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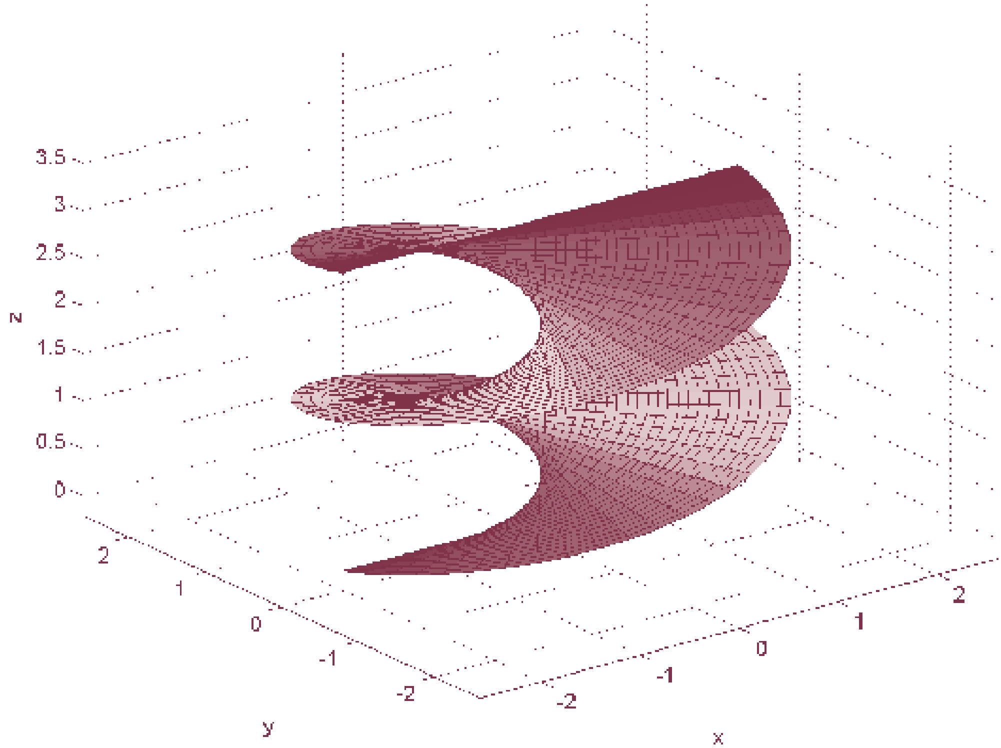

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CONTENTS

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Conformable fractional approximation by max-product operators

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Abstract. Here we study the approximation of functions by a big variety of Max-product operators under conformable fractional differentiability. These are positive sublinear operators. Our study is based on our general results about positive sublinear operators. We produce Jackson type inequalities under conformable fractional initial conditions. So our approach is quantitative by producing inequalities with their right hand sides involving the modulus of continuity of a high order conformable fractional derivative of the function under approximation.

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Keywords: positive sublinear operators, Max-product operators, modulus of continuity, conformable fractional derivative.

1. Introduction

The main motivation here is the monograph by B. Bede, L. Coroianu and S. Gal [4], 2016.

Let $N \in \mathbb{N}$, the well-known Bernstein polynomials ([7]) are positive linear operators, defined by the formula

$$B_N(f)(x) = \sum_{k=0}^{N} \binom{N}{k} x^k (1-x)^{N-k} f\left(\frac{k}{N}\right), \quad x \in [0,1], \ f \in C([0,1]). \tag{1.1}$$

T. Popoviciu in [8], 1935, proved for $f \in C([0,1])$ that

$$|B_N(f)(x) - f(x)| \leq \frac{5}{4}\omega_1\left(f, \frac{1}{\sqrt{N}}\right), \quad \forall\, x \in [0,1], \tag{1.2}$$

where

$$\omega_1(f,\delta) = \sup_{\substack{x,y\in[0,1]:\\ |x-y|\leq\delta}} |f(x) - f(y)|, \quad \delta > 0, \tag{1.3}$$

is the first modulus of continuity.

G.G. Lorentz in [7], 1986, p. 21, proved for $f \in C^1([0,1])$ that

$$|B_N(f)(x) - f(x)| \leq \frac{3}{4\sqrt{N}}\omega_1\left(f', \frac{1}{\sqrt{N}}\right), \quad \forall\, x \in [0,1], \tag{1.4}$$

In [4], p. 10, the authors introduced the basic Max-product Bernstein operators,

$$B_N^{(M)}(f)(x) = \frac{\bigvee_{k=0}^{N} p_{N,k}(x) f\left(\frac{k}{N}\right)}{\bigvee_{k=0}^{N} p_{N,k}(x)}, \quad N \in \mathbb{N}, \tag{1.5}$$

where $\bigvee$ stands for maximum, and

$$p_{N,k}(x) = \binom{N}{k} x^k (1-x)^{N-k}$$

and $f : [0,1] \to \mathbb{R}_+ = [0,\infty)$.

These are nonlinear and piecewise rational operators.

The authors in [4] studied similar such nonlinear operators such as: the Max-product Favard-Szász-Mirakjan operators and their truncated version, the Max-product Baskakov operators and their truncated version, also many other similar specific operators. The study in [4] is based on presented there general theory of sublinear operators. These Max-product operators tend to converge faster to the on hand function.

So we mention from [4], p. 30, that for $f : [0,1] \to \mathbb{R}_+$ continuous, we have the estimate

$$\left|B_N^{(M)}(f)(x) - f(x)\right| \leq 12\omega_1\left(f, \frac{1}{\sqrt{N+1}}\right), \quad \text{for all } N \in \mathbb{N},\ x \in [0,1], \tag{1.6}$$

Also from [4], p. 36, we mention that for $f : [0,1] \to \mathbb{R}_+$ being concave function we get that

$$\left|B_N^{(M)}(f)(x) - f(x)\right| \leq 2\omega_1\left(f, \frac{1}{N}\right), \quad \text{for all } x \in [0,1], \tag{1.7}$$

a much faster convergence.

In this article we expand the study in [4] by considering conformable fractional smoothness of functions. So our inequalities are with respect to $\omega_1(D_\alpha^n f, \delta)$, $\delta > 0$, $n \in \mathbb{N}$, where $D_\alpha^n f$ is the nth order conformable α-fractional derivative, $\alpha \in (0,1]$, see [1], [6].

We present at first some background and general related theory of sublinear operators and then we apply it to specific as above Max-product operators.

2. Background

We make

Definition 2.1. Let $f : [0,\infty) \to \mathbb{R}$ and $\alpha \in (0,1]$. We say that f is an α-fractional continuous function, iff $\forall\, \varepsilon > 0\ \exists\, \delta > 0$: for any $x, y \in [0,\infty)$ such that $|x^\alpha - y^\alpha| \leq \delta$ we get that $|f(x) - f(y)| \leq \varepsilon$.

We give

Theorem 2.2. *Over* $[a,b] \subseteq [0,\infty)$*,* $\alpha \in [0,1]$*, a* α*-fractional continuous function is a uniformly continuous function and vice versa, a uniformly continuous function is an* α*-fractional continuous function.*

(Theorem 2.2 is not valid over $[0,\infty)$.)

Note. Let $x, y \in [a,b] \subseteq [0,\infty)$, and $g(x) = x^{\alpha}$, $0 < \alpha \leq 1$, then

$$g'(x) = \alpha x^{\alpha-1} = \frac{\alpha}{x^{1-\alpha}}, \text{ for } x \in (0,\infty).$$

Since $a \leq x \leq b$, then $\frac{1}{x} \geq \frac{1}{b} > 0$ and $\frac{\alpha}{x^{1-\alpha}} \geq \frac{\alpha}{b^{1-\alpha}} > 0$.

Assume $y > x$. By the mean value theorem we get

$$y^{\alpha} - x^{\alpha} = \frac{\alpha}{\xi^{1-\alpha}}(y-x), \quad \text{where } \xi \in (x,y). \tag{2.1}$$

A similar to (2.1) equality when $x > y$ is true.

Then we obtain

$$\frac{\alpha}{b^{1-\alpha}}|y-x| \leq |y^{\alpha} - x^{\alpha}| = \frac{\alpha}{\xi^{1-\alpha}}|y-x|. \tag{2.2}$$

Thus, it holds

$$\frac{\alpha}{b^{1-\alpha}}|y-x| \leq |y^{\alpha} - x^{\alpha}|. \tag{2.3}$$

Proof of Theorem 2.2.

($\Rightarrow$) Assume that f is α-fractional continuous function on $[a,b] \subseteq [0,\infty)$. It means $\forall\, \varepsilon > 0\ \exists\, \delta > 0$: whenever $x, y \in [a,b] : |x^{\alpha} - y^{\alpha}| \leq \delta$, then $|f(x) - f(y)| \leq \varepsilon$. Let for $\{x_n\}_{n\in\mathbb{N}} \in [a,b] : \{x_n \to \lambda \in [a,b] \Leftrightarrow x_n^{\alpha} \to \lambda^{\alpha}\}$, it implies $f(x_n) \to f(\lambda)$, therefore f is continuous in λ. Therefore f is uniformly continuous over $[a,b]$.

For the converse we use the following criterion:

Lemma 2.3. *A necessary and sufficient condition that the function* f *is not* α*-fractional continuous (*$\alpha \in (0,1]$*) over* $[a,b] \subseteq [0,\infty)$ *is that there exist* $\varepsilon_0 > 0$*, and two sequences* $X = (x_n)$*,* $Y = (y_n)$ *in* $[a,b]$ *such that if* $n \in \mathbb{N}$*, then* $|x_n^{\alpha} - y_n^{\alpha}| \leq \frac{1}{n}$ *and* $|f(x_n) - f(y_n)| > \varepsilon_0$*.*

Proof. Obvious. □

(Proof of Theorem 2.2 continuous) ($\Leftarrow$) Uniform continuity implies α-fractional continuity on $[a,b] \subseteq [0,+\infty)$. Indeed: let f uniformly continuous on $[a,b]$, hence f continuous on $[a,b]$. Assume that f is not α-fractional continuous on $[a,b]$. Then by Lemma 2.3 there exist $\varepsilon_0 > 0$, and two sequences $X = (x_n)$, $Y = (y_n)$ in $[a,b]$ such that if $n \in \mathbb{N}$, then $|x_n^{\alpha} - y_n^{\alpha}| \leq \frac{1}{n}$ and

$$|f(x_n) - f(y_n)| > \varepsilon_0. \tag{2.4}$$

Since $[a,b]$ is compact, the sequences $\{x_n\}, \{y_n\}$ are bounded. By the Bolzano-Weierstrass theorem, there is a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ which converges to an element z. Since $[a,b]$ is closed, the limit $z \in [a,b]$, and f is continuous at z.

We have also that

$$\frac{\alpha}{b^{1-\alpha}}|x_n - y_n| \leq |x_n^{\alpha} - y_n^{\alpha}| \leq \frac{1}{n}, \tag{2.5}$$

hence

$$|x_n - y_n| \leq \frac{b^{1-\alpha}}{\alpha n}. \tag{2.6}$$

It is clear that the corrsponding subsequence $\left(y_{n(k)}\right)$ of Y also converges to z. Hence $f\left(x_{n(k)}\right) \to f(z)$, and $f\left(y_{n(k)}\right) \to f(z)$. Therefore, when k is sufficiently large we have $\left|f\left(x_{n(k)}\right) - f\left(y_{n(k)}\right)\right| < \varepsilon_0$, contradicting (2.4). □

We need

Definition 2.4. Let $[a,b] \subseteq [0,\infty)$, $\alpha \in [0,1]$. We define the α-fractional modulus of continuity:

$$\omega_1^{\alpha}(f,\delta) := \sup_{\substack{x,y\in[a,b]: \\ |x^{\alpha}-y^{\alpha}|\leq\delta}} |f(x) - f(y)|, \quad \delta > 0. \tag{2.7}$$

The same definition holds over $[0,\infty)$.

Properties.

1) $\omega_1^{\alpha}(f,0) = 0$.

2) $\omega_1^{\alpha}(f,\delta) \to 0$ as $\delta \downarrow 0$, iff f is in the set of all α-fractional continuous functions, denoted as $f \in C_{\alpha}([a,b],\mathbb{R})$ $(= C([a,b],\mathbb{R}))$.

Proof. $(\Rightarrow)$ Let $\omega_1^{\alpha}(f,\delta) \to 0$ as $\delta \downarrow 0$. Then $\forall\ \varepsilon > 0$, $\exists\ \delta > 0$ with $\omega_1^{\alpha}(f,\delta) \leq \varepsilon$, i.e. $\forall\ x,y \in [a,b] : |x^{\alpha} - y^{\alpha}| \leq \delta$ we get $|f(x) - f(y)| \leq \varepsilon$. That is $f \in C_{\alpha}([a,b],\mathbb{R})$.

$(\Leftarrow)$ Let $f \in C_{\alpha}([a,b],\mathbb{R})$. Then $\forall\ \varepsilon > 0$, $\exists\ \delta > 0$: whenever $|x^{\alpha} - y^{\alpha}| \leq \delta$, $x,y \in [a,b]$, it implies $|f(x) - f(y)| \leq \varepsilon$, i.e. $\forall\ \varepsilon > 0$, $\exists\ \delta > 0 : \omega_1^{\alpha}(f,\delta) \leq \varepsilon$. That is $\omega_1^{\alpha}(f,\delta) \to 0$, as $\delta \downarrow 0$. □

3) ω_1^{α} is ≥ 0 and non-decreasing on $\mathbb{R}_+$.

4) ω_1^{α} is subadditive:

$$\omega_1^{\alpha}(f,t_1+t_2) \leq \omega_1^{\alpha}(f,t_1) + \omega_1^{\alpha}(f,t_2). \tag{2.8}$$

Proof. If $|x^{\alpha} - y^{\alpha}| \leq t_1 + t_2$ $(x,y \in [a,b])$, there is a point $z \in [a,b]$ for which $|x^{\alpha} - z^{\alpha}| \leq t_1$, $|y^{\alpha} - z^{\alpha}| \leq t_2$, and $|f(x) - f(y)| \leq |f(x) - f(z)| + |f(z) - f(y)| \leq \omega_1^{\alpha}(f,t_1) + \omega_1^{\alpha}(f,t_2)$, implying $\omega_1^{\alpha}(f,t_1+t_2) \leq \omega_1^{\alpha}(f,t_1) + \omega_1^{\alpha}(f,t_2)$. □

5) ω_1^{α} is continuous on $\mathbb{R}_+$.

Proof. We get

$$|\omega_1^{\alpha}(f,t_1+t_2) - \omega_1^{\alpha}(f,t_1)| \leq \omega_1^{\alpha}(f,t_2). \tag{2.9}$$

By properties 2), 3), 4), we get that $\omega_1^{\alpha}(f,t)$ is continuous at each $t \geq 0$. □

6) Clearly it holds

$$\omega_1^{\alpha}(f,t_1+\ldots+t_n) \leq \omega_1^{\alpha}(f,t_1) + \ldots + \omega_1^{\alpha}(f,t_n), \tag{2.10}$$

for $t = t_1 = \ldots = t_n$, we obtain

$$\omega_1^{\alpha}(f,nt) = n\omega_1^{\alpha}(f,t). \tag{2.11}$$

7) Let $\lambda \geq 0$, $\lambda \notin \mathbb{N}$, we get

$$\omega_1^{\alpha}(f,\lambda t) \leq (\lambda+1)\,\omega_1^{\alpha}(f,t). \tag{2.12}$$

Proof. Let $n \in \mathbb{Z}_+ : n \leq \lambda < n+1$, we see that

$$\omega_1^{\alpha}(f, \lambda t) \leq \omega_1^{\alpha}(f, (n+1)t) \leq (n+1)\omega_1^{\alpha}(f,t) \leq (\lambda+1)\omega_1^{\alpha}(f,t). \qquad \square$$

Properties 1), 3), 4), 6), 7) are valid also for ω_1^{α} defined over $[0, \infty)$.

We notice that $\omega_1^{\alpha}(f, \delta)$ is finite when f is uniformly continuous on $[a, b]$. If $f : [0, \infty) \to \mathbb{R}$ is bounded then $\omega_1^{\alpha}(f, \delta)$ is again finite.

We need

Definition 2.5. ([1], [6]) Let $f : [0, \infty) \to \mathbb{R}$. The conformable α-fractional derivative for $\alpha \in (0, 1]$ is given by

$$D_{\alpha} f(t) := \lim_{\varepsilon \to 0} \frac{f\left(t + \varepsilon t^{1-\alpha}\right) - f(t)}{\varepsilon}, \tag{2.13}$$

$$D_{\alpha} f(0) = \lim_{t \to 0+} D_{\alpha} f(t). \tag{2.14}$$

If f is differentiable, then

$$D_{\alpha} f(t) = t^{1-\alpha} f'(t), \tag{2.15}$$

where f' is the usual derivative.

We define $D_{\alpha}^n f = D_{\alpha}^{n-1}(D_{\alpha} f)$.

If $f : [0, \infty) \to \mathbb{R}$ is α-differentiable at $t_0 > 0$, $\alpha \in (0, 1]$, then f is continuous at t_0, see [6].

We will use

Theorem 2.6. *(see [3]) (Taylor formula) Let $\alpha \in (0, 1]$ and $n \in \mathbb{N}$. Suppose f is $(n+1)$ times conformable α-fractional differentiable on $[0, \infty)$, and $s, t \in [0, \infty)$, and $D_{\alpha}^{n+1} f$ is assumed to be continuous on $[0, \infty)$. Then we have*

$$f(t) = \sum_{k=0}^{n} \frac{1}{k!} \left(\frac{t^{\alpha} - s^{\alpha}}{\alpha}\right)^k D_{\alpha}^k f(s) + \frac{1}{n!} \int_s^t \left(\frac{t^{\alpha} - \tau^{\alpha}}{\alpha}\right)^n D_{\alpha}^{n+1} f(\tau) \tau^{a-1} d\tau. \tag{2.16}$$

The case $n = 0$ follows.

Corollary 2.7. *Let $\alpha \in (0, 1]$. Suppose f is α-fractional differentiable on $[0, \infty)$, and $s, t \in [0, \infty)$. Assume that $D_{\alpha} f$ is continuous on $[0, \infty)$. Then*

$$f(t) = f(s) + \int_s^t D_{\alpha} f(\tau) \tau^{a-1} d\tau. \tag{2.17}$$

Note. Theorem 2.6 and Corollary 2.7 are also true for $f : [a, b] \to \mathbb{R}$, $[a, b] \subseteq [0, \infty)$, $s, t \in [a, b]$.

Proof of Corollary 2.7. Denote $I_{\alpha}^s(f)(t) := \int_s^t x^{\alpha-1} f(x) \, dx$. By [6] we get that

$$D_{\alpha} I_{\alpha}^s(f)(t) = f(t), \text{ for } t \geq s, \tag{2.18}$$

where f is any continuous function in the domain of I_{α}, $\alpha \in (0, 1)$.

Assume that $D_{\alpha} f$ is continuous, then

$$D_{\alpha} I_{\alpha}^s(D_{\alpha} f)(t) = (D_{\alpha} f)(t), \ \forall\, t \geq s. \tag{2.19}$$

Then, by [5], there exists a constant c such that

$$I_{\alpha}^s(D_{\alpha} f)(t) = f(t) + c. \tag{2.20}$$

Hence

$$0 = I_{\alpha}^{s} (D_{\alpha} f)(s) = f(s) + c, \tag{2.21}$$

then $c = -f(s)$.

Therefore

$$I_{\alpha}^{s} (D_{\alpha} f)(t) = f(t) - f(s) = \int_{s}^{t} (D_{\alpha} f)(\tau) \tau^{\alpha-1} d\tau. \tag{2.22}$$

The same proof applies for any $s \geq t$. □

3. Main results

We give

Theorem 3.1. *Let $\alpha \in (0,1]$ and $n \in \mathbb{Z}_+$. Suppose f is $(n+1)$ times conformable α-fractional differentiable on $[0,\infty)$, and $s,t \in [0,\infty)$, and $D_{\alpha}^{n+1} f$ is assumed to be continuous on $[0,\infty)$ and bounded. Then*

$$\left| f(t) - \sum_{k=0}^{n+1} \frac{1}{k!} \left(\frac{t^{\alpha} - s^{\alpha}}{\alpha} \right)^{k} D_{\alpha}^{k} f(s) \right| \leq \frac{\omega_1^{\alpha} \left(D_{\alpha}^{n+1} f, \delta \right)}{\alpha^{n+1} (n+1)!} |t^{\alpha} - s^{\alpha}|^{n+1} \left[1 + \frac{|t^{\alpha} - s^{\alpha}|}{(n+2)\delta} \right], \tag{3.1}$$

$\forall\ s,t \in [0,\infty)$, $\delta > 0$.

Note. Theorem 3.1 is valid also for $f : [a,b] \to \mathbb{R}$, $[a,b] \subseteq \mathbb{R}_+$, any $s,t \in [a,b]$.

Proof. We have that

$$\frac{1}{n!} \int_{s}^{t} \left(\frac{t^{\alpha} - \tau^{\alpha}}{\alpha} \right)^{n} D_{\alpha}^{n+1} f(s) \tau^{\alpha-1} d\tau = \frac{D_{\alpha}^{n+1} f(s)}{n!} \int_{s}^{t} \left(\frac{t^{\alpha} - \tau^{\alpha}}{\alpha} \right)^{n} \tau^{\alpha-1} d\tau$$

(by $\frac{d\tau^{\alpha}}{d\tau} = \alpha \tau^{\alpha-1} \Rightarrow d\tau^{\alpha} = \alpha \tau^{\alpha-1} d\tau \Rightarrow \frac{1}{\alpha} d\tau^{\alpha} = \tau^{\alpha-1} d\tau$)

$$= \frac{D_{\alpha}^{n+1} f(s)}{\alpha^{n+1} n!} \int_{s}^{t} (t^{\alpha} - \tau^{\alpha})^{n} d\tau^{\alpha} \tag{3.2}$$

(by $t \leq \tau \leq s \Rightarrow t^{\alpha} \leq \tau^{\alpha} (=: z) \leq s^{\alpha}$)

$$= \frac{D_{\alpha}^{n+1} f(s)}{\alpha^{n+1} n!} \int_{s^{\alpha}}^{t^{\alpha}} (t^{\alpha} - z)^{n} dz = \frac{D_{\alpha}^{n+1} f(s)}{\alpha^{n+1} n!} \frac{(t^{\alpha} - s^{\alpha})^{n+1}}{n+1}$$

$$= \frac{D_{\alpha}^{n+1} f(s)}{(n+1)!} \left(\frac{t^{\alpha} - s^{\alpha}}{\alpha} \right)^{n+1}. \tag{3.3}$$

Therefore it holds

$$\frac{1}{n!} \int_{s}^{t} \left(\frac{t^{\alpha} - \tau^{\alpha}}{\alpha} \right)^{n} D_{\alpha}^{n+1} f(s) \tau^{\alpha-1} d\tau = \frac{D_{\alpha}^{n+1} f(s)}{(n+1)!} \left(\frac{t^{\alpha} - s^{\alpha}}{\alpha} \right)^{n+1}. \tag{3.4}$$

By (2.16) and (2.17) we get:

$$f(t) = \sum_{k=0}^{n+1} \frac{1}{k!} \left(\frac{t^{\alpha} - s^{\alpha}}{\alpha} \right)^{k} D_{\alpha}^{k} f(s) + \tag{3.5}$$

$$\frac{1}{n!}\int_s^t\left(\frac{t^\alpha-\tau^\alpha}{\alpha}\right)^n\left(D_\alpha^{n+1}f(\tau)-D_\alpha^{n+1}f(s)\right)\tau^{\alpha-1}d\tau.$$

Call the remainder as

$$R_n(s,t):=\frac{1}{n!}\int_s^t\left(\frac{t^\alpha-\tau^\alpha}{\alpha}\right)^n\left(D_\alpha^{n+1}f(\tau)-D_\alpha^{n+1}f(s)\right)\tau^{\alpha-1}d\tau. \tag{3.6}$$

We estimate $R_n(s,t)$.
Cases:

1) Let $t\geq s$. Then

$$|R_n(s,t)|\leq\frac{1}{n!}\int_s^t\left(\frac{t^\alpha-\tau^\alpha}{\alpha}\right)^n\left|D_\alpha^{n+1}f(\tau)-D_\alpha^{n+1}f(s)\right|\tau^{\alpha-1}d\tau$$

$$\leq\frac{1}{\alpha n!}\int_s^t\left(\frac{t^\alpha-\tau^\alpha}{\alpha}\right)^n\omega_1^\alpha\left(D_\alpha^{n+1}f,\tau^\alpha-s^\alpha\right)d\tau^\alpha \tag{3.7}$$

$$=\frac{1}{\alpha^{n+1}n!}\int_s^t(t^\alpha-\tau^\alpha)^n\omega_1^\alpha\left(D_\alpha^{n+1}f,\frac{\delta(\tau^\alpha-s^\alpha)}{\delta}\right)d\tau^\alpha$$

$(\delta>0)$

$$\leq\frac{\omega_1^\alpha\left(D_\alpha^{n+1}f,\delta\right)}{\alpha^{n+1}n!}\int_s^t(t^\alpha-\tau^\alpha)^n\left(1+\frac{\tau^\alpha-s^\alpha}{\delta}\right)d\tau^\alpha$$

$$=\frac{\omega_1^\alpha\left(D_\alpha^{n+1}f,\delta\right)}{\alpha^{n+1}n!}\int_{s^\alpha}^{t^\alpha}(t^\alpha-z)^n\left(1+\frac{z-s^\alpha}{\delta}\right)dz$$

$$=\frac{\omega_1^\alpha\left(D_\alpha^{n+1}f,\delta\right)}{\alpha^{n+1}n!}\left[\int_{s^\alpha}^{t^\alpha}(t^\alpha-z)^n dz+\frac{1}{\delta}\int_{s^\alpha}^{t^\alpha}(t^\alpha-z)^{(n+1)-1}(z-s^\alpha)^{2-1}dz\right]$$

$$=\frac{\omega_1^\alpha\left(D_\alpha^{n+1}f,\delta\right)}{\alpha^{n+1}n!}\left[\frac{(t^\alpha-s^\alpha)^{n+1}}{n+1}+\frac{1}{\delta}\frac{\Gamma(n+1)\Gamma(2)}{\Gamma(n+3)}(t^\alpha-s^\alpha)^{n+2}\right] \tag{3.8}$$

$$=\frac{\omega_1^\alpha\left(D_\alpha^{n+1}f,\delta\right)}{\alpha^{n+1}n!}\left[\frac{(t^\alpha-s^\alpha)^{n+1}}{n+1}+\frac{1}{\delta}\frac{n!}{(n+2)!}(t^\alpha-s^\alpha)^{n+2}\right]$$

$$=\frac{\omega_1^\alpha\left(D_\alpha^{n+1}f,\delta\right)}{\alpha^{n+1}n!}\left[\frac{(t^\alpha-s^\alpha)^{n+1}}{n+1}+\frac{1}{\delta}\frac{(t^\alpha-s^\alpha)^{n+2}}{(n+1)(n+2)}\right] \tag{3.9}$$

$$=\frac{\omega_1^\alpha\left(D_\alpha^{n+1}f,\delta\right)}{\alpha^{n+1}(n+1)!}(t^\alpha-s^\alpha)^{n+1}\left[1+\frac{(t^\alpha-s^\alpha)}{(n+2)\delta}\right].$$

We have proved that (case of $t\geq s$)

$$|R_n(s,t)|\leq\frac{\omega_1^\alpha\left(D_\alpha^{n+1}f,\delta\right)}{\alpha^{n+1}(n+1)!}(t^\alpha-s^\alpha)^{n+1}\left[1+\frac{(t^\alpha-s^\alpha)}{(n+2)\delta}\right], \tag{3.10}$$

where $\delta>0$.

2) case of $t\leq s$: We have

$$|R_n(s,t)|\leq\frac{1}{n!}\left|\int_t^s\left(\frac{t^\alpha-\tau^\alpha}{\alpha}\right)^n\left(D_\alpha^{n+1}f(\tau)-D_\alpha^{n+1}f(s)\right)\tau^{\alpha-1}d\tau\right|$$

$$\frac{1}{n!}\left|\int_t^s\left(\frac{\tau^\alpha-t^\alpha}{\alpha}\right)^n\left(D_\alpha^{n+1}f(\tau)-D_\alpha^{n+1}f(s)\right)\tau^{\alpha-1}d\tau\right|$$

$$\leq\frac{1}{\alpha n!}\int_t^s\left(\frac{\tau^\alpha-t^\alpha}{\alpha}\right)^n\left|D_\alpha^{n+1}f(\tau)-D_\alpha^{n+1}f(s)\right|d\tau^\alpha \tag{3.11}$$

$$=\frac{1}{\alpha^{n+1}n!}\int_t^s(\tau^\alpha-t^\alpha)^n\,\omega_1^\alpha\left(D_\alpha^{n+1}f,s^\alpha-\tau^\alpha\right)d\tau^\alpha$$

$(\delta>0)$

$$\leq\frac{\omega_1^\alpha\left(D_\alpha^{n+1}f,\delta\right)}{\alpha^{n+1}n!}\int_t^s(\tau^\alpha-t^\alpha)^n\left(1+\frac{s^\alpha-\tau^\alpha}{\delta}\right)d\tau^\alpha$$

$$=\frac{\omega_1^\alpha\left(D_\alpha^{n+1}f,\delta\right)}{\alpha^{n+1}n!}\int_{t^\alpha}^{s^\alpha}(z-t^\alpha)^n\left(1+\frac{s^\alpha-z}{\delta}\right)dz$$

$$=\frac{\omega_1^\alpha\left(D_\alpha^{n+1}f,\delta\right)}{\alpha^{n+1}n!}\left[\int_{t^\alpha}^{s^\alpha}(z-t^\alpha)^n\,dz+\frac{1}{\delta}\int_{t^\alpha}^{s^\alpha}(s^\alpha-z)^{2-1}(z-t^\alpha)^{(n+1)-1}dz\right] \tag{3.12}$$

$$=\frac{\omega_1^\alpha\left(D_\alpha^{n+1}f,\delta\right)}{\alpha^{n+1}n!}\left[\frac{(s^\alpha-t^\alpha)^{n+1}}{n+1}+\frac{1}{\delta}\frac{\Gamma(2)\Gamma(n+1)}{\Gamma(n+3)}(s^\alpha-t^\alpha)^{n+2}\right]$$

$$=\frac{\omega_1^\alpha\left(D_\alpha^{n+1}f,\delta\right)}{\alpha^{n+1}n!}\left[\frac{(s^\alpha-t^\alpha)^{n+1}}{n+1}+\frac{1}{\delta}\frac{n!}{(n+2)!}(s^\alpha-t^\alpha)^{n+2}\right]$$

$$=\frac{\omega_1^\alpha\left(D_\alpha^{n+1}f,\delta\right)}{\alpha^{n+1}n!}\left[\frac{(s^\alpha-t^\alpha)^{n+1}}{n+1}+\frac{1}{\delta}\frac{(s^\alpha-t^\alpha)^{n+2}}{(n+1)(n+2)}\right] \tag{3.13}$$

$$=\frac{\omega_1^\alpha\left(D_\alpha^{n+1}f,\delta\right)}{\alpha^{n+1}(n+1)!}(s^\alpha-t^\alpha)^{n+1}\left[1+\frac{(s^\alpha-t^\alpha)}{(n+2)\delta}\right].$$

We have proved that $(t\leq s)$

$$|R_n(s,t)|\leq\frac{\omega_1^\alpha\left(D_\alpha^{n+1}f,\delta\right)}{\alpha^{n+1}(n+1)!}(s^\alpha-t^\alpha)^{n+1}\left[1+\frac{(s^\alpha-t^\alpha)}{(n+2)\delta}\right], \tag{3.14}$$

$\delta>0$.

Conclusion. We have proved that $(\delta>0)$

$$|R_n(s,t)|\leq\frac{\omega_1^\alpha\left(D_\alpha^{n+1}f,\delta\right)}{\alpha^{n+1}(n+1)!}|t^\alpha-s^\alpha|^{n+1}\left[1+\frac{|t^\alpha-s^\alpha|}{(n+2)\delta}\right],\quad\forall\ s,t\in[0,\infty). \tag{3.15}$$

The proof of the theorem now is complete. □

We proved that

Theorem 3.2. *Let $\alpha\in(0,1]$, $n\in\mathbb{N}$. Suppose f is n times conformable α-fractional differentiable on $[a,b]\subseteq[0,\infty)$, and let any $s,t\in[a,b]$. Assume that $D_\alpha^n f$ is continuous on $[a,b]$. Then*

$$\left|f(t)-\sum_{k=0}^n\frac{1}{k!}\left(\frac{t^\alpha-s^\alpha}{\alpha}\right)^kD_\alpha^kf(s)\right|\leq\frac{\omega_1^\alpha\left(D_\alpha^nf,\delta\right)}{\alpha^nn!}|t^\alpha-s^\alpha|^n\left[1+\frac{|t^\alpha-s^\alpha|}{(n+1)\delta}\right], \tag{3.16}$$

where $\delta>0$.

Proof. By Theorem 3.1. □

Corollary 3.3. *($n=1$ case of Theorem 3.2) Let $\alpha \in (0,1]$. Suppose f is α-conformable fractional differentiable on $[a,b] \subseteq [0,\infty)$, and let any $s,t \in [a,b]$. Assume that $D_\alpha f$ is continuous on $[a,b]$. Then*

$$\left| f(t) - f(s) - \left(\frac{t^\alpha - s^\alpha}{\alpha} \right) D_\alpha f(s) \right| \leq \frac{\omega_1^\alpha (D_\alpha f, \delta)}{\alpha} \left| t^\alpha - s^\alpha \right| \left[1 + \frac{\left| t^\alpha - s^\alpha \right|}{2\delta} \right], \quad (3.17)$$

where $\delta > 0$.

Corollary 3.4. *(to Theorem 3.2) Same assumptions as in Theorem 3.2. For specific $s \in [a,b]$ assume that $D_\alpha^k f(s) = 0$, $k = 1,...,n$. Then*

$$|f(t) - f(s)| \leq \frac{\omega_1^\alpha (D_\alpha^n f, \delta)}{\alpha^n n!} \left| t^\alpha - s^\alpha \right|^n \left[1 + \frac{\left| t^\alpha - s^\alpha \right|}{(n+1)\delta} \right], \quad \delta > 0. \quad (3.18)$$

The case $n=1$ follows:

Corollary 3.5. *(to Corollary 3.4) For specific $s \in [a,b]$ assume that $D_\alpha f(s) = 0$. Then*

$$|f(t) - f(s)| \leq \frac{\omega_1^\alpha (D_\alpha f, \delta)}{\alpha} \left| t^\alpha - s^\alpha \right| \left[1 + \frac{\left| t^\alpha - s^\alpha \right|}{2\delta} \right], \quad \delta > 0. \quad (3.19)$$

We make

Remark 3.6. For $0 < \alpha \leq 1$, $t,s \geq 0$, we have

$$2^{\alpha-1} (x^\alpha + y^\alpha) \leq (x+y)^\alpha \leq x^\alpha + y^\alpha. \quad (3.20)$$

Assume that $t > s$, then

$$t = t - s + s \Rightarrow t^\alpha = (t-s+s)^\alpha \leq (t-s)^\alpha + s^\alpha,$$

hence $t^\alpha - s^\alpha \leq (t-s)^\alpha$.

Similarly, when $s > t \Rightarrow s^\alpha - t^\alpha \leq (s-t)^\alpha$.

Therefore it holds

$$\left| t^\alpha - s^\alpha \right| \leq |t-s|^\alpha, \ \forall\, t,s \in [0,\infty). \quad (3.21)$$

Corollary 3.7. *(to Theorem 3.2) Same assumptions as in Theorem 3.2. For specific $s \in [a,b]$ assume that $D_\alpha^k f(s) = 0$, $k = 1,...,n$. Then*

$$|f(t) - f(s)| \leq \frac{\omega_1^\alpha (D_\alpha^n f, \delta)}{\alpha^n n!} |t-s|^{n\alpha} \left[1 + \frac{|t-s|^\alpha}{(n+1)\delta} \right], \quad \delta > 0, \quad (3.22)$$

$\forall\, t \in [a,b] \subseteq [0,\infty)$.

Corollary 3.8. *(to Corollary 3.3) Same assumptions as in Corollary 3.3. For specific $s \in [a,b]$ assume that $D_\alpha f(s) = 0$. Then*

$$|f(t) - f(s)| \leq \frac{\omega_1^\alpha (D_\alpha f, \delta)}{\alpha} |t-s|^\alpha \left[1 + \frac{|t-s|^\alpha}{2\delta} \right], \quad \delta > 0, \quad (3.23)$$

$\forall\, t \in [a,b] \subseteq [0,\infty)$.

We need

Definition 3.9. Here $C_{+}([a,b]) := \{f : [a,b] \subseteq [0,\infty) \to \mathbb{R}_{+}$, continuous functions$\}$. Let $L_N : C_{+}([a,b]) \to C_{+}([a,b])$, operators, $\forall\, N \in \mathbb{N}$, such that

(i)

$$L_N(\alpha f) = \alpha L_N(f), \ \forall \alpha \geq 0, \forall f \in C_{+}([a,b]), \tag{3.24}$$

(ii) if $f, g \in C_{+}([a,b]) : f \leq g$, then

$$L_N(f) \leq L_N(g), \ \ \forall N \in \mathbb{N}, \tag{3.25}$$

(iii)

$$L_N(f+g) \leq L_N(f) + L_N(g), \ \ \forall\, f, g \in C_{+}([a,b]). \tag{3.26}$$

We call $\{L_N\}_{N\in\mathbb{N}}$ positive sublinear operators.

We need a Hölder's type inequality, see next:

Theorem 3.10. *(see [2]) Let $L : C_{+}([a,b]) \to C_{+}([a,b])$, be a positive sublinear operator and $f, g \in C_{+}([a,b])$, furthermore let $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$. Assume that $L((f(\cdot))^p)(s_*), L((g(\cdot))^q)(s_*) > 0$ for some $s_* \in [a,b]$. Then*

$$L(f(\cdot)\, g(\cdot))(s_*) \leq \left(L((f(\cdot))^p)(s_*)\right)^{\frac{1}{p}} \left(L((g(\cdot))^q)(s_*)\right)^{\frac{1}{q}}. \tag{3.27}$$

We make

Remark 3.11. By [4], p. 17, we get: let $f, g \in C_{+}([a,b])$, then

$$|L_N(f)(x) - L_N(g)(x)| \leq L_N(|f-g|)(x), \ \ \forall\, x \in [a,b] \subseteq [0,\infty). \tag{3.28}$$

Furthermore, we also have that

$$|L_N(f)(x) - f(x)| \leq L_N(|f(\cdot) - f(x)|)(x) + |f(x)|\,|L_N(e_0)(x) - 1|, \tag{3.29}$$

$\forall\, x \in [a,b] \subseteq [0,\infty)$; $e_0(t) = 1$.

From now on we assume that $L_N(1) = 1$. Hence it holds

$$|L_N(f)(x) - f(x)| \leq L_N(|f(\cdot) - f(x)|)(x), \ \ \forall\, x \in [a,b] \subseteq [0,\infty). \tag{3.30}$$

Next we use Corollary 3.8.

Here $D_\alpha f(x) = 0$ for a specific $x \in [a,b] \subseteq [0,\infty)$. We also assume that $L_N\left(|\cdot - x|^{\alpha+1}\right)(x), L_N\left((\cdot - x)^{2(\alpha+1)}\right)(x) > 0$. By (3.23) we have

$$|f(\cdot) - f(x)| \leq \frac{\omega_1^\alpha(D_\alpha f, \delta)}{\alpha}\left[|\cdot - x|^\alpha + \frac{|\cdot - x|^{2\alpha}}{2\delta}\right], \ \ \delta > 0, \tag{3.31}$$

true over $[a,b] \subseteq [0,\infty)$.

By (3.30) we get

$$|L_N(f)(x) - f(x)| \leq \frac{\omega_1^\alpha(D_\alpha f, \delta)}{\alpha}\left[L_N(|\cdot - x|^\alpha)(x) + \frac{L_N\left(|\cdot - x|^{2\alpha}\right)(x)}{2\delta}\right] \tag{3.32}$$

$$\overset{(\text{by } (3.27))}{\leq} \frac{\omega_1^\alpha(D_\alpha f, \delta)}{\alpha}\left[\left(L_N\left(|\cdot - x|^{\alpha+1}\right)(x)\right)^{\frac{\alpha}{\alpha+1}} + \frac{\left(L_N\left((\cdot - x)^{2(\alpha+1)}\right)(x)\right)^{\frac{\alpha}{\alpha+1}}}{2\delta}\right] \tag{3.33}$$

(choose $\delta := \left(\left(L_N\left((\cdot - x)^{2(\alpha+1)}\right)(x)\right)^{\frac{\alpha}{\alpha+1}}\right)^{\frac{1}{2}} > 0$, hence
$\delta^2 = \left(L_N\left((\cdot - x)^{2(\alpha+1)}\right)(x)\right)^{\frac{\alpha}{\alpha+1}}$)

$$= \frac{\omega_1^{\alpha}\left(D_{\alpha}f, \left(L_N\left((\cdot - x)^{2(\alpha+1)}\right)(x)\right)^{\frac{\alpha}{2(\alpha+1)}}\right)}{\alpha} \cdot$$

$$\left[\left(L_N\left(|\cdot - x|^{\alpha+1}\right)(x)\right)^{\frac{\alpha}{\alpha+1}} + \frac{1}{2}\left(L_N\left((\cdot - x)^{2(\alpha+1)}\right)(x)\right)^{\frac{\alpha}{2(\alpha+1)}}\right]. \tag{3.34}$$

We have proved:

Theorem 3.12. *Let $\alpha \in (0,1]$, $[a,b] \subseteq [0,\infty)$. Suppose f is α-conformable fractional differentiable on $[a,b]$. $D_{\alpha}f$ is continuous on $[a,b]$. Let an $x \in [a,b]$ such that $D_{\alpha}f(x) = 0$, and $L_N : C_+([a,b])$ into itself, positive sublinear operators. Assume that $L_N(1) = 1$ and $L_N\left(|\cdot - x|^{\alpha+1}\right)(x)$, $L_N\left((\cdot - x)^{2(\alpha+1)}\right)(x) > 0$, $\forall\, N \in \mathbb{N}$.*

Then

$$|L_N(f)(x) - f(x)| \leq \frac{\omega_1^{\alpha}\left(D_{\alpha}f, \left(L_N\left((\cdot - x)^{2(\alpha+1)}\right)(x)\right)^{\frac{\alpha}{2(\alpha+1)}}\right)}{\alpha} \cdot$$

$$\left[\left(L_N\left(|\cdot - x|^{\alpha+1}\right)(x)\right)^{\frac{\alpha}{\alpha+1}} + \frac{1}{2}\left(L_N\left((\cdot - x)^{2(\alpha+1)}\right)(x)\right)^{\frac{\alpha}{2(\alpha+1)}}\right], \quad \forall\, N \in \mathbb{N}. \tag{3.35}$$

We make

Remark 3.13. By Theorem 3.10, we get that

$$L_N\left(|\cdot - x|^{\alpha+1}\right)(x) \leq \left(L_N\left((\cdot - x)^{2(\alpha+1)}\right)(x)\right)^{\frac{1}{2}}. \tag{3.36}$$

As $N \to +\infty$, by (3.35) and (3.36), and $L_N\left((\cdot - x)^{2(\alpha+1)}\right)(x) \to 0$, we obtain that $L_N(f)(x) \to f(x)$.

We continue with

Remark 3.14. In the assumptions of Corollary 3.7 and (3.22) we can write over $[a,b] \subseteq [0,\infty)$, that

$$|f(\cdot) - f(x)| \leq \frac{\omega_1^{\alpha}\left(D_{\alpha}^n f, \delta\right)}{\alpha^n n!}\left[|\cdot - x|^{n\alpha} + \frac{|\cdot - x|^{(n+1)\alpha}}{(n+1)\delta}\right], \quad \delta > 0. \tag{3.37}$$

By (3.30) we get

$$|L_N(f)(x) - f(x)| \leq \frac{\omega_1^{\alpha}\left(D_{\alpha}^n f, \delta\right)}{\alpha^n n!} \cdot$$

$$\left[L_N\left(|\cdot - x|^{n\alpha}\right)(x) + \frac{1}{(n+1)\delta} L_N\left(|\cdot - x|^{(n+1)\alpha}\right)(x)\right]$$

$$\overset{\text{(by (3.27))}}{\leq} \frac{\omega_1^{\alpha}\left(D_{\alpha}^n f, \delta\right)}{\alpha^n n!} \cdot \tag{3.38}$$

$$\left[\left(L_N\left(|\cdot-x|^{n(\alpha+1)}\right)(x)\right)^{\frac{\alpha}{\alpha+1}}+\frac{1}{(n+1)\delta}\left(L_N\left((\cdot-x)^{(n+1)(\alpha+1)}\right)(x)\right)^{\frac{\alpha}{\alpha+1}}\right]$$

[(here is assumed $L_N(1)=1$, and $L_N\left(|\cdot-x|^{n(\alpha+1)}\right)(x)$, $L_N\left((\cdot-x)^{(n+1)(\alpha+1)}\right)(x)>0$, $\forall\ N\in\mathbb{N}$),
(we take $\delta:=\left(L_N\left((\cdot-x)^{(n+1)(\alpha+1)}\right)(x)\right)^{\frac{\alpha}{(n+1)(\alpha+1)}}>0$, then
$\delta^{n+1}=\left(L_N\left((\cdot-x)^{(n+1)(\alpha+1)}\right)(x)\right)^{\frac{\alpha}{\alpha+1}}$)]

$$=\frac{\omega_1^{\alpha}\left(D_{\alpha}^{n}f,\left(L_N\left((\cdot-x)^{(n+1)(\alpha+1)}\right)(x)\right)^{\frac{\alpha}{(n+1)(\alpha+1)}}\right)}{\alpha^n n!}\cdot$$

$$\left[\left(L_N\left(|\cdot-x|^{n(\alpha+1)}\right)(x)\right)^{\frac{\alpha}{\alpha+1}}+\frac{1}{(n+1)}\left(L_N\left((\cdot-x)^{(n+1)(\alpha+1)}\right)(x)\right)^{\frac{n\alpha}{(n+1)(\alpha+1)}}\right]. \tag{3.39}$$

We have proved

Theorem 3.15. *Let $\alpha\in(0,1]$, $n\in\mathbb{N}$. Suppose f is n times conformable α-fractional differentiable on $[a,b]\subseteq[0,\infty)$, and $D_{\alpha}^{n}f$ is continuous on $[a,b]$. For a fixed $x\in[a,b]$ we have $D_{\alpha}^{k}f(x)=0$, $k=1,...,n$. Let positive sublinear operators $\{L_N\}_{N\in\mathbb{N}}$ from $C_+([a,b])$ into itself, such that $L_N(1)=1$, and $L_N\left(|\cdot-x|^{n(\alpha+1)}\right)(x)$, $L_N\left((\cdot-x)^{(n+1)(\alpha+1)}\right)(x)>0$, $\forall\ N\in\mathbb{N}$. Then*

$$|L_N(f)(x)-f(x)|\leq\frac{\omega_1^{\alpha}\left(D_{\alpha}^{n}f,\left(L_N\left((\cdot-x)^{(n+1)(\alpha+1)}\right)(x)\right)^{\frac{\alpha}{(n+1)(\alpha+1)}}\right)}{\alpha^n n!}\cdot \tag{3.40}$$

$$\left[\left(L_N\left(|\cdot-x|^{n(\alpha+1)}\right)(x)\right)^{\frac{\alpha}{\alpha+1}}+\frac{1}{(n+1)}\left(L_N\left((\cdot-x)^{(n+1)(\alpha+1)}\right)(x)\right)^{\frac{n\alpha}{(n+1)(\alpha+1)}}\right],$$

$\forall\ N\in\mathbb{N}$.

We make

Remark 3.16. By Theorem 3.10, we get that

$$L_N\left(|\cdot-x|^{n(\alpha+1)}\right)(x)\leq\left(L_N\left((\cdot-x)^{(n+1)(\alpha+1)}\right)(x)\right)^{\frac{n}{n+1}}. \tag{3.41}$$

As $N\to+\infty$, by (3.40), (3.41), and $L_N\left((\cdot-x)^{(n+1)(\alpha+1)}\right)(x)\to0$, we derive that $L_N(f)(x)\to f(x)$.

4. Applications

Here we apply Theorems 3.12 and 3.15 to well known Max-product operators. We make

Remark 4.1. The Max-product Bernstein operators $B_N^{(M)}(f)(x)$ are defined by (1.5), see also [4], p. 10; here $f:[0,1]\to\mathbb{R}_+$ is a continuous function.

We have $B_N^{(M)}(1)=1$, and

$$B_N^{(M)}(|\cdot - x|)(x) \leq \frac{6}{\sqrt{N+1}}, \quad \forall\, x\in[0,1], \forall\, N\in\mathbb{N},$$

see [4], p. 31.

$B_N^{(M)}$ are positive sublinear operators and thus they possess the monotonicity property, also since $|\cdot - x|\leq 1$, then $|\cdot - x|^{\beta}\leq 1$, $\forall\, x\in[0,1]$, $\forall\, \beta>0$.

Therefore it holds

$$B_N^{(M)}\left(|\cdot - x|^{1+\beta}\right)(x) \leq \frac{6}{\sqrt{N+1}}, \quad \forall\, x\in[0,1], \forall\, N\in\mathbb{N}, \forall\, \beta>0. \tag{4.1}$$

Furthermore, clearly it holds that

$$B_N^{(M)}\left(|\cdot - x|^{1+\beta}\right)(x) > 0, \ \forall\, N\in\mathbb{N}, \forall\, \beta\geq 0 \text{ and any } x\in(0,1). \tag{4.2}$$

The operator $B_N^{(M)}$ maps $C_+([0,1])$ into itself.

We have the following results:

Theorem 4.2. *Let $\alpha\in(0,1]$, f is α-conformable fractional differentiable on $[0,1]$, $D_\alpha f$ is continuous on $[0,1]$. Let $x\in(0,1)$ such that $D_\alpha f(x)=0$. Then*

$$\left|B_N^{(M)}(f)(x) - f(x)\right| \leq \frac{\omega_1^{\alpha}\left(D_\alpha f, \left(\frac{6}{\sqrt{N+1}}\right)^{\frac{\alpha}{2(\alpha+1)}}\right)}{\alpha}\cdot \tag{4.3}$$

$$\left[\left(\frac{6}{\sqrt{N+1}}\right)^{\frac{\alpha}{\alpha+1}} + \frac{1}{2}\left(\frac{6}{\sqrt{N+1}}\right)^{\frac{\alpha}{2(\alpha+1)}}\right], \quad \forall\, N\in\mathbb{N}.$$

Proof. By Theorem 3.12. ☐

Theorem 4.3. *Let $\alpha\in(0,1]$, f is n times conformable α-fractional differentiable on $[0,1]$, and $D_\alpha^n f$ is continuous on $[0,1]$. For a fixed $x\in(0,1)$ we have $D_\alpha^k f(x)=0$, $k=1,...,n\in\mathbb{N}$. Then*

$$\left|B_N^{(M)}(f)(x) - f(x)\right| \leq \frac{\omega_1^{\alpha}\left(D_\alpha^n f, \left(\frac{6}{\sqrt{N+1}}\right)^{\frac{\alpha}{(n+1)(\alpha+1)}}\right)}{\alpha^n n!}\cdot \tag{4.4}$$

$$\left[\left(\frac{6}{\sqrt{N+1}}\right)^{\frac{\alpha}{\alpha+1}} + \frac{1}{(n+1)}\left(\frac{6}{\sqrt{N+1}}\right)^{\frac{n\alpha}{(n+1)(\alpha+1)}}\right], \quad \forall\, N\in\mathbb{N}.$$

Proof. By Theorem 3.15. ☐

Note. By (4.3) and/or (4.4), as $N \to +\infty$, we get $B_N^{(M)}(f)(x) \to f(x)$.

We continue with

Remark 4.4. The truncated Favard-Szász-Mirakjan operators are given by

$$T_N^{(M)}(f)(x) = \frac{\bigvee_{k=0}^{N} s_{N,k}(x) f\left(\frac{k}{N}\right)}{\bigvee_{k=0}^{N} s_{N,k}(x)}, \quad x \in [0,1], \ N \in \mathbb{N}, \ f \in C_+([0,1]), \tag{4.5}$$

$s_{N,k}(x) = \frac{(Nx)^k}{k!}$, see also [4], p. 11.

By [4], p. 178-179, we get that

$$T_N^{(M)}(|\cdot - x|)(x) \le \frac{3}{\sqrt{N}}, \quad \forall\, x \in [0,1], \ \forall\, N \in \mathbb{N}. \tag{4.6}$$

Clearly it holds

$$T_N^{(M)}\left(|\cdot - x|^{1+\beta}\right)(x) \le \frac{3}{\sqrt{N}}, \quad \forall\, x \in [0,1], \ \forall\, N \in \mathbb{N}, \forall\, \beta > 0. \tag{4.7}$$

The operators $T_N^{(M)}$ are positive sublinear operators mapping $C_+([0,1])$ into itself, with $T_N^{(M)}(1) = 1$.

Furthermore it holds

$$T_N^{(M)}\left(|\cdot - x|^{\lambda}\right)(x) = \frac{\bigvee_{k=0}^{N} \frac{(Nx)^k}{k!} \left|\frac{k}{N} - x\right|^{\lambda}}{\bigvee_{k=0}^{N} \frac{(Nx)^k}{k!}} > 0, \quad \forall\, x \in (0,1], \ \forall\, \lambda \ge 1, \forall\, N \in \mathbb{N}. \tag{4.8}$$

We give the following results:

Theorem 4.5. *Let $\alpha \in (0,1]$, f is α-conformable fractional differentiable on $[0,1]$. $D_\alpha f$ is continuous on $[0,1]$. Let $x \in (0,1]$ such that $D_\alpha f(x) = 0$. Then*

$$\left|T_N^{(M)}(f)(x) - f(x)\right| \le \frac{\omega_1^{\alpha}\left(D_\alpha f, \left(\frac{3}{\sqrt{N}}\right)^{\frac{\alpha}{2(\alpha+1)}}\right)}{\alpha} \cdot \left[\left(\frac{3}{\sqrt{N}}\right)^{\frac{\alpha}{\alpha+1}} + \frac{1}{2}\left(\frac{3}{\sqrt{N}}\right)^{\frac{\alpha}{2(\alpha+1)}}\right], \quad \forall\, N \in \mathbb{N}. \tag{4.9}$$

Proof. By Theorem 3.12. □

Theorem 4.6. *Let $\alpha \in (0,1]$, f is n times conformable α-fractional differentiable on $[0,1]$, and $D_\alpha^n f$ is continuous on $[0,1]$. For a fixed $x \in (0,1]$ we have $D_\alpha^k f(x) = 0$, $k = 1, ..., n \in \mathbb{N}$. Then*

$$\left|T_N^{(M)}(f)(x) - f(x)\right| \le \frac{\omega_1^{\alpha}\left(D_\alpha^n f, \left(\frac{3}{\sqrt{N}}\right)^{\frac{\alpha}{(n+1)(\alpha+1)}}\right)}{\alpha^n n!} \cdot \left[\left(\frac{3}{\sqrt{N}}\right)^{\frac{\alpha}{\alpha+1}} + \frac{1}{(n+1)}\left(\frac{3}{\sqrt{N}}\right)^{\frac{n\alpha}{(n+1)(\alpha+1)}}\right], \quad \forall\, N \in \mathbb{N}. \tag{4.10}$$

Proof. By Theorem 3.15. □

Note. By (4.9) and/or (4.10), as $N \to +\infty$, we get $T_N^{(M)}(f)(x) \to f(x)$.

We continue with

Remark 4.7. Next we study the truncated Max-product Baskakov operators (see [4], p. 11)

$$U_N^{(M)}(f)(x) = \frac{\bigvee_{k=0}^{N} b_{N,k}(x) f\left(\frac{k}{N}\right)}{\bigvee_{k=0}^{N} b_{N,k}(x)}, \quad x \in [0,1], \ f \in C_+([0,1]), \ N \in \mathbb{N}, \tag{4.11}$$

where

$$b_{N,k}(x) = \begin{pmatrix} N+k-1 \\ k \end{pmatrix} \frac{x^k}{(1+x)^{N+k}}. \tag{4.12}$$

From [4], pp. 217-218, we get ($x \in [0,1]$)

$$\left(U_N^{(M)}(|\cdot - x|)\right)(x) \leq \frac{2\sqrt{3}\left(\sqrt{2}+2\right)}{\sqrt{N+1}}, \quad N \geq 2, \ N \in \mathbb{N}. \tag{4.13}$$

Let $\lambda \geq 1$, clearly then it holds

$$\left(U_N^{(M)}\left(|\cdot - x|^{\lambda}\right)\right)(x) \leq \frac{2\sqrt{3}\left(\sqrt{2}+2\right)}{\sqrt{N+1}}, \quad \forall \ N \geq 2, \ N \in \mathbb{N}. \tag{4.14}$$

Also it holds $U_N^{(M)}(1) = 1$, and $U_N^{(M)}$ are positive sublinear operators from $C_+([0,1])$ into itself. Furthermore it holds

$$U_N^{(M)}\left(|\cdot - x|^{\lambda}\right)(x) > 0, \quad \forall \ x \in (0,1], \ \forall \ \lambda \geq 1, \ \forall \ N \in \mathbb{N}. \tag{4.15}$$

We give

Theorem 4.8. *Let $\alpha \in (0,1]$, f is α-conformable fractional differentiable on $[0,1]$. $D_\alpha f$ is continuous on $[0,1]$. Let $x \in (0,1]$ such that $D_\alpha f(x) = 0$. Then*

$$\left|U_N^{(M)}(f)(x) - f(x)\right| \leq \frac{\omega_1^{\alpha}\left(D_\alpha f, \left(\frac{2\sqrt{3}\left(\sqrt{2}+2\right)}{\sqrt{N+1}}\right)^{\frac{\alpha}{2(\alpha+1)}}\right)}{\alpha} \cdot \tag{4.16}$$

$$\left[\left(\frac{2\sqrt{3}\left(\sqrt{2}+2\right)}{\sqrt{N+1}}\right)^{\frac{\alpha}{\alpha+1}} + \frac{1}{2}\left(\frac{2\sqrt{3}\left(\sqrt{2}+2\right)}{\sqrt{N+1}}\right)^{\frac{\alpha}{2(\alpha+1)}}\right], \quad \forall \ N \geq 2, \ N \in \mathbb{N}.$$

Proof. By Theorem 3.12. □

Theorem 4.9. *Let $\alpha \in (0,1]$, f is n times conformable α-fractional differentiable on $[0,1]$, and $D_\alpha^n f$ is continuous on $[0,1]$. For a fixed $x \in (0,1]$ we have $D_\alpha^k f(x) = 0$, $k = 1, ..., n \in \mathbb{N}$. Then*

$$\left|U_N^{(M)}(f)(x) - f(x)\right| \leq \frac{\omega_1^{\alpha}\left(D_\alpha^n f, \left(\frac{2\sqrt{3}\left(\sqrt{2}+2\right)}{\sqrt{N+1}}\right)^{\frac{\alpha}{(n+1)(\alpha+1)}}\right)}{\alpha^n n!} \cdot \tag{4.17}$$

$$\left[\left(\frac{2\sqrt{3}\left(\sqrt{2}+2\right)}{\sqrt{N+1}}\right)^{\frac{\alpha}{\alpha+1}} + \frac{1}{(n+1)}\left(\frac{2\sqrt{3}\left(\sqrt{2}+2\right)}{\sqrt{N+1}}\right)^{\frac{n\alpha}{(n+1)(\alpha+1)}}\right],$$

$\forall\ N \geq 2,\ N \in \mathbb{N}$.

Proof. By Theorem 3.15. □

Note. By (4.16) and/or (4.17), as $N \to +\infty$, we get that $U_N^{(M)}(f)(x) \to f(x)$.

We continue with

Remark 4.10. Here we study the Max-product Meyer-Köning and Zeller operators (see [4], p. 11) defined by

$$Z_N^{(M)}(f)(x) = \frac{\bigvee_{k=0}^{\infty} s_{N,k}(x) f\left(\frac{k}{N+k}\right)}{\bigvee_{k=0}^{\infty} s_{N,k}(x)}, \quad \forall\ N \in \mathbb{N},\ f \in C_+([0,1]), \tag{4.18}$$

$s_{N,k}(x) = \begin{pmatrix} N+k \\ k \end{pmatrix} x^k$, $x \in [0,1]$.

By [4], p. 253, we get that

$$Z_N^{(M)}(|\cdot - x|)(x) \leq \frac{8\left(1+\sqrt{5}\right)}{3} \frac{\sqrt{x}(1-x)}{\sqrt{N}}, \quad \forall\ x \in [0,1],\ \forall\ N \geq 4,\ N \in \mathbb{N}. \tag{4.19}$$

As before we get that (for $\lambda \geq 1$)

$$Z_N^{(M)}\left(|\cdot - x|^{\lambda}\right)(x) \leq \frac{8\left(1+\sqrt{5}\right)}{3} \frac{\sqrt{x}(1-x)}{\sqrt{N}} := \rho(x), \tag{4.20}$$

$\forall\ x \in [0,1]$, $N \geq 4$, $N \in \mathbb{N}$.

Also it holds $Z_N^{(M)}(1) = 1$, and $Z_N^{(M)}$ are positive sublinear operators from $C_+([0,1])$ into itself. Also it holds

$$Z_N^{(M)}\left(|\cdot - x|^{\lambda}\right)(x) > 0, \quad \forall\ x \in (0,1),\ \forall\ \lambda \geq 1, \forall\ N \in \mathbb{N}. \tag{4.21}$$

We give

Theorem 4.11. *Let $\alpha \in (0,1]$, f is α-conformable fractional differentiable on $[0,1]$. $D_\alpha f$ is continuous on $[0,1]$. Let $x \in (0,1)$ such that $D_\alpha f(x) = 0$. Then*

$$\left|Z_N^{(M)}(f)(x) - f(x)\right| \leq \frac{\omega_1^{\alpha}\left(D_\alpha f, (\rho(x))^{\frac{\alpha}{2(\alpha+1)}}\right)}{\alpha} \cdot \tag{4.22}$$

$$\left[(\rho(x))^{\frac{\alpha}{\alpha+1}} + \frac{1}{2}(\rho(x))^{\frac{\alpha}{2(\alpha+1)}}\right], \quad \forall\ N \geq 4,\ N \in \mathbb{N}.$$

Proof. By Theorem 3.12. □

Theorem 4.12. *Let $\alpha \in (0,1]$, f is n times conformable α-fractional differentiable on $[0,1]$, and $D_\alpha^n f$ is continuous on $[0,1]$. For a fixed $x \in (0,1)$ we have $D_\alpha^k f(x) = 0$, $k = 1, ..., n \in \mathbb{N}$. Then*

$$\left|Z_N^{(M)}(f)(x) - f(x)\right| \leq \frac{\omega_1^{\alpha}\left(D_\alpha^n f, (\rho(x))^{\frac{\alpha}{(n+1)(\alpha+1)}}\right)}{\alpha^n n!} \cdot \tag{4.23}$$

$$\left[(\rho(x))^{\frac{\alpha}{\alpha+1}} + \frac{1}{(n+1)}(\rho(x))^{\frac{n\alpha}{(n+1)(\alpha+1)}}\right], \quad \forall\ N \geq 4,\ N \in \mathbb{N}.$$

Proof. By Theorem 3.15. □

Note. By (4.22) and/or (4.23), as $N \to +\infty$, we get that $Z_N^{(M)}(f)(x) \to f(x)$.

We continue with

Remark 4.13. Here we deal with the Max-product truncated sampling operators (see [4], p. 13) defined by

$$W_N^{(M)}(f)(x) = \frac{\bigvee_{k=0}^{N} \frac{\sin(Nx-k\pi)}{Nx-k\pi} f\left(\frac{k\pi}{N}\right)}{\bigvee_{k=0}^{N} \frac{\sin(Nx-k\pi)}{Nx-k\pi}}, \tag{4.24}$$

and

$$K_N^{(M)}(f)(x) = \frac{\bigvee_{k=0}^{N} \frac{\sin^2(Nx-k\pi)}{(Nx-k\pi)^2} f\left(\frac{k\pi}{N}\right)}{\bigvee_{k=0}^{N} \frac{\sin^2(Nx-k\pi)}{(Nx-k\pi)^2}}, \tag{4.25}$$

$\forall\, x \in [0,\pi]$, $f : [0,\pi] \to \mathbb{R}_+$ a continuous function.

Following [4], p. 343, and making the convention $\frac{\sin(0)}{0} = 1$ and denoting $s_{N,k}(x) = \frac{\sin(Nx-k\pi)}{Nx-k\pi}$, we get that $s_{N,k}\left(\frac{k\pi}{N}\right) = 1$, and $s_{N,k}\left(\frac{j\pi}{N}\right) = 0$, if $k \neq j$, furthermore $W_N^{(M)}(f)\left(\frac{j\pi}{N}\right) = f\left(\frac{j\pi}{N}\right)$, for all $j \in \{0, ..., N\}$.

Clearly $W_N^{(M)}(f)$ is a well-defined function for all $x \in [0,\pi]$, and it is continuous on $[0,\pi]$, also $W_N^{(M)}(1) = 1$.

By [4], p. 344, $W_N^{(M)}$ are positive sublinear operators.

Call $I_N^+(x) = \{k \in \{0,1,...,N\}; s_{N,k}(x) > 0\}$, and set $x_{N,k} := \frac{k\pi}{N}$, $k \in \{0,1,...,N\}$.

We see that

$$W_N^{(M)}(f)(x) = \frac{\bigvee_{k \in I_N^+(x)} s_{N,k}(x) f(x_{N,k})}{\bigvee_{k \in I_N^+(x)} s_{N,k}(x)}. \tag{4.26}$$

By [4], p. 346, we have

$$W_N^{(M)}(|\cdot - x|)(x) \leq \frac{\pi}{2N}, \quad \forall\, N \in \mathbb{N},\ \forall\, x \in [0,\pi]. \tag{4.27}$$

Notice also $|x_{N,k} - x| \leq \pi$, $\forall\, x \in [0,\pi]$.

Therefore ($\lambda \geq 1$) it holds

$$W_N^{(M)}\left(|\cdot - x|^{\lambda}\right)(x) \leq \frac{\pi^{\lambda-1}\pi}{2N} = \frac{\pi^{\lambda}}{2N}, \quad \forall\, x \in [0,\pi], \forall\, N \in \mathbb{N}. \tag{4.28}$$

If $x \in \left(\frac{j\pi}{N}, \frac{(j+1)\pi}{N}\right)$, with $j \in \{0,1,...,N\}$, we obtain $nx - j\pi \in (0,\pi)$ and thus

$$s_{N,j}(x) = \frac{\sin(Nx - j\pi)}{Nx - j\pi} > 0,$$

see [4], pp. 343-344.

Consequently it holds ($\lambda \geq 1$)

$$W_N^{(M)}\left(|\cdot - x|^{\lambda}\right)(x) = \frac{\bigvee_{k \in I_N^+(x)} s_{N,k}(x) |x_{N,k} - x|^{\lambda}}{\bigvee_{k \in I_N^+(x)} s_{N,k}(x)} > 0, \quad \forall\, x \in [0,\pi], \tag{4.29}$$

such that $x \neq x_{N,k}$, for any $k \in \{0, 1, ..., N\}$.

We give

Theorem 4.14. *Let $\alpha \in (0,1]$, f is α-conformable fractional differentiable on $[0,\pi]$. $D_\alpha f$ is continuous on $[0,\pi]$. Let $x \in [0,\pi]$ be such that $x \neq \frac{k\pi}{N}$, $k \in \{0,1,...,N\}$, $\forall$ $N \in \mathbb{N}$, and $D_\alpha f(x) = 0$. Then*

$$\left| W_N^{(M)}(f)(x) - f(x) \right| \leq \frac{\omega_1^\alpha \left(D_\alpha f, \left(\frac{\pi^{2(\alpha+1)}}{2N} \right)^{\frac{\alpha}{2(\alpha+1)}} \right)}{\alpha} \cdot$$

$$\left[\left(\frac{\pi^{\alpha+1}}{2N} \right)^{\frac{\alpha}{\alpha+1}} + \frac{1}{2} \left(\frac{\pi^{2(\alpha+1)}}{2N} \right)^{\frac{\alpha}{2(\alpha+1)}} \right] =$$

$$\frac{\omega_1^\alpha \left(D_\alpha f, \frac{\pi^\alpha}{(2N)^{\frac{\alpha}{2(\alpha+1)}}} \right)}{\alpha} \left[\frac{\pi^\alpha}{(2N)^{\frac{\alpha}{(\alpha+1)}}} + \frac{\pi^\alpha}{2(2N)^{\frac{\alpha}{2(\alpha+1)}}} \right], \quad \forall\ N \in \mathbb{N}. \tag{4.30}$$

Proof. By Theorem 3.12. ☐

Theorem 4.15. *Let $\alpha \in (0,1]$, $n \in \mathbb{N}$. Suppose f is n times conformable α-fractional differentiable on $[0,\pi]$, and $D_\alpha^n f$ is continuous on $[0,\pi]$. For a fixed $x \in [0,\pi] : x \neq \frac{k\pi}{N}$, $k \in \{0,1,...,N\}$, $\forall$ $N \in \mathbb{N}$, we have $D_\alpha^k f(x) = 0$, $k = 1,...,n$. Then*

$$\left| W_N^{(M)}(f)(x) - f(x) \right| \leq \frac{\omega_1^\alpha \left(D_\alpha^n f, \frac{\pi^\alpha}{(2N)^{\frac{\alpha}{(n+1)(\alpha+1)}}} \right)}{\alpha^n n!} \cdot$$

$$\left[\frac{\pi^{n\alpha}}{(2N)^{\frac{\alpha}{(\alpha+1)}}} + \frac{\pi^{n\alpha}}{(n+1)(2N)^{\frac{n\alpha}{(n+1)(\alpha+1)}}} \right], \quad \forall\ N \in \mathbb{N}. \tag{4.31}$$

Proof. By Theorem 3.15. ☐

Note. (i) if $x = \frac{j\pi}{N}$, $j \in \{0,...,N\}$, then the left hand sides of (4.30) and (4.31) are zero, so these inequalities are trivially valid.

(ii) from (4.30) and/or (4.31), as $N \to +\infty$, we get that $W_N^{(M)}(f)(x) \to f(x)$.

We make

Remark 4.16. Here we continue with the Max-product truncated sampling operators (see [4], p. 13) defined by

$$K_N^{(M)}(f)(x) = \frac{\bigvee_{k=0}^N \frac{\sin^2(Nx-k\pi)}{(Nx-k\pi)^2} f\left(\frac{k\pi}{N}\right)}{\bigvee_{k=0}^N \frac{\sin^2(Nx-k\pi)}{(Nx-k\pi)^2}}, \tag{4.32}$$

$\forall$ $x \in [0,\pi]$, $f : [0,\pi] \to \mathbb{R}_+$ a continuous function.

Following [4], p. 350, and making the convention $\frac{\sin(0)}{0} = 1$ and denoting

$$s_{N,k}(x) = \frac{\sin^2(Nx - k\pi)}{(Nx - k\pi)^2},$$

we get that $s_{N,k}\left(\frac{k\pi}{N}\right)=1$, and $s_{N,k}\left(\frac{j\pi}{N}\right)=0$, if $k\neq j$, furthermore

$$K_N^{(M)}(f)\left(\frac{j\pi}{N}\right)=f\left(\frac{j\pi}{N}\right),$$

for all $j\in\{0,...,N\}$.

Since $s_{N,j}\left(\frac{j\pi}{N}\right)=1$ it follows that

$$\bigvee_{k=0}^{N} s_{N,k}\left(\frac{j\pi}{N}\right)\geq 1>0,$$

for all $j\in\{0,1,...,N\}$. Hence $K_N^{(M)}(f)$ is well-defined function for all $x\in[0,\pi]$, and it is continuous on $[0,\pi]$, also $K_N^{(M)}(1)=1$. By [4], p. 350, $K_N^{(M)}$ are positive sublinear operators.

Denote $x_{N,k}:=\frac{k\pi}{N}$, $k\in\{0,1,...,N\}$.

By [4], p. 352, we have

$$K_N^{(M)}(|\cdot-x|)(x)\leq\frac{\pi}{2N},\quad\forall\ N\in\mathbb{N},\ \forall\ x\in[0,\pi]. \tag{4.33}$$

Notice also $|x_{N,k}-x|\leq\pi$, $\forall\ x\in[0,\pi]$.

Therefore ($\lambda\geq 1$) it holds

$$K_N^{(M)}\left(|\cdot-x|^{\lambda}\right)(x)\leq\frac{\pi^{\lambda-1}\pi}{2N}=\frac{\pi^{\lambda}}{2N},\quad\forall\ x\in[0,\pi],\ \forall\ N\in\mathbb{N}. \tag{4.34}$$

If $x\in\left(\frac{j\pi}{N},\frac{(j+1)\pi}{N}\right)$, with $j\in\{0,1,...,N\}$, we obtain $nx-j\pi\in(0,\pi)$ and thus

$$s_{N,j}(x)=\frac{\sin^2(Nx-j\pi)}{(Nx-j\pi)^2}>0,$$

see [4], pp. 350.

Consequently it holds ($\lambda\geq 1$)

$$K_N^{(M)}\left(|\cdot-x|^{\lambda}\right)(x)=\frac{\bigvee_{k=0}^{N}s_{N,k}(x)\,|x_{N,k}-x|^{\lambda}}{\bigvee_{k=0}^{N}s_{N,k}(x)}>0,\quad\forall\ x\in[0,\pi], \tag{4.35}$$

such that $x\neq x_{N,k}$, for any $k\in\{0,1,...,N\}$.

We give

Theorem 4.17. *Let $\alpha\in(0,1]$, f is α-conformable fractional differentiable on $[0,\pi]$. $D_\alpha f$ is continuous on $[0,\pi]$. Let $x\in[0,\pi]$ be such that $x\neq\frac{k\pi}{N}$, $k\in\{0,1,...,N\}$, $\forall$ $N\in\mathbb{N}$, and $D_\alpha f(x)=0$. Then*

$$\left|K_N^{(M)}(f)(x)-f(x)\right|\leq\frac{\omega_1^{\alpha}\left(D_\alpha f,\left(\frac{\pi^{2(\alpha+1)}}{2N}\right)^{\frac{\alpha}{2(\alpha+1)}}\right)}{\alpha}$$

$$\cdot\left[\left(\frac{\pi^{\alpha+1}}{2N}\right)^{\frac{\alpha}{\alpha+1}}+\frac{1}{2}\left(\frac{\pi^{2(\alpha+1)}}{2N}\right)^{\frac{\alpha}{2(\alpha+1)}}\right]$$

$$= \frac{\omega_1^\alpha \left(D_\alpha f, \frac{\pi^\alpha}{(2N)^{\frac{\alpha}{2(\alpha+1)}}} \right)}{\alpha} \left[\frac{\pi^\alpha}{(2N)^{\frac{\alpha}{(\alpha+1)}}} + \frac{\pi^\alpha}{2(2N)^{\frac{\alpha}{2(\alpha+1)}}} \right], \quad \forall\, N \in \mathbb{N}. \tag{4.36}$$

Proof. By Theorem 3.12. □

Theorem 4.18. *Let $\alpha \in (0,1]$, $n \in \mathbb{N}$. Suppose f is n times conformable α-fractional differentiable on $[0,\pi]$, and $D_\alpha^n f$ is continuous on $[0,\pi]$. For a fixed $x \in [0,\pi] : x \neq \frac{k\pi}{N}$, $k \in \{0,1,...,N\}$, $\forall\, N \in \mathbb{N}$, we have $D_\alpha^k f(x) = 0$, $k = 1,...,n$. Then*

$$\left| K_N^{(M)}(f)(x) - f(x) \right| \leq \frac{\omega_1^\alpha \left(D_\alpha^n f, \frac{\pi^\alpha}{(2N)^{\frac{\alpha}{(n+1)(\alpha+1)}}} \right)}{\alpha^n n!}$$

$$\cdot \left[\frac{\pi^{n\alpha}}{(2N)^{\frac{\alpha}{(\alpha+1)}}} + \frac{\pi^{n\alpha}}{(n+1)(2N)^{\frac{n\alpha}{(n+1)(\alpha+1)}}} \right], \quad \forall\, N \in \mathbb{N}. \tag{4.37}$$

Proof. By Theorem 3.15. □

Note. (i) if $x = \frac{j\pi}{N}$, $j \in \{0,...,N\}$, then the left hand sides of (4.36) and (4.37) are zero, so these inequalities are trivially valid.

(ii) from (4.36) and/or (4.37), as $N \to +\infty$, we get that $K_N^{(M)}(f)(x) \to f(x)$.

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Generalizations of some fractional integral inequalities for m-convex functions via generalized Mittag-Leffler function

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Abstract. In this paper we are interested to present some general fractional integral inequalities for m-convex functions by involving generalized Mittag-Leffler function. In particular we produce inequalities for several kinds of fractional integrals. Also these inequalities have some connections with known integral inequalities.

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1. Introduction

Inequalities play an essential role in mathematical and other kinds of analysis, specially inequalities involving derivative and integral of functions are of great interest for researchers.

Convex functions are very special in the study of functions defined on real line, a lot of results, in particular inequalities in mathematical analysis based on their invention. A convex function $f : I \to \mathbb{R}$ is also equivalently defined by the Hadamard inequality

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(t)dt \leq \frac{f(a)+f(b)}{2}$$

where $a, b \in I$, $a < b$.

A close generalized form of convex functions is m-convex functions introduced by Toader [23].

Definition 1.1. A function $f : [0, b] \to \mathbb{R}$, $b > 0$ is said to be m-convex function if for all $x, y \in [0, b]$ and $t \in [0, 1]$

$$f(tx + m(1-t)y) \leq tf(x) + m(1-t)f(y)$$

holds for $m \in [0,1]$.

Every m-convex function is not convex function.

Example 1.2. [16] Let $f : [0,\infty] \to \mathbb{R}$ be defined by

$$g(t) = \frac{1}{12}(x^4 - 5x^3 + 9x^2 - 5x)$$

is $\frac{16}{17}$-convex function but it is not convex function.

For $m = 1$ the above definition becomes the definition of convex functions defined on $[0,b]$. If we take $m = 0$, then we obtain the concept of starshaped functions on $[0,b]$. A function $f : [0,b] \to \mathbb{R}$ is said to be starshaped if $f(tx) \leq tf(x)$ for all $t \in [0,1]$ and $x \in [0,b]$.
If set of m-convex functions on $[0,b]$ for which $f(0) < 0$ is denoted by $K_m(b)$, then we have

$$K_1(b) \subset K_m(b) \subset K_0(b)$$

whenever $m \in (0,1)$. In the class $K_1(b)$ there are convex functions $f : [0,b] \to \mathbb{R}$ for which $f(0) \leq 0$ (see, [2]). There are a number of results and inequalities obtained via m-convex functions for detail (see [2, 4, 7, 10]).
Recently, a number of authors are taking keen interest to obtain integral inequalities of the Hadamard type via fractional integral operators of different kinds in the various field of fractional calculus. For example one can see [5, 6, 11, 15, 17, 20, 22].

2. Preliminaries in fractional calculus and integral operators

Fractional calculus deals with the study of integral and differential operators of non-integral order. Many mathematicians like Liouville, Riemann and Weyl made major contributions to the theory of fractional calculus. The study on the fractional calculus continued with contributions from Fourier, Abel, Lacroix, Leibniz, Grunwald and Letnikov. For detail (see, [11, 13, 15]). Riemann-Liouville fractional integral operator is the first formulation of an integral operator of non-integral order.

Definition 2.1. [24] Let $f \in L_1[a,b]$. Then Riemann-Liouville fractional integral of f of order ν is defined by

$$I^{\nu}_{a^+} f(x) = \frac{1}{\Gamma(\nu)} \int_a^x (x-t)^{\nu-1} f(t)dt, \ x > a$$

and

$$I^{\nu}_{b\_} f(x) = \frac{1}{\Gamma(\nu)} \int_x^b (t-x)^{\nu-1} f(t)dt, \ x < b.$$

In fact these formulations of fractional integral operators have been established due to Letnikov [14], Sonin [21] and then by Laurent [12]. In these days a variety of fractional integral operators have been produced and many are under discussion. A number of generalized fractional integral operators are also very useful in generalizing the theory of fractional integral operators [1, 11, 15, 18, 22, 24].

Definition 2.2. [18] Let μ, ν, k, l, γ be positive real numbers and $\omega \in \mathbb{R}$. Then the generalized fractional integral operators containing Mittag-Leffler function $\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,a^+}$ and $\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,b_-}$ for a real valued continuous function f is defined by:

$$\left(\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,a^+} f\right)(x) = \int_a^x (x-t)^{\nu-1} E^{\gamma,\delta,k}_{\mu,\nu,l}(\omega(x-t)^{\mu}) f(t)dt, \tag{2.1}$$

and

$$\left(\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,b_-} f\right)(x) = \int_x^b (t-x)^{\nu-1} E^{\gamma,\delta,k}_{\mu,\nu,l}(\omega(t-x)^{\mu}) f(t)dt,$$

where the function $E^{\gamma,\delta,k}_{\mu,\nu,l}$ is generalized Mittag-Leffler function defined as

$$E^{\gamma,\delta,k}_{\mu,\nu,l}(t) = \sum_{n=0}^{\infty} \frac{(\gamma)_{kn} t^n}{\Gamma(\mu n+\nu)(\delta)_{ln}}, \tag{2.2}$$

$(a)_n$ is the Pochhammer symbol, it defined as

$$(a)_n = a(a+1)(a+2)...(a+n-1), \ (a)_0 = 1.$$

If $\delta = l = 1$ in (2.1), then integral operator $\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,a^+}$ reduces to an integral operator $\epsilon^{\gamma,1,k}_{\mu,\nu,1,\omega,a^+}$ containing generalized Mittag-Leffler function $E^{\gamma,1,k}_{\mu,\nu,1}$ introduced by Srivastava and Tomovski in [22]. Along with $\delta = l = 1$ in addition if $k = 1$ then (2.1) reduces to an integral operator defined by Prabhaker in [17] containing Mittag-Leffler function $E^{\gamma}_{\mu,\nu}$. For $\omega = 0$ in (2.1), integral operator $\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,a^+}$ reduces to the Riemann-Liouville fractional integral operator [18].

In [18, 22] properties of generalized integral operator and generalized Mittag-Leffler functions are studied in details. In [18] it is proved that $E^{\gamma,\delta,k}_{\mu,\nu,l}(t)$ is absolutely convergent for $k < l + \mu$. Let S be the sum of series of absolute terms of $E^{\gamma,\delta,k}_{\mu,\nu,l}(t)$. We will use this property of Mittag-Leffler function in sequal.

Now a days a number of authors are working on inequalities involving fractional integral operators and generalized fractional integral operators for example Riemann-Liouville, Caputo, Hilfer, Canvati etc [8, 20]. Actually, fractional integral inequalities are very useful to find the uniqueness of solutions for partial differential equations of non-integral order. In this paper we give some fractional integral inequalities for m-convex functions by involving generalized Mittag-Leffler function. Also we deduce some main results of [3, 9, 19].

3. Fractional integral inequalities

First we prove the following lemma which would be helpful to obtain the main results.

Lemma 3.1. *Let $f : I \to \mathbb{R}$ be a differentiable mapping on I, $a, b \in I$ with $0 \le a < b$ and also let $g : [a, mb] \to \mathbb{R}$ be a continuous function on $[a, mb]$. If $f', g \in L[a, mb]$,* *then the following equality holds for* $\nu > 0$

$$\begin{aligned}
&\left(\int_a^{mb} g(s)E^{\gamma,\delta,k}_{\mu,\nu,l}(\omega s^{\mu})ds\right)^{\nu}[f(a)+f(mb)] \qquad (3.1)\\
&-\nu\int_a^{mb}\left(\int_a^{t} g(s)E^{\gamma,\delta,k}_{\mu,\nu,l}(\omega s^{\mu})ds\right)^{\nu-1} g(t)E^{\gamma,\delta,k}_{\mu,\nu,l}(\omega t^{\mu})f(t)dt\\
&-\nu\int_a^{mb}\left(\int_t^{mb} g(s)E^{\gamma,\delta,k}_{\mu,\nu,l}(\omega s^{\mu})ds\right)^{\nu-1} g(t)E^{\gamma,\delta,k}_{\mu,\nu,l}(\omega t^{\mu})f(t)dt\\
&=\int_a^{mb}\left(\int_a^{t} g(s)E^{\gamma,\delta,k}_{\mu,\nu,l}(\omega s^{\mu})ds\right)^{\nu} f'(t)dt\\
&-\int_a^{mb}\left(\int_t^{mb} g(s)E^{\gamma,\delta,k}_{\mu,\nu,l}(\omega s^{\alpha})ds\right)^{\nu} f'(t)dt
\end{aligned}$$

where $E^{\gamma,\delta,k}_{\mu,\nu,l}$ *is generalized Mittag-Leffler function.*

Proof. One can have on integrating by parts

$$\begin{aligned}
&\int_a^{mb}\left(\int_a^{t} g(s)E^{\gamma,\delta,k}_{\mu,\nu,l}(\omega s^{\mu})ds\right)^{\nu} f'(t)dt \qquad (3.2)\\
&=\left(\int_a^{mb} g(s)E^{\gamma,\delta,k}_{\mu,\nu,l}(\omega s^{\mu})ds\right)^{\nu} f(mb)\\
&-\nu\int_a^{mb}\left(\int_a^{t} g(s)E^{\gamma,\delta,k}_{\mu,\nu,l}(\omega s^{\mu})ds\right)^{\nu-1} g(t)E^{\gamma,\delta,k}_{\mu,\nu,l}(\omega t^{\mu})f(t)dt.
\end{aligned}$$

And likewise

$$\begin{aligned}
&\int_a^{mb}\left(\int_t^{mb} g(s)E^{\gamma,\delta,k}_{\mu,\nu,l}(\omega s^{\mu})ds\right)^{\nu} f'(t)dt \qquad (3.3)\\
&=-\left(\int_a^{mb} g(s)E^{\gamma,\delta,k}_{\mu,\nu,l}(\omega s^{\mu})ds\right)^{\nu} f(a)\\
&+\nu\int_a^{mb}\left(\int_t^{mb} g(s)E^{\gamma,\delta,k}_{\mu,\nu,l}(\omega s^{\mu})ds\right)^{\nu-1} g(t)E^{\gamma,\delta,k}_{\mu,\nu,l}(\omega t^{\mu})f(t)dt.
\end{aligned}$$

On substracting equation (3.3) from (3.2), we get the result. □

We use Lemma 3.1 to establish the following fractional integral inequality.

Theorem 3.2. *Let* $f : I \to \mathbb{R}$ *be a differentiable mapping on* I, $a, b \in I$ *with* $0 \leq a < b$ *and also let* $g : [a, mb] \to \mathbb{R}$ *be a continuous function on* $[a, mb]$. *If* $|f'|$ *is* m*-convex* *function on* $[a, mb]$*, then the following inequality holds*

$$\left|\left(\int_a^{mb} g(s)E_{\mu,\nu,l}^{\gamma,\delta,k}(\omega s^{\mu})ds\right)^{\nu}(f(a)+f(mb))\right.$$
$$-\nu\int_a^{mb}\left(\int_a^{t} g(s)E_{\mu,\nu,l}^{\gamma,\delta,k}(\omega s^{\mu})ds\right)^{\nu-1} g(t)E_{\mu,\nu,l}^{\gamma,\delta,k}(\omega t^{\mu})f(t)dt$$
$$\left.-\nu\int_a^{mb}\left(\int_t^{mb} g(s)E_{\mu,\nu,l}^{\gamma,\delta,k}(\omega s^{\mu})ds\right)^{\nu-1} g(t)E_{\mu,\nu,l}^{\gamma,\delta,k}(\omega t^{\mu})f(t)dt\right|$$
$$\leq \frac{(mb-a)^{\nu+1}\|g\|_{\infty}^{\nu}S^{\nu}}{\nu+1}(|f'(a)|+m|f'(b)|)$$

for $k < l+\mu$*, where* $\| g \|_{\infty}= \sup_{t\in[a,b]} |g(t)|$.

Proof. By using Lemma 3.1, we have

$$\left|\left(\int_a^{mb} g(s)E_{\mu,\nu,l}^{\gamma,\delta,k}(\omega s^{\mu})ds\right)^{\nu}(f(a)+f(mb))\right. \tag{3.4}$$
$$-\nu\int_a^{mb}\left(\int_a^{t} g(s)E_{\mu,\nu,l}^{\gamma,\delta,k}(\omega s^{\mu})ds\right)^{\nu-1} g(t)E_{\mu,\nu,l}^{\gamma,\delta,k}(\omega t^{\mu})f(t)dt$$
$$\left.-\nu\int_a^{mb}\left(\int_t^{mb} g(s)E_{\mu,\nu,l}^{\gamma,\delta,k}(\omega s^{\mu})ds\right)^{\nu-1} g(t)E_{\mu,\nu,l}^{\gamma,\delta,k}(\omega t^{\mu})f(t)dt\right|$$
$$\leq \int_a^{mb}\left|\int_a^{t} g(s)E_{\mu,\nu,l}^{\gamma,\delta,k}(\omega s^{\mu})ds\right|^{\nu}|f'(t)|dt$$
$$+\int_a^{mb}\left|\int_t^{mb} g(s)E_{\mu,\nu,l}^{\gamma,\delta,k}(\omega s^{\mu})ds\right|^{\nu}|f'(t)|dt.$$

By using $\| g \|_{\infty}= \sup_{t\in[a,b]} |g(t)|$ and absolute convergence of Mittag-Leffler function, we have

$$\left|\left(\int_a^{mb} g(s)E_{\mu,\nu,l}^{\gamma,\delta,k}(\omega s^{\mu})ds\right)^{\nu}(f(a)+f(mb))\right. \tag{3.5}$$
$$-\nu\int_a^{mb}\left(\int_a^{t} g(s)E_{\mu,\nu,l}^{\gamma,\delta,k}(\omega s^{\mu})ds\right)^{\nu-1} g(t)E_{\mu,\nu,l}^{\gamma,\delta,k}(\omega t^{\mu})f(t)dt$$
$$\left.-\nu\int_a^{mb}\left(\int_t^{mb} g(s)E_{\mu,\nu,l}^{\gamma,\delta,k}(\omega s^{\mu})ds\right)^{\nu-1} g(t)E_{\mu,\nu,l}^{\gamma,\delta,k}(\omega t^{\mu})f(t)dt\right|$$
$$\leq \|g\|_{\infty}^{\nu}S^{\nu}\left(\int_a^{mb}(t-a)^{\nu}|f'(t)|dt+\int_a^{mb}(mb-t)^{\nu}|f'(t)|dt\right).$$

Since $|f'|$ is m-convex function, therefore it can be written as

$$|f'(t)| \leq \frac{mb-t}{mb-a}|f'(a)| + \frac{m(t-a)}{mb-a}|f'(b)| \tag{3.6}$$

for $t \in [a, mb]$.
Using (3.6) in (3.5), we have

$$\begin{aligned} &\Bigg| \left(\int_a^{mb} g(s) E_{\mu,\nu,l}^{\gamma,\delta,k}(\omega s^{\mu}) ds \right)^{\nu} (f(a) + f(mb)) \\ &-\nu \int_a^{mb} \left(\int_a^{t} g(s) E_{\mu,\nu,l}^{\gamma,\delta,k}(\omega s^{\mu}) ds \right)^{\nu-1} g(t) E_{\mu,\nu,l}^{\gamma,\delta,k}(\omega t^{\mu}) f(t) dt \\ &-\nu \int_a^{mb} \left(\int_t^{mb} g(s) E_{\mu,\nu,l}^{\gamma,\delta,k}(\omega s^{\mu}) ds \right)^{\mu-1} g(t) E_{\mu,\nu,l}^{\gamma,\delta,k}(\omega t^{\mu}) f(t) dt \Bigg| \\ &\leq \|g\|_{\infty}^{\nu} S^{\nu} \left(\int_a^{mb} (t-a)^{\nu} \left(\frac{mb-t}{mb-a}|f'(a)| + \frac{m(t-a)}{mb-a}|f'(b)| \right) dt \right. \\ &\left. + \int_a^{mb} (mb-t)^{\nu} \left(\frac{mb-t}{mb-a}|f'(a)| + \frac{m(t-a)}{mb-a}|f'(b)| \right) dt \right). \end{aligned} \tag{3.7}$$

After simplification of above inequality we get the result. □

Remark 3.3. By taking particular values of parameters used in Mittag-Leffler function in above theorem several fractional integral inequalities can be obtained for corresponding fractional integrals. For example see the following results.

Corollary 3.4. *If we take $m = 1$ in Theorem 3.2, then we get the following inequality*

$$\begin{aligned} &\Bigg| \left(\int_a^{b} g(s) E_{\mu,\nu,l}^{\gamma,\delta,k}(\omega s^{\mu}) ds \right)^{\nu} (f(a) + f(b)) \\ &-\nu \int_a^{b} \left(\int_a^{t} g(s) E_{\mu,\nu,l}^{\gamma,\delta,k}(\omega s^{\mu}) ds \right)^{\nu-1} g(t) E_{\mu,\nu,l}^{\gamma,\delta,k}(\omega t^{\mu}) f(t) dt \\ &-\nu \int_a^{b} \left(\int_t^{b} g(s) E_{\mu,\nu,l}^{\gamma,\delta,k}(\omega s^{\mu}) ds \right)^{\nu-1} g(t) E_{\mu,\nu,l}^{\gamma,\delta,k}(\omega t^{\mu}) f(t) dt \Bigg| \\ &\leq \frac{(b-a)^{\nu+1} \|g\|_{\infty}^{\nu} S^{\nu}}{\nu+1} (|f'(a)| + |f'(b)|). \end{aligned}$$

Remark 3.5. In Theorem 3.2, for $m = 1$.
(i) If we put $\omega = 0$, then we get [19, Theorem 6].
(ii) If we take $\omega = 0$, $\nu = \frac{\mu}{k}$ and $g(s) = 1$, then we get [9, Corollary 2.3].
(iii) For $g(s) = 1$ along with $\omega = 0$ and $\nu = \mu$, then we get [19, Corollary 2].

Next we give another fractional integral inequality.

Theorem 3.6. *Let $f : I \to \mathbb{R}$ be a differentiable mapping on I, $a, b \in I$ with $0 \leq a < b$ and also let $g : [a, mb] \to \mathbb{R}$ be a continuous function on $[a, mb]$. If $|f'|^q$ is m-convex function on $[a, mb]$ for $q > 1$, then the following inequality holds*

$$\begin{aligned}&\left|\left(\int_a^{mb} g(s)E_{\mu,\nu,l}^{\gamma,\delta,k}(\omega s^\mu)ds\right)^\nu (f(a)+f(mb))\right.\\&-\nu\int_a^{mb}\left(\int_a^t g(s)E_{\mu,\nu,l}^{\gamma,\delta,k}(\omega s^\mu)ds\right)^{\nu-1} g(t)E_{\mu,\nu,l}^{\gamma,\delta,k}(\omega t^\mu)f(t)dt\\&\left.-\nu\int_a^{mb}\left(\int_t^{mb} g(s)E_{\mu,\nu,l}^{\gamma,\delta,k}(\omega s^\mu)ds\right)^{\nu-1} g(t)E_{\mu,\nu,l}^{\gamma,\delta,k}(\omega t^\mu)f(t)dt\right|\\&\leq \frac{2(mb-a)^{\nu+1}\|g\|_\infty^\nu S^\nu}{(\nu p+1)^{\frac{1}{q}}}\left(\frac{|f'(a)|^q+m|f'(b)|^q}{2}\right)^{\frac{1}{q}}\end{aligned}$$

for $k < l + \mu$, where $\| g \|_\infty = \sup_{t\in[a,b]} |g(t)|$ and $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. By using Lemma 3.1, we have

$$\begin{aligned}&\left|\left(\int_a^{mb} g(s)E_{\mu,\nu,l}^{\gamma,\delta,k}(\omega s^\mu)ds\right)^\nu (f(a)+f(mb))\right.\\&-\nu\int_a^{mb}\left(\int_a^t g(s)E_{\mu,\nu,l}^{\gamma,\delta,k}(\omega s^\mu)ds\right)^{\nu-1} g(t)E_{\mu,\nu,l}^{\gamma,\delta,k}(\omega t^\mu)f(t)dt\\&\left.-\nu\int_a^{mb}\left(\int_t^{mb} g(s)E_{\mu,\nu,l}^{\gamma,\delta,k}(\omega s^\mu)ds\right)^{\nu-1} g(t)E_{\mu,\nu,l}^{\gamma,\delta,k}(\omega t^\mu)f(t)dt\right|\\&\leq \int_a^{mb}\left|\int_a^t g(s)E_{\mu,\nu,l}^{\gamma,\delta,k}(\omega s^\mu)ds\right|^\nu |f'(t)|dt\\&+\int_a^{mb}\left|\int_t^{mb} g(s)E_{\mu,\nu,l}^{\gamma,\delta,k}(\omega s^\mu)ds\right|^\nu |f'(t)|dt.\end{aligned} \tag{3.8}$$

Using Hölder inequality, we have

$$\begin{aligned}&\left|\left(\int_a^{mb} g(s)E_{\mu,\nu,l}^{\gamma,\delta,k}(\omega s^\mu)ds\right)^\nu (f(a)+f(mb))\right.\\&-\nu\int_a^{mb}\left(\int_a^t g(s)E_{\mu,\nu,l}^{\gamma,\delta,k}(\omega s^\mu)ds\right)^{\nu-1} g(t)E_{\mu,\nu,l}^{\gamma,\delta,k}(\omega t^\mu)f(t)dt\\&\left.-\nu\int_a^{mb}\left(\int_t^{mb} g(s)E_{\mu,\nu,l}^{\gamma,\delta,k}(\omega s^\mu)ds\right)^{\nu-1} g(t)E_{\mu,\nu,l}^{\gamma,\delta,k}(\omega t^\mu)f(t)dt\right|\end{aligned} \tag{3.9}$$

$$\leq \left(\int_a^{mb}\left|\int_a^t g(s)E_{\mu,\nu,l}^{\gamma,\delta,k}(\omega s^{\mu})ds\right|^{\nu p}dt\right)^{\frac{1}{p}}\left(\int_a^{mb}|f'(t)|^q dt\right)^{\frac{1}{q}}$$
$$+\left(\int_a^{mb}\left|\int_t^{mb} g(s)E_{\mu,\nu,l}^{\gamma,\delta,k}(\omega s^{\mu})ds\right|^{\nu p}dt\right)^{\frac{1}{p}}\left(\int_a^{mb}|f'(t)|^q dt\right)^{\frac{1}{q}}. \tag{3.10}$$

By using $\| g \|_{\infty}= \sup_{t\in[a,b]} |g(t)|$ and absolute convergence of Mittag-Leffler function, we have

$$\left|\left(\int_a^{mb} g(s)E_{\mu,\nu,l}^{\gamma,\delta,k}(\omega s^{\mu})ds\right)^{\nu}(f(a)+f(mb))\right. \tag{3.11}$$
$$-\nu\int_a^{mb}\left(\int_a^t g(s)E_{\mu,\nu,l}^{\gamma,\delta,k}(\omega s^{\mu})ds\right)^{\nu-1} g(t)E_{\mu,\nu,l}^{\gamma,\delta,k}(\omega t^{\mu})f(t)dt$$
$$\left.-\nu\int_a^{mb}\left(\int_t^{mb} g(s)E_{\mu,\nu,l}^{\gamma,\delta,k}(\omega s^{\mu})ds\right)^{\nu-1} g(t)E_{\mu,\nu,l}^{\gamma,\delta,k}(\omega t^{\mu})f(t)dt\right|$$
$$\leq \|g\|_{\infty}^{\nu}S^{\nu}\left(\left(\int_a^{mb}|t-a|^{\nu p}dt\right)^{\frac{1}{p}}\right.$$
$$\left.+\left(\int_a^{mb}|mb-t|^{\nu p}dt\right)^{\frac{1}{p}}\right)\left(\int_a^{mb}|f'(t)|^q dt\right)^{\frac{1}{q}}.$$

Since $|f'(t)|^q$ is m-convex, we have

$$|f'(t)|^q \leq \frac{mb-t}{mb-a}|f'(a)|^q + \frac{m(t-a)}{mb-a}|f'(b)|^q. \tag{3.12}$$

Using (3.12) in (3.11), we have

$$\left|\left(\int_a^{mb} g(s)E_{\mu,\nu,l}^{\gamma,\delta,k}(\omega s^{\mu})ds\right)^{\nu}(f(a)+f(mb))\right.$$
$$-\nu\int_a^{mb}\left(\int_a^t g(s)E_{\mu,\nu,l}^{\gamma,\delta,k}(\omega s^{\mu})ds\right)^{\nu-1} g(t)E_{\mu,\nu,l}^{\gamma,\delta,k}(\omega t^{\mu})f(t)dt$$
$$\left.-\nu\int_a^{mb}\left(\int_t^{mb} g(s)E_{\mu,\nu,l}^{\gamma,\delta,k}(\omega s^{\mu})ds\right)^{\nu-1} g(t)E_{\mu,\nu,l}^{\gamma,\delta,k}(\omega t^{\mu})f(t)dt\right| \tag{3.13}$$
$$\leq \|g\|_{\infty}^{\nu}S^{\nu}\left(\left(\int_a^{mb}|t-a|^{\nu p}dt\right)^{\frac{1}{p}}+\left(\int_a^{mb}|mb-t|^{\nu p}dt\right)^{\frac{1}{p}}\right)$$
$$\times\left(\int_a^{mb}\frac{mb-t}{mb-a}|f'(a)|^q+\frac{m(t-a)}{mb-a}|f'(b)|^q\right)^{\frac{1}{q}}.$$

After a simple calculation, we get the required result. □

Remark 3.7. It is remarkable that by taking particular values of parameters of Mittag-Leffler function in above theorem several fractional integral inequalities can be obtained for corresponding fractional integrals. For example some results are given below.

Corollary 3.8. *In Theorem* 3.6 *if we take* $m = 1$*, then we have the following integral inequality*

$$\begin{aligned}&\Bigg|\left(\int_a^b g(s)E_{\mu,\nu,l}^{\gamma,\delta,k}(\omega s^\mu)ds\right)^\nu (f(a)+f(b))\\&-\nu\int_a^b\left(\int_a^t g(s)E_{\mu,\nu,l}^{\gamma,\delta,k}(\omega s^\mu)ds\right)^{\nu-1} g(t)E_{\mu,\nu,l}^{\gamma,\delta,k}(\omega t^\mu)f(t)dt\\&-\nu\int_a^b\left(\int_t^b g(s)E_{\mu,\nu,l}^{\gamma,\delta,k}(\omega s^\mu)ds\right)^{\nu-1} g(t)E_{\mu,\nu,l}^{\gamma,\delta,k}(\omega t^\mu)f(t)dt\Bigg|\\&\leq \frac{2(b-a)^{\nu+1}\|g\|_\infty^\nu S^\nu}{(\nu p+1)^{\frac{1}{q}}}\left(\frac{|f'(a)|^q+|f'(b)|^q}{2}\right)^{\frac{1}{q}}.\end{aligned}$$

Remark 3.9. In Theorem 3.6, for $m = 1$.
(i) If we put $\omega = 0$, then we get [19, Theorem 7].
(ii) If we take $\omega = 0$ along with $\nu = \frac{\mu}{k}$, then we get [9, Theorem 2.5].
(iii) If we take $g(s) = 1$ and $\omega = 0$, then we get [3, Theorem 2.3].
(iv) If we put $\omega = 0$ and $\nu = 1$, then we get [3, Corollary 3].

In the next result we give the Hadamard type inequalities for m-convex functions via generalized fractional integral operator containing generalized Mittag-Leffler function.

Theorem 3.10. *Let* $f : [a, mb] \to \mathbb{R}$ *be a positive function with* $0 \leq a < b$ *and* $f \in L[a, mb]$*. If* f *is* m*-convex function, then the following inequalities for generalized fractional integral hold*

$$\begin{aligned}&f\left(\frac{a+mb}{2}\right)\left(\epsilon_{\mu,\nu,l,\omega',(\frac{a+mb}{2})^+}^{\gamma,\delta,k}1\right)(mb)\\&\leq\left(\epsilon_{\mu,\nu,l,\omega',(\frac{a+mb}{2})^+}^{\gamma,\delta,k}f\right)(mb)+\left(\epsilon_{\mu,\nu,l,m^\mu\omega',(\frac{a+mb}{2m})^-}^{\gamma,\delta,k}f\right)\left(\frac{a}{m}\right)\\&\leq\frac{1}{mb-a}\left[f(a)-mf\left(\frac{a}{m^2}\right)\right]\left(\epsilon_{\mu,\nu+1,l,\omega',(\frac{a+mb}{2})^+}^{\gamma,\delta,k}1\right)(mb)\\&+m^{\nu+1}\left(f(b)+mf\left(\frac{a}{m^2}\right)\right)\left(\epsilon_{\mu,\nu,l,m^\mu\omega',(\frac{a+mb}{2m})^+}^{\gamma,\delta,k}1\right)\left(\frac{a}{m}\right)\end{aligned}$$

where $\omega' = \frac{2^\mu\omega}{(mb-a)^\mu}$.

Proof. Using m-convexity of f, we have

$$f\left(\frac{x+my}{2}\right)\leq\frac{f(x)+mf(y)}{2} \tag{3.14}$$

for $x, y \in [a, mb]$.
By taking $x = \frac{t}{2}a + \frac{2-t}{2}mb$, $y = \frac{2-t}{2m}a + \frac{t}{2}b$ for $t \in [0,1]$ such that $x, y \in [a, mb]$, inequality (3.14) becomes

$$2f\left(\frac{a+mb}{2}\right) \leq f\left(\frac{t}{2}a + \frac{2-t}{2}mb\right) + mf\left(\frac{2-t}{2m}a + \frac{t}{2}b\right). \tag{3.15}$$

Multiplying both sides of (3.15) by $t^{\nu-1}E^{\gamma,\delta,k}_{\mu,\nu,l}(\omega t^{\mu})$ and integrating with respect to t on $[0,1]$

$$\begin{aligned} &2f\left(\frac{a+mb}{2}\right)\int_0^1 (t^{\nu-1})E^{\gamma,\delta,k}_{\mu,\nu,l}(\omega t^{\mu})dt \\ &\leq \int_0^1 (t^{\nu-1})E^{\gamma,\delta,k}_{\mu,\nu,l}(\omega t^{\mu})f\left(\frac{t}{2}a + \frac{2-t}{2}mb\right)dt \\ &+ m\int_0^1 (t^{\nu-1})E^{\gamma,\delta,k}_{\mu,\nu,l}(\omega t^{\mu})f\left(\frac{2-t}{2m}a + \frac{t}{2}b\right)dt. \end{aligned} \tag{3.16}$$

Setting $u = \frac{t}{2}a + \frac{2-t}{2}mb$ and $v = \frac{2-t}{2m}a + \frac{t}{2}b$ in (3.16), we have

$$\begin{aligned} &2f\left(\frac{a+mb}{2}\right)\int_{\frac{a+mb}{2}}^{mb} (mb-u)^{\nu-1}E^{\gamma,\delta,k}_{\mu,\nu,l}(\omega'(mb-u)^{\mu})du \\ &\leq \int_{\frac{a+mb}{2}}^{mb} (mb-u)^{\nu-1}E^{\gamma,\delta,k}_{\mu,\nu,l}(\omega'(mb-u)^{\mu})f(u)du \\ &+ m^{\nu+1}\int_{\frac{a}{m}}^{\frac{a+mb}{2m}} \left(v-\frac{a}{m}\right)^{\nu-1} E^{\gamma,\delta,k}_{\mu,\nu,l}\left(m^{\mu}\omega'(v-\frac{a}{m})^{\mu}\right)f(v)dv \end{aligned} \tag{3.17}$$

where $\omega' = \frac{2^{\mu}\omega}{(mb-a)^{\mu}}$.
This implies

$$\begin{aligned} &2f\left(\frac{a+mb}{2}\right)\left(\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega',(\frac{a+mb}{2})+}1\right)(mb) \\ &\leq \left(\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega',(\frac{a+mb}{2})+}f\right)(mb) + \left(\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,m^{\mu}\omega',(\frac{a+mb}{2m})_-}f\right)\left(\frac{a}{m}\right). \end{aligned} \tag{3.18}$$

To prove the second inequality from m-convexity of f, we have

$$\begin{aligned} &f\left(\frac{t}{2}a + m\frac{2-t}{2}b\right) + mf\left(\frac{2-t}{2m}a + \frac{t}{2}b\right) \\ &\leq \frac{t}{2}\left(f(a) - m^2 f\left(\frac{a}{m^2}\right)\right) + m\left(f(b) + mf\left(\frac{a}{m^2}\right)\right). \end{aligned} \tag{3.19}$$

Multiplying both sides of (3.19) by $t^{\nu-1}E^{\gamma,\delta,k}_{\alpha,\beta,l}(\omega t^{\alpha})$ and integrating with respect to t over $[0,1]$, we have

$$\int_0^1 t^{\nu-1}E^{\gamma,\delta,k}_{\mu,\nu,l}(\omega t^{\mu})f\left(\frac{t}{2}a+m\frac{2-t}{2}b\right)dt \tag{3.20}$$
$$+m\int_0^1 t^{\nu-1}E^{\gamma,\delta,k}_{\mu,\nu,l}(\omega t^{\mu})f\left(\frac{2-t}{2m}a+\frac{t}{2}b\right)$$
$$\leq \frac{1}{2}\left(f(a)-m^2 f\left(\frac{a}{m^2}\right)\right)\int_0^1 t^{\nu}E^{\gamma,\delta,k}_{\mu,\nu,l}(\omega t^{\mu})dt$$
$$+m\left(f(b)+mf\left(\frac{a}{m^2}\right)\right)\int_0^1 t^{\nu-1}E^{\gamma,\delta,k}_{\mu,\nu,l}(\omega t^{\mu})dt.$$

Setting $u=\frac{t}{2}a+m\frac{2-t}{2}b$ and $v=\frac{2-t}{2m}a+\frac{t}{2}b$ in (3.20), we have

$$\int_{\frac{a+mb}{2}}^{mb}(mb-u)^{\nu-1}E^{\gamma,\delta,k}_{\mu,\nu,l}(\omega'(mb-u)^{\mu})f(u)du \tag{3.21}$$
$$+\int_{\frac{a}{m}}^{\frac{a+mb}{2m}}\left(v-\frac{a}{m}\right)^{\nu-1}E^{\gamma,\delta,k}_{\mu,\nu,l}\left(m^{\mu}\omega'\left(v-\frac{a}{m}\right)^{\mu}\right)f(v)dv$$
$$\leq \frac{1}{2}\left(f(a)-m^2 f\left(\frac{a}{m^2}\right)\right)\int_{\frac{a+mb}{2}}^{mb}(mb-u)^{\nu}E^{\gamma,\delta,k}_{\mu,\nu,l}\left(\omega'(mb-u)^{\mu}\right)dt$$
$$+m^{\nu+1}\left(f(b)+mf\left(\frac{a}{m^2}\right)\right)\int_{\frac{a}{m}}^{\frac{a+mb}{2m}}\left(v-\frac{a}{m}\right)^{\nu-1}E^{\gamma,\delta,k}_{\mu,\nu,l}\left(m^{\mu}\omega'\left(v-\frac{a}{m}\right)^{\mu}\right)dt.$$

This implies

$$\left(\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega',(\frac{a+mb}{2})^+}f\right)(mb)+m^{\nu+1}\left(\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,m^{\mu}\omega',(\frac{a+mb}{2m})_-}f\right)\left(\frac{a}{m}\right) \tag{3.22}$$
$$\leq \frac{1}{mb-a}\left(f(a)-m^2 f\left(\frac{a}{m^2}\right)\right)\left(\epsilon^{\gamma,\delta,k}_{\mu,\nu+1,l,\omega',(\frac{a+mb}{2})^+}1\right)(mb)$$
$$+m^{\nu+1}\left(f(b)+mf\left(\frac{a}{m^2}\right)\right)\left(\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,m^{\mu}\omega',(\frac{a+mb}{2m})_-}1\right)\left(\frac{a}{m}\right).$$

Combining (3.18) and (3.22) we get the result. □

Corollary 3.11. *In Theorem* 3.10 *if we take* $\omega=0$*, then we get the following inequality for Riemann-Liouville fractional integral operator*

$$f\left(\frac{a+mb}{2}\right)\leq \frac{2^{\nu-1}\Gamma(\nu+1)}{(mb-a)^{\mu}}\left(I^{\nu}_{(\frac{a+mb}{2})^+}f(mb)+m^{\nu+1}I^{\nu}_{(\frac{a+mb}{2m})_-}f\left(\frac{a}{m}\right)\right) \tag{3.23}$$
$$\leq \frac{\nu}{4(\nu+1)}\left(f(a)-m^2 f\left(\frac{a}{m^2}\right)\right)+\frac{m}{2}\left(f(b)+mf\left(\frac{a}{m^2}\right)\right).$$

Remark 3.12. If we put $\omega=0$, $m=1$ and $\nu=1$ in Theorem 3.10, then we get the classical Hadamard inequality.

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Inequalities for the area balance of absolutely continuous functions

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Abstract. We introduce the *area balance* function associated to a Lebesgue integrable function $f:[a,b]\to\mathbb{C}$ by

$$AB_f(a,b,\cdot):[a,b]\to\mathbb{C},\ AB_f(a,b,x):=\frac{1}{2}\left[\int_x^b f(t)\,dt-\int_a^x f(t)\,dt\right].$$

We show amongst other that, if $f:I\to\mathbb{C}$ is an absolutely continuous function on the interval I and $[a,b]\subset\mathring{I}$, where $\mathring{I}$ is the interior of I and such that f' is of bounded variation on $[a,b]$, then we have the inequality

$$\left|AB_f(a,b,x)-\left(\frac{a+b}{2}-x\right)f(x)-\frac{f'(a)+f'(b)}{4}\left[\left(x-\frac{a+b}{2}\right)^2+\frac{1}{4}(b-a)^2\right]\right|$$
$$\le\frac{1}{4}\left[\frac{1}{4}(b-a)^2+\left(x-\frac{a+b}{2}\right)^2\right]\bigvee_a^b(f')$$

for any $x\in[a,b]$.

If there exists the real numbers m,M such that

$$m\le f'(t)\le M \text{ for a.e. } t\in[a,b],$$

then also

$$\left|AB_f(a,b,x)-\left(\frac{a+b}{2}-x\right)f(x)-\frac{m+M}{4}\left[\left(x-\frac{a+b}{2}\right)^2+\frac{1}{4}(b-a)^2\right]\right|$$
$$\le\frac{1}{4}\left[\frac{1}{4}(b-a)^2+\left(x-\frac{a+b}{2}\right)^2\right](M-m)$$

for any $x\in[a,b]$.

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1. Introduction

For a *Lebesgue integrable* function $f:[a,b]\to\mathbb{C}$ and a number $x\in(a,b)$ we can naturally ask how far the integral $\int_x^b f(t)\,dt$ is from the integral $\int_a^x f(t)\,dt$. If f is nonnegative and continuous on $[a,b]$, then the above question has the geometrical interpretation of comparing the area under the curve generated by f at the right of the point x with the area at the left of x. The point x will be called a *median point*, if

$$\int_x^b f(t)\,dt=\int_a^x f(t)\,dt.$$

Due to the above geometrical interpretation, we can introduce the *area balance* function associated to a Lebesgue integrable function $f:[a,b]\to\mathbb{C}$ defined as

$$AB_f(a,b,\cdot):[a,b]\to\mathbb{C},\ AB_f(a,b,x):=\frac{1}{2}\left[\int_x^b f(t)\,dt-\int_a^x f(t)\,dt\right].$$

Utilising the *cumulative function* notation $F:[a,b]\to\mathbb{C}$ given by

$$F(x):=\int_a^x f(t)\,dt$$

then we observe that

$$AB_f(a,b,x)=\frac{1}{2}F(b)-F(x),\ \ x\in[a,b].$$

If f is a *probability density*, i.e. f is nonnegative and $\displaystyle\int_a^b f(t)\,dt=1$, then

$$AB_f(a,b,x)=\frac{1}{2}-F(x),\ \ x\in[a,b].$$

In this paper we obtain some inequalities concerning the area balance for absolutely continuous. Applications for differentiable functions whose derivatives are Lipschitzian functions are provided. Bounds involving the *Jensen difference*

$$\frac{g(a)+g(b)}{2}-g\left(\frac{a+b}{2}\right)$$

are also established.

We notice that Jensen difference is closely related to the Hermite-Hadamard type inequalities where various bounds for the quantities

$$\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(t)\,dt$$

and

$$\frac{1}{b-a}\int_a^b f(t)\,dt-f\left(\frac{a+b}{2}\right)$$

are provided, see [1]-[6] and [8]-[18].

2. Preliminary results

The following representation result holds:

Theorem 2.1. *Let $f : [a,b] \to \mathbb{C}$ be an absolutely continuous function on $[a,b]$. Then we have the representation*

$$AB_f(a,b,x) = \left(\frac{a+b}{2} - x\right) f(x) + \frac{1}{2}\left[\int_a^x (t-a) f'(t)\,dt + \int_x^b (b-t) f'(t)\,dt\right] \tag{2.1}$$

and

$$AB_f(a,b,x) = \frac{bf(b) + af(a)}{2} - \frac{f(b)+f(a)}{2}x - \frac{1}{2}\int_a^b |t-x| f'(t)\,dt \tag{2.2}$$

for any $x \in [a,b]$, where the integrals in the right hand side are taken in the Lebesgue sense.

Proof. Since f is absolutely continuous on $[a,b]$, then f is differentiable almost everywhere (a.e.) on $[a,b]$ and the Lebesgue integrals in the right hand side of the equations (2.1) and (2.2) exist.

Utilising the integration by parts formula for the Lebesgue integral, we have

$$\begin{aligned} &\int_a^x (t-a) f'(t)\,dt + \int_x^b (b-t) f'(t)\,dt \\ &= (t-a) f(t)\Big|_a^x - \int_a^x f(t)\,dt + (b-t) f(t)\Big|_x^b + \int_x^b f(t)\,dt \\ &= (x-a) f(x) - \int_a^x f(t)\,dt - (b-x) f(x) + \int_x^b f(t)\,dt \\ &= (2x-a-b) f(x) + 2AB_f(a,b,x) \end{aligned} \tag{2.3}$$

for any $x \in [a,b]$.

Dividing (2.3) by 2 and rearranging the equation, we deduce (2.1).

Integrating by parts, we also have

$$\begin{aligned} &\int_a^b |t-x| f'(t)\,dt \\ &= \int_a^x (x-t) f'(t)\,dt + \int_x^b (t-x) f'(t)\,dt \\ &= (x-t) f(t)\Big|_a^x + \int_a^x f(t)\,dt + (t-x) f(t)\Big|_x^b - \int_x^b f(t)\,dt \\ &= -(x-a) f(a) + (b-x) f(b) - 2AB_f(a,b,x) \\ &= bf(b) + af(a) - [f(b)+f(a)]\,x - 2AB_f(a,b,x) \end{aligned} \tag{2.4}$$

for any $x \in [a,b]$.

Dividing (2.4) by 2 and rearranging the equation, we deduce (2.2). $\square$

Corollary 2.2. *Let $f:[a,b]\to\mathbb{R}$ be an absolutely continuous function on $[a,b]$. If $f'(t)\geq 0$ for a.e. $t\in[a,b]$, then*

$$\begin{aligned}\frac{bf(b)+af(a)}{2}-\frac{f(b)+f(a)}{2}x &\geq AB_f(a,b,x) \\ &\geq \left(\frac{a+b}{2}-x\right)f(x)\end{aligned} \tag{2.5}$$

for any $x\in[a,b]$.

In particular,

$$\frac{1}{4}(b-a)[f(b)-f(a)]\geq AB_f\left(a,b,\frac{a+b}{2}\right)\geq 0. \tag{2.6}$$

The constant $\frac{1}{4}$ is a best possible constant in the sense that it cannot be replaced by a smaller quantity.

Proof. The inequalities (2.5) follow from the representations (2.1) and (2.2) by taking into account that $f'(t)\geq 0$ for a.e. $t\in[a,b]$.

The inequality (2.6) follows by (2.5) for $x=\frac{a+b}{2}$.

Assume that the first inequality in (2.6) holds for a constant $C>0$, i.e.

$$C(b-a)[f(b)-f(a)]\geq AB_f\left(a,b,\frac{a+b}{2}\right) \tag{2.7}$$

Consider the function $f_n:[-1,1]\to\mathbb{R}$ given by

$$f_n(t)=\begin{cases}0 & \text{if } t\in[-1,0] \\ nt & \text{if } t\in\left(0,\frac{1}{n}\right) \\ 1 & \text{if } t\in\left[\frac{1}{n},1\right]\end{cases}$$

where $n\geq 2$, a natural number. This functions is absolutely continuous and $f_n'(t)\geq 0$ for any $t\in(-1,1)$. We have for $a=-1, b=1$

$$C(b-a)[f_n(b)-f_n(a)]=2C$$

and

$$\begin{aligned}AB_{f_n}\left(a,b,\frac{a+b}{2}\right) &= \frac{1}{2}\left[\int_0^1 f_n(t)\,dt-\int_{-1}^0 f_n(t)\,dt\right] \\ &= \frac{1}{2}\left(\int_0^{\frac{1}{n}} nt dt+\int_{\frac{1}{n}}^1 1dt\right) \\ &= \frac{1}{2}\left(\frac{1}{2n}+1-\frac{1}{n}\right)=\frac{1}{2}\left(1-\frac{1}{2n}\right).\end{aligned}$$

Replacing these values in (2.7) we get

$$2C \geq \frac{1}{2}\left(1-\frac{1}{2n}\right) \tag{2.8}$$

for any $n \geq 2$.

Taking the limit for $n \to \infty$ in (2.8) we get $C \geq \frac{1}{4}$, which proves that $\frac{1}{4}$ is best possible in the first inequality in (2.6) □

Remark 2.3. Let $f : [a,b] \to \mathbb{R}$ be an absolutely continuous function on $[a,b]$. If $f'(t) \geq 0$ for a.e. $t \in [a,b]$, then $AB_f(a,b,x) \geq 0$ for $x \in \left[a, \frac{a+b}{2}\right] \left(\left[\frac{a+b}{2}, b\right]\right)$.

Moreover, if $f(b) \neq -f(a)$ and

$$\frac{bf(b)+af(a)}{f(b)+f(a)} \in [a,b] \tag{2.9}$$

then

$$AB_f\left(a,b,\frac{bf(b)+af(a)}{f(b)+f(a)}\right) \leq 0. \tag{2.10}$$

Also, if $f(a), f(b) > 0$, then (2.9) holds and the inequality (2.10) is valid.

Corollary 2.4. *Let $f : [a,b] \to \mathbb{C}$ be an absolutely continuous function on $[a,b]$ and $\gamma \in \mathbb{C}$. Then we have the representation*

$$\begin{aligned} AB_f(a,b,x) &= \frac{1}{2}\gamma\left[\left(x-\frac{a+b}{2}\right)^2+\frac{1}{4}(b-a)^2\right]+\left(\frac{a+b}{2}-x\right)f(x) \\ &+\frac{1}{2}\left[\int_a^x (t-a)(f'(t)-\gamma)\,dt+\int_x^b (b-t)(f'(t)-\gamma)\,dt\right] \end{aligned} \tag{2.11}$$

and

$$\begin{aligned} AB_f(a,b,x) &= \frac{bf(b)+af(a)}{2}-\frac{f(b)+f(a)}{2}x \\ &-\frac{1}{2}\gamma\left[\left(x-\frac{a+b}{2}\right)^2+\frac{1}{4}(b-a)^2\right] \\ &-\frac{1}{2}\int_a^b |t-x|\,(f'(t)-\gamma)\,dt \end{aligned} \tag{2.12}$$

for any $x \in [a,b]$.

Proof. Let $e(t) = t, t \in [a,b]$. If we write the equality (2.1) for the function $f - \gamma e$ we have

$$\begin{aligned} AB_{f-\gamma e}(a,b,x) &= \left(\frac{a+b}{2}-x\right)(f(x)-\gamma x) \\ &+\frac{1}{2}\left[\int_a^x (t-a)(f'(t)-\gamma)\,dt+\int_x^b (b-t)(f'(t)-\gamma)\,dt\right] \end{aligned} \tag{2.13}$$

for any $x \in [a,b]$.

Observe that

$$AB_{f-\gamma e}(a,b,x) = AB_f(a,b,x) - \gamma AB_e(a,b,x)$$

and

$$\begin{aligned} AB_e(a,b,x) &= \frac{1}{2}\left(\int_x^b t dt - \int_a^x t dt\right) \\ &= \frac{1}{2}\left(\frac{b^2-x^2}{2} - \frac{x^2-a^2}{2}\right) = \frac{1}{2}\left(\frac{a^2+b^2}{2} - x^2\right). \end{aligned}$$

From (2.13) we have

$$\begin{aligned} AB_f(a,b,x) &= \left(\frac{a+b}{2} - x\right)(f(x) - \gamma x) + \frac{1}{2}\gamma\left(\frac{a^2+b^2}{2} - x^2\right) \qquad (2.14) \\ &+ \frac{1}{2}\left[\int_a^x (t-a)(f'(t)-\gamma)\,dt + \int_x^b (b-t)(f'(t)-\gamma)\,dt\right] \\ &= \left(\frac{a+b}{2} - x\right) f(x) + \frac{1}{2}\gamma\left(\frac{a^2+b^2}{2} - x^2\right) - \gamma\left(\frac{a+b}{2} - x\right) x \\ &+ \frac{1}{2}\left[\int_a^x (t-a)(f'(t)-\gamma)\,dt + \int_x^b (b-t)(f'(t)-\gamma)\,dt\right] \qquad (2.15) \\ &= \frac{1}{2}\gamma\left[x^2 - (a+b)x + \frac{a^2+b^2}{2}\right] + \left(\frac{a+b}{2} - x\right) f(x) \\ &+ \frac{1}{2}\left[\int_a^x (t-a)(f'(t)-\gamma)\,dt + \int_x^b (b-t)(f'(t)-\gamma)\,dt\right] \end{aligned}$$

for any $x \in [a,b]$.

Since

$$x^2 - (a+b)x + \frac{a^2+b^2}{2} = \left(x - \frac{a+b}{2}\right)^2 + \frac{1}{4}(b-a)^2$$

then from (2.14) we deduce the desired equality (2.11).

From (2.2) we have

$$\begin{aligned} AB_{f-\gamma e}(a,b,x) = \frac{bf(b)+af(a)}{2} - \gamma\frac{b^2+a^2}{2} - \frac{f(b)+f(a)}{2}x + \gamma\frac{a+b}{2}x \\ - \frac{1}{2}\int_a^b |t-x|(f'(t)-\gamma)\,dt \end{aligned}$$

and since

$$AB_{f-\gamma e}(a,b,x) = AB_f(a,b,x) - \gamma AB_e(a,b,x)$$

then

$$AB_f(a,b,x) = \frac{1}{2}\gamma\left(\frac{a^2+b^2}{2} - x^2\right) + \frac{bf(b)+af(a)}{2}$$
$$- \gamma\frac{b^2+a^2}{2} - \frac{f(b)+f(a)}{2}x + \gamma\frac{a+b}{2}x$$
$$- \frac{1}{2}\int_a^b |t-x|\left(f'(t)-\gamma\right)dt$$
$$= \frac{bf(b)+af(a)}{2} - \frac{f(b)+f(a)}{2}x$$
$$- \frac{1}{2}\gamma\left[x^2-(a+b)x+\frac{a^2+b^2}{2}\right] - \frac{1}{2}\int_a^b |t-x|\left(f'(t)-\gamma\right)dt$$

which proves the desired equality (2.12). □

Remark 2.5. We have the following equalities

$$AB_f\left(a,b,\frac{a+b}{2}\right) = \frac{1}{8}\gamma(b-a)^2 + \frac{1}{2}\left[\int_a^{\frac{a+b}{2}}(t-a)\left(f'(t)-\gamma\right)dt + \int_{\frac{a+b}{2}}^b (b-t)\left(f'(t)-\gamma\right)dt\right] \tag{2.16}$$

and

$$AB_f\left(a,b,\frac{a+b}{2}\right) = \frac{1}{4}(b-a)\left[f(b)-f(a)\right] - \frac{1}{8}\gamma(b-a)^2 - \frac{1}{2}\int_a^b \left|t-\frac{a+b}{2}\right|\left(f'(t)-\gamma\right)dt \tag{2.17}$$

for any $\gamma \in \mathbb{C}$.

3. Bounds for absolutely continuous functions

Now, for $\gamma,\Gamma \in \mathbb{C}$ and $[a,b]$ an interval of real numbers, define the sets of complex-valued functions

$$\bar{U}_{[a,b]}(\gamma,\Gamma) := \left\{f:[a,b]\to\mathbb{C}|\operatorname{Re}\left[(\Gamma-f(t))\left(\overline{f(t)}-\overline{\gamma}\right)\right] \geq 0 \text{ for each } t\in[a,b]\right\}$$

and

$$\bar{\Delta}_{[a,b]}(\gamma,\Gamma) := \left\{f:[a,b]\to\mathbb{C}|\ \left|f(t)-\frac{\gamma+\Gamma}{2}\right| \leq \frac{1}{2}|\Gamma-\gamma| \text{ for each } t\in[a,b]\right\}.$$

The following representation result may be stated.

Proposition 3.1. *For any $\gamma,\Gamma\in\mathbb{C}$, $\gamma\neq\Gamma$, we have that $\bar{U}_{[a,b]}(\gamma,\Gamma)$ and $\bar{\Delta}_{[a,b]}(\gamma,\Gamma)$ are nonempty, convex and closed sets and*

$$\bar{U}_{[a,b]}(\gamma,\Gamma) = \bar{\Delta}_{[a,b]}(\gamma,\Gamma). \tag{3.1}$$

Proof. We observe that for any $z \in \mathbb{C}$ we have the equivalence

$$\left| z - \frac{\gamma + \Gamma}{2} \right| \le \frac{1}{2} \left| \Gamma - \gamma \right|$$

if and only if

$$\operatorname{Re}\left[(\Gamma - z)(\bar{z} - \bar{\gamma}) \right] \ge 0.$$

This follows by the equality

$$\frac{1}{4} \left| \Gamma - \gamma \right|^2 - \left| z - \frac{\gamma + \Gamma}{2} \right|^2 = \operatorname{Re}\left[(\Gamma - z)(\bar{z} - \bar{\gamma}) \right]$$

that holds for any $z \in \mathbb{C}$.

The equality (3.1) is thus a simple consequence of this fact. □

On making use of the complex numbers field properties we can also state that:

Corollary 3.2. *For any $\gamma, \Gamma \in \mathbb{C}$, $\gamma \neq \Gamma$, we have that*

$$\begin{aligned} \bar{U}_{[a,b]}(\gamma, \Gamma) = \{ f : [a,b] \to \mathbb{C} \mid \ & (\operatorname{Re}\Gamma - \operatorname{Re} f(t))(\operatorname{Re} f(t) - \operatorname{Re}\gamma) \\ & + (\operatorname{Im}\Gamma - \operatorname{Im} f(t))(\operatorname{Im} f(t) - \operatorname{Im}\gamma) \ge 0 \textit{ for each } t \in [a,b] \}. \end{aligned} \tag{3.2}$$

Now, if we assume that $\operatorname{Re}(\Gamma) \ge \operatorname{Re}(\gamma)$ and $\operatorname{Im}(\Gamma) \ge \operatorname{Im}(\gamma)$, then we can define the following set of functions as well:

$$\begin{aligned} \bar{S}_{[a,b]}(\gamma, \Gamma) := \{ f : [a,b] \to \mathbb{C} \mid & \operatorname{Re}(\Gamma) \ge \operatorname{Re} f(t) \ge \operatorname{Re}(\gamma) \\ & \text{and } \operatorname{Im}(\Gamma) \ge \operatorname{Im} f(t) \ge \operatorname{Im}(\gamma) \ \text{ for each } \ t \in [a,b] \}. \end{aligned} \tag{3.3}$$

One can easily observe that $\bar{S}_{[a,b]}(\gamma, \Gamma)$ is closed, convex and

$$\emptyset \neq \bar{S}_{[a,b]}(\gamma, \Gamma) \subseteq \bar{U}_{[a,b]}(\gamma, \Gamma). \tag{3.4}$$

Theorem 3.3. *Let $f : [a,b] \to \mathbb{C}$ be an absolutely continuous function on $[a,b]$. If there exists $\gamma, \Gamma \in \mathbb{C}$, $\gamma \neq \Gamma$ such that $f' \in \bar{U}_{[a,b]}(\gamma, \Gamma)$ then*

$$\begin{aligned} & \left| AB_f(a,b,x) - \left(\frac{a+b}{2} - x \right) f(x) \right. \\ & \left. - \frac{\gamma + \Gamma}{4} \left[\left(x - \frac{a+b}{2} \right)^2 + \frac{1}{4}(b-a)^2 \right] \right| \\ & \le \frac{|\Gamma - \gamma|}{4} \left[\frac{1}{4}(b-a)^2 + \left(x - \frac{a+b}{2} \right)^2 \right] \end{aligned} \tag{3.5}$$

and

$$\begin{aligned} & \left| AB_f(a,b,x) - \frac{bf(b) + af(a)}{2} + \frac{f(b) + f(a)}{2} x \right. \\ & \left. + \frac{\gamma + \Gamma}{4} \left[\left(x - \frac{a+b}{2} \right)^2 + \frac{1}{4}(b-a)^2 \right] \right| \\ & \le \frac{|\Gamma - \gamma|}{4} \left[\frac{1}{4}(b-a)^2 + \left(x - \frac{a+b}{2} \right)^2 \right] \end{aligned} \tag{3.6}$$

for any $x \in [a,b]$.

Proof. From the equality (2.11) we have

$$AB_f(a,b,x) - \frac{\gamma+\Gamma}{4}\left[\left(x-\frac{a+b}{2}\right)^2+\frac{1}{4}(b-a)^2\right]-\left(\frac{a+b}{2}-x\right)f(x) = \frac{1}{2}\left[\int_a^x (t-a)\left(f'(t)-\frac{\gamma+\Gamma}{2}\right)dt+\int_x^b (b-t)\left(f'(t)-\frac{\gamma+\Gamma}{2}\right)dt\right] \tag{3.7}$$

for any $x \in [a,b]$.

If $f' \in \bar{U}_{[a,b]}(\gamma,\Gamma) = \bar{\Delta}_{[a,b]}(\gamma,\Gamma)$, then by taking the modulus in (3.7) we get

$$\begin{aligned}
&\Bigg|AB_f(a,b,x)-\left(\frac{a+b}{2}-x\right)f(x) \\
&-\frac{\gamma+\Gamma}{4}\left[\left(x-\frac{a+b}{2}\right)^2+\frac{1}{4}(b-a)^2\right]\Bigg| \\
&=\frac{1}{2}\left|\int_a^x (t-a)\left(f'(t)-\frac{\gamma+\Gamma}{2}\right)dt+\int_x^b (b-t)\left(f'(t)-\frac{\gamma+\Gamma}{2}\right)dt\right| \\
&\le \frac{1}{2}\left[\left|\int_a^x (t-a)\left(f'(t)-\frac{\gamma+\Gamma}{2}\right)dt\right|+\left|\int_x^b (b-t)\left(f'(t)-\frac{\gamma+\Gamma}{2}\right)dt\right|\right] \\
&\le \frac{1}{2}\left[\int_a^x (t-a)\left|f'(t)-\frac{\gamma+\Gamma}{2}\right|dt+\int_x^b (b-t)\left|f'(t)-\frac{\gamma+\Gamma}{2}\right|dt\right] \\
&\le \frac{|\Gamma-\gamma|}{4}\left[\int_a^x (t-a)\,dt+\int_x^b (b-t)\,dt\right] \\
&= \frac{|\Gamma-\gamma|}{4}\left[\frac{(x-a)^2+(b-x)^2}{2}\right] = \frac{|\Gamma-\gamma|}{4}\left[\frac{1}{4}(b-a)^2+\left(x-\frac{a+b}{2}\right)^2\right],
\end{aligned}$$

for any $x \in [a,b]$, which proves the inequality (3.5).

From the equality (2.12) we have

$$AB_f(a,b,x)-\frac{bf(b)+af(a)}{2}+\frac{f(b)+f(a)}{2}x + \frac{\gamma+\Gamma}{4}\left[\left(x-\frac{a+b}{2}\right)^2+\frac{1}{4}(b-a)^2\right] = -\frac{1}{2}\int_a^b |t-x|\left(f'(t)-\frac{\gamma+\Gamma}{2}\right)dt \tag{3.8}$$

for any $x \in [a,b]$.

Taking the modulus in (3.8) and using the fact that

$$f' \in \bar{U}_{[a,b]}(\gamma,\Gamma) = \bar{\Delta}_{[a,b]}(\gamma,\Gamma)$$

we have

$$\left|AB_f(a,b,x) - \frac{bf(b)+af(a)}{2} + \frac{f(b)+f(a)}{2}x \right.$$
$$\left. + \frac{\gamma+\Gamma}{4}\left[\left(x-\frac{a+b}{2}\right)^2 + \frac{1}{4}(b-a)^2\right]\right|$$
$$\leq \frac{1}{2}\int_a^b |t-x| \left|f'(t) - \frac{\gamma+\Gamma}{2}\right| dt$$
$$\leq \frac{|\Gamma-\gamma|}{4}\int_a^b |t-x|\,dt = \frac{|\Gamma-\gamma|}{4}\left[\int_a^x (x-t)\,dt + \int_x^b (t-x)\,dt\right]$$
$$= \frac{|\Gamma-\gamma|}{4}\left[\frac{(x-a)^2+(b-x)^2}{2}\right] = \frac{|\Gamma-\gamma|}{4}\left[\frac{1}{4}(b-a)^2 + \left(x-\frac{a+b}{2}\right)^2\right]$$

for any $x \in [a,b]$, which proves the desired inequality (3.6). □

Remark 3.4. Let $f : [a,b] \to \mathbb{R}$ be an absolutely continuous function on $[a,b]$. If there exists the real numbers m, M such that

$$m \leq f'(t) \leq M \text{ for a.e. } t \in [a,b],$$

then

$$\left|AB_f(a,b,x) - \left(\frac{a+b}{2} - x\right)f(x) \right. \tag{3.9}$$
$$\left. - \frac{m+M}{4}\left[\left(x-\frac{a+b}{2}\right)^2 + \frac{1}{4}(b-a)^2\right]\right|$$
$$\leq \frac{M-m}{4}\left[\frac{1}{4}(b-a)^2 + \left(x-\frac{a+b}{2}\right)^2\right]$$

and

$$\left|AB_f(a,b,x) - \frac{bf(b)+af(a)}{2} + \frac{f(b)+f(a)}{2}x \right. \tag{3.10}$$
$$\left. + \frac{m+M}{4}\left[\left(x-\frac{a+b}{2}\right)^2 + \frac{1}{4}(b-a)^2\right]\right|$$
$$\leq \frac{M-m}{4}\left[\frac{1}{4}(b-a)^2 + \left(x-\frac{a+b}{2}\right)^2\right]$$

for any $x \in [a,b]$.

Corollary 3.5. *With the assumptions of Theorem* 3.3 *we have*

$$\left|AB_f\left(a,b,\frac{a+b}{2}\right) - \frac{\gamma+\Gamma}{16}(b-a)^2\right| \leq \frac{|\Gamma-\gamma|}{16}(b-a)^2 \tag{3.11}$$

and

$$\left| \frac{1}{4}(b-a)[f(b)-f(a)] - \frac{\gamma+\Gamma}{16}(b-a)^2 - AB_f\left(a,b,\frac{a+b}{2}\right)\right| \le \frac{|\Gamma-\gamma|}{16}(b-a)^2 . \tag{3.12}$$

Theorem 3.6. *Let $f : I \to \mathbb{R}$ be an absolutely continuous function on the interval I and $[a,b] \subset \mathring{I}$, where $\mathring{I}$ is the interior of I and such that f' is of bounded variation on $[a,b]$. Then we have the inequalities*

$$\left| AB_f(a,b,x) - \left(\frac{a+b}{2} - x\right) f(x) - \frac{f'(a)+f'(b)}{4}\left[\left(x-\frac{a+b}{2}\right)^2 + \frac{1}{4}(b-a)^2\right]\right| \le \frac{1}{4}\left[\frac{1}{4}(b-a)^2 + \left(x-\frac{a+b}{2}\right)^2\right] \bigvee_a^b (f') \tag{3.13}$$

and

$$\left| AB_f(a,b,x) - \frac{bf(b)+af(a)}{2} + \frac{f(b)+f(a)}{2}x + \frac{f'(a)+f'(b)}{4}\left[\left(x-\frac{a+b}{2}\right)^2 + \frac{1}{4}(b-a)^2\right]\right| \le \frac{1}{4}\left[\frac{1}{4}(b-a)^2 + \left(x-\frac{a+b}{2}\right)^2\right] \bigvee_a^b (f') \tag{3.14}$$

for any $x \in [a,b]$.

Proof. From (2.11) for $\gamma = \frac{f'(a)+f'(b)}{2}$ we have the representation

$$\begin{aligned} &AB_f(a,b,x) - \frac{f'(a)+f'(b)}{4}\left[\left(x-\frac{a+b}{2}\right)^2 + \frac{1}{4}(b-a)^2\right] - \left(\frac{a+b}{2}-x\right) f(x) \\ &= \frac{1}{2}\left[\int_a^x (t-a)\left(f'(t) - \frac{f'(a)+f'(b)}{2}\right) dt + \int_x^b (b-t)\left(f'(t) - \frac{f'(a)+f'(b)}{2}\right) dt\right] \end{aligned} \tag{3.15}$$

for any $x \in [a,b]$.

Taking the modulus in (3.15) we get

$$\begin{aligned}&\left|AB_f(a,b,x)-\left(\frac{a+b}{2}-x\right)f(x)\right.\\&\left.-\frac{f'(a)+f'(b)}{4}\left[\left(x-\frac{a+b}{2}\right)^2+\frac{1}{4}(b-a)^2\right]\right|\\&\le\frac{1}{2}\left[\int_a^x(t-a)\left|f'(t)-\frac{f'(a)+f'(b)}{2}\right|dt\right.\\&\left.+\int_x^b(b-t)\left|f'(t)-\frac{f'(a)+f'(b)}{2}\right|dt\right]\end{aligned}\tag{3.16}$$

for any $x\in[a,b]$.

For $t\in[a,x]$ we have

$$\begin{aligned}\left|f'(t)-\frac{f'(a)+f'(b)}{2}\right|&=\left|\frac{f'(t)-f'(a)+f'(t)-f'(b)}{2}\right|\\&\le\frac{1}{2}\left[|f'(t)-f'(a)|+|f'(b)-f'(t)|\right]\\&\le\frac{1}{2}\bigvee_a^b(f')\end{aligned}$$

and similarly, for $t\in[x,b]$ we have

$$\left|f'(t)-\frac{f'(a)+f'(b)}{2}\right|\le\frac{1}{2}\bigvee_a^b(f')$$

and then by (3.16) we get

$$\begin{aligned}&\left|AB_f(a,b,x)-\left(\frac{a+b}{2}-x\right)f(x)\right.\\&\left.-\frac{f'(a)+f'(b)}{4}\left[\left(x-\frac{a+b}{2}\right)^2+\frac{1}{4}(b-a)^2\right]\right|\\&\le\frac{1}{4}\left[\int_a^x(t-a)\,dt+\int_x^b(b-t)\,dt\right]\bigvee_a^b(f')\\&=\frac{1}{4}\left[\frac{1}{4}(b-a)^2+\left(x-\frac{a+b}{2}\right)^2\right]\bigvee_a^b(f')\end{aligned}$$

for $t\in[a,b]$, and the inequality (3.13) is proved.

The second inequality goes along a similar way and we omit the details. □

Corollary 3.7. *With the assumptions of Theorem* 3.6 *we have*

$$\left|AB_f\left(a,b,\frac{a+b}{2}\right)-\frac{f'(a)+f'(b)}{16}(b-a)^2\right|\le\frac{1}{16}(b-a)^2\bigvee_a^b(f')\tag{3.17}$$

and

$$\left|\frac{1}{4}(b-a)[f(b)-f(a)]-\frac{f'(a)+f'(b)}{16}(b-a)^2-AB_f\left(a,b,\frac{a+b}{2}\right)\right| \tag{3.18}$$
$$\leq \frac{1}{16}(b-a)^2\bigvee_a^b(f').$$

4. Bounds for Lipschitzian derivatives

We say that v is *Lipschitzian* with the constant $L>0$, if

$$|v(t)-v(s)|\leq L|t-s|$$

for any $t,s\in[a,b]$.

Theorem 4.1. *Let $f:I\to\mathbb{R}$ be an absolutely continuous function on the interval I and $[a,b]\subset\mathring{I}$, where $\mathring{I}$ is the interior of I and such that f' is Lipschitzian with the constant $K>0$ on $[a,b]$. Then we have the inequalities*

$$\left|AB_f(a,b,x)-\left(\frac{a+b}{2}-x\right)f(x)\right. \tag{4.1}$$
$$\left.-\frac{1}{2}f'(x)\left[\frac{1}{4}(b-a)^2+\left(x-\frac{a+b}{2}\right)^2\right]\right|$$
$$\leq\frac{1}{12}(b-a)K\left[3\left(x-\frac{a+b}{2}\right)^2+\frac{1}{4}(b-a)^2\right]$$

for any $x\in[a,b]$.

In particular, we have

$$\left|AB_f\left(a,b,\frac{a+b}{2}\right)-\frac{1}{8}f'\left(\frac{a+b}{2}\right)(b-a)^2\right|\leq\frac{1}{48}K(b-a)^3. \tag{4.2}$$

The constant $\frac{1}{48}$ is best possible in (4.2).

Proof. We have from the equality (2.11) that

$$AB_f(a,b,x) \tag{4.3}$$
$$-\left(\frac{a+b}{2}-x\right)f(x)-\frac{1}{2}f'(x)\left[\frac{1}{4}(b-a)^2+\left(x-\frac{a+b}{2}\right)^2\right]$$
$$=\frac{1}{2}\left[\int_a^x(t-a)[f'(t)-f'(x)]\,dt+\int_x^b(b-t)[f'(t)-f'(x)]\,dt\right]$$

for any $x\in[a,b]$.

Taking the modulus on (4.3) we have

$$\begin{aligned}&\left|AB_f(a,b,x)-\left(\frac{a+b}{2}-x\right)f(x)\right.\\&\left.-\frac{1}{2}f'(x)\left[\frac{1}{4}(b-a)^2+\left(x-\frac{a+b}{2}\right)^2\right]\right|\\&\le\frac{1}{2}\left[\int_a^x(t-a)\left|f'(t)-f'(x)\right|dt+\int_x^b(b-t)\left|f'(t)-f'(x)\right|dt\right]\\&\le\frac{1}{2}K\left[\int_a^x(t-a)(x-t)\,dt+\int_x^b(b-t)(t-x)\,dt\right]\end{aligned}\tag{4.4}$$

for any $x\in[a,b]$.

Since a simple calculation shows that

$$\int_c^d(t-c)(d-t)\,dt=\frac{1}{6}(d-c)^3,$$

then

$$\begin{aligned}&\int_a^x(t-a)(x-t)\,dt+\int_x^b(b-t)(t-x)\,dt\\&=\frac{1}{6}\left[(x-a)^3+(b-x)^3\right]\\&=\frac{1}{6}(b-a)\left[3\left(x-\frac{a+b}{2}\right)^2+\frac{1}{4}(b-a)^2\right]\end{aligned}$$

for any $x\in[a,b]$.

Utilising (4.4) we get the desired inequality (4.1).

Consider the function $f:[a,b]\to\mathbb{R}$,

$$f(t):=\begin{cases}-\left(t-\frac{a+b}{2}\right)^2 & \text{if } t\in\left[a,\frac{a+b}{2}\right)\\ \left(t-\frac{a+b}{2}\right)^2 & \text{if } t\in\left[\frac{a+b}{2},b\right].\end{cases}$$

Then f is differentiable and

$$\begin{aligned}f'(t)&=\begin{cases}-2\left(t-\frac{a+b}{2}\right) & \text{if } t\in\left[a,\frac{a+b}{2}\right)\\ 2\left(t-\frac{a+b}{2}\right) & \text{if } t\in\left[\frac{a+b}{2},b\right].\end{cases}\\&=2\left|t-\frac{a+b}{2}\right|\end{aligned}$$

for $t\in[a,b]$.

Since

$$\begin{aligned}\left|f'(t)-f'(s)\right|&=2\left|\left|t-\frac{a+b}{2}\right|-\left|s-\frac{a+b}{2}\right|\right|\\&\le2\left|t-s\right|\end{aligned}$$

for any $t, s \in [a, b]$, we conclude that f' is Lipschitzian with the constant $K = 2$.

We have

$$\begin{aligned} AB_f\left(a, b, \frac{a+b}{2}\right) &= \frac{1}{2}\left[\int_{\frac{a+b}{2}}^{b} f(t)\,dt - \int_{a}^{\frac{a+b}{2}} f(t)\,dt\right] \\ &= \frac{1}{2}\left[\int_{\frac{a+b}{2}}^{b} \left(t - \frac{a+b}{2}\right)^2 dt + \int_{a}^{\frac{a+b}{2}} \left(t - \frac{a+b}{2}\right)^2 dt\right] \\ &= \frac{1}{2}\int_{a}^{b} \left(t - \frac{a+b}{2}\right)^2 dt = \frac{1}{24}(b-a)^3. \end{aligned}$$

If we replace these values in (4.2) we get in both sides the same quantity $\frac{1}{24}(b-a)^3$. □

The following result also holds:

Theorem 4.2. *With the assumptions of Theorem* 4.1 *we have the inequalities*

$$\begin{aligned} &\left| AB_f(a, b, x) - \frac{bf(b) + af(a)}{2} + \frac{f(b) + f(a)}{2}x \right. \\ &\left. + \frac{1}{2}f'(x)\left[\left(x - \frac{a+b}{2}\right)^2 + \frac{1}{4}(b-a)^2\right]\right| \\ &\leq \frac{1}{12}(b-a)K\left[3\left(x - \frac{a+b}{2}\right)^2 + \frac{1}{4}(b-a)^2\right] \end{aligned} \tag{4.5}$$

for any $x \in [a, b]$.

In particular, we have

$$\begin{aligned} &\left|\frac{1}{4}(b-a)[f(b) - f(a)] - \frac{1}{8}f'\left(\frac{a+b}{2}\right)(b-a)^2 - AB_f\left(a, b, \frac{a+b}{2}\right)\right| \\ &\leq \frac{1}{48}K(b-a)^3. \end{aligned} \tag{4.6}$$

The proof is similar to the above Theorem 4.1 and the details are omitted.

5. Inequalities for p-norms

For a Lebesgue measurable function $f : [c, d] \to \mathbb{C}$ we introduce the p-Lebesgue norms as

$$\|f\|_{[c,d],p} := \left(\int_a^b |f(t)|^p\,dt\right)^{1/p} \text{ if } p \geq 1$$

and

$$\|f\|_{[c,d],\infty} := ess \sup_{t\in[c,d]} |f(t)|$$

provided these quantities are finite. We denote $f \in L_p[c, d]$ and $f \in L_\infty[c, d]$.

Proposition 5.1. *Let $f : [a,b] \to \mathbb{C}$ be an absolutely continuous function on $[a,b]$. Then we have the inequalities*

$$\left| AB_f(a,b,x) - \left(\frac{a+b}{2} - x \right) f(x) \right| \tag{5.1}$$
$$\leq \frac{1}{2} \left[\int_a^x (t-a) |f'(t)| \, dt + \int_x^b (b-t) |f'(t)| \, dt \right] := B_1(x)$$

and

$$\left| \frac{bf(b) + af(a)}{2} - \frac{f(b) + f(a)}{2} x - AB_f(a,b,x) \right| \tag{5.2}$$
$$\leq \frac{1}{2} \int_a^b |t-x| \, |f'(t)| \, dt := B_2(x)$$

for any $x \in [a,b]$.

Moreover, we have

$$B_1(x) \leq \frac{1}{2} \times \begin{cases} \frac{1}{2}(x-a)^2 \|f'\|_{[a,x],\infty} & \text{if } f' \in L_\infty[a,x] \\ \frac{1}{(\alpha+1)^{1/\alpha}} (x-a)^{1+1/\alpha} \|f'\|_{[a,x],\beta} & \text{if } f' \in L_\beta[a,x], \ \frac{1}{\alpha} + \frac{1}{\beta} = 1, \alpha > 1 \\ (x-a) \|f'\|_{[a,x],1} \end{cases} \tag{5.3}$$
$$+ \frac{1}{2} \times \begin{cases} \frac{1}{2}(b-x)^2 \|f'\|_{[x,b],\infty} & \text{if } f' \in L_\infty[x,b] \\ \frac{1}{(\gamma+1)^{1/\gamma}} (b-x)^{1+1/\gamma} \|f'\|_{[x,b],\delta} & \text{if } f' \in L_\delta[x,b], \ \frac{1}{\gamma} + \frac{1}{\delta} = 1, \gamma > 1 \\ (b-x) \|f'\|_{[x,b],1} \end{cases}$$

and

$$B_2(x) \leq \frac{1}{2} \times \begin{cases} \frac{1}{2}(x-a)^2 \|f'\|_{[a,x],\infty} & \text{if } f' \in L_\infty[a,x] \\ \frac{1}{(\alpha+1)^{1/\alpha}} (x-a)^{1+1/\alpha} \|f'\|_{[a,x],\beta} & \text{if } f' \in L_\beta[a,x], \ \frac{1}{\alpha} + \frac{1}{\beta} = 1, \alpha > 1 \\ (x-a) \|f'\|_{[a,x],1} \end{cases} \tag{5.4}$$
$$+ \frac{1}{2} \times \begin{cases} \frac{1}{2}(b-x)^2 \|f'\|_{[x,b],\infty} & \text{if } f' \in L_\infty[x,b] \\ \frac{1}{(\gamma+1)^{1/\gamma}} (b-x)^{1+1/\gamma} \|f'\|_{[x,b],\delta} & \text{if } f' \in L_\delta[x,b], \ \frac{1}{\gamma} + \frac{1}{\delta} = 1, \gamma > 1 \\ (b-x) \|f'\|_{[x,b],1} \end{cases}$$

for any $x \in [a,b]$.

Proof. From (2.1) and (2.2) we have by taking the modulus

$$\begin{aligned}
&\left|AB_f\left(a,b,x\right)-\left(\frac{a+b}{2}-x\right)f\left(x\right)\right| && (5.5)\\
&\le \frac{1}{2}\left[\left|\int_a^x (t-a)\,f'(t)\,dt\right|+\left|\int_x^b (b-t)\,f'(t)\,dt\right|\right]\\
&\le \frac{1}{2}\left[\int_a^x (t-a)\left|f'(t)\right|dt+\int_x^b (b-t)\left|f'(t)\right|dt\right]
\end{aligned}$$

and

$$\begin{aligned}
&\left|\frac{bf(b)+af(a)}{2}-\frac{f(b)+f(a)}{2}x-AB_f\left(a,b,x\right)\right| && (5.6)\\
&\le \frac{1}{2}\int_a^b |t-x|\left|f'(t)\right|dt\\
&= \frac{1}{2}\left[\int_a^x (x-t)\left|f'(t)\right|dt+\int_x^b (t-x)\left|f'(t)\right|dt\right]
\end{aligned}$$

for any $x\in[a,b]$.

Using the Hölder inequality we have

$$\begin{aligned}
&B_1(x)\\
\le\ & \frac{1}{2}\times\begin{cases}
\frac{1}{2}(x-a)^2\left\|f'\right\|_{[a,x],\infty} & \text{if } f'\in L_\infty[a,x]\\[2ex]
\frac{1}{(\alpha+1)^{1/\alpha}}(x-a)^{1+1/\alpha}\left\|f'\right\|_{[a,x],\beta} & \text{if } f'\in L_\beta[a,x],\ \frac{1}{\alpha}+\frac{1}{\beta}=1,\alpha>1\\[2ex]
(x-a)\left\|f'\right\|_{[a,x],1}
\end{cases}\\
&+\frac{1}{2}\times\begin{cases}
\frac{1}{2}(b-x)^2\left\|f'\right\|_{[x,b],\infty} & \text{if } f'\in L_\infty[x,b]\\[2ex]
\frac{1}{(\gamma+1)^{1/\gamma}}(b-x)^{1+1/\gamma}\left\|f'\right\|_{[x,b],\delta} & \text{if } f'\in L_\delta[x,b],\ \frac{1}{\gamma}+\frac{1}{\delta}=1,\gamma>1\\[2ex]
(b-x)\left\|f'\right\|_{[x,b],1}
\end{cases}
\end{aligned}$$

and a similar inequality for B_2. □

Remark 5.2. We observe that

$$\begin{aligned}
B_1(x) &\le \frac{1}{4}(x-a)^2\left\|f'\right\|_{[a,x],\infty}+\frac{1}{4}(b-x)^2\left\|f'\right\|_{[x,b],\infty} && (5.7)\\
&\le \left[\frac{1}{4}(x-a)^2+\frac{1}{4}(b-x)^2\right]\max\left\{\left\|f'\right\|_{[a,x],\infty},\left\|f'\right\|_{[x,b],\infty}\right\}\\
&= \frac{1}{2}\left[\frac{1}{4}(b-a)^2+\left(x-\frac{a+b}{2}\right)^2\right]\left\|f'\right\|_{[a,b],\infty}
\end{aligned}$$

therefore

$$\left| AB_f(a,b,x) - \left(\frac{a+b}{2} - x \right) f(x) \right| \le \frac{1}{2} \left[\frac{1}{4}(b-a)^2 + \left(x - \frac{a+b}{2} \right)^2 \right] \|f'\|_{[a,b],\infty} \tag{5.8}$$

for any $x \in [a,b]$.

Similarly,

$$\left| \frac{bf(b)+af(a)}{2} - \frac{f(b)+f(a)}{2} x - AB_f(a,b,x) \right| \le \frac{1}{2} \left[\frac{1}{4}(b-a)^2 + \left(x - \frac{a+b}{2} \right)^2 \right] \|f'\|_{[a,b],\infty} \tag{5.9}$$

for any $x \in [a,b]$.

In particular, we have

$$\left| AB_f\left(a,b,\frac{a+b}{2}\right) \right| \le \frac{1}{8}(b-a)^2 \|f'\|_{[a,b],\infty} \tag{5.10}$$

and

$$\left| \frac{1}{4}(b-a)[f(b)-f(a)] - AB_f\left(a,b,\frac{a+b}{2}\right) \right| \le \frac{1}{8}(b-a)^2 \|f'\|_{[a,b],\infty}. \tag{5.11}$$

6. Applications for twice differentiable functions

If we write the equalities (2.11) and (2.12) for the function $f = g'$, where $g : I \to \mathbb{R}$ is a differentiable function on the interior of the interval I with the derivative absolutely continuous on $[a,b] \subset \mathring{I}$, then we get

$$\begin{aligned} AB_{g'}(a,b,x) &= \frac{1}{2}\gamma \left[\left(x - \frac{a+b}{2} \right)^2 + \frac{1}{4}(b-a)^2 \right] + \left(\frac{a+b}{2} - x \right) g'(x) \\ &+ \frac{1}{2} \left[\int_a^x (t-a)(g''(t)-\gamma)\,dt + \int_x^b (b-t)(g''(t)-\gamma)\,dt \right] \end{aligned} \tag{6.1}$$

and

$$\begin{aligned} AB_{g'}(a,b,x) = \frac{bg'(b)+ag'(a)}{2} - \frac{g'(b)+g'(a)}{2} x \\ - \frac{1}{2}\gamma \left[\left(x - \frac{a+b}{2} \right)^2 + \frac{1}{4}(b-a)^2 \right] \\ - \frac{1}{2} \int_a^b |t-x| (g''(t)-\gamma)\,dt \end{aligned} \tag{6.2}$$

and since

$$AB_f(a,b,x) = \frac{1}{2}F(b) - F(x),$$

where $F(x) := \int_a^x f(t)\,dt$, then

$$\begin{aligned} AB_{g'}(a,b,x) &= \frac{1}{2}[g(b) - g(a)] - g(x) + g(a) \\ &= \frac{g(a)+g(b)}{2} - g(x) \end{aligned}$$

and by (6.1) and (6.2) we get the representations

$$\begin{aligned} g(x) &= \frac{g(a)+g(b)}{2} \\ &- \frac{1}{2}\gamma\left[\left(x - \frac{a+b}{2}\right)^2 + \frac{1}{4}(b-a)^2\right] - \left(\frac{a+b}{2} - x\right)g'(x) \\ &- \frac{1}{2}\left[\int_a^x (t-a)(g''(t)-\gamma)\,dt + \int_x^b (b-t)(g''(t)-\gamma)\,dt\right] \end{aligned} \tag{6.3}$$

and

$$\begin{aligned} g(x) &= \frac{g(a)+g(b)}{2} - \frac{bg'(b)+ag'(a)}{2} + \frac{g'(b)+g'(a)}{2}x \\ &+ \frac{1}{2}\gamma\left[\left(x - \frac{a+b}{2}\right)^2 + \frac{1}{4}(b-a)^2\right] \\ &+ \frac{1}{2}\int_a^b |t-x|(g''(t)-\gamma)\,dt \end{aligned} \tag{6.4}$$

for any $x \in [a,b]$.

If we assume that $g'' \in \bar{U}_{[a,b]}(\psi,\Psi)$ for some $\psi, \Psi \in \mathbb{C}$, $\psi \neq \Psi$, then, as above, we have the inequalities

$$\begin{aligned} &\left| g(x) - \frac{g(a)+g(b)}{2} \right. \\ &\left. + \frac{\psi+\Psi}{4}\left[\left(x - \frac{a+b}{2}\right)^2 + \frac{1}{4}(b-a)^2\right] + \left(\frac{a+b}{2} - x\right)g'(x) \right| \\ &\leq \frac{|\Psi-\psi|}{4}\left[\frac{1}{4}(b-a)^2 + \left(x - \frac{a+b}{2}\right)^2\right] \end{aligned} \tag{6.5}$$

and

$$\left|g(x) - \frac{g(a)+g(b)}{2} + \frac{bg'(b)+ag'(a)}{2} - \frac{g'(b)+g'(a)}{2}x \right. \tag{6.6}$$
$$\left. - \frac{\psi+\Psi}{4}\left[\left(x - \frac{a+b}{2}\right)^2 + \frac{1}{4}(b-a)^2\right]\right|$$
$$\leq \frac{|\Psi-\psi|}{4}\left[\frac{1}{4}(b-a)^2 + \left(x - \frac{a+b}{2}\right)^2\right]$$

for any $x \in [a,b]$.

We have the particular inequalities

$$\left|g\left(\frac{a+b}{2}\right) - \frac{g(a)+g(b)}{2} + \frac{\psi+\Psi}{16}(b-a)^2\right| \tag{6.7}$$
$$\leq \frac{|\Psi-\psi|}{16}(b-a)^2$$

and

$$\left|g\left(\frac{a+b}{2}\right) - \frac{g(a)+g(b)}{2} + \frac{1}{4}(b-a)[g'(b)-g'(a)] \right. \tag{6.8}$$
$$\left. - \frac{\psi+\Psi}{16}(b-a)^2\right|$$
$$\leq \frac{|\Psi-\psi|}{16}(b-a)^2$$

Other similar results may be stated, however we do not present the details here.

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Applications of generalized fractional integral operator to unified subclass of prestarlike functions with negative coefficients

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Abstract. In this paper, we have introduced and studied various properties of unified class of prestarlike functions with negative coefficients in the unit disc U. Also distortion theorem involving a generalized fractional integral operator for functions in this class is established.

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1. Introduction

Let A denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n \, z^n \tag{1.1}$$

which are analytic in the unit disc U= $\{z : |z| < 1\}$. Let S denote the subclass of A, which consists of functions of the form (1.1) that are univalent in U.

A function $f \in S$ is said to be starlike of order $\mu(0 \leq \mu < 1)$ if and only if

$$Re\left\{\frac{zf'(z)}{f(z)}\right\} > \mu, \ z \in U$$

and convex of order $\mu(0 \leq \mu < 1)$ if and only if

$$Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \mu, \ z \in U.$$

Denote these classes respectively by $S^*(\mu)$ and $K(\mu)$.

Let T denote subclass of S consisting of functions of the form

$$f(z) = z - \sum_{n=2}^{\infty} a_n \, z^n, \; a_n \geq 0. \tag{1.2}$$

The classes obtained by taking intersections of the classes $S^*(\mu)$ and $K(\mu)$ with T are denoted by $T^*(\mu)$ and $K^*(\mu)$ respectively. The classes $T^*(\mu)$, $K^*(\mu)$ were studied by Silverman [9].

The function

$$S_\mu(z) = z(1-z)^{-2(1-\mu)} \;, \quad 0 \leq \mu < 1 \;, \tag{1.3}$$

is the familiar extremal function for the class $S^*(\mu)$, setting

$$C(\mu, n) = \frac{\prod_{i=2}^{n} (i - 2\mu)}{(n-1)!} \;, \quad n \in \mathbb{N}\backslash\{1\}, \; \mathbb{N} = \{1, 2, 3, \cdots\} \;, \tag{1.4}$$

then

$$S_\mu(z) = z + \sum_{n=2}^{\infty} C(\mu, n) \, z^n \;. \tag{1.5}$$

We note that $C(\mu, n)$ is a decreasing function in μ, and that

$$\lim_{n\to\infty} C(\mu, n) = \begin{cases} \infty \;, & \mu < \frac{1}{2} \\ 1 \;, & \mu = \frac{1}{2} \\ 0 \;, & \mu > \frac{1}{2} \end{cases} \;.$$

If $f(z)$ is given by (1.2) and $g(z)$ defined by

$$g(z) = z - \sum_{n=2}^{\infty} b_n \, z^n, \quad b_n \geq 0 \;,$$

belonging to T, then convolution or Hadamard product of f and g is defined by

$$(f * g)(z) = z - \sum_{n=2}^{\infty} a_n \, b_n \, z^n.$$

Let $R_\mu(\alpha, \beta, \gamma)$ be the subclass of A consisting functions $f(z)$ such that

$$\left| \frac{\frac{zh'(z)}{h(z)} - 1}{\gamma \frac{zh'(z)}{h(z)} + 1 - (1+\gamma)\alpha} \right| < \beta \;,$$

where, $h(z) = (f * S_\mu(z))$, $0 \leq \alpha < 1$, $0 < \beta \leq 1$, $0 \leq \gamma \leq 1$, $0 \leq \mu < 1$.

Also let $C_\mu(\alpha, \beta, \gamma)$ be the subclass of A consisting of functions $f(z)$, which satisfy the condition

$$zf'(z) \in R_\mu(\alpha, \beta, \gamma).$$

The classes $R_\mu(\alpha,\beta,\gamma)$ and $C_\mu(\alpha,\beta,\gamma)$ of prestarlike functions was investigated by Joshi [1]. In particular, the subclasses

$$R_\mu[\alpha,\beta,\gamma] = R_\mu(\alpha,\beta,\gamma)\cap T, \quad C_\mu[\alpha,\beta,\gamma] = C_\mu(\alpha,\beta,\gamma)\cap T \ ,$$

were also studied by Joshi [1].

The following results will be required for our investigation.

Lemma 1.1. [1]. *A function f defined by (1.2) is in the class $R_\mu[\alpha,\beta,\gamma]$ if and only if*

$$\sum_{n=2}^{\infty} C(\mu,n)\ \{(n-1)+\beta\,[\gamma n+1-(1+\gamma)\alpha]\}\ a_n \le \beta(1+\gamma)(1-\alpha). \tag{1.6}$$

The result (1.6) *is sharp.*

Lemma 1.2. [1]. *A function f defined by (1.2) is in the class $C_\mu[\alpha,\beta,\gamma]$ if and only if*

$$\sum_{n=2}^{\infty} C(\mu,n)\ n\,\{(n-1)+\beta\,[\gamma n+1-(1+\gamma)\alpha]\}\ a_n \le \beta(1+\gamma)(1-\alpha). \tag{1.7}$$

The result (1.7) *is sharp.*

Further we note that such type of classes were extensively studied by Sheil-Small *et al.* [8], Owa and Uralegaddi [4], Srivastava and Aouf [10] and Raina and Srivastava [7].

In view of Lemma 1.1 and Lemma 1.2, we present here a unified study of the classes $R_\mu[\alpha,\beta,\gamma]$ and $C_\mu[\alpha,\beta,\gamma]$ by introducing a new subclass $P_\mu(\alpha,\beta,\gamma,\sigma)$. Indeed, we say that a function $f(z)$ defined by (1.2) is in the class $P_\mu(\alpha,\beta,\gamma,\sigma)$ if and only if

$$\sum_{n=2}^{\infty} \frac{\{(n-1)+\beta\,[\gamma n+1-(1+\gamma)\alpha]\}\,(1-\sigma+\sigma n)}{\beta(1+\gamma)(1-\alpha)}\ C(\mu,n)a_n \le 1\ , \tag{1.8}$$

where, $0\le\alpha<1,\ 0<\beta\le 1,\ 0\le\gamma\le 1,\ 0\le\mu<1,\ 0\le\sigma\le 1.$
Then clearly we have,

$$P_\mu(\alpha,\beta,\gamma,\sigma) = (1-\sigma)R_\mu[\alpha,\beta,\gamma]+\sigma C_\mu[\alpha,\beta,\gamma]\ , \tag{1.9}$$

where, $0\le\sigma\le 1$. So that

$$P_\mu(\alpha,\beta,\gamma,0) = R_\mu[\alpha,\beta,\gamma], \quad P_\mu(\alpha,\beta,\gamma,1) = C_\mu[\alpha,\beta,\gamma]. \tag{1.10}$$

The main object of this paper is to investigate various interesting properties and characterization of the general class $P_\mu(\alpha,\beta,\gamma,\sigma)$. Also distortion theorem involving a generalized fractional integral operator for functions in this class are obtained.

2. Main results

Theorem 2.1. *A function f defined by (1.2) is in the class $P_\mu(\alpha,\beta,\gamma,\sigma)$ then*

$$a_n \le \frac{\beta(1+\gamma)(1-\alpha)}{C(\mu,n)\,\{(n-1)+\beta\,[\gamma n+1-(1+\gamma)\alpha]\}\,(1-\sigma+\sigma n)},\ n\in\mathbb{N}\backslash\{1\}. \tag{2.1}$$

Equality holds true for the function $f(z)$ given by

$$f(z) = z - \frac{\beta(1+\gamma)(1-\alpha)}{C(\mu,n)\left\{(n-1)+\beta\left[\gamma n+1-(1+\gamma)\alpha\right]\right\}(1-\sigma+\sigma n)} z^n, \ \ n \in \mathbb{N}\backslash\{1\}. \tag{2.2}$$

Proof. The proof of Theorem 2.1 is straightforward and hence details are omitted. □

A distortion theorem for function f in the class $P_\mu(\alpha,\beta,\gamma,\sigma)$ is given as follows:

Theorem 2.2. *If the function f defined by* (1.2) *is in the class $P_\mu(\alpha,\beta,\gamma,\sigma)$ then*

$$|z| - \frac{\beta(1+\gamma)(1-\alpha)}{2\left\{1+\beta\left[2\gamma+1-(1+\gamma)\alpha\right]\right\}(1-\mu)(1+\sigma)}|z|^2 \le |f(z)| \le |z| + \frac{\beta(1+\gamma)(1-\alpha)}{2\left\{1+\beta\left[2\gamma+1-(1+\gamma)\alpha\right]\right\}(1-\mu)(1+\sigma)}|z|^2, \tag{2.3}$$

and

$$1 - \frac{\beta(1+\gamma)(1-\alpha)}{\left\{1+\beta\left[2\gamma+1-(1+\gamma)\alpha\right]\right\}(1-\mu)(1+\sigma)}|z| \le |f'(z)| \le 1 + \frac{\beta(1+\gamma)(1-\alpha)}{\left\{1+\beta\left[2\gamma+1-(1+\gamma)\alpha\right]\right\}(1-\mu)(1+\sigma)}|z|. \tag{2.4}$$

Proof. Let

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n$$

Since $f(z) \in P_\mu(\alpha,\beta,\gamma,\sigma)$ and clearly $C(\mu,n)$ defined by (1.4) is non-decreasing for $0 \le \mu \le \frac{1}{2}$ and using (1.8) we get

$$\sum_{n=2}^{\infty} a_n \le \frac{\beta(1+\gamma)(1-\alpha)}{2\left\{1+\beta\left[2\gamma+1-(1+\gamma)\alpha\right]\right\}(1-\mu)(1+\sigma)}, \ \ n \in \mathbb{N}\backslash\{1\}. \tag{2.5}$$

Then using (1.2) and (2.5) we get (for $z \in U$),

$$\begin{aligned} |f(z)| &\le |z| + |z|^2 \sum_{n=2}^{\infty} |a_n| \\ &\le |z| + \frac{\beta(1+\gamma)(1-\alpha)}{2\left\{1+\beta\left[2\gamma+1-(1+\gamma)\alpha\right]\right\}(1-\mu)(1+\sigma)}|z|^2 \end{aligned}$$

and

$$\begin{aligned} |f(z)| &\ge |z| - |z|^2 \sum_{n=2}^{\infty} |a_n| \\ &\ge |z| - \frac{\beta(1+\gamma)(1-\alpha)}{2\left\{1+\beta\left[2\gamma+1-(1+\gamma)\alpha\right]\right\}(1-\mu)(1+\sigma)}|z|^2. \end{aligned}$$

which proves the assertion (2.3) of Theorem 2.2.

Also for $z \in U$, we find that

$$\begin{aligned} |f'(z)| &\leq 1 + |z| \sum_{n=2}^{\infty} n \ |a_n| \\ &\leq 1 + \frac{\beta(1+\gamma)(1-\alpha)}{\{1+\beta\,[2\gamma+1-(1+\gamma)\alpha]\}\,(1-\mu)(1+\sigma)} |z| \end{aligned}$$

and

$$\begin{aligned} |f'(z)| &\geq 1 - |z| \sum_{n=2}^{\infty} n \ |a_n| \\ &\geq 1 - \frac{\beta(1+\gamma)(1-\alpha)}{\{1+\beta\,[2\gamma+1-(1+\gamma)\alpha]\}\,(1-\mu)(1+\sigma)} |z|. \end{aligned}$$

which proves the assertion (2.4) of Theorem 2.2
This completes the proof. □

We note that results (2.3) and (2.4) is sharp for the function $f(z)$ given by

$$f(z) = z - \frac{\beta(1+\gamma)(1-\alpha)}{2\,\{1+\beta\,[2\gamma+1-(1+\gamma)\alpha]\}\,(1-\mu)(1+\sigma)} z^2. \tag{2.6}$$

3. Closure theorems

In this section, we shall prove that the class $P_\mu(\alpha, \beta, \gamma, \sigma)$ is closed under linear combination.

Theorem 3.1. *The class $P_\mu(\alpha, \beta, \gamma, \sigma)$ is closed under linear combination.*

Proof. Suppose $f(z),\ g(z) \in P_\mu(\alpha, \beta, \gamma, \sigma)$ and

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n$$

and

$$g(z) = z - \sum_{n=2}^{\infty} b_n z^n \ .$$

It is sufficient to prove that the function H defined by

$$H(z) = \lambda f(z) + (1-\lambda) g(z)\ , \quad (0 \leq \lambda \leq 1)$$

is also in the class $P_\mu(\alpha, \beta, \gamma, \sigma)$. Since

$$H(z) = z - \sum_{n=2}^{\infty} [\lambda a_n + (1-\lambda) b_n] z^n \ .$$

We observe that

$$\sum_{n=2}^{\infty} \frac{\{(n-1)+\beta\,[\gamma n+1-(1+\gamma)\alpha]\}\,(1-\sigma+\sigma n)}{\beta(1+\gamma)(1-\alpha)} \ C(\mu, n)[\lambda a_n + (1-\lambda) b_n] \leq 1.$$

Thus $H \in P_\mu(\alpha, \beta, \gamma, \sigma)$. This completes the proof. □

Theorem 3.2. *If*

$$f_1(z) = z$$

and

$$f_n(z) = z - \frac{\beta(1+\gamma)(1-\alpha)}{\{(n-1)+\beta\left[\gamma n+1-(1+\gamma)\alpha\right]\}(1-\sigma+\sigma n)C(\mu,n)}\ z^n\ , \quad (n \geq 2).$$

Then $f \in P_\mu(\alpha,\beta,\gamma,\sigma)$ *if and only if it can be expressed in the form*

$$f(z) = \sum_{n=1}^{\infty} \lambda_n f_n(z)$$

where $\lambda_n \geq 0$ *and* $\sum_{n=1}^{\infty} \lambda_n = 1$.

Proof. Let

$$\begin{aligned} f(z) &= \sum_{n=1}^{\infty} \lambda_n f_n(z) \\ &= z - \sum_{n=2}^{\infty} \frac{\beta(1+\gamma)(1-\alpha)}{\{(n-1)+\beta\left[\gamma n+1-(1+\gamma)\alpha\right]\}(1-\sigma+\sigma n)C(\mu,n)}\ \lambda_n z^n \\ &= z - \sum_{n=2}^{\infty} a_n\ z^n\ , \end{aligned}$$

where

$$a_n = \frac{\beta(1+\gamma)(1-\alpha)}{\{(n-1)+\beta\left[\gamma n+1-(1+\gamma)\alpha\right]\}(1-\sigma+\sigma n)C(\mu,n)}\ \lambda_n \geq 0,\ (n \geq 2).$$

Since,

$$\sum_{n=2}^{\infty}\left[\frac{\beta(1+\gamma)(1-\alpha)}{\{(n-1)+\beta\left[\gamma n+1-(1+\gamma)\alpha\right]\}(1-\sigma+\sigma n)C(\mu,n)} \frac{\{(n-1)+\beta\left[\gamma n+1-(1+\gamma)\alpha\right]\}(1-\sigma+\sigma n)C(\mu,n)}{\beta(1+\gamma)(1-\alpha)}\right]\lambda_n$$

$$= \sum_{n=2}^{\infty} \lambda_n = \sum_{n=1}^{\infty} \lambda_n - \lambda_1 = 1 - \lambda_1 \leq 1\ .$$

Therefore $f(z) \in P_\mu(\alpha,\beta,\gamma,\sigma)$.
Conversely, suppose that $f \in P_\mu(\alpha,\beta,\gamma,\sigma)$ and since

$$a_n = \frac{\beta(1+\gamma)(1-\alpha)}{\{(n-1)+\beta\left[\gamma n+1-(1+\gamma)\alpha\right]\}(1-\sigma+\sigma n)C(\mu,n)}\ \lambda_n \geq 0,\ (n \geq 2).$$

Setting

$$\lambda_n = \frac{\{(n-1)+\beta\left[\gamma n+1-(1+\gamma)\alpha\right]\}(1-\sigma+\sigma n)C(\mu,n)}{\beta(1+\gamma)(1-\alpha)}\ ,\ (n \geq 2)$$

and

$$\lambda_1 = 1 - \sum_{n=2}^{\infty} \lambda_n \ .$$

We get

$$f(z) = \sum_{n=1}^{\infty} \lambda_n f_n(z) \ .$$

This completes the proof. □

4. Generalized fractional integral operator

In recent years the theory of fractional calculus operator have been fruitfully applied to analytic functions. Moreover generalized operator of fractional integrals (or derivatives) having kernels of different types of special functions (including Fox's H-function) have generated keen interest in this area. For details one may refer to Kiryakova [2], Raina and Saigo [6], Srivastava and Owa [11] and Raina and Bolia [5]. Further we note that Riemann-Liouville fractional calculus operators have been used to obtain basic results which include coefficient estimates, boundedness properties for various subclasses of analytic and univalent functions.

A generalized fractional integral operator involving the celebrated Fox's H-function [2, 3] defined below.

Definition 4.1. *Let $m \in \mathbb{N}$, $\beta_k \in \mathbb{R}$ and γ_k, $\delta_k \in \mathbb{C}$, $\forall\ k = 1, 2, \cdots, m$. Then the integral operator*

$$\begin{aligned} I^{(\gamma_m),(\delta_m)}_{(\beta_m);m} f(z) &= I^{(\gamma_1,\cdots,\gamma_m),(\delta_1,\cdots,\delta_m)}_{(\beta_1,\cdots,\beta_m);m} f(z) \\ &= \frac{1}{z}\int_0^z H^{m,0}_{m,m}\left[\frac{t}{z}\left|\begin{matrix}\left(\gamma_k+\delta_k+1-\frac{1}{\beta_k},\frac{1}{\beta_k}\right)_{1,m}\\ \left(\gamma_k+1-\frac{1}{\beta_k},\frac{1}{\beta_k}\right)_{1,m}\end{matrix}\right.\right] f(t)dt, \\ &\qquad \text{for } \sum_i^m Re(\delta_k) > 0, \qquad (4.1) \\ &= f(z), \qquad \text{for } \delta_1 = \cdots = \delta_m = 0 \ , \end{aligned}$$

is said to be a multiple fractional integral operator of Riemann-Liouville type of multiorder $\delta = (\delta_1, \cdots, \delta_m)$.

Following [2], let Δ denote a complex domain starlike with respect to the origin $z = 0$, and $A(\Delta)$ denote the space of functions analytic in Δ. If $A_\rho(\Delta)$ denote the class of functions

$$A_\rho(\Delta) = \{f(z) = z^\rho \bar{f}(z) : \bar{f}(z) \in A(\Delta)\}, \qquad \rho \geq 0; \tag{4.2}$$

then clearly $A_\rho(\Delta) \subseteq A_v(\Delta) \subseteq A(\Delta)$ for $\rho \geq v \geq 0$.

The fractional integral operator (4.1) includes various useful and important fractional integral operators as special cases. For more details of these special cases, one may refer to Raina and Saigo [6]. Throughout this paper $(\lambda)_k$ stands for $\frac{\Gamma(\lambda+k)}{\Gamma(\lambda)}$.

The following results will be required for our investigation.

Lemma 4.2. [2]. *Let* $\gamma_k > -\frac{p}{\beta_k} - 1,\ \delta_k \leq 0\ (\forall\ k = 1, \cdots, m)$. *Then the operator* $I^{(\gamma_m),(\delta_m)}_{(\beta_m);m}$ *maps the class* $\Delta_p(G)$ *into itself preserving the power functions* $f(z) = z^p$ *(up to a constant multiplier):*

$$I^{(\gamma_m),(\delta_m)}_{(\beta_m);m}\{z^p\} = \prod_{k=1}^{m} \left\{ \frac{\Gamma\left(\frac{p}{\beta_k} + \gamma_k + 1\right)}{\Gamma\left(\frac{p}{\beta_k} + \gamma_k + \delta_k + 1\right)} \right\} z^p . \tag{4.3}$$

Theorem 4.3. *Let* $m \in \mathbb{N},\ h_k \in \mathbb{R}_+,$ *and* $\gamma_k,\ \delta_k \in \mathbb{R}$ *such that* $1 + \gamma_k + \delta_k > 0$ $(k = 1, \cdots, m)$, *and*

$$\prod_{k=1}^{m} \left\{ \frac{(1 + \gamma_k + 2h_k)_{h_k}}{(1 + \gamma_k + \delta_k + 2h_k)_{h_k}} \right\} \leq 1 \tag{4.4}$$

and $f(z)$ *defined by* (1.2) *be in the class* $P_\mu(\alpha, \beta, \gamma, \sigma)$ *with* $0 \leq \alpha < 1,\ 0 < \beta \leq 1,\ 0 \leq \gamma \leq 1,\ 0 \leq \mu \leq \frac{1}{2},\ 0 \leq \sigma \leq 1$. *Then*

$$\left| I^{(\gamma_m),(\delta_m)}_{(h_m^{-1});m} f(z) \right| \geq \left\{ \prod_{k=1}^{m} \left(\frac{\Gamma(1 + \gamma_k + h_k)}{\Gamma(1 + \gamma_k + \delta_k + h_k)} \right) \cdot \left[|z| - \frac{A^* \beta(1+\gamma)(1-\alpha)}{2\{1 + \beta[2\gamma + 1 - (1+\gamma)\alpha]\}(1-\mu)(1+\sigma)} |z|^2 \right] \right\} , \tag{4.5}$$

and

$$\left| I^{(\gamma_m),(\delta_m)}_{(h_m^{-1});m} f(z) \right| \leq \left\{ \prod_{k=1}^{m} \left(\frac{\Gamma(1 + \gamma_k + h_k)}{\Gamma(1 + \gamma_k + \delta_k + h_k)} \right) \cdot \left[|z| + \frac{A^* \beta(1+\gamma)(1-\alpha)}{2\{1 + \beta[2\gamma + 1 - (1+\gamma)\alpha]\}(1-\mu)(1+\sigma)} |z|^2 \right] \right\} , \tag{4.6}$$

for $z \in U$. *The inequalities in* (4.5) *and* (4.6) *are attained by the function* $f(z)$ *given by* (2.6), *where*

$$A^* = \prod_{k=1}^{m} \left\{ \frac{(1 + \gamma_k + h_k)_{h_k}}{(1 + \gamma_k + \delta_k + h_k)_{h_k}} \right\}. \tag{4.7}$$

Proof. By using lemma 4.2, we get

$$I^{(\gamma_m),(\delta_m)}_{(h_m^{-1});m} f(z) = \prod_{k=1}^{m} \left\{ \frac{\Gamma(1 + \gamma_k + h_k)}{\Gamma(1 + \gamma_k + \delta_k + h_k)} \right\} z - \sum_{n=2}^{\infty} \prod_{k=1}^{m} \left\{ \frac{\Gamma(1 + \gamma_k + n\ h_k)}{\Gamma(1 + \gamma_k + \delta_k + n\ h_k)} \right\} a_n z^n . \tag{4.8}$$

Letting

$$\begin{aligned} G(z) &= \prod_{k=1}^{m} \left\{ \frac{\Gamma(1+\gamma_k+\delta_k+h_k)}{\Gamma(1+\gamma_k+h_k)} \right\} I^{(\gamma_m),(\delta_m)}_{(h_m^{-1});m} f(z) \\ &= z - \sum_{n=2}^{\infty} \phi(n)\ a_n\ z^n \ , \end{aligned} \tag{4.9}$$

where,

$$\phi(n) = \prod_{k=1}^{m} \left\{ \frac{(1+\gamma_k+h_k)_{h_k(n-1)}}{(1+\gamma_k+\delta_k+h_k)_{h_k(n-1)}} \right\} \ , \quad (n \in \mathbb{N}\backslash\{1\}) \,. \tag{4.10}$$

Under the hypothesis of Theorem 4.3 (along with the conditions (4.4)), we can see that $\phi(n)$ is non-increasing for integers n $(n \geq 2)$, and we have

$$0 < \phi(n) \leq \phi(2) = \prod_{k=1}^{m} \left\{ \frac{(1+\gamma_k+h_k)_{h_k}}{(1+\gamma_k+\delta_k+h_k)_{h_k}} \right\} = A^* \ , \ (n \in \mathbb{N}\backslash\{1\}) \,. \tag{4.11}$$

Now in view equation (1.8) and (4.11), we have

$$\begin{aligned} |G(z)| &\geq |z| - \phi(2)\ |z|^2 \sum_{n=2}^{\infty} a_n \\ &\geq \left\{ \prod_{k=1}^{m} \left(\frac{\Gamma(1+\gamma_k+h_k)}{\Gamma(1+\gamma_k+\delta_k+h_k)} \right) \right. \\ &\quad \left. . \left[|z| - \frac{A^*\beta(1+\gamma)(1-\alpha)}{2\left\{1+\beta\left[2\gamma+1-(1+\gamma)\alpha\right]\right\}(1-\mu)(1+\sigma)} |z|^2 \right] \right\} \ . \end{aligned}$$

and

$$\begin{aligned} |G(z)| &\leq |z| + \phi(2)\ |z|^2 \sum_{n=2}^{\infty} a_n \\ &\leq \left\{ \prod_{k=1}^{m} \left(\frac{\Gamma(1+\gamma_k+h_k)}{\Gamma(1+\gamma_k+\delta_k+h_k)} \right) \right. \\ &\quad \left. . \left[|z| + \frac{A^*\beta(1+\gamma)(1-\alpha)}{2\left\{1+\beta\left[2\gamma+1-(1+\gamma)\alpha\right]\right\}(1-\mu)(1+\sigma)} |z|^2 \right] \right\} \ . \end{aligned}$$

It can be easily verified that the following inequalities are attained by the function $f(z)$ given by (2.6).

$$\begin{aligned} \left| I^{(\gamma_m),(\delta_m)}_{(h_m^{-1});m} f(z) \right| \geq & \left\{ \prod_{k=1}^{m} \left(\frac{\Gamma(1+\gamma_k+h_k)}{\Gamma(1+\gamma_k+\delta_k+h_k)} \right) \right. \\ & \left. . \left[|z| - \frac{A^*\beta(1+\gamma)(1-\alpha)}{2\left\{1+\beta\left[2\gamma+1-(1+\gamma)\alpha\right]\right\}(1-\mu)(1+\sigma)} |z|^2 \right] \right\} \ , \end{aligned}$$

and

$$\left| I^{(\gamma_m),(\delta_m)}_{(h_m^{-1});m} f(z) \right| \leq \left\{ \prod_{k=1}^{m} \left(\frac{\Gamma(1+\gamma_k+h_k)}{\Gamma(1+\gamma_k+\delta_k+h_k)} \right) \cdot \left[|z| + \frac{A^* \beta(1+\gamma)(1-\alpha)}{2\{1+\beta[2\gamma+1-(1+\gamma)\alpha]\}(1-\mu)(1+\sigma)} |z|^2 \right] \right\} .$$

Which are as desired in (4.5) and (4.6). This completes the proof of Theorem 4.3. □

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Starlike and convex properties for Poisson distribution series

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Abstract. In this paper, we find the necessary and sufficient conditions, inclusion relations for Poisson distribution series belonging to the classes $\mathscr{S}^*(\alpha,\beta)$ and $\mathscr{C}^*(\alpha,\beta)$. Further, we consider an integral operator related to Poisson Distribution series.

Mathematics Subject Classification (2010): 30C45.

Keywords: Starlike functions, convex functions, Poisson distribution series.

1. Introduction

Let $\mathscr{A}$ denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

which are analytic and univalent in the open disc $\mathbb{U} = \{z : z \in \mathbb{C}\ |z| < 1\}$. Let $\mathscr{T}$ be a subclass of $\mathscr{A}$ consisting of functions whose non-zero coefficients from second on is give by

$$f(z) = z - \sum_{n=2}^{\infty} |a_n| z^n, \qquad z \in \mathbb{U}. \tag{1.2}$$

In 2014, Porwal [4] introduced a power series whose coefficients are probabilities of Poisson distribution

$$\mathscr{K}(m,z) := z + \sum_{n=2}^{\infty} \frac{m^{n-1}}{(n-1)!} e^{-m} z^n, \quad z \in \mathbb{U},$$

where $m > 0$. By ratio test the radius of convergence of the above series is infinity. Further, Porwal [4] defined a series

$$\mathscr{F}(m,z) \quad = \quad 2z - \mathscr{K}(m,z) = z - \sum_{n=2}^{\infty} \frac{m^{n-1}}{(n-1)!} e^{-m} z^n, \qquad z \in \mathbb{U}.$$

Corresponding to the series $\mathscr{K}(m,z)$ using the Hadamard product for $f \in \mathscr{A}$, Porwal and Kumar [5] introduced a new linear operator $\mathscr{I}(m) : \mathscr{A} \to \mathscr{A}$ defined by

$$\begin{aligned} \mathscr{I}(m)f(z) : &= \mathscr{K}(m,z) * f(z) \\ &= z + \sum_{n=2}^{\infty} \frac{m^{n-1}}{(n-1)!} e^{-m} a_n z^n, \qquad z \in \mathbb{U}, \end{aligned}$$

where $*$ denotes the convolution (or Hadamard product) of two series

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \quad \text{and} \quad g(z) = \sum_{n=0}^{\infty} b_n z^n$$

is defined by

$$(f * g)(z) = \sum_{n=0}^{\infty} a_n b_n z^n.$$

Let $\mathscr{S}^*(\alpha,\beta)$ be the subclass of $\mathscr{T}$ consisting of functions which satisfy the condition:

$$\left| \frac{\frac{zf'(z)}{f(z)} - 1}{\frac{zf'(z)}{f(z)} + 1 - 2\alpha} \right| < \beta, \qquad z \in \mathbb{U},$$

where $0 \le \alpha < 1$ and $0 < \beta \le 1$.

Also, let $\mathscr{C}^*(\alpha,\beta)$ be the subclass of $\mathscr{T}$ consisting of functions which satisfy the condition:

$$\left| \frac{\frac{zf''(z)}{f'(z)}}{\frac{zf''(z)}{f'(z)} + 2(1-\alpha)} \right| < \beta, \qquad z \in \mathbb{U},$$

where $0 \le \alpha < 1$ and $0 < \beta \le 1$.

The classes $\mathscr{S}^*(\alpha,\beta)$ and $\mathscr{C}^*(\alpha,\beta)$, were introduced and studied by Gupta and Jain [2] (see [3]). Also, we note that for $\beta = 1$ the classes $\mathscr{S}^*(\alpha,\beta)$ and $\mathscr{C}^*(\alpha,\beta)$ reduce to the class of starlike and convex functions of order $\alpha (0 \le \alpha < 1)$ (see [6]).

A function $f \in \mathscr{A}$ is said to be in the class $\mathscr{R}^\tau(A,B)$, ($\tau \in \mathbb{C} \setminus \{0\}$, $-1 \le B < A \le 1$), if it satisfies the inequality

$$\left| \frac{f'(z) - 1}{(A-B)\tau - B[f'(z) - 1]} \right| < 1, \qquad z \in \mathbb{U}.$$

This class was introduced by Dixit and Pal [1].

Lemma 1.1. [2] *A function $f(z)$ of the form (1.2) is in $\mathscr{S}^*(\alpha,\beta)$ if and only if*

$$\sum_{n=2}^{\infty} [n(1+\beta) - 1 + \beta(1-2\alpha)]\, |a_n| \le 2\beta(1-\alpha). \tag{1.3}$$

Lemma 1.2. [2] *A function $f(z)$ of the form (1.2) is in $\mathscr{C}^*(\alpha,\beta)$ if and only if*

$$\sum_{n=2}^{\infty} n[n(1+\beta) - 1 + \beta(1-2\alpha)]\, |a_n| \le 2\beta(1-\alpha). \tag{1.4}$$

To obtain our main results, we need the following lemmas:

Lemma 1.3. [1] *If* $f \in \mathscr{R}^{\tau}(A,B)$ *is of the form* (1.1)*, then*

$$|a_n| \leq (A-B)\frac{|\tau|}{n}, \qquad n \in \mathbb{N} \setminus \{1\}. \tag{1.5}$$

In the present investigation, inspired by the works of Porwal [4] and Porwal and Kumar [5], we find the necessary and sufficient conditions for $\mathscr{F}(m,z)$ belonging to the classes $\mathscr{S}^*(\alpha,\beta)$ and $\mathscr{C}^*(\alpha,\beta)$. Also, we obtain inclusion relations for aforecited classes with $\mathscr{R}^{\tau}(A,B)$.

2. Necessary and sufficient conditions

Theorem 2.1. *If* $m > 0$, $0 \leq \alpha < 1$ *and* $0 < \beta \leq 1$, *then* $\mathscr{F}(m,z) \in \mathscr{S}^*(\alpha,\beta)$ *if and only if*

$$e^m m(1+\beta) \leq 2\beta(1-\alpha). \tag{2.1}$$

Proof. Since

$$\mathscr{F}(m,z) = z - \sum_{n=2}^{\infty} \frac{m^{n-1}}{(n-1)!} e^{-m} z^n,$$

in view of Lemma 1.1, it is enough to show that

$$\sum_{n=2}^{\infty} [n(1+\beta) - 1 + \beta(1-2\alpha)] \, \frac{m^{n-1}}{(n-1)!} e^{-m} \leq 2\beta(1-\alpha).$$

Let

$$T_1 = \sum_{n=2}^{\infty} [n(1+\beta) - 1 + \beta(1-2\alpha)] \, \frac{m^{n-1}}{(n-1)!} e^{-m}.$$

Now,

$$\begin{aligned}
T_1 &= \sum_{n=2}^{\infty} [n(1+\beta) - 1 + \beta(1-2\alpha)] \, \frac{m^{n-1}}{(n-1)!} e^{-m} \\
&= e^{-m} \sum_{n=2}^{\infty} [(n-1)(1+\beta) + 2\beta(1-\alpha)] \, \frac{m^{n-1}}{(n-1)!} \\
&= e^{-m} \left[(1+\beta) \sum_{n=2}^{\infty} \frac{m^{n-1}}{(n-2)!} + 2\beta(1-\alpha) \sum_{n=2}^{\infty} \frac{m^{n-1}}{(n-1)!} \right] \\
&= e^{-m} \left[(1+\beta) m e^m + 2\beta(1-\alpha)(e^m - 1) \right] \\
&= (1+\beta)m + 2\beta(1-\alpha)(1-e^{-m}).
\end{aligned}$$

But this last expression is bounded by $2\beta(1-\alpha)$, if and only if (2.1) holds. This completes the proof of Theorem 2.1. □

Theorem 2.2. *If* $m > 0$, $0 \leq \alpha < 1$ *and* $0 < \beta \leq 1$, *then* $\mathscr{F}(m,z) \in \mathscr{C}^*(\alpha,\beta)$ *if and only if*

$$e^m \left[(1+\beta)m^2 + 2(1+\beta(2-\alpha))m \right] \leq 2\beta(1-\alpha). \tag{2.2}$$

Proof. Since

$$\mathscr{F}(m,z) = z - \sum_{n=2}^{\infty} \frac{m^{n-1}}{(n-1)!} e^{-m} z^n,$$

in view of Lemma 1.2, it is enough to show that

$$\sum_{n=2}^{\infty} n[n(1+\beta) - 1 + \beta(1-2\alpha)] \frac{m^{n-1}}{(n-1)!} e^{-m} \leq 2\beta(1-\alpha).$$

Let

$$T_2 = \sum_{n=2}^{\infty} n[n(1+\beta) - 1 + \beta(1-2\alpha)] \frac{m^{n-1}}{(n-1)!} e^{-m}.$$

Therefore,

$$\begin{aligned} T_2 &= e^{-m} \left[\sum_{n=2}^{\infty} (n-1)(n-2)(1+\beta) \frac{m^{n-1}}{(n-1)!} \right. \\ &\quad \left. + \sum_{n=2}^{\infty} (n-1)[3(1+\beta) - 1 + \beta(1-2\alpha)] \frac{m^{n-1}}{(n-1)!} + \sum_{n=2}^{\infty} 2\beta(1-\alpha) \frac{m^{n-1}}{(n-1)!} \right] \\ &= e^{-m} \left[(1+\beta) \sum_{n=3}^{\infty} \frac{m^{n-1}}{(n-3)!} \right. \\ &\quad \left. + 2[1+\beta(2-\alpha)] \sum_{n=2}^{\infty} \frac{m^{n-1}}{(n-2)!} + 2\beta(1-\alpha) \sum_{n=2}^{\infty} \frac{m^{n-1}}{(n-1)!} \right] \\ &= e^{-m} \left[(1+\beta) m^2 e^m + 2(1+\beta(2-\alpha)) m e^m + 2\beta(1-\alpha)(e^m - 1) \right] \\ &= (1+\beta) m^2 + 2(1+\beta(2-\alpha)) m + 2\beta(1-\alpha)(1-e^{-m}). \end{aligned}$$

But this last expression is bounded by $2\beta(1-\alpha)$, if and only if (2.2) holds. This completes the proof of Theorem 2.2. □

3. Inclusion results

Theorem 3.1. *Let $m > 0$, $0 \leq \alpha < 1$ and $0 < \beta \leq 1$. If $f \in \mathscr{R}^\tau(A,B)$, then $\mathscr{I}(m)f \in \mathscr{S}^*(\alpha,\beta)$ if and only if*

$$(A-B)|\tau| \left[(1+\beta)(1-e^{-m}) + \frac{(\beta(1-2\alpha)-1)}{m}(1-e^{-m} - me^{-m}) \right] \leq 2\beta(1-\alpha). \tag{3.1}$$

Proof. In view of Lemma 1.1, it suffices to show that

$$P_1 = \sum_{n=2}^{\infty} [n(1+\beta) - 1 + \beta(1-2\alpha)] \frac{m^{n-1}}{(n-1)!} e^{-m} |a_n| \leq 2\beta(1-\alpha).$$

Since $f \in \mathscr{R}^\tau(A,B)$, then by Lemma 1.3, we have

$$|a_n| \leq \frac{(A-B)|\tau|}{n}.$$

Therefore,

$$\begin{aligned}
P_1 &\leq \sum_{n=2}^{\infty}[n(1+\beta)-1+\beta(1-2\alpha)]\ \frac{m^{n-1}}{(n-1)!}e^{-m}\frac{(A-B)|\tau|}{n}\\
&= (A-B)|\tau|e^{-m}\sum_{n=2}^{\infty}[n(1+\beta)-1+\beta(1-2\alpha)]\ \frac{m^{n-1}}{n!}\\
&= (A-B)|\tau|e^{-m}\left[(1+\beta)\sum_{n=2}^{\infty}\frac{m^{n-1}}{(n-1)!}+\frac{(\beta(1-2\alpha)-1)}{m}\sum_{n=2}^{\infty}\frac{m^n}{n!}\right]\\
&= (A-B)|\tau|e^{-m}\left[(1+\beta)(e^m-1)+\frac{(\beta(1-2\alpha)-1)}{m}(e^m-1-m)\right]\\
&= (A-B)|\tau|\left[(1+\beta)[1-e^{-m}]+\frac{(\beta(1-2\alpha)-1)}{m}(1-e^{-m}-me^{-m})\right].
\end{aligned}$$

But this last expression is bounded by $2\beta(1-\alpha)$, if (3.1) holds. This completes the proof of Theorem 3.1. □

Theorem 3.2. *Let $m>0$, $0\leq\alpha<1$ and $0<\beta\leq 1$. If $f\in\mathscr{R}^\tau(A,B)$, then $\mathscr{I}(m)f\in\mathscr{C}^*(\alpha,\beta)$ if and only if*

$$(A-B)|\tau|\left[m(1+\beta)+2\beta(1-\alpha)(1-e^{-m})\right]\leq 2\beta(1-\alpha). \tag{3.2}$$

Proof. In view of Lemma 1.2, it suffices to show that

$$P_2=\sum_{n=2}^{\infty}n[n(1+\beta)-1+\beta(1-2\alpha)]\ \frac{m^{n-1}}{(n-1)!}e^{-m}|a_n|\leq 2\beta(1-\alpha).$$

Since $f \in \mathscr{R}^\tau(A,B)$, then by Lemma 1.3, we have

$$|a_n| \leq \frac{(A-B)|\tau|}{n}.$$

Therefore,

$$\begin{aligned}
P_2 &\leq \sum_{n=2}^{\infty} n[n(1+\beta)-1+\beta(1-2\alpha)]\ \frac{m^{n-1}}{(n-1)!}e^{-m}\frac{(A-B)|\tau|}{n} \\
&= (A-B)|\tau|e^{-m}\sum_{n=2}^{\infty}[n(1+\beta)-1+\beta(1-2\alpha)]\ \frac{m^{n-1}}{(n-1)!} \\
&= (A-B)|\tau|e^{-m}\sum_{n=2}^{\infty}[(n-1)(1+\beta)+2\beta(1-\alpha)]\ \frac{m^{n-1}}{(n-1)!} \\
&= (A-B)|\tau|e^{-m}\left[\sum_{n=2}^{\infty}(1+\beta)\ \frac{m^{n-1}}{(n-2)!}+2\beta(1-\alpha)\sum_{n=2}^{\infty}\frac{m^{n-1}}{(n-1)!}\right] \\
&= (A-B)|\tau|e^{-m}\left[(1+\beta)\sum_{n=2}^{\infty}\ \frac{m^{n-1}}{(n-2)!}+2\beta(1-\alpha)\sum_{n=2}^{\infty}\frac{m^{n-1}}{(n-1)!}\right] \\
&= (A-B)|\tau|e^{-m}\left[me^{m}(1+\beta)+2\beta(1-\alpha)(e^{m}-1)\right].
\end{aligned}$$

But this last expression is bounded by $2\beta(1-\alpha)$, if (3.2) holds. This completes the proof of Theorem 3.2. □

4. An integral operator

Theorem 4.1. *If* $m>0$, $0\leq\alpha<1$ *and* $0<\beta\leq 1$, *then*

$$\mathscr{G}(m,z)=\int_0^z \frac{\mathscr{F}(m,t)}{t}dt$$

is in $\mathscr{C}^*(\alpha,\beta)$ *if and only if inequality* (2.1) *is satisfied.*

Proof. Since

$$\mathscr{G}(m,z) \;=\; z-\sum_{n=2}^{\infty}\frac{e^{-m}m^{n-1}}{(n-1)!}\frac{z^n}{n}=z-\sum_{n=2}^{\infty}\frac{e^{-m}m^{n-1}}{n!}z^n$$

by Lemma 1.2, we need only to show that

$$\sum_{n=2}^{\infty} n[n(1+\beta)-1+\beta(1-2\alpha)]\ \frac{m^{n-1}}{n!}e^{-m}\leq 2\beta(1-\alpha).$$

Let

$$Q_1=\sum_{n=2}^{\infty} n[n(1+\beta)-1+\beta(1-2\alpha)]\ \frac{m^{n-1}}{n!}e^{-m}.$$

Now,

$$\begin{aligned} Q_1 &= \sum_{n=2}^{\infty}[n(1+\beta)-1+\beta(1-2\alpha)]\ \frac{m^{n-1}}{(n-1)!}e^{-m} \\ &= e^{-m}\sum_{n=2}^{\infty}[(n-1)(1+\beta)+2\beta(1-\alpha)]\ \frac{m^{n-1}}{(n-1)!} \\ &= e^{-m}\left[\sum_{n=2}^{\infty}(n-1)(1+\beta)\ \frac{m^{n-1}}{(n-1)!}+\sum_{n=2}^{\infty}2\beta(1-\alpha)\ \frac{m^{n-1}}{(n-1)!}\right] \\ &= e^{-m}\left[(1+\beta)\sum_{n=2}^{\infty}\ \frac{m^{n-1}}{(n-2)!}+2\beta(1-\alpha)\sum_{n=2}^{\infty}\ \frac{m^{n-1}}{(n-1)!}\right] \\ &= e^{-m}\left[(1+\beta)me^{m}+2\beta(1-\alpha)(e^{m}-1)\right] \\ &= (1+\beta)m+2\beta(1-\alpha)(1-e^{-m}). \end{aligned}$$

But this last expression is bounded by $2\beta(1-\alpha)$, if and only if (2.1) holds. This completes the proof of Theorem 4.1. □

Theorem 4.2. *If* $m>0$, $0\le\alpha<1$ *and* $0<\beta\le 1$, *then*

$$\mathscr{G}(m,z)=\int_0^z \frac{\mathscr{F}(m,t)}{t}dt$$

is in $\mathscr{S}^*(\alpha,\beta)$ *if and only if*

$$(1+\beta)(1-e^{-m})+\frac{(\beta(1-2\alpha)-1)}{m}(1-e^{-m}-me^{-m})\le 2\beta(1-\alpha).$$

The proof of Theorem 4.2 is lines similar to the proof of Theorem 4.1, so we omitted the proof of Theorem 4.2.

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Solvability of BVPs for impulsive fractional differential equations involving the Riemann-Liouvile fractional derivatives

Yuji Liu

Abstract. In this paper, we study two classes of BVPs for impulsive fractional differential equations. Some existence results for these boundary value problems are established. Some comments on three published papers are made.

Mathematics Subject Classification (2010): 34K37, 34K45, 34B37, 34B15, 34B10, 92D25, 34A37, 34K15.

Keywords: Impulsive fractional differential equation, boundary value problem, Riemann-Liouville fractional derivative.

1. Introduction

Impulsive fractional differential equations is an important area of study [1]. In recent years, boundary value problems (BVPs) or initial value problems (IVPs) for impulsive fractional differential equations (IFDEs) have been studied by many authors. For example, in [2, 4, 3, 9, 11, 14, 15], the authors studied the existence or uniqueness of solutions of BVPs for IFDEs with Caputo type fractional derivatives and multiple starting points.

In [8], Kosmatov studied the following problem:

$$\begin{cases} D_{0^+}^{\alpha}x(t) = f(t,x(t)),\ t\in[0,1]\setminus\{t_1,t_2,\cdots,t_m\}, \\ D_{0^+}^{\beta}x(t_k^+) - D_{+}^{\beta}x(t_k^-) = J_k(x(t_k)),\ i=1,2,\cdots,m,\ I_{0^+}^{1-\alpha}x(0)=x_0, \end{cases} \tag{1.1}$$

where $\alpha\in(0,1)$, $D_{0^+}^{*}$ is the standard Riemann-Liouville fractional derivative of order $*$, $\beta\in(0,\alpha)$, $x_0\in\mathbb{R}$, $0=t_0<t_1<\cdots<t_m<t_{m+1}=1$, $J_k:\mathbb{R}\to\mathbb{R}$, $f:[0,1]\times\mathbb{R}\to\mathbb{R}$ are suitable functions.

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In [9], Liu studied the solvability of two classes of initial value problems of nonlinear impulsive multi-term fractional differential equations on half lines. One (IVP (1) for short) is as follows:

$$\begin{cases} D_{0^+}^{\alpha}x(t)+q(t)f(t,x(t),D_{0^+}^{p}x(t))=0, \quad t\in(t_i,t_{i+1}), i\in \mathbf{N}_0^m,\\ \lim\limits_{t\to 0^+} t^{1-\alpha}x(t)=x_0,\\ \lim\limits_{t\to t_i^+}(t-t_i)^{1-\alpha}x(t)=I(t_i,x(t_i),D_{0^+}^{p}x(t_i))), i\in \mathbf{N}_1^m, \end{cases} \tag{1.2}$$

and

$$\begin{cases} D_{t_i^+}^{\alpha}x(t)+q_1(t)f_1(t,x(t),D_{t_i^+}^{p}x(t))=0, \quad t\in(t_i,t_{i+1}), i\in \mathbf{N}_0^m,\\ \lim\limits_{t\to 0^+} t^{1-\alpha}x(t)=x_0,\\ \lim\limits_{t\to t_i^+}(t-t_i)^{1-\alpha}x(t)=I_1(t_i,x(t_i),D_{t_{i-1}^+}^{p}x(t_i)), i\in \mathbf{N}_1^m, \end{cases} \tag{1.3}$$

where $\alpha\in(0,1)$, $0<p<\alpha$, $D_{a^+}^{b}$ is the standard Riemann-Liouville fractional derivative of order $b>0$ with starting point a, $x_0\in\mathbf{R}$, ${\mathbf{N}_0}^m=\{0,1,2,\cdots,m\}$ and $\mathbf{N}_1^m=\{1,2,3,\cdots,m\}$, $0=t_0<t_1<t_2<t_3<\cdots<t_m<t_{m+1}=1$, $q:(0,1)\mapsto\mathbf{R}$ is continuous and satisfies that there exists $l\in(-1,-\alpha)$ such that $|q(t)|\le t^l$ for all $t\in(0,1)$, $q_1:\bigcup\limits_{i=0}^{m}(t_i,t_{i+1})\mapsto\mathbf{R}$ is continuous and satisfies that there exists $k_1>-1, l_1\le 0$ such that $|q_1(t)|\le (t-t_i)^{k_1}(t_{i+1}-t)^{l_1}$ for all $t\in(t_i,t_{i+1})(i\in\mathbf{N}_0)$, $f:(0,1)\times\mathbf{R}^2\mapsto\mathbf{R}$ is a I-Carathéodory function, $f_1:\left(\bigcup\limits_{i=0}^{m}(t_i,t_{i+1})\right)\times\mathbf{R}^2\mapsto\mathbf{R}$ is a II-Carathéodory function, $I,I_1:\{t_i:i\in\mathbf{N}\}\times\mathbf{R}^2\mapsto\mathbf{R}$ are discrete Carathéodory functions.

In [14], the authors studied the existence of solutions of the following BVP for IFDE

$$\begin{cases} D_{0^+}^{q}x(t)+\lambda(t)x(t)=f(t,x(t)),\ t\in[0,1]\setminus\{t_1,t_2,\cdots,t_m\},\\ I_{0^+}^{\alpha}x(t_i^+)-I_{0^+}^{\alpha}(t_i^-)=J_i(x(t_i)),\ i=1,2,\cdots,m,\ t^{1-q}x(t)|_{t=0}+t^{1-q}x(t)|_{t=1}=0, \end{cases} \tag{1.4}$$

where $q,\alpha\in(0,1)$, $D_{0^+}^{q}$ is the Riemann-Liouville fractional derivative, $I_{0^+}^{\alpha}$ is the Riemann-Liouville fractional integral, $0=t_0<t_1<\cdots<t_m<t_{m+1}=1$, $\lambda\in C^0([0,1],\mathbb{R})$ satisfies $\lambda_0=:\max\limits_{t\in[0,1]}\lambda(t)>0$, $J_k:\mathbf{R}\mapsto\mathbf{R}$ is continuous, f is a given piecewise continuous function. The following special case play a large role in the proof of the main theorem:

$$\begin{cases} D_{0^+}^{q}x(t)+\lambda_0x(t)=f(t,x(t)),\ t\in[0,1]\setminus\{t_1,t_2,\cdots,t_m\},\\ I_{0^+}^{\alpha}x(t_i^+)-I_{0^+}^{\alpha}(t_i^-)=J_i(x(t_i)),\ i=1,2,\cdots,m,\ t^{1-q}x(t)|_{t=0}+t^{1-q}x(t)|_{t=1}=0 \end{cases} \tag{1.5}$$

In [17], Zhao studied the following higher-order nonlinear Riemann-Liouville fractional differential equation with Riemann-Stieltjes integral boundary value conditions and impulses

$$\begin{cases} -D_{0+}^{\alpha}x(t) = \lambda a(t)f(t,x(t)), t \setminus \{t_i\}_{i=1}^{m}, \\ \Delta x(t_i) = I_i(x(t_i)), i = 1,2,\cdots,m, \\ x(0) = x'(0) = \cdots = x^{(n-2)}(0) = 0,\ x'(1) = \int_0^1 x(s)dH(s), \end{cases} \tag{1.6}$$

where D_{0+}^{α} is the standard Riemann-Liouville fractional derivative of order $n-1 < \alpha \le n$ with $n \ge 3$, the impulsive point sequence $0 = t_0 < t_1 < t_2 < \cdots < t_m < t_{m+1} = 1$,$\lambda > 0$, $f \in C([0,1] \times [0,+\infty), [0,+\infty))$, $a \in C((0,1),[0,+\infty))$, the integral $\int_0^1 x(s)dH(s)$ is the Riemann-Stieltjes integral with $H : [0,1] \to \mathbb{R}$ with

$$\delta =: \int_0^1 s^{\alpha-1}dH(s) \neq \alpha - 1.$$

Motivated by [8, 17], we investigate the solvability of the following two boundary value problems for impulsive fractional differential equations

$$\begin{cases} D_{0+}^{\alpha}x(t) - \lambda x(t) = f(t,x(t)), a.e., t \in (t_i, t_{i+1}], i \in \mathbb{N}_0^m, \\ \Delta I_{0+}^{\beta}x(t_i) =: I_{0+}^{\beta}x(t_i^+) - I_{0+}^{\beta}x(t_i) = I_n(t_i, x(t_i)), i \in \mathbb{N}_1^m, \\ \Delta D_{0+}^{\alpha-j}x(t_i) =: D_{0+}^{\alpha-j}x(t_i^+) - D_{0+}^{\alpha-j}x(t_i) = I_j(t_i, x(t_i)), i \in \mathbb{N}_1^m, j \in \mathbb{N}_1^{n-1}, \\ I_{0+}^{n-\alpha}x(0) = x_n,\ D_{0+}^{\alpha-j}x(0) = x_j, j \in \mathbb{N}_1^{n-1}, \end{cases} \tag{1.7}$$

and

$$\begin{cases} D_{0+}^{\alpha}x(t) = f(t,x(t)), t \setminus \{t_i\}_{i=1}^{m}, \\ \Delta D_{0+}^{\alpha-j}x(t_i) = I_j(t_i, x(t_i)), i \in \mathbb{N}_1^m, j \in \mathbb{N}_1^{n-1}, \\ \Delta I_{0+}^{n-\alpha}x(t_i = I_n(t_i, x(t_i)), i \in \mathbb{N}_1^m, \\ I_{0+}^{n-\alpha}x(0) = D_{0+}^{\alpha-j}x(0) = 0, j \in \mathbb{N}_1^{n-2}, D_{0+}^{\alpha-n+1}x(1) = \int_0^1 x(s)dH(s), \end{cases} \tag{1.8}$$

where m, n are positive integers, $\alpha \in (n-1, n)$, $\beta > 0$, $\lambda \in \mathbb{R}$, $0 = t_0 < t_1 < \cdots < t_m < t_{m+1} = 1$, $x_j \in \mathbb{R}(j \in \mathbb{N}_1^n$, $f : (0,1) \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory fraction, $I_j : \{t_i : i \in \mathbb{N}_1^m\} \times \mathbb{R} \to \mathbb{R}$ is a discrete Carathéodory function $(j \in \mathbb{N}_1^n)$.

The purposes of this paper are to establish existence results for solutions of IVP (1.7)$(\alpha - \beta - n = 0)$ and existence results for solutions of BVP(1.8) respectively. The method used is based upon the fixed point theorems. The results in this paper complement known ones in [8, 17] and generalize known ones [10].

A function $x : (0,1] \to \mathbb{N}$ is called a solution of IVP (1.7) (or IVP (1.8)) if $x \in C(t_i,t_{i+1}]$, $\lim_{t\to t_i^+}(t-t_i)^{n-\alpha}x(t)$ is finite for $i \in \mathbb{N}_0^m$ and $D_{0+}^{\alpha}x|_{(t_i,t_{i+1}]} - \lambda x|_{(t_i,t_{i+1}]} \in L^1(t_i,t_{i+1})$, and x satisfies all equations in (1.6) (or (1.8)).

The remainder of this paper is divided into three sections. In Section 2, we present related definitions and preliminary results. In Section 3, we establish existence results for IVP (1.7) and BVP(1.8) respectively. In Section 4, we give comments on some published papers.

2. Preliminary results

For the convenience of the readers, we firstly present the necessary definitions from the fractional calculus theory. These definitions can be found in the literature [5, 6, 7].

Let $a < b$. Denote $L^1(a,b)$ the set of all integrable functions on (a,b), $C^0(a,b]$ the set of all continuous functions on $(a,b]$. For $\phi \in L^1(a,b)$, denote

$$||\phi||_1 = \int_a^b |\phi(s)|ds.$$

For $\phi \in C^0[a,b]$, denote $||\phi||_0 = \max_{t\in[a,b]} |\phi(t)|$.

For two integers $a < b$, denote $\mathbb{N}_a^b = \{a, a+1, \cdots, b\}$.

Let the Gamma and beta functions $\Gamma(\alpha)$, $\mathbf{B}(p,q)$ and the Mitag-Leffler function $\mathbf{E}_{\alpha,\delta}(x)$ be defined by

$$\Gamma(\alpha) = \int_0^{+\infty} x^{\alpha-1}e^{-x}dx,$$

$$\mathbf{B}(p,q) = \int_0^1 x^{p-1}(1-x)^{q-1}dx,$$

$$\mathbf{E}_{\alpha,\delta}(x) = \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(k\alpha+\delta)}, \alpha, p, q, \delta > 0.$$

Definition 2.1. The Riemann-Liouville fractional integral of order $\alpha > 0$ of a function $g : (0,\infty) \mapsto \mathbf{R}$ (may be piecewise continuous) is given by

$$I_{0+}^{\alpha}g(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}g(s)ds,$$

provided that the right-hand side exists.

Definition 2.2. The Riemann-Liouville fractional derivative of order $\alpha > 0$ of a function $g : (0,\infty) \mapsto \mathbf{R}$ (may be piecewise continuous) is given by

$$D_{0+}^{\alpha}g(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_0^t \frac{g(s)}{(t-s)^{\alpha-n+1}}ds,$$

where $n-1 < \alpha < n$, provided that the right-hand side exists.

Remark 2.1. For a piecewise continuous function g which is continuous on $(t_i,t_{i+1}]$ $(i \in \mathbf{N}_0^m, 0 = t_0 < t_1 < \cdots < t_i < \cdots < t_m < t_{m+1} = 1)$, and $t \in (t_i,t_{i+1}]$, the

Riemann-Liouville fractional integral of order $\alpha > 0$ of g on $(0,t]$ with $t \in (t_i, t_{i+1}]$ is given by

$$I_{0^+}^{\alpha} g(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g(s) ds$$
$$= \sum_{j=0}^{i} \int_{t_j}^{t_{j+1}} \frac{(t-s)^{\alpha-1} g(s) ds}{\Gamma(\alpha)} + \int_{t_i}^{t} \frac{(t-s)^{\alpha-1} g(s) ds}{\Gamma(\alpha)},$$

provided that each term in the right-hand side exists.

Let $\alpha \in (n-1, n)$ with n being a positive integer. For a piecewise continuous function g which is continuous on $(t_i, t_{i+1}]$ $(i \in \mathbf{N}_0^m, 0 = t_0 < t_1 < \cdots < t_i < \cdots < t_m < t_{m+1} = 1)$, and $t \in (t_i, t_{i+1}]$, the Riemann-Liouville fractional derivative of order $\alpha > 0$ of g on $(0,t]$ with $t \in (t_i, t_{i+1}]$ is given by

$$D_{0^+}^{\alpha} g(t) = \frac{1}{\Gamma(n-\alpha)} \left[\int_0^t (t-s)^{n-\alpha-1} g(s) ds \right]^{(n)}$$
$$= \frac{\left[\sum_{j=0}^{i} \int_{t_j}^{t_{j+1}} (t-s)^{n-\alpha-1} g(s) ds + \int_{t_i}^{t} (t-s)^{n-\alpha-1} g(s) ds \right]^{(n)}}{\Gamma(n-\alpha)}$$

provided that each term in the right-hand side exists.

Definition 2.3. We call $F : (0,1) \times \mathbf{R} \mapsto \mathbf{R}$ a **Carathéodory function** if it satisfies the followings:

(i) $t \mapsto F(t, (t-t_i)^{n-\alpha} u)$ are measurable on $(t_i, t_{i+1}) (i \in \mathbf{N}_0^m)$ for any $u \in \mathbf{R}$,

(ii) $u \mapsto F(t, (t-t_i)^{n-\alpha} u)$ is continuous on $\mathbf{R}$ for all $t \in (t_i, t_{i+1}) (i \in \mathbf{N}_0^m)$,

(iii) for each $r > 0$, there exists $M_r \geq 0$ such that $|F(t, (t-t_i)^{n-\alpha} u)| \leq M_r$ for all $t \in (t_i, t_{i+1}) (i \in \mathbf{N}_0^m)$ and $|u| \leq r$.

Definition 2.4. We call $G : \{t_i : i \in \mathbf{N}_1^m\} \times \mathbf{R} \mapsto \mathbf{R}$ a discrete **I-Carathéodory function** if it satisfies the followings:

(i) $u \mapsto G(t_i, (t_i - t_{i-1})^{n-\alpha} u)$ is continuous on $\mathbf{R}$ for all $i \in \mathbf{N}_1^m$,

(ii) for each $r > 0$, there exists $M_r \geq 0$ such that $|G(t_i, (t_i - t_{i-1})^{n-\alpha} u)| \leq M_r$ for all $i \in \mathbf{N}_1^m$ and $|u| \leq r$.

Suppose that $\alpha \in (n-1, n)$, $0 = t_0 < t_1 < \cdots < t_{m+1} = 1$. Denote

$$PC_{n-\alpha}(0,T] = \left\{ x : (0,1] \mapsto \mathbf{R} : \begin{array}{c} x|_{(t_i, t_{i+1}]} \in C^0(t_i, t_{i+1}], i \in \mathbf{N}_0^m, \\ \lim_{t \to t_i^+} (t-t_i)^{n-\alpha} x(t), \ i \in \mathbf{N}_0^m \text{ are finite} \end{array} \right\}.$$

Define

$$||x|| = ||x||_{n-\alpha} = \max \left\{ \sup_{t \in (t_i, t_{i+1}]} (t-t_i)^{n-\alpha} |x(t)|, i \in \mathbf{N}_0^m \right\}.$$

Then $PC_{n-\alpha}$ is a Banach space with the norm $|| \cdot ||$ defined.

Theorem 2.1. *Suppose that $\alpha \in (n-1, n)$, $\lambda \in \mathbb{R}$, $h : (0,1) \to \mathbb{R}$ is continuous and satisfies $|h(t)| \leq t^k (1-t)^l$ for all $t \in (0,1)$, where $k > -1$, $l \leq 0$ with $1 + k + l > 0$.*

Then $x \in PC_{n-\alpha}(0,1]$ *is a piecewise continuous solution of*

$$D_{0+}^{\alpha}x(t) - \lambda x(t) = h(t), \ a.e., \ t \in (t_i, t_{i+1}], i \in \mathbb{N}_0^m \tag{2.1}$$

if and only if there exist constants $c_{\nu k} \in \mathbb{R}(\nu \in \mathbb{N}_1^n, k \in \mathbb{N}_0)$ *such that*

$$lx(t) = \int_0^t (t-s)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(\lambda(t-s)^{\alpha})h(s)ds + \sum_{k=0}^{j}\sum_{\nu=1}^{n} c_{\nu k}(t-t_k)^{\alpha-\nu}\mathbf{E}_{\alpha,\alpha-\nu+1}(\lambda(t-t_k)^{\alpha}), t \in (t_j, t_{j+1}], j \in \mathbb{N}_0^m. \tag{2.2}$$

Proof. We have

$$\begin{aligned} & t^{n-\alpha}\left|\int_0^t (t-s)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(\lambda(t-s)^{\alpha})h(s)ds\right| \\ & \le t^{n-\alpha}\sum_{\chi=0}^{\infty}\frac{|\lambda|^{\chi}}{\Gamma(\chi\alpha+\alpha)}\int_0^t (t-s)^{\alpha-1+\chi\alpha}s^k(1-s)^l ds \\ & = t^{n-\alpha}\sum_{\chi=0}^{\infty}\frac{|\lambda|^{\chi}}{\Gamma(\chi\alpha+\alpha)}t^{\chi\alpha+\alpha}\int_0^1 (1-w)^{\alpha+l-1+\chi\alpha}w^k dw \\ & \le t^{n-\alpha}\sum_{\chi=0}^{\infty}\frac{|\lambda|^{\chi}}{\Gamma(\chi\alpha+\alpha)}t^{\chi\alpha+k+\alpha+l}\int_0^1 (1-w)^{\alpha+l-1}w^k dw \\ & = \mathbf{B}(\alpha+l, k+1)\mathbf{E}_{\alpha,\alpha}(|\lambda|t^{\alpha})t^{n+k+l} \to 0 \text{ as } t \to 0. \end{aligned}$$

Then $t \to t^{n-\alpha}\int_0^t (t-s)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(\lambda(t-s)^{\alpha})h(s)ds$ is continuous on $[0,1]$.

Step 1. Assume $x \in PC_{n-\alpha}(0,1]$ is a solution of (2.1). We prove x satisfies (2.2).

From (5.1)-(5.3) in [7], there exist constants $c_{\nu 0} \in \mathbb{R}$ such that

$$x(t) = \int_0^t (t-s)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(\lambda(t-s)^{\alpha})h(s)ds + \sum_{\nu=1}^{n} c_{\nu 0}t^{\alpha-\nu}\mathbf{E}_{\alpha,\alpha-\nu+1}(\lambda t^{\alpha}), \ t \in (t_0, t_1].$$

It follows that (2.2) holds for $j=0$. Now suppose that (3.7) holds for $i = 0, 1, \cdots, \omega < m$, i.e.,

$$x(t) = \int_0^t (t-s)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(\lambda(t-s)^{\alpha})h(s)ds + \sum_{k=0}^{j}\sum_{\nu=1}^{n} c_{\nu k}(t-t_k)^{\alpha-\nu}\mathbf{E}_{\alpha,\alpha-\nu+1}(\lambda(t-t_k)^{\alpha}), t \in (t_j, t_{j+1}], j \in \mathbb{N}_0^{\omega}. \tag{2.3}$$

We will prove that (2.2) holds for $i = \omega+1$. Then by mathematical induction method, (2.2) holds for all $i \in \mathbb{N}_0^m$. In order to get the exact expression of x on $(t_{\omega+1}, t_{\omega+2}]$, we suppose that there exists Φ such that

$$x(t) = \Phi(t) + \int_0^t (t-s)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(\lambda(t-s)^{\alpha})h(s)ds + \sum_{k=0}^{\omega}\sum_{\nu=1}^{n} c_{\nu k}(t-t_k)^{\alpha-\nu}\mathbf{E}_{\alpha,\alpha-\nu+1}(\lambda(t-t_k)^{\alpha}), t \in (t_{\omega+01}, t_{\omega+2}]. \tag{2.4}$$

Using Definition 2.2, (2.3) and (2.4), we know for $t \in (t_{\omega+1}, t_{\omega+2}]$ by direct computation that

$$D_{0^+}^{\alpha} x(t) = \frac{\left[\int_0^t (t-s)^{n-\alpha-1} x(s) ds\right]^{(n)}}{\Gamma(n-\alpha)}$$

$$= \frac{\left[\sum\limits_{\rho=0}^{\omega} \int_{t_\rho}^{t_\rho+1} (t-s)^{n-\alpha-1} x(s) ds + \int_{t_{\omega+1}}^{t} (t-s)^{n-\alpha-1} x(s) ds\right]^{(n)}}{\Gamma(n-\alpha)}$$

$$= \left[\sum_{\rho=0}^{\omega} \int_{t_\rho}^{t_\rho+1} (t-s)^{n-\alpha-1} \left(\int_0^s (s-u)^{\alpha-1} \mathbf{E}_{\alpha,\alpha}(\lambda(s-u)^{\alpha}) h(u) du\right.\right.$$

$$\left.\left. + \sum_{k=0}^{\rho} \sum_{\nu=1}^{n} c_{\nu k} (s-t_k)^{\alpha-\nu} \mathbf{E}_{\alpha,\alpha-\nu+1}(\lambda(s-t_k)^{\alpha})\right) ds\right]^{(n)} \Big/ \Gamma(n-\alpha)$$

$$+ \left[\int_{t_{\omega+1}}^{t} (t-s)^{n-\alpha-1} \left(\Phi(s) + \int_0^s (s-u)^{\alpha-1} \mathbf{E}_{\alpha,\alpha}(\lambda(s-u)^{\alpha}) h(u) du\right.\right.$$

$$\left.\left. + \sum_{k=0}^{\omega} \sum_{\nu=1}^{n} c_{\nu k} (s-t_k)^{\alpha-\nu} \mathbf{E}_{\alpha,\alpha-\nu+1}(\lambda(s-t_k)^{\alpha})\right) ds\right]^{(n)} \Big/ \Gamma(n-\alpha)$$

$$= D_{t_{\omega+1}^+}^{\alpha} \Phi(t)$$

$$+ \frac{\left[\sum\limits_{\rho=0}^{\omega} \sum\limits_{k=0}^{\rho} \sum\limits_{\nu=1}^{n} c_{\nu k} \int_{t_\rho}^{t_\rho+1} (t-s)^{n-\alpha-1} (s-t_k)^{\alpha-\nu} \mathbf{E}_{\alpha,\alpha-\nu+1}(\lambda(s-t_k)^{\alpha}) ds\right]^{(n)}}{\Gamma(n-\alpha)}$$

$$+ \left[\int_0^t (t-s)^{n-\alpha-1} \int_0^s (s-u)^{\alpha-1} \mathbf{E}_{\alpha,\alpha}(\lambda(s-u)^{\alpha}) h(u) du ds\right.$$

$$\left. + \sum_{k=0}^{\omega} \sum_{\nu=1}^{n} c_{\nu k} \int_{t_{\omega+1}}^{t} (t-s)^{n-\alpha-1} (s-t_k)^{\alpha-\nu} \mathbf{E}_{\alpha,\alpha-\nu+1}(\lambda(s-t_k)^{\alpha}) ds\right]^{(n)} \Big/ \Gamma(n-\alpha)$$

$$= D_{t_{\omega+1}^+}^{\alpha} \Phi(t)$$

$$+ \frac{\left[\sum\limits_{k=0}^{\omega} \sum\limits_{\rho=k}^{\omega} \sum\limits_{\nu=1}^{n} c_{\nu k} \int_{t_\rho}^{t_\rho+1} (t-s)^{n-\alpha-1} (s-t_k)^{\alpha-\nu} \mathbf{E}_{\alpha,\alpha-\nu+1}(\lambda(s-t_k)^{\alpha}) ds\right]^{(n)}}{\Gamma(n-\alpha)}$$

$$+ \left[\int_0^t (t-s)^{n-\alpha-1} \int_0^s (s-u)^{\alpha-1} \mathbf{E}_{\alpha,\alpha}(\lambda(s-u)^{\alpha}) h(u) du ds\right.$$

$$\left. + \sum_{k=0}^{\omega} \sum_{\nu=1}^{n} c_{\nu k} \int_{t_{\omega+1}}^{t} (t-s)^{n-\alpha-1} (s-t_k)^{\alpha-\nu} \mathbf{E}_{\alpha,\alpha-\nu+1}(\lambda(s-t_k)^{\alpha}) ds\right]^{(n)} \Big/ \Gamma(n-\alpha)$$

$$= D^{\alpha}_{t^+_{\omega+1}}\Phi(t) + \frac{\left[\int_0^t (t-s)^{n-\alpha-1}\int_0^s (s-u)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(\lambda(s-u)^{\alpha})h(u)duds\right]^{(n)}}{\Gamma(n-\alpha)}$$

$$+\frac{\left[\sum\limits_{k=0}^{\omega}\sum\limits_{\nu=1}^{n} c_{\nu k}\int_{t_k}^t (t-s)^{n-\alpha-1}(s-t_k)^{\alpha-\nu}\mathbf{E}_{\alpha,\alpha-\nu+1}(\lambda(s-t_k)^{\alpha})ds\right]^{(n)}}{\Gamma(n-\alpha)}$$

$$= D^{\alpha}_{t^+_{\omega+1}}\Phi(t) + \frac{\left[\int_0^t \sum\limits_{\chi=0}^{\infty}\frac{\lambda^{\chi}}{\Gamma(\chi\alpha+\alpha)}\int_u^t (t-s)^{n-\alpha-1}(s-u)^{\alpha-1+\chi\alpha}dsh(u)du\right]^{(n)}}{\Gamma(n-\alpha)}$$

$$+\frac{\left[\sum\limits_{k=0}^{\omega}\sum\limits_{\nu=1}^{n} c_{\nu k}\sum\limits_{\chi=0}^{\infty}\frac{\lambda^{\chi}}{\Gamma(\chi\alpha+\alpha-\nu+1)}\int_{t_k}^t (t-s)^{n-\alpha-1}(s-t_k)^{\alpha-\nu+\chi\alpha}ds\right]^{(n)}}{\Gamma(n-\alpha)}$$

by

$$\frac{s-u}{t-u}=w, \frac{s-t_k}{t-t_k}=w$$

$$= D^{\alpha}_{t^+_{\omega+1}}\Phi(t) + \frac{\left[\int_0^t \sum\limits_{\chi=0}^{\infty}\frac{\lambda^{\chi}}{\Gamma(\chi\alpha+\alpha)}(t-u)^{\chi\alpha+n-1}\int_0^1 (1-w)^{n-\alpha-1}w^{\alpha-1+\chi\alpha}dwh(u)du\right]^{(n)}}{\Gamma(n-\alpha)}$$

$$+\frac{\left[\sum\limits_{k=0}^{\omega}\sum\limits_{\nu=1}^{n} c_{\nu k}\sum\limits_{\chi=0}^{\infty}\frac{\lambda^{\chi}}{\Gamma(\chi\alpha+\alpha-\nu+1)}(t-t_k)^{\chi\alpha+n-\nu}\int_0^1 (1-w)^{n-\alpha-1}w^{\alpha-\nu+\chi\alpha}dw\right]^{(n)}}{\Gamma(n-\alpha)}$$

$$= D^{\alpha}_{t^+_{\omega+1}}\Phi(t) + h(t) + \left[\int_0^t \sum_{\chi=0}^{\infty}\frac{\lambda^{\chi}(t-u)^{\chi\alpha+n-1}}{\Gamma(\chi\alpha+n)}h(u)du\right]^{(n)}$$

$$+\left[\sum_{k=0}^{\omega}\sum_{\nu=1}^{n} c_{\nu k}\sum_{\chi=0}^{\infty}\frac{\lambda^{\chi}(t-t_k)^{\chi\alpha+n-\nu}}{\Gamma(\chi\alpha+n-\nu+1)}\right]^{(n)}$$

$$-D^{\alpha}_{t^+_{\omega+1}}\Phi(t) + \int_0^t \sum_{\chi=1}^{\infty}\frac{\lambda^{\chi}(t-u)^{\chi\alpha+-1}}{\Gamma(\chi\alpha)}h(u)du + \sum_{k=0}^{\omega}\sum_{\nu=1}^{n} c_{\nu k}\sum_{\chi=1}^{\infty}\frac{\lambda^{\chi}(t-t_k)^{\chi\alpha-\nu}}{\Gamma(\chi\alpha-\nu+1)}.$$

Thus

$$D^{\alpha}_{0+}x(t)-\lambda x(t) = D^{\alpha}_{t^+_{\omega+1}}\Phi(t) + h(t) + \int_0^t \sum_{\chi=1}^{\infty}\frac{\lambda^{\chi}(t-u)^{\chi\alpha+-1}}{\Gamma(\chi\alpha)}h(u)du$$

$$+\sum_{k=0}^{\omega}\sum_{\nu=1}^{n} c_{\nu k}\sum_{\chi=1}^{\infty}\frac{\lambda^{\chi}(t-t_k)^{\chi\alpha-\nu}}{\Gamma(\chi\alpha-\nu+1)}$$

$$-\lambda\left[\Phi(t) + \int_0^t (t-s)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(\lambda(t-s)^{\alpha})h(s)ds\right.$$

$$+\sum_{k=0}^{\omega}\sum_{\nu=1}^{n} c_{\nu k}(t-t_k)^{\alpha-\nu}\mathbf{E}_{\alpha,\alpha-\nu+1}(\lambda(t-t_k)^{\alpha})\Bigg] = D^{\alpha}_{t^+_{\omega+1}}\Phi(t)-\lambda\Phi(t)+h(t).$$

From $D^{\alpha}_{0^+}x(t)-\lambda x(t)=h(t)$, we have $D^{\alpha}_{t^+_{\omega+1}}\Phi(t)-\lambda\Phi(t)=0$ on $(t_{\omega+1},t_{\omega+2}]$. By (5.1)-(5.3) in [7], we know that there exists constants $c_{\nu\omega+1}\in\mathbb{R}$ such that

$$\Phi(t)=\sum_{\nu=1}^{n} c_{\nu\omega+1}(t-t_{\omega+1})^{\alpha-\nu}\mathbf{E}_{\alpha,\alpha}(\lambda(t-t_{\omega+1})^{\alpha}).$$

Substituting Φ into (2.4). We know that (2.2) holds for $i=\omega+1$. By mathematical induction method, we know that (2.2) holds for $i\in\mathbb{N}_0^m$.

Step 2. We prove that x is a piecewise continuous solution of (2.1) if x satisfies (2.2). Since x satisfies (2.2), we know that $x|_{(t_i,t_{i+1}]}\in C^0(t_i,t_{i+1}]$ $(i\in\mathbb{N}_0^m)$ and

$$\lim_{t\to t_j^+}(t-t_j)^{n-\alpha}x(t)$$

exists and is finite for all $i\in\mathbb{N}_0^m$. Furthermore, by direct computation similarly to Step 1, by Definition 2.1, we can get for $t\in(t_j,t_{j+1}]$ that

$$I^{n-\alpha}_{0^+}x(t)=\frac{\int_0^t(t-s)^{n-\alpha-1}x(s)ds}{\Gamma(\alpha)}$$

$$=\frac{\sum_{\rho=0}^{j-1}\int_{t_\rho}^{t_\rho+1}(t-s)^{n-\alpha-1}x(s)ds+\int_{t_j}^{t}(t-s)^{n-\alpha-1}x(s)ds}{\Gamma(n-\alpha)}$$

$$=\int_0^t\sum_{\chi=0}^{\infty}\frac{\lambda^{\chi}}{\Gamma(\chi\alpha+n)}(t-u)^{\chi\alpha+n-1}h(u)du$$

$$+\sum_{k=0}^{j}\sum_{\nu=1}^{n}c_{\nu k}\sum_{\chi=0}^{\infty}\frac{\lambda^{\chi}}{\Gamma(\chi\alpha+n-\nu+1)}(t-t_k)^{\chi\alpha+n-\nu}$$

$$=\int_0^t(t-u)^{n-1}\mathbf{E}_{\alpha,n}(\lambda(t-s)^{\alpha})h(s)ds+\sum_{k=0}^{j}\sum_{\nu=1}^{n}c_{\nu k}(t-t_k)^{n-\nu}\mathbf{E}_{\alpha,n-\nu+1}(\lambda(t-t_k)^{\alpha}).$$

By Definition 2.2, we get for $i\in\mathbb{N}_1^{n-1}$, $t\in(t_jt_{j+1}]$ that

$$D^{\alpha-i}_{0^+}x(t)=\frac{\left[\int_0^t(t-s)^{n-\alpha-1}x(s)ds\right]^{(n-i)}}{\Gamma(\alpha)}$$

$$=\frac{\left[\sum_{\rho=0}^{j-1}\int_{t_\rho}^{t_\rho+1}(t-s)^{n-\alpha-1}x(s)ds+\int_{t_j}^{t}(t-s)^{n-\alpha-1}x(s)ds\right]^{(n-i)}}{\Gamma(n-\alpha)}$$

$$=\left[\int_0^t\sum_{\chi=0}^{\infty}\frac{\lambda^{\chi}(t-u)^{\chi\alpha+n-1}}{\Gamma(\chi\alpha+n)}h(u)du\right]^{(n-i)}+\left[\sum_{k=0}^{j}\sum_{\nu=1}^{n}c_{\nu k}\sum_{\chi=0}^{\infty}\frac{\lambda^{\chi}(t-t_k)^{\chi\alpha+n-\nu}}{\Gamma(\chi\alpha+n-\nu+1)}\right]^{(n-i)}$$

$$= \int_0^t \sum_{\chi=0}^{\infty} \frac{\lambda^\chi (t-u)^{\chi\alpha+i-1}}{\Gamma(\chi\alpha+i)} h(u)du$$

$$+\sum_{k=0}^{j}\sum_{\nu=1}^{n} c_{\nu k} \sum_{\chi=1}^{\infty} \frac{\lambda^\chi (t-t_k)^{\chi\alpha+i-\nu}}{\Gamma(\chi\alpha+i-\nu+1)} + \sum_{k=0}^{j}\sum_{\nu=1}^{i} c_{\nu k} \frac{(t-t_k)^{i-\nu}}{\Gamma(i-\nu+1)}$$

$$= \int_0^t (t-u)^{i-1}\mathbf{E}_{\alpha,i}(\lambda(t-u)^\alpha)h(u)du + \sum_{k=0}^{j}\sum_{\nu=1}^{i} c_{\nu k}(t-t_k)^{i-\nu}\mathbf{E}_{\alpha,i-\nu+1}(\lambda(t-t_k)^\alpha)$$

$$+\lambda \sum_{k=0}^{j}\sum_{\nu=i+1}^{n} c_{\nu k}(t-t_k)^{\alpha+i-\nu}\mathbf{E}_{\alpha,\alpha+i-\nu+1}(\lambda(t-t_k)^\alpha).$$

We see that

$$I_{0^+}^{n-\alpha}x|_{(t_j,t_{j+1}]}, D_{0^+}^{\alpha-i}x|_{(t_j,t_{j+1}]}(i \in \mathbb{N}_1^{n-1}, j \in \mathbb{N}_0^m)$$

are continuous and the limits

$$\lim_{t\to t_j^+} I_{0^+}^{n-\alpha}x(t), \quad \lim_{t\to t_j^+} D_{0^+}^{\alpha-i}x(t) \quad (i \in \mathbb{N}_1^{n-1},\ j \in \mathbb{N}_0^m)$$

are finite. By Definition 2.2, $\alpha \in (n-1,n)$, for $t \in (t_j,t_{j+1}]$, we have

$$D_{0^+}^{\alpha}x(t) = \frac{\left[\int_0^t (t-s)^{n-\alpha-1}x(s)ds\right]^{(n-i)}}{\Gamma(\alpha)}$$

$$= \frac{\left[\sum_{\rho=0}^{j-1}\int_{t_\rho}^{t_\rho+1}(t-s)^{n-\alpha-1}x(s)ds + \int_{t_j}^t (t-s)^{n-\alpha-1}x(s)ds\right]^{(n-i)}}{\Gamma(n-\alpha)}$$

$$= \left[\int_0^t \sum_{\chi=0}^{\infty} \frac{\lambda^\chi (t-u)^{\chi\alpha+n-1}}{\Gamma(\chi\alpha+n)} h(u)du\right]^{(n)} + \left[\sum_{k=0}^{j}\sum_{\nu=1}^{n} c_{\nu k}\sum_{\chi=0}^{\infty}\frac{\lambda^\chi (t-t_k)^{\chi\alpha+n-\nu}}{\Gamma(\chi\alpha+n-\nu+1)}\right]^{(n)}$$

$$= h(t) + \int_0^t \sum_{\chi=1}^{\infty} \frac{\lambda^\chi (t-u)^{\chi\alpha-1}}{\Gamma(\chi\alpha)} h(u)du + \sum_{k=0}^{j}\sum_{\nu=1}^{n} c_{\nu k}\sum_{\chi=1}^{\infty}\frac{\lambda^\chi (t-t_k)^{\chi\alpha+-\nu}}{\Gamma(\chi\alpha-\nu+1)}, t \in (t_j,t_{j+1}].$$

Then

$$D_{0^+}^{\alpha}x(t) - \lambda x(t) = h(t), \quad t \in (t_j,t_{j+1}],\ j \in \mathbb{N}_0^m.$$

So x is a piecewise continuous solution of (2.1). The proof is completed. □

Remark 2.1. Lemma 2.1 (when $\lambda = 0$) is one of the main results in [10] (see Theorem 3.2 in [10]). So our results generalizes the one in [10].

Theorem 2.2. *Suppose that x is a solution of (2.1) and is defined by (2.2). Then*

$$I_{0^+}^{\beta}x(t) = \int_0^t (t-s)^{\alpha+\beta-1}\mathbf{E}_{\alpha,\alpha+\beta}(\lambda(t-s)^\alpha)h(s)ds$$
$$+\sum_{k=0}^{j}\sum_{\nu=1}^{n} c_{\nu k}(t-t_k)^{\alpha+\beta-\nu}\mathbf{E}_{\alpha,\alpha+\beta-\nu+1}(\lambda(t-t_k)^\alpha),$$
$$t \in (t_j,t_{j+1}], j \in \mathbb{N}_0^m \tag{2.5}$$

$$I_{0^+}^{n-\alpha}x(t)=\sum_{k=0}^{i}\sum_{\nu=1}^{n}c_{\nu k}(t-t_k)^{n-\nu}\mathbf{E}_{\alpha,n-\nu+1}(\lambda(t-t_k)^{\alpha})$$
$$+\int_0^t(t-u)^{n-1}\mathbf{E}_{\alpha,n}(\lambda(t-u)^{\alpha})h(u)du,$$
$$t\in(t_i,t_{i+1}],\ i\in\mathbb{N}_0^m, \tag{2.6}$$

and

$$D_{0^+}^{\alpha-j}x(t)=\sum_{k=0}^{i}\sum_{\nu=1}^{j}c_{\nu k}(t-t_k)^{j-\nu}\mathbf{E}_{\alpha,j-\nu+1}(\lambda(t-t_k)^{\alpha})$$
$$+\lambda\sum_{k=0}^{i}\sum_{\nu=j+1}^{n}c_{\nu k}(t-t_k)^{\alpha+j-\nu}\mathbf{E}_{\alpha,\alpha+j-\nu+1}(\lambda(t-t_k)^{\alpha})$$
$$+\int_0^t(t-u)^{j-1}\mathbf{E}_{\alpha,j}(\lambda(t-u)^{\alpha})h(u)du,$$
$$t\in(t_i,t_{i+1}],\ i\in\mathbb{N}_0^m,\ j\in\mathbb{N}_1^{n-1}. \tag{2.7}$$

Proof. We firstly prove (2.5). For $t\in(t_i,t_{i+1}]$, by (2.2) and Definition 2.1, we get

$$I_{0^+}^{\beta}x(t)=\frac{\sum_{\mu=0}^{i-1}\int_{t_\mu}^{t_{\mu+1}}(t-s)^{\beta-1}x(s)ds+\int_{t_i}^{t}(t-s)^{\beta-1}x(s)ds}{\Gamma(\beta)}$$

$$=\sum_{\mu=0}^{i-1}\int_{t_\mu}^{t_{\mu+1}}(t-s)^{\beta-1}\left(\int_0^s(s-u)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(\lambda(s-u)^{\alpha})h(u)du\right.$$
$$\left.+\sum_{k=0}^{\mu}\sum_{\nu=1}^{n}c_{\nu k}(s-t_k)^{\alpha-\nu}\mathbf{E}_{\alpha,\alpha-\nu+1}(\lambda(s-t_k)^{\alpha})\right)ds\Big/\Gamma(\beta)$$
$$+\int_{t_i}^{t}(t-s)^{\beta-1}\left(\int_0^s(s-u)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(\lambda(s-u)^{\alpha})h(u)du\right.$$
$$\left.+\sum_{k=0}^{i}\sum_{\nu=1}^{n}c_{\nu k}(s-t_k)^{\alpha-\nu}\mathbf{E}_{\alpha,\alpha-\nu+1}(\lambda(s-t_k)^{\alpha})\right)ds\Big/\Gamma(\beta)$$

$$=\frac{\sum_{\mu=0}^{i-1}\int_{t_\mu}^{t_{\mu+1}}(t-s)^{\beta-1}\sum_{k=0}^{\mu}\sum_{\nu=1}^{n}c_{\nu k}(s-t_k)^{\alpha-\nu}\mathbf{E}_{\alpha,\alpha-\nu+1}(\lambda(s-t_k)^{\alpha})ds}{\Gamma(\beta)}$$
$$+\frac{\int_{t_i}^{t}(t-s)^{\beta-1}\sum_{k=0}^{i}\sum_{\nu=1}^{n}c_{\nu k}(s-t_k)^{\alpha-\nu}\mathbf{E}_{\alpha,\alpha-\nu+1}(\lambda(s-t_k)^{\alpha})ds}{\Gamma(\beta)}$$
$$+\frac{\int_0^t(t-s)^{\beta-1}\int_0^s(s-u)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(\lambda(s-u)^{\alpha})h(u)duds}{\Gamma(\beta)}$$

$$
\begin{aligned}
&= \frac{\sum_{k=0}^{i-1}\sum_{\mu=k}^{i-1}\sum_{\nu=1}^{n} c_{\nu k}\int_{t_\mu}^{t_{\mu+1}}(t-s)^{\beta-1}(s-t_k)^{\alpha-\nu}\sum_{\chi=0}^{\infty}\frac{\lambda^\chi}{\Gamma(\chi\alpha+\alpha-\nu+1)}(s-t_k)^{\chi\alpha}ds}{\Gamma(\beta)}\\
&\quad+\frac{\sum_{k=0}^{i}\sum_{\nu=1}^{n} c_{\nu k}\int_{t_i}^{t}(t-s)^{\beta-1}(s-t_k)^{\alpha-\nu}\sum_{\chi=0}^{\infty}\frac{\lambda^\chi}{\Gamma(\chi\alpha+\alpha-\nu+1)}(s-t_k)^{\chi\alpha}ds}{\Gamma(\beta)}\\
&\quad+\frac{\int_0^t\int_u^t(t-s)^{\beta-1}(s-u)^{\alpha-1}\sum_{\chi=0}^{\infty}\frac{\lambda^\chi}{\Gamma(\chi\alpha+\alpha)}(s-u)^{\chi\alpha}dsh(u)du}{\Gamma(\beta)}\\
&= \frac{\sum_{k=0}^{i-1}\sum_{\mu=k}^{i-1}\sum_{\nu=1}^{n} c_{\nu k}\sum_{\chi=0}^{\infty}\frac{\lambda^\chi}{\Gamma(\chi\alpha+\alpha-\nu+1)}\int_{t_\mu}^{t_{\mu+1}}(t-s)^{\beta-1}(s-t_k)^{\alpha-\nu+\chi\alpha}ds}{\Gamma(\beta)}\\
&\quad+\frac{\sum_{k=0}^{i}\sum_{\nu=1}^{n} c_{\nu k}\sum_{\chi=0}^{\infty}\frac{\lambda^\chi}{\Gamma(\chi\alpha+\alpha-\nu+1)}\int_{t_i}^{t}(t-s)^{\beta-1}(s-t_k)^{\alpha-\nu+\chi\alpha}ds}{\Gamma(\beta)}\\
&\quad+\frac{\sum_{\chi=0}^{\infty}\frac{\lambda^\chi}{\Gamma(\chi\alpha+\alpha)}\int_0^t\int_u^t(t-s)^{\beta-1}(s-u)^{\alpha-1+\chi\alpha}dsh(u)du}{\Gamma(\beta)}\\
&= \frac{\sum_{k=0}^{i-1}\sum_{\mu=k}^{i-1}\sum_{\nu=1}^{n} c_{\nu k}\sum_{\chi=0}^{\infty}\frac{\lambda^\chi}{\Gamma(\chi\alpha+\alpha-\nu+1)}t-t_k)^{\chi\alpha+\alpha+\beta-\nu}\int_{\frac{t_\mu-t_K}{t-t_k}}^{\frac{t_{\mu+1}-t_k}{t-t_k}}(1-w)^{\beta-1}w^{\alpha-\nu+\chi\alpha}dw}{\Gamma(\beta)}\\
&\quad+\frac{\sum_{k=0}^{i}\sum_{\nu=1}^{n} c_{\nu k}\sum_{\chi=0}^{\infty}\frac{\lambda^\chi}{\Gamma(\chi\alpha+\alpha-\nu+1)}(t-t_k)^{\chi\alpha+\alpha+\beta-\nu}\int_{\frac{t_i-t_k}{t-t_k}}^{1}(1-w)^{\beta-1}w^{\alpha-\nu+\chi\alpha}dw}{\Gamma(\beta)}\\
&\quad+\frac{\sum_{\chi=0}^{\infty}\frac{\lambda^\chi}{\Gamma(\chi\alpha+\alpha)}\int_0^t(t-u)^{\chi\alpha+\alpha+\beta-1}\int_0^1(1-w)^{\beta-1}w^{\alpha-1+\chi\alpha}dwh(u)du}{\Gamma(\beta)}\\
&= \frac{\sum_{k=0}^{i}\sum_{\nu=1}^{n} c_{\nu k}\sum_{\chi=0}^{\infty}\frac{\lambda^\chi}{\Gamma(\chi\alpha+\alpha-\nu+1)}(t-t_k)^{\chi\alpha+\alpha+\beta-\nu}\int_0^1(1-w)^{\beta-1}w^{\alpha-\nu+\chi\alpha}dw}{\Gamma(\beta)}\\
&\quad+\frac{\sum_{\chi=0}^{\infty}\frac{\lambda^\chi}{\Gamma(\chi\alpha+\alpha)}\int_0^t(t-u)^{\chi\alpha+\alpha+\beta-1}\int_0^1(1-w)^{\beta-1}w^{\alpha-1+\chi\alpha}dwh(u)du}{\Gamma(\beta)}\\
&= \sum_{k=0}^{i}\sum_{\nu=1}^{n} c_{\nu k}\sum_{\chi=0}^{\infty}\frac{\lambda^\chi}{\Gamma(\chi\alpha+\alpha+\beta-\nu+1)}(t-t_k)^{\chi\alpha+\alpha+\beta-\nu}\\
&\quad+\sum_{\chi=0}^{\infty}\frac{\lambda^\chi}{\Gamma(\chi\alpha+\alpha+\beta)}\int_0^t(t-u)^{\chi\alpha+\alpha+\beta-1}h(u)du
\end{aligned}
$$

$$= \int_0^t (t-s)^{\alpha+\beta-1}\mathbf{E}_{\alpha,\alpha+\beta}(\lambda(t-s)^\alpha)h(s)ds$$

$$+\sum_{k=0}^{i}\sum_{\nu=1}^{n} c_{\nu k}(t-t_k)^{\alpha+\beta-\nu}\mathbf{E}_{\alpha,\alpha+\beta-\nu+1}(\lambda(t-t_k)^\alpha), t\in(t_i,t_{i+1}], i\in\mathbb{N}_0^m$$

Thus (2.5) is proved. Hence (2.6) holds by $\beta = n-\alpha$.
Now, we prove (2.7). In fact, for $t\in(t_i,t_{i+1}]$, we have by using 2.2 and Definition 2.2 that

$$D_{0^+}^{\alpha-j}x(t) = \frac{\left[\int_0^t (t-s)^{n-\alpha-1}x(s)ds\right]^{(n-j)}}{\Gamma(n-\alpha)}$$

$$= \frac{\left[\sum\limits_{\mu=0}^{i-1}(t-s)^{n-\alpha-1}x(s)ds + \int_{t_i}^t (t-s)^{n-\alpha-1}x(s)ds\right]^{(n-j)}}{\Gamma(n-\alpha)}$$

$$= \left[\sum_{\mu=0}^{i-1}\int_{t_\mu}^{t_{\mu+1}}(t-s)^{n-\alpha-1}\left(\int_0^s (s-u)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(\lambda(s-u)^\alpha)h(u)du\right.\right.$$

$$\left.\left.+\sum_{k=0}^{\mu}\sum_{\nu=1}^{n} c_{\nu k}(s-t_k)^{\alpha-\nu}\mathbf{E}_{\alpha,\alpha-\nu+1}(\lambda(s-t_k)^\alpha)\right)ds\right]^{(n-j)}\Big/\Gamma(n-\alpha)$$

$$+\left[\int_{t_i}^t (t-s)^{n-\alpha-1}\left(\int_0^s (s-u)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(\lambda(s-u)^\alpha)h(u)du\right.\right.$$

$$\left.\left.+\sum_{k=0}^{i}\sum_{\nu=1}^{n} c_{\nu k}(s-t_k)^{\alpha-\nu}\mathbf{E}_{\alpha,\alpha-\nu+1}(\lambda(s-t_k)^\alpha)\right)ds\right]^{(n-j)}\Big/\Gamma(n-\alpha)$$

$$= \frac{\left[\sum\limits_{\mu=0}^{i-1}\int_{t_\mu}^{t_{\mu+1}}(t-s)^{n-\alpha-1}\sum\limits_{k=0}^{\mu}\sum\limits_{\nu=1}^{n} c_{\nu k}(s-t_k)^{\alpha-\nu}\mathbf{E}_{\alpha,\alpha-\nu+1}(\lambda(s-t_k)^\alpha)ds\right]^{(n-j)}}{\Gamma(n-\alpha)}$$

$$+\frac{\left[\int_{t_i}^t (t-s)^{n-\alpha-1}\sum\limits_{k=0}^{i}\sum\limits_{\nu=1}^{n} c_{\nu k}(s-t_k)^{\alpha-\nu}\mathbf{E}_{\alpha,\alpha-\nu+1}(\lambda(s-t_k)^\alpha)ds\right]^{(n-j)}}{\Gamma(n-\alpha)}$$

$$+\frac{\left[\int_0^t (t-s)^{n-\alpha-1}\int_0^s (s-u)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(\lambda(s-u)^\alpha)h(u)duds\right]^{(n-j)}}{\Gamma(n-\alpha)}$$

$$= \frac{\left[\sum\limits_{k=0}^{i-1}\sum\limits_{\mu=k}^{i-1}\sum\limits_{\nu=1}^{n} c_{\nu k}\int_{t_\mu}^{t_{\mu+1}}(t-s)^{n-\alpha-1}(s-t_k)^{\alpha-\nu}\sum\limits_{\chi=0}^{\infty}\frac{\lambda^\chi}{\Gamma(\chi\alpha+\alpha-\nu+1)}(s-t_k)^{\chi\alpha}ds\right]^{(n-j)}}{\Gamma(n-\alpha)}$$

$$+\frac{\left[\sum\limits_{k=0}^{i}\sum\limits_{\nu=1}^{n} c_{\nu k}\int_{t_i}^{t}(t-s)^{n-\alpha-1}(s-t_k)^{\alpha-\nu}\sum\limits_{\chi=0}^{\infty}\frac{\lambda^\chi}{\Gamma(\chi\alpha+\alpha-\nu+1)}(s-t_k)^{\chi\alpha}ds\right]^{(n-j)}}{\Gamma(n-\alpha)}$$

$$+\frac{\left[\int_0^t\int_u^t(t-s)^{n-\alpha-1}(s-u)^{\alpha-1}\sum_{\chi=0}^{\infty}\frac{\lambda^\chi}{\Gamma(\chi\alpha+\alpha)}(s-u)^{\chi\alpha}dsh(u)du\right]^{(n-j)}}{\Gamma(n-\alpha)}$$

$$=\frac{\left[\sum_{k=0}^{i-1}\sum_{\mu=k}^{i-1}\sum_{\nu=1}^{n}c_{\nu k}\sum_{\chi=0}^{\infty}\frac{\lambda^\chi}{\Gamma(\chi\alpha+\alpha-\nu+1)}\int_{t_\mu}^{t_{\mu+1}}(t-s)^{n-\alpha-1}(s-t_k)^{\alpha-\nu+\chi\alpha}ds\right]^{(n-j)}}{\Gamma(n-\alpha)}$$

$$+\frac{\left[\sum_{k=0}^{i}\sum_{\nu=1}^{n}c_{\nu k}\sum_{\chi=0}^{\infty}\frac{\lambda^\chi}{\Gamma(\chi\alpha+\alpha-\nu+1)}\int_{t_i}^{t}(t-s)^{n-\alpha-1}(s-t_k)^{\alpha-\nu+\chi\alpha}ds\right]^{(n-j)}}{\Gamma(n-\alpha)}$$

$$+\frac{\left[\sum_{\chi=0}^{\infty}\frac{\lambda^\chi}{\Gamma(\chi\alpha+\alpha)}\int_0^t\int_u^t(t-s)^{n-\alpha-1}(s-u)^{\alpha-1+\chi\alpha}dsh(u)du\right]^{(n-j)}}{\Gamma(n-\alpha)}$$

$$=\left[\sum_{k=0}^{i-1}\sum_{\mu=k}^{i-1}\sum_{\nu=1}^{n}c_{\nu k}\sum_{\chi=0}^{\infty}\frac{\lambda^\chi}{\Gamma(\chi\alpha+\alpha-\nu+1)}(t-t_k)^{\chi\alpha+n-\nu}\right.$$

$$\left.\times\int_{\frac{t_\mu-t_k}{t-t_k}}^{\frac{t_{\mu+1}-t_k}{t-t_k}}(1-w)^{n-\alpha-1}w^{\alpha-\nu+\chi\alpha}dw\right]^{(n-j)}\Big/\Gamma(n-\alpha)$$

$$+\frac{\left[\sum_{k=0}^{i}\sum_{\nu=1}^{n}c_{\nu k}\sum_{\chi=0}^{\infty}\frac{\lambda^\chi}{\Gamma(\chi\alpha+\alpha-\nu+1)}(t-t_k)^{\chi\alpha+n-\nu}\int_{\frac{t_i-t_k}{t-t_k}}^{1}(1-w)^{n-\alpha-1}w^{\alpha-\nu+\chi\alpha}dw\right]^{(n-j)}}{\Gamma(n-\alpha)}$$

$$+\frac{\left[\sum_{\chi=0}^{\infty}\frac{\lambda^\chi}{\Gamma(\chi\alpha+\alpha)}\int_0^t(t-u)^{\chi\alpha+n-1}\int_0^1(1-w)^{n-\alpha-1}w^{\alpha-1+\chi\alpha}dwh(u)du\right]^{(n-j)}}{\Gamma(n-\alpha)}$$

$$=\frac{\left[\sum_{k=0}^{i}\sum_{\nu=1}^{n}c_{\nu k}\sum_{\chi=0}^{\infty}\frac{\lambda^\chi}{\Gamma(\chi\alpha+\alpha-\nu+1)}(t-t_k)^{\chi\alpha+n-\nu}\int_0^1(1-w)^{n-\alpha-1}w^{\alpha-\nu+\chi\alpha}dw\right]^{(n-j)}}{\Gamma(n-\alpha)}$$

$$+\frac{\left[\sum_{\chi=0}^{\infty}\frac{\lambda^\chi}{\Gamma(\chi\alpha+\alpha)}\int_0^t(t-u)^{\chi\alpha+n-1}\int_0^1(1-w)^{n-\alpha-1}w^{\alpha-1+\chi\alpha}dwh(u)du\right]^{(n-j)}}{\Gamma(n-\alpha)}$$

$$=\left[\sum_{k=0}^{i}\sum_{\nu=1}^{n}c_{\nu k}\sum_{\chi=0}^{\infty}\frac{\lambda^\chi}{\Gamma(\chi\alpha+n-\nu+1)}(t-t_k)^{\chi\alpha+n-\nu}\right]^{(n-j)}$$

$$+\left[\sum_{\chi=0}^{\infty}\frac{\lambda^{\chi}}{\Gamma(\chi\alpha+n)}\int_0^t (t-u)^{\chi\alpha+n-1}h(u)du\right]^{(n-j)}$$

$$=\sum_{k=0}^{i}\sum_{\nu=1}^{j}c_{\nu k}\frac{1}{\Gamma(j-\nu+1)}(t-t_k)^{j-\nu}+\sum_{k=0}^{i}\sum_{\nu=1}^{n}c_{\nu k}\sum_{\chi=1}^{\infty}\frac{\lambda^{\chi}}{\Gamma(\chi\alpha+j-\nu+1)}(t-t_k)^{\chi\alpha+j-\nu}$$

$$+\sum_{\chi=0}^{\infty}\frac{\lambda^{\chi}}{\Gamma(\chi\alpha+j)}\int_0^t (t-u)^{\chi\alpha+j-1}h(u)du$$

$$=\sum_{k=0}^{i}\sum_{\nu=1}^{j}c_{\nu k}\sum_{\chi=0}^{\infty}\frac{\lambda^{\chi}}{\Gamma(\chi\alpha+j-\nu+1)}(t-t_k)^{\chi\alpha+j-\nu}$$

$$+\sum_{k=0}^{i}\sum_{\nu=j+1}^{n}c_{\nu k}\sum_{\chi=1}^{\infty}\frac{\lambda^{\chi}}{\Gamma(\chi\alpha+j-\nu+1)}(t-t_k)^{\chi\alpha+j-\nu}$$

$$+\int_0^t (t-u)^{j-1}\mathbf{E}_{\alpha,j}(\lambda(t-u)^{\alpha})h(u)du$$

$$=\sum_{k=0}^{i}\sum_{\nu=1}^{j}c_{\nu k}(t-t_k)^{j-\nu}\mathbf{E}_{\alpha,j-\nu+1}(\lambda(t-t_k)^{\alpha})$$

$$+\lambda\sum_{k=0}^{i}\sum_{\nu=j+1}^{n}c_{\nu k}(t-t_k)^{\alpha+j-\nu}\mathbf{E}_{\alpha,\alpha+j-\nu+1}(\lambda(t-t_k)^{\alpha})$$

$$+\int_0^t (t-u)^{j-1}\mathbf{E}_{\alpha,j}(\lambda(t-u)^{\alpha})h(u)du.$$

Banach space $PC_{n-\alpha}(0,1]$. Let n be a positive integer, $\alpha\in(n-1,n)$ and $0=t_0<t_1<\cdots<t_{m+1}=1$. Choose

$$PC_{n-\alpha}(0,1]=\left\{x:(0,1]\mapsto\mathbf{R}:\begin{array}{c}x|_{(t_i,t_{i+1}]}\in C^0(t_i,t_{i+1}],i\in\mathbf{N}_0^m,\\ \lim_{t\to t_i^+}(t-t_i)^{n-\alpha}x(t),\ i\in\mathbf{N}_0^m \text{ are finite}\end{array}\right\}.$$

Define

$$||x||=||x||_{n-\alpha}=\max\left\{\sup_{t\in(t_i,t_{i+1}]}(t-t_i)^{n-\alpha}|x(t)|,i\in\mathbf{N}_0^m\right\}.$$

Then $PC_{n-\alpha}(0,1]$ is a Banach space with the norm $||\cdot||$ defined.

Theorem 2.3. *Suppose that $\alpha+\beta=n$, $h\in L^1(0,1]$ $a_{ji},b_i\in\mathbb{R}$. Then $x\in PC_{n-\alpha}(0,1]$ is a solution of*

$$\begin{pmatrix} D_{0^+}^{\alpha}x(t)-\lambda x(t), a.e., t\in(t_i,t_{i+1}], i\in\mathbb{N}_0^m \\ \Delta I_{0^+}^{n-\alpha}x(t_i), i\in\mathbb{N}_1^m \\ \Delta D_{0^+}^{\alpha-j}x(t_i), i\in\mathbb{N}_1^m, j\in\mathbb{N}_1^{n-1} \\ I_{0^+}^{n-\alpha}x(0) \\ D_{0^+}^{\alpha-j}x(0), j\in\mathbb{N}_1^{n-1} \end{pmatrix} = \begin{pmatrix} h(t), a.e., t\in(t_i,t_{i+1}], i\in\mathbb{N}_0^m \\ a_{ni}, i\in\mathbb{N}_1^m \\ a_{ji}, i\in\mathbb{N}_1^m, j\in\mathbb{N}_1^{n-1} \\ b_n \\ b_j, j\in\mathbb{N}_1^{n-1} \end{pmatrix} \tag{2.8}$$

if and only if

$$x(t)=\int_0^t (t-s)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(\lambda(t-s)^{\alpha})h(s)ds+\sum_{\nu=1}^{n} b_\nu t^{\alpha-\nu}\mathbf{E}_{\alpha,\alpha-\nu+1}(\lambda t^{\alpha}) + \sum_{k=1}^{j}\sum_{\nu=1}^{n} a_{\nu k}(t-t_k)^{\alpha-\nu}\mathbf{E}_{\alpha,\alpha-\nu+1}(\lambda(t-t_k)^{\alpha}), t\in(t_j,t_{j+1}], j\in\mathbb{N}_0^m.$$

Proof. Suppose that x is a solution of (2.15). By Theorem 2.1, there exist constants $c_{\nu k}\in\mathbb{R}(\nu\in\mathbb{N}_1^n, k\in\mathbb{N}_0)$ such that

$$x(t)=\int_0^t (t-s)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(\lambda(t-s)^{\alpha})h(s)ds + \sum_{k=0}^{j}\sum_{\nu=1}^{n} c_{\nu k}(t-t_k)^{\alpha-\nu}\mathbf{E}_{\alpha,\alpha-\nu+1}(\lambda(t-t_k)^{\alpha}), \quad t\in(t_j,t_{j+1}],\ j\in\mathbb{N}_0^m. \tag{2.9}$$

Then Theorem 2.2 implies

$$I_{0^+}^{\beta}x(t)=\int_0^t (t-s)^{\alpha+\beta-1}\mathbf{E}_{\alpha,\alpha+\beta}(\lambda(t-s)^{\alpha})h(s)ds + \sum_{k=0}^{j}\sum_{\nu=1}^{n} c_{\nu k}(t-t_k)^{\alpha+\beta-\nu}\mathbf{E}_{\alpha,\alpha+\beta-\nu+1}(\lambda(t-t_k)^{\alpha}), \quad t\in(t_j,t_{j+1}],\ j\in\mathbb{N}_0^m \tag{2.10}$$

$$I_{0^+}^{n-\alpha}x(t)=\sum_{k=0}^{i}\sum_{\nu=1}^{n} c_{\nu k}(t-t_k)^{n-\nu}\mathbf{E}_{\alpha,n-\nu+1}(\lambda(t-t_k)^{\alpha}) + \int_0^t (t-u)^{n-1}\mathbf{E}_{\alpha,n}(\lambda(t-u)^{\alpha})h(u)du, \quad t\in(t_i,t_{i+1}],\ i\in\mathbb{N}_0^m, \tag{2.11}$$

and

$$D_{0^+}^{\alpha-j}x(t) = \sum_{k=0}^{i}\sum_{\nu=1}^{j} c_{\nu k}(t-t_k)^{j-\nu}\mathbf{E}_{\alpha,j-\nu+1}(\lambda(t-t_k)^\alpha)$$
$$+\lambda\sum_{k=0}^{i}\sum_{\nu=j+1}^{n} c_{\nu k}(t-t_k)^{\alpha+j-\nu}\mathbf{E}_{\alpha,\alpha+j-\nu+1}(\lambda(t-t_k)^\alpha)$$
$$+\int_0^t (t-u)^{j-1}\mathbf{E}_{\alpha,j}(\lambda(t-u)^\alpha)h(u)du,$$
$$t\in(t_i,t_{i+1}],\ i\in\mathbb{N}_0^m,\ j\in\mathbb{N}_1^{n-1}. \tag{2.12}$$

By $D_{0^+}^{\alpha-j}x(0)=b_j$ and (2.12), we get $c_{j0}=b_j, j\in\mathbb{N}_1^{n-1}$.
By $I_{0^+}^{n-\alpha}x(0)=b_n$ and (2.11), we get $c_{n0}=b_n$.
By $\Delta D_{0^+}^{\alpha-j}x(t_i)=a_{ji},\ i\in\mathbb{N}_1^m,\ j\in\mathbb{N}_1^{n-1}$ and (2.12), we get $c_{ji}=a_{ji}$ for $i\in\mathbb{N}_1^m$, $j\in\mathbb{N}_1^{n-1}$.
By $\Delta I_{0^+}^{n-\alpha}x(t_i)=a_{ni}, i\in\mathbb{N}_1^m$ and (2.11), we get $c_{ni}=a_{ni}, i\in\mathbb{N}_1^m$.
Hence

$$x(t)=\int_0^t (t-s)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(\lambda(t-s)^\alpha)h(s)ds+\sum_{\nu=1}^{n} b_\nu t^{\alpha-\nu}\mathbf{E}_{\alpha,\alpha-\nu+1}(\lambda t^\alpha)$$
$$+\sum_{k=1}^{j}\sum_{\nu=1}^{n} a_{\nu k}(t-t_k)^{\alpha-\nu}\mathbf{E}_{\alpha,\alpha-\nu+1}(\lambda(t-t_k)^\alpha), t\in(t_j,t_{j+1}], j\in\mathbb{N}_0^m.$$

The proof is completed. □

Theorem 2.4. *Suppose that*

$$M=:\frac{1}{\Gamma(\alpha-n+2)}+\sum_{\mu=0}^{m}\int_{t_\mu}^{t_{\mu+1}}\frac{s^{\alpha-n+1}}{\Gamma(\alpha-n+2)}dH(s)\neq 0,\ h\in L^1(0,1],\ a_{ji},b_i\in\mathbb{R}.$$

Then $x\in PC_{n-\alpha}(0,1]$ is a solution of

$$\begin{pmatrix} D_{0^+}^{\alpha}x(t), a.e., t\in(t_i,t_{i+1}], i\in\mathbb{N}_0^m \\ \Delta I_{0^+}^{n-\alpha}x(t_i), i\in\mathbb{N}_1^m \\ \Delta D_{0^+}^{\alpha-j}x(t_i), i\in\mathbb{N}_1^m, j\in\mathbb{N}_1^{n-1}, \\ I_{0^+}^{n-\alpha}x(0) \\ D_{0^+}^{\alpha-j}x(0)=0, j\in\mathbb{N}_1^{n-2} \\ D_{0^+}^{\alpha-n+1}x(1)-\int_0^1 x(s)dH(s) \end{pmatrix} = \begin{pmatrix} h(t), a.e., t\in(t_i,t_{i+1}], i\in\mathbb{N}_0^m \\ a_{ni}, i\in\mathbb{N}_1^m \\ a_{ji}, i\in\mathbb{N}_1^m, j\in\mathbb{N}_1^{n-1} \\ 0 \\ 0, j\in\mathbb{N}_1^{n-2} \\ 0 \end{pmatrix}$$

if and only if

$$x(t)=\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}h(s)ds+\frac{1}{M}\left[\int_0^1\left(\int_u^1\frac{(s-u)^{\alpha-1}}{\Gamma(\alpha)}dH(s)-\frac{(1-u)^{\alpha-1}}{\Gamma(\alpha)}\right)h(u)du\right.$$
$$\left.+\sum_{k=1}^{m}\sum_{\nu=1}^{n}a_{\nu k}\left(\int_{t_k}^1\frac{(s-t_k)^{\alpha-\nu}}{\Gamma(\alpha-\nu+1)}dH(s)-\frac{(1-t_k)^{\alpha-\nu}}{\Gamma(\alpha-\nu+1)}\right)\right]\frac{t^{\alpha-n+1}}{\Gamma(\alpha-n+2)}$$
$$+\sum_{k=1}^{j}\sum_{\nu=1}^{n}a_{\nu k}\frac{(t-t_k)^{\alpha-\nu}}{\Gamma(\alpha-\nu+1)},\ t\in(t_j,t_{j+1}],\ j\in\mathbb{N}_0^m.$$

Proof. By using Theorem 2.1 ($\lambda=0$), we get the proof similarly to that of Theorem 2.3 and the proof is omitted. □

Define the nonlinear operators T_1,T_2 on $PC_{n-\alpha}(0,1]$ by

$$(T_1x)(t)=\int_0^t(t-s)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(\lambda(t-s)^{\alpha})f(s,x(s))ds+\sum_{\nu=1}^{n}x_\nu t^{\alpha-\nu}\mathbf{E}_{\alpha,\alpha-\nu+1}(\lambda t^{\alpha})$$
$$+\sum_{\nu=1}^{n}\sum_{k=1}^{j}(t-t_k)^{\alpha-\nu}\mathbf{E}_{\alpha,\alpha-\nu+1}(\lambda(t-t_k)^{\alpha})I_\nu(t_k,x(t_k)),\ t\in(t_j,t_{j+1}],\ j\in\mathbb{N}_0^m$$

and

$$(T_2x)(t)=\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}f(s,x(s))ds$$
$$+\frac{1}{M}\left[\int_0^1\left(\int_u^1\frac{(s-u)^{\alpha-1}}{\Gamma(\alpha)}dH(s)-\frac{(1-u)^{\alpha-1}}{\Gamma(\alpha)}\right)f(u,x(u))du\right.$$
$$\left.+\sum_{\nu=1}^{n}\sum_{k=1}^{m}\left(\int_{t_k}^1\frac{(s-t_k)^{\alpha-\nu}}{\Gamma(\alpha-\nu+1)}dH(s)-\frac{(1-t_k)^{\alpha-\nu}}{\Gamma(\alpha-\nu+1)}\right)I_\nu(t_k,x(t_k))\right]\frac{t^{\alpha-n+1}}{\Gamma(\alpha-n+2)}$$
$$+\sum_{\nu=1}^{n}\sum_{k=1}^{j}\frac{(t-t_k)^{\alpha-\nu}}{\Gamma(\alpha-\nu+1)}I_\nu(t_k,x(t_k)),\ t\in(t_j,t_{j+1}],\ j\in\mathbb{N}_0^m.$$

Theorem 2.5. *Both $T_1:PC_{n-\alpha}(0,1]\to PC_{n-\alpha}(0,1]$ and $T_2:PC_{n-\alpha}(0,1]\to PC_{n-\alpha}(0,1]$ are well defined and are completely continuous and x is a solution of IVP (1.7) if and only if x is a fixed point of T_1, x is a solution of BVP(1.8) if and only if x is a fixed point of T_2.*

Proof. The proof is standard and is omitted. □

3. Main results

In this section, we establish existence results for IVP (1.6) when $\alpha+\beta\geq n$.

Theorem 3.1. *Suppose $\alpha+\beta=n$ and there exist constants $\sigma,A,B,C_i\geq 0$ and measurable function $\phi\in L^1(0,1)$ such that*

$$|f(t,(t-t_i)^{\alpha-n}u)-\phi(t)|\leq A|u|^{\sigma},t\in(t_i,t_{i+1}],i\in\mathbb{N}_0^m,u\in\mathbb{R},$$

$$|I_j(t_i,(t_i-t_{i-1})^{\alpha-n}u)-C_i|\leq B|u|^{\sigma},i\in\mathbb{N}_1^m,u\in\mathbb{R}.$$

Then IVP (1.7) has at least one solution if $\sigma \in [0,1)$ or $\sigma = 1$ with

$$\frac{\mathbf{E}_{\alpha,\alpha}(|\lambda|)}{\alpha} + m\sum_{\nu=1}^{n}\mathbf{E}_{\alpha,\alpha-\nu+1}(|\lambda|) < 1$$

or $\sigma > 1$ with

$$\left[\frac{\mathbf{E}_{\alpha,\alpha}(|\lambda|)}{\alpha} + m\sum_{\nu=1}^{n}\mathbf{E}_{\alpha,\alpha-\nu+1}(|\lambda|)\right]\left(\frac{\sigma}{\sigma-1}\right)^{\sigma}||\Phi||^{\sigma-1} < \frac{1}{\sigma-1}.$$

where

$$\Phi(t) = \int_0^t (t-s)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(\lambda(t-s)^{\alpha})\phi(s)ds + \sum_{\nu=1}^{n} x_\nu t^{\alpha-\nu}\mathbf{E}_{\alpha,\alpha-\nu+1}(\lambda t^{\alpha})$$
$$+\sum_{\nu=1}^{n}\sum_{k=1}^{j}(t-t_k)^{\alpha-\nu}\mathbf{E}_{\alpha,\alpha-\nu+1}(\lambda(t-t_k)^{\alpha})C_\nu,\ t\in(t_j,t_{j+1}],\ j\in\mathbb{N}_0^m.$$

Proof. By the definition of Φ, we know $\Phi \in PC_{n-\alpha}(0,1]$. For $r > 0$, denote

$$\Omega_r = \{x \in PC_{n-\alpha}(0,1] : ||x-\Phi|| \le r\}.$$

We will seek $r > 0$ such that $T_1\Omega_r \subseteq \Omega_r$. Then Schauder's fixed point theorem implies that T_1 has a fixed point in Ω_r. Thus IVP (1.7) has a solution by Theorem 2.5.

For $x \in \Omega_r$, we have $||x|| \le r + ||\Phi||$ and

$$|f(t,x(t)) - \phi(t)| = |f(t,(t-t_i)^{\alpha-n}(t-t_i)^{n-\alpha}x(t)) - \phi(t)|$$
$$\le A|(t-t_i)^{n-\alpha}x(t))|^{\sigma} \le A||x||^{\sigma} \le A[r+||\Phi||]^{\sigma},\ t\in(t_i,t_{i+1}],\ i\in\mathbb{N}_0^m,$$
$$|I_j(t_i,x(t_i)) - C_i| \le B||x||^{\sigma} \le B[r+||\Phi||]^{\sigma},\ i\in\mathbb{N}_1^m.$$

For $t \in (t_j, t_{j+1}]$, we have

$$(t-t_j)^{n-\alpha}|(T_1x)(t) - \Phi(t)|$$
$$\le (t-t_j)^{n-\alpha}\left|\int_0^t (t-s)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(\lambda(t-s)^{\alpha})[f(s,x(s))-\phi(s)]ds\right.$$
$$\left.+\sum_{\nu=1}^{n}\sum_{k=1}^{j}(t-t_k)^{\alpha-\nu}\mathbf{E}_{\alpha,\alpha-\nu+1}(\lambda(t-t_k)^{\alpha})|I_\nu(t_k,x(t_k)) - C_k|\right|$$
$$\le (t-t_j)^{n-\alpha}\int_0^t (t-s)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(|\lambda|)[r+||\Phi||]^{\sigma}ds + \sum_{\nu=1}^{n}\sum_{k=1}^{j}\mathbf{E}_{\alpha,\alpha-\nu+1}(|\lambda|)[r+||\Phi||]^{\sigma}$$
$$\le \left[\frac{\mathbf{E}_{\alpha,\alpha}(|\lambda|)}{\alpha} + m\sum_{\nu=1}^{n}\mathbf{E}_{\alpha,\alpha-\nu+1}(|\lambda|)\right][r+||\Phi||]^{\sigma}$$

Case 1. $\sigma \in [0,1)$. It is easy to see that there exists $r > 0$ such that

$$\left[\frac{\mathbf{E}_{\alpha,\alpha}(|\lambda|)}{\alpha} + m\sum_{\nu=1}^{n}\mathbf{E}_{\alpha,\alpha-\nu+1}(|\lambda|)\right][r+||\Phi||]^{\sigma} \le r.$$

Case 2. $\sigma = 1$. It is easy to see that there exists $r > 0$ such tha

$$\left[\frac{\mathbf{E}_{\alpha,\alpha}(|\lambda|)}{\alpha} + m\sum_{\nu=1}^{n}\mathbf{E}_{\alpha,\alpha-\nu+1}(|\lambda|)\right][r + ||\Phi||]^{\sigma} \leq r$$

by

$$\frac{\mathbf{E}_{\alpha,\alpha}(|\lambda|)}{\alpha} + m\sum_{\nu=1}^{n}\mathbf{E}_{\alpha,\alpha-\nu+1}(|\lambda|) < 1.$$

Case 3. $\sigma > 1$. Choose $r = \frac{||\Phi||}{\sigma-1} > 0$. By

$$\left[\frac{\mathbf{E}_{\alpha,\alpha}(|\lambda|)}{\alpha} + m\sum_{\nu=1}^{n}\mathbf{E}_{\alpha,\alpha-\nu+1}(|\lambda|)\right]\left(\frac{\sigma}{\sigma-1}\right)^{\sigma}||\Phi||^{\sigma-1} < \frac{1}{\sigma-1},$$

we know that

$$\left[\frac{\mathbf{E}_{\alpha,\alpha}(|\lambda|)}{\alpha} + m\sum_{\nu=1}^{n}\mathbf{E}_{\alpha,\alpha-\nu+1}(|\lambda|)\right][r + ||\Phi||]^{\sigma} \leq r.$$

From above discussion, we know $T_1\Omega_r \subset \Omega_r$. Then Schauder's fixed point theorem implies that T_1 has a fixed point in Ω_r. Thus IVP (1.7) has a solution by Theorem 2.5. The proof is completed. □

Theorem 3.2. *Suppose that there exist non-decreasing functions $\phi_f, \phi_I : [0,\infty) \to [0,\infty)$ such that*

$$|f(t,(t-t_i)^{\alpha-n}u)| \leq \phi_f(|u|),\ t \in (t_i, t_{i+1}],\ i \in \mathbb{N}_0^m,\ u \in \mathbb{R},$$

$$|I_j(t_i,(t_i-t_{i-1})^{\alpha-n}u)| \leq \phi_i(|u|),\ i \in \mathbb{N}_1^m,\ u \in \mathbb{R}.$$

Then VP (1.8) has at least one solution if there exists $r > 0$ such that

$$\left[\frac{1}{\Gamma(\alpha+1)} + \frac{1}{|M|}\frac{1}{\Gamma(\alpha-n+2)}\left|\int_0^1\left(\int_u^1\frac{(s-u)^{\alpha-1}}{\Gamma(\alpha)}dH(s) - \frac{(1-u)^{\alpha-1}}{\Gamma(\alpha)}\right)\right|\right]\phi_f(r)$$
$$+\left[\frac{1}{|M|}\frac{1}{\Gamma(\alpha-n+2)}\left|\sum_{\nu=1}^{n}\sum_{k=1}^{m}\left(\int_{t_k}^1\frac{(s-t_k)^{\alpha-\nu}}{\Gamma(\alpha-\nu+1)}dH(s) - \frac{(1-t_k)^{\alpha-\nu}}{\Gamma(\alpha-\nu+1)}\right)\right|\right.$$
$$\left.+\sum_{\nu=1}^{n}\frac{m}{\Gamma(\alpha-\nu+1)}\right]\phi_I(r) \leq r.$$

Proof. For $r > 0$, denote $\Omega_r = \{x \in PC_{n-\alpha}(0,1] : ||x|| \leq r\}$. We will seek $r > 0$ such that $T_2\Omega_r \subseteq \Omega_r$. Then Schauder's fixed point theorem implies that T_2 has a fixed point in Ω_r. Thus VP(1.6)8 has a solution by Theorem 2.5.

For $x \in \Omega_r$, we have

$$|f(t,x(t))| = |f(t,(t-t_i)^{\alpha-n}(t-t_i)^{n-\alpha}x(t))| \leq \phi_f(|(t-t_i)^{n-\alpha}x(t))|)$$
$$\leq \phi_f(||x||), t \in (t_i, t_{i+1}],\ i \in \mathbb{N}_0^m,$$
$$|I_j(t_i, x(t_i))| \leq \phi_i(||x||),\ i \in \mathbb{N}_1^m.$$

For $t \in (t_j, t_{j+1}]$, we have

$$(t-t_j)^{n-\alpha}|(T_2x)(t)| = (t-t_j)^{n-\alpha}\left|\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}f(s,x(s))ds\right.$$

$$+\frac{1}{M}\left[\int_0^1\left(\int_u^1\frac{(s-u)^{\alpha-1}}{\Gamma(\alpha)}dH(s)-\frac{(1-u)^{\alpha-1}}{\Gamma(\alpha)}\right)f(u,x(u))du\right.$$

$$\left.+\sum_{\nu=1}^{n}\sum_{k=1}^{m}\left(\int_{t_k}^1\frac{(s-t_k)^{\alpha-\nu}}{\Gamma(\alpha-\nu+1)}dH(s)-\frac{(1-t_k)^{\alpha-\nu}}{\Gamma(\alpha-\nu+1)}\right)I_\nu(t_k,x(t_k))\right]\frac{t^{\alpha-n+1}}{\Gamma(\alpha-n+2)}$$

$$\left.+\sum_{\nu=1}^{n}\sum_{k=1}^{j}\frac{(t-t_k)^{\alpha-\nu}}{\Gamma(\alpha-\nu+1)}I_\nu(t_k,x(t_k))\right|$$

$$\leq (t-t_j)^{n-\alpha}\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\phi_f(||x||)ds$$

$$+\frac{1}{|M|}\frac{1}{\Gamma(\alpha-n+2)}\int_0^1\left(\int_u^1\frac{(s-u)^{\alpha-1}}{\Gamma(\alpha)}dH(s)-\frac{(1-u)^{\alpha-1}}{\Gamma(\alpha)}\right)\phi_f(||x||)du$$

$$+\frac{1}{|M|}\frac{1}{\Gamma(\alpha-n+2)}\sum_{\nu=1}^{n}\sum_{k=1}^{m}\left(\int_{t_k}^1\frac{(s-t_k)^{\alpha-\nu}}{\Gamma(\alpha-\nu+1)}dH(s)-\frac{(1-t_k)^{\alpha-\nu}}{\Gamma(\alpha-\nu+1)}\right)\phi_I(||x||)$$

$$+(t-t_j)^{n-\alpha}\sum_{\nu=1}^{n}\sum_{k=1}^{j}\frac{(t-t_k)^{\alpha-\nu}}{\Gamma(\alpha-\nu+1)}\phi_I(||x||)$$

$$\leq\frac{1}{\Gamma(\alpha+1)}\phi_f(||x||)$$

$$+\frac{1}{|M|}\frac{1}{\Gamma(\alpha-n+2)}\left|\int_0^1\left(\int_u^1\frac{(s-u)^{\alpha-1}}{\Gamma(\alpha)}dH(s)-\frac{(1-u)^{\alpha-1}}{\Gamma(\alpha)}\right)\right|\phi_f(||x||)$$

$$+\frac{1}{|M|}\frac{1}{\Gamma(\alpha-n+2)}\left|\sum_{\nu=1}^{n}\sum_{k=1}^{m}\left(\int_{t_k}^1\frac{(s-t_k)^{\alpha-\nu}}{\Gamma(\alpha-\nu+1)}dH(s)-\frac{(1-t_k)^{\alpha-\nu}}{\Gamma(\alpha-\nu+1)}\right)\right|\phi_I(||x||)$$

$$+\sum_{\nu=1}^{n}\frac{m}{\Gamma(\alpha-\nu+1)}\phi_I(||x||)$$

$$\leq\left[\frac{1}{\Gamma(\alpha+1)}+\frac{1}{|M|}\frac{1}{\Gamma(\alpha-n+2)}\left|\int_0^1\left(\int_u^1\frac{(s-u)^{\alpha-1}}{\Gamma(\alpha)}dH(s)-\frac{(1-u)^{\alpha-1}}{\Gamma(\alpha)}\right)\right|\right]\phi_f(r)$$

$$+\left[\frac{1}{|M|}\frac{1}{\Gamma(\alpha-n+2)}\left|\sum_{\nu=1}^{n}\sum_{k=1}^{m}\left(\int_{t_k}^1\frac{(s-t_k)^{\alpha-\nu}}{\Gamma(\alpha-\nu+1)}dH(s)-\frac{(1-t_k)^{\alpha-\nu}}{\Gamma(\alpha-\nu+1)}\right)\right|\right.$$

$$\left.+\sum_{\nu=1}^{n}\frac{m}{\Gamma(\alpha-\nu+1)}\right]\phi_I(r).$$

By the assumption of theorem, we have $(t-t_j)^{n-\alpha}|(T_2x)(t)|\leq r$ for all $t\in(t_i,t_{i+1}]$, $i\in\mathbb{N}_0^m$. Then $||T_2x||\leq r$. Hence $T_2\Omega_r\subset\Omega_r$. The proof is completed. □

4. Comments on published paprs

We have the following result:

Theorem 4.1. *Consider the homogenous form of BVP(1.7):*

$$\begin{cases} D_{0+}^{\alpha}x(t)-\lambda x(t)=0, t\in(t_i,t_{i+1}], i\in\mathbb{N}_0^m,\\ \Delta I_{0+}^{\beta}x(t_i)=: I_{0+}^{\beta}x(t_i^+)-I_{0+}^{\beta}x(t_i)=0, i\in\mathbb{N}_1^m,\\ \Delta D_{0+}^{\alpha-j}x(t_i)=: D_{0+}^{\alpha-j}x(t_i^+)-D_{0+}^{\alpha-j}x(t_i)=0, i\in\mathbb{N}_1^m, j\in\mathbb{N}_1^{n-1},\\ I_{0+}^{n-\alpha}x(0)=D_{0+}^{\alpha-j}x(0)=0, j\in\mathbb{N}_1^{n-1}.\end{cases}\tag{4.1}$$

Then IVP (4.1) has infinitely many solutions if $\alpha+\beta>n$ and IVP (4.1) has a unique solution $x(t)=0$ if $\alpha+\beta=n$.

Proof. By Theorem 2.1 and $D_{0+}^{\alpha}x(t)-\lambda x(t)=0, t\in(t_i,t_{i+1}], i\in\mathbb{N}_0^m$, we get that there exist constants $c_{\nu k}\in\mathbb{R}$ such that

$$x(t)=\sum_{k=0}^{i}\sum_{\nu=1}^{n}c_{\nu k}(t-t_k)^{\alpha-\nu}\mathbf{E}_{\alpha,\alpha-\nu+1}(\lambda(t-t_k)^{\alpha}), t\in(t_i,t_{i+1}], i\in\mathbb{N}_0^m.$$

By Theorem 2.2, we get

$$I_{0+}^{n-\alpha}x(t)\sum_{k=0}^{i}\sum_{\nu=1}^{n}c_{\nu k}(t-t_k)^{n-\nu}\mathbf{E}_{\alpha,n-\nu+1}(\lambda(t-t_k)^{\alpha}), t\in(t_i,t_{i+1}], i\in\mathbb{N}_0^m,$$

$$I_{0+}^{\beta}x(t)=\sum_{k=0}^{i}\sum_{\nu=1}^{n}c_{\nu k}(t-t_k)^{\alpha+\beta-\nu}\mathbf{E}_{\alpha,\alpha+\beta-\nu+1}(\lambda(t-t_k)^{\alpha}), t\in(t_i,t_{i+1}], i\in\mathbb{N}_0^m,$$

$$D_{0+}^{\alpha-j}x(t)=\sum_{k=0}^{i}\sum_{\nu=1}^{j}c_{\nu k}(t-t_k)^{j-\nu}\mathbf{E}_{\alpha,j-\nu+1}(\lambda(t-t_k)^{\alpha})$$

$$+\lambda\sum_{k=0}^{i}\sum_{\nu=j+1}^{n}c_{\nu k}(t-t_k)^{\alpha+j-\nu}\mathbf{E}_{\alpha,\alpha+j-\nu+1}(\lambda(t-t_k)^{\alpha}),\ t\in(t_i,t_{i+1}],\ i\in\mathbb{N}_0^m,\ j\in\mathbb{N}_1^{n-1}.$$

(i) By $I_{0+}^{n-\alpha}x(0)=0$ and the expression of $I_{0+}^{n-\alpha}x$, we get $c_{n0}=0$.

(ii) By $D_{0+}^{\alpha-j}x(0)=0, j\in\mathbb{N}_1^{n-1}$ and and the expression of $D_{0+}^{\alpha-j}x$, we get $c_{j0}=0$ for all $j\in\mathbb{N}_1^{n-1}$.

(iii) By $\Delta D_{0+}^{\alpha-j}x(t_i)=0, i\in\mathbb{N}_1^m, j\in\mathbb{N}_1^{n-1}$ and the expression of $D_{0+}^{\alpha-j}x$, we get $c_{ji}=0$ for all $i\in\mathbb{N}_1^m, j\in\mathbb{N}_1^{n-1}$.

Then

$$x(t)=\sum_{k=1}^{i}c_{nk}(t-t_k)^{\alpha-n}\mathbf{E}_{\alpha,\alpha-n+1}(\lambda(t-t_k)^{\alpha}), t\in(t_i,t_{i+1}], i\in\mathbb{N}_0^m$$

and

$$I_{0+}^{\beta}x(t)=\sum_{k=1}^{i}c_{nk}(t-t_k)^{\alpha+\beta-n}\mathbf{E}_{\alpha,\alpha+\beta-n+1}(\lambda(t-t_k)^{\alpha}), t\in(t_i,t_{i+1}], i\in\mathbb{N}_0^m.$$

Case 1. $\alpha+\beta=n$.
By $\Delta I_{0+}^{\beta}x(t_i)=:I_{0+}^{\beta}x(t_i^+)-I_{0+}^{\beta}x(t_i)=0, i\in\mathbb{N}_1^m$, we get $c_{ni}=0$ for all $i\in\mathbb{N}_1^m$. It follows that $x(t)=0$ is a unique solution.

Case 2. $\alpha+\beta>n$.
By $\Delta I_{0+}^{\beta}x(t_i)=:I_{0+}^{\beta}x(t_i^+)-I_{0+}^{\beta}x(t_i)=0, i\in\mathbb{N}_1^m$, we get

$$-\sum_{k=1}^{i-1}c_{nk}(t_i-t_k)^{\alpha+\beta-n}\mathbf{E}_{\alpha,\alpha+\beta-n+1}(\lambda(t_i-t_k)^{\alpha})=0,\ i\in\mathbb{N}_1^m.$$

Hence $c_{ni}=0$ for all $i\in\mathbb{N}_1^{m-1}$.

Then

$$x(t)=\begin{cases}0,\ \ t\in(t_i,t_{i+1}], i\in\mathbb{N}_0^{m-1},\\ c_{nm}(t-t_m)^{\alpha-n}\mathbf{E}_{\alpha,\alpha-n+1}(\lambda(t-t_m)^{\alpha}), t\in(t_m,t_{m+1}].\end{cases}\tag{4.2}$$

Here $c_{nm}\in\mathbb{R}$ is a constants. Hence it has infinitely many solutions defined. □

In [8], Kosmatov studied the solvability of IVP (1.1).

Define the operator $T_\alpha: PC_\alpha(0,]\to PC_\alpha(0,1]$ by

$$(T_\alpha x)(t)=\frac{x_0}{\Gamma(\alpha)}t^{\alpha-1}+\frac{\Gamma(\alpha-\beta)}{\Gamma(\alpha)}\left(\sum_{0<t_k<t}t_k^{1+\beta-\alpha}J_k(x(t_k))\right)t^{\alpha-1}$$
$$+\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}f(s,x(s))ds,\ t\in(0,1].$$

Result 4.1. (see page 1296 in [8]). x is a solution of (1.1) if and only if x is a fixed point of T_α in $PC_\alpha(0,1]$.

Remark 4.1. By Lemma 2.1 ($n=1,\ \lambda=0$), x is a solution of $D_{0+}^{\alpha}x(t)=f(t,x(t))$, $t\in[0,1]\setminus\{t_1,t_2,\cdots,t_m\}$ if and only if there exist constants $c_i\in\mathbb{R}$ such that

$$x(t)=\sum_{j=0}^{i}c_j(t-t_j)^{\alpha-1}+\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}f(s,x(s))ds, t\in(t_i,t_{i+1}], i\in\mathbb{N}_0^m.$$

We can get by direct computation that

$$D_{0+}^{\beta}x(t)=\sum_{j=0}^{i}\frac{\Gamma(\alpha)}{\Gamma(\alpha-\beta)}c_j(t-t_j)^{\alpha-\beta-1}+\int_0^t\frac{(t-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)}f(s,x(s))ds,$$
$$t\in(t_i,t_{i+1}],\ i\in\mathbb{N}_0^m.$$

By $\beta\in(0,\alpha)$, we know $\alpha-\beta-1<0$. We find that $D_{0+}^{\beta}x(t)$ is singular at $t=t_i$. Hence the impulse functions are unsuitable.

From above discussion, we know that Result 4.1 [8] is unsuitable. □

Result 4.2. (Lemma 2.7 in [14]). Suppose that $q,\alpha\in(0,1)$. Then x is a solution of (1.3) if and only if x is a fixed point of the operator $T_q: PC_q(0,1]\mapsto PC_q(0,1]$, where T_q is defined by

$$(T_qx)(t)=\frac{\Gamma(q)t^{q-1}\mathbf{E}_{q,q}(-\lambda_0t^q)}{1+\Gamma(q)\mathbf{E}_{q,q}(-\lambda_0)}$$

$$\times\left[\sum_{i=1}^{m}\frac{J_i(x(t_i))}{\Gamma(q)t_i^{\alpha+q-1}\mathbf{E}_{q,q+\alpha}(-\lambda_0 t_i^q)}-\int_0^1(1-s)^{q-1}\mathbf{E}_{q,q}(-\lambda_0(1-s)^q)f(s,x(s))ds\right]$$
$$-t^{q-1}\mathbf{E}_{q,q}(-\lambda_0 t^q)\sum_{t\le t_i<1}\frac{J_i(x(t_i))}{t_i^{\alpha+q-1}\mathbf{E}_{q,q+\alpha}(-\lambda_0 t_i^q)}$$
$$+\int_0^t(t-s)^{q-1}\mathbf{E}_{q,q}(-\lambda_0(t-s)^q)f(s,x(s))ds. \tag{4.4}$$

Remark 4.2. Result 4.2 is incorrect.

Proof. By Lemma 2.1 ($(n=1)$), if x is a solution of BVP (4.3), then there exists constants c_i ($i\in\mathbf{N}_0^m$) such that

$$x(t)=\sum_{v=0}^{j}c_v(t-t_v)^{q-1}\mathbf{E}_{q,q}(-\lambda_0(t-t_v)^q)$$
$$+\int_0^t(t-s)^{q-1}\mathbf{E}_{q,q}(-\lambda_0(t-s)^q)f(s,x(s))ds,\ t\in(t_j,t_{j+1}],\ j\in\mathbf{N}_0^m. \tag{4.5}$$

By Definition 2.1, we get for $t\in(t_j,t_{j+1}]$ that

$$I_{0^+}^{\alpha}x(t)=\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}x(s)ds=\frac{\sum_{\tau=0}^{j-1}\int_{t_\tau}^{t_{\tau+1}}(t-s)^{\alpha-1}x(s)ds+\int_{t_j}^{t}(t-s)^{\alpha-1}x(s)ds}{\Gamma(\alpha)}$$
$$=\left(\sum_{\tau=0}^{j-1}\int_{t_\tau}^{t_{\tau+1}}(t-s)^{\alpha-1}\left[\sum_{v=0}^{\tau}c_v(s-t_v)^{q-1}\mathbf{E}_{q,q}(-\lambda_0(s-t_v)^q)\right.\right.$$
$$\left.\left.+\int_0^s(s-u)^{q-1}\mathbf{E}_{q,q}(-\lambda_0(s-u)^q)f(u,x(u))du\right]ds\right)\Big/\Gamma(\alpha)$$
$$+\left(\int_{t_j}^{t}(t-s)^{\alpha-1}\left[\sum_{v=0}^{j}c_v(s-t_v)^{q-1}\mathbf{E}_{q,q}(-\lambda_0(s-t_v)^q)\right.\right.$$
$$\left.\left.+\int_0^s(s-u)^{q-1}\mathbf{E}_{q,q}(-\lambda_0(s-u)^q)f(u,x(u))du\right]ds\right)\Big/\Gamma(\alpha)$$
$$=\frac{\sum_{\tau=0}^{j-1}\sum_{v=0}^{\tau}c_v\int_{t_\tau}^{t_{\tau+1}}(t-s)^{\alpha-1}(s-t_v)^{q-1}\mathbf{E}_{q,q}(-\lambda_0(s-t_v)^q)ds}{\Gamma(\alpha)}$$

change the order of sum and integral

$$+\left(\sum_{v=0}^{j}c_v\int_{t_j}^{t}(t-s)^{\alpha-1}(s-t_v)^{q-1}\mathbf{E}_{q,q}(-\lambda_0(s-t_v)^q)ds\right.$$
$$\left.+\int_0^t(t-s)^{\alpha-1}\int_0^s(s-u)^{q-1}\mathbf{E}_{q,q}(-\lambda_0(s-u)^q)f(u,x(u))duds\right)\Big/\Gamma(\alpha)$$
$$=\frac{\sum_{v=0}^{j-1}\sum_{\tau=v}^{j-1}c_v\int_{t_\tau}^{t_{\tau+1}}(t-s)^{\alpha-1}(s-t_v)^{q-1}\mathbf{E}_{q,q}(-\lambda_0(s-t_v)^q)ds}{\Gamma(\alpha)}$$

change the order of sum

$$+\left(\sum_{v=0}^{j} c_v \int_{t_j}^{t} (t-s)^{\alpha-1}(s-t_v)^{q-1}\mathbf{E}_{q,q}(-\lambda_0(s-t_v)^q)ds\right.$$

$$\left.+\int_0^t \int_u^t (t-s)^{\alpha-1}(s-u)^{q-1}\mathbf{E}_{q,q}(-\lambda_0(s-u)^q)dsf(u,x(u))du\right) \Big/ \Gamma(\alpha)$$

change the order of integral

$$=\left(\sum_{v=0}^{j} c_v \int_{t_v}^{t} (t-s)^{\alpha-1}(s-t_v)^{q-1}\mathbf{E}_{q,q}(-\lambda_0(s-t_v)^q)ds\right.$$

$$\left.+\int_0^t \int_u^t (t-s)^{\alpha-1}(s-u)^{q-1}\mathbf{E}_{q,q}(-\lambda_0(s-u)^q)dsf(u,x(u))du\right) \Big/ \Gamma(\alpha)$$

$$=\left(\sum_{v=0}^{j} c_v \sum_{\chi=0}^{\infty} \frac{(-\lambda_0)^\chi}{\Gamma(\chi q+q)} \int_{t_v}^{t} (t-s)^{\alpha-1}(s-t_v)^{\chi q+q-1}ds\right.$$

$$\left.+\int_0^t \sum_{\chi=0}^{\infty} \frac{(-\lambda_0)^\chi}{\Gamma(\chi q+q)} \int_u^t (t-s)^{\alpha-1}(s-u)^{\chi q+q-1}dsf(u,x(u))du\right) \Big/ \Gamma(\alpha)$$

by

$$\frac{s-t_v}{t-t_v}=w, \quad \frac{s-u}{t-u}=w$$

$$=\frac{\sum_{v=0}^{j} c_v \sum_{\chi=0}^{\infty} \frac{(-\lambda_0)^\chi (t-t_v)^{\chi q+\alpha+q-1}}{\Gamma(\chi q+q)} \int_0^1 (1-w)^{\alpha-1} w^{\chi q+q-1}dw}{\Gamma(\alpha)}$$

$$+\frac{\int_0^t \sum_{\chi=0}^{\infty} \frac{(-\lambda_0)^\chi (t-u)^{\chi q+\alpha+q-1}}{\Gamma(\chi q+q)} \int_0^1 (1-w)^{\alpha-1} w^{\chi q+q-1}dwf(u,x(u))du}{\Gamma(\alpha)}$$

$$=\sum_{v=0}^{j} c_v \sum_{\chi=0}^{\infty} \frac{(-\lambda_0)^\chi (t-t_v)^{\chi q+\alpha+q-1}}{\Gamma(\chi q+\alpha+q)}$$

$$+\int_0^t \sum_{\chi=0}^{\infty} \frac{(-\lambda_0)^\chi (t-u)^{\chi q+\alpha+q-1}}{\Gamma(\chi q+\alpha+q)} f(u,x(u))du$$

$$=\sum_{v=0}^{j} c_v (t-t_v)^{\alpha+q-1}\mathbf{E}_{q,\alpha+q}(-\lambda_0(t-t_v)^q)$$

$$+\int_0^t (t-u)^{\alpha+q-1}\mathbf{E}_{q,\alpha+q}(-\lambda_0(t-u)^q)f(u,x(u))du.$$

It follows that

$$
\begin{aligned}
I_{0+}^{\alpha}x(t) &= \sum_{v=0}^{j} c_v(t-t_v)^{\alpha+q-1}\mathbf{E}_{q,\alpha+q}(-\lambda_0(t-t_v)^q) \\
&\quad + \int_0^t (t-u)^{\alpha+q-1}\mathbf{E}_{q,\alpha+q}(-\lambda_0(t-u)^q)f(u,x(u))du, \\
&\quad t\in(t_j,t_{j+1}],\ j\in\mathbf{N}_0^m.
\end{aligned}
\tag{4.6}
$$

Case 1. $\alpha+q<1$. From $I_{0+}^{\alpha}x(t_i^+)-I_{0+}^{\alpha}(t_i^-)=J_i(x(t_i)), i=1,2,\cdots,m$, and (14.11), we get $J_i(x(t_i))=\infty(i\in\mathbf{N}_1^m)$. This is a contradiction. So BVP (4.2) is unsuitable proposed.

Case 2. $\alpha+q=1$. From $I_{0+}^{\alpha}x(t_i^+)-I_{0+}^{\alpha}(t_i^-)=J_i(x(t_i)), i=1,2,\cdots,m$, and (14.11), we get $J_i(x(t_i))=c_i(i\in\mathbb{N}_1^m)$. From $t^{1-q}x(t)|_{t=0}+t^{1-q}x(t)|_{t=1}=0$ and (14.10), we get

$$
c_0+\sum_{v=0}^{m} c_v(1-t_v)^{q-1}\mathbf{E}_{q,q}(-\lambda_0(1-t_v)^q)+\int_0^1(1-s)^{q-1}\mathbf{E}_{q,q}(-\lambda_0(1-s)^q)f(s,x(s))ds=0.
$$

Then

$$
\begin{aligned}
c_0 = -&\left(\sum_{v=1}^{m} J_v(x(t_v))(1-t_v)^{q-1}\mathbf{E}_{q,q}(-\lambda_0(1-t_v)^q)\right. \\
&\left.+\int_0^1(1-s)^{q-1}\mathbf{E}_{q,q}(-\lambda_0(1-s)^q)f(s,x(s))ds\right)\Big/(1+\mathbf{E}_{q,q}(-\lambda_0)).
\end{aligned}
$$

Hence x is a solution of BVP (4.2) if and only if

$$
\begin{aligned}
x(t) = -&\left(\sum_{v=1}^{m} J_v(x(t_v))(1-t_v)^{q-1}\mathbf{E}_{q,q}(-\lambda_0(1-t_v)^q)\right. \\
&\left.+\int_0^1(1-s)^{q-1}\mathbf{E}_{q,q}(-\lambda_0(1-s)^q)f(s,x(s))ds\right)\Big/(1+\mathbf{E}_{q,q}(-\lambda_0))t^{q-1}\mathbf{E}_{q,q}(-\lambda_0t^q) \\
&+\sum_{v=1}^{j} I_v(x(t_v))(t-t_v)^{q-1}\mathbf{E}_{q,q}(-\lambda_0(t-t_v)^q) \\
&+\int_0^t(t-s)^{q-1}\mathbf{E}_{q,q}(-\lambda_0(t-s)^q)f(s,x(s))ds,\ t\in(t_j,t_{j+1}],\ j\in\mathbf{N}_0^m.
\end{aligned}
$$

Case 3. $\alpha+q>1$. From $I_{0+}^{\alpha}x(t_i^+)-I_{0+}^{\alpha}(t_i^-)=J_i(x(t_i)), i=1,2,\cdots,m$, and (14.11), we get $J_i(x(t_i))=0(i\in\mathbb{N}_1^m)$. From $t^{1-q}x(t)|_{t=0}+t^{1-q}x(t)|_{t=1}=0$ and (14.10), we get similarly as in Case 2 that

$$
c_0=-\frac{\sum\limits_{v=1}^{m} c_v(1-t_v)^{q-1}\mathbf{E}_{q,q}(-\lambda_0(1-t_v)^q)+\int_0^1(1-s)^{q-1}\mathbf{E}_{q,q}(-\lambda_0(1-s)^q)f(s,x(s))ds}{1+\mathbf{E}_{q,q}(-\lambda_0)}.
$$

Hence x is a solution of BVP (4.2) if and only if

$$x(t)=-\frac{\sum\limits_{v=1}^{m} c_v(1-t_v)^{q-1}\mathbf{E}_{q,q}(-\lambda_0(1-t_v)^q)+\int_0^1(1-s)^{q-1}\mathbf{E}_{q,q}(-\lambda_0(1-s)^q)f(s,x(s))ds}{1+\mathbf{E}_{q,q}(-\lambda_0)}$$

$$\times t^{q-1}\mathbf{E}_{q,q}(-\lambda_0 t^q)+\sum_{v=1}^{j} c_v(t-t_v)^{q-1}\mathbf{E}_{q,q}(-\lambda_0(t-t_v)^q)$$

$$+\int_0^t (t-s)^{q-1}\mathbf{E}_{q,q}(-\lambda_0(t-s)^q)f(s,x(s))ds, t\in(t_j,t_{j+1}], j\in\mathbf{N}_0^m.$$

Then x is a solution of BVP (4.2) if and only if

$$x(t)=-\frac{\sum\limits_{v=1}^{m} c_v(t-t_v)^{q-1}\mathbf{E}_{q,q}(-\lambda_0(1-t_v)^q)+\int_0^1(1-s)^{q-1}\mathbf{E}_{q,q}(-\lambda_0(1-s)^q)f(s,x(s))ds}{1+\mathbf{E}_{q,q}(-\lambda_0)}$$

$$\times t^{q-1}\mathbf{E}_{q,q}(-\lambda_0 t^q)+\sum_{v=1}^{j} c_v(t-t_v)^{q-1}\mathbf{E}_{q,q}(-\lambda_0(t-t_v)^q)$$

$$+\int_0^t (t-s)^{q-1}\mathbf{E}_{q,q}(-\lambda_0(t-s)^q)f(s,x(s))ds, t\in(t_j,t_{j+1}], j\in\mathbf{N}_0^m$$

and $J_i(x(t_i))=0(i\in\mathbf{N}_1^m)$.

Hence from Case 1-Case 3 Result 4.2 is incorrect. □

In [17], Zhao studied the existence of solutions of BVP(1.7) for the higher-order nonlinear Riemann-Liouville fractional differential equation with Riemann-Stieltjes integral boundary value conditions and impulses Lemma 2.4 [17] claimed:

Result 4.3. If H is a function of bounded variation

$$\delta=\int_0^1 s^{\alpha-1}dH(s)\alpha-1$$

and $h\in C([0,1])$, then the unique solution of

$$-D_{0^+}^{\alpha}x(t)=h(t),\ t\setminus\{t_i\}_{i=1}^m,$$

$$\Delta x(t_i)=I_i(x(t_i)),\ i=1,2,\cdots,m,$$

$$x(0)=x'(0)=\cdots=x^{(n-2)}0=0,\ x'(1)=\int_0^1 x(s)dH(s),$$

is

$$x(t)=\int_0^1 G(t,s)h(s)ds+t^{\alpha-1}\sum_{t\le t_k<1} t_k^{1-\alpha}I_k(x(t_k)),\ t\in[0,1], \tag{4.7}$$

where $G(t,s)=G_1(t,s)+G_2(t,s)$ and

$$G_1(t,s)=\begin{cases}\frac{t^{\alpha-1}(1-s)^{\alpha-2}-(t-s)^{\alpha-1}}{\Gamma(\alpha)},\ 0\le s\le t\le 1,\\ \frac{t^{\alpha-1}(1-s)^{\alpha-2}}{\Gamma(\alpha)},\quad 0\le t\le s\le 1,\end{cases}$$

$$G_2(t,s)=\frac{t^{\alpha-1}}{\alpha-1-\delta}\int_0^1 G_1(\tau,s)dH(\tau).$$

Remark 4.3. Result 4.3 is wrong.

In fact, we re-write (4.7) by

$$x(t)=-\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}h(s)ds+M_k t^{\alpha-1}, t\in(t_{k-1},t_k], k\in\mathbb{N}_1^{m+1},$$

where

$$M_k=\sum_{j=k}^{m} t_j^{1-\alpha}I_j(x(t_j)+\int_0^1\frac{(1-s)^{\alpha-2}}{\Gamma(\alpha)}h(s)ds$$
$$+\frac{1}{\alpha-1-\delta}\int_0^1\int_0^1 G_1(\tau,s)dH(\tau)h(s)ds.$$

One finds from Definition 2.2 for $t\in(t_i,t_{i+1}]$ that

$$D_{0^+}^{\alpha}x(t)=\frac{\left[\int_0^t(t-s)^{n-\alpha-1}x(s)ds\right]^{(n)}}{\Gamma(n-\alpha)}$$

$$=\frac{\left[\sum\limits_{\mu=0}^{i-1}\int_{t_\mu}^{t_{\mu+1}}(t-s)^{n-\alpha-1}x(s)ds+\int_{t_i}^t(t-s)^{n-\alpha-1}x(s)ds\right]^{(n)}}{\Gamma(n-\alpha)}$$

$$=\frac{\left[\sum\limits_{\mu=0}^{i-1}\int_{t_\mu}^{t_{\mu+1}}(t-s)^{n-\alpha-1}\left(-\int_0^s\frac{(s-u)^{\alpha-1}}{\Gamma(\alpha)}h(u)du+M_{\mu+1}s^{\alpha-1}\right)ds\right]^{(n)}}{\Gamma(n-\alpha)}$$
$$+\frac{\left[\int_{t_i}^t(t-s)^{n-\alpha-1}\left(-\int_0^s\frac{(s-u)^{\alpha-1}}{\Gamma(\alpha)}h(u)du+M_{i+1}s^{\alpha-1}\right)ds\right]^{(n)}}{\Gamma(n-\alpha)}$$

$$=\frac{\left[\sum\limits_{\mu=0}^{i-1}M_{\mu+1}\int_{t_\mu}^{t_{\mu+1}}(t-s)^{n-\alpha-1}s^{\alpha-1}ds\right]^{(n)}}{\Gamma(n-\alpha)}+\frac{\left[M_{i+1}\int_{t_i}^t(t-s)^{n-\alpha-1}s^{\alpha-1}ds\right]^{(n)}}{\Gamma(n-\alpha)}$$
$$+\frac{\left[-\int_0^t(t-s)^{n-\alpha-1}\int_0^s\frac{(s-u)^{\alpha-1}}{\Gamma(\alpha)}h(u)duds\right]^{(n)}}{\Gamma(n-\alpha)}$$

$$=\frac{\left[\sum\limits_{\mu=0}^{i-1}M_{\mu+1}\int_{t_\mu}^{t_{\mu+1}}(t-s)^{n-\alpha-1}s^{\alpha-1}ds\right]^{(n)}}{\Gamma(n-\alpha)}+\frac{\left[M_{i+1}\int_{t_i}^t(t-s)^{n-\alpha-1}s^{\alpha-1}ds\right]^{(n)}}{\Gamma(n-\alpha)}$$
$$+\frac{\left[-\int_0^t\int_u^t(t-s)^{n-\alpha-1}\frac{(s-u)^{\alpha-1}}{\Gamma(\alpha)}dsh(u)du\right]^{(n)}}{\Gamma(n-\alpha)}$$

$$= \frac{\left[\sum_{\mu=0}^{i-1} M_{\mu+1} t^{n-1} \int_{\frac{t_\mu}{t}}^{\frac{t_{\mu+1}}{t}} (1-w)^{n-\alpha-1} w^{\alpha-1} dw\right]^{(n)}}{\Gamma(n-\alpha)}$$

$$+ \frac{\left[M_{i+1} \int_{\frac{t_i}{t}}^{1} (1-w)^{n-\alpha-1} w^{\alpha-1} dw\right]^{(n)}}{\Gamma(n-\alpha)}$$

$$+ \frac{\left[-\int_0^t (t-u)^{n-1} \int_0^1 (1-w)^{n-\alpha-1} \frac{w^{\alpha-1}}{\Gamma(\alpha)} dw h(u) du\right]^{(n)}}{\Gamma(n-\alpha)}$$

$$= -h(t) + \frac{\left[\sum_{\mu=0}^{i-1} M_{\mu+1} t^{n-1} \int_{\frac{t_\mu}{t}}^{\frac{t_{\mu+1}}{t}} (1-w)^{n-\alpha-1} w^{\alpha-1} dw\right]^{(n)}}{\Gamma(n-\alpha)}$$

$$+ \frac{\left[M_{i+1} \int_{\frac{t_i}{t}}^{1} (1-w)^{n-\alpha-1} w^{\alpha-1} dw\right]^{(n)}}{\Gamma(n-\alpha)}.$$

It is easy to see that $D_{0^+}^{\alpha} x(t) \neq -h(t)$ on $(t_1, t_2]$. In fact, we find that $D_{0^+}^{\alpha} x(t) \neq -h(t)$ on $(t_1, t_2]$ if and only if $M_1 = M_2$. □

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Generalizations of an asymptotic stability theorem of Bahyrycz, Páles and Piszczek on Cauchy differences to generalized cocycles

Árpád Száz

Abstract. We prove some straightforward analogues and generalizations of a recent asymptotic stability theorem of A. Bahyrycz, Zs. Páles and M. Piszczek on Cauchy differences to semi-cocycles and pseudo-cocycles introduced in a former paper by the present author.

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1. Introduction

The first results on a certain stability property of the Cauchy functional equation

$$f(x+y) = f(x) + f(y) \tag{1.1}$$

were proved by Pólya and Szegő [63, p. 171] in 1925 and Hyers [38] in 1941.

In particular, Pólya and Szegő proved the following statement in two rather difficult ways.

Theorem 1.1. *Suppose that the number sequence* $a_1, a_2, a_3, \dots$ *satisfies the condition*

$$a_m + a_n - 1 < a_{m+n} < a_m + a_n + 1.$$

Then, there exists the limit

$$\lim_{n\to\infty} \frac{a_n}{n} = \omega;$$

and even more ω *is finite, and for all* $n = 1, 2, 3, \dots$ *there holds*

$$\omega n - 1 < a_n < \omega n + 1.$$

Remark 1.2. The significance of this theorem was not recognized either by Pólya and Szegő or the mathematical community for a long time. It was first cited by Kuczma [50, p. 424] in 1985 at the suggestion of R. Ger. However, in contrast to [5, 11], several authors have still not been mentioning it.

By R. Ger [34, p. 4] and some communications with Ger and M. Laczkovich, his attention to this theorem was first drawn by Laczkovich, at an undetectable conference, who indicated that the real-valued particular case of Hyers's stability theorem can be derived from it. His proof, reconstructed with the help of Ger and M. Sablik, can be found in [77, p. 633].

Hyers, giving a partial answer to a general problem proposed by S. M. Ulam before the Mathematics Club of the University of Wisconsin in 1940, proved the following fundamental theorem in a quite simple way.

An obvious generalization of his theorem to a function of a commutative semigroup to a Banach space [30, p. 216] already includes Theorem 1.1. Moreover, by Remark 1.2 and [30, Theorem 3], the two theorems are actually equivalent.

Theorem 1.3. *Let E and E' be Banach spaces and let $f(x)$ be a δ-linear transformation of E into E'. Then the limit $l(x) = \lim_{n\to\infty} f(2^n x)/2^n$ exists for each x in E, $l(x)$ is a linear transformation, and the inequality $\|f(x) - l(x)\| \leq \delta$ is true for all x in E.*

Remark 1.4. Moreover, Hyers also stated that $l(x)$ is the only linear transformation satisfying this inequality.

Here, in contrast to the recent terminology, Hyers used the term "linear" instead of "additive". Thus, his δ–linearity of $f(x)$ means only that

$$\|f(x+y) - f(x) - f(y)\| \leq \delta \text{ for all } x, y \in E.$$

Now, because of $l(nx) = nl(x)$, we also have

$$\|f(nx)/n - l(x)\| = \|f(nx) - l(nx)\|/n \leq \delta/n$$

for all $n \in \mathbb{N}$ and $x \in E$. Therefore, analogously to Theorem 1.1 of Pólya and Szegő, we can also state that $l(x) = \lim_{n\to\infty} f(nx)/n$ for all $x \in X$.

The above basic Theorem 1.3 of Hyers has been generalized, in one direction, by Aoki [6], Th. M. Rassias [65], J. M. Rassias [64], Găvruţă [32] (see also [43]), and in other directions by several further mathematicians.

Moreover, some counterexamples, showing the necessity of certain extra assumptions in the corresponding stability theorems, such as commutativity and completeness for instance, have also been provided.

The interested reader can get a rapid overview on this enormous subject by consulting some of the numerous survey papers [39, 34, 31, 66, 67, 68, 69, 82, 56, 70, 71, 23, 21, 12, 17, 57] and the fundamental books [40, 44, 22, 49, 46].

The above works show that some local, restricted, asymptotic, super and hyper stability results have also been proved for the Cauchy equation. Moreover, some close relationships with invariant means, sandwich and fixed point theorems have also been established. And, quite early, some set-valued generalizations have also been given.

These set-valued generalizations show that Hyers's stability theorem is actually an additive selection theorem for a subadditive relation. Therefore, some of its generalizations should be derived from those of the Hahn–Banach extension theorems. (For some ideas in this respect, see [22, Chapter 34] and [35, 77, 36].)

However, it is now more important to note that recently Bahyrycz, Páles and Piszczek [9] have proved a new type, asymptotic stability theorem for the Cauchy functional equation by using metric Abelian groups instead of normed spaces.

They called a triple $(X, +, d)$ to be a metric abelian group if $(X, +)$ is an abelian group and d is a translation invariant metric on X in the sense that

$$d(x+z, y+z) = d(x, y) \text{ for all } x, y, z \in X.$$

In this case, they defined $\|x\|_d = d(x, 0)$ for all $x \in X$, and noticed that $\| \, \|_d$ is an even subadditive function function on X which is not, in general, even 2–homogeneous.

In [74, 83, 13, 48], metric groups and groupoids have been used in different senses. From Remarks 3.1 of [83, 13], we can see that a metric d on a group X is translation invariant if and only if $d(x+y, z+w) \leq d(x, z) + d(y, w)$ for all $x, y, z, w \in X$.

Thus, if in particular d is a translation invariant metric on a group X, then the addition in X is continuous. Moreover, in this case we can also note that

$$d(-x, -y) = d(0, x-y) = d(y, x) = d(x, y) \text{ for all } x, y \in X.$$

Thus, in particular the inversion in X is also continuous.

In particular, in [9], Bahyrycz, Páles and Piszczek have proved the following asymptotic stability theorem.

Theorem 1.5. *Let $(X, +, d)$ and $(Y, +, \rho)$ be metric abelian groups such that X is unbounded by d. Let $\varepsilon \geq 0$ and assume that $f : X \to Y$ possesses the asymptotic stability property*

$$\limsup_{\min(\|x\|_d, \|y\|_d) \to \infty} \|f(x+y) - f(x) - f(y)\|_\rho \leq \varepsilon,$$

then

$$\|f(x+y) - f(x) - f(y)\|_\rho \leq 5\varepsilon \qquad \text{for all} \quad x, y \in X.$$

Remark 1.6. Moreover, by taking $\varepsilon > 0$ and $x_0 \in X \setminus \{0\}$, and defining

$$f(x_0) = 3\varepsilon \qquad \text{and} \qquad f(x) = \varepsilon \quad \text{for} \quad x \in X \setminus \{x_0\},$$

they have also proved that 5 is the smallest possible constant in the above theorem.

The most closely related related results to Theorem 1.5 are [52, Theorem 1] of Losonczi with the same constant 5, and the results of Jung, Moslehian, and Sahoo [45, 47] and Chung [18, 19, 20] with some other natural constants in the concluded estimates.

The origins of these investigations go back to Skof [72, 73], Hyers, Isac and Rassias [41] and Găvruţă [33]. (See also [54].) In the real-valued case, Volkmann [84] proved the best estimate.

From Theorem 1.5, by taking $\varepsilon = 0$, Bahyrycz, Páles and Piszczek could immediately derive the following asymptotic hyperstability result.

Corollary 1.7. *Let* $(X, +, d)$ *and* $(Y, +, \rho)$ *be metric abelian groups such that* X *is unbounded by* d*. If* $f : X \to Y$ *satisfies*

$$\limsup_{\min(\|x\|_d, \|y\|_d) \to \infty} \|f(x+y) - f(x) - f(y)\|_\rho = 0,$$

then

$$f(x+y) = f(x) + f(y), \qquad x, y \in X.$$

Hyperstability results, for the Cauchy equation and its generalizations, have also been proved by Maksa and Páles [53], Najati and Rassias [59], Alimohammady and Sadeghi [3], and Brzdek [14, 15, 16], Piszczek [61, 62], Almahalebi, Charifi and Kabbaj [4], Bahyrycz and Olko [7, 8], Aiemsomboon and Sintunavarat [1, 2], Molaei, Najati and Park [55, 58].

Moreover, several interesting asymptotic stability and hyperstability theorems for additive functions have also been proved by using some other functional equations than the Cauchy and generalized Cauchy ones.

In the present paper, we shall improve and generalize Theorem 1.5 of Bahyrycz, Páles and Piszczek. For this, we shall use preseminormed groups instead of the metric ones. Moreover, we shall use generalized cocycles introduced in [78], instead of the Cauchy difference

$$F(x, y) = f(x+y) - f(x) - f(y). \tag{1.2}$$

Some basic definitions and results on these fundamental objects, which are certainly unfamiliar to the reader, will be briefly laid out in the next preparatory section.

2. A few basic facts on preseminorms and generalized cocycles

Motivated by the corresponding definitions of [76, 80] and the proofs of our forthcoming theorems, an even subadditive function $\| \ \|$ of a group X to $\mathbb{R}$ will be called a *preseminorm* on X.

Thus, under the notation $\|x\| = \| \ \|(x)$, we have $\|0\| = \|0+0\| \leq \|0\| + \|0\|$, and thus $0 \leq \|0\|$. And more generally, $\|0\| = \|x + (-x)\| \leq \|x\| + \|-x\| = 2\|x\|$, and thus $0 \leq \|x\|$ for all $x \in X$.

Therefore, if $\|0\| \neq 0$, then by defining $\|x\|^\star = 0$ for $x = 0$, and $\|x\|^\star = \|x\|$ for $x \in X \setminus \{0\}$, we can obtain a new preseminorm $\| \ \|^\star$ on X such that $\|0\|^\star = 0$ already holds.

By using induction and the corresponding definitions, we can also easily see that $\|nx\| \leq n\|x\|$, and thus $\|(-n)x\| = \|n(-x)\| \leq n\|-x\| = n\|x\|$ for all $n \in \mathbb{N}$ and $x \in X$.

Therefore, the preseminorm $\| \ \|$ may be naturally called a *seminorm* if $n\|x\| \leq \|nx\|$ for all $x \in X$. Namely, thus we have $\|kx\| = |k|\|x\|$ for all $x \in X$ and $k \in \mathbb{Z} \setminus \{0\}$. (If $\|0\| = 0$, then this also holds for $k = 0$.)

Note that a nonzero seminorm cannot be bounded. While, if $\| \ \|$ is a seminorm (preseminorm) on X, then for instance the function defined by $\|x\|^* = \min\{1, \|x\|\}$ for all $x \in X$ is a bounded preseminorm on X.

Now, a seminorm (preseminorm) $\| \ \|$ on X may be naturally called a *norm (prenorm)* if $\|x\| = 0$ implies $x = 0$ for all $x \in X$. If $X = \mathbb{Z}x$ for all $x \in X \setminus \{0\}$, then each nonzero preseminorm on X is a prenorm.

In [79, Remark 3.14], with the help of G. Horváth, it was proved that the latter condition is equivalent to the requirement that the cardinality of X is prime, or equivalently X has no nontrival proper subgroup.

Now, for instance, an ordered pair $X(\| \ \|) = (X, \| \ \|)$ consisting a group X and a preseminorm $\| \ \|$ on X, may be naturally called a *preseminormed group*. And, we may simply write X instead of $X(\| \ \|)$.

If X is a preseminormed group, then because of the subadditivity and evenness of the corresponding preseminorm, for any $x, y \in X$, we have

$$\|x\| = \|x + y - y\| \leq \|x + y\| + \| - y\| = \|x + y\| + \|y\|$$

and

$$\|y\| = \| - x + x + y\| \leq \| - x\| + \|x + y\| = \|x\| + \|x + y\|.$$

Therefore,

$$\big|\|x\| - \|y\|\big| \leq \|x + y\| \quad \text{and} \quad \big|\|x\| - \|y\|\big| = \big|\|x\| - \| - y\|\big| \leq \|x - y\|.$$

However, it is now more important to note that Bahyrycz, Páles and Piszczek, in the proof of their Theorem 1.5, have used, but not explicitly stated, the equality

$$\begin{aligned} f(x+y) - f(x) - f(y) &= f(x-u) + f(u) - f(x) \\ &+ f(y-v) + f(v) - f(y) + f(x+y-u-v) - f(x-u) - f(y-v) \\ &+ f(u+v) - f(u) - f(v) + f(x+y) - f(x+y-u-v) - f(u+v). \end{aligned} \tag{2.1}$$

In a former paper [78], by using the Cauchy difference (1.2), we have noticed that, instead of equation (2.1), it is more convenient to consider the equation

$$\begin{aligned} F(x,y) = F(u,v) &- F(x-u,u) - F(y-v,v) \\ &+ F(x-u,y-v) + F(x+y-u-v,u+v). \end{aligned} \tag{2.2}$$

Namely, thus Theorem 1.5 can be easily extended to the solutions of (2.2). Moreover, we can prove that every *symmetric cocycle* F on X to Y is a solution of this equation.

That is, if F is a function of X^2 to Y such that $F(x,y) = F(y,x)$ and

$$F(x,y) + F(x+y,z) = F(x,y+z) + F(y,z) \tag{2.3}$$

for all $x, y, z \in X$, then (2.2) also holds for all $x, y, u, v \in X$.

It is well-known that every Cauchy–difference is a symmetric cocycle. Moreover, Davison and Ebanks [24, Lemma 2] have proved that if F is a symmetric cocycle on X to Y, then

$$F(x+y,u+v) = F(x+u,y+v) + F(x,u) + F(y,v) - F(x,y) - F(u,v) \tag{2.4}$$

also holds for all $x, y, u, v \in X$.

At first seeing, I considered equations (2.2) and (2.4) to be very similar, but still quite independent. However, Gyula Maksa, my close colleague, has noticed that they are actually equivalent.

Namely, (2.4) can be immediately derived from (2.2) by replacing x by $x+u$ and y by $y+v$. And conversely, (2.2) can be immediately derived from (2.4) by replacing x by $x-u$ and y by $y-v$. Thus, equation (2.1) is a consequence of (2.4) too.

Inspired by the above observations, in our former paper [78], we have also considered the more difficult equations

$$\begin{aligned} &F(x,y)+F(u,y+v)+F(x+y,u+v) \\ &\qquad = F(x,u)+F(y,u+v)+F(x+u,y+v) \end{aligned} \tag{2.5}$$

and

$$\begin{aligned} &F(x,y)+F(x-u,u)+F(y-v,u)+F(y-v,v) \\ &\qquad = F(u,v)+F(u,y-v)+F(x-u,y-v)+F(x+y-u-v,u+v). \end{aligned} \tag{2.6}$$

Note that if in particular F is symmetric, then equation (2.6) is equivalent to (2.2), which is in turn equivalent to (2.4). Moreover, it can be easily shown that if F is additive in its second variable, then equations (2.5) and (2.6) are also equivalent.

In our former paper [78], by using some more difficult computations, we have also proved that equations (2.5) and (2.6) are also natural generalizations of (2.3) too. Therefore, their solutions may be naturally called *semi-cocycles* and *pseudo-cocycles*, respectively.

In the light of the above observations, it seems to be a reasonable research program to extend some of the basic theorems on cocycles to these generalized cocycles. And, to establish some deeper relationships among the various generalizations of cocycles mentioned in [78].

However, in the sequel, we shall only prove some straightforward analogues and generalizations of Theorem 1.5 to semi-cocycles and pseudo–cocycles.

3. Analogues of Theorem 1.5 for generalized cocycles

Notation 3.1. *In the sequel, we shall assume that F is a function of an unbounded, commutative preseminormed group X to a commutative preseminormed group Y.*

Remark 3.2. Note that now, by defining

$$(x,y)+(u,v)=(x+u,\ y+v)$$

and

$$\|(x,y)\|=\|x\|\vee\|y\|=\max\big\{\|x\|,\ \|y\|\big\}$$

for all $x,y,u,v\in X$, the set X^2 can also be turned into an unbounded commutative preseminormed group.

Thus, by using a more simple argument than that used by Bahyrycz, Páles and Piszczek in [9], we can prove the following natural analogue of Theorem 1.5.

Theorem 3.3. *If F is a semi-cocycle and*

$$\varepsilon=\overline{\lim_{\|z\|\to+\infty}}\ \|F(z)\|,$$

then

$$\|F(z)\| \leq 5\varepsilon$$

for all $z \in X^2$.

Proof. By the corresponding definitions, for any $\eta > \varepsilon$, we have

$$\inf_{r>0} \sup_{\|z\|>r} \|F(z)\| < \eta.$$

Therefore, there exists $r > 0$ such that $\sup_{\|z\|>r} \|F(z)\| < \eta$, and thus

$$\|F(z)\| < \eta$$

for all $z \in X^2$ with $\|z\| > r$.

Hence, since $\|z\| = \|(z_1, z_2)\| \geq \|z_i\|$ for $i = 1, 2$, it is clear that in particular we have

$$\|F(s, t)\| < \eta$$

for all $s, t \in X$ with either $\|s\| > r$ or $\|t\| > r$.

Now, by taking $x, y \in X$ and using equation (2.5), we can see that

$$\begin{aligned}\|F(x, y)\| &= \|F(x, u) + F(y, u + v) - F(u, y + v)\\ &\quad + F(x + u, y + v) - F(x + y, u + v)\|\\ &\leq \|F(x, u)\| + \|F(y, u + v)\| + \|F(u, y + v)\|\\ &\quad + \|F(x + u, y + v)\| + \|F(x + y, u + v)\| < 5\eta\end{aligned}$$

whenever for instance $u, v \in X$ such that

$$\|u\| > r, \qquad \|u + v\| > r, \qquad \|x + u\| > r.$$

Therefore, if such u and v exist, then

$$\|F(x, y)\| < 5\eta, \qquad \text{and thus} \qquad \|F(x, y)\| \leq 5\varepsilon$$

Now, to complete the proof, it remains to show only that the required u and v exist. For this, we can note that, because of the assumed unboundedness of X, there exist $u, v \in X$ such that

$$\|u\| > r + \|x\| \qquad \text{and} \qquad \|v\| > r + \|u\|.$$

Thus, we evidently have $\|u\| > r$. Moreover, by using the inequality $\|s+t\| \geq \|t\|-\|s\|$, we can also see that

$$\|x + u\| \geq \|u\| - \|x\| > r + \|x\| - \|x\| = r$$

and

$$\|u + v\| \geq \|v\| - \|u\| > r + \|u\| - \|u\| = r.$$ □

From equation (2.6) and the proof of Theorem 3.3, it is clear that we also have

Theorem 3.4. *If F is a pseudo-cocycle and*

$$\varepsilon = \overline{\lim_{\|z\|\to+\infty}} \|F(z)\|,$$

then

$$\|F(z)\| \leq 7\varepsilon$$

for all $z \in X^2$.

Now, as an immediate consequence of the above two theorems, we can also state

Corollary 3.5. *If Y is prenormed, F is either a semi or pseudo cocycle, and*

$$\overline{\lim_{\|z\| \to +\infty}} \|F(z)\| = 0,$$

then $F(z) = 0$ for all $z \in X^2$.

Remark 3.6. However, from Theorems 3.3 and 3.4 we cannot get proper generalizations of Theorem 1.5. Therefore, in the next section we shall prove some modification and improvement of Theorem 3.4.

4. Proper and partial generalizations of Theorem 1.5 to pseudo-cocycles

Remark 4.1. Because of the condition of Theorem 1.5, in the sequel we shall also use the quantity

$$\nparallel (x, y) \nparallel = \|x\| \wedge \|y\| = \min\big\{ \|x\|,\ \|y\| \big\},$$

for all $(x, y) \in X^2$ instead of the natural preseminorm considered in Remark 3.2. Thus, the function $\nparallel\ \nparallel$ is not a preseminorm on X^2. However, despite this, it can be well used to measure the magnitude of the points of X^2.

Moreover, it can as well be used to prove the following proper and partial generalizations of Theorem 1.5 to pseudo-cocycles. The proof of the first one is quite similar to the second one. Therefore, it will be omitted.

Theorem 4.2. *If F is a symmetric pseudo-cocycle and*

$$\varepsilon = \overline{\lim_{\nparallel z \nparallel \to +\infty}} \|F(z)\|,$$

then

$$\|F(z)\| \le 5\varepsilon$$

for all $z \in X^2$.

The proof of the following theorem is again quite similar, but a little more readable, than the one given by Bahyrycz, Páles and Piszczek in [9].

Theorem 4.3. *If F is a pseudo-cocycle and*

$$\varepsilon = \overline{\lim_{\nparallel z \nparallel \to +\infty}} \|F(z)\|,$$

then

$$\|F(z)\| \le 7\varepsilon$$

for all $z \in X^2$.

Proof. By the corresponding definitions, for any $\eta > \varepsilon$, we have

$$\inf_{r>0} \sup_{\nparallel z \nparallel > r} \|F(z)\| < \eta.$$

Therefore, there exists $r > 0$ such that $\sup_{\nparallel z \nparallel > r} \|F(z)\| < \eta$, and thus

$$\|F(z)\| < \eta$$

for all $z \in X^2$ with $\nparallel z \nparallel > r$.

Hence, since $\nparallel z \nparallel = \nparallel (z_1, z_2) \nparallel = \min\{\|z_1\|, \ \|z_2\|\}$, it is clear that in particular we have

$$\|F(s,t)\| < \eta$$

for all $s, t \in X$ with $\|s\| > r$ and $\|t\| > r$.

Now, by taking $x, y \in X$ and using equation (2.6), we can see that

$$\begin{aligned}\|F(x,y)\| &= \|F(u,v) + F(u, y-v) - F(y-v, u)\\ &\quad - F(x-u, u) - F(y-v, v) + F(x-u, y-v) + F(x+y-u-v, u+v)\|\\ &\leq \|F(u,v)\| + \|F(u, y-v)\| + \|F(y-v, u)\|\\ &\quad + \|F(x-u, u)\| + \|F(y-v, v)| + \|F(x-u, y-v)\|\\ &\quad + \|F(x+y-u-v, u+v)\| < 7\eta\end{aligned}$$

whenever $u, v \in X$ such that

$$\|u\| > r, \qquad \|v\| > r, \qquad \|x-u\| > r, \qquad \|y-v\| > r,$$
$$\|u+v\| > r, \qquad \|x+y-u-v\| > r.$$

Therefore, if such u and v exist, then

$$\|F(x,y)\| < 7\eta, \qquad \text{and thus} \qquad \|F(x,y)\| \leq 7\varepsilon.$$

Now, to complete the proof, it remains only to show that the required u and v exist. For this, following the arguments given [9], we can note that because of the assumed unboundedness of X there exist $u, v \in X$ such that

$$\|u\| > r + \|x\| \qquad \text{and} \qquad \|v\| > r + \|x\| + \|y\| + \|u\|.$$

Thus, we evidently have $\|u\| > r$ and $\|v\| > r$. Moreover, by using the facts that $\|s+t\| \geq \|t\| - \|s\|$ and $\|-t\| = \|t\|$, we can also see that

$$\|x-u\| \geq \|u\| - \|x\| > r + \|x\| - \|x\| = r,$$

$$\|y-v\| \geq \|v\| - \|y\| > r + \|x\| + \|y\| + \|u\| - \|x\| = r + \|y\| + \|u\| \geq r,$$

and

$$\underline{\|u+v\|} \geq \|v\| - \|u\| > r + \|x\| + \|y\| + \|u\| - \|u\| = \underline{r + \|x\| + \|y\|} \geq r,$$

$$\|x+y-u-v\| \geq \|u+v\| - \|x+y\| > r + \|x\| + \|y\| - \|x\| - \|y\| = r.$$

Namely, because of the inequality $\|x\| + \|y\| \geq \|x+y\|$, we also have

$$-\|x+y\| \geq -\|x\| - \|y\|. \qquad \square$$

From this theorem, we can immediately derive

Corollary 4.4. *If Y is prenormed, F is a pseudo-cocycle and*

$$\overline{\lim_{\nparallel z \nparallel \to +\infty}} \|F(z)\| = 0,$$

then $F(z) = 0$ for all $z \in X^2$.

Remark 4.5. Recall that a Cauchy-difference is a symmetric cocycle. Moreover, a cocycle is both a semi-cocycle and a pseudo-cocycle.

Therefore, in Theorem 4.2, F may, in particular, be a Cauchy-difference or a symmetric cocycle. While, in Theorems 3.3, 3.4 and 4.3 and their corollaries, F may already be an arbitrary cocycle.

5. Some supplementary notes

Remark 5.1. If for instance f is a function of X to Y and $\varepsilon \geq 0$ such that there exists $r > 0$ such that

$$\|f(x+y) - f(x) - f(y)\| \leq \varepsilon$$

for all $x, y \in X$ with $\|x\|, \|y\| > r$, then by Remark 4.1 and the definition of the upper limit we have

$$\overline{\lim_{\nparallel (x,y) \nparallel \to +\infty}} \|f(x+y) - f(x) - f(y)\| \leq \varepsilon.$$

Thus, by Remark 4.5 and Theorem 4.2, we can state that

$$\|f(x+y) - f(x) - f(y)\| \leq 5\varepsilon$$

for all $x, y \in X$. Therefore, Theorem 1.5 follows from Theorem 4.2.

Now, if in particular Y is the additive group of a Banach space, then by using a slight generation of Theorem 1.3 we can also state that there exists an additive function g of X to Y such that

$$\|f(x) - g(x)\| \leq 5\varepsilon$$

for all $x \in X$.

Remark 5.2. While, if f is an arbitrary and g is an additive function of X to Y such that

$$\|f(x) - g(x)\| \leq 5\varepsilon$$

for all $x \in X$, then we can only state that

$$\begin{aligned} \|f(x+y) - f(x) - f(y)\| &= \|f(x+y) - g(x+y) + g(x) - f(x) + g(y) - f(y)\| \\ &\leq \|f(x+y) - g(x+y)\| + \|g(x) - f(x)\| + \|g(y) - f(y)\| \\ &\leq 15\varepsilon \end{aligned}$$

for all $x, y \in X$. Therefore, Theorem 1.5 is sharper than the one derivable from the usual asymptotic stability theorems.

This clearly reveal that the corresponding theorems on restricted and assymptotic stabilities have to split into two parts. This idea is also apparent from the proofs of those theorems.

Remark 5.3. Concerning our former results, it is also worth mentioning that in Theorems 3.3 and 3.4 and Corollary 3.5, instead of the "supremum preseminorm"

$$\|(x,y)\| = \|(x,y)\|_\infty = \max\big\{\|x\|,\ \|y\|\big\}$$

considered in Remark 3.2, we may also naturally use an "L_p–preseminorm", defined by

$$\|(x,y)\| = \|(x,y)\|_p = \Big(\|x\|^p + \|y\|^p\Big)^{1/p}$$

for some $1 \le p < +\infty$ and all $(x,y) \in X^2$. Namely, this also has the important property that $\|z_i\| \le \|z\|$ for $i = 1,2$, whenever $z \in X^2$.

Remark 5.4. Moreover, we can also note that $\|z\| \geq \nparallel z \nparallel$ for all $z \in X^2$. Therefore,

$$\big\{F(z) :\ \nparallel z \nparallel > r\big\} \subseteq \big\{F(z) :\ \ \|z\| > r\big\},$$

and thus

$$\sup_{\nparallel z \nparallel > r} \|F(z)\| \le \sup_{\|z\|>r} \|F(z)\|$$

for all $r > 0$. Consequently,

$$\varlimsup_{\nparallel z \nparallel \to +\infty} \|F(z)\| = \inf_{r>0} \sup_{\nparallel z \nparallel > r} \|F(z)\| \le \inf_{r>0} \sup_{\|z\|>r} \|F(z)\| = \varlimsup_{\|z\| \to +\infty} \|F(z)\|.$$

Therefore, the results obtained with $\|\ \|$ are usually much weaker than that obtained with $\nparallel\ \nparallel$. However, the former ones are, in a certain sense, still more natural since $\nparallel\ \nparallel$ is not a preseminorm on X^2.

6. Suggestions for further investigations

Cauchy differences, in the theory of functional equations, were first characterized by Kurepa [51] and Erdős [29] with the help of the equation

$$F(x,y) + F(x+y,z) = F(x,y+z) + F(y,z) \tag{6.1}$$

(For the algebraic origins and several further developments on this cocycle equation, see the book [75] by Stetkaer.)

Quadratic differences were first characterized by Székelyhidi [81] with the help of the equation

$$F(x+y,z) + F(x-y,z) - 2F(y,z) = F(x,y+z) + F(x,y-z) - 2F(x,y). \tag{6.2}$$

(For some closely related results, see also Ebanks and Ng [25, 28].)

Some results on equations (6.1) and (6.2) were extended by Páles [60] to the more attractive equation

$$F(x,y) + \frac{1}{n}\sum_{i=1}^{n} F\big(x+\phi_i(y), z\big) = \frac{1}{n}\sum_{i=1}^{n} F\big(x,\ y+\phi_i(z)\big) + F(y,z). \tag{6.3}$$

(Later, this important equation has only been studied by Maksa and Páles [53].)

Recently, Leibniz differences has been characterized by Ebanks [27] with the help of the equation

$$F(xy,z) + xF(x,y) = F(x,yz) + xF(y,z). \tag{6.4}$$

(For some earlier results, see Jessen, Karpf and Thorup [42], Ebanks [26] and Gselmann and Páles [37].)

Moreover, affine differences has been characterized by Boros [10] with the help of the equation

$$F\big(s,\ rx+(1-r)y,\ tx+(1-t)y\big) = F\big(sr+(1-s)t, x, y\big) - sF(r,x,y) - (1-s)F(t,x,y). \qquad (6.5)$$

Thus, it is certainly true that several further important differences, such as for instance the Jensen one, can also be characterized with the help of some functional equations containing a little more variables than the corresponding differences.

Therefore, it seems to be a reasonable research program to prove some counterparts of the results of Bahyrycz, Páles and Piszczek [9] and the present author for such equations and their generalizations. First of all, some analogues and generalizations of [9, Theorem 2] could be proved.

Added in Proof. The original version of this paper (Tech. Rep., Inst. Math., Univ. Debrecen 20016/2, 12 pp.) was planned to be published in the Proceedings of the Conference on Ulam's Type Stability, 2016, Cluj-Napoca, Romania. However, it was later submitted to the present journal.

Here, to improve the presentation, some useful changes have been suggested by the referee and the editor. In particular, according to a general regulation of the journal, all items of References have to be cited in the text. Therefore, my original manuscript has been substantially rewritten.

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Korovkin type approximation for double sequences via statistical $\mathcal{A}$-summation process on modular spaces

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Abstract. In this work, we introduce the Korovkin type approximation theorems on modular spaces via statistical $\mathcal{A}$-summation process for double sequences of positive linear operators and we construct an example satisfying our new approximation theorem but does not satisfy the classical one.

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1. Introduction and preliminaries

Summability theory is the theory of the assignment of limits in the case of real or complex sequences which are divergent. There are many types of summability methods especially regular summability methods, for example, Abel and Borel methods [6]. Another regular summability method introduced by Fast ([8]) and which is not equivalent to any regular matrix method is called statistical convergence which is also known as $(C,1)$ statistical convergence. Furthermore, in recent years, various statistical approximation results and theorems have been proved via the concept of statistical convergence ([9, 11, 19]) and the motivation using this type of convergence comes from that the obtained results are more powerful than the classical version of the approximations. One of these frequently used approximation method is the Korovkin-type approximation theorems. As it is known Korovkin theorems allows us to check the convergence with a minimum of computations. In this paper, our main purpose is to study a further generalization of classical Korovkin theorem by considering certain matrix summability process in the frame of statistical convergence in abstract spaces (namely, modular spaces) for double sequences. We also introduce an example satisfying new approximation theorem but does not satisfy the classical one.

Now, let us mention the notion of statistical convergence for double sequences introduced by Moricz [15].

The double sequence $x = \{x_{i,j}\}$ is statistically convergent to L provided that for every $\varepsilon > 0$,

$$P - \lim_{m,n} \frac{1}{mn} \left| \{ i \leq m, \ j \leq n : \ \ |x_{i,j} - L| \geq \varepsilon \} \right| = 0,$$

where $P-$convergent denotes Pringsheim limit ([22]). In that case we write

$$st_2 - \lim_{i,j} x_{i,j} = L.$$

It can be easily seen that a $P-$convergent double sequence is statistically convergent to the same value but its converse is not always true. Also, it is crucial to state that a convergent single sequence needs to be bounded even though this necessity does not exist always for the double sequences. A convergent double sequence does not need to be bounded. For example, take into consideration the double sequence $x = \{x_{i,j}\}$ defined by

$$x_{i,j} = \begin{cases} ij, & i \text{ and } j \text{ are squares} \\ 1, & \text{otherwise.} \end{cases} .$$

Then, clearly $st_2 - \lim_{i,j} x_{i,j} = 1$ but not $P-$convergent and also, it is not bounded. The characterization for the statistical convergence for double sequences is given in [15] as indicated below :

A double sequence $x = \{x_{i,j}\}$ is statistically convergent to L if and only if there exists a set $S \subset \mathbb{N}^2$ such that the natural density of S is 1 and

$$P - \lim_{\substack{i,j \to \infty \\ \text{and } (i,j) \in S}} x_{i,j} = L.$$

In [7] the concepts of *statistical superior limit* and *inferior limit* for double sequences have been introduced by Çakan and Altay. For any real double sequence $x = \{x_{i,j}\}$, the statistical limit superior of x is defined by

$$st_2 - \lim_{i,j} \sup x_{i,j} = \begin{cases} \sup G_x, & \text{if } G_x \neq \varnothing, \\ -\infty, & \text{if } G_x = \varnothing, \end{cases}$$

where $G_x := \{C \in \mathbb{R} : \delta_2 (\{(i,j) : x_{i,j} > C\}) \neq 0\}$ and $\varnothing$ denotes the empty set. Note that, in general, by $\delta_2 (K) \neq 0$ we mean either $\delta_2 (K) > 0$ or K fails to have the double natural density. Similarly, the statistical limit inferior of x is given by

$$st_2 - \lim_{i,j} \inf x_{i,j} = \begin{cases} \inf F_x, & \text{if } F_x \neq \varnothing, \\ \infty, & \text{if } F_x = \varnothing, \end{cases}$$

where $F_x := \{D \in \mathbb{R} : \delta_2 (\{(i,j) : x_{i,j} < D\}) \neq 0\}$. As in the ordinary superior or inferior limit, it was proved that

$$st_2 - \lim_{i,j} \inf x_{i,j} \leq st_2 - \lim_{i,j} \sup x_{i,j}$$

and also that, for any double sequence $x = \{x_{i,j}\}$ satisfying

$$\delta_2 (\{(i,j) : |x_{i,j}| > M\}) = 0$$

for some $M > 0$,

$$st_2 - \lim_{i,j} x_{i,j} = L \text{ iff } st_2 - \liminf_{i,j} x_{i,j} = st_2 - \limsup_{i,j} x_{i,j} = L.$$

Let $A = [a_{k,l,i,j}]$, $k, l, i, j \in \mathbb{N}$, be a four-dimensional infinite matrix.
The $A-$transform of $x = \{x_{i,j}\}$, denoted by $Ax := \{(Ax)_{k,l}\}$, is defined by

$$(Ax)_{k,l} = \sum_{(i,j)\in\mathbb{N}^2} a_{k,l,i,j} x_{i,j}, \quad k, l \in \mathbb{N},$$

provided the double series converges in Pringsheim's sense for every $(k, l) \in \mathbb{N}^2$. Then, a double sequence x is $A-$summable to L if the $A-$transform of x exists for all $k, l \in \mathbb{N}$ and convergent in the Pringsheim's sense i.e.,

$$P - \lim_{p,q} \sum_{i=1}^{p} \sum_{j=1}^{q} a_{k,l,i,j} x_{i,j} = y_{k,l} \text{ and } P - \lim_{k,l} y_{k,l} = L.$$

Now let $\mathcal{A} := \left\{A^{(m,n)}\right\} = \left\{a^{(m,n)}_{k,l,i,j}\right\}$ be a sequence of four-dimensional infinite matrices with non-negative real entries. For a given double sequence of real numbers, $x = \{x_{i,j}\}$ is said to be $\mathcal{A}-$summable to L if

$$P - \lim_{k,l} \sum_{(i,j)\in\mathbb{N}^2} a^{(m,n)}_{k,l,i,j} x_{i,j} = L$$

uniformly in m and n. If $A^{(m,n)} = A$, four-dimensional infinite matrix, then $\mathcal{A}-$summability is the $A-$summability for four-dimensional infinite matrix. Some results concerning matrix summability method for double sequences may be attained in [9], [21], [24].

Now, we recall some definitions and notations on modular spaces.

Let $I = [a, b]$ be a bounded interval of the real line $\mathbb{R}$ provided with the Lebesgue measure. Then, let $X\left(I^2\right)$ denote the space of all real-valued measurable functions on $I^2 = [a, b] \times [a, b]$ provided with equality $a.e.$ As usual, let $C\left(I^2\right)$ denote the space of all continuous real-valued functions, and $C^\infty\left(I^2\right)$ denote the space of all infinitely differentiable functions on I^2. A functional $\rho : X\left(I^2\right) \to [0, +\infty]$ is called a *modular* on $X\left(I^2\right)$ if it satisfies the following conditions:

(i) $\rho(f) = 0$ if and only if $f = 0$ $a.e.$ in I^2,

(ii) $\rho(-f) = \rho(f)$ for every $f \in X\left(I^2\right)$,

(iii) $\rho(\alpha f + \beta g) \le \rho(f) + \rho(g)$ for every $f, g \in X(I^2)$ and for any $\alpha, \beta \ge 0$ with $\alpha + \beta = 1$.

A modular ρ is said to be $N-$*quasi convex* if there exists a constant $N \ge 1$ such that $\rho(\alpha f + \beta g) \le N\alpha\rho(Nf) + N\beta\rho(Ng)$ holds for every $f, g \in X\left(I^2\right)$, $\alpha, \beta \ge 0$ with $\alpha + \beta = 1$. In particular, if $N = 1$, then ρ is called *convex*.

A modular ρ is said to be $N-$*quasi semiconvex* if there exists a constant $N \ge 1$ such that $\rho(af) \le Na\rho(Nf)$ holds for every $f \in X\left(I^2\right)$ and $a \in (0, 1]$.

It is clear that every $N-$quasi convex modular is $N-$quasi semiconvex. Bardaro et. al. introduced and worked through the above two concepts in [3, 5].

We now present some acquired vector subspaces of $X(I^2)$ via a modular ρ as follows:

The modular space $L^\rho\left(I^2\right)$ generated by ρ is defined by

$$L^\rho\left(I^2\right) := \left\{ f \in X(I^2) : \lim_{\lambda \to 0^+} \rho(\lambda f) = 0 \right\},$$

and the space of the finite elements of $L^\rho\left(I^2\right)$ is given by

$$E^\rho\left(I^2\right) := \left\{ f \in L^\rho\left(I^2\right) : \rho(\lambda f) < +\infty \text{ for all } \lambda > 0 \right\}.$$

Observe that if ρ is $N-$quasi semiconvex, then the space

$$\left\{ f \in X\left(I^2\right) : \rho(\lambda f) < +\infty \text{ for some } \lambda > 0 \right\}$$

coincides with $L^\rho\left(I^2\right)$. The notions about modulars are introduced in [16] and widely discussed in [3] (see also [13, 17]).

Bardaro and Mantellini [4] introduced some Korovkin type approximation theorems via the notions of modular convergence and strong convergence. Afterwards Karakuş et al. [11] investigated the modular Korovkin-type approximation theorem via statistical convergence and then, Orhan and Demirci [20] extended these type of approximations to the spaces of double sequences of positive linear operators as follows:

Definition 1.1. [20] A function sequence $\{f_{i,j}\}$ in $L^\rho\left(I^2\right)$ is said to be statistical*ly modularly convergent* to a function $f \in L^\rho\left(I^2\right)$ iff

$$st_2 - \lim_{i,j} \rho\left(\lambda_0\left(f_{i,j} - f\right)\right) = 0 \text{ for some } \lambda_0 > 0. \tag{1.1}$$

Also, $\{f_{i,j}\}$ is statistically $F-$*norm convergent* (or, statistically *strongly convergent*) to f iff

$$st_2 - \lim_{i,j} \rho\left(\lambda\left(f_{i,j} - f\right)\right) = 0 \text{ for every } \lambda > 0. \tag{1.2}$$

It is known from [16] that (1.1) and (1.2) are equivalent if and only if the modular ρ satisfies the Δ_2-condition, i.e.

there exists a constant $M > 0$ such that $\rho(2f) \le M\rho(f)$ for every $f \in X\left(I^2\right)$.

Recently, Orhan and Demirci [19] have introduced the notion of $\mathcal{A}-$summation process on the one dimensional modular space $X(I)$. Now we introduce the notion of the $\mathcal{A}-$*summation process for double sequences* as follows:

A sequence $\mathbb{T} := \{T_{i,j}\}$ of positive linear operators of D into $X\left(I^2\right)$ is called an $\mathcal{A}-$summation process on D if $\{T_{i,j}(f)\}$ is $\mathcal{A}-$summable to f (with respect to modular ρ) for every $f \in D$, i.e.,

$$P - \lim_{k,l} \rho\left[\lambda\left(A^{\mathbb{T}}_{k,l,m,n}(f) - f\right)\right] = 0, \text{ uniformly in } m, n \tag{1.3}$$

for some $\lambda > 0$, where for all $k, l, m, n \in \mathbb{N}$, $f \in D$ the series

$$A^{\mathbb{T}}_{k,l,m,n}(f) := \sum_{(i,j) \in \mathbb{N}^2} a^{(m,n)}_{k,l,i,j} T_{i,j} f \tag{1.4}$$

is absolutely convergent almost everywhere with respect to Lebesgue measure and we denote the value of $T_{i,j}f$ at a point $(x,y) \in I^2$ by $T_{i,j}(f(u,v);x,y)$ or briefly, $T_{i,j}(f;x,y)$.

Our goal in the present work is to give the Korovkin theorem for double sequences of positive linear operators using *statistical $\mathcal{A}-$summation process* on a modular space. Some results concerning summation processes in the space $L_p[a,b]$ of Lebesgue integrable functions on a compact interval may be found in [18, 23].

It is required to give the following assumptions on a modular ρ :

A modular ρ is *monotone* if $\rho(f) \leq \rho(g)$ for $|f| \leq |g|$, ρ is said to be *finite* if $\chi_A \in L^\rho\left(I^2\right)$ whenever A is measurable subset of I^2 such that $\mu(A) < \infty$. If ρ is finite and, for every $\varepsilon > 0$, $\lambda > 0$, there exists a $\delta > 0$ such that $\rho(\lambda\chi_B) < \varepsilon$ for any measurable subset $B \subset I^2$ with $\mu(B) < \delta$, then ρ is *absolutely finite* and if $\chi_{I^2} \in E^\rho\left(I^2\right)$, then ρ is *strongly finite.* A modular ρ is *absolutely continuous* provided that there exists an $\alpha > 0$ such that, for every $f \in X\left(I^2\right)$ with $\rho(f) < +\infty$, the following condition holds:

- for every $\varepsilon > 0$ there is $\delta > 0$ such that $\rho(\alpha f\chi_B) < \varepsilon$ whenever B is any measurable subset of I^2 with $\mu(B) < \delta$.

Observe now that (see [4, 5]) if a modular ρ is monotone and finite, then we have $C(I^2) \subset L^\rho\left(I^2\right)$. Similarly, if ρ is monotone and strongly finite, then $C(I^2) \subset E^\rho\left(I^2\right)$. Also, if ρ is monotone, absolutely finite and absolutely continuous, then $\overline{C^\infty(I^2)} = L^\rho\left(I^2\right)$. (See for more details [2, 3, 14, 17]).

2. Main results

Let ρ be a monotone and finite modular on $X\left(I^2\right)$. Assume that D is a set satisfying $C^\infty\left(I^2\right) \subset D \subset L^\rho\left(I^2\right)$. (Such a subset D can be constructed when ρ is monotone and finite, see [4]). Also, assume that $\mathbb{T} := \{T_{i,j}\}$ is a sequence of positive linear operators from D into $X\left(I^2\right)$ for which there exists a subset $X_\mathbb{T} \subset D$ with $C^\infty\left(I^2\right) \subset X_\mathbb{T}$ such that

$$st_2 - \limsup_{k,l} \rho\left(\lambda\left(A^{\mathbb{T}}_{k,l,m,n}(h)\right)\right) \leq R\rho(\lambda h), \text{ uniformly in } m,n, \tag{2.1}$$

holds for every $h \in X_\mathbb{T}$, $\lambda > 0$ and for an absolute positive constant R.

We will use the test functions f_r $(r = 0,1,2,3)$ defined by $f_0(x,y) = 1$, $f_1(x,y) = x$, $f_2(x,y) = y$ and $f_3(x,y) = x^2 + y^2$ throughout the paper.

We now prove the following Korovkin type theorem.

Theorem 2.1. *Let $\mathcal{A} = \left\{A^{(m,n)}\right\}$ be a sequence of four dimensional infinite non-negative real matrices and let ρ be a monotone, strongly finite, absolutely continuous and $N-$quasi semiconvex modular on $X\left(I^2\right)$. Let $\mathbb{T} := \{T_{i,j}\}$ be a sequence of positive linear operators from D into $X\left(I^2\right)$ satisfying (2.1) for each $f \in D$. Suppose that*

$$st_2 - \lim_{k,l} \rho\left(\lambda\left(A^{\mathbb{T}}_{k,l,m,n}(f_r) - f_r\right)\right) = 0, \quad \textit{uniformly in } m,n \tag{2.2}$$

for every $\lambda > 0$ *and* $r = 0, 1, 2, 3$. *Now let* f *be any function belonging to* $L^{\rho}\left(I^2\right)$ *such that* $f - g \in X_{\mathbb{T}}$ *for every* $g \in C^{\infty}\left(I^2\right)$. *Then we have*

$$st_2 - \lim_{k,l} \rho\left(\lambda_0\left(A^{\mathbb{T}}_{k,l,m,n}(f) - f\right)\right) = 0, \quad \textit{uniformly in } m, n \tag{2.3}$$

for some $\lambda_0 > 0$.

Proof. We first claim that

$$st_2 - \lim_{k,l} \rho\left(\eta\left(A^{\mathbb{T}}_{k,l,m,n}(g) - g\right)\right) = 0 \text{ uniformly in } m, n \tag{2.4}$$

for every $g \in C(I^2) \cap D$ and $\eta > 0$ where

$$A^{\mathbb{T}}_{k,l,m,n}(g) = \sum_{(i,j)\in\mathbb{N}^2} a^{(m,n)}_{k,l,i,j} T_{i,j} g.$$

To see this assume that g belongs to $C\left(I^2\right) \cap D$ and η is any positive number. By the continuity of g on I^2 and in consequence of the linearity and positivity of the operators $T_{i,j}$, we can easily see that (see, for instance [20]), for a given $\varepsilon > 0$, there exists a number $\delta > 0$ such that for all $(u, v), (x, y) \in I^2$

$$|g(u, v) - g(x, y)| < \varepsilon + \frac{2M}{\delta^2}\left\{(u - x)^2 + (v - y)^2\right\}.$$

where $M := \sup_{(x,y)\in I^2} |g(x, y)|$. Since $T_{i,j}$ is linear and positive, we get

$$\begin{aligned}
&\left|\sum_{(i,j)\in\mathbb{N}^2} a^{(m,n)}_{k,l,i,j} T_{i,j}(g; x, y) - g(x, y)\right| \\
&= \left|\sum_{(i,j)\in\mathbb{N}^2} a^{(m,n)}_{k,l,i,j} T_{i,j}(g(.,.) - g(x, y); x, y)\right. \\
&\quad \left. + g(x, y)\left(\sum_{(i,j)\in\mathbb{N}^2} a^{(m,n)}_{k,l,i,j} T_{i,j}(f_0; x, y) - f_0(x, y)\right)\right| \\
&\leq \sum_{(i,j)\in\mathbb{N}^2} a^{(m,n)}_{k,l,i,j} T_{i,j}(|g(.,.) - g(x, y)|; x, y) \\
&\quad + |g(x, y)|\left|\sum_{(i,j)\in\mathbb{N}^2} a^{(m,n)}_{k,l,i,j} T_{i,j}(f_0; x, y) - f_0(x, y)\right| \\
&\leq \sum_{(i,j)\in\mathbb{N}^2} a^{(m,n)}_{k,l,i,j} T_{i,j}\left(\varepsilon + \frac{2M}{\delta^2}\left\{(. - x)^2 + (. - y)^2\right\}; x, y\right) \\
&\quad + |g(x, y)|\left|\sum_{(i,j)\in\mathbb{N}^2} a^{(m,n)}_{k,l,i,j} T_{i,j}(f_0; x, y) - f_0(x, y)\right|
\end{aligned}$$

$$= \varepsilon + (\varepsilon + M) \left| \sum_{(i,j)\in\mathbb{N}^2} a_{k,l,i,j}^{(m,n)} T_{i,j}(f_0; x, y) - f_0(x, y) \right|$$

$$+ \frac{2M}{\delta^2} \left| \sum_{(i,j)\in\mathbb{N}^2} a_{k,l,i,j}^{(m,n)} T_{i,j}(f_3; x, y) - f_3(x, y) \right|$$

$$+ \frac{4M}{\delta^2} \left(|f_1(x, y)| \left| \sum_{(i,j)\in\mathbb{N}^2}^{\infty} a_{k,l,i,j}^{(m,n)} T_{ij}(f_1; x, y) - f_1(x, y) \right| \right.$$

$$\left. + |f_2(x, y)| \left| \sum_{(i,j)\in\mathbb{N}^2}^{\infty} a_{k,l,i,j}^{(m,n)} T_{ij}(f_2; x, y) - f_2(x, y) \right| \right)$$

$$+ \frac{2M}{\delta^2} |f_3(x, y)| \left| \sum_{(i,j)\in\mathbb{N}^2}^{\infty} a_{k,l,i,j}^{(m,n)} T_{ij}(f_0; x, y) - f_0(x, y) \right|$$

for every $x, y \in I$ and $m, n \in \mathbb{N}$. Therefore, from the the last inequality we get

$$\left| \sum_{(i,j)\in\mathbb{N}^2} a_{k,l,i,j}^{(m,n)} T_{i,j}(g; x, y) - g(x, y) \right|$$

$$\leq \varepsilon + \left(\varepsilon + M + \frac{4Mc^2}{\delta^2} \right) \left| \sum_{(i,j)\in\mathbb{N}^2} a_{k,l,i,j}^{(m,n)} T_{i,j}(f_0; x, y) - f_0(x, y) \right|$$

$$+ \frac{4Mc}{\delta^2} \left| \sum_{(i,j)\in\mathbb{N}^2} a_{k,l,i,j}^{(m,n)} T_{i,j}(f_1; x, y) - f_1(x, y) \right|$$

$$+ \frac{4Mc}{\delta^2} \left| \sum_{(i,j)\in\mathbb{N}^2} a_{k,l,i,j}^{(m,n)} T_{i,j}(f_2; x, y) - f_2(x, y) \right|$$

$$+ \frac{2M}{\delta^2} \left| \sum_{(i,j)\in\mathbb{N}^2} a_{k,l,i,j}^{(m,n)} T_{i,j}(f_3; x, y) - f_3(x, y) \right|$$

where $c := \max \{ |f_1(x, y)|, |f_2(x, y)| \}$.
So, denoting by $K := \max \left\{ \varepsilon + M + \frac{4Mc^2}{\delta^2}, \frac{4Mc}{\delta^2}, \frac{2M}{\delta^2} \right\}$,

$$\left| \sum_{(i,j)\in\mathbb{N}^2} a_{k,l,i,j}^{(m,n)} T_{i,j}(g; x, y) - g(x, y) \right|$$

$$\leq \varepsilon + K \left\{ \left| \sum_{(i,j)\in\mathbb{N}^2} a_{k,l,i,j}^{(m,n)} T_{i,j}(f_0; x, y) - f_0(x, y) \right| \right.$$

$$+\left|\sum_{(i,j)\in\mathbb{N}^2} a_{k,l,i,j}^{(m,n)} T_{i,j}\left(f_1;x,y\right)-f_1\left(x,y\right)\right|$$
$$+\left|\sum_{(i,j)\in\mathbb{N}^2} a_{k,l,i,j}^{(m,n)} T_{i,j}\left(f_2;x,y\right)-f_2\left(x,y\right)\right|$$
$$\left.+\left|\sum_{(i,j)\in\mathbb{N}^2} a_{k,l,i,j}^{(m,n)} T_{i,j}\left(f_3;x,y\right)-f_3\left(x,y\right)\right|\right\}.$$

Hence, we obtain, for any $\eta>0$, that

$$\begin{aligned}
&\eta\left|\sum_{(i,j)\in\mathbb{N}^2} a_{k,l,i,j}^{(m,n)} T_{i,j}\left(g;x,y\right)-g\left(x,y\right)\right| \\
\leq\ & \eta\varepsilon+\eta K\left\{\left|\sum_{(i,j)\in\mathbb{N}^2} a_{k,l,i,j}^{(m,n)} T_{i,j}\left(f_0;x,y\right)-f_0\left(x,y\right)\right|\right. \\
&+\left|\sum_{(i,j)\in\mathbb{N}^2} a_{k,l,i,j}^{(m,n)} T_{i,j}\left(f_1;x,y\right)-f_1\left(x,y\right)\right| \\
&+\left|\sum_{(i,j)\in\mathbb{N}^2} a_{k,l,i,j}^{(m,n)} T_{i,j}\left(f_2;x,y\right)-f_2\left(x,y\right)\right| \\
&\left.+\left|\sum_{(i,j)\in\mathbb{N}^2} a_{k,l,i,j}^{(m,n)} T_{i,j}\left(f_3;x,y\right)-f_3\left(x,y\right)\right|\right\}.
\end{aligned}$$

Now we apply the modular ρ in the both-sides of the above inequality and since ρ is monotone, we get

$$\begin{aligned}
&\rho\left(\eta\left(A_{k,l,m,n}^{\mathbb{T}}\left(g\right)-g\right)\right)\leq\rho\left(\eta\varepsilon+\eta K\left(A_{k,l,m,n}^{\mathbb{T}}\left(f_0\right)-f_0\right)\right. \\
&\left.+\eta K\left(A_{k,l,m,n}^{\mathbb{T}}\left(f_1\right)-f_1\right)+\eta K\left(A_{k,l,m,n}^{\mathbb{T}}\left(f_2\right)-f_2\right)+\eta K\left(A_{k,l,m,n}^{\mathbb{T}}\left(f_3\right)-f_3\right)\right).
\end{aligned}$$

So, we may write that

$$\begin{aligned}
\rho\left(\eta\left(A_{k,l,m,n}^{\mathbb{T}}\left(g\right)-g\right)\right) &\leq \rho(5\eta\varepsilon)+\rho\left(5\eta K\left(A_{k,l,m,n}^{\mathbb{T}}\left(f_0\right)-f_0\right)\right) \\
&+\rho\left(5\eta K\left(A_{k,l,m,n}^{\mathbb{T}}\left(f_1\right)-f_1\right)\right) \\
&+\rho\left(5\eta K\left(A_{k,l,m,n}^{\mathbb{T}}\left(f_2\right)-f_2\right)\right) \\
&+\rho\left(5\eta K\left(A_{k,l,m,n}^{\mathbb{T}}\left(f_3\right)-f_3\right)\right).
\end{aligned}$$

Since ρ is $N-$quasi semiconvex and strongly finite, we have, assuming $0 < \varepsilon \leq 1$

$$\begin{aligned}\rho\left(\eta\left(A^{\mathbb{T}}_{k,l,m,n}(g)-g\right)\right) &\leq N\varepsilon\rho(5\eta N)+\rho\left(5\eta K\left(A^{\mathbb{T}}_{k,l,m,n}(f_0)-f_0\right)\right)\\ &\quad +\rho\left(5\eta K\left(A^{\mathbb{T}}_{k,l,m,n}(f_1)-f_1\right)\right)\\ &\quad +\rho\left(5\eta K\left(A^{\mathbb{T}}_{k,l,m,n}(f_2)-f_2\right)\right)\\ &\quad +\rho\left(5\eta K\left(A^{\mathbb{T}}_{k,l,m,n}(f_3)-f_3\right)\right).\end{aligned}$$

For a given $\varepsilon^* > 0$, choose an $\varepsilon \in (0,1]$ such that $N\varepsilon\rho(5\eta N) < \varepsilon^*$. Now we define the following sets:

$$\begin{aligned}G_\eta &:= \left\{(k,l) : \rho\left(\eta\left(A^{\mathbb{T}}_{k,l,m,n}(g)-g\right)\right) \geq \varepsilon^*\right\},\\ G_{\eta,r} &:= \left\{(k,l) : \rho\left(5\eta K\left(A^{\mathbb{T}}_{k,l,m,n}(f_r)-f_r\right)\right) \geq \frac{\varepsilon^* - N\varepsilon\rho(5\eta N)}{4}\right\},\end{aligned}$$

$r = 0,1,2,3$. Then, it is easy to see that $G_\eta \subseteq \bigcup_{r=0}^{3} G_{\eta,r}$. So, we can write that

$$\delta_2(G_\eta) \leq \sum_{r=0}^{3} \delta_2(G_{\eta,r}).$$

Using the hypothesis (2.2), we get

$$\delta_2(G_\eta) = 0,$$

which proves our claim (2.4). Obviously, (2.4) also holds for every $g \in C^\infty(I^2)$. Now let $f \in L^\rho(I^2)$ satisfying $f - g \in X_{\mathbb{T}}$ for every $g \in C^\infty(I^2)$. Since $\mu(I^2) < \infty$ and ρ is strongly finite and absolutely continuous, it can be seen that ρ is also absolutely finite on $X(I^2)$ (see [2]). So, it is known from [3, 14] that the space $C^\infty(I^2)$ is modularly dense in $L^\rho(I^2)$, i.e., there exists a sequence $\{g_{k,l}\} \subset C^\infty(I^2)$ such that

$$P - \lim_{k,l} \rho\left(3\lambda_0^*(g_{k,l} - f)\right) = 0 \quad \text{for some } \lambda_0^* > 0,$$

which means, for every $\varepsilon > 0$, there is a positive number $k_0 = k_0(\varepsilon)$ so that

$$\rho\left(3\lambda_0^*(g_{k,l} - f)\right) < \varepsilon \quad \text{for every } k,l \geq k_0. \tag{2.5}$$

In addition to that, because the operators $T_{i,j}$ are linear and positive, we can write that

$$\begin{aligned}&\lambda_0^*\left|\sum_{(i,j)\in\mathbb{N}^2} a^{(m,n)}_{k,l,i,j} T_{i,j}(f;x,y) - f(x,y)\right|\\ \leq\ &\lambda_0^*\left|\sum_{(i,j)\in\mathbb{N}^2} a^{(m,n)}_{k,l,i,j} T_{i,j}(f - g_{k_0,k_0};x,y)\right|\\ &+\lambda_0^*\left|\sum_{(i,j)\in\mathbb{N}^2} a^{(m,n)}_{k,l,i,j} T_{i,j}(g_{k_0,k_0};x,y) - g_{k_0,k_0}(x,y)\right|\\ &+\lambda_0^*\left|g_{k_0,k_0}(x,y) - f(x,y)\right|,\end{aligned}$$

holds for every $x, y \in I$ and $m, n \in \mathbb{N}$. Applying the modular ρ and moreover considering the monotonicity of ρ, we have

$$\begin{aligned}\rho\left(\lambda_0^*\left(A_{k,l,m,n}^{\mathbb{T}}(f)-f\right)\right) &\leq \rho\left(3\lambda_0^*\left(A_{k,l,m,n}^{\mathbb{T}}\left(f-g_{k_0,k_0}\right)\right)\right)\\ &+\rho\left(3\lambda_0^*\left(A_{k,l,m,n}^{\mathbb{T}}\left(g_{k_0,k_0}\right)-g_{k_0,k_0}\right)\right)\\ &+\rho\left(3\lambda_0^*\left(g_{k_0,k_0}-f\right)\right). \end{aligned} \tag{2.6}$$

Then, it follows from (2.5) and (2.6) that

$$\begin{aligned}\rho\left(\lambda_0^*\left(A_{k,l,m,n}^{\mathbb{T}}(f)-f\right)\right) &\leq \varepsilon+\rho\left(3\lambda_0^*\left(A_{k,l,m,n}^{\mathbb{T}}\left(f-g_{k_0,k_0}\right)\right)\right)\\ &+\rho\left(3\lambda_0^*\left(A_{k,l,m,n}^{\mathbb{T}}\left(g_{k_0,k_0}\right)-g_{k_0,k_0}\right)\right). \end{aligned} \tag{2.7}$$

So, taking statistical limit superior as $k, l \to \infty$ in the both-sides of (2.7) and also using the facts that $g_{k_0,k_0} \in C^{\infty}(I^2)$ and $f - g_{k_0,k_0} \in X_{\mathbb{T}}$, we obtain from (2.1) that

$$\begin{aligned}&st_2-\limsup_{k,l}\rho\left(\lambda_0^*\left(A_{k,l,m,n}^{\mathbb{T}}(f)-f\right)\right)\\ \leq\ &\varepsilon+R\rho\left(3\lambda_0^*(f-g_{k_0,k_0})\right)\\ &+st_2-\limsup_{k,l}\rho\left(3\lambda_0^*\left(A_{k,l,m,n}^{\mathbb{T}}\left(g_{k_0,k_0}\right)-g_{k_0,k_0}\right)\right),\end{aligned}$$

which gives

$$\begin{aligned}&st_2-\limsup_{k,l}\rho\left(\lambda_0^*\left(A_{k,l,m,n}^{\mathbb{T}}(f)-f\right)\right)\\ \leq\ &\varepsilon(R+1)+st_2-\limsup_{k,l}\rho\left(3\lambda_0^*\left(A_{k,l,m,n}^{\mathbb{T}}\left(g_{k_0,k_0}\right)-g_{k_0,k_0}\right)\right).\end{aligned} \tag{2.8}$$

By (2.4), since

$$st_2-\lim_{k,l}\rho\left(3\lambda_0^*\left(A_{k,l,m,n}^{\mathbb{T}}\left(g_{k_0,k_0}\right)-g_{k_0,k_0}\right)\right)=0,\ \text{uniformly in } m,n,$$

we get

$$st_2-\limsup_{k,l}\rho\left(3\lambda_0^*\left(A_{k,l,m,n}^{\mathbb{T}}\left(g_{k_0,k_0}\right)-g_{k_0,k_0}\right)\right)=0,\ \text{uniformly in } m,n. \tag{2.9}$$

From (2.8) and (2.9), we conclude that

$$st_2-\limsup_{k,l}\rho\left(\lambda_0^*\left(A_{k,l,m,n}^{\mathbb{T}}(f)-f\right)\right)\leq\varepsilon(R+1).$$

Since $\varepsilon > 0$ was arbitrary, we find

$$st_2-\limsup_{k,l}\rho\left(\lambda_0^*\left(A_{k,l,m,n}^{\mathbb{T}}(f)-f\right)\right)=0 \text{ uniformly in } m,n.$$

Furthermore, since $\rho\left(\lambda_0^*\left(A_{k,l,m,n}^{\mathbb{T}}(f)-f\right)\right)$ is non-negative for all $k, l, m, n \in \mathbb{N}$, we can easily see that

$$st_2-\lim_{k,l}\rho\left(\lambda_0^*\left(A_{k,l,m,n}^{\mathbb{T}}(f)-f\right)\right)=0,\ \text{uniformly in } m,n,$$

which completes the proof. □

If the modular ρ satisfies the Δ_2-condition, then one can get immediately the following result from Theorem 2.1.

Theorem 2.2. *Let $\mathcal{A} = \left\{A^{(m,n)}\right\}$ be a sequence of four dimensional infinite non-negative real matrices. Let ρ and $\mathbb{T} = \{T_{i,j}\}$ be the same as in Theorem 2.1. If ρ satisfies the Δ_2-condition, then the statements (a) and (b) are equivalent:*

(a) $st_2 - \lim_{k,l} \rho\left(\lambda\left(A^{\mathbb{T}}_{k,l,m,n}(f_r) - f_r\right)\right) = 0$ *uniformly in m, n, for every $\lambda > 0$ and $r = 0, 1, 2, 3,$*

(b) $st_2 - \lim_{k,l} \rho\left(\lambda\left(A^{\mathbb{T}}_{k,l,m,n}(f) - f\right)\right) = 0$ *uniformly in m, n, for every $\lambda > 0$ provided that f is any function belonging to $L^{\rho}(I^2)$ such that $f - g \in X_{\mathbb{T}}$ for every $g \in C^{\infty}\left(I^2\right)$.*

If one replaces the matrices $A^{(m,n)}$ by the identity matrix and taking $P-$limit, then the condition (2.1) reduces to

$$P - \limsup_{i,j} \rho\left(\lambda\left(T_{i,j}h\right)\right) \leq R\rho\left(\lambda h\right) \tag{2.10}$$

for every $h \in X_{\mathbb{T}}$, $\lambda > 0$ and for an absolute positive constant R. In this case, the next results which were obtained by Orhan and Demirci [20] immediately follows from our Theorems 2.1 and 2.2.

Corollary 2.3. ([20]) *Let ρ be a monotone, strongly finite, absolutely continuous and N-quasi semiconvex modular on $X\left(I^2\right)$. Let $\mathbb{T} := \{T_{i,j}\}$ be a sequence of positive linear operators from D into $X\left(I^2\right)$ satisfying (2.10). If $\{T_{i,j}f_r\}$ is strongly convergent to f_r for each $r = 0, 1, 2, 3,$ then $\{T_{i,j}f\}$ is modularly convergent to f provided that f is any function belonging to $L^{\rho}\left(I^2\right)$ such that $f - g \in X_{\mathbb{T}}$ for every $g \in C^{\infty}\left(I^2\right)$.*

Corollary 2.4. ([20]) *Let $\mathbb{T} = \{T_{i,j}\}$ and ρ be the same as in Corollary 2.3. If ρ satisfies the Δ_2-condition, then the following statements are equivalent:*

(a)$\{T_{i,j}f_r\}$ is strongly convergent to f_r for each $r = 0, 1, 2, 3,$

(b)$\{T_{i,j}f\}$ is strongly convergent to f provided that f is any function belonging to $L^{\rho}(I^2)$ such that $f - g \in X_{\mathbb{T}}$ for every $g \in C^{\infty}\left(I^2\right)$.

In the following, we construct an example of positive linear operators satisfying the conditions of Theorem 2.1.

Example 2.5. Take $I = [0, 1]$ and let $\varphi : [0, \infty) \to [0, \infty)$ be a continuous function for which the following conditions hold:

- φ is convex,
- $\varphi(0) = 0$, $\varphi(u) > 0$ for $u > 0$ and $\lim_{u \to \infty} \varphi(u) = \infty$.

Hence, let us consider the functional ρ^{φ} on $X(I^2)$ defined by

$$\rho^{\varphi}(f) := \int_0^1 \int_0^1 \varphi\left(|f(x, y)|\right) dxdy \quad \text{for } f \in X\left(I^2\right). \tag{2.11}$$

In this case, ρ^{φ} is a convex modular on $X\left(I^2\right)$, which satisfies all assumptions listed in Section 1 (see [4]). Let us consider the Orlicz space generated by φ as follows:

$$L^{\rho}_{\varphi}(I^2) := \left\{f \in X\left(I^2\right) : \rho^{\varphi}\left(\lambda f\right) < +\infty \quad \text{for some } \lambda > 0\right\}.$$

Then let us consider the following bivariate Bernstein-Kantorovich operator

$$\mathbb{U} := \{U_{i,j}\}$$

on the space $L_{\varphi}^{\rho}\left(I^2\right)$ which is defined by:

$$U_{i,j}(f;x,y) = \sum_{k=0}^{i}\sum_{l=0}^{j} p_{k,l}^{(i,j)}(x,y)(i+1)(j+1)\int_{k/(i+1)}^{(k+1)/(i+1)}\int_{l/(j+1)}^{(l+1)/(j+1)} f(t,s)\,dsdt \tag{2.12}$$

for $x,y \in I$, where $p_{k,l}^{(i,j)}(x,y)$ defined by

$$p_{k,l}^{(i,j)}(x,y) = \binom{i}{k}\binom{j}{l} x^k y^l (1-x)^{i-k}(1-y)^{j-l}.$$

Also, it is clear that,

$$\sum_{k=0}^{i}\sum_{l=0}^{j} p_{k,l}^{(i,j)}(x,y) = 1. \tag{2.13}$$

Observe that the operators $U_{i,j}$ map the Orlicz space $L_{\varphi}^{\rho}\left(I^2\right)$ into itself. Because of (2.13), as in the proof of [4] Lemma 5.1 and similar to Example 1[20], we obtain that for every $f \in L_{\varphi}^{\rho}\left(I^2\right)$ and $i,j \in \mathbb{N}$ there is an absolute constant $M > 0$ such that

$$\rho^{\varphi}(U_{i,j}f) \leq M\rho^{\varphi}(f).$$

Then, we know that, for any function $f \in L_{\varphi}^{\rho}\left(I^2\right)$ such that $f - g \in X_{\mathbf{U}}$ for every $g \in C^{\infty}\left(I^2\right)$, $\{U_{i,j}f\}$ is modularly convergent to f, with the choice of $X_{\mathbf{U}} := L_{\varphi}^{\rho}(I^2)$. Now define $\{s_{i,j}\}$ by

$$s_{i,j} = \begin{cases} 1, & \text{if } i,\ j \text{ are squares} \\ 0 & \text{otherwise.} \end{cases} \tag{2.14}$$

Now observe that, $st_2 - \lim_{i,j} s_{i,j} = 0$. Also, assume that

$$\mathcal{A} := \left\{A^{(m,n)}\right\} = \left\{a_{k,l,i,j}^{(m,n)}\right\}$$

is a sequence of four dimensional infinite matrices defined by

$$a_{k,l,i,j}^{(m,n)} = \frac{1}{kl} \text{ if } m \leq i \leq m+k-1,\ n \leq j \leq n+l-1,\ (m,n = 1,2,...)$$

and $a_{k,l,i,j}^{(m,n)} = 0$ otherwise. Then, using the operators $U_{i,j}$, we define the sequence of positive linear operators $\mathbb{V} := \{V_{i,j}\}$ *on* $L_{\varphi}^{\rho}(I^2)$ as follows:

$$V_{i,j}(f;x,y) = (1+s_{i,j})\,U_{i,j}(f;x,y) \quad \text{for } f \in L_{\varphi}^{\rho}(I^2),\ x,y \in [0,1] \text{ and } i,j \in \mathbb{N}. \tag{2.15}$$

As in the proof of Lemma 5.1 [4] and using the convexity of φ we get, for every $h \in X_{\mathbb{V}} := L_{\varphi}^{\rho}(I^2)$, $\lambda > 0$ and for positive constant M, that

$$st_2 - \limsup_{k,l} \rho^{\varphi}\left(\lambda\left(A_{k,l,m,n}^{\mathbb{V}}(h)\right)\right) \leq M\rho^{\varphi}(\lambda h),\ \text{uniformly in } m,n,$$

where

$$A^{\mathbb{V}}_{k,l,m,n}(h) = \sum_{(i,j)\in\mathbb{N}^2}^{\infty} a^{(m,n)}_{k,l,i,j} V_{i,j} h$$

as in (1.4). Therefore the condition (2.1) works for our operators $V_{i,j}$ given by (2.15) with the choice of $X_{\mathbb{V}} = X_{\mathbf{U}} = L^{\rho}_{\varphi}(I^2)$. We now claim that

$$st_2 - \lim_{k,l} \rho^{\varphi}\left(\lambda\left(A^{\mathbb{V}}_{k,l,m,n}(f_r) - f_r\right)\right) = 0, \text{ uniformly in } m,n;\ r = 0,1,2,3. \tag{2.16}$$

Observe that

$$U_{i,j}(f_0;x,y) = 1, \quad U_{i,j}(f_1;x,y) = \frac{ix}{i+1} + \frac{1}{2(i+1)},$$

$$U_{i,j}(f_2;x,y) = \frac{jy}{j+1} + \frac{1}{2(j+1)}$$

and

$$U_{i,j}(f_3;x,y) = \frac{i(i-1)x^2}{(i+1)^2} + \frac{2ix}{(i+1)^2} + \frac{1}{3(i+1)^2} + \frac{j(j-1)y^2}{(j+1)^2} + \frac{2jy}{(j+1)^2} + \frac{1}{3(j+1)^2}.$$

So, we can see,

$$\begin{aligned}
\rho^{\varphi}\left(\lambda\left(A^{\mathbb{V}}_{k,l,m,n}(f_0) - f_0\right)\right) &= \rho^{\varphi}\left(\lambda\left(\sum_{i=m}^{m+k-1}\sum_{j=n}^{n+l-1}\frac{1}{kl}(1+s_{i,j}) - 1\right)\right) \\
&= \int_0^1\int_0^1 \varphi\left(\left|\lambda\left(\sum_{i=m}^{m+k-1}\sum_{j=n}^{n+l-1}\frac{1}{kl}(1+s_{i,j}) - 1\right)\right|\right) dxdy \\
&= \varphi\left(\lambda\left(\sum_{i=m}^{m+k-1}\sum_{j=n}^{n+l-1}\frac{1}{kl}(1+s_{i,j}) - 1\right)\right),
\end{aligned}$$

because of

$$\frac{1}{kl}\sum_{i=m}^{m+k-1}\sum_{j=n}^{n+l-1}(1+s_{i,j}) = \begin{cases} 2, & \text{if } i,\ j \text{ are squares} \\ 1 & \text{otherwise.} \end{cases} \quad ,m,n = 1,2,...$$

and using continuity of φ, we get

$$st_2 - \lim_{k,l} \varphi\left(\lambda\left(\sum_{i=m}^{m+k-1}\sum_{j=n}^{n+l-1}\frac{1}{kl}(1+s_{i,j}) - 1\right)\right) = 0, \text{ uniformly in } m,n \tag{2.17}$$

and hence

$$st_2 - \lim_{k,l} \rho^{\varphi}\left(\lambda\left(A^{\mathbb{V}}_{k,l,m,n}(f_0) - f_0\right)\right) = 0, \quad \text{uniformly in } m,n,$$

which guarantees that (2.16) holds true for $r=0$. Also, since

$$\begin{aligned}
&\rho^{\varphi}\left(\lambda\left(A_{k,l,m,n}^{\mathbb{V}}\left(f_1\right)-f_1\right)\right)\\
= \; & \rho^{\varphi}\left(\lambda\left(\sum_{i=m}^{m+k-1}\sum_{j=n}^{n+l-1}\frac{1}{kl}(1+s_{i,j})\left(\frac{ix}{i+1}+\frac{1}{2(i+1)}\right)-x\right)\right)\\
\leq \; & \rho^{\varphi}\left(3\lambda\left(\frac{1}{kl}\sum_{i=m}^{m+k-1}\sum_{j=n}^{n+l-1}\frac{i}{i+1}-1\right)\right)+\rho^{\varphi}\left(3\lambda\left(\frac{1}{kl}\sum_{i=m}^{m+k-1}\sum_{j=n}^{n+l-1}\frac{1}{2(i+1)}\right)\right)\\
& +\rho^{\varphi}\left(3\lambda\left(\frac{1}{kl}\sum_{i=m}^{m+k-1}\sum_{j=n}^{n+l-1}s_{i,j}\left(\frac{i}{i+1}+\frac{1}{2(i+1)}\right)\right)\right)\\
= \; & \varphi\left(3\lambda\left(\frac{1}{kl}\sum_{i=m}^{m+k-1}\sum_{j=n}^{n+l-1}\frac{i}{i+1}-1\right)\right)+\varphi\left(3\lambda\left(\frac{1}{kl}\sum_{i=m}^{m+k-1}\sum_{j=n}^{n+l-1}\frac{1}{2(i+1)}\right)\right)\\
& +\varphi\left(3\lambda\left(\frac{1}{kl}\sum_{i=m}^{m+k-1}\sum_{j=n}^{n+l-1}s_{i,j}\left(\frac{i}{i+1}+\frac{1}{2(i+1)}\right)\right)\right)
\end{aligned}$$

Since

$$st_2-\lim_{k,l}\left(\sup_{m,n}\left(\frac{1}{kl}\sum_{i=m}^{m+k-1}\sum_{j=n}^{n+l-1}\frac{i}{i+1}-1\right)\right)=0,$$

$$st_2-\lim_{k,l}\left(\sup_{m,n}\frac{1}{kl}\sum_{i=m}^{m+k-1}\sum_{j=n}^{n+l-1}\frac{1}{2(i+1)}\right)=0$$

and

$$st_2-\lim_{k,l}\left(\sup_{m,n}\left(\frac{1}{kl}\sum_{i=m}^{m+k-1}\sum_{j=n}^{n+l-1}s_{i,j}\left(\frac{i}{i+1}+\frac{1}{2(i+1)}\right)\right)\right)=0$$

we have,

$$st_2-\lim_{k,l}\rho^{\varphi}\left(\lambda\left(A_{k,l,m,n}^{\mathbb{V}}\left(f_1\right)-f_1\right)\right)=0,\quad \text{uniformly in } m,n.$$

So (2.16) holds true for $r=1$. Similarly, we have

$$st_2-\lim_{k,l}\rho^{\varphi}\left(\lambda\left(A_{k,l,m,n}^{\mathbb{V}}\left(f_2\right)-f_2\right)\right)=0,\quad \text{uniformly in } m,n.$$

Finally, since

$$\begin{aligned}
&\rho^{\varphi}\left(\lambda\left(A_{k,l,m,n}^{\mathbb{V}}\left(f_3\right)-f_3\right)\right)\\
&=\rho^{\varphi}\left(\lambda\left(\sum_{i=m}^{m+k-1}\sum_{j=n}^{n+l-1}\frac{1}{kl}(1+s_{i,j})\left(\frac{i\left(i-1\right)x^2}{\left(i+1\right)^2}+\frac{2ix}{\left(i+1\right)^2}+\frac{1}{3\left(i+1\right)^2}\right.\right.\right.\\
&\left.\left.\left.+\frac{j\left(j-1\right)y^2}{\left(j+1\right)^2}+\frac{2jy}{\left(j+1\right)^2}+\frac{1}{3\left(j+1\right)^2}\right)-\left(x^2+y^2\right)\right)\right)
\end{aligned}$$

$$\leq \rho^{\varphi}\left(3\lambda\left(\frac{1}{kl}\sum_{i=m}^{m+k-1}\sum_{j=n}^{n+l-1}(1+s_{i,j})\frac{i\,(i-1)}{(i+1)^2}-1\right)\right)$$

$$+\rho^{\varphi}\left(3\lambda\left(\frac{1}{kl}\sum_{i=m}^{m+k-1}\sum_{j=n}^{n+l-1}(1+s_{i,j})\frac{j\,(j-1)}{(j+1)^2}-1\right)\right)$$

$$+\rho^{\varphi}\left(3\lambda\left(\frac{1}{kl}\sum_{i=m}^{m+k-1}\sum_{j=n}^{n+l-1}(1+s_{i,j})\left(\frac{2i}{(i+1)^2}+\frac{1}{3\,(i+1)^2}+\frac{2j}{(j+1)^2}+\frac{1}{3\,(j+1)^2}\right)\right)\right)$$

Hence we can easily see that

$$st_2-\lim_{k,l}\rho^{\varphi}\left(\lambda\left(A^{\mathbb{V}}_{k,l,m,n}(f_3)-f_3\right)\right)=0,\quad \text{uniformly in } m,n.$$

So, our claim (2.16) holds true for each $r=0,1,2,3$. $\{V_{i,j}\}$ satisfies all hypothesis of Theorem 2.1 and we immediately see that,

$$st_2-\lim_{k,l}\rho^{\varphi}\left(\lambda\left(A^{\mathbb{V}}_{k,l,m,n}(f)-f\right)\right)=0,\quad \text{uniformly in } m,n,$$

on $I^2=[0,1]\times[0,1]$ for all $f\in L^{\rho}_{\varphi}(I^2)$. Also, since $\{s_{i,j}\}$ does not converge modularly, $\{V_{i,j}\}$ does not satisfy Corollary 2.3.

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Quintic B-spline method for numerical solution of fourth order singular perturbation boundary value problems

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Abstract. In this communication, we have studied an efficient numerical approach based on uniform mesh for the numerical solutions of fourth order singular perturbation boundary value problems. Such type of problems arises in various fields of science and engineering, like electrical network and vibration problems with large Peclet numbers, Navier-Stokes flows with large Reynolds numbers in the theory of hydrodynamics stability, reaction-diffusion process, quantum mechanics and optimal control theory etc. In the present study, a quintic B-spline method has been discussed for the approximate solution of the fourth order singular perturbation boundary value problems. The convergence analysis is also carried out and the method is shown to have convergence of second order. The performance of present method is shown through some numerical tests. The numerical results are compared with other existing method available in the literature.

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1. Introduction

We consider the fourth order singular perturbation boundary value problem

$$-\varepsilon y^{iv}(t) - p(t)\, y'''(t) + q(t)\, y''(t) + r(t)\, y(t) = f(t), \quad t \in [a,\ b], \tag{1.1}$$

$$y(a) = \eta_1, \quad y(b) = \eta_2, \quad y''(a) = \eta_3, \quad y''(b) = \eta_4. \tag{1.2}$$

where $\eta_1,\ \eta_2,\ \eta_3$ and η_4 are finite real constants and ε is a small positive parameter, such that $0 < \varepsilon \ll 1$. Moreover, we assume that the functions $p(t)$,$q(t)$,$r(t)$ and $f(t)$ are sufficiently smooth. Further, the problem (1.1) is called non-turning point problem if $p(t) \geq \alpha > 0$ throughout the interval $[a,\ b]$, where α is some positive constant and boundary layer will be in the neighbourhood of $t = a$ [9]. In the same vein, if the $p(t)$ vanishes at $t = 0$, then it becomes a turning point problem. In that scenario, the boundary layer will be at both the end points $t = a$ and $t = b$ [2].

Singular perturbation problems are engendered by multiplication of a small positive parameter ε to highest derivative term of differential equation with boundary conditions. Many scholars have studied the analytical and numerical solutions of these problems, but sometimes they found that the classical numerical methods failed to get good approximate solutions of singular perturbation problems. That's why they have gone for the non classical methods. In the last few decades, many researchers have discussed the numerical solutions of singular perturbation problems. Most of the researchers have studied the numerical solutions of second order singular perturbation problems [5, 10, 11, 12, 13, 17, 19, 20, 21, 22, 29]. Only a few researchers have focused the numerical solutions of higher order singular perturbation problems [3, 24, 23, 28, 27]. Lodhi and Mishra [14, 15] have suggested the computational technique for numerical solutions of fourth order singular singularly perturbed and self adjoint boundary value problems. Raja and Tamilselvan [23] have designed a shooting method on a Shishkin mesh to solve reaction-diffusion type problems. Mishra and Saini [18] have used initial value technique for the numerical solution of fourth order singularly perturbed boundary value problems. Sarakhsi et al [25] have studied the existence of boundary layer problem. Parameter uniform numerical scheme to solve fourth order singularly perturbed turning points problems have been presented by Geetha and Tamilselvan [7]. Sharma et al. [26] have done the survey on singularly perturbed turning point and interior layers problem. Geetha et al. [8] have applied parameter uniform numerical method based on Shishkin mesh for third order singularly perturbed turning point problems exhibiting boundary layers.

This paper describes a quintic B-spline approach for the numerical solution of fourth order singular perturbation boundary value problems and it has been proved to be second order convergence. The paper is organized as follows: In section 2, we describe the quintic B-spline method. Convergence analysis is established in section 3. Quasilinearization method is discussed in section 4. Section 5 gives the numerical results which substantiate the theoretical aspects. Finally, we discuss the conclusions in section 6.

2. Quintic B-spline Method

We divide the interval $[a,\ b]$ into N equal subinterval and we choose piecewise uniform mesh points represented by $\pi = \{t_0, t_1, t_2, ..., t_N\}$,such that $t_0 = a$, $t_N = b$ and $h = \frac{b-a}{N}$ is the piecewise uniform spacing. We define $L_2[a,\ b]$ as a vector space of all the integrable functions on $[a,\ b]$, and X be the linear subspace of $L_2[a,\ b]$. Now we define

$$B_i(t) = \frac{1}{h^5}\begin{cases} (t - t_{i-3})^5, & \text{if } t \in [t_{i-3},\ t_{i-2}] \\ h^5 + 5h^4(t - t_{i-2}) + 10h^3(t - t_{i-2})^2 + 10h^2(t - t_{i-2})^3 & \\ +5h(t - t_{i-2})^4 - 5(t - t_{i-2})^5, & \text{if } t \in [t_{i-2},\ t_{i-1}] \\ 26h^5 + 50h^4(t - t_{i-1}) + 20h^3(t - t_{i-1})^2 - 20h^2(t - t_{i-1})^3 & \\ -20h(t - t_{i-1})^4 + 10(t - t_{i-1})^5, & \text{if } t \in [t_{i-1},\ t_i] \\ 26h^5 + 50h^4(t_{i+1} - t) + 20h^3(t_{i+1} - t)^2 - 20h^2(t_{i+1} - t)^3 & \\ -20h(t_{i+1} - x)^4 + 10(t_{i+1} - t)^5, & \text{if } t \in [t_i,\ t_{i+1}] \\ h^5 + 5h^4(t_{i+2} - t) + 10h^3(t_{i+2} - t)^2 + 10h^2(t_{i+2} - t)^3 & \\ +5h(t_{i+2} - t)^4 - 5(t_{i+2} - t)^5, & \text{if } t \in [t_{i+1},\ t_{i+2}] \\ (t_{i+3} - t)^5, & \text{if } t \in [t_{i+2},\ t_{i+3}] \\ 0 & \text{otherwise, for i} = 0, 1, 2, \ldots\text{N}. \end{cases} \tag{2.1}$$

We introduce six additional knots as $t_{-3} < t_{-2} < t_{-1} < t_0$ and $t_{N+3} > t_{N+2} > t_{N+1} > t_N$. From equation (2.1), we can easily check that each of the functions $B_i(t)$ is four times continuously differentiable on the entire real line. Also, the values of $B_i(t)$,$B_i'(t)$,$B_i''(t)$,$B_i'''(t)$ and $B_i^{iv}(t)$at the nodal points are given in Table 1.

Table 1. Quintic B-spline basis and its derivative function values at nodal points

$B(t)$	t_{i-3}	t_{i-2}	t_{i-1}	t	t_{i+1}	t_{i+2}	t_{i+3}
$B_i(t)$	0	1	26	66	26	1	0
$B_i'(t)$	0	$5/h$	$50/h$	0	$-50/h$	$-5/h$	0
$B_i''(t)$	0	$20/h^2$	$40/h^2$	$-120/h^2$	$40/h^2$	$20/h^2$	0
$B_i'''(t)$	0	$60/h^3$	$-120/h^3$	0	$120/h^3$	$-60/h^3$	0
$B_i^{iv}(t)$	0	$120/h^4$	$-480/h^4$	$720/h^4$	$-480/h^4$	$120/h^4$	0

Let $\Omega = \{B_{-2}, B_{-1}, B_0, B_1, \ldots\ldots, B_{N-1}, B_N, B_{N+1}, B_{N+2}\}$ and let $\phi_5(\pi) = \operatorname{span}\Omega$. The function Ω is linearly independent on $[a,\ b]$, thus $\phi_5(\pi)$ is $(N+5)$-dimensional. Even one can show that $\phi_5(\pi) \subseteq_{\text{subspace}} X$. Let L be a linear operator whose domain is X and whose range is also in X. Now we define

$$S(t) = \sum_{i=-2}^{N+2} c_i B_i(t), \tag{2.2}$$

be the approximate solution of the problem (1.1) with boundary conditions (1.2), where $c_i's$ is an unknown coefficient and $B_i(t)'s$ a fifth degree spline function. To solve fourth order singularly perturbed two point boundary value problems, the spline functions are evaluated at nodal points $t = t_i\ (i = 0, 1, 2, \ldots, N)$ which are needed for the solution.

From Table 1 and equation (2.2), we obtain the following relationships:

$$y(t_i) = S(t_i) = c_{i-2} + 26c_{i-1} + 66c_i + 26c_{i+1} + c_{i+2} \tag{2.3}$$

$$m(t_i) = S'(t_i) = \frac{1}{h}(-5c_{i-2} - 50c_{i-1} + 50c_{i+1} + 5c_{i+2}) \tag{2.4}$$

$$M_i = S''(t_i) = \frac{1}{h^2}(20c_{i-2} + 40c_{i-1} - 120c_i + 40c_{i+1} + 20c_{i+2}) \tag{2.5}$$

$$T_i = S'''(t_i) = \frac{1}{h^3}(-60c_{i-2} + 120c_{i-1} - 120c_{i+1} + 60c_{i+2}) \tag{2.6}$$

$$F_i = S^{iv}(t_i) = \frac{1}{h^4}(120c_{i-2} - 480c_{i-1} + 720c_i - 480c_{i+1} + 120c_{i+2}) \tag{2.7}$$

Moreover, m_i, M_i, T_i and F_i can be used to approximate values of $y'(t_i)$, $y''(t_i)$, $y'''(t_i)$ and $y^{iv}(t_i)$.
Since $S(t)$ is an approximate solution, it will satisfy equation (1.1) with boundary conditions (1.2). Hence we get

$$-\varepsilon S^{iv}(t) - p(t) S'''(t) + q(t) S''(t) + r(t) S(t) = f(t), \tag{2.8}$$

and

$$S(a) = \eta_1, \quad S(b) = \eta_2, \quad S''(a) = \eta_3, \quad S''(b) = \eta_4. \tag{2.9}$$

Discretizing equation (2.8) at the nodal points $t_i (i = 0, 1, \ldots, N)$, we have

$$-\varepsilon S^{iv}(t_i) - p(t_i) S'''(t_i) + q(t_i) S''(t_i) + r(t_i) S(t_i) = f(t_i),$$

Using equations (2.3)-(2.7) in above equation and simplifying, we obtain

$$\begin{array}{l} -\frac{\varepsilon}{h^4}\{120c_{i-2} - 480c_{i-1} + 720c_i - 480c_{i+1} + 120c_{i+2}\} \\ -\frac{p_i}{h^3}\{-60c_{i-2} + 120c_{i-1} - 120c_{i+1} + 60c_{i+2}\} \\ +\frac{q_i}{h^2}\{20c_{i-2} + 40c_{i-1} - 120c_i + 40c_{i+1} + 20c_{i+2}\} \\ +r_i\{c_{i-2} + 26c_{i-1} + 66c_i + 26c_{i+1} + c_{i+2}\} = f_i h^4, \end{array} \tag{2.10}$$

where $p_i = p(t_i)$, $q_i = q(t_i)$, $r_i = r(t_i)$ and $f_i = f(t_i)$. After simplifying above equation, we get

$$\gamma_1(t_i) c_{i-2} + \gamma_2(t_i) c_{i-1} + \gamma_3(t_i) c_i + \gamma_4(t_i) c_{i+1} + \gamma_5(t_i) c_{i+2} = f_i h^4, \tag{2.11}$$

where

$$\begin{array}{l} \gamma_1(t_i) = -120\varepsilon + 60p_i h + 20q_i h^2 + r_i h^4, \\ \gamma_2(t_i) = 480\varepsilon - 120p_i h + 40q_i h^2 + 26r_i h^4, \\ \gamma_3(t_i) = -720\varepsilon - 120q_i h^2 + 66r_i h^4, \\ \gamma_4(t_i) = 480\varepsilon + 120p_i h + 40q_i h^2 + 26r_i h^4, \\ \gamma_5(t_i) = -120\varepsilon - 60p_i h + 20q_i h^2 + r_i h^4, \quad \text{for} \quad i = 0, 1, \ldots, N. \end{array}$$

From the boundary conditions, we get the following equations

$$c_{-2} + 26c_{-1} + 66c_0 + 26c_1 + c_2 = \eta_1, \tag{2.12}$$

$$c_{N-2} + 26c_{N-1} + 66c_N + 26c_{N+1} + c_{N+2} = \eta_2, \tag{2.13}$$

$$20c_{-2} + 40c_{-1} - 120c_0 + 40c_1 + 20c_2 = \eta_3 h^2, \tag{2.14}$$

and

$$20c_{N-2} + 40c_{N-1} - 120c_N + 40c_{N+1} + 20c_{N+2} = \eta_4 h^2. \tag{2.15}$$

Coupling equations (2.11)-(2.15) lead to a system of $(N+5)$ linear equations $AY = D$ in the $(N+5)$ unknowns, where

$$Y = [c_{-2}, c_{-1}, c_0, c_1, \ldots, c_{N-1}, c_N, c_{N+1}, c_{N+2}]^T,$$

$$D = \left[\eta_1,\ \eta_3 h^2,\ f_0 h^4,\ f_1 h^4, \ldots,\ f_{N-1} h^4, f_N h^4,\ \eta_4 h^2,\ \eta_2\right]^T$$

and the coefficient matrix A is given by

$$A=\begin{bmatrix}
1 & 26 & 66 & 26 & 1 & 0 & 0 & 0 & \cdots & 0 & 0\\
20 & 40 & -120 & 40 & 20 & 0 & 0 & 0 & \cdots & 0 & 0\\
\gamma_1(t_0) & \gamma_2(t_0) & \gamma_3(t_0) & \gamma_4(t_0) & \gamma_5(t_0) & 0 & 0 & 0 & \cdots & 0 & 0\\
0 & \gamma_1(t_1) & \gamma_2(t_1) & \gamma_3(t_1) & \gamma_4(t_1) & \gamma_5(t_1) & 0 & 0 & \cdots & 0 & 0\\
0 & 0 & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\
0 & 0 & 0 & \gamma_1(t_i) & \gamma_2(t_i) & \gamma_3(t_i) & \gamma_4(t_i) & \gamma_5(t_i) & 0 & 0 & 0\\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & 0 & 0\\
0 & 0 & \cdots & 0 & 0 & \gamma_1(t_{N-1}) & \gamma_2(t_{N-1}) & \gamma_3(t_{N-1}) & \gamma_4(t_{N-1}) & \gamma_5(t_{N-1}) & 0\\
0 & 0 & \cdots & 0 & 0 & 0 & \gamma_1(t_N) & \gamma_2(t_N) & \gamma_3(t_N) & \gamma_4(t_N) & \gamma_5(t_N)\\
0 & 0 & \cdots & 0 & 0 & 0 & 20 & 40 & -120 & 40 & 20\\
0 & 0 & \cdots & 0 & 0 & 0 & 1 & 26 & 66 & 26 & 1
\end{bmatrix}. \tag{2.16}$$

Since A is a non-singular matrix, so we can solve the system $AY = D$ for c_{-2},c_{-1},c_0,c_1,c_2,...,c_{N-2},c_{N-1},c_N,c_{N+1},c_{N+2} substituting these values into equation (2.2), we get the required approximate solution.

3. Derivation for convergence

In this section, a technique is portrayed which will ascertain the truncation error for the quintic B-spline method over the whole range $a \le t \le b$. Here, we suppose that function $y(t)$ has continuous derivatives in the whole range.

We calculate the following relationships by comparing the coefficients of $c_i \quad (i = -2, -1, 0, 1, \ldots, N, N+1, N+2)$. From equations (2.3)-(2.7), we have

$$\begin{aligned}&S'(t_{i-2}) + 26S'(t_{i-1}) + 66S'(t_i) + 26S'(t_{i+1}) + S'(t_{i+2})\\ &\quad= \tfrac{1}{h}\left\{-5y(t_{i-2}) - 50y(t_{i-1}) + 50y(t_{i+1}) + 5y(i_{i+2})\right\}\end{aligned} \tag{3.1}$$

$$\begin{aligned}&S''(t_{i-2}) + 26S''(t_{i-1}) + 66S''(t_i) + 26S''(t_{i+1}) + S''(t_{i+2})\\ &\quad= \tfrac{1}{h^2}\left\{20y(t_{i-2}) + 40y(t_{i-1}) - 120y(t_i) + 40y(t_{i+1}) + 20y(t_{i+2})\right\}\end{aligned} \tag{3.2}$$

$$\begin{aligned}&S'''(t_{i-2}) + 26S'''(t_{i-1}) + 66S'''(t_i) + 26S'''(t_{i+1}) + S'''(t_{i+2})\\ &\quad= \tfrac{1}{h^3}\left\{-60y(t_{i-2}) + 120y(t_{i-1}) - 120y(t_{i+1}) + 60y(t_{i+2})\right\}\end{aligned} \tag{3.3}$$

$$\begin{aligned} &S^{iv}(t_{i-2}) + 26S^{iv}(t_{i-1}) + 66S^{iv}(t_i) + 26S^{iv}(t_{i+1}) + S^{iv}(t_{i+2}) \\ &= \tfrac{1}{h^4}\left\{120y(t_{i-2}) - 480y(t_{i-1}) + 720y(t_i) - 480y(t_{i+1}) + 120y(t_{i+2})\right\} \end{aligned} \tag{3.4}$$

Using the operator notation [6, 16], the equations (3.1)-(3.4) can we written as

$$S'(t_i) = \frac{1}{h}\left(\frac{-5E^{-2} - 50E^{-1} + 50E + 5E^2}{E^{-2} + 26E^{-1} + 66I + 26E + E^2}\right) y(t_i) \tag{3.5}$$

$$S''(t_i) = \frac{1}{h^2}\left(\frac{20E^{-2} + 40E^{-1} - 120I + 40E + 20E^2}{E^{-2} + 26E^{-1} + 66I + 26E + E^2}\right) y(t_i) \tag{3.6}$$

$$S'''(t_i) = \frac{1}{h^3}\left(\frac{-60E^{-2} + 120E^{-1} - 120E + 60E^2}{E^{-2} + 26E^{-1} + 66I + 26E + E^2}\right) y(t_i) \tag{3.7}$$

$$S^{iv}(t_i) = \frac{1}{h^4}\left(\frac{120E^{-2} - 480E^{-1} + 720I - 480E + 120E^2}{E^{-2} + 26E^{-1} + 66I + 26E + E^2}\right) y(t_i) \tag{3.8}$$

where the operators are defined as $Ey(t_i) = y(t_i + h)$, $Dy(t_i) = y'(t_i)$ and $Iy(t_i) = y(t_i)$. Let $E = e^{hD}$ and expand them in powers of hD, we get

$$S'(t_i) = y'(t_i) + \frac{1}{5040}h^6 y^7(t_i) - \frac{1}{21600}h^8 y^9(t_i) + \frac{1}{1036800}h^{10} y^{11}(t_i) + 0(h^{11}) \tag{3.9}$$

$$\begin{aligned} S''(t_i) &= y''(t_i) + \tfrac{1}{720}h^4 y^6(t_i) - \tfrac{1}{3360}h^6 y^8(t_i) + \tfrac{1}{86400}h^8 y^{10}(t_i) \\ &+ \tfrac{221}{239500800}h^{10} y^{12}(t_i) + 0(h^{11}) \end{aligned} \tag{3.10}$$

$$\begin{aligned} S'''(t_i) &= y'''(t_i) - \tfrac{1}{240}h^4 y^7(t_i) + \tfrac{11}{30240}h^6 y^9(t_i) - \tfrac{1}{28800}h^8 y^{11}(t_i) \\ &+ \tfrac{37}{11404800}h^{10} y^{13}(t_i) + 0(h^{11}) \end{aligned} \tag{3.11}$$

$$\begin{aligned} S^{iv}(t_i) &= y^{iv}(t_i) - \tfrac{1}{12}h^2 y^6(t_i) + \tfrac{1}{240}h^4 y^8(t_i) - \tfrac{1}{7560}h^6 y^{10}(t_i) \\ &- \tfrac{13}{907200}h^8 y^{12}(t_i) + \tfrac{643}{159667200}h^{10} y^{14}(t_i) + 0(h^{11}) \end{aligned} \tag{3.12}$$

We now define $e(t) = y(t) - S(t)$ and substitute equations (3.9)-(3.12) in the Taylor series expansion of $e(t_i + \theta h)$ we obtain

$$\begin{aligned} e(t_i + \theta h) &= \left(\tfrac{\theta^2}{1440} - \tfrac{5\theta^4}{1440}\right) h^6 y^6(t_i) + \left(\tfrac{\theta}{5040} - \tfrac{\theta^2}{1440}\right) h^7 y^7(t_i) \\ &+ \left(-\tfrac{\theta^2}{6720} + \tfrac{\theta^4}{5760}\right) h^8 y^8(t_i) + 0(h^9) \end{aligned} \tag{3.13}$$

where $a \le \theta \le b$. We abridge the above results in the following theorem:
Theorem 3.1. *Let $y(t)$ be the exact solution and $S(t)$ be the numerical solution of the singularly perturbed fourth order boundary value problem (1.1) with the boundary conditions (1.2) for sufficiently small h which further gives the truncation error of $O(h^6)$ and method of convergence of $O(h^2)$.*

4. Quasilinearization method

Let us consider the boundary value problem

$$-\varepsilon y^{iv}(t) = F(t, y, y'', y'''), \quad t = [a, b], \tag{4.1}$$

$$y(a) = \eta_1, \quad y(b) = \eta_2, \quad y''(a) = \eta_3, \quad y''(b) = \eta_4, \tag{4.2}$$

where $F(t, y, y'', y''')$ is a smooth function such that

$$\begin{cases} F_{y'''}(t, y, y'', y''') \geq \alpha > 0, \\ F_{y''}(t, y, y'', y''') \geq \beta > 0, \quad t \in [a, b], \\ 0 \geq F_y(t, y, y'', y''') \geq -\lambda, \quad \lambda > 0. \end{cases} \tag{4.3}$$

In order to obtain the numerical solution of the boundary value problem (4.1) and (4.2), Newton's method of quasilinearization [1, 4] is applied to generate the sequence of $\{y_k\}_0^\infty$of successive approximations with a proper selection of initial guess y_0. We define y_{k+1}, for each fixed non-negative integer k,to be solution of the following linear problem:

$$-\varepsilon y_{k+1}^{iv}(t) - p_k(t) y_{k+1}'''(t) + q_k(t) y_{k+1}''(t) + r_k(t) y_{k+1}(t) = f_k(t), \quad t \in [a, b], \tag{4.4}$$

$$y_{k+1}(a) = \eta_1, \quad y_{k+1}(b) = \eta_2, \quad y_{k+1}''(a) = \eta_3, \quad y_{k+1}''(b) = \eta_4, \tag{4.5}$$

where

$$\begin{aligned} p_k(t) &= F_{y'''}(t, y_k, y_k'', y_k'''), \ q_k(t) = F_{y''}(t, y_k, y_k'', y_k'''), \\ r_k(t) &= F_y(t, y_k, y_k'', y_k'''), \ f_k(t) = F_y(t, y_k, y_k'', y_k''') \\ &\quad - y_k F_y(t, y_k, y_k'', y_k''') - y_k'' F_{y''}(t, y_k, y_k'', y_k''') \\ &\quad - y_k''' F_{y'''}(t, y_k, y_k'', y_k'''). \end{aligned}$$

We make the following observations:

i) If the initial guess y_0 is sufficiently close to the solution $y(t)$ of (4.1) and (4.5), then the sequence $\{y_k\}_0^\infty$converges to $y(x)$. One can see the proof given in [4]. From (4.3), it follows that, for each fixed k,

$$\begin{aligned} &p_k(t) = F_{y'''}(t, y, y'', y''') \geq \alpha > 0, \\ &q_k(t) = F_{y''}(t, y, y'', y''') \geq \beta > 0, \\ &0 \geq r_k(t) = F_y(t, y, y'', y''') \geq -\lambda, \quad \lambda > 0. \end{aligned} \tag{4.6}$$

ii) Problem (4.4) with the boundary conditions (4.5), for each fixed k,is a linear fourth order boundary value problem which is in the form of (1.1) and (1.2). Hence it can be solved by the method described in section 2.

iii) The following convergence criterion is used to terminate the iteration:

$$\|y_{k+1}(t_i) - y_k(t_i)\| \leq \varepsilon, \quad t_i \in [a, b], \quad k \geq 0. \tag{4.7}$$

5. Numerical results

In the present section, we have presented numerical results of the considered examples with the help of MATLAB software which verifies theoretical estimates.
When the exact solutions of the considered examples are available then the maximum absolute errors E^N are evaluated using the following formula for the present method, which is given by

$$E^N = \max_{t_i \in [a, b]} \left| y_\varepsilon^N(t_i) - S_\varepsilon^N(t_i) \right|, \tag{5.1}$$

When the exact solutions of the considered examples are not available then the maximum absolute errors E_d^N are evaluated using the double mesh principle for the present method, which is given by

$$E_d^N = \max_{t_i \in [a,\, b]} \left| S_\varepsilon^N (t_i) - S_\varepsilon^{2N} (t_i) \right|, \tag{5.2}$$

The numerical order of convergence is computed using the following formula

$$\mathrm{Ord}^N = \frac{\ln E^N - \ln E^{2N}}{\ln 2}. \tag{5.3}$$

The exact and approximate solutions are denoted by y_ε^N and S_ε^N respectively.
Example 5.1 Consider the following singular perturbation boundary value problem [27]:

$$\begin{gathered} -\varepsilon y^{iv}(t) - 4y'''(t) = 1, \qquad t \in [0,\ 1], \\ y(0) = 1, \ \ y(1) = 1, \qquad y''(0) = -1, \qquad y''(1) = -1. \end{gathered}$$

The exact solution of Example 5.1 is given by

$$\begin{aligned} y(t) = \frac{1}{192\left(1 - e^{-\frac{4}{\varepsilon}}\right)} \Big\{ & -3\varepsilon^2 e^{-\frac{4t}{\varepsilon}} - 72t^2 - 8t^3 + 80t - 3t\varepsilon^2 + 192 + 3\varepsilon^2 \\ & + e^{-\frac{4}{\varepsilon}} \left(-192 + 96t^2 + 8t^3 - 104t + 3t\varepsilon^2 \right) \Big\}. \end{aligned}$$

Table 2. Maximum absolute errors and order of convergence of Example 5.1 for different values of ε and N.

N	$\varepsilon = 2^{-4}$	Ord	$\varepsilon = 2^{-5}$	Ord	$\varepsilon = 2^{-6}$	Ord	$\varepsilon = 2^{-7}$	Ord	$\varepsilon = 2^{-8}$	Ord
64	9.5153E-06	2.0442	1.1597E-05	2.2186	9.3841E-06	1.6594	7.6100E-06	1.6904	1.0217E-05	2.4174
128	2.3070E-06	2.0097	2.4916E-06	2.0418	2.9708E-06	2.2173	2.3579E-06	1.6494	1.9126E-06	1.6944
256	5.7289E-07	2.0010	6.0512E-07	2.0095	6.3885E-07	2.0421	7.5163E-07	2.2142	5.9098E-07	1.6435
512	1.4312E-07	2.1459	1.5029E-07	1.4799	1.5512E-07	2.0623	1.6198E-07	2.0885	1.8916E-07	2.2051
1024	3.2339E-08		5.3882E-08		3.7140E-08		3.8086E-08		4.1024E-08	

Example 5.2. Consider the following singular perturbation boundary value problem [7]:

$$\begin{gathered} -\varepsilon y^{iv}(t) + 5ty'''(t) + 4y''(t) + 2y(t) = 0, \qquad t \in [-1,\ 1], \\ y(-1) = 1, \ \ y(1) = 1, \qquad y''(-1) = 1, \qquad y''(1) = 1. \end{gathered}$$

Table 3. Comparison of maximum absolute error and order of convergence of Example 5.2 for different values of N and $\varepsilon = 2^{-4}$.

N	Geetha and Tamilselvan [7]		Present Method	
	E_d^N	Ord	E_d^N	Ord
64	3.9249E-2	0.9770	4.4948E-04	2.5794
128	1.9940E-2	0.9886	7.5204E-05	2.0864
256	1.0049E-2	0.9944	1.7708E-05	2.0177
512	5.0440E-3	0.9972	4.3731E-06	2.3071
1024	2.5269E-3		8.8366E-07	

Example 5.3. Consider the following singular perturbation boundary value problem [7]:

$$-\varepsilon y^{iv}(t)+5ty'''(t)+(4+t)\,y''(t)+\left(2+t^2\right)y(t)=-e^t+5,\quad t\in[-1,\ 1],$$

$$y(-1)=1,\quad y(1)=1,\qquad y''(-1)=2,\qquad y''(1)=2.$$

Table 4. Comparison of maximum absolute error and order of convergence of Example 5.3 for different values of N and $\varepsilon=2^{-4}$

N	Geetha and Tamilselvan [7]		Present Method	
	E_d^N	Ord	E_d^N	Ord
64	3.3778E-2	0.9823	4.1824E-04	2.5806
128	1.7097E-2	0.9913	6.9920E-05	2.0866
256	8.6002E-3	0.9957	1.6462E-05	2.0306
512	4.3130E-3	0.8693	4.0291E-06	2.1087
1024	2.3610E-3		9.3418E-07	

Example 5.4. Consider the following singular perturbation boundary value problem [7]:

$$-\varepsilon y^{iv}(t)+5ty'''(t)+(4+t)\,y''(t)+2y^2(t)=0,\qquad t\in[-1,\ 1],$$

$$y(-1)=1,\quad y(1)=1,\qquad y''(-1)=2,\qquad y''(1)=2.$$

Table 5. Comparison of maximum absolute error and order of convergence of Example 5.4 for different values of N and $\varepsilon=2^{-4}$

N	Geetha and Tamilselvan [7]		Present Method	
	E_d^N	Ord	E_d^N	Ord
64	7.5762E-02	0.9731	1.1620e-03	2.5795
128	3.8593E-02	0.9867	1.9440e-04	2.0863
256	1.9475E-02	0.9934	4.5777e-05	2.0215
512	9.7821E-03	0.9967	1.1275e-05	1.7449
1024	4.9021E-03	-	3.3639e-06	-

6. Conclusions

In this article, we have used the quintic B-spline method for finding the approximate solution of fourth order linear and non-linear singular perturbation boundary value problems. We linearised the non-linear boundary value problem via quasilinearization method and solved the problem. It is a computationally proficient technique and the algorithm can easily be applied on a computer. The results obtained through this method are better than the existing method [7] with the same number of nodal points.

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