order differential equation (1.1), we adopt the definition in Rus ([14]) for Ulam-Hyers stability, generalized Ulam-Hyers stability, Ulam-Hyers-Rassias stability and generalized Ulam-Hyers-Rassias stability.

Definition 2.16. The equation (1.1) is Ulam-Hyers stable if there exists a real number $c_f>0$ such that for each $\epsilon>0$ and for each solution $z\in C^{\gamma}_{1-\gamma}(J)$, of the inequality

$$|D^{\alpha,\beta}_{0^+}z(t)-f(t,z(t),D^{\alpha,\beta}_{0^+}z(t))|\le\epsilon,\ t\in(0,T],$$

there exists a solution $y\in C^{\gamma}_{1-\gamma}(J)$ of equation (1.1) with

$$|z(t)-y(t)|\le c_f\epsilon,\ t\in(0,T].$$

Definition 2.17. The equation (1.1) is generalized Ulam-Hyers stable if there exists $\psi_f\in C(\mathbb{R}_+,\mathbb{R}_+)$, $\psi_f(0)=0$, such that for each solution $z\in C^{\gamma}_{1-\gamma}(J)$ of the inequality

$$|D^{\alpha,\beta}_{0^+}z(t)-f(t,z(t),D^{\alpha,\beta}_{0^+}z(t))|\le\epsilon,\ t\in(0,T],$$

there exists a solution $y\in C^{\gamma}_{1-\gamma}(J)$ of the equation (1.1) with

$$|z(t)-y(t)|\le\psi_f(\epsilon),\ t\in(0,T].$$

Definition 2.18. The equation (1.1) is Ulam-Hyers-Rassias stable with respect to $\varphi \in C(J, \mathbb{R}_+)$ if there exists a real number $c_f > 0$ such that for each $\epsilon > 0$ and for each solution $z \in C^{\gamma}_{1-\gamma}(J)$ of the inequality

$$|D^{\alpha,\beta}_{0^+} z(t) - f(t, z(t), D^{\alpha,\beta}_{0^+} z(t))| \leq \epsilon\varphi(t), \ t \in (0, T],$$

there exists a solution $y \in C^{\gamma}_{1-\gamma}(J)$ of equation (1.1) with

$$|z(t) - y(t)| \leq c_f \epsilon \varphi(t), \ t \in (0, T].$$

Definition 2.19. The equation (1.1) is generalized Ulam-Hyers-Rassias stable with respect to $\varphi \in C(J, \mathbb{R}_+)$ if there exists a real number $c_{f,\varphi} > 0$ such that for each solution $z \in C^{\gamma}_{1-\gamma}(J)$ of the inequality

$$|D^{\alpha,\beta}_{0^+} z(t) - f(t, z(t), D^{\alpha,\beta}_{0^+} z(t))| \leq \varphi(t), \ t \in (0, T],$$

there exists a solution $y \in C^{\gamma}_{1-\gamma}(J)$ of equation (1.1) with

$$|z(t) - y(t)| \leq c_{f,\varphi} \varphi(t), \ t \in (0, T].$$

Remark 2.20. A function $z \in C^{\gamma}_{1-\gamma}(J)$ is a solution of the inequality

$$|D^{\alpha,\beta}_{0^+} z(t) - f(t, z(t), D^{\alpha,\beta}_{0^+} z(t))| \leq \epsilon, \ t \in (0, T],$$

if and only if there exists a function $\phi \in C^{\gamma}_{1-\gamma}(J)$ (which depends on solution y) such that

- **i).** $|\phi(t)| \leq \epsilon, \ t \in (0, T]$.
- **ii).** $D^{\alpha,\beta}_{0^+} z(t) = f(t, z(t), D^{\alpha,\beta}_{0^+} z(t)) + \phi(t), \ t \in (0, T]$.

Remark 2.21. Clearly,

- **i).** Definition (2.6) $\Rightarrow$ Definition (2.7)
- **ii).** Definition (2.8) $\Rightarrow$ Definition (2.9).

Remark 2.22. A solution of the implicit differential inequality

$$|D^{\alpha,\beta}_{0^+} z(t) - f(t, z(t), D^{\alpha,\beta}_{0^+} z(t))| \leq \epsilon, \ t \in (0, T],$$

with fractional order is called an fractional $\epsilon-$ solution of the implicit fractional differential equation (1.1).

3. Existence of solutions

Let $\gamma = \alpha + \beta - \alpha\beta$ where $0 < \alpha < 1$ and $0 \leq \beta \leq 1$, let $f : (0, T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a function such that $f(\cdot, y(\cdot), u(\cdot)) \in C_{1-\gamma}(J)$ for any $y, u \in C_{1-\gamma}(J)$ and let the operator $N : C_{1-\gamma}(J) \to C_{1-\gamma}(J)$ defined by

$$Ny(t) = w(t) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g(s) ds, \quad t \in (0, T], \tag{3.1}$$

where

$$w(t) = \frac{t^{\gamma-1}}{\Gamma(\alpha)\left(\Gamma(\gamma) - \sum_{i=1}^{m} \lambda_i \tau_i^{\gamma-1}\right)} \sum_{i=1}^{m} \lambda_i \int_0^{\tau_i} (\tau_i - s)^{\alpha-1} g(s) ds$$

and $g:(0,T]\to\mathbb{R}$ be a function satisfies the functional equation

$$g(t)=f(t,y(t),g(t)).$$

Clearly, $w\in C_{1-\gamma}(J)$ and $g\in C_{1-\gamma}(J)$. Also, by Lemma 2.3, $Ny\in C_{1-\gamma}(J)$.

Theorem 3.1. *If* $y\in C^{\gamma}_{1-\gamma}(J)$, *then* y *satisfies the problem* (1.1) − (1.2) *if and only if* y *is the fixed point of operator* N.

Proof. First, we prove the necessity. Let $y\in C^{\gamma}_{1-\gamma}(J)$ be a solution of problem (1.1) − (1.2). We want to prove that y is a fixed point of N. By the definition of $C^{\gamma}_{1-\gamma}(J)$, Lemma 2.4 and Definition 2.6, we have

$$I^{1-\gamma}_{0^+}y\in C(J)\text{ and }D^{\gamma}_{0^+}y=D(I^{1-\gamma}_{0^+}y)\in C_{1-\gamma}(J).$$

Thus, we have

$$I^{1-\gamma}_{0^+}y\in C^{1}_{1-\gamma}(J).$$

Now, applying Lemma 2.10 to obtain

$$I^{\gamma}_{0^+}D^{\gamma}_{0^+}y(t)=y(t)-\frac{I^{1-\gamma}_{0^+}y(0^+)}{\Gamma(\gamma)}t^{\gamma-1},\quad t\in(0,T]. \tag{3.2}$$

Since $D^{\gamma}_{0^+}y\in C_{1-\gamma}(J)$, Lemma 2.12 yields

$$\left(I^{\gamma}_{0^+}D^{\gamma}_{0^+}y\right)(t)=\left(I^{\alpha}_{0^+}D^{\alpha,\beta}_{0^+}y\right)(t)=I^{\alpha}_{0^+}g(t),\quad t\in(0,T] \tag{3.3}$$

From (3.2) and (3.3), we obtain

$$y(t)=\frac{I^{1-\gamma}_{0^+}y(0^+)}{\Gamma(\gamma)}t^{\gamma-1}+\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}g(s)ds,\quad t\in(0,T]. \tag{3.4}$$

Next, we substitute $t=\tau_i$ into the above equation

$$y(\tau_i)=\frac{I^{1-\gamma}_{0^+}y(0^+)}{\Gamma(\gamma)}\tau_i^{\gamma-1}+\frac{1}{\Gamma(\alpha)}\int_0^{\tau_i}(\tau_i-s)^{\alpha-1}g(s)ds,\quad t\in(0,T], \tag{3.5}$$

by multiplying λ_i to both sides of (3.5), we can write

$$\lambda_i y(\tau_i)=\frac{I^{1-\gamma}_{0^+}y(0^+)}{\Gamma(\gamma)}\lambda_i\tau_i^{\gamma-1}+\frac{\lambda_i}{\Gamma(\alpha)}\int_0^{\tau_i}(\tau_i-s)^{\alpha-1}g(s)ds.$$

Thus, we have

$$\begin{aligned}I^{1-\gamma}_{0^+}y(0^+)&=\sum_{i=1}^{m}\lambda_i y(\tau_i)\\&=\frac{I^{1-\gamma}_{0^+}y(0^+)}{\Gamma(\gamma)}\sum_{i=1}^{m}\lambda_i\tau_i^{\gamma-1}+\frac{1}{\Gamma(\alpha)}\sum_{i=1}^{m}\lambda_i\int_0^{\tau_i}(\tau_i-s)^{\alpha-1}g(s)ds,\end{aligned}$$

which implies

$$I^{1-\gamma}_{0^+}y(0^+)=\frac{\Gamma(\gamma)}{\Gamma(\alpha)\left(\Gamma(\gamma)-\sum_{i=1}^{m}\lambda_i\tau_i^{\gamma-1}\right)}\sum_{i=1}^{m}\lambda_i\int_0^{\tau_i}(\tau_i-s)^{\alpha-1}g(s)ds. \tag{3.6}$$

Submitting (3.6) to (3.4), we get for each $t \in (0, T]$

$$y(t) = w(t) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g(s) ds, \tag{3.7}$$

which is the fixed point of N. Now, we prove the sufficiency. Let $y \in C_{1-\gamma}^{\gamma}(J)$ the fixed point of operator N, which can be written as (3.7). Applying the operator $D_{0^+}^{\gamma}$ to both sides of (3.7), it follows from Lemmas 2.12 and 2.7 that

$$D_{0^+}^{\gamma} y = D_{0^+}^{\beta(1-\alpha)} g. \tag{3.8}$$

From (3.8), Definition 2.6 and $D_{0^+}^{\gamma} y \in C_{1-\gamma}(J)$, we have

$$D I_{0^+}^{1-\beta(1-\alpha)} g = D_{0^+}^{\beta(1-\alpha)} g \in C_{1-\gamma}(J). \tag{3.9}$$

Also, since $g \in C_{1-\gamma}(J)$, by Lemma 2.4, we have

$$I_{0^+}^{1-\beta(1-\alpha)} g \in C(J). \tag{3.10}$$

It follows from (3.9), (3.10) that

$$I_{0^+}^{1-\beta(1-\alpha)} g \in C_{1-\gamma}^{1}(J).$$

Thus, g and $I_{0^+}^{1-\beta(1-\alpha)} g$ satisfy the conditions of Lemma 2.10.
Now, applying $I_{0^+}^{\beta(1-\alpha)}$ to both sides of (3.8) and using Remark 2.11 and Lemma 2.10, we can write

$$D_{0^+}^{\alpha,\beta} y(t) = g(t) - \frac{\left[I_{0^+}^{1-\beta(1-\alpha)} g\right](0)}{\Gamma(\beta(1-\alpha))} t^{\beta(1-\alpha)-1}. \tag{3.11}$$

Since $1-\gamma < 1-\beta(1-\alpha)$, Lemma 2.5 implies that

$$\left[I_{0^+}^{1-\beta(1-\alpha)} g\right](0) = 0.$$

Hence, the relation (3.11) reduces to

$$D_{0^+}^{\alpha,\beta} y(t) = g(t), \quad t \in (0, T].$$

Now, we show that the initial condition (1.2) also holds.
We apply $I_{0^+}^{1-\gamma}$ to both sides of (3.7), we have

$$I_{0^+}^{1-\gamma} y(t) = \frac{I_{0^+}^{1-\gamma} t^{\gamma-1}}{\Gamma(\alpha)\left(\Gamma(\gamma) - \sum_{i=1}^{m} \lambda_i \tau_i^{\gamma-1}\right)} \sum_{i=1}^{m} \lambda_i \int_0^{\tau_i} (\tau_i - s)^{\alpha-1} g(s) ds + I_{0^+}^{1-\gamma} I_{0^+}^{\alpha} g(t),$$

using the Lemmas 2.10 and 2.11,

$$I_{0^+}^{1-\gamma} y(t) = \frac{\Gamma(\gamma)}{\Gamma(\alpha)\left(\Gamma(\gamma) - \sum_{i=1}^{m} \lambda_i \tau_i^{\gamma-1}\right)} \sum_{i=1}^{m} \lambda_i \int_0^{\tau_i} (\tau_i - s)^{\alpha-1} g(s) ds + I_{0^+}^{1-\beta(1-\alpha)} g(t).$$

Since $1-\gamma < 1-\beta(1-\alpha)$, Lemma 2.5 can be used when taking the limit as $t \to 0$,

$$I_{0^+}^{1-\gamma} y(0) = \frac{\Gamma(\gamma)}{\Gamma(\alpha)\left(\Gamma(\gamma) - \sum_{i=1}^{m} \lambda_i \tau_i^{\gamma-1}\right)} \sum_{i=1}^{m} \lambda_i \int_0^{\tau_i} (\tau_i - s)^{\alpha-1} g(s) ds. \tag{3.12}$$

Substituting $t = \tau_i$ into (3.7), we have

$$\begin{aligned} y(\tau_i) &= \frac{\tau_i^{\gamma-1}}{\Gamma(\alpha)\left(\Gamma(\gamma) - \sum_{i=1}^{m} \lambda_i \tau_i^{\gamma-1}\right)} \sum_{i=1}^{m} \lambda_i \int_0^{\tau_i} (\tau_i - s)^{\alpha-1} g(s) ds \\ &+ \frac{1}{\Gamma(\alpha)} \int_0^{\tau_i} (\tau_i - s)^{\alpha-1} g(s) ds. \end{aligned}$$

Then, we derive

$$\begin{aligned} \sum_{i=1}^{m} \lambda_i y(\tau_i) &= \frac{1}{\Gamma(\alpha)\left(\Gamma(\gamma) - \sum_{i=1}^{m} \lambda_i \tau_i^{\gamma-1}\right)} \sum_{i=1}^{m} \lambda_i \int_0^{\tau_i} (\tau_i - s)^{\alpha-1} g(s) ds \sum_{i=1}^{m} \lambda_i \tau_i^{\gamma-1} \\ &+ \frac{1}{\Gamma(\alpha)} \sum_{i=1}^{m} \lambda_i \int_0^{\tau_i} (\tau_i - s)^{\alpha-1} g(s) ds \\ &= \frac{1}{\Gamma(\alpha)} \sum_{i=1}^{m} \lambda_i \int_0^{\tau_i} (\tau_i - s)^{\alpha-1} g(s) ds \left(\frac{\sum_{i=1}^{m} \lambda_i \tau_i^{\gamma-1}}{\Gamma(\gamma) - \sum_{i=1}^{m} \lambda_i \tau_i^{\gamma-1}} + 1 \right). \end{aligned}$$

Thus

$$\sum_{i=1}^{m} \lambda_i y(\tau_i) = \frac{\Gamma(\gamma)}{\Gamma(\alpha)\left(\Gamma(\gamma) - \sum_{i=1}^{m} \lambda_i \tau_i^{\gamma-1}\right)} \sum_{i=1}^{m} \lambda_i \int_0^{\tau_i} (\tau_i - s)^{\alpha-1} g(s) ds. \tag{3.13}$$

It follows (3.12) and (3.13) that

$$I_{0^+}^{1-\gamma} y(0) = \sum_{i=1}^{m} \lambda_i y(\tau_i).$$

□

Theorem 3.2. *Let the hypotheses*

(H1). *The function $f : (0,T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ such that*

$$f(\cdot, u(\cdot), v(\cdot)) \in C_{1-\gamma}^{\beta(1-\alpha)} \quad \text{for any } u, v \in C_{1-\gamma}(J)$$

(H2). *There exist constants $K > 0$ and $0 < \overline{K} < 1$ such that*

$$|f(t,u,v) - f(t,\overline{u},\overline{v})| \le K|u-\overline{u}| + \overline{K}|v-\overline{v}| \quad \text{for any } u, v, \overline{u}, \overline{v} \in \mathbb{R} \quad \text{and } t \in (0,T].$$

If

$$\frac{K\Gamma(\gamma)}{(1-\overline{K})\Gamma(\alpha+\gamma)}\left[\frac{\sum_{i=1}^{m}\lambda_i\tau_i^{\alpha+\gamma-1}}{\left|\Gamma(\gamma)-\sum_{i=1}^{m}\lambda_i\tau_i^{\gamma-1}\right|}+T^{\alpha}\right]<1, \tag{3.14}$$

then, there exists a unique solution for the Cauchy-type problem (1.1) − (1.2) *in the space* $C^{\gamma}_{1-\gamma}(J)$.

Proof. The proof will be given in two steps.
Step 1. We show that the operator N defined in (3.1) has a unique fixed point y^* in $C_{1-\gamma}(J)$. Let $y, u \in C_{1-\gamma}(J)$ and $t \in (0,T]$, then, we have

$$\begin{aligned}|Ny(t)-Nu(t)| &\leq \frac{t^{\gamma-1}}{\Gamma(\alpha)\left|\Gamma(\gamma)-\sum_{i=1}^{m}\lambda_i\tau_i^{\gamma-1}\right|}\sum_{i=1}^{m}\lambda_i\int_0^{\tau_i}(\tau_i-s)^{\alpha-1}|g(s)-h(s)|ds\\ &+ \frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}|g(s)-h(s)|ds,\end{aligned}$$

where $g, h \in C_{1-\gamma}(J)$ such that

$$g(t) = f(t, y(t), g(t)).$$
$$h(t) = f(t, u(t), h(t)).$$

By (H2), we have

$$\begin{aligned}|g(t)-h(t)| &= |f(t,y(t),g(t))-f(t,u(t),h(t))|\\ &\leq K|y(t)-u(t)|+\overline{K}|g(t)-h(t)|.\end{aligned}$$

Then

$$|g(t)-h(t)| \leq \frac{K}{1-\overline{K}}|y(t)-u(t)|.$$

Therefore, for each $t \in (0,T]$

$$\begin{aligned}|Ny(t)-Nu(t)| \leq\ & \frac{Kt^{\gamma-1}}{(1-\overline{K})\Gamma(\alpha)\left|\Gamma(\gamma)-\sum_{i=1}^{m}\lambda_i\tau_i^{\gamma-1}\right|}\sum_{i=1}^{m}\lambda_i\int_0^{\tau_i}(\tau_i-s)^{\alpha-1}|y(s)-u(s)|ds\\ &+\frac{K}{(1-\overline{K})\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}|y(s)-u(s)|ds\\ =\ & \frac{Kt^{\gamma-1}}{(1-\overline{K})\Gamma(\alpha)\left|\Gamma(\gamma)-\sum_{i=1}^{m}\lambda_i\tau_i^{\gamma-1}\right|}\sum_{i=1}^{m}\lambda_i\int_0^{\tau_i}(\tau_i-s)^{\alpha-1}s^{\gamma-1}|s^{1-\gamma}\left[y(s)-u(s)\right]|ds\\ &+\frac{K}{(1-\overline{K})\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}s^{\gamma-1}|s^{1-\gamma}\left[y(s)-u(s)\right]|ds\end{aligned}$$

$$\leq \frac{Kt^{\gamma-1}\|y-u\|_{C_{1-\gamma}}}{(1-\overline{K})\left|\Gamma(\gamma)-\sum_{i=1}^{m}\lambda_i\tau_i^{\gamma-1}\right|}\sum_{i=1}^{m}\lambda_i I_{0^+}^{\alpha}(\tau_i^{\gamma-1})+\frac{K\|y-u\|_{C_{1-\gamma}}}{1-\overline{K}}I_{0^+}^{\alpha}(t^{\gamma-1}).$$

By Lemma 2.7, we have

$$\begin{aligned}|Ny(t)-Nu(t)| &\leq \frac{K\Gamma(\gamma)t^{\gamma-1}\|y-u\|_{C_{1-\gamma}}}{\Gamma(\alpha+\gamma)(1-\overline{K})\left|\Gamma(\gamma)-\sum_{i=1}^{m}\lambda_i\tau_i^{\gamma-1}\right|}\sum_{i=1}^{m}\lambda_i\tau_i^{\alpha+\gamma-1}\\ &+ \frac{K\Gamma(\gamma)t^{\alpha+\gamma-1}}{(1-\overline{K})\Gamma(\alpha+\gamma)}\|y-u\|_{C_{1-\gamma}},\end{aligned}$$

hence

$$\begin{aligned}|t^{1-\gamma}(Ny(t)-Nu(t))| &\leq \frac{K\Gamma(\gamma)}{(1-\overline{K})\Gamma(\alpha+\gamma)}\left[\frac{\sum_{i=1}^{m}\lambda_i\tau_i^{\alpha+\gamma-1}}{\left|\Gamma(\gamma)-\sum_{i=1}^{m}\lambda_i\tau_i^{\gamma-1}\right|}+t^{\alpha}\right]\|y-u\|_{C_{1-\gamma}}\\ &\leq \frac{K\Gamma(\gamma)}{(1-\overline{K})\Gamma(\alpha+\gamma)}\left[\frac{\sum_{i=1}^{m}\lambda_i\tau_i^{\alpha+\gamma-1}}{\left|\Gamma(\gamma)-\sum_{i=1}^{m}\lambda_i\tau_i^{\gamma-1}\right|}+T^{\alpha}\right]\|y-u\|_{C_{1-\gamma}},\end{aligned}$$

which implies that

$$\|Ny-Nu\|_{C_{1-\gamma}} \leq \frac{K\Gamma(\gamma)}{(1-\overline{K})\Gamma(\alpha+\gamma)}\left[\frac{\sum_{i=1}^{m}\lambda_i\tau_i^{\alpha+\gamma-1}}{\left|\Gamma(\gamma)-\sum_{i=1}^{m}\lambda_i\tau_i^{\gamma-1}\right|}+T^{\alpha}\right]\|y-u\|_{C_{1-\gamma}}.$$

By (3.14), the operator N is a contraction. Hence, by Banach's contraction principle, N has a unique fixed point $y^* \in C_{1-\gamma}(J)$.

Step 2. We show that such a fixed point $y^* \in C_{1-\gamma}(J)$ is actually in $C_{1-\gamma}^{\gamma}(J)$.
Since y^* is the unique fixed point of operator N in $C_{1-\gamma}(J)$, then, for each $t \in (0,T]$, we have

$$\begin{aligned}y^*(t) &= w(t)+I_{0^+}^{\alpha}g(t)\\ &= \frac{t^{\gamma-1}}{\Gamma(\alpha)\left(\Gamma(\gamma)-\sum_{i=1}^{m}\lambda_i\tau_i^{\gamma-1}\right)}\sum_{i=1}^{m}\lambda_i\int_0^{\tau_i}(\tau_i-s)^{\alpha-1}f(s,y^*(s),g(s))ds\\ &+ I_{0^+}^{\alpha}f(t,y^*(t),g(t)).\end{aligned}$$

Applying D^{γ}_{0+} to both sides and by Lemma 2.7, we have

$$\begin{aligned} D^{\gamma}_{0+}y^*(t) &= D^{\gamma}_{0+}\left[I^{\alpha}_{0+}f(t,y^*(t),g(t))\right] \\ &= D^{\gamma-\alpha}_{0+}f(t,y^*(t),g(t)) \\ &= D^{\beta(1-\alpha)}_{0+}f(t,y^*(t),g(t)). \end{aligned}$$

Since $\gamma \geq \alpha$, by (H1), the right hand side is in $C_{1-\gamma}(J)$ and thus $D^{\gamma}_{0+}y^* \in C_{1-\gamma}(J)$ which implies that $y^* \in C^{\gamma}_{1-\gamma}(J)$. As a consequence of Step 1 and 2 together with Theorem 3.1, we can conclude that the problem (1.1)-(1.2) has a unique solution in $C^{\gamma}_{1-\gamma}(J)$. □

4. Ulam-Hyers-Rassias stability

Theorem 4.1. *Assume that* $(H1), (H2)$ *and* (3.14) *are satisfied, then the problem* (1.1)*-*(1.2) *is Ulam-Hyers stable.*

Proof. Let $\epsilon > 0$ and let $z \in C^{\gamma}_{1-\gamma}(J)$ be a function which satisfies the inequality:

$$|D^{\alpha,\beta}_{0+}z(t) - f(t,z(t),D^{\alpha,\beta}_{0+}z(t))| \leq \epsilon \quad \text{for any } t \in (0,T] \tag{4.1}$$

and let $y \in C^{\gamma}_{1-\gamma}(J)$ be the unique solution of the following Cauchy problem

$$D^{\alpha,\beta}_{0+}y(t) = f(t,y(t),D^{\alpha,\beta}_{0+}y(t)), \quad \text{for every } t \in (0,T], T > 0,$$

$$I^{1-\gamma}_{0+}y(0^+) = I^{1-\gamma}_{0+}z(0^+) = \sum_{i=1}^{m}\lambda_i y(\tau_i).$$

Using Theorem 3.1, we obtain

$$y(t) = \frac{I^{1-\gamma}_{0+}y(0^+)}{\Gamma(\gamma)}t^{\gamma-1} + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}g(s)ds, \quad t \in (0,T],$$

where $g : (0,T] \to \mathbb{R}$ be a function satisfies the functional equation

$$g(t) = f(t,y(t),g(t)).$$

Now, applying I^{α}_{0+} to both sides of the inequality (4.1), we obtain

$$\left|I^{\alpha}_{0+}D^{\alpha,\beta}_{0+}z(t) - \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}h(s)ds\right| \leq \frac{\epsilon t^{\alpha}}{\Gamma(\alpha+1)} \leq \frac{\epsilon T^{\alpha}}{\Gamma(\alpha+1)}, \tag{4.2}$$

where $h : (0,T] \to \mathbb{R}$ be a function satisfies the functional equation

$$h(t) = f(t,z(t),h(t)).$$

By the definition of $C^{\gamma}_{1-\gamma}(J)$, Lemma 2.4 and Definition 2.6, we have

$$I^{1-\gamma}_{0+}z \in C(J) \text{ and } D^{\gamma}_{0+}z = D(I^{1-\gamma}_{0+}z) \in C_{1-\gamma}(J).$$

Thus, we have

$$I^{1-\gamma}_{0+}z \in C^{1}_{1-\gamma}(J).$$

Now, applying Lemma 2.10 to obtain

$$I_{0^+}^{\gamma} D_{0^+}^{\gamma} z(t) = z(t) - \frac{I_{0^+}^{1-\gamma} z(0^+)}{\Gamma(\gamma)} t^{\gamma-1}, \quad t \in (0,T]. \tag{4.3}$$

Since $D_{0^+}^{\gamma} z \in C_{1-\gamma}(J)$, Lemma 2.12 yields

$$\left(I_{0^+}^{\gamma} D_{0^+}^{\gamma} z\right)(t) = \left(I_{0^+}^{\alpha} D_{0^+}^{\alpha,\beta} z\right)(t), \quad t \in (0,T]. \tag{4.4}$$

From (4.3) and (4.4), we get

$$\left(I_{0^+}^{\alpha} D_{0^+}^{\alpha,\beta} z\right)(t) = z(t) - \frac{I_{0^+}^{1-\gamma} z(0^+)}{\Gamma(\gamma)} t^{\gamma-1}, \quad t \in (0,T] \tag{4.5}$$

By replacing (4.5) in (4.2), we have

$$\left| z(t) - \frac{I_{0^+}^{1-\gamma} z(0^+)}{\Gamma(\gamma)} t^{\gamma-1} - \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s) ds \right| \leq \frac{\epsilon T^{\alpha}}{\Gamma(\alpha+1)}.$$

We have for any $t \in (0,T]$

$$\begin{aligned} |z(t) - y(t)| &= \left| z(t) - \frac{I_{0^+}^{1-\gamma} y(0^+)}{\Gamma(\gamma)} t^{\gamma-1} - \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s) ds \right. \\ &\quad \left. + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} (h(s) - g(s)) ds \right| \\ &\leq \left| z(t) - \frac{I_{0^+}^{1-\gamma} z(0^+)}{\Gamma(\gamma)} t^{\gamma-1} - \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s) ds \right| \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |h(s) - g(s)| ds. \end{aligned}$$

Thus

$$|z(t) - y(t)| \leq \frac{\epsilon T^{\alpha}}{\Gamma(\alpha+1)} + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |h(s) - g(s)| ds, \quad t \in (0,T]. \tag{4.6}$$

By (H2), we have for each $t \in (0,T]$

$$\begin{aligned} |h(t) - g(t)| &= |f(t, z(t), h(t)) - f(t, y(t), g(t))| \\ &\leq K|z(t) - y(t)| + \overline{K}|h(t) - g(t)|. \end{aligned}$$

Then

$$|h(t) - g(t)| \leq \frac{K}{1 - \overline{K}} |z(t) - y(t)|. \tag{4.7}$$

Using (4.6) and (4.7), we obtain

$$|z(t) - y(t)| \leq \frac{\epsilon T^{\alpha}}{\Gamma(\alpha+1)} + \frac{K}{(1-\overline{K})\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |z(s) - y(s)| ds, \quad t \in (0,T].$$

By Lemma 2.15, we have

$$|z(t) - y(t)| \leq \frac{\epsilon T^{\alpha}}{\Gamma(\alpha+1)} \left[1 + \frac{\delta K T^{\alpha}}{(1-\overline{K})\Gamma(\alpha+1)} \right] := c\epsilon$$

where $\delta = \delta(\alpha)$ a constant, which completes the proof of the theorem. Moreover, if we set $\psi(\epsilon) = c\epsilon; \psi(0) = 0$, then the problem (1.1)-(1.2) is generalized Ulam-Hyers stable. □

Theorem 4.2. *Assume that (H1), (H2), (*3.14*) and*
(H3) there exists an increasing function $\varphi \in C(J, \mathbb{R}_+)$ and there exists $\lambda_\varphi > 0$ such that for any $t \in (0, T]$

$$I_{0+}^{\alpha} \varphi(t) \leq \lambda_\varphi \varphi(t)$$

*are satisfied, then, the problem (*1.1*)-(*1.2*) is Ulam-Hyers-Rassias stable.*

Proof. Let $z \in C_{1-\gamma}^{\gamma}(J)$ be a function which satisfies the inequality:

$$|D_{0^+}^{\alpha,\beta} z(t) - f(t, z(t), D_{0^+}^{\alpha,\beta} z(t))| \leq \epsilon\varphi(t) \quad \text{for any } t \in (0, T]\ ,\ \epsilon > 0 \tag{4.8}$$

and let $y \in C_{1-\gamma}^{\gamma}(J)$ be the unique solution of the following Cauchy problem

$$D_{0^+}^{\alpha,\beta} y(t) = f(t, y(t), D_{0^+}^{\alpha,\beta} y(t)), \quad \text{for every } t \in (0, T], T > 0,$$

$$I_{0^+}^{1-\gamma} y(0^+) = I_{0^+}^{1-\gamma} z(0^+) = \sum_{i=1}^{m} \lambda_i y(\tau_i).$$

Using Theorem 3.1, we obtain

$$y(t) = \frac{I_{0^+}^{1-\gamma} y(0^+)}{\Gamma(\gamma)} t^{\gamma-1} + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g(s) ds, \quad t \in (0, T],$$

where $g : (0, T] \to \mathbb{R}$ be a function satisfies the functional equation

$$g(t) = f(t, y(t), g(t)).$$

Now, applying I_{0+}^{α} to both sides of the inequality (4.8), we obtain

$$\left| I_{0+}^{\alpha} D_{0^+}^{\alpha,\beta} z(t) - \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s) ds \right| \leq \frac{\epsilon}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \varphi(s) ds,$$

where $h : (0, T] \to \mathbb{R}$ be a function satisfies the functional equation

$$h(t) = f(t, z(t), h(t)).$$

Using (H3), we have for each $t \in (0, T]$

$$\left| I_{0+}^{\alpha} D_{0^+}^{\alpha,\beta} z(t) - \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s) ds \right| \leq \epsilon \lambda_\varphi \varphi(t).$$

From the proof of Theorem 4.1, we obtain

$$\left| z(t) - \frac{I_{0^+}^{1-\gamma} z(0^+)}{\Gamma(\gamma)} t^{\gamma-1} - \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s) ds \right| \leq \epsilon \lambda_\varphi \varphi(t).$$

We have for any $t \in (0,T]$

$$\begin{aligned}|z(t)-y(t)| &= \left| z(t) - \frac{I_{0^+}^{1-\gamma}y(0^+)}{\Gamma(\gamma)}t^{\gamma-1} - \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}h(s)ds \right.\\ &\quad \left. + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\left[h(s)-g(s)\right]ds \right|\\ &\leq \left| z(t) - \frac{I_{0^+}^{1-\gamma}z(0^+)}{\Gamma(\gamma)}t^{\gamma-1} - \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}h(s)ds \right|\\ &\quad + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}|h(s)-g(s)|ds.\end{aligned}$$

Thus

$$|z(t)-y(t)| \leq \epsilon\lambda_\varphi\varphi(t) + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}|h(s)-g(s)|ds, \quad t\in(0,T]. \tag{4.9}$$

By (H2), we have for each $t \in (0,T]$

$$\begin{aligned}|h(t)-g(t)| &= |f(t,z(t),h(t)) - f(t,y(t),g(t))|\\ &\leq K|z(t)-y(t)| + \overline{K}|h(t)-g(t)|,\end{aligned}$$

then

$$|h(t)-g(t)| \leq \frac{K}{1-\overline{K}}|z(t)-y(t)|. \tag{4.10}$$

Using (4.9) and (4.10), we have

$$|z(t)-y(t)| \leq \epsilon\lambda_\varphi\varphi(t) + \frac{K}{(1-\overline{K})\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}|z(s)-y(s)|ds, \quad t\in(0,T].$$

By Lemma 2.15, we obtain

$$|z(t)-y(t)| \leq \epsilon\lambda_\varphi\varphi(t) + \frac{\delta_1\epsilon K\lambda_\varphi}{(1-\overline{K})\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\varphi(s)ds,$$

where $\delta_1 = \delta_1(\alpha)$ is constant, and by (H_2), we have

$$|z(t)-y(t)| \leq \epsilon\lambda_\varphi\varphi(t) + \frac{\delta_1\epsilon K\lambda_\varphi^2\varphi(t)}{1-\overline{K}} = \left(1+\frac{\delta_1 K\lambda_\varphi}{1-\overline{K}}\right)\epsilon\lambda_\varphi\varphi(t).$$

Then, for any $t \in (0,T]$

$$|z(t)-y(t)| \leq \left[\left(1+\frac{\delta_1 K\lambda_\varphi}{1-\overline{K}}\right)\lambda_\varphi\right]\epsilon\varphi(t) = c\epsilon\varphi(t),$$

which completes the proof of Theorem 4.2. □

5. Examples

Example 5.1. Consider the following problem of non-linear implicit fractional differential equations

$$D_{0^+}^{\frac{1}{2},0} y(t) = \frac{1}{10e^{-t+2}\left(1+|y(t)|+\left|D_{0^+}^{\frac{1}{2},0} y(t)\right|\right)} + \frac{1}{\sqrt{t}} \quad \text{for each} \quad t \in (0,1] \tag{5.1}$$

$$I_{0^+}^{\frac{1}{2}} y(0^+) = 3y\left(\frac{1}{3}\right) + 2y\left(\frac{1}{2}\right). \tag{5.2}$$

Set

$$f(t,u,v) = \frac{1}{10e^{-t+2}(1+|u|+|v|)} + \frac{1}{\sqrt{t}}, \quad t \in (0,1], \quad u,v \in \mathbb{R}.$$

We have

$$C_{1-\gamma}^{\beta(1-\alpha)}([0,1]) = C_{\frac{1}{2}}^{0}([0,1]) = \left\{h : (0,1] \to \mathbb{R} : t^{\frac{1}{2}} h \in C([0,1])\right\},$$

with $\gamma = \alpha = \dfrac{1}{2}$ and $\beta = 0$. Clearly, the function $f \in C_{\frac{1}{2}}([0,1])$.
Hence condition (H1) is satisfied.
For each $u, \bar{u}, v, \bar{v} \in \mathbb{R}$ and $t \in (0,1]$, we have

$$|f(t,u,v) - f(t,\bar{u},\bar{v})| \quad \leq \quad \frac{1}{10e}(|u-\bar{u}| + |v-\bar{v}|).$$

Hence condition (H2) is satisfied with $K = \overline{K} = \dfrac{1}{10e}$.
The condition

$$\frac{K\Gamma(\gamma)}{(1-\overline{K})\Gamma(\alpha+\gamma)} \left[\frac{\sum_{i=1}^{2} \lambda_i \tau_i^{\alpha+\gamma-1}}{\left|\Gamma(\gamma) - \sum_{i=1}^{2} \lambda_i \tau_i^{\gamma-1}\right|} + T^{\alpha} \right] \approx 0.122 < 1,$$

is satisfied with $\lambda_1 = 3$, $\lambda_2 = 2$, $\tau_1 = \dfrac{1}{3}$, $\tau_2 = \dfrac{1}{2}$ and $T = 1$. It follows from Theorem 3.2 that the problem (5.1)–(5.2) has a unique solution in the space $C_{\frac{1}{2}}^{\frac{1}{2}}([0,1])$. Moreover, Theorem 4.1, implies that the problem (5.1)–(5.2) is Ulam-Hyers stable.

Example 5.2. Consider the following initial value problem

$$D_{0^+}^{\frac{1}{2},0} y(t) = \frac{1}{9+e^{-t}} \left[\frac{|y(t)|}{1+|y(t)|} - \frac{|D_{0^+}^{\frac{1}{2},0} y(t)|}{1+|D_{0^+}^{\frac{1}{2},0} y(t)|} \right] + \frac{t+1}{\sqrt{t}}, \ t \in (0,1] \tag{5.3}$$

$$I_{0^+}^{\frac{1}{2}} y(0^+) = 2y\left(\frac{1}{2}\right). \tag{5.4}$$

Set

$$f(t,u,v) = \frac{1}{9+e^{-t}} \left[\frac{u}{1+u} - \frac{v}{1+v} \right] + \frac{t+1}{\sqrt{t}}, \ t \in (0,1], \ u,v \in [0,+\infty).$$

We have

$$C_{1-\gamma}^{\beta(1-\alpha)}([0,1]) = C_{\frac{1}{2}}^{0}([0,1]) = \left\{ h : (0,1] \to \mathbb{R} : t^{\frac{1}{2}} h \in C([0,1]) \right\},$$

with $\gamma = \alpha = \dfrac{1}{2}$ and $\beta = 0$. Clearly, the function $f \in C_{\frac{1}{2}}([0,1])$.
Hence condition (H1) is satisfied.
For each $u, \bar{u}, v, \bar{v} \in \mathbb{R}$ and $t \in (0,1]$:

$$\begin{aligned} |f(t,u,v) - f(t,\bar{u},\bar{v})| &\leq \frac{1}{9+e^{-t}} \left(|u-\bar{u}| + |v-\bar{v}|\right) \\ &\leq \frac{1}{9+e^{-1}} \left(|u-\bar{u}| + |v-\bar{v}|\right). \end{aligned}$$

Hence condition $(H2)$ is satisfied with $K = \overline{K} = \dfrac{1}{9+e^{-1}}$.
The condition

$$\frac{K\Gamma(\gamma)}{(1-\overline{K})\Gamma(\alpha+\gamma)} \left[\frac{\lambda_1 \tau_1^{\alpha+\gamma-1}}{\left|\Gamma(\gamma) - \lambda_1 \tau_1^{\gamma-1}\right|} + T^{\alpha} \right] \approx 0.6077 < 1,$$

is satisfied with $\lambda_1 = 2, \tau_1 = \dfrac{1}{2}$ and $T = 1$. It follows from Theorem 3.2 that the problem (5.3)-(5.4) has a unique solution in the space $C_{\frac{1}{2}}^{\frac{1}{2}}([0,1])$, and by Theorem 4.1, the problem (5.3)-(5.4) is Ulam-Hyers stable.

References

[1] Abbas, S., Benchohra, M., N'Guérékata, G.M., *Topics in Fractional Differential Equations*, Springer, New York, 2012.

[2] Abbas, S., Benchohra, M., N'Guérékata, G.M., *Advanced Fractional Differential and Integral Equations*, Nova Science Publishers, New York, 2015.

[3] Benchohra, M., Lazreg, J.E., *Existence and Ulam stability for nonlinear implicit fractional differential equations with Hadamard derivative*, Stud. Univ. Babeş-Bolyai Math., **62**(2017), no. 1, 27-38.

[4] Furati, K.M., Kassim, M.D., Tatar, N.-E., *Existence and uniqueness for a problem involving Hilfer fractional derivative*, Comput. Math. Appl., **64**(2012), 1616-1626.

[5] Furati, K.M., Kassim, M.D., Tatar, N.-E., *Non-existence of global solutions for a differential equation involving Hilfer fractional derivative*, Electron. J. Differential Equations, **235**(2013), 10 pp.

[6] Gerolymatou, E., Vardoulakis, I., Hilfer, R., *Modelling infiltration by means of a nonlinear fractional diffusion model*, J. Physics D, Applied Physics, **39**(2006), 4104-4110.

[7] Hilfer, R., *Applications of Fractional Calculus in Physics*, World Scientific, Singapore, 2000.

[8] Hilfer, R., *Experimental evidence for fractional time evolution in glass forming materials*, Chem. Phys., **284**(2002), 399-408.

[9] Hilfer, R., Anton, L., *Fractional master equations and fractal time random walks*, Physical Review E, **51**(1995), 848-851.

[10] Kilbas, A.A., Srivastava, H.M., Trujillo, J.J., *Theory and Applications of Fractional Differential Equations*, North-Holland Mathematics Studies, 204, Elsevier Science B.V., Amsterdam, 2006.

[11] Kou, C., Liu, J., Ye, Y., *Existence and uniqueness of solutions for the Cauchy-type problems of fractional differential equations*, Discrete Dyn. Nat. Soc., **2010**(2010), 1-15.

[12] Mainardi, F., Gorenflo, R., *Time-fractional derivatives in relaxation processes: a tutorial survey*, Fract. Calc. Appl. Anal., **10**(2007), no. 3, 269-308.

[13] Podlubny, I., *Fractional Differential Equations,* in: Mathematics in Science and Engineering, vol. 198, Acad. Press, 1999.

[14] Rus, I.A., *Ulam stabilities of ordinary differential equations in a Banach space*, Carpathian J. Math., **26**(2010), 103-107.

[15] Tomovski, Z., Sandev, T., Metzler, R., Dubbeldam, J., *Generalized space-time fractional diffusion equation with composite fractional time derivative*, Phys. A, **391**(2012), 2527-2542.

[16] Tan Wenchang, X.M., Pan, W., *A note on unsteady flows of a viscoelastic fluid with the fractional Maxwell model between two parallel plates*, Intern. J. Nonlin. Mech., **38**(2003), 645-650.

[17] Samko, S.G., Kilbas, A.A., Marichev, O.I., *Fractional Integrals and Derivatives, Theory and Applications,* Gordon and Breach, Amsterdam, 1987, Engl. Trans. from the Russian.

[18] Sandev, T., Metzler, R., Tomovski, Z., *Fractional diffusion equation with a generalized Riemann-Liouville time fractional derivative*, J. Phys. A, Math. Theor., **44**(2011), 255203.

[19] Wang, J., Zhang, Y., *Nonlocal initial value problems for differential equations with Hilfer fractional derivative*, Appl. Math. Comput., **266**(2015), 850-859.

[20] Ye, H., Gao, J., Ding, Y., *A generalized Gronwall inequality and its application to a fractional differential equation*, J. Math. Anal. Appl., **328**(2007), 1075-1081.

[21] Zhou, Y., *Basic Theory of Fractional Differential Equations*, World Scientific, Singapore, 2014.

Mouffak Benchohra
Laboratory of Mathematics, Djillali Liabes University of Sidi Bel-Abbes
P.O. Box 89 Sidi Bel Abbes 22000, Algeria
e-mail: `benchohra@univ-sba.dz`

Soufyane Bouriah
Laboratory of Mathematics, Djillali Liabes University of Sidi Bel-Abbes
P.O. Box 89 Sidi Bel Abbes 22000, Algeria
and
Department of Mathematics, Faculty of Exact Sciences and Informatics
Hassiba Benbouali University, P.O. Box 151 Chlef 02000, Algeria
e-mail: `s.bouriah@univ-chlef.dz` `Bouriahsoufiane@yahoo.fr`

Juan J. Nieto
Departamento de Análisis Matemático, Facultad de Matemáticas
Universidad de Santiago de Compostela
Santiago de Compostela, Spain
e-mail: `juanjose.nieto.roig@usc.es`

Stud. Univ. Babeş-Bolyai Math. 63(2018), No. 4, 465–474
DOI: 10.24193/subbmath.2018.4.04

Ostrowski-type fractional integral inequalities for mappings whose derivatives are h-convex via Katugampola fractional integrals

Ghulam Farid, Udita N. Katugampola and Muhammad Usman

Abstract. In this paper we generalize some Riemann-Liouville fractional integral inequalities of Ostrowski-type for h-convex functions via Katugampola fractional integrals, generalizations of the Riemann-Liouville and the Hadamard fractional integrals. Also we deduce some known results by using p-functions, convex functions and s-convex functions.

Mathematics Subject Classification (2010): 26A33, 26A51, 26D07, 26D10, 26D15.

Keywords: Ostrowski inequality, fractional integrals, convex functions, h-Convex functions.

1. Introduction

The following inequality is known as Ostrowski inequality [17] (see also, [16, page 468]) which gives an upper bound for approximation of the integral average by the value $f(x)$ at a point $x \in [a,b]$. It is proved by Ostrowski in 1938.

Theorem 1.1. *Let $f : I \to \mathbb{R}$, where I is an interval in $\mathbb{R}$, be a differentiable mapping in I°, the interior of I and $a, b \in I^\circ$, $a < b$. If $|f'(t)| \leq M$ for all $t \in [a,b]$, then we have*

$$\left| f(x) - \frac{1}{(b-a)} \int_a^b f(t)dt \right| \leq \left[\frac{1}{4} + \frac{(x - \frac{a+b}{2})^2}{(b-a)^2} \right] (b-a)M,$$

where $x \in [a,b]$.

Ostrowski and Ostrowski-type inequalities have great importance in numerical analysis as they provide the bounds of different quadrature rules [1]. Over the years researchers are working to obtain Ostrowski-type inequalities for different kinds of functions. Recently Ostrowski-type inequalities via Riemann-Liouville fractional integrals are in focus (see [3, 4, 5, 6, 14, 15] and references therein).

Definition 1.2. A function f is called convex function on the interval $[a,b]$ if for any two points $x, y \in [a,b]$ and any t, where $0 \le t \le 1$,

$$f(tx + (1-t)y) \le tf(x) + (1-t)f(y).$$

Definition 1.3. [2] A non-negative function $f : I \to \mathbb{R}$ is said to be p-function, if for any two points $x, y \in I$ and $t \in [0,1]$,

$$f\,(tx + (1-t)y) \le f(x) + f(y).$$

Definition 1.4. [7] A function $f : I \to \mathbb{R}$ is said to be Godunova-Levin function, if for any two points $x, y \in I$ and $t \in (0,1)$,

$$f\,(tx + (1-t)y) \le \frac{f(x)}{t} + \frac{f(y)}{1-t}.$$

s-convex functions in the second sense have been introduced by Hudzik and Maligranda in [10] as follows.

Definition 1.5. [10] A function $f : [0,\infty) \to \mathbb{R}$ is called $s-$convex in the second sense on the interval $[0,\infty)$ if for any two points $x, y \in [0,\infty)$ and any t where $0 \le t \le 1$ and for some fixed $s \in (0,1]$,

$$f(tx + (1-t)y) \le t^s f(x) + (1-t)^s f(y).$$

Definition 1.6. [19] Let $J \subseteq \mathbb{R}$ be an interval containing $(0,1)$ and let $h : J \to \mathbb{R}$ be a positive function. We say $f : I \to \mathbb{R}$ is a h-convex function, if f is non-negative and

$$f(tx + (1-t)y) \le h(t)f(x) + h(1-t)f(y) \tag{1.1}$$

for all $x, y \in I$ and $t \in (0,1)$. If above inequality is reversed, then f is called h-concave.

It is easy to see that
(i) If $h(t) = t$, then (1.1) gives non-negative convex function.
(ii) If $h(t) = \frac{1}{t}$, then (1.1) gives Godunova-levin function.
(iii) If $h(t) = 1$, then (1.1) gives p-function.
(iv) If $h(t) = t^s$ where $s \in (0,1)$, then (1.1) gives s-convex function in the second sense.
In a paper by Sonin in 1869 [18], he used the Cauchy's integral formula as a starting point to reach the differentiation with arbitrary index. Letnikov [13] extended the idea of Sonin a short time later in 1872. Both tried to define fractional derivatives by utilizing a closed contour. Finally, Laurent in [12] used a contour given as an open circuit instead of a closed circuit, led to the definition of the Riemann-Liouville fractional integral, which is due to a little known paper published by Holmgren in 1865 [9].

Definition 1.7. [12] Let $f \in L_1[a,b]$. The Riemann-Liouville fractional integrals $J^{\alpha}_{a+}f$ and $J^{\alpha}_{b-}f$ of order $\alpha > 0$ with $a \ge 0$ are defined by

$$J^{\alpha}_{a+}f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)dt, x > a$$

and

$$J_{b-}^{\alpha}f(x) = \frac{1}{\Gamma(\alpha)}\int_x^b (t-x)^{\alpha-1}f(t)dt, x < b,$$

where

$$\Gamma(\alpha) = \int_0^{\infty} e^{-u}u^{\alpha-1}du$$

is the integral representation of Euler gamma function. Here

$$J_{a+}^{0}f(x) = J_{b-}^{0}f(x) = f(x).$$

In case of $\alpha = 1$, the Riemann-Liouville fractional integrals reduces to the classical integral.

Definition 1.8. J. Hadamard introduced the Hadamard fractional integral in [8], and is given by

$$I_{a+}^{\alpha}f(x) = \frac{1}{\Gamma(\alpha)}\int_a^x \left(log\frac{x}{\tau}\right)^{\alpha-1} f(\tau)\frac{d\tau}{\tau},$$

for $Re(\alpha) > 0$, $x > a \geq 0$.

Recently Katugampola generalized Riemann-Liouville and Hadamard fractional integrals into a unique form as follows.

Definition 1.9. [11] Let $[a, b]$ be a finite interval in $\mathbb{R}$. Then the Katugampola fractional integrals of order $\alpha > 0$ for a real valued function f are defined by

$${}^{\rho}I_{a+}^{\alpha}f\left(x\right) = \frac{\rho^{1-\alpha}}{\Gamma\left(\alpha\right)}\int_a^x t^{\rho-1}\left(x^{\rho}-t^{\rho}\right)^{\alpha-1}f\left(t\right)dt$$

and

$${}^{\rho}I_{b-}^{\alpha}f\left(x\right) = \frac{\rho^{1-\alpha}}{\Gamma\left(\alpha\right)}\int_x^b t^{\rho-1}\left(t^{\rho}-x^{\rho}\right)^{\alpha-1}f\left(t\right)dt$$

with $a < x < b$ and $\rho > 0$, if the integrals exist, where $\Gamma(\alpha)$ is the Euler gamma function. For $\rho = 1$, Katugampola fractional integrals give Riemann-Liouville fractional integrals, while $\rho \to 0^{+}$ produces the Hadamard fractional integral. For its proof one can refer [11].

We organize the paper as follows:

In this paper we prove some Ostrowski-type inequalities for mappings whose derivatives are h-convex via Katugampola fractional integrals. We deduce some known results by using p-functions, convex functions and s-convex functions. In particular we find Ostrowski-type inequalities for Riemann-Liouville fractional integrals.

2. Ostrowski-type fractional inequalities for h-convex functions via Katugampola fractional integral

In this section we present some Ostrowski-type inequalities for h-convex functions via Katugampola fractional integrals. The following lemma is very useful to obtain our results.

Lemma 2.1. *Let $f : [a^\rho, b^\rho] \to \mathbb{R}$ be a differentiable mapping on (a^ρ, b^ρ) with $a < b$ such that $f' \in L_1[a,b]$. Then we have the following equality*

$$\begin{aligned} & f(x^\rho) - \frac{(\alpha\rho+\rho-1)\Gamma(\alpha)}{\rho^{1-\alpha}}\left[\frac{{}^\rho I^\alpha_{x-} f(a^\rho)}{2(x^\rho-a^\rho)^\alpha} + \frac{{}^\rho I^\alpha_{x+} f(b^\rho)}{2(b^\rho-x^\rho)^\alpha}\right] \\ &= \frac{\rho(x^\rho-a^\rho)}{2}\int_0^1 t^{\alpha\rho+\rho-1} f'(t^\rho x^\rho + (1-t^\rho)a^\rho)dt \\ &- \frac{(b^\rho-x^\rho)}{2}\int_0^1 t^{\alpha\rho+\rho-1} f'(t^\rho x^\rho + (1-t^\rho)b^\rho)dt;\ x\in(a,b), \end{aligned} \tag{2.1}$$

with $\alpha, \rho > 0$.

Proof. It is easy to see that

$$\begin{aligned} & \int_0^1 t^{\alpha\rho+\rho-1} f'(t^\rho x^\rho + (1-t^\rho)a^\rho)dt \\ &= \left.\frac{t^{\alpha\rho+\rho-1} f(t^\rho x^\rho + (1-t^\rho)a^\rho)}{\rho t^{\rho-1}(x^\rho-a^\rho)}\right|_0^1 \\ &- \frac{\alpha\rho+\rho-1}{\rho(x^\rho-a^\rho)}\int_0^1 t^{\alpha\rho-1} f(t^\rho x^\rho + (1-t^\rho)a^\rho)dt \\ &= \frac{f(x^\rho)}{\rho(x^\rho-a^\rho)} - \frac{\alpha\rho+\rho-1}{\rho(x^\rho-a^\rho)}\int_a^x \left(\frac{y^\rho-a^\rho}{x^\rho-a^\rho}\right)^{\alpha-1}\frac{y^{\rho-1}f(y^\rho)}{x^\rho-a^\rho}dy \\ &= \frac{f(x^\rho)}{\rho(x^\rho-a^\rho)} - \frac{{}^\rho I^\alpha_{x-} f(a^\rho)(\alpha\rho+\rho-1)\Gamma(\alpha)}{\rho^{2-\alpha}(x^\rho-a^\rho)^{\alpha+1}} \end{aligned} \tag{2.2}$$

and

$$\begin{aligned} & \int_0^1 t^{\alpha\rho+\rho-1} f'(t^\rho x^\rho + (1-t^\rho)b^\rho)dt \\ &= \left.\frac{t^{\alpha\rho+\rho-1} f(t^\rho x^\rho + (1-t^\rho)b^\rho)}{\rho t^{\rho-1}(x^\rho-b^\rho)}\right|_0^1 \\ &- \frac{\alpha\rho+\rho-1}{\rho(x^\rho-b^\rho)}\int_0^1 t^{\alpha\rho-1} f(t^\rho x^\rho + (1-t^\rho)b^\rho)dt \\ &= \frac{-f(x^\rho)}{\rho(b^\rho-x^\rho)} + \frac{\alpha\rho+\rho-1}{\rho(b^\rho-x^\rho)}\int_x^b \left(\frac{y^\rho-b^\rho}{x^\rho-b^\rho}\right)^{\alpha-1}\frac{y^{\rho-1}f(y^\rho)}{x^\rho-b^\rho}dy \\ &= \frac{-f(x^\rho)}{\rho(b^\rho-x^\rho)} + \frac{{}^\rho I^\alpha_{x+} f(b^\rho)(\alpha\rho+\rho-1)\Gamma(\alpha)}{\rho^{2-\alpha}(b^\rho-x^\rho)^{\alpha+1}}. \end{aligned} \tag{2.3}$$

Multiplying (2.2) by $\frac{\rho(x^\rho-a^\rho)}{2}$ and (2.3) by $\frac{\rho(b^\rho-x^\rho)}{2}$, then adding resulting equations we get (2.1). □

Theorem 2.2. *Let $f : [a^\rho, b^\rho] \subseteq [0,\infty) \to \mathbb{R}$ be a differentiable mapping on (a^ρ, b^ρ) with $a < b$ such that $f' \in L_1[a,b]$. If $|f'|$ is h-convex on $[a^\rho, b^\rho]$ and $|f'(x^\rho)| \le M$,*

$x \in [a,b]$, *then the following inequality for Katugampola fractional integrals holds*

$$\left| f(x^\rho) - \frac{(\alpha\rho+\rho-1)\Gamma(\alpha)}{\rho^{1-\alpha}} \left[\frac{{}^\rho I^\alpha_{x^-} f(a^\rho)}{2(x^\rho - a^\rho)^\alpha} + \frac{{}^\rho I^\alpha_{x^+} f(b^\rho)}{2(b^\rho - x^\rho)^\alpha} \right] \right|$$
$$\leq \frac{M\rho(b^\rho - a^\rho)}{2} \int_0^1 t^{\alpha\rho+\rho-1} \left[h(t^\rho) + h(1-t^\rho) \right] dt; x \in (a,b), \tag{2.4}$$

with $\alpha, \rho > 0$.

Proof. Using Lemma 2.1, h-convexity of $\left|f'\right|$, and upper bound of $\left|f'(x^\rho)\right|$ we have

$$\begin{aligned}
&\left| f(x^\rho) - \frac{(\alpha\rho+\rho-1)\Gamma(\alpha)}{\rho^{1-\alpha}} \left[\frac{{}^\rho I^\alpha_{x^-} f(a^\rho)}{2(x^\rho - a^\rho)^\alpha} + \frac{{}^\rho I^\alpha_{x^+} f(b^\rho)}{2(b^\rho - x^\rho)^\alpha} \right] \right| \\
&\leq \frac{\rho(x^\rho - a^\rho)}{2} \int_0^1 t^{\alpha\rho+\rho-1} \left| f'(t^\rho x^\rho + (1-t^\rho)a^\rho) \right| dt \\
&+ \frac{\rho(b^\rho - x^\rho)}{2} \int_0^1 t^{\alpha\rho+\rho-1} \left| f'(t^\rho x^\rho + (1-t^\rho)b^\rho) \right| dt \\
&\leq \frac{\rho(x^\rho - a^\rho)}{2} \int_0^1 t^{\alpha\rho+\rho-1} \left[h(t^\rho)\left|f'(x^\rho)\right| + h(1-t^\rho)\left|f'(a^\rho)\right| \right] dt \\
&+ \frac{\rho(b^\rho - x^\rho)}{2} \int_0^1 t^{\alpha\rho+\rho-1} \left[h(t^\rho)\left|f'(x^\rho)\right| + h(1-t^\rho)\left|f'(b^\rho)\right| \right] dt \\
&\leq \frac{M\rho(x^\rho - a^\rho)}{2} \int_0^1 t^{\alpha\rho+\rho-1} \left[h(t^\rho) + h(1-t^\rho) \right] dt \\
&+ \frac{M\rho(b^\rho - x^\rho)}{2} \int_0^1 t^{\alpha\rho+\rho-1} \left[h(t^\rho) + h(1-t^\rho) \right] dt \\
&= \frac{M\rho(b^\rho - a^\rho)}{2} \int_0^1 t^{\alpha\rho+\rho-1} \left[h(t^\rho) + h(1-t^\rho) \right] dt.
\end{aligned}$$

This completes the proof. □

Corollary 2.3. *In Theorem* 2.2, *if we take* $h(t) = 1$, *which means that* $\left|f'\right|$ *is p-function, then (*2.4*) becomes the following inequality*

$$\left| f(x^\rho) - \frac{(\alpha\rho+\rho-1)\Gamma(\alpha)}{\rho^{1-\alpha}} \left[\frac{{}^\rho I^\alpha_{x^-} f(a^\rho)}{2(x^\rho - a^\rho)^\alpha} + \frac{{}^\rho I^\alpha_{x^+} f(b^\rho)}{2(b^\rho - x^\rho)^\alpha} \right] \right|$$
$$\leq \frac{M(b^\rho - a^\rho)}{\alpha+1}; x \in [a,b], \tag{2.5}$$

with $\alpha, \rho > 0$.

Remark 2.4. (i) If we put $\rho = 1$ in (2.4) we get [15, Theorem 1].

(ii) If we put $\rho = 1$, $\alpha = 1$ and $x = \dfrac{a+b}{2}$ in (2.4) we get [15, Corollary 3].

(iii) If we put $\rho = 1$ and $h(t) = t$, which means that $\left|f'\right|$ is convex function in (2.4), then we get [15, Corollary 1].

(iv) If we put $\rho = 1$ and $h(t) = t^s$, which means that $|f'|$ is h-convex function in (2.4), then we get [15, Corollary 2].

Theorem 2.5. *Let $f : [a^\rho, b^\rho] \subseteq [0, \infty) \to \mathbb{R}$ be a differentiable mapping on (a^ρ, b^ρ) with $a < b$ such that $f' \in L_1[a, b]$. If $|f'|^q, q > 1$, is h-convex on $[a^\rho, b^\rho]$ and $|f'(x^\rho)| \leq M$, $x \in [a, b]$, then the following inequality for Katugampola fractional integrals holds*

$$\left| f(x^\rho) - \frac{(\alpha\rho + \rho - 1)\Gamma(\alpha)}{\rho^{1-\alpha}} \left[\frac{{}^\rho I^\alpha_{x-} f(a^\rho)}{2(x^\rho - a^\rho)^\alpha} + \frac{{}^\rho I^\alpha_{x+} f(b^\rho)}{2(b^\rho - x^\rho)^\alpha} \right] \right|$$

$$\leq \frac{M\rho(b^\rho - a^\rho)}{2\left(p(\alpha\rho + \rho - 1) + 1\right)^{\frac{1}{p}}} \left(\int_0^1 \left[h(t^\rho) + h(1 - t^\rho) \right] dt \right)^{\frac{1}{q}}; \; x \in (a, b), \tag{2.6}$$

with $\alpha, \rho > 0$ and $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. Using Lemma 2.1 and Holder's inequality we have

$$\left| f(x^\rho) - \frac{(\alpha\rho + \rho - 1)\Gamma(\alpha)}{\rho^{1-\alpha}} \left[\frac{{}^\rho I^\alpha_{x-} f(a^\rho)}{2(x^\rho - a^\rho)^\alpha} + \frac{{}^\rho I^\alpha_{x+} f(b^\rho)}{2(b^\rho - x^\rho)^\alpha} \right] \right|$$

$$\leq \frac{\rho(x^\rho - a^\rho)}{2} \int_0^1 t^{\alpha\rho+\rho-1} \left| f'(t^\rho x^\rho + (1 - t^\rho)a^\rho) \right| dt$$

$$+ \frac{\rho(b^\rho - x^\rho)}{2} \int_0^1 t^{\alpha\rho+\rho-1} \left| f'(t^\rho x^\rho + (1 - t^\rho)b^\rho) \right| dt$$

$$\leq \frac{\rho(x^\rho - a^\rho)}{2} \left(\int_0^1 t^{p(\alpha\rho+\rho-1)} dt \right)^{\frac{1}{p}} \left(\int_0^1 \left| f'(t^\rho x^\rho + (1 - t^\rho)a^\rho) \right|^q dt \right)^{\frac{1}{q}}$$

$$+ \frac{\rho(b^\rho - x^\rho)}{2} \left(\int_0^1 t^{p(\alpha\rho+\rho-1)} dt \right)^{\frac{1}{p}} \left(\int_0^1 \left| f'(t^\rho x^\rho + (1 - t^\rho)a^\rho) \right|^q dt \right)^{\frac{1}{q}}.$$

Since $|f'|^q$ is h-convex and $|f'(x^\rho)| \leq M$, $x \in [a, b]$, therefore we have for $x \in (a, b)$

$$\left| f(x^\rho) - \frac{(\alpha\rho + \rho - 1)\Gamma(\alpha)}{\rho^{1-\alpha}} \left[\frac{{}^\rho I^\alpha_{x-} f(a^\rho)}{2(x^\rho - a^\rho)^\alpha} + \frac{{}^\rho I^\alpha_{x+} f(b^\rho)}{2(b^\rho - x^\rho)^\alpha} \right] \right|$$

$$\leq \frac{\rho(x^\rho - a^\rho)}{2} \left(\int_0^1 t^{p(\alpha\rho+\rho-1)} dt \right)^{\frac{1}{p}} \left(\int_0^1 \left[h(t^\rho) \left| f'(x) \right|^q + h(1 - t^\rho) \left| f'(a^\rho) \right|^q \right] dt \right)^{\frac{1}{q}}$$

$$+ \frac{\rho(b^\rho - x^\rho)}{2} \left(\int_0^1 t^{p(\alpha\rho+\rho-1)} dt \right)^{\frac{1}{p}} \left(\int_0^1 \left[h(t^\rho) \left| f'(x) \right|^q + h(1 - t^\rho) \left| f'(b^\rho) \right|^q \right] dt \right)^{\frac{1}{q}}$$

$$\leq \frac{M\rho(x^\rho - a^\rho)}{2p(\alpha\rho + \rho - 1) + 1)^{\frac{1}{p}}} \left(\int_0^1 \left[h(t^\rho) + h(1 - t^\rho) \right] dt \right)^{\frac{1}{q}}$$

$$+ \frac{M\rho(b^\rho - x^\rho)}{2(p(\alpha\rho + \rho - 1) + 1)^{\frac{1}{p}}} \left(\int_0^1 \left[h(t^\rho) + h(1 - t^\rho) \right] dt \right)^{\frac{1}{q}}$$

$$= \frac{M\rho(b^\rho - a^\rho)}{2\left(p(\alpha\rho+\rho-1)+1\right)^{\frac{1}{p}}}\left(\int_0^1 \left[h(t^\rho)+h(1-t^\rho)\right]dt\right)^{\frac{1}{q}}.$$

This completes the proof. □

Corollary 2.6. *In Theorem* 2.5*, if we take* $h(t) = 1$*, which means that* $\left|f'\right|^q$ *is p-function, then* (2.6) *becomes the following inequality*

$$\left|f(x^\rho) - \frac{(\alpha\rho+\rho-1)\Gamma(\alpha)}{\rho^{1-\alpha}}\left[\frac{{}^\rho I^\alpha_{x-}f(a^\rho)}{2(x^\rho-a^\rho)^\alpha}+\frac{{}^\rho I^\alpha_{x+}f(b^\rho)}{2(b^\rho-x^\rho)^\alpha}\right]\right|$$
$$\le \frac{(2)^{\frac{1}{q}-1}M\rho(b^\rho-a^\rho)}{\left(p(\alpha\rho+\rho-1)+1\right)^{\frac{1}{p}}};\ x\in[a,b], \tag{2.7}$$

with $\alpha, \rho > 0$ *and* $\frac{1}{p}+\frac{1}{q}=1$.

Remark 2.7. (i) If we put $\rho = 1$ in (2.6) we get [15, Theorem 2].
(ii) If we put $\rho = 1$, $\alpha = 1$ and $x = \dfrac{a+b}{2}$ in (2.6) we get [15, Corollary 6].
(iii) If we put $\rho = 1$ and $h(t) = t$ in (2.6), which means that $\left|f'\right|$ is convex function, then we get [15, Corollary 4].
(iv) If we put $\rho = 1$ and $h(t) = t^s$, which means that $\left|f'\right|$ is h-convex function in (2.6), then we get [15, Corollary 5].

Theorem 2.8. *Let* $f : [a^\rho, b^\rho] \subseteq [0,\infty) \to \mathbb{R}$ *be a differentiable mapping on* (a^ρ, b^ρ) *such that* $f' \in L_1[a,b]$*, where* $a < b$*. If* $\left|f'\right|^q, q > 1$ *is h-convex on* $[a^\rho, b^\rho]$ *and* $\left|f'(x^\rho)\right| \le M$*,* $x \in [a,b]$*, then the following inequality for Katugampola fractional integrals holds for* $x \in (a,b)$

$$\left|f(x^\rho) - \frac{(\alpha\rho+\rho-1)\Gamma(\alpha)}{\rho^{1-\alpha}}\left[\frac{{}^\rho I^\alpha_{x-}f(a^\rho)}{2(x^\rho-a^\rho)^\alpha}+\frac{{}^\rho I^\alpha_{x+}f(b^\rho)}{2(b^\rho-x^\rho)^\alpha}\right]\right|$$
$$\le \frac{M\rho(b^\rho-a^\rho)}{2}\left(\frac{1}{\rho(\alpha+1)}\right)^{1-\frac{1}{q}}\left(\int_0^1 t^{\alpha\rho+\rho-1}\left[h(t^\rho)+h(1-t^\rho)\right]dt\right)^{\frac{1}{q}}, \tag{2.8}$$

with $\alpha, \rho > 0$.

Proof. Using Lemma 2.1 and power mean inequality we have

$$\left|f(x^\rho) - \frac{(\alpha\rho+\rho-1)\Gamma(\alpha)}{\rho^{1-\alpha}}\left[\frac{{}^\rho I^\alpha_{x-}f(a^\rho)}{2(x^\rho-a^\rho)^\alpha}+\frac{{}^\rho I^\alpha_{x+}f(b^\rho)}{2(b^\rho-x^\rho)^\alpha}\right]\right|$$
$$\le \frac{\rho(x^\rho-a^\rho)}{2}\int_0^1 t^{\alpha\rho+\rho-1}\left|f'(t^\rho x^\rho+(1-t^\rho)a^\rho)\right|dt$$
$$+\frac{\rho(b^\rho-x^\rho)}{2}\int_0^1 t^{\alpha\rho+\rho-1}\left|f'(t^\rho x^\rho+(1-t^\rho)b^\rho)\right|dt$$
$$\le \frac{\rho(x^\rho-a^\rho)}{2}\left(\int_0^1 t^{\alpha\rho+\rho-1}dt\right)^{1-\frac{1}{q}}\left(t^{\alpha\rho+\rho-1}\int_0^1\left|f'(t^\rho x^\rho+(1-t^\rho)a^\rho)\right|^q dt\right)^{\frac{1}{q}}$$

$$+\frac{\rho(b^\rho - x^\rho)}{2}\left(\int_0^1 t^{\alpha\rho+\rho-1}dt\right)^{1-\frac{1}{q}}\left(\int_0^1 t^{\alpha\rho+\rho-1}\left|f'(t^\rho x^\rho+(1-t^\rho)a^\rho)\right|^q dt\right)^{\frac{1}{q}}.$$

Since $\left|f'\right|^q$ is h-convex and $\left|f'(x^\rho)\right| \leq M$, $x \in [a,b]$, there for we have for $x \in (a,b)$

$$\left|f(x^\rho) - \frac{(\alpha\rho+\rho-1)\Gamma(\alpha)}{\rho^{1-\alpha}}\left[\frac{{}^\rho I^\alpha_{x-}f(a^\rho)}{2(x^\rho-a^\rho)^\alpha}+\frac{{}^\rho I^\alpha_{x+}f(b^\rho)}{2(b^\rho-x^\rho)^\alpha}\right]\right|$$

$$\leq \frac{\rho(x^\rho - a^\rho)}{2}\left(\frac{1}{\rho(\alpha+1)}\right)^{1-\frac{1}{q}} \times$$

$$\left(\int_0^1 t^{\alpha\rho+\rho-1}\left[h(t^\rho)\left|f'(x)\right|^q + h(1-t^\rho)\left|f'(a^\rho)\right|^q\right]dt\right)^{\frac{1}{q}}$$

$$+\frac{\rho(b^\rho - x^\rho)}{2}\left(\frac{1}{\rho(\alpha+1)}\right)^{1-\frac{1}{q}} \times$$

$$\left(\int_0^1 t^{\alpha\rho+\rho-1}\left[h(t^\rho)\left|f'(x)\right|^q + h(1-t^\rho)\left|f'(b^\rho)\right|^q\right]dt\right)^{\frac{1}{q}}$$

$$\leq \frac{M\rho(x^\rho - a^\rho)}{2}\left(\frac{1}{\rho(\alpha+1)}\right)^{1-\frac{1}{q}}\left(\int_0^1 t^{\alpha\rho+\rho-1}\left[h(t^\rho)+h(1-t^\rho)\right]dt\right)^{\frac{1}{q}}$$

$$+\frac{M\rho(b^\rho - x^\rho)}{2}\left(\frac{1}{\rho(\alpha+1)}\right)^{1-\frac{1}{q}}\left(\int_0^1 t^{\alpha\rho+\rho-1}\left[h(t^\rho)+h(1-t^\rho)\right]dt\right)^{\frac{1}{q}}$$

$$=\frac{M\rho(b^\rho - a^\rho)}{2}\left(\frac{1}{\rho(\alpha+1)}\right)^{1-\frac{1}{q}}\left(\int_0^1 t^{\alpha\rho+\rho-1}\left[h(t^\rho)+h(1-t^\rho)\right]dt\right)^{\frac{1}{q}}.$$

This completes the proof. □

Remark 2.9. (i) If we put $\rho = 1$ in (2.8) we get [15, Theorem 3].
(ii) If we put $\rho = 1$, $\alpha = 1$ and $x = \frac{a+b}{2}$ in (2.8), we get [15, Corollary 9].
(iii) If we put $\rho = 1$ and $h(t) = t$ in (2.8) which means that $\left|f'\right|$ is convex function, then we get [15, Corollary 7].
(iv) If we put $\rho = 1$ and $h(t) = t^s$, which means that $\left|f'\right|$ is h-convex function in (2.8), then we get [15, Corollary 8].

Corollary 2.10. *In Theorem 2.8, if we take $h(t) = 1$, which means that $\left|f'\right|^q$ is p-function, then (2.8) becomes the following inequality*

$$\left|f(x^\rho) - \frac{(\alpha\rho+\rho-1)\Gamma(\alpha)}{\rho^{1-\alpha}}\left[\frac{{}^\rho I^\alpha_{x-}f(a^\rho)}{2(x^\rho-a^\rho)^\alpha}+\frac{{}^\rho I^\alpha_{x+}f(b^\rho)}{2(b^\rho-x^\rho)^\alpha}\right]\right|$$

$$\leq \frac{(2)^{\frac{1}{q}-1}M(b^\rho - a^\rho)}{\rho(\alpha+1)};\ x \in (a,b), \tag{2.9}$$

with $\alpha, \rho > 0$.

Conclusion. Due to the fact that the Katugampola fractional integrals are the generalizations of both the Riemann-Liouville fractional integrals and Hadamard fractional integrals, so in our paper by taking $\rho = 1$ we have deduced the known results for Riemann-Liouville fractional integrals. All results proved in this research paper can also be deduced for the Hadamard fractional integrals by taking limits when parameter $\rho \to 0^+$.

Acknowledgement. The research work of Ghulam Farid is supported by the Higher Education Commission of Pakistan under NRPU 2016, Project No. 5421.
The research work of the second author was partially supported by the U.S. Army research office grant W911NF-15-1-0537.

References

[1] Dragomir, S.S., *Ostrowski-type inequalities for Lebesgue integral: A survey of recent results*, Aust. J. Math. Anal. Appl., **14**(2017), no. 1, 1-287.

[2] Dragomir, S.S., Pečarić, J., Persson, L.E., *Some inequalities of Hadamard type*, Soochow J. Math., **21**(1995), 335-341.

[3] Farid, G., *Some new Ostrowski-type inequalities via fractional integrals*, Int. J. Anal. App., **14**(2017), no. 1, 64-68.

[4] Farid, G., Katugampola, U.N., Usman, M., *Ostrowski-type fractional integral inequalities for S-Godunova-Levin functions via Katugampola fractional integrals*, Open J. Math. Sci., **1**(2017), no. 1, 97–110.

[5] Farid, G., Rafique, S., Rehman, A. Ur., *More on Ostrowski and Ostrowski-Gruss type inequalities*, Commun. Optim. Theory, 2017(2017), ArtID 15, 9 pages.

[6] Farid, G., Usman, M., *Ostrowski-type k-fractional integral inequalities for MT-convex and h-convex functions*, Nonlinear Funct. Anal. Appl., **22**(2017), no. 3, 627-639.

[7] Godunova, E.K., Levin, V.I., *Inequalities for functions of a broad class that contains convex, monotone and some other forms of functions*, Numerical Mathematics and Mathematical Physics (Russian), Moskov. Gos. Ped. Inst. Moscow, (1985), 138-142.

[8] Hadamard, J., *Essai sur l'etude des fonctions donnees par leur developpment de Taylor*, Journal de Mathématiques Pures et Appliquées, **8**(1892), 101–186.

[9] Holmgren, H.J., *Om differentialkalkylen med indices of hvad nature sam heist*, Kongliga Svenska Vetenskaps-Akademiens Handlinger, **5**(1865-1866), no. 11, 1-83.

[10] Hudzik, H., Maligranda, L., *Some remarks on s-convex functions*, Aequationes Math., **48**(1994), 100–111.

[11] Katugampola, U.N., *New approach to generalized fractional derivatives*, Bull. Math. Anal. Appl., **6**(2014), no. 4, 1–15.

[12] Laurent, H., *Sur le calcul des derivees a indicies quelconques*, Nouv. Annales de Mathematik, **3**(1884), no. 3, 240–252.

[13] Letnikov, A.V., *An explanation of the concepts of the theory of differentiation with arbitrary index*, Moscow Matem. Sbornik, **6**(1872), 413–445.

[14] Liu, W., *Ostrowski-type fractional integral inequalities for MT-convex function*, Miskole Mathematical Notes, **16**(2015), no. 1, 249–256.

[15] Matloka, M., *Ostrowski-type inequalities for functions whose derivatives are h-convex via fractional integrals*, Journal of Scientific Research and Reports, **3**(12)(2014), Article No. JSRR.2014.12.005, 1633–1641.

[16] Mitrinović, D.S., Pečarić, J., Fink, A.M., *Inequalities involving functions and their integrals and derivatives*, Kluwer Academic Publisher Group, Dordrecht, **53**(1991).

[17] Ostrowski, A., *Uber die Absolutabweichung einer differentierbaren funktion von ihrem integralmittelwert*, Comment. Math. Helv., **10**(1938), no. 1, 226–227.

[18] Sonin, N. Ya., *On differentiation with arbitrary index*, Moscow Matem. Sbornik, **6**(1869), no. 1, 1–38.

[19] Varosanec, S., *On h-convexity*, J. Math. Anal. Appl., **326**(2007), no. 1, 303–311.

Ghulam Farid
COMSATS University Islamabad
Attock Campus, Pakistan
e-mail: faridphdsms@hotmail.com ghlmfarid@cuiatk.edu.pk

Udita N. Katugampola
Department of Mathematics, Florida Polytechnic University
Lakeland, FL 33805, USA
e-mail: uditanalin@yahoo.com dkatugampola@floridapoly.edu

Muhammad Usman
COMSATS University Islamabad
Attock Campus, Pakistan
e-mail: usmanmani333@gmail.com

Stud. Univ. Babeş-Bolyai Math. 63(2018), No. 4, 475–482
DOI: 10.24193/subbmath.2018.4.05

Differential subordinations obtained by using a fractional operator

Eszter Szatmari

Abstract. We investigate several differential subordinations using the fractional operator $\mathbb{D}_\lambda^{\nu,n} : \mathcal{A} \to \mathcal{A}$, for $-\infty < \lambda < 2, \nu > -1, n \in \mathbb{N}_0 = \{0, 1, 2, \ldots\}$, introduced in [7].

Mathematics Subject Classification (2010): 30C45.

Keywords: Differential subordination, analytic function, fractional operator, convex function.

1. Introduction

Let $\mathcal{H}(U)$ denote the class of functions which are analytic in the open unit disk

$$U = \{z \in \mathbb{C} : |z| < 1\}.$$

For $a \in \mathbb{C}$ and $k \in \mathbb{N} = \{1, 2, \ldots\}$, let

$$\mathcal{H}[a, k] = \{f \in \mathcal{H}(U) : f(z) = a + a_k z^k + a_{k+1} z^{k+1} + \ldots\},$$

and

$$\mathcal{A} = \{f \in \mathcal{H}(U) : f(z) = z + a_2 z^2 + a_3 z^3 + \ldots, z \in U\}.$$

In [3], the fractional integral operator $D_z^{-\mu}$ of order $\mu, \mu > 0$, for the function $f \in \mathcal{A}$, is defined by

$$D_z^{-\mu} f(z) = \frac{1}{\Gamma(\mu)} \int_0^z \frac{f(t)}{(z - t)^{1-\mu}} dt, z \in U,$$

where the multiplicity of $(z - t)^{\mu-1}$ is removed by requiring $\log(z - t)$ to be real when $z - t > 0$.

Also, the fractional derivative operator D_z^λ of order $\lambda, \lambda \geq 0$, for the function $f \in \mathcal{A}$, is defined by

$$D_z^\lambda f(z) = \begin{cases} \frac{1}{\Gamma(1-\lambda)} \frac{d}{dz} \int\limits_0^z \frac{f(t)}{(z-t)^\lambda} dt, & 0 \leq \lambda < 1 \\ \frac{d^n}{dz^n} D_z^{\lambda-n} f(z), & n \leq \lambda < n+1 \end{cases}, n \in \mathbb{N}_0,$$

where the multiplicity of $(z-t)^{-\lambda}$ is understood in a similar way.

In [4] is defined the fractional differintegral operator $\Omega_z^\lambda : \mathcal{A} \to \mathcal{A}, -\infty < \lambda < 2$, by

$$\Omega_z^\lambda f(z) = \Gamma(2-\lambda) z^\lambda D_z^\lambda f(z), z \in U,$$

where $D_z^\lambda f(z)$ is the fractional integral of order$\lambda, -\infty < \lambda < 0$, and a fractional derivative of order $\lambda, 0 \leq \lambda < 2$.

In [6], the Sălăgean operator $\mathcal{D}^n$ of order n, $n \in \mathbb{N}_0$, for $f \in \mathcal{A}$, is defined by

$$\mathcal{D}^0 f(z) = f(z)$$

$$\mathcal{D}^1 f(z) = \mathcal{D} f(z) = z f'(z)$$

$$\mathcal{D}^n f(z) = \mathcal{D}(\mathcal{D}^{n-1} f(z)), n \in \mathbb{N}.$$

In [5], the Ruscheweyh operator $\mathcal{R}^\lambda : \mathcal{A} \to \mathcal{A}$ for $\lambda \geq -1$ is defined by

$$\mathcal{R}^\lambda f(z) = \frac{z}{(1-z)^{1+\lambda}} * f(z), z \in U,$$

where " $*$ " is the Hadamard product or convolution.

For $\lambda \in \mathbb{N}_0$ this operator is defined by

$$\mathcal{R}^\lambda f(z) = \frac{z(z^{\lambda-1} f(z))^\lambda}{\lambda!}, z \in U.$$

In [7], the fractional operator $\mathbb{D}_\lambda^{\nu,n} : \mathcal{A} \to \mathcal{A}$ for $-\infty < \lambda < 2, \nu > -1, n \in \mathbb{N}_0$, is defined as a composition of fractional differintegral operator, the Sălăgean operator and the Ruscheweyh operator:

$$\mathbb{D}_\lambda^{\nu,n} f(z) = \mathcal{R}^\nu \mathcal{D}^n \Omega_z^\lambda f(z).$$

The series expression of $\mathbb{D}_\lambda^{\nu,n} f(z)$ for $f \in \mathcal{A}$ of the form $f(z) = z + \sum\limits_{k=1}^{\infty} a_{k+1} z^{k+1}$ is given by

$$\mathbb{D}_\lambda^{\nu,n} f(z) = z + \sum_{k=1}^{\infty} \frac{(\nu+1)_k}{(2-\lambda)_k} (k+1)^{n+1} a_{k+1} z^{k+1},$$

$-\infty < \lambda < 2, \nu > -1, n \in \mathbb{N}_0, z \in U$, where the symbol $(\gamma)_k$ denotes the usual Pochhammer symbol, for $\gamma \in \mathbb{C}$, defined by

$$(\gamma)_k = \begin{cases} 1, k = 0 \\ \gamma(\gamma+1)\dots(\gamma+k-1), k \in \mathbb{N} \end{cases} = \frac{\Gamma(\gamma+k)}{\Gamma(\gamma)}, \gamma \in \mathbb{C} \setminus \mathbb{Z}_0^-.$$

Remark 1.1. [7] The fractional operator $\mathbb{D}_0^{\nu,0}$ is precisely the Ruscheweyh derivative operator $\mathcal{R}^\nu$ of order $\nu, \nu > -1$, and $\mathbb{D}_\lambda^{0,0}$ is the fractional differintegral operator Ω_z^λ of order $\lambda, -\infty < \lambda < 2$, while $\mathbb{D}_0^{0,n} = \mathcal{D}^n$ and $\mathbb{D}_\lambda^{1-\lambda,n} = \mathcal{D}^{n+1}$ are the Sălăgean operators, respectively, of order n and n+1, $n \in \mathbb{N}_0$.

Remark 1.2. [7] The operator $\mathbb{D}_{\lambda}^{\nu,n}$ satisfies the following identity:

$$\mathbb{D}_{\lambda}^{\nu+1,n} f(z) = \frac{\nu}{\nu+1}\mathbb{D}_{\lambda}^{\nu,n} f(z) + \frac{1}{\nu+1} z(\mathbb{D}_{\lambda}^{\nu,n} f(z))', \tag{1.1}$$

where $-\infty < \lambda < 2, \nu > -1, n \in \mathbb{N}_0$.

Remark 1.3. [8] The operator $\mathbb{D}_{\lambda}^{\nu,n}$ satisfies the following identities:

$$\mathbb{D}_{\lambda}^{\nu,n+1} f(z) = z(\mathbb{D}_{\lambda}^{\nu,n} f(z))', \tag{1.2}$$

where $-\infty < \lambda < 2, \nu > -1, n \in \mathbb{N}_0$, and

$$\mathbb{D}_{\lambda+1}^{\nu,n} f(z) = -\frac{\lambda}{1-\lambda}\mathbb{D}_{\lambda}^{\nu,n} f(z) + \frac{1}{1-\lambda} z(\mathbb{D}_{\lambda}^{\nu,n} f(z))', \tag{1.3}$$

where $-\infty < \lambda < 1, \nu > -1, n \in \mathbb{N}_0$.

Definition 1.4. [1, p. 4] *Let $f, F \in \mathcal{H}(U)$. The function f is said to be subordinate to F, written $f \prec F$, or $f(z) \prec F(z)$, if there exists a function $w \in \mathcal{H}(U)$, with $w(0) = 0$ and $\big|w(z)\big| < 1, z \in U$, such that $f(z) = F\big[w(z)\big], z \in U$.*

In order to prove our results we shall need the following lemma.

Lemma 1.5. [2] *Let q be a convex function in U and let*

$$h(z) = q(z) + n\alpha z q'(z),$$

where $\alpha > 0$ and n is a positive integer. If

$$p(z) = q(0) + p_n z^n + \ldots \in \mathcal{H}[q(0), n]$$

and

$$p(z) + \alpha z p'(z) \prec h(z)$$

then

$$p(z) \prec q(z),$$

and this result is sharp.

2. Main results

Theorem 2.1. *Let g be a convex function, $g(0) = 1$ and let h be a function such that*

$$h(z) = g(z) + \frac{1}{\nu+1} z g'(z), \nu > -1.$$

If $f \in \mathcal{A}$ verifies the differential subordination

$$\big(\mathbb{D}_{\lambda}^{\nu+1,n} f(z)\big)' \prec h(z) \tag{2.1}$$

then

$$\big(\mathbb{D}_{\lambda}^{\nu,n} f(z)\big)' \prec g(z).$$

The result is sharp.

Proof. If we denote by

$$p(z) = \left(\mathbb{D}_{\lambda}^{\nu,n} f(z)\right)',$$

where $p(z) \in \mathcal{H}[1,1]$, then, by (1.1), we get

$$\left(\mathbb{D}_{\lambda}^{\nu+1,n} f(z)\right)' = p(z) + \frac{1}{\nu+1} z p'(z), z \in U. \tag{2.2}$$

From (2.1) and (2.2) we obtain

$$p(z) + \frac{1}{\nu+1} z p'(z) \prec g(z) + \frac{1}{\nu+1} z g'(z) \equiv h(z).$$

Applying Lemma 1, we get

$$p(z) \prec g(z)$$

or

$$\left(\mathbb{D}_{\lambda}^{\nu,n} f(z)\right)' \prec g(z).$$

This result is sharp. □

Theorem 2.2. *Let g be a convex function, $g(0) = 1$ and let h be a function such that*

$$h(z) = g(z) + \frac{1}{1-\lambda} z g'(z), -\infty < \lambda < 1.$$

If $f \in \mathcal{A}$ verifies the differential subordination

$$\left(\mathbb{D}_{\lambda+1}^{\nu,n} f(z)\right)' \prec h(z) \tag{2.3}$$

then

$$\left(\mathbb{D}_{\lambda}^{\nu,n} f(z)\right)' \prec g(z).$$

The result is sharp.

Proof. If we denote by

$$p(z) = \left(\mathbb{D}_{\lambda}^{\nu,n} f(z)\right)',$$

where $p(z) \in \mathcal{H}[1,1]$, then, by (1.3), we get

$$\left(\mathbb{D}_{\lambda+1}^{\nu,n} f(z)\right)' = p(z) + \frac{1}{1-\lambda} z p'(z), z \in U. \tag{2.4}$$

From (2.3) and (2.4) we obtain

$$p(z) + \frac{1}{1-\lambda} z p'(z) \prec g(z) + \frac{1}{1-\lambda} z g'(z) \equiv h(z).$$

Applying Lemma 1, we get

$$p(z) \prec g(z)$$

or

$$\left(\mathbb{D}_{\lambda}^{\nu,n} f(z)\right)' \prec g(z).$$

This result is sharp. □

Theorem 2.3. *Let g be a convex function, $g(0) = 1$ and let h be a function such that*

$$h(z) = g(z) + zg'(z), n \in \mathbb{N}_0.$$

If $f \in \mathcal{A}$ verifies the differential subordination

$$\left(\mathbb{D}_\lambda^{\nu,n+1} f(z)\right)' \prec h(z) \tag{2.5}$$

then

$$\left(\mathbb{D}_\lambda^{\nu,n} f(z)\right)' \prec g(z).$$

The result is sharp.

Proof. If we denote by

$$p(z) = \left(\mathbb{D}_\lambda^{\nu,n} f(z)\right)',$$

where $p(z) \in \mathcal{H}[1,1]$, then, by (1.2), we get

$$\left(\mathbb{D}_\lambda^{\nu,n+1} f(z)\right)' = p(z) + zp'(z), z \in U. \tag{2.6}$$

From (2.5) and (2.6) we obtain

$$p(z) + zp'(z) \prec g(z) + zg'(z) \equiv h(z).$$

Applying Lemma 1, we get

$$p(z) \prec g(z)$$

or

$$\left(\mathbb{D}_\lambda^{\nu,n} f(z)\right)' \prec g(z).$$

This result is sharp. □

Theorem 2.4. *Let g be a convex function, $g(0) = 1$ and let h be a function such that*

$$h(z) = g(z) + zg'(z), z \in U.$$

If $f \in \mathcal{A}$ verifies the differential subordination

$$\left(\mathbb{D}_\lambda^{\nu,n} f(z)\right)' \prec h(z), z \in U \tag{2.7}$$

then

$$\frac{\mathbb{D}_\lambda^{\nu,n} f(z)}{z} \prec g(z).$$

The result is sharp.

Proof. Let

$$p(z) = \frac{\mathbb{D}_\lambda^{\nu,n} f(z)}{z}, z \in U.$$

Differentiating we obtain

$$p'(z) = \frac{\left(\mathbb{D}_\lambda^{\nu,n} f(z)\right)'}{z} - \frac{p(z)}{z}.$$

We get

$$\left(\mathbb{D}_\lambda^{\nu,n} f(z)\right)' = p(z) + zp'(z).$$

The subordination (2.7) becomes

$$p(z) + zp'(z) \prec g(z) + zg'(z).$$

Applying Lemma 1, we get

$$p(z) \prec g(z)$$

or

$$\frac{\mathbb{D}_{\lambda}^{\nu,n} f(z)}{z} \prec g(z).$$

This result is sharp. □

Theorem 2.5. *Let g be a convex function, $g(0) = 1$ and let h be a function such that*

$$h(z) = g(z) + zg'(z), z \in U.$$

If $f \in \mathcal{A}$ verifies the differential subordination

$$\left(\frac{z\mathbb{D}_{\lambda}^{\nu+1,n} f(z)}{\mathbb{D}_{\lambda}^{\nu,n} f(z)}\right)' \prec h(z), z \in U, \tag{2.8}$$

then

$$\frac{\mathbb{D}_{\lambda}^{\nu+1,n} f(z)}{\mathbb{D}_{\lambda}^{\nu,n} f(z)} \prec g(z), z \in U.$$

The result is sharp.

Proof. Let

$$p(z) = \frac{\mathbb{D}_{\lambda}^{\nu+1,n} f(z)}{\mathbb{D}_{\lambda}^{\nu,n} f(z)}.$$

We obtain

$$\left(\frac{z\mathbb{D}_{\lambda}^{\nu+1,n} f(z)}{\mathbb{D}_{\lambda}^{\nu,n} f(z)}\right)' = p(z) + zp'(z).$$

The subordination (2.8) becomes

$$p(z) + zp'(z) \prec g(z) + zg'(z).$$

Applying Lemma 1, we get

$$p(z) \prec g(z)$$

or

$$\frac{\mathbb{D}_{\lambda}^{\nu+1,n} f(z)}{\mathbb{D}_{\lambda}^{\nu,n} f(z)} \prec g(z), z \in U.$$

This result is sharp. □

Theorem 2.6. *Let g be a convex function, $g(0) = 1$ and let h be a function such that*

$$h(z) = g(z) + zg'(z), z \in U.$$

If $f \in \mathcal{A}$ verifies the differential subordination

$$\left(\frac{z\mathbb{D}_{\lambda+1}^{\nu,n} f(z)}{\mathbb{D}_{\lambda}^{\nu,n} f(z)}\right)' \prec h(z), z \in U, -\infty < \lambda < 1, \tag{2.9}$$

then

$$\frac{\mathbb{D}_{\lambda+1}^{\nu,n} f(z)}{\mathbb{D}_{\lambda}^{\nu,n} f(z)} \prec g(z), z \in U.$$

The result is sharp.

Proof. Let

$$p(z) = \frac{\mathbb{D}^{\nu,n}_{\lambda+1} f(z)}{\mathbb{D}^{\nu,n}_{\lambda} f(z)}.$$

We obtain

$$\left(\frac{z\mathbb{D}^{\nu,n}_{\lambda+1} f(z)}{\mathbb{D}^{\nu,n}_{\lambda} f(z)} \right)' = p(z) + zp'(z).$$

The subordination (2.9) becomes

$$p(z) + zp'(z) \prec g(z) + zg'(z).$$

Applying Lemma 1, we get

$$p(z) \prec g(z)$$

or

$$\frac{\mathbb{D}^{\nu,n}_{\lambda+1} f(z)}{\mathbb{D}^{\nu,n}_{\lambda} f(z)} \prec g(z), z \in U.$$

This result is sharp. □

Theorem 2.7. *Let g be a convex function, $g(0) = 1$ and let h be a function such that*

$$h(z) = g(z) + zg'(z), z \in U.$$

If $f \in \mathcal{A}$ verifies the differential subordination

$$\left(\frac{z\mathbb{D}^{\nu,n+1}_{\lambda} f(z)}{\mathbb{D}^{\nu,n}_{\lambda} f(z)} \right)' \prec h(z), z \in U, \tag{2.10}$$

then

$$\frac{\mathbb{D}^{\nu,n+1}_{\lambda} f(z)}{\mathbb{D}^{\nu,n}_{\lambda} f(z)} \prec g(z), z \in U.$$

The result is sharp.

Proof. Let

$$p(z) = \frac{\mathbb{D}^{\nu,n+1}_{\lambda} f(z)}{\mathbb{D}^{\nu,n}_{\lambda} f(z)}.$$

We obtain

$$\left(\frac{z\mathbb{D}^{\nu,n+1}_{\lambda} f(z)}{\mathbb{D}^{\nu,n}_{\lambda} f(z)} \right)' = p(z) + zp'(z).$$

The subordination (2.10) becomes

$$p(z) + zp'(z) \prec g(z) + zg'(z).$$

Applying Lemma 1, we get

$$p(z) \prec g(z)$$

or

$$\frac{\mathbb{D}^{\nu,n+1}_{\lambda} f(z)}{\mathbb{D}^{\nu,n}_{\lambda} f(z)} \prec g(z), z \in U.$$

This result is sharp. □

Acknowledgements. The author wish to express her sincere thanks to the reviewer of this paper for several useful comments and suggestions.

References

[1] Miller, S.S., Mocanu, P.T., *Differential Subordinations: Theory and Applications,* Pure and Applied Mathematics No. 225, Marcel Dekker, New York, 2000.

[2] Miller, S.S., Mocanu, P.T., *On some classes of first order differential subordinations,* Michigan Math. J., **32**(1985), 185-195.

[3] Owa, S., *On the distortion theorems I,* Kyungpook Math. J., **18**(1978), 53-59.

[4] Owa, S., Srivastava, H.M., *Univalent and starlike generalized hypergeometric functions,* Canad. J. Math., **39**(1987), 1057-1077.

[5] Ruscheweyh, S., *New criteria for univalent functions,* Proc. Amer. Math. Soc., **49**(1975), 109-115.

[6] Sălăgean, G.S., *Subclasses of univalent functions,* Lecture Notes in Math., Springer Verlag, **1013**(1983), 362-372.

[7] Sharma, P., Raina, R.K., Sălăgean, G.S., *Some geometric properties of analytic functions involving a new fractional operator,* Mediterr. J. Math., **13**(2016), 4591-4605.

[8] Szatmari, E., *On a class of analytic functions defined by a fractional operator,* Mediterr. J. Math., (2018), 15:158.

Eszter Szatmari
Babeş-Bolyai University
Faculty of Mathematics and Computer Sciences
1, Kogălniceanu Street, 400084 Cluj-Napoca, Romania
e-mail: `szatmari.eszter@math.ubbcluj.ro`

Stud. Univ. Babeş-Bolyai Math. 63(2018), No. 4, 483–501
DOI: 10.24193/subbmath.2018.4.06

Sufficient conditions of boundedness of L-index and analog of Hayman's Theorem for analytic functions in a ball

Andriy Bandura and Oleh Skaskiv

Abstract. We generalize some criteria of boundedness of **L**-index in joint variables for analytic in an unit ball functions. Our propositions give an estimate maximum modulus of the analytic function on a skeleton in polydisc with the larger radii by maximum modulus on a skeleton in the polydisc with the lesser radii. An analog of Hayman's Theorem for the functions is obtained. Also we established a connection between class of analytic in ball functions of bounded l_j-index in every direction $\mathbf{1}_j$, $j \in \{1,\ldots,n\}$ and class of analytic in ball of functions of bounded **L**-index in joint variables, where $\mathbf{L}(z) = (l_1(z),\ldots,l_n(z))$, $l_j : \mathbb{B}^n \to \mathbb{R}_+$ is continuous function, $\mathbf{1}_j = (0,\ldots,0,\underbrace{1}_{j\text{-th place}},0,\ldots,0) \in \mathbb{R}^n_+$, $z \in \mathbb{C}^n$.

Mathematics Subject Classification (2010): 32A05, 32A10, 32A30, 32A40, 30H99.

Keywords: Analytic function, unit ball, bounded **L**-index in joint variables, maximum modulus, partial derivative, bounded L-index in direction.

1. Introduction

Recently, there was introduced a concept of analytic function in a ball in $\mathbb{C}^n$ of bounded **L**-index in joint variables [8]. We also obtained criterion of boundedness of **L**-index in joint variables which describes a local behavior of partial derivatives on a skeleton in the polydisc and established other important properties of analytic functions in a ball of bounded **L**-index in joint variables. Those investigations used an idea of exhaustion of a ball in $\mathbb{C}^n$ by polydiscs.

The presented paper is a continuation of our investigations from [8]. We set the goal to prove new analogues of criteria of boundedness of **L**-index in joint variables for analytic in a ball functions. Particular, we prove an estimate of maximum modulus on a greater polydisc by maximum modulus on a lesser polydisc (Theorems 3.1, 3.2) and obtain an analog of Hayman's Theorem for analytic functions in a ball of bounded

L-index in joint variables (Theorems 4.1 and 4.2). For entire functions similar propositions were obtained by A. I. Bandura, M. T. Bordulyak, O. B. Skaskiv [4, 5] in a case $\mathbf{L}(z) = (l_1(z), \dots, l_n(z))$, $z \in \mathbb{C}^n$. Also A. I. Bandura, N.V. Petrechko, O. B. Skaskiv [6, 7] deduced same results for analytic in a polydisc functions. Hayman's Theorem and its generalizations for different classes of analytic functions [1, 3, 5, 7, 12, 15, 20, 21] are very important in theory of functions of bounded index. The criterion is helpful [1, 9] to investigate boundedness of index of entire solutions of ordinary or partial differential equations.

Note that the corresponding theorems for entire functions of bounded l-index and of bounded L-index in direction were also applied to investigate infinite products (see bibliography in [21, 1]). Thus, those generalizations for analytic in a ball functions are necessary to study **L**-index in joint variables of analytic solutions of PDE's, its systems and multidimensional counterparts of Blaschke products. At the end of the paper, we present a scheme of application of Hayman's Theorem to study properties of analytic solutions in the unit ball.

2. Main definitions and notations

We need some standard notations. Denote
$\mathbb{R}_+ = (0, +\infty)$, $\mathbf{0} = (0, \dots, 0)$, $\mathbf{1} = (1, \dots, 1)$,
$\mathbf{1}_j = (0, \dots, 0, \underbrace{1}_{j\text{–th place}}, 0, \dots, 0) \in \mathbb{R}^n_+$,
$R = (r_1, \dots, r_n) \in \mathbb{R}^n_+$, $z = (z_1, \dots, z_n) \in \mathbb{C}^n$, $|z| = \sqrt{\sum_{j=1}^n |z_j|^2}$.

For $A = (a_1, \dots, a_n) \in \mathbb{R}^n$, $B = (b_1, \dots, b_n) \in \mathbb{R}^n$ we will use formal notations without violation of the existence of these expressions
$AB = (a_1 b_1, \cdots, a_n b_n)$, $A/B = (a_1/b_1, \dots, a_n/b_n)$,
$A^B = a_1^{b_1} a_2^{b_2} \cdot \ldots \cdot a_n^{b_n}$, $\|A\| = a_1 + \cdots + a_n$,
and the notation $A < B$ means that $a_j < b_j$, $j \in \{1, \dots, n\}$; the relation $A \le B$ is defined similarly. For $K = (k_1, \dots, k_n) \in \mathbb{Z}^n_+$ denote $K! = k_1! \cdot \ldots \cdot k_n!$. The polydisc $\{z \in \mathbb{C}^n : \ |z_j - z_j^0| < r_j, \ j = 1, \dots, n\}$ is denoted by $\mathbb{D}^n(z^0, R)$, its skeleton $\{z \in \mathbb{C}^n : \ |z_j - z_j^0| = r_j, \ j = 1, \dots, n\}$ is denoted by $\mathbb{T}^n(z^0, R)$, and the closed polydisc $\{z \in \mathbb{C}^n : \ |z_j - z_j^0| \le r_j, \ j = 1, \dots, n\}$ is denoted by $\mathbb{D}^n[z^0, R]$. The open ball $\{z \in \mathbb{C}^n : \ |z - z^0| < r\}$ is denoted by $\mathbb{B}^n(z^0, r)$, its boundary is a sphere $\mathbb{S}^n(z^0, r) = \{z \in \mathbb{C}^n : \ |z - z^0| = r\}$, the closed ball $\{z \in \mathbb{C}^n : \ |z - z^0| \le r\}$ is denoted by $\mathbb{B}^n[z^0, r]$, $\mathbb{B}^n = \mathbb{B}^n(\mathbf{0}, 1)$, $\mathbb{D} = \mathbb{B}^1 = \{z \in \mathbb{C} : \ |z| < 1\}$.

For $K = (k_1, \dots, k_n) \in \mathbb{Z}^n_+$ and the partial derivatives of an analytic in $\mathbb{B}^n$ function $F(z) = F(z_1, \dots, z_n)$ we use the notation

$$F^{(K)}(z) = \frac{\partial^{\|K\|} F}{\partial z^K} = \frac{\partial^{k_1 + \cdots + k_n} F}{\partial z_1^{k_1} \ldots \partial z_n^{k_n}}.$$

Let $\mathbf{L}(z) = (l_1(z), \dots, l_n(z))$, where $l_j(z) : \mathbb{B}^n \to \mathbb{R}_+$ is a continuous function such that

$$(\forall z \in \mathbb{B}^n) \colon \ l_j(z) > \beta/(1 - |z|), \ j \in \{1, \dots, n\}, \tag{2.1}$$

where $\beta > \sqrt{n}$ is a some constant. For a polydisc A.I. Bandura, N.V. Petrechko and O.B. Skaskiv [6, 7] imposed the restriction $(\forall z \in \mathbb{D}^n(\mathbf{0},\mathbf{1}))\colon\ l_j(z) > \beta/(1-|z_j|)$, $j \in \{1,\dots,n\}$. A similar condition is used in one-dimensional case by S.N. Strochyk, M.M. Sheremeta, V.O. Kushnir [22, 14, 21].

Note that if $R \in \mathbb{R}^n_+$, $|R| \le \beta$, $z^0 \in \mathbb{B}^n$ and $z \in \mathbb{D}^n[z^0, R/\mathbf{L}(z^0)]$ then $z \in \mathbb{B}^n$ (see Remark 1 in [8]).

An analytic function $F\colon \mathbb{B}^n \to \mathbb{C}$ is said to be of *bounded* $\mathbf{L}$*-index (in joint variables),* if there exists $n_0 \in \mathbb{Z}_+$ such that for all $z \in \mathbb{B}^n$ and for all $J \in \mathbb{Z}^n_+$

$$\frac{|F^{(J)}(z)|}{J!\mathbf{L}^J(z)} \le \max\left\{\frac{|F^{(K)}(z)|}{K!\mathbf{L}^K(z)}:\ K \in \mathbb{Z}^n_+,\ \|K\| \le n_0\right\}. \tag{2.2}$$

The least such integer n_0 is called the $\mathbf{L}$*-index in joint variables of the function* F and is denoted by $N(F,\mathbf{L},\mathbb{B}^n)$ (see [8]). Entire and analytic in polydisc functions of bounded $\mathbf{L}$-index in joint variables are considered in [4, 5, 6, 7, 10, 13, 19, 18, 16, 17].

By $Q(\mathbb{B}^n)$ we denote the class of functions $\mathbf{L}$, which satisfy (2.1) and the following condition

$$(\forall R \in \mathbb{R}^n_+, |R| \le \beta,\ j \in \{1,\dots,n\})\colon\ 0 < \lambda_{1,j}(R) \le \lambda_{2,j}(R) < \infty, \tag{2.3}$$

where $\lambda_{1,j}(R) = \inf_{z^0 \in \mathbb{B}^n} \inf\left\{l_j(z)/l_j(z^0) : z \in \mathbb{D}^n\left[z^0, R/\mathbf{L}(z^0)\right]\right\}$,

$\lambda_{2,j}(R) = \sup_{z^0 \in \mathbb{B}^n} \sup\left\{l_j(z)/l_j(z^0) : z \in \mathbb{D}^n\left[z^0, R/\mathbf{L}(z^0)\right]\right\}$.

$\Lambda_1(R) = (\lambda_{1,1}(R),\dots,\lambda_{1,n}(R))$, $\Lambda_2(R) = (\lambda_{2,1}(R),\dots,\lambda_{2,n}(R))$.

We need the following results.

Theorem 2.1 ([8]). *Let* $\mathbf{L} \in Q(\mathbb{B}^n)$. *An analytic in* $\mathbb{B}^n$ *function* F *has bounded* $\mathbf{L}$*-index in joint variables if and only if for each* $R \in \mathbb{R}^n_+$, $|R| \le \beta$, *there exist* $n_0 \in \mathbb{Z}_+$, $p_0 > 0$ *such that for every* $z^0 \in \mathbb{B}^n$ *there exists* $K^0 \in \mathbb{Z}^n_+$, $\|K^0\| \le n_0$, *and*

$$\max\left\{\frac{|F^{(K)}(z)|}{K!\mathbf{L}^K(z)}\colon \|K\| \le n_0,\ z \in \mathbb{D}^n\left[z^0, R/\mathbf{L}(z^0)\right]\right\} \le p_0\frac{|F^{(K^0)}(z^0)|}{K^0!\mathbf{L}^{K^0}(z^0)}. \tag{2.4}$$

Denote $\widetilde{\mathbf{L}}(z) = (\widetilde{l}_1(z),\dots,\widetilde{l}_n(z))$. The notation $\mathbf{L} \asymp \widetilde{\mathbf{L}}$ means that there exist

$$\Theta_1 = (\theta_{1,j},\dots,\theta_{1,n}) \in \mathbb{R}^n_+,\ \Theta_2 = (\theta_{2,j},\dots,\theta_{2,n}) \in \mathbb{R}^n_+$$

such that $\forall z \in \mathbb{B}^n$ $\theta_{1,j}\widetilde{l}_j(z) \le l_j(z) \le \theta_{2,j}\widetilde{l}_j(z)$ for each $j \in \{1,\dots,n\}$.

Theorem 2.2 ([8]). *Let* $\mathbf{L} \in Q(\mathbb{B}^n)$, $\mathbf{L} \asymp \widetilde{\mathbf{L}}$, $\beta|\Theta_1| > \sqrt{n}$. *An analytic in* $\mathbb{B}^n$ *function* F *has bounded* $\widetilde{\mathbf{L}}$*-index in joint variables if and only if* F *has bounded* $\mathbf{L}$*-index in joint variables.*

3. Local behaviour of maximum modulus of analytic in ball function

For an analytic in $\mathbb{B}^n$ function F we put

$$M(r, z^0, F) = \max\{|F(z)|\colon z \in \mathbb{T}^n(z^0, r)\},$$

where $z^0 \in \mathbb{B}^n$, $r \in \mathbb{R}^n_+$. Then $M(R, z^0, F) = \max\{|F(z)|\colon z \in \mathbb{D}^n[z^0, R]\}$, because the maximum modulus for an analytic function in a closed polydisc is attained on its skeleton.

The following proposition uses an idea about the possibility of replacing universal quantifier by existential quantifier in sufficient conditions of index boundedness [2]. To prove an analog of Hayman's Theorem we need this theorem which has an independent interest.

Theorem 3.1. *Let $\mathbf{L} \in Q^n$, $F : \mathbb{B}^n \to \mathbb{C}$ be analytic function. If there exist R', $R'' \in \mathbb{R}^n_+$, $R' < R''$, $|R''| < \beta$ and $p_1 = p_1(R', R'') \geq 1$ such that for every $z^0 \in \mathbb{C}^n$*

$$M\left(\frac{R''}{\mathbf{L}(z^0)}, z^0, F\right) \leq p_1 M\left(\frac{R'}{\mathbf{L}(z^0)}, z^0, F\right) \tag{3.1}$$

then F is of bounded $\mathbf{L}$-index in joint variables.

Proof. At first, we assume that $\mathbf{0} < R' < \mathbf{1} < R''$.
Let $z^0 \in \mathbb{B}^n$ be an arbitrary point. We expand a function F in power series

$$F(z) = \sum_{K \geq \mathbf{0}} b_K (z - z^0)^K = \sum_{k_1, \ldots, k_n \geq 0} b_{k_1, \ldots, k_n} (z_1 - z_1^0)^{k_1} \ldots (z_n - z_n^0)^{k_n}, \tag{3.2}$$

where $b_K = b_{k_1, \ldots, k_n} = \frac{F^{(K)}(z^0)}{K!}$.
Let $\mu(R, z^0, F) = \max\{|b_K| R^K\colon\ K \geq \mathbf{0}\}$ be a maximal term of power series (3.2) and

$$\nu(R) = \nu(R, z^0, F) = (\nu_1^0(R), \ldots, \nu_n^0(R))$$

be a set of indices such that

$$\mu(R, z^0, F) = |b_{\nu(R)}| R^{\nu(R)},$$

$$\|\nu(R)\| = \sum_{j=1}^{n} \nu_j(R) = \max\{\|K\|\colon K \geq \mathbf{0},\ |b_K| R^K = \mu(R, z^0, F)\}.$$

In view of inequality (3.8) we obtain for any $|R| < 1 - |z^0|$,

$$\mu(R, z^0, F) \leq M(R, z^0, F).$$

Then for given R' and R'' with $0 < |R'| < 1 < |R''| < \beta$ we conclude

$$\begin{aligned} M(R'R, z^0, F) &\leq \sum_{k \geq \mathbf{0}} |b_k| (R'R)^k \leq \sum_{k \geq \mathbf{0}} \mu(R, z^0, F) (R')^k \\ &= \mu(R, z^0, F) \sum_{k \geq \mathbf{0}} (R')^k = \prod_{j=1}^{n} \frac{1}{1 - r'_j} \mu(R, z^0, F). \end{aligned}$$

Besides,

$$\begin{aligned}\ln \mu(R, z^0, F) &= \ln\{|b_{\nu(R)}|R^{\nu(R)}\} = \ln\left\{|b_{\nu(R)}|(RR'')^{\nu(R)}\frac{1}{(R'')^{\nu(R)}}\right\}\\ &= \ln\{|b_{\nu(R)}|(RR'')^{\nu(R)}\} + \ln\left\{\frac{1}{(R'')^{\nu(R)}}\right\}\\ &\le \ln \mu(R''R, z^0, F) - \|\nu(R)\| \ln \min_{1\le j\le n} r_j''.\end{aligned}$$

This implies that

$$\|\nu(R)\| \le \frac{1}{\ln \min_{1\le j\le n} r_j''}(\ln \mu(R''R, z^0, F) - \ln \mu(R, z^0, F))$$

$$\le \frac{1}{\ln \min_{1\le j\le n} r_j''}\left(\ln M(R''R, z^0, F) - \ln(\prod_{j=1}^{n}(1-r_j')M(R'R, z^0, F))\right)$$

$$\le \frac{1}{\ln \min_{1\le j\le n} r_j''}\left(\ln M(R''R, z^0, F) - \ln M(R'R, z^0, F)\right) - \frac{\sum_{j=1}^{n}\ln(1-r_j')}{\min_{1\le j\le n} r_j''}$$

$$= \frac{1}{\min_{1\le j\le n} r_j''}\ln\frac{M(R''R, z^0, F)}{M(R'R, z^0, F)} - \frac{\sum_{j=1}^{n}\ln(1-r_j)}{\min_{1\le j\le n} r_j''}. \tag{3.3}$$

Put $R = \frac{\mathbf{1}}{\mathbf{L}(z^0)}$. Now let $N(F, z^0, \mathbf{L})$ be the **L**-index of the function F in joint variables at point z^0 i. e. it is the least integer for which inequality (2.2) holds at point z^0. Clearly that

$$N(F, z^0, \mathbf{L}) \le \nu\left(\frac{1}{\mathbf{L}(z^0)}, z^0, F\right) = \nu(R, z^0, F). \tag{3.4}$$

But

$$M\left(R''/\mathbf{L}(z^0), z^0, F\right) \le p_1(R', R'')M\left(R'/\mathbf{L}(z^0), z^0, F\right). \tag{3.5}$$

Therefore, from (3.3), (3.4), (3.5) we obtain that $\forall z^0 \in \mathbb{B}^n$

$$N(F, z^0, \mathbf{L}) \le \frac{-\sum_{j=1}^{2}\ln(1-r_j')}{\ln \min\{r_1'', r_2''\}} + \frac{\ln p_1(R', R'')}{\ln \min\{r_1'', r_2''\}}.$$

This means that F has bounded **L**-index in joint variables, if $\mathbf{0} < R' < \mathbf{1} < R''$, $|R''| < \beta$.

Now we will prove the theorem for any $\mathbf{0} < R' < R''$, $|R''| < \beta$. From (3.1) with $\mathbf{0} < R_1 < R_2$ it follows that

$$\begin{aligned}&\max\left\{|F(z)| : z \in \mathbb{T}^n\left(z^0, \frac{2R''}{R'+R''}\frac{R'+R''}{2\mathbf{L}(z^0)}\right)\right\}\\ &\le P_1 \max\left\{|F(z)| : z \in \mathbb{T}^n\left(z^0, \frac{2R'}{R'+R''}\frac{R'+R''}{2\mathbf{L}(z^0)}\right)\right\}.\end{aligned}$$

Denoting $\widetilde{\mathbf{L}}(z)=\frac{2\mathbf{L}(z)}{R'+R''}$, we obtain

$$\max\left\{|F(z)|: z\in\mathbb{T}^n\left(z^0,\frac{2R''}{(R'+R'')\widetilde{\mathbf{L}}(z^0)}\right)\right\}$$
$$\leq P_1\max\left\{|F(z)|: z\in\mathbb{T}^n\left(z^0,\frac{2R''}{(R'+R'')\widetilde{\mathbf{L}}(z^0)}\right)\right\},$$

where $\mathbf{0}<\frac{2R'}{R'+R''}<\mathbf{1}<\frac{2R''}{R'+R''}$. Taking into account the first part of the proof, we conclude that the function F has bounded $\widetilde{\mathbf{L}}$-index in joint variables. By Theorem 2.2, the function F is of bounded $\mathbf{L}$-index in joint variables. □

Also the corresponding necessary conditions are valid.

Theorem 3.2. *Let $\mathbf{L}\in Q(\mathbb{B}^n)$. If an analytic in $\mathbb{B}^n$ function F has bounded $\mathbf{L}$-index in joint variables then for any R', $R''\in\mathbb{R}^n_+$, $R'<R''$, $|R''|\leq\beta$, there exists a number $p_1=p_1(R',R'')\geq 1$ such that for every $z^0\in\mathbb{B}^n$ inequality* (3.1) *holds.*

Proof. Let $N(F,\mathbf{L})=N<+\infty$. Suppose that inequality (3.1) does not hold i.e. there exist R', R'', $0<|R'|<|R''|<\beta$, such that for each $p_*\geq 1$ and for some $z^0=z^0(p_*)$

$$M\left(\frac{R''}{\mathbf{L}(z^0)},z^0,F\right)>p_*M\left(\frac{R'}{\mathbf{L}(z^0)},z^0,F\right). \tag{3.6}$$

By Theorem 2.1, there exists a number $p_0=p_0(R'')\geq 1$ such that for every $z^0\in\mathbb{B}^n$ and some $K^0\in\mathbb{Z}^n_+$, $\|K^0\|\leq N$, (i.e. $n_0=N$, see proof of necessity of Theorem 2.1 in [8]) one has

$$M\left(\frac{R''}{\mathbf{L}(z^0)},z^0,F^{(K^0)}\right)\leq p_0|F^{(K^0)}(z^0)|. \tag{3.7}$$

We put

$$b_1=p_0\left(\prod_{j=2}^n\lambda_{2,j}^N(R'')\right)(N!)^{n-1}\left(\sum_{j=1}^N\frac{(N-j)!}{(r_1'')^j}\right)\left(\frac{r_1''r_2''\dots r_n''}{r_1'r_2'\dots r_n'}\right)^N,$$

$$b_2=p_0\left(\prod_{j=3}^n\lambda_{2,j}^N(R'')\right)(N!)^{n-2}\left(\sum_{j=1}^N\frac{(N-j)!}{(r_2'')^j}\right)\left(\frac{r_2''\dots r_n''}{r_2'\dots r_n'}\right)^N\left\{1,\frac{1}{(r_1')^N}\right\},$$

$$\dots$$

$$b_{n-1}=p_0\lambda_{2,n}^N(R')N!\left(\sum_{j=1}^N\frac{(N-j)!}{(r_{n-1}'')^j}\right)\left(\frac{r_{n-1}''r_n''}{r_{n-1}'r_n'}\right)^N\max\left\{1,\frac{1}{(r_1'\dots r_{n-2}')^N}\right\},$$

$$b_n=p_0\left(\sum_{j=1}^N\frac{(N-j)!}{(r_n'')^j}\right)\left(\frac{r_n''}{r_n'}\right)^N\max\left\{1,\frac{1}{(r_1'\dots r_{n-1}')^N}\right\}$$

and

$$p_*=(N!)^np_0\left(\frac{r_1''r_2''\dots r_n''}{r_1'r_2'\dots r_n'}\right)^N+\sum_{k=1}^nb_k+1.$$

Let $z^0 = z^0(p_*)$ be a point for which inequality (3.6) holds and K^0 be such that (3.7) holds and

$$M\left(\frac{R'}{\mathbf{L}(z^0)}, z^0, F\right) = |F(z^*)|, \ M\left(\frac{r''}{\mathbf{L}(z^0)}, z^0, F^{(J)}\right) = |F^{(J)}(z^*_J)|$$

for every $J \in \mathbb{Z}^n_+$, $\|J\| \le N$. We apply Cauchy's inequality

$$|F^{(J)}(z^0)| \le J! \left(\frac{\mathbf{L}(z^0)}{R'}\right)^J |F(z^*)| \tag{3.8}$$

for estimate the difference

$$\begin{aligned} &|F^{(J)}(z^*_{J,1}, z^*_{J,2}, \dots, z^*_{J,n}) - F^{(J)}(z^0_1, z^*_{J,2}, \dots, z^*_{J,n})| \\ &= \left| \int_{z^0_1}^{z^*_{J,1}} \frac{\partial^{\|J\|+1} F}{\partial z_1^{j_1+1} \partial z_2^{j_2} \dots \partial z_n^{j_n}} (\xi, z^*_{J,2}, \dots, z^*_{J,n}) d\xi \right| \\ &\le \left| \frac{\partial^{\|J\|+1} F}{\partial z_1^{j_1+1} \partial z_2^{j_2} \dots \partial z_n^{j_n}} (z^*_{(j_1+1, j_2, \dots, j_n)}) \right| \frac{r''_1}{l_1(z^0)}. \end{aligned} \tag{3.9}$$

Taking into account $(z^0_1, z^*_{J,2}, \dots, z^*_{J,n}) \in \mathbb{D}^n[z^0, \frac{R''}{\mathbf{L}(z^0)}]$, for all $k \in \{1, \dots, n\}$,

$$|z^*_{J,k} - z^0_k| = \frac{r''_k}{l_k(z^0)}, \ l_k(z^0_1, z^*_{J,2}, \dots, z^*_{J,n}) \le \lambda_{2,k}(R'') l_k(z^0)$$

and (3.8) with $J = K^0$, by Theorem 2.1 we have

$$\begin{aligned} &|F^{(J)}(z^0_1, z^*_{J,2}, \dots, z^*_{J,n})| \\ &\le \frac{J! l_1^{j_1}(z^0_1, z^*_{J,2}, \dots, z^*_{J,n}) \prod_{k=2}^n l_k^{j_k}(z^0_1, z^*_{J,2}, \dots, z^*_{J,n})}{K^0! \mathbf{L}^{K^0}(z^0)} p_0 |F^{(K^0)}(z^0)| \\ &\le \frac{J! \mathbf{L}^J(z^0) \prod_{k=2}^n \lambda_{2,k}^{j_k}(R'')}{K^0! \mathbf{L}^{K^0}(z^0)} p_0 K^0! \left(\frac{\mathbf{L}(z^0)}{R'}\right)^{K^0} |F(z^*)| \\ &= \frac{p_0 J! \mathbf{L}^J(z^0) \prod_{k=2}^n \lambda_{2,k}^{j_k}(R'')}{(R')^{K^0}} |F(z^*)|. \end{aligned} \tag{3.10}$$

From inequalities (3.9) and (3.10) it follows that

$$\begin{aligned} &\left| \frac{\partial^{\|J\|+1} F}{\partial z_1^{j_1+1} \partial z_2^{j_2} \dots \partial z_n^{j_n}} (z^*_{(j_1+1, j_2, \dots, j_n)}) \right| \\ &\ge \frac{l_1(z^0)}{r''_1} \left\{ |F^{(J)}(z^*_j)| - |F^{(J)}(z^0_1, z^*_{J,2}, \dots, z^*_{J,n})| \right\} \\ &\ge \frac{l_1(z^0_1)}{r''_1} |F^{(J)}(z^*_j)| - \frac{p_0 J! \mathbf{L}^{(j_1+1, j_2, \dots, j_n)}(z^0) \prod_{k=2}^n \lambda_{2,k}^{j_k}(R'')}{r''_1 (R')^{K^0}} |F(z^*)|. \end{aligned}$$

Then

$$|F^{(K^0)}(z^*_{K^0})| \ge \frac{l_1(z^0)}{r''_1} \left| \frac{\partial^{\|K^0\|-1} f}{\partial z_1^{k_1^0-1} \partial z_2^{k_2^0} \dots \partial z_n^{k_n^0}} (z^*_{(k_1^0-1, k_2^0, \dots, k_n^0)}) \right|$$

$$-\frac{p_0(k_1^0-1)!k_2^0!\dots k_n^0!\mathbf{L}^{K^0}(z^0)\prod_{i=2}^n\lambda_{2,i}^{k_i^0}(R'')}{r_1''(R')^{K^0}}|F(z^*)|$$

$$\geq \frac{l_1^2(z^0)}{(r_1'')^2}\left|\frac{\partial^{\|K^0\|-2}f}{\partial z_1^{k_1^0-2}\partial z_2^{k_2^0}\dots\partial z_n^{k_n^0}}(z^*_{(k_1^0-2,k_2^0,\dots,k_n^0)})\right|$$

$$-\frac{p_0(k_1^0-2)!k_2^0!\dots k_n^0!\mathbf{L}^{K^0}(z^0)\prod_{i=2}^n\lambda_{2,i}^{k_i^0}(R'')}{(r_1'')^2(R')^{K^0}}|F(z^*)|$$

$$-\frac{p_0(k_1^0-1)!k_2^0!\dots k_n^0!\mathbf{L}^{K^0}(z^0)\prod_{i=2}^n\lambda_{2,i}^{k_i^0}(r_i'')}{r_1''(R')^{K^0}}|F(z^*)|$$

$$\dots$$

$$\geq \frac{l_1^{k_1^0}(z^0)}{(r_1'')^{k_1^0}}\left|\frac{\partial^{\|K^0\|-k_1^0}f}{\partial z_2^{k_2^0}\dots\partial z_n^{k_n^0}}(z^*_{(0,k_2^0,\dots,k_n^0)})\right|$$

$$-\frac{p_0}{(R')^{K^0}}\mathbf{L}^{K^0}(z^0)\left(\prod_{i=2}^n\lambda_{2,i}^{k_i^0}(R'')\right)k_2^0!\dots k_n^0!\sum_{j_1=1}^{k_1^0}\frac{(k_1^0-j_1)!}{(r_1'')^{j_1}}|F(z^*)|\dots$$

$$\geq \frac{l_1^{k_1^0}(z^0)}{(r_1'')^{k_1^0}}\frac{l_2^{k_2^0}(z^0)}{(r_2'')^{k_2^0}}\left|\frac{\partial^{\|K^0\|-k_1^0-k_2^0}f}{\partial z_3^{k_3^0}\dots\partial z_n^{k_n^0}}(z^*_{(0,0,k_3^0,\dots,k_n^0)})\right|$$

$$-\frac{l_1^{k_1^0}(z^0)p_0L^{(0,k_2^0,\dots,k_n^0)}(z^0)}{(r_1'')^{k_1^0}(R')^{K^0}}\left(\prod_{i=3}^n\lambda_{2,i}^{k_i^0}(R'')\right)k_3^0!\dots k_n^0!\sum_{i_2=1}^{k_2^0}\frac{(k_2^0-j_2)!}{(r_2'')^{j_2}}|F(z^*)|$$

$$-\frac{p_0}{(R')^{K^0}}\mathbf{L}^{K^0}(z^0)\left(\prod_{i=2}^n\lambda_{2,i}^{k_i^0}(R'')\right)k_2^0!\dots k_n^0!\sum_{j_1=1}^{k_1^0}\frac{(k_1^0-j_1)!}{(r_1'')^{j_1}}|F(z^*)|$$

$$\dots$$

$$\geq\left(\frac{L(z^0)}{R''}\right)|F(z^*_{\mathbf{0}})|-|F(z^*)|\sum_{i=1}^b\tilde{b}_i, \tag{3.11}$$

where in view of the inequalities $\lambda_{2,i}(R'')\geq 1$ and $R''\geq R'$ we have

$$\tilde{b}_1=\frac{p_0}{(R')^{K^0}}\mathbf{L}^{K^0}(z^0)\left(\prod_{i=2}^n\lambda_{2,i}^{k_i^0}(R'')\right)k_2^0!\dots k_n^0!\sum_{j_1=1}^{k_1^0}\frac{(k_1^0-j_1)!}{(r_1'')^{j_1}}$$

$$=\left(\frac{\mathbf{L}(z^0)}{R''}\right)^{K^0}\left(\frac{R''}{R'}\right)^{K^0}p_0\left(\prod_{i=2}^n\lambda_{2,i}^{k_i^0}(R'')\right)k_2^0!\dots k_n^0!\sum_{j_1=1}^{k_1^0}\frac{(k_1^0-j_1)!}{(r_1'')^{j_1}}\leq\left(\frac{\mathbf{L}(z^0)}{R''}\right)^{K^0}b_1,$$

$$\tilde{b}_2=\frac{p_0}{(R')^{K^0}}\mathbf{L}^{K^0}(z^0)\left(\prod_{i=3}^n\lambda_{2,i}^{k_i^0}(R'')\right)\frac{k_3^0!\dots k_n^0!}{(r_1'')^{k_1^0}}\sum_{j_2=1}^{k_2^0}\frac{(k_2^0-j_2)!}{(r_2'')^{j_2}}\leq\left(\frac{\mathbf{L}(z^0)}{R''}\right)^{K^0}b_2,$$

$$\dots$$

$$\tilde{b}_{n-1} = \frac{p_0}{(R')^{K^0}}\mathbf{L}^{K^0}(z^0)\lambda_{2,n}^{k_n^0}(R'')\frac{k_n^0!}{(r_1'')^{k_1^0}\dots(r_{n-2}'')^{k_{n-2}^0}}\times$$
$$\times\sum_{j_{n-1}=1}^{k_{n-1}^0}\frac{(k_{n-1}^0-j_{n-1})!}{(r_{n-1}'')^{j_{n-1}}}\le\left(\frac{\mathbf{L}(z^0)}{R''}\right)^{K^0} b_{n-1},$$
$$\tilde{b}_n = \frac{p_0}{(R')^{K^0}}\mathbf{L}^{K^0}(z^0)\frac{1}{(r_1'')^{k_1^0}\dots(r_{n-1}'')^{k_{n-1}^0}}\sum_{j_n=1}^{k_n^0}\frac{(k_n^0-j_n)!}{(r_n'')^{j_n}}\le\left(\frac{\mathbf{L}(z^0)}{R''}\right)^{K^0} b_n.$$

Thus, (3.11) implies that

$$|F^{(K^0)}(z^*_{K^0})|\ge\left(\frac{\mathbf{L}(z^0)}{R''}\right)^{K^0}|F(z^*)|\left\{\frac{|F(z^*_{\mathbf{0}})|}{|F(z^*)|}-\sum_{j=1}^n b_j\right\}.$$

But in view of (3.6) and a choice of p_* we have

$$\frac{|F(z^*_{\mathbf{0}})|}{|F(z^*)|}\ge p_*>\sum_{j=1}^n b_j.$$

Thus, (3.7) and (3.8) imply

$$|F^{(K^0)}(z^*_{K^0})|\ge\left(\frac{\mathbf{L}(z^0)}{R''}\right)^{K^0}|F(z^*)|\left\{p_*-\sum_{j=1}^n b_j\right\}$$
$$\ge\left(\frac{\mathbf{L}(z^0)}{R''}\right)^{K^0}\left\{p_*-\sum_{j=1}^n b_j\right\}\frac{|F^{(K^0)}(z^0)|(R')^{K^0}}{K^0!\mathbf{L}^{K^0}(z^0)}$$
$$\ge\left(\frac{r_1'\dots r_n'}{r_1''\dots r_n''}\right)^N\left\{p_*-\sum_{j=1}^n b_j\right\}\frac{|F^{(K^0)}(z^*_{K^0})|}{p_0(n!)^n}.$$

Hence, we have $p_*\le p_0\big(\frac{r_1'\dots r_n'}{r_1''\dots r_n''}\big)^N(N!)^n+\sum_{j=1}^n b_j$, but this contradicts the choice of p_*. □

4. Analogue of Theorem of Hayman for analytic in a ball function of bounded L-index in joint variables

Theorem 4.1. *Let $\mathbf{L}\in Q(\mathbb{B}^n)$. An analytic function F in $\mathbb{B}^n$ has bounded $\mathbf{L}$-index in joint variables if and only if there exist $p\in\mathbb{Z}_+$ and $c\in\mathbb{R}_+$ such that for each $z\in\mathbb{B}^n$*

$$\max\left\{\frac{|F^{(J)}(z)|}{\mathbf{L}^J(z)}:\ \|J\|=p+1\right\}\le c\cdot\max\left\{\frac{|F^{(K)}(z)|}{\mathbf{L}^K(z)}:\ \|K\|\le p\right\}. \tag{4.1}$$

Proof. Let $N=N(F,\mathbf{L},\mathbb{B}^n)<+\infty$. The definition of the boundedness of **L**-index in joint variables yields the necessity with $p=N$ and $c=((N+1)!)^n$.

We prove the sufficiency. For $F\equiv 0$ theorem is obvious. Thus, we suppose that $F\not\equiv 0$. Denote $\boldsymbol{\beta}=(\frac{\beta}{\sqrt{n}},\dots,\frac{\beta}{\sqrt{n}})$.

Assume that (4.1) holds, $z^0 \in \mathbb{B}^n$, $z \in \mathbb{D}^n[z^0, \frac{\boldsymbol{\beta}}{\mathbf{L}(z^0)}]$. For all $J \in \mathbb{Z}^n_+$, $\|J\| \le p+1$, one has

$$\frac{|F^{(J)}(z)|}{\mathbf{L}^J(z^0)} \le \Lambda_2^J(\boldsymbol{\beta}) \frac{|F^{(J)}(z)|}{\mathbf{L}^J(z)} \le c \cdot \Lambda_2^J(\boldsymbol{\beta}) \max\left\{ \frac{|F^{(K)}(z)|}{\mathbf{L}^K(z)} :\ \|K\| \le p \right\}$$

$$\le c \cdot \Lambda_2^J(\boldsymbol{\beta}) \max\left\{ \Lambda_1^{-K}(2) \frac{|F^{(K)}(z)|}{\mathbf{L}^K(z^0)} : \|K\| \le p \right\} \le BG(z), \tag{4.2}$$

where $B = c \cdot \max\{\Lambda_2^K(\boldsymbol{\beta}) :\ \|K\| = p+1\} \max\{\Lambda_1^{-K}(\boldsymbol{\beta}) :\ \|K\| \le p\}$, and

$$G(z) = \max\left\{ \frac{|F^{(K)}(z)|}{\mathbf{L}^K(z^0)} :\ \|K\| \le p \right\}.$$

We choose

$$z^{(1)} = (z_1^{(1)}, \dots, z_n^{(1)}) \in \mathbb{T}^n(z^0, \frac{\mathbf{1}}{2\beta\sqrt{n}\mathbf{L}(z^0)})$$

and

$$z^{(2)} = (z_1^{(2)}, \dots, z_n^{(2)}) \in \mathbb{T}^n(z^0, \frac{\boldsymbol{\beta}}{\mathbf{L}(z^0)})$$

such that $F(z^{(1)}) \ne 0$ and

$$|F(z^{(2)})| = M\left(\frac{\boldsymbol{\beta}}{\mathbf{L}(z^0)}, z^0, F \right) \ne 0. \tag{4.3}$$

These points exist, otherwise if $F(z) \equiv 0$ on skeleton

$$\mathbb{T}^n\left(z^0, \frac{\mathbf{1}}{2\beta\sqrt{n}\mathbf{L}(z^0)}\right) \quad \text{or} \quad \mathbb{T}^n\left(z^0, \frac{\boldsymbol{\beta}}{\mathbf{L}(z^0)}\right)$$

then by the uniqueness theorem $F \equiv 0$ in $\mathbb{B}^n$. We connect the points $z^{(1)}$ and $z^{(2)}$ with plane

$$\alpha = \begin{cases} z_2 = k_2 z_1 + c_2, \\ z_3 = k_3 z_1 + c_3, \\ \qquad \dots \\ z_n = k_n z_1 + c_n, \end{cases}$$

where

$$k_i = \frac{z_i^{(2)} - z_i^{(1)}}{z_1^{(2)} - z_1^{(1)}},\ c_i = \frac{z_i^{(1)} z_1^{(2)} - z_i^{(2)} z_1^{(1)}}{z_1^{(2)} - z_1^{(1)}},\ i = 2, \dots, n.$$

It is easy to check that $z^{(1)} \in \alpha$ and $z^{(2)} \in \alpha$. Let $\widetilde{G}(z_1) = G(z)|_\alpha$ be a restriction of the function G onto α.

For every $K \in \mathbb{Z}^n_+$ the function $F^{(K)}(z)\big|_\alpha$ is analytic function of variable z_1 and $\tilde{G}(z_1^{(1)}) = G(z^{(1)})\big|_\alpha \ne 0$ because $F(z^{(1)}) \ne 0$. Hence, all zeros of the function $F^{(K)}(z)\big|_\alpha$ are isolated as zeros of a function of one variable. Thus, zeros of the function $\tilde{G}(z_1)$ are isolated too. Therefore, we can choose piecewise analytic curve γ onto α as following

$$z = z(t) = (z_1(t), k_2 z_1(t) + c_2, \dots, k_n z_1(t) + c_n),\ t \in [0,1],$$

which connect the points $z^{(1)}$, $z^{(2)}$ and such that $G(z(t))\neq 0$ and

$$\int_0^1 |z_1'(t)|dt \leq \frac{2\beta}{\sqrt{n}l_1(z_1^0)}.$$

For a construction of the curve we connect $z_1^{(1)}$ and $z_1^{(2)}$ by a line

$$z_1^*(t)=(z_1^{(2)}-z_1^{(1)})t+z_1^{(1)},\ t\in[0,1].$$

The curve γ can cross points z_1 at which the function $\widetilde{G}(z_1)=0$. The number of such points $m=m(z^{(1)},z^{(2)})$ is finite. Let $(z_{1,k}^*)$ be a sequence of these points in ascending order of the value $|z_1^{(1)}-z_{1,k}^*|$, $k\in\{1,2,...,m\}$. We choose

$$r<\min_{1\leq k\leq m-1}\{|z_{1,k}^*-z_{1,k+1}^*|,|z_{1,1}^*-z_1^{(1)}|,|z_{1,m}^*-z_1^{(2)}|,\frac{2\beta^2-1}{2\pi\sqrt{n}\beta l_1(z^0)}\}.$$

Now we construct circles with centers at the points $z_{1,k}^*$ and corresponding radii $r_k' < \frac{r}{2^k}$ such that $\widetilde{G}(z_1)\neq 0$ for all z_1 on the circles. It is possible, because $F\not\equiv 0$.

Every such circle is divided onto two semicircles by the line $z_1^*(t)$. The required piecewise-analytic curve consists with arcs of the constructed semicircles and segments of line $z_1^*(t)$, which connect the arcs in series between themselves or with the points $z_1^{(1)},z_1^{(2)}$. The length of $z_1(t)$ in $\mathbb{C}$ (but not $z(t)$ in $\mathbb{C}^n$!) is lesser than

$$\frac{\beta/\sqrt{n}}{l_1(z^0)}+\frac{1}{2\sqrt{n}\beta l_1(z^0)}+\pi r\leq \frac{2\beta}{\sqrt{n}l_1(z^0)}.$$

Then

$$\int_0^1|z_s'(t)|dt=|k_s|\int_0^1|z_1'(t)|dt\leq \frac{|z_s^{(2)}-z_s^{(1)}|}{|z_1^{(2)}-z_1^{(1)}|}\frac{2\beta}{\sqrt{n}l_1(z^0)}$$

$$\leq\frac{2\beta^2+1}{2\sqrt{n}\beta l_s(z^0)}\frac{2\sqrt{n}\beta l_1(z^0)}{2\beta^2-1}\frac{2\beta}{\sqrt{n}l_1(z^0)}\leq\frac{2\beta(2\beta^2+1)}{(2\beta^2-1)\sqrt{n}l_s(z^0)},\ s\in\{2,\dots,n\}.$$

Hence,

$$\int_0^1\sum_{s=1}^n l_s(z^0)|z_s'(t)|dt\leq\frac{2\beta(2\beta^2+1)\sqrt{n}}{2\beta^2-1}=S. \tag{4.4}$$

Since the function $z=z(t)$ is piece-wise analytic on $[0,1]$, then for arbitrary $K\in\mathbb{Z}_+^n$, $J\in\mathbb{Z}_+^n$, $\|K\|\leq p$, either

$$\frac{|F^{(K)}(z(t))|}{\mathbf{L}^K(z^0)}\equiv\frac{|F(J)(z(t))|}{\mathbf{L}^J(z^0)}, \tag{4.5}$$

or the equality

$$\frac{|F^{(K)}(z(t))|}{\mathbf{L}^K(z^0)}=\frac{|F^{(J)}(z(t))|}{\mathbf{L}^J(z^0)} \tag{4.6}$$

holds only for a finite set of points $t_k\in[0;1]$.

Then for function $G(z(t))$ as maximum of such expressions $\frac{|F^{(J)}(z(t))|}{\mathbf{L}^J(z^0)}$ by all $\|J\|\leq p$ two cases are possible:

1. In some interval of analyticity of the curve γ the function $G(z(t))$ identically equals simultaneously to some derivatives, that is (4.5) holds. It means that $G(z(t)) \equiv \frac{|F^{(J)}(z(t))|}{\mathbf{L}^J(z^0)}$ for some J, $\|J\| \le p$. Clearly, the function $F^{(J)}(z(t))$ is analytic. Then $|F^{(J)}(z(t))|$ is continuously differentiable function on the interval of analyticity except points where this partial derivative equals zero $|F^{(j_1,j_2)}(z_1(t), z_2(t))| = 0$. However, there are not the points, because in the opposite case $G(z(t)) = 0$. But it contradicts the construction of the curve γ.
2. In some interval of analyticity of the curve γ the function $G(z(t))$ equals simultaneously to some derivatives at a finite number of points t_k, that is (4.6) holds. Then the points t_k divide interval of analyticity onto a finite number of segments, in which of them $G(z(t))$ equals to one from the partial derivatives, i. e. $G(z(t)) \equiv \frac{|F^{(J)}(z(t))|}{\mathbf{L}^J(z^0)}$ for some J, $\|J\| \le p$. As above, in each from these segments the functions $|F^{(J)}(z(t))|$, and $G(z(t))$ are continuously differentiable except the points t_k.

The inequality

$$\frac{d}{dt}|f(t)| \le \left|\frac{df(t)}{dt}\right|$$

holds for complex-valued functions of real argument outside a countable set of points. In view of this fact and (4.2) we have

$$\begin{aligned}\frac{d}{dt}G(z(t)) &\le \max\Big\{\frac{1}{\mathbf{L}^J(z^0)}\Big|\frac{d}{dt}F^{(J)}(z(t))\Big| : \ \|J\| \le p\Big\}\\
&\le \max\Big\{\sum_{s=1}^n \Big|\frac{\partial^{\|J\|+1}F}{\partial z_1^{j_1}\dots\partial z_s^{j_s+1}\dots\partial z_n^{j_n}}(z(t))\Big|\frac{|z_s'(t)|}{\mathbf{L}^j(z^0)} : \ \|J\| \le p\Big\}\\
&\le \max\Big\{\sum_{s=1}^n \Big|\frac{\partial^{\|J\|+1}F}{\partial z_1^{j_1}\dots\partial z_s^{j_s+1}\dots\partial z_n^{j_n}}(z(t))\Big|\frac{l_s(z^0)|z_s'(t)|}{l_1^{j_1}(z^0)\dots l_s^{j_1+1}(z^0)\dots l_n^{j_n}(z^0)} :\\
&\|J\| \le p\Big\} \le \Big(\sum_{s=1}^n l_s(z^0)|z_s'(t)|\Big)\max\Big\{\frac{|F^{(j)}(z(t))|}{\mathbf{L}^J(z^0)} : \ \|J\| \le p+1\Big\}\\
&\le \Big(\sum_{s=1}^n l_s(z^0)|z_s'(t)|\Big)BG(z(t)).\end{aligned}$$

Therefore, (4.4) yields

$$\Big|\ln\frac{G(z^{(2)})}{G(z^{(1)})}\Big| = \Big|\int_0^1 \frac{1}{G(z(t))}\frac{d}{dt}G(z(t))dt\Big| \le B\int_0^1 \sum_{s=1}^n l_s(z^0)|z_s'(t)|dt \le S\cdot B.$$

Using (4.3), we deduce

$$M\left(\frac{\boldsymbol{\beta}}{\mathbf{L}(z^0)}, z^0, F\right) \le G(z^{(2)}) \le G(z^{(1)})e^{SB}.$$

Since $z^{(1)} \in \mathbb{T}^n(z^0, \frac{\mathbf{1}}{2\beta\sqrt{n}\mathbf{L}(z^0)})$, the Cauchy inequality holds

$$\frac{|F^{(J)}(z^{(1)})|}{\mathbf{L}^J(z^0)} \le J!(2\beta\sqrt{n})^{\|J\|} M\left(\frac{\mathbf{1}}{2\beta\sqrt{n}\mathbf{L}(z^0)}, z^0, F\right).$$

for all $J \in \mathbb{Z}_+^n$. Therefore, for $\|J\| \le p$ we obtain

$$G(z^{(1)}) \le (p!)^n (2\beta\sqrt{n})^p M\left(\frac{\mathbf{1}}{2\beta\sqrt{n}\mathbf{L}(z^0)}, z^0, F\right),$$

$$M\left(\frac{\boldsymbol{\beta}}{\mathbf{L}(z^0)}, z^0, F\right) \le e^{SB}(p!)^n (2\beta\sqrt{n})^p M\left(\frac{\mathbf{1}}{2\beta\sqrt{n}\mathbf{L}(z^0)}, z^0, F\right).$$

Hence, by Theorem 3.1 the function F has bounded **L**-index in joint variables. □

The following result was also obtained for other classes of holomorphic functions in [21, 11, 7].

Theorem 4.2. *Let* $\mathbf{L} \in Q(\mathbb{B}^n)$. *An analytic function* F *in* $\mathbb{B}^n$ *has bounded* **L**-*index in joint variables if and only if there exist* $c \in (0; +\infty)$ *and* $N \in \mathbb{N}$ *such that for each* $z \in \mathbb{B}^n$ *the inequality*

$$\sum_{\|K\|=0}^{N} \frac{|F^{(K)}(z)|}{K!\mathbf{L}^K(z)} \ge c \sum_{\|K\|=N+1}^{\infty} \frac{|F^{(K)}(z)|}{K!\mathbf{L}^K(z)}. \tag{4.7}$$

Proof. Let $\frac{1}{\beta} < \theta_j < 1$, $j \in \{1, \dots, n\}$, $\Theta = (\theta_1, \dots, \theta_n)$. If the function F has bounded **L**-index in joint variables then by Theorem 2.2 F has bounded $\widetilde{\mathbf{L}}$-index in joint variables, where $\widetilde{\mathbf{L}} = (\widetilde{l}_1(z), \dots, \widetilde{l}_n(z))$, $\widetilde{l}_j(z) = \theta_j l_j(z)$, $j \in \{1, \dots, n\}$. Let $\widetilde{N} = N(F, \widetilde{L}, \mathbb{B}^n)$. Therefore,

$$\max\left\{\frac{|F^{(K)}(z)|}{K!\mathbf{L}^K(z)} : \|K\| \le \widetilde{N}\right\} = \max\left\{\frac{\Theta^K |F^{(K)}(z)|}{K!\widetilde{\mathbf{L}}^K(z)} : \|K\| \le \widetilde{N}\right\}$$

$$\ge \prod_{s=1}^{n} \theta_s^{\widetilde{N}} \max\left\{\frac{|F^{(K)}(z)|}{K!\widetilde{\mathbf{L}}^K(z)} : \|K\| \le \widetilde{N}\right\} \ge \prod_{s=1}^{n} \theta_s^{\widetilde{N}} \frac{|F^{(J)}(z)|}{J!\widetilde{\mathbf{L}}^J(z)} = \prod_{s=1}^{n} \theta_s^{\widetilde{N}-j_s} \frac{|F^{(J)}(z)|}{J!\mathbf{L}^J(z)}$$

for all $J \ge \mathbf{0}$ and

$$\sum_{\|J\|=\widetilde{N}+1}^{\infty} \frac{|F^{(J)}(z)|}{J!\mathbf{L}^j(z)} \le \max\left\{\frac{|F^{(K)}(z)|}{K!\mathbf{L}^K(z)} : \|K\| \le \widetilde{N}\right\} \sum_{\|J\|=\widetilde{N}+1}^{\infty} \theta_s^{j_s - \widetilde{N}}$$

$$= \prod_{i=1}^{n} \frac{\theta_s}{1-\theta_s} \max\left\{\frac{|F^{(K)}(z)|}{K!\mathbf{L}^K(z)} : \|K\| \le \widetilde{N}\right\} \le \prod_{i=1}^{n} \frac{\theta_s}{1-\theta_s} \sum_{\|K\|=0}^{\widetilde{N}} \frac{|F^{(K)}(z)|}{K!\mathbf{L}^K(z)}.$$

Hence, we obtain (4.7) with $N = \widetilde{N}$ and

$$c = \prod_{i=1}^{n} \frac{\theta_s}{1-\theta_s}.$$

On the contrary, inequality (4.7) implies

$$\max\left\{\frac{|F^{(J)}(z)|}{J!\mathbf{L}^J(z)}\colon \|J\|=N+1\right\}\leq \sum_{\|K\|=N+1}^{\infty}\frac{|F^{(K)}(z)|}{K!\mathbf{L}^K(z)}\leq \frac{1}{c}\sum_{\|K\|=0}^{N}\frac{|F^{(K)}(z)|}{K!\mathbf{L}^K(z)}$$

$$\leq \frac{1}{c}\sum_{i=0}^{N}C_{n+i-1}^{i}\max\left\{\frac{|F^{(K)}(z)|}{K!\mathbf{L}^K(z)}\colon \|K\|\leq N\right\}$$

and by Theorem 4.1 F is of bounded $\mathbf{L}$-index in joint variables. □

5. Some application for PDE: a scheme

Here we present a scheme of application of Hayman's Theorem to PDE. This is also appilcable in a more general situation.

Let us consider the following system of partial differential equations:

$$\begin{cases} F^{(2,0)}(z_1,z_2)=2\pi z_2\tan(\pi z_1z_2)F^{(1,0)}(z_1,z_2),\\ F^{(0,2)}(z_1,z_2)=2\pi z_1\tan(\pi z_1z_2)F^{(0,1)}(z_1,z_2).\end{cases}$$

Differentiate in variables z_1 and z_2 we deduce

$$\begin{cases} F^{(3,0)}(z_1,z_2)=\frac{2\pi^2z_2^2}{\cos^2(\pi z_1z_2)}F^{(1,0)}(z_1,z_2)+2\pi z_2\tan(\pi z_1z_2)F^{(2,0)}(z_1,z_2),\\ F^{(2,1)}(z_1,z_2)=2\pi\tan(\pi z_1z_2)F^{(1,0)}(z_1,z_2)+\frac{2\pi^2z_1z_2}{\cos^2(\pi z_1z_2)}F^{(1,0)}(z_1,z_2)+\\ \qquad +2\pi z_2\tan(\pi z_1z_2)F^{(1,1)}(z_1,z_2),\\ F^{(1,2)}(z_1,z_2)=2\pi\tan(\pi z_1z_2)F^{(0,1)}(z_1,z_2)+\frac{2\pi^2z_1z_2}{\cos^2(\pi z_1z_2)}F^{(1,0)}(z_1,z_2)+\\ \qquad +2\pi z_1\tan(\pi z_1z_2)F^{(1,1)}(z_1,z_2),\\ F^{(0,3)}(z_1,z_2)=\frac{2\pi^2z_1^2}{\cos^2(\pi z_1z_2)}F^{(1,0)}(z_1,z_2)+2\pi z_1\tan(\pi z_1z_2)F^{(2,0)}(z_1,z_2),\end{cases} \tag{5.1}$$

Let

$$\mathbf{L}(z_1,z_2)=(l_1(z_1,z_2),l_2(z_1,z_2))=\left(\frac{|z_2|+1}{(1-|z|)|\frac{1}{2}-z_1z_2|},\frac{|z_1|+1}{(1-|z|)|\frac{1}{2}-z_1z_2|}\right),$$

where $z=(z_1,z_2)$, $|z|=\sqrt{|z_1|^2+|z_2|^2}$. Now we will estimate all third order partial derivatives of the function $F(z_1,z_2)$ by its first and second order partial derivatives. From the first equation of system (5.1) we have for all $z\in\mathbb{B}^2$:

$$\frac{|F^{(3,0)}(z_1,z_2)|}{l_1^3(z_1,z_2)}\leq\frac{2\pi^2|z_2|^2|F^{(1,0)}(z_1,z_2)|}{|\cos^2(\pi z_1z_2)|l_1^3(z_1,z_2)}+2\pi|z_2\tan(\pi z_1z_2)|\frac{|F^{(2,0)}(z_1,z_2)|}{l_1^3(z_1,z_2)}$$

$$\leq\left(\frac{2\pi^2|z_2|^2}{|\cos^2(\pi z_1z_2)|l_1^2(z_1,z_2)}+\frac{2\pi|z_2\tan(\pi z_1z_2)|}{l_1(z_1,z_2)}\right)\max_{j\in\{1,2\}}\left\{\frac{|F^{(j,0)}(z_1,z_2)|}{l_1^j(z_1,z_2)}\right\}$$

$$\leq\left(2\pi^2\frac{(1-|z|)^2|\frac{1}{2}-z_1z_2|^2}{|\cos^2(\pi z_1z_2)|}+2\pi|\tan(\pi z_1z_2)|(1-|z|)\left|\frac{1}{2}-z_1z_2\right|\right)$$

$$\times \max\left\{\frac{|F^{(j,0)}(z_1,z_2)|}{l_1^j(z_1,z_2)} : j\in\{1,2\}\right\}$$

$$= \left(\frac{(1-|z|)^2|\frac{\pi}{2}-\pi z_1 z_2|^2}{|\sin^2(\frac{\pi}{2}-\pi z_1 z_2)|} + 2|\sin(\pi z_1 z_2)|(1-|z|)\frac{|\frac{\pi}{2}-\pi z_1 z_2|}{|\sin(\frac{\pi}{2}-\pi z_1 z_2)|}\right)$$

$$\times \max\left\{\frac{|F^{(j,0)}(z_1,z_2)|}{l_1^j(z_1,z_2)} : j\in\{1,2\}\right\} \le C\max\left\{\frac{|F^{(j,0)}(z_1,z_2)|}{l_1^j(z_1,z_2)} : j\in\{1,2\}\right\}.$$

Similarly, the second equation of system (5.1) yields

$$\frac{|F^{(2,1)}(z_1,z_2)|}{l_1^2(z_1,z_2)l_2(z_1,z_2)} \le \left(\frac{2\pi|\tan(\pi z_1 z_2)|}{l_1(z_1,z_2)l_2(z_1,z_2)} + \frac{2\pi^2|z_1 z_2|}{|\cos^2(\pi z_1 z_2)|l_1(z_1,z_2)l_2(z_1,z_2)}\right)$$

$$\times \frac{|F^{(1,0)}(z_1,z_2)|}{l_1(z_1,z_2)} + \frac{2\pi|z_2\tan(\pi z_1 z_2)|}{l_1(z_1,z_2)}\frac{|F^{(1,1)}(z_1,z_2)|}{l_1(z_1,z_2)l_2(z_1,z_2)}$$

$$\le \left(\frac{2\pi|\sin(\pi z_1 z_2)|(1-|z|)^2|\frac{1}{2}-z_1 z_2|^2}{|\cos(\pi z_1 z_2)|}\right.$$

$$\left. + \frac{2\pi^2(1-|z|)^2|\frac{1}{2}-z_1 z_2|^2}{|\cos^2(\pi z_1 z_2)|} + \frac{2\pi|\sin(\pi z_1 z_2)|(1-|z|)|\frac{1}{2}-z_1 z_2|}{|\cos(\pi z_1 z_2)|}\right)$$

$$\times \max\left\{\frac{|F^{(1,j)}(z_1,z_2)|}{l_1(z_1,z_2)l_2^j(z_1,z_2)} : j\in\{0,1\}\right\}$$

$$\le \left(\frac{2\pi|\sin(\pi z_1 z_2)|(1-|z|)^2|\frac{1}{2}-z_1 z_2|^2}{|\sin(\frac{\pi}{2}-\pi z_1 z_2)|} + \frac{2(1-|z|)^2|\frac{\pi}{2}-\pi z_1 z_2|^2}{|\sin^2(\frac{\pi}{2}-\pi z_1 z_2)|}\right.$$

$$\left. + \frac{2|\sin(\pi z_1 z_2)|(1-|z|)|\frac{\pi}{2}-\pi z_1 z_2|}{|\sin(\frac{\pi}{2}-\pi z_1 z_2)|}\right)\max\left\{\frac{|F^{(1,j)}(z_1,z_2)|}{l_1(z_1,z_2)l_2^j(z_1,z_2)} : j\in\{0,1\}\right\}$$

$$\le C\max\left\{\frac{|F^{(1,j)}(z_1,z_2)|}{l_1(z_1,z_2)l_2^j(z_1,z_2)} : j\in\{0,1\}\right\}.$$

By analogy, we can prove similar estimates for the third and the fourth equation of system (5.1). Combining all estimates, one has

$$\max\left\{\frac{|F^{(k,3-k)}(z_1,z_2)|}{l_1^k(z_1,z_2)l_2^{3-k}(z_1,z_2)} : k\in\{0,1,2,3\}\right\}$$

$$\le C\max\left\{\frac{|F^{(k,j)}(z_1,z_2)|}{l_1^k(z_1,z_2)l_2^j(z_1,z_2)} : 0\le k+j\le 2\right\}.$$

Hence, by Theorem 4.1 every analytic solution in $\mathbb{B}^2$ of system (5.1) has bounded **L**-index in joint variables with

$$\mathbf{L}(z_1,z_2) = \left(\frac{|z_2|+1}{(1-|z|)|\frac{1}{2}-z_1 z_2|}, \frac{|z_1|+1}{(1-|z|)|\frac{1}{2}-z_1 z_2|}\right).$$

Particularly, the function $F(z_1, z_2) = \tan(\pi z_1 z_2)$ has the bounded **L**-index in joint variables. Indeed, it is easy to see that the function F is analytic solution in $\mathbb{B}^2$ of system (5.1).

6. Boundedness of l_j-index in every direction $\mathbf{1}_j$

This section shows another application of Theorem 3.1. The boundedness of l_j-index of a function F in every variable z_j, generally speaking, does not imply the boundedness of **L**-index in joint variables (see example in [4]). But, if F has bounded l_j-index in every direction $\mathbf{1}_j$, $j \in \{1, \ldots, n\}$, then F is a function of bounded **L**-index in joint variables.

Let $\mathbf{b} = (b_1, \ldots, b_n) \in \mathbb{C}^n \setminus \{\mathbf{0}\}$ be a given direction, $L : \mathbb{B}^n \to \mathbb{R}_+$ be a continuous function such that for all $z \in \mathbb{B}^n$ $L(z) > \frac{\beta|\mathbf{b}|}{1-|z|}$, $\beta > 1$.

For $\eta \in [0, \beta]$, $z \in \mathbb{B}^n$, we define
$\lambda_1^{\mathbf{b}}(z, \eta, L) = \inf\{L(z + t\mathbf{b})/L(z) : |t| \le \frac{\eta}{L(z)}\}$,
$\lambda_1^{\mathbf{b}}(\eta, L) = \inf\{\lambda_1^{\mathbf{b}}(z, \eta, L) : z \in \mathbb{B}^n\}$,
$\lambda_2^{\mathbf{b}}(z, \eta, L) = \sup\{L(z + t\mathbf{b})/L(z) : |t| \le \frac{\eta}{L(z)}\}$,
$\lambda_2^{\mathbf{b}}(\eta, L) = \sup\{\lambda_2^{\mathbf{b}}(z, \eta, L) : z \in \mathbb{B}^n\}$.
By $Q_{\mathbf{b},\beta}(\mathbb{B}^n)$ we denote the class of all functions L satisfying $\forall \eta \in [0, \beta]$,

$$0 < \lambda_1^{\mathbf{b}}(\eta, L) \le \lambda_2^{\mathbf{b}}(\eta, L) < +\infty.$$

Analytic in $\mathbb{B}^n$ function $F(z)$ is called a function of *bounded L-index in the direction* $\mathbf{b}$, if there exists $m_0 \in \mathbb{Z}_+$ that for every $m \in \mathbb{Z}_+$ and for every $z \in \mathbb{B}^n$ the following inequality is valid

$$\frac{1}{m!L^m(z)}\left|\frac{\partial^m F(z)}{\partial \mathbf{b}^m}\right| \le \max\left\{\frac{1}{k!L^k(z)}\left|\frac{\partial^k F(z)}{\partial \mathbf{b}^k}\right| : 0 \le k \le m_0\right\}, \tag{6.1}$$

where

$$\frac{\partial^0 F(z)}{\partial \mathbf{b}^0} = F(z), \frac{\partial F(z)}{\partial \mathbf{b}} = \sum_{j=1}^{n} \frac{\partial F(z)}{\partial z_j} b_j, \ \overline{\mathbf{b}}\rangle, \frac{\partial^k F(z)}{\partial \mathbf{b}^k} = \frac{\partial}{\partial \mathbf{b}}\Big(\frac{\partial^{k-1} F(z)}{\partial \mathbf{b}^{k-1}}\Big), \ k \ge 2.$$

The least such integer m_0 is called the *L-index in the direction* $\mathbf{b}$ *of the analytic function F and is denoted by* $N_{\mathbf{b}}(F, L) = m_0$. In the case $n = 1$, $\mathbf{b} = 1$ and $L = l$ we obtain a definition of analytic in an unit disc function of bounded l-index [22, 21].

We need the following theorem.

Theorem 6.1 (**[3]**)**.** *Let $\beta > 1$, $L \in Q_{\mathbf{b},\beta}(\mathbb{B}^n)$. Analytic in $\mathbb{B}^n$ function $F(z)$ is of bounded L-index in the direction $\mathbf{b} \in \mathbb{C}^n$ if and only if for any r_1 and any r_2 with $0 < r_1 < r_2 \le \beta$, there exists number $P_1 = P_1(r_1, r_2) \ge 1$ such that for each $z^0 \in \mathbb{B}^n$*

$$\max\Big\{|F(z^0 + t\mathbf{b})| : |t| = \frac{r_2}{L(z^0)}\Big\} \le P_1 \max\Big\{|F(z^0 + t\mathbf{b})| : \ |t| = \frac{r_1}{L(z_0)}\Big\}. \tag{6.2}$$

It is easy to see that if $\mathbf{L}(z) = (l_1(z), \ldots, l_n(z))$ and $\mathbf{L} \in Q(\mathbb{B}^n)$, then

$$l_j \in Q_{\mathbf{1}_j,\beta/\sqrt{n}}(\mathbb{B}^n), \ j \in \{1, \ldots, n\}.$$

Theorem 6.2. *Let* $\mathbf{L}(z)=(l_1(z),\dots,l_n(z))$, $\mathbf{L}\in Q(\mathbb{B}^n)$. *If an analytic in* $\mathbb{B}^n$ *function* F *has bounded* l_j*-index in the direction* $\mathbf{1}_j$ *for every* $j\in\{1,\dots,n\}$, *then* F *is of bounded* **L***-index in joint variables.*

Proof. Let F be an analytic in $\mathbb{B}^n$ function of bounded l_j-index in every direction $\mathbf{1}_j$. Then by Theorem 6.1 for every $j\in\{1,\dots,n\}$ and arbitrary $0<r'_j<1<r''_j\le\frac{\beta}{\sqrt{n}}$ there exists a number $p_j=p_j(r',r'')$ such that for every $(z_1,\dots,z_{j-1},z_j^0,z_{j+1},\dots,z_n)\in\mathbb{B}^n$,

$$\max\left\{|F(z)|\colon |z_j-z_j^0|=\frac{r''_j}{l_j(z_1,\dots,z_{j-1},z_j^0,z_{j+1},\dots,z_n)}\right\}\le p_j(r'_j,r''_j)$$
$$\times\max\left\{|F(z)|\colon |z_j-z_j^0|=\frac{r'_j}{l_j(z_1,\dots,z_{j-1},z_j^0,z_{j+1},\dots,z_n)}\right\}. \qquad (6.3)$$

Obviously, if for every $j\in\{1,\dots,n\}$ $l_j\in Q_{\mathbf{1}_j,\beta/\sqrt{n}}(\mathbb{B}^n)$ then $\mathbf{L}\in Q(\mathbb{B}^n)$. Let z^0 be an arbitrary point in $\mathbb{B}^n$, and a point $z^*\in\mathbb{T}^n(z^0,\frac{R''}{\mathbf{L}(z^0)})$ is such that

$$M\left(\frac{R''}{\mathbf{L}(z^0)},z^0,F\right)=|F(z^*)|.$$

We choose R'' and R' such that $\mathbf{1}<R''\le(\frac{\beta}{\sqrt{n}},\dots,\frac{\beta}{\sqrt{n}})$ and $R'<\Lambda_1(R'')$. Then inequality (6.3) implies that

$$M\left(\frac{R''}{\mathbf{L}(z^0)},z^0,F\right)\le\max\left\{|F(z_1,z_2^*,z_3^*,\dots,z_n^*)|\colon\ |z_1-z_1^0|=\frac{r''_1}{l_1(z^0)}\right\}$$
$$=\max\left\{|F(z_1,z_2^*,\dots,z_n^*)|\colon |z_1-z_1^0|=\frac{r''_1}{l_1(z_1^0,z_2^*,\dots,z_n^*)}\frac{l_1(z_1^0,z_2^*,\dots,z_n^*)}{l_1(z^0)}\right\}$$
$$\le\max\left\{|F(z_1,z_2^*,\dots,z_n^*)|\colon\ |z_1-z_1^0|=\frac{r''_1\lambda_{2,1}(R'')}{l_1(z_1^0,z_2^*,\dots,z_n^*)}\right\}$$
$$\le p_1(r'_1,r''_1\lambda_{2,1}(R''))\max\left\{|F(z_1,z_2^*,\dots,z_n^*)|\colon |z_1-z_1^0|=\frac{r'_1}{l_1(z_1^0,z_2^*,\dots,z_n^*)}\right\}$$
$$=p_1(r'_1,r''_1\lambda_{2,1}(R''))$$
$$\times\max\left\{|F(z_1,z_2^*,\dots,z_n^*)|\colon |z_1-z_1^0|=\frac{r'_1}{l_1(z^0)}\frac{l_1(z^0)}{l_1(z_1^0,z_2^*,\dots,z_n^*)}\right\}$$
$$\le p_1(r'_1,r''_1\lambda_{2,1}(R''))\max\left\{|F(z_1,z_2^*,\dots,z_n^*)|\colon\ |z_1-z_1^0|=\frac{r'_1}{\lambda_{1,1}(R'')l_1(z^0)}\right\}$$
$$=p_1(r'_1,r''_1\lambda_{2,1}(R''))|F(z_1^{**},z_2^*,\dots,z_n^*)|\le p_1(r'_1,r''_1\lambda_{2,1}(R''))$$
$$\times\max\left\{|F(z_1^{**},z_2,z_3^*,\dots,z_n^*)|\colon\ |z_2-z_2^0|=\frac{r''_2}{l_2(z^0)}\right\}=p_1(r'_1,r''_1\lambda_{2,1}(R''))$$
$$\times\max\left\{|F(z_1^{**},z_2,\dots,z_n^*)|\colon |z_2-z_2^0|=\frac{r''_2}{l_2(z_1^{**},z_2^0,\dots,z_n^*)}\frac{l_2(z_1^{**},z_2^0,\dots,z_n^*)}{l_2(z^0)}\right\}$$
$$\le p_1(r'_1,r''_1\lambda_{2,1}(R''))\max\left\{|F(z_1^{**},z_2,\dots,z_n^*)|\colon |z_2-z_2^0|=\frac{r''_2\lambda_{2,2}(R'')}{l_2(z_1^{**},z_2^0,\dots,z_n^*)}\right\}$$

$$\leq \prod_{j=1}^{2} p_j(r'_j, r''_j\lambda_{2,j}(R''))$$

$$\times \max\left\{|F(z_1^{**}, z_2, \dots, z_n^*)|\colon |z_2 - z_2^0| = \frac{r'_2}{l_2(z_1^{**}, z_2^0, \dots, z_n^*)}\right\}$$

$$\leq \prod_{j=1}^{2} p_j(r'_j, r''_j\lambda_{2,j}(R''))\max\left\{|F(z_1^{**}, z_2, \dots, z_n^*)|\colon |z_2 - z_2^0| = \frac{r'_2}{\lambda_{1,2}(R'')l_2(z^0)}\right\}$$

$$= \prod_{j=1}^{2} p_j(r'_j, r''_j\lambda_{2,j}(R''))|F(z_1^{**}, z_2^{**}, z_3^*, \dots, z_n^*)| \leq \ldots \leq \prod_{j=1}^{n} p_j(r'_j, r''_j\lambda_{2,j}(R''))$$

$$\times \max\left\{|F(z_1, z_2, \dots, z_n)|\colon\ |z_j - z_j^0| = \frac{r'_j}{\lambda_{1,j}(R'')l_j(z^0)}, j \in \{1, \dots, n\}\right\}$$

$$= \prod_{j=1}^{n} p_j(r'_j, r''_j\lambda_{2,j}(R''))M\left(\frac{R'}{\Lambda_1(R'')\mathbf{L}(z^0)}, z^0, F\right).$$

Hence, by Theorem 3.1 the function F is of bounded $\mathbf{L}$-index in joint variables. □

References

[1] Bandura, A., Skaskiv, O., *Entire functions of several variables of bounded index*, Lviv, Publisher I.E. Chyzhykov, 2016, 128 p.

[2] Bandura, A.I., Skaskiv, O.B., *Directional logarithmic derivative and the distribution of zeros of an entire function of bounded L-index along the direction*, Ukrainian Mat. J., **69**(2017), no. 3, 500–508.

[3] Bandura, A., Skaskiv, O., *Analytic in the unit ball functions of bounded L-index in direciton,* (submitted to Rocky Mountain Journal of Mathematics).

[4] Bandura, A.I., Bordulyak, M.T., Skaskiv, O.B., *Sufficient conditions of boundedness of* $\mathbf{L}$*-index in joint variables*, Mat. Stud., **45**(2016), no. 1, 12–26.

[5] Bandura, A., *New criteria of boundedness of* $\mathbf{L}$*-index in joint variables for entire functions*, (Ukrainian), Math. Bull. Shevchenko Sci. Soc., **13**(2016), 58–67.

[6] Bandura, A.I., Petrechko, N.V., Skaskiv, O.B., *Analytic functions in a polydisc of bounded* $\mathbf{L}$*-index in joint variables*, Mat. Stud., **46**(2016), no. 1, 72–80.

[7] Bandura, A.I., Petrechko, N.V., Skaskiv, O.B., *Maximum modulus in a bidisc of analytic functions of bounded L-index and an analogue of Hayman's Theorem*, Mathematica Bohemica, doi: 10.21136/MB.2017.0110-16 (in print).

[8] Bandura, A., Skaskiv, O., *Functions analytic in a unit ball of bounded L-index in joint variables*, J. Math. Sci. (N.Y.), **227**(2017), no. 1, 1–12.

[9] Bandura, A., Skaskiv, O., Filevych, P., *Properties of entire solutions of some linear PDE's*, J. Appl. Math. Comput. Mech., **16**(2017), no. 2, 17–28.

[10] Chakraborty, B.C., Chanda, R., *A class of entire functions of bounded index in several variables*, J. Pure Math., **12**(1995), 16–21.

[11] Fricke, G.H., *Entire functions of locally slow growth*, J. Anal. Math., **28**(1975), no. 1, 101–122.

[12] Hayman, W.K., *Differential inequalities and local valency*, Pacific J. Math., **44**(1973), no. 1, 117–137.

[13] Krishna, G.J., Shah, S.M., *Functions of bounded indices in one and several complex variables*, In: Mathematical Essays Dedicated to A.J. Macintyre, Ohio Univ. Press, Athens, Ohio, 1970, 223–235.

[14] Kushnir, V.O., Sheremeta, M.M., *Analytic functions of bounded l-index*, Mat. Stud., **12**(1999), no. 1, 59–66.

[15] Kushnir, V.O., *Analogue of Hayman's theorem for analytic functions of bounded l-index*, (Ukrainian), Visn. Lviv Un-ty, Ser. Mekh.-Math., **53**(1999), 48–51.

[16] Nuray, F., Patterson, R.F., *Entire bivariate functions of exponential type*, Bull. Math. Sci., **5**(2015), no. 2, 171–177.

[17] Nuray, F., Patterson, R.F., *Multivalence of bivariate functions of bounded index*, Matematiche (Catania), **70**(2015), no. 2, 225–233.

[18] Patterson, R.F., Nuray, F., *A characterization of holomorphic bivariate functions of bounded index,* Math. Slovaca, **67**(2017), no. 3, 731–736.

[19] Salmassi, M., *Functions of bounded indices in several variables,* Indian J. Math., **31**(1989), no. 3, 249–257.

[20] Sheremeta, M.M., *Entire functions and Dirichlet series of bounded l-index*, Russian Math. (Iz. VUZ), **36**(1992), no. 9, 76–82.

[21] Sheremeta, M.M., *Analytic functions of bounded index*, Lviv, VNTL Publishers, 1999.

[22] Strochyk, S.N., Sheremeta, M.M., *Analytic in the unit disc functions of bounded index*, (Ukrainian), Dopov. Akad. Nauk Ukr., **1**(1993), 19–22.

Andriy Bandura
Ivano-Frankivsk National Technical University of Oil and Gas
Department of Advanced Mathematics
15, Karpatska Street, 76008 Ivano-Frankivsk, Ukraine
e-mail: `andriykopanytsia@gmail.com`

Oleh Skaskiv
Ivan Franko National University of Lviv
Department of Function Theory and Theory of Probability
1, Universytetska Street, 79000 Lviv, Ukraine
e-mail: `olskask@gmail.com`

Stud. Univ. Babeş-Bolyai Math. 63(2018), No. 4, 503–523
DOI: 10.24193/subbmath.2018.4.07

Existence and topological structure of solution sets for φ-Laplacian impulsive stochastic differential systems

Tayeb Blouhi and Mohamed Ferhat

Abstract. In this article, we present results on the existence and the topological structure of the solution set for initial-value problems relating to the first-order impulsive differential equation with infinite Brownian motions are proved. The approach is based on nonlinear alternative Leray-Schauder type theorem in generalized Banach spaces.

Mathematics Subject Classification (2010): 34A37, 34K45, 60H99, 60H05, 65C30, 47H10.

Keywords: ϕ-Laplacian stochastic differential equation, Wiener process, impulsive differential equations, matrix convergent to zero, generalized Banach space, fixed point.

1. Introduction

Differential equations with impulses were considered for the first time by Milman and Myshkis [18] and then followed by a period of active research which culminated with the monograph by Halanay and Wexler [13]. Many phenomena and evolution processes in physics, chemical technology, population dynamics, and natural sciences may change state abruptly or be subject to short-term perturbations. These perturbations may be seen as impulses. Impulsive problems arise also in various applications in communications, mechanics (jump discontinuities in velocity), electrical engineering, medicine and biology fields. A comprehensive introduction to the basic theory is well developed in the monographs by Benchohra et al [3], Graef *et al* [11], Laskshmikantham *et al.* [1], Samoilenko and Perestyuk [26].For instance, in the periodic treatment of some diseases, impulses correspond to the administration of a drug treatment or a missing product.In environmental sciences, impulses correspond to seasonal changes of the water level of artificial reservoirs.
Random differential and integral equations play an important role in characterizing

many social, physical, biological and engineering problems; see for instance the monograph of Da Prato and Zabczyk [7], Gard [9], Gikhman and Skorokhod [10], Sobzyk [27] and Tsokos and Padgett [28]. For example, a stochastic model for drug distribution in a biological system was described by Tsokos and Padgett [28] to a closed system with a simplified heat, one organ or capillary bed, and re-circulation of a blood with a constant rate of flow, where the heart is considered as a mixing chamber of constant volume. For the basic theory concerning stochastic differential equations see the monographs of Wu *et al* [30], Bharucha-Reid [4], Mao[16], Øksendal, [20], Tsokos and Padgett [28], Da Prato and Zabczyk [7].
In this paper, we study the existence theory for initial-value problems with impulse effects.

$$\begin{cases} (\phi(x'(t)))' &= f^1(t,x(t),y(t))dt \\ & \quad + \sum_{l=1}^{\infty} \sigma_l^1(t,x(t),y(t))dW^l(t),\ t\in[0,T], t\neq t_k, \\ (\phi(y'(t)))' &= f^2(t,x(t),y(t))dt \\ & \quad + \sum_{l=1}^{\infty} \sigma_l^2(t,x(t),y(t))dW^l(t),\ t\in[0,T], t\neq t_k, \\ \Delta x(t) &= I_k(x(t_k)),\ \Delta x'(t) = I_k^1(x'(t_k)),\ t=t_k,\ k=1,2,\dots,m, \\ \Delta y(t) &= \overline{I}_k(y(t_k)),\ \Delta y'(t) = \overline{I}_k^2(y'(t_k)),\ t=t_k,\ k=1,2,\dots,m, \\ x(0) &= A_0, \quad y(0)=B_0, \\ x'(0) &= A_1, \quad y'(0)=B_1, \end{cases} \tag{1.1}$$

where $0=t_0<t_1<\ldots<t_m<t_{m+1}=T,\ \ J:=[0,T].\ f_l^1, f_l^2 : J\times\mathbb{R}^2\to\mathbb{R}$ is a given function, $\sigma_l^1,\sigma_l^2 : J\times\mathbb{R}^2\to\mathbb{R}$ is a given function and W^l is an infinite sequence of independent standard Brownian motions, $l=1,2,\ldots$ and $\phi:\mathbb{R}\to\mathbb{R}$ is a suitable monotone homeomorphism, $I_k^1, \overline{I}_k^1, \overline{I}_k^2, I_k^2 \in C(\mathbb{R},\mathbb{R})$, $(k=1,2,\ldots,m)$ and $A_j, B_j\in\mathbb{R}$ for each $j=0,1$, $\Delta x|_{t=t_k} = x(t_k^+)-x(t_k^-), \Delta y|_{t=t_k} = y(t_k^+)-y(t_k^-)$ and $\Delta x'|_{t=t_k} = x'(t_k^+)-x'(t_k^-), \Delta y'|_{t=t_k} = y'(t_k^+)-y'(t_k^-)$. The notations $y(t_k^+)=\lim_{h\to 0^+} y(t_k+h)$ and $y(t_k^-)=\lim_{h\to 0^+} y(t_k-h)$ stand for the right and the left limits of the function y at $t=t_k$, respectively. Set

$$\begin{cases} f_i(.,x,y) = (f_1^i(.,x,y), f_2^i(.,x,y),\ldots), \\ \|f_i(.,x,y)\| = \Big(\sum_{l=1}^{\infty}(f_l^i)^2(.,x,y)\Big)^{\frac{1}{2}}, \end{cases} \tag{1.2}$$

where $i=1,2,\ f_i(.,x,y)\in l^2$ for all $x\in\mathbb{R}$.
This paper is organized as follows: In Section 2, we introduce all the back- ground material used in this paper such as stochastic calculus. In Section 3, to provide some existence results and to establish the compactness of solution sets to the above problems are quoted.

2. Preliminaries

In this section, we introduce notations, definitions, and preliminary facts which are used throughout this paper.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space with a filtration $(\mathcal{F} = \mathcal{F}_t)_{t\geq 0}$ satisfying the usual conditions (i.e. right continuous and $\mathcal{F}_0$ containing all $\mathbb{P}$-null sets). Assume $W(t)$ is an infinite sequence of independent standard Brownian motions, defined on $(\Omega, \mathcal{F}, \mathbb{P})$ that is, $W(t) = (W^1(t), W^2(t), \ldots)^T$. An $\mathbb{R}$-valued random variable is an $\mathcal{F}$-measurable function $x(t) : \Omega \to \mathbb{R}$ and the collection of random variables

$$S = \{x(t, \omega) : \Omega \to \mathbb{R} | \ t \in J\}$$

is called a stochastic process. Generally, we write $x(t)$ instead of $x(t, \omega)$.

Definition 2.1. An $\mathcal{F}$-adapted process X on $[0, T] \times \Omega$ is elementary processes if for a partition $\phi = \{t = 0 < t_1 < \ldots < t_n = T\}$ and $(\mathcal{F}_{t_i})$-measurable random variables $(X_{t_i})_{i<n}$, X_t satisfies

$$X_t(\omega) = \sum_{i=0}^{n-1} X_i(\omega)\chi_{[t_i,t_{i+1})}(t), \quad \text{for} \quad 0 \leq t \leq T, \quad \omega \in \Omega.$$

The Itô integral of the simple process X is defined as

$$\int_0^T X(s)dW^l(s) = \sum_{i=0}^{n-1} X_l(t_i)(W^l(t_{i+1}) - W^l(t_i)), \tag{2.1}$$

whenever $X_{t_i} \in L^2(\mathcal{F}_{t_i})$ for all $i \leq n$.

The following result is one of the elementary properties of square-integrable stochastic processes [20, 16].

Lemma 2.2. *(***Itô Isometry for Elementary Processes***) Let $(X_l)_{l\in\mathbb{N}}$ be a sequences of elementary processes. Assume that*

$$\int_0^T E|X(s)|^2 ds < \infty,$$

where $|X|^2 = \left(\sum_{l=1}^{\infty} X_l^2\right)$. *Then*

$$E\left(\sum_{l=1}^{\infty} \int_0^T X_l(s)dW^l(s)\right)^2 = E\left(\sum_{l=1}^{\infty} \int_0^T X_l^2(s)ds\right). \tag{2.2}$$

Remark 2.3. For a square integrable stochastic process X on $[0, T]$, its Itô integral is defined by

$$\int_0^T X(s)dW(s) = \lim_{n\to\infty} \int_0^T X_n(s)dW(s),$$

taking the limit in L^2, with X_n is defined in definition 2.1. Then the Itô isometry holds for all Itô-integrable X.

The next result is known as the Burholder-Davis-Gundy inequalities. It was first proved for discrete martingales and $p > 0$ by Burkholder [5] in 1966. In 1968, Millar [17] extended the result to continuous martingales. In 1970, Davis [8] extended the result for discrete martingales to $p = 1$. The extension to $p > 0$ was obtained independently by Burkholder and Gundy [6] in 1970 and Novikov [19] in 1971.

Theorem 2.4. [23] *For each $p > 0$ there exist constants $c_p, C_p \in (0, \infty)$, such that for any progressive process x with the property that for some $t \in [0, \infty)$, $\int_0^t X_s^2 ds < \infty$ a.s, we have*

$$c_p E\left(\int_0^t X_s^2 ds\right)^{\frac{p}{2}} \leq E\left(\sup_{s\in[0,t]} \int_0^t X_s dW(s)\right)^p \leq C_p E\left(\int_0^t X_s^2 ds\right)^{\frac{p}{2}}. \tag{2.3}$$

2.1. Some results on fixed point theorems and set-valued analysis

The classical Banach contraction principle was extended for contractive maps on spaces endowed with vector-valued metric space by Perov [21] in 1964 and Precup [22].

For $x, y \in \mathbb{R}^n$, $x = (x_1, \dots, x_n)$, $y = (y_1, \dots, y_n)$, by $x \leq y$ we mean $x_i \leq y_i$ for all $i = 1, \dots, n$.

Also $|x| = (|x_1|, \dots, |x_n|)$ and $\max(x, y) = \max(\max(x_1, y_1), \dots, \max(x_n, y_n))$.

If $c \in \mathbb{R}$, then $x \leq c$ means $x_i \leq c$ for each $i = 1, \dots, n$.

Definition 2.5. Let X be a nonempty set. A vector-valued metric on X is a map $d : X \times X \to \mathbb{R}^n$ with the following properties:

(i) $d(u, v) \geq 0$ for all $u, v \in X$; if $d(u, v) = 0$ then $u = v$;

(ii) $d(u, v) = d(v, u)$ for all $u, v \in X$;

(iii) $d(u, v) \leq d(u, w) + d(w, v)$ for all $u, v, w \in X$.

The pair (X, d) is said to be a generalized metric space.

For $r = (r_1, \dots, r_n) \in \mathbb{R}_+^n$, we will denote by

$$B(x_0, r) = \{x \in X : d(x_0, x) < r\},$$

the open ball centered in x_0 with radius r and

$$\overline{B(x_0, r)} = \{x \in X : d(x_0, x) \leq r\}$$

the closed ball centered in x_0 with radius r. We mention that for generalized metric space, the notation of open subset, closed set, convergence, Cauchy sequence and completeness are similar to those in usual metric spaces.

Definition 2.6. A generalized metric space (X, d), where

$$d(x, y) := \begin{pmatrix} d_1(x, y) \\ \cdots \\ d_n(x, y) \end{pmatrix},$$

is complete if (X, d_i) is a complete metric space for every $i = 1, \dots, n$.

Definition 2.7. The map $f : J \times X \to X$ is said to be L^2-Caratheodory if

i) $t \mapsto f(t, u)$ is measurable for each $u \in X$;

ii) $u \mapsto f(t,u)$ is continuous for almost all $t \in J$;
iii) For each $q > 0$, there exists $\alpha_q \in L^1(J, \mathbb{R}^+)$ such that

$$E|f(t,u)|_X^2 \leq \alpha_q \text{ ,for all } u \in X \text{ such that } E|u|_X^2 \leq q \text{ and for } \quad a.e. \quad t \in J.$$

Lemma 2.8 (Grönwall-Bihari [2]**).** *Let $I = [0,b]$ and let $u, g : I \to \mathbb{R}$ be positive continuous functions. Assume there exist $c > 0$ and a continuous nondecreasing function $h : [0,\infty) \to (0,+\infty)$ such that*

$$u(t) \leq c + g(s)h(u(s))ds, \quad \forall t \in I.$$

Then

$$u(t) \leq H^{-1}\Big(\int_p^t g(s)ds\Big), \quad \forall t \in I,$$

provided

$$\int_c^{+\infty} \frac{dy}{h(y)} > \int_p^q g(s)ds,$$

where H^{-1} refers to inverse of the function $H(u) = \int_c^u \frac{dy}{h(y)}$ for $u \geq c$.

In the paper [14], the case of a single system of differential equations was analyzed based on the technique of applying the nonlinear alternative of Leray-Schauder type. In the present paper we extend these results to the more general case of coupled stochastic differential systems with infinite Brownian motions, and we will apply a different technique to obtain our results.
Next, we quote the version of nonlinear alternative Leary-Schauder type theorem in generalized Banach space[29].

Theorem 2.9. *Let $C \subset E$ be a closed convex subset and $U \subset C$ a bounded open neighborhood of zero (with respect to topology of C). If $N : \overline{U} \to E$ is compact continuous then*

i) *Either N has a fixed point in $\overline{U}$, or*
ii) *There exists $x \in \partial U$ such that $x = \lambda N(x)$ for some $\lambda \in (0,1)$.*

3. Main results

Let $J_k = (t_k, t_{k+1}],\ k = 1, 2, \ldots, m$. In order to define a solution for Problem (1.1), consider the following space of piece-wise continuous functions.

Let us introduce the spaces

$$H_2([0,T]; L^2(\Omega,\mathbb{R})) = \{x : J \to L^2(\Omega,\mathbb{R}) \text{ },x\,|_{(t_k,t_{k+1}]} \in C((t_k,t_{k+1}], L^2(\Omega,\mathbb{R})),$$
$$k = 1,2,..,m \text{ and there exist } x(t_k^+) \text{ for } k = 1,2,..,m\},$$

and

$$H_2'([0,T]; L^2(\Omega,\mathbb{R})) = \{x : J \to L^2(\Omega,\mathbb{R}) \text{ },x\,|_{(t_k,t_{k+1}]} \in C^1((t_k,t_{k+1}], L^2(\Omega,\mathbb{R})),$$
$$k = 1,2,..,m \text{ and there exist } x(t_k^+) \text{ for } k = 1,2,..,m\}.$$

It is clear that $H_2([0,T];L^2(\Omega,\mathbb{R}))$ endowed with the norm

$$\|x\|_{H_2}=\sup_{s\in[0,T]}(E|x(s,.)|^2)^{\frac{1}{2}}.$$

It is easy to see that H_2' is a Banach space with the norm $\|x\|_{H_2'}=\|x\|_{H_2}+\|x'\|_{H_2}$. Finally, let the space

$$PC=\{x:[0,T]\to L^2(\Omega,\mathbb{R}) \quad \text{and}\, x\mid_J\in H_2' \text{ such that}$$
$$\sup_{t\in[0,T]}E|x(t,.)|^2<\infty \text{ almost surely}\},$$

endowed with the norm

$$\|x\|_{PC}=\sup_{s\in[0,T]}(E|x(s,.)|^2)^{\frac{1}{2}}.$$

It is not difficult to check that PC is a Banach space with norm $\|\cdot\|_{PC}$.

Let us now prove the existence and uniqueness of solutions to our problem which will be obtained by applying the Leary-Schauder fixed point theorem. To this end we first need to introduce the following hypotheses:

$(H1)$ $f^i,\sigma^i:[0,T]\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ is an Carathéodory function and $E|\phi^{-1}(X)|^2\le\phi^{-1}(E|X|^2)$ with $X\in\mathbb{R}$, $I_k,\bar{I}_k\in C(\mathbb{R},\mathbb{R})$.

$(H2)$ There exist constants $\overline{a}_i,\overline{b}_i,c_i\in\mathbb{R}^+$ such that each

$$|f^i(t,x,y)|^2\le\overline{a}_i|x|^2+\overline{b}_i|y|^2+c_i,\ i=1,2.$$

for all $x,y\in\mathbb{R}$, and a.e. $t\in J$.

$(H3)$ There exist constants $\overline{\alpha}_i\in\mathbb{R}^+$ and $\overline{\beta}_i,\overline{c}_i\in\mathbb{R}^+$ such that

$$\|\sigma^i(t,x,y)\|^2\le\overline{\alpha}_i|x|^2+\overline{\beta}_i|y|^2+\overline{c}_i,\quad i=1,2$$

for all $x,y\in\mathbb{R}$, and a.e. $t\in J$.

Theorem 3.1. *Assume that* $(H1)$-$(H3)$ *hold. Then, problem* (1.1) *has at least one solution and the solution set*

$$S_c=\{(x,y)\in PC\times PC:(x,y)\ \textit{is a solution of}\ (1.1)\}$$

is compact.

Proof. The proof involves several steps.

Step 1. Consider the problem

$$\begin{cases}(\phi(x'(t)))' &=f^1(t,x(t),y(t))dt+\sum_{l=1}^{\infty}\sigma_l^1(t,x(t)),y(t))dW^l(t),\ t\in[0,t_1],\\ (\phi(y'(t)))' &=f^2(t,x(t),y(t))dt+\sum_{l=1}^{\infty}\sigma_l^2(t,x(t)),y(t))dW^l(t),\ t\in[0,t_1],\\ x(0)=A_0 &,\quad y(0)=B_0,\\ x'(0)=A_1 &,\quad y'(0)=B_1.\end{cases}\tag{3.1}$$

Let

$$\widehat{C}_{t_0}=\{x:[0,t_1]\to L^2(\Omega,\mathbb{R})\quad,x\mid_{[0,t_1]}\in C^1([0,t_1],L^2(\Omega,\mathbb{R})),\quad k=1,2,..,m,$$

and there exists

$$x(t_1^+) \quad \text{for} \quad k = 1, 2, .., m\},$$

with

$$C_{t_0} = \{x : [0, t_1] \to L^2(\Omega, \mathbb{R}) \qquad \text{and}\, x\,|_{[0,t_1]} \in \widehat{C}_{t_0} \text{ such that}$$

$$\sup_{t\in[0,t_1]} E|x(t,.)|^2 < \infty \text{ almost surely}\},$$

Consider the operator

$$P^0 : C_{t_0} \times C_{t_0} \to C_{t_0} \times C_{t_0}$$

defined by

$$P^0(x, y) = (P_1^0(x, y), P_2^0(x, y)),\ (x, y) \in C_{t_0} \times C_{t_0}$$

where

$$\begin{cases} P_1^0(x,y) &= A_0 + \displaystyle\int_0^t \phi^{-1}\Big(\phi(A_1) + \int_0^s f^1(r, x(r), y(r))dr + \\ & \quad + \displaystyle\sum_{l=1}^{\infty} \int_0^s \sigma_l^1(r, x(r), y(r))dW^l(r)\Big)ds, \quad t \in [0, t_1],\ a.e.\ \omega \in \Omega. \\ P_2^0(x,y) &= B_0 + \displaystyle\int_0^t \phi^{-1}\Big(\phi(B_1) + \int_0^s f^2(r, x(r), y(r))dr + \\ & \quad + \displaystyle\sum_{l=1}^{\infty} \int_0^s \sigma_l^2(r, x(r), y(r))dW^l(r)\Big)ds, \quad t \in [0, t_1],\ a.e.\ \omega \in \Omega. \end{cases} \tag{3.2}$$

Clearly, the fixed points of $P^0 = (P_1^0, P_2^0)$ are solutions of the problem (3.1).
To apply the nonlinear alternative of Leray-Schauder type, we first show that P^0 is completely continuous. The proof will be given in several steps.
Claim 1. P^0 sends bounded sets into bounded sets in $C_{t_0} \times C_{t_0}$. Indeed, it is enough to show that for any $q > 0$, there exists a positive constant κ such that for each

$$(x, y) \in B_q = \{(x, y) \in C_{t_0} \times C_{t_0} : \sup_{t\in[0,t_1]} E|x(t,\cdot)|^2 \le q,\ \sup_{t\in[0,t_1]} E|x(t,\cdot)|^2 \le q\},$$

we have

$$\|P^0(x, y)\| \le \kappa = (\kappa_1, \kappa_2).$$

Then for each $t \in [0, t_1]$, we have

$$E|P_1^0(x,y)|^2 \le 2E|A_0|^2 + 2\int_0^t E\Big|\phi^{-1}\Big(\phi(A_1) + \int_0^s f^1(r, x(r), y(r))dr$$
$$+ \sum_{l=1}^{\infty} \int_0^s \sigma_l^1(r, x(r), y(r))dW^l(r)\Big)ds\Big|^2.$$

From Lemma 2.4, we obtain

$$\begin{aligned}
&E\Big|\phi(A_1)+\int_0^s f^1(r,x(r),y(r))dr+\sum_{l=1}^{\infty}\int_0^s \sigma_l^1(r,x(r),y(r))dW^l(r)\Big|^2\\
&\le 3|\phi(A_1)|_X^2+3t_1\int_0^s(\overline{a}_1|x(r)|_X^2+\overline{b}_1|y(r)|^2+c_1)dr\\
&+3C_2\int_0^s(\overline{\alpha}_1|x(r)|^2+\overline{\beta}_1|y(r)|^2+\overline{c}_1)dr,
\end{aligned}$$

it follows that

$$E\Big|\phi(A_1)+\int_0^s f^1(r,x(r),y(r))dr+\sum_{l=1}^{\infty}\int_0^s \sigma_l^1(r,x(r),y(r))dW(r)\Big|^2\in\overline{B}(0,l_1),$$

where

$$\begin{aligned}
l_1&=3E|\phi(A_1)|^2+3t_1\int_0^s(\overline{a}_1E|x(r)|^2+\overline{b}_1E|y(r)|^2+c_1)dr\\
&+3C_2\int_0^s(\overline{\alpha}_1E|x(r)|^2+\overline{\beta}_1E|y(r)|^2+\overline{c}_1)dr.
\end{aligned}$$

Since ϕ^{-1} is continuous,

$$\sup_{\eta_1\in\overline{B}(0,l_1)}|\phi^{-1}(\eta_1)|<\infty.$$

Thus

$$E|P_1^0(x,y)|^2\le 2E|A_0|^2+2t_1\sup_{\eta_1\in\overline{B}(0,l_1)}|\phi^{-1}(\eta_1)|:=\kappa_1.$$

Similarly,

$$E|P_2^0(x,y)|^2\le 2E|B_0|^2+2t_1\sup_{\eta_2\in\overline{B}(0,l_2)}|\phi^{-1}(\eta_2)|:=\kappa_2,$$

where

$$\begin{aligned}
l_2&=3E|\phi(B_1)|^2+3t_1\int_0^s(\overline{a}_2E|x(r)|^2+\overline{b}_2E|y(r)|^2+c_2)dr\\
&+3C_2\int_0^s(\overline{\alpha}_2E|x(r)|^2+\overline{\beta}_2E|y(r)|^2+\overline{c}_2)dr.
\end{aligned}$$

Since ϕ^{-1} is continuous,

$$\sup_{\eta_1\in\overline{B}(0,l_1)}|\phi^{-1}(\eta_1)|<\infty.$$

Claim 2. P^0 maps bounded sets into equicontinuous sets. Let $l_1, l_2 \in [0, t_1]$, $l_1 < l_2$ and B_q be a bounded set of $C_{t_0} \times C_{t_0}$ as in Claim 1. Let $(x, y) \in B_q$. Then

$$\begin{aligned}
E|(P_1^0(x,y))'(t)|^2 &= E\Big|\phi^{-1}\Big(\phi(A_1) + \int_0^t f^1(s, x(s), y(s))dr \\
&+ \sum_{l=1}^{\infty} \int_0^t \sigma_l^1(s, x(s), y(s))dW^l(s)\Big) - \phi^{-1}(\phi(A_1))\Big|^2 \\
&\leq E\Big|\phi^{-1}\Big(\phi(A_1) + \int_0^t f^1(s, x(s), y(s))dr \\
&+ \sum_{l=1}^{\infty} \int_0^t \sigma_l^1(s, x(s), y(s))dW^l(s)\Big)\Big|^2 + E\Big|A_1\Big|^2 \\
&\leq \sup_{\eta_1 \in \overline{B}(0,l_1)} |\phi^{-1}(\eta_1)| + E|A_1| := r'.
\end{aligned}$$

Using the mean value theorem, we obtain

$$E|(P_1^0(x,y))(l_2) - (P_1^0(x,y))(l_1)| = E|(P_1^0(x,y))'(\xi, \overline{\xi})(l_2 - l_1)| \leq r'|l_2 - l_1|.$$

As $l_2 \to l_1$ the right-hand side of the above inequality tends to zero.
Similarly,

$$\begin{aligned}
E|(P_2^0(x,y))'(t)|_X^2 &= E\Big|\phi^{-1}\Big(\phi(B_1) + \int_0^t f^2(s, x(s), y(s))dr \\
&+ \sum_{l=1}^{\infty} \int_0^t \sigma_l^2(s, x(s), y(s))dW^l(s)\Big) - \phi^{-1}(\phi(B_1))\Big|^2 \\
&\leq \Big|\phi^{-1}\Big(\phi(B_1) + \int_0^t f^2(s, x(s), y(s))dr \\
&+ \sum_{l=1}^{\infty} \int_0^t \sigma_l^2(s, x(s), y(s))dW^l(s)\Big)\Big|^2 + \Big|B_1\Big|^2 \\
&\leq \sup_{\eta_2 \in \overline{B}(0,l_2)} |\phi^{-1}(\eta_2)| + |B_1| := r'.
\end{aligned}$$

Using the mean value theorem, we obtain

$$E|(P_2^0(x,y))(l_2) - (P_2^0(x,y))(l_1)| = E|(P_2^0(x,y))'(\xi, \overline{\xi})(l_2 - l_1)| \leq r'|l_2 - l_1|.$$

As $l_2 \to l_1$ the right-hand side of the above inequality tends to zero.
Claim 3. P^0 is continuous. Let $(x_n, y_n)_{n\in\mathbb{N}}$ be a sequence such that $(x_n, y_n) \to (x, y)$ in $C_{t_0} \times C_{t_0}$. Then there is an integer q such that

$$\sup_{t\in[0,t_1]} E|x_n(t,\cdot)|^2 \leq q, \quad \sup_{t\in[0,t_1]} E|y_n(t,\cdot)|^2 \leq q \leq q \text{ for all } n \in \mathbb{N}$$

and

$$\sup_{t\in[0,t_1]} E|x(t,\cdot)|^2 \leq q, \quad \sup_{t\in[0,t_1]} E|y(t,\cdot)|^2 \leq q, \ (x_n, y_n) \in B_q \text{ and } (x, y) \in B_q.$$

Then for each $t \in [0, t_1]$, we have

$$\begin{aligned} E|P_1^0(x_n, y_n) - P_1^0(x, y)|_X^2 \quad &\leq \int_0^t E\Big|\phi^{-1}\Big(\phi(A_1) + \int_0^s f^1(r, x(r), y(r))dr \\ &+ \sum_{l=1}^{\infty} \int_0^s \sigma_l^1(r, x(r), y(r))dW^l(r)\Big) \\ &-\phi^{-1}\Big(\phi(A_1) + \int_0^s f^1(r, x(r), y(r))dr) \\ &+ \sum_{l=1}^{\infty} \int_0^s \sigma^1(r, x(r), y(r))dW^l(r)\Big)\Big|^2 ds. \end{aligned}$$

Using the dominated convergence theorem, we have

$$E\Big|\phi(A_1) + \int_0^s f^1(r, x_n(r), y_n(r))dr + \sum_{l=1}^{\infty} \int_0^s \sigma^1(r, x_n(r), y_n(r))dW^l(r),$$

$$-\phi(A_1) - \int_0^s f(r, x(r), y(r))dr - \sum_{l=1}^{\infty} \int_0^s \sigma^1(r, x(r), y(r))dW^l(r)\Big|_X^2 \to 0 \quad \text{as } n \to \infty,$$

since ϕ^{-1} is continuous. Then using the dominated convergence theorem, we have

$$\begin{aligned} &\sup_{t \in [0, t_1]} E|P_1^0(x_n, y_n) - P_1^0(x, y)|^2 \\ &\leq \int_0^{t_1} E|\phi^{-1}[\phi(B) + \int_0^s f^1(r, x_n, y_n)dr + \sum_{l=1}^{\infty} \int_0^s \sigma^1(r, x_n(r), y_n(r))dW^l(r)] \\ &- \phi^{-1}[\phi(B) + \int_0^s f^1(r, x, y)dr + \sum_{l=1}^{\infty} \int_0^s \sigma_l^1(r, x(r), y(r))dW^l(r)]^2 ds \to 0, \end{aligned}$$

as $n \to \infty$. Thus P_1^0 is continuous.
Similarly,

$$\begin{aligned} &\sup_{t \in [0, t_1]} E|P_2^0(x_n, y_n) - P_2^0(x, y)|^2 \\ &\leq \int_0^{t_1} E|\phi^{-1}[\phi(B) + \int_0^s f^2(r, x_n, y_n)dr + \sum_{l=1}^{\infty} \int_0^s \sigma_l^2(r, x_n(r), y_n(r))dW^l(r)] \\ &- \phi^{-1}[\phi(B) + \int_0^s f^2(r, x, y)dr + \sum_{l=1}^{\infty} \int_0^s \sigma_l^2(r, x(r), y(r))dW(r)]|^2 ds \to 0, \end{aligned}$$

as $n \to \infty$. Thus P_2^0 is continuous.
Claim 4. *Apriori* estimate. Now we show that there exists a constant M_0 such that $\sup_{t \in [0, t_1]} E|x(t, \cdot)|_X^2 \leq M_0$ where (x, y) is a solution of the problem (3.1). Let (x, y) a

solution of (3.1):

$$\begin{cases} x(t) &= A_0+\displaystyle\int_0^t \phi^{-1}\Big(\phi(A_1)+\int_0^s f^1(r,x(r),y(r))dr+ \\ &\quad +\displaystyle\sum_{l=1}^{\infty}\int_0^s \sigma_l^1(r,x(r),y(r))dW^l(r)\Big)ds, \quad t\in[0,t_1],\ a.e.\ \omega\in\Omega. \\ y(t) &= B_0+\displaystyle\int_0^t \phi^{-1}\Big(\phi(B_1)+\int_0^s f^2(r,x(r),y(r))dr+ \\ &\quad +\displaystyle\sum_{l=1}^{\infty}\int_0^s \sigma_l^2(r,x(r),y(r))dW^l(r)\Big)ds, \quad t\in[0,t_1],\ a.e.\ \omega\in\Omega. \end{cases} \tag{3.3}$$

From Lemma 2.4, we obtain

$$\begin{aligned} E|x(t)|^2 &\le E|A_0|^2+\int_0^t E\Big|\phi^{-1}\Big(\phi(A_1)+\int_0^s f^1(r,x(r),y(r))dr \\ &\quad +\sum_{l=1}^{\infty}\int_0^s \sigma^1(r,x(r),y(r))dW^l(r)\Big)\Big|^2 ds \\ &\le 2E|A_0|^2+2t_1\sup_{\eta_1\in\overline{B}(0,l_1)}|\phi^{-1}(\eta_1)| =: M_0 \end{aligned}$$

where

$$\begin{aligned} l_1 &= 3E|\phi(A_1)|^2+3t_1\int_0^s(\overline{a}_1E|x(r)|_X^2+\overline{b}_1E|y(r)|^2+c_1)dr \\ &\quad +3C_2\int_0^s(\overline{\alpha}_1E|x(r)|^2+\overline{\beta}_1E|y(r)|^2+\overline{c}_1)dr. \end{aligned}$$

Thus,

$$\sup_{t\in[0,t_1]}E|x(t)|^2\le M_0,$$

and

$$\begin{aligned} E|y(t)|^2 &\le E|B_0|^2+\int_0^t E\Big|\phi^{-1}\Big(\phi(B_1)+\int_0^s f^2(r,x(r),y(r))dr \\ &\quad +\sum_{l=1}^{\infty}\int_0^s \sigma^2(r,x(r),y(r))dW^l(r)\Big)\Big|^2 ds \\ &\le 2E|B_0|^2+2t_1\sup_{\eta_2\in\overline{B}(0,l_2)}|\phi^{-1}(\eta_2)| =: M_0 \end{aligned}$$

where

$$\begin{aligned} l_2 &= 3E|\phi(A_1)|^2+3t_1\int_0^s(\overline{a}_1E|x(r)|^2+\overline{b}_1E|y(r)|^2+c_1)dr \\ &\quad +3C_2\int_0^s(\overline{\alpha}_1E|x(r)|^2+\overline{\beta}_1E|y(r)|^2+\overline{c}_1)dr. \end{aligned}$$

Thus,

$$\sup_{t\in[0,t_1]} E|y(t)|^2 \leq M_0.$$

Set

$$U = \{y \in C([0,t_1],\mathbb{R}) : \sup_{t\in[0,t_1]} E|x(t)|^2 < M_0+1,\ \sup_{t\in[0,t_1]} E|y(t)|^2 < M_0+1\}.$$

As a consequence of Claims 1-4 and the Ascoli-Arzela theorem, we can conclude that the map $P^0 : \overline{U} \to C_{t_0} \times C_{t_0}$ is compact. From the choice of U there is no $(x,y) \in \partial U$ such that $(x,y) = \lambda P^0(x,y)$ for any $\lambda \in (0,1)$. And from the consequence of the nonlinear alternative of Leray-Schauder we deduce that P^0 has a fixed point denoted by $(x_0,y_0) \in \overline{U}$ which is solution of the problem (3.1).

Step 2. Now consider the problem

$$\begin{cases} (\phi(x'(t)))' &= f^1(t,x(t),y(t))dt + \sum_{l=1}^{\infty} \sigma_l^1(t,x(t)),y(t))dW(t),\ t \in (t_1,t_2], \\ (\phi(y'(t)))' &= f^2(t,x(t),y(t))dt + \sum_{l=1}^{\infty} \sigma_l^2(t,x(t)),y(t))dW(t),\ t \in (t_1,t_2], \\ x(t_1^+) &= x_0(t_1^-) + I_1(x_0(t_1^-)) \quad ,x'(t_1^+) = x_0'(t_1^-) + I_1^1(x_0(t_1^-)), \\ y(t_1^+) &= y_0(t_1^-) + \overline{I}_1(x_0(t_1^-)) \quad ,y'(t_1^+) = y_0'(t_1^-) + \overline{I}_1^2(y_0(t_1^-)). \end{cases} \tag{3.4}$$

Let

$$\widehat{C}_{t_1} = \{x : (t_1,t_2] \to L^2(\Omega,\mathbb{R}),\ x\mid_{(t_1,t_2]} \in C^1((t_1,t_2], L^2(\Omega,\mathbb{R})),\ k = 1,2,..,m$$

and there exists

$$x(t_2^+) \quad \text{for} \quad k = 1,2,..,m\},$$

with

$$D_{t_1} = \{x : (t_1,t_2] \to L^2(\Omega,\mathbb{R}) \qquad \text{and } x(t)\mid_{(t_1,t_2]} \in \widehat{C}_{t_1} \quad \text{such that}$$

$$\sup_{t\in(t_1,t_2]} E|x(t,.)|^2 < \infty \ \text{ almost surely}\}.$$

Set

$$C_1 = C_{t_0} \cap D_{t_1}.$$

Consider the operator $P^1 : C_1 \times C_1 \to C_1 \times C_1$ defined by

$$P^1(x,y) = (P_1^1(x,y), P_2^1(x,y)),\ (x,y) \in C_1 \times C_1.$$

It is clear that all solutions of (3.4) are fixed points of the multi-valued operator $P_i^1 : C_1 \times C_1 \to C_1$, for each $i = 1, 2$ defined by

$$\begin{cases} P_1^1(x,y) &= A_3 + \int_{t_1}^t \phi^{-1}\Big(\phi(A_4) + \int_{t_1}^s f^1(r, x(r), y(r))dr + \\ &\quad + \sum_{l=1}^{\infty} \int_0^s \sigma_l^1(r, x(r), y(r))dW^l(r)\Big)ds, \quad t \in (t_1, t_2], \ a.e.\ \omega \in \Omega. \\ \\ P_2^1(x,y) &= B_3 + \int_{t_1}^t \phi^{-1}\Big(\phi(B_4) + \int_{t_1}^s f^2(r, x(r), y(r))dr + \\ &\quad + \sum_{l=1}^{\infty} \int_0^s \sigma^2(r, x(r), y(r))dW^l(r)\Big)ds, \quad t \in (t_1, t_2], \ a.e.\ \omega \in \Omega. \end{cases} \tag{3.5}$$

and

$$A_3 = x_1(t_1) + I_1(x_1(t_1)), \quad A_4 = x_1'(t_1^-) + I_1^1(x_1(t_1^-)),$$
$$B_3 = y_1(t_1) + \overline{I}_1(y_1(t_1)), \quad B_4 = y_1'(t_1^-) + \overline{I}_1^2(y_1(t_1^-)).$$

As in Step 1, we can prove that P^1 has at least one fixed point which is a solution to (3.4).

Step 3. We continue this process taking into account that

$$(x_m, y_m) := (x\big|_{(t_m,T]}, y\big|_{(t_m,T]})$$

is a solution to the problem

$$\begin{cases} (\phi(x'(t)))' &= f^1(t, x(t), y(t))dt + \sum_{l=1}^{\infty} \sigma_l^1(t, x(t)), y(t))dW^l(t), \ t \in (t_m, T], \\ (\phi(y'(t)))' &= f^2(t, x(t), y(t))dt + \sum_{l=1}^{\infty} \sigma_l^2(t, x(t)), y(t))dW^l(t), \ t \in ((t_m, T], \\ x(t_m^+) &= x_{m-1}(t_m^-) + I_m(x_0(t_{m-1}^-)), \\ x'(t_m^+) &= x_{m-1}'(t_m^-) + I_m^1(x_{m-1}(t_m^-)), \\ y(t_m^+) &= y_{m-1}(t_m^-) + \overline{I}_m(x_0(t_{m-1}^-)), \\ y'(t_m^+) &= y_{m-1}'(t_m^-) + \overline{I}_m^2(y_{m-1}(t_m^-)). \end{cases} \tag{3.6}$$

A solution (x, y) of problem (3.6) is ultimately defined by

$$(x(t), y(t)) = \begin{cases} (x_0(t), y_0(t)), & \text{if } t \in [0, t_1], \\ (x_1(t), y_1(t)), & \text{if } t \in (t_1, t_2], \\ \dots \\ (x_m(t), y_m(t)), & \text{if } t \in (t_m, T]. \end{cases}$$

Step 4. Now we show that the set

$$S_c = \{(x, y) \in PC \times PC : (x, y) \text{ is a solution of (1.1)}\}$$

is compact. Let $(x_n, y_n)_{n \in \mathbb{N}}$ be a sequence in S_c. We put

$$B = \{(x_n, y_n) : n \in \mathbb{N}\} \subseteq PC \times PC.$$

Then from earlier parts of the proof of this theorem, we conclude that B is bounded and equicontinuous and from the Ascoli-Arzela theorem, we can also conclude that B is compact.
Recall that $J_0 = [0, t_1]$ and $J_k = (t_k, t_{k+1}]$, $k = 1, \ldots, m$. Hence:
• $(x_n, y_n)|_{J_0}$ has a subsequence

$$(x_{n_m}, y_{n_m})_{n_m \in \mathbb{N}} \subset S_{c_1} = \{(x, y) \in C_{t_0} \times C_{t_0} : (x, y) \text{ is a solution of } (3.1)\}$$

such that (x_{n_m}, y_{n_m}) converges to (x, y). Let

$$\begin{cases} z_0(t) = A_0 + \displaystyle\int_0^t \phi^{-1}\Big(\phi(A_1) + \int_0^s f^1(r, x(r), y(r))dr + \\ \qquad + \displaystyle\sum_{l=1}^{\infty} \int_0^s \sigma^1(r, x(r), y(r))dW^l(r)\Big)ds, \quad t \in [0, t_1], \ a.e.\ \omega \in \Omega. \\ \overline{z}_0(t) = B_0 + \displaystyle\int_0^t \phi^{-1}\Big(\phi(B_1) + \int_0^s f^2(r, x(r), y(r))dr + \\ \qquad + \displaystyle\sum_{l=1}^{\infty} \int_0^s \sigma^2(r, x(r), y(r))dW^l(r)\Big)ds, \quad t \in [0, t_1], \ a.e.\ \omega \in \Omega. \end{cases} \tag{3.7}$$

$$\begin{aligned} E\Big|x_{n_m}(t) - z_0(t)\Big|_X^2 \leq \int_0^t E\Big|&\phi^{-1}\Big(\phi(A_1) + \int_0^s f^1(r, x_{n_m}(r), y_{n_m}(r))dr \\ &+ \sum_{l=1}^{\infty} \int_0^s \sigma_l^1(r, x_{n_m}(r), y_{n_m}(r))dW^l(r)\Big) \\ &- \phi^{-1}\Big(\phi(A_1) + \int_0^s f^1(r, x(r), y(r))dr + \\ &+ \sum_{l=1}^{\infty} \int_0^s \sigma_l^1(r, x(r), y(r))dW^l(r)\Big)\Big|_X^2 ds, \end{aligned}$$

and

$$\begin{aligned} E\Big|y_{n_m}(t) - \overline{z}_0(t)\Big|_X^2 \leq \int_0^t E\Big|&\phi^{-1}\Big(\phi(B_1) + \int_0^s f^1(r, x_{n_m}(r), y_{n_m}(r))dr \\ &+ \sum_{l=1}^{\infty} \int_0^s \sigma_l^1(r, x_{n_m}(r), y_{n_m}(r))dW^l(r)\Big) \\ &- \phi^{-1}\Big(\phi(A_1) + \int_0^s f^1(r, x(r), y(r))dr + \\ &+ \sum_{l=1}^{\infty} \int_0^s \sigma_l^1(r, x(r), y(r))dW^l(r)\Big)\Big|_X^2 ds. \end{aligned}$$

As $n_m \to +\infty$, $(x_{n_m}, y_{n_m}) \to (z_0(t), \overline{z}_0(t))$, then

$$x(t) = A_0 + \int_0^t \phi^{-1}\Big(\phi(A_1) + \int_0^s f^1(r, x(r), y(r))dr + \sum_{l=1}^{\infty} \int_0^s \sigma_l^1(r, x(r), y(r))dW^l(r)\Big)ds,$$

and

$$y(t) = B_0 + \int_0^t \phi^{-1}\Big(\phi(B_1) + \int_0^s f^2(r, x(r), y(r))dr + \sum_{l=1}^{\infty} \int_0^s \sigma_l^2(r, x(r), y(r))dW^l(r)\Big)ds.$$

• $(x_n, y_n)|_{J_1}$ has a subsequence relabeled as $(x_{n_m}, y_{n_m}) \subset S_{c_2}$ converging to (x, y) in $C_1 \times C_1$ where

$$S_{c_2} = \{(x, y) \in C_1 \times C_1 : (x, y) \text{ is a solution of (3.4)}\}.$$

Let

$$z_1(t) = A_3 + \int_{t_1}^t \phi^{-1}\Big(\phi(A_4) + \int_{t_1}^s f^1(r, x(r), y(r))dr + \sum_{l=1}^{\infty} \int_{t_1}^s \sigma_l^1(r, x(r), y(r))dW^l(r)\Big)ds,$$

and

$$\overline{z}_1(t) = B_3 + \int_{t_1}^t \phi^{-1}\Big(\phi(B_4) + \int_{t_1}^s f^2(r, x(r), y(r))dr + \sum_{l=1}^{\infty} \int_{t_1}^s \sigma_l^2(r, x(r), y(r))dW^l(r)\Big)ds.$$

Then

$$\begin{aligned} E\Big|x_{n_m}(t) - z_1(t)\Big|^2 \le \int_{t_1}^t E\Big|&\phi^{-1}\Big(\phi(A_4) + \int_{t_1}^s f^1(r, x_{n_m}(r), y_{n_m}(r))dr \\ &+ \sum_{l=1}^{\infty} \int_{t_1}^s \sigma_l^1(r, x_{n_m}(r), y_{n_m}(r))dW^l(r)\Big) \\ &- \phi^{-1}\Big(\phi(A_4) + \int_{t_1}^s f^1(r, x(r), y(r))dr + \\ &+ \sum_{l=1}^{\infty} \int_{t_1}^s \sigma^1(r, x(r), y(r))dW^l(r)\Big)\Big|^2 ds, \end{aligned}$$

and

$$E\Big|y_{n_m}(t)-\overline{z}_1(t)\Big|^2 \leq \int_{t_1}^{t} E\Big|\phi^{-1}\Big(\phi(B_4)+\int_{t_1}^{s} f^1(r,x_{n_m}(r),y_{n_m}(r))dr + \sum_{l=1}^{\infty}\int_{t_1}^{s}\sigma^1(r,x_{n_m}(r),y_{n_m}(r))dW^l(r)\Big) - \phi^{-1}\Big(\phi(A_4)+\int_{t_1}^{s} f^1(r,x(r),y(r))dr+ + \sum_{l=1}^{\infty}\int_{t_1}^{s}\sigma^1(r,x(r),y(r))dW^l(r)\Big)\Big|^2 ds.$$

As $n_m \to +\infty$, $(x_{n_m}(t), y_{n_m}(t)) \to (z_1(t), \overline{z}_1(t))$, and then

$$x(t)=A_3+\int_{t_1}^{t}\phi^{-1}\Big(\phi(A_4)+\int_{t_1}^{s} f^1(r,x(r),y(r))dr + \sum_{l=1}^{\infty}\int_{t_1}^{s}\sigma_l^1(r,x(r),y(r))dW^l(r)\Big)ds,$$

and

$$y(t)=B_3+\int_{t_1}^{t}\phi^{-1}\Big(\phi(B_4)+\int_{t_1}^{s} f^2(r,x(r),y(r))dr + \sum_{l=1}^{\infty}\int_{t_1}^{s}\sigma^2(r,x(r),y(r))dW^l(r)\Big)ds.$$

• We continue this process, and we conclude that $\{(x_n, y_n) \mid n \in \mathbb{N}\}$ has a subsequence converging to

$$z_m(t)=A_{m+2}+\int_{t_m}^{t}\phi^{-1}\Big(\phi(A_{m+3})+\int_{t_m}^{s} f^1(r,x(r),y(r))dr + \sum_{l=1}^{\infty}\int_{t_m}^{s}\sigma_l^1(r,x(r),y(r))dW^l(r)\Big)ds,$$

and

$$\overline{z}_m(t)=B_{m+2}+\int_{t_m}^{t}\phi^{-1}\Big(\phi(B_{m+3})+\int_{t_m}^{s} f^2(r,x(r),y(r))dr + \sum_{l=1}^{\infty}\int_{t_m}^{s}\sigma_l^2(r,x(r),y(r))dW^l(r)\Big)ds.$$

Hence S_c is compact. □

Next we replace $(H2)$ and $(H3)$ in Theorem 3.1 by

$(H3)'$ Then there exist a function $p_i \in L^1(J, \mathbb{R}^+)$ and a continuous nondecreasing function $\psi_i : [0, \infty) \to [0, \infty)$for each $i = 1, 2$ such that

$$E|f^i(t, x, y)|^2 \leq p_i(t)\psi_i(E(|x|^2 + |y|^2)),$$

and

$$E||\sigma^i(t, x, y)||^2 \leq p_i(t)\psi_i(E(|x|^2 + |y|^2)).$$

Theorem 3.2. *Under assumption* $(H3)'$*, problem* (1.1) *has at least one solution and the solution set is compact.*

Proof. As in the proof of Theorem 3.1 we can show that (1.1) has at least one solution by applying the nonlinear alternative of Leray-Schauder. We show only the estimation of a solution (x, y) of (1.1).

• For $t \in [0, t_1]$, we have

$$\begin{cases} x(t) = A_0 + \int_0^t \phi^{-1}\Big(\phi(A_1) + \int_0^s f^1(r, x(r), y(r))dr + \\ \quad + \sum_{l=1}^{\infty} \int_0^s \sigma_l^1(r, x(r), y(r))dW^l(r)\Big)ds, \quad t \in [0, t_1], \ a.e.\ \omega \in \Omega. \\ \\ y(t) = B_0 + \int_0^t \phi^{-1}\Big(\phi(B_1) + \int_0^s f^2(r, x(r), y(r))dr + \\ \quad + \sum_{l=1}^{\infty} \int_0^s \sigma_l^2(r, x(r), y(r))dW^l(r)\Big)ds, \quad t \in [0, t_1], \ a.e.\ \omega \in \Omega. \end{cases} \tag{3.8}$$

Then

$$E|x(t)|^2 \leq 2E|A_0|^2 + 2\int_0^t E\Big|\phi^{-1}\Big(\phi(A_1) + \int_0^s f^1(r, x(r), y(r))dr + \sum_{l=1}^{\infty} \int_0^s \sigma_l^1(r, x(r), y(r))dW^l(r)\Big)\Big|^2 ds.$$

Consider functions $\mu, \overline{\mu}$ defined on $t \in [0, t_1]$ by

$$\mu(t) = \sup\{E|x(s)|^2 : 0 \leq s \leq t\}, \qquad \overline{\mu}(t) = \sup\{E|y(s)|^2 : 0 \leq s \leq t\}.$$

From Lemma 2.4, we obtain

$$\begin{aligned} &E\Big|\phi(A_1) + \int_0^s f^1(r, x(r), y(r))dr + \sum_{l=1}^{\infty} \int_0^s \sigma_l^1(r, x(r), y(r))dW^l(r)\Big|^2 \\ &\leq 3E|\phi(A_1)|_X^2 + 3\int_0^s p_1(r)\psi_1(E(|x(r)|^2 + |y(r)|^2))dr \\ &+3C_2\int_0^s p_1(r)\psi_1(E(|x(r)|_X^2 + |y(r)|^2))dr \\ &\leq 3E|\phi(A_1)|^2 + ||\overline{p}||_{L^1}\psi_1(\mu(s) + \overline{\mu}(s)), \end{aligned}$$

where $||\overline{p}||_{L^1} = (3t_1 + 3C_2)\|p_1\|_{L^1}$, and, consequently,

$$\mu(t) \leq 2E|A_0|^2 + \int_0^t \widehat{\psi}_1(\mu(s) + \overline{\mu}(s)), \quad t \in [0, t_1],$$

where $\widehat{\psi}_1 = (\phi^{-1} \circ \widetilde{\psi}_1)$ and $\widetilde{\psi}_1(u) = 3E|\phi(A_1)|^2 + ||\overline{p}_1||_{L^1}\psi_1(u)$. and similarly

$$\overline{\mu}(t) \leq 2E|B_0|^2 + \int_0^t \widehat{\psi}_2(\mu(s) + \overline{\mu}(s))ds, \quad t \in [0, t_1],$$

where $\widehat{\psi}_2 = (\phi^{-1} \circ \widetilde{\psi}_2)$ and $\widetilde{\psi}_2(u) = 3E|\phi(B_1)|^2 + ||p_2||_{L^1}\psi_1(u)$, combining $\mu(t)$ and $\overline{\mu}(t)$,

$$\mu(t) + \overline{\mu}(t) \leq 2E|A_0|^2 + 2E|B_0|^2 + \int_0^t \widehat{\psi}_1(\mu(s) + \overline{\mu}(s))ds + \int_0^t \widehat{\psi}_2(\mu(s) + \overline{\mu}(s))ds, \quad t \in [0, t_1].$$

Using the nonlinear Grönwall-Bihari inequality (Lemma 2.8), we infer the bound

$$\mu(t) + \overline{\mu}(t) \leq H^{-1}(t) \leq M_0.$$

Consequently, there exists a constant M_1 which only depends on t_1, t_2 such that

$$\sup_{t\in[0,t_1]} E|x(t)|^2 \leq M_0, \text{ and } \sup_{t\in[0,t_1]} E|y(t)|^2 \leq M_0,$$

where $H(t) = \displaystyle\int_{2E|A_0|_X^2 + 2E|B_0|_X^2}^t \frac{d\tau}{(\phi^{-1} \circ \widetilde{\psi}_1(\tau) + \phi^{-1} \circ \widetilde{\psi}_2(\tau)}$.

- For $t \in (t_1, t_2]$, we have

$$\begin{cases} x(t) = A_3 + \displaystyle\int_{t_1}^t \phi^{-1}\Big(\phi(A_4) + \int_{t_1}^s f^1(r, x(r), y(r))dr + \\ \qquad + \displaystyle\sum_{l=1}^{\infty} \int_{t_1}^s \sigma_l^1(r, x(r), y(r))dW^l(r)\Big)ds, \quad t \in [0, t_1]. \\ \\ y(t) = B_3 + \displaystyle\int_{t_1}^t \phi^{-1}\Big(\phi(B_4) + \int_{t_1}^s f^2(r, x(r), y(r))dr + \\ \qquad + \displaystyle\sum_{l=1}^{\infty} \int_{t_1}^s \sigma_l^2(r, x(r), y(r))dW^l(r)\Big)ds, \quad t \in [0, t_1]. \end{cases} \tag{3.9}$$

Then

$$E|x(t)|^2 \leq 2E|A_3|^2 + 2\int_{t_1}^t E\Big|\phi^{-1}\Big(\phi(A_4) + \int_{t_1}^s f^1(r, x(r), y(r))dr + \sum_{l=1}^{\infty} \int_{t_1}^s \sigma_l^1(r, x(r), y(r))dW^l(r)\Big)\Big|_X^2 ds.$$

Consider functions $\mu, \overline{\mu}$ defined on $t \in (t_1, t_2]$ by

$$\mu(t) = \sup\{E|x(s)|^2 : t_1 \leq s \leq t\}, \qquad \overline{\mu}(t) = \sup\{E|y(s)|^2 : t_1 \leq s \leq t\}.$$

$$E\Big|\phi(A_4)+\int_{t_1}^{s} f^1(r,x(r),y(r))dr+\sum_{l=1}^{\infty}\int_{t_1}^{s}\sigma_l^1(r,x(r),y(r))dW^l(r)\Big|_X^2$$
$$\le 3E|\phi(A_4)|^2+3\int_{t_1}^{s}p_1(r)\psi_1(E(|x(r)|^2+|y(r)|_X^2))dr$$
$$+3C_2\int_{t_1}^{s}p_1(r)\psi_1(E(|x(r)|_X^2+|y(r)|^2))dr$$
$$\le 3E|\phi(A_4)|^2+||\overline{p}||_{L^1}\psi_1(\mu(s)+\overline{\mu}(s)),$$

where $||\overline{p}||_{L^1}=(3t_2+3C_2)\|p_1\|_{L^1}$, and, consequently,

$$\mu(t)\le 2E|A_3|_X^2+\int_{t_1}^{t}\widehat{\psi}_1(\mu(s)+\overline{\mu}(s)),\quad t\in(t_1,t_2],$$

where $\widehat{\psi}_1=(\phi^{-1}\circ\widetilde{\psi}_1)$ and $\widetilde{\psi}_1(u)=3E|\phi(A_4)|_X^2+||\overline{p}_1||_{L^1}\psi_1(u)$. and similarly

$$\overline{\mu}(t)\le 2E|B_0|_X^2+\int_{t_1}^{t}\widehat{\psi}_2(\mu(s)+\overline{\mu}(s))ds,\quad t\in(t_1,t_2],$$

where $\widehat{\psi}_2=(\phi^{-1}\circ\widetilde{\psi}_2)$ and $\widetilde{\psi}_2(u)=3E|\phi(B_1)|_X^2+||p_2||_{L^1}\psi_1(u)$. Now, taking into account all the previous estimates we can write

$$\mu(t)+\overline{\mu}(t)\le 2E|A_3|^2+2E|B_3|^2+\int_{t_1}^{t}\widehat{\psi}_1(\mu(s)+\overline{\mu}(s))ds$$
$$+\int_{t_1}^{t}\widehat{\psi}_2(\mu(s)+\overline{\mu}(s))ds,\quad t\in(t_1,t_2],$$

By the nonlinear Grönwall-Bihari inequality (Lemma 2.8), we infer the bound

$$\mu(t)+\overline{\mu}(t)\le H^{-1}(t)\le M_1.$$

Consequently, there exists a constant M_1 which only depends on t_1,t_2 such that

$$\sup_{t\in(t_1,t_2]}E|x(t)|^2\le M_1,\ \text{and}\ \sup_{t\in(t_1,t_2]}E|y(t)|^2\le M_1.$$

where $H(t)=\displaystyle\int_{2E|A_3|^2+2E|B_3|^2}^{t}\frac{d\tau}{(\phi^{-1}\circ\widetilde{\psi}_1(\tau)+\phi^{-1}\circ\widetilde{\psi}_2(\tau)}$.

- For $t\in(t_m,T]$, we have

$$x(t)=A_{m+2}+\int_{t_m}^{t}\phi^{-1}\Big(\phi(A_{m+3})+\int_{t_m}^{s}f^1(r,x(r),y(r))dr+$$
$$\sum_{l=1}^{\infty}\int_{t_m}^{s}\sigma_l^1(r,x(r),y(r))dW^l(r)\Big)ds,$$

and

$$y(t)=B_{m+2}+\int_{t_m}^{t}\phi^{-1}\Big(\phi(B_{m+3})+\int_{t_m}^{s}f^2(r,x(r),y(r))dr$$
$$+\sum_{l=1}^{\infty}\int_{t_m}^{s}\sigma^2(r,x(r),y(r))dW^l(r)\Big)ds.$$

As in the pattern shown above, there exists $M_m > 0$ such that

$$\mu(t) + \overline{\mu}(t) \leq H^{-1}(t) \leq M_m.$$

Consequently, there exists a constant M_1 which only depends on t_m, T such that

$$\sup_{t\in(t_m,T]} E|x(t)|^2 \leq M_m, \text{ and } \sup_{t\in(t_m,T]} E|y(t)|^2 \leq M_m.$$

where $H(t) = \displaystyle\int_{2E|A_{m+2}|^2_X+2E|B_{m+2}|^2_X}^{t} \frac{d\tau}{(\phi^{-1}\circ\widetilde{\psi}_1(\tau)+\phi^{-1}\circ\widetilde{\psi}_2(\tau)}.$

Hence

$$\|x\|_{PC} \leq \max(M_0, M_1, \dots, M_m) = M,$$

and

$$\|y\|_{PC} \leq \max(M_0, M_1, \dots, M_m) = M.$$

The proof is complete. □

Acknowledgments. The authors would like to thank very much the anonymous referees for their careful reading and valuable comments on this work.

References

[1] Bainov, D.D., Lakshmikantham, V., Simeonov, P.S., *Theory of Impulsive Differential Equations*, World Sci., Singapore, 1989.

[2] Bainov, D.D., Simeonov, P.S., *Integral Inequalities and Applications, in: Mathematics and its Applications*, vol. 57, Kluwer Academic Publishers, Dordrecht, 1992.

[3] Benchohra, M., Henderson, J., Ntouyas, S.K., *Impulsive differential equations and inclusions*, Hindawi. Pub. Cor, New York, **2**(2006).

[4] Bharucha-Reid, A.T., *Random Integral Equations*, Academic Press, New York, 1972.

[5] Burkholder, D.L., *Martingale transforms*, Ann. Math. Statist., **37**(1966), 1494-1504.

[6] Burkholder, D.L., Gundy, R.F., *Extrapolation and interpolation of quasi-linear operators on martingales*, Acta. Math., **124**(1970), 249-304.

[7] Da Prato, G., Zabczyk, J., *Stochastic Equations in Infinite Dimensions*, Cambridge Univ. Press, Cambridge, 1992.

[8] Davis, B., *On the integrability of the martingale square function*, Israel. J. Math., **8**(1970), 187-190.

[9] Gard, T.C., *Introduction to Stochastic Differential Equations*, Marcel Dekker, New York, 1988.

[10] Gikhman, I.I., Skorokhod, A., *Stochastic Differential Equations*, Springer-Verlag, 1972.

[11] Graef, J.R., Henderson, J., Ouahab, A., *Impulsive differential inclusions. A fixed point approach*, De Gruy. Ser. Nonlinear. Anal. Appl 20. Berlin, de Gruyter, 2013.

[12] Guilan, C., Kai, H., *On a type of stochastic differential equations driven by countably many Brownian motions*, J. Funct. Anal., **203**(2003), 262-285.

[13] Halanay, A., Wexler, D., *Teoria calitativă a sistemelor cu impulsuri*, (Romanian), Editura Academiei Rep. Soc. Romania, Bucharest, 1968.

[14] Henderson, J., Ouahab, A., Youcefi, S., *Existence and topological structure of solution sets for ϕ-Laplacian impulsive differential equations*, Elec. J. Diff. Equ., **56**(2012), 1-16.

[15] Malliavin, P., *The canonic diffusion above the diffeomorphism group of the circle*, C.R. Acad. Sci. Paris Sér. I. Math., **329**(1999), 325-329.

[16] Mao, X., *Stochastic Differential Equations and Applications*, Horwood, Chichester, 1997.

[17] Millar, P., *Warwick Martingale integrals*, Trans. Amer. Math. Soc., **133**(1968), 145-166.

[18] Milman, V.D., Myshkis, A.A., *On the stability of motion in the presence of impulses,* (Russian), Sib. Math. J., **1**(1960), 233–237.

[19] Novikov, A.A., *The moment inequalities for stochastic integrals*, (Russian), *Teor. Verojatnost. i Primenen.*, **16**(1971), 548-551.

[20] Øksendal, B., *Stochastic Differential Equations: An Introduction with Applications*, Fourth Edition, Springer-Verlag, Berlin, 1995.

[21] Perov, A.I., *On the Cauchy problem for a system of ordinary differential equations*, (Russian), *Pviblizhen. Met. Reshen. Diff. Uvavn,* **2**(1964), 115-134.

[22] Precup, R., *Methods. Nonlinear Integral Equations*, Kluwer, Dordrecht, 2000.

[23] Revuz, D., Yor, M., *Continuous Martingales and Brownian Motion*, Springer, Third Edition, 1999.

[24] Rus, I.A., *The theory of a metrical fixed point theorem: theoretical and applicative relevances*, Fixed Point Theory, **9**(2008), 541-559.

[25] Sakthivel, R., Luo, J., *Asymptotic stability of nonlinear impulsive stochastic differential equations*, Statist. Probab. Lett., **79**(2009), 1219-1223.

[26] Samoilenko, A.M., Perestyuk, N.A., *Impulsive Differential Equations,* World Sci., Singapore, 1995.

[27] Sobczyk, H., *Stochastic Differential Equations with Applications to Physics and Engineering*, Kluwer Academic Publishers, London, 1991.

[28] Tsokos, C.P., Padgett, W.J., *Random Integral Equations with Applications to Life Sciences and Engineering*, Academic Press, New York, 1974.

[29] Viorel, A., *Contributions to the Study of Nonlinear Evolution Equations*, Ph.D. Thesis, Babeş-Bolyai Univ. Cluj-Napoca, Department of Mathematics, 2011.

[30] Wu, S.J., Guo, X.L., Lin, S.Q., *Existence and uniqueness of solutions to random impulsive differential systems*, Acta. Math. Appl. Sinica, **22**(2006), 595-600.

Tayeb Blouhi
Faculty of Mathematics and Computer Science, Department of Mathematics
University of Science and Technology Mohamed-Boudiaf El Mnaouar
BP 1505, Bir El Djir 31000, Oran, Algeria
e-mail: `blouhitayeb@yahoo.com`

Mohamed Ferhat
Faculty of Mathematics and Computer Science, Department of Mathematics
University of Science and Technology Mohamed-Boudiaf El Mnaouar
BP 1505, Bir El Djir 31000, Oran, Algeria
e-mail: `ferhat22@hotmail.fr`

Stud. Univ. Babeş-Bolyai Math. 63(2018), No. 4, 525–537
DOI: 10.24193/subbmath.2018.4.08

On Lupaş-Jain operators

Gülen Başcanbaz-Tunca, Murat Bodur and Dilek Söylemez

Abstract. In this paper, linear positive Lupaş-Jain operators are constructed and a recurrence formula for the moments is given. For the sequence of these operators; the weighted uniform approximation, also, monotonicity under convexity are obtained. Moreover, a preservation property of each Lupaş-Jain operator is presented.

Mathematics Subject Classification (2010): 41A36, 41A25.

Keywords: Lupaş operator, Jain operator, convexity, weighted uniform approximation, modulus of continuity function.

1. Introduction

In [13], Jain generalized the well known Százs-Mirakjan operators by constructing the linear positive operators given by

$$S_n^{\beta}(f)(x)=\sum_{k=0}^{\infty}\frac{nx\,(nx+k\beta)^{k-1}}{k!}e^{-(nx+k\beta)}f\left(\frac{k}{n}\right), \tag{1.1}$$

where $f:[0,\infty)\to\mathbb{R},\ n\in\mathbb{N},\ x>0$ and $0\leq\beta<1$, with β may depend only on n. For some interesting works related to Jain's operators we refer to [2], [1], [5], [8], [17], [18] and references cited therein.

In [3], Agratini studied some approximation properties of the following linear positive operators

$$L_n(f)(x)=2^{-nx}\sum_{k=0}^{\infty}\frac{(nx)_k}{2^k k!}f\left(\frac{k}{n}\right) \tag{1.2}$$

for $n\in\mathbb{N},\ x\geq 0$ and some suitable $f:[0,\infty)\to\mathbb{R}$ that the operator $L_n(f)$ makes sense. These operators are special form of the well-known operators defined by Lupaş in [15] and resemble the familiar Százs-Mirakjan operators. In the paper [3], the author obtained some estimates for the order of approximation on a finite interval as well as proved a Voronovskaya type theorem. Moreover, Agratini also considered the Kantorovich extension of $L_n(f)$ for f belonging to the class of local integrable functions on $[0,\infty)$ and studied the degree of approximation [4]. Some approximation results and basic history concerning Lupaş operators can be found in [9], [10], [7].

Recently, Patel and Mishra extended the Lupaş operators given by (1.2) as

$$L_n^{\beta}(f)(x)=\sum_{k=0}^{\infty}\frac{(nx+k\beta)_k}{2^k k!}2^{-(nx+k\beta)}f\left(\frac{k}{n}\right) \tag{1.3}$$

for real valued functions f on $[0,\infty)$, where they assumed that

$$(nx+k\beta)_0=1,\ (nx+k\beta)_1=nx$$

and

$$(nx+k\beta)_k=nx\,(nx+k\beta)\,(nx+k\beta+1)\ldots(nx+k\beta+k-1)\,,\ k\geq 2$$

[19]. Here, the authors studied direct approximation results and gave Kantorovich and Durrmeyer types modifications of (1.3).

In this work, we also construct a generalization of the Lupaş operators L_n in the sense of Jain in [13]. Here, we point out that our expression is different from L_n^{β} given by (1.3) in such a way that in the construction, we take the negative subscript "-1" of the Pochhammer symbol into consideration, in which case the calculations become simpler in a remarkable degree. By using analogous Abel and Jensen combinatorial formulas for factorial powers (see, e.g., [20]), we show the monotonicity property of these operators for n under the convexity of f. We investigate that the Lupaş-Jain operator can retain the properties of the modulus of continuity function. Moreover, we study the weighted uniform approximation of functions from the polynomial weighted space given in [11].

In what follows, let α and β be real parameters such that $0<\alpha<\infty$ and $0\leq\beta<1$. Then, as in [13], Taking into account of the Lagrange inversion formula

$$\phi(z)=\phi(0)+\sum_{k=1}^{\infty}\frac{1}{k!}\left[\frac{d^{k-1}}{dz^{k-1}}(f(z))^k\phi'(z)\right]_{z=0}\left(\frac{z}{f(z)}\right)^k$$

for

$$\phi(z)=\frac{1}{(1-z)^{\alpha}}\text{ and }f(z)=\frac{1}{(1-z)^{\beta}},\ |z|<1,$$

we obtain

$$\frac{1}{(1-z)^{\alpha}}=1+\sum_{k=1}^{\infty}\frac{\alpha\,(\alpha+1+k\beta)_{k-1}}{k!}z^k(1-z)^{k\beta}, \tag{1.4}$$

where

$$(a)_n=\begin{cases}a(a+1)\ldots(a+n-1) & n\in\mathbb{N}\\ 1 & n=0,\ a\neq 0,\end{cases}$$

is the well-known Pochhammer symbol, from which we have

$$(a)_{-n}=\frac{1}{(a-1)(a-2)\ldots(a-n)}=\frac{1}{(a-n)_n}=\frac{(-1)^n}{(1-a)_n}$$

for negative subscripts when $a \neq 1, 2, ..., n$ (see, e.g., p.5 of [12]). Hence, we immediately get that $(\alpha+1)_{-1} = \frac{1}{(\alpha)_1} = \frac{1}{\alpha}$. Now, we have

$$1 = \sum_{k=0}^{\infty} \frac{\alpha\,(\alpha+1+k\beta)_{k-1}}{2^k k!} 2^{-(\alpha+k\beta)} \tag{1.5}$$

for $0 < \alpha < \infty$ and $0 \leq \beta < 1$. So, denoting

$$L\,(0,\alpha,\beta) := \sum_{k=0}^{\infty} \frac{(\alpha+1+k\beta)_{k-1}}{2^k k!} 2^{-(\alpha+k\beta)} \tag{1.6}$$

it readily follows from (1.5) that

$$\alpha L\,(0,\alpha,\beta) = 1. \tag{1.7}$$

Hence, we present the following recurrence formula.

Lemma 1.1. *Let* $0 < \alpha < \infty,\ 0 \leq \beta < 1,\ r \in \mathbb{N}$ *and*

$$L\,(r,\alpha,\beta) := \sum_{k=0}^{\infty} \frac{(\alpha+1+k\beta)_{k+r-1}}{2^k k!} 2^{-(\alpha+k\beta)}. \tag{1.8}$$

Then we have

$$L\,(r,\alpha,\beta) = \sum_{k=0}^{\infty} \left(\frac{\beta+1}{2}\right)^k (\alpha+r-1+k\beta)\, L\,(r-1,\alpha+k\beta,\beta)\,.$$

Proof. Taking the fact

$$(\alpha+1+k\beta)_{k+r-1} = (\alpha+1+k\beta)_{k+r-2}\,(\alpha+r-1+k\,(\beta+1))$$

into consideration, then one finds

$$L\,(r,\alpha,\beta) = (\alpha+r-1)\,L\,(r-1,\alpha,\beta) + \frac{\beta+1}{2} L\,(r,\alpha+\beta,\beta)\,.$$

Recursive application of the last formula gives the result. □

For the calculation of moments of the operators, we can use the well-known property of the geometric series given below (see, e.g., [21]).

Remark 1.2. ([21]) Consider the geometric series

$$h_n\,(x) := \sum_{k=0}^{\infty} k^n x^k \qquad -1 < x < 1,\ n \in \mathbb{N}$$

and

$$h_0\,(x) := \frac{1}{1-x} = \sum_{k=0}^{\infty} x^k. \tag{1.9}$$

Term-wise differentiation gives that

$$h_n'\,(x) = \sum_{k=1}^{\infty} k^{n+1} x^{k-1},$$

which satisfies the following

$$xh_n'(x)=\sum_{k=1}^{\infty}k^{n+1}x^k=h_{n+1}(x).$$

From this recurrence, one has

$$h_1(x)=\frac{x}{(1-x)^2}=\sum_{k=1}^{\infty}kx^k, \tag{1.10}$$

$$h_2(x)=\frac{x^2+x}{(1-x)^3}=\sum_{k=1}^{\infty}k^2x^k. \tag{1.11}$$

Lemma 1.3. *For the auxiliary function $L(r,\alpha,\beta)$ defined by (1.8), one has*

$$\begin{aligned} L(1,\alpha,\beta) &= \frac{2}{1-\beta}, \\ L(2,\alpha,\beta) &= \frac{2^2(\alpha+1)}{(1-\beta)^2}+\frac{2^2\beta(\beta+1)}{(1-\beta)^3}. \end{aligned}$$

Proof. Since $0\le\beta<1$, then (1.9), (1.10) and (1.11), with $x=\frac{\beta+1}{2}$, give that

$$\begin{aligned} \sum_{k=0}^{\infty}\left(\frac{\beta+1}{2}\right)^k &= \frac{2}{1-\beta}, \\ \sum_{k=1}^{\infty}k\left(\frac{\beta+1}{2}\right)^k &= \frac{2(\beta+1)}{(1-\beta)^2}, \\ \sum_{k=1}^{\infty}k^2\left(\frac{\beta+1}{2}\right)^k &= \frac{2\left(\beta^2+4\beta+3\right)}{(1-\beta)^3}. \end{aligned}$$

Combining these results with (1.6), (1.7) and (1.8), it readily follows that

$$\begin{aligned} L(1,\alpha,\beta) &= \sum_{k=0}^{\infty}\left(\frac{\beta+1}{2}\right)^k(\alpha+k\beta)L(0,\alpha+k\beta,\beta) \\ &= \frac{2}{1-\beta}. \end{aligned} \tag{1.12}$$

Also, $L(2,\alpha,\beta)$ is obtained as

$$\begin{aligned} L(2,\alpha,\beta) &= \sum_{k=0}^{\infty}\left(\frac{\beta+1}{2}\right)^k(\alpha+1+k\beta)L(1,\alpha+k\beta,\beta) \\ &= \frac{2(\alpha+1)}{1-\beta}\sum_{k=0}^{\infty}\left(\frac{\beta+1}{2}\right)^k+\frac{2\beta}{1-\beta}\sum_{k=0}^{\infty}k\left(\frac{\beta+1}{2}\right)^k \\ &= \frac{4(\alpha+1)}{(1-\beta)^2}+\frac{4\beta(\beta+1)}{(1-\beta)^3}. \end{aligned} \tag{1.13}$$

□

2. Construction of the operators

Taking $\alpha = nx,\ n \in \mathbb{N},\ x > 0$ in (1.5), we consider the following linear positive operators

$$L_n^{\beta}(f)(x) = \sum_{k=0}^{\infty} \frac{nx\,(nx+1+k\beta)_{k-1}}{2^k k!} 2^{-(nx+k\beta)} f\left(\frac{k}{n}\right), \quad x \in (0,\infty) \tag{2.1}$$

and $L_n^{\beta}(f)(0) = f(0)$ for real valued bounded functions f on $[0,\infty)$, where $0 \leq \beta < 1$, depending only on n. We call the operators L_n^{β} as Lupaş-Jain. Obviously, Lupaş-Jain operators reduce to Lupaş operators in [3] when $\beta = 0$.

Lemma 2.1. *Let $e_i(t) := t^i,\ i = 0,1,2$. For the Lupaş-Jain operators, one has*

$$\begin{aligned} L_n^{\beta}(e_0)(x) &= 1, \\ L_n^{\beta}(e_1)(x) &= \frac{x}{1-\beta}, \\ L_n^{\beta}(e_2)(x) &= \frac{x^2}{(1-\beta)^2} + \frac{2x}{n(1-\beta)^3}. \end{aligned}$$

Proof. It is clear from (1.5) that $L_n^{\beta}(e_0)(x) = 1$. By taking $f = e_1$ in (2.1) and using (1.12) in the result, we easily get

$$\begin{aligned} L_n^{\beta}(e_1)(x) &= \sum_{k=1}^{\infty} \frac{nx\,(nx+1+k\beta)_{k-1}}{2^k k!} 2^{-(nx+k\beta)} \left(\frac{k}{n}\right) \\ &= x \sum_{k=0}^{\infty} \frac{(nx+\beta+1+k\beta)_k}{2^{k+1} k!} 2^{-(nx+\beta+k\beta)} \\ &= \frac{x}{2} L(1, nx+\beta, \beta) \\ &= \frac{x}{1-\beta}. \end{aligned}$$

By taking $f = e_2$ and using (1.12) and (1.13) we find

$$\begin{aligned} L_n^{\beta}(e_2)(x) &= \sum_{k=1}^{\infty} \frac{nx\,(nx+1+k\beta)_{k-1}}{2^k k!} 2^{-(nx+k\beta)} \left(\frac{k}{n}\right)^2 \\ &= \frac{x}{n} \sum_{k=0}^{\infty} \frac{(nx+\beta+1+k\beta)_k}{2^{k+1} k!} 2^{-(nx+\beta+k\beta)} (k+1) \\ &= \frac{x}{n} \left\{ \frac{1}{2^2} L(2, nx+2\beta, \beta) + \frac{1}{2} L(1, nx+\beta, \beta) \right\} \\ &= \frac{x}{n} \left\{ \frac{(nx+1+2\beta)}{(1-\beta)^2} + \frac{\beta(\beta+1)}{(1-\beta)^3} + \frac{1}{1-\beta} \right\} \\ &= \frac{x^2}{(1-\beta)^2} + \frac{2x}{n(1-\beta)^3}. \end{aligned}$$

$\square$

3. Weighted approximation

In this section, we deal with the weighted uniform approximation result of the sequence of the Lupaş-Jain operators L_n^{β} by using Gadjiev's theorem in [11], for which we have the following settings:

We take $\varphi(x) = 1 + x^2$ as the suitable weight function and, for simplicity, denote $\mathbb{R}^+ := [0, \infty)$. Related to φ, we take the space

$$B_{\varphi}(\mathbb{R}^+) = \left\{ f : \mathbb{R}^+ \to \mathbb{R} \, \middle| \, |f(x)| \leq M_f \varphi(x), \; x \in \mathbb{R}^+ \right\},$$

where M_f is a constant depending on f. $B_{\varphi}(\mathbb{R}^+)$ is a normed space with the norm

$$\|f\|_{\varphi} = \sup_{x \in \mathbb{R}^+} \frac{|f(x)|}{\varphi(x)}.$$

Moreover, we denote, as usual, by $C_{\varphi}(\mathbb{R}^+)$, $C_{\varphi}^k(\mathbb{R}^+)$ the following subspaces of $B_{\varphi}(\mathbb{R}^+)$

$$\begin{aligned} C_{\varphi}(\mathbb{R}^+) &: \left\{ f \in B_{\varphi}(\mathbb{R}^+) : f \text{ is continuous} \right\}, \\ C_{\varphi}^k(\mathbb{R}^+) &= \left\{ f \in C_{\varphi}(\mathbb{R}^+) \, \middle| \, \lim_{x \to \infty} \frac{f(x)}{\varphi(x)} = k_f \right\}, \end{aligned}$$

respectively, where k_f is a constant depending on f. We have the following two results due to Gadjiev in [11]:

Lemma 3.1. *The linear positive operators T_n, $n \in \mathbb{N}$, act from $C_{\varphi}(\mathbb{R}^+)$ to $B_{\varphi}(\mathbb{R}^+)$ if and only if*

$$|T_n(\varphi)(x)| \leq K\varphi(x),$$

where K is a positive constant.

Theorem 3.2. *Let $\{T_n\}_{n \in \mathbb{N}}$ be a sequence of linear positive operators mapping $C_{\varphi}(\mathbb{R}^+)$ into $B_{\varphi}(\mathbb{R}^+)$ and satisfying the conditions*

$$\lim_{n \to \infty} \|T_n(e_i) - e_i\|_{\varphi} = 0, \quad \text{for } i = 0, 1, 2.$$

Then for any $f \in C_{\varphi}^k(\mathbb{R}^+)$ one has

$$\lim_{n \to \infty} \|T_n(f) - f\|_{\varphi} = 0.$$

Now, we treat weighted uniform approximation for Lupaş-Jain operators L_n^{β} acting on $C_{\varphi}(\mathbb{R}^+)$. In order to get an approximation result, as in [13], we need to make an adjustment to the parameter β by taking it as a sequence such that $\beta = \beta_n$, $0 \leq \beta_n < 1$ and $\lim_{n \to \infty} \beta_n = 0$.

Theorem 3.3. *Let $\{\beta_n\}_{n \in \mathbb{N}}$ be a sequence such that $0 \leq \beta_n < 1$ and $\lim_{n \to \infty} \beta_n = 0$. Then for each $f \in C_{\varphi}^k(\mathbb{R}^+)$ we have*

$$\lim_{n \to \infty} \left\| L_n^{\beta_n}(f) - f \right\|_{\varphi} = 0.$$

Proof. According to Lemmas 2.1 and 3.1 we get that the operators $L_n^{\beta_n}$ act from $C_\varphi(\mathbb{R}^+)$ to $B_\varphi(\mathbb{R}^+)$. Now, it only remains to show the sufficient conditions of the Theorem 3.2 for $L_n^{\beta_n}$. Using Lemma 2.1 and the hypothesis on β_n, we obtain that

$$\lim_{n\to\infty} \left\| L_n^{\beta_n}(e_0) - e_0 \right\|_\varphi = 0$$

and that

$$\left\| L_n^{\beta_n}(e_1) - e_1 \right\|_\varphi \leq \frac{\beta_n}{1-\beta_n},$$

which gives

$$\lim_{n\to\infty} \left\| L_n^{\beta_n}(e_1) - e_1 \right\|_\varphi = 0.$$

Finally, since $2x \leq 1 + x^2$, we get

$$\begin{aligned} \left\| L_n^{\beta_n}(e_2) - e_2 \right\|_\varphi &= \sup_{x\in\mathbb{R}^+} \frac{\left| L_n^{\beta_n}(e_2) - e_2 \right|}{1+x^2} \\ &= \sup_{x\in\mathbb{R}^+} \left| \frac{1}{1+x^2} \left(\frac{x^2}{(1-\beta_n)^2} + \frac{2x}{n(1-\beta_n)^3} - x^2 \right) \right| \\ &= \sup_{x\in\mathbb{R}^+} \left| \frac{x^2}{1+x^2} \frac{2\beta_n - \beta_n^2}{(1-\beta_n)^2} + \frac{2x}{1+x^2} \frac{1}{n(1-\beta_n)^3} \right| \\ &\leq \frac{2\beta_n - \beta_n^2}{(1-\beta_n)^2} + \frac{1}{n(1-\beta_n)^3}, \end{aligned}$$

which clearly gives that

$$\lim_{n\to\infty} \left\| L_n^{\beta_n}(e_2) - e_2 \right\|_\varphi = 0.$$

This completes the proof. □

4. The monotonicity of the sequence of Lupaş-Jain operators

Recall that *a continuous function f is said to be convex in $D \subseteq \mathbb{R}$, if*

$$f\left(\sum_{i=1}^{n} \alpha_i t_i \right) \leq \sum_{i=1}^{n} \alpha_i f(t_i)$$

for every $t_1, t_2, ..., t_n \in D$ and for every nonnegative numbers $\alpha_1, \alpha_2, ..., \alpha_n$ such that $\alpha_1 + \alpha_2 + ... + \alpha_n = 1$.

For the proof of the main result of this section, we need the corresponding definition of the well-known Jensen and Abel combinatorial formulas for factorial powers. Below, we reproduce these formulas from the work of Stancu and Occorsio (pp.175-176 of [20]) for the increment -1, respectively.

$$\begin{aligned} &(u+v)(u+v+1+m\beta)_{m-1} \\ = &\sum_{k=0}^{m} \binom{m}{k} u(u+1+k\beta)_{k-1}\, v(v+1+(m-k)\beta)_{m-k-1} \end{aligned} \tag{4.1}$$

and

$$(u+v+m\beta)_m=\sum_{k=0}^{m}\binom{m}{k}(u+k\beta)_k\, v\,(v+1+(m-k)\,\beta)_{m-k-1}\,. \tag{4.2}$$

Note that the monotonicity of Százs-Mirakjan operators of convex function was proved by Cheney and Sharma [6]. On the other hand, the same result for the Lupaş operators was obtained by Erençin et al. [7]. Now, we present the monotonicity of each Lupaş-Jain operator $L_n^{\beta}(f)$ for n, when f is a convex function.

Theorem 4.1. *Let f be a convex function defined on $[0,\infty)$. Then, for all n, $L_n^{\beta}(f)$ is non-increasing in n.*

Proof. For $x=0$, the result is obvious. So, for $x>0$, we can write

$$2^x=\sum_{k=0}^{\infty}\frac{x\,(x+1+k\beta)_{k-1}}{2^k k!}2^{-k\beta}$$

by (1.5) with $\alpha=x$. Using this formula we can write

$$\begin{aligned}
&L_n^{\beta}(f)(x)-L_{n+1}^{\beta}(f)(x)\\
=\;& 2^x\sum_{k=0}^{\infty}\frac{nx\,(nx+1+k\beta)_{k-1}}{2^k k!}2^{-[(n+1)x+k\beta]}f\left(\frac{k}{n}\right)\\
&-\sum_{k=0}^{\infty}\frac{(n+1)\,x\,((n+1)\,x+1+k\beta)_{k-1}}{2^k k!}2^{-[(n+1)x+k\beta]}f\left(\frac{k}{n+1}\right)\\
=\;& \sum_{l=0}^{\infty}\frac{x\,(x+1+l\beta)_{l-1}}{2^l l!}2^{-l\beta}\sum_{k=0}^{\infty}\frac{nx\,(nx+1+k\beta)_{k-1}}{2^k k!}2^{-[(n+1)x+k\beta]}f\left(\frac{k}{n}\right)\\
&-\sum_{k=0}^{\infty}\frac{(n+1)\,x\,((n+1)\,x+1+k\beta)_{k-1}}{2^k k!}2^{-[(n+1)x+k\beta]}f\left(\frac{k}{n+1}\right)\\
=\;& \sum_{l=0}^{\infty}\frac{x\,(x+1+l\beta)_{l-1}}{2^l l!}2^{-l\beta}\\
&\times\sum_{k=l}^{\infty}\frac{nx\,(nx+1+(k-l)\,\beta)_{k-l-1}}{2^{k-l}\,(k-l)!}2^{-[(n+1)x+(k-l)\beta]}f\left(\frac{k-l}{n}\right)\\
&-\sum_{k=0}^{\infty}\frac{(n+1)\,x\,((n+1)\,x+1+k\beta)_{k-1}}{2^k k!}2^{-[(n+1)x+k\beta]}f\left(\frac{k}{n+1}\right).
\end{aligned}$$

Changing the order of the above summations, we obtain that

$$\begin{aligned}
&L_n^{\beta}(f)(x)-L_{n+1}^{\beta}(f)(x)\\
=&\sum_{k=0}^{\infty}\sum_{l=0}^{k}\frac{x\,(x+1+l\beta)_{l-1}}{l!}\frac{nx\,(nx+1+(k-l)\,\beta)_{k-l-1}}{2^k\,(k-l)!}2^{-[(n+1)x+k\beta]}f\left(\frac{k-l}{n}\right)\\
&-\sum_{k=0}^{\infty}\frac{(n+1)\,x\,((n+1)\,x+1+k\beta)_{k-1}}{2^k k!}2^{-[(n+1)x+k\beta]}f\left(\frac{k}{n+1}\right)
\end{aligned}$$

$$= \sum_{k=0}^{\infty} \left\{ \sum_{l=0}^{k} \frac{nx\,(nx+1+l\beta)_{l-1}}{l!} \frac{x\,(x+1+(k-l)\,\beta)_{k-l-1}}{2^k\,(k-l)!} f\left(\frac{l}{n}\right) \right.$$
$$\left. - \frac{(n+1)\,x\,((n+1)\,x+1+k\beta)_{k-1}}{2^k k!} f\left(\frac{k}{n+1}\right) \right\} 2^{-[(n+1)x+k\beta]} \tag{4.3}$$

Now, denote

$$\alpha_l := \binom{k}{l} \frac{nx\,(nx+1+l\beta)_{l-1}\,x\,(x+1+(k-l)\,\beta)_{k-l-1}}{(n+1)\,x\,((n+1)\,x+1+k\beta)_{k-1}} > 0$$

and

$$t_l := \frac{l}{n}.$$

Taking $u = nx,\ v = x$ and $m = k$ in (4.1) one has

$$\begin{aligned} &(n+1)\,x\,((n+1)\,x+1+k\beta)_{k-1} \\ = &\sum_{l=0}^{k} \binom{k}{l} nx\,(nx+1+l\beta)_{l-1}\,x\,(x+1+(k-l)\,\beta)_{k-l-1}\,, \end{aligned}$$

which clearly gives that

$$\sum_{l=0}^{k} \alpha_l = 1.$$

On the other hand, taking $u = nx+\beta+1,\ v = x$ and $m = k-1$ in (4.2), it follows that

$$\begin{aligned} &((n+1)\,x+1+k\beta)_{k-1} \\ = &(nx+\beta+1+x+(k-1)\,\beta)_{k-1} \\ = &\sum_{l=0}^{k-1} \binom{k-1}{l} (nx+\beta+1+l\beta)_l\,x\,(x+1+(k-1-l)\,\beta)_{k-l-2}\,. \end{aligned}$$

Taking into account of the above fact, it follows that

$$\begin{aligned} \sum_{l=0}^{k} \alpha_l t_l &= \frac{\sum_{l=1}^{k} \binom{k}{l} nx\,(nx+1+l\beta)_{l-1}\,x\,(x+1+(k-l)\,\beta)_{k-l-1}\left(\frac{l}{n}\right)}{(n+1)\,x\,((n+1)\,x+1+k\beta)_{k-1}} \\ &= \frac{k\sum_{l=0}^{k-1} \binom{k-1}{l} nx\,(nx+\beta+1+l\beta)_l\,x\,(x+1+(k-1-l)\,\beta)_{k-l-2}}{n\,(n+1)\,x\,((n+1)\,x+1+k\beta)_{k-1}} \\ &= \frac{k}{n+1} \frac{\sum_{l=0}^{k-1} \binom{k-1}{l} (nx+\beta+1+l\beta)_l\,x\,(x+1+(k-1-l)\,\beta)_{k-l-2}}{((n+1)\,x+1+k\beta)_{k-1}} \\ &= \frac{k}{n+1}. \end{aligned}$$

Hence, making use of the convexity of f, (4.3) gives that

$$L_n^{\beta}(f)(x) \geq L_{n+1}^{\beta}(f)(x)$$

for all $n \in \mathbb{N}$, which completes the proof. □

5. A preservation property

We recall the following definition for the subsequent result.

Definition 5.1. A continuous, and non-negative function ω defined on $[0,\infty)$ is called a function of modulus of continuity, if each of the following conditions is satisfied:

i) $\omega(u+v) \leq \omega(u) + \omega(v)$ for $u, v \in [0,\infty)$, i.e., ω is subadditive,

ii) $\omega(u) \geq \omega(v)$ for $u \geq v$, i.e., ω is non-decreasing,

iii) $\lim_{u\to 0^+} \omega(u) = \omega(0) = 0$ ([16]).

In [14], Li noticed a new preservation property that the Bernstein polynomials B_n, $n \in \mathbb{N}$ satisfy. Li proved that *if $\omega(x)$ is a modulus of continuity function, then for each $n \in \mathbb{N}$, $B_n(\omega; x)$ is also a modulus of continuity function.* The same result for the Lupaş operators was obtained in [7]. Below, we show that this result is satisfied by the Lupaş-Jain operators as well.

Theorem 5.2. *Let ω be a modulus of continuity function. Then, for all n, $L_n^{\beta}(\omega)$ is also a modulus of continuity function.*

Proof. Let $x, y \in [0,\infty)$ and $x \leq y$. Then from the definition of L_n^{β}, we have

$$L_n^{\beta}(\omega)(y) = \sum_{k=0}^{\infty} \frac{ny\,(ny+1+k\beta)_{k-1}}{2^k k!} 2^{-(ny+k\beta)} \omega\left(\frac{k}{n}\right).$$

Taking nx and $n(y-x)$ in place of u and v, respectively in (4.1), we obtain

$$\begin{aligned} & ny\,(ny+1+m\beta)_{m-1} \qquad (5.1) \\ = & \sum_{i=0}^{k} \binom{k}{i} nx\,(nx+1+i\beta)_{i-1}\, n(y-x)\,(n(y-x)+1+(k-i)\beta)_{k-i-1} \end{aligned}$$

which implies

$$\begin{aligned} & L_n^{\beta}(\omega)(y) \\ = & \sum_{k=0}^{\infty}\sum_{i=0}^{k} \omega\left(\frac{k}{n}\right)\binom{k}{i} \frac{nx\,(nx+1+i\beta)_{i-1}}{2^k k!} 2^{-(ny+k\beta)} \\ & \times n(y-x)\,(n(y-x)+1+(k-i)\beta)_{k-i-1}. \end{aligned}$$

Interchanging the order of the above summations gives that

$$\begin{aligned} & L_n^{\beta}(\omega)(y) \\ = & \sum_{i=0}^{\infty}\sum_{k=i}^{\infty} \omega\left(\frac{k}{n}\right) \frac{1}{i!\,(k-i)!} nx\,(nx+1+i\beta)_{i-1} \frac{2^{-(ny+k\beta)}}{2^k} \qquad (5.2) \\ & n(y-x)\,(n(y-x)+1+(k-i)\beta)_{k-i-1}. \end{aligned}$$

Taking $k-i=l$, (5.2) reduces to

$$\begin{aligned} & L_n^{\beta}(\omega)(y) \\ = & \sum_{i=0}^{\infty}\sum_{l=0}^{\infty}\omega\left(\frac{i+l}{n}\right)nx(nx+1+i\beta)_{i-1}\frac{2^{-(ny+(i+l)\beta)}}{2^{i+l}i!l!} \\ & \times n(y-x)(n(y-x)+1+l\beta)_{l-1}. \end{aligned} \tag{5.3}$$

On the other hand, $L_n^{\beta}(\omega)(x)$ can be written as

$$\begin{aligned} L_n^{\beta}(\omega)(x) &= \sum_{i=0}^{\infty}\omega\left(\frac{i}{n}\right)nx(nx+1+i\beta)_{i-1}\frac{2^{-(nx+i\beta)}}{2^i i!} \\ &= \sum_{i=0}^{\infty}\omega\left(\frac{i}{n}\right)nx(nx+1+i\beta)_{i-1}\frac{2^{-(ny+i\beta)}2^{n(y-x)}}{2^i i!}. \end{aligned} \tag{5.4}$$

Since

$$2^{n(y-x)}=\sum_{l=0}^{\infty}n(y-x)(n(y-x)+1+l\beta)_{l-1}\frac{2^{-l\beta}}{2^l l!}$$

then, one may write

$$\begin{aligned} L_n^{\beta}(\omega)(x) &= \sum_{i=0}^{\infty}\sum_{l=0}^{\infty}\omega\left(\frac{i}{n}\right)nx(nx+1+i\beta)_{i-1}\frac{2^{-(ny+(i+l)\beta)}}{2^{i+l}i!l!} \\ & \quad \times n(y-x)(n(y-x)+1+l\beta)_{l-1}. \end{aligned} \tag{5.5}$$

Subtracting (5.5) from (5.3)

$$\begin{aligned} & L_n^{\beta}(\omega)(y)-L_n^{\beta}(\omega)(x) \\ = & \sum_{i=0}^{\infty}\sum_{l=0}^{\infty}\left[\omega\left(\frac{i+l}{n}\right)-\omega\left(\frac{i}{n}\right)\right]nx(nx+1+i\beta)_{i-1}\frac{2^{-(ny+(i+l)\beta)}}{2^{i+l}i!l!} \\ & \times n(y-x)(n(y-x)+1+l\beta)_{l-1} \end{aligned} \tag{5.6}$$

and using the hypothesis that ω is a modulus of continuity function, one obtains

$$\begin{aligned} \leq & \sum_{i=0}^{\infty}\sum_{l=0}^{\infty}\omega\left(\frac{l}{n}\right)nx(nx+1+i\beta)_{i-1}\frac{2^{-(ny+(i+l)\beta)}}{2^{i+l}i!l!} \\ & \times n(y-x)(n(y-x)+1+l\beta)_{l-1} \\ = & \sum_{i=0}^{\infty}nx(nx+1+i\beta)_{i-1}\frac{2^{-i\beta}}{2^i i!} \\ & \times\sum_{l=0}^{\infty}\omega\left(\frac{l}{n}\right)n(y-x)(n(y-x)+1+l\beta)_{l-1}\frac{2^{-(ny+l\beta)}}{2^l l!} \\ = & \sum_{l=0}^{\infty}\omega\left(\frac{l}{n}\right)n(y-x)(n(y-x)+1+l\beta)_{l-1}\frac{2^{-(n(y-x)+l\beta)}}{2^l l!} \\ = & L_n^{\beta}(\omega)(y-x)). \end{aligned} \tag{5.7}$$

This shows that $L_n^{\beta}(\omega)$ satisfies the subadditivity property. Since ω is non-decreasing, then (5.6) provides that $L_n^{\beta}(\omega)(y) \geq L_n^{\beta}(\omega)(x)$ when $y \geq x$, namely, $L_n^{\beta}(\omega)$ is non-decreasing. From the definition of L_n^{β} it is obvious that $\lim_{x\to 0} L_n^{\beta}(\omega; x) = L_n^{\beta}(\omega; 0) = \omega(0) = 0$. Therefore, $L_n^{\beta}(\omega)$ is a function of modulus of continuity. □

References

[1] Abel, U., Agratini, O., *Asymptotic behaviour of Jain operators,* Numer. Algorithms, **71**(2016), 553–565.

[2] Abel, U., Ivan, M., *On a generalization of an approximation operator defined by A. Lupaş,* Gen. Math., **15**(2007), no. 1, 21–34.

[3] Agratini, O., *On a sequence of linear positive operators,* Facta Univ. Ser. Math. Inform., **14**(1999), 41–48.

[4] Agratini, O., *On the rate of convergence of a positive approximation process,* Nihonkai Math. J., **11**(2000), 47-56.

[5] Agratini, O., *Approximation properties of a class of linear operators,* Math. Methods Appl. Sci., **36**(2013), no. 17, 2353–2358.

[6] Cheney, E.W., Charma, A., *Bernstein power series,* Canad. J. Math., **16**(1964), 241-252.

[7] Erençin, A., Başcanbaz-Tunca, G., Taşdelen, F., *Some properties of the operators defined by Lupaş,* Rev. Anal. Numér. Théor. Approx., **43**(2014), no. 2, 168–174.

[8] Farcaş, A., *An asymptotic formula for Jain's operators,* Stud. Univ. Babeş-Bolyai Math., **57**(2012), no. 4, 511–517.

[9] Finta, Z., *Pointwise approximation by generalized Szász-Mirakjan operators,* Stud. Univ. Babeş-Bolyai Math., **46**(2001), no. 4, 61–67.

[10] Finta, Z., *Quantitative estimates for some linear and positive operators,* Stud. Univ. Babeş-Bolyai Math., **47**(2002), no. 3, 71–84.

[11] Gadzhiev, A.D., *Theorems of the type of P.P. Korovkin's type theorems,* Mat. Zametki, **20**(1976), no. 5, 781-786.

[12] Gasper, G., Rahman, M., *Basic Hypergeometric Series,* Cambridge University Press, 2004.

[13] Jain, G.C., *Approximation of functions by a new class of linear operators,* J. Aust. Math. Soc., **13**(1972), no. 3, 271–276.

[14] Li, Z., *Bernstein polynomials and modulus of continuity*, J. Approx. Theory, **102**(2000), no.1, 171-174.

[15] Lupaş, A., *The approximation by some positive linear operators,* In: Proceedings of the International Dortmund Meeting on Approximation Theory (M.W. Muller et al., eds.), Mathematical Research, Akademie Verlag, Berlin, **86**(1995), 201-229.

[16] Mhaskar, H.N., Pai, D.V., *Fundamentals of approximation theory*, CRC Press, Boca Raton, FL, Narosa Publishing House, New Delhi, 2000.

[17] Olgun, A., Taşdelen F., Erençin, A., *A generalization of Jain's operators,* Appl. Math. Comput., **266**(2015), 6–11.

[18] Özarslan, M.A., *Approximation Properties of Jain-Stancu Operators,* Filomat, **30**(2016), no. 4, 1081–1088.

[19] Patel, P., Mishra, V.N., *On new class of linear and positive operators,* Boll. Unione Mat. Ital., **8**(2015), no. 2, 81–96.

[20] Stancu, D.D., Occorsio, M.R., *On approximation by binomial operators of Tiberiu Popoviciu type,* Rev. Anal. Numér. Théor. Approx., **27**(1998), 167-181.

[21] Velleman, D.J., Call, G.S., *Permutation and combination locks,* Math. Mag., **68**(1995), no. 4, 243-253.

Gülen Başcanbaz-Tunca
Ankara University, Faculty of Science
Department of Mathematics
06100 Tandogan-Ankara, Turkey
e-mail: `tunca@science.ankara.edu.tr`

Murat Bodur
Ankara University, Faculty of Science
Department of Mathematics
06100 Tandogan-Ankara, Turkey
e-mail: `bodur@ankara.edu.tr`

Dilek Söylemez
Ankara University, Elmadag Vocational School
06780, Elmadag-Ankara, Turkey
e-mail: `dsoylemez@ankara.edu.tr`

Stud. Univ. Babeş-Bolyai Math. 63(2018), No. 4, 539–548
DOI: 10.24193/subbmath.2018.4.09

Partial averaging of discrete-time set-valued systems

Tatyana A. Komleva, Liliya I. Plotnikova and Andrej V. Plotnikov

Abstract. In the introduction of the article we given an overview of the results for set-valued equations. Further we considered the set-valued discrete-time dynamical systems and substantiates the averaging method for nonlinear set-valued discrete-time systems with a small parameter.

Mathematics Subject Classification (2010): 49M25, 34C29, 49J53.

Keywords: Averaging method, discrete-time system, set-valued mapping, Hukuhara difference.

1. Introduction

As it is well known, there are two main types of dynamical systems: differential equations and discrete-time equations. Differential equation describes the continuous time evaluation of the system, whereas discrete-time equation describes the discrete time evaluation of the system. The theory of discrete dynamical systems and difference equations developed greatly during the last decades (see [8, 18, 34] and references cited there).

In 1969, F.S. de Blasi and F. Iervolino [5] begun studying of set-valued differential equations in semilinear metric spaces. Later, the development of calculus in metric spaces became an object of attention of many researchers (see [7, 19, 20, 22, 30, 31, 27, 32, 40] and the references therein) and transformed into the theory of set-valued equations as an independent discipline. Set-valued equations are useful in other areas of mathematics. For example, set-valued differential equations are used as an auxiliary tool to prove the existence results for differential inclusions [19, 22, 27, 40]. Also, one can employ set-valued differential equations in the investigation of fuzzy differential equations [20, 30]. Moreover, set-valued differential equations are a natural generalization of usual ordinary differential equations in finite (or infinite) dimensional Banach spaces [40]. Clearly, in many cases, when modeling real-world phenomena, information about the behavior of a dynamical system is uncertain and one has to consider these uncertainties to gain better understanding of the full models. The set-valued equations can be used to model dynamical systems subjected to uncertainties.

This article deals with discrete set-valued dynamical systems, where time is measured by the number of iterations carried out, the dynamics are not continuous and values at each iteration is a set. In applications this would imply that the solutions are observed at discrete time intervals and also under uncertainty or interference effects [9, 13, 24, 35, 36, 38, 41]. Recurrence relations can be used to construct mathematical models of discrete systems under uncertainty. They are also used extensively to solve many differential equations with set-valued right-hand side which do not have an analytic solution; the set-valued differential equations are represented by recurrence relations (or difference equations) that can be solved numerically on a computer [1, 4, 24, 41].

Averaging theory for ordinary differential equations has a rich history, dating to back to the work of N.M. Krylov and N.N. Bogoliubov [17]. Also is well known, the averaging methods combined with the asymptotic representations began to be applied as the basic constructive tool for solving the complicated problems of analytical dynamics described by the differential equations [3, 27, 37] and the references therein. The possibility of using some averaging schemes for set-valued equations was studied in [11, 12, 14, 15, 16, 22, 23, 25, 30, 26, 29, 27, 39]. Throughout the years, many authors have published papers on averaging methods for different kinds of differential systems and discrete-time system [2, 21, 28]. The bulk of this article is concerned with the averaging method for nonlinear discrete-time set-valued systems.

2. Preliminaries

Let $conv(R^n)$ be a space of all nonempty convex compact subsets of R^n with the Hausdorff metric

$$h(A,B) = \min_{r\geq 0}\{B \subset S_r(A),\ \ A \subset S_r(B)\}$$

where $A, B \in conv(R^n)$, $S_r(A)$ be a r-neighborhood of the set A.

The usual set operations, i.e., well-known as Minkowski addition and scalar multiplication, are defined as follows

$$A + B = \{a + b \,:\, a \in A,\, b \in B\} \quad \text{and} \quad \lambda A = \{\lambda a \,:\, a \in A,\, \lambda \in R\}.$$

Lemma 2.1. [32] *The following properties hold:*

1. $(conv(R^n), h)$ *is a complete metric space,*
2. $h(A + C, B + C) = h(A, B)$,
3. $h(\lambda A, \lambda B) = |\lambda| h(A, B)$ *for all* $A, B, C \in conv(R^n)$ *and* $\lambda \in R$.

For any $A \in conv(R^n)$, it can be seen $A + (-1)A \neq \{0\}$ in general, thus the opposite of A is not the inverse of A with respect to the Minkowski addition unless $A = \{a\}$ is a singleton. To partially overcome this situation, the Hukuhara difference has been introduced [10].

Definition 2.2. [10] Let $X, Y \in conv(R^n)$. A set $Z \in conv(R^n)$ such that $X = Y + Z$ is called a Hukuhara difference of the sets X and Y and is denoted by $X \overset{h}{-} Y$.

An important property of Hukuhara difference is that $A \overset{h}{-} A = \{0\}, \forall A \in conv(R^n)$ and $(A+B) \overset{h}{-} B = A, \forall A, B \in conv(R^n)$; Hukuhara difference is unique, but a necessary condition for $A \overset{h}{-} B$ to exist is that A contains a translate $\{c\} + B$ of B.

Now consider the non-autonomous set-valued discrete-time equations

$$X_{i+1} = X_i + F(i, X_i), \tag{2.1}$$

and

$$X_{i+1} = X_i \overset{h}{-} F(i, X_i), \tag{2.2}$$

where $i \in I = \{0, 1, ..., N\}$, $X_i \in conv(R^n)$, $F : I \times conv(R^n) \to conv(R^n)$. If one starts with an initial value, say, X_0, then iteration of (2.1) (or (2.2)) leads to a sequence of the form

$$\{X_i : i = 0 \,\text{to}\, N\} = \{X_0, X_1, ..., X_N\}.$$

Definition 2.3. A solution to the set-valued discrete-time equation (2.1) (or (2.2)) is a discrete-time set-valued trajectory, $\{X_i\}_{i=0}^N$, that satisfies this equation at any point $i \in I$.

Remark 2.4. It is obvious that the solution of (2.1) exists for any $X_0 \in conv(R^n)$ and I.

Remark 2.5. Obviously, the differences in (2.2) may not always exist. For example,

1) let $n \geq 1$, $X_0 = \{a \in R^n : \|a\| \leq 1\}$, $F(i, X_i) = (i+2)\, X_i$, i.e. $F(0, X_0) = \{b \in R^n : \|b\| \leq 2\}$. In this case, the difference in (2.2) does not exist for $i = 0$;
2) let $n = 2$, $X_0 = \{a \in R^2 : |a_k| \leq 1,\ k = 1, 2\}$,

$$K(i) = \begin{pmatrix} cos(i+1) & sin(i+1) \\ -sin(i+1) & cos(i+1) \end{pmatrix},$$

$F(i, X_i) = K(i)\, X_i$. Also, the difference in (2.2) does not exist for $i = 0$.

Let $CC(R^n)$ $(n \geq 2)$ be a space of all nonempty strictly convex closed sets of R^n and all elements of R^n [33].

Remark 2.6. If $A, B \in CC(R^n)$ and $A + C = B$ then $C \in CC(R^n)$ [33].

Remark 2.7. If $A, B \in CC(R^n)$ and there exists $c \in R^n$ such that $A + c \subset B$, then there exists $C \in CC(R^n)$ such that $A + C = B$, i.e. $C = B \overset{h}{-} A$ [33].

Then the following theorem holds.

Theorem 2.8. *Let the following conditions hold:*

1) $F(i, X) \in CC(R^n)$ for any $i \in I$ and $X \in CC(R^n)$;

2) the following inequality

$$|C(X, \psi) + C(X, -\psi)| \geq |C(F(i, X), \psi) + C(F(i, X), -\psi)|$$

holds for all $\psi \in R^n$ $(\|\psi\| = 1)$, $i \in I$ and $X \in CC(R^n)$, where

$$C(A, \psi) = \max_{a \in A}(a_1\psi_1 + ... + a_n\psi_n),\ A \in CC(R^n).$$

Then the solution of (2.2) exists for any $X_0 \in CC(R^n)$ and I.

Proof. We put any set $X_0 \in CC(R^n)$. By condition 1) of the theorem, we have $F(0, X_0) \in CC(R^n)$. By condition 2) of the theorem, we obtain

$$|C(X_0, \psi) + C(X_0, -\psi)| \geq |C(F(0, X_0), \psi) + C(F(0, X_0), -\psi)|$$

for all $\psi \in R^n$, $\|\psi\| = 1$. Then, there exists $c \in R^n$ such that $F(0, X_0) + c \subset X_0$ [33]. By remark 2.7, we have the set $C \in CC(R^n)$ such that $F(0, X_0) + C = X_0$. Therefore, $X_1 = C = X_0 \overset{h}{-} F(0, X_0)$ and $X_1 \in CC(R^n)$. Further, applying the method of mathematical induction, we obtain $X_{i+1} = X_i \overset{h}{-} F(i, X_i)$ and $X_{i+1} \in CC(R^n)$ for all $i \in I$. The theorem is proved. □

3. The method of averaging

Now consider the non-autonomous set-valued discrete-time equations with a small parameter

$$X_{i+1} = X_i + \varepsilon F(i, X_i), \tag{3.1}$$

and

$$X_{i+1} = X_i \overset{h}{-} \varepsilon F(i, X_i), \tag{3.2}$$

where $\varepsilon > 0$ be a small parameter, $L > 0$ is any real number, $N = [L\varepsilon^{-1}]$, $[\cdot]$ is floor function.

3.1. Case (3.1).

In the beginning we consider the equation (3.1). We associate with the equation (3.1) the following averaged set-valued discrete-time equation with a small parameter

$$X_{i+1} = X_i + \varepsilon \overline{F}(i, X_i), \tag{3.3}$$

where $\overline{F}(i, X)$ such that

$$\lim_{n\to\infty} h\left(\frac{1}{n}\sum_{i=0}^{n-1} F(i, X), \quad \frac{1}{n}\sum_{i=0}^{n-1} \overline{F}(i, X)\right) = 0. \tag{3.4}$$

The main theorem of this subsection is on averaging for set-valued discrete-time equation with a small parameter. It establishes nearness of solutions of (3.1) and (3.3), and reads as follows.

Theorem 3.1. *Let in the domain $Q = \{\, (i, X) \,:\, i \in I, X \subset B \subset R^n \,\}$ the following conditions hold:*

1) mappings $F(i, X)$ and $\overline{F}(i, X)$ satisfy a Lipschitz condition, i.e. there is a constant $\lambda > 0$ such that

$$h(F(i, X'), F(i, X")) \leq \lambda h(X', X"), \quad h(\overline{F}(i, X'), \overline{F}(i, X")) \leq \lambda h(X', X"),$$

whenever $(i, X'), (i, X") \in Q$;

3) there exists $\gamma > 0$ such that $h(F(i, X), \{0\}) \leq \gamma$, $h(\overline{F}(i, X), \{0\}) \leq \gamma$ for every $(i, X) \in Q$;

4) limit (3.4) exists uniformly with respect to X in the domain B;

5) the solution of the problem (3.3) together with a $\rho-$neighborhood belong to the domain B for $\varepsilon \in (0, \bar{\varepsilon}]$.

Then for any $\eta \in (0, \rho]$ and $L > 0$ there exists $\varepsilon_0(\eta, L) \in (0, \bar{\varepsilon}]$ such that for all $\varepsilon \in (0, \varepsilon_0]$ and $i \in I$ the following inequality holds

$$h(X_i, \overline{X}_i) < \eta \tag{3.5}$$

where $\{X_i\}_{i=0}^N$, $\{\overline{X}_i\}_{i=0}^N$ are the solutions of initial and averaged problems.

Proof. We write the equations (3.1) and (3.3) in the form

$$X_{i+1} = X_0 + \varepsilon \sum_{j=0}^{i} F(j, X_j), \tag{3.6}$$

$$\overline{X}_{i+1} = X_0 + \varepsilon \sum_{j=0}^{i} \overline{F}(j, \overline{X}_j). \tag{3.7}$$

By (3.6) and (3.7), we have

$$h(X_{i+1},\ \overline{X}_{i+1}) = h\left(\varepsilon \sum_{j=0}^{i} F(j, X_j), \varepsilon \sum_{j=0}^{i} \overline{F}(j, \overline{X}_j)\right)$$

$$\le \varepsilon \sum_{j=0}^{i} h(F(j, X_j), F(j, \overline{X}_j)) + \varepsilon h\left(\sum_{j=0}^{i} F(j, \overline{X}_j), \sum_{j=0}^{i} \overline{F}(j, \overline{X}_j)\right)$$

$$\le \lambda \varepsilon \sum_{j=0}^{i} h(X_j, \overline{X}_j) + \phi, \tag{3.8}$$

where

$$\phi = \varepsilon h\left(\sum_{j=0}^{i} F(j, \overline{X}_j), \sum_{j=0}^{i} \overline{F}(j, \overline{X}_j)\right).$$

Now we will estimate ϕ on I. Divide the interval I into partial intervals by the points $t_k = kl(\varepsilon),\ k = \overline{0, m},\ t_{m-1} < L\varepsilon^{-1} \le t_m$, where $l(\varepsilon)$ is integer and

$$\lim_{\varepsilon \to 0} l(\varepsilon) = \infty,\ \lim_{\varepsilon \to 0} \varepsilon l(\varepsilon) = 0. \tag{3.9}$$

Let $kl(\varepsilon) < i \le (k+1)l(\varepsilon)$. Then we have

$$\phi = \varepsilon h\left(\sum_{j=0}^{i} F(i, \overline{X}_i), \sum_{j=0}^{i} \overline{F}(j, \overline{X}_j)\right)$$

$$\le \varepsilon \sum_{\zeta=0}^{k-1} h\left(\sum_{j=l(\varepsilon)\zeta}^{(\zeta+1)l(\varepsilon)-1} F(j, \overline{X}_j), \sum_{j=l(\varepsilon)\zeta}^{(\zeta+1)l(\varepsilon)-1} \overline{F}(j, \overline{X}_j)\right)$$

$$+\varepsilon h\left(\sum_{j=kl(\varepsilon)}^{i} F(j, \overline{X}_j),\quad \sum_{j=kl(\varepsilon)}^{i} \overline{F}(j, \overline{X}_j)\right)$$

$$\leq \varepsilon \sum_{\zeta=0}^{k-1} h\left(\sum_{j=l(\varepsilon)\zeta+1}^{(\zeta+1)l(\varepsilon)-1} F(j,\overline{X}_j), \sum_{j=l(\varepsilon)\zeta+1}^{(\zeta+1)l(\varepsilon)-1} F(j,\overline{X}_{\zeta l(\varepsilon)})\right)$$
$$+\varepsilon \sum_{\zeta=0}^{k-1} h\left(\sum_{j=l(\varepsilon)\zeta}^{(\zeta+1)l(\varepsilon)-1} F(j,\overline{X}_{\zeta l(\varepsilon)}), \sum_{j=l(\varepsilon)\zeta}^{(\zeta+1)l(\varepsilon)-1} \overline{F}(j,\overline{X}_{\zeta l(\varepsilon)})\right)$$
$$+\varepsilon \sum_{\zeta=0}^{k-1} h\left(\sum_{j=h\zeta+1}^{(\zeta+1)l(\varepsilon)-1} \overline{F}(j,\overline{X}_{\zeta l(\varepsilon)}), \sum_{j=l(\varepsilon)\zeta+1}^{(\zeta+1)l(\varepsilon)-1} \overline{F}(j,\overline{X}_j)\right)$$
$$+\varepsilon \sum_{j=kl(\varepsilon)}^{i} h(F(j,\overline{X}_j),\ \ \overline{F}(j,\overline{X}_j)). \tag{3.10}$$

Now we will estimate terms in (3.10)

$$\varepsilon h\left(\sum_{j=l(\varepsilon)\zeta+1}^{(\zeta+1)l(\varepsilon)-1} F(j,\overline{X}_j), \sum_{j=l(\varepsilon)\zeta+1}^{(\zeta+1)l(\varepsilon)-1} F(j,\overline{X}_{\zeta l(\varepsilon)})\right)$$
$$\leq \varepsilon \sum_{j=l(\varepsilon)\zeta+1}^{(\zeta+1)l(\varepsilon)-1} h(F(j,\overline{X}_j),\ \ F(j,\overline{X}_{\zeta l(\varepsilon)})) \leq \lambda \sum_{j=l(\varepsilon)\zeta+1}^{(\zeta+1)l(\varepsilon)-1} h(\overline{X}_j,\overline{X}_{\zeta l(\varepsilon)})$$
$$\leq \varepsilon^2\,\lambda \sum_{j=l(\varepsilon)\zeta+1}^{(\zeta+1)l(\varepsilon)-1}\ \sum_{r=k\zeta}^{j-1} \left\|\overline{F}(\overline{X}_j)\right\| \ \leq\ \varepsilon^2\lambda\gamma l(\varepsilon)^2/2. \tag{3.11}$$

Also, we obtain

$$\varepsilon h\left(\sum_{j=l(\varepsilon)\zeta}^{(\zeta+1)l(\varepsilon)-1} \overline{F}(j,\overline{X}_j), \sum_{j=l(\varepsilon)\zeta}^{(\zeta+1)l(\varepsilon)-1} \overline{F}(j,\overline{X}_{\zeta l(\varepsilon)})\ \right) \leq\ \varepsilon^2\lambda\gamma l(\varepsilon)^2/2. \tag{3.12}$$

Obviously,

$$\varepsilon \sum_{j=kl(\varepsilon)}^{i} \delta(F(j,\overline{X}_{kl(\varepsilon)}),\overline{F}(j,\overline{X}_{kl(\varepsilon)}))\ \leq 2\varepsilon\gamma l(\varepsilon). \tag{3.13}$$

From the condition 4) of the theorem there exists an increasing function $\mu(l)$, such that

1) $\lim\limits_{t\to\infty} \mu(t) = 0$;

2) $\varepsilon \sum\limits_{\zeta=0}^{k-1} h\left(\sum\limits_{j=l(\varepsilon)\zeta}^{(\zeta+1)l(\varepsilon)-1} F(j,\overline{X}_{\zeta l(\varepsilon)}), \sum\limits_{j=l(\varepsilon)\zeta}^{(\zeta+1)l(\varepsilon)-1} \overline{F}(j,\overline{X}_{\zeta l(\varepsilon)})\right)$

$$\leq m\varepsilon l(\varepsilon)\mu(l(\varepsilon)) \leq\ \ L\,\mu(l(\varepsilon)). \tag{3.14}$$

Combining (3.10) – (3.14), we obtain

$$\phi \leq \varepsilon l(\varepsilon)\gamma(\lambda L + 2) + L\mu(l(\varepsilon)). \tag{3.15}$$

By (3.9), we take $\varepsilon^0 \in (0,\rho]$ such that

$$e^{\lambda L}[\varepsilon l(\varepsilon)\gamma(\lambda L + 2) + L\phi(l(\varepsilon))] < \eta \tag{3.16}$$

for all $\varepsilon \in (0, \varepsilon^0]$.
From (3.8), (3.15), (3.16) we obtain (3.5). The theorem is proved. □

Remark 3.2. If $F(i, X_i) = \Delta \cdot G(t_0 + i\Delta, X_i)$, $G : R \times conv(R^n) \to conv(R^n)$, $X_i = X(t_0 + i\Delta)$, discrete-time equation (2.1) is a Euler polygonal curve for the differential equation with Hukuhara derivative [6]

$$D_h X(t) = G(t, X(t)), \quad X(t_0) = X_0,$$

where $X : R \to conv(R^n)$ is set-valued mapping, $D_h X(t)$ is Hukuhara derivative [6, 10]. Thus, Theorem 3.1 is a discrete analogue of the first Bogolyubov theorem for a differential equation with derivative Hukuhara [15, 16, 25, 30, 27].

3.2. Case (3.2).

We associate with the equation (3.2) the following averaged set-valued discrete-time equation with a small parameter

$$X_{i+1} = X_i \overset{h}{-} \varepsilon \overline{F}(i, X_i), \tag{3.17}$$

where $\overline{F}(i, X)$ such that limit (3.4) exists.

Theorem 3.3. *Let in the domain $Q = \{ (i, X) : i \in I, X \in CC(R^n), X \subset B \subset R^n \}$ the following conditions hold:*

1) mappings $F(i, X), \overline{F}(i, X) \in CC(R^n)$ for any $(i, X) \in Q$;

2) the inequality

$$|C(X, \psi) + C(X, -\psi)| \geq |C(\varepsilon F(i, X), \psi) + C(\varepsilon F(i, X), -\psi)|,$$

$$|C(X, \psi) + C(X, -\psi)| \geq |C(\varepsilon \overline{F}(i, X), \psi) + C(\varepsilon \overline{F}(i, X), -\psi)|$$

are true for all $\psi \in R^n$ ($\|\psi\| = 1$), $\varepsilon \in (0, \bar{\varepsilon}]$, $i \in I$ and $X \subset B$;

3) mappings $F(i, X)$ and $\overline{F}(i, X)$ satisfy a Lipschitz condition

$$h(F(i, X'), F(i, X")) \leq \lambda h(X', X"), \quad h(\overline{F}(i, X'), \overline{F}(i, X")) \leq \lambda h(X', X"),$$

with a Lipschitz constant $\lambda > 0$;

4) there exists $\gamma > 0$ such that $h(F(i, X), \{0\}) \leq \gamma$, $h(\overline{F}(i, X), \{0\}) \leq \gamma$ for every $(i, X) \in Q$;

5) limit (3.4) exists uniformly with respect to X in the domain B;

6) the solution of the problem (3.17) together with a $\rho-$neighborhood belong to the domain B for $\varepsilon \in (0, \bar{\varepsilon}]$.

Then for any $\eta \in (0, \rho]$ and $L > 0$ there exists $\varepsilon_0(\eta, L) \in (0, \bar{\varepsilon}]$ such that for all $\varepsilon \in (0, \varepsilon_0]$ and $i \in I$ the inequality (3.5) holds.

Proof. We write the equations (3.2) and (3.17) in the form

$$X_{i+1} = X_0 \overset{h}{-} \varepsilon \sum_{j=0}^{i} F(j, X_j), \quad \text{and} \quad \overline{X}_{i+1} = X_0 \overset{h}{-} \varepsilon \sum_{j=0}^{i} \overline{F}(j, \overline{X}_j). \tag{3.18}$$

By (3.18), we have

$$h(X_{i+1},\ \overline{X}_{i+1}) = h\left(\varepsilon\sum_{j=0}^{i} F(j, X_j), \varepsilon\sum_{j=0}^{i} \overline{F}(j, \overline{X}_j)\right).$$

Further, Theorem 3.3 is proved similarly to Theorem 3.1. This concludes the proof. □

Remark 3.4. If $\overline{F}(i, X) \equiv \overline{F}(X)$, i.e.

$$\lim_{n\to\infty} h\left(\frac{1}{n}\sum_{i=0}^{n-1} F(i, X), \frac{1}{n}\sum_{i=0}^{n-1} \overline{F}(X)\right) = 0,$$

then the validity of the full averaging scheme for (3.1) and (3.2) follows from the theorems 3.1 and 3.3.

References

[1] Baier, R., Donchev, T., *Discrete approximation of impulsive differential inclusions*, Numer. Funct. Anal. Optim., **31**(2010), no. 6, 653–678.

[2] Belan, E.P., *Averaging in the theory of finite-difference equations*, Ukr. Math. J., **19**(1967), no. 3, 319–323.

[3] Burd, V., *Method of Averaging for Differential Equations on an Infinite Interval. Theory and Applications*, Lect. Pure Appl. Math., **255**, Chapman & Hall/CRC, Boca Raton, FL, 2007.

[4] Chahma, I.A., *Set-valued discrete approximation of state-constrained differential inclusions*, Bayreuth. Math. Schr., **67**(2003), 3–162.

[5] de Blasi, F.S., Iervolino, F., *Equazioni differentiali con soluzioni a valore compatto convesso*, Boll. Unione Mat. Ital., **2**(4–5)(1969), 491–501.

[6] de Blasi, F., Iervolino, F., *Euler method for differential equations with set-valued solutions*, Boll. Unione Mat. Ital., **4**(1971), no. 4, 941–949.

[7] Deimling, K., *Multivalued Differential Equations*, De Gruyter series in nonlinear analysis and applications, **1**, Walter de Gruyter, 1992.

[8] Elaydi, S.N., *Discrete Chaos: With Applications in Science and Engineering, Second Edition*, Chapman and Hall/CRC, Boca Raton, FL, 2008.

[9] Gu, R.B., Guo, W.J., *On mixing properties in set valued discrete system*, Chaos Solitons Fractals, **28**(2006), no. 3, 747–754.

[10] Hukuhara, M., *Integration des applications mesurables dont la valeur est un compact convexe*, Funkc. Ekvacioj, Ser. Int., **10**(1967), 205–223.

[11] Janiak, T., Łuczak-Kumorek, E., *Method on partial averaging for functional-differential equations with Hukuhara's derivative*, Stud. Univ. Babeş-Bolyai, Math., **48**(2003), no. 2, 65–72.

[12] Janiak, T., Łuczak-Kumorek, E., *Bogolubov's type theorem for functional-differential inclusions with Hukuhara's derivative*, Stud. Univ. Babeş-Bolyai, Math., **36**(1991), no. 1, 41–55.

[13] Khan, A., Kumar, P., *Chaotic properties on time varying map and its set valued extension*, Adv. Pure Math., **3**(2013), 359–364.

[14] Kichmarenko, O.D., *Averaging of differential equations with Hukuhara derivative with maxima*, Int. J. Pure Appl. Math., **57**(2009), no. 3, 447–457.

[15] Kisielewicz, M., *Method of averaging for differential equations with compact convex valued solutions*, Rend. Mat., VI, Ser., **9**(1976), no. 3, 397–408.

[16] Komleva, T.A., Plotnikova L.I., Plotnikov A.V., Skripnik, N.V., *Averaging in fuzzy controlled systems*, Nonlinear Oscil., **14**(2012), no. 3, 342–349.

[17] Krylov, N.M., Bogoliubov, N.N., *Introduction to Nonlinear Mechanics*, Princeton University Press, Princeton, 1947.

[18] Kulenovic, M.R.s., Merino, O., *Discrete Dynamical Systems and Difference Equations with Mathematica,* Chapman and Hall / CRC, 2002.

[19] Lakshmikantham, V., Granna Bhaskar, T., Vasundhara Devi, J., *Theory of Set Differential Equations in Metric Spaces*, Cambridge Scientific Publishers, Cambridge, 2006.

[20] Lakshmikantham, V., Mohapatra, R.N., *Theory of Fuzzy Differential Equations and Inclusions*, Taylor and Francis, London, 2003.

[21] Martynyuk, D.I., Danilov, Ya., Pan'kov, G., *Second Bogolyubov theorem for systems of difference equations*, Ukr. Math. J., **48**(1996), no. 4, 516–529.

[22] Perestyuk, N.A., Plotnikov, V.A., Samoilenko, A.M., Skripnik, N.V., *Differential equations with impulse effects: multivalued right-hand sides with discontinuities*, de Gruyter Stud. Math. **40**, Berlin/Boston, Walter De Gruyter GmbH & Co, 2011.

[23] Perestyuk, N.A., Skripnik, N.V., *Averaging of set-valued impulsive systems*, Ukr. Math. J., **65**(2013), no. 1, 140–157.

[24] Petersen, I.R., Savkin, A.V., *Discrete-Time Set-Valued State Estimation*, In: Robust Kalman Filtering for Signals and Systems with Large Uncertainties. Control Engineering. Birkhauser, Boston, MA, 1999.

[25] Plotnikov, A.V., *Averaging differential embeddings with Hukuhara derivative*, Ukr. Math. J., **41**(1989), no. 1, 112–115.

[26] Plotnikov, V.A., Kichmarenko, O.D., *Averaging of controlled equations with the Hukuhara derivative*, Nonlinear Oscil., **9**(2006), no. 3, 365–374.

[27] Plotnikov, V.A., Plotnikov, A.V., Vityuk, A.N., *Differential equations with a multivalued right-hand side. Asymptotic methods*, AstroPrint, Odessa, 1999.

[28] Plotnikov, V.A., Plotnikova, L.I., Yarovoi, A.T., *Averaging method for discrete systems and its application to control problems*, Nonlinear Oscil., **7**(2004), no. 2, 240–253.

[29] Plotnikov, V.A., Rashkov, P.I., *Averaging in differential equations with hukuhara derivative and delay*, Funct. Differ. Equ., **8**(2001), no. 3-4, 371–381.

[30] Plotnikov, A.V., Skripnik, N.V. *Differential equations with "clear" and fuzzy multivalued right-hand side. Asymptotics methods*, AstroPrint, Odessa, 2009.

[31] Plotnikov, A.V., Skripnik, N.V., *An existence and uniqueness theorem to the cauchy problem for generalised set differential equations*, Dyn. Contin. Discrete Impuls. Syst. Ser. A Math. Anal., **20**(2013), no. 4, 433–445.

[32] Polovinkin, E.S., *Multivalued analysis and differential inclusions*, FIZMATLIT, Moscow, 2014.

[33] Polovinkin, E.S., *Strongly convex analysis*, Sb. Math., **187**(1996), 259–286.

[34] Robinson, R.C., *An Introduction to Dynamical Systems: Continuous and Discrete,* Pearson Education, Inc., 2004.

[35] Roman-Flores, H., *A Note on Transitivity in Set-Valued Discrete Systems*, Chaos Solution Fractals, **17**(2003), no. 1, 99–104.

[36] Roman-Flores, H., Chalco-Cano, Y., *Robinsons Chaos in Set-Valued Discrete Systems*, Chaos Solitons Fractals, **25**(2005), no. 1, 33–42.

[37] Sanders, J.A., Verhulst, F., *Averaging methods in nonlinear dynamical systems*, Applied Mathematical Sciences, **59**, Springer-Verlag, New York, 1985.

[38] Shi, Y.M., Chen, G.R., *Chaos of time-varying discrete dynamical systems*, J. Difference Equ. Appl., **15**(2009), no. 5, 429–449.

[39] Skripnik, N.V., *Averaging of impulsive differential inclusions with Hukuhara derivative*, Nonlinear Oscil., **10**(2007), no. 3, 422–438.

[40] Tolstonogov, A., *Differential Inclusions in a Banach Space*, Kluwer Academic Publishers, Dordrecht, 2000.

[41] Ungureanu, V., Lozan, V., *Linear discrete-time set-valued Pareto-Nash-Stackelberg control processes and their principles*, ROMAI J., **9**(2013), no. 1, 185–198.

Tatyana A. Komleva
Department of Mathematics
Odessa State Academy Civil Engineering and Architecture
4, Didrihsona Street, 65029 Odessa, Ukraine
e-mail: `t-komleva@ukr.net`

Liliya I. Plotnikova
Department of Mathematics
Odessa National Polytechnic University
1, Shevchenko Avenue, 65044 Odessa, Ukraine
e-mail: `liplotnikova@ukr.net`

Andrej V. Plotnikov
Department of Information Technology and Applied Mathematics
Odessa State Academy Civil Engineering and Architecture
4, Didrihsona Street, 65029 Odessa, Ukraine
e-mail: `a-plotnikov@ukr.net`

Stud. Univ. Babeş-Bolyai Math. 63(2018), No. 4, 549–559
DOI: 10.24193/subbmath.2018.4.10

Approximations of bi-criteria optimization problem

Traian Ionuţ Luca and Dorel I. Duca

Abstract. In this article we study approximation methods for solving bi-criteria optimization problems. Initial problem is approximated by a new one consisting of the second order approximation of feasible set and components of objective function might be initial function, first or second approximation of it. Conditions such that efficient solution of the approximate problem will remain efficient for initial problem and reciprocally are studied. Numerical examples are developed to emphasize the importance of these conditions.

Mathematics Subject Classification (2010): 90C46, 90C59.

Keywords: Efficient solution, bi-criteria optimization, η-approximation, invex and incave function.

1. Introduction

Bi-criteria optimization problems are quite often used to solve theoretical and practical problems from areas as portfolio theory [4], energy field [5], data analysis [3], logistics [6].
"Scalarization" methods [2] (weighting problem, k^{th} objective Lagrangian problem, k^{th} objective ε - constrained problem) are common methods for solving this type of problems. Highly complex mathematical models are reducing the efficiency of "scalarization" methods and approximation might represent a good alternative.
This article is analyzing conditions such that efficient solution of a certain approximate problem will remain efficient for the initial problem and reciprocally. Approximate problem consists of replacing components of objective function and also constraints with their approximate functions.

2. Basic concepts

Let X be a set in $\mathbb{R}^n$, x_0 an interior point of X, $\eta : X \times X \to X$ and $f : X \to \mathbb{R}$ functions. If f is differentiable at x_0 then we denote:

$$F^1(x) = f(x_0) + \nabla f(x_0)\, \eta(x, x_0)$$

and call it first $\eta-$approximation of f
and if f is twice differentiable at x_0 then we denote:

$$F^2(x) = f(x_0) + \nabla f(x_0)\, \eta(x, x_0) + \frac{1}{2}\eta(x, x_0)^T \nabla^2 f(x_0)\, \eta(x, x_0).$$

and call it second $\eta-$approximation of f.

Definition 2.1. Let X be a nonempty set of $\mathbb{R}^n$, x_0 an interior point of X, $f : X \to \mathbb{R}$ a function differentiable at x_0 and $\eta : X \times X \to X$. Then function f is:
invex *at* x_0 with respect to η if for all $x \in X$ we have:

$$f(x) - f(x_0) \geq \nabla f(x_0)\, \eta(x, x_0)$$

or equivalently:

$$f(x) \geq F^1(x);$$

incave at x_0 with respect to η if for all $x \in X$ we have:

$$f(x) - f(x_0) \leq \nabla f(x_0)\, \eta(x, x_0)$$

or equivalently

$$f(x) \leq F^1(x);$$

avex at x_0 with respect to η if it is both invex and incave at x_0 w.r.t. η.

If function f is invex, respectively incave or avex we denote invex^1, respectively incave^1 or avex^1.

Definition 2.2. Let X be a nonempty set of $\mathbb{R}^n$, x_0 an interior point of X, $f : X \to \mathbb{R}$ a function twice differentiable at x_0 and $\eta : X \times X \to X$. Then function f is:
second order invex *at* x_0 with respect to η if for all $x \in X$ we have:

$$f(x) - f(x_0) \geq \nabla f(x_0)\, \eta(x, x_0) + \frac{1}{2}\eta(x, x_0)^T \nabla^2 f(x_0)\, \eta(x, x_0)$$

or equivalently:

$$f(x) \geq F^2(x);$$

second order incave *at* x_0 with respect to η if for all $x \in X$ we have:

$$f(x) - f(x_0) \leq \nabla f(x_0)\, \eta(x, x_0) + \frac{1}{2}\eta(x, x_0)^T \nabla^2 f(x_0)\, \eta(x, x_0)$$

or equivalently:

$$f(x) \leq F^2(x);$$

second order avex at x_0 with respect to η if it is both second order invex and second order incave at x_0 w.r.t. η.

If function f is second order invex, respectively second order incave or second order avex we denote invex^2, respectively incave^2 or avex^2.

Let X be a nonempty set of $\mathbb{R}^n$, x_0 an interior point of $X, \eta : X \times X \to X$, T and S index sets, $f = (f_1, f_2) : X \to \mathbb{R}^2$ and $g_t,\ h_s : X \to \mathbb{R}, (t \in T,\ s \in S)$ functions.

We consider the bi-criteria optimization problem $\left(P_0^{0,0}\right)$, defined as:

$$\begin{cases} \min\left(f_1, f_2\right)(x) \\ x = (x_1, x_2, ...x_n) \in X \\ g_t(x) \leq 0,\ t \in T \\ h_s(x) = 0,\ s \in S. \end{cases}$$

Assuming that functions f_1, f_2, are differentiable of order $i,\ j \in \{1,\ 2\}$ and functions $g_t,\ (t \in T),\ h_s,\ (s \in S)$ are second order differentiable, we will approximate original problem $\left(P_0^{0,0}\right)$ by problems $\left(P_2^{i,j}\right)$:

$$\begin{cases} \min\left(F_1^i, F_2^j\right)(x) \\ x = (x_1, x_2, ...x_n) \in X \\ G_t^2(x) \leq 0,\ t \in T \\ H_s^2(x) = 0,\ s \in S \end{cases}$$

where $(i, j) \in \{(1,0),\ (1,1),\ (2,0),\ (2,1),\ (2,2)\}$ and $F_1^0 = f_1,\ F_2^0 = f_2$. We denote by

$$\mathcal{F}^k = \left\{x \in X :\ G_t^k(x) \leq 0,\ t \in T,\ H_s^k(x) = 0,\ s \in S\right\},\ k \in \{0,1,2\}$$

the set of feasible solutions for bi-criteria optimization problem $\left(P_k^{i,j}\right)$, where $(i, j) \in \{(1,0),\ (1,1),\ (2,0),\ (2,1),\ (2,2)\}$ and $k \in \{0,1,2\}$.

3. Approximate problems and relation to initial problem

In this section we will study the conditions such that efficient solution of approximated problems $\left(P_2^{1,0}\right)$, $\left(P_2^{2,0}\right)$, $\left(P_2^{2,1}\right)$ and $\left(P_2^{2,2}\right)$ will remain efficient also for original problem $\left(P_0^{0,0}\right)$ and reciprocally.

Case $\left(P_2^{1,1}\right)$ was studied in [1], where also conditions such that $\mathcal{F}^0 \subseteq \mathcal{F}^2$ and $\mathcal{F}^2 \subseteq \mathcal{F}^0$ were analyzed. We will use them in our work, so we will briefly present the Theorems stating these inclusions.

Theorem 3.1 (Boncea and Duca [1]**).** *Let X be a nonempty set of $\mathbb{R}^n$, x_0 an interior point of $X, \eta : X \times X \to X$, and $g_t,\ h_s : X \to \mathbb{R}, (t \in T,\ s \in S)$.*

Assume that:

- **a.** *for each $t \in T$, the function g_t is twice differentiable at x_0 and invex^2 at x_0 with respect to η,*
- **b.** *for each $s \in S$, the function h_s is twice differentiable at x_0 and avex^2 at x_0 with respect to η,*

then

$$\mathcal{F}^0 \subseteq \mathcal{F}^2.$$

Theorem 3.2 (Boncea and Duca [1]). *Let X be a nonempty set of $\mathbb{R}^n$, x_0 an interior point of $X, \eta : X \times X \to X$, and $g_t,\ h_s : X \to \mathbb{R}, (t \in T,\ s \in S)$.*

Assume that

- **a.** *for each $t \in T$, the function g_t is twice differentiable at x_0 and incave^2 at x_0 with respect to η,*
- **b.** *for each $s \in S$, the function h_s is twice differentiable at x_0 and avex^2 at x_0 with respect to η,*

then

$$\mathcal{F}^2 \subseteq \mathcal{F}^0.$$

Theorem 3.3. *Let X be a nonempty set of $\mathbb{R}^n$, x_0 an interior point of $X, \eta : X \times X \to X$, T and S index sets, $f = (f_1, f_2) : X \to \mathbb{R}^2$ and $g_t,\ h_s : X \to \mathbb{R}, (t \in T,\ s \in S)$ functions.*

Assume that:

- **a.** $x_0 \in \mathcal{F}^0$,
- **b.** *for each $t \in T$, the function g_t is twice differentiable at x_0 and invex^2 at x_0 with respect to η,*
- **c.** *for each $s \in S$, the function h_s is twice differentiable at x_0 and avex^2 at x_0 with respect to η,*
- **d.** *f_1 is twice differentiable at x_0 and invex^2 at x_0 with respect to η,*
- **e.** *f_2 is differentiable at x_0 and invex^1 at x_0 with respect to η,*
- **f.** $\eta(x_0,\ x_0) = 0$.

If x_0 is an efficient solution for $\left(P_2^{2,1}\right)$, then x_0 is an efficient solution for $\left(P_0^{0,0}\right)$.

Proof. x_0 being an efficient solution for $\left(P_2^{2,1}\right)$, implies that

$$\nexists x \in \mathcal{F}^2 \ s.t.\ \left(F_1^2(x), F_2^1(x)\right) \leq \left(F_1^2(x_0), F_2^1(x_0)\right).$$

Conditions b) and c) imply that

$$\mathcal{F}^0 \subseteq \mathcal{F}^2$$

and thus

$$\nexists x \in \mathcal{F}^0 \ s.t.\ \left(F_1^2(x), F_2^1(x)\right) \leq \left(F_1^2(x_0), F_2^1(x_0)\right). \tag{3.1}$$

Let's assume that x_0 is not an efficient solution for $\left(P_0^{0,0}\right)$. Then

$$\exists y \in \mathcal{F}^0 \ s.t.\ (f_1(y), f_2(y)) \leq (f_1(x_0), f_2(x_0))$$

which implies that $\exists y \in \mathcal{F}^0$ *s.t.*

$$\begin{cases} f_1(y) < f_1(x_0) \\ f_2(y) \leqq f_2(x_0) \end{cases} \tag{3.2}$$

or

$$\begin{cases} f_1(y) \leqq f_1(x_0) \\ f_2(y) < f_2(x_0). \end{cases} \tag{3.3}$$

Because f_1 is invex^2 at x_0 with respect to η we get $F_1^2(y) \leq f_1(y)$, $\forall y \in \mathcal{F}^0$.
Because f_2 is invex^1 at x_0 with respect to η we get $F_2^1(y) \leq f_2(y)$, $\forall y \in \mathcal{F}^0$.
Because $\eta(x_0\ x_0) = 0$ we get $f_1(x_0) = F_1^2(x_0)$ and $f_2(x_0) = F_2^1(x_0)$.
Thus from (3.2) we get that $\exists y \in \mathcal{F}^0$ *s.t.*

$$\begin{cases} F_1^2(y) < F_1^2(x_0) \\ F_2^1(y) \leqq F_2^1(x_0) \end{cases}$$

which contradicts (3.1) and from (3.3) we get that $\exists y \in \mathcal{F}^0$ *s.t.*

$$\begin{cases} F_1^2(y) \leqq F_1^2(x_0) \\ F_2^1(y) < F_2^1(x_0) \end{cases}$$

which contradicts (3.1).
In conclusion x_0 is an efficient solution for $\left(P_0^{0,0}\right)$. □

Theorem 3.4. *Let X be a nonempty set of $\mathbb{R}^n$, x_0 an interior point of $X, \eta : X \times X \to X$, T and S index sets, $f = (f_1, f_2) : X \to \mathbb{R}^2$ and $g_t,\ h_s : X \to \mathbb{R}, (t \in T,\ s \in S)$ functions.*

Assume that:

a. $x_0 \in \mathcal{F}^2$,
b. *for each $t \in T$, the function g_t is twice differentiable at x_0 and incave^2 at x_0 with respect to η,*
c. *for each $s \in S$, the function h_s is twice differentiable at x_0 and avex^2 at x_0 with respect to η,*
d. *f_1 is twice differentiable at x_0 and incave^2 at x_0 with respect to η,*
e. *f_2 is differentiable at x_0 and incave^1 at x_0 with respect to η,*
f. $\eta(x_0,\ x_0) = 0$.

If x_0 is an efficient solution for $\left(P_0^{0,0}\right)$, then x_0 is an efficient solution for $\left(P_2^{2,1}\right)$.

Proof. x_0 being an efficient solution for $\left(P_0^{0,0}\right)$, implies that

$$\nexists x \in \mathcal{F}^0 \ s.t.\ (f_1(x), f_2(x)) \leq (f_1(x_0), f_2(x_0)).$$

Conditions b) and c) imply that

$$\mathcal{F}^2 \subseteq \mathcal{F}^0$$

and thus

$$\nexists x \in \mathcal{F}^2 \ s.t.\ (f_1(x), f_2(x)) \leq (f_1(x_0), f_2(x_0)). \tag{3.4}$$

Let's assume that x_0 is not an efficient solution for $\left(P_2^{2,1}\right)$. Then

$$\exists y \in \mathcal{F}^2 \ s.t.\ \left(F_1^2(y), F_2^1(y)\right) \leq \left(F_1^2(x_0), F_2^1(x_0)\right)$$

which implies that $\exists y \in \mathcal{F}^2$ *s.t.*

$$\begin{cases} F_1^2(y) < F_1^2(x_0) \\ F_2^1(y) \leqq F_2^1(x_0) \end{cases} \tag{3.5}$$

or

$$\begin{cases} F_1^2(y) \leqq F_1^2(x_0) \\ F_2^1(y) < F_2^1(x_0). \end{cases} \tag{3.6}$$

Because f_1 is incave^2 at x_0 with respect to η we get $f_1(y) \leq F_1^2(y)$, $\forall y \in \mathcal{F}^2$.
Because f_2 is incave^1 at x_0 with respect to η we get $f_2(y) \leq F_2^1(y)$, $\forall y \in \mathcal{F}^2$.
Because $\eta(x_0\ x_0) = 0$ we get $f_1(x_0) = F_1^2(x_0)$ and $f_2(x_0) = F_2^1(x_0)$.
Thus from (3.5) we get that $\exists y \in \mathcal{F}^2$ *s.t.*

$$\begin{cases} f_1(y) < f_1(x_0) \\ f_2(y) \leqq f_2(x_0) \end{cases}$$

which contradicts (3.4) and from (3.6) we get that $\exists y \in \mathcal{F}^2$ *s.t.*

$$\begin{cases} f_1(y) \leqq f_1(x_0) \\ f_2(y) < f_2(x_0) \end{cases}$$

which contradicts (3.4).
In conclusion x_0 is an efficient solution for $\left(P_2^{2,1}\right)$. □

Theorem 3.5. *Let X be a nonempty set of $\mathbb{R}^n$, x_0 an interior point of $X, \eta : X \times X \to X$, T and S index sets, $f = (f_1, f_2) : X \to \mathbb{R}^2$ and $g_t,\ h_s : X \to \mathbb{R}, (t \in T,\ s \in S)$ functions.*

Assume that:

- **a.** $x_0 \in \mathcal{F}^0$,
- **b.** *for each $t \in T$, the function g_t is twice differentiable at x_0 and invex^2 at x_0 with respect to η,*
- **c.** *for each $s \in S$, the function h_s is twice differentiable at x_0 and avex^2 at x_0 with respect to η,*
- **d.** *f_1 is differentiable at x_0 and invex^1 at x_0 with respect to η,*
- **e.** $\eta(x_0,\ x_0) = 0$.

If x_0 is an efficient solution for $\left(P_2^{1,0}\right)$, then x_0 is an efficient solution for $\left(P_0^{0,0}\right)$.

Proof. Proof is similar with *Theorem* 3.3. □

Theorem 3.6. *Let X be a nonempty set of $\mathbb{R}^n$, x_0 an interior point of $X, \eta : X \times X \to X$, T and S index sets, $f = (f_1, f_2) : X \to \mathbb{R}^2$ and $g_t,\ h_s : X \to \mathbb{R}, (t \in T,\ s \in S)$ functions.*

Assume that:

- **a.** $x_0 \in \mathcal{F}^2$,
- **b.** *for each $t \in T$, the function g_t is twice differentiable at x_0 and incave^2 at x_0 with respect to η,*
- **c.** *for each $s \in S$, the function h_s is twice differentiable at x_0 and avex^2 at x_0 with respect to η,*
- **d.** *f_1 is differentiable at x_0 and incave^1 at $x_0$with respect to η,*
- **e.** $\eta(x_0,\ x_0) = 0$.

If x_0 is an efficient solution for $\left(P_0^{0,0}\right)$, then x_0 is an efficient solution for $\left(P_2^{1,0}\right)$.

Proof. Proof is similar with *Theorem* 3.4. □

Theorem 3.7. *Let X be a nonempty set of $\mathbb{R}^n$, x_0 an interior point of $X, \eta : X \times X \to X$, T and S index sets, $f = (f_1, f_2) : X \to \mathbb{R}^2$ and $g_t,\ h_s : X \to \mathbb{R}, (t \in T,\ s \in S)$ functions.*

Assume that:

- **a.** $x_0 \in \mathcal{F}^0$,
- **b.** *for each $t \in T$, the function g_t is twice differentiable at x_0 and invex^2 at x_0 with respect to η,*
- **c.** *for each $s \in S$, the function h_s is twice differentiable at x_0 and avex^2 at x_0 with respect to η,*
- **d.** *f_1 is twice differentiable at x_0 and invex^2 at x_0 with respect to η,*
- **e.** $\eta(x_0,\ x_0) = 0$.

If x_0 is an efficient solution for $\left(P_2^{2,0}\right)$, then x_0 is an efficient solution for $\left(P_0^{0,0}\right)$.

Proof. Proof is similar with *Theorem* 3.3. □

Theorem 3.8. *Let X be a nonempty set of $\mathbb{R}^n$, x_0 an interior point of $X, \eta : X \times X \to X$, T and S index sets, $f = (f_1, f_2) : X \to \mathbb{R}^2$ and $g_t,\ h_s : X \to \mathbb{R}, (t \in T,\ s \in S)$ functions.*

Assume that:

- **a.** $x_0 \in \mathcal{F}^2$,
- **b.** *for each $t \in T$, the function g_t is twice differentiable at x_0 and incave^2 at x_0 with respect to η,*
- **c.** *for each $s \in S$, the function h_s is twice differentiable at x_0 and avex^2 at x_0 with respect to η,*
- **d.** *f_1 is twice differentiable at x_0 and incave^2 at x_0 with respect to η,*
- **e.** $\eta(x_0,\ x_0) = 0$.

If x_0 is an efficient solution for $\left(P_0^{0,0}\right)$, then x_0 is an efficient solution for $\left(P_2^{2,0}\right)$.

Proof. Proof is similar with *Theorem* 3.4. □

Theorem 3.9. *Let X be a nonempty set of $\mathbb{R}^n$, x_0 an interior point of $X, \eta : X \times X \to X$, T and S index sets, $f = (f_1, f_2) : X \to \mathbb{R}^2$ and $g_t,\ h_s : X \to \mathbb{R}, (t \in T,\ s \in S)$ functions.*

Assume that:

- **a.** $x_0 \in \mathcal{F}^0$,
- **b.** *for each $t \in T$, the function g_t is twice differentiable at x_0 and invex^2 at x_0 with respect to η,*
- **c.** *for each $s \in S$, the function h_s is twice differentiable at x_0 and avex^2 at x_0 with respect to η,*
- **d.** *f_1 is twice differentiable at x_0 and invex^2 at x_0 with respect to η,*
- **e.** *f_2 is twice differentiable at x_0 and invex^2 at x_0 with respect to η,*
- **f.** $\eta(x_0,\ x_0) = 0$.

If x_0 is an efficient solution for $\left(P_2^{2,2}\right)$, then x_0 is an efficient solution for $\left(P_0^{0,0}\right)$.

Proof. Proof is similar with *Theorem* 3.3. □

Theorem 3.10. *Let X be a nonempty set of $\mathbb{R}^n$, x_0 an interior point of $X, \eta : X \times X \to X$, T and S index sets, $f = (f_1, f_2) : X \to \mathbb{R}^2$ and $g_t,\ h_s : X \to \mathbb{R}, (t \in T,\ s \in S)$ functions.*

Assume that:

- **a.** $x_0 \in \mathcal{F}^2$,
- **b.** *for each $t \in T$, the function g_t is twice differentiable at x_0 and incave^2 at x_0 with respect to η,*
- **c.** *for each $s \in S$, the function h_s is twice differentiable at x_0 and avex^2 at x_0 with respect to η,*
- **d.** *f_1 is twice differentiable at x_0 and incave^2 at x_0 with respect to η,*
- **e.** *f_2 is twice differentiable at x_0 and incave^2 at x_0 with respect to η,*
- **f.** $\eta(x_0,\ x_0) = 0$.

If x_0 is an efficient solution for $\left(P_0^{0,0}\right)$, then x_0 is an efficient solution for $\left(P_2^{2,2}\right)$.

Proof. Proof is similar with *Theorem* 3.4. □

4. Numerical examples

In the above theorems, conditions referring to invexity, incavity or avexity of functions are essential to ensure that efficient solution of the initial problem remains efficient for the approximate problem and reciprocally. If those conditions are not fulfill it is possible either that efficient solution of initial problem remains efficient for the approximate problem (and reciprocally) or it does not remain efficient.

Example 4.1. Let the initial bi-criteria optimization problem $\left(P_0^{0,0}\right)$ be:

$$\begin{cases} \min\left(-\left(x_1 - \frac{3\pi}{5}\right)^2 - \left(x_2 - \frac{2\pi}{5} - 1\right)^2; -x_1 + x_2\right) \\ -x_1 - \sin x_1 + x_2 \leq 0 \\ x_1 - \frac{5\pi}{2} \leq 0 \\ x_1; x_2 \geq 0 \end{cases}$$

An efficient solution of problem $\left(P_0^{0,0}\right)$ is $x_0 = (\frac{\pi}{2},\ 1 + \frac{\pi}{2}) \in \mathcal{F}^0$.
Second order approximate functions for the constraints are:

$$G_t^2(x) = g_t(x_0) + \nabla g_t(x_0)\,\eta(x, x_0) + \frac{1}{2}\eta(x, x_0)^T\,\nabla^2 g_t \eta(x, x_0), t \in \{1, 2, 3, 4\}$$

Considering $\eta(x, x_0) = x - x_0$ we get:

$$G_1^2(x) = -x_1 + x_2 + \frac{1}{2}\left(x_1 - \frac{\pi}{2}\right)^2 - 1,$$

$$G_2^2(x) = x_1 - \frac{5\pi}{2},$$

$$G_3^2(x) = x_1,\ G_4^2(x) = x_2.$$

Consequently, the approximate problem $\left(P_2^{0,0}\right)$ is:

$$\begin{cases} \min\left(-\left(x_1-\frac{3\pi}{5}\right)^2-\left(x_2-\frac{2\pi}{5}-1\right)^2;-x_1+x_2\right) \\ -x_1+x_2+\frac{1}{2}\left(x_1-\frac{\pi}{2}\right)^2-1\le 0 \\ x_1-\frac{5\pi}{2}\le 0 \\ x_1;x_2\ge 0 \end{cases}$$

Calculating the values of objective function for problem $\left(P_2^{0,0}\right)$ in

$$x_0=\left(\frac{\pi}{2},\ 1+\frac{\pi}{2}\right)\in\mathcal{F}^2 \text{ and } x=\left(\frac{3\pi}{4};\ \frac{3\pi}{4}+1-\frac{\pi^2}{32}\right)\in\mathcal{F}^2$$

we obtain:

$$f\left(\frac{3\pi}{4};\ \frac{3\pi}{4}+1-\frac{\pi^2}{32}\right)=\left(-\frac{58\pi^2}{400}+\frac{14\pi^3}{640}-\frac{\pi^4}{32};\ 1-\frac{\pi^2}{32}\right)$$

and

$$f\left(\frac{\pi}{2},\ 1+\frac{\pi}{2}\right)=\left(-\frac{\pi^2}{50},\ 1\right).$$

Because $(-\frac{58\pi^2}{400}+\frac{14\pi^3}{640}-\frac{\pi^4}{32};\ 1-\frac{\pi^2}{32})<(-\frac{\pi^2}{50},\ 1)$ it follows that $x_0=(\frac{\pi}{2},\ 1+\frac{\pi}{2})$ is not an efficient solution for approximate problem $\left(P_2^{0,0}\right)$.

Example 4.2. Let's consider the same initial problem as in *Example* 4.1. First order approximations for the components of the objective function are

$$F_p^1(x)=f_p(x_0)+\nabla f_p(x_0)\,\eta(x,x_0),\ p\in\{1,2\}.$$

Considering $\eta(x,x_0)=x-x_0$ we get:

$$F_1^1(x)=-\frac{\pi}{5}x_1-\frac{\pi}{5}x_2+\frac{9\pi^2}{50}+\frac{\pi}{5}$$

and

$$F_2^1(x)=-x_1+x_2.$$

Approximate functions for the constrains are the same computed at *Example* 4.1. Consequently the approximate problem $\left(P_2^{1,1}\right)$ is:

$$\begin{cases} \min\left(-\frac{\pi}{5}x_1-\frac{\pi}{5}x_2+\frac{9\pi^2}{50}+\frac{\pi}{5};\ -x_1+x_2\right) \\ -x_1+x_2+\frac{1}{2}\left(x_1-\frac{\pi}{2}\right)^2-1\le 0 \\ x_1-\frac{5\pi}{2}\le 0 \\ x_1;x_2\ge 0 \end{cases}$$

Calculating the values for the objective function of problem $\left(P_2^{1,1}\right)$ in

$$x_0=\left(\frac{\pi}{2},\ 1+\frac{\pi}{2}\right)\in\mathcal{F}^2 \text{ and } x=\left(\frac{3\pi}{4};\ \frac{3\pi}{4}+1-\frac{\pi^2}{32}\right)\in\mathcal{F}^2$$

we get that

$$F^1\left(\frac{3\pi}{4};\ \frac{3\pi}{4}+1-\frac{\pi^2}{32}\right) < F^1\left(\frac{\pi}{2},\ 1+\frac{\pi}{2}\right)$$

which proves that $x_0 = (\frac{\pi}{2},\ 1+\frac{\pi}{2})$ is not an efficient solution for problem $\left(P_2^{1,1}\right)$.

Example 4.3. Let's consider the same initial problem as in *Example* 4.1. Second order approximations for the components of the objective function are

$$F_p^2(x) = f_p(x_0) + \nabla f_p(x_0)\,\eta(x,x_0) + \frac{1}{2}\eta(x,x_0)^T\,\nabla^2 f_p(x_0)\,\eta(x,x_0),\ p\in\{1,2\}.$$

Considering $\eta(x,x_0) = x - x_0$ we get:

$$F_1^2(x) = -\frac{\pi}{2}\left(x_1-\frac{\pi}{2}\right)^2 - \frac{\pi+2}{2}\left(x_2-1-\frac{\pi}{2}\right)^2 - \frac{\pi}{5}x_1 - \frac{\pi}{5}x_2 + \frac{9\pi^2}{50} + \frac{\pi}{5}$$

and

$$F_2^2(x) = -x_1 + x_2.$$

Approximate functions for the constrains are the same computed at *Example* 4.1. Consequently the approximate problem $\left(P_2^{2,2}\right)$ is:

$$\begin{cases} \min\left(-\frac{\pi}{2}\left(x_1-\frac{\pi}{2}\right)^2 - \frac{\pi+2}{2}\left(x_2-1-\frac{\pi}{2}\right)^2 - \frac{\pi}{5}x_1 - \frac{\pi}{5}x_2 + \frac{9\pi^2}{50} + \frac{\pi}{5};\ -x_1+x_2\right) \\ -x_1 + x_2 + \frac{1}{2}\left(x_1-\frac{\pi}{2}\right)^2 - 1 \le 0 \\ x_1 - \frac{5\pi}{2} \le 0 \\ x_1; x_2 \ge 0 \end{cases}$$

Calculating the values for the objective function of problem $\left(P_2^{2,2}\right)$ in

$$x_0 = \left(\frac{\pi}{2},\ 1+\frac{\pi}{2}\right) \in \mathcal{F}^2 \text{ and } x = \left(\frac{3\pi}{4};\ \frac{3\pi}{4}+1-\frac{\pi^2}{32}\right) \in \mathcal{F}^2$$

we get that

$$F^2\left(\frac{3\pi}{4};\ \frac{3\pi}{4}+1-\frac{\pi^2}{32}\right) < F^2\left(\frac{\pi}{2},\ 1+\frac{\pi}{2}\right)$$

which proves that $x_0 = (\frac{\pi}{2},\ 1+\frac{\pi}{2})$ is not an efficient solution for problem $\left(P_2^{2,2}\right)$.

References

[1] Boncea, H., Duca, D.I., *On the $\eta-(1,2)$ approximated problems*, Carpathian J. Math., **28**(2012), no. 1, 17-24.

[2] Chankong, V., Haimes, Y., *Multiobjective Decision Making Theory and Methodology*, North-Holland, 1983.

[3] Chikalov, I., Hussain, S., Moshkov, M., *Bi-criteria optimization of decision trees with applications to data analysis*, European J. Oper. Res., **266**(2018), no. 2, 689-701.

[4] Konno, H., Yamazaki, H., *Mean absolute deviation portfolio optimization model and its applications to Tokyo Stock Market*, Manage. Sci., **37**(1991), no. 5, 519-531.

[5] Mahalov A., Luca T.I., *Minimax rule for energy optimization*, Computers & Fluids, **151**(2017), 35-45.

[6] Palacio, A., Adenso-Diaz, B., Lozano, S., Furio, S., *Bi-criteria optimization model for locating maritime container depots: application to the port of Valencia*, Netw. Spat. Econ., **16**(2016), no. 1, 331-348.

Traian Ionuţ Luca
Babeş-Bolyai University
Faculty of Business
7, Horea Street, Cluj-Napoca, Romania
e-mail: `ionut.luca@tbs.ubbcluj.ro`

Dorel I. Duca
Babeş-Bolyai University
Faculty of Mathematics and Computer Sciences
1, Kogălniceanu Street, 400084 Cluj-Napoca, Romania
e-mail: `dduca@math.ubbcluj.ro`

Stud. Univ. Babeş-Bolyai Math. 63(2018), No. 4, 561–562

Book reviews

Boris S. Mordukhovich; Variational Analysis and Applications,
Springer Monographs in Mathematics. Springer, Cham, 2018. xix+622 p.
ISBN: 978-3-319-92773-2/hbk; 978-3-319-92775-6/ebook.

Although variational principles in mathematical physics and mechanics were known since the 18th century, variational analysis, in its current acceptance, is a relatively new discipline. Its aim is to treat optimization and control problems via perturbations, approximations and generalized differentiation of nonsmooth or set-valued maps. As the author mentions in Preface, the first monograph dedicated to variational analysis in finite dimensions is that by R. T. Rockafellar and R. J.B. Wets, Springer 1998, where this very name was coined. The infinite-dimensional case is treated at large in the impressive two-volume monograph of the author, *Variational analysis and generalized differentiation.* I: *Basic theory* (579 p), II: *Applications* (610 p), Springer 2006 (a review of these volumes is published in vol. 52 (2007), no. 1, of the present journal).

The present book can be viewed as a companion to the two-volume monograph mentioned above. The first 6 chapters of the book are dedicated to a presentation of variational analysis in finite-dimensional spaces. This restriction allows to present simplified proofs (*ad usum Delphini*) of the main results, being accessible to graduate students in mathematics as well as to those in applied sciences and engineering. Each chapter is completed by a consistent section of exercises containing further results, including infinite dimensional ones. The most difficult of them are accompanied by hints or references.

The contents of this part is well illustrated by the headings of its chapters: 1. *Constructions of generalized differentiation*; 2. *Fundamental principles of variational analysis*; 3. *Well-posedness and coderivative calculus*; 4. *First-order subdifferential calculus*; 5. *Coderivatives of maximal monotone operators*; 6. *Nondifferentiable and bilevel optimization.*

The second part of the book, Chapters 7 to 10, is dedicated to applications of variational analysis to optimization and economics (in Ch. 10. *Set-valued optimization and economics*), and to other domains. Here the topics are treated in full infinite-dimensional generality, being addressed to researchers, graduate students and practitioners. As the author mention in Preface:

> The results obtained demonstrate the strength of variational analysis and dual-space constructions in solving concrete problems that may not even be of a variational nature.

Again, each chapter ends with a large number of exercises. As the author mentions, the exercise sections (containing some open problems and conjectures as well) play a crucial role in the organization of the book, providing the reader with a handy reference source to the enormous material available in first-order variational analysis, as well as with ideas for further research and developments.

Besides exercises, each chapter ends with a consistent section of *Commentaries*, containing references for the results included in the chapter or to other related results.

The book is very well organized – besides the *Subject Index*, it contains a *List of Statements* and a *Glossary of Notations and Acronyms.* The rich bibliography counts 790 items.

In conclusion, written by an expert in the areas of variational analysis and optimization and based on his didactic experience, this is an excellent textbook. By the wealth of information contained in the second part of the book and in exercises, it can be also used by researchers in optimization theory and its applications as a reference text.

S. Cobzaş