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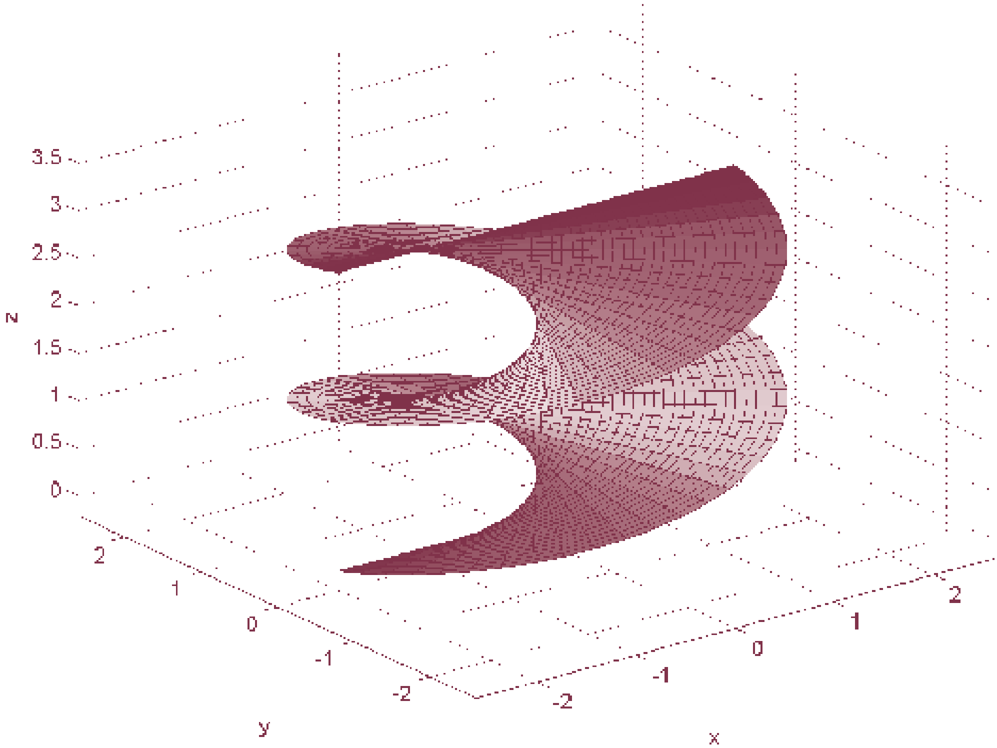

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Quadratic refinements of matrix means

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Abstract. The main target of this article is to present refinements of the matrix arithmetic-geometric mean inequality. The main difference between these refinements and the ones in the literature is the quadratic behavior of the refining terms. These refinements include the Löewner partial ordering, determinants, trace and unitarily invariant norms refinements.

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1. Introduction and motivation

Let $\mathbb{M}_n$ be the algebra of $n \times n$ complex matrices, $\mathbb{M}_n^+$ be the cone of positive semidefinite matrices in $\mathbb{M}_n$ and $\mathbb{M}_n^{++}$ be the cone of strictly positive matrices in $\mathbb{M}_n$. For two Hermitian matrices A and B, we write $A \geq B$ or $B \leq A$ to mean $A - B \in \mathbb{M}_n^+$, while we write $A > B$ or $B < A$ to mean $A - B \in \mathbb{M}_n^{++}$.

Comparison between Hermitian matrices is receiving a considerable attention these days, where the possible comparison between the means of these matrices is extensively considered.

In this article, we compare between matrices using the partial ordering $\leq$ defined above and using invariant norms. Recall that a norm $|||\ \ |||$ on $\mathbb{M}_n$ is called invariant, if $|||UAV||| = |||A|||$ for all $A \in \mathbb{M}_n$ and all unitary matrices U, V. Among the most useful invariant norms, is the Hilbert-Schmidt norm $\|\ \ \|_2$ defined as follows

$$\|A\|_2 = \left(\sum_{i,j=1}^{n} |a_{ij}|^2 \right)^{\frac{1}{2}}, \ A = [a_{ij}].$$

Notice that this is equivalent to $\|A\|_2 = \sqrt{\mathrm{tr}|A|^2}$, where $|A|^2 = A^*A$ and tr is the trace functional.

Another possible comparison between matrices is the comparison between their determinants, where the identity $\det A = \prod_{i=1}^{n} \lambda_i(A)$ becomes handy. In this context, $\{\lambda_i(A)\}$ refers to the set of eigenvalues of A.

Further, for $A, B \in \mathbb{M}_n^{++}$ and $0 \le t \le 1$, we define the weighted arithmetic and geometric means, respectively, as follows

$$A\nabla_t B = (1-t)A + tB \text{ and } A\#_t B = A^{\frac{1}{2}}\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^t A^{\frac{1}{2}}.$$

When $t = \frac{1}{2}$, it is customary to drop it from the notation. So we write $A\nabla B$ to denote $A\nabla_{\frac{1}{2}} B$, for example. Among the most well established comparisons between matrices is the following inequality known as the "arithmetic-geometric mean inequality"

$$A\#_t B \le A\nabla_t B, 0 \le t \le 1.$$

Then this means' inequality is refined and reversed in many ways. Before stating these refinements, we remind the reader that obtaining such matrix inequalities is done in different ways, but the most common technique is by a corresponding numerical inequality. For this, we need to look at the numerical forms of the above inequality. For the positive numbers a, b and $0 \le t \le 1$, we define the weighted means by $a\nabla_t b = (1-t)a + tb$ and $a\#_t b = a^{1-t}b^t$. The above matrix mean inequality can be simply proved using the known numerical inequality

$$a\#_t b \le a\nabla_t b, 0 \le t \le 1.$$

This inequality is well known by Young's inequality. We explain how to move from a numerical inequality to a matrix inequality in Theorem 2.10 below.

Since the matrix versions are obtained from numerical versions, refinements and reverses of numerical inequalities imply certain refinements and reverses of matrix inequalities. We mention here a few refinements known in the literature. In [5] it is proved that

$$a\#_t b + \min\{t, 1-t\}(\sqrt{a} - \sqrt{b})^2 \le a\nabla_t b \tag{1.1}$$

or simply

$$a\#_t b + L_1(t)f_1(a,b) \le a\nabla_t b,$$

for a piecewise linear function L_1 and some positive function $f_1(a,b)$. On the other hand, a two-term refinement has been proved in [16]

$$a\#_t b + L_1(t)f_1(a,b) + L_2(t)f_2(a,b) \le a\nabla_t b,$$

for another piecewise linear function L_2 and another positive function f_2. These refinements then were generalized in [13], [14] as

$$a\#_t b + \sum_{i=1}^{N} L_i(t)f_i(a,b) \le a\nabla_t b, N \in \mathbb{N}$$

for piecewise linear functions L_i and positive functions f_i.

Moreover, the reversed version

$$a\nabla_t b \le a\#_t b + \max\{t, 1-t\}(\sqrt{a} - \sqrt{b})^2, 0 \le t \le 1 \tag{1.2}$$

was proved in [6], and a generalization was presented recently in [13].

Further related results can be found in [1], [5], [12], [15], [16], [17].

What is common among these different refinements and reverses is the fact that all refining terms are linear in t.

In the recent paper [8], a quadratic refinement and reverse of Young's inequality were proved. Namely, it is shown that if $a, b > 0$ and $0 \le t \le 1$ are such that $(b-a)(2t-1) \ge 0$ then

$$a\#_t b + 2t(1-t)\left(\sqrt{a}-\sqrt{b}\right)^2 \le a\nabla_t b, \tag{1.3}$$

while we have the reversed inequality if $(b-a)(2t-1) \le 0$. Notice that the refining term in this inequality is $2t(1-t)\left(\sqrt{a}-\sqrt{b}\right)^2$ which is quadratic in t.

Our motivation of the current work begins with this observation. In fact, even (1.3) follows from a more general quadratic refinement. Our main target in this paper is to show that certain quadratic refinements and reverses can be shown for the arithmetic-geometric mean inequality, in both multiplicative and additive forms. Among many other matrix versions, we prove the following inequalities for $A, B \in \mathbb{M}_n^{++}$ and $X \in \mathbb{M}_n$ under some ordering condition,

$$\tau(1-\tau)\left(A\nabla_\nu B - A\#_\nu B\right) \le \nu(1-\nu)\left(A\nabla_\tau B - A\#_\tau B\right),$$

$$\det(A\#_\nu B)^{\frac{1}{n}} + \frac{\nu(1-\nu)}{\tau(1-\tau)}\det\left(A\nabla_\tau B - A\#_\tau B\right)^{\frac{1}{n}} \le \det(A\nabla_\nu B)^{\frac{1}{n}},$$

$$\mathrm{tr}|A^{1-\nu}B^\nu| + \frac{\nu(1-\nu)}{\tau(1-\tau)}\left(\mathrm{tr}(A\nabla_t B) - \mathrm{tr}A\#_\tau \mathrm{tr}B\right) \le \mathrm{tr}(A\nabla_\nu B),$$

and

$$\frac{\|(1-\nu)AX+\nu XB\|_2^2 - \|A^\nu XB^{1-\nu}\|_2^2}{\nu(1-\nu)} \le \frac{\|(1-\tau)AX+\tau XB\|_2^2 - \|A^\tau XB^{1-\tau}\|_2^2}{\tau(1-\tau)}.$$

A common aspect of all the refinements in this paper is the quadratic refining term $\nu(1-\nu)$ or $\tau(1-\tau)$.

We remark that in the recent paper [7], quadratic refinements of Heinz inequality have been shown.

The organization of this paper will be as follows. In the first part, we prove the needed numerical inequalities, and these will be done by some calculus computations. Then we present the matrix versions in the same order of the numerical ones to make it easier for the reader to follow.

2. Main results

Our main results section is divided into two parts. The first part will treat numerical versions needed to accomplish the matrix versions presented in the second part of the section.

2.1. The numerical versions

In the following computations, the reader must be careful about moving from one variable to another.

Lemma 2.1. *For $c>0$, let*

$$f(t)=\frac{1\nabla_t c-1\#_t c}{t(1-t)},\ 0<t<1.$$

Then

1. *f is increasing on $(0,1)$ if $c>1$.*
2. *f is decreasing on $(0,1)$ if $c<1$.*

Proof. Direct computations show that

$$f'(t)=\frac{g(c)}{t^2(1-t)^2},$$

where, considering t as a constant,

$$g(c)=-1+c^t(1-2t)+2t+(c-1)t^2+c^t(t-1)t\log c.$$

Then

$$g'(c)=t^2h(c),\ \text{where}\ h(c)=1-c^{t-1}(1+(1-t)\log c).$$

Furthermore,

$$h'(c)=(1-t)^2c^{t-2}\log c.$$

Now if $c>1$, then $h'(c)>0$ while $h'(c)<0$ when $c<1$. Therefore $h=h(c)$ attains its minimum at $c=1$. That is $h(c)\ge h(1)=0$. Consequently, $g'(c)>0$ and $g(c)$ is increasing on $(0,\infty)$.
If $c>1$ then $g(c)\ge g(1)=0$ and $f'(t)>0$. This proves the first statement.
On the other hand, if $c<1$, $g(c)\le g(1)=0$ and $f'(t)<0$. This proves the second statement. □

This entails the following quadratic refinement and reverse of Young's inequality.

Proposition 2.2. *Let $a,b>0$ and $0\le\nu,\tau\le 1$. If $(b-a)(\tau-\nu)\ge 0$, then*

$$\tau(1-\tau)(a\nabla_\nu b-a\#_\nu b)\le\nu(1-\nu)(a\nabla_\tau b-a\#_\tau b).$$

On the other hand, if $(b-a)(\tau-\nu)\le 0$ then

$$\tau(1-\tau)(a\nabla_\nu b-a\#_\nu b)\ge\nu(1-\nu)(a\nabla_\tau b-a\#_\tau b).$$

Proof. Letting $c=\frac{b}{a}$ in the function $f(t)=\frac{1\nabla_t c-1\#_t c}{t(1-t)}$ and using the monotonicity of Lemma 2.1 imply both inequalities. □

Letting $\nu=\frac{1}{2}$ in the above proposition implies the following [8].

Corollary 2.3. *Let $a,b>0$ and $0\le\nu\le 1$. If $(b-a)\left(\tau-\frac{1}{2}\right)\ge 0$ then*

$$a\#_\tau b+4\tau(1-\tau)(a\nabla b-a\#b)\le a\nabla_\tau b.$$

On the other hand, if $(b-a)\left(\tau-\frac{1}{2}\right)\le 0$ then

$$a\#_\tau b+4\tau(1-\tau)(a\nabla b-a\#b)\ge a\nabla_\tau b.$$

This explains the generality of these inequalities. Then squared versions that we can use to prove some Hilbert-Schmidt norm forms can be obtained as follows.

Proposition 2.4. *For $c > 0$, define $f : (0,1) \to [0,\infty)$ by*

$$f(t) = \frac{(1\nabla_t c)^2 - (1\#_t c)^2}{t(1-t)}.$$

1. *If $c < 1$, then f is decreasing on $(0,1)$ and*
2. *if $c > 1$, then f is increasing on $(0,1)$.*

Proof. Direct computations show that

$$f'(t) = \frac{g(c)}{t^2(1-t)^2},\ g(c) = -1 + 2t - t^2 + c^2t^2 + c^{2t}(1 - 2t + 2(-1+t)t\log c).$$

Further,

$$g'(c) = \frac{2t^2}{c}h(t),\ h(t) = c^2 + c^{2t}(-1 + 2(-1+t)\log c)$$

and

$$h'(t) = 4c^{2t}(t-1)\log^2 c.$$

Clearly $h'(t) < 0$, hence $h(t) \geq h(1) = 0$ and $g'(c) \geq 0$. If $c < 1$, then $g(c) \leq g(1) = 0$ and f is decreasing. On the other hand, if $c > 1$, $g(c) \geq g(1) = 0$ and f is increasing. □

Corollary 2.5. *Let $a, b > 0$ and $0 < \nu, \tau < 1$. If $(b-a)(\tau-\nu) \geq 0$ then*

$$\frac{(a\nabla_\nu b)^2 - (a\#_\nu b)^2}{\nu(1-\nu)} \leq \frac{(a\nabla_\tau b)^2 - (a\#_\tau b)^2}{\tau(1-\tau)}.$$

The inequality is reversed if $(b-a)(\tau-\nu) \leq 0$.

Again, letting $\tau = \frac{1}{2}$, we obtain the corresponding inequality from [8].

The above two refinements are "additive" versions, where the refining term is added to one side of the original inequality. Our next result presents a multiplicative form of these inequalities.

Lemma 2.6. *For $c > 0$, define $f : (0,1) \to [0,\infty)$ by*

$$f(t) = \left(\frac{1\nabla_t c}{1\#_t c}\right)^{\frac{1}{t(1-t)}}.$$

Then

1. *f is increasing on $(0,1)$ if $c < 1$ and*
2. *f is decreasing on $(0,1)$ if $c > 1$.*

Proof. Let $F(t) = \log f(t)$. That is

$$F(t) = \frac{\log(1-t+tc) - t\log c}{t(1-t)}.$$

Then

$$F'(t) = \frac{g(c)}{(1-t)^2t^2(1-t+tc)}$$

where

$$g(c)=t(c-1+t-tc+t(t-1-tc)\log c)+(2t-1)(1-t+tc)\log(1-t+tc).$$

Now

$$g'(c)=-\frac{t}{c}h(c) \text{ for } h(c)=(c-1)(t-1)t+ct^2\log c+c(1-2t)\log(1-t+tc).$$

Furthermore,

$$h'(c)=t^2\log c+(2t-1)\left[\frac{(c-1)(t-1)t}{1-t+tc}-\log(1-t+tc)\right]$$

and

$$h''(c)=\frac{(1-t)^2t(2c+(c-1)^2t)}{c(1-t+tc)^2}.$$

Now clearly $h''(c)\geq 0$, hence h' is increasing in c.

If $c<1$, then $h'(c)\leq h'(1)=0$ and h is decreasing when $c\leq 1$. That is, $h(c)\geq h(1)=0$ and $g'(c)\leq 0$ when $c\leq 1$. Thus, g is decreasing when $c\leq 1$, and hence $g(c)\geq g(1)=0$. Consequently, $F'(t)\geq 0$ and F is increasing in t, when $c\leq 1$. This proves the first assertion. When $c>1$, a similar argument implies that F is decreasing in t. □

As a consequence, we obtain the following multiplicative refinement and reverse of Young's inequality.

Corollary 2.7. *Let $a,b>0$ and $0<\nu,\tau<1$. If $(b-a)(\tau-\nu)>0$ then*

$$a\#_\tau b\left(\frac{a\nabla_\nu b}{a\#_\nu b}\right)^{\frac{\tau(1-\tau)}{\nu(1-\nu)}}\geq a\nabla_\tau b.$$

On the other hand, if $(b-a)(\tau-\nu)<0$ then

$$a\#_\tau b\left(\frac{a\nabla_\nu b}{a\#_\nu b}\right)^{\frac{\tau(1-\tau)}{\nu(1-\nu)}}\leq a\nabla_\tau b.$$

Proof. Let $c=\frac{b}{a}$ in

$$f(t)=\left(\frac{1\nabla_t c}{1\#_t c}\right)^{\frac{1}{t(1-t)}}.$$

If $b<a$, then f is increasing, by Lemma 2.6. Therefore, when $\nu<\tau$ we have $f(\nu)\leq f(\tau)$. This completes the proof of the first inequality. A similar argument implies the second inequality. □

Remark 2.8. Having introduced our numerical quadratic refinements and reverses, we compare these results with the linear inequalities. We have seen that, for $a,b>0$ and $0\leq t\leq 1$, one has

$$r(t)(\sqrt{a}-\sqrt{b})^2\leq a\nabla_t b-a\#_t b\leq R(t)(\sqrt{a}-\sqrt{b})^2,$$

where $r(t)=\min\{t,1-t\}$ and $R(t)=\max\{t,1-t\}$. On the other hand, under certain ordering conditions, we have the quadratic refinement or reverse

$$a\nabla_t b-a\#_t b\leq(\geq)2t(1-t)(\sqrt{a}-\sqrt{b})^2.$$

It is natural to ask about the advantage of introducing a quadratic refinement or reverse over the linear ones.
Direct calculations show that, for $0 \le t \le 1$, one has $r(t) \le 2t(1-t)$ and $R(t) \ge 2t(1-t)$. Therefore, when $(b-a)(2t-1) \ge 0$, we have

$$a\#_t b + r(t)(\sqrt{b}-\sqrt{a})^2 \le a\#_t b + 2t(1-t)(\sqrt{b}-\sqrt{a})^2 \le a\nabla_t b,$$

which is a refinement of the refinement (1.1). On the other hand, if $(b-a)(2t-1) \le 0$, we have the

$$a\#_t b + R(t)(\sqrt{b}-\sqrt{a})^2 \ge a\#_t b + 2t(1-t)(\sqrt{b}-\sqrt{a})^2 \ge a\nabla_t b,$$

which is a refinement of the reversed version (1.2). Therefore, introducing quadratic refinements serves as introducing one-term refinements of the already existing linear refinements.
A similar argument applies for the multiplicative versions.

We conclude this section by the following observation. The inequalities in Proposition 2.2 and Corollary 2.5 give rise to the following quotients

$$\frac{a\nabla_\nu b - a\#_\nu b}{a\nabla_\tau b - a\#_\tau b} \text{ and } \frac{(a\nabla_\nu b)^2 - (a\#_\nu b)^2}{(a\nabla_\tau b)^2 - (a\#_\tau b)^2}.$$

It is natural to ask about the relation between these quantities. Denoting these quotients by $A_{\nu,\tau}(a,b)$ and $A^{(2)}_{\nu,\tau}(a,b)$, respectively, we have the following comparison.

Proposition 2.9. *Let $a, b > 0$ and $0 \le \nu, \tau \le 1$. If $(b-a)(\tau-\nu) \ge 0$, then*

$$A^{(2)}_{\nu,\tau}(a,b) \le A_{\nu,\tau}(a,b).$$

On the other hand, if $(b-a)(\tau-\nu) \le 0$, then

$$A^{(2)}_{\nu,\tau}(a,b) \ge A_{\nu,\tau}(a,b).$$

Proof. Let $f(t) = a\nabla_t b + a\#_t b$. Then, clearly, f is increasing when $b > a$ and is decreasing if $b < a$. Now, if $(b-a)(\tau-\nu) \ge 0$, then $f(\tau) \ge f(\nu)$, whether $b > a$ or $b < a$. Simplifying the inequality $f(\nu) \le f(\tau)$ implies the inequality $A^{(2)}_{\nu,\tau}(a,b) \le A_{\nu,\tau}(a,b)$, when $(b-a)(\tau-\nu) \ge 0$. A similar argument implies the reversed inequality when $(b-a)(\tau-\nu) \le 0$. □

2.2. Matrix versions

Now we present the matrix versions one can obtain from the numerical versions proved above.

Theorem 2.10. *Let $A, B \in \mathbb{M}_n^{++}$ and $0 \le \nu, \tau \le 1$. If $(\tau-\nu)(B-A) \ge 0$ then*

$$\tau(1-\tau)\left(A\nabla_\nu B - A\#_\nu B\right) \le \nu(1-\nu)\left(A\nabla_\tau B - A\#_\tau B\right).$$

The inequality is reversed if $(\tau-\nu)(B-A) \le 0$.

Proof. If $(\tau-\nu)(B-A) \ge 0$, let $X = A^{-\frac{1}{2}}BA^{-\frac{1}{2}}$. Notice that if $\tau > \nu$ then $B \ge A$ and $\lambda_i(X) \ge 1, \forall i$. That is, $(\tau-\nu)(\lambda_i(X)-1) \ge 0$. A similar conclusion is achieved if $\tau<\nu$. Now since $(\tau-\nu)(\lambda_i(X)-1)\geq 0$, we may apply the first inequality of Proposition 2.2, using $a=1$ and $b=\lambda_i(X)$. This implies

$$\tau(1-\tau)\left(1\nabla_\nu\lambda_i(X)-1\#_\nu\lambda_i(X)\right)\leq\nu(1-\nu)\left(1\nabla_\tau\lambda_i(X)-1\#_\tau\lambda_i(X)\right),$$

which implies

$$\begin{aligned}&\tau(1-\tau)\left(I_n\nabla_\nu\text{diag}(\lambda_i(X))-I_n\#_\nu\text{diag}(\lambda_i(X))\right)\\&\qquad\leq\nu(1-\nu)\left(I_n\nabla_\tau\text{diag}(\lambda_i(X))-I_n\#_\tau\text{diag}(\lambda_i(X))\right).\end{aligned}$$

Now since X is Hermitian, $X=U\text{diag}(\lambda_i(X))U^*$ for some unitary matrix U. Conjugating the above inequality by U and noticing that conjugation is order preserving, we get

$$\tau(1-\tau)\left(I_n\nabla_\nu X-I_n\#_\nu X\right)\leq\nu(1-\nu)\left(I_n\nabla_\tau X-I_n\#_\tau X\right).$$

Now conjugating this inequality with $A^{\frac{1}{2}}$ implies the first desired inequality. The second inequality follows similarly. □

On the other hand, a determinant version may be obtained as follows. First, we recall Minkowski inequality, [3], p. 560,

$$\left(\prod_{i=1}^n a_i\right)^{\frac{1}{n}}+\left(\prod_{i=1}^n b_i\right)^{\frac{1}{n}}\leq\left(\prod_{i=1}^n(a_i+b_i)\right)^{\frac{1}{n}},\tag{2.1}$$

for the positive numbers $\{a_i,b_i:1\leq i\leq n\}$.

Theorem 2.11. *Let $A,B\in\mathbb{M}_n^{++}$ and $0<\nu,\tau<1$. If $(\tau-\nu)(B-A)\leq 0$ then*

$$\det(A\#_\nu B)^{\frac{1}{n}}+\frac{\nu(1-\nu)}{\tau(1-\tau)}\det\left(A\nabla_\tau B-A\#_\tau B\right)^{\frac{1}{n}}\leq\det(A\nabla_\nu B)^{\frac{1}{n}}.\tag{2.2}$$

Proof. Let $X=A^{-\frac{1}{2}}BA^{-\frac{1}{2}}$. If $(\tau-\nu)(B-A)\leq 0$ then $(\tau-\nu)(\lambda_i(X)-1)\leq 0$, which justifies the application of the second inequality of Proposition 2.2 in the following computations. Now

$$\begin{aligned}\det\left(I_n\nabla_\nu X\right)^{\frac{1}{n}}&=\prod_{i=1}^n\lambda_i\left((1-\nu)I_n+\nu X\right)^{\frac{1}{n}}\\&=\prod_{i=1}^n\left(1-\nu+\nu\lambda_i(X)\right)^{\frac{1}{n}}\ \text{(now apply Proposition 2.2 then (2.1))}\\&\geq\prod_{i=1}^n\left(\frac{\nu(1-\nu)}{\tau(1-\tau)}\left(1\nabla_\tau\lambda_i(X)-1\#_\tau\lambda_i(X)\right)+1\#_\nu\lambda_i(X)\right)^{\frac{1}{n}}\\&\geq\prod_{i=1}^n\left(\frac{\nu(1-\nu)}{\tau(1-\tau)}\left(1\nabla_\tau\lambda_i(X)-1\#_\tau\lambda_i(X)\right)\right)^{\frac{1}{n}}+\prod_{i=1}^n\left(1\#_\nu\lambda_i(X)\right)^{\frac{1}{n}}\\&=\frac{\nu(1-\nu)}{\tau(1-\tau)}\prod_{i=1}^n\lambda_i^{\frac{1}{n}}\left(I_n\nabla_\tau X-I_n\#_\tau X\right)+\prod_{i=1}^n\lambda_i^{\frac{1}{n}}\left(I_n\#_\nu X\right).\end{aligned}$$

Now multiplying both sides with $\det A^{\frac{1}{2}}$ and using basic properties of the determinant imply the desired inequality. $\square$

Notice that if we set $\tau = \frac{1}{2}$ in (2.2), we get

$$\det(A\#_\nu B)^{\frac{1}{n}} + 4\nu(1-\nu)\det\left(A\nabla B - A\#B\right)^{\frac{1}{n}} \le \det(A\nabla_\nu B)^{\frac{1}{n}},$$

when $(1-2\nu)(B-A) \le 0$. Raising both sides to the power n implies

$$\begin{aligned}\det(A^{1-\nu}B^\nu) &+ 4^n\nu^n(1-\nu)^n\det\left(A\nabla B - A\#B\right)\\ &\le \left(\det(A\#_\nu B)^{\frac{1}{n}} + 4\nu(1-\nu)\det\left(A\nabla B - A\#B\right)^{\frac{1}{n}}\right)^n\\ &\le \det((1-\nu)A + \nu B).\end{aligned}$$

In [8], it is proved that

$$\det(A^{1-\nu}B^\nu) + 4^n\nu^n(1-\nu)^n\det\left(A\nabla B - A\#B\right) \le \det((1-\nu)A + \nu B).$$

Therefore, (2.2) provides a refinement and a generalization of the corresponding result in this reference. On the other hand, Proposition 2.5 maybe used to obtain squared determinant versions as follows.

Proposition 2.12. *Let $A, B \in \mathbb{M}_n^{++}$ and let $0 < \nu, \tau < 1$. If $(\tau - \nu)(B - A) \ge 0$, then*

$$\det(A\#_\nu B)^{\frac{2}{n}} + \frac{\nu(1-\nu)}{\tau(1-\tau)}\det\left(A\nabla_\tau B - A\#_\tau B\right)^{\frac{2}{n}} \le \det(A\nabla_\nu B)^{\frac{2}{n}}.$$

Proof. Following the same notations of Theorem 2.11, we have

$$\begin{aligned}\det(I_n\nabla_\nu X)^{\frac{2}{n}} &= \left(\prod_{i=1}^n (1\nabla_\nu \lambda_i(X))^2\right)^{\frac{1}{n}} \quad \text{(apply Proposition 2.5 then (2.1))}\\ &\ge \prod_{i=1}^n \left((1\#_\nu\lambda_i(X))^2 + \frac{\nu(1-\nu)}{\tau(1-\tau)}\left[(1\nabla_\tau\lambda_i(X))^2 - (1\#_\tau\lambda_i(X))^2\right]\right)^{\frac{1}{n}}\\ &\ge \prod_{i=1}^n \left((1\#_\nu\lambda_i(X))^2\right)^{\frac{1}{n}} + \prod_{i=1}^n\left(\frac{\nu(1-\nu)}{\tau(1-\tau)}\left[(1\nabla_\tau\lambda_i(X))^2 - (1\#_\tau\lambda_i(X))^2\right]\right)^{\frac{1}{n}}\\ &\ge \prod_{i=1}^n \left(\lambda_i(I_n\#_\nu X)\right)^{\frac{2}{n}} + \frac{\nu(1-\nu)}{\tau(1-\tau)}\prod_{i=1}^n\left(1\nabla_\tau\lambda_i(X) - 1\#_\tau\lambda_i(X)\right)^{\frac{2}{n}}\\ &= \det(I_n\#_\nu X)^{\frac{2}{n}} + \frac{\nu(1-\nu)}{\tau(1-\tau)}\left[\det(I_n\nabla_\tau X) - \det(I_n\#_\tau X)\right]^{\frac{2}{n}},\end{aligned}$$

where we have used the simple inequality $(a^2 - b^2) \ge (a-b)^2$ when $a > b > 0$ to obtain the last inequality in the above proof. Now multiplying the last inequality with $\det A$ implies the desired inequality. $\square$

In the following result, we use the well known inequality [4]

$$\mathrm{tr}|A^{1-\nu}B^\nu| \le (\mathrm{tr}A)^{1-\nu}(\mathrm{tr}B)^\nu, A, B \in \mathbb{M}_n^+. \tag{2.3}$$

Proposition 2.13. *Let $A, B \in \mathbb{M}_n^{++}$ and let $0 < \nu, \tau < 1$. If $(\tau - \nu)(trB - trA) \le 0$, then*

$$tr|A^{1-\nu}B^{\nu}| + \frac{\nu(1-\nu)}{\tau(1-\tau)}\left(tr(A\nabla_t B) - trA\#_\tau trB\right) \le tr(A\nabla_\nu B).$$

Proof. Under the condition $(\tau - \nu)(\mathrm{tr}B - \mathrm{tr}A) \le 0$, we have

$$\begin{aligned}\mathrm{tr}(A\nabla_\nu B) &= \mathrm{tr}A\nabla_\nu \mathrm{tr}B \\ &\ge \frac{\nu(1-\nu)}{\tau(1-\tau)}\left(\mathrm{tr}A\nabla_\tau \mathrm{tr}B - \mathrm{tr}A\#_\tau \mathrm{tr}B\right) + \mathrm{tr}A\#_\nu \mathrm{tr}B \\ &\ge \mathrm{tr}|A^{1-\nu}B^{\nu}| + \frac{\nu(1-\nu)}{\tau(1-\tau)}\left(\mathrm{tr}(A\nabla_t B) - \mathrm{tr}A\#_\tau \mathrm{tr}B\right),\end{aligned}$$

where we have used (2.3) to obtain the last inequality and used Proposition 2.2 to obtain the first inequality. □

On the other hand, the squared version in Proposition 2.5 entails the following Hilbert-Schmidt norm inequality. For the next result, $\{\lambda_i\}$ will denote the eigenvalues of A arranged in a decreasing order and $\{\mu_j\}$ will denote the eigenvalues of B arranged in a decreasing order too. Moreover, the notation $X \circ Y$ will mean the Schur product of X and Y.

Theorem 2.14. *Let $A, B \in \mathbb{M}_n^{+}$ and $X \in \mathbb{M}_n$. If $\tau > \nu$ and $B \ge \lambda_1 I_n$, or if $\tau < \nu$ and $B \le \lambda_n I_n$ then*

$$\frac{\|(1-\nu)AX + \nu XB\|_2^2 - \|A^{\nu}XB^{1-\nu}\|_2^2}{\nu(1-\nu)} \le \frac{\|(1-\tau)AX + \tau XB\|_2^2 - \|A^{\tau}XB^{1-\tau}\|_2^2}{\tau(1-\tau)}.$$

The inequality is reversed if $\tau > \nu$ and $B \le \lambda_n I_n$ or if $\tau < \nu$ and $B \ge \lambda_1 I_n$.

Proof. Since $A, B \in \mathbb{M}_n^{+}$, there exist unitary matrices U, V and nonnegative numbers λ_i, μ_j such that

$$A = U\mathrm{diag}(\lambda_i)U^* \text{ and } B = V\mathrm{diag}(\mu_j)V^*.$$

Letting $Y = U^*XV$, we have

$$(1-\nu)AX + \nu XB = U\left([(1-\nu)\lambda_i + \nu\mu_j] \circ Y\right)V^*.$$

Notice that the condition $B \ge \lambda_1 I_n$ implies $\mu_j \ge \lambda_i, \forall i, j$ and the condition $B \le \lambda_n I_n$ implies $\mu_j \le \lambda_i, \forall i, j$. Therefore, the conditions $\tau > \nu$ and $B \ge \lambda_1 I_n$, or if $\tau < \nu$ and $B \le \lambda_n I_n$ imply $(\tau - \nu)(\mu_j - \lambda_i) \ge 0, \forall i, j$. Under these conditions, and noting that $\|\ \|_2$ is unitarily invariant, we have

$$\begin{aligned}&\|(1-\nu)AX + \nu XB\|_2^2 \\ &= \sum_{i,j}\left\{(\lambda_i\nabla_\nu\mu_j)^2|y_{ij}|^2\right\} \text{ (now apply Corollary 2.5)} \\ &\le \sum_{i,j}\left\{(\lambda_i\#_\nu\mu_j)^2 + \frac{\nu(1-\nu)}{\tau(1-\tau)}\left((\lambda_i\nabla_\tau\mu_j)^2 - (\lambda_i\#_\tau\mu_j)^2\right)\right\}|y_{ij}|^2 \\ &= \|A^{1-\nu}XB^{\nu}\|_2^2 + \frac{\nu(1-\nu)}{\tau(1-\tau)}\left(\|(1-\tau)AX + \tau XB\|_2^2 - \|A^{1-\tau}XB^{\tau}\|_2^2\right),\end{aligned}$$

which completes the proof for the first set of conditions. A similar argument implies the reversed inequality for the other conditions. □

Notice that the above inequalities provide a refinement and a reverse of the inequality $\|A^{1-\tau}XB^{\tau}\|_2 \leq \|(1-\tau)AX + \tau XB\|_2$.

Next, we present a matrix version of Corollary 2.7. For this result, we adopt the notation of Theorem 2.14.

Theorem 2.15. *Let $A, B \in \mathbb{M}_n^{++}, X \in \mathbb{M}_n, 0 < \nu, \tau < 1$ and let m, M be two positive numbers such that $mI_n \leq A, B \leq MI_n$. If $\tau > \nu$ and $B \geq \lambda_1 I_n$, or if $\tau < \nu$ and $B \leq \lambda_n I_n$ then*

$$\|A^{1-\tau}XB^{\tau}\|_2^2 \geq \left(\frac{m}{M}\right)^{\frac{2\tau(1-\tau)}{\nu(1-\nu)}} \|(1-\tau)AX + \tau XB\|_2^2.$$

Proof. Adopting the notation of Theorem 2.14, notice that the condition $mI_n \leq A$, $B \leq MI_n$ implies $m \leq \lambda_i, \mu_j \leq M$ and hence

$$\frac{m}{M} \leq \frac{\lambda_i \#_\nu \mu_j}{\lambda_i \nabla_\nu \mu_j} \leq \frac{M}{m}, \forall i, j.$$

Furthermore, the conditions $\tau > \nu$ and $B \geq \lambda_1 I_n$, or if $\tau < \nu$ and $B \leq \lambda_n I_n$ imply that $(\mu_j - \lambda_i)(\tau - \nu) \geq 0, \forall i, j$. Therefore, applying Corollary 2.7 we have

$$\begin{aligned}
\|A^{1-\tau}XB^{\tau}\|_2^2 &= \sum_{i,j} (\lambda_i \#_\tau \mu_j)^2 \, |y_{ij}|^2 \\
&\geq \sum_{i,j} \left(\frac{\lambda_i \#_\nu \mu_j}{\lambda_i \nabla_\nu \mu_j}\right)^{\frac{2\tau(1-\tau)}{\nu(1-\nu)}} (\lambda_i \nabla_\tau \mu_j)^2 |y_{ij}|^2 \\
&\geq \left(\frac{m}{M}\right)^{\frac{2\tau(1-\tau)}{\nu(1-\nu)}} \sum_{i,j} (\lambda_i \nabla_\tau \mu_j)^2 |y_{ij}|^2 \\
&= \left(\frac{m}{M}\right)^{\frac{2\tau(1-\tau)}{\nu(1-\nu)}} \|(1-\tau)AX + \tau XB\|_2^2.
\end{aligned}$$

This completes the proof. □

Notice that Theorem 2.15 provides a reverse of the well known inequality $\|A^{1-\tau}XB^{\tau}\|_2 \leq \|(1-\tau)AX + \tau XB\|_2$. Further, notice that the condition $B \geq \lambda_1 I_n$ means that $B \geq \|A\| \, I_n$ where $\|A\|$ is the operator norm, while the condition $B \leq \lambda_n I_n$ means that $A \geq \|B\| \, I_n$.

On the other hand, unitarily invariant norm inequalities can be obtained as follows. Recall first that for $A, B \in \mathbb{M}_n^{++}$ and $X \in \mathbb{M}_n$, we have the well known Hölder inequality [4]

$$|||A^{1-t}XB^t||| \leq |||AX\|^{1-t}|||XB|||^t, 0 \leq t \leq 1, \tag{2.4}$$

for any unitarily invariant norm $|||\ \ |||$ on $\mathbb{M}_n$. Applying Young's inequality on the left side implies the known matrix Young inequality

$$|||A^{1-t}XB^t||| \leq (1-t)|||AX||| + t|||XB|||.$$

We remark that the inequality $|||A^{1-t}XB^t||| \le |||(1-t)AX + tXB|||$ is not true in general, however, it is true for the norm $\|\ \|_2$.

In [11], it has been shown that the function $f(t) = |||A^{1-t}XB^t|||, 0 \le t \le 1$ is log-convex. We use this fact to present the following reverse of (2.4).

Lemma 2.16. *Let $A, B \in \mathbb{M}_n^{++}, X \in \mathbb{M}_n$ and $|||\ \ |||$ be a unitarily invariant norm on $\mathbb{M}_n$ such that $|||A^{1-t}XB^t||| \neq 0$ for any $0 \le t \le 1$. Then*

$$|||A^{1-t}XB^t||| \left(\frac{|||AX|||\ |||XB|||}{|||A^{\frac{1}{2}}XB^{\frac{1}{2}}|||^2}\right)^{R(t)} \ge |||AX\|^{1-t}|||XB|||^t, \tag{2.5}$$

where $R(t) = \max\{t, 1-t\}$.

Proof. Let $f(t) = |||A^{1-t}XB^t|||$. Then f is log-convex. For $0 \le t \le \frac{1}{2}$, notice that

$$\frac{1}{2} = \alpha t + (1-\alpha) \text{ where } \alpha = \frac{1}{2-2t}.$$

Using log-convexity of f, we have

$$f\left(\frac{1}{2}\right) \le f^{\alpha}(t) f^{1-\alpha}(1).$$

Simplifying this inequality implies the result for $0 \le t \le \frac{1}{2}$. Similar computations yield the result for $\frac{1}{2} \le t \le 1$. □

On the other hand, notice that the function $f(t) = |||(1-t)AX + tXB|||$ is convex. This fact follows immediately because $|||\ \ |||$ is a norm. This entails the following reverse of $|||(1-t)AX + tXB||| \le (1-t)|||AX||| + t|||XB|||$. The proof is similar to the above one. However, the reader is encouraged to look at [10] for a general discussion of these refinements and reverses of convex functions.

Lemma 2.17. *Let $A, B \in \mathbb{M}_n^{++}, X \in \mathbb{M}_n$ and $|||\ \ |||$ be a unitarily invariant norm on $\mathbb{M}_n$. Then*

$$\begin{aligned} &|||(1-t)AX + tXB||| + R(t)\left(|||AX||| + |||XB||| - |||AX + XB|||\right) \\ &\qquad \le (1-t)|||AX||| + t|||XB|||. \end{aligned} \tag{2.6}$$

Now we are ready to find quadratic refinements and reverses of

$$|||A^{1-t}XB^t||| \le |||AX|||^{1-t}|||XB|||^t \le (1-t)|||AX||| + t|||XB|||.$$

Theorem 2.18. *Let $A, B \in \mathbb{M}_n^{++}, X \in \mathbb{M}_n$ and $|||\ \ |||$ be a unitarily invariant norm on $\mathbb{M}_n$ such that $|||A^{1-t}XB^t||| \neq 0$ for any $0 \le t \le 1$. If $(|||XB||| - |||AX|||)(\tau - \nu) > 0$ then*

$$\begin{aligned} &|||(1-\tau)AX + \tau XB||| \\ &\le (1-\tau)|||AX||| + \tau|||XB||| \\ &\le |||AX|||^{1-\tau}|||XB|||^{\tau} \left(\frac{(1-\nu)|||AX||| + \nu|||XB|||}{|||AX|||^{1-\nu}|||XB|||^{\nu}}\right)^{\frac{\tau(1-\tau)}{\nu(1-\nu)}} \\ &\le |||A^{1-\tau}XB^{\tau}||| \left(\frac{|||AX|||\ |||XB|||}{|||A^{\frac{1}{2}}XB^{\frac{1}{2}}|||^2}\right)^{R(\tau)} \left(\frac{(1-\nu)|||AX||| + \nu|||XB|||}{|||A^{1-\nu}XB^{\nu}|||}\right)^{\frac{\tau(1-\tau)}{\nu(1-\nu)}}. \end{aligned}$$

On the other hand, if $(|||XB||| - |||AX|||)(\tau - \nu) < 0$, *then*

$$|||A^{1-\tau}XB^{\tau}||| \left(\frac{|||A^{\frac{1}{2}}XB^{\frac{1}{2}}|||^2}{|||AX|||\ |||XB|||}\right)^{R(\nu)} \left(\frac{|||(1-\nu)AX+\nu XB|||}{|||A^{1-\nu}XB^{\nu}|||}\right)^{\frac{\tau(1-\tau)}{\nu(1-\nu)}}$$
$$\leq |||AX|||^{1-\tau}|||XB|||^{\tau}\left(\frac{(1-\nu)|||AX|||+\nu|||XB|||}{|||AX|||^{1-\nu}|||XB|||^{\nu}}\right)^{\frac{\tau(1-\tau)}{\nu(1-\nu)}}$$
$$\leq (1-\tau)|||AX|||+\tau|||XB|||$$
$$\leq |||(1-\tau)AX+\tau XB||| + R(\tau)\left(|||AX|||+|||XB|||-|||AX+XB|||\right).$$

Proof. When $(|||XB||| - |||AX|||)(\tau - \nu) > 0$, the first inequality follows immediately because $|||\ \ |||$ is a norm. The second inequality follows from Corollary 2.7 on replacing (a, b) by $(|||AX|||, |||XB|||)$. Then the third inequality follows from (2.5) and the fact that $\|A^{1-\nu}XB^{\nu}||| \leq |||AX|||^{1-\nu}|||XB|||^{\nu}$.

Now when $(|||XB||| - |||AX|||)(\tau - \nu) < 0$, we apply Corollary 2.7, (2.4), (2.5) and (2.6) to obtain the desired inequalities. □

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Boundary value problems for fractional differential inclusions with Hadamard type derivatives in Banach spaces

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Abstract. The authors establish sufficient conditions for the existence of solutions to boundary value problems for fractional differential inclusions involving the Hadamard type fractional derivative of order $\alpha \in (1, 2]$ in Banach spaces. Their approach uses Mönch's fixed point theorem and the Kuratowski measure of noncompacteness.

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1. Introduction

In this paper we are concerned with the existence of solutions to boundary value problems (BVP for short) for fractional order differential inclusions. In particular, we consider the boundary value problem

$$^{H}D^{r}y(t) \in F(t, y(t)), \quad \text{for a.e. } t \in J = [1, T], \; 1 < r \leq 2, \tag{1.1}$$

$$y(1) = 0, \; y(T) = y_T, \tag{1.2}$$

where $^{H}D^{r}$ is the Hadamard fractional derivative, $(E, |\cdot|)$ is a Banach space, $\mathcal{P}(E)$ is the family of all nonempty subsets of E, $F : [1, T] \times E \to \mathcal{P}(E)$ is a multivalued map, and $y_T \in \mathbb{R}$.

Differential equations of fractional order are valuable in modeling phenomena in various fields of science and engineering. They can be found in viscoelasticity, electrochemistry, control, porous media, electromagnetism, etc. The monographs of Hilfer [18], Kilbas *et al.* [19], Podlubny [23], and Momani *et al.* [21] are very good sources on the background mathematics and various applications of fractional derivatives. The literature on Hadamard-type fractional differential equations has not undergone as much development as it has for the Caputo and Riemann-Liouville fractional derivatives; see, for example, the papers of Ahmed and Ntouyas [2], Benhamida, Graef, and Hamani [10], and Thiramanus, Ntouyas, and Tariboon [24].

The fractional derivative that Hadamard [16] introduced in 1892 differs from other fractional derivatives in the sense that the kernel of the integral in the definition of the Hadamard derivative contains a logarithmic function with an arbitrary exponent. A detailed description of the Hadamard fractional derivative and integral can be found in [11, 12, 13].

In this paper, we present existence results for the problem (1.1)-(1.2) in the case where the right hand side is convex valued. This result relies on the set-valued analog of Mönch's fixed point theorem combined with the technique of measure of noncompactness. Recently, this has proved to be a valuable tool in studying fractional differential equations and inclusions in Banach spaces; for additional details, see the papers of Laosta *et al.* [20], Agarwal *et al.* [1], and Benchohra *et al.* [7, 8, 9]. Our results here extend to the multivalued case some previous results in the literature and constitutes what we hope is an interesting contribution to this emerging field. We include an example to illustrate our main results.

2. Preliminaries

This section contains definitions, concepts, lemmas, and preliminary facts that will be used in the remainder of this paper. Let $C(J,E)$ be the Banach space of all continuous functions from J into E with the norm

$$\|y\|_\infty = \sup\{|y(t)| : t \in J\},$$

and let $L^1(J,E)$ be the Banach space of Lebesgue integrable functions $y : J \to E$ with the norm

$$\|y\|_{L^1} = \int_1^T |y(t)|dt.$$

The space $AC^1(J,E)$ is the space of functions $y : J \to E$ that are absolutely continuous and have an absolutely continuous first derivative.

For any Banach space X, we set

$P_{cl}(X) = \{Y \in \mathcal{P}(X) : Y \text{ is closed}\}$,
$P_b(X) = \{Y \in \mathcal{P}(X) : Y \text{ is bounded}\}$,
$P_{cp}(X) = \{Y \in \mathcal{P}(X) : Y \text{ is compact}\}$, and
$P_{cp,c}(X) = \{Y \in \mathcal{P}(X) : Y \text{ is compact and convex}\}$.

A multivalued map $G : X \to \mathcal{P}(X)$ is *convex (closed)* valued if $G(X)$ is convex (closed) for all $x \in X$. We say that G is *bounded on bounded sets* if $G(B) = \cup_{x\in B} G(x)$ is bounded in X for all $B \in P_b(X)$ (i.e., $\sup_{x\in B}\{\sup\{|y| : y \in G(x)\}\}$ is bounded).

The mapping G is *upper semi-continuous (u.s.c)* on X if for each $x_0 \in X$, the set $G(x_0)$ is a nonempty closed subset of X, and for each open set N of X containing $G(x_0)$, there exists an open neighborhood N_0 of x_0 such that $G(N_0) \subset N$. A map G is said to be *completely continuous* if $G(B)$ is relatively compact for every $B \in P_b(X)$.

If the multivalued map G is completely continuous with nonempty compact values, then G is u.s.c if and only if G has a closed graph (i.e., $x_n \to x_*$, $y_n \to y_*$,

$y_n \in G(x_n)$ imply $y_* \in G(x_*)$). The mapping $G : X \to \mathcal{P}(X)$ has a fixed point if there exists $x \in X$ such that $x \in G(x)$. The set of fixed points of the multivalued operator G will be denoted by $Fix\,G$. A multivalued map $G : J \to P_{cl}(X)$ is said to be measurable if for every $y \in X$, the function

$$t \to d(y, G(t)) = inf\{|y - z| : z \in G(t)\}$$

is measurable.

Definition 2.1. A multivalued map $F : J \times E \to \mathcal{P}(E)$ is said to be Carathéodory if:

(1) $t \to F(t, u)$ is measurable for each $u \in E$;
(2) $u \to F(t, u)$ is upper semicontinuous for a.e. $t \in J$.

For each $y \in AC^1(J, E)$, define the set of selections of F by

$$S_{F,y} = \{v \in L^1(J, E) : v(t) \in F(t, y(t)) \text{ a.e. } t \in J\}.$$

Let (X, d) be a metric space induced from the normed space $(X, |\cdot|)$. The function $H_d : \mathcal{P}(X) \times \mathcal{P}(X) \to \mathbb{R}_+ \cup \{\infty\}$ given by

$$H_d(A, B) = \max\{\sup_{a \in A} d(a, B), \sup_{b \in B} d(A, b)\}$$

is known as the Hausdorff-Pompeiu metric.

For more details on multivalued maps, see the books of Aubin and Cellina [4], Aubin and Frankowska [5], Castaing and Valadier [14], and Deimling [15].

For convenience, we first recall the definitions of the Kuratowski measure of noncompacteness and summarize the main properties of this measure.

Definition 2.2. ([3, 6]) Let E be a Banach space and let Ω_E be the bounded subsets of E. The Kuratowski measure of noncompactness is the map $\beta : \Omega_E \to [0, \infty)$ defined by

$$\beta(B) = \inf\{\epsilon > 0 : B \subset \bigcup_{j=1}^{m} B_j \text{ and } diam(B_j) \leq \epsilon\}.$$

Properties: The Kuratowski measure of noncompactness satisfies the following properties (for more details see [3, 6]):

(P_1) $\beta(B) = 0$ if and only if $\overline{B}$ is compact (B is relatively compact).

(P_2) $\beta(B) = \beta(\overline{B})$.

(P_3) $A \subset B$ implies $\beta(A) \leq \beta(B)$.

(P_4) $\beta(A + B) \leq \beta(A) + \beta(B)$.

(P_5) $\beta(cB) = |c|\beta(B)$, $c \in \mathbb{R}$.

(P_6) $\beta(convB) = \beta(B)$.

Here $\overline{B}$ and $conv\,B$ denote the closure and the convex hull of the bounded set B, respectively.

For a given set V of functions $u : J \to E$, we set

$$V(t) = \{u(t) : u \in V\}, t \in J,$$

and

$$V(J) = \{u(t) : u \in V(t),\, t \in J\}.$$

Theorem 2.3. ([17], [22, Theorem 1.3]) *Let E be a Banach space and let C be a countable subset of $L^1(J,E)$ such that there exists $h \in L^1(J,\mathbb{R}_+)$ with $|u(t)| \le h(t)$ for a.e. $t \in J$ and every $u \in C$. Then the function $\varphi(t) = \beta(C(t))$ belongs to $L^1(J,\mathbb{R}_+)$ and satisfies*

$$\beta\left(\left\{\int_0^T u(s)ds : u \in C\right\}\right) \le 2\int_0^T \beta(C(s))ds.$$

Lemma 2.4. ([20, Lemma 2.6]) *Let J be a compact real interval, F be a Carathéodory multivalued map, and let θ be a linear continuous map from $L^1(J,E) \to C(J,E)$. Then the operator*

$$\theta \circ S_{F,y} : L^1(J,E) \to P_{cp,c}(C(J,E)), \quad y \to (\theta \circ S_{F,y})(y) = \theta(S_{F,y})$$

is a closed graph operator in $L^1(J,E) \times C(J,E)$.

In what follows, $\log(\cdot) = \log_e(\cdot)$, and $n = [r] + 1$ where $[r]$ denotes the integer part of r.

Definition 2.5. ([19]) The Hadamard fractional integral of order r for a function $h : [1,+\infty) \to \mathbb{R}$ is defined by

$$I^r h(t) = \frac{1}{\Gamma(r)}\int_1^t \left(\log\frac{t}{s}\right)^{r-1}\frac{h(s)}{s}ds,\ r > 0,$$

provided the integral exists.

Definition 2.6. ([19]) For a function h on the interval $[1,+\infty)$, the Hadamard fractional derivative of h of order r is defined by

$$({}^H D^r h)(t) = \frac{1}{\Gamma(n-r)}\left(t\frac{d}{dt}\right)^n \int_1^t \left(\log\frac{t}{s}\right)^{n-r-1}\frac{h(s)}{s}ds,\ n-1 < r < n,\ n = [r]+1.$$

Let us now recall Mönch's fixed point theorem.

Theorem 2.7. ([22, Theorem 3.2]) *Let K be a closed and convex subset of a Banach space E, U be a relatively open subset of K, and $N : \overline{U} \to \mathcal{P}(K)$. Assume that graph N is closed, N maps compact sets into relatively compact sets, and for some $x_0 \in U$, the following two conditions are satisfied:*

(i) *$M \subset \overline{U}$, $M \subset conv(x_0 \cup N(M))$, $\overline{M} = \overline{C}$, with C a countable subset of M, implies $\overline{M}$ is compact;*
(ii) *$x \notin (1-\lambda)x_0 + \lambda N(x)$ for all $x \in \overline{U} \setminus U$, $\lambda \in (0,1)$.*

Then there exists $x \in \overline{U}$ with $x \in N(x)$.

3. Main results

Let us start by defining what we mean by a solution of the problem (1.1)-(1.2).

Definition 3.1. A function $y \in AC^1(J, E)$ is said to be a solution of (1.1)-(1.2) if there exist a function $v \in L^1(J, E)$ with $v(t) \in F(t, y(t))$ for a.e. $t \in J$, such that ${}^H D^\alpha y(t) = v(t)$ on J, and the conditions $y(1) = 0$ and $y(T) = y_T$ are satisfied.

Lemma 3.2. *Let $h : J \to E$ be a continuous function. A function y is a solution of the fractional integral equation*

$$y(t) = \frac{1}{\Gamma(r)} \int_1^t \left(\log \frac{t}{s}\right)^{r-1} h(s)\frac{ds}{s} + \frac{(\log t)^{r-1}}{(\log T)^{r-1}} \left[y_T - \frac{1}{\Gamma(r)} \int_1^T \left(\log \frac{T}{s}\right)^{r-1} h(s)\frac{ds}{s} \right] \tag{3.1}$$

if and only if y is a solution of the fractional BVP

$${}^H D^r y(t) = h(t), \text{ for a.e. } t \in J = [1, T], \quad 1 < r \le 2, \tag{3.2}$$

$$y(1) = 0, \;\; y(T) = y_T. \tag{3.3}$$

Proof. Applying the Hadamard fractional integral of order r to both sides of (3.2), we obtain

$$y(t) = c_1(\log t)^{r-1} + c_2(\log t)^{r-2} + {}^H I^r h(t). \tag{3.4}$$

From (3.3), we have $c_2 = 0$ and

$$c_1 = \frac{1}{(\log T)^{r-1}}[y_T - {}^H I^r h(T)].$$

Hence, we obtain (3.1). Conversely, it is clear that if y satisfies equation (3.1), then (3.2)-(3.3) hold. □

Theorem 3.3. *Let $R > 0$, $B = \{x \in E : \|x\| \le R\}$, $U = \{x \in C(J, E) : \|x\| \le R\}$, and assume that:*

(H1) *$F : J \times E \to \mathcal{P}_{cp,p}(E)$ is a Carathéodory multi-valued map;*

(H2) *For each $R > 0$, there exists a function $p \in L^1(J, E)$ such that*

$$\|F(t, u)\|_{\mathcal{P}} = \sup\{|v| : v(t) \in F(t, y)\} \le p(t)$$

for each $(t, y) \in J \times E$ with $|y| \ge R$, and

$$\liminf_{R\to\infty} \frac{\int_0^T p(t)dt}{R} = \delta < \infty;$$

(H3) *There exists a Carathéodory function $\psi : J \times [1, 2R] \to \mathbb{R}_+$ such that*

$$\beta(F(t, M)) \le \psi(t, \beta(M)) \text{ a.e. } t \in J \text{ and each } M \subset B;$$

(H4) *The function $\varphi = 0$ is the unique solution in $C(J, [1, 2R])$ of the inequality*

$$\varphi(t) \le 2\left\{ \frac{1}{\Gamma(r)} \int_1^t \left(\log \frac{t}{s}\right)^{r-1} \psi(s, \varphi(s))\frac{ds}{s} + \frac{(\log t)^{r-1}}{(\log T)^{r-1}} \left[y_T + \frac{1}{\Gamma(r)} \int_1^T \left(\log \frac{T}{s}\right)^{r-1} \psi(s, \varphi(s))\frac{ds}{s} \right] \right\} \text{ for } t \in J. \tag{3.5}$$

Then the BVP (1.1)-(1.2) has at least one solution in $C(J,B)$, provided that

$$\delta < \frac{\Gamma(r+1)}{(\log T)^r}. \tag{3.6}$$

Proof. We wish to transform the problem (1.1)-(1.2) into a fixed point problem, so consider the multivalued operator

$$N(y) = \left\{ h \in C(J,\mathbb{R}) : h(t) = \frac{1}{\Gamma(r)} \int_1^t \left(\log \frac{t}{s}\right)^{r-1} v(s)\frac{ds}{s} \right.$$
$$\left. + \frac{(\log t)^{r-1}}{(\log T)^{r-1}} \left[y_T - \frac{1}{\Gamma(r)} \int_1^T \left(\log \frac{T}{s}\right)^{r-1} v(s)\frac{ds}{s} \right], \quad v \in S_{F,y} \right\}.$$

Clearly, from Lemma 3.2, the fixed points of N are solutions to (1.1)-(1.2). We shall show that N satisfies the assumptions of Mönch's fixed point theorem. The proof will be given in several steps. First note that $\overline{U} = C(J,B)$.

Step 1: *$N(y)$ is convex for each $y \in C(J,B)$.*

Take h_1, $h_2 \in N(y)$; then there exist v_1, $v_2 \in S_{F,y}$ such that for each $t \in J$, we have

$$h_i(t) = \frac{1}{\Gamma(r)} \int_1^t \left(\log \frac{t}{s}\right)^{r-1} v_i(s)\frac{ds}{s}$$
$$+ \frac{(\log t)^{r-1}}{(\log T)^{r-1}} \left[y_T - \frac{1}{\Gamma(r)} \int_1^T \left(\log \frac{T}{s}\right)^{r-1} v_i(s)\frac{ds}{s} \right]$$

for $i = 1$, 2. Let $0 \le d \le 1$; then for each $t \in J$,

$$(dh_1 + (1-d)h_2)(t) = \frac{1}{\Gamma(r)} \int_1^t \left(\log \frac{t}{s}\right)^{r-1} [dv_1 + (1-d)v_2]\frac{ds}{s}$$
$$+ \frac{(\log t)^{r-1}}{(\log T)^{r-1}} \left[y_T - \frac{1}{\Gamma(r)} \int_1^T \left(\log \frac{T}{s}\right)^{r-1} [dv_1 + (1-d)v_2]\frac{ds}{s} \right].$$

Since $S_{F,y}$ is convex (because F has convex values), we have

$$dh_1 + (1-d)h_2 \in N(y).$$

Step 2: *$N(M)$ is relatively compact for each compact $M \subset \overline{U}$.*

Let $M \subset \overline{U}$ be a compact set and let $\{h_n\}$ be any sequence of elements of $N(M)$. We will show that $\{h_n\}$ has a convergent subsequence by using the Arzelà-Ascoli criterion of compactness in $C(J,B)$. Since $h_n \in N(M)$, there exist $y_n \in M$ and $v_n \in S_{F,y}$ such that

$$h_n(t) = \frac{1}{\Gamma(r)} \int_1^t \left(\log \frac{t}{s}\right)^{r-1} v_n(s)\frac{ds}{s}$$
$$+ \frac{(\log t)^{r-1}}{(\log T)^{r-1}} \left[y_T - \frac{1}{\Gamma(r)} \int_1^T \left(\log \frac{T}{s}\right)^{r-1} v_n(s)\frac{ds}{s} \right]$$

for $n \geq 1$. Using Theorem 2.3 and the properties of the Kuratowski measure of noncompactness, we have

$$\beta(\{h_n(t)\}) \leq 2\left\{\frac{1}{\Gamma(r)}\int_1^t \beta\left(\left\{\left(\log\frac{t}{s}\right)^{r-1}\frac{v_n(s)}{s} : n \geq 1\right\}\right)ds \right.$$
$$\left. + \frac{(\log t)^{r-1}}{(\log T)^{r-1}}\left[y_T + \frac{1}{\Gamma(r)}\int_1^T \beta\left(\left\{\left(\log\frac{T}{s}\right)^{r-1}\frac{v_n(s)}{s} : n \geq 1\right\}\right)ds\right]\right\}. \tag{3.7}$$

On the other hand, since $M(s)$ is compact in E, the set $\{v_n(s) : n \geq 1\}$ is compact. Consequently, $\beta(\{v_n(s) : n \geq 1\}) = 0$ for a.e. $s \in J$. Furthermore,

$$\beta\left(\left\{\left(\log\frac{t}{s}\right)^{r-1}\frac{v_n(s)}{s}\right\}\right) = \left(\log\frac{t}{s}\right)^{r-1}\frac{1}{s}\beta(\{v_n(s) : n \geq 1\}) = 0$$

and

$$\beta\left(\left\{\left(\log\frac{T}{s}\right)^{r-1}\frac{v_n(s)}{s}\right\}\right) = \left(\log\frac{T}{s}\right)^{r-1}\frac{1}{s}\beta(\{v_n(s) : n \geq 1\}) = 0$$

for a.e. t, $s \in J$. Hence, from this and (3.7), $\{h_n(t) : n \geq 1\}$ is relatively compact in B for each $t \in J$. In addition, for each t_1, $t_2 \in J$ with $t_1 < t_2$, we have

$$\begin{aligned}|h_n(t_2) - h_n(t_1)| &= \left|\frac{1}{\Gamma(\alpha)}\int_1^{t_1}\left[\left(\log\frac{t_2}{s}\right)^{\alpha-1} - \left(\log\frac{t_1}{s}\right)^{\alpha-1}\right]\frac{v_n(s)}{s}ds \right.\\ &\quad \left. + \frac{1}{\Gamma(\alpha)}\int_{t_1}^{t_2}\left(\log\frac{t_2}{s}\right)^{\alpha-1}\frac{v_n(s)}{s}ds\right|\\ &\leq \frac{p(t)}{\Gamma(\alpha)}\int_1^{t_1}\left[\left(\log\frac{t_2}{s}\right)^{\alpha-1} - \left(\log\frac{t_1}{s}\right)^{\alpha-1}\right]\frac{ds}{s}\\ &\quad + \frac{p(t)}{\Gamma(\alpha)}\int_{t_1}^{t_2}\left(\log\frac{t_2}{s}\right)^{\alpha-1}\frac{ds}{s}.\end{aligned}$$

As $t_1 \to t_2$, the right hand side of the above inequality tends to zero. This shows that $\{h_n : n \geq 1\}$ is equicontinuous. Consequently, $\{h_n : n \geq 1\}$ is relatively compact in $C(J, B)$.

Step 3: *N has a closed graph.*

Let $y_n \to y_*$, $h_n \in N(y_n)$, and $h_n \to h_*$. We need to show that $h_* \in N(y_*)$. Now $h_n \in N(y_n)$ means that there exists $v_n \in S_{F,y}$ such that, for each $t \in J$,

$$\begin{aligned}h_n(t) &= \frac{1}{\Gamma(r)}\int_1^t\left(\log\frac{t}{s}\right)^{r-1}v_n(s)\frac{ds}{s}\\ &\quad + \frac{(\log t)^{r-1}}{(\log T)^{r-1}}\left[y_T - \frac{1}{\Gamma(r)}\int_1^T\left(\log\frac{T}{s}\right)^{r-1}v_n(s)\frac{ds}{s}\right].\end{aligned}$$

Consider the continuous linear operator $\theta : L^1(J,E) \to C(J,E)$ defined by

$$\theta(v)(t) \to h_n(t) = \frac{1}{\Gamma(r)} \int_1^t \left(\log \frac{t}{s}\right)^{r-1} v_n(s)\frac{ds}{s} + \frac{(\log t)^{r-1}}{(\log T)^{r-1}} \left[y_T - \frac{1}{\Gamma(r)} \int_1^T \left(\log \frac{T}{s}\right)^{r-1} v_n(s)\frac{ds}{s}\right].$$

Clearly, $\|h_n(t) - h_*(t)\| \to 0$ as $n \to \infty$. From Lemma 2.4 it follows that $\theta \circ S_F$ is a closed graph operator. Moreover, $h_n(t) \in \theta(S_{F,y_n})$. Since $y_n \to y$, Lemma 2.4 implies

$$h(t) = \frac{1}{\Gamma(r)} \int_1^t \left(\log \frac{t}{s}\right)^{r-1} v(s)\frac{ds}{s} + \frac{(\log t)^{r-1}}{(\log T)^{r-1}} \left[y_T - \frac{1}{\Gamma(r)} \int_1^T \left(\log \frac{T}{s}\right)^{r-1} v(s)\frac{ds}{s}\right].$$

Step 4: *M is relatively compact in $C(J,B)$.*

Suppose $M \subset \overline{U}$, $M \subset conv(\{0\} \cup N(M))$, and $\overline{M} = \overline{C}$ for some countable set $C \subset M$. Using an argument similar to the one used in Step 2 shows that $N(M)$ is equicontinuous. Then, since $M \subset conv(\{0\} \cup N(M))$, we see that M is equicontinuous as well. To apply the Arzelà-Ascoli theorem, it remains to show that $M(t)$ is relatively compact in E for each $t \in J$. Since $C \subset M \subset conv(\{0\} \cup N(M))$ and C is countable, we can find a countable set $H = \{h_n : n \geq 1\} \subset N(M)$ with $C \subset conv(\{0\} \cup H)$. Then, there exist $y_n \in M$ and $v_n \in S_{F,y_n}$ such that

$$h_n(t) = \frac{1}{\Gamma(r)} \int_1^t \left(\log \frac{t}{s}\right)^{r-1} v_n(s)\frac{ds}{s} + \frac{(\log t)^{r-1}}{(\log T)^{r-1}} \left[y_T - \frac{1}{\Gamma(r)} \int_1^T \left(\log \frac{T}{s}\right)^{r-1} v_n(s)\frac{ds}{s}\right].$$

Since $M \subset C \subset conv(\{0\} \cup H))$, from the properties of the Kuratowski measure of noncompactness, we have

$$\beta(M(t)) \leq \beta(C(t)) \leq \beta(H(t)) = \beta(\{h_n(t) : n \geq 1\}).$$

Using (3.7) and the fact that $v_n(s) \in M(s)$, we obtain

$$\begin{aligned}
\beta(M(t)) \leq 2 \Bigg\{ & \frac{1}{\Gamma(r)} \int_1^t \beta\left(\left\{\left(\log \frac{t}{s}\right)^{r-1} \frac{v_n(s)}{s} : n \geq 1\right\}\right) ds \\
& + \frac{(\log t)^{r-1}}{(\log T)^{r-1}} \left[y_T + \frac{1}{\Gamma(r)} \int_1^T \beta\left(\left\{\left(\log \frac{T}{s}\right)^{r-1} \frac{v_n(s)}{s} : n \geq 1\right\}\right) ds \right] \Bigg\} \\
\leq 2 \Bigg\{ & \frac{1}{\Gamma(r)} \int_1^t \left(\log \frac{t}{s}\right)^{r-1} \beta(M(s)) \frac{ds}{s} \\
& + \frac{(\log t)^{r-1}}{(\log T)^{r-1}} \left[y_T + \frac{1}{\Gamma(r)} \int_1^T \left(\log \frac{T}{s}\right)^{r-1} \beta(M(s)) \frac{ds}{s} \right] \Bigg\} \\
\leq 2 \Bigg\{ & \frac{1}{\Gamma(r)} \int_1^t \left(\log \frac{t}{s}\right)^{r-1} \psi(s, \beta(M(s))) \frac{ds}{s} \\
& + \frac{(\log t)^{r-1}}{(\log T)^{r-1}} \left[y_T + \frac{1}{\Gamma(r)} \int_1^T \left(\log \frac{T}{s}\right)^{r-1} \psi(s, \beta(M(s))) \frac{ds}{s} \right] \Bigg\}.
\end{aligned}$$

We also have that the function φ given by $\varphi(t) = \beta(M(t))$ belongs to $C(J, [1, 2R])$. Consequently, by $(H4)$, $\varphi = 0$; that is, $\beta(M(t)) = 0$ for all $t \in J$. Now, by the Arzelà-Ascoli theorem, M is relatively compact in $C(J, B)$.

Step 5: Let $h \in N(y)$ with $y \in U$. We claim that $N(U) \subset U$. If this were not the case, then in view of $(H2)$, there exists functions $v \in S_{F,y}$ and $p \in L^1(J, E)$ such that

$$\begin{aligned}
h(t) = & \frac{1}{\Gamma(r)} \int_1^t \left(\log \frac{t}{s}\right)^{r-1} v(s) \frac{ds}{s} \\
& + \frac{(\log t)^{r-1}}{(\log T)^{r-1}} \left[y_T - \frac{1}{\Gamma(r)} \int_1^T \left(\log \frac{T}{s}\right)^{r-1} v(s) \frac{ds}{s} \right],
\end{aligned}$$

and

$$\begin{aligned}
R < \|N(y)\|_{\mathcal{P}} \leq & \frac{1}{\Gamma(r)} \int_1^t \left(\log \frac{t}{s}\right)^{r-1} |v(s)| \frac{ds}{s} \\
& + \frac{(\log t)^{r-1}}{(\log T)^{r-1}} \left[|y_T| + \frac{1}{\Gamma(r)} \int_1^T \left(\log \frac{T}{s}\right)^{r-1} |v(s)| \frac{ds}{s} \right] \\
\leq & \frac{(\log T)^r}{\Gamma(r+1)} \int_1^t p(s) ds + \frac{(\log T)^r}{\Gamma(r+1)} \int_1^T p(s) ds \\
\leq & 2 \frac{(\log T)^r}{\Gamma(r+1)} \int_1^t p(s) ds.
\end{aligned}$$

Dividing both sides by R and taking the $\liminf$ as $R \to \infty$, we have

$$2 \left[\frac{(\log T)^r}{\Gamma(r+1)} \right] \delta \geq 1$$

which contradicts (3.6). Hence, $N(U) \subset U$.

As a consequence of Steps 1-5 and Mönch's Theorem (Theorem 2.7 above), N has a fixed point $y \in C(J, B)$ that in turn is a solution of problem (1.1)-(1.2). $\square$

4. An example

We conclude this paper with an example to illustrate our main result, namely, Theorem 3.3 above.

Consider the fractional differential inclusion

$$ {}^{H}D^{\alpha}y(t) \in F(t, y(t)), \text{ for a.e. } t \in J = [1, e], \ 0 < \alpha \leq 1, \tag{4.1} $$

$$ y(1) = 0, \ y(e) = 1. \tag{4.2} $$

Here, $F : [1, e] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is a multivalued map satisfying

$$ F(t, y) = \{v \in \mathbb{R} : f_1(t, y) \leq v \leq f_2(t, y)\}, $$

where f_1, $f_2 : [1, e] \times \mathbb{R} \to \mathbb{R}$, $f_1(t, \cdot)$ is lower semi-continuous (i.e., the set $\{y \in \mathbb{R} : f_1(t, y) > \mu\}$ is open for each $\mu \in \mathbb{R}$), and $f_2(t, \cdot)$ is upper semi-continuous (i.e., the set the set $\{y \in \mathbb{R} : f_2(t, y) < \mu\}$ is open for each $\mu \in \mathbb{R}$). We assume that there is a function $p \in L^1(J, \mathbb{R})$ such that

$$ \begin{aligned} \|F(t, u)\|_{\mathcal{P}} &= \sup\{|v|, v(t) \in F(t, y)\} \\ &= \max(|f_1(t, y)|, |f_2(t, y)|\} \leq p(t), \ t \in [1, e], \ y \in \mathbb{R}. \end{aligned} $$

It is clear that F is compact and convex valued, and is upper semi-continuous.
Choose $C(s)$ to be the space of linear functions and choose $\varphi(t) = \beta(C(t))$ such that

$$ \beta(u(s)) = \frac{u(s)}{2} $$

with

$$ u(s) = as, \ a > 0, \ \frac{2}{a} \leq s \leq \frac{4R}{a}. $$

For $(t, y) \in J \times \mathbb{R}$ with $|y| \geq R$, we have

$$ \liminf_{R \to \infty} \frac{\int_0^e p(t)dt}{R} = \delta < \infty. $$

Finally, we assume that there exists a Carathéodory function $\psi : J[1, 2R] \to \mathbb{R}_+$ such that

$$ \beta(F(t, M)) \leq \psi(t, \beta(M)) \ a.e. \ t \in J \text{ and each } M \subset B = \{x \in \mathbb{R} : |x| \leq R\}, $$

and $\varphi = 0$ is the unique solution in $C(J, [1, 2R])$ of the inequality

$$ \begin{aligned} \varphi(t) \leq 2 \Bigg\{ &\frac{1}{\Gamma(r)} \int_1^t \left(\log \frac{t}{s}\right)^{r-1} \psi(s, \varphi(s)) \frac{ds}{s} \\ &+ (\log t)^{r-1} \left[1 + \frac{1}{\Gamma(r)} \int_1^e \left(\log \frac{e}{s}\right)^{r-1} \psi(s, \varphi(s)) \frac{ds}{s}\right] \Bigg\}. \end{aligned} $$

for $t \in J$.

Since all the conditions of Theorem 3.3 are satisfied, problem (4.1)-(4.2) has at least one solution y on $[1, e]$.

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Hermite-Hadamard type fractional integral inequalities for $\mathrm{MT}_{(m,\varphi)}$-preinvex functions

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Abstract. In the present paper, a new class of $\mathrm{MT}_{(m,\varphi)}$-preinvex functions is introduced and some new integral inequalities for the left-hand side of Gauss-Jacobi type quadrature formula involving $\mathrm{MT}_{(m,\varphi)}$-preinvex functions are given. Moreover, some generalizations of Hermite-Hadamard type inequalities for $\mathrm{MT}_{(m,\varphi)}$-preinvex functions that are twice differentiable via Riemann-Liouville fractional integrals are established. At the end, some applications to special means are given. These general inequalities give us some new estimates for Hermite-Hadamard type fractional integral inequalities.

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1. Introduction and preliminaries

The following notations are used throughout this paper. We use I to denote an interval on the real line $\mathbb{R} = (-\infty, +\infty)$ and I^0 to denote the interior of I. For any subset $K \subseteq \mathbb{R}^n, K^0$ is used to denote the interior of K. $\mathbb{R}^n$ is used to denote a n-dimensional vector space. The nonnegative real numbers are denoted by $\mathbb{R}_0 = [0, +\infty)$. The set of integrable functions on the interval $[a, b]$ is denoted by $L_1[a, b]$.

The following inequality, named Hermite-Hadamard inequality, is one of the most famous inequalities in the literature for convex functions.

Theorem 1.1. *Let $f : I \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ be a convex function on I and $a, b \in I$ with $a < b$. Then the following inequality holds:*

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}. \tag{1.1}$$

In (see [12]) and the references cited therein, Tunç and Yidirim defined the following so-called MT-convex function:

Definition 1.2. A function $f : I \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ is said to belong to the class of $\mathrm{MT}(I)$, if it is nonnegative and for all $x, y \in I$ and $t \in (0,1)$ satisfies the following inequality:

$$f(tx + (1-t)y) \leq \frac{\sqrt{t}}{2\sqrt{1-t}} f(x) + \frac{\sqrt{1-t}}{2\sqrt{t}} f(y). \tag{1.2}$$

In recent years, various generalizations, extensions and variants of such inequalities have been obtained. For other recent results concerning Hermite-Hadamard type inequalities through various classes of convex functions, (see [13]) and the references cited therein, also (see [3], [4], [5], [8], [9], [10], [17], [20]) and the references cited therein.

Fractional calculus (see [13]) and the references cited therein, was introduced at the end of the nineteenth century by Liouville and Riemann, the subject of which has become a rapidly growing area and has found applications in diverse fields ranging from physical sciences and engineering to biological sciences and economics.

Definition 1.3. Let $f \in L_1[a,b]$. The Riemann-Liouville integrals $J^{\alpha}_{a+}f$ and $J^{\alpha}_{b-}f$ of order $\alpha > 0$ with $a \geq 0$ are defined by

$$J^{\alpha}_{a+}f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)dt, \quad x > a$$

and

$$J^{\alpha}_{b-}f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t)dt, \quad b > x,$$

where $\Gamma(\alpha) = \displaystyle\int_0^{+\infty} e^{-u}u^{\alpha-1}du$. Here $J^0_{a+}f(x) = J^0_{b-}f(x) = f(x)$.
In the case of $\alpha = 1$, the fractional integral reduces to the classical integral.

Due to the wide application of fractional integrals, some authors extended to study fractional Hermite-Hadamard type inequalities for functions of different classes (see [8], [11], [13] [14], [15], [16], [18], [19]) and the references cited therein.

Definition 1.4. (see [7]) A nonnegative function $f : I \subseteq \mathbb{R} \longrightarrow \mathbb{R}_0$ is said to be P-function or P-convex, if

$$f(tx + (1-t)y) \leq f(x) + f(y), \quad \forall x, y \in I,\ t \in [0,1].$$

Definition 1.5. (see [1]) A set $K \subseteq \mathbb{R}^n$ is said to be invex with respect to the mapping $\eta : K \times K \longrightarrow \mathbb{R}^n$, if $x + t\eta(y,x) \in K$ for every $x, y \in K$ and $t \in [0,1]$.

Notice that every convex set is invex with respect to the mapping $\eta(y,x) = y - x$, but the converse is not necessarily true. For more details please see (see [1], [19]) and the references therein.

Definition 1.6. (see [15]) The function f defined on the invex set $K \subseteq \mathbb{R}^n$ is said to be preinvex with respect η, if for every $x, y \in K$ and $t \in [0,1]$, we have that

$$f\left(x + t\eta(y,x)\right) \leq (1-t)f(x) + tf(y).$$

The concept of preinvexity is more general than convexity since every convex function is preinvex with respect to the mapping $\eta(y,x) = y - x$, but the converse is not true.

The Gauss-Jacobi type quadrature formula has the following

$$\int_a^b (x-a)^p(b-x)^q f(x)dx = \sum_{k=0}^{+\infty} B_{m,k} f(\gamma_k) + R_m^\star|f|, \tag{1.3}$$

for certain $B_{m,k}, \gamma_k$ and rest $R_m^\star|f|$ (see [18]).

Recently, Liu (see [11]) obtained several integral inequalities for the left-hand side of (1.3) under the Definition 1.4 of P-function.
Also in (see [14]), Özdemir et al. established several integral inequalities concerning the left-hand side of (1.3) via some kinds of convexity.

Motivated by these results, in Section 2, the notion of $\mathrm{MT}_{(m,\varphi)}$-preinvex function is introduced and some new integral inequalities for the left-hand side of (1.3) involving $\mathrm{MT}_{(m,\varphi)}$-preinvex functions along with beta function are given. In Section 3, some generalizations of Hermite-Hadamard type inequalities for $\mathrm{MT}_{(m,\varphi)}$-preinvex functions that are twice differentiable via fractional integrals are given. In Section 4, some applications to special means, conclusions and future research are given. These general inequalities give us some new estimates for Hermite-Hadamard type fractional integral inequalities.

2. New integral inequalities for $\mathrm{MT}_{(m,\varphi)}$-preinvex functions

Definition 2.1. (see [6]) A set $K \subseteq \mathbb{R}^n$ is said to be m-invex with respect to the mapping $\eta : K \times K \times (0,1] \longrightarrow \mathbb{R}^n$ for some fixed $m \in (0,1]$, if $mx + t\eta(y, mx) \in K$ holds for each $x, y \in K$ and any $t \in [0,1]$.

Remark 2.2. In Definition 2.1, under certain conditions, the mapping $\eta(y, mx)$ could reduce to $\eta(y,x)$. For example when $m = 1$, then the m-invex set degenerates an invex set on K.

We next give new definition, to be referred as $MT_{(m,\varphi)}$-preinvex function.

Definition 2.3. Let $K \subseteq \mathbb{R}$ be an open m-invex set with respect to $\eta : K\times K\times(0,1] \longrightarrow K$ and $\varphi : I \longrightarrow K$ a continuous function. For $f : K \longrightarrow \mathbb{R}$ and some fixed $m \in (0,1]$, if

$$f(m\varphi(y) + t\eta(\varphi(x), \varphi(y), m)) \leq \frac{m\sqrt{t}}{2\sqrt{1-t}} f(\varphi(x)) + \frac{m\sqrt{1-t}}{2\sqrt{t}} f(\varphi(y)), \tag{2.1}$$

is valid for all $x, y \in I$ and $t \in (0,1)$, then we say that $f(x)$ belong to the class of $MT_{(m,\varphi)}(K)$ with respect to η.

Remark 2.4. In Definition 2.3, it is worthwhile to note that the class $MT_{(m,\varphi)}(K)$ is a generalization of the class $\mathrm{MT}(I)$ given in Definition 1.2 on $K = I$ with respect to $\eta(\varphi(x), \varphi(y), m) = \varphi(x) - m\varphi(y)$, $\varphi(x) = x$, $\forall x, y \in I$ and $m = 1$.

Let give below a nontrivial example for motivation of this new interesting class of $MT_{(m,\varphi)}$-preinvex functions.

Example 2.5. $f, g : (1,\infty) \longrightarrow \mathbb{R}$, $f(x) = x^p$, $g(x) = (1+x)^p$, $p \in \left(0, \frac{1}{1000}\right)$; $h : [1, 3/2] \longrightarrow \mathbb{R}$, $h(x) = (1+x^2)^k$, $k \in \left(0, \frac{1}{100}\right)$, are simple examples of the new class of $MT_{(m,x)}$-preinvex functions with respect to $\eta(\varphi(x), \varphi(y), m) = \varphi(x) - m\varphi(y)$, $\varphi(x) = x$, for any fixed $m \in (0, 1]$, but they are not convex.

In this section, in order to prove our main results regarding some new integral inequalities involving $MT_{(m,\varphi)}$-preinvex functions along with beta function, we need the following new interesting Lemma:

Lemma 2.6. *Let $\varphi : I \longrightarrow K$ be a continuous function. Assume that*

$$f : K = [m\varphi(a), m\varphi(a) + \eta(\varphi(b), \varphi(a), m)] \longrightarrow \mathbb{R}$$

is a continuous function on K^0 with $\eta(\varphi(b), \varphi(a), m) > 0$.
Then for some fixed $m \in (0, 1]$ and $p, q > 0$, we have

$$\int_{m\varphi(a)}^{m\varphi(a)+\eta(\varphi(b),\varphi(a),m)} (x - m\varphi(a))^p (m\varphi(a) + \eta(\varphi(b), \varphi(a), m) - x)^q f(x)dx$$

$$= \eta(\varphi(b), \varphi(a), m)^{p+q+1} \int_0^1 t^p (1-t)^q f(m\varphi(a) + t\eta(\varphi(b), \varphi(a), m))dt.$$

Proof. It is easy to observe that

$$\int_{m\varphi(a)}^{m\varphi(a)+\eta(\varphi(b),\varphi(a),m)} (x - m\varphi(a))^p (m\varphi(a) + \eta(\varphi(b), \varphi(a), m) - x)^q f(x)dx$$

$$= \eta(\varphi(b), \varphi(a), m) \int_0^1 (m\varphi(a) + t\eta(\varphi(b), \varphi(a), m) - m\varphi(a))^p$$
$$\times (m\varphi(a) + \eta(\varphi(b), \varphi(a), m) - m\varphi(a) - t\eta(\varphi(b), \varphi(a), m))^q$$
$$\times f(m\varphi(a) + t\eta(\varphi(b), \varphi(a), m))dt$$
$$= \eta(\varphi(b), \varphi(a), m)^{p+q+1} \int_0^1 t^p (1-t)^q f(m\varphi(a) + t\eta(\varphi(b), \varphi(a), m))dt.$$ □

The following definition will be used in the sequel.

Definition 2.7. The Euler Beta function is defined for $x, y > 0$ as

$$\beta(x, y) = \int_0^1 t^{x-1}(1-t)^{y-1}dt = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}.$$

Theorem 2.8. *Let $\varphi : I \longrightarrow K$ be a continuous function. Assume that*

$$f : K = [m\varphi(a), m\varphi(a) + \eta(\varphi(b), \varphi(a), m)] \longrightarrow \mathbb{R}$$

is a continuous function on K^0 with $\eta(\varphi(b), \varphi(a), m) > 0$.
If $k > 1$ and $|f|^{\frac{k}{k-1}}$ is a $MT_{(m,\varphi)}$-preinvex function on an open m-invex set K with respect to $\eta : K \times K \times (0, 1] \longrightarrow K$ for some fixed $m \in (0, 1]$, then for any fixed $p, q > 0$,

$$\int_{m\varphi(a)}^{m\varphi(a)+\eta(\varphi(b),\varphi(a),m)} (x - m\varphi(a))^p (m\varphi(a) + \eta(\varphi(b), \varphi(a), m) - x)^q f(x)dx$$

$$\leq \left(\frac{m\pi}{4}\right)^{\frac{k-1}{k}} \eta(\varphi(b),\varphi(a),m)^{p+q+1}\Big[\beta(kp+1,kq+1)\Big]^{\frac{1}{k}} \times\left(|f(\varphi(a))|^{\frac{k}{k-1}}+|f(\varphi(b))|^{\frac{k}{k-1}}\right)^{\frac{k-1}{k}}.$$

Proof. Since $|f|^{\frac{k}{k-1}}$ is a $MT_{(m,\varphi)}$-preinvex function on K, combining with Lemma 2.6, Definition 2.7 and Hölder inequality for all $t \in (0,1)$ and for some fixed $m \in (0,1]$, we get

$$\int_{m\varphi(a)}^{m\varphi(a)+\eta(\varphi(b),\varphi(a),m)} (x-m\varphi(a))^p(m\varphi(a)+\eta(\varphi(b),\varphi(a),m)-x)^q f(x)dx$$

$$\leq |\eta(\varphi(b),\varphi(a),m)|^{p+q+1}\left[\int_0^1 t^{kp}(1-t)^{kq}dt\right]^{\frac{1}{k}}$$

$$\times\left[\int_0^1 \left|f(m\varphi(a)+t\eta(\varphi(b),\varphi(a),m))\right|^{\frac{k}{k-1}}dt\right]^{\frac{k-1}{k}}$$

$$\leq \eta(\varphi(b),\varphi(a),m)^{p+q+1}\Big[\beta(kp+1,kq+1)\Big]^{\frac{1}{k}}$$

$$\times\left[\int_0^1\left(\frac{m\sqrt{t}}{2\sqrt{1-t}}|f(\varphi(b))|^{\frac{k}{k-1}}+\frac{m\sqrt{1-t}}{2\sqrt{t}}|f(\varphi(a))|^{\frac{k}{k-1}}\right)dt\right]^{\frac{k-1}{k}}$$

$$= \left(\frac{m\pi}{4}\right)^{\frac{k-1}{k}} \eta(\varphi(b),\varphi(a),m)^{p+q+1}\Big[\beta(kp+1,kq+1)\Big]^{\frac{1}{k}} \times\left(|f(\varphi(a))|^{\frac{k}{k-1}}+|f(\varphi(b))|^{\frac{k}{k-1}}\right)^{\frac{k-1}{k}}.$$

□

Theorem 2.9. *Let $\varphi : I \longrightarrow K$ be a continuous function. Assume that*

$$f : K = [m\varphi(a), m\varphi(a)+\eta(\varphi(b),\varphi(a),m)] \longrightarrow \mathbb{R}$$

is a continuous function on K^0 with $\eta(\varphi(b),\varphi(a),m) > 0$.
If $l \geq 1$ and $|f|^l$ is a $MT_{(m,\varphi)}$-preinvex function on an open m-invex set K with respect to $\eta : K \times K \times (0,1] \longrightarrow K$ for some fixed $m \in (0,1]$, then for any fixed $p,q > 0$,

$$\int_{m\varphi(a)}^{m\varphi(a)+\eta(\varphi(b),\varphi(a),m)} (x-m\varphi(a))^p(m\varphi(a)+\eta(\varphi(b),\varphi(a),m)-x)^q f(x)dx$$

$$\leq \left(\frac{m}{2}\right)^{\frac{1}{l}} \eta(\varphi(b),\varphi(a),m)^{p+q+1}\Big[\beta(p+1,q+1)\Big]^{\frac{l-1}{l}}$$

$$\times\left[|f(\varphi(a))|^l\beta\left(p+\frac{1}{2},q+\frac{3}{2}\right)+|f(\varphi(b))|^l\beta\left(p+\frac{3}{2},q+\frac{1}{2}\right)\right]^{\frac{1}{l}}.$$

Proof. Since $|f|^l$ is a $MT_{(m,\varphi)}$-preinvex function on K, combining with Lemma 2.6, Definition 2.7 and Hölder inequality for all $t \in (0,1)$ and for some fixed $m \in (0,1]$, we get

$$\int_{m\varphi(a)}^{m\varphi(a)+\eta(\varphi(b),\varphi(a),m)} (x-m\varphi(a))^p(m\varphi(a)+\eta(\varphi(b),\varphi(a),m)-x)^q f(x)dx$$

$$= \eta(\varphi(b),\varphi(a),m)^{p+q+1}$$

$$\times \int_0^1 \Big[t^p(1-t)^q\Big]^{\frac{l-1}{l}} \Big[t^p(1-t)^q\Big]^{\frac{1}{l}} f(m\varphi(a)+t\eta(\varphi(b),\varphi(a),m))dt$$

$$\leq |\eta(\varphi(b),\varphi(a),m)|^{p+q+1}\left[\int_0^1 t^p(1-t)^q dt\right]^{\frac{l-1}{l}}$$

$$\times\left[\int_0^1 t^p(1-t)^q \big|f(m\varphi(a)+t\eta(\varphi(b),\varphi(a),m))\big|^l dt\right]^{\frac{1}{l}}$$

$$\leq \eta(\varphi(b),\varphi(a),m)^{p+q+1}\Big[\beta(p+1,q+1)\Big]^{\frac{l-1}{l}}$$

$$\times\left[\int_0^1 t^p(1-t)^q\left(\frac{m\sqrt{t}}{2\sqrt{1-t}}|f(\varphi(b))|^l+\frac{m\sqrt{1-t}}{2\sqrt{t}}|f(\varphi(a))|^l\right)dt\right]^{\frac{1}{l}}$$

$$= \left(\frac{m}{2}\right)^{\frac{1}{l}}\eta(\varphi(b),\varphi(a),m)^{p+q+1}\Big[\beta(p+1,q+1)\Big]^{\frac{l-1}{l}}$$

$$\times\left[|f(\varphi(a))|^l\beta\left(p+\frac{1}{2},q+\frac{3}{2}\right)+|f(\varphi(b))|^l\beta\left(p+\frac{3}{2},q+\frac{1}{2}\right)\right]^{\frac{1}{l}}. \qquad \square$$

3. Hermite-Hadamard type fractional integral inequalities for $MT_{(m,\varphi)}$-preinvex functions

In this section, in order to prove our main results regarding some generalizations of Hermite-Hadamard type inequalities for $MT_{(m,\varphi)}$-preinvex functions via fractional integrals, we need the following new fractional integral identity:

Lemma 3.1. *Let $\varphi : I \longrightarrow K$ be a continuous function. Suppose $K \subseteq \mathbb{R}$ be an open m-invex subset with respect to $\eta : K\times K\times(0,1] \longrightarrow K$ for some fixed $m \in (0,1]$ and let $\eta(\varphi(b),\varphi(a),m) > 0$. Assume that $f : K \longrightarrow \mathbb{R}$ be a twice differentiable function on K^0 and f'' is integrable on $[m\varphi(a), m\varphi(a)+\eta(\varphi(b),\varphi(a),m)]$. Then for $\alpha > 0$, we have*

$$\frac{-\eta^{\alpha+1}(\varphi(x),\varphi(a),m)f'(m\varphi(a))-\eta^{\alpha+1}(\varphi(x),\varphi(b),m)f'(m\varphi(b))}{(\alpha+1)\eta(\varphi(b),\varphi(a),m)}$$

$$+\frac{\eta^{\alpha}(\varphi(x),\varphi(a),m)f(m\varphi(a)+\eta(\varphi(x),\varphi(a),m))+\eta^{\alpha}(\varphi(x),\varphi(b),m)f(m\varphi(b)+\eta(\varphi(x),\varphi(b),m))}{\eta(\varphi(b),\varphi(a),m)}$$

$$-\frac{\Gamma(\alpha+1)}{\eta(\varphi(b),\varphi(a),m)}$$

$$\times\Big[J^{\alpha}_{(m\varphi(a)+\eta(\varphi(x),\varphi(a),m))-}f(m\varphi(a))+J^{\alpha}_{(m\varphi(b)+\eta(\varphi(x),\varphi(b),m))-}f(m\varphi(b))\Big]$$

$$=\frac{\eta^{\alpha+2}(\varphi(x),\varphi(a),m)}{(\alpha+1)\eta(\varphi(b),\varphi(a),m)}\int_0^1(1-t^{\alpha+1})f''(m\varphi(a)+t\eta(\varphi(x),\varphi(a),m))dt$$

$$+\frac{\eta^{\alpha+2}(\varphi(x),\varphi(b),m)}{(\alpha+1)\eta(\varphi(b),\varphi(a),m)}\int_0^1(1-t^{\alpha+1})f''(m\varphi(b)+t\eta(\varphi(x),\varphi(b),m))dt, \tag{3.1}$$

where $\Gamma(\alpha)=\int_0^{+\infty}e^{-u}u^{\alpha-1}du$ *is the Euler Gamma function.*

Proof. A simple proof of the equality can be done by performing two integration by parts in the integrals from the right side and changing the variable. The details are left to the interested reader. □

Let us denote

$$I_{f,\eta,\varphi}(x;\alpha,m,a,b)$$

$$=\frac{\eta^{\alpha+2}(\varphi(x),\varphi(a),m)}{(\alpha+1)\eta(\varphi(b),\varphi(a),m)}\int_0^1(1-t^{\alpha+1})f''(m\varphi(a)+t\eta(\varphi(x),\varphi(a),m))dt$$

$$+\frac{\eta^{\alpha+2}(\varphi(x),\varphi(b),m)}{(\alpha+1)\eta(\varphi(b),\varphi(a),m)}\int_0^1(1-t^{\alpha+1})f''(m\varphi(b)+t\eta(\varphi(x),\varphi(b),m))dt. \tag{3.2}$$

Using Lemma 3.1 and the relation (3.2), the following results can be obtained for the corresponding version for power of the absolute value of the second derivative.

Theorem 3.2. *Let* $\varphi: I\longrightarrow A$ *be a continuous function. Suppose* $A\subseteq\mathbb{R}$ *be an open* m*-invex subset with respect to* $\eta: A\times A\times(0,1]\longrightarrow A$ *for some fixed* $m\in(0,1]$ *and let* $\eta(\varphi(b),\varphi(a),m)>0$*. Assume that* $f: A\longrightarrow\mathbb{R}$ *be a twice differentiable function on* A^0*. If* $|f''|^q$ *is a* $MT_{(m,\varphi)}$*-preinvex function on* $[m\varphi(a),m\varphi(a)+\eta(\varphi(b),\varphi(a),m)]$, $q>1$, $p^{-1}+q^{-1}=1$ *and* $|f''|\le M$*, then for* $\alpha>0$*, we have*

$$|I_{f,\eta,\varphi}(x;\alpha,m,a,b)|\le\frac{M}{(1+\alpha)^{1+\frac{1}{p}}}\left(\frac{m\pi}{2}\right)^{\frac{1}{q}}\left(\frac{\Gamma(p+1)\Gamma\left(\frac{1}{\alpha+1}\right)}{\Gamma\left(p+1+\frac{1}{\alpha+1}\right)}\right)^{\frac{1}{p}}$$

$$\times\left[\frac{|\eta(\varphi(x),\varphi(a),m)|^{\alpha+2}+|\eta(\varphi(x),\varphi(b),m)|^{\alpha+2}}{\eta(\varphi(b),\varphi(a),m)}\right]. \tag{3.3}$$

Proof. Suppose that $q>1$. Using Lemma 3.1, $MT_{(m,\varphi)}$-preinvexity of $|f''|^q$, Hölder inequality, the fact that $|f''|\le M$ and taking the modulus, we have

$$|I_{f,\eta,\varphi}(x;\alpha,m,a,b)|$$

$$\le\frac{|\eta(\varphi(x),\varphi(a),m)|^{\alpha+2}}{(\alpha+1)|\eta(\varphi(b),\varphi(a),m)|}\int_0^1|1-t^{\alpha+1}||f''(m\varphi(a)+t\eta(\varphi(x),\varphi(a),m))|dt$$

$$+\frac{|\eta(\varphi(x),\varphi(b),m)|^{\alpha+2}}{(\alpha+1)|\eta(\varphi(b),\varphi(a),m)|}\int_0^1|1-t^{\alpha+1}||f''(m\varphi(b)+t\eta(\varphi(x),\varphi(b),m))|dt$$

$$\le\frac{|\eta(\varphi(x),\varphi(a),m)|^{\alpha+2}}{(\alpha+1)\eta(\varphi(b),\varphi(a),m)}\left(\int_0^1(1-t^{\alpha+1})^p dt\right)^{\frac{1}{p}}$$

$$\times\left(\int_0^1 |f''(m\varphi(a)+t\eta(\varphi(x),\varphi(a),m))|^q dt\right)^{\frac{1}{q}}$$

$$+\frac{|\eta(\varphi(x),\varphi(b),m)|^{\alpha+2}}{(\alpha+1)\eta(\varphi(b),\varphi(a),m)}\left(\int_0^1 (1-t^{\alpha+1})^p dt\right)^{\frac{1}{p}}$$

$$\times\left(\int_0^1 |f''(m\varphi(b)+t\eta(\varphi(x),\varphi(b),m))|^q dt\right)^{\frac{1}{q}}$$

$$\leq \frac{|\eta(\varphi(x),\varphi(a),m)|^{\alpha+2}}{(\alpha+1)\eta(\varphi(b),\varphi(a),m)}\left(\int_0^1 (1-t^{\alpha+1})^p dt\right)^{\frac{1}{p}}$$

$$\times\left[\int_0^1 \left(\frac{m\sqrt{t}}{2\sqrt{1-t}}|f''(\varphi(x))|^q+\frac{m\sqrt{1-t}}{2\sqrt{t}}|f''(\varphi(a))|^q\right)dt\right]^{\frac{1}{q}}$$

$$+\frac{|\eta(\varphi(x),\varphi(b),m)|^{\alpha+2}}{(\alpha+1)\eta(\varphi(b),\varphi(a),m)}\left(\int_0^1 (1-t^{\alpha+1})^p dt\right)^{\frac{1}{p}}$$

$$\times\left[\int_0^1 \left(\frac{m\sqrt{t}}{2\sqrt{1-t}}|f''(\varphi(x))|^q+\frac{m\sqrt{1-t}}{2\sqrt{t}}|f''(\varphi(b))|^q\right)dt\right]^{\frac{1}{q}}$$

$$\leq \frac{M}{(1+\alpha)^{1+\frac{1}{p}}}\left(\frac{m\pi}{2}\right)^{\frac{1}{q}}\left(\frac{\Gamma(p+1)\Gamma\left(\frac{1}{\alpha+1}\right)}{\Gamma\left(p+1+\frac{1}{\alpha+1}\right)}\right)^{\frac{1}{p}}$$

$$\times\left[\frac{|\eta(\varphi(x),\varphi(a),m)|^{\alpha+2}+|\eta(\varphi(x),\varphi(b),m)|^{\alpha+2}}{\eta(\varphi(b),\varphi(a),m)}\right]. \qquad \square$$

Theorem 3.3. *Let $\varphi : I \longrightarrow A$ be a continuous function. Suppose $A \subseteq \mathbb{R}$ be an open m-invex subset with respect to $\eta : A \times A \times (0,1] \longrightarrow A$ for some fixed $m \in (0,1]$ and let $\eta(\varphi(b),\varphi(a),m) > 0$. Assume that $f : A \longrightarrow \mathbb{R}$ be a twice differentiable function on A^0. If $|f''|^q$ is a $MT_{(m,\varphi)}$-preinvex function on $[m\varphi(a), m\varphi(a)+\eta(\varphi(b),\varphi(a),m)]$, $q \geq 1$ and $|f''| \leq M$, then for $\alpha > 0$, we have*

$$|I_{f,\eta,\varphi}(x;\alpha,m,a,b)| \leq \frac{M}{\alpha+1}\left(\frac{\alpha+1}{\alpha+2}\right)^{1-\frac{1}{q}}\left(\frac{m}{2}\right)^{\frac{1}{q}}\left(\pi-\frac{\sqrt{\pi}(\alpha+1)\Gamma\left(\alpha+\frac{3}{2}\right)}{\Gamma(\alpha+3)}\right)^{\frac{1}{q}}$$

$$\times\left[\frac{|\eta(\varphi(x),\varphi(a),m)|^{\alpha+2}+|\eta(\varphi(x),\varphi(b),m)|^{\alpha+2}}{\eta(\varphi(b),\varphi(a),m)}\right]. \tag{3.4}$$

Proof. Suppose that $q \geq 1$. Using Lemma 3.1, $MT_{(m,\varphi)}$-preinvexity of $|f''|^q$, the well-known power mean inequality, the fact that $|f''| \leq M$ and taking the modulus, we have

$$|I_{f,\eta,\varphi}(x;\alpha,m,a,b)|$$

$$\leq \frac{|\eta(\varphi(x),\varphi(a),m)|^{\alpha+2}}{(\alpha+1)|\eta(\varphi(b),\varphi(a),m)|}\int_0^1 |1-t^{\alpha+1}||f''(m\varphi(a)+t\eta(\varphi(x),\varphi(a),m))|dt$$

$$
\begin{aligned}
&+\frac{|\eta(\varphi(x),\varphi(b),m)|^{\alpha+2}}{(\alpha+1)|\eta(\varphi(b),\varphi(a),m)|}\int_0^1 |1-t^{\alpha+1}||f''(m\varphi(b)+t\eta(\varphi(x),\varphi(b),m))|dt\\
&\le \frac{|\eta(\varphi(x),\varphi(a),m)|^{\alpha+2}}{(\alpha+1)\eta(\varphi(b),\varphi(a),m)}\left(\int_0^1(1-t^{\alpha+1})dt\right)^{1-\frac{1}{q}}\\
&\times\left(\int_0^1(1-t^{\alpha+1})|f''(m\varphi(a)+t\eta(\varphi(x),\varphi(a),m))|^q dt\right)^{\frac{1}{q}}\\
&+\frac{|\eta(\varphi(x),\varphi(b),m)|^{\alpha+2}}{(\alpha+1)\eta(\varphi(b),\varphi(a),m)}\left(\int_0^1(1-t^{\alpha+1})dt\right)^{1-\frac{1}{q}}\\
&\times\left(\int_0^1(1-t^{\alpha+1})|f''(m\varphi(b)+t\eta(\varphi(x),\varphi(b),m))|^q dt\right)^{\frac{1}{q}}\\
&\le \frac{|\eta(\varphi(x),\varphi(a),m)|^{\alpha+2}}{(\alpha+1)\eta(\varphi(b),\varphi(a),m)}\left(\int_0^1(1-t^{\alpha+1})dt\right)^{1-\frac{1}{q}}\\
&\times\left[\int_0^1(1-t^{\alpha+1})\left(\frac{m\sqrt{t}}{2\sqrt{1-t}}|f''(\varphi(x))|^q+\frac{m\sqrt{1-t}}{2\sqrt{t}}|f''(\varphi(a))|^q\right)dt\right]^{\frac{1}{q}}\\
&+\frac{|\eta(\varphi(x),\varphi(b),m)|^{\alpha+2}}{(\alpha+1)\eta(\varphi(b),\varphi(a),m)}\left(\int_0^1(1-t^{\alpha+1})dt\right)^{1-\frac{1}{q}}\\
&\times\left[\int_0^1(1-t^{\alpha+1})\left(\frac{m\sqrt{t}}{2\sqrt{1-t}}|f''(\varphi(x))|^q+\frac{m\sqrt{1-t}}{2\sqrt{t}}|f''(\varphi(b))|^q\right)dt\right]^{\frac{1}{q}}\\
&\le \frac{M}{\alpha+1}\left(\frac{\alpha+1}{\alpha+2}\right)^{1-\frac{1}{q}}\left(\frac{m}{2}\right)^{\frac{1}{q}}\left(\pi-\frac{\sqrt{\pi}(\alpha+1)\Gamma\left(\alpha+\frac{3}{2}\right)}{\Gamma(\alpha+3)}\right)^{\frac{1}{q}}\\
&\times\left[\frac{|\eta(\varphi(x),\varphi(a),m)|^{\alpha+2}+|\eta(\varphi(x),\varphi(b),m)|^{\alpha+2}}{\eta(\varphi(b),\varphi(a),m)}\right]. \qquad \square
\end{aligned}
$$

4. Applications to special means

In the following we give certain generalizations of some notions for a positive valued function of a positive variable.

Definition 4.1. (see [2]) A function $M : \mathbb{R}_+^2 \longrightarrow \mathbb{R}_+$, is called a Mean function if it has the following properties:

1. Homogeneity: $M(ax, ay) = aM(x, y)$, for all $a > 0$,
2. Symmetry: $M(x, y) = M(y, x)$,
3. Reflexivity: $M(x, x) = x$,
4. Monotonicity: If $x \le x'$ and $y \le y'$, then $M(x, y) \le M(x', y')$,
5. Internality: $\min\{x, y\} \le M(x, y) \le \max\{x, y\}$.

We consider some means for arbitrary positive real numbers α, β $(\alpha \neq \beta)$.

1. The arithmetic mean:
$$A := A(\alpha,\beta) = \frac{\alpha+\beta}{2}$$
2. The geometric mean:
$$G := G(\alpha,\beta) = \sqrt{\alpha\beta}$$
3. The harmonic mean:
$$H := H(\alpha,\beta) = \frac{2}{\frac{1}{\alpha}+\frac{1}{\beta}}$$
4. The power mean:
$$P_r := P_r(\alpha,\beta) = \left(\frac{\alpha^r+\beta^r}{2}\right)^{\frac{1}{r}}, \quad r \geq 1.$$
5. The identric mean:
$$I := I(\alpha,\beta) = \begin{cases} \frac{1}{e}\left(\frac{\beta^\beta}{\alpha^\alpha}\right), & \alpha \neq \beta; \\ \alpha, & \alpha = \beta. \end{cases}$$
6. The logarithmic mean:
$$L := L(\alpha,\beta) = \frac{\beta-\alpha}{\ln(\beta)-\ln(\alpha)}.$$
7. The generalized log-mean:
$$L_p := L_p(\alpha,\beta) = \left[\frac{\beta^{p+1}-\alpha^{p+1}}{(p+1)(\beta-\alpha)}\right]^{\frac{1}{p}}; \quad p \in \mathbb{R}\setminus\{-1,0\}.$$
8. The weighted p-power mean:
$$M_p\begin{pmatrix} \alpha_1, & \alpha_2, & \cdots & ,\alpha_n \\ u_1, & u_2, & \cdots & ,u_n \end{pmatrix} = \left(\sum_{i=1}^{n}\alpha_i u_i^p\right)^{\frac{1}{p}}$$
where $0 \leq \alpha_i \leq 1$, $u_i > 0\,(i=1,2,\dots,n)$ with $\sum_{i=1}^{n}\alpha_i = 1$.

It is well known that L_p is monotonic nondecreasing over $p \in \mathbb{R}$ with $L_{-1} := L$ and $L_0 := I$. In particular, we have the following inequality $H \leq G \leq L \leq I \leq A$. Now, let a and b be positive real numbers such that $a < b$. Consider the function $M := M(\varphi(a),\varphi(b)) : [\varphi(a),\varphi(a)+\eta(\varphi(b),\varphi(a))] \times [\varphi(a),\varphi(a)+\eta(\varphi(b),\varphi(a))] \longrightarrow \mathbb{R}_+$, which is one of the above mentioned means and $\varphi : I \longrightarrow A$ be a continuous function, therefore one can obtain various inequalities using the results of Section 3 for these means as follows: Replace $\eta(\varphi(x),\varphi(y),m)$ with $\eta(\varphi(x),\varphi(y))$ and setting $\eta(\varphi(x),\varphi(y)) = M(\varphi(x),\varphi(y))$, $\forall x,y \in I$ for value $m=1$ in (3.3) and (3.4), one can obtain the following interesting inequalities involving means:

$$\Bigg| \frac{-M^{\alpha+1}(\varphi(a),\varphi(x))f'(\varphi(a)) - M^{\alpha+1}(\varphi(b),\varphi(x))f'(\varphi(b))}{(\alpha+1)M(\varphi(a),\varphi(b))}$$
$$+\frac{M^{\alpha}(\varphi(a),\varphi(x))f(\varphi(a)+M(\varphi(a),\varphi(x))) + M^{\alpha}(\varphi(b),\varphi(x))f(\varphi(b)+M(\varphi(b),\varphi(x)))}{M(\varphi(a),\varphi(b))}$$

$$-\frac{\Gamma(\alpha+1)}{M(\varphi(a),\varphi(b))}\left[J^{\alpha}_{(\varphi(a)+M(\varphi(a),\varphi(x)))-}f(\varphi(a))+J^{\alpha}_{(\varphi(b)+M(\varphi(b),\varphi(x)))-}f(\varphi(b))\right]\Bigg|$$

$$\leq \frac{M}{(1+\alpha)^{1+\frac{1}{p}}}\left(\frac{\pi}{2}\right)^{\frac{1}{q}}\left(\frac{\Gamma(p+1)\Gamma\left(\frac{1}{\alpha+1}\right)}{\Gamma\left(p+1+\frac{1}{\alpha+1}\right)}\right)^{\frac{1}{p}}$$

$$\times\left[\frac{M^{\alpha+2}(\varphi(a),\varphi(x))+M^{\alpha+2}(\varphi(b),\varphi(x))}{M(\varphi(a),\varphi(b))}\right], \tag{4.1}$$

$$\Bigg|\frac{-M^{\alpha+1}(\varphi(a),\varphi(x))f'(\varphi(a))-M^{\alpha+1}(\varphi(b),\varphi(x))f'(\varphi(b))}{(\alpha+1)M(\varphi(a),\varphi(b))}$$

$$+\frac{M^{\alpha}(\varphi(a),\varphi(x))f(\varphi(a)+M(\varphi(a),\varphi(x)))+M^{\alpha}(\varphi(b),\varphi(x))f(\varphi(b)+M(\varphi(b),\varphi(x)))}{M(\varphi(a),\varphi(b))}$$

$$-\frac{\Gamma(\alpha+1)}{M(\varphi(a),\varphi(b))}\left[J^{\alpha}_{(\varphi(a)+M(\varphi(a),\varphi(x)))-}f(\varphi(a))+J^{\alpha}_{(\varphi(b)+M(\varphi(b),\varphi(x)))-}f(\varphi(b))\right]\Bigg|$$

$$\leq \frac{M}{\alpha+1}\left(\frac{\alpha+1}{\alpha+2}\right)^{1-\frac{1}{q}}\left(\frac{1}{2}\right)^{\frac{1}{q}}\left(\pi-\frac{\sqrt{\pi}(\alpha+1)\Gamma\left(\alpha+\frac{3}{2}\right)}{\Gamma(\alpha+3)}\right)^{\frac{1}{q}}$$

$$\times\left[\frac{M^{\alpha+2}(\varphi(a),\varphi(x))+M^{\alpha+2}(\varphi(b),\varphi(x))}{M(\varphi(a),\varphi(b))}\right]. \tag{4.2}$$

Letting $M(\varphi(x),\varphi(y))=A,G,H,P_r,I,L,L_p,M_p,\quad \forall x,y\in I$ in (4.1) and (4.2), we get the inequalities involving means for a particular choices of a twice differentiable $MT_{(1,\varphi)}$-preinvex function f. The details are left to the interested reader.

These general inequalities give us some new estimates for the left-hand side of Gauss-Jacobi type quadrature formula and Hermite-Hadamard type fractional integral inequalities.

Motivated by this new interesting class of $MT_{(m,\varphi)}$-preinvex functions we can indeed see to be vital for fellow researchers and scientists working in the same domain.

We conclude that our methods considered here may be a stimulant for further investigations concerning Hermite-Hadamard type integral inequalities for various kinds of preinvex functions involving classical integrals, Riemann-Liouville fractional integrals, k-fractional integrals and conformable fractional integrals.

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Hermite-Hadamard type inequalities for product of GA-convex functions via Hadamard fractional integrals

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Abstract. In this paper, some Hermite-Hadamard type inequalities for products of two GA-convex functions via Hadamard fractional integrals are established. Our results about GA-convex functions are analogous generalizations for some other results proved by Pachpette for convex functions.

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1. Introduction

In recent years, very large number of studies of error estimations have been done for Hermite-Hadamard type inequalities. It is known that Hermite-hadamard integral inequality was built on a convex function. In time, Hermite-Hadamrd inequality is developed other kinds of convex functions. For some results which generalize, improve, and extend the Hermite-Hadamard inequality see [1, 7, 10, 18, 20] and references therein.

Hermite-Hadamard type inequalities for products of two convex functions are interesting problem and firstly developed by Pachpatte in [16]. In [17], Pachpette also established Hermite-hadamard type inequalities involving two log-convex functions. In [11], Kırmacı et. al. proved several Hermite-Hadamard type inequalities for products of two convex and s-convex functions. In [19], Sarıkaya et. al. proved some Hermite-Hadamard type inequalities for products of two h-convex functions. In [2], Bakula et. al. established Hermite-Hadamard type inequalities for products of two m-convex and (α, m)-convex functions. In [4, 6], Chen and Wu obtained some Hermite-Hadamard type inequalities for products of two convex and harmonically s-convex functions. In [21], Yin and Qi established some Hermite-Hadamard type inequalities for products of two convex functions. In [5], Chen obtained some new Hermite-Hadamard type inequalities for products of two convex functions via Riemann-Liouville fractional integrals and in [3] he extended this problem to m-convex and (α, m)-convex functions.

In this work, we establish Hermite-Hadamard type inequalities for products of two GA-convex functions via Hadamard fractional integrals. Our results are analogous generalization for some results in [16].

2. Preliminaries

Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function defined on the interval I of real numbers and $a, b \in I$ with $a < b$. The inequality

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2} \tag{2.1}$$

is well known in the literature as Hermite-Hadamard's inequality [8].

In [16], Pachpette established following two Hermite-Hadamard type inequalities for products of convex functions as follows:

Theorem 2.1. *Let f and g be real-valued, non-negative and convex functions on $[a, b]$. Then*

$$\frac{1}{b-a}\int_a^b f(x)\,g(x)\,dx \leq \frac{1}{3}M(a,b) + \frac{1}{6}N(a,b) \tag{2.2}$$

and

$$2f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\,g(x)\,dx + \frac{1}{6}M(a,b) + \frac{1}{3}N(a,b) \tag{2.3}$$

where $M(a,b) = f(a)\,g(a) + f(b)\,g(b)$ and $N(a,b) = f(a)\,g(b) + f(b)\,g(a)$.

Definition 2.2. [14, 15]. A function $f : I \subseteq (0, \infty) \to \mathbb{R}$ is said to be GA-convex (geometric-arithmetically convex) if

$$f(x^t y^{1-t}) \leq t f(x) + (1-t)\,f(y)$$

for all $x, y \in I$ and $t \in [0, 1]$.

We will now give definitions of the right-hand side and left-hand side Hadamard fractional integrals which are used throughout this paper.

Definition 2.3. [12]. Let $f \in L[a, b]$. The right-hand side and left-hand side Hadamard fractional integrals $J_{a+}^{\alpha}f$ and $J_{b-}^{\alpha}f$ of order $\alpha > 0$ with $b > a \geq 0$ are defined by

$$J_{a+}^{\alpha}f(x) = \frac{1}{\Gamma(\alpha)}\int_a^x \left(\ln\frac{x}{t}\right)^{\alpha-1} f(t)\frac{dt}{t},\ x > a$$

and

$$J_{b-}^{\alpha}f(x) = \frac{1}{\Gamma(\alpha)}\int_x^b \left(\ln\frac{t}{x}\right)^{\alpha-1} f(t)\frac{dt}{t},\ x < b$$

respectively, where $\Gamma(\alpha)$ is the Gamma function defined by

$$\Gamma(\alpha) = \int_0^\infty e^{-t} t^{\alpha-1} dt.$$

In [9], İşcan represented Hermite-Hadamard's inequalities for GA-convex functions in fractional integral forms as follows.

Theorem 2.4. *Let $f : I \subseteq (0,\infty) \to \mathbb{R}$ be a function such that $f \in L[a,b]$ where $a, b \in I$ with $a < b$. If f is a GA-convex function on $[a,b]$, then the following inequalities for fractional integrals hold:*

$$f\left(\sqrt{ab}\right) \leq \frac{\Gamma(\alpha+1)}{2\left(\ln \frac{b}{a}\right)^{\alpha}} \left[J_{a+}^{\alpha} f(b) + J_{b-}^{\alpha} f(a)\right] \leq \frac{f(a)+f(b)}{2} \tag{2.4}$$

with $\alpha > 0$.

In [13], Kunt and İşcan established new Hermite-Hadamard type inequality for GA-convex function in fractional integral forms as follows:

Theorem 2.5. *Let $f : [a,b] \subseteq (0,\infty) \to \mathbb{R}$ be a GA-convex function with $a < b$ and $f \in L[a,b]$, then the following inequalities for fractional integrals hold:*

$$f\left(\sqrt{ab}\right) \leq \frac{\Gamma(\alpha+1)}{2^{1-\alpha}\left(\ln \frac{b}{a}\right)^{\alpha}} \left[J_{\sqrt{ab}-}^{\alpha} f(a) + J_{\sqrt{ab}+}^{\alpha} f(b)\right] \leq \frac{f(a)+f(b)}{2}. \tag{2.5}$$

3. General results

Theorem 3.1. *Let f and $g : [a,b] \subseteq (0,\infty) \to \mathbb{R}$ be non-negative and GA-convex functions with $a < b$ and $f \in L[a,b]$, then the following inequality for fractional integrals hold:*

$$\begin{aligned} &\frac{\Gamma(\alpha+1)}{2\left(\ln \frac{b}{a}\right)^{\alpha}} \left[J_{a+}^{\alpha} f(b)\, g(b) + J_{b-}^{\alpha} f(a)\, g(a)\right] \\ \leq\ & \left(\frac{\alpha}{\alpha+2} - \frac{\alpha}{\alpha+1} + \frac{1}{2}\right) M(a,b) + \frac{\alpha}{(\alpha+2)(\alpha+1)} N(a,b) \end{aligned} \tag{3.1}$$

where $\alpha > 0$, $M(a,b) = f(a)\, g(a) + f(b)\, g(b)$ and $N(a,b) = f(a)\, g(b) + f(b)\, g(a)$.

Proof. Since f and g are non-negative and GA-convex functions on $[a,b]$, we have for all $t \in [0,1]$

$$f(a^t b^{1-t}) \leq t f(a) + (1-t)\, f(b), \tag{3.2}$$

and

$$g(a^t b^{1-t}) \leq t g(a) + (1-t)\, g(b). \tag{3.3}$$

From products of (3.2) and (3.3), we have

$$\begin{aligned} f(a^t b^{1-t}) g(a^t b^{1-t}) \leq\ & t^2 f(a)\, g(a) + (1-t)^2 f(b)\, g(b) \\ & + t(1-t)\left[f(a)\, g(b) + f(b)\, g(a)\right]. \end{aligned} \tag{3.4}$$

Similarly (3.4), we have

$$f(a^{1-t}b^{t})g(a^{1-t}b^{t}) \leq (1-t)^2 f(a)g(a) + t^2 f(b)g(b) + t(1-t)[f(a)g(b) + f(b)g(a)]. \tag{3.5}$$

The sum of (3.4) and (3.5), we have

$$\begin{aligned} & f(a^{t}b^{1-t})g(a^{t}b^{1-t}) + f(a^{1-t}b^{t})g(a^{1-t}b^{t}) \\ \leq & \left(2t^2-2t+1\right)M(a,b) + 2t(1-t)N(a,b) \end{aligned} \tag{3.6}$$

Multiplying both sides of (3.6) by $t^{\alpha-1}\frac{\alpha}{2}$, then integrating the obtained inequality with respect to t over $[0,1]$, we have

$$\begin{aligned} & \frac{\alpha}{2}\left[\int_0^1 t^{\alpha-1} f(a^{t}b^{1-t})g(a^{t}b^{1-t})dt + \int_0^1 t^{\alpha-1} f(a^{1-t}b^{t})g(a^{1-t}b^{t})dt\right] \\ = & \frac{\alpha}{2}\left[\int_a^b \left(\frac{\ln\frac{b}{u}}{\ln\frac{b}{a}}\right)^{\alpha-1} f(u)g(u)\frac{du}{u\ln\frac{b}{a}} + \int_a^b \left(\frac{\ln\frac{v}{a}}{\ln\frac{b}{a}}\right)^{\alpha-1} f(v)g(v)\frac{du}{v\ln\frac{b}{a}}\right] \\ = & \frac{\alpha}{2\left(\ln\frac{b}{a}\right)^{\alpha}}\left[\int_a^b \left(\ln\frac{b}{u}\right)^{\alpha-1} f(u)g(u)\frac{du}{u} + \int_a^b \left(\ln\frac{v}{a}\right)^{\alpha-1} f(v)g(v)\frac{du}{v}\right] \\ = & \frac{\Gamma(\alpha+1)}{2\left(\ln\frac{b}{a}\right)^{\alpha}}\left[J_{a+}^{\alpha}f(b)g(b) + J_{b-}^{\alpha}f(a)g(a)\right] \\ \leq & \frac{\alpha}{2}\left[M(a,b)\int_0^1 t^{\alpha-1}\left(2t^2-2t+1\right)dt + N(a,b)\int_0^1 t^{\alpha-1}2t(1-t)\,dt\right] \\ = & \left(\frac{\alpha}{\alpha+2} - \frac{\alpha}{\alpha+1} + \frac{1}{2}\right)M(a,b) + \frac{\alpha}{(\alpha+2)(\alpha+1)}N(a,b) \end{aligned}$$

and this completes the proof. □

Remark 3.2. Theorem 3.1 is an analogous generalization of (2.2) for GA-convex functions.

Corollary 3.3. *In Theorem* 3.1, *if we take* $g:[a,b]\to\mathbb{R}$ *as* $g(x)=1$ *for all* $x\in[a,b]$, *then we have*

$$\frac{\Gamma(\alpha+1)}{2\left(\ln\frac{b}{a}\right)^{\alpha}}\left[J_{a+}^{\alpha}f(b) + J_{b-}^{\alpha}f(a)\right] \leq \frac{f(a)+f(b)}{2}$$

which is the right hand side of (2.4).

Corollary 3.4. *In Theorem* 3.1, *if we take* $\alpha=1$, *then we have*

$$\frac{1}{\ln b - \ln a}\int_a^b f(x)g(x)\frac{dx}{x} \leq \frac{1}{3}M(a,b) + \frac{1}{6}N(a,b)$$

for GA-convex functions.

Theorem 3.5. *Let f and $g : [a,b] \subseteq (0,\infty) \to \mathbb{R}$ be non-negative and GA-convex functions with $a < b$ and $f \in L[a,b]$, then the following inequality for fractional integrals hold:*

$$2f\left(\sqrt{ab}\right)g\left(\sqrt{ab}\right) \leq \frac{\Gamma(\alpha+1)}{2\left(\ln\frac{b}{a}\right)^{\alpha}}\left[J_{a+}^{\alpha}f(b)g(b)+J_{b-}^{\alpha}f(a)g(a)\right]$$
$$+\frac{\alpha}{(\alpha+2)(\alpha+1)}M(a,b)+\left(\frac{\alpha}{\alpha+2}-\frac{\alpha}{\alpha+1}+\frac{1}{2}\right)N(a,b) \tag{3.7}$$

where $\alpha > 0$, $M(a,b) = f(a)g(a) + f(b)g(b)$ and $N(a,b) = f(a)g(b) + f(b)g(a)$.

Proof. It is clear for all $t \in [0,1]$

$$\sqrt{ab} = \sqrt{a^t b^{1-t}.a^{1-t}b^t} = \sqrt{a^t b^{1-t}}\sqrt{a^{1-t}b^t}.$$

Since f and g are non-negative and GA-convex functions on $[a,b]$, we have for all $t \in [0,1]$

$$\begin{aligned}
f\left(\sqrt{ab}\right)g\left(\sqrt{ab}\right) &= f\left(\sqrt{a^tb^{1-t}}\sqrt{a^{1-t}b^t}\right)g\left(\sqrt{a^tb^{1-t}}\sqrt{a^{1-t}b^t}\right)\\
&\leq \frac{1}{4}\left[f\left(a^tb^{1-t}\right)+f\left(a^{1-t}b^t\right)\right]\left[g\left(a^tb^{1-t}\right)+g\left(a^{1-t}b^t\right)\right]\\
&= \frac{1}{4}\left[f\left(a^tb^{1-t}\right)g\left(a^tb^{1-t}\right)+f\left(a^{1-t}b^t\right)g\left(a^{1-t}b^t\right)\right]\\
&\quad+\frac{1}{4}\left[f\left(a^tb^{1-t}\right)g\left(a^{1-t}b^t\right)+f\left(a^{1-t}b^t\right)g\left(a^tb^{1-t}\right)\right]\\
&\leq \frac{1}{4}\left[f\left(a^tb^{1-t}\right)g\left(a^tb^{1-t}\right)+f\left(a^{1-t}b^t\right)g\left(a^{1-t}b^t\right)\right]\\
&\quad+\frac{1}{4}\left[tf(a)+(1-t)f(b)\right]\left[(1-t)g(a)+tg(b)\right]\\
&\quad+\frac{1}{4}\left[(1-t)f(a)+tf(b)\right]\left[tg(a)+(1-t)g(b)\right]\\
&= \frac{1}{4}\left[f\left(a^tb^{1-t}\right)g\left(a^tb^{1-t}\right)+f\left(a^{1-t}b^t\right)g\left(a^{1-t}b^t\right)\right]\\
&\quad+\frac{1}{4}\left\{2t(1-t)\left[f(a)g(a)+f(b)g(b)\right]\right.\\
&\quad\left.+\left(2t^2-2t+1\right)\left[f(a)g(b)+f(b)g(a)\right]\right\}
\end{aligned} \tag{3.8}$$

Multiplying both sides of (3.8) by $2\alpha t^{\alpha-1}$, then integrating the obtained inequality with respect to t over $[0,1]$, we have

$$\begin{aligned}
2f\left(\sqrt{ab}\right)g\left(\sqrt{ab}\right) &\leq \frac{\Gamma(\alpha+1)}{2\left(\ln\frac{b}{a}\right)^{\alpha}}\left[J_{a+}^{\alpha}f(b)g(b)+J_{b-}^{\alpha}f(a)g(a)\right]\\
&\quad+\frac{\alpha}{(\alpha+2)(\alpha+1)}M(a,b)+\left(\frac{\alpha}{\alpha+2}-\frac{\alpha}{\alpha+1}+\frac{1}{2}\right)N(a,b)
\end{aligned}$$

and this completes the proof. □

Remark 3.6. Theorem 3.5 is an analogous generalization of (2.3) for GA-convex functions.

Corollary 3.7. *In Theorem 3.5, if we take $g : [a,b] \to \mathbb{R}$ as $g(x) = 1$ for all $x \in [a,b]$, then we have*

$$2f\left(\sqrt{ab}\right) \le \frac{\Gamma(\alpha+1)}{2\left(\ln\frac{b}{a}\right)^{\alpha}}\left[J_{a+}^{\alpha} f(b) + J_{b-}^{\alpha} f(a)\right] + \frac{f(a)+f(b)}{2}.$$

Corollary 3.8. *In Theorem 3.5, if we take $\alpha = 1$, then we have*

$$2f\left(\sqrt{ab}\right) g\left(\sqrt{ab}\right) \le \frac{1}{\ln b - \ln a}\int_a^b f(x)\, g(x)\frac{dx}{x} + \frac{1}{6}M(a,b) + \frac{1}{3}N(a,b)$$

for GA-convex functions.

Theorem 3.9. *Let f and $g : [a,b] \subseteq (0,\infty) \to \mathbb{R}$ be non-negative and GA-convex functions with $a < b$ and $f \in L[a,b]$, then the following inequality for fractional integrals hold:*

$$\begin{aligned} &\frac{\Gamma(\alpha+1)}{2^{1-\alpha}\left(\ln\frac{b}{a}\right)^{\alpha}}\left[J_{\sqrt{ab}-}^{\alpha} f(a)\, g(a) + J_{\sqrt{ab}+}^{\alpha} f(b)\, g(b)\right] \\ \le\ &\left(\frac{\alpha}{4(\alpha+2)} - \frac{\alpha}{2(\alpha+1)} + \frac{1}{2}\right) M(a,b) + \frac{\alpha^2+3\alpha}{4(\alpha+2)(\alpha+1)} N(a,b) \end{aligned} \tag{3.9}$$

where $\alpha > 0$, $M(a,b) = f(a)\, g(a) + f(b)\, g(b)$ and $N(a,b) = f(a)\, g(b) + f(b)\, g(a)$.

Proof. Since f and g are non-negative and GA-convex functions on $[a,b]$, multiplying both sides of (3.6) by $t^{\alpha-1}\frac{\alpha}{2^{1-\alpha}}$, then integrating the obtained inequality with respect to t over $\left[0,\frac{1}{2}\right]$, we have

$$\begin{aligned} &\frac{\alpha}{2^{1-\alpha}}\left[\int_0^{\frac{1}{2}} t^{\alpha-1} f(a^t b^{1-t}) g(a^t b^{1-t}) dt + \int_0^{\frac{1}{2}} t^{\alpha-1} f(a^{1-t} b^t) g(a^{1-t} b^t) dt\right] \\ =\ &\frac{\alpha}{2^{1-\alpha}}\left[\int_{\sqrt{ab}}^{b}\left(\frac{\ln\frac{b}{u}}{\ln\frac{b}{a}}\right)^{\alpha-1} f(u)\, g(u)\frac{du}{u\ln\frac{b}{a}} + \int_a^{\sqrt{ab}}\left(\frac{\ln\frac{v}{a}}{\ln\frac{b}{a}}\right)^{\alpha-1} f(v)\, g(v)\frac{du}{v\ln\frac{b}{a}}\right] \\ =\ &\frac{\alpha}{2^{1-\alpha}\left(\ln\frac{b}{a}\right)^{\alpha}}\left[\int_{\sqrt{ab}}^{b}\left(\ln\frac{b}{u}\right)^{\alpha-1} f(u)\, g(u)\frac{du}{u} + \int_a^{\sqrt{ab}}\left(\ln\frac{v}{a}\right)^{\alpha-1} f(v)\, g(v)\frac{du}{v}\right] \\ =\ &\frac{\Gamma(\alpha+1)}{2^{1-\alpha}\left(\ln\frac{b}{a}\right)^{\alpha}}\left[J_{\sqrt{ab}+}^{\alpha} f(b)\, g(b) + J_{\sqrt{ab}-}^{\alpha} f(a)\, g(a)\right] \\ \le\ &\frac{\alpha}{2^{1-\alpha}}\left[M(a,b)\int_0^{\frac{1}{2}} t^{\alpha-1}\left(2t^2-2t+1\right) dt + N(a,b)\int_0^{\frac{1}{2}} t^{\alpha-1} 2t(1-t)\, dt\right] \\ =\ &\left(\frac{\alpha}{4(\alpha+2)} - \frac{\alpha}{2(\alpha+1)} + \frac{1}{2}\right) M(a,b) + \frac{\alpha^2+3\alpha}{4(\alpha+2)(\alpha+1)} N(a,b) \end{aligned}$$

and this completes the proof. □

Remark 3.10. Theorem 3.9 is an other analogous generalization of (2.2) for GA-convex functions.

Corollary 3.11. *In Theorem* 3.9*, if we take* $g : [a, b] \to \mathbb{R}$ *as* $g(x) = 1$ *for all* $x \in [a, b]$*, then we have*

$$\frac{\Gamma(\alpha+1)}{2^{1-\alpha}\left(\ln\frac{b}{a}\right)^{\alpha}}\left[J^{\alpha}_{\sqrt{ab}-}f(a) + J^{\alpha}_{\sqrt{ab}+}f(b)\right] \leq \frac{f(a)+f(b)}{2}$$

which is the right hand side of (2.5)*.*

Corollary 3.12. *In Theorem* 3.9*, if we take* $\alpha = 1$*, then we have*

$$\frac{1}{\ln b - \ln a}\int_a^b f(x)\,g(x)\,\frac{dx}{x} \leq \frac{1}{3}M(a,b) + \frac{1}{6}N(a,b)$$

for GA-convex functions.

Theorem 3.13. *Let* f *and* $g : [a, b] \subseteq (0,\infty) \to \mathbb{R}$ *be non-negative and GA-convex functions with* $a < b$ *and* $f \in L[a, b]$*, then the following inequality for fractional integrals hold:*

$$2f\left(\sqrt{ab}\right)g\left(\sqrt{ab}\right) \leq \frac{\Gamma(\alpha+1)}{2^{1-\alpha}\left(\ln\frac{b}{a}\right)^{\alpha}}\left[J^{\alpha}_{\sqrt{ab}-}f(a)\,g(a) + J^{\alpha}_{\sqrt{ab}+}f(b)\,g(b)\right]$$
$$+\frac{\alpha^2+3\alpha}{4(\alpha+2)(\alpha+1)}M(a,b) + \left(\frac{\alpha}{4(\alpha+2)} - \frac{\alpha}{2(\alpha+1)} + \frac{1}{2}\right)N(a,b) \qquad (3.10)$$

where $\alpha > 0$*,* $M(a,b) = f(a)\,g(a) + f(b)\,g(b)$ *and* $N(a,b) = f(a)\,g(b) + f(b)\,g(a)$*.*

Proof. Multiplying both sides of (3.8) by $2^{1+\alpha}\alpha t^{\alpha-1}$, then integrating the obtained inequality with respect to t over $\left[0, \frac{1}{2}\right]$, we have desired result. □

Remark 3.14. Theorem 3.13 is an other analogous generalization of (2.3) for GA-convex functions.

Corollary 3.15. *In Theorem* 3.13*, if we take* $g : [a, b] \to \mathbb{R}$ *as* $g(x) = 1$ *for all* $x \in [a, b]$*, then we have*

$$2f\left(\sqrt{ab}\right) \leq \frac{\Gamma(\alpha+1)}{2^{1-\alpha}\left(\ln\frac{b}{a}\right)^{\alpha}}\left[J^{\alpha}_{\sqrt{ab}-}f(a) + J^{\alpha}_{\sqrt{ab}+}f(b)\right] + \frac{f(a)+f(b)}{2}.$$

Corollary 3.16. *In Theorem* 3.13*, if we take* $\alpha = 1$*, then we have*

$$2f\left(\sqrt{ab}\right)g\left(\sqrt{ab}\right) \leq \frac{1}{\ln b - \ln a}\int_a^b f(x)\,g(x)\,\frac{dx}{x} + \frac{1}{6}M(a,b) + \frac{1}{3}N(a,b)$$

for GA-convex functions.

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Modified Hadamard product properties of certain class of analytic functions with varying arguments defined by Sălăgean and Ruscheweyh derivative

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Abstract. In this paper we study the modified Hadamard product properties of certain class of analytic functions with varying arguments defined by Sălăgean and Ruscheweyh derivative. The obtained results are sharp and they improve known results.

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Keywords: Analytic functions, modified Hadamard product, Sălăgean and Ruscheweyh derivative.

1. Introduction

Let $\mathcal{A}$ denote the class of functions of the form:

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \tag{1.1}$$

which are analytic in the open unit disc $U = \{z \in \mathbb{C} : |z| < 1\}$. Let $g \in \mathcal{A}$ where

$$g(z) = z + \sum_{k=2}^{\infty} b_k z^k. \tag{1.2}$$

Let $\mathbb{N} = \{0, 1, 2, \ldots, n, \ldots\}$.

Definition 1.1. [3] For $f \in \mathcal{A}, \lambda \geq 0$ and $n \in \mathbb{N}$, the operator $\mathscr{D}_\lambda^n$ is defined by $\mathscr{D}_\lambda^n : \mathcal{A} \to \mathcal{A}$,

$$\mathscr{D}_\lambda^0 f(z) = f(z),$$
$$\mathscr{D}_\lambda^1 f(z) = (1-\lambda) f(z) + \lambda z f'(z) = \mathscr{D}_\lambda f(z), \ldots$$
$$\mathscr{D}_\lambda^{n+1} f(z) = (1-\lambda) \mathscr{D}_\lambda^n f(z) + \lambda z \left(\mathscr{D}_\lambda^n f(z)\right)' = \mathscr{D}_\lambda \left(\mathscr{D}_\lambda^n f(z)\right), z \in U$$

Remark 1.2. If $f \in \mathcal{A}$ and $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$, then

$$\mathscr{D}_\lambda^n f(z) = z + \sum_{k=2}^{\infty} [1 + (k-1)\lambda]^n a_k z^k, z \in U.$$

Remark 1.3. For $\lambda = 1$ in the above definition we obtain the Sălăgean differential operator [10].

Definition 1.4. [8] For $f \in \mathcal{A}, n \in \mathbb{N}$, the operator $\mathscr{R}^n$ is defined by $\mathscr{R}^n : \mathcal{A} \to \mathcal{A}$,

$$\mathscr{R}^0 f(z) = f(z), \quad \mathscr{R}^1 f(z) = z f'(z), \ldots$$
$$(n+1)\mathscr{R}^{n+1} f(z) = z(\mathscr{R}^n f(z))' + n\mathscr{R}^n f(z), z \in U.$$

Remark 1.5. If $f \in \mathcal{A}$ and $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$, then

$$\mathscr{R}^n f(z) = z + \sum_{k=2}^{\infty} \frac{(n+k-1)!}{n!\,(k-1)!} a_k z^k, \ z \in U.$$

Definition 1.6. [1] Let $\gamma, \lambda \geq 0, n \in \mathbb{N}$. Denote by $\mathscr{L}^n$ the operator given by

$$\mathscr{L}^n : \mathcal{A} \to \mathcal{A}, \quad \mathscr{L}^n f(z) = (1-\gamma)\mathscr{R}^n f(z) + \gamma \mathscr{D}_\lambda^n f(z), z \in U.$$

Remark 1.7. If $f \in \mathcal{A}$ and $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$, then

$$\mathscr{L}^n f(z) = z + \sum_{k=2}^{\infty} \left\{ \gamma [1 + (k-1)\lambda]^n + (1-\gamma) \frac{(n+k-1)!}{n!\,(k-1)!} \right\} a_k z^k, \ z \in U.$$

Definition 1.8. [6] Let f and g be analytic functions in U. We say that the function f is subordinate to the function g, if there exists a function w, which is analytic in U and $w(0) = 0, |w(z)| < 1, z \in U$, such that $f(z) = g(w(z))$, $\forall z \in U$. We denote by $\prec$ the subordination relation.

Definition 1.9. For $\widetilde{\lambda} \geq 0; -1 \leq A < B \leq 1; 0 < B \leq 1; n \in \mathbb{N}$ let $L(n, \widetilde{\lambda}, A, B)$ denote the subclass of $\mathcal{A}$ which contain functions $f(z)$ of the form (1.1) such that

$$(1-\widetilde{\lambda})(\mathscr{L}^n f(z))' + \widetilde{\lambda}(\mathscr{L}^{n+1} f(z))' \prec \frac{1+Az}{1+Bz}. \tag{1.3}$$

Attiya and Aouf defined in [4] the class $\mathscr{R}(n, \lambda, A, B)$ with a condition like (1.3), but there instead of the operator $\mathscr{L}^n$ they used the Ruscheweyh operator.

Definition 1.10. [12],[9]A function $f(z)$ of the form (1.1) is said to be in the class $V(\theta_k)$ if $f \in \mathcal{A}$ and $arg(a_k) = \theta_k$,$\forall k \geq 2$. If $\exists \delta \in \mathbb{R}$ such that $\theta_k + (k-1)\delta \equiv \pi (mod\ 2\pi)$, $\forall k \geq 2$ then $f(z)$ is said to be in the class $V(\theta_k, \delta)$. The union of $V(\theta_k, \delta)$ taken over all possible sequences $\{\theta_k\}$ and all possible real numbers δ is denoted by V.

Let $VL(n, \widetilde{\lambda}, A, B)$ denote the subclass of V consisting of functions

$$f(z) \in L(n, \widetilde{\lambda}, A, B).$$

Definition 1.11. The modified Hadamard product of two functions f and g of the form (1.1) and (1.2), and which belong to $V(\theta_k, \delta)$ is defined by (see also [5], [9], [11])

$$(f * g)(z) = z - \sum_{k=2}^{\infty} a_k b_k z^k = (g * f)(z). \tag{1.4}$$

Theorem 1.12. [7] *Let the function $f(z)$ defined by* (1.1) *be in V. Then $f(z) \in VL(n, \widetilde{\lambda}, A, B)$, if and only if*

$$\sum_{k=2}^{\infty} kC_k\,(1+B)\,|a_k| \le (B-A), \tag{1.5}$$

where

$$C_k = \gamma\,[1+(k-1)\lambda]^n \left[1+\widetilde{\lambda}\lambda(k-1)\right] + \frac{(n+k-1)!}{n!\,(k-1)!}\,(1-\gamma)\left[1+\widetilde{\lambda}\frac{k-1}{n+1}\right].$$

The extremal functions are:

$$f(z) = z + \frac{B-A}{kC_k\,(1+B)} e^{i\theta_k} z^k, (k \ge 2).$$

2. Main results

Theorem 2.1. *If $f \in VL(n, \widetilde{\lambda}, A_1, B), g \in VL(n, \widetilde{\lambda}, A_2, B)$ then $f * g \in VL(n, \widetilde{\lambda}, A^*, B)$, where $A^* = B - \dfrac{(B-A_1)(B-A_2)}{2C_2\,(1+B)}$.*
The result is sharp.

Proof. Let $f \in VL(n, \widetilde{\lambda}, A_1, B), g \in VL(n, \widetilde{\lambda}, A_2, B)$ and suppose they have the form (1.1). Since $f \in VL(n, \widetilde{\lambda}, A_1, B)$ we have

$$\frac{\sum\limits_{k=2}^{\infty} kC_k\,(1+B)\,|a_k|}{B-A_1} \le 1 \tag{2.1}$$

and for $g \in VL(n, \widetilde{\lambda}, A_2, B)$ we have

$$\frac{\sum\limits_{k=2}^{\infty} kC_k\,(1+B)\,|b_k|}{B-A_2} \le 1. \tag{2.2}$$

We know from Theorem 1.12 that $f * g \in VL(n, \widetilde{\lambda}, A^*, B)$ if and only if

$$\frac{\sum\limits_{k=2}^{\infty} kC_k\,(1+B)\,|a_k b_k|}{B-A^*} \le 1. \tag{2.3}$$

By using the Cauchy-Schwarz inequality for (2.1) and (2.2) we have

$$\frac{\sum\limits_{k=2}^{\infty} kC_k\,(1+B)\,\sqrt{|a_k b_k|}}{\sqrt{(B-A_1)(B-A_2)}} \le 1.$$

We note that

$$\frac{\sum\limits_{k=2}^{\infty} kC_k\,(1+B)\,|a_k b_k|}{B-A^*} \leq \frac{\sum\limits_{k=2}^{\infty} kC_k\,(1+B)\,\sqrt{|a_k b_k|}}{\sqrt{(B-A_1)(B-A_2)}}$$

implies (2.3). But this is implied by

$$\frac{|a_k b_k|}{B-A^*} \leq \frac{\sqrt{|a_k b_k|}}{\sqrt{(B-A_1)(B-A_2)}}$$

or

$$\sqrt{|a_k b_k|} \leq \frac{B-A^*}{\sqrt{(B-A_1)(B-A_2)}}, (k \geq 2)\,. \tag{2.4}$$

From Theorem 1.12 we have:

$$|a_k| \leq \frac{B-A_1}{kC_k\,(1+B)} \text{ and } |b_k| \leq \frac{B-A_2}{kC_k\,(1+B)}, (k \geq 2)$$

this implies that

$$\sqrt{|a_k b_k|} \leq \frac{\sqrt{(B-A_1)(B-A_2)}}{kC_k\,(1+B)}, (k \geq 2). \tag{2.5}$$

From (2.5) we obtain that (2.4) holds if

$$\frac{\sqrt{(B-A_1)(B-A_2)}}{kC_k\,(1+B)} \leq \frac{B-A^*}{\sqrt{(B-A_1)(B-A_2)}}$$

or equivalently

$$A^* \leq B - \frac{(B-A_1)(B-A_2)}{kC_k\,(1+B)}.$$

But $kC_k < (k+1)C_{k+1}, (k \geq 2)$ so

$$B - \frac{(B-A_1)(B-A_2)}{kC_k\,(1+B)} \geq B - \frac{(B-A_1)(B-A_2)}{2C_2\,(1+B)}, (k \geq 2)$$

$$\Rightarrow A^* = B - \frac{(B-A_1)(B-A_2)}{2C_2\,(1+B)}.$$

The result is sharp, because if

$$f(z) = z + \frac{B-A_1}{2C_2\,(1+B)} e^{i\theta_1} z^2 \in VL(n, \widetilde{\lambda}, A_1, B)$$

$$g(z) = z + \frac{B-A_2}{2C_2\,(1+B)} e^{i\theta_2} z^2 \in VL(n, \widetilde{\lambda}, A_2, B)$$

then $f * g \in VL(n, \widetilde{\lambda}, A^*, B)$ and satisfy (1.5) with equality. Indeed,

$$2C_2\,(1+B)\,\frac{(B-A_1)(B-A_2)}{2^2 C_2^2\,(1+B)^2} = B - A^*$$

because

$$B - A^* = \frac{(B-A_1)(B-A_2)}{2C_2\,(1+B)}. \qquad \square$$

Corollary 2.2. *If* $f, g \in VL(n, \widetilde{\lambda}, A, B)$ *then* $f * g \in VL(n, \widetilde{\lambda}, A^*, B)$, *where*

$$A^* = B - \frac{(B-A)^2}{2C_2\,(1+B)}.$$

The result is sharp.

Theorem 2.3. *If* $f \in VL(n, \widetilde{\lambda}, A, B_1), g \in VL(n, \widetilde{\lambda}, A, B_2)$ *then* $f*g \in VL(n, \widetilde{\lambda}, A, B^*)$, *where*

$$B^* = A + \frac{(B_1-A)(B_2-A)\,(A+1)}{2C_2\,(1+B_1)\,(1+B_2) - (B_1-A)\,(B_2-A)}.$$

The result is sharp.

Proof. Let $f \in VL(n, \widetilde{\lambda}, A, B_1), g \in VL(n, \widetilde{\lambda}, A, B_2)$ and suppose they have the form (1.1). Since $f \in VL(n, \widetilde{\lambda}, A, B_1)$ we have

$$\frac{\sum\limits_{k=2}^{\infty} kC_k\,(1+B_1)\,|a_k|}{B_1 - A} \leq 1 \tag{2.6}$$

and for $g \in VL(n, \widetilde{\lambda}, A, B_2)$ we have

$$\frac{\sum\limits_{k=2}^{\infty} kC_k\,(1+B_2)\,|b_k|}{B_2 - A} \leq 1. \tag{2.7}$$

We know from Theorem 1.12 that $f * g \in VL(n, \widetilde{\lambda}, A, B^*)$ if and only if

$$\frac{\sum\limits_{k=2}^{\infty} kC_k\,(1+B^*)\,|a_k b_k|}{B^* - A} \leq 1. \tag{2.8}$$

By using the Cauchy-Schwarz inequality for (2.6) and (2.7) we have

$$\frac{\sum\limits_{k=2}^{\infty} kC_k\sqrt{(1+B_1)\,(1+B_2)}\sqrt{|a_k b_k|}}{\sqrt{(B_1-A)(B_2-A)}} \leq 1.$$

We note that

$$\frac{\sum\limits_{k=2}^{\infty} kC_k\,(1+B^*)\,|a_k b_k|}{B^* - A} \leq \frac{\sum\limits_{k=2}^{\infty} kC_k\sqrt{(1+B_1)\,(1+B_2)}\sqrt{|a_k b_k|}}{\sqrt{(B_1-A)(B_2-A)}}$$

implies (2.8). But this is implied by

$$\frac{|a_k b_k|\,(1+B^*)}{B^*-A} \leq \frac{\sqrt{|a_k b_k|}\sqrt{(1+B_1)\,(1+B_2)}}{\sqrt{(B_1-A)(B_2-A)}}$$

or

$$\sqrt{|a_k b_k|} \leq \frac{(B^*-A)\,\sqrt{(1+B_1)\,(1+B_2)}}{(1+B^*)\,\sqrt{(B_1-A)(B_2-A)}}, (k \geq 2)\,. \tag{2.9}$$

From Theorem 1.12 we have:

$$|a_k| \leq \frac{B_1 - A}{kC_k\,(1+B_1)} \text{ and } |b_k| \leq \frac{B_2 - A}{kC_k\,(1+B_2)}, (k \geq 2)$$

this implies that

$$\sqrt{|a_k b_k|} \leq \frac{\sqrt{(B_1 - A)(B_2 - A)}}{kC_k\sqrt{(1+B_1)\,(1+B_2)}}, (k \geq 2). \tag{2.10}$$

from (2.10) we obtain that (2.9) holds if

$$\frac{\sqrt{(B_1 - A)(B_2 - A)}}{kC_k\sqrt{(1+B_1)\,(1+B_2)}} \leq \frac{(B^* - A)\sqrt{(1+B_1)\,(1+B_2)}}{(1+B^*)\sqrt{(B_1 - A)(B_2 - A)}}$$

or equivalently

$$B^* \geq A + \frac{(B_1 - A)(B_2 - A)\,(A+1)}{kC_k\,(1+B_1)\,(1+B_2) - (B_1 - A)\,(B_2 - A)}.$$

But $kC_k < (k+1)C_{k+1}, (k \geq 2)$ so:

$$A + \frac{(B_1 - A)(B_2 - A)\,(A+1)}{kC_k\,(1+B_1)\,(1+B_2) - (B_1 - A)\,(B_2 - A)}$$

$$\leq A + \frac{(B_1 - A)(B_2 - A)\,(A+1)}{2C_2\,(1+B_1)\,(1+B_2) - (B_1 - A)\,(B_2 - A)}, \ (k \geq 2)$$

$$\Rightarrow B^* = A + \frac{(B_1 - A)(B_2 - A)\,(A+1)}{2C_2\,(1+B_1)\,(1+B_2) - (B_1 - A)\,(B_2 - A)}.$$

The result is sharp, because if

$$f(z) = z + \frac{B_1 - A}{2C_2\,(1+B_1)} e^{i\theta_1} z^2 \in VL(n, \widetilde{\lambda}, A, B_1)$$

$$g(z) = z + \frac{B_2 - A}{2C_2\,(1+B_2)} e^{i\theta_2} z^2 \in VL(n, \widetilde{\lambda}, A, B_2)$$

then $f * g \in VL(n, \widetilde{\lambda}, A, B^*)$ and satisfy (1.5) with equality. Indeed,

$$(1+B^*)\,2C_2 \frac{(B_1 - A)(B_2 - A)}{2^2 C_2^2\,(1+B_1)\,(1+B_2)} = B^* - A$$

because

$$B^* - A = \frac{(B_1 - A)(B_2 - A)\,(A+1)}{2C_2\,(1+B_1)\,(1+B_2) - (B_1 - A)\,(B_2 - A)}. \qquad \square$$

Corollary 2.4. *If $f, g \in VL(n, \widetilde{\lambda}, A, B)$ then $f * g \in VL(n, \widetilde{\lambda}, A, B^*)$, where*

$$B^* = A + \frac{(B - A)^2\,(A+1)}{2C_2\,(1+B)^2 - (B - A)^2}.$$

The result is sharp.

Theorem 2.5. *If* $f_j \in VL(n, \widetilde{\lambda}, A_j, B), \quad j = \overline{1, m}, \quad m \in \{2, 3, 4, \ldots\}$ *then*

$$f_1 * f_2 * \ldots * f_m \in VL(n, \widetilde{\lambda}, A^{(m-1)*}, B),$$

where

$$A^{(m-1)*} = B - \frac{\prod_{j=1}^{m} (B - A_j)}{2^{m-1} C_2^{m-1} (1 + B)^{m-1}}.$$

The result is sharp.

Proof. For the proof we use the mathematical induction method and suppose that $f_j, \forall j$ have the form (1.1).
Let $m = 2$. If $f_j \in VL(n, \widetilde{\lambda}, A_j, B), j = \overline{1, 2}$ then $f_1 * f_2 \in VL(n, \widetilde{\lambda}, A^*, B)$ where $A^* = B - \dfrac{(B - A_1)(B - A_2)}{2C_2 (1 + B)}$, from Theorem 2.1 is true.
Assume that the result is true for $m = k$, that is,

$$f_1 * f_2 * \ldots * f_k \in VL(n, \widetilde{\lambda}, A^{(k-1)*}, B)$$

where $A^{(k-1)*} = B - \dfrac{\prod_{j=1}^{k} (B - A_j)}{2^{k-1} C_2^{k-1} (1 + B)^{k-1}}$.
Next, we prove that the result is true for $k + 1$:
then $f_1 * f_2 * \ldots * f_k * f_{k+1} \in VL(n, \widetilde{\lambda}, A^{k*}, B)$, where

$$A^{k*} = B - \frac{(B - A^{(k-1)*})(B - A_{k+1})}{2C_2 (1 + B)}$$

$$= B - \frac{\dfrac{\prod_{j=1}^{k} (B - A_j)}{2^{k-1} C_2^{k-1} (1 + B)^{k-1}} (B - A_{k+1})}{2C_2 (1 + B)} = B - \frac{\prod_{j=1}^{k+1} (B - A_j)}{2^k C_2^k (1 + B)^k}.$$

The result is sharp, because if

$$f_j(z) = z + \frac{B - A_j}{2C_2 (1 + B)} e^{i\theta_j} z^2 \in VL(n, \widetilde{\lambda}, A_j, B), \quad j = \overline{1, m},$$

then

$$f_1 * f_2 * \ldots * f_m (z) = z + \frac{\prod_{j=1}^{m} (B - A_j)}{2^{m-1} C_2^{m-1} (1 + B)^{m-1}} e^{i(\theta_1 + \theta_2 + \ldots + \theta_m)} z^2$$

satisfy (1.5) with equality. Indeed,

$$2C_2 (1 + B) \prod_{j=1}^{m} (B - A_j) \frac{1}{2^m C_2^m (1 + B)^m} = B - A^{(m-1)*}. \qquad \square$$

Theorem 2.6. *If* $f_j \in VL(n, \widetilde{\lambda}, A, B_j), j = \overline{1, m}, m \in \{2, 3, 4, \ldots\}$ *then*

$$f_1 * f_2 * \ldots * f_m \in VL(n, \widetilde{\lambda}, A, B^{(m-1)*}),$$

where

$$B^{(m-1)*} = A + \frac{(A+1)\prod_{j=1}^{m}(B_j - A)}{2^{m-1}C_2^{m-1}\prod_{j=1}^{m}(1+B_j) - \prod_{j=1}^{m}(B_j - A)}.$$

The result is sharp.

Proof. The proof is similar to the demonstration for Theorem 2.5. □

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A certain subclass of analytic functions associated with the quasi-subordination

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Abstract. In the present paper we study quasi-subordination under a multivalent function and we consider a certain subclass of (normalized) analytic functions based on quasi-subordination. Applications and consequences of the main results are also cosidered which some of them extend the earlier issues.

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1. Introduction

Let $\mathcal{A}$ be the class of (normalized) analytic functions $f(z)$ in the open unit disk

$$\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\},$$

which have Taylor series expansion

$$f(z) = z + a_2 z^2 + \ldots ; \quad (z \in \mathbb{D}).$$

We denote by S the subclass of $\mathcal{A}$ containing univalent functions. For two analytic functions f, g we say that f is subordinate to g (or g is superordinate to f), and write $f \prec g$ (or$f(z) \prec g(z)$) if there exists an analytic function $w(z)$,

$$w(z) \in \Omega = \{w : |w(z)| \leqslant |z|, z \in \mathbb{D}\}$$

such that $f(z) = g(w(z))$. In the special case if g is univalent in $\mathbb{D}$, then we have the following equivalence

$$f(z) \prec g(z) \Longleftrightarrow f(0) = g(0) \;\; and \;\; f(\mathbb{D}) \subseteq g(\mathbb{D}).$$

A survey on articles shows that the notation of subordination was used frequently in the literature, see for example [4, 5, 6]. As an example, consider the following two classes of (normalized) analytic functions,

$$S^*(\phi) = \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec \phi(z), z \in \mathbb{D} \right\} \tag{1.1}$$

and

$$K(\phi) = \left\{ f \in \mathcal{A} : 1 + \frac{zf''(z)}{f'(z)} \prec \phi(z), z \in \mathbb{D} \right\}, \tag{1.2}$$

where $\phi(z)$ is analytic in $\mathbb{D}$ with $\phi(0) = 1$. For $\phi(z) = \frac{1+z}{1-z}$ we obtain the well-known classes S^* and K of starlike and convex functions, respectively. By taking

$$\phi(z) = \frac{1 + (1 - 2\alpha)z}{1 - z}, \; 0 \leq \alpha < 1$$

in (1.1) and (1.2) we obtain the class of starlike and convex functions of order α, respectively, while the choice

$$\phi(z) = \left(\frac{1+z}{1-z} \right)^{\alpha}$$

with $0 < \alpha \leq 1$ gives the class of strongly starlike and strongly conex functions of order α, respectively.

As an extension of subordination, Robertson [7] (see also [1]) introduced the concept of quasi-subordination. Let f, g be analytic functions. We say that f is quasi-subordinated to g, and write $f \prec_q g$ if there exist analytic functions ϕ and w with $|\phi(z)| \leq 1$ and $w(z) \in \Omega$ such that $f(z) = \phi(z)g(w(z))$. It is clear that for $\phi(z) = 1$ we have $f \prec g$. In [3] authors considered the following two classes:

$$S^*(n, A, B) = \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec \frac{1 + Az^n}{1 + Bz^n}, \; z \in \mathbb{D} \right\}$$

and

$$K(n, A, B) = \left\{ f \in \mathcal{A} : 1 + \frac{zf''(z)}{f'(z)} \prec \frac{1 + Az^n}{1 + Bz^n}, \; z \in \mathbb{D} \right\}$$

with $-1 \leq B < A \leq 1$, and proved certain results about the subordination properties of these two classes. Applying the notation of quasi-subordination we define the following two classes.

Definition 1.1. Let $n \in \mathbb{N}, \lambda \in \mathbb{C} - \{0\}$ and $-1 \leq B < A \leq 1$. We say that $f \in \mathcal{A}$ is in the class $S_q^*(n, \lambda, A, B)$ if there exists a $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$, such that

$$e^{i\theta} \left(1 + \frac{1}{\lambda} \left(\frac{zf'(z)}{f(z)} - 1 \right) \right) \prec_q \frac{1}{n} \sum_{k=1}^{n} \frac{1 + Az^k}{1 + Bz^k} \tag{1.3}$$

Definition 1.2. Let $n \in \mathbb{N}, \lambda \in \mathbb{C} - \{0\}$ and $-1 \leq B < A \leq 1$. We say that $f \in \mathcal{A}$ is in the class $K_q(n, \lambda, A, B)$ if there exists a θ, $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$, such that

$$e^{i\theta}(1 + \frac{1}{\lambda} \frac{zf''(z)}{f'(z)}) \prec_q \frac{1}{n} \sum_{k=1}^{n} \frac{1 + Az^k}{1 + Bz^k} \tag{1.4}$$

It is clear that

$$f \in K_q(n, \lambda, A, B) \Longleftrightarrow zf'(z) \in S_q^*(n, \lambda, A, B).$$

Note that the function $\psi(z) = \frac{1+Az^n}{1+Bz^n}, A \neq B, n \in \mathbb{N}$ is multivalent and maps $\mathbb{D}$ onto a disk or a half plane.

The classes $S_q^*(n, \lambda, A, B)$ and $K_q(n, \lambda, A, B)$ reduce to the classes which were introduced by Janowski [2] if we consider $\theta = 0, n = \lambda = 1 = \phi(z)$. Also, by taking $n = \lambda = 1, A = 1, B = -1$ and $\phi(z) = 1$ the class $S_q^*(n, \lambda, A, B)$ becomes the well-known class S_θ of $\theta-$ *spirallike* functions, (see [[6],p. 9]).

In the present paper, we aim to prove special results associated with the quasi-subordination for subclasses of (normalized) analytic functions. Some consequencees and applications are also considered.

In order to prove our main results, we shall use each of the following theorems.

Theorem 1.3. ([[6],p.70]) *Let h be conex in $\mathbb{D}$ and let $P : \mathbb{D} \longrightarrow \mathbb{C}$ with* Re$P(z) > 0$. *If $p(z)$ is analytic in $\mathbb{D}$, then*

$$p(z) + P(z)zp'(z) \prec h(z) \Longrightarrow p(z) \prec h(z).$$

Theorem 1.4. ([[6],p.86]) *Let $\beta, \gamma \in \mathbb{C}$ with $\beta \neq 0$ and $n \in \mathbb{N}$. Suppose that $R_{\beta a+\gamma,n}(z)$ is the "open door function" with* $\mathrm{Re}(\beta a + \gamma) > 0$,(see [[6],p. 46]), *and that $h(z)$ is analytic in $\mathbb{D}$ with $h(0) = a$. If*

$$\beta h(z) + \gamma \prec R_{\beta a+\gamma,n}(z)$$

then the solution $q(z)$ of

$$q(z) + \frac{nzq'(z)}{\beta q(z) + \gamma} = h(z) \tag{1.5}$$

with $q(0) = a$ is analytic in $\mathbb{D}$ and satisfies $\mathrm{Re}(\beta q(z) + \gamma) > 0$. *If $a \neq 0$, then the solution q is given by*

$$q(z) = \left[\frac{\beta}{n}\int_0^1 \left(\frac{H(tz)}{H(z)}\right)^{\frac{\beta a}{n}} \cdot t^{(\frac{\gamma}{n})-1}dt\right]^{-1} - \frac{\gamma}{\beta}$$

where

$$H(z) = z\exp\left(\int_0^z \frac{h(t) - a}{at}dt\right).$$

2. The classes $S_q^*(n, \lambda, A, B)$ and $K_q(n, \lambda, A, B)$

We begin this section with the following theorem, which gives a characterization of the functions in $S_q^*(n, \lambda, A, B)$.

Theorem 2.1. *Let the function $f(z)$ belongs to the class $S_q^*(n, \lambda, A, B)$. Then there exists an analytic function $p(z)$,*

$$p(z) \prec_q \frac{1}{n}\sum_{k=1}^{n} \frac{1 + Az^k}{1 + Bz^k}; \tag{2.1}$$

such that

$$f(z) = z\exp\left(\lambda\int_0^z \frac{e^{-i\theta}p(t)-1}{t}dt\right). \tag{2.2}$$

If, in addition, the analytic function $p(z)$ satisfies (2.1), then the function of the form (2.2) belongs to $S_q^*(n,\lambda,A,B)$.

Proof. Suppose that $f \in S_q^*(n,\lambda,A,B)$. For a fixed θ, $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$ the analytic function p defined by

$$p(z) = e^{i\theta}(1+\frac{1}{\lambda}(\frac{zf'(z)}{f(z)}-1)) \tag{2.3}$$

satisfies (2.1). An integration in (2.3) shows that

$$f(z) = z\exp\left(\lambda\int_0^z \frac{e^{-i\theta}p(t)-1}{t}dt\right). \tag{2.4}$$

Conversely, let f is given by (2.2), where $p(z)$ satisfies (2.1). By differentiating logarithmically of (2.4), we obtain:

$$p(z) = e^{i\theta}\left(1+\frac{1}{\lambda}\left(\frac{zf'}{f}-1\right)\right),$$

so,

$$e^{i\theta}\left(1+\frac{1}{\lambda}\left(\frac{zf'(z)}{f(z)}-1\right)\right) \prec_q \frac{1}{n}\sum_{k=1}^{n}\frac{1+Az^k}{1+Bz^k}$$

and $f \in S_q^*(n,\lambda,A,B)$. □

Corollary 2.2. *Let the function $f(z)$ belongs to the class $K_q(n,\lambda,A,B)$. Then there exists an analytic function p,*

$$p(z) \prec_q \frac{1}{n}\sum_{k=1}^{n}\frac{1+Az^k}{1+Bz^k},$$

such that

$$f(z) = \int_0^z \exp(\lambda\int_0^w \frac{e^{-i\theta}p(t)-1}{t}dt)dw, \quad (z \in \mathbb{D}) \tag{2.5}$$

Moreover, if the analytic function $p(z)$ satisfies

$$p(z) \prec_q \frac{1}{n}\sum_{k=1}^{n}\frac{1+Az^k}{1+Bz^k},$$

then the function f of the form (2.5) belongs to $K_q(n,\lambda,A,B)$.

Proof. This is a simple consequence of Theorem 2.1. In fact $f(z) \in K_q(n,\lambda,A,B)$ if and only if $zf'(z) \in S_q^*(n,\lambda,A,B)$. Equivalently $g \in S_q^*(n,\lambda,A,B)$ if and only if

$$f(z) = \int_0^z \frac{g(w)}{w}dw$$

is in the class $K_q(n,\lambda,A,B)$. □

Remark 2.3. Theorem 2.1 and Corollary 2.2 remain true for complex A, B and $A \neq B$ too.

Next, consider the class $S_q^*(n, \lambda, A, B)$ with complex A, B satisfying $|A| \leq 1$, $|B| \leq 1$ and $A \neq B$.

Theorem 2.4. *Let $A, B \in \mathbb{C}$, $|A| \leq 1$ and $|B| \leq 1$ with $A \neq B$. If the function f is in the class $S_q^*(n, \lambda, A, B)$, then there exist functions $f_k, f_k \in S_q^*(1, \lambda, A, B)$, $k = 1, 2, \cdots, n$, such that*

$$f^n(z) = \prod_{k=1}^{n} f_k(z), \quad (z \in \mathbb{D}).$$

On the other hand, if there exist functions $f_k \in S_q^*(1, \lambda, A, B)$ such that

$$f^n(z) = \prod_{k=1}^{n} f_k(z),$$

then $f \in S_q^*(n, \lambda, A, B)$.

Proof. Let $f \in S_q^*(n, \lambda, A, B)$. By Theorem 2.1 there exists analytic functions $p(z), \phi(z)$ and $w(z)$ with $w(z) \in \Omega$, such that

$$p(z) = \frac{1}{n}\phi(z)\sum_{k=1}^{n}\frac{1 + Aw^k(z)}{1 + Bw^k(z)}; \quad (z \in \mathbb{D}),$$

and

$$f^n(z) = z^n \exp\left(n\lambda \int_0^z \frac{e^{-i\theta}p(t) - 1}{t} dt\right).$$

As easy calculation yields

$$\begin{aligned} f^n(z) &= z^n \exp\left(\lambda \int_0^z \frac{e^{-i\theta}\phi(t)\sum_{k=1}^{n}\left\{\frac{1+Aw^k(t)}{1+Bw^k(t)} - n\right\}}{t} dt\right) \\ &= z^n \exp\left(\lambda \int_0^z \sum_{k=1}^{n}\left(e^{-i\theta}\phi(t)\frac{1 + Aw^k(t)}{1 + Bw^k(t)} - 1\right)\frac{dt}{t}\right) \\ &= \prod_{k=1}^{n} z \exp\left(\lambda \int_0^z \left(e^{-i\theta}\phi(t)\frac{1 + Aw^k(t)}{1 + Bw^k(t)} - 1\right)\frac{dt}{t}\right) \\ &= \prod_{k=1}^{n} f_k(z), \end{aligned}$$

where

$$f_k(z) = z \exp\left(\lambda \int_0^z \left(e^{-i\theta}\phi(t)\frac{1 + Aw^k(t)}{1 + Bw^k(t)} - 1\right)\frac{dt}{t}\right)$$

By taking

$$p_k(z) = \phi(z)\frac{1 + Aw^k(z)}{1 + Bw^k(z)}$$

it follows that

$$p_k(z) \prec_q \frac{1 + Az^k}{1 + Bz^k} \prec_q \frac{1 + Az}{1 + Bz}.$$

So, the function $f_k(z), (k = 1, 2, ..., n)$ are in the class $S_q^*(1, \lambda, A, B)$. On the other hand, if there exist functions $f_k \in S_q^*(1, \lambda, A, B)$ such that

$$f^n(z) = \prod_{k=1}^{n} f_k(z),$$

then by Theorem 2.1 the functions f_k must have the form:

$$f_k(z) = z \exp\left(\lambda \int_0^z \left(e^{-i\theta}\phi(t)\frac{1 + Aw^k(t)}{1 + Bw^k(t)} - 1\right)\frac{dt}{t}\right).$$

This shows that the product $\prod_{k=1}^{n} f_k(z)$ is a function $f^n(z)$ such that $f \in S_q^*(n, \lambda, A, B)$.

□

Here, we obtain another representation for the functions in $S_q^*(n, \lambda, A, B)$. Suppose that

$$p(z) \prec_q F(n, A, B) = \frac{1}{n}\sum_{k=1}^{n} \frac{1 + Az^k}{1 + Bz^k}$$

with complex parameters A, B, $|A| \le 1$, $|B| \le 1$, $A \ne B$ and $A \ne 0$, $B \ne 0$. By defination, there exist analytic functions $\phi(z), w(z)$, $(|\phi(z)| \le 1$ and $w(z) \in \Omega)$ such that

$$p(z) = \frac{1}{n}\phi(z)\sum_{k=1}^{n} \frac{1 + Aw^k(z)}{1 + Bw^k(z)}.$$

If $a_k^k = A, b_k^k = B$ and $\xi_{ik} = \sqrt[k]{-1}$ for $i = 1, 2, ..., k$ and $k = 1, 2, ..., n$, then

$$\begin{aligned}
p(z) &= \frac{1}{n}\phi(z)\sum_{k=1}^{n} \frac{1 + (a_k w(z)^k}{1 + (b_k w(z))^k} \\
&= \frac{1}{n}\phi(z)\sum_{k=1}^{n} \frac{(a_k w(z))^k - (\xi_{ik})^k}{(b_k w(z))^k - (\xi_{ik})^k} \\
&= \frac{1}{n}\phi(z)\sum_{k=1}^{n} \frac{(a_k w(z) - \xi_{1k})(a_k w(z) - \xi_{2k})...(a_k w(z) - \xi_{kk})}{(b_k w(z) - \xi_{1k})(b_k w(z) - \xi_{2k})...(b_k w(z) - \xi_{kk})}.
\end{aligned}$$

Therefore

$$\begin{aligned}
p(z) &= \frac{1}{n}\phi(z)\sum_{k=1}^{n}\prod_{i=1}^{k} \frac{a_k w(z) - \xi_{ik}}{b_k w(z) - \xi_{ik}} \\
&= \sum_{k=1}^{n} \frac{1}{n}\phi(z)\prod_{i=1}^{k} \frac{1 + A_{ik}w(z)}{1 + B_{ik}w(z)} \\
&= \sum_{k=1}^{n} p_k(z)
\end{aligned}$$

where $A_{ik} = a_k \xi_{ik}^{k-1}$, $B_{ik} = b_k \xi_{ik}^{k-1}$ and

$$p_k(z) = \frac{1}{n}\phi(z) \prod_{i=1}^{k} \frac{1 + A_{ik} w(z)}{1 + B_{ik} w(z)} \prec_q \prod_{i=1}^{k} \frac{1 + A_{ik} z}{1 + B_{ik} z}$$
$$= \prod_{i=1}^{k} F(1, A_{ik}, B_{ik}).$$

3. The class $M_q(n, \alpha, \lambda, A, B)$

It is interesting to consider the conditions in which $1 + \frac{1}{\lambda}(\frac{zf'}{f} - 1)$ and $1 + \frac{1}{\lambda}\frac{zf''}{f'}$ are joined. In [3] authors introduced the class $M(\alpha, n, A, B)$ as following

$$M(\alpha, n, A, B) = \left\{ f \in \mathcal{A} : \alpha \left(1 + \frac{zf''}{f'}\right) + (1 - \alpha)\frac{zf'}{f} \prec \frac{1 + Az^n}{1 + Bz^n},\ z \in \mathbb{D} \right\}$$

where $\alpha \in \mathbb{R}$ and $-1 \leq B < A \leq 1$. For the class $M(\alpha, n, A, B)$ we have

$$M(\alpha, n, A, B) \subseteq M(\alpha, 1, A, B) \subseteq M(\alpha)$$

where $M(\alpha)$ is the class of $\alpha - convex$ functions, (see [[6],p. 10]).

In the same way, we consider the class

$$M_q(n, \alpha, \lambda, A, B) = \left\{ f \in \mathcal{A} : \alpha \left(1 + \frac{1}{\lambda}\left(\frac{zf'}{f} - 1\right)\right) + (1 - \alpha)\left(1 + \frac{1}{\lambda}\frac{zf''}{f'}\right) \prec_q \frac{1 + Az^n}{1 + Bz^n} \right\} \tag{3.1}$$

where $\alpha \in \mathbb{R}$ and $-1 \leq B < A \leq 1$. By taking $\lambda = 1$ and $\phi(z) = 1$ (related to the definition of quasi-subordination) in (3.1) we obtain the class $M(\alpha, n, A, B)$. We aim to obtain the same result concerning this class. The next result involves a condition for finding a solution of Briot-Bouquet differential equation.

Theorem 3.1. *Let $-1 < B < A \leq 1$, $\alpha < 1$ and $\lambda \in \mathbb{C} - \{0\}$. In addition, assume that*

$$|\lambda| \frac{1 + A}{1 + B} + |1 - \lambda| \leq \sqrt{(1 - \alpha)(3 - \alpha)}. \tag{3.2}$$

If $f \in M_q(n, \alpha, \lambda, A, B)$, then f is starlike with respect to the origin. Also,

$$\frac{zf'(z)}{f(z)} = \left(\frac{1}{1 - \alpha} \int_0^1 \left(\frac{H(tz)}{H(z)}\right)^{\frac{\lambda}{1-\alpha}} t^{\frac{\alpha - \lambda}{1 - \alpha}} \right)^{-1}$$

where

$$H(z) = z \exp \int_0^z \frac{(\phi(t) - 1) + (A\phi(t) - B)w^n(t)}{t(1 + Bw^n(t))} dt,$$

in which $\phi(z)$ and $w(z)$ are analytic in $\mathbb{D}$, such that $|\phi(z)| \leq 1$ and $w(z) \in \Omega$. In the special case if $\phi(z) = 1$, then the condition $f \in M_q(n, \alpha, \lambda, A, B)$ implies $f \in S_q^*(1, \lambda, A, B)$.

Proof. Put

$$q(z) = 1 + \frac{1}{\lambda}\left(\frac{zf'(z)}{f(z)} - 1\right).$$

By a straightforward calculation we see that the equation

$$\alpha\left(1 + \frac{1}{\lambda}\left(\frac{zf'}{f} - 1\right)\right) + (1-\alpha)\left(1 + \frac{1}{\lambda}\frac{zf''}{f'}\right) = h(z)$$

becomes,

$$q(z) + \frac{zq'(z)}{\frac{\lambda}{1-\alpha}q(z) + \frac{1-\lambda}{1-\alpha}} = h(z). \tag{3.3}$$

By the assumption of the theorem, there exist analytic functions $\phi(z)$ and $w(z)$ in $\mathbb{D}$, $|\phi(z)| \leq 1$ and $w(z) \in \Omega$ such that

$$h(z) = \phi(z)\frac{1 + Aw^n(z)}{1 + Bw^n(z)}.$$

We have to verify conditions of the Theorem 1.4. We have

$$\beta = \frac{\lambda}{1-\alpha},\ \gamma = \frac{1-\lambda}{1-\alpha},\ h(0) = 1 = a$$

and

$$\mathrm{Re}(\beta a + \gamma) = \mathrm{Re}\left(\frac{1}{1-\alpha}\right) > 0.$$

Next, we investigate the condition

$$\beta h(z) + \gamma \prec R_{\beta a+\gamma,m}(z) = R_{\frac{1}{1-\alpha},1}(z). \tag{3.4}$$

We know that the set $R_{\frac{1}{1-\alpha},1}(\mathbb{D})$ is the complex plane with slits along the half-lines $\mathrm{Re}z = 0$ and $|\mathrm{Im}z| \geq \sqrt{1 + \frac{2}{1-\alpha}} = \sqrt{\frac{3-\alpha}{1-\alpha}}$ (see [[6],p. 46]). Easy calculations show that

$$\begin{aligned}|\beta h(z) + \gamma| &\leq \frac{|\lambda|}{1-\alpha}.\frac{|1 + Aw^n(z)|}{|1 + Bw^n(z)|} + \frac{|1-\lambda|}{1-\alpha} \\ &\leq \frac{|\lambda|}{1-\alpha}.\frac{1+A}{1+B} + \frac{|1-\lambda|}{1-\alpha} \\ &= \frac{1}{1-\alpha}(|\lambda|\frac{1+A}{1+B} + |1-\lambda|).\end{aligned}$$

So, in order to have (3.4) it must be

$$\frac{1}{1-\alpha}\left(|\lambda|\frac{1+A}{1+B} + |1-\lambda|\right) < \sqrt{\frac{3-\alpha}{1-\alpha}}$$

or equivalently,

$$|\lambda|\frac{1+A}{1+B} + |1-\lambda| < \sqrt{(1-\alpha)(3-\alpha)}. \tag{3.5}$$

If α and λ satisfy in (3.5), then we have $\beta h(z)+\gamma \prec R_{\frac{1}{1-\alpha},1}(z)$. So, all conditions of the Theorem 1.4 are satisfied and we obtain

$$0<\operatorname{Re}(\beta q(z)+\gamma)=\frac{1}{1-\alpha}\operatorname{Re}\left(\frac{zf'}{f}\right)$$

which means that $f\in S^*$. Also, we obtain the following representation for $\frac{zf'}{f}$:

$$\frac{zf'}{f}=\left(\frac{1}{1-\alpha}\int_0^1\left(\frac{H(tz)}{H(z)}\right)^{\frac{\lambda}{1-\alpha}}t^{\frac{\alpha-\lambda}{1-\alpha}}\right)^{-1}$$

where

$$\begin{aligned}H(z)&=z\exp\int_0^z\frac{h(t)-1}{t}dt\\&=z\exp\int_0^z\frac{(\phi(t)-1)+(A\phi(t)-B)w^n(t)}{t(1+Bw^n(t))}dt.\end{aligned}$$

In the special case if $\phi(z)=1$, then $h(z)=\frac{1+Aw^n(z)}{1+Bw^n(z)}$ and we have

$$q(z)+\frac{zq'(z)}{\frac{\lambda}{1-\alpha}q(z)+\frac{1-\lambda}{1-\alpha}}\prec\frac{1+Az}{1+Bz}=h_1(z),$$

where $h_1(z)$ is convex-univalent function in $\mathbb{D}$. Now by Theorem 1.3 we conclude that

$$q(z)=1+\frac{1}{\lambda}\left(\frac{zf'}{f}-1\right)\prec\frac{1+Az}{1+Bz},$$

hence $f\in S_q^*(1,\lambda,A,B)$. □

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Approximation with an arbitrary order by generalized Kantorovich-type and Durrmeyer-type operators on $[0, +\infty)$

Sorin Trifa

Abstract. Given an arbitrary sequence $\lambda_n > 0$, $n \in \mathbb{N}$, with the property that $\lim_{n\to\infty} \lambda_n = 0$ as fast we want, in this note we introduce modified/ generalized Szász-Kantorovich, Baskakov-Kantorovich, Szász-Durrmeyer-Stancu and Baskakov-Szász-Durrmeyer-Stancu operators in such a way that on each compact subinterval in $[0, +\infty)$ the order of uniform approximation is $\omega_1(f; \sqrt{\lambda_n})$. These modified operators uniformly approximate a Lipschitz 1 function, on each compact subinterval of $[0, \infty)$ with the arbitrary good order of approximation $\sqrt{\lambda_n}$. The results obtained are of a definitive character (that is are the best possible) and also have a strong unifying character, in the sense that for various choices of the nodes λ_n, one can recapture previous approximation results obtained for these operators by other authors.

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Keywords: Generalized Szász-Kantorovich operators, generalized Baskakov-Kantorovich operators, generalized Szász-Durrmeyer-Stancu operators, generalized Baskakov-Szász-Durrmeyer-Stancu operators, linear and positive operators, modulus of continuity, arbitrary order of approximation.

1. Introduction

It is known that the classical Baskakov operators are given by the formula (see, e.g., [2])

$$V_n(f)(x) = \sum_{j=0}^{\infty} \binom{n+j-1}{j} \frac{x^j}{(1+x)^{n+j}} f\left(\frac{j}{n}\right) = (1+x)^{-n} \sum_{j=0}^{\infty} \frac{(n+j-1)!}{j!(n-1)!} \frac{x^j}{(1+x)^j}$$

$$= (1+x)^{-n} \sum_{j=0}^{\infty} \frac{n(n+1)\dots(n+j-1)}{j!} \frac{x^j}{(1+x)^j}.$$

In the recent paper [9], this operator was modified by replacing n with $\frac{1}{\lambda_n}$, where $\lim_{n\to\infty}\lambda_n = 0$ as fast we want, and the approximation properties (of arbitrary good order depending on λ_n) of the new obtained Baskakov operator defined by the formula

$$V_n(f;\lambda_n)(x) = (1+x)^{\frac{-1}{\lambda_n}}\sum_{j=0}^{\infty}\frac{1}{j!}\frac{1}{\lambda_n}\left(1+\frac{1}{\lambda_n}\right)\ldots\left(j-1+\frac{1}{\lambda_n}\right)\left(\frac{x}{1+x}\right)^j f(j\lambda_n),\ x\geq 0,$$

were obtained. Above by convention,

$$\frac{1}{j!}\frac{1}{\lambda_n}\left(1+\frac{1}{\lambda_n}\right)\ldots\left(j-1+\frac{1}{\lambda_n}\right) = 1 \text{ for } j=0.$$

The complex variable case for $V_n(f;\lambda_n)$ was studied in [10]. Also, in [6], the above idea was applied to the Jakimovski-Leviatan-Ismail kind generalization of Szász-Mirakjan operators.

The goal of the present paper is that based on the above idea, to introduce modified/generalized Szász-Kantorovich, Baskakov-Kantorovich, Szász-Durrmeyer-Stancu and Baskakov-Szász-Durrmeyer-Stancu operators in such a way that on each compact subinterval in $[0,+\infty)$ the order of uniform approximation is $\omega_1(f;\sqrt{\lambda_n})$. These modified operators can uniformly approximate a Lipschitz 1 function, on each compact subinterval of $[0,\infty)$ with the arbitrary good order of approximation $\sqrt{\lambda_n}$ given at the beginning.

In conclusion, it is worth mentioning for these generalized operators that since λ_n ca be chosen with $\lambda_n \searrow 0$ arbitrary fast, in fact it follows that the order of convergence $\omega_1(f;\sqrt{\lambda_n})$ is arbitrary good. For this reason, the results obtained by this paper have a definitive character (that is they are the best possible). In the same time, the results also have a strong unifying character, in the sense that for various choices of the nodes λ_n one can recapture previous approximation results obtained by other authors.

2. Generalized Baskakov-Kantorovich operators

In this section we deal with the Baskakov-Kantorovich operators.

It is known that the classical Baskakov-Kantorovich operators are defined by (see, e.g., [3])

$$\begin{aligned} K_n(f)(x) &= \sum_{j=0}^{\infty}\binom{n+j-1}{j}\frac{x^j}{(1+x)^{n+j}}n\int_{j/n}^{(j+1)/n}f(v)dv \\ &= (1+x)^{-n}\sum_{j=0}^{\infty}\frac{n(n+1)\ldots(n+j-1)}{j!}\frac{x^j}{(1+x)^j}n\int_{j/n}^{(j+1)/n}f(v)dv. \end{aligned}$$

If we replace n with $\frac{1}{\lambda_n}$, then we obtain the generalized Baskakov-Kantorovich operators, defined by the formula

$$K_n(f;\lambda_n)(x) = (1+x)^{-\frac{1}{\lambda_n}}\sum_{j=0}^{\infty}\frac{1}{j!}\frac{1}{\lambda_n}\left(1+\frac{1}{\lambda_n}\right)\ldots\left(j-1+\frac{1}{\lambda_n}\right)\frac{x^j}{(1+x)^j}\frac{1}{\lambda_n}\int_{j\lambda_n}^{(j+1)\lambda_n}f(v)dv.$$

Denote everywhere in the paper $e_k(x) = x^k$, $k = 0, 1, 2, \dots$

This section deals with the approximation properties of the operator $K_n(f; \lambda_n)(x)$. For our purpose, firstly we need the following auxiliary result.

Lemma 2.1. *We have:*

(i) $K_n(e_0; \lambda_n)(x) = 1$; $K_n(e_1; \lambda_n)(x) = x + \frac{1}{2} \cdot \lambda_n$;

$$K_n(e_2; \lambda_n)(x) = x^2 + 2\lambda_n x + \lambda_n x^2 + \frac{1}{3} \cdot \lambda_n^2;$$

(ii) $K_n((t - x)^2; \lambda_n)(x) = \lambda_n \left(x^2 + x + \frac{1}{3} \cdot \lambda_n\right)$.

Proof. By using the formulas in Corollary 2.1 in [9],

$$V_n(e_0; \lambda_n)(x) = 1, \quad V_n(e_1; \lambda_n)(x) = x$$

and

$$V_n(e_2; \lambda_n)(x) = x^2 + \lambda_n x(1 + x),$$

we will calculate $K_n(e_0; \lambda_n)(x)$, $K_n(e_1; \lambda_n)(x)$, $K_n(e_2; \lambda_n)(x)$, $K_n((t - x)^2; \lambda_n)(x)$.

(i) Therefore,

$$K_n(e_0; \lambda_n)(x)$$

$$= (1+x)^{-\frac{1}{\lambda_n}} \sum_{j=0}^{\infty} \frac{1}{j!} \frac{1}{\lambda_n} \left(1 + \frac{1}{\lambda_n}\right) \dots \left(j - 1 + \frac{1}{\lambda_n}\right) \left(\frac{x}{1+x}\right)^j \frac{1}{\lambda_n} \int_{j\lambda_n}^{(j+1)\lambda_n} dv$$

$$= (1+x)^{-\frac{1}{\lambda_n}} \sum_{j=0}^{\infty} \frac{1}{j!} \frac{1}{\lambda_n} \left(1 + \frac{1}{\lambda_n}\right) \dots \left(j - 1 + \frac{1}{\lambda_n}\right) \left(\frac{x}{1+x}\right)^j \frac{1}{\lambda_n} (j\lambda_n + \lambda_n - j\lambda_n)$$

$$= (1+x)^{-\frac{1}{\lambda_n}} \sum_{j=0}^{\infty} \frac{1}{j!} \frac{1}{\lambda_n} \left(1 + \frac{1}{\lambda_n}\right) \dots \left(j - 1 + \frac{1}{\lambda_n}\right) \left(\frac{x}{1+x}\right)^j = V_n(e_0; \lambda_n)(x) = 1.$$

Also,

$$K_n(e_1; \lambda_n)(x)$$

$$= (1+x)^{-\frac{1}{\lambda_n}} \sum_{j=0}^{\infty} \frac{1}{j!} \frac{1}{\lambda_n} \left(1 + \frac{1}{\lambda_n}\right) \dots \left(j - 1 + \frac{1}{\lambda_n}\right) \left(\frac{x}{1+x}\right)^j \frac{1}{\lambda_n} \int_{j\lambda_n}^{(j+1)\lambda_n} v dv$$

$$= (1+x)^{-\frac{1}{\lambda_n}}$$

$$\cdot \sum_{j=0}^{\infty} \frac{1}{j!\lambda_n} \left(1 + \frac{1}{\lambda_n}\right) \dots \left(j - 1 + \frac{1}{\lambda_n}\right) \left(\frac{x}{1+x}\right)^j \frac{1}{2\lambda_n} (j^2\lambda_n^2 + \lambda_n^2 + 2j\lambda_n^2 - j^2\lambda_n^2)$$

$$= (1+x)^{-\frac{1}{\lambda_n}} \sum_{j=0}^{\infty} \frac{1}{j!} \frac{1}{\lambda_n} \left(1 + \frac{1}{\lambda_n}\right) \dots \left(j - 1 + \frac{1}{\lambda_n}\right) \left(\frac{x}{1+x}\right)^j \frac{1}{\lambda_n} \left(\frac{1}{2}\lambda_n^2 + j\lambda_n^2\right)$$

$$= (1+x)^{-\frac{1}{\lambda_n}} \sum_{j=0}^{\infty} \frac{1}{j!} \frac{1}{\lambda_n} \left(1 + \frac{1}{\lambda_n}\right) \dots \left(j - 1 + \frac{1}{\lambda_n}\right) \left(\frac{x}{1+x}\right)^j \frac{1}{\lambda_n} \frac{1}{2} \lambda_n^2$$

$$+ (1+x)^{-\frac{1}{\lambda_n}} \sum_{j=0}^{\infty} \frac{1}{j!} \frac{1}{\lambda_n} \left(1 + \frac{1}{\lambda_n}\right) \dots \left(j - 1 + \frac{1}{\lambda_n}\right) \left(\frac{x}{1+x}\right)^j \frac{1}{\lambda_n} j\lambda_n^2$$

$$= V_n(e_1; \lambda_n)(x) + \frac{1}{2}\lambda_n (1+x)^{-\frac{1}{\lambda_n}} \sum_{j=0}^{\infty} \frac{1}{j!} \frac{1}{\lambda_n} \left(1 + \frac{1}{\lambda_n}\right) \dots \left(j - 1 + \frac{1}{\lambda_n}\right) \left(\frac{x}{1+x}\right)^j$$

$$= V_n(e_1;\lambda_n)(x) + \frac{1}{2}\lambda_n V_n(e_0;\lambda_n)(x) = x + \frac{1}{2}\lambda_n.$$

Then,

$$K_n(e_2;\lambda_n)(x)$$

$$= (1+x)^{-\frac{1}{\lambda_n}} \sum_{j=0}^{\infty} \frac{1}{j!}\frac{1}{\lambda_n}\left(1+\frac{1}{\lambda_n}\right)\dots\left(j-1+\frac{1}{\lambda_n}\right)\left(\frac{x}{1+x}\right)^j \frac{1}{\lambda_n}\int_{j\lambda_n}^{(j+1)\lambda_n} v^2 dv$$

$$= (1+x)^{-\frac{1}{\lambda_n}} \sum_{j=0}^{\infty} \frac{1}{j!}\frac{1}{\lambda_n}\left(1+\frac{1}{\lambda_n}\right)\dots\left(j-1+\frac{1}{\lambda_n}\right)\left(\frac{x}{1+x}\right)^j \frac{1}{\lambda_n}\frac{1}{3}((j+1)^3\lambda_n^3 - j^3\lambda_n^3)$$

$$= (1+x)^{-\frac{1}{\lambda_n}}$$

$$\cdot \sum_{j=0}^{\infty} \frac{1}{j!}\frac{1}{\lambda_n}\left(1+\frac{1}{\lambda_n}\right)\dots\left(j-1+\frac{1}{\lambda_n}\right)\left(\frac{x}{1+x}\right)^j \frac{1}{\lambda_n}\frac{1}{3}(j^3\lambda_n^3 + 3j^2\lambda_n^3 + 3j\lambda_n^3 + \lambda_n^3 - j^3\lambda_n^3)$$

$$= (1+x)^{-\frac{1}{\lambda_n}} \sum_{j=0}^{\infty} \frac{1}{j!}\frac{1}{\lambda_n}\left(1+\frac{1}{\lambda_n}\right)\dots\left(j-1+\frac{1}{\lambda_n}\right)\left(\frac{x}{1+x}\right)^j \frac{1}{\lambda_n}(j^2\lambda_n^3 + j\lambda_n^3 + \frac{1}{3}\lambda_n^3)$$

$$= (1+x)^{-\frac{1}{\lambda_n}} \sum_{j=0}^{\infty} \frac{1}{j!}\frac{1}{\lambda_n}\left(1+\frac{1}{\lambda_n}\right)\dots\left(j-1+\frac{1}{\lambda_n}\right)\left(\frac{x}{1+x}\right)^j \frac{1}{\lambda_n} j^2\lambda_n^3$$

$$+ (1+x)^{-\frac{1}{\lambda_n}} \sum_{j=0}^{\infty} \frac{1}{j!}\frac{1}{\lambda_n}\left(1+\frac{1}{\lambda_n}\right)\dots\left(j-1+\frac{1}{\lambda_n}\right)\left(\frac{x}{1+x}\right)^j \frac{1}{\lambda_n} j\lambda_n^3$$

$$+ (1+x)^{-\frac{1}{\lambda_n}} \sum_{j=0}^{\infty} \frac{1}{j!}\frac{1}{\lambda_n}\left(1+\frac{1}{\lambda_n}\right)\dots\left(j-1+\frac{1}{\lambda_n}\right)\left(\frac{x}{1+x}\right)^j \frac{1}{\lambda_n}\frac{1}{3}\lambda_n^3$$

$$= (1+x)^{-\frac{1}{\lambda_n}} \sum_{j=0}^{\infty} \frac{1}{j!}\frac{1}{\lambda_n}\left(1+\frac{1}{\lambda_n}\right)\dots\left(j-1+\frac{1}{\lambda_n}\right)\left(\frac{x}{1+x}\right)^j j^2\lambda_n^2$$

$$+ (1+x)^{-\frac{1}{\lambda_n}} \sum_{j=0}^{\infty} \frac{1}{j!}\frac{1}{\lambda_n}\left(1+\frac{1}{\lambda_n}\right)\dots\left(j-1+\frac{1}{\lambda_n}\right)\left(\frac{x}{1+x}\right)^j j\lambda_n^2$$

$$+ \frac{1}{3}(1+x)^{-\frac{1}{\lambda_n}} \sum_{j=0}^{\infty} \frac{1}{j!}\frac{1}{\lambda_n}\left(1+\frac{1}{\lambda_n}\right)\dots\left(j-1+\frac{1}{\lambda_n}\right)\left(\frac{x}{1+x}\right)^j \lambda_n^2$$

$$= V_n(e_2;\lambda_n)(x) + \lambda_n V_n(e_1;\lambda_n)(x) + \frac{1}{3}\lambda_n^2 V_n(e_0;\lambda_n)(x)$$

$$= x^2 + 2\lambda_n x + \lambda_n x^2 + \frac{1}{3}\lambda_n^2.$$

(ii) Finally, we get

$$K_n((t-x)^2;\lambda_n)(x) = K_n(t^2 - 2tx + x^2;\lambda_n)(x)$$

$$= K_n(t^2;\lambda_n)(x) - K_n(2tx;\lambda_n)(x) + K_n(x^2;\lambda_n)(x)$$

$$= K_n(e_2;\lambda_n)(x) - 2xK_n(e_1;\lambda_n)(x) + x^2 K_n(e_0;\lambda_n)(x)$$

$$= x^2 + 2\lambda_n x + \lambda_n x^2 + \frac{1}{3}\lambda_n^2 - 2x^2 - x\lambda_n + x^2 = \lambda_n(x + x^2 + \frac{1}{3}\lambda_n). \qquad \square$$

The main result of this section is the following.

Theorem 2.2. *Let $\lambda_n \searrow 0$ (with $n \to \infty$) as fast we want and suppose that $f : [0, +\infty) \to \mathbb{R}$ is uniformly continuous on $[0, +\infty)$. For all $x \in [0, +\infty)$ and $n \in \mathbb{N}$, we have*

$$|K_n(f; \lambda_n)(x) - f(x)| \leq 2\omega_1(f; \sqrt{\lambda_n} \cdot \sqrt{x^2 + x + \lambda_n/3}),$$

where $\omega_1(f; \delta) = \sup\{|f(x) - f(y)|; x, y \in \mathbb{R}, |x - y| \leq \delta\}$ denotes the modulus of continuity of f with the step δ.

Proof. By the classical theory (see, e.g., Shisha-Mond [14] or, e.g., [1], Proposition 1.6.3) (where although the result is proved for continuous functions on compact intervals, the reasonings are similar for uniformly continuous functions on $[0, +\infty)$), for any positive and linear operator L defined on the set of uniformly continuous functions $UC[0, +\infty)$, we obtain

$$|L(f)(x) - f(x)| \leq (1 + \delta^{-1}\sqrt{L(\varphi_x^2)(x)})\omega(f; \delta),$$

for all $f \in UC[0, +\infty), x \in [0, +\infty), \delta > 0$, where $\varphi_x(t) = |t - x|$.
Replacing above L by K_n and taking into account that by Lemma 2.1, (ii) we have

$$\sqrt{K_n((t - x)^2; \lambda_n)(x)} = \sqrt{\lambda_n(x + x^2 + \frac{1}{3}\lambda_n)} = \sqrt{\lambda_n} \cdot \sqrt{x + x^2 + \frac{1}{3}\lambda_n},$$

this implies

$$|K_n(f; \lambda_n)(x) - f(x)| \leq (1 + \delta^{-1}\sqrt{\lambda_n} \cdot \sqrt{x + x^2 + \lambda_n/3})\omega_1(f; \delta).$$

Choosing now here $\delta = \sqrt{\lambda_n} \cdot \sqrt{x^2 + x + \lambda_n/3}$ we get

$$|K_n(f; \lambda_n)(x) - f(x)| \leq 2\omega_1(f; \sqrt{\lambda_n}\sqrt{x + x^2 + \lambda_n/3}),$$

which proves the estimate in the statement. □

As an immediate consequence of Theorem 2.2 we get the following.

Corollary 2.3. *Let $\lambda_n \searrow 0$ as fast we want and suppose that f is a Lipschitz function, that is there exists $M > 0$ such that $|f(x) - f(y)| \leq M|x - y|$, for all $x, y \in [0, \infty)$. Then, for all $x \in [0, +\infty)$ and $n \in \mathbb{N}$ we have*

$$|K_n(f; \lambda_n)(x) - f(x)| \leq 2M\sqrt{\lambda_n} \cdot \sqrt{x + x^2 + \lambda_n/3}.$$

Proof. Since by hypothesis f is a Lipschitz function, we easily get $\omega_1(f; \delta) \leq M\delta$, for all $\delta > 0$. Choosing now $\delta = \sqrt{\lambda_n} \cdot \sqrt{x + x^2 + \lambda_n/3}$ and applying Theorem 2.2, we get the desired estimate. □

Remarks. 1) Since $f \in UC[0, +\infty)$, it is well-known that we get $\lim_{\delta \searrow 0} \omega_1(f; \delta) = 0$. Therefore, since $\lambda_n \searrow 0$, passing to limit with $n \to \infty$ in the estimate in Theorem 2.2, it follows that $K_n(f; \lambda_n)(x) \to f(x)$, pointwise for any $x \in [0, +\infty)$. Now, in order to get uniform convergence in the above results, the expression $\sqrt{x + x^2 + \lambda_n/3}$ must be bounded, fact which holds when x belongs to a compact subinterval of $[0, +\infty)$.

2) If $f \in UC[0, +\infty)$, then $K_n(f; \lambda_n)(x)$ is well defined (that is $|K_n(f; \lambda_n)(x)| < +\infty$ for all $x \in [0, +\infty)$ and $n \in \mathbb{N}$). Indeed, if f is uniformly continuous on $[0, +\infty)$ then it is well known that its growth on $[0, +\infty)$ is linear, i.e. there exist $\alpha, \beta > 0$ such that $|f(x)| \leq \alpha x + \beta$, for all $x \in [0, +\infty)$ (see e.g. [4], p. 48, Problème 4, or [5]). This immediately implies

$$|K_n(f; \lambda_n)(x)| \leq K_n(|f|; \lambda_n)(x) \leq \alpha \cdot K_n(e_1; \lambda_n)(x) + \beta = \alpha(x + \lambda_n/2) + \beta,$$

for all $x \in [0, +\infty)$, $n \in \mathbb{N}$.

3) The optimality of the estimates in Theorem 2.2 and Corollary 2.3 consists in the fact that given an arbitrary sequence of strictly positive numbers $(\gamma_n)_n$, with $\lim_{n\to\infty} \gamma_n = 0$, we always can find a sequence λ_n satisfying

$$2\omega_1(f; \sqrt{\lambda_n} \cdot \sqrt{x + x^2 + \lambda_n/3}) \leq \gamma_n$$

for all $n \in \mathbb{N}$ and x belonging to a compact subinterval of $[0, +\infty)$ in the case of Theorem 2.2 and $\sqrt{\lambda_n} \cdot \sqrt{x + x^2 + \lambda_n/3}) \leq \gamma_n$ for all $n \in \mathbb{N}$ and x in a compact subinterval of $[0, +\infty)$, in the case of Corollary 2.3.

3. Generalized Szász-Kantorovich operators

The formula for the classic, linear and positive Szász-Kantorovich operators is given by (see, e.g., [16])

$$S_n(f)(x) = e^{-nx} \sum_{j=0}^{\infty} \frac{(nx)^j}{j!} n \int_{\frac{j}{n}}^{\frac{j+1}{n}} f(v)dv = e^{-nx} \sum_{j=0}^{\infty} \frac{(nx)^j}{j!} \int_0^1 f(\frac{t+j}{n})dt.$$

Replacing above n with $\frac{1}{\lambda_n}$, we obtain the generalized Szász-Kantorovich operators, defined by the formula

$$\begin{aligned} S_n(f; \lambda_n)(x) &= e^{-\frac{x}{\lambda_n}} \sum_{j=0}^{\infty} \frac{x^j}{\lambda_n^j j!} \frac{1}{\lambda_n} \int_{j\lambda_n}^{(j+1)\lambda_n} f(v)dv \\ &= e^{-\frac{x}{\lambda_n}} \sum_{j=0}^{\infty} \frac{x^j}{\lambda_n^j j!} \int_0^1 f(\lambda_n(t+j))dt. \end{aligned}$$

In this section we study the approximation properties of the operator $S_n(f; \lambda_n)(x)$. Firstly we need the following lemma.

Lemma 3.1. *We have:*

(i) $S_n(e_0; \lambda_n)(x) = 1$; $S_n(e_1; \lambda_n)(x) = x + \frac{1}{2} \cdot \lambda_n$;

$$S_n(e_2; \lambda_n)(x) = x^2 + 2\lambda_n x + \frac{1}{3} \cdot \lambda_n^2;$$

(ii) $S_n((t-x)^2; \lambda_n)(x) = \lambda_n \left(x + \frac{1}{3} \cdot \lambda_n\right)$.

Proof. (i) We have

$$S_n(e_0; \lambda_n)(x) = e^{-x/\lambda_n} \sum_{j=0}^{\infty} \frac{x^j}{j! \lambda_n^j} = 1,$$

for all $x \geq 0$ and $n \in \mathbb{N}$. Then,

$$\begin{aligned} S_n(e_1; \lambda_n)(x) &= e^{-x/\lambda_n} \sum_{j=0}^{\infty} \frac{x^j}{j! \lambda_n^j} \frac{1}{\lambda_n} \cdot \frac{1}{2} \left\{[(j+1)\lambda_n]^2 - (j\lambda_n)^2\right\} \\ &= e^{-x/\lambda_n} \sum_{j=0}^{\infty} \frac{x^j}{j! \lambda_n^j} \frac{1}{2\lambda_n} \cdot \left(2\lambda_n^2 j + \lambda_n^2\right) \end{aligned}$$

$$= e^{-x/\lambda_n} \sum_{j=0}^{\infty} \frac{x^j}{j!\lambda_n^j} \frac{\lambda_n}{2} + \lambda_n \cdot e^{-x/\lambda_n} \sum_{j=0}^{\infty} \frac{x^j \cdot j}{j!\lambda_n^j}$$

$$= \frac{\lambda_n}{2} + x \cdot e^{-x/\lambda_n} \sum_{j=1}^{\infty} \frac{x^{j-1}}{(j-1)!\lambda_n^{j-1}} = \frac{\lambda_n}{2} + x \cdot e^{-x/\lambda_n} \sum_{k=0}^{\infty} \frac{x^k}{k!\lambda_n^k} = \frac{\lambda_n}{2} + x.$$

Also,

$$S_n(e_2; \lambda_n)(x) = e^{-x/\lambda_n} \sum_{j=0}^{\infty} \frac{x^j}{j!\lambda_n^j} \frac{1}{\lambda_n} \cdot \frac{1}{3} \left\{ [(j+1)\lambda_n]^3 - (j\lambda_n)^3 \right\}$$

$$= e^{-x/\lambda_n} \sum_{j=0}^{\infty} \frac{x^j}{j!\lambda_n^j} \frac{1}{3\lambda_n} \cdot \left\{ (3j^2 + 3j + 1)\lambda_n^3 \right\}$$

$$= \frac{\lambda_n^2}{3} + e^{-x/\lambda_n} \sum_{j=1}^{\infty} \frac{x^j j^2 \lambda_n^2}{j!\lambda_n^j} + e^{-x/\lambda_n} \sum_{j=1}^{\infty} \frac{x^j j \lambda_n^2}{j!\lambda_n^j}$$

$$= \frac{\lambda_n^2}{3} + e^{-x/\lambda_n} \sum_{j=1}^{\infty} \frac{x^j j(j-1)\lambda_n^2}{j!\lambda_n^j} + 2e^{-x/\lambda_n} \sum_{j=1}^{\infty} \frac{x^j j \lambda_n^2}{j!\lambda_n^j} = x^2 + 2x\lambda_n + \frac{\lambda_n^2}{3}.$$

(ii) Concluding, we get

$$S_n((\cdot - x)^2; \lambda_n)(x) = K_n(\lambda_n; e_2)(x) - 2x \cdot K_n(\lambda_n; e_1)(x) + x^2$$
$$= x\lambda_n + \lambda_n^2/3 = \lambda_n(x + \lambda_n/3). \qquad \square$$

The main result of this section is the following.

Theorem 3.2. *Let $\lambda_n \searrow 0$ (with $n \to \infty$) as fast we want and suppose that $f : [0, +\infty) \to \mathbb{R}$ is uniformly continuous on $[0, +\infty)$. For all $x \in [0, +\infty)$ and $n \in \mathbb{N}$, we have*

$$|S_n(f; \lambda_n)(x) - f(x)| \le 2\omega_1(f; \sqrt{\lambda_n} \cdot \sqrt{x + \lambda_n/3}),$$

where $\omega_1(f; \delta) = \sup\{|f(x) - f(y)|; x, y \in \mathbb{R}, |x - y| \le \delta\}$ denotes the modulus of continuity of f with the step δ.

Proof. Reasoning exactly as in the proof of Theorem 2.2, we can write

$$|S_n(f; \lambda_n)(x) - f(x)| \le (1 + \delta^{-1} \sqrt{S_n(\varphi_x^2; \lambda_n)(x)})\omega_1(f; \delta).$$

Choosing here $\delta = \sqrt{S_n(\varphi_x^2; \lambda_n)(x)}$ an using Lemma 3.1, (ii), we obtain

$$|S_n(f; \lambda_n)(x) - f(x)| \le 2\omega_1 \left(f; \sqrt{\lambda_n} \cdot \sqrt{x + \frac{1}{3}\lambda_n} \right) \le 2\omega_1 \left(f; \sqrt{\lambda_n} \cdot \sqrt{x + \frac{1}{3}\lambda_n} \right),$$

which proves the theorem. $\square$

As an immediate consequence of Theorem 3.2 we get the following.

Corollary 3.3. *Let $\lambda_n \searrow 0$ as fast we want and suppose that f is a Lipschitz function, that is there exists $M > 0$ such that $|f(x) - f(y)| \le M|x - y|$, for all $x, y \in [0, \infty)$. Then, for all $x \in [0, +\infty)$ and $n \in \mathbb{N}$ we have*

$$|S_n(f; \lambda_n)(x) - f(x)| \le 2M\sqrt{\lambda_n} \cdot \sqrt{x + \lambda_n/3}.$$

Proof. Since by hypothesis f is a Lipschitz function, we easily get $\omega_1(f;\delta) \le M\delta$, for all $\delta > 0$. Choosing now $\delta = \sqrt{\lambda_n} \cdot \sqrt{x + \lambda_n/3}$ and applying Theorem 3.2, we get the desired estimate. □

Remark. All the Remarks 1)-3) made at the end of the previous section remain valid for the generalized Szász-Kantorovich operators too.

4. Generalized Szász-Durrmeyer-type operators

Let us recall that the classical Szász-Durrmeyer operators are given by the formula (see, e.g., [13])

$$SD_n(f)(x) = n\sum_{j=0}^{\infty} s_{n,j}(x)\int_0^{\infty} s_{n,j}(t)f(t)dt,$$

where $s_{n,j}(x) = e^{-nx}\frac{(nx)^j}{j!}$.

If we replace n with $\frac{1}{\lambda_n}$, then we obtain the generalized Szász-Durrmeyer operators, defined by the formula

$$SD_n(f;\lambda_n)(x) = \frac{1}{\lambda_n}\sum_{j=0}^{\infty} e^{-\frac{x}{\lambda_n}} \cdot \frac{x^j}{\lambda_n^j j!}\int_0^{\infty} e^{-\frac{t}{\lambda_n}} \cdot \frac{t^j}{\lambda_n^j j!} f(t)dt.$$

In the first part of this section we study the approximation properties of the operator $SD_n(f;\lambda_n)(x)$. Firstly we need the following lemma.

Lemma 4.1. *We have:*

(i) $SD_n(e_0;\lambda_n)(x) = 1$*;* $SD_n(e_1;\lambda_n)(x) = x + \lambda_n$*;*

$$SD_n(e_2;\lambda_n)(x) = x^2 + 4\lambda_n x + 2\lambda_n^2;$$

(ii) $SD_n((t-x)^2;\lambda_n)(x) = \lambda_n\,(2x + 2\lambda_n)$.

Proof. (i) Denoting

$$I_j(f) = \int_0^{\infty} e^{-\frac{t}{\lambda_n}}\frac{(\frac{t}{\lambda_n})^j}{j!}f(t)dt,$$

we can write

$$SD_n(f;\lambda_n)(x) = \frac{1}{\lambda_n}\sum_{j=0}^{\infty} e^{-\frac{x}{\lambda_n}}\frac{(\frac{x}{\lambda_n})^j}{j!} \cdot I_j(f).$$

Now, taking $f(t) = t^p$ and making the change of variable $v = \frac{t}{\lambda_n}$ it follows

$$I_j(e_p) = \lambda_n\int_0^{\infty} e^{-v} \cdot \frac{v^j}{j!} \cdot \lambda_n^p \cdot v^p dv = \frac{\lambda_n^{p+1}}{j!} \cdot \int_0^{\infty} e^{-v}v^{p+j}dv$$
$$= \frac{\lambda_n^{p+1}}{j!} \cdot \Gamma(p+j+1-1) = \frac{\lambda_n^{p+1}}{j!} \cdot (p+j)!,$$

where Γ is the Euler's gamma function.

So, for $p = 0$, we have $I_j(e_0) = \frac{\lambda_n}{j!}j! = \lambda_n$, which implies

$$SD_n(e_0,\lambda_n)(x) = \frac{1}{\lambda_n}\sum_{j=0}^{\infty} e^{-\frac{x}{\lambda_n}}\frac{(\frac{x}{\lambda_n})^j}{j!}\lambda_n = \frac{1}{\lambda_n}\lambda_n = 1.$$

Now, for $p = 1$ we have $I_j(e_1) = \frac{(\lambda_n)^2}{j!}(j+1)! = (j+1)\lambda_n^2$, which implies

$$SD_n(e_1, \lambda_n)(x) = \frac{1}{\lambda_n}\sum_{j=0}^{\infty} e^{-\frac{x}{\lambda_n}} \frac{(\frac{x}{\lambda_n})^j}{j!}(j+1)\lambda_n^2$$

$$= \frac{1}{\lambda_n}\sum_{j=0}^{\infty} e^{-\frac{x}{\lambda_n}} \frac{(\frac{x}{\lambda_n})^j}{j!} j\lambda_n^2 + \frac{1}{\lambda_n}\sum_{j=0}^{\infty} e^{-\frac{x}{\lambda_n}} \frac{(\frac{x}{\lambda_n})^j}{j!}\lambda_n^2 = x + \lambda_n.$$

Finally, for $p = 2$, we have $I_j(e_2) = \frac{(\lambda_n)^3}{j!}(j+2)! = (j+1)(j+2)\lambda_n^3$, which implies

$$SD_n(e_2, \lambda_n)(x) = \frac{1}{\lambda_n}\sum_{j=0}^{\infty} e^{-\frac{x}{\lambda_n}} \frac{(\frac{x}{\lambda_n})^j}{j!}(j^2 + 3j + 2)\lambda_n^3$$

$$= \frac{1}{\lambda_n}\sum_{j=0}^{\infty} e^{-\frac{x}{\lambda_n}} \frac{(\frac{x}{\lambda_n})^j}{j!} j^2\lambda_n^3 + \frac{1}{\lambda_n}\sum_{j=0}^{\infty} e^{-\frac{x}{\lambda_n}} \frac{(\frac{x}{\lambda_n})^j}{j!} 3j\lambda_n^3$$

$$+\frac{1}{\lambda_n}\sum_{j=0}^{\infty} e^{-\frac{x}{\lambda_n}} \frac{(\frac{x}{\lambda_n})^j}{j!} 2\lambda_n^3 = (x^2 + \lambda_n x) + 3\lambda_n x + 2\lambda_n^2 = x^2 + 4\lambda_n x + 2\lambda_n^2.$$

(ii) Concluding, we get

$$SD_n((t-x)^2; \lambda_n)(x) = SD_n(t^2, \lambda_n)(x) - SD_n(2tx, \lambda_n)(x) + SD_n(x^2, \lambda_n)(x)$$
$$= x^2 + 4\lambda_n x + 2\lambda_n^2 - 2x(x + \lambda_n) + x^2 = 2\lambda_n x + 2\lambda_n^2,$$

which proves the lemma. □

The first main result of this section is the following.

Theorem 4.2. *Let $\lambda_n \searrow 0$ as fast we want and suppose that $f : [0, +\infty) \to \mathbb{R}$ is uniformly continuous on $[0, +\infty)$. For all $x \in [0, +\infty)$ and $n \in \mathbb{N}$, we have*

$$|SD_n(f; \lambda_n)(x) - f(x)| \le 2\omega_1(f; \sqrt{\lambda_n} \cdot \sqrt{2x + 2\lambda_n}).$$

Proof. Reasoning exactly as in the proof of Theorem 2.2, we can write

$$|SD_n(f; \lambda_n)(x) - f(x)| \le (1 + \delta^{-1}\sqrt{SD_n(\varphi_x^2; \lambda_n)(x)})\omega_1(f; \delta).$$

Choosing here $\delta = \sqrt{SD_n(\varphi_x^2; \lambda_n)(x)}$ and using Lemma 4.1, (ii), we obtain

$$|S_n(f; \lambda_n)(x) - f(x)| \le 2\omega_1(f; \sqrt{\lambda_n} \cdot \sqrt{2x + 2\lambda_n}),$$

which proves the theorem. □

As an immediate consequence of Theorem 4.2 we get the following.

Corollary 4.3. *Let $\lambda_n \searrow 0$ as fast we want and suppose that f is a Lipschitz function, that is there exists $M > 0$ such that $|f(x) - f(y)| \le M|x - y|$, for all $x, y \in [0, \infty)$. Then, for all $x \in [0, +\infty)$ and $n \in \mathbb{N}$ we have*

$$|S_n(f; \lambda_n)(x) - f(x)| \le 2M\sqrt{\lambda_n} \cdot \sqrt{2x + 2\lambda_n}.$$

Proof. Since by hypothesis f is a Lipschitz function, we easily get $\omega_1(f; \delta) \le M\delta$, for all $\delta > 0$. Choosing now $\delta = \sqrt{\lambda_n} \cdot \sqrt{2x + 2\lambda_n}$ and applying Theorem 4.2, we get the desired estimate. □

Remark. All the Remarks 1)-3) made at the end of Section 2 remain valid for the generalized Szász-Durrmeyer, $SD_n(f; \lambda_n)$, operators too.

In what follows we will introduce and study the generalized Szász-Durrmeyer-Stancu operators. Thus it is well-known that the classical Szász-Durrmeyer-Stancu operators are given by the formula (see, e.g., [8])

$$SD_n^{(\alpha,\beta)}(f)(x) = n\sum_{j=0}^{\infty} s_{n,j}(x)\int_0^{\infty} s_{n,j}(t) f\left(\frac{nt+\alpha}{n+\beta}\right) dt,$$

where $0 \leq \alpha \leq \beta$ and $s_{n,j}(x) = e^{-nx}\frac{(nx)^j}{j!}$.

If we replace n with $\frac{1}{\lambda_n}$, we obtain:

$$SD_n^{(\alpha,\beta)}(f;\lambda_n)(x) = \frac{1}{\lambda_n}\sum_{j=0}^{\infty} e^{-\frac{x}{\lambda_n}}\frac{(\frac{x}{\lambda_n})^j}{j!}\int_0^{\infty} e^{-\frac{x}{\lambda_n}}\frac{(\frac{x}{\lambda_n})^j}{j!} f\left(\frac{\frac{t}{\lambda_n}+\alpha}{\frac{1}{\lambda_n+\beta}}\right) dt.$$

Firstly we prove the following lemma.

Lemma 4.4. *We have:*

(i) $SD_n^{(\alpha,\beta)}(e_0;\lambda_n)(x) = 1$; $SD_n^{(\alpha,\beta)}(e_1;\lambda_n)(x) = \frac{x}{1+\lambda_n\beta} + \frac{\lambda_n(\alpha+1)}{1+\lambda_n\beta}$;

$$SD_n^{(\alpha,\beta)}(e_2;\lambda_n)(x) = \frac{x^2}{(1+\lambda_n\beta)^2} + \frac{\lambda_n(2\alpha+3)}{(1+\lambda_n\beta)^2}x + \frac{\lambda_n^2(\alpha^2+2\alpha+2)}{(1+\lambda_n\beta)^2};$$

(ii) $SD_n^{(\alpha,\beta)}((t-x)^2;\lambda_n)(x)$

$$= \frac{\lambda_n^2\beta^2}{(1+\lambda_N\beta)^2}x^2 + \frac{\lambda_n(1-2\beta(\alpha+1)\lambda_n)}{(1+\lambda_n\beta)^2}x + \frac{\lambda_n^2(\alpha^2+2\alpha+2)}{(1+\lambda_n\beta)^2}.$$

Proof. (i) Firstly, we calculate $T_{n,k}^{(\alpha,\beta)}(x) := SD_n^{(\alpha,\beta)}(e_k)(x)$, $k = 0,1,2$. For this purpose, we will use the following formula in Lemma 2.1 in [8]

$$T_{n,k}^{(\alpha,\beta)} = \sum_{j=0}^{k}\binom{k}{j}\frac{n^j\alpha^{k-j}}{(n+\beta)^k}T_{n,j}(x), \tag{4.1}$$

where $T_{n,k}(x) = SD_n(e_k)(x)$.

Therefore, before that we need to calculate $T_{n,k}(x)$. For the calculation of $T_{n,k}(x)$, we use the recurrence formula in Lemma 2.2 in [7]

$$T'_{n,k}(x) = \frac{n}{x}T_{n,k+1}(x) - \left(n + \frac{k+1}{x}\right)T_{n,k}(x), \tag{4.2}$$

taking into account that $T_{n,0}(x) = 1$.

Thus, taking in (4.2) $k = 0$ we immediately get

$$0 = \frac{n}{x}T_{n,1}(x) - (n+1/x)T_{n,0}(x),$$

which implies

$$T_{n,1}(x) = (n+1/x)\cdot\frac{x}{n} = x + 1/n.$$

Taking in (4.2) $k = 1$, it follows

$$1 = \frac{n}{x}T_{n,2}(x) - \left(n + \frac{2}{x}\right)\left(x + \frac{1}{n}\right),$$

which implies

$$T_{n,2}(x) = \left(nx + 3 + \frac{2}{nx}\right)\left(x + \frac{1}{n}\right) = x^2 + \frac{3x}{n} + \frac{2}{n^2}.$$

Returning now to the formula (4.1), for $k = 0$ we obtain $T_{n,0}^{(\alpha,\beta)}(x) = 1$, for $k = 1$ we obtain

$$T_{n,1}^{(\alpha,\beta)} = \sum_{j=0}^{1} \binom{1}{j} \frac{n^j \alpha^{1-j}}{(n+\beta)^1} T_{n,j}(x) = \frac{\alpha}{n+\beta} + \frac{n}{n+\beta}\left(x + \frac{1}{n}\right)$$
$$= \frac{n}{n+\beta}x + \frac{\alpha+1}{n+\beta},$$

while for $k = 1$ we get

$$T_{n,2}^{(\alpha,\beta)}(x) = \sum_{j=0}^{2} \binom{2}{j} \frac{n^j \alpha^{2-j}}{(n+\beta)^2} T_{n,j}(x)$$
$$= \frac{\alpha^2}{(n+\beta)^2} + \frac{2n\alpha}{(n+\beta)^2}\left(x + \frac{1}{n}\right) + \frac{n^2}{(n+\beta)^2}\left(x^2 + \frac{3x}{n} + \frac{2}{n^2}\right)$$
$$= \frac{n^2}{(n+\beta)^2}x^2 + \frac{n(2\alpha+3)}{(n+\beta)^2}x + \frac{\alpha^2+2\alpha+2}{(n+\beta)^2}.$$

Now, if we replace n with $\frac{1}{\lambda_n}$ we easily obtain

$$SD_n^{(\alpha,\beta)}(e_0;\lambda_n)(x) = 1,$$
$$SD_n^{(\alpha,\beta)}(e_1;\lambda_n)(x) = \frac{x}{1+\lambda_n\beta} + \frac{\lambda_n(\alpha+1)}{1+\lambda_n\beta},$$
$$SD_n^{(\alpha,\beta)}(e_2;\lambda_n)(x) = \frac{x^2}{(1+\lambda_n\beta)^2} + \frac{\lambda_n(2\alpha+3)}{(1+\lambda_n\beta)^2}x + \frac{\lambda_n^2(\alpha^2+2\alpha+2)}{(1+\lambda_n\beta)^2}.$$

(ii) We have

$$SD_n^{(\alpha,\beta)}((t-x)^2;\lambda_n)(x)$$
$$= SD_n^{(\alpha,\beta)}(e_2;\lambda_n)(x) - 2xSD_n^{(\alpha,\beta)}(e_1;\lambda_n)(x) + x^2 SD_n^{(\alpha,\beta)}(e_0;\lambda_n)(x)$$
$$= x^2\left[\frac{1}{(1+\lambda_n\beta)^2} - \frac{2}{1+\lambda_n\beta} + 1\right] + x\left[\frac{\lambda_n(2\alpha+3)}{(1+\lambda_n\beta)^2} - \frac{2\lambda_n(\alpha+1)}{1+\lambda_n\beta}\right]$$
$$+\frac{\lambda_n^2(\alpha^2+2\alpha+2)}{(1+\lambda_n\beta)^2} = \frac{\lambda_n^2\beta^2}{(1+\lambda_n\beta)^2}x^2 + \frac{\lambda_n(1-2\beta(\alpha+1)\lambda_n)}{(1+\lambda_n\beta)^2}x$$
$$+\frac{\lambda_n^2(\alpha^2+2\alpha+2)}{(1+\lambda_n\beta)^2},$$

which proves the lemma. □

The second main result of this section is the following.

Theorem 4.5. *Let* $0 \le \alpha \le \beta$, $\lambda_n \searrow 0$ *as fast we want and suppose that* $f : [0,+\infty) \to \mathbb{R}$ *is uniformly continuous on* $[0,+\infty)$. *For all* $x \in [0,+\infty)$ *and* $n \in \mathbb{N}$, *we have*

$$|SD_n^{(\alpha,\beta)}(f;\lambda_n)(x) - f(x)| \le 2\omega_1(f;\sqrt{\lambda_n}\cdot\sqrt{E_n^{(\alpha,\beta)}(x)}),$$

where

$$E_n^{(\alpha,\beta)}(x) = \frac{\lambda_n \beta^2}{(1+\lambda_n\beta)^2}x^2 + \frac{1-2\beta(\alpha+1)\lambda_n}{(1+\lambda_n\beta)^2}x + \frac{\lambda_n(\alpha^2+2\alpha+2)}{(1+\lambda_n\beta)^2}.$$

Proof. Reasoning exactly as in the proof of Theorem 2.2, we can write

$$|SD_n^{(\alpha,\beta)}(f;\lambda_n)(x) - f(x)| \leq (1+\delta^{-1}\sqrt{SD_n^{(\alpha,\beta)}(\varphi_x^2;\lambda_n)(x)})\omega_1(f;\delta).$$

Choosing here $\delta = \sqrt{SD_n^{(\alpha,\beta)}(\varphi_x^2;\lambda_n)(x)}$ and using Lemma 4.1, (ii), we obtain

$$|SD_n^{(\alpha,\beta)}(f;\lambda_n)(x) - f(x)| \leq 2\omega_1(f;\sqrt{\lambda_n}\cdot\sqrt{E_n^{(\alpha,\beta)}(x)}),$$

which proves the theorem. □

As an immediate consequence of Theorem 4.5 we get the following.

Corollary 4.6. *Let $0 \leq \alpha \leq \beta$, $\lambda_n \searrow 0$ as fast we want and suppose that f is a Lipschitz function, that is there exists $M > 0$ such that $|f(x) - f(y)| \leq M|x-y|$, for all $x, y \in [0,\infty)$. Then, for all $x \in [0,+\infty)$ and $n \in \mathbb{N}$ we have*

$$|SD_n^{(\alpha,\beta)}(f;\lambda_n)(x) - f(x)| \leq 2M\sqrt{\lambda_n}\cdot\sqrt{E_n^{(\alpha,\beta)}(x)}.$$

Proof. Since by hypothesis f is a Lipschitz function, we easily get $\omega_1(f;\delta) \leq M\delta$, for all $\delta > 0$. Choosing now $\delta = \sqrt{\lambda_n}\cdot\sqrt{E_n^{(\alpha,\beta)}(x)}$ and applying Theorem 4.5, we get the desired estimate. □

Remark. All the Remarks 1)-3) made at the end of Section 2 remain valid for the generalized Szász-Durrmeyer-Stancu, $SD_n^{(\alpha,\beta)}(f;\lambda_n)$, operators too.

5. Generalized Baskakov-Szász-Durrmeyer-Stancu operators

For $0 \leq \alpha \leq \beta$, the classical Baskakov- Szász-Durrmeyer-Stancu operators are given by the formula (see, e.g., [12])

$$V_n^{(\alpha,\beta)}(f)(x) = n\sum_{j=0}^{\infty} b_{n,j}(x)\int_0^{\infty} s_{n,j}(t) f(\frac{nt+\alpha}{n+\beta}dt,$$

where, $s_{n,j}(x) = e^{-nx}\frac{(nx)^j}{j!}$ and

$$b_{n,j}(x) = \binom{n+j-1}{j}\frac{x^j}{(1+x)^{n+j}} = (1+x)^{-n}\frac{n(n+1)\dots(n+j-1)}{j!}\frac{x^j}{(1+x)^j}.$$

If we replace n with $\frac{1}{\lambda_n}$ we obtain the formula:

$$V_n^{(\alpha,\beta)}(f;\lambda_n)(x) = \frac{1}{\lambda_n}\sum_{j=0}^{\infty}(1+x)^{-\frac{1}{\lambda_n}}\frac{\frac{1}{\lambda_n}(\frac{1}{\lambda_n}+1)\dots(\frac{1}{\lambda_n}+j-1)}{j!}\frac{x^j}{(1+x)^j}$$

$$\cdot\int_0^{\infty} e^{-\frac{t}{\lambda_n}}\cdot\frac{(\frac{t}{\lambda_n})^j}{j!}f(\frac{\frac{t}{\lambda_n}+\alpha}{\frac{1}{\lambda_n}+\beta})dt.$$

Firstly we need the following auxiliary result.

Lemma 5.1. *We have:*

(i) $V_n^{(\alpha,\beta)}(e_0;\lambda_n)(x)=1$; $V_n^{(\alpha,\beta)}(e_1;\lambda_n)(x)=\frac{1}{1+\lambda_n\beta}x+\frac{\lambda_n+\lambda_n\alpha}{1+\lambda_n\beta}$;

$$V_n^{(\alpha,\beta)}(e_2;\lambda_n)(x)=\frac{1+\lambda_n}{(1+\lambda_n\beta)^2}x^2+\frac{4\lambda_n+2\lambda_n\alpha}{(1+\lambda_n\beta)^2}x+\frac{\lambda_n^2\alpha^2+2\lambda_n^2\alpha+2\lambda_n^2}{(1+\lambda_n\beta)^2};$$

(ii) $V_n^{(\alpha,\beta)}((t-x)^2;\lambda_n)(x)$

$$=\frac{\lambda_n+\lambda_n^2\beta^2}{(1+\lambda_n\beta)^2}x^2+\frac{2\lambda_n-2\lambda_n^2\beta-2\lambda_n^2\alpha\beta}{(1+\lambda_n\beta)^2}x+\frac{\lambda_n^2\alpha^2+2\lambda_n^2\alpha+2\lambda_n^2}{(1+\lambda_n\beta)^2}.$$

Proof. (i) We will make the calculations in three steps:

Step 1. We calculate $U_{n,k}^{(0,0)}(x):=V_n^{(0,0)}(e_k)(x)$, $k=1,2$ by using the recurrence formula (see, e.g., Lemma 2 in [11])

$$U_{n,k}^{(0,0)}(x)=\frac{x(1+x)}{n}\cdot\left[U_{n,k}^{(0,0)}(x)\right]'+\frac{nx+k+1}{n}U_{n,k}^{(0,0)}(x), \tag{5.1}$$

and by taking into account that $U_{n,0}^{(0,0)}(x)=1$. Taking $k=0$ in (5.1), we obtain

$$U_{n,1}^{(0,0)}(x)=\frac{x(1+x)}{n}\cdot(1)'+\frac{nx+1}{n}\cdot 1=\frac{nx+1}{n}=x+\frac{1}{n}.$$

For $k=1$ in (5.1), we get

$$U_{n,2}^{(0,0)}(x)=\frac{x(1+x)}{n}\cdot\left(\frac{nx+1}{n}\right)'+\frac{nx+2}{n}\cdot\frac{nx+1}{n}=\frac{x(1+x)}{n}+\frac{(nx+2)(nx+1)}{n^2}$$

$$=\frac{nx(1+x)+(nx+1)(nx+2)}{n^2}=x^2+\frac{x^2+4x}{n}+\frac{2}{n^2}.$$

Step 2. By direct calculation and based on the results obtained at Step 1, we will obtain the values for $V_n^{(\alpha,\beta)}(e_k)(x):=U_{n,k}^{(\alpha,\beta)}(x)$, $k=0,1,2$. Indeed, based on the formulas

$$\frac{nt+\alpha}{n+\beta}=\frac{n}{n+\beta}t+\frac{\alpha}{n+\beta}, U_{n,k}^{(\alpha,\beta)}(x)=n\sum_{j=0}^{\infty}b_{n,j}(x)\int_0^{\infty}s_{n,j}(t)f(\frac{nt+\alpha}{n+\beta})dt, \tag{5.2}$$

for $k=0$ in (5.2) we obtain

$$U_{n,0}^{(\alpha,\beta)}(x)=V_n^{(\alpha,\beta)}(e_0)(x)=n\sum_{j=0}^{\infty}b_{n,j}(x)\int_0^{\infty}s_{n,j}(t)dt=U_{n,0}^{(0,0)}(x)=1.$$

Then, for $k=1$ in (5.2) it follows

$$U_{n,1}^{(\alpha,\beta)}(x)=n\sum_{j=0}^{\infty}b_{n,j}(x)\int_0^{\infty}s_{n,j}(t)\frac{nt+\alpha}{n+\beta}dt$$

$$=n\sum_{j=0}^{\infty}b_{n,j}(x)\int_0^{\infty}s_{n,j}(t)\frac{n}{n+\beta}tdt+\frac{\alpha}{n+\beta}n\sum_{j=0}^{\infty}b_{n,j}(x)\int_0^{\infty}s_{n,j}(t)dt$$

$$=\frac{n}{n+\beta}\left(n\sum_{j=0}^{\infty}b_{n,j}(x)\int_0^{\infty}s_{n,j}(t)tdt\right)+\frac{\alpha}{n+\beta}\cdot U_{n,0}^{(0,0)}(x)$$

$$= \frac{n}{n+\beta} \cdot U_{n,1}^{(0,0)}(x) + \frac{\alpha}{n+\beta} = \frac{n}{n+\beta}\frac{nx+1}{n} + \frac{\alpha}{n+\beta} = \frac{n}{n+\beta}x + \frac{\alpha+1}{n+\beta}.$$

Finally, for $k = 2$ in (5.2) we obtain

$$U_{n,2}^{(\alpha,\beta)}(x) = V_n^{(\alpha,\beta)}(e_2)(x) = n\sum_{j=0}^{\infty} b_{n,j}(x)\int_0^{\infty} s_{n,j}(t)(\frac{nt+\alpha}{n+\beta})^2 dt$$

$$= \frac{n^2}{(n+\beta)^2}\left(n\sum_{j=0}^{\infty} b_{n,j}(x)\int_0^{\infty} s_{n,j}(t)t^2 dt\right)$$

$$+\frac{2n\alpha}{(n+\beta)^2}\left(n\sum_{j=0}^{\infty} b_{n,j}(x)\int_0^{\infty} s_{n,j}(t)t dt\right)$$

$$+\frac{\alpha^2}{(n+\beta)^2}\left(n\sum_{j=0}^{\infty} b_{n,j}(x)\int_0^{\infty} s_{n,j}(t) dt\right)$$

$$= \frac{n^2}{(n+\beta)^2} \cdot U_{n,2}^{(0,0)}(x) + \frac{2n\alpha}{(n+\beta)^2} \cdot U_{n,1}^{(0,0)}(x) + \frac{\alpha^2}{(n+\beta)^2} \cdot U_{n,0}^{(0,0)}(x)$$

$$= \frac{n^2}{(n+\beta)^2}\frac{nx(1+x)+(nx+1)(nx+2)}{n^2} + \frac{2n\alpha}{(n+\beta)^2}\frac{nx+1}{n} + \frac{\alpha^2}{(n+\beta)^2}$$

$$= \frac{nx(1+x)+(nx+1)(nx+2)+2\alpha(nx+1)+\alpha^2}{(n+\beta)^2}$$

$$= \frac{n^2+n}{(n+\beta)^2}x^2 + \frac{4n+2\alpha n}{(n+\beta)^2}x + \frac{\alpha^2+2\alpha+2}{(n+\beta)^2}.$$

Step 3. We calculate $U_{n,k}^{(\alpha,\beta)}(x)$, $k = 0, 1, 2$, by replacing at Step 2, n with $\frac{1}{\lambda_n}$. It immediately follows

$$V_n^{(\alpha,\beta)}(e_0;\lambda_n)(x) = U_{n,0}^{(\alpha,\beta)}(x;\lambda_n) = 1,$$

$$V_n^{(\alpha,\beta)}(e_1;\lambda_n)(x) = U_{n,1}^{(\alpha,\beta)}(x;\lambda_n) = \frac{1}{1+\lambda_n\beta}x + \frac{\lambda_n+\lambda_n\alpha}{1+\lambda_n\beta},$$

$$V_n^{(\alpha,\beta)}(e_2;\lambda_n)(x) = U_{n,2}^{(\alpha,\beta)}(x;\lambda_n) = \frac{1+\lambda_n}{(1+\lambda_n\beta)^2}x^2 + \frac{4\lambda_n+2\lambda_n\alpha}{(1+\lambda_n\beta)^2}x + \frac{\lambda_n^2\alpha^2+2\lambda_n^2\alpha+2\lambda_n^2}{(1+\lambda_n\beta)^2}.$$

(ii) We have $V_n^{(\alpha,\beta)}((t-x)^2;\lambda_n)(x)$

$$= V_n^{(\alpha,\beta)}(e_2;\lambda_n)(x) - 2xV_n^{(\alpha,\beta)}(e_1;\lambda_n)(x) + x^2V_n^{(\alpha,\beta)}(e_0;\lambda_n)(x)$$

$$= \frac{1+\lambda_n}{(1+\lambda_n\beta)^2}x^2 + \frac{4\lambda_n+2\lambda_n\alpha}{(1+\lambda_n\beta)^2}x + \frac{\lambda_n^2\alpha^2+2\lambda_n^2\alpha+2\lambda_n^2}{(1+\lambda_n\beta)^2}$$

$$-2x\left(\frac{1}{1+\lambda_n\beta}x + \frac{\lambda_n+\lambda_n\alpha}{1+\lambda_n\beta}\right) + x^2$$

$$= \frac{\lambda_n+\lambda_n^2\beta^2}{(1+\lambda_n\beta)^2}x^2 + \frac{2\lambda_n-2\lambda_n^2\beta-2\lambda_n^2\alpha\beta}{(1+\lambda_n\beta)^2}x + \frac{\lambda_n^2\alpha^2+2\lambda_n^2\alpha+2\lambda_n^2}{(1+\lambda_n\beta)^2},$$

which proves the lemma. □

The main result of this section is the following.

Theorem 5.2. *Let* $0 \le \alpha \le \beta$, $\lambda_n \searrow 0$ *as fast we want and suppose that* $f : [0, +\infty) \to \mathbb{R}$ *is uniformly continuous on* $[0, +\infty)$. *For all* $x \in [0, +\infty)$ *and* $n \in \mathbb{N}$, *we have*

$$|V_n^{(\alpha,\beta)}(f; \lambda_n)(x) - f(x)| \le 2\omega_1(f; \sqrt{\lambda_n} \cdot \sqrt{F_n^{(\alpha,\beta)}(x)}),$$

where

$$F_n^{(\alpha,\beta)}(x) = \frac{1 + \lambda_n\beta^2}{(1 + \lambda_n\beta)^2}x^2 + \frac{2 - 2\lambda_n\beta - 2\lambda_n\alpha\beta}{(1 + \lambda_n\beta)^2}x + \frac{\lambda_n\alpha^2 + 2\lambda_n\alpha + 2\lambda_n}{(1 + \lambda_n\beta)^2}.$$

Proof. Reasoning exactly as in the proof of Theorem 2.2, we can write

$$|V_n^{(\alpha,\beta)}(f; \lambda_n)(x) - f(x)| \le (1 + \delta^{-1}\sqrt{V_n^{(\alpha,\beta)}(\varphi_x^2; \lambda_n)(x)})\omega_1(f; \delta).$$

Choosing here $\delta = \sqrt{V_n^{(\alpha,\beta)}(\varphi_x^2; \lambda_n)(x)}$ and using Lemma 5.1, (ii), we obtain

$$|S_n^{(\alpha,\beta)}(f; \lambda_n)(x) - f(x)| \le 2\omega_1(f; \sqrt{\lambda_n} \cdot \sqrt{F_n^{(\alpha,\beta)}(x)}),$$

which proves the theorem. □

As an immediate consequence of Theorem 5.2 we get the following.

Corollary 5.3. *Let* $0 \le \alpha \le \beta$, $\lambda_n \searrow 0$ *as fast we want and suppose that* f *is a Lipschitz function, that is there exists* $M > 0$ *such that* $|f(x) - f(y)| \le M|x - y|$, *for all* $x, y \in [0, \infty)$. *Then, for all* $x \in [0, +\infty)$ *and* $n \in \mathbb{N}$ *we have*

$$|V_n^{(\alpha,\beta)}(f; \lambda_n)(x) - f(x)| \le 2M\sqrt{\lambda_n} \cdot \sqrt{F_n^{(\alpha,\beta)}(x)}.$$

Proof. Since by hypothesis f is a Lipschitz function, we easily get $\omega_1(f; \delta) \le M\delta$, for all $\delta > 0$. Choosing now $\delta = \sqrt{\lambda_n} \cdot \sqrt{F_n^{(\alpha,\beta)}(x)}$ and applying Theorem 5.2 we get the desired estimate. □

Remarks. 1) All the Remarks 1)-3) made at the end of Section 2 remain valid for the generalized Baskakov-Szász-Durrmeyer-Stancu, $V_n^{(\alpha,\beta)}(f; \lambda_n)$, operators too.

2) Note that in Theorems 2.2, 3.2, 4.2 and 5.2, for any $\delta > 0$ and $f : [0, +\infty) \to \mathbb{R}$ uniformly continuous, the modulus of continuity $\omega_1(f; \delta)$ is finite. For the reader's convenience, we present below the proof. Indeed, for a fixed ε_0, from the definition of the uniform continuity of f, there exists a $\delta_0 > 0$, such that $|f(x) - f(y)| < \varepsilon_0$, for all $x, y \in [0, +\infty)$ with $|x - y| \le \delta_0$. Passing here to supremum after these x, y, it immediately follows that $\omega_1(f; \delta_0) \le \varepsilon_0 < +\infty$. Let now $\delta > \delta_0$ be arbitrary. Evidently that there exists a sufficiently large $p \in \mathbb{N}$, such that $\delta \le p \cdot \delta_0$. Using now the monotonicity and the subadditivity of $\omega_1(f; \delta)$ as function of δ, we get

$$\omega_1(f; \delta) \le \omega_1(f; p\delta_0) \le p \cdot \omega_1(f; \delta_0) < +\infty.$$

Finally, we may conclude that the approximation results obtained for all the operators in this paper are of a definitive character, i.e. they furnish arbitrary good orders of approximation. It is also worth noting that the method in this paper does not work for the positive and linear operators expressed by finite sums (like Bernstein polynomials, Kantorovich polynomials, etc).

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Fixed point theorems for generalized contraction mappings on b-rectangular metric spaces

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Abstract. In the present article, we study some fixed point theorems for a hybrid class of generalized contractive operators in the context of b-rectangular metric spaces. Examples justifying theorems and an open problem regarding to further generalizations for this type of operators are also given.

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1. Introduction and preliminaries

In this section we shall present some useful lemmas and definitions regarding rectangular and b-rectangular metric spaces. Also, we shall present some recent results in the field of fixed point theory concerning expansive operators and some generalized contraction mappings.
In [6], A. Branciari introduced a new metric-type space, when triangle inequality is replaced by an inequality which involves four different elements. This is called a rectangular metric space or a generalized metric space (g.m.s.)

Definition 1.1. Let $X \neq \emptyset$, $d : X \times X \to [0, \infty)$, such that for each $x, y \in X$ and $u, v \in X$ (each distinct from x and y), we have that

(1) $d(x, y) = 0 \Longleftrightarrow x = y$,
(2) $d(x, y) = d(y, x)$,
(3) $d(x, y) \leq d(x, u) + d(u, v) + d(v, y)$.

Furthermore, from [10] we mention that convergent sequences and Cauchy sequences can be introduced in a similar manner as in metric spaces.
Also, from the same paper, we know that if (X, d) is a rectangular metric space and if (x_n) is a b-rectangular Cauchy sequence with the property that $x_n \neq x_m$, for each $n \neq m$, then (x_n) converge to at most one point, i.e. the property that (X, d) is

Haussdorf becomes superfluous.
Moreover, from [8], [9], [22], we recall the definition of b-rectangular metric spaces (or b-generalized metric spaces), briefly b-g.m.s.

Definition 1.2. Let $X \neq \emptyset$, $s \geq 1$ be a given real number and $d : X \times X \to [0, \infty)$, such that for each $x, y \in X$ and $u, v \in X$ (each distinct from x and y), we have that

(1) $d(x, y) = 0 \Longleftrightarrow x = y$,

(2) $d(x, y) = d(y, x)$,

(3) $d(x, y) \leq s\,[d(x, u) + d(u, v) + d(v, y)]$.

As in metric spaces, we recall the basic notions regarding sequences in b-g.m.s:

Definition 1.3. Let (X, d) be a b-g.m.s, $x \in X$ and $(x_n) \subset X$ be a given sequence. Then

(a) (x_n) is convergent in (X, d) to an element $x \in X$, if for each $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$, such that $d(x_n, x) < \varepsilon$, for each $n > n_0$. We denote this by $\lim\limits_{n\to\infty} x_n = x$.

(b) (x_n) is Cauchy in (X, d) (or b-rectangular Cauchy, briefly b-g.m.s.), if for each $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$, such that $d(x_n, x_{n+p}) < \varepsilon$, for each $n > n_0$ and for each $p > 0$. We denote this by $\lim\limits_{n\to\infty} d(x_n, x_{n+p}) = 0$, for each $p > 0$.

(c) (X, d) is said to be complete b-g.m.s, if every Cauchy sequence in X converges to some $x \in X$.

We recall the following important remark from [8]:

Remark 1.4. (1) Every metric space and every rectangular metric space (g.m.s) is b-g.m.s.

(2) The limit of a sequence in a b-rectangular metric space is not unique.

(3) Every convergent sequence in a b-g.m.s is not necessarily a b-g.m.s Cauchy.

For this, we recall a crucial lemma from [8], i.e. *(Lemma 1.5)*, that specify when a b-rectangular Cauchy sequence can't have two limits in a b-g.m.s.

Lemma 1.5. *Let (X, d) be a b-rectangular metric space, with the coefficient $s \geq 1$. Let (x_n) be a b-rectangular Cauchy sequence in X, such that $x_n \neq x_m$, for each $n \neq m$. Then (x_n) can converge to at most one point.*

Also, we recall from [12] and [8] the following crucial lemma.

Lemma 1.6. *Let (X, d) be a b-rectangular metric space, with the coefficient $s \geq 1$. Also, let (x_n) be a sequence for which $x_n \neq x_m$, for every $n \neq m$, with $\lim\limits_{n\to\infty} d(x_n, x_{n+1}) = 0$. If (x_n) is not a b-rectangular Cauchy sequence, then there exists $\varepsilon > 0$, such that for each $k \in \mathbb{N}$, there exists $(m(k))$ and $(n(k))$ two sequences of positive integers, such that*

$$\begin{aligned}
&d(x_{m(k)}, x_{n(k)}) \geq \varepsilon, \\
&\frac{\varepsilon}{s} \leq \limsup_{k\to\infty} d(x_{m(k)}, x_{n(k)-2}) \leq \varepsilon \text{ and} \\
&\frac{\varepsilon}{s} \leq \limsup_{k\to\infty} d(x_{m(k)+1}, x_{n(k)-1}).
\end{aligned}$$

In [22], another crucial lemma regarding sequences in b-rectangular metric spaces was presented. For convenience, we remind it below.

Lemma 1.7. *Let (X, d) be a b-g.m.s., with coefficient $s \geq 1$.*

(a) Consider two sequences (x_n) and (y_n), such that x_n converges to $x \in X$ and y_n converges to $y \in X$, with $x \neq y$. Also, suppose that for each $n \in \mathbb{N}$, $x_n \neq x$ and $y_n \neq y$. Then

$$\frac{1}{s}d(x,y) \leq \liminf_{n\to\infty} d(x_n, y_n) \leq \limsup_{n\to\infty} d(x_n, y_n) \leq sd(x,y).$$

(b) Consider an element $y \in X$ and a b-rectangular Cauchy sequence (x_n), such that $x_n \neq x_m$, for each $n \neq m$. Moreover, suppose that the sequence (x_n) converges to an element $x \neq y$. Then

$$\frac{1}{s}d(x,y) \leq \liminf_{n\to\infty} d(x_n, y) \leq \limsup_{n\to\infty} d(x_n, y) \leq sd(x,y).$$

Finally, for the convenience of the reader, we recall some important results in b-rectangular metric spaces. In [9], George et.al.studied basic contraction-type mappings in b-rectangular metric spaces, like Kannan operators, i.e.

$$d(Tx, Ty) \leq \lambda \left[d(x, Tx) + d(y, Ty)\right], \text{ with } \lambda \in \left[0, \frac{1}{s+1}\right].$$

In [8], Radenovic et.al. extended the results to mappings satisfying

$$d(fx, gy) \leq ad(gx, gy) + b\left[d(gx, fx) + d(gy, fy)\right],$$

for each $x, y \in X$ and studied unique coincidence and common fixed points for the pair of operators (f, g) that satisfies some additional assumptions.
Also, for more results in b-rectangular metric spaces and for a consistent survey on different generalized metric-type spaces, we recommend [11] and [12].
Now, regarding generalized contraction mappings we recall some recent advances in this subfield of fixed point theory.
In [13], Karapinar studied unique fixed points for some generalized contractions on cone Banach spaces satisfying the following contractive-type conditions

$$d(x, Tx) + d(y, Ty) \leq pd(x, y), \text{ where } p \in [0, 2)$$

and

$$ad(Tx, Ty) + b\left[d(x, Tx) + d(y, Ty)\right] \leq sd(x, y), \text{ with } 0 \leq s + |a| - 2b < 2(a + b).$$

Moreover, in 2009, Kumar [14] presented some theorems for two maps satisfying the following

$$d(fx, fy) \geq qd(gx, gy), \text{ with } q > 1,$$

where f is onto and g is one-to-one.
Moosaei, Azizi, Asadi and Wang generalized the results of Karapinar as follows
In [15], Moosaei used Krasnoselskii's iteration defined in convex metric spaces, for the following mappings, that satisfy

$$d(Tx, Ty) + d(x, Tx) + d(y, Ty) \leq rd(x, y), \text{ where } r \in [2, 5),$$

respectively

$$ad(x, fx) + bd(y, fy) + cd(fx, fy) \leq kd(x, y), \text{ with } 2b - |c| \leq k < 2(a + b + c) - |c|.$$

In [17], Moosaei and Azizi extended the results to generalized contraction-type operators, studying coincidence points for various mappings, such as

$$ad(Sx, Tx) + bd(Sy, Ty) + cd(Tx, Ty) \leq ed(x, y),$$

where $T(K) \subset S(K)$, K and $S(K)$ are closed and convex subsets of a convex metric space and the coefficients satisfy

$$2b - |c| \leq e < 2(a + b + c) - |c|.$$

Nevertheless, in 2014, Moosaei [16] studied a more generalized pair of contractions (S, T), where

$$\alpha d(Tx, Ty) + \beta \left[d(Sx, Tx) + d(Sy, Ty)\right] + \gamma \left[d(Sx, Ty) + d(Sy, Tx)\right] \leq \eta d(Sx, Sy),$$

with some assumptions on contractive-coefficients, i.e.

$$2\beta + \gamma - |\gamma| - \alpha \leq \eta < \alpha + 2\beta + 3\gamma - |\gamma| \text{ and } \beta + \gamma \leq 0.$$

Asadi in [3], using the same iteration (Krasnoselskii) on convex metric spaces, studied fixed points for generalized Hardy-Rogers type-mappings, as follows

$$ad(x, Tx) + bd(y, Ty) + cd(Tx, Ty) + ed(Ty, x) + fd(y, Tx) \leq kd(x, y),$$

where

$$\frac{b + e - |f|(1 - \lambda) - |c|\lambda}{1 - \lambda} \leq k < \frac{a + b + c + e + f - |c|\lambda - |f|(1 - \lambda)}{1 - \lambda},$$

and $\lambda \in [0, 1]$ is the coefficient of Krasnoselskii's iteration.
Furthermore, Wang and Zhang, in [23] extended the above results for pairs of generalized Hardy-Rogers type contractions.
Now, expansive and expansive-type mappings can be considered a particular case of generalized contractions. Regarding the former ones, we recall some recent development into the study of this type of operators.
In 2011, Aage [1] considered expansive mappings in cone metric spaces. The more general form of these mappings, with some underlying assumptions, are

$$d(Tx, Ty) \geq kd(x, y) + ld(x, Tx) + pd(y, Ty),$$

where T satisfies $K \geq -1$, $p < 1$, $l > 1$ and $k + l + p > 1$.
Aydi et.al. studied in [4] some interesting fixed point theorems for pairs of expansive mappings for spaces endowed with c-distances. We recall them using the standard notations for metric spaces, i.e.

$$d(Tx, Ty) \geq ad(fx, fy) + bd(Tx, fx) + cd(Ty, fy),$$

with $b < 1$, $a \neq 0$, $f(X) \subseteq T(X)$ and $(T(X), d) \subset (X, d)$ complete.
Also, in cone rectangular metric spaces, some fixed point theorems were developed. For example, in [20], pair of mappings satisfying

$$d(fx, fy) \geq \alpha d(gx, gy) + \beta d(fx, gx) + \gamma d(fy, gy)$$

were studied, with some assumptions on the coefficients α, β and γ and on the range of g and f.
These pairs of generalized mappings were extended by Olaoluwa and Olaleru in [18], but in the framework of b-metric spaces and for a pair of four mappings, as follows

$$d(fx, gy) \geq a_1 d(Sx, Ty) + a_2 d(fx, Sx) + a_3 d(gy, Ty) + a_4 d(fx, Ty) + a_5 d(gy, Sx).$$

Also, for the sake of convenience, we recall other studies in metric-type spaces and for expansive-type mappings, as follows: in [24] generalized mappings were studied on cone rectangular metric spaces using the technique of scalarizing, in [21] mappings that satisfy

$$d(Tx, Ty) \leq \varphi(d(x, y))$$

were studied on cone rectangular metric spaces and in [19], fixed point theorems for a general type of expansive mappings were developed, satisfying

$$\phi(d(S^2x, TSy)) \geq \frac{1}{3}\left[d(Sx, S^2x) + d(TSy, Sy) + d(Sx, Sy)\right].$$

Also, in the context of dislocated metric spaces, Daheriya et.al. [7] studied rational-type expansive mappings, and in [2] Alghamdi studied fixed points for generalized expansive mappings in b-metric like spaces.
The purpose of this work is to extend some fixed results for a hybrid class of generalized contractive-type mappings and for some expansive-type operators in the context of b-rectangular metric spaces. Moreover, at the end of the second section, we shall let and open problem.

2. Main results

Moosaei in [15] used Krasnoselskii iteration to develop fixed point theorems for generalized contractions on convex metric spaces. It is easily seen that we can use Picard instead of Krasnoselkii sequences in metric spaces.
In this section, our aim is to extend the results of Moosaei [15] for generalized contraction mappings from metric spaces to b-rectangular metric spaces. Also, we extend and develop the fixed point results of Aage [1] from cone metric spaces to b-g.m.s. Furthermore, we extend results from [20] of Patil, from rectangular metric spaces to b-rectangular ones (b-g.m.s).
Also, examples similar to those in [1], [12] and [20] justifying our theorems are given.
Now, let's consider generalized contractions $f : X \to X$ on a b-g.m.s. X, satisfying the following condition:

$$ad(x, fx) + bd(y, fy) + cd(fx, fy) \leq kd(x, y).$$

We will analyze two separate cases: when $c > 0$ and $c < 0$. Also, for expansive-type mappings, i.e. when $c < 0$, we consider two types of sequence, namely the classical Picard iteration $x_{n+1} = fx_n$, for each $n \in \mathbb{N}$ and the 'inverse' Picard iteration, i.e. $x_n = fx_{n+1}$, for each $n \in \mathbb{N}$, for which we require that the operator f is onto.
Our first result is a theorem for the existence and uniqueness of the fixed point of a mapping satisfying the contractive condition from above. The technique we will use is based on the (*Lemma 1.6*).

Theorem 2.1. *Let (X, d) be a complete b-rectangular metric space (b-gms), with coefficient $s > 1$. Consider a mapping $f : X \to X$, satisfying the following contractive condition*

$$ad(x, fx) + bd(y, fy) + cd(fx, fy) \le kd(x, y), \text{ where } 0 \le k - b < \frac{a + c}{s}.$$

Also, suppose the following assumptions are satisfied

(A) If $c > 0$ and $k \ge 0$, then $\frac{k}{c} < \frac{1}{s}$,

(B) If $c > 0$ and $k \le 0$, then we have no additional conditions,

(C) If $c < 0$ and $k < 0$, then $\frac{k}{c} > s^2$.

Then, the Picard sequence (x_n), defined as $x_{n+1} = fx_n$, for each $n \in \mathbb{N}$ converges to a fixed point of the mapping f.

Proof. We consider the Picard iterative process (x_n), defined as $x_{n+1} = fx_n$, for each $n \in \mathbb{N}$. Applying the contractive condition for the pair (x_{n-1}, x_n), we get that

$$\begin{aligned} &ad(x_n, fx_n) + bd(x_{n-1}, fx_{n-1}) + cd(fx_{n-1}, fx_n) \le kd(x_{n-1}, x_n) \\ &ad(x_n, x_{n+1}) + bd(x_{n-1}, x_n) + cd(x_n, x_{n+1}) \le kd(x_{n-1}, x_n) \\ &(a + c)d(x_n, x_{n+1}) \le (k - b)d(x_{n-1}, x_n) \end{aligned}$$

So $d(x_n, x_{n+1}) \le \delta d(x_{n-1}, x_n)$, where $\delta := \frac{k - b}{a + c} \in \left[0, \frac{1}{s}\right)$ from the theorem's assumptions, since $0 \le k - b < \frac{a + c}{s}$.

So $d(x_n, x_{n+1}) \le \delta^n d(x_0, x_1)$. Since $\delta \in \left[0, \frac{1}{s}\right)$, it follows that $\lim_{n\to\infty} d(x_n, x_{n+1}) = 0$.
Also, by a routine argument (by reductio ad absurdum), it follows easily that $x_n \ne x_{n+1}$, for each $n \in \mathbb{N}$ and that $x_n \ne x_m$, for each $n \ne m$.
The next step is to show that the sequence (x_n) is b-rectangular Cauchy. We will use *(Lemma 1.6)* and we shall apply it on three different cases
(1) Case $c > 0$: Let's suppose that the sequence (x_n) is not b-rectangular Cauchy. Then, there exists $\varepsilon > 0$ and two sequences of nonnegative real numbers $(m(k))$ and $(n(k))$, such that the assumptions from *(Lemma 1.6)* are satisfied.
Now, we will apply the contraction condition for $x = x_{m(k)}$ and $y = x_{n(k)-2}$. It follows that

$$ad(x_{m(k)}, x_{m(k)+1}) + bd(x_{n(k)-2}, x_{n(k)-1}) + cd(x_{m(k)+1}, x_{n(k)-1}) \le kd(x_{m(k)}, x_{n(k)-2})$$

$$cd(x_{m(k)+1}, x_{n(k)-1}) \le kd(x_{m(k)}, x_{n(k)-2}) - ad(x_{m(k)}, x_{m(k)+1}) - bd(x_{n(k)-2}, x_{n(k)-1}).$$

Because $c > 0$, we have that

$$d(x_{m(k)+1}, x_{n(k)-1}) \le \frac{k}{c}d(x_{m(k)}, x_{n(k)-2}) - \frac{a}{c}d(x_{m(k)}, x_{m(k)+1}) - \frac{b}{c}d(x_{n(k)-2}, x_{n(k)-1}).$$

Now, we want to apply the limit superior. We make the following necessary remark and consider the following cases
If $a \ge 0$, then $-\frac{a}{c} \le 0$, so $-\frac{a}{c}d(x_{m(k)}, x_{m(k)+1}) \le 0$, so an upper bound for this element is 0.

If $a \leq 0$, then $-\frac{a}{c} \geq 0$, so $-\frac{a}{c} d(x_{m(k)}, x_{m(k)+1}) \geq 0$. Applying the limit superior, we get that

$$\begin{aligned}\limsup_{k\to\infty}\left(-\frac{a}{c}\right) d(x_{m(k)}, x_{m(k)+1}) &= \left(-\frac{a}{c}\right)\limsup_{k\to\infty} d(x_{m(k)}, x_{m(k)+1}) \\ &= \left(-\frac{a}{c}\right)\lim_{k\to\infty} d(x_{m(k)}, x_{m(k)+1}) = 0.\end{aligned}$$

The same reasoning can be made about the sign of the coefficient b and about the limit superior of the sequence $(d(x_{n(k)-2}, x_{n(k)-1}))$ as a subsequence of $(d(x_n, x_{n-1}))$.
Case (A): When $k \geq 0$.
Since $k \geq 0$, we have that $\frac{k}{c} \geq 0$. We know that $\limsup_{k\to\infty} d(x_{m(k)}, x_{n(k)-2}) \leq \varepsilon$. Multiplying by $\left(\frac{k}{c}\right)$ and taking the limit superior, we get that

$$\begin{aligned}\limsup_{k\to\infty}\left(\frac{k}{c}\right) d(x_{m(k)}, x_{n(k)-2}) &= \limsup_{k\to\infty}\left|\frac{k}{c}\right| d(x_{m(k)}, x_{n(k)-2}) \\ &= \frac{k}{c}\limsup_{k\to\infty} d(x_{m(k)}, x_{n(k)-2}) \leq \frac{k}{c}\varepsilon.\end{aligned}$$

From *(Lemma 1.6)*, it follows that $\frac{\varepsilon}{s} \leq \limsup_{k\to\infty} d(x_{m(k)+1}, x_{m(k)-1}) \leq \frac{k}{c}\varepsilon$, so $\frac{1}{s} \leq \frac{k}{c}$. This is a contradiction with the assumption that in this case we have $\frac{k}{c} < \frac{1}{s}$.
Case (B): When $k \leq 0$.
In this case we have that $\frac{k}{c} \leq 0$, so $\frac{k}{c} d(x_{m(k)}, x_{n(k)-2}) \leq 0$, then we can take 0 as an upper bound for it. By *(Lemma 1.6)*, we have that $\frac{\varepsilon}{s} \leq 0$. Since $\varepsilon > 0$ and $s \geq 1$, we got a contradiction.
Now, in the two cases from above, we have shown that (x_n) is b-rectangular Cauchy. Moreover, we have said that $x_n \neq x_m$, for each $n \neq m$.
Since (X, d) is complete, it implies that there exists $u \in X$, such that $x_n \to u$, i.e.

$$\lim_{n\to\infty} d(x_n, u) = 0.$$

Now, we shall show that u is a fixed point for f

$$\begin{aligned}d(u, fu) &\leq s\left[d(u, x_n) + d(x_n, x_{n+1}) + d(x_{n+1}, fu)\right] \\ &= s\left[d(u, x_n) + d(x_n, x_{n+1}) + d(fx_n, fu)\right]\end{aligned}$$

Since $c > 0$, then

$$d(fx_n, fu) \leq \frac{k}{c} d(x_n, u) - \frac{b}{c} d(x_n, x_{n+1}) - \frac{a}{c} d(u, fu).$$

So

$$d(u, fu) \leq s\left[d(u, x_n) + d(x_n, x_{n+1}) + \frac{k}{c} d(x_n, u) - \frac{b}{c} d(x_n, x_{n+1}) - \frac{a}{c} d(u, fu)\right]$$

Taking the limit when $n \to \infty$, we get

$$\left(1 + s\frac{a}{c}\right) d(u, fu) \leq 0,$$

so $(c + sa)d(u, fu) \leq 0$. Furthermore, since $c > 0$ and $0 < (a + c) < (a + cs)$, then u is a fixed point for f.

(2) Case $c < 0$**:** We have that

$$\begin{aligned} &ad(x, fx) + bd(y, fy) + cd(fx, fy) \leq kd(x, y) \\ &cd(fx, fy) \leq kd(x, y) - ad(x, fx) - bd(y, fy) \end{aligned}$$

So

$$d(fx, fy) \geq \frac{k}{c}d(x, y) - \frac{a}{c}d(x, fx) - \frac{b}{c}d(y, fy).$$

This is a case of expansive-type mapping. By *(Lemma 1.6)*, there exists $\varepsilon > 0$, such that for every $k \in \mathbb{N}$, there exists $(m(k)), (n(k))$ two sequences of nonnegative real numbers such that the assumptions in the already mentioned lemma are true. By b-rectangular inequality, we have that

$$d(x_{n(k)-2}, x_{m(k)}) \leq s\left[d(x_{m(k)-1}, x_{n(k)-3}) + d(x_{n(k)-3}, x_{n(k)-2}) + d(x_{m(k)-1}, x_{m(k)})\right]$$

$$sd(x_{m(k)-1}, x_{n(k)-3}) \geq d(x_{n(k)-2}, x_{m(k)}) - sd(x_{n(k)-3}, x_{n(k)-2}) - sd(x_{m(k)-1}, x_{m(k)})$$

Dividing by $s \geq 1$, we obtain the following

$$d(x_{m(k)-1}, x_{n(k)-3}) \geq \frac{1}{s}d(x_{n(k)-2}, x_{m(k)}) - d(x_{n(k)-3}, x_{n(k)-2}) - d(x_{m(k)-1}, x_{m(k)}).$$

Case (C): When $k < 0$: Here we have that $\frac{k}{c} \geq 0$. Multiplying by $\left(\frac{k}{c}\right)$, it implies that

$$\begin{aligned} \frac{k}{c}d(x_{m(k)-1}, x_{n(k)-3}) \geq \frac{k}{cs}d(x_{n(k)-2}, x_{m(k)}) &- \frac{k}{c}d(x_{n(k)-3}, x_{n(k)-2}) \\ &- \frac{k}{c}d(x_{m(k)-1}, x_{m(k)}). \end{aligned}$$

Now, we apply the contractive condition for $x = x_{m(k)-1}$ and $y = x_{n(k)-3}$, i.e.

$$d(x_{m(k)}, x_{n(k)-2}) \geq \frac{k}{c}d(x_{m(k)-1}, x_{n(k)-3}) - \frac{a}{c}d(x_{m(k)-1}, x_{m(k)}) - \frac{b}{c}d(x_{n(k)-3}, x_{n(k)-2}).$$

So, combining the above inequalities, we get that

$$\begin{aligned} d(x_{m(k)}, x_{n(k)-2}) \geq \frac{k}{cs}d(x_{n(k)-2}, x_{m(k)}) &- \frac{k}{c}d(x_{n(k)-3}, x_{n(k)-2}) - \frac{k}{c}d(x_{m(k)-1}, x_{m(k)}) \\ &- \frac{a}{c}d(x_{m(k)-1}, x_{m(k)}) - \frac{b}{c}d(x_{n(k)-3}, x_{n(k)-2}). \end{aligned}$$

From the limit superior, we have get the following

$$\begin{aligned}\limsup_{k\to\infty}\left(-\frac{k}{c}\right)d(x_{n(k)-3},x_{n(k)-2}) &= \frac{k}{c}\limsup_{k\to\infty} -d(x_{n(k)-3},x_{n(k)-2})\\ &= -\frac{k}{c}\liminf_{k\to\infty} d(x_{n(k)-3},x_{n(k)-2})\\ &= \left(-\frac{k}{c}\right)\lim_{k\to\infty} d(x_{n(k)-3},x_{n(k)-2}) = 0\end{aligned}$$

We have the same reasoning for $d(x_{m(k)-1},x_{m(k)})$, with coefficient $-\frac{k}{c}$. Also, for coefficients a and b, we have that
If $a \geq 0$, then $-\frac{a}{c} \geq 0$, so $\left(-\frac{a}{c}\right)d(x_{m(k)-1},x_{m(k)}) \geq 0$, so we can make the lower bound 0.
If $a \leq 0$, then $-\frac{a}{c} \leq 0$, so $\left(-\frac{a}{c}\right)d(x_{m(k)-1},x_{m(k)}) \leq 0$, so taking the limit superior, it follows that:

$$\begin{aligned}\limsup_{k\to\infty}\left(-\frac{a}{c}\right)d(x_{m(k)-1},x_{m(k)}) &= \frac{a}{c}\limsup_{k\to\infty} -d(x_{m(k)-1},x_{m(k)})\\ &= -\frac{a}{c}\liminf_{k\to\infty} d(x_{m(k)-1},x_{m(k)})\\ &= -\frac{a}{c}\lim_{k\to\infty} d(x_{m(k)-1},x_{m(k)}) = 0\end{aligned}$$

Same remarks can be made about the coefficient b and for $d(x_{n(k)-3},x_{n(k)-2})$.
By *(Lemma 1.6)*, we get that

$$\varepsilon \geq \limsup_{k\to\infty} d(x_{m(k)},x_{n(k)-2}) \geq \frac{k}{cs}\limsup_{k\to\infty} d(x_{m(k)},x_{n(k)-2}) \geq \frac{\varepsilon k}{cs^2}.$$

So $\frac{1}{s^2} \leq \frac{c}{k}$. This is a contradiction with the fact that in this case $\frac{k}{c} > s^2$.
Now, since $x_n \neq x_m$, for each $n \neq m$, $d(x_n,x_{n+1}) \to 0$, (x_n) Cauchy b-rectangular and (X,d) is complete, then there exists $u \in X$, such that $x_n \to u$. We shall show that u is a fixed point for the mapping f.
Applying the contractive condition on the pair (u,x_n), we get

$$ad(u,fu) + bd(x_n,fx_n) + cd(fu,fx_n) \leq kd(u,x_n)$$
$$ad(u,fu) + bd(x_n,x_{n+1}) + cd(fu,x_{n+1}) \leq kd(u,x_n)$$

Letting $n \to \infty$, we have $(a+c)d(u,fu) \leq 0$ and since we know that $a+c > 0$, it follows that u is a fixed point for the mapping f. □

Relative to (*Theorem 2.1*), we give two examples that validate cases (A) and(C): From [12], we recall an example of a complete b-rectangular metric space.

Example 2.2. Let $X = A \cup B$, where $A = \left\{\frac{1}{n}\middle| n = \overline{2,5}\right\}$ and $B = [1,2]$. We define $d : X \times X \to [0,\infty)$, such that $d(x,y) = d(y,x)$ and

$$d\left(\frac{1}{2},\frac{1}{3}\right) = d\left(\frac{1}{4},\frac{1}{5}\right) = \frac{3}{100},$$

$$d\left(\frac{1}{2},\frac{1}{5}\right)=d\left(\frac{1}{3},\frac{1}{4}\right)=\frac{2}{100},$$
$$d\left(\frac{1}{4},\frac{1}{3}\right)=d\left(\frac{1}{5},\frac{1}{3}\right)=\frac{6}{100},$$

$d(x,y)=(x-y)^2$, otherwise.
Then (X,d) is a complete b-rectangular metric space, with coefficient $s=3$. Furthermore, (X,d) is not a metric space or a rectangular metric space.

Regarding case (A) of (*Theorem 2.1*), we give the following example.

Example 2.3. Let (X,d) be the b-rectangular metric space defined above, with $s=3$. Also, define $f:X\to X$, such as

$$f(x)=\begin{cases}\frac{1}{3}, & x\in A\\ \frac{1}{5}, & x\in B\end{cases}$$

It is easy to observe that f has a unique fixed point $\frac{1}{3}$. Moreover, we shall show that f satisfies

$$1\cdot d(fx,fy)\le\frac{1}{52}d(x,y)+\frac{1}{4}d(x,fx)+\frac{23}{100}d(y,fy),$$

for each $x,y\in X$.
Let's define: $a=\frac{-1}{4}$, $b=\frac{-23}{100}$, $k=\frac{1}{52}$, $c=1$ and $s=3$.
We have the following cases

1) $x\in A$ and $y\in A$: $d(fx,fy)=d\left(\frac{1}{3},\frac{1}{3}\right)=0$, so the above inequality is valid.

2) $x\in B$ and $y\in B$: $d(fx,fy)=d\left(\frac{1}{5},\frac{1}{5}\right)=0$, so the inequality of f is true.

Now, for the non-trivial cases, it follows that:

3) $x\in A$ and $y\in B$:

$$d(fx,fy)=\left(\frac{1}{3},\frac{1}{5}\right)=\frac{6}{100},$$
$$d(x,fx)=d\left(x,\frac{1}{3}\right)\ge\min_{x\in A}d\left(x,\frac{1}{3}\right)=\frac{1}{200},$$
$$d(y,fy)=d\left(y,\frac{1}{5}\right)=\left(y-\frac{1}{5}\right)^2=y^2-\frac{2}{5}y+\frac{1}{25}\ge\min_{y\in[1,2]}=1-\frac{1}{4}+\frac{1}{25}=\frac{6}{25}.$$

Also $d(x,y)=(y-x)^2=|y-x|^2$.
We have that

$$d(fx,fy)\le kd(x,y)+(-a)\min_{x\in A}d(x,fx)+(-b)\min_{y\in B}d(y,fy).$$

So

$$\frac{6}{100}\le\frac{1}{52}|y-x|^2+\frac{1}{4}\cdot\frac{1}{200}+\frac{23}{100}\cdot\frac{6}{25},$$

so $\frac{1}{52}|y-x|^2 \geq \frac{-6619}{12000}$, which is obviously true.

4) $x \in B$ and $y \in A$

$$d(fx, fy) = \left(\frac{1}{3}, \frac{1}{5}\right) = \frac{6}{100},$$

$$d(x, fx) \geq \min_{x \in B} d(x, fx) = \frac{6}{25},$$

$$d(y, fy) \geq \min_{y \in A} = \frac{1}{200}$$

and

$$d(x, y) = (y-x)^2 = |y-x|^2.$$

We have that

$$\frac{6}{100} \leq \frac{1}{52}|y-x|^2 + \frac{1}{4} \cdot \frac{6}{25} + \frac{23}{100} \cdot \frac{1}{200},$$

so $\frac{1}{52}|y-x|^2 \geq \frac{-419}{6000}$, which is also true.

Moreover, we show that the conditions from *(Theorem 2.1) - case (A)* on the coefficients are satisfied

$$c > 0 \Leftrightarrow 1 > 0$$

$$k > 0 \Leftrightarrow \frac{1}{52} > 0$$

$$a + c = 1 - \frac{1}{4} = \frac{3}{4} > 0$$

$$b \leq k \Leftrightarrow -\frac{23}{100} \leq \frac{1}{52}$$

$$\frac{k}{c} < \frac{1}{s} \Leftrightarrow k < \frac{1}{3} \Leftrightarrow 3 < 52$$

$$k < b + \frac{a+c}{s} \Leftrightarrow \frac{1}{52} + \frac{23}{100} < \frac{1}{4} \Leftrightarrow 324 < 325$$

Now, we construct an example of a complete b-rectangular metric space, which will be used further in this section.

Example 2.4. Let $X = \{1, 2, 3, 4\}$ and define $d : X \times X \to [0, \infty)$, such as

$$d(1,2) = d(2,1) = \frac{6}{10}$$

$$d(1,3) = d(3,1) = \frac{1}{10}$$

$$d(2,3) = d(3,2) = \frac{1}{10}$$

$$d(1,4) = d(4,1) = d(2,4) = d(4,2) = d(3,4) = d(4,3) = \frac{2}{10}$$

We will prove that (X, d) is a b-rectangular metric space with coefficient $s = \frac{3}{2}$, which is not a rectangular metric space.

For a b-rectangular metric space, we have that $d(x,y) \leq s\left[d(x,u)+d(u,v)+d(v,y)\right]$, for each $u,v \notin \{x,y\}$, with u,v being distinct. We have the following cases.
- When $x=y$, the right hand side is 0, so the above inequality remains valid.
- When $x \neq y$, we employ the following sub-cases

Case (1): If $x=1$ and $y=2$ ($x=2$ and $y=1$ by symmetry):

$$\frac{6}{10} \leq s\left[d(1,u)+d(u,v)+d(v,2)\right], \text{ for } u,v \notin \{1,2\}, \text{ i.e. } u,v \in I_1=\{3,4\}$$
$$\frac{6}{10} = d(1,2) \leq s\left[\min_{u\in I_1} d(1,u)+d(3,4)+\min_{v\in I_1} d(v,2)\right]$$
$$\frac{6}{10} \leq s\left[\frac{1}{10}+\frac{2}{10}+\frac{1}{10}\right], \text{ so } s \geq \frac{3}{2}$$

Case (2): If $x=3$ and $y=1$ ($x=1$ and $y=3$ by symmetry):

$$\frac{1}{10} \leq s\left[d(3,u)+d(u,v)+d(v,1)\right], \text{ for } u,v \notin \{1,3\}, \text{ i.e. } u,v \in I_2=\{2,4\}$$
$$\frac{1}{10} = d(3,1) \leq s\left[\min_{u\in I_2} d(3,u)+d(2,4)+\min_{v\in I_2} d(v,1)\right]$$
$$\frac{1}{10} \leq s\left[\frac{1}{10}+\frac{2}{10}+\frac{1}{10}\right], \text{ so } s \geq \frac{1}{4}$$

Case (3): If $x=4$ and $y=1$ ($x=1$ and $y=4$ by symmetry):

$$\frac{2}{10} \leq s\left[d(3,u)+d(u,v)+d(v,1)\right], \text{ for } u,v \notin \{1,4\}, \text{ i.e. } u,v \in I_3=\{2,3\}$$
$$\frac{2}{10} = d(4,1) \leq s\left[\min_{u\in I_3} d(4,u)+d(2,3)+\min_{v\in I_3} d(v,1)\right]$$
$$\frac{2}{10} \leq s\left[\frac{2}{10}+\frac{1}{10}+\frac{1}{10}\right], \text{ so } s \geq \frac{1}{2}$$

Case (4): If $x=2$ and $y=4$ ($x=4$ and $y=2$ by symmetry):

$$\frac{2}{10} \leq s\left[d(2,u)+d(u,v)+d(v,4)\right], \text{ for } u,v \notin \{2,4\}, \text{ i.e. } u,v \in I_4=\{1,3\}$$
$$\frac{2}{10} = d(4,2) \leq s\left[\min_{u\in I_4} d(2,u)+d(1,3)+\min_{v\in I_4} d(v,4)\right]$$
$$\frac{2}{10} \leq s\left[\frac{1}{10}+\frac{1}{10}+\frac{2}{10}\right], \text{ so } s \geq \frac{1}{2}$$

Case (5): If $x=3$ and $y=4$ ($x=4$ and $y=3$ by symmetry):

$$\frac{2}{10} \leq s\left[d(3,u)+d(u,v)+d(v,4)\right], \text{ for } u,v \notin \{3,4\}, \text{ i.e. } u,v \in I_5=\{1,2\}$$
$$\frac{2}{10} = d(3,4) \leq s\left[\min_{u\in I_5} d(3,u)+d(1,2)+\min_{v\in I_5} d(4,v)\right]$$
$$\frac{2}{10} \leq s\left[\frac{1}{10}+\frac{6}{10}+\frac{2}{10}\right], \text{ so } s \geq \frac{2}{9}$$

So $s \geq \frac{3}{2} > 1$,so we can take $s = \frac{3}{2}$.
Furthermore, (X, d) is not a b-g.m.s., because

$$\frac{6}{10} = d(1,2) > d(1,3) + d(3,u) + d(u,2) = \frac{1}{10} + \frac{2}{10} + \frac{2}{10} = \frac{5}{10},$$

so $6 > 5$, which is valid.

Now, we construct an example, justifying *case (C)* of *(Theorem 2.1)*.

Example 2.5. Let $X = \{1,2,3,4\}$ the b-rectangular metric space defined above, with coefficient $s = \frac{3}{2}$.

Let $f(x) = \begin{cases} 3, & x \neq 4 \\ 1, & x = 4 \end{cases}$ a self-mapping defined on X.

We shall show that f satisfies

$$d(fx, fy) \geq (-3)d(x,y) - 5d(x,fx) + 3d(y,fy)$$

and also the conditions from *case (C)* of *(Theorem 2.1)*.
Let f satisfy $cd(fx,fy) \geq kd(x,y) - ad(x,fx) - bd(y,fy)$. Let's normalize the contractive condition, by taking $c = -1 < 0$ We shall determine the coefficients k, a, b, with $k < 0, a > 0$ and $b < 0$. We have the following cases

1) If $x = y$, then $d(fx,fy) = d(fx,fx) = 0$, so the left hand side is 0. Now, the right hand side is $k \cdot 0 - ad(x,fx) - bd(x,fx) = -(a+b)d(x,f)$. This implies that $(a+b)d(x,fx) \geq 0$. We have two sub-cases:
If $x = 3$, then $d(x,fx) = d(3,3) = 0$, so the inequality is valid. Also, if $\neq 3$, then $d(x,fx) > 0$, so we have the condition that $-b \leq a$.

2) If $\neq y$, we have the following sub-cases

a) For $x = 4$ and $y \neq 4$, it follows that $d(fy,fx) = d(fy,1)$. Since $y \neq 4$, then $fy = 3$, so $d(fx,fy) = d(1,3) = \frac{1}{10}$.
Moreover, one can easily verify that $d(x,y) = d(4,y) = \frac{2}{10}$, for each $y \neq 4$, $d(x,fx) = d(4,fx) = \frac{2}{10}$, for each $x \in X$ and $d(y,fy) = d(y,3) \leq \max_{y \neq 4} d(y,3) = \frac{2}{10}$.

b) For $y = 4$ and $x \neq 4$, it follows that $d(fx,fy) = \frac{1}{10}$.
Moreover, we have that $d(x,y) = d(4,x) = \frac{2}{10}$, for each $x \neq 4$, $d(x,fx) = d(x,3) =\geq \min_{x \neq 4} d(x,3) = \frac{1}{10}$ and $d(y,fy) = d(4,fy) = \frac{2}{10}$, for each value of fy.

c) For $y \neq y \neq 4$ (simultaneously), it follows that $d(fx,fy) = d(3,3) = 0$. Also

$$kd(x,y) - ad(x,fx) - bd(y,fy) \leq 0, \quad \text{so} \quad kd(x,y) - bd(x,y) \leq ad(x,fx).$$

Now $d(x,y) \geq \min_{x,y\in X} d(x,y) = \frac{1}{10}$.
Furthermore, we have that

$$d(y,fy) = d(y,3) \leq \max_{y\neq 4} d(y,3) = \frac{2}{10} \quad \text{and} \quad d(x,fx) = d(x,3) \geq \min_{x\neq 4} d(x,3) = \frac{1}{10}.$$

Now, we analyze the conditions on f.
For the case (1), we get $-b \leq a$. For the case (2a), we get that

$$\begin{aligned} d(fx,fy) = \frac{1}{10} &\geq kd(x,y) - ad(x,fx) - b\max_{y\neq 4} d(y,fy) \\ &= \frac{2k}{10} - \frac{2a}{10} - \frac{2b}{10} \geq \frac{2k}{10} - \frac{2a}{10} - bd(y,fy), \end{aligned}$$

because $b<0$. So $k < a+b+\frac{1}{2}$.
For the case (2b), we obtain

$$\begin{aligned} d(fx,fy) = \frac{1}{10} &\geq kd(x,y) - a\min_{x\neq 4} d(x,fx) - bd(y,fy) \\ &= \frac{2k}{10} - \frac{a}{10} - \frac{2b}{10} \geq \frac{2k}{10} - \frac{2b}{10} - ad(x,fx), \end{aligned}$$

because $a>0$. So $k < \frac{a}{2}+b+\frac{1}{2}$.
For the case (2c), it follows that

$$\begin{aligned} d(fx,fy) = 0 &\geq k\min_{x\neq y\neq 4} d(x,y) - a\min_{x\neq 4} d(x,fx) - b\max_{y\neq 4} d(y,fy) \\ &= \frac{k}{10} - \frac{a}{10} - \frac{2b}{10} kd(x,y) - ad(x,fx) - bd(y,fy), \end{aligned}$$

because $b,k<0$ and $a>0$, so $k - sb \leq a$.
Additionally, f satisfies the conditions from *(Theorem 2.1) - Case (C)*.
Let's take $k=-3, c=-1, a=5, b=-3$, with $s=\frac{3}{2}$. We verify that the coefficients a,b,c,k verify all of the above conditions

$$\begin{cases} -b \leq a \Leftrightarrow 3 \leq 5,\ k < a+b+\frac{1}{2} \Leftrightarrow -3 < 2+\frac{1}{2} \\ k < \frac{a}{2}+b+\frac{1}{2} \Leftrightarrow 10+\frac{1}{2} > 0,\ k-2b \leq a \Leftrightarrow 3 > 1 \\ b \leq k \Leftrightarrow -3 \leq -3,\ \frac{k}{c} > s^2 \Leftrightarrow 12 > 9 \\ k < b+\frac{a+c}{s} \Leftrightarrow 6 > 0,\ a+c > 0 \Leftrightarrow 6 > 0 \end{cases}$$

Remark 2.6. We observe that the contractive condition when $c>0$, can be written as:

$$d(fx,fy) \leq \frac{k}{c} d(x,y) - \frac{a}{c} d(x,fx) - \frac{b}{c} d(y,fy), \text{ for each } x,y \in X.$$

Taking $k>0, a<0$ and $b<0$, it follows that the operator f is of Reich-type, so the above theorem (when $k>0$) is similar with the results of [8].

Now, we present an useful lemma for expansive-type mappings in b-rectangular metric spaces, following the technique used in [18].

Lemma 2.7. *Let (X, d) a b-rectangular metric space. Also, consider $\lambda \in \mathbb{R}$ and x, y, z, w arbitrary elements of X, each distinct from each other. Then*

$$\lambda d(x,z) \geq \left[\frac{1+s^2}{2s}\lambda + \frac{1-s^2}{2s}|\lambda|\right] d(x,y) + \left[\frac{s-1}{2}\lambda - \frac{s+1}{2}|\lambda|\right] d(z,w)$$
$$+ \left[\frac{s-1}{2}\lambda - \frac{s+1}{2}|\lambda|\right] d(w,y).$$

Proof. Let x, y, z, w arbitrary points from X, each distinct from each other. We analyze two cases for the parameter $\lambda \in \mathbb{R}$:
Case (1): Let $\lambda \geq 0$. From the b-rectangular inequality, we get that:

$$\begin{aligned} d(x,y) &\leq s\left[d(x,z) + d(z,w) + d(w,y)\right] \\ sd(x,z) &\geq d(x,y) - sd(z,w) - sd(w,y) \\ d(x,z) &\geq \frac{1}{s}d(x,y) - d(z,w) - d(w,y) \\ \lambda d(x,z) &\geq \frac{\lambda}{s}d(x,y) - \lambda d(z,w) - \lambda d(w,y) \end{aligned}$$

Case (2): Let $\lambda \leq 0$. From the b-rectangular inequality, it follows that:

$$\begin{aligned} d(x,z) &\leq s\left[d(x,y) + d(y,w) + d(w,z)\right] \\ \lambda d(x,z) &\geq \lambda sd(x,y) + \lambda sd(y,w) + \lambda sd(w,z) \end{aligned}$$

So, from the above inequality, we have that

$$\begin{cases} \lambda d(x,z) \geq \frac{\lambda}{s}d(x,y) - \lambda d(z,w) - \lambda d(w,y), & \lambda \geq 0 \\ \lambda d(x,z) \geq \lambda sd(x,y) + \lambda sd(y,w) + \lambda sd(w,z), & \lambda \leq 0 \end{cases}$$

Combining these cases, it follows that
$\lambda d(x,z) \geq \varphi(\lambda)d(x,y) + \psi(\lambda)d(z,w) + \psi(\lambda)d(w,y)$, where

$$\varphi(\lambda) := \begin{cases} \frac{\lambda}{s}, & \lambda \geq 0 \\ s\lambda, & \lambda \leq 0 \end{cases} \text{ and } \psi(\lambda) := \begin{cases} -\lambda, & \lambda \geq 0 \\ s\lambda, & \lambda \leq 0 \end{cases}$$

Similar to [18], we get that

$$\begin{cases} \varphi(\lambda) := \frac{1+s^2}{2s}\lambda + \frac{1-s^2}{2s}|\lambda| \\ \psi(\lambda) := \frac{s-1}{2}\lambda - \frac{s+1}{2}|\lambda| \end{cases}$$

Also, as a final remark, we observe that $\psi(\lambda) \leq 0$, for each $\lambda \in \mathbb{R}$. □

For expansive-type mappings, i.e. when $c < 0$, we make the following important remark.

Remark 2.8. We have studied contraction-type mappings, that satisfied

$$ad(x, fx) + bd(y, fy) + cd(fx, fy) \leq kd(x, y)$$
$$cd(fx, fy) \leq kd(x, y) - ad(x, fx) - bd(y, fy)$$
$$d(fx, fy) \geq \frac{k}{c}d(x, y) - \frac{a}{c}d(x, fx) - \frac{b}{c}d(y, fy)$$

By some substitutions we can make the mapping f satisfy

$$d(fx, fy) \geq \alpha d(x, y) + \beta d(x, fx) + \gamma d(y, fy),$$

where

$$\begin{cases} \alpha = \dfrac{k}{c} \\ \beta = -\dfrac{a}{c} \\ \gamma = -\dfrac{b}{c} \end{cases}$$

We will analyze the cases when $\alpha \leq 0$ and $\alpha \geq 0$, so, when $k \geq 0, c < 0$, respectively $k \leq 0, c < 0$.

Now, involving rate of convergence, we present a constructive fixed point theorem for expansive-type mappings in b-rectangular metric spaces, using Picard iterative process.

Theorem 2.9. *Let (X, d) a complete b-rectangular metric space, endowed with coefficient $s \geq 1$. Also, consider $f : X \to X$ a mapping satisfying*

$$d(fx, fy) \geq \alpha d(x, y) + \beta d(x, fx) + \gamma d(y, fy), \text{ for each } x, y \in X.$$

Moreover, suppose the following conditions are satisfied

(i) $\beta < 1 - s$, $\gamma > s$, $\alpha + \gamma < \dfrac{1-\beta}{s}$,

(ii) If $\alpha > \gamma$, then we have the additional assumptions $\alpha + 1 < \gamma\left(1 + \dfrac{1}{s}\right)$.

If $\alpha < \gamma$, then we have the additional assumptions $\alpha > 1$ and $1 - \alpha < \gamma\left(\dfrac{1}{s} - 1\right)$.

Then, the mapping f has a fixed point.

Proof. In the proof of *(Theorem 2.1)*, we have shown that the Picard sequence for generalized contraction satisfy $d(x_n, x_{n+1}) \leq \delta d(x_{n-1}, x_n)$, for each $n \in \mathbb{N}$, where $\delta = \dfrac{k-b}{a+c}$. This is also valid for the situation of expansive-type mappings, when $c < 0$. The condition that the Picard sequence is asymptotically regular was that $0 \leq k - b < \dfrac{a+c}{s}$.

In our case,

$$\delta = \frac{k-b}{a+c} = \frac{\frac{k}{c} - \frac{b}{c}}{\frac{a}{c} + 1} = \frac{\alpha + \gamma}{1 - \beta}.$$

Now $\delta \in \left[0, \frac{1}{s}\right)$, by hypothesis assumptions: $\beta < 1$, $\alpha + \gamma > 0$ and $\alpha + \gamma < \frac{1-\beta}{s}$.
By the contractive-type condition, we have that

$$d(fx, fy) \geq \alpha d(x, y) + \beta d(x, fx) + \gamma d(y, fy)$$

and applying it for the pair (x_{n-1}, x_{n+1}), we obtain

$$d(x_n, x_{n+2}) \geq \alpha d(x_{n-1}, x_{n+1}) + \beta d(x_{n-1}, x_n) + \gamma d(x_{n+1}, x_{n+2}) \qquad (2.1)$$

Now, we will try to evaluate an upper bound for $d(x_n, x_{n+2})$, for each $n \in \mathbb{N}$, i.e. using *(Lemma 2.7)*, we obtain that

$$\gamma d(x_{n+1}, x_{n+2}) \geq \varphi(\gamma) d(x_n, x_{n+2}) + \psi(\gamma) d(x_n, x_{n-1}) + \psi(\gamma) d(x_{n-1}, x_{n+1}).$$

Now, let's denote by $d_n^* := d(x_n, x_{n+2})$ and by $d_n := d(x_{n-1}, x_n)$, for each $n \in \mathbb{N}$.
From (2.1) we have

$$d_n^* \geq \alpha d_{n-1}^* + \beta d_n + \varphi(\gamma) d_n^* + \psi(\gamma) d_n + \psi(\gamma) d_{n-1}^*.$$

This means that

$$[\varphi(\gamma) - 1]\, d_n^* \leq [-\psi(\gamma) - \alpha]\, d_{n-1}^* + [-\psi(\gamma) - \beta]\, d_n \leq |\psi(\gamma) + \alpha| d_{n-1}^* + |\psi(\gamma) + \beta| d_n.$$

Let's denote by $a_2 := \frac{|\alpha + \psi(\gamma)|}{\varphi(\gamma) - 1}$ and by $a_1 := \frac{|\beta + \psi(\gamma)|}{\varphi(\gamma) - 1}$.
From the hypothesis, we know that $\varphi(\gamma) > 1$, i.e. $\gamma > s > 0$, since $\varphi(\gamma) = \frac{\gamma}{s}$. Then it follows that a_1 and a_2 are positive.
Furthermore, since $\gamma > 0$, we have that $\psi(\gamma) = -\gamma < 0$. So $a_2 = \frac{|\alpha - \gamma|}{\frac{\gamma}{s} - 1}$. For $a_2 < 1$, we get that $|\alpha - \gamma| < \frac{\gamma}{s} - 1$. So, we have two cases:

- When $\alpha > \gamma$, i.e. $\alpha - \gamma > 0$:

Then, the condition that $a_2 < 1$ becomes $\alpha + 1 < \frac{\gamma}{s} + \gamma$, i.e. $\alpha + 1 < \gamma\left(1 + \frac{1}{s}\right)$.
Now, since $\gamma + 1 < \alpha + 1 < \gamma\left(1 + \frac{1}{s}\right)$, then $s < \gamma$, which is true. Also, since $\gamma + 1 < \alpha + 1 < \gamma\left(1 + \frac{1}{s}\right) < 2\gamma$, then $1 < \gamma$, which is a valid assumption.
Moreover, from the hypothesis condition that $\alpha + \gamma < \frac{1-\beta}{s}$, we employ two sub-cases
If $\beta > 0$, then $1 - \beta < 1$, i.e. $\alpha + \gamma < \frac{1}{s} < 1$, so $\alpha + \gamma < 1$. Since $\alpha, \gamma > s > 1$, this is obviously not true.
If $\beta < 0$, then $\beta < 1$, so $1 - \beta > 0$ (the denominator in δ is positive, so δ is positive). Since $\beta < 0$, then $\frac{1-\beta}{s} > \frac{1}{s}$. Moreover, since $\alpha + \gamma > 1$, then we get $\beta < 1 - s$, which is valid from hypothesis (ii).
Finally, we can verify easily that since $s > 1$, then $\beta < 1$ and since $1 - s < 1$, then $s > 0$, which are evidently true.

- We know verify the case when $\alpha < \gamma$, i.e. $\alpha - \gamma < 0$:

Since $|\alpha-\gamma| = \gamma-\alpha < \frac{\gamma}{s} - 1$, then $1-\alpha < \gamma\left(\frac{1}{s}-1\right)$, which is true by hypothesis (ii).
Moreover, since $\frac{1}{s} - 1 < 0$, then $\alpha > 1$ is obviously true, also by hypothesis. Also, since $\gamma > \alpha > 1$, then $\gamma > 1$, which is valid by the fact that $\gamma > s$.
Also, as in previous case, by the assumption on δ that $\alpha+\gamma < \frac{1-\beta}{s}$, if $\beta > 0$, then $\alpha+\gamma < \frac{1-\beta}{s} < 1$, which contradicts the fact that $\alpha, \gamma > 1$.
So $\beta < 0$ and from the assumption that $\beta < 1-s$ means that the right hand side $\frac{1-\beta}{s} > 1$, so $1 < \alpha+\gamma < \frac{1-\beta}{s}$, which is valid.
So $d_n^* \leq a_2 d_{n-1}^* + a_1 d_n$, for each $n \in \mathbb{N}$. We know that

$$d_n = d(x_{n-1}, x_n) \leq \delta d(x_{n-1}, x_{n-2}) \leq \ldots \leq \delta^{n-1} D_0,$$

where $D_0 := d_1 = d(x_0, x_1)$, with x_0 an arbitrary fixed element.
So $d_n^* \leq a_2 d_{n-1}^* + a_1 \delta^{n-1} D_0$.
We take a major bound for d_n^* :

$$\begin{aligned}
d_n^* &\leq a_2 d_{n-1}^* + a_1 \delta^{n-1} D_0 \leq a_2 (a_2 d_{n-2}^* + a_1 \delta^{n-2} D_0) + a_1 \delta^{n-1} D_0 \\
&= a_2^2 d_{n-2}^* + a_2 a_1 \delta^{n-2} D_0 + a_1 \delta^{n-1} D_0 \\
&\leq a_2^2 (a_2 d_{n-3}^* + a_1 \delta^{n-3} D_0) + a_1 a_2 \delta_{n-2} D_0 + a_1 \delta^{n-1} D_0 \\
&= a_2^3 d_{n-3}^* + D_0 a_1 \left(\delta^{n-1} + a_2 \delta^{n-2} + a_2^2 \delta^{n-3}\right) D_0 \leq \ldots \\
&\leq a_2^k d_{n-k}^* + a_1 \left(\delta^{n-1} + a_2 \delta^{n-2} + \ldots + a_2^{k-1} \delta^{n-k}\right) D_0
\end{aligned}$$

The last term is $d_0^* = d(x_2, x_0)$, so $n-k = 0 \Longrightarrow k = n$. This means that

$$d_n^* \leq a_2^n d_0^* + a_1 D_0 \left(a_2^0 \delta^{n-1} + a_2 \delta^{n-2} + \ldots + a_2^{n-1} \delta^0\right)$$

Let's denote by $S := a_2^0 \delta^{n-1} + a_2 \delta^{n-2} + \ldots + a_2^{n-1} \delta^0$. The first term in the sum is δ^{n-1}. This is a geometric progression, with general term b_n and $\frac{b_3}{b_2} = a_2 \frac{\delta^{n-3}}{\delta^{n-2}} = \frac{a_2}{\delta}$, so

$$S = \frac{\delta^{n-1} \cdot \left(1 - \left(\frac{a_2}{\delta}\right)^n\right)}{1 - \left(\frac{a_2}{\delta}\right)} = \frac{\delta^n - a_2^n}{\delta - a_2}.$$

So $d_n^* \leq a_2^n d_0^* + \frac{\delta^n - a_2^n}{\delta - a_2} a_1 D_0$. Now we can show that the sequence (x_n) is b-rectangular Cauchy. We shall evaluate $d(x_n, x_{n+p})$, for each $n \in \mathbb{N}$ and $p > 0$ fixed. We divide in two cases: the first one, when $p = 2m$, with $m \geq 2$ and the second one, when $p = 2m+1$, with $m \geq 1$:
Case (i): When $p = 2m+1$, with $m \geq 1$. We evaluate

$$\begin{aligned}
&d(x_n, x_{n+p}) = d(x_n, x_{n+2m+1}) \leq s\left[d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+1}) + d(x_{n+2}, x_{n+2m+1})\right] \\
&\leq s\left[d_{n+2} + d_{n+1}\right] + s^2\left[d(x_{n+2}, x_{n+3}) + d(x_{n+3}, x_{n+4}) + d(x_{n+4}, x_{n+2m+1})\right] \\
&\leq s\left[d_{n+2} + d_{n+1}\right] + s^2\left[d_{n+3} + d_{n+4}\right] + s^3\left[d_{n+5} + d_{n+6}\right] + \ldots + s^m d_{n+2m},
\end{aligned}$$

where $d_{n+2m} = d(x_{n+2m}, x_{n+2m+1})$. So, we get the following estimation

$$\begin{aligned} d(x_n, x_{n+2m+1}) &\le s\left[\delta^n D_0 + \delta^{n+1} D_0\right] + s^2\left[\delta^{n+2} D_0 + \delta^{n+3} D_0\right] \\ &+ s^3\left[\delta^{n+4} D_0 + \delta^{n+5} D_0\right] + \ldots + s^m \delta^{n+2m} D_0 \\ &\le s\delta^n\left[1 + s\delta^2 + s^2\delta^4 + \ldots +\right] D_0 + s\delta^{n+1}\left[1 + s\delta^2 + s^2\delta^4 + \ldots +\right] D_0 \\ &= \frac{1+\delta}{1-s\delta^2} s\delta^n D_0, \end{aligned}$$

and by hypothesis we know that $s\delta^2 < 1$ is satisfied. So, $d(x_n, x_{n+2m+1}) \to 0$, when $n \to \infty$ and $m \ge 1$ fixed.

Case (ii): When $p = 2m$, with $m \ge 2$. We evaluate

$$\begin{aligned} d(x_n, x_{n+2m}) &\le s\left[d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + d(x_{n+2}, x_{n+2m})\right] \\ &\le s\left[d_{n+2} + d_{n+1}\right] + sd(x_{n+2}, x_{n+2m}) \\ &\le s\left[d_{n+2} + d_{n+1}\right] + s^2\left[d_{n+4} + d_{n+3}\right] + s^3\left[d_{n+6} + d_{n+5}\right] + \ldots + \\ &+ s^{m-1}\left[d_{2m-3} + d_{2m-2}\right] + s^{m-1} d(x_{n+2m-2}, x_{n+2m}) \\ &\le s\left[\delta^n D_0 + \delta^{n+1} D_0\right] + s^2\left[\delta^{n+2} D_0 + \delta^{n+3} D_0\right] + \ldots + \\ &+ s^{m-1}\left[\delta^{2m-4} D_0 + \delta^{2m-3} D_0\right] + s^{m-1} d(x_{n+2m-2}, x_{n+2m}) \\ &\le s\delta^n\left[1 + s\delta^2 + s^2\delta^4 + \ldots\right] D_0 \\ &+ s\delta^{n+1}\left[1 + s\delta^2 + s^2\delta^4 + \ldots\right] D_0 + s^{m-1} d^*_{n+2m} \\ &= \frac{1+\delta}{1-s\delta^2} s\delta^n D_0 + s^{m-1} d^*_{n+2m} \end{aligned}$$

Also, we have shown that $d^*_n \le a_2^n d^*_0 + \dfrac{\delta^n - a_2^n}{\delta - a_2} a_1 D_0$. So $d^*_{n+2m} \le a_2^{n+2m} d^*_0 + Q a_1 D_0$, where $Q := \dfrac{\delta^{n+2m} - a_2^{n+2m}}{\delta - a_2}$.

Now, we have two cases: if $\delta - a_2 > 0$, then $Q = \dfrac{\delta^{n+2m} - a_2^{n+2m}}{\delta - a_2} \le \dfrac{\delta^{n+2m}}{\delta - a_2}$ and this converge to 0 as $n \to \infty$. In a similar manner, if $\delta - a_2 < 0$, then

$$Q = \frac{a_2^{n+2m} - \delta^{n+2m}}{a_2 - \delta} \le \frac{a_2^{n+2m}}{a_2 - \delta},$$

and this converge to 0 as $n \to \infty$. This reasoning is valid, since, from the theorem's assumptions, we know that $0 \le a_2 < 1$ and $\delta < \dfrac{1}{s} < 1$. So, in this case, since $Q \to 0$, then $d(x_n, x_{n+2m}) \to 0$, as $n \to \infty$.

So, from both cases, we have shown that (x_n) is a b-rectangular Cauchy sequence. Also, we know that $x_n \ne x_m$, for each $n \ne m$ and that (X, d) is complete. This means that there exists $u \in X$, such that $\lim_{n\to\infty} x_n = u$.

Moreover, since the contractive condition can be reduced to the original form, i.e. $ad(x, fx) + bd(y, fy) + cd(fx, fy) \le kd(x, y)$, then, as in the proof of *(Theorem 2.1)*, there exists a unique point u of f, as long as $a + c > 0$ and $c < k$. □

Finally, we give an example regarding (*Theorem 2.9*).

Example 2.10. Let (X, d), with $X = \{1, 2, 3, 4\}$ be the b-rectangular metric space, endowed with the b-rectangular metric from (*Example 2.2*). Define a self-mapping f, by: $f(1) = 2$, $f(2) = 3$, $f(3) = 1$ and $f(4) = 4$. It is obviously that f has as a unique fixed point the element $4 \in X$. We will determine the coefficients α, β and γ, such that f satisfies $d(fx, fy) \geq \alpha d(x, y) + \beta d(x, fx) + \gamma d(y, fy)$:

$$\text{By } x = 2 \text{ and } y = 1, \text{ we get that } \frac{1}{10} \geq \alpha\frac{6}{10} + \beta\frac{1}{10} + \gamma\frac{6}{10} \tag{2.2}$$

$$\text{By } x = 1 \text{ and } y = 2, \text{ we get that } \frac{1}{10} \geq \alpha\frac{6}{10} + \beta\frac{6}{10} + \gamma\frac{1}{10} \tag{2.3}$$

$$\text{By } x = 1 \text{ and } y = 3, \text{ we get that } \frac{6}{10} \geq \alpha\frac{1}{10} + \beta\frac{6}{10} + \gamma\frac{1}{10} \tag{2.4}$$

$$\text{By } x = 3 \text{ and } y = 1, \text{ we get that } \frac{6}{10} \geq \alpha\frac{1}{10} + \beta\frac{1}{10} + \gamma\frac{6}{10} \tag{2.5}$$

$$\text{By } x = 1 \text{ and } y = 4, \text{ we get that } \frac{2}{10} \geq \alpha\frac{2}{10} + \beta\frac{6}{10} + \gamma\frac{2}{10} \tag{2.6}$$

$$\text{By } x = 4 \text{ and } y = 1, \text{ we get that } \frac{2}{10} \geq \alpha\frac{2}{10} + \beta\frac{2}{10} + \gamma\frac{6}{10} \tag{2.7}$$

$$\text{By } x = 3 \text{ and } y = 2, \text{ we get that } \frac{1}{10} \geq \alpha\frac{1}{10} + \beta\frac{1}{10} + \gamma\frac{1}{10} \tag{2.8}$$

$$\text{By } x = 2 \text{ and } y = 3, \text{ we get that } \frac{1}{10} \geq \alpha\frac{1}{10} + \beta\frac{1}{10} + \gamma\frac{1}{10} \tag{2.9}$$

$$\text{By } x = 4 \text{ and } y = 2, \text{ we get that } \frac{2}{10} \geq \alpha\frac{2}{10} + \beta\frac{2}{10} + \gamma\frac{1}{10} \tag{2.10}$$

$$\text{By } x = 2 \text{ and } y = 4, \text{ we get that } \frac{2}{10} \geq \alpha\frac{2}{10} + \beta\frac{1}{10} + \gamma\frac{2}{10} \tag{2.11}$$

$$\text{By } x = 4 \text{ and } y = 3, \text{ we get that } \frac{2}{10} \geq \alpha\frac{2}{10} + \beta\frac{2}{10} + \gamma\frac{1}{10} \tag{2.12}$$

$$\text{By } x = 3 \text{ and } y = 4, \text{ we get that } \frac{2}{10} \geq \alpha\frac{2}{10} + \beta\frac{1}{10} + \gamma\frac{2}{10} \tag{2.13}$$

$$\text{By } x = y, \text{ we get that } \beta + \gamma \leq 0 \tag{2.14}$$

Now, we observe that (2.11) and (2.14) are equivalent relations. Also, we shall employ the more restrictive conditions on the coefficients α, β and γ, i.e. inequalities (2.11), (2.3), (2.5), (2.7), (2.8) and (2.14). Furthermore, we shall impose more restrictive conditions such that the number of inequalities is reduced: instead of (2.11) and (2.3), we impose that $1 \geq 6\alpha + \beta + 2\gamma$, instead of (2.7) and (2.8) we require only (2.7) and instead of $1 \geq 6\alpha + \beta + 2\gamma$ and (2.5), we require $1 \geq 6\alpha + \beta + 6\gamma$. We mention that all of the above reasoning was made under the assumptions that $\beta \leq 0$ and $\gamma > 0$. Now, we have only two conditions, along with the conditions from (*Theorem 2.9*), when $\alpha > \gamma$

$$\begin{cases} \beta + \gamma \leq 0, 1 \geq 6\alpha + \beta + 6\gamma \\ \beta < 1 - s, \gamma > s, \alpha\gamma \\ \alpha + \gamma < \dfrac{1 - \beta}{s}, \alpha + 1 < \gamma\left(1 + \dfrac{1}{s}\right) \end{cases}$$

Now, taking account of the fact that $s = \frac{3}{2}$, we can find some values for the coefficients α, β and γ. For example, the inequalities are satisfied when $\alpha = \frac{9}{50}$, $\beta = -\frac{101}{5}$ and $\gamma = \frac{17}{100}$.

Now, we recall *(Lemma 2)* from [5], that is crucial for inequalities involving difference inequations.

Lemma 2.11. *Let (a_n) and (b_n) be two sequences of nonnegative real numbers, such that*

$$a_{n+1} \leq \alpha_1 a_n + \alpha_2 a_{n-1} + \ldots + \alpha_k a_{n-k+1} + b_n, \text{ where } n \geq k-1.$$

If $\alpha_1, \ldots, \alpha_k \in [0,1)$, $\sum_{i=1}^{k} \alpha_i < 1$ and $\lim_{n\to\infty} b_n = 0$, then it follows that $\lim_{n\to\infty} a_n = 0$.

Remark 2.12. In the previous proof, we have shown that the following estimation is valid

$$d_n^* = d(x_{n+2}, x_n) \leq a_2^n d_0^* + \frac{\delta^n - a_2^n}{\delta - a_2} a_1 D_0.$$

So, based on this lemma, we give a nonconstructive approach for evaluating (x_n) as a Cauchy sequence.
In the above lemma, let's take $k = 1$. Then, we get that $a_{n+1} \leq \alpha_1 a_n + b_n$, with $\alpha_1 \in [0,1)$ and $\lim_{n\to\infty} b_n = 0$. Then $\lim_{n\to\infty} a_n = 0$.
Now, we have proved that $d_n^* \leq a_2 d_{n-1}^* + a_1 \delta^{n-1} D_0$.
Let's define the following: $\alpha_1 := a_2$ and $b_n := a_1 D_0 \delta^{n-1}$. Since $\delta < \frac{1}{s} < 1$ and $a_2 \in [0,1)$, then apply *(Lemma 2)* from [5] with the particular case when $k = 1$, we get that $\lim_{n\to\infty} d_n^* = 0$.

Now, we give a proof for expansive-type mappings under the new assumption such that the mapping f is onto and we shall use the 'inverse' Picard iterative process.

Theorem 2.13. *Let (X,d) be a complete b-rectangular metric space and $f : X \to X$ a mapping satisfying*

$$d(fx, fy) \geq \alpha d(x,y) + \beta d(x, fx) + \gamma d(y, fy).$$

Let f continuous and onto. Suppose that

(i) $\beta < 1$, $\alpha + \gamma > 0$ and $1 - \beta < \frac{\alpha+\gamma}{s}$.

Also, suppose the following additional assumptions

Case (E1), i.e. $\alpha > 0$**:** *Suppose that the following assumptions are satisfied:*

(ii) $\alpha > 1$

Case (E2), i.e. $\alpha < 0$*: Suppose the following assumptions are satisfied:*

(ii) $\alpha < -1$, $\gamma > 0$

(iii) $s\left(1 - \frac{\alpha}{\gamma}\right) < 1 + \frac{1}{\alpha}$

Then, the mapping f has a fixed point in X.

Proof. Here, we know that f is continuous and onto. Let x_0 be an arbitrary point. As we have shown in the previous theorem, i.e. *(Theorem 2.9)*, we reduce the contractive condition to

$$d(fx, fy) \geq \alpha d(x, y) + \beta d(x, fx) + \gamma d(y, fy).$$

Because f is an onto mapping, by definition, we have that for each $y \in X$, there exists $x \in X$, such that $y = fx$.
Now, for $x_0 \in X$, there exists $x_1 \in X$, such that $x_0 = fx_1$. Also, for $x_1 \in X$, there exists $x_2 \in X$, such that $x_1 = fx_2$. Inductively, we get that $x_n = fx_{n+1}$, for each $n \in \mathbb{N}$.
Applying the contractive condition on the pair (x_{n+1}, x_n), it follows that:

$$\begin{aligned}
&d(fx_{n+1}, fx_n) \geq \alpha d(x_n, x_{n+1}) + \beta d(x_n, fx_n) + \gamma d(x_{n+1}, fx_{n+1})\\
&d(x_n, x_{n-1}) \geq \alpha d(x_n, x_{n+1}) + \beta d(x_n, x_{n-1}) + \gamma d(x_{n+1}, x_n)\\
&\implies (\alpha + \gamma) d(x_{n+1}, x_n) \leq (1 - \beta) d(x_{n-1}, x_n)\\
&\implies d(x_n, x_{n+1}) \leq \theta d(x_{n-1}, x_n),
\end{aligned}$$

where $\theta := \dfrac{1-\beta}{\alpha+\gamma}$. From the hypothesis,we know that $\theta \in \left[0, \dfrac{1}{s}\right)$, because $\beta < 1$, $\alpha+\gamma > 0$ and $1-\beta < \dfrac{\alpha+\gamma}{s}$. Furthermore, we have that $d_{n+1} := d(x_{n+1}, x_n) \leq \theta^n d_1$.
For simplicity, let's denote by $D_0 := d_1 = d(x_1, x_0)$.
Furthermore, as in the previous theorem, let $d_n^* := d(x_n, x_{n+2})$, for each $n \in \mathbb{N}$.
Now, we shall analyze two different cases for estimation of $d(x_n, x_{n+2})$

Case (E1): When $\alpha > 0$, or with the original notation, $\dfrac{k}{c} > 0$. Since $c < 0$, we get that $k < 0$.
Applying the expansive-type condition on the pair (x_n, x_{n+2}), it follows that

$$\begin{aligned}
&d(x_{n-1}, x_{n+1}) = d(fx_n, fx_{n+2}) \geq \alpha d(x_n, x_{n+2}) + \beta d(x_n, fx_n) + \gamma d(x_{n+2}, fx_{n+2})\\
&= \alpha d(x_n, x_{n+2}) + \beta d(x_n, x_{n-1}) + \gamma d(x_{n+1}, x_{n+2}) \implies\\
&\alpha d(x_n, x_{n+2}) \leq d(x_{n-1}, x_{n+1}) - \beta d(x_{n-1}, x_n) - \gamma d(x_{n+1}, x_{n+2})\\
&d(x_n, x_{n+2}) \leq \frac{1}{\alpha} d_{n-1}^* + \left(-\frac{\beta}{\alpha}\right) d_n + \left(-\frac{\gamma}{\alpha}\right) d_{n+2}\\
&d(x_n, x_{n+2}) \leq \frac{1}{\alpha} d_{n-1}^* + \left(\left|\frac{\beta}{\alpha}\right|\right) d_n + \left(\left|\frac{\gamma}{\alpha}\right|\right) d_{n+2}
\end{aligned}$$

Since $d_{n+1} \leq \theta^n D_0$, so $d_n \leq \theta^{n-1} D_0$, it follows that

$$d_n^* \leq \frac{1}{\alpha} d_{n-1}^* + \theta^{n-1} Q D_0, \text{ where } Q := \left|\frac{\beta}{\alpha}\right| + \left|\frac{\gamma}{\alpha}\right| \theta^3.$$

Since $\theta \in \left[0, \dfrac{1}{s}\right) \subset [0, 1)$ and $\alpha > 1$, we get, by *(Lemma 2)* in [5] and by *(Lemma 2.11)*, that $\lim_{n\to\infty} d_n^* = 0$. Now, as in the proof of *(Theorem 2.9)*, we give a constructive approach for the upper bound of $d(x_n, x_{n+p})$. Furthermore, we shall omit the details. We know that $d_n^* \leq a_2 d_{n-1}^* + a_1 \theta^{n-1} D_0$, briefly $d_n^* \leq a_2^n d_0^* + \dfrac{\theta^n - a_2^n}{\theta - a_2} a_1 D_0$, where

$a_1 := Q$ and $a_2 := \dfrac{1}{\alpha}$. When $p = 2m+1$, then $d(x_n, x_{n+2m+1}) \leq \dfrac{1+\theta}{1-s\theta^2} s\theta^n D_0$, and, by hypothesis, $s\theta^2 < 1$, then $d(x_n, x_{n+2m+1})$ converges to 0.

When $p = 2m$, then $d^*_{n+2m} \leq a_2^{n+2m} d^*_0 + \dfrac{\theta^{n+2m} - a_2^{n+2m}}{\theta - a_2} a_1 D_0$. Since $\theta < \dfrac{1}{s} < 1$ and $a_2 < 1$, by theorem's assumptions, then d^*_{n+2m} converges to 0.

Moreover, $d(x_n, x_{n+2m}) \leq \dfrac{1+\theta}{1-s\theta^2} s\theta^n D_0 + s^{m-1} d^*_{n+2m}$.

Case (E2): When $\alpha < 0$. We shall use *(Lemma 2.7)*:

We know that $d(x_n, x_{n+1}) \leq \theta d(x_{n-1}, x_n)$, for each $n \geq 1$.

As in the previous case, with the remark that we divide by $\alpha < 0$, we get that

$$d(x_n, x_{n+2}) \geq A d^*_{n-1} + B d_n + C d_{n+1}, \text{ where } A := \frac{1}{\alpha},\ B := \frac{\beta}{|\alpha|} \text{ and } C := \frac{\gamma}{|\alpha|}.$$

By *(Lemma 2.7)*, we get that

$$\begin{aligned}
&C d_{n+1} \geq \varphi(C) d^*_{n+1} + \psi(C) d_{n+3} + \psi(C) d^*_n \\
&d^*_n \geq A d^*_{n-1} + B d_n + \varphi(C) d^*_{n+1} + \psi(C) d_{n+3} + \psi(C) d^*_n \\
&\varphi(C) d^*_{n+1} \leq d^*_n \left[1 - \psi(C)\right] + (-A)\, d^*_{n-1} - \varphi(C) d_{n+3} - B d_n
\end{aligned}$$

Since, by theorem's assumptions, $\varphi(C) > 0$, we get that

$$\begin{aligned}
d^*_{n+1} &\leq \frac{1-\psi(C)}{\varphi(C)} d^*_n - A d^*_{n-1} - \left[\varphi(C) d_{n+3} + B d_n\right] \\
d^*_{n+1} &\leq \frac{1-\psi(C)}{\varphi(C)} d^*_n - A d^*_{n-1} + \left[|\varphi(C)| d_{n+3} + |B| d_n\right] \\
d^*_{n+1} &\leq \frac{1-\psi(C)}{\varphi(C)} d^*_n - A d^*_{n-1} + \left[|\varphi(C)|\theta^2 + |B|\right] \theta^n D_0
\end{aligned}$$

On the other hand, let's denote by $b_n := \left[|\varphi(C)|\theta^2 + |B|\right] \theta^n D_0$, $\alpha_1 := \dfrac{1-\psi(C)}{\varphi(C)}$ and by $\alpha_2 := -A$. Since $\gamma > 0$ and $C = \dfrac{\gamma}{|\alpha|} > 0$, then $\varphi(C) = \dfrac{C}{s} > 0$. Also, from $C > 0$, then $\psi(C) = -C < 0$. Now, $\alpha_1 > 0$ requires that $-C < 1$ and this is true since $C > 0$. Moreover, $\alpha_2 = -A = -\dfrac{1}{\alpha} > 0$, because $\alpha < 0$ and so $\dfrac{1}{\alpha} < 0$. This means that α_1 and α_2 are positive, so the sum of these two is positive. Now, we want to validate if the sum of α_1 and α_2 is less than 1.

$$\alpha_1 + \alpha_2 = \frac{1-\psi(C)}{\varphi(C)} - A = \frac{1+C}{\frac{C}{s}} - \frac{1}{\alpha}.$$

So $\alpha_1 + \alpha_2 < 1$ is equivalent to $s\left(\dfrac{1+C}{C}\right) < 1 + \dfrac{1}{\alpha}$. Since $C = \dfrac{\gamma}{|\alpha|} = \dfrac{\gamma}{-\alpha}$, then $s\left(1 - \dfrac{\alpha}{\gamma}\right) < 1 + \dfrac{1}{\alpha}$. Now, we have two sub-cases.

If $1 - \dfrac{\alpha}{\gamma} < 0$, then $\alpha - \gamma > 0$, i.e. $\alpha > \gamma$, so this is false, because $\alpha < 0$ and $\gamma > 0$.

So, the only valid case is when $1 - \frac{\alpha}{\gamma} > 0$, so $\alpha < \gamma$. Since α and γ have different signs, this is also valid. Now, because $s\left(1 - \frac{\alpha}{\gamma}\right) < 1 + \frac{1}{\alpha}$ and by the fact that the right hand side is positive, it follows that $1 + \frac{1}{\alpha} > 0$, i.e. $\alpha < -1$, which is valid by hypothesis assumptions. Since $\theta \in \left[0, \frac{1}{s}\right) \subset [0,1)$, then $\lim_{n\to\infty} b_n = 0$.
Also, since $\alpha_1 + \alpha_2 \in [0,1)$, $\alpha_1 \in [0,1)$ and $\alpha_2 \in [0,1)$, then $\lim_{n\to\infty} d_n^* = 0$. The rest of the proof follows as usual. □

Now, we give an example of a b-rectangular metric space, which is b-rectangular and validate *(Theorem 2.13)* through another example, showing that the hypotheses and conclusion of the already mentioned theorem are true also in b-metric spaces.

Example 2.14. Let $X = [0,\infty)$, endowed with $d : X \times X \to \mathbb{R}_+$, such that $d(x,y) = (x-y)^2$, for each $x, y \in X$. Then (X,d) is a complete b-metric space, with coefficient $s = 2$. Then, it is also a complete b-rectangular metric space, with coefficient $s = 4$.

Example 2.15. Let $X = [0,\infty)$, where d is the above b-rectangular metric, with $s = 4$. Define $f : X \to X$ as $f(x) = \frac{x+\delta_1}{\delta_2}$, with $\delta_1, \delta_2 \geq 0$. It is easy to see that f is continuous. Also, for each $y \in X$, there exists $x = y\delta_2 - \delta_1 \geq 0$,, since δ_1 and δ_2 are positive, so f is onto. Moreover:

$$d(fx,fy) = (fx - fy)^2 = \left|\frac{x+\delta_1}{\delta_2} - \frac{y+\delta_1}{\delta_2}\right| = \frac{1}{\delta_2}|x-y|^2 = \frac{1}{\delta_2}d(x,y).$$

Let's take $\beta = 0$, $\gamma = 0$ and $\alpha = 10$. Also, let $\delta < \frac{1}{s}$, i.e. $\delta_2 < \frac{1}{4}$. For example: $\delta_2 = \frac{1}{10}$ and $\delta_1 = 1$.
Then f satisfies $d(fx,fy) \geq 10d(x,y)$, for each $x, y \in X$.

As an open problem with respect to generalized contractions in b-rectangular metric spaces, we give the following.

Open Problem. Following [3], consider a self-mapping f defined on a complete b-rectangular space (X,d) with coefficient $s \geq 1$, that satisfy

$$ad(x,fx) + bd(y,fy) + cd(fx,fy) + ed(x,fy) + gd(y,fx) \leq kd(x,y).$$

Develop fixed point theorems for the self-mapping above, in the context of b-rectangular metric spaces, with suitable conditions on the coefficients a, b, c, e, g, k.

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Common fixed point theorem for generalized nonexpansive mappings on ordered orbitally complete metric spaces and application

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Abstract. We propose a common fixed point theorem for new notion of generalized nonexpansive mappings for two pairs of maps in an ordered orbitally complete metric space. To illustrate our result, we give throughout the paper two examples. Existence of solutions for certain system of functional equations arising in dynamic programming is also presented as application.

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1. Introduction

The significance of nonexpansive mappings was sketched, e.g., in 1980 by Bruck [8]. A nonexpansive mapping of a complete metric space need not have a fixed point (consider a translation operator $\mathcal{T}(x) = x + c$ in a Banach space). A fixed point of a nonexpansive mapping need not be unique (consider $\mathcal{T} = I$). To make certain the existence and/or uniqueness of fixed points we must assume supplementary conditions on $\mathcal{T}$ and/or the underlying space. Contraction mappings, isometries and orthogonal projection are all nonexpansive mappings. The study of nonexpansive mappings has been one of the main features in modern developments of fixed point theory–see for instance [7, 10]. Browder et al. [7] proved that every nonexpansive mapping $\mathcal{T}$ from a convex bounded closed subset C of a Hilbert space $\mathcal{X}$ into C has a fixed point. There are also several interesting unsolved problems. The existence fixed point results for nonexpansive mapping is discussed in the paper [10, 11, 14, 27, 30] and others.

In 1986, some near the beginning results in this direction were recognized in the papers of Turinici [31, 32]; note that their starting points were the "amorphous" contributions in the area due to Matkowski [15, 16]. These results have been revive by Ran and Reurings [26, Theorem 2.1], where they extended the Banach contraction principle in partially ordered sets with some applications to linear and nonlinear matrix equations. Subsequently, several authors obtained many fixed point theorems in the underlying space, see for more facts [1, 2, 13, 17, 20, 22, 25, 29] and the references cited therein. Recently, Nashine and Kadelburg [18] proved some results for two pairs of mapping for implicit type relations in ordered orbitally complete metric spaces.

We propose a new generalized nonexpansive mappings for two pairs of maps in ordered metric spaces and relevance to fixed point theorem on an ordered orbitally complete metric space. We furnish suitable examples to demonstrate the validity of the hypotheses of our result. Our result is extensions of the results of Ciric [10] and Nashine and Kadelburg [17] in the sense of considering two pairs of maps in an orbitally complete ordered metric space. In the final section, we apply the obtained result for proving the existence of solutions for certain system of functional equations arising in dynamic programming.

2. Preliminaries

We will bring into play the following notation and definitions. Consistent with Abbas et al. [1] the following definitions will be used all the way through the paper.

If $(\mathcal{X}, \preceq)$ is a partially ordered set then $x, y \in \mathcal{X}$ are called comparable if $x \preceq y$ or $y \preceq x$ holds. A subset $\mathcal{K}$ of $\mathcal{X}$ is said to be totally ordered if every two elements of $\mathcal{K}$ are comparable. If $\mathcal{T} : \mathcal{X} \to \mathcal{X}$ is such that, for $x, y \in \mathcal{X}$, $x \preceq y$ implies $\mathcal{T}x \preceq \mathcal{T}y$, then the mapping $\mathcal{T}$ is said to be nondecreasing.

Definition 2.1. Let $\mathcal{X}$ be a nonempty set. Then $(\mathcal{X}, d, \preceq)$ is called an ordered metric space if

(i) $(\mathcal{X}, d)$ is a metric space,
(ii) $(\mathcal{X}, \preceq)$ is a partially ordered set.

The space $(\mathcal{X}, d, \preceq)$ is called regular if the following hypothesis holds: if $\{z_n\}$ is a non-decreasing sequence in $\mathcal{X}$ with respect to $\preceq$ such that $z_n \to z \in \mathcal{X}$ as $n \to \infty$, then $z_n \preceq z$.

Definition 2.2. Let $(\mathcal{X}, \preceq)$ be a partially ordered set. A pair (f, g) of selfmaps of $\mathcal{X}$ is said to be weakly increasing if $fx \preceq gfx$ and $gx \preceq fgx$ for all $x \in \mathcal{X}$.

Now we give a definition of partially weakly increasing pair of mappings.

Definition 2.3. Let $(\mathcal{X}, \preceq)$ be a partially ordered set and f and g be two selfmaps on $\mathcal{X}$. An ordered pair (f, g) is said to be partially weakly increasing if $fx \preceq gfx$ for all $x \in \mathcal{X}$.

Note that a pair (f, g) is weakly increasing if and only if ordered pair (f, g) and (g, f) are partially weakly increasing.

Following is an example of an ordered pair (f, g) of selfmaps f and g which is partially weakly increasing but not weakly increasing.

Example 2.4. Let $\mathcal{X} = [0,1]$ be endowed with usual ordering and $f, g : \mathcal{X} \to \mathcal{X}$ be defined by $fx = x^2$ and $gx = \sqrt{x}$. Clearly, (f, g) is partially weakly increasing. But $gx = \sqrt{x} \neq x = fgx$ for $x \in (0,1)$ implies that (g, f) is not partially weakly increasing.

Definition 2.5. Let $(\mathcal{X}, \preceq)$ be a partially ordered set. A mapping f is a called weak annihilator of g if $fgx \preceq x$ for all $x \in \mathcal{X}$.

Example 2.6. Let $\mathcal{X} = [0,1]$ be endowed with usual ordering and $f, g : \mathcal{X} \to \mathcal{X}$ be defined by $fx = x^2$, $gx = x^3$. Obviously, $fgx = x^6 \leq x$ for all $x \in \mathcal{X}$. Thus f is a weak annihilator of g.

Definition 2.7. Let $(\mathcal{X}, \preceq)$ be a partially ordered set. A mapping f is called dominating if $x \preceq fx$ for each $x \in \mathcal{X}$.

Example 2.8. Let $\mathcal{X} = [0,1]$ be endowed with usual ordering and $f : \mathcal{X} \to \mathcal{X}$ be defined by $fx = x^{\frac{1}{3}}$. Since $x \leq x^{\frac{1}{3}} = fx$ for all $x \in \mathcal{X}$. Therefore f is a dominating map.

Example 2.9. Let $\mathcal{X} = [0,4]$, endowed with usual ordering. Let $f, g : \mathcal{X} \to \mathcal{X}$ be defined by

$$fx = \begin{cases} 0, & \text{if} \quad x \in [0,1) \\ 1, & \text{if} \quad x \in [1,3) \\ 3, & \text{if} \quad x \in (3,4) \\ 4, & \text{if} \quad x = 4. \end{cases} \qquad gx = \begin{cases} 0, & \text{if} \quad x = 0 \\ 1, & \text{if} \quad x \in (0,1] \\ 3, & \text{if} \quad x \in (1,3] \\ 4, & \text{otherwise.} \end{cases}$$

The pair (f, g) is partially weakly increasing and the dominating map g is a weak annihilator of f.

Recall that the notion of orbitally complete metric space and orbitally continuous mapping were introduced by Ćirić in [9]. These definitions were extended to the case of two or three mappings by Sastry et al. in [28]. Some common fixed point results in this situation were obtained in [12, 19]. We give now respective definitions for two pairs of mappings.

Definition 2.10. Let $\mathcal{A}, \mathcal{B}, \mathcal{S}, \mathcal{T}$ be four self-mappings defined on a metric space $(\mathcal{X}, d)$.

1. If for a point $x_0 \in \mathcal{X}$, there exist sequences $\{x_n\}$ and $\{y_n\}$ in $\mathcal{X}$ such that
$$y_{2n-1} = \mathcal{A}x_{2n-2} = \mathcal{T}x_{2n-1}, \quad y_{2n} = \mathcal{B}x_{2n-1} = \mathcal{S}x_{2n}, \qquad \forall\, n \in \mathbb{N}, \tag{2.1}$$
then the set $\mathcal{O}(x_0; \mathcal{A}, \mathcal{B}, \mathcal{S}, \mathcal{T}) = \{y_n : n = 1, 2, \dots\}$ is called the orbit of $(\mathcal{A}, \mathcal{B}, \mathcal{S}, \mathcal{T})$ at x_0.
2. The space $(\mathcal{X}, d)$ is said to be $(\mathcal{A}, \mathcal{B}, \mathcal{S}, \mathcal{T})$-orbitally complete at x_0 if every Cauchy sequence in $\mathcal{O}(x_0; \mathcal{A}, \mathcal{B}, \mathcal{S}, \mathcal{T})$ converges in $\mathcal{X}$.
3. The map $\mathcal{A}$ is said to be $(\mathcal{A}, \mathcal{B}, \mathcal{S}, \mathcal{T})$-orbitally continuous at x_0 if it is continuous on $\mathcal{O}(x_0; \mathcal{A}, \mathcal{B}, \mathcal{S}, \mathcal{T})$.
4. If $\mathcal{S} = \mathcal{T}$, we write $(\mathcal{A}, \mathcal{B}, \mathcal{S})$ in the previous definitions instead of $(\mathcal{A}, \mathcal{B}, \mathcal{S}, \mathcal{S})$,

3. Main results

First, we introduce the notion of generalized nonexpansive mapping for four mappings in ordered metric spaces.

Definition 3.1. Let $(\mathcal{X}, d, \preceq)$ be an ordered metric space. We call two pairs of mappings $\mathcal{A}, \mathcal{B}, \mathcal{T}, \mathcal{S} : \mathcal{X} \to \mathcal{X}$ as generalized nonexpansive (of Ćirić type) if

$$d(\mathcal{A}x, \mathcal{B}y) \leq a \max\left\{ d(\mathcal{S}x, \mathcal{T}y), d(\mathcal{S}x, \mathcal{A}x), d(\mathcal{T}y, \mathcal{B}y), \frac{1}{2}[d(\mathcal{S}x, \mathcal{B}y) + d(\mathcal{T}y, \mathcal{A}x)] \right\}$$
$$+ b \max\{d(\mathcal{S}x, \mathcal{A}x), d(\mathcal{T}y, \mathcal{B}y)\} + c[d(\mathcal{S}x, \mathcal{B}y) + d(\mathcal{T}y, \mathcal{A}x)], \qquad (3.1)$$

holds for all comparable $x, y \in \mathcal{X}$, where $a \geq 0$, $b, c > 0$ satisfy

$$a + b + 2c = 1.$$

Now, we state and prove our result.

Theorem 3.2. *Let $(\mathcal{X}, d, \preceq)$ be a ordered metric space. Suppose that $\mathcal{T}, \mathcal{S}, \mathcal{A}, \mathcal{B} : \mathcal{X} \to \mathcal{X}$ be given generalized nonexpansive mappings satisfying for every pair $x, y \in \overline{\mathcal{O}(x_0; \mathcal{A}, \mathcal{B}, \mathcal{S}, \mathcal{T})}$ (for some $x_0 \in \mathcal{X}$) such that x and y are comparable. We assume the following hypotheses:*

(i) *The space $(\mathcal{X}, d)$ is $(\mathcal{A}, \mathcal{B}, \mathcal{S}, \mathcal{T})$-orbitally complete at x_0;*
(ii) *$(\mathcal{T}, \mathcal{A})$ and $(\mathcal{S}, \mathcal{B})$ are partially weakly increasing on $\overline{\mathcal{O}(x_0; \mathcal{A}, \mathcal{B}, \mathcal{S}, \mathcal{T})}$;*
(iii) *$\mathcal{B}\mathcal{X} \subseteq \mathcal{S}\mathcal{X}$ and $\mathcal{A}\mathcal{X} \subseteq \mathcal{T}\mathcal{X}$;*
(iv) *$\mathcal{A}$ and $\mathcal{B}$ are dominating maps on $\overline{\mathcal{O}(x_0; \mathcal{A}, \mathcal{B}, \mathcal{S}, \mathcal{T})}$;*
(v) *$\mathcal{B}$ is a weak annihilator of $\mathcal{S}$ and $\mathcal{A}$ is a weak annihilator of $\mathcal{T}$ on $\overline{\mathcal{O}(x_0; \mathcal{A}, \mathcal{B}, \mathcal{S}, \mathcal{T})}$;*
(vi) *For each nondecreasing sequence $\{x_n\}$ in $\mathcal{X}$, with $x_n \preceq y_n$ for all n, $y_n \to u$ implies that $x_n \preceq u$.*

Assume either

(a) *$(\mathcal{A}, \mathcal{S})$ is compatible, $\mathcal{A}$ or $\mathcal{S}$ is $(\mathcal{A}, \mathcal{B}, \mathcal{S}, \mathcal{T})$-orbitally continuous and $(\mathcal{B}, \mathcal{T})$ is weakly compatible, or*
(b) *$(\mathcal{B}, \mathcal{T})$ is compatible, $\mathcal{B}$ or $\mathcal{T}$ is $(\mathcal{A}, \mathcal{B}, \mathcal{S}, \mathcal{T})$-orbitally continuous and $(\mathcal{A}, \mathcal{S})$ is weakly compatible.*

Then $\mathcal{A}, \mathcal{B}, \mathcal{S}$ and $\mathcal{T}$ have a common fixed point. Moreover, the set of common fixed points of $\mathcal{A}, \mathcal{B}, \mathcal{S}$ and $\mathcal{T}$ in $\overline{\mathcal{O}(x_0; \mathcal{A}, \mathcal{B}, \mathcal{S}, \mathcal{T})}$ is a singleton if and only if it is totally ordered.

Proof. Let $x_0 \in \mathcal{X}$ be a point given in (i). Since $\mathcal{B}\mathcal{X} \subseteq \mathcal{S}\mathcal{X}$ and $\mathcal{A}\mathcal{X} \subseteq \mathcal{T}\mathcal{X}$, we can consider sequences $\{x_n\}$ and $\{y_n\}$ in $\mathcal{X}$ given as in (2.1). By the given assumptions, $x_{2n-2} \preceq \mathcal{A}x_{2n-2} = \mathcal{T}x_{2n-1} \preceq \mathcal{A}\mathcal{T}x_{2n-1} \preceq x_{2n-1}$, and $x_{2n-1} \preceq \mathcal{B}x_{2n-1} = \mathcal{S}x_{2n} \preceq \mathcal{B}\mathcal{S}x_{2n} \preceq x_{2n}$. Thus, for all $n \geq 0$, we have

$$x_n \preceq x_{n+1}. \qquad (3.2)$$

Now we claim that $d(y_{n+1}, y_n) \leq d(y_n, y_{n-1})$ for all $n \geq 1$. Suppose this is not true, that is, there exists $n_0 \geq 1$ such that $d(y_{n_0+1}, y_{n_0}) > d(y_{n_0}, y_{n_0-1})$. Now since $x_{n_0-1} \preceq x_{n_0}$, we can use the inequality (3.1) for these elements.

Putting $x = x_{2n_0+1}$ and $y = x_{2n_0}$, from (3.2) and the considered contraction (3.1), we have

$$\begin{aligned}
&d(y_{2n_0+2}, y_{2n_0+1}) = d(\mathcal{A}x_{2n_0+1}, \mathcal{B}x_{2n_0})\\
&\leq a\max\left\{\begin{array}{c} d(\mathcal{S}x_{2n_0+1}, \mathcal{T}x_{2n_0}), d(\mathcal{S}x_{2n_0+1}, \mathcal{A}x_{2n_0+1}), d(\mathcal{T}x_{2n_0}, \mathcal{B}x_{2n_0}),\\ \frac{1}{2}[d(\mathcal{S}x_{2n_0+1}, \mathcal{B}x_{2n_0}) + d(\mathcal{T}x_{2n_0}, \mathcal{A}x_{2n_0+1})]\end{array}\right\}\\
&\quad + b\max\{d(\mathcal{S}x_{2n_0+1}, \mathcal{A}x_{2n_0+1}), d(\mathcal{T}x_{2n_0}, \mathcal{B}x_{2n_0})\}\\
&\quad + c[d(\mathcal{S}x_{2n_0+1}, \mathcal{B}x_{2n_0}) + d(\mathcal{T}x_{2n_0}, \mathcal{A}x_{2n_0+1})]\\
&= a\max\left\{\ d(y_{2n_0+1}, y_{2n_0}), \frac{1}{2}d(y_{2n_0}, y_{2n_0+2})\ \right\}\\
&\quad + b\max\{d(y_{2n_0+1}, y_{2n_0+2}), d(y_{2n_0}, y_{2n_0+1})\} + cd(y_{2n_0}, y_{2n_0+2}).
\end{aligned}$$

Using a triangular inequality, we have

$$\frac{1}{2}d(y_{2n_0}, y_{2n_0+2}) \leq \frac{1}{2}(d(y_{2n_0}, y_{2n_0+1}) + d(y_{2n_0+1}, y_{2n_0+2})) < d(y_{2n_0+1}, y_{2n_0+2}).$$

Since $c > 0$, this implies that

$$\begin{aligned}
d(y_{2n_0+2}, y_{2n_0+1}) &< (a+b)d(y_{2n_0+1}, y_{2n_0+2}) + 2cd(y_{2n_0+1}, y_{2n_0+2})\\
&= (a+b+2c)d(y_{2n_0+1}, y_{2n_0+2}) = d(y_{2n_0+2}, y_{2n_0+1}),
\end{aligned}$$

a contradiction. Thus $d(\mathcal{A}x_{2n+1}, \mathcal{B}x_{2n+1}) \leq d(\mathcal{A}x_{2n}, \mathcal{B}x_{2n})$. Hence

$$d(\mathcal{A}x_{n+1}, \mathcal{B}x_{n+1}) \leq d(\mathcal{A}x_0, \mathcal{B}x_0),\ \text{for all positive integers} n. \tag{3.3}$$

Using (3.1) and (3.3) and triangle inequality, we have

$$d(y_{2n-1}, \mathcal{B}x_{2n}) = d(\mathcal{A}x_{2n-2}, \mathcal{B}x_{2n}) \tag{3.4}$$

$$\begin{aligned}
&\leq a\max\left\{\begin{array}{c} d(\mathcal{S}x_{2n-2}, \mathcal{T}x_{2n}), d(\mathcal{S}x_{2n-2}, \mathcal{A}x_{2n-2}), d(\mathcal{T}x_{2n}, \mathcal{B}x_{2n}),\\ \frac{1}{2}[d(\mathcal{S}x_{2n-2}, \mathcal{B}x_{2n}) + d(\mathcal{T}x_{2n}, \mathcal{A}x_{2n-2})]\end{array}\right\}\\
&\quad + b\max\{d(\mathcal{S}x_{2n-2}, \mathcal{A}x_{2n-2}), d(\mathcal{T}x_{2n}, \mathcal{B}x_{2n})\}\\
&\quad + c[d(\mathcal{S}x_{2n-2}, \mathcal{B}x_{2n}) + d(\mathcal{T}x_{2n}, \mathcal{A}x_{2n-2})], \qquad (3.5)\\
&= a\max\left\{\begin{array}{c} d(y_{2n-2}, y_{2n}), d(y_{2n-2}, y_{2n-1}), d(y_{2n}, y_{2n+1}),\\ \frac{1}{2}[d(y_{2n-2}, y_{2n+1}) + d(y_{2n}, y_{2n-1})]\end{array}\right\}\\
&\quad + b\max\{d(y_{2n-2}, y_{2n-1}), d(y_{2n}, y_{2n+1})\} + c[d(y_{2n-2}, y_{2n+1}) + d(y_{2n}, y_{2n-1})].
\end{aligned}$$

From (3.3) and the triangle inequality we get

$$\begin{aligned}
&\tfrac{1}{2}[d(y_{2n-2}, y_{2n+1}) + d(y_{2n}, y_{2n-1})]\\
&\leq \tfrac{1}{2}[d(y_{2n-2}, y_{2n-1}) + d(y_{2n}, y_{2n-1}) + d(y_{2n}, y_{2n+1}) + d(y_{2n}, y_{2n-1}))]\\
&\leq 2d(y_{2n-2}, y_{2n-1}).
\end{aligned} \tag{3.6}$$

Substituting (3.6) in (3.4), we have

$$\begin{aligned}
d(y_{2n-1}, \mathcal{B}x_{2n}) &\leq 2ad(y_{2n-2}, y_{2n-1}) + bd(y_{2n-1}, y_{2n-1}) + 4cd(y_{2n-2}, y_{2n-1})\\
&= (2a+b+4c)d(y_{2n-2}, y_{2n-1}).
\end{aligned}$$

Hence $d(y_{2n-1}, \mathcal{B}x_{2n}) = (2-b)d(y_{2n-2}, y_{2n-1})$.

From (3.1), (3.3) and (3.6), we have

$$d(y_{2n}, \mathcal{B}x_{2n}) = d(\mathcal{A}x_{2n-1}, \mathcal{B}x_{2n})$$
$$\leq a \max\left\{ \begin{array}{c} d(\mathcal{S}x_{2n-1}, \mathcal{T}x_{2n}), d(\mathcal{S}x_{2n-1}, \mathcal{A}x_{2n-1}), d(\mathcal{T}x_{2n}, \mathcal{B}x_{2n}), \\ \frac{1}{2}[d(\mathcal{S}x_{2n-1}, \mathcal{B}x_{2n}) + d(\mathcal{T}x_{2n}, \mathcal{A}x_{2n-1})] \end{array} cc\right\}$$
$$+ b \max\{d(\mathcal{S}x_{2n-1}, \mathcal{A}x_{2n-1}), d(\mathcal{T}x_{2n}, \mathcal{B}x_{2n})\}$$
$$+ c[d(\mathcal{S}x_{2n-1}, \mathcal{B}x_{2n}) + d(\mathcal{T}x_{2n}, \mathcal{A}x_{2n-1})]$$
$$= a \max\left\{d(y_{2n-1}, y_{2n}), d(y_{2n-1}, y_{2n}), d(y_{2n}, y_{2n+1}), \tfrac{1}{2}[d(y_{2n-1}, y_{2n+1}) + d(y_{2n}, y_{2n})]\right\}$$
$$+ b \max\{d(y_{2n-1}, y_{2n}), d(y_{2n}, y_{2n+1})\} + c[d(y_{2n-1}, y_{2n+1}) + d(y_{2n}, y_{2n})]$$
$$\leq ad(y_{2n-2}, y_{2n-1}) + bd(y_{2n-2}, y_{2n-1}) + c(2-b)d(y_{2n-2}, y_{2n-1})$$

and hence

$$d(y_{2n}, \mathcal{B}x_{2n}) = (1-bc)d(y_{2n-2}, y_{2n-1}).$$

Proceeding in this manner we obtain

$$d(y_{2n}, \mathcal{B}x_{2n}) \leq (1-bc)^{[\frac{n}{2}]} d(y_0, y_1) \tag{3.7}$$

for all $n = 1, 2, \ldots$, where $\frac{n}{2}$ denotes the greatest integer not exceeding $\frac{n}{2}$. Since $1 - bc < 1$, from (3.7), we conclude that $\{y_n\}$ is a Cauchy sequence.

Finally, we prove the existence of a common fixed point of the four mappings $\mathcal{A}, \mathcal{B}, \mathcal{S}$ and $\mathcal{T}$.

Since $\{y_n\}$ is a Cauchy sequence, defined by (2.1) in an $(\mathcal{A}, \mathcal{B}, \mathcal{S}, \mathcal{T})$-orbitally complete metric space $(\mathcal{X}, d)$, there exists a point z in $\mathcal{X}$, such that y_n converges to z. Therefore,

$$y_{2n+1} = \mathcal{T}x_{2n+1} = \mathcal{A}x_{2n} \to z \text{ as } n \to \infty \tag{3.8}$$

and

$$y_{2n+2} = \mathcal{S}x_{2n+2} = \mathcal{B}x_{2n+1} \to z \text{ as } n \to \infty. \tag{3.9}$$

Suppose that (a) holds. Since $(\mathcal{A}, \mathcal{S})$ is compatible, we have

$$\lim_{n\to\infty} \mathcal{A}\mathcal{S}x_{2n+2} = \lim_{n\to\infty} \mathcal{S}\mathcal{A}x_{2n+2} = \mathcal{S}z.$$

Also, $x_{2n+1} \preceq \mathcal{B}x_{2n+1} = \mathcal{S}x_{2n+2}$. Now

$$d(\mathcal{A}\mathcal{S}x_{2n+2}, \mathcal{B}x_{2n+1})$$
$$\leq a \max\left\{ \begin{array}{c} d(\mathcal{S}\mathcal{S}x_{2n+2}, \mathcal{T}x_{2n+1}), d(\mathcal{S}\mathcal{S}x_{2n+2}, \mathcal{A}\mathcal{S}x_{2n+2}), d(\mathcal{T}x_{2n+1}, \mathcal{B}x_{2n+1}), \\ \frac{1}{2}[d(\mathcal{S}\mathcal{S}x_{2n+2}, \mathcal{B}x_{2n+1}) + d(\mathcal{T}x_{2n+1}, \mathcal{A}\mathcal{S}x_{2n+2})] \end{array} \right\}$$
$$+ b \max\{d(\mathcal{S}\mathcal{S}x_{2n+2}, \mathcal{A}\mathcal{S}x_{2n+2}), d(\mathcal{T}x_{2n+1}, \mathcal{B}x_{2n+1})\}$$
$$+ c[d(\mathcal{S}\mathcal{S}x_{2n+2}, \mathcal{B}x_{2n+1}) + d(\mathcal{T}x_{2n+1}, \mathcal{A}\mathcal{S}x_{2n+2})].$$

Assume that $\mathcal{S}$ is $(\mathcal{A}, \mathcal{B}, \mathcal{S}, \mathcal{T})$-orbitally continuous. Passing to the limit as $n \to \infty$, we obtain

$$d(\mathcal{S}z, z) \leq a \max\left\{d(\mathcal{S}z, z), 0, 0, \tfrac{1}{2}[d(\mathcal{S}z, z) + d(z, \mathcal{S}z)]\right\}$$
$$+ b \max\{0, 0\} + c[d(\mathcal{S}z, z) + d(z, \mathcal{S}z)],$$
$$\leq (a + 2c)d(\mathcal{S}z, z) = (1-b)d(\mathcal{S}z, z).$$

Since $b > 0$ and $(1-b) < 1$, this implies that

$$\mathcal{S}z = z. \tag{3.10}$$

Now, $x_{2n+1} \preceq \mathcal{B}x_{2n+1}$ and $\mathcal{B}x_{2n+1} \to z$ as $n \to +\infty$, so by assumption we have $x_{2n+1} \preceq z$ and (3.1) becomes

$$\begin{aligned} d(\mathcal{A}z, \mathcal{B}x_{2n+1}) \leq a \max &\left\{ \begin{array}{c} d(\mathcal{S}z, \mathcal{T}x_{2n+1}), d(\mathcal{S}z, \mathcal{A}z), d(\mathcal{T}x_{2n+1}, \mathcal{B}x_{2n+1}), \\ \frac{1}{2}[d(\mathcal{S}z, \mathcal{B}x_{2n+1}) + d(\mathcal{T}x_{2n+1}, \mathcal{A}z)] \end{array} \right\} \\ &+ b \max\{d(\mathcal{S}z, \mathcal{A}z), d(\mathcal{T}x_{2n+1}, \mathcal{B}x_{2n+1})\} \\ &+ c[d(\mathcal{S}z, \mathcal{B}x_{2n+1}) + d(\mathcal{T}x_{2n+1}, \mathcal{A}z)]. \end{aligned}$$

Passing to the limit $n \to +\infty$ in the above inequality and using (3.10),

$$\begin{aligned} d(\mathcal{A}z, z) &\leq a \max\left\{d(\mathcal{S}z, z), d(\mathcal{S}z, \mathcal{A}z), d(z, z), \tfrac{1}{2}[d(\mathcal{S}z, z) + d(z, \mathcal{A}z)]\right\} \\ &\quad + b \max\{d(\mathcal{S}z, \mathcal{A}z), d(z, z)\} + c[d(\mathcal{S}z, z) + d(z, \mathcal{A}z)]. \\ &= (a+b+c) d(z, \mathcal{A}z). \end{aligned}$$

Since $a, b, c > 0$ and $(a+b+c) < 1$, this implies that

$$\mathcal{A}z = z. \tag{3.11}$$

Since $\mathcal{A}(\mathcal{X}) \subseteq \mathcal{T}(\mathcal{X})$, there exists a point $\omega \in \mathcal{X}$ such that $\mathcal{A}z = \mathcal{T}\omega$. Suppose that $\mathcal{T}\omega \neq \mathcal{B}\omega$. Since $z \preceq \mathcal{A}z = \mathcal{T}\omega \preceq \mathcal{A}\mathcal{T}\omega \preceq \omega$ implies $z \preceq \omega$. From (3.1), we obtain

$$\begin{aligned} &d(\mathcal{T}\omega, \mathcal{B}\omega) = d(\mathcal{A}z, \mathcal{B}\omega) \\ &\leq a \max\left\{ d(\mathcal{S}z, \mathcal{T}\omega), d(\mathcal{S}z, \mathcal{A}z), d(\mathcal{T}\omega, \mathcal{B}\omega), \frac{1}{2}[d(\mathcal{S}z, \mathcal{B}\omega) + d(\mathcal{T}\omega, \mathcal{A}z)] \right\} \\ &\quad + b \max\{d(\mathcal{S}z, \mathcal{A}z), d(\mathcal{T}\omega, \mathcal{B}\omega)\} + c[d(\mathcal{S}z, \mathcal{B}\omega) + d(\mathcal{T}\omega, \mathcal{A}z)] \\ &\leq a \max\left\{d(z, \mathcal{T}\omega), 0, d(\mathcal{T}\omega, \mathcal{B}\omega), \tfrac{1}{2} d(z, \mathcal{B}\omega)\right\} + b \max\{0, d(\mathcal{T}\omega, \mathcal{B}\omega)\} + c d(z, \mathcal{B}\omega)], \\ &= (a+b+c) d(\mathcal{T}\omega, \mathcal{B}\omega) \end{aligned}$$

contradiction to the state $a + b + 2c = 1$. Hence, we get

$$\mathcal{T}\omega = \mathcal{B}\omega. \tag{3.12}$$

Since $\mathcal{B}$ and $\mathcal{T}$ are weakly compatible, $\mathcal{B}z = \mathcal{B}\mathcal{A}z = \mathcal{B}\mathcal{T}w = \mathcal{T}\mathcal{B}w = \mathcal{T}\mathcal{A}z = \mathcal{T}z$. Thus z is a coincidence point of $\mathcal{B}$ and $\mathcal{T}$.
Now, since $x_{2n} \preceq \mathcal{A}x_{2n}$ and $\mathcal{A}x_{2n} \to z$ as $n \to \infty$, implies that $x_{2n} \preceq z$, from (3.1)

$$\begin{aligned} &d(\mathcal{A}x_{2n}, \mathcal{B}z) \\ &\leq a \max\left\{ \begin{array}{c} d(\mathcal{S}x_{2n}, \mathcal{T}z), d(\mathcal{S}x_{2n}, \mathcal{A}x_{2n}), d(\mathcal{T}z, \mathcal{B}z), \\ \frac{1}{2}[d(\mathcal{S}x_{2n}, \mathcal{B}z) + d(\mathcal{T}z, \mathcal{A}x_{2n})] \end{array} \right\} \\ &\quad + b \max\{d(\mathcal{S}x_{2n}, \mathcal{A}x_{2n}), d(\mathcal{T}z, \mathcal{B}z)\} + c[d(\mathcal{S}x_{2n}, \mathcal{B}z) + d(\mathcal{T}z, \mathcal{A}x_{2n})]. \end{aligned}$$

Passing to the limit as $n \to +\infty$, we have

$$\begin{aligned} d(z, \mathcal{B}z) &\le a \max\left\{ d(z, \mathcal{B}z), 0, 0, \frac{1}{2}[d(z, \mathcal{B}z) + d(\mathcal{B}z, z)] \right\} \\ &\quad + b \max\{0, 0)\} + c[d(z, \mathcal{B}z) + d(\mathcal{B}z, z)] \\ &= (a + 2c)d(z, \mathcal{B}z) = (1 - b)d(z, \mathcal{B}z). \end{aligned}$$

Since $b > 0$ and $(1 - b) < 1$, which gives that

$$z = \mathcal{B}z. \tag{3.13}$$

Therefore, $\mathcal{A}z = \mathcal{B}z = \mathcal{S}z = \mathcal{T}z = z$, so z is a common fixed point of $\mathcal{A}, \mathcal{B}, \mathcal{S}$ and $\mathcal{T}$.

The proof is similar when $\mathcal{A}$ is orbitally continuous.

Similarly, the result follows when (b) holds.

Now, suppose that the set of common fixed points of $\mathcal{S}, \mathcal{T}, \mathcal{A}$ and $\mathcal{B}$ is totally ordered. We claim that there is a unique common fixed point of $\mathcal{A}, \mathcal{B}, \mathcal{S}$ and $\mathcal{T}$. Assume on contrary that $\mathcal{S}u = \mathcal{T}u = \mathcal{A}u = \mathcal{B}u = u$ and $\mathcal{S}\vartheta = \mathcal{T}\vartheta = \mathcal{A}\vartheta = \mathcal{B}\vartheta = \vartheta$ but $u \ne \vartheta$. By supposition, we can replace x by u and y by ϑ in (3.1) to obtain

$$\begin{aligned} d(u, \vartheta) &= d(\mathcal{A}u, \mathcal{B}\vartheta) \\ &\le a \max\left\{ d(\mathcal{S}u, \mathcal{T}\vartheta), d(\mathcal{S}u, \mathcal{A}u), d(\mathcal{T}\vartheta, \mathcal{B}\vartheta), \frac{1}{2}[d(\mathcal{S}u, \mathcal{B}\vartheta) + d(\mathcal{T}\vartheta, \mathcal{A}u)] \right\} \\ &\quad + b \max\{d(\mathcal{S}u, \mathcal{A}u), d(\mathcal{T}\vartheta, \mathcal{B}\vartheta)\} + c[d(\mathcal{S}u, \mathcal{B}\vartheta) + d(\mathcal{T}\vartheta, \mathcal{A}u)] \\ &= (a + 2c)d(u, \vartheta) = (1 - b)d(u, \vartheta). \end{aligned}$$

Since $b > 0$, this implies that $u = \vartheta$.

Conversely, if $\mathcal{A}, \mathcal{B}, \mathcal{S}$ and $\mathcal{T}$ have only one common fixed point, then the set of common fixed point of $\mathcal{S}, \mathcal{T}, \mathcal{A}$ and $\mathcal{B}$ being singleton is totally ordered. This completes the proof. □

As consequence of Theorem 3.2, we may state the following corollary.

Corollary 3.3. *Let $(\mathcal{X}, d, \preceq)$ be an ordered metric space. Let $\mathcal{A}, \mathcal{B}, \mathcal{S} : \mathcal{X} \to \mathcal{X}$ be given mappings satisfying for every pair $x, y \in \overline{\mathcal{O}(x_0; \mathcal{A}, \mathcal{B}, \mathcal{S})}$ (for some $x_0 \in \mathcal{X}$) such that x and y are comparable,*

$$\begin{aligned} d(\mathcal{A}x, \mathcal{B}y) &\le a \max\left\{ d(\mathcal{S}x, \mathcal{S}y), d(\mathcal{S}x, \mathcal{A}x), d(\mathcal{S}y, \mathcal{B}y), \frac{1}{2}[d(\mathcal{S}x, \mathcal{B}y) + d(\mathcal{S}y, \mathcal{A}x)] \right\} \\ &\quad + b \max\{d(\mathcal{S}x, \mathcal{A}x), d(\mathcal{S}y, \mathcal{B}y)\} + c[d(\mathcal{S}x, \mathcal{B}y) + d(\mathcal{S}y, \mathcal{A}x)], \end{aligned}$$

holds for all comparable $x, y \in \mathcal{X}$, where $a \ge 0$, $b, c > 0$ satisfy

$$a + b + 2c = 1.$$

The mappings $\mathcal{A}, \mathcal{B}, \mathcal{S}$ satisfy (i)-(vi) and (a) (or (b)) of Theorem 3.2. Then $\mathcal{A}, \mathcal{B}$ and $\mathcal{S}$ have a common fixed point. Moreover, the set of common fixed points of $\mathcal{A}, \mathcal{B}$ and $\mathcal{S}$ in $\overline{\mathcal{O}(x_0; \mathcal{A}, \mathcal{B}, \mathcal{S})}$ is a singleton if and only if it is totally ordered.

Proof. It follows by taking $\mathcal{T} = \mathcal{S}$ in (3.1) and Theorem 3.2. □

By choosing $\mathcal{A}, \mathcal{B}, \mathcal{S}$ and $\mathcal{T}$ suitably in Theorem 3.2, we can deduce some corollaries for a pair as well as for a triple of self mappings.

In what follows, we support the result of Theorem 3.2 by examples.

Following example is inspired by [18].

Example 3.4. Let $\mathcal{X} = [0, +\infty)$ be equipped with the standard metric and order. Consider the mappings $\mathcal{A}, \mathcal{B}, \mathcal{S}, \mathcal{T} : \mathcal{X} \to \mathcal{X}$ given by

$$\mathcal{A}x = \begin{cases} \frac{1+x}{2}, & 0 \le x \le 1 \\ 4x - 3, & x > 1, \end{cases} \qquad \mathcal{B}x = \begin{cases} \frac{2+x}{3}, & 0 \le x \le 1 \\ 3x - 2, & x > 1, \end{cases}$$

$$\mathcal{S}x = \begin{cases} 0, & 0 \le x \le \frac{5}{6} \\ 6x - 5, & x > \frac{5}{6}, \end{cases} \qquad \mathcal{T}x = \begin{cases} 0, & 0 \le x \le \frac{4}{5} \\ 5x - 4, & x > \frac{4}{5}. \end{cases}$$

Conditions (i)-(vi) and (a) (or (b)) of Theorem 3.2 are easy to check for $x_0 = \frac{5}{6}$. Then $\mathcal{O}(x_0; \mathcal{A}, \mathcal{B}, \mathcal{S}, \mathcal{T}) \subset [\frac{5}{6}, 1]$.

Note, though, that conditions (iii) and (v) are not satisfied on the entire space $\mathcal{X}$.

At present we will prove that condition (3.1) is fulfilled with $x_0 = \frac{5}{6}$, $a = \frac{2}{5}$, $b = \frac{1}{5}$, $c = \frac{1}{5}$. Then a, b, c undoubtedly accomplish all conditions, in particular $a + b + 2c = 1$.

Take $x, y \in \mathcal{O}(x_0; \mathcal{A}, \mathcal{B}, \mathcal{S}, \mathcal{T}) \subset [0, \frac{5}{6}]$. Then (3.1) converts to

$$\begin{aligned} \left| \frac{1+x}{2} - \frac{2+y}{3} \right| \le \frac{2}{5} \max & \left\{ \begin{array}{c} |6x - 5y - 1|, \frac{11(1-x)}{2}, \frac{17(y-1)}{3}, \\ \frac{1}{2}\left[\left|6x - 5 - \frac{2+y}{3}\right| + \left|5y - 4 - \frac{1+x}{2}\right| \right] \end{array} \right\} \\ & + \frac{1}{5} \max \left\{ \frac{11(1-x)}{2}, \frac{14(1-y)}{3} \right\} \\ & + \frac{1}{5} \left[\left| 6x - 5 - \frac{2+y}{3} \right| + \left| 5y - 4 - \frac{1+x}{2} \right| \right]. \end{aligned}$$

By means of the replacement $x = 1 - \xi$, $y = 1 - \xi t$, $0 \le \xi \le 1$, $t \ge 0$, the preceding inequality turn into

$$\begin{aligned} \left| \frac{t}{3} - \frac{1}{2} \right| \le \frac{2}{5} \max & \left\{ |5t - 6|, \frac{11}{2}, \frac{17t}{3}, \frac{1}{2}\left[\left| \frac{t}{3} - 6 \right| + \left| 5t - \frac{1}{2} \right| \right] \right\} \\ & + \frac{1}{5} \max \left\{ \frac{11}{2}, \frac{14t}{3} \right\} + \frac{1}{5} \left[\left| \frac{t}{3} - 6 \right| + \left| 5t - \frac{1}{2} \right| \right] \end{aligned}$$

and can be tested out by argument on feasible values of $t \ge 0$. It is remark that condition (3.1) does not hold exterior of $\mathcal{O}(x_0; \mathcal{A}, \mathcal{B}, \mathcal{S}, \mathcal{T})$. For instance, it is adequate to take $x = 2$ and $y = 3$.

Thus, $\mathcal{A}, \mathcal{B}, \mathcal{S}, \mathcal{T}$ have a (unique) common fixed point (which is $z = 1$).

Following is the another example, inspired by [23, 18].

Example 3.5. Let $\mathcal{X} = [0, \infty)$ with the usual distance and define an ordering $\preceq$ on $\mathcal{X}$ as follows:

$$x \preceq y \iff x = y \text{ or } (x, y \in [0, 1] \text{ and } y \le x).$$

Define $\mathcal{A},\mathcal{B},\mathcal{S},\mathcal{T}:\mathcal{X}\to\mathcal{X}$ by

$$\mathcal{A}x=\begin{cases}\ln(\frac{x}{2}+1), & 0\le x\le 1\\ 3x, & x>1,\end{cases}\qquad \mathcal{B}x=\begin{cases}\ln(\frac{x}{3}+1), & 0\le x\le 1\\ 2x, & x>1,\end{cases}$$

$$\mathcal{S}x=\frac{e^{6x}-1}{6},\quad \mathcal{T}x=\frac{e^{4x}-1}{6}.$$

Take $x_0=1$.
Then $\mathcal{O}(x_0;\mathcal{A},\mathcal{B},\mathcal{S},\mathcal{T})\subset(0,1)$ and $\overline{\mathcal{O}(x_0;\mathcal{A},\mathcal{B},\mathcal{S},\mathcal{T})}=\mathcal{O}(x_0;\mathcal{A},\mathcal{B},\mathcal{S},\mathcal{T})\cup\{0\}$.
It is easy to prove all conditions of Theorem 3.2 from (i)-(vi) and (a)-(b) along with condition (3.1) satisfy and 0 is the unique common fixed point of $\mathcal{A},\mathcal{B},\mathcal{S}$ and $\mathcal{T}$ in $\overline{\mathcal{O}(x_0;\mathcal{A},\mathcal{B},\mathcal{S},\mathcal{T})}$.

It is observed that the conditions of Theorem 3.2 do not hold on the complete space $\mathcal{X}$.

4. Application to functional equations arising in dynamic programming

The fundamental shape of the functional equation of dynamic programming is given by Bellman and Lee [5] as follows:

$$q(x)=\text{opt}_{y\in\mathcal{D}}\{G(x,y,q(\tau(x,y)))\},\quad x\in W,$$

where $\tau:W\times\mathcal{D}\to W$, $G:W\times\mathcal{D}\times\mathbb{R}\to\mathbb{R}$ are mappings, while $W\subseteq U$ is a state space, $\mathcal{D}\subseteq V$ is a decision space, and U, V are Banach spaces. Here x and y represent the state and decision vectors respectively, τ represents the transformation of the process and $q(x)$ represents the optimal return with initial state x (where opt denotes max or min).

Subsequently a lot of work have been done in this trend and existence and uniqueness outcome have been attained for solutions and common solutions of some functional equations, as well as systems of functional equations in dynamic programming with the use of fixed point results. For details see [6, 24] and the references therein.

Let $\mathcal{X}=B(W)$ be the set of all bounded real-valued functions on W. According to the ordinary addition of functions and scalar multiplication, and with the norm $\|.\|_\infty$ given by

$$\|h\|_\infty=\sup_{x\in W}|h(x)|\ \text{ for all } h\in\mathcal{X},$$

we have that $(\mathcal{X},\|\cdot\|_\infty)$ is a Banach space and the respective convergence is uniform. In fact, the distance in $\mathcal{X}$ is given by

$$d_\infty(u,v)=\sup_{x\in W}|u(x)-v(x)|\text{ for all } u,v\in\mathcal{X}.$$

Therefore, if we consider a Cauchy sequence $\{h_n\}$ in $\mathcal{X}$, then it converges uniformly to a function, say h^*, that is bounded. Therefore $h^*\in\mathcal{X}$.

Let $\sqsubseteq$ be the partial order relation on $\mathcal{X}$ defined by

$$x\sqsubseteq y \text{ if and only if } x(t)\le y(t) \text{ for any } t\in W.$$

Then $(\mathcal{X}, \sqsubseteq)$ is a partially ordered set. Moreover for any increasing sequence $\{x_n\}$ in $\mathcal{X}$ converging to $x^* \in \mathcal{X}$, we have $x_n(t) \sqsubseteq x^*(t)$ for any $t \in W$. Hence, the condition (vi) of Theorem 3.2 in $(\mathcal{X}, \|\cdot\|_\infty, \sqsubseteq)$ is fulfilled.

In this section, we study the existence and uniqueness of a common solution of the following functional equations arising in dynamic programming:

$$q(x) = \sup_{y\in\mathcal{D}}\{\mathcal{H}_i(x, y, q(\tau(x, y)))\}, \quad x \in W, \ \ i \in \{1, 2, 3, 4\}. \tag{4.1}$$

Consider the operators $\Im_i : \mathcal{X} \to \mathcal{X}$ given by

$$\Im_i h(x) = \sup_{y\in\mathcal{D}}\{\mathcal{H}_i(x, y, h(\tau(x, y)))\}, \tag{4.2}$$

for $h \in \mathcal{X}$, $x \in W$, where $i \in \{1, 2, 3, 4\}$; these mappings are well-defined if the functions $\mathcal{H}_i$ are bounded.

Theorem 4.1. *Let $\Im_i : \mathcal{X} \to \mathcal{X}$ be given by (4.2), where $i \in \{1, 2, 3, 4\}$. Suppose that the following hypotheses hold:*

(D1) *$\mathcal{H}_i : W \times \mathcal{D} \times \mathbb{R} \to \mathbb{R}$ are bounded functions, where $i \in \{1, 2, 3, 4\}$;*
(D2) *There exists $\lambda \geq 0$ such that, for all $x \in W$, $y \in \mathcal{D}$ and $\ell_i, \hbar_i \in \mathbb{R}$,*

$$|\mathcal{H}_i(x, y, \ell_i) - \mathcal{H}_i(x, y, \hbar_i)| \leq \lambda|\ell_i - \hbar_i| \quad \textit{for all} \quad i = 1, 2, 3, 4.$$

(D3) *for all $t \in W$, $s \in \mathcal{D}$, $h \in \mathcal{X}$, we have:*

$$h(t) \leq \mathcal{H}_1(t, s, h(s)) \ \textit{and}\ h(t) \leq \mathcal{H}_2(t, s, h(s));$$

(D4) *for all $(t, s) \in W \times \mathcal{D}$, $\varsigma \in W$ $h \in \mathcal{X}$, we have:*

$$\mathcal{H}_3(t, s, h(\varsigma)) \leq \mathcal{H}_1\left(t, s, \mathcal{H}_3(s, \tau, h(\varsigma))\right), \quad \mathcal{H}_4(t, s, h(\varsigma)) \leq \mathcal{H}_2\left(t, s, \mathcal{H}_4(s, \tau, h(\varsigma))\right);$$

(D5) *for all $t \in W$, $s \in \mathcal{D}$, $h \in \mathcal{X}$, we have:*

$$\mathcal{H}_1\left(t, s, \mathcal{H}_3(s, \tau, h(\tau))\right) \leq h(t), \quad \mathcal{H}_2\left(t, s, \mathcal{H}_4(s, \tau, h(\tau))\right) \leq h(t);$$

(D6)

$$\begin{cases} \textit{for all } t \in W, h \in \mathcal{X}, \Im_1\Im_4 h(t) = \Im_4\Im_1 h(t), \ \textit{whenever } \Im_1 h(t) = \Im_4 h(t), \ \textit{and} \\ \begin{cases} \textit{there exists } \{k_n\} \subset \mathcal{X} \textit{ such that } \lim_{n\to\infty} \Im_2 k_n = \lim_{n\to\infty} \Im_3 k_n = k^* \in \mathcal{X} \\ \textit{and } \lim_{n\to\infty} \sup_{x\in W} |\Im_2\Im_3 k_n - \Im_3\Im_2 k_n| = 0; \end{cases} \end{cases}$$

or

$$\begin{cases} \textit{for all } t \in W, h \in \mathcal{X}, \Im_2\Im_3 h(t) = \Im_3\Im_2 h(t), \ \textit{whenever } \Im_2 h(t) = \Im_3 h(t), \ \textit{and} \\ \begin{cases} \textit{there exists } \{h_n\} \subset \mathcal{X} \textit{ such that } \lim_{n\to\infty} \Im_1 h_n = \lim_{n\to\infty} \Im_4 h_n = h^* \in \mathcal{X} \\ \textit{and } \lim_{n\to\infty} \sup_{x\in W} |\Im_1\Im_4 h_n - \Im_4\Im_1 h_n| = 0; \end{cases} \end{cases}$$

(D7) *the functions* $\mathcal{H}_i : W \times \mathcal{D} \times \mathbb{R} \to \mathbb{R}$, $i \in \{1,2,3,4\}$, *satisfy*

$$\begin{aligned}
&|\mathcal{H}_1(x,y,h(x)) - \mathcal{H}_2(x,y,k(x))| \\
&\leq a \max\left\{\begin{array}{l} |\Im_4 h(s) - \Im_3 k(s)|), |\Im_3 h(s) - \Im_1 h(s)|, |\Im_4 k(s) - \Im_2 k(s)|, \\ \frac{1}{2}[|\Im_4 h(s) - \Im_2 k(s)| + |\Im_3 k(s) - \Im_1 h(s)|] \end{array}\right\} \\
&+ b \max\{|\Im_4 h(s) - \Im_1 h(s)|, |\Im_3 k(s) - \Im_2 h(s)|, \} \\
&+ c[|\Im_4 h(s) - \Im_2 k(s)| + |\Im_3 k(s) - \Im_1 h(s)|] \\
&:= R(h(s), k(s))
\end{aligned}$$

for all $h, k \in \mathcal{X}$, $s \in W$, *and some* $0 \leq a$, $b, c > 0$ *and* $a + b + 2c = 1$.

Then the system of functional equations (4.1) *has a bounded solution.*

Proof. First of all we prove that $\Im_i u$ is a bounded function on W, that is, $\Im_i u \in \mathcal{X}$ and the operators $\Im_i$ are well-defined.

We only need to prove that, for all $u \in \mathcal{X}$, the function $\Im_1 u : W \to \mathbb{R}$ is bounded. Indeed, let $u \in \mathcal{X}$ be arbitrary. As u is bounded, by hypothesis ($D1$), there exists $\lambda_1 > 0$ such that

$$|u(x)| \leq \lambda_1 \text{ for all } x \in W.$$

By hypothesis ($D1$), there exists $\lambda_2 > 0$ such that, for all $x \in W$ and all $y \in \mathcal{D}$,

$$|\mathcal{H}_1(x,y,0)| \leq \lambda_2.$$

Now by hypothesis ($D2$), for all $x \in W$ and all $y \in \mathcal{D}$,

$$\begin{aligned}
|\mathcal{H}_1(x,y,u(\tau(x,y))| &= |\mathcal{H}_1(x,y,u(\tau(x,y)) - \mathcal{H}_1(x,y,0)| + |\mathcal{H}_1(x,y,0)| \\
&\leq \lambda|u(\tau(x,y)| + \lambda_2 \leq \lambda\lambda_1 + \lambda_2.
\end{aligned}$$

As a result, for all $x \in W$, we have that

$$|\Im_1 h(x)| \leq \sup_{y \in \mathcal{D}} |\mathcal{H}_1(x,y,h_1(\tau(x,y)))| \leq \lambda\lambda_1 + \lambda_2.$$

That implies that $\Im_1 u$ is a bounded function on W, that is, $\Im_1 u \in \mathcal{X}$ and the operator $\Im_1$ is well-defined. Similarly we can show that other $\Im_i$ ($i = 2,3,4$) are well-defined.

Now, let λ be an arbitrary positive number, $x \in W$ and $h_1, h_2 \in \mathcal{X}$. Then there exist $y_1, y_2 \in \mathcal{D}$ such that

$$\Im_1 h_1(x) < \mathcal{H}_1(x, y_1, h_1(\tau(x,y_1))) + \lambda, \tag{4.3}$$

$$\Im_2 h_2(x) < \mathcal{H}_2(x, y_2, h_2(\tau(x,y_2))) + \lambda, \tag{4.4}$$

$$\Im_1 h_1(x) \geq \mathcal{H}_1(x, y_2, h_1(\tau(x,y_2))), \tag{4.5}$$

$$\Im_2 h_2(x) \geq \mathcal{H}_2(x, y_1, h_2(\tau(x,y_1))). \tag{4.6}$$

Let $h_1, h_2 \in \mathcal{X}$. Using hypothesis ($D3$), (4.5) and (4.6), for all $t \in W$, we have

$$h_1(t) \leq \Im_1 h_1(t) \text{ and } h_2(t) \leq \Im_2 h_2(t).$$

Then we have $h \sqsubseteq \Im_1 h$ and $h \sqsubseteq \Im_2 h$ for all $h \in \mathcal{X}$. This implies that $\Im_1$ and $\Im_2$ are dominating maps.

Let $h \in \mathcal{X}$. Using hypothesis $(D4)$, for all $t \in W$, we have

$$\begin{aligned}\Im_3 h(t) &= \sup_{s\in D} \mathcal{H}_3(t,s,h(\varsigma)) \le \sup_{s\in D} \mathcal{H}_1\left(t,s,\mathcal{H}_3(s,\tau,h(\varsigma))\right) \\ &\le \sup_{s\in D} \mathcal{H}_1(t,s,\Im_3 h(s)) = \Im_1\Im_3 h(t).\end{aligned}$$

Similarly, using hypothesis $(D4)$, for all $t \in W$, we have

$$\begin{aligned}\Im_4 h(t) &= \sup_{s\in D} \mathcal{H}_4(t,s,h(\varsigma)) \le \sup_{s\in D} \mathcal{H}_2\left(t,s,\mathcal{H}_4(s,\tau,h(\varsigma))\right) \\ &\le \sup_{s\in D} \mathcal{H}_2(t,s,\Im_4 h(s)) = \Im_2\Im_4 h(t).\end{aligned}$$

Then, we have $\Im_3 h \sqsubseteq \Im_1\Im_3 h$ and $\Im_4 h \sqsubseteq \Im_2\Im_4 h$ for all $h \in W$. This implies that the pairs $(\Im_3, \Im_1)$ and $(\Im_4, \Im_2)$ are partially weakly increasing.

Let $h \in \mathcal{X}$. Using hypothesis $(D5)$, for all $t \in W$, we have

$$\Im_1\Im_3 h(t) \le h(t) \text{ and } \Im_2\Im_4 h(t) \le h(t).$$

Then, we have $\Im_1\Im_3 h \sqsubseteq h$ and $\Im_2\Im_4 h \sqsubseteq h$ for all $h \in \mathcal{X}$. This implies that $\Im_1$ and $\Im_2$ are weak annihilators of $\Im_3$ and $\Im_4$ respectively.

From hypothesis $(D6)$, the pair $(\Im_1, \Im_4)$ is weakly compatible and $(\Im_2, \Im_3)$ is compatible, or the pair $(\Im_2, \Im_3)$ is weakly compatible and $(\Im_1, \Im_4)$ is compatible.

Now, by using (4.3), (4.6) and hypothesis (D7), we obtain

$$\begin{aligned}\Im_1 h_1(x) - \Im_1 h_2(x) &< \mathcal{H}_1(x,y_1,h_1(\tau(x,y_1))) - \mathcal{H}_2(x,y_1,h_2(\tau(x,y_1))) + \lambda \\ &\le |\mathcal{H}_1(x,y_1,h_1(\tau(x,y_1))) - \mathcal{H}_2(x,y_1,h_2(\tau(x,y_1)))| + \lambda \\ &\le R(h_1(x),h_2(x)) + \lambda\end{aligned}$$

and so we have

$$\Im_1 h_1(x) - \Im_2 h_2(x) < R(h_1(x),h_2(x)) + \lambda. \tag{4.7}$$

Analogously, by using (4.4) and (4.5), we get

$$\Im_1 h_2(x) - \Im_1 h_1(x) < R(h_1(x),h_2(x)) + \lambda \tag{4.8}$$

Finally, from (4.7) and (4.8), we deduce

$$|\Im_1 h_1(x) - \Im_2 h_2(x)| < R(h_1(x),h_2(x)) + \lambda,$$

implying that

$$d_\infty(\Im_1 h_1, \Im_2 h_2) \le R(h_1,h_2) + \lambda.$$

Notice that the last inequality does not depend on $x \in W$ and $\lambda > 0$ is taken arbitrarily, therefore we obtain that

$$\begin{aligned}&d_\infty(\Im_1 h_1, \Im_2 h_2) \\ &\le a \max\left\{\begin{array}{l} d_\infty(\Im_4 h(s), \Im_3 k(s)), d_\infty(\Im_3 h(s), \Im_1 h(s)), d_\infty(\Im_4 k(s), \Im_2 k(s)), \\ \frac{1}{2}[d_\infty(\Im_4 h(s), \Im_2 k(s)) + d_\infty(\Im_3 k(s), \Im_1 h(s))] \end{array}\right\} \\ &\quad + b\max\{d_\infty(\Im_4 h(s), \Im_1 h(s)), d_\infty(\Im_3 k(s), \Im_2 h(s))\} \\ &\qquad + c[d_\infty(\Im_4 h(s), \Im_2 k(s)) + d_\infty(\Im_3 k(s), \Im_1 h(s))].\end{aligned}$$

Hence Theorem 3.2 is applicable since all its hypotheses are satisfied for operators $\mathcal{A} = \Im_1, \mathcal{B} = \Im_2$, $\Im = \Im_3$ and $\mathcal{S} = \Im_4$. Thus, there exists a common fixed point of

$\mathcal{A}, \mathcal{B}, \mathcal{S}$ and $\Im$, i.e., a bounded solution $\nu^* \in \mathcal{X}$ such that $\Im_i \nu^* = \nu^*$. In other words, for all $x \in W$,

$$\nu^*(x) = \Im_i \nu^*(x) = \sup_{y \in \mathcal{D}} \{\mathcal{H}_i(x, y, \nu^*(\tau(x, y)))\}.$$

This completes the proof. □

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On Fryszkowski's problem

Andrei Comăneci

Abstract. In this paper we give two partial answers to Fryszkowski's problem which can be stated as follows: given $\alpha \in (0,1)$, an arbitrary non-empty set Ω and a set-valued mapping $F : \Omega \to 2^{\Omega}$, find necessary and (or) sufficient conditions for the existence of a (complete) metric d on Ω having the property that F is a Nadler set-valued α-contraction with respect to d. More precisely, on the one hand, we provide necessary and sufficient conditions for the existence of a complete and bounded metric d on Ω having the property that F is a Nadler set-valued α-contraction with respect to d, in the case that $\alpha \in (0, \frac{1}{2})$ and there exists $z \in \Omega$ such that $F(z) = \{z\}$ and, on the other hand, we give a sufficient condition for the existence of a complete metric d on Ω having the property that F is a Nadler set-valued α-contraction with respect to d, in the case that Ω is finite.

Mathematics Subject Classification (2010): 54C60, 54H25.

Keywords: Fixed point of a multi-valued map, Hausdorff-Pompeiu distance, α-contractions.

1. Introduction

The first version of a converse of the Banach-Caccioppoli-Picard principle is due to C. Bessaga (see [2]). For an application of Bessaga's converse see [20] and for some other converses of the contraction principle see [3], [7], [9], [12] and [17]. For more results along this line of research one can consult [1], [8], [13], [14], [15] and [23].

An extension of the contraction principle to set-valued mappings is due to J. T. Markin and S. B. Nadler Jr. (see [11] and [16]). For more information on this topic see [4], [5], [10], [18], [19], [21], and [22].

The last section of [6] consists of the following problem formulated by Professor Andrzej Fryszkowski at the 2nd Symposium on Nonlinear Analysis in Toruń, September 13-17, 1999, which asks for a converse of the contraction principle for set-valued mappings: *Given* $\alpha \in (0,1)$*, an arbitrary non-empty set* Ω *and a set-valued mapping*

$F : \Omega \to 2^{\Omega}$, find necessary and (or) sufficient conditions for the existence of a (complete) metric d on Ω having the property that F is a Nadler set-valued α-contraction with respect to d.

In this paper we give two partial answers to the above mentioned problem.

Our first result provides necessary and sufficient conditions for the existence of a complete and bounded metric d on Ω having the property that F is a Nadler set-valued α-contraction with respect to d, in the case that $\alpha \in (0, \frac{1}{2})$ and there exists $z \in \Omega$ such that $F(z) = \{z\}$.

Our second result gives a sufficient condition for the existence of a complete metric d on Ω having the property that F is a Nadler set-valued α-contraction with respect to d, in the case that Ω is finite.

2. Preliminaries

Definition 2.1. For a metric space (X, d), we consider the generalized Hausdorff-Pompeiu metric $H : 2^X \times 2^X \to [0, +\infty]$ described by

$$H(A, B) = \max\{\sup_{x \in A}(\inf_{y \in B} d(x, y)), \sup_{x \in B}(\inf_{y \in A} d(x, y))\},$$

for every $A, B \in 2^X$.

Definition 2.2. Given $\alpha \in (0, 1)$, an arbitrary non-empty set Ω and a metric d on Ω, a set-valued function $F : \Omega \to 2^{\Omega}$ is called Nadler set-valued α-contraction with respect to d if $H(F(x), F(y)) \leq \alpha d(x, y)$ for all $x, y \in \Omega$.

Definition 2.3. Given an arbitrary non-empty set Ω and a set-valued function $F : \Omega \to 2^{\Omega}$, $z \in \Omega$ is called a fixed point of F if $z \in F(z)$.

Definition 2.4. Given an arbitrary non-empty set Ω and a set-valued function $F : \Omega \to 2^{\Omega}$, one can consider the function $\widehat{F} : 2^{\Omega} \to 2^{\Omega}$ *given by*

$$\widehat{F}(P) = \bigcup_{x \in P} F(x)$$

for every $P \in 2^{\Omega}$.

Definition 2.5. Given an arbitrary non-empty set Ω, a function $f : \Omega \to \Omega$ and $n \in \mathbb{N}$, by f^n we mean the composition of f by itself n times, with the convention that $f^0 = \mathrm{Id}_{\Omega}$.

3. Main results

Lemma 3.1. *Given $\alpha \in (0, 1)$, an arbitrary non-empty set Ω and a set-valued function $F : \Omega \to 2^{\Omega}$ having a fixed point z such that $F(z) = \{z\}$, the following statements are equivalent:*

a) there exists a complete metric d on Ω such that F is a Nadler set-valued α-contraction with respect to d;

b) there exists a function $\varphi : \Omega \to [0, \infty)$ such that $\varphi^{-1}(\{0\}) = \{z\}$ and $\sup_{t \in F(x)} \varphi(t) \leq \alpha \varphi(x)$ for all $x \in \Omega$.

Proof. a)$\Rightarrow$b) We consider the function $\varphi : \Omega \to [0, \infty)$ given by $\varphi(x) = d(x, z)$ for all $x \in \Omega$. It is clear that $\varphi^{-1}(\{0\}) = \{z\}$. Moreover, we have

$$\sup_{t\in F(x)} \varphi(t) = \sup_{t\in F(x)} d(t, z) \leq H(F(x), \{z\}) = H(F(x), F(z)) \leq \alpha d(x, z) = \alpha\varphi(x)$$

for all $x \in \Omega$.

b)$\Rightarrow$a) Considering the metric $d : \Omega \times \Omega \to [0, \infty)$, given by

$$d(x, y) = \begin{cases} \varphi(x) + \varphi(y), & \text{if } x \neq y \\ 0, & \text{if } x = y \end{cases},$$

we have

$$\begin{aligned} \sup_{t\in F(x)} d(t, F(y)) &= \sup_{t\in F(x)} \inf_{u\in F(y)} d(t, u) \leq \sup_{t\in F(x)} \inf_{u\in F(y)} (\varphi(t) + \varphi(u)) \\ &= \sup_{t\in F(x)} (\varphi(t) + \inf_{u\in F(y)} \varphi(u)) = \sup_{t\in F(x)} \varphi(t) + \inf_{u\in F(y)} \varphi(u) \\ &\leq \alpha(\varphi(x) + \varphi(y)) = \alpha d(x, y) \end{aligned}$$

for all $x, y \in \Omega$, $x \neq y$. In a similar way we get $\sup_{t\in F(y)} d(t, F(x)) \leq \alpha d(x, y)$ for all $x, y \in \Omega$, $x \neq y$. Consequently we infer that

$$H(F(x), F(y)) = \max\{\sup_{t\in F(x)} d(t, F(y)), \sup_{t\in F(y)} d(t, F(x))\} \leq \alpha d(x, y)$$

for all $x, y \in \Omega$, $x \neq y$. Note that the last inequality is true for $x = y$. The proof of the fact that d is complete is identical to the one presented in Lemma 1 from [6]. $\square$

Corollary 3.2. *If $\alpha \in (0, 1)$, (Ω, d) is a complete metric space and $F : \Omega \to 2^\Omega$ is a Nadler set-valued α-contraction with respect to d having a fixed point z such that $F(z) = \{z\}$, then z is the unique fixed point of F.*

Proof. Let us suppose that y is another fixed point of F. Then, from Lemma 3.1, we obtain $\varphi(y) \leq \sup_{x\in F(y)} \varphi(x) \leq \alpha\varphi(y)$, so $\varphi(y) = 0$, i.e. $y \in \varphi^{-1}(\{0\}) = \{z\}$. Hence $y = z$. $\square$

Theorem 3.3. *Given $\alpha \in (0, \frac{1}{2})$, an arbitrary non-empty set Ω and a set-valued function $F : \Omega \to 2^\Omega$ having a fixed point z such that $F(z) = \{z\}$, the following statements are equivalent:*

a) $\bigcap_{n\in\mathbb{N}} \widehat{F}^n(\Omega) = \{z\}$;

b) there exists a bounded function $\varphi : \Omega \to [0, \infty)$ such that $\varphi^{-1}(\{0\}) = \{z\}$ and $\sup_{t\in F(x)} \varphi(t) \leq \alpha\varphi(x)$ for all $x \in \Omega$;

c) there exists a complete and bounded metric d on Ω such that F is a Nadler set-valued α-contraction with respect to d.

Proof. a)⇒b) Let us consider the bounded function $\varphi : \Omega \to [0, \infty)$ given by $\varphi(x) = \alpha^{n_x}$ for every $x \in \Omega$, where $n_x = \sup\{n \in \mathbb{N} \mid x \in \widehat{F}^n(\Omega)\}$ and we use the convention $\alpha^\infty = 0$. In the view of the hypothesis, $n_x \in \mathbb{N}$ for $x \neq z$ and $n_z = \infty$, so $\varphi^{-1}(\{0\}) = \{z\}$. Moreover, since, for $t \in F(x)$, we have $t \in \widehat{F}(\widehat{F}^{n_x}(\Omega)) = \widehat{F}^{n_x+1}(\Omega)$, so $n_t \geq n_x+1$, we infer that

$$\sup_{t\in F(x)} \varphi(t) = \sup_{t\in F(x)} \alpha^{n_t} \leq \sup_{t\in F(x)} \alpha^{n_x+1} = \alpha \cdot \alpha^{n_x} = \alpha\varphi(x)$$

for all $x \in \Omega$.

b)⇒c) The proof is the same with the one of b)⇒a) from Lemma 3.1, with the remark that

$$\operatorname{diam}(\Omega) = \sup_{x,y\in\Omega} d(x,y) \leq \sup_{x,y\in\Omega} (\varphi(x) + \varphi(y)) \leq 2 \sup_{x\in\Omega} \varphi(x).$$

c)⇒a) According to our hypothesis, we have $\{z\} \subseteq \bigcap_{n\in\mathbb{N}} \widehat{F}^n(\Omega)$.

Claim. $d(x,y) \leq (2\alpha)^n \operatorname{diam}(\Omega)$ for all $n \in \mathbb{N}^*$, $x, y \in \widehat{F}^n(\Omega)$.

Justification of the claim. We are going to prove the claim by using the method of mathematical induction. If $x, y \in \widehat{F}(\Omega)$, then there exist $u, v \in \Omega$ such that $x \in F(u)$ and $y \in F(v)$, so

$$\begin{aligned} d(x,y) &\leq d(x,z) + d(z,y) = d(x,F(z)) + d(y,F(z)) \\ &\leq H(F(u),F(z)) + H(F(v),F(z)) \\ &\leq \alpha d(u,z) + \alpha d(z,y) \leq 2\alpha \operatorname{diam}(\Omega). \end{aligned}$$

Thus the statement is valid for $n = 1$. Now, given $n \in \mathbb{N}^*$, we suppose that the statement is valid for $n-1$ and prove that it is true also for n. Indeed, if $x, y \in \widehat{F}^n(\Omega)$, then there exist $u, v \in \widehat{F}^{n-1}(\Omega)$ such that $x \in F(u)$ and $y \in F(v)$, so

$$\begin{aligned} d(x,y) &\leq d(x,z) + d(z,y) = d(x,F(z)) + d(y,F(z)) \\ &\leq H(F(u),F(z)) + H(F(v),F(z)) \\ &\leq \alpha d(u,z) + \alpha d(v,z). \end{aligned}$$

Because $u, v, z \in \widehat{F}^{n-1}(\Omega)$, we get

$$d(u,z) \leq (2\alpha)^{n-1} \operatorname{diam}(\Omega) \text{ and } d(v,z) \leq (2\alpha)^{n-1} \operatorname{diam}(\Omega).$$

So $d(x,y) \leq \alpha d(u,z) + \alpha d(v,z) \leq (2\alpha)^n \operatorname{diam}(\Omega)$. Consequently, the statement is valid for n. The proof of the claim is done.

Based on the claim, we conclude that $\lim_{n\to\infty} \operatorname{diam}(\widehat{F}^n(\Omega)) = 0$, so $\bigcap_{n\in\mathbb{N}} \widehat{F}^n(\Omega)$ is a singleton, namely $\bigcap_{n\in\mathbb{N}} \widehat{F}^n(\Omega) = \{z\}$. □

Theorem 3.4. *Let $\alpha \in (0,1)$, an arbitrary non-empty finite set Ω, $F : \Omega \to 2^\Omega$ a set-valued function and $z \in \Omega$ such that $\{z\}$ is the unique fixed point for $\widehat{F}$. Then there exists a complete metric d on Ω such that F is a Nadler set-valued α-contraction with respect to d.*

Proof. We have the following chain of inclusions:

$$\Omega = \widehat{F}^0(\Omega) \supseteq \widehat{F}^1(\Omega) = \widehat{F}(\Omega) \supseteq \widehat{F}^2(\Omega) \supseteq ... \supseteq \widehat{F}^n(\Omega) \supseteq ...,$$

where $n \in \mathbb{N}$ and $z \in \bigcap_{n\in\mathbb{N}} \widehat{F}^n(\Omega)$. Note that $\widehat{F}^n(\Omega) = \widehat{F}^{n+1}(\Omega)$ if and only if $\widehat{F}^n(\Omega) = \{z\}$. There exists $n \in \mathbb{N}$ such that $\widehat{F}^n(\Omega) = \{z\}$ otherwise we would get the following strictly decreasing sequence of non-negative integers:

$$|\Omega| > \left|\widehat{F}(\Omega)\right| > \left|\widehat{F}^2(\Omega)\right| > ... > \left|\widehat{F}^n(\Omega)\right| > ...$$

where $n \in \mathbb{N}$. This yields a contradiction with the fact that $\mathbb{N}$ is well-ordered. Thus we can consider the smallest $p \in \mathbb{N}$ having the property that $\widehat{F}^p(\Omega) = \{z\}$. To every $x \in \Omega \setminus \{z\}$ we associate $n_x = \max\{n \in \mathbb{N} \mid x \in \widehat{F}^n(\Omega)\} < p$. Moreover, we define $n_z = \infty$. Note that for $t \in F(x)$, we have $t \in \widehat{F}(\widehat{F}^{n_x}(\Omega)) = \widehat{F}^{n_x+1}(\Omega)$, so $n_t \geq n_x + 1$. Considering the function $\varphi : \Omega \to [0, \infty)$ given by $\varphi(x) = \alpha^{n_x}$ for every $x \in \Omega$, with the convention $\alpha^\infty = 0$, we have

$$\sup_{t\in F(x)} \varphi(t) = \sup_{t\in F(x)} \alpha^{n_t} \leq \sup_{t\in F(x)} \alpha^{n_x+1} = \alpha \cdot \alpha^{n_x} = \alpha\varphi(x)$$

for all $x \in \Omega$ and $\varphi^{-1}(\{0\}) = \{z\}$. Hence, the conclusion follows using Lemma 3.1. □

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Extended local convergence analysis of inexact Gauss-Newton method for singular systems of equations under weak conditions

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Abstract. A new local convergence analysis of the Gauss-Newton method for solving some optimization problems is presented using restricted convergence domains. The results extend the applicability of the Gauss-Newton method under the same computational cost given in earlier studies. In particular, the advantages are: the error estimates on the distances involved are tighter and the convergence ball is at least as large. Moreover, the majorant function in contrast to earlier studies is not necessarily differentiable. Numerical examples are also provided in this study.

Mathematics Subject Classification (2010): 65D10, 65D99, 65G99, 65K10, 90C30.

Keywords: Gauss-Newton method, local convergence, restricted convergence domains, majorant function, center-majorant function, convergence ball.

1. Introduction

In this study, we are concerned with the problem of approximating a solution of the equation

$$F(x) = 0, \tag{1.1}$$

where D is open and convex and $F : D \subset \mathbb{R}^j \to \mathbb{R}^m$ is a nonlinear operator with its Fréchet derivative denoted by F'. In the case $m = j$, the inexact Newton method (INM) was defined in [19] by:

$$x_{n+1} = x_n + s_n, \quad F'(x_n)s_n = -F(x_n) + r_n \quad \text{for each} \quad n = 0, 1, 2, \ldots, \tag{1.2}$$

where x_0 is an initial point, the residual control r_n satisfy

$$\|r_n\| \leq \lambda_n \|F(x_n)\| \quad \text{for each} \quad n = 0, 1, 2, \ldots, \tag{1.3}$$

and $\{\lambda_n\}$ is a sequence of forcing terms such that $0 \leq \lambda_n < 1$. Let x^* be a solution of (1.1) such that $F'(x^*)$ is invertible. As shown in [19], if $\lambda_n \leq \lambda < 1$, then, there exists $r > 0$ such that for any initial guess $x_0 \in U(x^*, r) := \{x \in \mathbb{R}^j : \|x - x^*\| < r\}$, the sequence $\{x_n\}$ is well defined and converges to a solution x^* in the norm $\|y\|_* := \|F'(x^*)y\|$, where $\|\cdot\|$ is any norm in $\mathbb{R}^j$. Moreover, the rate of convergence of $\{x_n\}$ to x^* is characterized by the rate of convergence of $\{\lambda_n\}$ to 0. It is worth noting that, in [19], no Lipschitz condition is assumed on the derivative F' to prove that $\{x_n\}$ is well defined and linearly converging. However, no estimate of the convergence radius r is provided. As pointed out by [16] the result of [19] is difficult to apply due to dependence of the norm $\|\cdot\|_*$, which is not computable.

In [41] Ypma used the affine invariant condition of residual control in the form:

$$\|F'(x_n)^{-1}r_n\| \leq \lambda_n\|F'(x_n)^{-1}F(x_n)\| \quad \text{for each} \quad n = 0, 1, 2, \ldots, \tag{1.4}$$

instead of (1.3) to study the local convergence of inexact Newton method (1.2). And the radius of convergent result are also obtained. Morini in [32] presented the following variation for the residual controls:

$$\|P_n r_n\| \leq \lambda_n\|P_n F(x_n)\| \quad \text{for each} \quad n = 0, 1, 2, \ldots, \tag{1.5}$$

where $\{P_n\}$ is a sequence of invertible operator from $\mathbb{R}^j$ to $\mathbb{R}^j$ and $\{\lambda_n\}$ is the forcing term. If $P_n = I$ and $P_n = F'(x_n)$ for each n, (1.5) reduces to (1.3) and (1.4), respectively.

Recently, several authors have studied the convergence behaviour of singular nonlinear systems by Gauss-Newton's method (GNM), which is defined by

$$x_{n+1} = x_n - F'(x_n)^\dagger F(x_n) \quad \text{for each} \quad n = 0, 1, 2, \ldots, \tag{1.6}$$

where $x_0 \in D$ is an initial point and $F'(x_n)^\dagger$ denotes the Moore-Penrose inverse of the linear operator (of matrix) $F'(x_n)$ [1, 12, 14, 15, 17, 18, 20, 21, 36].

In the present study, using the idea of restricted convergence domains, we provide a new local convergence analysis for GNM under the same computational cost and the following advantages: larger radius of convergence; tighter error estimates on the distances $\|x_n - x^*\|$ for each $n = 0, 1, \ldots$ and a clearer relationship between the majorant function (see (2.8) and the associated least squares problems (1.1)). These advantages are obtained because we use a center-type majorant condition (see (2.11)) for the computation of inverses involved which is more precise that the majorant condition used in [21, 22, 23, 24, 25, 26, 30, 31, 39, 40, 41, 42, 43]. Moreover, these advantages are obtained under the same computational cost, since as we will see in section 3 and section 4, the computation of the majorant function requires the computation of the center-majorant function. Furthermore, these advantages are very important in computational mathematics, since we have a wider choice of initial guesses x_0 and fewer computations to obtain a desired error tolerance on the distances $\|x_n - x^*\|$ for each $n = 0, 1, 2, \ldots$. Finally, the majorant functions (see ω and v) is not necessarily differentiable as in [21, 26, 30, 31, 39, 40, 41, 42, 43] but just differentiable. This is an improvement modification and extends the applicability of the method.

The rest of this study is structured as follows. In section 2, we introduce some preliminary notions and properties of the majorizing function. The main result about the local convergence are stated in section 3. In section 4, we prove the local convergence results given in section 3. Section 5 contains the numerical examples and section 6 the conclusion of this study.

2. Preliminaries

We present some standard results to make the study as self-contained as possible. More results can be found in [13, 28, 35].

Let $A : \mathbb{R}^j \to \mathbb{R}^m$ be a linear operator (or an $m \times j$ matrix). Recall that an operator (or $j \times m$ matrix) $A^\dagger : \mathbb{R}^m \to \mathbb{R}^j$ is the Moore-Penrose inverse of A if it satisfies the following four equations:

$$A^\dagger A A^\dagger = A^\dagger; \quad A A^\dagger A = A; \quad (AA^\dagger)^* = AA^\dagger; \quad (A^\dagger A) = A^\dagger A,$$

where A^* denotes the adjoint of A. Let $kerA$ and imA denote the kernel and image of A, respectively. For a subspace E of $\mathbb{R}^j$, we use Π_E to denote the projection onto E. Clearly, we have that

$$A^\dagger A = \Pi_{kerA^\perp} \quad \text{and} \quad AA^\dagger = \Pi_{imA}.$$

In particular, in the case when A is full row rank (or equivalently, when A is surjective), $AA^\dagger = I_{\mathbb{R}^m}$; when A is full column rank (or equivalently, when A is injective), $A^\dagger A = I_{\mathbb{R}^j}$.

The following lemma gives a Banach-type perturbation bound for Moore-Penrose inverse, which is stated in [25].

Lemma 2.1. ([25, Corollary 7.1.1 & Corollary 7.1.2]). *Let A and B be $m \times j$ matrices and let $r \leq \min\{m, j\}$. Suppose that $rankA = r$, $1 \leq rankB \leq A$ and $\|A^\dagger\|\|B - A\| < 1$. Then, $rankB = r$ and*

$$\|B^\dagger\| \leq \frac{\|A^\dagger\|}{1 - \|A^\dagger\|\|B - A\|}.$$

Also, we need the following useful lemma about elementary convex analysis.

Lemma 2.2. ([25, Proposition 1.3]). *Let $R > 0$. If $\varphi : [0, R] \to \mathbb{R}$ is continuously differentiable and convex, then, the following assertions hold:*

(a) $\dfrac{\varphi(t) - \varphi(\tau t)}{t} \leq (1 - \tau)\varphi'(t)$ *for each* $t \in (0, R)$ *and* $\tau \in [0, 1]$.

(b) $\dfrac{\varphi(u) - \varphi(\tau u)}{u} \leq \dfrac{\varphi(v) - \varphi(\tau v)}{v}$ *for each* $u, v \in [0, R)$, $u < v$ *and* $0 \leq \tau \leq 1$.

From now on we suppose that the (I) conditions listed below hold.

For a positive real $R \in \mathbb{R}^+$, let

$$\psi : [0, R] \times [0, 1) \times [0, 1) \to \mathbb{R}$$

be a continuous differentiable function of three of its arguments and satisfy the following properties:

(i) $\psi(0, \lambda, \theta) = 0$ and $\left.\dfrac{\partial}{\partial t}\psi(t, \lambda, \theta)\right|_{t=0} = -(1 + \lambda + \theta)$.

(ii) $\dfrac{\partial}{\partial t}\psi(t, \lambda, \theta)$ is convex and strictly increasing with respect to the argument t.

For fixed $\lambda, \theta \in [0, 1)$, we write $h_{\lambda,\theta}(t) \triangleq \psi(t, \lambda, \theta)$ for short below. Then the above two properties can be restated as follows.

(iii) $h_{\lambda,\theta}(0) = 0$ and $h'_{\lambda,\theta}(0) = -(1 + \lambda + \theta)$.

(*iv*) $h'_{\lambda,\theta}(t)$ is convex and strictly increasing.
(*v*) $\omega : [0, R] \longrightarrow \mathbb{R}$ is integrable, convex and strictly increasing with $\omega(0) = -1$.
(*vi*) $g : [0, R] \to \mathbb{R}$ is strictly increasing with $g(0) = 0$ and given by $g(t) = \int_0^t \omega(s)ds$.
(*vii*) $g(t) \leq h_{\lambda,\theta}(t)$, $\omega(t) \leq h'_{\lambda,\theta}(t)$ for each $t \in [0, R)$, λ, $\theta \in [0, 1]$.

Define

$$\zeta_0 := \sup\{t \in [0, R) : h'_{0,0}(t) < 0\}, \quad \zeta := \sup\{t \in [0, R) : \omega(t) < 0\}, \tag{2.1}$$

$$\begin{aligned} \rho_0 &:= \sup\left\{t \in [0, \zeta_0) : \left|\frac{h_{\lambda,\theta}(t)}{h'_{0,0}(t)} - t\right| < t\right\}, \\ \rho &= \sup\left\{t \in [0, \zeta) : \left|\frac{h_{\lambda,\theta}(t) - th'_{0,0}(t)}{\omega(t)}\right| < t\right\} \end{aligned} \tag{2.2}$$

$$\sigma := \sup\{t \in [0, R) : U(x^*, t) \subset D\}. \tag{2.3}$$

The next two lemmas show that the constants ζ and ρ defined in (2.1) and (2.2), respectively, are positive.

Lemma 2.3. *The constant ζ defined in (2.1) is positive and*

$$\frac{th'_{0,0}(t) - h_{\lambda,\theta}(t)}{\omega(t)} < 0$$

for each $t \in (0, \zeta)$.

Proof. Since $\omega(0) = -1$, there exists $\delta > 0$ such that $\omega(t) < 0$ for each $t \in (0, \delta)$. Then, we get $\zeta \geq \delta$ (> 0). We must show that $\dfrac{th'_{0,0}(t) - h_{\lambda,\theta}(t)}{\omega(t)} < 0$ for each $t \in (0, \zeta)$. By hypothesis, functions $h'_{\lambda,\theta}$, $\omega(t)$ are strictly increasing, then functions $h_{\lambda,\theta}$, $v(t)$ are strictly convex. It follows from Lemma 2.2 (*i*) and hypothesis (*vii*) that

$$\frac{h_{\lambda,\theta}(t) - h_{\lambda,\theta}(0)}{t} < h'_{\lambda,\theta}(t), \quad t \in (0, R).$$

In view of $h_{\lambda,\theta}(0) = 0$ and $\omega(t) < 0$ for all $t \in (0, \zeta)$. This together with the last inequality yields the desired inequality. □

Lemma 2.4. *The constant ρ defined in (2.2) is positive. Consequently,*

$$\left|\frac{th'_{0,0}(t) - h_{\lambda,\theta}(t)}{\omega(t)}\right| < t$$

for each $t \in (0, \rho)$.

Proof. Firstly, by Lemma 2.3, it is clear that $\left(\dfrac{h_{\lambda,\theta}(t)}{th'_{0,0}(t)} - 1\right)\dfrac{h'_{0,0}(t)}{\omega(t)} > 0$ for $t \in (0, \zeta)$. Secondly, we get from Lemma 2.2 (*i*) that

$$\lim_{t\to 0}\left(\frac{h_{\lambda,\theta}(t)}{th'_{0,0}(t)} - 1\right)\frac{h'_{0,0}(t)}{\omega(t)} = 0.$$

Hence, there exists a $\delta > 0$ such that

$$0 < \left(\frac{h_{\lambda,\theta}(t)}{t h'_{0,0}(t)} - 1 \right) \frac{h'_{0,0}(t)}{\omega(t)} < 1, \quad t \in (0, \zeta).$$

That is ρ is positive. □

Define

$$r := \min\{\rho, \delta\}, \tag{2.4}$$

where ρ and δ are given in (2.2) and (2.3), respectively.
For any starting point $x_0 \in U(x^*, r) \backslash \{x^*\}$, let $\{t_n\}$ be a sequence defined by:

$$t_0 = \|x_0 - x^*\|, \quad t_{n+1} = \left| \left(t_n - \frac{h_{\lambda,\theta}(t_n)}{h'_{0,0}(t_n)} \right) \frac{h'_{0,0}(t_n)}{\omega(t_n)} \right| \quad \text{for each} \quad n = 0, 1, 2, \ldots \tag{2.5}$$

Lemma 2.5. *The sequence $\{t_n\}$ given by (2.5) is well defined, strictly decreasing, remains in $(0, \rho)$ for each $n = 0, 1, 2, \ldots$ and converges to 0.*

Proof. Since $0 < t_0 = \|x_0 - x^*\| < r \le \rho$, using Lemma 2.4, we have that $\{t_n\}$ is well defined, strictly decreasing and remains in $[0, \rho)$ for each $n = 0, 1, 2, \ldots$ Hence, there exists $t^* \in [0, \rho)$ such that $\lim_{n \to +\infty} t_n = t^*$. That is, we have

$$0 \le t^* = \left(\frac{h_{\lambda,\theta}(t^*)}{h'_{0,0}(t^*)} - t^* \right) \frac{h'_{0,0}(t^*)}{\omega(t^*)} < \rho.$$

If $t^* \neq 0$, it follows from Lemma 2.4 that

$$\left(\frac{h_{\lambda,\theta}(t^*)}{h'_{0,0}(t^*)} - t^* \right) \frac{h'_{0,0}(t^*)}{\omega(t^*)} < t^*,$$

which is a contradiction. Hence, we conclude that $t_n \to 0$ as $n \to +\infty$. □

If $g(t) = h_{\lambda,\theta}(t)$, then Lemmas 2.3-2.5 reduce to the corresponding ones in [42, 43]. Otherwise, i. e., if $g(t) < h_{\lambda,\theta}(t)$, then our results are better, since

$$\zeta_0 < \zeta \quad \text{and} \quad \rho_0 < \rho.$$

Moreover, the scalar sequence used in [42, 43] is defined by

$$u_0 = \|x_0 - x^*\|, \quad u_{n+1} = \left| u_n - \frac{h_{\lambda,\theta}(u_n)}{h'_{0,0}(u_n)} \right| \quad \text{for each} \quad n = 0, 1, 2, \ldots \tag{2.6}$$

Using the properties of the functions $h_{\lambda,\theta}$, g, (2.5), (2.6) and a simple inductive argument we get that

$$t_0 = u_0, \quad t_1 = u_1, \quad t_n < u_n, \quad t_{n+1} - t_n < u_{n+1} - u_n \quad \text{for each} \quad n = 1, 2, \ldots$$

and

$$t^* \le u^* = \lim_{n \to +\infty} u_n,$$

which justify the advantages of our approach as claimed in the introduction of this study.

In Section 3 we shall show that $\{t_n\}$ is a majorizing sequence for $\{x_n\}$.

We state the following modified majorant condition for the convergence of various Newton-type methods in [10, 11, 12, 13].

Definition 2.6. Let $r > 0$ be such that $U(x^*, r) \subset D$. Then, F' is said to satisfy the majorant condition on $U(x^*, r)$ if

$$\|F'(x^*)^\dagger[F'(x) - F'(x^* + \tau(x - x^*))]\| \leq h'_{\lambda,\theta}(\|x - x^*\|) - h'_{\lambda,\theta}(\tau\|x - x^*\|) \quad (2.7)$$

for any $x \in U(x^*, r)$ and $\tau \in [0, 1]$.

In the case when $F'(x^*)$ is not surjective, the information on $imF'(x^*)^\perp$ may be lost. This is why the above notion was modified in [42, 43] to suit the case when $F'(x^*)$ is not surjective as follows:

Definition 2.7. Let $r > 0$ be such that $U(x^*, r) \subset D$. Then, F' is said to satisfy the modified majorant condition on $U(x^*, r)$, if

$$\|F'(x^*)^\dagger\|\|F'(x) - F'(x^* + \tau(x - x^*))\| \leq h'_{\lambda,\theta}(\|x - x^*\|) - h'_{\lambda,\theta}(\tau\|x - x^*\|) \quad (2.8)$$

for any $x \in U(x^*, r)$ and $\tau \in [0, 1]$.

If $\tau = 0$, condition (2.8) reduces to

$$\|F'(x^*)^\dagger\|\|F'(x) - F'(x^*)\| \leq h'_{\lambda,\theta}(\|x - x^*\|) - h'_{\lambda,\theta}(0). \quad (2.9)$$

In particular, for $\lambda = \theta = 0$, condition (2.9) reduces to

$$\|F'(x^*)^\dagger\|\|F'(x) - F'(x^*)\| \leq h'_{0,0}(\|x - x^*\|) - h'_{0,0}(0). \quad (2.10)$$

Condition (2.10) is used to produce the Banach-type perturbation Lemmas in [42, 43] for the computation of the upper bounds on the norms $\|F'(x)^\dagger\|$. In this study we use a more flexible function g than $h_{\lambda,\theta}$ function for the same purpose. This way the advantages as stated in the Introduction of this study can be obtained.

In order to achieve these advantages we introduce the following notion [2, 3, 7, 8, 4, 9, 5, 10, 11, 12].

Definition 2.8. Let $r > 0$ be such that $U(x^*, r) \subset D$. Then ω is said to satisfy the center-majorant condition on $U(x^*, r)$, if

$$\|F'(x^*)^\dagger\|\|F'(x) - F'(x^*)\| \leq \omega(\|x - x^*\|) - \omega(0). \quad (2.11)$$

Clearly,

$$\omega(t) \leq h'_{\lambda,\theta}(t) \quad \text{for each} \quad t \in [0, R], \quad \lambda, \theta \in [0, 1] \quad (2.12)$$

holds in general and $\dfrac{h'_{\lambda,\theta}(t)}{\omega(t)}$ can be arbitrarily large [2, 3, 7, 8, 4, 9, 5, 10, 11, 12].

It is worth noticing that (2.11) is not an additional condition to (2.8) since in practice the computation of function $h_{\lambda,\theta}$ requires the computation of g as a special case (see also the numerical examples).

3. Local convergence

In this section, we present local convergence for INM (1.2). Equation (1.1) is a surjective-undetermined (resp. injective-overdetermined) system if the number of equations is less (resp. greater) than the number of knowns and $F'(x)$ is of full rank for each $x \in D$. It is well known that, for surjective-underdetermined systems, the fixed points of the Newton operator $N_F(x) := x - F'(x)^\dagger F(x)$ are the zeros of F, while for injective-overdetermined systems, the fixed points of N_F are the least square solutions of (1.1), which, in general, are not necessarily the zeros of F.

We shall use the notation $D_0 = U(x^*, \xi)$ and $D = U(x^*, R)$ and set

$$D_1 = D_0 \cap U(x^*, r).$$

Next, we present the local convergence properties of INM for general singular systems with constant rank derivatives.

Theorem 3.1. *Let $F : D \subset \mathbb{R}^j \to \mathbb{R}^m$ be continuously Fréchet differentiable nonlinear operator and D is open and convex. Suppose that $F(x^*) = 0$, $F'(x^*) \neq 0$ and that F' satisfies the modified majorant condition (2.8) on D_1 and the center-majorant condition (2.11) on D, where r is given in (2.4). In addition, we assume that $\text{rank} F'(x) \leq \text{rank} F'(x^*)$ for any $x \in U(x^*, r)$ and that*

$$\|[I_{\mathbb{R}^j} - F'(x)^\dagger F'(x)](x - x^*)\| \leq \theta \|x - x^*\|, \quad x \in U(x^*, r), \tag{3.1}$$

where the constant θ satisfies $0 \leq \theta < 1$. Let sequence $\{x_n\}$ be generated by INM with any initial point $x_0 \in U(x^*, r) \backslash \{x^*\}$ and the conditions for the residual r_n and the forcing term λ_n:

$$\|r_n\| \leq \lambda_n \|F(x_n)\|, \quad 0 \leq \lambda_n F'(x_k) \leq \lambda \quad \textit{for each} \quad n = 0, 1, 2, \dots . \tag{3.2}$$

Then, sequence $\{x_n\}$ converges to x^* so that $F'(x^*)^\dagger F(x^*) = 0$. Moreover, we have the following estimate:

$$\|x_{n+1} - x^*\| \leq \frac{t_{n+1}}{t_n} \|x_n - x^*\| \quad \textit{for each} \quad n = 0, 1, 2, \dots, \tag{3.3}$$

where the sequence $\{t_n\}$ is defined by (2.5).

Remark 3.2. (a) If $g(t) = h_{\lambda,\theta}(t)$, then the results obtained in Theorem 3.1 reduce to the ones given in [42, 43].

(b) If $g(t)$ and $h_{\lambda,\theta}(t)$ are

$$g(t) = h_{\lambda,\theta}(t) = -(1 + \lambda + \theta)t + \int_0^t L(u)(t - u)\, du, \quad t \in [0, R], \tag{3.4}$$

then the results obtained in Theorem 3.1 reduce to the one given in [25]. Moreover, if taking $\lambda = 0$ (in this case $\lambda_n = 0$ and $r_n = 0$) in Theorem 3.1, we obtain the local convergence of Newton's method for solving the singular systems, which has been studied by Dedieu and Kim in [17] for analytic singular systems with constant rank derivatives and Li, Xu in [39] and Wang in [38] for some special singular systems with constant rank derivatives.

(c) If $g(t) < h_{\lambda,\theta}(t)$ then the improvements as mentioned in the Introduction of this study we obtained (see also the discussion above and below Definition 2.6)

If $F'(x)$ is full column rank for every $x \in U(x^*, r)$, then we have $F'(x)^\dagger F'(x) = I_{\mathbb{R}^j}$. Thus,

$$\|[I_{\mathbb{R}^m} - F'(x)^\dagger F'(x)](x - x^*)\| = 0,$$

i. e., $\theta = 0$. We immediately have the following corollary:

Corollary 3.3. *Suppose that* $rank F'(x) \leq rank F'(x^*)$ *and that*

$$\|[I_{\mathbb{R}^m} - F'^\dagger(x)F'(x)](x - x^*)\| = 0,$$

for any $x \in U(x^*, r)$. *Suppose that* $F(x^*) = 0$, $F'(x^*) \neq 0$ *and that* F' *satisfies the modifed majorant condition (*2.8*) on* D_1 *and the center-majorant condition (*2.11*) on* D. *Let sequence* $\{x_n\}$ *be generated by IGNM with any initial point* $x_0 \in U(x^*, r)\backslash\{x^*\}$ *and the condition (*3.2*) for the residual* r_n *and the forcing term* λ_n. *Then, sequence* $\{x_n\}$ *converges to* x^* *so that* $F'(x^*)^\dagger F(x^*) = 0$. *Moreover, we have the following estimate:*

$$\|x_{n+1} - x^*\| \leq \frac{t_{n+1}}{t_n}\|x_n - x^*\| \quad \textit{for each} \quad n = 0, 1, 2, \ldots, \tag{3.5}$$

where the sequence $\{t_n\}$ *is defined by (*2.5*) for* $\theta = 0$.

In the case when $F'(x^*)$ is full row rank, the modified majorant condition (2.8) can be replaced by the majorant condition (2.7).

Theorem 3.4. *Suppose that* $F(x^*) = 0$, $F'(x^*)$ *is full row rank, and that* F' *satisfies the majorant condition (*2.7*) on* D_1 *and the center-majorant condition (*2.11*) on* D, *where* r *is given in (*2.4*). In addition, we assume that* $rank F'(x) \leq rank F'(x^*)$ *for any* $x \in U(x^*, r)$ *and that condition (*3.1*) holds. Let sequence* $\{x_n\}$ *be generated by IGNM with any initial point* $x_0 \in U(x^*, r)\backslash\{x^*\}$ *and the conditions for the residual* r_n *and the forcing term* λ_n:

$$\|F'(x^*)^\dagger r_n\| \leq \lambda_n \|F'(x^*)^\dagger F(x_n)\|, 0 \leq \lambda_n F'(x^*)^\dagger F'(x_n) \leq \lambda \textit{ for each } n = 0, 1, 2, \ldots. \tag{3.6}$$

Then, sequence $\{x_n\}$ *converges to* x^* *so that* $F'(x^*)^\dagger F(x^*) = 0$. *Moreover, we have the following estimate:*

$$\|x_{n+1} - x^*\| \leq \frac{t_{n+1}}{t_n}\|x_n - x^*\| \quad \textit{for each} \quad n = 0, 1, 2, \ldots,$$

where the sequence $\{t_n\}$ *is defined by (*2.5*).*

Remark 3.5. Comments as in Remark 3.2 can follow for this case.

Theorem 3.6. *Suppose that* $F(x^*) = 0$, $F'(x^*)$ *is full row rank, and that* F' *satisfies the majorant condition (*2.7*) on* D_1 *and the center-majorant condition on* D, *where* r *is given in (*2.4*). In addition, we assume that* $rank F'(x) \leq rank F'(x^*)$ *for any* $x \in U(x^*, r)$ *and that condition (*3.1*) holds. Let sequence* $\{x_n\}$ *generated by IGNM with any initial point* $x_0 \in U(x^*, r)\backslash\{x^*\}$ *and the conditions for the control residual* r_n *and the forcing term* λ_n:

$$\|F'(x_n)^\dagger r_n\| \leq \lambda_n \|F'(x_n)^\dagger F(x_n)\|, \quad 0 \leq \lambda_n F'(x_n) \leq \lambda \quad \textit{for each} \quad n = 0, 1, 2, \ldots \tag{3.7}$$

Then, sequence $\{x_n\}$ *converges to* x^* *so that* $F'(x^*)^\dagger F(x^*) = 0$. *Moreover, we have the following estimate:*

$$\|x_{n+1} - x^*\| \leq \frac{t_{n+1}}{t_n}\|x_k - x^*\| \quad \textit{for each} \quad n = 0, 1, 2, \ldots,$$

where sequence $\{t_n\}$ *is defined by* (2.5).

Remark 3.7. In the case when $F'(x^*)$ is invertible in Theorem 3.6, $h_{\lambda,\theta}$ is given by (3.4) and $g(t) = -1 + \int_0^t L_0(t)(t-u)\,du$ for each $t \in [0, R]$, we obtain the local convergence results of IGNM for nonsingular systems, and the convergence ball r is this case satisfies

$$\frac{\int_0^r L(u)u\,du}{r\left((1-\lambda) - \int_0^r L_0(u)\,du\right)} \leq 1, \quad \lambda \in [0, 1). \tag{3.8}$$

In particular, if taking $\lambda = 0$, the convergence ball r determined in (3.8) reduces to the one given in [38] by Wang and the value r is the optimal radius of the convergence ball when the equality holds. That is our radius is r larger than the one obtained in [38], if $L_0 < L$ (see also the numerical examples). Notice that L is used in [38] for the estimate (3.8). Then, we can conclude that vanishing residuals, Theorem 3.6 merges into the theory of Newton's method.

4. Proofs

In this section, we prove our main results of local convergence for inexact Gauss-Newton method (1.2) given in Section 3.

4.1. Proof of Theorem 3.1

Lemma 4.1. *Suppose that* F' *satisfies the modified majorant condition on* $U(x^*, r)$ *and that* $\|x^* - x\| < \min\{\rho, x^*\}$, *where* r, ρ *and* x^* *are defined in* (2.4), (2.2) *and* (2.1), *respectively. Then,* $rankF'(x) = rankF'(x^*)$ *and*

$$\|F'(x)^\dagger\| \leq -\frac{\|F'(x^*)^\dagger\|}{\omega(\|x - x^*\|)}.$$

Proof. Since $\omega(0) = -1$, we have

$$\|F'(x^*)^\dagger\|\|F'(x) - F'(x^*)\| \leq \omega(\|x - x^*\|) - \omega(0) < -\omega(0) = 1.$$

It follows from Lemma (2.1) that $rankF'(x) = rankF'(x^*)$ and

$$\|F'(x)^\dagger\| \leq \frac{\|F'(x^*)^\dagger\|}{1 - (\omega(\|x - x^*\|) - \omega(0))} = -\frac{\|F'(x^*)^\dagger\|}{\omega(\|x - x^*\|)}. \qquad \square$$

Proof of Theorem 3.1. We shall prove by mathematical induction on n that $\{t_n\}$ is the majorizing sequence for $\{x_n\}$, i. e.,

$$\|x^* - x_j\| \leq t_j \quad \text{for each} \quad j = 0, 1, 2, \ldots \tag{4.1}$$

Because $t_0 = \|x_0 - x^*\|$, thus (4.1) holds for $j = 0$. Suppose that $\|x^* - x_j\| \le t_j$ for some $j = n \in \mathbb{N}$. For the case $j = n+1$, we first have that,

$$\begin{aligned} x_{n+1} - x^* &= x_n - x^* - F'(x_n)^\dagger [F(x_n) - F(x^*)] + F'(x_n)^\dagger r_n \\ &= F'(x_n)^\dagger [F(x^*) - F(x_n) - F'(x_n)(x^* - x_n)] + F'(x_n)^\dagger r_n \\ &\quad + [I_{\mathbb{R}^j} - F'(x_n)^\dagger F'(x_n)](x_n - x^*) \\ &= F'(x_n)^\dagger \int_0^1 [F'(x_n) - F'(x^* + \tau(x_n - x^*))](x_n - x^*)\, d\tau \\ &\quad + F'(x_n)^\dagger r_n + [I_{\mathbb{R}^j} - F'(x_n)^\dagger F'(x_n)](x_n - \zeta). \end{aligned} \tag{4.2}$$

By using the modified majorant condition (2.8), Lemma 2.4, the inductive hypothesis (4.1) and Lemma 2.2, we obtain in turn that

$$\begin{aligned} &\left\| F'(x_n)^\dagger \int_0^1 [F'(x_n) - F'(x^* + \tau(x_n - x^*))](x_n - x^*)\, d\tau \right\| \\ \le\ & -\frac{1}{\omega(\|x_n - x^*)\|} \int_0^1 \|F'(x^*)^\dagger\| \|F'(x_n) - F'(x^* + \tau(x_n - x^*))\| \|x_n - x^*\|\, d\tau \\ =\ & -\frac{1}{\omega(\|x_n - x^*\|)} \int_0^1 \frac{h'_{\lambda,0}(\|x_n - x^*\|) - h'_{\lambda,0}(\tau\|x_n - x^*\|)}{\|x_n - x^*\|}\, d\tau \cdot \|x_n - x^*\|^2 \\ \le\ & -\frac{1}{\omega(t_n)} \int_0^1 \frac{h'_{\lambda,0}(t_n) - h_{\lambda,0}(\tau t_n)}{t_n}\, d\tau \cdot \|x_n - x^*\|^2 \\ =\ & -\frac{1}{\omega(t_n)} (t_n h'_{\lambda,0}(t_n) - h_{\lambda,0}(t_n)) \frac{\|x_n - x^*\|^2}{t_n^2}. \end{aligned}$$

In view of (3.2),

$$\|F'(x_n)^\dagger r_n\| \le \|F'(x_n)^\dagger\| \|r_n\| \le \lambda_n \|F'(x_n)^\dagger\| \|F(x_n)\|. \tag{4.3}$$

We have that

$$\begin{aligned} -F(x_n) &= F(x^*) - F(x_n) - F'(x_n)(x^* - x_n) + F'(x_n)(x^* - x_n) \\ &= \int_0^1 [F'(x_n) - F'(x^* + \tau(x_n - x^*))](x_n - x^*)\, d\tau \\ &\quad + F'(x_n)(x^* - x_n). \end{aligned} \tag{4.4}$$

Then, combining Lemma 2.2, Lemma 4.1, the modified majorant condition (2.8), the inductive hypothesis (4.1) and the condition (3.2), we obtain in turn that

$$\begin{aligned} &\lambda_n \|F'(x_n)^\dagger\| \|F(x_n)\| \\ \le\ & \lambda_n \|F'(x_n)^\dagger\| \int_0^1 \|F'(x_n) - F'(x^* + \tau(x_n - x^*))\| \|x_n - x^*\|\, d\tau \\ & + \lambda_n \|F'(x_n)^\dagger\| \|F'(x_n)\| \|x_n - x^*\| \\ \le\ & -\frac{\lambda}{\omega(t_n)} (t_n h'_{\lambda,0}(t_n) - h_{\lambda,0}(t_n)) \frac{\|x_n - x^*\|^2}{t_n^2} + \lambda t_n \frac{\|x_n - x^*\|}{t_n} \\ \le\ & \lambda \frac{\lambda t_n + h_{\lambda,0}(t_n)}{\omega(t_n)} \frac{\|x_n - x^*\|}{t_n}. \end{aligned} \tag{4.5}$$

Combining (3.1), (4.3), (4.3) and (4.5), we get that

$$\begin{aligned}\|x_{n+1}-x^*\| &\leq \left[-\frac{t_n h'_{\lambda,0}(t_n)-h_{\lambda,0}(t_n)}{\omega(t_n)}+\lambda\frac{\lambda t_n+h_{\lambda,0}(t_n)}{\omega(t_n)}+\theta t_n\right]\frac{\|x_n-x^*\|}{t_n}\\ &= \left[-t_n+(1+\lambda)\left(\frac{\lambda t_n}{\omega(t_n)}+\frac{h_{\lambda,0}(t_n)}{\omega(t_n)}\right)+\theta t_n\right]\frac{\|x_n-x^*\|}{t_n}\cdot\end{aligned}$$

But, we have that $-1<\omega(t)<0$ for any $t\in(0,\rho)$, so

$$(1+\lambda)\left(\frac{\lambda t_n}{\omega(t_n)}+\frac{h_{\lambda,0}(t_n)}{\omega(t_n)}\right)+\theta t_n \leq \frac{h_{\lambda,0}(t_n)}{\omega(t_n)}+\theta_n \leq \frac{h_{\lambda,0}(t_n)-\theta t_n}{\omega(t_n)} = \frac{h_{\lambda,\theta}(t_n)}{\omega(t_n)}\cdot$$

Using the definition of $\{t_n\}$ given in (2.5), we get that

$$\|x_{n+1}-x^*\| \leq \frac{t_{n+1}}{t_n}\|x_n-x^*\|,$$

so we deduce that $\|x_{n+1}-x^*\|\leq t_{n+1}$, which completes the induction. In view of the fact that $\{t_n\}$ converges to 0 (by Lemma 2.5), it follows from (4.1) that $\{x_n\}$ converges to x^* and the estimate (3.3) holds for all $n\geq 0$. □

4.2. Proof of Theorem 3.4

Lemma 4.2. *Suppose that $F(x^*)=0$, $F'(x^*)$ is full row rank and that F' satisfies the majorant condition (2.7) on D_1. Then, for each $x\in U(x^*,r)$, we have $rankF'(x)=rankF'(x^*)$ and*

$$\|[I_{\mathbb{R}^j}-F'(x^*)^\dagger(F'(x^*)-F'(x))]^{-1}\|\leq -\frac{1}{\omega(\|x-x^*\|)}\cdot$$

Proof. Since $\omega(0)=-1$, we have

$$\|F'(x^*)^\dagger[F'(x)-F'(x^*)]\|\leq \omega(\|x-x^*\|)-\omega(0)<-\omega(0)=1.$$

It follows from Banach lemma that $[I_{\mathbb{R}^j}-F'(x^*)^\dagger(F'(x^*)-F'(x))]^{-1}$ exists and

$$\|[I_{\mathbb{R}^j}-F'(x^*)^\dagger(F'(x^*)-F'(x))]^{-1}\|\leq -\frac{1}{\omega(\|x-x^*\|)}\cdot$$

Since $F'(x^*)$ is full row rank, we have $F'(x^*)F'(x^*)^\dagger=I_{\mathbb{R}^m}$ and

$$F'(x)=F'(x^*)[I_{\mathbb{R}^j}-F'(z^*)^\dagger(F'(x^*)-F'(x))],$$

which implies that $F'(x)$ is full row, i. e., $rankF'(x)=rankF'(x^*)$. □

Proof of Theorem 3.4. Let $\widehat{F}:U(x^*,r)\to\mathbb{R}^m$ be defined by

$$\widehat{F}(x)=F'(x^*)^\dagger\widehat{F}(x),\quad x\in U(x^*,r),$$

with residual $\widehat{r}_k=F'(x^*)^\dagger r_n$. In view of

$$\widehat{F}'(x)^\dagger=[F'(x^*)^\dagger F'(x)]^\dagger=F'(x)^\dagger F'(x^*),\quad x\in U(x^*,r),$$

we have that $\{x_n\}$ coincides with the sequence generated by inexact Gauss-Newton method (1.2) for $\widehat{F}$. Moreover, we get that

$$\widehat{F}'(x^*)^\dagger=(F'(x^*)^\dagger F'(x^*))^\dagger=F'(x^*)^\dagger F'(x^*).$$

Consequently,

$$\|\widehat{F}'(x^*)^\dagger \widehat{F}'(x^*)\| = \|F'(x^*)^\dagger F'(x^*)F'(x^*)^\dagger F(x^*)\| = \|F'(x^*)^\dagger F(x^*)\|.$$

Because $\|F'(x^*)^\dagger F(x^*)\| = \|\Pi_{kerF'(x^*)^\perp}\| = 1$, thus, we have

$$\|\widehat{F}'(x^*)^\dagger\| = \|\widehat{F}'(x^*)^\dagger \widehat{F}'(x^*)\| = 1.$$

Therefore, by (2.7), we can obtain that

$$\begin{aligned}\|\widehat{F}'(x^*)^\dagger\|\|\widehat{F}'(x) - \widehat{F}'(x^* + \tau(x - x^*))\| &= \|F'(x^*)^\dagger(F'(x) - F'(x^* + \tau(x - x^*)))\| \\ &\leq h'_{\lambda,\theta}(\|x - x^*\|) - h_{\lambda,\theta}(\tau\|x - x^*\|).\end{aligned}$$

Hence, $\widehat{F}$ satisfies the modified majorant condition (2.8) on D_1. Then, Theorem 3.1 is applicable and $\{x_k\}$ converges to x^* follows. Note that, $\widehat{F}'(\cdot)^\dagger \widehat{F}(\cdot) = F'(\cdot)^\dagger F(\cdot)$ and $F(\cdot) = F'(\cdot)F'(\cdot)^\dagger F(\cdot)$. Hence, we conclude that x^* is a zero of F. □

4.3. Proof of Theorem 3.6

Lemma 4.3. *Suppose that $F(x^*) = 0$, $F'(x^*)$ is full row rank and that F' satisfies the majorant condition (2.7) on D_1. Then, we have*

$$\|F'(x)^\dagger F'(x^*)\| \leq -\frac{1}{\omega(\|x - x^*\|)} \quad \textit{for each} \quad x \in D_1.$$

Proof. Since $F'(x^*)$ is full row rank, we have $F'(x^*)F'(x^*)^\dagger = I_{R^m}$. Then, we get that

$$F'(x)^\dagger F'(x^*)(I_{\mathbb{R}^j} - F'(x^*)^\dagger(F'(x^*) - F'(x^*))) = F'(x)^\dagger F'(x), \quad x \in D_1.$$

By Lemma 4.2, $I_{\mathbb{R}^j} - F'(x^*)^\dagger(F'(x^*) - F'(x))$ is invertible for any $x \in D_1$. Thus, in view of the equality $A^\dagger A = \Pi_{kerA^\perp}$ for any $m \times j$ matrix A, we obtain that

$$F'(x)^\dagger F'(x^*) = \Pi_{kerF'(x)^\perp}[I_{\mathbb{R}^j} - F'(x^*)^\dagger(F'(x^*) - F'(x))]^{-1}.$$

Therefore, by Lemma 4.2 we deduce that

$$\begin{aligned}\|F'(x)^\dagger F'(x^*)\| &\leq \|\Pi_{kerF'(x)^\perp}\|\|[I_{\mathbb{R}^j} - F'(x^*)^\dagger(F'(x^*) - F'(x))]^{-1}\| \\ &\leq -\frac{1}{\omega(\|x - x^*\|)}.\end{aligned}$$

□

Proof of Theorem 3.6 Using Lemma 4.3, majorant condition (2.7) and the residual condition (3.7), respectively, instead of Lemma 4.1, modified majorant condition (2.8) and condition (3.2), one can complete the proof of Theorem 3.6 in an analogous way to the proof of Theorem 3.1. □

Remark 4.4. The results in [6] improved the corresponding ones in [21, 22, 23, 24, 25, 42, 43]. In the present study, we improved the results in [6], since $D_1 \subset U(x^*, r)$ leading to an at least as tight function $h'_{\lambda,\theta}$ than the one used in [6] (see also the Examples).

5. Numerical examples

We present some numerical examples, where

$$g(t) < h_{\lambda,\theta}(t) \tag{5.1}$$

and

$$\omega(t) < h'_{\lambda,\theta}(t). \tag{5.2}$$

For simplicity we take $F'(x)^\dagger = F'(x)^{-1}$ for each $x \in D$.

Example 5.1. Let $X = Y = (-\infty, +\infty)$ and define function $F : X \to Y$ by

$$F(x) = d_0 x - d_1 \sin(1) + d_2 \sin(e^{d_2 x})$$

where d_0, d_1, d_2 are given real numbers. Then $x^* = 0$. Define functions g and $h_{\lambda,\theta}$ by

$$g(t) = \frac{L_0}{2}t^2 - t \quad \text{and} \quad h_{\lambda,\theta}(t) = \frac{L}{2}t^2 - t.$$

Then, it can easily be seen that for d_2 sufficiently large and d_1 sufficiently small $\frac{L}{L_0}$ can be arbitrarily large. Hence, (5.1) and (5.2) hold.

Example 5.2. Let $F(x, y, z) = 0$ be a nonlinear system, where $F : D = U(0, 1) \subseteq \mathbb{R}^3 \to \mathbb{R}^3$ and $F(x, y, z) = \left(x, \frac{e-1}{2}y^2 + y, \mathrm{e}^z - 1\right)^T$. It is obvious that $(0, 0, 0)^T = \overline{x}^*$ is a solution of the system.

From F, we deduce

$$F'(\overline{x}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & (e-1)y & 0 \\ 0 & 0 & \mathrm{e}^z \end{pmatrix} \quad \text{and} \quad F'(\overline{x}^*) = \mathrm{diag}\{1, 1, 1\},$$

where $\overline{x} = (x, y, z)^T$. Hence, $[F'(\overline{x}^*)]^{-1} = \mathrm{diag}\{1, 1, 1\}$. Moreover, we can define for $L_0 = e - 1 < L = 1.78957239$, $g(t) = \frac{e-1}{2}t^2 - t$ and $h_{\lambda,\theta}(t) = \frac{1.78957239}{2}t^2 - t$. Then, again (5.1) and (5.2) hold.

Notice also that in [6] we used $L = e$ and $\bar{h}_{\lambda,\theta}(t) = \frac{e}{2}t^2 - t > h_{\lambda,\theta}(t)$. Hence, the present results improve the ones in [6].

Example 5.3. Let us consider the nonlinear least-squares problem

$$\min_{x \in \mathbb{R}} Q(x), \tag{5.3}$$

where $Q(x) = \frac{1}{2}F(x)^T F(x)$, and

$$F(x) = \begin{pmatrix} \frac{\mu}{2}x^2 - x + \mu_1 \\ \frac{\mu}{2}x^2 - x + \mu_2 \end{pmatrix}$$

with $\mu \neq 0, \mu_1, \mu_2$ being real parameters not all zero at the same time. If $\tilde{x}$ is a solution of (5.3), then $\tilde{x}$ is a solution of

$$\begin{aligned} \bigtriangledown Q(x) &= F'(x)^T F(x) = (1 - \mu x, 1 - \mu x)^T F(x) \\ &= (1 - \mu x)(\mu x^2 - 2x + \mu_1 + \mu_2). \end{aligned}$$

To obtain a global minimizer $\tilde{x}$ we must find the solutions of $\bigtriangledown Q(x) = 0$. Suppose that $\mu(\mu_1 + \mu_2) < 1$. Then, $\bigtriangledown Q(x) = 0$ has three distinct and positive solutions defined by $\frac{1}{\mu}$,

$$s_- = \frac{1 - \sqrt{1 - \mu(\mu_1 + \mu_2)}}{\mu} \quad \text{and} \quad s_+ = \frac{1 + \sqrt{1 - \mu(\mu_1 + \mu_2)}}{\mu}.$$

We have that $F'(x) = (1 - \mu x, 1 - \mu x)^T$. If $x = x^*$, then $F'(x^*)^\dagger = (0, 0)^T$ and

$$F'(x^*)^\dagger = \left(\frac{1}{2(1 - \mu x)}, \frac{1}{2(1 - \mu x)} \right)$$

for $x \neq \frac{1}{\mu}$ is the Moore-Penrose inverse of $F'(x)$. Having defined the Moore-Penrose inverse of $F'(x)$, we can now find the majorant functions along the lines of Example 5.3. We leave the details of the motivated reader.

Other examples can be found in [2, 8, 5, 10, 12].

6. Conclusion

We expanded the applicability of INM under a majorant and a center-majorant condition. The advantages of our analysis over earlier works such as [8, 9, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43] are also shown under the same computational cost for the functions and constants involved. These advantages include: a large radius of convergence and more precise error estimates on the distances $\|x_{n+1} - x^*\|$ for each $n = 0, 1, 2, \ldots$, leading to a wider choice of initial guesses and computation of less iterates x_n in order to obtain a desired error tolerance. Moreover, the differentiability of majorant function ω is not assumed as in earlier studies where $\omega = g'$ for some differentiable function g. Numerical examples show that the center-function can be smaller than the majorant function.

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