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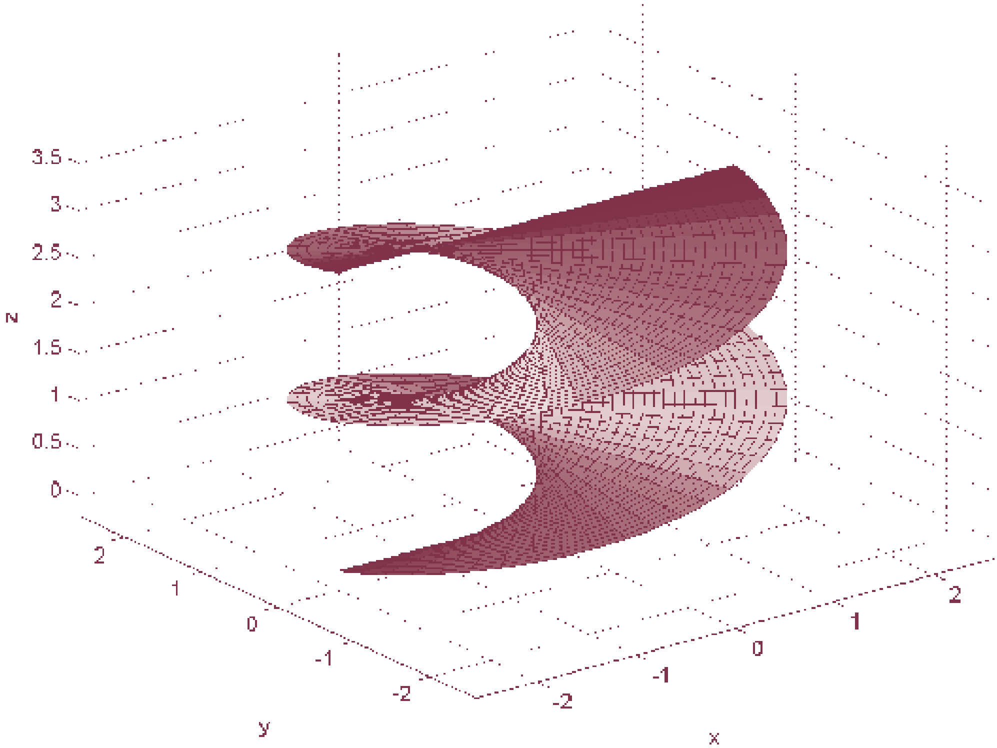

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Existence and stability of Langevin equations with two Hilfer-Katugampola fractional derivatives

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Abstract. In this note, we debate the existence, uniqueness and stability results for a general class of Langevin equations. We suggest the generalization via the Hilfer-Katugampola fractional derivative. We introduce some conditions for existence and uniqueness of solutions. We utilize the concept of fixed point theorems (Krasnoselskii fixed point theorem (KFPT), Banach contraction principle (BCP)). Moreover, we illustrate definitions of the Ulam type stability. These definitions generalize the fractional Ulam stability.

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Keywords: Fractional calculus, fractional differential operator, fractional differential equation, Ulam stability.

1. Introduction

The field of an arbitrary calculus (fractional calculus) is the extension of the ordinary calculus of fractional powers. It plays a significant field in the mathematical analysis. In addition, it is more than three centuries old, yet it only receives much attention and interest in last three decades [7, 17, 20]. The Langevin equation describes the stochastic problem in many fluctuating situations. A modified type of this equation used in various functional approaches of fractal mediums. Another modification requires replacing of ordinary differential equations into fractional differential equations (FDE), which yields the fractional Langevin equation [2, 4, 5, 6, 3, 21].

In recent times, Katugampola [13] introduced a new fractional differential operator which studied extensively by many researchers [14, 15, 25, 26]. Moreover, this operator has been compounded with Hilfer fractional differential operator introduced by Hilfer [7] to develop a new fractional differential operator, so called Hilfer-Katugampola fractional differential operator [19]. For the wide knowledge of fractional differential operators, one can refer to [22, 23, 24]. Rassias imposed the Hyers-Ulam stability (UHS) for both cases linear and nonlinear studies. This outcome of Rassias attracted many investigators worldwide who began to be motivated to study the stability problems of differential equations [1, 12, 18, 29]. The fractional Ulam stability (FUS) introduced by Wang [28], [27] and Ibrahim [8]-[11]. In our investigation, we focus on the following fractional differential equation containing two Hilfer-Katugampola fractional differential operators

$$\begin{cases} {}^{\rho}D^{\alpha_1,\beta}\left({}^{\rho}D^{\alpha_2,\beta}+\lambda\right)x(t)=f(t,x(t)), & t\in J:=(a,b], \\ I^{1-\gamma}x(a)=x_a, \quad \gamma=(\alpha_1+\alpha_2)(1-\beta)+\beta. \end{cases} \tag{1.1}$$

where ${}^{\rho}D^{\alpha_1,\beta}$ and ${}^{\rho}D^{\alpha_2,\beta}$ is Hilfer-Katugampola fractional differential operator of orders α_1 and α_2 and type β, $\rho>0$ and λ is any real number. Let $f:J\times R\to R$ is given continuous function.

The effort is systematic as follows: In Section 2, we submit preliminaries that utilized throughout the paper. In Section 3, we set up the existence and uniqueness for a special formula of multi-power FDE covering the Hilfer-Katugampola fractional differential operator. In Section 4, we discuss some types of fractional Ulam stability.

2. Preliminaries

Some basic definitions and results introduced in the recent section. The following observations selected from [17, 14, 19]. Let $C[a,b]$ be a space of all continuous functions subject to the sup. norm$\|\psi\|=\sup\{|\psi(t)|:t\in J\}$. The weighted space $C_{\gamma,\rho}[a,b]$ of functions f on $(a,b]$ is defined by

$$C_{\gamma,\rho}[a,b]=\left\{f:(a,b]\to R:\left(\frac{t^{\rho}-a^{\rho}}{\rho}\right)^{\gamma}f(x)\in C[a,b]\right\},0\le\gamma<1,$$

with the norm

$$\|g\|_{C_{\gamma,\rho}}=\left\|\left(\frac{t^{\rho}-a^{\rho}}{\rho}\right)^{\gamma}f(x)\right\|_{C}=\max_{t\in J}\left|\left(\frac{t^{\rho}-a^{\rho}}{\rho}\right)^{\gamma}f(x)\right|,C_{0,\rho}[a,b]=C[a,b].$$

Let $\delta_\rho=\left(t^{\rho}\frac{d}{dt}\right)$ and for $n\in N$, the notion $C^n_{\delta_{\rho,\gamma}}[a,b]$, be the Banach space of all functions f which are continuously differentiable. Suppose that the operator δ_ρ, is on $[a,b]$ of $(n-1)-$order and the derivative $\delta^n_\rho f$ of $n-$order on $(a,b]$ such that $\delta^n_\rho f\in C_{\gamma,\rho}[a,b]$. This leads to

$$C^n_{\delta_{\rho,\gamma}}[a,b]=\left\{\delta^k_\rho f\in C[a,b],\delta^n_\rho f\in C_{\gamma,\rho}[a,b],k=0,1,...,n-1\right\}$$

with the norm

$$\|f\|_{C^n_{\delta_{\rho,\gamma}}}=\sum_{k=0}^{n-1}\left\|\delta^k_\rho f\right\|_C+\left\|\delta^n_\rho f\right\|_{C_{\gamma,\rho}},\quad \|f\|_{C^n_{\delta_\rho}}=\sum_{k=0}^{n}\max_{x\in R}\left|\delta^k_\rho f(x)\right|.$$

For $n=0$, we have

$$C^0_{\delta_{\rho,\gamma}}[a,b]=C_{\gamma,\rho}[a,b].$$

Definition 2.1. Let $\alpha, c \in R$ with $\alpha > 0$ and $f \in X_c^p(a,b)$, where $f \in X_c^p(a,b)$ consists of the Lebesgue measurable functions. The generalized left-sided fractional integral ${}^\rho I_{a+}^\alpha f$ of order $\alpha \in C(\Re(\alpha))$ is defined by

$$\left({}^\rho I_{a+}^\alpha f\right)(t) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_a^t (t^\rho - s^\rho)^{\alpha-1} s^{\rho-1} f(s) ds, \ t > a. \tag{2.1}$$

The extended fractional derivative analog to the extended fractional integral (2.1), is given by

$$\left({}^\rho D_{a+}^\alpha f\right)(t) = \frac{\rho^{\alpha-n-1}}{\Gamma(n-\alpha)} \left(t^{1-\rho}\frac{d}{dt}\right)^n \int_a^t (t^\rho - s^\rho)^{n-\alpha+1} s^{\rho-1} f(s) ds. \tag{2.2}$$

Definition 2.2. The Hilfer-Katugampola fractional operator with respect to t, of order $\rho > 0$, is defined by

$$\begin{aligned} \left({}^\rho D_{a\pm}^{\alpha,\beta} f\right)(t) &= \left(\pm {}^\rho I_{a\pm}^\alpha \left(t^{\rho-1}\frac{d}{dt}\right) {}^\rho I_{a\pm}^{(1-\beta)(1-\alpha)}\right)(t) \\ &= \left(\pm {}^\rho I_{a\pm}^\alpha \delta_\rho {}^\rho I_{a\pm}^{(1-\beta)(1-\alpha)}\right)(t). \end{aligned} \tag{2.3}$$

- The operator ${}^\rho D_{a+}^{\alpha,\beta}$ can be written as

$${}^\rho D_{a+}^{\alpha,\beta} = {}^\rho I_{a+}^{\beta(1-\alpha)} \delta_\rho {}^\rho I_{a+}^{1-\gamma} = {}^\rho I_{a+}^{\beta(1-\alpha)} {}^\rho D_{a+}^{\gamma}, \quad \gamma = \alpha + \beta - \alpha\beta.$$

- The fractional derivative ${}^\rho D_{a+}^{\alpha,\beta}$ is considered as interpolation, with the convenient parameters, of the following fractional derivatives, Hilfer fractional operator when $\rho \to 1$, Hilfer-Hadamard operator when $\rho \to 0$, generalized fractional operator when $\beta = 0$, Caputo-type fractional derivative when $\beta = 1$, Riemann-Liouville fractional derivative when $\beta = 0, \rho \to 1$, Hadamard operator when $\beta = 0, \rho \to 0$, Caputo operator when $\beta = 1, \rho \to 1$, Caputo-Hadamard operator when $\beta = 1, \rho \to 0$, Liouville fractional derivative when $\beta = 0, \rho \to 1, a = 0$ and Hadamard fractional derivative when $\beta = 0, \rho \to 1, a = -\infty$. We consider the following parameters $\alpha, \beta, \gamma, \mu,$:

$$\gamma = \alpha + \beta - \alpha\beta, \ , \ 0 \le \mu < 1, \ \alpha > 0, \ \beta < 1, 0 \le \gamma < 1.$$

The following results can be found in [19]:

Lemma 2.3. *Let $\alpha, \beta > 0$, $0 < a < b < \infty$, $\rho, c \in R$, $1 \le p \le \infty$ and $\rho \ge c$. Then, for $f \in X_c^p(a,b)$ the semi group property is valid. This is,*

$$({}^\rho I_{a+}^\alpha {}^\rho I_{a+}^\beta f)(x) = ({}^\rho I_{a+}^{\alpha+\beta})(x),$$

and

$$({}^\rho D_{a+}^\alpha {}^\rho I_{a+}^\alpha f)(x) = f(x).$$

Lemma 2.4. *Assume that $x > a$, ${}^\rho I^\alpha_{a+}$ and ${}^\rho D^\alpha_{a+}$ are according on Eq.* (2.1) *and* (2.2)*, respectively. Then*

$$ {}^\rho I^\alpha_{a+}\left(\frac{t^\rho - a^\rho}{\rho}\right)^{\beta-1}(x) = \left(\frac{x^\rho - a^\rho}{\rho}\right)^{\alpha+\beta-1}\frac{\Gamma(\beta)}{\Gamma(\alpha+\beta)}, \alpha \geq 0, $$
$$ {}^\rho D^\alpha_{a+}\left(\frac{t^\rho - a^\rho}{\rho}\right)^{\beta-1}(x) = 0, \quad \alpha \in (0,1),\ \beta \in (0,\infty). $$

Lemma 2.5. *Let $0 < \alpha < 1$, $0 \leq \gamma < 1$. If $f \in C_\gamma[a,b]$ and ${}^\rho I^{1-\alpha}_{a+} f \in C^1_\gamma[a,b]$, then*

$$ \left({}^\rho I^\alpha_{a+}\, {}^\rho D^\alpha_{a+} f\right)(x) = f(x) - \frac{\left({}^\rho I^{1-\alpha}_{a+} f\right)(a)}{\Gamma(\alpha)}\left(\frac{x^\rho - a^\rho}{\rho}\right)^{\alpha-1}, $$

for all $x \in (a,b]$.

Lemma 2.6. *Let $0 < a < b < \infty$, $\alpha > 0$, $0 \leq \gamma < 1$ and $f \in C_{\gamma,\rho}[a,b]$. If $\alpha > \gamma$, then ${}^\rho I^\alpha_{a+} f$ is continuous on $[a,b]$ and*

$$ \left({}^\rho I^\alpha_{a+} f\right)(a) = \lim_{t\to a^+}\left({}^\rho I^\alpha_{a+} f\right)(t) = 0. $$

We present some spaces as follows:

$$ C^{\alpha,\beta}_{1-\gamma,\rho}[a,b] = \left\{ f \in C_{1-\gamma,\rho}[a,b], {}^\rho D^{\alpha,\beta}_{a+} f \in C_{\mu,\rho}[a,b] \right\} $$

and

$$ C^{\gamma}_{1-\gamma,\rho}[a,b] = \left\{ f \in C_{1-\gamma,\rho}[a,b], {}^\rho D^{\gamma}_{a+} f \in C_{1-\gamma,\rho}[a,b] \right\}. $$

Clearly, we have

$$ C^{\gamma}_{1-\gamma,\rho}[a,b] \subset C^{\alpha,\beta}_{1-\gamma,\rho}[a,b]. $$

Lemma 2.7. *If $C^\gamma_{1-\gamma}[a,b]$, then*

$$ {}^\rho I^\gamma_{a+}\, {}^\rho D^\gamma_{a+} f = {}^\rho I^\alpha_{a+}\, {}^\rho D^{\alpha,\beta}_{a+} f \tag{2.4} $$

and

$$ {}^\rho D^\gamma_{a+}\, {}^\rho I^\alpha_{a+} f = {}^\rho D^{\beta(1-\alpha)}_{a+} f. \tag{2.5} $$

Lemma 2.8. *If ${}^\rho D^{\beta(1-\alpha)}_{a+} f$ exists on $L^1(a,b)$, then*

$$ {}^\rho D^{\alpha,\beta}_{a+}\, {}^\rho I^\alpha_{a+} f = {}^\rho I^{\beta(1-\alpha)}_{a+}\, {}^\rho D^{\beta(1-\alpha)}_{a+} f. $$

Lemma 2.9. *Let $0 < \alpha < 1$, $0 \leq \beta \leq 1$ and $\gamma = \alpha + \beta - \alpha\beta$. If $f \in C_{1-\gamma}[a,b]$ and ${}^\rho I^{1-\beta(1-\alpha)}_{a+} \in C^1_{1-\gamma}[a,b]$, then ${}^\rho D^{\alpha,\beta}_{a+} I^\alpha_{a+}$ exists on $(a,b]$ and*

$$ {}^\rho D^{\alpha,\beta}_{a+} I^\alpha_{a+} f = f. $$

Lemma 2.10. [19] *Let $\gamma = \alpha + \beta - \alpha\beta$. If $f : (a,b] \times R \to R$ is a function such that $f(\cdot, x(\cdot)) \in C_{1-\gamma,\rho}[a,b]$ for all $x \in C_{1-\gamma,\rho}[a,b]$ then a function $x \in C^\gamma_{1-\gamma,\rho}[a,b]$ is the outcome of the problem*

$$ \begin{cases} {}^\rho D^{\alpha_1,\beta}_{a+}\left({}^\rho D^{\alpha_2,\beta}_{a+} + \lambda\right)x(t) = f(t,x(t)),\ t \in (a,b], \\ {}^\rho I^{1-\gamma}_{a_+} x(a) = x_a, \end{cases} $$

if and only if x achieves the following formula:

$$x(t) = \frac{x_a}{\Gamma(\gamma)}\left(\frac{t^\rho - a^\rho}{\rho}\right)^{\gamma-1} - \frac{\lambda}{\Gamma(\alpha_2)}\int_a^t \left(\frac{t^\rho - s^\rho}{\rho}\right)^{\alpha_2-1} s^{\rho-1}x(s)ds + \frac{1}{\Gamma(\alpha_1+\alpha_2)}\int_a^t \left(\frac{t^\rho - s^\rho}{\rho}\right)^{\alpha_1+\alpha_2-1} s^{\rho-1}f(s,x(s))ds.$$

The proof of the lemma is similar ([30], Lemma 3.1).

Theorem 2.11. *(KFPT) Suppose that Σ is a Banach space, Θ is a closed, bounded and convex subset of Σ and two functions $\Gamma_1, \Gamma_2 : \Theta \to \Sigma$ such that $\Gamma_1\chi + \Gamma_2\eta \in \Theta$ for all $\chi, \eta \in \Theta$. If Γ_1 is a contraction function and Γ_2 is completely continuous, then $\Gamma_1\chi + \Gamma_2\chi = \chi$ admits a solution in Θ.*

Theorem 2.12. *(Arzela-Ascoli theorem)* [16] *A subset F of $C(X)$ is relatively compact if and only if it is closed, bounded and equicontinuous.*

3. Existence and uniqueness results

For our setting, we deliver the following assumptions:

(H1) Let $f(\cdot, x(\cdot)) \in C^{\beta(1-\alpha)}_{1-\gamma,\rho}[a,b]$ for any $x \in C_{1-\gamma,\rho}[a,b]$. There exists a positive constant ℓ such that

$$|f(t,\chi) - f(t,\eta)| \le \ell\,|\chi - \eta|, \quad \text{for all } \chi, \eta \in R.$$

(H2) The constant

$$\Omega = \left(\frac{\ell B(\gamma, \alpha_1+\alpha_2)}{\Gamma(\alpha_1+\alpha_2)}\left(\frac{b^\rho - a^\rho}{\rho}\right)^{\alpha_1+\alpha_2} + \frac{\lambda B(\gamma,\alpha_2)}{\Gamma(\alpha_2)}\left(\frac{b^\rho - a^\rho}{\rho}\right)^{\alpha_2}\right) < 1.$$

(H3) There exist a nondecreasing function $\varphi : J \to R^+$ and $\lambda_\varphi > 0$ such that for $t \in J$,

$${}^\rho I^{\alpha_1+\alpha_2}_{a+}\varphi(t) \le \lambda_\varphi \varphi(t).$$

By applying Theorem 2.11, we have the following result:

Theorem 3.1. *(Existence) Suppose that [H1] and [H2] are achieved. Then, Eq.* (1.1) *admits at least one outcome in $C^\gamma_{1-\gamma,\rho}[a,b] \subset C^{\alpha,\beta}_{1-\gamma,\rho}[a,b]$.*

Proof. Define the operator $N : C_{1-\gamma,\rho}[a,b] \to C_{1-\gamma,\rho}[a,b]$, it is well defined and given by

$$(Nx)(t) = \begin{cases} \frac{x_a}{\Gamma(\gamma)}\left(\frac{t^\rho - a^\rho}{\rho}\right)^{\gamma-1} - \frac{\lambda}{\Gamma(\alpha_2)}\int_a^t \left(\frac{t^\rho - s^\rho}{\rho}\right)^{\alpha_2-1} s^{\rho-1}x(s)ds \\ +\frac{1}{\Gamma(\alpha_1+\alpha_2)}\int_a^t \left(\frac{t^\rho - s^\rho}{\rho}\right)^{\alpha_1+\alpha_2-1} s^{\rho-1} f(s,x(s))ds \end{cases} \tag{3.1}$$

Set $\widetilde{f}(s) = f(s,0)$ and

$$\omega = \left(\frac{b^\rho - a^\rho}{\rho}\right)^{\alpha_1+\alpha_2} \frac{B(\gamma, \alpha_1+\alpha_2)}{\Gamma(\alpha_1+\alpha_2)} \|f\|_{C_{1-\gamma,\rho}} + \frac{x_a}{\Gamma(\gamma)}$$

Consider the ball $B_r = \left\{\chi \in C_{1-\gamma,\rho}[a,b] : \|\chi\|_{C_{1-\gamma,\rho}} \leq r\right\}$.
Now we subdivide the operator N into two operator A and B on B_r as follows:

$$(Ax)(t) = \frac{1}{\Gamma(\alpha_1+\alpha_2)} \int_a^t \left(\frac{t^\rho - s^\rho}{\rho}\right)^{\alpha_1+\alpha_2-1} s^{\rho-1} f(s,x(s))ds$$

and

$$(Bx)(t) = \frac{x_a}{\Gamma(\gamma)} \left(\frac{t^\rho - a^\rho}{\rho}\right)^{\gamma-1} - \frac{\lambda}{\Gamma(\alpha_2)} \int_a^t \left(\frac{t^\rho - s^\rho}{\rho}\right)^{\alpha_2-1} s^{\rho-1} x(s)ds.$$

The proof is as follows:
Step 1. $Ax + By \in B_r$ for every $x, y \in B_r$.

$$\begin{aligned}
&\left|(Ax)(t)\left(\frac{t^\rho - a^\rho}{\rho}\right)^{1-\gamma}\right| \\
&\leq \left(\frac{t^\rho - a^\rho}{\rho}\right)^{1-\gamma} \frac{1}{\Gamma(\alpha_1+\alpha_2)} \int_a^t \left(\frac{t^\rho - s^\rho}{\rho}\right)^{\alpha_1+\alpha_2-1} s^{\rho-1} \times f(s,x(s))ds \\
&\leq \left(\frac{t^\rho - a^\rho}{\rho}\right)^{1-\gamma} \frac{1}{\Gamma(\alpha_1+\alpha_2)} \int_a^t \left(\frac{t^\rho - s^\rho}{\rho}\right)^{\alpha_1+\alpha_2-1} s^{\rho-1} \left(|f(s,x(s)) - f(s,0)| + |f(s,0)|\right) ds \\
&\leq \left(\frac{t^\rho - a^\rho}{\rho}\right)^{1-\gamma} \frac{1}{\Gamma(\alpha_1+\alpha_2)} \int_a^t \left(\frac{t^\rho - s^\rho}{\rho}\right)^{\alpha_1+\alpha_2-1} s^{\rho-1} \left(\ell\,|x(s)| + \left|\widetilde{f}(s)\right|\right) ds \\
&\leq \left(\frac{t^\rho - a^\rho}{\rho}\right)^{1-\gamma} \frac{B(\gamma,\alpha_1+\alpha_2)}{\Gamma(\alpha_1+\alpha_2)} \left(\frac{t^\rho - a^\rho}{\rho}\right)^{\alpha_1+\alpha_2+\gamma-1} \left(\ell\,\|x\|_{C_{1-\gamma}} + \left\|\widetilde{f}\right\|_{C_{1-\gamma,\rho}}\right).
\end{aligned}$$

This gives

$$\|Ax\|_{C_{1-\gamma,\rho}} \leq \frac{B(\gamma,\alpha_1+\alpha_2)}{\Gamma(\alpha_1+\alpha_2)} \left(\frac{b^\rho - a^\rho}{\rho}\right)^{\alpha_1+\alpha_2} \left(\ell\,\|x\|_{C_{1-\gamma,\rho}} + \left\|\widetilde{f}\right\|_{C_{1-\gamma,\rho}}\right). \tag{3.2}$$

For operator B

$$\begin{aligned}
&\left|(Bx)(t)\left(\frac{t^\rho - a^\rho}{\rho}\right)^{1-\gamma}\right| \\
&\leq \frac{x_a}{\Gamma(\gamma)} + \left(\frac{t^\rho - a^\rho}{\rho}\right)^{1-\gamma} \frac{\lambda}{\Gamma(\alpha_2)} \int_a^t \left(\frac{t^\rho - s^\rho}{\rho}\right)^{\alpha_2-1} s^{\rho-1} x(s)ds \\
&\leq \frac{x_a}{\Gamma(\gamma)} + \left(\frac{t^\rho - a^\rho}{\rho}\right)^{1-\gamma} \frac{\lambda B(\gamma,\alpha_2)}{\Gamma(\alpha_2)} \left(\frac{t^\rho - a^\rho}{\rho}\right)^{\alpha_2+\gamma-1} \|x\|_{C_{1-\gamma,\rho}}.
\end{aligned}$$

Thus, we obtain

$$\|(Bx)\|_{C_{1-\gamma}} \leq \frac{x_a}{\Gamma(\gamma)} + \frac{\lambda B(\gamma,\alpha_2)}{\Gamma(\alpha_2)} \left(\frac{b^\rho - a^\rho}{\rho}\right)^{\alpha_2} \|x\|_{C_{1-\gamma,\rho}}. \tag{3.3}$$

Linking (3.2) and (3.3), for every $x, y \in B_r$, we get

$$\|Ax + By\|_{C_{1-\gamma,\rho}} \leq \|Ax\|_{C_{1-\gamma,\rho}} + \|By\|_{C_{1-\gamma,\rho}} \leq \Omega r + \omega.$$

Step 2. A is a contraction mapping.
For any $x, y \in B_r$, we observe the conclusion

$$\left|((Ax)(t) - (Ay)(t))\left(\frac{t^\rho - a^\rho}{\rho}\right)^{1-\gamma}\right|$$
$$\leq \left(\frac{t^\rho - a^\rho}{\rho}\right)^{1-\gamma} \frac{1}{\Gamma(\alpha_1+\alpha_2)} \int_a^t \left(\frac{t^\rho - s^\rho}{\rho}\right)^{\alpha_1+\alpha_2-1} s^{\rho-1} |f(s,x(s)) - f(s,y(s))|\, ds$$
$$\leq \left(\frac{t^\rho - a^\rho}{\rho}\right)^{1-\gamma} \frac{\ell}{\Gamma(\alpha_1+\alpha_2)} \left(\frac{t^\rho - a^\rho}{\rho}\right)^{\alpha_1+\alpha_2+\gamma-1} \|x-y\|_{C_{1-\gamma,\rho}}.$$

This gives

$$\|(Ax) - (Ay)\| \leq \frac{\ell B(\gamma, \alpha_1+\alpha_2)}{\Gamma(\alpha_1+\alpha_2)} \left(\frac{b^\rho - a^\rho}{\rho}\right)^{\alpha_1+\alpha_2} \|x-y\|_{C_{1-\gamma,\rho}}.$$

In view of [H2], the operator A is a contraction mapping.

Step 3. The operator B is completely compact.
According to Step 1, we know that

$$\|(Bx)\|_{C_{1-\gamma,\rho}} \leq \frac{x_a}{\Gamma(\gamma)} + \frac{\lambda B(\gamma,\alpha_2)}{\Gamma(\alpha_2)} \left(\frac{b^\rho - a^\rho}{\rho}\right)^{\alpha_2} \|x\|_{C_{1-\gamma,\rho}}.$$

Thus, the operator B is uniformly bounded. Next, we show that the operator B is compact. A calculation implies

$$|(Bx)(t_1) - (Bx)(t_2)| \leq \frac{x_a}{\Gamma(\gamma)} \left| \left(\frac{t_1^\rho - a^\rho}{\rho}\right)^{\gamma-1} - \left(\frac{t_2^\rho - a^\rho}{\rho}\right)^{\gamma-1} \right|$$
$$+ \frac{\ell B(\gamma,\alpha_2)}{\Gamma(\alpha_2)} \|x\|_{C_{1-\gamma,\rho}} \left| \left(\frac{t_1^\rho - a^\rho}{\rho}\right)^{\alpha_2+\gamma-1} - \left(\frac{t_2^\rho - a^\rho}{\rho}\right)^{\alpha_2+\gamma-1} \right|,$$

which is tending to zero as $t_1 \to t_2$. Thus B is equicontinuous. Hence, in virtue of the Theorem 2.12, the operator B is compact on B_r. It leads by Krasnoselskii fixed point theorem, that the problem (1.1) admits a solution. □

Theorem 3.2. *If hypothesis (H1) and (H2) are fulfilled. Then, Eq.* (1.1) *admits a unique solution.*

4. Stability outcomes

In the recent section, we shall give the definitions and the criteria of (UHS) and (UHRS) for the generalized Langevin Eq. (1.1). Now for $\epsilon > 0$ and a continuous function $\varphi : J \to R^+$, we theorize the next inequalities:

$$\left| {}^\rho D_{a^+}^{\alpha_1,\beta} \left({}^\rho D_{a^+}^{\alpha_2,\beta} + \lambda \right) z(t) - f(t, z(t)) \right| \leq \epsilon, \quad t \in J, \tag{4.1}$$

$$\left| {}^\rho D_{a^+}^{\alpha_1,\beta} \left({}^\rho D_{a^+}^{\alpha_2,\beta} + \lambda \right) z(t) - f(t, z(t)) \right| \leq \epsilon\varphi(t), \quad t \in J, \tag{4.2}$$

$$\left|{}^{\rho}D_{a^+}^{\alpha_1,\beta}\left({}^{\rho}D_{a^+}^{\alpha_2,\beta}+\lambda\right)z(t)-f(t,z(t))\right|\leq\varphi(t),\quad t\in J. \tag{4.3}$$

Qualifier

The Eq. (1.1) is UHS if there occurs a number $C_f > 0$ and $\epsilon > 0$ such that for all outcome $z \in C_{1-\gamma,\rho}[a,b]$ of the inequality (4.1) there occurs an outcome $x \in C_{1-\gamma,\rho}[a,b]$ of Eq. (1.1) satisfying

$$|z(t)-x(t)|\leq C_f\ \epsilon,\quad t\in J.$$

The Eq. (1.1) is generalized UHS if there occurs a function $\varphi \in C_{1-\gamma,\rho}[a,b]$, $\varphi_f(0) = 0$ such that for all outcome $z \in C_{1-\gamma,\rho}[a,b]$ of the inequality (4.1) there occurs an outcome $x \in C_{1-\gamma,\rho}[a,b]$ of Eq. (1.1) achieving

$$|z(t)-x(t)|\leq \varphi_f\ \epsilon,\quad t\in J.$$

The Eq. (1.1) is UHRS esteeming by $\varphi \in C_{1-\gamma,\rho}[a,b]$ if there occurs a number $C_{f,\varphi} > 0$ for all $\epsilon > 0$ and for every outcome $z \in C_{1-\gamma,\rho}[a,b]$ of the inequality (4.2) there occurs an outcome $x \in C_{1-\gamma,\rho}[a,b]$ of Eq. (1.1) filing

$$|z(t)-x(t)|\leq C_{f,\varphi}\ \epsilon\varphi(t),\quad t\in J.$$

The Eq. (1.1) is generalized UHRS corresponding to $\varphi \in C_{1-\gamma,\rho}[a,b]$ if there occurs a real number $C_{f,\varphi} > 0$ whenever for every outcome $z \in C_{1-\gamma,\rho}[a,b]$ of the inequality (4.3) there occurs an outcome $x \in C_{1-\gamma,\rho}[a,b]$ of Eq. (1.1) satisfying

$$|z(t)-x(t)|\leq C_{f,\varphi}\varphi(t),\quad t\in J.$$

Remark 4.1. A function $z \in C_{1-\gamma,\rho}[a,b]$ is an outcome of the inequality (4.1) if and only if there exists a function $g \in C_{1-\gamma,\rho}[a,b]$ such that

$$\left|{}^{\rho}D_{a^+}^{\alpha_1,\beta}\left({}^{\rho}D_{a^+}^{\alpha_2,\beta}+\lambda\right)z(t)-f(t,z(t))\right|\leq\epsilon,\quad t\in J,$$

if and only if there occurs a function $g \in C_{1-\gamma,\rho}[a,b]$ such that

(i) $|g(t)| \leq \epsilon, t \in J$.
(ii) ${}^{\rho}D_{a^+}^{\alpha_1,\beta}\left({}^{\rho}D_{a^+}^{\alpha_2,\beta}+\lambda\right)z(t) = f(t,z(t)) + g(t),\ t \in J$.

Similarly, for the inequalities (4.2) and (4.3).

Remark 4.2. If z is an outcome of (4.1), then z is an outcome of the following formula:

$$\begin{aligned}&\Bigg|z(t)-\frac{z_a}{\Gamma(\gamma)}\left(\frac{t^\rho-a^\rho}{\rho}\right)^{\gamma-1}+\frac{\lambda}{\Gamma(\alpha_2)}\int_a^t\left(\frac{t^\rho-s^\rho}{\rho}\right)^{\alpha_2-1}s^{\rho-1}z(s)ds\\&-\frac{1}{\Gamma(\alpha_1+\alpha_2)}\int_a^t\left(\frac{t^\rho-s^\rho}{\rho}\right)^{\alpha_1+\alpha_2-1}s^{\rho-1}f(s,z(s))ds\Bigg|\\&\leq\frac{1}{\Gamma(\alpha_1+\alpha_2+1)}\left(\frac{b^\rho-a^\rho}{\rho}\right)^{\alpha_1+\alpha_2}\epsilon.\end{aligned}$$

It is clear that

$${}^{\rho}D_{a^+}^{\alpha_1,\beta}\left({}^{\rho}D_{a^+}^{\alpha_2,\beta}+\lambda\right)z(t)=\ f(t,z(t))+g(t),\ t\in J.$$

Then

$$z(t) = \frac{z_a}{\Gamma(\gamma)}\left(\frac{t^\rho - a^\rho}{\rho}\right)^{\gamma-1} - \frac{\lambda}{\Gamma(\alpha_2)}\int_a^t \left(\frac{t^\rho - s^\rho}{\rho}\right)^{\alpha_2-1} s^{\rho-1} z(s)ds$$
$$+ \frac{1}{\Gamma(\alpha_1+\alpha_2)}\int_a^t \left(\frac{t^\rho - s^\rho}{\rho}\right)^{\alpha_1+\alpha_2-1} s^{\rho-1}\left(f(s,z(s)) + g(s)\right)ds.$$

Consequently, we obtain

$$\left| z(t) - \frac{z_a}{\Gamma(\gamma)}\left(\frac{t^\rho - a^\rho}{\rho}\right)^{\gamma-1} + \frac{\lambda}{\Gamma(\alpha_2)}\int_a^t \left(\frac{t^\rho - s^\rho}{\rho}\right)^{\alpha_2-1} s^{\rho-1} z(s)ds \right.$$
$$\left. - \frac{1}{\Gamma(\alpha_1+\alpha_2)}\int_a^t \left(\frac{t^\rho - s^\rho}{\rho}\right)^{\alpha_1+\alpha_2-1} s^{\rho-1} f(s,z(s))ds \right|$$
$$\leq \frac{1}{\Gamma(\alpha_1+\alpha_2)}\int_a^t \left(\frac{t^\rho - s^\rho}{\rho}\right)^{\alpha_1+\alpha_2-1} s^{\rho-1}\left|g(s)\right| ds$$
$$\leq \frac{1}{\Gamma(\alpha_1+\alpha_2+1)}\left(\frac{b^\rho - a^\rho}{\rho}\right)^{\alpha_1+\alpha_2} \epsilon.$$

We have similar remarks for the inequality (4.2) and (4.3).

Our main result is as follows:

Theorem 4.3. *Suppose that the hypotheses [H1] and [H3] achieved. Then Eq.* (1.1) *is a generalized UHRS.*

Proof. Let z be a solution of 4.3. In view of Theorem 3.2, there x is a unique outcome of the problem satisfying

$$^{\rho}D_{a^+}^{\alpha_1,\beta}\left(^{\rho}D_{a^+}^{\alpha_2,\beta} + \lambda\right)x(t) = f(t,x(t)),$$
$$I_{a^+}^{1-\gamma}x(a) = I_{a^+}^{1-\gamma}z(a).$$

Then we have

$$x(t) = \frac{z_a}{\Gamma(\gamma)}\left(\frac{t^\rho - a^\rho}{\rho}\right)^{\gamma-1} - \frac{\lambda}{\Gamma(\alpha_2)}\int_a^t \left(\frac{t^\rho - s^\rho}{\rho}\right)^{\alpha_2-1} s^{\rho-1} x(s)ds$$
$$+ \frac{1}{\Gamma(\alpha_1+\alpha_2)}\int_a^t \left(\frac{t^\rho - s^\rho}{\rho}\right)^{\alpha_1+\alpha_2-1} s^{\rho-1} f(s,x(s))ds.$$

By differentiating inequality (4.3), we have

$$\left| z(t) - \frac{z_a}{\Gamma(\gamma)}\left(\frac{t^\rho - a^\rho}{\rho}\right)^{\gamma-1} + \frac{\lambda}{\Gamma(\alpha_2)}\int_a^t \left(\frac{t^\rho - s^\rho}{\rho}\right)^{\alpha_2-1} s^{\rho-1} z(s)ds \right.$$
$$\left. - \frac{1}{\Gamma(\alpha_1+\alpha_2)}\int_a^t \left(\frac{t^\rho - s^\rho}{\rho}\right)^{\alpha_1+\alpha_2-1} s^{\rho-1} f(s,z(s))ds \right|$$

$$\leq \left| \frac{1}{\Gamma(\alpha_1+\alpha_2)} \int_0^t \left(\frac{t^\rho - s^\rho}{\rho} \right)^{\alpha_1+\alpha_2-1} s^{\rho-1}\varphi(s)ds \right| \leq \lambda_\varphi \varphi(t).$$

Hence it follows that,

$$\begin{aligned}
&|z(t)-x(t)| \\
&\leq \left| z(t) - \frac{z_a}{\Gamma(\gamma)} \left(\frac{t^\rho - a^\rho}{\rho} \right)^{\gamma-1} + \frac{\lambda}{\Gamma(\alpha_2)} \int_a^t \left(\frac{t^\rho - s^\rho}{\rho} \right)^{\alpha_2-1} s^{\rho-1}x(s)ds \right. \\
&\left. - \frac{1}{\Gamma(\alpha_1+\alpha_2)} \int_a^t \left(\frac{t^\rho - s^\rho}{\rho} \right)^{\alpha_1+\alpha_2-1} s^{\rho-1} f(s,x(s))ds \right| \\
&\leq \left| z(t) - \frac{z_a}{\Gamma(\gamma)} \left(\frac{t^\rho - a^\rho}{\rho} \right)^{\gamma-1} + \frac{\lambda}{\Gamma(\alpha_2)} \int_a^t \left(\frac{t^\rho - s^\rho}{\rho} \right)^{\alpha_2-1} s^{\rho-1}z(s)ds \right. \\
&\left. - \frac{1}{\Gamma(\alpha_1+\alpha_2)} \int_a^t \left(\frac{t^\rho - s^\rho}{\rho} \right)^{\alpha_1+\alpha_2-1} s^{\rho-1} f(s,z(s))ds \right| \\
&+ \frac{\lambda}{\Gamma(\alpha_2)} \int_a^t \left(\frac{t^\rho - s^\rho}{\rho} \right)^{\alpha_2-1} s^{\rho-1} \left| x(s) - z(s) \right| ds \\
&+ \frac{1}{\Gamma(\alpha_1+\alpha_2)} \int_a^t \left(\frac{t^\rho - s^\rho}{\rho} \right)^{\alpha_1+\alpha_2-1} s^{\rho-1} \left| f(s,z(s)) - f(s,x(s)) \right\| ds \\
&\leq \lambda_\varphi \varphi(t) + \left(\frac{\lambda}{\Gamma(\alpha_2+1)} \left(\frac{b^\rho - a^\rho}{\rho} \right)^{\alpha_2} + \frac{\ell}{\Gamma(\alpha_1+\alpha_2+1)} \left(\frac{b^\rho - a^\rho}{\rho} \right)^{\alpha_1+\alpha_2} \right) |z-x|.
\end{aligned}$$

By Lemma 2.5, there occurs a constant $M^* > 0$ independent of $\lambda_\varphi \varphi(t)$, achieving

$$|z(t)-x(t)| \leq M^* \varphi(t).$$

Thus, Eq. (1.1) is generalized UHRS. □

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Applications of first order differential subordination for functions with positive real part

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Abstract. Several inclusions between the class of functions with positive real part and the class of starlike univalent functions associated with lemniscate of Bernoulli are obtained by making use of the well-known theory of differential subordination. Further, these inclusions give sufficient conditions for normalized analytic functions to belong to some subclasses of starlike functions. The results also provide sharp version of some previously known results.

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1. Introduction

Let $\mathcal{A}$ denote the class of analytic functions f on the disk $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ and normalized by the condition $f(0) = 0 = f'(0) - 1$. Let $\mathcal{S}$ be the subset of $\mathcal{A}$ of univalent functions. An analytic function f defined on $\mathbb{D}$ is subordinate to the analytic function g on $\mathbb{D}$ (or g is superordinate to f), if there exists an analytic function $w : \mathbb{D} \to \mathbb{D}$, with $w(0) = 0$, such that $f = g \circ w$. Furthermore, if g is univalent in $\mathbb{D}$, then $f \prec g$ is equivalent to $f(0) = g(0)$ and $f(\mathbb{D}) \subseteq g(\mathbb{D})$, see [15]. Let p be an analytic function in $\mathbb{D}$ with positive real part and $p(0) = 1$. The function $p(z) = (1 + z)/(1 - z)$ is a leading example of the function with positive real part such that $p(\mathbb{D})$ is the right-half plane. Goluzin [7] discussed the first order differential subordination $zp'(z) \prec zq'(z)$ and proved that, whenever $zq'(z)$ is convex, the subordination $p(z) \prec q(z)$ holds and the function q is the best dominant. After this basic result, many authors established several generalizations of first order differential subordination. The general theory of differential subordination is discussed in the monograph by Miller and Mocanu [14]. In 1989, Nunokawa *et al.* [16] proved that if subordination $1 + zp'(z) \prec 1 + z$ holds, then subordination $p(z) \prec 1 + z$ also holds.

In 2007, Ali *et al.* [2] extended this result and determined the estimates on β for which the subordination $1+\beta zp'(z)/p^j(z) \prec (1+Dz)/(1+Ez)$ $(j=0,1,2)$ implies the subordination $p(z) \prec (1+Az)/(1+Bz)$, where $A, B, D, E \in [-1,1]$. In 2013, Omar and Halim [17] determined the condition on β in terms of complex number D and real E with $-1 < E < 1$ and $|D| \leq 1$ such that $1+\beta zp'(z)/p^j(z) \prec (1+Dz)/(1+Ez)$ $(j=0,1,2)$ implies $p(z) \prec \sqrt{1+z}$. These results are not sharp. Recently, Kumar and Ravichandran [11] obtained sharp estimates on β for which the subordination $1+\beta zp'(z)/p^j(z)$ $(j=0,1,2)$ is subordinate to $\sqrt{1+z}, (1+Az)/(1+Bz)$ and some another functions with positive real part whenever the subordination $p(z) \prec e^z, (1+Az)/(1+Bz)$ holds. They further used these results to determine some sufficient conditions for the function $f \in \mathcal{A}$ to be in certain well-known subclasses of starlike functions. For more details, see [3, 5, 6, 20, 23, 25].
Motivated by all this work, we determine the sharp bound on β so that $p(z) \prec \mathcal{P}(z)$ where $\mathcal{P}(z)$ is a function with positive real part like $\sqrt{1+z}$, $(1+Az)/(1+Bz)$, e^z, $\varphi_s(z)$, $\varphi_q(z)$, $\varphi_0(z)$ and $\varphi_C(z)$, where $\varphi_0(z) := 1+\frac{z}{k}\left((k+z)/(k-z)\right)$ $(k=\sqrt{2}+1)$, $\varphi_s(z) := 1+\sin z$, $\varphi_c(z) := 1+\frac{4}{3}z+\frac{2}{3}z^2$ and $\varphi_q(z) := z+\sqrt{1+z^2}$, whenever $1+\beta zp'(z)/p^j(z) \prec \sqrt{1+z}$, $(j=0,1,2)$. Many of our subordination results in this paper improve the corresponding non-sharp results obtained by earlier authors in [1, 9, 13].

2. Subordination results

Our first result gives a bound on β so that $1+\beta zp'(z) \prec \sqrt{1+z}$ implies that the function p is subordinate to several well-known starlike functions.

Theorem 2.1. *Let the function p be analytic in $\mathbb{D}$, $p(0)=1$ and $1+\beta zp'(z) \prec \sqrt{1+z}$. Then the following subordination results hold:*

(a) *If $\beta \geq \frac{2(\sqrt{2}-1+\log 2-\log(1+\sqrt{2}))}{\sqrt{2}-1} \approx 1.09116$, then $p(z) \prec \sqrt{1+z}$.*
(b) *If $\beta \geq \frac{2(1-\log 2)}{3-2\sqrt{2}} \approx 3.57694$, then $p(z) \prec \varphi_0(z)$.*
(c) *If $\beta \geq \frac{2(1-\log 2)}{\sin(1)} \approx 0.729325$, then $p(z) \prec \varphi_s(z)$.*
(d) *If $\beta \geq (2+\sqrt{2})(1-\log 2) \approx 1.044766$, then $p(z) \prec \varphi_q(z)$.*
(e) *If $\beta \geq 3(1-\log 2) \approx 0.920558$, then $p(z) \prec \varphi_c(z)$.*
(f) *Let $-1 < B < A < 1$ and $B_0 = \frac{2-\log 4-\sqrt{2}+\log(1+\sqrt{2})}{\sqrt{2}-\log(1+\sqrt{2}+1)} \approx 0.151764$.*
If either
(i) *$B < B_0$ and $\beta \geq \frac{2(1-B)(1-\log 2)}{A-B} \approx 0.613706\frac{1-B}{A-B}$*
or
(ii) *$B > B_0$ and $\beta \geq \frac{2(1+B)(\sqrt{2}-1+\log 2-\log(1+\sqrt{2}))}{A-B} \approx 0.451974\frac{1+B}{A-B}$,*
then $p(z) \prec (1+Az)/(1+Bz)$.

The bounds on β are sharp.

The following lemma will be used in our investigations.

Lemma 2.2. [15, Theorem 3.4h, p. 132] *Let q be analytic in $\mathbb{D}$ and let ψ and ν be analytic in a domain U containing $q(\mathbb{D})$ with $\psi(w) \neq 0$ when $w \in q(\mathbb{D})$.*

Set $Q(z) := zq'(z)\psi(q(z))$ and $h(z) := \nu(q(z)) + Q(z)$. Suppose that (i) either h is convex, or Q is starlike univalent in $\mathbb{D}$ and (ii) $\mathrm{Re}\,(zh'(z)/Q(z)) > 0$ *for $z \in \mathbb{D}$. If p is analytic in $\mathbb{D}$, with $p(0) = q(0)$, $p(\mathbb{D}) \subseteq U$ and*

$$\nu(p(z)) + zp'(z)\psi(p(z)) \prec \nu(q(z)) + zq'(z)\psi(q(z)),$$

then $p(z) \prec q(z)$, and q is best dominant.

Proof of Theorem 2.1. The function $q_\beta : \overline{\mathbb{D}} \to \mathbb{C}$ defined by

$$q_\beta(z) = 1 + \frac{2}{\beta}(\sqrt{1+z} - \log(1 + \sqrt{1+z}) + \log 2 - 1).$$

is analytic and is a solution of the differential equation $1+\beta zq'_\beta(z) = \sqrt{1+z}$. Consider the functions $\nu(w) = 1$ and $\psi(w) = \beta$. The function $Q : \overline{\mathbb{D}} \to \mathbb{C}$ is defined by

$$Q(z) = zq'_\beta(z)\psi(q_\beta(z)) = \beta zq'_\beta(z).$$

Since $\sqrt{1+z} - 1$ is starlike function in $\mathbb{D}$, it follows that function Q is starlike. Also note that the function $h(z) = \nu(q_\beta(z)) + Q(z)$ satisfies $\mathrm{Re}(zh'(z)/Q(z)) > 0$ for $z \in \mathbb{D}$. Therefore, by making use of Lemma 2.2, it follows that

$$1 + \beta zp'(z) \prec 1 + \beta zq'_\beta(z) \text{ implies } p(z) \prec q_\beta(z).$$

Each of the conclusion in (a)-(f) is $p(z) \prec \mathcal{P}(z)$ for appropriate $\mathcal{P}$ and this holds if the subordination $q_\beta(z) \prec \mathcal{P}(z)$ holds.
If $q_\beta(z) \prec \mathcal{P}(z)$, then $\mathcal{P}(-1) < q_\beta(-1) < q_\beta(1) < \mathcal{P}(1)$. This gives a necessary condition for $p \prec \mathcal{P}$ to hold. Surprisingly, this necessary condition is also sufficient. This can be seen by looking at the graph of the respective functions.

(a) On taking $\mathcal{P}(z) = \sqrt{1+z}$, the inequalities $q_\beta(-1) \geq 0$ and $q_\beta(1) \leq \sqrt{2}$ reduce to $\beta \geq \beta_1$ and $\beta \geq \beta_2$, where

$$\beta_1 = 2(1 - \log 2) \text{ and } \beta_2 = 2(\sqrt{2} - 1 + \log 2 - \log(1 + \sqrt{2}))/(\sqrt{2} - 1)$$

respectively. Therefore, the subordination $q_\beta(z) \prec \sqrt{1+z}$ holds only if

$$\beta \geq \max\{\beta_1,\ \beta_2\} = \beta_2.$$

(b) Consider $\mathcal{P}(z) = \varphi_0(z)$. A simple calculation shows that the inequalities $q_\beta(-1) \geq \varphi_0(-1)$ and $q_\beta(1) \leq \varphi_0(1)$ reduce to $\beta \geq \beta_1$ and $\beta \geq \beta_2$, where

$$\beta_1 = 2(1 - \log 2)/(3 - 2\sqrt{2}) \text{ and } \beta_2 = 2(\sqrt{2} - 1 + \log 2 - \log(1 + \sqrt{2}))$$

respectively. Thus the subordination $q_\beta(z) \prec \varphi_0(z)$ holds only if

$$\beta \geq \max\{\beta_1, \beta_2\} = \beta_1.$$

(c) Consider $\mathcal{P}(z) = \varphi_s(z)$. The inequalities $q_\beta(-1) \geq \varphi_s(-1)$ and $q_\beta(1) \leq \varphi_s(1)$ reduce to $\beta \geq \beta_1$ and $\beta \geq \beta_2$, where

$$\beta_1 = \frac{2(1 - \log 2)}{\sin(1)} \text{ and } \beta_2 = \frac{2(\sqrt{2} - 1 + \log 2 - \log(1 + \sqrt{2}))}{\sin(1)}$$

respectively. The subordination $q_\beta(z) \prec \varphi_s(z)$ holds if $\beta \geq \max\{\beta_1, \beta_2\} = \beta_1$.

(d) Consider $\mathcal{P}(z) = \varphi_q(z)$. The inequalities $q_\beta(-1) \geq \varphi_q(-1)$ and $q_\beta(1) \leq \varphi_q(1)$ give $\beta \geq \beta_1$ and $\beta \geq \beta_2$, where

$$\beta_1 = (2+\sqrt{2})(1-\log 2) \text{ and } \beta_2 = \sqrt{2}(\sqrt{2}-1+\log 2 - \log(1+\sqrt{2}))$$

respectively. The subordination $q_\beta(z) \prec \varphi_q(z)$ holds if $\beta \geq \max\{\beta_1, \beta_2\} = \beta_1$.

(e) Consider $\mathcal{P}(z) = \varphi_c(z)$. From the inequalities

$$\varphi_c(-1) \leq q_\beta(-1) \text{ and } q_\beta(1) \leq \varphi_c(1),$$

we get

$$\beta \geq 3(1-\log 2) \text{ and } \beta \geq 2(\sqrt{2}-1+\log 2-\log(1+\sqrt{2}))$$

respectively. Thus the subordination $q_\beta(z) \prec \varphi_c(z)$ holds if

$$\beta \geq \max\left\{3(1-\log 2),\ 2(\sqrt{2}-1+\log 2-\log(1+\sqrt{2}))\right\} = 3(1-\log 2).$$

(f) Consider $\mathcal{P}(z) = (1+Az)/(1+Bz)$. From the inequalities

$$q_\beta(-1) \geq (1-A)/(1-B) \text{ and } q_\beta(1) \leq (1+A)/(1+B),$$

we note that $\beta \geq \beta_1$ and $\beta \geq \beta_2$, where

$$\beta_1 = \frac{2(1-B)(1-\log 2)}{A-B} \text{ and } \beta_2 = \frac{2(1+B)(\sqrt{2}-1+\log 2-\log(1+\sqrt{2}))}{A-B}$$

respectively. A simple calculation gives

$$\beta_1 - \beta_2 = 2(1-\log 2) + (1+B)(\log(1+\sqrt{2}) - \sqrt{2}).$$

We note that $\beta_1 - \beta_2 \geq 0$ if $B < B_0$ and $\beta_1 - \beta_2 \leq 0$ if $B > B_0$ where

$$B_0 = \frac{2-\log 4-\sqrt{2}+\log(1+\sqrt{2})}{\sqrt{2}-\log(1+\sqrt{2}+1)}.$$

The necessary subordination $p(z) \prec (1+Az)/(1+Bz)$ holds if $\beta \geq \max\{\beta_1,\ \beta_2\}$. ☐

Remark 2.3. The subordination results in part (a) and (f) in Theorem 2.1 were also investigated by the authors in [1, Lemma 2.1, p. 1019] and [9, Lemma 2.1, p. 3], but their results were non-sharp.

In 1985, Padmanabhan and Parvatham [18] introduced a unified classes of starlike and convex functions using convolution with the function of the form $z/(1-z)^\alpha$, $\alpha \in \mathbb{R}$. Later, Shanmugam [21] considered the class $\mathcal{S}^*_g(h)$ of all $f \in \mathcal{A}$ satisfying $z(f*g)'/(f*g) \prec h$ where h is a convex function, g is a fixed function in $\mathcal{A}$. Denote by $\mathcal{S}^*(h)$ and $\mathcal{K}(h)$, the subclass $\mathcal{S}^*_g(h)$, when g is $z/(1-z)$ and $z/(1-z)^2$ respectively. In 1992, Ma and Minda [12] considered a weaker assumption that h is a function with positive real part whose range is symmetric with respect to real axis and starlike with respect to $h(0)=1$ with $h'(0)>0$ and proved distortion, growth, and covering theorems. The class $\mathcal{S}^*(h)$ generalizes many subclasses of $\mathcal{A}$, for example, $\mathcal{S}^*[A, B] := \mathcal{S}^*((1+Az)/(1+Bz))$ $(-1 \leq B < A \leq 1)$ [8], $\mathcal{S}^*_e := \mathcal{S}^*(e^z)$ [13], $\mathcal{S}^*_s := \mathcal{S}^*(\varphi_s(z))$ [4], $\mathcal{S}^*_C := \mathcal{S}^*(\varphi_c(z))$ [22], $\mathcal{S}^*_R := \mathcal{S}^*(\varphi_0(z))$ [10], and $\mathcal{S}^*_q := \mathcal{S}^*(\varphi_q(z))$ [19]. The function $\sqrt{1+z}$ is associated with the class $\mathcal{S}^*_L := \mathcal{S}^*(\sqrt{1+z})$ [24], introduced by Sokół and Stankiewicz. This class consists of the function $f \in \mathcal{A}$ such that $w(z) := zf'(z)/f(z)$ lies in the region bounded by the right half of the lemniscate of

Bernoulli given by $|w^2 - 1| < 1$. The lemniscate of Bernoulli is a best known plane curve resembling the symbol ∞. It was named after James Bernoulli who considered it in elasticity theory in 1694. In geometry, the lemniscate is a plane curve defined by two given points F_1 and F_2, known as foci, at distance $2a$ from each other as the locus of points P so that $PF_1.PF_2 = a^2$. The equation of lemniscate may be written as $(x^2 + y^2)^2 = 2a^2(x^2 - y^2)$. The lemniscate in the complex plane is the locus of $z = x + iy$ such that $|z^2 - a^2| = a^2$.

Remark 2.4. Let the function $f \in \mathcal{A}$ satisfying the following subordination

$$1 + \beta \frac{zf'(z)}{f(z)} \left(1 - \frac{zf'(z)}{f(z)} + \frac{zf''(z)}{f'(z)} \right) \prec \sqrt{1+z}.$$

Then the following are sufficient conditions for f to be in various subclasses of $\mathcal{S}$.

(a) If $\beta \geq \frac{2(\sqrt{2}-1+\log 2-\log(1+\sqrt{2}))}{\sqrt{2}-1} \approx 1.09116$, then $f \in \mathcal{S}^*_L$.
(b) If $\beta \geq \frac{2(1-\log 2)}{3-2\sqrt{2}} \approx 3.57694$, then $f \in \mathcal{S}^*_R$.
(c) If $\beta \geq \frac{2(1-\log 2)}{\sin(1)} \approx 0.729325$, then $f \in \mathcal{S}^*_s$.
(d) If $\beta \geq (2+\sqrt{2})(1-\log 2) \approx 1.044766$, then $f \in \mathcal{S}^*_q$.
(e) If $\beta \geq 3(1-\log 2) \approx 0.920558$, then $f \in \mathcal{S}^*_c$.
(f) Let $-1 < B < A < 1$ and $B_0 = \frac{2-\log 4-\sqrt{2}+\log(1+\sqrt{2})}{\sqrt{2}-\log(1+\sqrt{2}+1)} \approx 0.151764$.
If either $B < B_0$ and $\beta \geq \frac{2(1-B)(1-\log 2)}{A-B} \approx 0.613706 \frac{1-B}{A-B}$ or $B > B_0$ and

$$\beta \geq \frac{2(1+B)(\sqrt{2}-1+\log 2-\log(1+\sqrt{2}))}{A-B} \approx 0.451974 \frac{1+B}{A-B},$$

then $f \in \mathcal{S}^*[A, B]$.

The bounds on β are sharp.

Next result provides a bound on β so that $1 + \beta zp'(z)/p(z) \prec \sqrt{1+z}$ implies p is subordinate to some well-known starlike functions.

Theorem 2.5. *Let the function p be analytic in $\mathbb{D}$, $p(0) = 1$ and*

$$1 + \beta zp'(z)/p(z) \prec \sqrt{1+z}.$$

Then the following subordination results hold:

(a) *If* $\beta \geq \frac{2(\log 2-1)}{\log(2\sqrt{2}-2})) \approx 3.26047$, *then* $p(z) \prec \varphi_0(z)$.
(b) *If* $\beta \geq \frac{2(\sqrt{2}-1+\log(2)-\log(\sqrt{2}+1))}{\log(1+\sin(1))} \approx 0.740256$, *then* $p(z) \prec \varphi_s(z)$.
(c) *If* $\beta \geq \frac{2(\log 2-1)}{\log(\sqrt{2}-1)} \approx 0.696306$, *then* $p(z) \prec \varphi_q(z)$.
(d) *If* $\beta \geq 2(1-\log 2) \approx 0.613706$, *then* $p(z) \prec e^z$.
(e) *If* $-1 < B < A < 1$ *and* $\beta \geq \max\{\beta_1, \beta_2\}$ *where*

$$\beta_1 = \frac{2(1-\log 2)}{\log(1-B)-\log(1-A)} \quad \textit{and} \quad \beta_2 = \frac{2(\sqrt{2}-1+\log 2-\log(1+\sqrt{2}))}{\log(1+A)-\log(1+B)},$$

then $p(z) \prec (1+Az)/(1+Bz)$.

The bounds on β are best possible.

Proof. The function $q_\beta : \overline{\mathbb{D}} \to \mathbb{C}$ defined by

$$q_\beta(z) = \exp\left(\frac{2}{\beta}(\sqrt{1+z} - \log(1+\sqrt{1+z}) + \log 2 - 1)\right)$$

is analytic and is a solution of the differential equation

$$1 + \beta z q_\beta'(z)/q_\beta(z) = \sqrt{1+z}.$$

Define the functions $\nu(w) = 1$ and $\psi(w) = \beta/w$.
The function $Q : \overline{\mathbb{D}} \to \mathbb{C}$ defined by

$$Q(z) := z q_\beta'(z)\psi(q_\beta(z)) = \beta z q_\beta'(z)/q_\beta(z) = \sqrt{1+z} - 1$$

is starlike in $\mathbb{D}$. The function $h(z) := \nu(q_\beta(z)) + Q(z) = 1 + Q(z)$ satisfies $\operatorname{Re}(zh'(z)/Q(z)) > 0$ for $z \in \mathbb{D}$. Therefore, by using Lemma 2.2, we see that the subordination

$$1 + \beta\frac{zp'(z)}{p(z)} \prec 1 + \beta\frac{zq_\beta'(z)}{q_\beta(z)}$$

implies $p(z) \prec q_\beta(z)$. As the similar lines of the proof of Theorem 2.1, the proofs of parts (a)-(e) are completed. ☐

Remark 2.6. The subordination in part (d) and (e) of Theorem 2.5 were earlier investigated in [13, Theorem 2.16(c), p. 10] and [9, Lemma 2.3, p. 5] where non-sharp results were obtained.

Remark 2.7. Let the function $f \in \mathcal{A}$ satisfies the following subordination

$$1 + \beta\left(1 - \frac{zf'(z)}{f(z)} + \frac{zf''(z)}{f'(z)}\right) \prec \sqrt{1+z}.$$

Then the following are sufficient conditions for f to be in various subclasses of $\mathcal{S}$.

(a) If $\beta \geq \frac{2(\log 2-1)}{\log(2\sqrt{2}-2})) \approx 3.26047$, then $f \in \mathcal{S}^*_R$.
(b) If $\beta \geq \frac{2(\sqrt{2}-1+\log(2)-\log(\sqrt{2}+1))}{\log(1+\sin(1))} \approx 0.740256$, then $f \in \mathcal{S}^*_s$.
(c) If $\beta \geq \frac{2(\log 2-1)}{\log(\sqrt{2}-1)} \approx 0.696306$, then $f \in \mathcal{S}^*_q$.
(d) If $\beta \geq 2(1-\log 2) \approx 0.613706$, then $f \in \mathcal{S}^*_e$.
(e) If $-1 < B < A < 1$ and $\beta \geq \max\{\beta_1, \beta_2\}$ where

$$\beta_1 = \frac{2(1-\log 2)}{\log(1-B) - \log(1-A)} \quad \text{and} \quad \beta_2 = \frac{2(\sqrt{2}-1+\log 2 - \log(1+\sqrt{2}))}{\log(1+A) - \log(1+B)},$$

then $f \in \mathcal{S}^*[A, B]$.

Next, we intend to determine a bound on β so that $1 + \beta zp'(z)/p^2(z) \prec \sqrt{1+z}$ implies p is subordinate to several well-known starlike functions.

Theorem 2.8. *Let the function p be analytic in $\mathbb{D}$, $p(0) = 1$ and*

$$1 + \beta zp'(z)/p^2(z) \prec \sqrt{1+z}.$$

Then the following subordination results hold for sharp bound of β:

(a) *If $\beta \geq 4(1+\sqrt{2})(1-\log 2) \approx 2.96323$, then $p(z) \prec \varphi_0(z)$.*

(b) *If* $\beta \geq \frac{2(1+\sin(1))(\sqrt{2}-\log(1+\sqrt{2})+\log 2-1)}{\sin(1)} \approx 0.989098$, *then* $p(z) \prec \varphi_s(z)$.
(c) *If* $\beta \geq (2+\sqrt{2})(\sqrt{2}-\log(1+\sqrt{2})+\log 2-1) \approx 0.771568$, *then* $p(z) \prec \varphi_q(z)$.
(d) *Let* $-1 < B < A < 1$ *and* $A_0 = \frac{2-\log 4-\sqrt{2}+\log(1+\sqrt{2})}{\sqrt{2}-\log(1+\sqrt{2}+1)} \approx 0.151764$. *If either*
  (i) $A > A_0$ *and* $\beta \geq \frac{2(1-A)(1-\log 2)}{A-B} \approx 0.613706\frac{1-A}{A-B}$ *or*
  (ii) $A < A_0$ *and* $\beta \geq \frac{2(1+A)(\sqrt{2}-1+\log 2-\log(1+\sqrt{2}))}{A-B} \approx 0.451974\frac{1+A}{A-B}$,
  then $p(z) \prec (1+Az)/(1+Bz)$.

Proof. The function $q_\beta : \overline{\mathbb{D}} \to \mathbb{C}$ defined by

$$q_\beta(z) = \left(1 - \frac{2}{\beta}\left(\sqrt{1+z} - \log(1+\sqrt{1+z}) + \log 2 - 1\right)\right)^{-1}$$

is clearly analytic and is a solution of the differential equation

$$1 + \beta z q_\beta'(z)/q_\beta^2(z) = \sqrt{1+z}.$$

Define the functions $\nu(w) = 1$ and $\psi(w) = \beta/w^2$. The function $Q : \overline{\mathbb{D}} \to \mathbb{C}$ defined by

$$Q(z) = zq_\beta'(z)\psi(q_\beta(z)) = \beta z q_\beta'(z)/q_\beta^2(z) = \sqrt{1+z} - 1$$

is starlike in $\mathbb{D}$, Q is starlike function.
The function $h(z) := \nu(q_\beta(z)) + Q(z) = \nu(q_\beta(z)) + Q(z)$ satisfies the inequality $\operatorname{Re}(zh'(z)/Q(z)) > 0$ for $z \in \mathbb{D}$. Therefore, by using Lemma 2.2, we see that the subordination

$$1 + \beta\frac{zp'(z)}{p^2(z)} \prec 1 + \beta\frac{zq_\beta'(z)}{q_\beta^2(z)}$$

implies $p(z) \prec q_\beta(z)$. As the similar lines of the proof of Theorem 2.1, the proofs of parts (a)-(d) are obtained. □

Remark 2.9. The subordination in part (d) of Theorem 2.8 was earlier investigated in [9, Lemma 2.4, p. 6] where non-sharp result was obtained.

Remark 2.10. Let the function $f \in \mathcal{A}$ satisfies the following subordination

$$1 + \beta\left(\frac{zf'(z)}{f(z)}\right)^{-1}\left(1 - \frac{zf'(z)}{f(z)} + \frac{zf''(z)}{f'(z)}\right) \prec \sqrt{1+z}.$$

Then the following are sufficient conditions for f to be in various subclasses of $\mathcal{S}$.
(a) If $\beta \geq 4(1+\sqrt{2})(1-\log 2) \approx 2.96323$, then $f \in \mathcal{S}_R^*$.
(b) If $\beta \geq \frac{2(1+\sin(1))(\sqrt{2}-\log(1+\sqrt{2})+\log 2-1)}{\sin(1)} \approx 0.989098$, then $f \in \mathcal{S}_s^*$.
(c) If $\beta \geq (2+\sqrt{2})(\sqrt{2}-\log(1+\sqrt{2})+\log 2-1) \approx 0.771568$, then $f \in \mathcal{S}_q^*$.
(d) Let $-1 < B < A < 1$ and $A_0 = \frac{2-\log 4-\sqrt{2}+\log(1+\sqrt{2})}{\sqrt{2}-\log(1+\sqrt{2}+1)} \approx 0.151764$.
  If either $A > A_0$ and $\beta \geq \frac{2(1-A)(1-\log 2)}{A-B} \approx 0.613706\frac{1-A}{A-B}$ or $A < A_0$ and

$$\beta \geq \frac{2(1+A)(\sqrt{2}-1+\log 2-\log(1+\sqrt{2}))}{A-B} \approx 0.451974\frac{1+A}{A-B},$$

  then $f \in \mathcal{S}^*[A,B]$.

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Subclasses of p-valent meromorphic functions involving certain operator

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Abstract. In this paper we investigate some inclusion relationships of two new subclassses of meromorphically p-valent functions, defined by means of a linear operator. We also study some integral preserving properties and convolution properties of these classes.

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Keywords: Analytic, p-valent, meromorphic, linear operator, differential subordination, inclusion relationships.

1. Introduction

Let $\sum_p$ denote the class of all meromorphic functions f defined by:

$$f(z) = z^{-p} + \sum_{k=1}^{\infty} a_{k-p} z^{k-p} \qquad (p \in \mathbb{N} = \{1, 2, ...\}), \tag{1.1}$$

which are analytic and p-valent in a punctured unit disk $\mathbb{U}^* = \{z : z \in \mathbb{C}$ and $0 < |z| < 1\} = \mathbb{U} \backslash \{0\}$.

The class of analytic functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \ , z \in \mathbb{U},$$

is denoted by $\mathbb{A}$. The functions of this class is called starlike of order $\gamma, 0 \leq \gamma < 1$ if

$$\Re \frac{zf'(z)}{f(z)} > \gamma$$

and called prestarlike of order $\gamma, \gamma < 1$ if

$$\frac{z}{(1-z)^{2(1-\gamma)}} * f(z) \in S^*(\gamma),$$

we denote by $S^*(\gamma)$ and $R(\gamma)$ the classes of stalike and prestarlike of order γ.

If f and g are analytic functions in $\mathbb{U}$, we say that f is subordinate to g, written $f \prec g$ if there exists a Schwarz function w, which (by definition) is analytic in $\mathbb{U}$ with $w(0) = 0$ and $|w(z)| < 1$ for all $z \in \mathbb{U}$, such that $f(z) = g(w(z))$, $z \in \mathbb{U}$. Furthermore, if the function g is univalent in $\mathbb{U}$, then we have the following equivalence (see [2, 5, 6]):

$$f(z) \prec g(z) \Leftrightarrow f(0) = g(0) \text{ and } f(\mathbb{U}) \subset g(\mathbb{U}).$$

For functions $f(z) \in \sum_p$ given by (1.1) and $g(z) \in \sum_p$ given by

$$g(z) = z^{-p} + \sum_{k=1}^{\infty} b_{k-p} z^{k-p},$$

the Hadamard product of $f(z)$ and $g(z)$ is given by:

$$(f * g)(z) = z^{-p} + \sum_{k=1}^{\infty} a_{k-p} b_{k-p} z^{k-p} = (g * f)(z). \tag{1.2}$$

Using the operator $Q^{\alpha}_{\beta,p} : \sum_p \to \sum_p$ defined by (see [1]):

$$Q^{\alpha}_{\beta,p} f(z) = \begin{cases} z^{-p} + \frac{\Gamma(\alpha+\beta)}{\Gamma(\beta)} \sum\limits_{k=1}^{\infty} \frac{\Gamma(k+\beta)}{\Gamma(k+\beta+\alpha)} a_{k-p} z^{k-p} & (\alpha > 0; \beta > -1) \\ f(z) & (\alpha = 0\,; \beta > -1)\,. \end{cases}$$

Mostafa [8] defined the operator $H^{\alpha}_{p,\beta,\mu} : \Sigma_p \to \Sigma_p$ as follows:
First put

$$G^{\alpha}_{\beta,p}(z) = z^{-p} + \frac{\Gamma(\alpha+\beta)}{\Gamma(\beta)} \sum_{k=1}^{\infty} \frac{\Gamma(k+\beta)}{\Gamma(k+\beta+\alpha)} z^{k-p} \quad (p \in \mathbb{N}) \tag{1.3}$$

and let $G^{\alpha *}_{\beta,p,\mu}$ be defined by

$$G^{\alpha}_{\beta,p}(z) * G^{\alpha *}_{\beta,p,\mu}(z) = \frac{1}{z^p(1-z)^{\mu}} \quad (\mu > 0; p \in \mathbb{N})\,. \tag{1.4}$$

Then

$$H^{\alpha}_{p,\beta,\mu} f(z) = G^{\alpha *}_{\beta,p}(z) * f(z) \quad (f \in \Sigma_p)\,. \tag{1.5}$$

Using (1.3)-(1.5), we have

$$H^{\alpha}_{p,\beta,\mu} f(z) = z^{-p} + \frac{\Gamma(\beta)}{\Gamma(\alpha+\beta)} \sum_{k=1}^{\infty} \frac{\Gamma(k+\beta+\alpha)(\mu)_k}{\Gamma(k+\beta)(1)_k} a_{k-p} z^{k-p}\,, \tag{1.6}$$

where $f \in \Sigma_p$ is in the form (1.1) and $(\nu)_n$ denotes the Pochhammer symbol given by

$$(\nu)_n = \frac{\Gamma(\nu+n)}{\Gamma(\nu)} = \begin{cases} 1 & (n = 0) \\ \nu(\nu+1)...(\nu+n-1) & (n \in \mathbb{N})\,. \end{cases}$$

It is readily verified from (1.6) that (see [8])

$$z(H^{\alpha}_{p,\beta,\mu} f(z))' = (\alpha+\beta) H^{\alpha+1}_{p,\beta,\mu} f(z) - (\alpha+\beta+p) H^{\alpha}_{p,\beta,\mu} f(z) \tag{1.7}$$

and

$$z(H^{\alpha}_{p,\beta,\mu} f(z))' = \mu H^{\alpha}_{p,\beta,\mu+1} f(z) - (\mu+p) H^{\alpha}_{p,\beta,\mu} f(z)\,. \tag{1.8}$$

It is noticed that, putting $\mu = 1$ in (1.6), we obtain the operator

$$H^{\alpha}_{p,\beta,1}f(z) = H^{\alpha}_{p,\beta}f(z) = z^{-p} + \frac{\Gamma(\beta)}{\Gamma(\alpha+\beta)}\sum_{k=1}^{\infty}\frac{\Gamma(k+\alpha+\beta)}{\Gamma(k+\beta)}a_{k-p}z^{k-p}. \qquad (1.9)$$

Let $\mathbb{P}$ be the class of functions $h(z)$ with $h(0) = 1$, $\text{Re}h(z) > 0$ which are convex univalent in $\mathbb{U}$.

For $p, n \in \mathbb{N}$, $\in_n = e^{2\pi/n}$, let

$$f^{\mu}_n(\alpha)(z) = \frac{1}{n}\sum_{j=0}^{n-1}\varepsilon_n^{jp}H^{\alpha}_{p,\beta,\mu}f(\varepsilon_n^{j}z) = z^{-p} + ..., f \in \sum\nolimits_p . \qquad (1.10)$$

By (1.7) and (1.8), $f^{\mu}_n(\alpha)(z)$ satisfies:

$$z(f^{\mu}_n(\alpha)(z))' = (\alpha+\beta)f^{\mu}_n(\alpha+1)(z) - (\alpha+\beta+p)f^{\mu}_n(\alpha)(z) \qquad (1.11)$$

and

$$z(f^{\mu}_n(\alpha)(z))' = \mu f^{\mu+1}_n(\alpha)(z) - (\mu+p)f^{\mu}_n(\alpha)(z). \qquad (1.12)$$

Definition 1.1. For $h \in \mathbb{P}$,$f \in \sum_p$, $f^{\mu}_n(\alpha)(z) \neq 0, z \in \mathbb{U}^*$, $S^{\mu}_n(\alpha, h)$ is the class of functions f satisfying:

$$-\frac{z(H^{\alpha}_{p,\beta,\mu}f(z))'}{pf^{\mu}_n(\alpha)(z)} \prec h(z) \qquad (1.13)$$

and $K^{\mu}_n(\alpha, h)$ is the class of functions f satisfying:

$$-\frac{z(H^{\alpha}_{p,\beta,\mu}f(z))'}{pg^{\mu}_n(\alpha)(z)} \prec h(z), \qquad (1.14)$$

where $g^{\mu}_n(\alpha)(z) \neq 0$, is defined as in (1.10).

To prove our results, we need the following Lemmas.

Lemma 1.2. [3] *Let $\beta, \gamma \in \mathbb{C}, \beta \neq 0, h$ be convex univalent with $\Re\{\beta h(z) + \gamma\} > 0$ and q be an analytic function such that $q(0) = h(0)$. If*

$$q(z) + \frac{zq'(z)}{\beta q(z) + \gamma} \prec h(z),$$

then

$$q(z) \prec h(z).$$

Lemma 1.3. [7] *Let h be convex univalent and w be analytic, $\Re w \geq 0$. If the analytic function q satisfies $q(0) = h(0)$ and*

$$q(z) + w(z)zq'(z) \prec h(z),$$

then $q(z) \prec h(z)$.

Lemma 1.4. [9] *For $\alpha < 1, f \in R(\alpha)$ and $\varphi \in S^*(\alpha)$, we have for any analytic function F in $\mathbb{U}$,*

$$\frac{f * (\varphi F)}{f * \varphi}(\mathbb{U}) \subset \overline{co}(F(\mathbb{U}),$$

where $\overline{co}(F(\mathbb{U})$ is the convex hull of $(F(\mathbb{U})$.

2. Main results

Theorem 2.1. *If $f \in S_n^{\mu}(\alpha, h)$, then*

$$-\frac{z\left(f_n^{\mu}(\alpha)(z)\right)'}{pf_n^{\mu}(\alpha)(z)} \prec h(z), \tag{2.1}$$

*where $f_n^{\mu}(\alpha)(z)$ is defined as in (*1.10*).*

Proof. From (1.10), we have:

$$\begin{aligned} f_n^{\mu}(\alpha)(\in_n^j z) &= \frac{1}{n}\sum_{t=0}^{n-1} \varepsilon_n^{jt} H_{p,\beta,\mu}^{\alpha} f(\varepsilon_n^{j+t} z) \\ &= \frac{\varepsilon_n^{-jp}}{n}\sum_{t=0}^{k-1} \varepsilon_n^{(j+t)p} H_{p,\beta,\mu}^{\alpha} f(\varepsilon_n^{j+t} z) \\ &= \varepsilon_n^{-jp} f_n^{\mu}(\alpha)(z) \end{aligned} \tag{2.2}$$

and

$$\left(f_n^{\mu}(\alpha)(z)\right)' = \frac{1}{n}\sum_{j=0}^{n-1} \varepsilon_n^{j(p+1)} \left(H_{p,\beta,\mu}^{\alpha} f(\varepsilon_n^{j+t} z)\right)'. \tag{2.3}$$

By (2.2) and (2.3), we have

$$\begin{aligned} -\frac{z\left(f_n^{\mu}(\alpha)(z)\right)'}{pf_n^{\mu}(\alpha)(z)} &= -\frac{1}{n}\sum_{j=0}^{n-1} \frac{\varepsilon_n^{j(p+1)}\left(H_{p,\beta,\mu}^{\alpha} f(\varepsilon_k^{j} z)\right)'}{pf_n^{\mu}(\alpha)(z)} \\ &= -\frac{1}{n}\sum_{j=0}^{n-1} \frac{\varepsilon_n^{j}\left(H_{p,\beta,\mu}^{\alpha} f(\varepsilon_n^{j} z)\right)'}{pf_n^{\mu}(\alpha)(z)}. \end{aligned} \tag{2.4}$$

Since $f \in S_n^{\mu}(\alpha, h)$, we have,

$$-\frac{\varepsilon_n^{j}\left(H_{p,\beta,\mu}^{\alpha} f(\varepsilon_n^{j} z)\right)'}{pf_n^{\mu}(\alpha)(z)} \prec h(z),$$

which leads to (2.1). □

Theorem 2.2. *For $\alpha + \beta > 0$, $h \in \mathbb{P}$ with $\Re\{\alpha + \beta + p - ph(z)\} > 0$ and for $f \in S_n^{\mu}(\alpha + 1, h)$, $g_n^{\mu}(\alpha) \neq 0$, we have, $f \in S_n^{\mu}(\alpha, h)$.*

Proof. Since $f \in S_n^{\mu}(\alpha + 1, h)$, then the function

$$q(z) = -\frac{z\left(H_{p,\beta,\mu}^{\alpha} f(z)\right)'}{pf_n^{\mu}(\alpha)(z)}, \tag{2.5}$$

is analytic and $q(0) = 1$. Applying (1.8) in (2.5), we have

$$q(z) f_n^{\mu}(\alpha_1)(z) = -\frac{1}{p}[(\alpha + \beta) H_{p,\beta,\mu}^{\alpha+1} f(z) - (\alpha + \beta + p) H_{p,\beta,\mu}^{\alpha} f(z)]. \tag{2.6}$$

Differentiating (2.6) and using (1.8) again, we have

$$\left(\alpha+\beta+p+\frac{z\left(f_n^{\mu}(\alpha)(z)\right)'}{f_n^{\mu}(\alpha)(z)}\right)q(z)+zq'(z)=-\frac{(\alpha+\beta)z\left(H_{p,\beta,\mu}^{\alpha+1}f(z)\right)'}{pf_n^{\mu}(\alpha)(z)}. \tag{2.7}$$

Taking

$$\phi(z)=-\frac{z\left(f_n^{\mu}(\alpha)(z)\right)'}{pf_n^{\mu}(\alpha)(z)}, \tag{2.8}$$

we see that $\phi(z)$ is analytic, $\phi(0)=1$ and (2.7) can be written as

$$(\alpha+\beta+p-p\phi(z))\,q(z)+zq'(z)=-\frac{(\alpha+\beta)z\left(H_{p,\beta,\mu}^{\alpha+1}f(z)\right)'}{pf_n^{\mu}(\alpha)(z)}, \tag{2.9}$$

that is

$$q(z)+\frac{zq'(z)}{\alpha+\beta+p-p\phi(z)}=-\frac{z\left(H_{p,\beta,\mu}^{\alpha+1}f(z)\right)'}{pf_n^{\mu}(\alpha+1)(z)}. \tag{2.10}$$

Since $f\in S_n^{\mu}(\alpha+1,h)$, (2.10) implies

$$q(z)+\frac{zq'(z)}{\alpha+\beta+p-p\phi(z)}\prec h(z). \tag{2.11}$$

Combining (2.11) and (2.8), we have

$$\alpha+\beta+p-p\phi(z)=\frac{(\alpha+\beta)f_n^{\mu}(\alpha+1)(z)}{pf_n^{\mu}(\alpha)(z)}. \tag{2.12}$$

Differentiating (2.12), we get

$$\phi(z)+\frac{z\phi'(z)}{\alpha+\beta+p-p\phi(z)}=-\frac{z\left(f_n^{\mu}(\alpha+1)(z)\right)'}{pf_n^{\mu}(\alpha+1)(z)}. \tag{2.13}$$

By Theorem 2.1, we have

$$-\frac{z\left(f_n^{\mu}(\alpha+1)(z)\right)'}{pf_n^{\mu}(\alpha+1)(z)}\prec h(z),$$

which yields

$$\phi(z)+\frac{z\phi'(z)}{\alpha+\beta+p-p\phi(z)}\prec h(z).$$

Since $\Re\{\alpha+\beta+p-ph(z)\}>0$, by Lemma 1.2, we have $\phi(z)\prec h(z)$, which implies $\Re\{\alpha+\beta+p-p\phi(z)\}>0$. Applying Lemma 1.3 and from (2.10), we have $q(z)\prec h(z)$ that is $f\in S_n^{\mu}(\alpha,h)$. □

Theorem 2.3. *Let $\alpha+\beta>0$, $h\in\mathbb{P}$ with $\Re\{\alpha+\beta+p-ph(z)\}>0$ and $f\in K_n^{\mu}(\alpha+1,h)$ with $g\in S_n^{\mu}(\alpha+1,h)$. Then, $f\in K_n^{\mu}(\alpha,h)$ provided $g_n^{\mu}(\alpha)(z)\neq 0$.*

Proof. By Theorem 2.2, $g\in S_n^{\mu}(\alpha+1,h)\Rightarrow g\in S_n^{\mu}(\alpha,h)$ and by Theorem 2.1, we have

$$\psi(z)=-\frac{z\left(g_n^{\mu}(\alpha)(z)\right)'}{pg_n^{\mu}(\alpha)(z)}\prec h(z). \tag{2.14}$$

Let

$$q(z) = -\frac{z\left(H^{\alpha}_{p,\beta,\mu}f(z)\right)'}{pg^{\mu}_{n}(\alpha)(z)}. \tag{2.15}$$

Then, from (1.8), we have

$$q(z)g^{\mu}_{n}(\alpha)(z) = -\frac{1}{p}[(\alpha+\beta)H^{\alpha+1}_{p,\beta,\mu}f(z) - (\alpha+\beta+p)H^{\alpha}_{p,\beta,\mu}f(z)]. \tag{2.16}$$

Differentiating (2.16), we have

$$(\alpha+\beta+p-p\psi(z))\,q(z) + zq'(z) = -\frac{(\alpha+\beta)z\left(H^{\alpha+1}_{p,\beta,\mu}f(z)\right)'}{pg^{\mu}_{n}(\alpha)(z)}. \tag{2.17}$$

Applying (1.11) for g, (2.17) is equivalent to

$$q(z) + \frac{zq'(z)}{\alpha_1+p-p\psi(z)} = -\frac{z\left(H^{\alpha+1}_{p,\beta,\mu}f(z)\right)'}{pg^{\mu}_{n}(\alpha+1)(z)}. \tag{2.18}$$

Since $f \in K^{\mu}_{n}(\alpha+1,h)$, the above equation leads to

$$q(z) + \frac{zq'(z)}{\alpha_1+p-p\psi(z)} \prec h(z). \tag{2.19}$$

We have $\Re\{\alpha+\beta+p-p\psi(z)\} > 0$ because $\Re\{\alpha+\beta+p-ph(z)\} > 0$. Applying Lemma 1.3, for (2.19), we have $q(z) \prec h(z)$. That is $f \in K^{\mu}_{n}(\alpha,h)$. □

Theorem 2.4. *Let $h \in \mathbb{P}, \Re\{\mu+p-ph(z)\} > 0$ and $f \in S^{\mu+1}_{n}(\alpha,h)$ such that $f^{\mu+1}_{n}(\alpha)(z) \neq 0$. Then $f \in S^{\mu}_{n}(\alpha,h)$.*

Proof. Let $f \in S^{\mu+1}_{n}(\alpha,h)$,

$$q(z) = -\frac{z\left(H^{\alpha}_{p,\beta,\mu}f(z)\right)'}{pf^{\mu}_{n}(\alpha)(z)}. \tag{2.20}$$

Applying (1.9) in (2.20), we have

$$q(z)f^{\mu}_{n}(\alpha)(z) = -\frac{\mu}{p}[H^{\alpha}_{p,\beta,\mu+1}f(z) + (\frac{\mu+p}{p})H^{\alpha}_{p,\beta,\mu}f(z)]. \tag{2.21}$$

Differentiating (2.21) and putting

$$\Phi(z) = -\frac{z\left(f^{\mu}_{n}(\alpha)(z)\right)'}{pf^{\mu}_{n}(\alpha)(z)}, \tag{2.22}$$

simple computations leads to

$$[\mu+p-p\Phi(z)]\,q(z) + zq'(z) = -\left(\frac{\mu}{p}\right)\frac{z\left(H^{\alpha}_{p,\beta,\mu+1}f(z)\right)'}{pf^{\mu}_{n}(\alpha)(z)}. \tag{2.23}$$

Using (1.12), we have

$$\mu+p-p\Phi(z) = \frac{\mu f^{\mu+1}_{n}(\alpha)(z)}{f^{\mu}_{n}(\alpha)(z)}. \tag{2.24}$$

So, (2.23), reduces to

$$q(z)+\frac{zq'(z)}{\mu+p-p\Phi(z)}=-\frac{z\left(H^{\alpha}_{p,\beta,\mu+1}f(z)\right)'}{pf^{\mu+1}_n(\alpha)(z)}\prec h(z), \tag{2.25}$$

where $f\in S^{\mu+1}_n(\alpha,h)$. Also differentating (2.24), we have

$$\Phi(z)+\frac{z\Phi'(z)}{\mu+p-p\Phi(z)}=-\frac{z\left(f^{\mu+1}_n(\alpha)f(z)\right)'}{pf^{\mu+1}_n(\alpha)(z)}. \tag{2.26}$$

By Theorem 2.1, we have

$$-\frac{z\left(f^{\mu+1}_n(\alpha)f(z)\right)'}{pf^{\mu+1}_n(\alpha)(z)}\prec h(z). \tag{2.27}$$

Combining (2.26), (2.27) and the condition $\Re\{\mu+p-ph(z)\}>0$, we have $\Phi(z)\prec h(z)$, which leads to $\Re\{\mu+p-p\Phi(z)\}>0$ and so applying Lemma 1.3 to (2.25). we have $q(z)\prec h(z)$ which completes the proof of Theorem 2.4. □

Theorem 2.5. *Let $h\in\mathbb{P}$ with $\Re\{\mu+p-ph(z)\}>0$ and $f\in K^{\mu+1}_n(\alpha,h)$ with $g\in S^{\mu+1}_n(\alpha,h)$. Then, $f\in K^{\mu}_n(\alpha,h)$ provided $g^{\mu}_n(\alpha)(z)\neq 0$.*

Proof. By Theorem 2.4, $g\in S^{\mu+1}_n(\alpha,h)\Rightarrow g\in S^{\mu}_n(\alpha,h)$ and by Theorem 2.1, we have

$$\Psi(z)=-\frac{z\left(g^{\mu}_n(\alpha)(z)\right)'}{pg^{\mu}_n(\alpha)(z)}\prec h(z),$$

and letting

$$q(z)=-\frac{z\left(H^{\alpha}_{p,\beta,\mu}f(z)\right)'}{pg^{\mu}_n(\alpha)(z)},$$

we can complete the proof as in Theorem 2.4. Next, let

$$F_{p,\delta}(f(z))=\frac{\delta-p}{z^{\delta}}\int_0^z t^{\delta-1}f(t)dt \quad (\delta>0), \tag{2.28}$$

which by using (1.6) gives

$$\delta H^{\alpha}_{p,\beta,\mu}F_{p,\delta}f(z)+z\left(H^{\alpha}_{p,\beta,\mu+1})F_{p,\delta}f(z)\right)'=(\delta-p)H^{\alpha}_{p,\beta,\mu}f(z). \tag{2.29}$$

The operator $F_{p,\delta}$ was investigated by many authors (see [10, 11]). □

Theorem 2.6. *Let $h\in\mathbb{P}$ with $\Re\{\delta-ph(z)\}>0$ and $f\in S^{\mu}_n(\alpha,h)$, then $F_{p,\delta}(f)\in S^{\mu}_n(\alpha,h)$ provided $F^{\mu}_n(\alpha)\neq 0$, where $F^{\mu}_n(\alpha)$ is defined as in (*1.10*) .*

Proof. From (2.29), we have

$$\delta F^{\mu}_n(\alpha)(z)+z\left(F^{\mu}_n(\alpha)(z)\right)'=(\delta-p)f^{\mu}_n(\alpha)(z). \tag{2.30}$$

Let

$$q(z)=-\frac{z\left(H^{\alpha}_{p,\beta,\mu}F_{p,\delta}(f(z))\right)'}{pF^{\mu}_n(\alpha)(z)}$$

and

$$w(z) = -\frac{z\left(F_n^{\mu}(\alpha)(z)\right)'}{pF_n^{\mu}(\alpha)(z)}. \tag{2.31}$$

Using (2.30) in (2.31), we have

$$\delta - pw(z) = (\delta - p)\frac{f_n^{\mu}(\alpha)(z)}{F_n^{\mu}(\alpha)(z)}.$$

Differentiating and using Theorem 2.1, we obtain

$$w(z) + \frac{zw'(z)}{\delta - pw(z)} = -\frac{z\left(f_n^{\mu}(\alpha)(z)\right)'}{pf_n^{\mu}(\alpha)(z)} \prec h(z). \tag{2.32}$$

By Lemma 1.2, (2.32) implies $w(z) \prec h(z)$. The remaining part of the proof is similar to that of Theorem 2.2, so we omit it. □

The proof of the following theorem is similar to that of Theorems 2.3 and 2.5, so we omit it.

Theorem 2.7. *Let $h \in \mathbb{P}$ with $\Re\{\delta - ph(z)\} > 0$ and $f \in K_n^{\mu}(\alpha, h)$, with respect to $g_n^{\mu} \in S_n^{\mu}(\alpha, h)$, then, $F_{p,\delta}(f) \in K_n^{\mu}(\alpha, h)$ with respect to $G = F_{p,\delta}(g)$ provided $G_n^{\mu}(\alpha)(z) \neq 0$.*

Note that for $h(z) = \frac{1+Az}{1+Bz}, -1 \leq B < A \leq 1$, we have $\Re h(z) = \frac{1+A}{1+B}$.

Remark 2.8. Taking $h(z) = \frac{1+Az}{1+Bz}$, in Theorems 2.2-2.7 we get corresponding results for the classes $S_n^{\mu}(\alpha, A, B)$ and $K_n^{\mu}(\alpha, A, B)$.

Theorem 2.9. *If $h \in \mathbb{P}$, with $\Re\{p + 1 - \gamma - ph(z)\} > 0$, $f \in S_n^{\mu}(\alpha, h), \varphi \in \sum_p$ and $z^{p+1}\varphi(z) \in R(\gamma), \gamma < 1$, then $f * \varphi \in S_n^{\mu}(\alpha, h)$.*

Proof. For $f \in S_n^{\mu}(\alpha, h)$, we have

$$F(z) = -\frac{z\left(H_{p,\beta,\mu}^{\alpha} f(z)\right)'}{pf_n^{\mu}(\alpha)(z)} \prec h(z). \tag{2.33}$$

Let

$$\psi(z) = z^{p+1} f_n^{\mu}(\alpha)(z),$$

then $\psi \in \mathbb{A}$ and

$$\frac{z\psi'(z)}{\psi(z)} = p + 1 + \frac{z\left(f_n^{\mu}(\alpha)(z)\right)'}{f_n^{\mu}(\alpha)(z)} \prec p + 1 - ph(z). \tag{2.34}$$

From the hypotheses of the theorem, we see that

$$\Re\frac{z\psi'(z)}{\psi(z)} > \gamma, \tag{2.35}$$

that is $\psi \in S^*\ (\gamma), \gamma < 1$.For $\varphi \in \sum_p$ it is easy to get

$$z^{p+1} H_{p,\beta,\mu}^{\alpha}(f * \varphi)(\varepsilon_k^j z) = (z^{p+1}\varphi(z)) * H_{p,\beta,\mu}^{\alpha} f(\varepsilon_k^j z)$$

and

$$z^{p+2}(H_{p,\beta,\mu}^{\alpha}(f * \varphi)(z))' = (z^{p+1}\varphi(z)) * (z^{p+2} H_{p,\beta,\mu}^{\alpha} f(z))'.$$

So, we have

$$\begin{aligned}\Psi(z) &= -\frac{(H^{\alpha}_{p,\beta,\mu}(f*\varphi)(z))'}{\frac{p}{k}\sum\limits_{j=0}^{k-1}\varepsilon_k^{jp}H^{\alpha}_{p,\beta,\mu}(f*\varphi)(\varepsilon_k^{j}z)}\\ &= -\frac{(z^{p+1}\varphi(z))*z^{p+2}(H^{\alpha}_{p,\beta,\mu}f(z))'}{pz^{p+1}\varphi(z)*(z^{p+1}f_n^{\mu}(\alpha)(z))}\\ &= \frac{z^{p+1}\varphi(z)*(\psi(z)F(z))}{z^{p+1}\varphi(z)*\psi(z)}. \end{aligned} \tag{2.36}$$

Since h is convex, univalent, applying Lemma 1.4, it follows $\Psi(z) \prec h(z)$, that is $f*\varphi \in S_n^{\mu}(\alpha, h)$. □

Remark 2.10. Taking $\mu = 1$, in the above results we obtain results concerning the operator $H^{\alpha}_{p,\beta}f(z)$ defined by (1.9).

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Fekete-Szegö problem for a class of analytic functions defined by Carlson-Shaffer operator

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Abstract. In the present paper, authors study a Fekete-Szegö problem for a class of analytic functions defined by Carlson-Shaffer operator. Relevant connections of the results presented here with various known results are briefly indicated.

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1. Introduction

Let A denote the class of functions f of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \tag{1.1}$$

which are analytic in the open unit disk $U = \{z : z \in C \text{ and } |z| < 1\}$ and S denote the subclass of A that are univalent in U. Fekete and Szegö [10] proved a interesting result that the estimate

$$\left|a_3 - \lambda a_2^2\right| \leq 1 + 2\exp\left(\frac{-2\lambda}{1-\lambda}\right) \tag{1.2}$$

holds for any normalized univalent function $f(z)$ of the form (1.1) in the open unit disk U for $0 \leq \lambda \leq 1$. This inequality is sharp for each λ.

The coefficient functional

$$\phi_\lambda(f) = a_3 - \lambda a_2^2 = \frac{1}{6}\left(f'''(0) - \frac{3\lambda}{2}\left[f''(0)\right]^2\right), \tag{1.3}$$

on normalized analytic functions f in the unit disk represents various geometric quantities, for example, when $\lambda = 1$, $\phi_\lambda(f) = a_3 - a_2^2$, becomes $\frac{S_f(0)}{6}$, where S_f denote the Schwarzian derivative $(f'''/f')' - (f''/f')^2/2$ of locally univalent functions f in

U. The problem of maximising the absolute value of the functional $\phi_\lambda(f)$ is called the Fekete-Szegö problem.

The Fekete-Szegö problem is one of the interesting problems in Geometric Function Theory. This attracts many researchers (see the work of [1]-[5], [7]-[9], [12], [13], [16], [17], [20] and [3]) to study the Fekete-Szegö problem for the various classes of analytic univalent functions. Very recently, Bansal [4] introduced the class $R_\gamma^\tau(\phi)$ of functions in $f \in S$ for which

$$1 + \frac{1}{\tau}\left(f'(z) + \gamma z f''(z) - 1\right) \prec \phi(z), \ z \in U$$

where $0 \leq \gamma < 1$, $\tau \in C \setminus \{0\}$, $\phi(z)$ is an analytic function with positive real part on U with $\phi(0) = 0$, $\phi'(0) > 0$ which maps the unit like disk U onto a starlike region with respect to 1 which is symmetric with respect to the real axis and $\prec$ denotes the subordination between analytic functions and studied the Fekete-Szegö problem for this class.

Now, by using the Carlson-Shaffer operator we introduce a new subclass $R_\gamma^\tau(\phi, a, c)$ for functions $f \in A$ and $0 \leq \gamma < 1$, $\tau \in C \backslash \{0\}$, $a, c \in C$, $\{c \neq 0, -1, -2, ...\}$ satisfying the condition

$$1 + \frac{1}{\tau}\big((L(a,c)f(z))' + \gamma z (L(a,c)f(z))'' - 1\big) \prec \phi(z) \ \ (z \in U) \tag{1.4}$$

where $\phi(z)$ is defined the same as above and $L(a, c)$ denotes the Carlson-Shaffer operator introduced in [6] and defined in the following way:

$$L(a,c)f(z) = \quad f(z) * zh(a,c;z),$$

where

$$h(a,c;z) = \quad 1 + \sum_{n=1}^{\infty} \frac{(a)_n}{(c)_n} z^n$$

$L(a, c)$ maps A into itself. $L(c, c)$is the identity and if $a \neq 0, -1, -2...$, then $L(a, c)$ has a continuous inverse $L(c, a)$ and is an one-to-one mapping of A onto itself. $L(a, c)$ provides a convenient representation of differentiation and integration. If $g(z) = zf'(z)$, then $g = L(2, 1)f$ and $f = L(1, 2)g$. If we set

$$\phi(z) = \frac{1 + Az}{1 + Bz}, \qquad (-1 \leq B < A \leq 1; z \in U),$$

in (1.4), we obtain

$$R_\gamma^\tau\left(\frac{1+Az}{1+Bz}, a, c\right) = R_\gamma^\tau(A, B, a, c)$$

$$= \left\{ f \in A : \left| \frac{(L(a,c)f(z))' + \gamma z (L(a,c)f(z))'' - 1}{\tau(A-B) - B\big((L(a,c)f(z))' + \gamma z (L(a,c)f(z))'' - 1\big)} \right| < 1 \right\}$$

which is again a new class.

By specializing parameters in the subclass $R_\gamma^\tau(A, B, a, c)$ we obtain the following known subclasses studied earlier by various authors.

1. $R_\gamma^\tau(A, B, a, a) \equiv R_\gamma^\tau(A, B)$ studied by Bansal [4].
2. $R_\gamma^\tau(1 - 2\beta, -1, a, a) \equiv R_\gamma^\tau(\beta)$ for $0 \leq \beta < 1$, studied by Swaminathan [21].

3. $R_\gamma^\tau(1-2\beta,-1,a,a) \equiv R_\gamma^\tau(\beta)$ for $\tau = e^{i\eta}cos\eta, 0 \le \beta < 1$, where $-\pi/2 < \eta < \pi/2$ introduced by Ponnusamy and Rønning [19], (see also [18]).
4. $R_1^\tau(0,-1,a,a) \equiv R^\tau(\beta)$ for $\tau = e^{i\eta}cos\eta$ was considered in [14].

To prove our main result, we shall require the following lemma.

Lemma 1.1. *(see* [11], [15]*). If* $p(z) = 1+c_1z+c_2z^2+c_3z^3+...$ $(z \in U)$ *is a function with positive real part, then for any complex number* μ *,*

$$\left|c_2 - \mu c_1^2\right| \le \;2\;\max\{1,\;|2\mu-1|\} \tag{1.5}$$

and the result is sharp for the functions given by

$$p(z) = \frac{1+z^2}{1-z^2},\quad p(z) = \frac{1+z}{1-z}\quad (z \in U). \tag{1.6}$$

2. Main results

Our main result is contained in the following theorem.

Theorem 2.1. *Let* $\phi(z) = 1+B_1z+B_2z^2+B_3z^3+\ldots$*, where* $\phi(z) \in A$ *with* $\phi'(0) > 0$*. If* $f(z)$ *given by* (1.1) *belongs to* $R_\gamma^\tau(\phi,a,c)$ $(0 \le \gamma \le 1,\ \tau \in C\setminus\{0\},\ a,c \in C,\ \{c \ne 0,-1,-2,....\},\ z \in U)$*, then for any complex number* μ

$$\left|a_3 - \mu a_2^2\right| \le \frac{B_1\,|\tau|\,c(c+1)}{3a(a+1)(1+2\gamma)}\max\left\{1, \left|\frac{B_2}{B_1} - \frac{3\mu\tau B_1 c(a+1)(1+2\gamma)}{4a(c+1)(1+\gamma)^2}\right|\right\}. \tag{2.1}$$

This result is sharp.

Proof. If $f(z) \in R_\gamma^\tau(\phi,a,c)$, then there exists a Schwarz function $w(z)$ analytic in U with $w(0) = 0$ and $|w(z)| < 1$ in U such that

$$1 + \frac{1}{\tau}\left((L(a,c)f(z))' + \gamma z(L(a,c)f(z))'' - 1\right) = \phi(w(z)),\quad (z \in U). \tag{2.2}$$

Define the function $p_1(z)$ by

$$p_1(z) = \frac{1+w(z)}{1-w(z)} = 1 + c_1z + c_2z^2 + ... \tag{2.3}$$

Since $w(z)$ is a Schwarz function, we see that $Re\{p_1(z)\} > 0$ and $p_1(0) = 1$.

Define the function $p(z)$ by,

$$p(z) = 1 + \frac{1}{\tau}\left((L(a,c)f(z))' + \gamma z(L(a,c)f(z))'' - 1\right) = 1 + b_1z + b_2z^2 + ... \tag{2.4}$$

In view of (2.2), (2.3), (2.4), we have

$$\begin{aligned} p(z) = \phi\left(\frac{p_1(z)-1}{p_1(z)+1}\right) &= \phi\left(\frac{c_1z + c_2z^2 + ...}{2 + c_1z + c_2z^2 + ...}\right) \\ &= \phi\left(\frac{1}{2}c_1z + \frac{1}{2}\left(c_2 - \frac{c_1^2}{2}\right)z^2 + ...\right) \\ = 1 + B_1\frac{1}{2}c_1z &+ B_1\frac{1}{2}\left(c_2 - \frac{c_1^2}{2}\right)z^2 + B_2\frac{1}{4}c_1^2z^2 + ... \end{aligned} \tag{2.5}$$

Thus,

$$b_1 = \frac{1}{2}B_1c_1; \quad b_2 = \frac{1}{2}B_1\left(c_2 - \frac{c_1^2}{2}\right) + \frac{1}{4}B_2c_1^2. \tag{2.6}$$

From (2.4), we obtain

$$a_2 = \frac{\tau\, B_1c_1c}{4a\,(1+\gamma)}; \quad a_3 = \frac{\tau\, c(c+1)}{6a(a+1)\,(1+2\gamma)}\left[B_1\left(c_2 - \frac{c_1^2}{2}\right) + \frac{1}{2}B_2c_1^2\right]. \tag{2.7}$$

Therefore, we have

$$a_3 - \mu a_2^2 = \frac{B_1\tau\, c(c+1)}{6a(a+1)\,(1+2\gamma)}\left(c_2 - \nu c_1^2\right) \tag{2.8}$$

where

$$\nu = \frac{1}{2}\left(1 - \frac{B_2}{B_1} + \frac{3\tau\mu B_1 c(a+1)\,(1+2\gamma)}{4a(c+1)(1+\gamma)^2}\right). \tag{2.9}$$

Our result now is followed by an application of Lemma 1.1. Also, by the application of Lemma 1.1 equality in (2.1) is obtained when

$$p_1\,(z) = \frac{1+z^2}{1-z^2} \text{ or } p_1\,(z) = \frac{1+z}{1-z} \tag{2.10}$$

but

$$p\,(z) = 1 + \frac{1}{\tau}\left((L(a,c)f(z))' + \gamma z(L(a,c)f(z))' - 1\right) = \phi\left(\frac{p_1\,(z)-1}{p_1\,(z)+1}\right). \tag{2.11}$$

Putting value of $p_1(z)$ we get the desired results. Thus the proof of Theorem 2.1 is established. □

For the class $R_\gamma^\tau\,(A,\ B,a,c)$,

$$\phi\,(z) = \frac{1+Az}{1+Bz} = (1+Az)\,(1+Bz)^{-1} = 1 + (A-B)\,z - \left(AB - B^2\right)z^2 + ... \tag{2.12}$$

Thus, putting $B_1 = A-B$ and $B_2 = -B\,(A-B)$ in Theorem 2.1, we get the following corollary.

Corollary 2.2. *If $f(z)$ given by (1.1) belongs to $R_\gamma^\tau\,(A,\ B,a,c)$, then*

$$\left|a_3 - \mu a_2^2\right| \le \frac{(A-B)\ \ |\tau|\,c(c+1)}{3a(a+1)\,(1+2\gamma)}\max\left\{1,\ \left|B + \frac{3\tau\mu c(a+1)\,(A-B)\,(1+2\gamma)}{4a(c+1)(1+\gamma)^2}\right|\right\}. \tag{2.13}$$

If we put $a = c$ in Theorem 2.1, then we obtain the following result of Bansal [4].

Corollary 2.3. *Let $\phi(z) = 1 + B_1z + B_2z^2 + B_3z^3 + ...$, where $\phi(z) \in A$ with $\phi'(0) > 0$. If $f(z)$ given by (1.1) belongs to $R_\gamma^\tau(\phi)(0 \le \gamma \le 1, \tau \in C\backslash\{0\}, z \in U)$ then for any complex number μ*

$$\left|a_3 - \mu a_2^2\right| \le \frac{B_1\,|\tau|}{3(1+2\gamma)}\max\left\{1, \left|\frac{B_2}{B_1} - \frac{3\mu\tau B_1(1+2\gamma)}{4(1+\gamma)^2}\right|\right\}.$$

This result is sharp.

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Fekete-Szegö problems for generalized Sakaguchi type functions associated with quasi-subordination

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Abstract. In the present paper, the authors introduce a generalized Sakaguchi type non-Bazilevic function class $\mathcal{M}_q^{\lambda,\beta}(\phi, s, t)$ of analytic functions involving quasi-subordination and obtain bounds for the Fekete-Szegö functional $|a_3 - \mu a_2^2|$ for the functions belonging to the above and associated classes. Some important and useful special cases of the main results are also pointed out.

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1. Introduction and preliminaries

Let $\mathcal{A}$ be the class of analytic functions in the open unit disk:

$$\mathbb{U} := \{z \in \mathbb{C} : |z| < 1\}$$

having the normalized power series expansion given by

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad (z \in \mathbb{U}). \tag{1.1}$$

A function $f(z) \in \mathcal{A}$ is said to be univalent in $\mathbb{U}$ if $f(z)$ is one-to-one in $\mathbb{U}$. As usual, we denote by $\mathcal{S}$ the subclass of $\mathcal{A}$ consisting of univalent functions in $\mathbb{U}$ (see [3]). For two functions f and g in $\mathcal{A}$, we say that f is *subordinate* to g in $\mathbb{U}$, and write as

$$f \prec g \text{ in } \mathbb{U} \quad \text{or} \quad f(z) \prec g(z) \quad (z \in \mathbb{U}),$$

if there exists a Schwarz function $w(z)$, analytic in $\mathbb{U}$ with $w(0) = 0$ and $|w(z)| < 1$, $z \in \mathbb{U}$ such that

$$f(z) = g(w(z)) \quad (z \in \mathbb{U}). \tag{1.2}$$

If the function g is univalent in $\mathbb{U}$, then

$$f(z) \prec g(z) \quad (z \in \mathbb{U}) \Longleftrightarrow f(0) = g(0) \text{ and } f(\mathbb{U}) \subset g(\mathbb{U}).$$

For a brief survey on the concept of subordination, we refer to the works in [3, 10, 13, 27].

Further, a function $f(z)$ is said to be quasi-subordinate to $g(z)$ in the unit disk $\mathbb{U}$ if there exists the functions $\varphi(z)$ and $w(z)$ (with constant coefficient zero) which are analytic and bounded by one in the unit disk $\mathbb{U}$ such that

$$\frac{f(z)}{\varphi(z)} \prec g(z) \quad (z \in \mathbb{U}). \tag{1.3}$$

We denote the quasi-subordination by

$$f(z) \prec_q g(z) \quad (z \in \mathbb{U}). \tag{1.4}$$

Also, we note that quasi-subordination (1.4) is equivalent to

$$f(z) = \varphi(z) g(w(z)) \quad (z \in \mathbb{U}). \tag{1.5}$$

One may observe that when $\varphi(z) \equiv 1$ $(z \in \mathbb{U})$, the quasi-subordination $\prec_q$ becomes the usual subordination $\prec$. If we put $w(z) = z$ in (1.5), then the quasi-subordination (1.5) becomes the majorization. In this case, we have

$$f(z) \prec_q g(z) \Longrightarrow f(z) = \varphi(z) g(z) \Longrightarrow f(z) \ll g(z) \quad (z \in \mathbb{U}).$$

The concept of majorization is due to MacGreogor [12] and quasi-subordination is thus a generalization of the usual subordination as well as the majorization. The work on quasi-subordination is quite extensive which includes some recent expository investigations in [1, 7, 9, 14, 21, 22].

Recently, Frasin [5] introduced and studied a generalized Sakaguchi type classes $\mathcal{S}(\alpha, s, t)$ and $\mathcal{T}(\alpha, s, t)$. A function $f(z) \in \mathcal{A}$ is said to be in the class $\mathcal{S}(\alpha, s, t)$ if it satisfies

$$\Re\left[\frac{(s-t)zf'(z)}{f(sz) - f(tz)}\right] > \alpha \tag{1.6}$$

for some α $(0 \leq \alpha < 1)$, $s,\ t \in \mathbb{C}$, $|s - t| \leq 1$, $s \neq t$ and $z \in \mathbb{U}$.

We also denote by $\mathcal{T}(\alpha, s, t)$, the subclass of $\mathcal{A}$ consisting of all functions $f(z)$ such that $zf'(z) \in \mathcal{S}(\alpha, s, t)$. For $s = 1$, the class $\mathcal{S}(\alpha, 1, t)$ becomes the subclass $\mathcal{S}^*(\alpha, t)$ studied by Owa et al. [17, 18]. If $t = -1$ in $\mathcal{S}(\alpha, 1, t)$, then the class $\mathcal{S}(\alpha, 1, -1) = \mathcal{S}_s(\alpha)$ was introduced by Sakaguchi [23] and is called Sakaguchi function of order α (see [2, 17]), whereas $\mathcal{S}_s(0) \equiv \mathcal{S}_s$ is the class of starlike functions with respect to symmetrical points in $\mathbb{U}$. Further, $\mathcal{S}(\alpha, 1, 0) \equiv \mathcal{S}^*(\alpha)$ and $\mathcal{T}(\alpha, 1, 0) \equiv \mathcal{C}(\alpha)$ are the familiar classes of starlike functions of order α $(0 \leq \alpha < 1)$ and convex function of order α $(0 \leq \alpha < 1)$, respectively.

Obradovic [16] introduced a class of functions $f \in \mathcal{A}$ which satisfies the inequality:

$$\Re\left[f'(z)\left(\frac{z}{f(z)}\right)^{1+\lambda}\right] > 0 \quad (0 < \lambda < 1; z \in \mathbb{U}), \tag{1.7}$$

and he calls such functions as functions of non-Bazilevič type.

By $\mathcal{P}$, we denote the class of functions ϕ analytic in $\mathbb{U}$ such that $\phi(0) = 1$ and $\Re(\phi(z)) > 0$.

Ma and Minda [11] unified various subclasses of starlike and convex functions for which either of the quantity $\frac{zf'(z)}{f(z)}$ or $1 + \frac{zf''(z)}{f'(z)}$ is subordinate to a more general subordination function. They introduced a class $S^*(\phi)$ defined by

$$S^*(\phi) = \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec \phi(z) \quad (z \in \mathbb{U}) \right\}, \tag{1.8}$$

where $\phi \in \mathcal{P}$ and $\phi(\mathbb{U})$ is symmetrical about the real axis and $\phi'(0) > 0$. A function $f \in S^*(\phi)$ is called a Ma and Minda starlike function with respect to ϕ.
Recently, Sharma and Raina [25] introduced and studied a generalized Sakaguchi type non-Bazilevic function class $\mathcal{G}_q^\lambda(\phi, b)$. A function $f(z) \in \mathcal{A}$ is said to be in the class $\mathcal{G}_q^\lambda(\phi, b)$ if it satisfies the condition that

$$\left[f'(z) \left(\frac{(1-b)z}{f(z) - f(bz)} \right)^\lambda - 1 \right] \prec_q (\phi(z) - 1) \quad (1 \neq b \in \mathbb{C}, |b| \leq 1, \lambda \geq 0; z \in \mathbb{U}). \tag{1.9}$$

Motivated by aforementioned works, we introduce here a new subclass of $\mathcal{A}$ which is defined as follows:
Definition 1.1. Let $\phi \in \mathcal{P}$ be univalent and $\phi(\mathbb{U})$ symmetrical about the real axis and $\phi'(0) > 0$. For $s,\ t \in \mathbb{C},\ s \neq t,\ |s - t| \leq 1,\ \lambda,\ \beta \geq 0$, a function $f \in \mathcal{A}$ is said to be in the class $\mathcal{M}_q^{\lambda,\beta}(\phi, s, t)$ if it satisfies the condition that

$$\left[(1-\beta) f'(z) + \beta \frac{f(z)}{z} \left(1 + \frac{zf''(z)}{f'(z)} \right) \right] \left[\frac{(s-t)z}{f(sz) - f(tz)} \right]^\lambda - 1 \prec_q (\phi(z) - 1) \ (z \in \mathbb{U}), \tag{1.10}$$

where the powers are considered to be having only principal values.

By specializing the parameters $\lambda,\ \beta,\ s,$ and t in Definition 1.1 above, we obtain various subclasses which have been studied recently. To illustrate these subclasses, we observe the following:
(i) When $\beta = 0, s = 1$, then the class $\mathcal{M}_q^{\lambda,0}(\phi, 1, t) = \mathcal{G}_q^\lambda(\phi, t)$ which was studied recently by Sharma and Raina [25].
(ii) Next, when $\beta = t = 0$, $\lambda = s = 1$; $\beta = \lambda = t = 0$, $s = 1$ and $\lambda = \beta = s = 1$, $t = 0$; then the classes $\mathcal{M}_q^{1,0}(\phi, 1, 0)$, $\mathcal{M}_q^{0,0}(\phi, 1, 0)$ and $\mathcal{M}_q^{1,,1}(\phi, 1, 0)$ which, respectively, reduce to the classes $\mathcal{S}_q^*(\phi)$, $\mathcal{R}_q(\phi)$ and $\mathcal{C}_q(\phi)$ were studied earlier by Mohd and Darus [14].

From the Definition 1.1, it follows that $f \in \mathcal{M}_q^{\lambda,\beta}(\phi, s, t)$ if and only if there exists an analytic function $\varphi(z)$ with $|\varphi(z)| \leq 1$ $(z \in \mathbb{U})$ such that

$$\frac{\left[(1-\beta) f'(z) + \beta \frac{f(z)}{z} \left(1 + \frac{zf''(z)}{f'(z)} \right) \right] \left[\frac{(s-t)z}{f(sz) - f(tz)} \right]^\lambda - 1}{\varphi(z)} \prec (\phi(z) - 1) \ (z \in \mathbb{U}). \tag{1.11}$$

If we set $\varphi(z) \equiv 1$ $(z \in \mathbb{U})$ in (1.11), then the class $\mathcal{M}_q^{\lambda,\beta}(\phi, s, t)$ is denoted by $\mathcal{M}^{\lambda,\beta}(\phi, s, t)$ satisfying the condition that

$$\left[(1-\beta) f'(z) + \beta \frac{f(z)}{z} \left(1 + \frac{zf''(z)}{f'(z)} \right) \right] \left[\frac{(s-t)z}{f(sz) - f(tz)} \right]^\lambda \prec \phi(z) \quad (z \in \mathbb{U}). \tag{1.12}$$

It may be noted that for $\beta = 0$, $s = \lambda = 1$ and for real t, the class $\mathcal{M}^{1,0}(\phi, 1, t) = \mathcal{S}^*(\phi, t)$ which was studied by Goyal and Goswami [6].

It is well-known (see [3]) that for $f \in \mathcal{S}$ given by (1.1), there holds a sharp inequality for the functional $|a_3 - a_2^2|$. Fekete-Szegö [4] obtained sharp upper bounds for $|a_3 - \mu a_2^2|$ for $f \in \mathcal{S}$ when μ is real and thus the determination of the sharp upper bounds for such a nonlinear functional for any compact family $\mathcal{F}$ of functions in $\mathcal{S}$ is popularly known as the Fekete-Szegö problem for $\mathcal{F}$. Fekete-Szegö problems for several subclasses of $\mathcal{S}$ have been investigated by many authors including [19, 20, 24]; see also [26].

The aim of this paper is to obtain the coefficient estimates including a Fekete-Szegö inequality of functions belonging to the classes $\mathcal{M}_q^{\lambda,\beta}(\phi, s, t)$ and $\mathcal{M}^{\lambda,\beta}(\phi, s, t)$ and the class involving the majorization. Some consequences of the main results are also pointed out.

We need the following lemma in our investigations.

Lemma 1.2. ([8, p.10]) *Let the Schwarz function $w(z)$ be given by*

$$w(z) = w_1 z + w_2 z^2 + w_3 z^3 + \cdots \quad (z \in \mathbb{U}), \tag{1.13}$$

then

$$|w_1| \le 1,\ |w_2 - \mu w_1^2| \le 1 + (|\mu| - 1)|w_1|^2 \le \max\{1, |\mu|\},$$

where $\mu \in \mathbb{C}$. The result is sharp for the function $w(z) = z$ or $w(z) = z^2$.

2. Main results

Let $f \in \mathcal{A}$ of the form (1.1), then for $s,\ t \in \mathbb{C}$, $|s - t| \le 1, s \neq t$, we may write that

$$\frac{f(sz) - f(tz)}{s - t} = z + \sum_{n=2}^{\infty} \gamma_n a_n z^n, \tag{2.1}$$

where

$$\gamma_n = \frac{s^n - t^n}{s - t} = s^{n-1} + s^{n-2}t + \cdots + t^{n-1} \quad (n \in \mathbb{N}). \tag{2.2}$$

Therefore for $\lambda \ge 0$, we have

$$\left[\frac{(s-t)z}{f(sz) - f(tz)}\right]^{\lambda} = 1 - \lambda\gamma_2 a_2 z + \lambda\left[\frac{\lambda+1}{2}\gamma_2^2 a_2^2 - \gamma_3 a_3\right] z^2 + \cdots. \tag{2.3}$$

Unless otherwise stated, throughout the sequel, we assume that

$$\lambda\gamma_n \neq (n-1)^2\beta + n;$$

and that for real s, t:

$$\lambda\gamma_n < (n-1)^2\beta + n,\ n = 2, 3, 4, \ldots .$$

Let the function $\phi \in \mathcal{P}$ be of the form

$$\phi(z) = 1 + B_1 z + B_2 z^2 + \cdots \quad (B_1 \in \mathbb{R}, B_1 > 0), \tag{2.4}$$

and $\varphi(z)$ analytic in $\mathbb{U}$ be of the form

$$\varphi(z) = c_0 + c_1 z + c_2 z^2 + \cdots (c_0 \neq 0). \tag{2.5}$$

We now state and prove our first main result.

Theorem 2.1. *Let the function $f \in \mathcal{A}$ of the form (1.1) be in the class $\mathcal{M}_q^{\lambda,\beta}(\phi, s, t)$, then*

$$|a_2| \leq \frac{B_1}{|2+\beta-\lambda\gamma_2|}, \tag{2.6}$$

and for any $\mu \in \mathbb{C}$:

$$|a_3 - \mu a_2^2| \leq \frac{B_1}{|3+4\beta-\lambda\gamma_3|} \max\left\{1, \left|\frac{B_2}{B_1} - B_1 R\right|\right\}, \tag{2.7}$$

where

$$R = \frac{(3+4\beta-\lambda\gamma_3)\mu}{(2+\beta-\lambda\gamma_2)^2} - \frac{\lambda\left(4+2\beta-(1+\lambda)\gamma_2\right)\gamma_2+4\beta}{2(2+\beta-\lambda\gamma_2)^2} \tag{2.8}$$

and γ_n $(n \in \mathbb{N})$ is given by (2.2). The result is sharp.

Proof. Let $f \in \mathcal{M}_q^{\lambda,\beta}(\phi, s, t)$. In view of Definition 1.1, there exists then a Schwarz function $w(z)$ given by (1.13) and an analytic function $\varphi(z)$ given by (2.5) such that

$$\left[(1-\beta)f'(z) + \beta\frac{f(z)}{z}\left(1+\frac{zf''(z)}{f'(z)}\right)\right]\left[\frac{(s-t)z}{f(sz)-f(tz)}\right]^{\lambda} - 1 = \varphi(z)(\phi(w(z))-1), \tag{2.9}$$

which can be expressed as

$$\begin{aligned}\varphi(z)(\phi(w(z))-1) &= (c_0 + c_1 z + c_2 z^2 + \cdots)\left(B_1 w_1 z + (B_1 w_2 + B_2 w_1^2)z^2 + \cdots\right)\\ &= c_0 B_1 w_1 z + \left\{c_0(B_1 w_2 + B_2 w_1^2) + c_1 B_1 w_1\right\} z^2 + \cdots .\end{aligned} \tag{2.10}$$

Using now the series expansions for $f'(z)$ and $f''(z)$ from (1.1), we obtain that

$$(1-\beta)f'(z) + \beta\frac{f(z)}{z}\left[1+\frac{zf''(z)}{f'(z)}\right] = 1 + (2+\beta)a_2 z + \left((3+4\beta)a_3 - 2\beta a_2^2\right)z^2 + \cdots . \tag{2.11}$$

Thus, it follows from (2.3) and (2.11) that

$$\begin{aligned}&\left[(1-\beta)f'(z) + \beta\frac{f(z)}{z}\left(1+\frac{zf''(z)}{f'(z)}\right)\right]\left[\frac{(s-t)z}{f(sz)-f(tz)}\right]^{\lambda} - 1\\ &= (2+\beta-\lambda\gamma_2)a_2 z + \left[(3+4\beta-\lambda\gamma_3)a_3 - \lambda\left(2+\beta-\frac{1+\lambda}{2}\gamma_2\right)\gamma_2 a_2^2 - 2\beta a_2^2\right]z^2 + \cdots .\end{aligned} \tag{2.12}$$

Making use of (2.10) and (2.12) in (2.9) and equating the coefficients of z and z^2 in the resulting expression, we get

$$(2+\beta-\lambda\gamma_2)a_2 = c_0 B_1 w_1 \tag{2.13}$$

and

$$(3+4\beta-\lambda\gamma_3)a_3 - \lambda\left[2+\beta-\frac{1+\lambda}{2}\gamma_2\right]\gamma_2 a_2^2 - 2\beta a_2^2 = c_0(B_1 w_2 + B_2 w_1^2) + c_1 B_1 w_1. \tag{2.14}$$

Now (2.13) yields that

$$a_2 = \frac{c_0 B_1 w_1}{2+\beta-\lambda\gamma_2}. \tag{2.15}$$

From (2.14), we have

$$a_3 = \frac{B_1}{3+4\beta-\lambda\gamma_3}\left[c_1 w_1 + c_0\left\{w_2 + \left(\frac{c_0\lambda\left(1+\frac{2+\beta-\gamma_2}{2+\beta-\lambda\gamma_2}\right)\gamma_2 B_1}{2(2+\beta-\lambda\gamma_2)}\frac{2\beta c_0 B_1}{(2+\beta-\lambda\gamma_2)^2}+\frac{B_2}{B_1}\right)w_1^2\right\}\right]. \tag{2.16}$$

Hence, for any complex number μ, we have

$$\begin{aligned} a_3-\mu a_2^2 &= \frac{B_1}{3+4\beta-\lambda\gamma_3}\left[c_1 w_1 + c_0\left\{w_2+\left(\frac{\lambda(4+2\beta-(1+\lambda)\gamma_2)\gamma_2+4\beta}{2(2+\beta-\lambda\gamma_2)^2}c_0 B_1+\frac{B_2}{B_1}\right)w_1^2\right\}\right]-\mu\frac{c_0^2B_1^2w_1^2}{(2+\beta-\lambda\gamma_2)^2} \\ &= \frac{B_1}{3+4\beta-\lambda\gamma_3}\left[c_1w_1+\left(w_2+\frac{B_2}{B_1}w_1^2\right)c_0-B_1c_0^2w_1^2R\right], \end{aligned} \tag{2.17}$$

where R is given by (2.8).
Since $\varphi(z)$ given by (2.5) is analytic and bounded in the open unit disk $\mathbb{U}$, hence upon using [15, p. 172], we have for some y ($|y| \leq 1$):

$$|c_0| \leq 1 \text{ and } c_1 = (1-c_0^2)y. \tag{2.18}$$

Putting the value of c_1 from (2.18) into (2.17), we finally get

$$a_3-\mu a_2^2 = \frac{B_1}{3+4\beta-\lambda\gamma_3}\left[yw_1+\left(w_2+\frac{B_2}{B_1}w_1^2\right)c_0-(B_1w_1^2R+w_1y)c_0^2\right]. \tag{2.19}$$

If $c_0 = 0$, then (2.19) gives

$$|a_3-\mu a_2^2| \leq \frac{B_1}{|3+4\beta-\lambda\gamma_3|}. \tag{2.20}$$

On the other hand, if $c_0 \neq 0$, then we consider

$$T(c_0) = yw_1+\left(w_2+\frac{B_2}{B_1}w_1^2\right)c_0-(B_1w_1^2R+w_1y)c_0^2. \tag{2.21}$$

The expression (2.21) is a quadratic polynomial in c_0 and hence analytic in $|c_0| \leq 1$. The maximum value of $|T(c_0)|$ is attained at $c_0 = e^{i\theta}$ $(0 \leq \theta < 2\pi)$, and hence, we have

$$\begin{aligned} \max|T(c_0)| &= \max_{0\leq\theta<2\pi}|T(e^{i\theta})| = |T(1)| \\ &= \left|w_2-\left(B_1R-\frac{B_2}{B_1}\right)w_1^2\right|. \end{aligned}$$

Thus from (2.19), we get

$$|a_3-\mu a_2^2| \leq \frac{B_1}{|3+4\beta-\lambda\gamma_3|}\left|w_2-\left(B_1R-\frac{B_2}{B_1}\right)w_1^2\right|, \tag{2.22}$$

and in view of Lemma 1.2, we obtain that

$$|a_3-\mu a_2^2| \leq \frac{B_1}{|3+4\beta-\lambda\gamma_3|}\max\left\{1,\left|B_1R-\frac{B_2}{B_1}\right|\right\}. \tag{2.23}$$

The desired assertion (2.7) follows now from (2.20) and (2.23).
The result is sharp for the function $f(z)$ given by

$$\left[(1-\beta)f'(z)+\beta\frac{f(z)}{z}\left(1+\frac{zf''(z)}{f'(z)}\right)\right]\left[\frac{(s-t)z}{f(sz)-f(tz)}\right]^{\lambda}=\phi(z)$$

or

$$\left[(1-\beta)f'(z)+\beta\frac{f(z)}{z}\left(1+\frac{zf''(z)}{f'(z)}\right)\right]\left[\frac{(s-t)z}{f(sz)-f(tz)}\right]^{\lambda}=\phi(z^2).$$

This completes the proof of Theorem 2.1. □

By setting $\beta=t=0,\ \lambda=s=1$ in the Theorem 2.1, we obtain the following sharp results for the subclass $\mathcal{S}_q^*(\phi)$.

Corollary 2.2. *Let $f\in\mathcal{A}$ of the form (1.1) be in the class $\mathcal{S}_q^*(\phi)$, then*

$$|a_2|\le B_1,$$

and for any $\mu\in\mathbb{C}$:

$$|a_3-\mu a_2^2|\le\frac{B_1}{2}\max\left\{1,\left|\frac{B_2}{B_1}+(1-2\mu)B_1\right|\right\}.$$

The result is sharp.

Next, putting $\beta=\lambda=s=1$ and $t=0$ in Theorem 2.1, we obtain the following sharp results for the class $\mathcal{C}_q(\phi)$.

Corollary 2.3. *Let $f\in\mathcal{A}$ of the form (1.1) belong to the class $\mathcal{C}_q(\phi)$, then*

$$|a_2|\le\frac{B_1}{2},$$

and for any $\mu\in\mathbb{C}$:

$$|a_3-\mu a_2^2|\le\frac{B_1}{6}\max\left\{1,\left|\frac{B_2}{B_1}+\left(1-\frac{3\mu}{2}\right)B_1\right|\right\}.$$

The result is sharp.

Further, by putting $\beta=\lambda=t=0$ and $s=1$ in Theorem 2.1, we get the following sharp results for the class $\mathcal{R}_q(\phi)$.

Corollary 2.4. *Let $f\in\mathcal{A}$ of the form (1.1) belong to the class $\mathcal{R}_q(\phi)$, then*

$$|a_2|\le\frac{B_1}{2},$$

and for any $\mu\in\mathbb{C}$:

$$|a_3-\mu a_2^2|\le\frac{B_1}{3}\max\left\{1,\left|\frac{B_2}{B_1}-\frac{3\mu}{4}B_1\right|\right\}.$$

The result is sharp.

Remark 2.5. The Fekete-Szegö type inequalities mentioned above for the classes $\mathcal{S}_q^*(\phi)$, $\mathcal{C}_q(\phi)$ and $\mathcal{R}_q(\phi)$ improve similar results obtained earlier in [14].

The next theorem gives the result for the class $\mathcal{M}^{\lambda,\beta}(\phi,s,t)$.

Theorem 2.6. *Let $f \in \mathcal{A}$ of the form* (1.1) *belong to the class $\mathcal{M}^{\lambda,\beta}(\phi, s, t)$, then*

$$|a_2| \leq \frac{B_1}{|2+\beta-\lambda\gamma_2|},$$

and for any $\mu \in \mathbb{C}$:

$$|a_3 - \mu a_2^2| \leq \frac{B_1}{|3+4\beta-\lambda\gamma_3|} \max\left\{1, \left|\frac{B_2}{B_1} - B_1 R\right|\right\},$$

where R is given by (2.8) *and γ_n $(n \in \mathbb{N})$ is given by* (2.2). *The result is sharp.*

Proof. The proof is similar to Theorem 2.1. Let $f \in \mathcal{M}^{\lambda,\beta}(\phi, s, t)$. If $\varphi(z) \equiv 1$, then (2.5) gives $c_0 = 1$ and $c_n = 0$ $(n \in \mathbb{N})$. Therefore, in view of (2.15), (2.17) and by an application of Lemma 1.2, we obtain the desired assertion. The result is sharp for the function $f(z)$ given by

$$\left[(1-\beta)f'(z) + \beta\frac{f(z)}{z}\left(1 + \frac{zf''(z)}{f'(z)}\right)\right]\left[\frac{(s-t)z}{f(sz)-f(tz)}\right]^{\lambda} = \phi(z)$$

or

$$\left[(1-\beta)f'(z) + \beta\frac{f(z)}{z}\left(1 + \frac{zf''(z)}{f'(z)}\right)\right]\left[\frac{(s-t)z}{f(sz)-f(tz)}\right]^{\lambda} = \phi(z^2).$$

□

The next theorem gives the result based on majorization.

Theorem 2.7. *Let $s, t \in \mathbb{C}$, $s \neq t$, $|s-t| \leq 1$.*

If a function $f \in \mathcal{A}$ of the form (1.1) *satisfies*

$$\left[(1-\beta)f'(z) + \beta\frac{f(z)}{z}\left(1 + \frac{zf''(z)}{f'(z)}\right)\right]\left[\frac{(s-t)z}{f(sz)-f(tz)}\right]^{\lambda} - 1 \ll \phi(z) - 1 \quad (z \in \mathbb{U}), \tag{2.24}$$

then

$$|a_2| \leq \frac{B_1}{|2+\beta-\lambda\gamma_2|},$$

and for any $\mu \in \mathbb{C}$:

$$|a_3 - \mu a_2^2| \leq \frac{B_1}{|3+4\beta-\lambda\gamma_3|} \max\left\{1, \left|\frac{B_2}{B_1} - B_1 R\right|\right\},$$

where R is given by (2.8) *and γ_n $(n \in \mathbb{N})$ is defined as* (2.2). *The result is sharp.*

Proof. Assume that (2.24) holds true. Hence, by the definition of majorization there exists an analytic function $\varphi(z)$ given by (2.5) such that for $z \in \mathbb{U}$ we have

$$\left[(1-\beta)f'(z) + \beta\frac{f(z)}{z}\left(1 + \frac{zf''(z)}{f'(z)}\right)\right]\left[\frac{(s-t)z}{f(sz)-f(tz)}\right]^{\lambda} - 1 = \varphi(z)(\phi(z)-1). \tag{2.25}$$

Following similar steps as in the proof of Theorem 2.1 and by setting $w(z) \equiv 1$, so that $w_1 = 1$ and $w_n = 0$, $n \geq 2$, we obtain

$$a_2 = \frac{c_0 B_1}{2+\beta-\lambda\gamma_2},$$

so that

$$|a_2| \leq \frac{B_1}{|2+\beta-\lambda\gamma_2|}$$

and

$$a_3 - \mu a_2^2 = \frac{B_1}{3+4\beta-\lambda\gamma_3}\left[c_1 + \frac{B_2}{B_1}c_0 - B_1 c_0^2 R\right]. \tag{2.26}$$

On putting the value of c_1 from (2.18) in (2.26), we get

$$a_3 - \mu a_2^2 = \frac{B_1}{3+4\beta-\lambda\gamma_3}\left[y + \frac{B_2}{B_1}c_0 - (B_1 R + y)c_0^2\right]. \tag{2.27}$$

If $c_0 = 0$, then (2.27) yields

$$|a_3 - \mu a_2^2| \leq \frac{B_1}{|3+4\beta-\lambda\gamma_3|}. \tag{2.28}$$

But if $c_0 \neq 0$, then we define the function

$$H(c_0) := y + \frac{B_2}{B_1}c_0 - (B_1 R + y)c_0^2. \tag{2.29}$$

The expression (2.29) is a polynomial in c_0 and hence analytic in $|c_0| \leq 1$. The maximum value of $|H(c_0)|$ occurs at $c_0 = e^{i\theta}$ $(0 \leq \theta < 2\pi)$, and we have

$$\max_{0\leq\theta<2\pi} H(e^{i\theta}) = |H(1)|.$$

From (2.27), we get

$$|a_3 - \mu a_2^2| \leq \frac{B_1}{|3+4\beta-\lambda\gamma_3|}\left|B_1 R - \frac{B_2}{B_1}\right|. \tag{2.30}$$

Thus, the assertion of Theorem 2.7 follows from (2.28) and (2.30). The result is sharp for the function given by

$$\left[(1-\beta)f'(z) + \beta\frac{f(z)}{z}\left(1 + \frac{zf''(z)}{f'(z)}\right)\right]\left[\frac{(s-t)z}{f(sz)-f(tz)}\right]^{\lambda} = \phi(z) \quad (z \in \mathbb{U}).$$

This completes the proof of Theorem 2.7. ☐

Next, we determine the bounds for the functional $|a_3 - \mu a_2^2|$ for real μ, s and t for the class $\mathcal{M}_q^{\lambda,\beta}(\phi, s, t)$.

Corollary 2.8. *Let the function $f \in \mathcal{A}$ given by* (1.1) *be in the class $\mathcal{M}_q^{\lambda,\beta}(\phi, s, t)$, then (for real values of μ, s, t):*

$$|a_3 - \mu a_2^2| \leq \begin{cases} \frac{B_1}{3+4\beta-\lambda\gamma_3}\left[B_1 Q + \frac{B_2}{B_1}\right] & \mu \leq \alpha_1, \\ \frac{B_1}{3+4\beta-\lambda\gamma_3} & \alpha_1 \leq \mu \leq \alpha_1 + 2\rho, \\ -\frac{B_1}{3+4\beta-\lambda\gamma_3}\left[B_1 Q + \frac{B_2}{B_1}\right] & \mu \geq \alpha_1 + 2\rho, \end{cases} \tag{2.31}$$

where

$$\alpha_1 = \frac{\lambda(4+2\beta-(1+\lambda)\gamma_2)\gamma_2 + 4\beta}{2(3+4\beta-\lambda\gamma_3)} - \frac{(2+\beta-\lambda\gamma_2)^2}{(3+4\beta-\lambda\gamma_3)}\left(\frac{1}{B_1} - \frac{B_2}{B_1^2}\right), \tag{2.32}$$

$$\rho = \frac{(2+\beta-\lambda\gamma_2)^2}{(3+4\beta-\lambda\gamma_3)B_1}, \tag{2.33}$$

$$Q = \frac{\lambda\{4+2\beta-(1+\lambda)\gamma_2\}\gamma_2+4\beta-2\mu(3+4\beta-\lambda\gamma_3)}{2(2+\beta-\lambda\gamma_2)^2}$$

and γ_n $(n \in \mathbb{N})$ *is given by (*2.2*). Each of the estimates in (*2.31*) is sharp.*

Proof. For $s,\ t,\ \mu \in \mathbb{R}$, the above bounds can be obtained from (2.7), respectively, under the following cases:

$$B_1R - \frac{B_2}{B_1} \leq -1,\ -1 \leq B_1R - \frac{B_2}{B_1} \leq 1 \text{ and } B_1R - \frac{B_2}{B_1} \geq 1,$$

where R is given by (2.8). We also note the following:
(i) When $\mu < \alpha_1$ or $\mu > \alpha_1 + 2\rho$, then the equality holds if and only if $w(z) = z$ or one of its rotations.
(ii) When $\alpha_1 < \mu < \alpha_1 + 2\rho$, then the inequality holds if and only if $w(z) = z^2$ or one of its rotation.
(iii) Equality holds for $\mu = \alpha_1$ if and only if $w(z) = \frac{z(z+\epsilon)}{1+\epsilon z}$ $(0 \leq \epsilon \leq 1)$ or one of its rotations, while for $\mu = \alpha_1 + 2\rho$, the equality holds if and only if $w(z) = -\frac{z(z+\epsilon)}{1+\epsilon z}$ $(0 \leq \epsilon \leq 1)$, or one of its rotations. □

The second part of assertion in (2.31) can be improved further.

Theorem 2.9. *Let* $f \in \mathcal{A}$ *of the form (*1.1*) belong to the class* $\mathcal{M}_q^{\lambda,\beta}(\phi, s, t)$*, then (for* $s, t, \mu \in \mathbb{R}$ $(\alpha_1 \leq \mu \leq \alpha_1 + 2\rho)$*)*

$$|a_3 - \mu a_2^2| + (\mu - \alpha_1)|a_2|^2 \leq \frac{B_1}{3+4\beta-\lambda\gamma_3} \quad (\alpha_1 \leq \mu \leq \alpha_1 + \rho) \tag{2.34}$$

and

$$|a_3 - \mu a_2^2| + (\alpha_1 + 2\rho - \mu)|a_2|^2 \leq \frac{B_1}{3+4\beta-\lambda\gamma_3} \quad (\alpha_1 + \rho, < \mu < \alpha_1 + 2\rho) \tag{2.35}$$

where α_1 *and* ρ *are given by (*2.32*) and (*2.33*), respectively, and* γ_3 *is given by (*2.2*).*

Proof. Let $f \in \mathcal{M}_q^{\lambda,\beta}(\phi, s, t)$. For $s, t, \mu \in \mathbb{R}$ and $\alpha_1 \leq \mu \leq \alpha_1 + \rho$, and in view of (2.15) and (2.22), we get

$$|a_3 - \mu a_2^2| + (\mu - \alpha_1)|a_2|^2 \leq \frac{B_1}{3+4\beta-\lambda\gamma_3}$$

$$\cdot \left[|w_2| - \frac{B_1(3+4\beta-\lambda\gamma_3)}{(2+\beta-\lambda\gamma_2)^2}(\mu-\alpha_1-\rho)|w_1|^2 + \frac{B_1(3+4\beta-\lambda\gamma_3)}{(2+\beta-\lambda\gamma_2)^2}(\mu-\alpha_1)|w_1|^2\right].$$

Hence, by virtue of Lemma 1.2, we have

$$|a_3 - \mu a_2^2| + (\mu - \alpha_1)|a_2|^2 \leq \frac{B_1}{3+4\beta-\lambda\gamma_3}\left[1 - |w_1|^2 + |w_1|^2\right],$$

which yields the assertion (2.34).
If $\alpha_1 + \rho < \mu < \alpha_1 + 2\rho$, then again from (2.15) and (2.22) and Lemma 1.2, we obtain

$$|a_3 - \mu a_2^2| + (\alpha_1 + 2\rho - \mu)|a_2|^2 \leq \frac{B_1}{3+4\beta-\lambda\gamma_3}$$

$$\cdot \left[|w_2| + \frac{B_1(3+4\beta-\lambda\gamma_3)}{(2+\beta-\lambda\gamma_2)^2}(\mu-\alpha_1-\rho)|w_1|^2 + \frac{B_1(3+4\beta-\lambda\gamma_3)}{(2+\beta-\lambda\gamma_2)^2}(\alpha_1+2\rho-\mu)|w_1|^2\right]$$
$$\leq \frac{B_1}{3+4\beta-\lambda\gamma_3}[1-|w_1|^2+|w_1|^2],$$

which gives the estimate (2.35). □

We conclude this paper by remarking that the above theorems include several previously established results for particular values of the parameters λ, s, t and β. Thus, if we set $\beta = 0, s = 1$ in Theorems 2.1 and 2.6 above, we arrive at the Fekete-Szegö type inequalities for the classes $\mathcal{G}_q^\lambda(\phi, t)$ and $\mathcal{G}^\lambda(\phi, t)$, respectively, studied by Sharma and Raina [25]. Further, the majorization result and improvement of bounds given by Theorems 2.7 and 2.9 provide extensions of similar results due to Sharma and Raina [25].

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Certain sufficient conditions for starlikeness and convexity using a multiplier transformation

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Abstract. In the present paper, we study a differential subordination involving a multiplier transformation. Selecting different dominants to our main result, we obtain certain sufficient conditions for starlikeness and convexity of analytic functions. In particular, we obtain the sufficient conditions for parabolic starlikeness and uniform convexity. Some known results appear as particular cases of our main result.

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1. Introduction

Let $\mathcal{A}_p$ denote the class of functions of the form

$$f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k, \ (p \in \mathbb{N}; z \in \mathbb{E})$$

which are analytic and p-valent in the open unit disk $\mathbb{E} = \{z : |z| < 1\}$. Obviously, $\mathcal{A}_1 = \mathcal{A}$, the class of all analytic functions f, normalized by the conditions $f(0) = f'(0) - 1 = 0$. Let the functions f and g be analytic in $\mathbb{E}$. We say that f is subordinate to g in $\mathbb{E}$ (written as $f \prec g$), if there exists a Schwarz function ϕ in $\mathbb{E}$ (i.e. ϕ is regular in $|z| < 1, \phi(0) = 0$ and $|\phi(z)| < 1$ for all $z \in \mathbb{E}$) such that

$$f(z) = g(\phi(z)), \ |z| < 1.$$

Let $\Phi : \mathbb{C}^2 \times \mathbb{E} \to \mathbb{C}$ be an analytic function, p an analytic function in $\mathbb{E}$ with $(p(z), zp'(z); z) \in \mathbb{C}^2 \times \mathbb{E}$ for all $z \in \mathbb{E}$ and h be univalent in $\mathbb{E}$. Then the function p is said to satisfy first order differential subordination if

$$\Phi(p(z), zp'(z); z) \prec h(z), \Phi(p(0), 0; 0) = h(0). \tag{1.1}$$

A univalent function q is called a dominant of the differential subordination (1.1) if $p(0) = q(0)$ and $p(z) \prec q(z)$ for all p satisfying (1.1). A dominant $\tilde{q}$ that satisfies $\tilde{q}(z) \prec q(z)$ for all dominants q of (1.1), is said to be the best dominant of (1.1). The best dominant is unique up to a rotation of $\mathbb{E}$.
A function $f \in \mathcal{A}_p$ is said to be p-valent starlike of order $\alpha(0 \leq \alpha < p)$ in $\mathbb{E}$ if

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > \alpha, z \in \mathbb{E}.$$

Let $\mathcal{S}_p^*(\alpha)$ denote the class of p-valent starlike functions of order α. Write $\mathcal{S}_p^*(0) = \mathcal{S}_p^*$, the class of p-valent starlike functions.
A function $f \in \mathcal{A}_p$ is said to be p-valent convex of order $\alpha(0 \leq \alpha < p)$ in $\mathbb{E}$ if it satisfies the condition

$$\Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha, z \in \mathbb{E}.$$

Let the class of such functions be denoted by $\mathcal{K}_p(\alpha)$. Let $\mathcal{K}_p(0) = \mathcal{K}_p$, the class of p-valent convex functions.
A function $f \in \mathcal{A}_p$ is said to be p-valent parabolic starlike in $\mathbb{E}$ if

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > \left|\frac{zf'(z)}{f(z)} - p\right|, z \in \mathbb{E}. \tag{1.2}$$

Let $\mathcal{S}_{\mathcal{P}}^p$ denote the class of p-valent parabolic starlike functions. Write $\mathcal{S}_{\mathcal{P}}^1 = \mathcal{S}_{\mathcal{P}}$, the class of parabolic starlike functions.
A function $f \in \mathcal{A}_p$ is said to be uniformly p-valent convex in $\mathbb{E}$ if

$$\Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \left|\frac{zf''(z)}{f'(z)} - (p-1)\right|, z \in \mathbb{E}, \tag{1.3}$$

and is denoted by UCV_p, the class of uniformly p-valent convex functions and let $UCV_1 = UCV$, the class of uniformly convex functions.
Define the parabolic domain Ω as under

$$\Omega = \{u + iv : u > \sqrt{(u-p)^2 + v^2}\}.$$

Define the function

$$q(z) = p + \frac{2p}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2 \tag{1.4}$$

by considering only the principal values of logarithmic function. Clearly q maps the unit disk $\mathbb{E}$ onto the domain Ω. Hence the conditions (1.2) and (1.3) are equivalent to

$$\frac{zf'(z)}{f(z)} \prec q(z),$$

and

$$1 + \frac{zf''(z)}{f'(z)} \prec q(z),$$

respectively, where q is given by (1.4).
Ronning [16] and Ma and Minda [12] studied the domain Ω and the above function

q in a special case where $p = 1$. For $f \in \mathcal{A}_p$, we define the multiplier transformation $I_p(n, \lambda)$ as

$$I_p(n,\lambda)[f](z) = z^p + \sum_{k=p+1}^{\infty} \left(\frac{k+\lambda}{p+\lambda}\right)^n a_k z^k, \text{where } \lambda \geq 0, n \in \mathbb{Z}.$$

Recently, Billing [2, 3, 4, 5, 6], Singh et al. [18, 19], Brar and Billing [7] investigated the operator $I_p(n, \lambda)$ and obtained certain sufficient conditions for starlike and convex functions. Earlier, this operator was studied by Aghalary et al. [1]. In 2003, Cho and Srivastava [9] and Cho and Kim [8] investigated the operator $I_1(n, \lambda)$, whereas Uralegaddi and Somanatha [20] studied the operator $I_1(n, 1)$. The operator $I_1(n, 0)$ is the well-known Sălăgean [17] derivative operator

$$D^n f(z) = z + \sum_{k=2}^{\infty} k^n a_k z^k,\ n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$$

and $f \in \mathcal{A}$.
Let $\mathcal{S}_n(\alpha)$ denote the class of functions $f \in \mathcal{A}$ for which

$$\Re\left(\frac{D^{n+1}[f](z)}{D^n[f](z)}\right) > \alpha, z \in \mathbb{E}, 0 \leq \alpha < 1.$$

In 1989, Owa, Shen and Obradović [15] studied this class and obtained some sufficient conditions for $f \in \mathcal{A}$ to be a member of the class $\mathcal{S}_n(\alpha)$ in terms of differential inequality. Later on, Li and Owa [11] extended the result of Obradović .
Let $\mathcal{S}_n(p, \lambda, \alpha)$ denote the class of functions $f \in \mathcal{A}_p$ for which

$$\Re\left(\frac{I_p(n+1,\lambda)[f](z)}{I_p(n,\lambda)[f](z)}\right) > \frac{\alpha}{p}, z \in \mathbb{E}, 0 \leq \alpha < p.$$

In 2008, Billing et al. [18] investigated the above class and proved the sufficient condition for a multivalent function to be a member of this class.
In the present paper, we study a differential subordination involving multiplier transformation $I_p(n, \lambda)$ defined above. In particular cases to our main result, we obtain sufficient conditions for parabolic starlikeness, starlikeness, uniform convexity and convexity of multivalent/univalent analytic functions. Some known results are also obtained as particular cases of our main result.
To prove our main results, we shall use the following lemma of Miller and Mocanu ([13], [14], p.132).

Lemma 1.1. *Let q be a univalent in $\mathbb{E}$ and let θ and ϕ be analytic in a domain $\mathbb{D}$ containing $q(\mathbb{E})$, with $\phi(w) \neq 0$, when $w \in q(\mathbb{E})$. Set*

$$Q(z) = zq'(z)\phi[q(z)],\ h(z) = \theta[q(z)] + Q(z)$$

and suppose that either

(i) h is convex, or

(ii) Q is starlike.

In addition, assume that

(iii) $\Re\left(\dfrac{zh'(z)}{Q(z)}\right) > 0$ *for all* z *in* $\mathbb{E}$.

If p *is analytic in* $\mathbb{E}$, *with* $p(0) = q(0)$, $p(\mathbb{E}) \subset \mathbb{D}$ *and*

$$\theta[p(z)] + zp'(z)\phi[p(z)] \prec \theta[q(z)] + zq'(z)\phi[q(z)],$$

then $p(z) \prec q(z)$ *and* q *is the best dominant.*

2. Main results

Theorem 2.1. *Let* q *be a univalent function in* $\mathbb{E}$ *such that*

(i) $\Re\left[1 + \dfrac{zq''(z)}{q'(z)} - \dfrac{zq'(z)}{q(z)}\right] > 0$

(ii) $\Re\left[1 + \dfrac{zq''(z)}{q'(z)} - \dfrac{zq'(z)}{q(z)} + \dfrac{p+\lambda}{\alpha}q(z)\right] > 0.$

If $f \in \mathcal{A}_p$ *satisfies*

$$(1-\alpha)\left(\frac{I_p(n+1,\lambda)[f](z)}{I_p(n,\lambda)[f](z)}\right) + \alpha\left(\frac{I_p(n+2,\lambda)[f](z)}{I_p(n+1,\lambda)[f](z)}\right) \prec q(z) + \frac{\alpha}{p+\lambda}\frac{zq'(z)}{q(z)}, \quad (2.1)$$

then

$$\frac{I_p(n+1,\lambda)[f](z)}{I_p(n,\lambda)[f](z)} \prec q(z),$$

where $\lambda \geq 0, n \in \mathbb{N}_0$ *and* α *is a non zero complex number*

Proof. On writing $\dfrac{I_p(n+1,\lambda)[f](z)}{I_p(n,\lambda)[f](z)} = u(z)$, in (2.1), we obtain:

$$u(z) + \frac{\alpha}{p+\lambda}\frac{zu'(z)}{u(z)} \prec q(z) + \frac{\alpha}{p+\lambda}\frac{zq'(z)}{q(z)},$$

Let us define the functions θ and ϕ as follows:

$$\theta(w) = w$$

and

$$\phi(w) = \frac{\alpha}{p+\lambda}\frac{1}{w}.$$

Therefore,

$$Q(z) = \phi(q(z))zq'(z) = \frac{\alpha}{p+\lambda}\frac{zq'(z)}{q(z)}$$

and

$$h(z) = \theta(q(z)) + Q(z) = q(z) + \frac{\alpha}{p+\lambda}\frac{zq'(z)}{q(z)}.$$

On differentiating, we obtain $\dfrac{zQ'(z)}{Q(z)} = 1 + \dfrac{zq''(z)}{q'(z)} - \dfrac{zq'(z)}{q(z)}$ and

$$\frac{zh'(z)}{Q(z)} = 1 + \frac{zq''(z)}{q'(z)} - \frac{zq'(z)}{q(z)} + \frac{p+\lambda}{\alpha}q(z).$$

In view of the given conditions, we see that Q is starlike and $\Re\left(\frac{zh'(z)}{Q(z)}\right) > 0.$ Therefore, the proof, now follows from Lemma 1.1. □

3. Special cases and applications

Setting $\lambda = 0$ *and* $p = 1$ in Theorem 2.1, we obtain the following result.

Corollary 3.1. *If $f \in \mathcal{A}$ satisfies*

$$(1-\alpha)\left(\frac{D^{n+1}[f](z)}{D^n[f](z)}\right) + \alpha\left(\frac{D^{n+2}[f](z)}{D^{n+1}[f](z)}\right) \prec q(z) + \alpha\frac{zq'(z)}{q(z)},$$

then

$$\frac{D^{n+1}[f](z)}{D^n[f](z)} \prec q(z),$$

where $n \in \mathbb{N}_0$ and α is a non zero complex number.

Setting $\alpha = 1$ in Theorem 2.1, we get the following result of Shivaprasad Kumar et. al. [10]:

Corollary 3.2. *Let ψ be univalent in $\mathbb{E}$, $\psi(0) = 1, \Re\psi(z) > 0$ and $\frac{z\psi'(z)}{\psi(z)}$ be starlike in $\mathbb{E}$. Suppose $f \in \mathcal{A}_p$ satisfies*

$$\frac{I_p(n+2,\lambda)[f](z)}{I_p(n+1,\lambda)[f](z)} \prec \psi(z) + \frac{z\psi'(z)}{(p+\lambda)\psi(z)},$$

then

$$\frac{I_p(n+1,\lambda)[f](z)}{I_p(n,\lambda)[f](z)} \prec \psi(z).$$

When we select the dominant $q(z) = 1 + \frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2$ in Theorem 2.1, a simple calculation yields that

$$1 + \frac{zq''(z)}{q'(z)} - \frac{zq'(z)}{q(z)} = \frac{1+z}{2(1-z)} + \frac{\sqrt{z}}{(1-z)\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)} - \frac{\frac{4\sqrt{z}}{\pi^2(1-z)}\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)}{1+\frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2}$$

$$1 + \frac{zq''(z)}{q'(z)} - \frac{zq'(z)}{q(z)} + \frac{p+\lambda}{\alpha}q(z)$$

$$= \frac{1+z}{2(1-z)} + \frac{\sqrt{z}}{(1-z)\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)} - \frac{\frac{4\sqrt{z}}{\pi^2(1-z)}\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)}{1+\frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2}$$
$$+\frac{p+\lambda}{\alpha}\left[1+\frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2\right].$$

Thus for a positive real number α we notice that q satisfies the conditions (i) and (ii) of Theorem 2.1. Therefore, we immediately arrive at the following result.

Corollary 3.3. *Let α be a positive real number. If $f \in \mathcal{A}_p$ satisfies*

$$(1-\alpha)\left(\frac{I_p(n+1,\lambda)[f](z)}{I_p(n,\lambda)[f](z)}\right)+\alpha\left(\frac{I_p(n+2,\lambda)[f](z)}{I_p(n+1,\lambda)[f](z)}\right) \prec 1+\frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2$$
$$+\frac{\alpha}{p+\lambda}\left\{\frac{\frac{4\sqrt{z}}{\pi^2(1-z)}\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)}{1+\frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2}\right\},$$

then

$$\frac{I_p(n+1,\lambda)[f](z)}{I_p(n,\lambda)[f](z)} \prec 1+\frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2,$$

where $\lambda \geq 0, n \in \mathbb{N}_0$.

Setting $\lambda = n = 0$ in Corollary 3.3, we have the following result.

Example 3.4. Let α be a positive real number. If $f \in \mathcal{A}_p$ satisfies

$$(1-\alpha)\left(\frac{zf'(z)}{f(z)}\right)+\alpha\left(1+\frac{zf''(z)}{f'(z)}\right) \prec p+\frac{2p}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2$$
$$+\frac{\frac{4\alpha\sqrt{z}}{\pi^2(1-z)}\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)}{1+\frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2},$$

then $f \in \mathcal{S}_{\mathcal{P}}^p$.

Setting $p = 1$ in above example, we get the following result of Brar and Billing [7]:

Example 3.5. Let α be a positive real number. If $f \in \mathcal{A}$ satisfies

$$(1-\alpha)\left(\frac{zf'(z)}{f(z)}\right)+\alpha\left(1+\frac{zf''(z)}{f'(z)}\right) \prec 1+\frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2$$
$$+\frac{\frac{4\alpha\sqrt{z}}{\pi^2(1-z)}\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)}{1+\frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2},$$

then $f \in \mathcal{S}_{\mathcal{P}}$.

Setting $\lambda = 0$, $n = 1$ in Corollary 3.3, we obtain:

Example 3.6. Let α be a positive real number. If $f \in \mathcal{A}_p$ satisfies

$$(1-\alpha)\left(1+\frac{zf''(z)}{f'(z)}\right)+\alpha\left(1+\frac{2zf''(z)+z^2f'''(z)}{f'(z)+zf''(z)}\right)$$
$$\prec p+\frac{2p}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2+\frac{\frac{4\alpha\sqrt{z}}{\pi^2(1-z)}\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)}{1+\frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2},$$

then $f \in UCV_p$.

Setting $p = 1$ in above example, we get:

Example 3.7. Let α be a positive real number. If $f \in \mathcal{A}$ satisfies

$$(1-\alpha)\left(1+\frac{zf''(z)}{f'(z)}\right)+\alpha\left(1+\frac{2zf''(z)+z^2f'''(z)}{f'(z)+zf''(z)}\right) \prec 1+\frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2$$
$$+\frac{\frac{4\alpha\sqrt{z}}{\pi^2(1-z)}\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)}{1+\frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2},$$

then $f \in UCV$.

When we select the dominant $q(z)=\dfrac{1+(1-2\beta)z}{1-z}, 0 \le \beta < 1$ in Theorem 2.1, an easy calculation yields that

$$1+\frac{zq''(z)}{q'(z)}-\frac{zq'(z)}{q(z)}=\frac{1+(1-2\beta)z^2}{(1-z)(1+(1-2\beta)z)},$$

$$1+\frac{zq''(z)}{q'(z)}-\frac{zq'(z)}{q(z)}+\frac{p+\lambda}{\alpha}q(z)=\frac{1+(1-2\beta)z^2}{(1-z)(1+(1-2\beta)z)}+\frac{p+\lambda}{\alpha}\left(\frac{1+(1-2\beta)z}{1-z}\right).$$

Thus for a positive real number α we notice that q satisfies the conditions (i) and (ii) of Theorem 2.1. Therefore, we get the following result.

Corollary 3.8. *Let α be a positive real number. If $f \in \mathcal{A}_p$ satisfies*

$$(1-\alpha)\left(\frac{I_p(n+1,\lambda)[f](z)}{I_p(n,\lambda)[f](z)}\right)+\alpha\left(\frac{I_p(n+2,\lambda)[f](z)}{I_p(n+1,\lambda)[f](z)}\right) \prec \frac{1+(1-2\beta)z}{1-z}$$
$$+\frac{2\alpha(1-\beta)z}{(p+\lambda)(1-z)(1+(1-2\beta)z)}, 0 \le \beta < 1,$$

then

$$\frac{I_p(n+1,\lambda)[f](z)}{I_p(n,\lambda)[f](z)} \prec \frac{1+(1-2\beta)z}{1-z},$$

where $\lambda \ge 0, n \in \mathbb{N}_0$.

Setting $\lambda = n = 0$ in Corollary 3.8, we obtain the following criterion for starlikeness.

Example 3.9. Let α be a positive real number. If $f \in \mathcal{A}_p$ satisfies

$$(1-\alpha)\left(\frac{zf'(z)}{f(z)}\right)+\alpha\left(1+\frac{zf''(z)}{f'(z)}\right) \prec \frac{p+p(1-2\beta)z}{1-z}$$
$$+\frac{2\alpha(1-\beta)z}{(1-z)(1+(1-2\beta)z)}, 0 \le \beta < 1,$$

then

$$\frac{zf'(z)}{f(z)} \prec \frac{p+p(1-2\beta)z}{1-z}.$$

The selection $p = 1$ in above example, yields the following result:

Example 3.10. Let α be a positive real number. If $f \in \mathcal{A}$ satisfies

$$(1-\alpha)\left(\frac{zf'(z)}{f(z)}\right)+\alpha\left(1+\frac{zf''(z)}{f'(z)}\right) \prec \frac{1+(1-2\beta)z}{1-z}$$
$$+\frac{2\alpha(1-\beta)z}{(1-z)(1+(1-2\beta)z)},\ 0 \leq \beta < 1,$$

then $f \in \mathcal{S}^*(\beta)$.

Setting $\lambda = 0, n = 1$ in Corollary 3.8, we obtain the following result of convexity.

Example 3.11. Let α be a positive real number. If $f \in \mathcal{A}_p$ satisfies

$$(1-\alpha)\left(1+\frac{zf''(z)}{f'(z)}\right)+\alpha\left(1+\frac{2zf''(z)+z^2f'''(z)}{f'(z)+zf''(z)}\right) \prec \frac{p+p(1-2\beta)z}{1-z}$$
$$+\frac{2\alpha(1-\beta)z}{(1-z)(1+(1-2\beta)z)},\ 0 \leq \beta < 1,$$

then $1+\frac{zf''(z)}{f'(z)} \prec \frac{p+p(1-2\beta)z}{1-z}$.

Setting $p = 1$ in above example, we obtain:

Example 3.12. Let α be a positive real number. If $f \in \mathcal{A}$ satisfies

$$(1-\alpha)\left(1+\frac{zf''(z)}{f'(z)}\right)+\alpha\left(1+\frac{2zf''(z)+z^2f'''(z)}{f'(z)+zf''(z)}\right) \prec \frac{1+(1-2\beta)z}{1-z}$$
$$+\frac{2\alpha(1-\beta)z}{(1-z)(1+(1-2\beta)z)}, 0 \leq \beta < 1,$$

then $f \in \mathcal{K}(\beta)$.

When we select the dominant $q(z) = e^z$ in Theorem 2.1, a simple calculation gives that

$$1+\frac{zq''(z)}{q'(z)}-\frac{zq'(z)}{q(z)} = 1$$
$$1+\frac{zq''(z)}{q'(z)}-\frac{zq'(z)}{q(z)}+\frac{p+\lambda}{\alpha}q(z) = 1+\frac{p+\lambda}{\alpha}e^z.$$

Thus for a positive real number α we notice that q satisfies the conditions (i) and (ii) of Theorem 2.1. We obtain the following result.

Corollary 3.13. *Let α be a positive real number. If $f \in \mathcal{A}_p$ satisfies*

$$(1-\alpha)\left(\frac{I_p(n+1,\lambda)[f](z)}{I_p(n,\lambda)[f](z)}\right)+\alpha\left(\frac{I_p(n+2,\lambda)[f](z)}{I_p(n+1,\lambda)[f](z)}\right) \prec e^z+\frac{\alpha z}{p+\lambda},$$

then

$$\frac{I_p(n+1,\lambda)[f](z)}{I_p(n,\lambda)[f](z)} \prec e^z,$$

where $\lambda \geq 0, n \in \mathbb{N}_0$.

The selection $\lambda = n = 0$ in Corollary 3.13, yields the following result.

Example 3.14. Let α be a positive real number. If $f \in \mathcal{A}_p$ satisfies

$$(1-\alpha)\left(\frac{zf'(z)}{f(z)}\right)+\alpha\left(1+\frac{zf''(z)}{f'(z)}\right) \prec pe^z+\alpha z,$$

then $f \in \mathcal{S}_p^*$.

Setting $p=1$ in above example, we get:

Example 3.15. Let α be a positive real number. If $f \in \mathcal{A}$ satisfies

$$(1-\alpha)\left(\frac{zf'(z)}{f(z)}\right)+\alpha\left(1+\frac{zf''(z)}{f'(z)}\right) \prec e^z+\alpha z,$$

then $f \in \mathcal{S}^*$.

Setting $\lambda=0, n=1$ in Corollary 3.13, we obtain the following result.

Example 3.16. Let α be a positive real number. If $f \in \mathcal{A}_p$ satisfies

$$(1-\alpha)\left(1+\frac{zf''(z)}{f'(z)}\right)+\alpha\left(1+\frac{2zf''(z)+z^2f'''(z)}{f'(z)+zf''(z)}\right) \prec pe^z+\alpha z,$$

then $f \in \mathcal{K}_p$.

Setting $p=1$ in above example, we get:

Example 3.17. Let α be a positive real number. If $f \in \mathcal{A}$ satisfies

$$(1-\alpha)\left\{1+\frac{zf''(z)}{f'(z)}\right\}+\alpha\left(1+\frac{2zf''(z)+z^2f'''(z)}{f'(z)+zf''(z)}\right) \prec e^z+\alpha z,$$

then $f \in \mathcal{K}$.

When we select the dominant $q(z)=\dfrac{\alpha'(1-z)}{\alpha'-z}$, $\alpha'>1$ in Theorem 2.1, a simple calculation yields that

$$1+\frac{zq''(z)}{q'(z)}-\frac{zq'(z)}{q(z)}=\frac{1}{1-z}+\frac{z}{\alpha'-z}$$

$$1+\frac{zq''(z)}{q'(z)}-\frac{zq'(z)}{q(z)}+\frac{p+\lambda}{\alpha}q(z)=\frac{1}{1-z}+\frac{z}{\alpha'-z}+\frac{p+\lambda}{\alpha}\left(\frac{\alpha'(1-z)}{\alpha'-z}\right).$$

Thus for a positive real number α we notice that q satisfies the conditions (i) and (ii) of Theorem 2.1. Therefore, we, immediately arrive at the following result.

Corollary 3.18. *Let α be a positive real number. If $f \in \mathcal{A}_p$ satisfies*

$$(1-\alpha)\left(\frac{I_p(n+1,\lambda)[f](z)}{I_p(n,\lambda)[f](z)}\right)+\alpha\left(\frac{I_p(n+2,\lambda)[f](z)}{I_p(n+1,\lambda)[f](z)}\right)$$
$$\prec \frac{\alpha'(1-z)}{\alpha'-z}+\frac{\alpha z(1-\alpha')}{(p+\lambda)(1-z)(\alpha'-z)},\ \alpha'>1,$$

then

$$\frac{I_p(n+1,\lambda)[f](z)}{I_p(n,\lambda)[f](z)} \prec \frac{\alpha'(1-z)}{\alpha'-z},$$

where $\lambda \geq 0, n \in \mathbb{N}_0$.

Setting $\lambda = n = 0$ in Corollary 3.18, we obtain the following result for starlikeness.

Example 3.19. Let α be a positive real number. If $f \in \mathcal{A}_p$ satisfies

$$(1-\alpha)\left(\frac{zf'(z)}{f(z)}\right) + \alpha\left(1 + \frac{zf''(z)}{f'(z)}\right) \prec \frac{p\alpha'(1-z)}{\alpha'-z} + \frac{\alpha z(1-\alpha')}{(1-z)(\alpha'-z)},\ \alpha' > 1,$$

then $f \in \mathcal{S}_p^*$.

Setting $p = 1$ in above example, we have:

Example 3.20. Let α be a positive real number. If $f \in \mathcal{A}$ satisfies

$$(1-\alpha)\left(\frac{zf'(z)}{f(z)}\right) + \alpha\left(1 + \frac{zf''(z)}{f'(z)}\right) \prec \frac{\alpha'(1-z)}{\alpha'-z} + \frac{\alpha z(1-\alpha')}{(1-z)(\alpha'-z)},\ \alpha' > 1,$$

then $f \in \mathcal{S}^*$.

Setting $\lambda = 0, n = 1$ in Corollary 3.18, we obtain the following result.

Example 3.21. Let α be a positive real number. If $f \in \mathcal{A}_p$ satisfies

$$(1-\alpha)\left(1 + \frac{zf''(z)}{f'(z)}\right) + \alpha\left(1 + \frac{2zf''(z) + z^2 f'''(z)}{f'(z) + zf''(z)}\right)$$
$$\prec \frac{p\alpha'(1-z)}{\alpha'-z} + \frac{\alpha z(1-\alpha')}{(1-z)(\alpha'-z)}, \alpha' > 1,$$

then $f \in \mathcal{K}_p$.

Setting $p = 1$ in above example, we get:

Example 3.22. Let α be a positive real number. If $f \in \mathcal{A}$ satisfies

$$(1-\alpha)\left(1 + \frac{zf''(z)}{f'(z)}\right) + \alpha\left(1 + \frac{2zf''(z) + z^2 f'''(z)}{f'(z) + zf''(z)}\right)$$
$$\prec \frac{\alpha'(1-z)}{\alpha'-z} + \frac{\alpha z(1-\alpha')}{(1-z)(\alpha'-z)},\ \alpha' > 1,$$

then $f \in \mathcal{K}$.

When we select the dominant $q(z) = \left(\frac{1+z}{1-z}\right)^\gamma$, $0 < \gamma \leq 1$ such that $q(0) = 1$ in Theorem 2.1, an easy calculation yields that

$$1 + \frac{zq''(z)}{q'(z)} - \frac{zq'(z)}{q(z)} = \frac{1+z^2}{1-z^2}$$

$$1 + \frac{zq''(z)}{q'(z)} - \frac{zq'(z)}{q(z)} + \frac{(p+\lambda)}{\alpha} q(z) = \frac{1+z^2}{1-z^2} + \frac{(p+\lambda)}{\alpha}\left(\frac{1+z}{1-z}\right)^\gamma.$$

Thus for a positive real number α we notice that q satisfies the conditions (i) and (ii) of Theorem 2.1. Therefore we obtain the following result.

Corollary 3.23. *Let α be a positive real number. If $f \in \mathcal{A}_p$ satisfies*

$$(1-\alpha)\left(\frac{I_p(n+1,\lambda)[f](z)}{I_p(n,\lambda)[f](z)}\right)+\alpha\left(\frac{I_p(n+2,\lambda)[f](z)}{I_p(n+1,\lambda)[f](z)}\right) \prec \frac{(1+z)^\gamma}{(1-z)^\gamma}+\frac{2\alpha\gamma z}{(p+\lambda)(1-z^2)},\ 0<\gamma\le 1,$$

then

$$\frac{I_p(n+1,\lambda)[f](z)}{I_p(n,\lambda)[f](z)} \prec \left(\frac{1+z}{1-z}\right)^\gamma,$$

where $\lambda \ge 0, n \in \mathbb{N}_0$.

Setting $\lambda = n = 0$ in Corollary 3.23, we get the following result.

Example 3.24. Let α be a positive real number. If $f \in \mathcal{A}_p$ satisfies

$$(1-\alpha)\left(\frac{zf'(z)}{f(z)}\right)+\alpha\left(1+\frac{zf''(z)}{f'(z)}\right) \prec p\frac{(1+z)^\gamma}{(1-z)^\gamma}+\frac{2\alpha\gamma z}{(1-z^2)},\ 0<\gamma\le 1,$$

then

$$\frac{zf'(z)}{f(z)} \prec p\left(\frac{1+z}{1-z}\right)^\gamma.$$

Setting $p = 1$ in above example, we get:

Example 3.25. Let α be a positive real number. If $f \in \mathcal{A}$ satisfies

$$(1-\alpha)\left(\frac{zf'(z)}{f(z)}\right)+\alpha\left(1+\frac{zf''(z)}{f'(z)}\right) \prec \frac{(1+z)^\gamma}{(1-z)^\gamma}+\frac{2\alpha\gamma z}{1-z^2},\ 0<\gamma\le 1,$$

then

$$\frac{zf'(z)}{f(z)} \prec \left(\frac{1+z}{1-z}\right)^\gamma$$

Setting $\lambda = 0, n = 1$ in Corollary 3.23, we obtain the following result.

Example 3.26. Let α be a positive real number. If $f \in \mathcal{A}_p$ satisfies

$$(1-\alpha)\left(1+\frac{zf''(z)}{f'(z)}\right)+\alpha\left(1+\frac{2zf''(z)+z^2f'''(z)}{f'(z)+zf''(z)}\right) \prec \frac{p(1+z)^\gamma}{(1-z)^\gamma}+\frac{2\alpha\gamma z}{(1-z^2)},\ 0<\gamma\le 1,$$

then

$$1+\frac{zf''(z)}{f'(z)} \prec p\left(\frac{1+z}{1-z}\right)^\gamma.$$

Setting $p = 1$ in above example, we get:

Example 3.27. Let α be a positive real number. If $f \in \mathcal{A}$ satisfies

$$(1-\alpha)\left(1+\frac{zf''(z)}{f'(z)}\right)+\alpha\left(1+\frac{2zf''(z)+z^2f'''(z)}{f'(z)+zf''(z)}\right) \prec \frac{(1+z)^\gamma}{(1-z)^\gamma}+\frac{2\alpha\gamma z}{1-z^2},\ 0<\gamma\le 1,$$

then

$$1+\frac{zf''(z)}{f'(z)} \prec \left(\frac{1+z}{1-z}\right)^\gamma.$$

When we select the dominant $q(z) = 1 + az;\ 0 \le a < 1$ in Theorem 2.1, a simple calculation yields that

$$1 + \frac{zq''(z)}{q'(z)} - \frac{zq'(z)}{q(z)} = \frac{1}{1+az}$$

$$1 + \frac{zq''(z)}{q'(z)} - \frac{zq'(z)}{q(z)} + \frac{p+\lambda}{\alpha}q(z) = \frac{1}{1+az} + \frac{p+\lambda}{\alpha}(1+az).$$

Thus for a positive real number α we notice that q satisfies the conditions (i) and (ii) of Theorem 2.1. Therefore, we immediately deduce the following result.

Corollary 3.28. *Let α be a positive real number. If $f \in \mathcal{A}_p$ satisfies*

$$(1-\alpha)\left(\frac{I_p(n+1,\lambda)[f](z)}{I_p(n,\lambda)[f](z)}\right) + \alpha\left(\frac{I_p(n+2,\lambda)[f](z)}{I_p(n+1,\lambda)[f](z)}\right)$$
$$\prec 1 + az + \frac{\alpha az}{(p+\lambda)(1+az)},\ 0 \le a < 1,$$

then

$$\frac{I_p(n+1,\lambda)[f](z)}{I_p(n,\lambda)[f](z)} \prec 1 + az,$$

where $n \in \mathbb{N}_0$.

Setting $\lambda = n = 0$ in Corollary 3.28, we get the following result.

Example 3.29. Let α be a positive real number. If $f \in \mathcal{A}_p$ satisfies

$$(1-\alpha)\left(\frac{zf'(z)}{f(z)}\right) + \alpha\left(1 + \frac{zf''(z)}{f'(z)}\right) \prec p(1+az) + \frac{\alpha az}{1+az},\ 0 \le a < 1,$$

then $f \in \mathcal{S}_p^*$.

The substitution $p = 1$ in above example, yields the following result:,

Example 3.30. Let α be a positive real number. If $f \in \mathcal{A}$ satisfies

$$(1-\alpha)\left(\frac{zf'(z)}{f(z)}\right) + \alpha\left(1 + \frac{zf''(z)}{f'(z)}\right) \prec 1 + az + \frac{\alpha az}{1+az},\ 0 \le a < 1,$$

then $f \in \mathcal{S}^*$.

Setting $\lambda = 0, n = 1$ in Corollary 3.28, we obtain the following result.

Example 3.31. Let α be a positive real number. If $f \in \mathcal{A}_p$ satisfies

$$(1-\alpha)\left(1 + \frac{zf''(z)}{f'(z)}\right) + \alpha\left(1 + \frac{2zf''(z) + z^2 f'''(z)}{f'(z) + zf''(z)}\right)$$
$$\prec p(1+az) + \frac{\alpha az}{1+az},\ 0 \le a < 1,$$

then $f \in \mathcal{K}_p$.

Setting $p = 1$ in above example, we get:

Example 3.32. Let α be a positive real number. If $f \in \mathcal{A}$ satisfies

$$(1-\alpha)\left(1+\frac{zf''(z)}{f'(z)}\right)+\alpha\left(1+\frac{2zf''(z)+z^2f'''(z)}{f'(z)+zf''(z)}\right) \prec 1+az+\frac{\alpha az}{1+az},\ 0 \leq a < 1,$$

then $f \in \mathcal{K}$.

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Meromorphic functions with small Schwarzian derivative

Vibhuti Arora and Swadesh Kumar Sahoo

Abstract. We consider the family of all meromorphic functions f of the form

$$f(z) = \frac{1}{z} + b_0 + b_1 z + b_2 z^2 + \cdots$$

analytic and locally univalent in the puncture disk $\mathbb{D}_0 := \{z \in \mathbb{C} : 0 < |z| < 1\}$. Our first objective in this paper is to find a sufficient condition for f to be meromorphically convex of order α, $0 \leq \alpha < 1$, in terms of the fact that the absolute value of the well-known Schwarzian derivative $S_f(z)$ of f is bounded above by a smallest positive root of a non-linear equation. Secondly, we consider a family of functions g of the form $g(z) = z + a_2 z^2 + a_3 z^3 + \cdots$ analytic and locally univalent in the open unit disk $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$, and show that g is belonging to a family of functions convex in one direction if $|S_g(z)|$ is bounded above by a small positive constant depending on the second coefficient a_2. In particular, we show that such functions g are also contained in the starlike and close-to-convex family.

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1. Introduction

Recall that a function f which is analytic in a region, except possibly at poles, is said to be meromorphic in that region. Hence, analytic functions are by default meromorphic without poles. In this paper, we consider the family of all meromorphic functions f of the form

$$f(z) = \frac{1}{z} + b_0 + b_1 z + b_2 z^2 + \cdots$$

defined in the open unit disk $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$. Clearly, f has a simple pole at the origin, and hence it is analytic in the puncture disk $\mathbb{D}_0 := \{z \in \mathbb{C} : 0 < |z| < 1\}$. Let us denote this family of meromorphic functions by $\mathcal{B}$. The set of all univalent functions in $\mathcal{B}$ is usually denoted by Σ. We also consider the family $\mathcal{A}$, of functions g analytic in $\mathbb{D}$ of the form

$$g(z) = z + a_2 z^2 + a_3 z^3 + \cdots .$$

A quick observation which can easily be verified that

$$f \in \mathcal{B} \iff g = 1/f \in \mathcal{A}. \tag{1.1}$$

A single valued function f is said to be *univalent* (or schlicht) in a domain $D \subset \mathbb{C}$ if it never takes the same value twice: $f(z_1) \neq f(z_2)$ for all $z_1 \neq z_2$ in D. The family of all univalent functions in $\mathcal{A}$ is denoted by $\mathcal{S}$. Such functions g are of interest because they appear in the Riemann mapping theorem. The study of the family $\mathcal{S}$ became popular when the Bieberbach conjecture was first posed in 1916 and remained as a challenge to all mathematicians until 1985 when it was solved by de Branges. Since then the conjecture is known as the de Branges Theorem. This problem has been attracted to many mathematicians in introducing certain subclasses of $\mathcal{S}$ and developing important new methods in geometric function theory. The de Branges theorem gives a necessary condition for a function g to be in $\mathcal{S}$ in terms of its Taylor's coefficient. On the other hand, several important sufficient conditions for functions to be in $\mathcal{S}$ were also introduced by several researchers to generate its subclasses having interesting geometric properties. Part of this development is the family of convex functions, starlike functions, close-to-convex functions, etc. Later, counterpart of this development for the family Σ of meromorphic univalent functions were also studied extensively. We refer to the standard books by Duren [3], Goodman [5], Lehto [8], and Pommerenke [17] for the literature on the topic. Therefore, the study of sufficient conditions for functions to be in $\mathcal{S}$, in particular, in its subfamilies are important in this context. In this paper, we mainly deal with such properties in terms of the well-known Schwarzian derivative of locally univalent functions.

First let us recall the definition of the Schwarzian derivative. Let f be a *meromorphic function* and $f'(z) \neq 0$ in $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ (in other words, we say, f is locally univalent in $\mathbb{D}$), then the Schwarzian derivative of f at z is defined as

$$S_f(z) = \left(\frac{f''}{f'}\right)' - \frac{1}{2}\left(\frac{f''}{f'}\right)^2 .$$

It is appropriate here to recall from texts that $S_f = 0$ if and only if f is a Möbius transformation (see for instance, [8, p 51]). A simple computation through (1.1) yields the useful relation

$$S_f(z) = S_g(z)$$

for all locally univalent meromorphic functions $f \in \mathcal{B}$ and $g = 1/f \in \mathcal{A}$.

The study of necessary and sufficient conditions for functions to be univalent, in particular to be starlike, convex, close-to-convex, in terms of Schwarzian derivatives are attracted by a number of mathematicians. It is a surprising fact is that most of such necessary conditions are proved using standard theorems in complex variables, whereas sufficient conditions are proved through initial value problems of differential equations; see for instance [3, 8]. The conditions of the form

$$|S_g(z)| \leq \frac{C_0}{(1-|z|^2)^2}, \tag{1.2}$$

for a positive constant C_0, have been most popular to many mathematicians. For instance, Nehari in 1949 first proved that if g is an analytic and locally univalent function in $\mathbb{D}$ satisfying (1.2) with $C_0 = 2$ then g is univalent in $\mathbb{D}$. This condition becomes necessary when the constant $C_0 = 6$; see [12]. Hille [6] showed that the constant 2 in the sufficient condition of Nehari is the best possible constant. Related problems are also investigated in [13, 14, 16]. Thus, applications of the Schwarzian derivative can be seen in second order linear differential equations, univalent functions, and also in Teichmüller spaces [3, 17]. Note that if $g \in \mathcal{A}$ is univalent then (1.1) leads to the useful coefficient relation $|a_2^2 - a_3| = |S_g(0)|/6$; see [3, p. 263].

Another form of sufficient condition for univalency in terms of Schwarzian derivative attracted by many researchers in this field is

$$|S_g(z)| \leq 2C_1, \tag{1.3}$$

for some positive constant C_1. If $g \in \mathcal{A}$ satisfies (1.3) with $C_1 = \pi^2/4$, then it is proved by Nehari [12] that g is univalent in $\mathbb{D}$. Gabriel [4] studied a sufficient condition for a function $g \in \mathcal{A}$ to be starlike in the form (1.3) for some optimal constant C_1. Sufficient condition in the form (1.3) for convexity of order α is investigated by Chiang in [2]. However, the best possible constant is not yet known in this case. Kim and Sugawa in [7] obtained the sufficient condition in the form (1.3) for starlikeness of order α by fixing the second coefficient of the function.

Our main objective in this paper is to study the sufficient conditions of the form (1.3) for meromorphically convex functions of order α and for functions in a family that are convex in one direction, in particular in the starlike and close-to-convex family. Rest of the structure of this paper is as follows. Section 2 is devoted to the definitions of the classes of functions and statements of our main results. Section 3 deals with some preliminary results those are used to prove our main results. Finally, the proof of our main results are given in Section 4 followed by examples of functions satisfying these results.

2. Definitions and main results

This section is divided into two subsections. The first subsection concerns about the definition of a subclass of the class $\mathcal{B}$, namely, the meromorphically convex functions of order α having simple pole at $z = 0$, and the main results associated with these functions. The second subsection deals with some well-known analytic functions convex in one direction, in particular, functions in the starlike and close-to-convex families. Sufficient conditions in the form (1.3) for functions to be in these families are also stated.

2.1. Meromorphic functions in $\mathbb{D}$ with a simple pole at $z = 0$

If $f \in \mathcal{B}$ satisfies $f(z) \neq 0$ in $\mathbb{D}_0$ and

$$-\mathrm{Re}\Big(\frac{zf'(z)}{f(z)}\Big) > \alpha \quad (z \in \mathbb{D},\, 0 \leq \alpha < 1),$$

then f is said to be *meromorphically starlike of order* α. A function $f \in \mathcal{B}$ is said to be *meromorphically starlike (of order* 0*)* if and only if complement of $f(\mathbb{D}_0)$ is starlike with respect to the origin (see [5, p. 265, Vol. 2]). Note that meromorphically starlike functions are univalent and hence they lie on the class Σ. Similarly, if $f \in \mathcal{B}$ satisfies $f(z) \neq 0$ in $\mathbb{D}_0$ and

$$-\mathrm{Re}\Big(1 + \frac{zf''(z)}{f'(z)}\Big) > \alpha \quad (z \in \mathbb{D},\, 0 \leq \alpha < 1), \tag{2.1}$$

then f is said to be *meromorphically convex of order* α. If $\alpha = 0$, the inequality (2.1) is equivalent to the definition of meromorphically convex functions. That is, f maps $\mathbb{D}$ onto the complement of a convex region [4, 15]. In this case, we say f is *meromorphically convex*. Note that meromorphically convex functions are also univalent and hence they lie in the class Σ. For more geometric properties of these classes, we refer to the standard books [5, 10].

Main results of this paper deal with functions whose Schwarzian derivatives are bounded above by some constant, that is, functions satisfy (1.3). Note that if $S_f(z)$ is uniformly bounded in $\mathbb{C}$, then the Schwarzian derivative is still well defined. Hence the assumption that f is locally univalent at a point z (or $f'(z) \neq 0$), in (1.3) is not chosen; see also Tichmarsh [22, p. 198].

Gabriel modified Nehari's technique to show univalency and convexity property of functions $f \in \mathcal{B}$ and proved the following:

Theorem A. [4, Theorem 1] *If $f \in \mathcal{B}$ satisfies*

$$|S_f(z)| \leq 2c_0 \quad \textit{for } |z| < 1, \tag{2.2}$$

where c_0 is the smallest positive root of the equation

$$2\sqrt{x} - \tan\sqrt{x} = 0,$$

then f is univalent in the punctured disk and maps the interior of each circle $|z| = r < 1$ onto the exterior of a convex region. The constant c_0 is the largest possible constant satisfying (2.2)*.*

An analog to this result for meromorphically convex functions of order α is one of our main results which is stated below.

Theorem 2.1. *Let $0 \leq \alpha < 1$. If $f \in \mathcal{B}$ satisfies*

$$|S_f(z)| \leq 2c_\alpha \quad \textit{for } |z| < 1, \tag{2.3}$$

where c_α is the smallest positive root of the equation

$$2\sqrt{x} - (1+\alpha)\tan\sqrt{x} = 0 \tag{2.4}$$

depending on α, then

a. *f is meromorphically convex of order α; and*

b. *the quantity* c_α *is the largest possible constant satisfying* (2.3)*.*

In particular, if $\alpha = 0$, Theorem 2.1 reduces to Theorem A.

2.2. Analytic functions in $\mathbb{D}$

A function $g \in \mathcal{A}$ is said to be *convex of order* β, $0 \le \beta < 1$, if and only if

$$\mathrm{Re}\left(1 + \frac{zg''(z)}{g'(z)}\right) > \beta, \quad z \in \mathbb{D}.$$

Chiang proved the following sufficient condition for convex functions of order β in terms of small Schwarzian derivative:

Theorem B. [2, Theorem 2] *Let* $g \in \mathcal{A}$ *and* $|a_2| = \eta < 1/3$*. Suppose that*

$$\sup_{z\in\mathbb{D}} |S_g(z)| = 2\delta,$$

where $\delta = \delta(\eta)$ *satisfies the inequality*

$$6\eta + 5\delta(1+\eta)e^{\delta/2} < 2.$$

Then g *is convex of order*

$$\frac{2 - 6\eta - 5(1+\eta)\delta e^{\delta/2}}{2 - 2\eta - (1+\eta)\delta e^{\delta/2}}.$$

Our aim in this subsection is to state results similar to Theorem B for certain functions convex in one direction, in particular, for functions in the family of starlike and close-to-convex functions.

For $\beta \ge 3/2$, we consider the class $\mathcal{C}_\beta$ introduced by Shah in [21] as follows:

$$\mathcal{C}_\beta = \left\{ g \in \mathcal{A} : \frac{-\beta}{2\beta - 3} < \mathrm{Re}\left(1 + \frac{zg''(z)}{g'(z)}\right) < \beta, \ z \in \mathbb{D} \right\}.$$

This originally follows from a sufficient condition for a function g to be convex in one direction studied by Umeraza in [23]. Note that the special cases $\mathcal{C}_{3/2}$ and $\mathcal{C}_\infty$ are contained in the family of starlike and close-to-convex functions respectively (see the detailed discussion below in this section). It is a natural question to ask for functions belonging to the family $\mathcal{C}_\beta$ for all $\beta \ge 3/2$. Such functions can be generated in view of [21, Theorem 12], which says that for all functions $g \in \mathcal{A}$ satisfying

$$\frac{\beta}{3 - 2\beta} < \mathrm{Re}\left(\frac{zg'(z)}{g(z)}\right) < \beta,$$

the Alexander transform of g belongs to the family $\mathcal{C}_\beta$, $\beta \ge 3/2$.

We now state our second main result which provides a sufficient condition for functions to be in $\mathcal{C}_\beta$ with respect to its small Schwarzian derivative.

Theorem 2.2. *For* $\beta \ge 3/2$*, set*

$$\phi(\beta) = \min\left\{\frac{\beta - 1}{\beta + 1}, \frac{6(\beta - 1)}{2(7\beta - 9)}\right\} \quad \text{and} \quad \psi(\beta) = \max\left\{\frac{\beta + 3}{\beta + 1}, \frac{11\beta - 15}{7\beta - 9}\right\}.$$

Let $g \in \mathcal{A}$ and $|a_2| = \eta < \phi(\beta)$. Suppose that

$$\sup_{z\in\mathbb{D}} |S_g(z)| = 2\delta,$$

where $\delta = \delta(\eta)$ satisfies the inequality

$$2\eta + \psi(\beta)\delta(1+\eta)e^{\delta/2} < 2\phi(\beta). \tag{2.5}$$

Then $g \in \mathcal{C}_\beta$. In particular, g is convex in one direction.

A function $g \in \mathcal{A}$ is said to be *starlike of order* β, $0 \le \beta < 1$, if and only if

$$\mathrm{Re}\left(\frac{zg'(z)}{g(z)}\right) > \beta, \quad z \in \mathbb{D}.$$

In particular, for $\beta = 0$, we simply call such functions g as *starlike functions*. Recall the sufficient condition for starlike functions $g \in \mathcal{A}$ from [18, (16)] which tells us that

$$\mathrm{Re}\left(1 + \frac{zg''(z)}{g'(z)}\right) < \frac{3}{2} \implies \left|\frac{zg'(z)}{g(z)} - \frac{2}{3}\right| < \frac{2}{3}.$$

This generates the following subclass of the class of starlike functions:

$$\mathcal{C}_{3/2} := \left\{ g \in \mathcal{A} :\, \mathrm{Re}\left(1 + \frac{zg''(z)}{g'(z)}\right) < \frac{3}{2} \right\}.$$

This particular class of functions is also studied in different contexts in [19].

The following corollary immediately follows from Theorem 2.2 for the class $\mathcal{C}_{3/2}$.

Corollary 2.3. *Let $g \in \mathcal{A}$ and $|a_2| = \eta < 1/5$. Suppose*

$$\sup_{z\in\mathbb{D}} |S_g(z)| = 2\delta$$

where $\delta = \delta(\eta)$ satisfies the inequality

$$10\eta + 9\delta(1+\eta)e^{\delta/2} < 2. \tag{2.6}$$

Then $g \in \mathcal{C}_{3/2}$. In particular, g is starlike.

We next recall what is close-to-convex function followed by a subclass of the class of close-to-convex functions and then state the corresponding result which is again an easy consequence of Theorem 2.2.

We here adopt the well-known Kaplan characterization for close-to-convex functions. Let $g \in \mathcal{A}$ be locally univalent. Then g is *close-to-convex* if and only if

$$\int_{\theta_1}^{\theta_2} \mathrm{Re}\left(1 + \frac{zg''(z)}{g'(z)}\right) d\theta > -\pi, \quad z = re^{i\theta},$$

for each r $(0 < r < 1)$ and for each pair of real numbers θ_1 and θ_2 with $\theta_1 < \theta_2$. If a locally univalent analytic function g defined in $\mathbb{D}$ satisfies

$$\mathrm{Re}\left(1 + \frac{zg''(z)}{g'(z)}\right) > -1/2,$$

then by the Kaplan characterization it follows easily that g is close-to-convex in $\mathbb{D}$ (here θ_1 and θ_2 are chosen as 0 and 2π respectively) and hence g is univalent in $\mathbb{D}$. This generates the following subclass of the class of close-to-convex (univalent) functions:

$$\mathcal{C}_\infty := \left\{ g \in \mathcal{A} : \operatorname{Re}\Big(1 + \frac{zg''(z)}{g'(z)}\Big) > -\frac{1}{2} \right\}.$$

This class of functions is also studied recently by several authors in different contexts; for instance see [1, 9, 11, 20] and references therein.

Now we are ready to state our sufficient condition for functions g to be in $\mathcal{C}_\infty$ in terms of their Schwarzian derivatives bounded by small quantity.

Corollary 2.4. *Let $g \in \mathcal{A}$ and $|a_2| = \eta < 3/7$. Suppose that*

$$\sup_{z\in\mathbb{D}} |S_g(z)| = 2\delta$$

where $\delta = \delta(\eta)$ satisfies the inequality

$$14\eta + 11\delta(1+\eta)e^{\delta/2} < 6. \tag{2.7}$$

Then $g \in \mathcal{C}_\infty$ and hence g is close-to-convex function.

3. Preliminary results

Connection with a linear differential equation

In this section we study a relationship between Schwarzian derivative of a meromorphic function f and solution of a second order linear differential equation depending on f.

Recall the following lemma from Duren [3, p. 259].

Lemma 3.1. *For a given analytic function $p(z)$, a meromorphic function f has the Schwarzian derivative of the form $S_f(z) = 2p(z)$ if and only if $f(z) = w_1(z)/w_2(z)$ for any pair of linearly independent solutions $w_1(z)$ and $w_2(z)$ of the linear differential equation*

$$w'' + p(z)w = 0. \tag{3.1}$$

Note that an example satisfying Lemma 3.1 is described in the proof of Theorem 2.1(b). Assume now that $w_1(z)$ and $w_2(z)$ satisfy the following conditions:

$$w_1(0) = 1,\ w_2(0) = 0;$$
$$w_1'(0) = 0,\ w_2'(0) = 1.$$

Clearly $w_1(0)$ and $w_2(0)$ are linearly independent since the Wronskian $W(w_1(0), w_2(0))$ is non-vanishing. Recall that

$$f(z) = \frac{w_1(z)}{w_2(z)} = \frac{1}{z} + b_0 + b_1 z + \cdots. \tag{3.2}$$

Hence, a simple computation on logarithmic derivative of $f'(z)$ leads to

$$\frac{f''(z)}{f'(z)} = \frac{w_2(z)w_1''(z) - w_1(z)w_2''(z)}{w_2(z)w_1'(z) - w_1(z)w_2'(z)} - 2\frac{w_2'(z)}{w_2(z)}.$$

Since $w_1(z)$ and $w_2(z)$ satisfy (3.1), it follows that

$$\frac{f''(z)}{f'(z)} = -2\frac{w_2'(z)}{w_2(z)},$$

and hence we have the relation

$$\mathrm{Re}\Big(1+\frac{zf''(z)}{f'(z)}\Big) = 1-2\mathrm{Re}\Big(\frac{zw_2'(z)}{w_2(z)}\Big). \tag{3.3}$$

The function $2x-(1+\alpha)\tan x$

For $0\le\alpha<1$, we set

$$h(x) := 2x-(1+\alpha)\tan x.$$

Derivative test for $h(x)$ tells us that $h(x)$ is decreasing in

$$(\arctan(\sqrt{(1-\alpha)/(1+\alpha)}), \pi/2).$$

Then the following lemma is useful.

Lemma 3.2. *Let $\beta<\pi/2$ be the smallest positive root of $h(x)=2x-(1+\alpha)\tan x=0$ for some $\alpha>0$. Then*

$$\beta \ge \arctan\sqrt{(1-\alpha)/(1+\alpha)}$$

holds true.

Proof. Given that $h(\beta)=0=2\beta-(1+\alpha)\tan\beta$. This gives

$$\alpha = \frac{2\beta}{\tan\beta}-1. \tag{3.4}$$

On contrary, suppose that $0<\beta<\arctan\sqrt{(1-\alpha)/(1+\alpha)}<\pi/2$. This implies that

$$\tan^2\beta < \frac{1-\alpha}{1+\alpha}.$$

Substituting the value of α in (3.4), we obtain

$$\tan^2\beta < \frac{\tan\beta}{\beta}-1$$

equivalently,

$$\sec^2\beta < \frac{\tan\beta}{\beta} \iff 2\beta<\sin 2\beta,$$

which is a contradiction. Thus, the proof of our lemma is complete. □

Let c_α be the smallest positive root of the equation (2.4). Since $h(\sqrt{c_\alpha})=0$, it follows by Lemma 3.2 that

$$h(x)\begin{cases} \ge 0, & \text{for } 0\le x\le\sqrt{c_\alpha};\\ <0, & \text{for } \sqrt{c_\alpha}<x<\pi/2. \end{cases} \tag{3.5}$$

If we replace x by $x\sqrt{c}$, $c>0$, in (3.5), we obtain

$$h(x\sqrt{c}) = 2x\sqrt{c}-(1+\alpha)\tan(x\sqrt{c}) \ge 0 \text{ for } 0\le x\sqrt{c}\le\sqrt{c_\alpha} \tag{3.6}$$

and

$$h(x\sqrt{c}) = 2x\sqrt{c} - (1+\alpha)\tan(x\sqrt{c}) < 0 \text{ for } \sqrt{c_\alpha} < x\sqrt{c} < \pi/2. \tag{3.7}$$

We may have the following two cases when $h(x\sqrt{c})$ is negative.
Case 1: If $c \le c_\alpha$, then (3.7) gives that $h(x\sqrt{c})$ is also negative in $[1, \pi/2\sqrt{c})$.
Case 2: If $c > c_\alpha$, then (3.7) gives that $h(x\sqrt{c})$ is also negative in $(\sqrt{c_\alpha/c}, 1)$.
In the sequel, we collect the following lemmas to be used in the proof of Theorem 2.1.

Lemma 3.3. *A function $f \in \mathcal{B}$ in the form* (3.2) *is meromorphically convex of order α if and only if $w_2(z)$ is starlike of order* $(\alpha+1)/2$.

Proof. Condition (3.3) is equivalent to

$$-\mathrm{Re}\Big(1 + \frac{zf''(z)}{f'(z)}\Big) = -1 + 2\mathrm{Re}\Big(\frac{zw_2'(z)}{w_2(z)}\Big),$$

which yields

$$-\mathrm{Re}\Big(1 + \frac{zf''(z)}{f'(z)}\Big) > \alpha \iff \mathrm{Re}\Big(\frac{zw_2'(z)}{w_2(z)}\Big) > \frac{\alpha+1}{2}\ .$$

Since $w_2(0) = 0$ and $w_2'(0) = 1$, $w_2(z)$ is starlike of order $(\alpha+1)/2$. Thus, completing the proof of our lemma. □

Remark 3.4. A simple computation using the identity (3.3) yields

$$\mathrm{Re}\left(\frac{zw_1'(z)}{w_1(z)}\right) = \frac{1}{2} + \mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right) - \frac{1}{2}\mathrm{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right).$$

Therefore, the function w_1 is not necessarily starlike when the function f is meromorphically convex.

Lemma 3.5. *For $0 \le \alpha < 1$, let c_α $(0 < c \le c_\alpha)$ be the root of the equation given by* (2.4)*. Then we have*

$$\mathrm{Re}(z\sqrt{c}\cot(z\sqrt{c})) > \frac{\alpha+1}{2}, \quad |z| < 1. \tag{3.8}$$

Proof. Substituting $z = x+iy$ in (3.8), we see that the desired inequality is equivalent to

$$2\mathrm{Re}\Big(\sqrt{c}(x+iy)\frac{\cos(\sqrt{c}(x+iy))}{\sin(\sqrt{c}(x+iy))}\Big) > \alpha + 1.$$

This is, using the basic identities

$$2\mathrm{Re}\, w = w + \overline{w},\ \cos(iy) = \cosh(y),\ \text{and}\ \sin(iy) = i\sinh(y),$$

we see that it is equivalent to proving

$$2x\sqrt{c}\sin(\sqrt{c}x)\cos(\sqrt{c}x) + 2y\sqrt{c}\sinh(\sqrt{c}y)\cosh(\sqrt{c}y) > (1+\alpha)(\sin^2(\sqrt{c}x) + \sinh^2(\sqrt{c}y)).$$

So, it suffices to prove the inequality

$$\sin(\sqrt{c}x)\cos(\sqrt{c}x)[2\sqrt{c}x - (1+\alpha)\tan(\sqrt{c}x)] > \sinh(\sqrt{c}y)\cosh(\sqrt{c}y)[(1+\alpha)\tanh(\sqrt{c}y) - 2\sqrt{c}y] \tag{3.9}$$

for $0 < c \le c_\alpha$ and $x^2 + y^2 < 1$. First consider the points $x,\ y$ in the first quadrant. Then we see that $\sin(\sqrt{c}x)$, $\cos(\sqrt{c}x)$, $\sinh(\sqrt{c}y)$ and $\cosh(\sqrt{c}y)$ are all positive since $c < c_\alpha < \pi^2/4$. Also $2x\sqrt{c} - (1+\alpha)\tan(\sqrt{c}x)$ is positive which follows from (3.6). On the other hand, $(1+\alpha)\tanh(\sqrt{c}y) - 2(\sqrt{c}y)$ is non-positive because

$$g(y) = (1+\alpha)\tanh(\sqrt{c}y) - 2(\sqrt{c}y)$$

is decreasing, hence for $y \ge 0$ we obtain

$$g(y) = (1+\alpha)\tanh(\sqrt{c}y) - 2\sqrt{c}y \le 0.$$

Hence, the inequality (3.9) holds true in the first quadrant. Now if we replace x by $-x$ and y by $-y$ then the inequality (3.9) remains same in all the other quadrants of $\mathbb{D}$. The desired inequality thus follows. □

The following results of Gabriel are also useful.

Lemma 3.6. [4, Lemma 4.1] *If $w(z)$ satisfies (3.1) with $w(0) = 0$ and $w'(0) = 1$, then for $0 < \rho \le r < 1$ and for a fixed $\theta \in [0, 2\pi]$, we have*

$$|w(re^{i\theta})|^2 \mathrm{Re}\Big(\frac{re^{i\theta}w'(re^{i\theta})}{w(re^{i\theta})}\Big) = r\int_0^r |w'(\rho e^{\theta})|^2 d\rho - r\int_0^r \mathrm{Re}(\rho^2 e^{2i\theta}p(\rho e^{i\theta}))\frac{|w(\rho e^{i\theta})|^2}{\rho^2} d\rho. \tag{3.10}$$

Lemma 3.7. [4, Lemma 4.2] *Let $y(\rho)$ and $y'(\rho)$ be continuous real functions of ρ for $0 \le \rho < 1$. For small values of ρ let $y(\rho) = O(\rho)$. Then*

$$r\int_0^r [y'(\rho)]^2 d\rho - cr\int_0^r [y^2(\rho)]d\rho - r\sqrt{c}\cot(r\sqrt{c})\cdot y^2(r) \ge 0 \tag{3.11}$$

for $0 < r < 1$ and $c > 0$. Equality holds for

$$y(\rho) = c^{-1/2}\sin(\rho\sqrt{c}), \quad c > 0.$$

4. Proof of the main results

4.1. Proof of Theorem 2.1

Given that $f \in \mathcal{B}$ satisfies (2.3) and c_α is the smallest positive root of the equation (2.4). A simple computation yields

$$\alpha = \frac{2\sqrt{c_\alpha} - \tan\sqrt{c_\alpha}}{\tan\sqrt{c_\alpha}}.$$

Differentiating α with respect to c_α, we obtain

$$\frac{d\alpha}{dc_\alpha} = \frac{\tan\sqrt{c_\alpha} - \sqrt{c_\alpha}\sec^2\sqrt{c_\alpha}}{\sqrt{c_\alpha}\tan^2\sqrt{c_\alpha}}.$$

Since $\tan x - x\sec^2 x \le 0$ is equivalent to $\sin 2x \le 2x$, which is always true for all $x \in \mathbb{R}$, it follows that c_α increases if and only if α decreases.

Now we proceed for completing the proof of (a) and (b).

a. In this part we prove that f is meromorphically convex of order α, $0 \leq \alpha < 1$, that is f satisfies (2.1).
Set $S_f(z) = 2p(z)$ for a given analytic function $p(z)$. Then by (2.3), it follows that $|p(z)| \leq c_\alpha$, and hence we have

$$\mathrm{Re}(z^2 p(z)) \leq c_\alpha |z|^2 \quad \text{for } |z| < 1.$$

By Lemma 3.1, the function has the form $f(z) = w_1(z)/w_2(z)$ for any pair of linearly independent solutions $w_1(z)$ and $w_2(z)$ of the linear differential equation (3.1). Clearly, the particular solution $w_2(z)$ satisfies the hypothesis of Lemma 3.6. Since $\mathrm{Re}(z^2 p(z)) \leq c_\alpha |z|^2$ holds, (3.10) implies

$$|w_2(re^{i\theta})|^2 \mathrm{Re}\Big(\frac{re^{i\theta} w_2'(re^{i\theta})}{w_2(re^{i\theta})}\Big) \geq r \int_0^r |w_2'(\rho e^{i\theta})|^2 d\rho - r c_\alpha \int_0^r |w_2(\rho e^{i\theta})|^2 d\rho, \qquad (4.1)$$

for $0 < \rho \leq r < 1$ and for some fixed θ.
Putting $w_2(\rho e^{i\theta}) = u_2(\rho, \theta) + i v_2(\rho, \theta)$. For a constant ray θ, w_2 will become a function of ρ only. Note that $u_2(\rho)$ and $v_2(\rho)$ satisfies the hypothesis of Lemma 3.7. We obtain the following two inequalities after substituting $u_2(\rho)$ and $v_2(\rho)$ in (3.11) and replacing c by c_α

$$r \int_0^r [u_2'(\rho)]^2 d\rho - c_\alpha r \int_0^r [u_2^2(\rho)] d\rho - \sqrt{c_\alpha} r \cot(\sqrt{c_\alpha} r) \cdot u_2^2(r) \geq 0, \qquad (4.2)$$

and

$$r \int_0^r [v_2'(\rho)]^2 d\rho - c_\alpha r \int_0^r [v_2^2(\rho)] d\rho - \sqrt{c_\alpha} r \cot(\sqrt{c_\alpha} r) \cdot v_2^2(r) \geq 0. \qquad (4.3)$$

Since $w_2(\rho e^{i\theta}) = u_2(\rho, \theta) + i v_2(\rho, \theta)$, addition of (4.2) and (4.3) leads to

$$r \int_0^r |w_2'(\rho e^{i\theta})|^2 d\rho - r c_\alpha \int_0^r |w_2(\rho e^{i\theta})|^2 d\rho \geq \sqrt{c_\alpha} r \cot(\sqrt{c_\alpha} r) |w_2|^2. \qquad (4.4)$$

Comparing (4.1) with (4.4), we obtain

$$|w_2(re^{i\theta})|^2 \mathrm{Re}\Big(\frac{z w_2'(re^{i\theta})}{w_2(re^{i\theta})}\Big) \geq \sqrt{c_\alpha} r \cot(\sqrt{c_\alpha} r) |w_2(re^{i\theta})|^2,$$

that is,

$$\mathrm{Re}\Big(\frac{z w_2'(z)}{w_2(z)}\Big) \geq \sqrt{c_\alpha} r \cot(\sqrt{c_\alpha} r) \quad \text{for } |z| = r < 1. \qquad (4.5)$$

It follows from Lemma 3.5 that

$$\sqrt{c_\alpha} r \cot(\sqrt{c_\alpha} r) = \mathrm{Re}(\sqrt{c_\alpha} r \cot(\sqrt{c_\alpha} r)) > \frac{\alpha + 1}{2}. \qquad (4.6)$$

Comparison of (4.5) with (4.6) yields

$$\mathrm{Re}\Big(\frac{z w_2'(z)}{w_2(z)}\Big) > \frac{\alpha + 1}{2},$$

and hence it follows from Lemma 3.3 that f is meromorphically convex of order α.

b. We prove that the quantity c_α is the largest possible constant satisfying (2.3), i.e. we can not replace c_α by a larger quantity. We prove this by contradiction. If we replace c_α by a larger number $c = c_\alpha + \epsilon$ for some $\epsilon > 0$, then we observe that there exists a function $f \in \mathcal{B}$ satisfying

$$|S_f(z)| \leq 2(c_\alpha + \epsilon), \quad |z| < 1, \tag{4.7}$$

but f is not meromorphically convex of order α. For this, we consider the function

$$f(z) = \frac{w_1(z)}{w_2(z)}, \quad |z| < 1,$$

with the two linearly independent solutions

$$w_1(z) = \cos(\sqrt{c}z) \text{ and } w_2(z) = \frac{\sin(\sqrt{c}z)}{\sqrt{c}}$$

of the differential equation $w'' + cw = 0$. Clearly, by a simple computation, the function $f(z) = \sqrt{c}\cot(\sqrt{c}z)$ satisfies $S_f(z) = 2c$. It remains to show that this function f is not meromorphically convex of order α, equivalently, by definition, we prove that

$$-\mathrm{Re}\Big(1 + \frac{z_0 f''(z_0)}{f'(z_0)}\Big) \leq \alpha$$

for some $z_0 \in \mathbb{D}$. By Lemma 3.3, it is equivalently to proving

$$\mathrm{Re}\Big(\frac{z_0 w_2'(z_0)}{w_2(z_0)}\Big) = \mathrm{Re}\Big(\frac{\sqrt{c}z_0 \cos(\sqrt{c}z_0)}{\sin(\sqrt{c}z_0)}\Big) \leq \frac{\alpha+1}{2}. \tag{4.8}$$

for some non-zero $z_0 \in \mathbb{D}$, since for $z_0 = 0$ the relation (4.8) contradicts to the assumption $\alpha < 1$. Substituting $0 \neq z_0 = x_0 + iy_0 \in \mathbb{D}$ in (4.8) and simplifying, we obtain

$$\begin{aligned} 2x_0\sqrt{c}\sin(\sqrt{c}x_0)\cos(\sqrt{c}x_0) &+ 2y_0\sqrt{c}\sinh(\sqrt{c}y_0)\cosh(\sqrt{c}y_0) \\ &\leq (1+\alpha)(\sin^2(\sqrt{c}x_0) + \sinh^2(\sqrt{c}y_0)), \end{aligned}$$

or

$$\begin{aligned} &\sin(\sqrt{c}x_0)\cos(\sqrt{c}x_0)[2x_0\sqrt{c} - (1+\alpha)\tan(\sqrt{c}x_0)] \\ &\leq \sinh(\sqrt{c}y_0)\cosh(\sqrt{c}y_0)[(1+\alpha)\tanh(\sqrt{c}y_0) - 2(\sqrt{c}y_0)], \end{aligned}$$

for $0 < c = c_\alpha + \epsilon$ and $x_0^2 + y_0^2 < 1$. Choose $y_0 = 0$. Then to obtain our desired inequality, we have to find $x_0 \in (-1, 1)$, $x_0 \neq 0$, such that

$$\sin(\sqrt{c}x_0)\cos(\sqrt{c}x_0)[2x_0\sqrt{c} - (1+\alpha)\tan(\sqrt{c}x_0)] \leq 0 \tag{4.9}$$

holds. Now, we see that $\sin(\sqrt{c}x_0)$ and $\cos(\sqrt{c}x_0)$ are positive in $(0, \pi/2\sqrt{c})$, and $2x_0\sqrt{c} - (1+\alpha)\tan(\sqrt{c}x_0)$ is negative in $(\sqrt{c_\alpha}/\sqrt{c}, 1)$, where the latter part follows by (3.7). Therefore, (4.9) holds true for some x_0 in the intersection

$$(0, \pi/2\sqrt{c}) \cap (\sqrt{c_\alpha}/\sqrt{c}, 1) \subset (0, 1),$$

since $c_\alpha < c$. This completes the proof of our first main theorem. □

In the following example, we construct a function meromorphically convex of order α satisfies the hypothesis of Theorem 2.1.

Example 4.1. For a constant $c > 0$, consider the function f defined by

$$f(z) = \frac{w_1(z)}{w_2(z)} = \sqrt{c}\cot(\sqrt{c}z),$$

where $w_1(z) = \cos(\sqrt{c}z)$ and $w_2(z) = (1/\sqrt{c})\sin(\sqrt{c}z)$ that satisfy the differential equation

$$w'' + cw = 0.$$

By Lemma 3.1, it follows that $S_f(z) = 2c$. Now, for any such constant $c \le c_\alpha$, where c_α is the smallest positive root of the equation (2.4), one clearly sees that

$$|S_f(z)| \le 2c_\alpha.$$

Next, by comparing with Lemma 3.5, we see that

$$\mathrm{Re}\Big(\frac{zw_2'(z)}{w_2(z)}\Big) = \mathrm{Re}(z\sqrt{c}\cot(\sqrt{c}z)) > \frac{1+\alpha}{2}.$$

This is equivalent to saying that f is meromorphically convex of order α, by Lemma 3.3. Thus, Theorem 2.1 is satisfied by the function $f(z) = \sqrt{c}\cot(\sqrt{c}z)$.

4.2. Proof of Theorem 2.2

We adopt the idea from the proof of [2, Theorem 2]. Suppose that $u(z)$ and $v(z)$ are two linearly independent solutions of the differential equation (3.1) with $S_g(z) = 2p(z)$, where $u(0) = v'(0) = 0$ and $u'(0) = v(0) = 1$. Then by a similar analysis as in the proof of [2, Theorem 2], we obtain

$$g(z) = \frac{u(z)}{cu(z) + v(z)},$$

where $c = -a_2$. An easy computation yields

$$1 + \frac{zg''(z)}{g'(z)} = 1 - 2z\frac{cu'(z) + v'(z)}{cu(z) + v(z)}. \tag{4.10}$$

Now, by the hypothesis, it is easy to see that

$$\phi(\beta) = \min\left\{\frac{\beta-1}{\beta+1}, \frac{6(\beta-1)}{2(7\beta-9)}\right\} < 1 \text{ and } \psi(\beta) = \max\left\{\frac{\beta+3}{\beta+1}, \frac{11\beta-15}{7\beta-9}\right\} > 1.$$

Also, we note that

$$2\eta + (1+\eta)\delta e^{\delta/2} < 2\eta + \psi(\beta)\delta(1+\eta)e^{\delta/2} < 2\phi(\beta) < 2$$

follows from the assumption (2.5). Hence $\eta + (1+\eta)\delta e^{\delta/2}/2 < 1$. Now [2, (13)] also satisfied by our hypothesis. Thus, it follows from the similar argument as in the proof of [2, Theorem 2] that

$$\Big|\frac{cu'(z) + v'(z)}{cu(z) + v(z)}\Big| \le \frac{2(\eta + (1+\eta)\delta e^{\delta/2})}{2 - 2\eta - (1+\eta)\delta e^{\delta/2}},$$

which yields

$$\mathrm{Re}\Big(\frac{z(cu'(z) + v'(z))}{cu(z) + v(z)}\Big) > -\Big|\frac{z(cu'(z) + v'(z))}{cu(z) + v(z)}\Big| > -\frac{2(\eta + (1+\eta)\delta e^{\delta/2})}{2 - 2\eta - (1+\eta)\delta e^{\delta/2}}, \tag{4.11}$$

and

$$\mathrm{Re}\Big(\frac{z(cu'(z)+v'(z))}{cu(z)+v(z)}\Big) \le \Big|\frac{z(cu'(z)+v'(z))}{cu(z)+v(z)}\Big| < \frac{2(\eta+(1+\eta)\delta e^{\delta/2})}{2-2\eta-(1+\eta)\delta e^{\delta/2}}. \tag{4.12}$$

The relations (4.10), (4.11) and (4.12) together lead to

$$\frac{2-6\eta-5(1+\eta)\delta e^{\delta}/2}{2-2\eta-(1+\eta)\delta e^{\delta}/2} < \mathrm{Re}\Big(1+\frac{zg''(z)}{g'(z)}\Big) < \frac{2+2\eta+3(1+\eta)\delta e^{\delta}/2}{2-2\eta-(1+\eta)\delta e^{\delta}/2}.$$

The hypothesis (2.5) thus obtains

$$\frac{2+2\eta+3(1+\eta)\delta e^{\delta}/2}{2-2\eta-(1+\eta)\delta e^{\delta}/2} < \beta$$

and

$$\frac{2-6\eta-5(1+\eta)\delta e^{\delta}/2}{2-2\eta-(1+\eta)\delta e^{\delta}/2} > \frac{-\beta}{2\beta-3},$$

completing the proof. □

Remark 4.2. The constant $\phi(\beta)$ in the statement of Theorem 2.2 is not sharp. For instance, the function $g(z)=\dfrac{2z-z^2}{2(1-z)^2}\in\mathcal{C}_\infty$ for which $|a_2|=3/2>1$.

In the following example we construct a function that agree with Theorem 2.2 for some $\beta\ge 3/2$.

Example 4.3. For any constant c with $|c|<3/7$, consider the function g defined by

$$g(z)=\frac{z}{1-cz}, \quad |z|<1.$$

Although it directly follows from Corollary 2.4 that $g\in\mathcal{C}_\infty$, one can also show that $g\in\mathcal{C}_{5/2}$ and it satisfies the hypothesis of Theorem 2.2.

Indeed, first we note that g is a Möbius transformation and hence $S_g=0$. Therefore, it trivially satisfies the hypothesis of Theorem 2.2.

Secondly, an easy computation yields

$$1+\frac{zg''(z)}{g'(z)}=\frac{1+cz}{1-cz}.$$

From this, we have

$$\mathrm{Re}\Big(1+\frac{zg''(z)}{g'(z)}\Big)=\frac{1-|c|^2|z|^2}{|1-cz|^2}.$$

By the usual triangle inequalities, it follows that

$$\frac{1-|c||z|}{1+|c||z|}\le\frac{1-|c|^2|z|^2}{|1-cz|^2}\le\frac{1+|c||z|}{1-|c||z|}.$$

Since $|c|<3/7$, for $|z|<1$, it is easy to verify that

$$\frac{1+|c||z|}{1-|c||z|}<\frac{5}{2} \quad \text{and} \quad -\frac{5}{4}<\frac{1-|c||z|}{1+|c||z|}$$

hold true. Thus, $g\in\mathcal{C}_{5/2}$.

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A note on the degree of approximation of functions belonging to certain Lipschitz class by almost Riesz means

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Abstract. The problem of obtaining degree of approximation for the $2\pi-$periodic functions in the weighted Lipschitz class $W(L^p, \xi(t))$ $(p \geq 1)$ by Riesz means of the Fourier series have been studied by various investigators under L^p-norm. Recently, Deepmala and Piscoran [Approximation of signals(functions) belonging to certain Lipschitz classes by almost Riesz means of its Fourier series, J. Inequal. Appl., (2016), 2016:163. DOI 10.1186/s13660-016-1101-5] obtained a result on degree of approximation for weighted Lipschitz class by Riesz means. In this note, we extend this study to the weighted L^p-norm which in turn improves some of the previous results. We also derive some corollaries from our result.

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1. Introduction

Let f be a $2\pi-$periodic function belonging to the space $L^p: = L^p[0, 2\pi](p \geq 1)$. Then the trigonometric Fourier series of f is defined as

$$f(x) \sim \frac{a_0}{2} + \sum_{k=1}^{\infty}(a_k \cos kx + b_k \sin kx) \tag{1.1}$$

and the n^{th} partial sum of the Fourier series of f, given by

$$s_n(f; x) = \frac{a_0}{2} + \sum_{k=1}^{n}(a_k \cos kx + b_k \sin kx), n \in N \text{ with } s_0(f; x) = a_0/2,$$

is called the trigonometric polynomial of degree or order n (see [11]). Throughout this paper $||.||_p$ will denote the L^p norm defined by

$$||f||_p = \begin{cases} \left(\frac{1}{2\pi}\int_0^{2\pi}|f(x)|^p\,dx\right)^{\frac{1}{p}}, & p \geq 1; \\ ess \sup\limits_{0\leq x\leq 2\pi} |f(x)|, & p = \infty. \end{cases} \tag{1.2}$$

The following subclasses of $L^p[0,2\pi]-$ space are well known in the literature.
A function $f \in \mathrm{Lip}\alpha$, if

$$f(x+t) - f(x) = O(t^\alpha), \text{ for } 0 < \alpha \leq 1,\ t > 0.$$

A function $f \in Lip(\alpha, p)$, if

$$||f(x+t) - f(x)||_p = O(t^\alpha), \text{ for } 0 < \alpha \leq 1,\ p \geq 1,\ t > 0.$$

Given a positive increasing function $\xi(t)$, a function $f \in \mathrm{Lip}(\xi(t), p)$, if

$$||f(x+t) - f(x)||_p = O(\xi(t)), \text{ for } p \geq 1,\ t > 0,$$

and $f \in W(L^p, \xi(t))$, if

$$\left|\left|\big(f(x+t) - f(x)\big)sin^\beta\left(\frac{x}{2}\right)\right|\right|_p = O(\xi(t)), \text{ for } \beta \geq 0,\ p \geq 1,\ t > 0. \tag{1.3}$$

We have the following inclusions:

$$Lip\alpha \subseteq Lip(\alpha, p) \subseteq Lip(\xi(t), p) \subseteq W(L^p, \xi(t))$$

for all $0 < \alpha \leq 1$ and $p \geq 1$. Khan [2] was the first to use the weight function of the form $\sin^{\beta p}(x/2)$.
We obtain the degree of approximation of a function $f \in L^p-$space by a trigonometric polynomial $\tau_n(f;x)$ of degree n in L^p-norm by measuring the deviation $||\tau_n(f;x) - f(x)||_p$. This method of approximation is called the trigonometric Fourier approximation. The $\tau_n(f;x)$ is called Fourier approximant of f.
A bounded sequence $\{s_n\}$ is said to be almost convergent to a limit s, if

$$\lim_{n\to\infty} S_{n,m} = \lim_{n\to\infty}\left(\frac{1}{n+1}\sum_{k=m}^{m+n} s_k\right) = s \tag{1.4}$$

uniformly with respect to m (see [4]).
It can be easily verified that a convergent sequence is almost convergent and both the limits are same.
An infinite series $\sum u_n$ with the sequence of partial sums $\{s_n\}$ is said to be almost Riesz summable to s, if

$$\tau_{n,m} = \frac{1}{P_n}\sum_{k=0}^{n} p_k S_{k,m} \to s \ as \ n \to \infty \tag{1.5}$$

uniformly with respect to m (see [9]).
We also use the following notations

$$\psi(x,t) = \psi(t) = f(x+t) - 2f(x) + f(x-t),$$

$$\text{and } M_{n,m}(t) = \frac{1}{2\pi}\sum_{k=0}^{n}\frac{p_k}{(k+1)P_n}\frac{\sin((k+2m+1)\frac{t}{2})\sin((k+1)\frac{t}{2})}{\sin^2(t/2)}. \tag{1.6}$$

2. Known results

Lorentz [4] was the first who introduced the concept of almost convergence of sequences. King [3] investigated the regularity conditions for the almost summability matrices. Mazhar and Siddiqui [5] applied the concept of almost convergence of sequences to almost convergence of trigonometric sequences. In [7], Nanda introduced the spaces of strongly almost summable sequence spaces which happened to be complete paranormed spaces under certain conditions. The concept of almost convergence led to the formulation of various almost summability methods. After the definition of almost summability methods, Sharma and Qureshi [9] and Qureshi [8] determined the degree of approximation of certain functions by almost Riesz and almost Nörlund means of their Fourier series. Working in the same direction, Mishra et. al. [6] determined the degree of approximation of functions belonging to $Lip(\alpha, p)$ class by almost Riesz means.
Recently, Deepmala and Piscoran [1] proved a theorem on the degree of approximation for functions belnging to $W(L^p, \xi(t))(p \geq 1)-$class using almost Riesz means of its Fourier series with non-negative, non-decreasing weights p_n. They proved the following theorem:

Theorem 2.1. [1] *Assume f is a $2\pi-$periodic signal $(function)$ and integrable in the sense of Lebesgue over $[0, 2\pi]$. Then the degree of approximation of $f \in W(L^p, \xi(t))$ $(p \geq 1)-$class with $0 \leq \beta \leq 1 - 1/p$ by an almost Riesz means of its Fourier series is given by*

$$||\tau_{n,m}(f;x) - f(x)||_p = O\left(P_n^{\beta+1/p}\xi(P_n^{-1})\right),\ \forall n > 0, \tag{2.1}$$

provided that the positive increasing function $\xi(t)$ has the following features:

$$\{\xi(t)/t\} \text{is non-increasing in } t, \tag{2.2}$$

$$\left(\int_0^{\pi/P_n}\left(\frac{|\psi_x(t)|}{\xi(t)}\right)^p sin^{\beta p}(t/2)dt\right)^{1/p} = O(1) \tag{2.3}$$

and

$$\left(\int_{\pi/P_n}^{\pi}\left(\frac{t^{-\delta}|\psi_x(t)|}{\xi(t)}\right)^p dt\right)^{1/p} = O(P_n^{\delta}), \tag{2.4}$$

where δ is an arbitrary number such that $(\beta-\delta)q-1 > 0,\ p^{-1}+q^{-1} = 1,\ 1 \leq p \leq \infty$, and conditions (2.3) and (2.4) holds uniformly in x.

Remark 2.2. We note that in the statement of Theorem 2.1, the authors have taken $p \geq 1$, but in the proof they have used the Hölder's inequality for $p > 1$. Therefore, the proof is not valid for $p = 1$.

Remark 2.3. For $p = \infty$, conditions (2.3) and (2.4) will not hold in the present form.

Remark 2.4. In view of the remarks of Zhang [[10], p.1140], we note that the assumption conditions $0 \leq \beta \leq 1 - 1/p$ with $1/p + 1/q = 1$ and $(\beta - \delta)q - 1 > 0$ of Theorem 2.1 imply that $\delta < 0$. In this case, from condition (2.4), Theorem 2.1 is true for the function f which is a constant almost everywhere and thus the result is trivial.

3. Reformulation of the problem and main result

Being motivated by the above remarks, we reconsider the problem of Theorem 2.1 and note that the authors defined the function class $W(L^p, \xi(t))(p \geq 1)$ with the weight function $sin^{\beta p}(x/2)$ whereas the deviation $||\tau_{n,m}(f;x) - f(x)||_p$ is measured in ordinary L_p-norm. Actually, the function class $W(L^p, \xi(t))$ defined in (1.3) is a subclass of the weighted $L^p[0, 2\pi]-$space with the weight function $sin^{\beta p}(x/2)$, so it is pertinent to measure the deviation in the weighted norm defined as

$$||f||_{p,\beta} = \left(\frac{1}{2\pi} \int_0^{2\pi} |f(x)|^p sin^{\beta p}(x/2) dx \right)^{1/p}, \; p \geq 1. \tag{3.1}$$

In this paper, we reformulate the problem of Theorem 2.1 for the almost Riesz means and measure the deviation in the weighted norm defined in (3.1). More precisely, we prove

Theorem 3.1. *Let f be a $2\pi-$periodic function in $W(L^p, \xi(t))(p \geq 1)-$class and let $\{p_n\}$ be a non-negative, monotonic sequence such that*

$$(n + 1) \max\{p_0, p_n\} = O(P_n). \tag{3.2}$$

Then the degree of approximation of f by almost Riesz means of its Fourier series is given by

$$||\tau_{n,m}(f;x) - f(x)||_{p,\beta} = O\left(\xi\left(\frac{\pi}{n+1}\right) + (n+1)^{-\sigma} \right), \tag{3.3}$$

where the positive increasing function $\xi(t)$ satisfies the condition

$$t^{-\sigma}\xi(t) \text{ is non-decreasing for some } 0 < \sigma < 1. \tag{3.4}$$

Note that the conditions (2.3) and (2.4) of Theorem 2.1 have been relaxed in Theorem 3.1. Also, we prove the theorem for both non-decreasing and non-increasing sequence $\{p_n\}$ with condition (3.2).

4. Lemmas

We need the following lemmas for the proof of our theorem:

Lemma 4.1. *Let $M_{n,m}(t)$ be given by* (1.6)*. Then*

$$M_{n,m}(t) = O(n+1), \ \textit{for} \ \ 0 < t \leq \frac{\pi}{(n+1)}.$$

Proof. For $0 < t \leq \frac{\pi}{n+1}$, using $\sin(t/2) \geq t/\pi$ and $\sin nt \leq n \sin t$, we have

$$\begin{aligned} |M_{n,m}(t)| &= \left| \frac{1}{2\pi} \sum_{k=0}^{n} \frac{p_k}{(k+1)P_n} \frac{\sin((k+2m+1)\frac{t}{2}) sin((k+1)\frac{t}{2})}{sin^2(\frac{t}{2})} \right| \\ &\leq \frac{1}{2\pi P_n} \sum_{k=0}^{n} \frac{p_k}{(k+1)} \frac{(k+1)(k+2m+1)\sin^2(\frac{t}{2})}{\sin^2(\frac{t}{2})} \\ &= \frac{1}{2\pi P_n} \sum_{k=0}^{n} p_k(k+2m+1) \\ &= O(n+1). \end{aligned}$$

□

Lemma 4.2. *Let $M_{n,m}(t)$ be given by* (1.6)*. Then*

$$M_{n,m}(t) = O\left(\frac{1}{(n+1)t^2}\right), \ \textit{for} \ \frac{\pi}{(n+1)} < t \leq \pi.$$

Proof. For $\pi/(n+1) < t \leq \pi$, using $\sin(t/2) \geq t/\pi$ and $\sin nt \leq n \sin t$, we have

$$\begin{aligned} |M_{n,m}(t)| &= \left| \frac{1}{2\pi} \sum_{k=0}^{n} \frac{p_k}{(k+1)P_n} \frac{\sin((k+2m+1)\frac{t}{2}) \sin((k+1)\frac{t}{2})}{\sin^2(\frac{t}{2})} \right| \\ &\leq \frac{1}{2\pi P_n} \left| \sum_{k=0}^{n} \frac{p_k}{(k+1)} \frac{(k+1)\sin(\frac{t}{2})\sin((k+2m+1)\frac{t}{2})}{\sin(\frac{t}{2})} \frac{\pi}{t} \right| \\ &= \frac{1}{2\pi P_n t} \left| \sum_{k=0}^{n} p_k \sin\left((k+2m+1)\frac{t}{2}\right) \right|. \end{aligned}$$

Then, using condition (3.2), monotonicity of $\{p_n\}$ and Abel's lemma, we have

$$\left|M_{n,m}(t)\right| = O\left(\frac{1}{(n+1)t^2}\right),$$

in view of

$$\left| \sum_{k=0}^{n} \sin\left((k+2m+1)\frac{t}{2}\right) \right| = O(1/t).$$

□

Lemma 4.3. *Let $g(x,t) \in L^p\left([a,b] \times [c,d]\right), \ p \geq 1$. Then,*

$$\left\{ \int_a^b \left| \int_c^d g(x,t) dt \right|^p dx \right\}^{1/p} \leq \int_c^d \left(\int_a^b \left|g(x,t)\right|^p dx \right)^{1/p} dt.$$

This inequality is also known as the generalized form of Minkowski's inequality [[11], *p.*19].

5. Proof of Theorem 3.1

Proof. Using the integral representation of $S_{k,m}(f;x)$ and definition of $\tau_{n,m}(f;x)$ given in (1.5), we have

$$\begin{aligned}\tau_{n,m}(f;x)-f(x) &= \frac{1}{P_n}\sum_{k=0}^{n} p_k\{S_{k,m}(f;x)-f(x)\}\\ &= \frac{1}{2\pi P_n}\int_0^{\pi}\psi(t)\sum_{k=0}^{n}\frac{p_k}{(k+1)}\frac{[\cos mt-\cos(k+m+1)t]}{2\sin^2(\frac{t}{2})}dt\\ &= \frac{1}{2\pi P_n}\int_0^{\pi}\psi(t)\sum_{k=0}^{n}\frac{p_k}{(k+1)}\frac{\sin((k+2m+1)t/2)\sin((k+1)t/2)}{\sin^2(\frac{t}{2})}dt\\ &= \int_0^{\pi}\psi(t)M_{n,m}(t)dt,\end{aligned}$$

which on applying Lemma 4.3 gives

$$\begin{aligned}||\tau_{n,m}(f;x)-f(x)||_{p,\beta} &= \left\{\frac{1}{2\pi}\int_0^{2\pi}\left|\int_0^{\pi}\psi(t)M_{n,m}(t)dt\right|^p sin^{\beta p}(x/2)dx\right\}^{1/p}\\ &\le \int_0^{\pi}\left(\frac{1}{2\pi}\int_0^{2\pi}|\psi(t)|^p sin^{\beta p}(x/2)dx\right)^{1/p}|M_{n,m}(t)|dt.\end{aligned}$$

Using the fact that $\psi(t)\in W(L^p,\xi(t))$ due to $f\in W(L^p,\xi(t))$, we have

$$\begin{aligned}||\tau_{n,m}(f;x)-f(x)||_{p,\beta} &= \int_0^{\pi} O(\xi(t))|M_{n,m}(t)|dt\\ &= O(1)\left[\int_0^{\pi/(n+1)}\xi(t)|M_{n,m}(t)|dt+\int_{\pi/(n+1)}^{\pi}\xi(t)|M_{n,m}(t)|dt\right]\\ &= I_1+I_2\ (say). \qquad (5.1)\end{aligned}$$

Using Lemma 4.1, increasing nature of $\xi(t)$ and the mean value theorem for integrals, we have

$$\begin{aligned}I_1 &= O(1)\int_0^{\pi/(n+1)}\xi(t)|M_{n,m}(t)|dt = O(1)\int_0^{\pi/(n+1)}(n+1)\xi(t)dt\\ &= O\left(\xi\left(\frac{\pi}{n+1}\right)\right). \qquad (5.2)\end{aligned}$$

Using Lemma 4.2, condition (3.4) and the mean value theorem for integrals, we have

$$\begin{aligned} I_2 &= O(1)\int_{\pi/(n+1)}^{\pi} \xi(t)|M_{n,m}(t)|dt \\ &= O(1)\frac{1}{(n+1)}\int_{\pi/(n+1)}^{\pi} \frac{t^\sigma}{t^2}\frac{\xi(t)}{t^\sigma}dt = O(1)\frac{\xi(\pi)\pi^{-\sigma}}{(n+1)}\left(\frac{\pi}{n+1}\right)^{\sigma-1} \\ &= O\big((n+1)^{-\sigma}\big). \end{aligned} \tag{5.3}$$

Collecting (5.1) – (5.3), we have

$$||\tau_{n,m}(f;x) - f(x)||_{p,\beta} = O\bigg(\xi\left(\frac{\pi}{n+1}\right) + (n+1)^{-\sigma}\bigg). \qquad \square$$

6. Corollaries

For $\beta = 0$, the weighted class $W(L^p, \xi(t))$ reduces to $Lip(\xi(t), p)$. Thus, we have the following corollary:

Corollary 6.1. *Let f be a $2\pi-$periodic function in $Lip(\xi(t), p)(p \geq 1)-$class and let $\{p_n\}$ be a non-negative, monotonic sequence such that*

$$(n+1)\max\{p_0, p_n\} = O(P_n).$$

Then the degree of approximation of f by almost Riesz means of its Fourier series is given by

$$||\tau_{n,m}(f;x) - f(x)||_p = O\bigg(\xi\left(\frac{\pi}{n+1}\right) + (n+1)^{-\sigma}\bigg),$$

where the positive increasing function $\xi(t)$ satisfies the condition

$$t^{-\sigma}\xi(t) \text{ is non-decreasing for some } 0 < \sigma < 1.$$

If $\beta = 0$ and $\xi(t) = t^\alpha$, then the weighted class $W(L^p, \xi(t))(p \geq 1)$ reduces to the class $\text{Lip}(\alpha, p)(p \geq 1)$. In this case, the function $t^{-\sigma}\xi(t) = t^{\alpha-\sigma}$ is increasing for $0 < \sigma < \alpha \leq 1$. Thus, we have the following corollary:

Corollary 6.2. *Let f be a $2\pi-$periodic function in $Lip(\alpha, p)(p \geq 1)-$class and let $\{p_n\}$ be a non-negative, monotonic sequence such that*

$$(n+1)\max\{p_0, p_n\} = O(P_n).$$

Then the degree of approximation of f by almost Riesz means of its Fourier series is given by

$$||\tau_{n,m}(f;x) - f(x)||_p = O\big((n+1)^{-\sigma}\big),\ 0 < \sigma < \alpha.$$

However, we can obtain the degree of approximation of a function $f \in Lip(\alpha, p)$ independently as under:
Putting $\xi(t) = t^\alpha$ in (5.1), we have

$$I_1 = O(1)\int_0^{\pi/(n+1)} (n+1)t^\alpha dt = O\big((n+1)^{-\alpha}\big),\ 0 < \alpha \leq 1, \tag{6.1}$$

and

$$I_2 = O\big(1/(n+1)\big)\int_{\pi/(n+1)}^{\pi}\frac{t^\alpha}{t^2}dt = \begin{cases} O((n+1)^{-\alpha}), & 0<\alpha<1; \\ O\big(\frac{log(n+1)}{n+1}\big), & \alpha = 1. \end{cases} \tag{6.2}$$

Combining (6.1) and (6.2), we have

$$||\tau_{n,m}(f;x) - f(t)||_p = \begin{cases} O((n+1)^{-\alpha}), & 0<\alpha<1; \\ O\big(\frac{log(n+1)}{n+1}\big), & \alpha = 1. \end{cases}$$

If $\beta = 0$, $\xi(t) = t^\alpha$ and $p \to \infty$, then the weighted class $W(L^p, \xi(t))(p \geq 1)$ reduces to the class Lip(α). Thus, we have the following corollary:

Corollary 6.3. *Let f be a $2\pi-$periodic function in $Lip(\alpha)$class and let $\{p_n\}$ be a non-negative, monotonic sequence such that*

$$(n+1)\max\{p_0, p_n\} = O(P_n).$$

Then the degree of approximation of f by almost Riesz means of its Fourier series is given by

$$||\tau_{n,m}(f;x) - f(x)||_\infty = O\big((n+1)^{-\sigma}\big),\ 0<\sigma<\alpha.$$

Independently, we can obtain

$$||\tau_{n,m}(f;x) - f(x)||_\infty = \begin{cases} O((n+1)^{-\alpha}), & 0<\alpha<1; \\ O\big(\frac{log(n+1)}{n+1}\big), & \alpha = 1. \end{cases}$$

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Variable Hardy and Hardy-Lorentz spaces and applications in Fourier analysis

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Abstract. We summarize some results about the variable Hardy and Hardy-Lorentz spaces $H_{p(\cdot)}(\mathbb{R}^d)$ and $H_{p(\cdot),q}(\mathbb{R}^d)$ and about the θ-summability of multi-dimensional Fourier transforms. We prove that the maximal operator of the θ-means is bounded from $H_{p(\cdot)}(\mathbb{R}^d)$ to $L_{p(\cdot)}(\mathbb{R}^d)$ and from $H_{p(\cdot),q}(\mathbb{R}^d)$ to $L_{p(\cdot),q}(\mathbb{R}^d)$. This implies some norm and almost everywhere convergence results for the Riesz, Bochner-Riesz, Weierstrass, Picard and Bessel summations.

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1. Introduction

It was proved in Stein, Taibleson and Weiss [27] that the Bochner-Riesz means

$$\sigma_T^\alpha f(x) := \int_{-T}^{T} \left(1 - \left(\frac{|t|}{T}\right)^2\right)^\alpha \widehat{f}(t) e^{2\pi\imath xt}\, dt \qquad (x \in \mathbb{R}, T > 0)$$

converge almost everywhere to f, whenever the one-dimensional function $f \in L_p(\mathbb{R}^d)$ $(1 \le p < \infty, 0 < \alpha < \infty)$. Here $\widehat{f}$ denotes the Fourier transform of f. Moreover, the maximal operator σ_*^θ of the Bochner-Riesz means is bounded from the Hardy space $H_p(\mathbb{R})$ to $L_p(\mathbb{R})$ if $p > 1/(\alpha+1)$ (see also Grafakos [12] and Lu [20] or Weisz [33]).

In this paper, we generalize these results to multi-dimensional functions, to Lebesgue and Hardy spaces with variable exponents and to a general method of summation, to the θ-summability. The θ-summation is generated by a single function θ and includes all well known summations. This topic is studied intensively in the literature (see e.g. Butzer and Nessel [2], Trigub and Belinsky [29], Gát [8, 9], Goginava

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[10, 11], Persson, Tephnadze and Wall [22], Simon [24, 25] and Feichtinger and Weisz [7, 31, 32, 33] and the references therein).

Let $p(\cdot) : \mathbb{R}^n \to (0,\infty)$ be a variable exponent function satisfying the globally log-Hölder condition and $0 < q \leq \infty$. We introduce the variable Lebesgue, Lorentz, Hardy and Hardy-Lorentz spaces $L_{p(\cdot)}(\mathbb{R}^d)$, $L_{p(\cdot),q}(\mathbb{R}^d)$, $H_{p(\cdot)}(\mathbb{R}^d)$ and $H_{p(\cdot),q}(\mathbb{R}^d)$. These spaces are investigated very intensively in the literature nowadays (see e.g. Cruz-Uribe and Fiorenza [5], Diening at al. [6], Kempka and Vybral [16], Nakai and Sawano [21, 23], Jiao at al. [14, 15], Yan at al. [35] and Liu at al. [18, 19]). We give the atomic decomposition of these Hardy spaces just mentioned. If $p(\cdot)$ is a constant, then we get back the classical Lebesgue and Hardy spaces. Under some conditions on θ, we will prove that the maximal operator σ_*^θ is bounded from $H_{p(\cdot)}(\mathbb{R}^d)$ to $L_{p(\cdot)}(\mathbb{R}^d)$ and from $H_{p(\cdot),q}(\mathbb{R}^d)$ to $L_{p(\cdot),q}(\mathbb{R}^d)$ for all $p(\cdot) > p_0$. As a consequence, we obtain some norm and almost everywhere convergence results for the θ-means. As special cases of the θ-summation, we consider the Riesz, Bochner-Riesz, Weierstrass, Picard and Bessel summations. This paper was the base of my talk given at the 12th Joint Conference on Mathematics and Computer Science, Cluj-Napoca, June 2018.

2. θ-summability of Fourier transforms

For a constant p, the $L_p(\mathbb{R}^d)$ space is equipped with the quasi-norm

$$\|f\|_{L_p(\mathbb{R}^d)} := \left(\int_{\mathbb{R}^d} |f(x)|^p \, dx\right)^{1/p} \qquad (0 < p < \infty),$$

with the usual modification for $p = \infty$. Here we integrate with respect to the Lebesgue measure λ. The Lebesgue measure of a set H will be denoted also by $|H|$.

The *Fourier transform* of a function $f \in L_1(\mathbb{R}^d)$ is defined by

$$\widehat{f}(x) := \int_{\mathbb{R}^d} f(t) e^{-2\pi \imath x \cdot t} \, dt \qquad (x \in \mathbb{R}^d),$$

where $\imath = \sqrt{-1}$ and $x \cdot t := \sum_{k=1}^d x_k t_k$. Suppose first that $f \in L_p(\mathbb{R}^d)$ for some $1 \leq p \leq 2$. The Fourier inversion formula

$$f(x) = \int_{\mathbb{R}^d} \widehat{f}(t) e^{2\pi \imath x \cdot t} \, dt \qquad (x \in \mathbb{R}^d)$$

holds if $\widehat{f} \in L_1(\mathbb{R}^d)$. This motivates the definition of the Dirichlet integral $s_T f$ defined by

$$s_T f(x) := \int_{\mathbb{R}^d} \chi_{\{|t| \leq T\}} \widehat{f}(t) e^{2\pi \imath x \cdot t} \, dt \qquad (x \in \mathbb{R}^d). \tag{2.1}$$

It is known that, when $d = 1$ and $1 < p < \infty$, for any one-dimensional function $f \in L_p(\mathbb{R})$,

$$\lim_{T \to \infty} s_T f = f \qquad \text{in the } L_p(\mathbb{R})\text{-norm and almost everywhere.} \tag{2.2}$$

The almost everywhere convergence result in (2.2), due to Carleson [3] and Hunt [13] (see also Grafakos [12]), is one of the deepest results in harmonic analysis. It is also known that the convergence in (2.2) does not hold true for any higher dimensional function $f \in L_p(\mathbb{R}^d)$, except the norm convergence for $p = 2$ (see Stein and Weiss [28] or Grafakos [12]). On the other hand, the convergence in (2.2) does not hold true for $p = 1$ even when $d = 1$. This motivates one to replace the Dirichlet integrals by some summability means, which are defined via replacing the characteristic function in (2.1) by various functions with higher regularity. Via this replacement of the Dirichlet integrals by some summability means, we will extend (2.2) to the case $p \leq 1$.

Now we introduce the definition of θ-summability, which is a general summation generated by a single function $\theta : [0,\infty) \to \mathbb{R}$. Let $\theta_0(x) := \theta(|x|)$ and suppose that

$$\theta \in C_0[0,\infty), \qquad \theta(0) = 1, \qquad \theta_0 \in L_1(\mathbb{R}^d), \qquad \widehat{\theta_0} \in L_1(\mathbb{R}^d), \tag{2.3}$$

where $C_0[0,\infty)$ denotes the spaces of continuous functions vanishing at infinity and $|\cdot|$ denotes the Euclidean norm. For $T > 0$, the Tth *θ-mean* of the function $f \in L_p(\mathbb{R}^d)$ $(1 \leq p \leq 2)$ is given by

$$\sigma_T^\theta f(x) := \int_{\mathbb{R}^d} \theta\left(\frac{|t|}{T}\right) \widehat{f}(t) e^{2\pi\imath x\cdot t}\, dt \qquad (x \in \mathbb{R}^d, T > 0).$$

This integral is well defined because $\theta_0 \in L_p(\mathbb{R}^d)$ and $\widehat{f} \in L_{p'}(\mathbb{R}^d)$, where

$$1/p + 1/p' = 1.$$

For an integrable function f, it is known that we can rewrite $\sigma_T^\theta f$ as

$$\sigma_T^\theta f(x) = \int_{\mathbb{R}^d} f(x-t) K_T^\theta(t)\, dt = f * K_T^\theta(x) \qquad (x \in \mathbb{R}^d, T > 0),$$

where $*$ denotes the convolution and the Tth θ-kernel is given by

$$K_T^\theta(x) := \int_{\mathbb{R}^d} \theta\left(\frac{|t|}{T}\right) e^{2\pi\imath x\cdot t}\, dt = T^d \widehat{\theta}_0(Tx) \qquad (x \in \mathbb{R}^d, T > 0).$$

We can extend the θ-means to all function spaces investigated in this paper by

$$\sigma_T^\theta f := f * K_T^\theta \qquad (T > 0).$$

The maximal θ-operator is introduced by

$$\sigma_*^\theta f := \sup_{T>0} \left|\sigma_T^\theta f\right|.$$

3. One-dimensional Hardy spaces $H_p(\mathbb{R})$

For a Schwartz function $\psi \in S(\mathbb{R})$ with $\int_{\mathbb{R}} \psi\, d\lambda \neq 0$ and a tempered distribution $f \in S'(\mathbb{R})$ let the maximal function f^+ be defined by

$$f^+(x) := \sup_{0<t<\infty} |(f * \psi_t)(x)| \qquad (x \in \mathbb{R}),$$

where

$$\psi_t(x) := t^{-1}\psi(x/t) \qquad (t > 0).$$

A tempered distribution $f \in S'(\mathbb{R}^d)$ is in the *Hardy spaces* $H_p(\mathbb{R})$ $(0 < p \leq \infty)$ if

$$\|f\|_{H_p(\mathbb{R})} := \|f^+\|_{L_p(\mathbb{R})} < \infty.$$

For different Schwartz functions ψ, we get the same Hardy space with equivalent norms. The following theorem is well known (see e.g. Stein [26] or Weisz [33]).

Theorem 3.1. *If* $1 < p \leq \infty$, *then the Hardy space* $H_p(\mathbb{R})$ *is equivalent to* $L_p(\mathbb{R})$, *i.e.,*

$$H_p(\mathbb{R}) \sim L_p(\mathbb{R}).$$

The atomic decomposition is a useful characterization of the Hardy spaces by the help of which some boundedness results, duality theorems, inequalities and interpolation results can be proved. The atomic decomposition of Hardy spaces were proved e.g. in Latter [17], Lu [20], Wilson [34], Stein [26] and Weisz [33].

Definition 3.2. Let $0 < p < \infty$ and fix an integer $1/p - 1 < s < \infty$. A measurable function a is called a p-atom if there exists an interval $B \subset \mathbb{R}$ such that

(a) $\operatorname{supp} a \subset B$,
(b) $\|a\|_{L_\infty(\mathbb{R})} \leq |B|^{-1/p}$,
(c) $\int_{\mathbb{R}} a(x) x^\alpha dx = 0$ for all natural numbers $\alpha \leq s$.

Every function from the Hardy space $H_p(\mathbb{R})$ $(0 < p \leq 1)$ can be decomposed into the sum of atoms.

Theorem 3.3. *Let* $0 < p \leq 1$. *A tempered distribution* $f \in S'(\mathbb{R})$ *is in* $H_p(\mathbb{R})$ *if and only if there exist a sequence* $\{a_i\}_{i\in\mathbb{N}}$ *of p-atoms and a sequence* $\{\lambda_i\}_{i\in\mathbb{N}}$ *of positive numbers such that*

$$f = \sum_{i\in\mathbb{N}} \lambda_i a_i \qquad \text{in} \qquad S'(\mathbb{R}).$$

Moreover,

$$\|f\|_{H_p(\mathbb{R})} \sim \inf \left(\sum_{i\in\mathbb{N}} \lambda_i^p \right)^{1/p}, \tag{3.1}$$

where the infimum is taken over all decompositions of f *as above.*

In the present form the theorem does not hold for $1 < p < \infty$ and it cannot be extended to variable Hardy spaces. However, using the following ideas, we will extend the atomic decomposition to all $0 < p < \infty$ and to variable Hardy spaces in Section 5. First of all observe that (ii) of Definition 3.2 is the same as

$$\|a\|_{L_\infty(\mathbb{R})} \leq \frac{1}{\|\chi_B\|_{L_p(\mathbb{R})}}.$$

Secondly, for $0 < p \leq 1$, (3.1) can be written as

$$\|f\|_{H_p(\mathbb{R})} \sim \inf \left(\sum_{i\in\mathbb{N}} \lambda_i^p \right)^{1/p} = \inf \left\| \left(\sum_{i\in\mathbb{N}} \left(\frac{\lambda_i \chi_{B_i}}{\|\chi_{B_i}\|_{L_p(\mathbb{R})}} \right)^p \right)^{1/p} \right\|_{L_p(\mathbb{R})},$$

where B_i is the support of the p-atom a_i. This form of the atomic decomposition can already be extended to variable Hardy spaces.

It is an important problem as to whether a sublinear operator V is bounded from the Hardy space $H_p(\mathbb{R})$ to $L_p(\mathbb{R})$. If this boundedness holds for at least one p with $p < 1$ and for at least one p with $p > 1$, then we obtain by interpolation that V

is of weak type $(1,1)$, which is a basic inequality in harmonic analysis. The following (falls) theorem can be found several times in the literature.

Theorem 3.4. *Suppose that $0 < p \leq 1$, V is a sublinear operator and*

$$\|Va\|_{L_p(\mathbb{R})} \leq C_p \tag{3.2}$$

for all p-atoms a. Then

$$\|Vf\|_{L_p(\mathbb{R})} \leq C_p \|f\|_{H_p(\mathbb{R})} \qquad (f \in H_p(\mathbb{R})).$$

Here, we give a typical proof of this theorem. Usually, we take an atomic decomposition

$$f = \sum_{i \in \mathbb{N}} \lambda_i a_i,$$

where each a_i is a p-atom and

$$\left(\sum_{i \in \mathbb{N}} \lambda_i^p \right)^{1/p} \leq C_p \|f\|_{H_p(\mathbb{R})}.$$

Then

$$|Vf| \leq \sum_{i \in \mathbb{N}} \lambda_i |Va_i| \tag{3.3}$$

and

$$\|Vf\|_p^p \leq \sum_{i \in \mathbb{N}} \lambda_i^p \|Va_i\|_p^p \leq C_p \|f\|_{H_p(\mathbb{R})}^p \qquad (0 < p \leq 1).$$

The problem is that this proof is falls because the inequality (3.3) does not necessarily hold. Indeed, Bownik [1] have given an operator V for which (3.3) and Theorem 3.4 do not hold.

Now we present one correct version of Theorem 3.4 (see Weisz [33]). In summability theory, we investigate often the operators

$$V_t f(x) = f * K_t(x) := \int_{\mathbb{R}} f(u) K_t(x-u)\, du \qquad (t > 0),$$

where $K_t \in L_1(\mathbb{R})$ are summability kernels. Then $V_t : L_1(\mathbb{R}) \to L_1(\mathbb{R})$ are bounded linear operators. Set

$$V_* f := \sup_{t>0} |V_t f|.$$

Let us denote by $2I$ the interval with the same center as I and the radius of $2I$ is two times the radius of I.

Theorem 3.5. *Let $0 < p \leq 1$, $K_t \in L_1(\mathbb{R})$ and $V_t f = f * K_t$. Suppose that*

$$\int_{\mathbb{R} \setminus 2B} |V_* a|^p \, d\lambda \leq C_p \tag{3.4}$$

for all p-atoms a with support B. If V_* is bounded from $L_\infty(\mathbb{R})$ to $L_\infty(\mathbb{R})$, then

$$\|Vf\|_{L_p(\mathbb{R})} \leq C_p \|f\|_{H_p(\mathbb{R})} \qquad (f \in H_p(\mathbb{R})).$$

It is easy to see that (3.4) implies (3.2). Indeed, using (ii) of Definition 3.2 and the boundedness of V_* on $L_\infty(\mathbb{R})$, we obtain

$$\int_{\mathbb{R}} |V_* a|^p \, d\lambda \leq \int_{2B} |V_* a|^p \, d\lambda + C_p \leq C_p.$$

One of the most investigated summability is the Bochner-Riesz summability, defined by

$$\theta_0(t) = \begin{cases} (1-|t|^2)^\alpha, & \text{if } |t| > 1; \\ 0, & \text{if } |t| \leq 1 \end{cases} \qquad (t \in \mathbb{R}),$$

where $0 < \alpha < \infty$. It was introduced already in the Introduction. The next result was proved in Stein, Taibleson and Weiss [27], Grafakos [12] and Lu [20]. [27] contains a counterexample which shows that the theorem is not true for $p \leq 1/\alpha + 1$.

Corollary 3.6. *If $\frac{1}{\alpha+1} < p < \infty$, then for the Bochner-Riesz means we have*

$$\left\|\sigma_*^\theta f\right\|_{L_p(\mathbb{R})} \lesssim \|f\|_{H_p(\mathbb{R})} \qquad (f \in H_p(\mathbb{R})).$$

In the next sections, we will generalize these theorems to higher dimensional functions and to Hardy spaces with variable exponents.

4. Variable Lebesgue and Lorentz spaces

We are going to generalize the $L_p(\mathbb{R})$ spaces. A measurable function $p(\cdot) : \mathbb{R}^d \to (0,\infty)$ is called a *variable exponent* if

$$0 < p_- := \operatorname*{ess\,inf}_{x\in\mathbb{R}^d} p(x) \qquad \text{and} \qquad p_+ := \operatorname*{ess\,sup}_{x\in\mathbb{R}^d} p(x) < \infty.$$

The *variable Lebesgue space* $L_{p(\cdot)}(\mathbb{R}^d)$ contains all measurable functions f, for which

$$\|f\|_{L_{p(\cdot)}(\mathbb{R}^d)} := \inf\left\{\rho \in (0,\infty) : \int_{\mathbb{R}^d} \left(\frac{|f(x)|}{\rho}\right)^{p(x)} dx \leq 1\right\} < \infty.$$

If $p(\cdot)$ is a constant, then we get back the $L_p(\mathbb{R}^d)$ spaces. Usually, we cannot compute exactly the $L_{p(\cdot)}(\mathbb{R}^d)$-norm of a function or even of a characteristic function. However, we know the following inequalities due to Cruz-Uribe and Fiorenza [5, Corollary 2.23]:

$$\left(\int_{\mathbb{R}^d} |f(x)|^{p(x)} \, dx\right)^{1/p_+} \leq \|f\|_{L_{p(\cdot)}(\mathbb{R}^d)} \leq \left(\int_{\mathbb{R}^d} |f(x)|^{p(x)} \, dx\right)^{1/p_-}$$

if $\|f\|_{L_{p(\cdot)}(\mathbb{R}^d)} \geq 1$ and

$$\left(\int_{\mathbb{R}^d} |f(x)|^{p(x)} \, dx\right)^{1/p_-} \leq \|f\|_{L_{p(\cdot)}(\mathbb{R}^d)} \leq \left(\int_{\mathbb{R}^d} |f(x)|^{p(x)} \, dx\right)^{1/p_+}$$

if $\|f\|_{L_{p(\cdot)}(\mathbb{R}^d)} \leq 1$.

We denote by $C^{\log}(\mathbb{R}^d)$ the set of all variable exponents $p(\cdot)$ satisfying the so-called *globally log-Hölder continuous condition*, namely, there exist two positive constants $C_{\log}(p)$ and C_∞, and $p_\infty \in \mathbb{R}$ such that, for any $x, y \in \mathbb{R}^d$,

$$|p(x) - p(y)| \leq \frac{C_{\log}(p)}{\log(e + 1/|x-y|)} \tag{4.1}$$

and

$$|p(x) - p_\infty| \leq \frac{C_\infty}{\log(e + |x|)}.$$

It is easy to see that a Lipschitz function of order α $(0 < a \leq 1)$ satisfies (4.1).

Given a locally integrable function f, the *Hardy-Littlewood maximal operator* M is defined by

$$Mf(x) := \sup_{x \in B} \frac{1}{|B|} \int_B |f(y)| dy \qquad (x \in \mathbb{R}^d),$$

where the supremum is taken over all balls B of $\mathbb{R}^d$ containing x. It is known that M is bounded on $L_p(\mathbb{R}^d)$ if $1 < p < \infty$ and is of weak type $(1, 1)$. This is extended to the variable Lebesgue spaces as follows (Cruz-Uribe and Fiorenza [5, Theorem 3.16]).

Theorem 4.1. *Suppose that* $p(\cdot) \in C^{\log}(\mathbb{R}^d)$ *and* $f \in L_{p(\cdot)}(\mathbb{R}^d)$*. If* $p_- \geq 1$*, then*

$$\sup_{\rho \in (0,\infty)} \left(\rho \left\| \chi_{\{x \in \mathbb{R}^d :\, Mf(x) > \rho\}} \right\|_{L_{p(\cdot)}(\mathbb{R}^d)} \right) \leq \|f\|_{L_{p(\cdot)}(\mathbb{R}^d)}.$$

If in addition $p_- > 1$*, then*

$$\|Mf\|_{L_{p(\cdot)}(\mathbb{R}^d)} \leq C \|f\|_{L_{p(\cdot)}(\mathbb{R}^d)}. \tag{4.2}$$

We recall the Fefferman-Stein vector-valued inequality on variable Lebesgue spaces, which is a generalization of inequality (4.2) and is used in the proof of Theorem 6.1 (for the proof see Cruz-Uribe at al. [4, Corollary 2.1]).

Theorem 4.2. *If* $p(\cdot) \in C^{\log}(\mathbb{R}^d)$ *with* $p_- > 1$ *and* $1 < r < \infty$*, then*

$$\left\| \left(\sum_{j=1}^\infty (Mf_j)^r \right)^{1/r} \right\|_{L_{p(\cdot)}(\mathbb{R}^d)} \leq C \left\| \left(\sum_{j=1}^\infty |f_j|^r \right)^{1/r} \right\|_{L_{p(\cdot)}(\mathbb{R}^d)}.$$

The *variable Lorentz spaces* were introduced and investigated by Kempka and Vybral [16]. $L_{p(\cdot),q}(\mathbb{R}^d)$ is defined to be the space of all measurable functions f such that

$$\|f\|_{L_{p(\cdot),q}(\mathbb{R}^d)} := \begin{cases} \left(\int_0^\infty \rho^q \left\| \chi_{\{x \in \mathbb{R}^d :\, |f(x)| > \rho\}} \right\|^q_{L_{p(\cdot)}(\mathbb{R}^d)} \frac{d\rho}{\rho} \right)^{1/q}, & \text{if } 0 < q < \infty; \\ \sup_{\rho \in (0,\infty)} \rho \left\| \chi_{\{x \in \mathbb{R}^d :\, |f(x)| > \rho\}} \right\|_{L_{p(\cdot)}(\mathbb{R}^d)}, & \text{if } q = \infty \end{cases}$$

is finite. If $p(\cdot)$ is a constant, we get back the classical Lorentz spaces and $L_{p,p}(\mathbb{R}^d) = L_p(\mathbb{R}^d)$. This last equality is not true for variable spaces.

5. Variable Hardy and Hardy Lorentz spaces

Now we introduce the *variable Hardy* and *Hardy-Lorentz spaces* and give their atomic decompositions. Denote by $S(\mathbb{R}^d)$ the space of all Schwartz functions and by $S'(\mathbb{R}^d)$ the space of all tempered distributions. For $t \in (0,\infty)$ and $x \in \mathbb{R}^d$, let

$$\psi_t(x) := t^{-d}\psi(x/t).$$

Let $\psi \in S(\mathbb{R}^d)$ be a fixed Schwartz function with $\int_{\mathbb{R}^d} \psi \, d\lambda \neq 0$. The maximal function of a tempered distribution $f \in S'(\mathbb{R}^d)$ is defined by

$$f^+(x) := \sup_{0<t<\infty} |f * \psi_t(x)| \qquad (x \in \mathbb{R}^d).$$

The variable Hardy and Hardy-Lorentz spaces $H_{p(\cdot)}(\mathbb{R}^d)$ and $H_{p(\cdot),q}(\mathbb{R}^d)$ are consisting of all tempered distributions $f \in S'(\mathbb{R}^d)$ such that

$$\|f\|_{H_{p(\cdot)}(\mathbb{R}^d)} := \|f^+\|_{L_{p(\cdot)}(\mathbb{R}^d)} < \infty, \qquad \|f\|_{H_{p(\cdot),q}(\mathbb{R}^d)} := \|f^+\|_{L_{p(\cdot),q}(\mathbb{R}^d)} < \infty,$$

respectively. It is known that different functions ψ give the same space with equivalent norms. Moreover, all $f \in H_{p(\cdot)}(\mathbb{R}^d)$ and $f \in H_{p(\cdot),q}(\mathbb{R}^d)$ are bounded distributions, i.e. $f * \phi \in L_\infty(\mathbb{R}^d)$ for all $\phi \in S(\mathbb{R}^d)$.

Theorem 5.1. *If $p_- > 1$, then*

$$H_{p(\cdot)}(\mathbb{R}^d) \sim L_{p(\cdot)}(\mathbb{R}^d), \qquad H_{p(\cdot),q}(\mathbb{R}^d) \sim L_{p(\cdot),q}(\mathbb{R}^d).$$

For variable Hardy and Hardy-Lorentz spaces see the references Nakai and Sawano [21, 23], Yan at al. [35], Liu at al. [18, 19] and Jiao at al. [15]. If $p(\cdot)$ is a constant, then we get back the classical Hardy and Hardy-Lorentz spaces $H_p(\mathbb{R}^d)$ and $H_{p,q}(\mathbb{R}^d)$.

The atomic decomposition of variable Hardy spaces were studied in Nakai and Sawano [21, 23], Yan at al. [35], Liu at al. [18, 19] and Jiao at al. [15].

Definition 5.2. Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^d)$ and fix an integer $d(1/p_- - 1) < s < \infty$. A measurable function a is called a $p(\cdot)$-atom if there exists a ball $B \subset \mathbb{R}^d$ such that

(a) $\operatorname{supp} a \subset B$,

(b) $\|a\|_{L_\infty(\mathbb{R}^d)} \leq \frac{1}{\|\chi_B\|_{L_{p(\cdot)}(\mathbb{R}^d)}}$,

(c) $\int_{\mathbb{R}^d} a(x)x^\alpha dx = 0$ for all multi-indices α with $|\alpha| \leq s$.

Theorem 5.3. *Let $p(\cdot) \in C^{\log}(\mathbb{R}^d)$. A tempered distribution $f \in S'(\mathbb{R}^d)$ is in $H_{p(\cdot)}(\mathbb{R}^d)$ if and only if there exist a sequence $\{a_i\}_{i\in\mathbb{N}}$ of $p(\cdot)$-atoms with support $\{B_i\}_{i\in\mathbb{N}}$ and a sequence $\{\lambda_i\}_{i\in\mathbb{N}}$ of positive numbers such that*

$$f = \sum_{i\in\mathbb{N}} \lambda_i a_i \qquad in \qquad S'(\mathbb{R}^d).$$

Moreover,

$$\|f\|_{H_{p(\cdot)}(\mathbb{R}^d)} \sim \inf \left\| \left(\sum_{i\in\mathbb{N}} \left(\frac{\lambda_i \chi_{B_i}}{\|\chi_{B_i}\|_{L_{p(\cdot)}(\mathbb{R}^d)}} \right)^{\underline{p}} \right)^{1/\underline{p}} \right\|_{L_{p(\cdot)}(\mathbb{R}^d)},$$

where the infimum is taken over all decompositions of f as above.

Here we use the notation $\underline{p} = \min\{p_-, 1\}$.

Theorem 5.4. *Let $p(\cdot) \in C^{\log}(\mathbb{R}^d)$ and $0 < q \leq \infty$. A tempered distribution $f \in S'(\mathbb{R}^d)$ is in $H_{p(\cdot),q}(\mathbb{R}^d)$ if and only if there exist a sequence $\{a_{i,j}\}_{i\in\mathbb{Z},j\in\mathbb{N}}$ of $p(\cdot)$-atoms with support $\{B_{i,j}\}_{i\in\mathbb{Z},j\in\mathbb{N}}$ such that*

$$f = \sum_{i\in\mathbb{Z}} \sum_{j\in\mathbb{N}} \lambda_{i,j} a_{i,j} \qquad \text{in} \qquad S'(\mathbb{R}^d),$$

where $\sum_{j\in\mathbb{N}} \chi_{B_{i,j}}(x) \leq A$ for all $x \in \mathbb{R}^d$ and $i \in \mathbb{Z}$ and $\lambda_{i,j} := C2^i \|\chi_{B_{i,j}}\|_{L_{p(\cdot)}(\mathbb{R}^d)}$ $(i \in \mathbb{Z}, j \in \mathbb{N})$ with A and C being positive constants. Moreover,

$$\|f\|_{H_{p(\cdot),q}(\mathbb{R}^d)} \sim \inf \left(\sum_{i\in\mathbb{Z}} \left\| \left(\sum_{j\in\mathbb{N}} \left(\frac{\lambda_{i,j}\chi_{B_{i,j}}}{\|\chi_{B_{i,j}}\|_{L_{p(\cdot)}(\mathbb{R}^d)}} \right)^{\underline{p}} \right)^{1/\underline{p}} \right\|^q_{L_{p(\cdot)}(\mathbb{R}^d)} \right)^{1/q},$$

where the infimum is taken over all decompositions of f as above.

6. Summability in $H_{p(\cdot)}(\mathbb{R}^d)$ and $H_{p(\cdot),q}(\mathbb{R}^d)$

In this section, we will investigate the boundedness of some operators from $H_{p(\cdot)}(\mathbb{R}^d)$ to $L_{p(\cdot)}(\mathbb{R}^d)$ and from $H_{p(\cdot),q}(\mathbb{R}^d)$ to $L_{p(\cdot),q}(\mathbb{R}^d)$ (see Weisz [30]).

Theorem 6.1. *Let $p(\cdot) \in C^{\log}(\mathbb{R}^d)$, $0 < q < \infty$, $\gamma > 1$ and $p_- > 1/\gamma$. For each $t > 0$ let $K_t \in L_1(\mathbb{R}^d)$ and $V_t f = f * K_t$. Suppose that*

$$|V_* a(x)| \leq C \, \|\chi_B\|^{-1}_{L_{p(\cdot)}(\mathbb{R}^d)} \, |M\chi_B(x)|^\gamma \qquad (x \notin 2B) \tag{6.1}$$

for all $p(\cdot)$-atoms a with support B. If V_* is bounded from $L_\infty(\mathbb{R}^d)$ to $L_\infty(\mathbb{R}^d)$, then

$$\|V_* f\|_{L_{p(\cdot)}(\mathbb{R}^d)} \leq C \, \|f\|_{H_{p(\cdot)}(\mathbb{R}^d)} \qquad (f \in H_{p(\cdot)}(\mathbb{R}^d))$$

and

$$\|V_* f\|_{L_{p(\cdot),q}(\mathbb{R}^d)} \lesssim \|f\|_{H_{p(\cdot),q}(\mathbb{R}^d)} \qquad (f \in H_{p(\cdot),q}(\mathbb{R}^d)).$$

Using Theorem 4.1, we can show easily that (6.1) implies (3.4). Now let us apply the result for the summability of Fourier transforms. It is known that σ_T^θ is bounded from $L_1(\mathbb{R}^d)$ to $L_1(\mathbb{R}^d)$ for all $T > 0$ and σ_*^θ is bounded from $L_\infty(\mathbb{R}^d)$ to $L_\infty(\mathbb{R}^d)$ (see e.g. Weisz [31]). The next theorem shows that the additional condition (6.2) implies (6.1) (see Weisz [30]).

Theorem 6.2. *Let (2.3) be satisfied. Assume that $\widehat{\theta_0}$ is $(N+1)$-times differentiable for some $N \in \mathbb{N}$ and there exists $d + N < \beta \leq d + N + 1$ such that*

$$\left| \partial_1^{i_1} \dots \partial_d^{i_d} \widehat{\theta_0}(x) \right| \leq C|x|^{-\beta} \qquad (x \neq 0) \tag{6.2}$$

whenever $i_1 + \ldots + i_d = N$ or $i_1 + \ldots + i_d = N + 1$. If $p(\cdot) \in C^{\log}(\mathbb{R}^d)$, then

$$\left| \sigma_*^\theta a(x) \right| \leq C \, \|\chi_B\|^{-1}_{L_{p(\cdot)}(\mathbb{R}^d)} \, |M\chi_B(x)|^{\beta/d}$$

for all $p(\cdot)$-atoms a and all $x \notin 2B$, where the ball B is the support of the atom.

The following result follows from Theorems 6.1 and 6.2. Note that $\beta > d$.

Corollary 6.3. *Let (2.3) and (6.2) be satisfied. If $p(\cdot) \in C^{\log}(\mathbb{R}^d)$, $0 < q < \infty$ and $p_- > d/\beta$, then*

$$\left\|\sigma_*^\theta f\right\|_{L_{p(\cdot)}(\mathbb{R}^d)} \lesssim \|f\|_{H_{p(\cdot)}(\mathbb{R}^d)} \qquad (f \in H_{p(\cdot)}(\mathbb{R}^d))$$

and

$$\left\|\sigma_*^\theta f\right\|_{L_{p(\cdot),q}(\mathbb{R}^d)} \lesssim \|f\|_{H_{p(\cdot),q}(\mathbb{R}^d)} \qquad (f \in H_{p(\cdot),q}(\mathbb{R}^d)).$$

Note that if $p(\cdot)$ is a constant, then we get back the classical result (see Weisz [31, 32] as well as Corollary 3.6). Using Corollary 6.3 and a usual density argument, we obtain the next convergence results (see Weisz [30]).

Corollary 6.4. *Suppose that (2.3) and (6.2) are satisfied, $p(\cdot) \in C^{\log}(\mathbb{R}^d)$, $0 < q < \infty$ and $p_- > d/\beta$. If $f \in H_{p(\cdot)}(\mathbb{R}^d)$ (resp. $f \in H_{p(\cdot),q}(\mathbb{R}^d)$), then $\sigma_T^\theta f$ converges almost everywhere as well as in the $L_{p(\cdot)}(\mathbb{R}^d)$-norm (resp. in the $L_{p(\cdot),q}(\mathbb{R}^d)$-norm) as $T \to \infty$.*

Corollary 6.5. *Suppose that (2.3) and (6.2) are satisfied, $p(\cdot) \in C^{\log}(\mathbb{R}^d)$ and $1 \le q < \infty$. If $p_- > 1$ and $f \in L_{p(\cdot)}(\mathbb{R}^d)$ (resp. $f \in L_{p(\cdot),q}(\mathbb{R}^d)$), then*

$$\lim_{T\to\infty} \sigma_T^\theta f(x) = f(x) \qquad \text{for a.e. } x \in \mathbb{R}^d$$

as well as in the $L_{p(\cdot)}(\mathbb{R}^d)$-norm (resp. in the $L_{p(\cdot),q}(\mathbb{R}^d)$-norm). The almost everywhere convergence holds also if $f \in L_{p(\cdot)}(\mathbb{R}^d)$ with $p_- \ge 1$.

7. Some summability methods

As special cases, we consider some summability methods.

7.1. Riesz summation

The function

$$\theta_0(t) = \begin{cases} (1 - |t|^\gamma)^\alpha, & \text{if } |t| > 1; \\ 0, & \text{if } |t| \le 1 \end{cases} \qquad (t \in \mathbb{R}^d)$$

defines the *Riesz summation* if $0 < \alpha < \infty$ and γ is a positive integer. It is called *Bochner-Riesz summation* if $\gamma = 2$ and *Fejr summation* if $\alpha = \gamma = 1$. The following result follows from Corollaries 6.3–6.5.

Corollary 7.1. *If $p(\cdot) \in C^{\log}(\mathbb{R}^d)$, $0 < q < \infty$ and*

$$\alpha > \frac{d-1}{2}, \qquad \frac{d}{d/2 + \alpha + 1/2} < p_- < \infty,$$

then

$$\left\|\sigma_*^\theta f\right\|_{L_{p(\cdot)}(\mathbb{R}^d)} \lesssim \|f\|_{H_{p(\cdot)}(\mathbb{R}^d)} \qquad (f \in H_{p(\cdot)}(\mathbb{R}^d))$$

and

$$\left\|\sigma_*^\theta f\right\|_{L_{p(\cdot),q}(\mathbb{R}^d)} \lesssim \|f\|_{H_{p(\cdot),q}(\mathbb{R}^d)} \qquad (f \in H_{p(\cdot),q}(\mathbb{R}^d)).$$

Moreover, the corresponding Corollaries 6.4–6.5 hold as well.

7.2. Weierstrass summation

The *Weierstrass summation* is defined by

$$\theta_0(t) = e^{-|t|^2/2} \qquad \text{or} \qquad \theta_0(t) = e^{-|t|} \qquad (t \in \mathbb{R}^d).$$

Corollary 7.2. *If $p(\cdot) \in C^{\log}(\mathbb{R}^d)$ and $0 < q < \infty$, then*

$$\left\|\sigma_*^\theta f\right\|_{L_{p(\cdot)}(\mathbb{R}^d)} \lesssim \|f\|_{H_{p(\cdot)}(\mathbb{R}^d)} \qquad (f \in H_{p(\cdot)}(\mathbb{R}^d))$$

and

$$\left\|\sigma_*^\theta f\right\|_{L_{p(\cdot),q}(\mathbb{R}^d)} \lesssim \|f\|_{H_{p(\cdot),q}(\mathbb{R}^d)} \qquad (f \in H_{p(\cdot),q}(\mathbb{R}^d)).$$

Moreover, the corresponding Corollaries 6.4–6.5 *hold as well.*

7.3. Picard–Bessel summation

Corollary 7.2 holds for the summability method defined by

$$\theta_0(t) = (1 + |t|^2)^{-(d+1)/2} \qquad (t \in \mathbb{R}^d).$$

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A note on the Wang-Zhang and Schwarz inequalities

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Abstract. In this note we show that the Wang-Zhang inequality can be naturally applied to obtain an elegant reverse for the classical Schwarz inequality in complex inner product spaces.

Mathematics Subject Classification (2010): 46C05, 26D15.

Keywords: Schwarz inequality, inner products, inequalities for sums.

1. Introduction

Let $(H, \langle \cdot, \cdot \rangle)$ be a complex inner product space and $x, y \in H$ two nonzero vectors. One can define the *angle* between the vectors x, y either by

$$\Phi_{x,y} = \arccos \left(\frac{\operatorname{Re} \langle x, y \rangle}{\|x\| \|y\|} \right) \text{ or by } \Psi_{x,y} = \arccos \left(\frac{|\langle x, y \rangle|}{\|x\| \|y\|} \right).$$

The function $\Psi_{x,y}$ is a natural metric on complex projective space [6].

In 1969 M. K. Kreĭn [5] obtained the following inequality for angles between two vectors

$$\Phi_{x,y} \leq \Phi_{x,z} + \Phi_{z,y} \tag{1.1}$$

for any $x, y, z \in H \setminus \{0\}$.

By using the representation

$$\Psi_{x,y} = \inf_{\alpha,\beta \in \mathbb{C} \setminus \{0\}} \Phi_{\alpha x, \beta y} = \inf_{\alpha \in \mathbb{C} \setminus \{0\}} \Phi_{\alpha x, y} = \inf_{\beta \in \mathbb{C} \setminus \{0\}} \Phi_{x, \beta y} \tag{1.2}$$

and Kreĭn's inequality (1.1), M. Lin [6] has shown recently that the following triangle inequality is also valid

$$\Psi_{x,y} \leq \Psi_{x,z} + \Psi_{z,y} \tag{1.3}$$

for any $x, y, z \in H \setminus \{0\}$.

The following inequality has been obtained by Wang and Zhang in [9] (see also [11, p. 195])

$$\sqrt{1-\frac{|\langle x,y\rangle|^2}{\|x\|^2\|y\|^2}} \le \sqrt{1-\frac{|\langle x,z\rangle|^2}{\|x\|^2\|z\|^2}} + \sqrt{1-\frac{|\langle y,z\rangle|^2}{\|y\|^2\|z\|^2}} \tag{1.4}$$

for any $x,y,z \in H\setminus\{0\}$. Using the above notations it can be written as [6]

$$\sin\Psi_{x,y} \le \sin\Psi_{x,z} + \sin\Psi_{z,y} \tag{1.5}$$

for any $x,y,z \in H\setminus\{0\}$. It also provides another triangle type inequality complementing the Kreĭn and Lin inequalities above.

In this note we show that the Wang-Zhang inequality can be naturally applied to obtain an elegant reverse for the classical Schwarz inequality in complex inner product spaces.

2. Reverse of Schwarz inequality

In the sequel we assume that $(H,\langle\cdot,\cdot\rangle)$ is a complex inner product space. The inequality

$$|\langle x,y\rangle|^2 \le \|x\|^2\|y\|^2 \text{ for } x,y\in H \tag{2.1}$$

is well know in the literature as the *Schwarz inequality.* The equality holds in (2.1) iff x and y are linearly dependent.

Theorem 2.1. *Let $x,y,z\in H$ with $\|z\|=1$ and $\alpha,\beta\in\mathbb{C}$, $r,s>0$ such that*

$$\|x-\alpha z\|\le r \text{ and } \|y-\beta z\|\le s. \tag{2.2}$$

Then

$$(0\le)\,\|x\|^2\|y\|^2-|\langle x,y\rangle|^2 \le (r\|y\|+s\|x\|)^2. \tag{2.3}$$

Proof. If we multiply (1.4) by $\|x\|\,\|y\|\,\|z\|>0$, then we get

$$\begin{aligned} &\|z\|\sqrt{\|x\|^2\|y\|^2-|\langle x,y\rangle|^2} \\ &\le \|y\|\sqrt{\|x\|^2\|z\|^2-|\langle x,z\rangle|^2}+\|x\|\sqrt{\|y\|^2\|z\|^2-|\langle y,z\rangle|^2} \end{aligned} \tag{2.4}$$

for any $x,y,z\in H\setminus\{0\}$.

We observe that, if either $x=0$ or $y=0$, then the inequality (2.4) reduces to an equality.

Let $z\in H$ with $\|z\|=1$, and since (see for instance [2, Lemma 2.4])

$$\|x\|^2-|\langle x,z\rangle|^2=\inf_{\lambda\in\mathbb{C}}\|x-\lambda z\|^2 \text{ and } \|y\|^2-|\langle y,z\rangle|^2=\inf_{\mu\in\mathbb{C}}\|y-\mu z\|^2$$

then by (2.4) we have

$$\sqrt{\|x\|^2\|y\|^2-|\langle x,y\rangle|^2}\le\|y\|\inf_{\lambda\in\mathbb{C}}\|x-\lambda z\|+\|x\|\inf_{\mu\in\mathbb{C}}\|y-\mu z\|, \tag{2.5}$$

for any $x,y,z\in H$ with $\|z\|=1$.

Since, by (2.2)

$$\inf_{\lambda\in\mathbb{C}}\|x-\lambda z\|\leq\|x-\alpha z\|\leq r \text{ and } \inf_{\mu\in\mathbb{C}}\|y-\mu z\|\leq\|y-\beta z\|\leq s,$$

then by (2.5) we obtain the desired result (2.3). □

Corollary 2.2. *Let $x,y,z\in H$ with $\|z\|=1$ and $\lambda,\Lambda,\gamma,\Gamma\in\mathbb{C}$ with $\lambda\neq\Lambda$, $\gamma\neq\Gamma$ and such that either*

$$\operatorname{Re}\langle\Lambda z-x,x-\lambda z\rangle\geq 0 \textit{ and } \operatorname{Re}\langle\Gamma z-y,y-\gamma z\rangle\geq 0 \tag{2.6}$$

or, equivalently

$$\left\|x-\frac{\lambda+\Lambda}{2}z\right\|\leq\frac{1}{2}|\Lambda-\lambda| \textit{ and } \left\|y-\frac{\gamma+\Gamma}{2}z\right\|\leq\frac{1}{2}|\Gamma-\gamma|$$

are valid. Then

$$(0\leq)\,\|x\|^2\|y\|^2-|\langle x,y\rangle|^2\leq\frac{1}{4}\left(|\Lambda-\lambda|\,\|y\|+|\Gamma-\gamma|\,\|x\|\right)^2. \tag{2.7}$$

Proof. Follows by Theorem 2.1 on observing that

$$\operatorname{Re}\langle\Delta e-u,u-\delta e\rangle=\frac{1}{4}|\Delta-\delta|^2-\left\|u-\frac{\delta+\Delta}{2}e\right\|^2$$

for any $\delta,\Delta\in\mathbb{C}$ with $\delta\neq\Delta$ and $u,e\in H$ with $\|e\|=1$. □

We give an example for n-tuples of complex numbers.

Let $x=(x_1,...,x_n)$, $y=(y_1,...,y_n)$ and $z=(z_1,...,z_n)$ be n-tuples of complex numbers, $p=(p_1,...,p_n)$ a probability distribution, i.e. $p_i>0$ $i\in\{1,...,n\}$ and $\sum_{i=1}^n p_i=1$, with $\sum_{i=1}^n p_i|z_i|^2=1$ and $\lambda,\Lambda,\gamma,\Gamma\in\mathbb{C}$ with $\lambda\neq\Lambda$, $\gamma\neq\Gamma$ and such that

$$\operatorname{Re}\left[(\Lambda z_i-x_i)\left(\overline{x}_i-\overline{\lambda}\overline{z}_i\right)\right]\geq 0 \text{ and } \operatorname{Re}\left[(\Gamma z_i-\overline{y}_i)\left(\overline{y}_i-\overline{\gamma z_i}\right)\right]\geq 0$$

or, equivalently

$$\left|x_i-\frac{\lambda+\Lambda}{2}z_i\right|\leq\frac{1}{2}|\Lambda-\lambda| \text{ and } \left|y_i-\frac{\gamma+\Gamma}{2}z_i\right|\leq\frac{1}{2}|\Gamma-\gamma|$$

for any $i\in\{1,...,n\}$. Then

$$\sum_{i=1}^n p_i\operatorname{Re}\left[(\Lambda z_i-x_i)\left(\overline{x}_i-\overline{\lambda}\overline{z}_i\right)\right]\geq 0 \text{ and } \sum_{i=1}^n p_i\operatorname{Re}\left[(\Gamma z_i-\overline{y}_i)\left(\overline{y}_i-\overline{\gamma z_i}\right)\right]\geq 0$$

and by applying Corollary 2.2 for the inner product $\langle\cdot,\cdot\rangle_p:\mathbb{C}^n\times\mathbb{C}^n\to\mathbb{C}$ with

$$\langle x,y\rangle_p=\sum_{i=1}^n p_i x_i\overline{y}_i,$$

we have

$$0 \le \sum_{i=1}^{n} p_i |x_i|^2 \sum_{i=1}^{n} p_i |y_i|^2 - \left| \sum_{i=1}^{n} p_i x_i \overline{y}_i \right|^2 \tag{2.8}$$
$$\le \frac{1}{4} \left[|\Lambda - \lambda| \left(\sum_{i=1}^{n} p_i |y_i|^2 \right)^{1/2} + |\Gamma - \gamma| \left(\sum_{i=1}^{n} p_i |x_i|^2 \right)^{1/2} \right]^2 .$$

If $0 < a \le a_i \le A < \infty$ and $0 < b \le b_i \le B < \infty$ for any $i \in \{1, ..., n\}$ then by (2.8) we have for any $p = (p_1, ..., p_n)$ a probability distribution that

$$0 \le \sum_{i=1}^{n} p_i a_i^2 \sum_{i=1}^{n} p_i b_i^2 - \left(\sum_{i=1}^{n} p_i a_i b_i \right)^2 \tag{2.9}$$
$$\le \frac{1}{4} \left[(A - a) \left(\sum_{i=1}^{n} p_i b_i^2 \right)^{1/2} + (B - b) \left(\sum_{i=1}^{n} p_i a_i^2 \right)^{1/2} \right]^2 .$$

The interested reader may compare this new result with the classical reverses of Schwarz inequality obtained by Diaz and Metcalf [1], Ozeki [4], G. Pólya and G. Szegö [7], Shisha and Mond [8] and Cassels [10].

For other reverses of Schwarz inequality in complex inner product spaces see the monograph [3] and the references therein.

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A comparative analysis of the convergence regions for different parallel affine projection algorithms

Irina Maria Artinescu

Abstract. This paper analysis the dimension and the shape of convergence regions of three algorithms used to solve the convex feasibility problem in bidimensional space: the *Parallel Projection Method (PPM)*, the classical *Extrapolated Method of Parallel Projections (EMOPP)* and a modified version of EMOPP.

Mathematics Subject Classification (2010): 52A10.

Keywords: Convex feasibility problem, parallel projection method, affine projection.

1. Introduction

The convex feasibility problem is one the classical problem in computational mathematics and can be simplify described as the problem to find a solution that satisfied a given set of inequalities. The projection methods were used in the past to solve some systems of linear equations in Euclidean spaces [4] and were modified to be applied to systems of linear inequalities in [1], [11], [12]. The algorithmic steps in these first algorithms consists of projections onto some affine subspaces or a half-spaces. Later, the method become more sophisticated [8], [9], [10], being adapted to solve the more general problem of finding a point in the intersection of a family of closed convex sets in a Hilbert space [2], [5].

The affine projection methods have numerous practical applications in data compression, in tomography, neural networks or in image filtering (see also [3]). While the mathematical analysis of week or strong convergence of different projection methods was largely studied in the past ([2], [5], [6], [7]), an explicit analysis of the convergence

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regions of the convex feasibility algorithms inspired by the projections methods where rarely approached.

In this paper we tested the convergence in finite number of steps for two of these projection methods: the *Parallel Projection Method (PPM)* and the classical *Extrapolated Method of Parallel Projections (EMOPP)*, and a variant of the EMOPP that uses variable affine combinations, in order to determine theirs convergence in finit number of steps and the shapes of theirs convergence regions, defined by the staring points for witch the algorithms converge in a given number of steps.

2. The convex feasibility problem and the parallel projections method

The convex feasibility problem (CFP) was formulated in [5] as : Given m closed convex sets $C_1, C_2, \ldots, C_m \subseteq \mathcal{R}^n$, with nonempty intersection, $\cap C_i \neq \emptyset$, defined by $C_i = \{x \in \mathcal{R}^n \mid f_i(x) \leq 0\}$, with $f_i : \mathcal{R}^n \to \mathcal{R}$ a convex function, the CFP is to find a point $x \in C = \bigcap_{i=1}^m C_i$.

Consider $\mathcal{P} = \{P_1, P_2, \ldots, P_n\}$ a convex polygon and consider the *Parallel Projection Method (PPM)*, which is governed by the iteration ([7]):

$$(\forall n \in \mathcal{N}) Q_{j+1} = Q_j + \Lambda(\sum_{i\in 1..n} w_i P_i(Q_j) - Q_j), \tag{2.1}$$

where $1+\epsilon \leq \Lambda \leq 2-\epsilon$ are the *relaxation* parameters, $0 < \epsilon < 1$ and $\sum_{i\in 1..n} w_i = 1$, with the fixed weight w_i.

A variation of the PPM, called *Extrapolated Method of Parallel Projections (EMOPP)* is obtained involving involving different classes of control index sets $\{I_n\}$ [5]. The iteration of this method are similar to (2.1):

$$(\forall j \in \mathcal{N}) Q_{j+1} = Q_j + \Lambda(\sum_{i\in I_n} w_i P_i(Q_j) - Q_j), \tag{2.2}$$

where the indices set $\{I_n\}$, called *control* sequence, are variable from one iteration to another. Many variants of the control sequences where studied in [5].

The *modified Extrapolated Method of Parallel Projections (mEMOPP)* was obtained introducing variable weight w_i that depend inverse proportionally to the distance from Q_n to his projections on the considered semi-planes. If we denote:

$$M_{j,i} = pr(Q_j, P_i P_{i+1}),$$

by convention ($P_{n+1} = P_1$) and $d_{j,i} = dist(Q_j, P_i P_{i+1})$ and the weights are defined by

$$w_{j,i} = \frac{1/(d_{j,i}+1)}{\sum_{i\in I_j} 1/(d_{j,i}+1)}$$

where I_j are the set of indexes i for witch $P_i P_{i+1}$ separates Q_j and the interior of the polygon $\mathcal{P}$ for the case of mEMOPP.

The determination of the shape of convergence regions is equivalent to inverse the CFP: *determine for a given point Q of the plane the set of the points that are transported in Q using the different versions of the PPM associate transformation.*

For the simplification of the calculus, we choused as convex $\mathcal{P}$ a regular rectangle, defined by the relations $x = -a$, $x = a$, $y = -b$, $y = b$, where $a, b > 0$:

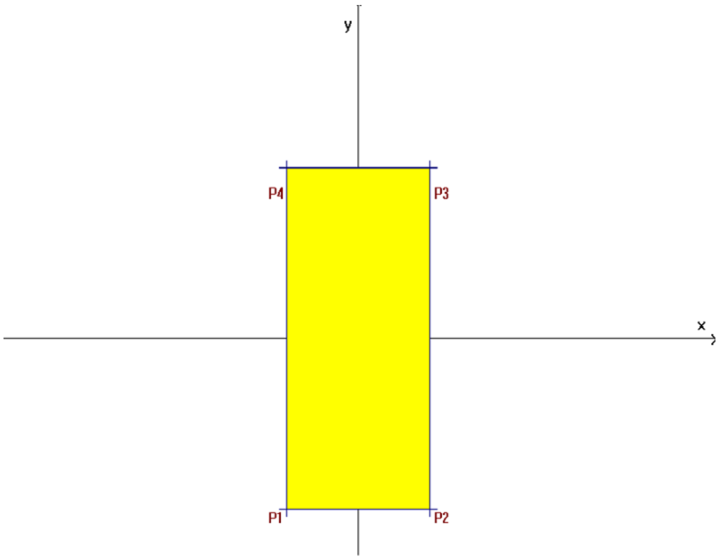

FIGURE 1. The initial polygon.

3. The convergence regions for parallel projection method

In the case of PPM algorithm, any point $Q(x, y)$ from the plane has the projection on the lines $y = -a$, $y = a$, $x = -b$, $x = b$ formed by the points $M_1(x, -b)$, $M_2(x, b)$, $M_3(-a, y)$ respectively $M_4(a, y)$. The transform $S = f_{PPM}(Q)$ move the point Q in

$$S(m, n) = (1 - \Lambda)(x, y) + \Lambda \left(\frac{x}{2}, \frac{y}{2} \right) = \frac{2 - \Lambda}{2}(x, y) \tag{3.1}$$

The transform f_{PPM} is a continuous bijection with the inverse:

$$Q(x, y) = f^{-1}_{PPM}(m, n) = \frac{2}{2 - \Lambda}(m, n) \tag{3.2}$$

The convergence regions formed by the starting points Q for which the algorithm stop in k steps, are given by

Theorem 3.1. *If we denote by L_k the rectangle defined with the relations $x \geq -\frac{2(k+1)}{2-\Lambda}b$, $x \leq \frac{2(k+1)}{2-\Lambda}b$, $y \geq -\frac{2(k+1)}{2-\Lambda}a$ and $y \leq \frac{2(k+1)}{2-\Lambda}a$, then the k convergence region for PPM algorithm are defined by $L_k \setminus Ł_{k-1}$.*

The proof is immediate from (3.2).

4. The convergence regions for extrapolated method of parallel projections

The first convergence region is formed by starting points for witch the algorithms converge in a one steps.

One consider the case of the EMOPP algorithm. The plane is separated in eight regions, function of the orientation of each point relatives to the sides of the rectangle, as in the Figure 2.

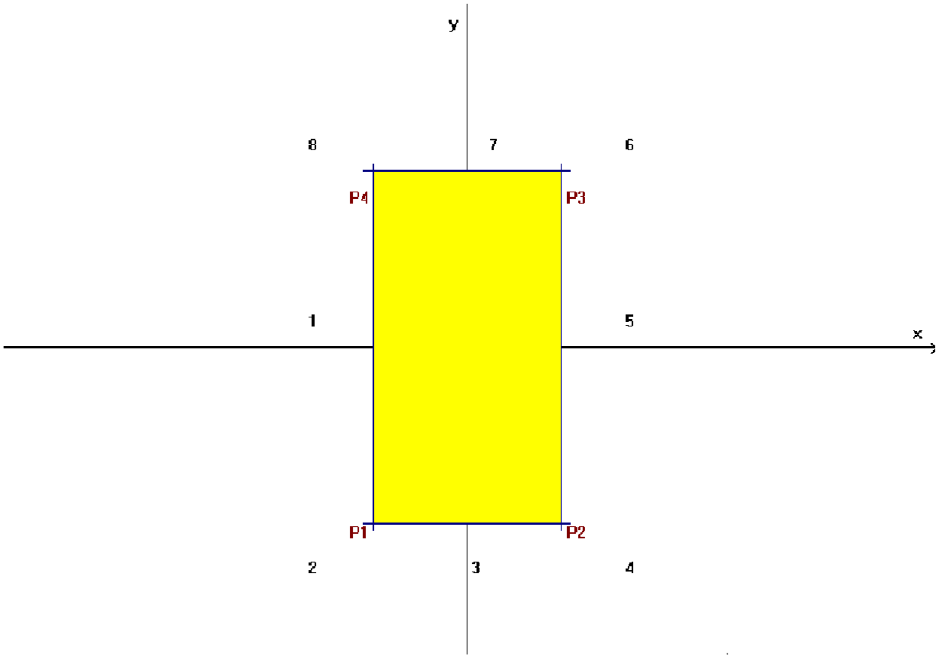

FIGURE 2. The eight sectors of the plane defined by the edges of the rectangle.

Consider a generic point $Q(x,y) \in ext(\mathcal{P})$. For Q in one of the sectors 1, 3, 5 and 7, the EMOPP step in involve only one projection $M = Pr(Q, P_iP_{i+1})$ on the nearest side $[P_iP_{i+1}]$ of the rectangle.

Defining the step of EMOPP as $Q(x,y) \to S(m,n) := f_{EMOPP}(Q)$, we have for Q in the sector 1:

$$S = f_{EMOPP}(Q) = Q + \Lambda(Pr(Q, P_1P_4) - Q). \qquad (4.1)$$

With this notation we have (for the fist sector)

$$f_{EMOPP}(x,y) = (-x(\Lambda - 1) - \Lambda a, y) \qquad (4.2)$$

The inverse transform is

$$f^{-1}_{EMOPP}(m,n) = \left(\frac{-\Lambda a - m}{\Lambda - 1}, n\right) \qquad (4.3)$$

For the sector 5, one obtain

$$f_{EMOPP}(x,y) = (-x(\Lambda-1)+\Lambda a, y) \tag{4.4}$$

The inverse transform is

$$f^{-1}_{EMOPP}(m,n) = \left(\frac{\Lambda a - m}{\Lambda - 1}, n\right) \tag{4.5}$$

For the sectors 3 and 7, similar formulas are obtained. For example, in the sector 3:

$$f_{EMOPP}(x,y) = (x, -y(\Lambda-1)-\Lambda b) \tag{4.6}$$

The inverse transform is

$$f^{-1}_{EMOPP}(m,n) = \left(m, \frac{-\Lambda b - n}{\Lambda - 1}\right) \tag{4.7}$$

and in the sector 7:

$$f_{EMOPP}(x,y) = (x, -y(\Lambda-1)+\Lambda b) \tag{4.8}$$

The inverse transform is

$$f^{-1}_{EMOPP}(m,n) = \left(m, \frac{\Lambda b - n}{\Lambda - 1}\right). \tag{4.9}$$

The preimage of each point inside the rectangle is formed then by at least four points from the sectors 1, 3, 5, 7. The preimage of entire rectangle in the sectors 1 and 5 is obtained by imposing the condition $-a < m < a$, and the preimage of the rectangle in the sectors 3 and 7 is obtained by imposing the condition $-b < n < b$. A direct calculus give that

$$\begin{aligned}
Q &\in \left[-a\frac{\Lambda+1}{\Lambda-1}, -a\right) \times (-b,b) \text{ for the sector 1;}\\
Q &\in (-a,a) \times \left[-b\frac{\Lambda+1}{\Lambda-1}, -b\right) \text{ for the sector 3;}\\
Q &\in \left(a, a\frac{\Lambda+1}{\Lambda-1}\right] \times (-b,b) \text{ for the sector 5;}\\
Q &\in (-a,a) \times \left(b, b\frac{\Lambda+1}{\Lambda-1}\right] \text{ for the sector 7;}
\end{aligned} \tag{4.10}$$

In the case of the other sectors, the transform f involve two projection at each step. Choosing, for example, the sector 6 (for $Q(x,y)$ having $x > a,\ y > b$), the projections of Q on the line $y = b$ give the point $M_1 = (x,b)$, the projections of Q on the line $x = a$ give the point $M_2 = (a,y)$ and the algorithm produces $Q(x,y) \to S(m,n)$ with

$$m = \frac{(2+\Lambda)x - \Lambda a}{2}, \ n = \frac{(2+\Lambda)y - \Lambda b}{2}. \tag{4.11}$$

The conditions $-a \le m \le a$ and $-b \le n \le b$ produce

$$\frac{\lambda-2}{\lambda+2}a \le x \le a \text{ and } \frac{\lambda-2}{\lambda+2}b \le y \le b \tag{4.12}$$

witch are in contradiction with the supposition $x > a,\ y > b$.

The image of every point of the sector 6 become to the sector 6 (see Figure 3).

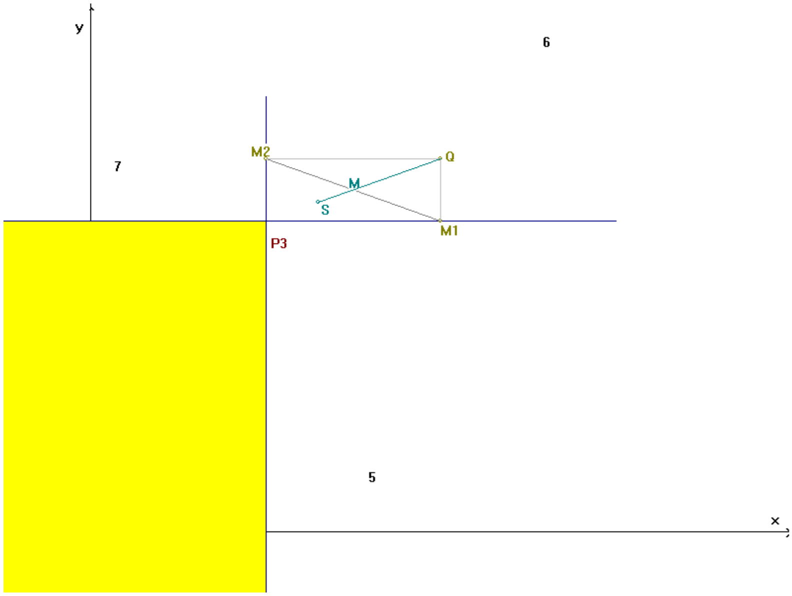

FIGURE 3. The EMOPP transform $Q \to S$ for a starting point Q inside the sector 6.

In conclusion, the EMOPP algorithm do not converges in finite number of steps for any starting point of the sector 6, if $\Lambda < 2$.

Similar results are obtained for the starting points of the sectors 2, 4, and 8. The infinite series defined by $Q_0 = Q$, $Q_{n+1} = f_{EMOPP}(Q)$ converges to the vertice P_3 of the rectangle.

The second order convergence regions, formed by the points for witch the algorithm stop in two steps ($Q(x, y) \to Q_1 \to S \in \mathcal{P}$), are determined by imposing the conditions

- $a < -x(\Lambda - 1) - \Lambda a < a\frac{\Lambda+1}{\Lambda-1}$ for Q in the sector 1,
- $b < -y(\Lambda - 1) - \Lambda b < b\frac{\Lambda+1}{\Lambda-1}$ for Q in the sector 3,
- $-a\frac{\Lambda+1}{\Lambda-1} < -x(\Lambda - 1) - \Lambda a < -a$ for Q in the sector 5,
- $-b\frac{\Lambda+1}{\Lambda-1} < -y(\Lambda - 1) - \Lambda b < -b$ for Q in the sector 7.

A direct calculus give:

$$\begin{aligned} Q &\in \left[-a\frac{\Lambda^2+1}{(\Lambda-1)^2}, -a\frac{\Lambda+1}{\Lambda-1}\right) \times (-b,b) \text{ for the sector 1;} \\ Q &\in (-a,a) \times \left[-b\frac{\Lambda^2+1}{(\Lambda-1)^2}, -b\frac{\Lambda+1}{\Lambda-1}\right) \text{ for the sector 3;} \\ Q &\in \left(a\frac{\Lambda+1}{\Lambda-1}, a\frac{\Lambda^2+1}{(\Lambda-1)^2}\right] \times (-b,b) \text{ for the sector 5;} \\ Q &\in (-a,a) \times \left(b\frac{\Lambda+1}{\Lambda-1}, b\frac{\Lambda^2+1}{(\Lambda-1)^2}\right] \text{ for the sector 7;} \end{aligned} \tag{4.13}$$

In general, we have:

Theorem 4.1. *The k-th convergence regions for the EMOPP algorithm (the regions of starting point for witch the algorithm stop in k steps) are defined by:*

$$\begin{aligned} Q &\in \left[-a\frac{R_k(\Lambda)}{(\Lambda-1)^k}, -a\frac{R_{k-1}(\Lambda)}{(\Lambda-1)^{k-1}}\right) \times (-b,b) \textit{ for the sector 1;} \\ Q &\in (-a,a) \times \left[-b\frac{R_k(\Lambda)}{(\Lambda-1)^k}, -b\frac{R_{k-1}(\Lambda)}{(\Lambda-1)^{k-1}}\right) \textit{ for the sector 3;} \\ Q &\in \left(a\frac{R_{k-1}(\Lambda)}{(\Lambda-1)^{k-1}}, a\frac{R_k(\Lambda)}{(\Lambda-1)^k}\right] \times (-b,b) \textit{ for the sector 5;} \\ Q &\in (-a,a) \times \left(b\frac{R_{k-1}(\Lambda)}{(\Lambda-1)^{k-1}}, b\frac{R_k(\Lambda)}{(\Lambda-1)^k}\right] \textit{ for the sector 7;} \end{aligned} \tag{4.14}$$

for any $k > 1$, where the polynomial R_k is given by:

$$R_k(\Lambda) = 1 + \frac{\Lambda}{2-\Lambda}[1-(\Lambda-1)^k] \tag{4.15}$$

Proof. He have $R_1 = 1 + \Lambda$ and the recursion

$$R_k = R_{k-1} + \Lambda(\Lambda-1)^{k-1} \tag{4.16}$$

can be proved by mathematical induction. Indeed, for $k = 2$ the formulas (4.15) was verified before. Now, supposing that the point $S(m,n)$ verifies (for the (k+1)-convergence region in sector 5)

$$a\frac{R_k(\Lambda)}{(\Lambda-1)^k} < m \le a\frac{R_{k+1}(\Lambda)}{(\Lambda-1)^{k+1}} \ , \ -b \le y \le b$$

a short computation give for $Q(x,y) = f^{-1}_{EMOPP}(S)$, using that $x = \frac{\Lambda a - m}{\Lambda - 1}$ and $y = n$:

$$-a\frac{R_k(\Lambda)}{(\Lambda-1)^k} \le x < -a\frac{R_{k-1}(\Lambda)}{(\Lambda-1)^{k-1}}.$$

witch is a point inside the k-convergence region of the sector 1. A similar formulas permit to pass from the (k+1)-convergence region of the sector 3 to the k- convergence region of the sector 7 and reciprocally.

Finally

$$R_k = 1 + \sum_{i=1}^{k} \Lambda(\Lambda - 1)^{i-1}$$

and the relation(4.15) is immediate. □

5. The convergence regions of modified extrapolated method of parallel projections

In the case of modified Extrapolated Method of Parallel Projection, the weights w_i are not constants, but are inverse proportionates to the distances from starting points Q to the nearest edges of the rectangle. We consider now that the relaxation parameter $\Lambda \in (1, 2)$.

If the starting point Q become to one of the regions 1, 3, 5, or 7, there are a single projection involved, then the weight involved will be only $w_1 = 1$. In this case, the same formulas as in (4.14) can be deduced.

The situation of the sectors 2, 4, 6 and 8 are therefore different. Studying as example only the sector 6 (formed by the points $Q(x, y)$ with $x > a$ and $y > b$), the projection of nearest edges of the rectangle $y = b$, respectively $x = a$ give the points $M_1(x, b)$ and $M_2(a, y)$.

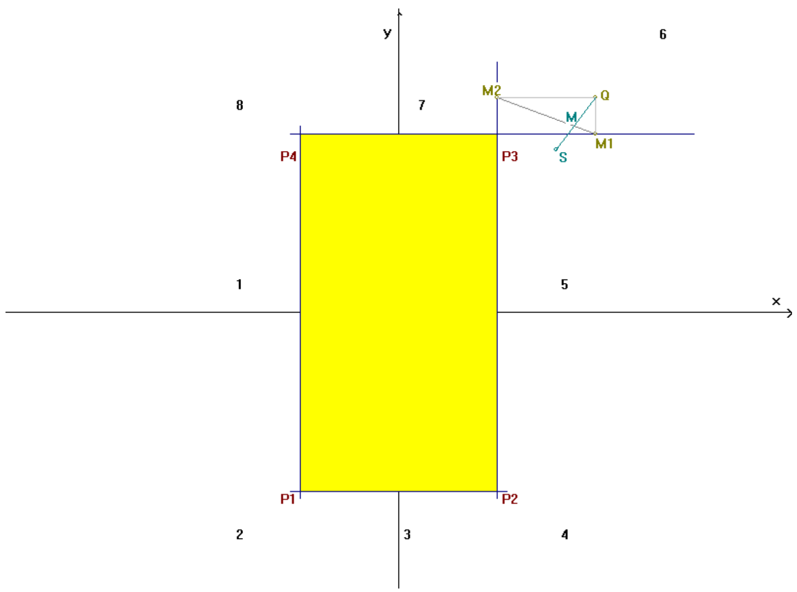

FIGURE 4. The mEMOPP transform $Q \to S$ for a starting point Q from the sector 6.

The weights are computed from $d_1 = QM_1 = y - b$ and $d_2 = QM_2 = x - a$:

$$\begin{cases} w_1 & = & \frac{\frac{1}{d_1+1}}{\frac{1}{d_1+1}+\frac{1}{d_2+1}} = \frac{x-a+1}{x+y-a-b+2} \\ w_2 & = & \frac{\frac{1}{d_2+1}}{\frac{1}{d_1+1}+\frac{1}{d_2+1}} = \frac{y-b+1}{x+y-a-b+2} \end{cases} \quad (5.1)$$

and we obtain the points

$$M = w_1M_1 + w_2M_2 = (w_1x + w_2a, w_1b + w_2y) \text{ and } S(m, n) = f_{mEMOPP}(Q) :$$

$$\begin{cases} m & = & (1-\Lambda)x + \Lambda(w_1x + w_2a) \\ n & = & (1-\Lambda)y + \Lambda(w_1b + w_2y) \end{cases} \quad (5.2)$$

We supposed at this stage that the distances $d_1 < d_2$ (see the Figure 4). The point S can belong to the sector 6 or can jump to the sector 5. For any point in the sector 5, the mEMOPP algorithm converges in a finite number of steps, conform to the Theorem 4.1. We wish to determine the sub-region of the sector 6 that have the image from f_{mEMOPP} inside the sector 5.

The condition that S become to the sector 5 writes as $m > a$ and $-b \le n \le b$. The condition $m > a$ is always verified because $QS < QP_3$. The condition $n \le b$ rewrites as:

$$\begin{aligned} (1-\Lambda)y + \Lambda(w_1b + (1-w_1)y) & \le b \\ (1-\Lambda w_1)y & \le b(1-\lambda w_1) \text{and because } y > b \Rightarrow \\ (1-\Lambda w_1) & \le 0 \end{aligned} \quad (5.3)$$

The weight where computet in the formulas (5.1), then we have

$$\Lambda\frac{y-b+1}{x+y-a-b+2} \ge 1,$$

and finally

$$y \le (x-a)(\Lambda - 1) + b - 2. \quad (5.4)$$

The condition $-b \le n$ become:

$$\begin{aligned} (1-\Lambda)y + \Lambda(w_1b + (1-w_1)y) & \ge -b \\ (1-\Lambda w_1)y & \ge -b(1+\lambda w_1) \text{and because } 1-\Lambda w_1 < 0 \Rightarrow \\ y & \le b(1+\frac{2}{\Lambda w_1 - 1}) \end{aligned} \quad (5.5)$$

and $\Lambda w_1 < 2$ give $y < 3b$.

In conclusion if the starting point $Q(x, y)$ verifies $b < y < 3b$ and

$$y \le (x-a)(\Lambda - 1) + b - 2,$$

the mEMOPP algorithm produce a point $S = f_{mEMOPP}(Q)$ inside the section 5. The algorithm will stop in $k+1$ steps if we have the supplementary condition

$$a\frac{R_{k-1}(\Lambda)}{(\Lambda-1)^{k-1}} < (1-\Lambda)x + \Lambda(w_1x + w_2a) \le a\frac{R_k(\Lambda)}{(\Lambda-1)^k} \quad (5.6)$$

where

$$R_k(\Lambda) = 1 + \frac{\Lambda}{2-\Lambda}[1 - (\Lambda-1)^k]$$

(from the Theorem 4.1).

With the notation

$$S_k = a\frac{1 + \frac{\Lambda}{2-\Lambda}[1 - (\Lambda - 1)^k]}{(\Lambda - 1)^k}$$

the limits between the $k+1$ and the $k+2$ -convergence regions inside the surface delimited by $b < y < 3b$ and $y \leq (x-a)(\Lambda - 1) + b - 2$ verifies the equation (from (5.6))

$$(1 - \lambda w_2)x + \Lambda w_2 a = S_k. \tag{5.7}$$

After some computations, the equation (5.7) become

$$\begin{aligned} x^2 - (\Lambda - 1)xy - [a + b - 2 + \Lambda(1 - b) + S_k]x - \\ -(S_k - \Lambda a)y + S_k(a + b - 2) + \Lambda a(b - 1) \end{aligned} \tag{5.8}$$

and the substitution $z = x - \frac{\Lambda-1}{2}y$ give the equation

$$\begin{aligned} z^2 - \frac{(\Lambda - 1)^2}{4}y^2 - [a + b - 2 + \Lambda(1 - b) + S_k](z + \frac{\Lambda - 1}{2}y) - \\ -(S_k - \Lambda a)y + S_k(a + b - 2) + \Lambda a(b - 1) \end{aligned} \tag{5.9}$$

The equation (5.9) is obviously a hyperbola equation, denoted by $\mathcal{H}_{k+1}$ in the next. We proved

Theorem 5.1. *In the case of the mEMOPP algorithm applied for the rectangle* $P_1P_2P_3P_4$ *, for any* $k > 1$ *there are a region of starting points inside the sector 6, delimited by* $b < y < 3b$ *and* $y \leq (x-a)(\Lambda-1) + b - 2$ *and the hyperbolas* $\mathcal{H}_{k-1}$, $\mathcal{H}_k$*, from where the algorithm stop in exactly k steps.*

This result imply that the algorithm mEMOPP have a better convergence that the not-modified version EMOPP. Similar results can be obtained for the case of the sectors 2, 4 and 8.

6. Conclusions

In this paper we have shown that while the PPM algorithm that solve the convex feasibility problem converges always for a regular quadrilateral convex, his newest version EMOPP do not converges in finite number of steps for large regions of starting points. The modified version of EMOPP that involves variable weights in the affine combination used at each step, depending on the relative position of the point to the convex, permits to significative extend the convergence regions of the algorithm. Explicit determination of these regions where also presented.

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Book reviews

Daniel Li and Hervé Queffélec, Introduction to Banach spaces: analysis and probability (2 volumes, translated from the French by Danièle Gibbons and Greg Gibbons), Cambridge Studies in Advanced Mathematics:
no. 166: vol. 1, xxx+431 p., ISBN: 978-1-107-16051-4/hbk ($ 94.99), 978-1-316-67576-2/ebook;
no. 167: vol. 2, xxx+374 p., ISBN: 978-1-107-16262-4/hbk ($ 99.99), 978-1-316-67739-1/ebook, ISBN 978-1-107-16263-1/set ($ 165.00),
Cambridge University Press, Cambridge, 2017.

The subject of the book is the interplay between the Banach space theory and probability theory, which has a long and fruitful history. Although the study of Banach space valued random variables started in the 1950s, their importance in the study of Banach spaces became clear only after the introduction, at the beginning of 1970s, of the notions of type and cotype (presented in volume I, Chapter 5), proving their intimately connections with Banach spaces. On the other hand, probability methods allow the proof of some deep results in Banach space theory as, for instance, Dvoretzky's theorem on the Euclidean sections of convex bodies in Banach spaces, the relevance of martingales in the study of Radon-Nikodým property, Davie's proof on the existence of Banach spaces without approximation property, Gowers' dichotomy theorem.

The characteristic of this textbook is that full proofs are given to all results, both from Banach spaces and from probability theory as well. As the authors mention in Preface – the proofs are given "from scratch", without referring in the proof to a "well-known result" or admitting an auxiliary difficult result. For this reason, proofs of some theorems in analysis or functional analysis as, for instance, Marcel Riesz' theorem on the continuity of Hilbert transform, Riesz-Thorin's interpolation theorem, Rademacher's theorem on the a.e. differentiability of Lipschitz functions, Eberlein-Shmulian and Krein-Milman theorems, etc, are included.

The book contains the necessary material from probability theory: Ch.1, *Fundamental notions of probability* (including Khinchin's inequality and martingales), Ch.4, *Banach space valued random variables* (Lévy's symmetry principle, Kahane's inequalities), in volume I, and Ch.3, *Gaussian processes* (including Brownian motion, Dudley's majoration and Fernique's minoration theorems), from volume II.

Concerning Banach spaces the following topics are treated in the first volume: Ch.2, *Bases in Banach space*, Ch.3, *Unconditional convergence* (Orlicz-Pettis' theorem, Gowers' dichotomy theorem), Ch.5, *Type and cotype of Banach spaces*, Ch.6, *p-Summing operators*, Ch.7, *Some properties of L^p-spaces*, Ch.8, *The space ℓ_1* (dedicated to Rosenthal's ℓ_1-theorem). In the second volume: Ch.1, *Euclidean sections* (Dvoretzky's theorem), Ch.2, *Separable Banach spaces without the approximation property* (the counterexamples of Enflo and Davie), Ch.4, *Reflexive subspaces of L^1* (the Kadec-Pełczyński theorem, Maurey's factorization theorem), Ch.5, *The method of selectors* (contains three results of Bourgain illustrating the method of selectors), Ch.6, *The Pisier space of almost surely continuous functions.*

In fact, the contents is explained in details in Preface (30 pages). The bibliography is divided in two sections - books and papers.

Each chapter ends with a section of Comments and one of Exercises. The comments refer to the origin of some results presented in the chapter or to complementary results. Many of exercises propose proofs of recent and important results. These proofs are decomposed in several steps, so that the reader can fill up in the details and, in most cases, the sources (an article or a book) are indicated.

The book was published as one "thick" volume (627 pages) in the collection *Cours Spécialisés de la Société Mathématique de France.* Danièle and Greg Gibbons provided an excellent translation of the French text, keeping the lively and pleasant style of the French original (e.g. quotations from George Brassens, or a reference to "ensemble flirtant" of Bourbaki). With respect to the French edition some mistakes were corrected and some missing arguments were added. At the end of the second volume four appendices were added, three surveys – A. *News in the theory of infinite-dimensional Banach spaces in the past 20 years*, by G. Godefroy (7 pages), B. *An update on some problems in high-dimensional convex geometry and related probabilistic results*, by O. Guédon (8 pages), C. *A few updates and pointers*, by G. Pisier (9 pages) – and a research paper, D. *On the mesh condition for Sidon sets*, by L. Rodríguez-Piazza (8 pages).

The book contains a lot of interesting and deep results on Banach spaces and harmonic analysis treated, with the methods of probability theory. It can be used for advanced courses in functional analysis, but also by professional mathematicians as a valuable source of information.

S. Cobzaş

A. R. Alimov and I. G. Tsar'kov; Geometric theory of approximation Geometricheskaya teoriya priblizhenii) (Russian).
Part I. **Classical notions and constructions in the approximation by sets** (Klassicheskie ponyatiya i kontruktsii priblizheniya mnozhestvami), 346 p, OntoPrint Moscow, 2017, ISBN 978-5-906886-91-0;
Part II. **Approximation by classes of sets, further developments of basic questions of the geometric approximation theory** (Priblizhenie klassami mnozhestv, dal'neishee razvitie osnovnykh voprosov geometricheskoi teorii priblizhenya), 350 p, OntoPrint Moscow, 2018, ISBN 978-5-00121-053-5.

The book is about the best approximation in normed linear spaces in connection with the geometric properties of the underlying space. For a long time the standard reference in this area was Ivan Singer, *Best approximation in normed linear spaces by elements of linear subspaces*, Springer 1970 (an updated translation of the Romanian version from 1967, see also, I. Singer, *The theory of best approximation and functional analysis*, SIAM, Philadelphia, PA, 1974). Singer's books stimulated the research in this area and, since ten, a lot of results were obtained, new notions emerged and many challenging problems were solved. But one, considered by some researchers the most important in best approximation theory, resisted to all attempts to solve it – the problem of the convexity of Chebyshev sets – is any Chebyshev subset of a Hilbert space convex? The authors of this book have important contributions to this problem, mainly concerning the class of the so-called solar sets (or suns), a recurrent theme of the book and an essential tool in the characterization of best approximation (e.g. Kolmogorov's criterium).

The authors treat best approximation problems both in abstract and in concrete normed spaces. For instance, the first volume contains some classical results on Chebyshev problem of best approximation – alternation results, Haar theorem, Chebyshev systems, rational approximation – in $C[a,b]$ and in L^p-spaces. Among the abstract problems studied in the first volume we mention: best approximation in Euclidean spaces (characterization, Phelps theorem on the nonexpansiveness of the metric projection), the role of approximative compactness and of Efimov-Stechkin spaces in the study of the existence of best approximation and continuity of the metric projection, solarity and characterization of best approximation. Five proofs (of Berdyshev-Klee-Vlasov, Asplund, Konyagin, Vlasov and Brosowski) are given for the convexity of Chebyshev sets in $\mathbb{R}^n$. Solarity and connectedness of Chebyshev sets in connection with continuity and selection properties of metric projection are also studied.

In the first chapter of the second volume, *Approximation of vector-valued functions*, one presents some results of Zukhovickii, Stechkin, Tsar'kov, Garkavi, Koshcheev, a.o., on the extension of the results from the first volume (characterization, Haar condition, Chebyshev systems, etc) to the case of the space $C(Q,X)$, where Q is a compact Hausdorff space and X a Banach space. The second chapter is devoted to a detailed study of Jung constant defined as the radius of the smallest set covering an arbitrary set of diameter 1. This is a very important tool in the geometry of Banach spaces with applications to fixed point theory for nonexpansive (the inverse of Jung constant is called the coefficient of normality of the corresponding Banach space) and condensing mappings. A consistent chapter, Chapter 3 (102 pages), contains a detailed study of Chebyshev centers, a notion related to best approximation (simultaneous approximation) and having important applications as, for instance, to optimal location problems. One studies the existence and uniqueness of Chebyshev centers, continuity, stability and selections for the Chebyshev center map, algorithms for finding Chebyshev centers and applications.

Chapter 6 is concerned with widths in the sense of Kolmogorov, a notion strongly related to approximation theory – one studies the approximation by classes of functions, comparing the efficiency of the approximation by various classes of approximating sets (e.g. algebraic or trigonometric polynomials, rational functions, etc). The last chapter of the second volume, Chapter 7, *Approximation properties of arbitrary sets in linear normed spaces. Almost Chebyshev sets and sets of almost uniqueness,* is concerned with genericity properties (in the sense of Baire category) and porosity results in best approximation problems and in the study of farthest points (existence and uniqueness), a direction of research initiated by S. B. Stechkin in 1963.

Written by two experts with substantial contributions to the domain, this two volume book incorporates a lot of results (including authors' results), both classical but also new ones situated in the focus of current research, the book is of interest to a large community of mathematicians interested in the applications of Banach space geometry and applications (and reading Russian). The book is clearly written, in a pleasant style, reflecting the erudition of the authors (not only in mathematics, but also in other areas of human knowledge, as illustrated by the mottos and the beginning of some chapters and sections).

S. Cobzaş